(十五) 离散数学: 复习 (Review)

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$$\vdash \qquad \models$$

Theorem

$$\Sigma \vdash \alpha \iff \Sigma \models \alpha$$

$\rightarrow \qquad \Longrightarrow$

$\leftrightarrow \qquad \Longleftrightarrow$

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"$\rightarrow$" and "$\leftrightarrow$" are used in a single formula.

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$$\begin{aligned}
& x \in A \setminus B \\
\Longleftrightarrow\ & x \in A \land x \notin B \\
\Longleftrightarrow\ & x \in A \land (x \in U \land x \notin B) \\
\Longleftrightarrow\ & x \in A \land x \in \overline{B} \\
\Longleftrightarrow\ & x \in A \cap \overline{B}
\end{aligned}$$

$$
\begin{aligned}
p \oplus q &\triangleq (p \vee q) \wedge \neg(p \wedge q) \\
&= (p \wedge \neg q) \vee (\neg q \wedge q)
\end{aligned}
$$

p	q	$p \oplus q$
0	0	0
0	1	1
1	0	1
1	1	0

$$
\begin{aligned}
p \oplus q &\triangleq (p \lor q) \land \neg(p \land q) \\
&= (p \land \neg q) \lor (\neg q \land q)
\end{aligned}
$$

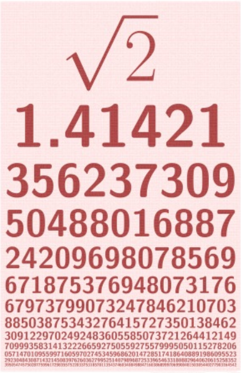
p	q	$p \oplus q$
0	0	0
0	1	1
1	0	1
1	1	0

$$p \oplus q = q \oplus r$$

$$(p \oplus q) \oplus r = p \oplus (q \oplus r)$$

Theorem

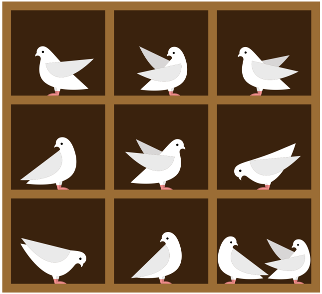
$\sqrt{2}$ is irrational.



The First Crisis in Mathematics

Theorem (Bézout's Identity)

$$(a, b) = d \implies \exists u, v \in \mathbb{Z}.\ au + bv = d$$



Theorem (Pigeonhole Principle)

If n objects are placed in r boxes, where $r < n$, then at least one of the boxes contains ≥ 2 ($\geq \lceil \frac{n}{r} \rceil$) object.

Numbers

Consider the numbers $1, 2, \ldots, 2n$, and take any $n+1$ of them.
There are two among these $n+1$ numbers which are relatively prime.

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There must be two numbers which are only 1 apart.

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$$a = 2^k m, \quad (1 \leq m \leq 2n-1 \text{ is odd})$$

There $n+1$ numbers have only n different odd parts.

There must be two numbers with the same odd part.

Hand-shaking

If there are $n > 1$ people who can shake hands with one another, there are two people who shake hands with the same number of people.

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Either the '0' hole or the 'n − 1' hole or both must be empty.

Sums

Suppose we are given n integers $a_1, a_2, \ldots, a_n$.
Then there is a set of consecutive numbers $a_{k+1}, a_{k+2}, \ldots, a_l$ whose sum $\sum_{i=k+1}^{l} a_i$ is a multiple of n.

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$$A_j - A_i = a_{i+1} + \cdots + a_j = 0 \mod n$$

Championship Match

“胡司令” (胡荣华) 要安排一次长达 77 天的象棋练习赛。

他想每天至少要有一场比赛, 但是总共不超过 132 场比赛。

请证明, 无论如何安排, 他都要在连续的若干天内恰好完成 21 场比赛。

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There must be ≥ 2 elements having the same value.

It must be $a_i + 21 = a_j$.

Sequences

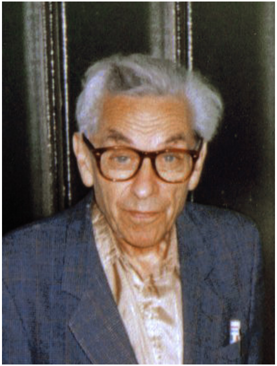
In any sequence $a_1, a_2, \dots, a_{mn+1}$ of $mn+1$ distinct numbers, there exists an increasing subsequence

$$a_{i_1} < a_{i_2} < \cdots < a_{i_{m+1}} \quad (i_1 < i_2 < \cdots < i_{m+1})$$

of length $m+1$, or a decreasing subsequence

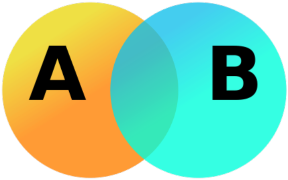
$$a_{j_1} > a_{j_2} > \cdots > a_{j_{n+1}} \quad (j_1 > i_2 < \cdots > j_{n+1})$$

of length $n+1$, or both.

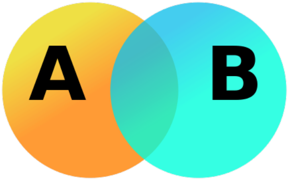


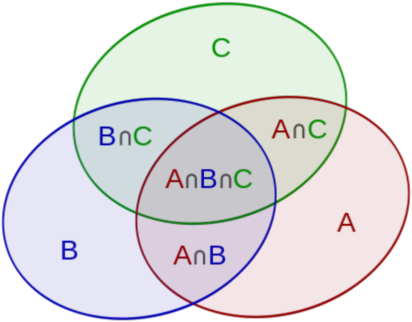
Paul Erdős (1913 $\sim$ 1996)

Chapter 28 of "Proofs from THE Book"



$$|A \cup B| = |A| + |B| - |A \cap B|$$





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$$\begin{aligned} |A \cup B \cup C| = &|A| + |B| + |C| \\ &- |A \cap B| - |A \cap C| - |B \cap C| \\ &+ |A \cap B \cap C| \end{aligned}$$

Theorem (Inclusion-Exclusion Principle)

$$\begin{aligned}\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - &\sum_{1 \leqslant i < j \leqslant n} |A_i \cap A_j| \\ &+ \sum_{1 \leqslant i < j < k \leqslant n} |A_i \cap A_j \cap A_k| \\ &- \cdots \\ &+ (-1)^{n-1} |A_1 \cap \cdots \cap A_n| .\end{aligned}$$

Theorem (Inclusion-Exclusion Principle)

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$$\begin{aligned}\left|\bigcap_{i=1}^{n} \bar{A}_i\right| = \left|S - \bigcup_{i=1}^{n} A_i\right| = |S| - \sum_{i=1}^{n} |A_i| + \sum_{1\leqslant i<j\leqslant n} |A_i \cap A_j| \\ - \cdots + (-1)^n |A_1 \cap \cdots \cap A_n|.\end{aligned}$$

Counting Integers

How many integers in $1, \ldots, 100$ are not divisible by 2, 3 or 5?

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$$100 - (50 + 33 + 20) + (16 + 10 + 6) - 3 = 26.$$

Counting Derangements (错排)

Suppose there is a deck of n cards numbered from 1 to n.
Suppose a card numbered i is in the correct position if it is the i-th card in the deck. How many ways can the cards be shuffled without any cards being in the correct position?

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$$S_k \triangleq \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| =$$

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$$S_k = \frac{n!}{k!}$$

$$\begin{aligned}\left|\bigcap_{i=1}^{n} \overline{A_i}\right| &= n! - \frac{n!}{1!} + \frac{n!}{2!} - \cdots + (-1)^n \frac{n!}{n!} \\ &= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}\end{aligned}$$

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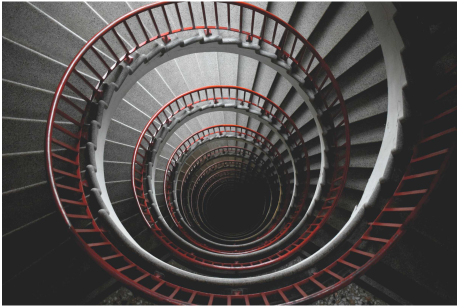
$$n \to \infty \implies \sum_{k=0}^{n} \frac{(-1)^k}{k!} \to e^{-1} \approx 0.368$$



$$\boxed{a_n = f(a_{n-1}, a_{n-2}, \ldots, a_{n-t}) + g(n)}$$

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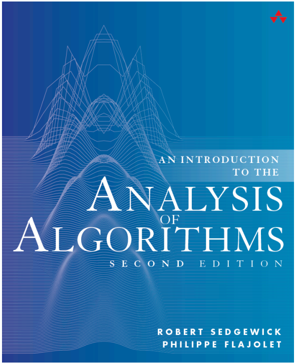

recurrence type	typical example
first-order	
linear	$a_n = na_{n-1} - 1$
nonlinear	$a_n = 1/(1 + a_{n-1})$
second-order	
linear	$a_n = a_{n-1} + 2a_{n-2}$
nonlinear	$a_n = a_{n-1}a_{n-2} + \sqrt{a_{n-2}}$
variable coefficients	$a_n = na_{n-1} + (n-1)a_{n-2} + 1$
tth order	$a_n = f(a_{n-1}, a_{n-2}, \ldots, a_{n-t})$
full-history	$a_n = n + a_{n-1} + a_{n-2} \ldots + a_1$
divide-and-conquer	$a_n = a_{\lfloor n/2 \rfloor} + a_{\lceil n/2 \rceil} + n$

Table 2.1 Classification of recurrences

Homogeneous Linear Recurrence Relations with Constant Coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_t a_{n-t}$$

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2020 年课程录屏 (Recurrence)

$$R \subseteq A \times A$$

$$\begin{cases} R^0 = I_A \\ R^{n+1} = R \circ R^n \end{cases}$$

Representing Relations as Matrices/Digraphs

$$A = \{1, 2, 3, 4\}$$

$$R = \{(1,1), (1,2), (2,1), (2,2), (2,3), (2,4), (3,4), (4,1)\}$$

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$$R^2 \qquad R^3$$

$$R+ = \bigcup_{i=1}^{\infty} R \qquad R^* = \bigcup_{i=0}^{\infty} R$$

Definition (Reflexive Closure (自反闭包))

The reflexive closure $\mathsf{cl}_{\text{ref}}(R)$ of a relation $R \subseteq X \times X$ is the smallest reflexive relation on X that contains R.

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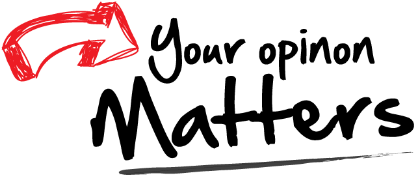
If T is any transitive relation containing R, then $R^+ \subset T$.

By induction on i, we can show that $R^i \subseteq T$.


Thank
You!



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