(八) 集合: 无穷 (Infinity)

魏恒峰

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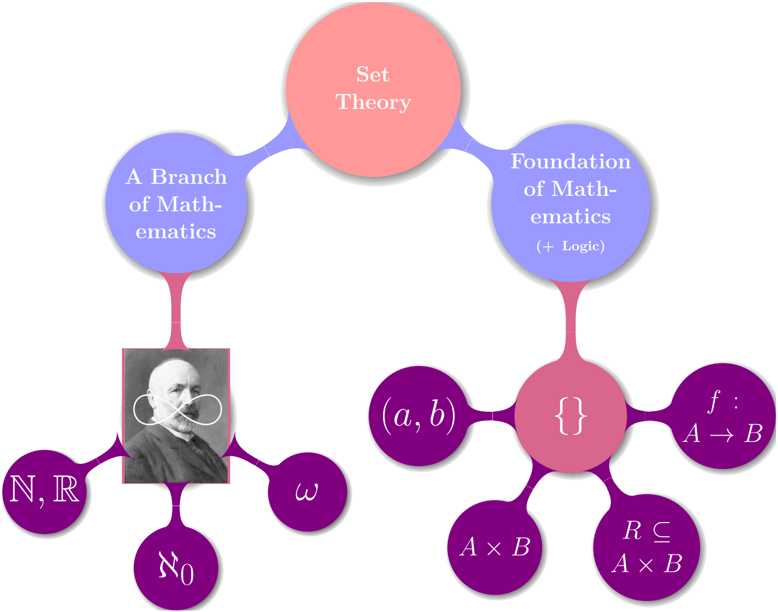
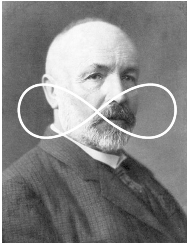
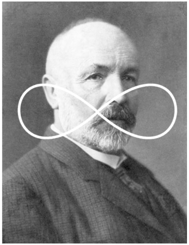
2021 年 04 月 29 日


Set Theory
A Branch of Math- ematics
Foundation of Math- ematics
(+ Logic)
$\mathbb{N}, \mathbb{R}$
$\aleph_0$
ω
(a, b)
$\{\}$
$f : A \to B$
$A \times B$
$R \subseteq A \times B$



Georg Cantor (1845 – 1918)



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Leopold Kronecker
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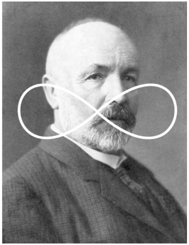

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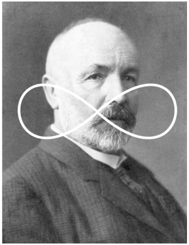
Henri Poincaré
(1854 – 1912)



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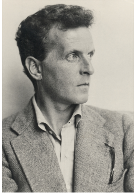
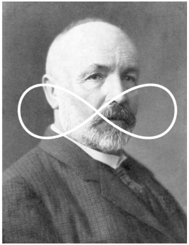


Ludwig Wittgenstein
(1889 – 1951)



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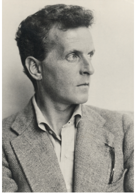

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— *David Hilbert*

"没有人能把我们从 *Cantor* 创造的乐园中驱逐出去"

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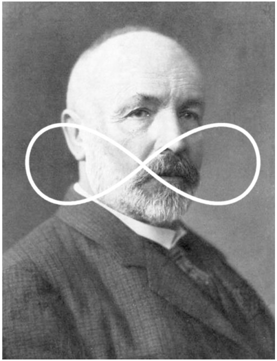
MATHEMATIKER
BEGRÜNDER DER MENGENLEHRE
1845-1918
DAS WESEN DER MATHEMA
TIK LIEGT IN IHRER FREIHEIT
GEORG CANTOR

"das wesen der mathematik liegt in ihrer freiheit"

"das wesen der mathematik liegt in ihrer freiheit"

"The essence of mathematics lies in its freedom"

Before Cantor




决定经典
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几何原本
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Euclid's Elements
13卷图释全本


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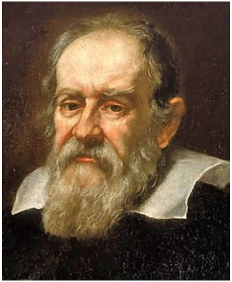
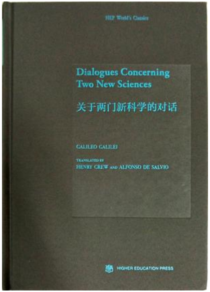
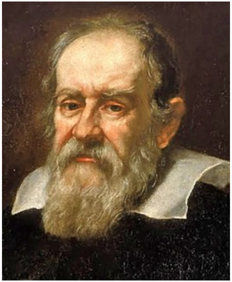
公理: “整体大于部分”



Galileo Galilei (1564 – 1642)



“关于两门新科学的对话” (1638)



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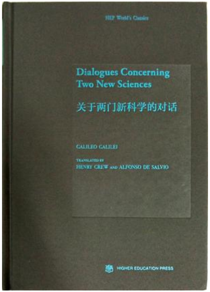

“关于两门新科学的对话” (1638)

“用我们有限的心智来讨论无限 · · · ”

$$S_1 = \{1, 2, 3, \cdots, n, \cdots\}$$
$$S_2 = \{1, 4, 9, \cdots, n^2, \cdots\}$$

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说到底，“等于”、“大于”和“小于”诸性质不能用于无限，而只能用于有限的数量。 — *Galileo Galilei*

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无穷数是不可能的。 — *Gottfried Wilhelm Leibniz*

这些证明一开始就期望那些数要具有有穷数的一切性质，或者甚至于把有穷数的性质强加于无穷。

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这些性质完全依赖于事物的本性，$\cdots$ 而并非来自我们的主观任意性或我们的偏见。

— *Georg Cantor (1885)*

Definition (Dedekind-infinite & Dedekind-finite (Dedekind, 1888))

A set A is *Dedekind-infinite* if there is a bijective function from A to some proper subset B of A.

A set is *Dedekind-finite* if it is not Dedekind-infinite.

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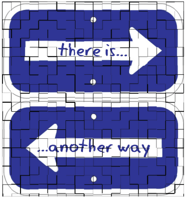

This is a theorem in our theory of infinity.

We have not defined “finite” and “infinite”!

Comparing Sets

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$$= \qquad\qquad <$$

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$$=\qquad\qquad\qquad <$$

Function

$$=$$

Definition ($|A| = |B|$ ($A \approx B$) (1878))

A and B are *equipotent (等势)* if there exists a *bijection* from A to B.

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Abstract from order: $\{1, 2, 3, \cdots\} \quad \textit{vs.} \quad \{1, 3, 5, \cdots, 2, 4, 6, \cdots\}$

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Theorem ( )

For any sets A, B, C:

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(b) $A \approx B \implies B \approx A$

(c) $A \approx B \land B \approx C \implies A \approx C$

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Do not care too much.

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X is finite if

$$\exists n \in \mathbb{N} : |X| = |n| = |\{0, 1, \ldots, n-1\}|.$$

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集合 X 是有穷的当且仅当它与某个自然数等势。

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$\mathbb{N}$ is infinite. (So are $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$.)

By Contradiction.

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Countably Infinite

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Countable

(finite $\vee$ countably infinite)

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Countably Infinite

$$|X| = |\mathbb{N}| \triangleq \aleph_0$$

Countable

$$(\text{finite} \lor \text{countably infinite})$$

Uncountable

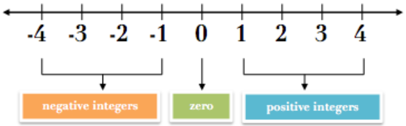
$$(\neg \text{ countable})$$

$$(\text{infinite}) \land \Big(\neg \, (\text{countably infinite})\Big)$$

$$\aleph_0$$

Theorem ($\mathbb{Z}$ is Countable.)

$$|\mathbb{Z}| = |\mathbb{N}|$$



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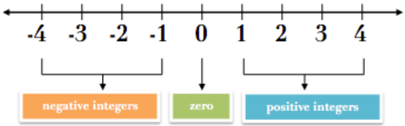

$$0 \quad 1 \quad -1 \quad 2 \quad -2 \quad \cdots$$

Theorem ($\mathbb{Q}$ is Countable. (Cantor 1873-11; Published in 1874))

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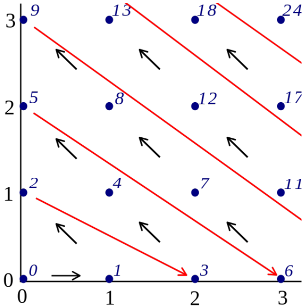
1/1	1/2 →	1/3	1/4 →	1/5	1/6 →	1/7	1/8 →	⋯
2/1	2/2	2/3	2/4	2/5	2/6	2/7	2/8	⋯
3/1	3/2	3/3	3/4	3/5	3/6	3/7	3/8	⋯
4/1	4/2	4/3	4/4	4/5	4/6	4/7	4/8	⋯
5/1	5/2	5/3	5/4	5/5	5/6	5/7	5/8	⋯
6/1	6/2	6/3	6/4	6/5	6/6	6/7	6/8	⋯
7/1	7/2	7/3	7/4	7/5	7/6	7/7	7/8	⋯
8/1	8/2	8/3	8/4	8/5	8/6	8/7	8/8	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Theorem ($\mathbb{N} \times \mathbb{N}$ is Countable.)

$$|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$$



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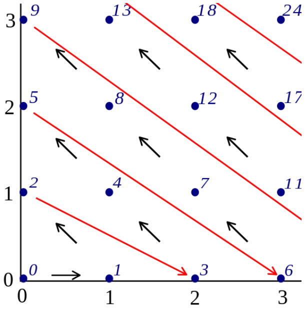

$$\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

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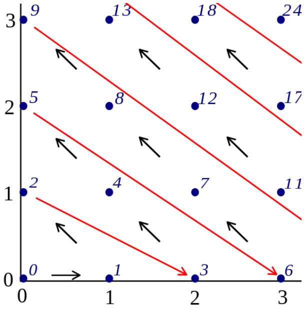

$$\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

$$\pi(k_1, k_2) = \frac{1}{2}(k_1 + k_2)(k_1 + k_2 + 1) + k_2$$

$$\pi(2, 1) = 7$$

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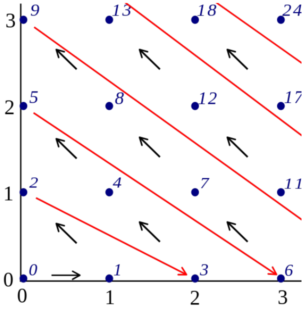

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Cantor Pairing Function

Theorem ($\mathbb{N}^n$ is Countable.)

$$|\mathbb{N}^n| = |\mathbb{N}|$$

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$$\pi^{(n)}(k_1, \ldots, k_{n-1}, k_n) = \pi(\pi^{(n-1)}(k_1, \ldots, k_{n-1}), k_n) \quad (n \geq 3)$$

Theorem

Any finite union of countable sets is countable.

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$$A = \{a_n \mid n \in \mathbb{N}\} \quad B = \{b_n \mid n \in \mathbb{N}\} \quad C = \{c_n \mid n \in \mathbb{N}\}$$

Theorem

Any *finite* *union of countable sets is countable.*

$$A = \{a_n \mid n \in \mathbb{N}\} \quad B = \{b_n \mid n \in \mathbb{N}\} \quad C = \{c_n \mid n \in \mathbb{N}\}$$

$$a_0 \quad b_0 \quad c_0 \quad a_1 \quad b_1 \quad c_1 \cdots$$

Theorem

The union of countably many countable sets is countable.

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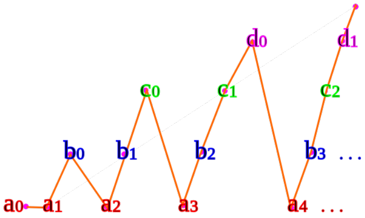

Counting by Diagonals

Beyond

$$\aleph_0$$

Theorem ($\mathbb{R}$ is Uncountable. (Cantor 1873-12; Published in 1874))

$$|\mathbb{R}| \neq |\mathbb{N}|$$

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Different “Sizes” of Infinity

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Cantor's Diagonal Argument (1890)

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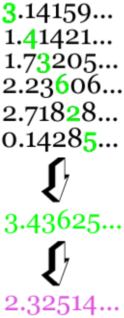

$$\mathfrak{c} \triangleq |\mathbb{R}|$$

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$\mathbb{R}$ 是一个连续统 (Continuum)

https://en.wikipedia.org/wiki/Continuum_(set_theory)

Theorem ($|\mathbb{R}|$ (Cantor 1877))

$$|(0,1)| = |\mathbb{R}| = |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}^{n \in \mathbb{N}}|$$

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$$f(x) = \tan \frac{(2x-1)\pi}{2}$$

$$|(0,1)| = |(-\frac{\pi}{2}, \frac{\pi}{2})| = |\mathbb{R}|$$

$$\mathbb{R} \times \mathbb{R} \approx \mathbb{R}$$

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Was Cantor Surprised?

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"Je le vois, mais je ne le crois pas !"

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Theorem (Brouwer (Topological Invariance of Dimension))

There is no continuous bijections between $\mathbb{R}^m$ and $\mathbb{R}^n$ for $m \neq n$.

Beyond

$$\mathfrak{c}$$

Theorem (Cantor's Theorem (1891))

$$|A| \neq |\mathcal{P}(A)|$$

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Theorem (Cantor Theorem)

If $f : A \to \mathcal{P}(A)$*, then* f *is not onto.*

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There is no largest infinity.

$$<$$

Definition ($|A| \leq |B|$)

$|A| \leq |B|$ if there exists an *one-to-one* function f from A into B.

Definition ($|A| < |B|$)

$$|A| < |B| \iff |A| \le |B| \land |A| \ne |B|$$

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$$|\mathbb{N}| < |\mathbb{R}|$$

$$|X| < |2^X|$$

$$|\mathbb{N}| < |2^{\mathbb{N}}|$$

Definition (Countable Revisited)

X is countable:

$$(\exists n \in \mathbb{N} : |X| = n) \lor |X| = |\mathbb{N}|$$

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$$g = f|_B$$

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$$(\mathbb{Q}, \mathbb{R})$$

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$$|\mathbb{R}| \le |\mathbb{Q} \times \mathbb{R}| \le |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$$

Q : Is "$\leq$" a partial order?

Q : *Is "$\leq$" a partial order?*

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$$|X| \leq |Y| \land |Y| \leq |X| \implies |X| = |Y|$$

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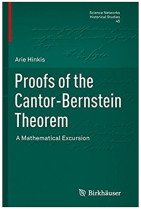

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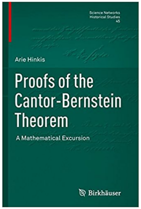

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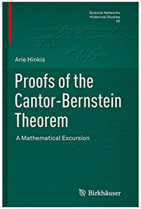

Schröder–Bernstein theorem @ wiki

Q : Is "$\leq$" a total order?

Q : Is "$\leq$" a total order?

Theorem (PCC)

Principle of Cardinal Comparability (PCC) $\iff$ *Axiom of Choice*

Theorem

$$|\mathbb{R}| = |\mathcal{P}(\mathbb{N})| = |\mathcal{P}(\mathbb{Q})|$$

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$$|\mathbb{R}| = |\mathcal{P}(\mathbb{N})| = |\mathcal{P}(\mathbb{Q})|$$

$$|\mathbb{R}| \leq |\mathcal{P}(\mathbb{Q})| \qquad |\mathcal{P}(\mathbb{Q})| \leq |\mathbb{R}|$$

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$$\mathfrak{c} \triangleq |\mathbb{R}| = |\mathcal{P}(\mathbb{N})| = |2^{\mathbb{N}}| \triangleq 2^{\aleph_0}$$

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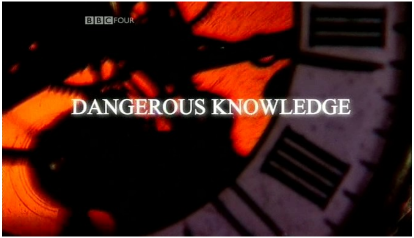
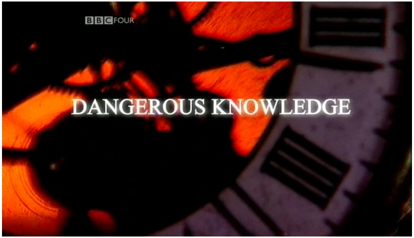
Continuum Hypothesis (CH)

$$\boxed{\nexists A : \aleph_0 < |A| < \mathfrak{c}}$$



☞ Dangerous Knowledge (22:20; BBC 2007)



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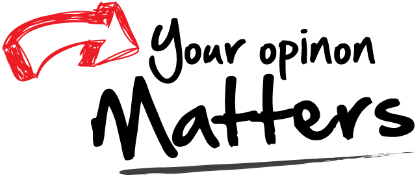
Independence from ZFC:

Kurt Gödel (1940) CH cannot be disproved from ZF.

Paul Cohen (1964) CH cannot be proven from the ZFC axioms.


Thank
You!



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