# (九) 图论: 路径与圈 (Paths and Cycles)

魏恒峰

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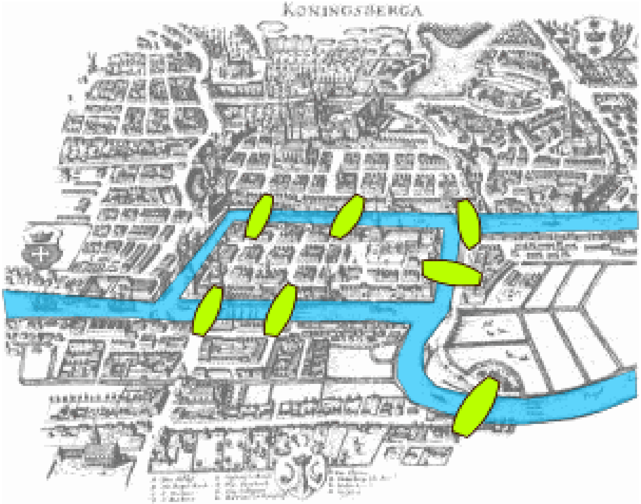
2021 年 05 月 06 日


Definitions　　Theorems　　Proofs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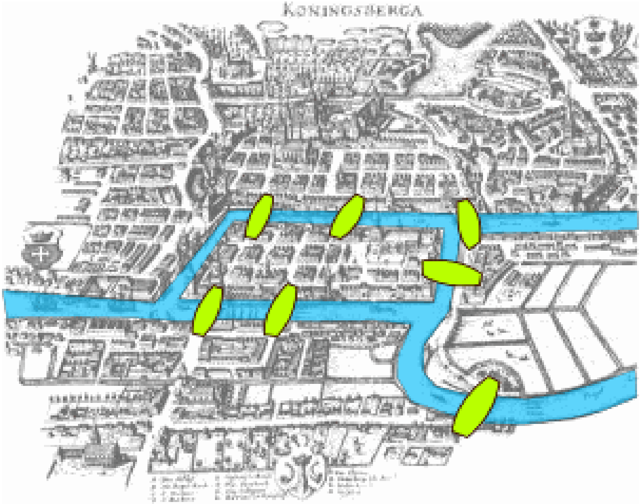
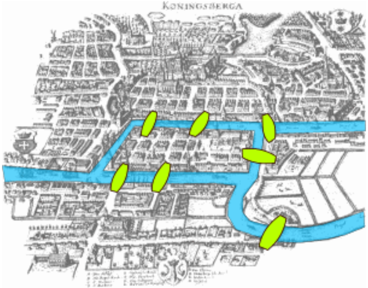



*"to devise a walk through the city*
*that would cross each of those bridges once and only once"*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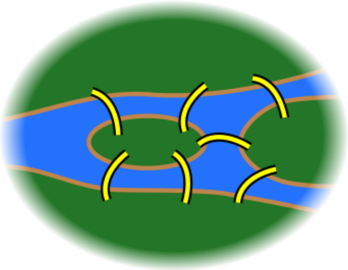
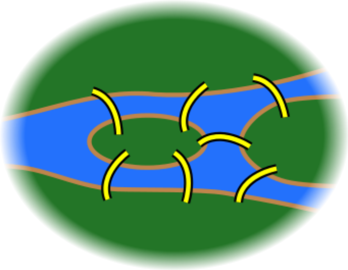
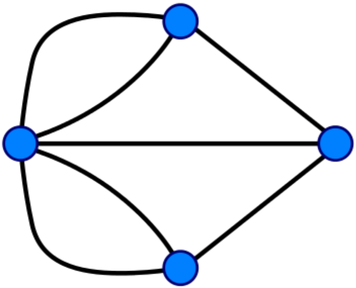
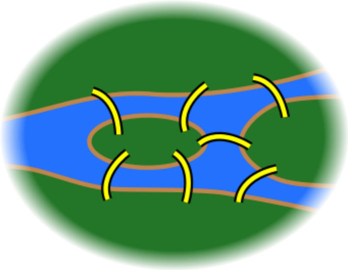
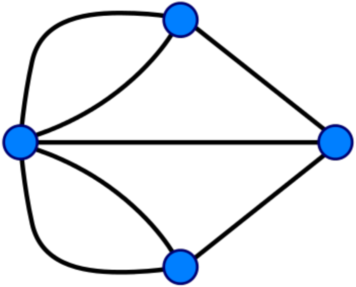








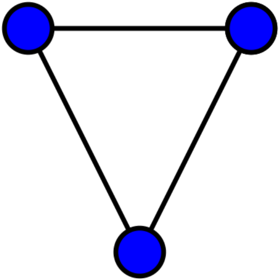


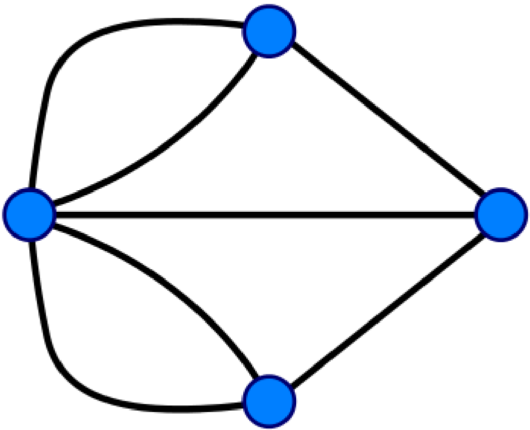
*"to devise a walk through the graph*
*that would cross each of those edges once and only once"*

### Definition (Graph (图))

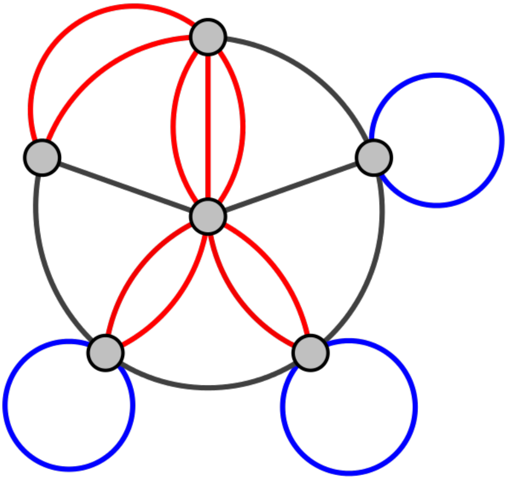
An (undirected simple) graph is a pair $G = (V, E)$ where

- $V$ is a set of vertices (顶点);
- $E \subseteq \{\{x, y\} \mid x, y \in V \land x \neq y\}$ is a set of edges





Undirected Multigraph



Undirected Multigraph Permitting Loops

## Definition (Walk (道路))

Given a graph $G$, a (finite) walk in $G$ is a sequence of edges of the form

$$\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{m-1}, v_m\}.$$

$$(v_0 \to v_1 \to v_2 \to \cdots \to v_m)$$

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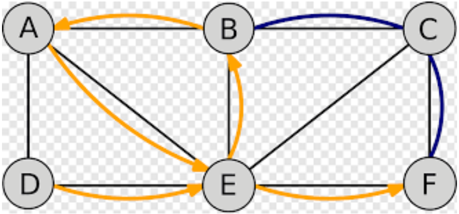
$$D \to E \to B \to A \to E \to F$$



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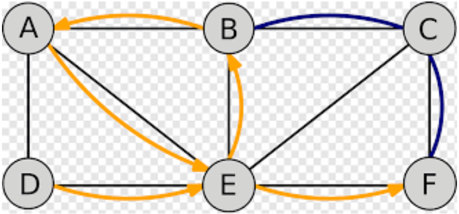


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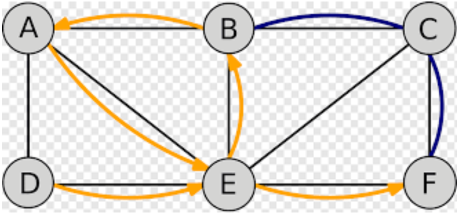


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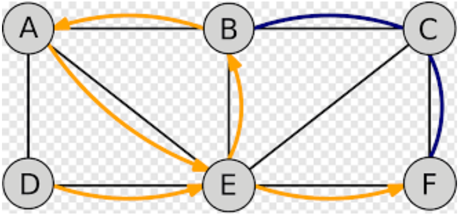


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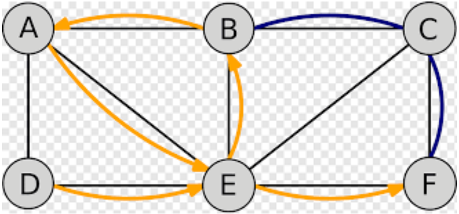

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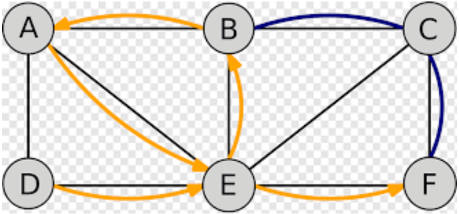


### Definition (Cycle)

A cycle is a closed path with at least one edge.

*"to devise a walk through the graph*
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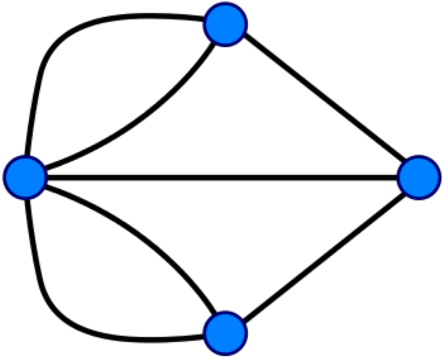

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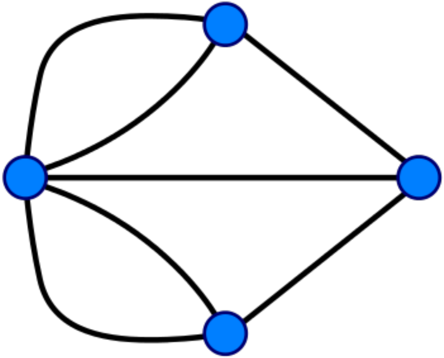
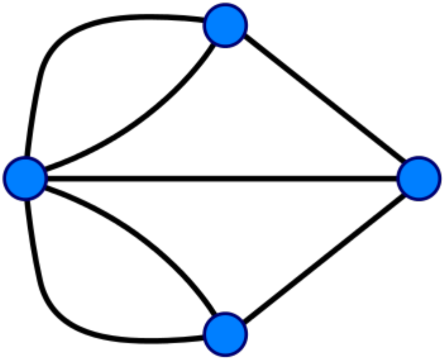


*to find a trail that contains all edges of the graph*

$$v_0 \to v_1 \to \cdots \to \textcolor{red}{v_i} \to \cdots \to v_m$$



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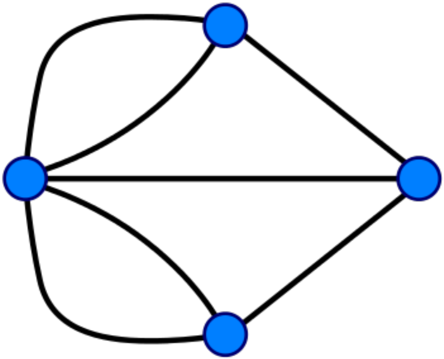

$$v_i \notin \{v_0, v_m\} \implies \color{red}{\deg(v_i)} \text{ is even}$$

**Lemma (Necessary Condition for Eulerian Trails)**

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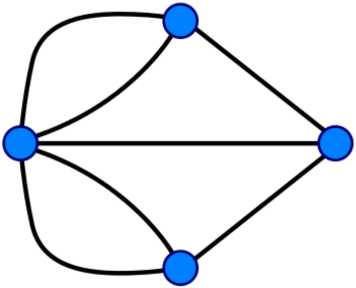

4 vertices of odd degree $\implies$ has no Eulerian trails

### Theorem (Euler's Theorem)

*A graph has Eulerian trails iff it contains zero or two vertices that have an odd degree.*

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Carl Hierholzer (1840 $\sim$ 1871) gave the first complete proof in 1873.

Pierre-Henry Fleury gave another proof in 1883.

**Theorem (Euler's Theorem (Carl Hierholzer))**

*A connected graph has Eulerian cycles iff every vertex has even degree.*

*(Such graphs are called Eulerian graphs.)*

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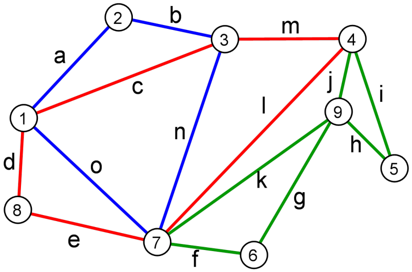
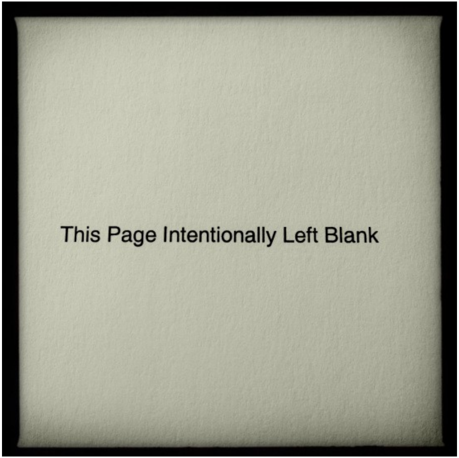



cycle + cycle + cycle + . . .




Can you repeat the Carl Hierholzer's algorithm?

**Lemma**

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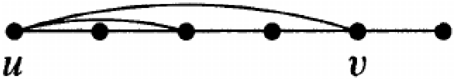

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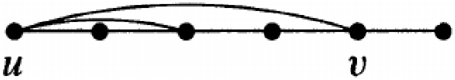


$\deg(u) \geq 2$

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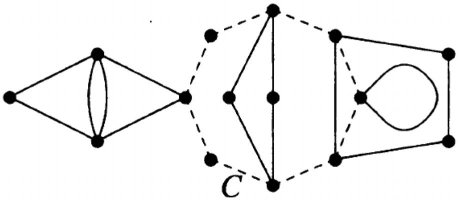
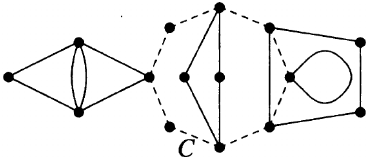
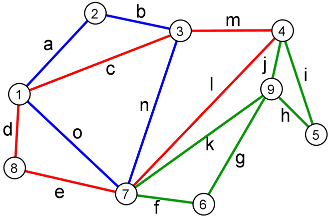



$$G' = G - C$$

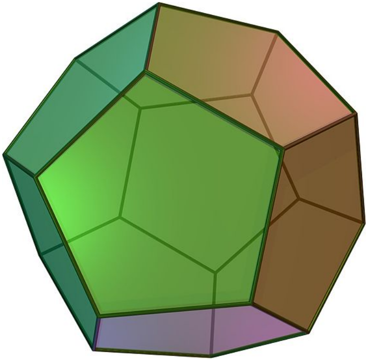
### Theorem

*Every Eulerian graph decomposes into cycles.*



Dodecahedron: 12 faces, 20 vertices, and 30 edges



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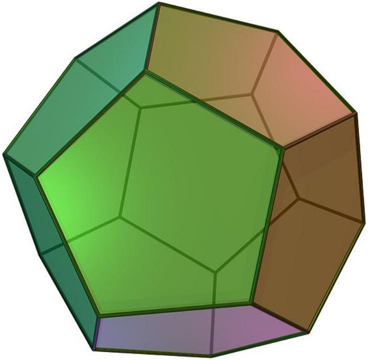
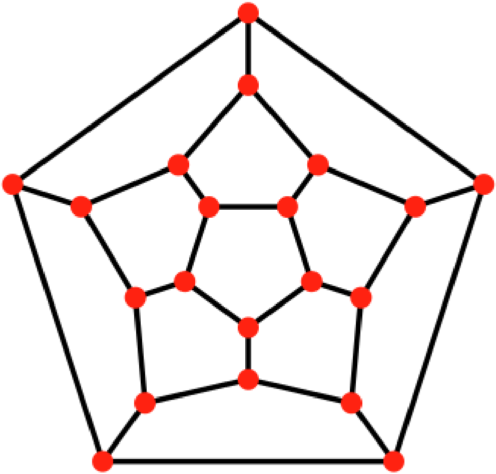


Is there a cycle that visits each vertex exactly once?



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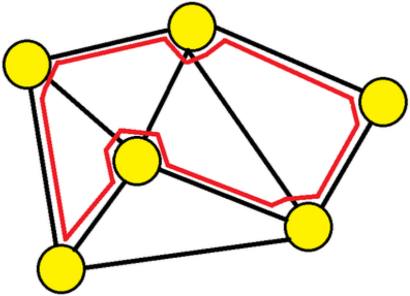
## Definition (Hamiltonian Path)

A Hamiltonian path is a path that visits each vertex exactly once.



## Definition (Hamiltonian Cycle)

A Hamiltonian cycle is a Hamiltonian path that is a cycle.

**Definition (Hamiltonian Graph)**

A graph is a Hamiltonian graph if it has a Hamiltonian cycle.

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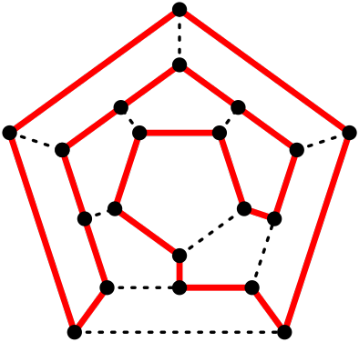
**Definition (Semi-Hamiltonian Graph)**

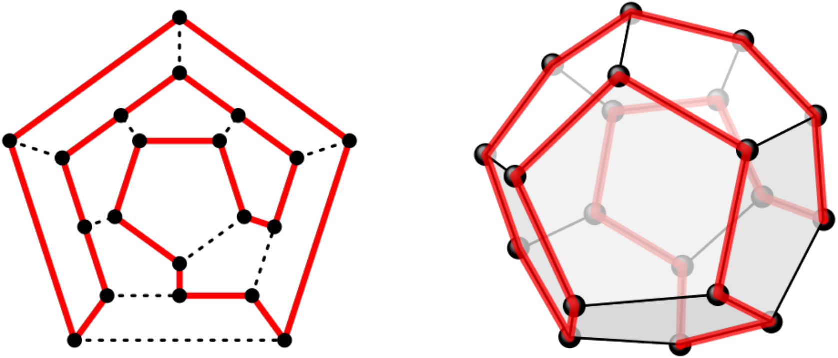
A non-Hamiltonian graph is semi-Hamiltonian if it has a Hamiltonian path.

William Rowan Hamilton
(1805 $\sim$ 1865)


(October 16, 1843)

$$i^2 = j^2 = k^2 = ijk = -1$$





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Nobody knows.

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Nobody knows.

We will probably never know it.


**Theorem**

*The Hamiltonian Path/Cycle problem is NP-complete.*

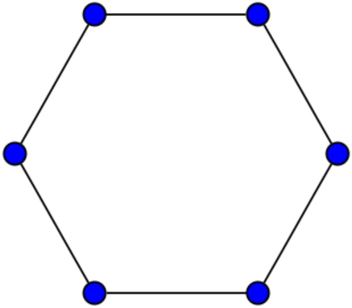
Typical (Positive/Negative) Graph Examples

Sufficient Conditions

Necessary Conditions


EXAMPLES

- Every cycle is Hamiltonian



$C_6$

- A complete graph (完全图) with $|V| > 2$ is Hamiltonian.

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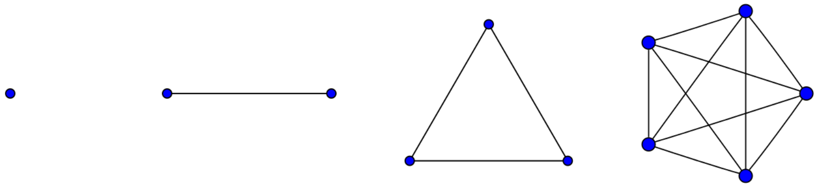
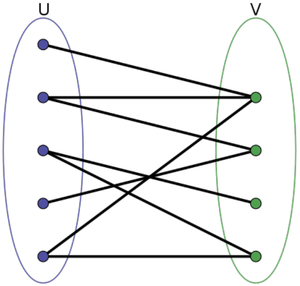


$K_1$ $K_2$ $K_3$ $K_5$

- A complete bipartite graph $K_{m,n}$ is Hamiltonian iff $m = n \geq 2$.

## Definition (Bipartite Graph (Bigraph; 二部图))

A bipartite graph $G = (U, V, E)$ is a graph whose vertices can be divided into two disjoint sets $U$ and $V$ such that every edge connects a vertex in $U$ to one in $V$.



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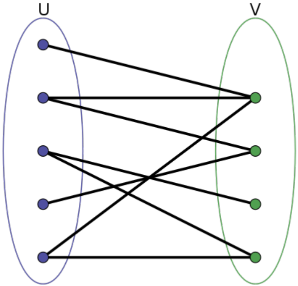


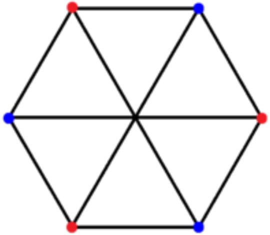

**Definition (Complete Bipartite Graph (Biclique; 完全二部图))**

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$$K_{m,n} : m = |U|, n = |V|$$

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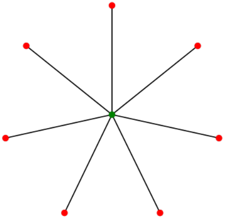
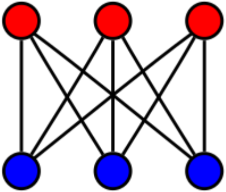
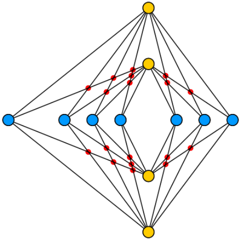


$K_{1,5}$ (star)



$K_{3,3}$ (utility graph)



$K_{4,7}$

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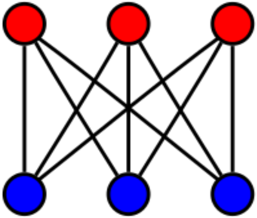

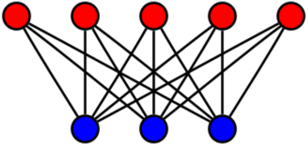

- Every platonic solid (正多面体), considered as a graph, is Hamiltonian.

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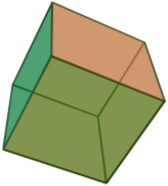
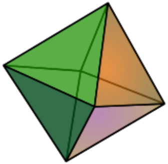
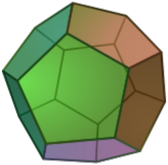
Tetrahedron



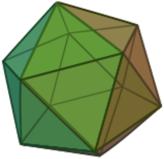
Cube



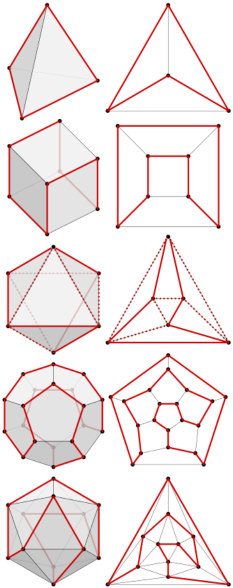
Octahedron



Dodecahedron



Icosahedron



## Theorem

- *Petersen graph is not Hamiltonian.*

*Julius Petersen* $(1839 \sim 1910)$

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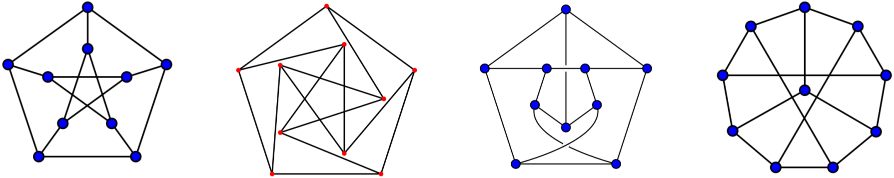


"If $G$ has enough edges, then $G$ is Hamiltonian."

## Theorem (Ore's Theorem, 1960)

*Let $G$ be a simple graph with $n \geq 3$ vertices. If*

$$deg(u) + deg(v) \geq n$$

*for each pair of non-adjacent vertices $u$ and $v$, then $G$ is Hamiltonian.*

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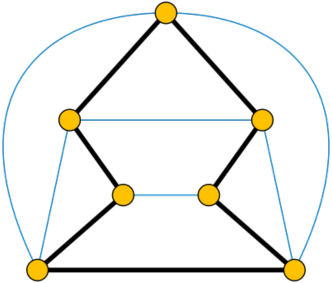

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Adding edges cannot violate the Ore's Condition.

Thus we may consider only *maximal* non-Hamiltonian graphs:

adding any edge gives a Hamiltonian graph.

By its “maximality”, $G$ contains a Hamiltonian path

($G$ is a semi-Hamiltonian graph)

$$v_1 \to v_2 \to \cdots \to v_n$$

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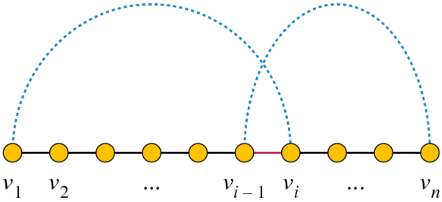


There must be some vertex $v_i$ adjacent to $v_1$

such that $v_{i-1}$ is adjacent to $v_n$.

Theorem (Dirac's Theorem (1952; Gabriel Andrew Dirac))

*A simple graph* $G = (V, E)$ *with* $n \geq 3$ *vertices is Hamiltonian*

$$\forall v \in V.\ deg(v) \geq n/2.$$

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## Family [ edit | edit source ]

He was born Balázs Gábor in Budapest, to Richárd Balázs, a military officer and businessman, and Margit "Manci" Wigner (sister of Eugene Wigner).[5] When his mother married Paul Dirac in 1937, he and his sister resettled in England and were formally adopted, changing their family name to Dirac.[6]

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Counterexample: $C_{\lfloor (n+1)/2 \rfloor}$ and $C_{\lceil (n+1)/2 \rceil}$ sharing a vertex

"If $G$ is Hamiltonian, then $G$ must be somewhat connected."

"If $G$ is Hamiltonian, then $G$ must be somewhat connected."

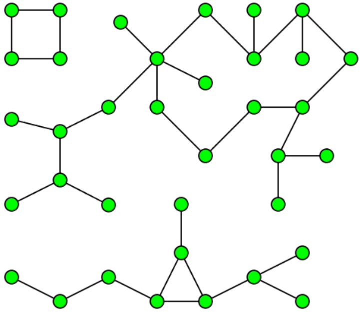

"If $G$ is not so connected, then $G$ is non-Hamiltonian."

**Theorem**

*If* $G = (V, E)$ *is Hamiltonian, then for each nonempty set* $S \subset V$*, the graph* $G - S$ *has* $\leq |S|$ *components.*

## Definition (Components (连通分支))

A component of an undirected graph is an subgraph in which any two vertices are connected to each other by paths, and which is connected to no additional vertices in the rest of the graph.



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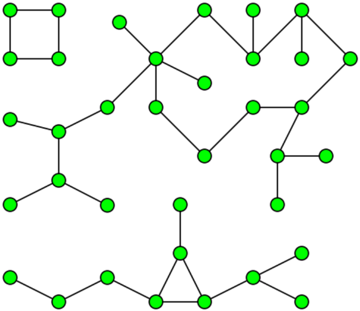

$$c(G) = 3$$

**Theorem**

*If* $G = (V, E)$ *is Hamiltonian, then*

$$\forall S \subset V.\ c(G - S) \le |S|.$$

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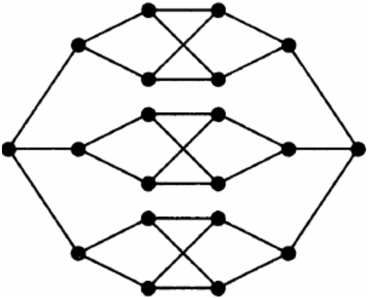

A non-Hamiltonian bipartite graph with $m = n = 10$

## Theorem

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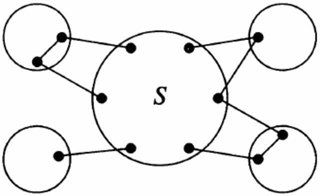

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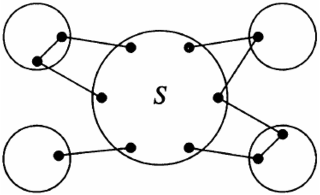


"When a Hamiltonian cycle leaves a component of $G - S$, it can go only to a distinct vertex in $S$."

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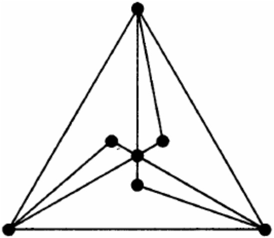

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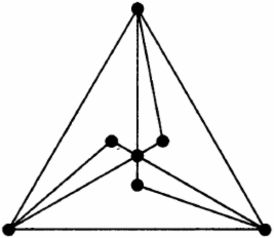
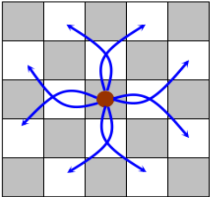
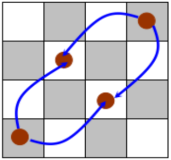


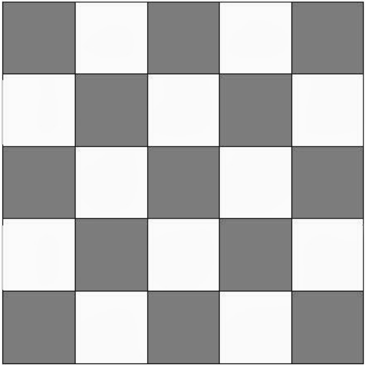
All edges incident to vertices of degree 2 must be used.

## Chessboard Problem (“马踏棋盘” 问题)

Is it possible for a “knight” to visit every field of a $4 \times 4$ or $5 \times 5$ chessboard exactly once and return to the starting point?



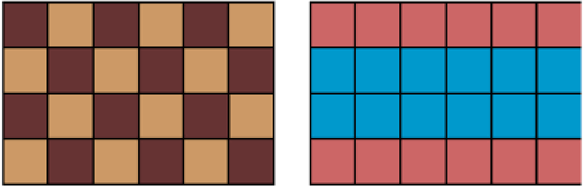




$$G = (U, V, E) : |U| = 12, |V| = 13$$

Removing the middle 4 squares leaves $\geq 5$ components.

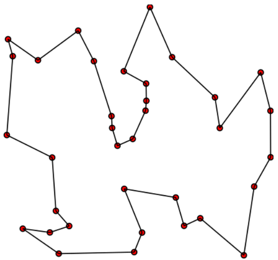
## Chessboard Problem

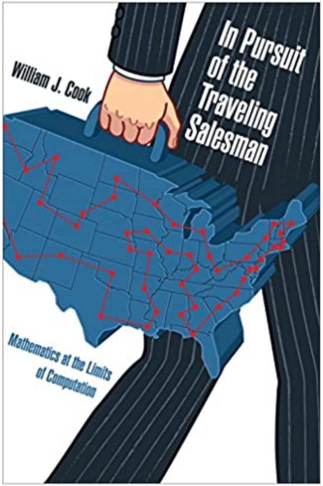


$4 \times n$

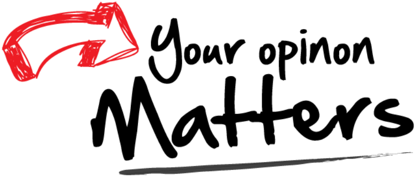
Definition (Travelling Salesman/Salesperson Problem (TSP; 旅行商问题))

Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city exactly once and returns to the origin city?




William J. Cook
In Pursuit of the Traveling Salesman
Mathematics at the Limits of Computation


Thank
You!

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