(十二) 图论: 匹配与网络流
(Matching and Network Flow)

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3 Theorems + 1 Algorithm

To maximize the size of a mathematical structure $\mathcal{S}$ in G

To maximize the size of a mathematical structure $\mathcal{S}$ in G

Theorem

$\mathcal{S}$ is maximum iff G does not contain $\mathcal{S}$-augmenting objects.

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Algorithm

Repeatedly finding $\mathcal{S}$-augmenting objects until no more ones exist.

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Theorem (Weak Duality Theorem)

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Theorem (Strong Duality Theorem)

The size of a maximum $\mathcal{S}$ $=$ *The size of a minimum* $\mathcal{S}'$


let's get
married
today

The Marriage Problem (Philip Hall, 1935)

If there is a finite set of girls, each of whom knows several boys, under what conditions can all the girls marry boys in such a way that each girl marries a boy that she knows?

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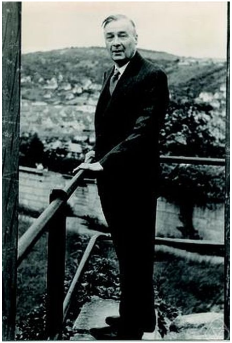
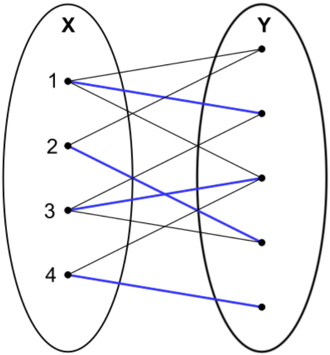


Philip Hall (1904 $\sim$ 1982)


X
Y
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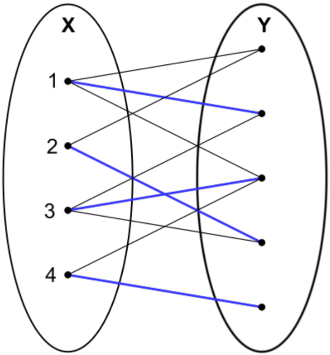
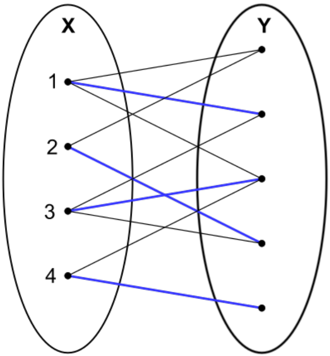


Definition (Matching (匹配))

A matching in a graph G is a set of edges with no shared endpoints.

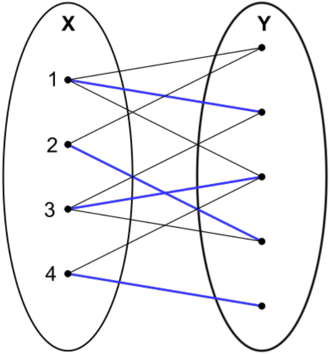


Definition (X-Perfect Matching (X-Saturating Matching))

Let $G = (X, Y, E)$ be a bipartite graph.

An X-perfect matching of G is a matching which covers each vertex in X.

$$|X| \leq |Y|$$



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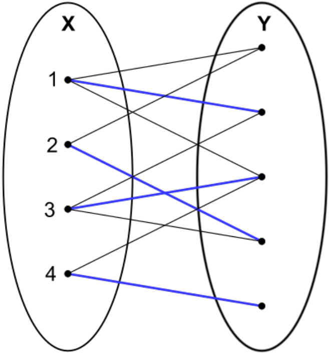

$$\forall W \subseteq X.\ |W| \leq |N(W)|$$

Theorem (Hall Theorem; 1935)

Let $G = (X, Y, E)$ be a bipartite graph. There is an X-perfect matching of G iff

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Induction Hypothesis: Suppose that it holds if $\left|X\right| < m$.

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Theorem (Hall Theorem; 1935)

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Definition (M-augmenting Paths)

An M-augmenting path is an M-alternating path whose endpoints are unsaturated by M.

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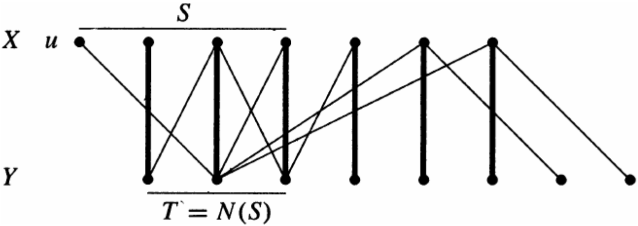

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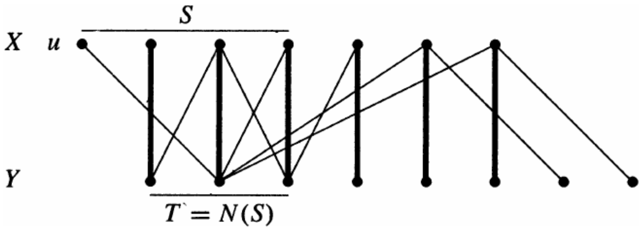

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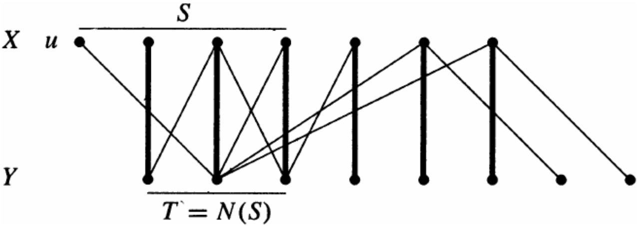

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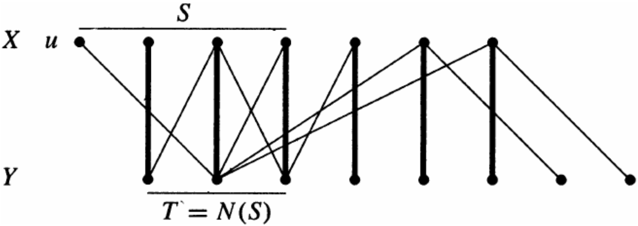

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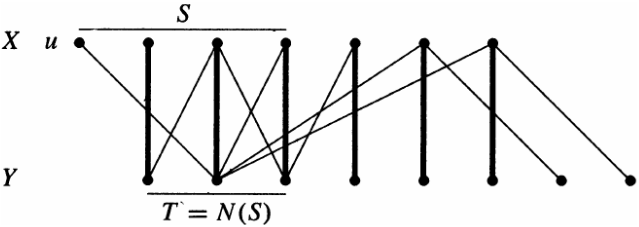
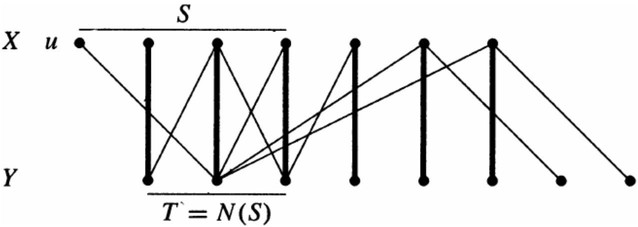
Consider all *M-alternating paths* starting from u.



$T \triangleq$ the set of vertices in Y reachable from u by M-alternating paths.



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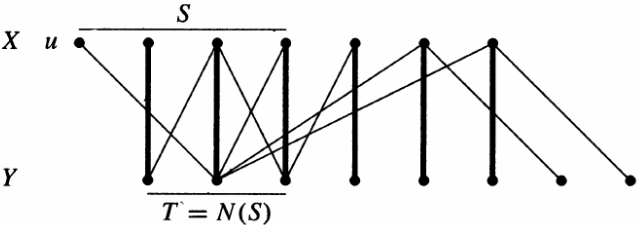

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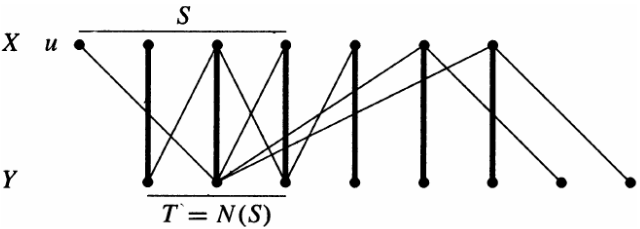
We will show that

$$T = N(S) \wedge \left|T\right| = \left|S - \{u\}\right|$$



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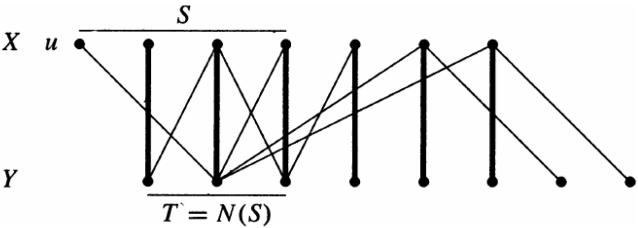

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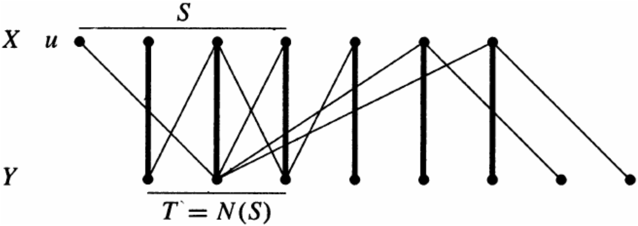

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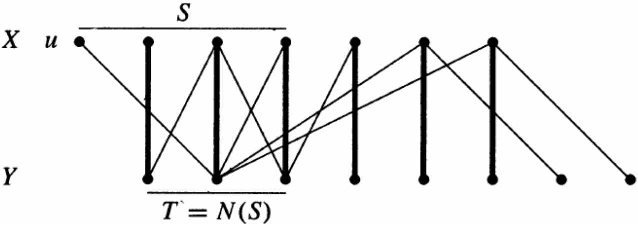

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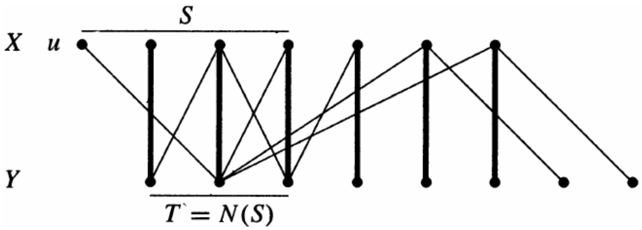

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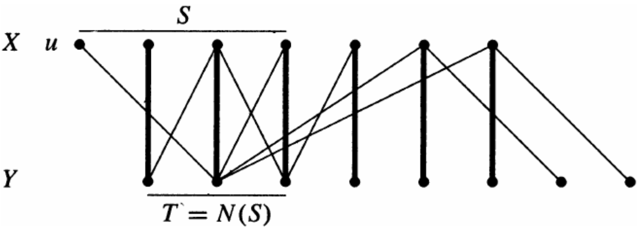

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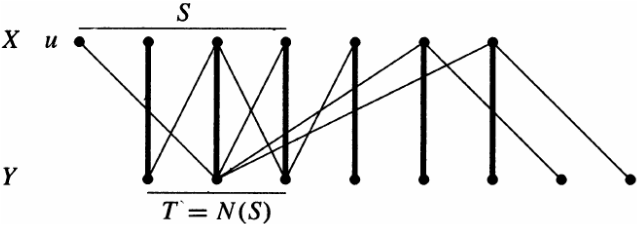

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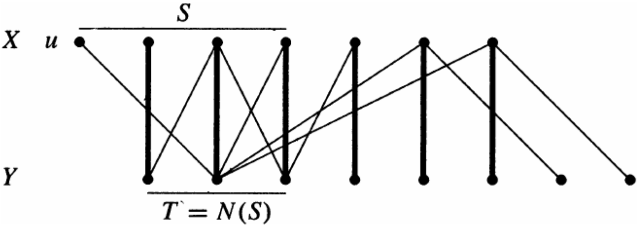

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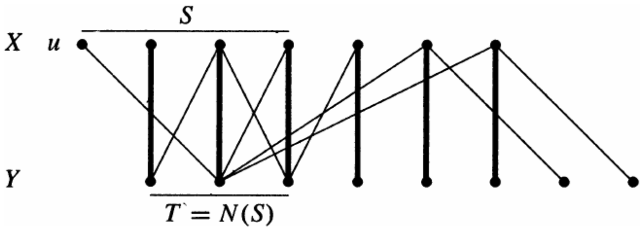

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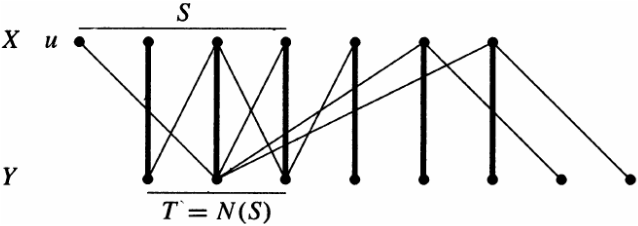

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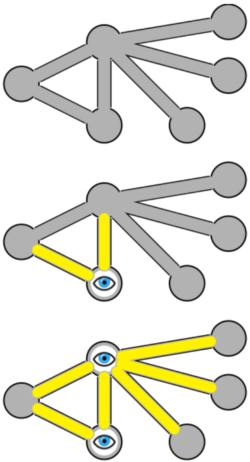
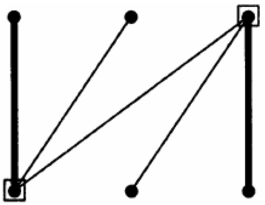
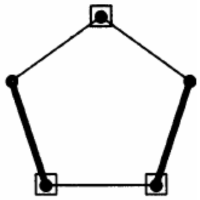
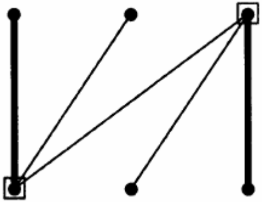
Definitions (Vertex Cover (点覆盖))

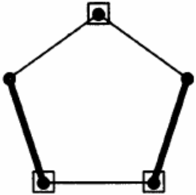
A vertex cover of a graph G is a set $Q \subseteq V(G)$ that covers all edges.

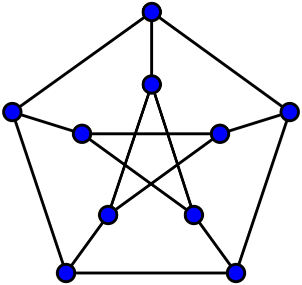












Theorem (Weak Duality Theorem (弱对偶定理))

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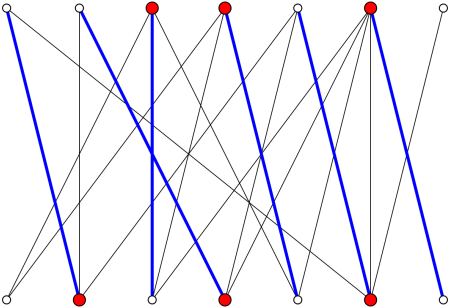

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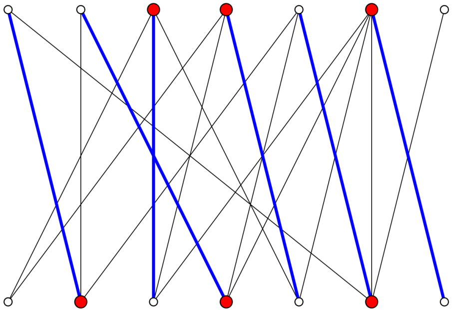

Theorem (König (1931), Egerváry (1931))

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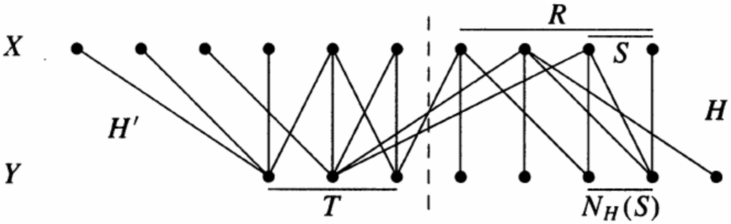

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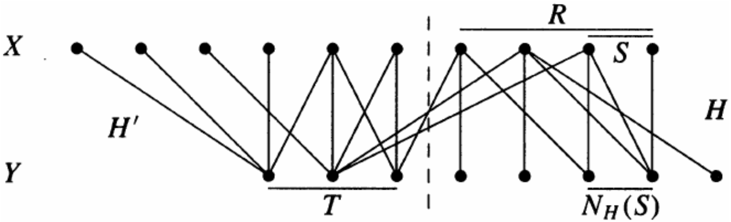

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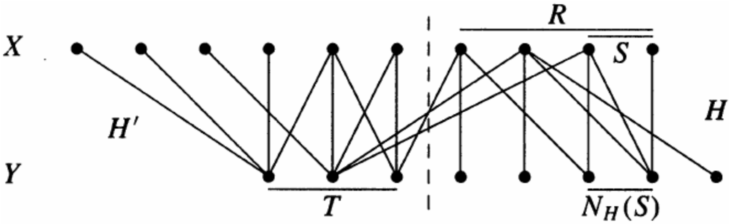

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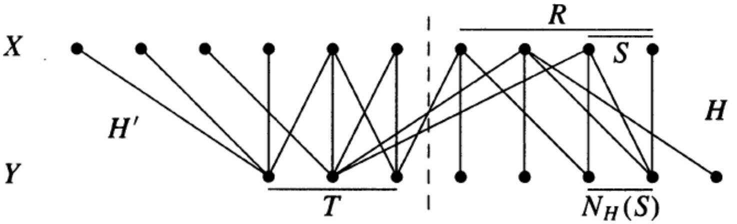

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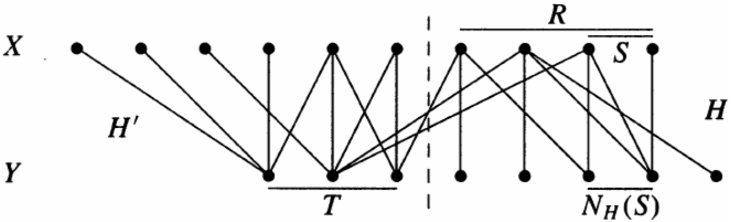

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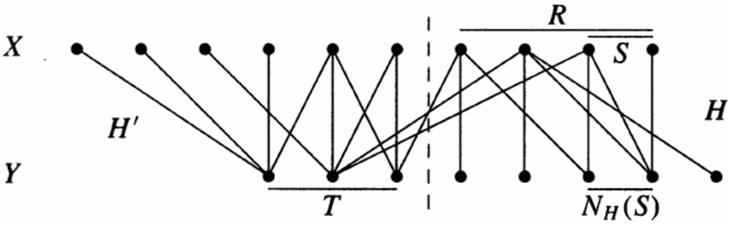
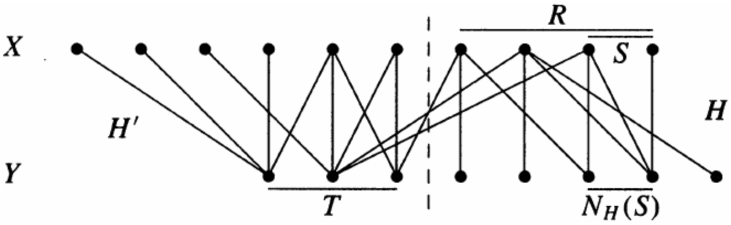
$\boxed{H \text{ has a } R\text{-perfect matching and } H' \text{ has a } T\text{-perfect matching.}}$

H has a R-perfect matching and H' has a T-perfect matching.

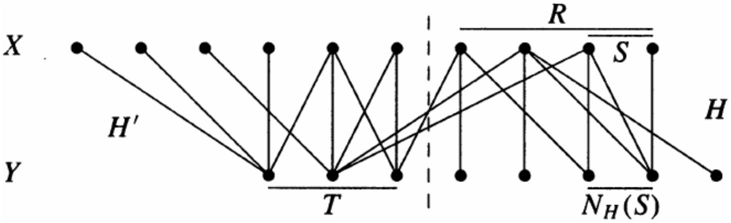


H has a R-perfect matching and H' has a T-perfect matching.



By contradiction.

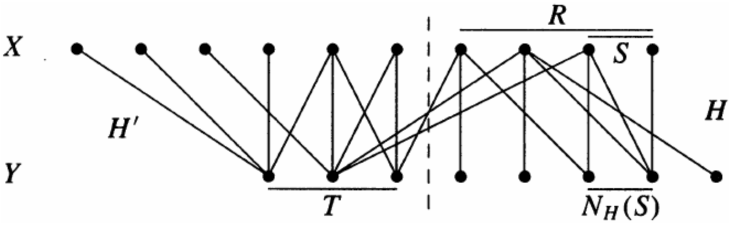
H has a R-perfect matching and H' has a T-perfect matching.



By contradiction.

$$\exists S \subseteq R. \left|N_H(S)\right| < \left|S\right|$$

H has a R-perfect matching and H' has a T-perfect matching.



By contradiction.

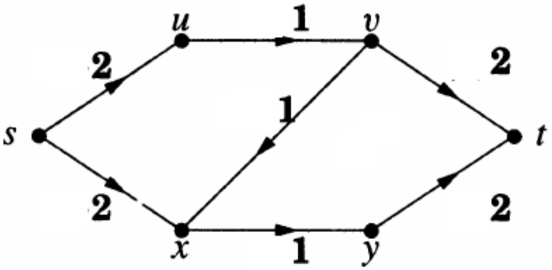
$$\exists S \subseteq R.\ \left|N_H(S)\right| < \left|S\right|$$

$$T \cup (R - S + N_H(S)) \text{ is a smaller vertex cover than } Q$$

Definition (Network (网络))

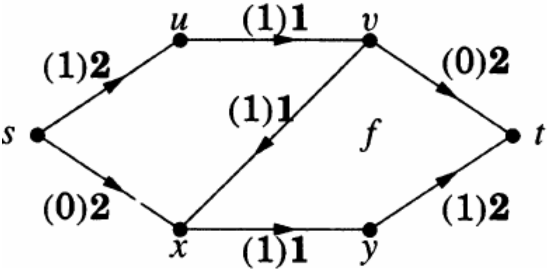
A network is a digraph with

- a distinguished source vertex s,
- a distinguished sink vertex t,
- a capacity $c(e) \geq 0$ on each edge e



Definition (Flow (流))

A flow f is a function that assigns a value $f(e)$ to each edge e.



Definition (Feasible Flow (可行流))

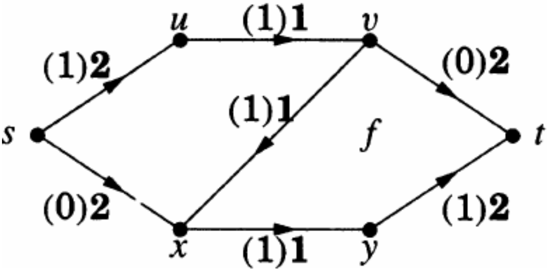
A flow f is feasible if it satisfies

Capacity Constraints:

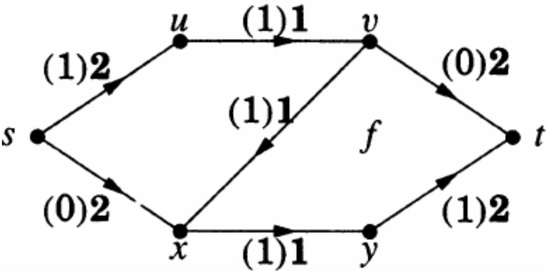
$$\forall e \in E.\ 0 \le f(e) \le c(e)$$

Flow Conservation:

$$\forall v \in V.\ f^+(v) = f^-(v)$$



$$f^{+}(v) = \sum_{(v,w)\,\in\, E} f(v,w) \qquad f^{-}(v) = \sum_{(u,v)\,\in\, E} f(u,v)$$



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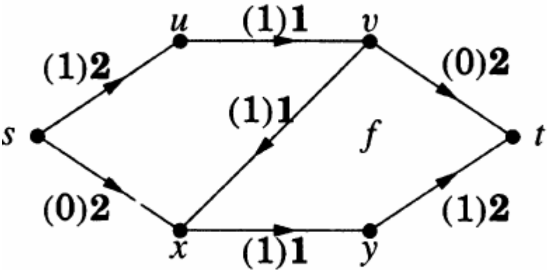
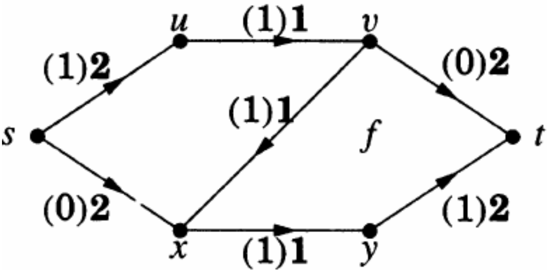
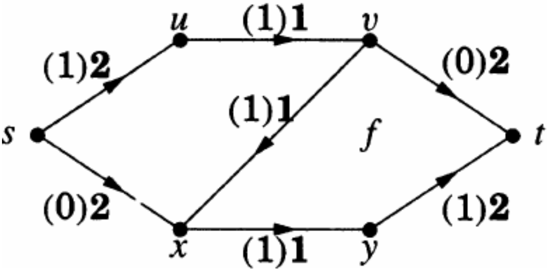


$$f^+(U) = \sum_{u\in U, v\in \overline{U}, (u,v)\in E} f(u,v)$$

$$f^+(v) = \sum_{(v,w) \in E} f(v,w) \qquad f^-(v) = \sum_{(u,v) \in E} f(u,v)$$



$$f^+(U) = \sum_{u \in U, v \in \overline{U}, (u,v) \in E} f(u,v) \qquad f^-(U) = \sum_{v \in \overline{U}, u \in U, (v,u) \in E} f(v,u)$$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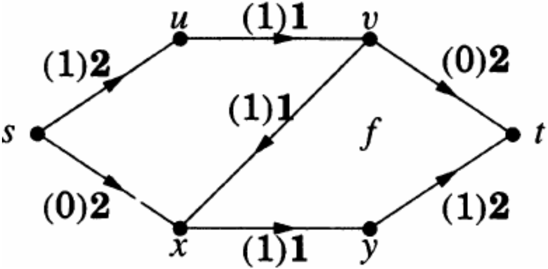
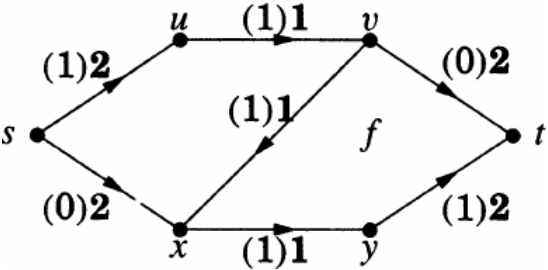
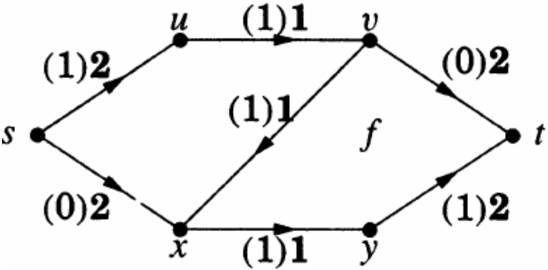


$$\forall U \subseteq (V - \{s,t\}).\ f^{+}(U) = f^{-}(U)$$



$$\forall U \subseteq (V - \{s, t\}).\ f^{+}(U) = f^{-}(U)$$

$$s \in U \wedge t \notin U \implies f^{+}(U) - f^{-}(U) =$$



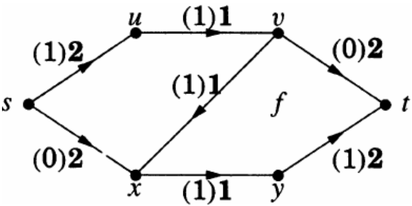
$$\forall U \subseteq (V - \{s, t\}).\ f^{+}(U) = f^{-}(U)$$

$$s \in U \wedge t \notin U \implies f^{+}(U) - f^{-}(U) = f^{+}(s)$$

Definition (Value (值))

The value $\mathsf{val}(f)$ of a flow f is

$$\mathsf{val}(f) = f^{-}(t) = f^{+}(s).$$



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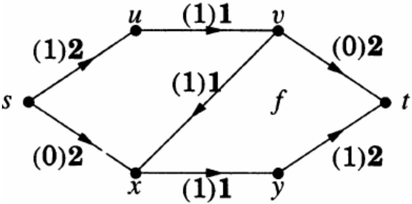

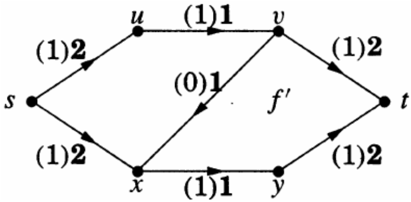

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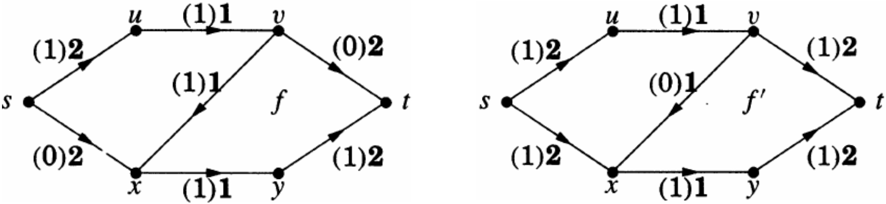
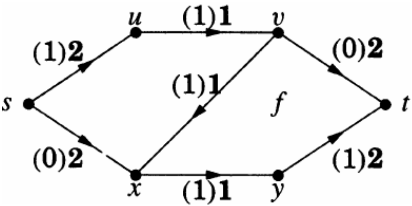
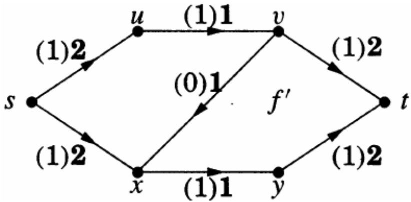


Definition (Maximum Flow (最大流))

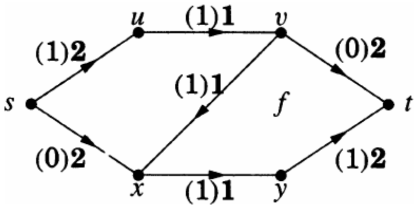
A maximum flow is a feasible flow of maximum value.

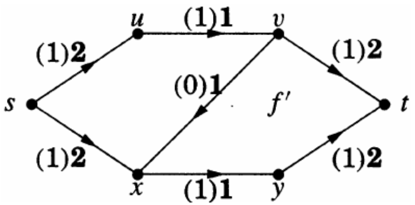
$$s - x - v - t$$





$$s - x - v - t$$





Definition (f-augmenting Paths (增广路径))

When f is a feasible flow, an f-augmenting path is a $s \sim t$ path P in the underlying graph such that for each edge $e \in E(P)$,

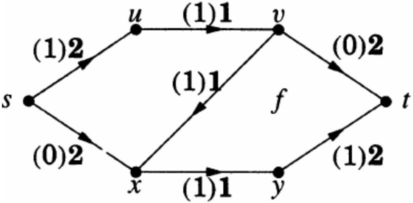
(a) if P follows e in the forward direction, then $f(e) < c(e)$;

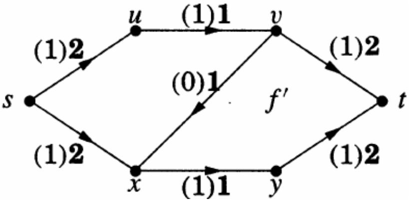
(b) if P follows e in the backward direction, then $f(e) > 0$.

Definition (f-augmenting Paths)

Let P be an f-augmenting path.

$$\epsilon(e) = \begin{cases} c(e) - f(e) & \text{if } e \text{ is forward on } P \\ f(e) & \text{if } e \text{ is backward on } P \end{cases}$$





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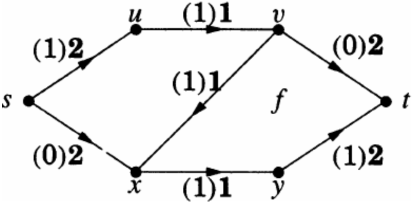
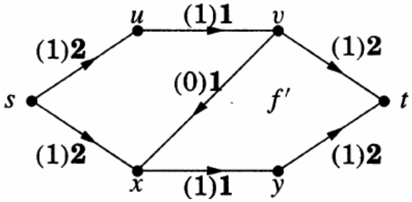
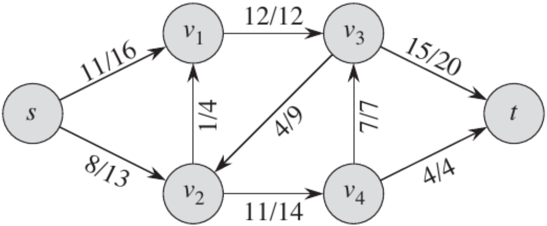




An f-augmenting path leads to a flow with larger value.

$$\min_{e \in E(P)} \epsilon(e)$$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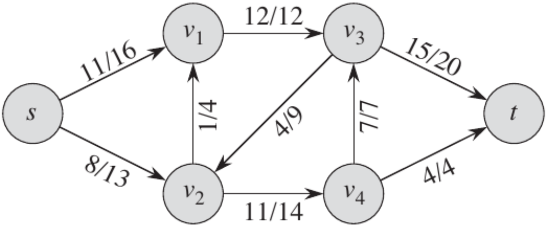

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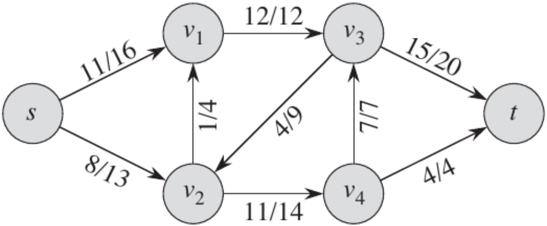

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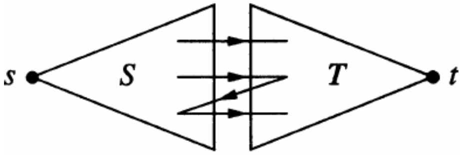
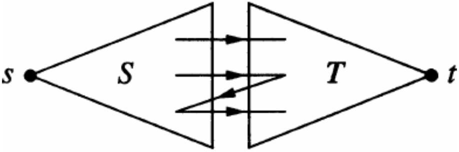
Definition (Source/Sink Cut (割))

In a network, a source/sink cut $[S, T]$ consists of the edges from a source set S to a sink set T, where

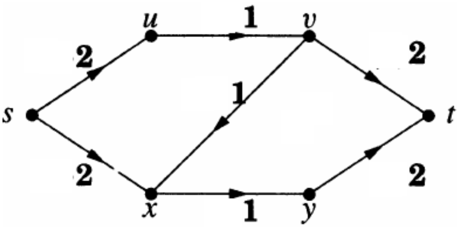
$$(T = V - S) \wedge (s \in S) \wedge (t \in T)$$





Definition (Capacity of Cut (割的容量))

$$\mathsf{cap}(S,T) = \sum_{u \in S, v \in T, uv \in E} c(u,v)$$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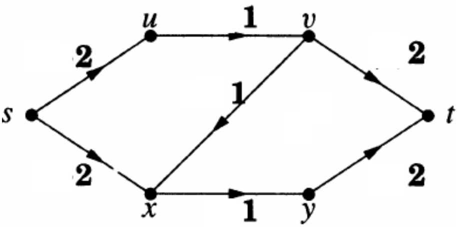
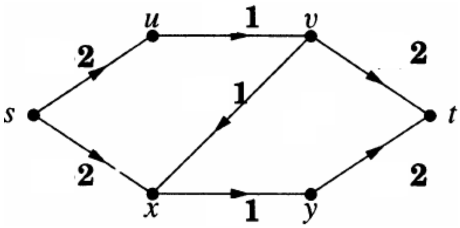


Definition (Minimum Cut (最小割))

A minimum cut is a cut of minimum value.



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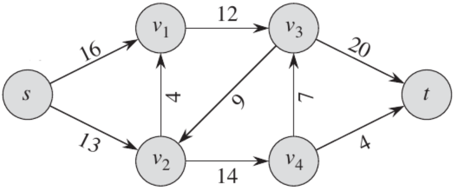
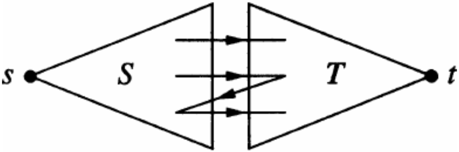


Theorem (Weak Duality (弱对偶定理))

Let f be any feasible flow and $[S, T]$ be any source/sink cut.

$$\mathit{val}(f) \leq \mathit{cap}(S, T).$$



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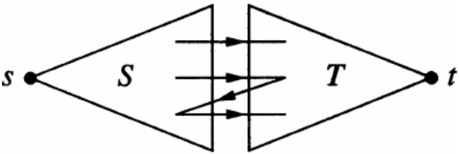

$$\mathrm{val}(f) = f^{+}(S) - f^{-}(S)$$

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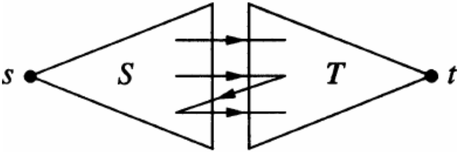

$$\mathrm{val}(f) = f^+(S) - f^-(S) \leq f^+(S)$$

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Let f be any feasible flow and $[S,T]$ be any source/sink cut.

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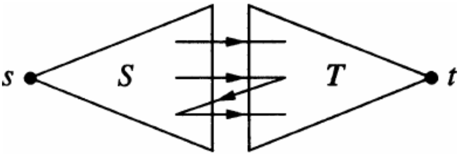

$$\mathrm{val}(f) = f^{+}(S) - f^{-}(S) \le f^{+}(S) \le \mathrm{cap}(S,T)$$

Lemma

$$\max_{f} \mathit{val}(f) \le \min_{[S,T]} \mathit{cap}(S,T)$$

Lemma

$$\max_f val(f) \le \min_{[S,T]} cap(S,T)$$

What if $\mathsf{val}(f) = \mathsf{cap}(S,T)$ for some flow f and some cut $[S,T]$?

Lemma

$$\max_{f} \mathit{val}(f) \leq \min_{[S,T]} \mathit{cap}(S,T)$$

What if $\mathsf{val}(f) = \mathsf{cap}(S,T)$ for some flow f and some cut $[S,T]$?

f is maximum and $[S,T]$ is minimum

$$\mathsf{val}(f) = f^+(S) - f^-(S) = f^+(S) = \mathsf{cap}(S, T)$$

$$\mathsf{val}(f) = f^{+}(S) - f^{-}(S) = f^{+}(S) = \mathsf{cap}(S, T)$$

$$\boxed{f^{-1}(S) = 0 \wedge f^{+}(S) = \mathsf{cap}(S, T)}$$

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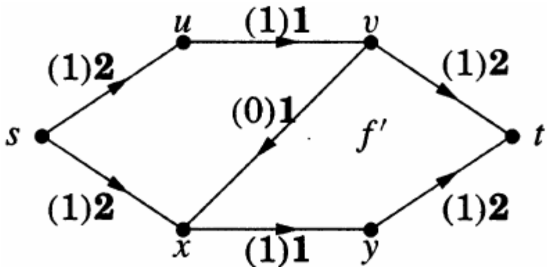

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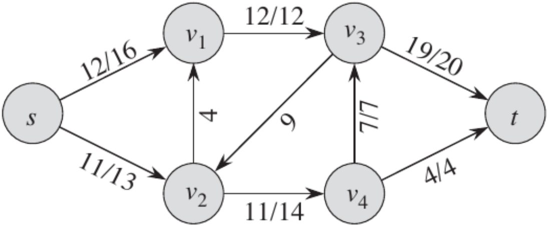


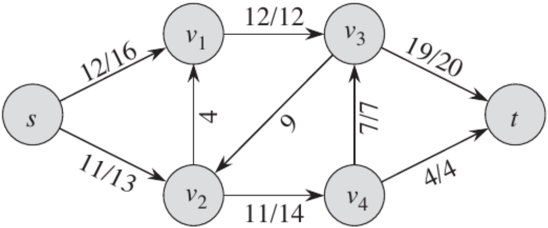
Theorem (Max-flow Min-cut Theorem (Ford and Fulkerson; 1956))

$$\max_{f} \mathit{val}(f) = \min_{[S,T]} \mathit{cap}(S,T)$$

(Strong Duality)

L. R. Ford Jr. $(1927 \sim 2017)$

D. R. Fulkerson $(1924 \sim 1976)$



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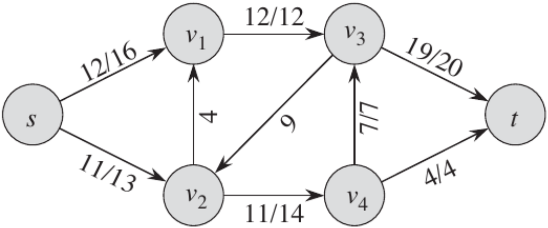
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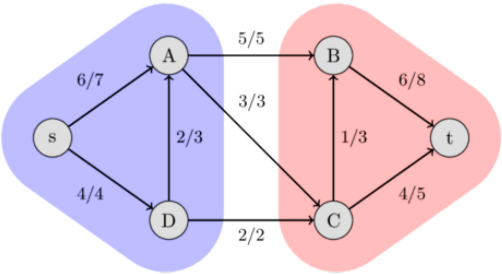
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Theorem

A feasible flow f is maximum iff there are no f-augmenting paths.

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We construct a cut $[S, T]$ with $\mathsf{val}(f) = \mathsf{cap}(S, T)$.

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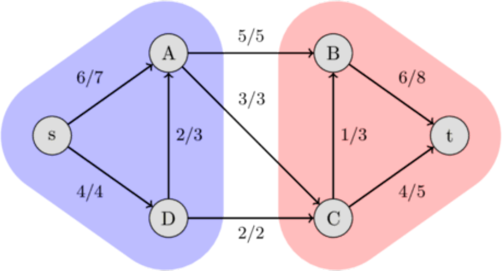
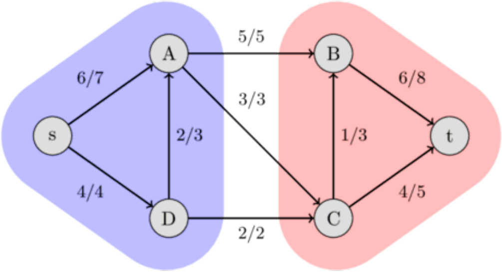


$S \triangleq \{$the vertices reachable from s along partial f-augmenting paths$\}$

$$S \triangleq \{\text{the vertices reachable from } s \text{ along } \textcolor{red}{\text{partial}}\ f\text{-augmenting paths}\}$$

$$T \triangleq V - S$$



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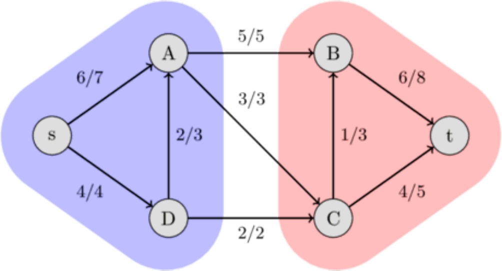

$$f^{-1}(S) = 0 \wedge f^{+}(S) = \mathsf{cap}(S, T)$$

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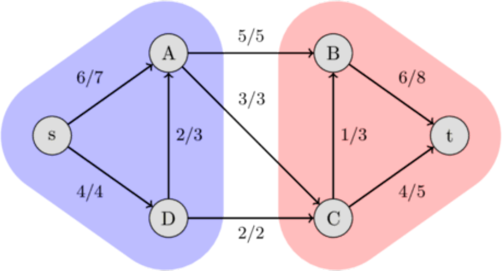

$$f^{-1}(S) = 0 \wedge f^{+}(S) = \mathsf{cap}(S, T)$$

$$\mathsf{val}(f) = f^{+}(S) - f^{-}(S) = f^{+}(S) = \mathsf{cap}(S, T)$$

The Ford-Fulkerson Method

Repeatedly finding f-augmenting paths until no more ones exist.

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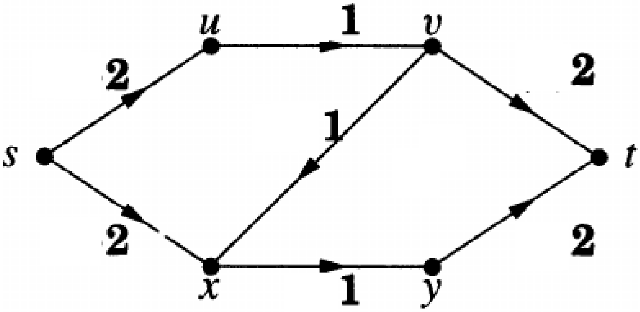

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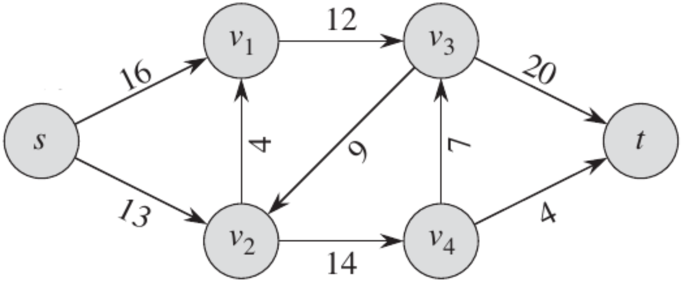

The Edmonds-Karp Algorithm

Using BFS (Breadth-first Search) to find f-augmenting paths.

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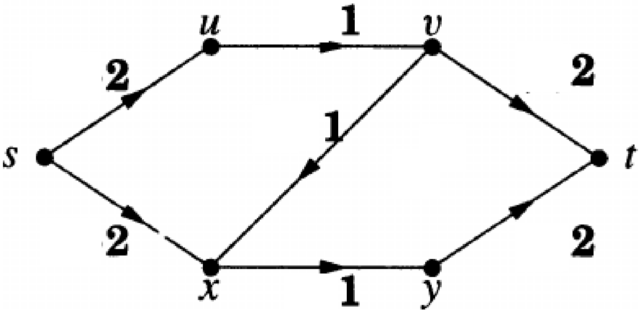

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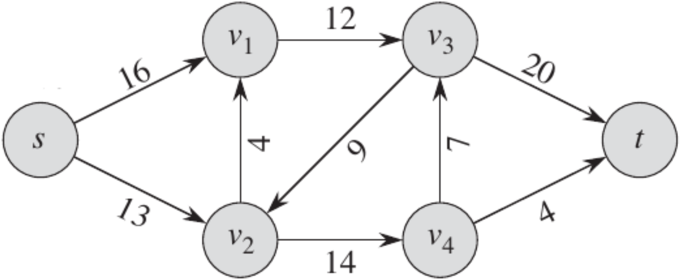

Theorem (Hall Theorem; 1935)

There is an X-perfect matching of G iff

$$\forall W \subseteq X. \left|W\right| \le \left|N_G(W)\right|$$

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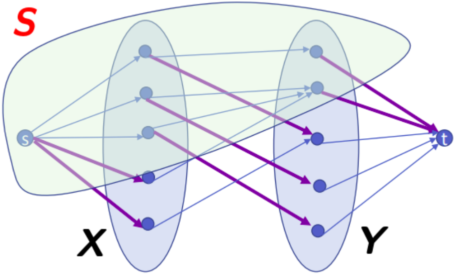

$\forall x \in X.\ c(s,x) = 1 \quad \forall y \in Y.\ c(y,t) = 1 \quad \forall x \in X, y \in Y.\ c(x,y) = \infty$

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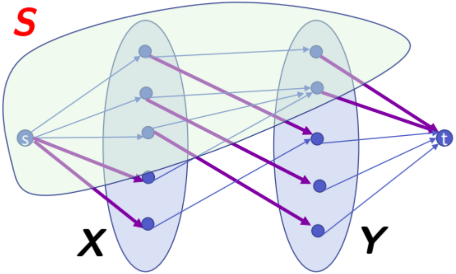

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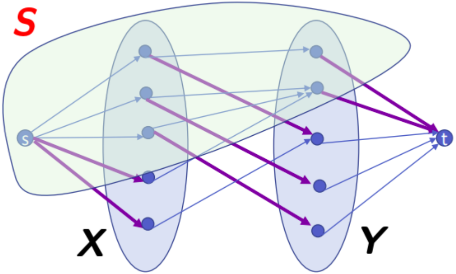
We need to show that $\max_f \mathsf{val}(f) = |X|$.

We need to show that $\min\limits_{[S,\overline{S}]} \mathsf{cap}(S,\overline{S}) = \left|X\right|$.



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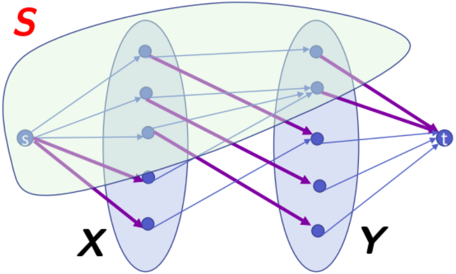

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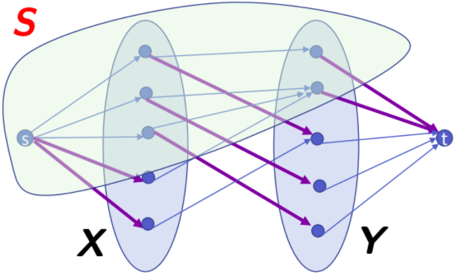
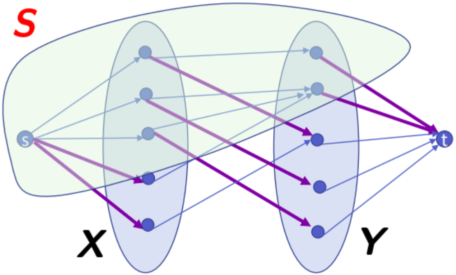


$$\min_{[S,\overline{S}]} \mathsf{cap}(S,\overline{S}) \le \left|X\right|$$

Therefore, we need to show that $\min\limits_{[S,\overline{S}]} \mathsf{cap}(S,\overline{S}) \ge \left|X\right|$.

Let $[S, \overline{S}]$ be a minimum cut. We need to show that $\mathsf{cap}(S, \overline{S}) = \left|X\right|$.



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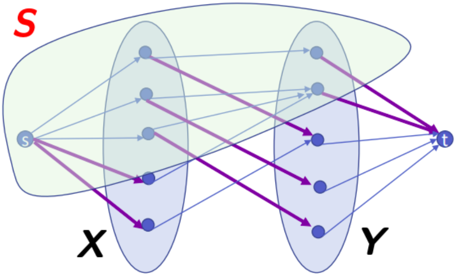

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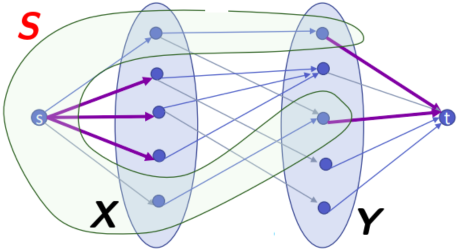

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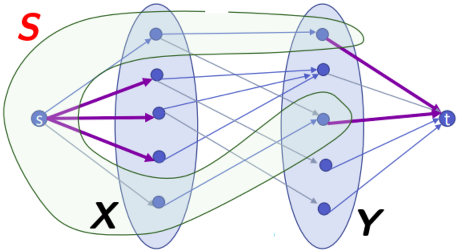
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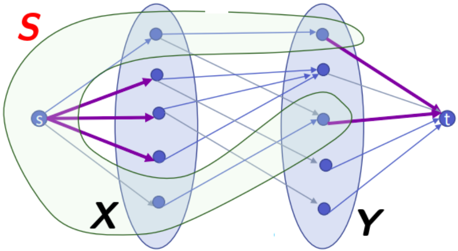
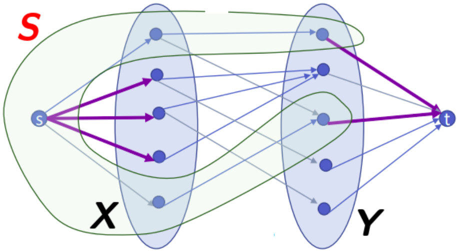


$$\mathsf{cap}(S, \overline{S}) = \sum_{u \in S, v \in \overline{S}} c(x, y)$$



$$\mathsf{cap}(S,\overline{S}) = \sum_{u\in S, v\in\overline{S}} c(x,y)$$
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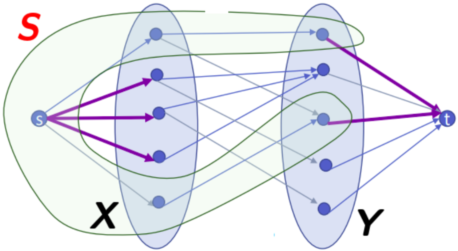

$$
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&= \sum_{v \in \overline{S} \cap X} c(s, v) + \sum_{u \in S \cap Y} c(u, t) \\
&= \left|X\right| - \left|S \cap X\right| + \left|S \cap Y\right|
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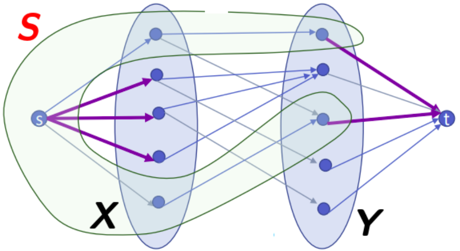

$$
\begin{aligned}
\mathsf{cap}(S,\overline{S}) &= \sum_{u\in S, v\in \overline{S}} c(x,y) \\
&= \sum_{v\in \overline{S}\cap X} c(\textcolor{red}{s}, v) + \sum_{u\in S\cap Y} c(u, \textcolor{red}{t}) \\
&= \left|X\right| - \left|S\cap X\right| + \left|S\cap Y\right| \\
&\textcolor{red}{\geq} \left|X\right| - \left|S\cap X\right| + \textcolor{red}{\left|N(S\cap X)\right|}
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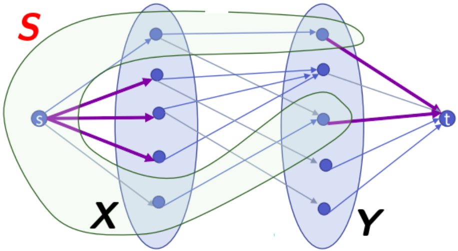

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If G is a bipartite graph, then the maximum size of a mathching in G equals the minimum size of a vertex cover of G

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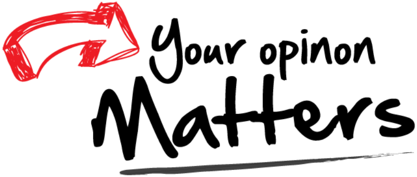
Thank You!



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