Valu3s robustness parameters changes

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1 Setting

Suppose we have a hybrid dynamical system of the form

$$\dot{w} = \begin{cases} A_0 w + B_0 r, & \text{if } hw < c \\ A_1 w + B_1 r, & \text{otherwise} \end{cases}$$

where:

- w is a vector in $\mathbb{R}^n$;
- A_0, A_1 are real stable matrices (*i.e.*, the real parts of their eigenvalues are negative) of dimension $n \times n$;
- B_0, B_1 are real matrices of dimension $n \times m$;
- r is a vector in $\mathbb{R}^m$;
- h is a row vector in $\mathbb{R}^n$;
- c is a real constant.

We call M_0 or *mode 0* the set $\{w|hw < c\}$, and M_1 or *mode 1* its complement. We focus only on mode 0, all the results can be generalized to mode 1 in a straightforward way.

Let

$$ss_0^r = -A_0^{-1} B_0 r$$

be the stable point of mode 0. Suppose we have an affine quadratic form given by

$$V_0^r(x) = (x - ss_0^r)^T P_0 (x - ss_0^r),$$

that is a Lyapunov function for mode 0. The matrix P_0 depends only on A_0.

We already know how to find a stable region for the hybrid dynamical system: let

$$C_0^r = \{x \in \mathbb{R}^n | hx = c\} \cap \{x \in \mathbb{R}^n | h(A_0 x + B_0 r) > 0\},$$

or, equivalently

$$C_0^r = \{x \in \mathbb{R}^n | hx = c\} \cap \{x \in \mathbb{R}^n | hA_0 x > -hB_0 r\};$$

and define

$$k_0^r = \sup\{k \in \mathbb{R} \mid \{V_0^r(x) \leq k\} \cap C_0^r = \emptyset\} = \inf_{x \in C_0^r} \{V_0^r(x)\}.$$

We know that

$$S_0^r = \{w \in \mathbb{R}^n | V_0(w) \leq k_0 \wedge hw < c\}$$

is a stable region.

2 Assumptions

We will assume that the stable point of each region lies in the interior part of the region. For mode 0, this means that

$$-hA_0^{-1} B_0 r < c.$$

3 The goal

We want to prove that there exist a $\varepsilon > 0$ such that, if

$$\|r' - r\| < \varepsilon$$

then

$$ss_0^r \in S_0^{r'}.$$

4 The method

When we substitute r with r', there are many objects that change. In particular, the stable point ss_0^r becomes

$$ss_0^{r'} = -A_0^{-1} B_0 r';$$

notice that if $\|r' - r\| < \varepsilon$, then

$$\|ss_0^{r'} - ss_0^r\| = \|A_0^{-1} B_0 (r' - r)\| \leq \|A_0^{-1} B_0\|_2 \|r' - r\| < \|A_0^{-1} B_0\|_2 \varepsilon,$$

where by $\|\cdot\|_2$ we mean the spectral norm of a matrix.

We now consider two cases. Let $hA_0 = \sigma h + g$ be the orthogonal decomposition of hA_0 in the spaces $\langle h \rangle, h^{\perp}$.

4.1 First case: $g = 0$

It follows that C_0^r is empty, since the map

$$x \longmapsto h(A_0 x + B_0 r) = \sigma h x + h B_0 r$$

is constant on $\{x | hx = c\}$, and using the fact that $ss_0^r \in M_0$ its value is

$$\sigma c + h B_0 r = \sigma c + \sigma h A_0^{-1} B_0 r < 0.$$

In this case, the stable region S_0^r is the whole M_0. This happens for every reference value r' for which the stable point $ss_0^{r'}$ is in M_0, hence we just have to check this condition. Since

$$\|ss_0^{r'} - ss_0^r\| = \|A_0^{-1} B_0 (r' - r)\| \leq \|A_0^{-1} B_0\|_2 \|r' - r\|,$$

we can make sure that $ss_0^{r'} \in M_0$ by taking

$$\varepsilon = \frac{\mathrm{dist}(ss_0^r, \{x \in \mathbb{R}^n \mid hx = c\})}{\|A_0^{-1} B_0\|_2}.$$

4.2 Second case: $g \neq 0$

The set C_0^r changes in the following way:

$$C_0^{r'} = \{x \in \mathbb{R}^n | hx = c\} \cap \{x \in \mathbb{R}^n | hA_0 x > -hB_0 r'\}.$$

Notice that if $\|r' - r\| < \varepsilon$, then $\|hB_0 r' - hB_0 r\| < \|hB_0\|\varepsilon$, hence

$$C_0^{r'} \subset D_0^r = \{x \in \mathbb{R}^n | hx = c\} \cap \{x \in \mathbb{R}^n | hA_0 x > -hB_0 r - \|hB_0\|\varepsilon\}.$$

We notice the following:

Observation 1. It holds:

$$\max_{x \in D_0^r} \mathrm{dist}(x, C_0^r) \leq \frac{\|hB_0\|}{\|g\|}\varepsilon.$$

Proof. Recall that $hA_0 = \sigma h + g$ where g is a vector in $h^\perp$ different from 0. For an element in D_0^r, it holds

$$hA_0 x = \sigma h x + g x = \sigma c + g x.$$

Take $x \in D_0^r$ and consider $y = x + \frac{g^T}{\|g\|^2}\|hB_0\|\varepsilon$. We can show directly that $y \in C_0^r$, since

$$hA_0 y = hA_0 x + \sigma \frac{hg^T}{\|g\|^2}\|hB_0\|\varepsilon + \frac{gg^T}{\|g\|^2}\|hB_0\|\varepsilon =$$

$$= \sigma c + g x + \|hB_0\|\varepsilon > -hB_0 r - \|hB_0\|\varepsilon + \|hB_0\|\varepsilon = -hB_0 r.$$

Since the norm of $\frac{g^T}{\|g\|^2}\|hB_0\|\varepsilon$ is exactly $\frac{\|hB_0\|\varepsilon}{\|g\|}$, the proof is finished. $\square$

We also need another ingredient to conclude. Recall that P_0 is the matrix that defines the Lyapunov function V_0. Recall that exists an orthogonal change of coordinates such that the matrix P_0 is diagonal. In this basis, the function $V_0(x)$ is of the form:

$$V_0(x + ss_0^r) = \mu_1 x(1)^2 + \mu_2 x(2)^2 + \ldots + \mu_n x(n)^2,$$

where $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n > 0$ are the eigenvalues of P_0.

Observation 2. Consider a sublevel of the function $V_0^{r'}(x)$:

$$L_k = \{x \in \mathbb{R}^n \mid V_0^{r'}(x) \leq k\}.$$

If L_k contains a point at distance a from ss_0^r, then it contains the whole ball centered in ss_0^r of radius

$$a\sqrt{\frac{\mu_n}{\mu_1}}.$$

Proof. Notice that

$$V_0^{r'}(x + ss_0^{r'}) = \mu_1 x(1)^2 + \mu_2 x(2)^2 + \ldots + \mu_n x(n)^2 \geq \mu_n \|x\|^2.$$

Since L_k contains a point at distance a from ss_0^r, it follows that

$$k \geq \mu_n a^2.$$

On the other hand

$$V_0^{r'}(x + ss_0^{r'}) = \mu_1 x(1)^2 + \mu_2 x(2)^2 + \ldots + \mu_n x(n)^2 \leq \mu_1 \|x\|^2,$$

hence for all x with norm squared less or equal than $a^2 \frac{\mu_n}{\mu_1}$ we have that

$$V_0^{r'}(x + ss_0^{r'}) \leq \mu_1 \|x\|^2 \leq \mu_1 a^2 \frac{\mu_n}{\mu_1} = a^2 \mu_n \leq k.$$

This implies the thesis. $\square$

4.2.1 Conclusion

We fix some notation:

- let $\alpha > 0$ be a positive real number such that the ball centered in ss_0 with radius α is contained in S_0^r. There are various effective ways to compute the optimal such α (knowing P_0, k_0 and h);
- let $\beta = \|A_0^{-1}B_0\|_2 > 0$;
- let $\gamma = \frac{\|hB_0\|}{\|g\|} > 0$;
- let $\delta = \mathrm{dist}(ss_0^r, \{x \in \mathbb{R}^n \mid hx = c\}) > 0$;

- let $\mu = \sqrt{\frac{\mu_n}{\mu_1}} > 0$.

Using the previous consideration we can prove the following:

Proposition 3. Let

$$\varepsilon = \min\left\{ \frac{\alpha\mu}{\mu(\beta+\gamma)+\beta}, \frac{\delta}{\beta} \right\} > 0.$$

If $\|r - r'\| < \varepsilon$, then $ss_0^r \in S_0^{r'}$.

Proof. Since $\|r - r'\| < \frac{\delta}{\beta}$, we know that $\|ss_0^{r'} - ss_0^r\| < \beta\frac{\delta}{\beta} = \delta$, hence $ss_0^{r'}$ is in mode 0.

Since $B(ss_0^r, \alpha)$ is contained in S_0^r, we know that $\operatorname{dist}(ss_0^r, C_0^r) > \alpha$. We already noticed that $\operatorname{dist}(ss_0^r, ss_0^{r'}) < \beta\varepsilon$. Since $C_0^{r'} \subset D_0^r$, we know that $\operatorname{dist}(x, C_0^{r'}) > \operatorname{dist}(x, D_0^r)$.

We now prove that the sublevel $\{V_0^{r'}(w) \leq k_0\}$ contains a point at distance $\alpha - \varepsilon(\gamma+\beta)$ from $ss_0^{r'}$.

Taking together the previous results with Observation 1 we notice that:

$$\operatorname{dist}(ss_0^{r'}, C_0^{r'}) > \operatorname{dist}(ss_0^{r'}, D_0^r) > \operatorname{dist}(ss_0^r, D_0^r) - \operatorname{dist}(ss_0^{r'}, ss_0^r) >$$

$$> (\operatorname{dist}(ss_0^r, C_0^r) - \gamma\varepsilon) - \beta\varepsilon > \alpha - \gamma\varepsilon - \beta\varepsilon = \alpha - \varepsilon(\gamma+\beta).$$

This tells us that the sublevel

$$\{V_0^{r'}(w) \leq k_0\}$$

contains a point at distance at least $\alpha - \varepsilon(\gamma+\beta)$. Using Observation 2 we can conclude that it contains the whole ball centered in $ss_0^{r'}$ of radius

$$\mu(\alpha - \varepsilon(\gamma+\beta)).$$

It is now left to prove that this radius is greater or equal than the distance $\operatorname{dist}(ss_0^r, ss_0^{r'})$. In particular we want to show that

$$\mu(\alpha - \varepsilon(\gamma+\beta)) - \operatorname{dist}(ss_0^r, ss_0^{r'}) > 0.$$

We proceed as follows:

$$\mu(\alpha - \varepsilon(\gamma+\beta)) - \operatorname{dist}(ss_0^r, ss_0^{r'}) > \mu(\alpha - \varepsilon(\gamma+\beta)) - \beta\varepsilon = \mu\alpha - \varepsilon(\mu(\gamma+\beta)+\beta) \geq$$

$$\geq \mu\alpha - \frac{\alpha\mu}{\mu(\beta+\gamma)+\beta}(\mu(\gamma+\beta)+\beta) = 0;$$

this concludes the proof. $\square$