(四) 集合: 关系 (Relation)

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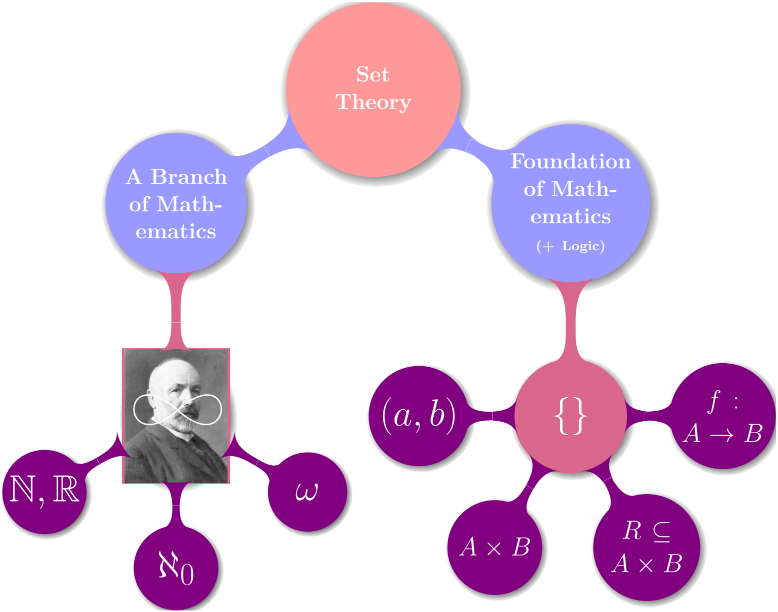

Set Theory
A Branch of Mathematics
Foundation of Mathematics
(+ Logic)
$\mathbb{N}, \mathbb{R}$
$\aleph_0$
ω
(a, b)
$\{\}$
$f : A \to B$
$A \times B$
$R \subseteq A \times B$

Figure 13. A selection of consistency axioms over an execution $(E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar})$

Auxiliary relations

$\mathsf{sameobj}(e, f) \iff \mathsf{obj}(e) = \mathsf{obj}(f)$

Per-object causality (aka happens-before) order:
$\mathsf{hbo} = ((\mathsf{ro} \cap \mathsf{sameobj}) \cup \mathsf{vis})^+$

Causality (aka happens-before) order: $\mathsf{hb} = (\mathsf{ro} \cup \mathsf{vis})^+$

Axioms

EVENTUAL:
$\forall e \in E. \neg(\exists \text{ infinitely many } f \in E. \mathsf{sameobj}(e, f) \wedge \neg(e \xrightarrow{\mathsf{vis}} f))$

THINAIR: $\mathsf{ro} \cup \mathsf{vis}$ is acyclic

POCV (Per-Object Causal Visibility): $\mathsf{hbo} \subseteq \mathsf{vis}$

POCA (Per-Object Causal Arbitration): $\mathsf{hbo} \subseteq \mathsf{ar}$

COCV (Cross-Object Causal Visibility): $(\mathsf{hb} \cap \mathsf{sameobj}) \subseteq \mathsf{vis}$

COCA (Cross-Object Causal Arbitration): $\mathsf{hb} \cup \mathsf{ar}$ is acyclic

Figure 17. Optimized state-based multi-value register and its simulation

$$
\begin{aligned}
\Sigma &= \mathsf{ReplicaID} \times \mathcal{P}(\mathbb{Z} \times (\mathsf{ReplicaID} \to \mathbb{N}_0)) \\
\vec{\sigma}_0 &= \langle r, \emptyset \rangle \\
M &= \mathcal{P}(\mathbb{Z} \times (\mathsf{ReplicaID} \to \mathbb{N}_0)) \\
\mathsf{do}(\mathtt{wr}(a), \langle r, V\rangle, t) &= \\
&(\langle r, \{(a, (\lambda s. \text{if } s \neq r \text{ then } \max\{v(s) \mid (\_, v) \in V\} \\
&\qquad \text{else } \max\{v(s) \mid (\_, v) \in V\} + 1))\}\rangle, \bot) \\
\mathsf{do}(\mathtt{rd}, \langle r, V\rangle, t) &= (\langle r, V\rangle, \{a \mid (a, \_) \in V\}) \\
\mathsf{send}(\langle r, V\rangle) &= (\langle r, V\rangle, V) \\
\mathsf{receive}(\langle r, V\rangle, V') &= \langle r, \{(a, v) \in V'' \mid \\
&\qquad v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'. (a', v') \in V'' \wedge a \neq a'\}\}\rangle, \\
\text{where } V'' &= \{(a, \bigsqcup\{v' \mid (a, v') \in V \cup V'\}) \mid (a, \_) \in V \cup V'\}
\end{aligned}
$$

$\langle s, V\rangle\ [\mathcal{R}_r]\ I \iff (r = s) \wedge (V\ [\mathcal{M}]\ I)$

$$
\begin{aligned}
&V\ [\mathcal{M}]\ ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar}), \mathsf{info}) \iff \\
&\quad (\forall (a, v), (a', v') \in V. (a = a' \implies v = v')) \wedge \\
&\quad (\forall (a, v) \in V. \exists s. v(s) > 0) \wedge \\
&\quad (\forall (a, v) \in V. v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'. (a', v') \in V \wedge a \neq a'\}) \wedge \\
&\quad \exists \text{ distinct } e_{s,k}. \\
&\quad (\{e \in E \mid \exists a. \mathsf{oper}(e) = \mathtt{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicaID} \wedge \\
&\qquad 1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V\}\}) \wedge \\
&\quad (\forall s, j, k. (\mathsf{repl}(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}} e_{s,k} \iff j < k)) \wedge \\
&\quad (\forall (a, v) \in V. \forall q. \{j \mid \mathsf{oper}(e_{q,j}) = \mathtt{wr}(a)\} \cup \\
&\qquad \{j \mid \exists s, k. e_{q,j} \xrightarrow{\mathsf{vis}} e_{s,k} \wedge \mathsf{oper}(e_{s,k}) = \mathtt{wr}(a)\} = \\
&\qquad \{j \mid 1 \leq j \leq v(q)\}) \wedge \\
&\quad (\forall e \in E. (\mathsf{oper}(e) = \mathtt{wr}(a) \wedge \\
&\qquad \neg\exists f \in E. \mathsf{oper}(f) = \mathtt{wr}(\_) \wedge e \xrightarrow{\mathsf{vis}} f) \implies (a, \_) \in V)
\end{aligned}
$$

the former. The only non-trivial obligation is to show that if

$$V\ [\mathcal{M}]\ ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}), \mathsf{info}),$$

then

$$
\begin{aligned}
\{a \mid (a, \_) \in V\} \subseteq \{a \mid \exists e \in E. \mathsf{oper}(e) = \mathtt{wr}(a) \wedge \\
\neg\exists f \in E. \exists a'. \mathsf{oper}(e) = \mathtt{wr}(a') \wedge e \xrightarrow{\mathsf{vis}} f\}
\end{aligned} \tag{13}
$$

(the reverse inclusion is straightforwardly implied by $\mathcal{R}_r$).

Take $(a, v) \in V$. We have $\forall (a, v) \in V. \exists s. v(s) > 0$,

$$v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'. (a', v') \in V \wedge a \neq a'\}$$

and

$$
\begin{aligned}
&\forall (a, v) \in V. \forall q. \{j \mid \mathsf{oper}(e_{q,j}) = \mathtt{wr}(a)\} \cup \\
&\quad \{j \mid \exists s, k. e_{q,j} \xrightarrow{\mathsf{vis}} e_{s,k} \wedge \mathsf{oper}(e_{s,k}) = \mathtt{wr}(a)\} = \\
&\quad \{j \mid 1 \leq j \leq v(q)\}.
\end{aligned}
$$

From this we get that for some $e \in E$

$$
\begin{aligned}
&\mathsf{oper}(e) = \mathtt{wr}(a) \wedge \neg\exists f \in E. \exists a'. a' \neq a \wedge \\
&\qquad\qquad \mathsf{oper}(e) = \mathtt{wr}(a') \wedge e \xrightarrow{\mathsf{vis}} f.
\end{aligned}
$$

Since vis is acyclic, this implies that for some $e' \in E$

$$\mathsf{oper}(e') = \mathtt{wr}(a) \wedge \neg\exists f \in E. \mathsf{oper}(e') = \mathtt{wr}(\_) \wedge e' \xrightarrow{\mathsf{vis}} f,$$

which establishes (13).

Let us now discharge RECEIVE. Let $\mathsf{receive}(\langle r, V\rangle, V') = \langle r, V'''\rangle$, where

$$
\begin{aligned}
V'' &= \{(a, \bigsqcup\{v' \mid (a, v') \in V \cup V'\}) \mid (a, \_) \in V \cup V'\}; \\
V''' &= \{(a, v) \in V'' \mid v \not\sqsubseteq \bigsqcup\{(a', v') \in V'' \mid a \neq a'\}\}.
\end{aligned}
$$

Assume $\langle r, V\rangle\ [\mathcal{R}_r]\ I, V'\ [\mathcal{M}]\ J$ and

$$
\begin{aligned}
I &= ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar}), \mathsf{info}); \\
J &= ((E', \mathsf{repl}', \mathsf{obj}', \mathsf{oper}', \mathsf{rval}', \mathsf{ro}', \mathsf{vis}', \mathsf{ar}'), \mathsf{info}'); \\
I \sqcup J &= ((E'', \mathsf{repl}'', \mathsf{obj}'', \mathsf{oper}'', \mathsf{rval}'', \mathsf{ro}'', \mathsf{vis}'', \mathsf{ar}''), \mathsf{info}'').
\end{aligned}
$$

By agree we have $I \sqcup J \in \mathsf{IEx}$. Then

$$
\begin{aligned}
&(\forall (a, v), (a', v') \in V. (a = a' \implies v = v')) \wedge \\
&(\forall (a, v) \in V. \exists s. v(s) > 0) \wedge \\
&(\forall (a, v) \in V. v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'. (a', v') \in V \wedge a \neq a'\}) \wedge \\
&\exists \text{ distinct } e_{s,k}. \\
&(\{e \in E \mid \exists a. \mathsf{oper}''(e) = \mathtt{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicaID} \wedge \\
&\quad 1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V\}\}) \wedge \\
&(\forall s, j, k. (\mathsf{repl}''(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}} e_{s,k} \iff j < k)) \wedge \\
&(\forall (a, v) \in V. \forall q. \{j \mid \mathsf{oper}''(e_{q,j}) = \mathtt{wr}(a)\} \cup \\
&\quad \{j \mid \exists s, k. e_{q,j} \xrightarrow{\mathsf{vis}} e_{s,k} \wedge \mathsf{oper}''(e_{s,k}) = \mathtt{wr}(a)\} = \\
&\quad \{j \mid 1 \leq j \leq v(q)\}) \wedge \\
&(\forall e \in E. (\mathsf{oper}''(e) = \mathtt{wr}(a) \wedge \\
&\quad \neg\exists f \in E. \mathsf{oper}''(f) = \mathtt{wr}(\_) \wedge e \xrightarrow{\mathsf{vis}} f) \implies (a, \_) \in V)
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&(\forall (a, v), (a', v') \in V'. (a = a' \implies v = v')) \wedge \\
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&(\forall e \in E'. (\mathsf{oper}''(e) = \mathtt{wr}(a) \wedge \\
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\end{aligned}
$$

The agree property also implies

$$
\begin{aligned}
\forall s, k. 1 \leq k \leq \min\{&\max\{v(s) \mid \exists a. (a, v) \in V\}, \\
&\max\{v(s) \mid \exists a. (a, v) \in V'\}\} \implies e_{s,k} = e'_{s,k}.
\end{aligned}
$$

Hence, there exist distinct

$$e''_{s,k} \text{ for } s \in \mathsf{ReplicaID},\ k = 1..(\max\{v(s) \mid \exists a. (a, v) \in V'''\}),$$

such that

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&(\forall s, k. 1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V\} \implies e''_{s,k} = e_{s,k}) \wedge \\
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\end{aligned}
$$

and

$$
\begin{aligned}
&(\{e \in E \cup E' \mid \exists a. \mathsf{oper}''(e) = \mathtt{wr}(a)\} = \\
&\{e''_{s,k} \mid s \in \mathsf{ReplicaID} \wedge 1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V'''\}\}) \\
&\wedge (\forall s, j, k. (\mathsf{repl}(e''_{s,k}) = s) \wedge (e''_{s,j} \xrightarrow{\mathsf{ro}''} e''_{s,k} \iff j < k)).
\end{aligned}
$$

By the definition of V'' and V''' we have

$$\forall (a, v), (a', v') \in V'''. (a = a' \implies v = v').$$

We also straightforwardly get

$$\forall (a, v) \in V'. \exists s. v(s) > 0$$

and

$$
\begin{aligned}
&(\forall (a, v) \in V''. \forall q. \{j \mid \mathsf{oper}''(e''_{q,j}) = \mathtt{wr}(a)\} \cup \\
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\end{aligned} \tag{14}
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Figure 13. A selection of consistency axioms over an execution $(E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar})$

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&\qquad v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'.\ (a', v') \in V'' \wedge a \neq a'\}\}\rangle,
\end{aligned}
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where $V'' = \{(a, \bigsqcup\{v' \mid (a, v') \in V \cup V'\}) \mid (a, \_) \in V \cup V'\}$

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&\qquad 1 \leq k \leq \max\{v(s) \mid \exists a.\ (a, v) \in V\}\}) \wedge\\
&\quad (\forall s, j, k.\ (\mathsf{repl}(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}} e_{s,k} \iff j < k)) \wedge\\
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\end{aligned}
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the former. The only non-trivial obligation is to show that if

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then

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\{a \mid (a, \_) \in V\} \subseteq \{a \mid \exists e \in E.\ &\mathsf{oper}(e) = \mathtt{wr}(a) \wedge\\
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(the reverse inclusion is straightforwardly implied by $\mathcal{R}_r$).

Take $(a, v) \in V$. We have $\forall (a, v) \in V.\ \exists s.\ v(s) > 0$,

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From this we get that for some $e \in E$

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\end{aligned}
$$

Since vis is acyclic, this implies that for some $e' \in E$

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\end{aligned}
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Assume $\langle r, V\rangle\ [\mathcal{R}_r]\ I$, $V'\ [\mathcal{M}]\ J$ and

$$
\begin{aligned}
I &= ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar}), \mathsf{info});\\
J &= ((E', \mathsf{repl}', \mathsf{obj}', \mathsf{oper}', \mathsf{rval}', \mathsf{ro}', \mathsf{vis}', \mathsf{ar}'), \mathsf{info}');\\
I \sqcup J &= ((E'', \mathsf{repl}'', \mathsf{obj}'', \mathsf{oper}'', \mathsf{rval}'', \mathsf{ro}'', \mathsf{vis}'', \mathsf{ar}''), \mathsf{info}'').
\end{aligned}
$$

By agree we have $I \sqcup J \in \mathsf{IEx}$. Then

$$
\begin{aligned}
&(\forall (a, v), (a', v') \in V.\ (a = a' \implies v = v')) \wedge\\
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&(\forall s, j, k.\ (\mathsf{repl}''(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}} e_{s,k} \iff j < k)) \wedge\\
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The agree property also implies

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$$
\begin{aligned}
&(\{e \in E \cup E' \mid \exists a.\ \mathsf{oper}''(e) = \mathtt{wr}(a)\} =\\
&\{e''_{s,k} \mid s \in \mathsf{ReplicaID} \wedge 1 \leq k \leq \max\{v(s) \mid \exists a.\ (a, v) \in V'''\}\})\\
&\wedge (\forall s, j, k.\ (\mathsf{repl}(e''_{s,k}) = s) \wedge (e''_{s,j} \xrightarrow{\mathsf{ro}''} e''_{s,k} \iff j < k)).
\end{aligned}
$$

By the definition of V'' and V''' we have

$$\forall (a, v), (a', v') \in V'''.\ (a = a' \implies v = v').$$

We also straightforwardly get

$$\forall (a, v) \in V'.\ \exists s.\ v(s) > 0$$

and

$$
\begin{aligned}
&(\forall (a, v) \in V''.\ \forall q.\ \{j \mid \mathsf{oper}''(e''_{q,j}) = \mathtt{wr}(a)\} \cup\\
&\quad \{j \mid \exists s, k.\ e''_{q,j} \xrightarrow{\mathsf{vis}''} e''_{s,k} \wedge \mathsf{oper}''(e''_{s,k}) = \mathtt{wr}(a)\} = \qquad (14)\\
&\quad \{j \mid 1 \leq j \leq v(q)\}).
\end{aligned}
$$


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AN ANTHOLOGY FOR CHILDREN BY WELL-LOVED AUTHORS & ARTISTS


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I'm so excited.

Definition (关系 (Relations))

A *relation* R from A to B is a subset of $A \times B$:

$$R \subseteq A \times B$$

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Theorem (Ordered Pairs)

$$(a, b) = (c, d) \iff a = c \land b = d$$

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$$(a, b) \triangleq \{\{a\}, \{a, b\}\}$$

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Proof.

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Proof.

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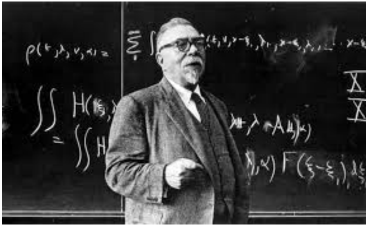
CASE I : $a = b$

CASE II : $a \neq b$

$\square$

Definition (Ordered Pairs (Norbert Wiener; 1914))

$$(a, b) \triangleq \Big\{\{\{a\}, \emptyset\}, \{\{b\}\}\Big\}$$



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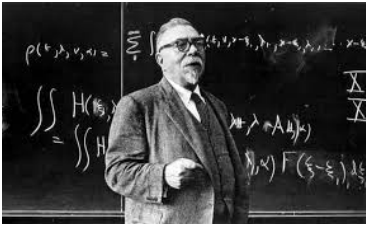

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Definition (n-元组 (n-ary tuples))

$$(x, y, z) \triangleq ((x, y), z)$$

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Theorem

$$(x_1, \ldots, x_n) = (y_1, \ldots, y_n) \iff x_1 = y_1 \land \ldots x_n = y_n$$

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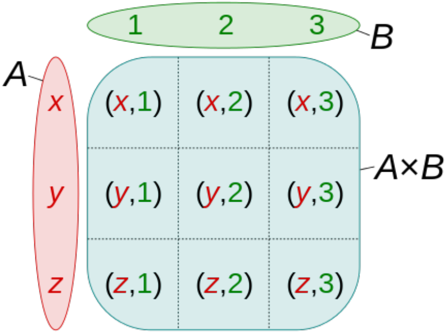
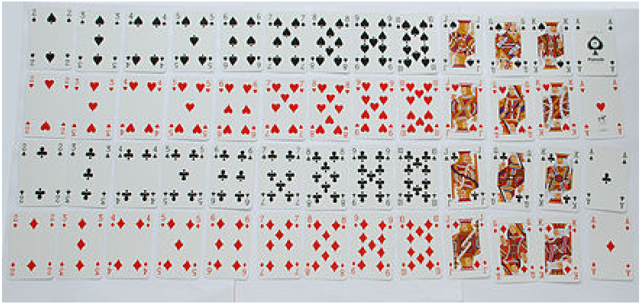
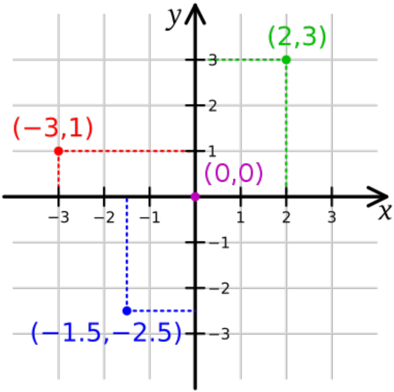


$$\text{Ranks} = \{2, \ldots, 10, J, Q, K, A\}$$



$$\text{Suits} = \{\}$$



$$\mathbb{Z}^2 \triangleq \mathbb{Z} \times \mathbb{Z}$$

$$X \times \emptyset = \emptyset \times X$$

$$X \times \emptyset = \emptyset \times X$$

$$X \times Y \neq Y \times X$$

$$X \times \emptyset = \emptyset \times X$$

$$X \times Y \neq Y \times X$$

$$(X \times Y) \times Z \neq X \times (Y \times Z)$$

$$X \times \emptyset = \emptyset \times X$$

$$X \times Y \neq Y \times X$$

$$(X \times Y) \times Z \neq X \times (Y \times Z)$$

$$A = \{1\} \qquad (A \times A) \times A \neq A \times (A \times A)$$

Theorem (分配律 (Distributivity))

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

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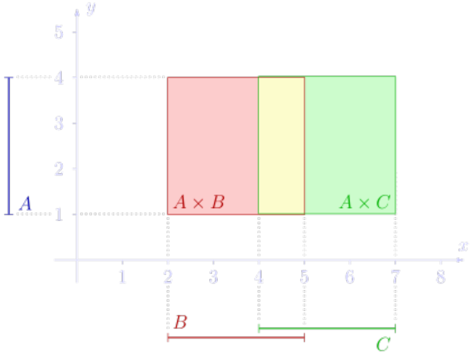

Definition (n-元笛卡尔积 (n-ary Cartesian Product))

$$X_1 \times X_2 \times X_3 \triangleq (X_1 \times X_2) \times X_3$$

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$$X^n \triangleq \underbrace{X \times \cdots \times X}_{n}$$

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Definition (Notations)

$$(a, b) \in R \qquad R(a, b) \qquad aRb$$

Definition (Relations)

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Examples

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- $$D = \{(a, b) \in \mathbb{N} \times \mathbb{N} \mid \exists q \in \mathbb{N} : a \cdot q = b\}$$
- P : the set of people

$$M = \{(a, b) \in P \times P \mid a \text{ is the mother of } b\}$$

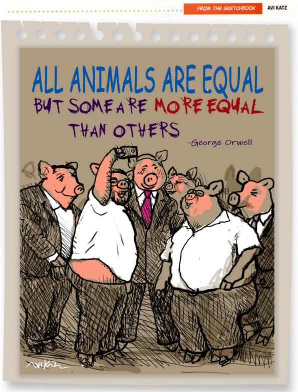
$$B = \{(a, b) \in P \times P \mid a \text{ is the brother of } b\}$$

Important Relations:

Equivalence Relations

Ordering Relations

Functions (next class)



Outline:

3 Definitions

$$\mathrm{dom}(R) \qquad \mathrm{ran}(R) \qquad \mathrm{fld}(R)$$

Definition (定义域 (Domain))

$$\mathrm{dom}(R) = \{a \mid \exists b.\ (a, b) \in R\}$$

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$$\mathrm{dom}(R) = \{a \mid \exists b.\ (a, b) \in R\}$$

Definition (值域 (Range))

$$\mathrm{ran}(R) = \{b \mid \exists a : (a, b) \in R\}$$

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Definition (域 (Field))

$$\mathrm{fld}(R) = \mathrm{dom}(R) \cup \mathrm{ran}(R)$$

$$R = \{(x, y) \mid x = y\} \subseteq \mathbb{R} \times \mathbb{R}$$

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$$\mathrm{dom}(R) = \mathbb{R} \qquad \mathrm{ran}(R) = \mathbb{R} \qquad \mathrm{fld}(R) = \mathbb{R}$$

$$R = \{(x, y) \mid x^2 + y^2 = 1\} \subseteq \mathbb{R} \times \mathbb{R}$$

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$$\mathrm{dom}(R) = [1, 1] \qquad \mathrm{ran}(R) = [-1, 1] \qquad \mathrm{fld}(R) = [-1, 1]$$

Theorem

$$dom(R) \subseteq \bigcup\bigcup R \qquad ran(R) \subseteq \bigcup\bigcup R$$

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对任意 a,

$$a \in \mathrm{dom}(R) \tag{1}$$

$$\tag{6}$$

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\begin{align}
& a \in \mathrm{dom}(R) \tag{1} \\
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\Longrightarrow\ & \exists b.\ \{\{a\}, \{a, b\}\} \in R \tag{3} \\
\Longrightarrow\ & \exists b.\ \{a, b\} \in \bigcup R \tag{4} \\
& \tag{6}
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$$\Longrightarrow \exists b.\ a \in \bigcup\bigcup R \tag{5}$$

$$\tag{6}$$

Theorem

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对任意 a,

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\begin{aligned}
& a \in \mathrm{dom}(R) && (1)\\
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\Longrightarrow\ & \exists b.\ a \in \bigcup\bigcup R && (5)\\
\Longrightarrow\ & a \in \bigcup\bigcup R && (6)
\end{aligned}
$$

5 Operations

R^{-1} $\qquad R|_X$ $\qquad R[X]$ $\qquad R^{-1}[Y]$ $\qquad R \circ S$

Definition (逆 (Inverse))

The *inverse* of R is the relation

$$R^{-1} = \{(a, b) \mid (b, a) \in R\}$$

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$$\leq = \{(x,y) \mid x \leq y\} \subseteq \mathbb{R} \times \mathbb{R} \qquad \leq^{-1} = \ \geq \triangleq \{(x,y) \mid x \geq y\}$$

Theorem

$$(R^{-1})^{-1} = R$$

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对任意 (a, b),

$$(a, b) \in (R^{-1})^{-1} \tag{1}$$

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Theorem

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对任意 (a, b),

$$(a, b) \in (R^{-1})^{-1} \tag{1}$$

$$\iff (b, a) \in R^{-1} \tag{2}$$

$$\tag{3}$$

Theorem

$$(R^{-1})^{-1} = R$$

对任意 (a, b),

$$
\begin{aligned}
& (a, b) \in (R^{-1})^{-1} && (1)\\
\iff & (b, a) \in R^{-1} && (2)\\
\iff & (a, b) \in R && (3)
\end{aligned}
$$

Theorem (关系的逆)

$$R, S \subseteq A \times B$$

$$(R \cup S)^{-1} = R^{-1} \cup S^{-1}$$
$$(R \cap S)^{-1} = R^{-1} \cap S^{-1}$$
$$(R \setminus S)^{-1} = R^{-1} \setminus S^{-1}$$

Definition (左限制 (Left-Restriction))

Suppose $R \subseteq X \times Y$ and $S \subseteq X$. The *left-restriction* relation of R to S is

$$R|_S = \{(x, y) \in R \mid x \in S\}$$

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Definition (右限制 (Right-Restriction))

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Suppose $R \subseteq X \times Y$ and $S \subseteq X$. The *left-restriction* relation of R to S is

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Definition (右限制 (Right-Restriction))

Suppose $R \subseteq X \times Y$ and $S \subseteq Y$. The *right-restriction* relation of R to S is

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Definition (限制 (Restriction))

Suppose $R \subseteq X \times X$ and $S \subseteq X$. The *restriction* relation of R to S is

$$R|_S = \{(x, y) \in R \mid x \in S \wedge y \in S\}$$

example

Definition (像 (Image))

The *image* of X under R is the set

$$R[X] = \{b \in \mathrm{ran}(R) \mid \exists a \in X.\ (a, b) \in R\}$$

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$$R[a] \triangleq R[\{a\}] = \{b \mid (a, b) \in R\}$$

Definition (逆像 (Inverse Image))

The *inverse image* of Y under R is the set

$$R^{-1}[Y] = \{a \in \mathrm{dom}(R) \mid \exists b \in Y : (a, b) \in R\}$$

Definition (逆像 (Inverse Image))

The *inverse image* of Y under R is the set

$$R^{-1}[Y] = \{a \in \mathrm{dom}(R) \mid \exists b \in Y : (a, b) \in R\}$$

$$R^{-1}[b] \triangleq R^{-1}[\{b\}] = \{a \mid (a, b) \in R\}$$

$$R \subseteq A \times B \qquad X \subseteq A \qquad Y \subseteq B$$

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$$R^{-1}[R[X]] \ ? \ X$$

$$R[R^{-1}[Y]] \ ? \ Y$$

$$R \subseteq A \times B \qquad X \subseteq A \qquad Y \subseteq B$$

$$R^{-1}[R[X]] \; {\color{red}?} \; X$$

$$R[R^{-1}[Y]] \; {\color{red}?} \; Y$$

Theorem

$$R[X_1 \cup X_2] = R[X_1] \cup R[X_2]$$

$$R[X_1 \cap X_2] \subseteq R[X_1] \cap R[X_2]$$

$$R[X_1 \setminus X_2] \supseteq R[X_1] \setminus R[X_2]$$

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对任意 b,

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对任意 b,

$$b \in R[X_1 \cup X_2]$$

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对任意 b,

$$b \in R[X_1 \cup X_2]$$

$$\iff \exists a \in X_1 \cup X_2 : (a, b) \in R$$

Theorem

$$R[X_1 \cup X_2] = R[X_1] \cup R[X_2]$$

$$R[X_1 \cap X_2] \subseteq R[X_1] \cap R[X_2]$$

$$R[X_1 \setminus X_2] \supseteq R[X_1] \setminus R[X_2]$$

对任意 b,

$$
\begin{aligned}
& b \in R[X_1 \cup X_2] \\
\iff \; & \exists a \in X_1 \cup X_2 : (a, b) \in R \\
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Definition (复合 (Composition; $R \circ S$, $R; S$))

The *composition* of relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ is the relation

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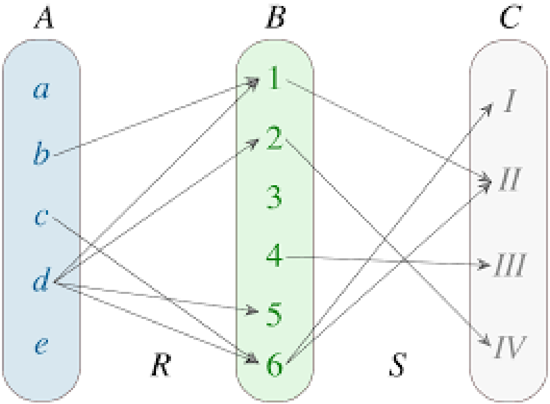
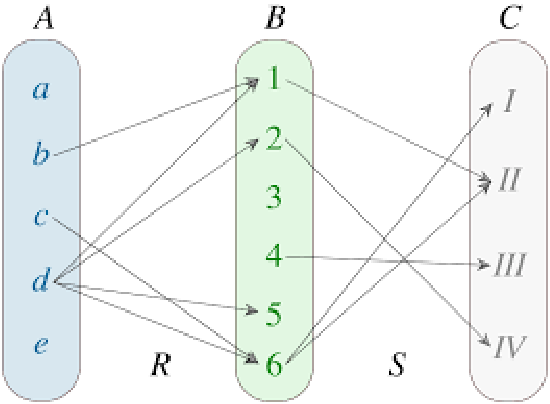


$$|R \circ S| =$$



$$|R \circ S| = 7$$

$$\leq \circ \leq \; =$$

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$$\leq \circ \leq \; = \; \leq$$

$$\leq \circ \geq \; =$$

$$\leq \circ \leq \; = \; \leq$$

$$\leq \circ \geq \; = \; \mathbb{R} \times \mathbb{R}$$

Theorem

$$(R \circ S)^{-1} = S^{-1} \circ R^{-1}$$

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对任意 (a, b),

$$(a, b) \in (R \circ S)^{-1} \tag{1}$$

$$\tag{5}$$

Theorem

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对任意 (a, b),

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$$\iff (b, a) \in R \circ S \tag{2}$$

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$$(a, b) \in (R \circ S)^{-1} \tag{1}$$

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$$\tag{5}$$

Theorem

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对任意 (a, b),

$$
\begin{aligned}
& (a, b) \in (R \circ S)^{-1} && (1) \\
\iff & (b, a) \in R \circ S && (2) \\
\iff & \exists c.\ (b, c) \in R \land (c, a) \in S && (3) \\
\iff & \exists c.\ (c, b) \in R^{-1} \land (a, c) \in S^{-1} && (4) \\
\iff & (a, c) \in S^{-1} \circ R^{-1} && (5)
\end{aligned}
$$

Theorem

$$(R \circ S) \circ T = R \circ (S \circ T)$$

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对任意 (a, b),

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Theorem

$$(R \circ S) \circ T = R \circ (S \circ T)$$

对任意 (a, b),

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对任意 (a, b),

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Theorem

$$(R \circ S) \circ T = R \circ (S \circ T)$$

对任意 (a, b),

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$$\iff \exists d. \Big((a, d) \in S \circ T \land (d, b) \in R\Big) \tag{6}$$

Theorem

$$(R \circ S) \circ T = R \circ (S \circ T)$$

对任意 (a, b),

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燕小六: “帮我照顾好我七舅姥爷和我外甥女”

“舅姥爷”：姥姥/外婆的兄弟

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“舅姥爷”：妈妈的舅舅

Theorem (关系的复合)

$$(X \cup Y) \circ Z = (X \circ Z) \cup (Y \circ Z)$$

$$(X \cap Y) \circ Z \subseteq (X \circ Z) \cap (Y \circ Z)$$

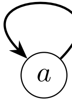
7 Properties

$$R \subseteq X \times X$$

Definition (自反的 (Reflexive))

$R \subseteq X \times X$ is *reflexive* if

$$\forall a \in X : (a, a) \in R$$



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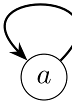

$$\leq \subseteq \mathbb{R} \times \mathbb{R} \text{ is reflexive}$$

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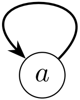

$$\leq \; \subseteq \mathbb{R} \times \mathbb{R} \text{ is reflexive}$$

三角形上的全等关系是自反的

Definition (反自反 (Irreflexive))

$R \subseteq X \times X$ is *irreflexive* if

$$\forall a \in X. \ (a, a) \notin R$$

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$R \subseteq X \times X$ is *irreflexive* if

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$R \subseteq X \times X$ is *irreflexive* if

$$\forall a \in X.\ (a, a) \notin R$$

$$< \subseteq \mathbb{R} \times \mathbb{R} \text{ is irreflexive}$$

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$$A = \{1, 2, 3\} \qquad R \subseteq A \times A$$

$$\{(1,1), (1,3), (2,1), (2,2), (3,3)\}$$

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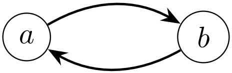
Definition (对称 (Symmetric))

$R \subseteq X \times X$ is *symmetric* if

$$\forall a, b \in X.\ aRb \to bRa$$



Definition (对称 (Symmetric))

$R \subseteq X \times X$ is *symmetric* if

$$\forall a, b \in X.\ aRb \to bRa$$

$$\begin{tikzcd} a \arrow[r, bend left] & b \arrow[l, bend left] \end{tikzcd}$$

$$\forall a, b \in X.\ aRb \leftrightarrow bRa$$

$$A = \{1, 2, 3\} \qquad R \subseteq A \times A$$

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$$\{(1,1),(2,2),(3,3)\}$$

$$\{(1,2),(2,1),(2,3)\}$$

Definition (反对称 (AntiSymmetric))

$R \subseteq X \times X$ is *antisymmetric* if

$$\forall a, b \in X.\ (aRb \land bRa) \to a = b$$

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$R \subseteq X \times X$ is *antisymmetric* if

$$\forall a, b \in X.\ (aRb \land bRa) \to a = b$$

$\geq$ *is* antisymmetric

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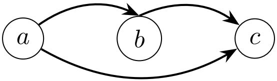
Definition (传递的 (Transitive))

$R \subseteq X \times X$ is *transitive* if

$$\forall a, b, c \in X. \ (aRb \land bRc \to aRc)$$



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Definition (三分的 (Trichotomous))

$R \subseteq X \times X$ is *trichotomous* if

$$\forall a, b \in X. \ (\text{exactly one of } aRb, bRa, \text{ or } a = b \text{ holds})$$

Theorem

$$R \textit{ is reflexive} \iff I \subseteq R$$

$$I = \{(a, a) \in A \times A \mid a \in A\}$$

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Theorem

$$R \text{ is symmetric and transitive} \iff R = R^{-1} \circ R$$

Equivalence Relations

Definition (Equivalence Relation)

$R \subseteq X \times X$ is an *equivalence relation* on X iff R is

- reflexive: $\forall a \in X.\ aRa$
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Why are equivalence relations important?

Equivalence Relations as Abstractions

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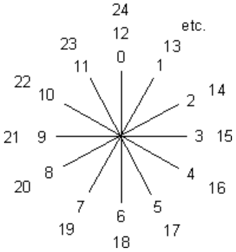

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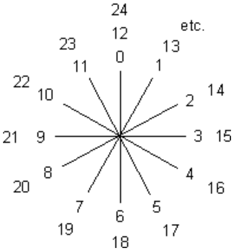

"全国人民代表大会各省代表团"

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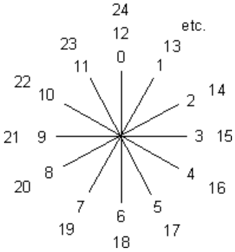
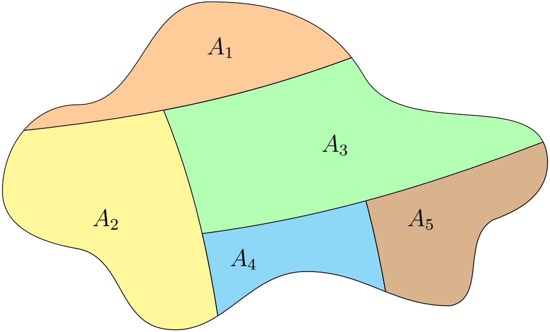


“全国人民代表大会各省代表团”

Equivalence Relation $\iff$ Partition

Partition



"不空、不漏、不重"

Definition (划分 (Partition))

A family of sets $\Pi = \{A_\alpha \mid \alpha \in I\}$ is a *partition* of X if

(i) (不空)

$$\forall \alpha \in I.\ A_\alpha \neq \emptyset$$

(ii) (不漏)

$$\bigcup_{\alpha \in I} A_\alpha = X$$

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$$\forall \alpha, \beta \in I.\ A_\alpha \cap A_\beta = \emptyset \lor A_\alpha = A_\beta$$

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Partition Π of $X \implies$ Equivalence Relation $R \subseteq X \times X$

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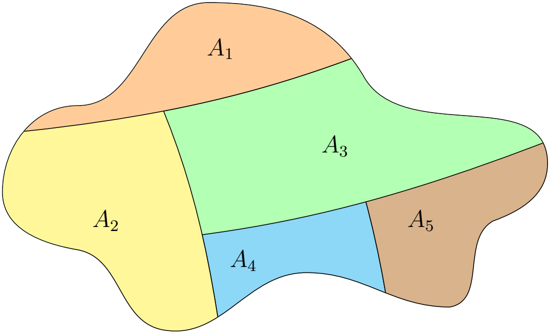

Equivalence Relation $\iff$ Partition

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Q : What is $\mathbb{N} \times \mathbb{N} / \sim$?

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Definition ($\mathbb{Z}$)

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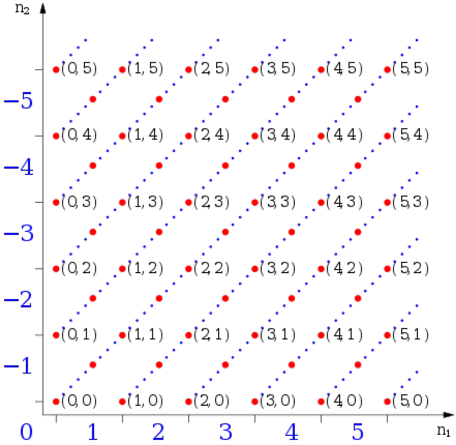
Definition ($\mathbb{Z}$)

$$\mathbb{Z} \triangleq \mathbb{N} \times \mathbb{N} / \sim$$

$$[(1,3)]_{\sim} = \{(0,2), (1,3), (2,4), (3,5), \cdots\} \triangleq -2 \in \mathbb{Z}$$



$$\mathbb{Z} \triangleq \mathbb{N} \times \mathbb{N} / \sim$$

Definition ($+_{\mathbb{Z}}$)

$$[(m_1, n_1)] +_{\mathbb{Z}} [(m_2, n_2)] = [m_1 +_{\mathbb{N}} m_2, n_1 +_{\mathbb{N}} n_2]$$

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Definition

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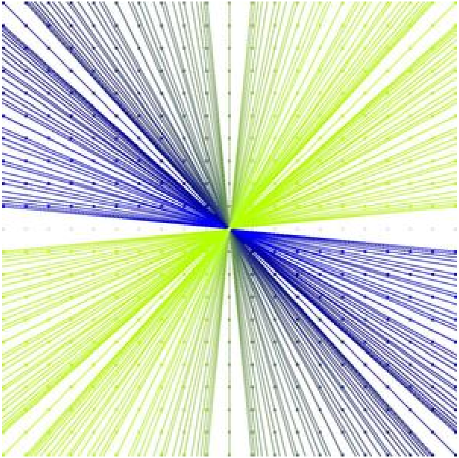

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如何用有理数定义实数?

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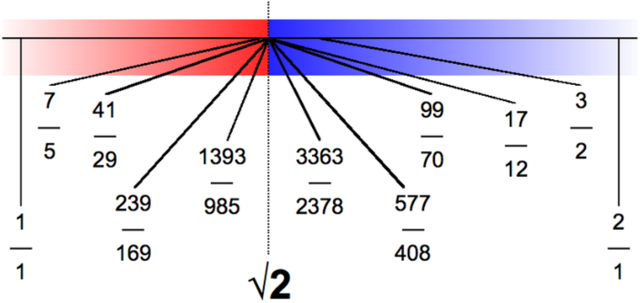
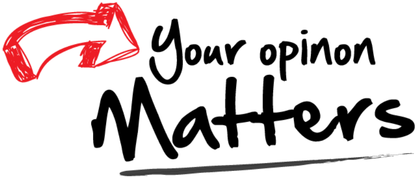


Dedekind Cut (戴德金分割)


Thank You!



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