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Problem 1

Door 1,2,3 , Contestant Choose Door 1
Host opens door 2

A = door A contains the prize
B = Host to open door 2

$P(A) = \frac{1}{3}$

$P(A|B) = \frac{P(A,B)}{P(B)}$, $P(A,B) = \frac{1}{3} \times \frac{1}{2} = 1/6$

$P(B) = \sum_{A} P(A,B) = \frac{1}{3} \times 1 \times \frac{1}{3} \times 0$
$= 3/6$

$\Rightarrow P(A|B) = \frac{1/6}{3/6} = \frac{1}{3}$

C = Prize is behind door 3

$P(C|B) = \frac{2}{3} \Rightarrow$ switch Selection

Problem 2

Likelihood: $P(x|\pi) = \prod_{j=1}^{j} \left(\pi_j^{\sum_{i=1}^{n} x_{ij}}\right)$

Prior: $\pi \sim P(\theta)$
We assume Dirichlet as Prior and calculate the posterior

$Dir(\pi|\alpha) = \prod_{j=1}^{K} \left(\pi^{\alpha_j - 1}\right) \cdot \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_j)}$

Posterior: $P(\pi|x,\alpha) \propto P(x|\pi) \cdot P(\pi|\alpha)$
$\propto \prod^{j} \left(\pi^{\sum_{i=1}^{n} x_{ij}}\right) \cdot \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_j)} \cdot \pi^{\alpha_j - 1}$

we drop $\frac{\Gamma(\alpha_0)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_j)}$ since it doesn't contain π

$$\Rightarrow P(\pi \mid x, \alpha) \propto \prod_{j=1}^{k} \left(\pi^{\alpha_j + \sum_{i=1}^{n} x_{ij} - 1} \right)$$

By comparison this is proportional to Dirichlet with parameter $Dir(\pi \mid \alpha + \sum_{i=1}^{n} x_i)$ by substitution

Posterior $= P(\pi \mid x, \alpha) =$

$$\frac{\Gamma(\alpha_0 + N)}{\Gamma(\alpha + \sum_{i=1}^{n} x_i) \cdots \Gamma(\alpha_j + \sum_{i=1}^{n} x_i)} \prod_{j=1}^{n} \left(\pi^{\alpha_j + \sum_{i=1}^{n} x_{ij} - 1} \right)$$

which is obviosly Dirichlet with paramter $\alpha_j + \sum x_{ij} - 1$

Problem 3:

Ignore

a) $\lambda \sim Gamma(b,c) \Rightarrow f(\lambda \mid b,c) = \frac{c^b \lambda^{b-1} e^{-c\lambda}}{\Gamma(b)}$

$$L(b,c \mid \lambda) = \frac{c^{nb} \prod_{i=1}^{n} \lambda^{b-1} e^{-c \sum_{i=1}^{n} \lambda}}{\Gamma(b)^n}$$

Assume prior of $p(b,c \mid p,q,r,s) \propto \frac{p^{b-1} e^{-cq}}{\Gamma(b)^r c^{-bs}}$

$$\Rightarrow p(b,c \mid \lambda, p, q, r, s) = p(\lambda \mid b,c) \cdot p(b,c \mid p,q,r,s)$$

$$\propto \frac{c^{nb} \prod_{i=1}^{n} \lambda^{b-1} e^{-c \sum_{i=1}^{n} \lambda}}{\Gamma(b)^n} \times \frac{p^{b-1} e^{-cq}}{\Gamma(b)^r c^{-bs}}$$

$$\propto \frac{\left(p \prod_{i=1}^{n} \lambda\right)^{b-1} e^{-c \sum_{i=1}^{n} \lambda - cq}}{\Gamma(b)^{r+n} c^{-bs-nb}} \propto \frac{\left(p \prod_{i=1}^{n} \lambda\right)^{b-1} e^{-c(\sum \lambda + q)}}{\Gamma(b)^{r+n} c^{-b(s+n)}}$$

$P(\mu, \lambda | x) = P(x|\mu, \lambda) \cdot P(\mu, \lambda)$

Likelihood $= P(x|\mu,\lambda) = \prod_{i=1}^{n} P(x_i|\mu,\lambda)$

Note: $\sigma^2 = 1/\lambda$

$$\Rightarrow \prod_{i=1}^{n} P(x_i|\mu,\lambda) \propto \prod_{i=1}^{n} \left\{ \frac{1}{\sqrt{1/\lambda}} \cdot \exp\left\{ -\frac{\lambda}{2}(x_i-\mu)^2 \right\} \right\}$$

$$\propto \sqrt{\lambda}^{\,n} \cdot \exp\left\{ -\frac{\lambda}{2} \sum_{i=1}^{n} (x_i - \mu)^2 \right\}$$

$$\propto \sqrt{\lambda}^{\,n} \cdot \exp\left\{ -\frac{\lambda}{2} \sum_{i=1}^{n} \left[(x_i - \bar{x} + \bar{x} - \mu)^2 \right] \right\}$$

$$\propto \sqrt{\lambda}^{\,n} \cdot \exp\left\{ -\frac{\lambda}{2} \sum_{i=1}^{n} \left[(x_i - \bar{x})^2 + (\bar{x} - \bar{\mu}) \right] \right\}$$

$$\propto \lambda^{n} \cdot \exp\left\{ -\frac{\lambda}{2} \left(\sum_{i=1}^{n} (x_i - \bar{x})^2 + n(\bar{x} - \bar{x}) \right.\right.$$

Conjugate prior is normal-Gamma (μ_0, λ_0, b, c)

$$f(\mu, \lambda | \mu_0, \lambda_0, b, c) = \frac{c^b \sqrt{\lambda_0}}{\Gamma(b)\sqrt{2\pi}} \cdot \lambda^{b-\frac{1}{2}} \cdot e^{-c\lambda} \cdot e^{-\frac{\lambda_0 \lambda (\mu}{2}}$$

$$\propto \lambda^{b-\frac{1}{2}} \cdot e^{-c\lambda} \cdot e^{-\frac{\lambda_0 \lambda (\mu - \mu_0)}{2}}$$

Posterior:

$$P(\mu, \lambda | x) \propto \sqrt{\lambda} \cdot \exp\left\{ -\frac{\lambda}{2} \left(\sum (x_i - \bar{x})^2 + n(\bar{x} - \bar{\mu}) \right) \right\} \cdot \lambda^{b-\frac{1}{2}} \cdot e^{-c\lambda} \cdot \exp\{$$

$$\propto \lambda^{n/2 + b - \frac{1}{2}} \cdot e^{-\frac{\lambda}{2}\sum(x_i - \bar{x})^2 - c\lambda} \cdot e^{-\frac{n\lambda}{2}(\bar{x} - \bar{\mu})^2 + \lambda_0 \lambda \frac{(\mu - \mu_0)^2}{2}}$$

$$\propto \lambda^{n/2 + b - \frac{1}{2}} \; e^{-\lambda \left[\frac{1}{2} \sum (x_i - \bar{x}) + c \right]} \cdot e^{-\frac{\lambda}{2} \left[(\bar{x} - \bar{\mu})^2 + \lambda_0 (\mu - \mu_0) \right.}$$

$$\propto \lambda^{n/2+b-\frac{1}{2}} \cdot \exp\left[-\lambda\left(\frac{1}{2}\sum_{i=1}^{n}(x_i-\bar{x})+c\right)\right]\cdot\exp\left[-\frac{\lambda}{2}(\lambda_0+n)\cdot\left(\mu-\frac{\lambda_0\mu_0+n\bar{x}}{\lambda_0+n}+\frac{\lambda_0 n(\bar{x}-\mu_0)}{\lambda_0+n}\right)\right]$$

$$\lambda^{n/2+b-\frac{1}{2}} \cdot \exp\left[-\lambda\left(\frac{1}{2}\sum_{i=1}^{n}(x-\bar{x})^2+c+\frac{\lambda_0 n(\bar{x}-\mu_0)^2}{2\lambda_0+n}\right)\right] \propto \exp\left\{-\frac{\lambda}{2}(\lambda_0+n)\left(\mu-\frac{\lambda_0\mu_0+n\bar{x}}{\lambda_0+n}\right)^2\right]$$

$$\Rightarrow P(\lambda,\mu\mid X) = \text{NormalGam}\left(\frac{\lambda_0\mu_0+n\bar{x}}{\lambda_0+n}, \lambda_0+n, b+\frac{n}{2}, c+\frac{1}{2}\sum x_i-x + \frac{\lambda_0 n(\bar{x}-\mu_0)^2}{\lambda_0+n}\right)$$

b)

$$p(x^*\mid x_1\ldots x_n) = \int_0^\infty\int_{-\infty}^{\infty} P(x^*\mid\mu,\lambda)\,P(\mu,\lambda\mid x\ldots x_n)$$

$$p(x^*\mid\mu_0,\lambda_0,b,c) = \int_0^\infty\int_{-\infty}^{\infty} p(x^*\mid\mu,\lambda)\cdot p(\mu\mid\mu_0,\lambda_0,\lambda)\cdot p(\lambda\mid b,c)\,d\mu\,d\lambda$$

$$= \int_0^\infty p(\lambda\mid b,c)\left[\int p(x^*\mid\mu,\lambda)\cdot p(\mu\mid\mu_0,\lambda_0,\lambda)\,d\mu\right]d\lambda$$

Integrating over μ

$$\int_{-\infty}^{+\infty} p(x^*\mid\mu,\lambda)\cdot p(\mu\mid\mu_0,\lambda_0,\lambda)$$

$$= \int_{-\infty}^{\infty}\frac{\sqrt{\lambda}}{2\pi}e^{\left\{-\frac{1}{2}\lambda(x^*-\mu)^2\right.}\cdot\frac{\sqrt{\lambda_0\lambda}}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\lambda_0\lambda(\mu-\mu_0)^2\right\}d\mu$$

$$= \int_{-\infty}^{\infty}\frac{\sqrt{\lambda}}{\sqrt{2\pi}}\frac{\sqrt{\lambda_0\lambda}}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\lambda\left[(x^*-\mu)^2+\lambda_0(\mu-\mu_0)^2\right]\right\}d\mu$$

for $(x^*-\mu)^2 + \lambda_0(\mu-\mu_0)^2$

$= 2x^* - 2x^*\mu + \mu^2 + \lambda_0\mu_0^2 + 2\lambda_0\mu\lambda_0 + \lambda_0\mu_0^2$

$= \lambda_0\mu^2 + \mu^2 - 2x^*\mu + \lambda_0 2\mu\mu_0 + x^{*2} + \lambda_0\mu_0^2$

$= (\lambda_0+1)\mu^2 - 2\mu(x^* + \mu_0\lambda_0) + x^{*2} + \lambda_0\mu_0^2$

$= (\lambda_0+1)\left[\mu_p^2 - \frac{2\mu(x^*+\mu_0\lambda_0)}{(\lambda_0+1)}\right] + x^{*2} + \lambda_0\mu_0^2$

$= (\lambda_0+1)\left[\mu_p^2 - \frac{2\mu(x^*+\mu_0\lambda_0)}{(\lambda_0+1)} + \frac{(x^*+\mu_0\lambda_0)^2}{(\lambda_0+1)^2} \frac{(x^*+\mu_0\lambda_0)^2}{(\lambda_0+1)^2}\right]$

$+ x^{*2} + \lambda_0\mu_0$

$= (\lambda_0+1)\left[\mu_p^2 - \frac{2\mu(x^*+\mu_0\lambda_0)}{(\lambda_0+1)} + \frac{(x+\mu_0\lambda_0)^2}{(\lambda_0+1)^2}\right] - \frac{(x^*+\mu_0\lambda_0)^2}{(\lambda_0+1)} + x^{*2} + \lambda\mu_0$

$= (\lambda_0+1)\left[\mu_p - \frac{x+\lambda_0\mu_0}{\lambda_0+1}\right]^2 - \frac{x^*+\mu_0\lambda_0}{\lambda_0+1} + x^{*2} + \lambda_0\mu_0$

$\vdots$

$(\lambda_0+1)\left(\mu - \frac{x+\lambda_0\mu_0}{\lambda_0+1}\right) + \frac{-2x^*\lambda_0\mu_0 + \lambda_0 x^2 + \lambda_0\mu_0}{\lambda_0+1}$

$(\lambda_0+1)\left(\mu - \frac{x^*+\lambda_0\mu_0}{\lambda_0+1}\right) + \frac{\lambda_0(x^*-\mu_0)^2}{\lambda_0+1}$

Substitute with original expression

$= \int_{-\infty}^{\infty} \frac{\sqrt{\lambda}}{\sqrt{2\pi}} \frac{\sqrt{\lambda_0\lambda}}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\lambda\left[(\lambda_0+1)\left(\mu - \frac{x^*+\lambda_0\mu_0}{\lambda_0+1}\right) + \frac{\lambda_0(x-\mu_0)}{\lambda_0+1}\right]^2\right\} d\mu$

$= \int_{-\infty} d \frac{\sqrt{\lambda}}{\sqrt{2\pi}} \frac{\sqrt{\lambda_0\lambda}}{2\pi} \exp\left\{-\frac{1}{2}\lambda\left[(\lambda_0+1)\left(\mu - \frac{x^*+\lambda_0\mu_0}{\lambda_0+1}\right)\right]\right\} \cdot \exp\left\{-\frac{1}{2}\lambda\left(\frac{\lambda_0(x}{(\lambda_0+1)}\right.\right.$

$d\mu$

~~Remove~~ take constant outside the integral

$$\frac{\sqrt{\lambda_0\lambda}}{(2\pi(\lambda_0+1)}\exp\left\{-\frac{1}{2}\lambda\left(\frac{\lambda_0(x^*-\mu_0)^2}{(\lambda_0+1)}\right)\right\}\underbrace{\int_{-\infty}^{\infty}\frac{\sqrt{\lambda(\lambda_0+1)}}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\lambda(\lambda_0+1)\left(\mu-\left(\frac{x^*+\lambda_0\mu_0}{\lambda_0+1}\right)\right)^2\right\}d\mu}_{=1\text{, since normal}}$$

$$\Rightarrow \frac{\sqrt{\lambda_0\lambda}}{2\pi(\lambda_0+1)}\exp\left\{-\frac{1}{2}\lambda\frac{(\lambda_0(x^*-\mu_0)^2}{(\lambda_0+1)}\right\}$$

Going back to $p(x^*|x_1..x_n)$

$$= \int p(\lambda|b,c)\cdot p(\mu|\mu_0,\lambda_0,\lambda)$$

$$= p(\lambda|b,c)\cdot\frac{\sqrt{\lambda_0\lambda}}{2\pi(\lambda_0+1)}\exp\left\{-\frac{1}{2}\frac{\lambda(\lambda_0(x^*-\mu_0)^2}{(\lambda_0+1)}\right\}$$

since $\lambda\sim\text{Gamma}(b,c)$

$$\Rightarrow \int\frac{c^b}{\Gamma(b)}\lambda^{b-1}\exp\{-c\lambda\}\cdot\frac{\sqrt{\lambda_0\lambda}}{\sqrt{2\pi(\lambda_0+1)}}\exp\left\{-\frac{1}{2}\frac{\lambda(\lambda_0(x-\mu_0)^2}{(\lambda_0+1)}\right\}$$

$$\Rightarrow\int\frac{c^b}{\Gamma(b)}\frac{\sqrt{\lambda_0}}{\sqrt{2\pi(\lambda_0+1)}}\cdot\lambda^{b+1/2}\cdot\exp\left\{-c\lambda-\frac{1}{2}\frac{\lambda(\lambda_0(x^*-\mu_0)^2}{\lambda_0+1}\right\}$$

let $b' = b+\frac{1}{2}$

$$c' = \left(c+\frac{\lambda_0}{2}\left(\frac{(x^*-\mu_0)^2}{\lambda_0+1}\right)\right)$$

$$\Rightarrow\int\frac{c^b}{\Gamma(b)}\frac{\sqrt{\lambda_0}}{\sqrt{2\pi(\lambda_0+1)}}\lambda^{b'-1}\exp\{-\lambda c'\}\,d\lambda$$

$$=\frac{c^b}{\Gamma(b)}\frac{\sqrt{\lambda_0}}{\sqrt{2\pi(\lambda_0+1)}}\int\lambda^{b'-1}\exp\{-\lambda c'\}$$

$$= \frac{c^b}{\Gamma(b)} \frac{\sqrt{\lambda_0}}{\sqrt{2\pi(\lambda_0+1)}} \int_0 \frac{c'^{b'} \cdot \Gamma(b')}{c'^{b'} \cdot \Gamma(b')} \cdot \lambda^{b'-1} \cdot \exp\{-\lambda c'\}$$

$$= \frac{\Gamma(b')}{c'^{b'}} \cdot \frac{c^b}{\Gamma(b)} \frac{\sqrt{\lambda_0}}{\sqrt{2\pi(\lambda_0+1)}} \int \underbrace{\frac{c'^{b'}}{\Gamma(b')} \cdot \lambda^{b'-1} \exp\{-\lambda c'\}}_{=1 \text{ Since Gamma}}$$

$$\Rightarrow \frac{\Gamma(b')}{c'^{b'}} \cdot \frac{c^b}{\Gamma(b)} \frac{\sqrt{\lambda_0}}{2\pi(\lambda_0+1)}$$

$$\Rightarrow \frac{\Gamma(b+1/2)}{\Gamma(b)} \cdot \frac{c^b}{\left(c + \frac{\lambda_0}{2}\left(\frac{(\hat{x}-\mu_0)^2}{\lambda_0+1}\right)\right)^{b+1/2}}$$

$\Rightarrow$ Student's t distribution.

Problem 4:

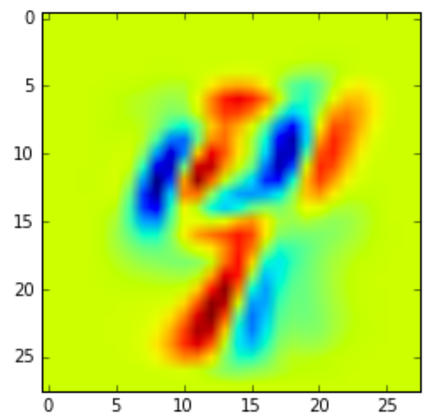
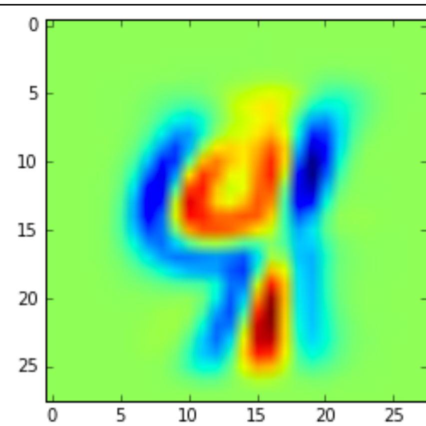
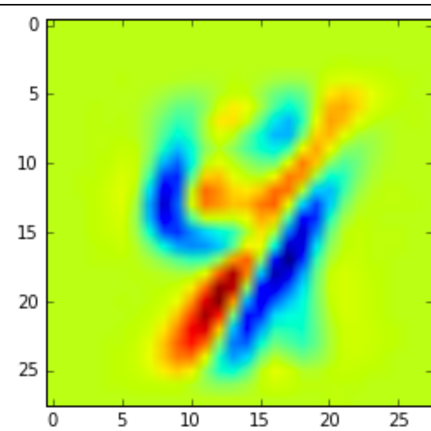
Part a

- Code submitted

Part b

```
Confusion Matrix:
[[930  52]
 [ 82 927]]
Accuracy:  0.932697137117
```

Part c – Misclassified Images

0 0.264645 1 0.735355	
0 0.480633 1 0.519367	
0 0.273412 1 0.726588	

d- Ambiguous Images

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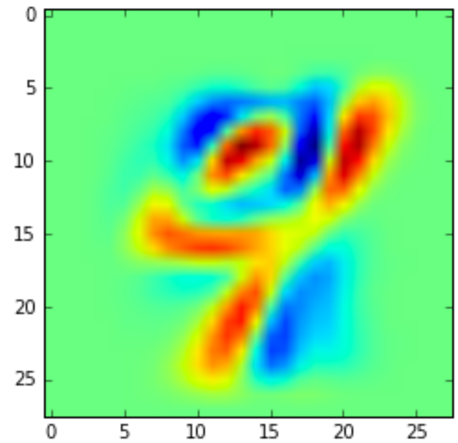
diff 0.000550</td><td>
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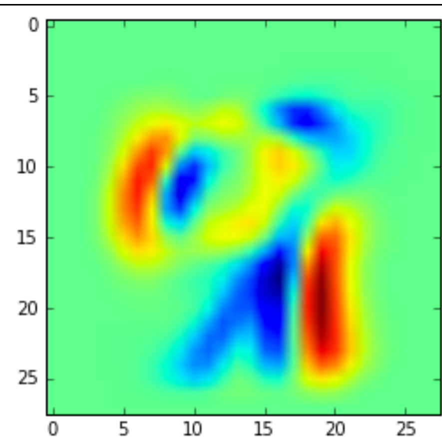
diff 0.004616</td><td>
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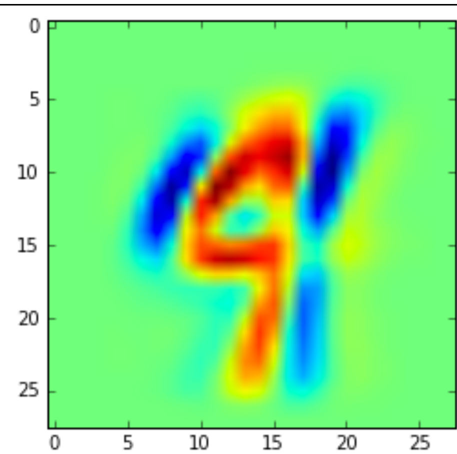
diff 0.007185</td><td>
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