

(九) 图论: 路径与圈 (Paths and Cycles)

魏恒峰

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2021 年 05 月 06 日

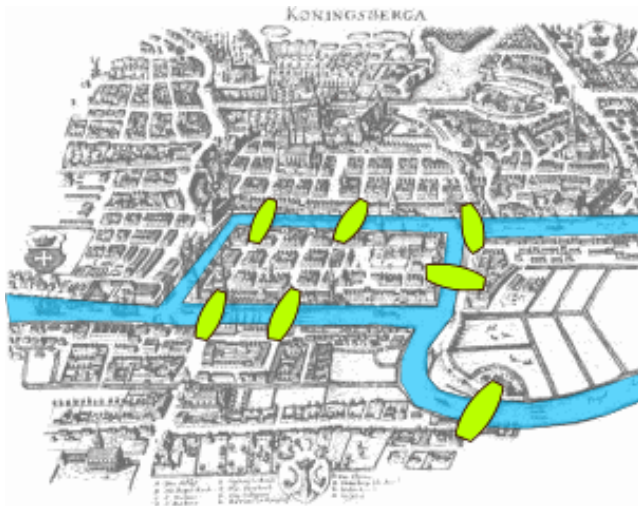


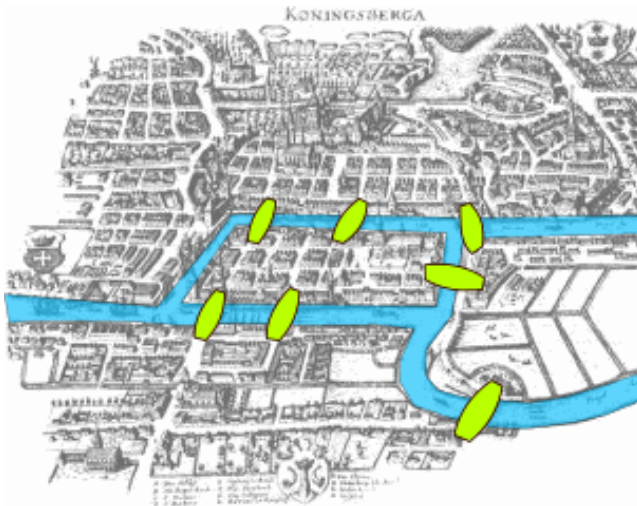


Definitions

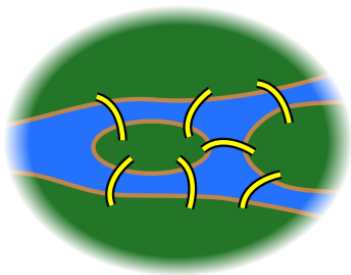
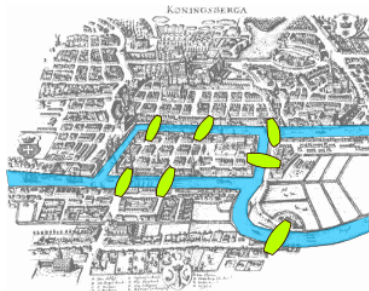
Theorems

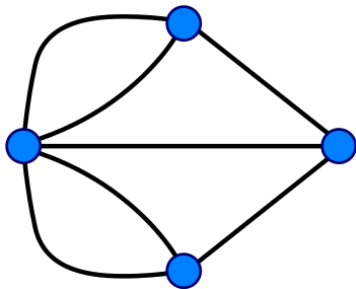
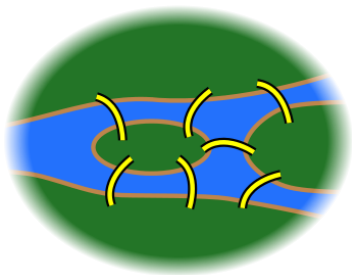
Proofs

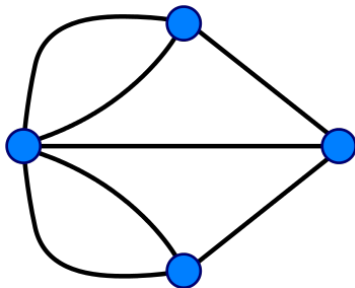
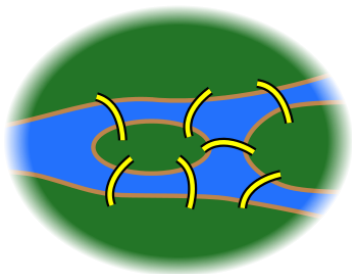




*“to devise a walk through the **city**
that would cross each of those **bridges** **once and only once**”*





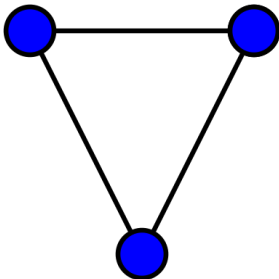


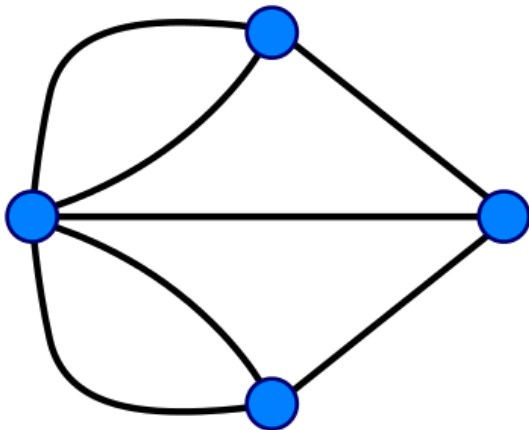
*“to devise a walk through the **graph**
that would cross each of those **edges** once and only once”*

Definition (Graph (图))

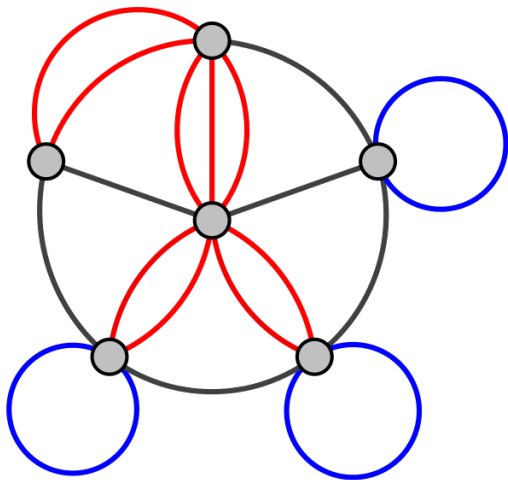
An **(undirected simple) graph** is a **pair** $G = (V, E)$ where

- ▶ V is a set of **vertices** (顶点);
- ▶ $E \subseteq \{\{x, y\} \mid x, y \in V \wedge x \neq y\}$ is a set of **edges**





Undirected **Multigraph**



Undirected Multigraph Permitting **Loops**

Definition (Walk (道路))

Given a graph G , a (finite) **walk** in G is a sequence of edges of the form

$$\{v_0, v_1\}, \{v_1, v_2\}, \dots, \{v_{m-1}, v_m\}.$$

$$(v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_m)$$

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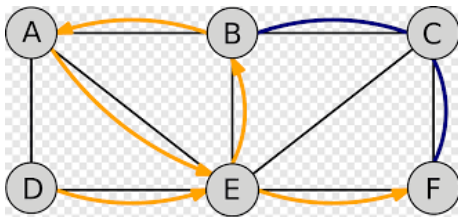
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$$D \rightarrow E \rightarrow B \rightarrow A \rightarrow E \rightarrow F$$



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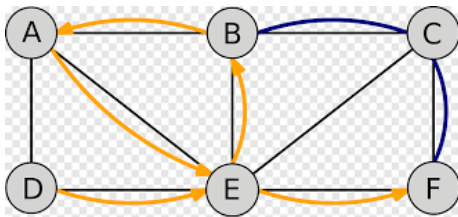
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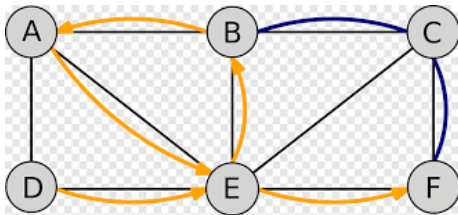
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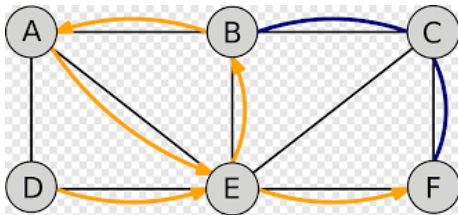
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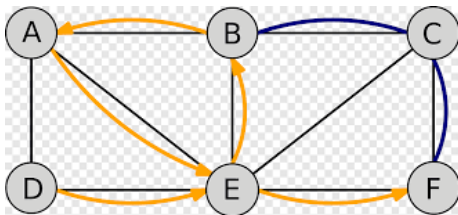
$$D \rightarrow E \rightarrow F$$

Definition (Closed Walk/Trail/Path)

A walk, trail, or path is **closed** if $v_0 = v_m$.

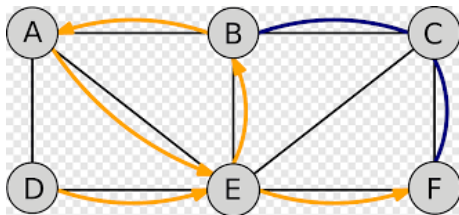
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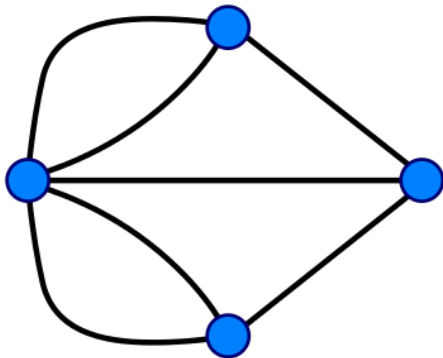
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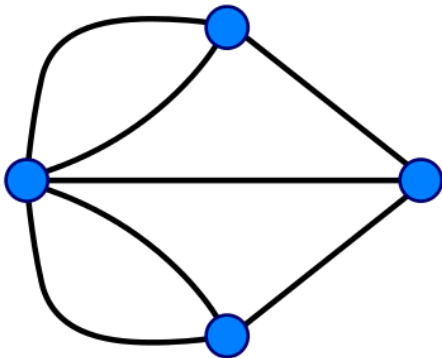
Definition (Cycle)

A **cycle** is a **closed path** with at least one edge.

*“to devise a walk through the **graph**
that would cross each of those **edges** once and only once”*

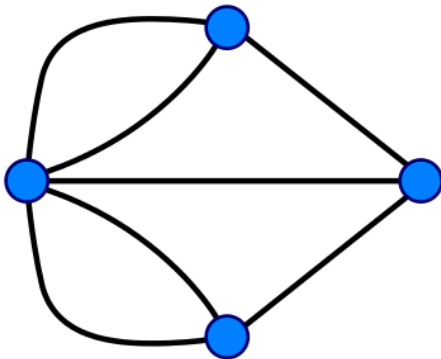


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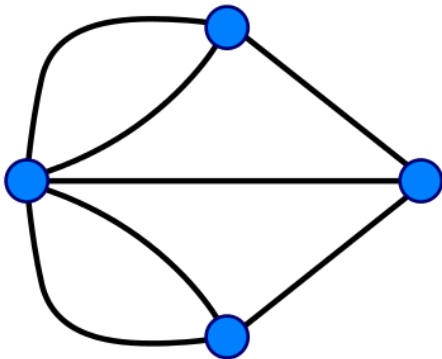


*to find a **trail** that contains all **edges** of the **graph***

$$v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_i \rightarrow \cdots \rightarrow v_m$$



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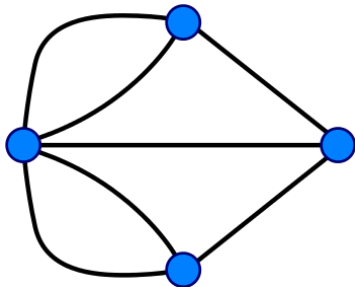
$$v_i \notin \{v_0, v_m\} \implies \deg(v_i) \text{ is even}$$

Lemma (Necessary Condition for Eulerian Trails)

*If a graph has **Eulerian trails**, then **zero or two** vertices have an **odd** degree.*

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If a graph has *Eulerian trails*, then *zero or two* vertices have an *odd* degree.



4 vertices of odd degree \implies has no Eulerian trails

Theorem (Euler's Theorem)

A graph has *Eulerian trails* iff it contains *zero or two* vertices that have an *odd* degree.

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Pierre-Henry Fleury gave another proof in 1883.

Theorem (Euler's Theorem (Carl Hierholzer))

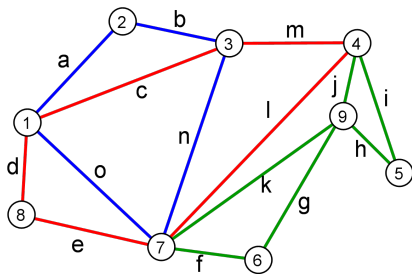
A *connected* graph has *Eulerian cycles/circuits* iff *every vertex* has *even degree*.

(Such graphs are called Eulerian graphs.)

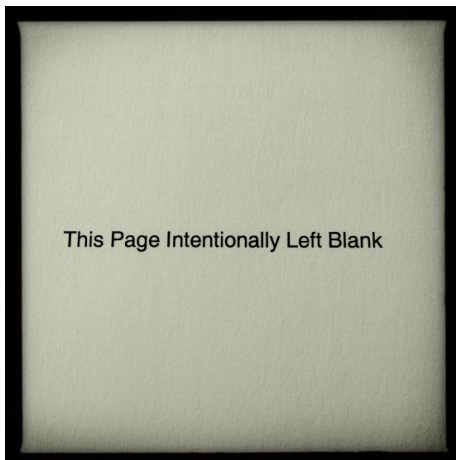
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cycle + cycle + cycle + ...



Can you repeat the Carl Hierholzer's algorithm?

Lemma

If every vertex of a graph G has degree ≥ 2 , then G contains a cycle.

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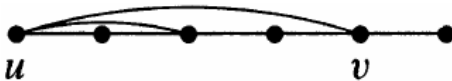
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Let $P : u \rightarrow \dots$ be a *maximal* path in G .

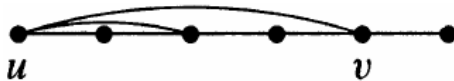


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$$\deg(u) \geq 2$$

Theorem (Euler's Theorem (Carl Hierholzer))

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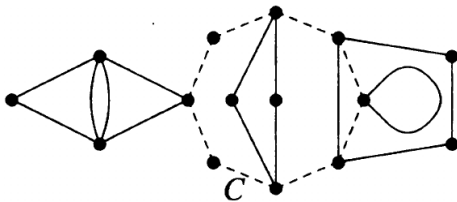
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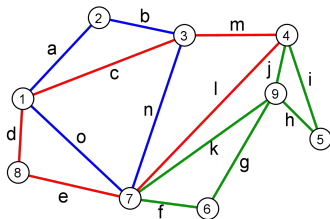
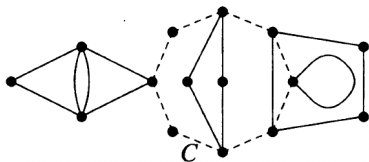
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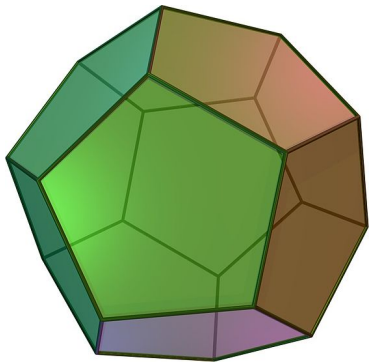
$$G' = G - C$$

Theorem

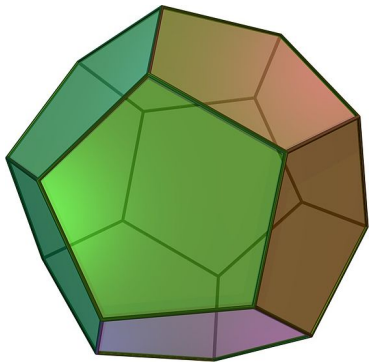
Every Eulerian graph decomposes into cycles.



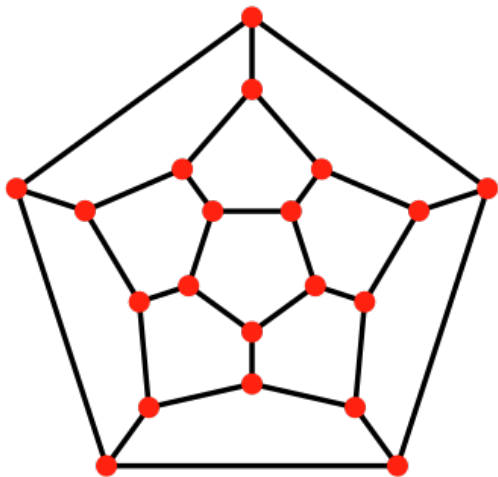
Dodecahedron: 12 faces, 20 vertices, and 30 edges



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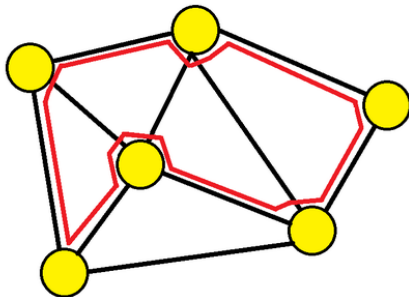
Is there a cycle that visits each vertex exactly once?



Is there a cycle that visits each vertex exactly once?

Definition (Hamiltonian Path)

A **Hamiltonian path** is a **path** that visits each **vertex** exactly once.



Definition (Hamiltonian Cycle)

A **Hamiltonian cycle** is a **Hamiltonian path** that is a **cycle**.

Definition (Hamiltonian Graph)

A graph is a **Hamiltonian graph** if it has a **Hamiltonian cycle**.

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Definition (Semi-Hamiltonian Graph)

A **non-Hamiltonian** graph is **semi-Hamiltonian** if it has a **Hamiltonian path**.

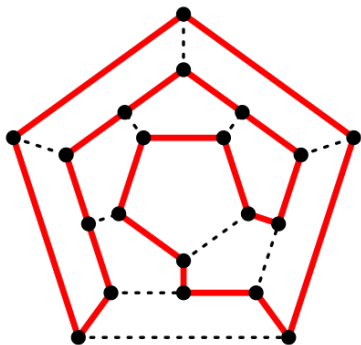


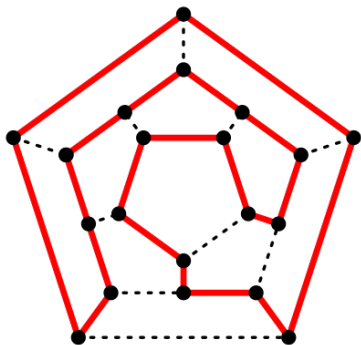
William Rowan Hamilton
(1805 ~ 1865)



(October 16, 1843)

$$i^2 = j^2 = k^2 = ijk = -1$$





What is “THE” theorem for finding a Hamiltonian path/cycle
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I do not know.

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Nobody knows.

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Nobody knows.

We will probably never know it.



Theorem

The Hamiltonian Path/Cycle problem is NP-complete.

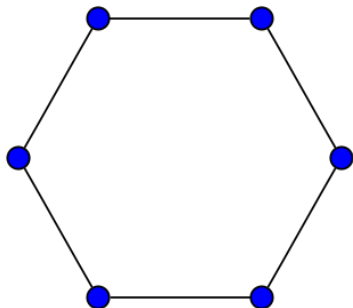
Typical (Positive/Negative) Graph Examples

Sufficient Conditions

Necessary Conditions



- Every **cycle** is Hamiltonian



C_6

- ▶ A **complete** graph (完全图) with $|V| > 2$ is Hamiltonian.

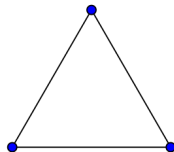
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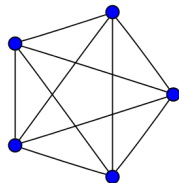
K_1



K_2



K_3

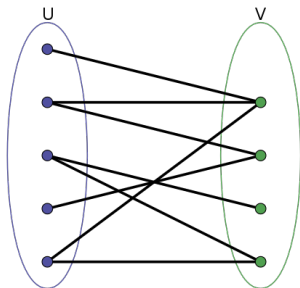


K_5

- ▶ A complete bipartite graph $K_{m,n}$ is Hamiltonian iff $m = n \geq 2$.

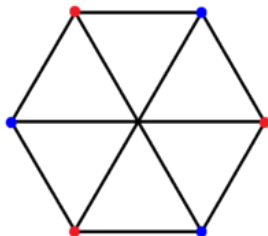
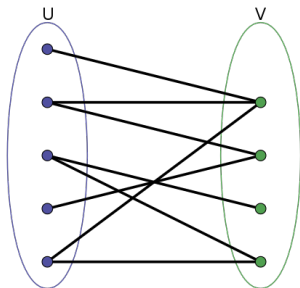
Definition (Bipartite Graph (Bigraph; 二部图))

A **bipartite graph** $G = (U, V, E)$ is a graph whose **vertices can** be divided into two disjoint sets U and V such that every edge connects a vertex in U to one in V .



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Definition (Complete Bipartite Graph (Biclique; 完全二部图))

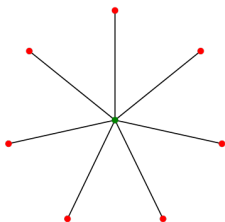
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$$K_{m,n} : m = |U|, n = |V|$$

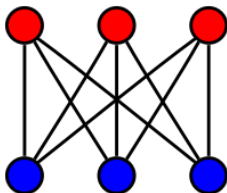
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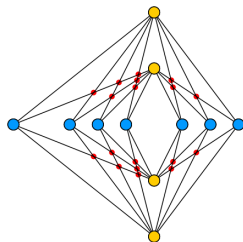
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$K_{1,5}$ (star)



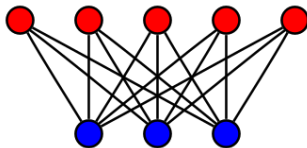
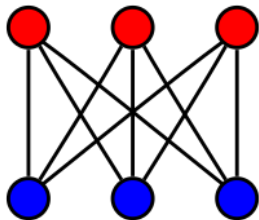
$K_{3,3}$ (utility graph)



$K_{4,7}$

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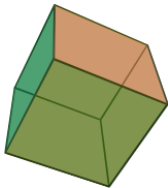


- ▶ Every **platonic solid** (正多面体), considered as a graph, is Hamiltonian.

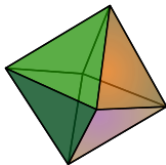
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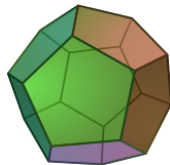
Tetrahedron



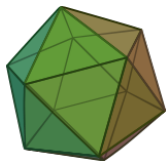
Cube



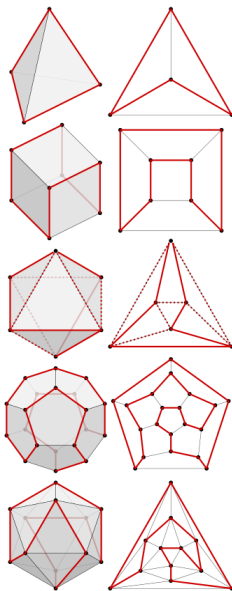
Octahedron



Dodecahedron



Icosahedron



Theorem

- ▶ *Petersen graph* is *not* Hamiltonian.



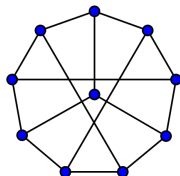
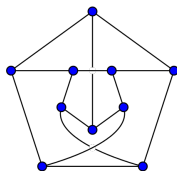
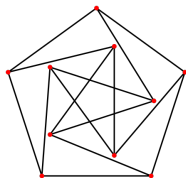
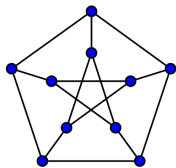
Julius Petersen (1839 ~ 1910)

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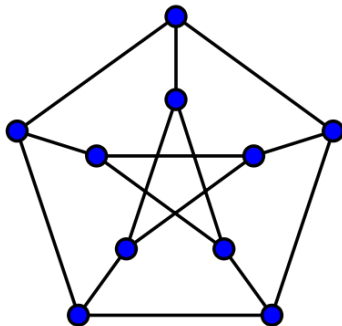


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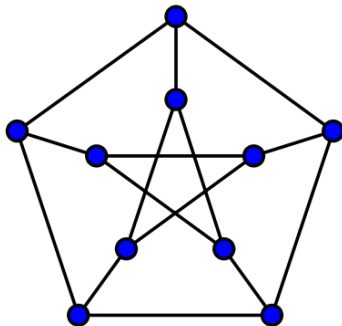
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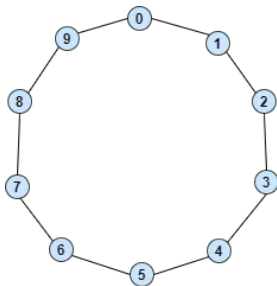


The Petersen graph has *no* cycles of length < 5 .

By Contradiction.

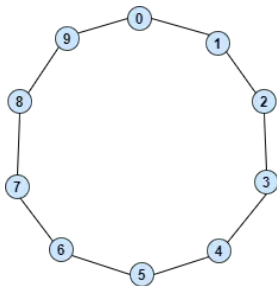
By Contradiction.

Suppose that it has a Hamiltonian cycle C .



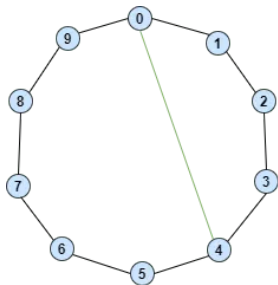
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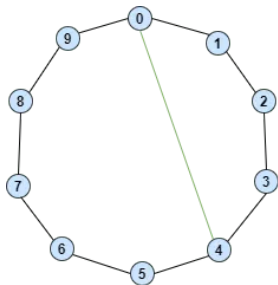


v_0 is adjacent to v_4 , v_5 , or v_6 .

CASE I: v_0 is adjacent to v_4

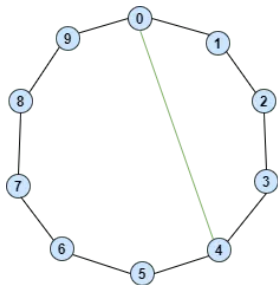


CASE I: v_0 is adjacent to v_4



v_5 is adjacent to v_1 or v_9

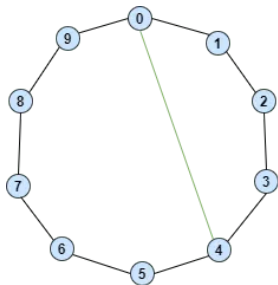
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cycle of length 4!

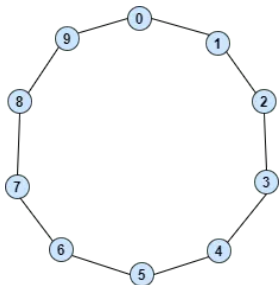
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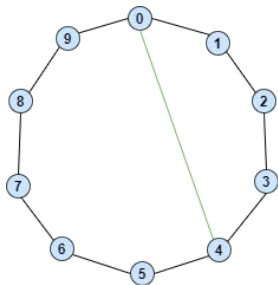
v_5 is adjacent to v_1 or v_9

cycle of length 4!

CASE II: v_0 is adjacent to v_5



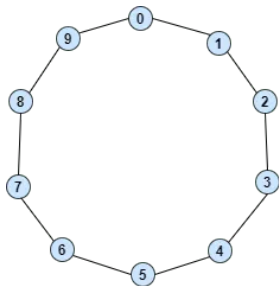
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v_5 is adjacent to v_1 or v_9

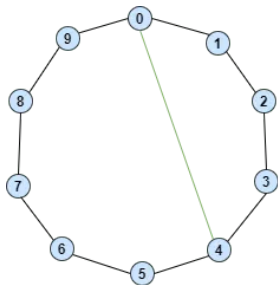
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CASE II: v_0 is adjacent to v_5



v_1 is adjacent to v_7

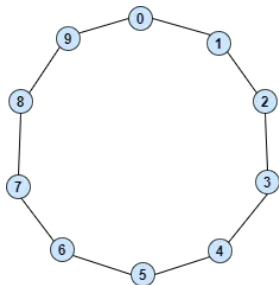
CASE I: v_0 is adjacent to v_4



v_5 is adjacent to v_1 or v_9

cycle of length 4!

CASE II: v_0 is adjacent to v_5



v_1 is adjacent to v_7

back to CASE I!



“If G has enough edges, then G is Hamiltonian.”

Theorem (Ore's Theorem, 1960)

Let G be a *simple* graph with $n \geq 3$ vertices. If

$$\deg(u) + \deg(v) \geq n$$

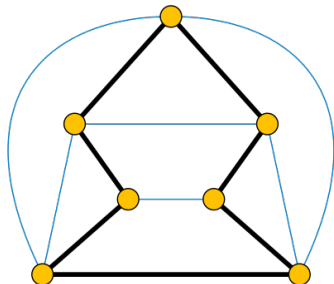
for *each pair* of *non-adjacent* vertices u and v , then G is *Hamiltonian*.

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Adding edges cannot violate the **Ore's Condition**.

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We need to derive a contradiction.

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Adding edges cannot violate the *Ore's Condition*.

Thus we may consider only *maximal* non-Hamiltonian graphs:
adding any edge gives a Hamiltonian graph.

By its “maximality”, G contains a **Hamiltonian path**
(G is a **semi-Hamiltonian graph**)

$$v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_n$$

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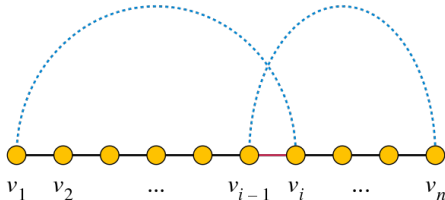
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v_1 and v_n are **non-adjacent**

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There must be some vertex v_i adjacent to v_1
such that v_{i-1} is adjacent to v_n .

Theorem (Dirac's Theorem (1952; Gabriel Andrew Dirac))

A *simple* graph $G = (V, E)$ with $n \geq 3$ vertices is *Hamiltonian*

$$\forall v \in V. \deg(v) \geq n/2.$$

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Family [\[edit | edit source \]](#)

He was born Balázs Gábor in Budapest, to Richárd Balázs, a military officer and businessman, and Margit "Manci" Wigner (sister of Eugene Wigner).^[5] When his mother married Paul Dirac in 1937, he and his sister resettled in England and were formally adopted, changing their family name to Dirac.^[6]

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Counterexample: $K_{\lfloor (n+1)/2 \rfloor}$ and $K_{\lceil (n+1)/2 \rceil}$ sharing a vertex

“If G is Hamiltonian, then G must be somewhat connected.”



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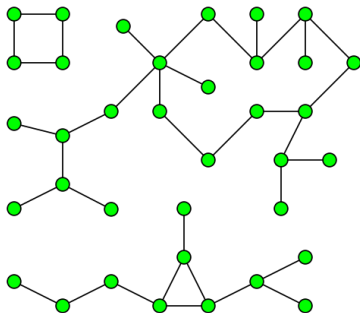
“If G is not so connected, then G is non-Hamiltonian.”

Theorem

If $G = (V, E)$ is Hamiltonian, then for each nonempty set $S \subset V$, the graph $G - S$ has $\leq |S|$ components.

Definition (Components (连通分支))

A **component** of an **undirected graph** is an **subgraph** in which any two vertices are connected to each other by paths, and which is connected to no additional vertices in the rest of the graph.



$$c(G) = 3$$

Theorem

If $G = (V, E)$ is Hamiltonian, then

$$\forall S \subset V. c(G - S) \leq |S|.$$

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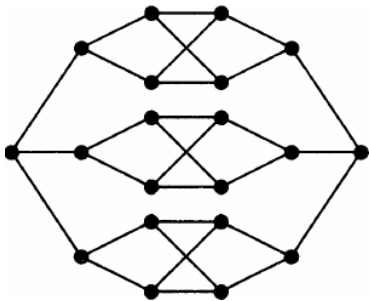
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- ▶ A complete bipartite graph $K_{m,n}$ is Hamiltonian iff $m = n \geq 2$.

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A non-Hamiltonian bipartite graph with $m = n = 10$

Theorem

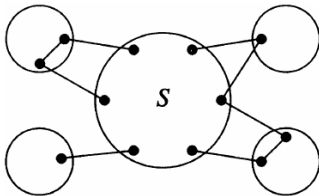
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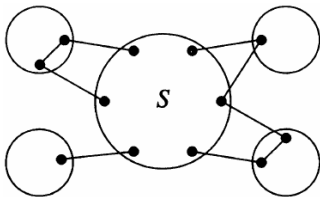
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Theorem

If $G = (V, E)$ is Hamiltonian,

$$\forall S \subset V. c(G - S) \leq |S|.$$



“When a Hamiltonian cycle **leaves a component** of $G - S$,
it can go only to a **distinct vertex** in S .”

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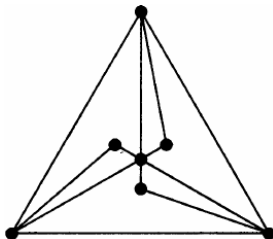
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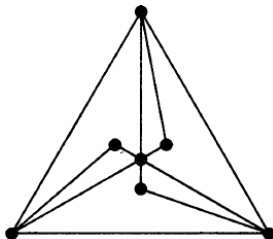


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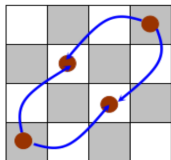
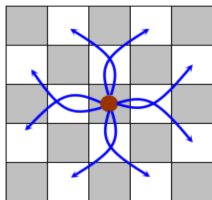
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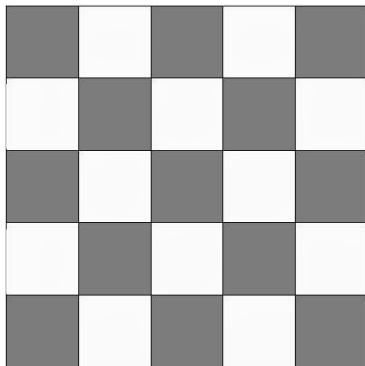


All edges incident to vertices of **degree 2** must be used.

Chessboard Problem (“马踏棋盘” 问题)

Is it possible for a “knight” to visit every field of a 4×4 or 5×5 chessboard exactly once and return to the starting point?





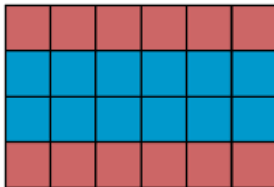
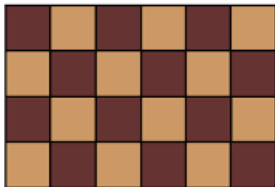
$$G = (U, V, E) : |U| = 12, |V| = 13$$





Removing the middle 4 squares leaves ≥ 5 components.

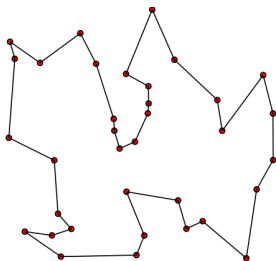
Chessboard Problem

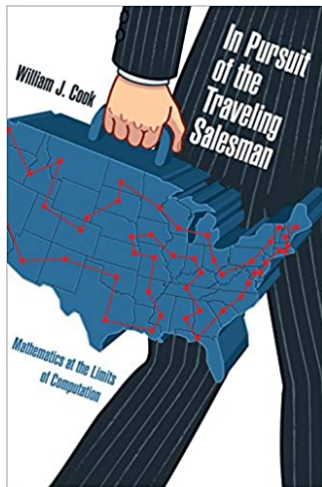


$$4 \times n$$

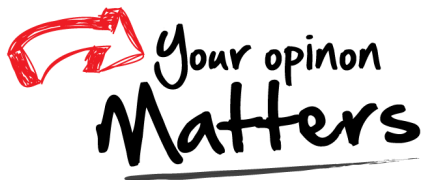
Definition (Travelling Salesman/Salesperson Problem (TSP; 旅行商问题))

Given a list of cities and the distances between each pair of cities, what is the **shortest** possible route that visits each city **exactly once** and returns to the origin city?





Thank
You!



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