

(十五) 离散数学: 复习 (Review)

魏恒峰

hfwei@nju.edu.cn

2021 年 06 月 17 日



\vdash \models

Theorem

$$\Sigma \vdash \alpha \iff \Sigma \models \alpha$$



\rightarrow \Rightarrow

\leftrightarrow \Leftrightarrow

“ \rightarrow ” and “ \leftrightarrow ” are used in a **single** formula.

“ \Rightarrow ” and “ \Leftrightarrow ” are used to connect **two** formulas.

\rightarrow \Rightarrow

\leftrightarrow \Longleftrightarrow

“ \rightarrow ” and “ \leftrightarrow ” are used in a **single** formula.

“ \Rightarrow ” and “ \Longleftrightarrow ” are used to connect **two** formulas.

$$x \in A \setminus B$$

$$\Longleftrightarrow x \in A \wedge x \notin B$$

$$\Longleftrightarrow x \in A \wedge (x \in U \wedge x \notin B)$$

$$\Longleftrightarrow x \in A \wedge x \in \overline{B}$$

$$\Longleftrightarrow x \in A \cap \overline{B}$$

$$\begin{aligned} p \oplus q &\triangleq (p \vee q) \wedge \neg(p \wedge q) \\ &= (p \wedge \neg q) \vee (\neg q \wedge q) \end{aligned}$$

p	q	$p \oplus q$
0	0	0
0	1	1
1	0	1
1	1	0

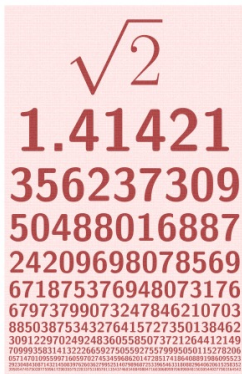
$$\begin{aligned}
 p \oplus q &\triangleq (p \vee q) \wedge \neg(p \wedge q) \\
 &= (p \wedge \neg q) \vee (\neg q \wedge q)
 \end{aligned}$$

p	q	$p \oplus q$
0	0	0
0	1	1
1	0	1
1	1	0

$$p \oplus q = q \oplus p$$

$$(p \oplus q) \oplus r = p \oplus (q \oplus r)$$

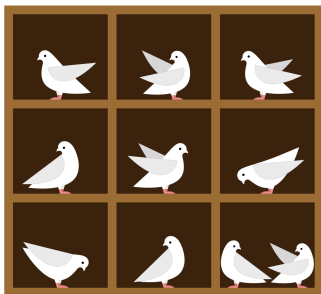
$\sqrt{2}$ is irrational.



The First Crisis in Mathematics

Theorem (Bézout's Identity)

$$(a, b) = d \implies \exists u, v \in \mathbb{Z}. au + bv = d$$



Theorem (Pigeonhole Principle)

If n **objects** are placed in r **boxes**, where $r < n$, then at least one of the boxes contains ≥ 2 ($\geq \lceil \frac{n}{r} \rceil$) object.

Numbers

Consider the numbers $1, 2, \dots, 2n$, and take any $n + 1$ of them.

There are two among these $n + 1$ numbers which are **relatively prime**.

Numbers

Consider the numbers $1, 2, \dots, 2n$, and take any $n + 1$ of them.

There are two among these $n + 1$ numbers which are **relatively prime**.

There must be two numbers which are **only 1 apart**.

Numbers

Consider the numbers $1, 2, \dots, 2n$, and take any $n + 1$ of them. There are two among these $n + 1$ numbers such as one **divides** the other.

Numbers

Consider the numbers $1, 2, \dots, 2n$, and take any $n + 1$ of them. There are two among these $n + 1$ numbers such as one **divides** the other.

$$a = 2^k m, \quad (1 \leq m \leq 2n - 1 \text{ is odd})$$

There $n + 1$ numbers have only n different odd parts.

There must be two numbers **with the same odd part**.

Hand-shaking

If there are $n > 1$ people who can shake hands with one another, there are two people who shake hands with the same number of people.

Hand-shaking

If there are $n > 1$ people who can shake hands with one another, there are two people who shake hands with the same number of people.

$$0 \sim n - 1$$

Hand-shaking

If there are $n > 1$ people who can shake hands with one another, there are two people who shake hands with the same number of people.

$$0 \sim n - 1$$

Either the '0' hole or the ' $n - 1$ ' hole or both must be empty.

Sums

Suppose we are given n integers a_1, a_2, \dots, a_n .

Then there is a set of **consecutive numbers** $a_{k+1}, a_{k+2}, \dots, a_l$ whose sum $\sum_{i=k+1}^l a_i$ is a multiple of n .

Sums

Suppose we are given n integers a_1, a_2, \dots, a_n .

Then there is a set of **consecutive numbers** $a_{k+1}, a_{k+2}, \dots, a_l$ whose sum $\sum_{i=k+1}^l a_i$ is a multiple of n .

$$A_i = \sum_{k=1}^{k=i} a_i$$

Sums

Suppose we are given n integers a_1, a_2, \dots, a_n .

Then there is a set of **consecutive numbers** $a_{k+1}, a_{k+2}, \dots, a_l$ whose sum $\sum_{i=k+1}^l a_i$ is a multiple of n .

$$A_i = \sum_{k=1}^{k=i} a_i$$

$$A_0, A_1, A_2, \dots, A_n$$

Sums

Suppose we are given n integers a_1, a_2, \dots, a_n .

Then there is a set of **consecutive numbers** $a_{k+1}, a_{k+2}, \dots, a_l$ whose sum $\sum_{i=k+1}^l a_i$ is a multiple of n .

$$A_i = \sum_{k=1}^{k=i} a_i$$

$$A_0, A_1, A_2, \dots, A_n$$

$$\exists 0 \leq i < j \leq n. A_i = A_j \pmod n$$

Sums

Suppose we are given n integers a_1, a_2, \dots, a_n .

Then there is a set of **consecutive numbers** $a_{k+1}, a_{k+2}, \dots, a_l$ whose sum $\sum_{i=k+1}^l a_i$ is a multiple of n .

$$A_i = \sum_{k=1}^{k=i} a_i$$

$$A_0, A_1, A_2, \dots, A_n$$

$$\exists 0 \leq i < j \leq n. A_i = A_j \pmod n$$

$$A_j - A_i = a_{i+1} + \dots + a_j = 0 \pmod n$$

Championship Match

“胡司令” (胡荣华) 要安排一次长达 77 天的象棋练习赛。

他想每天至少要有一场比赛, 但是总共不超过 132 场比赛。

请证明, 无论如何安排, 他都要在连续的若干天内恰好完成 21 场比赛。

Championship Match

“胡司令” (胡荣华) 要安排一次长达 77 天的象棋练习赛。

他想每天至少要有一场比赛, 但是总共不超过 132 场比赛。

请证明, 无论如何安排, 他都要在连续的若干天内恰好完成 21 场比赛。

Let a_i denote the number of games he plays up through the i -th day.

Championship Match

“胡司令” (胡荣华) 要安排一次长达 77 天的象棋练习赛。

他想每天至少要有一场比赛, 但是总共不超过 132 场比赛。

请证明, 无论如何安排, 他都要在连续的若干天内恰好完成 21 场比赛。

Let a_i denote the number of games he plays up through the i -th day.

$$a_1, a_2, \dots, a_{76}, a_{77}, a_1 + 21, a_2 + 21, \dots, a_{76} + 21, a_{77} + 21$$

Championship Match

“胡司令” (胡荣华) 要安排一次长达 77 天的象棋练习赛。

他想每天至少要有一场比赛, 但是总共不超过 132 场比赛。

请证明, 无论如何安排, 他都要在连续的若干天内恰好完成 21 场比赛。

Let a_i denote the number of games he plays up through the i -th day.

$$a_1, a_2, \dots, a_{76}, a_{77}, a_1 + 21, a_2 + 21, \dots, a_{76} + 21, a_{77} + 21$$

There must be ≥ 2 elements having the same value.

Championship Match

“胡司令” (胡荣华) 要安排一次长达 77 天的象棋练习赛。

他想每天至少要有一场比赛, 但是总共不超过 132 场比赛。

请证明, 无论如何安排, 他都要在连续的若干天内恰好完成 21 场比赛。

Let a_i denote the number of games he plays up through the i -th day.

$$a_1, a_2, \dots, a_{76}, a_{77}, a_1 + 21, a_2 + 21, \dots, a_{76} + 21, a_{77} + 21$$

There must be ≥ 2 elements having the same value.

It must be $a_i + 21 = a_j$.

Sequences

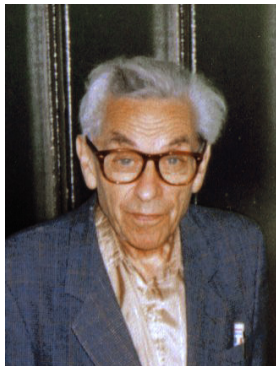
In any sequence $a_1, a_2, \dots, a_{mn+1}$ of $mn + 1$ **distinct** numbers, there exists an **increasing** subsequence

$$a_{i_1} < a_{i_2} < \dots < a_{i_{m+1}} \quad (i_1 < i_2 < \dots < i_{m+1})$$

of length $m + 1$, or a **decreasing** subsequence

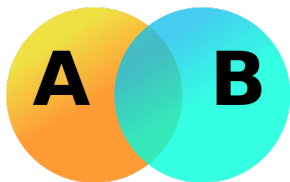
$$a_{j_1} > a_{j_2} > \dots > a_{j_{n+1}} \quad (j_1 > i_2 < \dots > j_{n+1})$$

of length $n + 1$, or both.

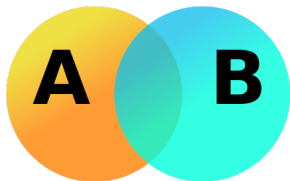


Paul Erdős (1913 ~ 1996)

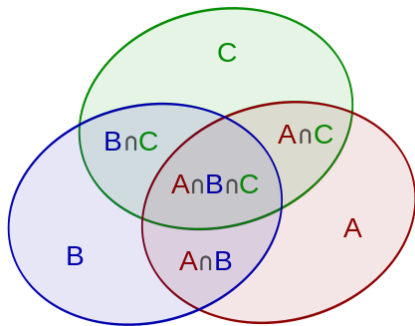
Chapter 28 of “Proofs from THE Book”



$$|A \cup B| = |A| + |B| - |A \cap B|$$



$$|A \cup B| = |A| + |B| - |A \cap B|$$



$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| \\ &\quad - |A \cap B| - |A \cap C| - |B \cap C| \\ &\quad + |A \cap B \cap C| \end{aligned}$$

Theorem (Inclusion-Exclusion Principle)

$$\begin{aligned} \left| \bigcup_{i=1}^n A_i \right| &= \sum_{i=1}^n |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| \\ &\quad + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| \\ &\quad - \dots \\ &\quad + (-1)^{n-1} |A_1 \cap \dots \cap A_n|. \end{aligned}$$

Theorem (Inclusion-Exclusion Principle)

$$\begin{aligned}\left| \bigcup_{i=1}^n A_i \right| &= \sum_{i=1}^n |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| \\ &\quad + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| \\ &\quad - \dots \\ &\quad + (-1)^{n-1} |A_1 \cap \dots \cap A_n|.\end{aligned}$$

$$\begin{aligned}\left| \bigcap_{i=1}^n \bar{A}_i \right| &= \left| S - \bigcup_{i=1}^n A_i \right| = |S| - \sum_{i=1}^n |A_i| + \sum_{1 \leq i < j \leq n} |A_i \cap A_j| \\ &\quad - \dots + (-1)^n |A_1 \cap \dots \cap A_n|.\end{aligned}$$

Counting Integers

How many integers in $1, \dots, 100$ are not divisible by 2, 3 or 5?

Counting Integers

How many integers in $1, \dots, 100$ are not divisible by 2, 3 or 5?

$$100 - (50 + 33 + 20) + (16 + 10 + 6) - 3 = 26.$$

Counting Derangements (错排)

Suppose there is a deck of n cards numbered from 1 to n .

Suppose a card numbered i is in the **correct** position if it is the i -th card in the deck. How many ways can the cards be shuffled **without any cards** being in the correct position?

Counting Derangements (错排)

Suppose there is a deck of n cards numbered from 1 to n .

Suppose a card numbered i is in the **correct** position if it is the i -th card in the deck. How many ways can the cards be shuffled **without any cards** being in the correct position?

A_m : all of the orderings of cards with the m -th card correct

Counting Derangements (错排)

Suppose there is a deck of n cards numbered from 1 to n .

Suppose a card numbered i is in the **correct** position if it is the i -th card in the deck. How many ways can the cards be shuffled **without any cards** being in the correct position?

A_m : all of the orderings of cards with the m -th card correct

$$\left| \bigcap_{i=1}^n \overline{A_i} \right| = \left| S - \bigcup_{i=1}^n A_i \right| = n! - \sum_{i=1}^n |A_i| + \sum_{1 \leq i < j \leq n} |A_i \cap A_j| \\ - \cdots + (-1)^n |A_1 \cap \cdots \cap A_n|.$$

Counting Derangements (错排)

Suppose there is a deck of n cards numbered from 1 to n .

Suppose a card numbered i is in the **correct** position if it is the i -th card in the deck. How many ways can the cards be shuffled **without any cards** being in the correct position?

A_m : all of the orderings of cards with the m -th card correct

$$\left| \bigcap_{i=1}^n \overline{A_i} \right| = \left| S - \bigcup_{i=1}^n A_i \right| = n! - \sum_{i=1}^n |A_i| + \sum_{1 \leq i < j \leq n} |A_i \cap A_j| \\ - \cdots + (-1)^n |A_1 \cap \cdots \cap A_n|.$$

$$S_k \triangleq \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| =$$

Counting Derangements (错排)

Suppose there is a deck of n cards numbered from 1 to n .

Suppose a card numbered i is in the **correct** position if it is the i -th card in the deck. How many ways can the cards be shuffled **without any cards** being in the correct position?

A_m : all of the orderings of cards with the m -th card correct

$$\left| \bigcap_{i=1}^n \overline{A_i} \right| = \left| S - \bigcup_{i=1}^n A_i \right| = n! - \sum_{i=1}^n |A_i| + \sum_{1 \leq i < j \leq n} |A_i \cap A_j| \\ - \cdots + (-1)^n |A_1 \cap \cdots \cap A_n|.$$

$$S_k \triangleq \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = \binom{n}{k} (n-k)! = \frac{n!}{k!}$$

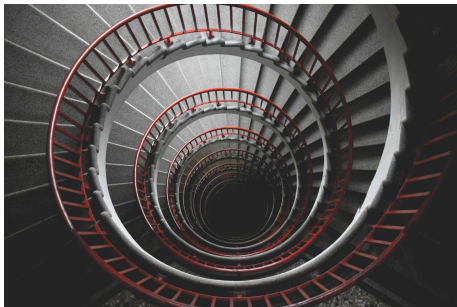
$$S_k = \frac{n!}{k!}$$

$$\begin{aligned} \left| \bigcap_{i=1}^n \overline{A_i} \right| &= n! - \frac{n!}{1!} + \frac{n!}{2!} - \cdots + (-1)^n \frac{n!}{n!} \\ &= n! \sum_{k=0}^n \frac{(-1)^k}{k!} \end{aligned}$$

$$S_k = \frac{n!}{k!}$$

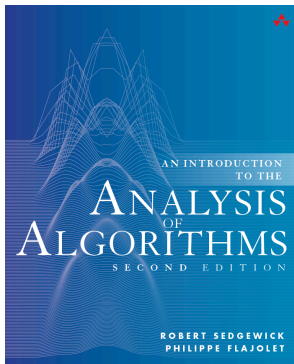
$$\begin{aligned} \left| \bigcap_{i=1}^n \overline{A_i} \right| &= n! - \frac{n!}{1!} + \frac{n!}{2!} - \cdots + (-1)^n \frac{n!}{n!} \\ &= n! \sum_{k=0}^n \frac{(-1)^k}{k!} \end{aligned}$$

$$n \rightarrow \infty \implies \sum_{k=0}^n \frac{(-1)^k}{k!} \rightarrow e^{-1} \approx 0.368$$



$$a_n = f(a_{n-1}, a_{n-2}, \dots, a_{n-t}) + g(n)$$

$$a_n = f(a_{n-1}, a_{n-2}, \dots, a_{n-t}) + g(n)$$



recurrence type	typical example
first-order	
linear	$a_n = na_{n-1} - 1$
nonlinear	$a_n = 1/(1 + a_{n-1})$
second-order	
linear	$a_n = a_{n-1} + 2a_{n-2}$
nonlinear	$a_n = a_{n-1}a_{n-2} + \sqrt{a_{n-2}}$
variable coefficients	$a_n = na_{n-1} + (n-1)a_{n-2} + 1$
t th order	$a_n = f(a_{n-1}, a_{n-2}, \dots, a_{n-t})$
full-history	$a_n = n + a_{n-1} + a_{n-2} \dots + a_1$
divide-and-conquer	$a_n = a_{\lfloor n/2 \rfloor} + a_{\lceil n/2 \rceil} + n$

Table 2.1 Classification of recurrences

Homogeneous Linear Recurrence Relations with Constant Coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_t a_{n-t}$$

Homogeneous Linear Recurrence Relations with Constant Coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_t a_{n-t}$$

2020 年课程录屏 (Recurrence)

$$R \subseteq A \times A$$

$$\begin{cases} R^0 = I_A \\ R^{n+1} = R \circ R^n \end{cases}$$

Representing Relations as Matrices/Digraphs

$$A = \{1, 2, 3, 4\}$$

$$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4), (4, 1)\}$$

Representing Relations as Matrices/Digraphs

$$A = \{1, 2, 3, 4\}$$

$$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4), (4, 1)\}$$

$$R^2 \quad R^3$$

Representing Relations as Matrices/Digraphs

$$A = \{1, 2, 3, 4\}$$

$$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4), (4, 1)\}$$

$$R^2 \quad R^3$$

$$R^+ = \bigcup_{i=1}^{\infty} R \quad R^* = \bigcup_{i=0}^{\infty} R$$

Definition (Reflexive Closure (自反闭包))

The **reflexive closure** $\text{cl}_{\text{ref}}(R)$ of a relation $R \subseteq X \times X$ is the **smallest** reflexive relation on X that contains R .

Definition (Reflexive Closure (自反闭包))

The **reflexive closure** $\text{cl}_{\text{ref}}(R)$ of a relation $R \subseteq X \times X$ is the **smallest** reflexive relation on X that contains R .

$$\text{cl}_{\text{ref}}(R) = R \cup I_X$$

Definition (Symmetric Closure (对称闭包))

The **symmetric closure** $\text{cl}_{\text{sym}}(R)$ of a relation $R \subseteq X \times X$ is the **smallest** symmetric relation on X that contains R .

Definition (Symmetric Closure (对称闭包))

The **symmetric closure** $\text{cl}_{\text{sym}}(R)$ of a relation $R \subseteq X \times X$ is the **smallest** symmetric relation on X that contains R .

$$\text{cl}_{\text{sym}}(R) = R \cup R^{-1}$$

Definition (Transitive Closure (传递闭包))

The **transitive closure** $\text{cl}_{\text{trn}}(R)$ of a relation $R \subseteq X \times X$ is the **smallest** transitive relation on X that contains R .

Definition (Transitive Closure (传递闭包))

The **transitive closure** $\text{cl}_{\text{trn}}(R)$ of a relation $R \subseteq X \times X$ is the **smallest** transitive relation on X that contains R .

$$\text{cl}_{\text{trn}}(R) = R^+$$

Definition (Transitive Closure (传递闭包))

The **transitive closure** $\text{cl}_{\text{trn}}(R)$ of a relation $R \subseteq X \times X$ is the **smallest** transitive relation on X that contains R .

$$\text{cl}_{\text{trn}}(R) = R^+$$

- ▶ R^+ contains R
- ▶ R^+ is transitive
- ▶ R^+ is minimal

Definition (Transitive Closure (传递闭包))

The **transitive closure** $\text{cl}_{\text{trn}}(R)$ of a relation $R \subseteq X \times X$ is the **smallest** transitive relation on X that contains R .

$$\text{cl}_{\text{trn}}(R) = R^+$$

- ▶ R^+ contains R
- ▶ R^+ is transitive
- ▶ R^+ is minimal

If T is any transitive relation containing R , then $R^+ \subset T$.

Definition (Transitive Closure (传递闭包))

The **transitive closure** $\text{cl}_{\text{trn}}(R)$ of a relation $R \subseteq X \times X$ is the **smallest** transitive relation on X that contains R .

$$\text{cl}_{\text{trn}}(R) = R^+$$

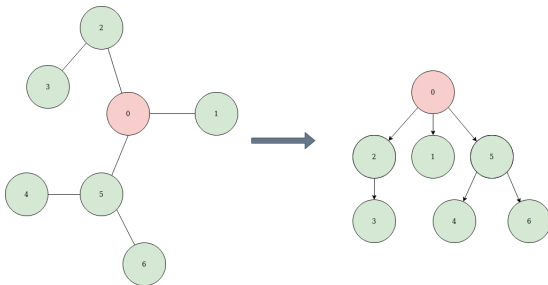
- ▶ R^+ contains R
- ▶ R^+ is transitive
- ▶ R^+ is minimal

If T is any transitive relation containing R , then $R^+ \subseteq T$.

By induction on i , we can show that $R^i \subseteq T$.

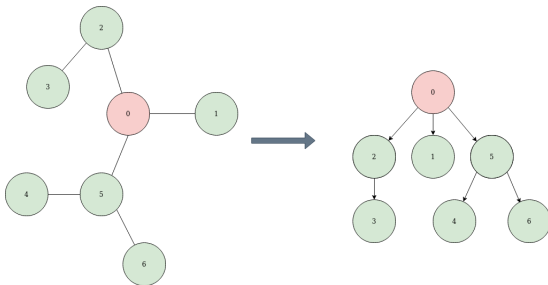
Definition (Rooted Tree (有根树))

A **rooted tree** is a **tree** where one vertex has been **designated the root**.



Definition (Rooted Tree (有根树))

A **rooted tree** is a **tree** where one vertex has been **designated the root**.



Definition (Directed Rooted Tree (有向有根树))

A **directed rooted tree** is a **rooted tree** where all edges directed **away from** or **towards** the root.

Definition

Parent, Child; Sibling; Ancestor, Descendant

Definition

Parent, Child; Sibling; Ancestor, Descendant

Definition (k -ary Trees (k -叉树))

A k -ary tree is a rooted tree in which each vertex has $\leq k$ children.

2-ary trees are often called binary trees.

Definition

Parent, Child; Sibling; Ancestor, Descendant

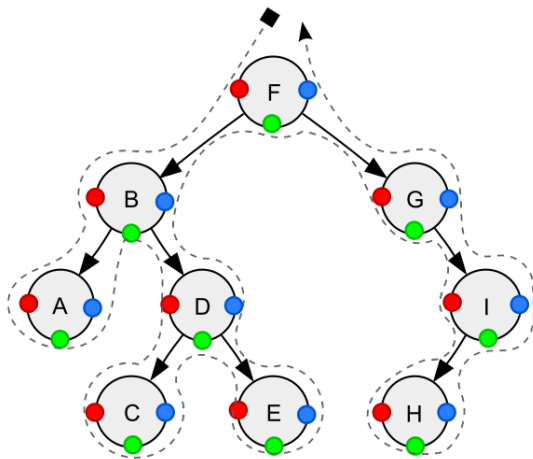
Definition (k -ary Trees (k -叉树))

A k -ary tree is a rooted tree in which each vertex has $\leq k$ children.

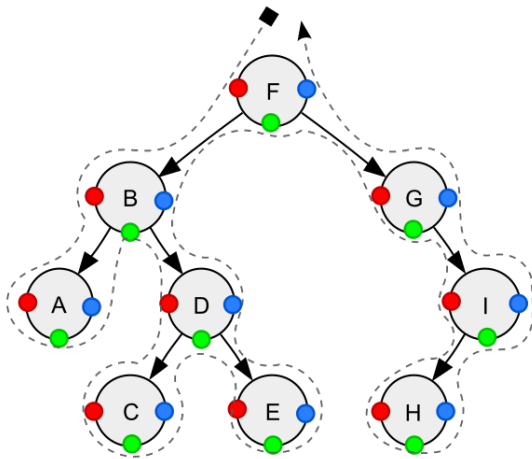
2-ary trees are often called **binary trees**.

Definition (Complete k -Tree (完全 k -叉树))

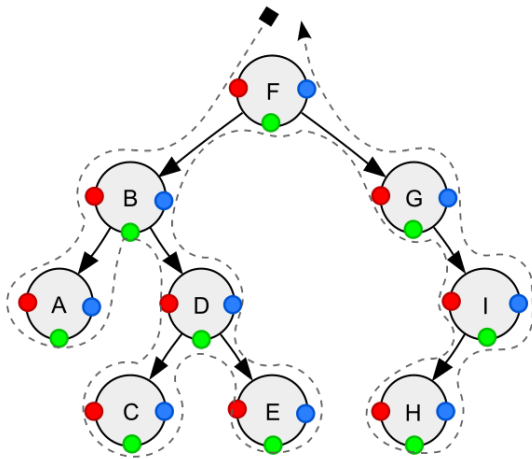
A **complete k -tree** is a k -ary tree in which each vertex, other than leaves, has $= k$ children.



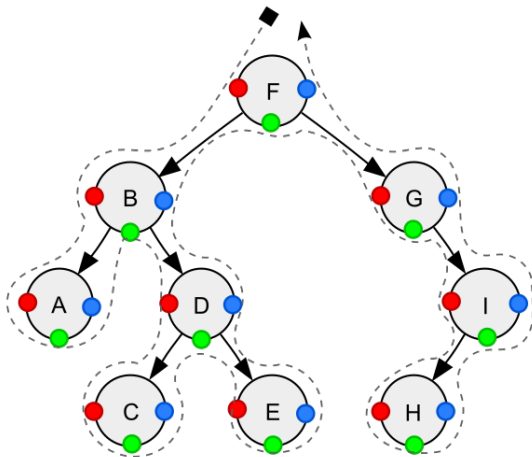
Depth-First Search (DFS)



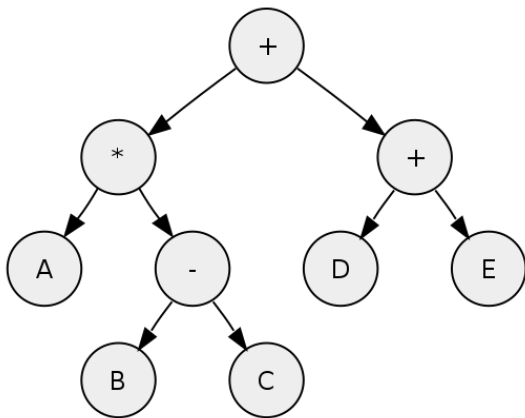
Pre-order (前序) Traversal: $F, B, A, D, C, E, G, I, H$



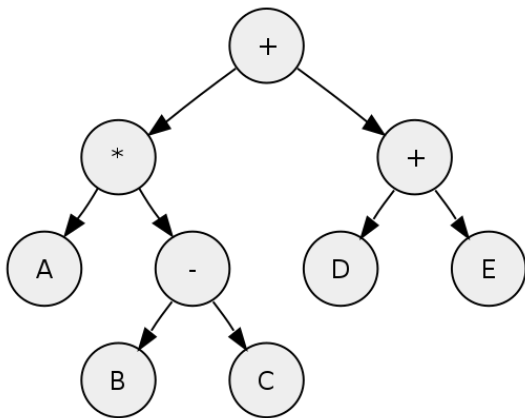
In-order (中序) Traversal: $A, B, C, D, E, F, G, H, I$



Post-order (后序) Traversal: $A, C, E, D, B, H, I, G, F$

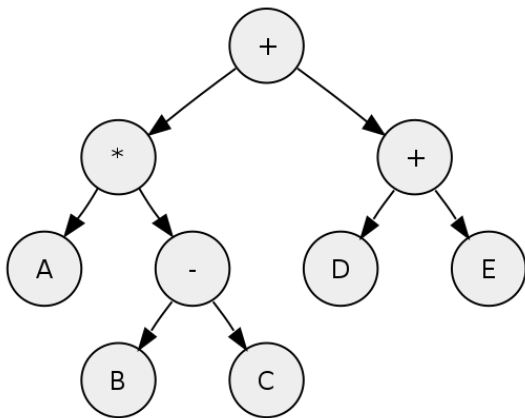


Prefix Expression (前缀表达式): $+ * A - BC + DE$



Prefix Expression (前缀表达式): $+ * A - BC + DE$

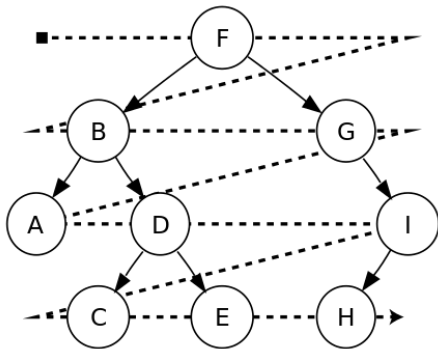
Infix Expression (中缀表达式): $A * (B - C) + (D + E)$



Prefix Expression (前缀表达式): $+ * A - BC + DE$

Infix Expression (中缀表达式): $A * (B - C) + (D + E)$

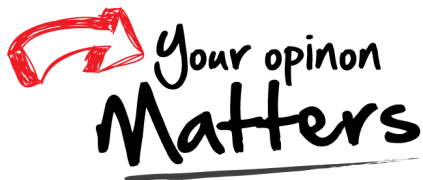
Postfix Expression (后缀表达式): $ABC - * DE + +$



Breadth-First Search (BFS): $F, B, G, A, D, I, C, E, H$

Huffman Code

Thank
You!



Office 302

Mailbox: H016

hfwei@nju.edu.cn