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SCHOOL OF MATHEMATICS

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# Spin models on random bipartite graphs

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## **Abstract**

This is my abstract.

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# 1 Introduction

This project is centered around doing Ising Model simulations on random bipartite graphs. As such, it is useful to know more the Ising model and graph theory before we get started.

## 1.1 The Ising Model

### 1.1.1 What is the Ising Model?

The Ising Model is a mathematical model that was invented by Wilhelm Lenz, but developed by Ernst Ising in 1925. It is used to to explore ferromagnetism in statistical physics.

### 1.1.2 Analytical tools used for studying the Ising Model

### 1.1.3 How it is normally studied in serial and parallel simulations

The two dimensional Ising model is usually represented as a rectangular lattice, sometimes with periodic boundary conditions and sometimes not, depending on the case in question. Each point on the lattice is assigned a spin at the start of the simulation, usually at random. Each point only ‘feels’ the interaction of its nearest neighbours (each point has four neighbours). These interactions allow a Hamiltonian to be calculated for each point.

#### FILL ABOUT METROPOLIS HASTINGS AND UPDATING POINTS

In parallel we divide the grid up into smaller subgrids. In the figure, the grid is divided into four subgrids but obviously more subdivisions are possible. Normally we have one subgrid per processor that we intend to run the program on. The need for *Message Passing Interface* (MPI) function calls arises from the fact that on the edges of our subgrids there are points needed for the update step whose values are stored on another processor. These values need to be identified and passed to the relevant processor. MPI is used for this task.

## 1.2 Graph Theory

Graph theory refers to the mathematical study of *graphs*. A graph is a visual representation of set of objects, known as *vertices*. Some pairs of these objects are then connected by links, known as *edges*. If the edges are said to have orientation (if edge  $(a, b) \neq (b, a)$ , where  $a, b$  are vertices), then we call the graph a directed graph. If the

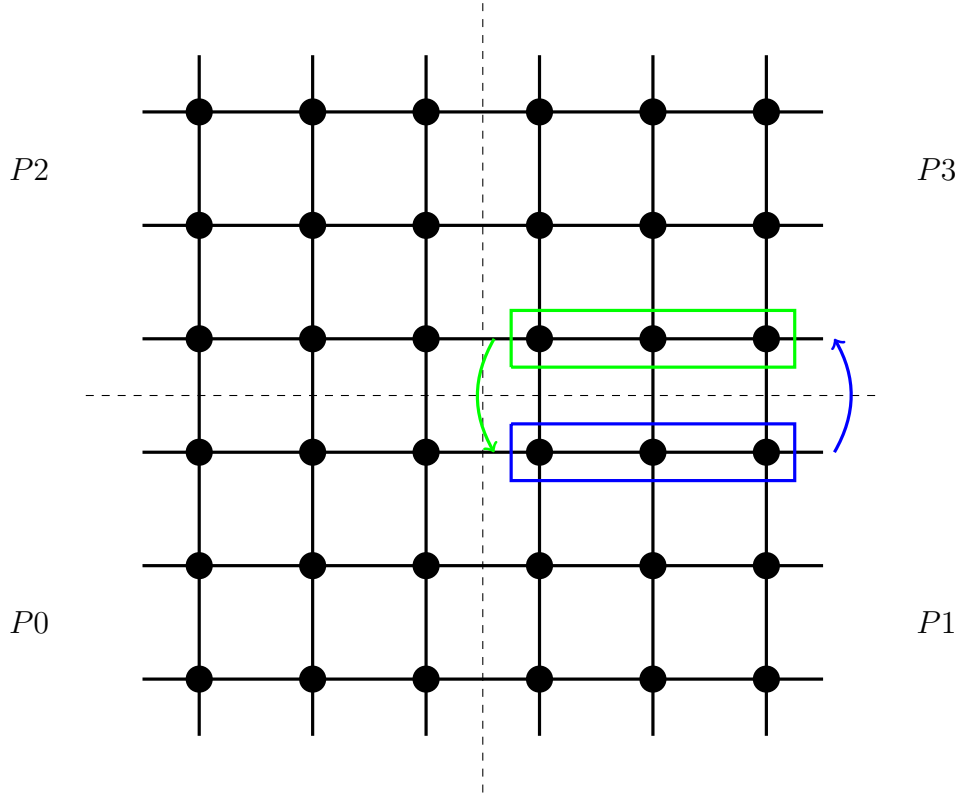


Figure 1: Regular square lattice Ising model.

edges don't have orientation (if  $(a, b) = (b, a)$ ) then we call the graph an undirected graph. We will deal only with undirected graphs in this report.

For this project we're not going to consider disconnected graphs. That is, graphs where there are no nodes connecting a vertex, or set of connected vertices, to the rest of the graph. Only connected graphs are considered.

All graphs considered will be random, *bipartite* graphs. A bipartite graph is a graph in which we can divide its vertices into two disjoint sets,  $A$  and  $B$ , such that vertices in  $A$  are only connected to vertices in  $B$ , and vice versa. Disjoint means that the two sets have no element in common.

A random graph is a graph where the edges connect vertices at random; there is no pattern or order to how vertices are connected.

*Trivalent graphs* are another class of the graphs we will be dealing with a lot. For a graph to be trivalent it means that every vertex has exactly three edges connecting it to three other distinct vertices.

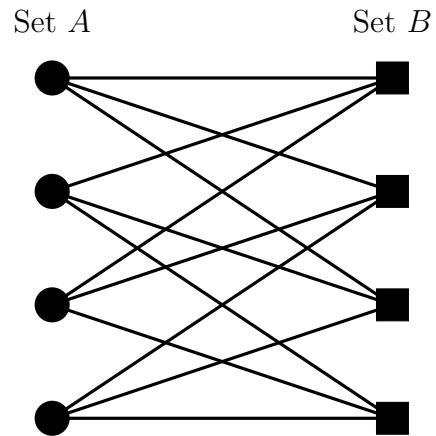


Figure 2: A random bipartite trivalent graph.

### 1.3 Motivation

#### 1.3.1 Why this is an interesting MPI problem

#### 1.3.2 Physical uses of these simulations

### 1.4 Aims

## 2 Software Architecture

In this section I will discuss the various techniques used in creating the simulations that were run in the duration of this project. The main areas these fall under are: graph generating - in both serial and parallel, as well as associated swap algorithms; MPI communication structure; and, finally, how the update step worked.

### 2.1 Random graph generation

Over the course of this project numerous different methods were implemented and compared for the ‘best’ way to create a random bipartite graph. These methods were written in both serial and parallel.

#### 2.1.1 Generating random graphs in serial

There were two methods employed to generate random graphs over the course of this project. The first method (which we will call *simple generation*) was not used as often as it could not be generalised as easily to parallel code and is only applicable for generating bipartite, trivalent graphs. The second method is the swap algorithm which is discussed in detail in the following subsection.

The simple generation method works by building the set of neighbours of one of the bipartite sets and then using that to reverse engineer the other. The first set, set  $A$ , is generated using the simple rules:

1. Each node in set  $A$  must have exactly three unique neighbours.
2. In the set of all neighbours of  $A$  each node index of nodes in  $B$  must appear exactly three time.

The first rule ensure that the graph is trivalent, from  $A$ ’s point of view, with three unique nodes in  $B$  as neighbours. The second rule ensures that each point in  $B$  has three neighbours, as it only allows three points in  $A$  to connect to each point in  $B$ .

This method is works quite well in serial, as it starts at node 0 and fills in its neighbours and then continues all the way down to node  $n - 1$ . The only complication is in filling in the final three nodes where some fairly simple checks must be preformed to ensure that the last three ‘slots’ to fill in aren’t ‘illegal’ moves. By this, it is meant that filling the slots with the remaining available values would break one of the rules above.

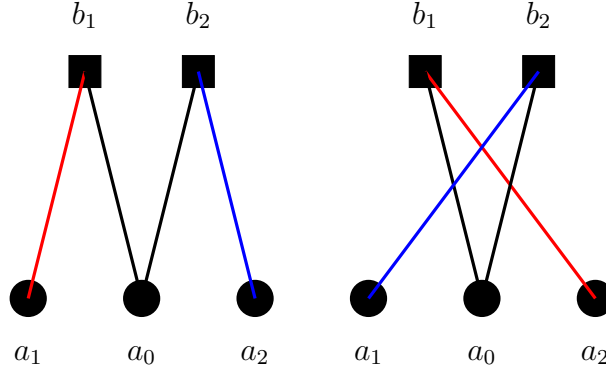


Figure 3: Swap alg.

### 2.1.2 Swap algorithms in serial and parallel

The swap algorithm is used for two purposes in this project. The main use is in the generation of random graphs, and a secondary use is as a proposal step during an update (instead of proposing a spin flip a swap is proposed instead) MIGHT DO THIS COME BACK TO.

In the simplest terms the swap algorithm takes two points in one bipartite set and chooses a neighbour of each of the chosen points and then swaps the connectivity, so that the first point is now a neighbour to the point who was neighbouring the second point, and vice versa. The main advantage of using this swap algorithm is that it preserves the properties of the graph. If we start with a bipartite, trivalent graph and do 1000 swaps, say, then the graph will still remain bipartite and trivalent.

The algorithm preserves this because each swap ensures that points in set  $A$  remain only connected to points in set  $B$ . It also does not change the number of edges coming out of any vertices, this is because it simply two edges are directly swapped with each other between pairs of points so all points involved retain the number of connections they have.

#### The algorithm

The algorithm works by first choosing a node in set  $A$ , at random, which we will call  $a_0$ . From this node two of its neighbours in set  $B$  are chosen (again, at random), we call these  $b_1$  and  $b_2$ . The next step is to choose one neighbour each of  $b_1$  and  $b_2$ , ensuring not to choose  $a_0$  again. Label the neighbour of  $b_1$  as  $a_1$  and the neighbour of  $b_2$  as  $a_2$ . At this point we have the nodes selected that we wish to swap. We want to have  $b_1$  not connected to  $a_1$  anymore, but instead have it connected to  $a_2$ , and similarly



no longer have  $b_2$  connected to  $a_2$ , but connected to  $a_1$ .

Before we can do this however we must preform some checks. We must ensure that  $a_1$  is not connected to  $b_2$  by any other edge that we have not considered so far, and similarly we must ensure that  $a_2$  is in no way connected to  $b_1$ . The reason for this check is that we may already have the nodes we wish to swap connected to each other by another edge, so if the swap is performed then it would mean having two edges connecting the same two points to each other, which is not desirable.

## Serial Implementation

Implementing the algorithm in serial is a rather straightforward. It follows the description given above and there is no complication involving which point is hosted on which process. The hardest part of writing the code for this algorithm was doing the checks at the end to ensure that the points you were swapping weern't already neighbours through another edge that hadn't yet been considered.

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### Algorithm 1 Serial swap algorithm

---

```

Pick  $a_0 = \text{rand} \in A$ 
while  $b_1 \neq b_2$  do
     $b_1 = \text{rand} \in N(a_0)$ 
     $b_2 = \text{rand} \in N(a_0)$ 
end while
while  $flag = 1$  do
    while  $a_1 \neq a_2, a_1 \neq a_0, a_2 \neq a_0$  do
         $a_1 = \text{rand} \in N(b_1)$ 
         $a_2 = \text{rand} \in N(b_2)$ 
    end while
    if  $a_1 \in N(b_2)$  then
         $flag = 1$ 
    end if
    if  $a_2 \in N(b_1)$  then
         $flag = 1$ 
    end if
    Do swap:
         $a_2 \in N(b_1)$  and  $a_1 \in N(b_2)$ 
         $b_2 \in N(a_1)$  and  $b_1 \in N(a_2)$ 
    end while

```

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## Parallel Implementation

The parallel implementation of this algorithm is, naturally, a little more tricky. It is also a problem that does not parallise well at all. In fact, the serial algorithm is much faster and more efficient. The main reason for this is that in the parallel version nodes, and information about what connections they have, are stored on different processors. This means that every time it is needed to pick another node or check a node's neighbours every other processor must wait while one processor does the work. These tasks can't be done simultaneously during the algorithm since each step is directly dependent on the previous step. The problem also can't be avoided by running the algorithm twice (or more) at the same time. This is because, again the whole geometry of the graph must be know in order for the swap to happen, and if one version changes the graph then the other versions don't know about this change yet and so could make a swap that no longer exists on the graph.

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**Algorithm 2** Parallel swap algorithm

---

```
if  $rank = 0$  then
    Pick  $a_0 = \text{rand} \in A$ 
    Determine rank of proc  $a_0$  is hosted on,  $rank(a_0)$ 
end if
MPI_Bcast( $a_0, \dots, 0, \dots$ )
MPI_Bcast( $rank(a_0), \dots, 0, \dots$ )
if  $rank = rank(a_0)$  then
    while  $b_1 \neq b_2$  do
         $b_1 = \text{rand} \in N(a_0)$ 
         $b_2 = \text{rand} \in N(a_0)$ 
    end while
end if
MPI_Bcast( $b_1, \dots, rank(a_0), \dots$ )
MPI_Bcast( $b_2, \dots, rank(a_0), \dots$ )
Determine rank of proc  $b_1$  is hosted on,  $rank(b_1)$ 
Determine rank of proc  $b_2$  is hosted on,  $rank(b_2)$ 
while  $flag = 1$  do
    while  $a_1 \neq a_2, a_1 \neq a_0, a_2 \neq a_0$  do
        if  $rank = rank(b_1)$  then
             $a_1 = \text{rand} \in N(b_1)$ 
        end if
```

```

    if  $rank = rank(b_2)$  then
         $a_2 = \text{rand} \in N(b_2)$ 
    end if
end while
if  $a_1 \in N(b_2)$  then
     $flag = 1$ 
end if
if  $a_2 \in N(b_1)$  then
     $flag = 1$ 
end if
end while
MPI_Bcast(a_1, ..., rank(b_1), ...)
MPI_Bcast(a_2, ..., rank(b_2), ...)
Determine rank of proc  $a_1$  is hosted on,  $rank(a_1)$ 
Determine rank of proc  $a_2$  is hosted on,  $rank(a_2)$ 
Do swap:
if  $rank = rank(a_1)$  then
     $b_2 \in N(a_1)$ 
end if
if  $rank = rank(a_2)$  then
     $b_1 \in N(a_2)$ 
end if
if  $rank = rank(b_1)$  then
     $a_2 \in N(b_1)$ 
end if
if  $rank = rank(b_2)$  then
     $a_1 \in N(b_2)$ 
end if

```

---

## 2.2 MPI communications

### 2.2.1 Data division and load balancing

The problem of how to split the data most efficiently between computing cores will be handled in this section. When we want to solve the system in parallel we have to give each processor a certain number of points on the graph. The method used to divide the data was to simply divide the nodes equally among the processors.

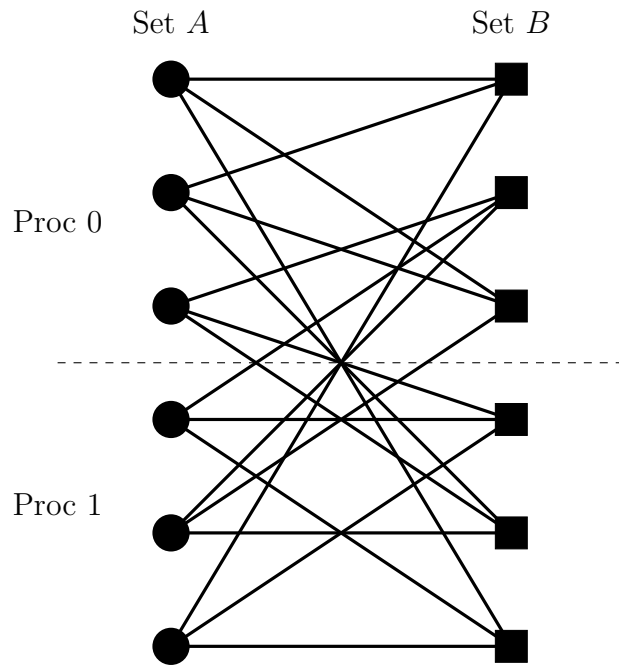


Figure 4: Data division for a random trivalent, bipartite graph.

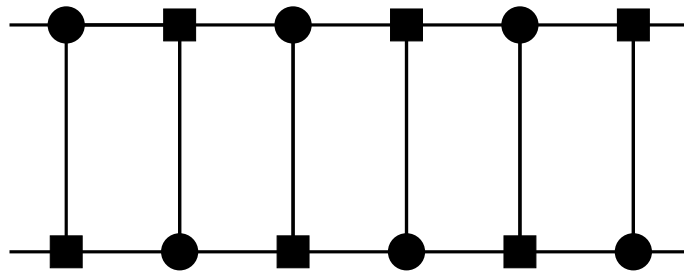


Figure 5: Double ring bipartite graph.

## 3 Results

### 3.1 Serial code

#### 3.1.1 Double Ring Graph

#### 3.1.2 Random Bipartite Trivalent graph

### 3.2 Swap Algorithm

In this section the performance of the swap algorithm is analysed. The serial version of the algorithm will be compared to the parallel algorithm for a varying number of cores. The number of swaps required to make a graph random will also be studied.

#### 3.2.1 How Many Swaps makes a Random Graph?

Before a graph is randomised for a simulation it is first initialised in an order that is fast and easy to compute. This is normally just the double ring configuration, where every node has neighbours of the node with the same index, one ahead and one behind in the other bipartite set. From this initialisation we have to randomise the graph using the swap algorithm.

The problem here is knowing how many swaps need to be made before all traces of this original configuration are lost and the graph can be considered random. We know that initially every points neighbours are the three closest to it in our indexing system. So, if we have point  $n$  in  $A$  we know that its neighbours are  $n - 1$ ,  $n$  and  $n + 1$  in set  $B$ .

If a graph is truly random then it is expected that given any node in set  $A$ , the probability of it being connected to any given node in  $B$  is simply  $\frac{1}{n}$ , where  $n$  is the number of nodes in  $B$ . So this means that if we divide the graph in half horizontally, as in the data division figure, we would expect, on average, half of our connections to cross this horizontal line.

In other words, if we think of how

#### 3.2.2 Serial Vs. Parallel

### 3.3 Speedup

### 3.4 How

## 4 Conclusion

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## 5 Future Work