EENG307: Rotational and Fluid Systems*

Lecture 14

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Contents

1 Pre-requisite Material

This lecture assumes that the reader is familiar with the following material:

- Lecture 5: Impedance and Transfer Functions

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2 Overview: More System Analogies

In Lecture 5, we introduced the concept of *system analogies* based on "across" and "through" variables, i.e., those that must be measured across a component (e.g. voltage, position) and those that flow through an element (e.g., current, force). In this lecture, we continue to develop analogies for two more types of systems: rotating mechanical systems and fluid systems. Although some differential equations are included, we focus on impedance-based modeling techniques.

The following table lists the across and through variables for the modeling domains that we have discussed and will discuss in this course.

Across and Through Variables

Domain	Across Variable	Through Variable
Electrical	Voltage	Current
Translational Mechanical	Position	Force
Fluid	Pressure	Flow
Rotational Mechanical	Angular Position	Torque

By utilizing variables that have the same function in different domains, we can come up with system analogies. Analogous systems are systems in different domains that have the same equations (with different parameters) describing their behavior.

In this lecture, we'll demonstrate impedance-based techniques for modeling fluid and rotational mechanical systems. In each of the two following article sections, we'll

- note the across and through variables,
- list the components with their component equations,
- define the connection rules and boundary conditions,
- give a table of the equivalent impedances for each component, and finally
- give an impedance example using the same steps you have previously seen for creating an impedance network:
  - identifying and sketching nodes (including ground),
  - connecting components between nodes, and
  - incorporating sources.

3 Rotational Mechanical Systems

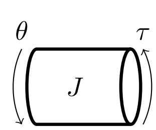
Previously, we modeled translational motion in one direction. When objects rotate, the modeling process is similar. For rotational mechanical systems, the variables that we are modeling are

- torque, which has units of Newton meters [N m], and
- angular position, which has units of radians [rad].

Unless otherwise specified, we will always assume that the torque and position variables have these units. We will use the following elements to describe rotational mechanical systems.

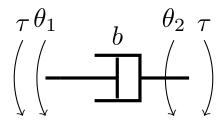
3.1 Components

When objects rotate, they follow the rotational version of Newton's law, which relates applied torque to angular acceleration. The scaling parameter in this case is the rotational inertia, J which has units of [kg m^2].



$$\tau = J\ddot{\theta}$$

Viscous friction also applies in the rotational case. In this case, each of the ends of the damper can rotate in response to an applied torque. The scaling parameter b has units of [N m s rad^{-1}].

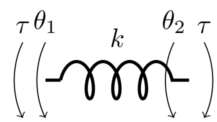


$$\dot{\theta} = \dot{\theta}_1 - \dot{\theta}_2$$

$$\tau = b\dot{\theta}$$

Note that the applied torque is always *equal and opposite* on both ends of the damper.

A rotational spring. Each of the ends of the spring can rotate in response to an applied torque. The scaling parameter has units of [N m rad^{-1}].



$$\theta = \theta_1 - \theta_2$$

$$\tau = k\theta$$

Like the damper, the applied torque is always *equal and opposite* on both ends of the spring.

3.2 Connection Rules and Boundary Conditions

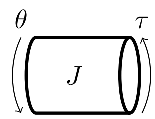
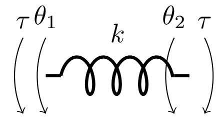
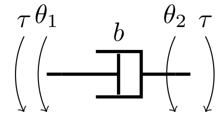
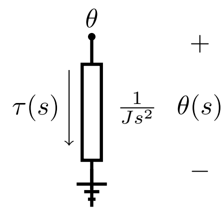
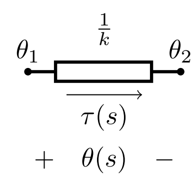
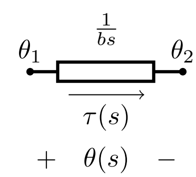
The connection rules and boundary conditions are the same as in the translational case.

- When two elements are connected, the two components share the same *angular position*.
- When two elements are connected, the torques at the connection *sum to zero*.
- A ground fixes the position of that terminal to zero.
- A force or position input prescribes the force or position of that terminal.

3.3 Impedances

We note that the algebraic laws that relate rotational mechanical elements are *identical* to that of translational mechanical elements, with angular position as the across variable and torque as the through variable. We can use the following impedance elements.

Rotational Impedance

	mass	spring	damper
Component		 $\theta = \theta_1 - \theta_2$	 $\dot{\theta} = \dot{\theta}_1 - \dot{\theta}_2$
Component Law	$J\ddot{\theta} = \tau$	$\tau = k\theta$	$\tau = b\dot{\theta}$
Laplace Transform	$\theta(s) = \frac{1}{Js^2}\tau(s)$	$\theta(s) = \frac{1}{k}\tau(s)$	$\theta(s) = \frac{1}{bs}\tau(s)$
Impedance Component (force direction agrees with positive direction)			

3.4 Example

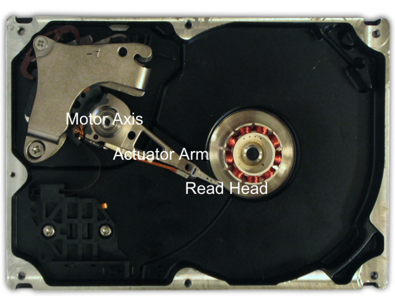
The process of finding the transfer function using impedances is straightforward.

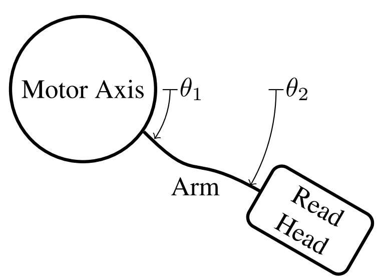
0. Draw a diagram of ideal elements.
1. Identify all of the independent nodes (across variables) which for mechanical systems are positions. Add a ground node or identify a fixed node as ground.
2. Connect elements between nodes, with masses and inertias always connected between one node and ground
3. Add boundary conditions as sources.
4. Use circuit techniques to find the transfer function.

Let's try this process to model a hard disk drive read head, which is an important application of control systems. The control system must move the read head over the correct track within a few milliseconds, and hold it there despite disturbances such as external shocks or disk irregularities.

Hard Disk Drive Read Head

In order to move the read head to the correct track and hold it there, we need to be able to predict the relationship between the motor torque τ and the angular position of the read head θ_2. First, find the equivalent impedance model. Then, find the transfer function $\frac{\theta_2}{\tau}$.





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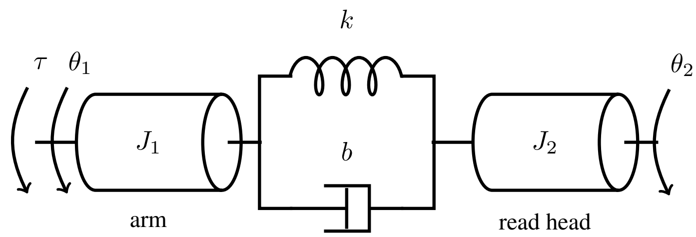
Since the arm is thin, it is flexible. This can be modeled with ideal elements by splitting the inertia between the arm and the read head, and connecting these inertias with a spring and damper. The motor applies the torque τ. J_1 is the inertia of the motor axis, while $J_2 = mr^2$, where m is the mass of the read head, and r is the radius at which it rotates. k and b are constants that describe the stiffness and damping characteristics of the actuator arm.

Step 0 Diagram of ideal elements

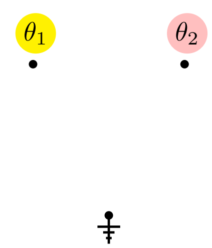
Hard Disk Drive Ideal Elements



Step 1 For this problem, there are two independent positions: θ_1 and θ_2.

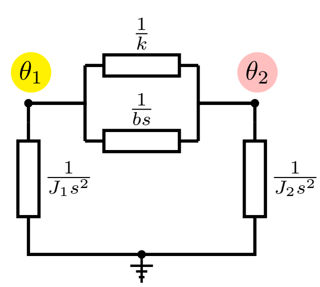
Step 2 Neither of these are fixed, so we also add a ground node.

Nodes



Step 3.1 Connect the correct impedance elements between nodes

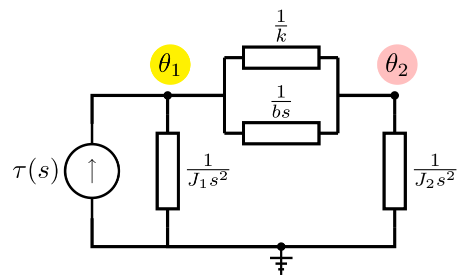
Disk Drive Impedance Network



Step 3.2 The applied torque is a boundary condition. Since torque is a through variable, we should add a current source to the θ_1 node.

> Rule for assigning sign of sources: The current source goes *into* the node if a positive torque would tend to cause a positive $\theta_1(s)$.

Disk Drive Complete Circuit



Step 4 Find the transfer function

We can now find the transfer function from $\tau(s)$ to either $\theta_1(s)$ or $\theta_2(s)$ using our standard circuit methods. If we wanted to find $\theta_2(s)/\tau(s)$, the easiest way is to set up the nodal equations, which is KCL at each node.

Nodal equations:

$$\begin{bmatrix} \tau(s) \\ 0 \end{bmatrix} = \begin{bmatrix} J_1 s^2 + bs + k & -(bs+k) \\ -(bs+k) & J_2 s^2 + bs + k \end{bmatrix} \begin{bmatrix} \theta_1(s) \\ \theta_2(s) \end{bmatrix}$$

Solve for θ_1 using second equation:

$$\theta_1(s) = \frac{J_2 s^2 + bs + k}{bs + k} \theta_2(s)$$

Plug into first equation:

$$\begin{aligned} \tau(s) &= (J_1 s^2 + bs + k) \frac{J_2 s^2 + bs + k}{bs + k} \theta_2(s) - (bs + k)\theta_2(s) \\ &= \frac{J_1 J_2 s^4 + J_1 s^2 (bs + k) + J_2 s^2 (bs + k) + (bs + k)^2 - (bs + k)^2}{(bs + k)} \theta_2(s) \\ &= \frac{J_1 J_2 s^4 + b(J_1 + J_2) s^3 + k(J_1 + J_2) s^2}{(bs + k)} \theta_2(s) \end{aligned}$$

Thus

$$\frac{\theta_2(s)}{\tau(s)} = \frac{bs + k}{J_1 J_2 s^4 + b(J_1 + J_2) s^3 + k(J_1 + J_2) s^2}$$

4 Incompressible Fluid Systems

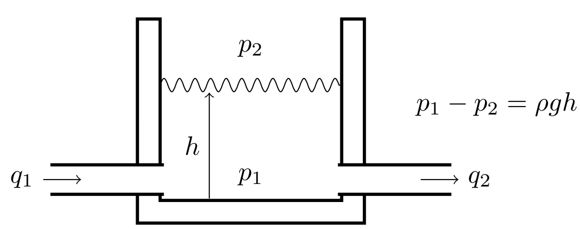
For fluid systems, the variables that we are modeling are

- Pressure, which has units of Newtons per meter squared [Nm^{-2}] or equivalently [$\mathrm{kg\ m^{-1} s^{-2}}$]
- Volumetric Flow, which has units of meters cubed per second [$\mathrm{m^3 s^{-1}}$].

4.1 Components

Tank

The change in the volume V of fluid inside the tank is equal to the difference between the input and output volumetric flow rates, q_1 and q_2, respectively, in this diagram:



Tank Area: A

The change in the volume of fluid in the tank with respect to time is equal to the difference between the input and output volumetric flows, so that

$$\frac{dV}{dt} = q_1 - q_2. \tag{1}$$

For a tank with constant cross-sectional area A [m^2], $V = Ah$, and taking the derivative of both sides with respect to time gives us

$$\frac{dV}{dt} = A\frac{dh}{dt}. \tag{2}$$

Setting (1) equal to (2) gives us the equation

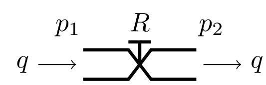
$$A\frac{dh}{dt} = q_1 - q_2.$$

Another way of considering this tank model is in terms of the pressure difference between the bottom and top of the tank, since $p_1 - p_2 = \rho g h$, where ρ is the fluid density [kg m^{-3}], and g is gravitational acceleration [m s^{-2}], both of which are constant. Substituting $h = \frac{p_1 - p_2}{\rho g}$ results in

$$\frac{A}{\rho g}\frac{d(p_1 - p_2)}{dt} = q_1 - q_2.$$

Linear Valve

A valve causes a restriction that causes the pressure on one end of the value to be higher than the other end. When this pressure drop is proportional to the flow, the valve is linear with valve constant R. (Most valves are nonlinear, however.) By unit analysis, the units of valve resistance are [N s m^{-5}] or equivalently [kg m^{-4}s^{-1}].



$$p_1 - p_2 = Rq$$

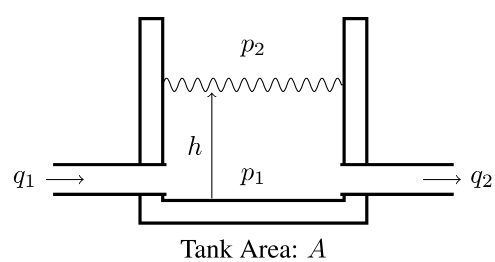
Why only two elements? We have not considered fluid inertia.

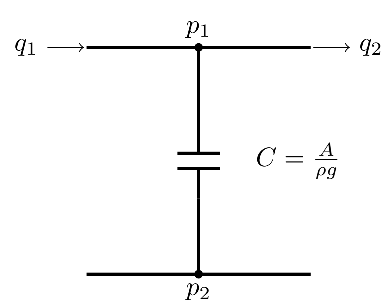
4.2 Connection Rules and Boundary Conditions

The connection rules and boundary conditions will sound familiar

- When elements are connected, the two components share the same *pressure*.
- When elements are connected, the *flows* sum to zero.
- Boundary conditions can set either the pressure or flow of one side of a component

Electrical Analogy for Fluid Elements:
Tank

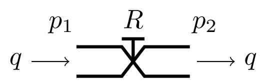


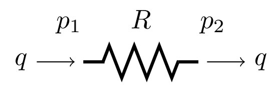


$\frac{A}{\rho g}\frac{d(p_1 - p_2)}{dt} = q_1 - q_2$

$C\frac{d(p_1 - p_2)}{dt} = q_1 - q_2$

Electrical Analogy for Fluid Elements:
Valve





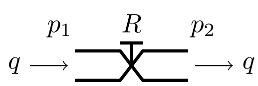
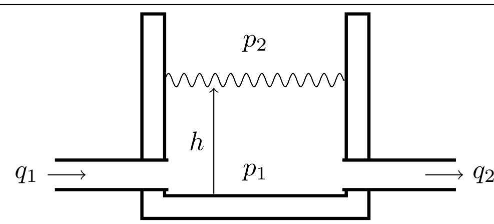
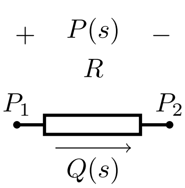
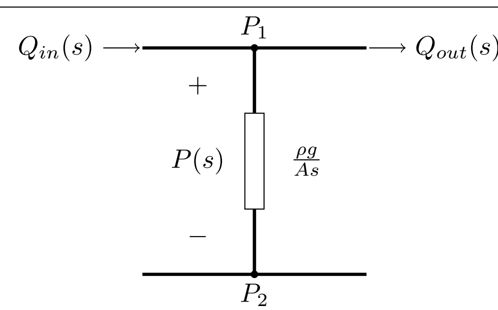
$p_1 - p_2 = Rq$ $\qquad\qquad$ $p_1 - p_2 = Rq$

4.3 Fluid Impedances

With these two analogies to electrical components (tank as capacitor and valve as resistor), it is very clear what the impedance of fluid components are when we want to use impedance methods to find transfer functions.

For fluid systems, we have the following impedances

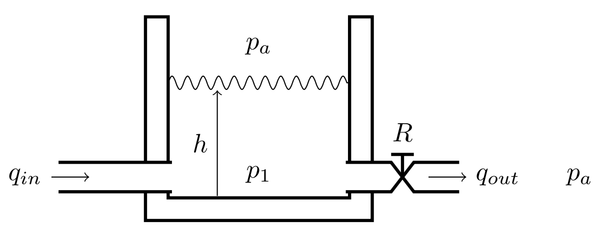
Fluid Impedances

	valve	tank
Component	$p = p_1 - p_2$ 	 Tank Area: A
Component law	$p = Rq$	$\frac{A}{\rho g}\frac{dp}{dt} = q_{in} - q_{out}$
Laplace Transform	$P(s) = RQ(s)$	$\frac{A}{\rho g}sP(s) = Q_{in}(s) - Q_{out}(s)$
Impedance Component		

Let's return to the earlier tank and valve system and go through the modeling process with analogous elements.

Tank and Valve Problem

Example: Find the equivalent impedance model of the tank and valve system below with input flow q_{in} and output flow q_{out}.



Tank Area: A

Step 1 **Identify all node variables.** For a fluid problem, the node variables are pressure. There are two unique pressure variables: p_1, the pressure at the bottom of the tank, and p_a atmospheric pressure. Our circuit should thus have two identifiable voltage nodes.

Tank system nodes

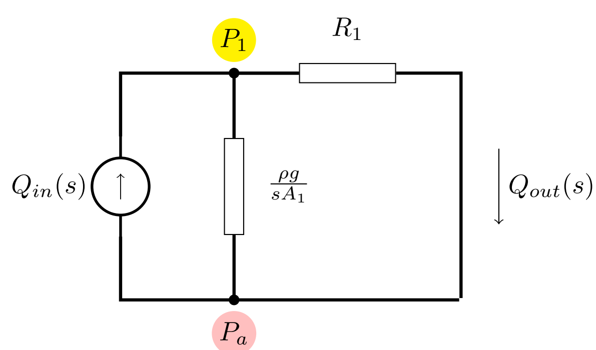
Step 2 **Identify one node as ground, or add a ground node.** Most node variables are relative - for example, consider the ground on a circuit diagram. All voltages are measured with respect to ground, so the ground voltage is simply what we will consider to be 0 V. Negative voltages are lower than ground, positive voltages are higher than ground. If we identify a node as the ground, then we will measure all node variables relative to this node. For fluid systems, there are two typical choices: Either choose absolute zero pressure (i.e. a vacuum) as ground, or choose atmospheric pressure as ground. Pressure measured with respect to atmospheric pressure is called *gauge* pressure, and the translation is

$$\text{Absolute Pressure} = \text{Gauge Pressure} + \text{Atmospheric Pressure}$$

For this problem, we will choose to measure pressure using gauge pressure, so that node p_a will be considered ground.

Step 3 **Connect components between nodes.** Since flow is a through variable, setting the input flow is done with a current source. The flow is into the node representing pressure p_1, and all sources representing boundary conditions have the other end connected to ground. We also connect a tank (capacitance) impedance between p_1 and p_a, because the pressure at the top of the tank is p_a. A valve (resistance) impedance is connected between p_1 and p_a because the valve transmits flow from a node at pressure p_1 to a node at pressure p_a.

Tank system impedance model



You can now use your circuit simplifications to find any transfer function of interest.

Again, it should be emphasized that in this case we have chosen the ambient pressure p_a as the reference pressure. This means that all pressure nodes in this "circuit" are measured with respect to ambient pressure, so that the pressure at node p_1 is in fact $p_1 + p_a$.

5 Application Example

Consider a wind turbine as a rotating mechanical system. Occasionally it is desirable to use the generator in reverse (as a motor providing torque τ) to turn the rotor to a certain angular position θ so that maintenance can be performed. An automatic controller can be designed to position the rotor, but before designing the controller first we need to be able *to model the turbine in transfer function form*.

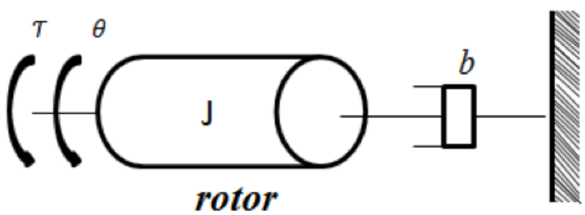
Step 0: to model the turbine's rotor, first determine which ideal components we need:

a) rotational mass with inertia J

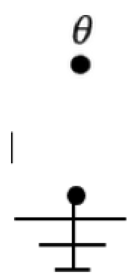
b) rotational damper with damping constant b (to represent friction)

Note that a wind turbine would not have a rotational spring k. If it did, the spring would get more and more extended as the rotor rotated, and eventually it would need to unwind.

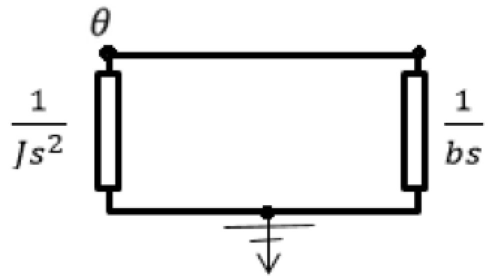
Step 1: sketch the ideal element diagram



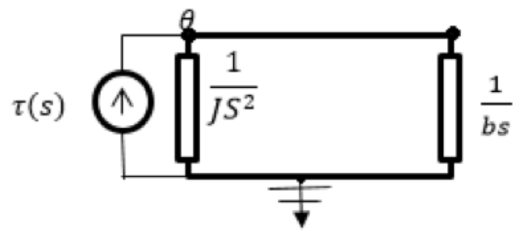
Step 2: sketch the required nodes (in this case: just one since there is only one rotational position)



Step 3: connect the node to the ground with impedances for both the mass and damper



Step 4: add the source (torque is analogous to a current source). The current source goes into the node if a positive torque would tend to cause a positive θ.



Now that we have an impedance network, we can use circuit methods to find the transfer function. Note that the two impedances are in parallel; therefore, the total impedance is

$$Z_{eq}(s) = \frac{\left(\frac{1}{Js^2}\right)\left(\frac{1}{bs}\right)}{\frac{1}{Js^2} + \frac{1}{bs}} = \frac{1}{bs + Js^2}$$

From the impedance version of Ohm's law, we know that $\theta(s) = \tau(s)Z_{eq}(s)$. Therefore, our transfer function from torque to position is

$$\frac{\theta(s)}{\tau(s)} = \frac{1}{Js^2 + bs}$$

Now we have our "plant" (turbine) transfer function from our input signal τ to our output signal θ. Later on this semester, we'll use information in this transfer function to design a proportional-integral-derivative (PID) feedback controller to achieve the desired position.

6 Lecture Highlights

The primary takeaways from this article include

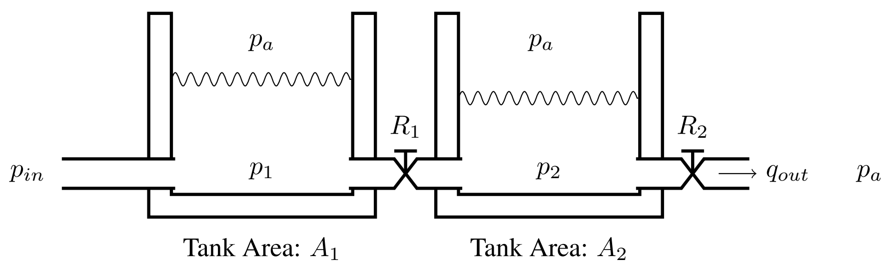
1. Rotational mechanical systems are very similar to translational mechanical systems, with rotational inertia replacing mass, rotational dampers replacing translational dampers, rotational springs replacing translational springs, torque replacing force, and angular position replacing position. If you're struggling with a sign convention when determining a system's differential equation, you might try replacing each rotational system component with its analogous translational component to see if the signs make more sense.
2. In this class, we model fluid systems using two ideal components: tanks and linear valves.
3. The component law for a tank is a differential equation and the component law for a linear valve is an algebraic equation. You can take the Laplace transform of each equation to find the component's impedance: the transfer function from the flow to the pressure.
4. Individual impedances can be connected together according to the connection rules to form an equivalent circuit, which enables circuit-based system analysis.

5. The process for sketching the equivalent impedance circuit is to identify and sketch all nodes (including the ground), then connect nodes using component impedances and sources.
6. As with all linear dynamic systems, system transfer functions can also be found by taking the Laplace transform of the system's differential equations, but many students find these impedance-based techniques more straightforward after some practice.

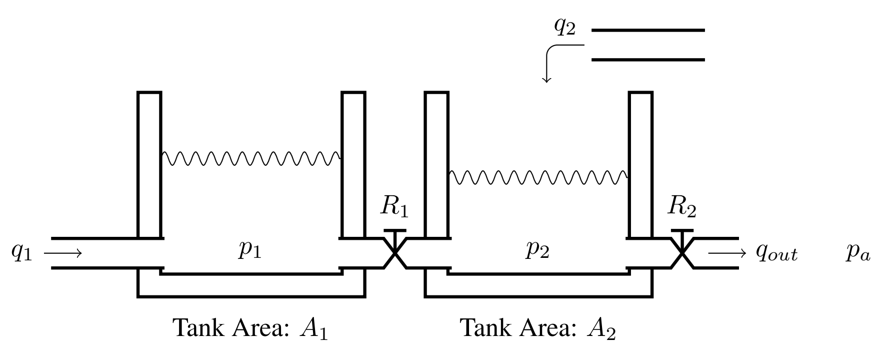
7 Quiz Yourself

7.1 Questions

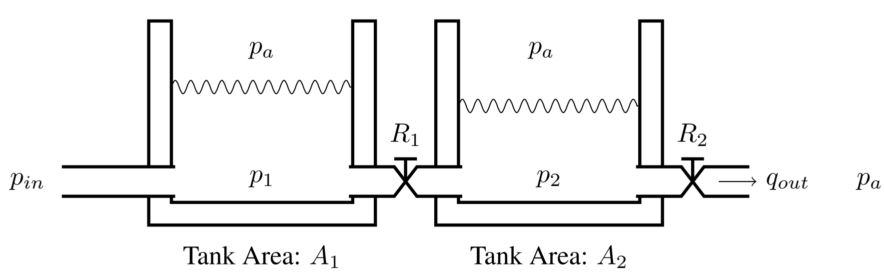
1. Find an equivalent circuit using current and/or voltage sources, resistor(s), and capacitor(s) (not Laplace-domain impedance blocks) for the following system. Note that



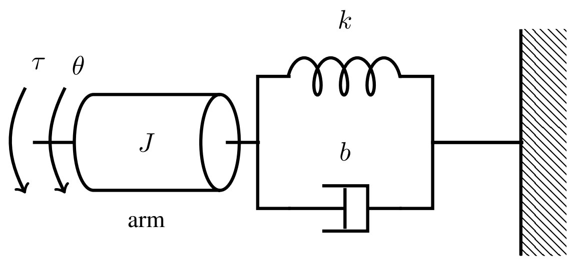
2. Find an equivalent circuit using current and/or voltage sources, resistor(s), and capacitor(s) (not Laplace-domain impedance blocks) for the following fluid system. The fluid density is ρ. Note that there are two sources.



3. Two tanks are connected in series, with the supply at pressure p_{in}, measured relative to atmospheric pressure. Use impedance network concepts to find the Laplace-domain equations that describe the behavior of the tanks, including the height of the second tank, $H_2(s)$.

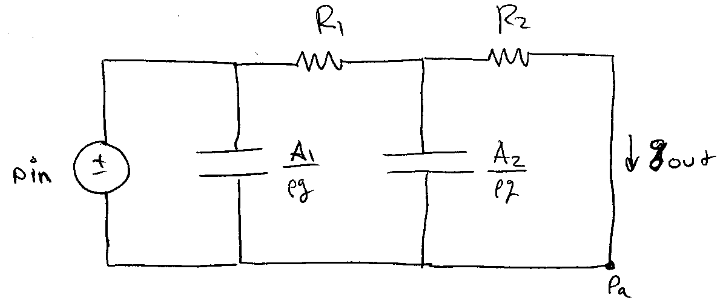


4. Find the transfer function relating τ to θ for the following system

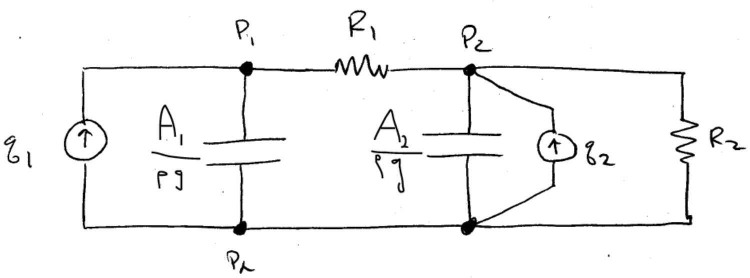


7.2 Solutions

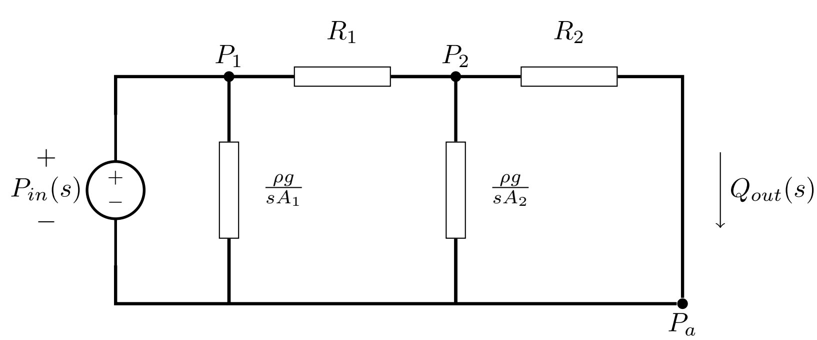
1.



2.



3. To find the equations, we can replace each element with the impedance equivalent, along with a "pressure source". Note that there are three identifiable pressures - the pressure at the bottom of each tank, p_1 and p_2 and the atmospheric pressure p_a. Our circuit should thus have three identifiable voltage nodes (one of which will be ground)



The mesh equations for this system are

$$\begin{bmatrix} P_{in}(s) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\rho g}{sA_1} & -\frac{\rho g}{sA_1} & 0 \\ -\frac{\rho g}{sA_1} & \frac{\rho g}{sA_1} + R_1 + \frac{\rho g}{sA_2} & -\frac{\rho g}{sA_2} \\ 0 & -\frac{\rho g}{sA_2} & \frac{\rho g}{sA_2} + R_2 \end{bmatrix} \begin{bmatrix} Q_1(s) \\ Q_2(s) \\ Q_{out}(s) \end{bmatrix}$$

To find the relationship with h_2, we can write

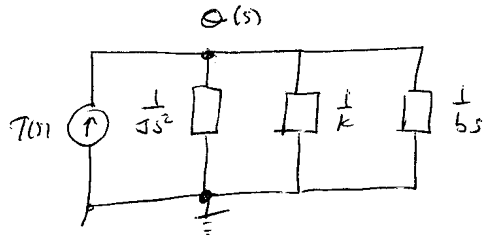
$$P_2(s) = R_2 Q_{out}(s),$$

and use the fact that

$$H_2(s) = \frac{P_2(s)}{\rho g}.$$

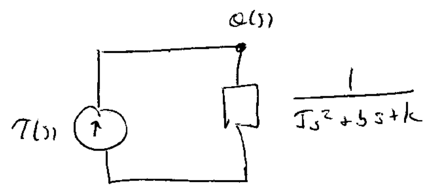
4.

mechanical circuit:



using parallel combination rule:

$$Z_{eq} = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}} = \frac{1}{Js^2 + bs + k}$$



Thus: $$\Theta(s) = \frac{1}{Js^2+bs+k} T(s)$$

or $$\frac{\Theta(s)}{T(s)} = \frac{1}{Js^2+bs+k}$$

8 Resources

8.1 Books

- Fluid systems occur in the feedback control context when using hydraulic actuators and also in chemical processing. The following textbooks discuss modeling fluid systems.
  - Gene F. Franklin, J. David Powell and Abbas Emami-Naeini, *Feedback Control of Dynamic Systems*, Pearson, 6th and 7th edition: Section 2.4
- Mechanical impedance is nothing more than using the Laplace Transform of the mechanical component laws, but this connection is not always emphasized in introductory textbooks. This is often done via the graphical modeling technique of bond graphs, which are an alternate way of drawing impedance networks. For those who are interested, the following textbook covers modeling from this point of view.
  - Javier A. Kypuros *System Dynamics and Control with Bond Graph Modeling*, CRC Press, 2013

8.2 Web resources

There are a variety of web resources, with differing levels of modeling for both fluid and rotating mechanical systems. In this lecture, we did not consider fluid inertia, which you may find in some on-line resources. If you find something useful, or if you find a link that no longer works, please inform your instructor!

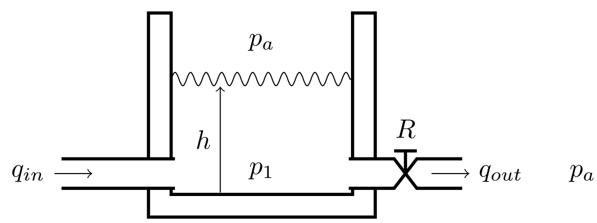
- http://web.mit.edu/2.151/www/Handouts/Impedances.pdf: A handout from a system dynamics course at MIT. Note that they use velocity as the across variable, but this is just the derivative of position.
- A YouTube video on lumped modeling of fluid systems: https://www.youtube.com/watch?v=_cpubJ94ugc

9 Appendix: Fluid Example with Differential Equations

Although experience has shown that most students prefer to use the impedance or equivalent circuit methods to model fluid systems, for completeness we include this example: find the differential equation that describes the relationship between q_{in}, and h for the following system:

Tank and Valve Example

Assume the valve is linear with valve constant R and that the density of the fluid is ρ. The valve empties to atmospheric pressure, p_a, which is the same as the pressure at the top of the tank.



Tank Area: A

We use the following steps:

- Write down the component laws for the tank and valve, along with connection laws where applicable.
- Write down the relationship between the pressures and the tank height.
- Eliminate unwanted variables.

The component laws are:

$$A\frac{dh}{dt} = q_{in} - q_{out},$$
$$p_1 - p_2 = Rq_{out}.$$

Note that we used a connection law to establish that the flow out of the tank is q_{out}, the pressure at the left side of the valve is p_1, and the pressure on the right side of the valve is p_a where p_a corresponds to p_2 in the component law above. We also know that

$$\rho g h = p_1 - p_2.$$

Thus,

$$A\frac{dh}{dt} = q_{in} - q_{out},$$
$$\rho g h = Rq_{out}.$$

Eliminating q_{out},

$$A\frac{dh}{dt} = q_{in} - \frac{\rho g}{R}h,$$

or

$$A\frac{dh}{dt} + \frac{\rho g}{R}h = q_{in}.$$