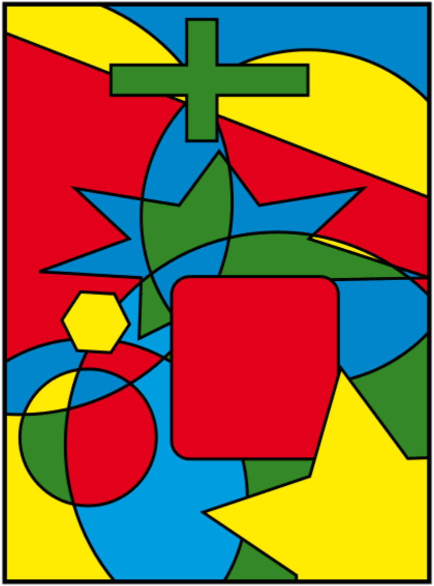
(十一) 图论: 平面图与图着色 (Planarity and Coloring)

魏恒峰

hfwei@nju.edu.cn

2021 年 05 月 20 日



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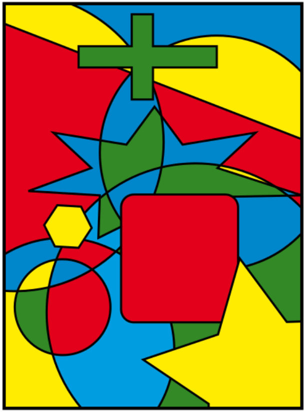

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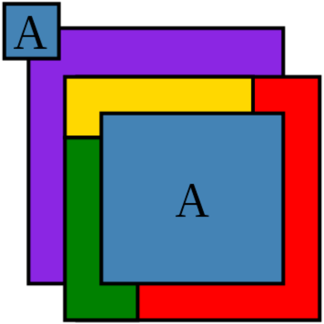

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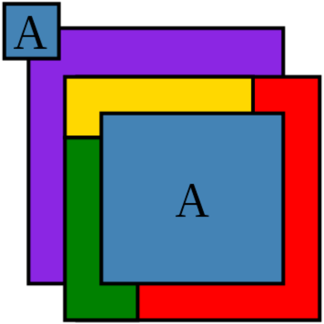

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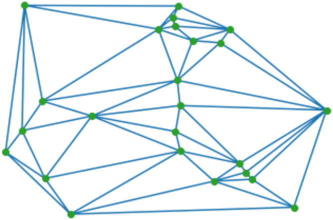

Adjacent regions share a segment.

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What if we have a map in which every region is adjacent to ≥ 5 other regions?

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What does it to do with GRAPH THEORY?

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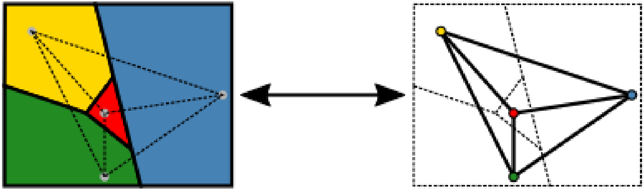

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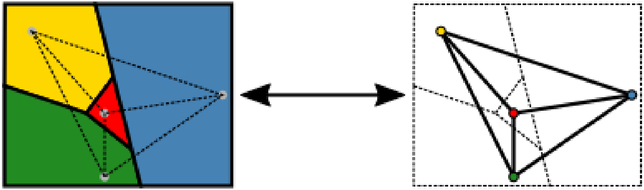

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I will *not* show its proof (which I don't understand either)!

Theorem

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Theorem (Percy John Heawood)

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Definition (k-Colorable (k-可着色的))

If G is a connected undirected graph without loops,
then G is k-colorable if its vertices can be colored in $\leq k$ colors
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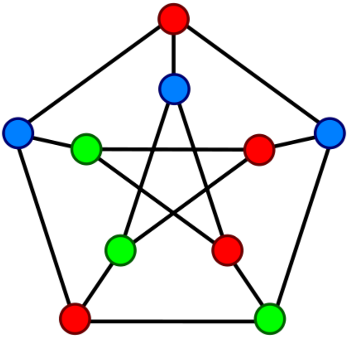

The Petersen graph is ≥ 3-colorable.

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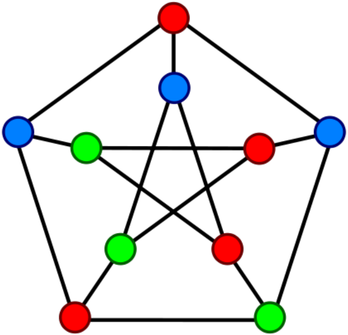

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Lemma

The empty graph (null graph) is 1*-chromatic.*

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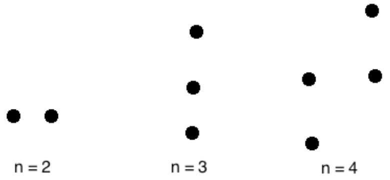

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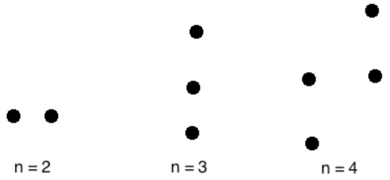

Lemma

K_n is n-chromatic.

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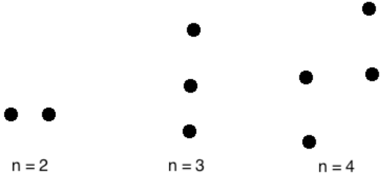

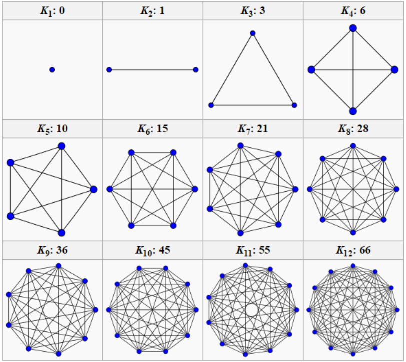

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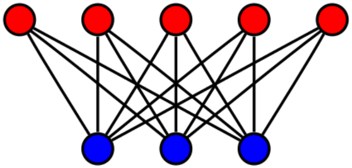

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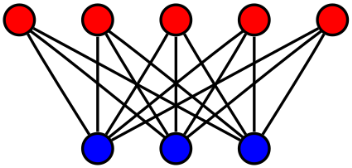

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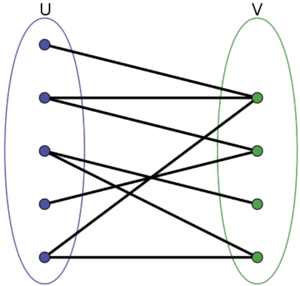

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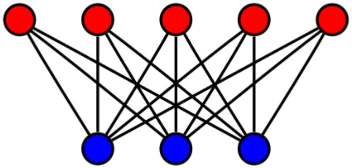

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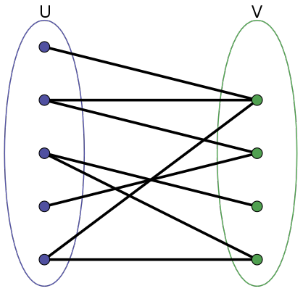

Lemma

Every tree is bipartite and is thus 2-colorable.

Lemma (Characterization of Bipartite Graphs ($\Longrightarrow$))

If a graph is bipartite, then it does not contain any odd cycles.

Lemma (Characterization of Bipartite Graphs ($\Longleftarrow$))

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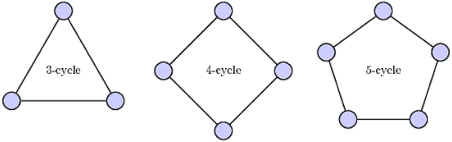

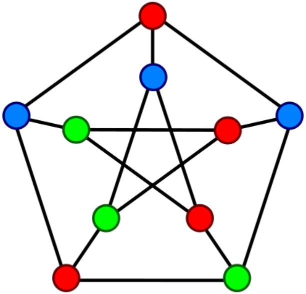

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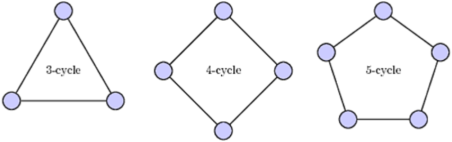

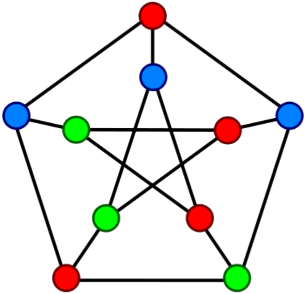

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The 4-coloring problem is also NP-complete.

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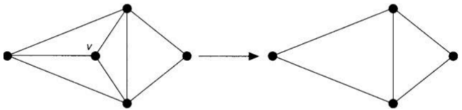

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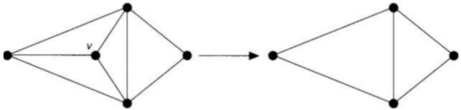

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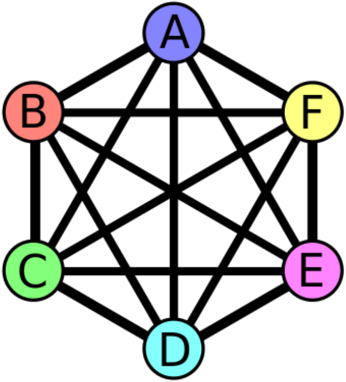

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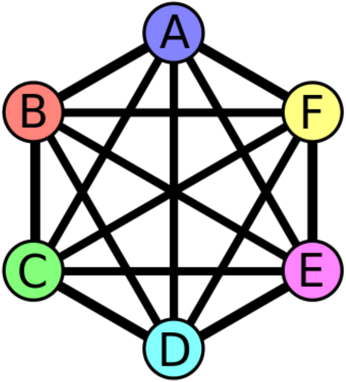

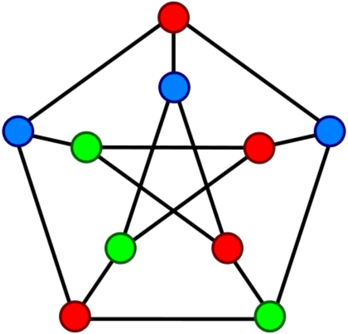

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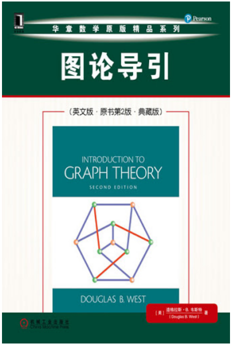

Theorem 5.1.22

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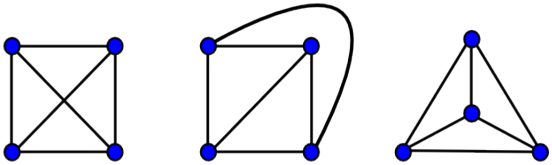

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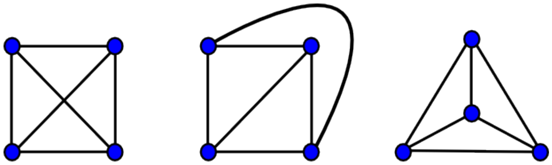
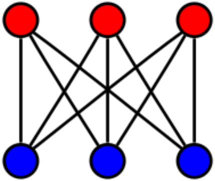


Theorem (K. Wagner (1936); I. Fáry (1948))

Every simple planar graph can be drawn with straight lines.

Theorem (Kazimierz Kuratowski, 1930)

The utility graph $K_{3,3}$ is non-planar.



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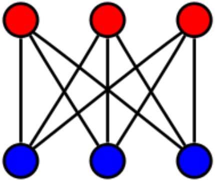

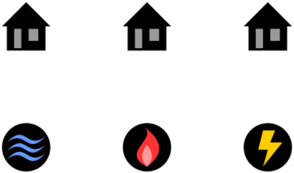

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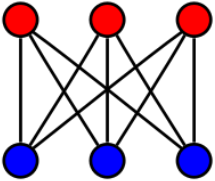
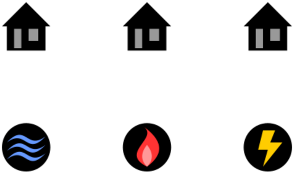
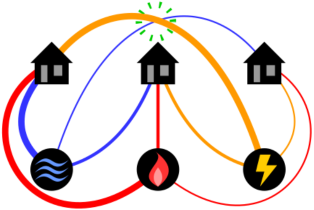
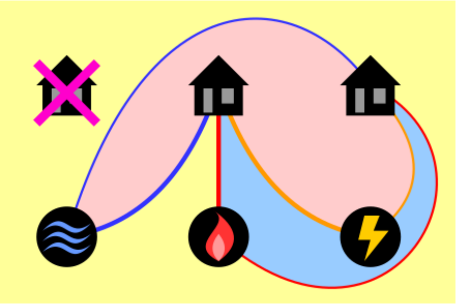
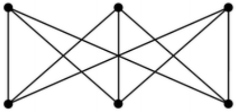




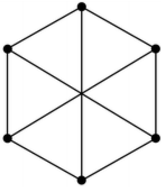


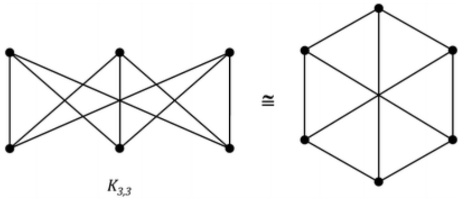




$K_{3,3}$

$\cong$




$K_{3,3}$
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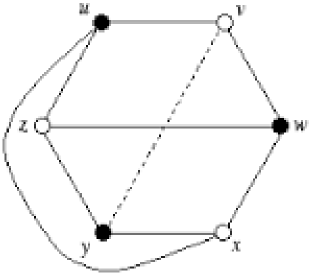
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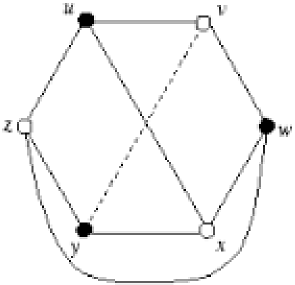
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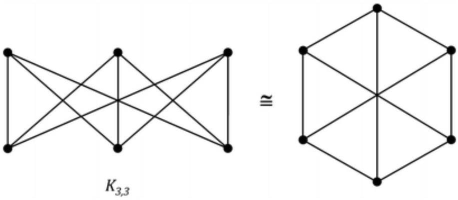
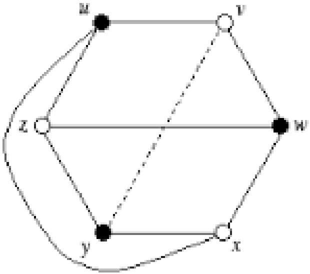
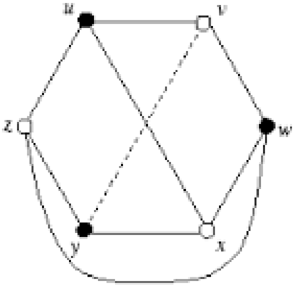


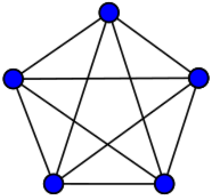




$$\textcolor{red}{\mathrm{cr}}(K_{3,3}) = 1$$

Theorem

K_5 is non-planar.



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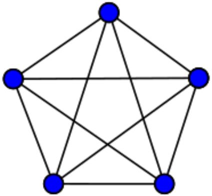

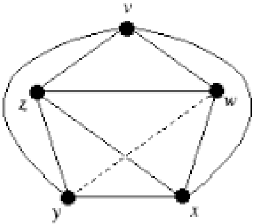

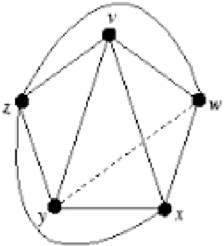

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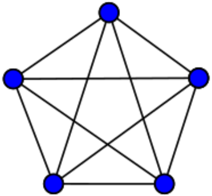
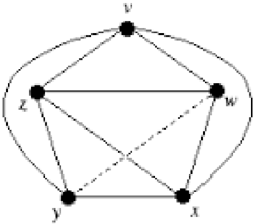
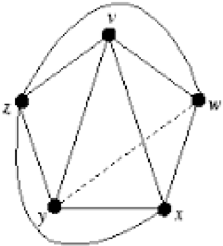


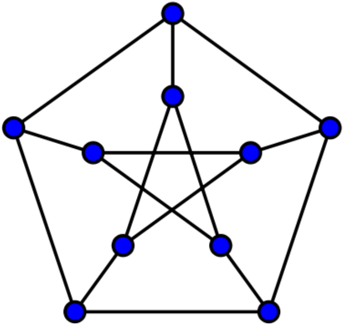




$$\textcolor{red}{\mathrm{cr}}(K_5) = 1$$

Theorem

The Petersen graph is non-planar.



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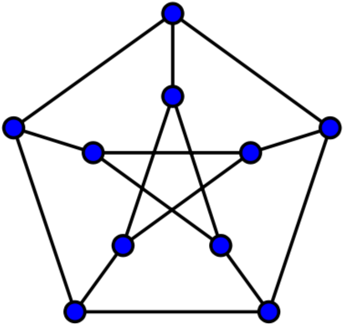

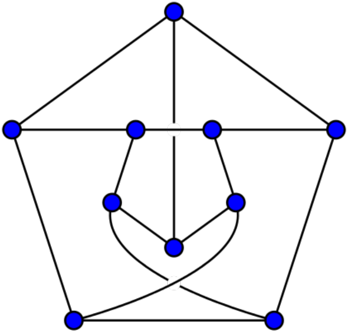

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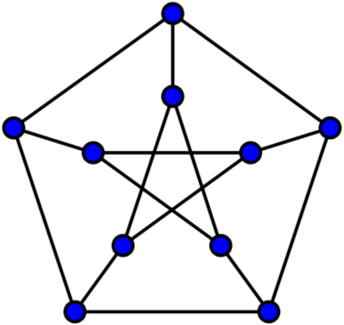

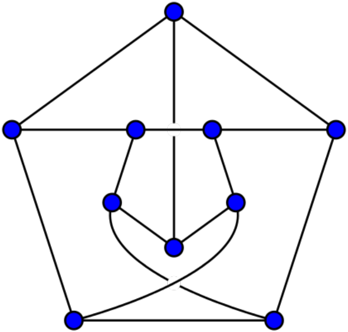

$$\color{red}{\mathrm{cr}}(\text{Petersen Graph}) = 2$$

Theorem (Kazimierz Kuratowski, 1930)

A graph is planar iff it contains no subgraph homeomorphic to K_5 or $K_{3,3}$.

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"The K in K_5 *stands for Kazimierz,*
and the K in $K_{3,3}$ *stands for Kuratowski."*

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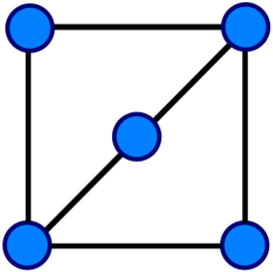

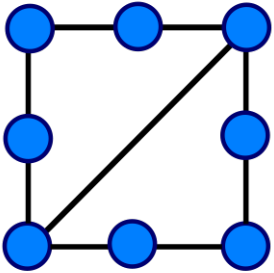

Definition (Homeomorphic)

Two graphs are homeomorphic if one can be obtained from another by inserting or contracting vertices of degree 2.

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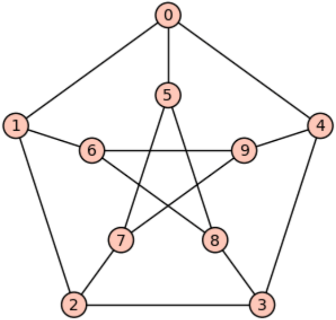

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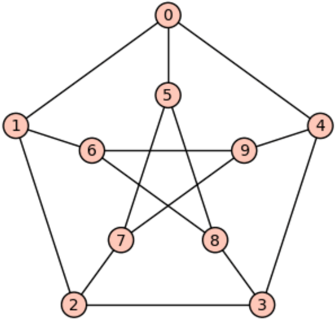

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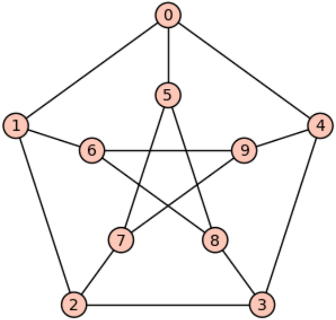

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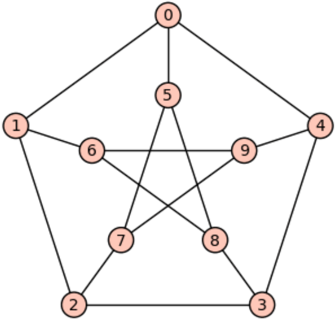

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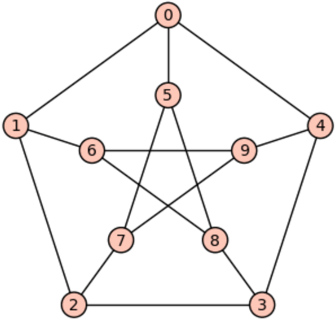

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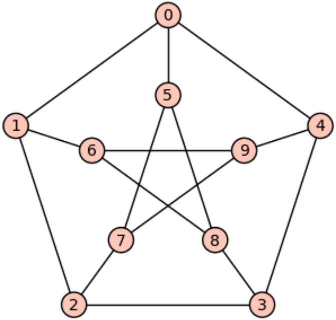

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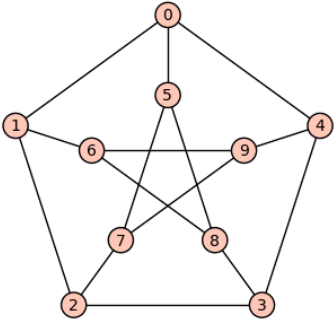

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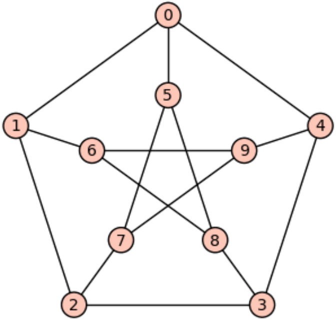

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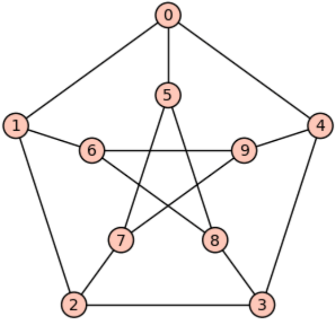

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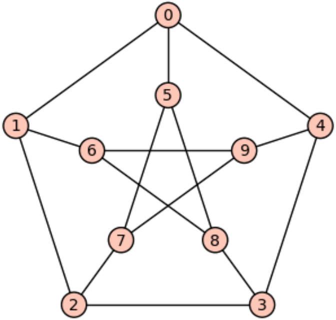

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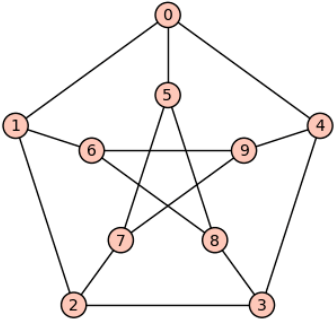

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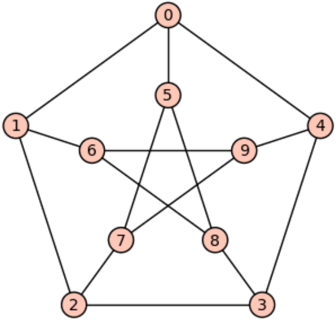

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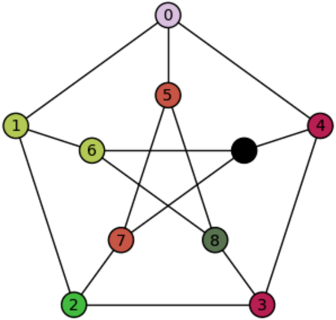

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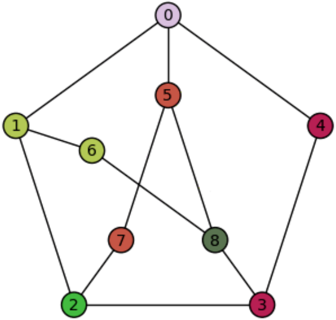

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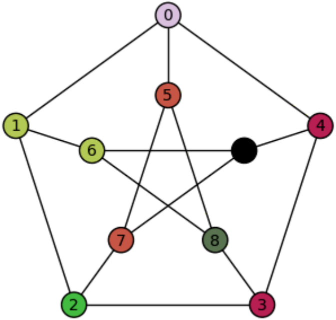

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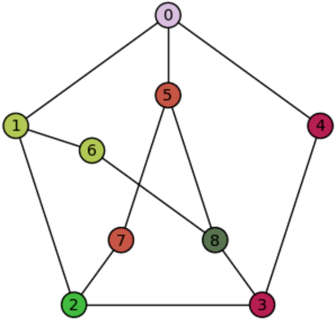

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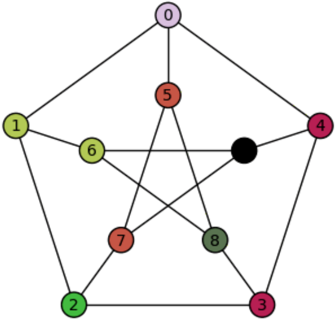

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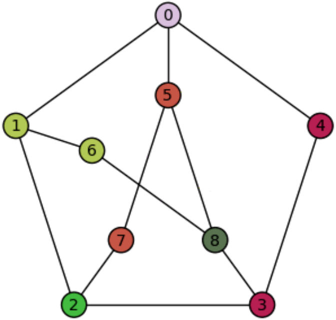

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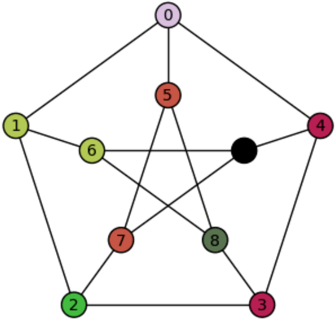

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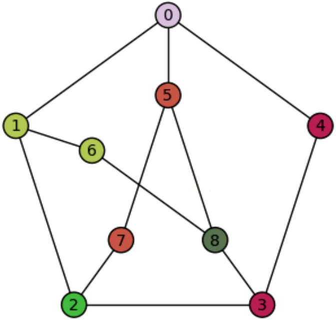

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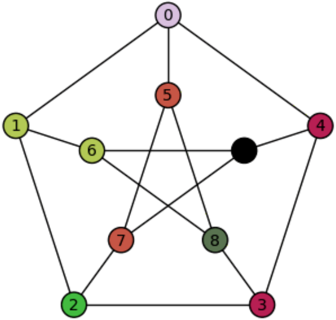

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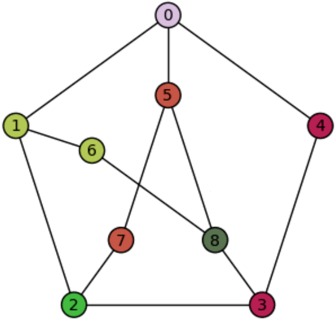

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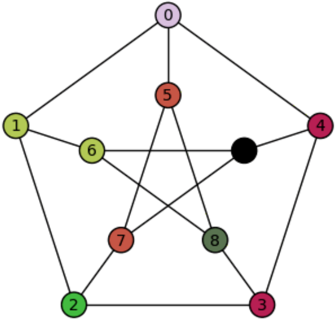

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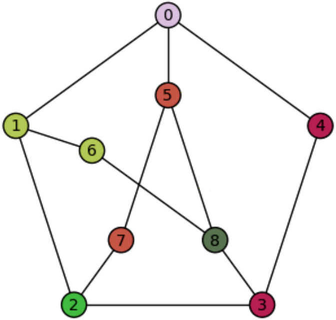

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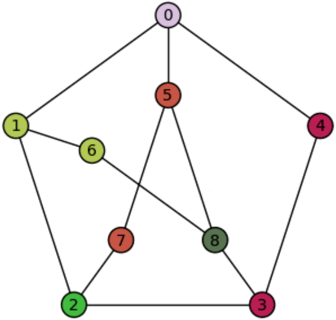

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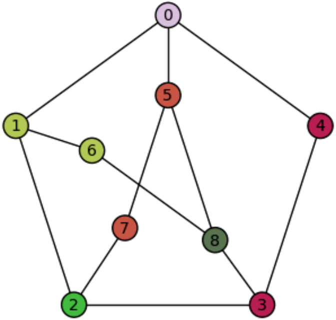

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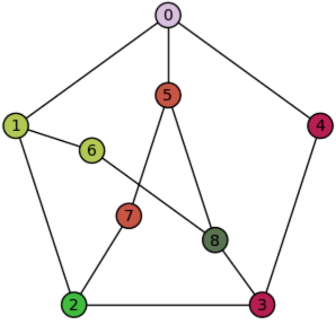

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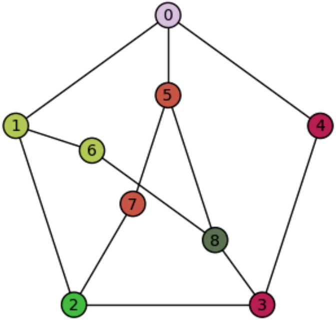

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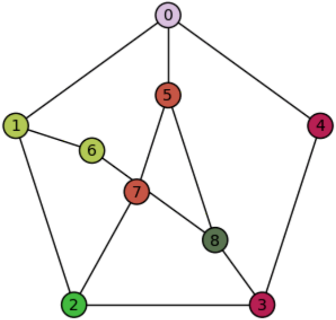

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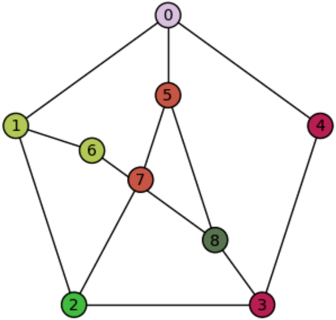

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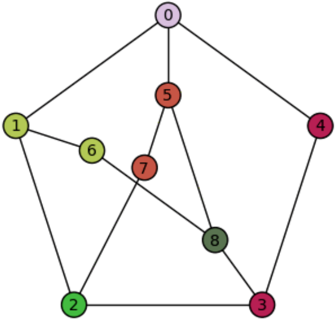

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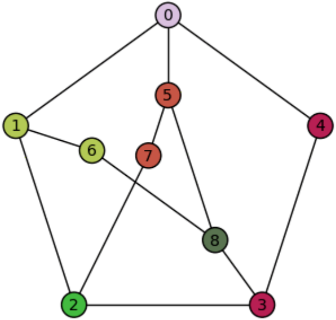

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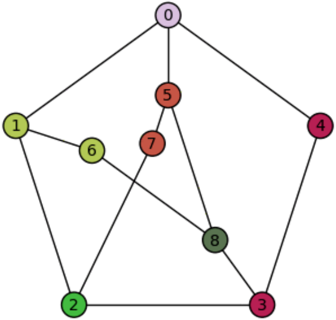

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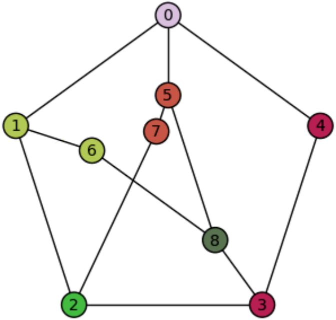

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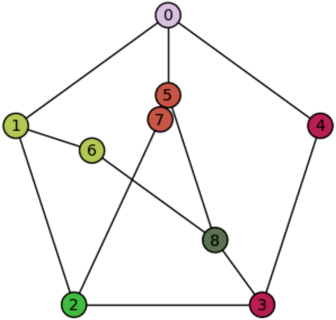

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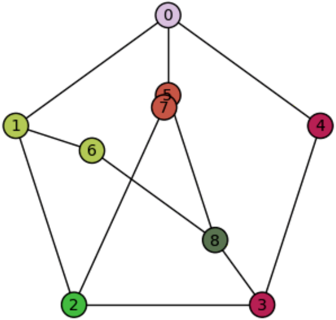

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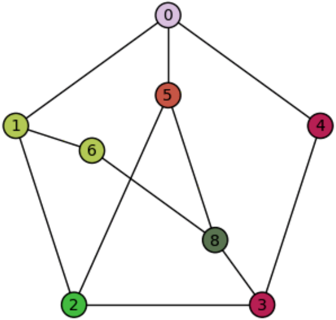

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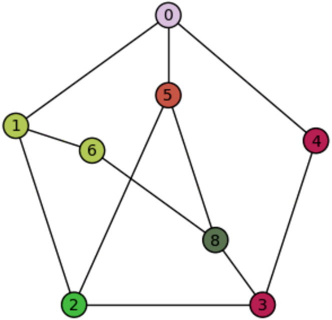

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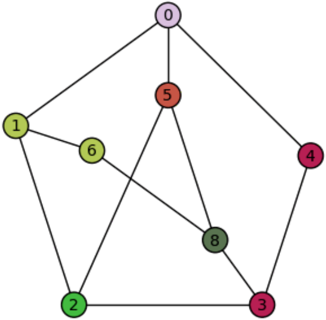

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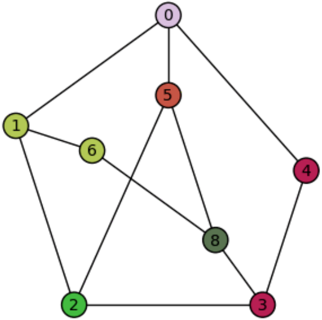

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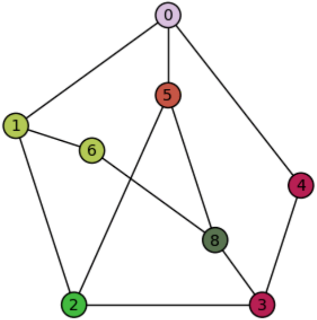

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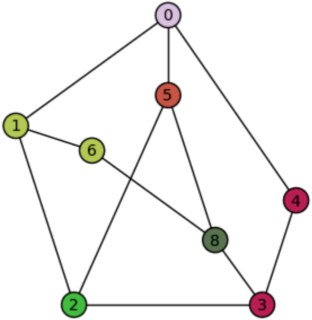

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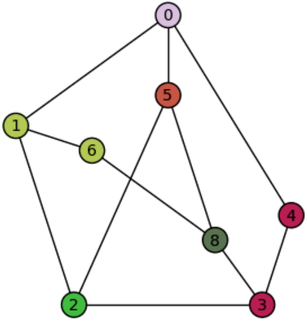

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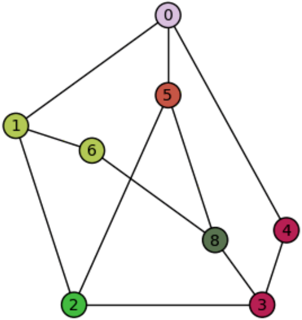

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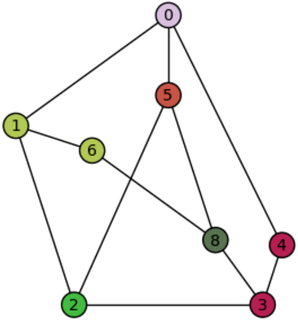

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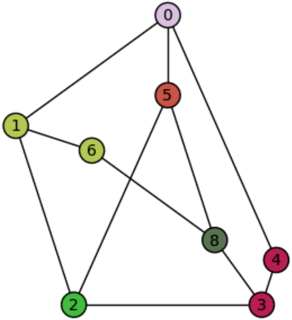

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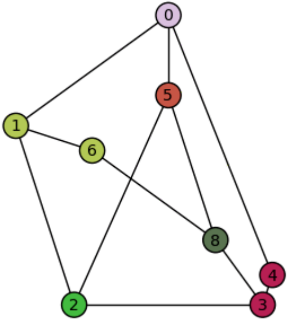

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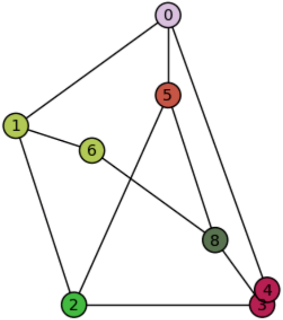

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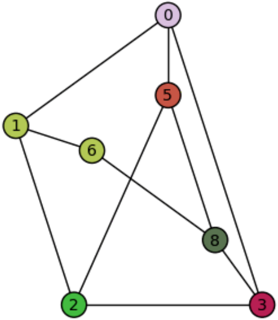

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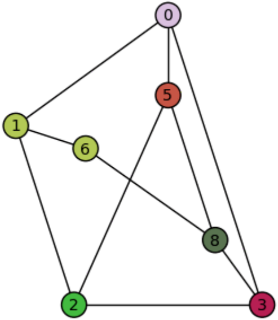

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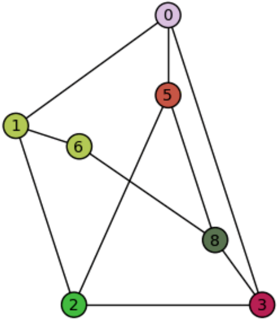

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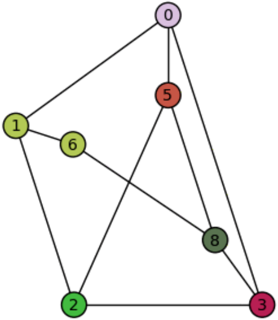

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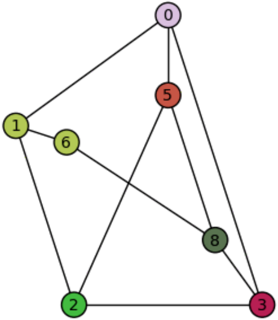

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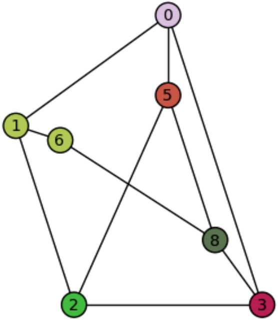

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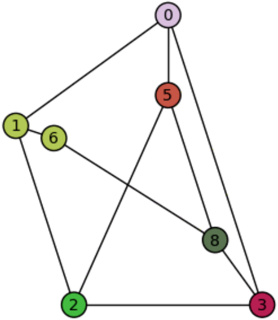

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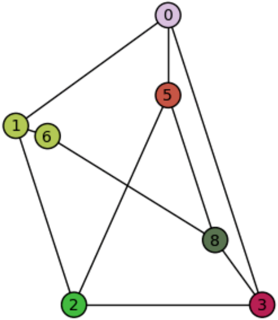

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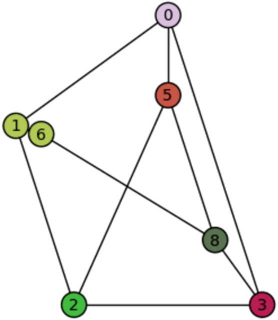

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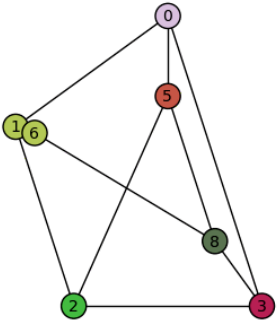

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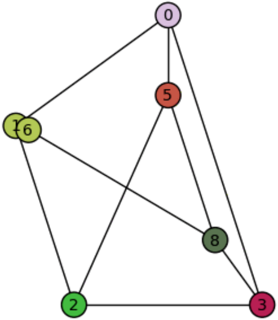

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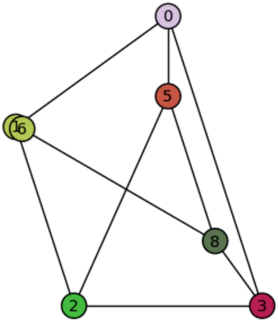

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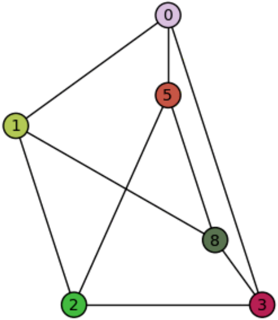

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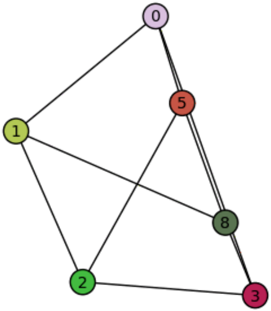

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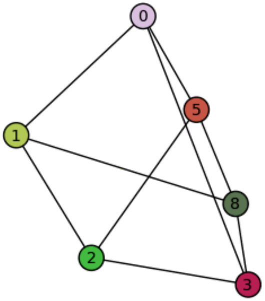

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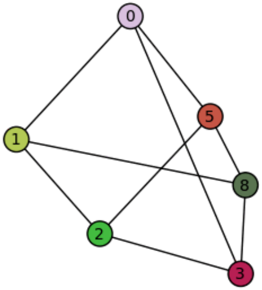

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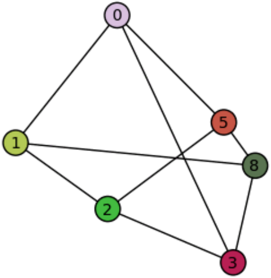

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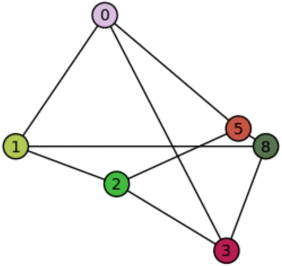

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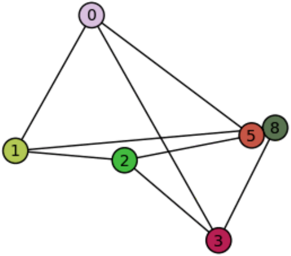

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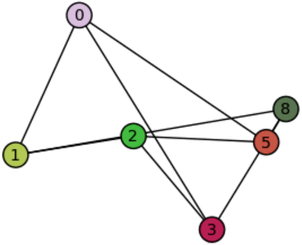

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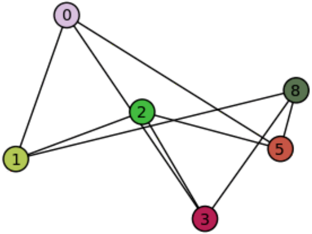

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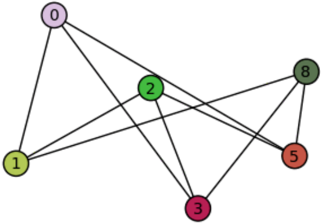

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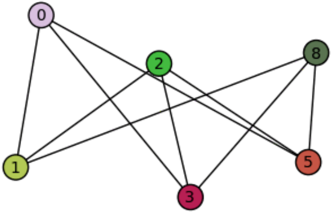

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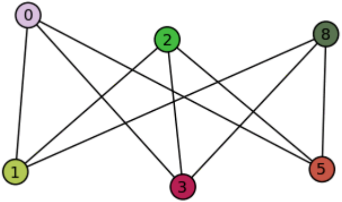

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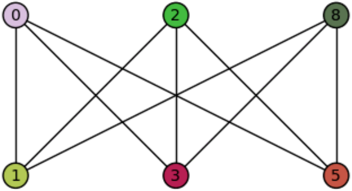

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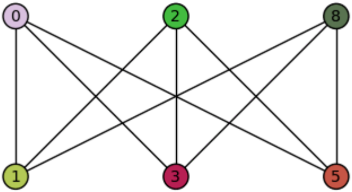

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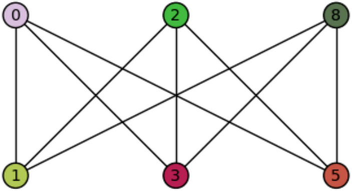

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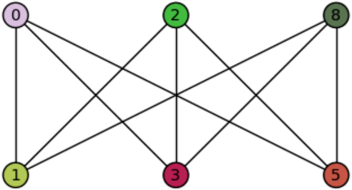

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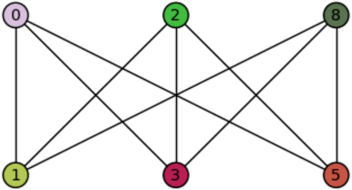

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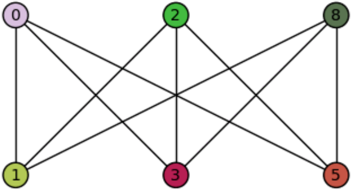

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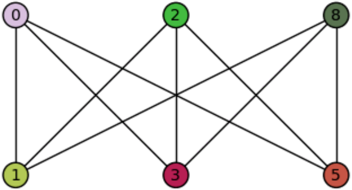

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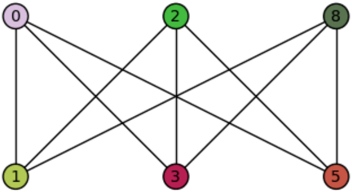

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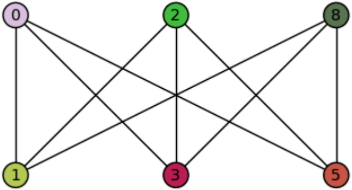

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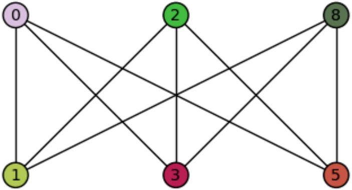

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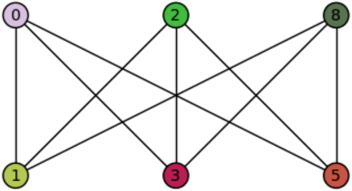

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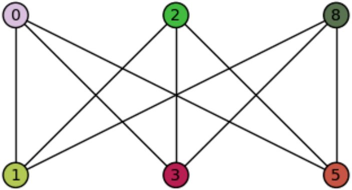

A planar graph should not has too many edges.

Theorem (Euler's Formula, 1750)

Let G be a plane drawing of a connected planar graph, and let n, m, and f denote respectively the number of vertices, edges, and faces of G.

$$n - m + f = 2$$

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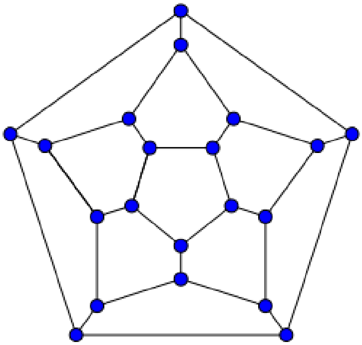

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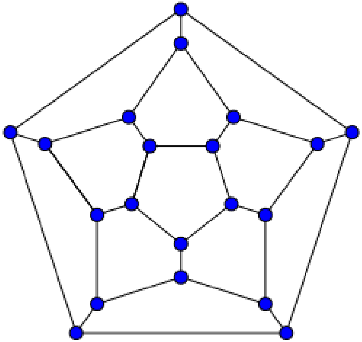

$$n - m + f = 20 - 30 + 12 = 2$$

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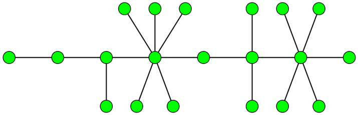

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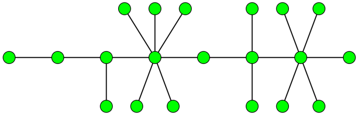

$$n - m + f = n - (n-1) + 1 = 2$$

By induction on the number of edges of G.

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Basis Step: $m = 0$. We have $n = 1$ and $f = 1$.

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Consider $G' = G - e$.

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Therefore,

$$n - m + f = 2$$

Theorem

Let G be a simple connected planar graph with $n \geq 3$ vertices and m edges. Then

$$m \leq 3n - 6.$$

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Double Counting:

each face is bounded by ≥ 3 edges;
each edge bounds 2 faces

Theorem

K_5 is non-planar.

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$$m \leq 3n - 6$$

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$$m \leq 3n - 6$$

$$10 \leq 3 \times 5 - 6$$

Theorem

$K_{3,3}$ is non-planar.

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$$9 \leq 3 \times 6 - 6$$

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Every simple planar graph contains a vertex of degree ≤ 5.

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Suppose that, by contradiction, $\delta(G) \geq 6$.

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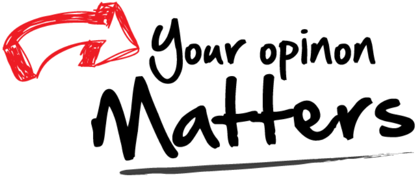
$$6n \leq 2m$$

$$3n \leq m \leq 3n - 6$$


Thank
You!



Office 302

Mailbox: H016

hfwei@nju.edu.cn