Taylor series

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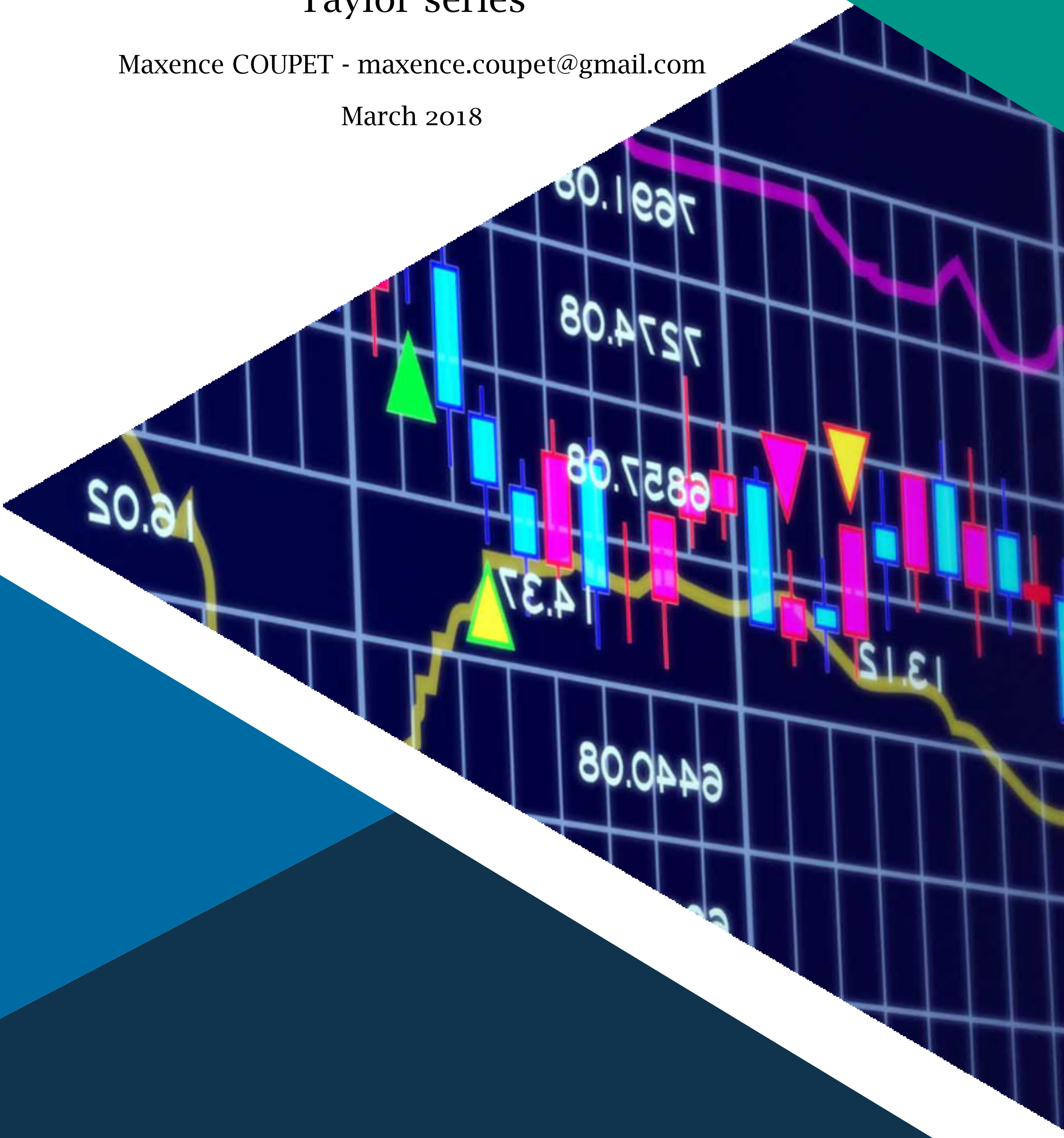

1 Taylor's theorem

Let I be an interval, $a \in I$ and f a function which is $n > 1$ times derivable.

$$\begin{aligned}\forall x \in I, \quad f(x) &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \\ &\qquad \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x) \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n(x)\end{aligned} \tag{1}$$

By using a change of variable, we also have :

$$\begin{aligned}f(a+h) &= f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f^{(2)}(a) + \cdots + \\ &\qquad \frac{h^n}{n!}f^{(n)}(a) + R_n(h) \\ &= \sum_{k=0}^{n} \frac{h^k}{k!}f^{(k)}(a) + R_n(h)\end{aligned} \tag{2}$$

2 Usual functions

2.1 Exponential

$$\begin{aligned} \forall x \in \mathbb{R}, \quad e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \end{aligned} \tag{3}$$

2.2 Logarithm

$$\begin{aligned} \forall x \in]-1,1[, \quad ln(1-x) &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots \\ &= -\sum_{n=1}^{\infty} \frac{x^n}{n} \end{aligned} \tag{4}$$

$$\begin{aligned} \forall x \in]-1,1[, \quad ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \end{aligned} \tag{5}$$

2.3 Geometric series

$$\begin{aligned} \forall x \in]-1,1[, \quad \frac{1}{1-x} &= 1 + x + x^2 + x^3 + \ldots \\ &= \sum_{n=0}^{\infty} x^n \end{aligned} \tag{6}$$

$$\begin{aligned} \forall x \in]-1,1[, \quad \frac{1}{(1-x)^2} &= 1 + 2x + 3x^2 + 4x^3 + \ldots \\ &= \sum_{n=1}^{\infty} n x^{n-1} \end{aligned} \tag{7}$$

$$\begin{aligned} \forall x \in]-1,1[, \quad \frac{1}{(1-x)^3} &= 1 + 3x + 4x^2 + 10x^3 + \ldots \\ &= \sum_{n=2}^{\infty} \frac{(n-1)n}{2} x^{n-2} \end{aligned} \tag{8}$$

2.4 Binomial series

$$\forall x \in]-1,1[, \quad (1-x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \ldots = \sum_{n=0}^{\infty}\left(\prod_{k=1}^{n}\frac{\alpha-k-1}{k}\right)x^n = \sum_{n=0}^{\infty}\binom{\alpha}{n}x^n \tag{9}$$

Two particular values of α are interesting : $\alpha = \frac{1}{2}$ and $\alpha = -\frac{1}{2}$, which gives use the square root function and its inverse :

$$\forall x \in]-1,1[, \quad \sqrt{1-x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \ldots \tag{10}$$

$$\forall x \in]-1,1[, \quad \frac{1}{\sqrt{1-x}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \ldots \tag{11}$$

2.5 Trigonometric functions

$$\forall x \in \mathbb{R}, \quad sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} \tag{12}$$

$$\forall x \in \mathbb{R}, \quad cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \ldots = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} \tag{13}$$

$$\forall x \in \left]-\frac{\Pi}{2},\frac{\Pi}{2}\right[, \quad tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots \tag{14}$$

$$\forall x \in]-1,1[, \quad arcsin(x) = x + \frac{x^3}{6} + \frac{3x^5}{40} + \ldots = \sum_{n=0}^{\infty}\frac{(2n)!}{4^n(n!)^2(2n+1)}x^{2n+1} \tag{15}$$

$$\begin{aligned}\forall x \in]-1,1[, \quad arccos(x) &= \frac{\Pi}{2} - arcsin(x) \\ &= \frac{\Pi}{2} - x - \frac{x^3}{6} - \frac{3x^5}{40} - \ldots \\ &= \frac{\Pi}{2} - \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1}\end{aligned} \tag{16}$$

$$\begin{aligned}\forall x \in]-1,1[, \quad arctan(x) &= x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}\end{aligned} \tag{17}$$

2.6 Hyperbolic functions

$$\begin{aligned}\forall x \in \mathbb{R}, \quad sinh(x) &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \\ &= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}\end{aligned} \tag{18}$$

$$\begin{aligned}\forall x \in \mathbb{R}, \quad cosh(x) &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \\ &= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}\end{aligned} \tag{19}$$

$$\forall x \in \mathbb{R}, \quad tanh(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} - \ldots \tag{20}$$