(十二) 图论: 匹配与网络流
(Matching and Network Flow)

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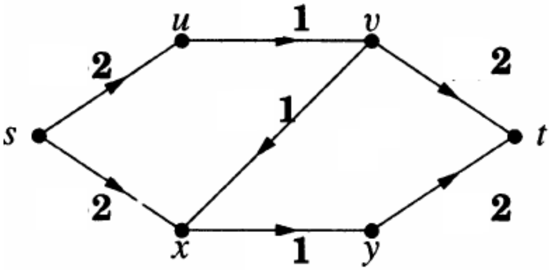
Definition (Network (网络))

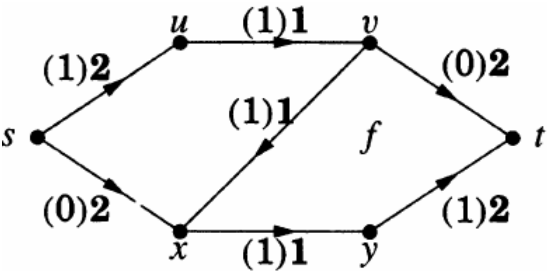
A network is a digraph with

- a distinguished source vertex s,
- a distinguished sink vertex t,
- a capacity $c(e) \geq 0$ on each edge e



Definition (Flow (流))

A flow f is a function that assigns a value $f(e)$ to each edge e.



Definition (Feasible Flow)

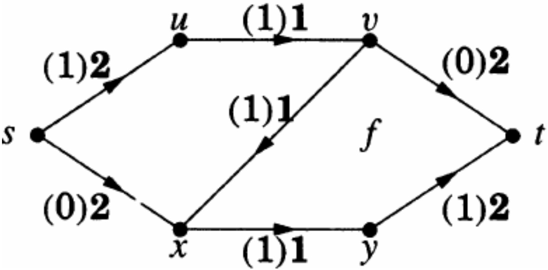
A flow f is feasible if it satisfies

Capacity Constraints:

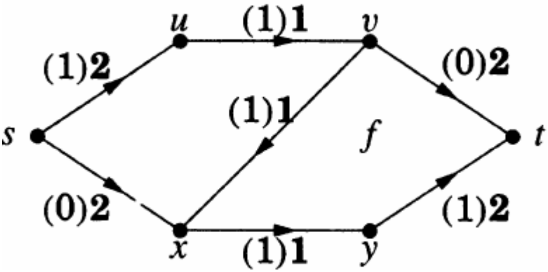
$$\forall e \in E.\ 0 \le f(e) \le c(e)$$

Flow Conservation:

$$\forall v \in V.\ f^{+}(v) = f^{-}(v)$$



$$f^{+}(v) = \sum_{(v,w)\,\in\, E} f(v,w) \qquad f^{-}(v) = \sum_{(u,v)\,\in\, E} f(u,v)$$



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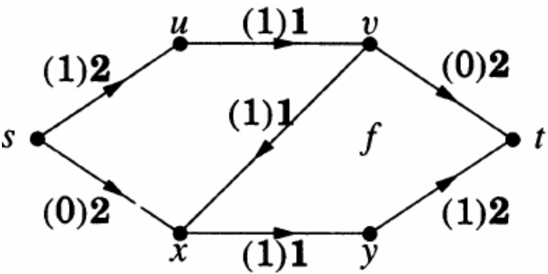

$$f^+(U) = \sum_{u\in U, v\in \overline{U}, (u,v)\in E} f(u,v)$$

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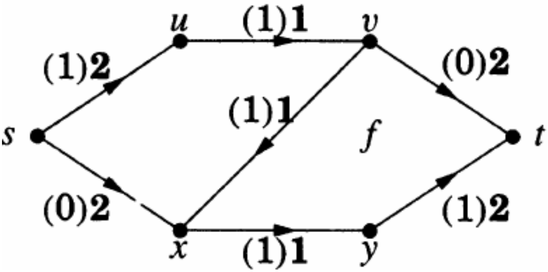

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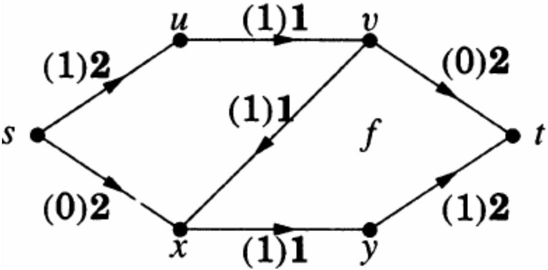
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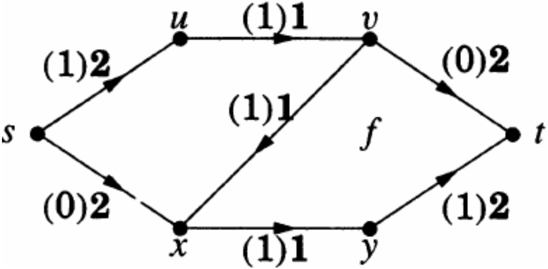
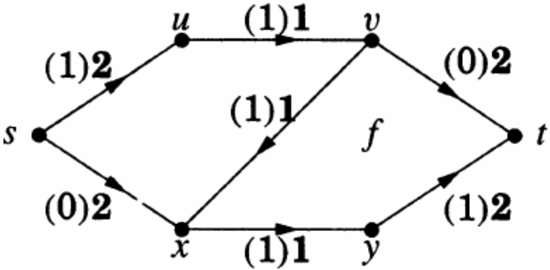


$$\forall U \subseteq (V - \{s, t\}).\ f^{+}(U) = f^{-}(U)$$



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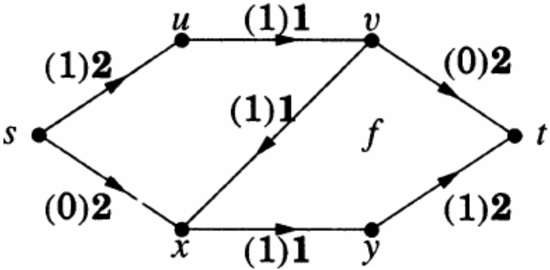
$$s \in U \land t \notin U \implies f^+(U) - f^-(U) =$$



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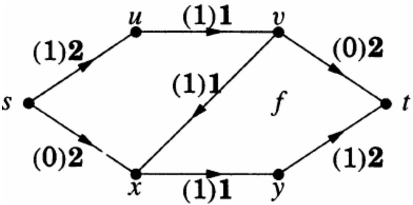
Definition (Value (值))

The value $\mathsf{val}(f)$ of a flow f is

$$\mathsf{val}(f) = f^{-}(t) = f^{+}(s).$$



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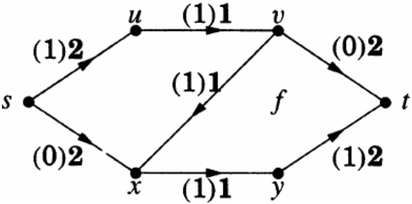

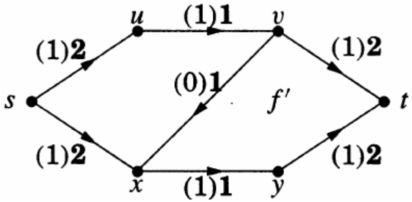

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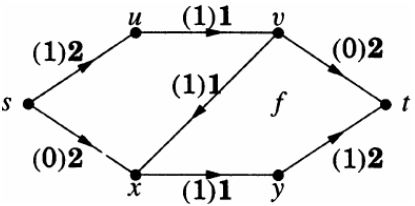
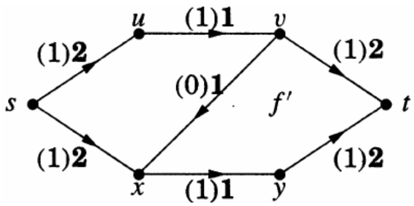
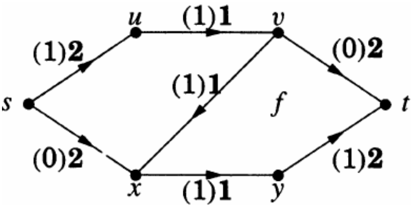


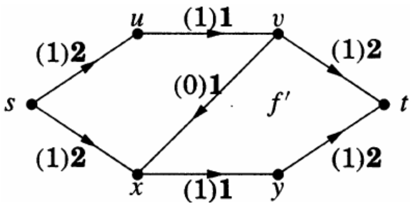


Definition (Maximum Flow (最大流))

A maximum flow is a feasible flow of maximum value.

$$s - -x \textcolor{red}{--} v - -t$$





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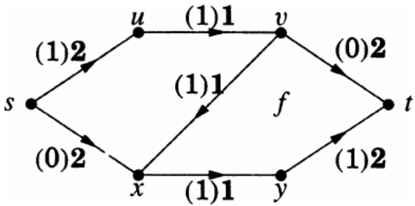
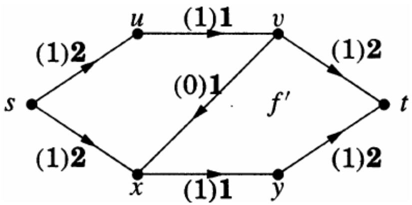




Definition (f-augmenting Paths (增广路径))

When f is a feasible flow, an f-augmenting path is a $s \sim t$ path P in the underlying graph such that for each edge $e \in E(P)$,

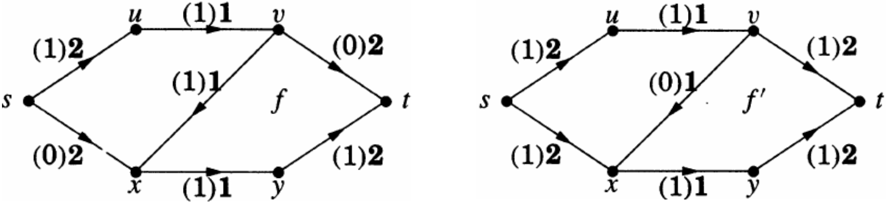
(a) if P follows e in the forward direction, then $f(e) < c(e)$;

(b) if P follows e in the backward direction, then $f(e) > 0$.

Definition (f-augmenting Paths)

Let P be an f-augmenting path.

$$\epsilon(e) = \begin{cases} c(e) - f(e) & \text{if } e \text{ is forward on } P \\ f(e) & \text{if } e \text{ is backward on } P \end{cases}$$



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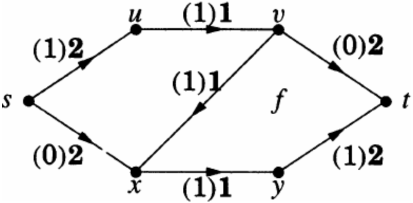
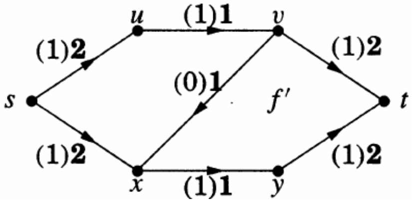
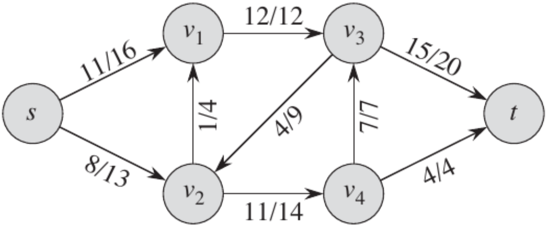




An f-augmenting path leads to a flow with larger value.

$$\min_{e \in E(P)} \epsilon(e)$$



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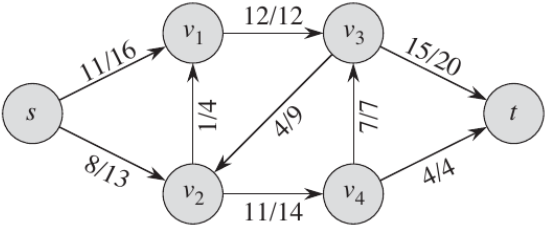

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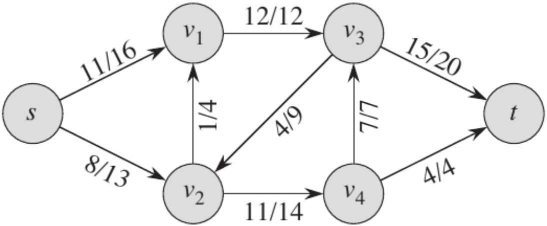

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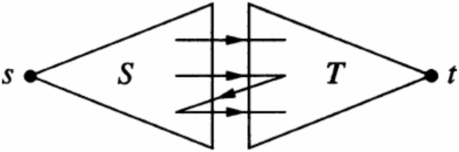
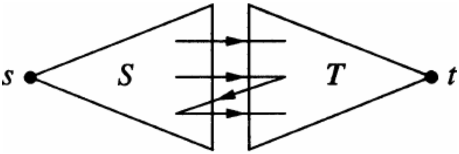
Definition (Source/Sink Cut (割))

In a network, a source/sink cut $[S, T]$ consists of the edges from a source set S to a sink set T, where

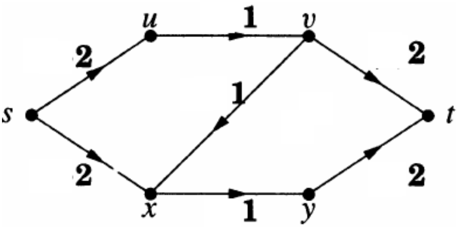
$$(T = V - S) \wedge (s \in S) \wedge (t \in T)$$





Definition (Capacity of Cut (割的容量))

$$\mathsf{cap}(S,T)=\sum_{u\in S,v\in T,uv\in E}c(u,v)$$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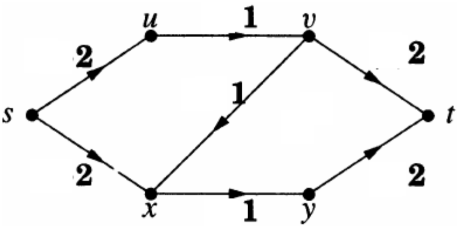
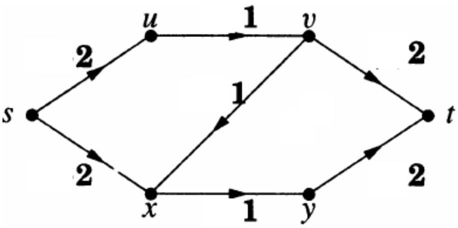


Definition (Minimum Cut (最小割))

A minimum cut is a cut of minimum value.



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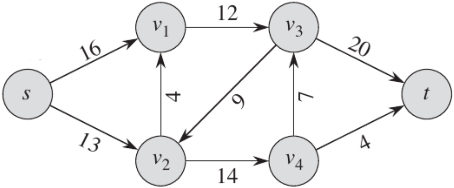
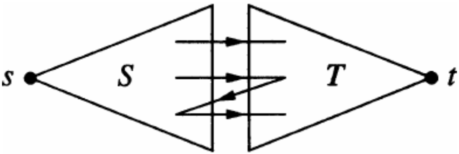


Theorem (Weak Duality (弱对偶定理))

Let f be any feasible flow and $[S, T]$ be any source/sink cut.

$$\mathsf{val}(f) \leq \mathsf{cap}(S, T).$$



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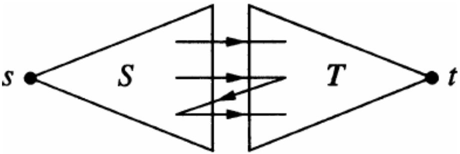

$$\mathrm{val}(f) = f^{+}(S) - f^{-}(S)$$

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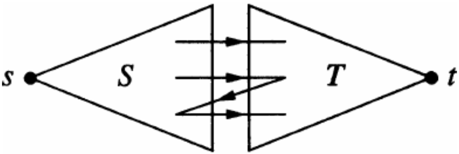

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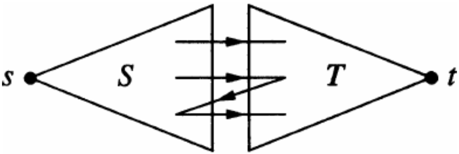

$$\mathsf{val}(f) = f^+(S) - f^-(S) \leq f^+(S) \leq \mathsf{cap}(S, T)$$

Lemma

$$\max_f \mathit{val}(f) \le \min_{[S,T]} \mathit{cap}(S, T)$$

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What if $\mathsf{val}(f) = \mathsf{cap}(S,T)$ for some flow f and some cut $[S,T]$?

Lemma

$$\max_{f} \mathit{val}(f) \leq \min_{[S,T]} \mathit{cap}(S,T)$$

What if $\mathsf{val}(f) = \mathsf{cap}(S,T)$ for some flow f and some cut $[S,T]$?

f is maximum and $[S,T]$ is minimum

$$\mathsf{val}(f) = f^+(S) - f^-(S) = f^+(S) = \mathsf{cap}(S, T)$$

$$\mathsf{val}(f) = f^{+}(S) - f^{-}(S) = f^{+}(S) = \mathsf{cap}(S, T)$$

$$\boxed{f^{-1}(S) = 0 \land f^{+}(S) = \mathsf{cap}(S, T)}$$

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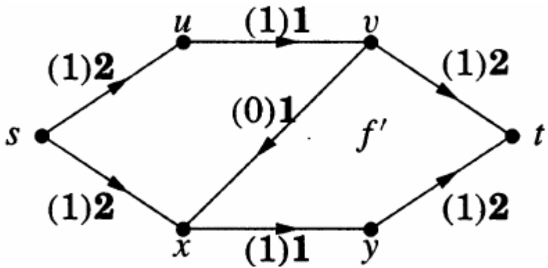

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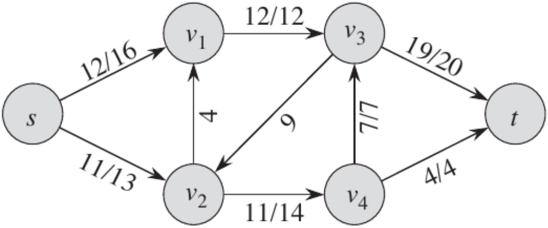


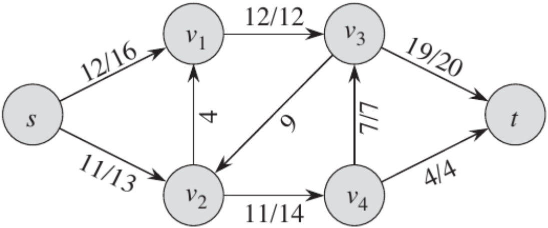
Theorem (Max-flow Min-cut Theorem (Ford and Fulkerson; 1956))

$$\max_{f} \mathit{val}(f) = \min_{[S,T]} \mathit{cap}(S,T)$$

(Strong Duality)

L. R. Ford Jr. (1927 ∼ 2017)

D. R. Fulkerson (1924 ∼ 1976)



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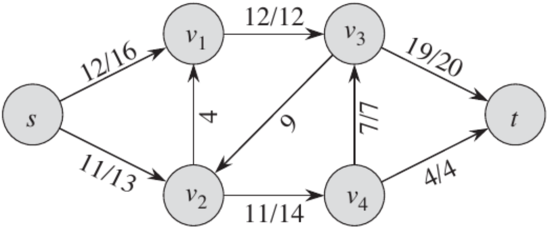
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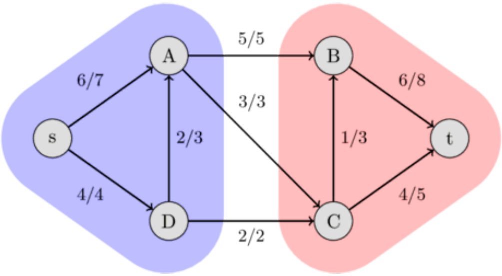
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Theorem

A feasible flow f is maximum iff there are no f-augmenting paths.

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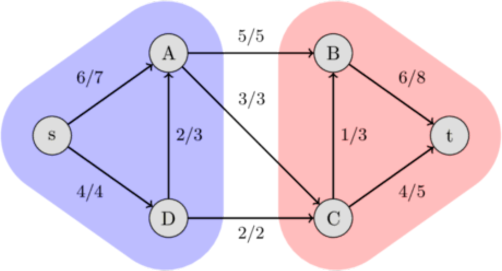

$S \triangleq \{$the vertices reachable from s along partial f-augmenting paths$\}$

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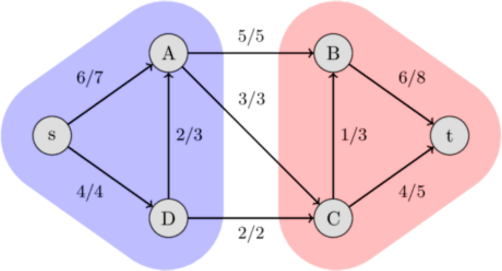
$$T \triangleq V - S$$



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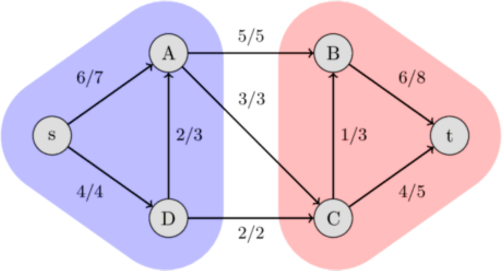

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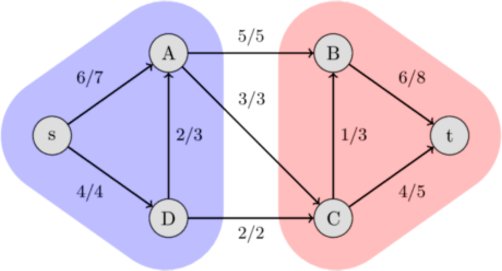

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Repeatedly finding f-augmenting paths until no more ones exist.

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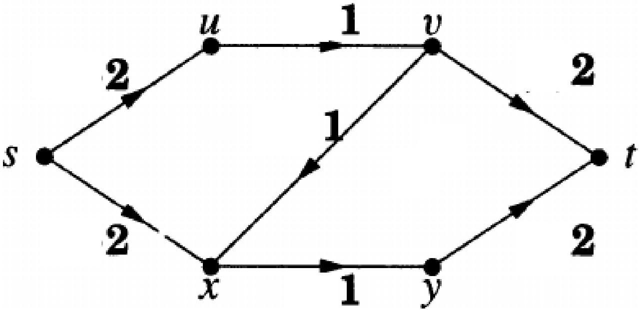

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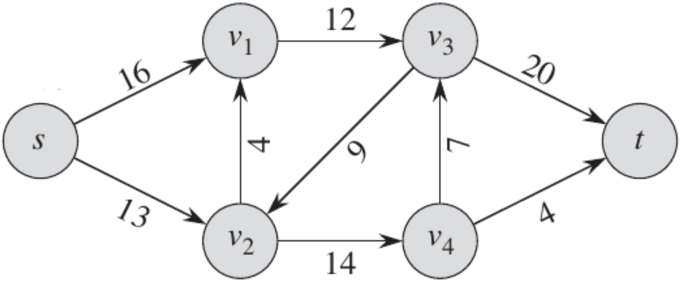

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Using BFS (Breadth-first Search) to find f-augmenting paths.

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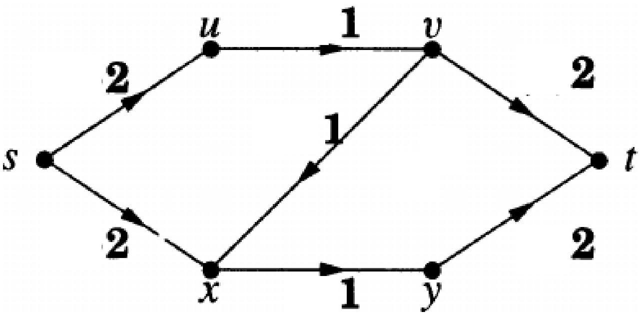
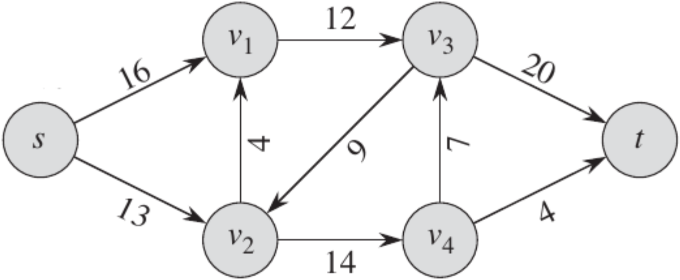


The Edmonds-Karp Algorithm

Using BFS (Breadth-first Search) to find f-augmenting paths.



Theorem (Hall Theorem; 1935)

There is an X-perfect matching of G iff

$$\forall W \subseteq X.\ \left|W\right| \le \left|N_G(W)\right|$$

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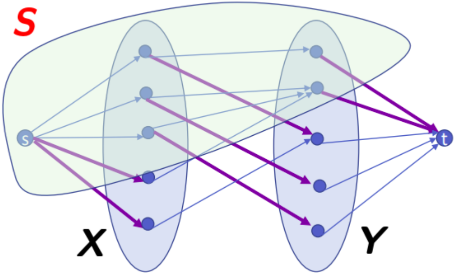

$$\forall x \in X.\ c(s,x) = 1 \quad \forall y \in Y.\ c(y,t) = 1 \quad \forall x \in X, y \in Y.\ c(x,y) = \infty$$

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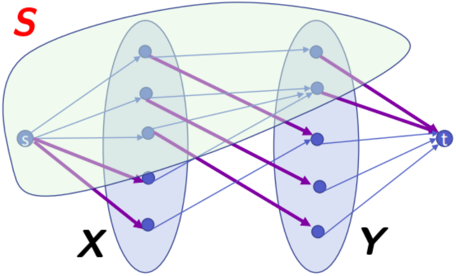

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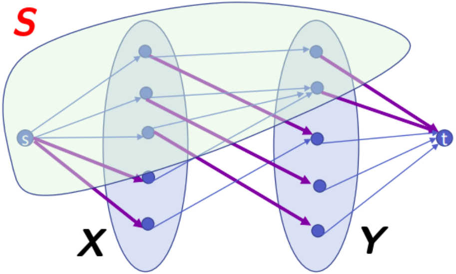
We need to show that $\max\limits_f \mathsf{val}(f) = \left|X\right|$.

$$\text{We need to show that } \min_{[S,\overline{S}]} \mathsf{cap}(S,\overline{S}) = \left|X\right|.$$



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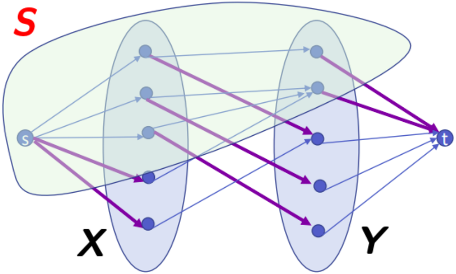

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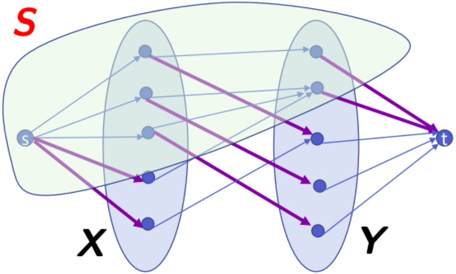
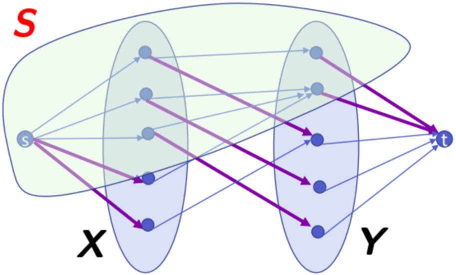


$$\min_{[S,\overline{S}]} \mathsf{cap}(S, \overline{S}) \le \left|X\right|$$

Therefore, we need to show that $\min\limits_{[S,\overline{S}]} \mathsf{cap}(S, \overline{S}) \ge \left|X\right|$.

Let $[S, \overline{S}]$ be a minimum cut. We need to show that $\mathsf{cap}(S, \overline{S}) = \left|X\right|$.



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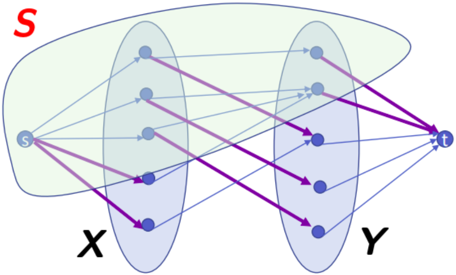

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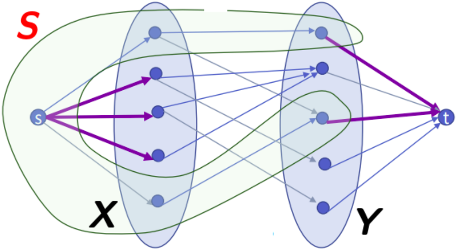

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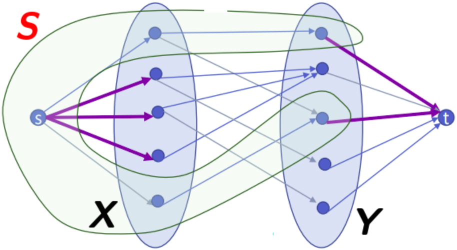
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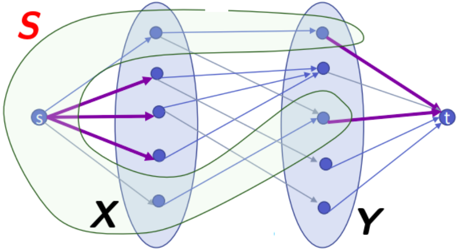
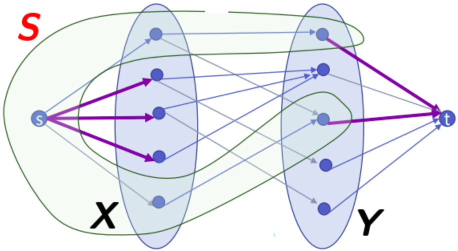


$$\mathrm{cap}(S, \overline{S}) = \sum_{u \in S, v \in \overline{S}} c(x, y)$$



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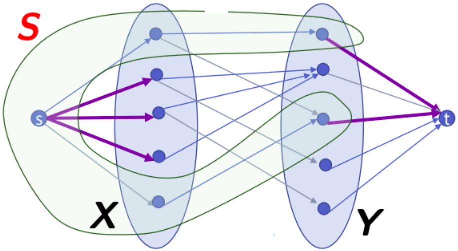

$$\begin{aligned}
\text{cap}(S, \overline{S}) &= \sum_{u \in S, v \in \overline{S}} c(x, y) \\
&= \sum_{v \in \overline{S} \cap X} c({\color{red}s}, v) + \sum_{u \in S \cap Y} c(u, {\color{red}t}) \\
&= \left|X\right| - \left|S \cap X\right| + \left|S \cap Y\right|
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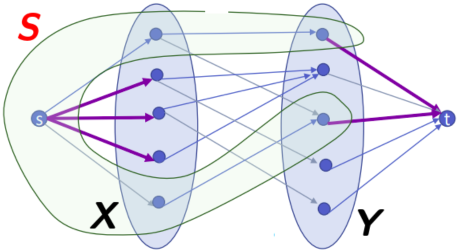

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&\geq \left|X\right| - \left|S \cap X\right| + \left|N(S \cap X)\right|
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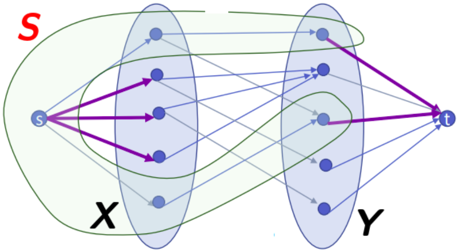

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&= \left|X\right| - \left|S\cap X\right| + \left|S\cap Y\right| \\
&{\color{red}\geq} \left|X\right| - \left|S\cap X\right| + {\color{red}\left|N(S\cap X)\right|} \\
&{\color{blue}\geq} \left|X\right|
\end{aligned}
$$

Theorem (König (1931), Egerváry (1931))

If G is a bipartite graph, then the maximum size of a mathching in G equals the minimum size of a vertex cover of G

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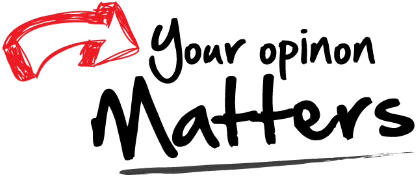
Thank
You!



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