

EENG307 Unit1: Lecture Summaries

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1 L07: Solving Differential Equations Using Laplace

What we know:

- Input signal in time-domain (we will call this $f(t)$)
- Initial conditions (we will call these $x(0^-)$ and $\dot{x}(0^-)$)
- Our system (represented with a schematic)

What we want:

- System output $x(t)$ in time-domain for a given input $f(t)$ and initial conditions

How we get what we want:

1. **Step 1:** Get differential equation describing system (using process in Lectures 2 and 3)
 - Make sure to organize this differential equation with like terms on each side (all x terms on left side and all other terms on right side)
 - Ex: $\ddot{x} + 3\dot{x} + 2x = f(t)$
2. **Step 2:** Plug in given input signal
 - Ex: $\ddot{x} + 3\dot{x} + 2x = e^{-3t}$
3. **Step 3:** Take Laplace Transform of each side and plug in initial conditions
 - $\mathcal{L}(\ddot{x}) = s^2 X(s) - sx(0^-) - \dot{x}(0^-)$
 - $\mathcal{L}(\dot{x}) = sX(s) - x(0^-)$
 - $\mathcal{L}(x) = X(s)$
4. **Step 4:** Isolate $X(s)$
 - $X(s) = \frac{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s^1 + a_0}{b_n s^n + b_{n-1} s^{n-1} + \dots + b_1 s^1 + b_0}$ where a and b represent the coefficients for each s term.
5. **Step 5:** Take the inverse Laplace of each side to get $x(t)$

1.1 Partial Fraction Decomposition (PFD)

We use partial fraction decomposition (PFD) to get our fraction into a recognizable form so we can easily take the inverse Laplace. There are three different *cases* for PFD and three different methods to solve for your residuals.

The 3 cases depend on the type of *roots* of the denominator (aka - *poles*). You find the roots of the denominator by factoring it and finding what s is equal to when you set the denominator equal to zero. For example, if your denominator is $s^3 + 6s^2 + 11s + 6$ this factors into $(s+1)(s+2)(s+3)$ which has poles $s = -1, -2, -3$. If the poles are complex conjugates, you may need to use the quadratic formula to solve for them OR use MATLAB (see end of article). Cases for partial fraction decomposition are found in Table 1.

Table 1: L07: Cases for PFD

Case	Pole Format	Generic PFD Equation	Notes
Simple real poles	$s = a_1, a_2$	$\frac{A}{(s-a_1)} + \frac{B}{(s-a_2)}$	\mathcal{L} is probably e^{-at} or similar
Simple complex conjugate poles	$s = a \pm jb$	$\frac{Bs+C}{(s-(a+jb))(s-(a-jb))}$	\mathcal{L} is sinusoid or damped sinusoid
Repeated Roots	$s = a, a$	$\frac{A}{(s-a)} + \frac{B}{(s-a)^2}$	\mathcal{L} is sinusoid or exponential multiplied by t or t^n

1.1.1 Simple Real Poles - Additional Notes

- Should be able to use “Equate Coefficients” method to solve for residuals
- Poles are not repeated and only have real parts

1.1.2 Simple Complex Poles - Additional Notes

- May have to use a combination of methods to solve for residuals
- May have to complete the square on the denominator to get fraction into a recognizable format to do inverse Laplace
- May have to modify the numerator to correspond with denominator to correctly apply the inverse Laplace
- Likely Laplace transforms for this case are found in Table 2.

Table 2: Laplace Transforms for Simple Complex Poles

$f(t)$	$F(s)$
$\sin(\omega t)u(t)$	$\frac{\omega}{s^2 + \omega^2}$
$\cos(\omega t)u(t)$	$\frac{s}{s^2 + \omega^2}$
$e^{-at} \sin(\omega t)u(t)$	$\frac{\omega}{(s+a)^2 + \omega^2}$
$e^{-at} \cos(\omega t)u(t)$	$\frac{s+a}{(s+a)^2 + \omega^2}$

1.1.3 Repeated Poles - Additional Notes

- Cannot use “Find Residue Directly” method for anything but the higher power of the repeated root.
- Likely Laplace transforms for this case are found in Table 3

Table 3: Laplace Transforms for Repeated Poles

$f(t)$	$F(s)$
$tu(t)$	$\frac{1}{s^2}$
$\frac{1}{2}t^2u(t)$	$\frac{1}{s^3}$
$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$
$\frac{1}{2}t^2e^{-at}u(t)$	$\frac{1}{(s+a)^3}$
$\frac{t^n e^{-at}}{n!}u(t)$	$\frac{1}{(s+a)^{n+1}}$
$t \sin(\omega t)u(t)$	$\frac{2\omega s}{(s^2 + \omega^2)^2}$
$te^{-at} \sin(\omega t)u(t)$	$\frac{2\omega(s+a)}{((s+a)^2 + \omega^2)^2}$

1.2 Methods to solve for residuals

Note on notation: 'LHS' means "Left-hand side" of an equation and 'RHS' means "Right-hand side" of an equation.

1. Option 1: Equate Coefficients (primarily for cases with simple real poles)

- Get common denominator and have one fraction with all residuals
- Equate numerator coefficients of s terms on LHS and RHS of equation
- Solve equations for residuals (MATLAB Tip: put system of equations into matrix form and use function *linsolve*)

Example:

$$\begin{aligned}\frac{s+4}{s^3+6s^2+11s+6} &= \frac{A}{(s+1)} + \frac{B}{(s+2)} + \frac{C}{(s+3)} \\ \text{Get Common Denominator} \\ \frac{s+4}{s^3+6s^2+11s+6} &= \frac{A(s+2)(s+3) + B(s+1)(s+3) + C(s+1)(s+2)}{(s+1)(s+2)(s+3)} \\ \frac{s+4}{s^3+6s^2+11s+6} &= \frac{A(s^2+5s+6) + B(s^2+4s+3) + C(s^2+3s+2)}{(s+1)(s+2)(s+3)} \\ \text{Equate Numerator Coefficients} \\ s^2 : 0 &= A + B + C \\ s^1 : 1 &= 5A + 4B + 3C \\ s^0 : 4 &= 6A + 3B + 2C \\ \text{Solve system of equations.}\end{aligned}$$

2. Option 2: Equate Coefficients for Specific Values of s

- Choose a value for s (usually 0 is a good starting point as long as the denominator doesn't have a root at 0)
- Substitute that value of s into each side of the equation
- Equate coefficients similar to Option 1
- Note: Usually used in combination with Option 3 (once you already know some of the residuals)

Example:

$$\begin{aligned}\frac{s+4}{s^3+6s^2+11s+6} &= \frac{A}{(s+1)} + \frac{B}{(s+2)} + \frac{C}{(s+3)} \\ \text{Get Common Denominator} \\ \frac{s+4}{s^3+6s^2+11s+6} &= \frac{A(s+2)(s+3) + B(s+1)(s+3) + C(s+1)(s+2)}{(s+1)(s+2)(s+3)} \\ \frac{s+4}{s^3+6s^2+11s+6} &= \frac{A(s^2+5s+6) + B(s^2+4s+3) + C(s^2+3s+2)}{(s+1)(s+2)(s+3)} \\ \text{Set } s \text{ to some value:} \\ \left. \frac{s+4}{s^3+6s^2+11s+6} \right|_{s=0,1,2} &= \left. \frac{A(s^2+5s+6) + B(s^2+4s+3) + C(s^2+3s+2)}{(s+1)(s+2)(s+3)} \right|_{s=0} \\ \text{Equate Numerator Coefficients:} \\ s=0 : 4 &= 6A + 3B + 2C \\ s=1 : 5 &= 12A + 8B + 6C \\ s=2 : 6 &= 20A + 15B + 12C \\ \text{Solve System of Equations}\end{aligned}$$

3. Option 3: Find Residue Directly (plugging in values of s)

- Multiply each side of the equation by the denominator corresponding to the residual you want to solve for
- Substitute s equal to the pole of the denominator you just multiplied each side of the equation by
- Solve for the residual
- *Note: does not work for lower powers of repeated roots. If you have $(s + a)^3$ then it will only work for the residual with $(s + a)^3$ in the denominator and not the residuals with $(s + a)^2$ or $(s + a)$ in the denominators.*

Example:

$$\frac{s + 4}{s^3 + 6s^2 + 11s + 6} = \frac{A}{(s + 1)} + \frac{B}{(s + 2)} + \frac{C}{(s + 3)}$$

Solve for A

$$A = (s + 1) \frac{s + 4}{s^3 + 6s^2 + 11s + 6} \Big|_{s=-1}$$

Solve for B

$$B = (s + 2) \frac{s + 4}{s^3 + 6s^2 + 11s + 6} \Big|_{s=-2}$$

Solve for C

$$C = (s + 3) \frac{s + 4}{s^3 + 6s^2 + 11s + 6} \Big|_{s=-3}$$

1.3 Complete the Square

Completing the square is commonly used to help us get our fraction in a form that we can take the inverse Laplace transform of (specifically, sine, cosine, damped sine, and damped cosine forms).

Steps to Complete the Square:

1. Get coefficient of s^2 term to equal 1: $s^2 + bs + c$
2. Get all terms with s on LHS and all other terms on RHS: $s^2 + bs = -c$
3. Add $(\frac{b}{2})^2$ to each side: $s^2 + bs + (\frac{b}{2})^2 = -c + (\frac{b}{2})^2$
4. Factor LHS: $(s + \frac{b}{2})^2 = -c + (\frac{b}{2})^2$
5. Move RHS to LHS: $(s + \frac{b}{2})^2 + (c - (\frac{b}{2})^2)$

1.4 MATLAB TIPS

To find the poles of your system, there are two ways to use MATLAB to do it for you.

Option 1 (preferred for all systems):

`s=tf('s')` tf means that s is of type "transfer function"

`sys=(2*s+5)/(3*s^2+2*s+1)` sys is your fraction for your system

`P=pole(sys)` P is a vector of your poles, it will give you the complex poles

Option 2 (only good for simple real poles):

`syms s` syms means that s is of type "symbolic"

`denom=(3*s^2+2*s+1)` denom in the denominator of your fraction

`P=factor(denom,s)` P is the vector of your poles, it **will not** give you the complex poles

You can use MATLAB to solve a system of equations. If you use Option 1: Equate Coefficients to solve for your residuals, this can be useful. Suppose your equations are:

$$A + B + C = 0$$

$$5A + 4B + 3C = 1$$

$$6A + 3B + 2C = 4$$

Put the coefficients on the LHS into a matrix (we will call it 'M' in MATLAB). Each row of M corresponds to each equation, and each column corresponds to the residual. Put the values on the RHS into a vector (we will call it 'N' in MATLAB). We can use the *linsolve* function to give us a vector of the residuals (we will call it 'Resid' in MATLAB, where 'Resid'=[A;B;C]).

```
M=[1 1 1;5 4 3;6 3 2]
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```
N=[0;1;4]
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```
Resid=linsolve(M,N)
```

$$\begin{bmatrix} 1 & 1 & 1 \\ 5 & 4 & 3 \\ 6 & 3 & 2 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} \text{ is the same as } M * \text{Resid} = N$$