1 Affine and Convex Sets

1. Line

Definition 1.1 (Line and line segment) *Suppose $x_1 \neq x_2$ are two points $\in \mathbb{R}^n$, Points of the form $y = \theta x_1 + (1-\theta)x_2$ is called a line.*

If we restrict $\theta \in [0,1]$, call it a line segment.

Remark Also we write it as

$$\begin{aligned} y &= \theta x_1 + (1-\theta)x_2 \\ &= \underbrace{x_2}_{\text{base point}} + \theta \underbrace{(x_1 - x_2)}_{(x_2 \to x_1) \text{ direction scale by factor } \theta.} \end{aligned}$$

2. Affine sets

Definition 1.2 *A set of points $C \subseteq \mathbb{R}^n$ is affine if the line through any two distinct points in C lies in C.*

Example 1.1 *The set of line is a affine set.*

Definition 1.3 (Affine Combination) *Call a point of the form $\theta_1 x_1 + \cdots + \theta_k x_k$ as an affine combination where $\theta_1 + \cdots + \theta_k = 1$.*

Remark
- Affine Combination is the special form of linear combination, with each coeffs summing up to 1. This will be provided with some useful scenarios.

- Each θs has no restrictions as $\theta > 0$.

Proposition 1.1 (An affine set contains every affine combination of its points) *If C is an affine set, $x_1, x_2, \cdots, x_k \in C$, and $\theta_1 + \cdots + \theta_k = 1$, then $\theta_1 x_1 + \cdots + \theta_k x_k \in C$.*

Proof. (rough idea): by induction, i.e. as $n = 3$, and $\theta_1 x_1 + \theta_2 x_2 + (1 - \theta_1 - \theta_2) x_3 \in C$, recombining the terms, getting $(\theta_1 x_1 + (1 - \theta_1) x_3) + \theta_2 x_2 - \theta_2 x_3 \in C$. The first term is a point on C, and this forms the form in the remark.

Definition 1.4 (Subspace) *If C is an affine set and $x_0 \in C$,then the set*

$$V = C - x_0 = \{x - x_0 \mid x \in C\}$$

is a subspace.

Proposition 1.2 *It is closed under addition and scalar multiplication.*

Proof. Done in *Advanced Algebra II*.

Hence we use the notation

$$C = V + x_0 = \{v + x_0 \mid v \in V\}.$$

Analogy:

- V: null space of a matrix/linear eq

- x_0 : special solution of a matrix/linear eq.
- $V + x_0 \Rightarrow$ General solution.

Definition 1.5 (Affine Hull) *Affine hull The set of all affine combination of points in some set (not possibly affine) $C \subseteq \mathbb{R}^n$ is called the affine hull of C, and denoted as of C.*

Formally, $\mathbf{aff}\, C = \{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_1, \cdots, x_k \in C, \theta_1 + \cdots + \theta_k = 1\}$

Proposition 1.3 (Affine hull is the smallest affine set that contains C.) *If S is any affine set with $C \subseteq S$, then aff $C \subseteq S$.*

3. Affine dimension, Relative interior

Definition 1.6 (Affine dimension, Relative interior) *Affine Dimension: The Affine dimension of a set C as the dimension of its affine hull.*

Remark Not possibly consistent with other definition of dimensions.

Example 1.2 *The dimension of unit circle is 2, for its affine hull ($\mathbb{R}^2$) is 2 dimension.*

Definition 1.7 (Relative interior) *If affine dimension of a set $C \subseteq \mathbb{R}^n$ is less than n, then the set lies in the affine set of $C \neq \mathbb{R}^n$. We define relative interior of set C, denoted as reliant C, as its interior relative to of C.*

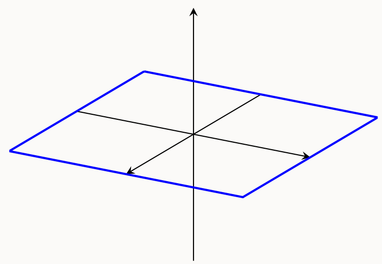
$$\mathbf{relint} C = \{x \in C \mid (B(x, r) \cap \mathbf{aff} C) \subseteq C \quad (\exists r > 0)\}$$

.

where $B(x, r) = \{y \mid \|y - x\| < r\}$, *a sphere with radius* r.

Definition 1.8 (Relative Boundary) *Define relative boundary of a set* C *as cl*$C \backslash \mathbf{relint} C$, *where cl* C *is the closure (all combination mentioned above) of* C,

Example 1.3 *Consider a square in* (x_1, x_2) *plane in* $\mathbb{R}^3$, *defined as* $C = \big\{x \in \mathbb{R}^3 \mid -1 \leqslant x_1 \leqslant 1, -1 \leqslant x_2 \leqslant 1, x_3 = 0$



- $\mathbf{relint} C = \big\{x \in \mathbb{R}^3 \mid -1 < x_1 < 1, -1 < x_2 < 1, x_3 = 0\big\}$, *the boundary is filtered.*
- *reason: at boundary, no ball can satisfy the condition.*
- *Affine hull of this is* $\mathbb{R}^2$.

4. Convex sets

Definition 1.9 (Convex) *A set* C *is convex if the line segment between any two points in* C *lies in* C. *That is, if for any* $x_1, x_2 \in C$, *and any* θ *with* $0 \leqslant \theta \leqslant 1$, *we have*

$$\theta x_1 + (1 - \theta) x_2 \in C$$

Remark • Every affine set is convex;

• Note the restriction here for θ is greater than 0.

Definition 1.10 (Convex hull) *The convex mull of a set C, denoted* **conv**C*, is the set of all convex combinations of points in C.*

$$\mathbf{conv}C = \left\{ \theta_1 x_1 + \cdots + \theta_k x_k \;\middle|\; \begin{array}{l} x_i \in C, \theta_i \geqslant 0, \\ i = 1, 2, \cdots, k. \\ \theta_1 + \cdots + \theta_k = 1 \end{array} \right\}$$

Remark We can generalize it to infinite case: Suppose $\theta_1, \theta_2, \cdots$ satisfies $\theta_i \geqslant 0, i = 1, 2, \cdots$, and $\sum_{i=1}^{\infty} \theta_i = 1$, and $x_1, x_2, \cdots \in C$, where $C \subseteq \mathbb{R}^n$ is convex, then $\sum_{i=1}^{\infty} \theta_i x_i \in C$, if the series converges.

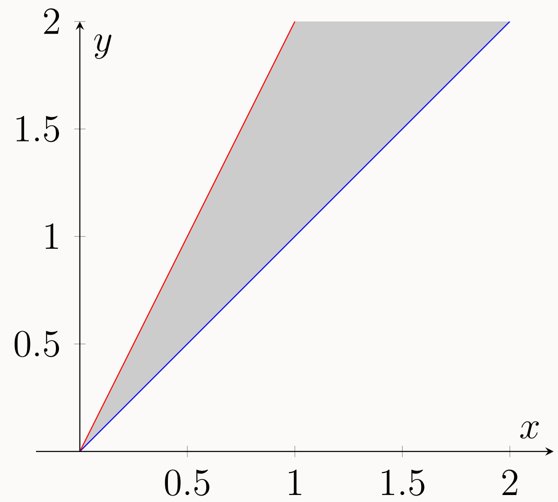
We can also replace sum with intergal: Suppose $p : \mathbb{R}^n \to \mathbb{R}$, satisfies $p(x) \geqslant 0$ for all $x \in C$ and $\int_C P(x)dx = 1$, where $C \subseteq \mathbb{R}^n$ is convex, then $\int_c P(x)xdx \in C$, if the internal exists.

The most general form is: Suppose $C \subseteq \mathbb{R}^n$ is convex and x is a random vector with $x \in C$ with probability 1. Then $\mathbb{E}x \in C$.

5. Cones

Definition 1.11 (Cones(nonnegative homogeneous)) *A set C is called Cones(nonnegative homogeneous) if $\forall x \in C, \theta \geqslant 0$, we have $\theta x \in C$.*

Definition 1.12 (Convex cone) *A set C is a convex cone, if it is a convex and a cone, which means that for any $x_1, x_2 \in C$ and $\theta_1, \theta_2 \geqslant 0$, we have $\theta_1 x_1 + \theta_2 x_2 \in C$.*



Definition 1.13 (Conic combination) *A point of the form $\theta_1 x_1 + \cdots + \theta_k x_k$ with $\theta_1, \cdots, \theta_k \geqslant 0$ is called a conic combination (nonnegative linear combination). of $x_1, \ldots, x_k$. (This can also generlize to infinite cases).*

Definition 1.14 (Convex hull) *The Convex hull of set C is the set of all conic combinations of points in C, ice,* $\{\theta_1 x_1 + \cdots + \theta_k x_k \,(x_i \in C, \theta_i \geqslant 0, i = 1, k\}$.

2 Some important examples

0. Some simple examples:

- $\varnothing$, any single point $\{x_0\}$, whole space $\mathbb{R}^n$ affine, convex subsets of $\mathbb{R}^n$
- Line affine. + pass O convex.

- Line segment convex, not affine.
- Ray. $\{x_0 + \theta v \mid \theta > 0\}$, $v \neq 0$, convex, not affine convex cone if $x_0 = 0$.
- $\forall$ subspace affine, convex cone.

1. Hyperplane and halfplane

Definition 2.1 (Hyperplane) *A hyperplane is a set of the form* $\left\{x \mid a^\top x = b\right\}$ *where* $a \in \mathbb{R}^n, b \in \mathbb{R}$.

Remark The meaning of a and b:

- a set of points with a cont inner prod to a given vector a.
- $b \in \mathbb{R}$: how far it is from the origin.Hence can write as

$$\left\{x \mid a^\top (x - x_0) = 0\right\}.$$

Remark We define $\left\{x \mid a^\top (x - x_0) = 0\right\} =: x_0 + a^\perp$.

- $a^\perp$: the orthogonal complement of a.
- i.e. (the set of all vectors orthogonal to it $a^\perp := \left\{v \mid a^\top v = 0\right\}$.)

Definition 2.2 (Half planes) *A hyperplane divides* $\mathbb{R}^n$ *into 2 halfspaces. A closed(or open) form of halfspace is by the form* $\left\{x \mid a^\top x \leqslant b\right\}$ *(or* $\left\{x \mid a^\top x < b\right\}$*) where* $a \neq 0$.

There is a property that it is convex but not affine.

Remark Alternative form: $\left\{x \mid a^\top (x - x_0) \leqslant 0\right\}$.

2. Euclidean balls and ellipsoids

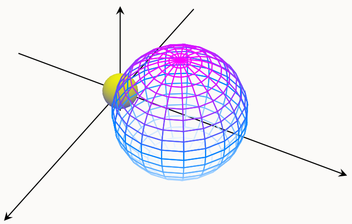
Definition 2.3 ((Euclidean balls)) *Euclidean balls in $\mathbb{R}^n$ has the form*

$$
\begin{aligned}
B\left(x_c, r\right) &= \left\{x \mid \left\|x-x_c\right\|_2 \leqslant r\right\} \\
&= \left\{x \mid \left(x-x_c\right)^{\top}\left(x-x_c\right) \leqslant r^2\right\}
\end{aligned}
$$

. *where $r>0, \|u\|_2=\sqrt{u^{\top} u}. x_c \in \mathbb{R}^n$ is the center of the ball, scalar r is its radius.*

Remark Another common repr for ball is

$$
B\left(x_c, r\right)=\{x_c+r \underbrace{u}_{\text{normalize to 1}} \mid \|u\|_2 \leqslant 1\}
$$



Proposition 2.1 (A Euclidean ball is a convex set) *If $\left\|x_1-x_c\right\|_2 \leqslant r$, $\left\|x_2-x_c\right\| \leqslant r$, and $0 \leqslant \theta \leqslant 1$, then $\left\|\theta x_1+(1-\theta) x_2-x_c\right\|_2 \leqslant r$.*

Proof.

$$
\begin{aligned}
& \left\|\theta x_1+(1-\theta) x_2-x_c\right\|_2 \\
= & \left\|\theta\left(x_1-x_c\right)+(1-\theta)\left(x_2-x_c\right)\right\|_2 \\
\leqslant & \theta\left\|x_1-x_c\right\|_2+(1-\theta)\left\|x_2-x_c\right\|_2 \\
\leqslant & r .
\end{aligned}
$$

Definition 2.4 (Ellipsoid) *Ellipsoid is a set of points that* $\varepsilon=\left\{x \mid\left(x-x_c\right)^T P^{-1}\left(x-x_c\right) \leqslant 1\right\}$, *where* $P=P^{\top} \succ 0$, *i.e. symmetric and positive definite.*

Remark Each parameter in the formula:

- The vector $x_c \in \mathbb{R}^n$ is the center of ellipsoid.
- The matrix P determines how far the ellipsoid extends in every direction from x_c.
- The lengths of semi-axes of $\mathcal{E}$ are given by $\sqrt{\lambda_i}, \lambda_i$: eigenvalues of P.

Remark A ball in the form is $P=r^2 I$.

Remark We may also represent as

$$
\varepsilon=\left\{x_c+A u \mid\|u\|_2 \leqslant 1\right\} .
$$

where A is square and nonsingular. still symmetric and pos. def.

And If sym. pos. semidef, singular, it is called a degenerlized ellipsoid.

The affine dimension is defined as $r(A)$, and it is convex.

3. Norm balls and Norm cones Suppose $\|\cdot\|$ is any norm on $\mathbb{R}^n$,

Definition 2.5 (Norm balls & Norm Cones) *A norm ball of radius r and centre x_c is given by*

$$\{x \mid \|x - x_d\| \leqslant r\}.$$

A norm cone associated with the norm $\|1\|$ is the set

$$C = \{(x,t) \mid \|x\| \leqslant t\} \subseteq \mathbb{R}^{n+1}.$$

Proposition 2.2 *Norm balls and Norm cone are convex.*

Example 2.1 *The 2nd order cone is the norm cone of the Euclidean norm, i.e.*

$$\begin{aligned} C &= \left\{(x,t) \in \mathbb{R}^{n+1} \mid \|x\|_2 \leqslant t\right\} \\ &= \left\{\begin{bmatrix} x \\ t \end{bmatrix} \mid \begin{bmatrix} x \\ t \end{bmatrix}^\top \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \leqslant 0, t \geqslant 0\right\}. \end{aligned}$$

4. Polyhedra

Definition 2.6 (Polyhedron) *A polyhedron is defined as the solution set of a finite number of linear equalities and inequalities.*

$$\mathscr{P} = \left\{x \mid \begin{matrix} a_j^\top x \leqslant b_j, & c_j^\top x = d_j \\ (1 \leqslant j \leqslant m), & (1 \leqslant j \leqslant p) \end{matrix}\right\}$$

Remark a more compact notation looks like

$$Q = \{x \mid Ax \preceq b, C_x = d\}$$

$$A = \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix}, \quad C = \begin{bmatrix} c_1^T \\ \vdots \\ c_m^T \end{bmatrix}$$

where $u \preceq v$ means $u_i \preceq v_i$ for $i = 1, \cdots, m$. We will see how this come into effect later.

We take a look at important series of poly hedra:

Definition 2.7 *Suppose $k+1$ points $v_0, \cdots, v_k \in \mathbb{R}^n$ are affinely independent: (meaning $v_i - v_0, \cdots, v_k - v_0$ are linearly independent) The simplex determined by*

$$C = \operatorname{conv}\{v_0, \cdots, v_k\} = \left\{\theta_0 v_0 + \cdots + \theta_k v_k \mid \theta \geq 0, 1^\top \theta = 1\right\}$$

where 1 means vector with all entries 1. The affine dimension of this simplex is k.

Remark This is sometimes referred to as a R-dimensional simplex in $\mathbb{R}^n$.

- Unit simplex: n dimensional simplex determined by zero vector $\&$ unit vectors, that is $0, \underbrace{e_1, \cdots, e_n}_{\text{Unit vectors}} \in \mathbb{R}^n$. satisfying $x \geq 0, 1^\top x \leqslant 1$.
- Probability simplex: $(n-1)$ dimensional simplex determined by unit vector $e_1, \cdots, e_n \in \mathbb{R}^n$. It is set of vecs:

$$x \geq 0, \quad I^\top x = 1$$

Example 2.2 *Describe the simplex in form of polyhedron.*

- Write the form of Polyhedron: $\mathcal{P} = \{x \mid Ax \preceq b, Cx = d\}$
- Write the form of simplex: $x \in C \Leftrightarrow x = \theta_0 v_0 + \cdots + \theta_k v_k, \exists \theta \geq 0, 1^\top \theta = 1.$
- Define

$$y = (\theta_1, \theta_2, \cdots, \theta_k)$$

$$B = [v_1 - v_0, v_2 - v_0, \cdots, v_k - v_0] \in \mathbb{R}^{n \times k} \quad (\text{rank } B = k)$$

- $x \in C \Leftrightarrow \underbrace{x = v_0 + By}_{(*_1)}, \exists y \geq 0, 1^\top y \leqslant 1.$

Hence $\exists$ ivertable matrix $A := (A_1, A_2) \in \mathbb{R}^{n \times n}$, s.t.

$$AB = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} B = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

Multiply $*_1$ left on A, getting

$$A_1 x = A_1 v_0 + y$$

$$A_2 x = A_2 v_0 + \underbrace{A_2 B_y}_{\text{here is } 0}$$

Hence we see that

$$x \in C \Leftrightarrow (A_2 x = A_2 v_0) \wedge \Big(y = A_1 x - A_1 v_0, y \geq 0, 1^\top y \leqslant 1\Big)$$

Conclusion: $x \in C$ iff

$$\cdot A_2 x = A_2 v_0 \quad \cdot A_1 x \succeq A_1 v_0 \quad \cdot I^\top A_1 x \leqslant 1 + 1^\top A_1 v_0$$

Example 2.3 *Convex hull description of polynedra.*

First we Generlize convex hull's definition. $\operatorname{conv}\{v_1 \cdots v_k\} = \left\{\theta_1 v_1 + \cdots + \theta_k v_k \mid \theta \succeq 0, 1^T \theta = 1\right\}$

We make only first m values add up to 1, i.e.

$$\{\theta_1 v_1 + \cdots + \theta_k v_k \mid \theta_1 + \cdots + \theta_m = 1, \quad \theta_i \geqslant 0, \quad i = 1 \ldots k\}.$$

as $m < k$.

Example 2.4 *Unit ball in l_∞-norm in $\mathbb{R}^n$.*

$$\begin{aligned} C &= \{x \mid |x_i| \leqslant 1, i = 1, 2, \cdots, n\}. \\ &= \left\{x \mid \pm e_i^\top x \leqslant 1\right\}, e_i = i_{th} \textit{ unit vector} \end{aligned}$$

To describe it in convex nl form needs 2^n points, each of them chooses from $\{-1, 1\}$.

$$C = \{v_1, \cdots, v_{2^n}\}.$$

5. The pos. def. cone. We use $\mathbb{S}^n$ to denote set of sym $n \times n$ mats. and $\mathbb{S}_+^n = \{X \in \mathbb{S}^n (X \succeq 0\}$

$$\mathbb{S}_{++}^n = \{X \in \mathbb{S}^n \mid X \succ 0\}.$$

Proposition 2.3 *The set $\mathbb{S}_+^n$ is a cone: if $\theta_1, \theta_2 \geqslant 0$ and $A, B \in \mathbb{S}_+^n$, then $\theta_1 A + \theta_2 B \in \mathbb{S}_+^n$.*

Proof. $\forall x \in \mathbb{R}^n$, we have

$$x^\top (\theta_1 A + \theta_2 B)\, x = \theta_1 x^\top A x + \theta_2 x^\top B x$$

$$\text{if } A \geq 0, B \succeq 0, \theta_1, \theta_2 > 0.$$

3 Ops that preserve convexity

1. Intersection

Theorem 3.1 *If S_1 and S_2 convex, then $S_1 \cap S_2$ convex. So is infinite set.*

Idea. This is trivial. Let two points chosen in $S_1 \cap S_2$.

Example 3.1

- *Positive semidefinite cone $\mathbb{S}_+^n$: $\bigcap_{z \neq 0} \underbrace{\left\{x \in \mathbb{S}^n \mid z^\top x z \geqslant 0\right\}}_{\textit{half planes } \in \mathbb{S}^n}$*
- *Consider $S = \{x \in \mathbb{R}^m || p(t) | \leqslant 1$ for $|t| \leqslant \frac{\pi}{3}\}$ where $p(t) = \sum_{k-1}^m x_k \cos kt$, $S = \bigcap_{|t| \leqslant \frac{\pi}{3}} S_t, S_t := \left\{x \mid -1 \leqslant (\cos t, \cdots, \cos mt)^\top x\right\}$ t,$S = \bigcap \{H \mid \mathcal{H}$ Halfspace, $S \subseteq \mathcal{H}\}$.*

2. Affine Functions Recall. Affine function $f : \mathbb{R}^n \to \mathbb{R}^m$ is affine if it is a sum of linear transform and a constant

- Linear transform: has the form $f(x) = Ax + b\ b \in \mathbb{R}^m$

Theorem 3.2 *Suppose $S \subseteq \mathbb{R}^n$ is convex and $f : \mathbb{R}^n \to \mathbb{R}^m$ is affine function. Then the image of S under f $f(s) = \{f(x) \mid x \in S\}$ is convex. If $f : \mathbb{R}^k \to \mathbb{R}^n$ is an affine function, the inverse image of S under f :*

$$f^{-1}(s) = \{x \mid f(x) \in S\}$$

Reason. We may cancel out the difference between the croodinates.

Theorem 3.3 *The projection of a convex set onto some of its croodinates is convex, ie. if $S \subseteq \mathbb{R}^m \times \mathbb{R}^n$ is convex, then $T = \{x_1 \in \mathbb{R}^m \,|(x_1, x_2) \in S \text{ for some } x_2 \in \mathbb{R}^n\}$.*

i.e. if $S \subseteq \mathbb{R}^m \times \mathbb{R}^n$ is convex, then $T = \{x_1 \in \mathbb{R}^m | \,(x_1, x_2) \in S$ for some $x_2 \in \mathbb{R}^n\}$.

Reason[**proj-v**]: Affine map preserves convexity. Suppose $x \in \mathbb{R}^n, y \in \mathbb{R}^m$, then given some point $P = (x_1, y_1) \in \mathbb{R}^{n+m}$, if you wanted to project P onto the X-space, you can simply obtain the projection of P i.e. $(\text{Proj}_X(P))$ by multiplying $[I_n; [0]_{n\times m}]\,P$, where I_n is the $n \times n$ identity matrix, and $[0]_{n\times m}$ is the $n \times m 0$ matrix. The $n \times (n+m)$ matrix $[I_n; [0]_{n\times m}]$ can be thought of as the affine map.

Definition 3.1 *(Sum of sets) The sum of 2 sets is defined as $S_1 + S_2 = \{x + y \mid x \in S_1, y \in S_2\}$.*

Theorem 3.4 *If S_1, S_2 convex, then $S_1 + S_2$ is convex.*

Proof. If S_1 and S_2 convex, so is the direct Cartesian product $S_1 \times S_2$. It follows directly Let linear function $f(x_1, x_2) = x_1 + x_2$. From the definition we see the case.

Definition 3.2 *(Partial Sum) If $S_1 \in \mathbb{R}^n, S_2 \in \mathbb{R}^m$, the partial sum of S_1 and $S_2 \in \mathbb{R}^n \times \mathbb{R}^m$ defined as*

$$S = \{(x, y_1 + y_2) \mid (x, y_1) \in S_1, (x, y_2) \in S_2\}.$$

Example 3.2 *a) Polyhedron* $\{x \mid Ax \leqslant b, C_x = d\}$ *can be expressed as inv ing of Cartesian product of the nonnegative orthant and the origin under Affine function* $f(x) = (b - Ax, d - Cx))$

$$\{x \mid Ax \leq b, Cx = d\} = \{x \mid f(x) \in \mathbb{R}_{\downarrow}^{m} \times \{0\}\}.$$

b) Solution set of a linear matrix inequality in x

$$A(x) = x_1 A_1 + \cdots + x_n A_n \preceq B$$

where $B, A_i \in \mathbb{S}^m$, Denoted as $\{x \mid A(x) \preceq B\}$. is convex. for it is the inverse ing of positive semidefinite cone under affine transf. $f : \mathbb{R}^n \to \mathbb{S}^m$ by $f(x) = B - A(x)$.

c) Hyperbolic cone. The set

$$\left\{x \mid x^\top p_{n\times n} x \leqslant \left(c^\top x\right)^2, c^\top x \geqslant 0\right\}$$

where $P \in \mathbb{S}_{+}^{n}$, and $c \in \mathbb{R}^n$ is convex, for it's inverse ing of second order cone $\left\{(z,t) \mid z^\top z \leq t^2, t \geqslant 0\right\}$ under affine function $f(x) = \left(\left(P^{1/2}x, C^T x\right)\right.$

d) Ellipsoid. The ellipsoid

$$\varepsilon = \left\{x \mid \left(x - x_c\right)^\top P^{-1} \left(x - x_c\right) \leq 1\right\}$$

as $P \in \mathbb{S}_{++}^{n}$, is the ing of Euclidean ball $\{u \mid \|u\|_2 \leqslant 1\}$ under affine mapping

$$f(u) = p^{1/2}u + x_c.$$

3. Linear fractional and persp funcs

Definition 3.3 *(The persp func). $P : \mathbb{R}^{n+1} \to \mathbb{R}^n$ with domain* $\operatorname{dom} P = \mathbb{R}^n \times \mathbb{R}_{++}$ *as* $P(z,t) = z/t$

- $\mathbb{R}_{++}$ *– Set of positive int.*

Remark
- The perspective function scales or normalizes vectors s.t. last component is 1 , then drops the last component.
- This is actually describing a pin-hole camera.

Theorem 3.5 *If $C \subseteq$ dom P is convex, then its ing*

$$P(C) = \{P(x) \mid x \in C\}$$

is convex.

Proof. Suppose $x = (\tilde{x}, x_{n+1})\,, y = (\tilde{y}, y_{n+1}) \in \mathbb{R}^{n+1}$ with $x_{n+1} > 0 . y_{n+1} > 0$. Then for $\theta \in [0,1]$

$$P(\underbrace{\theta x + (1-\theta)y}_{\text{def of convex}}) = \frac{\theta\tilde{x} + (1-\theta)\tilde{y}}{\theta x_{n+1} + (1-\theta)y_{n+1}}$$

as $\mu = \frac{\theta x_{n+1}}{\theta x_{n+1} + (r-\theta)y_{n+1}} \in [0,1]$. and the corrspondance of θ and μ is monotonic.

Remark This Thm's converse

Th 2's converse. The inv ing of a convex set under the perse funct, is also convex. if $C \subseteq \mathbb{R}^n$ is convex, then

$$P^{-1}(c) = \left\{(x,t) \in \mathbb{R}^{n+1} \left(\frac{x}{t} = c, t > 0\right\}\right.$$

Definition 3.4 *(Linear fractional functions) A linear fractional function is formed by composing persp. f_n. and affine f_n. Suppose $g : \mathbb{R}^n \to \mathbb{R}^{m+1}$*

$$g(x) = \begin{bmatrix} A \\ c^\top \end{bmatrix} x + \begin{bmatrix} b \\ d \end{bmatrix}$$

where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$ and $d \in \mathbb{R}$.

Remark a) linear-fractional function as a matrix

$$Q = \begin{bmatrix} A & b \\ C^\top & d \end{bmatrix} \in \mathbb{R}^{(m+1)x(n+1)}$$

b) It's persp. func plus an affine transf.

4 Generlized inequalities

1. Proper cones and General inequalities

Definition 4.1 *a convex cone $K \subseteq \mathbf{R}^n$ is a proper cone if*

- *K is closed (contains its boundary)*
- *K is solid (has nonempty interior)*
- *K is pointed (contains no line) – $x \in K, -x \in K \implies x = 0$*

Example 4.1 *a) nonnegative orthant $K = \mathbf{R}^n_+ = \{x \in \mathbf{R}^n \mid x_i \geq 0, i = 1, \ldots, n\}$*

b) positive semidefinite cone $K = \mathbf{S}^n_+$

c) nonnegative polynomials on $[0,1]$:

$$K = \left\{x \in \mathbf{R}^n \mid x_1 + x_2 t + x_3 t^2 + \cdots + x_n t^{n-1} \geq 0 \textit{ for } t \in [0,1]\right\}$$

Definition 4.2 *(Generalized inequalities)(nonstrict and strict) generalized inequality defined by a proper cone K :*

$$x \preceq_K y \quad \Longleftrightarrow y - x \in K, \quad x \prec_K y \quad \Longleftrightarrow \quad y - x \in \textit{ int } K$$

Example 4.2

- *componentwise inequality $(K = \mathbf{R}^n_+) : x\mathbf{R}^n_+ y \Longleftrightarrow x_i \leq y_i, \quad i = 1, \ldots, n$*
- *matrix inequality $(K = \mathbf{S}^n_+) : X \preceq^n_+ Y \Longleftrightarrow Y - X$ positive semidefinite*

these two types are so common that we drop the subscript in $\preceq_K$

Many properties are similar to $<$ on $\mathbb{R}$.

2. Minimum and minimal elements

Definition 4.3 *A point $x \in S$ is the minimum element of S if and only if*

$$S \subseteq x + K$$

where $x + K$ denotes all the points that are comparable to x and greater than or equal to x (according to $\preceq_K$). A point $x \in S$ is a minimal element if and only if

$$(x - K) \cap S = \{x\}.$$

Here $x - K$ denotes all the points that are comparable to x and less than or equal to x (according to $\preceq_K$); the only point in common with S is x.

5 Separating and supporting hyperplanes

1. Separating hyperplane theorem Idea: the use of hyperplanes or affine functions to separate convex sets that do not intersect

Theorem 5.1 *if C and D are nonempty disjoint (i.e., $C \cap D = \emptyset$) convex sets, there exist $a \neq 0, b$ s.t.*

$$a^T x \leq b \text{ for } x \in C, \quad a^T x \geq b \text{ for } x \in D$$