EENG307: Translational Mechanical Impedance* & Rotational Mechanical Impedance†

Lecture 8 & 9

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Contents

1 Pre-requisite Material

This lecture assumes that the reader is familiar with the following material:

- Lecture 7: Impedance and Transfer Functions
- Lecture 10: Fluid Systems and System Analogies

2 Mechanical Impedance

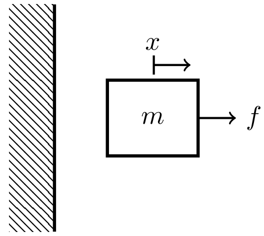
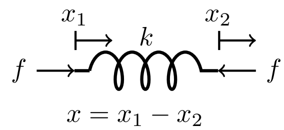
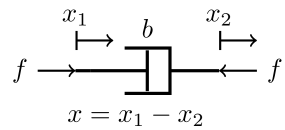
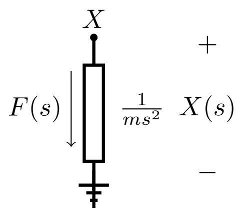
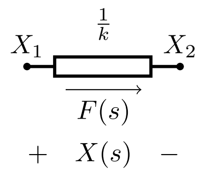
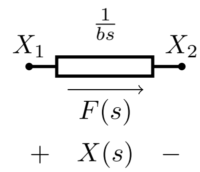
We have seen how impedance is used to find transfer functions for circuits in Lecture 7. Impedance can also be used in other modeling domains. While some textbooks define mechanical impedance as the ratio of the Laplace transform of force to position, we will *not* use this definition, as it is not compatible with electrical systems. Instead, we will make the following general definition of impedance.

Definition 1. The *impedance* of an element is the ratio of the Laplace Transform of the across variable over the Laplace Transform of the through variable

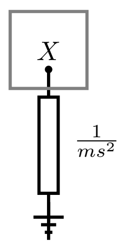
In mechanical systems, the across variable is position, while the through variable is force, so we have the impedances in the following table.

Mechanical Impedance

‡Developed and edited by Tyrone Vincent and Kathryn Johnson, Colorado School of Mines, with contributions from Salman Mohagheghi, Chris Coulston, Kevin Moore, CSM and Matt Kupilik, University of Alaska, Anchorage

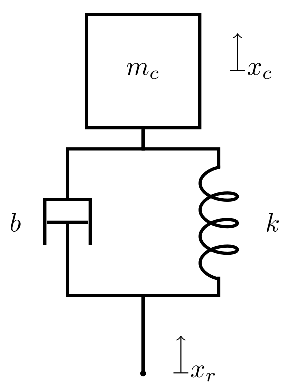
	mass	spring	damper
Component			
Component law	$m\ddot{x} = f$	$f = kx$	$f = b\dot{x}$
Laplace Transform	$X(s) = \frac{1}{ms^2}F(s)$	$X(s) = \frac{1}{k}F(s)$	$X(s) = \frac{1}{bs}F(s)$
Impedance Component (positive f direction agrees with positive x direction)			

Note that since the position of the mass is relative to a fixed measurement frame, one end of the mass impedance element is *always* connected to ground. When translating between mechanical drawings and impedance networks, it is helpful to visualize the mass sitting on top of the mass impedance, like this:



2.1 Example

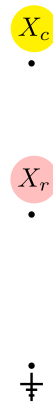
Let's model the automobile suspension system using impedance. The system description using ideal elements is the following, where x_r is the road position (input) and x_c is the car position (output)



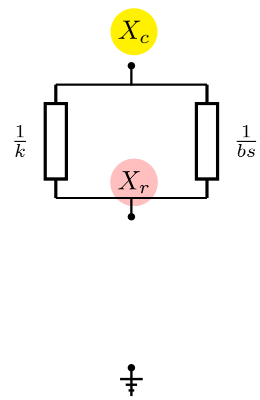
The process is the same as we followed for fluid systems

Step 1 **Identify all node variables.** For mechanical systems, positions are the across variables, so positions are the nodes. For this system, we see two independent positions: x_c and x_r. **Note:** In impedance diagrams, we don't have the flexibility to define a different direction as positive at each node. When drawing the diagrams all the nodes refer to positions that are defined with the same direction as positive. For example, in the above diagram, both x_c and x_r are defined with positive being upward.

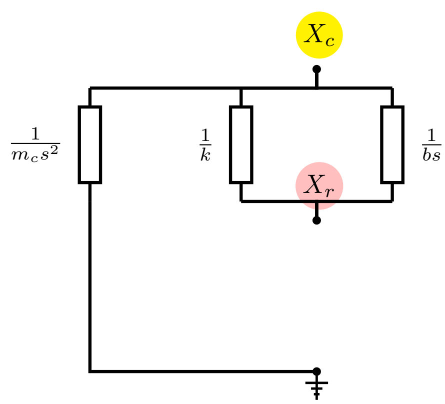
Step 2 **Identify one node as ground, or add a ground node.** Since both x_c and x_r will change, neither can be considered a ground. We must add a ground node. The system description thus far is the following:



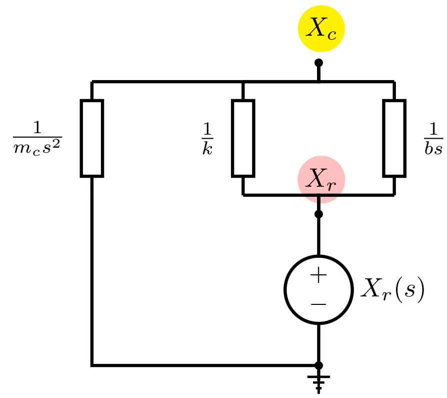
Step 3 **Connect components between nodes.** There is a spring and a damper between x_c and x_r.



There is a mass at node x_c. For a mass, the other end of the impedance *must* be connected to ground



Finally, we apply the boundary condition x_r using a voltage source



3 Rotational Mechanical Systems

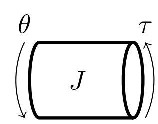
Previously, we modeled translational motion in one direction. When objects rotate, the modeling process is similar. For rotational mechanical systems, the variables that we are modeling are

- torque, which has units of Newton meters [N m], and
- angular position, which has units of radians [rad].

Unless otherwise specified, we will always assume that the torque and position variables have these units. We will use the following elements to describe rotational mechanical systems.

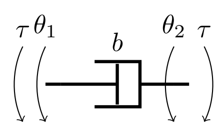
3.1 Components

When objects rotate, they follow the rotational version of Newton's law, which relates applied torque to angular acceleration. The scaling parameter in this case is the rotational inertia, J which has units of [kg m^2].



$$\tau = J\ddot{\theta}$$

Viscous friction also applies in the rotational case. In this case, each of the ends of the damper can rotate in response to an applied torque. The scaling parameter b has units of [N m s rad^{-1}].

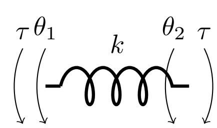


$$\dot{\theta} = \dot{\theta}_1 - \dot{\theta}_2$$

$$\tau = b\dot{\theta}$$

Note that the applied torque is always *equal and opposite* on both ends of the damper.

A rotational spring. Each of the ends of the spring can rotate in response to an applied torque. The scaling parameter has units of [N m rad^{-1}].



$$\theta = \theta_1 - \theta_2$$

$$\tau = k\theta$$

Like the damper, the applied torque is always *equal and opposite* on both ends of the spring.

3.2 Connection Rules and Boundary Conditions

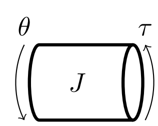
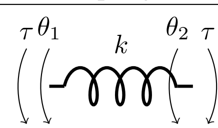
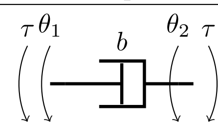
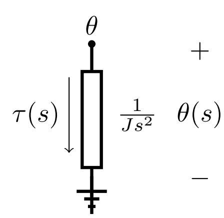
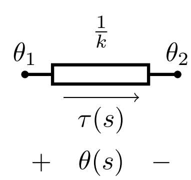
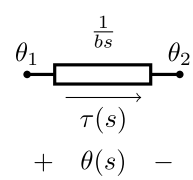
The connection rules and boundary conditions are the same as in the translational case.

- When two elements are connected, the two components share the same *angular position*.
- When two elements are connected, the torques at the connection *sum to zero*.
- A ground fixes the position of that terminal to zero.
- A force or position input prescribes the force or position of that terminal.

3.3 Impedances

We note that the algebraic laws that relate rotational mechanical elements are *identical* to that of translational mechanical elements, with angular position as the across variable and torque as the through variable. We can use the following impedance elements.

Rotational Impedance

	mass	spring	damper
Component		 $\theta = \theta_1 - \theta_2$	 $\dot\theta = \dot\theta_1 - \dot\theta_2$
Component Law	$J\ddot\theta = \tau$	$\tau = k\theta$	$\tau = b\dot\theta$
Laplace Transform	$\theta(s) = \frac{1}{Js^2}\tau(s)$	$\theta(s) = \frac{1}{k}\tau(s)$	$\theta(s) = \frac{1}{bs}\tau(s)$
Impedance Component (force direction agrees with positive direction)			

3.4 Example

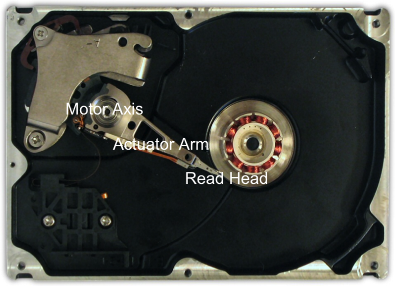
The process of finding the transfer function using impedances is straightforward.

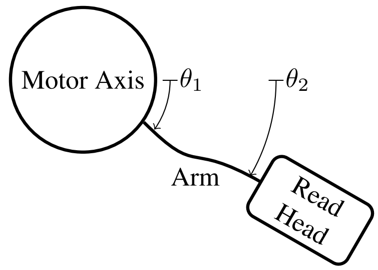
0. Draw a diagram of ideal elements.
1. Identify all of the independent nodes (across variables) which for mechanical systems are positions. Add a ground node, or identify a fixed node as ground.
2. Connect elements between nodes, with masses and inertias always connected between one node and ground
3. Add boundary conditions as sources.
4. Use circuit techniques to find the transfer function.

Let's try this process to model a hard disk drive read head, which is an important application of control systems. The control system must move the read head over the correct track within a few milliseconds, and hold it there despite disturbances such as external shocks or disk irregularities.

Hard Disk Drive Read Head

In order to move the read head to the correct track and hold it there, we need to be able to predict the relationship between the motor torque τ and the angular position of the read head θ_2. First, find the equivalent impedance model. Then, find the transfer function $\frac{\theta_2}{\tau}$.





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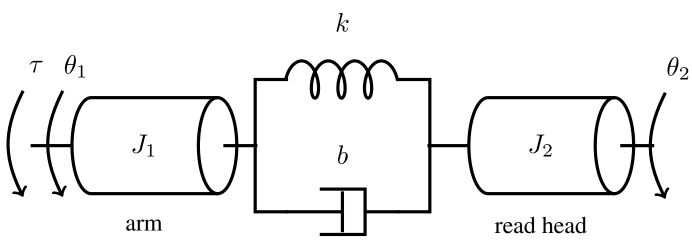
Since the arm is thin, it is flexible. This can be modeled with ideal elements by splitting the inertia between the arm and the read head, and connecting these inertias with a spring and damper. The motor applies the torque τ. J_1 is the inertia of the motor axis, while $J_2 = mr^2$, where m is the mass of the read head, and r is the radius at which it rotates. k and b are constants that describe the stiffness and damping characteristics of the actuator arm.

Step 0 Diagram of ideal elements

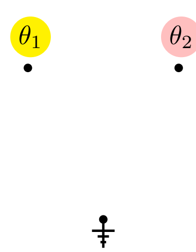
Hard Disk Drive Ideal Elements



Step 1 For this problem, there are two independent positions: θ_1 and θ_2.

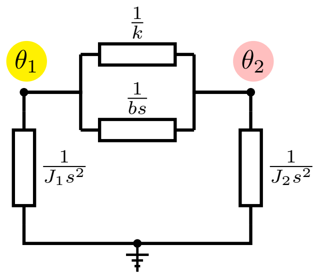
Step 2 Neither of these are fixed, so we also add a ground node.

Nodes



Step 3.1 Connect the correct impedance elements between nodes

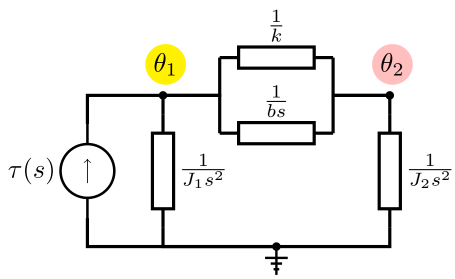
Disk Drive Impedance Network



Step 3.2 The applied torque is a boundary condition. Since torque is a through variable, we should add a current source to the θ_1 node.

> Rule for assigning sign of sources: The current source goes *into* the node if a positive torque would tend to cause a positive $\theta_1(s)$.

Disk Drive Complete Circuit



Step 4 Find the transfer function

We can now find the transfer function from $\tau(s)$ to either $\theta_1(s)$ or $\theta_2(s)$ using our standard circuit methods. If we wanted to find $\theta_2(s)/\tau(s)$, the easiest way is to set up the nodal equations, which is KCL at each node.

Nodal equations:

$$\begin{bmatrix} \tau(s) \\ 0 \end{bmatrix} = \begin{bmatrix} J_1 s^2 + bs + k & -(bs+k) \\ -(bs+k) & J_2 s^2 + bs + k \end{bmatrix} \begin{bmatrix} \theta_1(s) \\ \theta_2(s) \end{bmatrix}$$

Solve for θ_1 using second equation:

$$\theta_1(s) = \frac{J_2 s^2 + bs + k}{bs + k} \theta_2(s)$$

Plug into first equation:

$$\begin{aligned} \tau(s) &= (J_1 s^2 + bs + k) \frac{J_2 s^2 + bs + k}{bs + k} \theta_2(s) - (bs + k)\theta_2(s) \\ &= \frac{J_1 J_2 s^4 + J_1 s^2 (bs + k) + J_2 s^2 (bs + k) + (bs + k)^2 - (bs + k)^2}{(bs + k)} \theta_2(s) \\ &= \frac{J_1 J_2 s^4 + b(J_1 + J_2)s^3 + k(J_1 + J_2)s^2}{(bs + k)} \theta_2(s) \end{aligned}$$

Thus

$$\frac{\theta_2(s)}{\tau(s)} = \frac{bs + k}{J_1 J_2 s^4 + b(J_1 + J_2)s^3 + k(J_1 + J_2)s^2}$$

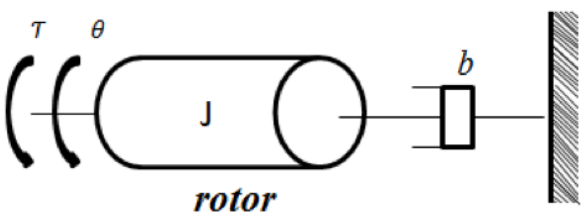
4 Application Example

Recall the wind turbine from the Application Example in Lecture 6: Solving Differential Equations II (some information reprinted here for your convenience). Occasionally it is desirable to use the generator in reverse (as a motor providing torque τ) to turn the rotor to a certain angular position θ so that maintenance can be performed. An automatic controller can be designed to position the rotor, but before designing the controller first we need to be able *to model the turbine in transfer function form*.

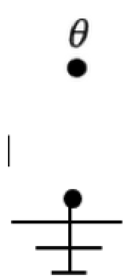
Step 0: to model the turbine's rotor, first determine which ideal components we need:
a) rotational mass with inertia J
b) rotational damper with damping constant b (to represent friction)

Note that a wind turbine would not have a rotational spring k. If it did, the spring would get more and more extended as the rotor rotated, and eventually it would need to unwind.

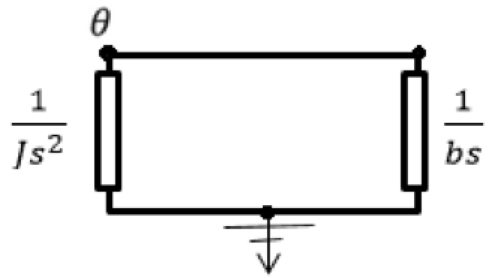
Step 1: sketch the ideal element diagram



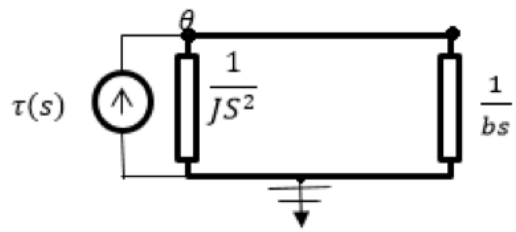
Step 2: sketch the required nodes (in this case: just one since there is only one rotational position)



Step 3: connect the node to the ground with impedances for both the mass and damper



Step 4: add the source (torque is analogous to a current source). The current source goes into the node if a positive torque would tend to cause a positive θ.



Now that we have an impedance network, we can use circuit methods to find the transfer function. Note that the two impedances are in parallel; therefore, the total impedance is

$$Z_{eq}(s) = \frac{\left(\frac{1}{Js^2}\right)\left(\frac{1}{bs}\right)}{\frac{1}{Js^2} + \frac{1}{bs}} = \frac{1}{bs + Js^2}$$

From the impedance version of Ohm's law, we know that $\theta(s) = \tau(s)Z_{eq}(s)$. Therefore, our transfer function from torque to position is

$$\frac{\theta(s)}{\tau(s)} = \frac{1}{Js^2 + bs}$$

Now we have our "plant" (turbine) transfer function from our input signal τ to our output signal θ. Later on this semester, we'll use information in this transfer function to design a proportional-integral-derivative (PID) feedback controller to achieve the desired position.

5 Lecture Highlights

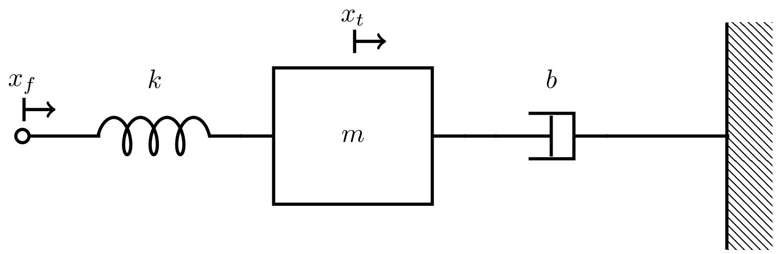
The primary takeaways from this article include

1. Just like fluid systems, mechanical systems can be modeled using impedances, where the impedance is the Laplace transform of the across variable (position) divided by the Laplace transform of the through variable (force).
2. Rotational mechanical systems are very similar to translational mechanical systems, with rotational inertia replacing mass, rotational dampers replacing translational dampers, rotational springs replacing translational springs, torque replacing force, and angular position replacing position. If you're struggling with a sign convention when determining a system's differential equation, you might try replacing each rotational system component with its analogous translational component to see if the signs make more sense.
3. As with fluid systems, the process for creating an equivalent impedance circuit starts with identifying and drawing nodes, then connecting nodes according to the component impedances, then including boundary conditions (sources).

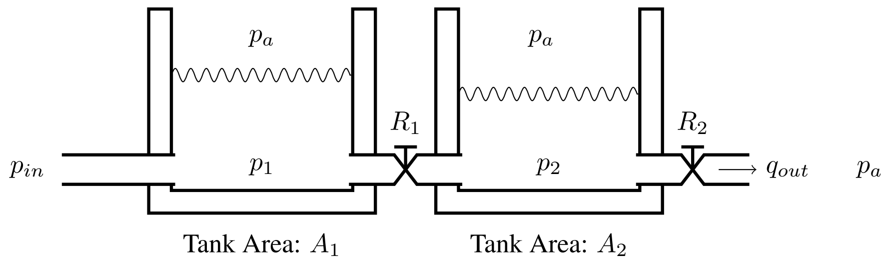
6 Quiz Yourself

6.1 Questions

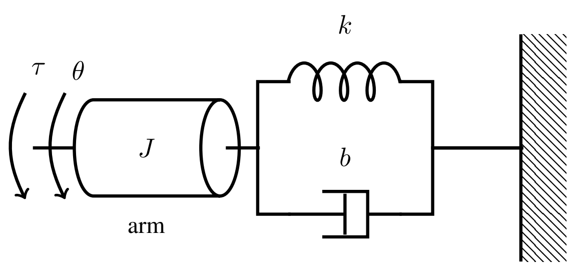
1. The following is a model for the lateral motion of a building subject to an earthquake. x_f is the position of the foundation of the building relative to a fixed point. During an earthquake, x_f becomes a function of time, and will be considered the input. The horizontal position of the top of the building is x_t. Find the transfer function from x_f to x_t.



2. Two tanks are connected in series, with the supply at pressure p_{in}, measured relative to atmospheric pressure. Find the equations that describe the behavior of the tanks, including the height of the second tank, h_2.

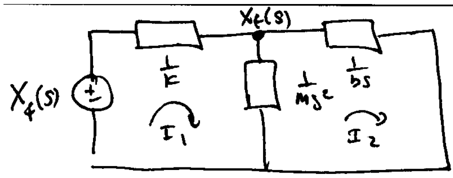


3. Find the transfer function relating τ to θ for the following system



6.2 Solutions

1.



$$\begin{bmatrix} X_p(s) \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{k} + \frac{1}{ms^2} & -\frac{1}{ms^2} \\ -\frac{1}{ms^2} & \frac{1}{ms^2} + \frac{1}{bs} \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix}$$

$$X_t(s) = \frac{1}{bs} I_2(s)$$

$$\frac{1}{ms^2} I_1(s) = \left(\frac{1}{ms^2} + \frac{1}{bs}\right) I_2(s)$$

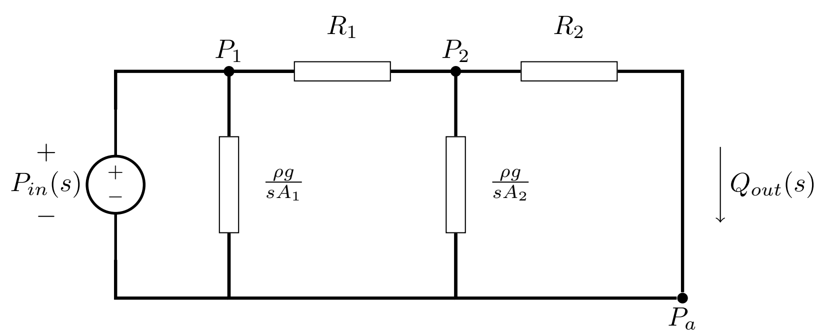
$$X_p(s) = \left[\left(\frac{1}{k} + \frac{1}{ms^2}\right)\left(\frac{1}{ms^2} + \frac{1}{bs}\right) ms^2 - \frac{1}{ms^2}\right] I_2(s)$$

$$= \left(\frac{1}{kms^2} + \cancel{\frac{1}{(ms^2)^2}} + \frac{ms^2}{kbs} + \frac{1}{ms^2 bs}\right) ms^2 - \cancel{\frac{1}{ms^2}}\Big] I_2(s)$$

$$X_f(s) = \frac{bs + ms^2 + k}{kbs} I_2(s)$$

$$X_t(s) = \frac{1}{b\cancel{s}} \cdot \frac{kb\cancel{s}}{ms^2 + bs + k} X_f(s)$$

2. To find the equations, we can replace each element with the impedance equivalent, along with a "pressure source". Note that there are three identifiable pressures - the pressure at the bottom of each tank, p_1 and p_2 and the atmospheric pressure p_a. Our circuit should thus have three identifiable voltage nodes (one of which will be ground)



The mesh equations for this system are

$$\begin{bmatrix} P_{in}(s) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\rho g}{sA_1} & -\frac{\rho g}{sA_1} & 0 \\ -\frac{\rho g}{sA_1} & \frac{\rho g}{sA_1} + R_1 + \frac{\rho g}{sA_2} & -\frac{\rho g}{sA_2} \\ 0 & -\frac{\rho g}{sA_2} & \frac{\rho g}{sA_2} + R_2 \end{bmatrix} \begin{bmatrix} Q_1(s) \\ Q_2(s) \\ Q_{out}(s) \end{bmatrix}$$

To find the relationship with h_2, we can write

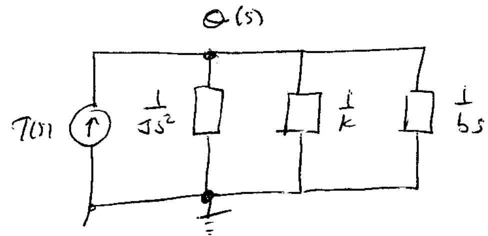
$$P_2(s) = R_2 Q_{out}(s),$$

and use the fact that

$$H_2(s) = \frac{P_2(s)}{\rho g}.$$

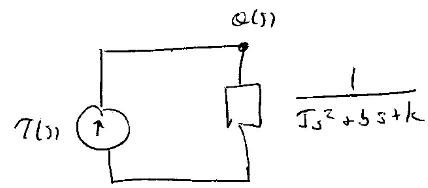
3.

mechanical circuit:



using parallel combination rule:

$$Z_{eq} = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}} = \frac{1}{Js^2 + bs + k}$$



Thus:
$$\Theta(s) = \frac{1}{Js^2+bs+k} T(s)$$

or
$$\frac{\Theta(s)}{T(s)} = \frac{1}{Js^2+bs+k}$$

7 Resources

7.1 Books

Mechanical impedance is nothing more than using the Laplace Transform of the mechanical component laws, but this connection is not always emphasized in introductory textbooks. This is often done via the graphical modeling technique of bond graphs, which are an alternate way of drawing impedance networks. For those who are interested, the following textbook covers modeling from this point of view.

- Javier A. Kypuros *System Dynamics and Control with Bond Graph Modeling*, CRC Press, 2013

7.2 Web resources

There are also some web resources that cover modeling mechanical systems. If you find something useful, or if you find a link that no longer works, please inform your instructor!

- `http://web.mit.edu/2.151/www/Handouts/Impedances.pdf`: A handout from a system dynamics course at MIT. Note that they use velocity as the across variable, but this is just the derivative of position.