(三) 数学归纳法 (Mathematical Induction)

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2021 年 03 月 25 日

数学归纳法真得很简单吗?

Theorem (第一数学归纳法 (The First Mathematical Induction))

设 $P(n)$ 是关于自然数的一个性质。如果

(i) $P(0)$ 成立;

(ii) 对任意自然数 n, 如果 $P(n)$ 成立, 则 $P(n+1)$ 成立。

那么, $P(n)$ 对所有自然数 n 都成立。

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$$\frac{P(0) \qquad \forall n \in \mathbb{N}. \Big(P(n) \to P(n+1)\Big)}{\forall n \in \mathbb{N}. P(n)} \quad (\text{第一数学归纳法})$$

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$$\Big(P(0) \land \forall n \in \mathbb{N}.\ \big(P(n) \to P(n+1)\big)\Big) \to \forall n \in \mathbb{N}.\ P(n).$$

Theorem (第二数学归纳法 (The Second Mathematical Induction))

设 $Q(n)$ 是关于自然数的一个性质。如果

(i) $Q(0)$ 成立;

(ii) 对任意自然数 n, 如果 $Q(0), Q(1), \ldots, Q(n)$ 都成立, 则 $Q(n+1)$ 成立。

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$$\frac{Q(0) \quad \forall n \in \mathbb{N}. \Big((Q(0) \land \cdots \land Q(n)) \to Q(n+1) \Big)}{\forall n \in \mathbb{N}. Q(n)} \quad (第二数学归纳法)$$

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Theorem (数学归纳法)

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Q : 第二数学归纳法也被称为“**强**” **(Strong)** 数学归纳法，它强在何处？

Lemma

第二数学归纳法蕴含第一数学归纳法。

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$$Q(n) \triangleq P(n)$$

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$$P(n) \triangleq Q(0) \land \cdots \land Q(n)$$

数学归纳法为何成立?

Peano 公理体系刻画了**自然数的递归结构**

Definition (Peano Axioms)

(1) 0 是自然数;

(2) 如果 n 是自然数, 则它的后继 $\mathbf{S}n$ 也是自然数;

(3) 0 不是任何自然数的后继;

(4) 两个自然数相等当且仅当它们的后继相等;

(5) **数学归纳原理:** 如果

  (i) $P(0)$ 成立;

  (ii) 对任意自然数 n, 如果 $P(n)$ 成立, 则 $P(n+1)$ 成立。

那么, $P(n)$ 对所有自然数 n 都成立。

Definition (良序原理 (The Well-Ordering Principle))

自然数集的任意非空子集都有一个最小元。

Theorem

良序原理与 (第一) 数学归纳法等价。

Lemma

(第一) 数学归纳法蕴含良序原理。

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Proof.

By mathematical induction on the size n of non-empty subsets of $\mathbb{N}$.

$$P(k): \text{All subsets of size } k \text{ contain a minimum.}$$

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Basis Step: $P(1)$

Inductive Hypothesis: $P(n)$

Inductive Step: $P(n) \to P(n+1)$

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- $A' \leftarrow A \setminus a$
- $x \leftarrow \min A'$
- Compare x with a

$\square$

Lemma

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$$\begin{aligned} \text{Basis Step: } & P(1) \\ \text{Inductive Hypothesis: } & P(n) \\ \text{Inductive Step: } & P(n) \to P(n+1) \end{aligned}$$

$$\forall n \in \mathbb{N} : P(n) \quad vs. \quad P(\infty)$$

$\square$

Lemma

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$P(n)$: 任何一个含有 $\leq n$ 的某个自然数的自然数子集都有最小元

Lemma

良序原理蕴含 *(第一)* 数学归纳法。

反证法

设 $P(0)$ 成立且 $\forall n \in \mathbb{N}.\ P(n) \to P(n+1)$ 成立,
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$$m \triangleq \min A \quad (\text{by 良序原理})$$

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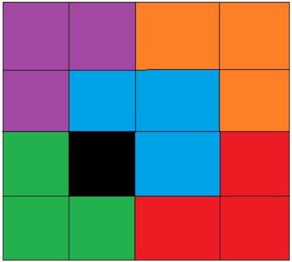
Learn by Examples

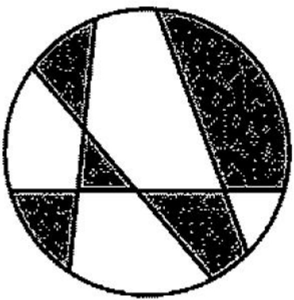
Tiling Puzzle

任何一个缺失了一格的 $2^n \times 2^n$ 的网格都可以被 L 型填满。



Definition (Line Map)

- A blank circle is a line map;
- A line map with a chord (弦) is a line map.



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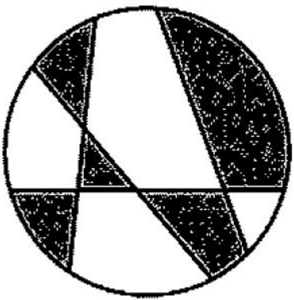

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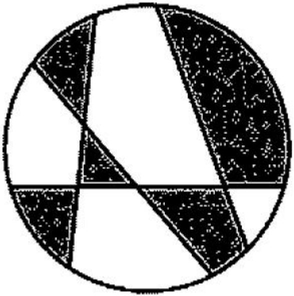

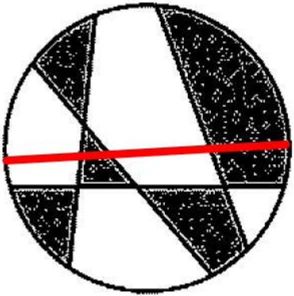

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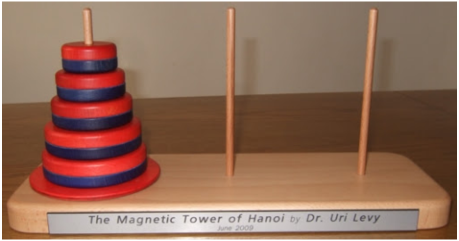
The Tower of Hanoi



$\textsc{Hanoi}(n, A, B, C)$: 借助于 B 柱, 将 n 个盘子从 A 柱移到 C 柱

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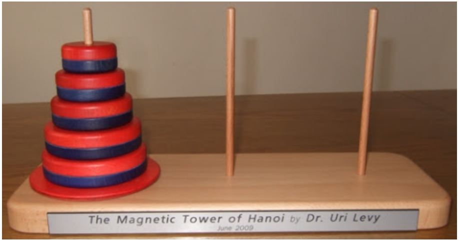

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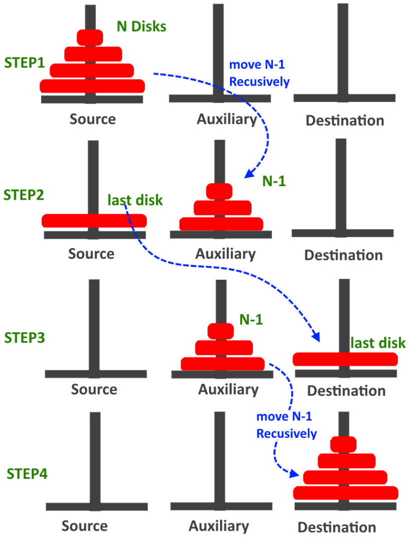
T_n : the **minimum** number of moves for n disks


N Disks
STEP1
move N-1
Recusively
Source
Auxiliary
Destination
STEP2
last disk
N-1
Source
Auxiliary
Destination
STEP3
N-1
last disk
Source
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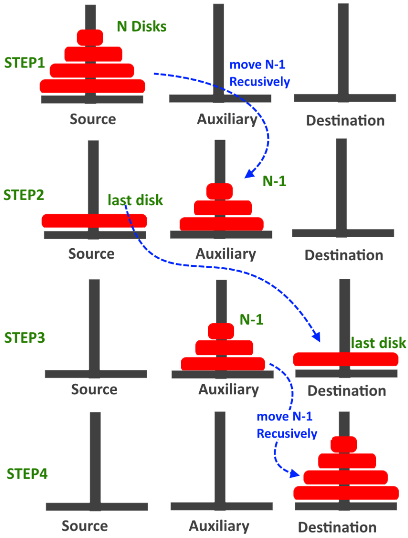
Destination
move N-1
Recusively
STEP4
Source
Auxiliary
Destination



$$T(n) \leq 2T(n-1) + 1 \qquad (n \geq 1)$$

考虑第一次以及最后一次移动最大盘时的情况

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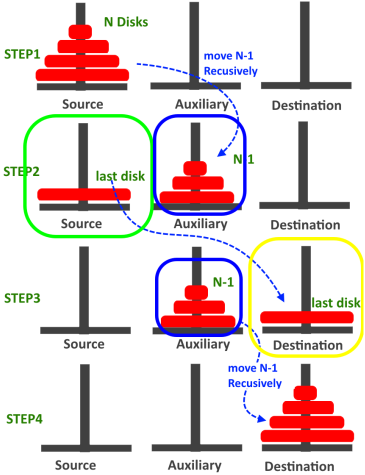

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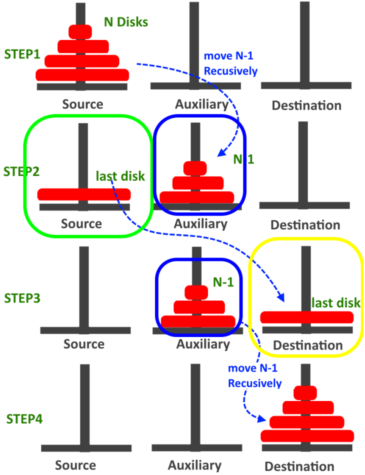

$$T(n) \geq 2T(n-1)+1 \qquad (n \geq 1)$$

$$
\begin{aligned}
T(0) &= 0, \\
T(n) &= 2T(n-1) + 1, \quad n \geq 1
\end{aligned}
$$

$$T(0) = 0,$$
$$T(n) = 2T(n-1) + 1, \quad n \geq 1$$

$$T(n) = 2^n - 1, \quad n \geq 0$$

Theorem (Fermat's Little Theorem)

对于任意自然数 a 与任意素数 p,

$$a^p \equiv a \pmod{p}.$$

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$$(a+1)^p \equiv a + 1 \pmod{p}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \in \mathbb{N} \quad (0 \le k \le n)$$

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$$k = 0 \qquad k = n+1 \qquad 1 \le k \le n$$

Horse Paradox

所有马的颜色都相同。

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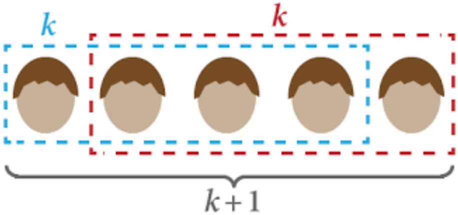

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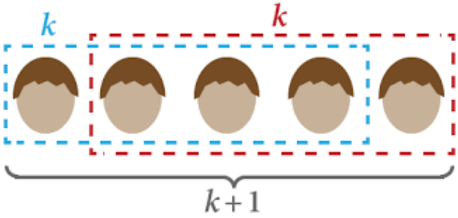

$$n = 1 \not\Rightarrow n = 2$$

$F(n)$ 是偶数, 当且仅当 $F(n+3)$ 是偶数。

设 $*$ 是一个满足结合律的二元运算符。请证明，

$$a_1 * a_2 * \cdots * a_n$$

的值与括号的使用方式无关。

算术基本定理

Prove that every integer greater than 2 can be written as product of primes.

请证明, 只用 4 分与 5 分邮票, 就可以组成 12 分及以上的每种邮资。

(每个不小于 12 的整数都可以写成若干个 4 或 5 的和。)

堆盒子游戏

现有 n 个盒子堆在一起。你可以移动这些盒子，每次移动只能将一堆盒子分成不为空的两堆盒子，最后得到 n 堆盒子，即每堆只有一个盒子时，游戏结束。

每次移动盒子时，如果将高度为 $a+b$ 的盒子堆拆分成高度为 a 和 b 的两堆，玩家可以得 ab 分。

玩家的总得分是每次移动盒子得分的总和。请问，如何才能得到最高分?

+fig

Lemma

任何一种平铺 n 个盒子的方法，得分都是 $\frac{n(n-1)}{2}$。

只用以下三种图示拼出 $2 \times n$ 的形状, 有几种不同的拼法?

$$T(n) = ?T(n-1) + ??T(n-2) + \dots$$

The Blue-eyed Islanders Puzzle

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The Blue-eyed Islanders Puzzle

There is an island upon which a tribe resides. The tribe consists of 1000 people, with various eye colours. Yet, **their religion forbids them to know their own eye color, or even to discuss the topic**; thus, each resident can (and does) see the eye colors of all other residents, but has no way of discovering his or her own (there are no reflective surfaces). If a tribesperson does discover his or her own eye color, then their religion compels them to **commit ritual suicide at noon the following day in the village square for all to witness**.

The Blue-eyed Islanders Puzzle

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Of the 1000 islanders, it turns out that **100 of them have blue eyes and 900 of them have brown eyes**, although the islanders are not initially aware of these statistics (each of them can of course only see 999 of the 1000 tribespeople).

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What effect, if anything, does this *faux pas* have on the tribe?

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(everyone in the tribe can already see that

there are several blue-eyed people in their tribe).

100 days after the address,

all the blue eyed people commit suicide.

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Theorem (The Blue-eyed Islanders Puzzle)

Suppose that the tribe had $n > 0$ blue-eyed people.

Then n days after the traveller's address,

all n blue-eyed people commit suicide.

By induction on the number n of blue-eyed people in the tribe.

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基础步骤: $n = 1$.

归纳假设: 假设命题对 n 个蓝眼人的情况也成立。

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这个唯一的蓝眼人的内心独白: 你直接念我身份证吧

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归纳步骤:

Theorem

对于任何自然数 n, 13^n 都可以写成两个自然数的平方之和。

$$
\begin{aligned}
13^{n+1} &= 13 \cdot 13^n \\
&= (2^2 + 3^2)(a^2 + b^2) \\
&= (\underbrace{2a + 3b}_{x})^2 + (\underbrace{3a - 2b}_{y})^2 \\
&= x^2 + y^2
\end{aligned}
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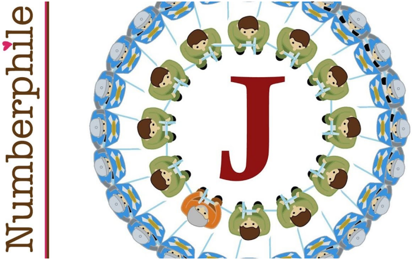
Josephus Problem



$$
\begin{aligned}
f(1,1) &= 2 \\
f(m+1,n) &= f(m,n) + 2(m+n) \\
f(m,n+1) &= f(m,n) + 2(m+n-1)
\end{aligned}
$$

请证明,

$$\forall m,n \in \mathbb{N}^{+}.\ f(m,n) = (m+n)^2 - (m+n) - 2n + 2$$

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Double Induction (Induciton Twice)

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$$f(k,1) \to f(k+1,1)$$

$$f(h,k) \to f(h,k+1) \text{ for any } h$$

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对 $m+n$ 作归纳

gcd

Thank You!

Your opinon

Matters

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