(四) 集合: 基本概念与运算 (Set Theory)

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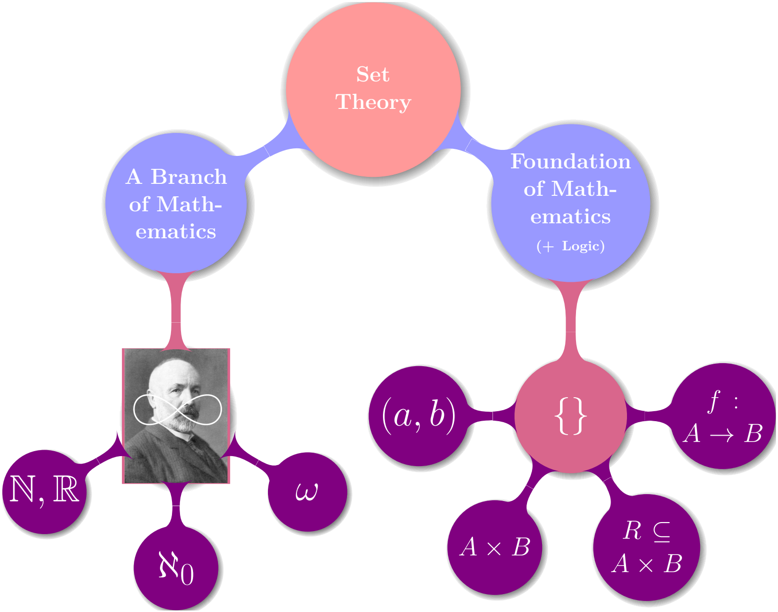
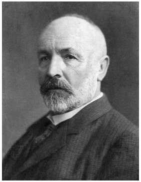

Set Theory
A Branch of Math-ematics
Foundation of Math-ematics
(+ Logic)
$\mathbb{N}, \mathbb{R}$
$\aleph_0$
ω
(a, b)
$\{\}$
$f : A \to B$
$A \times B$
$R \subseteq A \times B$

我们将集合理解为任何将我们思想中那些确定而彼此独立的对象放在一起而形成的聚合。

— *Georg Cantor*《超穷数理论基础》



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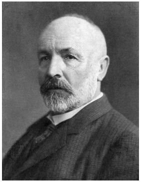

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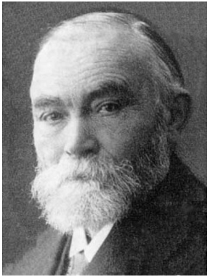
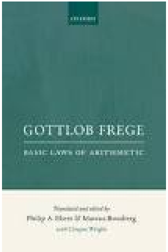
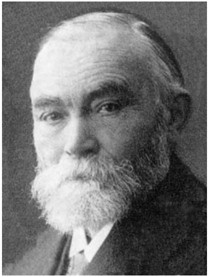


Gottlob Frege (1848–1925)



“Basic Laws of Arithmetic”
(1893 & 1903)



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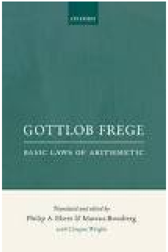

“Basic Laws of Arithmetic”
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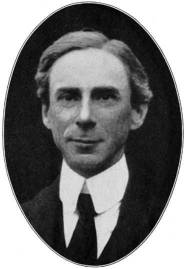
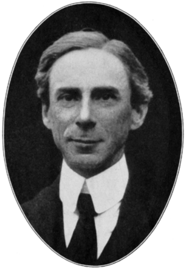
对于一个科学工作者来说，最不幸的事情莫过于：当他的工作接近完成时，却发现那大厦的基础已经动摇。

— 《附录二》，*1902*



Bertrand Russell (1872–1970)



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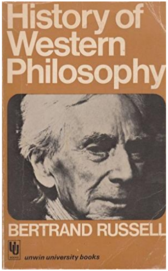

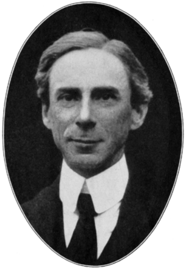

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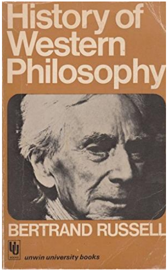

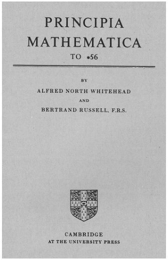

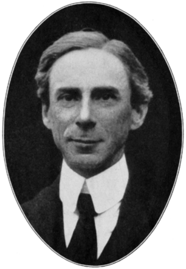

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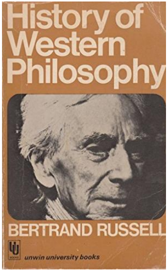

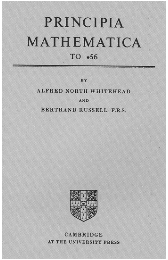

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$$Q : R \in R\,?$$

Q： 既然朴素集合论存在悖论，你是如何做作业的?


YOU HAVE
TO FIX IT!

Theorem (Russell's Paradox)

$$\{x \mid x \notin x\} \text{ is } \textit{not} \text{ a set.}$$

Axiomatic Set Theory (ZFC)

Ernst Zermelo (1871–1953)

Abraham Fraenkel (1891–1965)

First-order Language for Sets $\mathcal{L}_{Set} = \{\in\}$

Parentheses: $($, $)$

Variables: $x, y, z, \cdots$

Connectives: $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$

Quantifiers: $\forall$, $\exists$

Equality: $=$

Constants:

Functions:

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Everything we consider in $\mathcal{L}_{Set}$ is a set.

Q : What is "$\in$"?

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We don't define them directly.

We only describe their properties in an axiomatic way.

(1) To draw a straight line from any point to any point.

(2) To extend a finite straight line continuously in a straight line.

(3) To describe a circle with any center and radius.

(4) That all right angles are equal to one another.

(5) The parallel postulate.

Definition ($\notin$)

$$x \notin A \triangleq \neg(x \in A).$$

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Definition ($\subseteq$)

$$A \subseteq B \triangleq \forall x (x \in A \implies x \in B)$$

Definition ("$\bigcup A$" (Arbitrary Union))

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$$\bigcup \emptyset = \emptyset.$$

Theorem ("$\bigcap A$" (Arbitrary Intersection))

For any nonempty set A, there is a unique set B such that

$$\forall x \, (x \in B \iff x \textit{ belongs to every member of } A).$$

$$\forall x \, \big(x \in B \iff \forall y \in A (x \in y)\big).$$

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Let c be a fixed member of A.

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"$\bigcap \emptyset$"

$$\bigcap \emptyset \text{ is } \textit{not} \text{ a set.}$$

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There is no universal set.

$$\nexists B(\forall x(x \in B)).$$

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$$B \in A \implies (B \in B \iff B \notin B)$$

□

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We can never look for objects "not in B" unless we know where to start looking. — *UD (Chapter 6; Page 64)*

Definition (Power Set Axiom)

For any set A, there is a set whose members are the subsets of A:

$$\forall A \, \exists B \, \forall x (x \in B \iff x \subseteq A).$$

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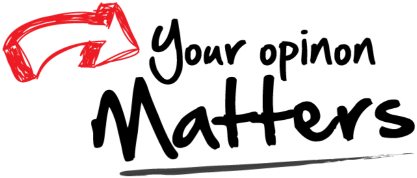
The is *not* correct!

$$\mathcal{P}(A) \triangleq \{x \mid x \subseteq A\}$$


Thank
You!



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