(三) 数学归纳法

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Theorem (第一数学归纳法 (The First Mathematical Induction))

令 $P(n)$ 表示关于自然数 n 的某个性质。如果

(i) $P(0)$ 成立;

(ii) 对任意自然数 n, 如果 $P(n)$ 成立, 则 $P(n+1)$ 成立。

那么, $P(n)$ 对所有自然数 n 都成立。

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Q : 第二数学归纳法也被称为**“强” (Strong)** 数学归纳法, 它强在何处?

Lemma

第二数学归纳法蕴含第一数学归纳法。

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数学归纳法为何成立？

Peano 公理体系刻画了自然数的递归结构

Definition

Peano Axioms

(1)

Definition (良序原理 (The Well-Ordering Principle))

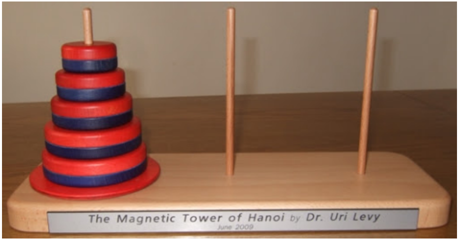
自然数集的任意非空子集都有一个最小元。

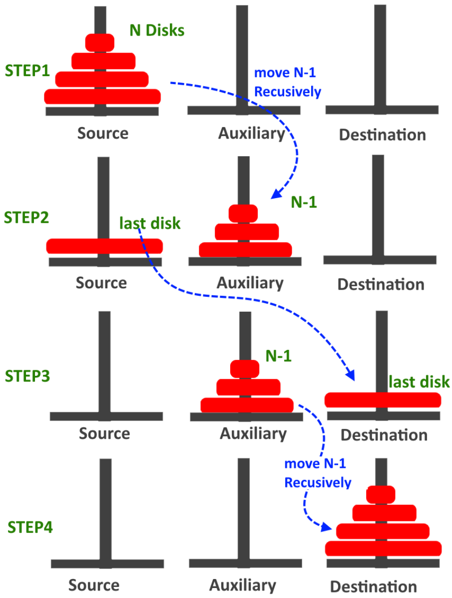
Theorem

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

knuth draw lines

The Tower of Hanoi




N Disks
STEP1
move N-1
Recusively
Source
Auxiliary
Destination
STEP2
last disk
N-1
Source
Auxiliary
Destination
STEP3
N-1
last disk
Source
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move N-1
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STEP4
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Auxiliary
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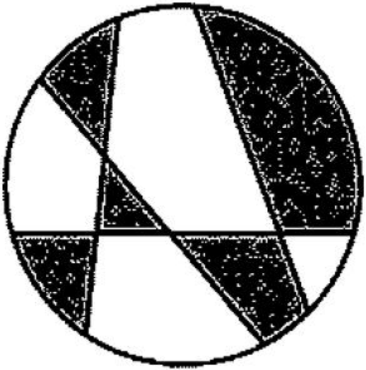

所有马的颜色都相同。

$F(n)$ 是偶数, 当且仅当 $F(n+3)$ 是偶数。

算术基本定理

Prove that every integer greater than 2 can be written as product of primes.

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Prove that every integer greater than 2 can be written as product of primes.

Prove that every integer greater than 12 can be made as sum of 4 and 5s.
请证明, 只用 4 分与 5 分邮票, 就可以组成 12 分及以上的每种邮资。

堆盒子游戏

现有 n 个盒子堆在一起。你可以移动这些盒子, 每次移动只能将一堆盒子分成不为空的两堆盒子, 最后得到 n 堆盒子, 即每堆只有一个盒子时, 游戏结束。

每次移动盒子时, 如果将高度为 $a+b$ 的盒子堆拆分成高度为 a 和 b 的两堆, 玩家可以得 ab 分。

玩家的总得分是每次移动盒子得分的总和。请问, 如何才能得到最高分?

+fig

Lemma

任何一种平铺 n 个盒子的方法，得分都是 $\frac{n(n-1)}{2}$。

只用以下三种图示拼出 $2 \times n$ 的形状, 有几种不同的拼法?

$$T(n) = ?T(n-1) + ??T(n-2) + \dots$$

The Blue-eyed Islanders Puzzle

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What effect, if anything, does this *faux pas* have on the tribe?

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there are several blue-eyed people in their tribe).

100 days after the address,

all the blue eyed people commit suicide.

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Theorem (The Blue-eyed Islanders Puzzle)

Suppose that the tribe had $n > 0$ blue-eyed people.

Then n days after the traveller's address,

all n blue-eyed people commit suicide.

By induction on the number n of blue-eyed people in the tribe.

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基础步骤: $n = 1$.

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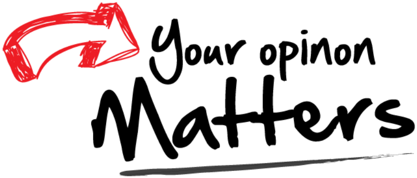
Thank
You!



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