

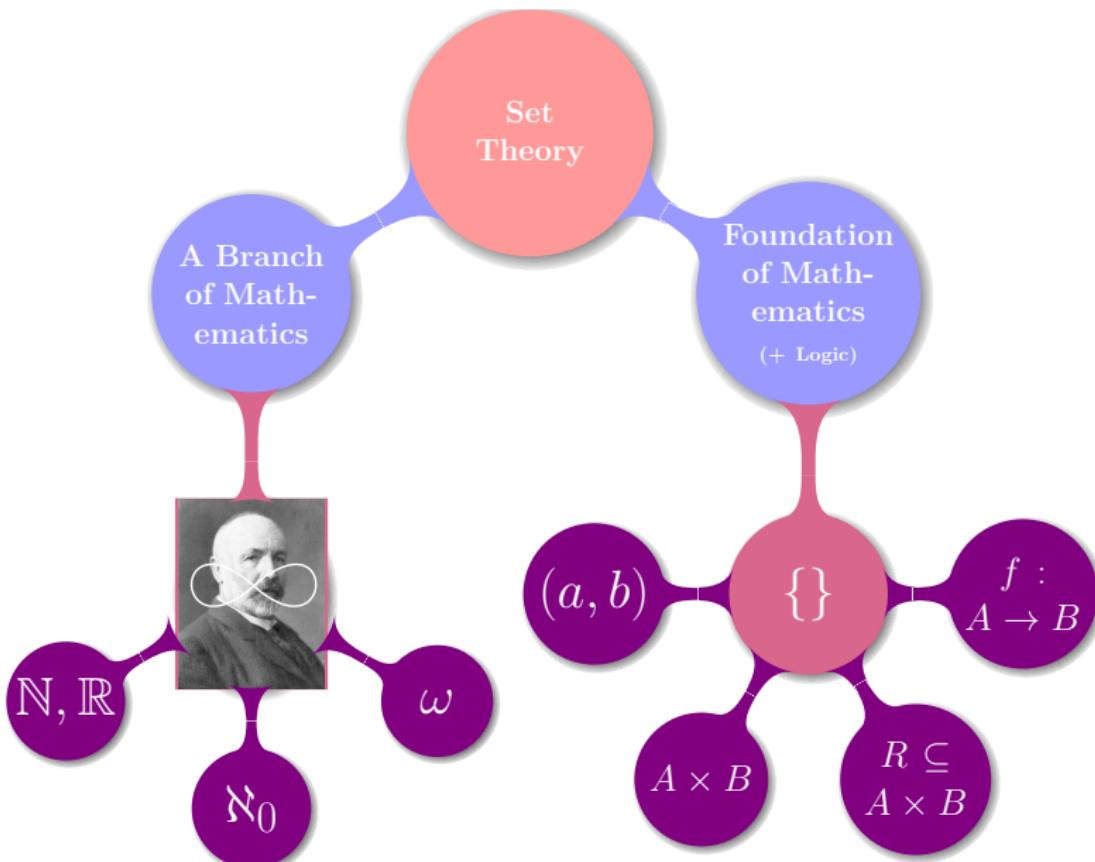
## (五) 集合: 关系 (Relation)

魏恒峰

hfwei@nju.edu.cn

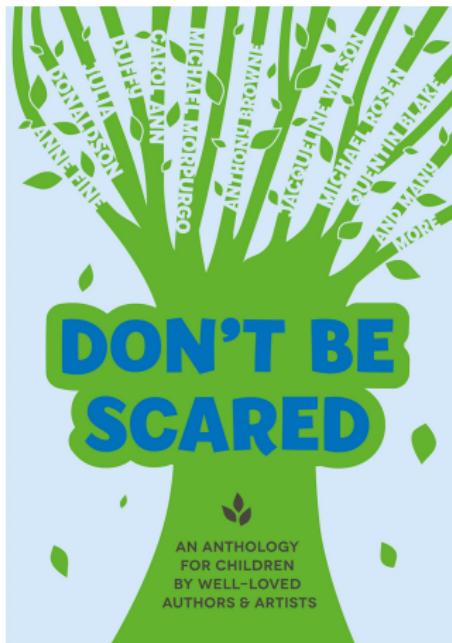
2021 年 04 月 08 日

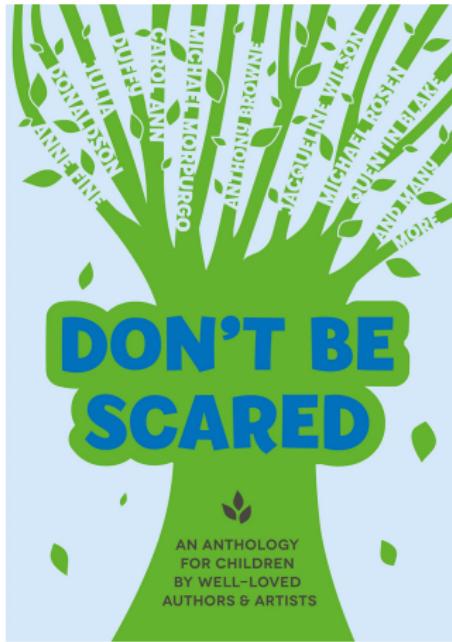








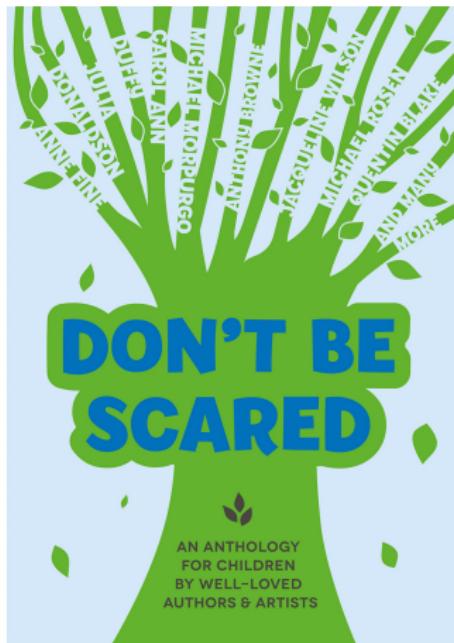




I'm so excited.



离散数学学得好不好，  
一个重要的衡量标准就是是否完成了这种转变



I'm so excited.



# The Relational Data Model

## A Relational Model of Data for Large Shared Data Banks

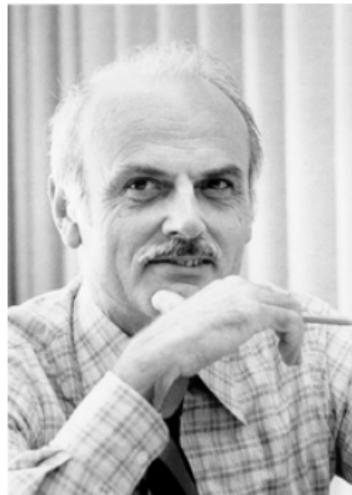
E. F. CODD

*IBM Research Laboratory, San Jose, California*

Future users of large data banks must be protected from having to know how the data is organized in the machine (the internal representation). A prompting service which supplies such information is not a satisfactory solution. Activities of users at terminals and most application programs should remain unaffected when the internal representation of data is changed and even when some aspects of the external representation are changed. Changes in data representation will often be needed as a result of changes in query, update, and report traffic and natural growth in the types of stored information.

Existing noninferential, formatted data systems provide users with tree-structured files or slightly more general network models of the data. In Section 1, inadequacies of these models are discussed. A model based on  $n$ -ary relations, a normal form for data base relations, and the concept of a universal data sublanguage are introduced. In Section 2, certain operations on relations (other than logical inference) are discussed and applied to the problems of redundancy and consistency in the user's model.

Codd@CACM'1970  
(Turing Award'1981)



Edgar F. Codd (1923 – 2003)

# The Relational Data Model — 如何靠“关系”赢得图灵奖?

## A Relational Model of Data for Large Shared Data Banks

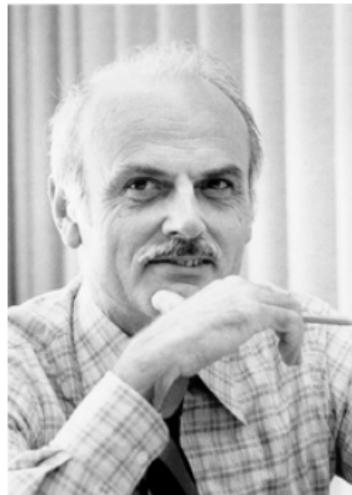
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# $\mathbb{R}$ : 实数集

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“Near” 关系:  $|a - b| < 1$

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自反性 + 对称性 = 相容关系

$$X = \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$$

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$X$  上的整除关系

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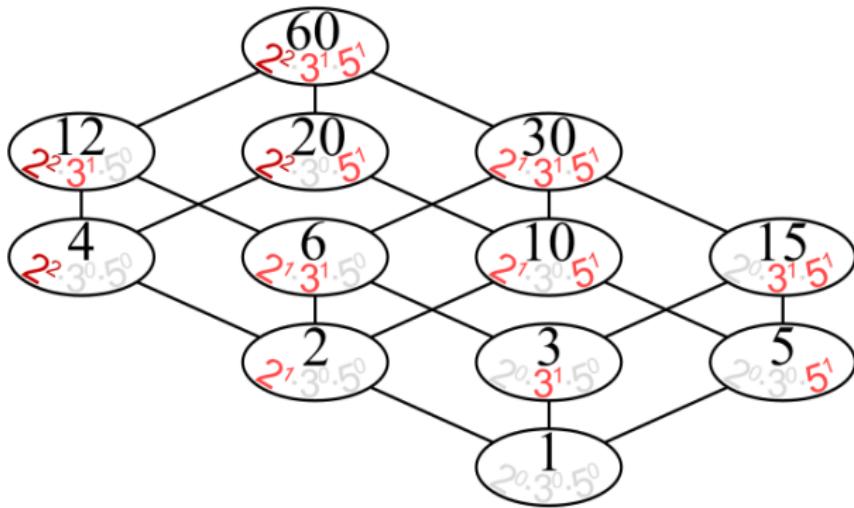
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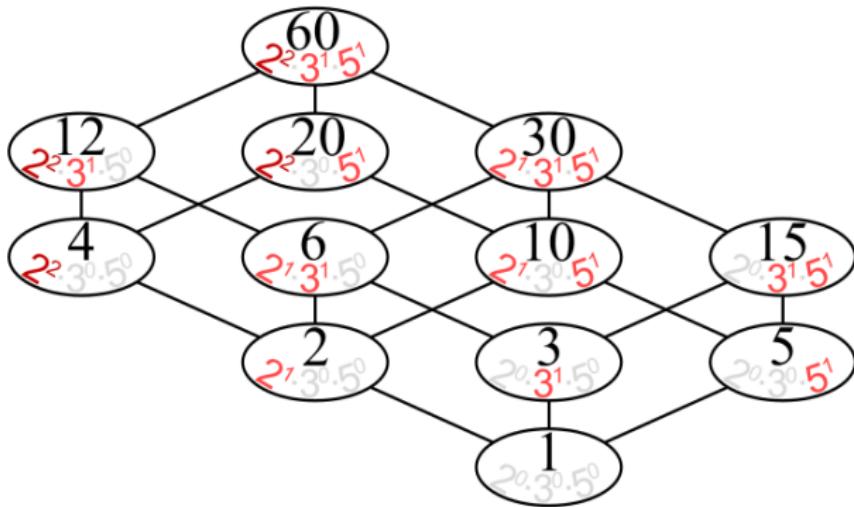
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自反性 + 反对称性 + 传递性 = 偏序关系

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“偏序”严格刻画了人类对于“序”的直观理解

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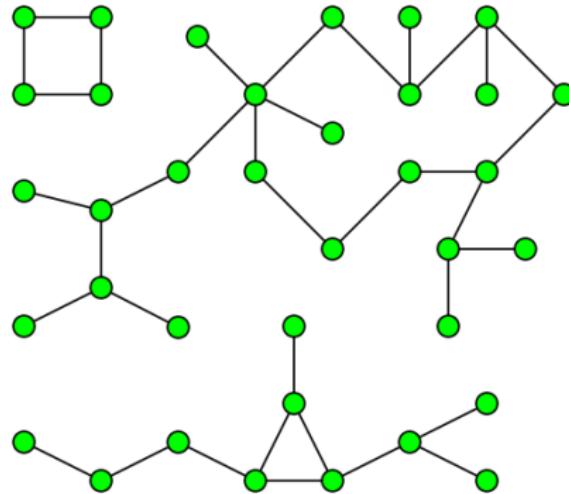
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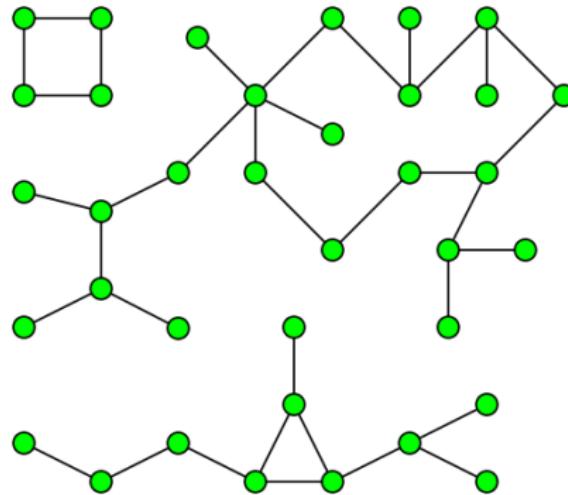
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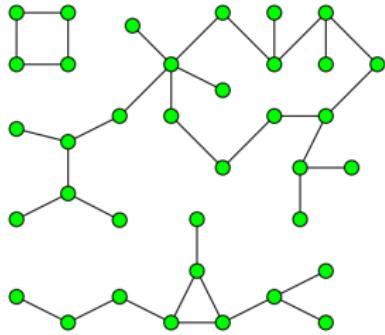
考虑无向图中的**顶点**集合

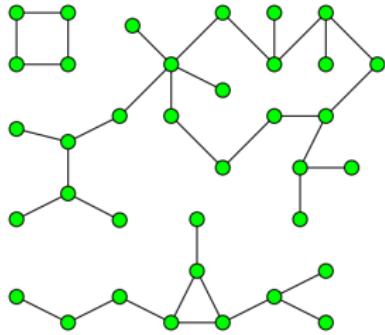


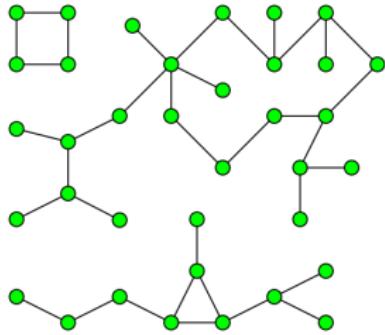
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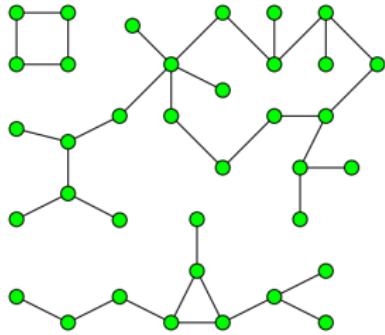
顶点间的“可达 (Reachability) 关系”:  $R = \{(a, b) \mid a \sim b\}$




$$\forall a \in X. (a, a) \in R$$

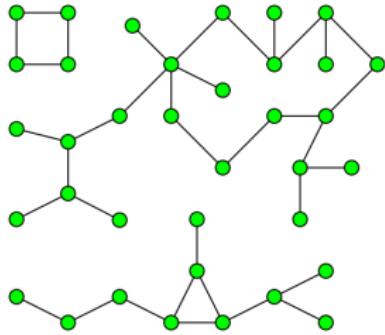


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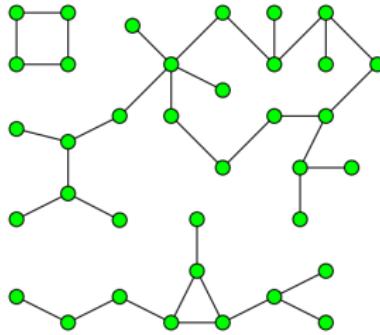
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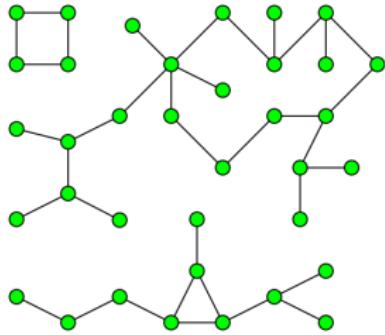
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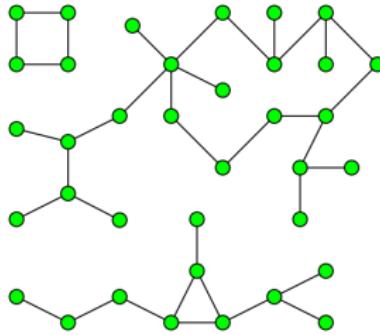
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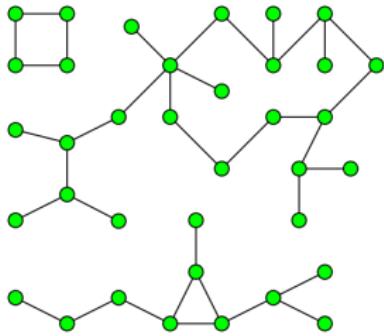


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自反性 + 对称性 + 传递性 = 等价关系



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“可达关系”将顶点划分成相互独立的“连通分量”

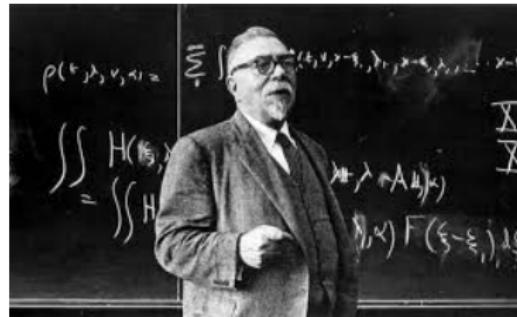


## Definition (有序对**公理** (Ordered Pairs))

$$(a, b) = (c, d) \iff a = c \wedge b = d$$

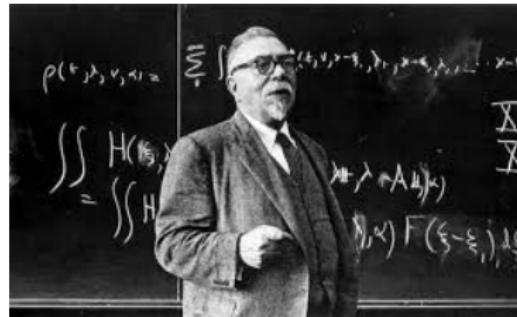
## Definition (Ordered Pairs (Norbert Wiener; 1914))

$$(a, b) \triangleq \left\{ \{\{a\}, \emptyset\}, \{\{b\}\} \right\}$$



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## Definition ( $n$ -元组 (n-ary tuples))

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$$(x_1, x_2, \dots, x_{n-1}, x_n) \triangleq ((x_1, x_2, \dots, x_{n-1}), x_n)$$

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多数情况下, 我们仅处理“二元关系”, 因此也仅使用“有序对”

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The *Cartesian product*  $A \times B$  of  $A$  and  $B$  is defined as

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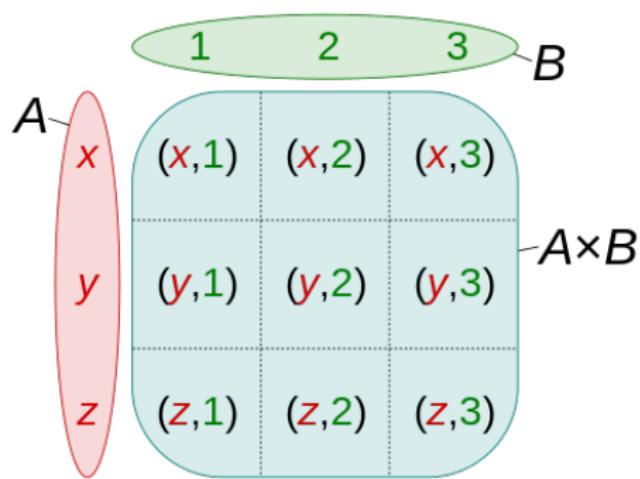
$$X^2 \triangleq X \times X$$

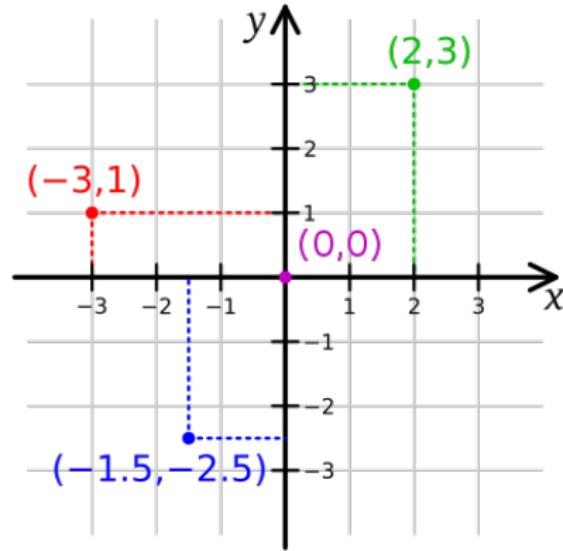
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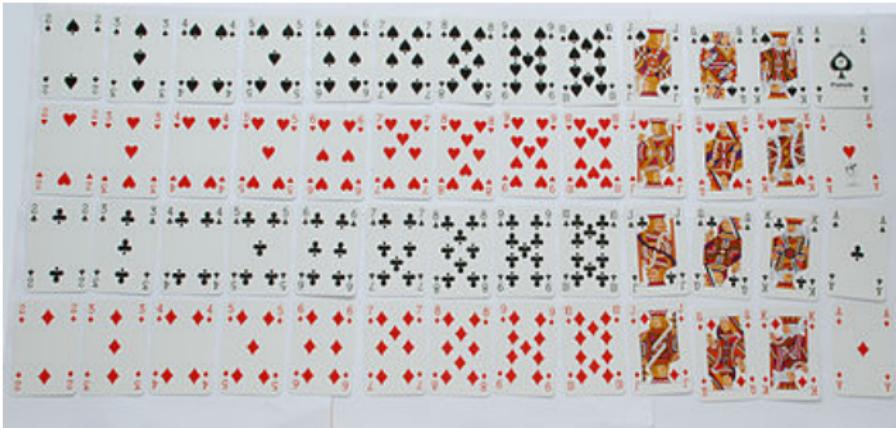
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$$\mathbb{Z}^2 \triangleq \mathbb{Z} \times \mathbb{Z}$$

Ranks = {2, ..., 10, J, Q, K, A}



Suits = {♠, ♥, ♣, ♦}

$$X \times \emptyset = \emptyset \times X$$

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$$X \times Y \neq Y \times X$$

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$$(X \times Y) \times Z \neq X \times (Y \times Z)$$

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$$A = \{1\} \quad (A \times A) \times A \neq A \times (A \times A)$$

## Theorem (分配律 (Distributivity))

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

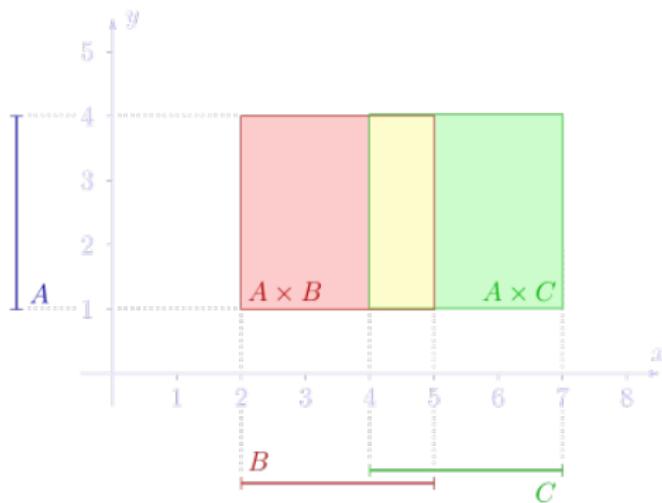
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$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

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对任意有序对  $(a, b)$ ,

$$(a, b) \in A \times (B \cap C) \quad (1)$$

(6)

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对任意有序对  $(a, b)$ ,

$$(a, b) \in A \times (B \cap C) \quad (1)$$

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多数情况下，我们仅处理“二元关系”，因此也仅使用“二元笛卡尔积”

## Definition (关系 (Relations))

A *relation*  $R$  from  $A$  to  $B$  is a **subset** of  $A \times B$ :

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## Definition (Notations)

$$(a, b) \in R \quad aRb$$

$$(a, b) \notin R \quad a\overline{R}b$$

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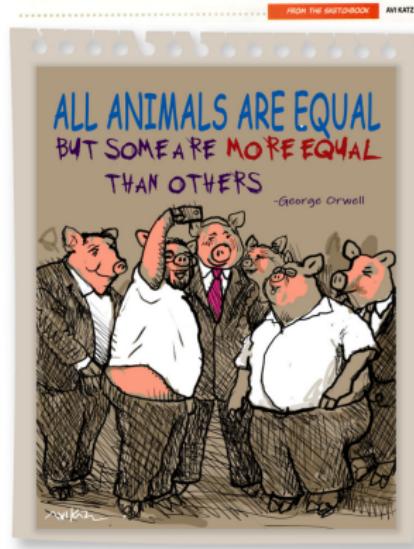
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 $D = \{(a, b) \in \mathbb{N} \times \mathbb{N} \mid \exists q \in \mathbb{N}. a \cdot q = b\}$
- ▶  $P$  : the set of people
  - $M = \{(a, b) \in P \times P \mid a \text{ is the mother of } b\}$
  - $B = \{(a, b) \in P \times P \mid a \text{ is the brother of } b\}$

# Important Relations:

Equivalence Relations

Ordering Relations (next class)

Functions (next class)



## Outline:

3 Definitions

5 Operations

7 Properties

### 3 Definitions

$\text{dom}(R)$        $\text{ran}(R)$        $\text{fld}(R)$

## Definition (定义域 (Domain))

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Definition (域 (Field))

$$\text{fld}(R) = \text{dom}(R) \cup \text{ran}(R)$$



$$R = \{(x, y) \mid x = y\} \subseteq \mathbb{R} \times \mathbb{R}$$

$$\text{dom}(R) = \mathbb{R} \quad \text{ran}(R) = \mathbb{R} \quad \text{fld}(R) = \mathbb{R}$$

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$$\text{dom}(R) = [1, 1] \quad \text{ran}(R) = [-1, 1] \quad \text{fld}(R) = [-1, 1]$$

## Theorem

$$dom(R) \subseteq \bigcup \bigcup R \quad ran(R) \subseteq \bigcup \bigcup R$$

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对任意  $a$ ,

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## 5 Operations

$R^{-1}$        $R|_X$        $R[X]$        $R^{-1}[Y]$        $R \circ S$

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## Theorem

$$(R^{-1})^{-1} = R$$

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对任意  $(a, b)$ ,

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$$\iff (b, a) \in R^{-1} \quad (2)$$

(3)

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## Theorem (关系的逆)

$R, S$  均为关系

$$(R \cup S)^{-1} = R^{-1} \cup S^{-1}$$

$$(R \cap S)^{-1} = R^{-1} \cap S^{-1}$$

$$(R \setminus S)^{-1} = R^{-1} \setminus S^{-1}$$

## Definition (左限制 (Left-Restriction))

Suppose  $R \subseteq X \times Y$  and  $S \subseteq X$ . The *left-restriction* relation of  $R$  to  $S$  over  $X$  and  $Y$  is

$$R|_S = \{(x, y) \in R \mid x \in S\}$$

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## Definition (右限制 (Right-Restriction))

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Suppose  $R \subseteq X \times Y$  and  $S \subseteq X$ . The *left-restriction* relation of  $R$  to  $S$  over  $X$  and  $Y$  is

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$$R = \{(x, y) \mid x^2 + y^2 = 1\} \subseteq \mathbb{R} \times \mathbb{R}$$

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$R|^{\mathbb{R}^+}$  (right restriction)

## Definition (像 (Image))

The *image* of  $X$  under  $R$  is the set

$$R[X] = \{b \in \text{ran}(R) \mid \exists a \in X. (a, b) \in R\}$$

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$$R[a] \triangleq R[\{a\}] = \{b \mid (a, b) \in R\}$$

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$$R^{-1}[b] \triangleq R^{-1}[\{b\}] = \{a \mid (a, b) \in R\}$$

$$R \subseteq A \times B \quad X \subseteq A \quad Y \subseteq B$$

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$$R^{-1}[R[X]] \textcolor{red}{?} X$$

$$R[R^{-1}[Y]] \textcolor{red}{?} Y$$

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$$R^{-1}[R[X]] \textcolor{red}{?} X$$

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## Theorem

$$R[X_1 \cup X_2] = R[X_1] \cup R[X_2]$$

$$R[X_1 \cap X_2] \subseteq R[X_1] \cap R[X_2]$$

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$$b \in R[X_1 \cup X_2]$$

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## Definition (复合 (Composition; $R \circ S$ , $R; S$ ))

The *composition* of relations  $R \subseteq X \times Y$  and  $S \subseteq Y \times Z$  is the **relation**

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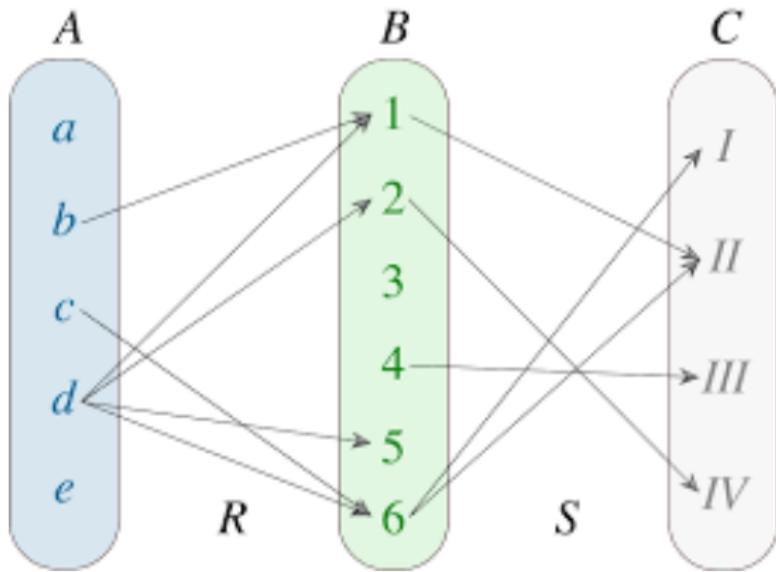
$$R = \{(1, 2), (3, 1)\} \quad S = \{(1, 3), (2, 2), (2, 3)\}$$

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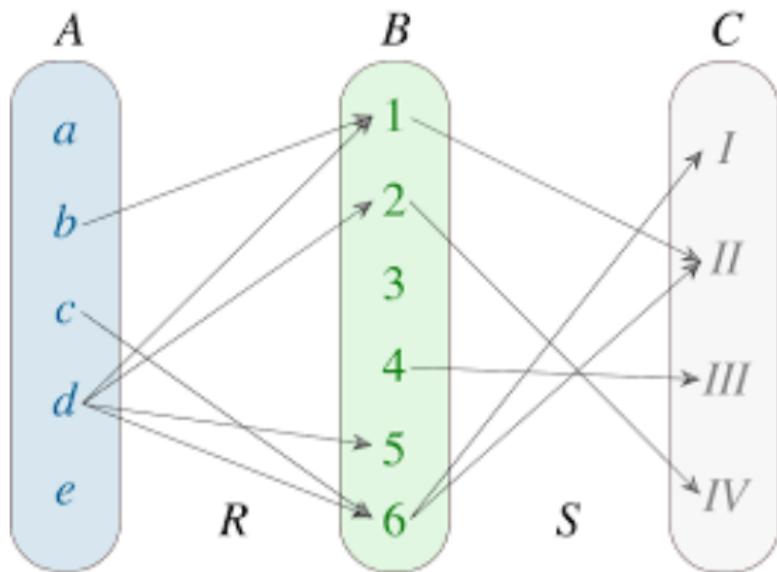
$$S \circ R = \{(1, 2), (1, 3), (3, 3)\}$$

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$$|S \circ R| =$$



$$|S \circ R| = 7$$

$\leq \circ \leq =$

$\leq \circ \leq = \leq$

$\leq \circ \leq = \leq$  $\geq \circ \leq =$

$$\leq \circ \leq = \leq$$

$$\geq \circ \leq = \mathbb{R} \times \mathbb{R}$$

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$$(a, |a| + |b|) \in \leq \quad (|a| + |b|, b) \in \geq$$

## Theorem

$$(R \circ S)^{-1} = S^{-1} \circ R^{-1}$$

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对任意  $(a, b)$ ,

$$(a, b) \in (R \circ S)^{-1} \quad (1)$$

(5)

## Theorem

$$(R \circ S)^{-1} = S^{-1} \circ R^{-1}$$

对任意  $(a, b)$ ,

$$(a, b) \in (R \circ S)^{-1} \quad (1)$$

$$\iff (b, a) \in R \circ S \quad (2)$$

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$$(a, b) \in (R \circ S)^{-1} \quad (1)$$

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## Theorem

$$(R \circ S) \circ T = R \circ (S \circ T)$$

对任意  $(a, b)$ ,

$$(a, b) \in (R \circ S) \circ T \quad (1)$$

$$\iff \exists c. ((a, c) \in T \wedge (c, b) \in R \circ S) \quad (2)$$

$$\iff \exists c. ((a, c) \in T \wedge (\exists d. (c, d) \in S \wedge (d, b) \in R)) \quad (3)$$

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帮我照顾好我七舅姥爷和我外甥女

**燕小六:** “帮我照顾好我七舅姥爷和我外甥女”

## “舅姥爷”: 姥姥/外婆的兄弟

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“舅姥爷”: 妈妈的舅舅

## Theorem (关系的复合)

$$(X \cup Y) \circ Z = (X \circ Z) \cup (Y \circ Z)$$

$$(X \cap Y) \circ Z \subseteq (X \circ Z) \cap (Y \circ Z)$$

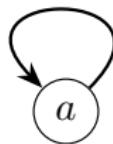
## 7 Properties

$$R \subseteq X \times X$$

Definition (自反的 (Reflexive))

$R \subseteq X \times X$  is *reflexive* if

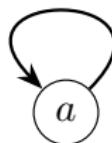
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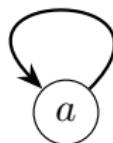


$\leq \subseteq \mathbb{R} \times \mathbb{R}$  is reflexive

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三角形上的全等关系是自反的

## Definition (反自反 (Irreflexive))

$R \subseteq X \times X$  is *irreflexive* if

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$$A = \{1, 2, 3\} \quad R \subseteq A \times A$$

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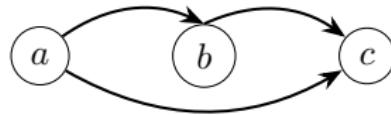
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## Definition (三分的 (Trichotomous))

$R \subseteq X \times X$  is *trichotomous* if

$$\forall a, b \in X. (\text{exactly one of } aRb, bRa, \text{ or } a = b \text{ holds})$$

## Theorem

$R$  is reflexive  $\iff I \subseteq R$

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(3)

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## Theorem

$R$  is symmetric and transitive  $\iff R = R^{-1} \circ R$

# Equivalence Relations

## Definition (Equivalence Relation)

$R \subseteq X \times X$  is an *equivalence relation* on  $X$  iff  $R$  is

- ▶ reflexive:  $\forall a \in X. aRa$
- ▶ symmetric:  $\forall a, b \in X. (aRb \leftrightarrow bRa)$
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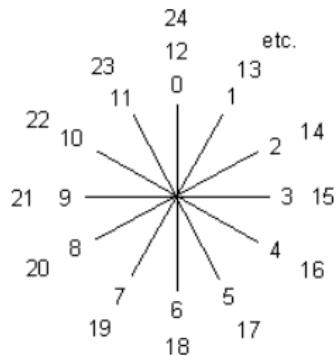
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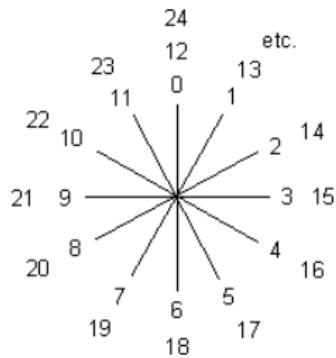
Why are equivalence relations important?

# Equivalence Relations as Abstractions

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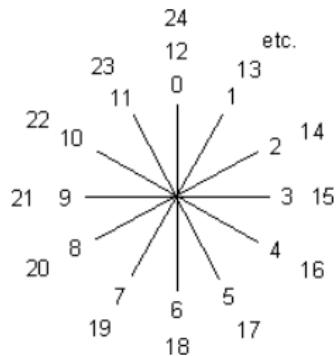


## Equivalence Relations as Abstractions



“全国人民代表大会各省代表团”

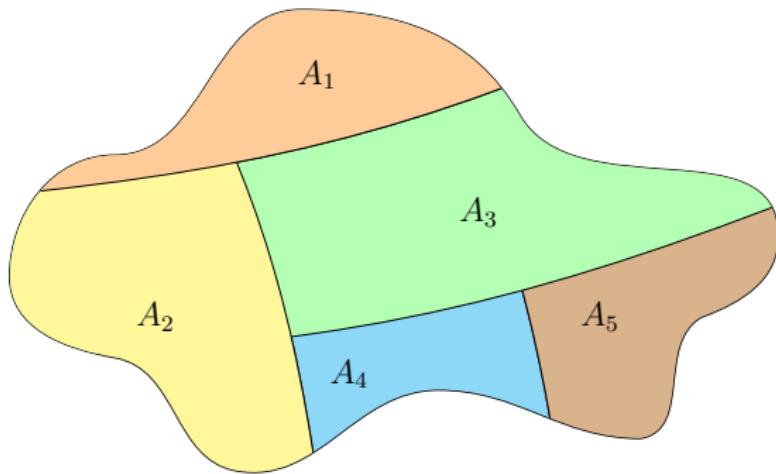
## Equivalence Relations as Abstractions



“全国人民代表大会各省代表团”

Equivalence Relation  $\iff$  Partition

# Partition



“不空、不漏、不重”

## Definition (划分 (Partition))

A family of sets  $\Pi = \{A_\alpha \mid \alpha \in I\}$  is a *partition* of  $X$  if

(i) (不空)

$$\forall \alpha \in I. A_\alpha \neq \emptyset$$

(ii) (不漏)

$$\bigcup_{\alpha \in I} A_\alpha = X$$

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$$\forall \alpha, \beta \in I. A_\alpha \cap A_\beta = \emptyset \vee A_\alpha = A_\beta$$

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## Theorem

$$\forall a, b \in X. ([a]_R = [b]_R \leftrightarrow aRb)$$

Partition  $\Pi$  of  $X \implies$  Equivalence Relation  $R \subseteq X \times X$

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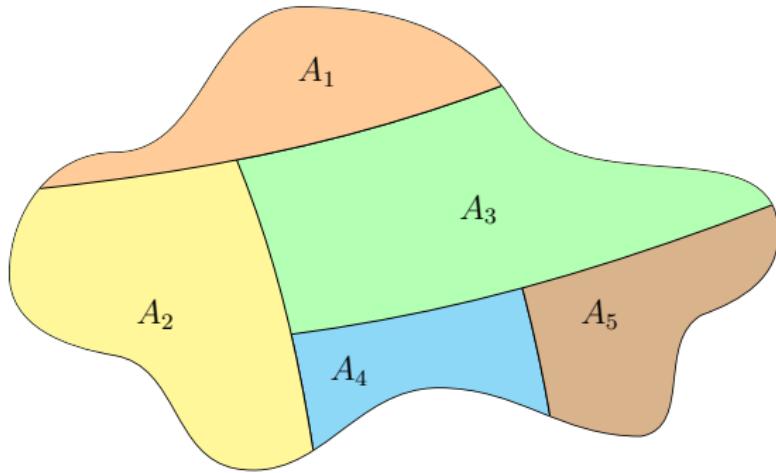
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$$\forall x, y, z \in X. xRy \wedge yRz \rightarrow xRz$$



Equivalence Relation  $\iff$  Partition

## Definition

$$\sim \subseteq \mathbb{N} \times \mathbb{N}$$

$$(a, b) \sim (c, d) \iff a +_{\mathbb{N}} d = b +_N c$$

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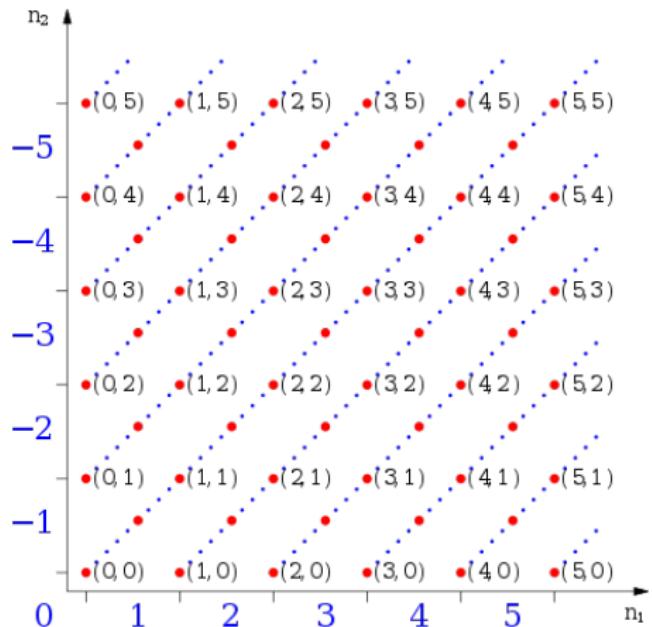
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$$[(1, 3)]_{\sim} = \{(0, 2), (1, 3), (2, 4), (3, 5), \dots\} \triangleq -2 \in \mathbb{Z}$$



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## Definition ( $+_{\mathbb{Z}}$ )

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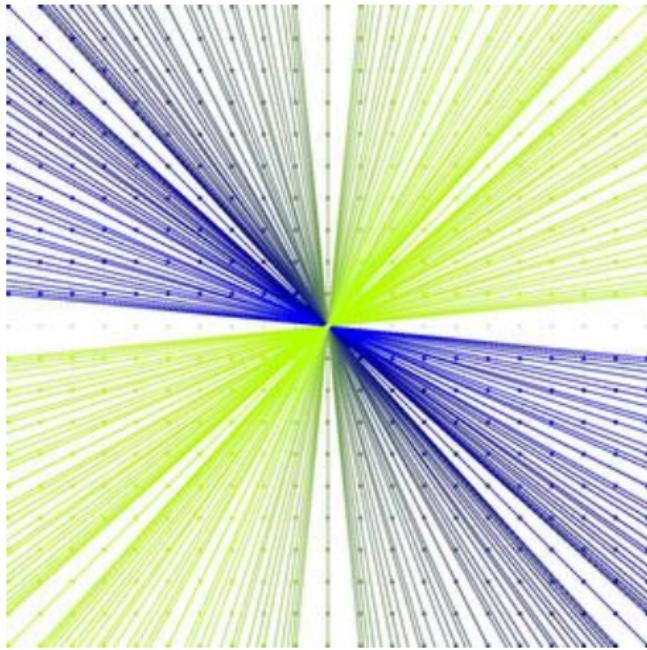
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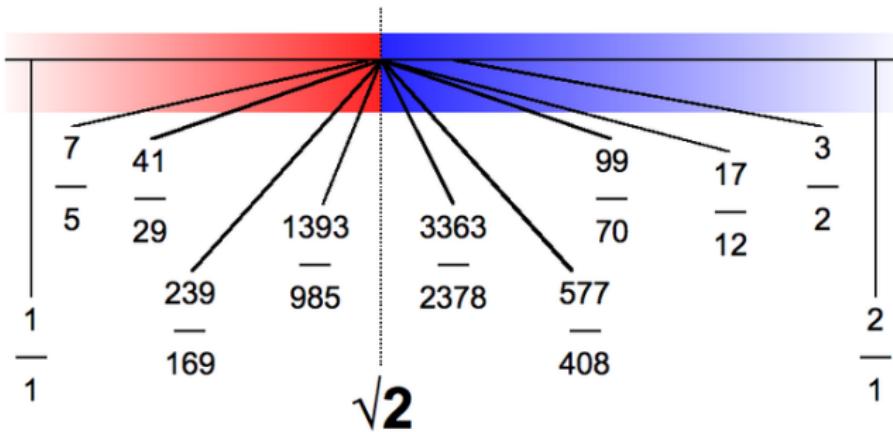
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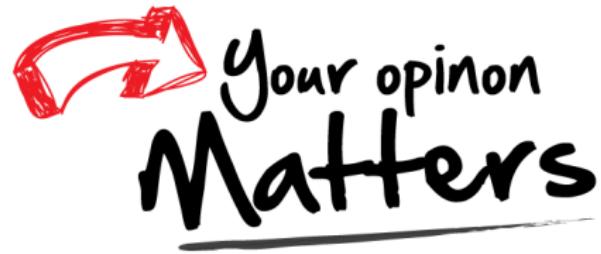
# 如何用有理数定义实数?

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Dedekind Cut (戴德金分割)

# Thank You!



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