EEENG307: Impedance and Transfer Functions[1]

Lecture 5

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$$\ddot{y} + 3\dot{y} + 2y = r \quad y(0) = 0, \dot{y}(0) = 0$$

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Laplace Transform Derivative Property: $\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - \cancelto{0}{y(0)}$

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$$s^2Y(s) + 3sY(s) + 2Y(s) = R(s)$$
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$$Y(s) = \frac{1}{s^2 + 3s + 2}R(s)$$

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$$s^2 Y(s) + 3sY(s) + 2Y(s) = R(s)$$
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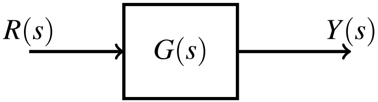
Transfer Function

$$Y(s) = \frac{1}{s^2 + 3s + 2} R(s)$$


$R(s)$
$G(s)$
$Y(s)$

Transfer function definition

Definition

The *Transfer Function* ($G(s)$) of a (linear, time invariant) system is the ratio of the Laplace Transform of the output signal over the Laplace Transform of the input signal with zero initial conditions.

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$$\frac{Y(s)}{R(s)} = G(s) \qquad Y(s) = G(s)R(s)$$

Finding the Transfer Function

- Current Method:
  - Write down all the (differential) equations describing the system
  - Eliminate all variables except input and output
  - Find the Laplace Transform of the resulting differential equation
- Impedance Method:
  - Write down all the (differential) equations describing the system
  - Take the Laplace Transform of these equations
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Electrical Components

	resistor	capacitor	inductor
Component law	$v = iR$	$C\frac{dv}{dt} = i$	$v = L\frac{di}{dt}$
Laplace Transform			

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Laplace Transform	$V(s) = RI(s)$		

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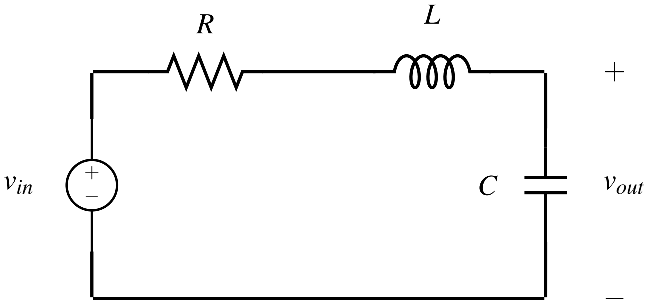
Component law	$v = iR$	$C\frac{dv}{dt} = i$	$v = L\frac{di}{dt}$
Laplace Transform	$V(s) = RI(s)$	$CsV(s) = I(s)$	$V(s) = LsI(s)$

Definition

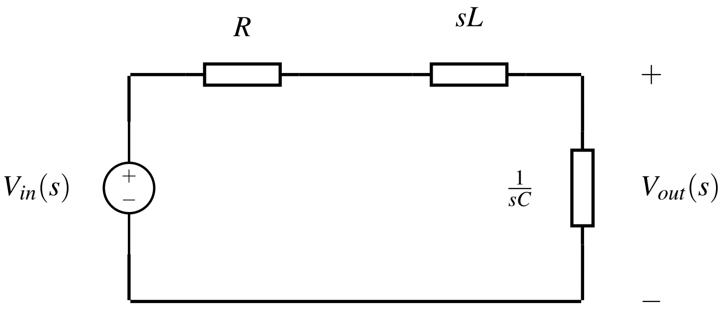
The *impedance* of an element is the ratio of the Laplace Transform of the across variable (voltage) over the Laplace Transform of the through variable (current)

Electrical Impedance

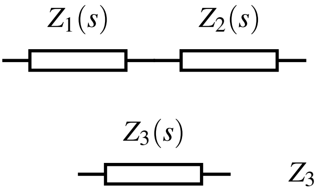
	resistor	capacitor	inductor
Component law	$v = iR$	$C\frac{dv}{dt} = i$	$v = L\frac{di}{dt}$
Laplace Transform	$V(s) = RI(s)$	$CsV(s) = I(s)$	$V(s) = LsI(s)$
Impedance	$\frac{V(s)}{I(s)} = R$	$\frac{V(s)}{I(s)} = \frac{1}{sC}$	$\frac{V(s)}{I(s)} = Ls$


R
L
v_{in}
+
−
C
v_{out}
+
−

Impedance Circuit

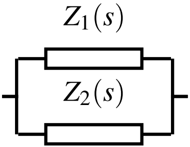


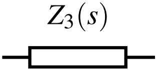
Impedances in series



$$Z_3(s) = Z_1(s) + Z_2(s)$$

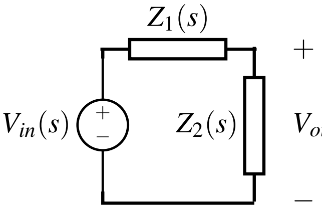
Impedances in parallel





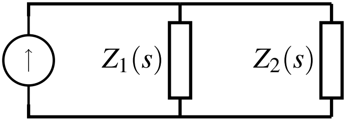
$$Z_3(s) = \frac{Z_1(s)Z_2(s)}{Z_1(s) + Z_2(s)}$$

Voltage Divider



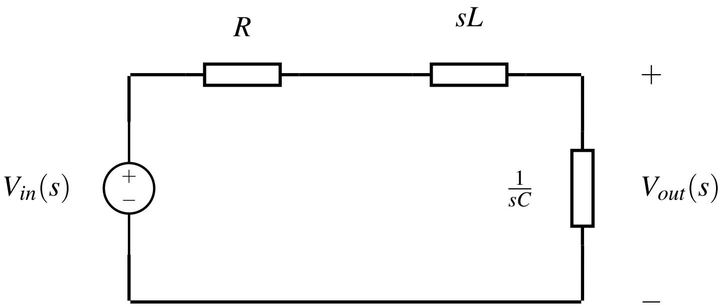
$$V_{out}(s) = \frac{Z_2(s)}{Z_1(s) + Z_2(s)} V_{in}(s)$$

Current Divider

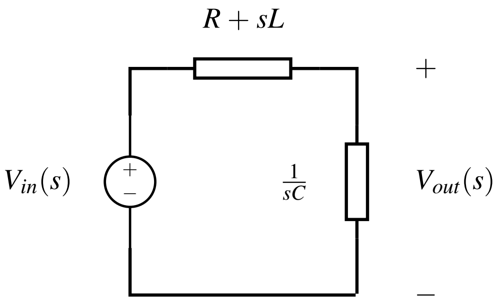


$$\downarrow I_{out}(s) = \frac{Z_1(s)}{Z_1(s) + Z_2(s)} I_{in}(s)$$

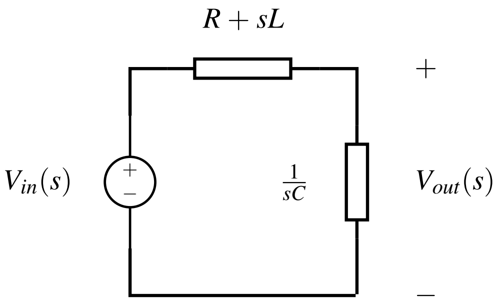
Impedance Circuit



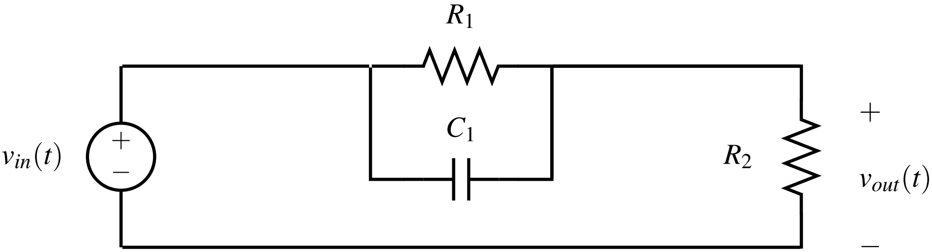
Circuit with Equivalent Impedance

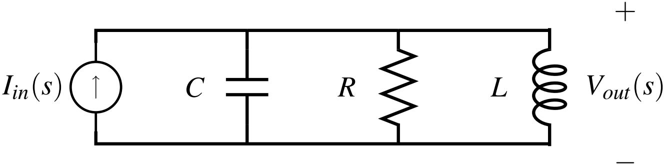


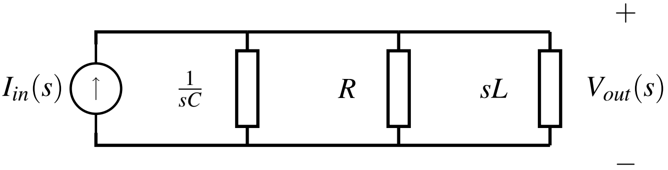
Circuit with Equivalent Impedance

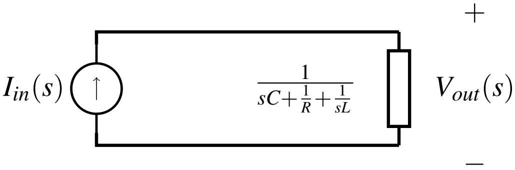


$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{\frac{1}{sC}}{\frac{1}{sC} + R + sL} = \frac{1}{s^2LC + sRC + 1}.$$


R_1
C_1
$v_{in}(t)$
R_2
$+$
$v_{out}(t)$
$-$

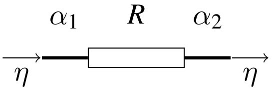

$I_{in}(s)$
C
R
L
+
$V_{out}(s)$
−


$I_{in}(s)$
$\frac{1}{sC}$
R
sL
$+$
$V_{out}(s)$
$-$


$I_{in}(s)$
$\frac{1}{sC+\frac{1}{R}+\frac{1}{sL}}$
+
$V_{out}(s)$
−

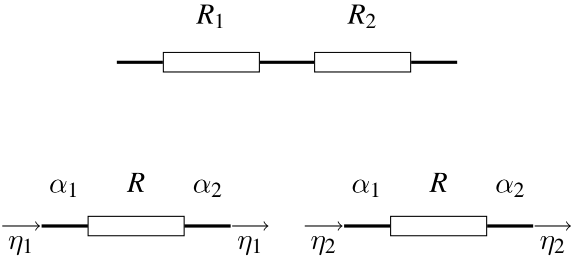
Generic Lumped Element

across variable: α
through variable: η



component law: $\alpha_1 - \alpha_2 = R\eta$
or $\frac{d(\alpha_1 - \alpha_2)}{dt} = R\eta$
or $\alpha_1 - \alpha_2 = R\frac{d\eta}{dt}$
or $\cdots$

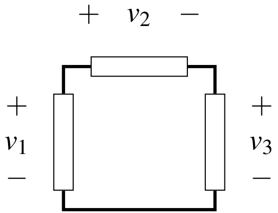
Generic Connection Rules



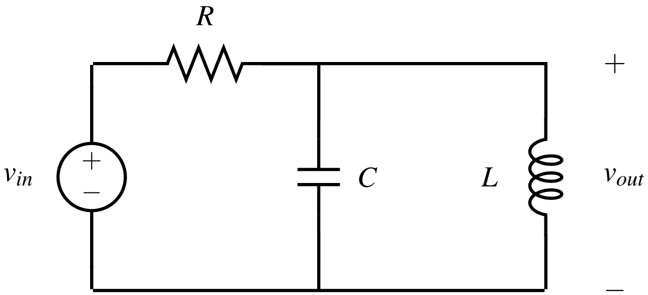
Across and Through Variables

Domain	Across Variable	Through Variable
Electrical	Voltage	Current
Translational Mechanical	Position	Force
Fluid	Pressure	Flow
Rotational Mechanical	Angular Position	Torque
Thermal	Temperature	Heat Flow

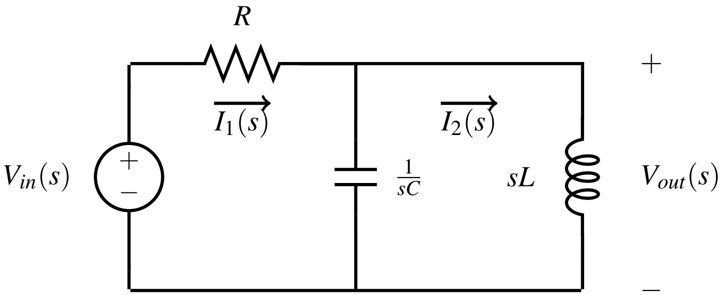
Mesh example



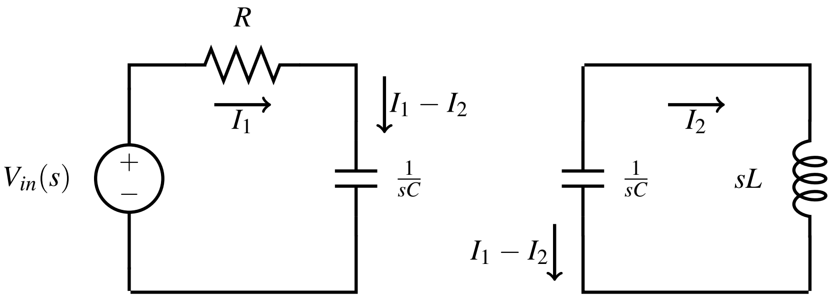
Circuit problem



Circuit problem in impedance form



Two meshes



Patterns of mesh equations

- the matrix is symmetric
- the diagonal terms are positive
- the off-diagonal terms are negative

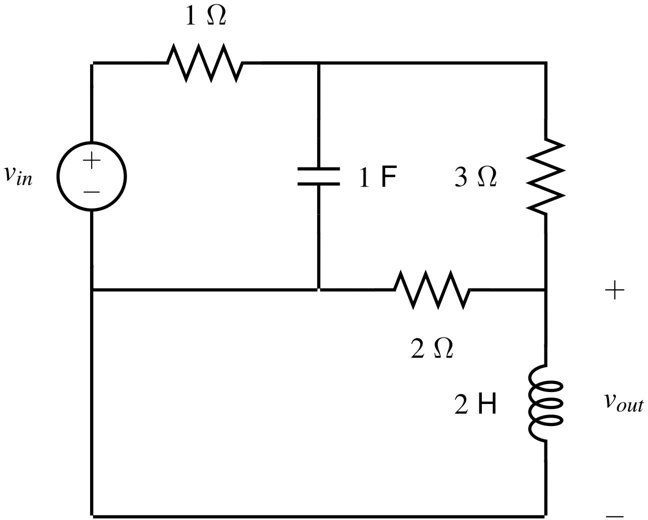
sum of impedances on mesh 1

impedance shared between mesh 1 and 2

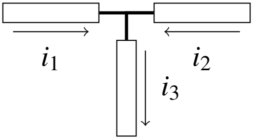
$$\begin{array}{r} \text{Source on mesh 1} \rightarrow \\ \text{Source on mesh 2} \rightarrow \end{array} \begin{bmatrix} V_{in}(s) \\ 0 \end{bmatrix} = \begin{bmatrix} R + \frac{1}{sC} & -\frac{1}{sC} \\ -\frac{1}{sC} & \frac{1}{sC} + sL \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix}$$

sum of impedances on mesh 2

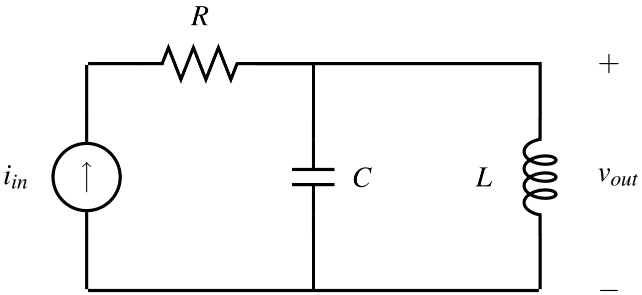
A Circuit with three meshes



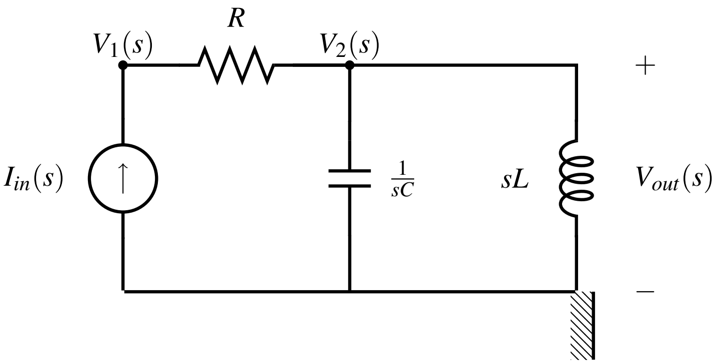
Node example



Circuit problem



Circuit problem in impedance form



Patterns of node equations

- the matrix is symmetric
- the diagonal terms are positive
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sum of admittances that touch node 1

admittances that touch both nodes 1 and 2

$$\begin{array}{r} \text{Source into node 1} \longrightarrow \\ \text{Source into node 2} \longrightarrow \end{array} \begin{bmatrix} I_{in}(s) \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{R} & -\frac{1}{R} \\ -\frac{1}{R} & \frac{1}{R} + sC + \frac{1}{sL} \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix}$$

sum of admittances that touch node 2