EEENG307: Sinusoidal Steady State[*]

Lecture 23

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Fall 2020

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1 Pre-lecture Math Facts

Suppose you have the following rational transfer function:

$$G(s) = \frac{s^2 + 2s + 3}{s^2 + 5s + 4}.$$

In the following lecture, we will see that evaluating a transfer function at $s = j\omega$ will be useful

$$G(j\omega) = \frac{(j\omega)^2 + 2(j\omega) + 3}{(j\omega)^2 + 5(j\omega) + 4} = \frac{3 - \omega^2 + j2\omega}{4 - \omega^2 + j5\omega}$$

Now, suppose we instead evaluate $G(-j\omega)$:

$$G(-j\omega) = \frac{(-j\omega)^2 + 2(-j\omega) + 3}{(-j\omega)^2 + 5(-j\omega) + 4} = \frac{3 - \omega^2 - j2\omega}{4 - \omega^2 - j5\omega}$$

Notice that both the numerator and denominator are the *complex conjugate* of the numerator and denominator of $G(j\omega)$, because the real parts are the same, but the imaginary parts have opposite signs:

$$(3 - \omega^2 - j2\omega)^* = 3 - \omega^2 + j2\omega$$
$$(4 - \omega^2 - j5\omega)^* = 4 - \omega^2 + j5\omega$$

Since conjugation and division commute, i.e.

$$\frac{a^*}{b^*} = \left(\frac{a}{b}\right)^*$$

we actually have the following:

$$\boxed{G(-j\omega) = G(j\omega)^*}$$

Although we showed this for a specific case, this is true in general for any rational transfer function with real coefficients (just use the property that sum and conjugation also commute: $(a^* + b^*) = (a + b)^*$)

In addition, recall that if s_1 and s_2 are complex numbers

$$\left|\frac{s_1}{s_2}\right| = \frac{|s_1|}{|s_2|}$$
$$\angle\frac{s_1}{s_2} = \angle s_1 - \angle s_2$$

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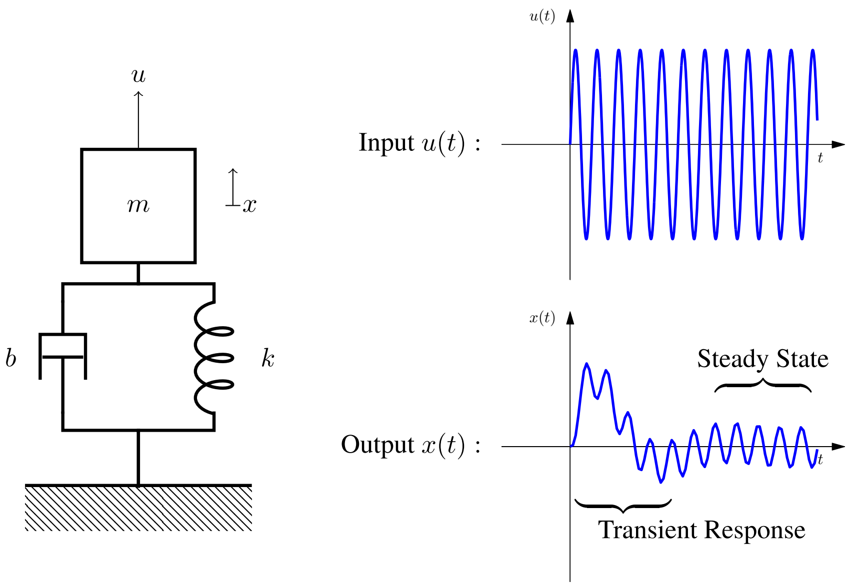
2 Response of Linear Systems to Sinusoidal Inputs

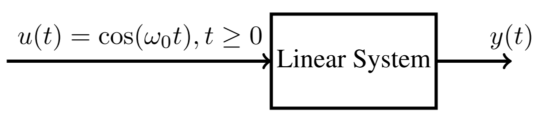
You are probably already aware of a basic property of **stable** linear systems: If the input is a sinusoid, then the *steady state* output (also called the steady state response) will be a sinusoid of the same frequency.

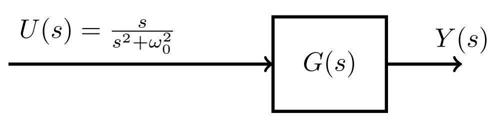
System Response to Sinusoidal Input



This can be proven by using Laplace Transforms.

Step 1: Multiply $G(s)$ by the Laplace Transform of a cosine





$$Y(s) = G(s)\tfrac{s}{s^2+\omega_0^2}$$

Step 2: Write partial fraction expansion, splitting poles at $\pm j\omega_0$

- Let $p_1, p_2, \cdots, p_n$ be the poles of $G(s)$ (assume simple poles for now, but result also holds if $G(s)$ has repeated poles)

$$Y(s) = G(s)\frac{s}{s^2+\omega_0^2} = \frac{A}{s-j\omega_0} + \frac{B}{s+j\omega_0} + \frac{C}{s-p_1} + \frac{D}{s-p_2} + \cdots$$

- By residue formula

$$A = \cancel{(s - j\omega_0)}G(s)\frac{s}{\cancel{(s - j\omega_0)}(s + j\omega_0)}\bigg|_{s=j\omega_0} = G(j\omega_0)\frac{1}{2}$$
$$B = \cancel{(s + j\omega_0)}G(s)\frac{s}{(s - j\omega_0)\cancel{(s + j\omega_0)}}\bigg|_{s=-j\omega_0} = G(-j\omega_0)\frac{1}{2}$$

Step 3: Find Inverse Laplace Transform

- Since
$$Y(s) = \frac{G(j\omega_0)\frac{1}{2}}{s - j\omega_0} + \frac{G(-j\omega_0)\frac{1}{2}}{s + j\omega_0} + \frac{C}{s - p_1} + \frac{D}{s - p_2} + \cdots$$

- Using $e^{-at}u \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{a+s}$ the time response is
$$y(t) = \left(G(j\omega_0)\frac{1}{2}e^{j\omega_0 t} + G(-j\omega_0)\frac{1}{2}e^{-j\omega_0 t} + Ce^{p_1 t} + De^{p_2 t}\right)u(t)$$

Step 4: Use property of stable poles

- If $\text{Re}(p_i) = -a < 0$,
$$\lim_{t\to\infty} Ce^{p_i t}u(t) = \lim_{t\to\infty} Ce^{\text{Re}(p_i)t}e^{j\text{Im}(p_i)t}u(t) = \lim_{t\to\infty} Ce^{-at}e^{j\text{Im}(p_i)t}u(t) = 0$$

- Thus
$$\lim_{t\to\infty} y(t) = \left(G(j\omega_0)\frac{1}{2}e^{j\omega_0 t} + G(-j\omega_0)\frac{1}{2}e^{-j\omega_0 t}\right)$$

Step 5: Use symmetry property $G(-j\omega) = G(j\omega)^*$ and Euler's formula

- Substitute for $G(-j\omega_0)$
$$\lim_{t\to\infty} y(t) = \left(\frac{1}{2}G(j\omega_0)e^{j\omega_0 t} + \frac{1}{2}G(j\omega_0)^* e^{-j\omega_0 t}\right)$$

- Write in polar form, using $a^* = |a|e^{-j\angle a}$
$$\lim_{t\to\infty} y(t) = \left(\frac{1}{2}|G(j\omega_0)|e^{j\angle G(j\omega_0)}e^{j\omega_0 t} + \frac{1}{2}|G(j\omega_0)|e^{-j\angle G(j\omega_0)}e^{-j\omega_0 t}\right)$$

- Euler's formula: $\frac{1}{2}Ae^{j\theta} + \frac{1}{2}Ae^{-j\theta} = A\cos(\theta)$
$$\lim_{t\to\infty} y(t) = |G(j\omega_0)|\cos(\omega_0 t + \angle G(j\omega_0))$$

What we have shown is if the input is a cosine with frequency ω_0, then the output, at steady state, is also a cosine with frequency ω_0, but with an amplitude gain of $G(j\omega_0)$, and a phase shift of $\angle G(j\omega_0)$. This is summarized below in the general case.

Sinusoidal Steady State is the Frequency Response

Theorem 1. *Given a stable system with transfer function $G(s)$, the sinusoidal steady state response is defined by the input/output relationship*

$$\boxed{\begin{aligned} u(t) &= A\cos(\omega_0 t + \theta), \\ y_{ss}(t) &= |G(j\omega_0)|A\cos\left(\omega_0 t + \theta + \angle G(j\omega_0)\right). \end{aligned}}$$

Definition 2. $G(j\omega)$ is the *frequency response function* of the system with transfer function $G(s)$.

Note: since $\sin(\omega t) = \cos(\omega t - 90°)$ the same is true for sine functions.

3 Examples

Example 3. Find the steady state response

$$G(s) = \frac{1}{s+1}$$
$$u(t) = 3\cos(2t + 30^\circ), t \geq 0$$

Solution: Calculate the magnitude and phase of $G(j2)$:

$$\begin{aligned} |G(j2)| &= \left|\frac{1}{j2+1}\right| \\ &= \frac{|1|}{|j2+1|} \\ &= \frac{1}{\sqrt{1+2^2}} \\ &= \frac{1}{\sqrt{5}} \end{aligned} \qquad \begin{aligned} \angle G(j2) &= \angle\frac{1}{j2+1} \\ &= \angle 1 - \angle j2 + 1 \\ &= 0 - \tan^{-1}\left(\frac{2}{1}\right) \\ &= -63.4^\circ \end{aligned}$$

The solution is thus

$$y_{ss}(t) = \frac{3}{\sqrt{5}}\cos(2t - 33.4^\circ)$$

Example 4. Find an expression for the magnitude and phase frequency response functions for a system with transfer function

$$G(s) = \frac{\sigma}{s+\sigma}$$

where $\sigma > 0$. **Solution:** The magnitude of the frequency response function is given by

$$\begin{aligned} |G(j\omega)| &= \left|\frac{\sigma}{j\omega+\sigma}\right| \\ &= \frac{|\sigma|}{|j\omega+\sigma|} \\ &= \frac{\sigma}{\sqrt{\omega^2+\sigma^2}} \end{aligned}$$

The phase of the frequency response function is given by

$$\begin{aligned} \angle G(j\omega) &= \angle\frac{\sigma}{j\omega+\sigma} \\ &= \angle\sigma - \angle(j\omega+\sigma) \\ &= 0 - \tan^{-1}\left(\frac{\omega}{\sigma}\right) \\ &= -\tan^{-1}\left(\frac{\omega}{\sigma}\right) \end{aligned}$$

4 Lecture Highlights

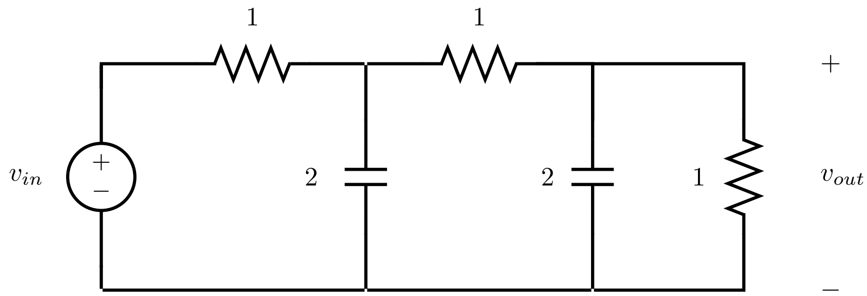
The primary takeaways from this article include

1. After an initial transient response period, the output of a dynamic system with a sinusoidal input will be a steady-state sinusoid that has the same frequency ω_0 as the input signal.
2. The output sinusoid will be scaled (amplitude changed) and shifted (phase angle changed) according to properties of the system's transfer function $G(s)$.
3. By evaluating the magnitude and phase angle of the system's transfer function $G(s)$ at $s = j\omega_0$, we can determine the steady-state output signal without having to go through the full inverse Laplace transform process.

5 Quiz Yourself

5.1 Questions

1. The following system models a transmission line



and has the transfer function

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{4s^2 + 8s + 3}$$

Find the steady state output if the following sinusoids are applied as input

(a) $\cos(0.1t)$

(b) $\cos(t)$

(c) $\cos(10t)$

5.2 Solutions

1(a)

if $v_{in}(t) = \cos(0.1t)$,

$$G(j.1) = \frac{1}{4(.1j)^2 + 8(.1j) + 3} = \frac{1}{-.04 + .8j + 3}$$

$$= \frac{1}{2.96 + 0.8j}$$

$$|G(j.1)| = \frac{1}{|2.96 + 0.8j|} = \frac{1}{3.07} = .326$$

$$\angle G(j.1) = \angle 1 - \angle 2.96 + 0.8j = 0 - 15.1^\circ = -15.1^\circ$$

$V_{out}(t)$ at steady state $= .326\cos(0.1t - 15.1^\circ)$

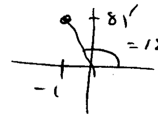
1(b)

) if $V_{in}(t) = \cos(t)$,

$$G(j1) = \frac{1}{4(j)^2 + 8(j) + 3} = \frac{1}{-4 + 8j + 3} = \frac{1}{-1 + 8j}$$

$$|G(j1)| = \frac{1}{|-1 + 8j|} = \frac{1}{8.06} = 0.124$$

$$\angle G(j1) = \angle 1 - \angle -1 + 8j = 0 - 97.1^\circ = -97.1^\circ$$



$= 180^\circ - \tan^{-1}\left(\frac{8}{1}\right) = 97.1^\circ$

at steady state

$$V_{out}(t) = 0.124 \cos(t - 97.1^\circ)$$

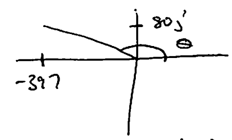
1(c)

$$G(j10) = \frac{1}{4(j10)^2 + 8(10j) + 3} = \frac{1}{-400 + 80j + 3} = \frac{1}{-397 + 80j}$$

$$|G(j10)| = \frac{1}{|-397 + 80j|} = \frac{1}{\sqrt{397^2 + 80^2}} = 0.0025$$

$$\angle G(j10) = \angle 0 - \angle -397 + 80j$$

$$= -168.6^\circ$$



$$\theta = 180 - \tan^{-1}\left(\frac{80}{397}\right) =$$

$$= 180 - 11.4 = 168.6^\circ$$

at steady state

$$V_{out}(t) = 0.0025 \cos(10t - 168.6^\circ)$$