Double Ring Ising model

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Define the 4 possible spins as s_i where

$$s_i \in \left\{ \begin{pmatrix} +1 \\ +1 \end{pmatrix}, \begin{pmatrix} +1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ +1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\}$$

The Hamiltonian is given by:

$$H = -J \sum_{\langle i,j \rangle} s_i \cdot s_j$$

we set $J = 1$ for simplicity and take the dot product of the two spin vectors. The partition function, Z, is given by the trace of the matrix P which has entries $e^{H(s_i,s_j)}$ for all possible combinations of s_i, s_j.

$$P = \begin{pmatrix} e^{-2\beta} & e^0 & e^0 & e^{2\beta} \\ e^0 & e^{-2\beta} & e^{2\beta} & e^0 \\ e^0 & e^{2\beta} & e^{-2\beta} & e^0 \\ e^{2\beta} & e^0 & e^0 & e^{-2\beta} \end{pmatrix} \tag{1}$$

$$= \begin{pmatrix} e^{-2\beta} & 1 & 1 & e^{2\beta} \\ 1 & e^{-2\beta} & e^{2\beta} & 1 \\ 1 & e^{2\beta} & e^{-2\beta} & 1 \\ e^{2\beta} & 1 & 1 & e^{-2\beta} \end{pmatrix} \tag{2}$$

$$\tag{3}$$

Now find the eigenvalues.. $det(P - \lambda I)$.

$$\begin{vmatrix} e^{-2\beta} - \lambda & 1 & 1 & e^{2\beta} \\ 1 & e^{-2\beta} - \lambda & e^{2\beta} & 1 \\ 1 & e^{2\beta} & e^{-2\beta} - \lambda & 1 \\ e^{2\beta} & 1 & 1 & e^{-2\beta} - \lambda \end{vmatrix} \tag{4}$$

$$\tag{5}$$

$$
\begin{aligned}
&= (e^{-2\beta}-\lambda)\begin{vmatrix} e^{-2\beta}-\lambda & e^{2\beta} & 1 \\ e^{2\beta} & e^{-2\beta}-\lambda & 1 \\ 1 & 1 & e^{-2\beta}-\lambda \end{vmatrix} - \begin{vmatrix} 1 & e^{2\beta} & 1 \\ 1 & e^{-2\beta}-\lambda & 1 \\ e^{2\beta} & 1 & e^{-2\beta}-\lambda \end{vmatrix} && (6)\\
&+ \begin{vmatrix} 1 & e^{-2\beta}-\lambda & 1 \\ 1 & e^{2\beta} & 1 \\ e^{2\beta} & 1 & e^{-2\beta}-\lambda \end{vmatrix} - e^{2\beta}\begin{vmatrix} 1 & e^{-2\beta}-\lambda & e^{2\beta} \\ 1 & e^{2\beta} & e^{-2\beta}-\lambda \\ e^{2\beta} & 1 & 1 \end{vmatrix} && (7)\\
&= (e^{-2\beta}-\lambda)\left[(e^{-2\beta}-\lambda)\left((e^{-2\beta}-\lambda)^2-1\right) - e^{2\beta}\left(e^{2\beta}(e^{-2\beta}-\lambda)-1\right) + (e^{2\beta}-(e^{-2\beta}-\lambda))\right] && (8)\\
&- \left[\left((e^{-2\beta}-\lambda)^2-1\right) - \left(e^{2\beta}(e^{-2\beta}-\lambda)-1\right) + e^{2\beta}\left(e^{2\beta}-(e^{-2\beta}-\lambda)\right)\right] && (9)\\
&+ \left[e^{2\beta}\left((e^{-2\beta}-\lambda)-1\right) - \left((e^{-2\beta}-\lambda)^2-1\right) + e^{2\beta}\left((e^{-2\beta}-\lambda)-e^{2\beta}\right)\right] && (10)\\
&-e^{2\beta}\left[\left(e^{2\beta}-(e^{-2\beta}-\lambda)\right) - \left((e^{-2\beta}-\lambda)-e^{2\beta}\right) + e^{2\beta}\left((e^{-2\beta}-\lambda)^2-e^{4\beta}\right)\right] && (11)\\
&= (e^{-2\beta}-\lambda)\left[(e^{-2\beta}-\lambda)^3 - (e^{-2\beta}-\lambda) - e^{4\beta}(e^{-2\beta}-\lambda) + e^{2\beta} + e^{2\beta} - (e^{-2\beta}-\lambda)\right] && (12)\\
&- \left[(e^{-2\beta}-\lambda) - 1 - e^{2\beta}(e^{-2\beta}-\lambda) + 1 - e^{2\beta}(e^{-2\beta}-\lambda) + e^{4\beta}\right] && (13)\\
&+ \left[e^{2\beta}(e^{-2\beta}-\lambda) - e^{2\beta} - (e^{-2\beta}-\lambda)^2 + 1 + e^{2\beta}(e^{-2\beta}-\lambda) - e^{4\beta}\right] && (14)\\
&-e^{2\beta}\left[e^{2\beta} - (e^{-2\beta}-\lambda) - (e^{-2\beta}-\lambda) + e^{2\beta} + e^{2\beta}(e^{-2\beta}-\lambda) - e^{6\beta}\right] && (15)\\
&= (e^{-2\beta}-\lambda)^4 - 2(e^{-2\beta}-\lambda)^2 - e^{4\beta}(e^{-2\beta}-\lambda)^2 + 2e^{2\beta}(e^{-2\beta}-\lambda) - (e^{-2\beta}-\lambda)^2 && (16)\\
&+2e^{2\beta}(e^{-2\beta}-\lambda) - e^{4\beta} + 2e^{2\beta}(e^{-2\beta}-\lambda) - (e^{-2\beta}-\lambda)^2 && (17)\\
&-e^{2\beta} + 1 - 2e^{4\beta} - e^{4\beta} + 2e^{2\beta}(e^{-2\beta}-\lambda) - e^{4\beta}(e^{-2\beta}-\lambda)^2 + e^{8\beta} && (18)
\end{aligned}
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