EEENG307: Impedance and Transfer Functions[1]

Lecture 7

Colorado School of Mines[2]

Spring 2022

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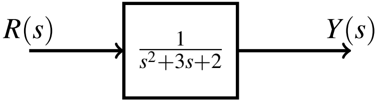
Transfer Function

$$Y(s) = \frac{1}{s^2 + 3s + 2}R(s)$$


$R(s)$
$\frac{1}{s^2+3s+2}$
$Y(s)$

Transfer function definition

Definition

The *Transfer Function* $(G(s))$ of a (linear, time invariant) system is the ratio of the Laplace Transform of the output over the Laplace Transform of the input with zero initial conditions.

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$$\frac{Y(s)}{R(s)} = G(s) \qquad Y(s) = G(s)R(s)$$

Finding the Transfer Function

- Current Method:
  - Write down all the (differential) equations describing the system
  - Eliminate all variables except input and output
  - Find the Laplace Transform of the resulting differential equation
- Impedance Method:
  - Write down all the (differential) equations describing the system
  - Take the Laplace Transform of these equations
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Electrical Components

	resistor	capacitor	inductor
Component law	$v = iR$	$C\frac{dv}{dt} = i$	$v = L\frac{di}{dt}$
Laplace Transform			

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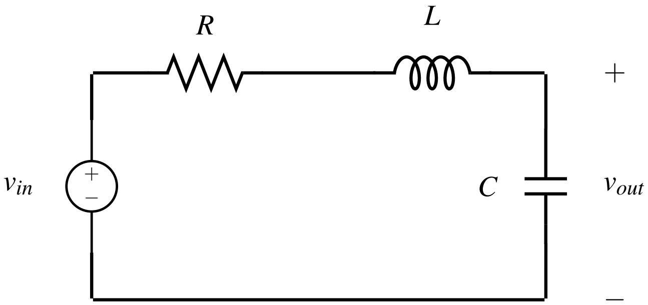
Laplace Transform	$V(s) = RI(s)$	$CsV(s) = I(s)$	$V(s) = LsI(s)$

Definition

The *impedance* of an element is the ratio of the Laplace Transform of the across variable (voltage) over the Laplace Transform of the through variable (current)

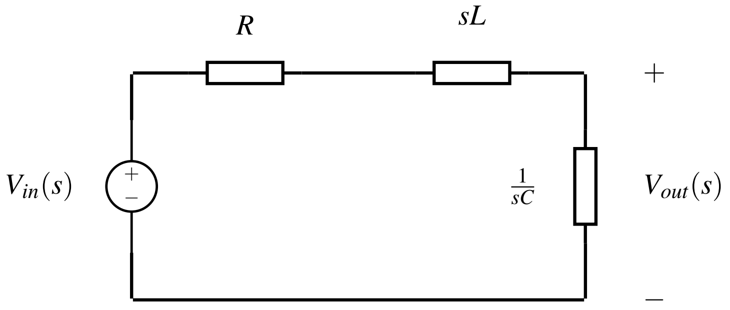
Electrical Impedance

	resistor	capacitor	inductor
Component law	$v = iR$	$C\frac{dv}{dt} = i$	$v = L\frac{di}{dt}$
Laplace Transform	$V(s) = RI(s)$	$CsV(s) = I(s)$	$V(s) = LsI(s)$
Impedance	$\frac{V(s)}{I(s)} = R$	$\frac{V(s)}{I(s)} = \frac{1}{sC}$	$\frac{V(s)}{I(s)} = Ls$

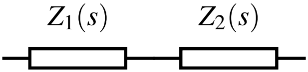


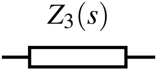
R
L
v_{in}
C
v_{out}
+
−

Impedance Circuit



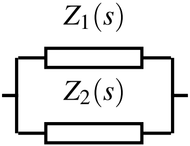
Impedances in series

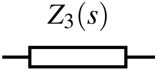




$$Z_3(s) = Z_1(s) + Z_2(s)$$

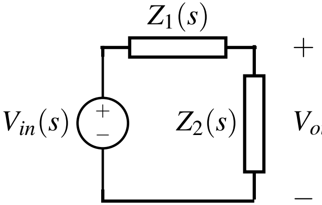
Impedances in parallel





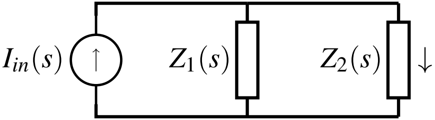
$$Z_3(s) = \frac{Z_1(s)Z_2(s)}{Z_1(s) + Z_2(s)}$$

Voltage Divider



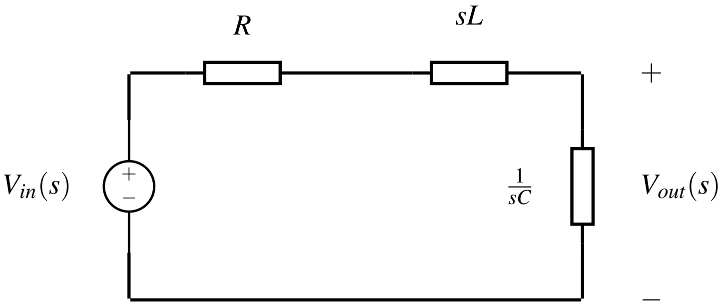
$$V_{out}(s) = \frac{Z_2(s)}{Z_1(s) + Z_2(s)} V_{in}(s)$$

Current Divider

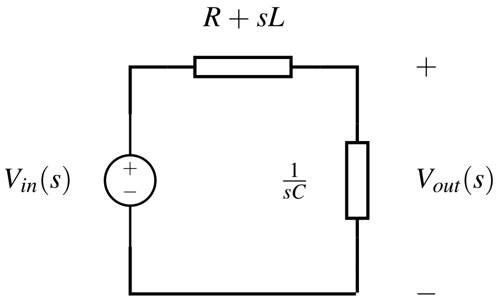


$$I_{out}(s) = \frac{Z_1(s)}{Z_1(s) + Z_2(s)} I_{in}(s)$$

Impedance Circuit



Circuit with Equivalent Impedance



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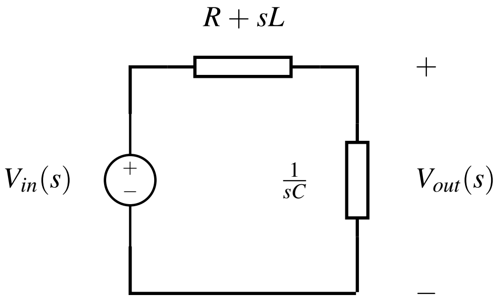
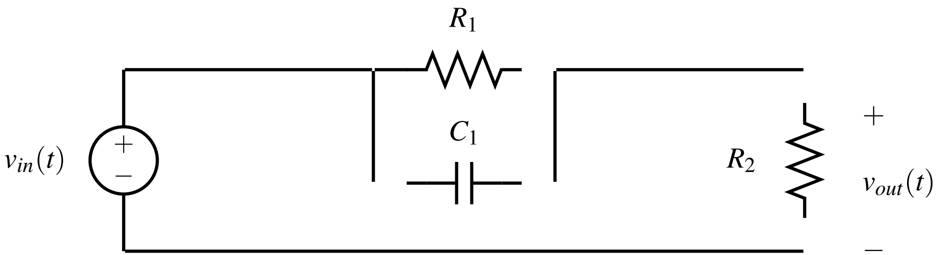
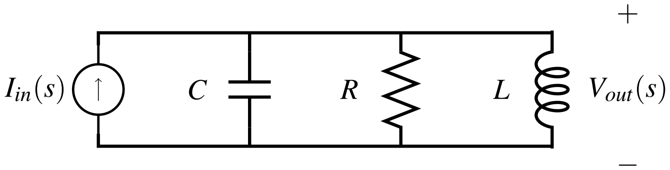
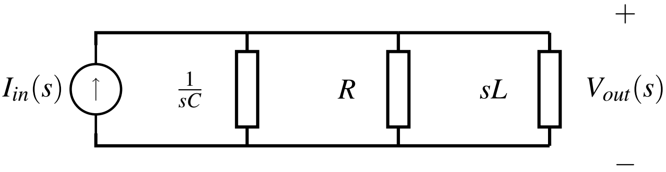


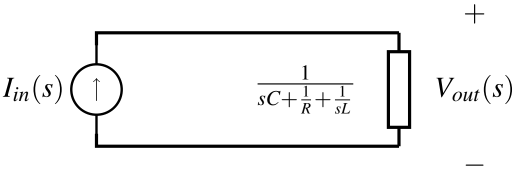
$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{\frac{1}{sC}}{\frac{1}{sC} + R + sL} = \frac{1}{s^2LC + sRC + 1}.$$

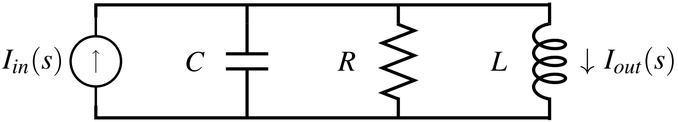

R_1
C_1
$v_{in}(t)$
+
−
R_2
+
$v_{out}(t)$
−


$I_{in}(s)$
C
R
L
$V_{out}(s)$
$+$
$-$

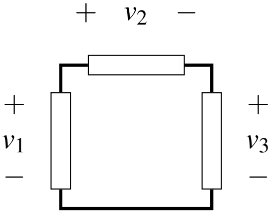


$I_{in}(s)$
$\frac{1}{sC}$
R
sL
$V_{out}(s)$
+
−

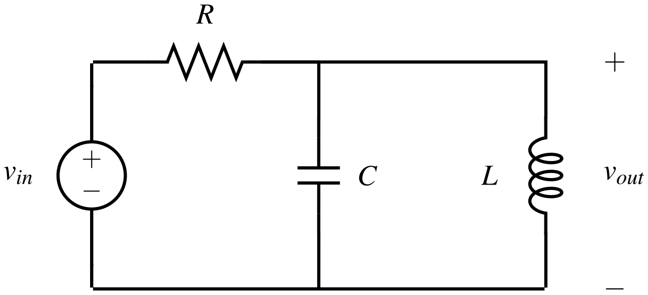

$I_{in}(s)$
$\frac{1}{sC+\frac{1}{R}+\frac{1}{sL}}$
$+$
$V_{out}(s)$
$-$


$I_{in}(s)$
C
R
L
$\downarrow I_{out}(s)$

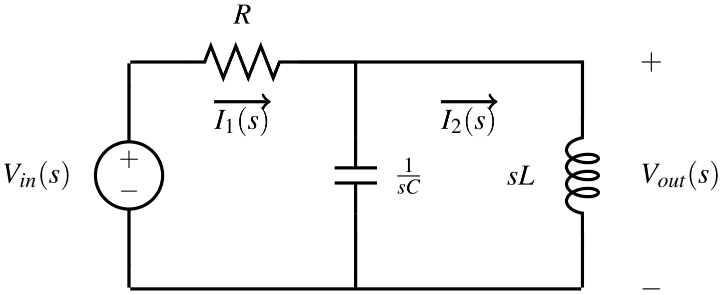
Mesh example



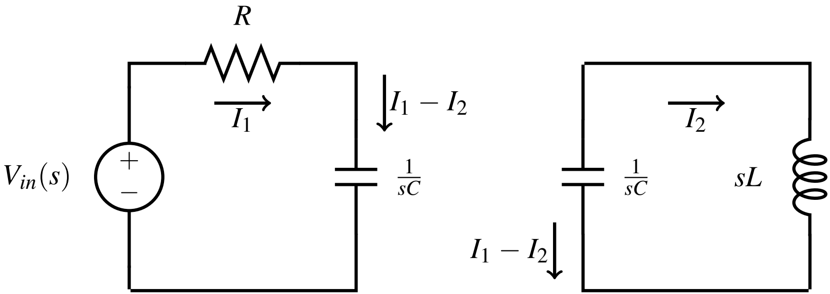
Circuit problem



Circuit problem in impedance form



Two meshes



Patterns of mesh equations

- the matrix is symmetric
- the diagonal terms are positive
- the off-diagonal terms are negative

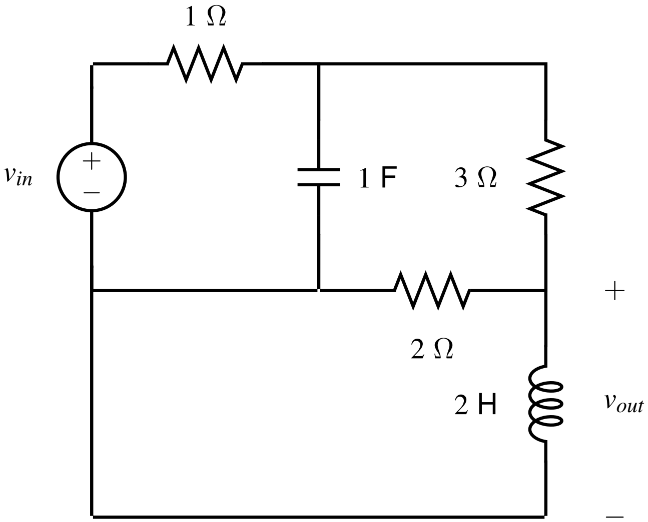
sum of impedances on mesh 1

impedance shared between mesh 1 and 2

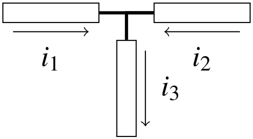
$$\begin{array}{l} \text{Source on mesh 1} \\ \text{Source on mesh 2} \end{array} \begin{bmatrix} V_{in}(s) \\ 0 \end{bmatrix} = \begin{bmatrix} R+\frac{1}{sC} & -\frac{1}{sC} \\ -\frac{1}{sC} & \frac{1}{sC}+sL \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix}$$

sum of impedances on mesh 2

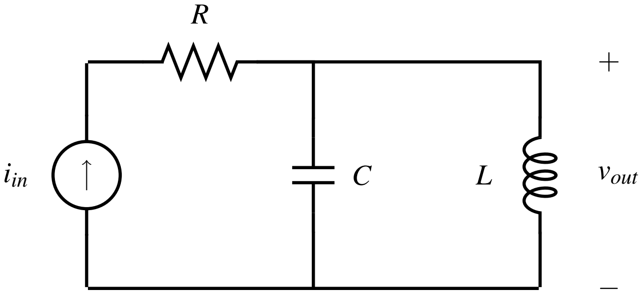
A Circuit with three meshes



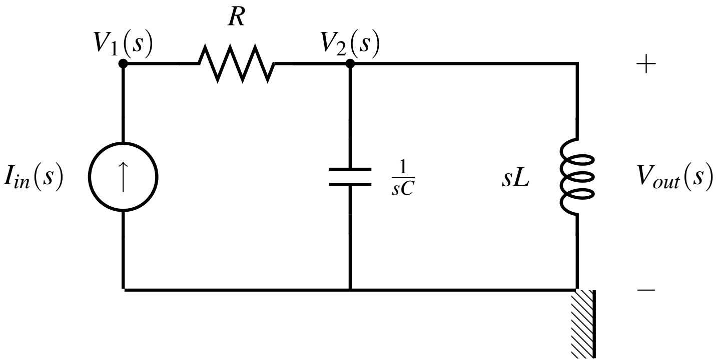
Node example



Circuit problem



Circuit problem in impedance form



Patterns of node equations

- the matrix is symmetric
- the diagonal terms are positive
- the off-diagonal terms are negative

sum of admittances that touch node 1

admittances that touch both nodes 1 and 2

$$\begin{array}{l} \text{Source into node 1} \longrightarrow \\ \text{Source into node 2} \longrightarrow \end{array} \begin{bmatrix} I_{in}(s) \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{R} & -\frac{1}{R} \\ -\frac{1}{R} & \frac{1}{R} + sC + \frac{1}{sL} \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix}$$

sum of admittances that touch node 2