Chapter 4.
Number Theory
Discussion on CMath: A Foundation for CS

AUGPath

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The Divisible

Definition (Divisibility)

$m \mid n \iff m > 0 \land n = mk$ for some int k.

- That is, n is a multiple of m, and it is not possibly positive.

Greatest Common Divisor

Definition (GCD and LCM)

- Defn. $\gcd(m,n) = \max\{k : k \mid m \wedge k \mid n\}$,
- Defn. $\mathsf{lcm}(m,n) = \max\{k : m > 0 \wedge m \mid k \wedge n \mid k\}$,

The Euclid Algorithm

We assert that $\gcd(m, n) = \gcd(n, m \bmod n)$ (proof later).
Extended: Compute integers m' and n' s.t.

$$m'm + n'n = \gcd(m, n).$$

- At the end of the formula, $m = 0, n = \gcd(m, n)$.
- take $m' = 0, n' = 1$.
- Otherwise, keep an eye on the derivation process:

The Euclid Algorithm

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the derivation process, (q=quotient, r=remainder)

$$\begin{array}{ll}
b = rq_1 + r_1 & 0 \leqslant r_1 < r \\
r = r_1 q_2 + r_2 & 0 \leqslant r_2 < r_1 \\
r_1 = r_2 q_3 + r_3 & 0 \leqslant r_3 < r_2 \\
\cdots & \\
r_{n-3} = r_{n-2} q_{n-1} + r_{n-1} & 0 \leqslant r_{n-1} < r_{n-2} \\
r_{n-2} = r_{n-1} q_n + r_n & 0 \leqslant r_n < r_{n-1} \\
r_{n-1} = r_n q_{n+1} &
\end{array}$$

Euclid Algo: Substitution back

$$\begin{aligned}
&b = rq_1 + r_1 && 0 \leqslant r_1 < r \\
&r = r_1 q_2 + r_2 && 0 \leqslant r_2 < r_1 \\
&r_1 = r_2 q_3 + r_3 && 0 \leqslant r_3 < r_2 \\
&\cdots && \\
&r_{n-3} = r_{n-2} q_{n-1} + r_{n-1} && 0 \leqslant r_{n-1} < r_{n-2} \\
&r_{n-2} = r_{n-1} q_n + r_n && 0 \leqslant r_n < r_{n-1} \\
&r_{n-1} = r_n q_{n+1} &&
\end{aligned}$$

Hence,

$$\begin{aligned}
d = 1 r_n + 0 \times 0 &= 1 r_{n-2} - q_n r_{n-1} \\
&= 1 r_{n-2} - (r_{n-3} - q_n r_{n-2} q_{n-1}) \\
&= -q_n r_{n-3} + (1 + q_{n-1} q_n) r_{n-2} \\
&= \cdots \\
&= xa + yb \quad (x, y \in \mathbb{Z}).
\end{aligned}$$

Euclid Algo: Code

- Use RECURSION to maintain the relation.

  EXTENDED-EUCLID (a, b)

  if $b == 0$
  $\text{return}(a, 1, 0)$
  else $(d', x', y') = \text{EXTENDED} - \text{EUCLID}(b, a \bmod b)$
  $(d, x, y) = (d', y', x' - \lfloor a/b \rfloor y')$
  $\text{return}(d, x, y)$

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Why $(d', y', x' - \lfloor a/b \rfloor y')$?

- Condition:
  - $ax_1 + by_1 = \gcd(a, b)$
  - $bx_2 + (a \bmod b)y_2 = \gcd(b, a \bmod b)$
- Derivation:
  - $ax_1 + by_1 = bx_2 + (a \bmod b)y_2$
  - and we have that $a \bmod b = a - \left(\left\lfloor \frac{a}{b} \right\rfloor \times b\right)$
  - So we get $ax_1 + by_1 = bx_2 + \left(a - \left(\left\lfloor \frac{a}{b} \right\rfloor \times b\right)\right) y_2$
  - $ax_1 + by_1 = ay_2 + bx_2 - \left\lfloor \frac{a}{b} \right\rfloor \times by_2 = ay_2 + b\left(x_2 - \left\lfloor \frac{a}{b} \right\rfloor y_2\right)$
  - Compare the coeffs.

Euclid Algo: Code

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  EXTENDED-EUCLID (a, b)

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  $(d, x, y) = (d', y', x' - \lfloor a/b \rfloor y')$
  $\text{return}(d, x, y)$

An Example:

a	b	$\lfloor a/b \rfloor$	d	x	y
99	78	1	3	-11	14
78	21	3	3	3	-11
21	15	1	3	-2	3
15	6	2	3	1	-2
6	3	2	3	0	1
3	0	-	3	1	0

Props about Divisions

Theorem

$$(k \mid m) \wedge (k \mid n) \Leftrightarrow k \mid \gcd(m, n)$$

- follows directly from definition.

Theorem (The conjuctuce of factors)

$$\sum_{m|n} a_m = \sum_{m|n} a_{n|m}, \textit{ integer } n > 0.$$

- Are there anything similar to this?

Motivation for this (by definition):

$$\sum_{m|n} a_m = \sum_{k} \sum_{m>0} a_m [n = mk]$$

Props about Divisions

Theorem

$$(k \mid m) \wedge (k \mid n) \Leftrightarrow k \mid \gcd(m, n)$$

$$\sum_{m|n} a_m = \sum_k \sum_{m>0} a_m[n = mk]$$

Following above, we have

Theorem (Interchange summation order)

$$\sum_{m|n} \sum_{k|m} a_{k,m} = \sum_{k|n} \sum_{l|(n/k)} a_{k,kl}$$

Proof for Interchange the order

Theorem

$$\sum_{m|n}\sum_{k|m} a_{k,m} = \sum_{k|n}\sum_{l|(n/k)} a_{k,kl}$$

Consider LHS:

Proof for Interchange the order

Theorem

$$\sum_{m|n}\sum_{k|m} a_{k,m} = \sum_{k|n}\sum_{l|(n/k)} a_{k,kl}$$

Consider LHS:

$$\sum_{j,l}\sum_{k,m>0} a_{k,m}[n = jm][m = kl] =$$

Proof for Interchange the order

Theorem

$$\sum_{m|n} \sum_{k|m} a_{k,m} = \sum_{k|n} \sum_{l|(n/k)} a_{k,kl}$$

Consider LHS:

$$\sum_{j,l} \sum_{k,m>0} a_{k,m}[n = jm][m = kl] = \sum_{j} \sum_{k,l>0} a_{k,kl}[n = jkl]$$

Proof for Interchange the order

Theorem

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Consider LHS:

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Consider RHS:

Proof for Interchange the order

Theorem

$$\sum_{m|n}\sum_{k|m} a_{k,m} = \sum_{k|n}\sum_{l|(n/k)} a_{k,kl}$$

Consider LHS:

$$\sum_{j,l}\sum_{k,m>0} a_{k,m}[n = jm][m = kl] = \sum_{j}\sum_{k,l>0} a_{k,kl}[n = jkl]$$

Consider RHS:

$$\sum_{j,m}\sum_{k,l>0} a_{k,kl}[n = jk][n/k = ml] = \sum_{m}\sum_{k,l>0} a_{k,kl}[n = mlk]$$

They are the same, standing the same meaning.

Interchange of Order Example

If $k = 12$:

$$\begin{array}{r|llllll}
m = 1 & k = 1 & & & & & \\
m = 2 & k = 1 & k = 2 & & & & \\
m = 3 & k = 1 & k = 3 & & & & \\
m = 4 & k = 1 & k = 2 & k = 4 & & & \\
m = 6 & k = 1 & k = 2 & k = 3 & k = 6 & & \\
m = 12 & k = 1 & k = 2 & k = 3 & k = 4 & k = 6 & k = 12
\end{array}$$

to

$$\begin{array}{rrrrrr}
k = 1 & k = 2 & k = 3 & k = 4 & k = 6 & k = 12 \\
\hline
m = 1 & m = 2 & m = 3 & m = 4 & m = 6 & m = 12 \\
m = 2 & m = 4 & m = 6 & m = 12 & m = 12 & \\
m = 3 & m = 6 & m = 12 & & & \\
m = 4 & m = 12 & & & & \\
m = 6 & & & & & \\
m = 12 & & & & &
\end{array}$$

Prime

Definition

A positive integer p is called prime if it has just two divisors, namely 1 and p. We will also take p to represent some prime in this chapter.

Example:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, \ldots$$

Prime Factorlization

Theorem

Any positive integer n can be written as a product of primes.

$$n = p_1 \dots p_m = \prod_{k=1}^{m} p_k, \quad p_1 \leqslant \dots \leqslant p_m$$

Proof idea.

- By Contradiction, assume $n = p_1 \dots p_m = q_1 \dots q_k, \quad p_1 \leqslant \dots \leqslant p_m$ and $q_1 \leqslant \dots \leqslant q_k$
- Prove $p_1 = q_1$
  - assume $p_1 < q_1$, and they are primes, their gcd must be 1.
  - Using Euclid's Algo, we get $ap_1 + bq_1 = 1$
  - we will get $ap_1 q_2 \dots q_k + bq_1 q_2 \dots q_k = q_2 \dots q_k$.
  - teal has factor of q_1
  - but $q_2 \dots q_k < n$, contradiction, unless eq.

Alternative means for GCD and LCM

$$n = \prod_p p^{n_p}, \qquad \text{where each } n_p \geqslant 0$$

- Unique!
- linear combination!
- just like coordinate system
- infinite dimensions

We can formally describe like this:

- $\langle n_2, n_3, n_5, \ldots \rangle$
- $12 = \langle 2, 1, 0, 0, \ldots \rangle$

Alternative means for GCD and LCM

$$n = \prod_p p^{n_p}, \qquad \text{where each } n_p \geqslant 0$$

$$k = mn \quad \Longleftrightarrow \quad k_p = m_p + n_p \quad \text{for all } p.$$

$$m \mid n \quad \Longleftrightarrow \quad m_p \leqslant n_p \text{ for all } p$$

$$k = \gcd(m, n) \quad \Longleftrightarrow \quad k_p = \min(m_p, n_p) \quad \text{for all } p;$$

$$k = \operatorname{lcm}(m, n) \quad \Longleftrightarrow \quad k_p = \max(m_p, n_p) \quad \text{for all } p.$$

There Are Infinitely Many primes

”Oi πρω̃τοι ἀριθμο` πλείους εἰσὶπαντὰςτοῦ πλὴθουςπρὼτων ἀριθμω̃ν.” — *Euler*

- Notice that $\gcd(m, m+1) = 1$.

List:

$$
\begin{aligned}
e_1 &= 1 + 1 = 2; \\
e_2 &= 2 + 1 = 3; \\
e_3 &= 2 \cdot 3 + 1 = 7 \\
e_4 &= 2 \cdot 3 \cdot 7 + 1 = 43
\end{aligned}
$$

Prime density

- the nth prime, P_n, is about n times the natural $\log$ of n:

$$P_n \sim n \ln n$$

- the number of primes $\pi(x)$ not exceeding x is

$$\pi(x) \sim \frac{x}{\ln x}$$

Factorial

Definition (Factorial)

$$n! = 1 \cdot 2 \cdot \ldots \cdot n = \prod_{k=1}^{n} k, \quad \text{integer } n \geqslant 0,$$

and we define that $0! = 1$.

Some fun properties:

- the number of digits in $n!$ exceeds n when $n \geq 25$
- 1×10^9 at around 10.

How fast is factorial growing?

- Take the idea of Gaussian's trick
- we have $(n!)^2 = \prod_{k=1}^{n} k(n+1-k)$,
- hence

$$n \leqslant k(n+1-k) \leqslant \frac{1}{4}(n+1)^2$$

Factorial: Example

Example

For any given prime p, the largest power of p divides $n!$. We denote this number by $\epsilon_p(n!)$. Pattern of $\epsilon_p(n!)$?

- Observation on $p = 2, n = 10$: