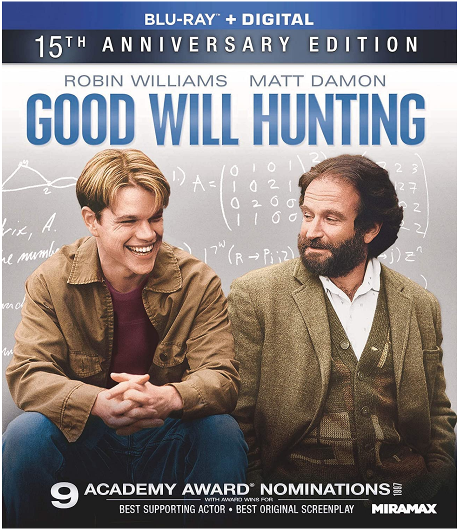
(十) 图论: 树 (Trees)

魏恒峰

hfwei@nju.edu.cn

2021 年 05 月 13 日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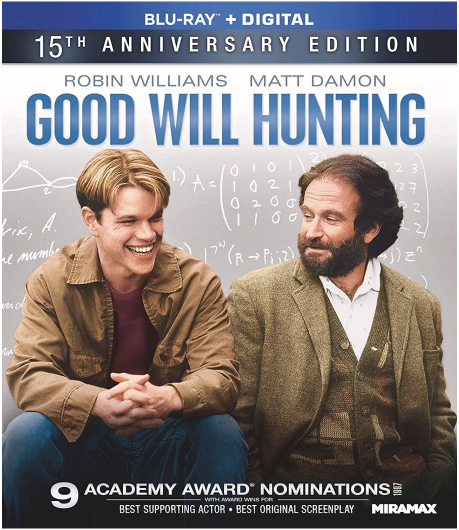
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你, 真得, 看懂了吗?

Definition (Tree (树))

A tree is a connected acyclic undirected graph.

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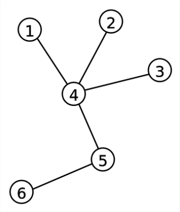
Definition (Forest (森林))

A forest is a acyclic undirected graph.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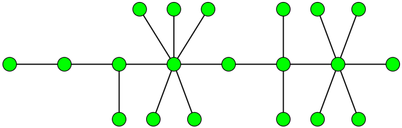

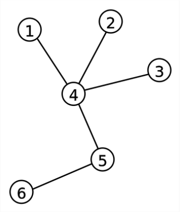
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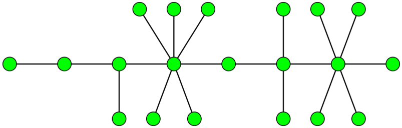

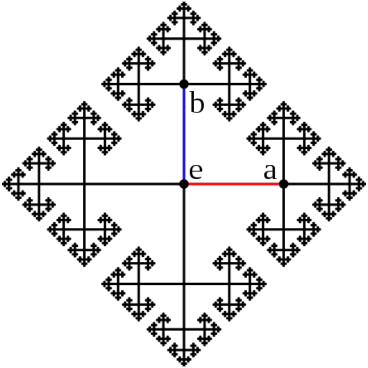
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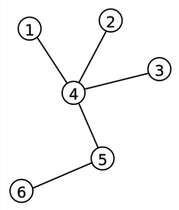
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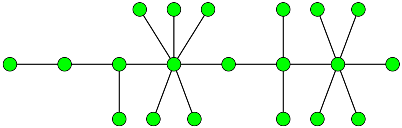

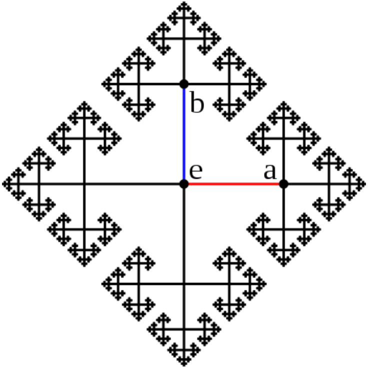
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Definition (Internal Vertex (内部顶点); Leaf (叶子))

In a tree T with ≥ 2 vertices, for a vertex v in T, if

$$\deg(v) = 1$$

then v is called a leaf; otherwise, v is an internal vertex.

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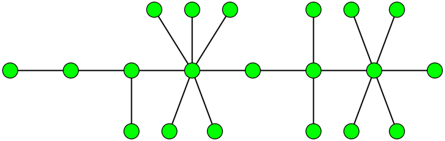

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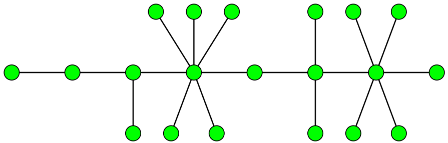

Lemma

Any tree T with ≥ 2 vertices contains ≥ 1 leaf.

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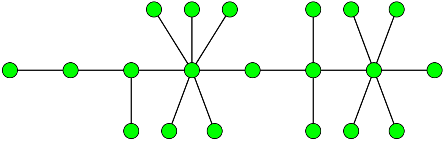

Lemma

Any tree T with ≥ 2 vertices contains ≥ 1 leaf.

$$\text{Otherwise, } \forall v \in V.\ \deg(v) \geq 2 \implies T \text{ has cycles.}$$

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Consider the two endpoints of any maximal (nontrivial) path in T.

They are leaves of T.

Lemma

Deleting a leaf from a tree T with n vertices produces a tree with $n-1$ vertices.

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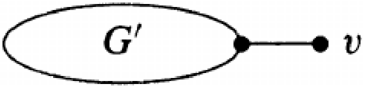

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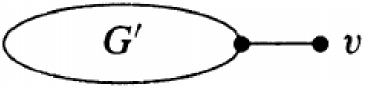

$G' = G - v$ is connected and acyclic.

A leaf does *not* belong to any paths connecting two other vertices.

Definition (Irreducible Tree)

An irreducible tree is a tree T where

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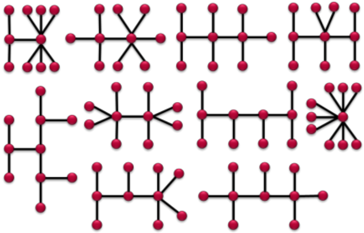

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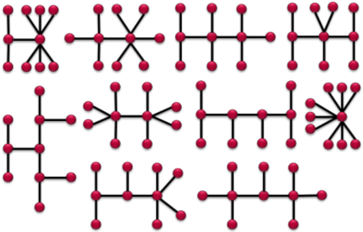

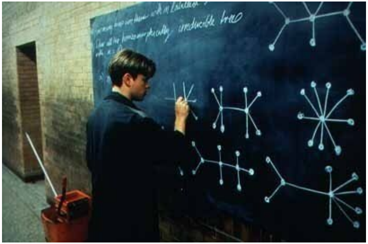

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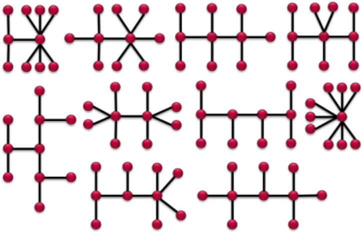

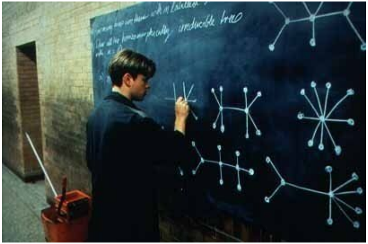

Homeomorphically Irreducible Trees of size $n = 10$

Theorem ((We call it) Tree Theorem)

Let T be an undirected graph with n vertices.

Then the following statements are equivalent:

(1) *T is a tree;*

(2) *T is acyclic, and has $n-1$ edges;*

(3) *T is connected, and has $n-1$ edges;*

(4) *T is connected, and each edge is a bridge;*

(5) *Any two vertices of T are connected by exactly one path;*

(6) *T is acyclic, but the addition of any edge creates exactly one cycle.*


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Definition (Spanning Tree (生成树))

A spanning tree T of an undirected graph G is a subgraph that is a tree with all vertices of G.

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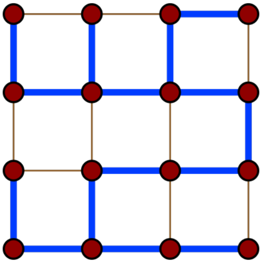

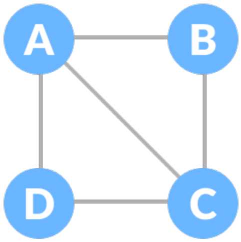

Definition (Subgraph (子图))

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Definition (Induced Subgraph (诱导子图))

Theorem

Every connected undirected graph G admits a spanning tree.

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Every connected undirected graph G admits a spanning tree.

Repeatedly deleting vertices in cycles until the graph is acyclic.

Definition (Minimum Spanning Tree (MST; 最小生成树))

A minimum spanning tree T of an edge-weighted undirected graph G is a spanning tree with minimum total weight of edges.

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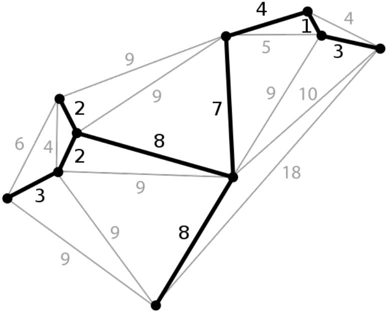

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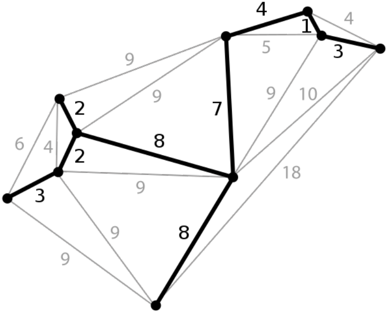

Existence? Uniqueness? Algorithms?

Theorem

Every weighted connected undirected graph G admits a minimum spanning tree.

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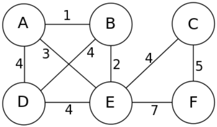
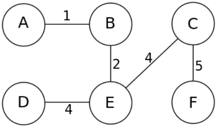
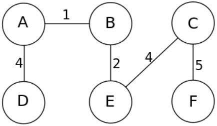
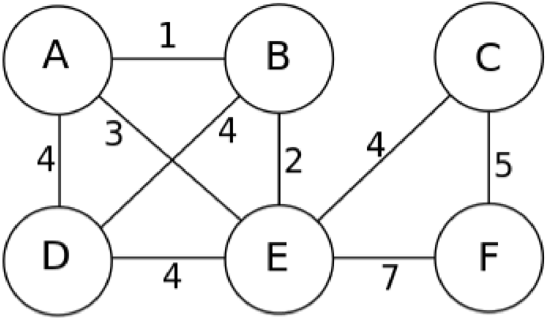






Joseph Kruskal (1928 ~ 2010)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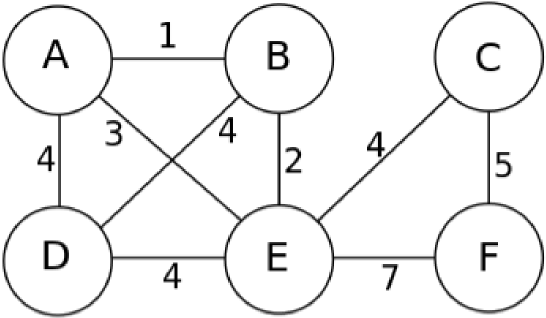
Robert C. Prim (1921 ∼)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Cut Property

Cut Property (Version I)

X : A part of some MST T_1 of G

$(S, V \setminus S)$: A *cut* such that X does *not* cross $(S, V \setminus S)$

e : A lightest edge across $(S, V \setminus S)$

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Then $X \cup \{e\}$ is a part of some MST T_2 of G.

Cut Property (Version I)

$$
\begin{aligned}
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e &: \text{ A lightest edge across } (S, V \setminus S)
\end{aligned}
$$

Then $X \cup \{e\}$ is a part of some MST T_2 of G.

Correctness of Prim's and Kruskal's algorithms.

By Exchange Argument.

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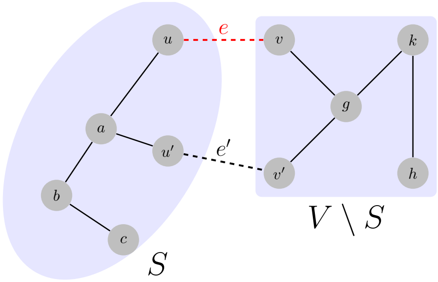

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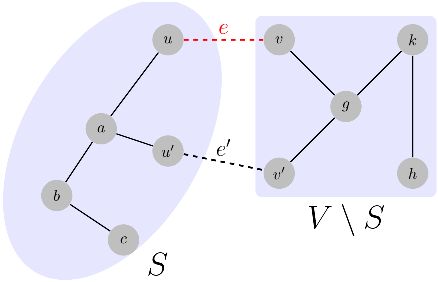

$$T' = \underbrace{\underbrace{T}_{X \subseteq T} + \{e\}}_{\text{if } e \notin T} - \{e'\}$$

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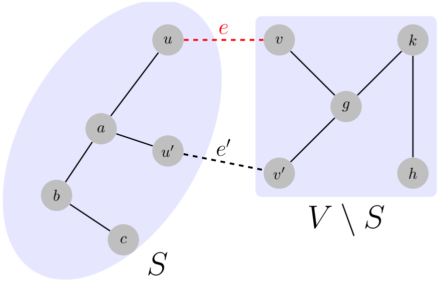

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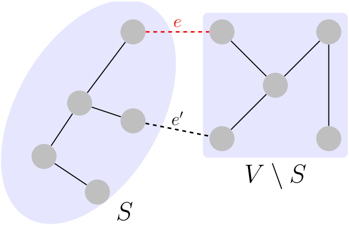
"a" $\rightarrow$ "the" $\Longrightarrow$ "some" $\rightarrow$ "all"

Cut Property (Version II)

A cut $(S, V \setminus S)$

Let $e = (u, v)$ be a lightest edge across $(S, V \setminus S)$

$\exists$ MST T of $G : e \in T$



Cut Property (Version II)

$$\text{A cut } (S, V \setminus S)$$

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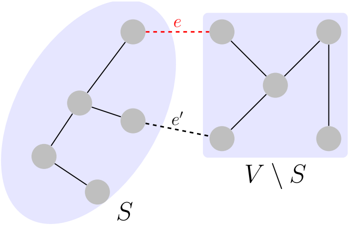

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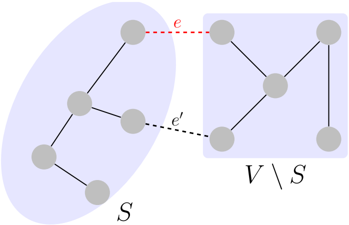

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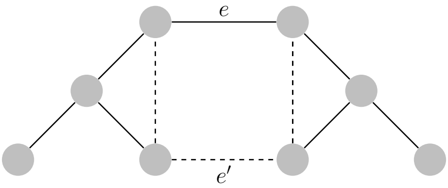
"a" $\to$ "the" $\implies$ "$\exists$" $\to$ "$\forall$"

Cycle Property

Cycle Property

- Let C be any cycle in G
- Let $e = (u, v)$ be *a* maximum-weight edge in C

Then $\exists$ MST T of $G : e \notin T$.



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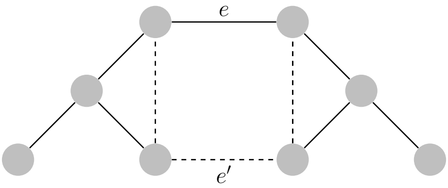

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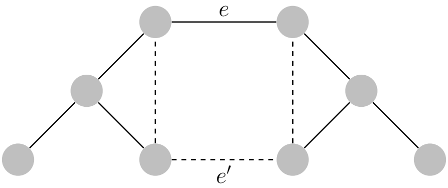

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Joseph Kruskal (1928 $\sim$ 2010)

Anti-Kruskal Algorithm

Reverse-delete algorithm (wiki; clickable)

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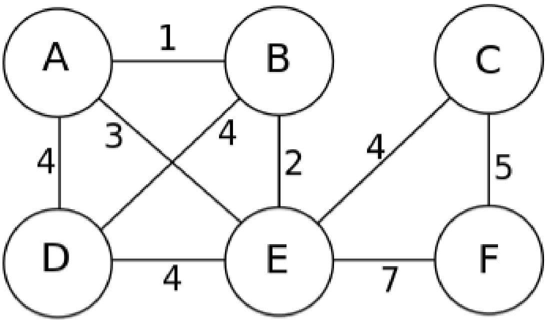
"On the Shortest Spanning Subtree of a Graph and the Traveling Salesman Problem"

— Kruskal, 1956.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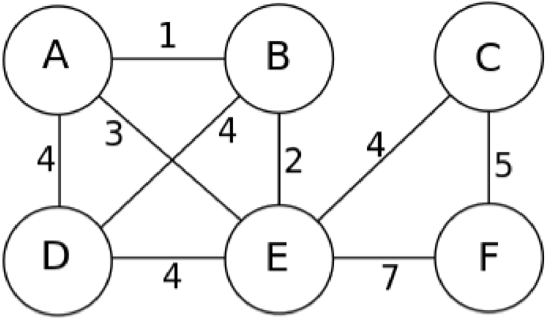
Otakar Borůvka ($1899 \sim 1995$)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Theorem (Uniqueness of MST)

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If each edge has a distinct weight, then there is a unique MST of G.

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$$\Delta E = \{e \mid e \in T_1 \setminus T_2 \vee e \in T_2 \setminus T_1\}$$

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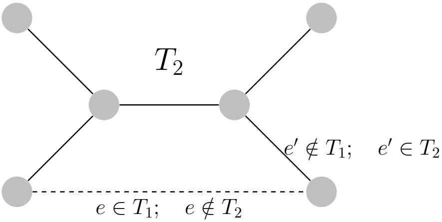
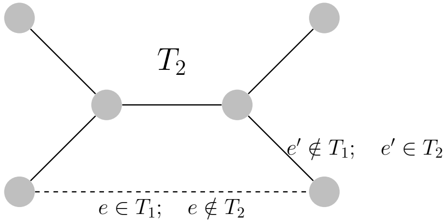
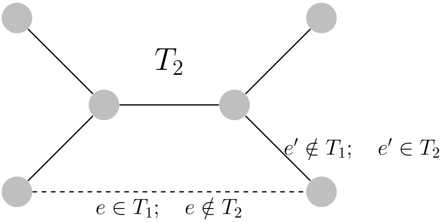
$$e \in T_1 \setminus T_2 \ \textit{(w.l.o.g)}$$



T_2
$e' \notin T_1; \quad e' \in T_2$
$e \in T_1; \quad e \notin T_2$

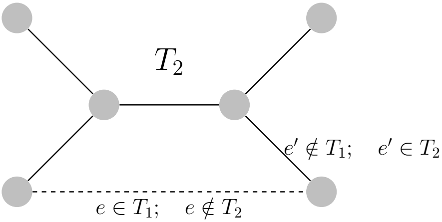


$$T_2 + \{e\} \implies C$$



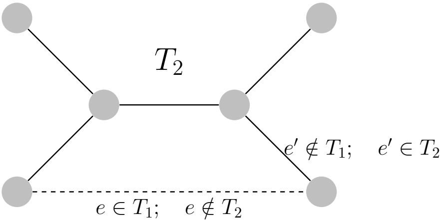
$$T_2 + \{e\} \implies C$$

$\exists (e' \in C) \notin T_1$



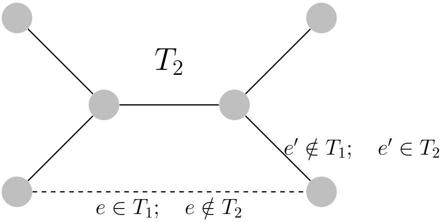
$$T_2 + \{e\} \implies C$$

$$\exists (e' \in C) \notin T_1 \implies e' \in T_2 \setminus T_1 \implies e' \in \Delta E$$



$$T_2 + \{e\} \implies C$$

$$\exists (e' \in C) \notin T_1 \implies e' \in T_2 \setminus T_1 \implies e' \in \Delta E \implies w(e') > w(e)$$



$$T_2 + \{e\} \implies C$$

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$$T' = T_2 + \{e\} - \{e'\} \implies w(T') < w(T_2)$$

Condition for Uniqueness of MST

$$\text{Unique MST} \not\Longrightarrow \text{Distinct weights}$$

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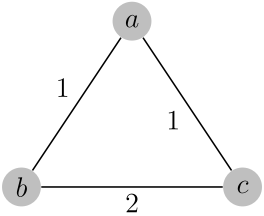
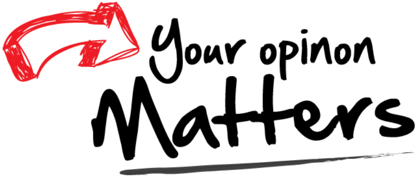



Thank
You!



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