EEENG307: Solving Differential Equations using Laplace Transforms, Part II*

Lecture 6

Christopher Coulston, Kathryn Johnson and Hisham Sager†

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Contents

1 Pre-requisite Material

This lecture assumes that the reader is familiar with the following material:

- Lecture 2: Modeling Mechanical Systems
- Lecture 3: Modeling Electrical Systems
- Lecture 5: Solving Differential Equations using Laplace Transforms, Part I

2 Inverse Laplace Transform with Repeated roots

Since polynomials can have repeated roots, we need to determine how to take the inverse Laplace Transform when the denominator roots are repeated.

2.1 Laplace Transform Pairs with repeated roots

Laplace Transform Pairs with repeated roots

Repeated roots occur when an exponential or sinusoid is multiplied by t or t^n. First, let's derive the Laplace Transform of t^n. We already have established the Laplace Transform pair

$$u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s},$$

and the integration theorem

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}\mathcal{L}\left\{f(t)\right\}.$$

†Developed and edited by Tyrone Vincent and Kathryn Johnson, Colorado School of Mines, with contributions from Salman Mohagheghi, Chris Coulston, Kevin Moore, CSM and Matt Kupilik, University of Alaska, Anchorage

Laplace Transform Pairs with repeated roots

Note that a ramp is the integral of a step. That is, the ramp function $tu(t)$ can be defined via

$$tu(t) = \int_{0^-}^{t} u(t)dt.$$

Using the integration theorem gives us

$$tu(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s^2}.$$

Thus, a ramp has two poles at $s = 0$.

Higher Powers

The Laplace Transform for higher powers of t can be found via further integration. For example,

$$\frac{1}{2}t^2u(t) = \int_{0^-}^{t} tu(t)dt,$$

which implies

$$\frac{1}{2}t^2u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s^3}.$$

and even higher orders of t can be found similarly. What we see is that powers of t give us a denominator with repeated roots at $s = 0$. Now, we can use the frequency shift theorem

$$\mathcal{L}\left\{e^{-s_0t}f(t)\right\} = F(s + s_0)$$

to show that

$$te^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+a)^2}.$$

Thus the product of a ramp and exponential will give a repeated real root.

Repeated imaginary or complex roots occur when t is multiplied by a sinusoid or decaying sinusoid. The following table is a partial list of the Laplace Transform pairs with repeated roots

$f(t)$	$F(s)$
te^{-at}	$\frac{1}{(s+a)^2}$
$\frac{t^ne^{-at}}{n!}$	$\frac{1}{(s+a)^{n+1}}$
$t\sin(\omega t)$	$\frac{2\omega s}{(s^2+\omega^2)^2}$
$te^{-at}\sin(\omega t)$	$\frac{2\omega(s+a)}{((s+a)^2+\omega^2)^2}$

2.2 Partial Fraction Expansion with Repeated Roots

Suppose we had the Laplace Transform

$$X(s) = \frac{s+3}{(s^2+2s+1)(s+2)},$$

and we want to find the inverse Laplace Transform. The denominator of this function has roots at $-1, -1$ and -2. Thus, the root at -1 is repeated. In order to apply the partial fraction expansion correctly, we need one term for each root:

$$\frac{s+3}{(s+1)^2(s+2)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+2}$$

Since we have expanded our Laplace Transform pairs to include repeated roots, once we find the residues, the inverse Laplace Transform is easy. Thus, we have the general rule:

If the term $(s + a)^n$ appears in the denominator, then the partial fraction expansion will include the terms $(s + a)$, $(s + a)^2$, $\cdots$, $(s + a)^n$.

Of course, we still have to solve for the residues A, B, C. The good news is that the residue formula will hold for the highest power, as well as all other terms. So we have

$$B = (s+1)^2 \frac{s+3}{(s+1)^2(s+2)}\bigg|_{s=-1} = \frac{2}{1} = 2$$
$$C = (s+2) \frac{s+3}{(s+1)^2(s+2)}\bigg|_{s=-2} = \frac{1}{1} = 1$$

However, the residue formula will *not* hold for A. There is an alternate residue formula, but it is usually easier to simply return to putting the right hand side over a common denominator, substituting in the values for B and C found above.

$$\begin{aligned}\frac{s+3}{(s+1)^2(s+2)} &= \frac{A(s+1)(s+2) + 2(s+2) + 1(s+1)^2}{(s+1)^2(s+2)} \\ &= \frac{As^2 + 3As + 2A + 2s + 4 + s^2 + 2s + 1}{(s+1)^2(s+2)}\end{aligned}$$

Solve for A by equating coefficients for powers of s. For example, using the coefficients for s^2,

$$0 = A + 1$$

which implies $A = -1$. The final partial fraction expansion becomes

$$X(s) = \frac{-1}{s+1} + \frac{2}{(s+1)^2} + \frac{1}{s+2}$$

The inverse Laplace Transform is then

$$x(t) = \left(-e^{-t} + 2te^{-t} + e^{-2t}\right) u(t)$$

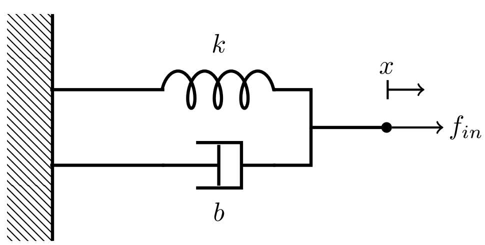
3 Differential Equation Examples

Let's try some examples to solidify the process of solving differential equations using Laplace Transforms.

3.1 Example 1

Spring and Damper

A spring with spring constant $k = 4$ N/m and damper with damping coefficient $b = 2$ Ns/m is connected in parallel to a wall. A force of $f_{in} = 1$ N is applied for $t \geq 0$. If the initial displacement of the right side of the spring and damper is $x = 1$ m at $t = 0$, find $x(t)$ for $t \geq 0$



This system is governed by the differential equation

$$b\dot{x} + kx = f_{in}$$

Thus, we can ask: what is the solution to the differential equation

$$2\dot{x} + 4x = 1 \qquad x(0) = 1$$

Take the Laplace Transform of both sides:

$$2(sX(s) - 1) + 4X(s) = \frac{1}{s}$$

Solve for X(s):

$$(2s+4)X(s) = \frac{2s+1}{s}$$
$$X(s) = \frac{s+\frac{1}{2}}{s(s+2)}$$

Find partial fraction expansion:

$$\frac{s+\frac{1}{2}}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2}$$

where

$$A = \left.\frac{s+\frac{1}{2}}{s+2}\right|_{s=0} = \frac{1}{4}$$
$$B = \left.\frac{s+\frac{1}{2}}{s}\right|_{s=-2} = \frac{3}{4}$$

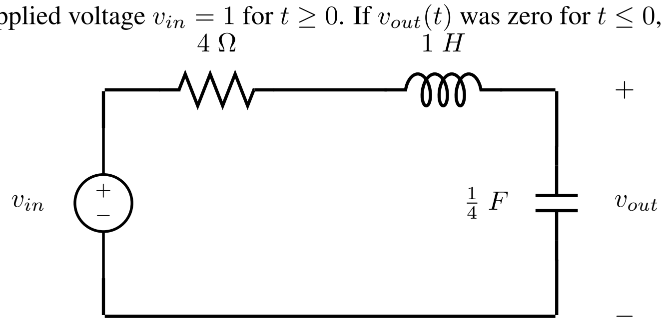
Giving inverse Laplace Transform

$$x(t) = \left(\frac{1}{4} + \frac{3}{4}e^{-2t}\right)u(t)$$

3.2 Example 2

Circuit Problem

An LRC circuit has applied voltage $v_{in} = 1$ for $t \geq 0$. If $v_{out}(t)$ was zero for $t \leq 0$, find $v_{out}(t)$ for $t \geq 0$.



Previously, we found that this system was governed by the differential equation

$$CL\ddot{v}_{out} + CR\dot{v}_{out} + v_{out} = v_{in}$$

Thus, we can ask: what is the solution to the differential equation

$$\frac{1}{4}\ddot{v}_{out} + \dot{v}_{out} + v_{out} = 1 \qquad v_{out}(0) = 0,\ \dot{v}_{out}(0) = 0$$

First, take the Laplace Transform of both sides (noting that initial conditions are zero):

$$\frac{1}{4}s^2V_{out}(s) + sV_{out}(s) + V_{out}(s) = \frac{1}{s}.$$

Then, solve for $V_{out}(s)$:

$$\begin{aligned} V_{out}(s) &= \frac{4}{(s^2+4s+4)s} \\ &= \frac{4}{s(s+2)^2} \end{aligned}$$

Find partial fraction expansion:

$$V_{out}(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

Residues:

$$\begin{aligned} A &= \left.\frac{4}{(s+2)^2}\right|_{s=0} = 1 \\ C &= \left.\frac{4}{s}\right|_{s=-2} = -2 \end{aligned}$$

Solve for remaining coefficients

$$\frac{4}{s(s+2)^2} = \frac{(s^2+4s+4) + B(s^2+2s) - 2s}{s(s+2)^2}$$

Thus $B = -1$, and

$$v_{out}(t) = (1 - e^{-2t} - 2te^{-2t})u(t)$$

4 Laplace Transform Table

Here is a summary of the Laplace Transforms and other results derived in the last few lectures.

Laplace transform pairs	
$f(t)$	$F(s)$
Unit impulse $\delta(t)$	1
Unit step $u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$\frac{1}{2}t^2u(t)$	$\frac{1}{s^3}$
$Ae^{at}u(t)$	$\frac{A}{s-a}$
$te^{at}u(t)$	$\frac{1}{(s-a)^2}$
$\frac{1}{2}t^2e^{at}u(t)$	$\frac{1}{(s-a)^3}$
$\sin(\omega t)u(t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)u(t)$	$\frac{s}{s^2+\omega^2}$
$e^{-at}\sin(\omega t)u(t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos(\omega t)u(t)$	$\frac{s+a}{(s+a)^2+\omega^2}$
$\frac{df}{dt}$	$sF(s) - f(0^-)$
$\frac{d^nf}{dt^n}$	$s^nF(s) - \sum_{k=1}^{n} s^{n-k}\frac{d^{k-1}f}{dt^{k-1}}(0^-)$
$\int_0^t f(\tau)d\tau$	$\frac{F(s)}{s}$

5 Application Example

A wind turbine is a rotational mechanical system that has a spinning “rotor” (blades plus their connections) with a large rotational inertia *J*. Although most of the time the turbine is expected to spin continuously to produce electricity, occasionally it is desirable to use the generator in reverse (as a motor providing torque τ)to turn the rotor to a certain angular position θ so that maintenance can be performed.

The differential equation relating the input torque to the output position (angular acceleration and velocity) is given by

$$J\ddot{\theta} + b\dot{\theta} = \tau,$$

where b is the rotational damping coefficient. You decide to try to achieve a constant angular position $\theta(t) = \theta_{setpoint}$ by applying a unit step torque input $\tau = u(t), t \geq 0$, as shown in this block diagram. Is this a successful approach?

Solution

Replace the torque τ with the unit step function and then take the Laplace transform of the equation:

$$\theta(s)(Js^2 + bs) = \frac{1}{s}$$

Solve for $\theta(s)$:

$$\theta(s) = \frac{1}{s(Js^2 + bs)} = \frac{1}{s^2(Js + b)}$$

Use partial fraction expansion

$$\frac{1}{s^2(Js + b)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{Js + b}$$

and solve for the numerator coefficients by picking useful values of s:

$$1 = As(Js + b) + B(Js + b) + Cs^2$$

$$s = 0: \qquad 1 = Bb \qquad \Rightarrow B = \frac{1}{b}$$

$$s = \frac{-b}{J}: \qquad 1 = A(0) + B(0) + C\left(\frac{-b}{J}\right)^2 \qquad \Rightarrow C = \frac{J^2}{b^2}$$

$$s = 1: \qquad 1 = A(J + b) + B(J + b) + C \qquad \Rightarrow 1 = A(J + b) + \frac{1}{b}(J + b) + \frac{J^2}{b^2} \qquad \Rightarrow A = \frac{-J}{b^2}$$

Plug these values for A, B, and C into the expanded $\theta(s)$

$$\begin{aligned}\theta(s) &= \frac{\frac{-J}{b^2}}{s} + \frac{\frac{1}{b}}{s^2} + \frac{\frac{J^2}{b^2}}{Js + b} \\ &= \frac{\frac{-J}{b^2}}{s} + \frac{\frac{1}{b}}{s^2} + \frac{\frac{J}{b^2}}{s + \frac{b}{J}}\end{aligned}$$

and finally take the inverse Laplace Transform:

$$\theta(t) = \frac{-J}{b^2} + \frac{1}{b}t + \frac{J}{b^2}e^{-\frac{b}{J}t}, t \geq 0$$

Although the first term in the solution $\theta(t)$ is a constant and the third term decays toward zero, the second term, $\frac{1}{b}t$, grows toward infinity as time goes to infinity. Therefore, *using a unit step torque input is not an effective way to make a wind turbine go to a constant angular position.* This result makes sense: as long as there is a constant, nonzero torque applied to this rotational system, we expect it to keep moving; in fact, that's what's happening when the wind is applying a torque to the system for power production.

Challenge

How does the answer change if the turbine has a non-zero initial velocity $\dot{\theta}(0^-) = -\pi$ rad/s?

Challenge Approach

The initial velocity is handled when we first take the Laplace transform, recalling that

$$L\left\{\frac{d^2}{dt^2}f(t)\right\} = s^2F(s) - sf(0^-) - \frac{df}{dt}(0^-)$$

Therefore, we have

$$s^2J\theta(s) + \pi + sb\theta(s) = \frac{1}{s}$$

$$\theta(s)(Js^2 + bs) = \frac{1 - \pi s}{s}$$

$$\theta(s) = \frac{1 - \pi s}{s(Js^2 + bs)}$$

And we complete the problem in the same manner as the initial problem - using partial fraction expansion, solving for numerator coefficients, and taking the inverse Laplace Transform with this new numerator.

6 Lecture Highlights

The primary takeaways from this article include

1. When the Laplace domain function has repeated roots in the denominator, the structure of the partial fraction expansion requires a term for each root. In this case, solving for the coefficients in the partial fraction expansion requires a little more care than in the previous lecture.
2. If there are repeated roots in the Laplace domain, you will find the multiplier t in your time domain solution.

7 Quiz Yourself

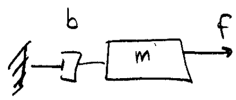
7.1 Questions

1. A motor on a car can be considered an input force for the car
   (a) Find a differential equation relating the input force and the velocity of the car. Include the mass of the car and viscous friction (air and rolling resistance.) Assume the car is on level ground.
   (b) Assume that the velocity of the car is 40 m/s and the car starts to coast at $t = 0$. Find the velocity $v(t)$ for $t \geq 0$ if the mass of the car is 1000 kg and the viscous friction is 125 N sec/m.
   (c) Determine how long it will take the car to slow to $5m/s$.
   (d) Sketch the output as a function of time.

7.2 Solutions

1.

(a)



$$\underline{m\ddot{x} + b\dot{x} = f}$$

-or-

let $v = \dot{x}$

$$\underline{m\dot{v} + bv = f}$$

(b)

$$1000\dot{v} + 125v = 0 \qquad v(0) = 40$$

$$1000(sV(s) - 40) + 125V(s) = 0$$

$$(1000s + 125)V(s) = 40\,000$$

$$V(s) = \frac{40\,000}{1000s + 125} = \frac{40}{s + .125}$$

$$\underline{v(t) = 40e^{-.125t}u(t)}$$

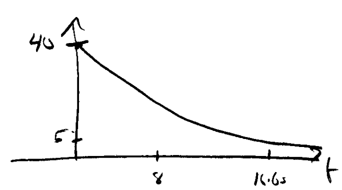
(c)

$$5 = 40e^{-0.125t}$$

$$\ln\frac{1}{8} = -.125t$$

$$t = \frac{-\ln\left(\frac{1}{8}\right)}{.125} = 16.6\text{ s}$$

(d)



8 Resources

Since this lecture is a continuation of Lecture 5, Solving Differential Equations using Laplace Transforms, Part I, see this lecture for the relevant resources.