(九) 图论: 路径与圈 (Paths and Cycles)

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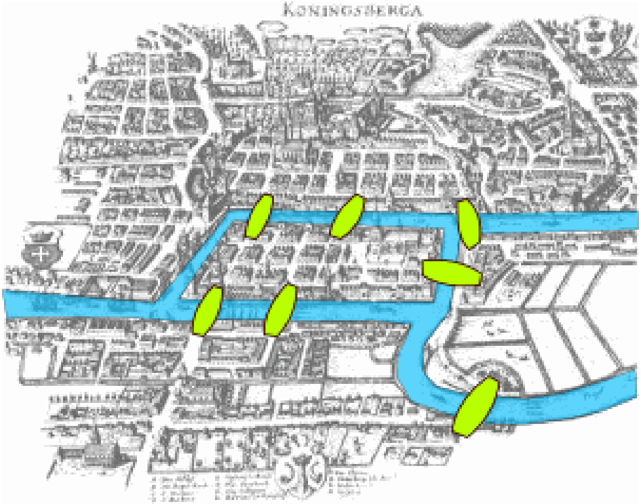
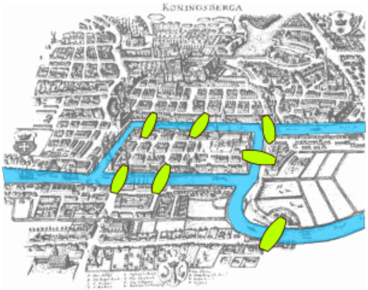
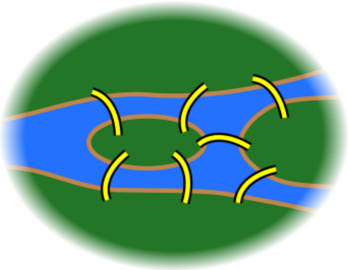
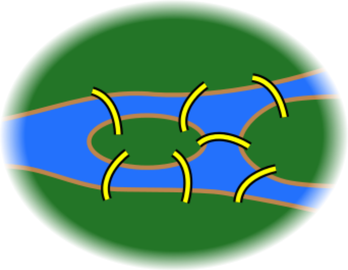
Definitions Theorems Proofs

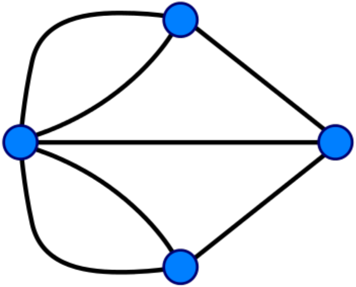


"to devise a walk through the city
that would cross each of those bridges once and only once"


KONINGSBERGA





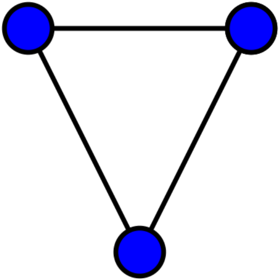


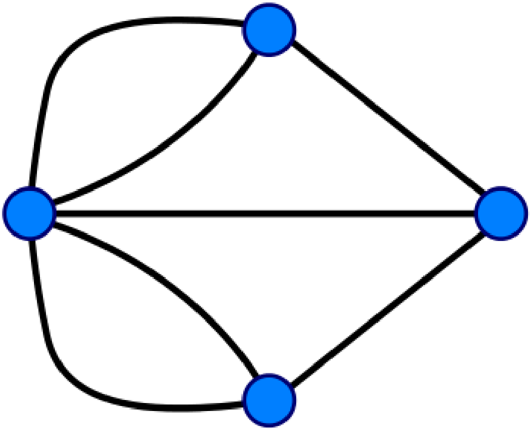
"to devise a walk through the graph
that would cross each of those edges once and only once"

Definition (Graph (图))

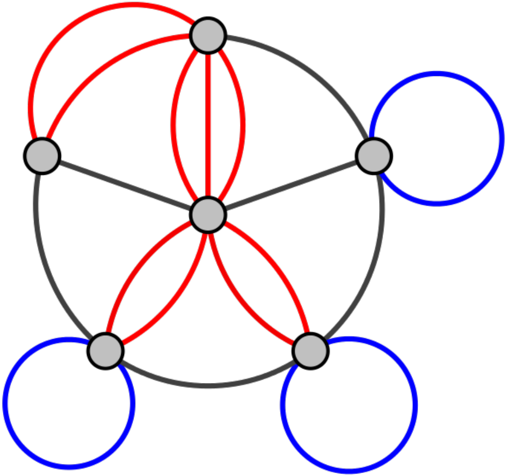
An (undirected simple) graph is a pair $G = (V, E)$ where

- V is a set of vertices (顶点);
- $E \subseteq \{\{x, y\} \mid x, y \in V \land x \neq y\}$ is a set of edges





Undirected Multigraph



Undirected Multigraph Permitting Loops

Definition (Walk (道路))

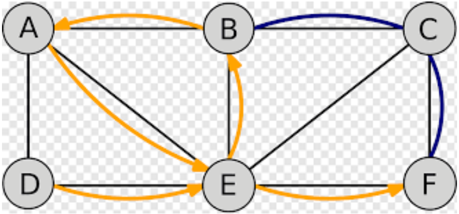
Given a graph G, a (finite) walk in G is a sequence of edges of the form

$$\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{m-1}, v_m\}.$$

$$(v_0 \to v_1 \to v_2 \to \cdots \to v_m)$$

It is a walk from the initial vertex v_0 to the final vertex v_m.

$$D \to E \to B \to A \to E \to F$$

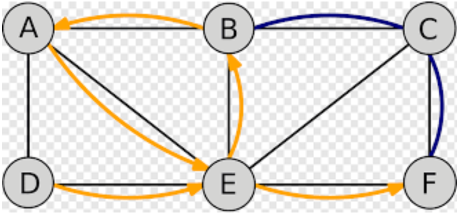


$$D \to E \to B \to E \to F$$

Definition (Trail (迹))

A trail is a walk in which all the edges are distinct.

$$D \to E \to B \to A \to E \to F$$

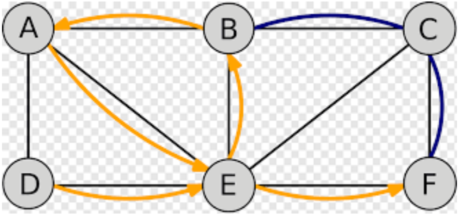


$$D \to E \to B \to E \to F$$

Definition (Path (路径))

A path is a trial in which all vertices are distinct.

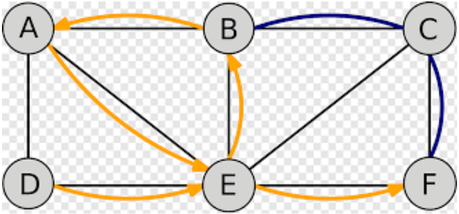
$$D \to E \to B \to A \to E \to F$$



$$D \to E \to F$$

Definition (Closed Walk/Trail/Path)

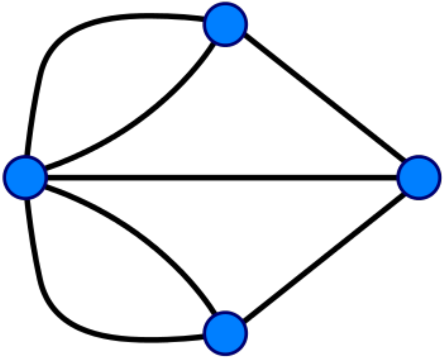
A walk, trail, or path is closed if $v_0 = v_m$.



Definition (Cycle)

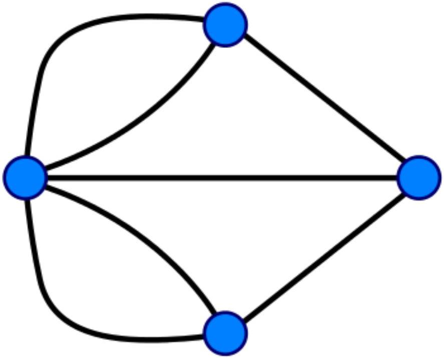
A cycle is a closed path with at least one edge.

"to devise a walk through the graph
that would cross each of those edges once and only once"



to find a trail that contains all edges of the graph

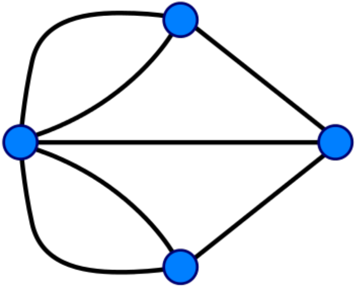
$$v_0 \to v_1 \to \cdots \to v_i \to \cdots \to v_m$$



$$v_i \notin \{v_0, v_m\} \implies \deg(v_i) \text{ is even}$$

Lemma (Necessary Condition for Eulerian Trails)

If a graph has Eulerian trails, then zero or two vertices have an odd degree.



4 vertices of odd degree $\implies$ has no Eulerian trails

Theorem (Euler's Theorem)

A graph has Eulerian trails iff it contains zero or two vertices that have an odd degree.

Euler stated but did *not* prove the "$\Longleftarrow$ (*if*)" direction.

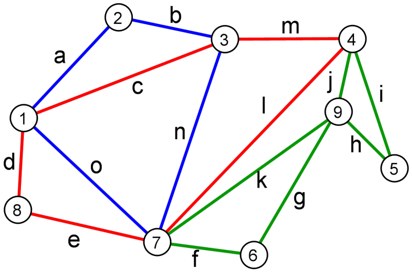
Carl Hierholzer $(1840 \sim 1871)$ gave the first complete proof in 1873.

Pierre-Henry Fleury gave another proof in 1883.

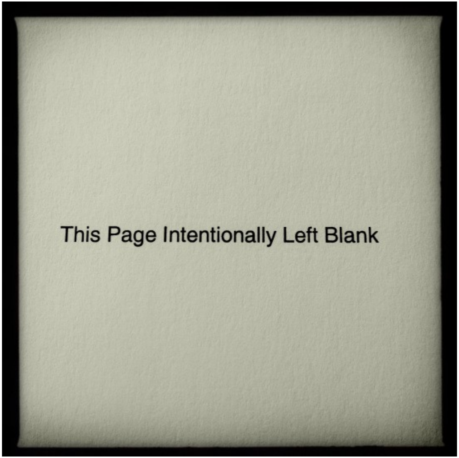
Theorem (Euler's Theorem (Carl Hierholzer))

A connected graph has Eulerian cycles iff every vertex has even degree.

(Such graphs are called Eulerian graphs.)



cycle + cycle + cycle + . . .



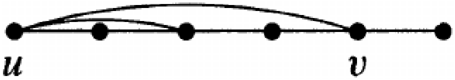
Can you repeat the Carl Hierholzer's algorithm?

Lemma

If every vertex of a graph G has degree ≥ 2, then G contains a cycle.

By Extremality.

Let $P : u \to \dots$ be a *maximal* path in G.

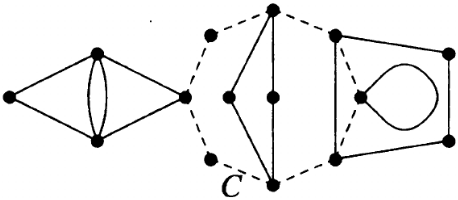


$$\deg(u) \geq 2$$

Theorem (Euler's Theorem (Carl Hierholzer))

A connected graph has Eulerian cycles iff every vertex has even degree.

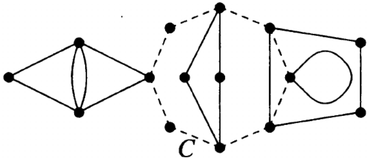
By induction on the number of edges m.

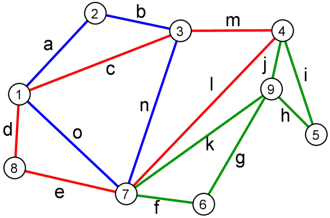


$$G' = G - C$$

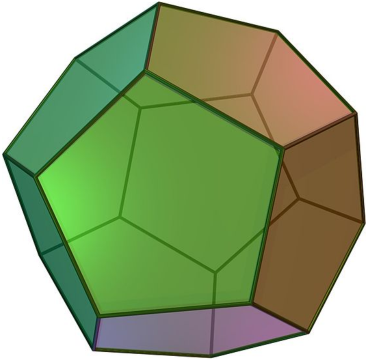
Theorem

Every Eulerian graph decomposes into cycles.

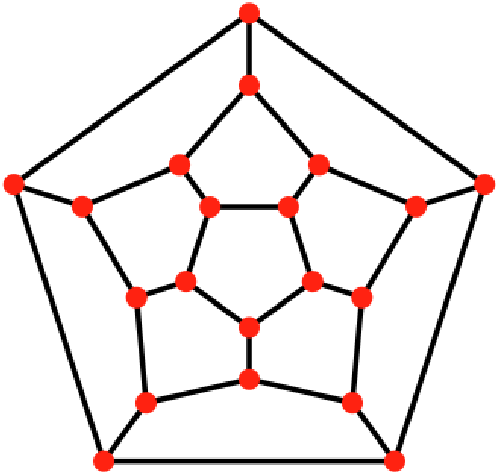




Dodecahedron: 12 faces, 20 vertices, and 30 edges



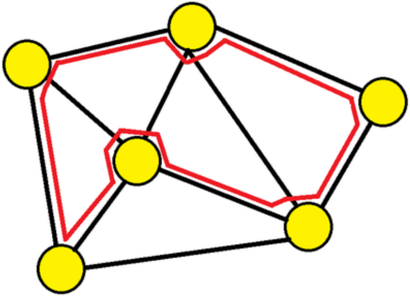
Is there a cycle that visits each vertex exactly once?



Is there a cycle that visits each vertex exactly once?

Definition (Hamiltonian Path)

A Hamiltonian path is a path that visits each vertex exactly once.



Definition (Hamiltonian Cycle)

A Hamiltonian cycle is a Hamiltonian path that is a cycle.

Definition (Hamiltonian Graph)

A graph is a Hamiltonian graph if it has a Hamiltonian cycle.

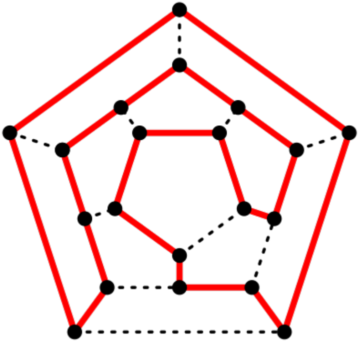
Definition (Semi-Hamiltonian Graph)

A non-Hamiltonian graph is semi-Hamiltonian if it has a Hamiltonian path.

William Rowan Hamilton
(1805 ~ 1865)

(October 16, 1843)

$$i^2 = j^2 = k^2 = ijk = -1$$



What is “THE” theorem for finding a Hamiltonian path/cycle or determining its existence?

I do not know.

Nobody knows.

We will probably never know it.

Theorem

The Hamiltonian Path/Cycle problem is NP-complete.

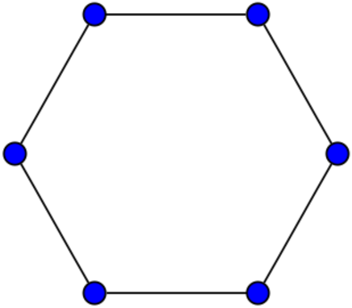
Typical (Positive/Negative) Graph Examples

Sufficient Conditions

Necessary Conditions


EXAMPLES

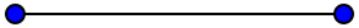
- Every cycle is Hamiltonian



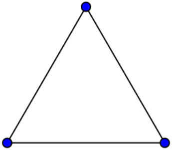
C_6

- A complete graph (完全图) with $|V| > 2$ is Hamiltonian.

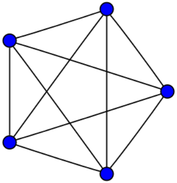
K_1



K_2



K_3

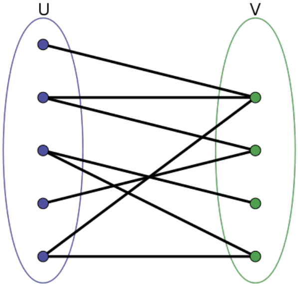


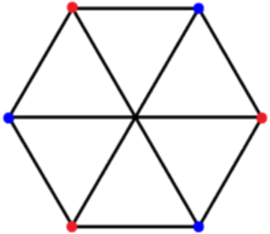
K_5

- A complete bipartite graph $K_{m,n}$ is Hamiltonian iff $m = n \geq 2$.

Definition (Bipartite Graph (Bigraph; 二部图))

A bipartite graph $G = (U, V, E)$ is a graph whose vertices can be divided into two disjoint sets U and V such that every edge connects a vertex in U to one in V.

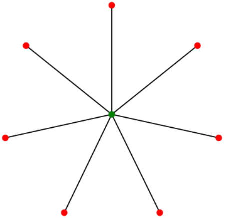




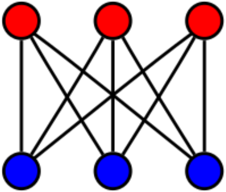
Definition (Complete Bipartite Graph (Biclique; 完全二部图))

A complete bipartite graph $G = (U, V, E)$ is bipartite graph where every vertex of U is connected to every vertex of V.

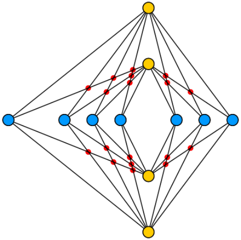
$$K_{m,n} : m = |U|, n = |V|$$



$K_{1,5}$ (star)

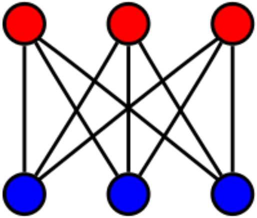


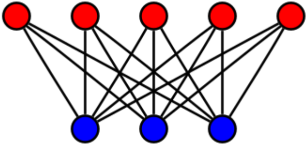
$K_{3,3}$ (utility graph)



$K_{4,7}$

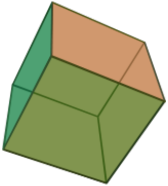
- A complete bipartite graph $K_{m,n}$ is Hamiltonian iff $m = n \geq 2$.



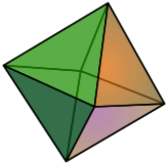


- Every platonic solid (正多面体), considered as a graph, is Hamiltonian.

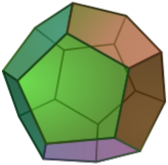
Tetrahedron



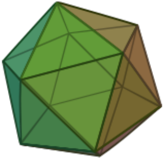
Cube



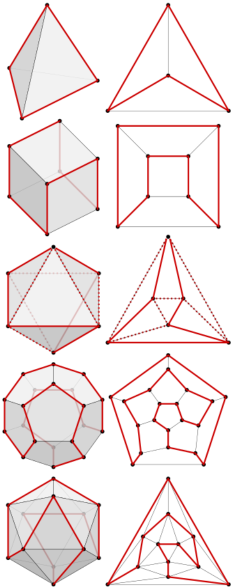
Octahedron



Dodecahedron



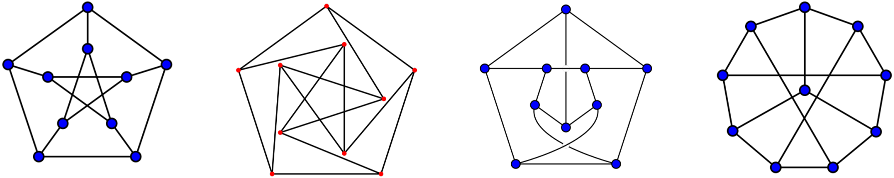
Icosahedron



Theorem

- *Petersen graph is not Hamiltonian.*

Julius Petersen $(1839 \sim 1910)$



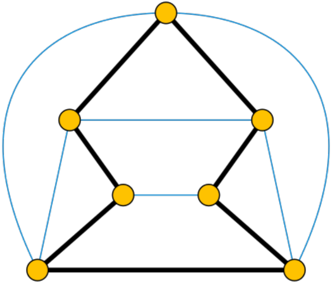
"If G has enough edges, then G is Hamiltonian."

Theorem (Ore's Theorem, 1960)

Let G be a simple graph with $n \geq 3$ vertices. If

$$deg(u) + deg(v) \geq n$$

for each pair of non-adjacent vertices u and v, then G is Hamiltonian.



By Contradiction.

Let G be a *non-Hamiltonian* (simple) graph with $n \geq 3$ vertices.

Suppose that G meets the Ore's Condition.

We need to derive a contradiction.

By Extremality.

Adding edges cannot violate the Ore's Condition.

Thus we may consider only *maximal* non-Hamiltonian graphs:

adding any edge gives a Hamiltonian graph.

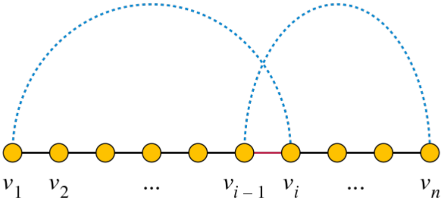
By its “maximality”, G contains a Hamiltonian path

(G is a semi-Hamiltonian graph)

$$v_1 \to v_2 \to \cdots \to v_n$$

v_1 and v_n are non-adjacent

$$\deg(v_1) + \deg(v_2) \geq n$$



There must be some vertex v_i adjacent to v_1

such that v_{i-1} is adjacent to v_n.

Theorem (Dirac's Theorem (1952; Gabriel Andrew Dirac))

A simple graph $G = (V, E)$ with $n \geq 3$ vertices is Hamiltonian

$$\forall v \in V.\ deg(v) \geq n/2.$$

$$\delta(G) \triangleq \min_{v \in V} \deg(v) \qquad \Delta(G) \triangleq \max_{v \in V} \deg(v)$$

$$\delta(G) \geq n/2$$

Family [edit | edit source]

He was born Balázs Gábor in Budapest, to Richárd Balázs, a military officer and businessman, and Margit "Manci" Wigner (sister of Eugene Wigner).[5] When his mother married Paul Dirac in 1937, he and his sister resettled in England and were formally adopted, changing their family name to Dirac.[6]

Theorem (Dirac's Theorem (1952))

A *simple* *graph* $G = (V, E)$ *with* $n \geq 3$ *vertices is* *Hamiltonian*

$$\delta(G) \geq n/2$$

$$\delta(G) = \lfloor (n-1)/2 \rfloor$$

Counterexample: $C_{\lfloor (n+1)/2 \rfloor}$ and $C_{\lceil (n+1)/2 \rceil}$ sharing a vertex

“If G is Hamiltonian, then G must be somewhat connected.”

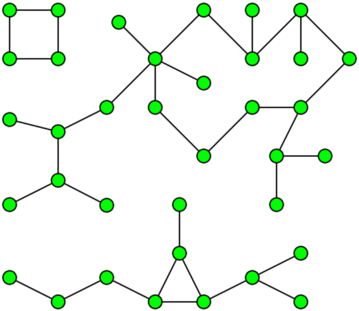
“If G is not so connected, then G is non-Hamiltonian.”

Theorem

If $G = (V, E)$ *is Hamiltonian, then for each nonempty set* $S \subset V$, *the graph* $G - S$ *has* $\leq |S|$ *components.*

Definition (Components (连通分支))

A component of an undirected graph is an subgraph in which any two vertices are connected to each other by paths, and which is connected to no additional vertices in the rest of the graph.



$c(G) = 3$

Theorem

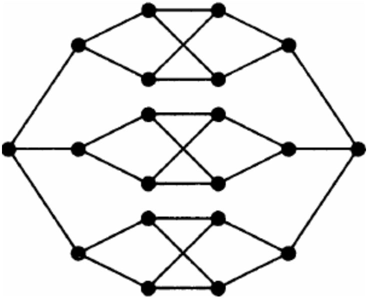
If $G = (V, E)$ is Hamiltonian, then

$$\forall S \subset V.\ c(G - S) \le |S|.$$

$$S \neq V$$

$C_{\lfloor (n+1)/2 \rfloor}$ and $C_{\lceil (n+1)/2 \rceil}$ sharing a vertex

- A complete bipartite graph $K_{m,n}$ is Hamiltonian iff $m = n \geq 2$.

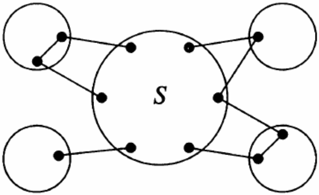


A non-Hamiltonian bipartite graph with $m = n = 10$

Theorem

If $G = (V, E)$ is Hamiltonian,

$$\forall S \subset V.\ c(G - S) \le |S|.$$



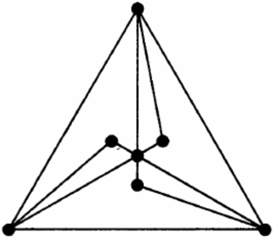
"When a Hamiltonian cycle leaves a component of $G - S$,
it can go only to a distinct vertex in S."

Theorem

If $G = (V, E)$ is Hamiltonian,

$$\forall S \subset V.\ c(G - S) \leq |S|.$$

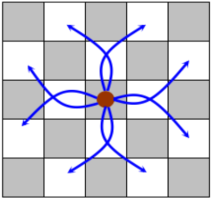
The condition is *not* sufficient.

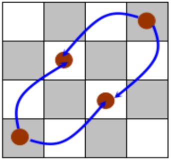


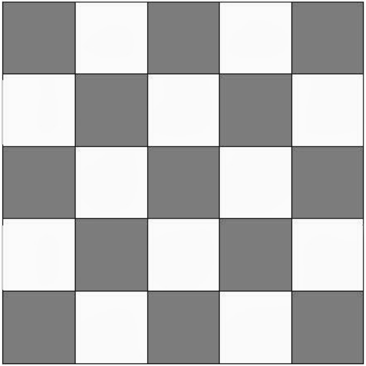
All edges incident to vertices of degree 2 must be used.

Chessboard Problem (“马踏棋盘” 问题)

Is it possible for a “knight” to visit every field of a 4×4 or 5×5 chessboard exactly once and return to the starting point?



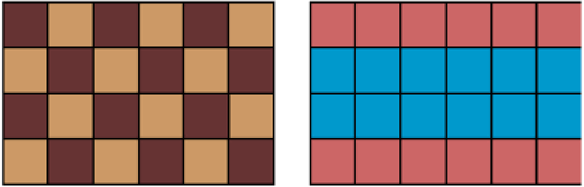




$$G = (U, V, E) : |U| = 12, |V| = 13$$

Removing the middle 4 squares leaves ≥ 5 components.

Chessboard Problem



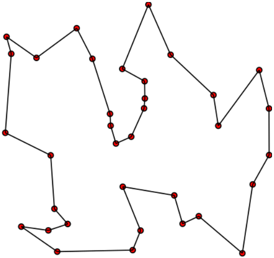
$$4 \times n$$

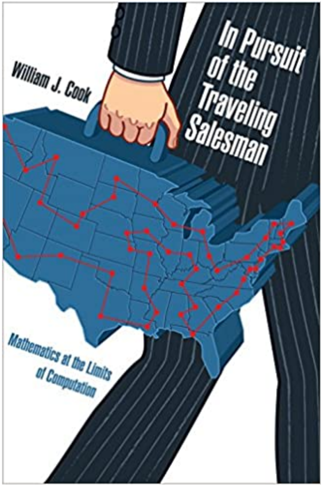
Definition (Travelling Salesman/Salesperson Problem (TSP; 旅行商问题))

Given a list of cities and the distances between each pair of cities,

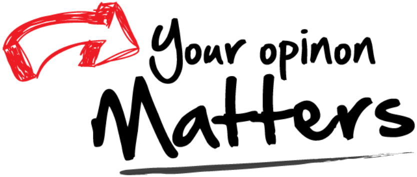
what is the shortest possible route

that visits each city exactly once and returns to the origin city?




William J. Cook
In Pursuit of the Traveling Salesman
Mathematics at the Limits of Computation


Thank
You!



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