EENG307: Introduction to Control Concepts and More on Stability*

Lecture 9

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Contents

1 Pre-requisite Material

This lecture assumes that the reader is familiar with the following material:

- Lecture 6: Mechanical Impedance and Introduction to Modeling with Simulink
- Lecture 5: Impedance and Transfer Functions
- Lecture 7: Solving Differential Equations using Laplace Transforms and Stability
- Lecture 8: Block Diagrams

2 Transfer Functions and Stability

2.1 Poles and Zeros of Transfer Functions

So far, most of the transfer functions we have seen have been ratios of polynomials. A function that is a ratio of polynomials is called a *rational function*, so these kinds of transfer functions are called *rational transfer functions*

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Rational and Irrational Transfer Functions

$G(s) = \frac{s^2+6}{s^3+3s+5}$	$G(s) = \frac{e^{-2s}}{s^2}$
rational transfer function	irrational transfer function

A polynomial is completely specified by its roots and a multiplicative gain. That is, if I tell you a polynomial has roots at -1, -2, and -3, and the highest coefficient is 6, you can figure out what the polynomial is:

$$\begin{aligned} p(s) &= 6(s+1)(s+2)(s+3) \\ &= 6(s^3+6s^2+11s+6) \\ &= 6s^3+36s^2+66s+36 \end{aligned}$$

In the same way, a rational transfer function is completely specified by the roots of the numerator and denominator, and a gain.

Poles and Zeros

Given rational transfer function $G(s) = \frac{N(s)}{D(s)}$ where $N(s)$ and $D(s)$ are polynomials in s with no common factors.

Definition 1. The *zeros* of $G(s)$ are the roots of $N(s)$. These are the values of s for which $G(s) = 0$

Definition 2. The *poles* of $G(s)$ are the roots of $D(s)$. These are the values of s for which $G(s) = \infty$

Definition 3. The *system order* is the number of poles of $G(s)$.

If a rational transfer function has poles p_1, p_2, zeros z_1, z_2 and multiplicative gain K, then the transfer function is

$$G(s) = K\frac{(s-z_1)(s-z_2)}{(s-p_1)(s-p_2)}$$

2.2 Stability of Transfer Functions

Recall from Lecture 7 that we examined the solutions (output signals) to differential equations in terms of the *structure* of the solution with respect to the specified input signal. In particular, we asked for several cases whether the solutions converged to a constant value as time $t \to \infty$ and/or remained bounded. Finally, we introduced the definition of BIBO stability, which we reprint here for convenience.

Definition 4. A system is *Bounded Input Bounded Output* (BIBO) *stable* if every bounded input results in a bounded output

In the context of differential equations, we noticed that positive terms in the denominators of the Laplace-domain signals $F(s)$ corresponded to the time-domain signal $f(t)$ being bounded. We can formalize this observation with the following test for stability:

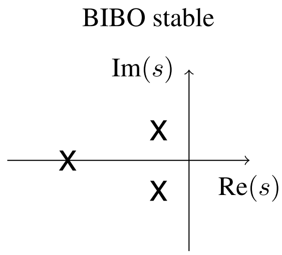
Fact 5. *An LTI system with transfer function $G(s)$ is BIBO stable if and only if $G(s)$ is proper and all poles p_i satisfy Re$\{p_i\} < 0$.*

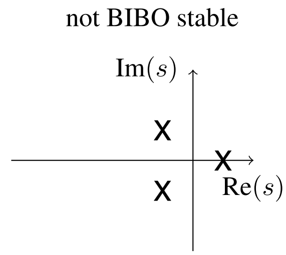
where

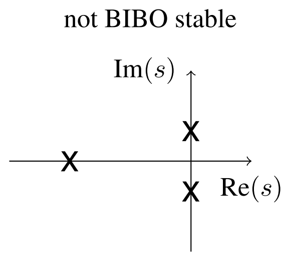
Definition 6. A rational transfer function $G(s)$ is *proper* if the number of poles is greater than or equal to the number of zeros

The following BIBO stability examples show plots of transfer function pole locations in the complex plane that are or are not BIBO stable. A common point of confusion is cases with poles on the imaginary axis, as in the 3rd plot shown. Note that *any pole on the imaginary axis (real part = 0) indicates that a system is not BIBO stable*, even if all other pole(s) are in the left half plane (real part less than zero).

BIBO Stability Examples



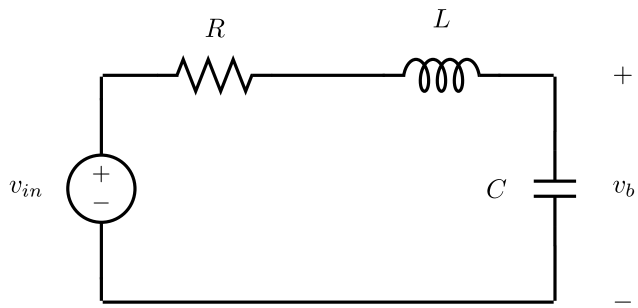




3 Connecting Concepts

In this section, we connect many of the concepts from the previous lectures, including transfer functions, stability, solving differential equations, and modeling systems in Simulink. We make these connections using a circuit example from a previous lecture.

Circuit Example



In a previous lecture, we found that the differential equation representing this circuit with input $v_{in}(t)$ and output voltage across the capacitor $v_b(t)$ was

$$CL\frac{d^2v_b(t)}{dt^2} + CR\frac{dv_b(t)}{dt} + v_b(t) = v_{in}(t).$$

Let's now work through the following tasks:

Circuit Example Tasks

1. Find the transfer function from $V_{in}(s)$ to $V_b(s)$
2. Calculate the poles of the transfer function when $L = 10$ H, $C = 0.1$ F, and $R = 5\ \Omega$.
3. Is this circuit BIBO stable?
4. Use "solving differential equations" concepts to calculate $v_b(t)$ when $v_{in}(t)$ is a unit step, i.e., $v_{in}(t) = u(t)$.
5. Does your answer to #4 make sense given your answer to #3?

6. Use Simulink to model the circuit given the transfer function you found in #1. Simulate the system with a step input. Is your answer consistent with your calculation from #4?

Task 1: Transfer Function

The first task uses the differentiation property of the Laplace transform and the fact that transfer functions have zero initial conditions (by definition). Therefore, the differential equation can be found by first taking the Laplace transform and collecting the terms multiplying the input and output signals

$$\begin{aligned} LCs^2V_b(s) + CRsV_b(s) + V_b(s) &= V_{in}(s) \\ \left(LCs^2 + CRs + 1\right) V_b(s) &= V_{in}(s) \end{aligned}$$

and then finding the transfer function as

$$\begin{aligned} \frac{V_b(s)}{V_{in}(s)} &= \frac{1}{LCs^2 + CRs + 1} \\ &= G(s) \end{aligned}$$

Tasks 2-3: Poles and Stability

Given the component values specified, the denominator of the transfer function is $s^2 + 0.5s + 1$. Using Matlab's `roots` function, we find that the poles of $G(s)$ are $-0.25 \pm j0.97$. Both of these poles p_i are in the left half plane ($\text{Re}\{p_i\} < 0$), so the system is BIBO stable.

Task 4: Solving the Differential Equation

When the input signal $v_{in}(t) = u(t)$, we have

$$\begin{aligned} V_b(s) &= \frac{1}{s^2 + 0.5s + 1} V_{in}(s) \\ &= \frac{1}{s^2 + 0.5s + 1} \left(\frac{1}{s}\right) \\ &= \frac{1}{s\left(s^2 + 0.5s + 1\right)} \end{aligned}$$

Partial fraction expansion results in

$$V_b(s) = \frac{A}{s} + \frac{Bs + C}{s^2 + 0.5s + 1}$$

and using any of the residue calculation methods gives us $A = 1$, $B = -1$, and $C = -0.5$. Finally, using a table of Laplace transform pairs we can solve for the time-domain output signal as

$$v_b(t) = \left(1 + -e^{-0.25t}\cos(0.968t) - 0.258e^{-0.25t}\sin(0.968t)\right) u(t)$$

Task 5: Sense-Making

We had concluded that the system is BIBO stable, and we can see from Task 4 that the output signal $v_{out}(t)$ is bounded. Therefore, yes, the answer makes sense given the definition of BIBO stability.

Task 6: Simulink Modeling Using the Simulink model shown in Figure 1 with step time of 0 s and simulation time of 30 s gives us the output voltage shown as the solid blue line in Figure 2. Superimposed on the Simulink result is the calculation from the solution to the differential equation (Task 4), which is indistinguishable at the scale shown. Thus, we can confirm that using Simulink to model the circuit system as a transfer function gives the same result as solving the differential equation.

The Matlab commands used to generate the curve from the differential equation solution and add it to the existing Simulink plot are

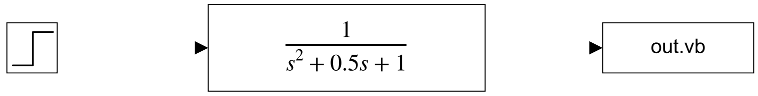


Figure 1: Simulink model for electrical circuit example.

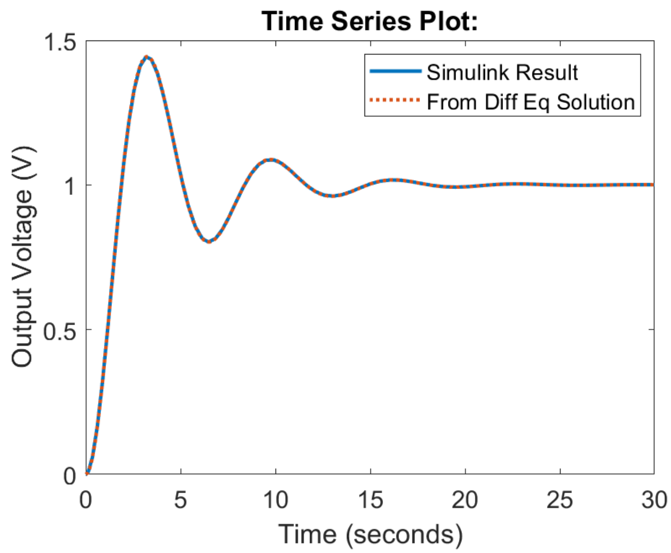


Figure 2: Output voltage vs. time for electrical circuit example when the input voltage is a unit step function.

```
t=[0:0.1:30];
vb=1-exp(-.25*t).*cos(.968*t)-0.258*exp(-.25*t).*sin(.968*t);
hold on;plot(t,vb,':')
legend('Simulink Result','From Diff Eq Solution')
```

4 Closing the Loop: Feedback Control

You'll notice that the results shown in Figure 2 show an output voltage v_{out} that has some oscillations and then converges to 1 V. On the bright side, as discussed in Task 3, this circuit system is stable. However, it might be that we want to explore ways to either decrease the oscillations or make the output voltage go to a different value. We could spend time laboriously testing different possible input voltages $v_{in}(t)$ until we could get the desired output for a given situation, but it's much easier to *design a feedback controller to do that for us*.

Figure 3 shows the closed-loop system with a feedback loop, reference input, and controller (triangular "Gain" block with value of 10 in this example). Let's follow around the loop and talk about what happens in this feedback control system.

What Happens in Feedback Control?

1. First, the designer specifies the *reference input*, which is the value that we want to see the output track. In other words, we want $v_b \to r$, where r is whatever we design it to be. Let's say we want a 12 V system. In that case, we let $r = 12$ V.
2. The automatic controller then calculates the error $e = r - v_b$ to see how different the actual voltage v_b is from the desired value.

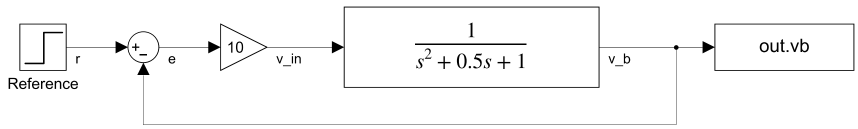


Figure 3: Adding closed-loop control to the electric circuit example in Simulink. The output signal, v_b, is measured (sensor assumed to be perfect and therefore not shown) and "fed back" via the feedback path for comparison to the reference signal r, which is the desired output voltage. The error $e = r - v_b$ is multiplied by the controller gain 10 to automatically calculate v_{in}.

3. If $e > 0$, that means $r > v_b$ and therefore v_{in} needs to be higher so that v_b is also higher (closer to the reference). The automatic controller does this automatically by multiplying e by the proportional gain value of 10 in the triangular "Gain" block.
4. Similarly, if $e < 0$, that means that $r < v_b$ and v_{in} needs to be smaller, which would also be calculated automatically.
5. These calculations continue for as long as the simulation runs, with v_{in} continuing to be calculated throughout as the controller tries to achieve the desired output voltage v_b.

Figure 4 shows the simulation results in terms of both the output signal $v_b(t)$ and the plant (electrical circuit) input signal $v_{in}(t)$ automatically calculated by the controller to try to get $v_b(t) \to r(t)$ when $r(t) = 12u(t)$. In other words, we are trying to achieve a 12 V output.

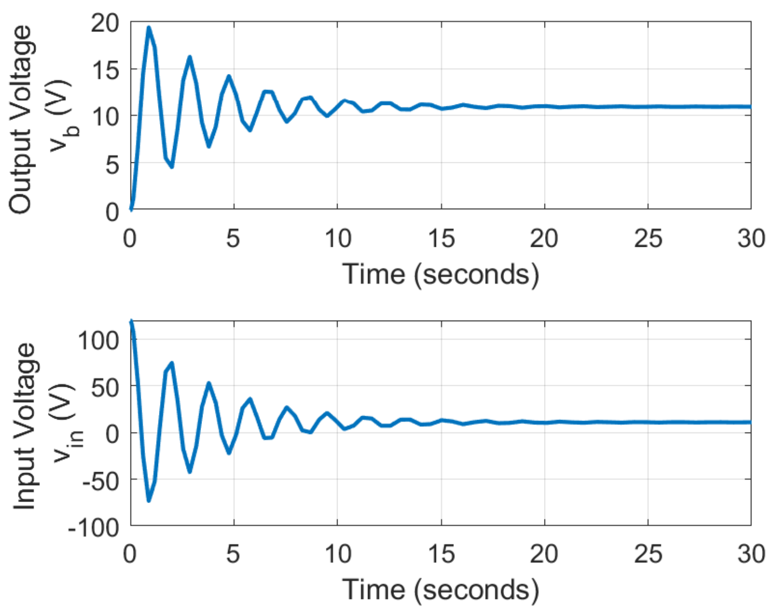


Figure 4: Plant input v_{in} and output v_b signals when the reference is a scaled step $r(t) = 12u(t)$.

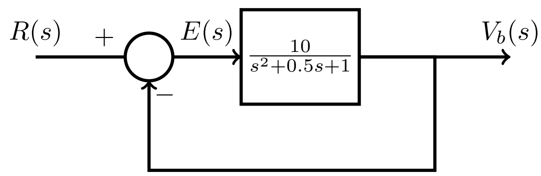
Finally, let's take a look at the stability of our closed-loop system and the performance of our feedback controller via the following questions:

Closed Loop Stability and Performance Assessment

1. Is the closed loop system from reference input $R(s)$ to output $V_b(s)$ BIBO stable?
2. Does the output $v_b(t)$ reach the desired value $r(t)$?

3. What does the behavior of the output $v_b(t)$ look like along the way?
4. How much "control authority" (amount of voltage $v_{in}(t)$) is needed to achieve the results?

Q1: Stability
We can assess the stability of the closed-loop system using techniques we learned in Lecture 8: Block Diagrams. Since the two blocks in the forward path, the "Gain" and "Transfer Function," are in series, their equivalent combined transfer function is just the product of each individual transfer function, or



Using the feedback property of block diagrams we see that

$$\begin{aligned}\frac{V_b(s)}{R(s)} &= \frac{\frac{10}{s^2+0.5s+1}}{1+\frac{10}{s^2+0.5s+1}} \\ &= \frac{10}{(s^2+0.5s+1)+10} \\ &= \frac{10}{s^2+0.5s+11}\end{aligned}$$

Using Matlab's `roots` function, we can determine that the closed loop transfer function's poles, i.e., the roots of $s^2+0.5s+11$, are $-0.25 \pm j3.3$. Since these poles are in the left half plane (LHP), we conclude that the closed-loop system is BIBO stable.

Q2-Q4: Performance
As can be seen from Figure 4, the overall performance doesn't look great: there are lots of oscillations in $v_b(t)$, it doesn't look like $v_b(t)$ quite reaches the 12 V reference (it settles in at around 11 V), and there is a lot of input voltage v_{in} needed to achieve even this questionable performance - over 100 V!

In the coming lectures, we'll be learning more about how to characterize and control for (first) the *transient performance* – in this case, the oscillatory behavior in the first 15 s – and (next) the *steady-state performance*, i.e., the discrepancy between the 11 V output we achieve and the 12 V we want.

A Note on Controller Units
In most engineering classes, there's a lot of attention paid to units, but convention in control theory can often be for units to be invisible, and therefore we don't use units in many lecture articles. In all cases, we can determine controller units by looking at the units on the input and output signal to the controller.

For this example, the input signal to the controller ("Gain" block with value 10) is the voltage error $e(t)$, which has units of volts. The output signal from the controller, $v_{in}(t)$, is also measured in volts, so the controller has units of $\frac{V}{V}$ and is therefore unitless. However, we will also see cases where the input and output to the controller block do not have the same units. For example, if our electrical circuit example had a current source instead of a voltage source, the plant input (which is also the controller output) would have units of amps, so the controller would have units of $\frac{A}{V}$.

5 Lecture Highlights

The primary takeaways from this article include

1. In this lecture, we first solidified the concept of stability. A key result is that we can test for BIBO stability by calculating the transfer function poles. If all of the poles are in the left half plane, the transfer function is stable.
2. This stability test can be used for any transfer function. In this article, we applied it to both the electrical circuit plant (open loop) and the closed-loop transfer function with feedback control.

3. We also illustrated closed-loop control in Simulink by randomly testing a proportional controller with gain of 10 and examined the signals around the loop.

4. Finally, we assessed the control design by considering both stability and performance observations (and found that this controller isn't great!), motivating future lessons.

6 Quiz Yourself

6.1 Questions

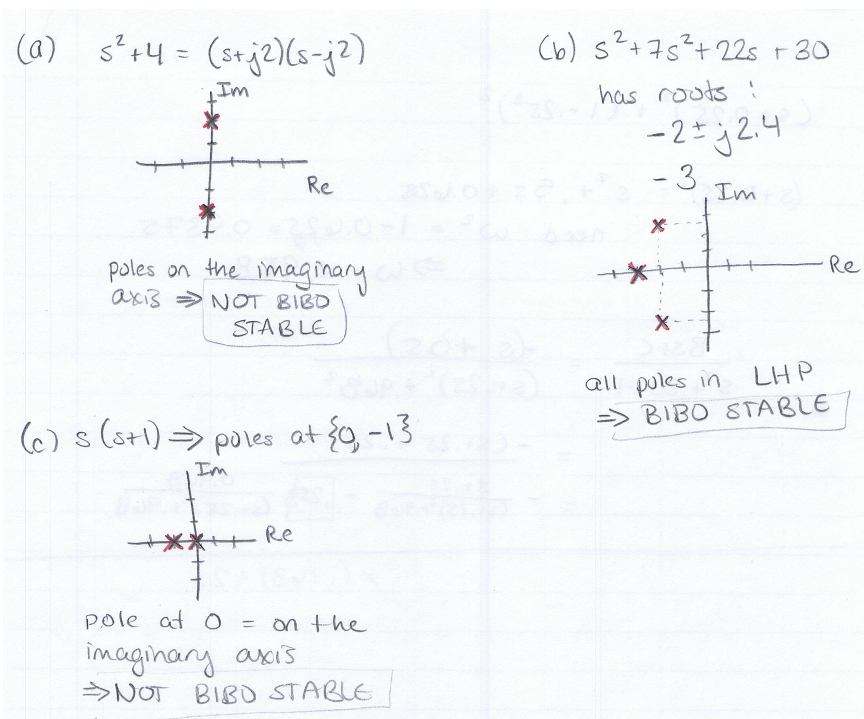
1. For the following systems with transfer functions given, plot the poles and indicate whether each is BIBO stable or not.

   (a) $G(s) = \frac{s-1}{s^2+4}$
   (b) $G(s) = \frac{30}{s^3+7s^2+22s+30}$
   (c) $G(s) = \frac{2}{s(s+1)}$

2. Consider again the circuit example in this lecture. How would the stability analysis change if the circuit had negligible resistance, i.e., $R = 0\Omega$? Does this answer make sense given that neither inductors nor capacitors dissipate energy?

6.2 Solutions

1.



2. From the open loop transfer function $G(s) = \frac{1}{LCs^2+CRs+1}$, when $R = 0\Omega$ and using the same inductance and capacitance in the lecture, we have $G(s) = \frac{1}{s^2+1}$. The poles of this open loop transfer function are $s_{1,2} = \pm j$, which are both on the imaginary axis and not in the open LHP, which means that the transfer function is unstable.

   Similarly, the closed-loop transfer function becomes $\frac{V_b(s)}{R(s)} = \frac{10}{s^2+11}$, which also has purely imaginary poles on the imaginary axis and is therefore not BIBO stable.
   It makes sense that a circuit with a resistor would be “more stable” in a sense than one without, given that resistors can dissipate energy, which means that the output signal is less likely to increase to a magnitude of infinity.