

EENG307: Sinusoidal Steady State*

Lecture 24

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1 Pre-lecture Math Facts

Suppose you have the following rational transfer function:

$$G(s) = \frac{s^2 + 2s + 3}{s^2 + 5s + 4}.$$

In the following lecture, we will see that evaluating a transfer function at $s = j\omega$ will be useful

$$G(j\omega) = \frac{(j\omega)^2 + 2(j\omega) + 3}{(j\omega)^2 + 5(j\omega) + 4} = \frac{3 - \omega^2 + j2\omega}{4 - \omega^2 + j5\omega}$$

Now, suppose we instead evaluate $G(-j\omega)$:

$$G(-j\omega) = \frac{(-j\omega)^2 + 2(-j\omega) + 3}{(-j\omega)^2 + 5(-j\omega) + 4} = \frac{3 - \omega^2 - j2\omega}{4 - \omega^2 - j5\omega}$$

Notice that both the numerator and denominator are the *complex conjugate* of the numerator and denominator of $G(j\omega)$, because the real parts are the same, but the imaginary parts have opposite signs:

$$(3 - \omega^2 - j2\omega)^* = 3 - \omega^2 + j2\omega$$
$$(4 - \omega^2 - j5\omega)^* = 4 - \omega^2 + j5\omega$$

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Since conjugation and division commute, i.e.

$$\frac{a^*}{b^*} = \left(\frac{a}{b}\right)^*$$

we actually have the following:

$$G(-j\omega) = G(j\omega)^*$$

Although we showed this for a specific case, this is true in general for any rational transfer function with real coefficients (just use the property that sum and conjugation also commute: $(a^* + b^*) = (a + b)^*$)

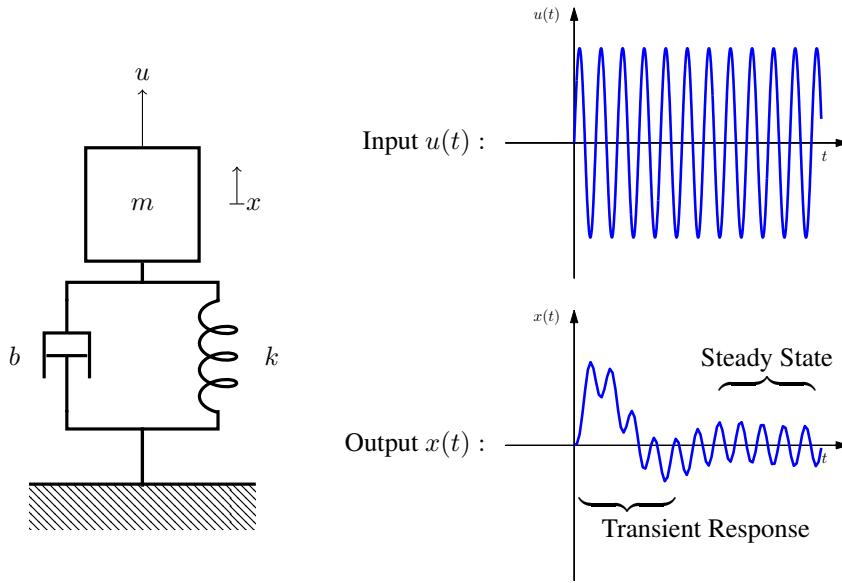
In addition, recall that if s_1 and s_2 are complex numbers

$$\begin{aligned} \left| \frac{s_1}{s_2} \right| &= \frac{|s_1|}{|s_2|} \\ \angle \frac{s_1}{s_2} &= \angle s_1 - \angle s_2 \end{aligned}$$

2 Response of Linear Systems to Sinusoidal Inputs

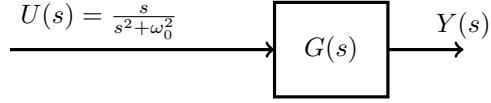
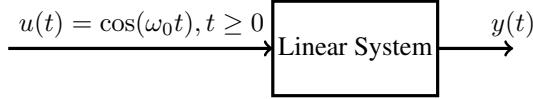
You are probably already aware of a basic property of **stable** linear systems: If the input is a sinusoid, then the *steady state* output (also called the steady state response) will be a sinusoid of the same frequency.

System Response to Sinusoidal Input



This can be proven by using Laplace Transforms.

Step 1: Multiply $G(s)$ by the Laplace Transform of a cosine



$$Y(s) = G(s) \frac{s}{s^2 + \omega_0^2}$$

Step 2: Write partial fraction expansion, splitting poles at ±jω₀

- Let p_1, p_2, \dots, p_n be the poles of $G(s)$ (assume simple poles for now, but result also holds if $G(s)$ has repeated poles)

$$Y(s) = G(s) \frac{s}{s^2 + \omega_0^2} = \frac{A}{s - j\omega_0} + \frac{B}{s + j\omega_0} + \frac{C}{s - p_1} + \frac{D}{s - p_2} + \dots$$

- By residue formula

$$\begin{aligned} A &= \cancel{(s - j\omega_0)} G(s) \frac{s}{\cancel{(s - j\omega_0)}(s + j\omega_0)} \Big|_{s=j\omega_0} = G(j\omega_0) \frac{1}{2} \\ B &= \cancel{(s + j\omega_0)} G(s) \frac{s}{(s - j\omega_0) \cancel{(s + j\omega_0)}} \Big|_{s=-j\omega_0} = G(-j\omega_0) \frac{1}{2} \end{aligned}$$

Step 3: Find Inverse Laplace Transform

- Since

$$Y(s) = \frac{G(j\omega_0) \frac{1}{2}}{s - j\omega_0} + \frac{G(-j\omega_0) \frac{1}{2}}{s + j\omega_0} + \frac{C}{s - p_1} + \frac{D}{s - p_2} + \dots$$

- Using $e^{-at} u \xleftarrow{\mathcal{L}} \frac{1}{a+s}$ the time response is

$$y(t) = \left(G(j\omega_0) \frac{1}{2} e^{j\omega_0 t} + G(-j\omega_0) \frac{1}{2} e^{-j\omega_0 t} + C e^{p_1 t} + D e^{p_2 t} \right) u(t)$$

Step 4: Use property of stable poles

- If $\operatorname{Re}(p_i) = -a < 0$,

$$\lim_{t \rightarrow \infty} C e^{p_i t} u(t) = \lim_{t \rightarrow \infty} C e^{\operatorname{Re}(p_i)t} e^{j\operatorname{Im}(p_i)t} u(t) = \lim_{t \rightarrow \infty} C e^{-at} e^{j\operatorname{Im}(p_i)t} u(t) = 0$$

- Thus

$$\lim_{t \rightarrow \infty} y(t) = \left(G(j\omega_0) \frac{1}{2} e^{j\omega_0 t} + G(-j\omega_0) \frac{1}{2} e^{-j\omega_0 t} \right)$$

Step 5: Use symmetry property $G(-j\omega) = G(j\omega)^*$ and Euler's formula

- Substitute for $G(-j\omega_0)$

$$\lim_{t \rightarrow \infty} y(t) = \left(\frac{1}{2} G(j\omega_0) e^{j\omega_0 t} + \frac{1}{2} G(j\omega_0)^* e^{-j\omega_0 t} \right)$$

- Write in polar form, using $a^* = |a|e^{-j\angle a}$

$$\lim_{t \rightarrow \infty} y(t) = \left(\frac{1}{2} |G(j\omega_0)| e^{j\angle G(j\omega_0)} e^{j\omega_0 t} + \frac{1}{2} |G(j\omega_0)| e^{-j\angle G(j\omega_0)} e^{-j\omega_0 t} \right)$$

- Euler's formula: $\frac{1}{2} A e^{j\theta} + \frac{1}{2} A e^{-j\theta} = A \cos(\theta)$

$$\lim_{t \rightarrow \infty} y(t) = |G(j\omega_0)| \cos(\omega_0 t + \angle G(j\omega_0))$$

What we have shown is if the input is a cosine with frequency ω_0 , then the output, at steady state, is also a cosine with frequency ω_0 , but with an amplitude gain of $|G(j\omega_0)|$, and a phase shift of $\angle G(j\omega_0)$. This is summarized below in the general case.

Sinusoidal Steady State is the Frequency Response

Theorem 1. Given a stable system with transfer function $G(s)$, the sinusoidal steady state response is defined by the input/output relationship

$u(t) = A \cos(\omega_0 t + \theta),$ $y_{ss}(t) = G(j\omega_0) A \cos(\omega_0 t + \theta + \angle G(j\omega_0)).$

Definition 2. $G(j\omega)$ is the *frequency response function* of the system with transfer function $G(s)$.

Note: since $\sin(\omega t) = \cos(\omega t - 90^\circ)$ the same is true for sine functions.

3 Examples

Example 3. Find the steady state response

$$G(s) = \frac{1}{s+1}$$

$$u(t) = 3 \cos(2t + 30^\circ), t \geq 0$$

Solution: Calculate the magnitude and phase of $G(j2)$:

$$\begin{aligned} |G(j2)| &= \left| \frac{1}{j2+1} \right| & \angle G(j2) &= \angle \frac{1}{j2+1} \\ &= \frac{|1|}{|j2+1|} & &= \angle 1 - \angle j2+1 \\ &= \frac{1}{\sqrt{1+2^2}} & &= 0 - \tan^{-1} \left(\frac{2}{1} \right) \\ &= \frac{1}{\sqrt{5}} & &= -63.4^\circ \end{aligned}$$

The solution is thus

$$y_{ss}(t) = \frac{3}{\sqrt{5}} \cos(2t - 33.4^\circ)$$

Example 4. Find an expression for the magnitude and phase frequency response functions for a system with transfer function

$$G(s) = \frac{\sigma}{s + \sigma}$$

where $\sigma > 0$. **Solution:** The magnitude of the frequency response function is given by

$$\begin{aligned} |G(j\omega)| &= \left| \frac{\sigma}{j\omega + \sigma} \right| \\ &= \frac{|\sigma|}{|j\omega + \sigma|} \\ &= \frac{\sigma}{\sqrt{\omega^2 + \sigma^2}} \end{aligned}$$

The phase of the frequency response function is given by

$$\begin{aligned} \angle G(j\omega) &= \angle \frac{\sigma}{j\omega + \sigma} \\ &= \angle \sigma - \angle(j\omega + \sigma) \\ &= 0 - \tan^{-1} \left(\frac{\omega}{\sigma} \right) \\ &= -\tan^{-1} \left(\frac{\omega}{\sigma} \right) \end{aligned}$$

4 Lecture Highlights

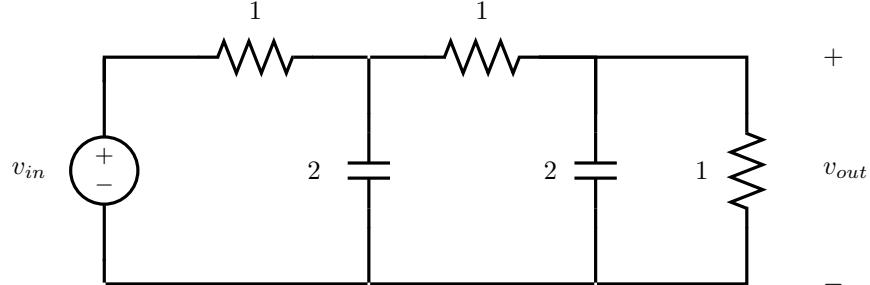
The primary takeaways from this article include

1. After an initial transient response period, the output of a dynamic system with a sinusoidal input will be a steady-state sinusoid that has the same frequency ω_0 as the input signal.
2. The output sinusoid will be scaled (amplitude changed) and shifted (phase angle changed) according to properties of the system's transfer function $G(s)$.
3. By evaluating the magnitude and phase angle of the system's transfer function $G(s)$ at $s = j\omega_0$, we can determine the steady-state output signal without having to go through the full inverse Laplace transform process.

5 Quiz Yourself

5.1 Questions

1. The following system models a transmission line



and has the transfer function

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{4s^2 + 8s + 3}$$

Find the steady state output if the following sinusoids are applied as input

- (a) $\cos(0.1t)$
- (b) $\cos(t)$
- (c) $\cos(10t)$

5.2 Solutions

1(a)

$$i.e. \quad V_{in}(z) = \cos(0.1t),$$

$$\begin{aligned} G(j\omega) &= \frac{1}{4(j\omega)^2 + 8(j\omega) + 3} = \frac{1}{-0.4 + 8j + 3} \\ &= \frac{1}{2.96 + 0.8j}, \end{aligned}$$

$$|G(j\omega)| = \frac{1}{\sqrt{2.96^2 + 0.8^2}} = \frac{1}{\sqrt{3.07}} = 0.326$$

$$\angle G(j\omega) = 11 - \angle(2.96 + 0.8j) = 0 - 15.1^\circ = -15.1^\circ$$

$$V_{out}(t) \text{ at steady state} = 0.326 \cos(0.1t - 15.1^\circ)$$

1(b)

) if $V_{in}(t) = \cos(\omega t)$,

$$G(j\omega) = \frac{1}{4(\omega^2 + 8\omega + 3)} = \frac{1}{-\omega^2 + 8\omega + 3} = \frac{1}{-\omega^2 + 8\omega + 3}$$

$$|G(j\omega)| = \frac{1}{\sqrt{-\omega^2 + 8\omega + 3}} = \frac{1}{\sqrt{8.06}} = 0.124$$

$$\angle G(j\omega) = \angle 1 - \angle -\omega^2 + 8\omega + 3 = 0 - 97.1^\circ = -97.1^\circ$$

$$\angle = 180^\circ - 97.1^\circ = 97.1^\circ$$

at steady state

$$V_{out}(t) = 0.124 \cos(\omega t - 97.1^\circ)$$

1(c)

$$G(j\omega) = \frac{1}{4(j\omega)^2 + 8(j\omega) + 3} = \frac{1}{-400 + 80j + 3} = \frac{1}{-397 + 80j}$$

$$|G(j\omega)| = \frac{1}{|-397 + 80j|} = \frac{1}{\sqrt{397^2 + 80^2}} = 0.0025$$

$$\angle G(j\omega) = \angle 0 - \angle -397 + 80j$$

$= -168.6^\circ$

at steady state



$$\theta = 180 - \tan^{-1}\left(\frac{80}{397}\right) = \\ = 180 - 11.4 = 168.6^\circ$$

$$V_{out}(t) = 0.0025 \cos(10t - 168.6^\circ)$$