EEENG307: Modeling Electrical Systems*

Lecture 3

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Contents

1 Pre-requisite Material

This lecture assumes that the reader is familiar with the following material:

- Lecture 2: Modeling Mechanical Systems
- Circuits

2 Idealized components for Electrical Systems

Feedback control in the context of electrical systems used to be very important, as it is used to implement analog amplification and filtering systems using operational amplifiers. While much of these tasks have been taken over by digital systems, we still encounter electrical components in electric motors and other electro-mechanical systems. We will also see that techniques used in circuit analysis can be very useful for analyzing other types of linear systems as well.

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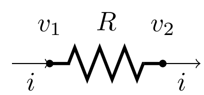
When finding the behavior of electrical systems, we at first limit ourselves to *linear* components. However, this will be enough to model electromechanical objects such as motors or speakers. Electrical systems are described by the two variables

- Voltage, which has units of Volts [V]
- Current, which has units of Amperes [A] or amps. Current is the rate of flow of charge, so amps represent coulombs per second.

2.1 Resistor

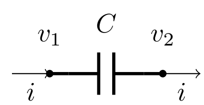
A resistor is a linear element that converts electrical energy to heat. There is a linear relationship between voltage across the resistor and current through the resistor, with a slope called the resistance R. By unit analysis, the units of R are [VA^{-1}], but this has been named ohms [Ω] in honor of German physicist Georg Simon Ohm.



$$v_1 - v_2 = Ri$$

2.2 Capacitor

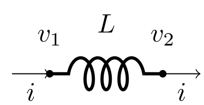
A capacitor is a storage element. It can store electric charge, which in turn sets up a voltage between the terminals. There is a linear relationship between the voltage and the stored charge, called the capacitance C, which has units of coulombs per volt that are named farads [F], after the English physicist Michael Faraday. Taking the derivative with respect to time, this becomes a linear relationship between the derivative of the voltage and the current.



$$C\tfrac{d(v_1 - v_2)}{dt} = i$$

2.3 Inductor

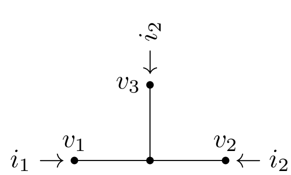
The inductor is also a storage element: it stores energy in a magnetic field. In this case, the linear relationship is between the derivative of the current and the voltage, and the slope of the relationship (volts per amp) is the inductance in henries [H], named after American scientist Joseph Henry.



$$v_1 - v_2 = L\tfrac{di}{dt}$$

2.4 Connection Rules

When components are combined, then we apply the connection rules for electrical systems:



- The voltage at a connection point is the *same* (Kirchhoff's voltage law)

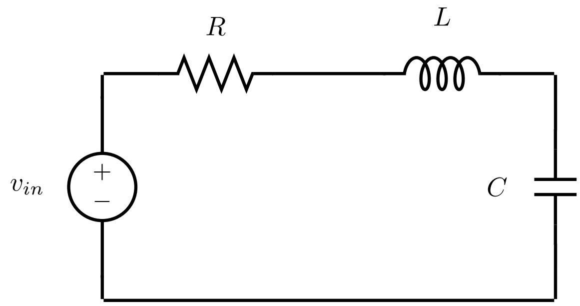
$$v_1 = v_2 = v_3$$

- The currents into a connection point *sum to zero* (Kirchhoff's current law)

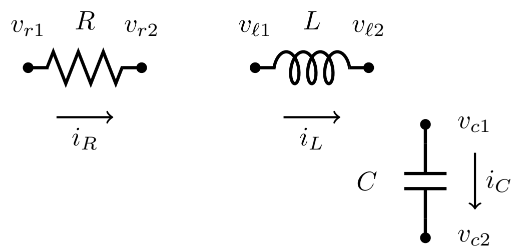
$$i_1 + i_2 + i_3 = 0$$

We also apply boundary conditions to an electrical system by specifying a *current source* or a *voltage source*.

Example 1. As an example, let's find the equations that describe the following circuit:



We could consider a "free body diagram," similar to the mechanical case, by separating each element as follows:



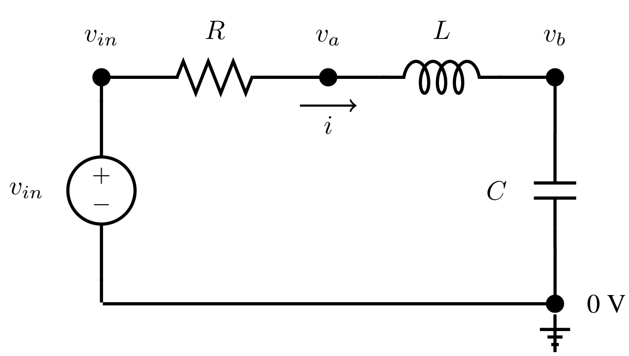
Our connection laws would tell us that

$$\begin{aligned} v_{r2} &= v_{\ell 1}, \\ v_{\ell 2} &= v_{c1}, \\ i_R &= i_L = i_C, \end{aligned}$$

while the boundary condition is

$$v_{in} = v_{r1} - v_{c2}.$$

However, rather than using a free body diagram, we usually find it easier to apply the connection rules and boundary conditions more directly. This starts by identifying the voltages at each node (i.e. connection point) using a unique name. Since all of the voltages are relative, one of the nodes can be identified as the ground node. The voltage at this node is *defined* to be 0V. By doing this, boundary condition of v_{in} implies the upper left most node has voltage of v_{in}.



We then define the current through each element. In this case, the connection rules tell us that the current through each element is the same, so we use only one current variable: i. The component equations are then

$$v_{in} - v_a = iR,$$
$$v_a - v_b = L\frac{di}{dt},$$
$$C\frac{dv_b}{dt} = i.$$

Just as in the case of mechanical systems, we now have to ask ourselves which variables are of interest. We should be able to find an equation that relates a boundary condition (the input) to any other variable (the output). Let's suppose we want to designate v_b as the output. We have three equations, so we can eliminate the other two variables: v_a and i. Solving for v_a using the second equation and plugging into the first results in the two equations

$$v_{in} - L\frac{di}{dt} - v_b = iR,$$
$$C\frac{dv_b}{dt} = i.$$

Now, use the last equation to eliminate i

$$v_{in} - CL\frac{d^2v_b}{dt^2} - v_b = CR\frac{dv_b}{dt},$$

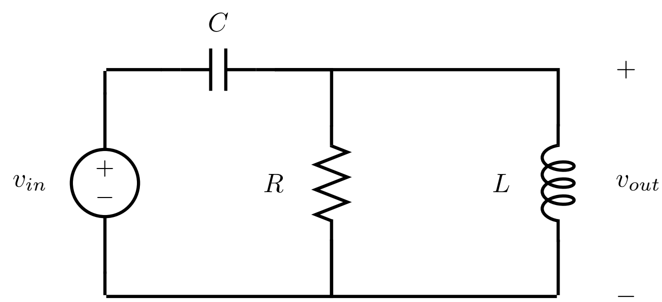
and rearranging,

$$v_{in} = CL\frac{d^2v_b}{dt^2} + CR\frac{dv_b}{dt} + v_b.$$

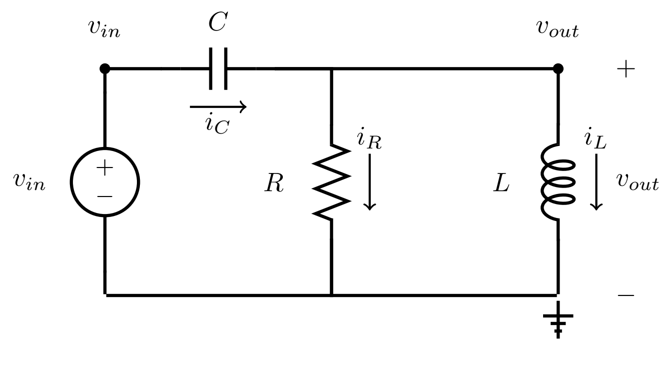
Those of you that have previously taken circuit analysis classes may find this derivation a bit odd. What about using impedance, equivalent resistances, voltage dividers, etc? We will get to this in the next lectures...

3 Example

Example 2. Find a differential equation that describes the circuit that includes only v_{in} and v_{out} as variables.



Step 1: Select a ground and name the nodes and currents:



Step 2: Write down component laws and any remaining connection laws:

$$\begin{aligned}
i_c &= C\frac{d(v_{in} - v_{out})}{dt} \\
v_{out} &= Ri_R \\
v_{out} &= L\frac{di_L}{dt} \\
i_C &= i_R + i_L
\end{aligned}$$

Step 3: Eliminate unwanted variables: i_R, i_C and i_L. First, substitute in for i_C:

$$\begin{aligned}
i_R + i_L &= C\frac{d(v_{in} - v_{out})}{dt} \\
v_{out} &= Ri_R \\
v_{out} &= L\frac{di_L}{dt}
\end{aligned}$$

Then, plug in for i_R:

$$\begin{aligned}
\frac{v_{out}}{R} + i_L &= C\frac{d(v_{in} - v_{out})}{dt} \\
v_{out} &= L\frac{di_L}{dt}
\end{aligned}$$

Solve for i_L using the first equation, then take the derivative with respect to both sides.

$$\begin{aligned}
i_L &= -\frac{v_{out}}{R} + C\frac{d(v_{in} - v_{out})}{dt} \\
\frac{di_L}{dt} &= -\frac{dv_{out}}{dt}\frac{1}{R} + C\frac{d^2(v_{in} - v_{out})}{dt^2}
\end{aligned}$$

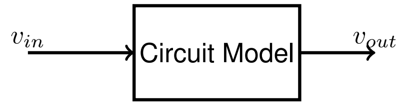
Now we can plug this into the second equation:

$$v_{out} = L\left(-\frac{dv_{out}}{dt}\frac{1}{R} + C\frac{d^2(v_{in} - v_{out})}{dt^2}\right)$$

We can clean this up by collecting like terms on the same side of the equation:

$$LC\frac{d^2v_{out}}{dt^2} + \frac{L}{R}\frac{dv_{out}}{dt} + v_{out} = LC\frac{d^2v_{in}}{dt^2}$$

Connecting back to the Modeling Mechanical Systems article and its visualization of signals and systems, we would represent this system with its input and output signals as follows:



The upcoming Laplace Transform Review article will provide a foundation for a more rigorous mathematical interpretation of this block diagram visualization.

4 Lecture Highlights

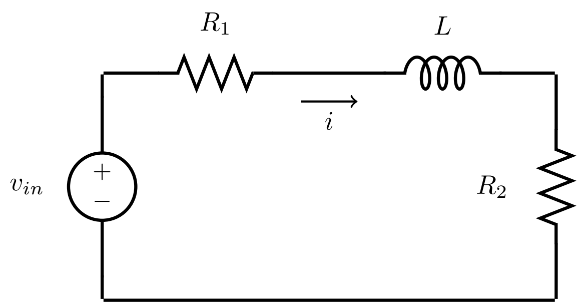
The primary takeaways from this article include

1. The primary idealized components for electrical systems are resistors, inductors, and capacitors.
2. As with mechanical systems, electrical idealized components are described by component laws and connected via connection rules. These connection rules should sound familiar from your circuits class: they are Kirchhoff's voltage and current laws.

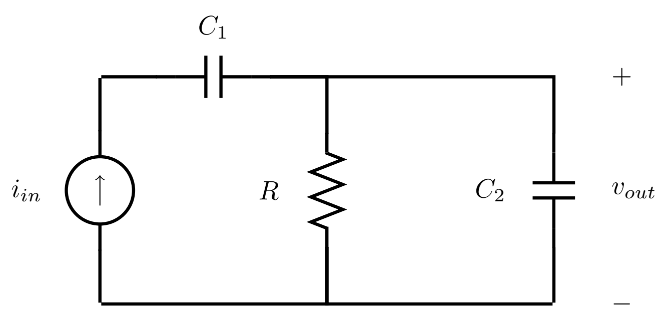
5 Quiz Yourself

5.1 Questions

1. Find a differential equation relating v_{in} to i for the following system



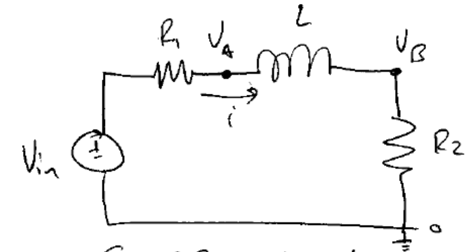
2. Find a differential equation that describes the circuit that includes only i_{in} and v_{out} as variables.



5.2 Solutions

1.

(1)



component laws
$$\begin{cases} iR_1 = V_{in} - V_A & (1) \\ L\dfrac{di}{dt} = V_A - V_B & (2) \\ iR_2 = V_B & (3) \end{cases}$$

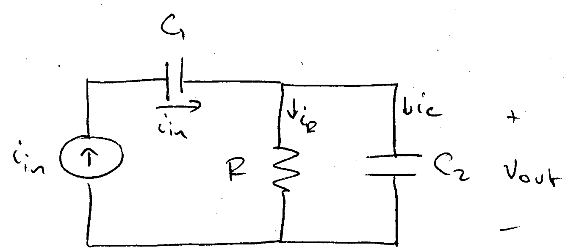
eliminate V_A, V_B

using (2) $\quad L\dfrac{di}{dt} = V_A - iR_2 \quad \Rightarrow \quad V_A = L\dfrac{di}{dt} + iR_2$

using (1) $\quad iR_1 = V_{in} - L\dfrac{di}{dt} - iR_2$

$$\Rightarrow \quad L\frac{di}{dt} + i(R_1 + R_2) = V_{in}$$

2.



$i_{in} = i_R + i_C$

$V_{out} = i_R R$

$C_2 \dfrac{dV_{out}}{dt} = i_C$

$i_R = \dfrac{1}{R} V_{out}$

$$i_{in} = \frac{1}{R} V_{out} + C_2 \frac{dV_{out}}{dt}$$

6 Resources

6.1 Books

Electrical systems are of interest in control on their own (for example, amplifiers), but more commonly in combination with mechanical systems (for example, motors). Modeling of electrical systems can be found in many introductory control systems textbooks, although usually they are introduced directly with impedances, which we will introduce soon. The following textbook includes examples with differential equations, similar to what is covered in this lecture.

- Gene F. Franklin, J. David Powell and Abbas Emami-Naeini, *Feedback Control of Dynamic Systems*, Pearson
  - 6th and 7th edition: Section 2.2

6.2 Web resources

There are also some web resources that cover electrical systems. If you find something useful, or if you find a link that no longer works, please inform your instructor!

- https://www.youtube.com/watch?v=nz2rY79QhsQ. This is a very complete (1.5 hour) video from Rick Hill at University of Detroit, Mercy. The first 30 minutes of the video covers the same material as in this lecture. Laplace transforms are also introduced as a method for solving differential equations, which we will introduce starting in Lecture 4.