Black-Scholes Formula

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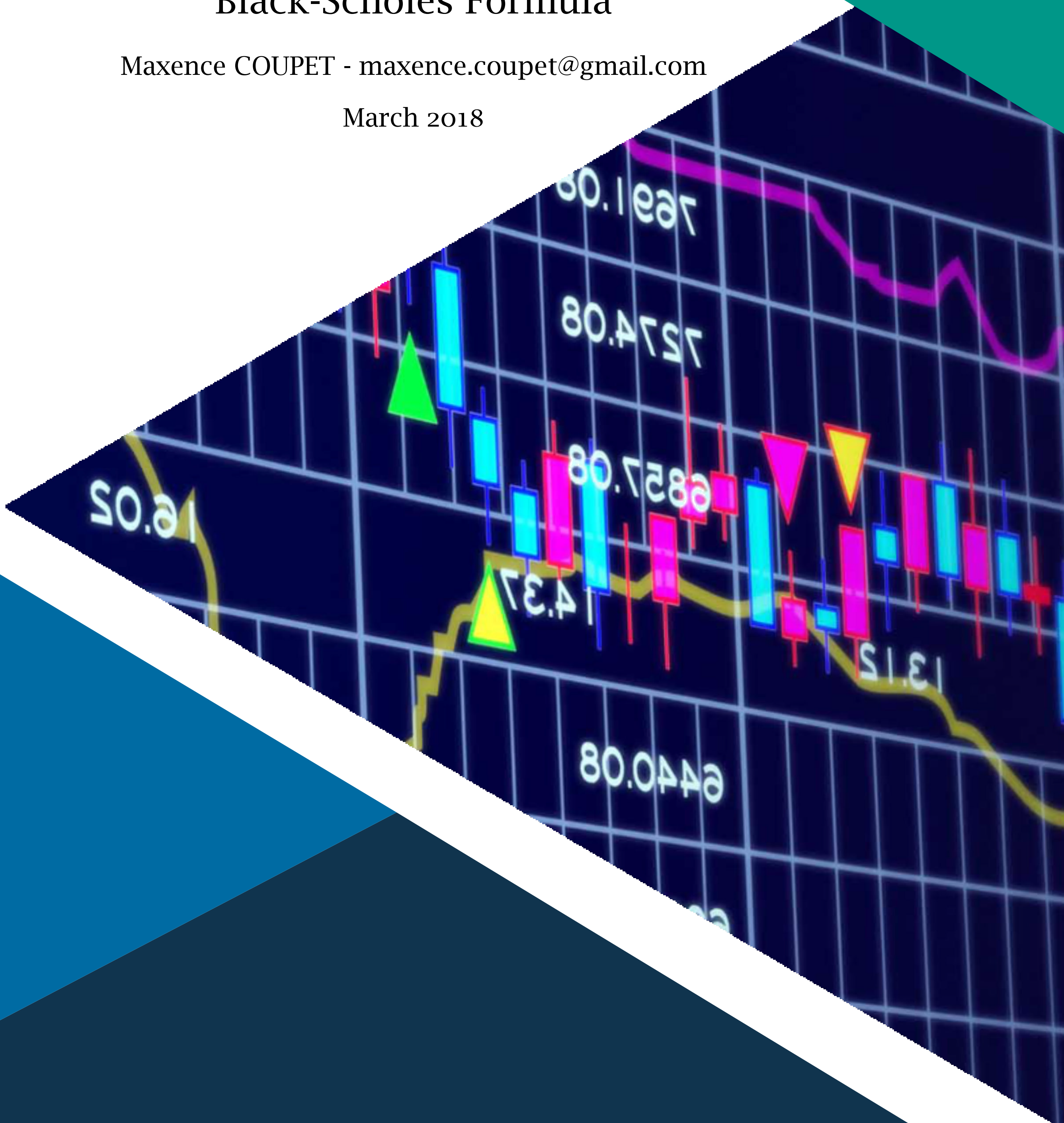
March 2018



We want to compute the the fair price for an european call option on a stock which doesn't pay dividends.

1 Hypothesis

Let the interest rate r and the volatility $\sigma > 0$ be constant.

Let the stock price be a geometric Brownian motion (i.e. stock price returns follow a log-normal distribution) :

$$\forall t > 0, \quad S_t = S_0 \cdot e^{(r-\frac{\sigma^2}{2})t+\sigma \cdot W_t}$$

whith the initial price $S_0 > 0$ and $(W_t)_{t>0}$ a Brownian motion, thus $W_t \sim \mathcal{N}(0,t)$.

Let $K > 0$ and $T > 0$.

2 Calculation

We will compute $\mathbb{E}[e^{-rT}(S_T - K)^+]$ using stochastic integral with respect to the Brownian motion $(W_t)_{t>0}$.

$$\begin{aligned}
\mathbb{E}\left[e^{-rT}(S_T - K)^+\right] &= e^{-rT}\int_{\{S_T>K\}}^{\infty}(S_T - K)\cdot\frac{e^{-\frac{x^2}{2T}}}{\sqrt{2\Pi T}}dx \\
&= e^{-rT}\int_{\{S_T>K\}}^{\infty}(S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma x} - K)\cdot\frac{e^{-\frac{x^2}{2T}}}{\sqrt{2\Pi T}}dx
\end{aligned} \tag{1}$$

In order to determine the integral bound, we make to following caculations :

$$\begin{aligned}
S_T > K &\Leftrightarrow ln\,(S_T) > ln\,(K) \\
&\Leftrightarrow ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T > \sigma \\
&\Leftrightarrow W_T > \frac{ln\left(\frac{K}{S_0}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma}
\end{aligned} \tag{2}$$

We can now replace the bounds :

$$\mathbb{E}\left[e^{-rT}(S_T - K)^+\right] = e^{-rT}\int_{\frac{ln\left(\frac{K}{S_0}\right)-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma}}^{\infty}(S_T - K)\cdot\frac{e^{-\frac{x^2}{2T}}}{\sqrt{2\Pi T}}dx \tag{3}$$

Let's use the change of variable $y = \frac{x}{\sqrt{T}} \Rightarrow dy = \frac{dx}{\sqrt{T}}$:

$$\begin{aligned}
\mathbb{E}\left[e^{-rT}(S_T - K)^+\right] &= e^{-rT}\int_{\frac{ln\left(\frac{K}{S_0}\right)-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}}^{\infty} S_0 \cdot e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}y} \cdot \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\Pi}}dy \\
&\quad - Ke^{-rT}\int_{\frac{ln\left(\frac{K}{S_0}\right)-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}}^{\infty} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\Pi}}dy \\
&= S_0\int_{\frac{ln\left(\frac{K}{S_0}\right)-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}}^{\infty} \frac{1}{\sqrt{2\Pi}}e^{-\frac{1}{2}\left(y^2-2\sigma\sqrt{T}y+\sigma^2T\right)}dy \\
&\quad - Ke^{-rT}\int_{\frac{ln\left(\frac{K}{S_0}\right)-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}}^{\infty} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\Pi}}dy \\
&= S_0\int_{\frac{ln\left(\frac{K}{S_0}\right)-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}}^{\infty} \frac{1}{\sqrt{2\Pi}}e^{-\frac{1}{2}\left(y-\sigma\sqrt{T}\right)^2}dy \\
&\quad - Ke^{-rT}\int_{\frac{ln\left(\frac{K}{S_0}\right)-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}}^{\infty} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\Pi}}dy
\end{aligned} \tag{4}$$

We now have to remark that :

$$\begin{aligned}
\int_a^{\infty} \frac{1}{\sqrt{2\Pi}}e^{-\frac{y^2}{2}}dy &= \int_{-\infty}^{-a} \frac{1}{\sqrt{2\Pi}}e^{-\frac{y^2}{2}}dy \\
&= N(-a)
\end{aligned} \tag{5}$$

With $x \mapsto N(x)$ the cumulative standard normal distribution fonction. We have :

$$\begin{aligned}
\mathbb{E}\left[e^{-rT}(S_T - K)^+\right] &= S_0\int_{\frac{ln\left(\frac{K}{S_0}\right)-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}}^{\infty} \frac{1}{\sqrt{2\Pi}}e^{-\frac{1}{2}\left(y-\sigma\sqrt{T}\right)^2}dy \\
&\quad - Ke^{-rT}\int_{-\infty}^{\frac{ln\left(\frac{S_0}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\Pi}}dy
\end{aligned} \tag{6}$$

Let's use the change of variable $\xi = y - \sigma\sqrt{T} \Rightarrow d\xi = dy$:

$$\begin{aligned}
\mathbb{E}\left[e^{-rT}(S_T - K)^+\right] &= S_0\int_{\frac{ln\left(\frac{K}{S_0}\right)-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}-\sigma\sqrt{T}}^{\infty} \frac{1}{\sqrt{2\Pi}}e^{-\frac{\xi^2}{2}}d\xi \\
&\quad - Ke^{-rT}N\left(\frac{ln\left(\frac{S_0}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)
\end{aligned} \tag{7}$$

Using again the transformation with the bounds of the integral, we have :

$$\begin{aligned}\mathbb{E}\left[e^{-rT}(S_T - K)^+\right] &= S_0 \int_{-\infty}^{\frac{ln\left(\frac{S_0}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}} \frac{1}{\sqrt{2\Pi}} e^{-\frac{\xi^2}{2}} d\xi \\ &\qquad - Ke^{-rT} N\left(\frac{ln\left(\frac{S_0}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) \\ &= S_0 N\left(\frac{ln\left(\frac{S_0}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) \\ &\qquad - Ke^{-rT} N\left(\frac{ln\left(\frac{S_0}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)\end{aligned} \tag{8}$$

Let $d_1 = \frac{ln\left(\frac{S_0}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$ and $d_2 = \frac{ln\left(\frac{S_0}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$, we now have the Black-Scholes formula for an european call on a stock paying no dividends :

$$\mathbb{E}\left[e^{-rT}(S_T - K)^+\right] = S_0 N(d_1) - Ke^{-rT} N(d_2) \tag{9}$$

□