(九) 图论: 路径与圈 (Paths and Cycles)

魏恒峰

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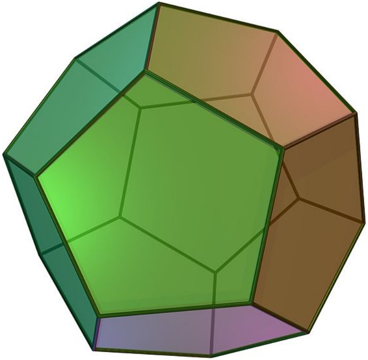
2021 年 05 月 06 日

Dodecahedron: 12 faces, 20 vertices, and 30 edges



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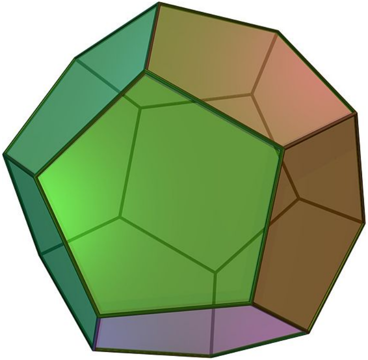
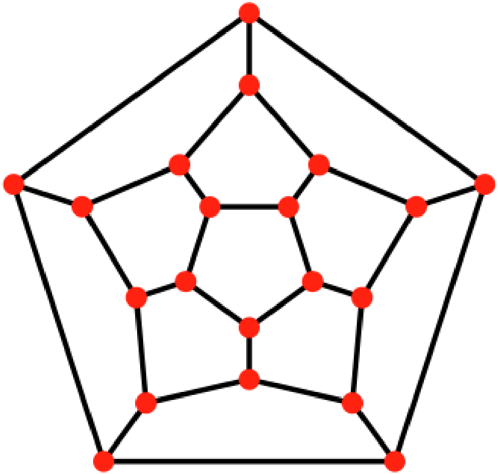


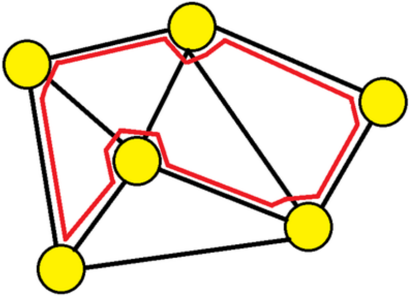
Is there a cycle that visits each vertex exactly once?



Is there a cycle that visits each vertex exactly once?

Definition (Hamiltonian Path)

A Hamiltonian path is a path that visits each vertex exactly once.



Definition (Hamiltonian Cycle)

A Hamiltonian cycle is a Hamiltonian path that is a cycle.

Definition (Hamiltonian Graph)

A graph is a Hamiltonian graph if it has a Hamiltonian cycle.

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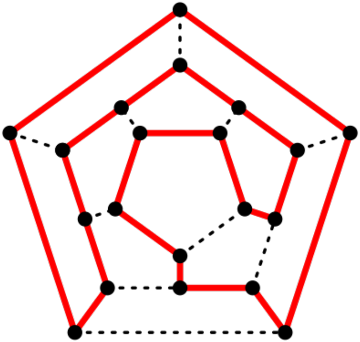
Definition (Semi-Hamiltonian Graph)

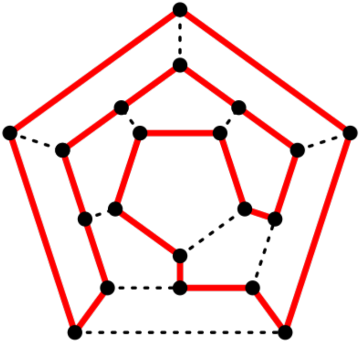
A non-Hamiltonian graph is semi-Hamiltonian if it has a Hamiltonian path.

William Rowan Hamilton
(1805 ~ 1865)

(October 16, 1843)

$$i^2 = j^2 = k^2 = ijk = -1$$





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Nobody knows.

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Nobody knows.

We will probably never know it.

Theorem

The Hamiltonian Path/Cycle problem is NP-complete.

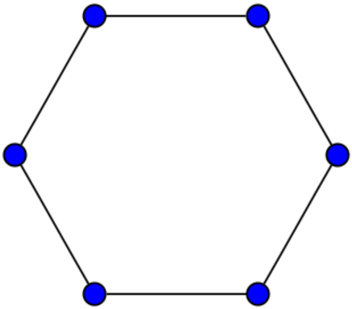
Typical (Positive/Negative) Graph Examples

Sufficient Conditions

Necessary Conditions


EXAMPLES

- Every cycle is Hamiltonian



C_6

- A complete graph (完全图) with $|V| > 2$ is Hamiltonian.

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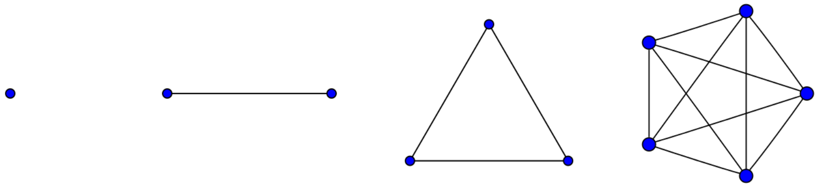
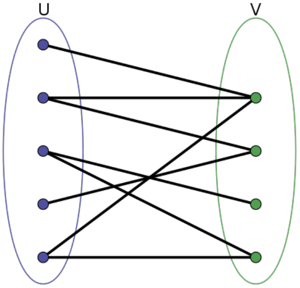


K_1 K_2 K_3 K_5

- A complete bipartite graph $K_{m,n}$ is Hamiltonian iff $m = n$.

Definition (Bipartite Graph (Bigraph; 二部图))

A bipartite graph $G = (U, V, E)$ is a graph whose vertices can be divided into two disjoint sets U and V such that every edge connects a vertex in U to one in V.



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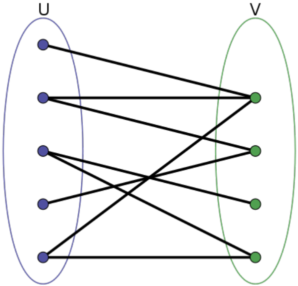

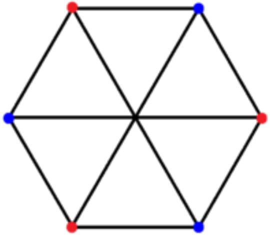

Definition (Complete Bipartite Graph (Biclique; 完全二部图))

A complete bipartite graph $G = (U, V, E)$ is bipartite graph where every vertex of U is connected to every vertex of V.

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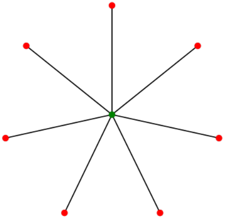
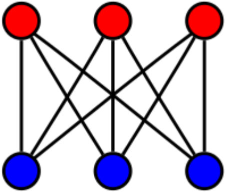
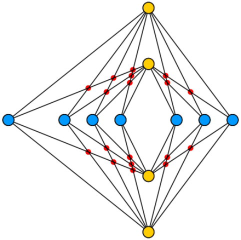


$K_{1,5}$ (star)



$K_{3,3}$ (utility graph)



$K_{4,7}$

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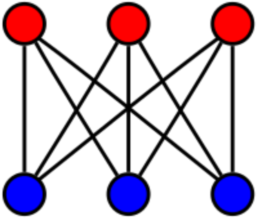

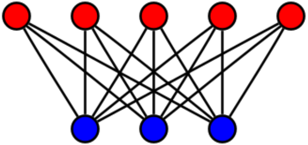

- Every platonic solid (正多面体), considered as a graph, is Hamiltonian.

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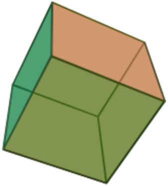
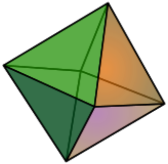
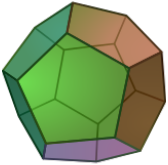
Tetrahedron



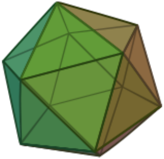
Cube



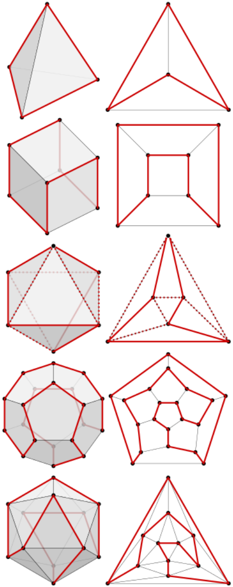
Octahedron



Dodecahedron



Icosahedron



Theorem

- *Petersen graph is not Hamiltonian.*

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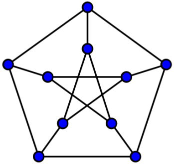

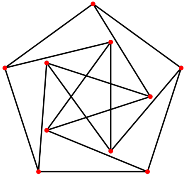

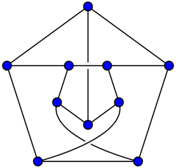

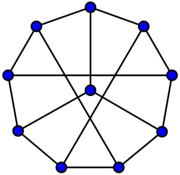

"If G has enough edges, then G is Hamiltonian."

Theorem (Ore's Theorem, 1960)

Let G be a simple graph with $n \geq 3$ vertices. If

$$deg(u) + deg(v) \geq n$$

for each pair of non-adjacent vertices u and v, then G is Hamiltonian.

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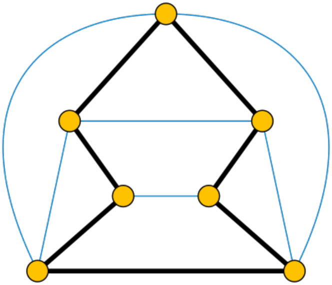

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Thus we may consider only *maximal* non-Hamiltonian graphs:

adding any edge gives a Hamiltonian graph.

By its “maximality”, G contains a Hamiltonian path

$$v_1 \to v_2 \to \cdots \to v_n$$

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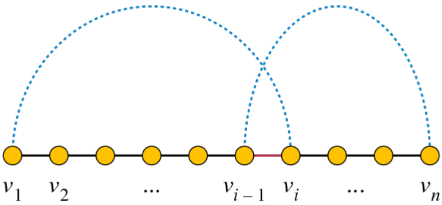

There must be some vertex v_i adjacent to v_1

such that v_{i-1} is adjacent to v_n.

Theorem (Dirac's Theorem (1952; Gabriel Andrew Dirac))

A simple graph $G = (V, E)$ *with* $n \geq 3$ *vertices is Hamiltonian*

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Family [edit | edit source]

He was born Balázs Gábor in Budapest, to Richárd Balázs, a military officer and businessman, and Margit "Manci" Wigner (sister of Eugene Wigner).[5] When his mother married Paul Dirac in 1937, he and his sister resettled in England and were formally adopted, changing their family name to Dirac.[6]

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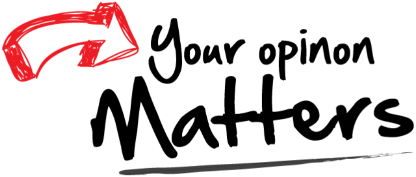
$$\delta(G) = \lfloor (n-1)/2 \rfloor$$

Counterexample: $C_{\lfloor (n+1)/2 \rfloor}$ and $C_{\lceil (n+1)/2 \rceil}$ sharing a vertex


NECESSARY


NECESSARY


Thank
You!



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