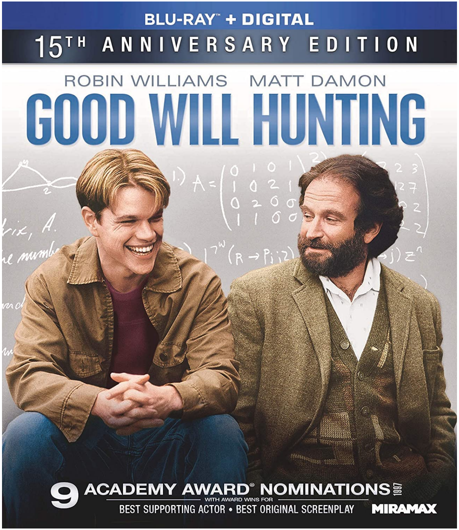
(十) 图论: 树 (Trees)

魏恒峰

hfwei@nju.edu.cn

2021 年 05 月 13 日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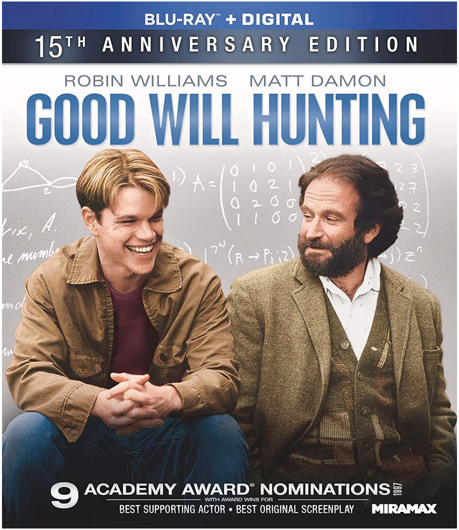
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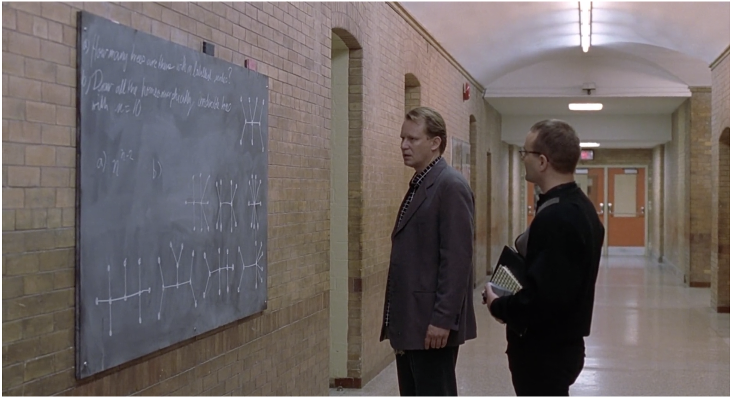
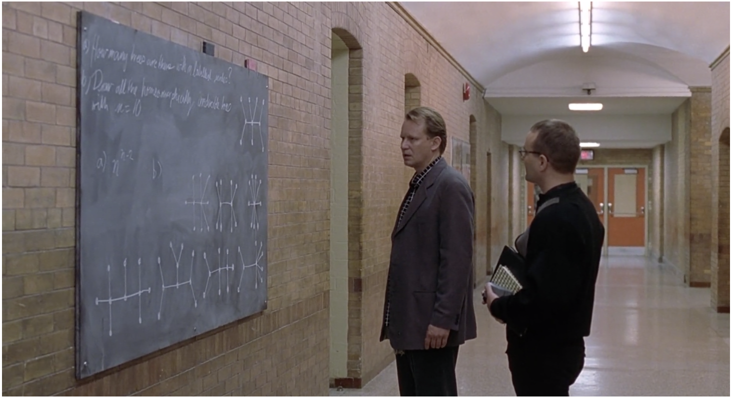
你, 真得, 看懂了吗?





(1) How many trees are there with n labeled vertices?

(2) Draw all homomorphically irreducible trees with $n = 10$.

Definition (Tree (树))

A tree is a connected acyclic undirected graph.

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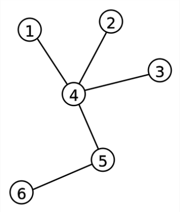
Definition (Forest (森林))

A forest is a acyclic undirected graph.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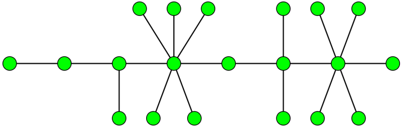

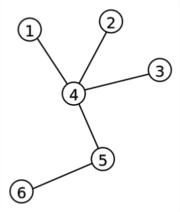
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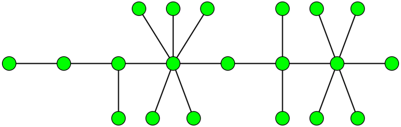

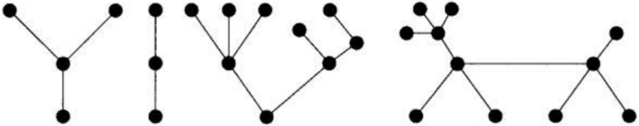

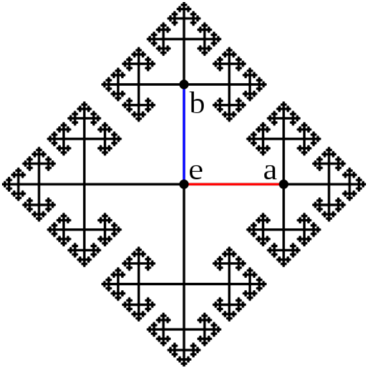

Cayley Graph (4-regular tree)

Definition (Internal Vertex (内部顶点); Leaf (叶子))

In a tree T with ≥ 2 vertices, for a vertex v in T, if

$$\deg(v) = 1$$

then v is called a leaf; otherwise, v is an internal vertex.

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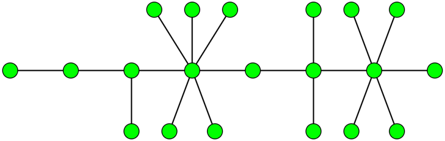

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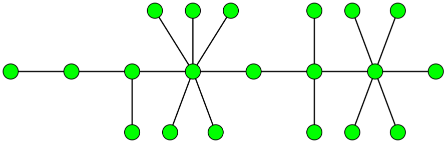

Lemma

Any tree T with ≥ 2 vertices contains ≥ 1 leaf.

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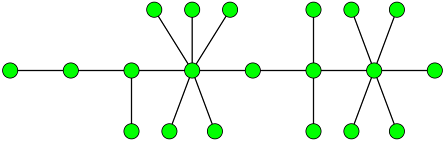

Lemma

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$$\text{Otherwise, } \forall v \in V.\ \deg(v) \geq 2 \implies T \text{ has cycles.}$$

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Consider the two endpoints of any maximal path in T.

They are leaves of T.

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Deleting a leaf from a tree T with n vertices produces a tree with $n-1$ vertices.

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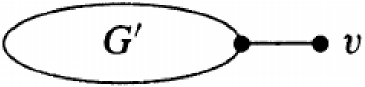

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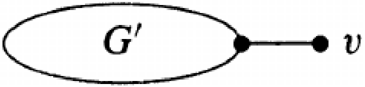

$G' = G - v$ is connected and acyclic.

A leaf does *not* belong to any paths connecting two other vertices.

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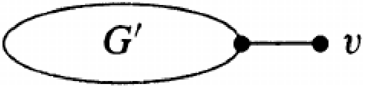

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A leaf does *not* belong to any paths connecting two other vertices.

This lemma can be used in induction for trees!

Theorem ((We call it) Characterization of Trees)

Let T be an undirected graph with n vertices.

Then the following statements are equivalent:

(1) *T is a tree;*

(2) *T is acyclic, and has $m = n - 1$ edges;*

(3) *T is connected, and has $m = n - 1$ edges;*

(4) *T is connected, and each edge is a bridge;*

(5) *Any two vertices of T are connected by exactly one path;*

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$$(1) \implies (2) \implies (3) \implies (4) \implies (5) \implies (6) \implies (1)$$

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For $T' = T - v$, $m(T') = (n - 1) - 1 = n - 2$.

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$$m(T) = (n - 2) + 1 = n - 1.$$

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$$m(T) = \sum_{i=1}^{k} m(T_i) = n - k \neq n - 1.$$

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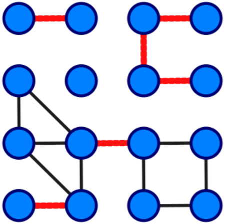

Definition (Bridge (桥))

A bridge of a graph G is an edge e such that

$$c(G-e) > c(G).$$

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(4) *T is connected, and each edge is a bridge.*

Consider any edge e of T.

$$m(T-e) = (n-1) - 1 = n-2.$$

$T - e$ must be disconnected.

Theorem (Characterization of Trees)

(4) *T is connected, and each edge is a bridge;*

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If u and v are connected by two paths,

the edges on these two paths are not bridges.

Theorem (Characterization of Trees)

(5) *Any two vertices of T are connected by exactly one path;*

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If T has a cycle C,

any two vertices in C is connected by ≥ 2 paths.

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It creates a cycle, consisting of $\{u, v\}$ and the path from u to v.

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Lemma

If two distinct cycles of a graph G share a common edge e,

then G has a cycle that does not contain e.

Theorem (Characterization of Trees)

(6) *T is acyclic, but the addition of any edge creates exactly one cycle;*

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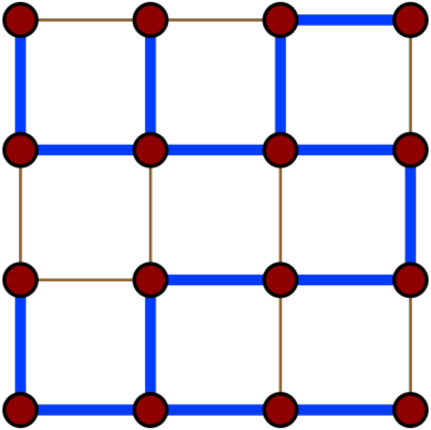
Choose $u \in V(T_1)$, $v \in V(T_2)$.

$T + \{u, v\}$ does not create cycles.



Spanning Trees (trees in graphs)

Definition (Subgraph (子图))

A graph S is a subgraph of G if

$$V(S) \subseteq V(G) \land E(S) \subseteq E(G) \land \bigcup E(S) \subseteq V(S)$$

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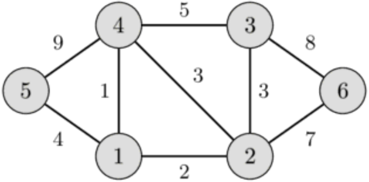

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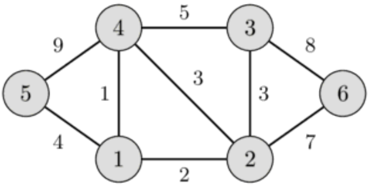

Definition (Induced Subgraph (诱导子图))

A graph S is an induced subgraph of G if S is a subgraph of G such that

$$\{\{u, v\} \in E(G) \mid u \in V(S), v \in V(S)\} \subseteq E(S).$$

Definition (Spanning Tree (生成树))

A spanning tree T of an undirected graph G is a subgraph that is a tree with all vertices of G.

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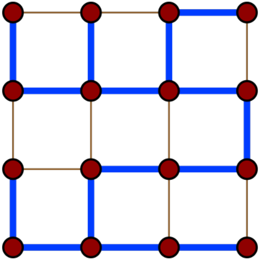

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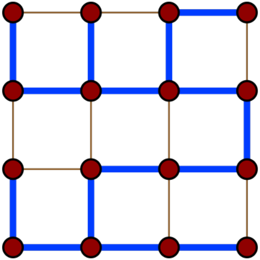

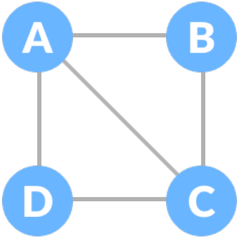

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Every connected undirected graph G admits a spanning tree.

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Repeatedly deleting edges in cycles until the graph is acyclic.

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Every connected undirected graph G admits a spanning tree.

Repeatedly deleting edges in cycles until the graph is acyclic.

The remaining graph is a spanning tree of G.

Definition (Minimum Spanning Tree (MST; 最小生成树))

A minimum spanning tree T of an edge-weighted undirected graph G is a spanning tree with minimum total weight of edges.

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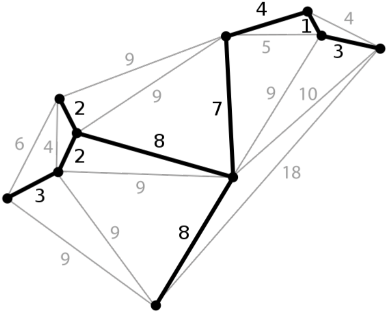

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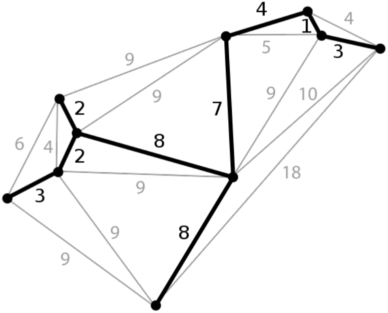

Existence? Uniqueness? Algorithms?

Theorem

Every weighted connected undirected graph G admits a minimum spanning tree.

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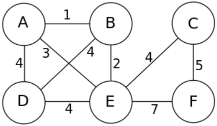

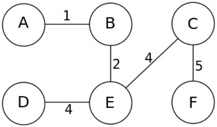

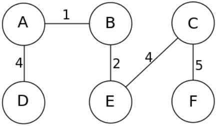

Joseph Kruskal (1928 ~ 2010)

Repeatedly adding the next lowest-weight edge
that will not form a cycle until $n-1$ edges are added.

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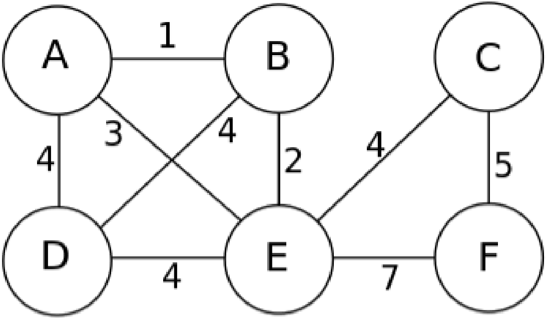


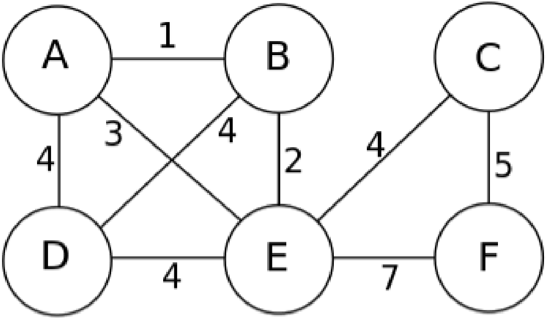
Robert C. Prim (1921 ∼)

Repeatedly adding the cheapest possible edge

from the partially built tree to another vertex,

until $n-1$ edges are added.

Repeatedly adding the cheapest possible edge from the partially built tree to another vertex, until $n-1$ edges are added.



Theorem (Cut Property)

$$X: \textit{A part of some MST } T \textit{ of } G$$

$$(S, V \setminus S): \textit{A cut such that } X \textit{ does not cross } (S, V \setminus S)$$

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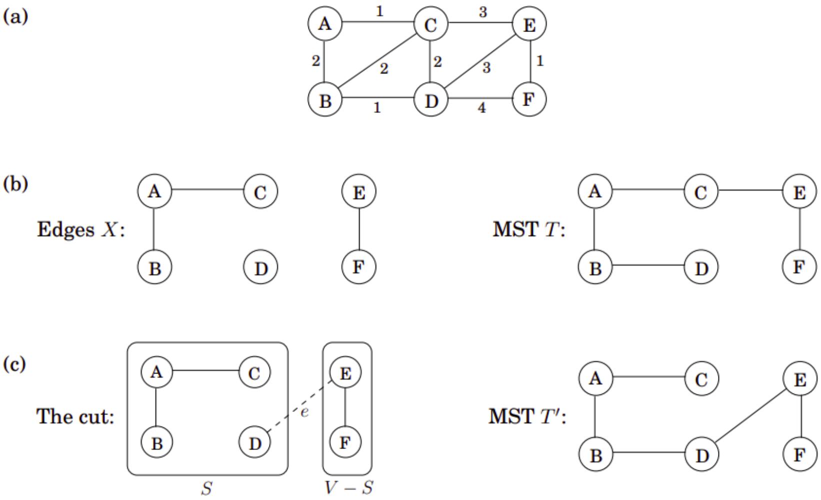
Then $X \cup \{e\}$ *is a part of some MST* T' *of* G.



(a)
(b)
Edges X:
MST T:
(c)
The cut:
e
S
V − S
MST T′:

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$$
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Correctness of Kruskal's and Prim's algorithms.

By Exchange Argument.

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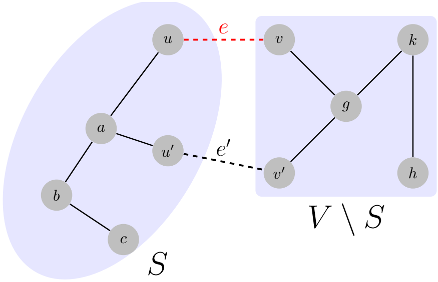

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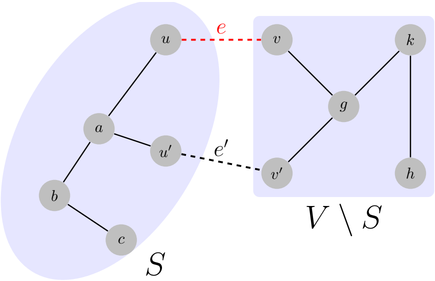

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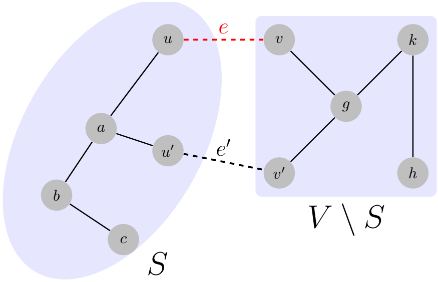

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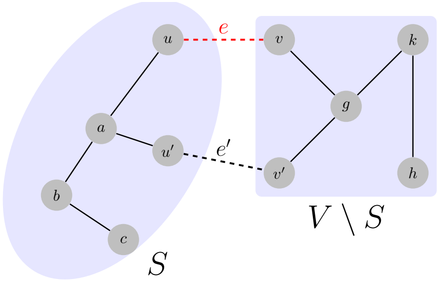

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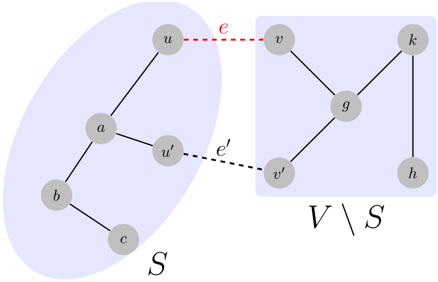

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$w(e) \le w(e') \implies T'$ is also an MST

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“a” $\to$ “the” $\Longrightarrow$ “some” $\to$ “all”

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- *Let C be any cycle in connected undirected G*
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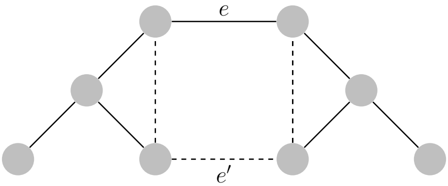

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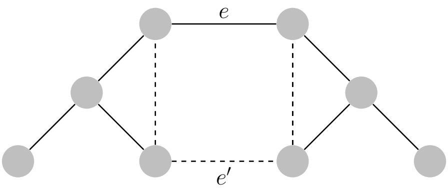

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Let G be an edge-weighted undirected graph.
If each edge has a distinct weight, then there is a unique MST of G.

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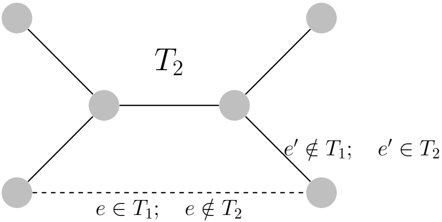
$$\text{Suppose that } e \in T_1 \setminus T_2 \ (w.l.o.g)$$


T_2
$e' \notin T_1; \quad e' \in T_2$
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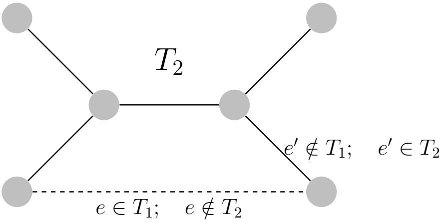
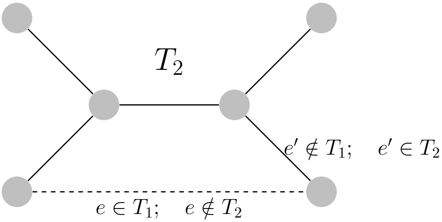


$$T_2 + \{e\} \implies C$$



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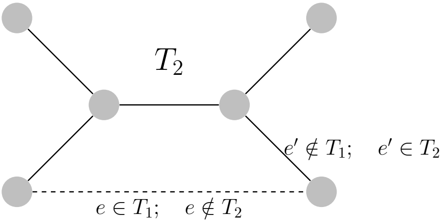

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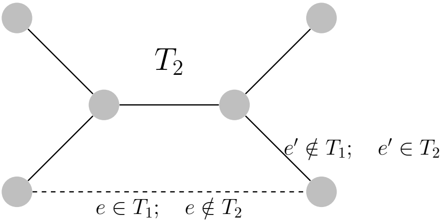

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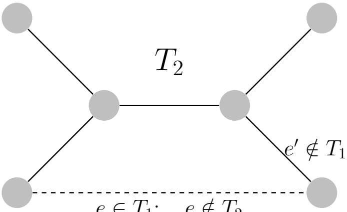

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$$T' = T_2 + \{e\} - \{e'\} \implies w(T') < w(T_2)$$

Condition for Uniqueness of MST

$$\text{Unique MST} \;\not\Longrightarrow\; \text{Distinct weights}$$

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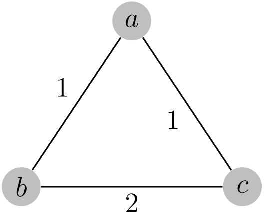

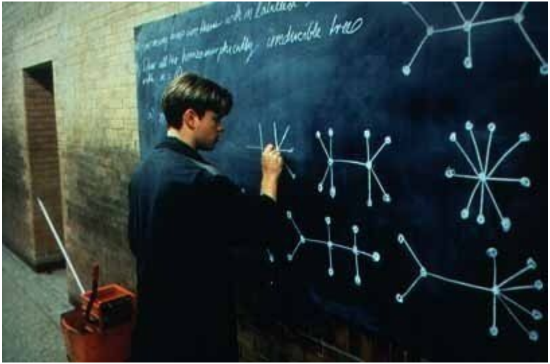

Theorem (Cayley's Formula)

The number T_n of labeled trees on $n \geq 2$ vertices is n^{n-2}.

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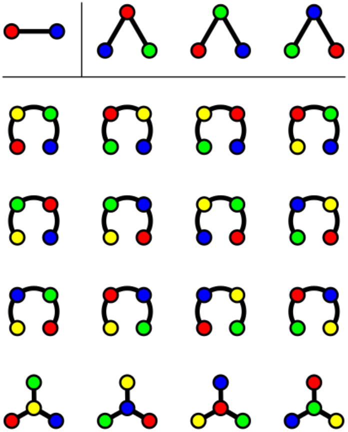
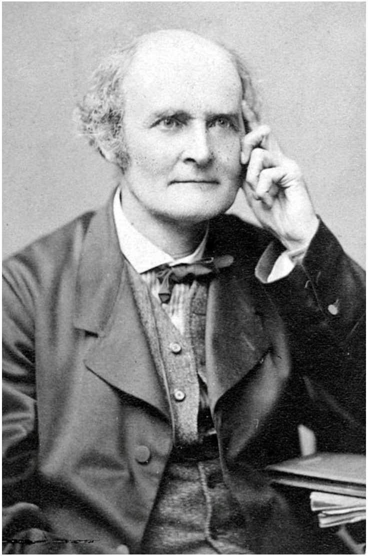
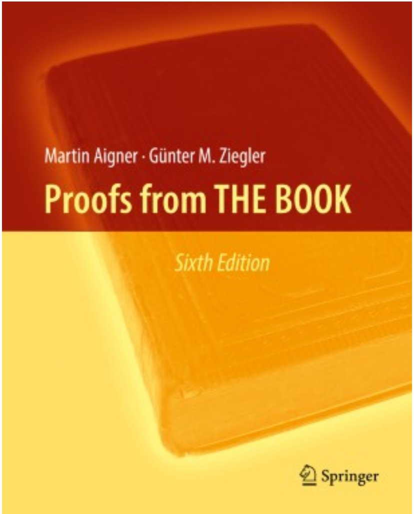




Arthur Cayley $(1821 \sim 1895)$



Chapter 33: Cayley's formula for the number of trees

By Double Counting.

— Jim Pitman

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https://en.wikipedia.org/wiki/Double_counting_(proof_technique)#Counting_trees

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How many ways are there of forming a directed rooted tree from an empty graph by adding directed edges one by one?

Definition (Rooted Tree (有根树))

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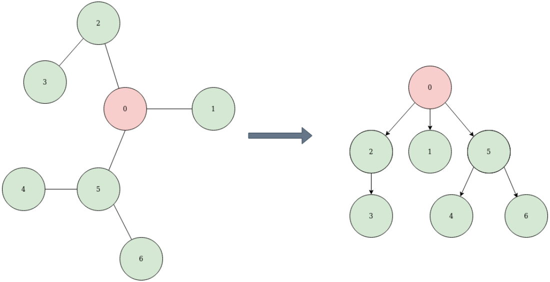

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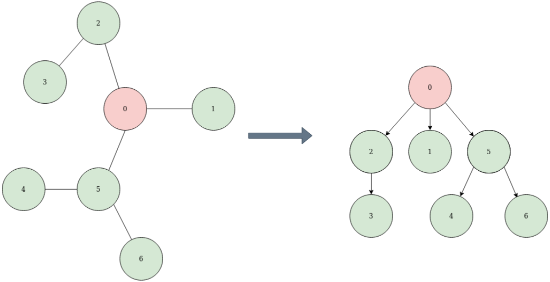

Definition (Directed Rooted Tree (有向有根树))

A directed rooted tree is a rooted tree where all edges directed away from or towards the root.

Choose one of the T_n labeled trees on n vertices.

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Choose one of its n vertices as root.

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Choose one of the $(n-1)!$ possible sequences in which to add its $n-1$ directed edges.

$$T_n n(n-1)! = T_n n!$$

Suppose that we have added $n - k$ directed edges.

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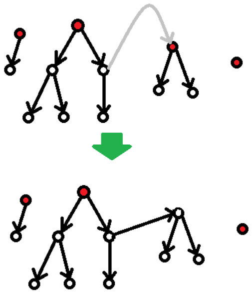

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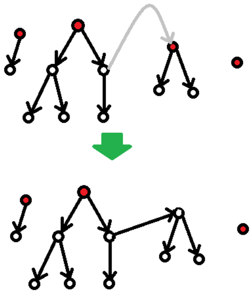

$$\prod_{k=2}^{n} n(k-1) = n^{n-1}(n-1)! = n^{n-2}n!$$

$$T_n n! = n^{n-2} n!$$

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$$\boxed{T_n = n^{n-2}}$$

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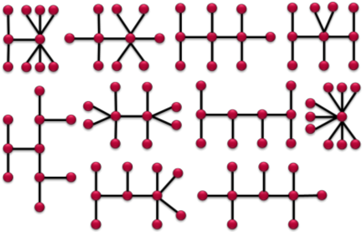

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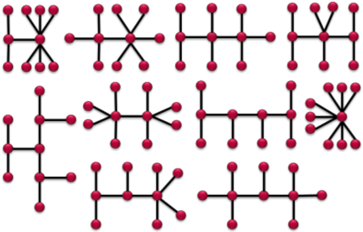

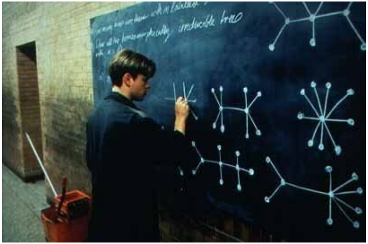

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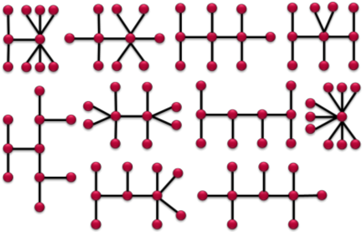
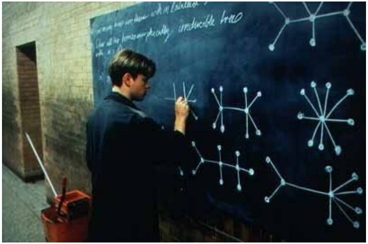
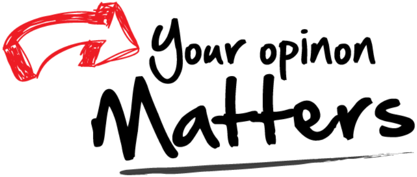




Homeomorphically Irreducible Trees of size $n = 10$


Thank
You!



Office 302

Mailbox: H016

hfwei@nju.edu.cn