

The Lanczos Algorithm

Shane Harding

February 18, 2015

Outline of talk

- 1 Introduction
- 2 The Algorithm
- 3 Parallel Lanczos Algorithm
- 4 Applications
- 5 Questions

- 1 Introduction
- 2 The Algorithm
- 3 Parallel Lanczos Algorithm
- 4 Applications
- 5 Questions

Introduction

The Lanczos Algorithm is a very efficient method of finding
‘extreme’ eigenvalues of symmetric matrices.

By extreme, we mean the eigenvalues with the largest and smallest
absolute values.

What is an Eigenvalue

If we consider the matrix equation: $Ax = b$. Then x is an eigenvector if, and only if, applying A to x yields a scalar multiple of x . The factor x is rescaled by is called the eigenvalue and is denoted λ .

$$Ax = \lambda x$$

Brief History

The Lanczos algorithm was developed by Cornelius Lanczos. He developed the method while working at the National Bureau of Standards in Washington DC. It was then forgotten for a number of years due to the lack of computers meaning that it couldn't be properly utilised. When it was then “rediscovered” it was modified multiple times to fix some instability issues.

Brief History

Lanczos later joined the Institute of Numerical Analysis (INA). While working there he entered correspondence with Erwin Schrödinger, who was the director of the Dublin Institute of Advanced Studies (DIAS) at the time. Schrödinger gave him one year visiting professorship at DIAS. In 1954 Eamon de Valera invited Lanczos to be a senior professor in the School of Theoretical Physics.

- 1 Introduction
- 2 The Algorithm
- 3 Parallel Lanczos Algorithm
- 4 Applications
- 5 Questions

The Lanczos Algorithm is used to solve large scale eigenvalue problems. So give a large $n \times n$ matrix A we have:

Eigenvalues and eigenvectors

$$Av_i = \lambda_i v_i$$

Where the vectors v_i are the eigenvectors and the scalars λ_i are the eigenvalues.

Power Iteration

- The Lanczos algorithm is an adaptation of the power method.
- The power methods (or power iteration or Von Mises iteration) is an eigenvalue algorithm.
- This algorithm will only produce one eigenvalue and may converge slowly.
- It finds the eigenvalue with the greatest absolute value.

Power Iteration

The power method can be summarised as follows: If x_0 is some random vector and $x_{n+1} = Ax_n$ then if we consider the limit of n being large we find that $\frac{x_n}{\|x_n\|}$ approaches the normed eigenvector with the greatest eigen value.

- Then if $A = U \text{diag}(\sigma_i) U'$ is the eigendecomposition of A then $A^n = U \text{diag}(\sigma_i^n) U'$.
- As n gets big this will be dominated by the largest eigenvalue

The Lanczos Algorithm

We wish to find the eigenvalues of the matrix A .

Where $A \in \mathbb{R}^{N \times N}$. Let $v_1 \in \mathbb{R}^N$ be a random vector, with norm one.

We will calculate the tridiagonal, symmetric matrix

$T_{mm} = V_m^* A V_m$. With the diagonal elements denoted $\alpha_i = t_{ii}$ and the off diagonal elements given by $\beta_i = t_{i+1,i}$. And note that $t_{i-1,i} = t_{i,i-1}$ due to symmetry.

```
1. v_1 = random vec; v_0 = 0; beta_1 = 0;
2. for i = 1,...,m -1do
    w_i = A v_i
    alpha_i = (r_i,v_i)
    w_i = w_i - alpha_i v_i - beta_i v_{i=1}
    beta_{i+1} = abs(w_i)
    v_{i+1} = w_i/beta_{i+1}
endfor
3. w_m = A v_m
   alpha_m = (w_m,v_m)
   return
```

- 1 Introduction
- 2 The Algorithm
- 3 Parallel Lanczos Algorithm**
- 4 Applications
- 5 Questions

- 1 Introduction
- 2 The Algorithm
- 3 Parallel Lanczos Algorithm
- 4 Applications**
- 5 Questions

- 1 Introduction
- 2 The Algorithm
- 3 Parallel Lanczos Algorithm
- 4 Applications
- 5 Questions

Questions?

Thank you!