# The Lanczos Algorithm

Shane Harding

February 18, 2015

## Outline of talk

1. Introduction

2. The Algorithm

3. Parallel Lanczos Algorithm

4. Applications

5. Questions

1 Introduction

2 The Algorithm

3 Parallel Lanczos Algorithm

4 Applications

5 Questions

## Introduction

The Lanczos Algorithm is a very efficient method of finding
'extreme' eigenvalues of symmetric matrices.
By extreme, we mean the eigenvalues with the largest and smallest
ablsoute values.

## What is an Eigenvalue

If we consider the matrix equation: $Ax = b$. Then $x$ is an eigenvector if, and only if, applying $A$ to $x$ yields a scalar multiple of $x$. The factor $x$ is rescaled by is called the eigenvalue and is denoted $\lambda$.

$$Ax = \lambda x$$

## Brief History

The Lanczos algorithm was developed by Cornelius Lanczos. He developed the method while working at the National Bureau of Standards in Washington DC. It was then forgotten for a number of years due to the lack of computers meaning that it couldn't be properly utilised. When it was then "rediscovered" it was modified multiple times to fix some instability issues.

## Brief History

Lanczos later joined the Institute of Numerical Analysis (INA).
While working there he entered correspondence with Erwin
Schrödinger, who was the director of the Dublin Institute of
Advanced Studies (DIAS) at the time. Schroödinger gave him one
year visiting professorship at DIAS. In 1954 Eamon de Valera
invited Lanczos to be a senior professor in the School of
Theoretical Physics.

1 Introduction

2 The Algorithm

3 Parallel Lanczos Algorithm

4 Applications

5 Questions

The Lanczos Algorithm is used to solve large scale eigenvalue problems. So give a large $n \times n$ matrix $A$ we have:

## Eigenvalues and eigenvectors

$$Av_i = \lambda_i v_i$$

Where the vectors $v_i$ are the eigenvectors and the scalars $\lambda_i$ are the eigenvalues.

## Power Iteration

- The Lanczos algorithm is an adaptation of the power method.

- The power methods (or power iteration or Von Mises iteration) is an eigenvalue algorithm.

- This algorithm will only produce one eigenvalue and may converge slowly.

- It finds the eigenvalue with the greatest absolute value.

## Power Iteration

The power method can be summarised as follows: If $x_0$ is some random vector and $x_{n+1} = Ax_n$ then if we consider the limit of $n$ being large we find that $\frac{x_n}{||x_n||}$ approaches the normed eigenvector with the greatest eigen value.

- Then if $A = Udiag(\sigma_i)U'$ is the eigendeecomposition of $A$ then $A^n = Udiag(\sigma_i^n)U'$.
- As n gets big this will be dominated by the largest eigenvalue

## The Lanczos Algorithm

- We wish to find the eigenvalues of the matrix $A$.
- Where $A \in \mathbb{R}^{N \times N}$ and $v_1 \in \mathbb{R}^N$ be a random vector, with norm one.
- We will calculate the tridiagonal, symmetric matrix $T_{mm} = V_m^* A V_m$. With the diagonal elements denoted $\alpha_i = t_{ii}$ and the off diagonal elements given by $\beta_i = t + i - 1, i$. And note that $t_{i-1,i} = t_{i,i-1}$ due to symmetry.

## The Lanczos Algorithm

```
1. v_1 = random vec; v_0 = 0; beta_1 = 0;
2. for i = 1,...,m -1do
     w_i = A v_i
     alpha_i = (r_i,v_i)
     w_i = w_i - alpha_i v_i - beta_i v_{i=1}
     beta_{i+1} = abs(w_i)
     v_{i+1} = w_i/beta_{i+1}
   endfor
3. w_m = A v_m
   alpha_m = (w_m,v_m)
   return
```

Where $(x, y)$ represents the dot product of $x$ and $y$.
$m$ is chosen by the user. It is the number of eigenvalues that want
to be obtained.

## The Lanczos Algorithm

The previous routine yields the matrix:

$$T_{mm} = \begin{pmatrix} \alpha_1 & \beta_2 & & 0 \\ \beta_2 & \ddots & \ddots & \\ & \ddots & \ddots & \beta_m \\ 0 & & \beta_{m,} & \alpha_m \end{pmatrix}$$

The vectors $v_i$ are called the Lanczos vectors. They are then used to construct the transformation matrix $V_m = (v_1, v_2, \ldots, v_m)$. This is very useful for computing the eigenvectors.

## The Lanczos Algorithm

Once $T_{mm}$ is obtained, its eigenvalues $\lambda(m)_i$ and their matching eigenvectors $u(m)_i$ can be calculated. There are a few commonly used methods to do this. Two examples are: multiple relatively robust representations (MRRR); and QR algorithm.

It can be proved that the eigenvalues of $T_{mm}$ are approximate eigenvalues of the original matrix $A$.

1 Introduction

2 The Algorithm

3 Parallel Lanczos Algorithm

4 Applications

5 Questions

1. **Introduction**

2. **The Algorithm**

3. **Parallel Lanczos Algorithm**

4. **Applications**

5. **Questions**

Questions?

Thank you!