

(十五) 离散数学: 复习 (Review)

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\vdash \models

Theorem

$$\Sigma \vdash \alpha \iff \Sigma \models \alpha$$



\rightarrow \Rightarrow

\leftrightarrow \Longleftrightarrow

“ \rightarrow ” and “ \leftrightarrow ” are used in a **single** formula.

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$$x \in A \setminus B$$

$$\Longleftrightarrow x \in A \wedge x \notin B$$

$$\Longleftrightarrow x \in A \wedge (x \in U \wedge x \notin B)$$

$$\Longleftrightarrow x \in A \wedge x \in \overline{B}$$

$$\Longleftrightarrow x \in A \cap \overline{B}$$

$$\begin{aligned} p \oplus q &\triangleq (p \vee q) \wedge \neg(p \wedge q) \\ &= (p \wedge \neg q) \vee (\neg q \wedge q) \end{aligned}$$

p	q	$p \oplus q$
0	0	0
0	1	1
1	0	1
1	1	0

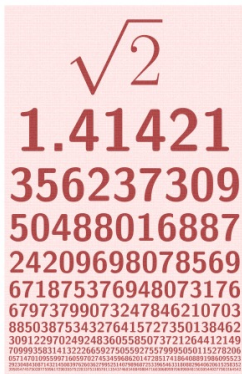
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$$p \oplus q = q \oplus r$$

$$(p \oplus q) \oplus r = p \oplus (q \oplus r)$$

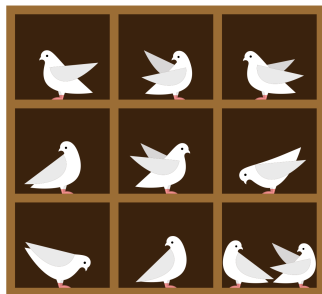
$\sqrt{2}$ is irrational.



The First Crisis in Mathematics

Theorem (Bézout's Identity)

$$(a, b) = d \implies \exists u, v \in \mathbb{Z}. au + bv = d$$



Theorem (Pigeonhole Principle)

If n **objects** are placed in r **boxes**, where $r < n$, then at least one of the boxes contains ≥ 2 ($\geq \lceil \frac{n}{r} \rceil$) object.

Numbers

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There must be two numbers which are **only 1 apart**.

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$$a = 2^k m, \quad (1 \leq m \leq 2n - 1 \text{ is odd})$$

There $n + 1$ numbers have only n different odd parts.

There must be two numbers **with the same odd part**.

Hand-shaking

If there are $n > 1$ people who can shake hands with one another, there are two people who shake hands with the same number of people.

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Either the '0' hole or the ' $n - 1$ ' hole or both must be empty.

Sums

Suppose we are given n integers a_1, a_2, \dots, a_n .

Then there is a set of **consecutive numbers** $a_{k+1}, a_{k+2}, \dots, a_l$ whose sum $\sum_{i=k+1}^l a_i$ is a multiple of n .

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$$A_j - A_i = a_{i+1} + \dots + a_j = 0 \pmod n$$

Championship Match

“胡司令” (胡荣华) 要安排一次长达 77 天的象棋练习赛。

他想每天至少要有一场比赛, 但是总共不超过 132 场比赛。

请证明, 无论如何安排, 他都要在连续的若干天内恰好完成 21 场比赛。

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$$a_1, a_2, \dots, a_{76}, a_{77}, a_1 + 21, a_2 + 21, \dots, a_{76} + 21, a_{77} + 21$$

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It must be $a_i + 21 = a_j$.

Sequences

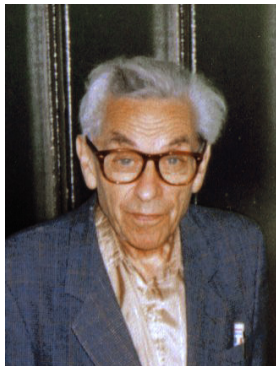
In any sequence $a_1, a_2, \dots, a_{mn+1}$ of $mn + 1$ **distinct** numbers, there exists an **increasing** subsequence

$$a_{i_1} < a_{i_2} < \cdots < a_{i_{m+1}} \quad (i_1 < i_2 < \cdots < i_{m+1})$$

of length $m + 1$, or a **decreasing** subsequence

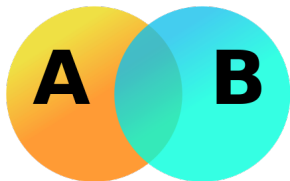
$$a_{j_1} > a_{j_2} > \cdots > a_{j_{n+1}} \quad (j_1 > i_2 < \cdots > j_{n+1})$$

of length $n + 1$, or both.

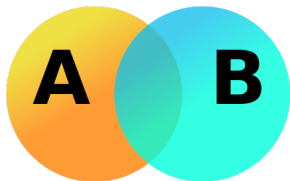


Paul Erdős (1913 ~ 1996)

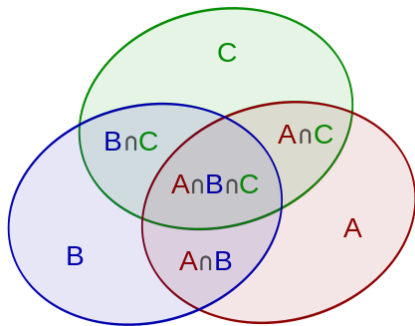
Chapter 28 of “Proofs from THE Book”



$$|A \cup B| = |A| + |B| - |A \cap B|$$



$$|A \cup B| = |A| + |B| - |A \cap B|$$



$$\begin{aligned}
 |A \cup B \cup C| &= |A| + |B| + |C| \\
 &\quad - |A \cap B| - |A \cap C| - |B \cap C| \\
 &\quad + |A \cap B \cap C|
 \end{aligned}$$

Theorem (Inclusion-Exclusion Principle)

$$\begin{aligned} \left| \bigcup_{i=1}^n A_i \right| &= \sum_{i=1}^n |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| \\ &\quad + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| \\ &\quad - \dots \\ &\quad + (-1)^{n-1} |A_1 \cap \dots \cap A_n|. \end{aligned}$$

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$$\begin{aligned}\left| \bigcap_{i=1}^n \bar{A}_i \right| &= \left| S - \bigcup_{i=1}^n A_i \right| = |S| - \sum_{i=1}^n |A_i| + \sum_{1 \leq i < j \leq n} |A_i \cap A_j| \\ &\quad - \dots + (-1)^n |A_1 \cap \dots \cap A_n|.\end{aligned}$$

Counting Integers

How many integers in $1, \dots, 100$ are not divisible by 2, 3 or 5?

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$$100 - (50 + 33 + 20) + (16 + 10 + 6) - 3 = 26.$$

Counting Derangements (错排)

Suppose there is a deck of n cards numbered from 1 to n .

Suppose a card numbered i is in the **correct** position if it is the i -th card in the deck. How many ways can the cards be shuffled **without any cards** being in the correct position?

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$$S_k \triangleq \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| =$$

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$$S_k \triangleq \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = \binom{n}{k} (n-k)! = \frac{n!}{k!}$$

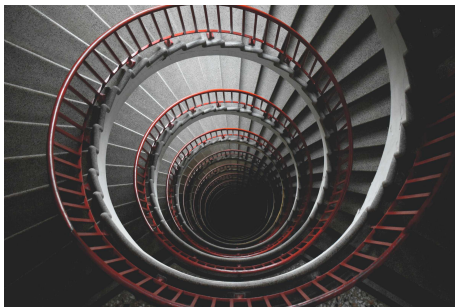
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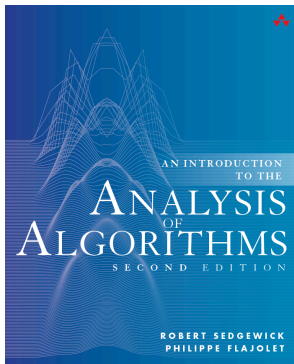
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$$n \rightarrow \infty \implies \sum_{k=0}^n \frac{(-1)^k}{k!} \rightarrow e^{-1} \approx 0.368$$



$$a_n = f(a_{n-1}, a_{n-2}, \dots, a_{n-t}) + g(n)$$

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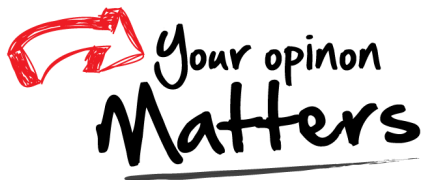
recurrence type	typical example
first-order	
linear	$a_n = na_{n-1} - 1$
nonlinear	$a_n = 1/(1 + a_{n-1})$
second-order	
linear	$a_n = a_{n-1} + 2a_{n-2}$
nonlinear	$a_n = a_{n-1}a_{n-2} + \sqrt{a_{n-2}}$
variable coefficients	$a_n = na_{n-1} + (n-1)a_{n-2} + 1$
t th order	$a_n = f(a_{n-1}, a_{n-2}, \dots, a_{n-t})$
full-history	$a_n = n + a_{n-1} + a_{n-2} \dots + a_1$
divide-and-conquer	$a_n = a_{\lfloor n/2 \rfloor} + a_{\lceil n/2 \rceil} + n$

Table 2.1 Classification of recurrences

Homogeneous Linear Recurrence Relations with Constant Coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_t a_{n-t}$$

Thank
You!



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