EEENG307: Laplace Transform Review[1]

Lecture 4

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Fall 2022

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Complex Representation Transform

Rectangular Form	Polar Form
$s = a + jb$	$s = r\angle\theta$

Given a complex number represented in rectangular form, we can *transform* it to polar form (and vice-versa)

$$(a, b) \leftrightarrow (r, \theta)$$

Laplace Transform Definition

Let $f(t)$ be a function of time defined for $t > 0$. The one-sided Laplace Transform of $f(t)$ is defined to be

$$\boxed{F(s) := \mathcal{L}[f(t)] := \int_{0^-}^{\infty} f(t)e^{-st}dt}$$

- 0^- is shorthand for the lower limit approaching 0 from the left (always negative)
- The Laplace Transform exists if the integral converges for *any* value of s
  - Region of convergence is not as important for "one-sided" Laplace transforms

Important facts you should already know

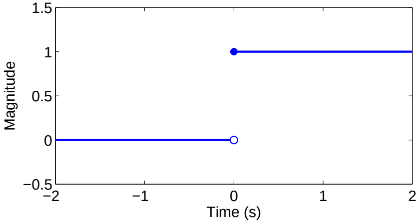
- Complex Exponentials (Euler's Formula): $e^{j\theta} = \cos(\theta) + j\sin(\theta)$.
- Differentiation and integration with s as a complex number:

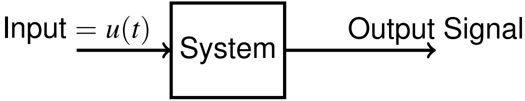
$$\frac{d}{dt}e^{st} = se^{st}$$

$$\int_a^b e^{st}dt = \frac{1}{s}e^{st}\Big|_a^b$$

Unit Step Function

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$




Input $= u(t)$
System
Output Signal

Our First Laplace Transform Pair

$$3u(t) \xleftrightarrow{\mathcal{L}} \frac{3}{s}$$

where $u(t)$ is the unit step function

Second Laplace Transform Pair

$$Ae^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{A}{a+s}$$

Scaling $\mathcal{L}\{Af(t)\} = A\mathcal{L}\{f(t)\}$

Linearity $\mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$

Sine and Cosine Laplace Transform Pair

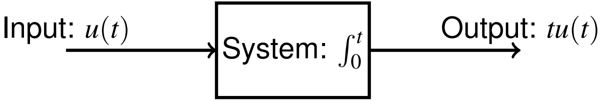
$$\cos(\omega t)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s}{s^2+\omega^2}$$
$$\sin(\omega t)u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{\omega}{s^2+\omega^2}$$

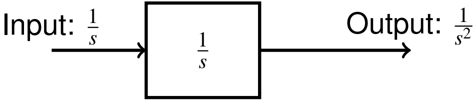
Time shift $\mathcal{L}\left\{f(t-t_0)u(t-t_0)\right\} = e^{-st_0}F(s)$

Frequency shift $\mathcal{L}\left\{e^{-s_0 t}f(t)\right\} = F(s+s_0)$

Differentiation $\mathcal{L}\left\{\frac{d}{dt}f(t)\right\} = s\mathcal{L}\left\{f(t)\right\} - f(0^-)$

Integration $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}\mathcal{L}\left\{f(t)\right\}$

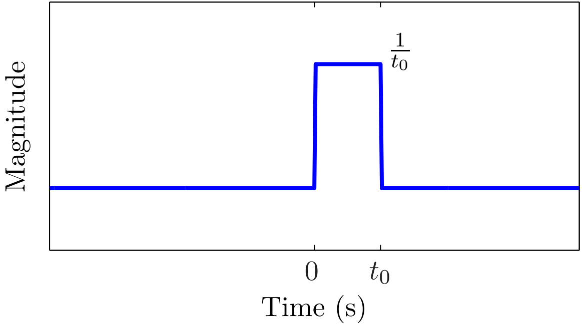

Input: $u(t)$
System: $\int_0^t$
Output: $tu(t)$


Input: $\frac{1}{s}$
$\frac{1}{s}$
Output: $\frac{1}{s^2}$

Definition of pulse function

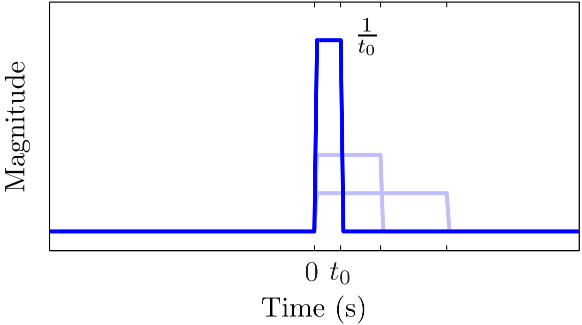
$$p_{t_0}(t) = \begin{cases} \frac{1}{t_0} & 0 < t < t_0 \\ 0 & \text{otherwise} \end{cases}$$

where t_0 is a constant.



Definition of impulse function

$$\delta(t) := \lim_{t_0 \to 0} p_{t_0}(t)$$



The simplest Laplace Transform pair

$$\delta(t) \overset{\mathcal{L}}{\longleftrightarrow} 1$$