(三) 数学归纳法 (Mathematical Induction)

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数学归纳法真得很简单吗?

Theorem (第一数学归纳法 (The First Mathematical Induction))

设 $P(n)$ 是关于自然数的一个性质。如果

(i) $P(0)$ 成立;

(ii) 对任意自然数 n, 如果 $P(n)$ 成立, 则 $P(n+1)$ 成立。

那么, $P(n)$ 对所有自然数 n 都成立。

$$\frac{P(0) \qquad \forall n \in \mathbb{N}.\ \Big(P(n) \to P(n+1)\Big)}{\forall n \in \mathbb{N}.\ P(n)} \quad (\text{第一数学归纳法})$$

$$\Bigg(P(0) \land \forall n \in \mathbb{N}.\ \Big(P(n) \to P(n+1)\Big)\Bigg) \to \forall n \in \mathbb{N}.\ P(n).$$

Theorem (第二数学归纳法 (The Second Mathematical Induction))

设 $Q(n)$ 是关于自然数的一个性质。如果

(i) $Q(0)$ 成立;

(ii) 对任意自然数 n, 如果 $Q(0), Q(1), \ldots, Q(n)$ 都成立,
则 $Q(n+1)$ 成立。

那么, $Q(n)$ 对所有自然数 n 都成立。

$$\frac{Q(0) \quad \forall n \in \mathbb{N}.\ \Big((Q(0) \land \cdots \land Q(n)) \to Q(n+1) \Big)}{\forall n \in \mathbb{N}.\ Q(n)} \quad (\text{第二数学归纳法})$$

$$\Bigg(Q(0) \land \forall n \in \mathbb{N}.\ \Big((Q(0) \land \cdots \land Q(n)) \to Q(n+1) \Big) \Bigg) \to \forall n \in \mathbb{N}.\ Q(n).$$

Theorem (数学归纳法)

第一数学归纳法与第二数学归纳法等价。

Q : 第二数学归纳法也被称为**"强" (Strong)** 数学归纳法, 它强在何处?

Lemma

第二数学归纳法蕴含第一数学归纳法。

$$Q(n) \triangleq P(n)$$

Lemma

第一数学归纳法蕴含第二数学归纳法。

$$P(n) \triangleq Q(0) \land \cdots \land Q(n)$$

数学归纳法为何成立?

Peano 公理体系刻画了**自然数的递归结构**

Definition (Peano Axioms)

(1) 0 是自然数;

(2) 如果 n 是自然数, 则它的后继 $\mathbf{S}n$ 也是自然数;

(3) 0 不是任何自然数的后继;

(4) 两个自然数相等当且仅当它们的后继相等;

(5) **数学归纳原理:** 如果

  (i) $P(0)$ 成立;

  (ii) 对任意自然数 n, 如果 $P(n)$ 成立, 则 $P(n+1)$ 成立。

那么, $P(n)$ 对所有自然数 n 都成立。

Definition (良序原理 (The Well-Ordering Principle))

自然数集的任意非空子集都有一个最小元。

Theorem

良序原理与 (第一) 数学归纳法等价。

Lemma

(第一) 数学归纳法蕴含良序原理。

Proof.

By mathematical induction on the size n of non-empty subsets of $\mathbb{N}$.

$$P(n): \text{All subsets of size } n \text{ contain a minimum.}$$

Basis Step: $P(1)$

Inductive Hypothesis: $P(n)$

Inductive Step: $P(n) \to P(n+1)$

- $A' \leftarrow A \setminus a$
- $x \leftarrow \min A'$
- Compare x with a

$$\forall n \in \mathbb{N} : P(n) \quad \textit{vs.} \quad P(\infty)$$

Lemma

(第一) 数学归纳法蕴含良序原理。

$P(n)$: 任何一个含有 $\leq n$ 的某个自然数的自然数子集都有最小元

$$P(0)$$

$$P(n) \to P(n+1)$$

$$\exists x.\ x \leq n \qquad \forall x.\ x \geq n+1$$

Lemma

良序原理蕴含 (第一) 数学归纳法。

反证法

设 $P(0)$ 成立且 $\forall n \in \mathbb{N}.\ P(n) \to P(n+1)$ 成立，
但 $\exists n \in \mathbb{N}.\ P(n)$ 不成立

$$A = \{k \in \mathbb{N} \mid \neg P(k)\} \neq \emptyset$$

$$m \triangleq \min A \quad (\text{by 良序原理}) \qquad \neg P(m)$$

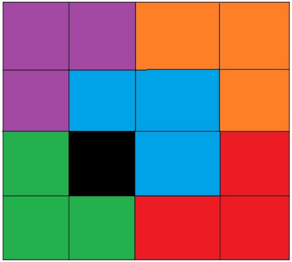
$$m \neq 0 \qquad m \geq 1$$

$$n \triangleq m - 1 \qquad P(n) \qquad P(n+1) \qquad P(m)$$

Learn by Examples

Tiling Puzzle

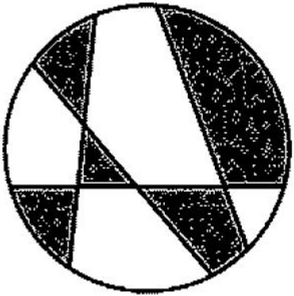
任何一个缺失了一格的 $2^n \times 2^n$ 的网格都可以被 L 型填满。

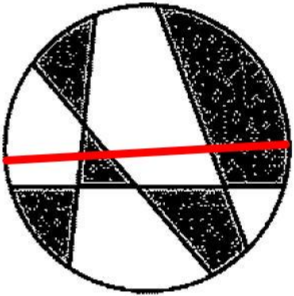


对自然数 n 作归纳

Definition (Line Map)

- A blank circle is a line map;
- A line map with a chord (弦) is a line map.

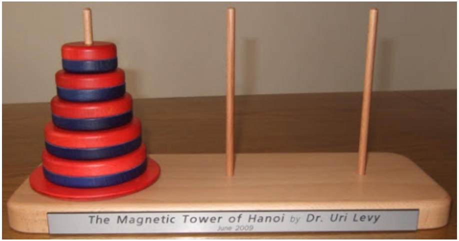




Theorem

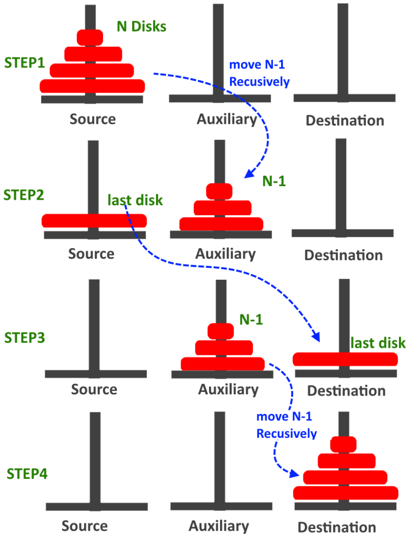
Any line map can be two-colored.

The Tower of Hanoi



$\textsc{Hanoi}(n, A, B, C)$: 借助于 B 柱, 将 n 个盘子从 A 柱移到 C 柱

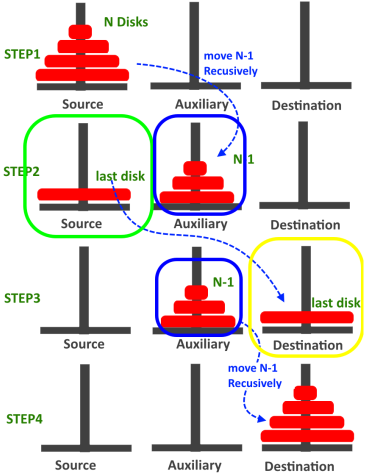
T_n : the **minimum** number of moves for n disks



$$T(n) \leq 2T(n-1) + 1 \qquad (n \geq 1)$$

考虑第一次以及最后一次移动最大盘时的情况

另外 $(n-1)$ 个盘子一定在同一个柱子上



$$T(n) \geq 2T(n-1)+1 \qquad (n \geq 1)$$

$$
\begin{aligned}
T(0) &= 0, \\
T(n) &= 2T(n-1) + 1, \quad n \ge 1
\end{aligned}
$$

$$T(n) = 2^n - 1, \quad n \ge 0$$

Theorem (Fermat's Little Theorem)

对于任意自然数 a 与任意素数 p,

$$a^p \equiv a \pmod p.$$

对自然数 a 作归纳 (对于任意素数 p)

$$(a+1)^p = a^p + \binom{p}{1}a^{p-1} + \binom{p}{2}a^{p-2} + \cdots + \binom{p}{p-1}a + 1$$

$$\binom{p}{k} = \frac{p(p-1)\dots(p-k+1)}{k!} \equiv 0 \pmod p \quad (1 \le k \le p-1)$$

$$(a+1)^p \equiv a + 1 \pmod p$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \in \mathbb{N} \quad (0 \le k \le n)$$

对自然数 n 作归纳 (对于任意的 $0 \le k \le n$)

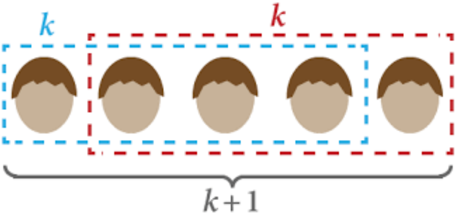
$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

$$k = 0 \qquad k = n+1 \qquad 1 \le k \le n$$

Horse Paradox

所有马的颜色都相同。

对马的数目 $n \geq 1$ 作归纳



$$n = 1 \not\Longrightarrow n = 2$$

算术基本定理 (The Fundamental Theorem of Arithmetic)

任何一个 ≥ 2 的自然数都可以(唯一)写为若干素数的乘积。

对自然数 n 作强数学归纳

设 $*$ 是一个满足结合律的二元运算符, 即

$$(a * b) * c = a * (b * c).$$

请证明, $a_1 * a_2 * \cdots * a_n \ (n \geq 3)$ 的值与括号的使用方式无关。

对 n 作强数学归纳

$$(\ldots) * (\ldots)$$

$$F_0 = 0, \qquad F_1 = 1,$$

$$F_n = F(n-1) + F(n-2) \quad (n \geq 2)$$

请证明: $F(n)$ 是偶数当且仅当 $F(n+3)$ 是偶数。

对 n 作归纳

基础步骤: 命题对 $n = 0, 1$ 成立

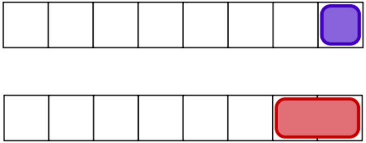
归纳假设: 对任意 $n \leq k$, $F(n)$ 是偶数当且仅当 $F(n+3)$ 是偶数

$$F(k+1) = F(k) + F(k-1)$$

$$F(k+4) = F(k+3) + F(k+2)$$

Tiling Puzzle

只用 1×1 与 1×2 两种矩形, 拼出 $1 \times n$ 的形状, 有几种不同的拼法?



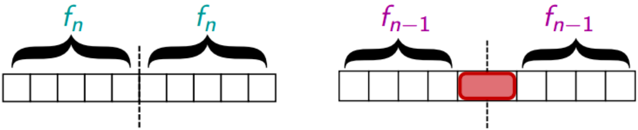
$$T_0 = 1, \qquad T_1 = 1,$$

$$T_n = {\color{blue}T(n-1)} + {\color{red}T(n-2)} \quad (n \geq 2)$$

$$\boxed{F_n = T_n}$$

$$\boxed{F_n = T_n}$$

$$F_{2n} = (F_n)^2 + (F_{n-1})^2$$



Blue Eyes:

The Hardest Logic Puzzle in the World

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The blue-eyed islanders puzzle

5 February, 2008 in diversions, math.GM, math.IT, math.LO | Tags: blue-eyed islander puzzle, common information, logic puzzle, mathematical induction

Given that there has recently been a lot of discussion on this blog about this logic puzzle, I thought I would make a dedicated post for it (and move all the previous comments to this post). The text here is adapted from an earlier web page of mine from a few years back.

The Blue-eyed Islanders Puzzle

There is an island upon which a tribe resides. The tribe consists of 1000 people, with various eye colours. Yet, **their religion forbids them to know their own eye color, or even to discuss the topic**; thus, each resident can (and does) see the eye colors of all other residents, but has no way of discovering his or her own (there are no reflective surfaces). **If** a tribesperson does discover his or her own eye color, then their religion compels them to **commit ritual suicide at noon the following day in the village square for all to witness**. All the tribespeople are **highly logical and devout**, **and** they all know that each other is also highly logical and devout (and they all know that they all know that each other is highly logical and devout, and so forth).

Of the 1000 islanders, it turns out that **100 of them have blue eyes** and **900 of them have brown eyes**, although the islanders are not initially aware of these statistics (each of them can of course only see 999 of the 1000 tribespeople).

One day, a **blue-eyed foreigner** visits to the island and wins the complete trust of the tribe.

One evening, he addresses the entire tribe to thank them for their hospitality.

However, not knowing the customs, the foreigner makes the mistake of mentioning eye color in his address, remarking "**how unusual it is to see another blue-eyed person like myself in this region of the world**".

What effect, if anything, does this *faux pas (失礼)* have on the tribe?

The foreigner **has no effect**,

because his comments do **not** tell the tribe anything

that they do not already know

(everyone in the tribe can already see that

there are several blue-eyed people in their tribe).

100 days after the address,

all the blue eyed people commit suicide.

Theorem (The Blue-eyed Islanders Puzzle)

Suppose that the tribe had $n > 0$ blue-eyed people.

Then n days after the traveller's address,

all n blue-eyed people commit suicide.

By induction on the number n of blue-eyed people in the tribe.

基础步骤: $n = 1$.

这个**唯一的蓝眼人**的内心独白: **"你直接念我身份证吧"**

归纳假设: 有 n 个蓝眼人时, 前 $n-1$ 天无人自杀, 第 n 天集体自杀。

归纳步骤: 考虑恰有 $n+1$ 个蓝眼人的情况。

每个**蓝眼人**都如此推理: 我看到了 n 个蓝眼人, 他们应该在第 n 天集体自杀。

但是, 每个蓝眼人都在等其它 n 个蓝眼人自杀, 因此, 第 n 天无人自杀。

每个**蓝眼人**继续推理: 一定不止 n 个蓝眼人, 但是我看到的其余人都不是蓝眼。

所以, **"小丑竟是我自己"**。

“how unusual it is to see another blue-eyed person like myself in this region of the world”.

考虑 $n = 1, n = 2$ 的简单情况

“我不知道 ...”

“我知道 ...”

“我知道你知道 ...”

“我知道你知道我知道 ...”

Theorem

对于任何自然数 n, 13^n 都可以写成两个自然数的平方之和。

$$
\begin{aligned}
13^{n+1} &= 13 \cdot 13^n \\
&= (2^2 + 3^2)(a^2 + b^2) \\
&= (\underbrace{2a + 3b}_{x})^2 + (\underbrace{3a - 2b}_{y})^2 \\
&= x^2 + y^2
\end{aligned}
$$

Theorem

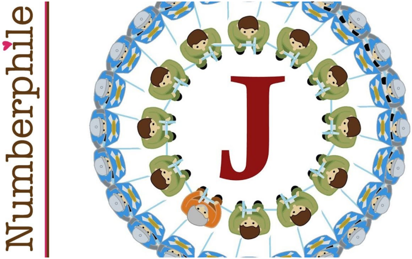
对于任何自然数 n, 13^n 都可以写成两个自然数的平方之和。

$$13^0 = 1^2 + 0^2$$

$$13^1 = 2^2 + 3^2$$

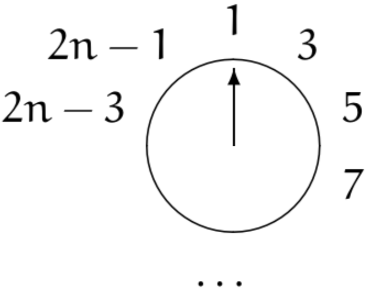
$$\begin{aligned}
13^{n+2} &= 13^2 \cdot 13^n \\
&= 13^2(a^2 + b^2) \\
&= (\underbrace{13a}_{x})^2 + (\underbrace{13b}_{y})^2 \\
&= x^2 + y^2
\end{aligned}$$

Josephus Problem



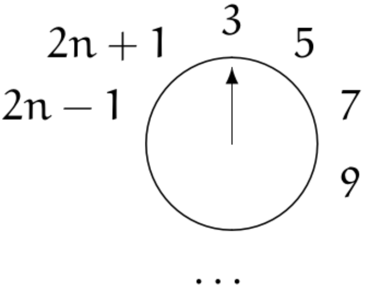
$$J(12) = 9$$

$2n$ 个人



$$J(2n) = 2J(n) - 1, n \geq 1$$

$2n + 1$ 个人

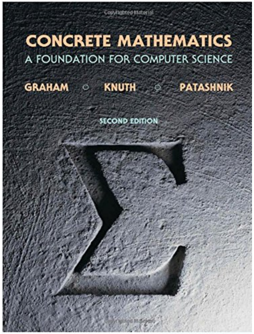


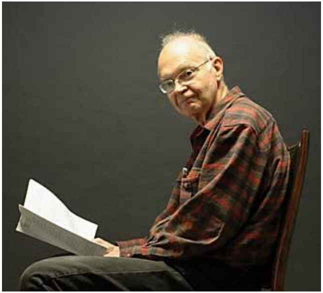
$$J(2n+1) = 2J(n) + 1, n \geq 1$$

$$J(1) = 1$$

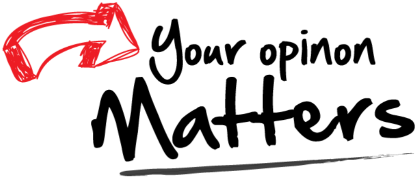
$$J(2n) = 2J(n) - 1, n \geq 1$$

$$J(2n+1) = 2J(n) + 1, n \geq 1$$






Thank
You!



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