(四) 集合: 关系 (Relation)

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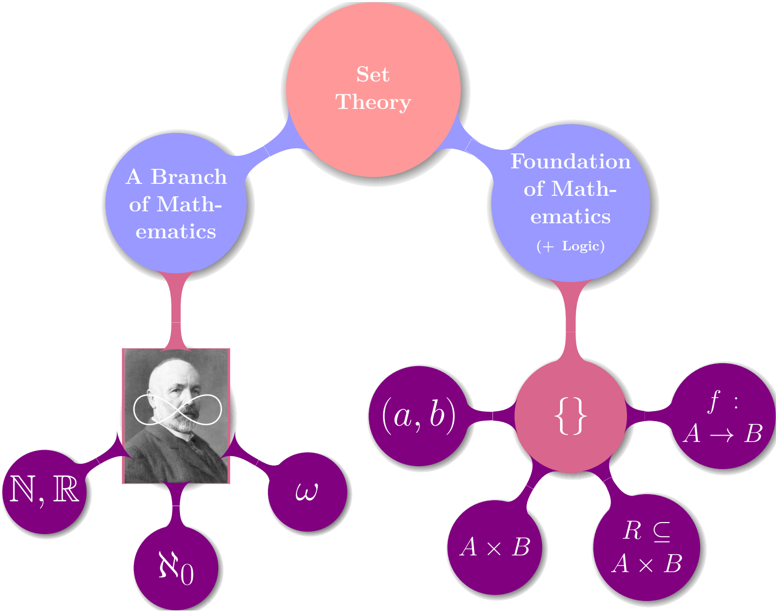

Set Theory
A Branch of Mathematics
Foundation of Mathematics
(+ Logic)
$\mathbb{N}, \mathbb{R}$
$\aleph_0$
ω
(a, b)
$\{\}$
$f : A \to B$
$A \times B$
$R \subseteq A \times B$

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Figure 13. A selection of consistency axioms over an execution $(E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar})$

Auxiliary relations

$\mathsf{sameobj}(e, f) \iff \mathsf{obj}(e) = \mathsf{obj}(f)$

Per-object causality (aka happens-before) order:
$\mathsf{hbo} = ((\mathsf{ro} \cap \mathsf{sameobj}) \cup \mathsf{vis})^+$

Causality (aka happens-before) order: $\mathsf{hb} = (\mathsf{ro} \cup \mathsf{vis})^+$

Axioms

EVENTUAL:
$\forall e \in E.\ \neg(\exists \text{ infinitely many } f \in E.\ \mathsf{sameobj}(e, f) \wedge \neg(e \xrightarrow{\mathsf{vis}} f))$

THINAIR: $\mathsf{ro} \cup \mathsf{vis}$ is acyclic

POCV (Per-Object Causal Visibility): $\mathsf{hbo} \subseteq \mathsf{vis}$

POCA (Per-Object Causal Arbitration): $\mathsf{hbo} \subseteq \mathsf{ar}$

COCV (Cross-Object Causal Visibility): $(\mathsf{hb} \cap \mathsf{sameobj}) \subseteq \mathsf{vis}$

COCA (Cross-Object Causal Arbitration): $\mathsf{hb} \cup \mathsf{ar}$ is acyclic

Figure 17. Optimized state-based multi-value register and its simulation

$$
\begin{array}{ll}
\Sigma & = \mathsf{ReplicaID} \times \mathcal{P}(\mathbb{Z} \times (\mathsf{ReplicaID} \to \mathbb{N}_0)) \\
\vec{\sigma}_0 & = \langle r, \emptyset \rangle \\
M & = \mathcal{P}(\mathbb{Z} \times (\mathsf{ReplicaID} \to \mathbb{N}_0)) \\
\mathsf{do}(\mathtt{wr}(a), \langle r, V \rangle, t) = & \\
\quad (\langle r, \{(a, (\lambda s.\ \text{if } s \neq r \text{ then } \max\{v(s) \mid (\_, v) \in V\} & \\
\qquad \text{else } \max\{v(s) \mid (\_, v) \in V\} + 1))\}\rangle, \bot) & \\
\mathsf{do}(\mathtt{rd}, \langle r, V \rangle, t) & = (\langle r, V \rangle, \{a \mid (a, \_) \in V\}) \\
\mathsf{send}(\langle r, V \rangle) & = (\langle r, V \rangle, V) \\
\mathsf{receive}(\langle r, V \rangle, V') & = \langle r, \{(a, v) \in V'' \mid \\
 & \quad v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'.\, (a', v') \in V'' \wedge a \neq a'\}\}\rangle, \\
\multicolumn{2}{l}{\text{where } V'' = \{(a, \bigsqcup\{v' \mid (a, v') \in V \cup V'\}) \mid (a, \_) \in V \cup V'\}}
\end{array}
$$

$$\langle s, V \rangle\ [\mathcal{R}_r]\ I \iff (r = s) \wedge (V\ [\mathcal{M}]\ I)$$

$$
\begin{array}{l}
V\ [\mathcal{M}]\ ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar}), \mathsf{info}) \iff \\
\quad (\forall (a, v), (a', v') \in V.\ (a = a' \implies v = v')) \wedge \\
\quad (\forall (a, v) \in V.\ \exists s.\ v(s) > 0) \wedge \\
\quad (\forall (a, v) \in V.\ v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'.\, (a', v') \in V \wedge a \neq a'\}) \wedge \\
\quad \exists \text{ distinct } e_{s,k}. \\
\quad (\{e \in E \mid \exists a.\, \mathsf{oper}(e) = \mathtt{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicaID} \wedge \\
\qquad 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V\}\}) \wedge \\
\quad (\forall s, j, k.\ (\mathsf{repl}(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}} e_{s,k} \iff j < k)) \wedge \\
\quad (\forall (a, v) \in V.\ \forall q.\ \{j \mid \mathsf{oper}(e_{q,j}) = \mathtt{wr}(a)\} \cup \\
\qquad \{j \mid \exists s, k.\, e_{q,j} \xrightarrow{\mathsf{vis}} e_{s,k} \wedge \mathsf{oper}(e_{s,k}) = \mathtt{wr}(a)\} = \\
\qquad \{j \mid 1 \leq j \leq v(q)\}) \wedge \\
\quad (\forall e \in E.\ (\mathsf{oper}(e) = \mathtt{wr}(a) \wedge \\
\qquad \neg\exists f \in E.\, \mathsf{oper}(f) = \mathtt{wr}(\_) \wedge e \xrightarrow{\mathsf{vis}} f) \implies (a, \_) \in V)
\end{array}
$$

the former. The only non-trivial obligation is to show that if

$$V\ [\mathcal{M}]\ ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}), \mathsf{info}),$$

then

$$
\begin{aligned}
\{a \mid (a, \_) \in V\} \subseteq \{a \mid \exists e \in E.\, & \mathsf{oper}(e) = \mathtt{wr}(a) \wedge \\
& \neg\exists f \in E.\, \exists a'.\, \mathsf{oper}(e) = \mathtt{wr}(a') \wedge e \xrightarrow{\mathsf{vis}} f\}
\end{aligned} \tag{13}
$$

(the reverse inclusion is straightforwardly implied by $\mathcal{R}_r$).
Take $(a, v) \in V$. We have $\forall (a, v) \in V.\ \exists s.\ v(s) > 0$,

$$v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'.\, (a', v') \in V \wedge a \neq a'\}$$

and

$$
\begin{array}{l}
\forall (a, v) \in V.\ \forall q.\ \{j \mid \mathsf{oper}(e_{q,j}) = \mathtt{wr}(a)\} \cup \\
\qquad \{j \mid \exists s, k.\ e_{q,j} \xrightarrow{\mathsf{vis}} e_{s,k} \wedge \mathsf{oper}(e_{s,k}) = \mathtt{wr}(a)\} = \\
\qquad \{j \mid 1 \leq j \leq v(q)\}.
\end{array}
$$

From this we get that for some $e \in E$

$$
\begin{aligned}
\mathsf{oper}(e) = \mathtt{wr}(a) \wedge \neg\exists f \in E.\, \exists a'.\, a' \neq a \wedge & \\
\mathsf{oper}(e) = \mathtt{wr}(a') \wedge e \xrightarrow{\mathsf{vis}} f. &
\end{aligned}
$$

Since vis is acyclic, this implies that for some $e' \in E$

$$\mathsf{oper}(e') = \mathtt{wr}(a) \wedge \neg\exists f \in E.\, \mathsf{oper}(e') = \mathtt{wr}(\_) \wedge e' \xrightarrow{\mathsf{vis}} f,$$

which establishes (13).
Let us now discharge RECEIVE. Let $\mathsf{receive}(\langle r, V \rangle, V') = \langle r, V''' \rangle$, where

$$
\begin{aligned}
V'' &= \{(a, \bigsqcup\{v' \mid (a, v') \in V \cup V'\}) \mid (a, \_) \in V \cup V'\}; \\
V''' &= \{(a, v) \in V'' \mid v \not\sqsubseteq \bigsqcup\{(a', v') \in V'' \mid a \neq a'\}\}.
\end{aligned}
$$

Assume $\langle r, V \rangle\ [\mathcal{R}_r]\ I$, $V'\ [\mathcal{M}]\ J$ and

$$
\begin{aligned}
I &= ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar}), \mathsf{info}); \\
J &= ((E', \mathsf{repl}', \mathsf{obj}', \mathsf{oper}', \mathsf{rval}', \mathsf{ro}', \mathsf{vis}', \mathsf{ar}'), \mathsf{info}'); \\
I \sqcup J &= ((E'', \mathsf{repl}'', \mathsf{obj}'', \mathsf{oper}'', \mathsf{rval}'', \mathsf{ro}'', \mathsf{vis}'', \mathsf{ar}''), \mathsf{info}'').
\end{aligned}
$$

By agree we have $I \sqcup J \in \mathsf{IEx}$. Then

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\begin{array}{l}
(\forall (a, v), (a', v') \in V.\ (a = a' \implies v = v')) \wedge \\
(\forall (a, v) \in V.\ \exists s.\ v(s) > 0) \wedge \\
(\forall (a, v) \in V.\ v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'.\, (a', v') \in V \wedge a \neq a'\}) \wedge \\
\exists \text{ distinct } e_{s,k}. \\
(\{e \in E \mid \exists a.\, \mathsf{oper}''(e) = \mathtt{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicaID} \wedge \\
\quad 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V\}\}) \wedge \\
(\forall s, j, k.\ (\mathsf{repl}''(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}} e_{s,k} \iff j < k)) \wedge \\
(\forall (a, v) \in V.\ \forall q.\ \{j \mid \mathsf{oper}''(e_{q,j}) = \mathtt{wr}(a)\} \cup \\
\quad \{j \mid \exists s, k.\, e_{q,j} \xrightarrow{\mathsf{vis}} e_{s,k} \wedge \mathsf{oper}''(e_{s,k}) = \mathtt{wr}(a)\} = \\
\quad \{j \mid 1 \leq j \leq v(q)\}) \wedge \\
(\forall e \in E.\ (\mathsf{oper}''(e) = \mathtt{wr}(a) \wedge \\
\quad \neg\exists f \in E.\, \mathsf{oper}''(f) = \mathtt{wr}(\_) \wedge e \xrightarrow{\mathsf{vis}} f) \implies (a, \_) \in V)
\end{array}
$$

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$$
\begin{array}{l}
(\forall (a, v), (a', v') \in V'.\ (a = a' \implies v = v')) \wedge \\
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\end{array}
$$

The agree property also implies

$$
\begin{aligned}
\forall s, k.\ 1 \leq k \leq \min\{ & \max\{v(s) \mid \exists a.\, (a, v) \in V\}, \\
& \max\{v(s) \mid \exists a.\, (a, v) \in V'\}\} \implies e_{s,k} = e'_{s,k}.
\end{aligned}
$$

Hence, there exist distinct

$$e''_{s,k} \text{ for } s \in \mathsf{ReplicaID},\ k = 1..(\max\{v(s) \mid \exists a.\, (a, v) \in V'''\}),$$

such that

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\begin{array}{l}
(\forall s, k.\ 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V\} \implies e''_{s,k} = e_{s,k}) \wedge \\
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\end{array}
$$

and

$$
\begin{array}{l}
(\{e \in E \cup E' \mid \exists a.\, \mathsf{oper}''(e) = \mathtt{wr}(a)\} = \\
\{e''_{s,k} \mid s \in \mathsf{ReplicaID} \wedge 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V'''\}\}) \\
\wedge (\forall s, j, k.\ (\mathsf{repl}(e''_{s,k}) = s) \wedge (e''_{s,j} \xrightarrow{\mathsf{ro}''} e''_{s,k} \iff j < k)).
\end{array}
$$

By the definition of V'' and V''' we have

$$\forall (a, v), (a', v') \in V'''.\ (a = a' \implies v = v').$$

We also straightforwardly get

$$\forall (a, v) \in V'.\ \exists s.\ v(s) > 0$$

and

$$
\begin{array}{l}
(\forall (a, v) \in V''.\ \forall q.\ \{j \mid \mathsf{oper}''(e''_{q,j}) = \mathtt{wr}(a)\} \cup \\
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Figure 13. A selection of consistency axioms over an execution $(E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar})$

Auxiliary relations

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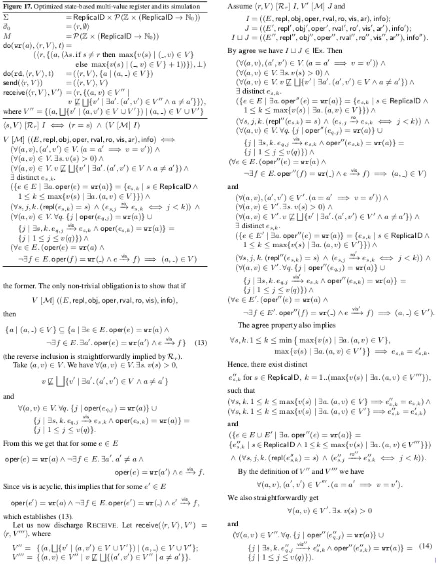

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&\quad (\{e \in E \mid \exists a.\ \mathsf{oper}(e) = \mathtt{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicaID} \wedge\\
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&\quad (\forall s, j, k.\ (\mathsf{repl}(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}} e_{s,k} \iff j < k)) \wedge\\
&\quad (\forall (a, v) \in V.\ \forall q.\ \{j \mid \mathsf{oper}(e_{q,j}) = \mathtt{wr}(a)\} \cup\\
&\qquad \{j \mid \exists s, k.\ e_{q,j} \xrightarrow{\mathsf{vis}} e_{s,k} \wedge \mathsf{oper}(e_{s,k}) = \mathtt{wr}(a)\} =\\
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&\quad (\forall e \in E.\ (\mathsf{oper}(e) = \mathtt{wr}(a) \wedge\\
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then

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\{a \mid (a, \_) \in V\} \subseteq \{a \mid \exists e \in E.\ \mathsf{oper}(e) = \mathtt{wr}(a) \wedge&\\
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(the reverse inclusion is straightforwardly implied by $\mathcal{R}_r$).
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and

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&\quad \{j \mid 1 \leq j \leq v(q)\}.
\end{aligned}
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From this we get that for some $e \in E$

$$\mathsf{oper}(e) = \mathtt{wr}(a) \wedge \neg\exists f \in E.\ \exists a'.\ a' \neq a \wedge \mathsf{oper}(e) = \mathtt{wr}(a') \wedge e \xrightarrow{\mathsf{vis}} f.$$

Since vis is acyclic, this implies that for some $e' \in E$

$$\mathsf{oper}(e') = \mathtt{wr}(a) \wedge \neg\exists f \in E.\ \mathsf{oper}(e') = \mathtt{wr}(\_) \wedge e' \xrightarrow{\mathsf{vis}} f,$$

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Let us now discharge RECEIVE. Let $\mathsf{receive}(\langle r, V\rangle, V') = \langle r, V'''\rangle$, where

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\end{aligned}
$$

Assume $\langle r, V\rangle\ [\mathcal{R}_r]\ I$, $V'\ [\mathcal{M}]\ J$ and

$$
\begin{aligned}
I &= ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar}), \mathsf{info});\\
J &= ((E', \mathsf{repl}', \mathsf{obj}', \mathsf{oper}', \mathsf{rval}', \mathsf{ro}', \mathsf{vis}', \mathsf{ar}'), \mathsf{info}');\\
I \sqcup J &= ((E'', \mathsf{repl}'', \mathsf{obj}'', \mathsf{oper}'', \mathsf{rval}'', \mathsf{ro}'', \mathsf{vis}'', \mathsf{ar}''), \mathsf{info}'').
\end{aligned}
$$

By agree we have $I \sqcup J \in \mathsf{IEx}$. Then

$$
\begin{aligned}
&(\forall (a, v), (a', v') \in V.\ (a = a' \implies v = v')) \wedge\\
&(\forall (a, v) \in V.\ \exists s.\ v(s) > 0) \wedge\\
&(\forall (a, v) \in V.\ v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'.\ (a', v') \in V \wedge a \neq a'\}) \wedge\\
&\exists \text{ distinct } e_{s,k}.\\
&(\{e \in E \mid \exists a.\ \mathsf{oper}''(e) = \mathtt{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicaID} \wedge\\
&\quad 1 \leq k \leq \max\{v(s) \mid \exists a.\ (a, v) \in V\}\}) \wedge\\
&(\forall s, j, k.\ (\mathsf{repl}''(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}} e_{s,k} \iff j < k)) \wedge\\
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The agree property also implies

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\forall s, k.\ 1 \leq k \leq \min\{&\max\{v(s) \mid \exists a.\ (a, v) \in V\},\\
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such that

$$
\begin{aligned}
&(\forall s, k.\ 1 \leq k \leq \max\{v(s) \mid \exists a.\ (a, v) \in V\} \implies e''_{s,k} = e_{s,k}) \wedge\\
&(\forall s, k.\ 1 \leq k \leq \max\{v(s) \mid \exists a.\ (a, v) \in V'\} \implies e''_{s,k} = e'_{s,k})
\end{aligned}
$$

and

$$
\begin{aligned}
&(\{e \in E \cup E' \mid \exists a.\ \mathsf{oper}''(e) = \mathtt{wr}(a)\} =\\
&\{e''_{s,k} \mid s \in \mathsf{ReplicaID} \wedge 1 \leq k \leq \max\{v(s) \mid \exists a.\ (a, v) \in V'''\}\})\\
&\wedge (\forall s, j, k.\ (\mathsf{repl}(e''_{s,k}) = s) \wedge (e''_{s,j} \xrightarrow{\mathsf{ro}''} e''_{s,k} \iff j < k)).
\end{aligned}
$$

By the definition of V'' and V''' we have

$$\forall (a, v), (a', v') \in V'''.\ (a = a' \implies v = v').$$

We also straightforwardly get

$$\forall (a, v) \in V'.\ \exists s.\ v(s) > 0$$

and

$$
\begin{aligned}
&(\forall (a, v) \in V''.\ \forall q.\ \{j \mid \mathsf{oper}''(e''_{q,j}) = \mathtt{wr}(a)\} \cup\\
&\quad \{j \mid \exists s, k.\ e''_{q,j} \xrightarrow{\mathsf{vis}''} e''_{s,k} \wedge \mathsf{oper}''(e''_{s,k}) = \mathtt{wr}(a)\} = \quad (14)\\
&\quad \{j \mid 1 \leq j \leq v(q)\}).
\end{aligned}
$$


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I'm so excited.

离散数学学得好不好，

一个重要的衡量标准就是是否完成了这种转变

The Relational Data Model

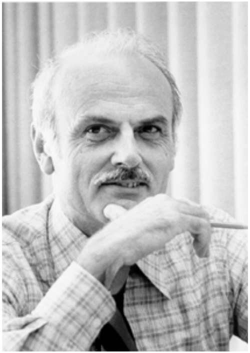
A Relational Model of Data for Large Shared Data Banks

E. F. Codd
IBM Research Laboratory, San Jose, California

Future users of large data banks must be protected from having to know how the data is organized in the machine (the internal representation). A prompting service which supplies such information is not a satisfactory solution. Activities of users at terminals and most application programs should remain unaffected when the internal representation of data is changed and even when some aspects of the external representation are changed. Changes in data representation will often be needed as a result of changes in query, update, and report traffic and natural growth in the types of stored information.

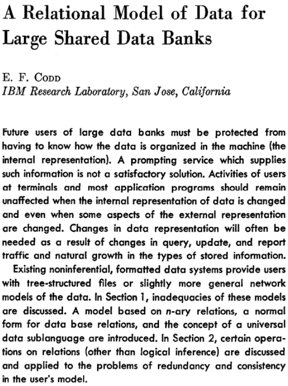
Existing noninferential, formatted data systems provide users with tree-structured files or slightly more general network models of the data. In Section 1, inadequacies of these models are discussed. A model based on *n*-ary relations, a normal form for data base relations, and the concept of a universal data sublanguage are introduced. In Section 2, certain operations on relations (other than logical inference) are discussed and applied to the problems of redundancy and consistency in the user's model.

Codd@CACM'1970
(Turing Award'1981)



Edgar F. Codd (1923 – 2003)

The Relational Data Model
— 如何靠“关系”赢得图灵奖?



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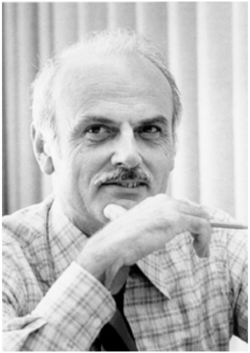

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"Near" 关系: $|a - b| < 1$

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$$R = \{(a, b) \mid |a - b| < 1\}$$

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$$\boxed{\text{自反性} + \text{反对称性} = \textbf{相容关系}}$$

$$X = \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$$

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X 上的整除关系

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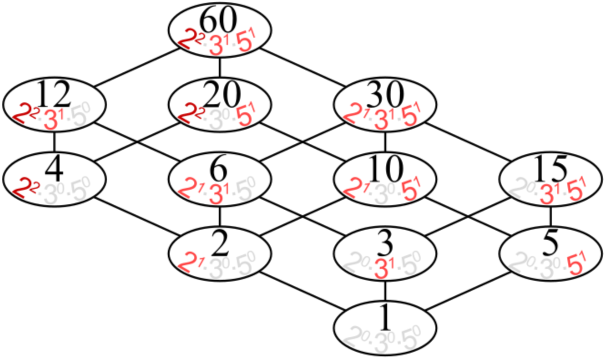

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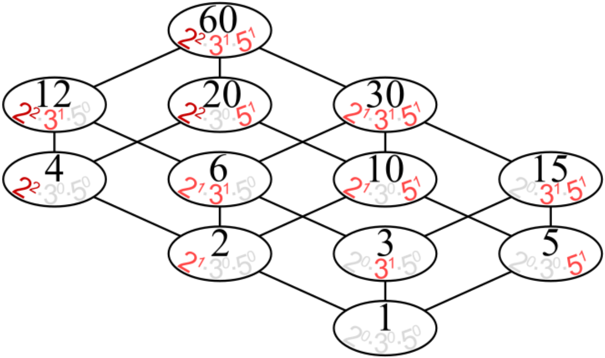

“偏序” 严格刻画了人类对于“序”的直观理解

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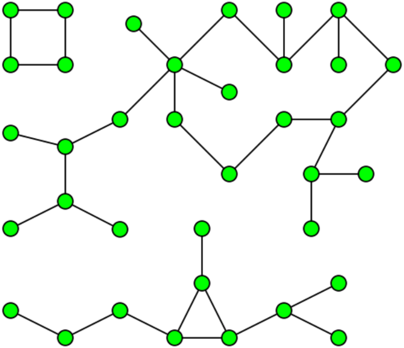
考虑无向图中的顶点集合



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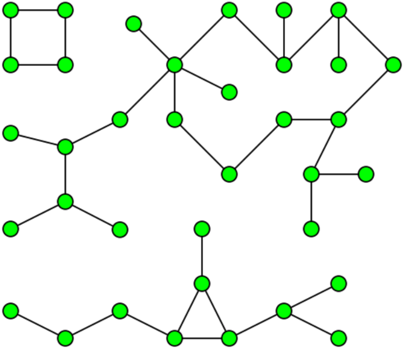
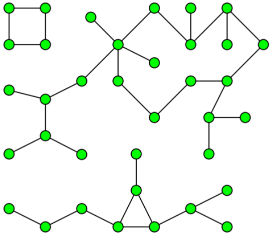
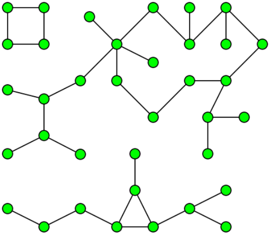


顶点间的 “可达 (Reachability) 关系”: $R = \{(a, b) \mid a \rightsquigarrow b\}$





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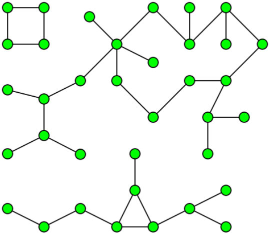

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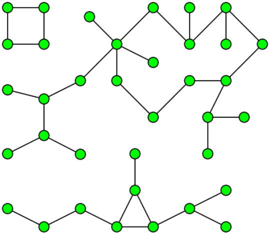

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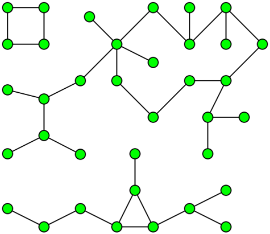

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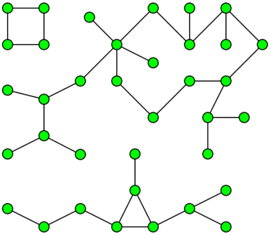

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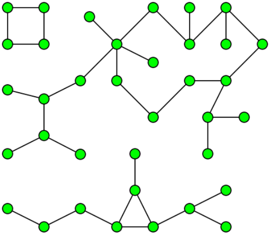

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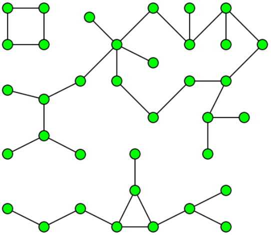

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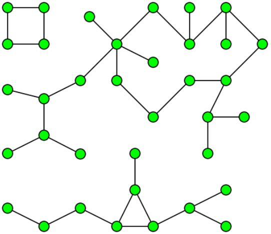

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$\boxed{\text{自反性 + 对称性 + 传递性 = 等价关系}}$

“可达关系” 将顶点划分成相互独立的 “连通分量”

Definition (有序对 (Ordered Pairs))

$$(a, b) = (c, d) \iff a = c \land b = d$$

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Definition (有序对 (Ordered Pairs) (Kazimierz Kuratowski; 1921))

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$$(a, b) \triangleq \{\{a\}, \{a, b\}\}$$

Theorem

$$(a, b) = (c, d) \iff a = c \land b = d$$

Proof.

$$\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$$

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$$(a, b) = (c, d) \iff a = c \land b = d$$

Proof.

$$\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$$

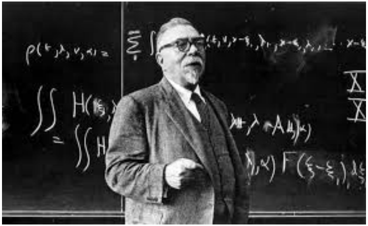
Case I : $a = b$

Case II : $a \neq b$

$\square$

Definition (Ordered Pairs (Norbert Wiener; 1914))

$$(a, b) \triangleq \Big\{\{\{a\}, \emptyset\}, \{\{b\}\}\Big\}$$



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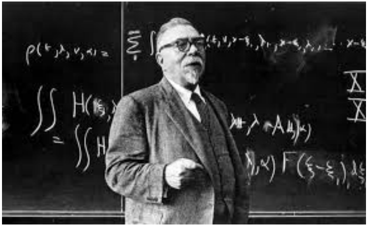

Theorem

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Definition (n-元组 (n-ary tuples))

$$(x, y, z) \triangleq ((x, y), z)$$

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Theorem

$$(x_1, \ldots, x_n) = (y_1, \ldots, y_n) \iff x_1 = y_1 \land \ldots x_n = y_n$$

Definition (笛卡尔积 (Cartesian Products))

The *Cartesian product* $A \times B$ of A and B is defined as

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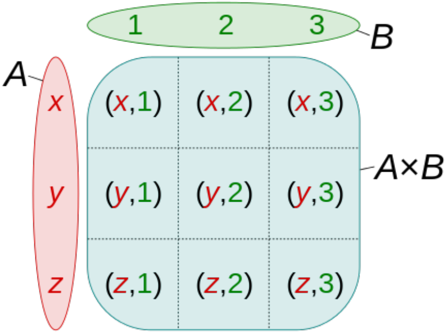
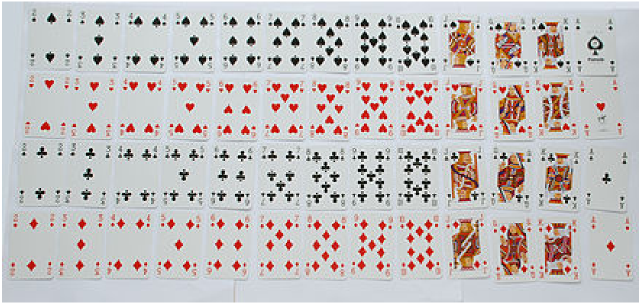
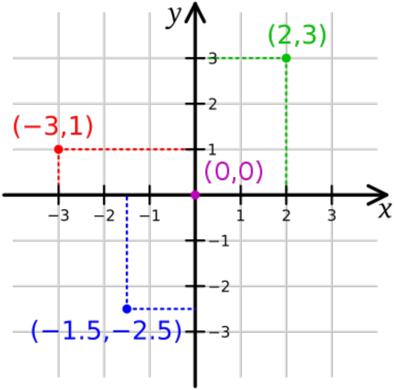


$$\text{Ranks} = \{2, \ldots, 10, J, Q, K, A\}$$



$$\text{Suits} = \{\}$$



$$\mathbb{Z}^2 \triangleq \mathbb{Z} \times \mathbb{Z}$$

$$X \times \emptyset = \emptyset \times X$$

$$X \times \emptyset = \emptyset \times X$$

$$X \times Y \neq Y \times X$$

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$$(X \times Y) \times Z \neq X \times (Y \times Z)$$

$$X \times \emptyset = \emptyset \times X$$

$$X \times Y \neq Y \times X$$

$$(X \times Y) \times Z \neq X \times (Y \times Z)$$

$$A = \{1\} \qquad (A \times A) \times A \neq A \times (A \times A)$$

Theorem (分配律 (Distributivity))

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

$$A \times (B \setminus C) = (A \times B) \setminus (A \times C)$$

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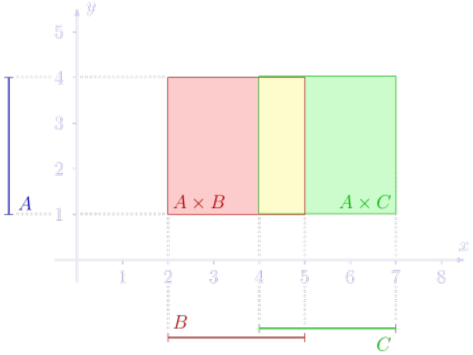

Definition (n-元笛卡尔积 (n-ary Cartesian Product))

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$$X^n \triangleq \underbrace{X \times \cdots \times X}_{n}$$

Definition (关系 (Relations))

A *relation* R from A to B is a subset of $A \times B$:

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Definition (Notations)

$$(a, b) \in R \qquad R(a, b) \qquad aRb$$

Definition (Relations)

A *relation* R from A to B is a subset of $A \times B$:

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Examples

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- $$D = \{(a, b) \in \mathbb{N} \times \mathbb{N} \mid \exists q \in \mathbb{N} : a \cdot q = b\}$$
- P : the set of people

$$M = \{(a, b) \in P \times P \mid a \text{ is the mother of } b\}$$

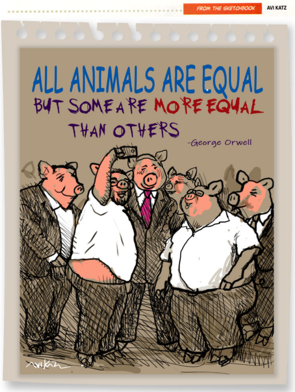
$$B = \{(a, b) \in P \times P \mid a \text{ is the brother of } b\}$$

Important Relations:

Equivalence Relations

Ordering Relations

Functions (next class)



Outline:

3 Definitions

$$\mathrm{dom}(R) \qquad \mathrm{ran}(R) \qquad \mathrm{fld}(R)$$

Definition (定义域 (Domain))

$$\mathrm{dom}(R) = \{a \mid \exists b.\ (a, b) \in R\}$$

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$$\text{dom}(R) = \{a \mid \exists b.\ (a, b) \in R\}$$

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Definition (域 (Field))

$$\mathrm{fld}(R) = \mathrm{dom}(R) \cup \mathrm{ran}(R)$$

$$R = \{(x, y) \mid x = y\} \subseteq \mathbb{R} \times \mathbb{R}$$

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$$\mathrm{dom}(R) = \mathbb{R} \qquad \mathrm{ran}(R) = \mathbb{R} \qquad \mathrm{fld}(R) = \mathbb{R}$$

$$R = \{(x, y) \mid x^2 + y^2 = 1\} \subseteq \mathbb{R} \times \mathbb{R}$$

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$$\mathrm{dom}(R) = [1, 1] \qquad \mathrm{ran}(R) = [-1, 1] \qquad \mathrm{fld}(R) = [-1, 1]$$

Theorem

$$dom(R) \subseteq \bigcup\bigcup R \qquad ran(R) \subseteq \bigcup\bigcup R$$

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对任意 a,

$$a \in \mathrm{dom}(R) \tag{1}$$

$$\tag{6}$$

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$$
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& && (6)
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$$\Longrightarrow \exists b.\ a \in \bigcup\bigcup R \tag{5}$$

$$\Longrightarrow a \in \bigcup\bigcup R \tag{6}$$

5 Operations

R^{-1} $\qquad$ $R|_X$ $\qquad$ $R[X]$ $\qquad$ $R^{-1}[Y]$ $\qquad$ $R \circ S$

Definition (逆 (Inverse))

The *inverse* of R is the relation

$$R^{-1} = \{(a, b) \mid (b, a) \in R\}$$

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$$\leq = \{(x, y) \mid x \leq y\} \subseteq \mathbb{R} \times \mathbb{R} \qquad \leq^{-1} = \ \geq \triangleq \{(x, y) \mid x \geq y\}$$

Theorem

$$(R^{-1})^{-1} = R$$

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对任意 (a, b),

$$(a, b) \in (R^{-1})^{-1} \tag{1}$$

$$\tag{3}$$

Theorem

$$(R^{-1})^{-1} = R$$

对任意 (a, b),

$$(a, b) \in (R^{-1})^{-1} \tag{1}$$

$$\Longleftrightarrow (b, a) \in R^{-1} \tag{2}$$

$$\tag{3}$$

Theorem

$$(R^{-1})^{-1} = R$$

对任意 (a, b),

$$
\begin{aligned}
& (a, b) \in (R^{-1})^{-1} && (1) \\
\iff & (b, a) \in R^{-1} && (2) \\
\iff & (a, b) \in R && (3)
\end{aligned}
$$

Theorem (关系的逆)

$$R, S \subseteq A \times B$$

$$(R \cup S)^{-1} = R^{-1} \cup S^{-1}$$

$$(R \cap S)^{-1} = R^{-1} \cap S^{-1}$$

$$(R \setminus S)^{-1} = R^{-1} \setminus S^{-1}$$

Definition (左限制 (Left-Restriction))

Suppose $R \subseteq X \times Y$ and $S \subseteq X$. The *left-restriction* relation of R to S is

$$R|_S = \{(x, y) \in R \mid x \in S\}$$

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Definition (右限制 (Right-Restriction))

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Definition (左限制 (Left-Restriction))

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Definition (右限制 (Right-Restriction))

Suppose $R \subseteq X \times Y$ and $S \subseteq Y$. The *right-restriction* relation of R to S is

$$R|_S = \{(x, y) \in R \mid y \in S\}$$

Definition (限制 (Restriction))

Suppose $R \subseteq X \times X$ and $S \subseteq X$. The *restriction* relation of R to S is

$$R|_S = \{(x, y) \in R \mid x \in S \wedge y \in S\}$$

example

Definition (像 (Image))

The *image* of X under R is the set

$$R[X] = \{b \in \mathrm{ran}(R) \mid \exists a \in X.\ (a, b) \in R\}$$

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$$R[a] \triangleq R[\{a\}] = \{b \mid (a, b) \in R\}$$

Definition (逆像 (Inverse Image))

The *inverse image* of Y under R is the set

$$R^{-1}[Y] = \{a \in \text{dom}(R) \mid \exists b \in Y : (a, b) \in R\}$$

Definition (逆像 (Inverse Image))

The *inverse image* of Y under R is the set

$$R^{-1}[Y] = \{a \in \mathrm{dom}(R) \mid \exists b \in Y : (a, b) \in R\}$$

$$R^{-1}[b] \triangleq R^{-1}[\{b\}] = \{a \mid (a, b) \in R\}$$

$$R \subseteq A \times B \qquad X \subseteq A \qquad Y \subseteq B$$

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$$R^{-1}[R[X]] \ ? \ X$$

$$R[R^{-1}[Y]] \ ? \ Y$$

$$R \subseteq A \times B \qquad X \subseteq A \qquad Y \subseteq B$$

$$R^{-1}[R[X]] \; ? \; X$$

$$R[R^{-1}[Y]] \; ? \; Y$$

Theorem

$$R[X_1 \cup X_2] = R[X_1] \cup R[X_2]$$

$$R[X_1 \cap X_2] \subseteq R[X_1] \cap R[X_2]$$

$$R[X_1 \setminus X_2] \supseteq R[X_1] \setminus R[X_2]$$

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对任意 b,

$$b \in R[X_1 \cup X_2]$$

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对任意 b,

$$\begin{aligned} & b \in R[X_1 \cup X_2] \\ \iff & \exists a \in X_1 \cup X_2 : (a, b) \in R \\ \iff & \exists a \in X_1 : (a, b) \in R \lor \exists a \in X_2 : (a, b) \in R \end{aligned}$$

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\iff\ & b \in R[X_1] \lor b \in R[X_2] \\
\iff\ & b \in R[X_1] \cup R[X_2]
\end{aligned}$$

Definition (复合 (Composition; $R \circ S$, $R; S$))

The *composition* of relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ is the relation

$$R \circ S = \{(a, c) \mid \exists b : (a, b) \in S \land (b, c) \in R\}$$

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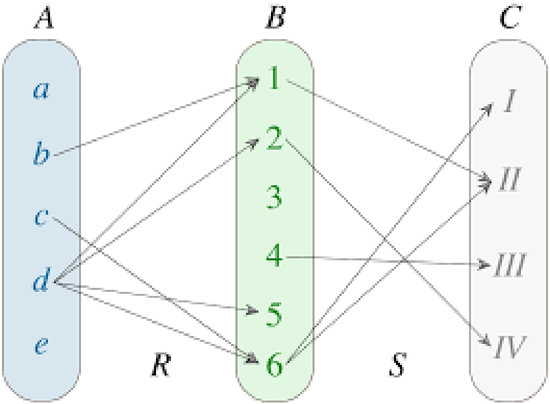
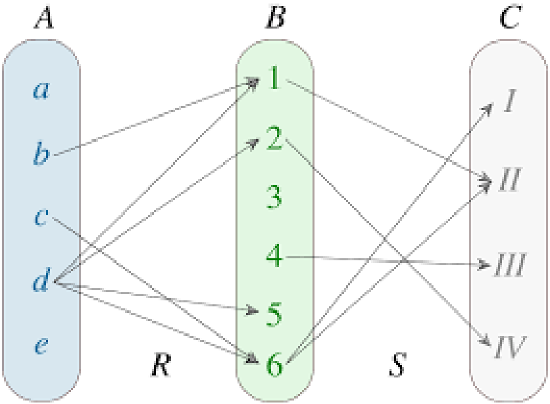


$$|R \circ S| =$$



$$|R \circ S| = 7$$

$$\leq \circ \leq \; =$$

$$\leq \circ \leq \;=\; \leq$$

$$\leq \circ \leq \; = \; \leq$$

$$\leq \circ \geq \; =$$

$$\leq \circ \leq \;=\; \leq$$

$$\leq \circ \geq \;=\; \mathbb{R} \times \mathbb{R}$$

Theorem

$$(R \circ S)^{-1} = S^{-1} \circ R^{-1}$$

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对任意 (a, b),

$$(a, b) \in (R \circ S)^{-1} \tag{1}$$

(5)

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对任意 (a, b),

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$$\tag{5}$$

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$$\tag{5}$$

Theorem

$$(R \circ S)^{-1} = S^{-1} \circ R^{-1}$$

对任意 (a, b),

$$
\begin{align}
& (a, b) \in (R \circ S)^{-1} \tag{1} \\
\iff & (b, a) \in R \circ S \tag{2} \\
\iff & \exists c.\ (b, c) \in R \wedge (c, a) \in S \tag{3} \\
\iff & \exists c.\ (c, b) \in R^{-1} \wedge (a, c) \in S^{-1} \tag{4} \\
\iff & (a, c) \in S^{-1} \circ R^{-1} \tag{5}
\end{align}
$$

Theorem

$$(R \circ S) \circ T = R \circ (S \circ T)$$

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对任意 (a, b),

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Theorem

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对任意 (a, b),

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$$(R \circ S) \circ T = R \circ (S \circ T)$$

对任意 (a, b),

$$
\begin{align}
& (a, b) \in (R \circ S) \circ T \tag{1} \\
\iff & \exists c.\ \Big((a, c) \in T \land (c, b) \in R \circ S\Big) \tag{2} \\
\iff & \exists c.\ \Big((a, c) \in T \land (\exists d : (c, d) \in S \land (d, b) \in R)\Big) \tag{3} \\
\iff & \exists d.\ \exists c.\ \Big((a, c) \in T \land (c, d) \in S \land (d, b) \in R\Big) \tag{4}
\end{align}
$$

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对任意 (a, b),

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$$\iff \exists d. \Big((a, d) \in S \circ T \land (d, b) \in R\Big) \tag{6}$$

Theorem

$$(R \circ S) \circ T = R \circ (S \circ T)$$

对任意 (a, b),

$$
\begin{aligned}
& (a, b) \in (R \circ S) \circ T && (1) \\
\iff & \exists c.\ \Big((a, c) \in T \land (c, b) \in R \circ S\Big) && (2) \\
\iff & \exists c.\ \Big((a, c) \in T \land (\exists d : (c, d) \in S \land (d, b) \in R)\Big) && (3) \\
\iff & \exists d.\ \exists c.\ \Big((a, c) \in T \land (c, d) \in S \land (d, b) \in R\Big) && (4) \\
\iff & \exists d.\ \Big((\exists c.\ (a, c) \in T \land (c, d) \in S) \land (d, b) \in R\Big) && (5) \\
\iff & \exists d.\ \Big((a, d) \in S \circ T \land (d, b) \in R\Big) && (6)
\end{aligned}
$$

燕小六: “帮我照顾好我七舅姥爷和我外甥女”

“舅姥爷”：姥姥/外婆的兄弟

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"舅姥爷": 妈妈的舅舅

Theorem (关系的复合)

$$(X \cup Y) \circ Z = (X \circ Z) \cup (Y \circ Z)$$

$$(X \cap Y) \circ Z \subseteq (X \circ Z) \cap (Y \circ Z)$$

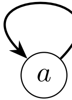
7 Properties

$$R \subseteq X \times X$$

Definition (自反的 (Reflexive))

$R \subseteq X \times X$ is *reflexive* if

$$\forall a \in X : (a, a) \in R$$



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$$\leq \subseteq \mathbb{R} \times \mathbb{R} \text{ is reflexive}$$

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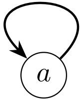

$$\leq \subseteq \mathbb{R} \times \mathbb{R} \text{ is reflexive}$$

三角形上的全等关系是自反的

Definition (反自反 (Irreflexive))

$R \subseteq X \times X$ is *irreflexive* if

$$\forall a \in X.\ (a, a) \notin R$$

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$$\forall a \in X.\ (a, a) \notin R$$

$$< \subseteq \mathbb{R} \times \mathbb{R} \text{ is irreflexive}$$

$$> \subseteq \mathbb{R} \times \mathbb{R} \text{ is irreflexive}$$

$$A = \{1, 2, 3\} \qquad R \subseteq A \times A$$

$$\{(1,1), (1,3), (2,1), (2,2), (3,3)\}$$

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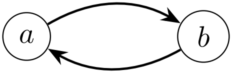
Definition (对称 (Symmetric))

$R \subseteq X \times X$ is *symmetric* if

$$\forall a, b \in X.\ aRb \to bRa$$



Definition (对称 (Symmetric))

$R \subseteq X \times X$ is *symmetric* if

$$\forall a, b \in X.\ aRb \to bRa$$

$$\begin{tikzcd} a \arrow[r, bend left] & b \arrow[l, bend left] \end{tikzcd}$$

$$\forall a, b \in X.\ aRb \leftrightarrow bRa$$

$$A = \{1, 2, 3\} \qquad R \subseteq A \times A$$

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$$\{(1,2),(2,1),(2,3)\}$$

Definition (反对称 (AntiSymmetric))

$R \subseteq X \times X$ is *antisymmetric* if

$$\forall a, b \in X.\ (aRb \wedge bRa) \to a = b$$

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$R \subseteq X \times X$ is *antisymmetric* if

$$\forall a, b \in X.\ (aRb \wedge bRa) \to a = b$$

$\geq$ *is* antisymmetric

$$A = \{1, 2, 3\} \qquad R \subseteq A \times A$$

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$$\{(1,1), (1,2), (1,3), (2,1), (3,1), (3,3)\}$$

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$$\{(1,2),(2,1),(2,3)\}$$

Definition (传递的 (Transitive))

$R \subseteq X \times X$ is *transitive* if

$$\forall a, b, c \in X.\ (aRb \wedge bRc \rightarrow aRc)$$

$$\begin{tikzcd} a \arrow[r, bend left] \arrow[rr, bend right] & b \arrow[r, bend left] & c \end{tikzcd}$$

$$A = \{1, 2, 3\} \qquad R \subseteq A \times A$$

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$$A = \{1, 2, 3\} \qquad R \subseteq A \times A$$

$$\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\}$$

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$$\{(1,3)\}$$

$$A = \{1, 2, 3\} \qquad R \subseteq A \times A$$

$$\{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$$

$$\{(1,2), (2,3), (3,1)\}$$

$$\{(1,3)\}$$

$$\emptyset$$

Definition (连接的 (Connex))

$R \subseteq X \times X$ is *connex* if

$$\forall a, b \in X.\ (aRb \lor bRa)$$

Definition (连接的 (Connex))

$R \subseteq X \times X$ is *connex* if

$$\forall a, b \in X. \ (aRb \vee bRa)$$

Definition (三分的 (Trichotomous))

$R \subseteq X \times X$ is *trichotomous* if

$$\forall a, b \in X. \ (\text{exactly one of } aRb, bRa, \text{ or } a = b \text{ holds})$$

Theorem

$$R \textit{ is reflexive} \iff I \subseteq R$$

$$I = \{(a, a) \in A \times A \mid a \in A\}$$

Theorem

$$R \textit{ is reflexive} \iff I \subseteq R$$

$$I = \{(a,a) \in A \times A \mid a \in A\}$$

Theorem

$$R \textit{ is symmetric} \iff R^{-1} = R$$

Theorem

$$R \textit{ is transitive} \iff R \circ R \subseteq R$$

Theorem

$$R \textit{ is transitive} \iff R \circ R \subseteq R$$

$$R = \{(1,2),(2,3),(1,3),(4,4)\}$$

Theorem

$$R \text{ is symmetric and transitive} \iff R = R^{-1} \circ R$$

Equivalence Relations

Definition (Equivalence Relation)

$R \subseteq X \times X$ is an *equivalence relation* on X iff R is

- reflexive: $\forall a \in X.\ aRa$
- symmetric: $\forall a, b \in X.\ (aRb \leftrightarrow bRa)$
- transitive: $\forall a, b, c \in X.\ (aRb \land bRc \to aRc)$

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- reflexive: $\forall a \in X.\ aRa$
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Definition (Equivalence Relation)

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Why are equivalence relations important?

Equivalence Relations as Abstractions

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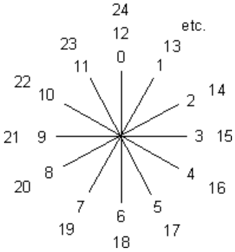

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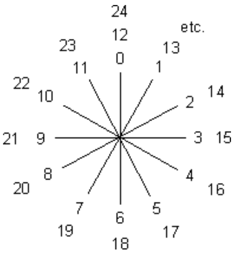

“全国人民代表大会各省代表团”

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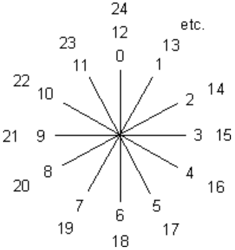
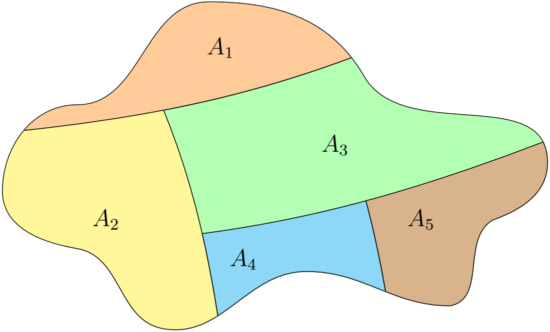


“全国人民代表大会各省代表团”

Equivalence Relation $\Longleftrightarrow$ Partition

Partition



“不空、不漏、不重”

Definition (划分 (Partition))

A family of sets $\Pi = \{A_\alpha \mid \alpha \in I\}$ is a *partition* of X if

(i) (不空)

$$\forall \alpha \in I.\ A_\alpha \neq \emptyset$$

(ii) (不漏)

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$$\forall \alpha, \beta \in I.\ A_\alpha \cap A_\beta = \emptyset \lor A_\alpha = A_\beta$$

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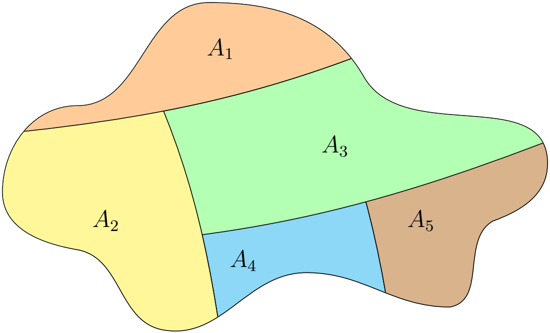

Equivalence Relation $\iff$ Partition

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Q : What is $\mathbb{N} \times \mathbb{N} / \sim$?

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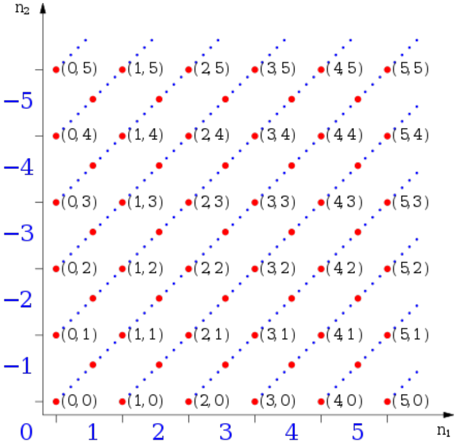
$$[(1, 3)]_{\sim} = \{(0, 2), (1, 3), (2, 4), (3, 5), \cdots\} \triangleq -2 \in \mathbb{Z}$$



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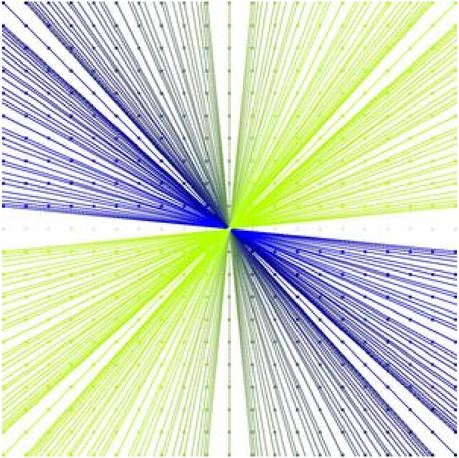

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如何用有理数定义实数?

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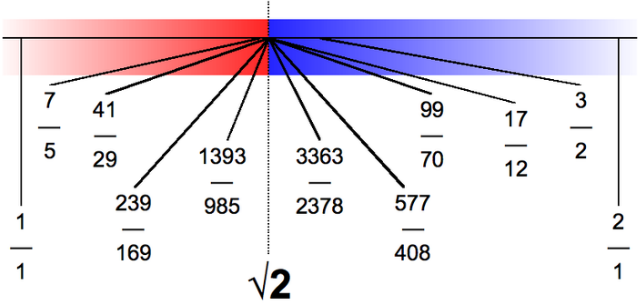
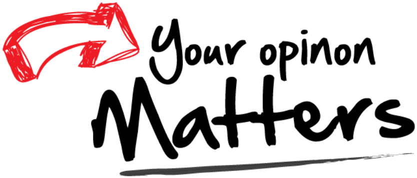


Dedekind Cut (戴德金分割)


Thank
You!



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