I.1 The column Space of A contains All vectors Ax

1. Matrix multiplication revisited (by a vector)

$$\begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ 5 & 7 & 12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

A_1

① row: → ↓ dot product

row $\cdot x$

② column:

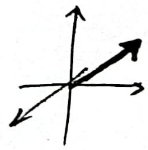
$$= x_1\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 1 \\ 7 \end{bmatrix} + x_3\begin{bmatrix} 3 \\ 4 \\ 12 \end{bmatrix}$$

combination of vectors in col of A.

Imagine all combinations. (all $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$), the answer altogether is $\boxed{}$?

2. Column Space: all Ax = Column space $:= C(A)$.
   - depending on the matrix.

$$A_2 = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 3 & 8 \\ 1 & 3 & 8 \end{bmatrix}$$ the column space $C(A_2) =$ line, $r(A_2) = 1$.



and $C(A_1) =$ plane, $r(A_1) = 2. =$ #independent columns.

3. • Rank 1 matrices can be rewritten of this form.

$$A_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}[1\ 3\ 8] = \underset{\text{col}}{u}\ \underset{\text{row}}{v^T}$$ is a rank 1 mat.

- basis for column space. everything else is linear comb. of basis.

- ▷ find a basis for A_1. (Selection method).

Once we pick up a matrix, C.

$$\underset{\begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ 5 & 7 & 12 \end{bmatrix}}{A} = \underset{\begin{bmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 7 \end{bmatrix}}{C} \underset{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}}{R}$$

C ← basis for col space

R ← basis for row space

$\boxed{A = CR}$

(3x2) ← Column rank

(2x3) ← Row rank

This suggests $\boxed{\text{Column rank} = \text{Row rank.}}$

- Also define ~~Colu~~ Row space as $C(A^T)$.

  Why R is basis for row space?

  1) no dependent rows.
  2) combination produces all rows

3. $A = \overset{\text{actual}}{C}R \leftarrow$ reduced

   is $ABCx$ in col sp? Yes! $A(BCx)$.

   or $A = \underset{2\times 2}{C U} R \leftarrow$ actual

4. Matrix multiplication by a matrix

$$\underset{\text{row}A}{\overset{A}{\left[\;\text{---}\;\right]}} \underset{\text{col}\,B}{\overset{B}{\left[\;|\;\right]}} = \underset{r \cdot c}{\text{dot product}}$$

$$\underset{\substack{\text{col}\\k}}{\left[\;\updownarrow\;\right]} \underset{\substack{\text{row}\\k}}{\left[\longleftrightarrow\right]} = \sum_{i=1}^{n} \begin{pmatrix}\text{col } k\\ \text{of } A\end{pmatrix}\begin{pmatrix}\text{row } k\\ \text{of } B\end{pmatrix}$$ "sum of outer products"

How many mults do we do? for $(m\times n)(n\times p)$ mat.

r, c $\quad n \cdot \underbrace{mp}_{\text{answer scale.}}$ $\quad \rightarrow mnp$ multiplies.

$c, r.$ $\quad \underbrace{mp}_{\text{1 piece}} \cdot n$

Eg. $\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}1&2&3\\1&3&2\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}\begin{bmatrix}1&2&3\end{bmatrix} + \begin{bmatrix}2\\4\end{bmatrix}\begin{bmatrix}1&3&2\end{bmatrix}$

$= \begin{bmatrix}1&2&3\\3&6&9\end{bmatrix} + \begin{bmatrix}2&6&4\\4&12&8\end{bmatrix} = \begin{bmatrix}3&8&7\\7&18&17\end{bmatrix}$

. Homework problem

1. The example is $A=\underbrace{\begin{bmatrix} 1 & 2 & -3 \\ 1 & 3 & -4 \\ 1 & 1 & -2 \\ 2 & 2 & -4 \end{bmatrix}}_{4\times 3}$, and $x=\underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{3\times 1}$.

4. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. The third one can be combined through the first and the second one.

9. $m, n \geq 3$. $r \geq 3$.

18. $A = CR$, then the matrix of $\begin{bmatrix} 0 & A \\ 0 & A \end{bmatrix}$ is

Letting $C' = \begin{bmatrix} A \\ A \end{bmatrix}$, $R' = [0, 1]$. Hence $C' = \begin{bmatrix} CR \\ CR \end{bmatrix} = CR\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$R' = [0, 1]$.