(十一) 图论: 平面图与图着色
(Planarity and Coloring)

魏恒峰

hfwei@nju.edu.cn

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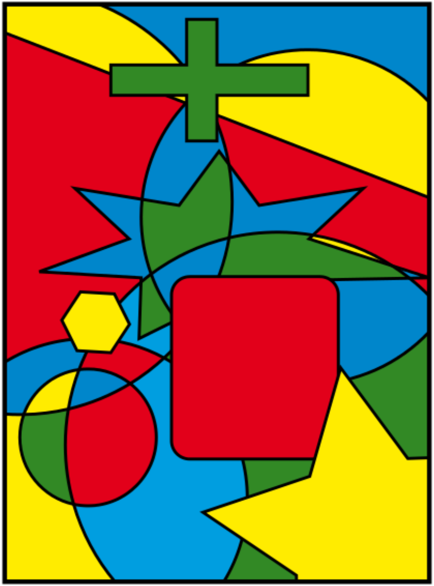
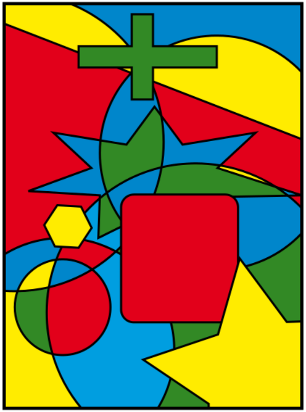
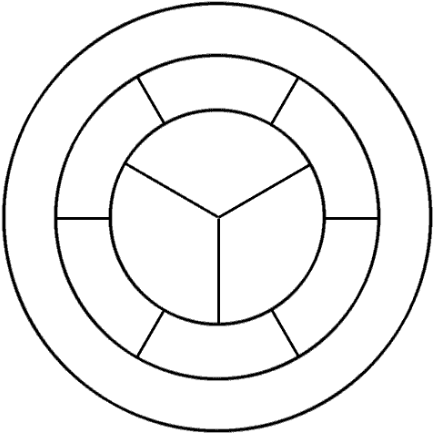
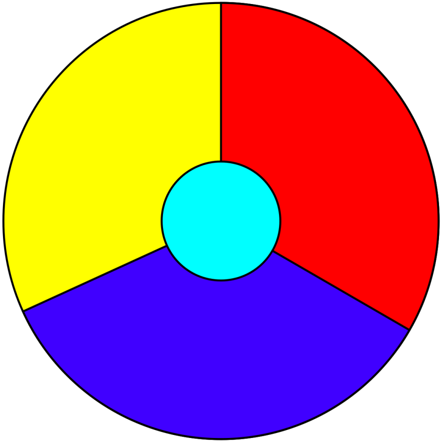


Theorem (Four Color (Map) Theorem (informal))

Every map can be colored with only four colors such that no two adjacent regions share the same color.





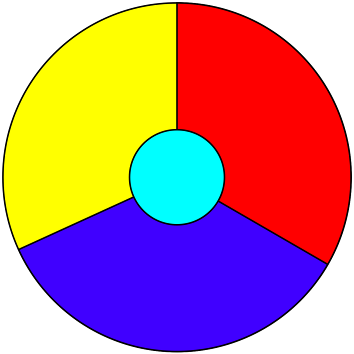


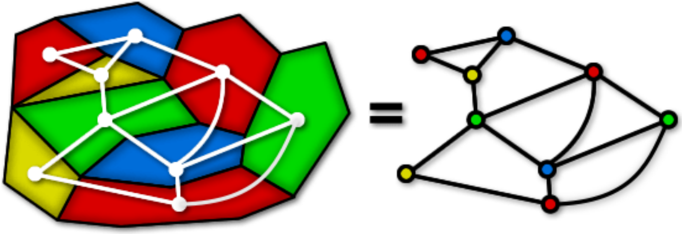
Every region is adjacent to the other 3 regions.

What if we have a map which contains 5 regions so that every region is adjacent to the other 4 regions?

IMPOSSIBLE™

What does Four Color Theorem to do with Graph Theory?





Every map produces a planar graph.

Theorem (Four Color Theorem (Kenneth Appel, Wolfgang Haken; 1976))

Every simple planar graph is 4-colorable.

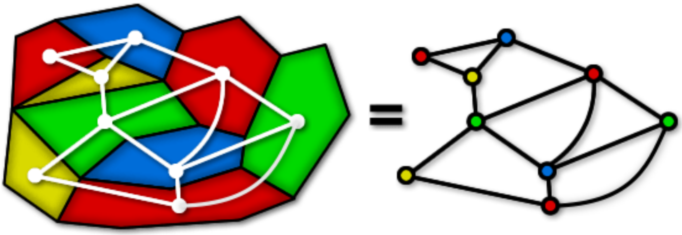
I will *not* show its proof (which I don't understand either)!

Theorem

Every simple planar graph is 6-colorable.

Theorem (Percy John Heawood (1890))

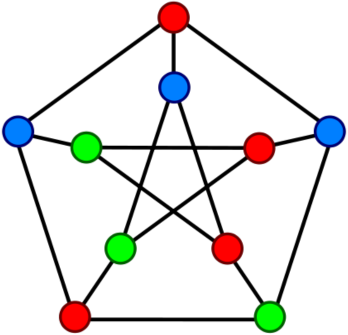
Every simple planar graph is 5-colorable.



Graph Coloring Problem

Definition (k-Colorable (k-可着色的))

If G is a connected undirected graph without loops,
then G is k-colorable if its vertices can be colored in k colors
so that adjacent vertices have different colors.

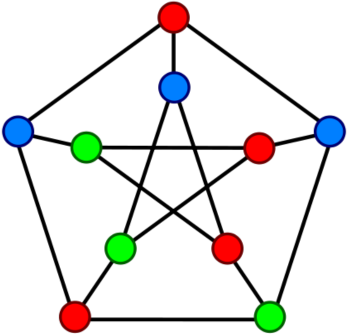


The Petersen graph is ≥ 3-colorable.

Definition (k-Chromatic (k-色数的))

If G is k-colorable, but is not $(k-1)$-colorable, then G is k-chromatic.

$$\chi(G) = k$$



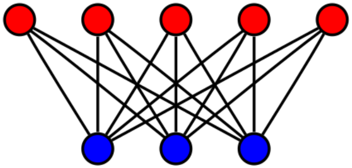
The Petersen graph is 3-chromatic.
(It contains an odd cycle.)

Lemma

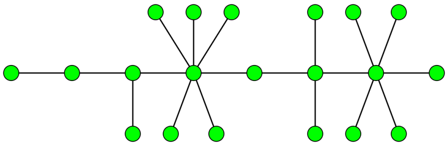
A graph is 1*-colorable iff it is the the empty graph with 1 vertex.*

Theorem

A graph is 2-colorable iff it is bipartite.



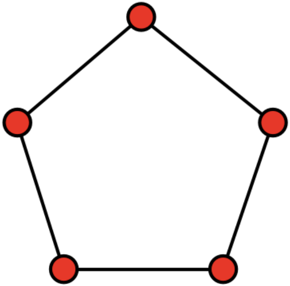
$K_{5,3}$

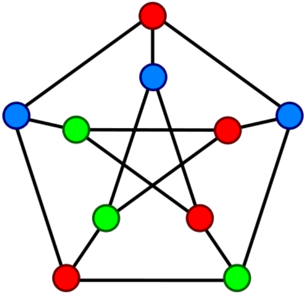


Trees are bipartite.

Theorem

The 3-coloring problem (i.e., testing whether a graph is 3-colorable or not) is NP-complete *(HARD!).*





Theorem

The 4-coloring problem is NP-complete *(HARD!).*

Theorem (Four Color Theorem (Kenneth Appel, Wolfgang Haken; 1976))

Every simple planar graph is 4-colorable.

Theorem

Let G be a simple connected graph. Then,

$$\chi(G) \leq \Delta(G) + 1.$$

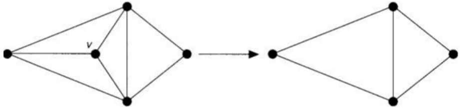
By induction on the number of vertices of G.

Basis Step: $n = 1$. $\chi(G) = 1$ and $\Delta(G) = 0$.

Induction Hypothesis: Suppose that for any simple connected graph G with n vertices,

$$\chi(G) \leq \Delta(G) + 1.$$

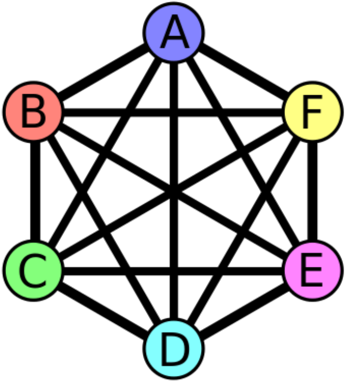
Induction Step: Consider a simple connected graph G with $n + 1$ vertices. $\deg(v) \leq \Delta(G)$

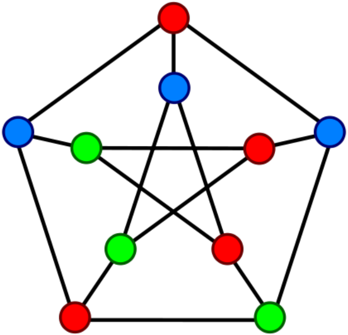


Theorem (Brooks's Theorem (R. Leonard Brooks; 1941))

Let G be a simple connected graph other than a complete graph or an odd cycle. Then

$$\chi(G) \le \Delta(G).$$





Theorem (Brooks's Theorem (R. Leonard Brooks; 1941))

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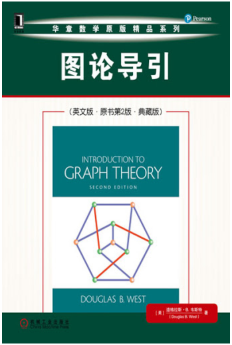
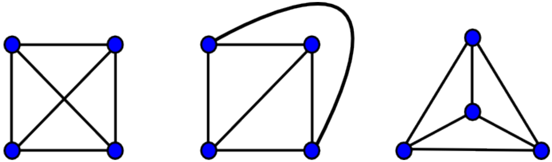


Theorem 5.1.22

Definition (Planar Graph (平面图))

A planar graph is a graph that can be drawn in the plane without edge crossings.

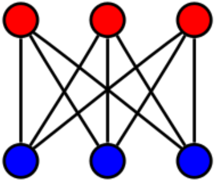


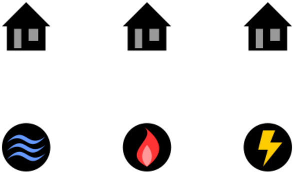
Theorem (K. Wagner (1936); I. Fáry (1948))

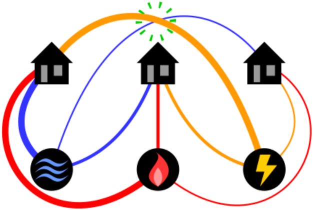
Every simple planar graph can be drawn with straight lines.

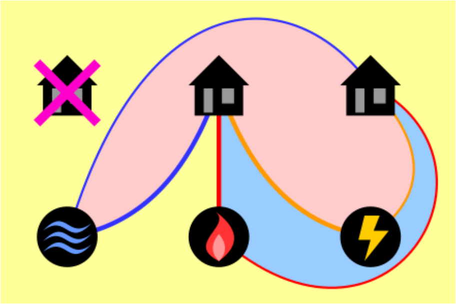
Theorem (Kazimierz Kuratowski, 1930)

The utility graph $K_{3,3}$ is non-planar.

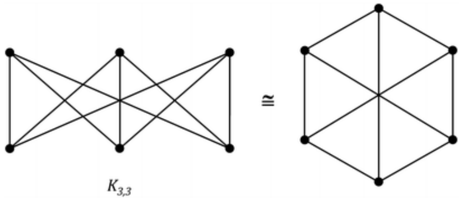


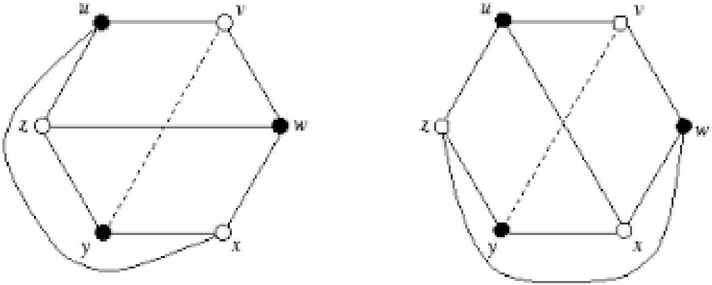






Proof without Words

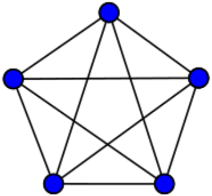


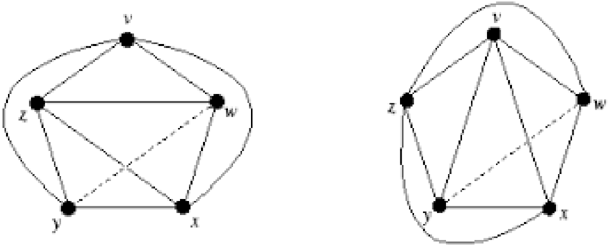


$\color{red}{\mathrm{cr}}(K_{3,3}) = 1 \quad$ (crossing number)

Theorem

K_5 is non-planar.





$$\textcolor{red}{\mathrm{cr}}(K_5) = 1$$

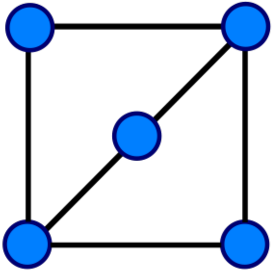
Theorem (Kazimierz Kuratowski, 1930)

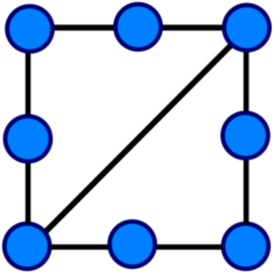
A graph is planar iff it contains no subgraph homeomorphic to K_5 or $K_{3,3}$.

"The K in K_5 stands for Kazimierz,
and the K in $K_{3,3}$ stands for Kuratowski."

Definition (Homeomorphic)

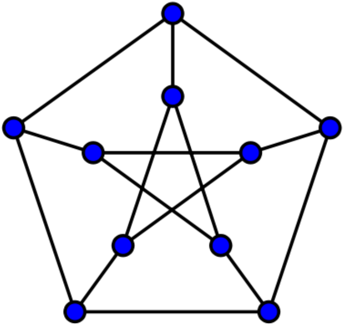
Two graphs are homeomorphic if one can be obtained from another by inserting or contracting vertices of degree 2.



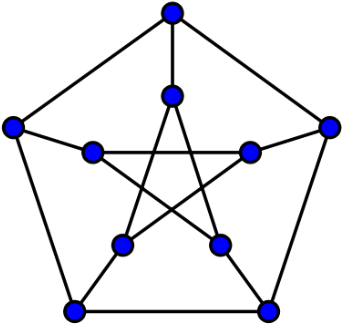
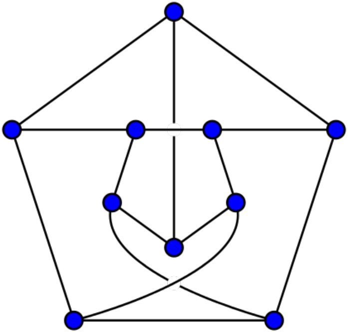


Theorem

The Petersen graph is non-planar.



https://github.com/courses-at-nju-by-hfwei/discrete-math-lectures/blob/main/11-planarity-coloring/figs/Kuratowski-Petersen.gif

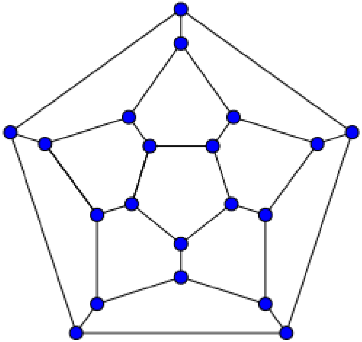
$$\textcolor{red}{\mathrm{cr}}(\text{Petersen Graph}) = 2$$

A planar graph should not has too many edges.

Theorem (Euler's Formula, 1750)

Let G be a plane drawing of a connected planar graph, and let n, m, and f denote respectively the number of vertices, edges, and faces of G.

$$n - m + f = 2$$

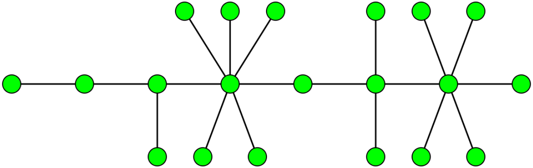


$$n - m + f = 20 - 30 + 12 = 2$$

Theorem (Euler's Formula, 1750)

Let G be a plane drawing of a connected planar graph, and let n, m, and f denote respectively the number of vertices, edges, and faces of G.

$$n - m + f = 2$$



$$n - m + f = n - (n-1) + 1 = 2$$

By induction on the number of edges of G.

Basis Step: $m = 0$. We have $n = 1$ and $f = 1$.

Induction Hypothesis: It holds for plane graphs with m edges.

Induction Step: Consider a plane graph G with $m + 1$ edges.

If G is a tree, we are done.

Otherwise, G contains a cycle.

Let e be an edge in some cycle of G.

Consider $G' = G - e$.

$$n - (m - 1) + (f - 1) = 2$$

Therefore,

$$n - m + f = 2$$

Theorem

Let G be a simple connected planar graph with $n \geq 3$ vertices and m edges. Then

$$m \leq 3n - 6.$$

$$n - m + f = 2$$

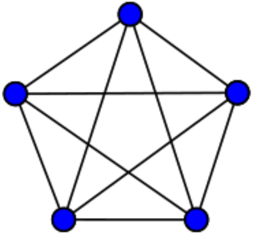
$$3f \leq 2m$$

Double Counting:

each face is bounded by ≥ 3 edges;
each edge bounds 2 faces

Theorem

K_5 is non-planar.

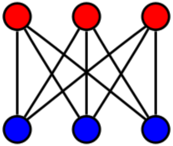


$$m \leq 3n - 6$$

$$10 \leq 3 \times 5 - 6$$

Theorem

$K_{3,3}$ is non-planar.



$$m \leq 3n - 6$$

$$9 \leq 3 \times 6 - 6$$

Theorem

Let G be a simple connected planar graph with $n \geq 3$ vertices and m edges. If G has no triangles, then

$$m \leq 2n - 4.$$

$$n - m + f = 2$$

$$4f \leq 2m$$

Theorem

$K_{3,3}$ is non-planar.

$$m \leq 2n - 4$$

$$9 \leq 2 \times 6 - 4$$

Theorem

Every simple planar graph contains a vertex of degree ≤ 5.

$$m \leq 3n - 6$$

Suppose that, by contradiction, $\delta(G) \geq 6$.

$$6n \leq 2m$$

$$3n \leq m \leq 3n - 6$$

Theorem

Every simple planar graph is 6-colorable.

By induction on the number of vertices.

Basis Step: $n = 1$. Trivial.

Induction Hypothesis: Suppose that it holds for simple planar graphs with $n \geq 1$ vertices.

Induction Step: Consider a simple planar graph G with $n + 1$ vertices.

G contains a vertex v of degree ≤ 5.

$G' = G - v$ is 6-colorable.

Thus, G is 6-colorable. $(\deg(v) \leq 5)$

Theorem (Percy John Heawood (1890))

Every simple planar graph is 5-colorable.

By induction on the number of vertices.

Basis Step: $n = 1$. Trivial.

Induction Hypothesis: Suppose that it holds for simple planar graphs with $n \geq 1$ vertices.

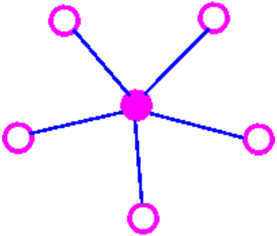
Induction Step: Consider a simple planar graph G with $n + 1$ vertices.

G contains a vertex v of degree ≤ 5.

$G' = G - v$ is 5-colorable.

If $\deg(v) < 5$, G is 5-colorable.

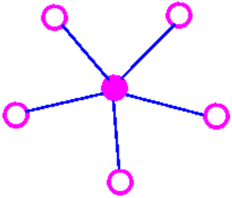
Now assume that $\deg(v) = 5$.



$$\{v, v_1\}, \{v, v_2\}, \{v, v_3\}, \{v, v_4\}, \{v, v_5\}$$

If v_1, v_2, v_3, v_4, and v_5 uses < 5 colors, we are done.

Now assume that v_1, v_2, v_3, v_4, and v_5 uses 5 colors.



Suppose that there is *no* $v_1 \sim v_3$ path in $G' = G - v$, all of whose vertices are colored red or blue.

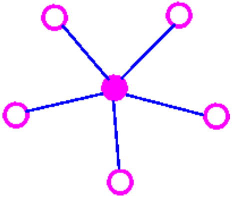
Let S be the set of all red or blue vertices of $G - v$ connected to v_1 by a red-blue path.

$$v_1 \in S, \quad v_3 \notin S$$

Interchange the colors of the vertices in S

Coloring v red produces a 5-coloring of G.

Now assume that there *is* a $v_1 \sim v_3$ path in $G' = G - v$,
all of whose vertices are colored red or blue.

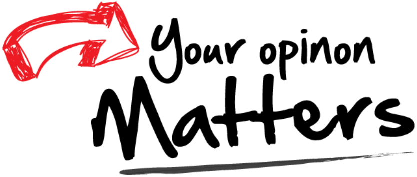


There cannot be $v_2 \sim v_4$ path in $G' = G - v$,
all of whose vertices are colored green or purple.

(Otherwise, G' and thus G is non-planar.)

By similar argument, G is 5-colorable.


Thank
You!



Office 302

Mailbox: H016

hfwei@nju.edu.cn