

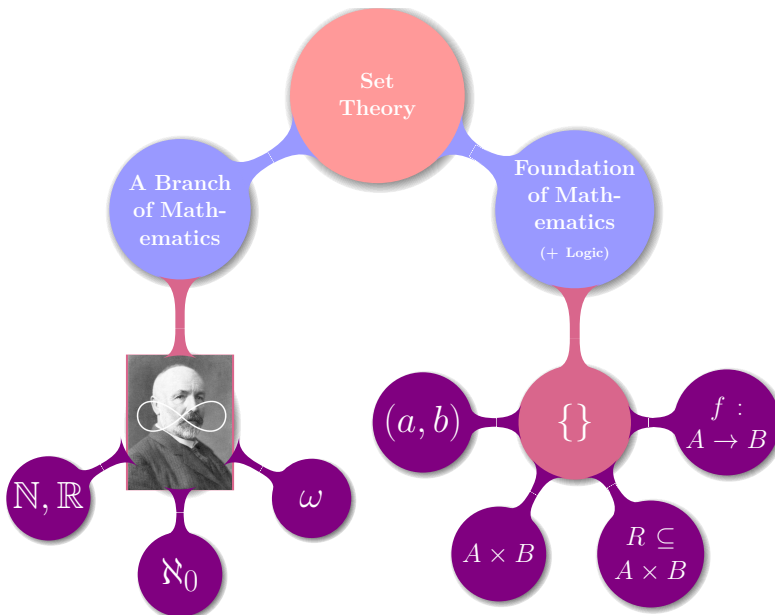
(六) 集合: 函数 (Functions)

魏恒峰

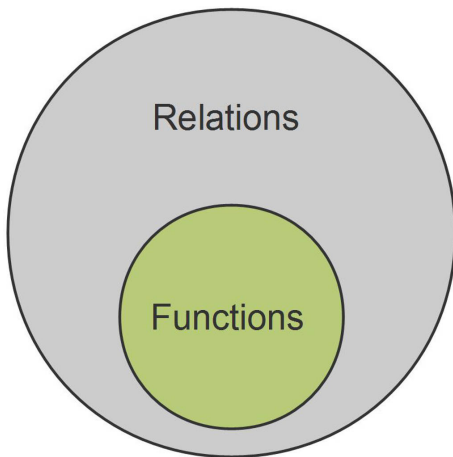
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2021 年 04 月 15 日






从“关系”的角度理解“函数”



$$f(x) = 2x + 1$$

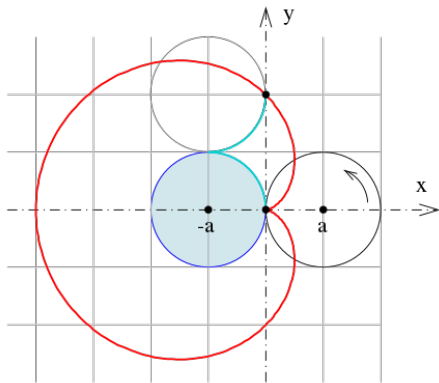


$f(x)$

“函数”也是“关系”

$\{\dots, (-2, -3), (-1, -1), (0, 1), (1, 3), \dots\}$

$$(x^2 + y^2)^2 + 4ax(x^2 + y^2) - 4a^2y^2 = 0$$



“函数”不允许“一对多”

Functions

Functions



PROOF!

Definition of Functions

$$R \subseteq A \times B$$

is a *relation* from A to B

Definition (Function)

$f \subseteq A \times B$ is a *function* from A to B if

$$\forall a \in A. \exists! b \in B. (a, b) \in f.$$

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$$\text{dom}(f) = A \quad \text{cod}(f) = B$$

$$\text{ran}(f) = f(A) \subseteq B$$

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$$f : a \mapsto b$$

$$f(a) \triangleq b$$

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$$\exists! b \in B.$$

$$\forall b, b' \in B. (a, b) \in f \wedge (a, b') \in f \implies b = b'$$

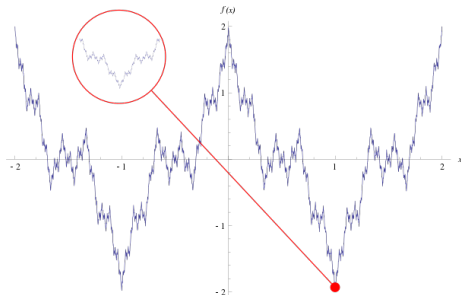
$$I_X : X \rightarrow X$$

X 上的恒等函数

$$\forall x \in X. I_X(x) = x$$

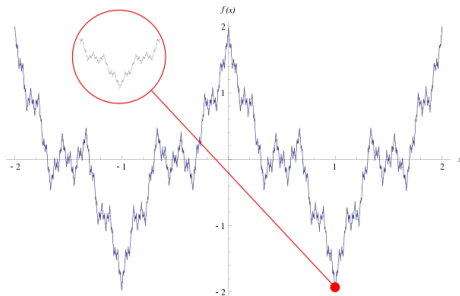
$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$

$0 < a < 1$, b is a positive odd integer, $ab > 1 + \frac{3}{2}\pi$



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Weierstrass Function (1872)

“处处连续, 但处处不可导”

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$$\bigcup_{I_X \in A} \text{dom}(I_X)$$

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For every set X , there exists a function $I_X : \{X\} \rightarrow \{X\}$.

$\bigcup_{I_X \in A} \text{dom}(I_X)$ would be the *universe* that does not exist!

Functions as Sets

Axiom (Axiom of Extensionality)

$$\forall A \forall B \forall x : (x \in A \iff x \in B) \iff A = B.$$

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Theorem (The Principle of Functional Extensionality)

f, g are functions :

$$f = g \iff \text{dom}f = \text{dom}g \wedge \left(\forall x \in \text{dom}f : f(x) = g(x) \right)$$

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It may be that $\text{cod} f \neq \text{cod} g$.

$$f : A \rightarrow B \quad g : C \rightarrow D$$

Q : Is $f \cap g$ a function?

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Theorem (Intersection of Functions)

$$f \cap g : (A \cap C) \rightarrow (B \cap D)$$

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Theorem (Union of Functions)

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UD Problem 14.3 (g)

$$f : \mathbb{Q} \rightarrow$$

$$f(x) = \begin{cases} x + 1 & \text{if } x \in 2\mathbb{Z} \\ x - 1 & \text{if } x \in 3\mathbb{Z} \\ 2 & \text{otherwise} \end{cases}$$

$$f : A \rightarrow B \quad g : C \rightarrow D$$

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$$x \in 6\mathbb{Z}$$

$$D : \rightarrow$$

$$D(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Dirichlet Function

Special Functions (-jectivity)

Definition (Injective (one-to-one; 1-1) 单射函数)

$$f : A \rightarrow B$$

$$\forall a_1, a_2 \in A : a_1 \neq a_2 \implies f(a_1) \neq f(a_2)$$

Definition (Injective (one-to-one; 1-1) 单射函数)

$$f : A \rightarrow B \quad f : A \rightarrowtail B$$

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For Proof:

► To prove that f *is* 1-1:

$$\forall a_1, a_2 \in A : f(a_1) = f(a_2) \implies a_1 = a_2$$

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- ▶ To show that f *is not* 1-1:

$$\exists a_1, a_2 \in A : a_1 \neq a_2 \wedge f(a_1) = f(a_2)$$

Definition (Surjective (onto) 满射函数)

$$f : A \rightarrow B$$

$$\operatorname{ran} f = B$$

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$$f : A \rightarrow B \quad f : A \twoheadrightarrow B$$

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For Proof:

► To prove that f *is* onto:

$$\forall b \in B \left(\exists a \in A : f(a) = b \right)$$

Definition (Surjective (onto) 满射函数)

$$f : A \rightarrow B \quad f : A \twoheadrightarrow B$$

$$\text{ran } f = B$$

For Proof:

- To prove that f *is* onto:

$$\forall b \in B \left(\exists a \in A : f(a) = b \right)$$

- To show that f *is not* onto:

$$\exists b \in B \left(\forall a \in A : f(a) \neq b \right)$$

Definition (Bijective (one-to-one correspondence) 一一对应)

$$f : A \rightarrow B$$

1-1 & onto

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$$f : A \rightarrow B \quad f : A \xrightarrow[\text{onto}]{1-1} B$$

1-1 & onto

Functions as Relations

$$f|_X \quad f(A) \quad f^{-1}(B) \quad f^{-1} \quad f \circ g$$

Definition (Restriction)

The *restriction* of a function f to X is the **function**:

$$f|_X = \{(x, y) \in f \mid x \in X\}$$

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$$f : A \rightarrow B$$

$$f|_X : A \cap X \rightarrow B$$

$$f|_X(x) = f(x), \forall x \in A \cap X$$

Definition (Image)

The *image* of X under a function f is the **set**

$$f(X) = \{b \mid \exists a \in X : (a, b) \in f\}$$

Definition (Inverse Image)

The *inverse image* of Y under a function f is the **set**

$$f^{-1}(Y) = \{a \mid \exists b \in Y : (a, b) \in f\}$$

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$X \subseteq \text{dom} f$, $Y \subseteq \text{ran} f$ are not necessary

f may not be **invertible** in $f^{-1}(Y)$

$$y \in f(X) \iff \exists x \in \operatorname{dom} f \cap X : y = f(x)$$

$$y \in f(X) \iff \exists x \in X : y = f(x)$$

$$x \in f^{-1}(Y) \iff f(x) \in Y$$

Theorem (Properties of f and f^{-1} (UD Theorem 17.7))

$$f : A \rightarrow B \quad A_1, A_2 \subseteq A, \quad B_1, B_2 \subseteq B$$

(i) f preserves only \subseteq and \cup :

$$(1) \quad A_1 \subseteq A_2 \implies f(A_1) \subseteq f(A_2)$$

$$(2) \quad f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$$

$$(3) \quad f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$$

$$(4) \quad f(A_1 \setminus A_2) \supseteq f(A_1) \setminus f(A_2)$$

(ii) f^{-1} preserves \subseteq, \cup, \cap , and \setminus :

$$(5) \quad B_1 \subseteq B_2 \implies f^{-1}(B_1) \subseteq f^{-1}(B_2)$$

$$(6) \quad f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$$

$$(7) \quad f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$$

$$(8) \quad f^{-1}(B_1 \setminus B_2) = f^{-1}(B_1) \setminus f^{-1}(B_2)$$

Theorem (UD Problem 17.5)

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$$b \in f(A_1 \cap A_2)$$

Theorem (UD Problem 17.5)

$$f : A \rightarrow B$$

$$f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$$

$$b \in f(A_1 \cap A_2)$$

$$\implies \exists a \in A_1 \cap A_2 \cap A : b = f(a)$$

Theorem (UD Problem 17.5)

$$f : A \rightarrow B$$

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Q : When does $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ hold?

Theorem (UD Problem 17.5)

$$f : A \rightarrow B$$

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Q : When does $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ hold?

f is injective.

Theorem (Properties of f and f^{-1} (UD Theorem 17.7))

$$f : A \rightarrow B$$

(iii) f and f^{-1} :

$$(9) \quad A_0 \subseteq A \implies A_0 \subseteq f^{-1}(f(A_0))$$

$$(10) \quad B_0 \supseteq f(f^{-1}(B_0))$$

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Theorem (UD Problem 17.8)

$$A_0 \subseteq A \implies A_0 \subseteq f^{-1}(f(A_0))$$

Theorem

$$f : A \rightarrow B$$

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Theorem

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$$b \in f(f^{-1}(B_0))$$

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Theorem

$$f : A \rightarrow B$$

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$$b \in f(f^{-1}(B_0))$$

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$$\implies \exists a \in A : f(a) \in B_0 \wedge b = f(a)$$

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Q: When does $B_0 = f(f^{-1}(B_0))$ hold?

f is surjective and $B_0 \subseteq B$.

Theorem

$$f : A \rightarrow B$$

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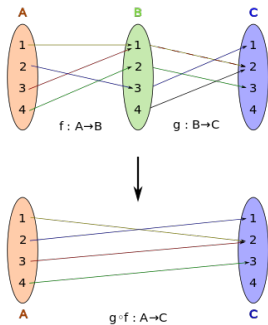
$$\implies b \in B_0$$

Q: When does $B_0 = f(f^{-1}(B_0))$ hold?

f is surjective and $B_0 \subseteq B$.

$$B_0 \subseteq \text{ran } f$$

Function Composition



Definition (Composition)

$$f : A \rightarrow B \quad g : C \rightarrow D$$

$$\text{ran } f \subseteq C$$

The *composite function* $g \circ f : A \rightarrow D$ is defined as

$$(g \circ f)(x) = g(f(x))$$

Definition (Composition)

$$f : A \rightarrow B \quad g : C \rightarrow D$$

$$\text{ran } f \subseteq C$$

The *composite function* $g \circ f : A \rightarrow D$ is defined as

$$(g \circ f)(x) = g(f(x))$$

Why not “ $\exists b$ ” as below?

Definition (Composition)

The *composition* of relations R and S is the relation

$$R \circ S = \{(a, c) \mid \exists b : (a, b) \in S \wedge (b, c) \in R\}$$

Theorem (Associative Property for Composition)

$$f : A \rightarrow B \quad g : B \rightarrow C \quad h : C \rightarrow D$$

$$h \circ (g \circ f) = (h \circ g) \circ f$$

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Proof.

(i)

$$\text{dom } h \circ (g \circ f) = \text{dom } (h \circ g) \circ f$$

(ii)

$$(h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x)$$



Theorem (UD Theorem 16.7)

$$f : A \rightarrow B \quad g : B \rightarrow C$$

- (i) *If f, g are injective, then $g \circ f$ is injective.*
- (ii) *If f, g are surjective, then $g \circ f$ is surjective.*
- (iii) *If f, g are bijective, then $g \circ f$ is bijective.*

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Proof for (i).

$$\forall a_1, a_2 \in A : ((g \circ f)(a_1) = (g \circ f)(a_2) \implies a_1 = a_2)$$



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Proof for (i).

$$\forall a_1, a_2 \in A : ((g \circ f)(a_1) = (g \circ f)(a_2) \implies a_1 = a_2)$$



Proof for (ii).

$$\forall c \in C : (\exists a \in A : (g \circ f)(a) = c)$$



Theorem (UD Theorem 16.8)

$$f : A \rightarrow B \quad g : B \rightarrow C$$

- (i) *If $g \circ f$ is surjective, then g is surjective.*
- (ii) *If $g \circ f$ is injective, then f is injective.*

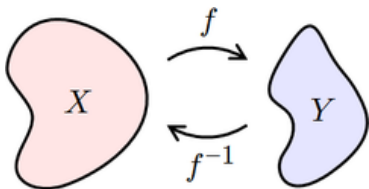
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You can also prove it by contradiction.

Inverse Functions



Definition (Inverse)

Let $f : A \rightarrow B$ be a **bijective** function.

The *inverse* of f is the **function** $f^{-1} : B \rightarrow A$ defined by

$$f^{-1}(b) = a \iff f(a) = b.$$

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Definition (Invertible)

$f : X \rightarrow Y$ is *invertible* if there exists $g : Y \rightarrow X$ such that

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To show that g defined above is indeed a function from Y to X .

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$$f^{-1} \triangleq g$$

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By Contradiction

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$$f(x) = y \iff f^{-1}(y) = x$$

Theorem (UD Theorem 16.4)

$f : A \rightarrow B$ is bijective

(i) $f \circ f^{-1} = I_B$

(ii) $f^{-1} \circ f = I_A$

(iii) f^{-1} is bijective.

(iv) $g : B \rightarrow A \wedge f \circ g = I_B \implies g = f^{-1}$

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The ways to find/check f^{-1} .

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The ways to find/check f^{-1} .

$$g = f^{-1} \circ (f \circ g) = f^{-1} \circ I_B = f^{-1}$$

Theorem (Inverse of Composition (UD Theorem 16.6))

$f : A \rightarrow B$ $g : B \rightarrow C$ are *bijective*

- (i) $g \circ f$ is *bijective*
- (ii) $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Proof for (ii).

It suffices to check **either** one of the following identities:

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = I_A$$

$$(g \circ f) \circ (f^{-1} \circ g^{-1}) = I_C$$



Theorem (UD Theorem 16.8)

$$f : A \rightarrow B \quad g : B \rightarrow A$$

$$(iii) \quad f \circ g = I_B \wedge g \circ f = I_A \implies g = f^{-1}$$

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You need to check **both** identities.

Theorem (UD Theorem 16.8)

$$f : A \rightarrow B \quad g : B \rightarrow A$$

$$(iii) \quad f \circ g = I_B \wedge g \circ f = I_A \implies g = f^{-1}$$

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Theorem (UD Theorem 16.8)

$$f : A \rightarrow B \quad g : B \rightarrow C$$

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Theorem (UD Theorem 16.8)

$$f : A \rightarrow B \quad g : B \rightarrow A$$

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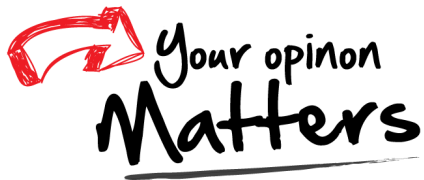
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First show that f is bijective, and then use Theorem 16.4.

Thank
You!



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