(三) 数学归纳法

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数学归纳法真得很简单吗?

Theorem (第一数学归纳法 (The First Mathematical Induction))

设 $P(n)$ 是关于自然数的一个性质。如果

(i) $P(0)$ 成立;

(ii) 对任意自然数 n, 如果 $P(n)$ 成立, 则 $P(n+1)$ 成立。

那么, $P(n)$ 对所有自然数 n 都成立。

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$$\Bigg(P(0) \land \forall n \in \mathbb{N}.\ \Big(P(n) \to P(n+1)\Big)\Bigg) \to \forall n \in \mathbb{N}.\ P(n).$$

Theorem (第二数学归纳法 (The Second Mathematical Induction))

设 $Q(n)$ 是关于自然数的一个性质。如果

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Theorem (数学归纳法)

第一数学归纳法与第二数学归纳法等价。

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Q：第二数学归纳法也被称为**“强” (Strong)** 数学归纳法，它强在何处？

Lemma

第二数学归纳法蕴含第一数学归纳法。

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$$Q(n) \triangleq P(n)$$

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$$P(n) \triangleq Q(0) \wedge \cdots \wedge Q(n)$$

数学归纳法为何成立?

Peano 公理体系刻画了**自然数的递归结构**

Definition (Peano Axioms)

(1) 0 是自然数;

(2) 如果 n 是自然数, 则它的后继 $\mathbf{S}n$ 也是自然数;

(3) 0 不是任何自然数的后继;

(4) 两个自然数相等当且仅当它们的后继相等;

(5) **数学归纳原理:** 如果

  (i) $P(0)$ 成立;

  (ii) 对任意自然数 n, 如果 $P(n)$ 成立, 则 $P(n+1)$ 成立。

那么, $P(n)$ 对所有自然数 n 都成立。

Definition (良序原理 (The Well-Ordering Principle))

自然数集的任意非空子集都有一个最小元。

Theorem

良序原理与 (第一) 数学归纳法等价。

Lemma

(第一) 数学归纳法蕴含良序原理。

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Proof.

By mathematical induction on the size n of non-empty subsets of $\mathbb{N}$.

$$P(k): \text{All subsets of size } k \text{ contain a minimum.}$$

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Basis Step: $P(1)$

Inductive Hypothesis: $P(n)$

Inductive Step: $P(n) \to P(n+1)$

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- $A' \leftarrow A \setminus a$
- $x \leftarrow \min A'$
- Compare x with a

□

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$$\forall n \in \mathbb{N} : P(n) \quad \textit{vs.} \quad P(\infty)$$

□

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$P(n)$: 任何一个含有 $\leq n$ 的某个自然数的自然数子集都有最小元

Lemma

良序原理蕴含 *(第一)* 数学归纳法。

反证法

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Learn by Examples

Theorem (Fermat's Little Theorem)

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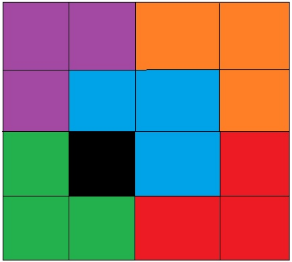
$$(a+1)^p \equiv a + 1 \pmod{p}$$

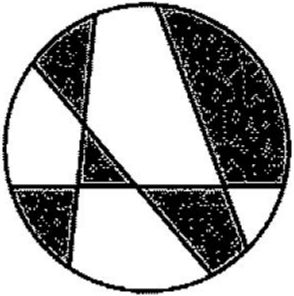
Tiling Puzzle

任何一个缺失了一格的 $2^n \times 2^n$ 的网格都可以被 L 型填满。



Definition (Line Map)

- A blank circle is a line map;
- A line map with a chord (弦) is a line map.



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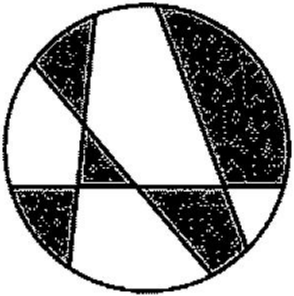
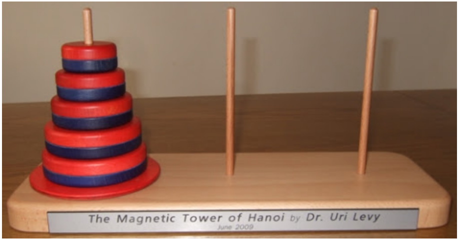
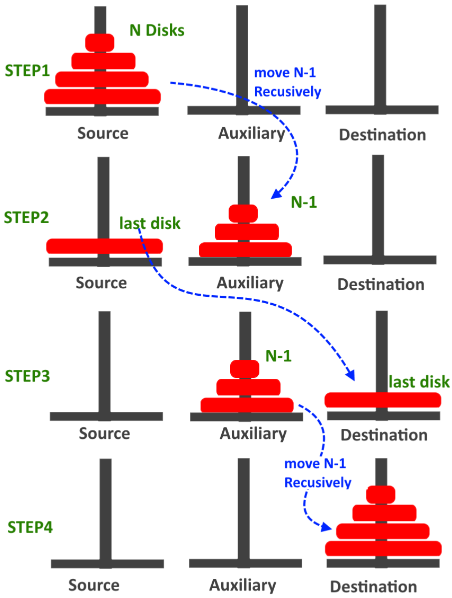


Theorem

Any line map can be two-colored.

The Tower of Hanoi




STEP1
N Disks
move N-1
Recusively
Source
Auxiliary
Destination
STEP2
last disk
N-1
Source
Auxiliary
Destination
STEP3
N-1
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Source
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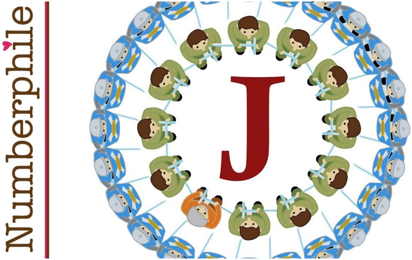
Josephus Problem



$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \in \mathbb{N} \quad (0 \leq k \leq n)$$

Horse Paradox

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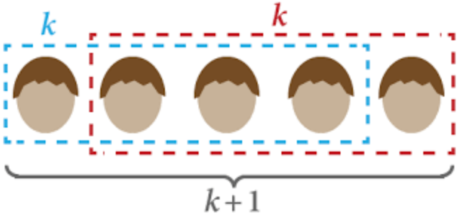

$F(n)$ 是偶数, 当且仅当 $F(n+3)$ 是偶数。

设 $*$ 是一个满足结合律的二元运算符。请证明，

$$a_1 * a_2 * \cdots * a_n$$

的值与括号的使用方式无关。

算术基本定理

Prove that every integer greater than 2 can be written as product of primes.

请证明, 只用 4 分与 5 分邮票, 就可以组成 12 分及以上的每种邮资。

(每个不小于 12 的整数都可以写成若干个 4 或 5 的和。)

堆盒子游戏

现有 n 个盒子堆在一起。你可以移动这些盒子, 每次移动只能将一堆盒子分成不为空的两堆盒子, 最后得到 n 堆盒子, 即每堆只有一个盒子时, 游戏结束。

每次移动盒子时, 如果将高度为 $a+b$ 的盒子堆拆分成高度为 a 和 b 的两堆, 玩家可以得 ab 分。

玩家的总得分是每次移动盒子得分的总和。请问, 如何才能得到最高分?

+fig

Lemma

任何一种平铺 n 个盒子的方法，得分都是 $\frac{n(n-1)}{2}$。

只用以下三种图示拼出 $2 \times n$ 的形状, 有几种不同的拼法?

$$T(n) = ?T(n-1) + ??T(n-2) + \dots$$

The Blue-eyed Islanders Puzzle

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One evening, he addresses the entire tribe to thank them for their hospitality.

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However, not knowing the customs, the foreigner makes the mistake of mentioning eye color in his address, remarking "**how unusual it is to see another blue-eyed person like myself in this region of the world**".

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However, not knowing the customs, the foreigner makes the mistake of mentioning eye color in his address, remarking "**how unusual it is to see another blue-eyed person like myself in this region of the world**".

What effect, if anything, does this *faux pas* have on the tribe?

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(everyone in the tribe can already see that

there are several blue-eyed people in their tribe).

100 days after the address,

all the blue eyed people commit suicide.

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Theorem (The Blue-eyed Islanders Puzzle)

Suppose that the tribe had $n > 0$ blue-eyed people.

Then n days after the traveller's address,

all n blue-eyed people commit suicide.

By induction on the number n of blue-eyed people in the tribe.

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基础步骤: $n = 1$.

归纳假设: 假设命题对 n 个蓝眼人的情况也成立。

归纳步骤:

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Theorem

对于任何自然数 n, 13^n 都可以写成两个自然数的平方之和。

$$
\begin{aligned}
13^{n+1} &= 13 \cdot 13^n \\
&= (2^2 + 3^2)(a^2 + b^2) \\
&= (\underbrace{2a + 3b}_{x})^2 + (\underbrace{3a - 2b}_{y})^2 \\
&= x^2 + y^2
\end{aligned}
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$$13^0 = 1^2 + 0^2$$

$$13^1 = 2^2 + 3^2$$

$$\begin{aligned}
13^{n+2} &= 13^2 \cdot 13^n \\
&= 13^2(a^2 + b^2) \\
&= (\underbrace{13a}_{x})^2 + (\underbrace{13b}_{y})^2 \\
&= x^2 + y^2
\end{aligned}$$

$$
\begin{aligned}
f(1,1) &= 2 \\
f(m+1,n) &= f(m,n) + 2(m+n) \\
f(m,n+1) &= f(m,n) + 2(m+n-1)
\end{aligned}
$$

请证明,

$$\forall m,n \in \mathbb{N}^+.\ f(m,n) = (m+n)^2 - (m+n) - 2n + 2$$

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$$f(k,1) \to f(k+1,1)$$

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Double Induction (Inducitor Twice)

$$f(1,1)$$

$$f(k,1) \to f(k+1,1)$$

$$f(h,k) \to f(h,k+1) \text{ for any } h$$

$$
\begin{aligned}
f(1,1) &= 2 \\
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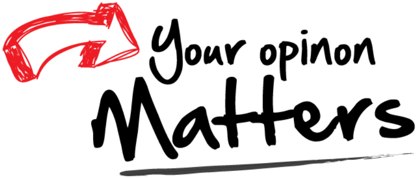
$$\forall m,n \in \mathbb{N}^{+}.\ f(m,n) = (m+n)^2 - (m+n) - 2n + 2$$

对 $m+n$ 作归纳


Thank You!



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