Binomial pricing formula

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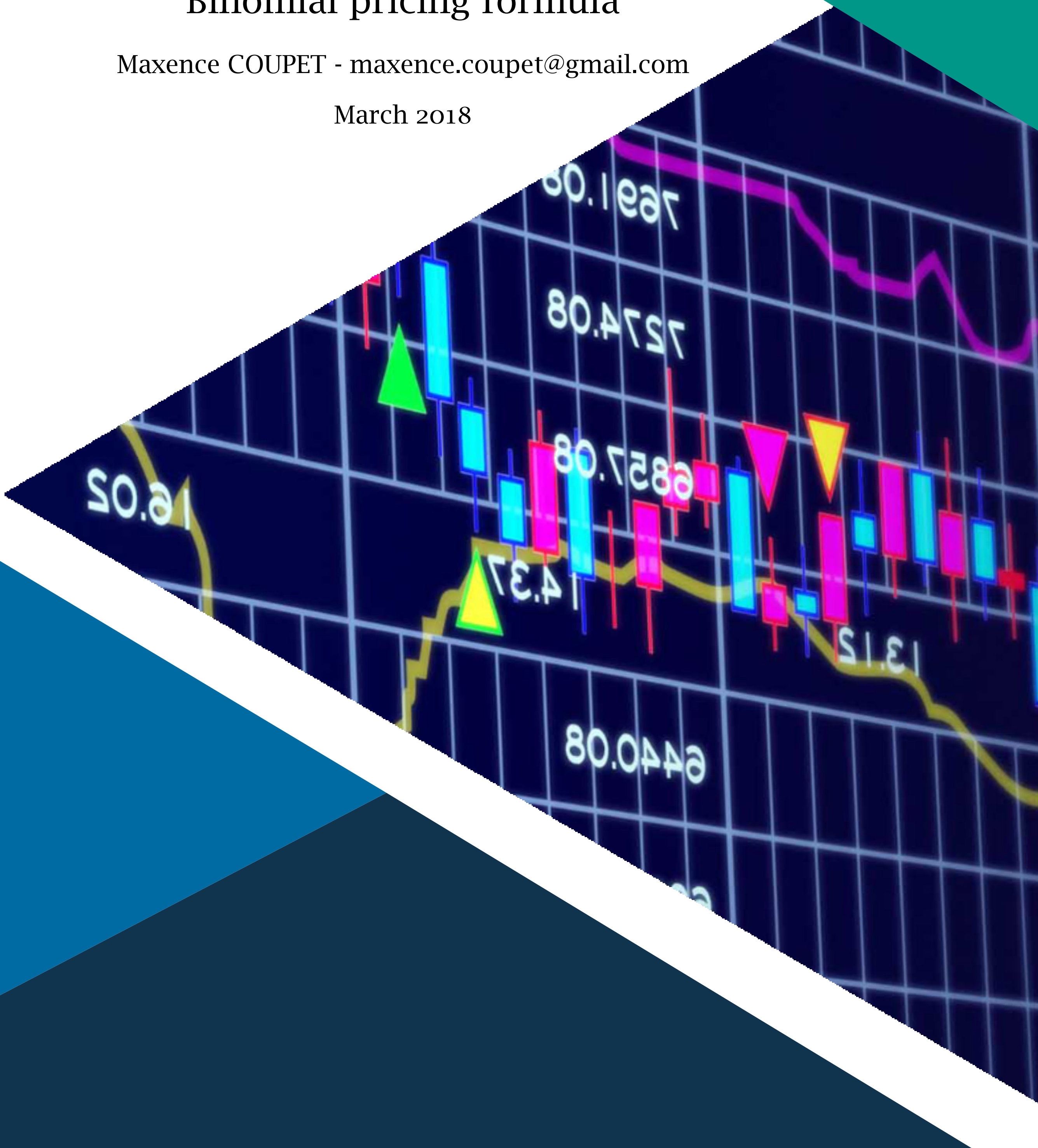

The main idea behind binomial pricing for a derivative is to consider that at one point in time, the underlying price can only moove in two direction : up or down. We will use the following notation : S_0 is the underlying price, T is the maturirty of the derivative, f is the fair price of the derivative, f_u is the price of the derivative after an upward movement and f_d is the price of the derivative after a downward movement, r is the risk-free rate, u is the muliplicative rate of the underlying for an upward movement (the price after an upward movement is then S_0u), d is the multiplicative rate of the underlying for an downward movement and p the probability of an upward movement.

For this document, we will only consider the one period model, which means that we consider that at time $t = 0$, the underlying price is S_0 and at time $t = T$ the underlying price can only be S_0u or S_0d. Figure 1 summarize the one step binomial model. Even if the one step binomial model is too simple to give a good approximation of the price of a derivative, a multi steps binomial method is very accurate and one can even show the the price of the mutli steps binomial method converges to the Black-Scholes price. Thus the binomial pricing method is a discret version of the Black-Scholes formula and one can see a multi steps binomial method as a succession of one step binomial method, therefore the formula for a one step binomial method can be generalized.

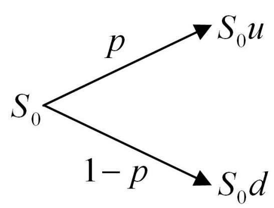

FIGURE 1 – One-step binomial tree pricing method

Let's consider a portfolio with a short position in the derivative and a long position in Δ underlyings (eventually, if $\Delta < 0$, we will take a short postion of $|\Delta|$ underlyings). The value of our portfolio will thus be $\Delta S_0 - f$ at $t = 0$, $\Delta S_0 u - f_u$ at time $t = T$ after an upward movement and $\Delta S_0 d - f_d$ at time $t = T$ after an downward movement.

We want our portfolio to be risk-free, which means that we want to be sure of his value a time $t = T$, so the value after an upward and downward movement should equal and we have :

$$\Delta S_0 u - f_u = \Delta S_0 d - f_d \Rightarrow \Delta = \frac{f_u - f_d}{S_0(u - d)} \tag{1}$$

We now have an analytic formula for the Δ. Since our portfolio is risk-free, its return should not be higher or lower than the risk-free rate, otherwise there would be an arbitrage to perform. The arbitrage-free condition allow use to write that the value of the portfolio at time $t = 0$ should be equal to the value of the portfolio at time $t = T$ (the upward or dorward value, remember that they are equal), discounted with the risk-free rate. We then have :

$$\Delta S_0 - f = e^{-rT}(\Delta S_0 u - f_u) \tag{2}$$

By combining this equation to the analytic formula for Δ, we can now proove the binomial model pricing formula :

$$\begin{aligned} f &= \Delta S_0 - e^{-rT}\Delta S_0 u + e^{-rT} f_u \\ &= \Delta S_0 \left(1 - ue^{-rT}\right) + e^{-rT} f_u \\ &= S_0 \frac{f_u - f_d}{S_0(u-d)} \left(1 - ue^{-rT}\right) + \frac{e^{-rT} f_u (u-d)}{u-d} \\ &= \frac{f_u \left(1 - ue^{-rT} + ue^{-rT} - de^{-rT}\right) + f_d \left(ue^{-rT} - 1\right)}{u-d} \\ &= f_u \frac{1 - de^{-rT}}{u-d} + f_d \frac{ue^{-rT} - 1}{u-d} \\ &= e^{-rT} \left(f_u \frac{e^{rT} - d}{u-d} + f_d \frac{u - e^{rT}}{u-d} \right) \end{aligned} \tag{3}$$

Let $p = \frac{e^{rT}-d}{u-d}$, we also note that $1 - p = \frac{u-e^{rT}}{u-d}$. Therefore, we have :

$$f = e^{-rT}\left[p f_u + (1-p) f_d\right] \tag{4}$$

Now the only unknown parameters are u and d. We have an arbitrage-free condition on those parameters :

$$0 < d < e^{rT} < u \tag{5}$$

Otherwise, one could take advantage of the market with an arbitrage and the pricing will not be a good representation of the market.

We will define u and d as a function of the volatility of the underlying. It is important to remember that the volatility of the return of an underlying following a log-normal law (the geometric Brownian motion) is $\sigma\sqrt{T}$ and that the formula giving the variance for a random variable X is : $Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$.

Let's consider consider the one-step binomial model, on the period of length T, we have a probability p that the return will be $u - 1$ and a probability $1 - p$ that the return will be $d - 1$. The variance is therfore :

$$Var = p(u-1)^2 + (1-p)(d-1)^2 - \left[p(u-1) + (1-p)(d-1)\right]^2$$

Combining what we know about the volatility of the returns with the above formula will lead to :

$$pu^2 + (1-p)d^2 - \left[pu + (1-p)d\right]^2 = \sigma^2\sqrt{T}$$

Now we use the explicit formula for p in the previous equation and present the equation that any u and d must respect :

$$e^{rT}(u+d) - ud - e^{2rT} = \sigma^2\sqrt{T} \tag{6}$$

Cox, Ross and Rubenstein proposed one solution for this equation, which is recombining ($ud = 1$) and very popular :

$$\begin{aligned} u &= e^{\sigma\sqrt{T}} \\ d &= e^{-\sigma\sqrt{T}} \end{aligned} \tag{7}$$