(十一) 图论: 平面图与图着色
(Planarity and Coloring)

魏恒峰

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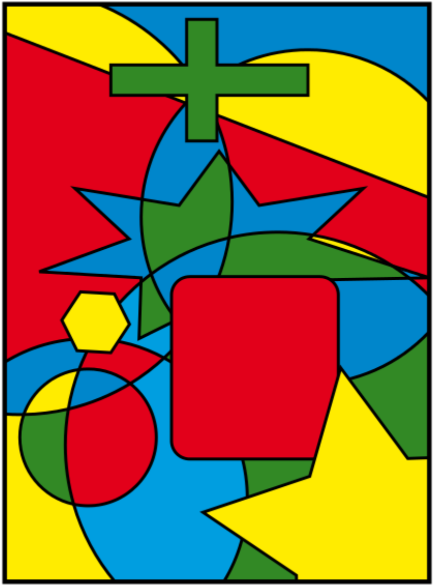
2021 年 05 月 20 日



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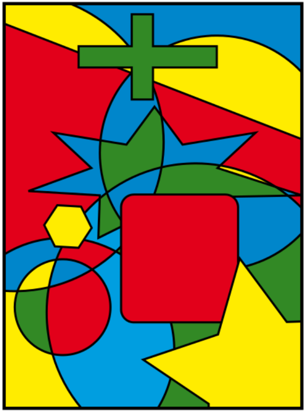

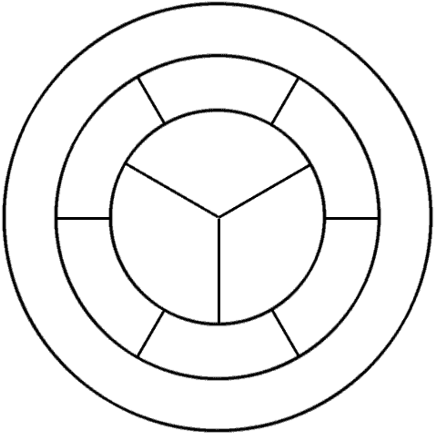

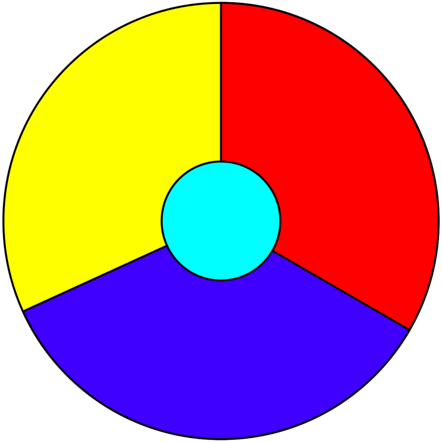

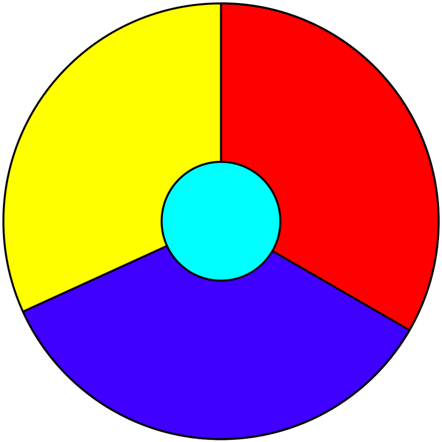

Every region is adjacent to the other 3 regions.

What if we have a map which contains 5 regions so that every region is adjacent to the other 4 regions?

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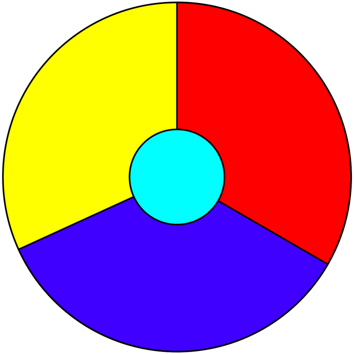
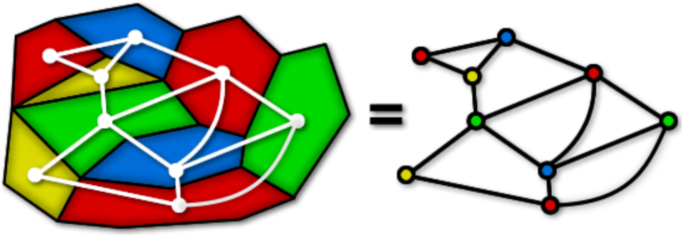
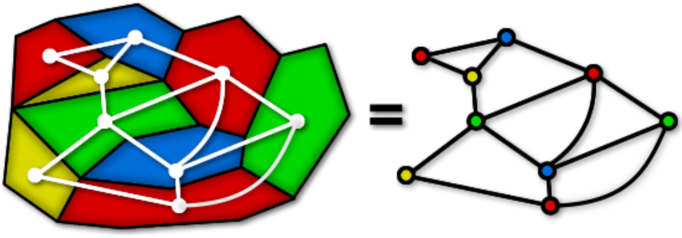
What does Four Color Theorem to do with Graph Theory?







Every map produces a planar graph.

Theorem (Four Color Theorem (Kenneth Appel, Wolfgang Haken; 1976))

Every simple planar graph is 4-colorable.

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I will *not* show its proof (which I don't understand either)!

Theorem

Every simple planar graph is 6-colorable.

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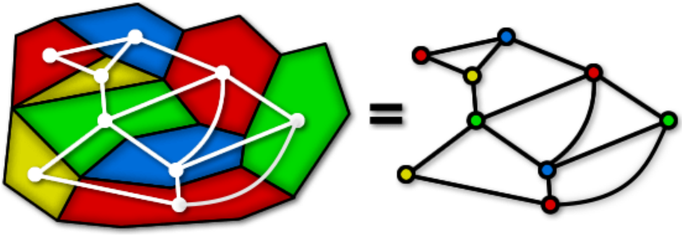
Theorem (Percy John Heawood (1890))

Every simple planar graph is 5-colorable.



Graph Coloring Problem

Definition (k-Colorable (k-可着色的))

If G is a connected undirected graph without loops,
then G is k-colorable if its vertices can be colored in k colors
so that adjacent vertices have different colors.

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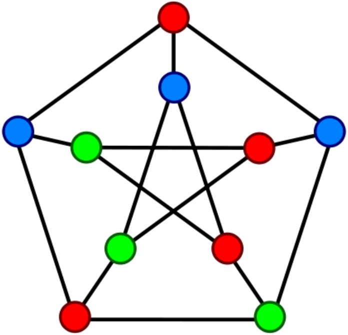

The Petersen graph is ≥ 3-colorable.

Definition (k-Chromatic (k-色数的))

If G is k-colorable, but is not $(k-1)$-colorable, then G is k-chromatic.

$$\chi(G) = k$$

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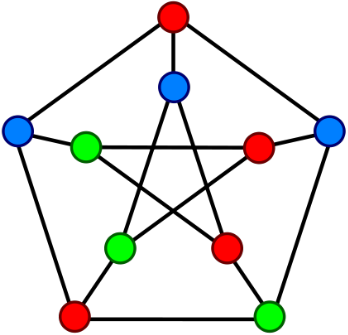

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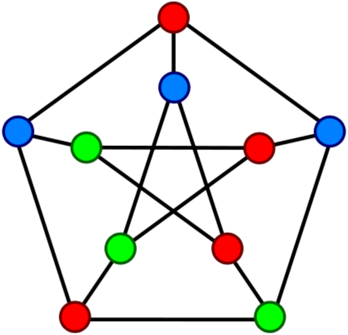

The Petersen graph is 3-chromatic.
(It contains an odd cycle.)

Lemma

A graph is 1*-colorable iff*

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A graph is 1*-colorable iff it is the the empty graph with 1 vertex.*

Theorem

A graph is 2-colorable iff

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A graph is 2-colorable iff it is bipartite.

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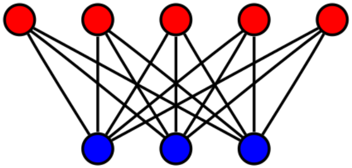
A graph is 2-colorable iff it is *bipartite*.



$K_{5,3}$

Theorem

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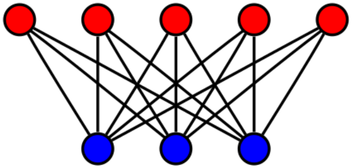

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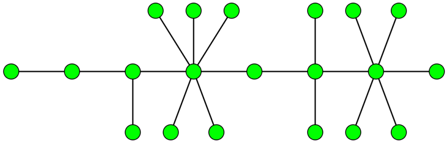

Trees are bipartite.

Theorem

The 3-coloring problem (i.e., testing whether a graph is 3-colorable or not) is NP-complete *(HARD!).*

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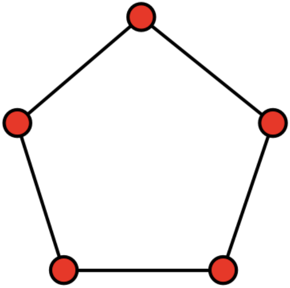

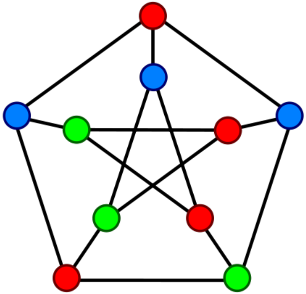

Theorem

The 4-coloring problem is NP-complete *(HARD!)*.

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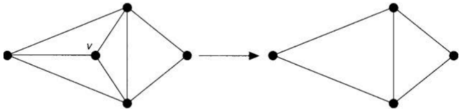

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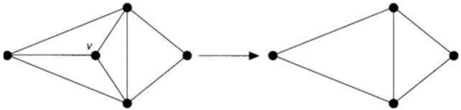

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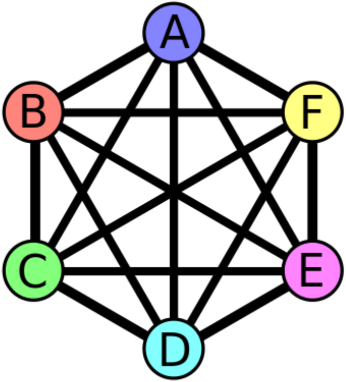

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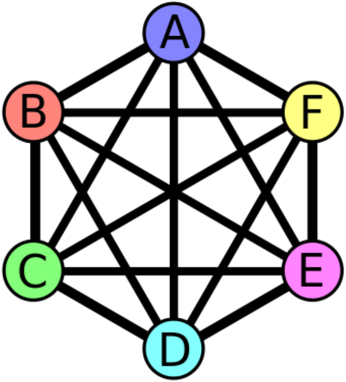

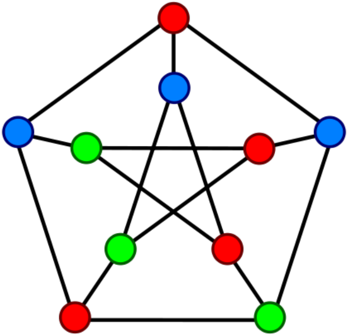

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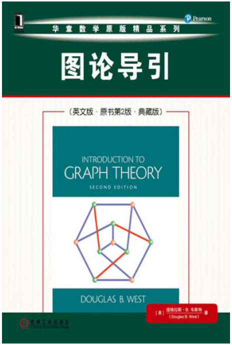

Theorem 5.1.22

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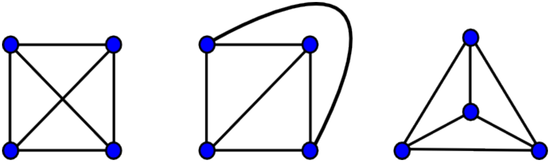

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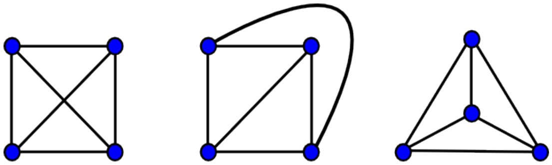
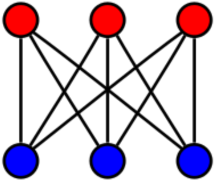


Theorem (K. Wagner (1936); I. Fáry (1948))

Every simple planar graph can be drawn with straight lines.

Theorem (Kazimierz Kuratowski, 1930)

The utility graph $K_{3,3}$ *is non-planar.*



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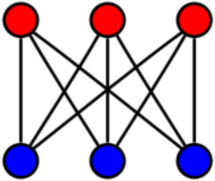

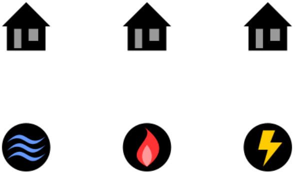

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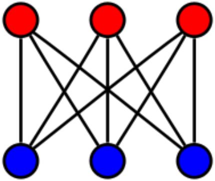
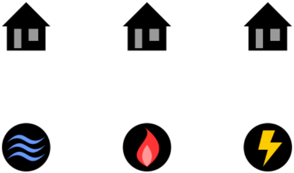
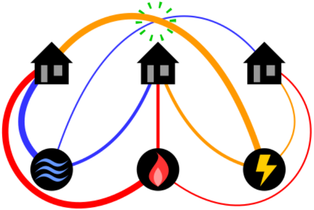
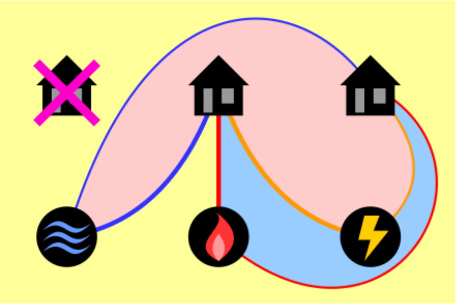
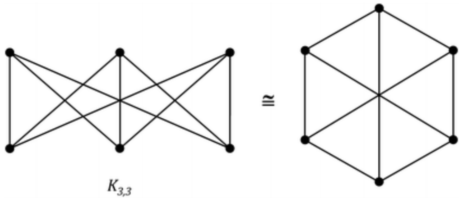
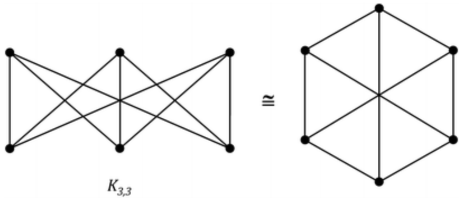









$K_{3,3}$
$\cong$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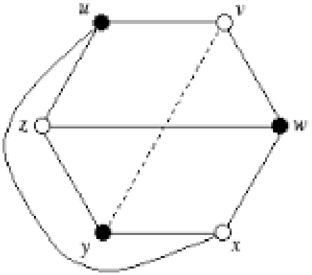
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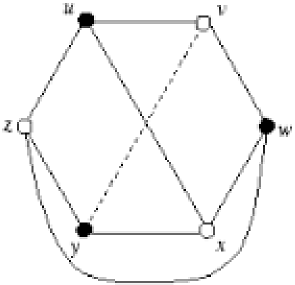
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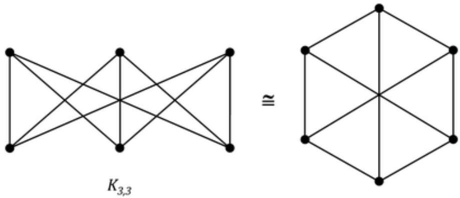
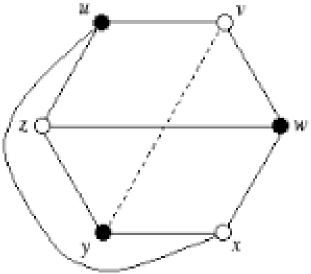
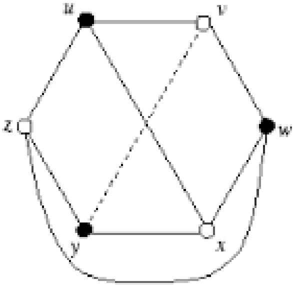


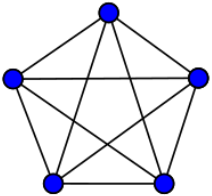




$$\textcolor{red}{\mathrm{cr}}(K_{3,3}) = 1$$

Theorem

K_5 is non-planar.



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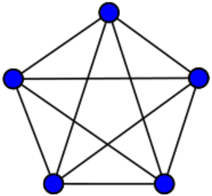

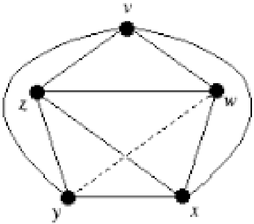

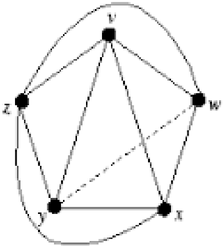

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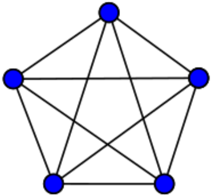

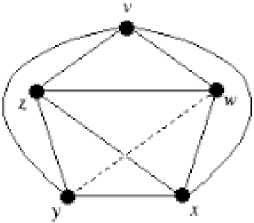

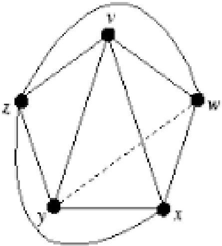

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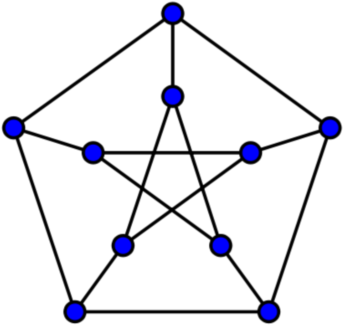
The Petersen graph is non-planar.



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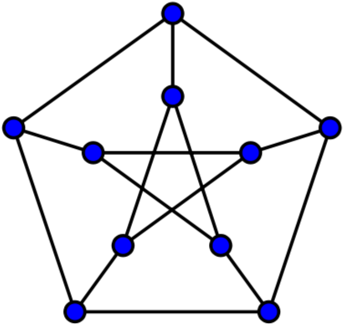

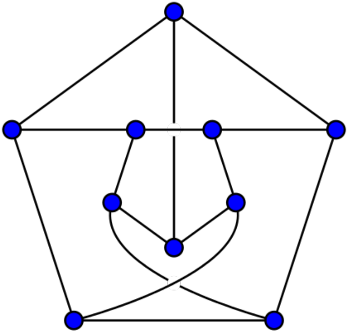

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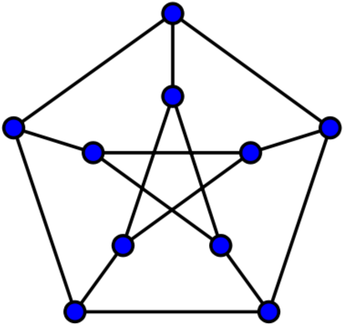

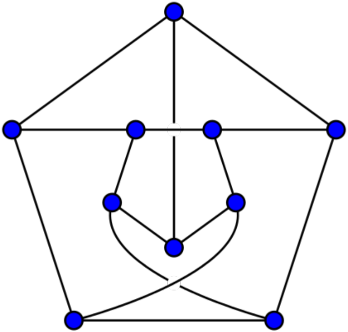

$$\textcolor{red}{\mathrm{cr}}(\text{Petersen Graph}) = 2$$

Theorem (Kazimierz Kuratowski, 1930)

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"The K in K_5 *stands for Kazimierz,*
and the K in $K_{3,3}$ *stands for Kuratowski."*

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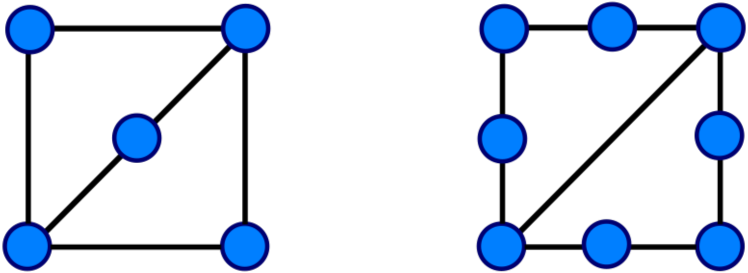

Definition (Homeomorphic)

Two graphs are homeomorphic if one can be obtained from another by inserting or contracting vertices of degree 2.

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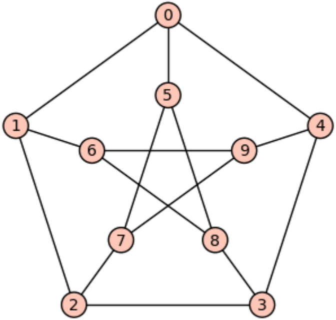

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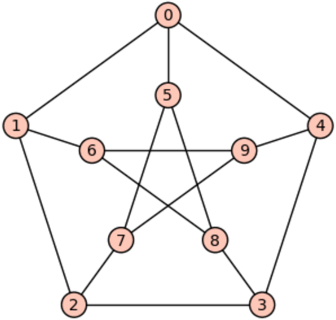

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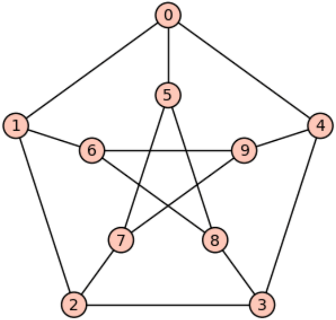

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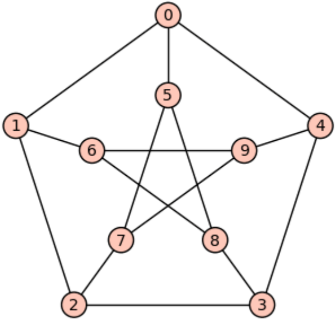

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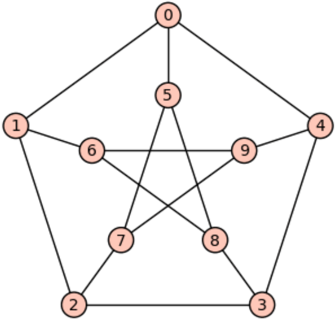

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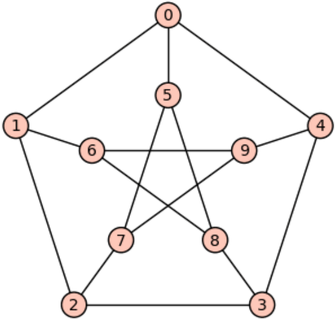

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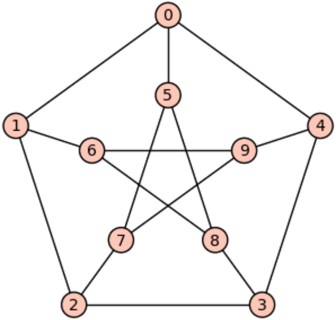

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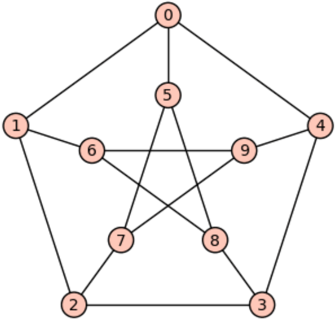

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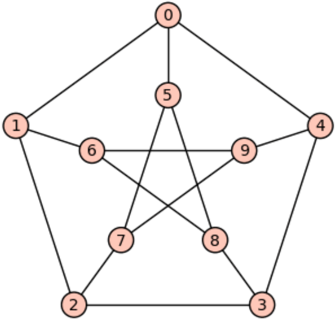

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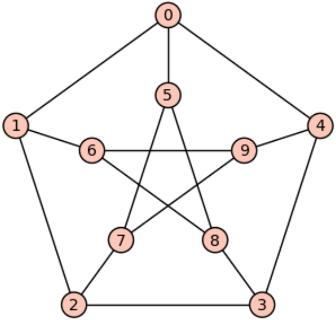

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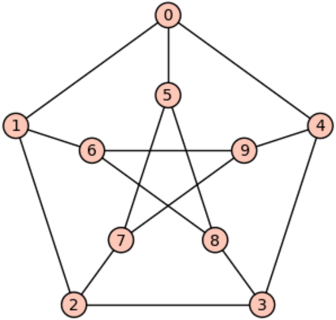

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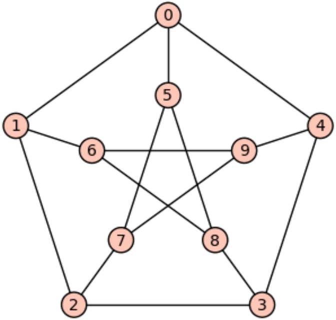

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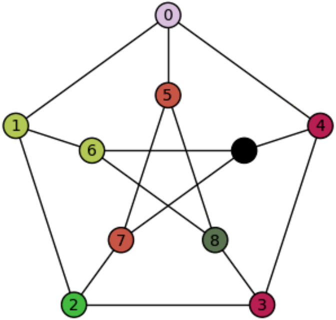

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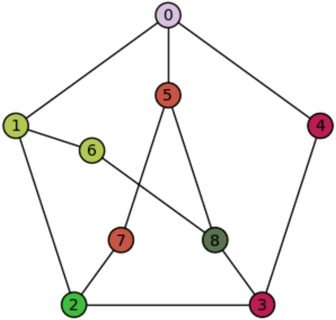

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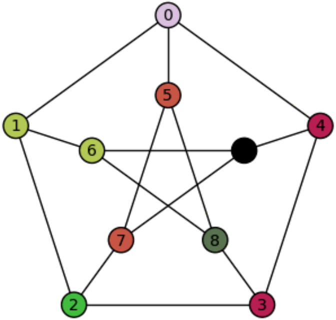

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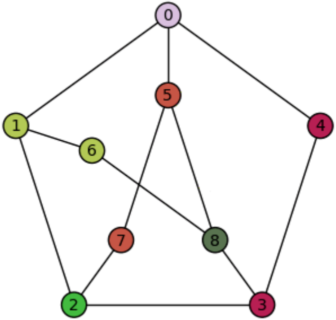

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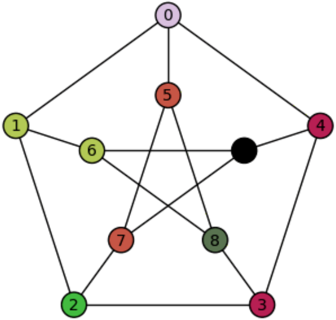

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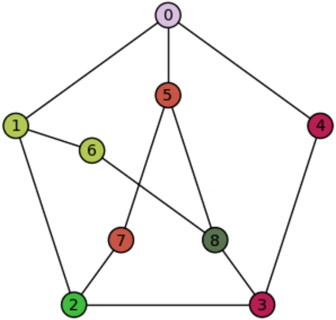

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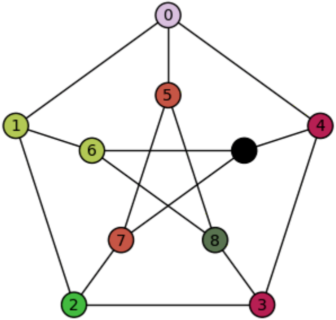

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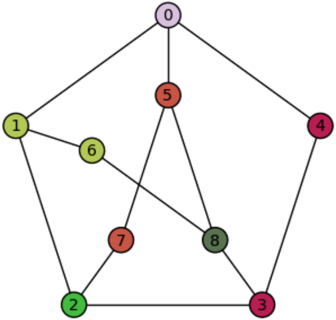

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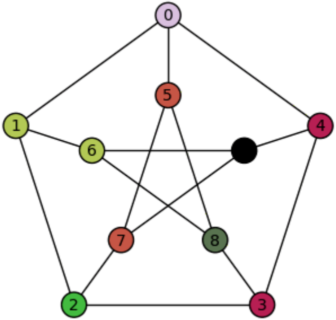

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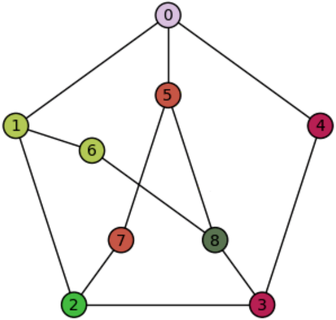

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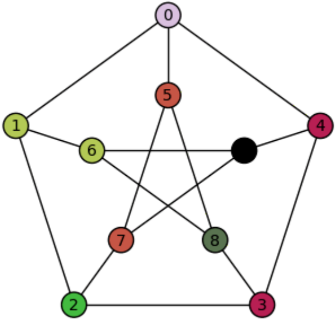

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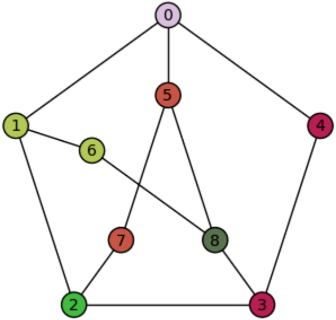

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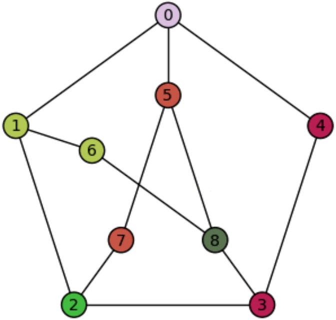

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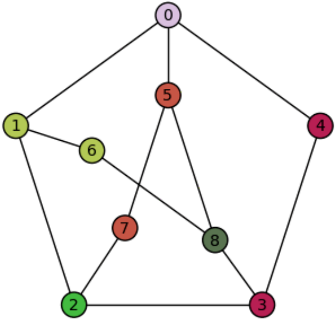

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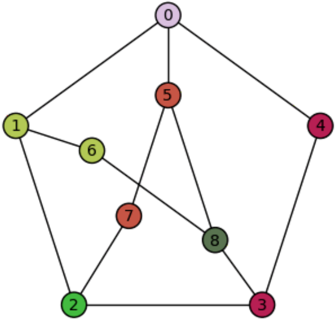

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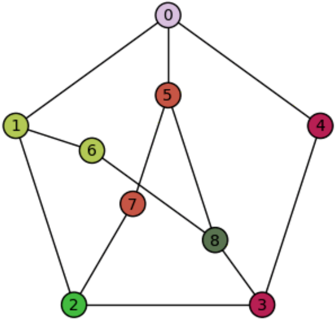

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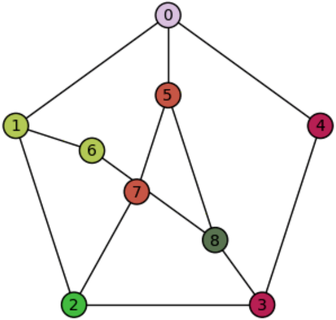

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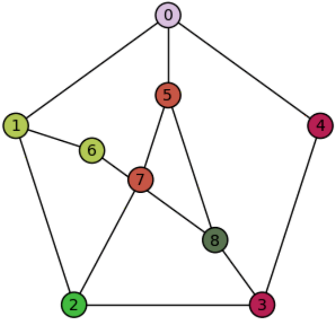

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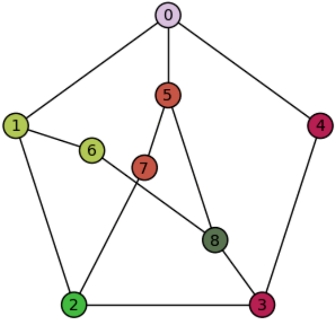

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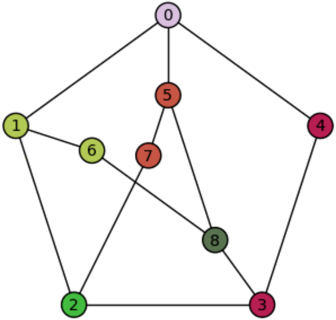

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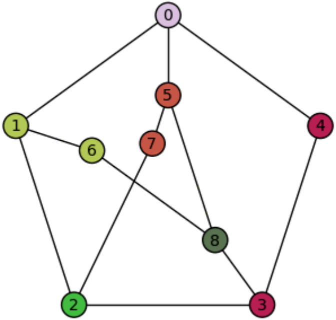

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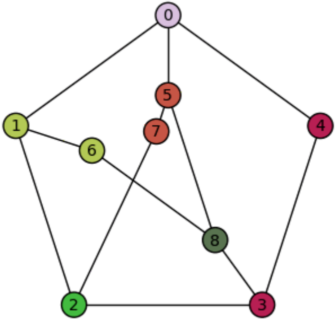

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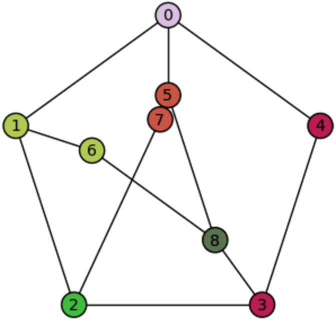

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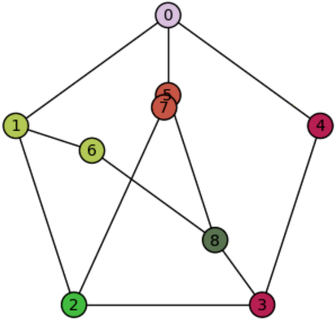

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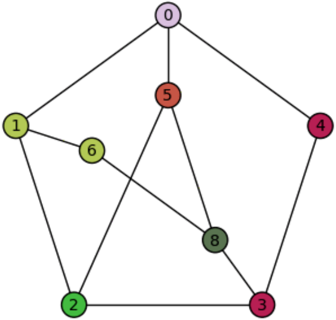

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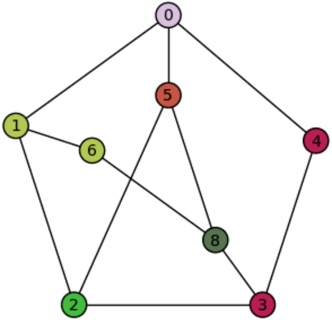

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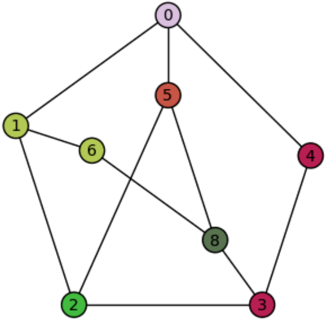

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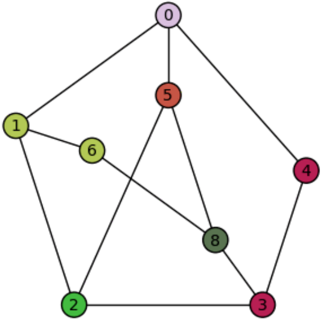

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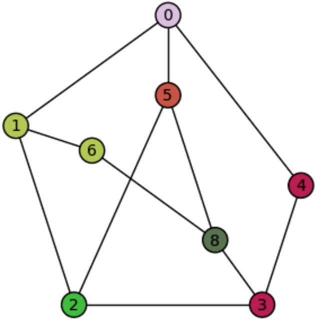

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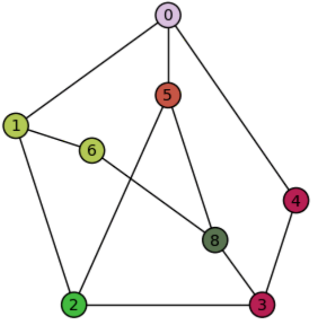

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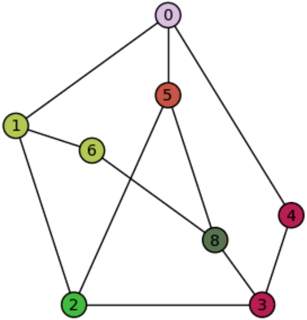

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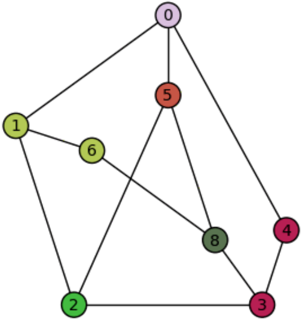

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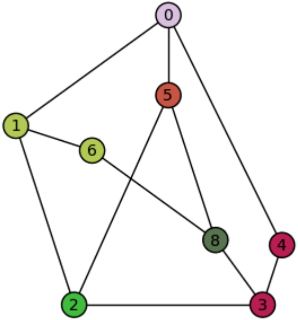

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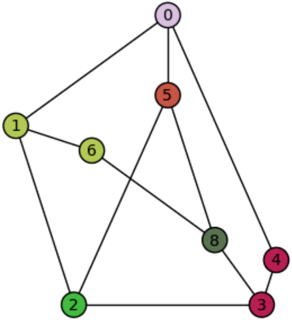

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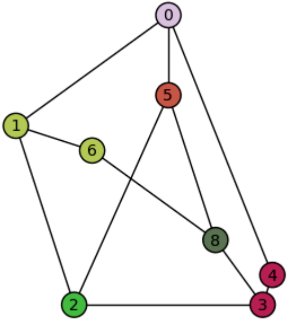

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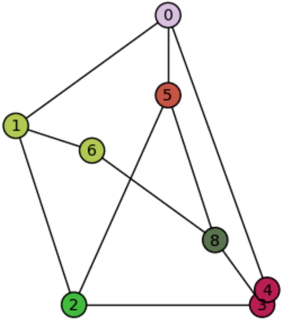

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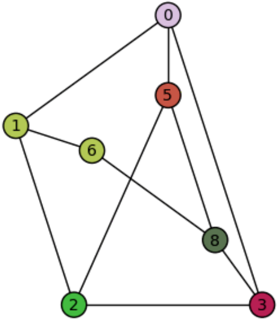

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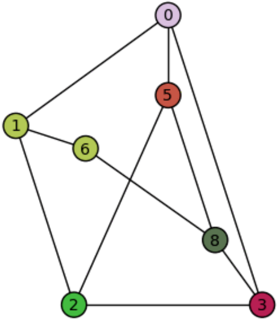

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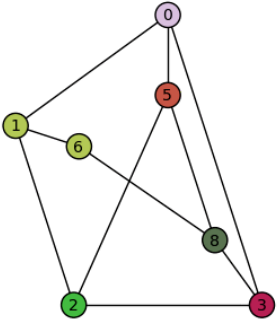

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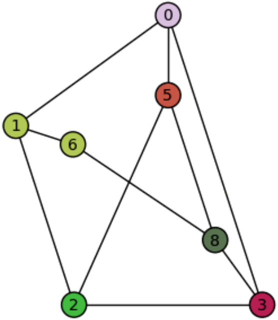

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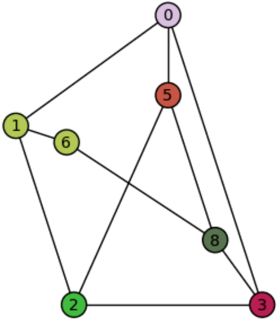

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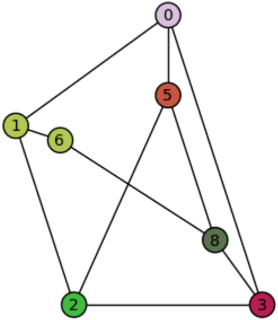

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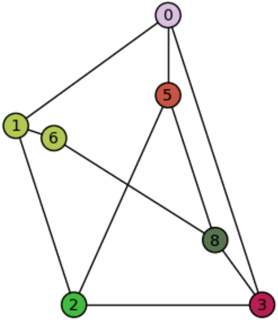

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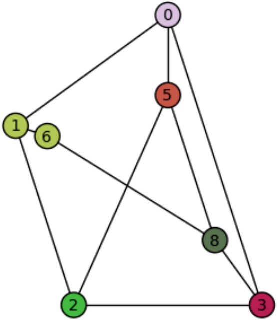

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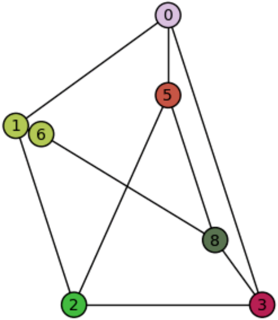

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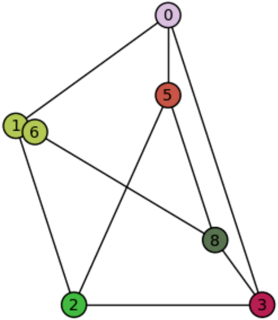

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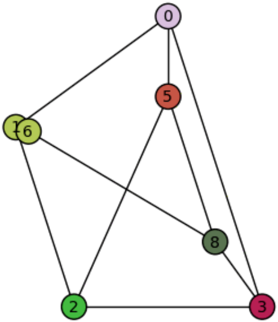

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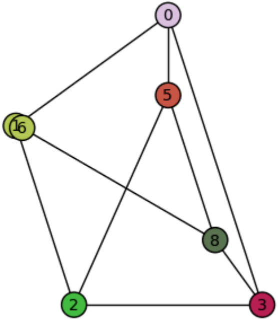

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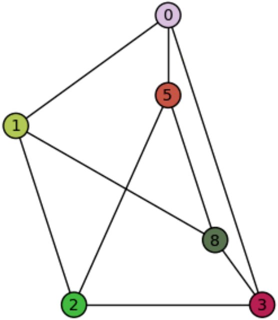

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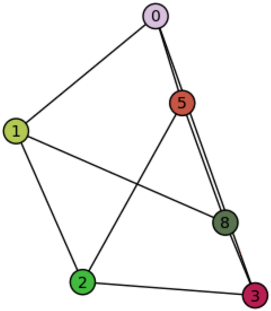

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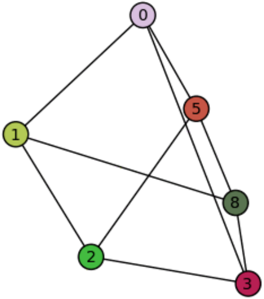

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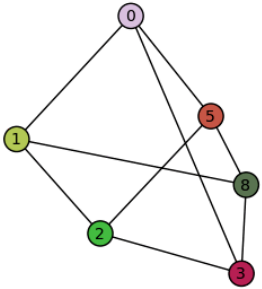

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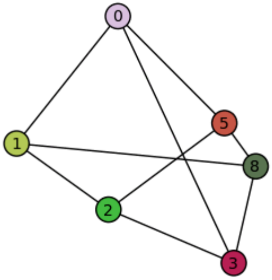

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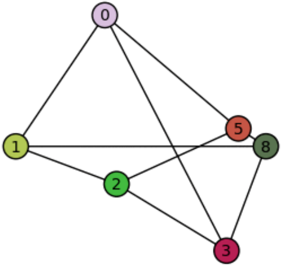

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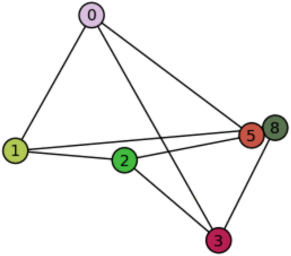

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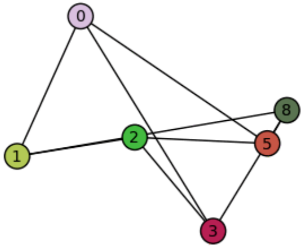

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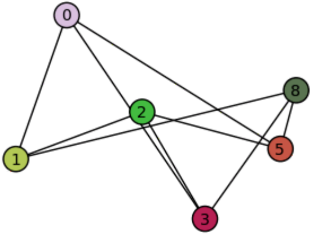

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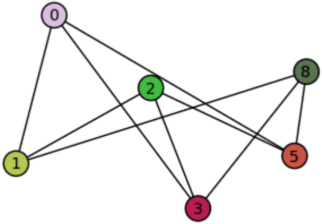

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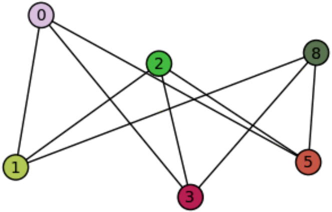
The Petersen graph is non-planar.



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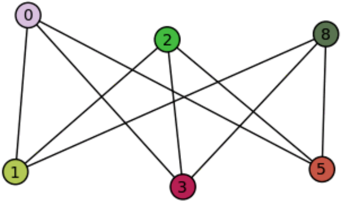

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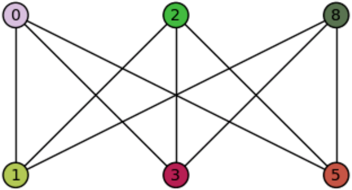

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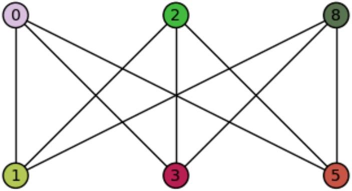

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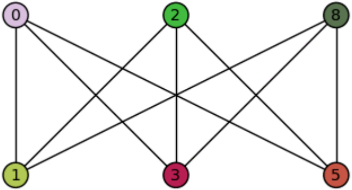

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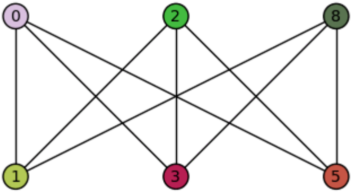

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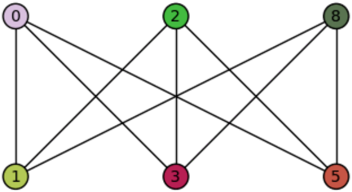

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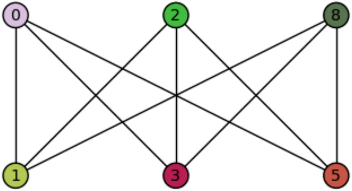

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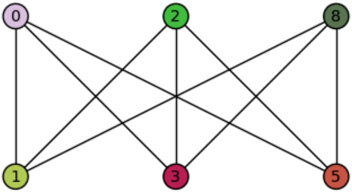

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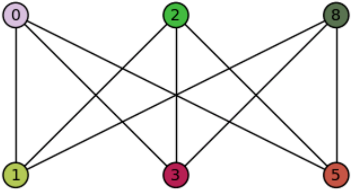

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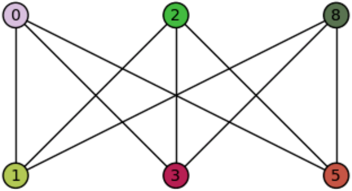

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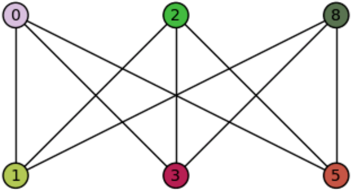

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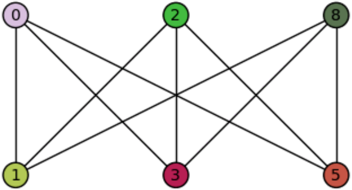

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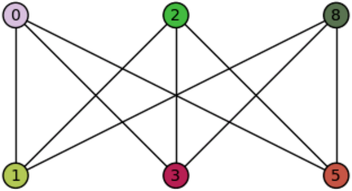

A planar graph should not has too many edges.

Theorem (Euler's Formula, 1750)

Let G be a plane drawing of a connected planar graph, and let n, m, and f denote respectively the number of vertices, edges, and faces of G.

$$n - m + f = 2$$

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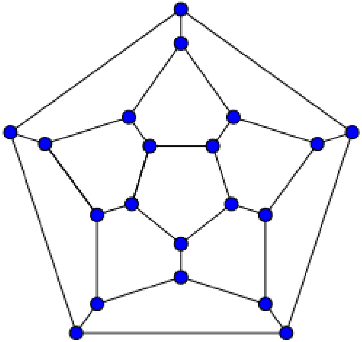

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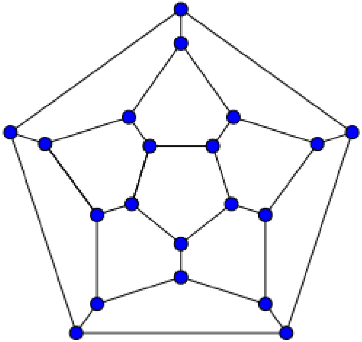

$$n - m + f = 20 - 30 + 12 = 2$$

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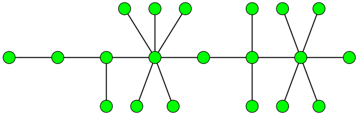

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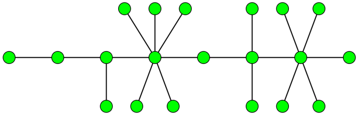

$$n - m + f = n - (n-1) + 1 = 2$$

By induction on the number of edges of G.

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Basis Step: $m = 0$. We have $n = 1$ and $f = 1$.

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Otherwise, G contains a cycle.

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Consider $G' = G - e$.

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Therefore,

$$n - m + f = 2$$

Theorem

Let G be a simple connected planar graph with $n \geq 3$ vertices and m edges. Then

$$m \leq 3n - 6.$$

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Double Counting:

each face is bounded by ≥ 3 edges;
each edge bounds 2 faces

Theorem

K_5 is non-planar.

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$$m \leq 3n - 6$$

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$$m \le 3n - 6$$

$$10 \le 3 \times 5 - 6$$

Theorem

$K_{3,3}$ is non-planar.

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$$9 \le 3 \times 6 - 6$$

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Let G be a simple connected planar graph with $n \geq 3$ vertices and m edges. If G has no triangles, then

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$$4f \leq 2m$$

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Every simple planar graph contains a vertex of degree ≤ 5.

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Suppose that, by contradiction, $\delta(G) \geq 6$.

$$6n \leq 2m$$

$$3n \leq m \leq 3n - 6$$

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G contains a vertex v of degree ≤ 5.

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$G' = G - v$ is 6-colorable.

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$G' = G - v$ is 6-colorable.

Thus, G is 6-colorable.

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$G' = G - v$ is 6-colorable.

Thus, G is 6-colorable. $(\deg(v) \leq 5)$

Theorem (Percy John Heawood (1890))

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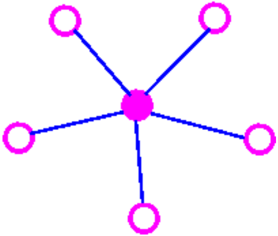
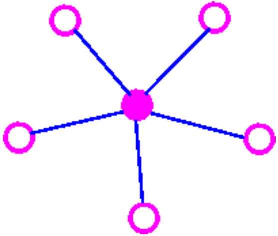
If $\deg(v) < 5$, G is 5-colorable.

Now assume that $\deg(v) = 5$.



$$\{v, v_1\}, \{v, v_2\}, \{v, v_3\}, \{v, v_4\}, \{v, v_5\}$$

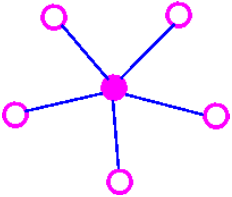
Now assume that $\deg(v) = 5$.



$$\{v, v_1\}, \{v, v_2\}, \{v, v_3\}, \{v, v_4\}, \{v, v_5\}$$

If v_1, v_2, v_3, v_4, and v_5 uses < 5 colors, we are done.

Now assume that v_1, v_2, v_3, v_4, and v_5 uses 5 colors.



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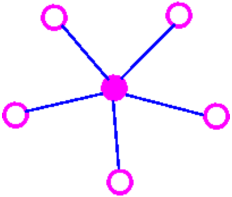

Suppose that there is *no* $v_1 \sim v_3$ path in $G' = G - v$, all of whose vertices are colored red or blue.

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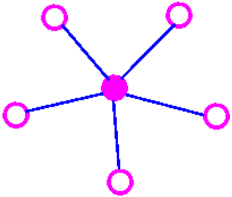

Suppose that there is *no* $v_1 \sim v_3$ path in $G' = G - v$, all of whose vertices are colored red or blue.

Let S be the set of all red or blue vertices of $G - v$ connected to v_1 by a red-blue path.

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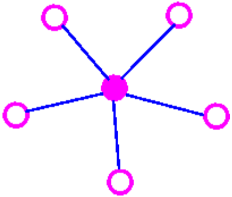

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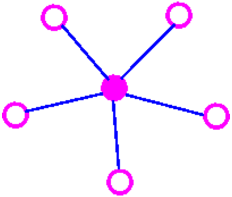
$$v_1 \in S, \quad v_3 \notin S$$

Now assume that v_1, v_2, v_3, v_4, and v_5 uses 5 colors.



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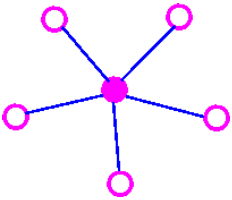
Interchange the colors of the vertices in S

Now assume that v_1, v_2, v_3, v_4, and v_5 uses 5 colors.



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Interchange the colors of the vertices in S

Coloring v red produces a 5-coloring of G.

Now assume that there *is* a $v_1 \sim v_3$ path in $G' = G - v$,
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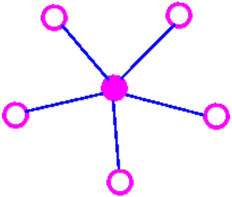

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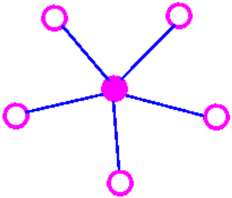

There cannot be $v_2 \sim v_4$ path in $G' = G - v$, all of whose vertices are colored green or purple.

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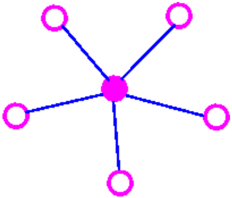

There cannot be $v_2 \sim v_4$ path in $G' = G - v$,
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(Otherwise, G' and thus G is non-planar.)

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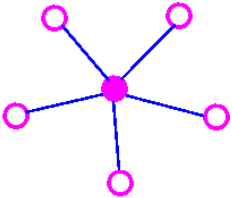

There cannot be $v_2 \sim v_4$ path in $G' = G - v$, all of whose vertices are colored green or purple.

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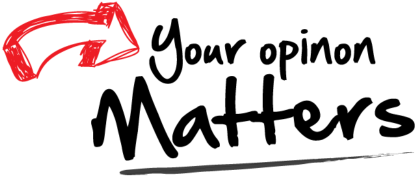
By similar argument, G is 5-colorable.


Thank
You!



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