(十二) 图论: 对偶
(Duality: Matching, Network Flow, and Connectivity)

魏恒峰

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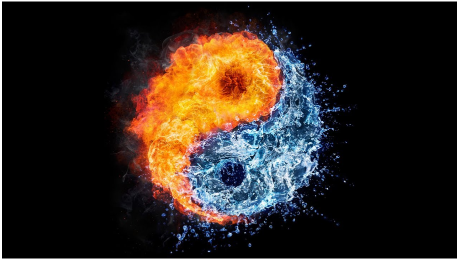
2021 年 05 月 27 日




let's get
married
today

The Marriage Problem (1935)

If there is a finite set of girls, each of whom knows several boys, under what conditions can all the girls marry boys in such a way that each girl marries a boy that she knows?

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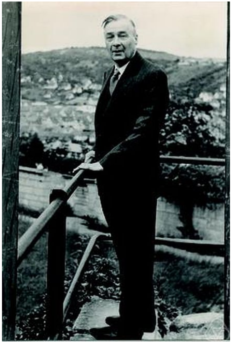
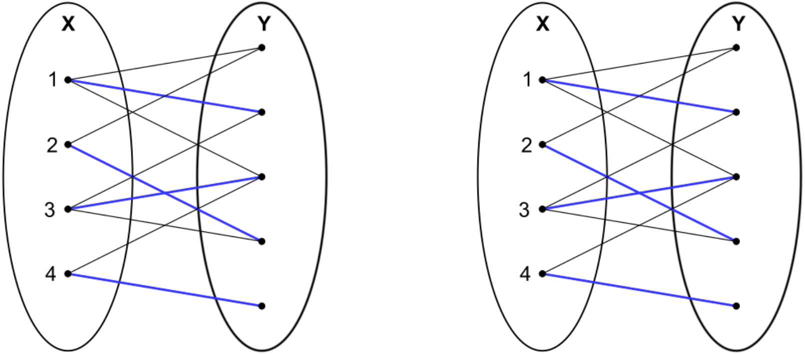


Philip Hall ($1904 \sim 1982$)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X
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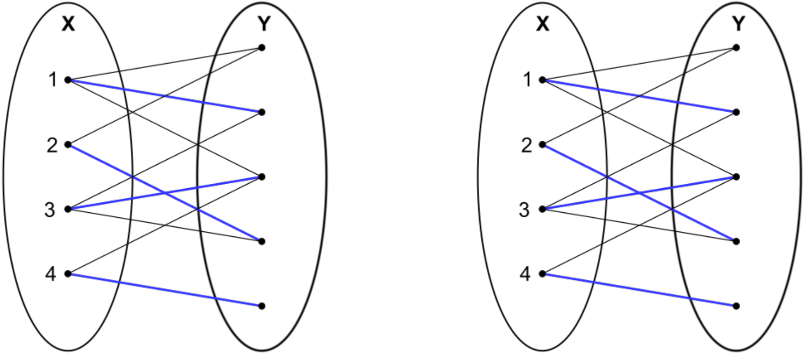
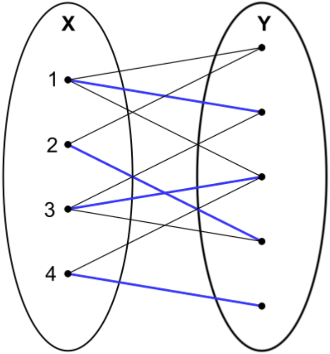


Definition (X-Perfect Matching (X-Saturating Matching))

Let $G = (X, Y, E)$ be a bipartite graph.

An X-perfect matching of G is a matching which covers each vertex in X.

$$\left|X\right| \leq \left|Y\right|$$



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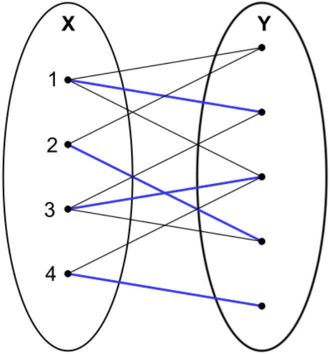

$$\forall W \subseteq X.\ \left|W\right| \le \left|N(W)\right|$$

Theorem (Hall Theorem; 1935)

Let $G = (X, Y, E)$ be a bipartite graph. There is a X-perfect matching of G iff

$$\forall W \subseteq X.\ \left|W\right| \le \left|N_G(W)\right|$$

By induction on the number $\left|X\right|$ of vertices in X.

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Basis Step: $\left|X\right| = 1$. $\left|X\right| \le \left|N_G(X)\right|$.

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Induction Hypothesis: Suppose that it holds if $\Big|X\Big| < m$.

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Basis Step: $\left|X\right| = 1$. $\left|X\right| \le \left|N_G(X)\right|$. I am married!

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- CASE I: Every $k < m$ girls in X know $\geq k+1$ boys in Y.

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$$G' = G - \{x, y\}$$

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There is a $(X - \{x\})$-perfer matching in G'.

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- CASE II: There is a set of $k < m$ girls in X who know k boys in Y.

Theorem (Hall Theorem; 1935)

Let $G = (X, Y, E)$ be a bipartite graph. There is a X-perfect matching of G iff

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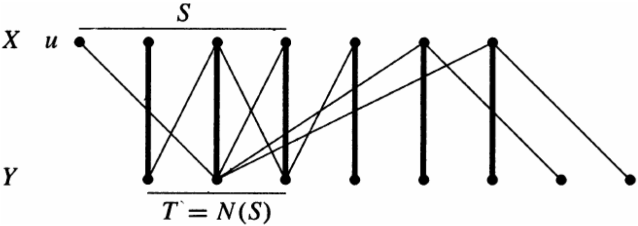

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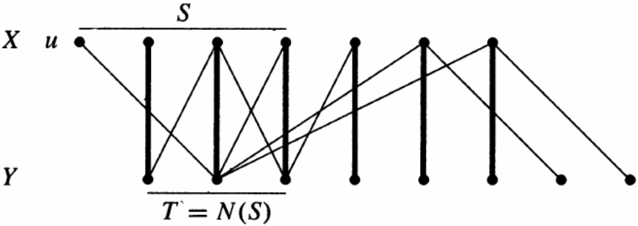

Let M be a *maximum* matching.

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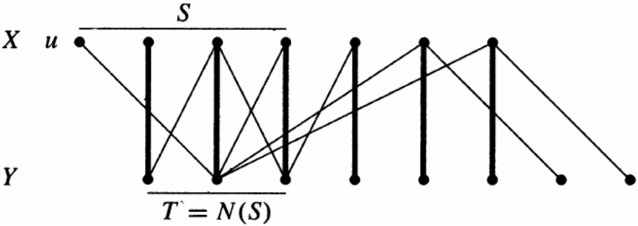

Let M be a *maximum* matching.

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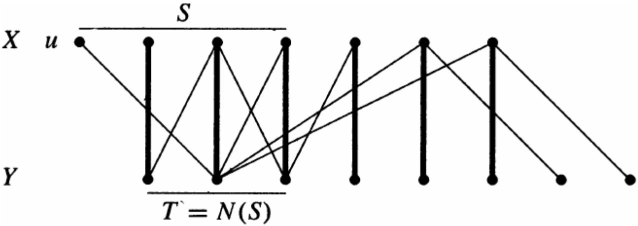
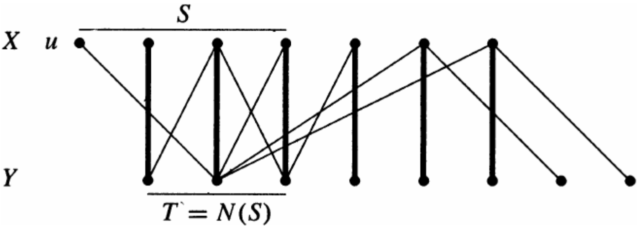


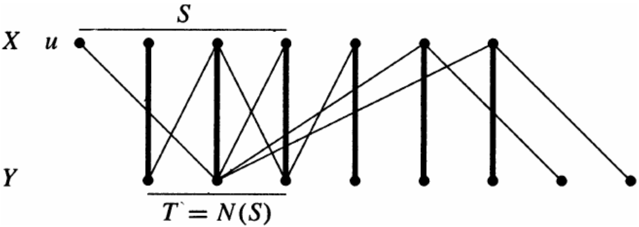
Let M be a *maximum* matching.

Let $u \in X$ be a vertex of X not saturated by M.

Consider all *M-alternating paths* starting from u.



$T \triangleq$ the set of vertices in Y reachable from u by M-alternating paths.



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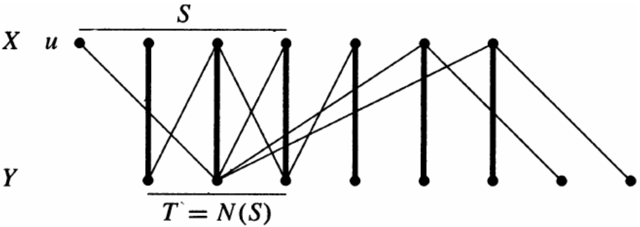

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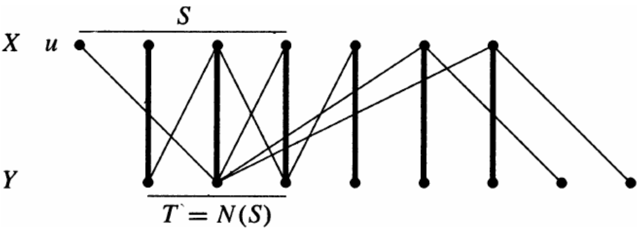
We will show that

$$T = N(S) \land |T| = |S - \{u\}|$$



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$$T = N(S) \wedge \left|T\right| = \left|S - \{u\}\right|$$

$$\left|N(S)\right| = \left|T\right| = \left|S\right| - 1 < \left|S\right|$$

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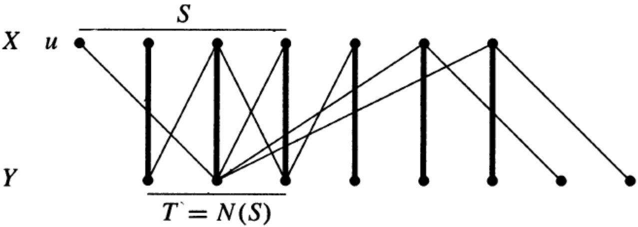

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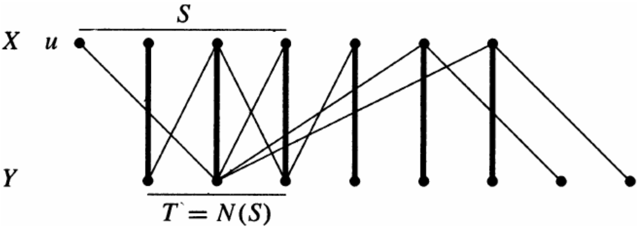

We show that there is a bijection from T to $S - \{u\}$.

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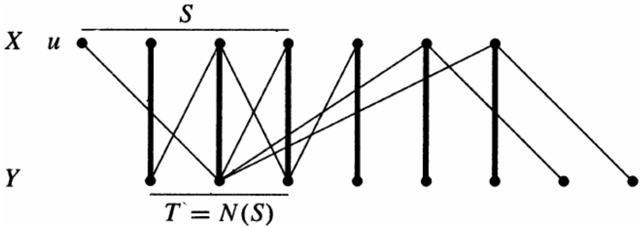

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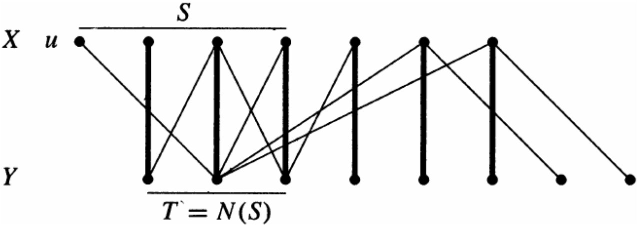
$$T = N(S)$$



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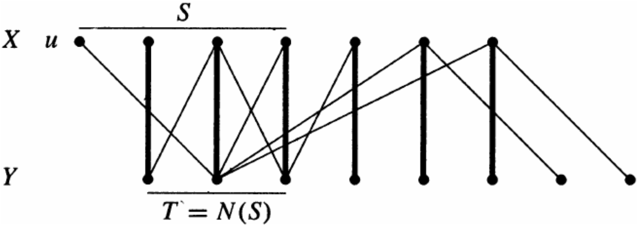

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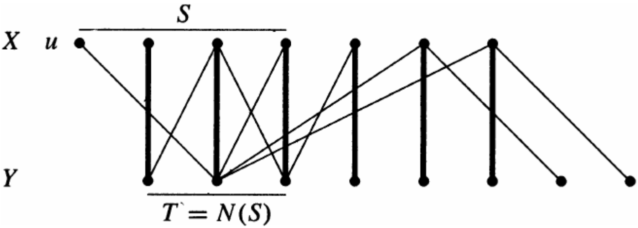

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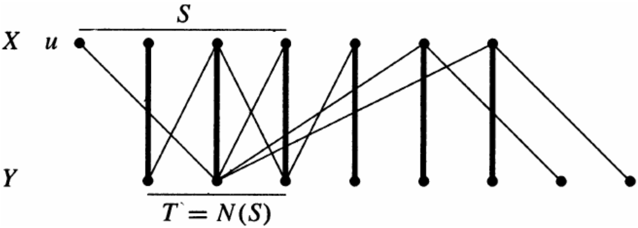

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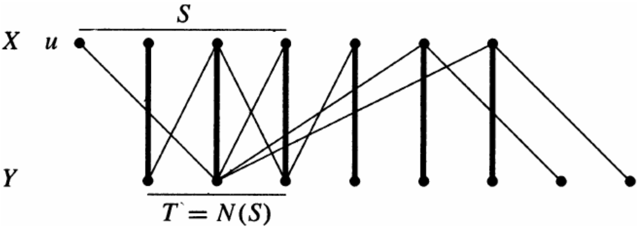

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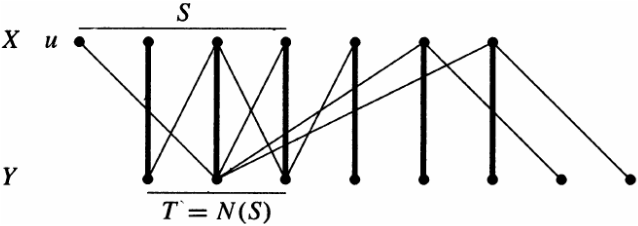

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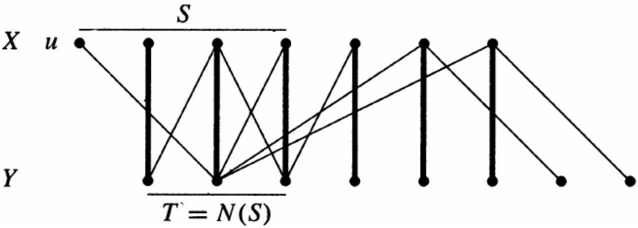

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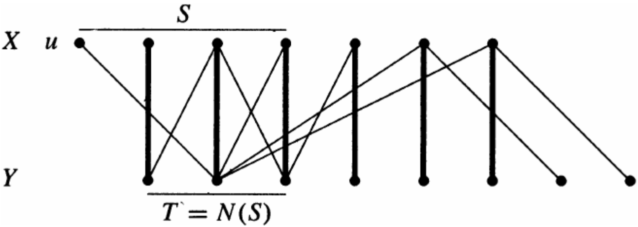

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$$s \neq u \implies s \in S - \{u\} \implies \{s, y\} \notin M$$

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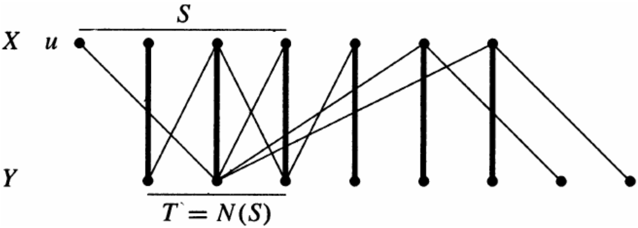

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Theorem (Hall Theorem; 1935)

Let $G = (X, Y, E)$ be a bipartite graph. There is a X-perfect matching of G iff

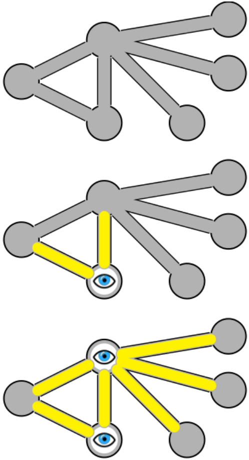
$$\forall W \subseteq X.\ \left|W\right| \leq \left|N_G(W)\right|$$

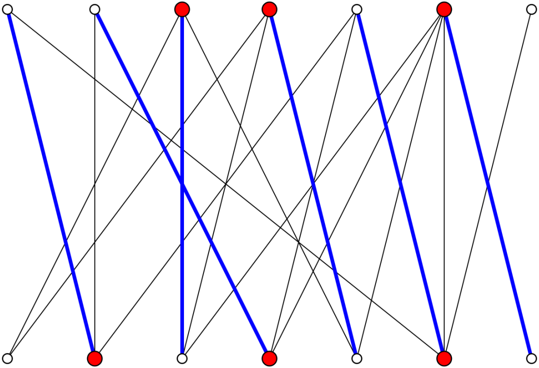
algorithm

Definitions (Vertex Cover (点覆盖))

A vertex cover of a graph G is a set $Q \subseteq V(G)$ that covers all edges.

$$\forall e \in E(G).\ e \cap Q \neq \emptyset$$

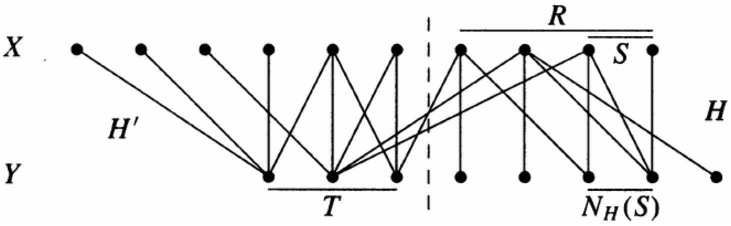




examples

Theorem (König (1931), Egerváry (1931))

If G is a bipartite graph, then the maximum size of a mathching in G equals the minimum size of a vertex cover of G


R
X
S
H
H'
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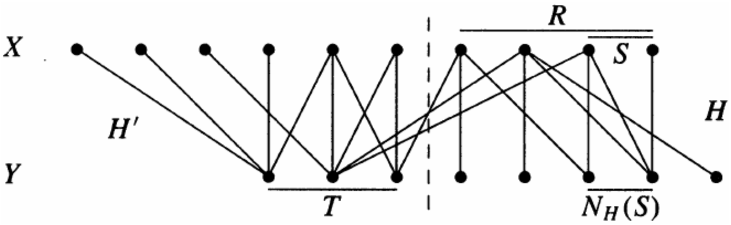
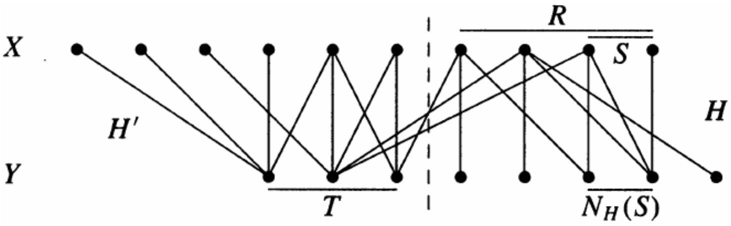
T
N_H(S)

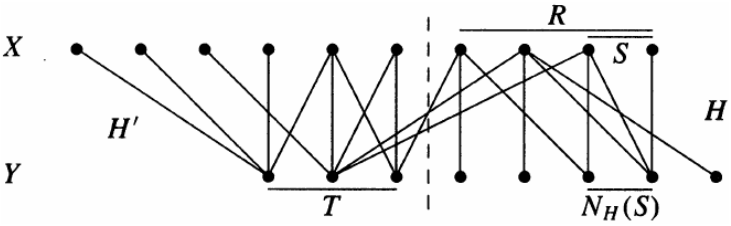


$$R = Q \cap X \qquad T = Q \cap Y$$



$$R = Q \cap X \qquad T = Q \cap Y$$

$$H \triangleq (R \cup (Y - T))\text{-induced subgraph of } G$$
$$H' \triangleq (T \cup (X - R))\text{-induced subgraph of } G$$



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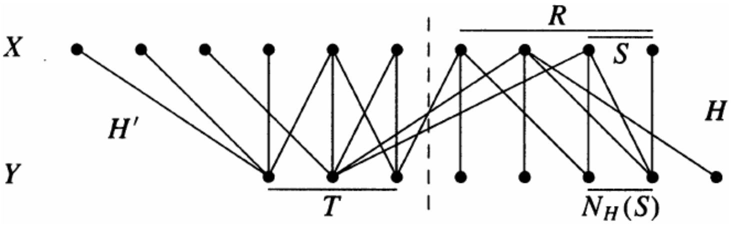
G has no edges from $X - R$ to $Y - T$.



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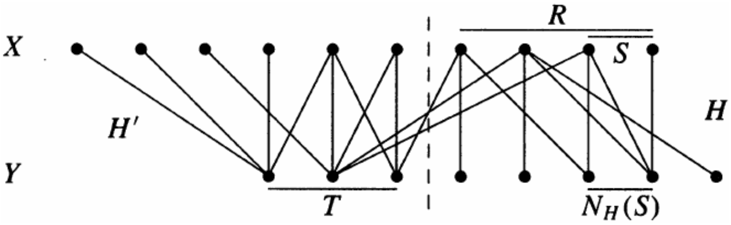
H has a R-perfect matching and H' has a T-perfect matching.

$\boxed{H \text{ has a } R\text{-perfect matching and } H' \text{ has a } T\text{-perfect matching.}}$



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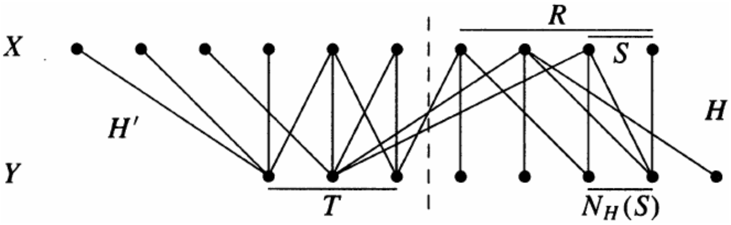

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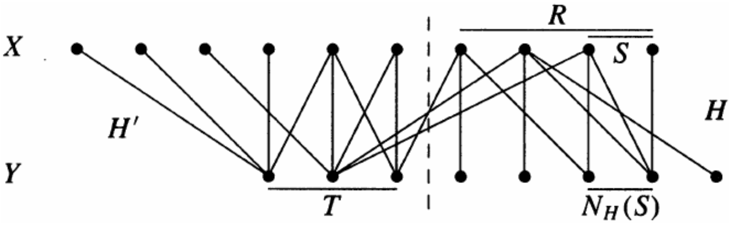

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$$\exists S \subseteq R.\ \left|N_H(S)\right| < \left|S\right|$$

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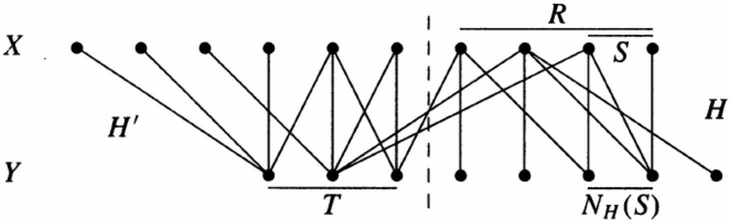

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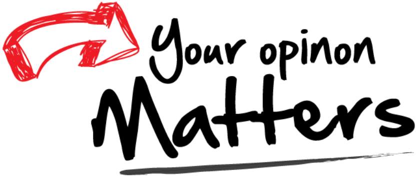
$T \cup (R - S + N_H(S))$ is a smaller vertex cover than Q

Thank You!



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