

Complex Number Review*

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Elements to Review

These notes assume that you are familiar with the following facts:

- Quadratic Formula: How do we find the roots of the polynomial

$$p(x) = ax^2 + bx + c?$$

- Trig Functions: How can we use $\tan^{-1}(\theta)$ to find the angles of a right triangle?
- Algebra of exponentials: $e^a e^b = e^{a+b}$, $e^a / e^b = e^{a-b}$, $(e^a)^n = e^{an}$
- Taylor Series Expansion (MATH112). What are the Taylor Series expansions of $\sin(x)$, $\cos(x)$ and e^x at $x = 0$?

Lesson Objectives

- Convert from rectangular to polar form and vice versa
- Perform the operations of complex conjugate, addition, subtraction, multiplication, division.
- Use Euler's formula to represent a complex number in polar form
- Find roots and powers of a complex number using polar form

1 Complex Number Definition and Representation

Definition of Complex Numbers

- Complex numbers are needed to express all roots of polynomials.

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- What is the solution to

$$x^2 = -1?$$

- By definition

$$j := \sqrt{-1}$$

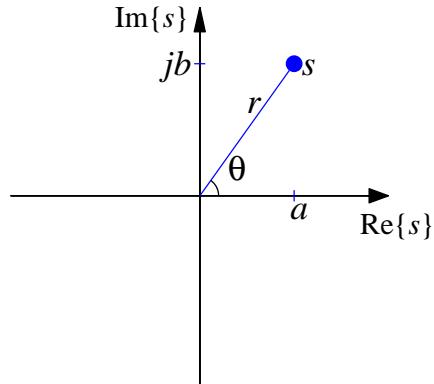
Complex Numbers

$$s = a + jb$$

where a and b are real numbers

Complex Plane

Visualize complex numbers in the complex plane



Two Representations

- Rectangular: $s = a + jb$
- Polar: $s = r\angle\theta$

Notation

- Magnitude: $r = |s|$
- Angle: $\theta = \angle s$

1.1 Converting between Rectangular and Polar Form

The two representations of a complex number are

- Rectangular: $s = a + jb$
- Polar: $s = r\angle\theta$

Where $r = |s|$ is called the magnitude of the complex number, and $\theta = \angle s$ is called the angle or phase of the complex number. In Figure 1 a complex number is plotted in the complex plane with both representations noted. Our first objective is to use trigonometry to find the relationships between a, b, r, θ .

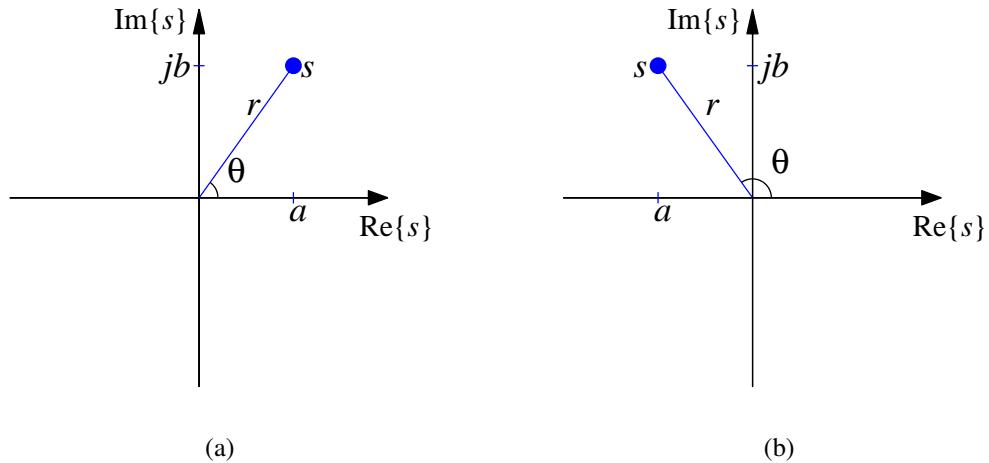


Figure 1: The complex plane. s marks a particular location, which can be identified using either rectangular or polar coordinates. (a) s in first quadrant. (b) s in second quadrant.

- **From Polar to Rectangular:** From Figure 1 we see that

$$\begin{aligned} a &= r \cos(\theta), \\ b &= r \sin(\theta), \end{aligned}$$

giving the transformation from polar to rectangular coordinates. This relationship is valid in all four quadrants.

- **From Rectangular to Polar:** For points in the first quadrant, we see that

$$\begin{aligned} \tan(\theta) &= \left(\frac{b}{a} \right) \\ r^2 &= a^2 + b^2 \end{aligned}$$

Thus the transformation from rectangular to polar becomes

$$\begin{aligned} \theta &= \tan^{-1} \left(\frac{b}{a} \right) \\ r &= \sqrt{a^2 + b^2} \end{aligned}$$

The equation for r is valid for all points, but the equation for θ is not valid 2nd and 3rd quadrants. Note that in these quadrants

$$\begin{aligned} \tan(\pm\pi - \theta) &= \left(\frac{b}{-a} \right) \\ r^2 &= a^2 + b^2 \end{aligned}$$

Thus, in general, the four quadrant inverse tangent rule must be applied. The general rectangular to polar transformation is

$$r = \sqrt{a^2 + b^2},$$

$$\theta = \begin{cases} \tan^{-1}(b/a), & \text{if } a \geq 0 \\ \pm\pi + \tan^{-1}(b/a), & \text{if } a < 0. \end{cases}$$

Note that we have used the fact that $\tan^{-1}(-\theta) = -\tan^{-1}(\theta)$.

Here are some examples to try:

Example 1. Convert from polar to rectangular

$$1. \ 1\angle\pi/3$$

$$2. \ 2\angle\pi$$

Solution:

$$1. \ 1\angle\pi/3$$

$$1\angle\pi/3 = \cos(\pi/3) + j \sin(\pi/3) = \frac{1}{2} + \frac{\sqrt{3}}{2}j$$

$$2. \ 2\angle\pi$$

$$2\angle\pi = 2\cos(\pi) + j2\sin(\pi) = -2$$

Example 2. Convert from rectangular to polar

$$1. \ 1+j$$

$$2. \ -1+j$$

Solution:

$$1. \ 1+j$$

$$\begin{aligned} r &= \sqrt{1+1} = \sqrt{2} \\ \theta &= \tan^{-1}(1) = \pi/4 \end{aligned}$$

$$2. \ -1+j$$

$$\begin{aligned} r &= \sqrt{1+1} = \sqrt{2} \\ \theta &= \pi - \tan^{-1}(1) = 3\pi/4 \end{aligned}$$

2 Complex Algebra I - Rectangular Form

- Complex Conjugate: $s = a + jb$

$$s^* = a - jb$$

The complex conjugate of s has the same real part, but negative imaginary part.

- Addition, Subtraction, Multiplication and Division: Rectangular Form: $s_1 = a_1 + jb_1$, $s_2 = a_2 + jb_2$

$$\begin{aligned} s_1 + s_2 &= (a_1 + a_2) + j(b_1 + b_2) \\ s_1 - s_2 &= (a_1 - a_2) + j(b_1 - b_2) \\ s_1 s_2 &= (a_1 + jb_1)(a_2 + jb_2) \\ &= (a_1 a_2 - b_1 b_2) + j(a_1 b_2 + b_1 a_2) \\ \frac{s_1}{s_2} &= \frac{(a_1 + jb_1)}{(a_2 + jb_2)} \\ &= \frac{(a_1 + jb_1)(a_2 - jb_2)}{(a_2 + jb_2)(a_2 - jb_2)} \\ &= \frac{(a_1 a_2 + b_1 b_2)}{(a_2^2 + b_2^2)} + j \frac{(b_1 a_2 - a_1 b_2)}{(a_2^2 + b_2^2)} \end{aligned}$$

Addition and subtraction are easy in rectangular form, but multiplication and division are easier in polar form, which we discuss later.

Example 3. Evaluate the following if $s_1 = 1 + j$, $s_2 = e^{j\pi/3}$

1. $s_1 + s_1^*$

2. $s_1 - s_2$

Solution:

- 1.

$$s_1 + s_1^* = (1 + j) + (1 - j) = 2$$

2. Since we are subtracting, we need to convert both numbers to rectangular form. Note that

$$s_2 = e^{j\pi/3} = \cos(\pi/3) + j \sin(\pi/3) = \frac{1}{2} + j \frac{\sqrt{3}}{2}$$

Thus,

$$s_1 - s_2 = (1 + j) - \left(\frac{1}{2} + j \frac{\sqrt{3}}{2} \right) = \frac{1}{2} + j \frac{2 - \sqrt{3}}{2}$$

3 Euler's Formula and Polar Form

- A very important equation that will help make complex multiplication and division easier is Euler's formula. It helps define the extension of the exponential function to complex numbers.

$$e^{j\theta} = \cos(\theta) + j \sin(\theta)$$

The function e^x is very important, as it appears in many places in engineering subjects. The reason that it is so ubiquitous is that it satisfies the relationship

$$\frac{d}{dx} e^x = e^x$$

and thus appears in situations when quantities increase in proportion to how much is already there. We want to extend this function to complex numbers. In this case, we will need to calculate e^z when z is complex. We can build e^z through its Taylor series expansion around $z = 0$, since we know $e^0 = 1$, and we want $\frac{d}{dz} e^z = e^z$. (Item to be swept under the rug: what exactly is the complex derivative of a complex function? Answer: same definition as for real derivatives, but with some extra care: See advanced engineering math!) Thus,

$$\begin{aligned} e^z &= e^0 + \frac{d}{dz} e^z \Big|_{z=0} z + \frac{1}{2!} \frac{d^2}{dz^2} e^z \Big|_{z=0} z^2 + \frac{1}{3!} \frac{d^3}{dz^3} e^z \Big|_{z=0} z^3 + \dots \\ &= e^0 + e^0 z + \frac{1}{2!} e^0 z^2 + \frac{1}{3!} e^0 z^3 + \dots \\ &= 1 + z + \frac{1}{2!} z^2 + \frac{1}{3!} z^3 + \dots \end{aligned}$$

We won't verify this, but you can show with a bit of algebra that this function satisfies the laws of exponents

$$e^{z_1+z_2} = e^{z_1} e^{z_2}.$$

Thus, if we plug in $z = a + jb$,

$$e^{a+jb} = e^a e^{jb}.$$

We already know e^a , so let's find out what e^{jb} is. Using the series expansion

$$\begin{aligned} e^{jb} &= 1 + jb + \frac{1}{2!}(jb)^2 + \frac{1}{3!}(jb)^3 + \frac{1}{4!}(jb)^4 \dots \\ &= 1 + jb - \frac{1}{2!}(b)^2 - j\frac{1}{3!}(b)^3 + \frac{1}{4!}(b)^4 \dots \\ &= (1 - \frac{1}{2!}(b)^2 + \frac{1}{4!}(b)^4 - \dots) + j(b - \frac{1}{3!}(b)^3 + \dots) \end{aligned}$$

But wait, we recognize the following Taylor Series Expansions:

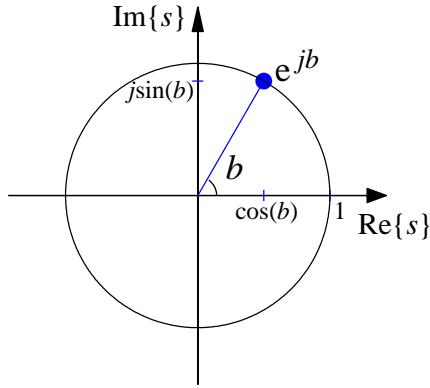
$$\begin{aligned} \cos(b) &= 1 - \frac{1}{2!}(b)^2 + \frac{1}{4!}(b)^4 - \dots \\ \sin(b) &= b - \frac{1}{3!}(b)^3 + \dots \end{aligned}$$

Plugging in gives us **Euler's Formula**:

$$e^{jb} = \cos(b) + j \sin(b)$$

If we plot e^{jb} , we see that it lies on the unit circle in the complex plane, at an angle of b .

Plotting Euler's formula



Thus, $e^{jb} = 1 \angle b$. This gives us an alternate way of writing polar form:

$$r \angle \theta = r e^{j\theta}$$

Example 4. Use Euler's formula to verify that $\frac{d}{d\theta} e^{j\theta} = j e^{j\theta}$

Solution:

$$\frac{de^{j\theta}}{d\theta} = \frac{d(\cos(\theta) + j \sin(\theta))}{d\theta} = -\sin(\theta) + j \cos(\theta) = j(\cos(\theta) + j \sin(\theta)) = j e^{j\theta}$$

Example 5. Use Euler's formula to find j^j

Solution:

$$j^j = (e^{j\pi/2})^j = (e^{j(j\pi/2)}) = e^{-\pi/2}$$

4 Complex Algebra II - Polar Form

4.1 Multiplication and Division

Addition and Subtraction must be done in rectangular form. But the other operations are easier when working in polar form.

- Conjugate: Polar Form: $s = re^{j\theta}$

$$s^* = re^{-j\theta}$$

- Multiplication and Division: Polar Form: $s_1 = r_1 e^{j\theta_1}, s_2 = r_2 e^{j\theta_2}$

$$\begin{aligned} s_1 s_2 &= r_1 e^{j\theta_1} r_2 e^{j\theta_2} \\ &= r_1 r_2 e^{j\theta_1} e^{j\theta_2} \\ &= r_1 r_2 e^{j(\theta_1 + \theta_2)} \\ \frac{s_1}{s_2} &= \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} \\ &= \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)} \end{aligned}$$

Example 6. Find the following in polar form: $(1 + j)2e^{j\pi/2}$

First, note that

$$(1 + j) = \sqrt{2}e^{j\pi/4}$$

Thus

$$(1 + j)2e^{j\pi/2} = \sqrt{2}e^{j\pi/4}2e^{j\pi/2} = 2\sqrt{2}e^{j3\pi/4}$$

Example 7. Find the magnitude of the following complex number: $\frac{(1+j)(2+2j)}{3+4j}$

The fact that when complex numbers multiply, their magnitudes multiply (and similarly for division) means that we can solve this problem by finding the magnitudes of each term in the expression *first* and then multiplying.

$$\begin{aligned} \left| \frac{(1 + j)(2 + 2j)}{3 + 4j} \right| &= \frac{|1 + j| |2 + 2j|}{|3 + 4j|} \\ &= \frac{\sqrt{1^2 + 1^2} \sqrt{2^2 + 2^2}}{\sqrt{3^2 + 4^2}} \\ &= \frac{\sqrt{2}\sqrt{8}}{\sqrt{9 + 16}} \\ &= \frac{\sqrt{16}}{\sqrt{25}} = \frac{4}{5} \end{aligned}$$

Example 8. Find the phase of the following complex number: $\frac{(1+j)(2+2j)}{3+4j}$

The fact that when complex numbers multiply, their magnitudes add (and subtract for division) means that we can solve this problem by finding the phases of each term in the expression *first* and then adding (if multiplied) or subtracting (if

divided).

$$\begin{aligned}\angle \frac{(1+j)(2+2j)}{3+4j} &= \angle(1+j) + \angle(2+2j) - \angle(3+4j) \\ &= \frac{\pi}{4} + \frac{\pi}{4} - 0.927 = .644 \text{ (rad)} \\ &= 45^\circ + 45^\circ - 53 = 37^\circ\end{aligned}$$

4.2 Powers and Roots of Complex Numbers

Rounding out our complex algebra:

- Powers: Polar Form: $s = re^{j\theta}$

$$s^n = r^n e^{jn\theta}$$

- Roots: Polar Form: solve

$$z^n = c.$$

Let's substitute $z = re^{j\theta}$ for z . Then

$$r^n e^{jn\theta} = c,$$

Since the magnitude and angle of the left and right hand sides must be equal, we get the two equations

$$\begin{aligned}r &= |c|^{1/n} \\ n\theta &= \angle c + 2\pi\ell \text{ for any integer } \ell.\end{aligned}$$

There are n unique solutions to the angle equation, so that

$$z = |c|^{1/n} e^{j(\frac{\angle c}{n} + \frac{2\pi\ell}{n})} \quad \ell = 0, 1, \dots, n$$

Example 9.

Solve the following:

$$z^3 = -1$$

Solutions:

$$\begin{aligned}z &= (e^{-j\pi})^{1/3} \quad (e^{-j3\pi})^{1/3} \quad (e^{-j5\pi})^{1/3} \\ z &= (e^{-j\pi/3}) \quad (e^{-j\pi}) \quad (e^{-j5\pi/3})\end{aligned}$$