(六) 集合: 函数 (Functions)

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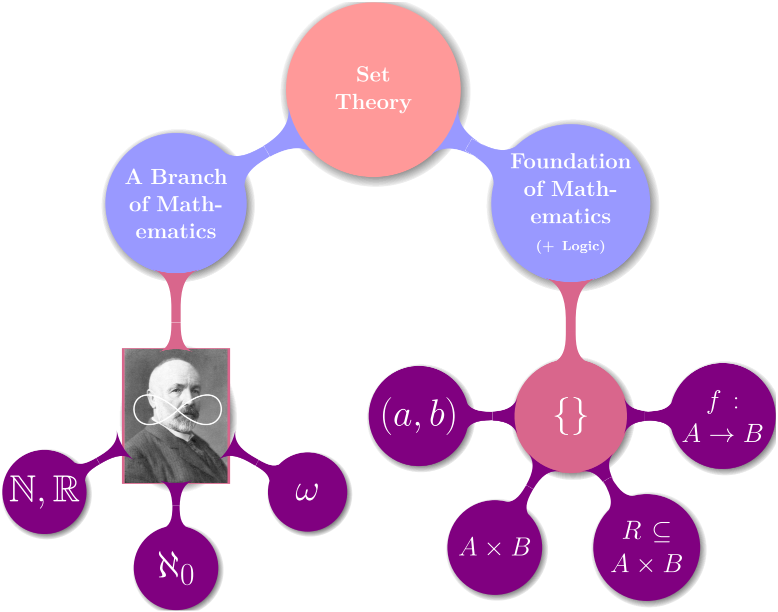
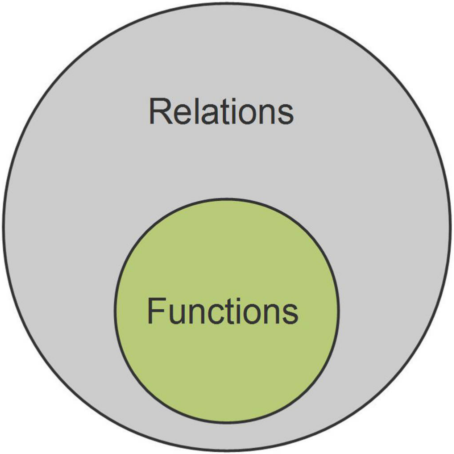

Set Theory
A Branch of Mathematics
Foundation of Mathematics (+ Logic)
$\mathbb{N}, \mathbb{R}$
$\aleph_0$
ω
(a, b)
$\{\}$
$f : A \to B$
$A \times B$
$R \subseteq A \times B$

从“关系”的角度理解“函数”

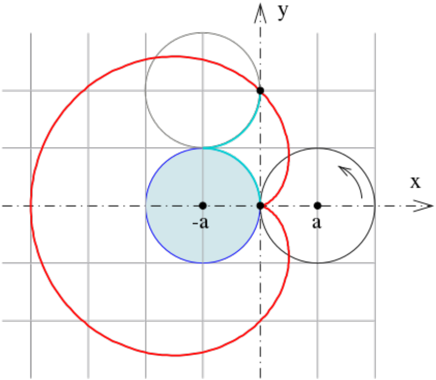


$$f(x) = 2x + 1$$

“函数” 也是 “关系”

$$\{\ldots, (-2, -3), (-1, -1), (0, 1), (1, 3), \ldots\}$$

$$(x^2+y^2)^2+4ax(x^2+y^2)-4a^2y^2=0$$



“函数”不允许“一对多”

Functions

PROOF!

Definition of Functions

$$R \subseteq A \times B$$

is a *relation* from A to B

Definition (Function)

$f \subseteq A \times B$ is a *function* from A to B if

$$\forall a \in A.\ \exists!\, b \in B.\ (a, b) \in f.$$

$$f : A \to B$$

$$\mathrm{dom}(f) = A \qquad \mathrm{cod}(f) = B$$

$$\mathrm{ran}(f) = f(A) \subseteq B$$

$$f : a \mapsto b$$

$$f(a) \triangleq b$$

Definition (Function)

$f \subseteq A \times B$ is a *function* from A to B if

$$\forall a \in A.\ \exists!\ b \in B.\ (a, b) \in f.$$

For Proof:

$$\forall a \in A.$$

$$\forall a \in A.\ \exists b \in B. (a, b) \in f$$

$$\exists! b \ \in B.$$

$$\forall b, b' \in B.\ (a, b) \in f \land (a, b') \in f \implies b = b'$$

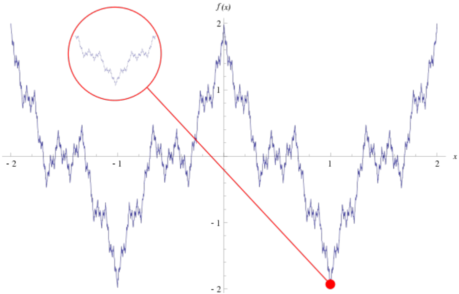
$$I_X : X \to X$$

X 上的恒等函数

$$\forall x \in X.\ I_X(x) = x$$

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$

$$0 < a < 1,\ b \text{ is a positive odd integer},\ ab > 1 + \frac{3}{2}\pi$$



Weierstrass Function (1872)

"处处连续, 但处处不可导"

Definition (Y^X)

The *set* of all functions from X to Y:

$$Y^X = \{f \mid f : X \to Y\}$$

$$|X| = x \quad |Y| = y, \qquad |Y^X| = y^x$$

Definition (Y^X)

The *set* of all functions from X to Y:

$$Y^X = \{f \mid f : X \to Y\}$$

$$\forall Y.\ Y^{\emptyset} = \{\emptyset\}$$

$$\emptyset^{\emptyset} = \{\emptyset\}$$

$$\forall X \neq \emptyset.\ \emptyset^X = \emptyset$$

Definition (Y^X)

The *set* of all functions from X to Y:

$$Y^X = \{f \mid f : X \to Y\}$$

Q : Is there a set consisting of all functions?

Theorem

There is no set consisting of all functions.

Suppose by contradiction that A is the set of all functions.

For every set X, there exists a function $I_{\{X\}} : \{X\} \to \{X\}$.

$\bigcup_{I_X \in A} \text{dom}(I_X)$ would be the universe that does not exist!

Functions as Sets

Theorem (函数的外延性原理 (The Principle of Functional Extensionality))

$$f, g \ \mathit{are\ functions}:$$

$$f = g \iff \mathit{dom}(f) = \mathit{dom}(g) \wedge (\forall x \in \mathit{dom}(f).\ f(x) = g(x))$$

$$f = g \iff \forall (a, b).\ ((a, b) \in f \leftrightarrow (a, b) \in g).$$

It may be that $\mathrm{cod}(f) \neq \mathrm{cod}(g)$.

$$f : A \to B \qquad g : C \to D$$

Q : Is $f \cap g$ a function?

Theorem (Intersection of Functions)

$$f \cap g : (A \cap C) \to (B \cap D)$$

$$f : A \to B \qquad g : C \to D$$

Q : Is $f \cup g$ a function?

Theorem (Union of Functions)

$$f \cup g : (A \cup C) \to (B \cup D) \iff \forall x \in dom(f) \cap dom(g).\ f(x) = g(x)$$

$$f : \mathbb{Q} \to \mathbb{R}$$

$$f(x) = \begin{cases} x+1, & \text{if } x \in 2\mathbb{Z} \\ x-1, & \text{if } x \in 3\mathbb{Z} \\ 2, & \text{otherwise} \end{cases}$$

$$f : \mathcal{P}(\mathbb{R}) \to \mathbb{Z}$$

$$f(A) = \begin{cases} \min(A \cap \mathbb{N}) & \text{if } A \cap \mathbb{N} \neq \emptyset \\ -1 & \text{if } A \cap \mathbb{N} = \emptyset \end{cases}$$

$$\mathrm{dom}(f) \cap \mathrm{dom}(g) = \emptyset$$

By the *Well-Ordering Principle* of $\mathbb{N}$

$$D : \mathbb{R} \to \mathbb{R}$$

$$D(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Dirichlet Function

“处处不连续”

Special Functions (*-jectivity*)

Definition (Injective (one-to-one; 1-1) 单射函数)

$$f : A \to B \qquad f : A \rightarrowtail B$$

$$\forall a_1, a_2 \in A.\ a_1 \neq a_2 \to f(a_1) \neq f(a_2)$$

For Proof:

- To prove that f *is* 1-1:

$$\forall a_1, a_2 \in A.\ f(a_1) = f(a_2) \to a_1 = a_2$$

- To show that f *is not* 1-1:

$$\exists a_1, a_2 \in A.\ a_1 \neq a_2 \wedge f(a_1) = f(a_2)$$

Definition (Surjective (onto) 满射函数)

$$f : A \to B \qquad f : A \twoheadrightarrow B$$

$$\mathrm{ran}(f) = B$$

For Proof:

- To prove that f *is* onto:

$$\forall b \in B.\ (\exists a \in A.\ f(a) = b)$$

- To show that f *is not* onto:

$$\exists b \in B.\ (\forall a \in A.\ f(a) \neq b)$$

Definition (Bijective (one-to-one correspondence) 双射; 一一对应)

$$f : A \to B \qquad f : A \xleftrightarrow[onto]{1-1} B$$

1-1 & onto

$$f : \mathbb{Z} \to \mathbb{N}, \qquad f(x) = x^2 + 1$$

$$f : \mathbb{N} \to \mathbb{Q}, \qquad f(x) = \frac{1}{x}$$

$$f : \mathbb{N} \to \mathbb{N}, \qquad f(x) = 2^x$$

$$f : \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}, \qquad f(z, n) = \frac{z}{n+1}$$

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \qquad f(x, y) = (x+1, y+1)$$

Theorem (Cantor Theorem)

If $f : A \to 2^A$, then f is not onto.

Proof. Let A be a set and let $f : A \to 2^A$. To show that f is not onto, we must find a $B \in 2^A$ (i.e., $B \subseteq A$) for which there is no $a \in A$ with $f(a) = B$. In other words, B is a set that f "misses." To this end, let

$$B = \{x \in A : x \notin f(x)\}.$$

We claim there is no $a \in A$ with $f(a) = B$.

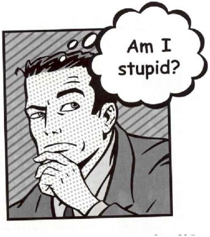
Suppose, for the sake of contradiction, there is an $a \in A$ such that $f(a) = B$. We ponder: Is $a \in B$?

- If $a \in B$, then, since $B = f(a)$, we have $a \in f(a)$. So, by definition of B, $a \notin f(a)$; that is, $a \notin B$.$\Rightarrow\Leftarrow$
- If $a \notin B = f(a)$, then, by definition of B, $a \in B$.$\Rightarrow\Leftarrow$

Both $a \in B$ and $a \notin B$ lead to contradictions, and hence our supposition [there is an $a \in A$ with $f(a) = B$] is false, and therefore f is not onto. ■

Theorem (Cantor Theorem)

If $f : A \to 2^A$, then f is not onto.



Theorem (Cantor Theorem)

If $f : A \to 2^A$, then f is not onto.

Understanding this problem:

$$A = \{1, 2, 3\}$$

$$2^A = \mathcal{P}(A) = \Big\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\Big\}$$

Onto

$$\forall B \in 2^A. \Big(\exists a \in A. \; f(a) = B\Big)$$

Not Onto

$$\exists B \in 2^A. \Big(\forall a \in A. \; f(a) \neq B\Big)$$

$$
\begin{aligned}
f(1) &= \{1, 2\} \\
f(2) &= \{1, 3\} \\
f(3) &= \emptyset
\end{aligned}
$$

$$B = \{2, 3\}$$

$$B = \{x \in \{1, 2, 3\} \mid x \notin f(x)\} = \{2, 3\}$$

Theorem (Cantor Theorem)

If $f : A \to 2^A$, then f is not onto.

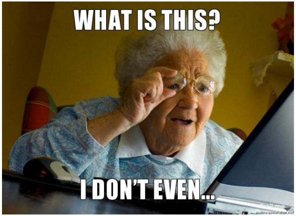
$$\exists B \in 2^A. \left(\forall a \in A.\ f(a) \neq B\right)$$

- Constructive proof ($\exists$):

$$B = \{a \in A \mid a \notin f(a)\}$$

- By contradiction ($\forall$):

$$\exists a \in A.\ f(a) = B.$$



$$Q : a \in B?$$

$$a \in B \iff a \notin B$$

Theorem (Cantor Theorem)

If $f : A \to 2^A$, then f is not onto.

对角线论证 (Cantor's diagonal argument) (以下仅适用于可数集合 A).

a	$f(a)$					
	1	2	3	4	5	$\cdots$
1	1	1	0	0	1	$\cdots$
2	0	0	0	0	0	$\cdots$
3	1	0	0	1	0	$\cdots$
4	1	1	1	1	1	$\cdots$
5	0	1	0	1	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$

$$B = \{0, 1, 1, 0, 1\}$$

□

Functions as Relations

$$f|_X \qquad f(A) \qquad f^{-1}(B) \qquad f \circ g$$

Definition (Restriction)

The *restriction* of a function $f : A \to B$ to X is the function:

$$f|_X = \{(x, y) \in f \mid x \in X\}$$

It is unnecessary that $X \subseteq A$, though it is usually the case.

$$f|_X : A \cap X \to B$$

$$\forall x \in A \cap X.\ f|_X(x) = f(x)$$

Definition (像 (Image))

The *image* of X under a function $f : A \to B$ is the set

$$f(X) = \{y \mid \exists x \in X.\ (x, y) \in f\}$$

It is unnecessary that $X \subseteq \mathrm{dom}(f) = A$, though it is usually the case.

$$f(X) = f(A \cap X)$$

$$f(\{a\}) = \{b\} \quad \text{简记为} \quad f(a) = b$$

Definition (像 (Image))

The *image* of X under a function $f : A \to B$ is the set

$$f(X) = \{y \mid \exists x \in X.\ (x, y) \in f\}$$

$$\boxed{y \in f(X) \iff \exists x \in X.\ y = f(x)}$$

Definition (逆像 (Inverse Image))

The *inverse image* of Y under a function $f : A \to B$ is the set

$$f^{-1}(Y) = \{x \mid \exists y \in Y.\ (x, y) \in f\}$$

It is unnecessary that $Y \subseteq \mathrm{ran}(f)$, though it is usually the case.

$$f^{-1}(Y) = f^{-1}(Y \cap \mathrm{ran}(f))$$

$f^{-1}(\{b\}) = \{a\}$ 可简记为 $f^{-1}(b) = \{a\}$ 不能简记为 $f^{-1}(b) = a$

Definition (逆像 (Inverse Image))

The *inverse image* of Y under a function $f : A \to B$ is the set

$$f^{-1}(Y) = \{x \mid \exists y \in Y.\ (x, y) \in f\}$$

$$\boxed{x \in f^{-1}(Y) \iff f(x) \in Y}$$

$$\boxed{y \in f(X) \iff \exists x \in X.\, y = f(x)}$$

$$\boxed{x \in f^{-1}(Y) \iff f(x) \in Y}$$

$$f : A \to B$$

$$a \in A_0 \;\not\Longrightarrow\; f(a) \in f(A_0)$$

$$a \in A_0 \cap A \;\Longrightarrow\; f(a) \in f(A_0)$$

Theorem (Properties of f and f^{-1})

$$f : A \to B \qquad \cancel{A_1, A_2 \subseteq A \quad B_1, B_2 \subseteq B}$$

(i) *f preserves only $\subseteq$ and $\cup$:*

(1) $A_1 \subseteq A_2 \implies f(A_1) \subseteq f(A_2)$

(2) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$

(3) $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$

(4) $f(A_1 \setminus A_2) \supseteq f(A_1) \setminus f(A_2)$

Theorem

$$f : A \to B$$

$$A_1 \subseteq A_2 \implies f(A_1) \subseteq f(A_2)$$

对任意 b,

$$\begin{aligned}
& b \in f(A_1) && (1)\\
\iff & \exists a \in A_1.\ b = f(a) && (2)\\
\implies & \exists a \in A_2.\ b = f(a) && (3)\\
\iff & b \in f(A_2) && (4)
\end{aligned}$$

Theorem

$$f : A \to B$$

$$f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$$

对任意 b,

$$b \in f(A_1 \cap A_2) \tag{1}$$

$$\iff \exists a \in A_1 \cap A_2.\ b = f(a) \tag{2}$$

$$\implies (\exists a \in A_1.\ b = f(a)) \land (\exists a \in A_2.\ b = f(a)) \tag{3}$$

$$\iff b \in f(A_1) \land b = f(A_2) \tag{4}$$

$$\iff b \in f(A_1) \cap f(A_2) \tag{5}$$

Q : When does $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ hold?

f is injective.

Theorem

$$f : A \to B$$

$$f(A_1 \setminus A_2) \supseteq f(A_1) \setminus f(A_2)$$

对任意 b,

$$
\begin{align}
& b \in f(A_1) \setminus f(A_2) \tag{1}\\
\iff & b \in f(A_1) \land b \notin f(A_2) \tag{2}\\
\iff & (\exists a_1 \in A_1 .\ b = f(a_1)) \land (\forall a_2 \in A_2 .\ b \neq f(a_2)) \tag{3}\\
\implies & \exists a \in A_1 \setminus A_2 .\ b = f(a) \tag{4}\\
\iff & b \in f(A_1 \setminus A_2) \tag{5}
\end{align}
$$

Theorem (Properties of f and f^{-1})

$$f : A \to B \qquad \cancel{A_1, A_2 \subseteq A, B_1, B_2 \subseteq B}$$

(ii) *f^{-1} preserves $\subseteq$, $\cup$, $\cap$, and $\setminus$:*

(5) $B_1 \subseteq B_2 \implies f^{-1}(B_1) \subseteq f^{-1}(B_2)$

(6) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$

(7) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$

(8) $f^{-1}(B_1 \setminus B_2) = f^{-1}(B_1) \setminus f^{-1}(B_2)$

Theorem

$$f : A \to B$$

$$B_1 \subseteq B_2 \implies f^{-1}(B_1) \subseteq f^{-1}(B_2)$$

对任意 a,

$$a \in f^{-1}(B_1) \tag{1}$$

$$\iff f(a) \in B_1 \tag{2}$$

$$\implies f(a) \in B_2 \tag{3}$$

$$\iff a \in f^{-1}(B_2) \tag{4}$$

Theorem

$$f : A \to B$$

$$f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$$

对任意 a,

$$a \in f^{-1}(B_1 \cap B_2) \tag{1}$$

$$\Longleftrightarrow f(a) \in B_1 \cap B_2 \tag{2}$$

$$\Longleftrightarrow f(a) \in B_1 \land f(a) \in B_2 \tag{3}$$

$$\Longleftrightarrow a \in f^{-1}(B_1) \land a \in f^{-1}(B_2) \tag{4}$$

$$\Longleftrightarrow a \in f^{-1}(B_1) \cap f^{-1}(B_2) \tag{5}$$

Theorem (Properties of f and f^{-1})

$$f : A \to B$$

(iii) *f and f^{-1}:*

(9) $A_0 \subseteq A \implies A_0 \subseteq f^{-1}(f(A_0))$

(10) $B_0 \supseteq f(f^{-1}(B_0))$

Theorem

$$f : A \to B$$

$$A_0 \subseteq A \implies A_0 \subseteq f^{-1}(f(A_0))$$

对任意 b,

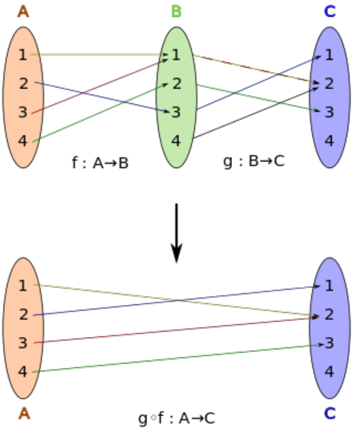
$$a \in A_0 \tag{1}$$

$$\implies a \in A_0 \cap A \tag{2}$$

$$\implies f(a) \in f(A_0) \tag{3}$$

$$\iff a \in f^{-1}(f(A_0)) \tag{4}$$

Function Composition



Definition (Composition)

$$f : A \to B \qquad g : C \to D$$

$$\mathrm{ran}(f) \subseteq C$$

The *composite function* $g \circ f : A \to D$ is defined as

$$(g \circ f)(x) = g(f(x))$$

Why not "$\exists b$" as below?

Definition (Composition)

The *composition* of relations R and S is the relation

$$R \circ S = \{(a, c) \mid \exists b : (a, b) \in S \land (b, c) \in R\}$$

Theorem (Associative Property for Composition)

$$f : A \to B \quad g : B \to C \quad h : C \to D$$

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Proof.

(i)

$$\mathrm{dom}(h \circ (g \circ f)) = \mathrm{dom}((h \circ g) \circ f)$$

(ii)

$$\forall x \in A.\ (h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x)$$

□

$$(h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x)$$

$$\begin{aligned} & (h \circ (g \circ f))(x) && (1) \\ = \; & h((g \circ f)(x)) && (2) \\ = \; & h(g(f(x))) && (3) \end{aligned}$$

$$\begin{aligned} & ((h \circ g) \circ f)(x) && (1) \\ = \; & ((h \circ g)(f(x))) && (2) \\ = \; & h(g(f(x))) && (3) \end{aligned}$$

Theorem

$$f: A \to B \qquad g: B \to C$$

(i) *If f, g are injective, then $g \circ f$ is injective.*

(ii) *If f, g are surjective, then $g \circ f$ is surjective.*

(iii) *If f, g are bijective, then $g \circ f$ is bijective.*

Theorem

$$f : A \to B \qquad g : B \to C$$

If f, g *are injective, then* $g \circ f$ *is injective.*

$$\forall a_1, a_2 \in A. \Big((g \circ f)(a_1) = (g \circ f)(a_2) \to a_1 = a_2 \Big)$$

$$\begin{aligned} & (g \circ f)(a_1) = (g \circ f)(a_2) && (1) \\ \iff & g(f(a_1)) = g(f(a_2)) && (2) \\ \implies & f(a_1) = f(a_2) && (3) \\ \implies & a_1 = a_2 && (4) \end{aligned}$$

Theorem

$$f : A \to B \qquad g : B \to C$$

(i) *If* $g \circ f$ *is injective, then* f *is injective.*

(ii) *If* $g \circ f$ *is surjective, then* g *is surjective.*

Theorem

$$f : A \to B \qquad g : B \to C$$

If $g \circ f$ is surjective, then g is surjective.

对任意 a_1, a_2,

$$g \circ f \text{ is surjective} \tag{1}$$

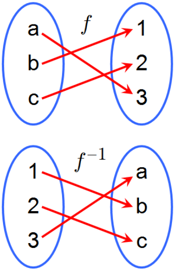
$$\iff \forall c \in C.\ \exists a \in A.\ (g \circ f)(a) = c \tag{2}$$

$$\iff \forall c \in C.\ \exists a \in A.\ g(f(a)) = c \tag{3}$$

$$\implies \forall c \in C.\ \exists b \in B.\ g(b) = c \tag{4}$$

$$\iff g \text{ is surjective} \tag{5}$$

Inverse Functions



$$f : A \to B$$

What is the condition for f^{-1} to be a function from B to A?

$$f^{-1} : B \to A$$

Definition (反函数 (Inverse Function))

Let $f : A \to B$ be a function.

The *inverse* of f is a function from B to A, denoted $f^{-1} : B \to A$

if f is bijective.

We call f^{-1} the *inverse function* of f.

$$f \text{ is bijective} : \boxed{f(x) = y \iff f^{-1}(y) = x}$$

Theorem

Suppose that $f : A \to B$ is bijective. Then, its inverse function $f^{-1} : B \to A$ is unique.

By Contradiction

$$f(x) = y \iff f^{-1}(y) = x$$

$$f(x) = y \iff g(y) = x$$

$$\forall y \in B.\ f^{-1}(y) = g(y)$$

examples

Theorem

$$\boxed{f : A \to B \ \textit{is bijective}}$$

(i) $f \circ f^{-1} = I_B$

(ii) $f^{-1} \circ f = I_A$

(iii) f^{-1} *is bijective*

(iv) $g : B \to A \land f \circ g = I_B \implies g = f^{-1}$

(v) $g : B \to A \land g \circ f = I_A \implies g = f^{-1}$

The ways to find/check f^{-1}.

Theorem

$$f : A \to B \text{ is bijective}$$

$$f \circ f^{-1} = I_B$$

对任意 $b \in B$,

$$(f \circ f^{-1})(b) = f(f^{-1}(b))$$

$$\text{Suppose that } a = f^{-1}(b)$$

$$\boxed{a = f^{-1}(b) \iff f(a) = b}$$

$$(f \circ f^{-1})(b) = f(f^{-1}(b)) = f(a) = b$$

Theorem

$$f : A \to B \textit{ is bijective}$$

$$g : B \to A \land f \circ g = I_B \implies g = f^{-1}$$

$$g = (f^{-1} \circ f) \circ g = f^{-1} \circ (f \circ g) = f^{-1} \circ I_B = f^{-1}$$

Theorem (Inverse of Composition)

$$\textit{Both } f : A \to B \textit{ and } g : B \to C \textit{ are bijective}$$

(i) $g \circ f$ *is bijective*

(ii) $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Proof for (ii).

It suffices to check either one of the following identities:

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = I_A$$

$$(g \circ f) \circ (f^{-1} \circ g^{-1}) = I_C$$

□

Theorem

$$f : A \to B \quad g : B \to A$$

(iii) $f \circ g = I_B \land g \circ f = I_A \implies g = f^{-1}$

You do *not* know that f is bijective.

You need to check both identities.

Use two previous theorems


Thank
You!

Your opinon

Matters

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