(九) 图论: 路径与圈 (Paths and Cycles)

魏恒峰

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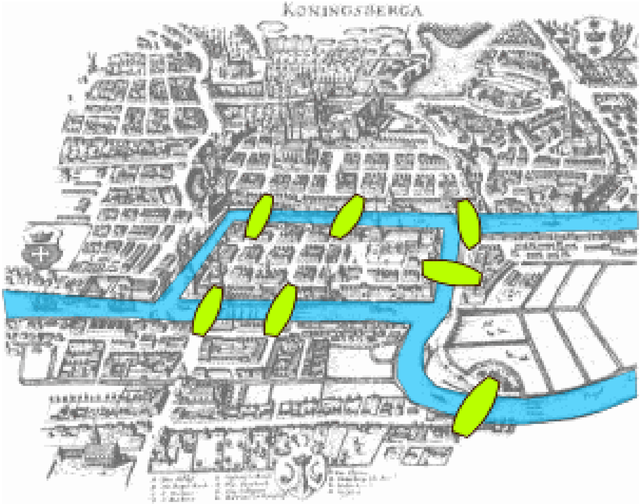
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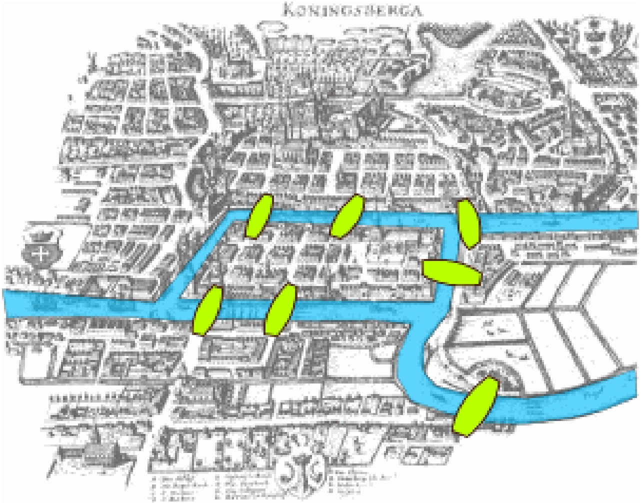
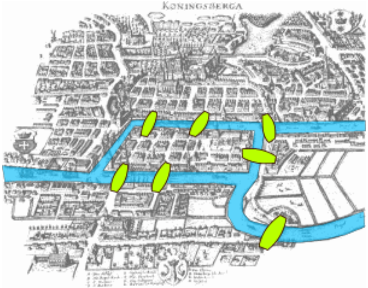


"to devise a walk through the city that would cross each of those bridges once and only once"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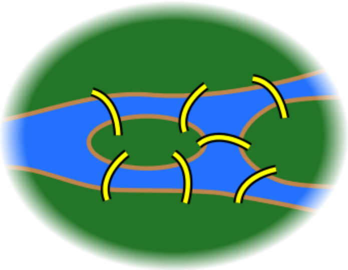

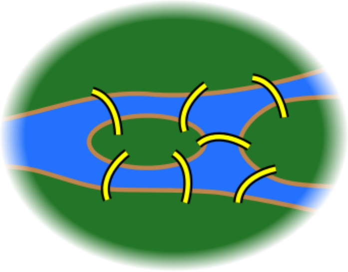

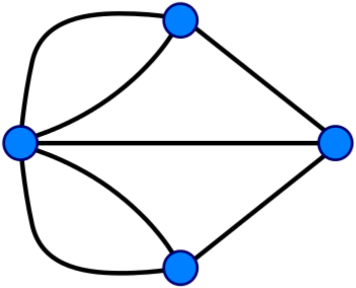

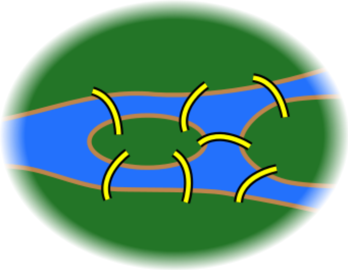

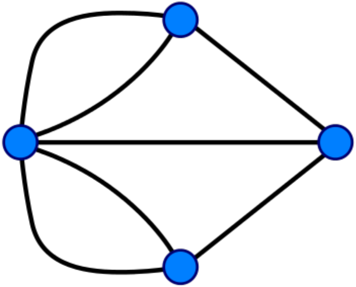

"to devise a walk through the graph
that would cross each of those edges once and only once"

Definition (Graph (图))

An (undirected simple) graph is a pair $G = (V, E)$ where

- V is a set of vertices (顶点);
- $E \subseteq \{\{x, y\} \mid x, y \in V \land x \neq y\}$ is a set of edges

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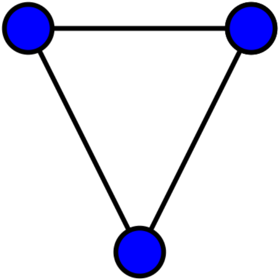
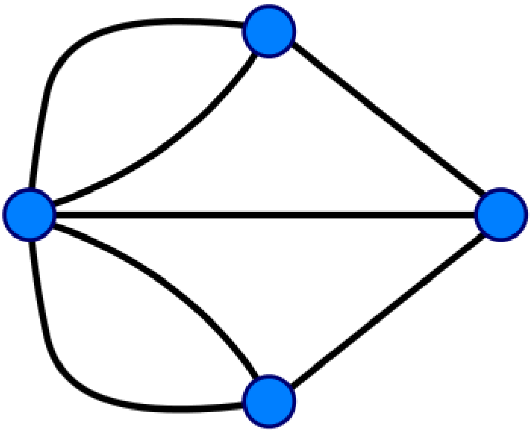
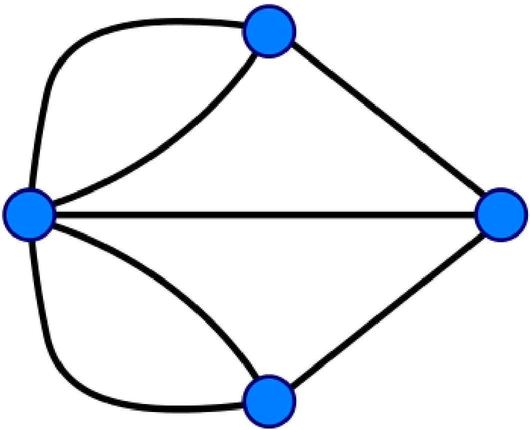
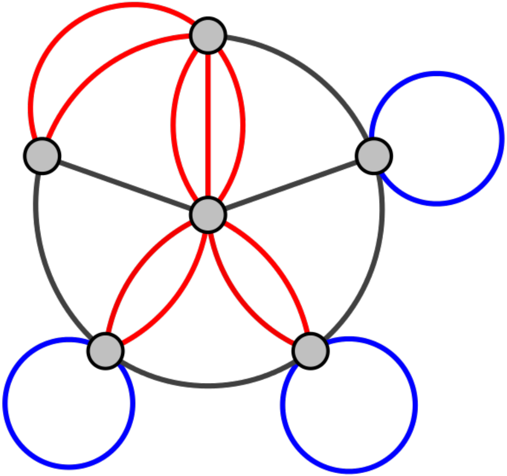
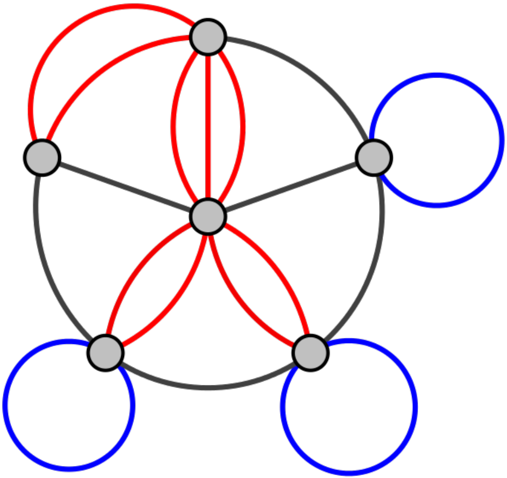






Undirected Multigraph





Undirected Multigraph Permitting Loops

Definition (Walk (道路))

Given a graph G, a (finite) walk in G is a sequence of edges of the form

$$\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{m-1}, v_m\}.$$

$$(v_0 \to v_1 \to v_2 \to \cdots \to v_m)$$

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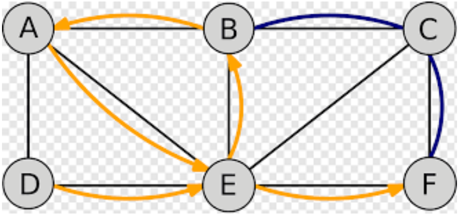
$$D \to E \to B \to A \to E \to F$$



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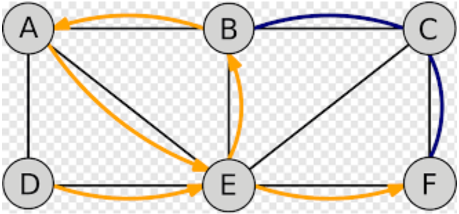

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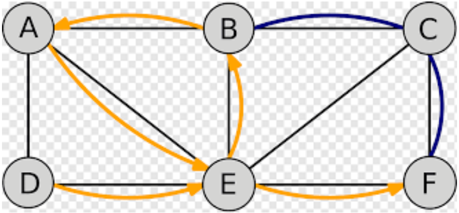

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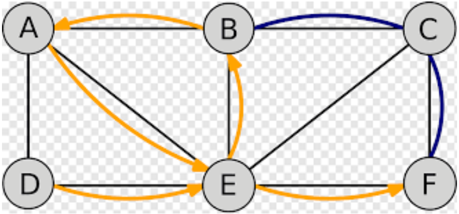

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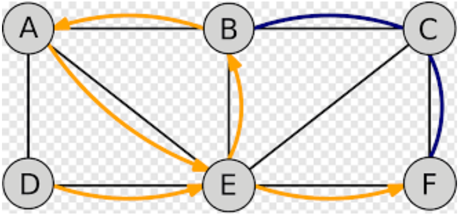

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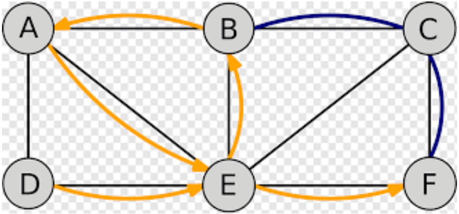

Definition (Cycle)

A cycle is a closed path with at least one edge.

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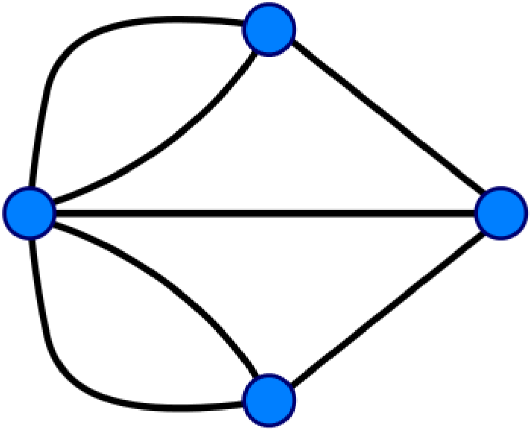

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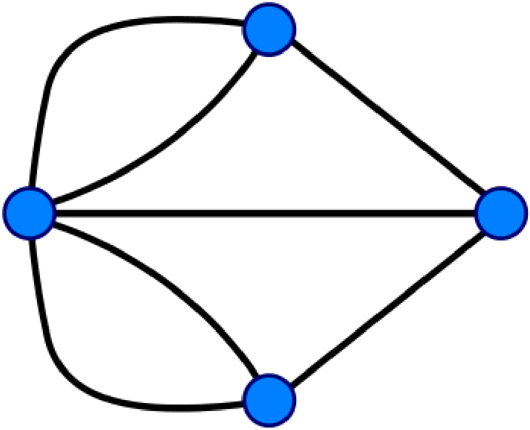
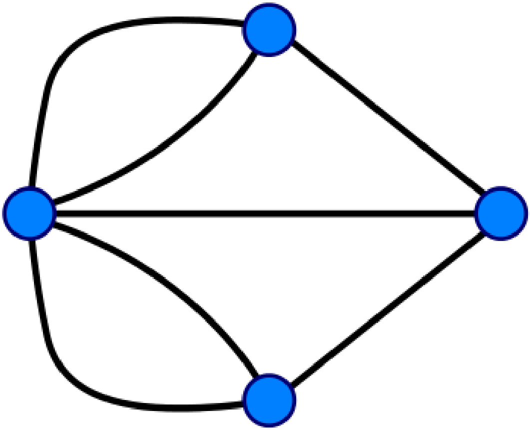


to find a trail that contains all edges of the graph

$$v_0 \to v_1 \to \cdots \to {\color{red}v_i} \to \cdots \to v_m$$



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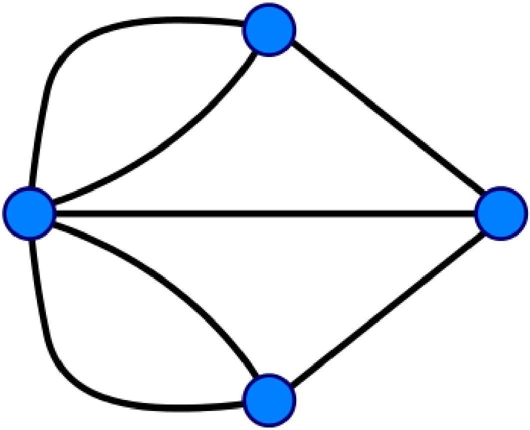

$$v_i \notin \{v_0, v_m\} \implies {\color{red} \deg(v_i)} \text{ is even}$$

Lemma (Necessary Condition for Eulerian Trails)

If a graph has Eulerian trails, then zero or two vertices have an odd degree.

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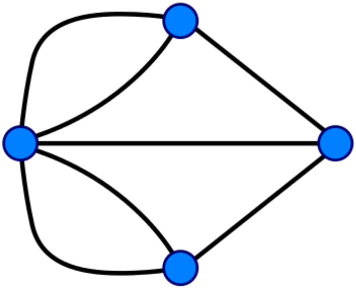

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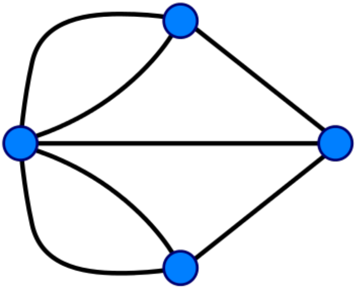

4 vertices of odd degree $\implies$ has no Eulerian trails

Theorem (Euler's Theorem)

A graph has Eulerian trails iff it contains zero or two vertices that have an odd degree.

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Pierre-Henry Fleury gave another proof in 1883.

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A connected graph has Eulerian cycles iff every vertex has even degree.

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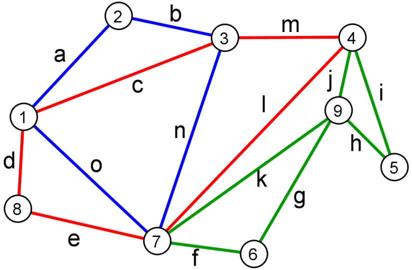
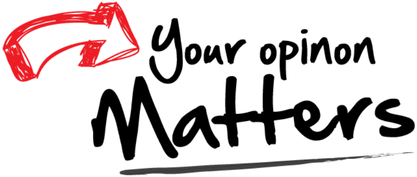


alg

edge-disjoint cycle decomposition

directed graphs


Thank
You!



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