(十二) 图论: 对偶
(Duality: Matching, Network Flow, and Connectivity)

魏恒峰

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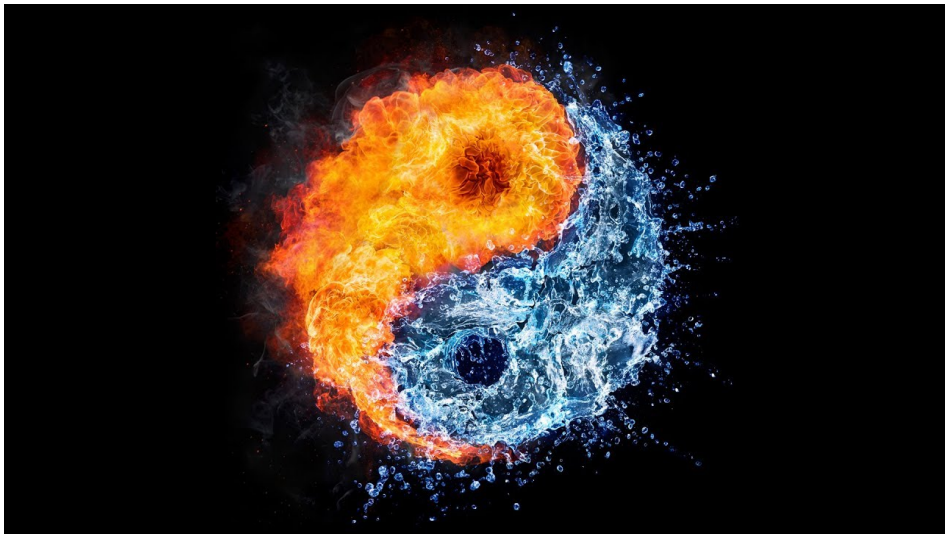
2021 年 05 月 27 日



let's get married today

The Marriage Problem (Philip Hall, 1935)

If there is a finite set of girls, each of whom knows several boys, under what conditions can all the girls marry boys in such a way that each girl marries a boy that she knows?

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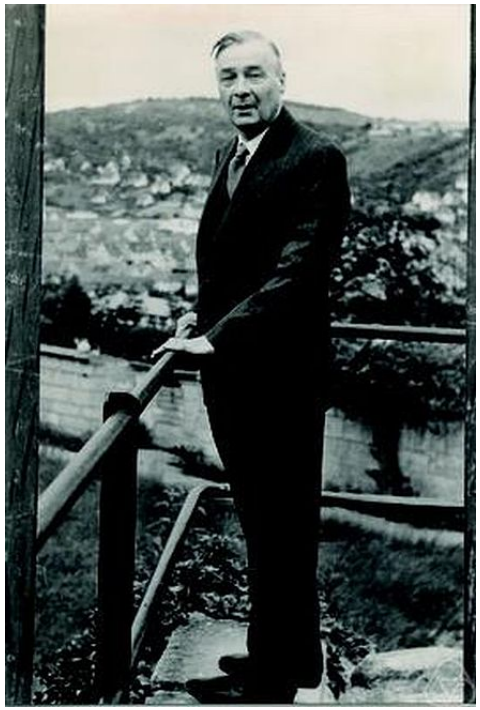
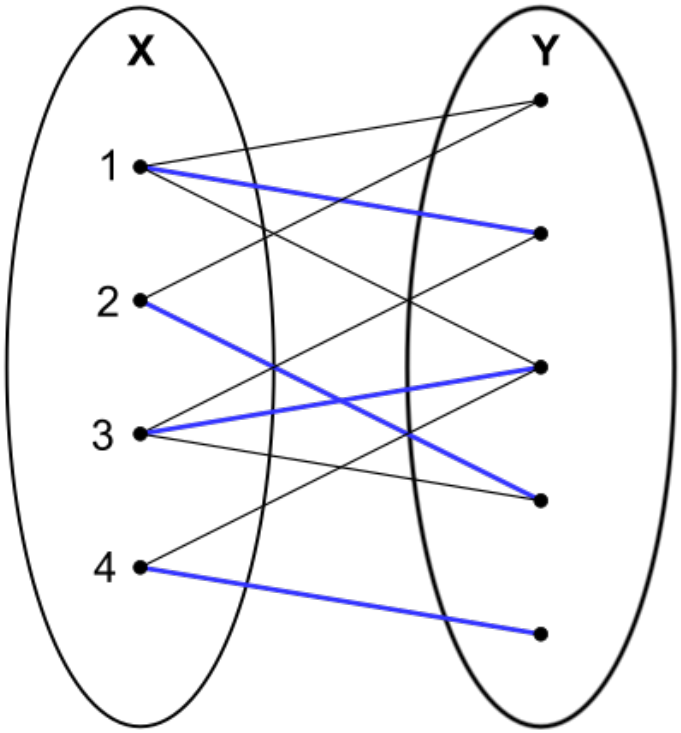
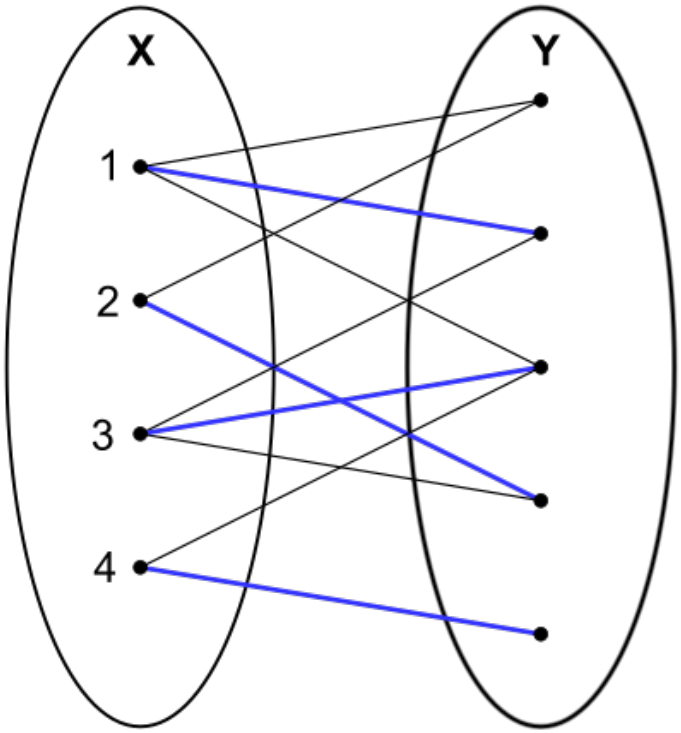


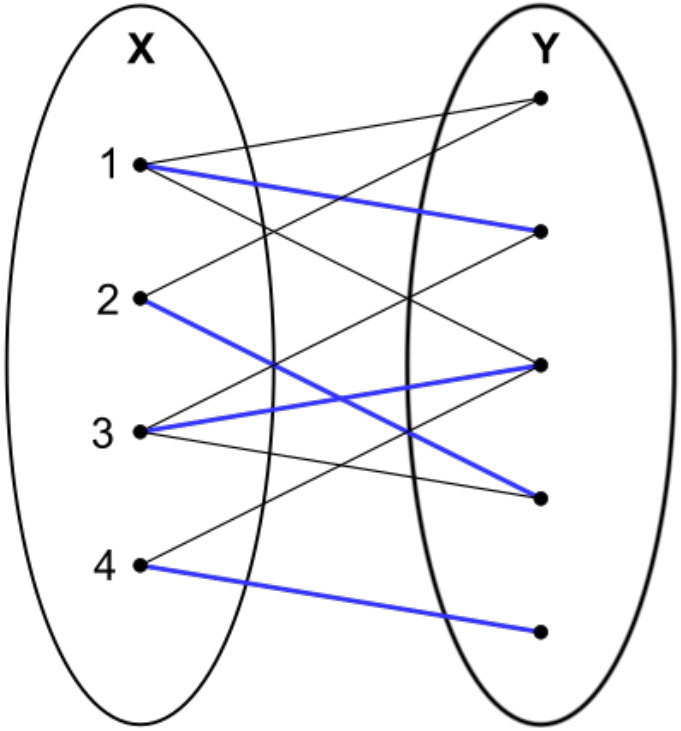
Philip Hall (1904 ~ 1982)


X
Y
1
2
3
4



Definition (Matching (匹配))

A matching in a graph G is a set of edges with no shared endpoints.

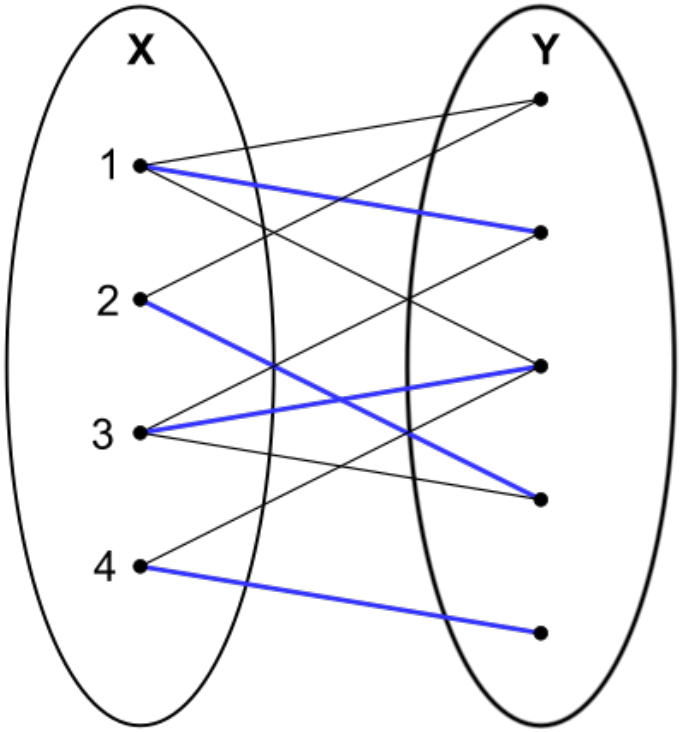


Definition (X-Perfect Matching (X-Saturating Matching))

Let $G = (X, Y, E)$ be a bipartite graph.

An X-perfect matching of G is a matching which covers each vertex in X.

$$\left|X\right| \leq \left|Y\right|$$



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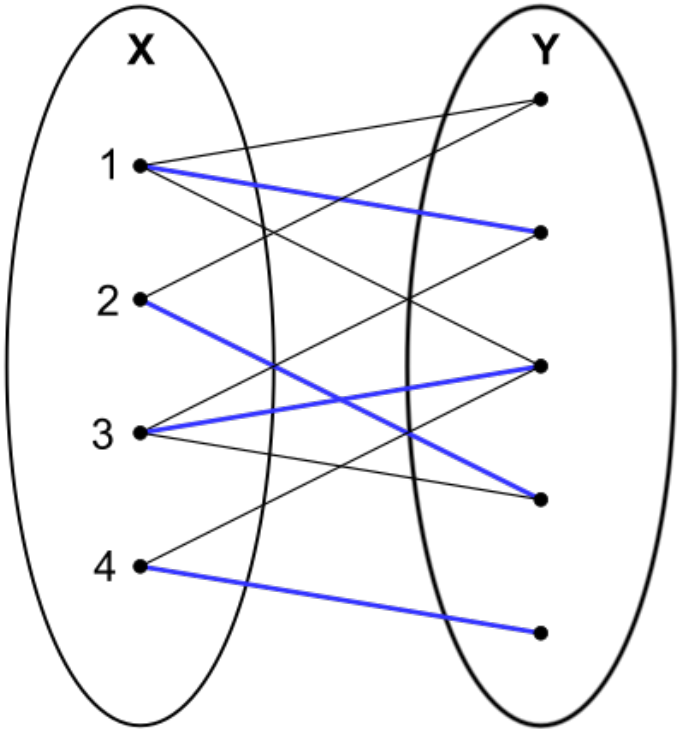

$$\forall W \subseteq X.\ \left|W\right| \leq \left|N(W)\right|$$

Theorem (Hall Theorem; 1935)

Let $G = (X, Y, E)$ be a bipartite graph. There is an X-perfect matching of G iff

$$\forall W \subseteq X. \left|W\right| \leq \left|N_G(W)\right|$$

By induction on the number $\left|X\right|$ of vertices in X.

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Induction Hypothesis: Suppose that it holds if $\left|X\right| < m$.

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- Case II: There is a set of $k < m$ girls in X who know k boys in Y.

Theorem (Hall Theorem; 1935)

Let $G = (X, Y, E)$ be a bipartite graph. There is a X-perfect matching of G iff

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Definition (M-alternating Paths)

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Definition (M-augmenting Paths)

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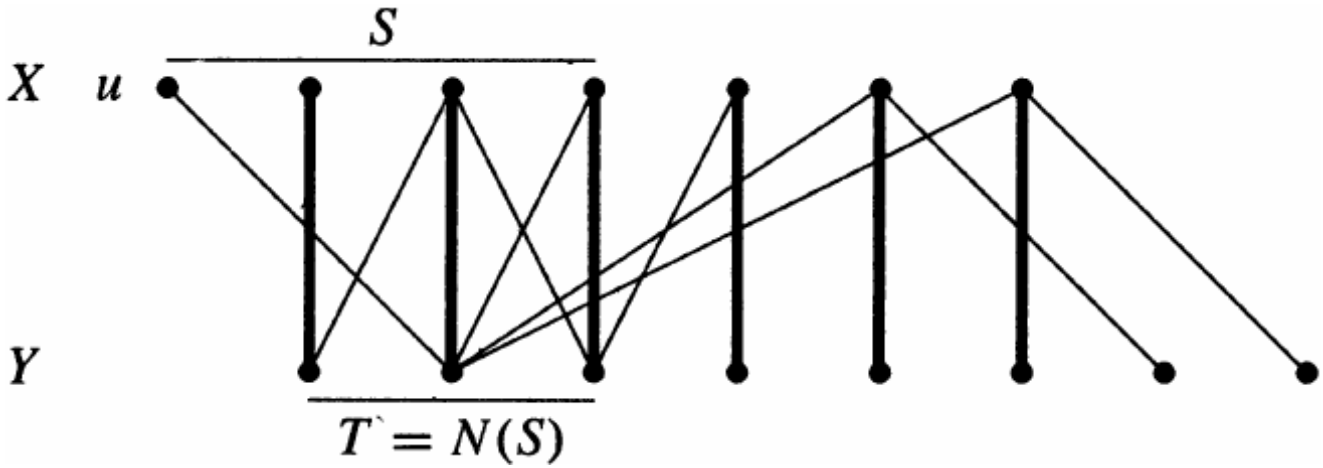

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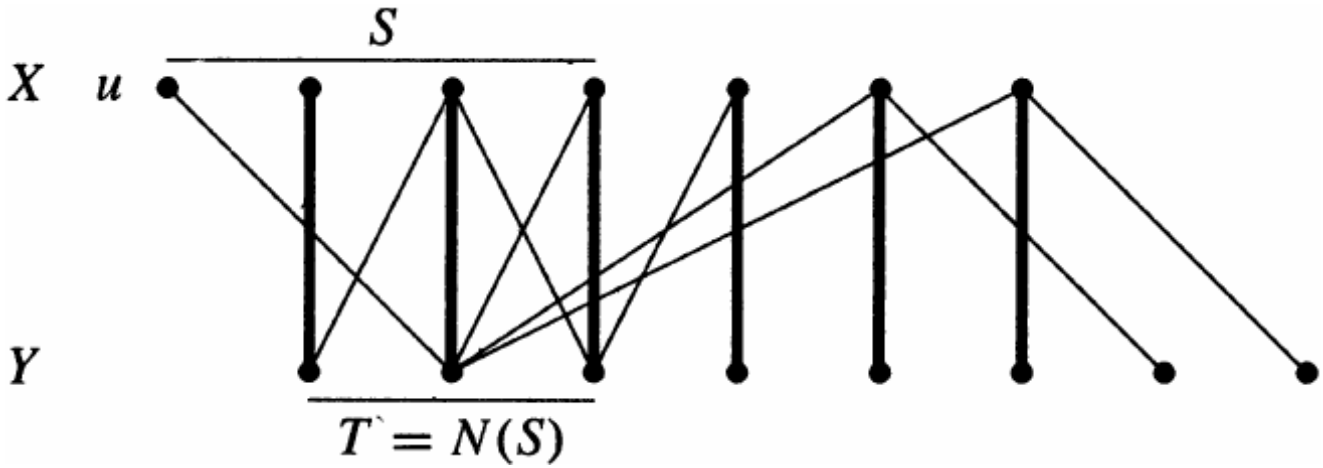

Let M be a *maximum* matching.

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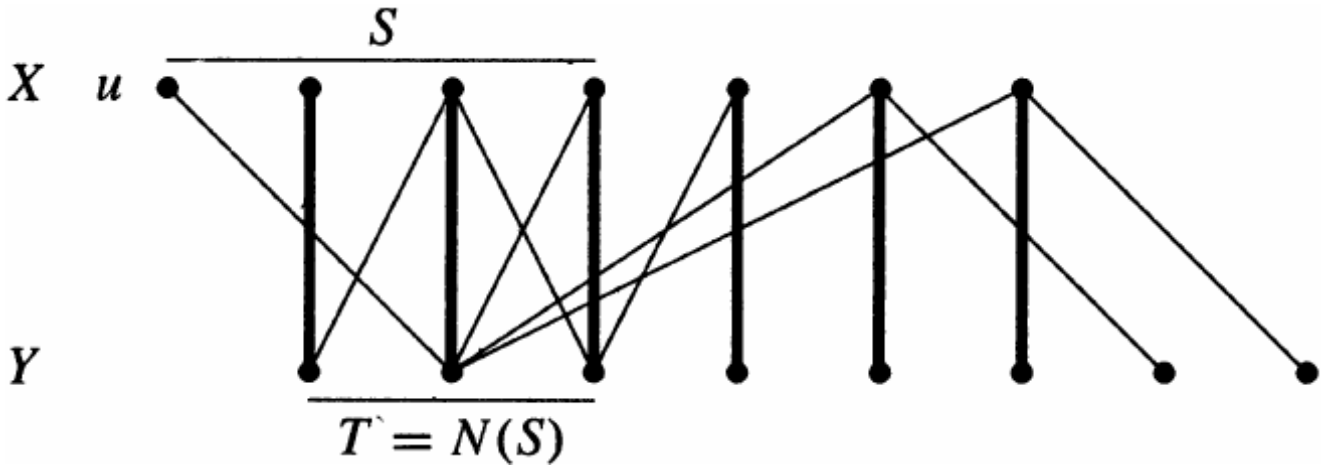

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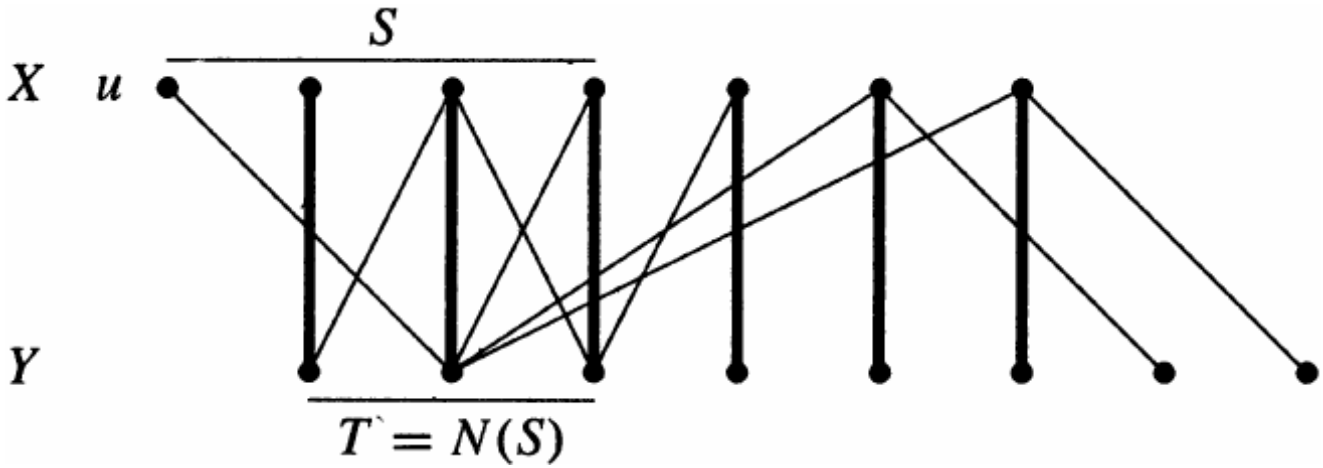

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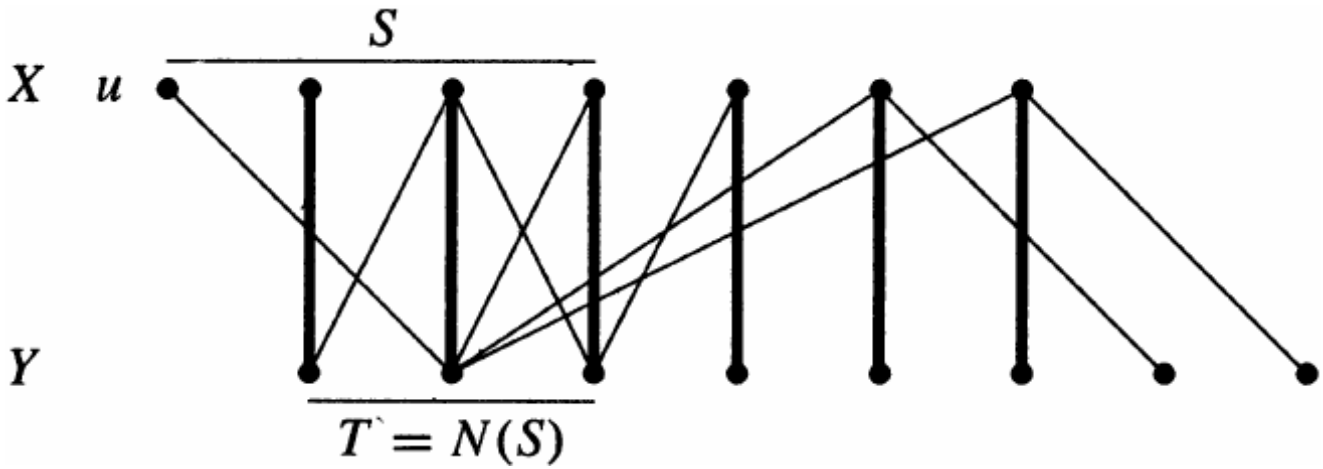
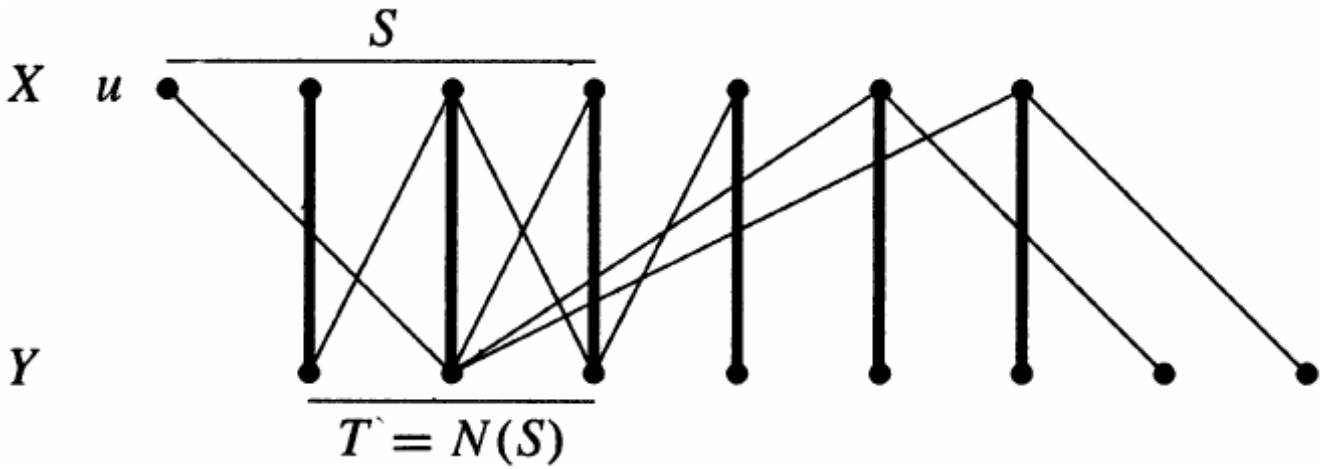
Consider all *M-alternating paths* starting from u.



$T \triangleq$ the set of vertices in Y reachable from u by M-alternating paths.



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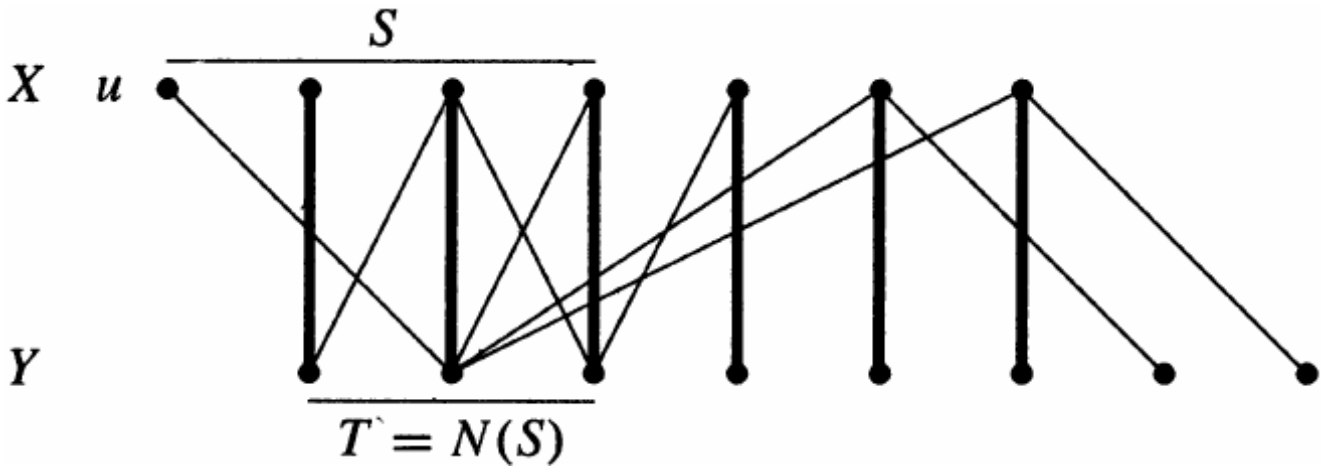

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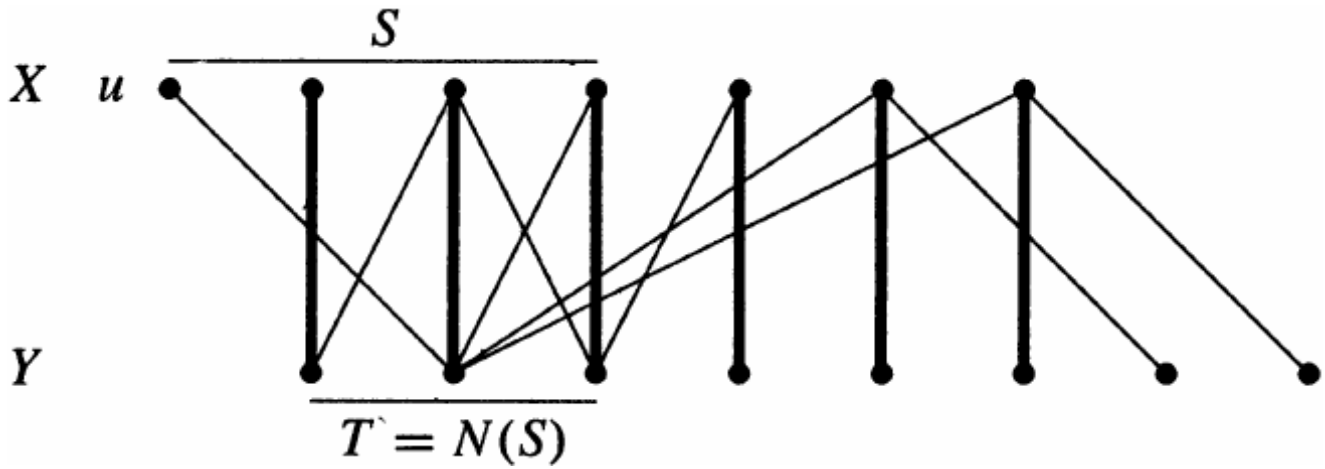
We will show that

$$T = N(S) \wedge \left|T\right| = \left|S - \{u\}\right|$$



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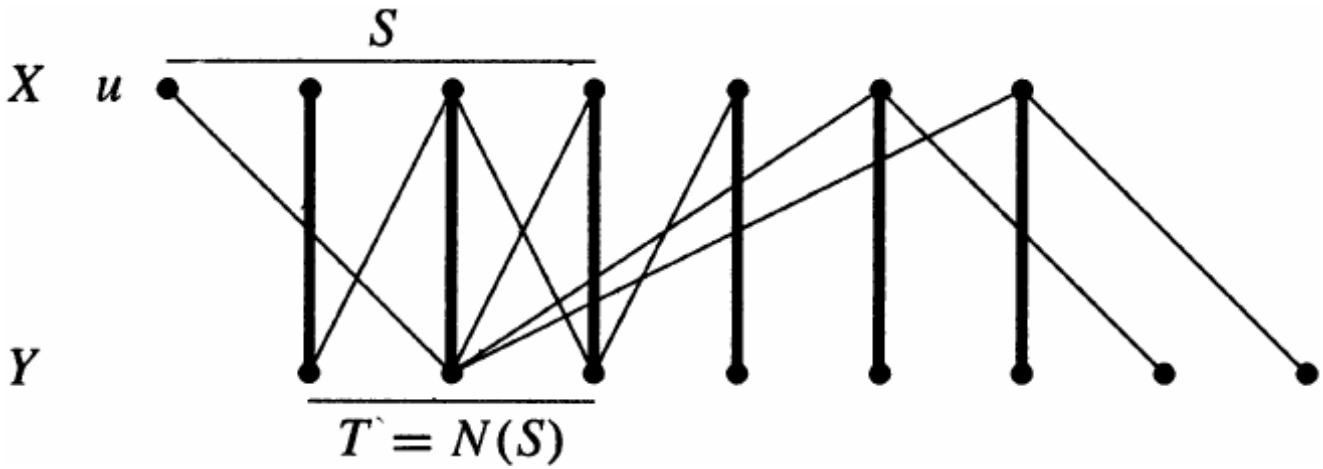

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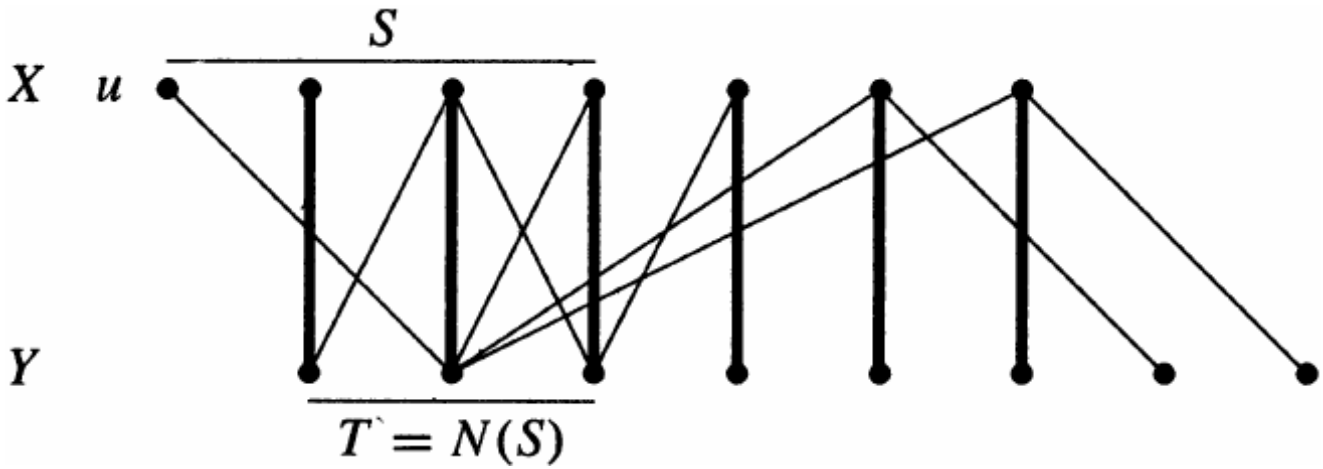

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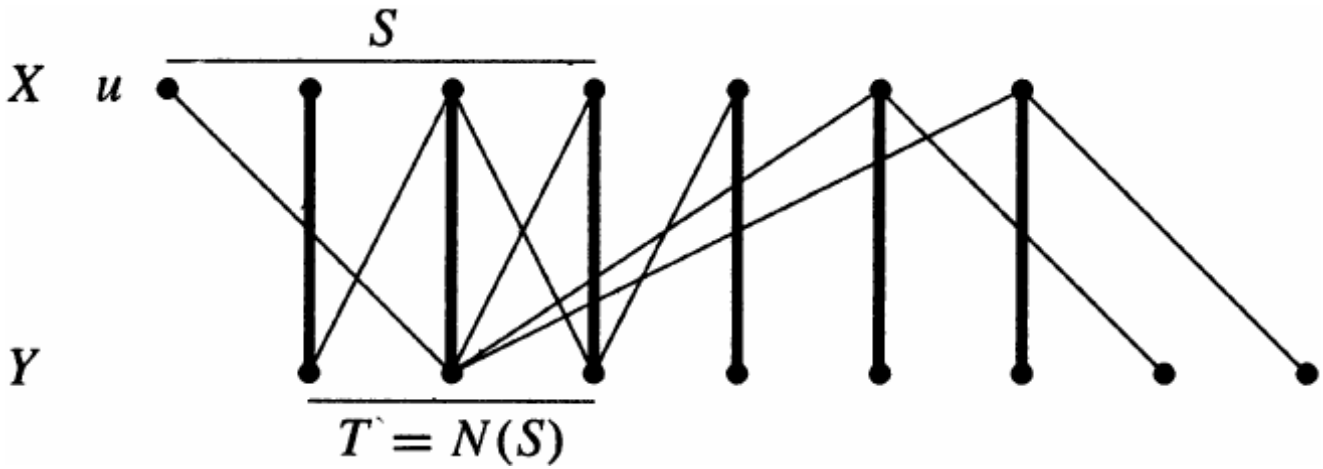

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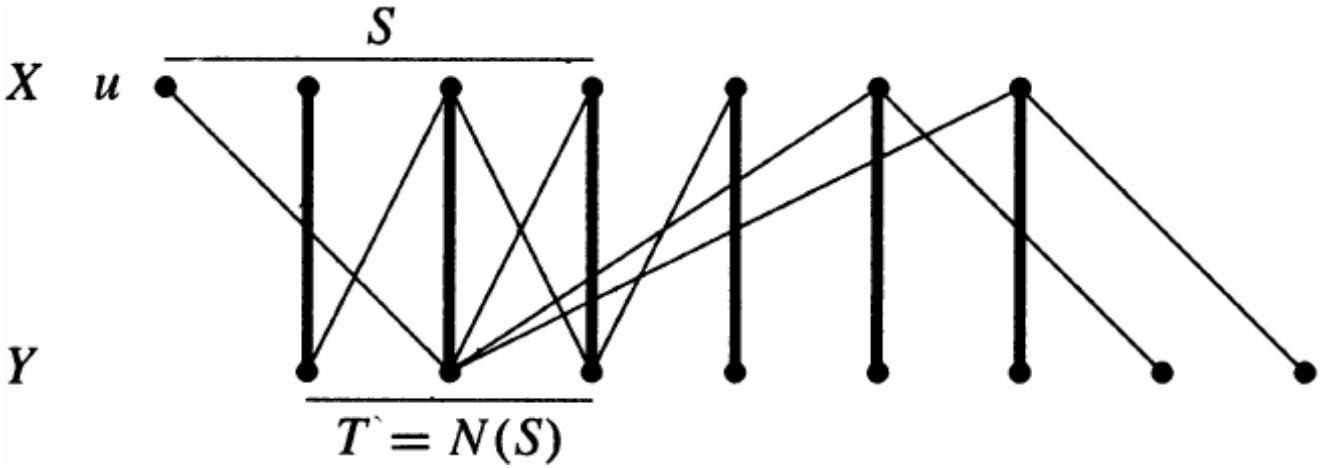

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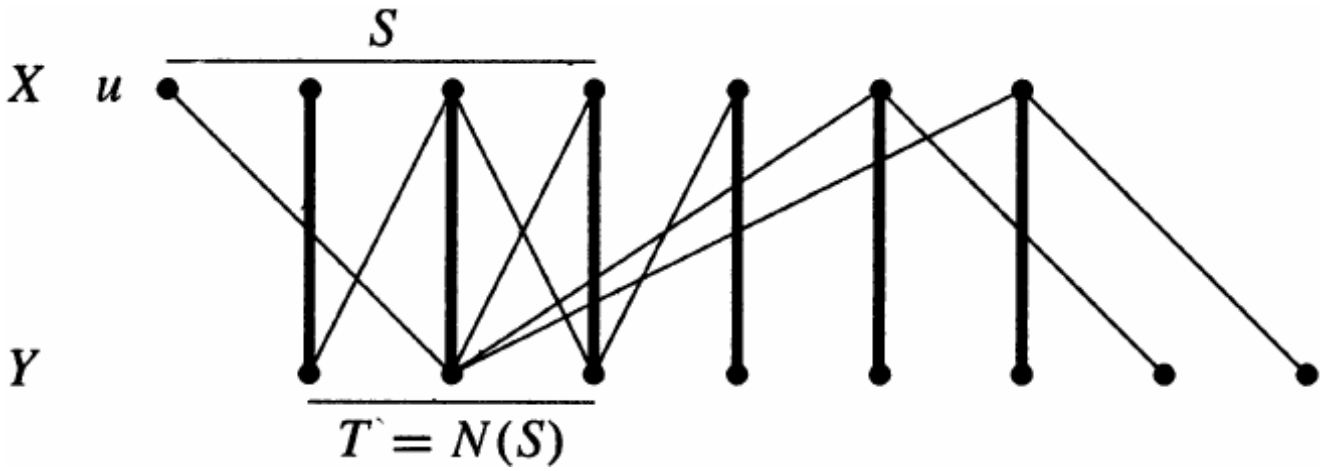

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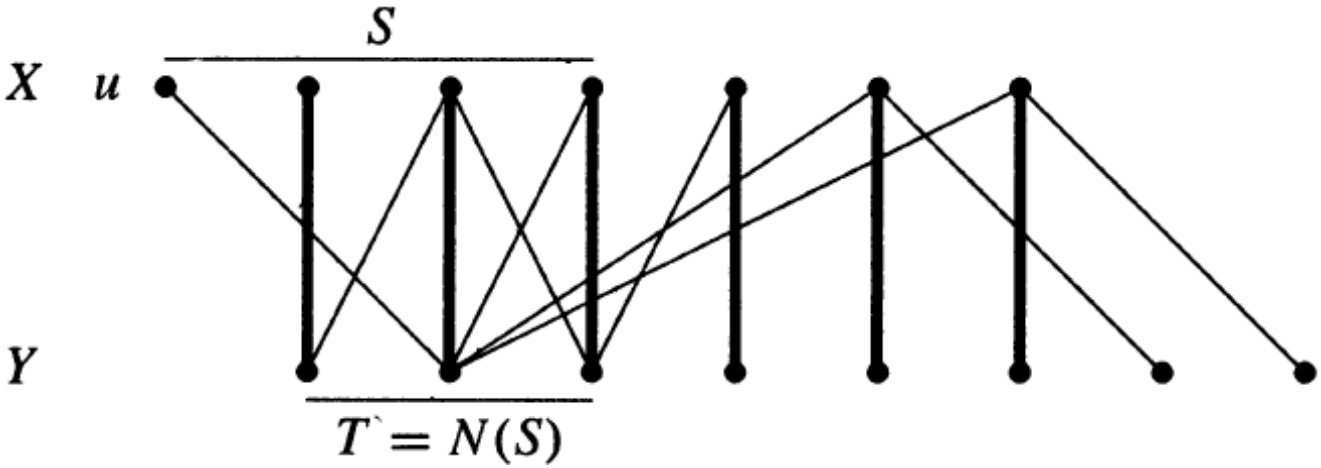

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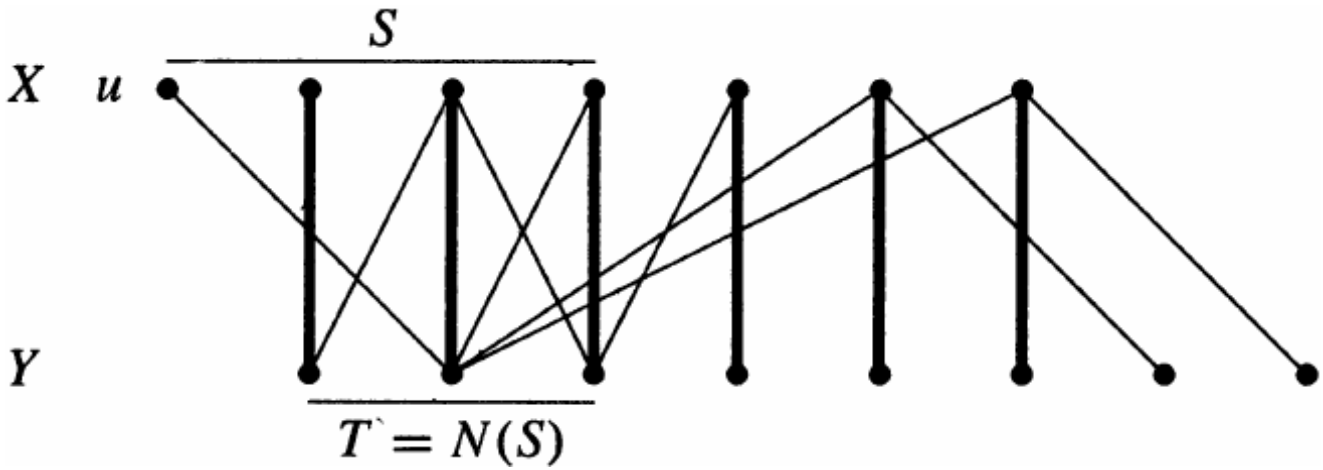

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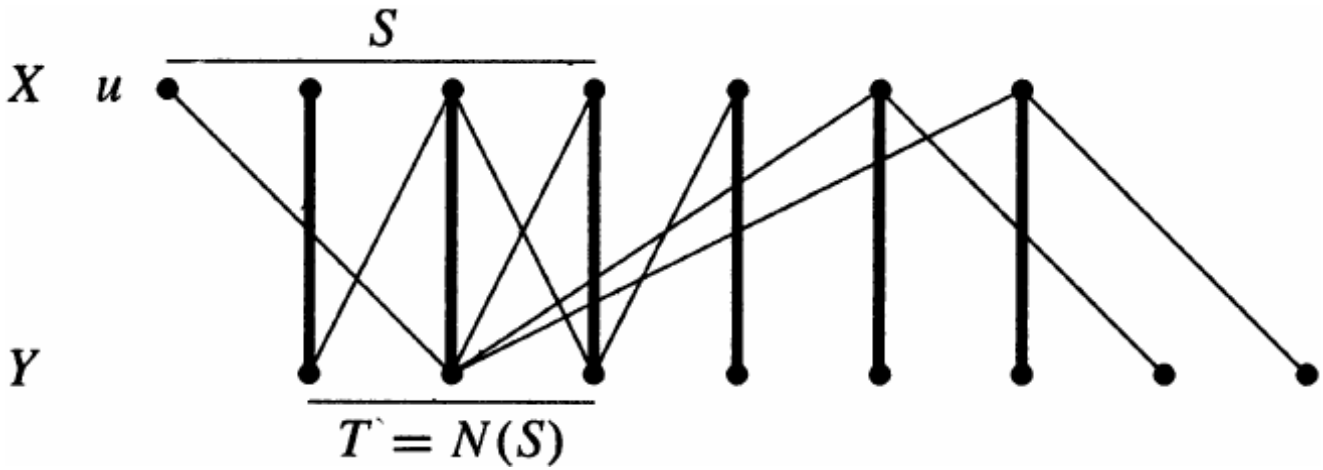

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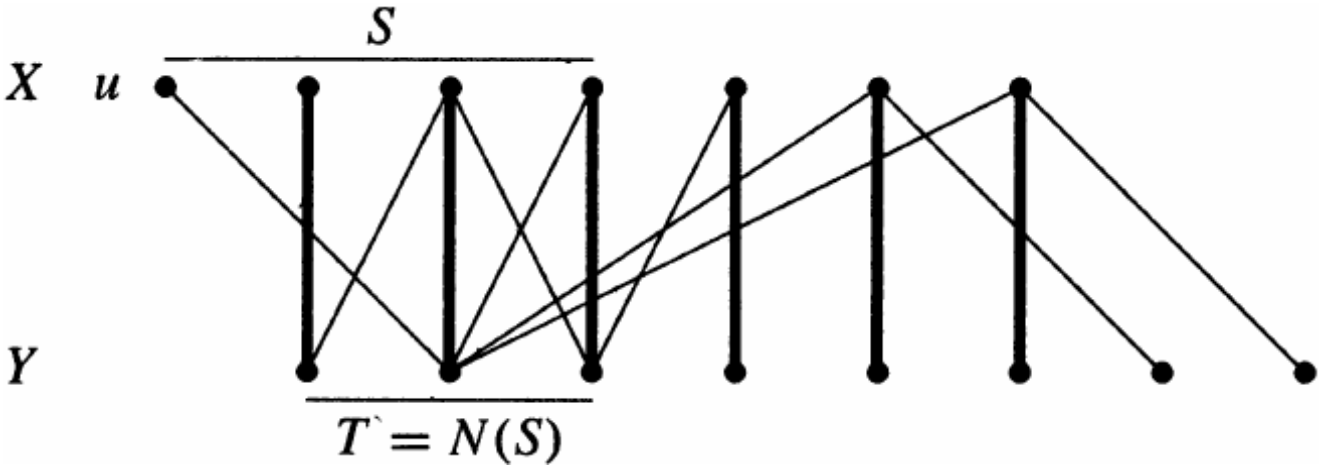

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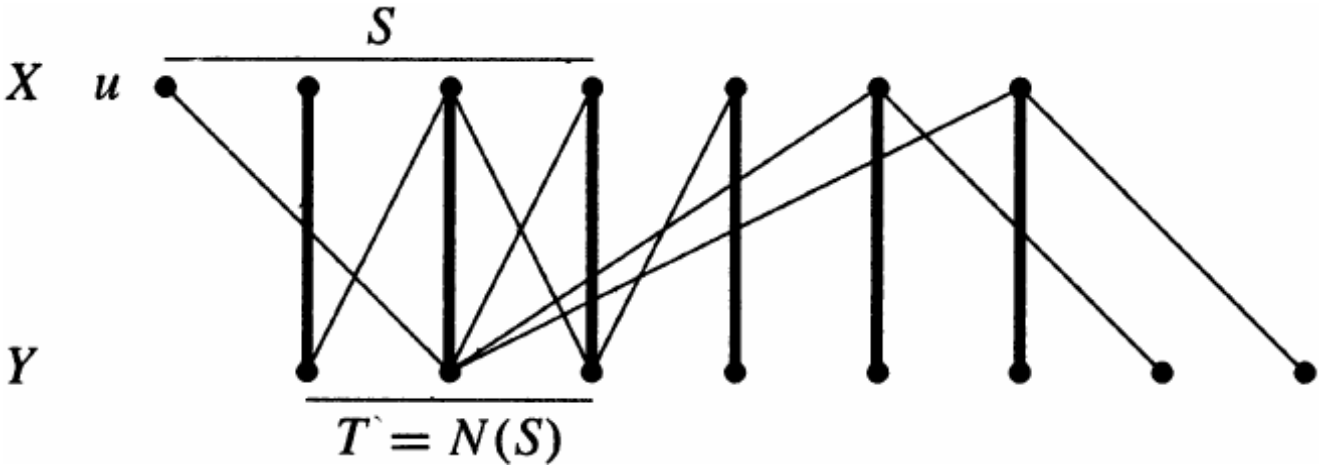

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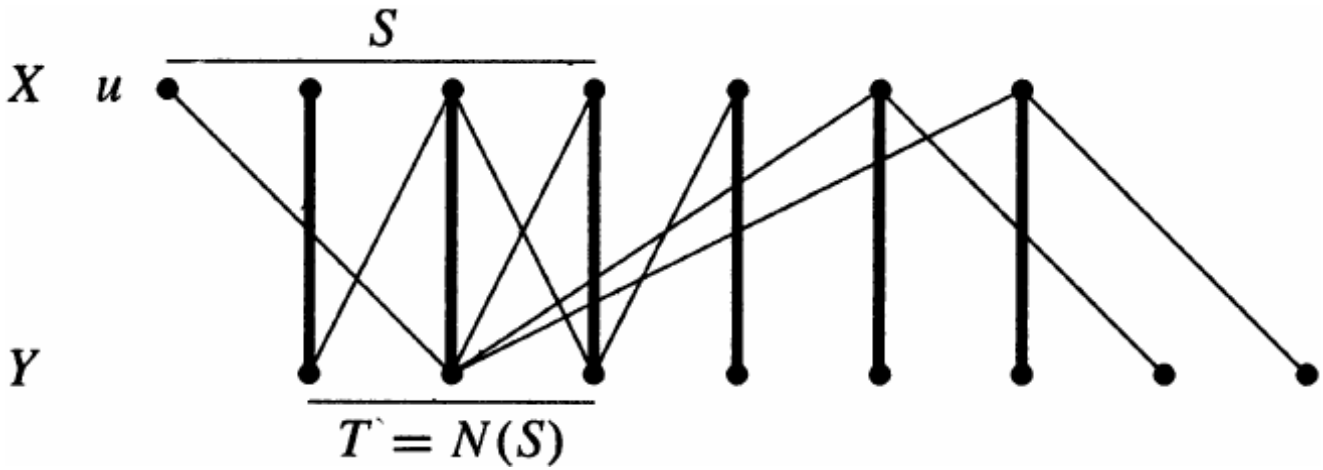

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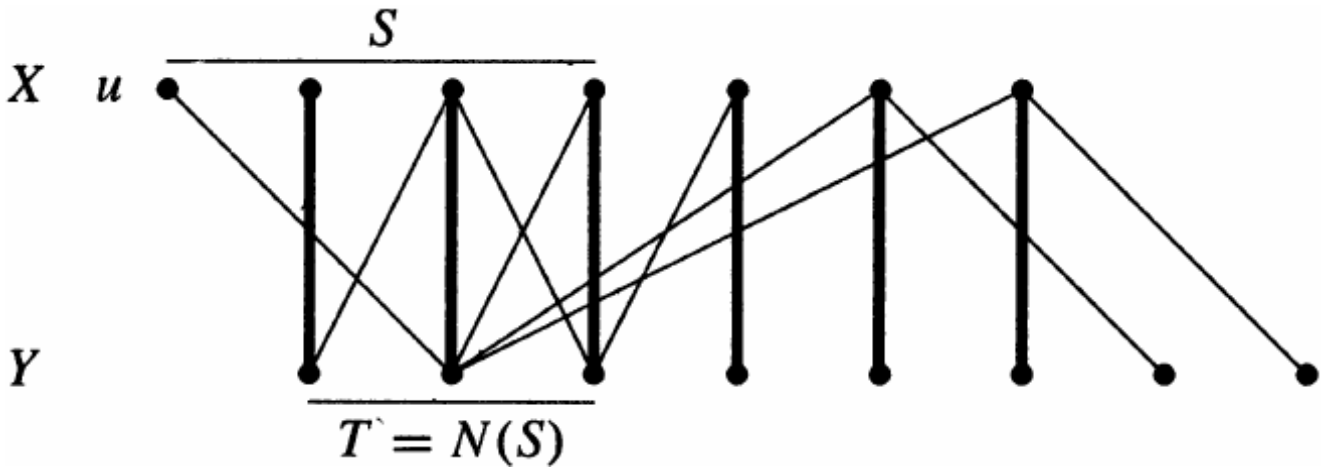

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Theorem (Hall Theorem; 1935)

Let $G = (X, Y, E)$ be a bipartite graph. There is a X-perfect matching of G iff

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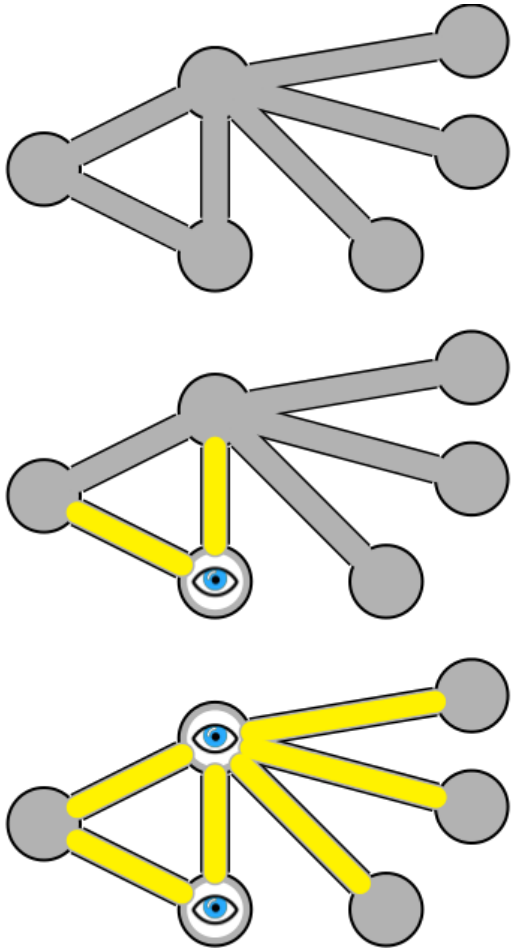
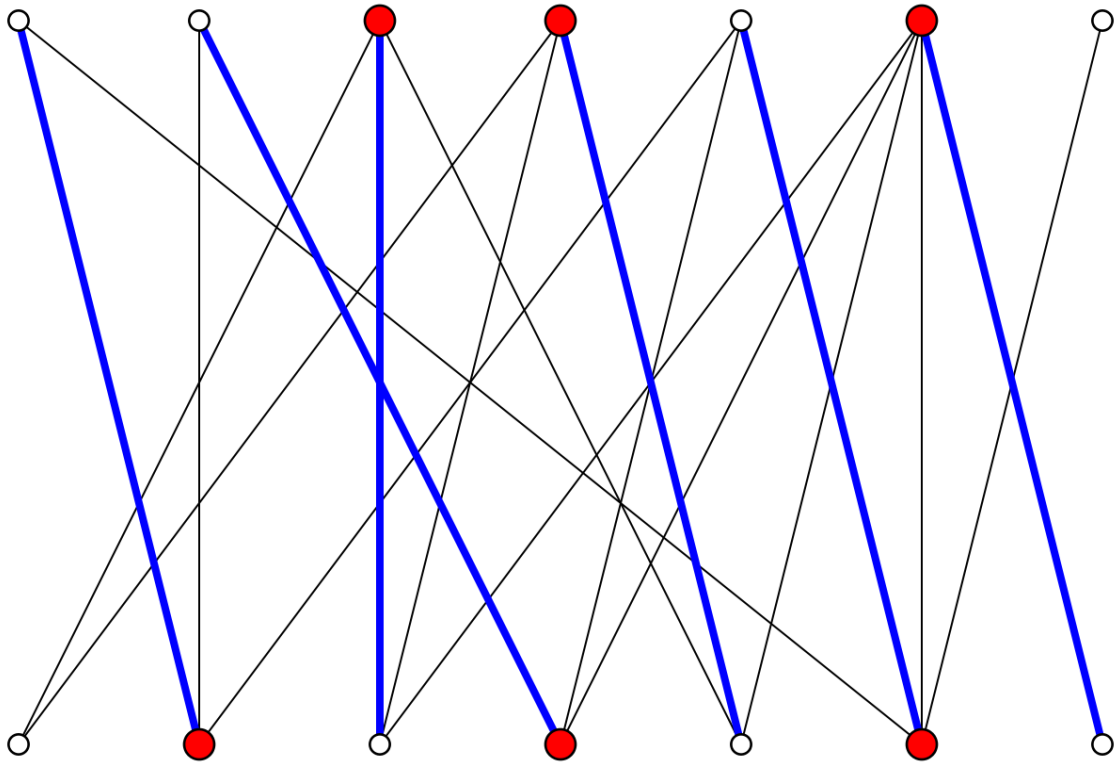
algorithm

Definitions (Vertex Cover (点覆盖))

A vertex cover of a graph G is a set $Q \subseteq V(G)$ that covers all edges.

$$\forall e \in E(G).\ e \cap Q \neq \emptyset$$

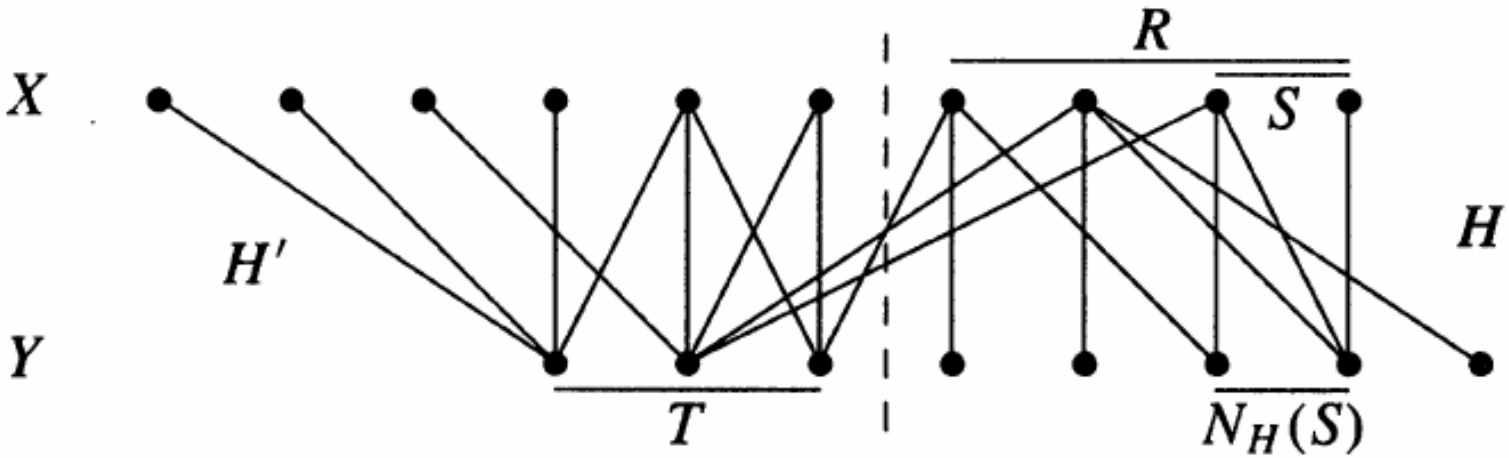


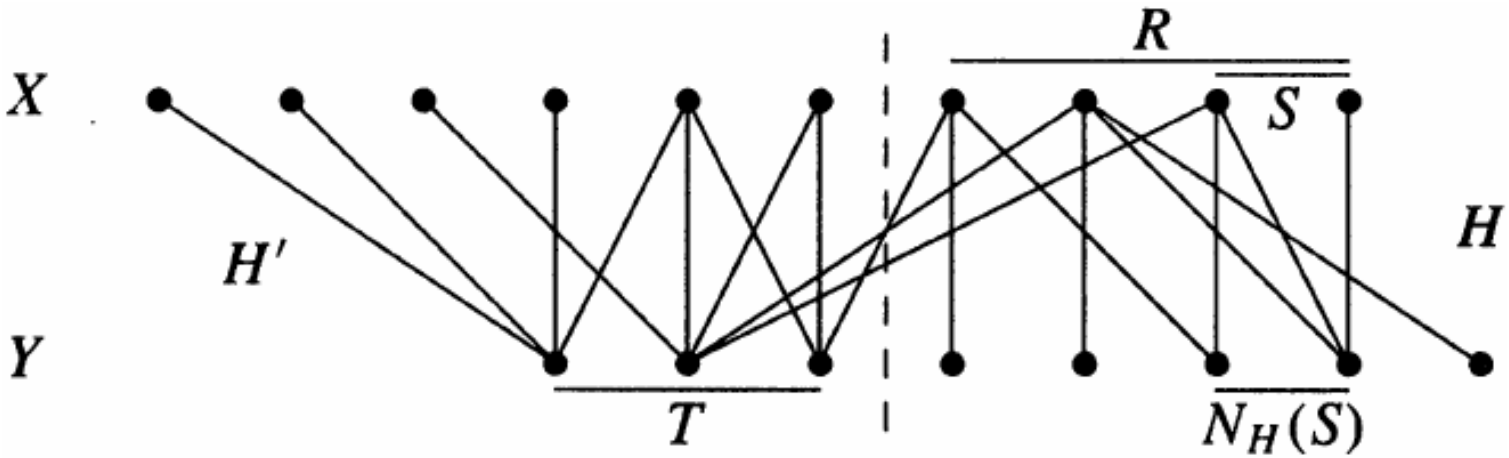


examples

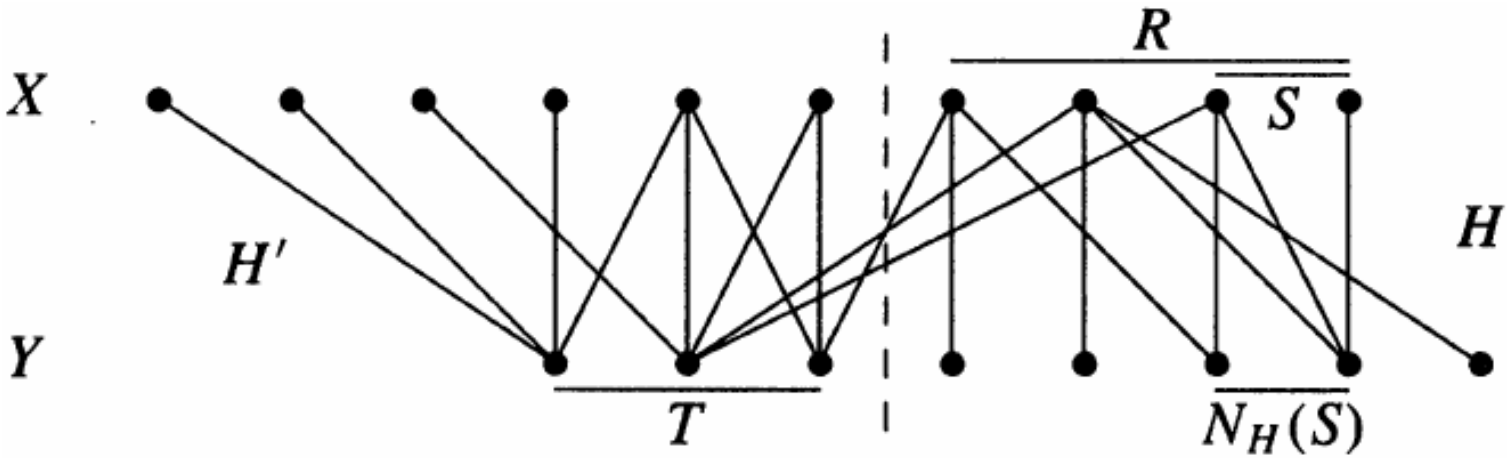
Theorem (König (1931), Egerváry (1931))

If G is a bipartite graph, then the maximum size of a mathching in G equals the minimum size of a vertex cover of G


X
Y
H′
T
R
S
H
$N_H(S)$



$$R = Q \cap X \qquad T = Q \cap Y$$



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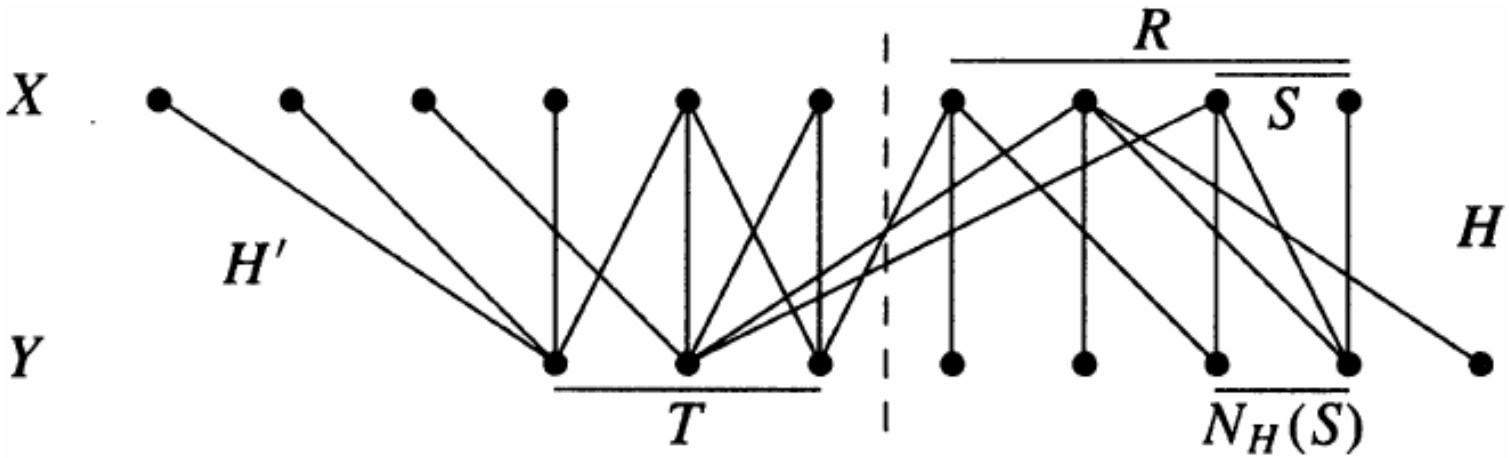
$$H \triangleq (R \cup (Y - T))\text{-induced subgraph of } G$$

$$H' \triangleq (T \cup (X - R))\text{-induced subgraph of } G$$



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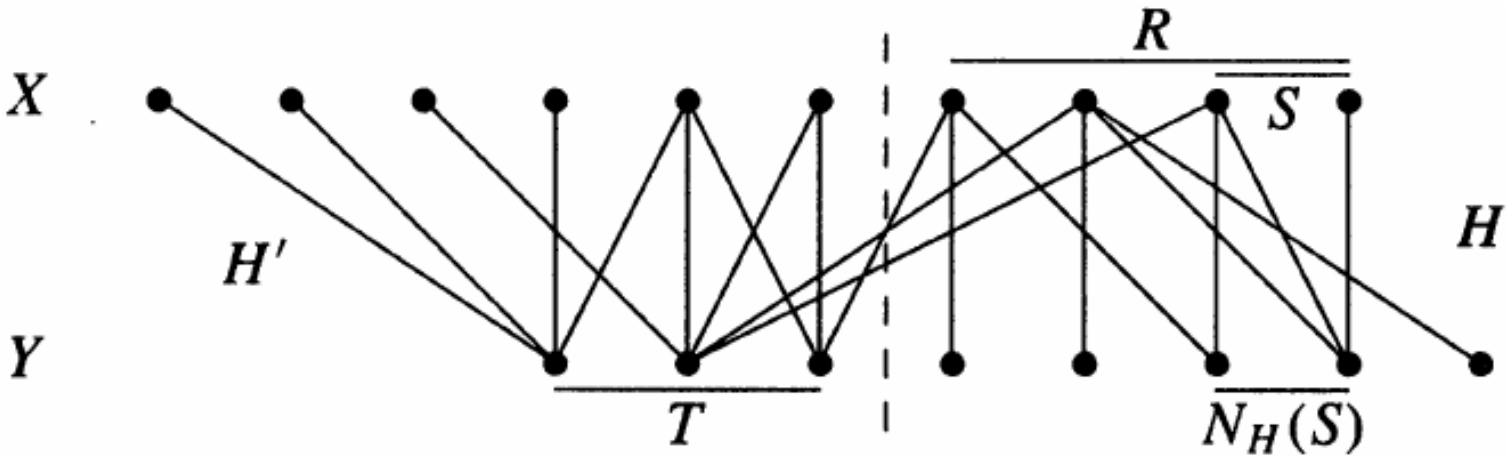
G has no edges from $X - R$ to $Y - T$.



$$R = Q \cap X \qquad T = Q \cap Y$$

$$H \triangleq (R \cup (Y - T))\text{-induced subgraph of } G$$

$$H' \triangleq (T \cup (X - R))\text{-induced subgraph of } G$$

$$G \text{ has no edges from } X - R \text{ to } Y - T.$$

H has a R-perfect matching and H' has a T-perfect matching.

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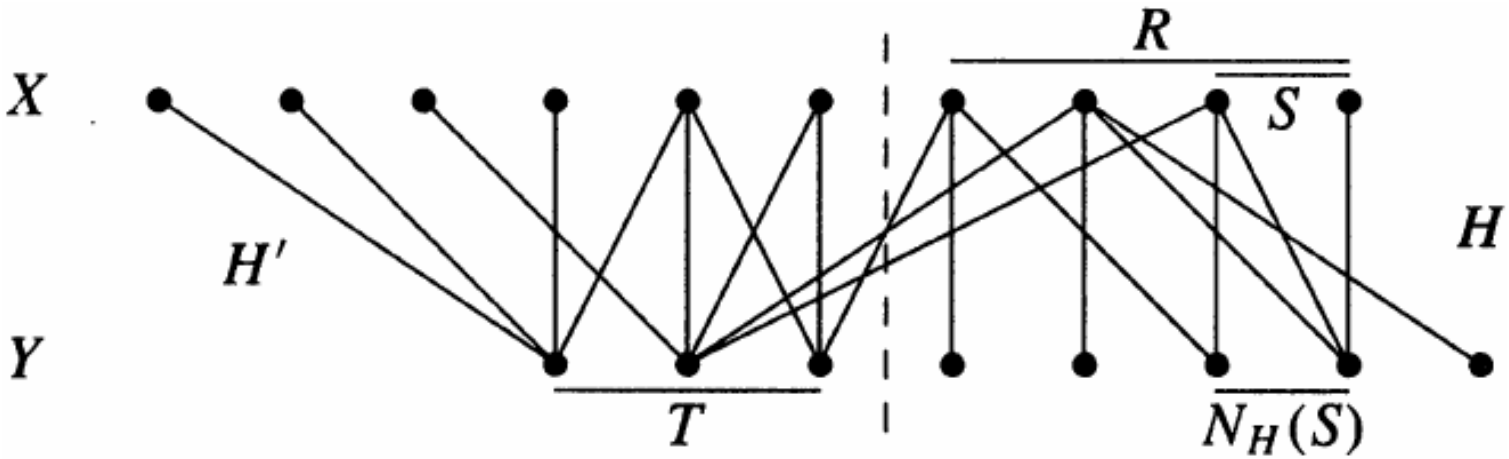

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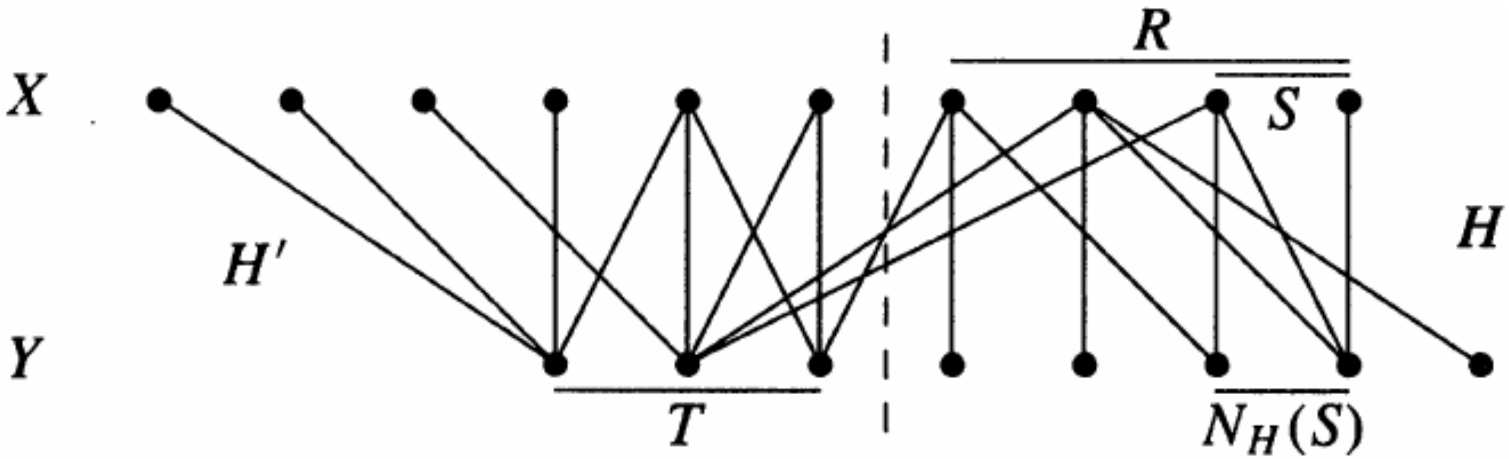

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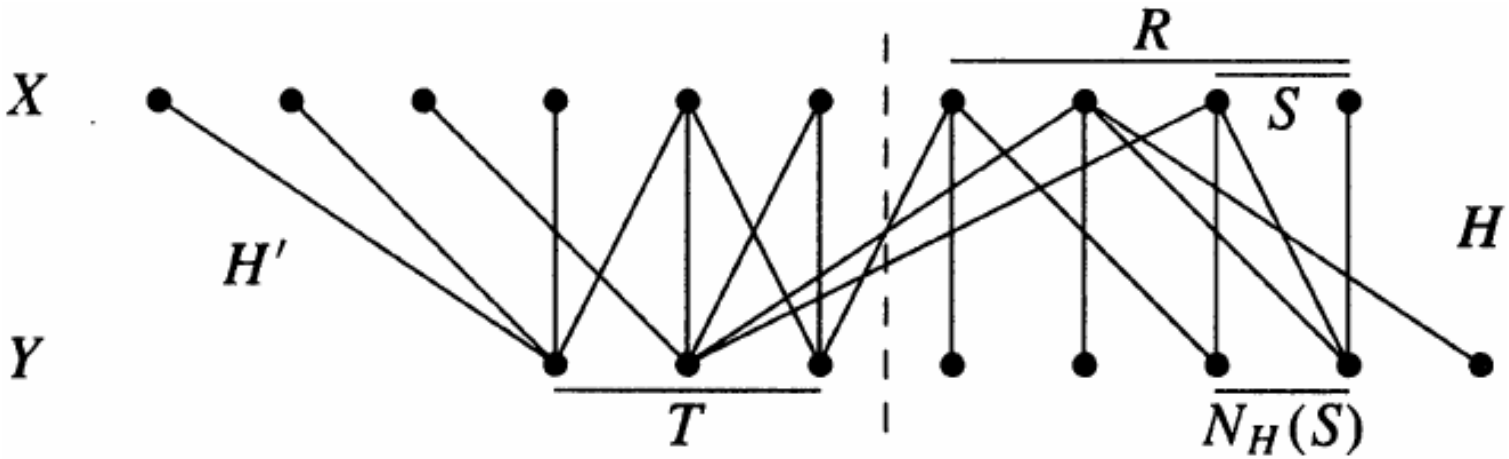

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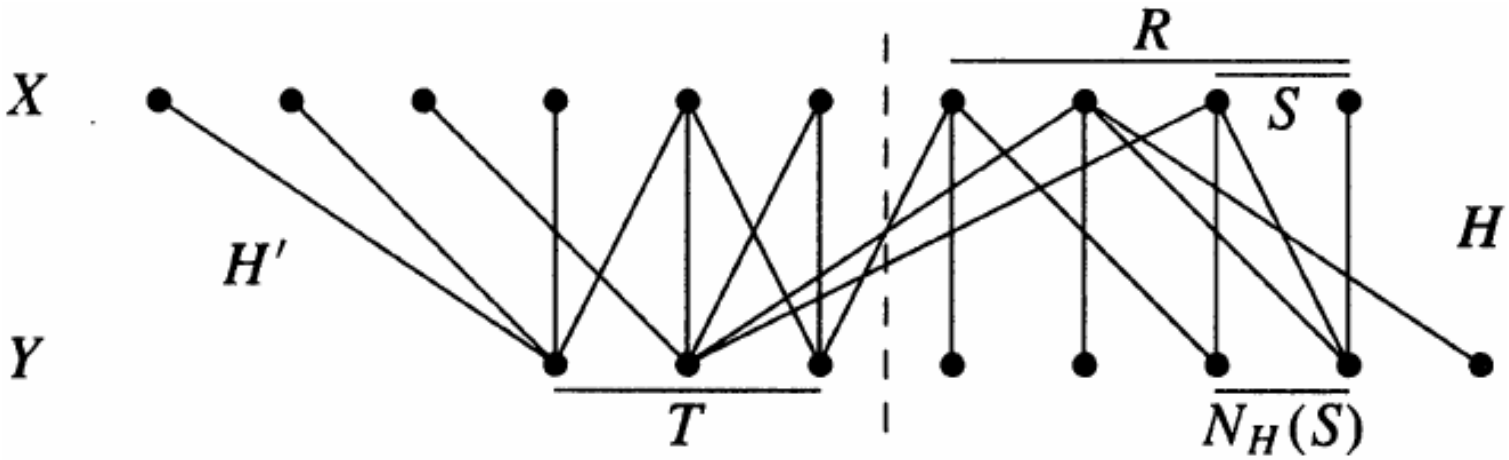


By contradition.

$$\exists S \subseteq R.\ \left|N_H(S)\right| < \left|S\right|$$

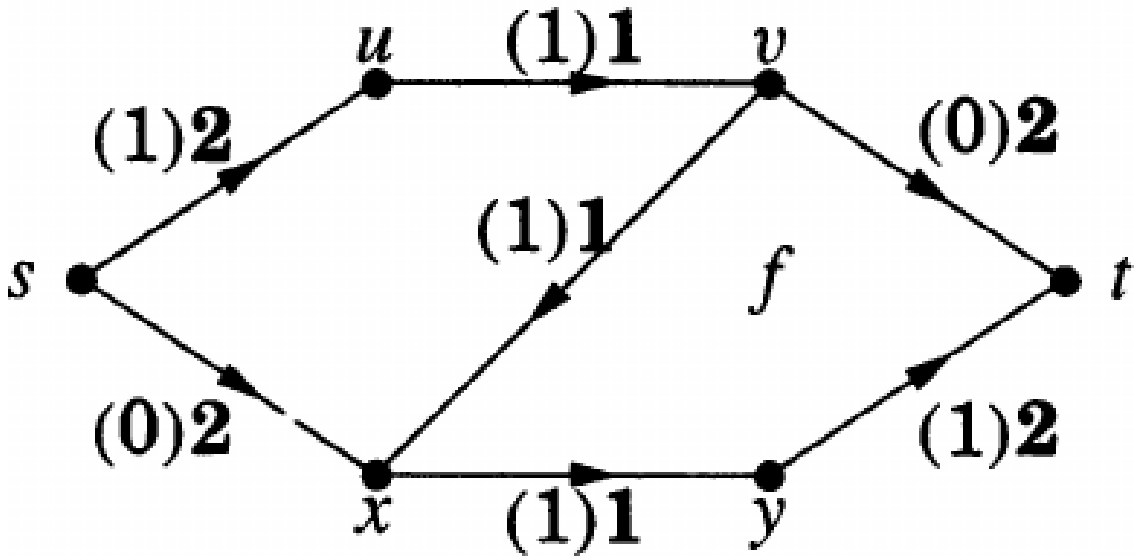
$T \cup (R - S + N_H(S))$ is a smaller vertex cover than Q

independent sets and covers

Definition (Network (网络))

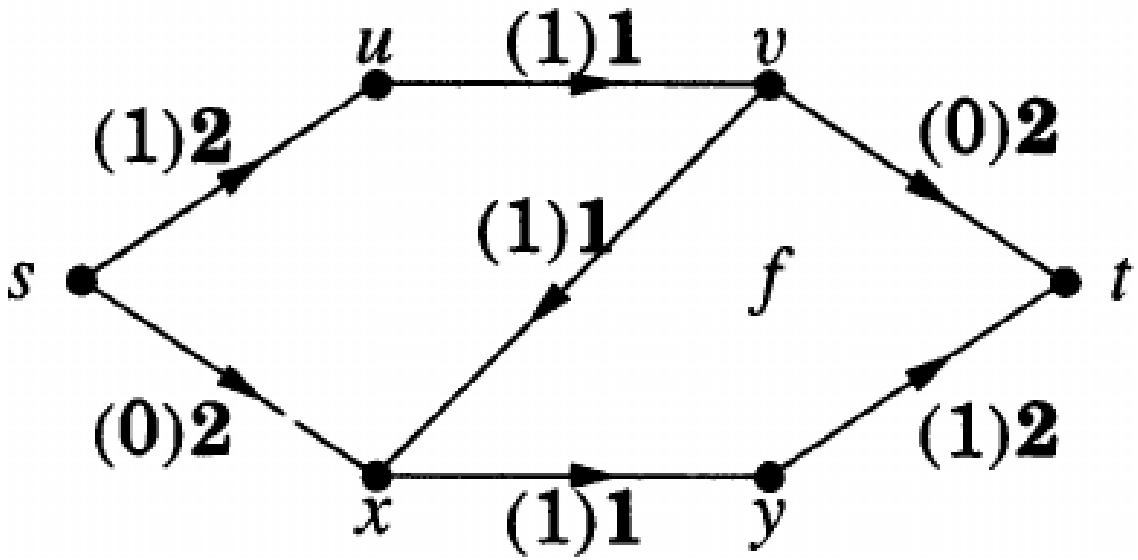
A network is a digraph with

- a distinguished source vertex s,
- a distinguished sink vertex t,
- a capacity $c(e) \geq 0$ on each edge e



Definition (Flow (流))

A flow f is a function that assigns a value $f(e)$ to each edge e.



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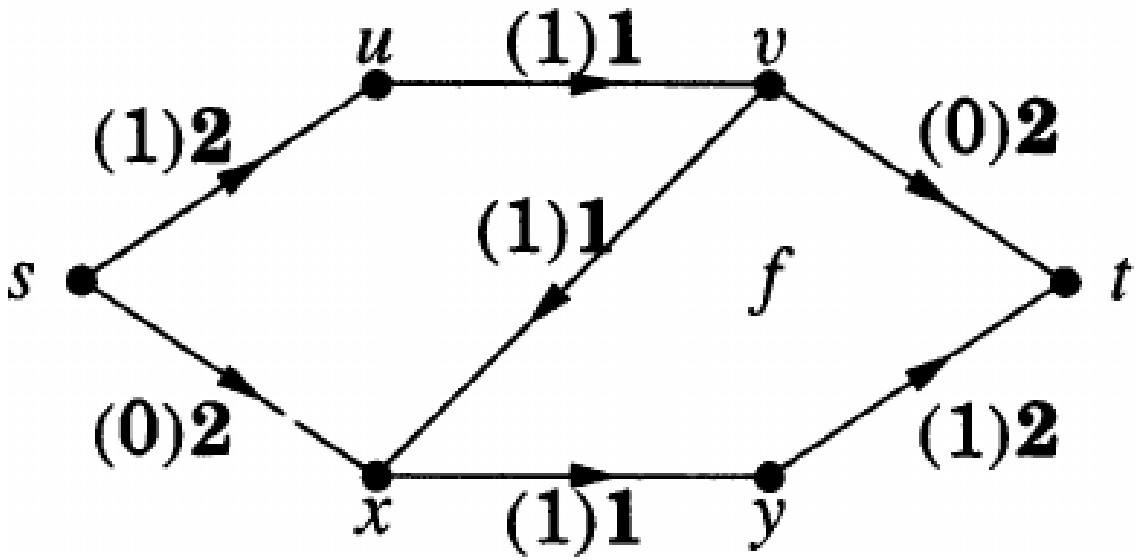

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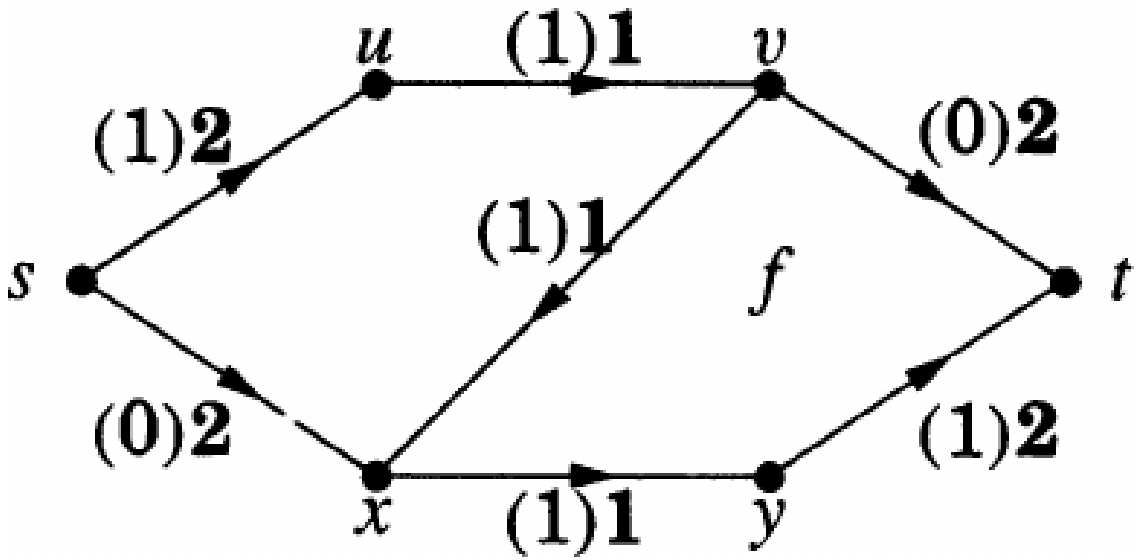


$$f^+(v) = \sum_{vw \in E} f(vw) \qquad f^-(v) = \sum_{uv \in E} f(uv)$$

Definition (Feasible)

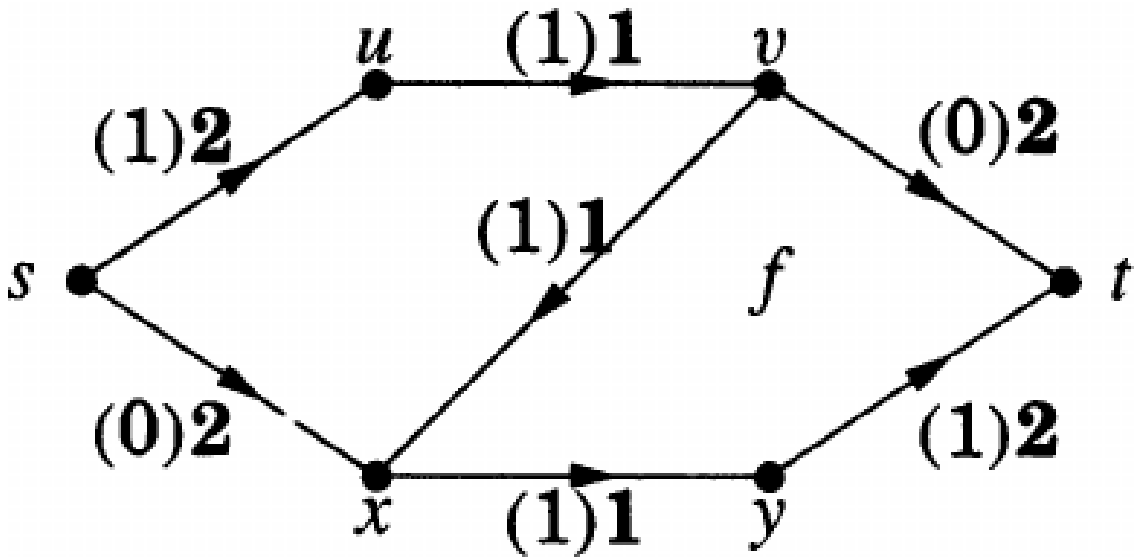
A flow f is feasible if it satisfies

Capacity Constraints:

$$\forall e \in E(G).\ 0 \leq f(e) \leq c(e)$$

Conservation Constraints:

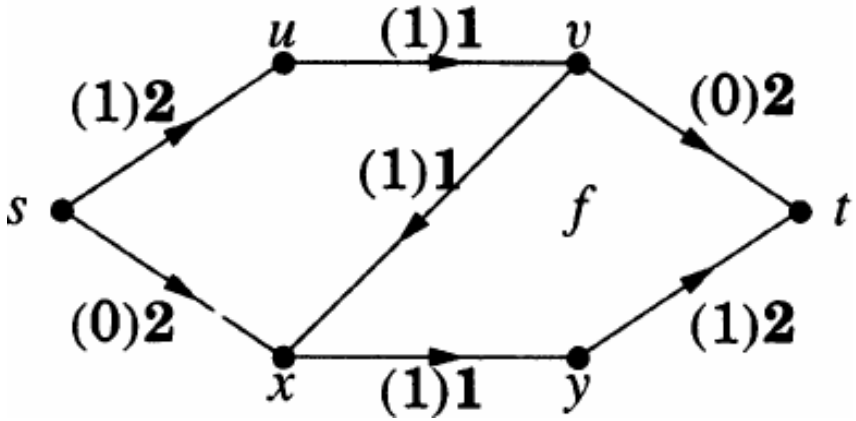
$$\forall v \in V(G) - \{s, t\}.\ f^+(v) = f^-(v)$$



Definition (Value (值))

The value $\mathsf{val}(f)$ of a flow f is

$$\mathsf{val}(f) = f^-(t) = f^+(s).$$



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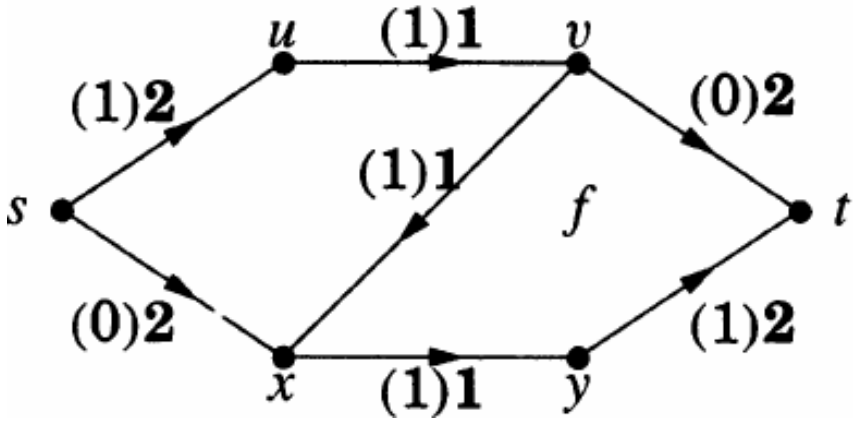

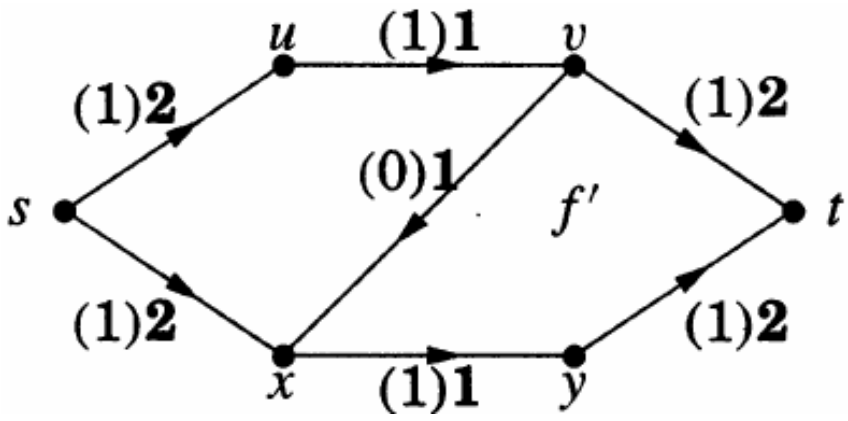

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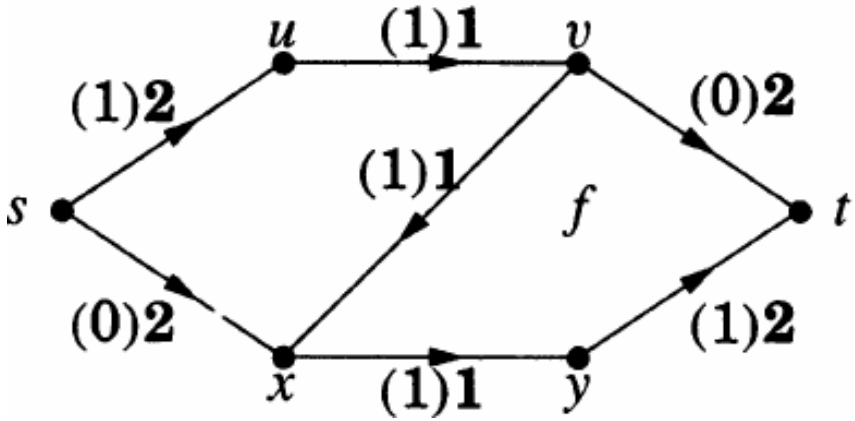
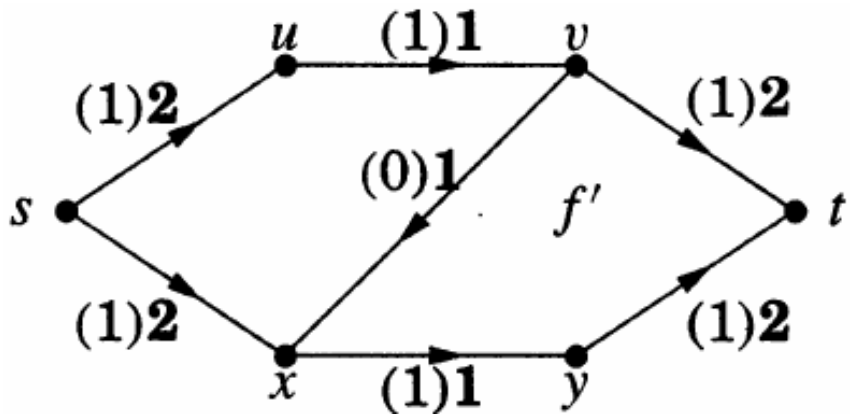
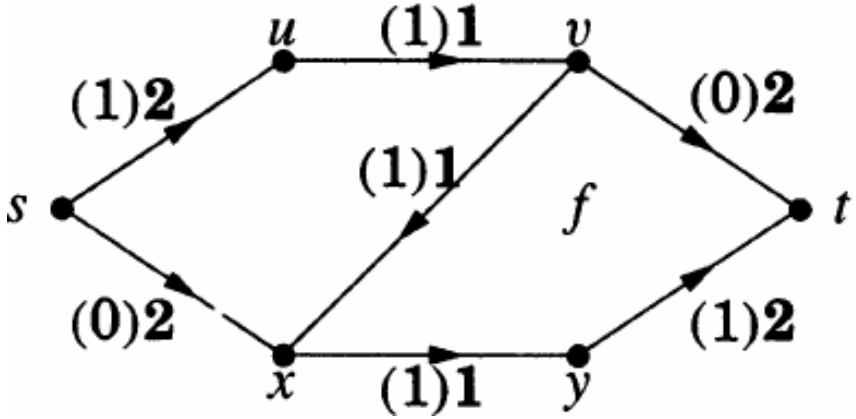




Definition (Maximum Flow (最大流))

A maximum flow is a feasible flow of maximum value.


u
(1)1
v
(1)2
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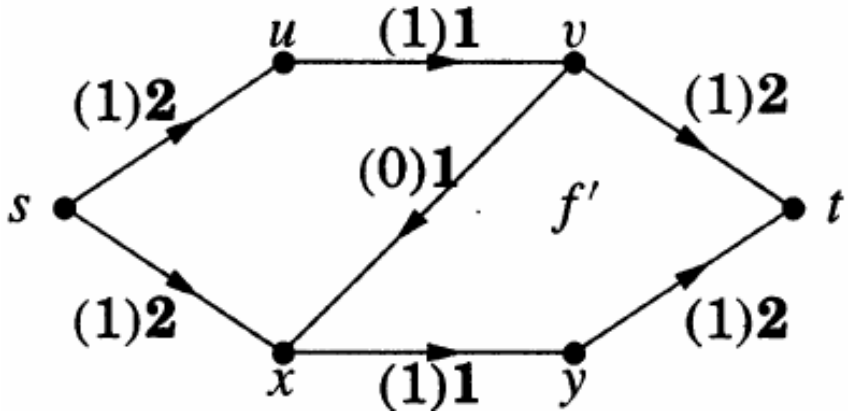
f
s
t
(0)2
(1)2
x
(1)1
y


u
(1)1
v
(1)2
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f′
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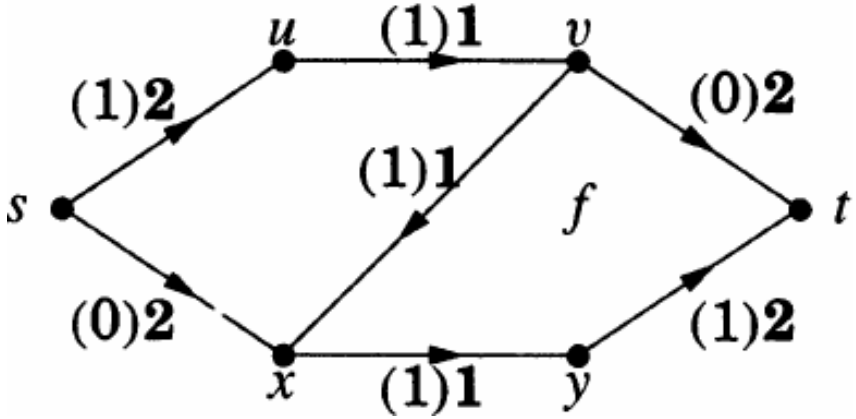
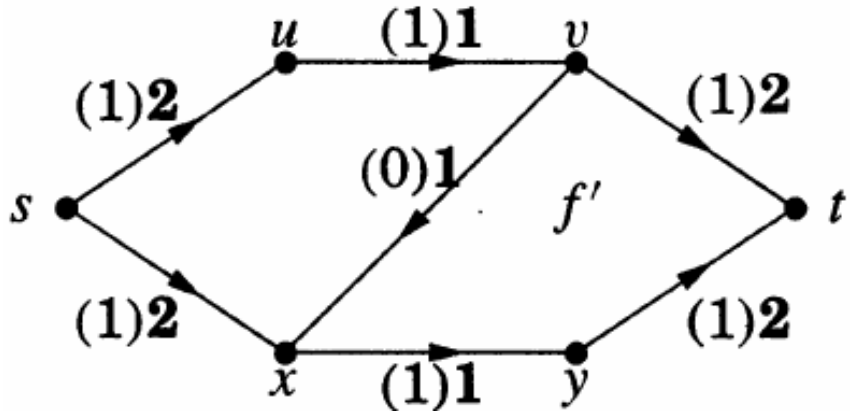
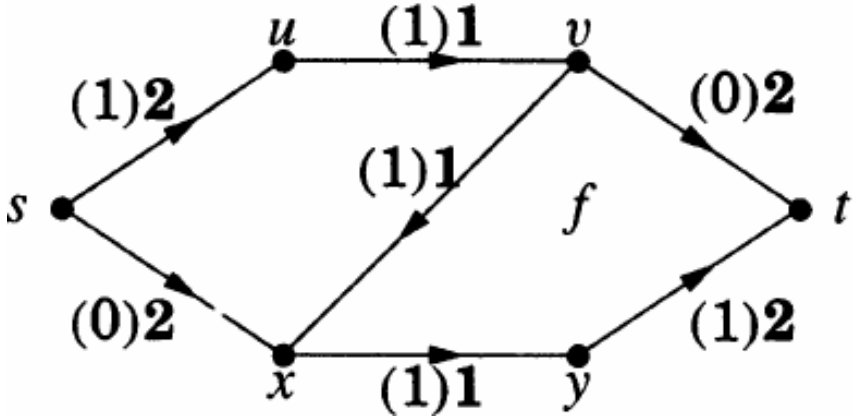
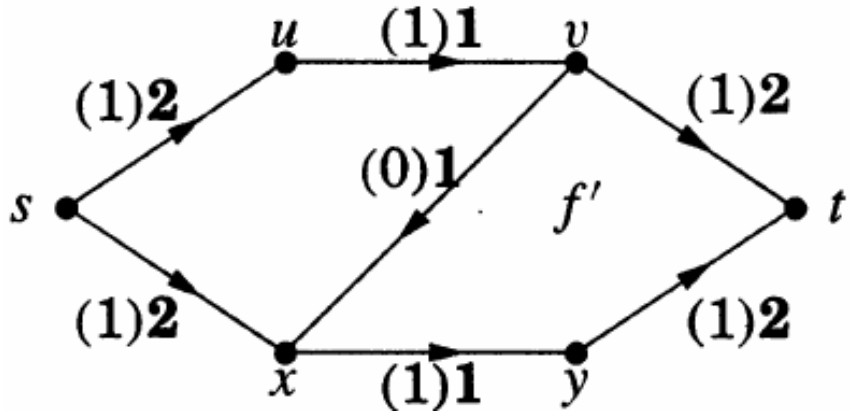
(1)2
x
(1)1
y





$$s \to x \to v \to t$$





$$s \to x \to v \to t$$

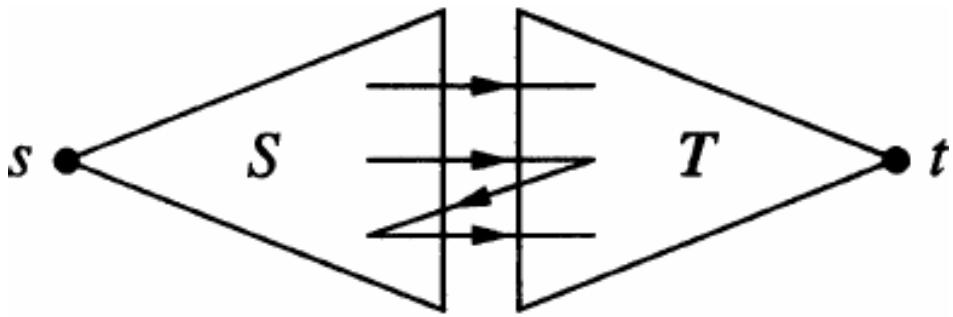
Definition (f-augmenting Paths (增广路径))

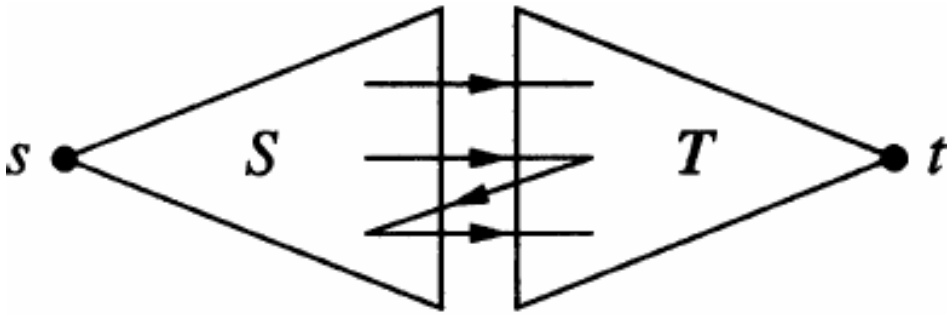
$$\min_{e \in E(P)} \epsilon(e)$$

Definition (Source/Sink Cut (割))

In a network, a source/sink cut $[S, T]$ consists of the edges from a source set S to a sink set T, where

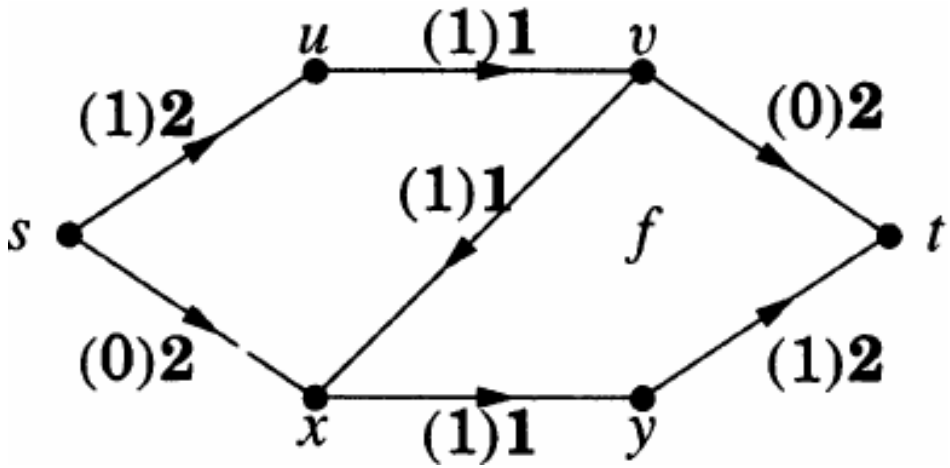
$$V = S \uplus T \wedge s \in S \wedge t \in T$$





Definition (Capacity of Cut (割的容量))

$$\mathsf{cap}(S,T) = \sum_{u\in S, v\in T, uv\in E} c(uv)$$



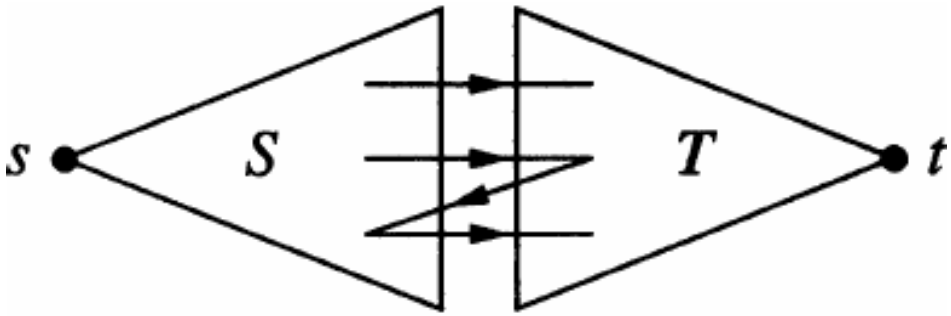
Definition (Minimum Cut (最小割))

A minimum cut is a cut of minimum value.

Theorem (Weak Duality (弱对偶定理))

Let f be any feasible flow and $[S,T]$ be any source/sink cut.

$$val(f) \leq cap(S,T).$$



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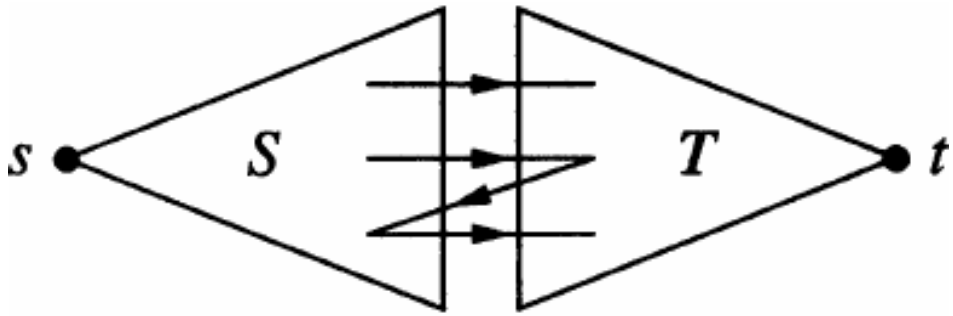

$$\mathsf{val}(f) = f^+(S) - f^-(S)$$

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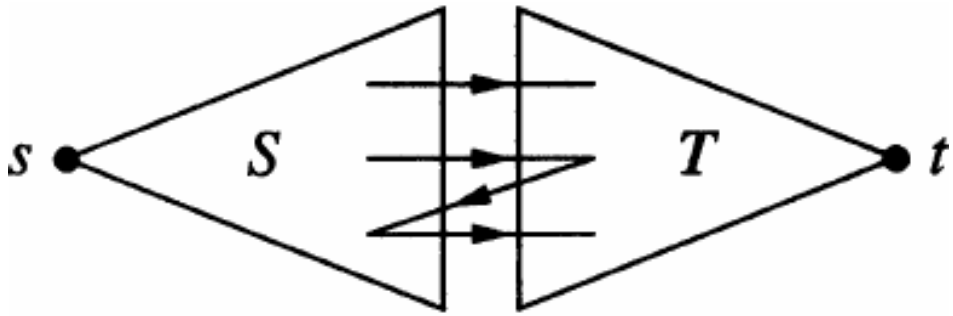

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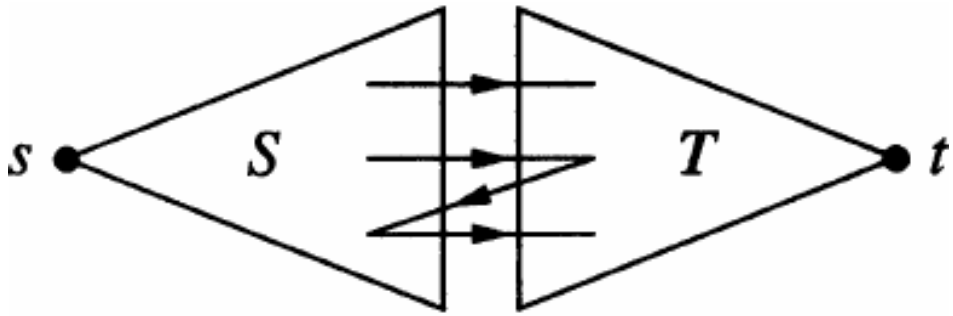
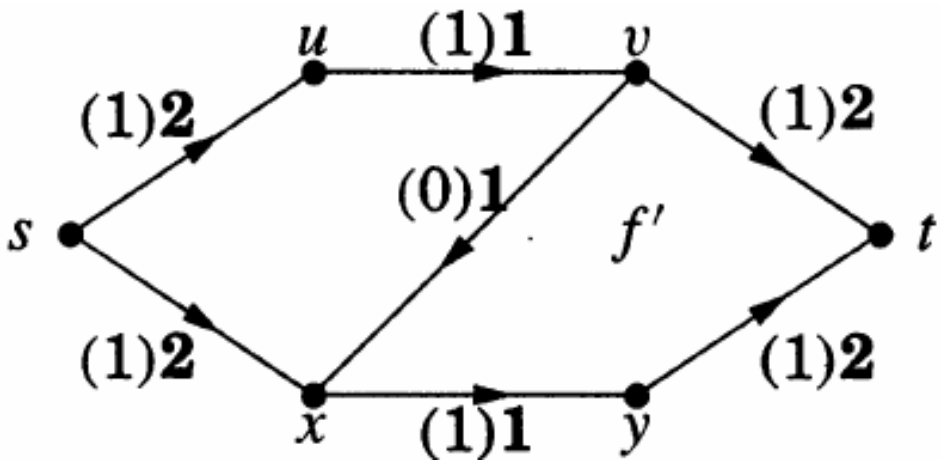


$$\mathsf{val}(f) = f^+(S) - f^-(S) \leq f^+(S) \leq \mathsf{cap}(S,T)$$

Lemma

$$\max_{f} \mathit{val}(f) \leq \min_{[S,T]} \mathit{cap}(S,T)$$



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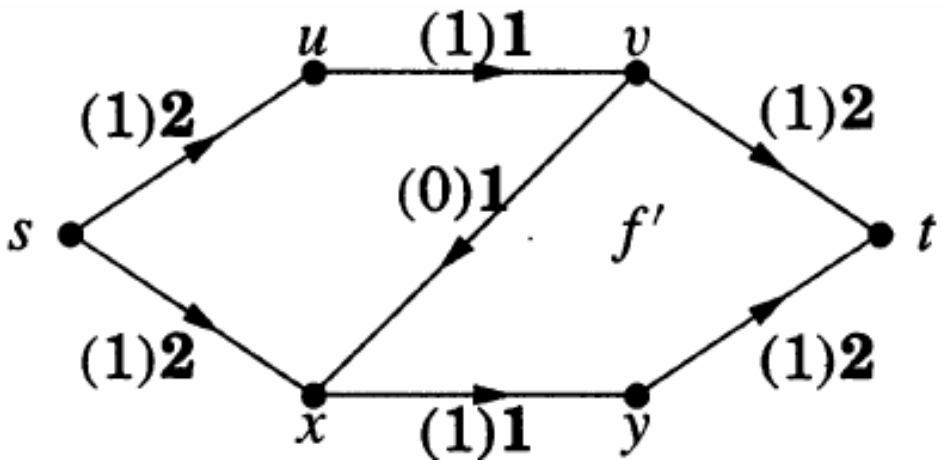


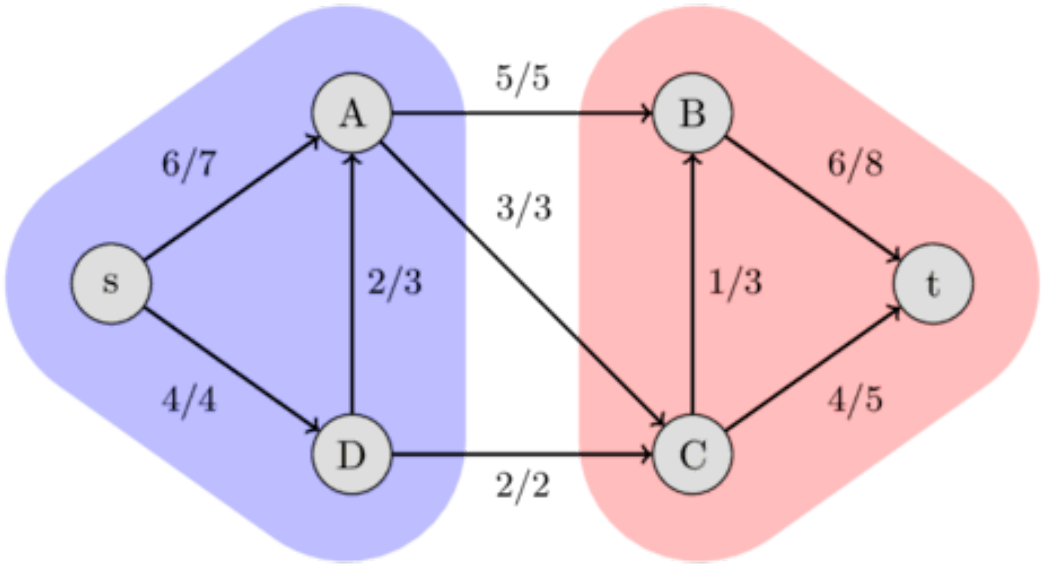
Theorem (Max-flow Min-cut Theorem (Ford and Fulkerson; 1956))

$$\max_{f} \mathit{val}(f) = \min_{[S,T]} \mathit{cap}(S,T)$$

(Strong Duality)

L. R. Ford Jr. (1927 ∼ 2017)

D. R. Fulkerson (1924 ∼ 1976)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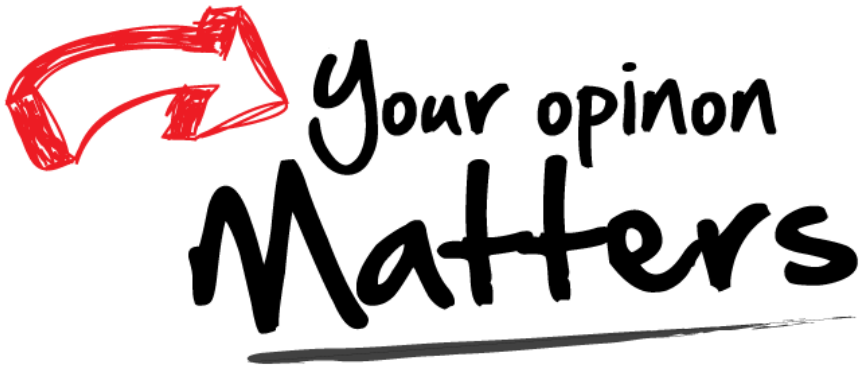
Thank
You!



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