(十四) 群论：子群 (Subgroup)

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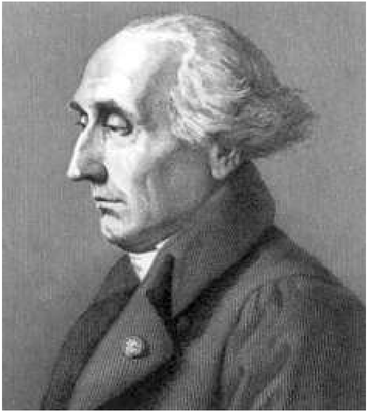
Lagrange's Theorem



Joseph-Louis Lagrange (1736 $\sim$ 1813)

Fundamental Homomorphism Theorem

Emmy Noether ($1882 \sim 1935$)

Lagrange's Theorem

Help us understand the structure of a group
via its subgroups/normal subgroups

Fundamental Homomorphism Theorem

Definition (Subgroup (子群))

Let $(G, *)$ be a group and $\emptyset \neq H \subseteq G$.

If $(H, *)$ is a group, then we call H a subgroup of G, denoted $H \leq G$.

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$H = G, H = \{e\}$ are two trivial (平凡) subgroups.

If $H \subset G$, then H is a proper subgroup (真子群).

$$(H = \{mz \mid z \in \mathbb{Z}\}, +) \leq (\mathbb{Z}, +)$$

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$$H = \{1, 2, 4\} \le G = \mathbb{Z}_7^* = \{1, 2, 3, 4, 5, 6\}$$

Theorem

Suppose that $H \leq G$.

(1) *The identity of H is the same with that of G.*

$$e_H = e_G$$

(2) *The inversion of a in H is the same with that in G.*

$$\forall a \in H.\ a_H^{-1} = a_G^{-1}$$

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$$e_H e_H = e_H = e_H e_G \implies e_H = e_G$$

$$a a_H^{-1} = e_H = e_G = a a_G^{-1} \implies a_H^{-1} = a^{-1}(G)$$

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Let G be a group and $\emptyset \neq H \subseteq G$. $H \leq G$ iff

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$$H_1 \cup H_2?$$

Definition (Symmetric Group (对称群; Sym(M)))

Let $M \neq \emptyset$ be a set.

All the permutations/bijective functions of M, together with the composition operation, is a group, called the symmetric group of M.

$$M = \{1, 2, \ldots, n\}$$

$$S_n \triangleq \mathrm{Sym}(M)$$

$$S_3$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

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$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \qquad \tau = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

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$$\sigma\tau \neq \tau\sigma$$

Cyclic Notation (轮换表示法) & Transposition (对换)

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 6 & 1 & 5 & 2 \end{pmatrix}$$

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$$(1) \quad (1\ 2) \quad (1\ 3) \quad (2\ 3) \quad (1\ 2\ 3) \quad (1\ 3\ 2)$$

Definition (Permutation Group (置换群))

Let $M \neq \emptyset$ be a set.
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Definition (Coset (陪集))

Suppose that $H \leq G$. For $a \in G$,

$$aH = \{ah \mid h \in H\}, \quad Ha = \{ha \mid h \in H\},$$

is called the left coset (左陪集) and right coset of H in G, respectively.

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$$(1)H = (1\ 2\ 3)H = (1\ 3\ 2)H = H \leq S_3$$

$$(1\ 2)H = (1\ 3)H = (2\ 3)H = \{(1\ 2), (1\ 3), (2\ 3)\}$$

Theorem

Suppose that $H \leq G$, $a, b \in G$.

(1)

$$|aH| = |H| = |bH|$$

(2)

$$a \in aH$$

(3)

$$aH = H \iff a \in H \iff aH \leq G$$

(4)

$$aH = bH \iff a^{-1}b \in H$$

(5)

$$\forall a, b \in G. \; (aH = bH) \vee (aH \cap bH = \emptyset)$$

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$$aH = bH \implies a^{-1}aH = a^{-1}bH \implies a^{-1}bH = H \implies a^{-1}b \in H$$

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$$aH = bH \implies a^{-1}aH = a^{-1}bH \implies a^{-1}bH = H \implies a^{-1}b \in H$$

$$a^{-1}bH = H \implies a(a^{-1}bH) = aH \implies bH = aH$$

$$\forall a, b \in G.\ (aH = bH) \vee (aH \cap bH = \emptyset)$$

$$\forall a, b \in G. \ (aH = bH) \vee (aH \cap bH = \emptyset)$$

$$\forall a, b \in G. \ (aH \cap bH \neq \emptyset \to aH = bH)$$

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Take any $g \in aH \cap bH$.

$$\forall a, b \in G. \ (aH = bH) \lor (aH \cap bH = \emptyset)$$

$$\forall a, b \in G. \ (aH \cap bH \neq \emptyset \to aH = bH)$$

Take any $g \in aH \cap bH$.

$$\exists h_1, h_2 \in H. \ (ah_1 = g = ah_2) \land (h_1 H = H = h_2 H)$$

$$\forall a, b \in G. \ (aH = bH) \vee (aH \cap bH = \emptyset)$$

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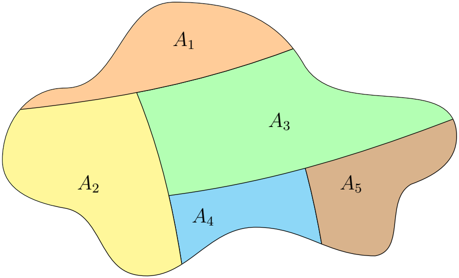
Take any $g \in aH \cap bH$.

$$\exists h_1, h_2 \in H. \ (ah_1 = g = ah_2) \wedge (h_1 H = H = h_2 H)$$

$$aH = a(h_1 H) = (ah_1)H = (bh_2)H = b(h_2 H) = bH$$

A balanced partition of G by its subgraph H



Theorem (Lagrange's Theorem)

Suppose that $H \le G$. Then

$$|G| = [G : H] \cdot |H|$$

Definition (Index (指标))

$$G/H = \{gH \mid g \in G\}$$

$$[G : H] \triangleq |G/H|$$

$$H \leq G \implies |H| \mid |G|$$

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There are *no* subgraphs of order 5, 7, or 8 of a group of order 12.

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Theorem

- *There are only 2 groups of order 4.*
- *There are only 2 groups of order 6.*

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It is possible that $aH \neq Ha$.

$$S_3 = \{(1), (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$$

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$$\forall a \in S_3.\ aH = Ha$$

Definition (Normal Subgroup (正规子群))

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$$aH = Ha \not\Rightarrow \forall h \in H.\ ah = ha$$

$$aH = Ha \Longrightarrow \forall h \in H.\ \exists h' \in H.\ ah = h'a$$

Theorem

$$H \triangleleft G \iff \forall a \in G, h \in H.\ aha^{-1} \in H$$

Theorem

$$H \lhd G \iff \forall a \in G, h \in H.\ aha^{-1} \in H$$

$$\begin{aligned} aH = Ha &\implies aHa^{-1} = (Ha)a^{-1} = H(aa^{-1}) = H \\ &\implies aHa^{-1} \subseteq H \\ &\implies \forall h \in H.\ aha^{-1} \in H \end{aligned}$$

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aH = Ha &\implies aHa^{-1} = (Ha)a^{-1} = H(aa^{-1}) = H \\
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\end{aligned}$$

$$aha^{-1} \in H \implies ah = (aha^{-1})a \in Ha \implies aH \subseteq Ha$$

$$a^{-1}ha = a^{-1}h(a^{-1})^{-1} \in H \implies ha \in aH \implies Ha \subseteq aH$$

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$$H = \{(1), (1\ 2\ 3), (1\ 3\ 2)\} \lhd S_3$$

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Theorem

$$\sigma\tau\sigma^{-1} = \begin{pmatrix} \sigma(1) & \sigma(2) & \dots & \sigma(n) \\ \sigma(\tau(1)) & \sigma(\tau(2)) & \dots & \sigma(\tau(n)) \end{pmatrix}$$

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$$\forall \sigma \in S_3, \tau \in H.\ \sigma \tau \sigma^{-1} \in H$$

Theorem

$$\sigma \tau \sigma^{-1} = \begin{pmatrix} \sigma(1) & \sigma(2) & \dots & \sigma(n) \\ \sigma(\tau(1)) & \sigma(\tau(2)) & \dots & \sigma(\tau(n)) \end{pmatrix}$$

$$(1\ 2)(1\ 2\ 3)(1\ 2)^{-1} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (1\ 3\ 2)$$

Definition (正规子群的陪集)

Suppose that $H \lhd G$.

$$G/H = \{aH \mid a \in G\}$$

is the coset of H in G.

Definition (Quotient Group (商群))

Suppose that $H \lhd G$. Define

$$aH \cdot bH = (ab)H.$$

Then $(G/H, \cdot)$ is a group, called the quotient group of G by H (denoted G/H).

$aH \cdot bH = (ab)H$ is well-defined

$$aH \cdot bH = (ab)H \text{ is well-defined}$$

$$aH = a'H \land bH = b'H \implies aH \cdot bH = a'H \cdot b'H$$

结果与代表元的选取无关

Definition (Isomorphism (同构))

Let $(G, \cdot)$ and $(G', *)$ be two groups. Let ϕ be a bijection such that

$$\forall a, b \in G.\ \phi(a \cdot b) = \phi(a) * \phi(b).$$

Then ϕ is an isomorphism from G to G'.

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G and G' are isomorphic

$$\phi : G \cong G'$$

$$(\mathbb{R}, +) \cong (\mathbb{R}^{+}, *)$$

$$(\mathbb{R}, +) \cong (\mathbb{R}^+, *)$$

$$\phi(x) = e^x$$

Theorem

Suppose that $\phi : G \cong G'$. *Let* e *and* e' *are identities of* G *and* G', *respectively.*

(1) $\phi(e) = e'$

(2) $\phi(a^{-1}) = (\phi(a))^{-1}$

(3) $\phi^{-1} : G' \cong G$

Theorem

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$$ea = a \implies \phi(e)\phi(a) = \phi(ea) = \phi(a) = e'\phi(a)$$

Klein Four-group (四元群; $K_4; V$)

*	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$$a^2 = b^2 = c^2 = (ab)^2 = e$$

$$ab = c = ba \quad ac = b = ca \quad bc = a = cb$$

Klein Four-group (四元群; $K_4; V$)

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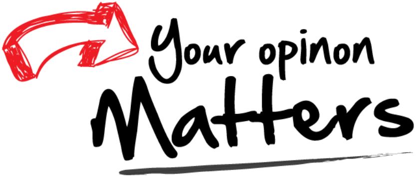
$$U(8) = \{1, 3, 5, 7\}$$


Thank You!



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