(十二) 图论: 对偶
(Duality: Matching, Network Flow, and Connectivity)

魏恒峰

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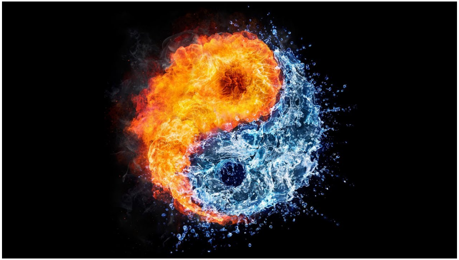
2021 年 05 月 27 日




let's get
married
today

The Marriage Problem (1935)

If there is a finite set of girls, each of whom knows several boys, under what conditions can all the girls marry boys in such a way that each girl marries a boy that she knows?

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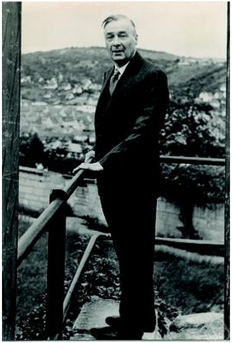
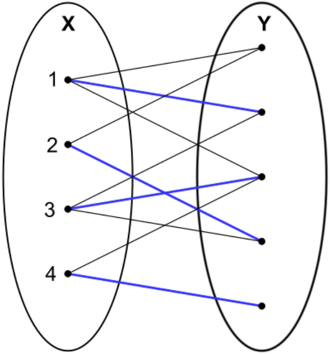


Philip Hall (1904 ~ 1982)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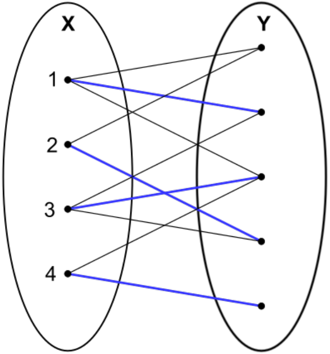
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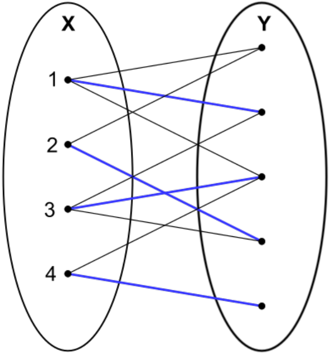
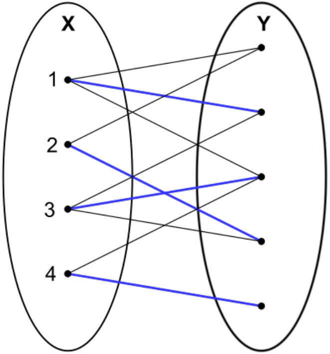


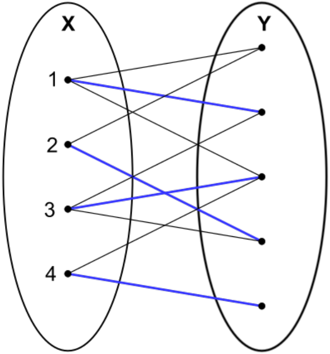


Definition (X-Perfect Matching (X-Saturating Matching))

Let $G = (X, Y, E)$ be a bipartite graph.

An X-perfect matching of G is a matching which covers each vertex in X.

$$|X| \le |Y|$$



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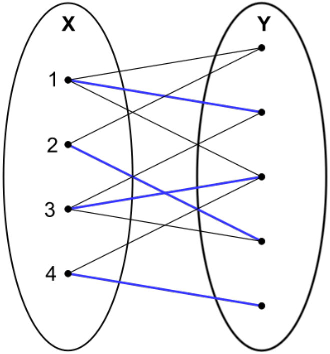

$$\forall W \subseteq X.\ \left|W\right| \le \left|N(W)\right|$$

Theorem (Hall Theorem; 1935)

Let $G = (X, Y, E)$ be a bipartite graph. There is a X-perfect matching of G iff

$$\forall W \subseteq X. \ \left|W\right| \le \left|N_G(W)\right|$$

By induction on the number $\left|X\right|$ of vertices in X.

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Induction Hypothesis: Suppose that it holds if $\left|X\right| < m$.

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- Case I: Every $k < m$ girls in X know $\geq k + 1$ boys in Y.

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- CASE II: There is a set of $k < m$ girls in X who know k boys in Y.

Theorem (Hall Theorem; 1935)

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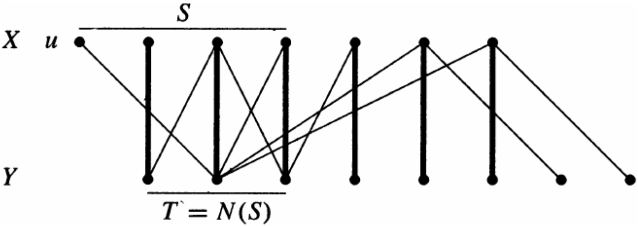

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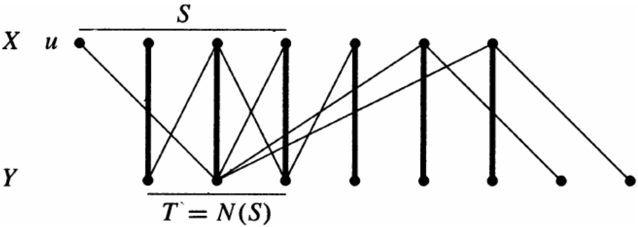

Let M be a *maximum* matching.

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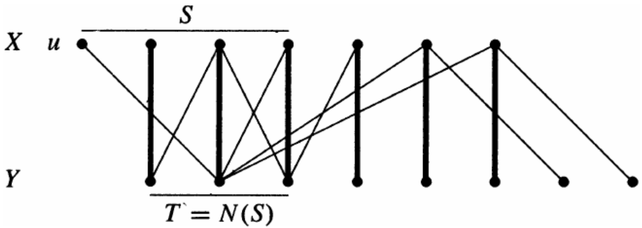

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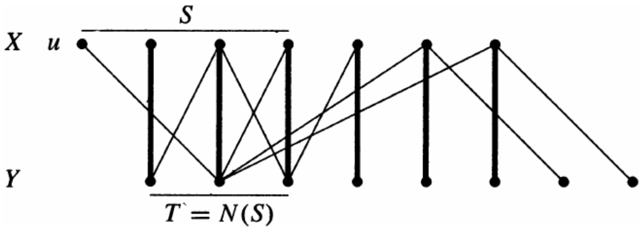

Let M be a *maximum* matching.

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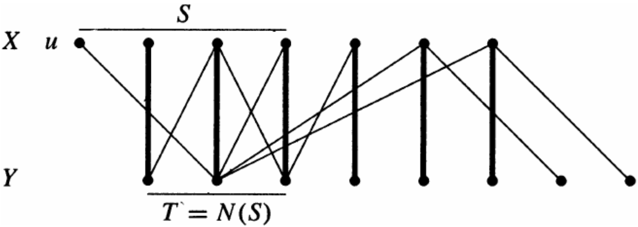
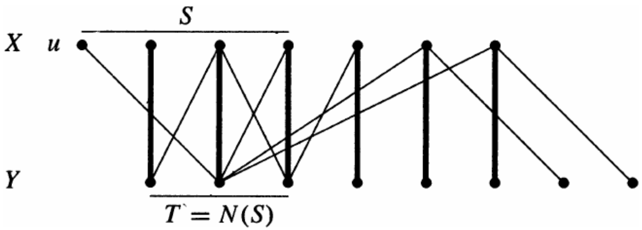
Consider all *M-alternating paths* starting from u.



$T \triangleq$ the set of vertices in Y reachable from u by M-alternating paths.



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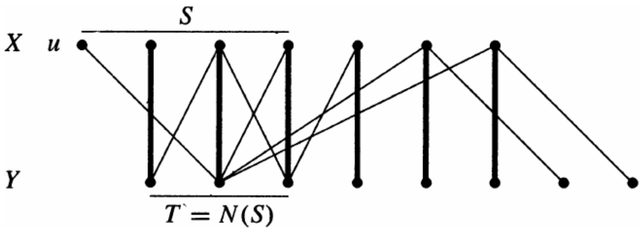

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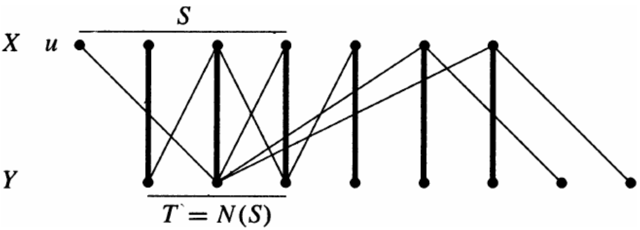
$$T = N(S) \wedge \left|T\right| = \left|S - \{u\}\right|$$



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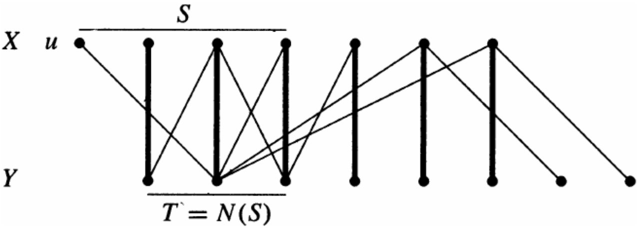

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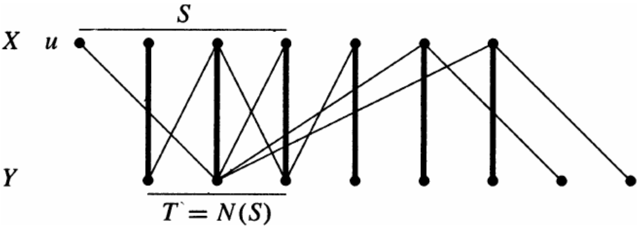

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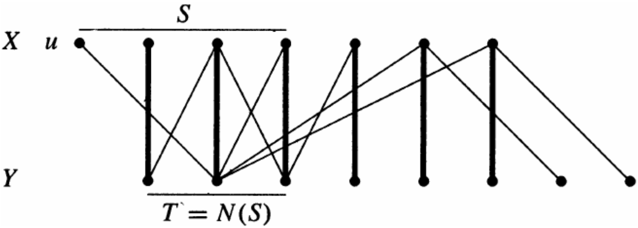

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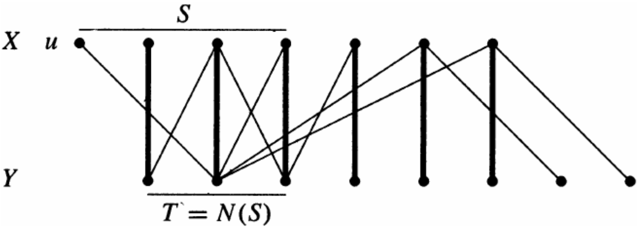

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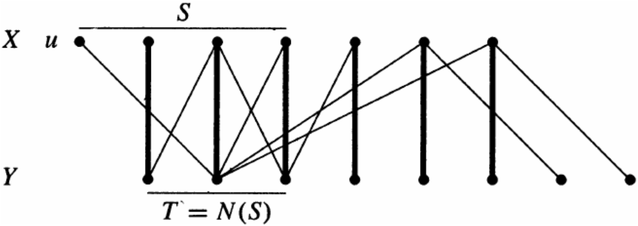

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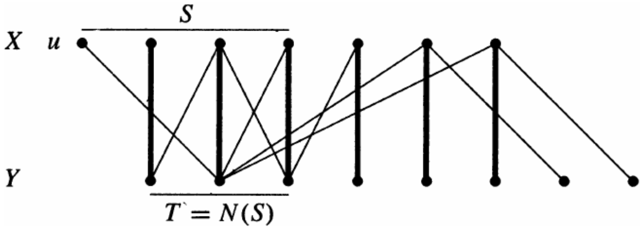

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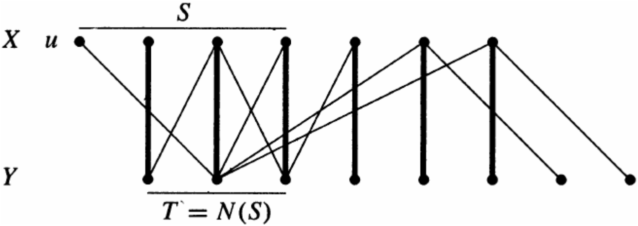

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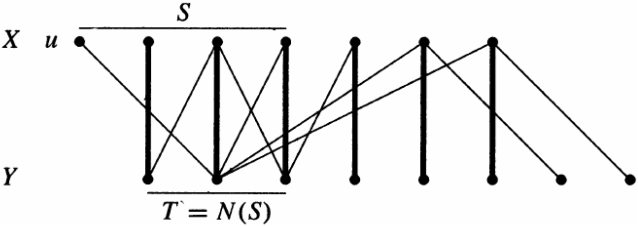

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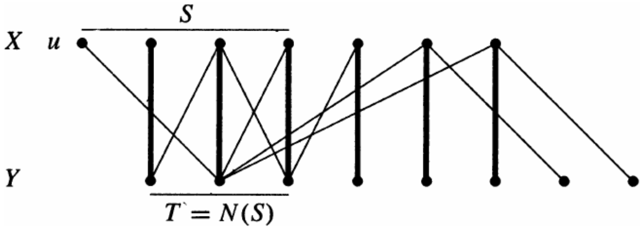

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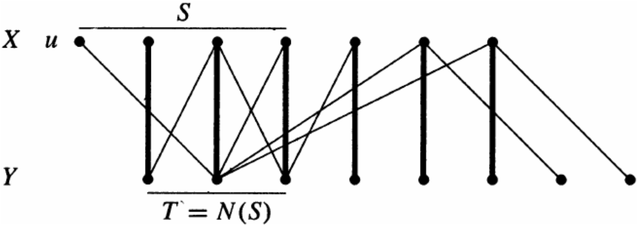

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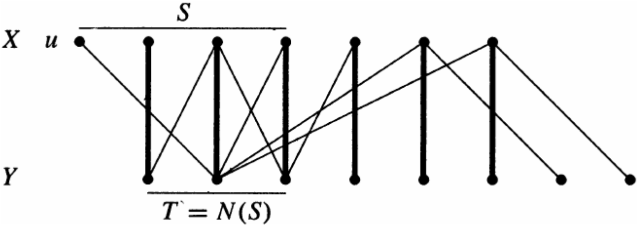

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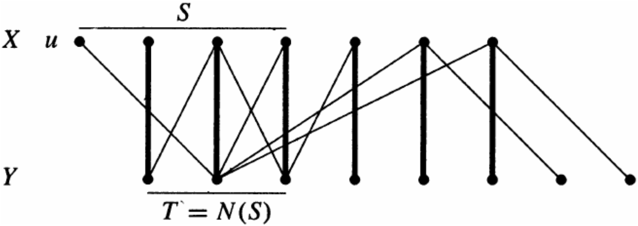

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algorithm


Thank
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Your opinon Matters

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