(十) 图论: 树 (Trees)

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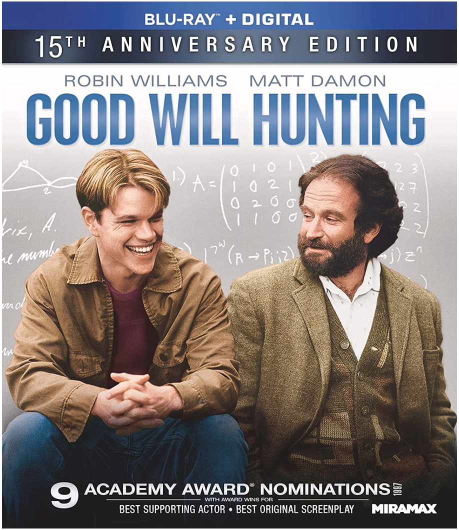
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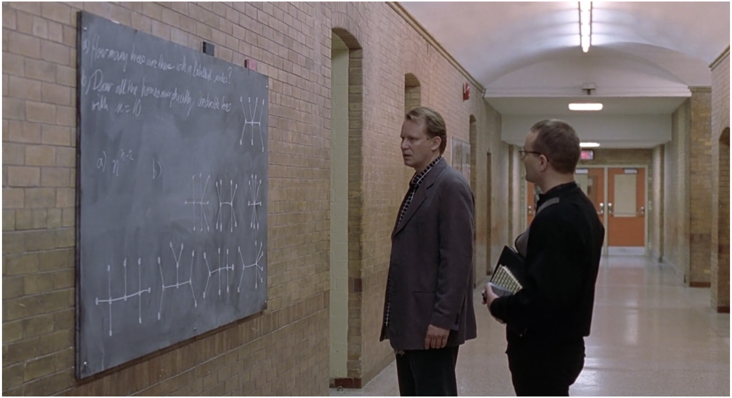
你, 真得, 看懂了吗?



(1) How many trees are there with n labeled vertices?

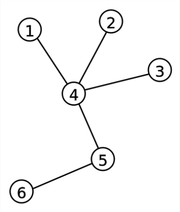
(2) Draw all homomorphically irreducible trees with $n = 10$.

Definition (Tree (树))

A tree is a connected acyclic undirected graph.

Definition (Forest (森林))

A forest is a acyclic undirected graph.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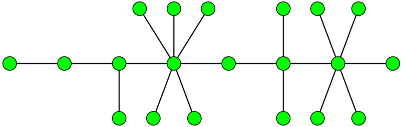
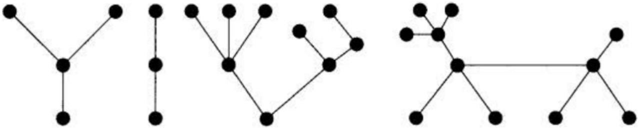
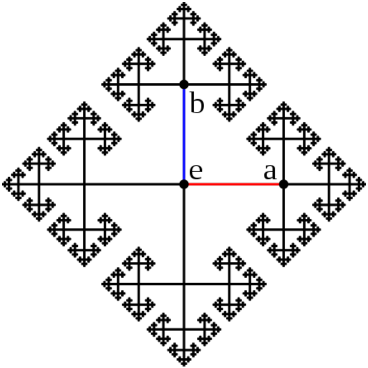






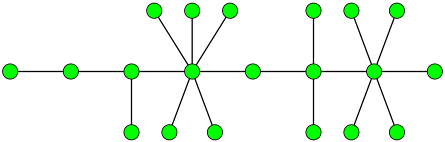
Cayley Graph (4-regular tree)

Definition (Internal Vertex (内部顶点); Leaf (叶子))

In a tree T with ≥ 2 vertices, for a vertex v in T, if

$$\deg(v) = 1$$

then v is called a leaf; otherwise, v is an internal vertex.



Lemma

Any tree T with ≥ 2 vertices contains ≥ 1 leaf.

$$\text{Otherwise, } \forall v \in V.\ \deg(v) \geq 2 \implies T \text{ has cycles.}$$

Lemma

Any tree T with ≥ 2 vertices contains ≥ 2 leaves.

$$\sum_{v \in V} \deg(v) = 2n - 2$$

Consider the two endpoints of any maximal path in T.

They are leaves of T.

Lemma

Deleting a leaf from a tree T with n vertices produces a tree with $n-1$ vertices.

$G' = G - v$ is connected and acyclic.

A leaf does *not* belong to any paths connecting two other vertices.

This lemma can be used in induction for trees!

Theorem ((We call it) Characterization of Trees)

Let T be an undirected graph with n vertices.

Then the following statements are equivalent:

(1) *T is a tree;*

(2) *T is acyclic, and has $m = n - 1$ edges;*

(3) *T is connected, and has $m = n - 1$ edges;*

(4) *T is connected, and each edge is a bridge;*

(5) *Any two vertices of T are connected by exactly one path;*

(6) *T is acyclic, but the addition of any edge creates exactly one cycle.*

$$(1) \implies (2) \implies (3) \implies (4) \implies (5) \implies (6) \implies (1)$$

Theorem (Characterization of Trees)

(1) *T is a tree;*

(2) *T is acyclic, and has $m = n - 1$ edges.*

By induction on the number n of vertices of trees.

Basis Step: $n = 1$. $m = 0 = n - 1$.

Induction Hypothesis: Any trees with $n - 1$ vertices has $n - 2$ edges.

Induction Step: Consider a tree T with $n \geq 2$ vertices.

T has a leaf v.

For $T' = T - v$, $m(T') = (n - 1) - 1 = n - 2$.

$$m(T) = (n - 2) + 1 = n - 1.$$

Theorem (Characterization of Trees)

(2) *T is acyclic, and has $n-1$ edges;*

(3) *T is connected, and has $n-1$ edges.*

By Contradiction.

Suppose that T is *disconnected*.

T is a forest, consisting of $k \geq 2$ trees $T_1, T_2, \ldots$.

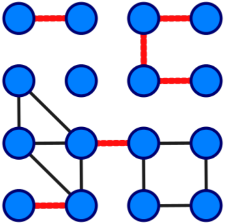
By (2), for each T_i, $m(T_i) = n(T_i) - 1$.

$$m(T) = \sum_{i=1}^{k} m(T_i) = n - k \neq n - 1.$$

Theorem (Characterization of Trees)

(3) T *is connected, and has* $n-1$ *edges;*

(4) T *is connected, and each edge is a bridge.*



Definition (Bridge (桥))

A bridge of a graph G is an edge e such that

$$c(G-e) > c(G).$$

Theorem (Characterization of Trees)

(3) *T is connected, and has $n-1$ edges;*

(4) *T is connected, and each edge is a bridge.*

Consider any edge e of T.

$$m(T-e) = (n-1) - 1 = n-2.$$

$T - e$ must be disconnected.

Theorem (Characterization of Trees)

(4) *T is connected, and each edge is a bridge;*

(5) *Any two vertices of T are connected by exactly one path.*

Consider any two vertices u and v.

T is connected $\implies$ u and v are connected by ≥ 1 path.

If u and v are connected by two paths,

the edges on these two paths are not bridges.

Theorem (Characterization of Trees)

(5) *Any two vertices of T are connected by exactly one path;*

(6) *T is acyclic, but the addition of any edge creates exactly one cycle.*

If T has a cycle C,

any two vertices in C is connected by ≥ 2 paths.

Consider the addition of edge $\{u, v\}$ to T.

It creates a cycle, consisting of $\{u, v\}$ and the path from u to v.

Lemma

If two distinct cycles of a graph G share a common edge e, then G has a cycle that does not contain e.

Theorem (Characterization of Trees)

(6) *T is acyclic, but the addition of any edge creates exactly one cycle;*

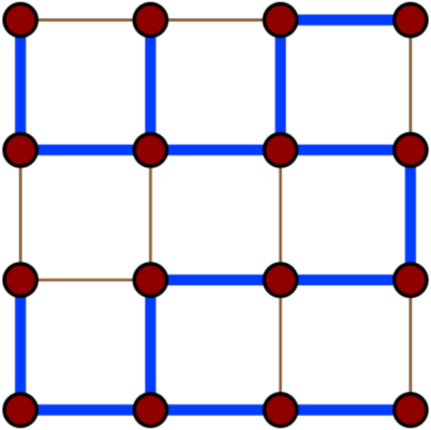
(1) *T is a tree.*

Suppose that T is disconnected.

T is a forest, consisting of ≥ 2 trees $T_1, T_2, \ldots$

Choose $u \in V(T_1)$, $v \in V(T_2)$.

$T + \{u, v\}$ does not create cycles.

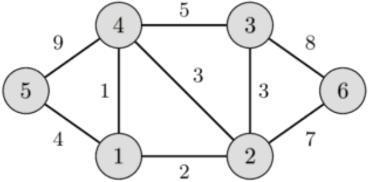


Spanning Trees (trees in graphs)

Definition (Subgraph (子图))

A graph S is a subgraph of G if

$$V(S) \subseteq V(G) \land E(S) \subseteq E(G) \land \bigcup E(S) \subseteq V(S)$$



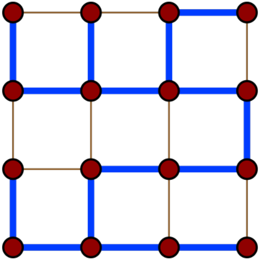
Definition (Induced Subgraph (诱导子图))

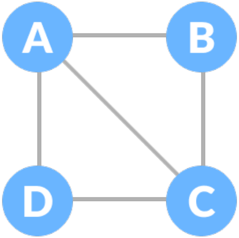
A graph S is an induced subgraph of G if S is a subgraph of G such that

$$\{\{u, v\} \in E(G) \mid u \in V(S), v \in V(S)\} \subseteq E(S).$$

Definition (Spanning Tree (生成树))

A spanning tree T of an undirected graph G is a subgraph that is a tree with all vertices of G.





Theorem

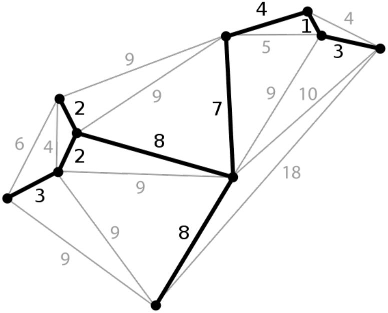
Every connected undirected graph G admits a spanning tree.

Repeatedly deleting edges in cycles until the graph is acyclic.

The remaining graph is a spanning tree of G.

Definition (Minimum Spanning Tree (MST; 最小生成树))

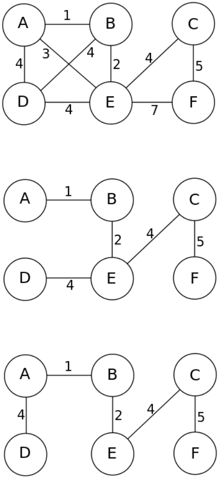
A minimum spanning tree T of an edge-weighted undirected graph G is a spanning tree with minimum total weight of edges.



Existence? Uniqueness? Algorithms?

Theorem

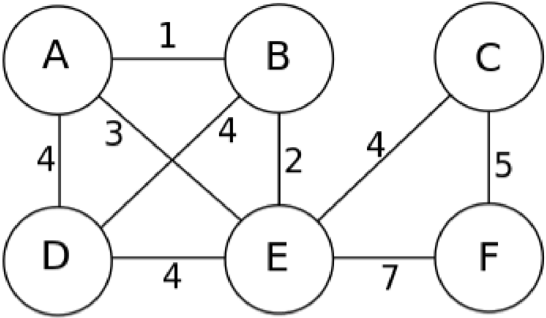
Every weighted connected undirected graph G admits a minimum spanning tree.



Joseph Kruskal ($1928 \sim 2010$)

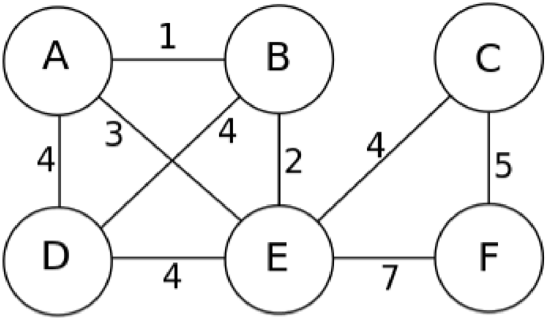
Repeatedly adding the next lowest-weight edge

that will not form a cycle until $n-1$ edges are added.



Robert C. Prim (1921 ∼)

Repeatedly adding the cheapest possible edge from the partially built tree to another vertex, until $n-1$ edges are added.

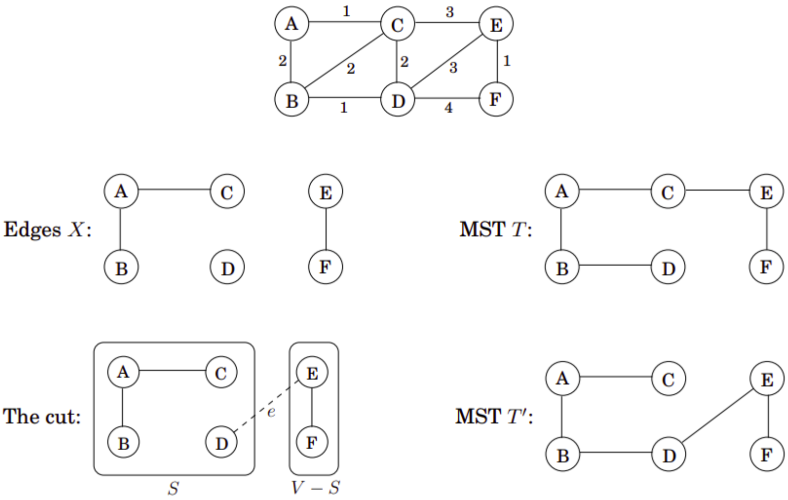


Theorem (Cut Property)

$$
\begin{aligned}
X &: \textit{ A part of some MST } T \textit{ of } G \\
(S, V \setminus S) &: \textit{ A cut such that } X \textit{ does not cross } (S, V \setminus S) \\
e &: \textit{ A lightest edge across } (S, V \setminus S)
\end{aligned}
$$

Then $X \cup \{e\}$ *is a part of some MST* T' *of* G.

(a) (b) (c)



Theorem (Cut Property)

$$X: \textit{A part of some MST } T \textit{ of } G$$

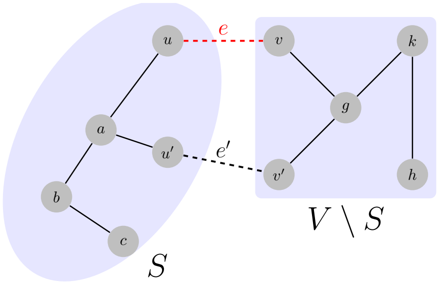
$$(S, V \setminus S): \textit{A cut such that } X \textit{ does not cross } (S, V \setminus S)$$

$$e: \textit{A lightest edge across } (S, V \setminus S)$$

$$\textit{Then } X \cup \{e\} \textit{ is a part of some MST } T' \textit{ of } G.$$

Correctness of Kruskal's and Prim's algorithms.

By Exchange Argument.



If $e \in T$, we are done. Otherwise, construct T'.

$$T' = \underbrace{\underbrace{T}_{X \subseteq T} + \{e\} - \{e'\}}_{\text{if } e \notin T}$$

$$w(e) \leq w(e') \implies T' \text{ is also an MST}$$

Theorem (Cut Property)

$$\begin{aligned} X &: \textit{A part of some MST } T \textit{ of } G \\ (S, V \setminus S) &: \textit{A cut such that } X \textit{ does not cross } (S, V \setminus S) \\ e &: \textit{A lightest edge across } (S, V \setminus S) \end{aligned}$$

Then $X \cup \{e\}$ *is a part of some MST* T' *of* G.

"a" $\to$ "the" $\implies$ "some" $\to$ "all"

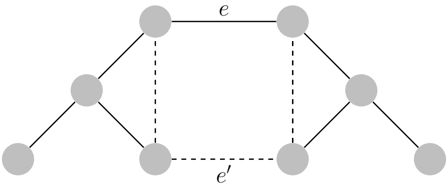
Theorem (Cycle Property)

- *Let C be any cycle in connected undirected G*
- *Let $e = (u, v)$ be a maximum-weight edge in C*

$$\textit{Then } \exists \textit{ MST } T \textit{ of } G : e \notin T.$$

Choose any MST T of G.

If $e \notin T$, we are done. Otherwise, construct T'.



$$T' = \underbrace{T - \{e\}}_{\text{if } e \in T} + \{e'\}$$

Theorem (Cycle Property)

- *Let C be any cycle in connected undirected G*
- *Let $e = (u, v)$ be a maximum-weight edge in C*

$$\textit{Then } \exists \textit{ MST } T \textit{ of } G : e \notin T.$$

$$\text{“a”} \to \text{“the”} \implies \text{“}\exists\text{”} \to \text{“}\forall\text{”}$$

Theorem (Uniqueness of MST)

Let G be an edge-weighted undirected graph.
If each edge has a distinct weight, then there is a unique MST of G.

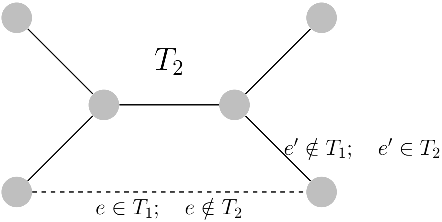
By Contradiction.

$$\exists \text{ MSTs } T_1 \neq T_2$$

$$\Delta E = \{e \mid e \in T_1 \setminus T_2 \lor e \in T_2 \setminus T_1\}$$

$$e = \min \Delta E$$

$$\text{Suppose that } e \in T_1 \setminus T_2 \ \textit{(w.l.o.g)}$$



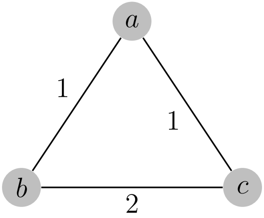
$$T_2 + \{e\} \implies C$$

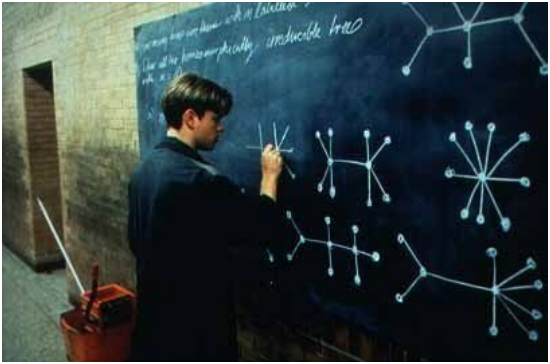
$$\exists (e' \in C) \notin T_1 \implies e' \in T_2 \setminus T_1 \implies e' \in \Delta E \implies w(e') > w(e)$$

$$T' = T_2 + \{e\} - \{e'\} \implies w(T') < w(T_2)$$

Condition for Uniqueness of MST

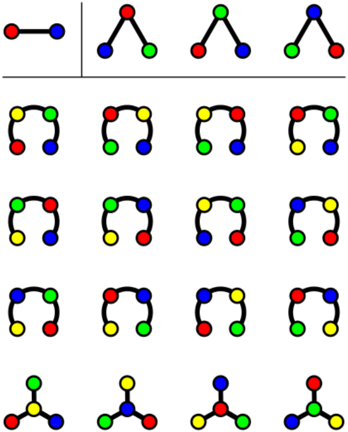
$$\text{Unique MST} \;\not\Longrightarrow\; \text{Distinct weights}$$

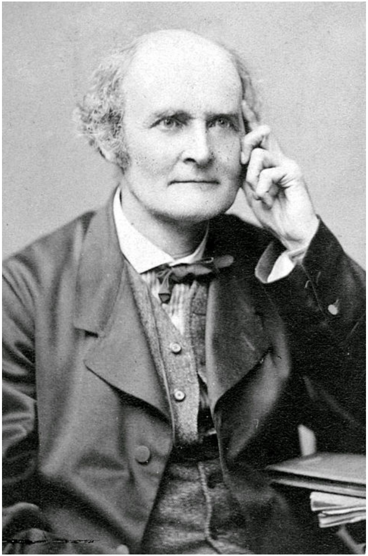



irreducible tree

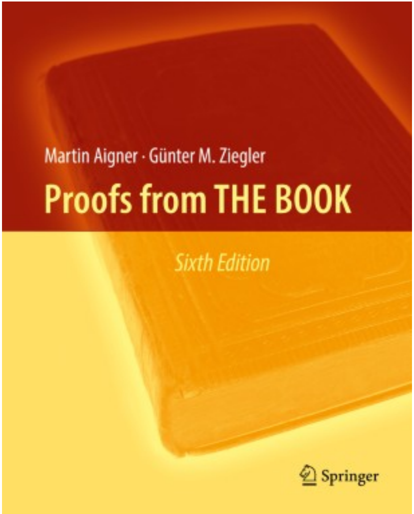
Theorem (Cayley's Formula)

The number T_n of labeled trees on $n \geq 2$ vertices is n^{n-2}.





Arthur Cayley (1821 ~ 1895)



Chapter 33: Cayley's formula for the number of trees

By Double Counting.

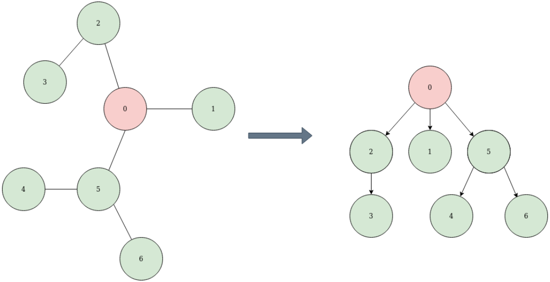
— Jim Pitman

https://en.wikipedia.org/wiki/Double_counting_(proof_technique)#Counting_trees

How many ways are there of forming a directed rooted tree from an empty graph by adding directed edges one by one?

Definition (Rooted Tree (有根树))

A rooted tree is a tree where one vertex has been designated the root.



Definition (Directed Rooted Tree (有向有根树))

A directed rooted tree is a rooted tree where all edges directed away from or towards the root.

Choose one of the T_n labeled trees on n vertices.

Choose one of its n vertices as root.

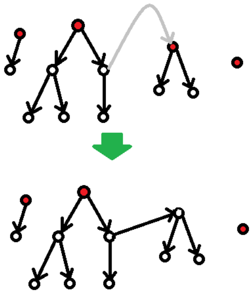
Choose one of the $(n-1)!$ possible sequences in which to add its $n-1$ directed edges.

$$T_n n(n-1)! = T_n n!$$

Suppose that we have added $n-k$ directed edges.

We obtain a rooted forest with k trees.

There are $n(k-1)$ choices for the next edge to add.



$$\prod_{k=2}^{n} n(k-1) = n^{n-1}(n-1)! = n^{n-2}n!$$

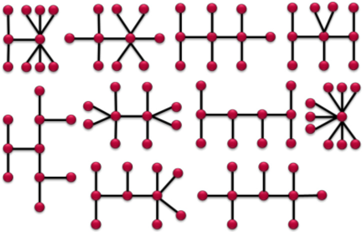
$$T_n n! = n^{n-2} n!$$

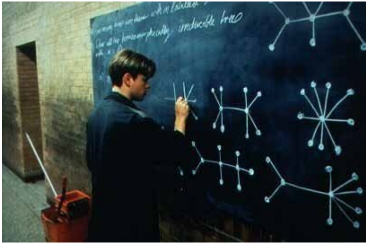
$$\boxed{T_n = n^{n-2}}$$

Definition (Irreducible Tree)

An irreducible tree is a tree T where

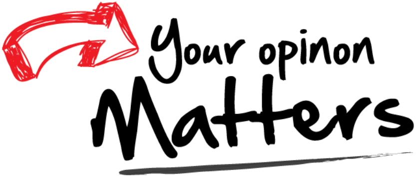
$$\forall v \in V(T).\ \deg(v) \neq 2.$$





Homeomorphically Irreducible Trees of size $n = 10$


Thank
You!



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