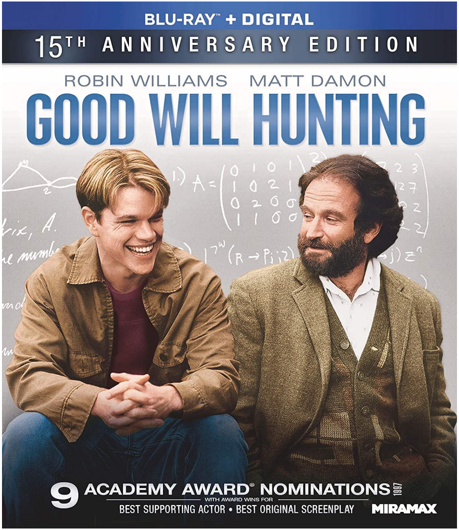
(十) 图论: 树 (Trees)

魏恒峰

hfwei@nju.edu.cn

2021 年 05 月 13 日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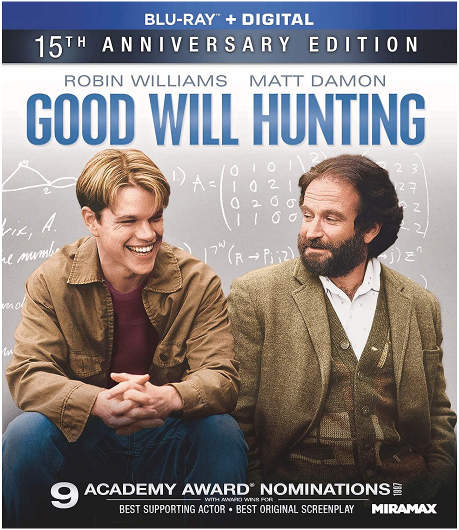
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你, 真得, 看懂了吗?

Definition (Tree (树))

A tree is a connected acyclic undirected graph.

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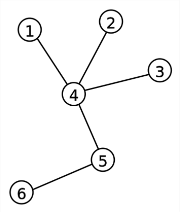
Definition (Forest (森林))

A forest is a acyclic undirected graph.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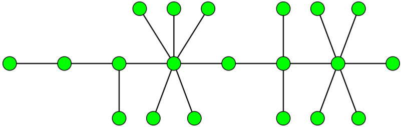

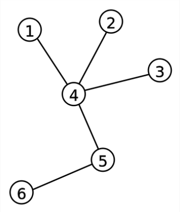
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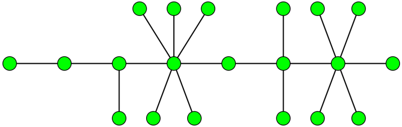

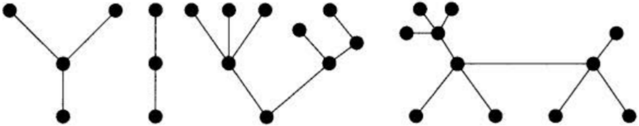

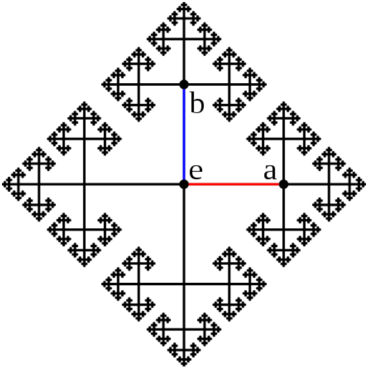

Cayley Graph (4-regular tree)

Definition (Internal Vertex (内部顶点); Leaf (叶子))

In a tree T with ≥ 2 vertices, for a vertex v in T, if

$$\deg(v) = 1$$

then v is called a leaf; otherwise, v is an internal vertex.

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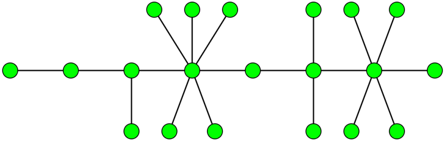

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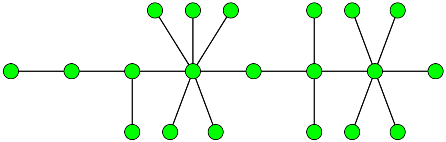

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Any tree T with ≥ 2 vertices contains ≥ 1 leaf.

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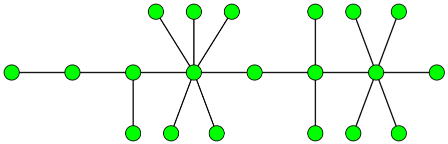

Lemma

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$$\text{Otherwise, } \forall v \in V.\ \deg(v) \geq 2 \implies T \text{ has cycles.}$$

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Consider the two endpoints of any maximal (nontrivial) path in T.

They are leaves of T.

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Deleting a leaf from a tree T with n vertices produces a tree with $n-1$ vertices.

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A leaf does *not* belong to any paths connecting two other vertices.

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A leaf does *not* belong to any paths connecting two other vertices.

This lemma can be used in induction for trees!

Theorem ((We call it) Characterization of Trees)

Let T be an undirected graph with n vertices.

Then the following statements are equivalent:

(1) *T is a tree;*

(2) *T is acyclic, and has $m = n - 1$ edges;*

(3) *T is connected, and has $m = n - 1$ edges;*

(4) *T is connected, and each edge is a bridge;*

(5) *Any two vertices of T are connected by exactly one path;*

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$$(1) \implies (2) \implies (3) \implies (4) \implies (5) \implies (6) \implies (1)$$

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$$m(T) = (n - 2) + 1 = n - 1.$$

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By (2), for each T_i, $m(T_i) = n(T_i) - 1$.

$$m(T) = \sum_{i=1}^{k} m(T_i) = n - k \neq n - 1.$$

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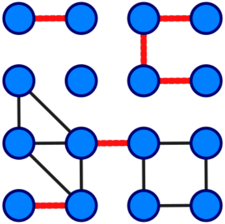

Definition (Bridge (桥))

A bridge of a graph G is an edge e such that

$$c(G-e) > c(G).$$

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$$m(T-e) = (n-1) - 1 = n - 2.$$

$T - e$ must be disconnected.

Theorem (Characterization of Trees)

(4) *T is connected, and each edge is a bridge;*

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If u and v are connected by two paths,

the edges on these two paths are not bridges.

Theorem (Characterization of Trees)

(5) *Any two vertices of* T *are connected by exactly one path;*

(6) T *is acyclic, but the addition of any edge creates exactly one cycle.*

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If T has a cycle C,

any two vertices in C is connected by ≥ 2 paths.

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It creates a cycle, consisting of $\{u, v\}$ and the path from u to v.

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Lemma

If two distinct cycles of a graph G share a common edge e,
then G has a cycle that does not contain e.

Theorem (Characterization of Trees)

(6) *T is acyclic, but the addition of any edge creates exactly one cycle;*

(1) *T is a tree.*

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Choose $u \in V(T_1)$, $v \in V(T_2)$.

$T + \{u, v\}$ does not create cycles.

Definition (Spanning Tree (生成树))

A spanning tree T of an undirected graph G is a subgraph that is a tree with all vertices of G.

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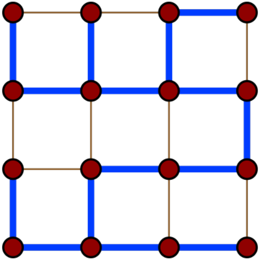

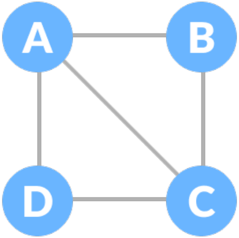

Definition (Subgraph (子图))

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Definition (Induced Subgraph (诱导子图))

Theorem

Every connected undirected graph G admits a spanning tree.

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Every connected undirected graph G admits a spanning tree.

Repeatedly deleting vertices in cycles until the graph is acyclic.

Definition (Minimum Spanning Tree (MST; 最小生成树))

A minimum spanning tree T of an edge-weighted undirected graph G is a spanning tree with minimum total weight of edges.

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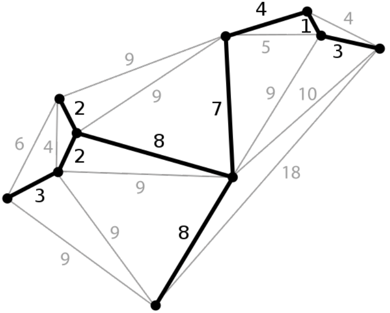

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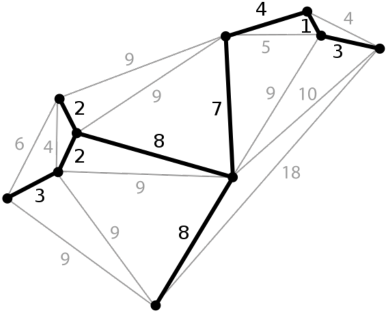

Existence? Uniqueness? Algorithms?

Theorem

Every weighted connected undirected graph G admits a minimum spanning tree.

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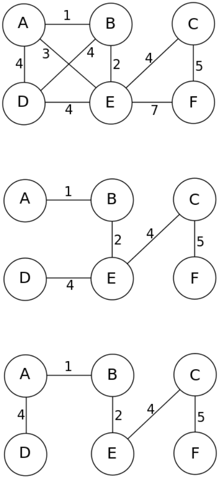
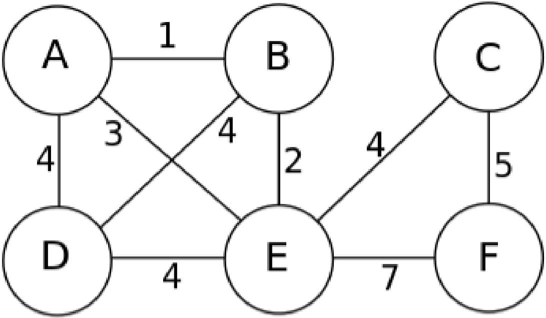


Joseph Kruskal (1928 ~ 2010)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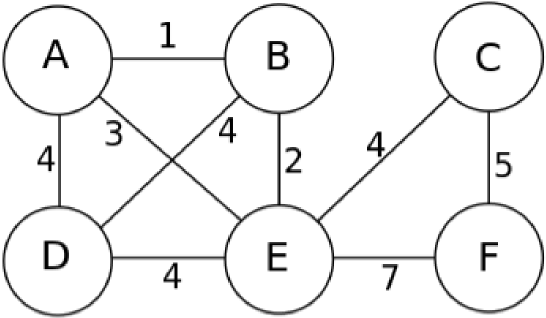
Robert C. Prim (1921 ∼)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Cut Property

Cut Property (Version I)

X : A part of some MST T_1 of G

$(S, V \setminus S)$: A *cut* such that X does *not* cross $(S, V \setminus S)$

e : A lightest edge across $(S, V \setminus S)$

Cut Property (Version I)

X : A part of some MST T_1 of G

$(S, V \setminus S)$: A *cut* such that X does *not* cross $(S, V \setminus S)$

e : A lightest edge across $(S, V \setminus S)$

Then $X \cup \{e\}$ is a part of some MST T_2 of G.

Cut Property (Version I)

$$\begin{aligned} X &: \text{A part of some MST } T_1 \text{ of } G \\ (S, V \setminus S) &: \text{A } \textit{cut} \text{ such that } X \text{ does } \textit{not} \text{ cross } (S, V \setminus S) \\ e &: \text{A lightest edge across } (S, V \setminus S) \end{aligned}$$

Then $X \cup \{e\}$ is a part of some MST T_2 of G.

Correctness of Prim's and Kruskal's algorithms.

By Exchange Argument.

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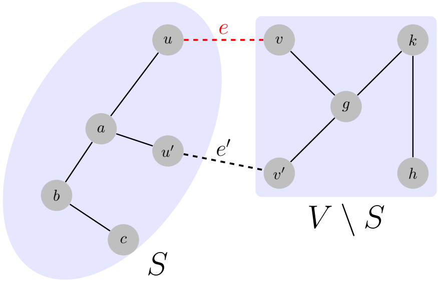

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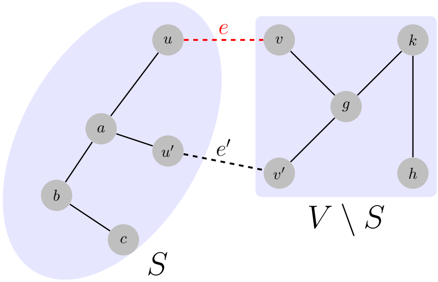
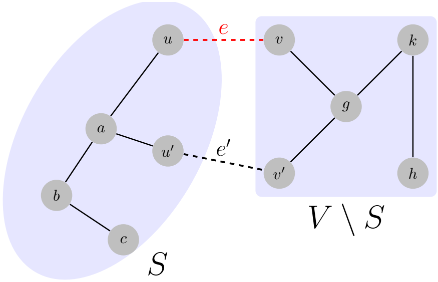


$$T' = \underbrace{\underbrace{T}_{X \subseteq T} + \{e\}}_{\text{if } e \notin T} - \{e'\}$$

By Exchange Argument.



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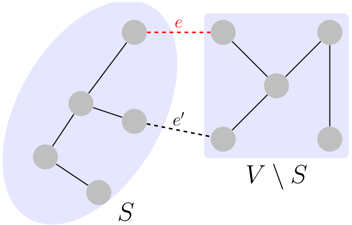
"a" $\to$ "the" $\Longrightarrow$ "some" $\to$ "all"

Cut Property (Version II)

A cut $(S, V \setminus S)$

Let $e = (u, v)$ be a lightest edge across $(S, V \setminus S)$

$\exists$ MST T of $G : e \in T$

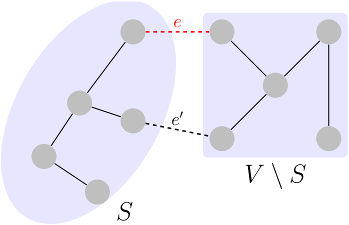


Cut Property (Version II)

$$\text{A cut } (S, V \setminus S)$$

$$\text{Let } e = (u, v) \text{ be } \underline{\text{a}} \text{ lightest edge across } (S, V \setminus S)$$

$$\exists \text{ MST } T \text{ of } G : e \in T$$



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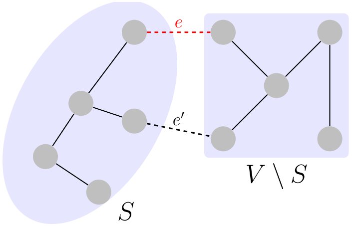


$$T' = \underbrace{T + \{e\}}_{\text{if } e \notin T} - \{e'\}$$

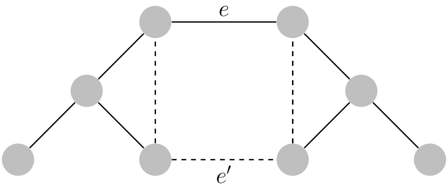
"a" $\rightarrow$ "the" $\Longrightarrow$ "$\exists$" $\rightarrow$ "$\forall$"

Cycle Property

Cycle Property

- Let C be any cycle in G
- Let $e = (u, v)$ be *a* maximum-weight edge in C

$$\text{Then } \exists \text{ MST } T \text{ of } G : e \notin T.$$



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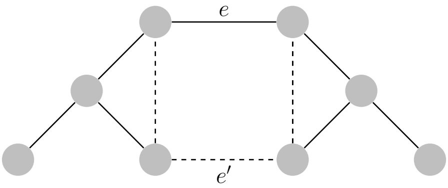

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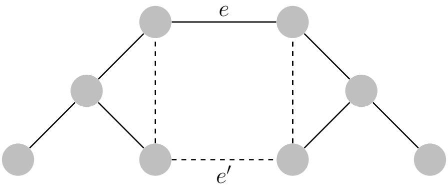

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"a" $\to$ "the" $\implies$ "$\exists$" $\to$ "$\forall$"

Joseph Kruskal ($1928 \sim 2010$)

Anti-Kruskal Algorithm

Reverse-delete algorithm (wiki; clickable)

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Cycle Property

$$T \subseteq F \implies \exists\, T' : T' \subseteq F - \{e\}$$

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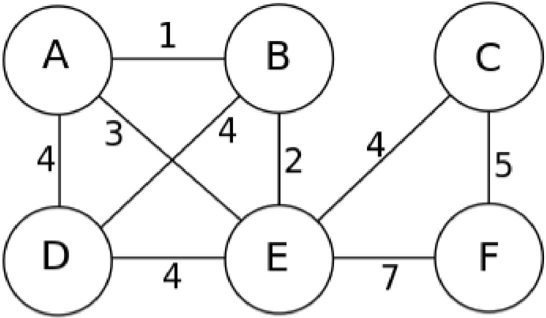
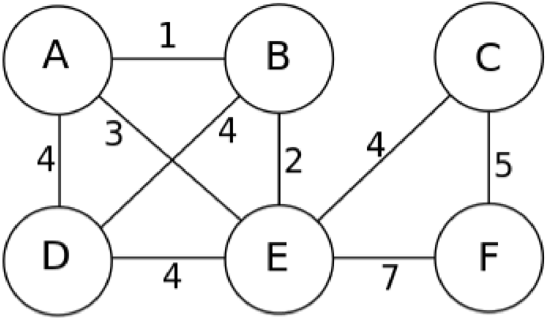
"On the Shortest Spanning Subtree of a Graph and the Traveling Salesman Problem"

— Kruskal, 1956.


A
1
B
C
3
4
4
2
4
5
D
E
F
4
7

Otakar Borůvka ($1899 \sim 1995$)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Theorem (Uniqueness of MST)

Let G be an edge-weighted undirected graph.
If each edge has a distinct weight, then there is a unique MST of G.

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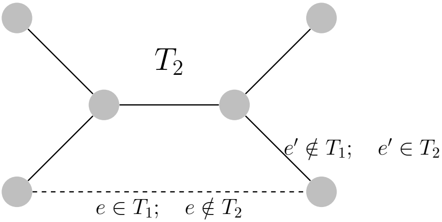
$$e \in T_1 \setminus T_2 \ (w.l.o.g)$$


T_2
$e' \notin T_1; \quad e' \in T_2$
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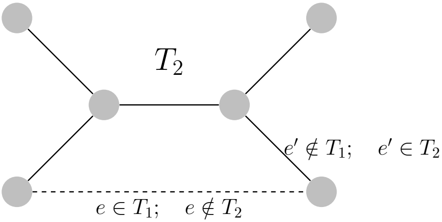
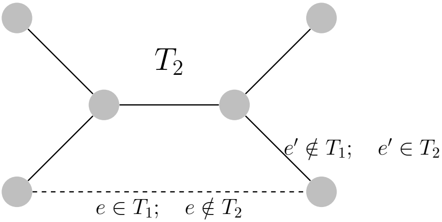


$$T_2 + \{e\} \implies C$$



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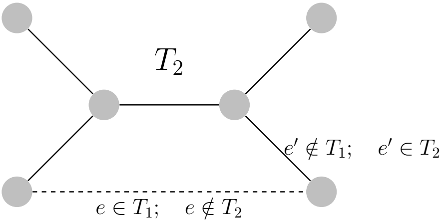
$\exists (e' \in C) \notin T_1$



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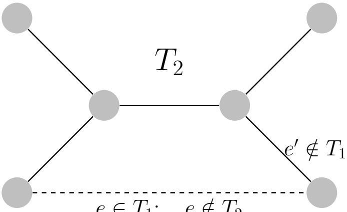

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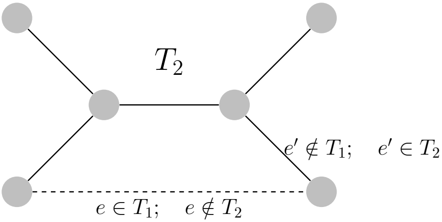

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$$T' = T_2 + \{e\} - \{e'\} \implies w(T') < w(T_2)$$

Condition for Uniqueness of MST

$$\text{Unique MST} \;\not\Longrightarrow\; \text{Distinct weights}$$

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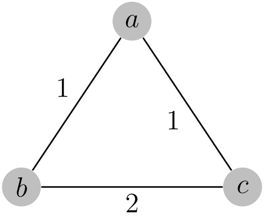


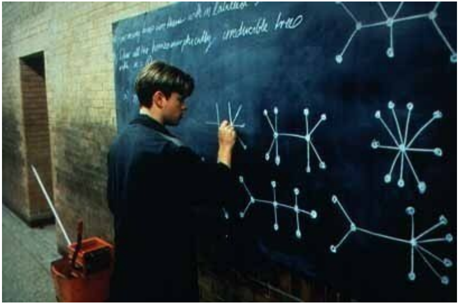
Rooted Trees in Computer Science

Definition (Rooted Trees (有根树))

bfs

dfs: in-order, pre-order, post-order

search trees



Theorem (Cayley's Formula)

The number T_n of labeled trees on $n \geq 2$ vertices is n^{n-2}.

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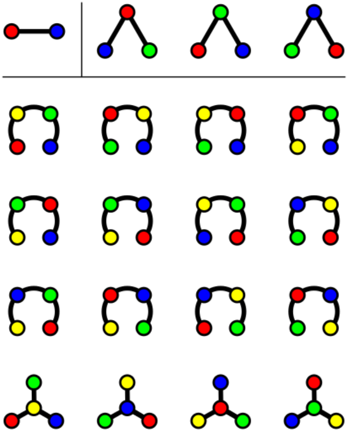
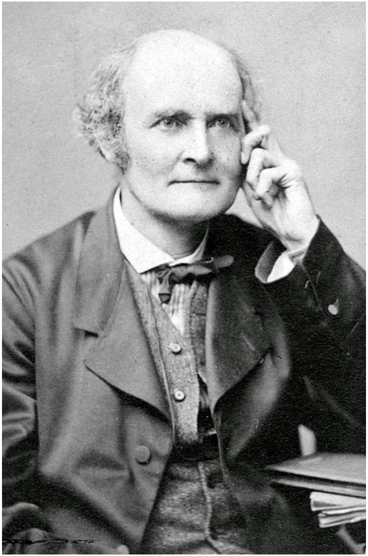
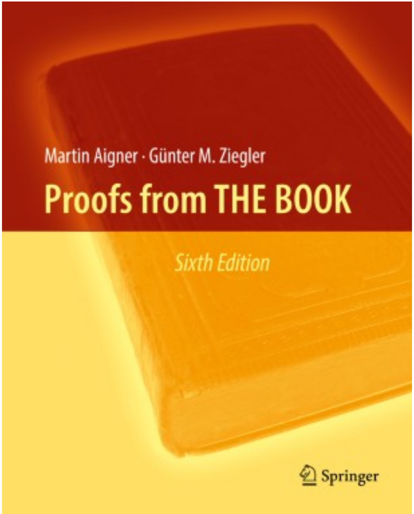




Arthur Cayley (1821 $\sim$ 1895)



Chapter 33: Cayley's formula for the number of trees

By Double Counting.

— Jim Pitman

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https://en.wikipedia.org/wiki/Double_counting_(proof_technique)#Counting_trees

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How many ways are there of forming a rooted tree from an empty graph by adding directed edges one by one?

Choose one of the T_n labeled trees on n vertices.

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Choose one of its n vertices as root.

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Choose one of the $(n-1)!$ possible sequences
in which to add its $n-1$ directed edges.

$$T_n n(n-1)! = T_n n!$$

Suppose that we have added $n - k$ directed edges.

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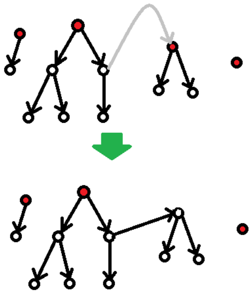

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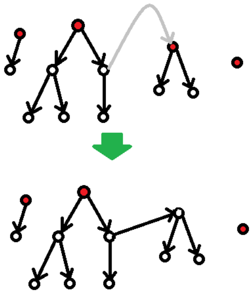

$$\prod_{k=2}^{n} n(k-1) = n^{n-1}(n-1)! = n^{n-2}n!$$

$$T_n n! = n^{n-2} n!$$

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$$T_n = n^{n-2}$$

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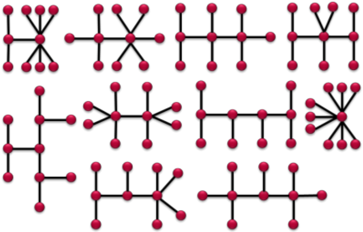

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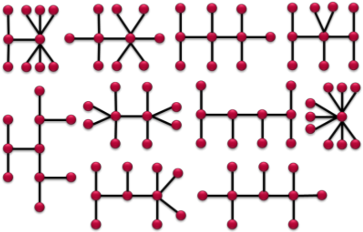

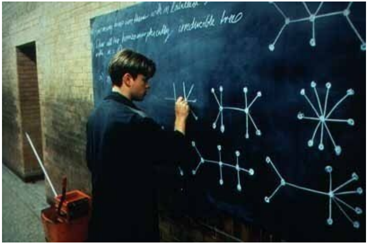

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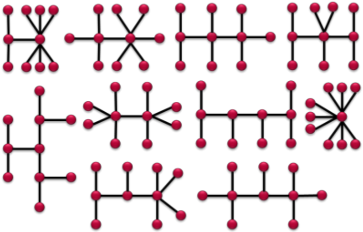
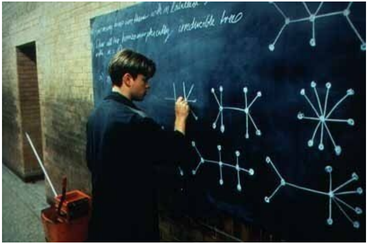
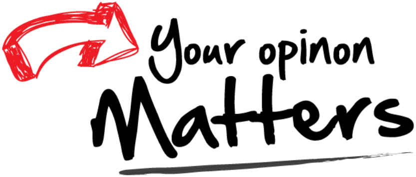




Homeomorphically Irreducible Trees of size $n = 10$


Thank
You!



Office 302

Mailbox: H016

hfwei@nju.edu.cn