(十二) 图论: 匹配与网络流

(Matching and Network Flow)

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3 Theorems + 1 Algorithm

To maximize the size of a mathematical structure $\mathcal{S}$ in G

Theorem

$\mathcal{S}$ is maximum iff G does not contain $\mathcal{S}$-augmenting objects.

Algorithm

Repeatedly finding $\mathcal{S}$-augmenting objects until no more ones exist.

To maximize the size of a mathematical structure $\mathcal{S}$ in G

To minimize the size of its dual mathematical structure $\mathcal{S}'$ in G

Theorem (Weak Duality Theorem)

The size of a maximum $\mathcal{S}$ $\leq$ *The size of a minimum* $\mathcal{S}'$

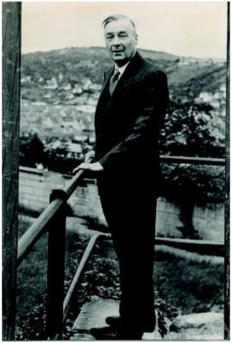
Theorem (Strong Duality Theorem)

The size of a maximum $\mathcal{S}$ $=$ *The size of a minimum* $\mathcal{S}'$

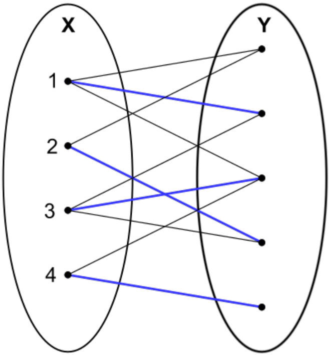

let's get
married
today

The Marriage Problem (Philip Hall, 1935)

If there is a finite set of girls, each of whom knows several boys, under what conditions can all the girls marry boys in such a way that each girl marries a boy that she knows?

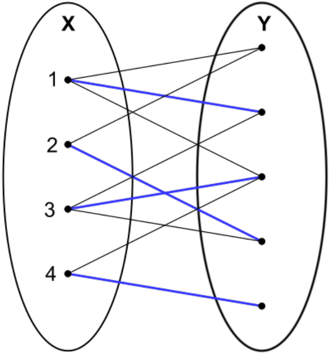


Philip Hall (1904 $\sim$ 1982)



Definition (Matching (匹配))

A matching in a graph G is a set of edges with no shared endpoints.

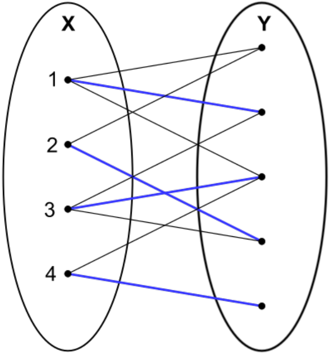


Definition (X-Perfect Matching (X-Saturating Matching))

Let $G = (X, Y, E)$ be a bipartite graph.

An X-perfect matching of G is a matching which covers each vertex in X.

$$|X| \le |Y|$$



$$\forall W \subseteq X.\ |W| \le |N(W)|$$

Theorem (Hall Theorem; 1935)

Let $G = (X, Y, E)$ *be a bipartite graph. There is an* X*-perfect matching of* G *iff*

$$\forall W \subseteq X. \; \left|W\right| \leq \left|N_G(W)\right|$$

By induction on the number $\left|X\right|$ of vertices in X.

Basis Step: $\left|X\right| = 1$. $\left|X\right| \le \left|N_G(X)\right|$. I am married!

Induction Hypothesis: Suppose that it holds if $\left|X\right| < m$.

Induction Step: Consider the case $\left|X\right| = m$.

$$\text{Consider the case } \left|X\right| = m.$$

- Case I: Every $k < m$ girls in X know $\geq k+1$ boys in Y.

Take any girl x and marry her to any boy y she knows.

$$G' = G - \{x, y\}$$

The Hall's Condition still holds for G'.

$$\forall W \subseteq X - \{x\}.\ \left|W\right| \leq \left|N_{G'}(W)\right|$$

There is a $(X - \{x\})$-perfer matching in G'.

Therefore, there is a $(X - \{x\})$-perfer matching in G.

- CASE II: There is a set of $k < m$ girls in X who know k boys in Y.

There k girls can be married by induction to the k boys.

$$G' = G - \{\text{these } k \text{ girls}\} - \{\text{these } k \text{ boys}\}$$

G' satisfies the Hall's Condition.

By contradition:

Suppose that in G', there are $l \leq m - k$ girls who know $< l$ boys.

Then, in G, there are $k + l$ girls who know $< k + l$ boys.

There is a $(X - \{\text{these } k \text{ girls}\})$-perfer matching in G'.

Therefore, there is a $(X - \{x\})$-perfer matching in G.

Theorem (Hall Theorem; 1935)

Let $G = (X, Y, E)$ be a bipartite graph. There is a X-perfect matching of G iff

$$\forall W \subseteq X. \left|W\right| \le \left|N_G(W)\right|$$

Definition (M-alternating Paths)

Let M be a matching. An M-alternating path is a path that alternates between edges in M and edges not in M.

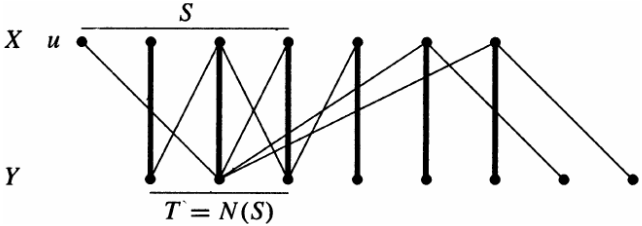
Definition (M-augmenting Paths)

An M-augmenting path is an M-alternating path whose endpoints are unsaturated by M.

By contradiction.

Suppose that there is *no* X-perfect matching.

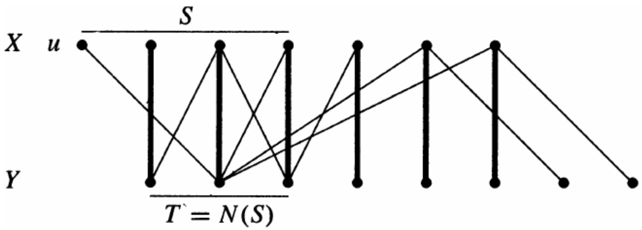
We show that Hall's Condition is violated for some $S \subseteq X$.



Let M be a *maximum* matching.

Let $u \in X$ be a vertex of X not saturated by M.

Consider all *M-alternating paths* starting from u.



$T \triangleq$ the set of vertices in Y reachable from u by M-alternating paths.

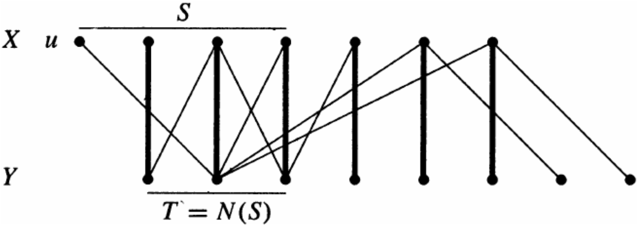
$S \triangleq$ the set of vertices in X reachable from u by M-alternating paths.

We will show that

$$T = N(S) \land \left|T\right| = \left|S - \{u\}\right|$$

$$\left|N(S)\right| = \left|T\right| = \left|S\right| - 1 < \left|S\right|$$

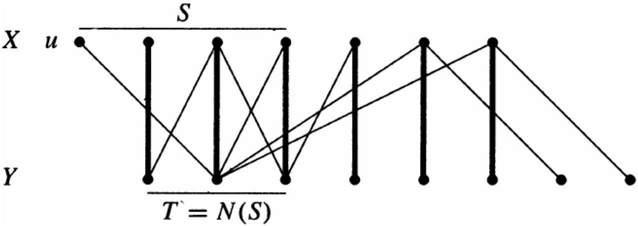
$$\left|T\right| = \left|S - \{u\}\right|$$



We show that there is a bijection from T to $S - \{u\}$.

M matches T with $S - \{u\}$.

$$T = N(S)$$



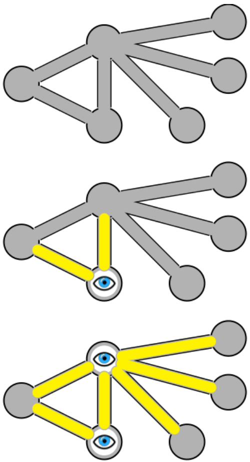
$$\left|T\right| = \left|S - \{u\}\right| \implies T \subseteq N(S)$$

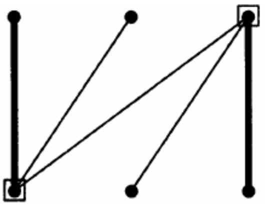
We need to show that $N(S) \subseteq T$

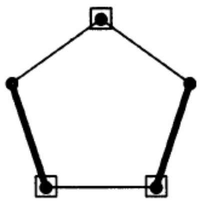
Consider the neighbors of $x \in S$: $x = u, \quad x \in S - \{u\}$

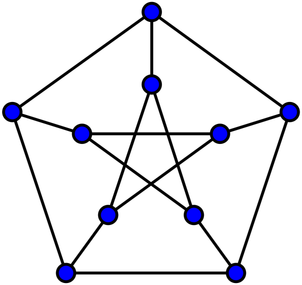
Definitions (Vertex Cover (点覆盖))

A vertex cover of a graph G is a set $Q \subseteq V(G)$ that covers all edges.



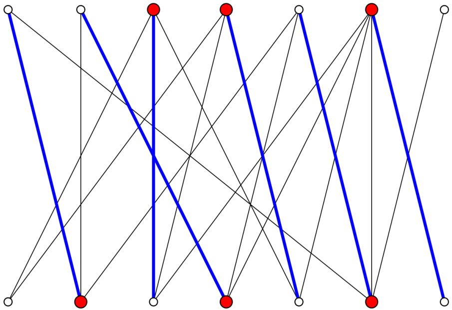






Theorem (Weak Duality Theorem (弱对偶定理))

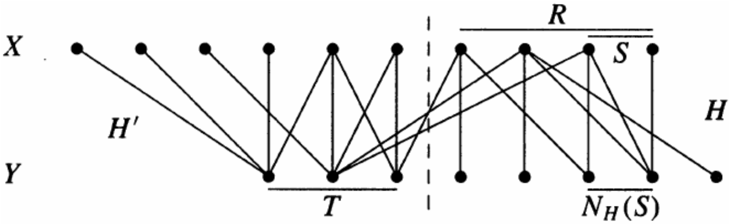
Let G be a graph. The maximum size of a mathching in G $\leq$ the minimum size of a vertex cover of G.



Theorem (König (1931), Egerváry (1931))

Let G be a bipartite graph. The maximum size of a mathching in G equals the minimum size of a vertex cover of G

Given a vertex cover Q, we construct a matching M of the same size.



$$R = Q \cap X \qquad T = Q \cap Y$$

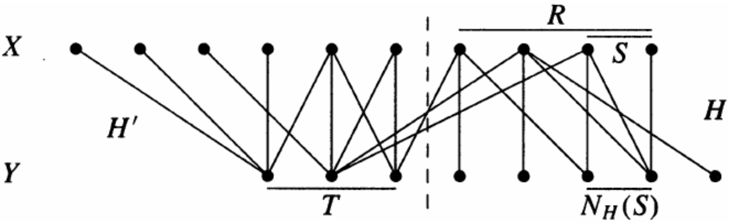
$$H \triangleq (R \cup (Y - T))\text{-induced subgraph of } G$$

$$H' \triangleq (T \cup (X - R))\text{-induced subgraph of } G$$

G has no edges from $X - R$ to $Y - T$.

H has a R-perfect matching and H' has a T-perfect matching.

$\boxed{H \text{ has a } R\text{-perfect matching and } H' \text{ has a } T\text{-perfect matching.}}$



By contradiction.

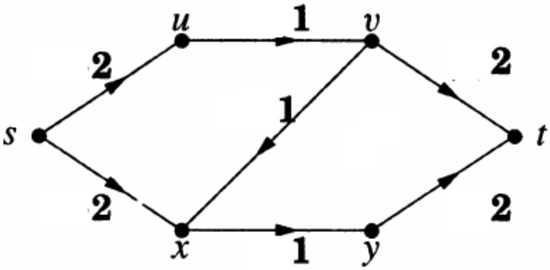
$$\exists S \subseteq R.\ \left|N_H(S)\right| < \left|S\right|$$

$T \cup (R - S + N_H(S))$ is a smaller vertex cover than Q

Definition (Network (网络))

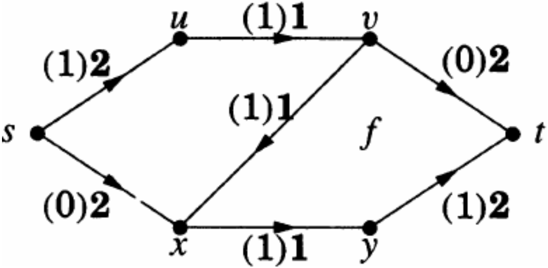
A network is a digraph with

- a distinguished source vertex s,
- a distinguished sink vertex t,
- a capacity $c(e) \geq 0$ on each edge e



Definition (Flow (流))

A flow f is a function that assigns a value $f(e)$ to each edge e.



Definition (Feasible Flow (可行流))

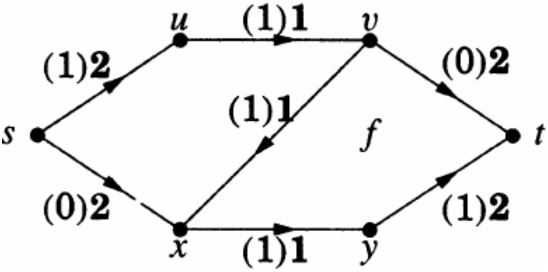
A flow f is feasible if it satisfies

Capacity Constraints:

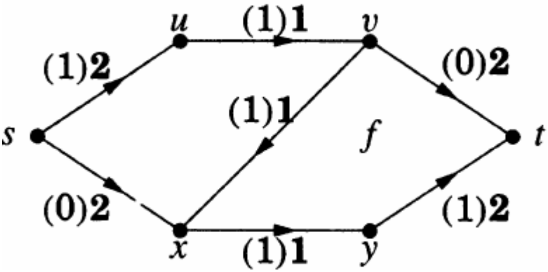
$$\forall e \in E.\ 0 \le f(e) \le c(e)$$

Flow Conservation:

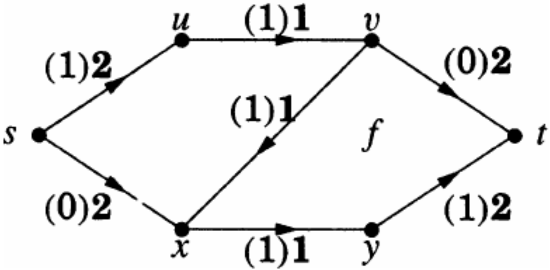
$$\forall v \in V.\ f^{+}(v) = f^{-}(v)$$



$$f^+(v) = \sum_{(v,w)\,\in\, E} f(v,w) \qquad f^-(v) = \sum_{(u,v)\,\in\, E} f(u,v)$$



$$f^+(U) = \sum_{u\in U, v\in \overline{U}, (u,v)\in E} f(u,v) \qquad f^-(U) = \sum_{v\in \overline{U}, u\in U, (v,u)\in E} f(v,u)$$



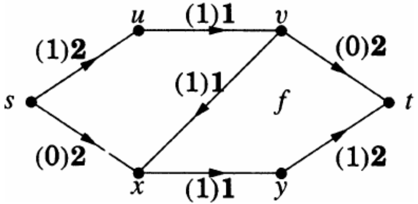
$$\forall U \subseteq (V - \{s,t\}).\ f^+(U) = f^-(U)$$

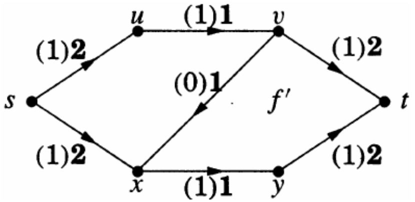
$$s \in U \wedge t \notin U \implies f^+(U) - f^-(U) = f^+(s)$$

Definition (Value (值))

The value $\mathsf{val}(f)$ of a flow f is

$$\mathsf{val}(f) = f^{-}(t) = f^{+}(s).$$

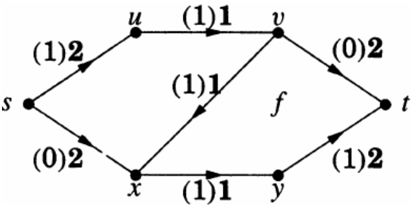


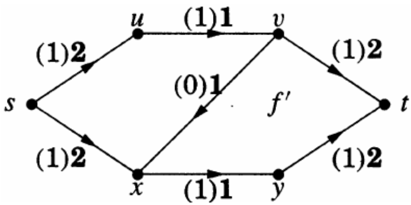


Definition (Maximum Flow (最大流))

A maximum flow is a feasible flow of maximum value.

$$s - x - v - t$$





Definition (f-augmenting Paths (增广路径))

When f is a feasible flow, an f-augmenting path is a $s \sim t$ path P in the underlying graph such that for each edge $e \in E(P)$,

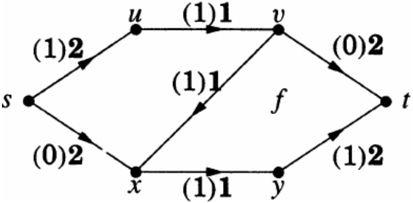
(a) if P follows e in the forward direction, then $f(e) < c(e)$;

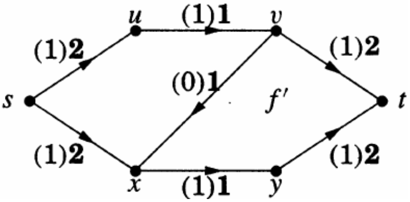
(b) if P follows e in the backward direction, then $f(e) > 0$.

Definition (f-augmenting Paths)

Let P be an f-augmenting path.

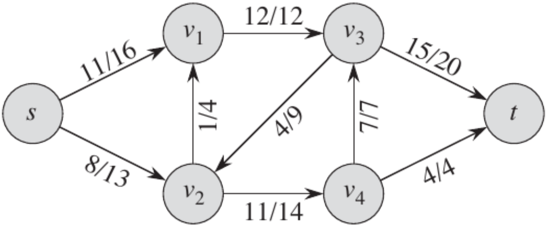
$$\epsilon(e) = \begin{cases} c(e) - f(e) & \text{if } e \text{ is forward on } P \\ f(e) & \text{if } e \text{ is backward on } P \end{cases}$$





An f-augmenting path leads to a flow with larger value.

$$\min_{e \in E(P)} \epsilon(e)$$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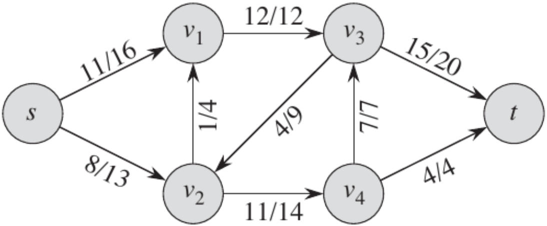
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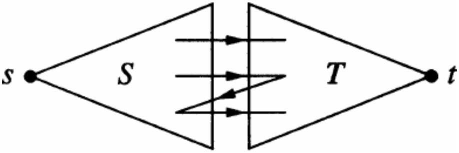
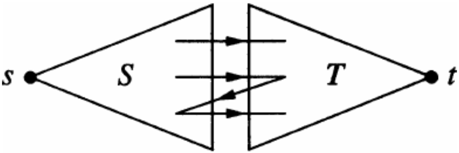
Definition (Source/Sink Cut (割))

In a network, a source/sink cut $[S, T]$ consists of the edges from a source set S to a sink set T, where

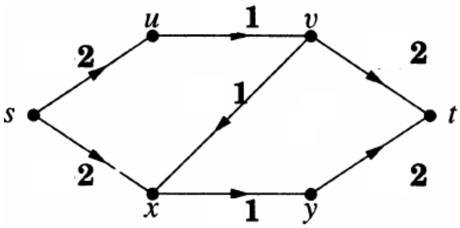
$$(T = V - S) \wedge (s \in S) \wedge (t \in T)$$





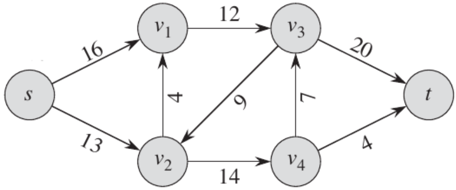
Definition (Capacity of Cut (割的容量))

$$\mathsf{cap}(S,T) = \sum_{u \in S, v \in T, uv \in E} c(u,v)$$



Definition (Minimum Cut (最小割))

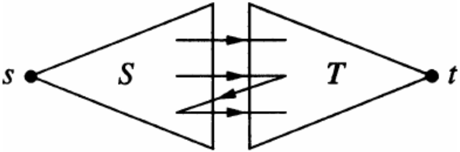
A minimum cut is a cut of minimum value.



Theorem (Weak Duality (弱对偶定理))

Let f be any feasible flow and $[S,T]$ be any source/sink cut.

$$\mathsf{val}(f) \le \mathsf{cap}(S,T).$$



$$\mathrm{val}(f) = f^+(S) - f^-(S) \le f^+(S) \le \mathrm{cap}(S,T)$$

Lemma

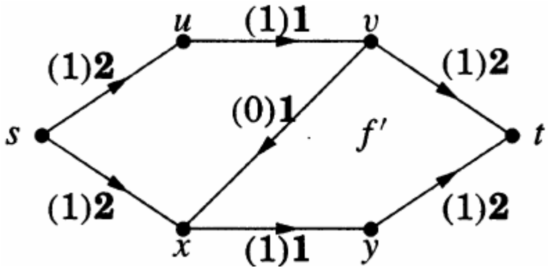
$$\max_{f} \mathit{val}(f) \le \min_{[S,T]} \mathit{cap}(S,T)$$

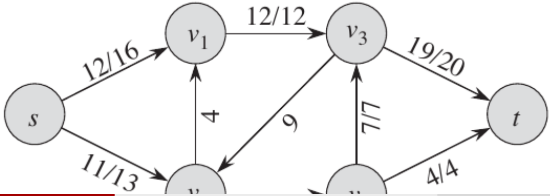
What if $\mathsf{val}(f) = \mathsf{cap}(S,T)$ for some flow f and some cut $[S,T]$?

f is maximum and $[S,T]$ is minimum

$$\mathsf{val}(f) = f^+(S) - f^-(S) = f^+(S) = \mathsf{cap}(S,T)$$

$$\boxed{f^{-1}(S) = 0 \land f^+(S) = \mathsf{cap}(S,T)}$$





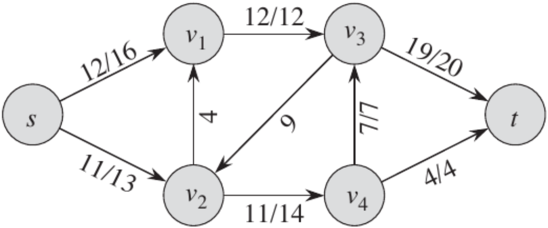
Theorem (Max-flow Min-cut Theorem (Ford and Fulkerson; 1956))

$$\max_{f} \mathit{val}(f) = \min_{[S,T]} \mathit{cap}(S,T)$$

(Strong Duality)

L. R. Ford Jr. (1927 ∼ 2017)

D. R. Fulkerson (1924 ∼ 1976)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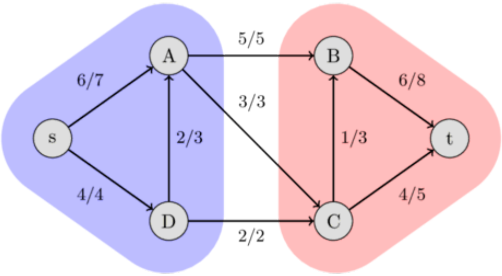
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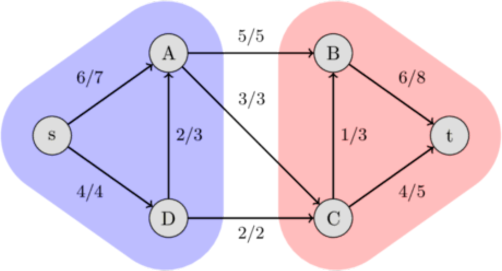
Theorem

A feasible flow f is maximum iff there are no f-augmenting paths.

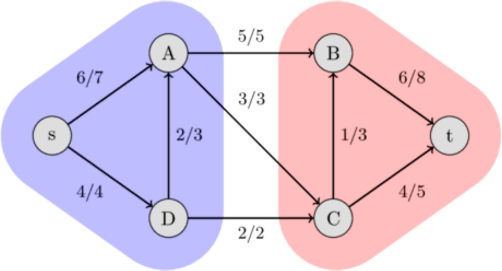
We construct a cut $[S, T]$ with $\mathsf{val}(f) = \mathsf{cap}(S, T)$.



$S \triangleq \{$the vertices reachable from s along partial f-augmenting paths$\}$

$$S \triangleq \{\text{the vertices reachable from } s \text{ along } \textcolor{red}{\text{partial}}\ f\text{-augmenting paths}\}$$

$$T \triangleq V - S$$

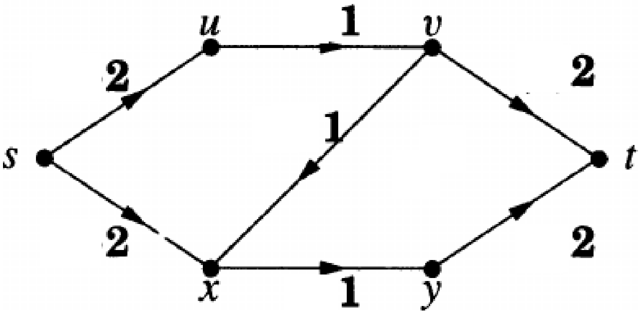


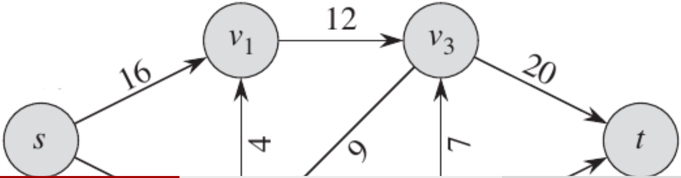
$$f^{-1}(S) = 0 \land f^{+}(S) = \mathsf{cap}(S, T)$$

$$\mathsf{val}(f) = f^{+}(S) - f^{-}(S) \textcolor{red}{=} f^{+}(S) \textcolor{red}{=} \mathsf{cap}(S, T)$$

The Ford-Fulkerson Method

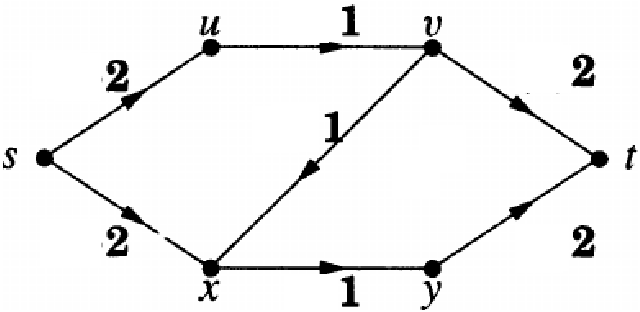
Repeatedly finding f-augmenting paths until no more ones exist.

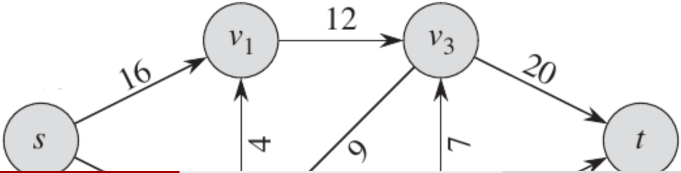




The Edmonds-Karp Algorithm

Using BFS (Breadth-first Search) to find f-augmenting paths.

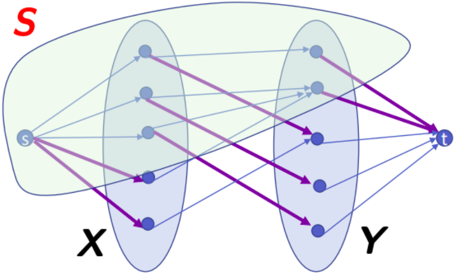




Theorem (Hall Theorem; 1935)

There is an X-perfect matching of G iff

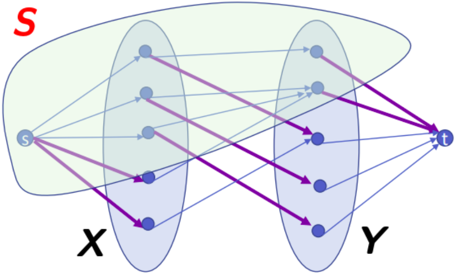
$$\forall W \subseteq X.\ \left|W\right| \le \left|N_G(W)\right|$$



$$\forall x \in X.\ c(s,x) = 1 \quad \forall y \in Y.\ c(y,t) = 1 \quad \forall x \in X, y \in Y.\ c(x,y) = \infty$$

We need to show that $\max\limits_{f} \mathsf{val}(f) = \left|X\right|$.

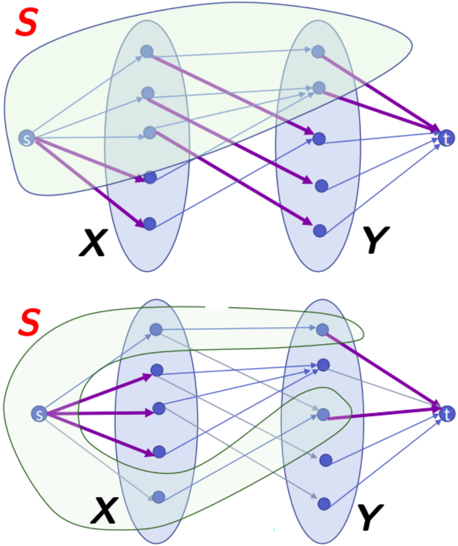
We need to show that $\min\limits_{[S,\overline{S}]} \mathsf{cap}(S,\overline{S}) = \left|X\right|$.

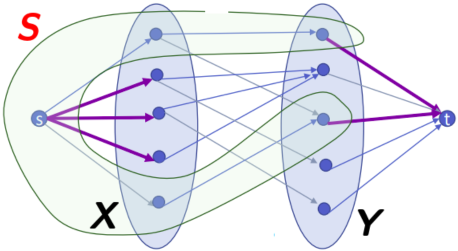


$$\min_{[S,\overline{S}]} \mathsf{cap}(S,\overline{S}) \le \left|X\right|$$

Therefore, we need to show that $\min\limits_{[S,\overline{S}]} \mathsf{cap}(S,\overline{S}) \ge \left|X\right|$.

Let $[S, \overline{S}]$ be a minimum cut. We need to show that $\mathsf{cap}(S, \overline{S}) = \left|X\right|$.



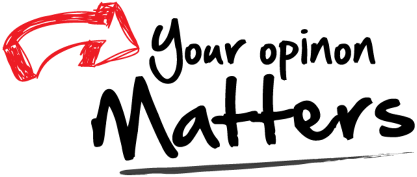


$$
\begin{aligned}
\mathsf{cap}(S, \overline{S}) &= \sum_{u \in S, v \in \overline{S}} c(x, y) \\
&= \sum_{v \in \overline{S} \cap X} c({\color{red}s}, v) + \sum_{u \in S \cap Y} c(u, {\color{red}t}) \\
&= \left|X\right| - \left|S \cap X\right| + \left|S \cap Y\right| \\
&{\color{red}\geq} \left|X\right| - \left|S \cap X\right| + {\color{red}\left|N(S \cap X)\right|} \\
&{\color{blue}\geq} \left|X\right|
\end{aligned}
$$

Theorem (König (1931), Egerváry (1931))

If G is a bipartite graph, then the maximum size of a mathching in G equals the minimum size of a vertex cover of G


Thank You!



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