EEENG307: Time/Frequency Relationships and Control Performance Measures*

Lecture 31

Elenya Grant, Kathryn Johnson, and Hisham Sager†

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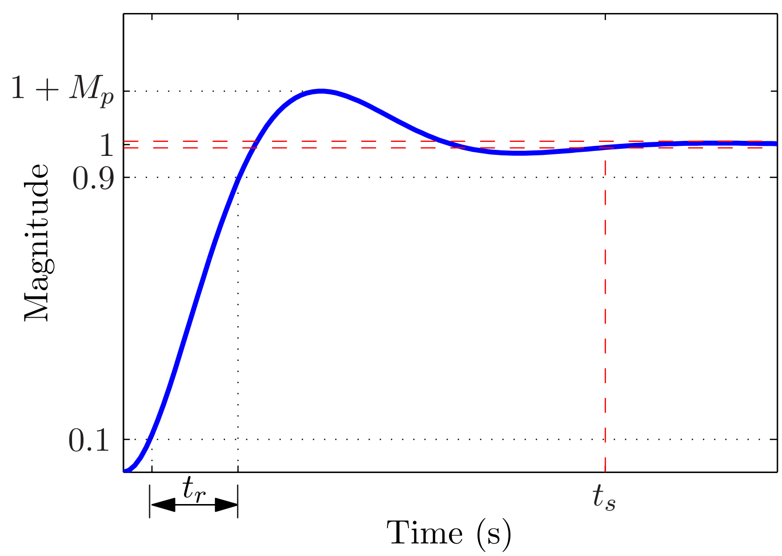
1 Time Domain Specifications

Recall that if the dominant (closed loop) poles of a system have natural frequency ω_n and damping ratio ζ, then the step response has the following characteristics:

Step Response Specifications

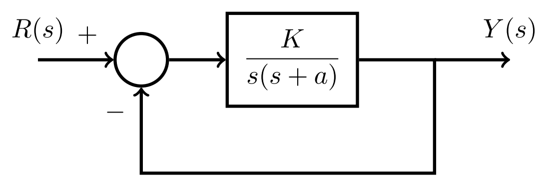
†Developed and edited by Tyrone Vincent and Kathryn Johnson, Colorado School of Mines, with contributions from Salman Mohagheghi, Chris Coulston, Kevin Moore, CSM and Matt Kupilik, University of Alaska, Anchorage



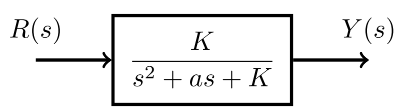
- 1% Settling time: $t_s = \frac{4.6}{\zeta\omega_n}$
- Overshoot: $\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100\%$
- 10% to 90% Rise time: $t_r = \frac{2.2}{\omega_n}$.

2 Bandwidth

To translate these specifications to the frequency domain, we will look at how the time domain and frequency domain characteristics of a canonical feedback system are related.



Closing the loop, we get the following closed loop system



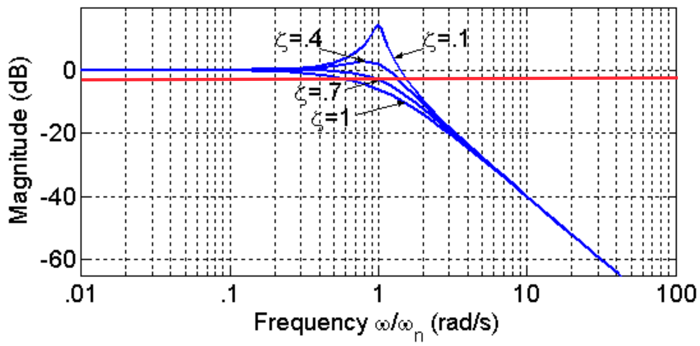
Let $K = \omega_n^2$ and $a = 2\zeta\omega_n$.

The *closed loop* system is second order, and it has the typical second order transfer function

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2},$$

with frequency response shown below

Second Order System Frequency Response



The speed of the step response of this system is mainly determined by ω_n. To get a measure of the frequency response that is related to ω_n, we use the system bandwidth

Definition 1. The *bandwidth*, ω_{BW}, of a linear time-invariant system is the frequency at which the magnitude response drops by 3 dB below the DC gain

Because of resonance, for fixed ω_n (=1 in the figure above) the bandwidth varies a bit. However, we see that for $0 < \zeta < 1$,

$$0.5\omega_n < \omega_{BW} < 2\omega_n \tag{1}$$

For most control design situations, we will use the approximation

$$\omega_{BW} \approx \omega_n \tag{2}$$

and remember to double-check our results and iterate our design since this is an approximation, not a precise equality.

3 Relationship Between Open and Closed Loop Systems

3.1 Frequency Response

Let's see how the *closed loop* frequency response $T(j\omega)$ depends on the *open loop* frequency response $G_F(j\omega)$, which we use in this article to indicate the forward gain of the feedback loop (might include both a plant and a controller) as we did in Lecture 16. Since the transfer functions are related by the feedback simplification law $T(s) = \frac{G_F(s)}{1+G_F(s)}$, we have

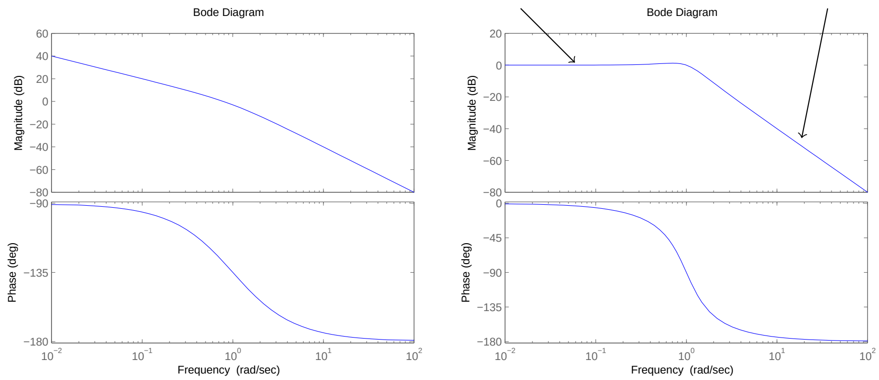
$$T(j\omega) = \frac{G_F(j\omega)}{1 + G_F(j\omega)} \tag{3}$$

This algebraic relationship has some simple implications when $G_F(j\omega)$ is big and small. Clearly, when $G_F(j\omega)$ is big – much bigger than 1 – then because $G_F(j\omega)$ appears in both the numerator and denominator, $T(j\omega) \approx 1$. Similarly, when $G_F(j\omega)$ is small, because it is much smaller than the 1 in the denominator, $T(j\omega) \approx G_F(j\omega)$. These cases are illustrated below:

Open and closed loop frequency response:
Main characteristics

$$G_F(s) = \frac{1}{s(s+1)} \qquad\qquad T(s) = \frac{G_F(s)}{1+G_F(s)}$$

$|G_F(j\omega)|$ big, $T(j\omega) \approx 1$ $\quad$ $|G_F(j\omega)|$ small, $T(j\omega) \approx G_F(j\omega)$



The closed-loop bandwidth is going to occur at a frequency that is intermediate between these two regions, i.e., when $|G_F(j\omega)|$ is of medium magnitude. When $G_F(j\omega) = 1\angle - 135$, then from the equation for the closed-loop transfer function (3) we can calculate that $|T(j\omega)| = \frac{1}{\sqrt{2}}$, which corresponds to -3dB. This drop of -3dB defines the bandwidth (in this case, the closed-loop bandwidth, since we are referring to a -3dB drop in $T(j\omega)$). As an approximation, if $|G_F(j\omega)| = 1$, the frequency at which this magnitude of 1 occurs will be close to the closed-loop bandwidth. Since the frequency at which $|G_F(j\omega)| = 1$ is an important frequency, we give it a special name

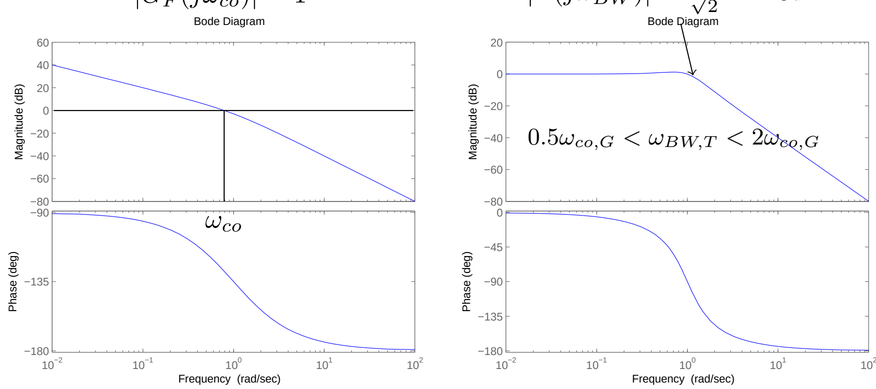
Definition 2. The frequency where the magnitude of a transfer function $|H(j\omega_{co})| = 1$ is called the *crossover frequency*, and is denoted ω_{co}. This is the frequency where the magnitude Bode plot crosses between positive and negative dB.

The relationship between the open-loop crossover frequency $\omega_{co,G}$ and closed loop bandwidth $\omega_{BW,T}$ is illustrated below.

Open and closed loop frequency response:
Bandwidth and Crossover Frequency

$$G_F(s) = \frac{1}{s(s+1)} \qquad\qquad T(s) = \frac{G_F(s)}{1+G_F(s)}$$

$$|G_F(j\omega_{co})| = 1 \qquad\qquad |T(j\omega_{BW})| \approx \frac{1}{\sqrt{2}} = -3\text{dB}$$



Combined with the previous relationship between ω_n and ω_{BW} shown in (1), we have

$$0.25\omega_{co,G} < \omega_{n,T} < 4\omega_{co,G} \tag{4}$$

or as an approximation

$$\boxed{\omega_{n,T} \approx \omega_{co,G}}. \tag{5}$$

The take home message is thus:

> To achieve a particular *closed loop* $\omega_{n,T}$ (e.g., to achieve desired closed-loop rise time or settling time to a step response), set the *open loop* crossover frequency $\omega_{co,G} = \omega_{n,T}$.

This result, which can sometimes be achieved as simply as changing the gain of the controller, establishes that the crossover frequency of $G_F(s)$ affects the speed $\omega_{n,T}$ of the closed loop step response. As you know from past controller designs to achieve $\%OS$ and settling time specifications, we also need to be concerned with how oscillatory that response is. The oscillatory behavior of a transfer function's step response is determined by its damping ratio ζ. In the frequency domain, a small damping ratio can cause a resonance to occur.

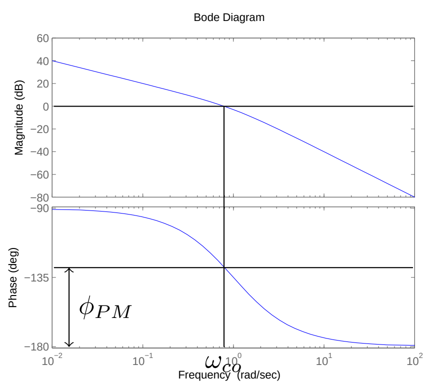
Let's see how the open loop frequency response $G_F(j\omega)$ affects the closed loop resonance.

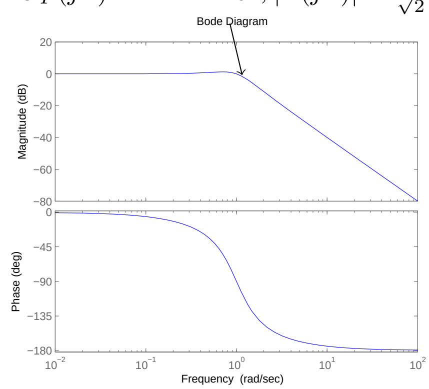
Below we have two examples of open and closed loop frequency response. Compare the two sets of plots, one with a small resonance appearing in the Bode plot of $T(j\omega)$, and one with a large resonance appearing in the Bode plot of $T(j\omega)$. What do you notice about the phase margin ϕ_{PM} of the open-loop Bode plot of $G_F(s)$ in each case?

Open and closed loop frequency response:
Small resonance

$$G_F(s) = \frac{1}{s(s+1)} \qquad\qquad T(s) = \frac{G_F(s)}{1+G_F(s)}$$

$$G_F(j\omega) = 1\angle -128^\circ,\ |T(j\omega)| \approx \tfrac{1}{\sqrt{2}}$$

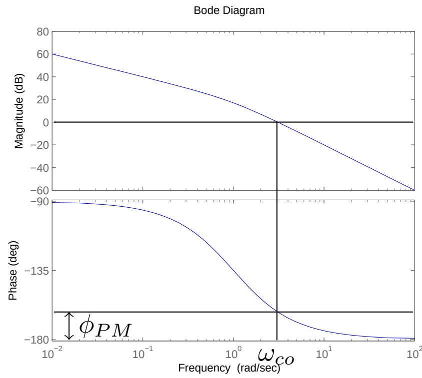


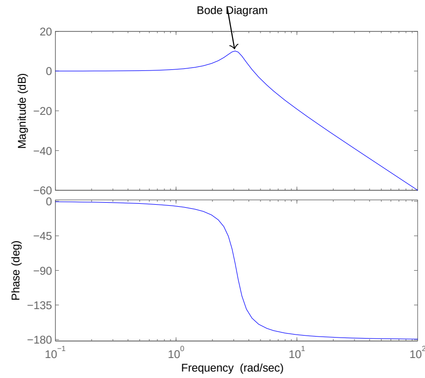


Open and closed loop frequency response:
Large resonance

$$G(s) = \frac{10}{s(s+1)} \qquad\qquad T(s) = \frac{G(s)}{1+G(s)}$$

$G(j\omega) \approx -1$, $|T(j\omega)| \approx$ large number



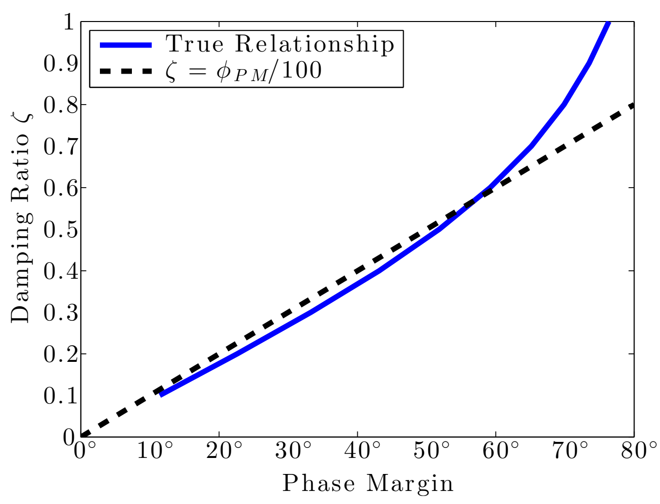


The difference between these two cases lies in the phase angle, and thus the phase margin, when $|G_F(j\omega)| = 1$. Note that as the phase of $G_F(j\omega)$ approaches -180°, $G_F(j\omega) \approx -1$, implying

$$|T(j\omega)| = \left|\frac{G_F(j\omega)}{1+G_F(j\omega)}\right| \approx \left|\frac{-1}{0}\right| = \infty.$$

The main result is that a low phase margin of the open-loop system $G_F(s)$ will give rise to a resonance in the closed-loop system $T(s)$. By plotting the phase margin $\phi_{PM,G}$ vs the corresponding closed loop damping ratio ζ_T (as determined by the size of the resonance) we get the following graph

Phase margin vs closed loop damping ratio



A useful approximation is

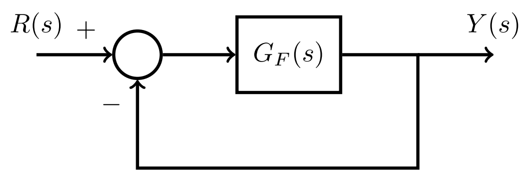
$$\boxed{\zeta_T \approx \frac{\phi_{PM,G}}{100}}$$

Again, we should emphasize that ζ_T is associated with the *closed loop* system, while $\phi_{PM,G}$ is measured on the *open loop* system.

3.2 Transient Response

Using the relationships $\omega_{n,T} \approx \omega_{BW} \approx \omega_{co,G}$ and $\zeta_T \approx \phi_{PM,G}/100$, we can estimate the closed loop transient response from the open loop frequency response.

Example 3. For the feedback system



Assume we want to achieve a closed loop rise time of approximately 22 seconds to a step reference input. What should the crossover frequency of $G_F(s)$ be?

Solution:

$$t_r = \frac{2.2}{\omega_n} = 22 \text{ seconds.}$$

Solving for $\omega_{n,T}$,

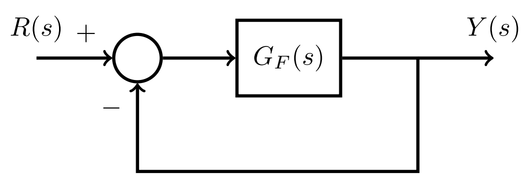
$$\omega_{n,T} = 0.1 \text{ rad/s}$$

Thus, from (5), we should design our controller so that

$$\omega_{co,G} = 0.1 \text{ rad/s}$$

□

Example 4. For the feedback system



find the necessary crossover frequency $\omega_{co,G}$ and phase margin $\phi_{PM,G}$ to obtain a closed loop step response with approximately $t_s > .1$s and %OS<10%.

Solution: Since

$$\zeta = \frac{-\ln(0.1)}{\sqrt{\pi^2 + \ln(0.1)^2}} = .59, \quad t_s = \frac{4.6}{\zeta\omega_n} = 0.1$$

thus

$$\zeta\omega_n = \frac{4.6}{0.1} = 46, \quad \omega_n = \frac{46}{.59} = 78.$$

Using our approximations, we should choose

$$\omega_{co,G} = 78 \text{ rad/s}, \quad \phi_{PM,G} = 59^\circ$$

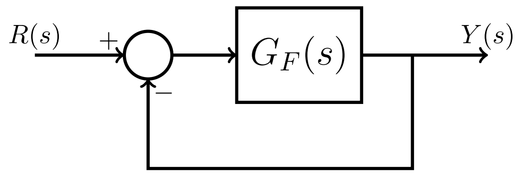
□

3.3 Steady State Error

The open loop frequency response can also give us information about the performance of the closed loop system steady state error relative to reference commands. Recall the following unity gain feedback system and its relationship to the steady-state error (depending on the form of the reference command) from Lecture 16.

For a unity gain feedback system with forward gain $G_F(s)$, such as shown below,

Unity Gain Feedback System



$\frac{Y(s)}{R(s)} = T(s) = \frac{G_F(s)}{1+G_F(s)}$

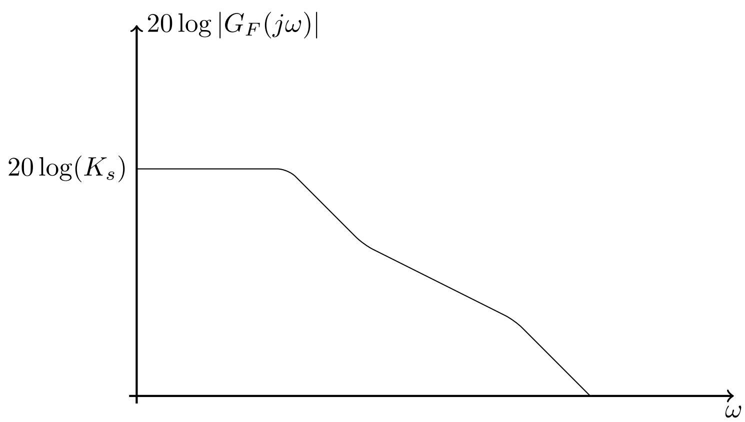
the response to a reference command of a unity step or ramp can be obtained from the following table.

	Steady State Error to a...		
System Type	Step Input $r(t) = Au(t)$	Ramp Input $r(t) = Atu(t)$	Error Constants
0	$e_{ss} = \frac{A}{1+K_s}$	$e_{ss} = \infty$	$K_s = \lim_{s\to 0} G_F(s) = G_F(0)$
1	$e_{ss} = 0$	$e_{ss} = \frac{A}{K_v}$	$K_v = \lim_{s\to 0} sG_F(s)$

Note that the key elements are the error constants $G_F(0)$ or $\lim_{s\to 0} sG_F(s)$. Both of these values can be estimated from the Bode plot of $G_F(j\omega)$.

First, note that $K_s = G_F(0)$, the DC gain of $G_F(s)$. Although the frequency of 0 does not show up on a Bode plot, we can infer it from the behavior at low frequency. If $G_F(s)$ has no pure integrators (i.e, $G(s)$ is Type 0) and no derivatives, i.e., it has a low-frequency slope of 0 dB/dec, then the DC gain is simply the gain at low frequency.

Magnitude Bode plot if $G_F(s)$ is Type 0



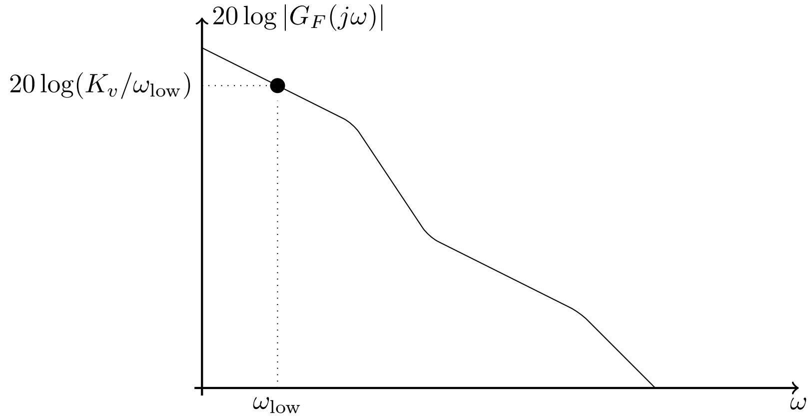
On the other hand, since $G_F(0)$ is bounded, $\lim_{s\to 0} sG_F(s) = 0$, so $K_v = 0$ for a Type 0 system

If $G_F(s)$ is Type 1, then its magnitude gain keeps increasing as $\omega \to 0$ (i.e., as we move from *right to left* on the magnitude Bode plot). Thus the DC gain – the gain at 0 rad/s – is infinite, so that $K_s = \infty$. This explains why $e_{ss} = 0$ when the system Type = 1 and the reference input is a step.

To find the error constant K_v, we first verify that the low frequency slope is -20dB/dec.[1] Then, find the magnitude of the Bode plot at some low frequency ω_{low}. The magnitude at this frequency will be K_v/ω_{low}.

Magnitude Bode plot if $G_F(s)$ is Type 1

[1] If the slope has larger magnitude, e.g., -40dB/dec or more, then $K_v = \infty$.



4 Lecture Highlights

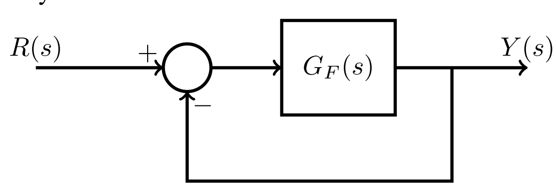
The primary takeaways from this article include

1. It is possible to related the *open loop* frequency response to the *closed loop* system's transient behavior (rise time, settling time, percent overshoot) according to the approximations derived in this lecture article.
2. Using the approximations $\omega_{n,T} \approx \omega_{co,G}$ (where $\omega_{n,T}$ is the desired closed-loop natural frequency and $\omega_{co,G}$ is the design specification for the open loop Bode plot) and $\zeta_T = \phi_{PM,G}/100$ (where ζ_T is the desired closed-loop damping ratio and $\phi_{PM,G}$ is the design specification for the open loop Bode plot) enables us to perform control design without having to compute the closed-loop transfer function.
3. The steady state error due to reference commands can be determined from the low frequency behavior of the open-loop transfer function, including whether the system is Type 0, Type 1, or higher.

5 Quiz Yourself

5.1 Questions

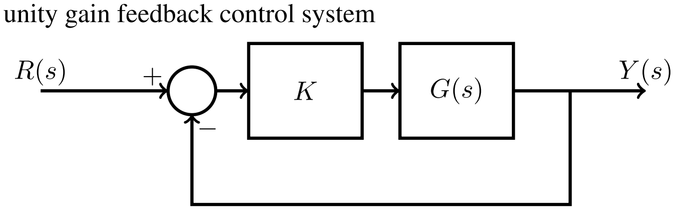
1. Consider a unity gain feedback system



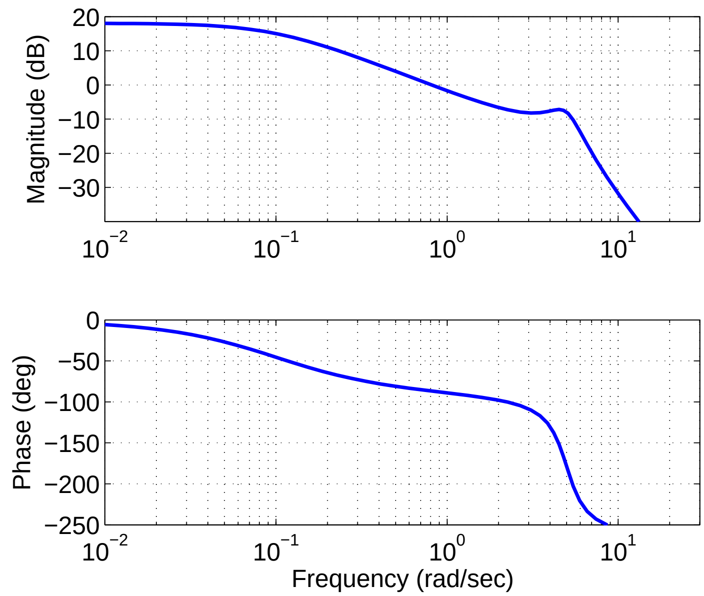
The specifications for the closed loop behavior are
- Step response rise time $t_r < .5s$
- Step response $\%OS < 10\%$.
- Steady state error for reference step input < 0.1

Translate these specifications to requirements on the low frequency gain, crossover frequency and phase margin for the forward gain $G_F(s)$.

2. Consider the following unity gain feedback control system



The bode plot for the plant $G(s)$ is shown below. By varying the controller gain K, what is the smallest reference steady state error that can be achieved while keeping maximum overshoot less than 10%?



5.2 Solutions

1.

$t_r < .5s$

$\%OS < 10\%$

$sse < 0.1$

$t_r = \frac{2.2}{\omega_n} < .5 \Rightarrow \omega_n > 4.4$ rad/s

$\zeta \geq \frac{-\ln(.1)}{\sqrt{\ln(.1)^2 + \pi^2}} \Rightarrow \zeta > .59$

$sse = \frac{1}{1+G(0)} < 0.1 \Rightarrow \underline{G(0) > 9}$

$\omega_n > 4.4 \Rightarrow \underline{\omega_{co} > 4.4}$

$\zeta > .59 \Rightarrow \underline{\phi_{pm} > 59^\circ}$

2.

$\%OS < 10\% \Rightarrow \phi_m > 59^\circ$ (by problem 2) $\Rightarrow$ need phase at crossover $> -121^\circ$

from Bode plot, -121° occurs at 4 rad/s

$\Rightarrow$ cannot make ω_{co} greater than 4 rad/s.

Magnitude at 4 rad/s is ~ -8dB $\Rightarrow$ cannot increase gain more than 8dB

current gain at low freq is 18dB Max gain at low freq is 26dB

$G(0) < 10^{26/20} = 20$

$sse > \frac{1}{1+20} = \underline{0.048}$