EEENG307: Proportional-Integral-Derivative (PID) Control Design, Simulation, and Evaluation[1]

Lecture 17

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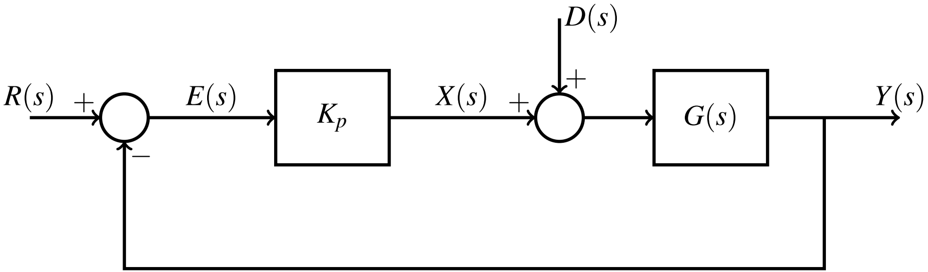
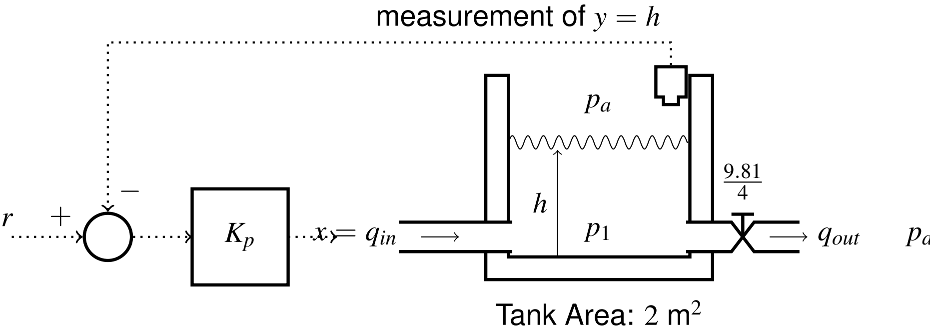
Proportional Control

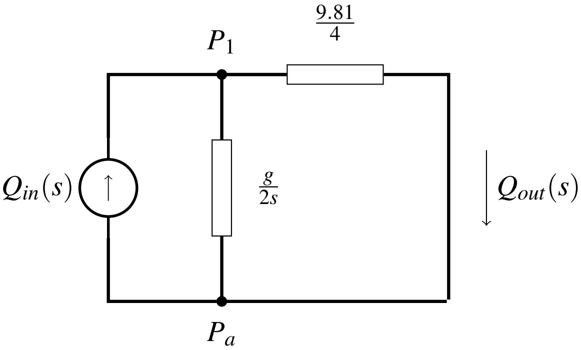


1. **Collect the design specifications.** Design specifications could be in terms of transient response (rise time, settling time, overshoot) or steady state response (reference tracking or disturbance rejection).
2. **Find the relevant *closed loop* transfer functions for your design specifications.**

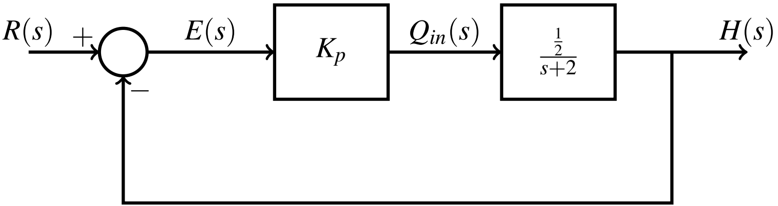
$$
\begin{aligned}
\frac{Y(s)}{R(s)} &= \frac{K_pG(s)}{1+K_pG(s)}, \\
\frac{E(s)}{R(s)} &= \frac{1}{1+K_pG(s)}, \\
\frac{E(s)}{D(s)} &= -\frac{G(s)}{1+K_pG(s)}.
\end{aligned}
$$

3. **Select K_p so that the design specifications are met, or determine that no K_p exists to meet the specifications.**


measurement of $y = h$
r
$+$
$-$
K_p
$x = q_{in}$
p_a
h
p_1
$\frac{9.81}{4}$
q_{out}
p_a
Tank Area: 2 m^2



$$\frac{Q_{out}(s)}{Q_{in}(s)} = \frac{\frac{1}{2}}{s+2}$$


R(s)
+
−
E(s)
Kp
Qin(s)
1/2 / (s+2)
H(s)

1. Design specifications
   - $t_s \leq 0.2$ s
   - $e_{ss} \leq 0.1$ for unit step reference
2. Closed loop transfer functions

$$\frac{H(s)}{R(s)} = \frac{K_p/2}{s + \frac{K_p}{2} + 2}$$

$$\frac{E(s)}{R(s)} = \frac{s+2}{s + \frac{K_p}{2} + 2}$$

Note that the closed loop transfer functions are also first order.

$$\frac{H(s)}{R(s)} = \frac{K_p/2}{s+\frac{K_p}{2}+2} \qquad \frac{E(s)}{R(s)} = \frac{s+2}{s+\frac{K_p}{2}+2}$$

2. Select K_p.
   - $t_s \leq 0.2$.

$$t_s = \frac{4.6}{\sigma} \leq 0.2 \implies \sigma \geq \frac{4.6}{0.2} = 23$$

$$\sigma = \frac{K_p}{2} + 2 \geq 23 \implies K_p \geq 42$$

   - $e_{ss} \leq 0.1$.

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{E(s)}{R(s)}\frac{1}{s} = s\frac{s+2}{s+\frac{K_p}{2}+2}\frac{1}{s} = \frac{2}{2+\frac{K_p}{2}}$$

$$e_{ss} = \frac{2}{2+\frac{K_p}{2}} \leq 0.1 \implies K_p \geq 36$$

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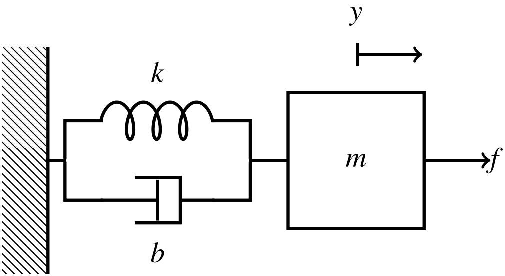
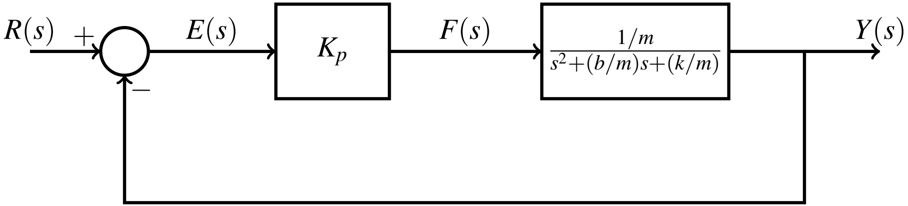


$$\frac{Y(s)}{F(s)} = \frac{1/m}{s^2 + (b/m)s + (k/m)}$$



$$F(s) = K_p(R(s) - Y(s))$$

$$f(t) = K_p(r(t) - y(t))$$

$$\frac{Y(s)}{R(s)} = \frac{K_p/m}{s^2 + (b/m)s + (k + K_p)/m}.$$

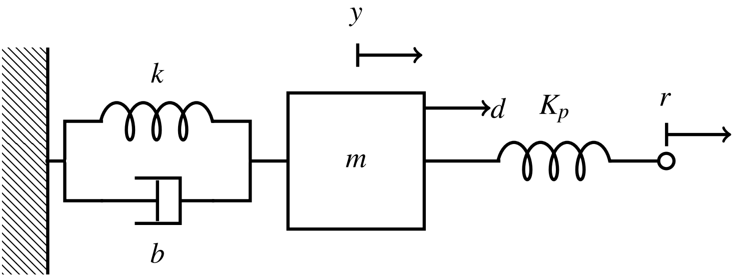
$$\frac{Y(s)}{R(s)} = \frac{K_p/m}{s^2 + (b/m)s + (k + K_p)/m}.$$

$$\omega_n = \sqrt{\frac{k + K_p}{m}}, \quad \zeta = \frac{b}{2\sqrt{m}\sqrt{k + K_p}}, \quad \zeta\omega_n = \frac{b}{2m}.$$

$$\frac{Y(s)}{R(s)} = \frac{K_p/m}{s^2 + (b/m)s + (k+K_p)/m}.$$

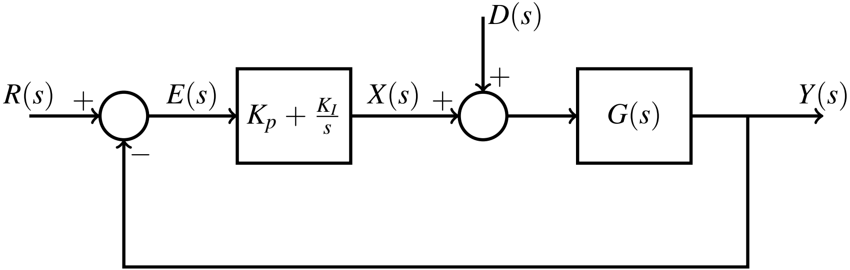
$$\omega_n = \sqrt{\frac{k+K_p}{m}}, \quad \zeta = \frac{b}{2\sqrt{m}\sqrt{k+K_p}}, \quad \zeta\omega_n = \frac{b}{2m}.$$

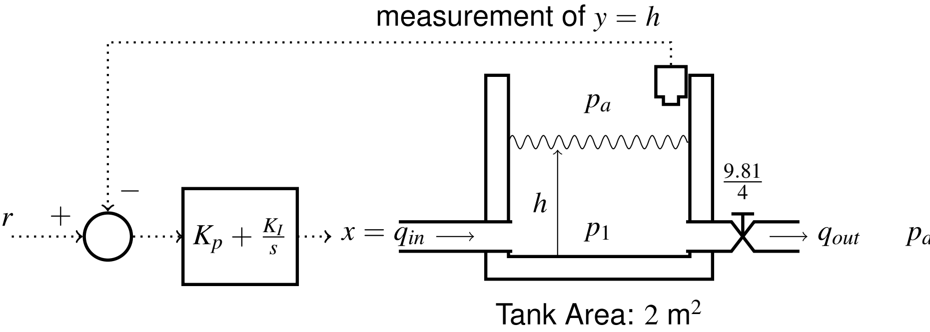
$$\uparrow K_p \implies \begin{cases} \uparrow \omega_n & \implies \text{reduced rise time} \\ \downarrow \zeta & \implies \text{increased overshoot} \\ \leftrightarrow t_s & \implies \text{no change in settling time} \end{cases}$$

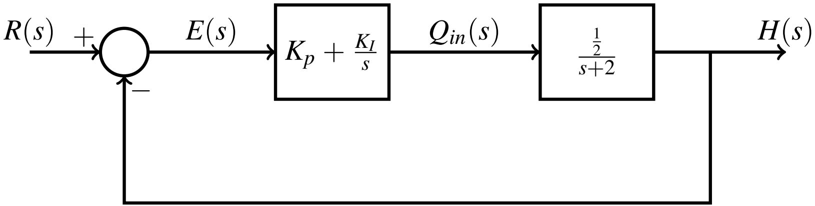


$$f(t) = K_p(r(t) - y(t))$$

Proportional/Integral (PI) Control




measurement of $y = h$
r
$+$
$-$
$K_p + \frac{K_I}{s}$
$x = q_{in} \longrightarrow$
p_a
h
p_1
$\frac{9.81}{4}$
$\longrightarrow q_{out}$
p_a
Tank Area: 2 m^2


$R(s)$
+
−
$E(s)$
$K_p + \frac{K_I}{s}$
$Q_{in}(s)$
$\frac{\frac{1}{2}}{s+2}$
$H(s)$

The closed loop transfer functions are

$$\frac{H(s)}{R(s)} = \frac{\frac{K_p}{2}s + \frac{K_I}{2}}{s^2 + \frac{4+K_p}{2}s + \frac{K_I}{2}}$$
$$\frac{E(s)}{R(s)} = \frac{s^2 + 2s}{s^2 + \frac{4+K_p}{2}s + \frac{K_I}{2}}$$

Note that since $E(s)/R(s)$ has a zero at $s = 0$, we automatically meet the steady state error specification, as

$$e_{ss} = \lim_{s \to 0} s \frac{s^2 + 2s}{s^2 + \frac{4+K_p}{2}s + \frac{K_I}{2}} \frac{1}{s} = 0$$

We need only choose K_p and K_I to meet our settling time specification.

$$\frac{H(s)}{R(s)} = \frac{\frac{K_p}{2}s + \frac{K_I}{2}}{s^2 + \frac{4+K_p}{2}s + \frac{K_I}{2}}$$

Since $2\zeta\omega_n = \frac{4+K_p}{2}$, $\zeta\omega_n = 1 + \frac{K_p}{4}$. Thus, we require

$$t_s = \frac{4.6}{\zeta\omega_n} = \frac{4.6}{1 + \frac{K_p}{4}} \leq 0.2$$

so we should choose

$$K_p \geq 88$$

K_I can be chosen as desired. For example, we may wish to have a reasonable damping ratio.

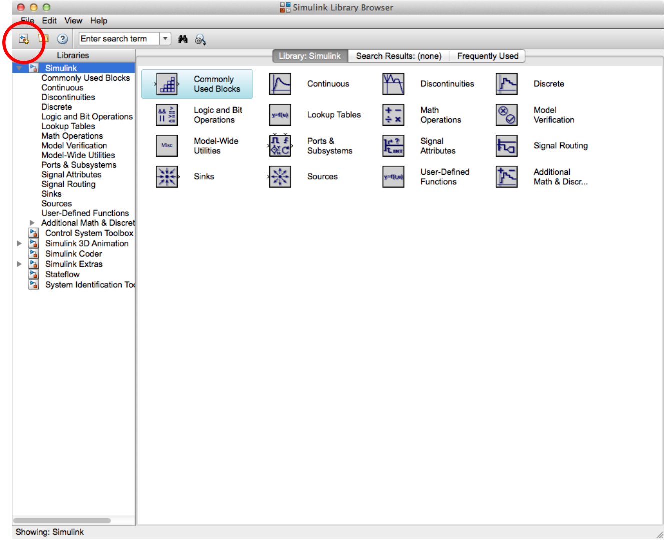
$$\frac{H(s)}{R(s)} = \frac{\frac{K_p}{2}s+\frac{K_I}{2}}{s^2+\frac{4+K_p}{2}s+\frac{K_I}{2}}$$

From above, $\zeta = \frac{1+\frac{K_p}{4}}{\omega_n}$, and from $\frac{H(s)}{R(s)}$, $\omega_n = \sqrt{\frac{K_I}{2}}$. With $K_p = 88$,

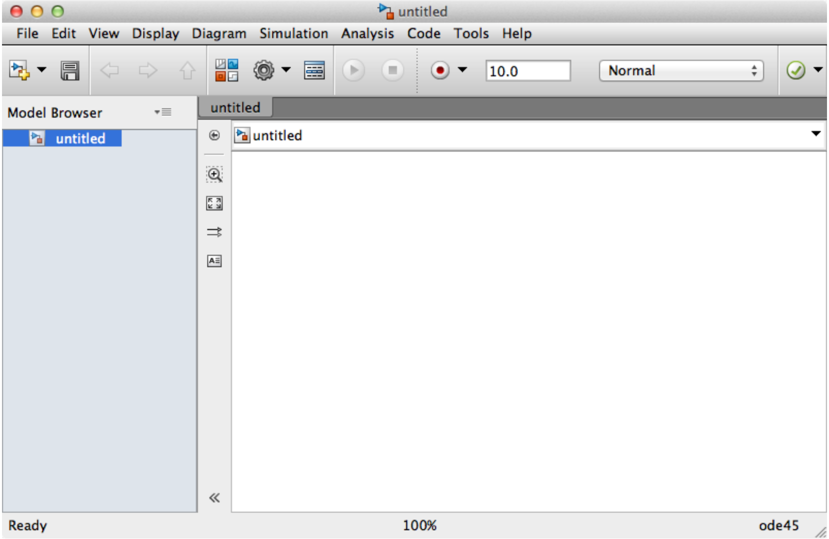
$$\zeta = \frac{(1+88/4)}{\omega_n} = \frac{23}{\sqrt{K_I/2}}$$

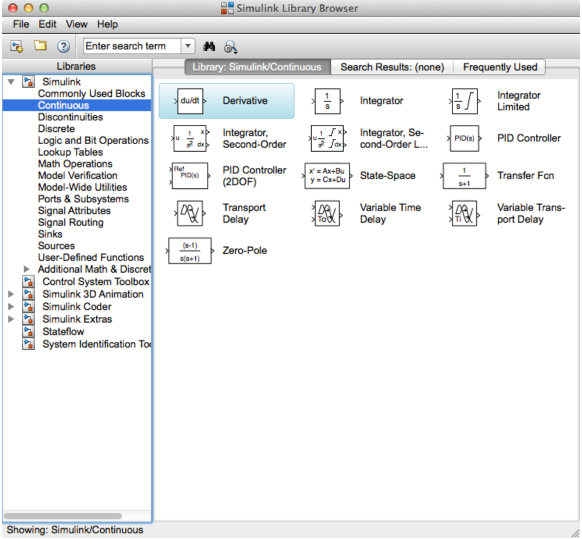
So $K_I = 4232$ will give a damping ratio of 0.5.

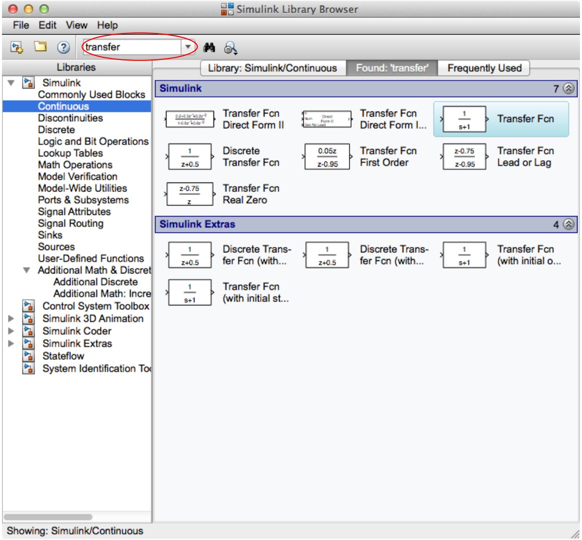
Simulink Library



Blank Simulink Model




Simulink Library Browser
File Edit View Help
Enter search term
Libraries
Library: Simulink/Continuous
Search Results: (none)
Frequently Used
Simulink
Commonly Used Blocks
Continuous
Discontinuities
Discrete
Logic and Bit Operations
Lookup Tables
Math Operations
Model Verification
Model-Wide Utilities
Ports & Subsystems
Signal Attributes
Signal Routing
Sinks
Sources
User-Defined Functions
Additional Math & Discret
Control System Toolbox
Simulink 3D Animation
Simulink Coder
Simulink Extras
Stateflow
System Identification To
Derivative
Integrator
Integrator Limited
Integrator, Second-Order
Integrator, Second-Order L...
PID Controller
PID Controller (2DOF)
State-Space
Transfer Fcn
Transport Delay
Variable Time Delay
Variable Transport Delay
Zero-Pole
Showing: Simulink/Continuous


Simulink Library Browser
File Edit View Help
transfer
Libraries
Simulink
Commonly Used Blocks
Continuous
Discontinuities
Discrete
Logic and Bit Operations
Lookup Tables
Math Operations
Model Verification
Model-Wide Utilities
Ports & Subsystems
Signal Attributes
Signal Routing
Sinks
Sources
User-Defined Functions
Additional Math & Discret
Additional Discrete
Additional Math: Incre
Control System Toolbox
Simulink 3D Animation
Simulink Coder
Simulink Extras
Stateflow
System Identification To
Library: Simulink/Continuous
Found: 'transfer'
Frequently Used
Simulink
7
Transfer Fcn Direct Form II
Transfer Fcn Direct Form I...
Transfer Fcn
Discrete Transfer Fcn
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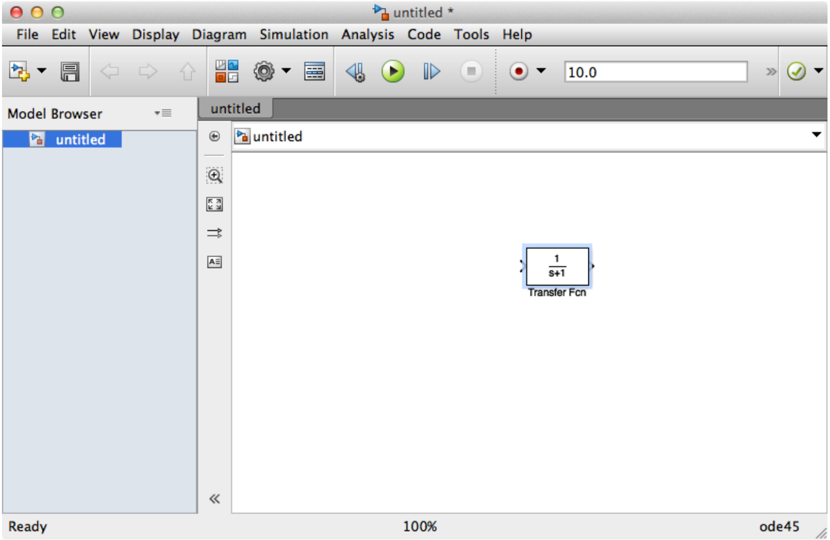
untitled *
File Edit View Display Diagram Simulation Analysis Code Tools Help
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Model Browser
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s+1
Transfer Fcn
Ready
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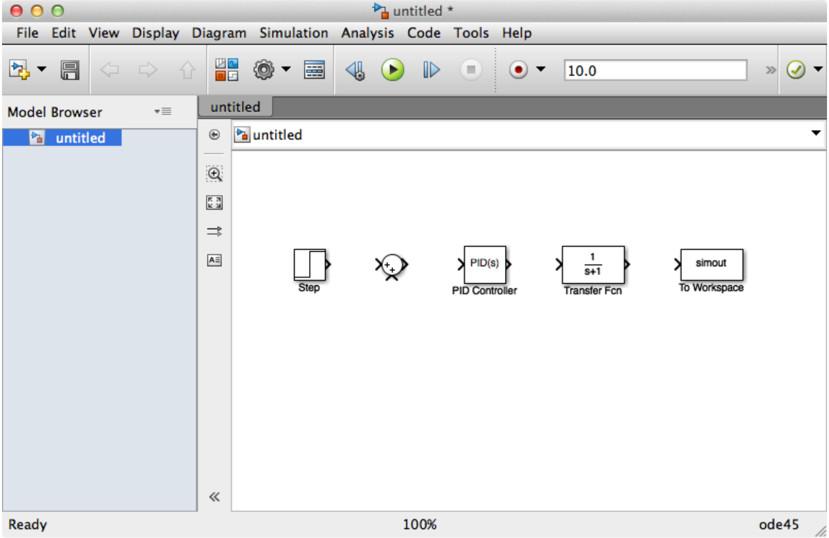
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Model Browser
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Step
PID(s)
PID Controller
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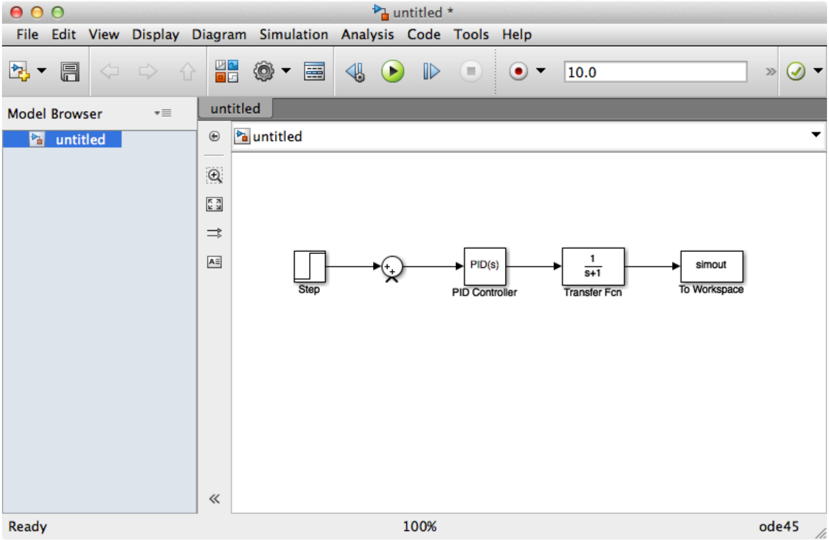
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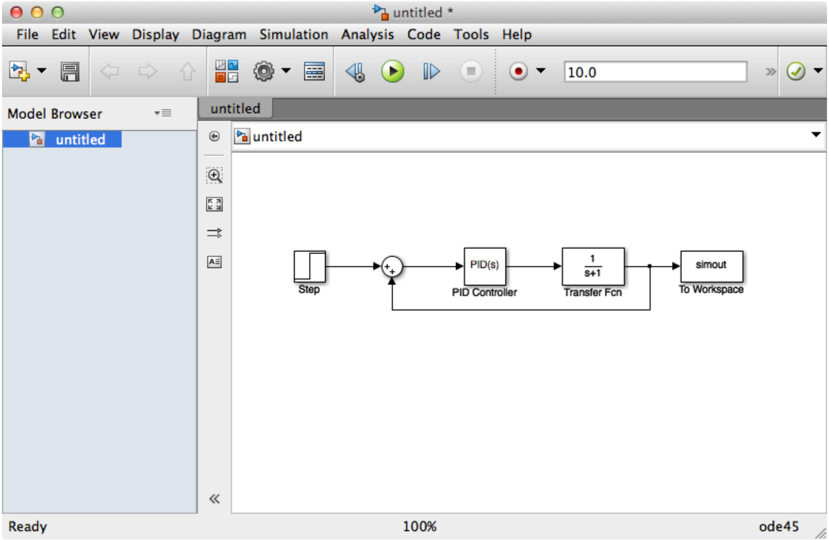
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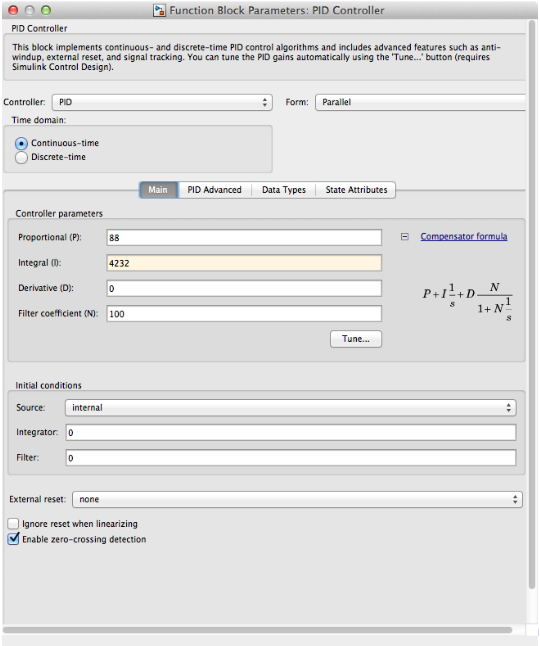
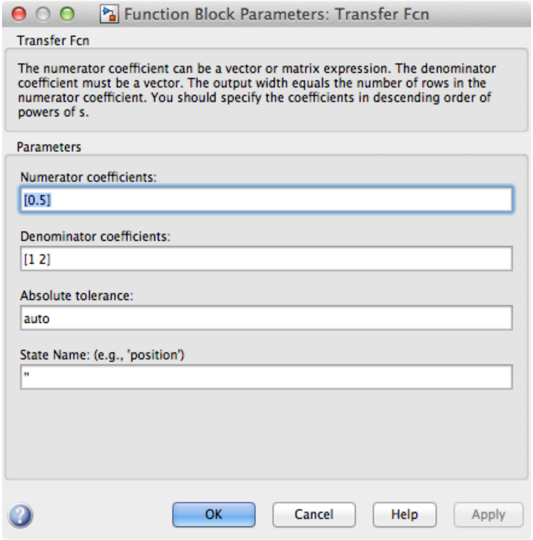
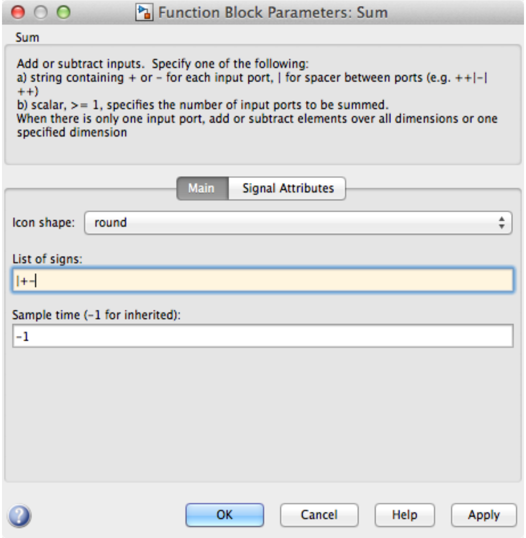
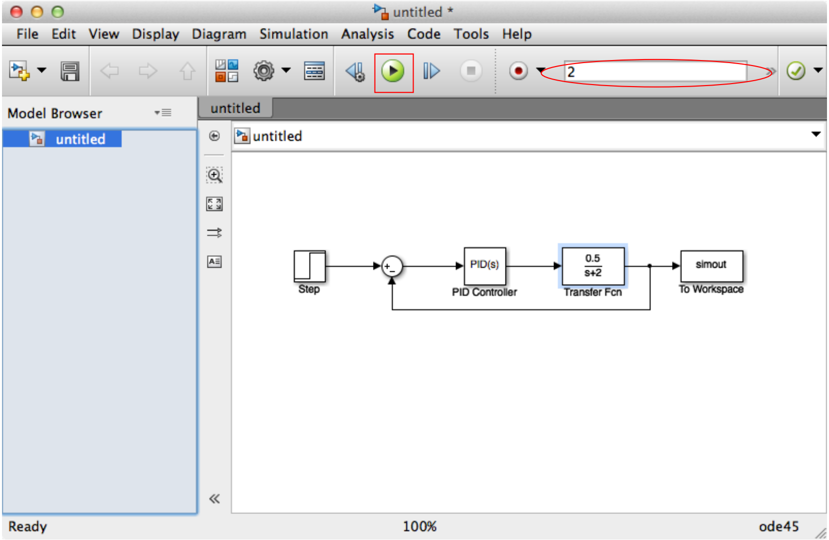

Function Block Parameters: PID Controller
PID Controller
This block implements continuous- and discrete-time PID control algorithms and includes advanced features such as anti-windup, external reset, and signal tracking. You can tune the PID gains automatically using the 'Tune...' button (requires Simulink Control Design).
Controller: PID
Form: Parallel
Time domain:
Continuous-time
Discrete-time
Main
PID Advanced
Data Types
State Attributes
Controller parameters
Proportional (P): 88
Integral (I): 4232
Derivative (D): 0
Filter coefficient (N): 100
Compensator formula
$P+I\frac{1}{s}+D\frac{N}{1+N\frac{1}{s}}$
Tune...
Initial conditions
Source: internal
Integrator: 0
Filter: 0
External reset: none
Ignore reset when linearizing
Enable zero-crossing detection


Function Block Parameters: Transfer Fcn
Transfer Fcn
The numerator coefficient can be a vector or matrix expression. The denominator coefficient must be a vector. The output width equals the number of rows in the numerator coefficient. You should specify the coefficients in descending order of powers of s.
Parameters
Numerator coefficients:
[0.5]
Denominator coefficients:
[1 2]
Absolute tolerance:
auto
State Name: (e.g., 'position')
''
OK
Cancel
Help
Apply
;


Function Block Parameters: Sum
Sum
Add or subtract inputs. Specify one of the following:
a) string containing + or - for each input port, | for spacer between ports (e.g. ++|-|++)
b) scalar, >= 1, specifies the number of input ports to be summed.
When there is only one input port, add or subtract elements over all dimensions or one specified dimension
Main
Signal Attributes
Icon shape:
round
List of signs:
|+-
Sample time (-1 for inherited):
-1
OK
Cancel
Help
Apply

;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File Edit View Display Diagram Simulation Analysis Code Tools Help
2
Model Browser
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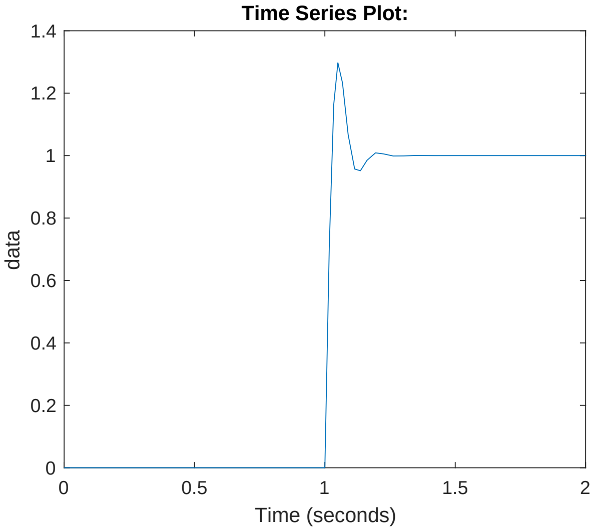
Step
PID(s)
PID Controller
0.5
s+2
Transfer Fcn
simout
To Workspace
Ready
100%
ode45


Time Series Plot:
data
Time (seconds)
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2

;