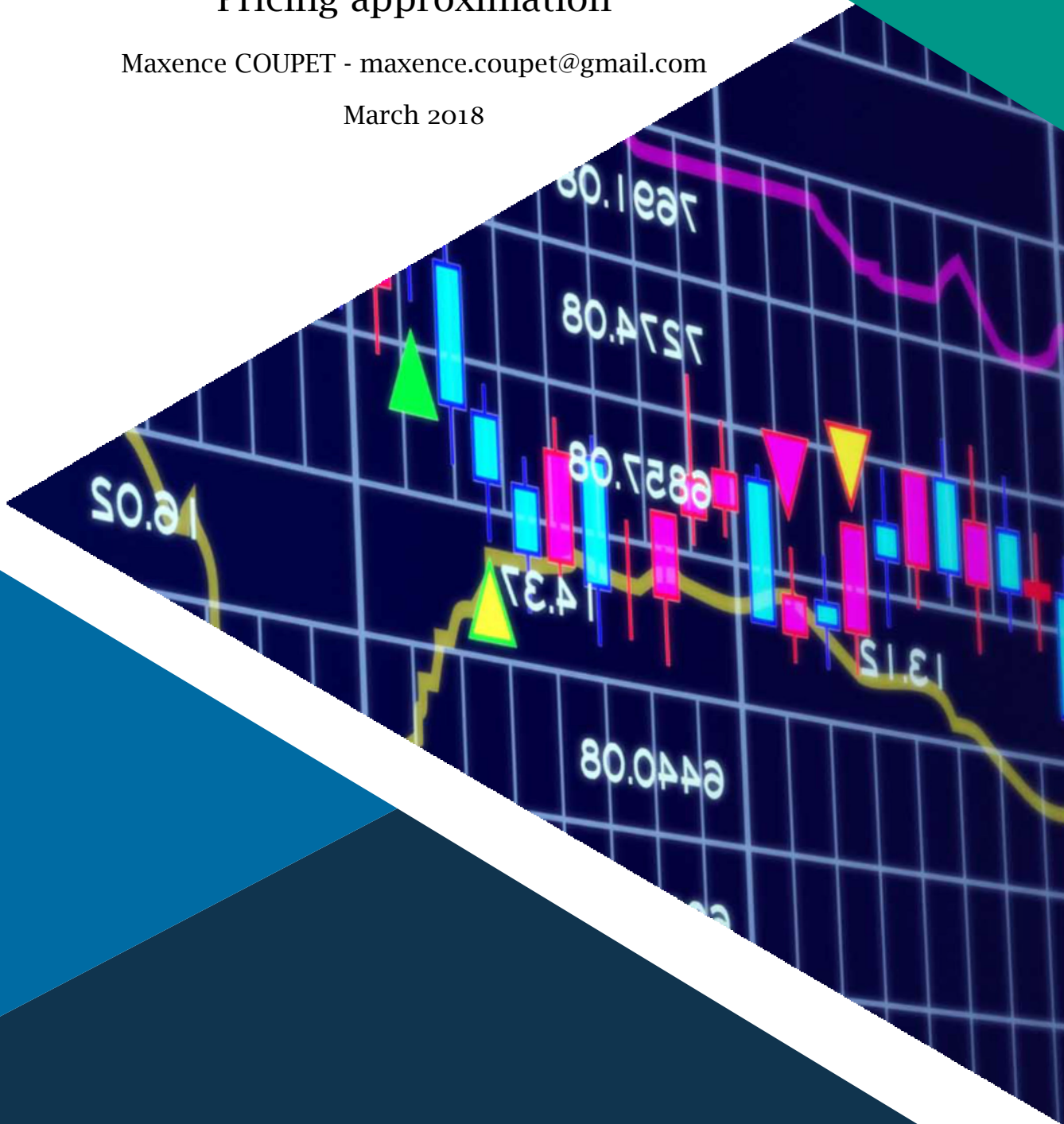


Pricing approximation

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1 Option price

A powerfull tool in finance is the approximation of an option price which allow one to mentally compute an option price under certain conditions. As usuals, we will use the following notation : S_0 is the underlying price, r is the risk-free interest rate, T the maturity of the option, σ is the volatility of the underlying and K is the strike price of the option.

Remember that the price of an european call is given by :

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

with,

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}},$$

$$d_2 = d_1 - \sigma\sqrt{T},$$

and $N(\cdot)$ the standard normal cumulative function which is defined by the following formula : $N(x) = \frac{1}{2} \left(1 + \frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2} dt\right)$.

Let's consider an ATM forward option on an underlying paying no dividend, which means that the strike is equal to the forward price : $K = S_0 e^{rT}$. If we consider the call-put parity relation we can show that in this case the price of a call is equal to the price of a put :

$$\begin{aligned} c - p &= S_0 - Ke^{-rT} \\ &= S_0 - S_0 e^{rT} e^{-rT} \\ &= 0 \end{aligned} \tag{1}$$

We will then only reffer to calls in the following document, but the relations are also true for puts.

In the case of an ATM forward option, it is easy to show that we have $d_1 = \frac{\sigma\sqrt{T}}{2}$ and $d_2 = -\frac{\sigma\sqrt{T}}{2} = -d_1$. Thus the Black-Scholes formula is now :

$$\begin{aligned} c &= S_0 N\left(\frac{\sigma\sqrt{T}}{2}\right) - S_0 e^{rT} e^{-rT} N\left(-\frac{\sigma\sqrt{T}}{2}\right) \\ &= S_0 \left(N\left(\frac{\sigma\sqrt{T}}{2}\right) - N\left(-\frac{\sigma\sqrt{T}}{2}\right) \right) \\ &= S_0 \left(2N\left(\frac{\sigma\sqrt{T}}{2}\right) - 1 \right) \end{aligned} \tag{2}$$

While this expression is already quite simple, it is not really easy to mentally compute a cumulative normal value. We will then use the Taylor formula to the first order for the standard normal cumulative function :

$$\forall x \in \mathbb{R}, \quad N(x) \approx \frac{1}{2} + \frac{1}{\sqrt{2\pi}}x$$

Another very useful approximation is : $\frac{1}{\sqrt{2\pi}} \approx 0.4$.

By combining all those results, we now have a very simple approximation for the price of an ATM forward call :

$$\begin{aligned} c &\approx S_0 \left(2 \left(\frac{1}{2} + \frac{1}{\sqrt{2\pi}} \frac{\sigma\sqrt{T}}{2} \right) - 1 \right) \\ &\approx S_0 \frac{1}{\sqrt{2\pi}} \sigma \sqrt{T} \\ &\approx 0.4 S_0 \sigma \sqrt{T} \end{aligned} \tag{3}$$

Let's make a simple numerical application with $S_0 = 100$, $\sigma = 20\%$, $T = 1$, $r = 10\%$ and $K = 110$. A discerning reader will note that $110 \neq 100e^{0.10 \times 1}$ so we do not have an exact ATM forward call, but remember that the goal is to make mental calculus and it is most unlikely that one compute an exponential without any computer help. The results of comparison between the Black-Scholes formula and the approximation are in the following table and show that it is quite accurate for a mental calculus on an option that is not exactly an ATM forward.

Theoretical value	Approximation	Error
8.183	8.0	2.22 %

2 Implied volatility approximation

The computation of implied volatility is the process of a time consuming iterative method since the Black-Scholes formula can not be easily inverted. But with the above approximation, it is quite trivial to compute the implied volatility given a market observation of the option price :

$$\begin{aligned} c \approx 0.4S_0\sigma\sqrt{T} \Rightarrow \sigma &\approx \frac{c}{0.4S_0\sqrt{T}} \\ &\approx \frac{2.5c}{S_0\sqrt{T}} \end{aligned} \quad (4)$$

3 Delta

Remember that the formula for the delta of a call is : $\Delta_{call} = N(d_1)$ and for a put it is : $\Delta_{put} = N(d_1) - 1$. By using the Taylor formula to the first order for the standard normal cumulative function, we have :

$$\begin{aligned} \Delta_{call} &\approx 0.2\sigma\sqrt{T} + \frac{1}{2} \\ \Delta_{put} &\approx 0.2\sigma\sqrt{T} - \frac{1}{2} \end{aligned} \quad (5)$$

4 Gamma

The gamma of a call or a put is given by :

$$\Gamma = \frac{N'(d_1)}{\sqrt{2\pi}S_0\sigma\sqrt{T}} = \frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}S_0\sigma\sqrt{T}}$$

But in most cases we have $\frac{d_1^2}{2} \approx 0$ we then have the following approximation for the gamma :

$$\Gamma \approx \frac{0.4}{S_0\sigma\sqrt{T}} \quad (6)$$

5 Vega

The vega of a call or a put is given by :

$$v = S_0\sqrt{T}N'(d_1)$$

Using the same approximation that for the gamma, we have :

$$v \approx 0.4S_0\sqrt{T} \quad (7)$$