

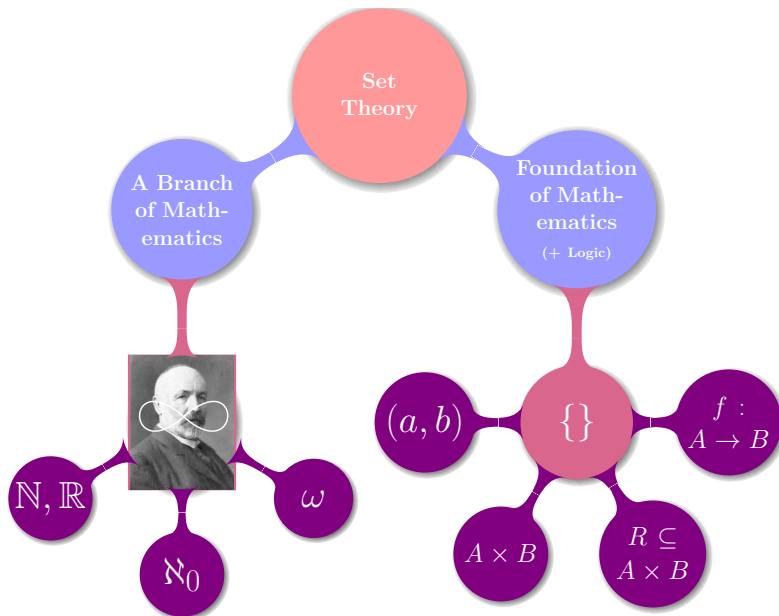
## (六) 集合: 函数 (Functions)

魏恒峰

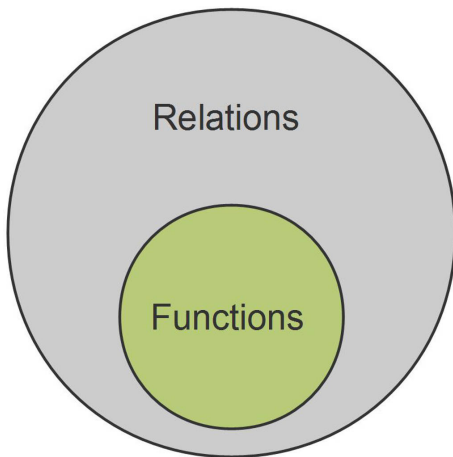
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2021 年 04 月 15 日





## 从“关系”的角度理解“函数”



$$f(x) = 2x + 1$$

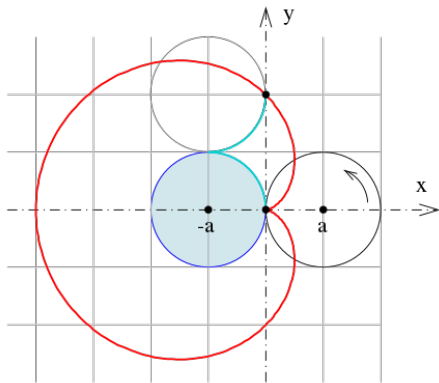


$f(x)$

“函数”也是“关系”

$\{\dots, (-2, -3), (-1, -1), (0, 1), (1, 3), \dots\}$

$$(x^2 + y^2)^2 + 4ax(x^2 + y^2) - 4a^2y^2 = 0$$



“函数”不允许“一对多”

# Functions

# Functions



PROOF!

# Definition of Functions



$$R \subseteq A \times B$$

is a *relation* from  $A$  to  $B$

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$$\text{dom}(f) = A \quad \text{cod}(f) = B$$

$$\text{ran}(f) = f(A) \subseteq B$$

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$$f : a \mapsto b$$

$$f(a) \triangleq b$$

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$$\forall b, b' \in B. (a, b) \in f \wedge (a, b') \in f \implies b = b'$$



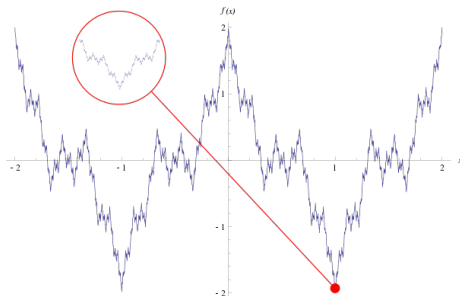
$$I_X : X \rightarrow X$$

$X$  上的恒等函数

$$\forall x \in X. I_X(x) = x$$

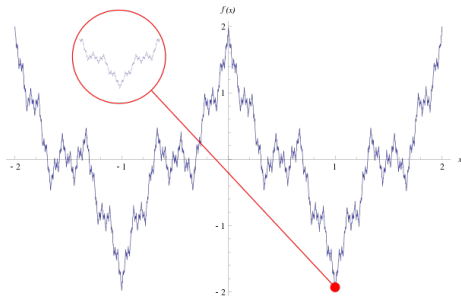
$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$

$0 < a < 1$ ,  $b$  is a positive odd integer,  $ab > 1 + \frac{3}{2}\pi$



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Weierstrass Function (1872)

“处处连续, 但处处不可导”

## Definition ( $Y^X$ )

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$$\bigcup_{I_X \in A} \text{dom}(I_X)$$

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$\bigcup_{I_X \in A} \text{dom}(I_X)$  would be the *universe* that does not exist!

# Functions as Sets

## Theorem (函数的外延性原理 (The Principle of Functional Extensionality))

*f, g are functions :*

$$f = g \iff \text{dom}(f) = \text{dom}(g) \wedge (\forall x \in \text{dom}(f). f(x) = g(x))$$

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It may be that  $\text{cod}(f) \neq \text{cod}(g)$ .

$$f : A \rightarrow B \quad g : C \rightarrow D$$

$Q$  : Is  $f \cap g$  a function?

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Theorem (Intersection of Functions)

$$f \cap g : (A \cap C) \rightarrow (B \cap D)$$



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Theorem (Union of Functions)

$$f \cup g : (A \cup C) \rightarrow (B \cup D) \iff \forall x \in \text{dom}(f) \cap \text{dom}(g). f(x) = g(x)$$

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### Theorem (Union of Functions)

$$f \cup g : (A \cup C) \rightarrow (B \cup D) \iff \forall x \in \text{dom}(f) \cap \text{dom}(g). f(x) = g(x)$$

$$f : \mathbb{Q} \rightarrow \mathbb{R}$$

$$f(x) = \begin{cases} x + 1, & \text{if } x \in 2\mathbb{Z} \\ x - 1, & \text{if } x \in 3\mathbb{Z} \\ 2, & \text{otherwise} \end{cases}$$

$$f : \mathcal{P}(\mathbb{R}) \rightarrow \mathbb{Z}$$

$$f(A) = \begin{cases} \min(A \cap \mathbb{N}) & \text{if } A \cap \mathbb{N} \neq \emptyset \\ -1 & \text{if } A \cap \mathbb{N} = \emptyset \end{cases}$$

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$$\text{dom}(f) \cap \text{dom}(g) = \emptyset$$

By the *Well-Ordering Principle* of  $\mathbb{N}$

$$D : \mathbb{R} \rightarrow \mathbb{R}$$

$$D(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Dirichlet Function

“处处不连续”

# Special Functions (*-jectivity*)

Definition (Injective (one-to-one; 1-1) 单射函数)

$$f : A \rightarrow B$$

$$\forall a_1, a_2 \in A. a_1 \neq a_2 \rightarrow f(a_1) \neq f(a_2)$$



Definition (Injective (one-to-one; 1-1) 单射函数)

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For Proof:

► To prove that  $f$  *is* 1-1:

$$\forall a_1, a_2 \in A. f(a_1) = f(a_2) \rightarrow a_1 = a_2$$

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- ▶ To prove that  $f$  *is* 1-1:

$$\forall a_1, a_2 \in A. f(a_1) = f(a_2) \rightarrow a_1 = a_2$$

- ▶ To show that  $f$  *is not* 1-1:

$$\exists a_1, a_2 \in A. a_1 \neq a_2 \wedge f(a_1) = f(a_2)$$

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- ▶ To show that  $f$  *is not* onto:

$$\exists b \in B. (\forall a \in A. f(a) \neq b)$$

Definition (Bijective (one-to-one correspondence) 双射; 一一对应)

$$f : A \rightarrow B$$

1-1 & onto



Definition (Bijective (one-to-one correspondence) 双射; 一一对应)

$$f : A \rightarrow B \quad f : A \xrightarrow[\text{onto}]{1-1} B$$

1-1 & onto

$$f : \mathbb{Z} \rightarrow \mathbb{N}, \quad f(x) = x^2 + 1$$

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$$f : \mathbb{Z} \times \mathbb{N} \rightarrow \mathbb{Q}, \quad f(z, n) = \frac{z}{n+1}$$

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$$f : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad f(x, y) = (x+1, y+1)$$

## Theorem (Cantor Theorem)

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**Proof.** Let  $A$  be a set and let  $f : A \rightarrow 2^A$ . To show that  $f$  is not onto, we must find a  $B \in 2^A$  (i.e.,  $B \subseteq A$ ) for which there is no  $a \in A$  with  $f(a) = B$ . In other words,  $B$  is a set that  $f$  “misses.” To this end, let

$$B = \{x \in A : x \notin f(x)\}.$$

We claim there is no  $a \in A$  with  $f(a) = B$ .

Suppose, for the sake of contradiction, there is an  $a \in A$  such that  $f(a) = B$ .

We ponder: Is  $a \in B$ ?

- If  $a \in B$ , then, since  $B = f(a)$ , we have  $a \in f(a)$ . So, by definition of  $B$ ,  $a \notin f(a)$ ; that is,  $a \notin B \Rightarrow \Leftarrow$
- If  $a \notin B = f(a)$ , then, by definition of  $B$ ,  $a \in B \Rightarrow \Leftarrow$

Both  $a \in B$  and  $a \notin B$  lead to contradictions, and hence our supposition [there is an  $a \in A$  with  $f(a) = B$ ] is false, and therefore  $f$  is not onto. ■



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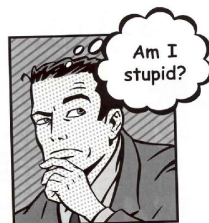
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$$\forall B \in 2^A. (\exists a \in A. f(a) = B)$$

Not Onto

$$\exists B \in 2^A. (\forall a \in A. f(a) \neq B)$$



$$f(1) = \{1, 2\}$$

$$f(2) = \{1, 3\}$$

$$f(3) = \emptyset$$

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► Constructive proof ( $\exists$ ):

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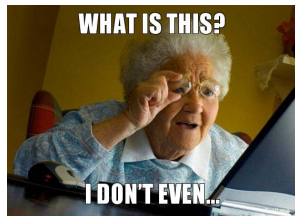
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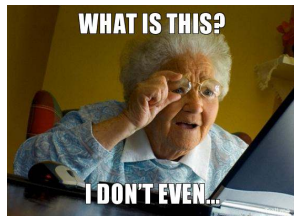
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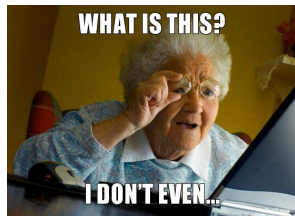
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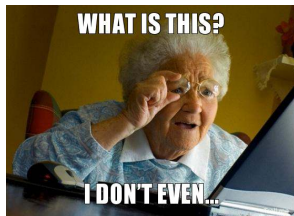
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$$a \in B \iff a \notin B$$

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$a$	$f(a)$					
	1	2	3	4	5	...
1	1	1	0	0	1	...
2	0	0	0	0	0	...
3	1	0	0	1	0	...
4	1	1	1	1	1	...
5	0	1	0	1	0	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	...



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3	1	0	0	1	0	...
4	1	1	1	1	1	...
5	0	1	0	1	0	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	...



## Theorem (Cantor Theorem)

If  $f : A \rightarrow 2^A$ , then  $f$  is *not* onto.

对角线论证 (Cantor's diagonal argument) .

$a$	$f(a)$					
	1	2	3	4	5	...
1	1	1	0	0	1	...
2	0	0	0	0	0	...
3	1	0	0	1	0	...
4	1	1	1	1	1	...
5	0	1	0	1	0	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	...

$$B = \{0, 1, 1, 0, 1\}$$



## Theorem (Cantor Theorem)

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对角线论证 (Cantor's diagonal argument) (以下仅适用于可数集合  $A$ ).

$a$	$f(a)$					
	1	2	3	4	5	...
1	1	1	0	0	1	...
2	0	0	0	0	0	...
3	1	0	0	1	0	...
4	1	1	1	1	1	...
5	0	1	0	1	0	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	...

$$B = \{0, 1, 1, 0, 1\}$$



# Functions as Relations

$$f|_X \quad f(A) \quad f^{-1}(B) \quad f \circ g$$

## Definition (Restriction)

The *restriction* of a function  $f : A \rightarrow B$  to  $X$  is the **function**:

$$f|_X = \{(x, y) \in f \mid x \in X\}$$



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It is **unnecessary** that  $X \subseteq A$ , though it is **usually** the case.

$$f|_X : A \cap X \rightarrow B$$

$$\forall x \in A \cap X. f|_X(x) = f(x)$$

## Definition (像 (Image))

The *image* of  $X$  under a function  $f : A \rightarrow B$  is the set

$$f(X) = \{y \mid \exists x \in X. (x, y) \in f\}$$

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$$f(\{a\}) = \{b\} \quad \text{简记为} \quad f(a) = b$$

## Definition (像 (Image))

The *image* of  $X$  under a function  $f : A \rightarrow B$  is the set

$$f(X) = \{y \mid \exists x \in X. (x, y) \in f\}$$

$$y \in f(X) \iff \exists x \in X. y = f(x)$$

## Definition (逆像 (Inverse Image))

The *inverse image* of  $Y$  under a function  $f : A \rightarrow B$  is the set

$$f^{-1}(Y) = \{x \mid \exists y \in Y. (x, y) \in f\}$$



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$$f^{-1}(Y) = f^{-1}(Y \cap \text{ran}(f))$$

$$f^{-1}(\{b\}) = \{a\} \quad \text{可简记为} \quad f^{-1}(b) = \{a\} \quad \text{不能简记为} \quad f^{-1}(b) = a$$

## Definition (逆像 (Inverse Image))

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$$f^{-1}(Y) = \{x \mid \exists y \in Y. (x, y) \in f\}$$

$$x \in f^{-1}(Y) \iff f(x) \in Y$$

$$y \in f(X) \iff \exists x \in X. y = f(x)$$

$$x \in f^{-1}(Y) \iff f(x) \in Y$$

$$f : A \rightarrow B$$

$$a \in A_0 \not\Rightarrow f(a) \in f(A_0)$$

$$f : A \rightarrow B$$

$$a \in A_0 \not\Rightarrow f(a) \in f(A_0)$$

$$a \in A_0 \cap A \Rightarrow f(a) \in f(A_0)$$

## Theorem (Properties of $f$ and $f^{-1}$ )

$$f : A \rightarrow B \quad \cancel{A_1, A_2 \subseteq A, B_1, B_2 \subseteq B}$$

(i)  $f$  preserves only  $\subseteq$  and  $\cup$ :

$$(1) A_1 \subseteq A_2 \implies f(A_1) \subseteq f(A_2)$$

$$(2) f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$$

$$(3) f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$$

$$(4) f(A_1 \setminus A_2) \supseteq f(A_1) \setminus f(A_2)$$



## Theorem

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对任意  $b$ ,

$$b \in f(A_1) \tag{1}$$

$$\tag{4}$$

## Theorem

$$f : A \rightarrow B$$

$$A_1 \subseteq A_2 \implies f(A_1) \subseteq f(A_2)$$

对任意  $b$ ,

$$b \in f(A_1) \tag{1}$$

$$\iff \exists a \in A_1. b = f(a) \tag{2}$$

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## Theorem

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$Q$  : When does  $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$  hold?

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$Q$  : When does  $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$  hold?

$f$  is injective.

## Theorem

$$f : A \rightarrow B$$

$$f(A_1 \setminus A_2) \supseteq f(A_1) \setminus f(A_2)$$

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(5)

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对任意  $b$ ,

$$b \in f(A_1) \setminus f(A_2) \quad (1)$$

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$$b \in f(A_1) \setminus f(A_2) \tag{1}$$

$$\iff b \in f(A_1) \wedge b \notin f(A_2) \tag{2}$$

$$\iff (\exists a_1 \in A_1. b = f(a_1)) \wedge (\forall a_2 \in A_2. b \neq f(a_2)) \tag{3}$$

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对任意  $b$ ,

$$b \in f(A_1) \setminus f(A_2) \tag{1}$$

$$\iff b \in f(A_1) \wedge b \notin f(A_2) \tag{2}$$

$$\iff (\exists a_1 \in A_1. b = f(a_1)) \wedge (\forall a_2 \in A_2. b \neq f(a_2)) \tag{3}$$

$$\implies \exists a \in A_1 \setminus A_2. b = f(a) \tag{4}$$

$$\iff b \in f(A_1 \setminus A_2) \tag{5}$$

## Theorem (Properties of $f$ and $f^{-1}$ )

$$f : A \rightarrow B \quad \cancel{A_1}, \cancel{A_2} \not\subseteq \cancel{A}, \cancel{B_1}, \cancel{B_2} \not\subseteq \cancel{B}$$

(ii)  $f^{-1}$  preserves  $\subseteq$ ,  $\cup$ ,  $\cap$ , and  $\setminus$ :

$$(5) \quad B_1 \subseteq B_2 \implies f^{-1}(B_1) \subseteq f^{-1}(B_2)$$

$$(6) \quad f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$$

$$(7) \quad f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$$

$$(8) \quad f^{-1}(B_1 \setminus B_2) = f^{-1}(B_1) \setminus f^{-1}(B_2)$$

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对任意  $a$ ,

$$a \in f^{-1}(B_1) \tag{1}$$

(4)

## Theorem

$$f : A \rightarrow B$$

$$B_1 \subseteq B_2 \implies f^{-1}(B_1) \subseteq f^{-1}(B_2)$$

对任意  $a$ ,

$$a \in f^{-1}(B_1) \tag{1}$$

$$\iff f(a) \in B_1 \tag{2}$$

$$\tag{4}$$

## Theorem

$$f : A \rightarrow B$$

$$B_1 \subseteq B_2 \implies f^{-1}(B_1) \subseteq f^{-1}(B_2)$$

对任意  $a$ ,

$$a \in f^{-1}(B_1) \tag{1}$$

$$\iff f(a) \in B_1 \tag{2}$$

$$\implies f(a) \in B_2 \tag{3}$$

$$\tag{4}$$



## Theorem

$$f : A \rightarrow B$$

$$B_1 \subseteq B_2 \implies f^{-1}(B_1) \subseteq f^{-1}(B_2)$$

对任意  $a$ ,

$$a \in f^{-1}(B_1) \tag{1}$$

$$\iff f(a) \in B_1 \tag{2}$$

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$$\iff a \in f^{-1}(B_2) \tag{4}$$

## Theorem

$$f : A \rightarrow B$$

$$f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$$

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对任意  $a$ ,

$$a \in f^{-1}(B_1 \cap B_2) \tag{1}$$

(5)

## Theorem

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对任意  $a$ ,

$$a \in f^{-1}(B_1 \cap B_2) \tag{1}$$

$$\iff f(a) \in B_1 \cap B_2 \tag{2}$$

(5)

## Theorem

$$f : A \rightarrow B$$

$$f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$$

对任意  $a$ ,

$$a \in f^{-1}(B_1 \cap B_2) \quad (1)$$

$$\iff f(a) \in B_1 \cap B_2 \quad (2)$$

$$\iff f(a) \in B_1 \wedge f(a) \in B_2 \quad (3)$$

(5)

## Theorem

$$f : A \rightarrow B$$

$$f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$$

对任意  $a$ ,

$$a \in f^{-1}(B_1 \cap B_2) \quad (1)$$

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$$\iff f(a) \in B_1 \wedge f(a) \in B_2 \quad (3)$$

$$\iff a \in f^{-1}(B_1) \wedge a \in f^{-1}(B_2) \quad (4)$$

$$(5)$$

## Theorem

$$f : A \rightarrow B$$

$$f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$$

对任意  $a$ ,

$$a \in f^{-1}(B_1 \cap B_2) \quad (1)$$

$$\iff f(a) \in B_1 \cap B_2 \quad (2)$$

$$\iff f(a) \in B_1 \wedge f(a) \in B_2 \quad (3)$$

$$\iff a \in f^{-1}(B_1) \wedge a \in f^{-1}(B_2) \quad (4)$$

$$\iff a \in f^{-1}(B_1) \cap f^{-1}(B_2) \quad (5)$$

## Theorem (Properties of $f$ and $f^{-1}$ )

$$f : A \rightarrow B$$

(iii)  $f$  and  $f^{-1}$ :

$$(9) \quad A_0 \subseteq A \implies A_0 \subseteq f^{-1}(f(A_0))$$

$$(10) \quad B_0 \supseteq f(f^{-1}(B_0))$$



## Theorem

$$f : A \rightarrow B$$

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## Theorem

$$f : A \rightarrow B$$

$$A_0 \subseteq A \implies A_0 \subseteq f^{-1}(f(A_0))$$

对任意  $b$ ,

$$a \in A_0 \tag{1}$$

$$\tag{4}$$

## Theorem

$$f : A \rightarrow B$$

$$A_0 \subseteq A \implies A_0 \subseteq f^{-1}(f(A_0))$$

对任意  $b$ ,

$$a \in A_0 \tag{1}$$

$$\implies a \in A_0 \cap A \tag{2}$$

$$\tag{4}$$

## Theorem

$$f : A \rightarrow B$$

$$A_0 \subseteq A \implies A_0 \subseteq f^{-1}(f(A_0))$$

对任意  $b$ ,

$$a \in A_0 \tag{1}$$

$$\implies a \in A_0 \cap A \tag{2}$$

$$\implies f(a) \in f(A_0) \tag{3}$$

$$\tag{4}$$

## Theorem

$$f : A \rightarrow B$$

$$A_0 \subseteq A \implies A_0 \subseteq f^{-1}(f(A_0))$$

对任意  $b$ ,

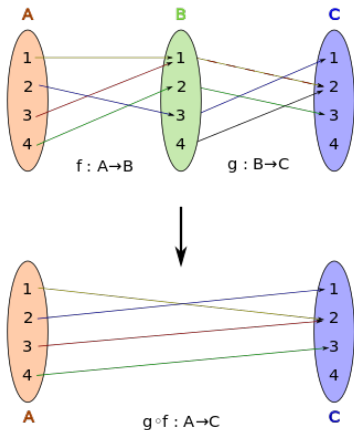
$$a \in A_0 \tag{1}$$

$$\implies a \in A_0 \cap A \tag{2}$$

$$\implies f(a) \in f(A_0) \tag{3}$$

$$\iff a \in f^{-1}(f(A_0)) \tag{4}$$

# Function Composition



## Definition (Composition)

$$f : A \rightarrow B \quad g : C \rightarrow D$$

$$\text{ran}(f) \subseteq C$$

The *composite function*  $g \circ f : A \rightarrow D$  is defined as

$$(g \circ f)(x) = g(f(x))$$

## Definition (Composition)

$$f : A \rightarrow B \quad g : C \rightarrow D$$

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The *composite function*  $g \circ f : A \rightarrow D$  is defined as

$$(g \circ f)(x) = g(f(x))$$

## Definition (Composition)

The *composition* of relations  $R$  and  $S$  is the relation

$$R \circ S = \{(a, c) \mid \exists b : (a, b) \in S \wedge (b, c) \in R\}$$



## Definition (Composition)

$$f : A \rightarrow B \quad g : C \rightarrow D$$

$$\text{ran}(f) \subseteq C$$

The *composite function*  $g \circ f : A \rightarrow D$  is defined as

$$(g \circ f)(x) = g(f(x))$$

Why not “ $\exists b$ ” as below?

## Definition (Composition)

The *composition* of relations  $R$  and  $S$  is the relation

$$R \circ S = \{(a, c) \mid \exists b : (a, b) \in S \wedge (b, c) \in R\}$$

## Theorem (Associative Property for Composition)

$$f : A \rightarrow B \quad g : B \rightarrow C \quad h : C \rightarrow D$$

$$h \circ (g \circ f) = (h \circ g) \circ f$$

## Theorem (Associative Property for Composition)

$$f : A \rightarrow B \quad g : B \rightarrow C \quad h : C \rightarrow D$$

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Proof.

## Theorem (Associative Property for Composition)

$$f : A \rightarrow B \quad g : B \rightarrow C \quad h : C \rightarrow D$$

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Proof.

(i)

$$\text{dom}(h \circ (g \circ f)) = \text{dom}((h \circ g) \circ f)$$

(ii)

$$\forall x \in A. (h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x)$$



$$(h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x)$$

$$(h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x)$$

$$(h \circ (g \circ f))(x) \quad (1)$$

$$= h((g \circ f)(x)) \quad (2)$$

$$= h(g(f(x))) \quad (3)$$

$$(h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x)$$

$$(h \circ (g \circ f))(x) \quad (1) \qquad ((h \circ g) \circ f)(x) \quad (1)$$

$$= h((g \circ f)(x)) \quad (2) \qquad = ((h \circ g)(f(x))) \quad (2)$$

$$= h(g(f(x))) \quad (3) \qquad = h(g(f(x))) \quad (3)$$

## Theorem

$$f : A \rightarrow B \quad g : B \rightarrow C$$

- (i) *If  $f, g$  are injective, then  $g \circ f$  is injective.*
- (ii) *If  $f, g$  are surjective, then  $g \circ f$  is surjective.*
- (iii) *If  $f, g$  are bijective, then  $g \circ f$  is bijective.*



## Theorem

$$f : A \rightarrow B \quad g : B \rightarrow C$$

*If  $f, g$  are injective, then  $g \circ f$  is injective.*

## Theorem

$$f : A \rightarrow B \quad g : B \rightarrow C$$

*If  $f, g$  are injective, then  $g \circ f$  is injective.*

$$\forall a_1, a_2 \in A. \left( (g \circ f)(a_1) = (g \circ f)(a_2) \rightarrow a_1 = a_2 \right)$$

## Theorem

$$f : A \rightarrow B \quad g : B \rightarrow C$$

*If  $f, g$  are injective, then  $g \circ f$  is injective.*

$$\forall a_1, a_2 \in A. \left( (g \circ f)(a_1) = (g \circ f)(a_2) \rightarrow a_1 = a_2 \right)$$

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$$f : A \rightarrow B \quad g : B \rightarrow C$$

- (i) *If  $g \circ f$  is injective, then  $f$  is injective.*
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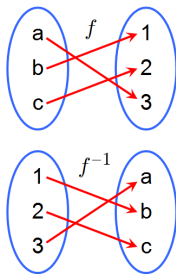
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# Inverse Functions



$$f : A \rightarrow B$$

What is the condition for  $f^{-1}$  to be a function from  $B$  to  $A$ ?

$$f^{-1} : B \rightarrow A$$

## Definition (反函数 (Inverse Function))

Let  $f : A \rightarrow B$  be a function.

The *inverse* of  $f$  is a function from  $B$  to  $A$ , denoted  $f^{-1} : B \rightarrow A$  if  $f$  is bijective.

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$$f \text{ is bijective : } f(x) = y \iff f^{-1}(y) = x$$

## Theorem

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$$\forall y \in B. f^{-1}(y) = g(y)$$

examples

## Theorem

$f : A \rightarrow B$  is bijective

(i)  $f \circ f^{-1} = I_B$

(ii)  $f^{-1} \circ f = I_A$

(iii)  $f^{-1}$  is bijective

(iv)  $g : B \rightarrow A \wedge f \circ g = I_B \implies g = f^{-1}$

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The ways to find/check  $f^{-1}$ .



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## Theorem (Inverse of Composition)

*Both  $f : A \rightarrow B$  and  $g : B \rightarrow C$  are bijective*

- (i)  *$g \circ f$  is bijective*
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Proof for (ii).

It suffices to check **either** one of the following identities:

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = I_A$$

$$(g \circ f) \circ (f^{-1} \circ g^{-1}) = I_C$$



## Theorem

$$f : A \rightarrow B \quad g : B \rightarrow A$$

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You do *not* know that  $f$  is bijective.

You need to check **both** identities.

## Theorem

$$f : A \rightarrow B \quad g : B \rightarrow A$$

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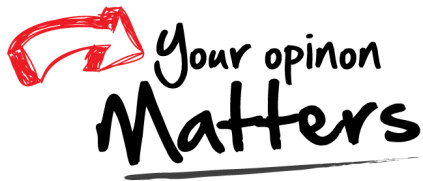
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Use two previous theorems

Thank  
You!



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