(十三) 群论: 群的基本概念 (What are Groups?)

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$$\exists e \in G.\ \forall a \in G.\ e * a = a * e = a$$

Inverse (逆元): Let e be the identity of G.

$$\forall a \in G.\ \exists b \in G.\ a * b = b * a = e$$

The inverse of a is denoted a^{-1}.

Definition (Commutative Group (交换群); Abelian Group (阿贝尔群))

Let $(G, *)$ be a group. If $*$ is commutative,

$$\forall a, b \in G.\ a * b = b * a,$$

then $(G, *)$ is a commutative group.

$$(\mathbb{Z}, +)$$

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$$(1, -1, \mathbf{i}, -\mathbf{i})$$

Group of n-th Roots of Unity (n 次单位根群)

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\begin{aligned}
U_n &= \{z \in \mathbb{C} \mid z^n = 1\} \\
&= \{\cos \frac{2k\pi}{n} + \mathbf{i} \sin \frac{2k\pi}{n} \mid k = 0, 1, \ldots, n-1\}
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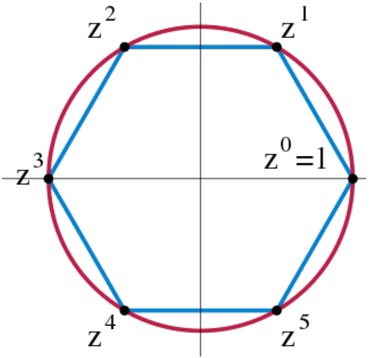

Quaternion Group (四元群)

$$(1, i, j, k, -1, -i, -j, -k)$$

×	e	$\bar{e}$	i	$\bar{i}$	j	$\bar{j}$	k	$\bar{k}$
e	e	$\bar{e}$	i	$\bar{i}$	j	$\bar{j}$	k	$\bar{k}$
$\bar{e}$	$\bar{e}$	e	$\bar{i}$	i	$\bar{j}$	j	$\bar{k}$	k
i	i	$\bar{i}$	$\bar{e}$	e	k	$\bar{k}$	$\bar{j}$	j
$\bar{i}$	$\bar{i}$	i	e	$\bar{e}$	$\bar{k}$	k	j	$\bar{j}$
j	j	$\bar{j}$	$\bar{k}$	k	$\bar{e}$	e	i	$\bar{i}$
$\bar{j}$	$\bar{j}$	j	k	$\bar{k}$	e	$\bar{e}$	$\bar{i}$	i
k	k	$\bar{k}$	j	$\bar{j}$	$\bar{i}$	i	$\bar{e}$	e
$\bar{k}$	$\bar{k}$	k	$\bar{j}$	j	i	$\bar{i}$	e	$\bar{e}$

Cayley Table

$$i^2 = j^2 = k^2 = 1 \qquad ij = k, ji = -k, jk = i, kj = -i, ki = j, ik = -j$$

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(6) $\forall a, b \in G.\ \exists!\ x \in G.\ ax = b \land ya = b.$

$$(\mathcal{P}(A), \cup)$$

Additive Group of Integers Modulo m (模 m 剩余类加群)

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$$(\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}, \times_6)$$

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$$(u, m) = 1 \qquad ua = au = au + mv = 1 \mod m$$

When p is a prime,

$$\mathbb{Z}_p^* \triangleq U(p) = \{1, 2, \ldots, p-1\}$$

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$$U(20) = \{1, 3, 7, 9, 11, 13, 17, 19\}$$

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$$a^n a_1 \ldots a_n = a_1 \ldots a_n \implies a^n = e$$

Theorem (Euler Theorem (1736))

Let $m \in \mathbb{Z}^+$ *and* $a \in \mathbb{Z}$. *If* $(a, m) = 1$, *then*

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$$7^{222} \equiv 7^{4 \times 55 + 2} \equiv 7^2 \equiv 9 \mod 10$$

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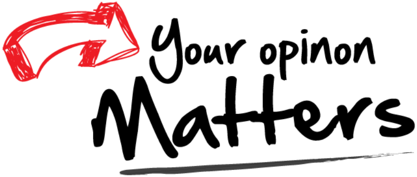
$$\varphi(p) = p - 1$$


Thank
You!



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