EENG307 Unit 2: Lecture Summaries

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1 L10-L12: Systems

The relationship between the system output $Y(s)$ and input $R(s)$ is described by the transfer function $G(s)$ where $Y(s) = G(s)R(s)$. Common inputs are the impulse, step, and ramp. These are described in Table 1.

Table 1: Common System Inputs

Name	Time Domain	Laplace Domain
Impulse	$r(t) = \delta$	$R(s) = 1$
Step	$r(t) = u(t)$	$R(s) = \frac{1}{s}$
Ramp	$r(t) = tu(t)$	$R(s) = \frac{1}{2}$

1.1 First Order Systems

Open-loop transfer function of a first order system is $G(s) = K\frac{\sigma}{s+\sigma}$. K is the DC gain, we have a pole at $s = -\sigma$ and no zeros.

Impulse Response:

The impulse response of a system is the magnitude of the *output signal* $y(t)$ *in response to* an impulse input. An impulse signal is $r(t) = \delta$ or $R(s) = 1$. The output signal of a system in response to an impulse input is $Y(s) = G(s)R(s) = G(s)(1) = K\frac{\sigma}{s+\sigma}$. In the time-domain this is $y(t) = K\sigma e^{-\sigma t}$. At time $t = 0$ the impulse response has a magnitude of $K\sigma$. At time $t = \frac{1}{\sigma}$ the impulse response has a magnitude of $0.36K$. The time that our system decays by 64% (or has a magnitude of $0.36K$) is called the *time constant* and is denoted τ where $\tau = \frac{1}{\sigma}$.

Step Response is the magnitude of the *output signal* $y(t)$ *in response to* a step input. A step signal is $r(t) = u(t)$ or $R(s) = \frac{1}{s}$. The output signal of a system in response to an impulse input is $Y(s) = G(s)R(s) = G(s)\frac{1}{s} = K\frac{\sigma}{s(s+\sigma)}$. To get this in the time-domain, we need to do some partial fraction decomposition.

$$\frac{K\sigma}{s(s+\sigma)} = \frac{A}{s} + \frac{B}{s+\sigma}$$

Solve for A

$$A = (s)\frac{K\sigma}{s(s+\sigma)}\bigg|_{s=0} = \frac{K\sigma}{\sigma} = K$$

Solve for B

$$B = (s)\frac{K\sigma}{s(s+\sigma)}\bigg|_{s=-\sigma} = \frac{K\sigma}{-\sigma} = -K$$

Plug in Residuals

$$\frac{K\sigma}{s(s+\sigma)} = \frac{A=K}{s} - \frac{K}{s+\sigma}$$

Take inverse Laplace

$$y(t) = (K - Ke^{-\sigma t})u(t)$$

Settling Time is the time it takes to get to 99% of our final value, our final value is the DC gain K. The time to get to $0.99K$ is denoted t_s where $t_s = \frac{4.6}{\sigma}$. The derivation is found in the lecture articles.

Rise Time is the time it takes to get from 10% of our final value to 90% of our final value. The time to get to from $0.1K$ to $0.9K$ is denoted t_r where $t_r = \frac{2.2}{\sigma}$. The derivation is found in the lecture articles.

1.2 Second Order Systems

Open-loop transfer function of a second order system is $G(s) = K\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$. K is the DC gain, ζ is the damping ratio, and ω_n is the natural frequency. The poles are found using the quadratic formula so we have poles at: $s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$.

Intuition with Poles:

- $\zeta \geq 1$: *over-damped* our poles are real (no complex part). As ζ increases, our step response becomes more over-damped (purple and blue line in Figure 1).
- $0 < \zeta < 1$: *under-damped* our poles are complex. As ζ increases, our step response becomes more damped (green and yellow lines in Figure 1)
- $\zeta \leq 0$: *unstable* the real part of the poles are positive which means the output signal goes to infinity as time goes to infinity.

Let's make sense of the under-damped case. Let's say $\omega_n = 1$ so the denominator of $G(s)$ is $s^2 + 2\zeta s + 1$ and our poles are $s = -\zeta \pm j\sqrt{1-\zeta^2}$. We know that we will have to do PFD to get the inverse Laplace, and because the poles are complex conjugates that we will have to complete the square and the inverse Laplace will have the form of damped sinusoids, found in Table 2.

When we complete the square on the denominator, it is in the form $(s+\zeta)^2 + (\zeta^2 - 1)$. When we look at Table 2, we see that $a = \zeta$ and $\omega^2 = (\zeta^2 - 1)$. When we increase ζ, the a in the e^{-at} term increases, which means that our signal will converge faster. We also see that when we increase ζ the ω found in the $\cos(\omega t)$ term or $\sin(\omega t)$ term decreases, which means the frequency of oscillations decreases. Hence, as ζ increases in the under-damped case, we expect the step-response of this system to be slower and have lower frequency oscillations.

Table 2: Laplace Transforms for Damped Sinusoids

$f(t)$	$F(s)$
$e^{-at}\sin(\omega t)u(t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos(\omega t)u(t)$	$\frac{s+a}{(s+a)^2+\omega^2}$

Step Response is the magnitude of the *output signal* $y(t)$ *in response to* a step input. A step signal is $r(t) = u(t)$ or $R(s) = \frac{1}{s}$. The output signal of a system in response to an impulse input is $Y(s) = G(s)R(s) = G(s)\frac{1}{s} = K\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\frac{1}{s}$. To get this in the time-domain, we need to do some partial fraction decomposition which is done in the lecture article. Essentially, in the under-damped case, this yields $y(t) = K - Ke^{-\zeta\omega_n}\left(\cos(\omega_n\sqrt{1-\zeta^2}t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_n\sqrt{1-\zeta^2}t)\right)$. If we say $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and $\sigma = \zeta\omega_n$ we can rewrite this as $y(t) = K - Ke^{-\sigma}\left(\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right)$.

Settling Time is the time it takes to get to 99% of our final value *and stay there*. Our final value is the DC gain K. The time to get to $0.99K$ is denoted t_s where $t_s = \frac{4.6}{\zeta\omega_n}$. The derivation is found in the lecture articles.

Rise Time is the time it takes to get from 10% of our final value to 90% of our final value. The time to get to from $0.1K$ to $0.9K$ is denoted t_r where $t_r = \frac{2.2}{\omega_n}$. The derivation is found in the lecture articles.

Overshoot is the percent which we overshoot our final value or DC Gain K. The magnitude of the peak of the overshoot is denoted $K + M_p$. Our percent overshoot is $\%OS = \frac{M_p}{K} * 100\% = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} * 100\%$

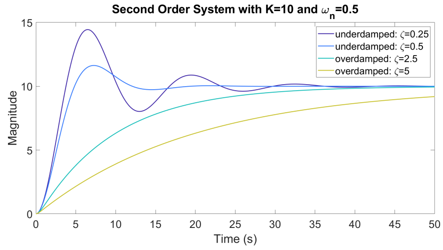

Figure 1: How ζ impacts step response in over-damped and under-damped case

How Terms Impact Step Response with under-damped Poles:

In Figure 2, we see the step response of a second order system when we hold ω_n and ζ constant but vary K. As K increases that the final value increases but that the rise time, settling time, and percent overshoot remain the same.

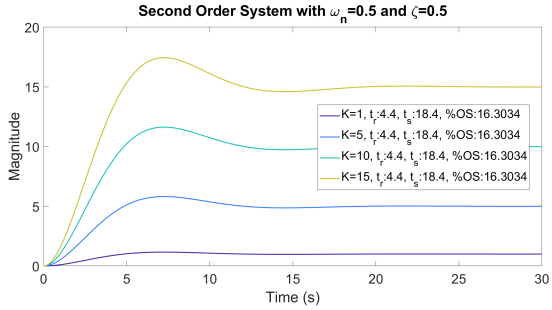

Figure 2: How K impacts step response in and under-damped case

In Figure 3, we see the step response of a second order system when we hold K and ζ constant but vary ω_n. As ω_n increases, the rise time and settling time decrease but the percent overshoot remains the same.

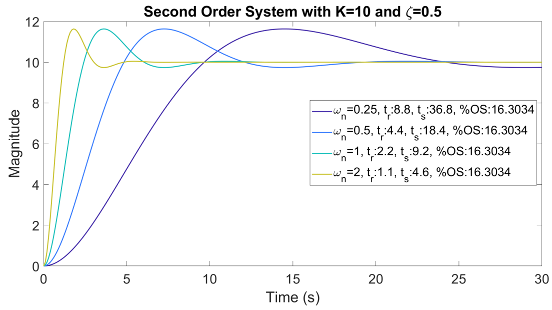

Figure 3: How ω_n impacts step response in and under-damped case

In Figure 4, we see the step response of a second order system when we hold ω_n and K constant but vary ζ. As ζ increases, the settling time and percent overshoot decrease but the rise time remains the same.

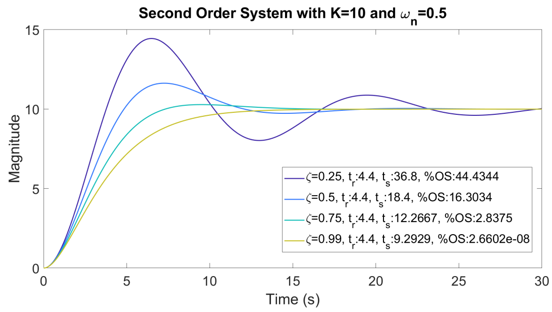

Figure 4: How ζ impacts step response in and under-damped case

Check for yourself that this makes sense given our equations for rise time, settling time, and percent overshoot for second order systems.