(十四) 群论：子群 (Subgroup)

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Definition (Subgroup (子群))

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If $H \subset G$, then H is a proper subgroup (真子群).

$$(H = \{mz \mid z \in \mathbb{Z}\}, +) \leq (\mathbb{Z}, +)$$

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$$H = \{1, 2, 4\} \leq G = \mathbb{Z}_7^* = \{1, 2, 3, 4, 5, 6\}$$

Theorem

Suppose that $H \le G$.

(1) *The identity of* H *is the same with that of* G.

$$e_H = e_G$$

(2) *The inversion of* a *in* H *is the same with that in* G.

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$$a a_H^{-1} = e_H = e_G = a a_G^{-1} \implies a_H^{-1} = a^{-1}(G)$$

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$$H_1 \cup H_2 ?$$

Definition (Symmetric Group (对称群; Sym(M)))

Let $M \neq \emptyset$ be a set.
All the permutations/bijective functions of M, together with the composition operation, is a group, called the symmetric group of M.

$$M = \{1, 2, \ldots, n\}$$

$$S_n \triangleq \mathrm{Sym}(M)$$

$$S_3$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

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$$\sigma\tau \neq \tau\sigma$$

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$$(1) \quad (1\ 2) \quad (1\ 3) \quad (2\ 3) \quad (1\ 2\ 3) \quad (1\ 3\ 2)$$

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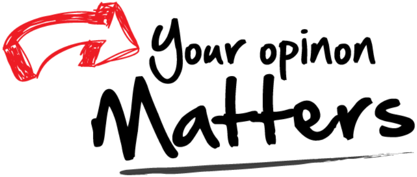
$$H = \{(1), (1\ 2)\} \leq S_3$$


Thank You!



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