§3.2 Finite difference approximations

We showed. $\delta f = f(x+\delta x) - f(x) = f'(x)\delta x + o(\|\delta x\|)$.

1. Finite difference approximations

$f(x+\delta x) - f(x)$ fwd difference

$f(x) - f(x-\delta x)$ backward difference ← have no relation with fwd differentiatio

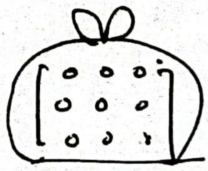
Example. $f(A) = A^2$.

$$\frac{\|\text{approx} - \text{exact}\|}{\|\text{exact}\|} := \text{relative error}.$$

where $\|\cdot\|$ is a norm., like that in the vector case.

2. Norm of a matrix — the simplest one.

$\|x\| = \sqrt{x^T x} = \sqrt{\sum_i x_i^2}$ (L_2 norm).

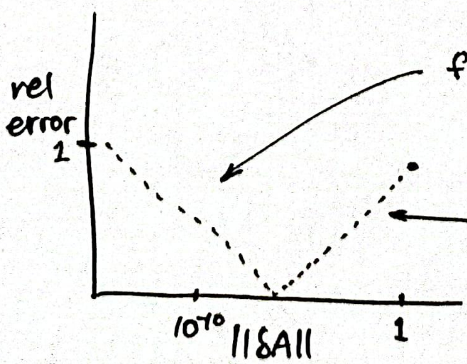


big bag of nu

(Generlized dot product is $B \cdot A = \mathrm{tr}(B^T A)$).

3. Accuracy of finite differences

- plot (relative error – $\|\delta A\|$) in log-log scale.



floating point roundoffs.

First order accuracy.
decr linearly with $\|\delta A\|$.
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What's the ~~ratio~~ portion ~~of the~~ ?

- Order of accuracy.

using Taylor Series, getting 2 order accuracy.

$$f(x+\delta x) = f(x) + f'(x)\,\delta x + \frac{f''(x)}{2!}\delta^2 x + \cdots$$

$o(\|\delta x\|^2)$.
higher-order terms

Then our rel. error $= \dfrac{\| f(x+\delta x) - f(x) - f'(x)\,\delta x \|}{\| f'(x)\,\delta x \|}$

grab off

$= \dfrac{(\text{terms prop. to } \|\delta x\|^2) + o(\|\delta x\|^2)}{\text{proposional to } \|\delta x\|}$.

$=$ (terms proportional to $\|\delta x\|$) $+ o(\|\delta x\|)$.

truncation error (why it is linear).

- Roundoff error. When δx is too small,

  $f(x+\delta x) - f(x)$ get rounded off to 0.

  round to 0.

i.e. $\sin(x)$ at $x=1$, $f'(x) = \dfrac{f(x+\delta x) - f(x)}{\delta x} + O(\delta x)$.

3. Rule of thumb for finite difference

The number of significant digits stored by pc is expred by machine epsilion ε.

$\varepsilon|x|$ is the sz of last significant digit of x.

choose half of significant digits,

$$\|\delta x\| \approx \sqrt{\varepsilon}\,\|x\|.$$

if x is of order ~ 1, δx should be of order $\sqrt{\varepsilon} = 10^{-8}$.