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Black Hole Dynamics, Hawking Radiation as Tunnelling, and Thermal Phase Transition

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30 March 2018

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SPA6776 Extended Independent Project
30 Credit Units

Submitted in partial fulfilment of the requirements for the degree of
BSc Physics from Queen Mary University of London

Declaration

I hereby certify that this project report, which is approximately 13,500 words in length, has been written by me at the School of Physics and Astronomy, Queen Mary University of London, that all material in this dissertation which is not my own work has been properly acknowledged, and that it has not been submitted in any previous application for a degree.

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Abstract

Black holes are perhaps one of the most fascinating objects to study as a Physics Undergraduate as they come with many counterintuitive and thought-provoking results. This paper will aim to provide an Undergraduate review of black holes, studying their classical, quantum and thermodynamic descriptions. It will begin with a brief review of general relativity, providing brief derivations of black holes in flat and in anti-de Sitter space, followed by a review of geodesic motion in Schwarzschild space looking at Mercury's perihelion shift and the deflection of light. Next, will be a derivation of Hawking radiation as a quantum mechanical tunnelling effect, resulting in thermodynamic quantities such as the temperature and entropy of the black holes both in flat and anti-de Sitter space. Finally, it is shown that a black hole in anti-de Sitter space undergoes a thermal phase transition allowing for a more involved thermodynamic analysis of these black holes. Addressing these various concepts should provide a more precise understanding of the elementary physics of black holes.

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1 Introduction

Albert Einstein in 1916 published the paper which completely transformed our understanding of gravity since Newton, with his new Theory of General Relativity. The new theory gave strange predictions which had to be tested and in 1919 the astronomer Arthur Eddington confirmed that Einstein's theory correctly predicted the deflection of light passing near to our sun. Since then there have been many new experiments giving results in accordance with predictions of Einstein's theory of gravity. The earliest and most notable of these being the deflection of light and the perihelion shift of Mercury's orbit around the sun. Soon after Einstein published his general theory of relativity in 1916, physicists realised that in places of extreme density like the core of a dead star, space and time could be 'dragged inwards' to create a black hole, a boundary in spacetime we now call an event horizon, that after entering, nothing could return. This boundary encloses a region in spacetime, in which null geodesics (path light takes in spacetime) have inwards paths, but there are no null geodesic paths from inside to the outside. There are now many solutions to Einstein's field equation which contain these regions, we call black holes, that classically there is no escaping and must therefore look completely black to an outside observer.

Until 1974, when Hawking published his paper named Black Hole Explosions and another follow up a year after, in which he showed that by applying quantum mechanics one can show black holes are not so black after all. Steven Hawking proposed that a black hole can radiate energy from its horizon giving rise to a black hole temperature then found to be $T_H = \frac{\hbar c^3}{8\pi M G k_B}$ (for the Schwarzschild solution), and further thermodynamic analysis showed they had negative specific heat and consequently being thermodynamically unstable. In 1873, after much interest had arisen in the field of black hole physics, Hawking published a paper [5] on Schwarzschild-AdS black holes, in which through thermodynamic properties, he showed they not only can they be thermodynamically stable but can also be favorable over thermal radiation if large enough. In that paper, he showed they undergo a first order phase transition as the temperature increases past some critical temperature. Hawking radiation, though initially controversial became an inescapable phenomenon of black holes and was quickly accepted by physicists, new methods of calculating the temperature were arising. A particularly elegant method was found by M. K. Parikh and F. Wilczek in 2000 in their paper Hawking radiation as tunneling [1] which they use the phenomenon of quantum tunnelling to find the Hawking temperature for both the Schwarzschild and Reissner-Nordström solutions. Shortly after S. Hemming and E. Keski-Vakkuri [2] then using the same method found the Hawking temperature for the Schwarzschild-AdS black hole. According to Parikh and Wilczek's picture, pair production occurred just beneath the horizon of the black hole, from which point the positive energy particle tunnelled out of the black hole, whilst the negative energy particle would combine with the black hole, decreasing its mass, hence energy conservation is central to their model.

In this project, we briefly review of the main ideas behind Special and General Relativity. We then guide the derivation of three black hole solutions, the Schwarzschild, the Reissner-Nordström and the Schwarzschild-AdS solutions. Allowing us to use the Schwarzschild solution to describe the motion of massive and massless particles in that geometry, in particular we find the perihelion shift of mercury around the sun, light deflection and radial fall into a black hole. Next comes the central part of the paper, which is a full review of M. K. Parikh and F. Wilczek's [1] paper where we find the Hawking temperature of the Schwarzschild and Charged black holes, and then using the same methods find the Hawking temperature for Schwarzschild-AdS black hole by following [2]. As we shall see, by following Hawking and Page [5], we can find further thermodynamic quantities for black holes when in anti-de Sitter space, in which Hawking considered a finite action difference between empty AdS space and AdS with a black hole sitting in it, from which Hawking could then find more interesting properties such as a phase transition.

2 General Relativity

2.1 Flat Space

Special relativity was developed by Einstein in 1905, it revolutionised the way we think about classical mechanics and how we think about space and time. Only it had a big constriction, is that it could only be applied to inertial frames and Einstein would later work for 11 years to generalize this to non-inertial frames and develop the theory of general relativity. Here we give a short introduction to the main ideas of special relativity, then allowing for ideas of general relativity to also be introduced. We begin by introducing the metric tensor, which is an indispensable mathematical object to study for both special and general relativity.

The Metric

With special relativity, Einstein dismissed the idea of absolute time and introduced 4-dimensional coordinates (t, x, y, z), establishing the idea of spacetime. Begin then, by introducing the idea of an interval in spacetime, which will be infinitesimal and measured in an inertial frame S, with rectangular coordinates (t, x, y, z), and we then find the interval has the form

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \tag{2.1}$$

For simplicity, we will set $c = G = 1$ from here on and shall only restore them if it is necessary to complete the calculation in hand. The metric then can tell us how this spacetime interval varies as we move in the space, described by a geometry, which in this case is describing flat space. The metric is the mathematical object which contains information about the causal structure of the geometry of spacetime it describes, whether it is curved or flat space. For the interval (2.1), the metric is known as the Minkowski metric $\eta_{\mu\nu}$, which can be used to re-write the interval in tensor form given by

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$$

The indices μ and ν run from 0 to 3 where so that in this case $dx^0 = dt$, $dx^1 = dx$, $dx^2 = dy$, $dx^3 = dz$, however, 0 will always be the time component and 1 to 3 will always be the spacial components. It can easily be deduced then, that in these rectangular coordinates in flat space, the metric is given by

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We also call this Minkowski space as this is the Minkowski metric, who was the first to introduce this metric formalism of Special Relativity. With this mathematical formalism and a profound postulate made by Einstein in his 1905 paper, special relativity can be stated in surprisingly few words.

Invariant Transformations

Special relativity is based on the postulate that Physical laws do not change under a set of transformations which preserve the spacetime interval given by (2.1), in other words, there is a set of transformations connecting inertial frames which also keep (2.1) invariant. We shall see that together, these transformations make up the Lorentz group which constitutes of the identity, three spacial rotations and three boosts.

Mathematically, a transformation Λ will keep the interval ds^2 invariant if it satisfies both of the following

$$\Lambda^{\mathrm{T}}\eta\Lambda = \eta \qquad \text{and} \qquad \det(\Lambda) = +1 \tag{2.2}$$

The set of Λ's which satisfy both these conditions make up the $SO(1,3)$ group, which in special relativity is also known as the Proper Lorenz Group, explicitly given by $G = \{I, \Lambda_{yx}, \Lambda_{zx}, \Lambda_{xy}, \Lambda_{xt}, \Lambda_{yt}, \Lambda_{zt}\}$ all of which satisfy (2.2). All of these, transform the coordinates of our observers inertial frame S to another frame S', and as the postulate demands, the laws of physics in frame S' are unchanged from S. The first four transformations on the group are uninteresting since it had long been known the laws of classical dynamics and electromagnetism along with their equations were invariant to rotations $(\Lambda_{yx}, \Lambda_{zx}, \Lambda_{xy})$ and the trivial identity (I) transformation. It was the three boosts $(\Lambda_{xt}, \Lambda_{yt}, \Lambda_{zt})$ that took the interest of physicists at the time because by acting on S, the boosts gives the coordinate frame an instantaneous velocity. The speed at which electromagnetic waves propagate through the universe is a prediction from Maxwell equations in terms of the electric and magnetic natural constants, explicitly given as $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$. Consequently then, from the invariance of Maxwell's equations under the boost transformations the speed of light must be equal for all observers in inertial frame, so contrary to popular belief the constancy of the speed of light is not a postulate of special relativity, but rather, this phenomenon can be used to test special relativity.

Now returning to the metric, is also possible to re-write the Minkowski metric with other coordinate systems, for example, spherical polar coordinates, and it would take different from but still be describing flat space. The transformations would also take a different form but the same principles apply, however classical dynamics equations and laws of physics take their simplest form written with rectangular coordinates. And so special relativity has simple, preferred coordinates and as will be seen this is not the case for general relativity.

Special to General Relativity

We must now move on to some curved space and general relativity, to do this we first look at the how the metric changes as we move from flat to curved spaces:

- If we are working with Galilean relativity, we are in flat space, we have absolute time and dimension of $d = 3$. The metric is just the delta function $g_{ij} = \delta_{ij}$, with a euclidean signature $(+, +, +)$. Consequently, with absolute time, we can parametrize the path a particle takes in space by the absolute time $x^i(t)$.

- If we are working with Special relativity, we are now in flat spacetime, but non-absolute time as our dimensions are now $d = 4$ with time t included. The metric is the Minkowski metric, $g_{\mu\nu} = \eta_{\mu\nu}$ with the Lorentzian signature $(-, +, +, +)$ and because of constrictions of special relativity, it cannot undergo general coordinate transformations, only those in Proper Lorentz group. What is more, the metric has the property $\eta^{\mu\nu} = \eta_{\mu\nu}$ i.e. in matrix notation $\eta^{-1} = \eta$. There is also no absolute time, so the path particles take in spacetime is parametrised by the new parameter called the proper time. The proper time is choosen because it dictates the length of the world line, which crucially is itself invariant under coordinate transformations from the Lorenz group. The action of a free relativistic particle in Minkowski space is then $S = -m \int \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu}$ and can easily enough be shown to result in to $\frac{d^2 x^\mu}{d\tau^2} = 0$

- If we are working with General relativity, we are now in curved spacetime, time is also non-absolute and we are in dimension $d = 4$. The metric is now a function of position, it therefore varies as the coordinates vary. This means the spacetime interval changes as a function of position and the metric is $g_{\mu\nu}(x^\mu)$, which is non-diagonal having 10 independent components. Unlike the Minkowski metric and special relativity, general relativity is a theory of general covariance which means this new metric must be able to undergo general coordinate transformations. The proper time is chosen to parametrise motion since it is also invariant under general coordinate transformations, which we shall apply this later in chapter 3. To then find the action of a particle in curved spacetime, we need to simply replace the simple $\eta_{\mu\nu}$ by the general metric $g_{\mu\nu}(x^\mu)$ changing the action to be $S = -m \int \sqrt{-g_{\mu\nu} dx^\mu dx^\nu}$, to which varying gives the geodesic equation.

To summarise, in special relativity there are a preferred set of coordinates which simplify equations, and the chosen transformations are those which preserve this simplicity. When moving to general relativity there are no proffered set of coordinates so the transformations that will be chosen are a general or arbitrary set of transformations. As expected then, the metric describing curved space is a lot more involved than the simple Minkowski metric.

2.2 Curved Spaces

General relativity is a vast area of physics with most books being around a thousand pages long, so one cannot hope to learn it in a section. However, the main ideas can be simplified as a reminder for those undergrads who have already learned the subject. A good place to start is with Einstein's Equivalence Principle (EEP), nicely quoted by Sean M. Carroll in the beginning of section 4 of his lecture notes on General Relativity [11], "In small enough regions of spacetime, the laws of physics reduce to those of special relativity; it is impossible to detect the existence of a gravitational field." EEP can be said in a few different ways, but the two important implications it has are; (i) that in a free-falling frame special relativity can correctly describe physics in deep space (flat space), and (ii) that when looking at what spacetimes are allowed in general relativity, they must be locally flat, i.e. a manifold.

So we find, that in a small enough lab, the metric $g_{\mu\nu}(x^\mu)$ must reduce to flat Minkowski space, with some metric $\eta_{\mu\nu}$, however, if this region is too big then curvature cannot be ignored. So unlike special relativity, in general relativity, we cannot extend the same local coordinate system to a global coordinate system, and the best to expect is local flatness giving rise to a lack of preferred coordinates like in special relativity which means there are no change of coordinates we should restrict ourselves to. Therefore we need a general coordinate transformation, a theory with general covariance in the same way in there is Lorenz covariance for special relativity. Though we can't go much further in here, many useful resources can be found such as Lecture Notes on General Relativity [11] and Einstein Gravity in a Nutshell [3]. After all most of the concepts have been covered, we can state the main content of general relativity, where the first is how spacetime gets curved in the first place which is described by Einstein's field equations given by

$$R_{\mu\nu} - Rg_{\mu\nu} = 8\pi T_{\mu\nu} \tag{2.3}$$

The LHS is commonly written using the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - Rg_{\mu\nu}$. The first thing to note is that this is symmetric under μ and ν, therefore there are 10 independent equations in 4 dimensions to solve for, however, the Bianchi identity can further reduce it to 6 equations for 10 unknowns. On the RHS is the source term given by the energy momentum tensor $T_{\mu\nu}$ where we introduce the source that we wish to find a curved geometry for. To summarise then, for a given $T_{\mu\nu}$ there is one unique geometry, where this geometry can be represented by many different $g_{\mu\nu}$, all of which are connected by coordinate transformations. Once we have a geometry, we will likely be interested understanding motion in it, for that we use the geodesic equation given by

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\delta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{2.4}$$

This equation describes the motion of particles in curved space which are in free fall with respect to some other observer.

2.3 Solutions to Einstein Equation

We now move on to find solutions to the Einstein equation, in particular, we find the three solutions which will later be used to find the Hawking radiation for. These are the Schwarzschild, the Reissner-Nordström, and the AdS_{d+1} solutions. To begin, recall the metric $g_{\mu\nu}(x^\mu)$ is what needs to be found, as it describes the geometries of the spacetimes which will contain horizons and therefore the black holes. In it's most general form, the metric has some 10 independent functions given by the elements on and above the diagonal in (2.5). Turning to Einstein's equation, what is desired is to find a spherically symmetric solution in a source free region (excluding the charged black hole as $T_{\mu\nu} \neq 0$). Therefore two steps are now taken; (i) we greatly simplify the general metric by imposing spherical symmetry in the space, (ii) we use Einstein's equation for three different cases to find three different solutions, all of which contain horizons.

2.3.1 Spherical Symmetry

Beginning in 4-dimensional spacetime using spherical polar coordinates, the metric in its most general form

$$g_{\mu\nu}(x^\mu) = \begin{pmatrix} g_{tt} & g_{tr} & g_{t\theta} & g_{t\phi} \\ g_{rt} & g_{rr} & g_{r\theta} & g_{r\phi} \\ g_{\theta t} & g_{\theta r} & g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi t} & g_{\phi r} & g_{\phi\theta} & g_{\phi\phi} \end{pmatrix} \tag{2.5}$$

Because the sources we will solve for will not be rotating but will be simply static spherical objects, spherically symmetric is then enforced on the geometry of spacetime. One way to impose spherical symmetry is by imagining the spacetime in terms of an S^2-foliation. This means we build up the spacetime by stacking concentric 2-spheres (like an onion), except in this case they are stacked along the radial direction r (moving along r changes the 'size' of each S^2) and also lined up along t. Now the consequences this now has are; (i) if we focus on one of the 2-spheres (fix t and r) then the metric should take that form of $d\theta^2 + \sin^2\theta d\phi^2 = d\Omega^2$ and force $g_{\theta\phi} = g_{\phi\theta} = 0$ and (ii) the spheres can be aligned so that a radial geodesic goes through the same value for θ and ϕ and likewise motion along t also keep them unchanged, which will now force $g_{t\theta} = g_{\theta t} = g_{\theta r} = g_{r\theta} = g_{t\phi} = g_{\phi t} = g_{r\phi} = g_{\phi r} = 0$. The metric now has some 5 independent functions, each with 2 variables, where from now on coordinate transformations are made to reduce further the metric. Coordinate transformations can then simplify $g_{tt}(r,t), g_{tr}(r,t), g_{rt}(r,t)$ and $g_{rr}(r,t)$ down to $g_{tt}(r',t'), g_{rr}(r',t')$. Then writing the full reduced line element we have

$$ds^2 = -g_{tt}(r',t')dt^2 + g_{rr}(r',t')dr^2 + r^2 d\Omega^2 \tag{2.6}$$

We can further simplify (2.6) by forcing the geometry to be static spacetime, eliminating any dependence of time t from the metric. If we then re-write $g_{tt} = A$ and $g_{rr} = B$ and re-labelling r as just r' for simplicity, we find

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\Omega^2 \tag{2.7}$$

Spherical symmetry is important as it can describe curved spaces around planets and stars. The simplification is commonly performed, and detailed versions of this process are shown by S. Carroll and A. Zee in [11, 3]. Summing up, we have started with 10 independent functions each with four variables (r, t, θ, ϕ in this case)(2.5), and reduced it down to 2 equation with 1 variable. To go any further from this point, Einstein's field equation (2.3) is needed, to understand how matter curves space.

2.3.2 Schwarzschild Solution

For the simplest case, the Schwarzschild solution, we are simply solving the Einstein equation exterior to a spherically symmetric body of mass M and radius R. This reduces Einstein's equation to $G_{\mu\nu} = 0$, however, by writing (2.3) in trace reverse form (shown in appendix 5.5, and $T_{\mu\nu} = 0 \therefore T = 0$), reduces Einstein's equation to

$$R_{\mu\nu} = 0 \tag{2.8}$$

To which after substituting (2.7) into this we find four independent differential equations, three of which, are explicitly given by

$$2 - \frac{2 + \frac{rA'}{B}}{B} + \frac{rB}{B^2} = 0 \tag{2.9}$$

$$\frac{2A'}{r} - \frac{A'^2}{2A} - \frac{A'B'}{2B} + A'' = 0 \tag{2.10}$$

$$\frac{A'^2}{2A} + \frac{2AB'}{rB} + \frac{A'B'}{2B} - A'' = 0 \tag{2.11}$$

If we then multiply (2.10) by $B(r)$ and (2.11) by $A(r)$ then sum them to find $0 = BA' + AB' \implies A(r) = \frac{1}{B(r)}$. Then we substitute this result into (2.9) and find $1 - A - rA' = 0$ which is simple enough to solve and find

$$A(r) = 1 + \frac{C_0}{r} \tag{2.12}$$

With C_0 being the integration constant that can be found to be $C_0 = -2M$ through the boundary condition at $r \to \infty$, we then find the final form of the Schwarzschild line element to be

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{(1 - \frac{2M}{r})} + r^2 d\Omega^2 \tag{2.13}$$

If c and G are restored, the integration constant would take the form $C_0 = -\frac{2MG}{c^2}$. By looking at (2.13) we see it gets interesting for $r = 2M$ and $r = 0$, note that for small astrophysical objects (earth, sun) these are always inside the object and since this solution is for $R_{\mu\nu} = 0$ (2.13) no longer holds inside the object. With a black hole we can reach $r = 2M$ without reaching the 'surface' of the object, at which point the causal structure of spacetime force interesting phenomena. At $r = 2M$ the metric seems to be singular, however this is due to the choice of coordinates, we therefore call it a coordinate singularity. We can contract the Riemann curvature tensor which will result in an invariant value for local curvature in the Schwarzschild geometry.

$$R^{\mu\nu\rho\alpha} R_{\mu\nu\rho\alpha} = \frac{48M^2}{r^6} \tag{2.14}$$

Hence we find that at $r = 2M$ there is finite curvature (coordinate singularity), we call this the horizon. However, at $r = 0$ we see even (2.14) is singular, therefore this point is a true curvature singularity, we then call it the singularity.

2.3.3 Reissner - Nordström Solution

Now that we have found the simplest solution, we turn to the Reissner-Nordström solution also known as the charged black hole. In this case, we are solving Einstein's equation exterior to a spherical mass of mass M and radius R. This solution, however, is not in empty space but is the presence of an electric field emitted by the spherical mass, which is now charged with charge Q. Consequently, the source term $T_{\mu\nu}$ is no longer zero and Einstein's equation to find the solution to has the form

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

Though Maxwell wrote down his equations in the form of 4 equations in 3-dimensional Euclidean space, they can be reduced to 2 equations if written in 4-dimensional tensor form. Incidentally, these equations can easily be shown to be invariant under the Lorentz transformations, so had Maxwell written them in the 4-dimensional tensor form he might have discovered Special Relativity. The energy momentum tensor is given by

$$T_{\mu\nu} = F_{\mu\lambda}F_{\nu}^{\lambda} - \frac{1}{4}g^{\mu\nu}F_{\sigma\rho}F^{\sigma\rho} \tag{2.15}$$

and the black hole itself has electric filed given by $E = \frac{Q\sqrt{AB}}{r^2}$ which reduces to $E = \frac{Q}{r^2}$ since this solution also has the form $A = \frac{1}{B}$. The solution to these equations using a static and spherically symmetric metric (The solution is carried out detailed enough by A. Zee in chapter VII.6 [3]), will result in the Reissner-Nordström solution given by

$$ds^2 = -\left(1 - \frac{2GM}{r} + \frac{Q^2}{r^2}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2GM}{r} + \frac{Q^2}{r^2}\right)} + r^2 d\Omega^2 \tag{2.16}$$

The first trivial thing to notice is how this tends to the Schwarzschild solution as the charge $Q \to 0$, as expected. Secondly, there exists values for $A(r) = 0 = 1 - \frac{2GM}{r} + \frac{Q^2}{r^2}$ therefore this solution contains horizon and therefore a black hole. This solution however, is more bizarre, as it can have multiple horizons found by solving solve the corresponding quadratic to find $r_+ = M + \sqrt{M^2 - Q^2}$ and $r_- = M - \sqrt{M^2 - Q^2}$. As a result, this black hole has some interesting properties, namely, this black can have two, one or no horizons.

Naked Singularity

If the charge satisfies $Q > M$ then we have what is called a naked singularity, notice both the horizons lose any real solution because in this case there is none, hence a naked singularity. This can be seen by observing the time component never becomes negative, always maintaining a negative sign, unlike the Schwarzschild case (where for $r < r_H$, the time component $g_{tt} > 0$). Likewise, the radial component of the metric never becomes time like (i.e. negative) and for all r we have $g_{rr} > 0$ therefore, unlike the Schwarzschild case light can reach an observer at $r = \infty$ from a signal near or at the singularity $r = 0$. This as we shall see, results in an eternal singularity as there is no radiation being emitted by the charged object.

Regular and Extremal Black Holes

As the charge of the black hole is decreased, we find two cases, it can reach $Q = M$ (Extremal black hole) or get bellow $Q < M$ (Regular black hole). For the extremal case, we find the two horizons r_+ and r_- meet to form just one horizon at $r_+ = r_- = M = Q$ which is interesting since this is when gravitational attraction and charge repulsion can reach a point of zero net force. Consequently, if there are two charged black holes which have reached this distance of zero net force they will sit in the space as two static objects, allowing for static solutions to be found (A. Zee also finds this in chapter VII.6 appendix 1 [3]). When it comes to Hawking temperature, we will find for extremal black holes, the temperature also loses physical meaning for $Q = M$, on the other hand, the entropy quantity will remain valid for this case.

For the regular case ($Q < M$) we have two horizons given by r_+ and r_- therefore, there are three regions to look at, (i) outside the outer horizon $r > r_+$, (ii) between the inner and outer horizon $r_+ > r > r_-$ and (iii) inside the inner horizon $r_- > r$. To find causal meaning from these cases, look at the spacelike and timelike components of the metric and how they change in each case. For (i), trivially the time component (g_{tt}) is timelike as expected outside a black hole, for (ii) g_{tt} becomes spacelike and g_{rr} becomes timelike, just like a Schwarzschild black hole, therefore, the same causal structure applies, finally for (iii) g_{tt} again becomes timelike which can be seen as for small r $-\frac{Q^2}{r^2}$ dominates. To clarify understanding of the complete causal structure of a charged black hole, a Penrose diagram would be the next steps to take.

2.3.4 Schwarzschild-AdS Solution

Let's now find the anti-de Sitter black hole solution, which has recently greatly increased in interest in theoretical physics due to the rise of string theory. Anti-de Sitter space is different in that it is embedded in higher dimensional space of Minkowski form, so AdS_4 which we will solve for here is embedded in 5-dimensional spacetime $M^{3,2}$ with two time like components i.e. a metric signature of $(-,+,+,+,-)$. In arbitrary dimensions, we are looking at a d-dimensional anti-de Sitter peacetime embedded in $(d+1)$-dimensional Minkowskian spacetime. Explicitly, this embedding looks like

$$-x_0^2 + \sum_{i=1}^{d-1} x_i^2 - x_d^2 = b^2$$

We can see the space is embedded in the form of a hyperbola, only in arbitrary dimensional space. Furthermore, the space is bounded by b, where the whole space is in $(d+1)$-dimensions however the embedded AdS space is of course in d-dimensions. We will not look any deeper into this space from here (A. Zee covers it extensively [3], J. Gath also gives a good review in appendix A [12]). To continue we just look to find the solution from Einstein equation however we first need to side on maximally symmetric spaces.

Maximal Symmetry

Before continuing with the solution, maximal symmetric spaces will be useful to understand how to arrive at the solution more smoothly. As previously mentioned, a geometry representing curved spacetime in general relativity is mathematically represented as a manifold, as one property spacetime must have, is the property of local flatness. Manifolds have symmetries associated with them, these can be found using an equation called the killing equation, where solutions to this equation result in killing vectors. killing vectors, therefore, represent a symmetry in the geometry (which describes some spacetime), in addition, since killing vectors must be coordinate invariant, a geometry will have the same symmetries regardless of the coordinates describing the space. In physics symmetries represent conserved quantities (one such example is the independence of ϕ in the metrics in this paper, resulting in conserved angular momentum, therefore orbits are constrained to one plane of motion), hence the importance of killing vectors is to find such conserved quantities. If a manifold of dimension d, contains $\frac{d(d+1)}{2}$ killing vectors (i.e. symmetries) then the space is called a maximally symmetric space and all three solutions we are here finding are such spaces with this property. Furthermore, for purposes in this paper, these spaces have the property of having constant curvature R. A maximally symmetric space has a constraint on the Riemann curvature tensor $R_{\rho\alpha\mu\nu}$, in that they solve the expression given by

$$R_{\rho\alpha\mu\nu} = \frac{R}{d(d-1)} \left(g_{\rho\mu} g_{\alpha\nu} - g_{\rho\nu} g_{\alpha\mu} \right) \tag{2.17}$$

Proof of this equation and further clarification on maximally symmetric spaces are shown in [11], and further clarification of curvature tensors are given in Appendix. If we then force a Riemann tensor to have this local property given by (2.17) then contract it (as done in appendix 5.4) or trace it we can find the following Ricci tensor

$$R_{\mu\nu} = \frac{(d-2)}{2d} R g_{\mu\nu} \tag{2.18}$$

Where R is the constant curvature governing the constant curvature of the spacetime. Comparing (2.18) to Einstein's equation, we find it is the same but in empty space with no source term ($T_{\mu\nu} = 0$) just like for the Schwarzschild solution (for Schwarzschild, $R = 0$ resulting in flat space, i.e. set $M = 0$ the space becomes Minkowski space, flat). The difference here is if we add a negative cosmological constant given by Λ to Einstein's equation (i.e. $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$) which then forces out a new form of the Ricci tensor

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \tag{2.19}$$

Solving this equation will results in maximally symmetric spaces with constant curvature, Λ determines whether space will be flat ($\Lambda = 0$), positively curved de-Sitter space ($\Lambda > 0$) or negatively curved anti-de Sitter space ($\Lambda < 0$). We can also compare (2.18) with (2.19) and find that they are fixed by $R = \frac{2d}{d-2}\Lambda$.(Equations (2.17,2.18,2.19) are also further motivated in [11, 12])

Black Hole in Anti-de Sitter space

At this point, similarly to the previous two solutions we impose spherical symmetry, then substituting the line element (2.7) in (2.19) we find four differential equations, given by $R_{t,t}, R_{r,r}, R_{\theta,\theta}, R_{\phi,\phi}$ ((i) we are in $d = 4$, (ii) all other Ricci components vanish), however we shall only need to show $R_{t,t}$ and $R_{r,r}$ to find the final solution, explicitly

$$R_{t,t} \ ; \ \frac{2rABA'' - A(rBA' + A(rB' - 4B))}{4rAB^2} = 0 \tag{2.20}$$

$$R_{r,r} \ ; \ \frac{A(4A + rA')B' + rB(A'^2 - 2AA')}{4rBA^2} = 0 \tag{2.21}$$

From here we add (2.20) and (2.21) as follows, $(4rAB^2)R_{t,t} + (4rBA^2)R_{r,r}$ consequently simplify to the differential equation $B(r)A'(r) + A(r)B'(r) = 0 \implies A(r) = \frac{1}{B(r)}$ the same form as the previous two solutions. We therefore find all three solutions have the same form of

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2 d\Omega^2 \tag{2.22}$$

The only change is the function of $F(r)$, so to continue finding the Schwarzschild-AdS solution lets substitute this in (2.19) again, to which we find

$$1 - \Lambda r^2 - F(r) - rF'(r) = 0 \implies F(r) = 1 - \frac{\Lambda r^2}{3} + \frac{C_0}{r^2} \tag{2.23}$$

Finally, we introduce the radius of curvature b where after normalizing we find it is equal to $b^2 = -\frac{3}{\Lambda}$ (again in AdS $\Lambda < 0$) after which we find that the full Schwarzschild-AdS line element in 4 dimensions reads

$$ds^2 = -\left(1 + \frac{r^2}{b^2} - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{\left(1 + \frac{r^2}{b^2} - \frac{2M}{r}\right)} + r^2 d\Omega^2 \tag{2.24}$$

Where we again have the integration constant to be $C_0 = 2M$. We will not limit ourselves to this four-dimensional line element but will move to an arbitrary dimensional Schwarzschild-AdS black hole. When moving to arbitrary dimensions the cosmological constant now becomes

$$\Lambda = -\frac{(d-1)(d-2)}{2b^2} \tag{2.25}$$

Which then fixes the Ricci curvature to take the form

$$R = -\frac{d(d-1)}{b^2} \tag{2.26}$$

The line element for an arbitrary dimensional Schwarzschild-AdS black hole is given by

$$ds^2 = -\left(1 + \frac{r^2}{b^2} - \frac{\mu}{r^{d-2}}\right) dt^2 + \frac{dr^2}{\left(1 + \frac{r^2}{b^2} - \frac{\mu}{r^{d-2}}\right)} + r^2 d\Omega_{d-1}^2 \tag{2.27}$$

With regard to notation, $d\Omega_{d-1}^2$ is the metric of a unit $(d-1)$-dimensional sphere and μ is not an index, but a generalization of the quantity $2GM$ (from 4-dimensional spacetime metrics) for arbitrary $(d+1)$ dimensional spacetime, it is given by

$$\mu = \frac{16\pi G_{d+1}}{(d-1)A_{d-1}} M \tag{2.28}$$

Here we have G_{d+1} which is the Newtonian constant in $(d+1)$ dimensions which from here on we shall also set $G_{d+1} = 1$ (note that when we are in arbitrary dimensions this is the Newtonian constant to be restored), and A_{d-1} is the volume of a $(d-1)$-dimensional unit hypersphere, given explicitly as $A_{d-1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$.

For $d > 2$ there will always be a root for $F(r) = 0$ we call the horizon r_+ i.e. $F(r_+) = 0$ and therefore there exists a black hole in this space. Contracting the Riemann curvature tensor for this metric, we find

$$R^{\mu\nu\rho\alpha} R_{\mu\nu\rho\alpha} = \frac{24}{b^2} + \frac{g(d)}{b^2}\left(\frac{M}{r^d}\right) + f(d)\left(\frac{M}{r^d}\right)^2 \tag{2.29}$$

Where $f(d)$ and $g(d)$ simply are messy functions of d, but the important part is to see that it is again, only singular at the singularity of the black hole with infinite curvature. There will be finite curvature at the horizon therefore change of coordinates will be needed when looking at Hawking radiation as tunnelling, furthermore, note that as $r \to \infty$ there is still curvature contrary to the Schwarzschild geometry. The black hole can be removed, by setting $M = 0$ at which the space would not be flat like in the Schwarzschild solution but the space would have constant curvature given by $R = -\frac{d(d-1)}{b^2}$ and the line element would now take the explicit form

$$ds^2 = -\left(1 + \frac{r^2}{b^2}\right) dt^2 + \frac{dr^2}{\left(1 + \frac{r^2}{b^2}\right)} + r^2 d\Omega_{d-1}^2 \tag{2.30}$$

Note that asymptotically $r \to \infty$ the black hole solution also tends to the anti-de Sitter space, a property that will be central for finding thermodynamic properties for a black hole sitting in anti-de Sitter space.

Bekenstein-Hawking area entropy law

Before moving on, one thermodynamic law of black holes will be stated so that we can see a consistency of results when finding the entropy (via different methods [1, 2, 5]) for each of the black hole solutions. There exists a relation between the entropy of a black hole and its area, found by Stephen Hawking and Jacob Bekenstein, restoring c and G the entropy is given by

$$S_{B-H} = \frac{Ac^3}{4G\hbar} \tag{2.31}$$

Where A is the area of the black hole's horizon. We will be calculating entropies of black holes and using this famous relation to show they all follow it.

3 Motion Around a Black Hole

3.1 Motion in Curved Space

In curved spaces, particles follow paths known as geodesics, where massive and massless particles do, of course, have slightly different geodesics. Given we now want to describe moving particles in curved spaces, the geodesic is indispensable for us. There are a few different ways the geodesic can be derived, an elegant method is by extremizing the proper time $\tau_{AB} = \int_A^B \sqrt{-g_{\mu\nu}dx^\mu dx^\nu}$ (A to B is the proper time elapsed in space-time and we will find it convenient later to make it a unit length). This method of extremization results in the geodesic given by (3.3) [7], however, we need not derive the equation explicitly here since it shall be more straightforward to substitute our metric in (3.2) to obtain the equations of motion by solving the Euler-Lagrange equations directly.

$$
\begin{aligned}
I &= \int \sqrt{-g_{\mu\nu}dx^\mu dx^\nu} \\
&= \int d\tau \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} \qquad (3.1) \\
&= \int d\tau L \qquad (3.2)
\end{aligned}
$$

Begin by introducing an arbitrary parameter which must parametrize a curve in space-time. There are two important properties this parameter must have, the first is that increments of this parameter must always be positive, that is it must not change sign in its evolution (formally called a monotonic function), number two is that the parameter must keep the curve invariant under coordinate transformations. Fortunately, we know that the proper time fits this perfectly, therefore we parametrize the curve in space-time by the proper time (τ), and we find the length of this curve (world line) to be (3.1). Again, in the absence of external forces, particles follow geodesics which are the maximization of the length of the world line, and so by extremizing the action, we find the path particles would follow in the geometry described by $g_{\mu\nu}$, likewise the geodesic would result. Since we are free to choose the length of this curve we force it to be 1 and can, therefore, set the Lagrangian in the action to $L = 1$ later on, which we shall make our lives much easier.

At this point (3.2), if we maximize the action by solving Euler-Lagrange equations we will obtain the famous geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\delta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{3.3}$$

We now have a choice to either sub our metric in (3.3) or by solving Euler-Lagrange from (3.2), either choice would clearly result in the same equation of motions. By personal preferences we go for the latter method, taking the line element (2.7) we find the Lagrangian

$$L = \sqrt{A(r)\left(\frac{dt}{d\tau}\right)^2 - B(r)\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\theta}{d\tau}\right)^2 - r^2\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2} \tag{3.4}$$

As mentioned before we can take the Lagrangian above and set it to 1, giving us an extra equation of motion, then solving the Euler-Lagrange we explicitly get

$$1 = A(r)\left(\frac{dt}{d\tau}\right)^2 - B(r)\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\theta}{d\tau}\right)^2 - r^2\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2 \tag{3.5}$$

$$\frac{dt}{d\tau} = \frac{\alpha}{A(r)} \tag{3.6}$$

$$\frac{d}{d\tau}\left(-B(r)\frac{dr}{d\tau}\right) = \frac{A'(r)}{2}\left(\frac{dt}{d\tau}\right)^2 - \frac{B'(r)}{2}\left(\frac{dr}{d\tau}\right)^2 - r\left(\frac{d\theta}{d\tau}\right)^2 - r\sin^2\theta\left(\frac{dr}{d\tau}\right)^2 \tag{3.7}$$

$$\frac{d}{d\tau}\left(r^2\frac{d\theta}{d\tau}\right) = r^2\sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2 \tag{3.8}$$

$$\frac{d\phi}{d\tau} = \frac{\beta}{r^2\sin^2\theta} \tag{3.9}$$

Equations (3.6,3.9) have both have been further simplified from $\frac{d}{d\tau}\left(A(r)\frac{dt}{d\tau}\right) = 0$ and $\frac{d}{d\tau}\left(r^2\sin^2\theta\frac{d\phi}{d\tau}\right) = 0$, where α and β are simply integration constants we are yet to interpret.

Our goal now is to force an equation of the form $E = KE + V$ out of these differential equations, at a first glance we can see (3.5) will prove more useful, being a first order differential equation, rather than the second order (3.7).

Before continue with the maths let's think about some physics. Firstly, due to conservation of angular momentum, we know planets orbit on fixed planes, this can be seen from the independence of ϕ in (3.4). Because of this, we can set θ to a constant of our choice, which for simplicity we will choose $\theta = \frac{\pi}{2}$, consequently $\sin(\frac{\pi}{2}) = 1$ and $\frac{d\theta}{d\tau} = 0$. Secondly, to find the integration constants we take a glance at (3.9) and see it's exactly of the form of the angular momentum equation with $\beta = l$, for α we look at (3.6) and recognise $\frac{dt}{d\tau}$ from the 4-Momentum or 4-Velocity and deduce it is related to the energy of the system and we shall call it $\alpha = \epsilon$.

Following all these simplifications, the steps to take now is to collect equations (3.6), (3.9) and $\frac{d\theta}{d\tau} = 0$ then substituting them in (3.5), resulting in the expression

$$0 = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2B(r)} + \frac{l^2}{2r^2B(r)} - \frac{\epsilon^2}{2A(r)B(r)} \tag{3.10}$$

At this point, it will not be of much use to go any further without a function for $A(r)$ and $B(r)$ to which we could use any of the solutions from section 2.3, for simplicity however only the Schwarzschild solution will be used for the rest of section 2. So let's end this section by giving ourselves $A(r) = \frac{1}{B(r)} = \left(1 - \frac{2GM}{r}\right)$ and consequently (3.10) becomes

$$\frac{\epsilon^2 - 1}{2} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{l^2}{2r}\left(1 - \frac{2M}{r}\right) - \frac{M}{r} \tag{3.11}$$

Where we have rearranged it further to take the form of $E = KE + V(r)$ which will be the principal equation for the next two sections. Now $A(r)$ and $B(r)$ are given, to interpret ϵ further, take (3.6) and deduce that $\lim_{r\to\infty} \epsilon = \frac{dt}{d\tau}$, then from the first term in the 4-Momentum given by $E = p^0 = m^0 \frac{dt}{d\tau}$ we can deduce that $\epsilon = \frac{E}{m^0} \sim energy$ at very large distances r.

3.2 Falling into a Black Hole

We now look at a radially falling observer towards the singularity of a Schwarzschild black hole from on initial radial distance r_i. We are interested in the time as measured by the falling observer i.e. the proper time τ, therefore we are interested in $\frac{dr}{d\tau}$. Considering then, a free falling observer stationary from $r = \infty$ to the singularity at $r = 0$ and giving him no initial angular momentum, forcing $l = 0$, we also set $\epsilon = 1$ so that net energy $E = 0$. Applying all these conditions and substituting into the equation (3.11) we find explicitly

$$\frac{dr}{d\tau} = \pm\left(\frac{2M}{r}\right)^{\frac{1}{2}} \tag{3.12}$$

We are interested in the $(-ve)$ ingoing geodesic i.e. radially decreasing. Next, we solve this very simple differential equation to find τ given by

$$\frac{3\tau}{4M} = \left(\frac{r_i}{2M}\right)^{\frac{3}{2}} - \left(\frac{r}{2M}\right)^{\frac{3}{2}} \tag{3.13}$$

We see immediately it takes finite time to reach $r = 0$ if you are the falling object for any initial radius (r_i at $\tau = 0$), however, if we are interested in finding whether it is a harmless journey one would have to look at calculating the tidal forces on the way down, which we could expect to be very strong of course. One could now also calculate the time taken as measured by an observer from $r = \infty$ looking down at the free-falling particle through the horizon, the integral not as straightforward but nevertheless it is interesting to compare the two, by finding the coordinate time t as a function of r, so take $\frac{\frac{dt}{d\tau}}{\frac{dr}{d\tau}}$ and again we set $\epsilon = 1$ and find we would have to solve $\frac{dt}{dr} = \frac{\frac{dt}{d\tau}}{\frac{dr}{d\tau}} = -\left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{2M}{r}\right)^{-\frac{1}{2}}$

Then by making the substitution $u^2 = \frac{1}{r}$ and then separating the factor through partial fractions, one can obtain $-\int dr\left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{2M}{r}\right)^{-\frac{1}{2}} \to \int 2du\left(\frac{1}{u^2} - \frac{2M}{(2M)u^2-1}\right)$ after which we can easily enough integrate and replace r back to obtain

$$t = 2M\left(-\frac{2}{3}\left(\frac{r}{2M}\right)^{\frac{3}{2}} + 2\left(\frac{r}{2M}\right)^{\frac{1}{2}} + ln\left|\frac{\left(\frac{r}{2M}\right)^{\frac{1}{2}} + 1}{\left(\frac{r}{2M}\right)^{\frac{1}{2}} - 1}\right|\right) \tag{3.14}$$

To see what is going on and also more easily compare the two results, we plot the two results seen in the figure below 3.1. The first is time measured by the falling observer through his own falling clock (i.e. proper time) and the second is time measured by an observer at $r = \infty$ looking down at the falling observer's clock, we can immediately see by merely looking at the equations that the two are very different.

Both are falling from the same initial radial position we called r_i and therefore are both following the same trajectory. The three main points to take from this plot are; firstly, the proper time τ reaches $r = 0$ in a finite time, secondly, the Schwarzschild coordinate time t takes an infinite amount of time, appearing to never cross the horizon, which is of course due to the singular point $r = 2M$ in Schwarzschild coordinates. And thirdly, the fact that the falling observer himself falls through

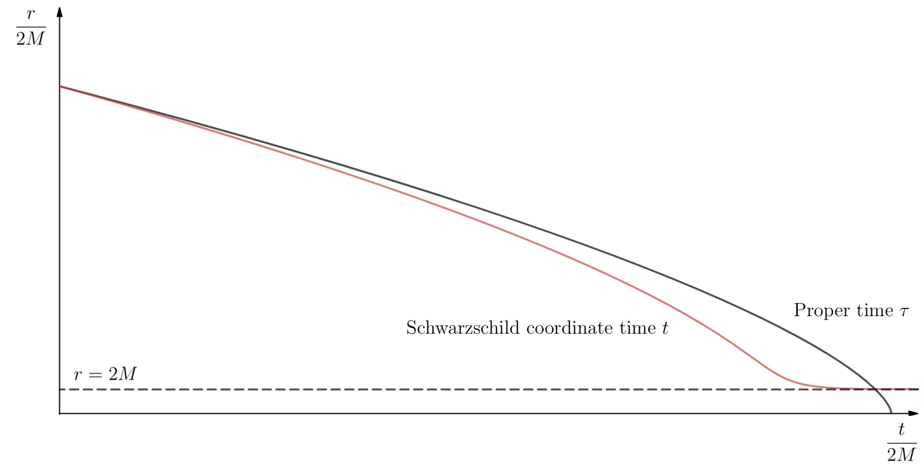

Figure 3.1: A Plot of $\frac{r}{2M}$ against $\frac{t}{2M}$, comparing the time trajectory from the proper time τ and the Schwarzschild coordinate time t. The dotted line shows the horizon at $r = 2M$.

the horizon in the proper time case, tells us that the point $r = 2M$ is a coordinate singularity only dependent on the chosen coordinates. We shall see later that in chapter 4, that Parikh had to change coordinates to Painlevé to remove this singular point $r = 2M$.

3.3 Orbiting a Black Hole

Now let us give the particle some angular momentum so it can avoid falling through the horizon if it can. We will continue to use Schwarzschild coordinates here so the values for $A(r)$ and $B(r)$ are unchanged from section 3.2 and hence we still have equation (3.11) in it's exact form. Given we are working with something of the form $E = KE + V$, we can, just like in a Newtonian problem, read off the effective potential from this and plot it to find out more about the system. Plot is shown in figure 3.2 with the explicit potential given as

$$V_{eff}(r) = \frac{l^2}{2r^2}\left(1 - \frac{2M}{r}\right) - \frac{M}{r} \tag{3.15}$$

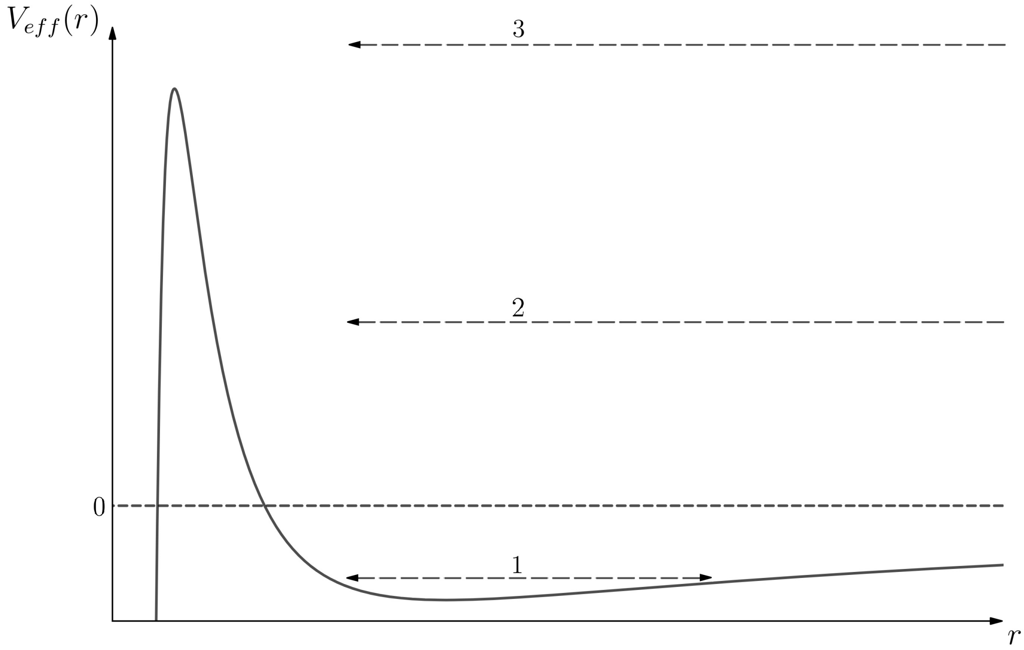

Figure 3.2: This shows a plot of $V_{eff}(r)$. Dotted line represents $V(r) = 0$ and the three lines represent the types of orbit a particle can have in this potential. r_U and r_S are the turning points representing the stable and unstable orbits.

From figure 3.2 we can deduce four classes of orbits in a Schwarzschild potential. Label (1) corresponds to a particle with energy bellow $V(r) = 0$ therefore it cannot escape the gravitational bound of the black hole, it is in orbit around the black hole as in 4.1. Label (2) corresponds to a particle coming in with more energy of that needed to escape gravitational (escape velocity) pull of the black hole, therefore this particle comes close to the black hole then experiencing its gravitational pull but is not bound consequently escaping it again. Label (3) we see the particle moves above the maxima in the graph which means in reality this particle will reach $r = 0$ and therefore fall into the black hole. There is, of course, the fourth class of orbit indicated by r_U and r_S where the particle would have a fixed potential energy consequently having circular orbits at these points, however as we see from the graph, only r_S is a stable point allowing for a stable circular orbit, a particle at r_U is at risk of falling into the black hole or falling back out for larger r. These two can of course be easily obtained by $\frac{dV}{dr} = 0 = \frac{M}{r^2} - \frac{l^2}{r^3} + \frac{3Ml^2}{r^4}$, algebraically solving for r, obtaining

$$r_S = \frac{l^2}{2M}\left(1 + \sqrt{1 - 12\left(\frac{M^2}{l^2}\right)}\right) \; ; r_U = \frac{l^2}{2M}\left(1 - \sqrt{1 - 12\left(\frac{M^2}{l^2}\right)}\right) \quad (3.16)$$

We see immediately from the expression inside the square root that there is a minimum possible value that l, the angular momentum, can take if we solve $1 - 12\left(\frac{M^2}{l^2}\right) = 0$ we find $l = 2\sqrt{3}M$ being the value for l such that $r_U = r_S$ consequently it is the innermost stable circular orbit (ISCO) r_{ISCO}, because at this point r_S is no longer stable or for any $l \ < 2\sqrt{3}M$. If we then sub this into r_S, restore c and G we find

$$r_{\text{ISCO}} = 3\left(\frac{2MG}{c^2}\right) = 3r_H \quad (3.17)$$

Where r_H is the horizon of the black hole

3.3.1 Perihelion shift

We now consider the orbital class in which we have labeled as (1) in figure 3.2 where the particle is in orbit around the black holes, with radii oscillating around the minimum potential r_S. This description would apply equally to a Newtonian potential, however, the difference between the two is subtle. In a Newtonian orbit, the particle orbiting completely closes its orbit after an orbital sweep of 2π, this is shown as an ellipse in figure 3.3. The Schwarzschild orbit we can see does not close its orbit after a 2π sweep, as every time it sweeps 2π it does not return to the same value of r. This is known as a perihelion shift and it was one the first predictions correctly tested for Einstein's theory of Relativity in which we show the prediction of the sweep given here.

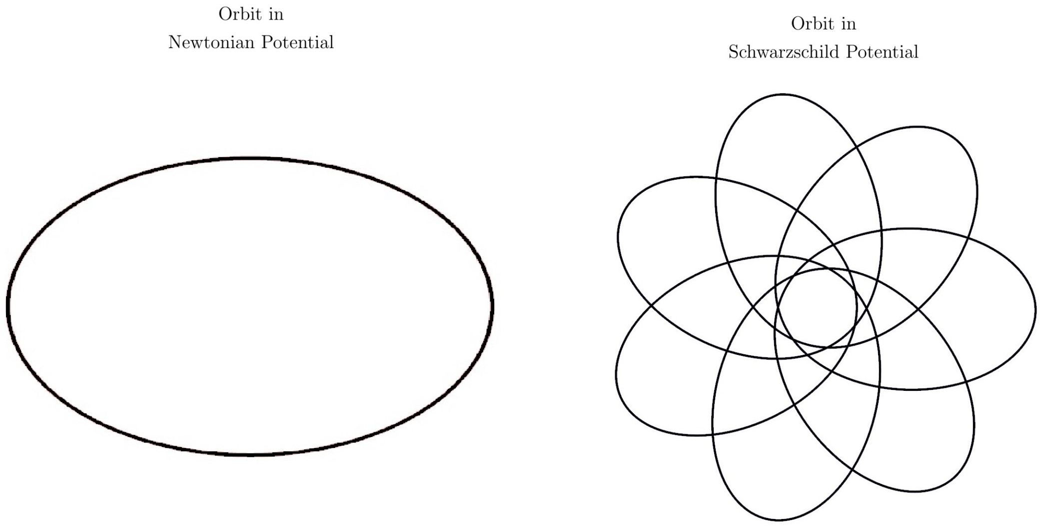

Figure 3.3: Left shows an elliptical Newtonian orbit with the 2π orbit completion, on the right the Einstein equivalent showing the very exaggerated shift of perihelion the theory predicts.

Before continuing it is important to clarify what is needed to find this shift, the orbit is described by the differential equations of motion given by (3.5 - 3.9). We are interested in the variations of $d\phi$ and dr, given we want to calculate the amount of $d\phi$ that is swept after the value of any given r returns back to the same value. The answer will be an angle ($\Delta\phi$) larger than 2π, and therefore we are after $\delta\phi \equiv \Delta\phi - 2\pi$, where $\delta\phi$ is the shift we are looking for. (For the following calculation for the perihelion shift of Mercury, we motivate from A. Zee Part VI.3, Appendix 1 of [3], and from [10]). Clearly then we must find $\frac{dr}{d\phi} = \frac{dr}{d\tau}\left(\frac{d\phi}{d\tau}\right)^{-1}$, which can be obtained by dividing (3.10) by (3.9) (with $\beta = l$ of course), with further simple algebraic manipulations one then obtains

$$2E = \left(\frac{1}{r^2}\frac{dr}{d\phi}\right)^2 + \frac{l^2}{r^2} - \frac{2Ml^2}{r^3} - \frac{2M}{r} \tag{3.18}$$

We make the substitution given by $y = \frac{1}{r} \implies dy = -\frac{dr}{r^2}$ to (3.18), where we find

$$2E = l^2\left(\frac{dy}{d\phi}\right)^2 + l^2y^2 - (2M)y - (2Ml^2)y^3 \tag{3.19}$$

Next, this equation is differentiated and further simplified, taking the form

$$0 = \frac{d^2y}{d\phi^2} + y - \frac{M}{l^2} - 3My^2 \tag{3.20}$$

The term $(3M)y^2$ in reality turns out to be very small, even for mercury [10]. Without this extra term, the equation (3.20) equates it's Newtonian counterpart, to which the solution can easily enough be found, explicitly as

$$0 = \frac{d^2y}{d\phi^2} + y - \frac{M}{l^2} \implies y_0 = \frac{M}{l^2}(1 + e \ \cos\phi) \tag{3.21}$$

Where we have set $m = 1$ of course, so l is unit (constant) angular momentum, e is given by $e = \sqrt{1 - \frac{2El^2}{M^2G^2}}$, where we have restored G. Given the solution to the Newtonian equation (3.21) is known explicitly, and the goal is to find the solution to (3.20) with the insignificant extra term $(3M)y^2$, we can treat it as a perturbation and solve it using perturbation methods. So we introduce the term $\epsilon = \frac{3M^2}{l^2}$ and also re-write $\frac{dy}{d\phi} = y'$, then

$$0 = y'' + y - \frac{M}{l^2} - \epsilon\frac{l^2y^2}{M} \tag{3.22}$$

Now this needs to be solved, assuming the solution is given by $y = y_0 + \epsilon y_1 + O(\epsilon^2)$ to which y_0 is already known. To verify this solution, substitute it back into (3.22) after differentiating it twice, to which we find

$$\begin{aligned} 0 &= y_0'' + \epsilon y_1'' - \frac{M}{l^2} + y_0 + \epsilon y_1 - \frac{3l^2\epsilon}{M}\left(y_0^2 + 2\epsilon y_0 y_1 + \epsilon^2 y_1^2\right) + O(\epsilon^2) \\ &= y_0'' + y_0 - \frac{M}{l^2} + \epsilon\left(y_1'' + y_1 - \frac{l^2y_0^2}{M}\right) + O(\epsilon^2) \end{aligned}$$

Now the powers of ϵ are separated and solved, but it can immediately be seen that the zeroth power of ϵ is given by (3.21), as expected. The only term left is of ϵ's first order $y_1'' + y_1 - \frac{l^2y_0^2}{M}$, which comes out as

$$\begin{aligned} y_1'' + y_1 &= \frac{l^2}{M}y_0^2 \\ &= \frac{M}{l^2}\left(1 + 2e\cos\phi + e^2cos^2\phi\right) \\ &= \frac{Me^2}{2l^2}\cos 2\phi + \frac{2Me}{l^2}\cos\phi + \frac{M}{l^2}\left(1 + \frac{e^2}{2}\right) \\ &= U_1\cos 2\phi + U_2\cos\phi + U_3 \end{aligned}$$

solving this through Mathematica, the differential comes to

$$y = U_3 + \left[\frac{U_2}{2} + C_0\right]\cos\phi - \frac{U_1}{3}\cos\phi - \frac{U_2\phi}{2}\sin\phi + C_1\sin\phi \tag{3.23}$$

Here all the terms oscillate on fixed value, except for the term $\phi sin\phi$ which increases as ϕ evolves around, consequently this will dominate the solution (3.23) and hence we can ignore all the other terms in this solution. After substituting this back in $y = y_0 + \epsilon y_1$ and replacing all other known parameters, we find

$$\begin{aligned} y &\approx y_0 + y_1 \\ &= \frac{M}{l^2}(1 + e\ \cos\phi) + \frac{Me\epsilon\phi}{l^2}\sin\phi \\ &= \frac{M}{l^2}\left(1 + e\ \cos\phi + e\epsilon\phi\ \sin\phi\right) \\ \implies \frac{1}{r} &\approx \frac{M}{l^2}\left(1 + e\ \cos[\phi(1-\epsilon)]\right) \end{aligned} \tag{3.24}$$

Comparing both (3.21) and (3.24) shows the difference between the Newtonian and GR orbits. In the Newtonian case, the function was periodic and closes its orbit, meaning if we shift the orbit by $\phi \to \phi + 2\pi$ we end up at the same value for r. In the GR case, equation (3.24) is still periodic, but no longer completes its orbit implying a change of period, therefore shifting this by 2π we get a different value for r. Hence we find $r(\phi + 2\pi) = r(\phi - \delta\phi)$, which means the particle ends up at the position of r it started with, shifted by some angle $\delta\phi$ which is exactly what we are here looking for. Using $r(\phi + 2\pi) = r(\phi - \delta\phi)$ we can find

$$\begin{aligned} \delta\phi &\approx 2\pi\epsilon \\ &\approx \frac{6M^2}{l^2} \end{aligned}$$

From the classical equation $L = mr^2\omega$ we deduce

$$l^2 = \left(\frac{L}{m}\right)^2 = \left(r^2\frac{d\phi}{d\tau}\right)^2 \simeq \left(r^2_-\frac{d\phi}{dt}\right)^2 = GMa(1-e^2)$$

$GMa(1-e^2)$ coming from the famous Keplers laws with a as the semimajor axis and e as the eccentricity. By restoring the speed of light c and G we then arrive at a more intuitive, modified version the of shift factor.

$$\delta\phi = \frac{6\pi GM}{ac^2(1-e^2)}$$

We can see from this to get large $\delta\phi$ we need large e, small a and large M, all of which can be easily achieved near a Black Hole. We also find from this, that the perihelion shift of mercury comes out as $\phi = 43$ arcseconds/century

3.4 Light around a Black Hole

This is a situation very similar to a massive particle but with a few subtle but very important differences, namely for massless particles we cannot use proper time as a parameter we therefore use any parameter we may like. We are therefore now varying the action $S = \int d\gamma \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\gamma}\frac{dx^\nu}{d\gamma}}$, where γ is the parameter we have chosen. Contrary to the planetary orbit case, our Lagrangian is not in the form $L = 1$ since we are not parametrizing with the length of the world line any more, but with an arbitrary parameter γ. Furthermore, light follows geodesics know as null geodesics $ds^2 = 0$ where a massive particles must follow time like geodesics $ds^2 < 0$, and so both their equations of motion are largely alike, except that for light particles we have

$$g_{\mu\nu}\frac{dx^\mu}{d\gamma}\frac{dx^\nu}{d\gamma} = 0$$

This is the same for massive particles, except that for massive we had $g_{\mu\nu}\frac{dx^\mu}{d\gamma}\frac{dx^\nu}{d\gamma} = -1$. Continuing from here, the equations of motion for light will be exactly the same as (3.6 - 3.9) except with the parameter changing from $\tau \to \gamma$, and the Lagrangian now becoming $L = 0$ which results (3.5) in taking the new form

$$A(r)\left(\frac{dt}{d\gamma}\right)^2 - B(r)\left(\frac{dr}{d\gamma}\right)^2 - r^2\left(\frac{d\theta}{d\gamma}\right)^2 - r^2\sin^2\theta\left(\frac{d\phi}{d\gamma}\right)^2 = 0 \tag{3.25}$$

Again we also want to work on the plane $\theta = \frac{\pi}{2}$ forcing $\frac{d\theta}{d\gamma} = 0$, then the same substitutions taken to find (3.10) are performed, however, we substitute them into (3.25) instead of the former equation. Furthermore, the all the parameters are changed ($\tau \to \gamma$). Following these simple manipulations, we find

$$\frac{1}{l^2}\left(\frac{dr}{d\gamma}\right)^2 + \frac{1}{B(r)}\left(\frac{1}{r^2} - \frac{\epsilon^2}{l^2 A(r)}\right) = 0$$

To continue, define the identity $b = \frac{l^2}{\epsilon^2}$, then find $\frac{dr}{d\phi}$ by using the expressions $\frac{dr}{d\gamma}$ and $\frac{d\phi}{d\gamma}$ as follows

$$\left(\frac{dr}{d\gamma}\right)^2 \left(\frac{d\gamma}{d\phi}\right)^2 = \frac{l^2}{B(r)} \left(\frac{1}{b^2 A(r)} - \frac{1}{r^2}\right) \left(\frac{r^4}{l^2}\right) \tag{3.26}$$

$$= \frac{r^4}{b^2 A(r) B(r)} - \frac{r^2}{B(r)}$$

$$= \frac{r^4}{b^2} - r^2 \left(1 - \frac{2M}{r}\right)$$

$$\therefore \left(\frac{dr}{d\phi}\right)^2 = \frac{r^4}{b^2} - r^2 + (2M)r \tag{3.27}$$

Where the $A(r) = \frac{1}{B(r)} = \left(1 - \frac{2M}{r}\right)$ have been substituted in the final line. To find the physical meaning of b, turn back to (3.26) to which we can see that by letting $r \to \infty$ we get

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4}{B(r)} \left(\frac{1}{b^2 A(r)} - \frac{1}{r^2}\right)$$

$$r \to \infty \implies \frac{dr}{d\phi} \simeq \frac{r^2}{b}$$

$$\implies b \simeq r\phi$$

Which we can deduce from simple geometry that b can be interpreted Figure 3.4, which we shall call the impact parameter.

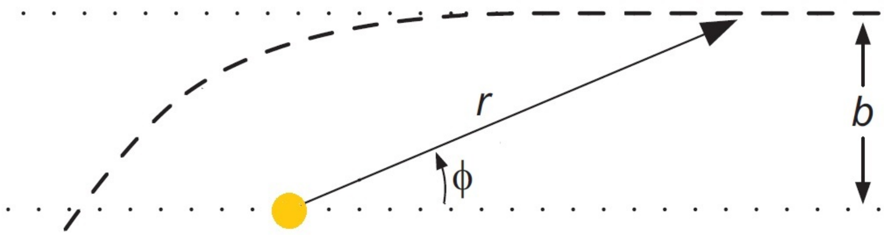

Figure 3.4: Diagram of an incoming photon from $r = \infty$ deflecting around a star, showing the role of parameter b in the process

Continuing with the derivation by solving the differential equation (3.27), where again, motivated A. Zee (chapter VI.3) [3] we make a substitution $y = \frac{1}{r}$ and re-writing $y' = \frac{dy}{d\phi}$ to find

$$
\begin{aligned}
\left(\frac{dy}{d\phi}\right)^2 &= \left(\frac{dy}{dr}\right)^2 \left(\frac{dr}{d\phi}\right)^2 \\
&= (-y^2)^2 \left(\frac{1}{y^4 b^2} - \frac{1}{y^2} + \frac{2M}{y}\right) \\
\therefore y' &= \frac{1}{b^2} - y^2 + 2My^3 \qquad (3.28) \\
\therefore y'' &= 3My^2 - y \qquad (3.29)
\end{aligned}
$$

In the last step, the equation was differentiated again, so that in the absence of the term $3My^2$ the equation (3.29) becomes the harmonic oscillator, to which the solution is explicitly known as $y_0 = \frac{\sin\phi}{b}$ (b comes from solving (3.28) without $2My^3$). Perturbation method then is the clear path to solving this, with perturbation parameter now $\epsilon = 3M$. Assuming the solution to be $y = y_0 + \epsilon y_1 + O(\epsilon^2)$ then differentiating this twice and substituting back in the original differential equation we find

$$
y_0'' + \epsilon y_1'' + y_0 + \epsilon y_1 = \epsilon(y_0^2 + 2\epsilon y_0 y_1 + \epsilon^2 y_1^2) + O(\epsilon^2)
$$

Then bringing the zero and first terms of ϵ together, where we will only need to solve the first order

$$
\begin{aligned}
\epsilon^0 &: \; y_0'' + y_0 = 0 \\
\epsilon^1 &: \; y_1'' + y_1 = \frac{\sin^2\phi}{b^2}
\end{aligned}
$$

To which solving through Mathematica, ignoring $O(\epsilon^2)$ terms and with some further algebraic manipulations we find

$$
\begin{aligned}
y &\simeq C_0\cos\phi + \frac{1}{2b^2} + \frac{\cos 2\phi}{6b^2} + C_1\sin\phi \\
&\simeq \frac{\sin\phi}{b} + \frac{M}{b^2}\left[2 - \sin^2\phi + 3b^2(C_0\cos\phi + C_1\sin\phi)\right] \qquad (3.30)
\end{aligned}
$$

Where C_0 and C_1 are integration constants, being a second order there are two boundary conditions, however they end up being $C_0 = C_1 = 0$ giving us the final expression

$$y \simeq \frac{\sin\phi}{b} + \frac{M}{b^2}\left[2 - \sin^2\phi\right]$$

Then if we were to apply this to the sun, we want the impact parameter to be $b = R_{sun}$ then to find the angle of deflection we find just need to solve for the roots ϕ as the photon comes from $r \to \infty$ then passing the sun. Then in this limit of r, clearly $y \to 0$ so we set $y = 0$ in (3.30) then solve the simple quadratic choosing the solution $\sin\phi = -\frac{2M}{b}$. If we plot $z = \sin\phi$ we find there are two points $z = -\frac{2M}{b}$ cross the $\sin\phi$ curve, one just before the origin, at $\phi \simeq -\frac{2M}{b}$ and the next just after $\phi = \pi$ at $\phi \simeq \pi + \frac{2M}{b}$, in physical terms it is the photon coming from infinity, passing the sun then leaving to infinity, therefore the deflection angle is $\delta\phi \simeq \frac{4M}{b}$ which after restoring c and G we find $\delta\phi = \frac{4GM}{c^2b} \simeq 1.75$ arcseconds for our sun.

4 Hawking Radiation as a Tunneling Effect

In this section, we calculate the transmission coefficients for all our three black holes so that we can then get to the position of finding thermodynamic state variables by using the strange yet potentially profound connection between the partition function in statistical physics and the path integral from QFT. The derivations and methods in this section are completely based on the methods introduced by Parikh and Wilczek in [1], where they derived the tunnelling probability for the Schwarzschild and Reissner - Nordström black hole. Fairly soon after their publication S. Hemming and K. Vakkuri in [2], motivated by Parikh and Wilczek's methods, derive the tunnelling probability for the Schwarzschild-AdS black hole which we shall also repeat in this section.

4.1 Transmission coefficient

Since we are concerned with the tunnelling of particles it is common sense to begin by finding the transmission coefficient (Γ), which is, of course, our chosen means of finding hawking radiation of the three black holes. However the standard expression of the transmission coefficient (Γ) that Undergraduates first encounter on a quantum mechanics course, will not be suitable for our porpuses, consequently we must find an expression for Γ we can work with. Similar to [1, 2] we could jump straight to the explicit expression for (Γ) that we require then continue with our derivation, explicitly it is given as

$$\Gamma = e^{-\frac{2}{\hbar}\mathrm{Im}\ I} \tag{4.1}$$

However, it will be practical for our purposes to show some motivation for obtaining (4.1), which to do so we must turn to the famous result known as Feynman's path integral. In short the path integral is an equivalent, but different, formalism to the Heinsenberg picture of quantum mechanics $\langle q_I|e^{-\frac{i}{\hbar}HT}|q_F\rangle$, where we have the initial (q_F) and final (q_I) states with the evolution of their amplitudes (from q_I to q_F) completetly dictated by the operator $e^{-\frac{i}{\hbar}HT}$, consisting of the time taken for the transition T and the Hamiltonian H. The proof in which the path integral is forced out of $\langle q_I|e^{-\frac{i}{\hbar}HT}|q_F\rangle$ is elegantly shown by A. zee in chapter I.2 of [4]. It consists of dividing the time taken T into many portions of length δt then using the completed set of states for q result $\int |q\rangle\langle q|dq = 1$, by further manipulations and identities one could show

$$\langle q_I|e^{-\frac{i}{\hbar}HT}|q_F\rangle = \int Dq(t)e^{-\frac{i}{\hbar}\int_0^T L(\dot{q},q)dt} \tag{4.2}$$

This expression describes the integral over possible paths in space $Dq(t)$ with their phases $e^{-\frac{i}{\hbar}\int_0^T L(\dot{q},q)dt}$, here we also notice the functional $\int_0^T L(\dot{q},q)dt$ which we know very well to be the action $I[q]$, allowing us to re-write the phase as $e^{-\frac{i}{\hbar}I[q]}$. It now seems straightforward why (4.2) is useful in our case since we can easily find expressions for the action from the metrics of the black holes, however, it still does not represent any transmission coefficient as in (4.1). It is important to add that we disregard the integral over the paths $\int Dq(t)$ as we will be dealing with metrics that maximize the action, forcing one path to dominate the integral, and we therefore pull out $e^{-\frac{i}{\hbar}I[q]}$ on its own. We can think of the term $e^{-\frac{i}{\hbar}I[q]}$ like a quantum mechanic wave function so that in the end we must mod it and square it to find the probability. Parikh noticed that $I[q]$ becomes complex as the particle tunnels through the horizon, this then greatly simplifies things for us, because we can now write the action as $I[q] = \text{Re } I[q] + \text{Im } I[q]$ then substituting it back in and taking the square of the mod of $e^{-\frac{i}{\hbar}I[q]}$ so that we are computing the probability of tunnelling Γ, we find

$$\begin{aligned}
\Gamma &= \left| e^{-\frac{i}{\hbar}(\text{ Re } I[q] \ + \ i \text{ Im } I[q] \ )} \right|^2 \\
&= \left(e^{-\frac{i}{\hbar}(\text{ Re } I[q] \ + \ i \text{ Im } I[q] \ )} \right)^* \left(e^{-\frac{i}{\hbar}(\text{ Re } I[q] \ + \ i \text{ Im } I[q] \ )} \right) \\
&= e^{-\frac{2}{\hbar}\text{Im } I[q]}
\end{aligned}$$

We have arrived at (4.1), but we are not yet finished as we must re-write the action as something easier to work with, at the moment we have $I[q] = \int_0^T L(\dot{q},q)dt$ and we know the Hamiltonian can be written as $H = p_q\left(\frac{dq}{dt}\right) - L$, since we are interested in the net energy (energy conservation through the process is important) we force H to be a constant which for convenience we choose to be zero, this allows us to solve for L and substitute this back in the action and re-write the action as $I[q] = \int p_q dq$. To find the transmission coefficient we must solve the expression

$$\text{Im } S = \text{Im } \int_{q_I}^{q_F} p_q dq \tag{4.3}$$

4.1.1 The Tunnelling Model

Before continuing, it will be somewhat handy to revise the model we are using through the help of a diagram and some short explanations. We are considering the same scenario in all three black holes, the model Parikh consider in [1] is pair production (of a positive and negative mass particle of energy ω each) occurring just beneath the horizon, the positive particle then tunnels out and the negative recombines with the black hole, causing it to lose mass ω. This is shown in the diagram below which will help us to find what the limits are for the integral in the action,

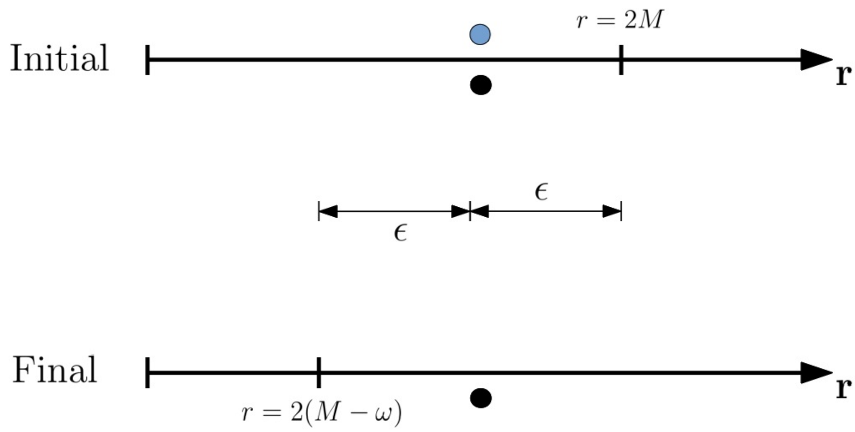

Figure 4.1: This is a diagram showing the tunnelling model we are using, it shows on the top the initial condition of pair production occurring just beneath the horizon (*Black* $= +ve$ energy particle and *Blue* $= -ve$ energy particle). The lower is the final condition which shows the tunnelled $+ve$ energy particle and the shifted horizon to the lower radii. We can see the total mass of the black hole decreasing by ω and the small shift in the radius of the horizon by ϵ.

The particle itself cannot of course escape through any classical trajectory through the horizon, and we are not assuming it does in this model. It is the decreasing radial movement of the horizon past the created ($+ve$ energy) particle that we assume allows the particle to tunnel past the horizon. It is also important to note, as we did in the above section, that it is as the horizon passes the particle that the action acquires its complex part, conveniently allowing us to get a real answer for the tunnelling or transmission coefficient. This was the elegant mathematical trick Parikh [1] noticed allowing him to easily perform the calculation we are now repeating.

We can now do some manipulations to (4.3) and jump into the calculations, first we can replace q with r since it is the only variable we are interested in and we can then expand the momentum p_r into its own integral, explicitly we then get

$$\begin{aligned} Im\ S &= Im \int_{r_{in}}^{r_{out}} p_r dr \\ &= Im \int_{r_{in}}^{r_{out}} dr \int_0^p dp' \end{aligned}$$

We can use Hamilton's equations $\frac{\partial H}{\partial p_r} = \dot{r}$ and $\frac{\partial H}{\partial r} = \dot{p_r}$, then clearly $\dot{r} = \frac{dH}{dr}|_r$ giving us $dp = \frac{dH}{\dot{r}}$. Then, we can use the equation $H = M - \omega$ (since we know the net energy is the loss of the particle by the black hole) to find that $dH = -d\omega$, substituting these we get

$$\begin{aligned} Im\ S &= Im \int_{r_{in}}^{r_{out}} dr \int_M^{M-\omega} \frac{dH}{\dot{r}} \\ &= -Im \int_{r_{in}}^{r_{out}} dr \int_0^{\omega} \frac{d\omega'}{\dot{r}} \end{aligned} \tag{4.4}$$

From the diagram, one can see for example that $r_{out} = 2(M - \omega) + \epsilon$ and $r_{in} = 2M - \epsilon$, we have arrived at the point which we must return to our black hole solutions to find what $\dot{r}$ is for each so that we can solve the integral and find the tunnelling probability.

4.2 General Non-singular Coordinates

Now that we have revised the model and know explicitly what we must find and solve, we must "shut up and calculate" as Richard Feynman famously said. We start in this section by finding a general radial null geodesic $\dot{r}$ to avoid repetition and so that we can then perform the relevant integrals for our three black holes. The coordinates introduced for all three Black Holes contain singularities across their horizons, which will be no good when performing integrals, so the first thing to do is change coordinates to more suitable ones. Fortunately, all three metrics have the same general form of

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2 d\Omega_{d-2}^2$$

We perform the transformation $t = \hat{t} + f(r)$ to the above metric, these were first introduced by Painlevé in 1921, and conveniently used in by Parikh and Wilczek in [1] and as shall see it successfully removes the coordinate singularities for all three black hole's. Performing this transformation gives:

$$ds^2 = -F(r)d\hat{t}^2 - 2f'(r)F(r)d\hat{t}dr + \left(\frac{1}{F(r)} - F(r)(f'(r))^2\right)dr^2 + r^2 d\Omega_{d-2}^2 \quad (4.5)$$

To avoid repeating the methods used in [1] and for the three cases, we find one general expression which can then be applied to all three more quickly later on. The next step would be to find $f'(r)$ for our three black holes, to do this for a general case we demand that for constant time slices ($d\hat{t} = 0$) the metric reduces to the form

$$ds^2 = g(r)dr^2 + r^2 d\Omega_{d-2}^2 \quad (4.6)$$

The Schwarzschild and the charged black holes are both asymptotically flat so for constant time slices, we simply have $g(r) = 1$. The AdS black hole however, is not asymptotically flat, it is at this point where the methods of Hemming and K. Vakkuri [2] take a slight detour from [1]. Asymptotically, the AdS black hole becomes AdS global giving us the value of $g(r)$ to be $g(r) = \frac{1}{1+r^2}$.

Continuing with our goal to find a general expression, by forcing (4.5) to equate to (4.6) we need to simple solve the algebraic expression for $f'(r)$ given by:

$$g(r) = \frac{1}{F(r)} - F(r)(f'(r))^2$$

Substituting $f'(r)$ into (4.5) we get our desired non singular metric, explicitly given by

$$ds^2 = -F(r)d\hat{t}^2 + 2\sqrt{1 - F(r)g(r)}d\hat{t}dr + g(r)dr^2 + r^2 d\Omega_{d-2}^2$$

This clearly reduces to (4.6) for constant time slices, furthermore, the singularity has also successfully been removed.

The next and last thing we find for the general case is the null, radial geodesics which for obvious reasons will be necessary when we look at particles tunnelling out. To do this we set $ds = 0$ and solve for $\dot{r} \equiv \frac{dr}{d\hat{t}}$, giving us

$$\dot{r} = \frac{1}{g(r)} \left(\pm 1 - \sqrt{1 - F(r)g(r)} \right) \tag{4.7}$$

This is the radial null geodesic, with a $(-ve)$ ingoing geodesic and our desired $(+ve)$ outgoing geodesic. It is not yet clear here because of $F(r)g(r)$ but bellow the horizon for all three cases, $\dot{r}$ is decreasing for both $+ve$ and $-ve$ geodesics. So using the methods of both [1] and [2] we have arrived at a general expression for the radial null geodesic, from here on $g(r) = 1$ for the Reissner - Nordström and Schwarzschild cases or $g(r) = \frac{1}{1+r^2}$ for the AdS case so we shall not repeat the steps just taken for the rest of the paper but jump straight to the expression of $\dot{r}$ for the relevant black hole.

4.3 The Schwarzschild Black Hole

We are now in a position to find the transmission coefficient (Γ) for the Schwarzschild black hole, from the line element (2.13) we deduce that $F(r) = \left(1 - \frac{2M}{r}\right)$. We also know, that asymptotically Schwarzschild spacetime tends to flat space, therefore for constant time slices we find $g(r) = 1$ back in (4.6). If we substitute these in the radial null geodesic (4.7) we find

$$\dot{r} = \pm 1 - \sqrt{\frac{2M}{r}}$$

We can then substitute this into the integral (4.4) to find

$$\text{Im } S = -\text{Im} \int_0^{\omega} d\omega' \int_{2M}^{2(M-\omega)} \frac{1}{1 - \sqrt{\frac{2(M-\omega')}{r}}} dr$$

There are several methods of going about this integral, one such method is by the substitution $r = u^2$ and $u_0 = \sqrt{2(M-\omega)}$, after which we obtain $2\int \frac{u^2}{u-u_0} du$. Recalling that we are in search of the imaginary part of the integral and that the contour goes over a pole at $u = u_0$. Therefore the next step, is to deform the contour over and around u_0 with a semicircle through the substitution $u - u_0 = Re^{i\theta}$ and run the integral from π to 0 ignoring any other part of the contour on the real line, since it will not contribute with any imaginary factor. Lastly we take limit as $R \to 0$ since we are only interested on the pole i.e. the horizon. Explicitly it goes

$$
\begin{aligned}
\text{Im } S &= -\text{Im } 2i \int_0^{\omega} d\omega' \int_{\pi}^{0} (Re^{i\theta} + u_0)^2 d\theta \\
&= \pi \int_0^{\omega} 4(M - \omega')d\omega' \\
&= -Im \int_0^{\omega} 4(-i\pi)(M - \omega')d\omega' \\
&= 4\pi\omega \left(M - \frac{\omega}{2}\right)
\end{aligned}
$$

We have arrived at what we wanted, a value for Im S, we finish by substituting back into (4.1) to find our tunnelling probability to be

$$
\Gamma = e^{-8\pi\omega\left(M - \frac{\omega}{2}\right)} \tag{4.8}
$$

In search of the temperature, we now ignore the quadratic term for ω we find it reduces to the familiar Boltzmann factor $e^{-\beta E}$. Small energies are also the probable energy as the process of pair production is unlikely to occur here for any massive particle. Comparing the tunnelling factor with the Boltzmann factor, we then get the Hawking temperature

$$
T_H = \frac{1}{8\pi M}
$$

Parikh and Wilczek further stated that the entropy of the black hole can be obtained through the relation they stated as

$$
\Gamma \sim e^{-2\text{Im}S} = e^{\Delta S_{B-H}}
$$

The entropy can be found by setting $\omega = M$ in (4.8), (Fleming goes in more detail here [9]) then the entropy is found as

$$
S_{B-H} = 4\pi M^2 = \pi(2M)^2
$$

Which follows, as expected the Bekenstein-Hawking entropy $S = \frac{A}{4}$ as $r_H = 2M$

4.4 The Reissner - Nordström Black Hole

Ans now we repeat the process with a slight difference from 4.3. First thing is we know $F(r) = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)$ from (2.16). We also remind ourselves that this black hole is also asymptotically flat for constant time slices forcing $g(r) = 1$ in (4.6), it is also important to note here the horizon of the black hole has of course changed to $r_H = M \pm \sqrt{M^2 - Q^2}$. Consequently then, from $F(r)$ and $g(r)$ the outgoing radial null geodesic becomes

$$\dot{r} = 1 - \sqrt{\frac{2M}{r} - \frac{Q^2}{r^2}}$$

Which we can immediately substitute in (4.4) to find

$$\begin{aligned} \text{Im } S &= -\text{Im} \int_{r_{\text{in}}}^{r_{\text{out}}} \int_0^{\omega} \frac{1}{1 - \sqrt{\frac{2(M-\omega')}{r} - \frac{Q^2}{r^2}}} d\omega' \; dr \\ &= -\text{Im} \int_{r_{\text{in}}}^{r_{\text{out}}} \int_0^{\omega} \frac{r}{r - \sqrt{2r(M-\omega') - Q^2}} d\omega' \; dr \end{aligned}$$

Then, following Parikh [1], set $u = \sqrt{2(M-\omega')r - Q^2}$ giving $du = \frac{-r}{u} d\omega'$, allowing us to then repeat the method used in section 4.3, only here we integrate with respect to energy ω first. By deforming the contour over and around the pole at $u = r$ with a semicircle, consequently arising in the factor of $i\pi$ just as before. Explicitly we find

$$\text{Im } S = \text{Im} \int_{r_{\text{in}}}^{r_{\text{out}}} \int_{\sqrt{2Mr - Q^2}}^{\sqrt{2(M-\omega)r - Q^2}} \frac{u}{r - u} du \; dr$$

Here we sub $u - r = Re^{i\theta}$, apply the new limits of the semicircle contour then take the limit $R \to 0$ as before and find

$$\begin{aligned} \text{Im } S &= \text{Im } i \int_{r_{\text{in}}}^{r_{\text{out}}} \int_{\pi}^{0} (Re^{i\theta} + r) d\theta \; dr \\ &= -\pi \int_{r_{\text{in}}}^{r_{\text{out}}} r \; dr \\ &= -\frac{\pi}{2} r^2 |_{r_{\text{in}}}^{r_{\text{out}}} \end{aligned}$$

Finally, recalling that in this case $r_{\text{out}} = (M - \omega) + \sqrt{(M-\omega)^2 - Q^2}$ and $r_{\text{in}} = M + \sqrt{M^2 - Q^2}$, we substitute our result for ImS in (4.1) to find the same result as Parikh [1], explicitly

$$\Gamma = e^{-2\pi\left(2\omega(M - \frac{\omega}{2}) - \left((M-\omega)\sqrt{(M-\omega)^2 - Q^2} - M\sqrt{M^2 - Q^2}\right)\right)}$$

We can now Taylor expand the exponent in ω, and take the first order

$$f(\omega) = -2\pi\left(2\omega\left(M - \frac{\omega}{2}\right) - \left((M-\omega)\sqrt{(M-\omega)^2 - Q^2} - M\sqrt{M^2 - Q^2}\right)\right)$$

$$\therefore \frac{df}{d\omega} = -2\pi\left(2M - 2\omega + \sqrt{(M-Q)^2 - Q^2} + \frac{(M-\omega)^2}{\sqrt{(M-\omega)^2 - Q^2}}\right)$$

$$\implies \omega\frac{df}{d\omega}\bigg|_{\omega=0} = -2\pi\omega\frac{\left(M + \sqrt{M^2 - Q^2}\right)^2}{\sqrt{M^2 - Q^2}}$$

Again to find the temperature, we equate this first order term to $e^{-\beta E}$ to find the hawking temperature of the Reissner-Nordström black hole to be

$$T_H = \frac{\sqrt{M^2 - Q^2}}{2\pi(M + \sqrt{M^2 - Q^2})^2}$$

First notice, by setting $Q = 0$ again we get to the Schwarzschild temperature $T = \frac{1}{8\pi M}$ as expected. Secondly, for the naked singularity $Q > M$, the black hole with no horizon, we find there is no real solution for the temperature, which is also expected as there is no horizon for any radiation to arise. Finally, to calculate the entropy, we rearrange $f(\omega)$ as follows

$$\begin{aligned} f(\omega) + 2\pi M^2 - 2\pi M^2 + \pi Q^2 - \pi Q^2 = &-\pi\left[M + \sqrt{M^2 - Q^2}\right]^2 \\ &+ \pi\left[(M-\omega) + (M-\omega)\sqrt{(M-\omega)^2 - Q^2}\right]^2 \end{aligned}$$

To which we know this equates to the difference in entropy before and after the particle leaving the black hole as $\Delta S_{B-H} = S_{B-H}(M) - S_{B-H}(M-\omega)$ to which the entropy can then be found to be

$$S_{B-H} = \pi\left(M + \sqrt{M^2 - Q^2}\right)^2$$

Which we can see, given $r_+ = M + \sqrt{M^2 - Q^2}$, that like the previous case this also follows the thermodynamic law $S_{B-H} = \frac{4\pi r_+^2}{4G} = \frac{A}{4G}$. An interesting observation, is that for extremal black holes ($Q = M$) the temperature is zero and therefore of no physical meaning, however the entropy does not go to zero and still applies.

4.5 Schwarzschild-AdS Black Hole

And now for the last black hole, we find the same result for the AdS_{d+1} black hole, now using the methods from Hemming [2]. In this last case we are dealing with an arbitrary dimensional black hole, so there will be some differences to the previous two. We would normally begin again by finding the radial null geodesic, by the substitution of $F(r) = 1 - \frac{\mu}{r^{d-2}} + \frac{r^2}{b^2}$ and $g(r) = \frac{1}{1+r^2}$ into the general radial null equation (4.7). Following Hemming, we set the AdS radius $b = 1$, obtaining

$$F(r) = 1 - \frac{\mu}{r^{d-2}} + r^2 \tag{4.9}$$

Where again, μ is here given by (2.28). Then, by using this new equation (4.9) we find the radial null to be

$$\dot{r} = -\sqrt{\frac{\mu(1+r^2)}{r^{d-2}}} \pm (1+r^2)$$

Continuing with the usual procedure, we need to replace μ by μ' in order to take care of the loss in energy of the black hole of the leaving particle, found from Per Kraus and Frank Wilczek [Self-Interaction Correction to Black Hole Radiance], the Self-Interaction Correction to Black Hole Radiance. Therefore we have

$$\mu' = \frac{16\pi}{(d-1)A_{d-1}}(M - \omega)$$

And the integral consequently becomes

$$\begin{aligned}
\text{Im } S &= -\text{Im} \int_0^{\omega} \int_{r_{\text{in}}}^{r_{\text{out}}} \frac{1}{(1+r^2) - \sqrt{\frac{\mu'(1+r^2)}{r^{d-2}}}}\, dr\, d\omega' \\
&= -\text{Im} \int_0^{\omega} \int_{r_{\text{in}}}^{r_{\text{out}}} \frac{1 + \sqrt{\frac{\mu'}{r^{d-2}}(1+r^2)^{-1}}}{F(r)}\, dr\, d\omega'
\end{aligned} \tag{4.10}$$

Like both the previous times the only imaginary factor this integral picks up is at the horizon $F(r_H) = 0$, so we know we can ignore the integral in any other point on the real line and focus on the pole at r_H. Evidently we must then expand $F(r)$ around the pole to find the answer, and of course not forget the arising of $-i\pi$ as we deform the contour. Expanding $F(r)$ we find

$$F(r_H) \simeq F(r_H) + (r - r_H)\frac{dF}{dr}\bigg|_{r=r_H} + \sigma((r-r_H)^2)$$

We can immediately simplify as we know $F(r_H) = 0$, then ignoring any terms of order two or higher we find

$$\begin{aligned} F(r_H) &\simeq (d-2)\frac{\mu}{r_H{}^{d-1}} + 2r_H \\ &= \frac{1}{r_H}\left((d-2) + r_H{}^2 d\right) \end{aligned}$$

Following from (4.10), we substitute the above, along with the factor of $-i\pi$ and also note that the numerator $1 + \sqrt{\frac{\mu'}{r_H{}^{d-2}}(1+r_H{}^2)^{-1}} = 2$. We then we find

$$\begin{aligned} \text{Im } S &= -\text{Im} \int_0^\omega (-2i\pi)\frac{r_H}{(d-2) + r_H{}^2 d}\, d\omega' \\ &= \pi \int_0^\omega \frac{2r_H}{(d-2) + r_H{}^2 d}\, d\omega' \end{aligned} \tag{4.11}$$

At this point, following Hemming [2], we performed a change of variables. First we find μ as a function of r_H from $F(r_H)$, to then find $\mu = r_H{}^{d-2} + r_H{}^d$. Then, we can find the expression

$$\frac{2r_H}{(d-2) + r_H{}^2 d} = 2r_H{}^{d-2}\frac{dr_H}{d\mu}$$

Next using equation (4.5) we can find write $d\omega'$ as

$$d\omega' = -\frac{(d-1)A_{d-1}}{16\pi} d\mu' \tag{4.12}$$

Finally then, by substituting this back in (4.11) and finding the new limits from (4.5) we find

$$\text{Im } S = -\frac{(d-1)A_{d-1}}{16} \int_\mu^{\mu - \frac{16\pi}{(d-1)A_{d-1}}\omega} 2r_H{}^{d-2}\, \frac{dr_H}{d\mu'} d\mu'$$

Which we can easily enough solve to find

$$\begin{aligned}
\text{Im } I &= -\left(\frac{A_{d-1}}{8}\right) {r_H}^{d-1}\Big|_{\mu}^{\mu-\frac{16\pi}{(d-1)A_{d-1}}\omega} \\
&= \left(\frac{A_{d-1}}{8}\right)\left(\frac{16\pi}{(d-1)A_{d-1}}\right)^{d-1}\left(M^{d-1}-(M-\omega)^{d-1}\right) \\
&= \frac{2\pi}{d-1}\left(\frac{16\pi}{(d-1)A_{d-1}}\right)^{d-2}\left(M^{d-1}-(M-\omega)^{d-1}\right)
\end{aligned}$$

To which we could also Taylor expand then take the first order like in the previous section to compare with the Boltzmann factor and find the Hawking temperature, to which we will calculate an explicit expression for in the next section. For the entropy, rewrite the result as Hemming [2] showed

$$e^{-2\text{Im } I} = e^{-\frac{A_{d-1}}{4}\left(r^{d-1}(M)-r^{d-1}(M-\omega)\right)} = e^{\Delta S_{B-H}}$$

Where, as stated by Hemming $\Delta S_{B-H} = S_{B-H}(M) - S_{B-H}(M-\omega)$ giving us the entropy of the black hole at mass M as

$$S_{B-H} = \frac{A_{d-1}}{4} r^{d-1}$$

It is so that the term $r_+^{d-1}A_{d-1}$ turns out to be the surface area of a d-sphere with radius r_+ therefore the surface area of the horizon A. Thus, again the famous Bekenstein-Hawking entropy is retained as $S_{B-H} = \frac{A}{4}$

5 Thermal Phase Transition of Schwarzschild-AdS Black Holes

So far in this paper, Hawking radiation has been derived via the tunnelling methods introduced by Parikh and Wilczek [1], where the radiation was derived for a charged and non charged black hole sitting in flat space (Reissner-Nordström and Schwarzschild solutions respectively). Additionally from Hemmings methods, Hawking radiation was found for a black hole sitting in anti-de Sitter space (Schwarschild-AdS Solution), however, a thermodynamic analysis has not yet been addressed here. In this section then, we find thermodynamic properties of the more interesting of these three cases, the black hole in anti-de Sitter space where it was found by Hawking and Page that black holes undergo a phase transition for a given temperature, can be in thermal equilibrium with and, as we shall see, at hight enough temperatures be favoured over the surrounding thermal radiation. The methods in the paper from Hawking and Page we shall here review was calculating the finite difference of the Einstein action for AdS and Schwarszchild-AdS metrics, from which thermodynamic properties can consequently be found from the action difference. So before continuing, the connection between statistical physics and the path integral will be necessary if we are to find thermodynamic state variables in the end. This connection is well known and reviewed by [3, 4, 6, 8].

5.1 Statistical Physics and the Path Integral

Boltzmann and statistical physics

We will later be in search of thermodynamic state variables, and when in thermal equilibrium these variables can be understood from the point of view of the particles that make up such thermodynamic systems, in particular we understand them using statistical mechanics by defining ensembles and subsequently finding the relevant partition function Z. The canonical ensemble, in statistical physics, describes the possible states a system can have when it is in thermodynamic equilibrium with the heat bath it lies in, therefore it can exchange energy with the heat bath it lies in. A partition function Z, can be calculated for these ensembles, where the partition function themselves are functions of thermodynamic state variables, i.e. Temperature T or Entropy S. As a result, it then allows us to calculate thermodynamic properties using partition functions, which is why they are so useful in statistical physics. Note there are three such ensembles, the microcanonical, the canonical and the grand canonical ensembles, however we shall only be talking about the canonical ensemble here.

If the system is enclosed in a heat bath with a temperature T, which itself is not affected by the system enclosed inside it, then we can focus only in the microstates of the system μ_n, each with energy E_n and probability $P(\mu_n)$. Then by enforcing that

$$\sum_{\mu_n} P(\mu_n) = 1 \tag{5.1}$$

We are therefore enforcing the system to be in a state so that the sum of all the microstates must satisfy (5.1). This then implies the introduction of a normalization function called the partition function Z, to which we then have

$$P(\mu_n) = \frac{1}{Z} e^{-\beta E_n} \qquad \text{where} \qquad Z = \sum_{\mu_n} e^{-\beta E_n}$$

From this function, many thermodynamic quantities can be derived and is therefore essential. For latter, we can find the energy and entropy using the partition function, where the average energy by summing all the E_n each with their equivalent probability, it can be found to be

$$\langle E \rangle = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} = -\frac{\partial \log Z}{\partial \beta} \tag{5.2}$$

In similar ways the entropy S and free energy F can also be found, explicitly they are

$$S = \beta \langle E \rangle + \log Z \qquad\qquad F = -T \log Z \tag{5.3}$$

The reason for stating all these here is due to the connection logZ will have to the action, which we shall see. A further note, is that if we take the system to be quantum mechanical then the energy becomes the Hamiltonian H and the partition function can be stated in the forms

$$Z = \sum_n e^{-\beta H} = \sum_n \langle q_n | e^{-\beta H} | q_n \rangle = \text{Tr } e^{-\beta H} \tag{5.4}$$

Heisenberg and the path integral

Now we move to quantum mechanics, in particular lets go back to section 4.1 where we looked at Heisenberg formulation of an evolving quantum mechanical system governed by e^{-iHT}, where the probability to go from an initial state q_I to a final state q_F, which can also be written as the path integral form, all given by

$$Z = \langle q_F | e^{-iHT} | q_I \rangle = \int Dq(t) e^{-\frac{i}{\hbar} I[q]} \tag{5.5}$$

Now the next step is beyond undergraduate level as it is dealing with quantum field theory and quantum gravity, but nevertheless, let's try tackle the basics. When moving to quantum field theory (QFT), special relativity is added to quantum mechanics, and the path integral looks parallel to (5.5) except that wave functions q are replaced by fields ϕ. From 4.1 we change, $Dq \to \mathcal{D}\phi$ where we are now dealing and fields ϕ, and the action similarly becomes $I[q] \to I[\phi]$ where the Lagrangian in the action has changed to the Lagrangian density given by $\mathcal{L}(\phi, \partial_\mu \phi)$. Therefore by simply adding special relativity we have moved from q to ϕ and instead of simply taking the time derivative $\dot{q}$ for $L(\dot{q}, q)$ we now have a partial ∂_μ giving rise the the new Euler-Lagrange equations to take the new form

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \qquad \to \qquad \frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right)$$

We can then rewrite the path integral as the following

$$Z = \langle \phi_F | e^{-iHT} | \phi_I \rangle = \int \mathcal{D}\phi e^{-\frac{i}{\hbar} I[\phi]} \tag{5.6}$$

At which point there is a problem, the action is given by the integral of the Lagrangian $\mathcal{L}(\phi, \partial_\mu \phi)$ which is given by a real number, therefore the term $e^{-\frac{i}{\hbar} I[\phi]}$ will be oscillatory and could cause divergent problems. To ensure it converges we perform a Wick rotation, a rotation on t by $\frac{\pi}{2}$ to the complex plane $t = -i\tau$ which results in the Lagrangian to become imaginary which further results in $e^{-\frac{1}{\hbar} I[\phi]}$ where the action is now Euclidean as it has positive signature, and this term now converges. Another important change after the Wick rotation was what happened to the time T which is the time taken to go from ϕ_I (at time t_I) to ϕ_F (at time t_F) therefore

$T = t_F - t_I$, consequently after the rotation $T = -i(\tau_F - \tau_I)$.

Relation between these two

We now look at both cases, and we see that to force one into the other we must do a slightly strange transformation. First we set the time to be imaginary $T \rightarrow -i\beta$ then we set the final and initial state to be come back on to itself $|\phi_I\rangle = |\phi_F\rangle = |\phi_n\rangle$. This procedure forces time to be cyclic, as $T = -i(\tau_F - \tau_I) = -i\beta$ with the time taken to complete the cycle being given by β, therefore we find that in this way the period is given by the inverse temperature. We can explicitly see that under this transformation (5.6) becomes the partition function if we sum over all ϕ

$$Z = \mathrm{Tr} e^{-\beta H} = \int \mathcal{D}\phi e^{-I[\phi]} \tag{5.7}$$

Thus the strange relation between the path integral and partition function is found. From (5.7), connected to statistical physics is the partition function Z and the trace element, and equating to these is the path integral, governed by the action, which is itself dependent from general relativity and gravity. A potentially profound connection then. However, as a final step lets force the path integral into something we can work with, the biggest contributions to the path integral will be the semi-classical cases, as $\hbar \rightarrow 0$. This is the saddle point approximations, where we are assuming the only contributions to the path integral will be near the extremum of the action. If then, the path integral is only evaluated at these extremal points, (5.7) then takes the simpler form

$$Z = e^{-I} \qquad \Longrightarrow \qquad \log Z = -I \tag{5.8}$$

From which can be seen immediately that the thermodynamic equations obtained through the partition function as (5.2) and (5.3) can written in terms of the action $I[\phi]$ as

$$\langle E \rangle = \frac{\partial I}{\partial \beta} \qquad ; \qquad S = \beta \langle E \rangle - I \qquad ; \qquad F = TI \tag{5.9}$$

The next sections will all be dedicated to finding an explicit expression for I so that the thermodynamic quantities can be calculated and used to analyze the phase transition found by Hawking and Page [5]. We will find that for the wick rotation to give a sensible result, the period of the Euclidean metric must be fixed to one value for the space to be smooth through the horizon, and that the fixed value for the period (which is given by β) result to be the temperature of the Schwarzschild-AdS black hole.

5.2 Phase Transition in Anti-de Sitter Space

5.2.1 The Euclidean Metric and Periodic Time

There is a second, simpler method of calculating the temperature of all of the black holes in this paper, and here the temperature of the AdS_d black hole is found by forcing the Euclidean metric to be smooth at $r+$. This was first done by Hawking and Page with $d = 3$, [5] but here we follow E. Witten, who based on Hawking's methods found the same result for arbitrary d AdS black holes. The metric we have been using so far all have the Lorentzian signature $(-,+,+,+)$ however by performing a wick rotation $t \to -i\tau$ therefore rotating time into the complex plane, the metric now takes a Euclidean signature $(+,+,+,+)$ since $-dt^2 = d\tau^2$. What is more the new time τ now has a period $(\tau \sim \tau + \beta)$ associated with it given by $\beta = \frac{1}{k_B T}$ (from now, $k_B = 1$ unless mentioned otherwise), and as shall now be seen, this will need to be fixed to a fixed value so that the new Euclidean metric is smooth at the horizon r_+.

To introduce the concept, take the familiar polar coordinates given by the line element

$$ds^2 = d\varrho^2 + \varrho^2 d\theta^2 \tag{5.10}$$

In this space, the angle θ can take the values $[0, 2\pi]$ at which the space would clearly be flat, and $\varrho = 0$ is non-singular. However, if the values of theta are reduced to $(0, 2\pi - \delta]$ for $\delta > 0$ then this space would now represent a cone shaped surface, and with infinite curvature at $r = 0$, in other words the origin would now be singular. So in this simpler space the period is constrained to $\delta = 0$.

The AdS black hole line element after transformation $t \to -i\tau$ has been perform takes the form

$$ds^2 = F(r)d\tau^2 + \frac{dr^2}{F(r)} + r^2 d\Omega_{d-2}^2$$

Exactly the same idea applies in this space, there is a value for the period β which allows this Euclidean metric to be smooth at r_+. To find, begin by expanding $F(r) = 1 + \frac{r^2}{b^2} - \frac{\mu}{r^{d-2}}$ around the horizon and taking the linear terms

$$
\begin{aligned}
F(r) &= F(r_+) + (r - r_+)\frac{dF}{dr}\bigg|_{r+r+} + O[(r - r_+^2)] \\
F(r) &\simeq (r - r_+)U(r - r_+)
\end{aligned}
$$

The first term $F(r_+) = 0$ by definition, and we are ignoring terms higher than first order. We have introduced $U = \frac{dF}{dr}|_{r+r+}$, and to continue let's set $r - r_+ = \varrho^2 \implies dr = 2\varrho d\varrho$ to which the line element will take the new form

$$
\begin{aligned}
ds^2 &\simeq U\varrho^2 d\tau^2 + \frac{4}{U}\varrho + r^2 d\Omega_{d-2}^2 \\
&= \frac{4}{U}\left(d\varrho^2 + \frac{U^2}{4}\varrho^2 d\tau^2\right) + r^2 d\Omega_{d-2}^2 \\
&= \frac{4}{U}\left(d\varrho^2 + \varrho^2\left(\frac{U}{2}d\tau\right)^2\right) + r^2 d\Omega_{d-2}^2
\end{aligned} \tag{5.11}
$$

Here we can compare the bracket term, with the line element (5.10) which we know must have a period of 2π to be smooth, allowing us to deduce that for (5.11) to also be smooth at $r = r_+$ the period must be $\frac{U}{2}\tau \sim \frac{U}{2}\tau + 2\pi$ therefore we have $\tau \sim \tau + \frac{4\pi}{U}$ and

$$
\beta = \frac{4\pi}{U}
$$

To which we can deduce from the definition of U that

$$
\beta = \frac{4\pi b^2 r_+}{r_+^2 d + b^2(d-2)} \tag{5.12}
$$

Obtained with $U = \frac{dF}{dr}|_{r+r+}$ and with the expression of μ in terms of the horizon r_+. This results agrees with Hawking's result for $d = 3$ [5], which is distinct from the results found for the asymptotically flat black holes ($g(r) = 1$). The black hole with radius r_+ is sitting in anti-de Sitter space with radius b, so by finding the roots of r_+, interesting results can be found

$$
r_+ = \frac{2\pi b^2}{\beta d} \pm \frac{b}{\beta d}\sqrt{4\pi^2 b^2 - d(d-2)\beta^2} \tag{5.13}
$$

To ensure there are only real roots

$$
\begin{aligned}
0 &\leq 4\pi^2 b^2 - d(d-2)\beta^2 \\
\therefore \beta &\leq \frac{2b\pi}{\sqrt{d(d-2)}} \\
\implies T_0 &= \frac{\sqrt{d(d-2)}}{2\pi b}
\end{aligned}
\tag{5.14}
$$

To which we can find that there is a maximum value of β, therefore a minimum value for the temperature given by T_0 of the black hole, given by the inverse. We can substitute the max value of β from (5.14) into (5.13) to find the corresponding radius as $r_0 = \frac{b\sqrt{d(d-2)}}{d}$. We can plot (5.12) against r to more easily visualise this relationship

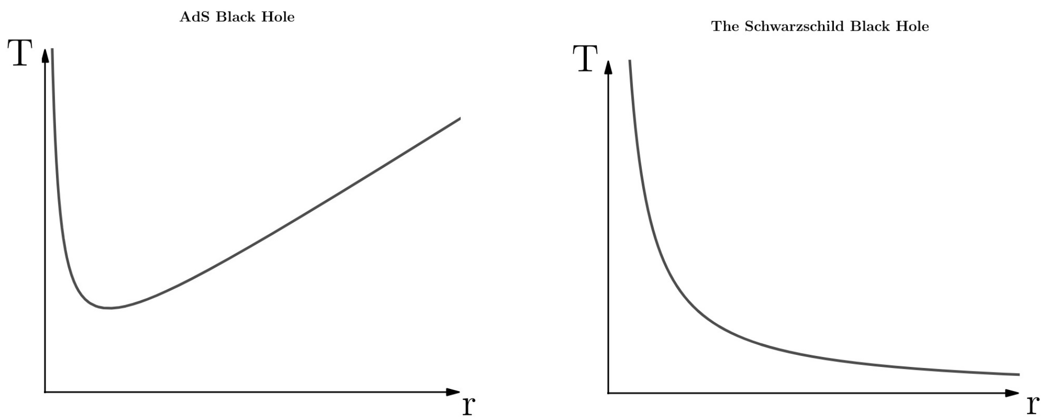

Figure 5.1: A plot comparison between the AdS and Schwarzschild black holes, both are plots of temperature T against the radius r.

Thus, we see there exists a phase transition in the AdS black hole at the minimum, which is $r = r_0$. The plot is against $r \propto M = \langle E \rangle$ and therefore the slope of the curves is proportional to the specific heat $\frac{\partial \langle E \rangle}{\partial \beta}$, which shows us that as r goes from $r < r_0$ to $r > r_0$ the specific heat of the AdS black hole changes sign, becoming positive and therefore thermodynamically stable, where as when $r < r_0$ it has negative specific heat hence thermodynamically unstable, like the Scwarzschild black hole is for all value of r.

5.2.2 AdS Thermodynamics from the Action Difference

So far from, just from the temperature of the Schawarzschild-AdS black hole, more interesting results have can be deduced than the other two black holes. Nevertheless, finding the temperature of the black hole was not the central result Hawking and Page found in their paper [5]. They showed once a critical temperature is reached, a Schwarzschild-AdS black holes undergoes a first order phase transition. This was by showing that for a given temperature, the difference in free energy F_{AdS} (AdS space with no) and F_{BH} (AdS space with black hole) changes sign. This meant not only black holes could be stable, but above some critical temperature they are a preferred state over radiation, likewise below a critical temperature they are not the preferred state whilst radiation is. Their methods were taking the difference of the Eucledian action between AdS and Schwarzschild-AdS spaces, as both metrics minimize the action (5.15) the integral will consequently represent volumes in the two geometries. Individually, the volumes will diverge to infinity (Integrating over the whole volume) however the difference between the two converge to a finite number, as a result it allowed Hawking and Page to deduce further thermodynamic quantities (using relationship found by (5.9))we shall here repeat in arbitrary dimensions by following E. Witten, who based on Hawking's methods found the same result for the AdS_{d+1} [6].

By keeping the normalisation for d-dimensions found from the Schwarzschild-AdS solution (2.25) and (2.26) only here we are in the usual (d+1)-dimensions. Folowing [5, 6, 8], we begin with the Einstein Hilbert action

$$\begin{aligned} I &= -\frac{1}{16\pi}\int d^{d+1}x\sqrt{g}(R-2\Lambda) \\ &= -\frac{1}{16\pi}\int d^{d+1}x\sqrt{g}(-\frac{d(d+1)}{b^2}+\frac{d(d-1)}{b^2}) \\ &= \frac{d}{8\pi b^2}\int d^{d+1}x\sqrt{g} \\ &= \frac{d}{8\pi b^2}(V_{BH}-V_{AdS}) \end{aligned} \tag{5.15}$$

So the action I is given by $\frac{d}{8\pi b^2}$ multiplied by a volume of some hyperspace, where the volume we will calculate is the volume difference $(V_{BH}-V_{AdS})$. After finding the finite volume difference, the action difference will be found. We are looking to find the difference between the actions from (2.30) and (2.27), to put it simply we are differentiating between the black hole and empty space. Both spaces are Euclidian and have periodic time given by the inverse temperature β' (for AdS) and β (for Schwarzschild-AdS), where only β is constrained to a fixed value given by (5.12), whilst β' is unconstrained to any value. To perform the integral, introduce a cutoff for the radial integral R since we only take $\lim R \to \infty$ later, explicitly then we have

$$\begin{aligned} V_{AdS} &= \int_0^{\beta'} dt \int_0^R dr \int_{S^{d-1}} d\Omega\ r^{d-1} \\ &= \frac{\beta' A_{d+1}}{d} R^d \\ V_{BH} &= \int_0^{\beta} dt \int_{r_+}^R dr \int_{S^{d-1}} d\Omega\ r^{d-1} \\ &= \frac{\beta A_{d+1}}{d} \left(R^d - r_+^d\right) \end{aligned}$$

Before taking the difference of the two, the two hyperspaces must be adjusted so that the time periods β' and β are equal at the cutoff $r = R$, consequently forcing both spaces to equate asymptotically arriving a meaningful final result. Because the black hole metric must be smooth, forcing β to (5.12), the only option is to change β' to equate to the fixed β. To relate the two, note that asymptotically both spaces are identical in angular and radial components exepc in the time component where they will miss align. Therefore performing an integral over all the spaces and equating them will cancel out any non-time component part of the integral and we can therefore relate the two periods by

$$\begin{aligned} \int_0^{\beta} \sqrt{1 + \frac{R^2}{b^2} - \frac{\mu}{R^{d-2}}} dt &= \int_0^{\beta'} \sqrt{1 + \frac{R^2}{b^2}} dt \\ \beta \sqrt{1 + \frac{R^2}{b^2} - \frac{\mu}{R^{d-2}}} &= \beta' \sqrt{1 + \frac{R^2}{b^2}} \\ \beta' &= \beta \sqrt{1 - \frac{b^2 \mu}{R^d + b^2 R^{d-2}}} \\ \therefore \beta' &= \beta \sqrt{1 - \frac{b^2 \mu}{R^d \left(1 + \frac{b^2}{R^2}\right)}} \end{aligned} \tag{5.16}$$

We now Taylor expand (5.16) in R and because we are interested in taking the limit of R to infinity we ignore any second order terms $O(R^2)$

$$\beta' \approx \beta \left(1 - \frac{1}{2} \frac{b^2 \mu}{R^d \left(1 + \frac{b^2}{R^2}\right)}\right)$$

Now using the large R limit $\left(1 + \frac{b^2}{R^2}\right) \to 1$ and the value for μ at the horizon $F(r_+) = 0 \implies \mu = r_+^{d-2} \left(1 + \frac{r_+^2}{b^2}\right)$ we find

$$\beta' \approx \beta \left(1 - \frac{1}{2}\left(1 + \frac{r_+^2}{b^2}\right)\frac{b^2 r_+^{d-2}}{R^d}\right)$$

From which the action difference can them be found

$$\begin{aligned}
I &= \frac{d}{8\pi b^2}\lim_{R\to\infty}\left(V_{BH} - V_{AdS}\right) \\
&= \frac{\beta A_{d-1}}{8\pi b^2}\lim_{R\to\infty}\left(\beta(R^d - r_+^d) - \beta' R^d\right) \\
&= \frac{\beta A_{d-1}}{8\pi b^2}\left(\frac{1}{2}\left(1 + \frac{r_+^2}{b^2}\right) b^2 r_+^{d-2} - r_+^d\right) \\
&= \frac{\beta A_{d-1}}{16\pi b^2}\left(b^2 r_+^{d-2} - r_+^d\right) \qquad (5.17) \\
&= \frac{A_{d-1}}{4}\left(\frac{b^2 r_+^{d-1} - r_+^{d+1}}{n r_+^2 + (n-2)b^2}\right) \qquad (5.18)
\end{aligned}$$

In the next three parts we find thermodynamic quantities using this action difference

The Free Energy and Phase transition

Firstly, the simplest way to see the phase transition is through the free energy F, which will reveal what the point along r_+ and T that each of the two actions (I_{BH} or I_{AdS}) will dominate the partition function. We know $F = IT = \frac{I}{\beta}$ then we simply have

$$F = \frac{A_{d-1}}{16\pi b^2}\left(b^2 r_+^{d-2} - r_+^d\right)$$

We see that the free energy goes from positive to negative at the point $r_+ = b$, found simply by setting $F = 0$. So there is a phase transition occurring from $r_+ < b$ to $r_+ > b$ but to understand what it means, let's look at action difference again. What we have calculated is $I = I_{AdS} - I_{BH} = \log Z_{BH} - \log Z_{AdS}$ so a change in sign consequently means a change of which action is dominating the partition function (5.8). Given that at $r_+ = b$ the temperature is $T_1 = \frac{d-1}{2\pi b}$, the following can be deduced. If $r_+ < b$ therefore $T < T_1$, then $I < 0$ which means that thermal radiation is favoured as this will dominate the partition (I_{AdS}). On the other hand, if $r_+ > b$ therefore $T > T_1$, then $I > 0$ which means that black holes are favoured over thermal radiation.

P. Zhao [8] reviewing Hawking and Page's paper showed a nice diagram, where he included in the story what was also found from the previous section, T_0, the minimum temperature Schwarzschild-AdS black holes can have to be thermodynamically stable (5.14). It can be shown that for all dimensions $d > 2$, that $T_0 \leq T_1$ and so if a black hole is at $T < T_0$, it is thermodynamically unstable and less favourable over thermal radiation in the anti-de Sitter space (the only stable equilibrium for $T < T_0$). If $T_0 < T < T_1$ then the black hole is thermodynamically stable, however it is still the unfavourable state, over thermal radiation in the anti-de Sitter space. And finally if $T > T_1$ than the black hole is both thermodynamically stable and favourable over thermal radiation. Hawking showed a further temperature T_2, which if $T > T_2$ then thermal radiation in anti-de Sitter space will always collapse into a black hole.

Average Energy

The energy $\langle E \rangle$ is what we now find, we know it is given by $\langle E \rangle = \frac{\partial I}{\partial \beta} = \frac{\partial I}{\partial r}\frac{\partial r}{\partial \beta}$, therefore we find $\frac{\partial I}{\partial r}$. We see that (5.17) is of the form $I \sim f(\beta)g(r_+)$ so we simply find that $\frac{\partial I}{\partial \beta} = \left(g\frac{\partial f}{\partial \beta} + f\frac{\partial g}{\partial r}\right)\frac{\partial r}{\partial \beta}$ and so we have

$$\begin{aligned}
\langle E \rangle &= \frac{\partial I}{\partial r_+}\frac{\partial r_+}{\partial \beta} \\
&= \frac{A_{d-1}}{16\pi b^2 G_{d+1}}\left(\frac{\partial \beta}{\partial r_+}(b^2 r_+^{d-2} - r_+^d) + \beta(b^2(n-2)r_+^{d-3} - dr_+^{d-1})\right)\frac{\partial r_+}{\partial \beta} \\
&= \frac{A_{d-1}}{16\pi b^2 G_{d+1}}\left((b^2 r_+^{d-2} - r_+^d) + \frac{\partial r_+}{\partial \beta}\beta(b^2(d-2)r_+^{d-3} - nr_+^{d-1})\right)
\end{aligned}$$

Now let's find $\frac{\partial r_+}{\partial \beta}$

$$\begin{aligned}
\beta &= \frac{4\pi b^2 r_+}{dr_+^2 + b^2(d-2)} \\
\therefore \frac{\partial \beta}{\partial r_+} &= \beta\left(\frac{1}{r_+} - \frac{2r_+ d}{r_+^2 d + b^2(d-2)}\right) \\
\therefore \frac{\partial r_+}{\partial \beta} &= \frac{r_+}{\beta}\left(\frac{r_+^2 d + (d-2)b^2}{(d-2)b^2 - r_+^2 d}\right)
\end{aligned}$$

Which when substituted back to find $\langle E \rangle$, and using $\mu = r_+^{d-2}\left(1 + \frac{r_+^2}{b^2}\right)$ and (2.28) we find

$$
\begin{aligned}
\langle E \rangle &= \frac{(d-1)A_{d-1}}{16\pi}\left(\frac{r_+^d}{b^2} + r_+^{d-2}\right) \\
\therefore \langle E \rangle &= M
\end{aligned}
$$

We can now use this result to find the entropy of the black hole, at which point we could check against the Bekenstein-Hawking entropy.

The B-H Entropy

The last thermodynamic quantity we find for Schwarzschild black holes is the entropy, given by $S = \beta \langle E \rangle - I$, therefore we find

$$
\begin{aligned}
S &= \beta \langle E \rangle - I \\
&= \frac{\beta(d-1)A_{d-1}}{16\pi}\left(\frac{r_+^2}{b^2} + r_+^d\right) - \frac{\beta A_{d-1}}{16\pi b^2}\left(r_+^{d-2} - \frac{r_+^d}{b^2}\right) \\
&= \frac{A_{d-1}}{4} r_+^{d-1}
\end{aligned}
$$

It is so that the term $r_+^{d-1}A_{d-1}$ turns out to be the surface area of a d-sphere with radius r_+ therefore the surface area of the horizon A. Thus, again the famous Bekenstein-Hawking entropy is retained as $S_{B-H} = \frac{A}{4}$

Appendices

5.3 Group and Tensors in Special Relativity

In SR, we deal with transformations which are continuously connected to the identity, that is we deal with a continuous set of transformations. This is important as we have calculus on the space of transformations, furthermore they give rise to conserved quantities. Examples is, continuous translations symmetry in time, give rise to conservation of energy. Also, continuous transformation of rotation symmetry give rise to conservation of angular momentum. Therefore continuous transformations are important.

Tensors

Tensors represent physical quantities that are invariant but often when expressed with coordinates, they have components that transform. We label them as $(tanjent, cotanjent)$, some examples are a scalar:$(0, 0)$, a vector:$(1, 0) \sim V^\mu$, a dual vector:$(0, 1) \sim W_\mu$, the metric:$(0, 2) \sim \eta_{\mu\nu}$ and the inverse metric:$(2, 0) \sim \eta^{\mu\nu}$. A exaggerated example to get the idea is a $(3, 4)$ tensor given as $T^{\mu\nu\alpha}{}_{\lambda\beta\gamma\delta}$

So a tensor is something that transforms like a tensor, they are multilinear maps from the space of vectors and dual vectors in to the real numbers. Extreme example again

$$T^{\mu\nu\alpha}{}_{\lambda\beta\gamma\delta} V^\alpha V^\beta V^\gamma V^\delta W_\mu W_\nu W_\alpha \tag{5.19}$$

We can see the tensor 'eats up' the vectors indices, resulting in a real number. If the indices do not add up, then the result will be another tensor with fewer indices. An operation like above is know as a contraction and all the indices add up therefore the result is a real number. We have further used the Einstein summation convention, in which we simply remove the sum sign $\sum$ and sum over the indices, since we know we will always be in 4 dimensions for spacetime therefore indices always run from 0 to 3.

5.4 Curvature Tensors

To describe curvature in curved space, begin by introducing the concept of transporting (moving) a vector in the space. First, take some vector and infinitesimally move it from some position A to another position $B = A(x^\mu + dx^\mu)$, then move the same vector from same initial position A through some different path to the same end infinitesimal location B. If the two vectors in location B are equal the space is flat, if not the space is deemed curved. The tensor measuring this curvature is called the Riemann curvature tensor given by

$$[D_\mu, D_\nu] V^\lambda = R^\lambda_{\rho\mu\nu} \tag{5.20}$$

Where on the LHS, described the infinitesimal movement of the 4-vector V^λ. The tensor has some 256 independent components, which by symmetries reduce them to 20 independent components [11]. If $R^\lambda_{\rho\mu\nu} = 0$ then space is flat and since this tensors must be invariant under general coordinated transformations, it must be flat in any coordinate system. It is a local expression for curvature, not a global which can be used to prove that the horizon of a black hole in Schwarzschild coordinates is non-singular. By taking the trace of the Riemann curvature tensor, the Ricci tensor can be obtained

$$R^\lambda_{\rho\lambda\nu} = g^{\lambda\alpha} R_{\alpha\rho\lambda\nu} = R_{\rho\nu} \tag{5.21}$$

The Ricci tenso

5.5 Trace Reverse for of EE

Einstein's equation can be re-written in a form called trace reverse form, this can be done as follows

$$
\begin{aligned}
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= 8\pi G T_{\mu\nu} \\
\therefore g_{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) &= g_{\mu\nu}\left(8\pi G T_{\mu\nu}\right) \\
\therefore R - \frac{1}{2}(4)R &= -8\pi G T \\
\implies R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(-8\pi G T) &= 8\pi G T_{\mu\nu} \\
\therefore R_{\mu\nu} &= 8\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)
\end{aligned}
$$

Acknowledgement; With thanks to my personal supervisor Andreas Brandhuber for guidance and teaching throughout this project

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