EEENG307: Sinusoidal Steady State[1]

Lecture 23

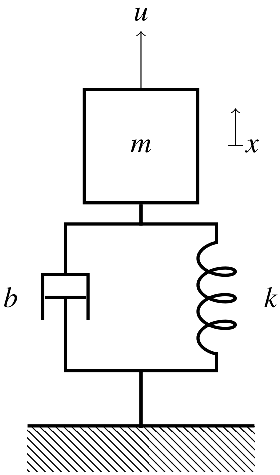
Christopher Coulston, Salman Mohagheghi and Jeffrey Schowalter[2]

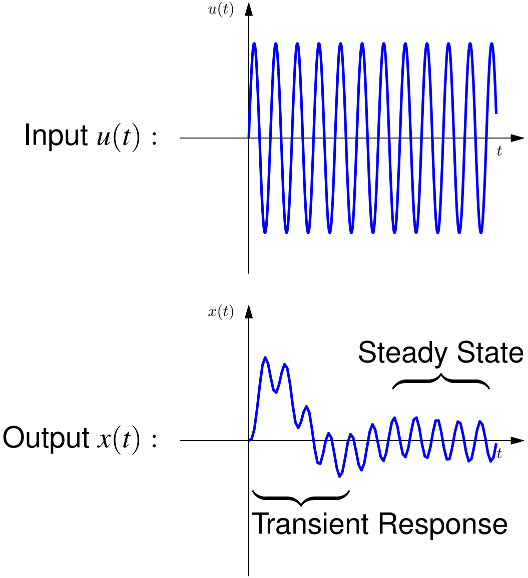
Department of Electrical Engineering
Colorado School of Mines

Fall 2020

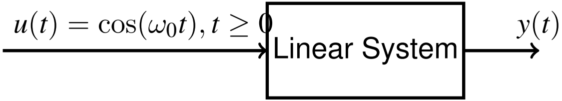
[2] Developed and edited by Tyrone Vincent and Kathryn Johnson, Colorado School of Mines, with contributions from Salman Mohagheghi, Chris Coulston, Kevin Moore, CSM and Matt Kupilik, University of Alaska, Anchorage

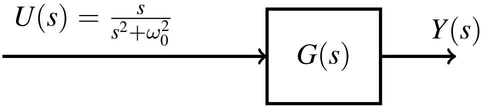
System Response to Sinusoidal Input





Step 1: Multiply $G(s)$ by the Laplace Transform of a cosine





$$Y(s) = G(s)\frac{s}{s^2+\omega_0^2}$$

Step 2: Write partial fraction expansion, splitting poles at $\pm j\omega_0$

- Let $p_1, p_2, \cdots, p_n$ be the poles of $G(s)$ (assume simple poles for now, but result also holds if $G(s)$ has repeated poles)

$$Y(s) = G(s)\frac{s}{s^2+\omega_0^2} = \frac{A}{s-j\omega_0} + \frac{B}{s+j\omega_0} + \frac{C}{s-p_1} + \frac{D}{s-p_2} + \cdots$$

- By residue formula

$$A = \cancel{(s-j\omega_0)}G(s)\frac{s}{\cancel{(s-j\omega_0)}(s+j\omega_0)}\bigg|_{s=j\omega_0} = G(j\omega_0)\frac{1}{2}$$

$$B = \cancel{(s+j\omega_0)}G(s)\frac{s}{(s-j\omega_0)\cancel{(s+j\omega_0)}}\bigg|_{s=-j\omega_0} = G(-j\omega_0)\frac{1}{2}$$

Step 3: Find Inverse Laplace Transform

- Since

$$Y(s) = \frac{G(j\omega_0)\frac{1}{2}}{s - j\omega_0} + \frac{G(-j\omega_0)\frac{1}{2}}{s + j\omega_0} + \frac{C}{s - p_1} + \frac{D}{s - p_2} + \cdots$$

- Using $e^{-at}u \xleftrightarrow{\mathcal{L}} \frac{1}{a+s}$ the time response is

$$y(t) = \left(G(j\omega_0)\frac{1}{2}e^{j\omega_0 t} + G(-j\omega_0)\frac{1}{2}e^{-j\omega_0 t} + Ce^{p_1 t} + De^{p_2 t} \right) u(t)$$

Step 4: Use property of stable poles

- If $\text{Re}(p_i) = -a < 0$,

$$\lim_{t\to\infty} Ce^{p_i t}u(t) = \lim_{t\to\infty} Ce^{\text{Re}(p_i)t}e^{j\text{Im}(p_i)t}u(t) = \lim_{t\to\infty} Ce^{-at}e^{j\text{Im}(p_i)t}u(t) = 0$$

- Thus

$$\lim_{t\to\infty} y(t) = \left(G(j\omega_0)\frac{1}{2}e^{j\omega_0 t} + G(-j\omega_0)\frac{1}{2}e^{-j\omega_0 t}\right)$$

Step 5: Use symmetry property $G(-j\omega) = G(j\omega)^*$ and Euler's formula

- Substitute for $G(-j\omega_0)$

$$\lim_{t\to\infty} y(t) = \left(\frac{1}{2}G(j\omega_0)e^{j\omega_0 t} + \frac{1}{2}G(j\omega_0)^* e^{-j\omega_0 t}\right)$$

- Write in polar form, using $a^* = |a|e^{-j\angle a}$

$$\lim_{t\to\infty} y(t) = \left(\frac{1}{2}|G(j\omega_0)|e^{j\angle G(j\omega_0)}e^{j\omega_0 t} + \frac{1}{2}|G(j\omega_0)|e^{-j\angle G(j\omega_0)}e^{-j\omega_0 t}\right)$$

- Euler's formula: $\frac{1}{2}Ae^{j\theta} + \frac{1}{2}Ae^{-j\theta} = A\cos(\theta)$

$$\lim_{t\to\infty} y(t) = |G(j\omega_0)|\cos(\omega_0 t + \angle G(j\omega_0))$$

Sinusoidal Steady State is the Frequency Response

Theorem

Given a stable system with transfer function $G(s)$, the sinusoidal steady state response is defined by the input/output relationship

$$\boxed{\begin{aligned} u(t) &= A\cos(\omega_0 t + \theta), \\ y_{ss}(t) &= |G(j\omega_0)| A\cos\left(\omega_0 t + \theta + \angle G(j\omega_0)\right). \end{aligned}}$$

Definition

$G(j\omega)$ is the *frequency response function* of the system with transfer function $G(s)$.