(四) 集合: 关系 (Relation)

魏恒峰

hfwei@nju.edu.cn

2021 年 04 月 08 日

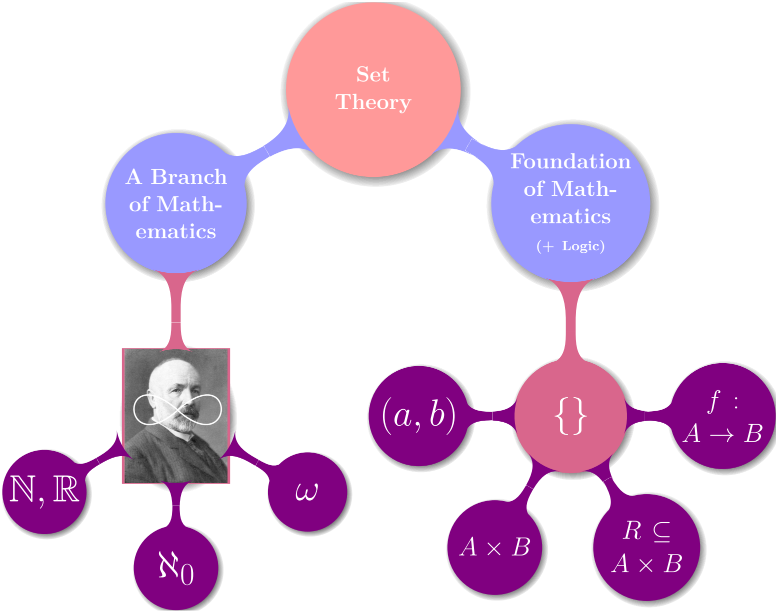

Set Theory
A Branch of Mathematics
Foundation of Mathematics
(+ Logic)
$\mathbb{N}, \mathbb{R}$
$\aleph_0$
ω
(a, b)
$\{\}$
$f : A \to B$
$A \times B$
$R \subseteq A \times B$

Figure 13. A selection of consistency axioms over an execution $(E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar})$

Auxiliary relations

$\mathsf{sameobj}(e, f) \iff \mathsf{obj}(e) = \mathsf{obj}(f)$

Per-object causality (aka happens-before) order:

$\mathsf{hbo} = ((\mathsf{ro} \cap \mathsf{sameobj}) \cup \mathsf{vis})^+$

Causality (aka happens-before) order: $\mathsf{hb} = (\mathsf{ro} \cup \mathsf{vis})^+$

Axioms

EVENTUAL:

$\forall e \in E.\, \neg(\exists \text{ infinitely many } f \in E.\, \mathsf{sameobj}(e, f) \wedge \neg(e \xrightarrow{\mathsf{vis}} f))$

THINAIR: $\mathsf{ro} \cup \mathsf{vis}$ is acyclic

POCV (Per-Object Causal Visibility): $\mathsf{hbo} \subseteq \mathsf{vis}$

POCA (Per-Object Causal Arbitration): $\mathsf{hbo} \subseteq \mathsf{ar}$

COCV (Cross-Object Causal Visibility): $(\mathsf{hb} \cap \mathsf{sameobj}) \subseteq \mathsf{vis}$

COCA (Cross-Object Causal Arbitration): $\mathsf{hb} \cup \mathsf{ar}$ is acyclic

Figure 17. Optimized state-based multi-value register and its simulation

$$
\begin{aligned}
\Sigma &= \mathsf{ReplicaID} \times \mathcal{P}(\mathbb{Z} \times (\mathsf{ReplicaID} \to \mathbb{N}_0))\\
\vec{\sigma}_0 &= \langle r, \emptyset \rangle\\
M &= \mathcal{P}(\mathbb{Z} \times (\mathsf{ReplicaID} \to \mathbb{N}_0))\\
\mathsf{do}(\mathtt{wr}(a), \langle r, V\rangle, t) &=\\
&(\langle r, \{(a, (\lambda s.\, \text{if } s \neq r \text{ then } \max\{v(s) \mid (\_, v) \in V\}\\
&\qquad \text{else } \max\{v(s) \mid (\_, v) \in V\} + 1))\}\rangle, \bot)\\
\mathsf{do}(\mathtt{rd}, \langle r, V\rangle, t) &= (\langle r, V\rangle, \{a \mid (a, \_) \in V\})\\
\mathsf{send}(\langle r, V\rangle) &= (\langle r, V\rangle, V)\\
\mathsf{receive}(\langle r, V\rangle, V') &= \langle r, \{(a, v) \in V'' \mid\\
&\qquad v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'.\, (a', v') \in V'' \wedge a \neq a'\}\}\rangle,\\
\text{where } V'' &= \{(a, \bigsqcup\{v' \mid (a, v') \in V \cup V'\}) \mid (a, \_) \in V \cup V'\}
\end{aligned}
$$

$$\langle s, V\rangle\ [\mathcal{R}_r]\ I \iff (r = s) \wedge (V\ [\mathcal{M}]\ I)$$

$$
\begin{aligned}
&V\ [\mathcal{M}]\ ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar}), \mathsf{info}) \iff\\
&\quad (\forall (a, v), (a', v') \in V.\, (a = a' \implies v = v')) \wedge\\
&\quad (\forall (a, v) \in V.\, \exists s.\, v(s) > 0) \wedge\\
&\quad (\forall (a, v) \in V.\, v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'.\, (a', v') \in V \wedge a \neq a'\}) \wedge\\
&\quad \exists \text{ distinct } e_{s,k}.\\
&\quad (\{e \in E \mid \exists a.\, \mathsf{oper}(e) = \mathtt{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicaID} \wedge\\
&\qquad 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V\}\}) \wedge\\
&\quad (\forall s, j, k.\, (\mathsf{repl}(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}} e_{s,k} \iff j < k)) \wedge\\
&\quad (\forall (a, v) \in V.\, \forall q.\, \{j \mid \mathsf{oper}(e_{q,j}) = \mathtt{wr}(a)\} \cup\\
&\qquad \{j \mid \exists s, k.\, e_{q,j} \xrightarrow{\mathsf{vis}} e_{s,k} \wedge \mathsf{oper}(e_{s,k}) = \mathtt{wr}(a)\} =\\
&\qquad \{j \mid 1 \leq j \leq v(q)\}) \wedge\\
&\quad (\forall e \in E.\, (\mathsf{oper}(e) = \mathtt{wr}(a) \wedge\\
&\qquad \neg\exists f \in E.\, \mathsf{oper}(f) = \mathtt{wr}(\_) \wedge e \xrightarrow{\mathsf{vis}} f) \implies (a, \_) \in V)
\end{aligned}
$$

the former. The only non-trivial obligation is to show that if

$$V\ [\mathcal{M}]\ ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}), \mathsf{info}),$$

then

$$
\begin{aligned}
\{a \mid (a, \_) \in V\} \subseteq \{a \mid \exists e \in E.\, &\mathsf{oper}(e) = \mathtt{wr}(a) \wedge\\
&\neg\exists f \in E.\, \exists a'.\, \mathsf{oper}(e) = \mathtt{wr}(a') \wedge e \xrightarrow{\mathsf{vis}} f\} \qquad (13)
\end{aligned}
$$

(the reverse inclusion is straightforwardly implied by $\mathcal{R}_r$).

Take $(a, v) \in V$. We have $\forall (a, v) \in V.\, \exists s.\, v(s) > 0$,

$$v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'.\, (a', v') \in V \wedge a \neq a'\}$$

and

$$
\begin{aligned}
&\forall (a, v) \in V.\, \forall q.\, \{j \mid \mathsf{oper}(e_{q,j}) = \mathtt{wr}(a)\} \cup\\
&\quad \{j \mid \exists s, k.\, e_{q,j} \xrightarrow{\mathsf{vis}} e_{s,k} \wedge \mathsf{oper}(e_{s,k}) = \mathtt{wr}(a)\} =\\
&\quad \{j \mid 1 \leq j \leq v(q)\}.
\end{aligned}
$$

From this we get that for some $e \in E$

$$\mathsf{oper}(e) = \mathtt{wr}(a) \wedge \neg\exists f \in E.\, \exists a'.\, a' \neq a \wedge \mathsf{oper}(e) = \mathtt{wr}(a') \wedge e \xrightarrow{\mathsf{vis}} f.$$

Since vis is acyclic, this implies that for some $e' \in E$

$$\mathsf{oper}(e') = \mathtt{wr}(a) \wedge \neg\exists f \in E.\, \mathsf{oper}(e') = \mathtt{wr}(\_) \wedge e' \xrightarrow{\mathsf{vis}} f,$$

which establishes (13).

Let us now discharge RECEIVE. Let $\mathsf{receive}(\langle r, V\rangle, V') = \langle r, V'''\rangle$, where

$$
\begin{aligned}
V'' &= \{(a, \bigsqcup\{v' \mid (a, v') \in V \cup V'\}) \mid (a, \_) \in V \cup V'\};\\
V''' &= \{(a, v) \in V'' \mid v \not\sqsubseteq \bigsqcup\{(a', v') \in V'' \mid a \neq a'\}\}.
\end{aligned}
$$

Assume $\langle r, V\rangle\ [\mathcal{R}_r]\ I$, $V'\ [\mathcal{M}]\ J$ and

$$
\begin{aligned}
I &= ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar}), \mathsf{info});\\
J &= ((E', \mathsf{repl}', \mathsf{obj}', \mathsf{oper}', \mathsf{rval}', \mathsf{ro}', \mathsf{vis}', \mathsf{ar}'), \mathsf{info}');\\
I \sqcup J &= ((E'', \mathsf{repl}'', \mathsf{obj}'', \mathsf{oper}'', \mathsf{rval}'', \mathsf{ro}'', \mathsf{vis}'', \mathsf{ar}''), \mathsf{info}'').
\end{aligned}
$$

By agree we have $I \sqcup J \in \mathsf{IEx}$. Then

$$
\begin{aligned}
&(\forall (a, v), (a', v') \in V.\, (a = a' \implies v = v')) \wedge\\
&(\forall (a, v) \in V.\, \exists s.\, v(s) > 0) \wedge\\
&(\forall (a, v) \in V.\, v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'.\, (a', v') \in V \wedge a \neq a'\}) \wedge\\
&\exists \text{ distinct } e_{s,k}.\\
&(\{e \in E \mid \exists a.\, \mathsf{oper}''(e) = \mathtt{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicaID} \wedge\\
&\quad 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V\}\}) \wedge\\
&(\forall s, j, k.\, (\mathsf{repl}''(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}} e_{s,k} \iff j < k)) \wedge\\
&(\forall (a, v) \in V.\, \forall q.\, \{j \mid \mathsf{oper}''(e_{q,j}) = \mathtt{wr}(a)\} \cup\\
&\quad \{j \mid \exists s, k.\, e_{q,j} \xrightarrow{\mathsf{vis}} e_{s,k} \wedge \mathsf{oper}''(e_{s,k}) = \mathtt{wr}(a)\} =\\
&\quad \{j \mid 1 \leq j \leq v(q)\}) \wedge\\
&(\forall e \in E.\, (\mathsf{oper}''(e) = \mathtt{wr}(a) \wedge\\
&\quad \neg\exists f \in E.\, \mathsf{oper}''(f) = \mathtt{wr}(\_) \wedge e \xrightarrow{\mathsf{vis}} f) \implies (a, \_) \in V)
\end{aligned}
$$

and

$$
\begin{aligned}
&(\forall (a, v), (a', v') \in V'.\, (a = a' \implies v = v')) \wedge\\
&(\forall (a, v) \in V'.\, \exists s.\, v(s) > 0) \wedge\\
&(\forall (a, v) \in V'.\, v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'.\, (a', v') \in V' \wedge a \neq a'\}) \wedge\\
&\exists \text{ distinct } e_{s,k}.\\
&(\{e \in E' \mid \exists a.\, \mathsf{oper}''(e) = \mathtt{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicaID} \wedge\\
&\quad 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V'\}\}) \wedge\\
&(\forall s, j, k.\, (\mathsf{repl}''(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}'} e_{s,k} \iff j < k)) \wedge\\
&(\forall (a, v) \in V'.\, \forall q.\, \{j \mid \mathsf{oper}''(e_{q,j}) = \mathtt{wr}(a)\} \cup\\
&\quad \{j \mid \exists s, k.\, e_{q,j} \xrightarrow{\mathsf{vis}'} e_{s,k} \wedge \mathsf{oper}''(e_{s,k}) = \mathtt{wr}(a)\} =\\
&\quad \{j \mid 1 \leq j \leq v(q)\}) \wedge\\
&(\forall e \in E'.\, (\mathsf{oper}''(e) = \mathtt{wr}(a) \wedge\\
&\quad \neg\exists f \in E'.\, \mathsf{oper}''(f) = \mathtt{wr}(\_) \wedge e \xrightarrow{\mathsf{vis}'} f) \implies (a, \_) \in V').
\end{aligned}
$$

The agree property also implies

$$
\begin{aligned}
\forall s, k.\, 1 \leq k \leq \min\{&\max\{v(s) \mid \exists a.\, (a, v) \in V\},\\
&\max\{v(s) \mid \exists a.\, (a, v) \in V'\}\} \implies e_{s,k} = e'_{s,k}.
\end{aligned}
$$

Hence, there exist distinct

$$e''_{s,k} \text{ for } s \in \mathsf{ReplicaID},\ k = 1..(\max\{v(s) \mid \exists a.\, (a, v) \in V'''\}),$$

such that

$$
\begin{aligned}
&(\forall s, k.\, 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V\} \implies e''_{s,k} = e_{s,k}) \wedge\\
&(\forall s, k.\, 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V'\} \implies e''_{s,k} = e'_{s,k})
\end{aligned}
$$

and

$$
\begin{aligned}
&(\{e \in E \cup E' \mid \exists a.\, \mathsf{oper}''(e) = \mathtt{wr}(a)\} =\\
&\{e''_{s,k} \mid s \in \mathsf{ReplicaID} \wedge 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V'''\}\})\\
&\wedge (\forall s, j, k.\, (\mathsf{repl}(e''_{s,k}) = s) \wedge (e''_{s,j} \xrightarrow{\mathsf{ro}''} e''_{s,k} \iff j < k)).
\end{aligned}
$$

By the definition of V'' and V''' we have

$$\forall (a, v), (a', v') \in V'''.\, (a = a' \implies v = v').$$

We also straightforwardly get

$$\forall (a, v) \in V'.\, \exists s.\, v(s) > 0$$

and

$$
\begin{aligned}
&(\forall (a, v) \in V''.\, \forall q.\, \{j \mid \mathsf{oper}''(e''_{q,j}) = \mathtt{wr}(a)\} \cup\\
&\quad \{j \mid \exists s, k.\, e''_{q,j} \xrightarrow{\mathsf{vis}''} e''_{s,k} \wedge \mathsf{oper}''(e''_{s,k}) = \mathtt{wr}(a)\} = \qquad (14)\\
&\quad \{j \mid 1 \leq j \leq v(q)\}).
\end{aligned}
$$

Figure 13. A selection of consistency axioms over an execution $(E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar})$

Auxiliary relations

$\mathsf{sameobj}(e, f) \iff \mathsf{obj}(e) = \mathsf{obj}(f)$

Per-object causality (aka happens-before) order:

$\mathsf{hbo} = ((\mathsf{ro} \cap \mathsf{sameobj}) \cup \mathsf{vis})^+$

Causality (aka happens-before) order: $\mathsf{hb} = (\mathsf{ro} \cup \mathsf{vis})^+$

Axioms

EVENTUAL:

$\forall e \in E.\, \neg(\exists \text{ infinitely many } f \in E.\, \mathsf{sameobj}(e, f) \wedge \neg(e \xrightarrow{\mathsf{vis}} f))$

THINAIR: $\mathsf{ro} \cup \mathsf{vis}$ is acyclic

POCV (Per-Object Causal Visibility): $\mathsf{hbo} \subseteq \mathsf{vis}$

POCA (Per-Object Causal Arbitration): $\mathsf{hbo} \subseteq \mathsf{ar}$

COCV (Cross-Object Causal Visibility): $(\mathsf{hb} \cap \mathsf{sameobj}) \subseteq \mathsf{vis}$

COCA (Cross-Object Causal Arbitration): $\mathsf{hb} \cup \mathsf{ar}$ is acyclic

Figure 17. Optimized state-based multi-value register and its simulation

$$
\begin{aligned}
\Sigma &= \mathsf{ReplicaID} \times \mathcal{P}(\mathbb{Z} \times (\mathsf{ReplicaID} \to \mathbb{N}_0))\\
\vec{\sigma}_0 &= \langle r, \emptyset \rangle\\
M &= \mathcal{P}(\mathbb{Z} \times (\mathsf{ReplicaID} \to \mathbb{N}_0))\\
\mathsf{do}(\mathtt{wr}(a), \langle r, V \rangle, t) &= \\
&(\langle r, \{(a, (\lambda s.\, \text{if } s \neq r \text{ then } \max\{v(s) \mid (\_, v) \in V\}\\
&\qquad \text{else } \max\{v(s) \mid (\_, v) \in V\} + 1))\}\rangle, \bot)\\
\mathsf{do}(\mathtt{rd}, \langle r, V \rangle, t) &= (\langle r, V \rangle, \{a \mid (a, \_) \in V\})\\
\mathsf{send}(\langle r, V \rangle) &= (\langle r, V \rangle, V)\\
\mathsf{receive}(\langle r, V \rangle, V') &= \langle r, \{(a, v) \in V'' \mid\\
&\qquad v \not\sqsubseteq \textstyle\bigsqcup\{v' \mid \exists a'.\, (a', v') \in V'' \wedge a \neq a'\}\}\rangle,\\
\text{where } V'' &= \{(a, \textstyle\bigsqcup\{v' \mid (a, v') \in V \cup V'\}) \mid (a, \_) \in V \cup V'\}
\end{aligned}
$$

$$\langle s, V \rangle\ [\mathcal{R}_r]\ I \iff (r = s) \wedge (V\ [\mathcal{M}]\ I)$$

$$
\begin{aligned}
&V\ [\mathcal{M}]\ ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar}), \mathsf{info}) \iff\\
&\quad (\forall (a, v), (a', v') \in V.\, (a = a' \implies v = v')) \wedge\\
&\quad (\forall (a, v) \in V.\, \exists s.\, v(s) > 0) \wedge\\
&\quad (\forall (a, v) \in V.\, v \not\sqsubseteq \textstyle\bigsqcup\{v' \mid \exists a'.\, (a', v') \in V \wedge a \neq a'\}) \wedge\\
&\quad \exists \text{ distinct } e_{s,k}.\\
&\quad (\{e \in E \mid \exists a.\, \mathsf{oper}(e) = \mathtt{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicaID} \wedge\\
&\qquad 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V\}\}) \wedge\\
&\quad (\forall s, j, k.\, (\mathsf{repl}(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}} e_{s,k} \iff j < k)) \wedge\\
&\quad (\forall (a, v) \in V.\, \forall q.\, \{j \mid \mathsf{oper}(e_{q,j}) = \mathtt{wr}(a)\} \cup\\
&\qquad \{j \mid \exists s, k.\, e_{q,j} \xrightarrow{\mathsf{vis}} e_{s,k} \wedge \mathsf{oper}(e_{s,k}) = \mathtt{wr}(a)\} =\\
&\qquad \{j \mid 1 \leq j \leq v(q)\}) \wedge\\
&\quad (\forall e \in E.\, (\mathsf{oper}(e) = \mathtt{wr}(a) \wedge\\
&\qquad \neg\exists f \in E.\, \mathsf{oper}(f) = \mathtt{wr}(\_) \wedge e \xrightarrow{\mathsf{vis}} f) \implies (a, \_) \in V)
\end{aligned}
$$

the former. The only non-trivial obligation is to show that if

$$V\ [\mathcal{M}]\ ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}), \mathsf{info}),$$

then

$$
\begin{aligned}
\{a \mid (a, \_) \in V\} \subseteq \{a \mid \exists e \in E.\, \mathsf{oper}(e) = \mathtt{wr}(a) \wedge\\
\neg\exists f \in E.\, \exists a'.\, \mathsf{oper}(e) = \mathtt{wr}(a') \wedge e \xrightarrow{\mathsf{vis}} f\}
\end{aligned}
\tag{13}
$$

(the reverse inclusion is straightforwardly implied by $\mathcal{R}_r$).

Take $(a, v) \in V$. We have $\forall (a, v) \in V.\, \exists s.\, v(s) > 0$,

$$v \not\sqsubseteq \textstyle\bigsqcup\{v' \mid \exists a'.\, (a', v') \in V \wedge a \neq a'\}$$

and

$$
\begin{aligned}
&\forall (a, v) \in V.\, \forall q.\, \{j \mid \mathsf{oper}(e_{q,j}) = \mathtt{wr}(a)\} \cup\\
&\quad \{j \mid \exists s, k.\, e_{q,j} \xrightarrow{\mathsf{vis}} e_{s,k} \wedge \mathsf{oper}(e_{s,k}) = \mathtt{wr}(a)\} =\\
&\quad \{j \mid 1 \leq j \leq v(q)\}.
\end{aligned}
$$

From this we get that for some $e \in E$

$$
\begin{aligned}
\mathsf{oper}(e) = \mathtt{wr}(a) \wedge \neg\exists f \in E.\, \exists a'.\, a' \neq a \wedge\\
\mathsf{oper}(e) = \mathtt{wr}(a') \wedge e \xrightarrow{\mathsf{vis}} f.
\end{aligned}
$$

Since vis is acyclic, this implies that for some $e' \in E$

$$\mathsf{oper}(e') = \mathtt{wr}(a) \wedge \neg\exists f \in E.\, \mathsf{oper}(e') = \mathtt{wr}(\_) \wedge e' \xrightarrow{\mathsf{vis}} f,$$

which establishes (13).

Let us now discharge RECEIVE. Let $\mathsf{receive}(\langle r, V \rangle, V') = \langle r, V''' \rangle$, where

$$
\begin{aligned}
V'' &= \{(a, \textstyle\bigsqcup\{v' \mid (a, v') \in V \cup V'\}) \mid (a, \_) \in V \cup V'\};\\
V''' &= \{(a, v) \in V'' \mid v \not\sqsubseteq \textstyle\bigsqcup\{(a', v') \in V'' \mid a \neq a'\}\}.
\end{aligned}
$$

Assume $\langle r, V \rangle\ [\mathcal{R}_r]\ I$, $V'\ [\mathcal{M}]\ J$ and

$$
\begin{aligned}
I &= ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar}), \mathsf{info});\\
J &= ((E', \mathsf{repl}', \mathsf{obj}', \mathsf{oper}', \mathsf{rval}', \mathsf{ro}', \mathsf{vis}', \mathsf{ar}'), \mathsf{info}');\\
I \sqcup J &= ((E'', \mathsf{repl}'', \mathsf{obj}'', \mathsf{oper}'', \mathsf{rval}'', \mathsf{ro}'', \mathsf{vis}'', \mathsf{ar}''), \mathsf{info}'').
\end{aligned}
$$

By agree we have $I \sqcup J \in \mathsf{IEx}$. Then

$$
\begin{aligned}
&(\forall (a, v), (a', v') \in V.\, (a = a' \implies v = v')) \wedge\\
&(\forall (a, v) \in V.\, \exists s.\, v(s) > 0) \wedge\\
&(\forall (a, v) \in V.\, v \not\sqsubseteq \textstyle\bigsqcup\{v' \mid \exists a'.\, (a', v') \in V \wedge a \neq a'\}) \wedge\\
&\exists \text{ distinct } e_{s,k}.\\
&(\{e \in E \mid \exists a.\, \mathsf{oper}''(e) = \mathtt{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicaID} \wedge\\
&\quad 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V\}\}) \wedge\\
&(\forall s, j, k.\, (\mathsf{repl}''(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}} e_{s,k} \iff j < k)) \wedge\\
&(\forall (a, v) \in V.\, \forall q.\, \{j \mid \mathsf{oper}''(e_{q,j}) = \mathtt{wr}(a)\} \cup\\
&\quad \{j \mid \exists s, k.\, e_{q,j} \xrightarrow{\mathsf{vis}} e_{s,k} \wedge \mathsf{oper}''(e_{s,k}) = \mathtt{wr}(a)\} =\\
&\quad \{j \mid 1 \leq j \leq v(q)\}) \wedge\\
&(\forall e \in E.\, (\mathsf{oper}''(e) = \mathtt{wr}(a) \wedge\\
&\quad \neg\exists f \in E.\, \mathsf{oper}''(f) = \mathtt{wr}(\_) \wedge e \xrightarrow{\mathsf{vis}} f) \implies (a, \_) \in V)
\end{aligned}
$$

and

$$
\begin{aligned}
&(\forall (a, v), (a', v') \in V'.\, (a = a' \implies v = v')) \wedge\\
&(\forall (a, v) \in V'.\, \exists s.\, v(s) > 0) \wedge\\
&(\forall (a, v) \in V'.\, v \not\sqsubseteq \textstyle\bigsqcup\{v' \mid \exists a'.\, (a', v') \in V' \wedge a \neq a'\}) \wedge\\
&\exists \text{ distinct } e_{s,k}.\\
&(\{e \in E' \mid \exists a.\, \mathsf{oper}''(e) = \mathtt{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicaID} \wedge\\
&\quad 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V'\}\}) \wedge\\
&(\forall s, j, k.\, (\mathsf{repl}''(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}'} e_{s,k} \iff j < k)) \wedge\\
&(\forall (a, v) \in V'.\, \forall q.\, \{j \mid \mathsf{oper}''(e_{q,j}) = \mathtt{wr}(a)\} \cup\\
&\quad \{j \mid \exists s, k.\, e_{q,j} \xrightarrow{\mathsf{vis}'} e_{s,k} \wedge \mathsf{oper}''(e_{s,k}) = \mathtt{wr}(a)\} =\\
&\quad \{j \mid 1 \leq j \leq v(q)\}) \wedge\\
&(\forall e \in E'.\, (\mathsf{oper}''(e) = \mathtt{wr}(a) \wedge\\
&\quad \neg\exists f \in E'.\, \mathsf{oper}''(f) = \mathtt{wr}(\_) \wedge e \xrightarrow{\mathsf{vis}'} f) \implies (a, \_) \in V').
\end{aligned}
$$

The agree property also implies

$$
\begin{aligned}
\forall s, k.\, 1 \leq k \leq \min\{&\max\{v(s) \mid \exists a.\, (a, v) \in V\},\\
&\max\{v(s) \mid \exists a.\, (a, v) \in V'\}\} \implies e_{s,k} = e'_{s,k}.
\end{aligned}
$$

Hence, there exist distinct

$$e''_{s,k} \text{ for } s \in \mathsf{ReplicaID},\ k = 1..(\max\{v(s) \mid \exists a.\, (a, v) \in V'''\}),$$

such that

$$
\begin{aligned}
&(\forall s, k.\, 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V\} \implies e''_{s,k} = e_{s,k}) \wedge\\
&(\forall s, k.\, 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V'\} \implies e''_{s,k} = e'_{s,k})
\end{aligned}
$$

and

$$
\begin{aligned}
&(\{e \in E \cup E' \mid \exists a.\, \mathsf{oper}''(e) = \mathtt{wr}(a)\} =\\
&\{e''_{s,k} \mid s \in \mathsf{ReplicaID} \wedge 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V'''\}\})\\
&\wedge (\forall s, j, k.\, (\mathsf{repl}(e''_{s,k}) = s) \wedge (e''_{s,j} \xrightarrow{\mathsf{ro}''} e''_{s,k} \iff j < k)).
\end{aligned}
$$

By the definition of V'' and V''' we have

$$\forall (a, v), (a', v') \in V'''.\, (a = a' \implies v = v').$$

We also straightforwardly get

$$\forall (a, v) \in V'.\, \exists s.\, v(s) > 0$$

and

$$
\begin{aligned}
&(\forall (a, v) \in V''.\, \forall q.\, \{j \mid \mathsf{oper}''(e''_{q,j}) = \mathtt{wr}(a)\} \cup\\
&\quad \{j \mid \exists s, k.\, e''_{q,j} \xrightarrow{\mathsf{vis}''} e''_{s,k} \wedge \mathsf{oper}''(e''_{s,k}) = \mathtt{wr}(a)\} =\\
&\quad \{j \mid 1 \leq j \leq v(q)\}).
\end{aligned}
\tag{14}
$$


ANNE FINE
JULIA DONALDSON
DUFFU
CAROL ANN
MICHAEL MORPURGO
ANTHONY BROWNE
JACQUELINE WILSON
MICHAEL ROSEN
QUENTIN BLAKE
AND MANY MORE
DON'T BE SCARED
AN ANTHOLOGY
FOR CHILDREN
BY WELL-LOVED
AUTHORS & ARTISTS


ANNE FINE
JULIA DONALDSON
DUFFU
CAROL ANN
MICHAEL MORPURGO
ANTHONY BROWNE
JACQUELINE WILSON
MICHAEL ROSEN
QUENTIN BLAKE
AND MANY MORE
DON'T BE SCARED
AN ANTHOLOGY
FOR CHILDREN
BY WELL-LOVED
AUTHORS & ARTISTS


I'm so excited.

Definition (关系 (Relations))

A *relation* R from A to B is a subset of $A \times B$:

$$R \subseteq A \times B$$

Definition (关系 (Relations))

A *relation* R from A to B is a subset of $A \times B$:

$$R \subseteq A \times B$$

Definition (Cartesian Products)

The *Cartesian product* $A \times B$ of A and B is defined as

$$A \times B \triangleq \{(a, b) \mid a \in A \land b \in B\}$$

Definition (关系 (Relations))

A *relation* R from A to B is a subset of $A \times B$:

$$R \subseteq A \times B$$

Definition (Cartesian Products)

The *Cartesian product* $A \times B$ of A and B is defined as

$$A \times B \triangleq \{(a,b) \mid a \in A \wedge b \in B\}$$

Theorem (Ordered Pairs)

$$(a,b) = (c,d) \iff a = c \wedge b = d$$

Definition (有序对 (Ordered Pairs))

$$(a, b) = (c, d) \iff a = c \land b = d$$

Definition (有序对 (Ordered Pairs))

$$(a, b) = (c, d) \iff a = c \land b = d$$

Definition (有序对 (Ordered Pairs) (Kazimierz Kuratowski; 1921))

$$(a, b) \triangleq \{\{a\}, \{a, b\}\}$$

Definition (Ordered Pairs (Kazimierz Kuratowski; 1921))

$$(a, b) \triangleq \{\{a\}, \{a, b\}\}$$

Theorem

$$(a, b) = (c, d) \iff a = c \land b = d$$

Definition (Ordered Pairs (Kazimierz Kuratowski; 1921))

$$(a, b) \triangleq \{\{a\}, \{a, b\}\}$$

Theorem

$$(a, b) = (c, d) \iff a = c \wedge b = d$$

Proof.

$$\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$$

Definition (Ordered Pairs (Kazimierz Kuratowski; 1921))

$$(a, b) \triangleq \{\{a\}, \{a, b\}\}$$

Theorem

$$(a, b) = (c, d) \iff a = c \land b = d$$

Proof.

$$\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$$

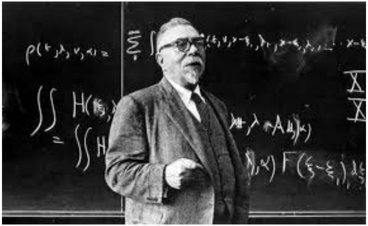
Case I : $a = b$

Case II : $a \neq b$

□

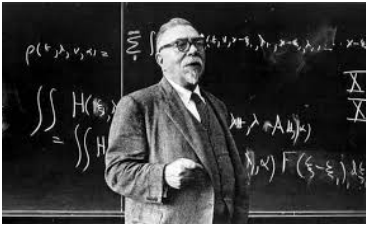
Definition (Ordered Pairs (Norbert Wiener; 1914))

$$(a, b) \triangleq \Big\{\{\{a\}, \emptyset\}, \{\{b\}\}\Big\}$$



Definition (Ordered Pairs (Norbert Wiener; 1914))

$$(a, b) \triangleq \Big\{ \{\{a\}, \emptyset\}, \{\{b\}\} \Big\}$$



Theorem

$$(a, b) = (c, d) \iff a = c \land b = d$$

Definition (n-元组 (n-ary tuples))

$$(x, y, z) \triangleq ((x, y), z)$$

Definition (n-元组 (n-ary tuples))

$$(x, y, z) \triangleq ((x, y), z)$$

$$(x_1, x_2, \dots, x_{n-1}, x_n) \triangleq ((x_1, x_2, \dots, x_{n-1}), x_n)$$

Definition (n-元组 (n-ary tuples))

$$(x, y, z) \triangleq ((x, y), z)$$

$$(x_1, x_2, \ldots, x_{n-1}, x_n) \triangleq ((x_1, x_2, \ldots, x_{n-1}), x_n)$$

Theorem

$$(x_1, \ldots, x_n) = (y_1, \ldots, y_n) \iff x_1 = y_1 \land \ldots x_n = y_n$$

Definition (笛卡尔积 (Cartesian Products))

The *Cartesian product* $A \times B$ of A and B is defined as

$$A \times B \triangleq \{(a, b) \mid a \in A \land b \in B\}$$

Definition (笛卡尔积 (Cartesian Products))

The *Cartesian product* $A \times B$ of A and B is defined as

$$A \times B \triangleq \{(a, b) \mid a \in A \wedge b \in B\}$$

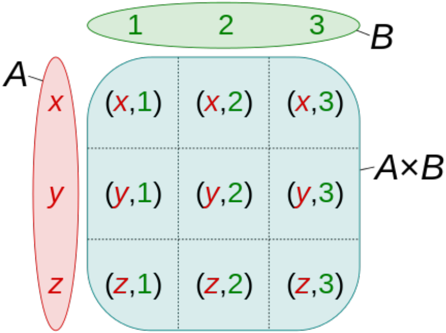
$$X^2 \triangleq X \times X$$

Definition (笛卡尔积 (Cartesian Products))

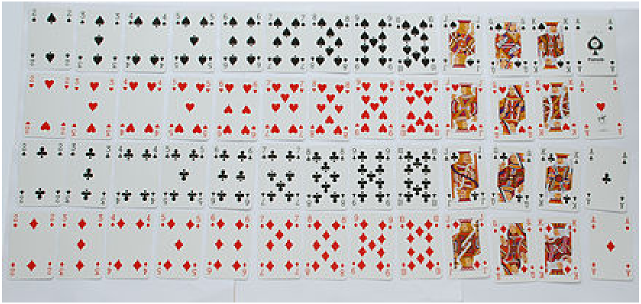
The *Cartesian product* $A \times B$ of A and B is defined as

$$A \times B \triangleq \{(a, b) \mid a \in A \wedge b \in B\}$$

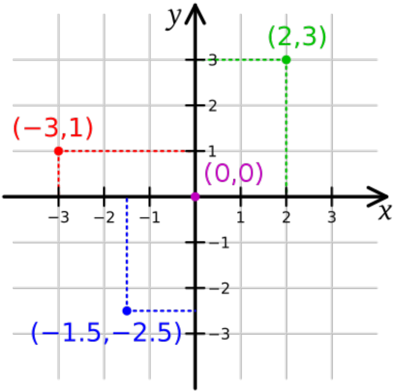
$$X^2 \triangleq X \times X$$



$$\text{Ranks} = \{2, \ldots, 10, J, Q, K, A\}$$



$$\text{Suits} = \{\}$$



$$\mathbb{Z}^2 \triangleq \mathbb{Z} \times \mathbb{Z}$$

$$X \times \emptyset = \emptyset \times X$$

$$X \times \emptyset = \emptyset \times X$$

$$X \times Y \neq Y \times X$$

$$X \times \emptyset = \emptyset \times X$$

$$X \times Y \neq Y \times X$$

$$(X \times Y) \times Z \neq X \times (Y \times Z)$$

$$X \times \emptyset = \emptyset \times X$$

$$X \times Y \neq Y \times X$$

$$(X \times Y) \times Z \neq X \times (Y \times Z)$$

$$A = \{1\} \qquad (A \times A) \times A \neq A \times (A \times A)$$

Theorem (分配律 (Distributivity))

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

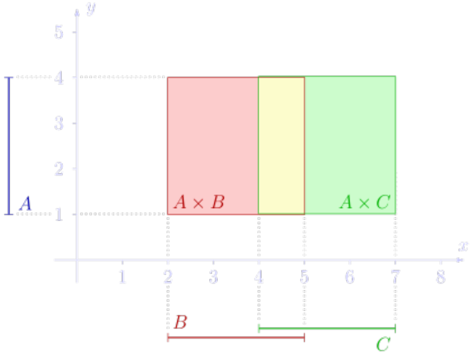
$$A \times (B \setminus C) = (A \times B) \setminus (A \times C)$$

Theorem (分配律 (Distributivity))

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

$$A \times (B \setminus C) = (A \times B) \setminus (A \times C)$$



Definition (n-元笛卡尔积 (n-ary Cartesian Product))

$$X_1 \times X_2 \times X_3 \triangleq (X_1 \times X_2) \times X_3$$

Definition (n-元笛卡尔积 (n-ary Cartesian Product))

$$X_1 \times X_2 \times X_3 \triangleq (X_1 \times X_2) \times X_3$$

$$X_1 \times X_2 \times \cdots \times X_n \triangleq (X_1 \times X_2 \times \cdots \times X_{n-1}) \times X_n$$

Definition (n-元笛卡尔积 (n-ary Cartesian Product))

$$X_1 \times X_2 \times X_3 \triangleq (X_1 \times X_2) \times X_3$$

$$X_1 \times X_2 \times \cdots \times X_n \triangleq (X_1 \times X_2 \times \cdots \times X_{n-1}) \times X_n$$

$$X^n \triangleq \underbrace{X \times \cdots \times X}_{n}$$

Definition (关系 (Relations))

A *relation* R from A to B is a subset of $A \times B$:

$$R \subseteq A \times B$$

Definition (关系 (Relations))

A *relation* R from A to B is a subset of $A \times B$:

$$R \subseteq A \times B$$

If $A = B$, R is called a relation on A.

Definition (关系 (Relations))

A *relation* R from A to B is a subset of $A \times B$:

$$R \subseteq A \times B$$

If $A = B$, R is called a relation on A.

Definition (Notations)

$$(a, b) \in R \qquad R(a, b) \qquad aRb$$

Definition (Relations)

A *relation* R from A to B is a subset of $A \times B$:

$$R \subseteq A \times B$$

Examples

Definition (Relations)

A *relation* R from A to B is a subset of $A \times B$:

$$R \subseteq A \times B$$

Examples

- Both $A \times B$ and $\emptyset$ are relations from A to B.

Definition (Relations)

A *relation* R from A to B is a subset of $A \times B$:

$$R \subseteq A \times B$$

Examples

- Both $A \times B$ and $\emptyset$ are relations from A to B.
-
$$< \; = \{(a, b) \in \mathbb{R} \times \mathbb{R} \mid a \text{ is less than } b\}$$

Definition (Relations)

A *relation* R from A to B is a subset of $A \times B$:

$$R \subseteq A \times B$$

Examples

- Both $A \times B$ and $\emptyset$ are relations from A to B.
- $$< \; = \{(a, b) \in \mathbb{R} \times \mathbb{R} \mid a \text{ is less than } b\}$$
- $$D = \{(a, b) \in \mathbb{N} \times \mathbb{N} \mid \exists q \in \mathbb{N} : a \cdot q = b\}$$

Definition (Relations)

A *relation* R from A to B is a subset of $A \times B$:

$$R \subseteq A \times B$$

Examples

- Both $A \times B$ and $\emptyset$ are relations from A to B.
- $$< \, = \{(a, b) \in \mathbb{R} \times \mathbb{R} \mid a \text{ is less than } b\}$$
- $$D = \{(a, b) \in \mathbb{N} \times \mathbb{N} \mid \exists q \in \mathbb{N} : a \cdot q = b\}$$
- P : the set of people

$$M = \{(a, b) \in P \times P \mid a \text{ is the mother of } b\}$$

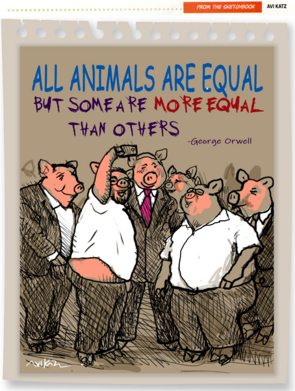
$$B = \{(a, b) \in P \times P \mid a \text{ is the brother of } b\}$$

Important Relations:

Equivalence Relations

Ordering Relations

Functions (next class)



Outline:

3 Definitions

$$\mathrm{dom}(R) \qquad \mathrm{ran}(R) \qquad \mathrm{fld}(R)$$

Definition (定义域 (Domain))

$$\mathrm{dom}(R) = \{a \mid \exists b.\ (a, b) \in R\}$$

Definition (定义域 (Domain))

$$\mathrm{dom}(R) = \{a \mid \exists b.\ (a, b) \in R\}$$

Definition (值域 (Range))

$$\mathrm{ran}(R) = \{b \mid \exists a : (a, b) \in R\}$$

Definition (定义域 (Domain))

$$\mathrm{dom}(R) = \{a \mid \exists b.\ (a, b) \in R\}$$

Definition (值域 (Range))

$$\mathrm{ran}(R) = \{b \mid \exists a : (a, b) \in R\}$$

Definition (域 (Field))

$$\mathrm{fld}(R) = \mathrm{dom}(R) \cup \mathrm{ran}(R)$$

$$R = \{(x, y) \mid x = y\} \subseteq \mathbb{R} \times \mathbb{R}$$

$$R = \{(x, y) \mid x = y\} \subseteq \mathbb{R} \times \mathbb{R}$$

$$\mathrm{dom}(R) = \mathbb{R} \qquad \mathrm{ran}(R) = \mathbb{R} \qquad \mathrm{fld}(R) = \mathbb{R}$$

$$R = \{(x, y) \mid x^2 + y^2 = 1\} \subseteq \mathbb{R} \times \mathbb{R}$$

$$R = \{(x, y) \mid x^2 + y^2 = 1\} \subseteq \mathbb{R} \times \mathbb{R}$$

$$\mathrm{dom}(R) = [1, 1] \qquad \mathrm{ran}(R) = [-1, 1] \qquad \mathrm{fld}(R) = [-1, 1]$$

Theorem

$$dom(R) \subseteq \bigcup \bigcup R \qquad ran(R) \subseteq \bigcup \bigcup R$$

Theorem

$$dom(R) \subseteq \bigcup\bigcup R \qquad ran(R) \subseteq \bigcup\bigcup R$$

对任意 a,

$$a \in \mathrm{dom}(R) \tag{1}$$

$$\tag{6}$$

Theorem

$$dom(R) \subseteq \bigcup\bigcup R \qquad ran(R) \subseteq \bigcup\bigcup R$$

对任意 a,

$$a \in \mathrm{dom}(R) \tag{1}$$

$$\Longrightarrow \exists b.\ (a, b) \in R \tag{2}$$

$$\tag{6}$$

Theorem

$$dom(R) \subseteq \bigcup\bigcup R \qquad ran(R) \subseteq \bigcup\bigcup R$$

对任意 a,

$$a \in \mathrm{dom}(R) \tag{1}$$

$$\implies \exists b.\ (a, b) \in R \tag{2}$$

$$\tag{6}$$

Theorem

$$dom(R) \subseteq \bigcup\bigcup R \qquad ran(R) \subseteq \bigcup\bigcup R$$

对任意 a,

$$a \in \mathrm{dom}(R) \tag{1}$$

$$\Longrightarrow \exists b.\ (a, b) \in R \tag{2}$$

$$\Longrightarrow \exists b.\ \{\{a\}, \{a, b\}\} \in R \tag{3}$$

$$\tag{6}$$

Theorem

$$dom(R) \subseteq \bigcup\bigcup R \qquad ran(R) \subseteq \bigcup\bigcup R$$

对任意 a,

$$a \in \mathrm{dom}(R) \tag{1}$$

$$\Longrightarrow \exists b.\ (a, b) \in R \tag{2}$$

$$\Longrightarrow \exists b.\ \{\{a\}, \{a, b\}\} \in R \tag{3}$$

$$\Longrightarrow \exists b.\ \{a, b\} \in \bigcup R \tag{4}$$

$$\tag{6}$$

Theorem

$$dom(R) \subseteq \bigcup\bigcup R \qquad ran(R) \subseteq \bigcup\bigcup R$$

对任意 a,

$$
\begin{aligned}
& a \in \mathrm{dom}(R) && (1)\\
\Longrightarrow\ & \exists b.\ (a, b) \in R && (2)\\
\Longrightarrow\ & \exists b.\ \{\{a\}, \{a, b\}\} \in R && (3)\\
\Longrightarrow\ & \exists b.\ \{a, b\} \in \bigcup R && (4)\\
\Longrightarrow\ & \exists b.\ a \in \bigcup\bigcup R && (5)\\
& && (6)
\end{aligned}
$$

Theorem

$$dom(R) \subseteq \bigcup\bigcup R \qquad ran(R) \subseteq \bigcup\bigcup R$$

对任意 a,

$$
\begin{aligned}
& a \in \mathrm{dom}(R) && (1)\\
\Longrightarrow\ & \exists b.\ (a, b) \in R && (2)\\
\Longrightarrow\ & \exists b.\ \{\{a\}, \{a, b\}\} \in R && (3)\\
\Longrightarrow\ & \exists b.\ \{a, b\} \in \bigcup R && (4)\\
\Longrightarrow\ & \exists b.\ a \in \bigcup\bigcup R && (5)\\
\Longrightarrow\ & a \in \bigcup\bigcup R && (6)
\end{aligned}
$$

5 Operations

$$R^{-1} \qquad R|_X \qquad R[X] \qquad R^{-1}[Y] \qquad R \circ S$$

Definition (逆 (Inverse))

The *inverse* of R is the relation

$$R^{-1} = \{(a, b) \mid (b, a) \in R\}$$

Definition (逆 (Inverse))

The *inverse* of R is the relation

$$R^{-1} = \{(a, b) \mid (b, a) \in R\}$$

$$R = \{(x, y) \mid x = y\} \subseteq \mathbb{R} \times \mathbb{R} \qquad R^{-1} =$$

Definition (逆 (Inverse))

The *inverse* of R is the relation

$$R^{-1} = \{(a, b) \mid (b, a) \in R\}$$

$$R = \{(x, y) \mid x = y\} \subseteq \mathbb{R} \times \mathbb{R} \qquad R^{-1} = R$$

Definition (逆 (Inverse))

The *inverse* of R is the relation

$$R^{-1} = \{(a, b) \mid (b, a) \in R\}$$

$$R = \{(x, y) \mid x = y\} \subseteq \mathbb{R} \times \mathbb{R} \qquad R^{-1} = R$$

$$R = \{(x, y) \mid y = \sqrt{x}\} \subseteq \mathbb{R} \times \mathbb{R} \qquad R^{-1} =$$

Definition (逆 (Inverse))

The *inverse* of R is the relation

$$R^{-1} = \{(a, b) \mid (b, a) \in R\}$$

$$R = \{(x, y) \mid x = y\} \subseteq \mathbb{R} \times \mathbb{R} \qquad R^{-1} = R$$

$$R = \{(x, y) \mid y = \sqrt{x}\} \subseteq \mathbb{R} \times \mathbb{R} \qquad R^{-1} = \{(x, y) \mid y = x^2 \land x > 0\}$$

Definition (逆 (Inverse))

The *inverse* of R is the relation

$$R^{-1} = \{(a, b) \mid (b, a) \in R\}$$

$$R = \{(x, y) \mid x = y\} \subseteq \mathbb{R} \times \mathbb{R} \qquad R^{-1} = R$$

$$R = \{(x, y) \mid y = \sqrt{x}\} \subseteq \mathbb{R} \times \mathbb{R} \qquad R^{-1} = \{(x, y) \mid y = x^2 \land x > 0\}$$

$$\leq = \{(x, y) \mid x \leq y\} \subseteq \mathbb{R} \times \mathbb{R} \qquad \leq^{-1} =$$

Definition (逆 (Inverse))

The *inverse* of R is the relation

$$R^{-1} = \{(a, b) \mid (b, a) \in R\}$$

$$R = \{(x, y) \mid x = y\} \subseteq \mathbb{R} \times \mathbb{R} \qquad R^{-1} = R$$

$$R = \{(x, y) \mid y = \sqrt{x}\} \subseteq \mathbb{R} \times \mathbb{R} \qquad R^{-1} = \{(x, y) \mid y = x^2 \land x > 0\}$$

$$\leq = \{(x, y) \mid x \leq y\} \subseteq \mathbb{R} \times \mathbb{R} \qquad \leq^{-1} = \ \geq \triangleq \{(x, y) \mid x \geq y\}$$

Theorem

$$(R^{-1})^{-1} = R$$

Theorem

$$(R^{-1})^{-1} = R$$

对任意 (a, b),

$$(a, b) \in (R^{-1})^{-1} \tag{1}$$

$$\tag{3}$$

Theorem

$$(R^{-1})^{-1} = R$$

对任意 (a, b),

$$(a, b) \in (R^{-1})^{-1} \tag{1}$$

$$\iff (b, a) \in R^{-1} \tag{2}$$

$$\tag{3}$$

Theorem

$$(R^{-1})^{-1} = R$$

对任意 (a, b),

$$\begin{aligned}
& (a, b) \in (R^{-1})^{-1} && (1) \\
\Longleftrightarrow\ & (b, a) \in R^{-1} && (2) \\
\Longleftrightarrow\ & (a, b) \in R && (3)
\end{aligned}$$

Theorem (关系的逆)

$$R, S \subseteq A \times B$$

$$(R \cup S)^{-1} = R^{-1} \cup S^{-1}$$
$$(R \cap S)^{-1} = R^{-1} \cap S^{-1}$$
$$(R \setminus S)^{-1} = R^{-1} \setminus S^{-1}$$

Definition (左限制 (Left-Restriction))

Suppose $R \subseteq X \times Y$ and $S \subseteq X$. The *left-restriction* relation of R to S is

$$R|_S = \{(x, y) \in R \mid x \in S\}$$

Definition (左限制 (Left-Restriction))

Suppose $R \subseteq X \times Y$ and $S \subseteq X$. The *left-restriction* relation of R to S is

$$R|_S = \{(x, y) \in R \mid x \in S\}$$

Definition (右限制 (Right-Restriction))

Suppose $R \subseteq X \times Y$ and $S \subseteq Y$. The *right-restriction* relation of R to S is

$$R|_S = \{(x, y) \in R \mid y \in S\}$$

Definition (左限制 (Left-Restriction))

Suppose $R \subseteq X \times Y$ and $S \subseteq X$. The *left-restriction* relation of R to S is

$$R|_S = \{(x, y) \in R \mid x \in S\}$$

Definition (右限制 (Right-Restriction))

Suppose $R \subseteq X \times Y$ and $S \subseteq Y$. The *right-restriction* relation of R to S is

$$R|_S = \{(x, y) \in R \mid y \in S\}$$

Definition (限制 (Restriction))

Suppose $R \subseteq X \times X$ and $S \subseteq X$. The *restriction* relation of R to S is

$$R|_S = \{(x, y) \in R \mid x \in S \wedge y \in S\}$$

example

Definition (像 (Image))

The *image* of X under R is the set

$$R[X] = \{b \in \mathrm{ran}(R) \mid \exists a \in X.\ (a, b) \in R\}$$

Definition (像 (Image))

The *image* of X under R is the set

$$R[X] = \{b \in \text{ran}(R) \mid \exists a \in X.\ (a, b) \in R\}$$

$$R[a] \triangleq R[\{a\}] = \{b \mid (a, b) \in R\}$$

Definition (逆像 (Inverse Image))

The *inverse image* of Y under R is the set

$$R^{-1}[Y] = \{a \in \text{dom}(R) \mid \exists b \in Y : (a, b) \in R\}$$

Definition (逆像 (Inverse Image))

The *inverse image* of Y under R is the set

$$R^{-1}[Y] = \{a \in \mathrm{dom}(R) \mid \exists b \in Y : (a, b) \in R\}$$

$$R^{-1}[b] \triangleq R^{-1}[\{b\}] = \{a \mid (a, b) \in R\}$$

$$R \subseteq A \times B \qquad X \subseteq A \qquad Y \subseteq B$$

$$R \subseteq A \times B \qquad X \subseteq A \qquad Y \subseteq B$$

$$R^{-1}[R[X]] \ ? \ X$$

$$R[R^{-1}[Y]] \ ? \ Y$$

$$R \subseteq A \times B \qquad X \subseteq A \qquad Y \subseteq B$$

$$R^{-1}[R[X]] \; ? \; X$$

$$R[R^{-1}[Y]] \; ? \; Y$$

Theorem

$$R[X_1 \cup X_2] = R[X_1] \cup R[X_2]$$

$$R[X_1 \cap X_2] \subseteq R[X_1] \cap R[X_2]$$

$$R[X_1 \setminus X_2] \supseteq R[X_1] \setminus R[X_2]$$

Theorem

$$R[X_1 \cup X_2] = R[X_1] \cup R[X_2]$$

$$R[X_1 \cap X_2] \subseteq R[X_1] \cap R[X_2]$$

$$R[X_1 \setminus X_2] \supseteq R[X_1] \setminus R[X_2]$$

对任意 b,

Theorem

$$R[X_1 \cup X_2] = R[X_1] \cup R[X_2]$$

$$R[X_1 \cap X_2] \subseteq R[X_1] \cap R[X_2]$$

$$R[X_1 \setminus X_2] \supseteq R[X_1] \setminus R[X_2]$$

对任意 b,

$$b \in R[X_1 \cup X_2]$$

Theorem

$$R[X_1 \cup X_2] = R[X_1] \cup R[X_2]$$

$$R[X_1 \cap X_2] \subseteq R[X_1] \cap R[X_2]$$

$$R[X_1 \setminus X_2] \supseteq R[X_1] \setminus R[X_2]$$

对任意 b,

$$b \in R[X_1 \cup X_2]$$

$$\iff \exists a \in X_1 \cup X_2 : (a, b) \in R$$

Theorem

$$R[X_1 \cup X_2] = R[X_1] \cup R[X_2]$$

$$R[X_1 \cap X_2] \subseteq R[X_1] \cap R[X_2]$$

$$R[X_1 \setminus X_2] \supseteq R[X_1] \setminus R[X_2]$$

对任意 b,

$$
\begin{aligned}
& b \in R[X_1 \cup X_2] \\
\iff\ & \exists a \in X_1 \cup X_2 : (a, b) \in R \\
\iff\ & \exists a \in X_1 : (a, b) \in R \lor \exists a \in X_2 : (a, b) \in R
\end{aligned}
$$

Theorem

$$R[X_1 \cup X_2] = R[X_1] \cup R[X_2]$$

$$R[X_1 \cap X_2] \subseteq R[X_1] \cap R[X_2]$$

$$R[X_1 \setminus X_2] \supseteq R[X_1] \setminus R[X_2]$$

对任意 b,

$$\begin{aligned}
& b \in R[X_1 \cup X_2] \\
\iff & \exists a \in X_1 \cup X_2 : (a, b) \in R \\
\iff & \exists a \in X_1 : (a, b) \in R \lor \exists a \in X_2 : (a, b) \in R \\
\iff & b \in R[X_1] \lor b \in R[X_2]
\end{aligned}$$

Theorem

$$R[X_1 \cup X_2] = R[X_1] \cup R[X_2]$$

$$R[X_1 \cap X_2] \subseteq R[X_1] \cap R[X_2]$$

$$R[X_1 \setminus X_2] \supseteq R[X_1] \setminus R[X_2]$$

对任意 b,

$$\begin{aligned}
& b \in R[X_1 \cup X_2] \\
\iff & \exists a \in X_1 \cup X_2 : (a, b) \in R \\
\iff & \exists a \in X_1 : (a, b) \in R \lor \exists a \in X_2 : (a, b) \in R \\
\iff & b \in R[X_1] \lor b \in R[X_2] \\
\iff & b \in R[X_1] \cup R[X_2]
\end{aligned}$$

Definition (复合 (Composition; $R \circ S$, $R; S$))

The *composition* of relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ is the relation

$$R \circ S = \{(a, c) \mid \exists b : (a, b) \in S \land (b, c) \in R\}$$

Definition (复合 (Composition; $R \circ S$, $R; S$))

The *composition* of relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ is the relation

$$R \circ S = \{(a, c) \mid \exists b : (a, b) \in S \land (b, c) \in R\}$$

$$R = \{(1,2), (3,1)\} \qquad S = \{(1,3), (2,2), (2,3)\}$$

Definition (复合 (Composition; $R \circ S$, $R; S$))

The *composition* of relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ is the relation

$$R \circ S = \{(a, c) \mid \exists b : (a, b) \in S \land (b, c) \in R\}$$

$$R = \{(1, 2), (3, 1)\} \qquad S = \{(1, 3), (2, 2), (2, 3)\}$$

$$R \circ S =$$

Definition (复合 (Composition; $R \circ S$, $R;S$))

The *composition* of relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ is the relation

$$R \circ S = \{(a, c) \mid \exists b : (a, b) \in S \wedge (b, c) \in R\}$$

$$R = \{(1,2),(3,1)\} \qquad S = \{(1,3),(2,2),(2,3)\}$$

$$R \circ S = \{(1,1),(2,1)\}$$

Definition (复合 (Composition; $R \circ S$, $R;S$))

The *composition* of relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ is the relation

$$R \circ S = \{(a, c) \mid \exists b : (a, b) \in S \land (b, c) \in R\}$$

$$R = \{(1, 2), (3, 1)\} \qquad S = \{(1, 3), (2, 2), (2, 3)\}$$

$$R \circ S = \{(1, 1), (2, 1)\}$$

$$S \circ R =$$

Definition (复合 (Composition; $R \circ S$, $R; S$))

The *composition* of relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ is the relation

$$R \circ S = \{(a, c) \mid \exists b : (a, b) \in S \land (b, c) \in R\}$$

$$R = \{(1, 2), (3, 1)\} \qquad S = \{(1, 3), (2, 2), (2, 3)\}$$

$$R \circ S = \{(1, 1), (2, 1)\}$$

$$S \circ R = \{(1, 2), (1, 3), (3, 3)\}$$

Definition (复合 (Composition; $R \circ S$, $R; S$))

The *composition* of relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ is the relation

$$R \circ S = \{(a, c) \mid \exists b : (a, b) \in S \land (b, c) \in R\}$$

$$R = \{(1,2),(3,1)\} \qquad S = \{(1,3),(2,2),(2,3)\}$$

$$R \circ S = \{(1,1),(2,1)\}$$

$$S \circ R = \{(1,2),(1,3),(3,3)\}$$

$$R^{(2)} \triangleq R \circ R =$$

Definition (复合 (Composition; $R \circ S$, $R; S$))

The *composition* of relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ is the relation

$$R \circ S = \{(a, c) \mid \exists b : (a, b) \in S \land (b, c) \in R\}$$

$$R = \{(1,2), (3,1)\} \qquad S = \{(1,3), (2,2), (2,3)\}$$

$$R \circ S = \{(1,1), (2,1)\}$$

$$S \circ R = \{(1,2), (1,3), (3,3)\}$$

$$R^{(2)} \triangleq R \circ R = \{(3,2)\}$$

Definition (复合 (Composition; $R \circ S$, $R; S$))

The *composition* of relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ is the relation

$$R \circ S = \{(a, c) \mid \exists b : (a, b) \in S \land (b, c) \in R\}$$

$$R = \{(1,2), (3,1)\} \qquad S = \{(1,3), (2,2), (2,3)\}$$

$$R \circ S = \{(1,1), (2,1)\}$$

$$S \circ R = \{(1,2), (1,3), (3,3)\}$$

$$R^{(2)} \triangleq R \circ R = \{(3,2)\} \qquad (R \circ R) \circ R =$$

Definition (复合 (Composition; $R \circ S$, $R; S$))

The *composition* of relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ is the relation

$$R \circ S = \{(a, c) \mid \exists b : (a, b) \in S \land (b, c) \in R\}$$

$$R = \{(1, 2), (3, 1)\} \qquad S = \{(1, 3), (2, 2), (2, 3)\}$$

$$R \circ S = \{(1, 1), (2, 1)\}$$

$$S \circ R = \{(1, 2), (1, 3), (3, 3)\}$$

$$R^{(2)} \triangleq R \circ R = \{(3, 2)\} \qquad (R \circ R) \circ R = \emptyset$$

Definition (复合 (Composition; $R \circ S$, $R; S$))

The *composition* of relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ is the relation

$$R \circ S = \{(a, c) \mid \exists b : (a, b) \in S \land (b, c) \in R\}$$

$$R = \{(1,2), (3,1)\} \qquad S = \{(1,3), (2,2), (2,3)\}$$

$$R \circ S = \{(1,1), (2,1)\}$$

$$S \circ R = \{(1,2), (1,3), (3,3)\}$$

$$R^{(2)} \triangleq R \circ R = \{(3,2)\} \qquad (R \circ R) \circ R = \emptyset$$

$$S^{(2)} \triangleq S \circ S =$$

Definition (复合 (Composition; $R \circ S$, $R; S$))

The *composition* of relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ is the relation

$$R \circ S = \{(a, c) \mid \exists b : (a, b) \in S \land (b, c) \in R\}$$

$$R = \{(1, 2), (3, 1)\} \qquad S = \{(1, 3), (2, 2), (2, 3)\}$$

$$R \circ S = \{(1, 1), (2, 1)\}$$

$$S \circ R = \{(1, 2), (1, 3), (3, 3)\}$$

$$R^{(2)} \triangleq R \circ R = \{(3, 2)\} \qquad (R \circ R) \circ R = \emptyset$$

$$S^{(2)} \triangleq S \circ S = \{(2, 2), (2, 3)\}$$

Definition (复合 (Composition; $R \circ S$, $R; S$))

The *composition* of relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ is the relation

$$R \circ S = \{(a, c) \mid \exists b : (a, b) \in S \land (b, c) \in R\}$$

$$R = \{(1,2),(3,1)\} \qquad S = \{(1,3),(2,2),(2,3)\}$$

$$R \circ S = \{(1,1),(2,1)\}$$

$$S \circ R = \{(1,2),(1,3),(3,3)\}$$

$$R^{(2)} \triangleq R \circ R = \{(3,2)\} \qquad (R \circ R) \circ R = \emptyset$$

$$S^{(2)} \triangleq S \circ S = \{(2,2),(2,3)\} \qquad (S \circ S) \circ S =$$

Definition (复合 (Composition; $R \circ S$, $R; S$))

The *composition* of relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ is the relation

$$R \circ S = \{(a, c) \mid \exists b : (a, b) \in S \land (b, c) \in R\}$$

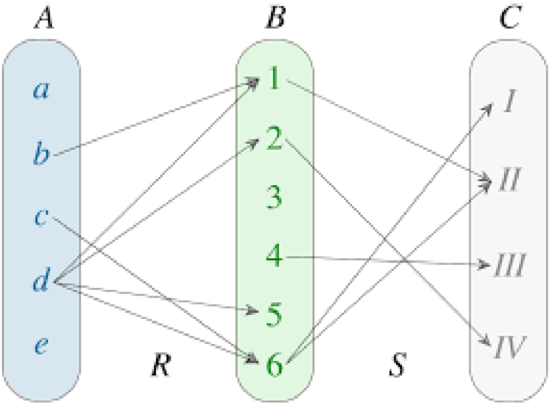
$$R = \{(1,2),(3,1)\} \qquad S = \{(1,3),(2,2),(2,3)\}$$

$$R \circ S = \{(1,1),(2,1)\}$$

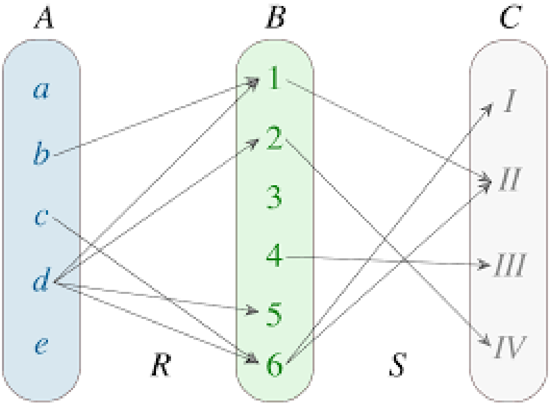
$$S \circ R = \{(1,2),(1,3),(3,3)\}$$

$$R^{(2)} \triangleq R \circ R = \{(3,2)\} \qquad (R \circ R) \circ R = \emptyset$$

$$S^{(2)} \triangleq S \circ S = \{(2,2),(2,3)\} \qquad (S \circ S) \circ S = \{(2,2),(2,3)\}$$



$$|R \circ S| =$$



$$|R \circ S| = 7$$

$$\leq \circ \leq \; =$$

$$\leq \circ \leq \;=\; \leq$$

$$\leq \circ \leq \; = \; \leq$$

$$\leq \circ \geq \; =$$

$$\leq \circ \leq \; = \; \leq$$

$$\leq \circ \geq \; = \mathbb{R} \times \mathbb{R}$$

Theorem

$$(R \circ S)^{-1} = S^{-1} \circ R^{-1}$$

Theorem

$$(R \circ S)^{-1} = S^{-1} \circ R^{-1}$$

对任意 (a, b),

$$(a, b) \in (R \circ S)^{-1} \tag{1}$$

(5)

Theorem

$$(R \circ S)^{-1} = S^{-1} \circ R^{-1}$$

对任意 (a, b),

$$(a, b) \in (R \circ S)^{-1} \tag{1}$$

$$\iff (b, a) \in R \circ S \tag{2}$$

$$\tag{5}$$

Theorem

$$(R \circ S)^{-1} = S^{-1} \circ R^{-1}$$

对任意 (a, b),

$$(a, b) \in (R \circ S)^{-1} \tag{1}$$

$$\iff (b, a) \in R \circ S \tag{2}$$

$$\iff \exists c.\ (b, c) \in R \land (c, a) \in S \tag{3}$$

$$\tag{5}$$

Theorem

$$(R \circ S)^{-1} = S^{-1} \circ R^{-1}$$

对任意 (a, b),

$$
\begin{align}
& (a, b) \in (R \circ S)^{-1} \tag{1} \\
\iff & (b, a) \in R \circ S \tag{2} \\
\iff & \exists c.\ (b, c) \in R \wedge (c, a) \in S \tag{3} \\
\iff & \exists c.\ (c, b) \in R^{-1} \wedge (a, c) \in S^{-1} \tag{4} \\
& \tag{5}
\end{align}
$$

Theorem

$$(R \circ S)^{-1} = S^{-1} \circ R^{-1}$$

对任意 (a, b),

$$
\begin{aligned}
& (a, b) \in (R \circ S)^{-1} && (1) \\
\iff & (b, a) \in R \circ S && (2) \\
\iff & \exists c.\ (b, c) \in R \land (c, a) \in S && (3) \\
\iff & \exists c.\ (c, b) \in R^{-1} \land (a, c) \in S^{-1} && (4) \\
\iff & (a, c) \in S^{-1} \circ R^{-1} && (5)
\end{aligned}
$$

Theorem

$$(R \circ S) \circ T = R \circ (S \circ T)$$

Theorem

$$(R \circ S) \circ T = R \circ (S \circ T)$$

对任意 (a, b),

$$(a, b) \in (R \circ S) \circ T \tag{1}$$

Theorem

$$(R \circ S) \circ T = R \circ (S \circ T)$$

对任意 (a, b),

$$(a, b) \in (R \circ S) \circ T \tag{1}$$

$$\iff \exists c. \Big((a, c) \in T \land (c, b) \in R \circ S\Big) \tag{2}$$

Theorem

$$(R \circ S) \circ T = R \circ (S \circ T)$$

对任意 (a, b),

$$(a, b) \in (R \circ S) \circ T \tag{1}$$

$$\iff \exists c. \Big((a, c) \in T \land (c, b) \in R \circ S\Big) \tag{2}$$

$$\iff \exists c. \Big((a, c) \in T \land (\exists d : (c, d) \in S \land (d, b) \in R)\Big) \tag{3}$$

Theorem

$$(R \circ S) \circ T = R \circ (S \circ T)$$

对任意 (a, b),

$$(a, b) \in (R \circ S) \circ T \tag{1}$$

$$\iff \exists c. \Big((a, c) \in T \land (c, b) \in R \circ S\Big) \tag{2}$$

$$\iff \exists c. \Big((a, c) \in T \land (\exists d : (c, d) \in S \land (d, b) \in R)\Big) \tag{3}$$

$$\iff \exists d. \exists c. \Big((a, c) \in T \land (c, d) \in S \land (d, b) \in R\Big) \tag{4}$$

Theorem

$$(R \circ S) \circ T = R \circ (S \circ T)$$

对任意 (a, b),

$$
\begin{align}
& (a, b) \in (R \circ S) \circ T \tag{1} \\
\iff & \exists c. \Big((a, c) \in T \land (c, b) \in R \circ S \Big) \tag{2} \\
\iff & \exists c. \Big((a, c) \in T \land (\exists d : (c, d) \in S \land (d, b) \in R) \Big) \tag{3} \\
\iff & \exists d.\ \exists c. \Big((a, c) \in T \land (c, d) \in S \land (d, b) \in R \Big) \tag{4} \\
\iff & \exists d. \Big((\exists c.\ (a, c) \in T \land (c, d) \in S) \land (d, b) \in R \Big) \tag{5}
\end{align}
$$

Theorem

$$(R \circ S) \circ T = R \circ (S \circ T)$$

对任意 (a, b),

$$(a, b) \in (R \circ S) \circ T \tag{1}$$

$$\iff \exists c. \Big((a, c) \in T \land (c, b) \in R \circ S\Big) \tag{2}$$

$$\iff \exists c. \Big((a, c) \in T \land (\exists d : (c, d) \in S \land (d, b) \in R)\Big) \tag{3}$$

$$\iff \exists d. \exists c. \Big((a, c) \in T \land (c, d) \in S \land (d, b) \in R\Big) \tag{4}$$

$$\iff \exists d. \Big((\exists c. (a, c) \in T \land (c, d) \in S) \land (d, b) \in R\Big) \tag{5}$$

$$\iff \exists d. \Big((a, d) \in S \circ T \land (d, b) \in R\Big) \tag{6}$$

Theorem

$$(R \circ S) \circ T = R \circ (S \circ T)$$

对任意 (a, b),

$$
\begin{aligned}
& (a, b) \in (R \circ S) \circ T && (1) \\
\iff & \exists c. \Big((a, c) \in T \land (c, b) \in R \circ S \Big) && (2) \\
\iff & \exists c. \Big((a, c) \in T \land \big(\exists d : (c, d) \in S \land (d, b) \in R\big) \Big) && (3) \\
\iff & \exists d.\ \exists c. \Big((a, c) \in T \land (c, d) \in S \land (d, b) \in R \Big) && (4) \\
\iff & \exists d. \Big(\big(\exists c.\ (a, c) \in T \land (c, d) \in S\big) \land (d, b) \in R \Big) && (5) \\
\iff & \exists d. \Big((a, d) \in S \circ T \land (d, b) \in R \Big) && (6)
\end{aligned}
$$

燕小六: “帮我照顾好我七舅姥爷和我外甥女”

“舅姥爷”：姥姥/外婆的兄弟

“舅姥爷”：姥姥/外婆的兄弟

$$G = \{(a, b) : a \text{ 是 } b \text{ 的舅姥爷}\}$$

“舅姥爷”：姥姥/外婆的兄弟

$$G = \{(a, b) : a \text{ 是 } b \text{ 的舅姥爷}\}$$

$$B = \{(a, b) \mid a \text{ is the brother of } b\}$$

$$M = \{(a, b) \mid a \text{ is the mother of } b\}$$

“舅姥爷”：姥姥/外婆的兄弟

$$G = \{(a,b) : a \text{ 是 } b \text{ 的舅姥爷}\}$$

$$B = \{(a,b) \mid a \text{ is the brother of } b\}$$

$$M = \{(a,b) \mid a \text{ is the mother of } b\}$$

$$G = B \circ (M \circ M)$$

“舅姥爷”: 姥姥/外婆的兄弟

$$G = \{(a, b) : a \text{ 是 } b \text{ 的舅姥爷}\}$$

$$B = \{(a, b) \mid a \text{ is the brother of } b\}$$

$$M = \{(a, b) \mid a \text{ is the mother of } b\}$$

$$G = B \circ (M \circ M)$$

$$G = B \circ (M \circ M) = (B \circ M) \circ M$$

"舅姥爷": 姥姥/外婆的兄弟

$$G = \{(a,b) : a \text{ 是 } b \text{ 的舅姥爷}\}$$

$$B = \{(a,b) \mid a \text{ is the brother of } b\}$$

$$M = \{(a,b) \mid a \text{ is the mother of } b\}$$

$$G = B \circ (M \circ M)$$

$$G = B \circ (M \circ M) = (B \circ M) \circ M$$

"舅姥爷": 妈妈的舅舅

Theorem (关系的复合)

$$(X \cup Y) \circ Z = (X \circ Z) \cup (Y \circ Z)$$

$$(X \cap Y) \circ Z \subseteq (X \circ Z) \cap (Y \circ Z)$$

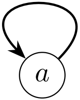
7 Properties

$$R \subseteq X \times X$$

Definition (自反的 (Reflexive))

$R \subseteq X \times X$ is *reflexive* if

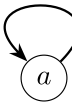
$$\forall a \in X : (a, a) \in R$$



Definition (自反的 (Reflexive))

$R \subseteq X \times X$ is *reflexive* if

$$\forall a \in X : (a, a) \in R$$

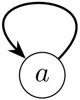


$$\leq \subseteq \mathbb{R} \times \mathbb{R} \text{ is reflexive}$$

Definition (自反的 (Reflexive))

$R \subseteq X \times X$ is *reflexive* if

$$\forall a \in X : (a, a) \in R$$



$$\leq \subseteq \mathbb{R} \times \mathbb{R} \text{ is reflexive}$$

三角形上的全等关系是自反的

Definition (反自反 (Irreflexive))

$R \subseteq X \times X$ is *irreflexive* if

$$\forall a \in X. \ (a, a) \notin R$$

Definition (反自反 (Irreflexive))

$R \subseteq X \times X$ is *irreflexive* if

$$\forall a \in X.\ (a, a) \notin R$$

$$< \subseteq \mathbb{R} \times \mathbb{R} \text{ is irreflexive}$$

Definition (反自反 (Irreflexive))

$R \subseteq X \times X$ is *irreflexive* if

$$\forall a \in X.\ (a, a) \notin R$$

$$< \subseteq \mathbb{R} \times \mathbb{R} \text{ is irreflexive}$$

$$> \subseteq \mathbb{R} \times \mathbb{R} \text{ is irreflexive}$$

$$A = \{1, 2, 3\} \qquad R \subseteq A \times A$$

$$\{(1,1), (1,3), (2,1), (2,2), (3,3)\}$$

$$A = \{1, 2, 3\} \qquad R \subseteq A \times A$$

$$\{(1,1),(1,3),(2,1),(2,2),(3,3)\}$$

$$\{(1,2),(2,3),(3,1)\}$$

$$A = \{1, 2, 3\} \qquad R \subseteq A \times A$$

$$\{(1,1),(1,3),(2,1),(2,2),(3,3)\}$$

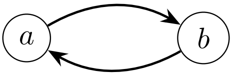
$$\{(1,2),(2,3),(3,1)\}$$

$$\{(1,2),(2,2),(2,3),(3,1)\}$$

Definition (对称 (Symmetric))

$R \subseteq X \times X$ is *symmetric* if

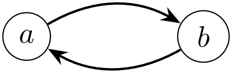
$$\forall a, b \in X.\ aRb \to bRa$$



Definition (对称 (Symmetric))

$R \subseteq X \times X$ is *symmetric* if

$$\forall a, b \in X.\ aRb \to bRa$$



$$\forall a, b \in X.\ aRb \leftrightarrow bRa$$

$$A = \{1, 2, 3\} \qquad R \subseteq A \times A$$

$$A = \{1, 2, 3\} \qquad R \subseteq A \times A$$

$$\{(1, 1), (1, 2), (1, 3), (2, 1), (3, 1), (3, 3)\}$$

$$A = \{1, 2, 3\} \qquad R \subseteq A \times A$$

$$\{(1, 1), (1, 2), (1, 3), (2, 1), (3, 1), (3, 3)\}$$

$$\{(1, 2), (2, 3), (2, 2), (3, 1)\}$$

$$A = \{1, 2, 3\} \qquad R \subseteq A \times A$$

$$\{(1,1), (1,2), (1,3), (2,1), (3,1), (3,3)\}$$

$$\{(1,2), (2,3), (2,2), (3,1)\}$$

$$\{(1,1), (2,2), (3,3)\}$$

$$A = \{1, 2, 3\} \qquad R \subseteq A \times A$$

$$\{(1,1), (1,2), (1,3), (2,1), (3,1), (3,3)\}$$

$$\{(1,2), (2,3), (2,2), (3,1)\}$$

$$\{(1,1), (2,2), (3,3)\}$$

$$\{(1,2), (2,1), (2,3)\}$$

Definition (反对称 (AntiSymmetric))

$R \subseteq X \times X$ is *antisymmetric* if

$$\forall a, b \in X. \ (aRb \wedge bRa) \rightarrow a = b$$

Definition (反对称 (AntiSymmetric))

$R \subseteq X \times X$ is *antisymmetric* if

$$\forall a, b \in X.\ (aRb \land bRa) \to a = b$$

$\geq$ *is* antisymmetric

$$A = \{1, 2, 3\} \qquad R \subseteq A \times A$$

$$A = \{1, 2, 3\} \qquad R \subseteq A \times A$$

$$\{(1,1), (1,2), (1,3), (2,1), (3,1), (3,3)\}$$

$$A = \{1,2,3\} \qquad R \subseteq A \times A$$

$$\{(1,1),(1,2),(1,3),(2,1),(3,1),(3,3)\}$$

$$\{(1,2),(2,3),(2,2),(3,1)\}$$

$$A = \{1, 2, 3\} \qquad R \subseteq A \times A$$

$$\{(1,1), (1,2), (1,3), (2,1), (3,1), (3,3)\}$$

$$\{(1,2), (2,3), (2,2), (3,1)\}$$

$$\{(1,1), (2,2), (3,3)\}$$

$$A = \{1, 2, 3\} \qquad R \subseteq A \times A$$

$$\{(1,1), (1,2), (1,3), (2,1), (3,1), (3,3)\}$$

$$\{(1,2), (2,3), (2,2), (3,1)\}$$

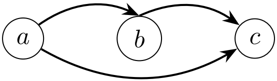
$$\{(1,1), (2,2), (3,3)\}$$

$$\{(1,2), (2,1), (2,3)\}$$

Definition (传递的 (Transitive))

$R \subseteq X \times X$ is *transitive* if

$$\forall a, b, c \in X.\ (aRb \land bRc \to aRc)$$



$$A = \{1, 2, 3\} \qquad R \subseteq A \times A$$

$$A = \{1, 2, 3\} \qquad R \subseteq A \times A$$

$$\{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$$

$$A = \{1, 2, 3\} \qquad R \subseteq A \times A$$

$$\{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$$

$$\{(1,2), (2,3), (3,1)\}$$

$$A = \{1, 2, 3\} \qquad R \subseteq A \times A$$

$$\{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$$

$$\{(1,2), (2,3), (3,1)\}$$

$$\{(1,3)\}$$

$$A = \{1,2,3\} \qquad R \subseteq A \times A$$

$$\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\}$$

$$\{(1,2),(2,3),(3,1)\}$$

$$\{(1,3)\}$$

$$\emptyset$$

Definition (连通的 (Connex))

$R \subseteq X \times X$ is *connex* if

$$\forall a, b \in X. \, (aRb \lor bRa)$$

Definition (连通的 (Connex))

$R \subseteq X \times X$ is *connex* if

$$\forall a, b \in X.\ (aRb \lor bRa)$$

Definition (三分的 (Trichotomous))

$R \subseteq X \times X$ is *trichotomous* if

$$\forall a, b \in X.\ (\text{exactly one of } aRb, bRa, \text{ or } a = b \text{ holds})$$

Theorem

$$R \textit{ is reflexive} \iff I \subseteq R$$

$$I = \{(a, a) \in A \times A \mid a \in A\}$$

Theorem

$$R \textit{ is reflexive} \iff I \subseteq R$$

$$I = \{(a, a) \in A \times A \mid a \in A\}$$

Theorem

$$R \textit{ is symmetric} \iff R^{-1} = R$$

Theorem

$$R \text{ is transitive} \iff R \circ R \subseteq R$$

Theorem

$$R \text{ is transitive } \iff R \circ R \subseteq R$$

$$R = \{(1,2),(2,3),(1,3),(4,4)\}$$

Theorem

$$R \textit{ is symmetric and transitive } \iff R = R^{-1} \circ R$$

Equivalence Relations

Definition (Equivalence Relation)

R is an *equivalence relation* on X iff R is

- reflexive
- symmetric
- transitive

Definition (Equivalence Relation)

R is an *equivalence relation* on X iff R is

- reflexive
- symmetric
- transitive

$$= \in \mathbb{R} \times \mathbb{R}$$

Definition (Equivalence Relation)

R is an *equivalence relation* on X iff R is

- reflexive
- symmetric
- transitive

$$= \in \mathbb{R} \times \mathbb{R}$$

$$\parallel \in \mathbb{L} \times \mathbb{L}$$

Definition (Equivalence Relation)

R is an *equivalence relation* on X iff R is

- reflexive
- symmetric
- transitive

$$= \in \mathbb{R} \times \mathbb{R}$$

$$\parallel \in \mathbb{L} \times \mathbb{L}$$

$$a \sim b \iff a \% 12 = b \% 12$$

Definition (Equivalence Relation)

R is an *equivalence relation* on X iff R is

- reflexive
- symmetric
- transitive

$$= \in \mathbb{R} \times \mathbb{R}$$

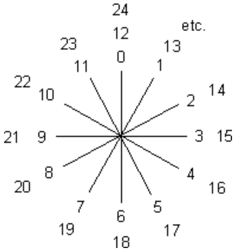
$$\parallel \in \mathbb{L} \times \mathbb{L}$$

$$a \sim b \iff a \,\%\, 12 = b \,\%\, 12$$

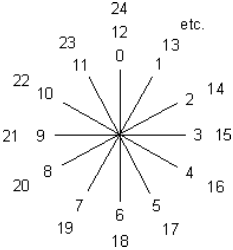
Why are equivalence relations important?

Equivalence Relations as Abstractions

Equivalence Relations as Abstractions

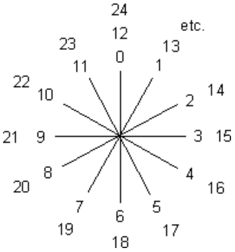


Equivalence Relations as Abstractions



“全国人民代表大会各省代表团”

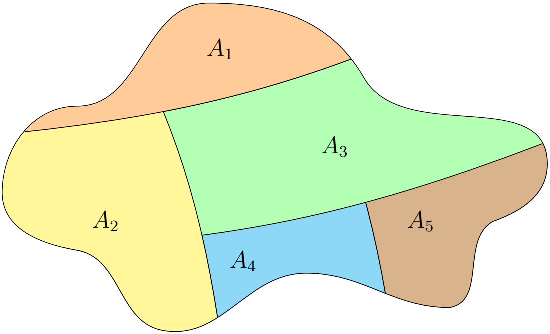
Equivalence Relations as Abstractions



“全国人民代表大会各省代表团”

Equivalence Relation $\Longleftrightarrow$ Partition

Partition



“不空、不漏、不重”

Definition (Partition)

A family of sets $\{A_\alpha : \alpha \in I\}$ is a *partition* of X if

(i)

$$\forall \alpha \in I : A_\alpha \neq \emptyset$$

(ii)

$$\bigcup_{\alpha \in I} A_\alpha = X$$

(iii)

$$\forall \alpha, \beta \in I : A_\alpha \cap A_\beta = \emptyset \lor A_\alpha = A_\beta$$

Definition (Partition)

A family of sets $\{A_\alpha : \alpha \in I\}$ is a *partition* of X if

(i)

$$\forall \alpha \in I : A_\alpha \neq \emptyset$$

$$(\forall \alpha \in I\ \exists x \in X : x \in A_\alpha)$$

(ii)

$$\bigcup_{\alpha \in I} A_\alpha = X$$

(iii)

$$\forall \alpha, \beta \in I : A_\alpha \cap A_\beta = \emptyset \lor A_\alpha = A_\beta$$

Definition (Partition)

A family of sets $\{A_\alpha : \alpha \in I\}$ is a *partition* of X if

(i)

$$\forall \alpha \in I : A_\alpha \neq \emptyset$$

$$(\forall \alpha \in I \; \exists x \in X : x \in A_\alpha)$$

(ii)

$$\bigcup_{\alpha \in I} A_\alpha = X$$

$$(\forall x \in X \; \exists \alpha \in I : x \in A_\alpha)$$

(iii)

$$\forall \alpha, \beta \in I : A_\alpha \cap A_\beta = \emptyset \lor A_\alpha = A_\beta$$

Definition (Partition)

A family of sets $\{A_\alpha : \alpha \in I\}$ is a *partition* of X if

(i)

$$\forall \alpha \in I : A_\alpha \neq \emptyset$$

$$(\forall \alpha \in I\ \exists x \in X : x \in A_\alpha)$$

(ii)

$$\bigcup_{\alpha \in I} A_\alpha = X$$

$$(\forall x \in X\ \exists \alpha \in I : x \in A_\alpha)$$

(iii)

$$\forall \alpha, \beta \in I : A_\alpha \cap A_\beta = \emptyset \lor A_\alpha = A_\beta$$

$$(\forall \alpha, \beta \in I : A_\alpha \cap A_\beta \neq \emptyset \implies A_\alpha = A_\beta)$$

Equivalence Relation $R \subseteq X \times X \implies$ Partition Π of X

Equivalence Relation $R \subseteq X \times X \implies$ Partition Π of X

Definition (Equivalence Class)

The *equivalence class* of a *modulo* R is a set:

$$[a]_R = \{b \in X : aRb\}$$

Equivalence Relation $R \subseteq X \times X \implies$ Partition Π of X

Definition (Equivalence Class)

The *equivalence class* of *a modulo R* is a set:

$$[a]_R = \{b \in X : aRb\}$$

Definition (Quotient Set)

The *quotient set* is a set:

$$X/R = \{[a]_R \mid a \in X\}$$

Theorem

$$X/R = \{[a]_R \mid a \in X\} \text{ is a partition of } X.$$

Theorem

$$X/R = \{[a]_R \mid a \in X\} \textit{ is a partition of } X.$$

$$\forall a \in X : [a]_R \neq \emptyset$$

Theorem

$$X/R = \{[a]_R \mid a \in X\} \textit{ is a partition of } X.$$

$$\forall a \in X : [a]_R \neq \emptyset$$

$$\forall a \in X : \exists b \in X : a \in [b]_R$$

Theorem

$$X/R = \{[a]_R \mid a \in X\} \textit{ is a partition of } X.$$

$$\forall a \in X : [a]_R \neq \emptyset$$

$$\forall a \in X : \exists b \in X : a \in [b]_R$$

Theorem

$$\forall a \in X, b \in X : [a]_R \cap [b]_R = \emptyset \vee [a]_R = [b]_R$$

Theorem

$$X/R = \{[a]_R \mid a \in X\} \textit{ is a partition of } X.$$

$$\forall a \in X : [a]_R \neq \emptyset$$

$$\forall a \in X : \exists b \in X : a \in [b]_R$$

Theorem

$$\forall a \in X, b \in X : [a]_R \cap [b]_R = \emptyset \lor [a]_R = [b]_R$$

$$\forall a \in X, b \in X : [a]_R \cap [b]_R \neq \emptyset \implies [a]_R = [b]_R$$

Partition Π of $X \implies$ Equivalence Relation $R \subseteq X \times X$

Partition Π of $X \implies$ Equivalence Relation $R \subseteq X \times X$

Definition

$$(a, b) \in R \iff \exists S \in \Pi : a \in S \land b \in S$$

Partition Π of $X \implies$ Equivalence Relation $R \subseteq X \times X$

Definition

$$(a, b) \in R \iff \exists S \in \Pi : a \in S \land b \in S$$

$$R = \{(a, b) \in X \times X \mid \exists S \in \Pi : a \in S \land b \in S\}$$

Partition Π of $X \implies$ Equivalence Relation $R \subseteq X \times X$

Definition

$$(a, b) \in R \iff \exists S \in \Pi : a \in S \land b \in S$$

$$R = \{(a, b) \in X \times X \mid \exists S \in \Pi : a \in S \land b \in S\}$$

Theorem

R is an equivalence relation on X.

Partition Π of $X \implies$ Equivalence Relation $R \subseteq X \times X$

Definition

$$(a, b) \in R \iff \exists S \in \Pi : a \in S \land b \in S$$

$$R = \{(a, b) \in X \times X \mid \exists S \in \Pi : a \in S \land b \in S\}$$

Theorem

R is an equivalence relation on X.

$$\forall x \in X : xRx$$

Partition Π of $X \implies$ Equivalence Relation $R \subseteq X \times X$

Definition

$$(a, b) \in R \iff \exists S \in \Pi : a \in S \land b \in S$$

$$R = \{(a, b) \in X \times X \mid \exists S \in \Pi : a \in S \land b \in S\}$$

Theorem

R is an equivalence relation on X.

$$\forall x \in X : xRx$$

$$\forall x, y \in X : xRy \implies yRx$$

Partition Π of $X \implies$ Equivalence Relation $R \subseteq X \times X$

Definition

$$(a, b) \in R \iff \exists S \in \Pi : a \in S \land b \in S$$

$$R = \{(a, b) \in X \times X \mid \exists S \in \Pi : a \in S \land b \in S\}$$

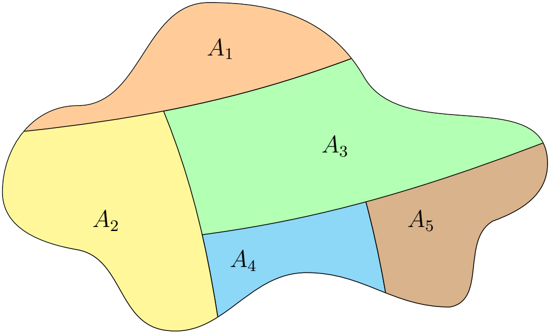
Theorem

R is an equivalence relation on X.

$$\forall x \in X : xRx$$

$$\forall x, y \in X : xRy \implies yRx$$

$$\forall x, y, z \in X : xRy \land yRz \implies xRz$$



Equivalence Relation $\iff$ Partition

Definition

$$\sim \subseteq \mathbb{N}^2 \times \mathbb{N}^2$$

$$(a, b) \sim (c, d) \iff a +_{\mathbb{N}} d = b +_{\mathbb{N}} c$$

Definition

$$\sim \subseteq \mathbb{N}^2 \times \mathbb{N}^2$$

$$(a, b) \sim (c, d) \iff a +_{\mathbb{N}} d = b +_{\mathbb{N}} c$$

Theorem

$\sim$ *is an equivalence relation.*

Definition

$$\sim \subseteq \mathbb{N}^2 \times \mathbb{N}^2$$

$$(a, b) \sim (c, d) \iff a +_{\mathbb{N}} d = b +_{\mathbb{N}} c$$

Theorem

$\sim$ *is an equivalence relation.*

Q : What is $\mathbb{N} \times \mathbb{N} / \sim$?

Definition

$$\sim \subseteq \mathbb{N}^2 \times \mathbb{N}^2$$

$$(a, b) \sim (c, d) \iff a +_{\mathbb{N}} d = b +_{\mathbb{N}} c$$

Theorem

$\sim$ *is an equivalence relation.*

Q : What is $\mathbb{N} \times \mathbb{N} / \sim$?

Definition ($\mathbb{Z}$)

$$\mathbb{Z} \triangleq \mathbb{N} \times \mathbb{N} / \sim$$

Definition

$$\sim \subseteq \mathbb{N}^2 \times \mathbb{N}^2$$

$$(a, b) \sim (c, d) \iff a +_{\mathbb{N}} d = b +_{\mathbb{N}} c$$

Theorem

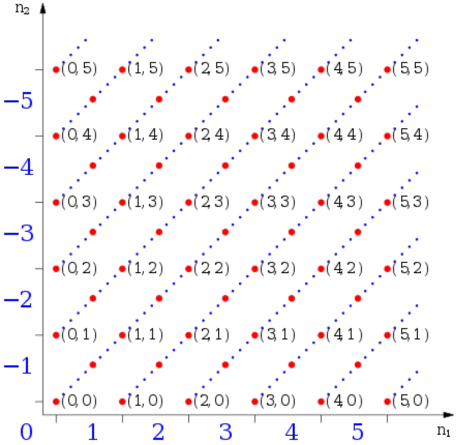
$\sim$ is an equivalence relation.

Q : What is $\mathbb{N} \times \mathbb{N} / \sim$?

Definition ($\mathbb{Z}$)

$$\mathbb{Z} \triangleq \mathbb{N} \times \mathbb{N} / \sim$$

$$[(1, 3)]_{\sim} = \{(0, 2), (1, 3), (2, 4), (3, 5), \cdots\} \triangleq -2 \in \mathbb{Z}$$



$$\mathbb{Z} \triangleq \mathbb{N} \times \mathbb{N} / \sim$$

Definition ($+_{\mathbb{Z}}$)

$$[(m_1, n_1)] +_{\mathbb{Z}} [(m_2, n_2)] = [m_1 +_{\mathbb{N}} m2, n_1 +_{\mathbb{N}} n_2]$$

Definition ($+_{\mathbb{Z}}$)

$$[(m_1, n_1)] +_{\mathbb{Z}} [(m_2, n_2)] = [m_1 +_{\mathbb{N}} m2, n_1 +_{\mathbb{N}} n_2]$$

Definition ($\cdot_{\mathbb{Z}}$)

$$\begin{aligned}
&[(m_1, n_1)] \cdot_{\mathbb{Z}} [(m_2, n_2)] \\
= &[m_1 \cdot_{\mathbb{N}} m_2 +_{\mathbb{N}} n_1 \cdot_{\mathbb{N}} n_2, m_1 \cdot_{\mathbb{N}} n_2 +_{\mathbb{N}} n_1 \cdot_{\mathbb{N}} m_2]
\end{aligned}$$

Definition

$$\sim \subseteq (\mathbb{Z} \times \mathbb{Z} \setminus \{0_{\mathbb{Z}}\})^2$$

$$(a, b) \sim (c, d) \iff a \cdot_{\mathbb{Z}} d = b \cdot_{\mathbb{Z}} c$$

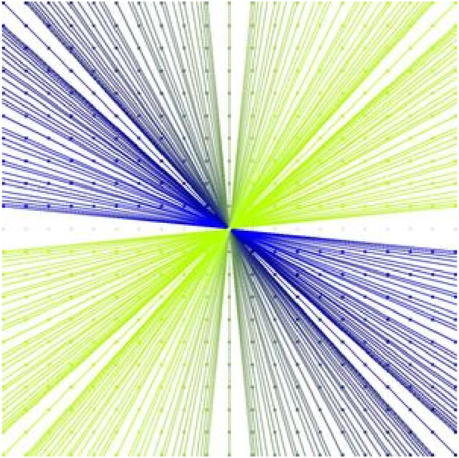
Definition

$$\sim \subseteq (\mathbb{Z} \times \mathbb{Z} \setminus \{0_{\mathbb{Z}}\})^2$$

$$(a, b) \sim (c, d) \iff a \cdot_{\mathbb{Z}} d = b \cdot_{\mathbb{Z}} c$$

Definition ($\mathbb{Q}$)

$$\mathbb{Q} \triangleq \mathbb{Z} \times \mathbb{Z} / \sim$$

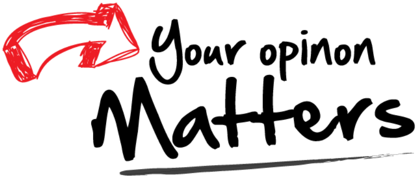


$$\mathbb{Q} \triangleq \mathbb{Z} \times \mathbb{Z} / \sim$$

How to define $\mathbb{R}$ as equivalence classes of ordered pairs of $\mathbb{Q}$?

How to define $\mathbb{R}$ as equivalence classes of ordered pairs of $\mathbb{Q}$?


Thank You!



Office 926

hfwei@nju.edu.cn