

# EENG307: Solving Differential Equations using Laplace Transforms and Stability<sup>1</sup>

## Lecture 7

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# Laplace Transform: Differentiation Property

$$\mathcal{L} \left\{ \frac{d}{dt} f(t) \right\} = sF(s) - f(0^-),$$

$$\begin{aligned} \mathcal{L} \left\{ \frac{d^2}{dt^2} f(t) \right\} &= s \mathcal{L} \left\{ \frac{d}{dt} f(t) \right\} - \frac{df}{dt}(0^-), \\ &= s^2 F(s) - sf(0^-) - \frac{df}{dt}(0^-), \end{aligned}$$

$$\begin{aligned} \mathcal{L} \left\{ \frac{d^n}{dt^n} f(t) \right\} &= s^n F(s) - s^{n-1} f(0^-) - \\ &\quad \dots - sf^{(n-2)}(0^-) - f^{(n-1)}(0^-). \end{aligned}$$

Here is a simple differential equation:

$$\frac{dy}{dt} = z(t)$$

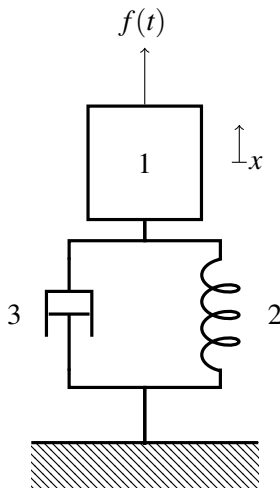
where the input  $z(t)$  is the unit step function, i.e.,

$$z(t) = 1 \quad t \geq 0.$$

What is  $y(t)$ ?

# Mass-spring-damper System

At time  $t = 0$ , this system is at position  $x = 0$ , but with an initial velocity of 1 m/s. Beginning at this time, the force  $f(t) = e^{-3t}$  is applied. Find  $x(t)$  for  $t \geq 0$ .



# Partial Fraction Expansion

We can use a partial fraction expansion to break a rational function up into easily recognizable parts that appear in a typical Laplace transform pair table. For example, we can determine from observation that the inverse Laplace transform of

$$X(s) = \frac{3}{s - 5}$$

is given by

$$x(t) = 3e^{5t}u(t).$$

# Equating Coefficients on Powers of $s$

$$\frac{s+4}{(s+1)(s+2)(s+3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}.$$

# Solution to the mass-spring-damper example

$$x(t) = (1.5e^{-t} - 2e^{-2t} + 0.5e^{-3t})u(t).$$

What do you notice about the form of the solution to this mass-spring-damper example, given that the input force is a decaying exponential  $f(t) = e^{-3t}$ ?

- Does the input force pull on the mass forever or does it eventually drop to (near) zero?
- Does the mass eventually return to its starting position ( $x(t) = 0$ ) after enough time has passed?
- Do these results make sense given your understanding of masses, springs, and dampers?

# Interpreting the form of the solution

*Is the solution bounded or does it go to infinity as time goes to infinity?*

# Laplace Transform Pairs with repeated roots

Repeated roots occur when an exponential or sinusoid is multiplied by  $t$  or  $t^n$ . First, let's derive the Laplace Transform of  $t^n$ . We already have established the Laplace Transform pair

$$u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s},$$

and the integration theorem

$$\mathcal{L} \left\{ \int_0^t f(\tau) d\tau \right\} = \frac{1}{s} \mathcal{L} \{f(t)\}.$$

# Laplace Transform Pairs with Repeated Roots

Note that a ramp is the integral of a step. That is, the ramp function  $tu(t)$  can be defined via

$$tu(t) = \int_{0^-}^t u(t) dt.$$

Using the integration theorem gives us

$$tu(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s^2}.$$

Thus, a ramp has two poles at  $s = 0$ .

# Higher Powers

The Laplace Transform for higher powers of  $t$  can be found via further integration. For example,

$$\frac{1}{2}t^2u(t) = \int_{0^-}^t tu(t)dt,$$

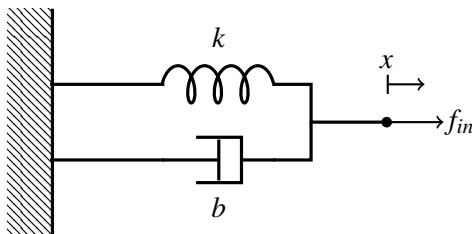
which implies

$$\frac{1}{2}t^2u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s^3}.$$

and even higher orders of  $t$  can be found similarly. What we see is that powers of  $t$  give us a denominator with repeated roots at  $s = 0$ .

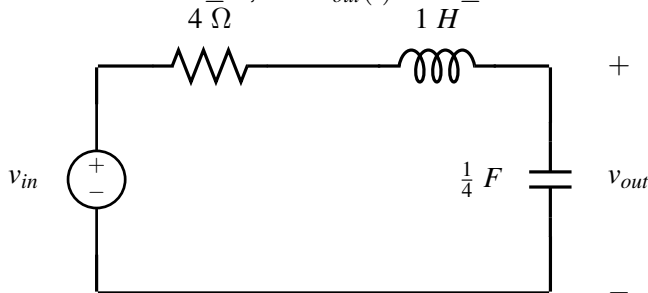
# Spring and Damper

A spring with spring constant  $k = 4$  N/m and damper with damping coefficient  $b = 2$  Ns/m is connected in parallel to a wall. A force of  $f_{in} = 1$  N is applied for  $t \geq 0$ . If the initial displacement of the right side of the spring and damper is  $x = 1$  m at  $t = 0$ , find  $x(t)$  for  $t \geq 0$



# Circuit Problem

An LRC circuit has applied voltage  $v_{in} = 1$  for  $t \geq 0$ . If  $v_{out}(t)$  was zero for  $t \leq 0$ , find  $v_{out}(t)$  for  $t \geq 0$ .



# Definition of BIBO Stability

## Definition

A system is *Bounded Input Bounded Output* (BIBO) *stable* if every bounded input results in a bounded output

How does this definition relate to the concepts in this lecture?