(十一) 图论: 平面图与图着色
(Planarity and Coloring)

魏恒峰

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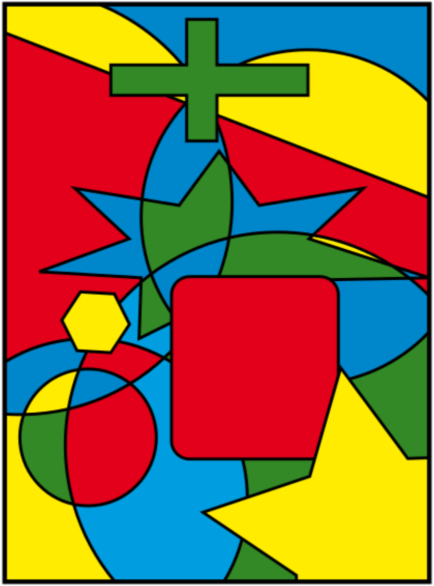
2021 年 05 月 20 日



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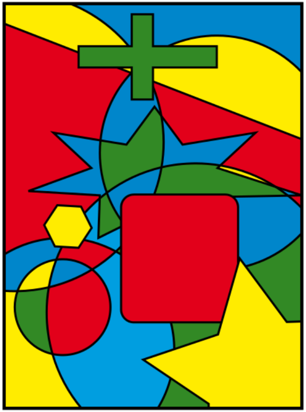

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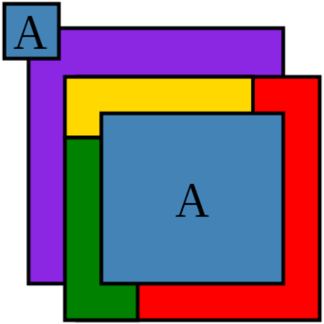

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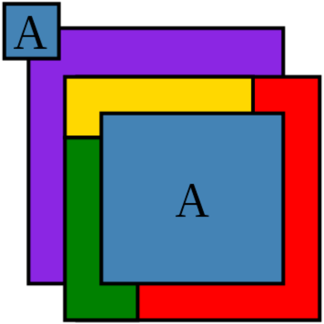

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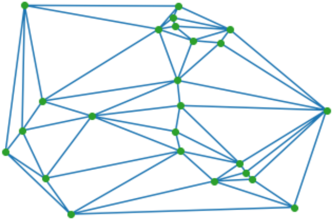

Adjacent regions share a segment.

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What if we have a map in which every region is adjacent to ≥ 5 other regions?

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What does it to do with GRAPH THEORY?

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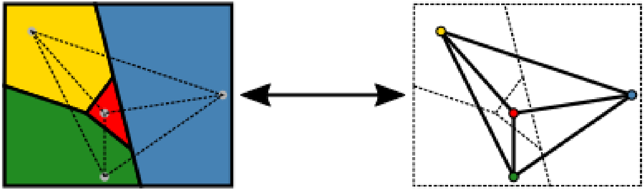

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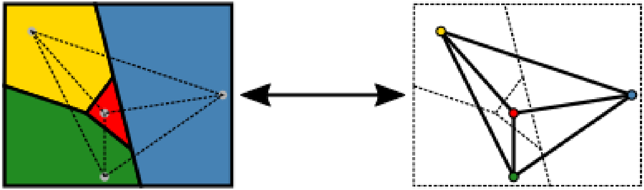

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Every simple planar graph is 4-colorable.

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I will *not* show its proof (which I don't understand either)!

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Every simple planar graph is 6-colorable.

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Theorem (Percy John Heawood)

Every simple planar graph is 5-colorable.

Definition (k-Colorable (k-可着色的))

If G is a connected undirected graph without loops,
then G is k-colorable if its vertices can be colored in $\leq k$ colors
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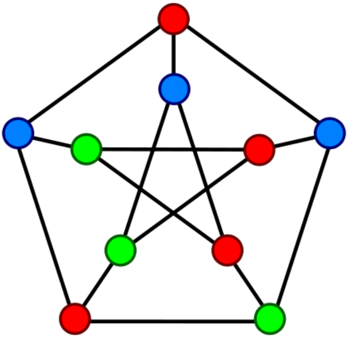

The Petersen graph is ≥ 3-colorable.

Definition (k-Chromatic (k-色数的))

If G is k-colorable, but is not $(k-1)$-colorable, then G is k-chromatic.

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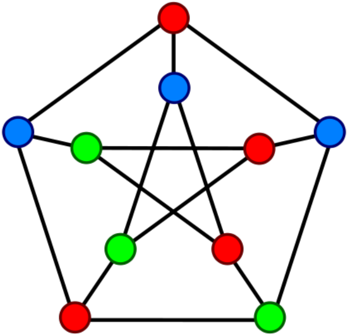

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Lemma

The empty graph (null graph) is 1*-chromatic.*

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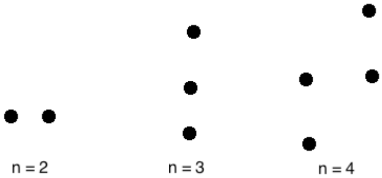

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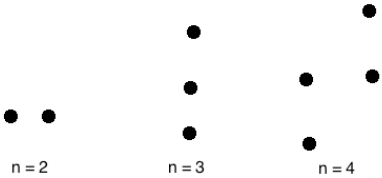

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K_n is n-chromatic.

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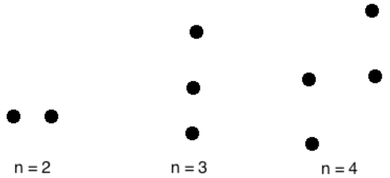

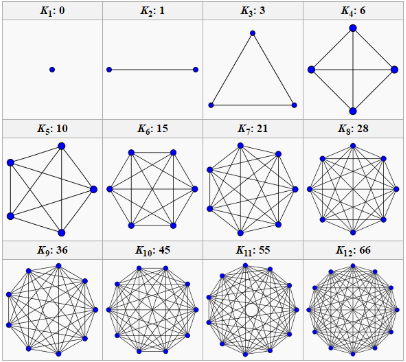

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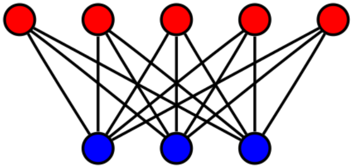

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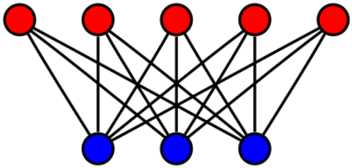

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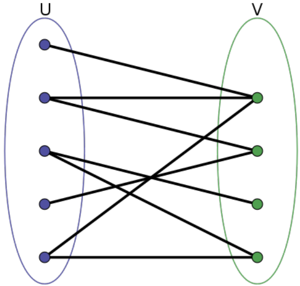

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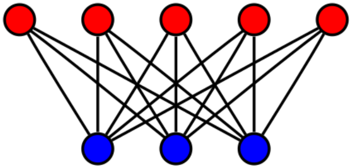

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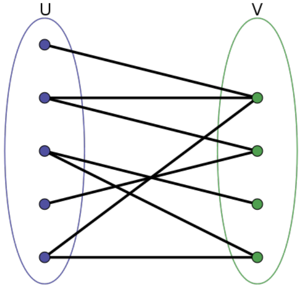

Lemma

Every tree is bipartite and is thus 2-colorable.

Lemma (Characterization of Bipartite Graphs ($\Longrightarrow$))

If a graph is bipartite, then it does not contain any odd cycles.

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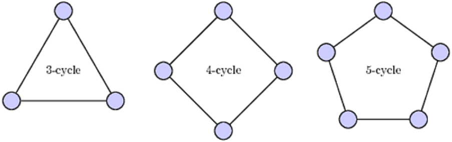

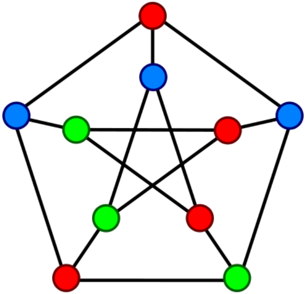

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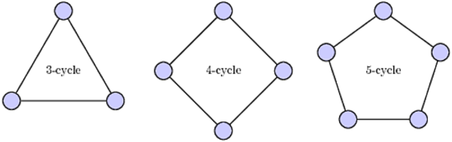

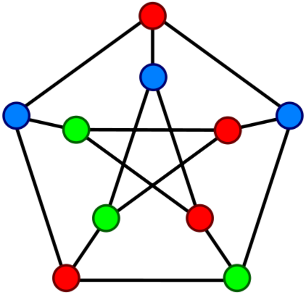

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The 4-coloring problem is also NP-complete.

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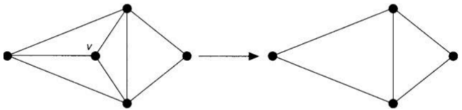

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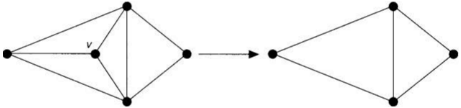

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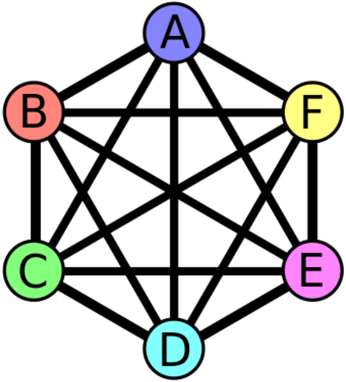
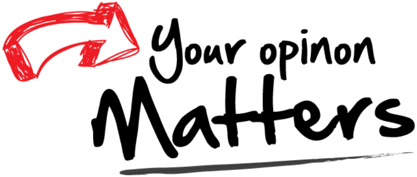



Thank You!



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