

# EENG307: Sinusoidal Steady State\*

## Lecture 24

Elenya Grant, Kathryn Johnson, and Hisham Sager†

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## 1 Pre-lecture Math Facts

Suppose you have the following rational transfer function:

$$G(s) = \frac{s^2 + 2s + 3}{s^2 + 5s + 4}.$$

In the following lecture, we will see that evaluating a transfer function at  $s = j\omega$  will be useful

$$G(j\omega) = \frac{(j\omega)^2 + 2(j\omega) + 3}{(j\omega)^2 + 5(j\omega) + 4} = \frac{3 - \omega^2 + j2\omega}{4 - \omega^2 + j5\omega}$$

Now, suppose we instead evaluate  $G(-j\omega)$ :

$$G(-j\omega) = \frac{(-j\omega)^2 + 2(-j\omega) + 3}{(-j\omega)^2 + 5(-j\omega) + 4} = \frac{3 - \omega^2 - j2\omega}{4 - \omega^2 - j5\omega}$$

Notice that both the numerator and denominator are the *complex conjugate* of the numerator and denominator of  $G(j\omega)$ , because the real parts are the same, but the imaginary parts have opposite signs:

$$\begin{aligned}(3 - \omega^2 - j2\omega)^* &= 3 - \omega^2 + j2\omega \\ (4 - \omega^2 - j5\omega)^* &= 4 - \omega^2 + j5\omega\end{aligned}$$

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†Developed and edited by Tyrone Vincent and Kathryn Johnson, Colorado School of Mines, with contributions from Salman Mohagheghi, Chris Coulston, Kevin Moore, CSM and Matt Kupilik, University of Alaska, Anchorage

Since conjugation and division commute, i.e.

$$\frac{a^*}{b^*} = \left(\frac{a}{b}\right)^*$$

we actually have the following:

$$\boxed{G(-j\omega) = G(j\omega)^*}$$

Although we showed this for a specific case, this is true in general for any rational transfer function with real coefficients (just use the property that sum and conjugation also commute:  $(a^* + b^*) = (a + b)^*$ )

In addition, recall that if  $s_1$  and  $s_2$  are complex numbers

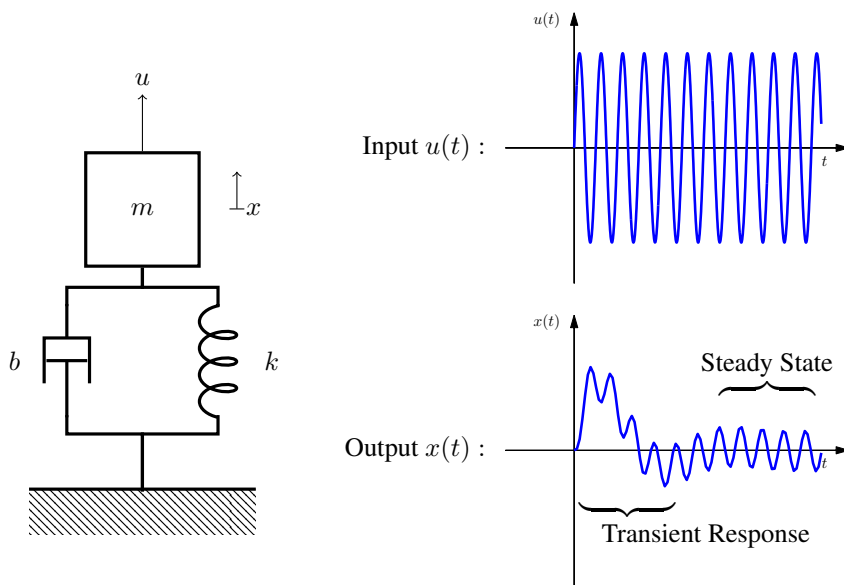
$$\left|\frac{s_1}{s_2}\right| = \frac{|s_1|}{|s_2|}$$

$$\angle \frac{s_1}{s_2} = \angle s_1 - \angle s_2$$

## 2 Response of Linear Systems to Sinusoidal Inputs

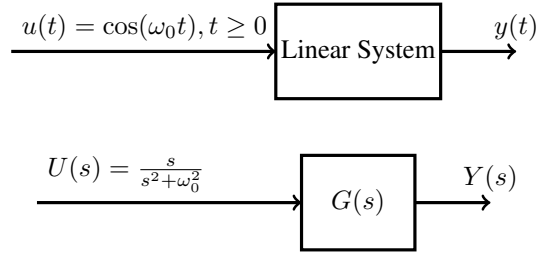
You are probably already aware of a basic property of **stable** linear systems: If the input is a sinusoid, then the *steady state* output (also called the steady state response) will be a sinusoid of the same frequency.

### System Response to Sinusoidal Input



This can be proven by using Laplace Transforms.

**Step 1: Multiply  $G(s)$  by the Laplace Transform of a cosine**



$$Y(s) = G(s) \frac{s}{s^2 + \omega_0^2}$$

**Step 2: Write partial fraction expansion, splitting poles at  $\pm j\omega_0$**

- Let  $p_1, p_2, \dots, p_n$  be the poles of  $G(s)$  (assume simple poles for now, but result also holds if  $G(s)$  has repeated poles)

$$Y(s) = G(s) \frac{s}{s^2 + \omega_0^2} = \frac{A}{s - j\omega_0} + \frac{B}{s + j\omega_0} + \frac{C}{s - p_1} + \frac{D}{s - p_2} + \dots$$

- By residue formula

$$A = \cancel{(s - j\omega_0)} G(s) \frac{s}{\cancel{(s - j\omega_0)}(s + j\omega_0)} \Big|_{s=j\omega_0} = G(j\omega_0) \frac{1}{2}$$

$$B = \cancel{(s + j\omega_0)} G(s) \frac{s}{(s - j\omega_0)\cancel{(s + j\omega_0)}} \Big|_{s=-j\omega_0} = G(-j\omega_0) \frac{1}{2}$$

**Step 3: Find Inverse Laplace Transform**

- Since

$$Y(s) = \frac{G(j\omega_0) \frac{1}{2}}{s - j\omega_0} + \frac{G(-j\omega_0) \frac{1}{2}}{s + j\omega_0} + \frac{C}{s - p_1} + \frac{D}{s - p_2} + \dots$$

- Using  $e^{-at}u \xleftrightarrow{\mathcal{L}} \frac{1}{a+s}$  the time response is

$$y(t) = \left( G(j\omega_0) \frac{1}{2} e^{j\omega_0 t} + G(-j\omega_0) \frac{1}{2} e^{-j\omega_0 t} + C e^{p_1 t} + D e^{p_2 t} \right) u(t)$$

**Step 4: Use property of stable poles**

- If  $\text{Re}(p_i) = -a < 0$ ,

$$\lim_{t \rightarrow \infty} C e^{p_i t} u(t) = \lim_{t \rightarrow \infty} C e^{\text{Re}(p_i)t} e^{j\text{Im}(p_i)t} u(t) = \lim_{t \rightarrow \infty} C e^{-at} e^{j\text{Im}(p_i)t} u(t) = 0$$

- Thus

$$\lim_{t \rightarrow \infty} y(t) = \left( G(j\omega_0) \frac{1}{2} e^{j\omega_0 t} + G(-j\omega_0) \frac{1}{2} e^{-j\omega_0 t} \right)$$

**Step 5: Use symmetry property  $G(-j\omega) = G(j\omega)^*$  and Euler's formula**

- Substitute for  $G(-j\omega_0)$

$$\lim_{t \rightarrow \infty} y(t) = \left( \frac{1}{2} G(j\omega_0) e^{j\omega_0 t} + \frac{1}{2} G(j\omega_0)^* e^{-j\omega_0 t} \right)$$

- Write in polar form, using  $a^* = |a|e^{-j\angle a}$

$$\lim_{t \rightarrow \infty} y(t) = \left( \frac{1}{2} |G(j\omega_0)| e^{j\angle G(j\omega_0)} e^{j\omega_0 t} + \frac{1}{2} |G(j\omega_0)| e^{-j\angle G(j\omega_0)} e^{-j\omega_0 t} \right)$$

- Euler's formula:  $\frac{1}{2} A e^{j\theta} + \frac{1}{2} A e^{-j\theta} = A \cos(\theta)$

$$\lim_{t \rightarrow \infty} y(t) = |G(j\omega_0)| \cos(\omega_0 t + \angle G(j\omega_0))$$

What we have shown is if the input is a cosine with frequency  $\omega_0$ , then the output, at steady state, is also a cosine with frequency  $\omega_0$ , but with an amplitude gain of  $|G(j\omega_0)|$ , and a phase shift of  $\angle G(j\omega_0)$ . This is summarized below in the general case.

**Sinusoidal Steady State is the Frequency Response**

**Theorem 1.** *Given a stable system with transfer function  $G(s)$ , the sinusoidal steady state response is defined by the input/output relationship*

$$\begin{aligned} u(t) &= A \cos(\omega_0 t + \theta), \\ y_{ss}(t) &= |G(j\omega_0)| A \cos(\omega_0 t + \theta + \angle G(j\omega_0)). \end{aligned}$$

**Definition 2.**  $G(j\omega)$  is the *frequency response function* of the system with transfer function  $G(s)$ .

Note: since  $\sin(\omega t) = \cos(\omega t - 90^\circ)$  the same is true for sine functions.

### 3 Examples

*Example 3.* Find the steady state response

$$\begin{aligned} G(s) &= \frac{1}{s+1} \\ u(t) &= 3 \cos(2t + 30^\circ), t \geq 0 \end{aligned}$$

**Solution:** Calculate the magnitude and phase of  $G(j2)$ :

$$\begin{aligned} |G(j2)| &= \left| \frac{1}{j2+1} \right| & \angle G(j2) &= \angle \frac{1}{j2+1} \\ &= \frac{|1|}{|j2+1|} & &= \angle 1 - \angle j2 + 1 \\ &= \frac{1}{\sqrt{1+2^2}} & &= 0 - \tan^{-1} \left( \frac{2}{1} \right) \\ &= \frac{1}{\sqrt{5}} & &= -63.4^\circ \end{aligned}$$

The solution is thus

$$y_{ss}(t) = \frac{3}{\sqrt{5}} \cos(2t - 33.4^\circ)$$

*Example 4.* Find an expression for the magnitude and phase frequency response functions for a system with transfer function

$$G(s) = \frac{\sigma}{s + \sigma}$$

where  $\sigma > 0$ . **Solution:** The magnitude of the frequency response function is given by

$$\begin{aligned} |G(j\omega)| &= \left| \frac{\sigma}{j\omega + \sigma} \right| \\ &= \frac{|\sigma|}{|j\omega + \sigma|} \\ &= \frac{\sigma}{\sqrt{\omega^2 + \sigma^2}} \end{aligned}$$

The phase of the frequency response function is given by

$$\begin{aligned} \angle G(j\omega) &= \angle \frac{\sigma}{j\omega + \sigma} \\ &= \angle \sigma - \angle(j\omega + \sigma) \\ &= 0 - \tan^{-1} \left( \frac{\omega}{\sigma} \right) \\ &= -\tan^{-1} \left( \frac{\omega}{\sigma} \right) \end{aligned}$$

## 4 Lecture Highlights

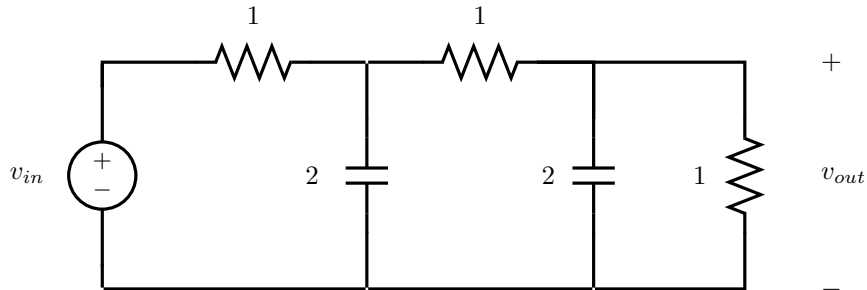
The primary takeaways from this article include

1. After an initial transient response period, the output of a dynamic system with a sinusoidal input will be a steady-state sinusoid that has the same frequency  $\omega_0$  as the input signal.
2. The output sinusoid will be scaled (amplitude changed) and shifted (phase angle changed) according to properties of the system's transfer function  $G(s)$ .
3. By evaluating the magnitude and phase angle of the system's transfer function  $G(s)$  at  $s = j\omega_0$ , we can determine the steady-state output signal without having to go through the full inverse Laplace transform process.

## 5 Quiz Yourself

### 5.1 Questions

1. The following system models a transmission line



and has the transfer function

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{4s^2 + 8s + 3}$$

Find the steady state output if the following sinusoids are applied as input

- (a)  $\cos(0.1t)$
- (b)  $\cos(t)$
- (c)  $\cos(10t)$

## 5.2 Solutions

1(a)

$$\text{if } v_{in}(t) = \cos(0.1t),$$

$$G(j.1) = \frac{1}{4(.1j)^2 + 8(.1j) + 3} = \frac{1}{-.04 + .8j + 3}$$

$$= \frac{1}{2.96 + 0.8j}$$

$$|G(j.1)| = \frac{1}{|2.96 + 0.8j|} = \frac{1}{3.07} = .326$$

$$\angle G(j.1) = \angle 1 - \angle 2.96 + 0.8j = 0 - 15.1^\circ = -15.1^\circ$$

$$v_{out}(t) \text{ at steady state} = .326 \cos(0.1t - 15.1^\circ)$$

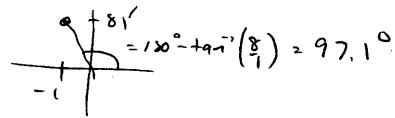
1(b)

if  $V_{in}(t) = \cos(t)$ ,

$$G(s) = \frac{1}{4(s)^2 + 8(s) + 3} = \frac{1}{-4 + 8s + 3} = \frac{1}{-1 + 8s}$$

$$|G(j1)| = \frac{1}{|-1 + 8j|} = \frac{1}{8.06} = 0.124$$

$$\angle G(j1) = \angle 1 - \angle -1 + 8j = 0 - 97.1^\circ = -97.1^\circ$$



$$= 100^\circ - \tan^{-1}\left(\frac{1}{8}\right) = 97.1^\circ$$

at steady state

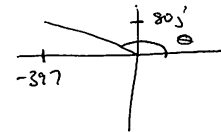
$$V_{out}(t) = 0.124 \cos(t - 97.1^\circ)$$

1(c)

$$G(j10) = \frac{1}{4(j10)^2 + 8(j10) + 3} = \frac{1}{-400 + 80j + 3} = \frac{1}{-397 + 80j}$$

$$|G(j10)| = \frac{1}{|-397 + 80j|} = \frac{1}{\sqrt{397^2 + 80^2}} = 0.0025$$

$$\begin{aligned} \angle G(j10) &= \angle 0 - \angle -397 + 80j \\ &= -168.6^\circ \end{aligned}$$



$$\begin{aligned} \theta &= 180 - \tan^{-1}\left(\frac{80}{397}\right) = \\ &= 180 - 11.4 = 168.6^\circ \end{aligned}$$

at steady state

$$V_{out}(t) = 0.0025 \cos(10t - 168.6^\circ)$$