Chapter 3.
Integer Functions
Discussion on CMath: A Foundation for CS

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Section 1

Basic concepts

Floors and Ceilings

Definition

- $\lfloor x \rfloor$ = the greatest integer less than or equal to x.
- $\lceil x \rceil$ = the least integer greater than or equal to x.

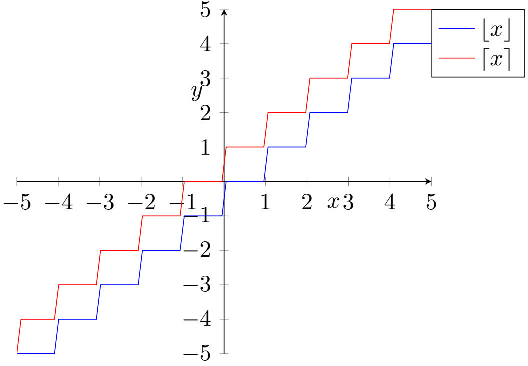

Figure: Floor and ceiling

Basic Properties

- Inequality: $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$.
- Negation: $\lceil -x \rceil = -\lfloor x \rfloor$, $\lfloor -x \rfloor = -\lceil x \rceil$.
- **Convert**:
  - $\lfloor x \rfloor = n \iff$ (with respect to x);
  - $\lfloor x \rfloor = n \iff$ (with respect to n);
  - $\lceil x \rceil = n \iff$ (with respect to x);
  - $\lceil x \rceil = n \iff$ (with respect to n);
- Moving integers: For integer n, $\lfloor x + n \rfloor = \lfloor x \rfloor + n$.

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  - $\lfloor x \rfloor = n \iff n \le x < n + 1$ (with respect to x);
  - $\lfloor x \rfloor = n \iff x - 1 \le n < x$ (with respect to n);
  - $\lceil x \rceil = n \iff n - 1 < x \le n$ (with respect to x);
  - $\lceil x \rceil = n \iff x \le n < x + 1$ (with respect to n);
- Moving integers: For integer n, $\lfloor x + n \rfloor = \lfloor x \rfloor + n$.

Example. An identity

Example

Prove that $\lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor$.

Example. An identity

Example

Prove that $\lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor$.

- Let $m = \lfloor \sqrt{\lfloor x \rfloor} \rfloor$. What is the range of m?
- $m \leq \sqrt{\lfloor x \rfloor} < m + 1$.
- Squaring to get the answer.

Example. Switching counting number

Example

Let $f(x)$ be any cont. mono. increasing func. with prop. that $f(x) = \text{integer} \implies x = \text{integer}$, prove that $\lfloor f(x) \rfloor = f(\lfloor x \rfloor)$, same as the ceiling.

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- If $x = \lfloor x \rfloor$, trivial.
- Otherwise $x > \lfloor x \rfloor \implies f(x) > f(\lfloor x \rfloor)$. What about $f(\lfloor x \rfloor)$ and $\lfloor f(x) \rfloor$?
  - Assume this is true: $f(\lfloor x \rfloor) < \lfloor f(x) \rfloor$, continuous,
  - must be a number y s.t. $x \le y < \lceil x \rceil$ and $f(y) = \lceil f(x) \rceil$. By the special property of x
  - y integer, no number between $x \le y < \lceil x \rceil$, hence they are equal.

Same problem: cont. mono. decreasing, what's that?

Counting the Integer Points

Count the integer points on a number line.

- if $a, b \in \mathbb{Z}$, integer point in $[a, b]$ is $b - a + 1$.
- More general case
  - $[\alpha, \beta]$
  - $(\alpha, \beta]$
  - $[\alpha, \beta)$
  - (α, β)

Helpful when handling summations by counting.

Counting the Integer Points

Count the integer points on a number line.

- if $a, b \in \mathbb{Z}$, integer point in $[a, b]$ is $b - a + 1$.
- More general case
  - $[\alpha, \beta]$ $\lfloor \beta \rfloor - \lceil \alpha \rceil + 1$
  - $(\alpha, \beta]$ $\lfloor \beta \rfloor - \lfloor \alpha \rfloor$
  - $[\alpha, \beta)$ $\lceil \beta \rceil - \lceil \alpha \rceil$
  - (α, β) $\lceil \beta \rceil - \lfloor \alpha \rfloor - 1$

Helpful when handling summations by counting.

Example. Computing a sum

Example

Compute

$$W = \sum_{i=1}^{1000} [\lceil \sqrt[3]{n} \rceil \, | n]$$

Example. Computing a sum

Example

Compute

$$W = \sum_{i=1}^{1000} [\lceil \sqrt[3]{n} \rceil \, | n]$$

- Make a new one name for $k = \sqrt[3]{n}$, getting $k \mid n, 1 \leq n \leq 1000$.
- The range for k is $k \leq \sqrt[3]{n} < k + 1$
- $k|n$ means that there is a m so that $n = km$.
- then becomes $1 + \sum_{k,m} [k^3 \leq km \leq (k+1)^3][1 \leq k < 10]$.

Example cont'd. Computing a sum

Example

Compute

$$W = \sum_{i=1}^{K} [\lceil \sqrt[3]{n} \rceil \, | n], K \in \mathbb{Z}.$$

Example cont'd. Computing a sum

Example

Compute

$$W = \sum_{i=1}^{K} [\lceil \sqrt[3]{n} \rceil \mid n], K \in \mathbb{Z}.$$

- We should care about $\sum_m [k^3 \leq Km \leq N]$.
- this part become $\sum_m [m \in [k^2..N/K]]$.
- the estimation will be $3/2N^{2/3} + O(N^{1/3})$.

Example. The Spectra Example

Example

The spectrum of a real number α to be an infinite multiset of integers. That is

$$\text{Spec}(\alpha) = \{\lceil\alpha\rceil, \lceil 2\alpha\rceil, \cdots\}$$

We can prove that (1) no two spectrum are equal; (2) $\text{Spec}(2) \cup \text{Spec}(2+\sqrt{2}) = \mathbb{Z}$.

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We can prove that (1) no two spectrum are equal; (2) $\text{Spec}(2) \cup \text{Spec}(2+\sqrt{2}) = \mathbb{Z}$.

- We define $N(\alpha, n) = \sum_{k>0}[\lfloor k\alpha \rfloor \leq n]$.
- which is $\lceil (n+1)/\alpha \rceil - 1$.
- proving $\left\lceil n + 1/\sqrt{2} \right\rceil - 1 + \left\lceil n + 1/2 + \sqrt{2} \right\rceil - 1 = n$.

This equality will be helpful:

$$a \leq b \implies a < b - 1 \text{ for floors and ceiling func.}$$

Section 2

Solving Recurrences

The first example: Knuth Number(KN)

We have the following example:

$$\begin{aligned} K_0 &= 1; \\ K_{n+1} &= 1 + \min(2K_{\lfloor n/2 \rfloor}, 3K_{\lfloor n/3 \rfloor}). \end{aligned}$$

Prove or disproof that for $n \geq 0, K_n \geq n$.

- List small vals for k.
- Proof by induction.
- Base case: $K = 0$ satisfy the condition.
- Induction

KN: Induction Step

$$K_0 = 1;$$
$$K_{n+1} = 1 + \min(2K_{\lfloor n/2 \rfloor}, 3K_{\lfloor n/3 \rfloor}).$$

- Assume the inequality hold for all vals up to some non negative vals n,
- Goal: show that $K_{n+1} \geq n + 1$.
- Given $K_{n+1} = 1 + \min(2K_{\lfloor n/2 \rfloor}, 3K_{\lfloor n/3 \rfloor})$, and $2K_{\lfloor n/2 \rfloor} \geq 2 \lfloor n/2 \rfloor, 3K_{\lfloor n/3 \rfloor} \geq 3 \lfloor n/3 \rfloor$(by hypothesis)
- But $2 \lfloor n/2 \rfloor$ can be as small as $n - 1$, $3 \lfloor n/3 \rfloor$ can be as small as $n - 2$, breaking the induction.
- Or really? This case jumps fast.

KN: The special case

We can prove by contradiction:

- Assume we can find a value m s.t. $K_m \leq m$
- finding m's origin, say $m = n' + 1$
- requires $K_{\lfloor n'/2 \rfloor} \leq \lfloor n'/2 \rfloor$, and $K_{\lfloor n'/3 \rfloor} \leq \lfloor n'/3 \rfloor$.
- This implies $K_0 \leq 0$, but $K_0 = 1$, contradiction.

About Math. Induction

In trying to devise a proof by mathematical induction, you may fail for two opposite reasons. You may fail because you try to prove too much: Your $P(n)$ *is too heavy a burden. Yet you may also fail because you try to prove too little: Your* $P(n)$ *is too weak a support.*
In general, you have to balance the statement of your theorem so that the support is just enough for the burden."

Figure: G. Polya

Jospher's Problem Generlized(JPG)

Idea: Whenever a person is passed over, give it a new number.
Demonstrate:

Jospher's Problem Generlized(JPG)

Idea: Whenever a person is passed over, give it a new number.
Demonstrate:

1	2	3	4	5	6	7	8	9	10
11	12		13	14		15	16		17
18			19	20			21		22
			23	24					25
			26						27
			28						
			29						
			30						

JPG: Findings

1	2	3^1	4	5	6^2	7	8	9^3	10
11	12^4		13	14		15^5	16		17
18^6			19	20			21^7		22
			23	24^8					25
			26						27^9
			28						
			29						
			30^{10}						

What will the id become?

- 1, 2 become ;
- 3 executed;
- 4, 5 become ;
- 6 is executed;
- $3k+1, 3k+2$ will become ;
- $3k+3$ is executed.

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1	2	3^1	4	5	6^2	7	8	9^3	10
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18^6			19	20			21^7		22
			23	24^8					25
			26						27^9
			28						
			29						
			30^{10}						

What will the id become?

- 1, 2 become $n+1, n+2$;
- 3 executed;
- 4, 5 become $n+3, n+4$;
- 6 is executed;
- $3k+1, 3k+2$ will become $n+2k+1, n+2k+2$;
- $3k+3$ is executed.

JPG: Findings

1	2	3^1	4	5	6^2	7	8	9^3	10
11	12^4		13	14		15^5	16		17
18^6			19	20			21^7		22
			23	24^8					25
			26						27^9
			28						
			29						
			30^{10}						

- Counting is consistent, no jumping over someone.
- The k-th person eliminated ends up with number $3k$.
- To find the survivor = figure out the original number $3N$.

JPG: Findings

1	2	3^1	4	5	6^2	7	8	9^3	10
11	12^4		13	14		15^5	16		17
18^6			19	20			21^7		22
			23	24^8					25
			26						27^9
			28						
			29						
			30^{10}						

- What is $3N$ originally?
- $N(N > n)$ has a form of $N = n + 2k + 1$ or $N = n + 2k + 2$, in a single round.
- for two ks, getting $k_1 = (N - 1 - n)/2, k_2 = (N - 2 - n)/2$.
- $= \lfloor (N - n - 1)/2 \rfloor$.

JPG: Findings

1	2	3^1	4	5	6^2	7	8	9^3	10
11	12^4		13	14		15^5	16		17
18^6			19	20			21^7		22
			23	24^8					25
			26						27^9
			28						
			29						
			30^{10}						

An algorithm for this:

- Let $N \leftarrow 3n$;
- while $N > n$, let $N \leftarrow \lfloor (N - n - 1)/2 \rfloor + N - n$;
- Answer$\leftarrow N$.

JPG: Findings

1	2	3^1	4	5	6^2	7	8	9^3	10
11	12^4		13	14		15^5	16		17
18^6			19	20			21^7		22
			23	24^8					25
			26						27^9
			28						
			29						
			30^{10}						

Simplifying this algorithm: like treating arithmetic series.

- Assign the numbers from largest to smallest
- yielding $\lceil 3/2D \rceil$.

JPG: Findings

1	2	3^1	4	5	6^2	7	8	9^3	10
11	12^4		13	14		15^5	16		17
18^6			19	20			21^7		22
			23	24^8					25
			26						27^9
			28						
			29						
			30^{10}						

Generalized: $D = \lceil q/(q-1)D \rceil$ for general qs, i.e. q-kill one.

Section 3

Mod: The binary Op

Mod: definition

We may rewrite the quotient and remainder as follows:
If n is an integer, then

$$n = m \lfloor n/m \rfloor + n \bmod m.$$

for $y \neq 0$.

- generalize it to negative integers
- $5 \bmod 3 = 5 - 3 \lfloor 5/3 \rfloor = 2.$
- $5 \bmod -3 = 5 - (-3) \lfloor 5/-3 \rfloor = -1.$
- $-5 \bmod 3 = -5 - (-3) \lfloor -5/3 \rfloor = 1.$
- $-5 \bmod -3 = -5 - (-3) \lfloor -5/-3 \rfloor = -2.$

Mod: definition

We may rewrite the quotient and remainder as follows:
If n is an integer, then

$$n = m \lfloor n/m \rfloor + n \bmod m.$$

for $y \neq 0$.

- Observation: In any case the result number is exactly in between 2 numbers.
- Special definition: if $y = 0$, then $x \bmod 0 = x$.
- preserves the property that x and y always differs from x by a multiple of y.

Another notation: Mumble

We have $n \bmod m = n - \lfloor n/m \rfloor m$
Alternative definition: mumble.

$$x \,\mathrm{mumble}\, y = y \left\lceil \frac{x}{y} \right\rceil - x$$

Properties

- Distributive: $c(x \bmod y) = (cx) \bmod (cy)$ for $c, x, y \in \mathbb{R}$.
- reason: $c(x \bmod y) = c(x - y\lceil x/y\rceil) = cx - cy(\lfloor cx/cy\rfloor) = (cx) \bmod (cy)$.

Example: Even partition problem(EPP)

Problem: Partition n things into m groups as equally as possible.
An example:

1	9	17	25	33
2	10	18	26	34
3	11	19	27	35
4	12	20	28	36
5	13	21	29	37
6	14	22	30	
7	15	23	31	
8	16	24	32	

- the final row has only 5 elems, can we do better?

Example: Even partition problem(EPP)

Problem: Partition n things into m groups as equally as possible.
A evener example: An example:

1	9	17	24	31
2	10	18	25	32
3	11	19	26	33
4	12	20	27	34
5	13	21	28	35
6	14	22	29	36
7	15	23	30	37
8	16			

Example: Even partition problem(EPP)

Problem: Partition n things into m groups as equally as possible.

- Division: a row by row arrange not always good.
- it tells us how many lines to put
  - Some of the short one put $\lceil n/m \rceil$ columns, others put $\lfloor n/m \rfloor$ cols.
  - There will be exactly $n \bmod m$ cols, and exactly $m - n \bmod m = n \,\mathrm{mumble}\, m$short ones.

Example: Even partition problem(EPP)

Problem: Partition n things into m groups as equally as possible.
Procedure:

- To distribute n things into m groups as even as possible,
- when $m > 0$, put $\lceil n/m \rceil$ things into one group
- then use this procedure to recursively
- i.e. put put the remaining $n' = n - \lceil n - m \rceil$ things into $m' = m - 1$ groups.

Proof:

- Suppose that $n = qm + r$
- If $r = 0$, We put $\lfloor n/m \rfloor = q$ things into the first, $n' = n - q, m' = m - 1$.
- If $r > 0$, put $\lfloor n/m \rfloor = q + 1$ into first group, leaving $n' = n - q - 1 = qm' + r - 1$.

Example: Even partition problem(EPP)

Problem: Partition n things into m groups as equally as possible. A closed form for the formula?

- Effect: the quotient stays the same, but the remainder decrease by 1.
- That is there are $\lceil n/m \rceil$ things when $k \leq n \bmod m$, and $\lfloor n/m \rfloor$ things o.w.
- So the closed form is $\lceil n - k + 1/m \rceil$.

Since we are arrange n elems, we have the following identity:

$$n = \left\lfloor \frac{n}{m} \right\rfloor + \left\lfloor \frac{n+1}{m} \right\rfloor + \cdots + \left\lfloor \frac{n+(m-1)}{m} \right\rfloor$$

Replace n by mx we get

$$mx = \lfloor x \rfloor + \left\lfloor x + \frac{1}{m} \right\rfloor + \cdots + \left\lfloor x + \frac{m-1}{m} \right\rfloor$$

Example: A Weird Sum(WS)

Find

$$\sum_{0\le k\le n} \left\lfloor \sqrt{k} \right\rfloor$$

where a is a perfect square.
Solution:

$$\begin{aligned} &\sum_{0\le k\le n} \left\lceil \sqrt{k} \right\rceil \\ &= \sum_{k,m\ge 0} m[k<n][m=\lceil k \rceil] \\ &= \sum_{k,m\ge 0} m[k<n][m\le \sqrt{k} < m+1] \end{aligned}$$

Then we calculate the total number of this.

Example: A Weird Sum(WS)

Find

$$\sum_{0 \le k \le n} \left\lfloor \sqrt{k} \right\rfloor$$

where a is a perfect square.
Solution:

$$
\begin{aligned}
&= \sum_{k,m \ge 0} m[k < n][m \le \sqrt{k} < m+1] \\
&= \sum_{k,m \ge 0} m[m \le k \le (m+1)^2 \le a^2] \\
&= \sum_{m \ge 0} m((m+1)^2 - m^2)[m+1 \le a] \\
&= \sum_{m \ge 0, m \le a} m(2m+1)
\end{aligned}
$$

Example: A Weird Sum(WS)

Find

$$\sum_{0 \le k \le n} \left\lfloor \sqrt{k} \right\rfloor$$

where a is a perfect square.
Solution:
That is

$$\sum_{0}^{a} (2m^{\underline{2}} + 3m^{\underline{1}}) \delta m$$

Using the integration rule, we get
$2/3a(a-1)(a-2) + 3/2a(a-1)$.

Example: A Weird Sum(WS)

Find

$$\sum_{0 \leq k \leq n} \left\lfloor \sqrt{k} \right\rfloor$$

where a is a perfect square.
Solution:
Removing the perfect square condition

- do the partition from $[0..a^2]$ and $[a^2..n]$.
- this will use O notation to express its increament.

Example: an Integrated Example(IE)

Find the closed form for

$$\sum_{0 \le k < m} \left\lfloor \frac{nk + x}{m} \right\rfloor$$

for integer $m > 0$, integer n.
We first look at some observations

- $n = 1$, yields $\sum_{0 \le k < m} \lfloor (k + x)/m \rfloor$, where we found at the EPP problem.
- $m = 1$, this will be $\lfloor x \rfloor$;
- $m = 2$, we look at $\lfloor x/2 \rfloor + \lfloor (x + n)/2 \rfloor$.
  - n even, $n/2$ integer. $\lfloor x/2 \rfloor + \lfloor (x + n)/2 \rfloor = 2 \lfloor x/2 \rfloor + n/2$.
  - n odd, $(n - 1)/2$ integer.
    $\lfloor x/2 \rfloor + (\lfloor (x + 1)/2 \rfloor + (n - 1)/2) = \lfloor x \rfloor + (n - 1)/2$.

Example: an Integrated Example(IE)

Find the closed form for

$$\sum_{0 \le k < m} \left\lfloor \frac{nk+x}{m} \right\rfloor$$

for integer $m > 0$, integer n.
Have a look at $m = 3$:

- $n \bmod 3 = 0, n/3$ and $2n/3$ integers:
  $\left\lfloor \frac{x}{3} \right\rfloor + \left\lfloor \frac{x}{3} + \frac{n}{3} \right\rfloor + \left\lfloor \frac{x}{3} + \frac{2n}{3} \right\rfloor = 3 \lfloor x/3 \rfloor + n.$
- $n \bmod 3 = 1, n - 1/3$ and $2n - 2/3$ integers:
  $\left\lfloor \frac{x}{3} \right\rfloor + \left\lfloor \frac{x+1}{3} + \frac{n-1}{3} \right\rfloor + \left\lfloor \frac{x+2}{3} + \frac{2n-2}{3} \right\rfloor = \lfloor x \rfloor + n - 1.$
- $n \bmod 3 = 2, n - 2/3$ and $2n - 4/3$ integers:
  $\left\lfloor \frac{x}{3} \right\rfloor + \left\lfloor \frac{x+2}{3} + \frac{n-2}{3} \right\rfloor + \left\lfloor \frac{x+4}{3} + \frac{2n-4}{3} \right\rfloor = \lfloor x \rfloor + n - 1.$

Example: an Integrated Example(IE)

Find the closed form for

$$\sum_{0\le k<m} \left\lfloor \frac{nk+x}{m} \right\rfloor$$

for integer $m > 0$, integer n.
Look at $n = 4$,

- $n \bmod 4 = 0, 4\lfloor x/4 \rfloor + 3n/2;$
- $n \bmod 4 = 1, \lfloor x \rfloor + 3n/2 - 3/2;$
- $n \bmod 4 = 0, \lfloor x \rfloor + 3n/2 - 3/2;$
- $n \bmod 4 = 0, 2\lfloor x \rfloor + 3n/2 - 1;$

Example: an Integrated Example(IE)

Find the closed form for

$$\sum_{0\leq k<m} \left\lfloor \frac{nk+x}{m} \right\rfloor$$

for integer $m > 0$, integer n.

We make a small table for this:

m	$n \bmod m = 0$	$n \bmod m = 1$	$n \bmod m = 2$	$n \bmod m == 3$
1	$\lfloor x \rfloor$			
2	$2\left\lfloor \frac{x}{2} \right\rfloor + \frac{n}{2}$	$\lfloor x \rfloor + \frac{n}{2} - \frac{1}{2}$		
3	$3\left\lfloor \frac{x}{3} \right\rfloor + n$	$\lfloor x \rfloor + n - 1$	$\lfloor x \rfloor + n - 1$	
4	$4\left\lfloor \frac{x}{4} \right\rfloor + \frac{3n}{2}$	$\lfloor x \rfloor + \frac{3n}{2} - \frac{3}{2}$	$2\left\lfloor \frac{x}{2} \right\rfloor + \frac{3n}{2} - 1$	$\lfloor x \rfloor + \frac{3n}{2} - \frac{3}{2}$

It looks that:

$$\left\lfloor \frac{x + kn \bmod m}{m} + \frac{kn}{m} - \frac{kn \bmod m}{m} \right\rfloor$$

Example: an Integrated Example(IE)

Find the closed form for

$$\sum_{0\le k<m} \left\lfloor \frac{nk+x}{m} \right\rfloor$$

for integer $m > 0$, integer n.
This can be extracted from

$$\left\lfloor \frac{x + kn \bmod m}{m} \right\rfloor + \frac{kn}{m} - \frac{kn \bmod m}{m}$$

Example: an Integrated Example(IE)

Find the closed form for

$$\sum_{0\le k<m}\left\lfloor\frac{nk+x}{m}\right\rfloor$$

for integer $m>0$, integer n.

$$
\begin{aligned}
&\left\lfloor\frac{x}{m}\right\rfloor && +\frac{0}{m}-\frac{0 \bmod m}{m}\\
+&\left\lfloor\frac{x+n \bmod m}{m}\right\rfloor && +\frac{n}{m}-\frac{n \bmod m}{m}\\
+&\left\lfloor\frac{x+2n \bmod m}{m}\right\rfloor && +\frac{2n}{m}-\frac{2n \bmod m}{m}\\
&\vdots && \vdots\\
&\underbrace{+\left\lfloor\frac{x+(m-1)n \bmod m}{m}\right\rfloor}_{a\left\lfloor\frac{x}{a}\right\rfloor} && +\underbrace{\frac{(m-1)n}{m}}_{bn}-\underbrace{\frac{(m-1)n \bmod m}{m}}_{C}.
\end{aligned}
$$

Example: an Integrated Example(IE)

Find the closed form for

$$\sum_{0 \le k < m} \left\lfloor \frac{nk + x}{m} \right\rfloor$$

for integer $m > 0$, integer n.
Looking at the table:

- The second column is $\frac{1}{2}\left(0 + \frac{(m-1)n}{m}\right)m$
- The first column: See what $0 \bmod m, \quad n \bmod m, 2n \bmod m, \cdots, (m-1)n \bmod m$ will get.

Example: an Integrated Example(IE)

Find the closed form for

$$\sum_{0 \le k < m} \left\lfloor \frac{nk + x}{m} \right\rfloor$$

for integer $m > 0$, integer n.
Look at the first row of that one, recall

$$n = \left\lfloor \frac{n}{m} \right\rfloor + \left\lfloor \frac{n+1}{m} \right\rfloor + \cdots + \left\lfloor \frac{n+(m-1)}{m} \right\rfloor$$

- We will encounter the remainder from 1 to n one time(we will show at Chapt. 4)

Example: an Integrated Example(IE)

Find the closed form for

$$\sum_{0\le k<m} \left\lfloor \frac{nk+x}{m} \right\rfloor$$

for integer $m > 0$, integer n.
So we have:

$$\begin{aligned} &d\left(\left\lfloor \frac{x}{m} \right\rfloor + \left\lfloor \frac{x+d}{m} \right\rfloor + \cdots + \left\lfloor \frac{x+m-d}{m} \right\rfloor\right) \\ =&d\left(\left\lfloor \frac{x/d}{m/d} \right\rfloor + \left\lfloor \frac{x/d+1}{m/d} \right\rfloor + \cdots + \left\lfloor \frac{x/d+m/d-1}{m/d} \right\rfloor\right) \\ =&d\left\lfloor \frac{x}{d} \right\rfloor ., \text{ and hence, } a = d = \gcd(m,n). \end{aligned}$$

Example: an Integrated Example(IE)

Find the closed form for

$$\sum_{0\le k<m} \left\lfloor \frac{nk+x}{m} \right\rfloor$$

for integer $m > 0$, integer n.
The third column: $d\left(\frac{1}{2}\left(0+\frac{m-d}{m}\right)\cdot\frac{m}{d}\right)=\frac{m-d}{2}$

- $c=\frac{d-m}{2}$.

Example: an Integrated Example(IE)

Find the closed form for

$$\sum_{0\leq k<m} \left\lfloor \frac{nk+x}{m} \right\rfloor$$

for integer $m > 0$, integer n.
Putting altogether:

$$\sum_{0\leqslant k<m} \left\lfloor \frac{nk+x}{m} \right\rfloor = d\left\lfloor \frac{x}{d} \right\rfloor + \frac{m-1}{2}n + \frac{d-m}{2}.$$

where $d = \gcd(m, n)$.

Example: an Integrated Example(IE)

Find the closed form for

$$\sum_{0\leq k<m} \left\lfloor \frac{nk+x}{m} \right\rfloor$$

for integer $m > 0$, integer n.
In fact, m and n are symmetric:

$$\begin{aligned}
\sum_{0\leqslant k<m} \left\lfloor \frac{nk+x}{m} \right\rfloor &= d\left\lfloor \frac{x}{d} \right\rfloor + \frac{m-1}{2}n + \frac{d-m}{2} \\
&= d\left\lfloor \frac{x}{d} \right\rfloor + \frac{(m-1)(n-1)}{2} + \frac{m-1}{2} + \frac{d-m}{2} \\
&= d\left\lfloor \frac{x}{d} \right\rfloor + \frac{(m-1)(n-1)}{2} + \frac{d-1}{2}
\end{aligned}$$

saying,

$$\sum_{0\leqslant k<m} \left\lfloor \frac{nk+x}{m} \right\rfloor = \sum_{0\leqslant k<n} \left\lfloor \frac{mk+x}{n} \right\rfloor, \quad \text{integers } m, n > 0.$$

Thanks