(十五) 离散数学: 复习 (Review)

魏恒峰

hfwei@nju.edu.cn

2021 年 06 月 17 日

$$\vdash \qquad \models$$

Theorem

$$\Sigma \vdash \alpha \iff \Sigma \models \alpha$$

$\rightarrow$ $\Longrightarrow$

$\leftrightarrow$ $\Longleftrightarrow$

$\rightarrow \qquad \Longrightarrow$

$\leftrightarrow \qquad \Longleftrightarrow$

"$\rightarrow$" and "$\leftrightarrow$" are used in a single formula.

" $\Longrightarrow$ " and " $\Longleftrightarrow$ " are used to connect two formulas.

$$\rightarrow \qquad \Longrightarrow$$

$$\leftrightarrow \qquad \Longleftrightarrow$$

"$\rightarrow$" and "$\leftrightarrow$" are used in a single formula.

"$\Longrightarrow$" and "$\Longleftrightarrow$" are used to connect two formulas.

$$\begin{aligned}
& x \in A \setminus B \\
\Longleftrightarrow\ & x \in A \land x \notin B \\
\Longleftrightarrow\ & x \in A \land (x \in U \land x \notin B) \\
\Longleftrightarrow\ & x \in A \land x \in \overline{B} \\
\Longleftrightarrow\ & x \in A \cap \overline{B}
\end{aligned}$$

$$
\begin{aligned}
p \oplus q &\triangleq (p \vee q) \wedge \neg(p \wedge q) \\
&= (p \wedge \neg q) \vee (\neg q \wedge q)
\end{aligned}
$$

p	q	$p \oplus q$
0	0	0
0	1	1
1	0	1
1	1	0

$$
\begin{aligned}
p \oplus q &\triangleq (p \lor q) \land \neg(p \land q) \\
&= (p \land \neg q) \lor (\neg q \land q)
\end{aligned}
$$

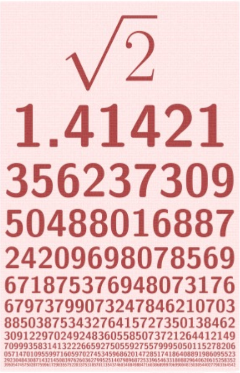
p	q	$p \oplus q$
0	0	0
0	1	1
1	0	1
1	1	0

$$p \oplus q = q \oplus r$$

$$(p \oplus q) \oplus r = p \oplus (q \oplus r)$$

Theorem

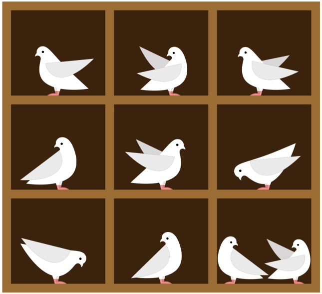
$\sqrt{2}$ is irrational.



The First Crisis in Mathematics

Theorem (Bézout's Identity)

$$(a, b) = d \implies \exists u, v \in \mathbb{Z}.\ au + bv = d$$



Theorem (Pigeonhole Principle)

If n objects are placed in r boxes, where $r < n$, then at least one of the boxes contains ≥ 2 $(\geq \lceil \frac{n}{r} \rceil)$ object.

Numbers

Consider the numbers $1, 2, \ldots, 2n$, and take any $n+1$ of them.
There are two among these $n+1$ numbers which are relatively prime.

Numbers

Consider the numbers $1, 2, \ldots, 2n$, and take any $n+1$ of them.
There are two among these $n+1$ numbers which are relatively prime.

There must be two numbers which are only 1 apart.

Numbers

Consider the numbers $1, 2, \ldots, 2n$, and take any $n+1$ of them.
There are two among these $n+1$ numbers such as one divides the other.

Numbers

Consider the numbers $1, 2, \ldots, 2n$, and take any $n+1$ of them.
There are two among these $n+1$ numbers such as one divides the other.

$$a = 2^k m, \quad (1 \leq m \leq 2n-1 \text{ is odd})$$

There $n+1$ numbers have only n different odd parts.

There must be two numbers with the same odd part.

Hand-shaking

If there are $n > 1$ people who can shake hands with one another, there are two people who shake hands with the same number of people.

Hand-shaking

If there are $n > 1$ people who can shake hands with one another, there are two people who shake hands with the same number of people.

$$0 \sim n - 1$$

Hand-shaking

If there are $n > 1$ people who can shake hands with one another, there are two people who shake hands with the same number of people.

$$0 \sim n-1$$

Either the '0' hole or the 'n − 1' hole or both must be empty.

Sums

Suppose we are given n integers $a_1, a_2, \ldots, a_n$.
Then there is a set of consecutive numbers $a_{k+1}, a_{k+2}, \ldots, a_l$ whose sum $\sum_{i=k+1}^{l} a_i$ is a multiple of n.

Sums

Suppose we are given n integers $a_1, a_2, \dots, a_n$.
Then there is a set of consecutive numbers $a_{k+1}, a_{k+2}, \dots, a_l$ whose sum $\sum_{i=k+1}^{l} a_i$ is a multiple of n.

$$A_i = \sum_{k=1}^{k=i} a_i$$

Sums

Suppose we are given n integers $a_1, a_2, \ldots, a_n$.
Then there is a set of consecutive numbers $a_{k+1}, a_{k+2}, \ldots, a_l$ whose sum $\sum_{i=k+1}^{l} a_i$ is a multiple of n.

$$A_i = \sum_{k=1}^{k=i} a_i$$

$$A_0, A_1, A_2, \ldots, A_n$$

Sums

Suppose we are given n integers $a_1, a_2, \ldots, a_n$.
Then there is a set of consecutive numbers $a_{k+1}, a_{k+2}, \ldots, a_l$ whose sum $\sum_{i=k+1}^{l} a_i$ is a multiple of n.

$$A_i = \sum_{k=1}^{k=i} a_i$$

$$A_0, A_1, A_2, \ldots, A_n$$

$$\exists 0 \le i < j \le n.\ A_i = A_j \mod n$$

Sums

Suppose we are given n integers $a_1, a_2, \ldots, a_n$.
Then there is a set of consecutive numbers $a_{k+1}, a_{k+2}, \ldots, a_l$ whose sum $\sum_{i=k+1}^{l} a_i$ is a multiple of n.

$$A_i = \sum_{k=1}^{k=i} a_i$$

$$A_0, A_1, A_2, \ldots, A_n$$

$$\exists 0 \leq i < j \leq n.\ A_i = A_j \mod n$$

$$A_j - A_i = a_{i+1} + \cdots + a_j = 0 \mod n$$

Championship Match

“胡司令” (胡荣华) 要安排一次长达 77 天的象棋练习赛。

他想每天至少要有一场比赛, 但是总共不超过 132 场比赛。

请证明, 无论如何安排, 他都要在连续的若干天内恰好完成 21 场比赛。

Championship Match

“胡司令” (胡荣华) 要安排一次长达 77 天的象棋练习赛。

他想每天至少要有一场比赛, 但是总共不超过 132 场比赛。

请证明, 无论如何安排, 他都要在连续的若干天内恰好完成 21 场比赛。

Let a_i denote the number of games he plays up through the i-th day.

Championship Match

“胡司令” (胡荣华) 要安排一次长达 77 天的象棋练习赛。

他想每天至少要有一场比赛, 但是总共不超过 132 场比赛。

请证明, 无论如何安排, 他都要在连续的若干天内恰好完成 21 场比赛。

Let a_i denote the number of games he plays up through the i-th day.

$$a_1, a_2, \ldots, a_{76}, a_{77}, a_1 + 21, a_2 + 21, \ldots, a_{76} + 21, a_{77} + 21$$

Championship Match

“胡司令” (胡荣华) 要安排一次长达 77 天的象棋练习赛。

他想每天至少要有一场比赛, 但是总共不超过 132 场比赛。

请证明, 无论如何安排, 他都要在连续的若干天内恰好完成 21 场比赛。

Let a_i denote the number of games he plays up through the i-th day.

$$a_1, a_2, \ldots, a_{76}, a_{77}, a_1 + 21, a_2 + 21, \ldots, a_{76} + 21, a_{77} + 21$$

There must be ≥ 2 elements having the same value.

Championship Match

“胡司令” (胡荣华) 要安排一次长达 77 天的象棋练习赛。
他想每天至少要有一场比赛, 但是总共不超过 132 场比赛。
请证明, 无论如何安排, 他都要在连续的若干天内恰好完成 21 场比赛。

Let a_i denote the number of games he plays up through the i-th day.

$$a_1, a_2, \ldots, a_{76}, a_{77}, a_1 + 21, a_2 + 21, \ldots, a_{76} + 21, a_{77} + 21$$

There must be ≥ 2 elements having the same value.

It must be $a_i + 21 = a_j$.

Sequences

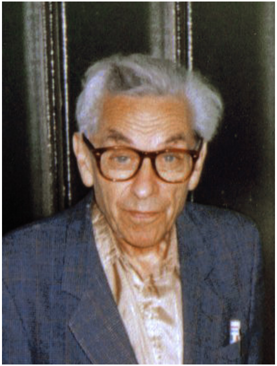
In any sequence $a_1, a_2, \ldots, a_{mn+1}$ of $mn+1$ distinct numbers, there exists an increasing subsequence

$$a_{i_1} < a_{i_2} < \cdots < a_{i_{m+1}} \quad (i_1 < i_2 < \cdots < i_{m+1})$$

of length $m+1$, or a decreasing subsequence

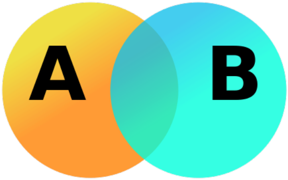
$$a_{j_1} > a_{j_2} > \cdots > a_{j_{n+1}} \quad (j_1 > i_2 < \cdots > j_{n+1})$$

of length $n+1$, or both.

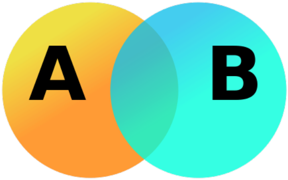


Paul Erdős (1913 ~ 1996)

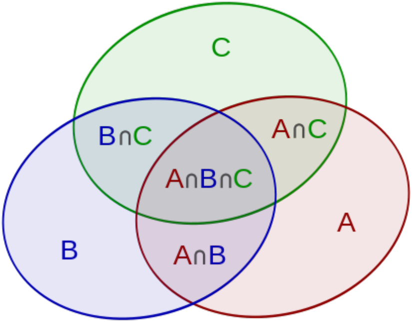
Chapter 28 of “Proofs from THE Book”



$$|A \cup B| = |A| + |B| - |A \cap B|$$



$$|A \cup B| = |A| + |B| - |A \cap B|$$



$$\begin{aligned}|A \cup B \cup C| = & |A| + |B| + |C| \\ & - |A \cap B| - |A \cap C| - |B \cap C| \\ & + |A \cap B \cap C|\end{aligned}$$

Theorem (Inclusion-Exclusion Principle)

$$
\begin{aligned}
\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| &- \sum_{1\leqslant i<j\leqslant n} |A_i \cap A_j| \\
&+ \sum_{1\leqslant i<j<k\leqslant n} |A_i \cap A_j \cap A_k| \\
&- \cdots \\
&+ (-1)^{n-1} |A_1 \cap \cdots \cap A_n| .
\end{aligned}
$$

Theorem (Inclusion-Exclusion Principle)

$$\begin{aligned}\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| &- \sum_{1\leqslant i<j\leqslant n} |A_i \cap A_j| \\ &+ \sum_{1\leqslant i<j<k\leqslant n} |A_i \cap A_j \cap A_k| \\ &- \cdots \\ &+ (-1)^{n-1} |A_1 \cap \cdots \cap A_n| .\end{aligned}$$

$$\begin{aligned}\left|\bigcap_{i=1}^{n} \bar{A}_i\right| = \left|S - \bigcup_{i=1}^{n} A_i\right| = |S| - \sum_{i=1}^{n} |A_i| + \sum_{1\leqslant i<j\leqslant n} |A_i \cap A_j| \\ - \cdots + (-1)^n |A_1 \cap \cdots \cap A_n|.\end{aligned}$$

Counting Integers

How many integers in $1, \ldots, 100$ are not divisible by 2, 3 or 5?

Counting Integers

How many integers in $1, \ldots, 100$ are not divisible by 2, 3 or 5?

$$100 - (50 + 33 + 20) + (16 + 10 + 6) - 3 = 26.$$

Counting Derangements (错排)

Suppose there is a deck of n cards numbered from 1 to n. Suppose a card numbered i is in the correct position if it is the i-th card in the deck. How many ways can the cards be shuffled without any cards being in the correct position?

Counting Derangements (错排)

Suppose there is a deck of n cards numbered from 1 to n. Suppose a card numbered i is in the correct position if it is the i-th card in the deck. How many ways can the cards be shuffled without any cards being in the correct position?

$$A_m : \text{all of the orderings of cards with the } m\text{-th card correct}$$

Counting Derangements (错排)

Suppose there is a deck of n cards numbered from 1 to n. Suppose a card numbered i is in the correct position if it is the i-th card in the deck. How many ways can the cards be shuffled without any cards being in the correct position?

A_m : all of the orderings of cards with the m-th card correct

$$\left|\bigcap_{i=1}^{n} \overline{A_i}\right| = \left|S - \bigcup_{i=1}^{n} A_i\right| = n! - \sum_{i=1}^{n} |A_i| + \sum_{1 \leqslant i < j \leqslant n} |A_i \cap A_j|$$

$$- \cdots + (-1)^n |A_1 \cap \cdots \cap A_n|.$$

Counting Derangements (错排)

Suppose there is a deck of n cards numbered from 1 to n. Suppose a card numbered i is in the correct position if it is the i-th card in the deck. How many ways can the cards be shuffled without any cards being in the correct position?

A_m : all of the orderings of cards with the m-th card correct

$$\left|\bigcap_{i=1}^{n} \overline{A_i}\right| = \left|S - \bigcup_{i=1}^{n} A_i\right| = n! - \sum_{i=1}^{n} |A_i| + \sum_{1 \leqslant i < j \leqslant n} |A_i \cap A_j| - \cdots + (-1)^n |A_1 \cap \cdots \cap A_n|.$$

$$S_k \triangleq \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| =$$

Counting Derangements (错排)

Suppose there is a deck of n cards numbered from 1 to n.
Suppose a card numbered i is in the correct position if it is the i-th card in the deck. How many ways can the cards be shuffled without any cards being in the correct position?

A_m : all of the orderings of cards with the m-th card correct

$$\left|\bigcap_{i=1}^{n} \overline{A_i}\right| = \left|S - \bigcup_{i=1}^{n} A_i\right| = n! - \sum_{i=1}^{n} |A_i| + \sum_{1 \leqslant i < j \leqslant n} |A_i \cap A_j| - \cdots + (-1)^n |A_1 \cap \cdots \cap A_n|.$$

$$S_k \triangleq \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = \binom{n}{k}(n-k)! = \frac{n!}{k!}$$

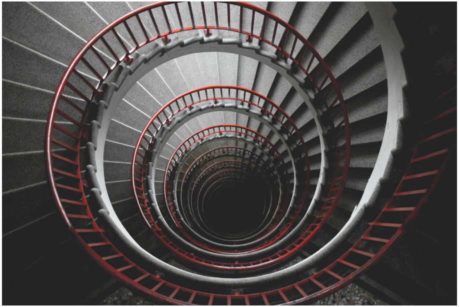
$$S_k = \frac{n!}{k!}$$

$$\begin{aligned}\left|\bigcap_{i=1}^{n} \overline{A_i}\right| &= n! - \frac{n!}{1!} + \frac{n!}{2!} - \cdots + (-1)^n \frac{n!}{n!} \\ &= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}\end{aligned}$$

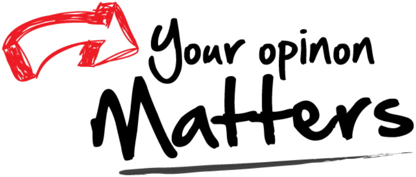
$$S_k = \frac{n!}{k!}$$

$$\begin{aligned}\left|\bigcap_{i=1}^{n} \overline{A_i}\right| &= n! - \frac{n!}{1!} + \frac{n!}{2!} - \cdots + (-1)^n \frac{n!}{n!} \\ &= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}\end{aligned}$$

$$n \to \infty \implies \sum_{k=0}^{n} \frac{(-1)^k}{k!} \to e^{-1} \approx 0.368$$




Thank
You!



Office 302

Mailbox: H016

hfwei@nju.edu.cn