(四) 集合: 基本概念与运算 (Naive Set Theory)

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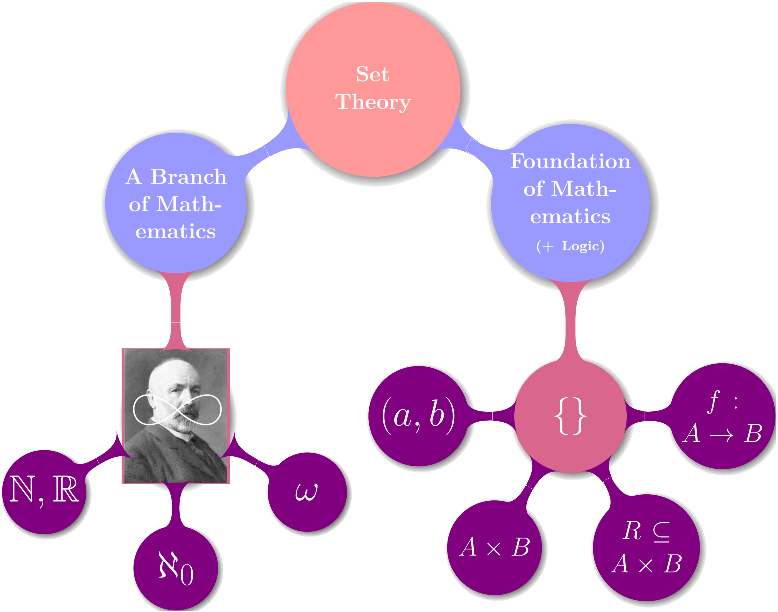
Set Theory
A Branch of Math-ematics
Foundation of Math-ematics
(+ Logic)
$\mathbb{N}, \mathbb{R}$
$\aleph_0$
ω
(a, b)
$\{\}$
$f : A \to B$
$A \times B$
$R \subseteq A \times B$

Definition (集合)

集合就是任何一个有明确定义的对象的整体。

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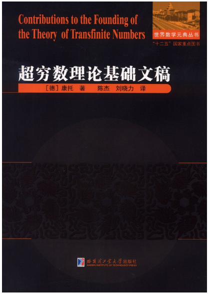

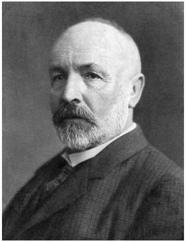

Georg Cantor (1845–1918)

Definition (集合)

我们将集合理解为任何将我们思想中那些确定而彼此独立的对象放在一起而形成的聚合。

Theorem (概括原则)

对于任意性质/谓词 $P(x)$, 都存在一个集合 X:

$$X = \{x \mid P(x)\}$$

$$A = \{2, 3, 5, 7\}$$

$$B = \{x \mid x < 10 \land \mathrm{Prime}(x)\}$$

$$C = \{x \mid x \text{ 是方程 } x^4 - 17x^3 + 101x^2 - 247x + 210 = 0 \text{ 的根}\}$$

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$$3 \in A$$

Definition (外延性原理 (Extensionality))

两个集合相等当且仅当它们有相同的元素。

$$\forall A.\, \forall B.\, \Big(\forall x.\, (x \in A \leftrightarrow x \in B) \leftrightarrow A = B\Big)$$

集合完全由它的元素决定

Definition (子集)

设 A、B 是任意两个集合。

$A \subseteq B$ 表示 A 是 B 的子集 (subset)

$$A \subseteq B \iff \forall x \in A. (x \in A \to x \in B)$$

$A \subset B$ 表示 A 是 B 的真子集 (proper subset)

$$A \subset B \iff A \subseteq B \land A \neq B$$

$$\{1,2\} \subseteq \{1,2,3\} \qquad \{1,2\} \subset \{1,2,3\} \qquad \{1,4\} \neg \subseteq \{1,2,3\}$$

Theorem

两个集合相等当且仅当它们互为子集。

$$A = B \iff A \subseteq B \land B \subseteq A$$

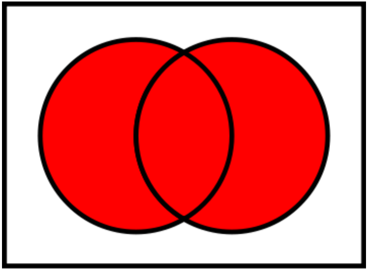
这是证明两个集合相等的常用方法

集合的运算 (I)

$\cap \qquad \cup \qquad \setminus \qquad \Delta$

Definition (集合的并 (Union))

$$A \cup B \triangleq \{x \mid x \in A \vee x \in B\}$$



$$A \cup \emptyset = A$$

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$$A \cup B = B \cup A$$

$$(A \cup B) \cup C = A \cup (B \cup C)$$

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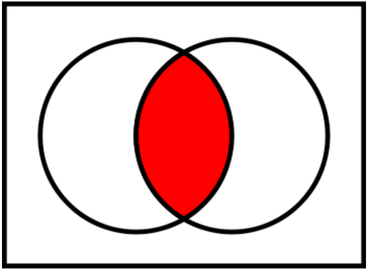
$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$A \subseteq A \cup B$$

$$B \subseteq A \cup B$$

Definition (集合的交 (Intersection))

$$A \cup B \triangleq \{x \mid x \in A \land x \in B\}$$



$$A \cap \emptyset = \emptyset$$

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$$A \cap A = A$$

$$A \cap B = B \cap A$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

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$$(A \cap B) \cap C = A \cap (B \cap C)$$

$$A \cap B \subseteq A$$

$$A \cap B \subseteq B$$

Theorem (分配律 (Distributive Law))

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Theorem (分配律 (Distributive Law))

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Proof.

If $x \in A \cup (B \cap C)$, then $x \in A$ or $x \in B \cap C$. Suppose first that $x \in A$. Then $x \in A \cup B$ and $x \in A \cup C$. In this first case, we see that $x \in (A \cup B) \cap (A \cup C)$. Now suppose that $x \in B \cap C$. Then $x \in B$ and $x \in C$. Since $x \in B$, we see that $x \in A \cup B$. Since we also have $x \in C$, we see that $x \in A \cup C$. Therefore, $x \in (A \cup B) \cap (A \cup C)$ in this case as well. In either case $x \in (A \cup B) \cap (A \cup C)$ and we may conclude that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

To complete the proof, we must now show that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. So if $x \in (A \cup B) \cap (A \cup C)$, then $x \in A \cup B$ and $x \in A \cup C$. It is, once again, helpful to break this into two cases, since we know that either $x \in A$ or $x \notin A$. Now if $x \in A$, then $x \in A \cup (B \cap C)$. If $x \notin A$, then the fact that $x \in A \cup B$ implies that x must be in B. Similarly, the fact that $x \in A \cup C$ implies that x must be in C. Therefore, $x \in B \cap C$. Hence $x \in A \cup (B \cap C)$. In either case $x \in A \cup (B \cap C)$ and we may conclude that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

Since we proved containment in both directions we may conclude that the two sets are equal. ■

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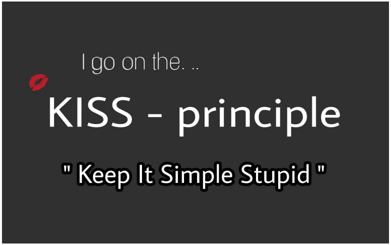


Theorem (分配律 (Distributive Law))

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Proof.

对于任意 x,

$$x \in A \cup (B \cap C) \tag{1}$$

□

Theorem (吸收律 (Absorption Law))

$$A \cup (A \cap B) = A$$

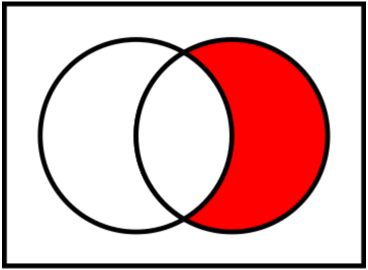
$$A \cap (A \cup B) = A$$

Theorem

$$A \subseteq B \iff A \cup B = B \iff A \cap B = A$$

Definition (集合的差 (Set Difference); 相对补 (Relative Complement))

$$A \setminus B = \{x \mid x \in A \wedge x \notin B\}$$



$$A = \{2, 5, 6\} \quad B = \{1, 2, 4, 7, 9\}$$

$$A \setminus B = \{5, 6\} \qquad B \setminus A = \{1, 4, 7, 9\}$$

Definition (绝对补 (Absolute Complement))

设全集为 U。

$$\overline{A} = U \setminus A = \{x \mid x \in U \land x \notin A\}$$

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不存在绝对的全集 (Universal Set)!!!

设全集为 U

$$\overline{\overline{A}} = A$$

$$\overline{U} = \emptyset$$

$$\overline{\emptyset} = U$$

$$A \cup \overline{A} = U$$

$$A \cap \overline{A} = \emptyset$$

Theorem

设全集为 U。

$$A \setminus B = A \cap \overline{B}$$

$$A \setminus B = A \setminus (A \cap B)$$

Theorem (德摩根律)

设全集为 U。

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Theorem (德摩根律)

设全集为 U。

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Theorem (德摩根律)

$$C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$$

Theorem

$$A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$$

Theorem

$$A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$$

Proof.

$\square$

Theorem

$$A \subseteq B \implies \overline{B} \subseteq \overline{A}$$

$$A \subseteq B \implies (B \setminus A) \cup A = B$$

Definition (对称差 (Symmetric Difference))

$$A \mathbin{\Delta} B = (A \setminus B) \cup (B \setminus A)$$

$$A \mathbin{\Delta} B = \{x \mid (x \in A) \oplus (x \in B)\}$$

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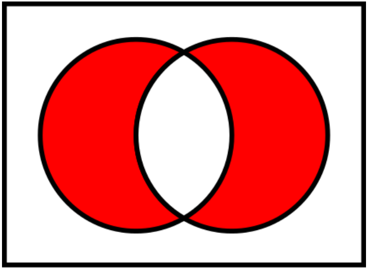

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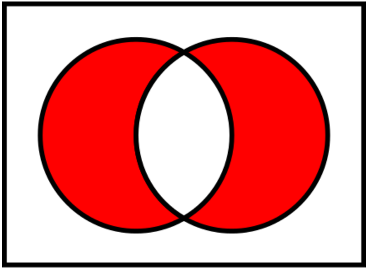

$$A \,\Delta\, B = (A \cup B) \setminus (A \cap B)$$

$$A \,\Delta\, B \,\Delta\, C$$

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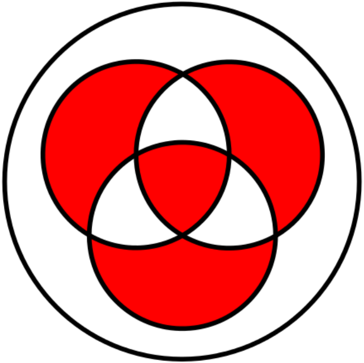

$$A \oplus \emptyset = A$$

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$$A \oplus A = \emptyset$$

$$A \oplus B = B \oplus A$$

$$(A \oplus B) \oplus C = A \oplus (B \oplus C)$$

$$(A \oplus B) \oplus C = A \oplus (B \oplus C)$$

Proof.

$\square$

$$A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$$

$$A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$$

$$A \cup (B \oplus C) \neq (A \cup B) \oplus (A \cup C)$$

$$A \cup B = A \cup C \implies B = C$$

$$A \cap B = A \cap C \implies B = C$$

$$A \oplus B = A \oplus C \implies B = C$$

集合的运算 (II)

$\cap \quad \cup$

Definition (广义并 (Arbitrary Union))

设 $\mathbb{M}$ 是一组集合 (a *collection* of sets)

$$\bigcup \mathbb{M} = \left\{ x \mid \exists A \in \mathbb{M}.\ x \in A \right\}$$

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$$\mathbb{M} = \{\{1, 2\}, \{\{1, 2\}, 3\}, \{4, 5\}\}$$

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$$\mathbb{M} = \{\{1,2\}, \{\{1,2\},3\}, \{4,5\}\}$$

$$\bigcup \mathbb{M} = \{1,2,3,4,5,\{1,2\}\}$$

$$\bigcup \emptyset = \emptyset$$

$$\bigcup_{j=1}^{n} A_j \triangleq A_1 \cup A_2 \cup \cdots \cup A_n$$

$$\bigcup_{j=1}^{n} A_j \triangleq A_1 \cup A_2 \cup \cdots \cup A_n$$

$$\bigcup_{j=1}^{\infty} A_j \triangleq A_1 \cup A_2 \cup \cdots$$

$$\bigcup_{j=1}^{n} A_j \triangleq A_1 \cup A_2 \cup \cdots \cup A_n$$

$$\bigcup_{j=1}^{\infty} A_j \triangleq A_1 \cup A_2 \cup \cdots$$

$$\bigcup_{\alpha \in I} A_\alpha \triangleq \Big\{ x \mid \exists \alpha \in I : x \in A_\alpha \Big\}$$

Definition (广义交 (Arbitrary Intersection))

设 $\mathbb{M}$ 是一组集合 (a *collection* of sets)

$$\bigcap \mathbb{M} = \Big\{ x \mid \forall A \in \mathbb{M}.\ x \in A \Big\}$$

$$\bigcap_{j=1}^{n} A_j \triangleq A_1 \cap A_2 \cap \cdots \cap A_n$$

$$\bigcap_{j=1}^{n} A_j \triangleq A_1 \cap A_2 \cap \cdots \cap A_n$$

$$\bigcap_{j=1}^{\infty} A_j \triangleq A_1 \cap A_2 \cap \cdots$$

$$\bigcap_{j=1}^{n} A_j \triangleq A_1 \cap A_2 \cap \cdots \cap A_n$$

$$\bigcap_{j=1}^{\infty} A_j \triangleq A_1 \cap A_2 \cap \cdots$$

$$\bigcap_{\alpha \in I} A_\alpha \triangleq \Big\{ x \mid \forall \alpha \in I : x \in A_\alpha \Big\}$$

Theorem (德摩根律)

$$X \setminus \bigcup_{\alpha \in I} A_\alpha = \bigcap_{\alpha \in I} (X \setminus A_\alpha)$$

$$X \setminus \bigcap_{\alpha \in I} A_\alpha = \bigcup_{\alpha \in I} (X \setminus A_\alpha)$$

Theorem (德摩根律)

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Proof.

□

德摩根律的应用

$$A = \mathbb{R} \setminus \bigcap_{n \in \mathbb{Z}^+} \left(\mathbb{R} \setminus \{-n, -n+1, \cdots, 0, \cdots, n-1, n\} \right)$$

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$$X_n = \{-n, -n+1, \cdots, 0, \cdots, n-1, n\}$$

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$$A = \mathbb{R} \setminus \bigcap_{n \in \mathbb{Z}^+} (\mathbb{R} \setminus X_n)$$

德摩根律的应用

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$$X_n = \{-n, -n+1, \cdots, 0, \cdots, n-1, n\}$$

$$\begin{aligned} A &= \mathbb{R} \setminus \bigcap_{n \in \mathbb{Z}^+} (\mathbb{R} \setminus X_n) \\ &= \mathbb{R} \setminus \left(\mathbb{R} \setminus \bigcup_{n \in \mathbb{Z}^+} X_n\right) \end{aligned}$$

德摩根律的应用

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德摩根律的应用

$$A = \mathbb{R} \setminus \bigcap_{n \in \mathbb{Z}^+} (\mathbb{R} \setminus \{-n, -n+1, \cdots, 0, \cdots, n-1, n\})$$

$$X_n = \{-n, -n+1, \cdots, 0, \cdots, n-1, n\}$$

$$\begin{aligned}
A &= \mathbb{R} \setminus \bigcap_{n \in \mathbb{Z}^+} (\mathbb{R} \setminus X_n) \\
&= \mathbb{R} \setminus \left(\mathbb{R} \setminus \bigcup_{n \in \mathbb{Z}^+} X_n\right) \\
&= \mathbb{R} \setminus \left(\mathbb{R} \setminus \mathbb{Z}\right) \\
&= \mathbb{Z}
\end{aligned}$$

集合的运算 (III)

$\mathcal{P}(X)$

Definition (幂集 (Powerset))

$$\mathcal{P}(()A) = \left\{X \mid X \subseteq A\right\}$$

$$A = \{1, 2, 3\}$$

$$\mathcal{P}(()A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

$$|A| = n$$

$$|A| = n$$

$$|\mathcal{P}(A)| = 2^n$$

$$|A| = n$$

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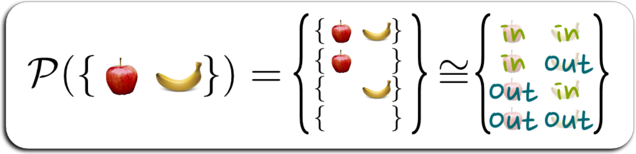
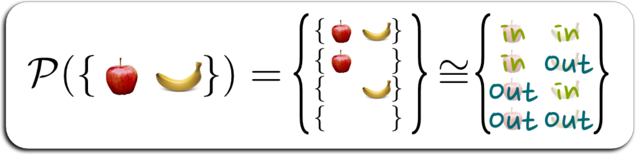


$$|A| = n$$

$$|\mathcal{P}(A)| = 2^n$$



$$\mathcal{P}(A) = 2^A = \{0, 1\}^A$$

$$\boxed{S \in \mathcal{P}(X) \iff S \subseteq X}$$

请证明

$$\{\emptyset, \{\emptyset\}\} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(S)))$$

Proof.

请证明

$$\{\emptyset, \{\emptyset\}\} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(S)))$$

Proof.

$$\{\emptyset, \{\emptyset\}\} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(S))) \iff \{\emptyset, \{\emptyset\}\} \subseteq \mathcal{P}(\mathcal{P}(S))$$

请证明

$$\{\emptyset, \{\emptyset\}\} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(S)))$$

Proof.

$$\{\emptyset, \{\emptyset\}\} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(S))) \iff \{\emptyset, \{\emptyset\}\} \subseteq \mathcal{P}(\mathcal{P}(S))$$

$$\{\emptyset\} \in \mathcal{P}(\mathcal{P}(S))$$

$$\emptyset \in \mathcal{P}(\mathcal{P}(S))$$

□

请证明

$$\{\emptyset, \{\emptyset\}\} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(S)))$$

Proof.

$$\{\emptyset, \{\emptyset\}\} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(S))) \iff \{\emptyset, \{\emptyset\}\} \subseteq \mathcal{P}(\mathcal{P}(S))$$

$$\{\emptyset\} \in \mathcal{P}(\mathcal{P}(S))$$

$$\emptyset \in \mathcal{P}(\mathcal{P}(S))$$

$$\iff \emptyset \subseteq \mathcal{P}(S)$$

□

请证明

$$\{\emptyset, \{\emptyset\}\} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(S)))$$

Proof.

$$\{\emptyset, \{\emptyset\}\} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(S))) \iff \{\emptyset, \{\emptyset\}\} \subseteq \mathcal{P}(\mathcal{P}(S))$$

$$\{\emptyset\} \in \mathcal{P}(\mathcal{P}(S))$$

$$\emptyset \in \mathcal{P}(\mathcal{P}(S)) \qquad \iff \{\emptyset\} \subseteq \mathcal{P}(S)$$

$$\iff \emptyset \subseteq \mathcal{P}(S)$$

□

请证明

$$\{\emptyset, \{\emptyset\}\} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(S)))$$

Proof.

$$\{\emptyset, \{\emptyset\}\} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(S))) \iff \{\emptyset, \{\emptyset\}\} \subseteq \mathcal{P}(\mathcal{P}(S))$$

$$\{\emptyset\} \in \mathcal{P}(\mathcal{P}(S))$$

$$\emptyset \in \mathcal{P}(\mathcal{P}(S)) \qquad \iff \{\emptyset\} \subseteq \mathcal{P}(S)$$

$$\iff \emptyset \subseteq \mathcal{P}(S) \qquad \iff \emptyset \in \mathcal{P}(S)$$

□

请证明

$$\{\emptyset, \{\emptyset\}\} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(S)))$$

Proof.

$$\frac{\{\emptyset, \{\emptyset\}\} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(S))) \iff \{\emptyset, \{\emptyset\}\} \subseteq \mathcal{P}(\mathcal{P}(S))}{\emptyset \in \mathcal{P}(\mathcal{P}(S)) \qquad \{\emptyset\} \in \mathcal{P}(\mathcal{P}(S))}$$

$$\emptyset \in \mathcal{P}(\mathcal{P}(S)) \iff \emptyset \subseteq \mathcal{P}(S)$$

$$\{\emptyset\} \in \mathcal{P}(\mathcal{P}(S)) \iff \{\emptyset\} \subseteq \mathcal{P}(S) \iff \emptyset \in \mathcal{P}(S) \iff \emptyset \subseteq S$$

□

请证明

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对于任意 x,

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对于任意 x,

$$x \in \mathcal{P}(A) \cap \mathcal{P}(B)$$

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请证明

$$\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$$

对于任意 x,

$$\begin{aligned} & x \in \mathcal{P}(A) \cap \mathcal{P}(B) \\ \iff & x \in \mathcal{P}(A) \land x \in \mathcal{P}(B) \\ \iff & x \subseteq A \land x \subseteq B \end{aligned}$$

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$$\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$$

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& x \in \mathcal{P}(A) \cap \mathcal{P}(B) \\
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\iff & x \subseteq A \land x \subseteq B \\
\iff & x \subseteq A \cap B
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$$\begin{aligned}
& x \in \mathcal{P}(A) \cap \mathcal{P}(B) \\
\iff & x \in \mathcal{P}(A) \land x \in \mathcal{P}(B) \\
\iff & x \subseteq A \land x \subseteq B \\
\iff & x \subseteq A \cap B \\
\iff & x \in \mathcal{P}(A \cap B)
\end{aligned}$$

请证明

$$\bigcap_{\alpha \in I} \mathcal{P}(A_\alpha) = \mathcal{P}(\bigcap_{\alpha \in I} A_\alpha)$$

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对于任意 x,

$$x \in \bigcap_{\alpha \in I} \mathcal{P}(A_\alpha)$$

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\iff & \forall \alpha \in I : x \in \mathcal{P}(A_\alpha) \\
\iff & \forall \alpha \in I : x \subseteq A_\alpha \\
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\begin{aligned}
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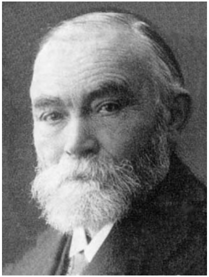
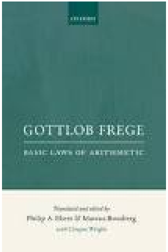
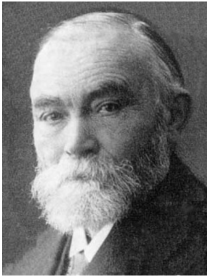
\iff\ & \forall \alpha \in I : x \subseteq A_\alpha \\
\iff\ & x \subseteq \bigcap_{\alpha\in I} A_\alpha \\
\iff\ & x \in \mathcal{P}(\bigcap_{\alpha\in I} A_\alpha)
\end{aligned}
$$



Gottlob Frege (1848–1925)



“Basic Laws of Arithmetic”
(1893 & 1903)



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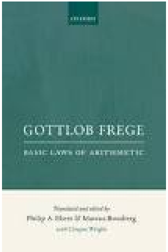
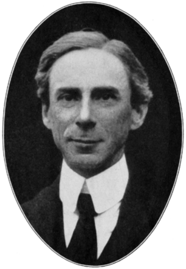


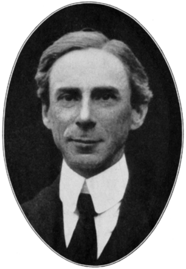
"Basic Laws of Arithmetic"
(1893 & 1903)

对于一个科学工作者来说，最不幸的事情莫过于：当他的工作接近完成时，却发现那大厦的基础已经动摇。

— 《附录二》，*1902*



Bertrand Russell (1872–1970)



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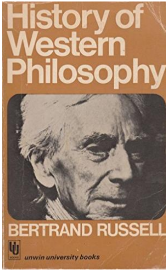

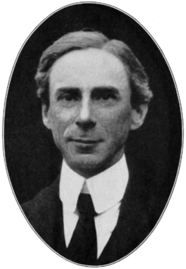

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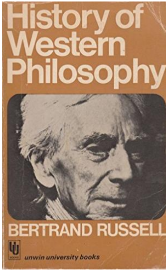

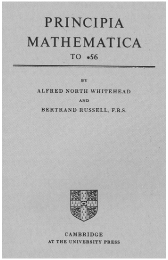

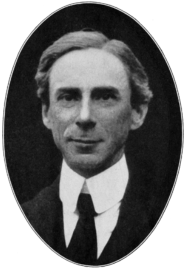

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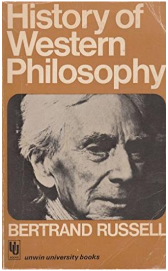

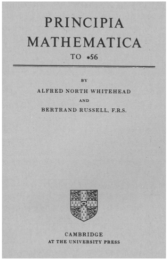

Theorem (概括原则)

For any predicate $\psi(x)$, *there is a set* X:

$$X = \{x \mid \psi(x)\}.$$

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$$Q : R \in R\,?$$

Q： 既然朴素集合论存在悖论，你是如何做作业的?


YOU HAVE
TO FIX IT!

Theorem (Russell's Paradox)

$$\{x \mid x \notin x\} \text{ is } \textit{not} \text{ a set.}$$

Axiomatic Set Theory (ZFC)

Ernst Zermelo (1871–1953)

Abraham Fraenkel (1891–1965)

First-order Language for Sets $\mathcal{L}_{Set} = \{\in\}$

Parentheses: $(,)$

Variables: $x, y, z, \cdots$

Connectives: $\wedge$, $\vee$, $\neg$, $\to$, $\leftrightarrow$

Quantifiers: $\forall$, $\exists$

Equality: $=$

Constants:

Functions:

Predicates: $\in$

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Everything we consider in $\mathcal{L}_{Set}$ is a set.

Q : What is "$\in$"?

Q : What are "sets"?

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Q : What are "sets"?

We don't define them directly.

We only describe their properties in an axiomatic way.

(1) To draw a straight line from any point to any point.

(2) To extend a finite straight line continuously in a straight line.

(3) To describe a circle with any center and radius.

(4) That all right angles are equal to one another.

(5) The parallel postulate.

Definition ($\notin$)

$$x \notin A \triangleq \neg(x \in A).$$

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Definition ($\subseteq$)

$$A \subseteq B \triangleq \forall x(x \in A \implies x \in B)$$

Definition ("$\bigcup A$" (Arbitrary Union))

$$\bigcup A \triangleq \text{ the unique set obtained by unioning } A.$$

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Theorem

$$\bigcup \emptyset = \emptyset.$$

Theorem ("$\bigcap A$" (Arbitrary Intersection))

For any nonempty set A, there is a unique set B such that

$$\forall x \; (x \in B \iff x \textit{ belongs to every member of } A).$$

$$\forall x \; \big(x \in B \iff \forall y \in A (x \in y)\big).$$

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$\square$

"$\bigcap \emptyset$"

$\bigcap \emptyset$ is *not* a set.

Theorem (No Universal Set)

There is no universal set.

$$\nexists B (\forall x (x \in B)).$$

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$$B \in A \implies (B \in B \iff B \notin B)$$

□

Definition ("$u \setminus v$" (Relative Complement))

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We can never look for objects “not in B” unless we know where to start looking. — *UD (Chapter 6; Page 64)*

Definition (Power Set Axiom)

For any set A, there is a set whose members are the subsets of A:

$$\forall A \; \exists B \; \forall x (x \in B \iff x \subseteq A).$$

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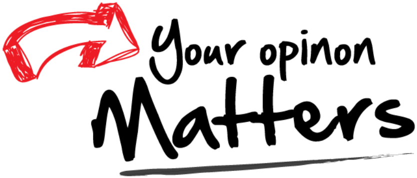
The is *not* correct!

$$\mathcal{P}(A) \triangleq \{x \mid x \subseteq A\}$$


Thank
You!



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