(五) 集合: 关系 (Relation)

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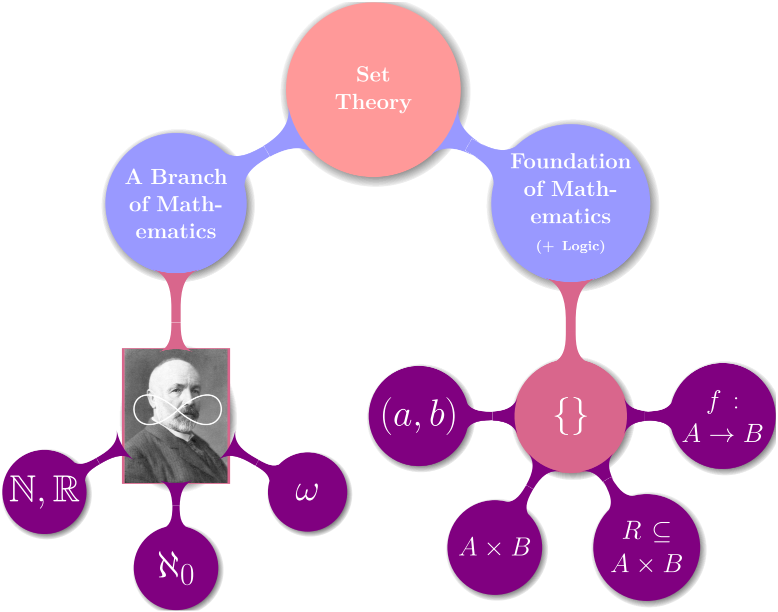

Set Theory
A Branch of Mathematics
Foundation of Mathematics
(+ Logic)
$\mathbb{N}, \mathbb{R}$
$\aleph_0$
ω
(a, b)
$\{\}$
$f : A \to B$
$A \times B$
$R \subseteq A \times B$

我的工作日常 . . .

Figure 13. A selection of consistency axioms over an execution $(E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar})$

Auxiliary relations

$\mathsf{sameobj}(e, f) \iff \mathsf{obj}(e) = \mathsf{obj}(f)$

Per-object causality (aka happens-before) order:

$\quad \mathsf{hbo} = ((\mathsf{ro} \cap \mathsf{sameobj}) \cup \mathsf{vis})^+$

Causality (aka happens-before) order: $\mathsf{hb} = (\mathsf{ro} \cup \mathsf{vis})^+$

Axioms

EVENTUAL:

$\quad \forall e \in E.\, \neg(\exists \text{ infinitely many } f \in E.\, \mathsf{sameobj}(e, f) \wedge \neg(e \xrightarrow{\mathsf{vis}} f))$

THINAIR: $\mathsf{ro} \cup \mathsf{vis}$ is acyclic

POCV (Per-Object Causal Visibility): $\mathsf{hbo} \subseteq \mathsf{vis}$

POCA (Per-Object Causal Arbitration): $\mathsf{hbo} \subseteq \mathsf{ar}$

COCV (Cross-Object Causal Visibility): $(\mathsf{hb} \cap \mathsf{sameobj}) \subseteq \mathsf{vis}$

COCA (Cross-Object Causal Arbitration): $\mathsf{hb} \cup \mathsf{ar}$ is acyclic

Figure 17. Optimized state-based multi-value register and its simulation

$$
\begin{aligned}
\Sigma &= \mathsf{ReplicaID} \times \mathcal{P}(\mathbb{Z} \times (\mathsf{ReplicaID} \to \mathbb{N}_0)) \\
\vec{\sigma}_0 &= \langle r, \emptyset \rangle \\
M &= \mathcal{P}(\mathbb{Z} \times (\mathsf{ReplicaID} \to \mathbb{N}_0)) \\
\mathsf{do}(\mathtt{wr}(a), \langle r, V\rangle, t) &= \\
&\quad (\langle r, \{(a, (\lambda s.\, \text{if } s \neq r \text{ then } \max\{v(s) \mid (\_, v) \in V\} \\
&\qquad \text{else } \max\{v(s) \mid (\_, v) \in V\} + 1))\}\rangle, \bot) \\
\mathsf{do}(\mathtt{rd}, \langle r, V\rangle, t) &= (\langle r, V\rangle, \{a \mid (a, \_) \in V\}) \\
\mathsf{send}(\langle r, V\rangle) &= (\langle r, V\rangle, V) \\
\mathsf{receive}(\langle r, V\rangle, V') &= \langle r, \{(a, v) \in V'' \mid \\
&\qquad v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'.\, (a', v') \in V'' \wedge a \neq a'\}\}\rangle, \\
\text{where } V'' &= \{(a, \bigsqcup\{v' \mid (a, v') \in V \cup V'\}) \mid (a, \_) \in V \cup V'\}
\end{aligned}
$$

$\langle s, V\rangle\ [\mathcal{R}_r]\ I \iff (r = s) \wedge (V\ [\mathcal{M}]\ I)$

$$
\begin{aligned}
&V\ [\mathcal{M}]\ ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar}), \mathsf{info}) \iff \\
&\quad (\forall (a, v), (a', v') \in V.\, (a = a' \implies v = v')) \wedge \\
&\quad (\forall (a, v) \in V.\, \exists s.\, v(s) > 0) \wedge \\
&\quad (\forall (a, v) \in V.\, v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'.\, (a', v') \in V \wedge a \neq a'\}) \wedge \\
&\quad \exists \text{ distinct } e_{s,k}. \\
&\quad (\{e \in E \mid \exists a.\, \mathsf{oper}(e) = \mathtt{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicaID} \wedge \\
&\qquad 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V\}\}) \wedge \\
&\quad (\forall s, j, k.\, (\mathsf{repl}(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}} e_{s,k} \iff j < k)) \wedge \\
&\quad (\forall (a, v) \in V.\, \forall q.\, \{j \mid \mathsf{oper}(e_{q,j}) = \mathtt{wr}(a)\} \cup \\
&\qquad \{j \mid \exists s, k.\, e_{q,j} \xrightarrow{\mathsf{vis}} e_{s,k} \wedge \mathsf{oper}(e_{s,k}) = \mathtt{wr}(a)\} = \\
&\qquad \{j \mid 1 \leq j \leq v(q)\}) \wedge \\
&\quad (\forall e \in E.\, (\mathsf{oper}(e) = \mathtt{wr}(a) \wedge \\
&\qquad \neg\exists f \in E.\, \mathsf{oper}(f) = \mathtt{wr}(\_) \wedge e \xrightarrow{\mathsf{vis}} f) \implies (a, \_) \in V)
\end{aligned}
$$

the former. The only non-trivial obligation is to show that if

$$V\ [\mathcal{M}]\ ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}), \mathsf{info}),$$

then

$$
\begin{aligned}
\{a \mid (a, \_) \in V\} \subseteq \{a \mid \exists e \in E.\, \mathsf{oper}(e) = \mathtt{wr}(a) \wedge \\
\neg\exists f \in E.\, \exists a'.\, \mathsf{oper}(e) = \mathtt{wr}(a') \wedge e \xrightarrow{\mathsf{vis}} f\}
\end{aligned}
\tag{13}
$$

(the reverse inclusion is straightforwardly implied by $\mathcal{R}_r$).

Take $(a, v) \in V$. We have $\forall (a, v) \in V.\, \exists s.\, v(s) > 0$,

$$v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'.\, (a', v') \in V \wedge a \neq a'\}$$

and

$$
\begin{aligned}
&\forall (a, v) \in V.\, \forall q.\, \{j \mid \mathsf{oper}(e_{q,j}) = \mathtt{wr}(a)\} \cup \\
&\quad \{j \mid \exists s, k.\, e_{q,j} \xrightarrow{\mathsf{vis}} e_{s,k} \wedge \mathsf{oper}(e_{s,k}) = \mathtt{wr}(a)\} = \\
&\quad \{j \mid 1 \leq j \leq v(q)\}.
\end{aligned}
$$

From this we get that for some $e \in E$

$$
\begin{aligned}
\mathsf{oper}(e) = \mathtt{wr}(a) \wedge \neg\exists f \in E.\, \exists a'.\, a' \neq a \wedge \\
\mathsf{oper}(e) = \mathtt{wr}(a') \wedge e \xrightarrow{\mathsf{vis}} f.
\end{aligned}
$$

Since vis is acyclic, this implies that for some $e' \in E$

$$\mathsf{oper}(e') = \mathtt{wr}(a) \wedge \neg\exists f \in E.\, \mathsf{oper}(e') = \mathtt{wr}(\_) \wedge e' \xrightarrow{\mathsf{vis}} f,$$

which establishes (13).

Let us now discharge RECEIVE. Let $\mathsf{receive}(\langle r, V\rangle, V') = \langle r, V'''\rangle$, where

$$
\begin{aligned}
V'' &= \{(a, \bigsqcup\{v' \mid (a, v') \in V \cup V'\}) \mid (a, \_) \in V \cup V'\}; \\
V''' &= \{(a, v) \in V'' \mid v \not\sqsubseteq \bigsqcup\{(a', v') \in V'' \mid a \neq a'\}\}.
\end{aligned}
$$

Assume $\langle r, V\rangle\ [\mathcal{R}_r]\ I$, $V'\ [\mathcal{M}]\ J$ and

$$
\begin{aligned}
I &= ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar}), \mathsf{info}); \\
J &= ((E', \mathsf{repl}', \mathsf{obj}', \mathsf{oper}', \mathsf{rval}', \mathsf{ro}', \mathsf{vis}', \mathsf{ar}'), \mathsf{info}'); \\
I \sqcup J &= ((E'', \mathsf{repl}'', \mathsf{obj}'', \mathsf{oper}'', \mathsf{rval}'', \mathsf{ro}'', \mathsf{vis}'', \mathsf{ar}''), \mathsf{info}'').
\end{aligned}
$$

By agree we have $I \sqcup J \in \mathsf{IEx}$. Then

$$
\begin{aligned}
&(\forall (a, v), (a', v') \in V.\, (a = a' \implies v = v')) \wedge \\
&(\forall (a, v) \in V.\, \exists s.\, v(s) > 0) \wedge \\
&(\forall (a, v) \in V.\, v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'.\, (a', v') \in V \wedge a \neq a'\}) \wedge \\
&\exists \text{ distinct } e_{s,k}. \\
&(\{e \in E \mid \exists a.\, \mathsf{oper}''(e) = \mathtt{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicaID} \wedge \\
&\quad 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V\}\}) \wedge \\
&(\forall s, j, k.\, (\mathsf{repl}''(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}} e_{s,k} \iff j < k)) \wedge \\
&(\forall (a, v) \in V.\, \forall q.\, \{j \mid \mathsf{oper}''(e_{q,j}) = \mathtt{wr}(a)\} \cup \\
&\quad \{j \mid \exists s, k.\, e_{q,j} \xrightarrow{\mathsf{vis}} e_{s,k} \wedge \mathsf{oper}''(e_{s,k}) = \mathtt{wr}(a)\} = \\
&\quad \{j \mid 1 \leq j \leq v(q)\}) \wedge \\
&(\forall e \in E.\, (\mathsf{oper}''(e) = \mathtt{wr}(a) \wedge \\
&\quad \neg\exists f \in E.\, \mathsf{oper}''(f) = \mathtt{wr}(\_) \wedge e \xrightarrow{\mathsf{vis}} f) \implies (a, \_) \in V)
\end{aligned}
$$

and

$$
\begin{aligned}
&(\forall (a, v), (a', v') \in V'.\, (a = a' \implies v = v')) \wedge \\
&(\forall (a, v) \in V'.\, \exists s.\, v(s) > 0) \wedge \\
&(\forall (a, v) \in V'.\, v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'.\, (a', v') \in V' \wedge a \neq a'\}) \wedge \\
&\exists \text{ distinct } e_{s,k}. \\
&(\{e \in E' \mid \exists a.\, \mathsf{oper}''(e) = \mathtt{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicaID} \wedge \\
&\quad 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V'\}\}) \wedge \\
&(\forall s, j, k.\, (\mathsf{repl}''(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}'} e_{s,k} \iff j < k)) \wedge \\
&(\forall (a, v) \in V'.\, \forall q.\, \{j \mid \mathsf{oper}''(e_{q,j}) = \mathtt{wr}(a)\} \cup \\
&\quad \{j \mid \exists s, k.\, e_{q,j} \xrightarrow{\mathsf{vis}'} e_{s,k} \wedge \mathsf{oper}''(e_{s,k}) = \mathtt{wr}(a)\} = \\
&\quad \{j \mid 1 \leq j \leq v(q)\}) \wedge \\
&(\forall e \in E'.\, (\mathsf{oper}''(e) = \mathtt{wr}(a) \wedge \\
&\quad \neg\exists f \in E'.\, \mathsf{oper}''(f) = \mathtt{wr}(\_) \wedge e \xrightarrow{\mathsf{vis}'} f) \implies (a, \_) \in V').
\end{aligned}
$$

The agree property also implies

$$
\begin{aligned}
\forall s, k.\, 1 \leq k \leq \min\{&\max\{v(s) \mid \exists a.\, (a, v) \in V\}, \\
&\max\{v(s) \mid \exists a.\, (a, v) \in V'\}\} \implies e_{s,k} = e'_{s,k}.
\end{aligned}
$$

Hence, there exist distinct

$$e''_{s,k} \text{ for } s \in \mathsf{ReplicaID},\ k = 1..(\max\{v(s) \mid \exists a.\, (a, v) \in V'''\}),$$

such that

$$
\begin{aligned}
&(\forall s, k.\, 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V\} \implies e''_{s,k} = e_{s,k}) \wedge \\
&(\forall s, k.\, 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V'\} \implies e''_{s,k} = e'_{s,k})
\end{aligned}
$$

and

$$
\begin{aligned}
&(\{e \in E \cup E' \mid \exists a.\, \mathsf{oper}''(e) = \mathtt{wr}(a)\} = \\
&\{e''_{s,k} \mid s \in \mathsf{ReplicaID} \wedge 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V'''\}\}) \\
&\wedge (\forall s, j, k.\, (\mathsf{repl}(e''_{s,k}) = s) \wedge (e''_{s,j} \xrightarrow{\mathsf{ro}''} e''_{s,k} \iff j < k)).
\end{aligned}
$$

By the definition of V'' and V''' we have

$$\forall (a, v), (a', v') \in V'''.\, (a = a' \implies v = v').$$

We also straightforwardly get

$$\forall (a, v) \in V'.\, \exists s.\, v(s) > 0$$

and

$$
\begin{aligned}
&(\forall (a, v) \in V''.\, \forall q.\, \{j \mid \mathsf{oper}''(e''_{q,j}) = \mathtt{wr}(a)\} \cup \\
&\quad \{j \mid \exists s, k.\, e''_{q,j} \xrightarrow{\mathsf{vis}''} e''_{s,k} \wedge \mathsf{oper}''(e''_{s,k}) = \mathtt{wr}(a)\} = \\
&\quad \{j \mid 1 \leq j \leq v(q)\}).
\end{aligned}
\tag{14}
$$

离散数学学得好不好，

一个重要的衡量标准就是是否完成了这种转变

The Relational Data Model

— 如何靠“关系”赢得图灵奖?

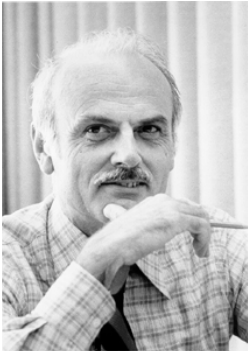
A Relational Model of Data for Large Shared Data Banks

E. F. Codd
IBM Research Laboratory, San Jose, California

Future users of large data banks must be protected from having to know how the data is organized in the machine (the internal representation). A prompting service which supplies such information is not a satisfactory solution. Activities of users at terminals and most application programs should remain unaffected when the internal representation of data is changed and even when some aspects of the external representation are changed. Changes in data representation will often be needed as a result of changes in query, update, and report traffic and natural growth in the types of stored information.

Existing noninferential, formatted data systems provide users with tree-structured files or slightly more general network models of the data. In Section 1, inadequacies of these models are discussed. A model based on *n*-ary relations, a normal form for data base relations, and the concept of a universal data sublanguage are introduced. In Section 2, certain operations on relations (other than logical inference) are discussed and applied to the problems of redundancy and consistency in the user's model.

Codd@CACM’1970
(Turing Award’1981)



Edgar F. Codd (1923 – 2003)

$\mathbb{R}$: 实数集

"Near" 关系: $|a-b|<1$

$$R = \{(a,b) \mid |a-b| < 1\}$$

$$(0, 0.618) \in R \qquad (-0.618, 0.618) \notin R$$

$$\forall a \in X.\ (a,a) \in R \qquad (\text{自反性})$$

$$\forall a,b \in X.\ ((a,b) \in R \to (b,a) \in R) \qquad (\text{对称性})$$

$$\forall a,b,c \in X.\ ((a,b) \in R \land (b,c) \in R \to (a,c) \in R) \qquad (\text{传递性})$$

$\boxed{\text{自反性} + \text{对称性} = \textbf{相容关系}}$

$$X = \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$$

X 上的整除关系

$$R = \{(1, 2), \ldots, (4, 12), \ldots, (12, 60), \ldots, (4, 60), \ldots, (60, 60)\}$$

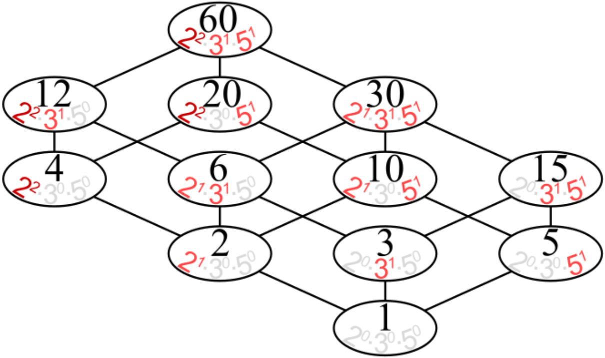
$\forall a \in X.\ (a, a) \in R$ （自反性）

$\forall a, b \in X.\ ((a, b) \in R \land (b, a) \in R \to a = b)$ （反对称性）

$\forall a, b, c \in X.\ ((a, b) \in R \land (b, c) \in R \to (a, c) \in R)$ （传递性）

自反性 + 反对称性 + 传递性 = **偏序关系**

$$X = \{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$$



“偏序” 严格刻画了人类对于“序”的直观理解

$$\mathbb{N}: \text{自然数集}$$

$$\leq = \{(a,b) \mid a \leq b\}$$

$$\forall a \in X.\ (a,a) \in R \qquad \text{(自反性)}$$

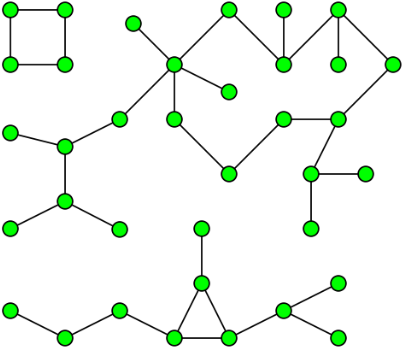
$$\forall a,b \in X.\ ((a,b) \in R \land (b,a) \in R \to a = b) \qquad \text{(反对称性)}$$

$$\forall a,b,c \in X.\ ((a,b) \in R \land (b,c) \in R \to (a,c) \in R) \qquad \text{(传递性)}$$

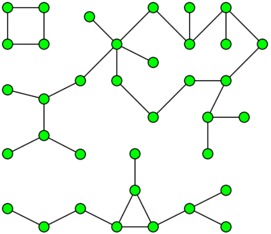
$$\forall a,b \in X.\ ((a,b) \in R \lor (b,a) \in R) \qquad \text{(连接性)}$$

$$\boxed{\text{自反性} + \text{反对称性} + \text{传递性} + \text{连接性} = \textbf{全序关系}}$$

考虑无向图中的顶点集合



顶点间的 “可达 (Reachability) 关系”: $R = \{(a, b) \mid a \rightsquigarrow b\}$



$$\forall a \in X.\ (a, a) \in R \qquad \text{(自反性)}$$

$$\forall a, b \in X.\ ((a, b) \in R \to (b, a) \in R) \qquad \text{(对称性)}$$

$$\forall a, b, c \in X.\ ((a, b) \in R \land (b, c) \in R \to (a, c) \in R) \qquad \text{(传递性)}$$

$$\boxed{\text{自反性} + \text{对称性} + \text{传递性} = \textbf{等价关系}}$$

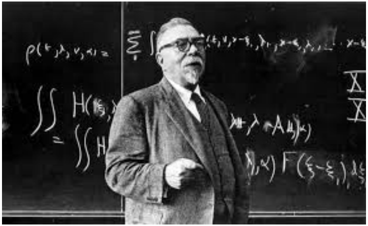
“可达关系”将顶点划分成相互独立的“连通分量”

Definition (有序对公理 (Ordered Pairs))

$$(a, b) = (c, d) \iff a = c \land b = d$$

Definition (Ordered Pairs (Norbert Wiener; 1914))

$$(a, b) \triangleq \Big\{\{\{a\}, \emptyset\}, \{\{b\}\}\Big\}$$



Theorem

$$(a, b) = (c, d) \iff a = c \land b = d$$

Definition (Ordered Pairs (Kazimierz Kuratowski; 1921))

$$(a, b) \triangleq \{\{a\}, \{a, b\}\}$$

Theorem

$$(a, b) = (c, d) \iff a = c \land b = d$$

Theorem

$$(a, b) = (c, d) \iff a = c \land b = d$$

$$\Big(\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}\Big) \iff (a = c \land b = d)$$

$$\begin{aligned}
& \{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\} \\
\iff & (\{a\} = \{c\} \lor \{a\} = \{c, d\}) \land (\{a, b\} = \{c\} \lor \{a, b\} = \{c, d\}) \\
\iff & (\{a\} = \{c\} \land \{a, b\} = \{c\}) \lor \\
& (\{a\} = \{c\} \land \{a, b\} = \{c, d\}) \lor \\
& (\{a\} = \{c, d\} \land \{a, b\} = \{c\}) \lor \\
& (\{a\} = \{c, d\} \land \{a, b\} = \{c, d\})
\end{aligned}$$

Definition (n-元组 (n-ary tuples))

$$(x, y, z) \triangleq ((x, y), z)$$

$$(x_1, x_2, \ldots, x_{n-1}, x_n) \triangleq ((x_1, x_2, \ldots, x_{n-1}), x_n)$$

Theorem

$$(x_1, \ldots, x_n) = (y_1, \ldots, y_n) \iff x_1 = y_1 \land \ldots x_n = y_n$$

By mathematical induction.

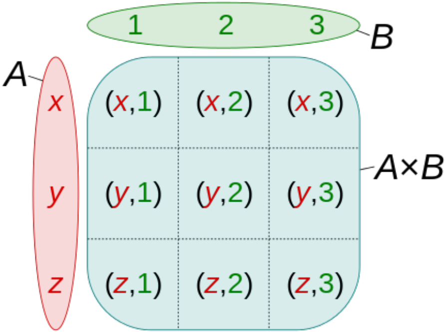
多数情况下, 我们仅处理 “二元关系”, 因此也仅使用 “有序对”

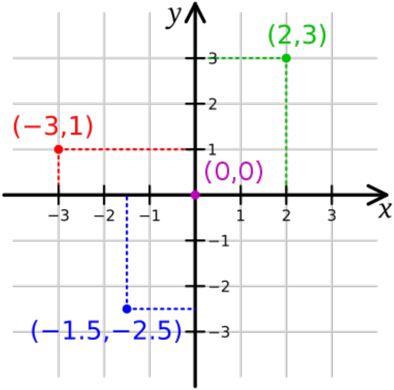
Definition (笛卡尔积 (Cartesian Products))

The *Cartesian product* $A \times B$ of A and B is defined as

$$A \times B \triangleq \{(a, b) \mid a \in A \land b \in B\}$$

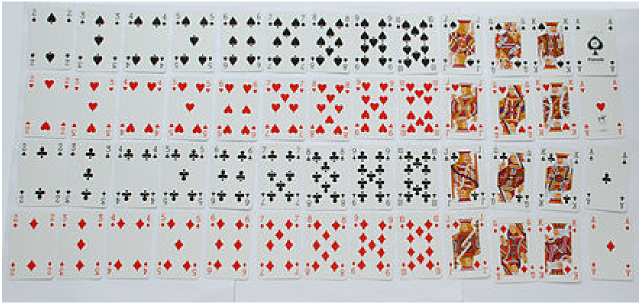
$$X^2 \triangleq X \times X$$





$$\mathbb{Z}^2 \triangleq \mathbb{Z} \times \mathbb{Z}$$

$$\text{Ranks} = \{2, \ldots, 10, J, Q, K, A\}$$



$$\text{Suits} = \{\spadesuit, \heartsuit, \clubsuit, \diamondsuit\}$$

$$X \times \emptyset = \emptyset \times X$$

$$X \times Y \neq Y \times X$$

$$(X \times Y) \times Z \neq X \times (Y \times Z)$$

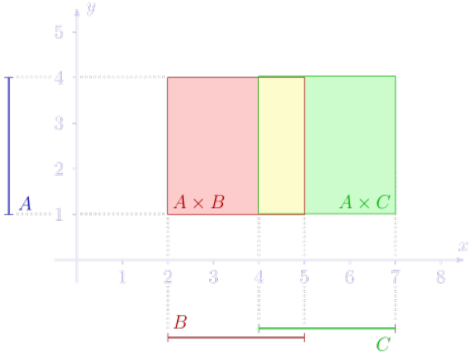
$$A = \{1\} \qquad (A \times A) \times A \neq A \times (A \times A)$$

Theorem (分配律 (Distributivity))

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

$$A \times (B \setminus C) = (A \times B) \setminus (A \times C)$$



$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

对任意有序对 (a, b),

$$
\begin{aligned}
& (a, b) \in A \times (B \cap C) && (1) \\
\iff & a \in A \land b \in (B \cap C) && (2) \\
\iff & a \in A \land b \in B \land b \in C && (3) \\
\iff & (a \in A \land b \in B) \land (a \in A \land b \in C) && (4) \\
\iff & (a, b) \in A \times B \land (a, b) \in A \times C && (5) \\
\iff & (a, b) \in (A \times B) \cap (A \times C) && (6)
\end{aligned}
$$

Definition (n-元笛卡尔积 (n-ary Cartesian Product))

$$X_1 \times X_2 \times X_3 \triangleq (X_1 \times X_2) \times X_3$$

$$X_1 \times X_2 \times \cdots \times X_n \triangleq (X_1 \times X_2 \times \cdots \times X_{n-1}) \times X_n$$

$$X^n \triangleq \underbrace{X \times \cdots \times X}_{n}$$

多数情况下, 我们仅处理 “二元关系”, 因此也仅使用 “二元笛卡尔积”

Definition (关系 (Relations))

A *relation* R from A to B is a subset of $A \times B$:

$$R \subseteq A \times B$$

If $A = B$, R is called a relation on A.

Definition (Notations)

$$(a, b) \in R \qquad aRb$$
$$(a, b) \notin R \qquad a\overline{R}b$$

Definition (Relations)

A *relation* R from A to B is a subset of $A \times B$:

$$R \subseteq A \times B$$

Examples

- Both $A \times B$ and $\emptyset$ are relations from A to B.
- $$< \;= \{(a, b) \in \mathbb{R} \times \mathbb{R} \mid a \text{ is less than } b\}$$
- $$D = \{(a, b) \in \mathbb{N} \times \mathbb{N} \mid \exists q \in \mathbb{N}.\ a \cdot q = b\}$$
- P : the set of people

$$M = \{(a, b) \in P \times P \mid a \text{ is the mother of } b\}$$

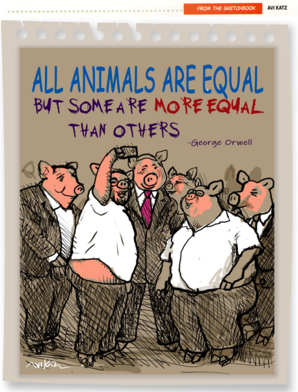
$$B = \{(a, b) \in P \times P \mid a \text{ is the brother of } b\}$$

Important Relations:

Equivalence Relations

Ordering Relations (next class)

Functions (next class)



Outline:

3 Definitions

$$\mathrm{dom}(R) \qquad \mathrm{ran}(R) \qquad \mathrm{fld}(R)$$

Definition (定义域 (Domain))

$$\mathrm{dom}(R) = \{a \mid \exists b.\ (a, b) \in R\}$$

Definition (值域 (Range))

$$\mathrm{ran}(R) = \{b \mid \exists a.\ (a, b) \in R\}$$

Definition (域 (Field))

$$\mathrm{fld}(R) = \mathrm{dom}(R) \cup \mathrm{ran}(R)$$

$$R = \{(x, y) \mid x = y\} \subseteq \mathbb{R} \times \mathbb{R}$$

$$\mathrm{dom}(R) = \mathbb{R} \qquad \mathrm{ran}(R) = \mathbb{R} \qquad \mathrm{fld}(R) = \mathbb{R}$$

$$R = \{(x, y) \mid x^2 + y^2 = 1\} \subseteq \mathbb{R} \times \mathbb{R}$$

$$\mathrm{dom}(R) = [1, 1] \qquad \mathrm{ran}(R) = [-1, 1] \qquad \mathrm{fld}(R) = [-1, 1]$$

Theorem

$$dom(R) \subseteq \bigcup\bigcup R \qquad ran(R) \subseteq \bigcup\bigcup R$$

对任意 a,

$$\begin{aligned}
& a \in \mathrm{dom}(R) && (7)\\
\Longrightarrow\ & \exists b.\ (a,b) \in R && (8)\\
\Longrightarrow\ & \exists b.\ \{\{a\},\{a,b\}\} \in R && (9)\\
\Longrightarrow\ & \exists b.\ \{a,b\} \in \bigcup R && (10)\\
\Longrightarrow\ & \exists b.\ a \in \bigcup\bigcup R && (11)\\
\Longrightarrow\ & a \in \bigcup\bigcup R && (12)
\end{aligned}$$

5 Operations

R^{-1} $\qquad R|_X$ $\qquad R[X]$ $\qquad R^{-1}[Y]$ $\qquad R \circ S$

Definition (逆 (Inverse))

The *inverse* of R is the relation

$$R^{-1} = \{(a, b) \mid (b, a) \in R\}$$

$$R = \{(x, y) \mid x = y\} \subseteq \mathbb{R} \times \mathbb{R} \qquad R^{-1} = R$$

$$R = \{(x, y) \mid y = \sqrt{x}\} \subseteq \mathbb{R} \times \mathbb{R} \qquad R^{-1} = \{(x, y) \mid y = x^2 \land x > 0\}$$

$$\leq = \{(x, y) \mid x \leq y\} \subseteq \mathbb{R} \times \mathbb{R} \qquad \leq^{-1} = \; \geq \triangleq \{(x, y) \mid x \geq y\}$$

Theorem

$$(R^{-1})^{-1} = R$$

对任意 (a,b),

$$
\begin{align}
& (a,b) \in (R^{-1})^{-1} \tag{1} \\
\iff & (b,a) \in R^{-1} \tag{2} \\
\iff & (a,b) \in R \tag{3}
\end{align}
$$

Theorem (关系的逆)

R, S 均为关系

$$(R \cup S)^{-1} = R^{-1} \cup S^{-1}$$
$$(R \cap S)^{-1} = R^{-1} \cap S^{-1}$$
$$(R \setminus S)^{-1} = R^{-1} \setminus S^{-1}$$

Definition (左限制 (Left-Restriction))

Suppose $R \subseteq X \times Y$ and $S \subseteq X$. The *left-restriction* relation of R to S over X and Y is

$$R|_S = \{(x, y) \in R \mid x \in S\}$$

Definition (右限制 (Right-Restriction))

Suppose $R \subseteq X \times Y$ and $S \subseteq Y$. The *right-restriction* relation of R to S over X and Y is

$$R|^S = \{(x, y) \in R \mid y \in S\}$$

Definition (限制 (Restriction))

Suppose $R \subseteq X \times X$ and $S \subseteq X$. The *restriction* relation of R to S over X is

$$R|_S = \{(x, y) \in R \mid x \in S \land y \in S\}$$

$$R = \{(x, y) \mid x^2 + y^2 = 1\} \subseteq \mathbb{R} \times \mathbb{R}$$

$$R|_{\mathbb{R}^+} \qquad \text{(left restriction, restriction)}$$

$$R|^{\mathbb{R}^+} \qquad \text{(right restriction)}$$

Definition (像 (Image))

The *image* of X under R is the set

$$R[X] = \{b \in \mathrm{ran}(R) \mid \exists a \in X.\ (a, b) \in R\}$$

$$R[a] \triangleq R[\{a\}] = \{b \mid (a, b) \in R\}$$

Definition (逆像 (Inverse Image))

The *inverse image* of Y under R is the set

$$R^{-1}[Y] = \{a \in \mathrm{dom}(R) \mid \exists b \in Y.\ (a, b) \in R\}$$

$$R^{-1}[b] \triangleq R^{-1}[\{b\}] = \{a \mid (a, b) \in R\}$$

$$R \subseteq A \times B \qquad X \subseteq A \qquad Y \subseteq B$$

$$R^{-1}[R[X]] \ ? \ X$$

$$R[R^{-1}[Y]] \ ? \ Y$$

Theorem

$$R[X_1 \cup X_2] = R[X_1] \cup R[X_2]$$
$$R[X_1 \cap X_2] \subseteq R[X_1] \cap R[X_2]$$
$$R[X_1 \setminus X_2] \supseteq R[X_1] \setminus R[X_2]$$

对任意 b,

$$\begin{aligned}
& b \in R[X_1 \cup X_2] \\
\iff & \exists a \in X_1 \cup X_2.\ (a, b) \in R \\
\iff & \exists a \in X_1.\ (a, b) \in R \vee \exists a \in X_2.\ (a, b) \in R \\
\iff & b \in R[X_1] \vee b \in R[X_2] \\
\iff & b \in R[X_1] \cup R[X_2]
\end{aligned}$$

Definition (复合 (Composition; $R \circ S$, $R; S$))

The *composition* of relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ is the relation

$$R \circ S = \{(a, c) \mid \exists b.\ (a, b) \in S \land (b, c) \in R\}$$

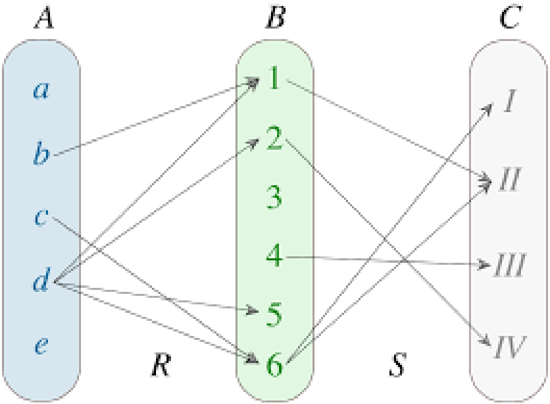
$$R = \{(1,2),(3,1)\} \qquad S = \{(1,3),(2,2),(2,3)\}$$

$$R \circ S = \{(1,1),(2,1)\}$$

$$S \circ R = \{(1,2),(1,3),(3,3)\}$$

$$R^{(2)} \triangleq R \circ R = \{(3,2)\} \qquad (R \circ R) \circ R = \emptyset$$

$$S^{(2)} \triangleq S \circ S = \{(2,2),(2,3)\} \qquad (S \circ S) \circ S = \{(2,2),(2,3)\}$$



$$|S \circ R| = 7$$

$$\leq \circ \leq \; = \leq$$

$$\geq \circ \leq \; = \mathbb{R} \times \mathbb{R}$$

$$\forall a, b \in \mathbb{R}.\ (a, b) \in \geq \circ \leq$$

$$(a, |a| + |b|) \in \leq \qquad (|a| + |b|, b) \in \geq$$

Theorem

$$(R \circ S)^{-1} = S^{-1} \circ R^{-1}$$

对任意 (a, b),

$$
\begin{aligned}
& (a, b) \in (R \circ S)^{-1} && (1) \\
\iff & (b, a) \in R \circ S && (2) \\
\iff & \exists c.\ (b, c) \in S \land (c, a) \in R && (3) \\
\iff & \exists c.\ (c, b) \in S^{-1} \land (a, c) \in R^{-1} && (4) \\
\iff & (a, b) \in S^{-1} \circ R^{-1} && (5)
\end{aligned}
$$

Theorem

$$(R \circ S) \circ T = R \circ (S \circ T)$$

对任意 (a, b),

$$
\begin{aligned}
& (a, b) \in (R \circ S) \circ T && (1) \\
\iff & \exists c. \Big((a, c) \in T \land (c, b) \in R \circ S\Big) && (2) \\
\iff & \exists c. \Big((a, c) \in T \land (\exists d. (c, d) \in S \land (d, b) \in R)\Big) && (3) \\
\iff & \exists d. \exists c. \Big((a, c) \in T \land (c, d) \in S \land (d, b) \in R\Big) && (4) \\
\iff & \exists d. \Big((\exists c. (a, c) \in T \land (c, d) \in S) \land (d, b) \in R\Big) && (5) \\
\iff & \exists d. \Big((a, d) \in S \circ T \land (d, b) \in R\Big) && (6) \\
\iff & (a, b) \in R \circ (S \circ T) && (7)
\end{aligned}
$$

燕小六: “帮我照顾好我七舅姥爷和我外甥女”

"舅姥爷": 姥姥/外婆的兄弟

$$G = \{(a,b) \mid a \text{ 是 } b \text{ 的舅姥爷}\}$$

$$B = \{(a,b) \mid a \text{ is the brother of } b\}$$

$$M = \{(a,b) \mid a \text{ is the mother of } b\}$$

$$G = (M \circ M) \circ B$$

$$G = (M \circ M) \circ B = M \circ (M \circ B)$$

"舅姥爷": 妈妈的舅舅

Theorem (关系的复合)

$$(X \cup Y) \circ Z = (X \circ Z) \cup (Y \circ Z)$$

$$(X \cap Y) \circ Z \subseteq (X \circ Z) \cap (Y \circ Z)$$

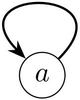
7 Properties

$$R \subseteq X \times X$$

Definition (自反的 (Reflexive))

$R \subseteq X \times X$ is *reflexive* if

$$\forall a \in X.\ (a, a) \in R$$



$$\leq \subseteq \mathbb{R} \times \mathbb{R} \text{ is reflexive}$$

三角形上的全等关系是自反的

Definition (反自反 (Irreflexive))

$R \subseteq X \times X$ is *irreflexive* if

$$\forall a \in X.\ (a, a) \notin R$$

$$< \subseteq \mathbb{R} \times \mathbb{R} \text{ is irreflexive}$$

$$> \subseteq \mathbb{R} \times \mathbb{R} \text{ is irreflexive}$$

$$A = \{1, 2, 3\} \qquad R \subseteq A \times A$$

$$\{(1,1), (1,3), (2,1), (2,2), (3,3)\}$$

$$\{(1,2), (2,3), (3,1)\}$$

$$\{(1,2), (2,2), (2,3), (3,1)\}$$

Definition (对称 (Symmetric))

$R \subseteq X \times X$ is *symmetric* if

$$\forall a, b \in X.\ aRb \to bRa$$

$$\begin{tikzcd} a \arrow[r, bend left] & b \arrow[l, bend left] \end{tikzcd}$$

$$\forall a, b \in X.\ aRb \leftrightarrow bRa$$

$$A = \{1, 2, 3\} \qquad R \subseteq A \times A$$

$$\{(1,1), (1,2), (1,3), (2,1), (3,1), (3,3)\}$$

$$\{(1,2), (2,3), (2,2), (3,1)\}$$

$$\{(1,1), (2,2), (3,3)\}$$

$$\{(1,2), (2,1), (2,3)\}$$

Definition (反对称 (AntiSymmetric))

$R \subseteq X \times X$ is *antisymmetric* if

$$\forall a, b \in X.\ (aRb \wedge bRa) \to a = b$$

$\geq$ is antisymmetric

$\mid$ is antisymmetric

$$A = \{1, 2, 3\} \qquad R \subseteq A \times A$$

$$\{(1,1), (1,2), (1,3), (2,1), (3,1), (3,3)\}$$

$$\{(1,2), (2,3), (2,2), (3,1)\}$$

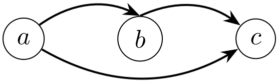
$$\{(1,1), (2,2), (3,3)\}$$

$$\{(1,2), (2,1), (2,3)\}$$

Definition (传递的 (Transitive))

$R \subseteq X \times X$ is *transitive* if

$$\forall a, b, c \in X. \ (aRb \wedge bRc \to aRc)$$



$$A = \{1, 2, 3\} \qquad R \subseteq A \times A$$

$$\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\}$$

$$\{(1,2),(2,3),(3,1)\}$$

$$\{(1,3)\}$$

$$\emptyset$$

Definition (连接的 (Connex))

$R \subseteq X \times X$ is *connex* if

$$\forall a, b \in X. \ (aRb \lor bRa)$$

Definition (三分的 (Trichotomous))

$R \subseteq X \times X$ is *trichotomous* if

$$\forall a, b \in X. \ (\text{exactly one of } aRb, bRa, \text{ or } a = b \text{ holds})$$

Theorem

$$R \textit{ is reflexive} \iff I \subseteq R$$

$$I = \{(a,a) \in A \times A \mid a \in A\}$$

Theorem

$$R \textit{ is symmetric} \iff R^{-1} = R$$

Theorem

$$R \textit{ is transitive} \iff R \circ R \subseteq R$$

$$R = \{(1,2),(2,3),(1,3),(4,4)\}$$

对任意 (a,b),

$$(a,b) \in R \circ R \tag{1}$$

$$\implies \exists c.\ (a,c) \in R \land (b,c) \in R \tag{2}$$

$$\implies (a,b) \in R \tag{3}$$

对任意 a, b, c

$$(a,b) \in R \land (b,c) \in R \implies (a,c) \in R \circ R \implies (a,c) \in R$$

Theorem

$$R \textit{ is symmetric and transitive } \iff R = R^{-1} \circ R$$

Equivalence Relations

Definition (Equivalence Relation)

$R \subseteq X \times X$ is an *equivalence relation* on X iff R is

- reflexive: $\forall a \in X.\ aRa$
- symmetric: $\forall a, b \in X.\ (aRb \leftrightarrow bRa)$
- transitive: $\forall a, b, c \in X.\ (aRb \land bRc \to aRc)$

$$= \in \mathbb{R} \times \mathbb{R}$$

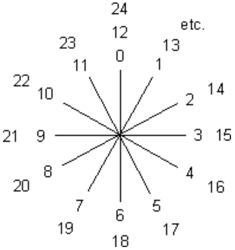
$$\parallel \in \mathbb{L} \times \mathbb{L}$$

三角形的相似关系

$$R = \{(x, y) \mid x \equiv y \pmod{k}\} \subseteq \mathbb{Z} \times \mathbb{Z}$$

Why are equivalence relations important?

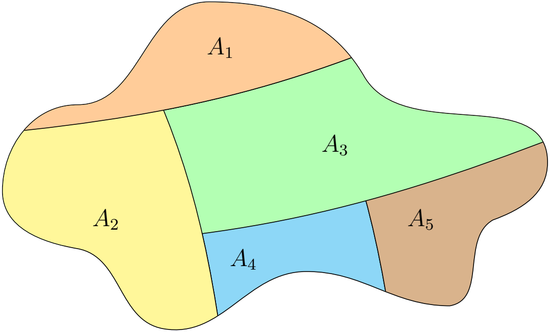
Equivalence Relations as Abstractions



“全国人民代表大会各省代表团”

Equivalence Relation $\iff$ Partition

Partition



“不空、不漏、不重”

Definition (划分 (Partition))

A family of sets $\Pi = \{A_\alpha \mid \alpha \in I\}$ is a *partition* of X if

(i) (不空)

$$\forall \alpha \in I.\ A_\alpha \neq \emptyset$$

$$(\forall \alpha \in I.\ \exists x \in X.\ x \in A_\alpha)$$

(ii) (不漏)

$$\bigcup_{\alpha \in I} A_\alpha = X$$

$$(\forall x \in X.\ \exists \alpha \in I.\ x \in A_\alpha)$$

(iii) (不重)

$$\forall \alpha, \beta \in I.\ A_\alpha \cap A_\beta = \emptyset \lor A_\alpha = A_\beta$$

$$(\forall \alpha, \beta \in I.\ A_\alpha \cap A_\beta \neq \emptyset \implies A_\alpha = A_\beta)$$

Equivalence Relation $R \subseteq X \times X \implies$ Partition Π of X

Definition (等价类 (Equivalence Class))

The *equivalence class* of a *modulo* R is a set:

$$[a]_R = \{b \in X.\ aRb\}$$

$$= \in \mathbb{R} \times \mathbb{R}$$

$$\parallel \in \mathbb{L} \times \mathbb{L}$$

三角形的相似关系

$$R = \{(x, y) \mid x \equiv y \pmod{k}\} \subseteq \mathbb{Z} \times \mathbb{Z}$$

Definition (商集 (Quotient Set))

The *quotient set* of X by R (X modulo R) is a set:

$$X/R = \{[a]_R \mid a \in X\}$$

$$= \in \mathbb{R} \times \mathbb{R}$$

$$\| \in \mathbb{L} \times \mathbb{L}$$

三角形的相似关系

$$R = \{(x, y) \mid x \equiv y \pmod{k}\} \subseteq \mathbb{Z} \times \mathbb{Z}$$

Theorem

$$X/R = \{[a]_R \mid a \in X\} \textit{ is a partition of } X.$$

$$\forall a \in X.\ [a]_R \neq \emptyset$$

$$\forall a \in X.\ \exists b \in X.\ a \in [b]_R$$

Theorem

$$\forall a \in X, b \in X.\ [a]_R \cap [b]_R = \emptyset \vee [a]_R = [b]_R$$

$$\forall a \in X, b \in X.\ [a]_R \cap [b]_R \neq \emptyset \to [a]_R = [b]_R$$

$$\forall a \in X.\ b \in X.\ [a]_R \cap [b]_R \neq \emptyset \to [a]_R = [b]_R$$

不妨设 $x \in [a]_R \wedge [b]_R$

$$\begin{aligned} & x \in [a]_R \wedge [b]_R && (1) \\ \Longrightarrow\ & aRx \wedge xRb && (2) \\ \Longrightarrow\ & aRb && (3) \end{aligned}$$

对于任意 x,

$$\begin{aligned} & x \in [a]_R && (1) \\ \Longleftrightarrow\ & xRa && (2) \\ \Longleftrightarrow\ & xRb && (3) \\ \Longleftrightarrow\ & x \in [b]_R && (4) \end{aligned}$$

Theorem

$$\forall a, b \in X. \ ([a]_R = [b]_R \leftrightarrow aRb)$$

Partition Π of $X \implies$ Equivalence Relation $R \subseteq X \times X$

Definition

$$(a, b) \in R \iff \exists S \in \Pi.\ a \in S \land b \in S$$

$$R = \{(a, b) \in X \times X \mid \exists S \in \Pi.\ a \in S \land b \in S\}$$

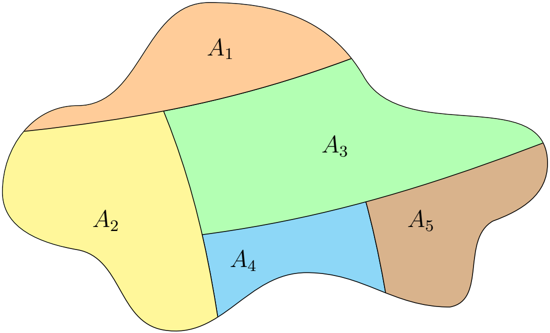
Theorem

R is an equivalence relation on X.

$$\forall x \in X.\ xRx$$

$$\forall x, y \in X.\ xRy \to yRx$$

$$\forall x, y, z \in X.\ xRy \land yRz \to xRz$$



Equivalence Relation $\iff$ Partition

Definition

$$\sim \subseteq \mathbb{N} \times \mathbb{N}$$

$$(a, b) \sim (c, d) \iff a +_{\mathbb{N}} d = b +_{N} c$$

Theorem

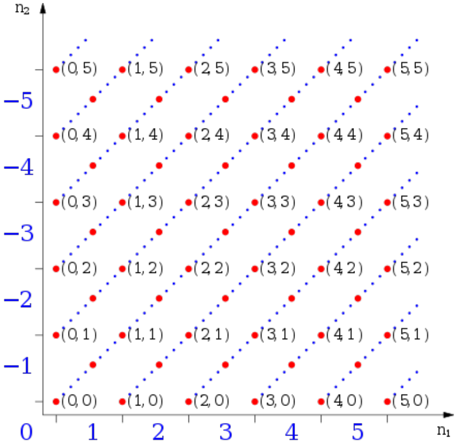
$\sim$ is an equivalence relation.

Q : What is $\mathbb{N} \times \mathbb{N} / \sim$?

Definition ($\mathbb{Z}$)

$$\mathbb{Z} \triangleq \mathbb{N} \times \mathbb{N} / \sim$$

$$[(1,3)]_{\sim} = \{(0,2), (1,3), (2,4), (3,5), \cdots\} \triangleq -2 \in \mathbb{Z}$$



$$\mathbb{Z} \triangleq \mathbb{N} \times \mathbb{N} / \sim$$

Definition ($+_{\mathbb{Z}}$)

$$[(m_1, n_1)] +_{\mathbb{Z}} [(m_2, n_2)] = [m_1 +_{\mathbb{N}} m_2, n_1 +_{\mathbb{N}} n_2]$$

Definition ($\cdot_{\mathbb{Z}}$)

$$\begin{aligned} &[(m_1, n_1)] \cdot_{\mathbb{Z}} [(m_2, n_2)] \\ = &[m_1 \cdot_{\mathbb{N}} m_2 +_{\mathbb{N}} n_1 \cdot_{\mathbb{N}} n_2, m_1 \cdot_{\mathbb{N}} n_2 +_{\mathbb{N}} n_1 \cdot_{\mathbb{N}} m_2] \end{aligned}$$

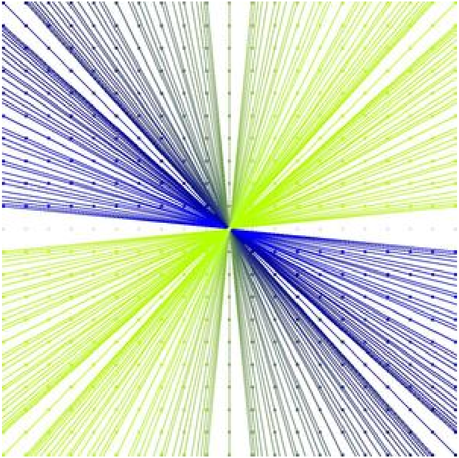
Definition

$$\sim \subseteq (\mathbb{Z} \times \mathbb{Z} \setminus \{0_{\mathbb{Z}}\})^2$$

$$(a, b) \sim (c, d) \iff a \cdot_{\mathbb{Z}} d = b \cdot_{\mathbb{Z}} c$$

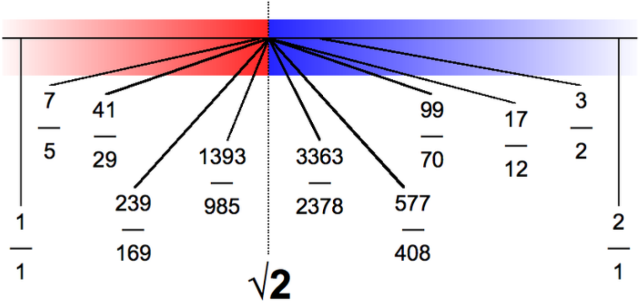
Definition ($\mathbb{Q}$)

$$\mathbb{Q} \triangleq \mathbb{Z} \times \mathbb{Z} / \sim$$



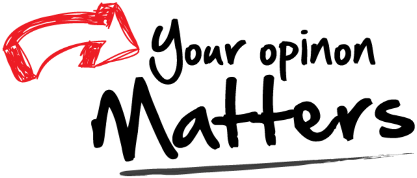
$$\mathbb{Q} \triangleq \mathbb{Z} \times \mathbb{Z} / \sim$$

如何用有理数定义实数?



Dedekind Cut (戴德金分割)


Thank
You!



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