(十三) 群论: 群的基本概念 (What are Groups?)

魏恒峰

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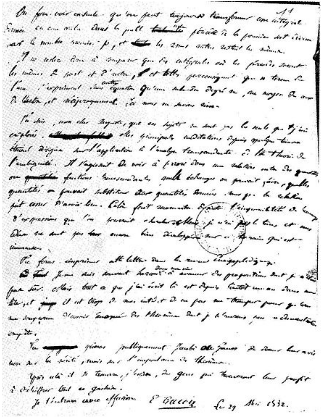
“这里需要补充说明，可是我没有时间了！”

Évariste Galois
(伽罗瓦; 1811 ∼ 1832)



May 29, 1832

“论五次方程的代数解法问题” (1929)

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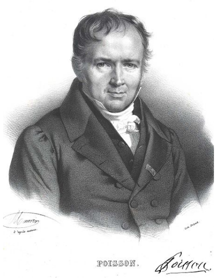

Siméon Denis Poisson
(1781 ∼ 1840)

"Ask Jacobi or Gauss publicly to give their opinion,

not as to the truth, but as to the importance of these theorems."

“Is there a formula for the roots of a ≥ 5 degree polynomial equation

in terms of its coefficients, using only $+, -, \times, \div, \sqrt[r]{\ }$?

$$x^3 + px + q = 0$$

$$x_1 = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$$

$$x_2 = \omega\sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \omega^2\sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}$$

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其中$\omega = \frac{-1+\sqrt{3}i}{2}$。

Girolamo Cardano
(1501 ~ 21/09/1576)

对于一元四次方程

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

记

$$\begin{cases} \Delta_1 = c^2 - 3bd + 12ae \\ \Delta_2 = 2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace \end{cases}$$

并记

$$\Delta = \frac{\sqrt[3]{2}\Delta_1}{3a\sqrt[3]{\Delta_2 + \sqrt{-4\Delta_1^3 + \Delta_2^2}}} + \frac{\sqrt[3]{\Delta_2 + \sqrt{-4\Delta_1^3 + \Delta_2^2}}}{3\sqrt[3]{2}a}$$

则有

$$\begin{cases}
x_1 = -\dfrac{b}{4a} - \dfrac{1}{2}\sqrt{\dfrac{b^2}{4a^2} - \dfrac{2c}{3a} + \Delta} - \dfrac{1}{2}\sqrt{\dfrac{b^2}{2a^2} - \dfrac{4c}{3a} - \Delta - \dfrac{-\dfrac{b^3}{a^3} + \dfrac{4bc}{a^2} - \dfrac{8d}{a}}{4\sqrt{\dfrac{b^2}{4a^2} - \dfrac{2c}{3a} + \Delta}}} \\
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x_3 = -\dfrac{b}{4a} + \dfrac{1}{2}\sqrt{\dfrac{b^2}{4a^2} - \dfrac{2c}{3a} + \Delta} - \dfrac{1}{2}\sqrt{\dfrac{b^2}{2a^2} - \dfrac{4c}{3a} - \Delta + \dfrac{-\dfrac{b^3}{a^3} + \dfrac{4bc}{a^2} - \dfrac{8d}{a}}{4\sqrt{\dfrac{b^2}{4a^2} - \dfrac{2c}{3a} + \Delta}}} \\
x_4 = -\dfrac{b}{4a} + \dfrac{1}{2}\sqrt{\dfrac{b^2}{4a^2} - \dfrac{2c}{3a} + \Delta} + \dfrac{1}{2}\sqrt{\dfrac{b^2}{2a^2} - \dfrac{4c}{3a} - \Delta + \dfrac{-\dfrac{b^3}{a^3} + \dfrac{4bc}{a^2} - \dfrac{8d}{a}}{4\sqrt{\dfrac{b^2}{4a^2} - \dfrac{2c}{3a} + \Delta}}}
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Theorem (Abel-Ruffini Thoerem)

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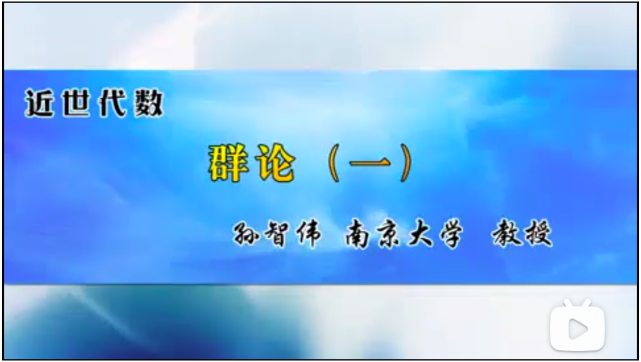
Niels Henrik Abel (1802 ~ 1929)

Theorem (Galois Theorem)

An equation is solvable in terms of radicals iff the Galois group of its splitting field is solvable.



https://www.bilibili.com/video/BV1Ex411k7wk?share_source=copy_web

"我看出了 *Galois* 用来证明这个美妙定理的方法是完全正确的。在那个瞬间，我体验到一种强烈的愉悦。"

— *J. Liouville (刘维尔; 1846)*

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Identity (单位元):

$$\exists e \in G.\ \forall a \in G.\ e * a = a * e = a$$

Inverse (逆元): Let e be the identity of G.

$$\forall a \in G.\ \exists b \in G.\ a * b = b * a = e$$

The inverse of a is denoted a^{-1}.

$$\forall n \in \mathbb{Z}^+ .\ a^n \triangleq \underbrace{a * a * \cdots * a}_{\# = n}$$

$$a^0 \triangleq e$$

$$a^{-n} \triangleq (a^{-1})^n$$

Definition (Commutative Group (交换群); Abelian Group (阿贝尔群))

Let $(G, *)$ be a group. If $*$ is commutative,

$$\forall a, b \in G.\ a * b = b * a,$$

then $(G, *)$ is a commutative group.

$$(\mathbb{Z}, +)$$

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$$(1, -1, \mathbf{i}, -\mathbf{i})$$

Group of n-th Roots of Unity (n 次单位根群)

$$
\begin{aligned}
U_n &= \{z \in \mathbb{C} \mid z^n = 1\} \\
&= \{\cos \frac{2k\pi}{n} + \mathbf{i} \sin \frac{2k\pi}{n} \mid k = 0, 1, \ldots, n-1\}
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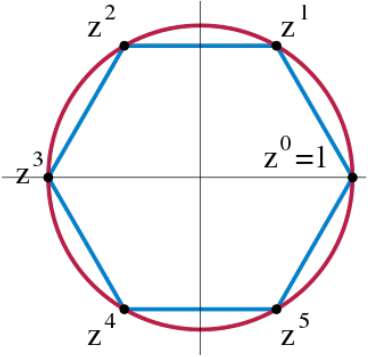

Quaternion Group (四元数群)

$$(1, i, j, k, -1, -i, -j, -k)$$

×	e	$\bar{e}$	i	$\bar{i}$	j	$\bar{j}$	k	$\bar{k}$
e	e	$\bar{e}$	i	$\bar{i}$	j	$\bar{j}$	k	$\bar{k}$
$\bar{e}$	$\bar{e}$	e	$\bar{i}$	i	$\bar{j}$	j	$\bar{k}$	k
i	i	$\bar{i}$	$\bar{e}$	e	k	$\bar{k}$	$\bar{j}$	j
$\bar{i}$	$\bar{i}$	i	e	$\bar{e}$	$\bar{k}$	k	j	$\bar{j}$
j	j	$\bar{j}$	$\bar{k}$	k	$\bar{e}$	e	i	$\bar{i}$
$\bar{j}$	$\bar{j}$	j	k	$\bar{k}$	e	$\bar{e}$	$\bar{i}$	i
k	k	$\bar{k}$	j	$\bar{j}$	$\bar{i}$	i	$\bar{e}$	e
$\bar{k}$	$\bar{k}$	k	$\bar{j}$	j	i	$\bar{i}$	e	$\bar{e}$

Cayley Table

$$i^2 = j^2 = k^2 = 1 \qquad ij = k, ji = -k, jk = i, kj = -i, ki = j, ik = -j$$

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(6) $\forall a, b \in G.\ \exists!\ x \in G.\ ax = b \land ya = b.$

Additive Group of Integers Modulo m (模 m 剩余类加群)

$$(\mathbb{Z}_m = \{0, 1, \ldots, m-1\}, +_m)$$

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$$(\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}, \times_6)$$

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$$(u, m) = 1 \qquad ua = au = au + mv = 1 \pmod m$$

When p is a prime,

$$\mathbb{Z}_p^* \triangleq U(p) = \{1, 2, \ldots, p-1\}$$

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$$U(20) = \{1, 3, 7, 9, 11, 13, 17, 19\}$$

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Theorem (Euler Theorem (1736))

Let $m \in \mathbb{Z}^+$ *and* $a \in \mathbb{Z}$. *If* $(a, m) = 1$, *then*

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$$7^{222} \equiv 7^{4 \times 55 + 2} \equiv 7^2 \equiv 9 \mod 10$$

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If $H \subset G$, then H is a proper subgroup (真子群).

$$(H = \{mz \mid z \in \mathbb{Z}\}, +) \leq (\mathbb{Z}, +)$$

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$$H = \{1, 2, 4\} \le G = \mathbb{Z}_7^* = \{1, 2, 3, 4, 5, 6\}$$

Theorem

Suppose that $H \leq G$.

(1) *The identity of* H *is the same with that of* G.

$$e_H = e_G$$

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$$e_H e_H = e_H = e_H e_G \implies e_H = e_G$$

$$a a_H^{-1} = e_H = e_G = a a_G^{-1} \implies a_H^{-1} = a^{-1}(G)$$

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$$a^{-1} = ea^{-1} \in H$$

$$ab = a(b^{-1})^{-1} \in H$$

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$$H_1 \cup H_2?$$

Center (中心)

Let G be a group. Let

$$C(G) \triangleq \{g \in G \mid gx = xg, \forall x \in G\}.$$

Then $C(G) \leq G$.

Definition (Isomorphism (同构))

Let $(G, \cdot)$ and $(G', *)$ be two groups. Let ϕ be a bijection such that

$$\forall a, b \in G.\ \phi(a \cdot b) = \phi(a) * \phi(b).$$

Then ϕ is an isomorphism from G to G'.

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G and G' are isomorphic

$$\phi : G \cong G'$$

$$(\mathbb{R}, +) \cong (\mathbb{R}^+, *)$$

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$$\phi(x) = e^x$$

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Suppose that $\phi : G \cong G'$*. Let* e *and* e' *are identities of* G *and* G'*, respectively.*

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Klein Four-group (四元群; $K_4; V$)

*	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$$a^2 = b^2 = c^2 = (ab)^2 = e$$

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$$U(8) = \{1, 3, 5, 7\}$$

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The order of e is the smallest positive integer r such that $a^r = e$.

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$$\text{ord } a = r$$

If such r does not exist, then $\text{ord } a = \infty$.

$$\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$$

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Definition (Cyclic Group (循环群))

Let G be a group. If

$$\exists a \in G.\ G = \langle a \rangle \triangleq \{a^0 = e, a, a^2, a^3, \dots\},$$

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If $G = \langle a \rangle$, then a is a generator (生成元) of G.

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(2) *Let* $G = \{e, a, a^2, \dots, a^{n-1}\}$ *be a finite cyclic group of order* n.

$$\forall k, l \in \mathbb{Z}.\ (a^k = a^l \leftrightarrow n \mid (k - l)).$$

Theorem (Structure Theorem of Cyclic Groups (循环群结构定理))

Let $G = \langle a \rangle$ be a cyclic group.

(1) *If $|G| = \infty$, then $G \cong (\mathbb{Z}, +)$.*

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$$\operatorname{ord} a^r = \frac{n}{(n, r)}$$

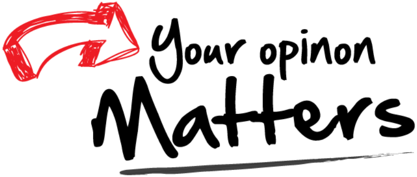
$$(\mathbb{Z}_{12}, +)$$

$$\text{Generators} : 1, 5, 7, 11$$

Theorem (Subgroups of Cyclic Groups)

Every subgroup of a cyclic group is cyclic.


Thank You!



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