EEENG307: References, Steady State Error, and System Type*

Lecture 16

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Contents

1 Pre-requisite Material

This lecture assumes that the reader is familiar with the following material:

- Lecture 8: Block Diagrams
- Lecture 15: Disturbances and Steady State Error

2 Reference Input Tracking

One major reason to implement a control system is to get the system output to follow a prescribed *reference trajectory*. A typical, basic feedback control configuration would have a controller $C(s)$ and plant $G(s)$ in the forward path of the feedback loop.

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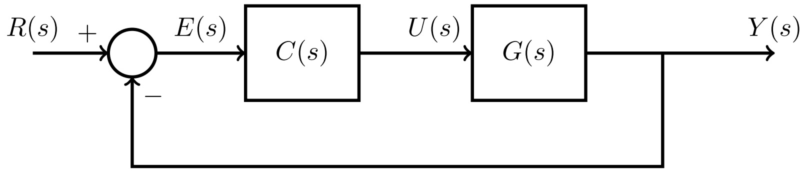
Feedback Control Configuration



In this basic feedback control diagram, $R(s)$ is the reference trajectory, which is our way of defining what we want the plant output $Y(s)$ to do, and $E(s) = R(s) - Y(s)$ is the error between the reference and the output. This error is fed into the controller $C(s)$, which decides on the actuator commands $U(s)$ to the plant $G(s)$.

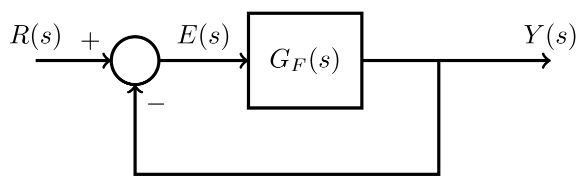
The controller will be designed to meet many specifications, for example, the closed loop system from $R(s)$ to $Y(s)$ has to be stable, and there may be specific specifications in terms of rise time, settling time, or overshoot for a step command. But we also have a basic question: if we change the set-point command to 72°, does the output settle out to 72° or something else? Thus, we are also interested in the steady state response to a command signal. This question can be answered using the same tool we used in the last lecture: the final value theorem.

3 Reference Tracking Analysis

In this case our question is the following. If the reference signal $r(t)$ is a step input, what is the steady state error $e(t) = r(t) - y(t)$? Conveniently, for unity gain feedback (as shown in this lecture article), the signal $E(s)$ is easily found as the output of the summer block.

For generic analysis, we can combine the controller with the plant (i.e., all gain blocks in the forward path are combined and represented as $G_F(s)$) and examine the following system:

Reference Error Analysis



Since $E(s) = R(s) - Y(s)$, and $Y(s) = \frac{G_F(s)}{1+G_F(s)} R(s)$, the transfer function from $R(s)$ to $E(s)$ is

$$\begin{aligned} E(s) &= R(s) - \frac{G_F(s)}{1 + G_F(s)} R(s) \\ &= \left(1 - \frac{G_F(s)}{1 + G_F(s)}\right) R(s) \\ &= \frac{1}{1 + G_F(s)} R(s) \end{aligned}$$

In order to apply the final value theorem, a necessary condition is that the Laplace Transform $\frac{s}{1+G_F(s)} R(s)$ has all poles in the left half plane. For the derivation in this section, we will assume this condition holds, but you'll want to remember it for future designs. Then, the final value theorem tells us that

$$\boxed{\lim_{t \to \infty} e(t) = \lim_{s \to 0} \frac{s}{1 + G_F(s)} R(s)}$$

We will examine the consequences of this for two different inputs

3.1 Step Input

If the reference is a step of magnitude A, then $R(s) = \frac{A}{s}$. In this case,

$$\begin{aligned}
\lim_{t\to\infty} e(t) &= \lim_{s\to 0} \frac{s}{1+G_F(s)} \frac{A}{s} \\
&= \frac{A}{1+G_F(0)} \\
&= \frac{A}{1+K_s}
\end{aligned}$$

Where $K_s = G_F(0)$. Conclusion: In order to have a small steady state error, the value $G_F(0)$ should be large. We will see later that this corresponds to the *DC gain* of $G_F(s)$, so we can say that a small steady state error requires a large DC gain. If $G_F(s)$ were to have a pole at $s = 0$, (which is usually stated as "$G_F(s)$ contains a **pure integrator**") then the steady state error would be zero.

3.2 Ramp Input

If the reference is a ramp of slope A, then $R(s) = \frac{A}{s^2}$. In this case,

$$\begin{aligned}
\lim_{t\to\infty} e(t) &= \lim_{s\to 0} \frac{s}{1+G_F(s)} \frac{A}{s^2} \\
&= \lim_{s\to 0} \frac{A}{s+sG_F(s)} \\
&= \frac{A}{K_v}
\end{aligned}$$

where $K_v = \lim_{s\to 0} sG_F(s)$. For a small steady state error, we need to have K_v large. Note that $G_F(s)$ *must* contain a pure integrator in order for K_v to be nonzero. If we want zero steady state error to a ramp input, then $G_F(s)$ must have two pure integrators (i.e. two roots at $s = 0$)

3.3 System Type

Since the number of integrators is an important factor in steady state error, this has been given a special name.

Definition 1. The number of pure integrators – i.e., poles of $G_F(s)$ equal to zero, a system contains is the *system type*

For example:

- $\frac{2}{s^2+1}$ is a Type 0 system
- $\frac{2}{s(s^2+1)}$ is a Type 1 system
- $\frac{2}{s^2(s^2+1)}$ is a Type 2 system

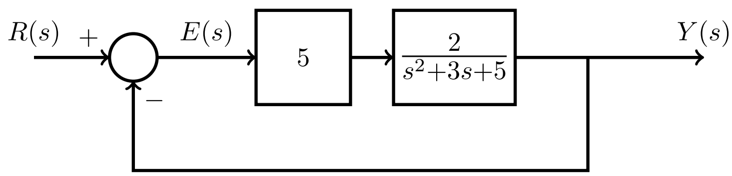
This gives us the following table for steady state error

	Steady State Error to a...		
System Type	Step Input $r(t) = Au(t)$	Ramp Input $r(t) = Atu(t)$	Error Constants
0	$\frac{A}{1+K_s}$	∞	$K_s = \lim_{s\to 0} G_F(s)$
1	0	$\frac{A}{K_v}$	$K_v = \lim_{s\to 0} sG_F(s)$

4 Examples

Example 2. Consider the following feedback control system

Control System



If the reference input is $r(t) = 2u(t)$ where $u(t)$ is a unit step function, what is the steady state error $\lim_{t\to\infty} e(t)$?

Solution: Step 1 is to find the transfer function from $R(s)$ to $E(s)$. This is

$$\frac{E(s)}{R(s)} = \frac{1}{1 + 5\frac{2}{s^2+3s+5}} = \frac{s^2 + 3s + 5}{s^2 + 3s + 15}$$

Step 2 is to apply the input and use the final value theorem. First, we check that the closed loop system is stable, which we do by verifying that $s^2 + 3s + 15$ has two roots with negative real part. Then

$$\begin{aligned}
\lim_{t\to\infty} e(t) &= \lim_{s\to 0} sE(s) \\
&= \lim_{s\to 0} s\frac{s^2 + 3s + 5}{s^2 + 3s + 15}\frac{2}{s} \\
&= \frac{5(2)}{15} = \frac{10}{15}
\end{aligned}$$

If we would like the steady state error to be smaller, we can increase the DC gain, or add an integrator.

Example 3. Consider the same system as above, but find the steady state error when $r(t) = tu(t)$ (a ramp).

Solution: We have already found the transfer function to $E(s)$, so we just need to apply the final value theorem

$$\begin{aligned}
\lim_{t\to\infty} e(t) &= \lim_{s\to 0} sE(s) \\
&= \lim_{s\to 0} s\frac{s^2 + 3s + 5}{s^2 + 3s + 15}\frac{1}{s^2} \\
&= \lim_{s\to 0} \frac{s^2 + 3s + 5}{(s^2 + 3s + 15)s} \\
&= \infty
\end{aligned}$$

The steady state error is infinite! To check this, let's find the response using MATLAB. The closed loop response is

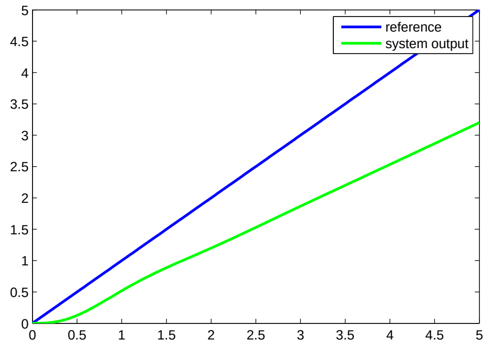
$$\frac{Y(s)}{R(s)} = \frac{\frac{10}{s^2+3s+5}}{1 + \frac{10}{s^2+3s+5}} = \frac{10}{s^2 + 3s + 15}$$

The ramp response of $G_F(s)$ is the same as the step response of $G_F(s)/s$, so we can do the following to simulate:

```
sys=tf([10],[1 3 15 0])
[y,t]=step(sys,5)
plot(t,t,'b-',t,y,'g-','linewidth',2)
legend('reference','system output')
```

The results are shown below.

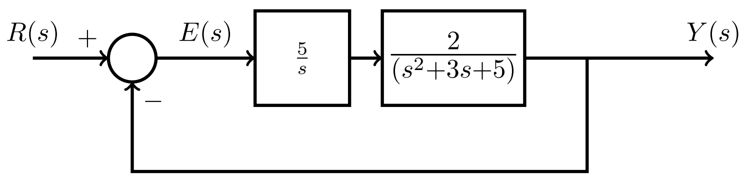
Ramp Response



The system output looks like a ramp, but the difference – i.e., the error $e(t)$ – gets bigger with time.

Example 4. Things will get better if we increase the system type of the controller. Let's repeat the analysis for the following control system:

Second Control System



Note that the product of the control and system is now type 1. To find the steady state error, we first find the transfer function from $R(s)$ to $E(s)$

$$\frac{E(s)}{R(s)} = \frac{1}{1 + \frac{5}{s}\frac{2}{s^2+3s+5}} = \frac{s^3+3s^2+5s}{s^3+3s^2+5s+10}$$

The closed loop is stable since each of the three roots of $s^3 + 3s^2 + 5s + 10$ has a negative real part. Then for the step input $r(t) = 2u(t)$,

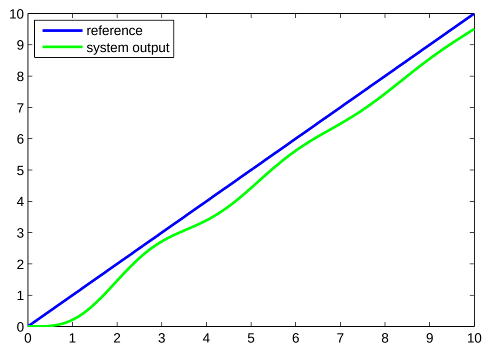
$$\begin{aligned}
\lim_{t\to\infty} e(t) &= \lim_{s\to 0} sE(s) \\
&= \lim_{s\to 0} s\frac{s^3+3s^2+5s}{s^3+3s^2+5s+10}\frac{2}{s} \\
&= \lim_{s\to 0} \frac{s^3+3s^2+5s}{s^3+3s^2+5s+10}2 \\
&= 0
\end{aligned}$$

and for the ramp input $r(t) = tu(t)$,

$$\begin{aligned}
\lim_{t\to\infty} e(t) &= \lim_{s\to 0} sE(s) \\
&= \lim_{s\to 0} s\frac{s^3+3s^2+5s}{s^3+3s^2+5s+10}\frac{1}{s^2} \\
&= \lim_{s\to 0} \frac{s^2+3s+5}{s^3+3s^2+5s+10} \\
&= \frac{5}{10}
\end{aligned}$$

The steady state error to a ramp input is bounded, which is confirmed via simulation below

Ramp Response



5 Lecture Highlights

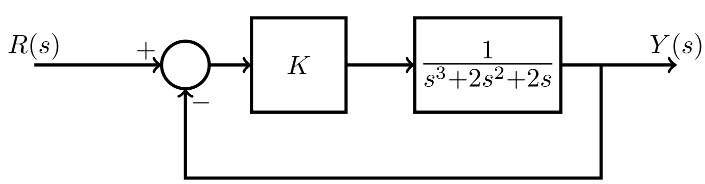
The primary takeaways from this article include

1. In control systems, we often want a system's output to track a "reference" (desired) input trajectory. For example, we may want an automobile to travel at the speed set by the driver in the cruise control system, in which case the reference would be a step input scaled by the value of the desired speed.
2. System Type theory enables us to determine the order (step, ramp...) of the reference input that a system can track with zero error without having to compute the full time-domain response $y(t)$ or apply the Final Value Theorem to $Y(s)$ for each input type separately.
3. System Type theory also enables us to determine the final value of the error $e(t)$ when the system cannot track the reference with zero error.
4. A system's Type is determined by its number of "pure integrators"; that is, the number of poles equal to zero. A common mistake is to confuse the number of pure integrators with the order of the system; see Definition 1 for some examples.

6 Quiz Yourself

6.1 Questions

1. Determine the smallest steady state error to a step and ramp input achievable for the following control configuration (hint: don't forget about stability.)



6.2 Solutions

1.

$$\frac{E(s)}{R(s)} = \frac{1}{1+K\cdot\frac{1}{s^3+2s^2+2s}} = \frac{s^3+2s^2+2s}{s^3+2s^2+2s+K}$$

stability:

$$\begin{array}{lll} s^3: & 1 & 2 \\ s^2: & 2 & K \\ s^1 & \frac{4-K}{2} & \\ s^0 & K & \end{array} \qquad \Rightarrow \text{require } 0<K<4$$

step:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s\cdot\frac{s^3+2s^2+2s}{s^3+2s^2+2s+K}\cdot\frac{1}{\cancel{s}} = \underline{0}$$

ramp:

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} \cancel{s}\,\frac{s^{\cancel{3}\,2}+2s^{\cancel{2}}+\cancel{2s}\,2}{s^3+2s^2+2s+K}\cdot\frac{1}{\cancel{s^2}} = \frac{2}{K} > \frac{2}{4} = \underbrace{\frac{1}{2}}_{\text{min error}}$$

7 Resources

7.1 Books

- Norman S. Nise, *Control Systems Engineering*, Wiley
  - 7th edition: Sections 7.2-7.3
- Richard C. Dorf and Robert H. Bishop, *Modern Control Systems*, Pearson
  - 13th edition: Section 4.6
- Gene F. Franklin, J. David Powell and Abbas Emami-Naeini, *Feedback Control of Dynamic Systems*, Pearson
  - 6th and 7th edition: Section 4.2.1

7.2 Web resources

If you find any useful web resources, please contact your instructor.