(十三) 群论: 群的基本概念 (What are Groups?)

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2021 年 06 月 03 日

Definition (Group (群))

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Identity (单位元):

$$\exists e \in G.\ \forall a \in G.\ e * a = a * e = a$$

Inverse (逆元): Let e be the identity of G.

$$\forall a \in G.\ \exists b \in G.\ a * b = b * a = e$$

The inverse of a is denoted a^{-1}.

Definition (Commutative Group (交换群); Abelian Group (阿贝尔群))

Let $(G, *)$ be a group. If $*$ is commutative,

$$\forall a, b \in G.\ a * b = b * a,$$

then $(G, *)$ is a commutative group.

$$(\mathbb{Z}, +)$$

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$$(\mathbb{Q} \setminus \{0\}, \times)$$

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$$(1, -1, \mathbf{i}, -\mathbf{i})$$

Group of n-th Roots of Unity (n 次单位根群)

$$
\begin{aligned}
U_n &= \{z \in \mathbb{C} \mid z^n = 1\} \\
&= \{\cos \frac{2k\pi}{n} + \mathbf{i} \sin \frac{2k\pi}{n} \mid k = 0, 1, \ldots, n-1\}
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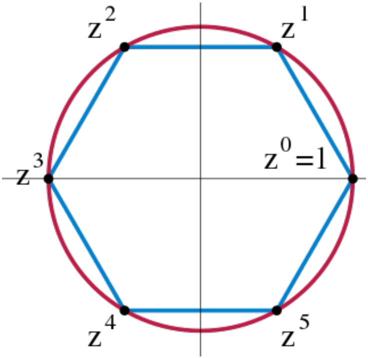

Quaternion Group (四元数群)

$$(1, i, j, k, -1, -i, -j, -k)$$

×	e	$\bar{e}$	i	$\bar{i}$	j	$\bar{j}$	k	$\bar{k}$
e	e	$\bar{e}$	i	$\bar{i}$	j	$\bar{j}$	k	$\bar{k}$
$\bar{e}$	$\bar{e}$	e	$\bar{i}$	i	$\bar{j}$	j	$\bar{k}$	k
i	i	$\bar{i}$	$\bar{e}$	e	k	$\bar{k}$	$\bar{j}$	j
$\bar{i}$	$\bar{i}$	i	e	$\bar{e}$	$\bar{k}$	k	j	$\bar{j}$
j	j	$\bar{j}$	$\bar{k}$	k	$\bar{e}$	e	i	$\bar{i}$
$\bar{j}$	$\bar{j}$	j	k	$\bar{k}$	e	$\bar{e}$	$\bar{i}$	i
k	k	$\bar{k}$	j	$\bar{j}$	$\bar{i}$	i	$\bar{e}$	e
$\bar{k}$	$\bar{k}$	k	$\bar{j}$	j	i	$\bar{i}$	e	$\bar{e}$

Cayley Table

$$i^2 = j^2 = k^2 = 1 \qquad ij = k, ji = -k, jk = i, kj = -i, ki = j, ik = -j$$

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Let G be a group.

(1) *The identity is unique.*

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(6) $\forall a, b \in G.\ \exists!\ x \in G.\ ax = b \land ya = b.$

Additive Group of Integers Modulo m (模 m 剩余类加群)

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$$(\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}, \times_6)$$

Multiplicative Group of Integers Modulo m (模 m 剩余类乘法群)

$$U(m) = \{a \in \mathbb{Z}_m \mid (a, m) = 1\}$$

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$$(a, b) = d \implies \exists u, v \in \mathbb{Z}.\ au + bv = d$$

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$$(u, m) = 1 \qquad ua = au = au + mv = 1 \mod m$$

When p is a prime,

$$\mathbb{Z}_p^* \triangleq U(p) = \{1, 2, \ldots, p-1\}$$

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$$U(20) = \{1, 3, 7, 9, 11, 13, 17, 19\}$$

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Theorem (Euler Theorem (1736))

Let $m \in \mathbb{Z}^+$ *and* $a \in \mathbb{Z}$. *If* $(a, m) = 1$, *then*

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$$7^{222} \equiv 7^{4 \times 55 + 2} \equiv 7^2 \equiv 9 \mod 10$$

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$$\varphi(p) = p - 1$$

permutation

Definition (Subgroup (子群))

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If $(H, *)$ is a group, then we call H a subgroup of G, denoted $H \leq G$.

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If $H \subset G$, then H is a proper subgroup (真子群).

$$(H = \{mz \mid z \in \mathbb{Z}\}, +) \leq (\mathbb{Z}, +)$$

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$$H = \{1, 2, 4\} \leq G = \mathbb{Z}_7^* = \{1, 2, 3, 4, 5, 6\}$$

Theorem

Suppose that $H \leq G$.

(1) *The identity of H is the same with that of G.*

$$e_H = e_G$$

(2) *The inversion of a in H is the same with that in G.*

$$\forall a \in H.\ a_H^{-1} = a_G^{-1}$$

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Let G be a group and $\emptyset \neq H \subseteq G$. $H \leq G$ iff

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$$H_1 \cup H_2?$$

Center (中心)

Let G be a group. Let

$$C(G) \triangleq \{g \in G \mid gx = xg, \forall x \in G\}.$$

Then $C(G) \leq G$.

Definition (Isomorphism (同构))

Let $(G, \cdot)$ and $(G', *)$ be two groups. Let ϕ be a bijection such that

$$\forall a, b \in G.\ \phi(a \cdot b) = \phi(a) * \phi(b).$$

Then ϕ is an isomorphism from G to G'.

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Then ϕ is an isomorphism from G to G'.

G and G' are isomorphic

$$\phi : G \cong G'$$

$$(\mathbb{R}, +) \cong (\mathbb{R}^{+}, *)$$

$$(\mathbb{R}, +) \cong (\mathbb{R}^+, *)$$

$$\phi(x) = e^x$$

Theorem

Suppose that $\phi : G \cong G'$. *Let* e *and* e' *are identities of* G *and* G', *respectively.*

(1) $\phi(e) = e'$

(2) $\phi(a^{-1}) = (\phi(a))^{-1}$

(3) $\phi^{-1} : G' \cong G$

Klein Four-group (四元群; K_4; V)

*	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$$a^2 = b^2 = c^2 = (ab)^2 = e$$

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$$U(8) = \{1, 3, 5, 7\}$$

permutation?

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The order of e is the smallest positive integer r such that $a^r = e$.

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If such r does not exist, then $\text{ord } a = \infty$.

$$\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$$

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$$\mathbb{Z}_5^* = \{1, 2, 3, 4\}$$

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$$(\mathbb{Z}, +)$$

Theorem

Let G be a group and e be the identity of G.

$$\big((\mathit{ord}\, a = n) \land (\exists m \in \mathbb{Z}.\ a^m = e)\big) \implies n \mid m.$$

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If $r > 0$,

$$a^r = a^{m-nq} = a^m \cdot (a^n)^{-q} = e \cdot e = e$$

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$$\text{ord}\, a \neq n$$

Definition (Cyclic Group (循环群))

Let G be a group. If

$$\exists a \in G. \ G = \langle a \rangle \triangleq \{a^0 = e, a, a^2, a^3, \ldots\},$$

then G is a cyclic group.

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then G is a cyclic group.

If $G = \langle a \rangle$, then a is a generator (生成元) of G.

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$$(\mathbb{Z}_m, +) = \langle 1 \rangle$$

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$$\mathbb{Z}_5^* = \langle 2 \rangle = \langle 3 \rangle$$

Theorem

(1) *Let* $G = \{e, a, a^{-1}, a^2, a^{-2}, \dots\}$ *be an infinite cyclic group.*

$$\forall k, l \in \mathbb{Z}.\ (a^k = a^l \to k = l).$$

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$$\forall k, l \in \mathbb{Z}. \ (a^k = a^l \to k = l).$$

(2) *Let* $G = \{e, a, a^2, \dots, a^{n-1}\}$ *be a finite cyclic group of order* n.

$$\forall k, l \in \mathbb{Z}. \ (a^k = a^l \leftrightarrow n \mid (k - l)).$$

Theorem (Structure Theorem of Cyclic Groups (循环群结构定理))

Let $G = \langle a \rangle$ be a cyclic group.

(1) *If $G = \langle a \rangle$ is an infinite cyclic group, then $G \cong (\mathbb{Z}, +)$.*

(2) *If $G = \langle a \rangle$ is an finite cyclic group of order n, then $G \cong (\mathbb{Z}_n, +)$.*

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Let $G = \langle a \rangle$ be a cyclic group.

(1) *If $G = \langle a \rangle$ is an infinite cyclic group, then $G \cong (\mathbb{Z}, +)$.*

(2) *If $G = \langle a \rangle$ is an finite cyclic group of order n, then $G \cong (\mathbb{Z}_n, +)$.*

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$$\text{ord } a^r = \frac{n}{(n, r)}$$

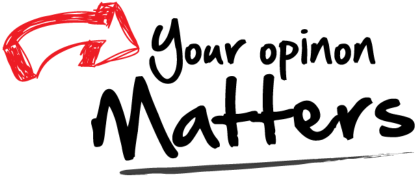
$$(\mathbb{Z}_{12}, +)$$

$$\text{Generators} : 1, 5, 7, 11$$

Theorem (Subgroups of Cyclic Groups)

Every subgroup of a cyclic group is cyclic.


Thank You!



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