(三) 数学归纳法 (Mathematical Induction)

魏恒峰

hfwei@nju.edu.cn

2021 年 03 月 25 日

数学归纳法真得很简单吗?

Theorem (第一数学归纳法 (The First Mathematical Induction))

设 $P(n)$ 是关于自然数的一个性质。如果

(i) $P(0)$ 成立;

(ii) 对任意自然数 n, 如果 $P(n)$ 成立, 则 $P(n+1)$ 成立。

那么, $P(n)$ 对所有自然数 n 都成立。

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$$\frac{P(0) \qquad \forall n \in \mathbb{N}.\ \Big(P(n) \to P(n+1)\Big)}{\forall n \in \mathbb{N}.\ P(n)} \quad \text{(第一数学归纳法)}$$

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$$\Big(P(0) \land \forall n \in \mathbb{N}.\ \big(P(n) \to P(n+1)\big)\Big) \to \forall n \in \mathbb{N}.\ P(n).$$

Theorem (第二数学归纳法 (The Second Mathematical Induction))

设 $Q(n)$ 是关于自然数的一个性质。如果

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Theorem (数学归纳法)

第一数学归纳法与第二数学归纳法等价。

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Q : 第二数学归纳法也被称为**"强" (Strong)** 数学归纳法, 它强在何处?

Lemma

第二数学归纳法蕴含第一数学归纳法。

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$$Q(n) \triangleq P(n)$$

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$$P(n) \triangleq Q(0) \wedge \cdots \wedge Q(n)$$

数学归纳法为何成立？

Peano 公理体系刻画了**自然数的递归结构**

Definition (Peano Axioms)

(1) 0 是自然数;

(2) 如果 n 是自然数, 则它的后继 $\mathbf{S}n$ 也是自然数;

(3) 0 不是任何自然数的后继;

(4) 两个自然数相等当且仅当它们的后继相等;

(5) **数学归纳原理:** 如果

  (i) $P(0)$ 成立;

  (ii) 对任意自然数 n, 如果 $P(n)$ 成立, 则 $P(n+1)$ 成立。

那么, $P(n)$ 对所有自然数 n 都成立。

Definition (良序原理 (The Well-Ordering Principle))

自然数集的任意非空子集都有一个最小元。

Theorem

良序原理与 (第一) 数学归纳法等价。

Lemma

(第一) 数学归纳法蕴含良序原理。

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Proof.

By mathematical induction on the size n of non-empty subsets of $\mathbb{N}$.

$$P(k): \text{All subsets of size } k \text{ contain a minimum.}$$

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Basis Step: $P(1)$

Inductive Hypothesis: $P(n)$

Inductive Step: $P(n) \to P(n+1)$

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- $A' \leftarrow A \setminus a$
- $x \leftarrow \min A'$
- Compare x with a

$\square$

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$$\begin{aligned} \text{Basis Step:} &\quad P(1) \\ \text{Inductive Hypothesis:} &\quad P(n) \\ \text{Inductive Step:} &\quad P(n) \to P(n+1) \end{aligned}$$

$$\forall n \in \mathbb{N} : P(n) \quad vs. \quad P(\infty)$$

□

Lemma

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$P(n)$: 任何一个含有 $\leq n$ 的某个自然数的自然数子集都有最小元

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良序原理蕴含 (第一) 数学归纳法。

反证法

设 $P(0)$ 成立且 $\forall n \in \mathbb{N}.\ P(n) \to P(n+1)$ 成立,
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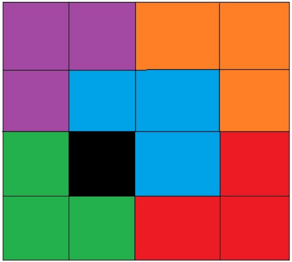
Learn by Examples

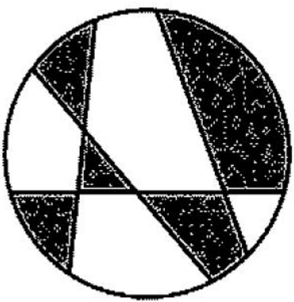
Tiling Puzzle

任何一个缺失了一格的 $2^n \times 2^n$ 的网格都可以被 L 型填满。



Definition (Line Map)

- A blank circle is a line map;
- A line map with a chord (弦) is a line map.



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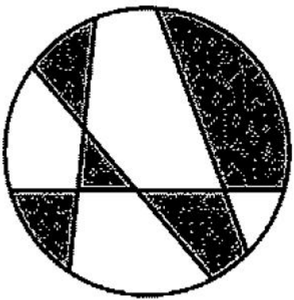

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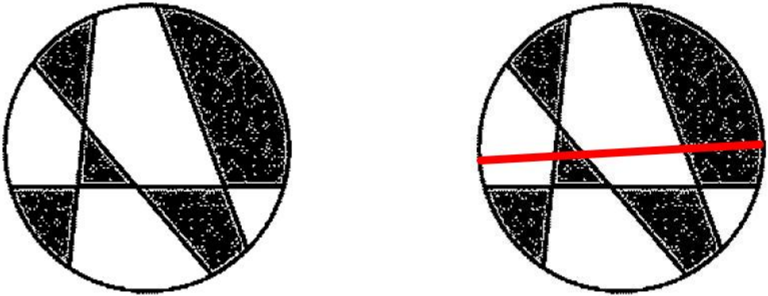

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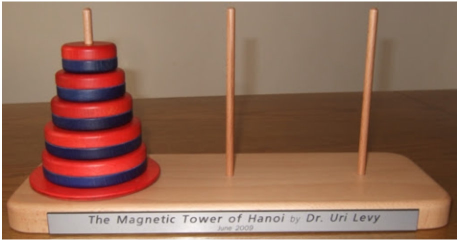
The Tower of Hanoi



$\textsc{Hanoi}(n, A, B, C)$: 借助于 B 柱, 将 n 个盘子从 A 柱移到 C 柱

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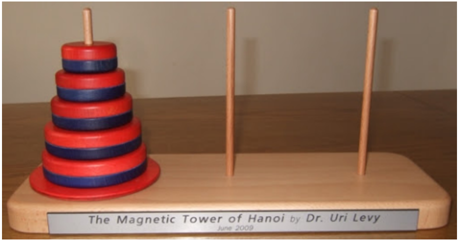

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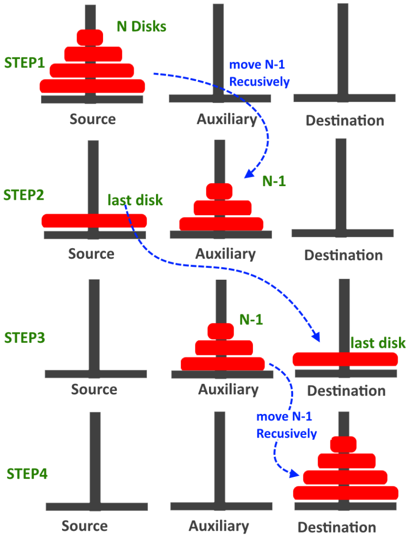
T_n : the **minimum** number of moves for n disks


STEP1
N Disks
move N-1
Recusively
Source
Auxiliary
Destination
STEP2
last disk
N-1
Source
Auxiliary
Destination
STEP3
N-1
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Destination
move N-1
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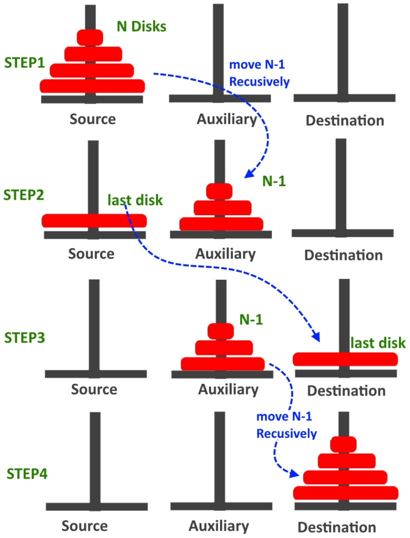
STEP4
Source
Auxiliary
Destination



$$T(n) \leq 2T(n-1) + 1 \qquad (n \geq 1)$$

考虑第一次以及最后一次移动最大盘时的情况

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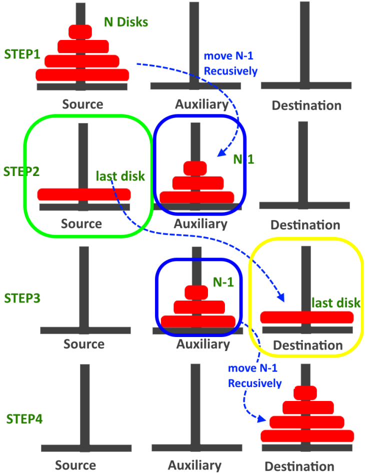

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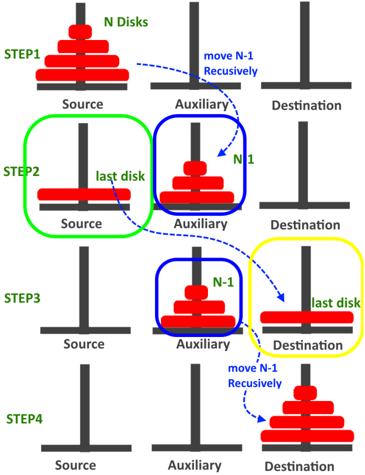

$$T(n) \geq 2T(n-1) + 1 \qquad (n \geq 1)$$

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T(0) &= 0, \\
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T(n) = 2^n - 1, \quad n \geq 0
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Theorem (Fermat's Little Theorem)

对于任意自然数 a 与任意素数 p,

$$a^p \equiv a \pmod{p}.$$

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$$(a+1)^p \equiv a + 1 \pmod{p}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \in \mathbb{N} \quad (0 \leq k \leq n)$$

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$$k = 0 \qquad k = n+1 \qquad 1 \le k \le n$$

Horse Paradox

所有马的颜色都相同。

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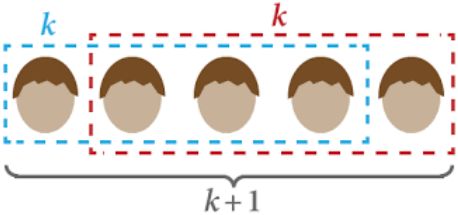

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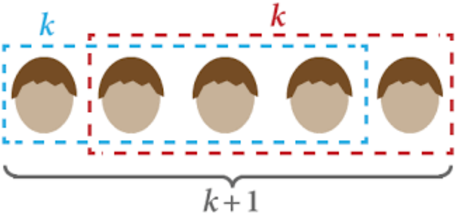

$$n = 1 \not\Longrightarrow n = 2$$

算术基本定理 (The Fundamental Theorem of Arithmetic)

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设 $*$ 是一个满足结合律的二元运算符, 即

$$(a * b) * c = a * (b * c).$$

请证明, $a_1 * a_2 * \cdots * a_n$ $(n \geq 3)$ 的值与括号的使用方式无关。

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对 n 作强数学归纳

$$(\dots) * (\dots)$$

$$F_0 = 0, \qquad F_1 = 1,$$

$$F_n = F(n-1) + F(n-2) \quad (n \geq 2)$$

请证明: $F(n)$ 是偶数当且仅当 $F(n+3)$ 是偶数。

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对 n 作归纳

基础步骤: 命题对 $n = 0, 1$ 成立

归纳假设: $F(n)$ 是偶数当且仅当 $F(n+3)$ 是偶数

$$F(n+1) = F(n) + F(n-1)$$

$$F(n+4) = F(n+3) + F(n+2)$$

Tiling Puzzle

只用 1×1 与 1×2 两种矩形, 拼出 $1 \times n$ 的形状, 有几种不同的拼法?

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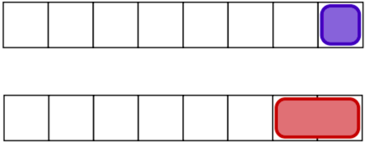

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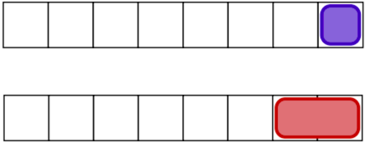

$$T_0 = 0, \qquad T_1 = 1,$$

$$T_n = {\color{blue}T(n-1)} + {\color{red}T(n-2)} \quad (n \geq 2)$$

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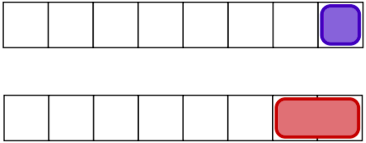

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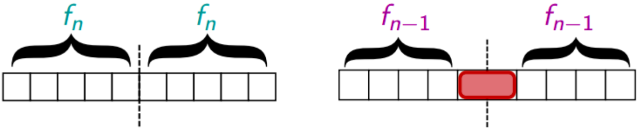

The Blue-eyed Islanders Puzzle

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What effect, if anything, does this *faux pas* have on the tribe?

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there are several blue-eyed people in their tribe).

100 days after the address,

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Theorem (The Blue-eyed Islanders Puzzle)

Suppose that the tribe had $n > 0$ blue-eyed people.

Then n days after the traveller's address,

all n blue-eyed people commit suicide.

By induction on the number n of blue-eyed people in the tribe.

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基础步骤: $n = 1$.

归纳假设: 假设命题对 n 个蓝眼人的情况也成立。

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归纳步骤:

Theorem

对于任何自然数 n, 13^n 都可以写成两个自然数的平方之和。

$$
\begin{aligned}
13^{n+1} &= 13 \cdot 13^n \\
&= (2^2 + 3^2)(a^2 + b^2) \\
&= (\underbrace{2a + 3b}_{x})^2 + (\underbrace{3a - 2b}_{y})^2 \\
&= x^2 + y^2
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&= (\underbrace{13a}_{x})^2 + (\underbrace{13b}_{y})^2 \\
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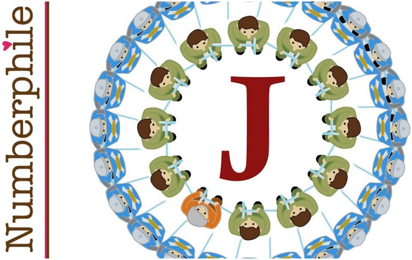
\end{aligned}$$

Josephus Problem



$$
\begin{aligned}
f(1,1) &= 2 \\
f(m+1,n) &= f(m,n) + 2(m+n) \\
f(m,n+1) &= f(m,n) + 2(m+n-1)
\end{aligned}
$$

请证明,

$$\forall m, n \in \mathbb{N}^+.\ f(m,n) = (m+n)^2 - (m+n) - 2n + 2$$

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$$f(1,1)$$

$$f(k,1) \to f(k+1,1)$$

$$f(h,k) \to f(h,k+1) \text{ for any } h$$

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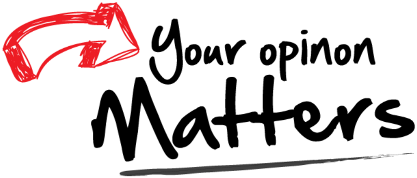
对 $m+n$ 作归纳

gcd


Thank
You!



Office 926

hfwei@nju.edu.cn