EENG307: Fluid Systems and System Analogies*

Lecture 10

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1 Pre-requisite Material

This lecture assumes that the reader is familiar with the following material:

- Lecture 7: Impedance and Transfer Functions

2 Incompressible Fluid Systems

For fluid systems, the variables that we are modeling are

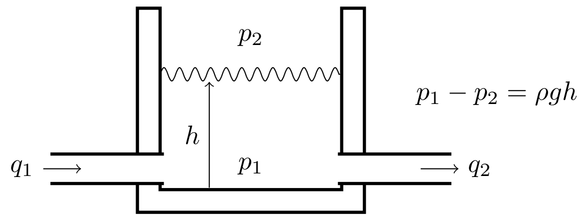
- Pressure, which has units of Newtons per meter squared [Nm^{-2}] or equivalently [$\mathrm{kg\ m}^{-1}\mathrm{s}^{-2}$]
- Volumetric Flow, which has units of meters cubed per second [$\mathrm{m}^3\mathrm{s}^{-1}$].

†Developed and edited by Tyrone Vincent and Kathryn Johnson, Colorado School of Mines, with contributions from Salman Mohagheghi, Chris Coulston, Kevin Moore, CSM and Matt Kupilik, University of Alaska, Anchorage

2.1 Components

Tank

The change in the volume of fluid inside the tank is equal to the difference between the input and output volumetric flow.



Tank Area: A

$A\frac{dh}{dt} = q_1 - q_2$ $\qquad$ $\frac{A}{\rho g}\frac{d(p_1-p_2)}{dt} = q_1 - q_2$

If V is the volume of fluid, then the change in the amount of fluid in the tank over time is equal to the volumetric flow in minus the volumetric flow out, so that

$$\frac{dV}{dt} = q_1 - q_2. \tag{1}$$

For a tank with constant cross-sectional area A m^2, $V = Ah$, so taking the derivative of both sides with respect to time gives us

$$\frac{dV}{dt} = A\frac{dh}{dt}. \tag{2}$$

Setting (1) equal to (2) gives us the equation

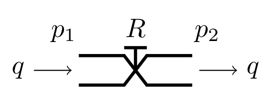
$$A\frac{dh}{dt} = q_1 - q_2.$$

This can also be written in terms of the pressure difference between the bottom and top of the tank, since $p_1 - p_2 = \rho g h$, where ρ is the fluid density [kg m^{-3}], and g is gravitational acceleration [m s^{-2}]. Substituting for h results in

$$\frac{A}{\rho g}\frac{d(p_1 - p_2)}{dt} = q_1 - q_2.$$

Linear Valve

A valve causes a restriction that causes the pressure on one end of the value to be higher than the other end. When this pressure drop is proportional to the flow, the valve is linear with valve constant R. (Most valves are nonlinear, however.) By unit analysis, the units of valve resistance are [N s m^{-5}] or equivalently [kg m^{-4}s^{-1}].



$$p_1 - p_2 = Rq$$

Why only two elements? We have not considered fluid inertia.

2.2 Connection Rules and Boundary Conditions

The connection rules and boundary conditions will sound familiar

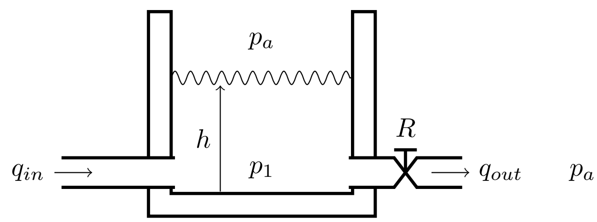
- When elements are connected, the two components share the same *pressure*.
- When elements are connected, the *flows* sum to zero.
- Boundary conditions can set either the pressure or flow of one side of a component

2.3 Example

Find the differential equation that describes the relationship between q_{in}, and h for the following system:

Tank and Valve Example

Assume the valve is linear with valve constant R and that the density of the fluid is ρ. The valve empties to atmospheric pressure, p_a, which is the same as the pressure at the top of the tank.



Tank Area: A

We use the following steps:

- Write down the component laws for the tank and valve, along with connection laws where applicable.
- Write down the relationship between the pressures and the tank height.
- Eliminate unwanted variables.

The component laws are:

$$A\frac{dh}{dt} = q_{in} - q_{out},$$
$$p_1 - p_2 = Rq_{out}.$$

Note that we used a connection law to establish that the flow out of the tank is q_{out}, the pressure at the left side of the valve is p_1, and the pressure on the right side of the valve is p_a where p_a corresponds to p_2 in the component law above. We also know that

$$\rho g h = p_1 - p_2.$$

Thus,

$$A\frac{dh}{dt} = q_{in} - q_{out},$$
$$\rho g h = Rq_{out}.$$

Eliminating q_{out},

$$A\frac{dh}{dt} = q_{in} - \frac{\rho g}{R}h,$$

or

$$A\frac{dh}{dt} + \frac{\rho g}{R}h = q_{in}.$$

3 System Analogies

Thus far, we have modeled translational mechanical systems, electrical systems, and fluid systems using lumped, linear elements. We have seen a lot of commonalities:

- Two variables are important.
- Each element relates these two variables via a linear algebraic or differential relationship.
- There are two connection rules that govern how the two variables are related at a connection point.

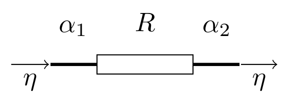
We can use these commonalities to come up with a modeling method that could applied to all of the domains we have discussed so far.

3.1 Generic Modeling Elements

Physical system modeling requires us to keep track of two variables. The one that is measured on each side of an element is called the *across* variable, and the one whose magnitude is the same on each side is the *through* variable. The component law is a linear algebraic or differential relationship between these two variables.

Generic Lumped Element

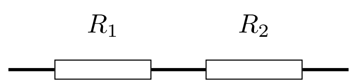
across variable: α
through variable: η

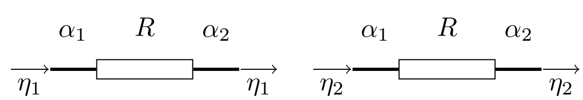


component law: $\alpha_1 - \alpha_2 = R\eta$
or $\frac{d(\alpha_1 - \alpha_2)}{dt} = R\eta$
or $\alpha_1 - \alpha_2 = R\frac{d\eta}{dt}$
or $\cdots$

When the components are connected, we apply generic connection rules to the across and through variables.

Generic Connection Rules





- When elements are connected, the two components share the same *across variable*.
- When elements are connected, the *through variables* sum to zero.
- Boundary conditions can set either the across or through variables on one side of a component

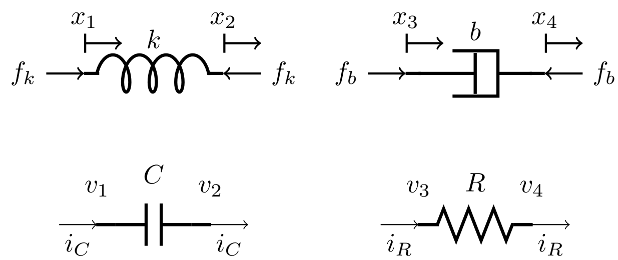
The following table lists the across and through variables for the modeling domains that we will discuss in this course.

Across and through variables

Domain	Across Variable	Through Variable
Electrical	Voltage	Current
Translational Mechanical	Position	Force
Fluid	Pressure	Flow
Rotational Mechanical	Angular Position	Torque
Thermal	Temperature	Heat Flow

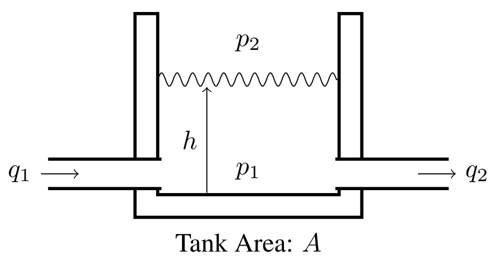
Remark 1. *One detail that may seem odd when considering force as a flow variable is the fact that force is defined as acting in opposite directions on the ends of each element. However, despite this, the interconnection rules are exactly the same. That is, for the example below, if the spring and damper are interconnected, we have $f_k = f_b$, while if the capacitor and inductor are connected, $i_C = i_R$. So in both cases, the flow variable on the left hand side of one element "flows" to the left hand side of the other element.*

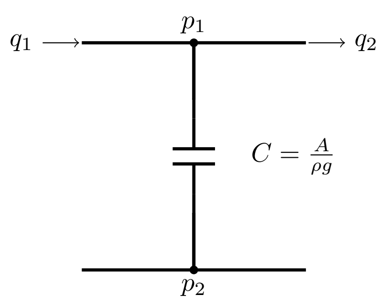
Force flow vs Current flow



By utilizing variables that have the same function in different domains, we can come up with system analogies. Analogous systems are systems in different domains that have the same equations describing their behavior.

Electrical Analogy for Fluid Elements:
Tank

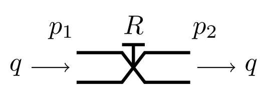


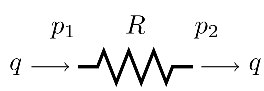


$\frac{A}{\rho g}\frac{d(p_1 - p_2)}{dt} = q_1 - q_2$

$C\frac{d(p_1 - p_2)}{dt} = q_1 - q_2$

Electrical Analogy for Fluid Elements:
Valve





$p_1 - p_2 = Rq$

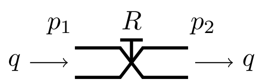
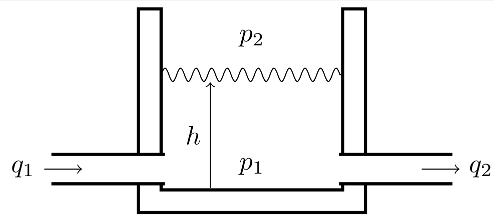
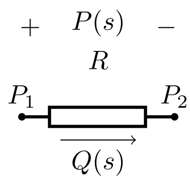
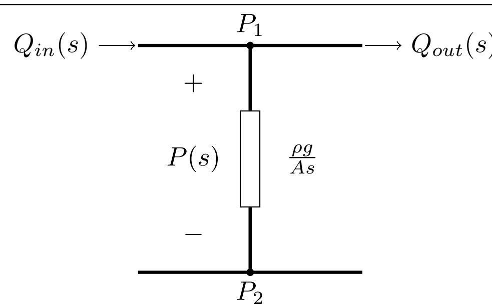
$p_1 - p_2 = Rq$

4 Fluid Impedances

With analogies to electrical components, it is very clear what the impedance of fluid components are when we want to use impedance methods to find transfer functions.

For fluid systems, we have the following impedances

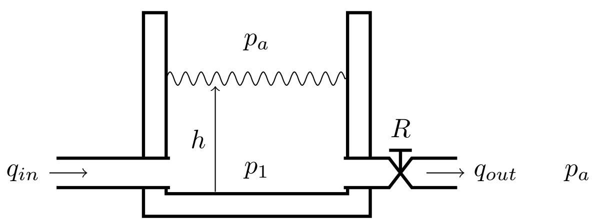
Fluid Impedances

	valve	tank
Component	$p = p_1 - p_2$ 	 Tank Area: A
Component law	$p = Rq$	$\frac{A}{\rho g}\frac{dp}{dt} = q_{in} - q_{out}$
Laplace Transform	$P(s) = RQ(s)$	$\frac{A}{\rho g}sP(s) = Q_{in}(s) - Q_{out}(s)$
Impedance Component		

Let's return to the earlier tank and valve system and go through the modeling process with analogous elements.

Tank and Valve Problem

Example: Find the equivalent impedance model of the tank and valve system below with input flow q_{in} and output flow q_{out}.



Tank Area: A

Step 1 **Identify all node variables.** For a fluid problem, the node variables are pressure. There are two unique pressure variables: p_1, the pressure at the bottom of the tank, and p_a atmospheric pressure. Our circuit should thus have two identifiable voltage nodes.

Tank system nodes

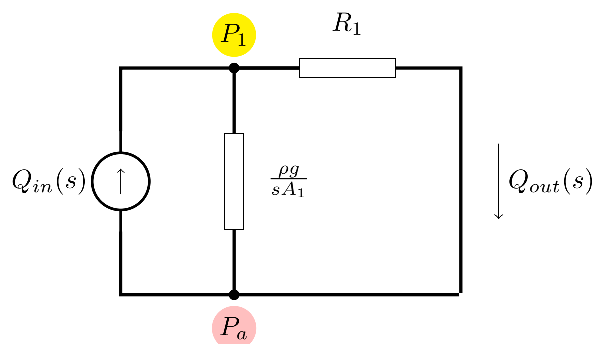
Step 2 **Identify one node as ground, or add a ground node.** Most node variables are relative - for example, consider the ground on a circuit diagram. All voltages are measured with respect to ground, so the ground voltage is simply what we will consider to be 0 V. Negative voltages are lower than ground, positive voltages are higher than ground. If we identify a node as the ground, then we will measure all node variables relative to this node. For fluid systems, there are two typical choices: Either choose absolute zero pressure (i.e. a vacuum) as ground, or choose atmospheric pressure as ground. Pressure measured with respect to atmospheric pressure is called *gauge* pressure, and the translation is

$$\text{Absolute Pressure} = \text{Gauge Pressure} + \text{Atmospheric Pressure}$$

For this problem, we will choose to measure pressure using gauge pressure, so that node p_a will be considered ground.

Step 3 **Connect components between nodes.** Since flow is a through variable, setting the input flow is done with a current source. The flow is into the node representing pressure p_1, and all sources representing boundary conditions have the other end connected to ground. We also connect a tank (capacitance) impedance between p_1 and p_a, because the pressure at the top of the tank is p_a. A valve (resistance) impedance is connected between p_1 and p_a because the valve transmits flow from a node at pressure p_1 to a node at pressure p_a.

Tank system impedance model



You can now use your circuit simplifications to find any transfer function of interest.

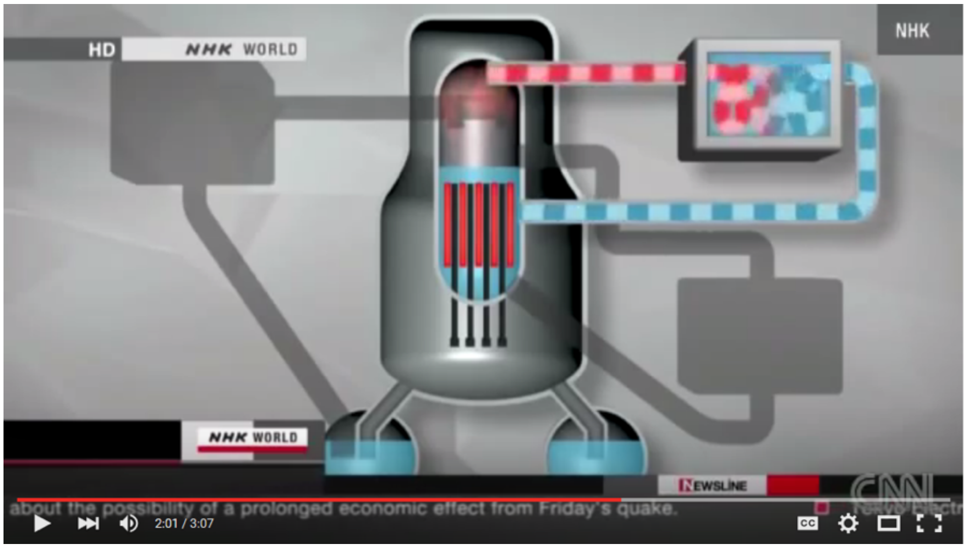
Again, it should be emphasized that in this case we have chosen the ambient pressure p_a as the reference pressure. This means that all pressure nodes in this "circuit" are measured with respect to ambient pressure, so that the pressure at node p_1 is in fact $p_1 + p_a$.

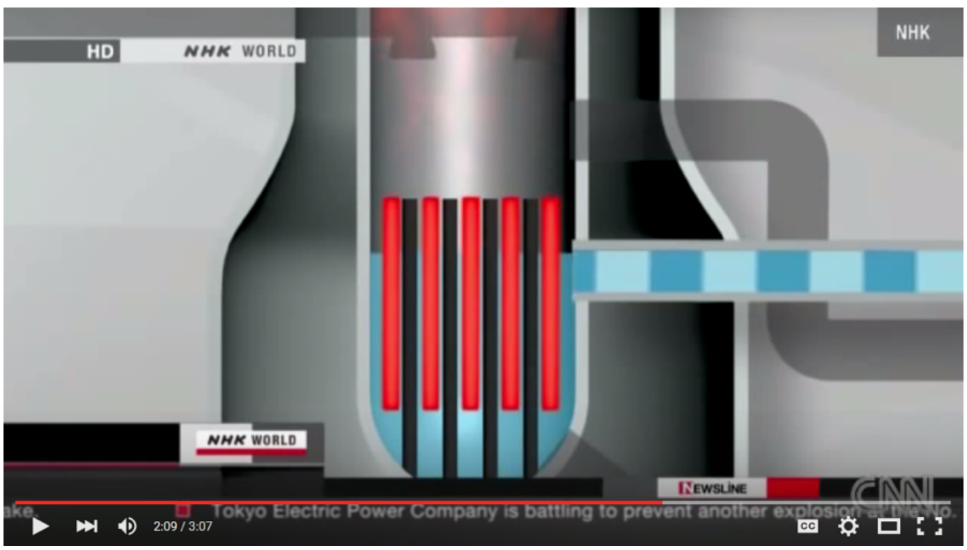
5 Application Example

A real world example information related to a fluid system that must be carefully controlled

A video explaining what went wrong at the Fukushima Nuclear Reactor in 2011 is available at https://www.youtube.com/watch?v=BdbitRlbLDc

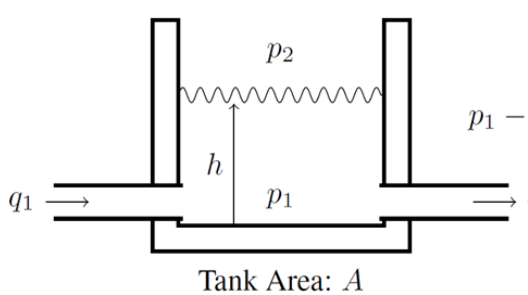
These two snapshots are from the video and show (top) one of the safety feedback loops and (bottom) the water level dropping in the reactor rod cooling system.



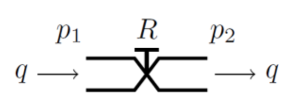


What do we need to know before we can begin to design the three separate water-cooling-based safety control loops for the nuclear reactor cooling system?

Compare the tank example in Lecture 10

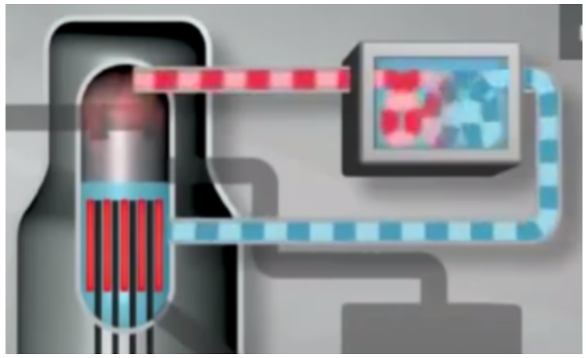


$$p_1 - p_2 = \rho g h$$



$$A\frac{dh}{dt} = q_1 - q_2 \qquad \frac{A}{\rho g}\frac{d(p_1-p_2)}{dt} = q_1 - q_2 \qquad p_1 - p_2 = Rq$$

to the Fukushima set-up



What are some key simplifications?

6 Lecture Highlights

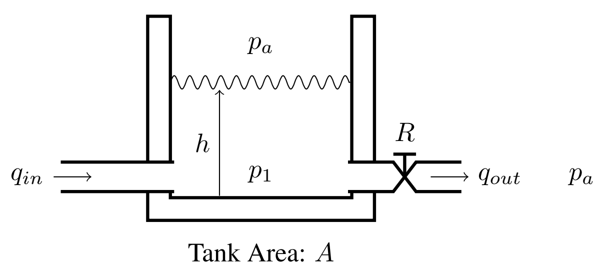
The primary takeaways from this article include

1. In this class, we model fluid systems using two ideal components: tanks and linear valves.
2. The component law for a tank is a differential equation and the component law for a linear valve is an algebraic equation. You can take the Laplace transform of each equation to find the component's impedance: the transfer function from the flow to the pressure.
3. Individual impedances can be connected together according to the connection rules to form an equivalent circuit, which enables circuit-based system analysis.
4. The process for sketching the equivalent impedance circuit is to identify and sketch all nodes (including the ground), then connect nodes using component impedances and sources.
5. As with all linear dynamic systems, system transfer functions can also be found by taking the Laplace transform of the system's differential equations, but many students find these impedance-based techniques more straightforward after some practice.

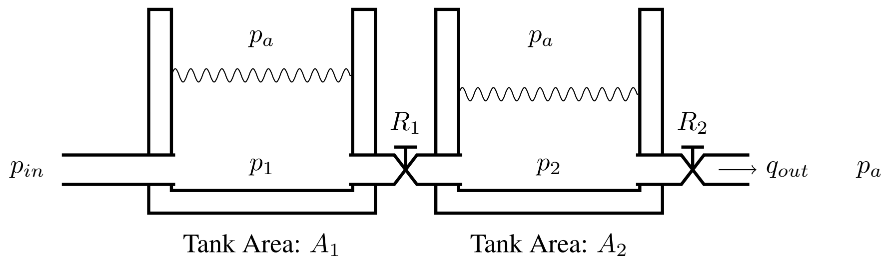
7 Quiz Yourself

7.1 Questions

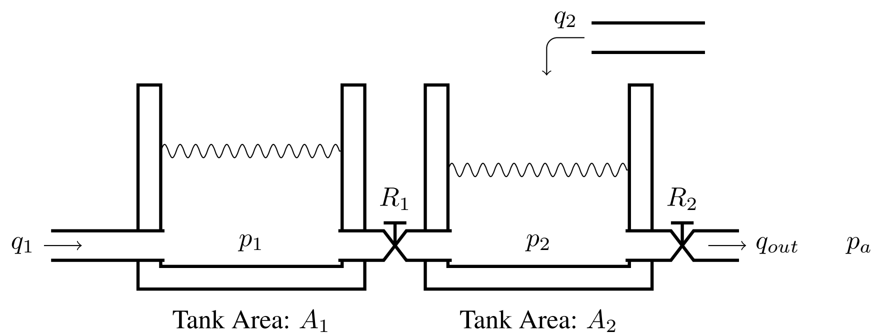
1. Find a differential equation that describes this system in terms of p_1, p_a and q_{in}. The fluid density is ρ.



2. Find an analogous circuit for the following system

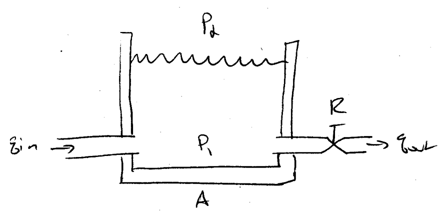


3. Find an equivalent circuit for the following fluid system. The fluid density is ρ. Note that there are two sources.



7.2 Solutions

1.



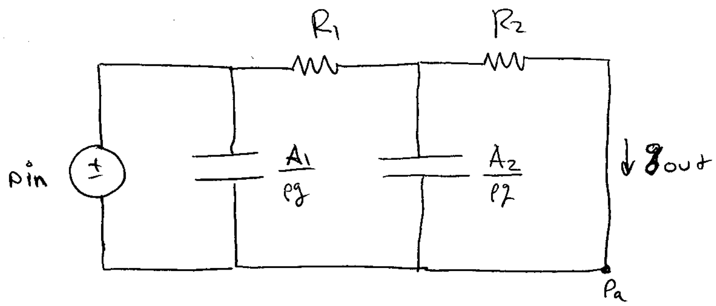
$$\frac{A}{\rho g}\frac{d(P_1 - P_a)}{dt} = q_{in} - q_{out} \qquad P_1 - P_a = R q_{out}$$

$$q_{out} = \frac{P_1 - P_a}{R}$$

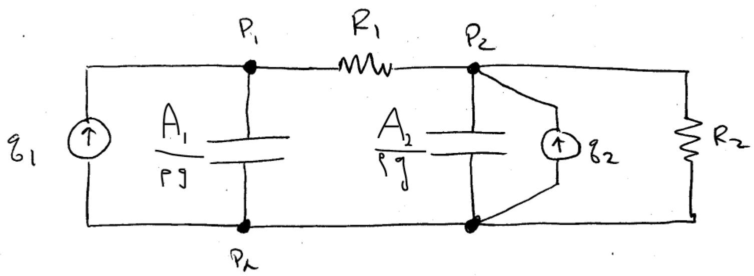
$$\frac{A}{\rho g}\frac{d(P_1 - P_a)}{dt} = q_{in} - \frac{P_1 - P_a}{R}$$

$$\frac{A}{\rho g}\frac{d(P_1 - P_a)}{dt} + \frac{P_1 - P_a}{R} = q_{in}$$

2.



3.



8 Resources

8.1 Books

Fluid systems occur in the feedback control context when using hydraulic actuators and also in chemical processing. The following textbooks discuss modeling fluid systems.

- Gene F. Franklin, J. David Powell and Abbas Emami-Naeini, *Feedback Control of Dynamic Systems*, Pearson
  - 6th and 7th edition: Section 2.4

8.2 Web resources

There are a variety of web resources, with differing levels of modeling. In this lecture, we did not consider fluid inertia, which you may find in some on-line resources. If you find something useful, or if you find a link that no longer works, please inform your instructor!

- A YouTube video on lumped modeling of fluid systems: https://www.youtube.com/watch?v=_cpubJ94ugc