The Logistic Map

A Journey into Chaos

Nicholas Cotton

April 23, 2024

Presentation Overview

What is the Logistic map?

The Logistic Map

The Logistic Map is a function $f : [0,1] \to [0,1]$ defined as

$$f(x) = rx(1-x)$$

or more commonly

$$x_{n+1} = rx_n(1-x_n)$$

where

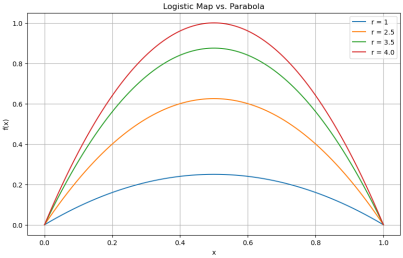
- x_n is the (normalized) population at a time n
- r is a parameter that controls the rate of population growth. Here we focus on $0 \leq r \leq 4$

- Note that the Logistic Map consists of a growth term rx_n and a limiting factor $(1-x_n)$

Just a parabola?

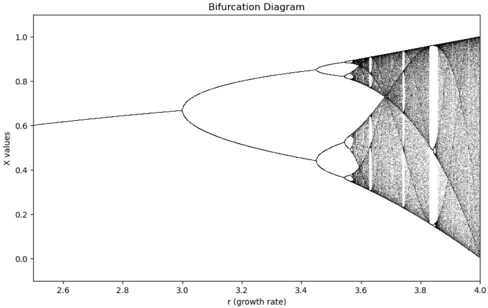
You might notice that the Logistic Map is just a quadratic of form $f(x) = rx - rx^2$

- The graph of a single iteration is simply a parabola, regardless of initial conditions
- We can clearly see that this achieves its maximum value at $x_{max} = \frac{1}{2}$
  - Plugging in x_{max}, we can see that the maximum value is $\frac{r}{4}$
  - This explains why we focus on $0 \leq r \leq 4$



Long Term Behaviors

- Of course, this is a dynamical system, so we are interested in the *long term* behavior of the map
- We do this by fixing r and seeing what sort of behaviors occur after iterating the map many times
  - Fixed and periodic points
  - Stability at those points
  - Denseness of orbits
  - Chaos?



Stability of Fixed/Periodic Points

For various distinct regions of r, we want find fixed/periodic points and study their stability.

Checking stability of a point

1. We first consider the first order Taylor Series expansion around a fixed or periodic point x^*

$$x_{n+1} = f'(x^*)(x_n - x^*) + x^*$$

   where $f'(x^*)$ is the derivative of the Logistic Map at x^*.
   - $f'(x^*) = r(1 - 2x^*)$
2. We say that x^* is *stable* if $|f'(x^*)| < 1$ since this means that points around x^* will converge to x^*.

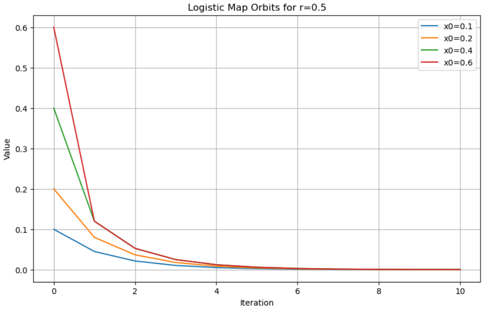
$r < 1$

Note that we have an obvious fixed point at $x^* = 0$.

- Taking the derivitive gives us

$$f'(x^*) = r(1 - 2x^*) = r$$

- Since $r < 1$, $|r| < 1$ and so $x^* = 0$ is a stable fixed point.
- We also see convergence to 0 in the $r = 1$ case



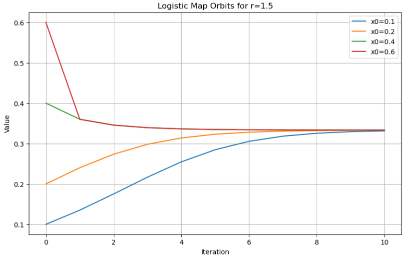
$1 < r < 2$

When $1 < r < 2$, we still have our fixed point at 0; however, checking for stability shows

$$|f'(0)| = |r| > 1$$

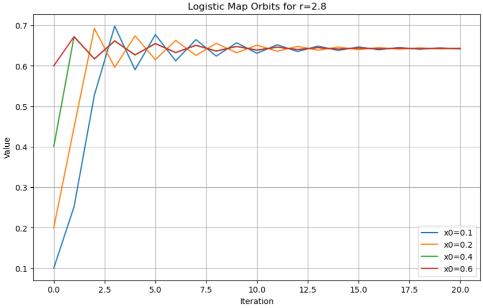
Thus, 0 is not a stable fixed point.
Checking for other fixed points, we find one at $x^* = \frac{r-1}{r}$

- This time
$$f'(x^*) = r(1 - 2x^*) = \frac{2-r}{r}$$
- Clearly, $|f'(x^*)| < 1$ and so $x^* = \frac{r-1}{r}$ is a stable fixed point.



$2 \leq r < 3$

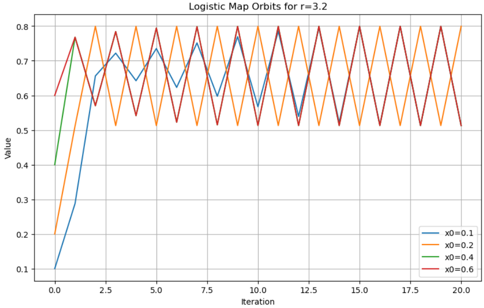
- For this case we perform the same analysis for $x^* = \frac{r-1}{r}$
- $|f'(x^*)|$ is still less than 1
- However, in this region of r, while no points are periodic (except the fixed point), we see some oscillations before converging to x^*
    - This hints at possible periodicity in other regions of r



$3 \leq r < 4$

- This is where things get interesting
- We see many different behaviors in this region of r
- For $3 \leq r < 1 + \sqrt{6}$ we have 2 stable periodic points at

$$x_{\pm} = \frac{1}{2r}\left(r + 1 \pm \sqrt{(r-3)(r+1)}\right)$$



$3 \leq r < 4$

Immediately after $r = 1 + \sqrt{6}$ we find period 2 orbits, then period 4 orbits, then period 8 orbits and so on...

- This period doubling continues until we reach $r \approx 3.56995$
  - This region of $3 \leq r < 3.56995$ is commonly referred to as the *period doubling cascade*
- $r > 3.56995$ is characterized by chaotic behavior

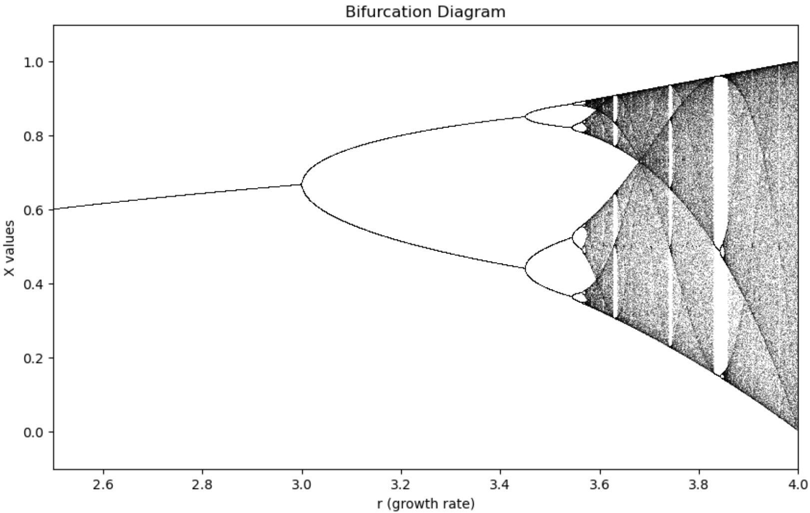
Bifurcation Diagram

The bifurcation diagram is an interesting numerical result that helps us visualize the change in behavior across different values of r

Generating the diagram

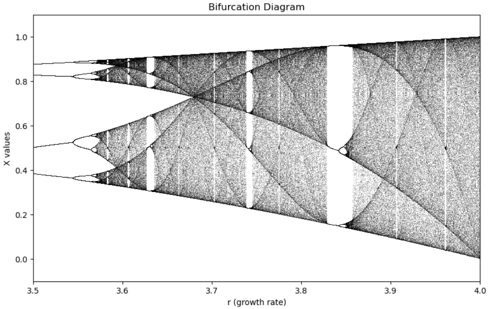
1. Fix x_0
2. Iterate the map 1000 times to ensure we are looking at long term behavior
3. Plot the next 100 iterations
4. Repeat 2-3 for each value of r

Bifurcation Diagram



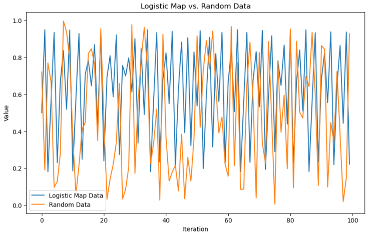
Chaos

- In this context we define Chaos as satisfying the following conditions
  1. Sensitive dependence on initial conditions
  2. Topological Mixing
  3. Dense orbits
- For logistic map, we look at the region $r > 3.56995$ as showing chaotic behavior.



Randomness vs. Chaos

- Plotting logistic map orbits in this region against random data may lead us to believe that the orbits in this region are completely random
- However, the orbit of a point is entirely determined by its initial conditions.
- Plotting x_n against x_{n+1} in whats known as a *Poincare plot*, the logistic map shows behavior determined entirely by the equation itself while the random data just looks like noise



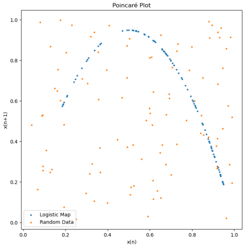


Figure: Generated with $x_0 = 0.5$ and $r = 3.8$

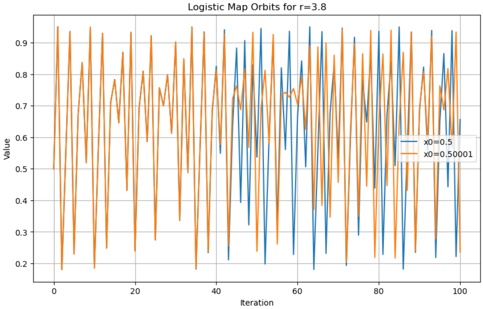
Sensitive Dependence on Initial Conditions

Sensitive Dependance on Initial Conditions

For any two initial values x_0 and x_0', the distance between their orbits grows exponentially. More specifically, there exists $\lambda > 0$ s.t.

$$|x_n - x_n'| = e^{\lambda n}|x_0 - x_0'|$$

λ is called Lyapunov Exponent



Lyapunov Exponents

Lyapunov Exponents measure how quickly initially nearby points diverge.

- A positive Lyapunov Exponent indicates exponential divergence
- A negative Lyapunov Exponent indicates exponential convergence.
- Computed using:

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln |f'(x_i)|$$

Topological Mixing

Topological Mixing

A map $f : X \to X$ is topologically mixing if for any non-empty open sets $U, V \subseteq X$,

$$f^n(U) \cap V \neq \emptyset$$

for sufficiently large n.

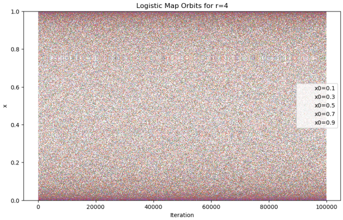
- This can easily be proven for $r = 4$.
  1. Define semi-conjugacy with $E_2(x) = 2x \mod 1$ by $h = \sin^2 \pi x$
  2. Take orbits of logistic map to orbits of E_2
  3. We have already proved E_2 is Topologically Mixing in Class

Dense Orbits

Dense Orbits

A point has orbit dense in [0, 1] if it visits every open interval in [0, 1].

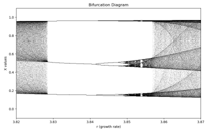
- Here we can see this property for the case of $r = 4$.

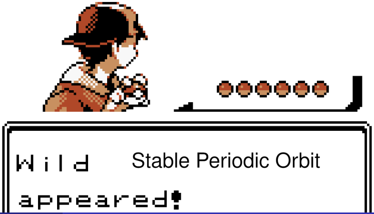


Stable Period-3 Orbit Appears!

Remarkably, suddenly at $r = 1 + \sqrt{8}$, three stable period-3 periodic points appear.

- We can check the Lyapunov Exponents in this region and see that they are negative
- We can see this on the bifurcation diagram
- We see period doubling before quickly returning to chaos





Attractors

On the bifurcation diagram, we can see that chaotic orbits, while not converging to periodic points, still remain in some bounded region after many iterations

- This region is called an attractor

Attractor

For a function $f : X \to X$, a attractor is a subset A of X s.t

1. $a \in A \implies f^n(a) \in A$ for all $n \in \mathbb{N}$
2. There exists a neighborhood of A, denoted $B(A)$, defined as the set of all $b \in X$ s.t. for any open set V containing A, $g^n(b) \in V$ for sufficiently large n.
3. There is no smaller subset of A satisfying (1) and (2)

Video

https://youtu.be/PtfPDfoF-iY?si=NTYnzAEWu-ngK0pb