(十二) 图论：匹配与网络流
(Matching and Network Flow)

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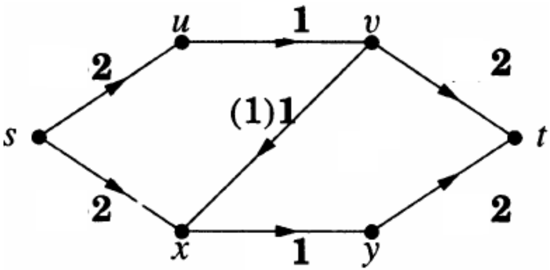
3 Theorems + 1 Algorithm

Definition (Network (网络))

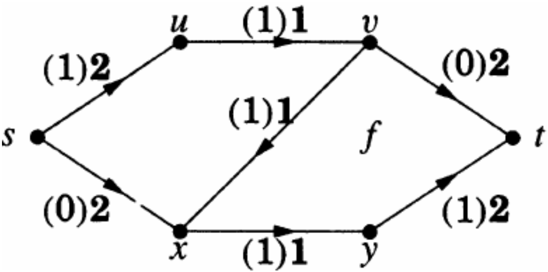
A network is a digraph with

- a distinguished source vertex s,
- a distinguished sink vertex t,
- a capacity $c(e) \geq 0$ on each edge e



Definition (Flow (流))

A flow f is a function that assigns a value $f(e)$ to each edge e.



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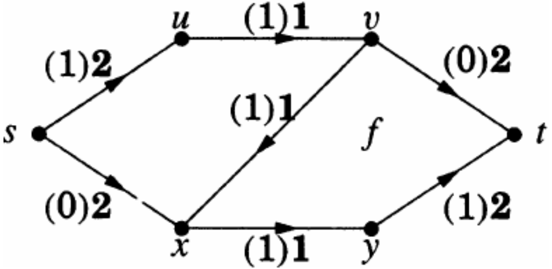

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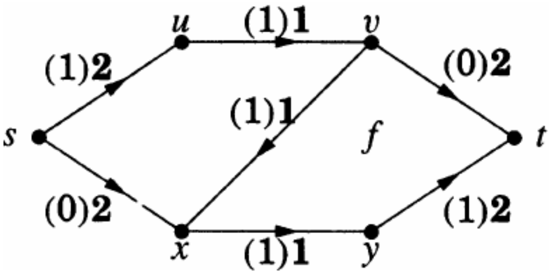


$$f^+(v) = \sum_{vw \in E} f(vw) \qquad f^-(v) = \sum_{uv \in E} f(uv)$$

Definition (Feasible)

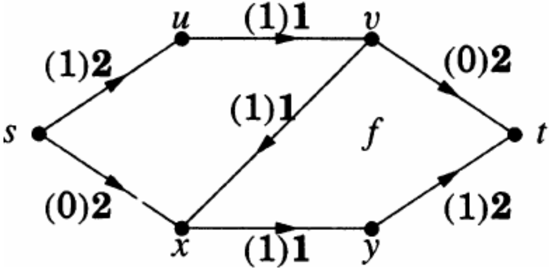
A flow f is feasible if it satisfies

Capacity Constraints:

$$\forall e \in E(G).\ 0 \leq f(e) \leq c(e)$$

Conservation Constraints:

$$\forall v \in V(G) - \{s, t\}.\ f^{+}(v) = f^{-}(v)$$



Definition (Value (值))

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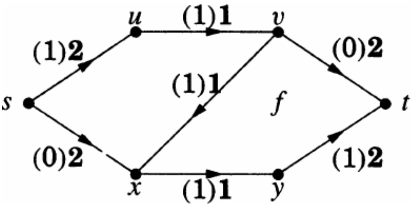

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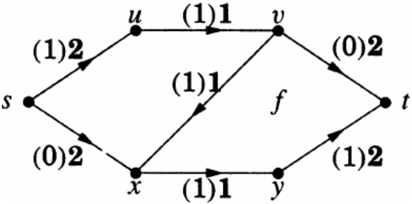

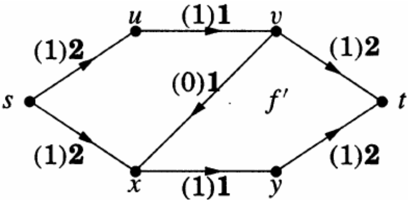

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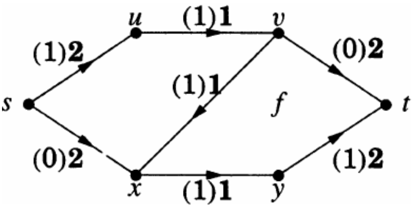
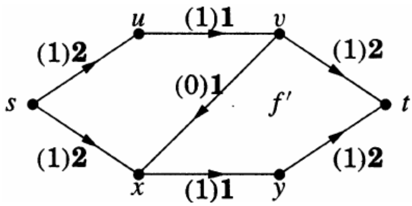
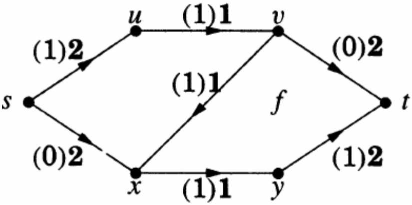




Definition (Maximum Flow (最大流))

A maximum flow is a feasible flow of maximum value.


u
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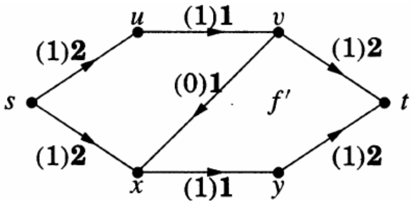
y


u
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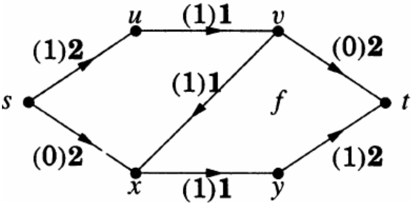
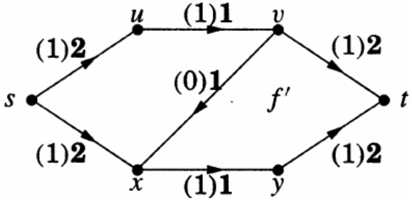
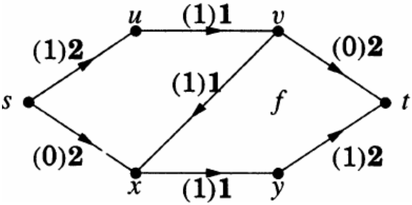
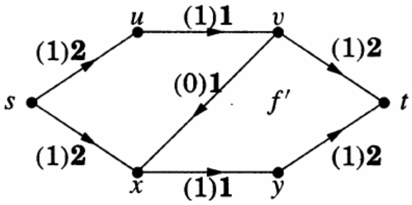




$$s \to x \textcolor{red}{\to} v \to t$$





$$s \to x \to v \to t$$

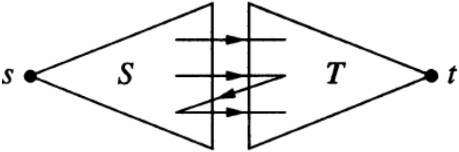
Definition (f-augmenting Paths (增广路径))

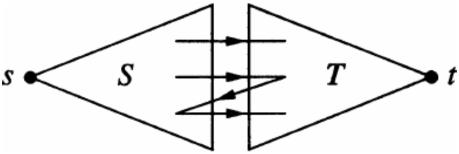
$$\min_{e \in E(P)} \epsilon(e)$$

Definition (Source/Sink Cut (割))

In a network, a source/sink cut $[S, T]$ consists of the edges from a source set S to a sink set T, where

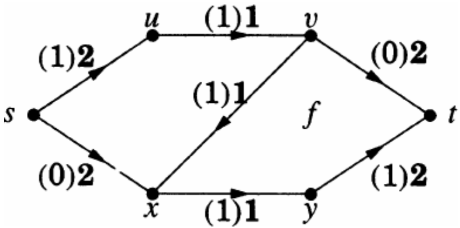
$$V = S \uplus T \land s \in S \land t \in T$$





Definition (Capacity of Cut (割的容量))

$$\mathsf{cap}(S,T) = \sum_{u \in S, v \in T, uv \in E} c(uv)$$



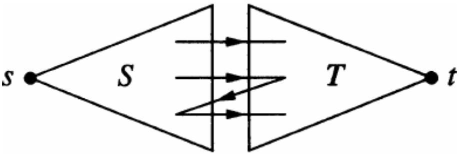
Definition (Minimum Cut (最小割))

A minimum cut is a cut of minimum value.

Theorem (Weak Duality (弱对偶定理))

Let f be any feasible flow and $[S, T]$ be any source/sink cut.

$$\mathsf{val}(f) \leq \mathsf{cap}(S, T).$$



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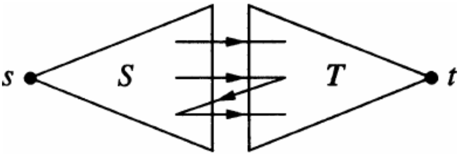

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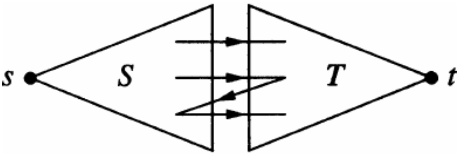

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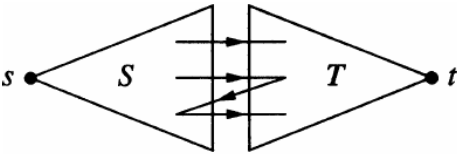
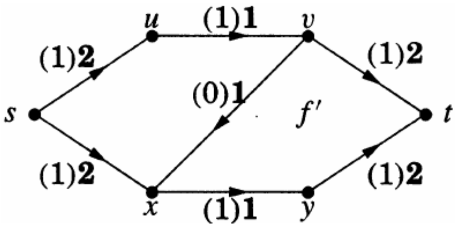


$$\mathrm{val}(f) = f^{+}(S) - f^{-}(S) \le f^{+}(S) \le \mathrm{cap}(S,T)$$

Lemma

$$\max_{f} \mathit{val}(f) \leq \min_{[S,T]} \mathit{cap}(S,T)$$



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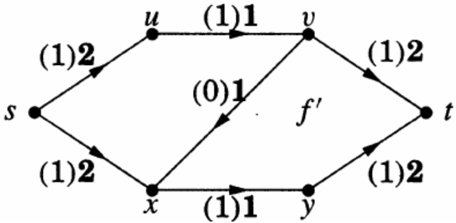

Theorem (Max-flow Min-cut Theorem (Ford and Fulkerson; 1956))

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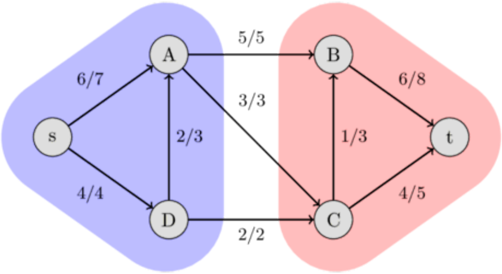
(Strong Duality)

L. R. Ford Jr. (1927 ~ 2017)

D. R. Fulkerson (1924 ~ 1976)


A
B
s
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C
5/5
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4/4
4/5
2/2

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Repeatedly finding f-augmenting paths until no more ones exist.

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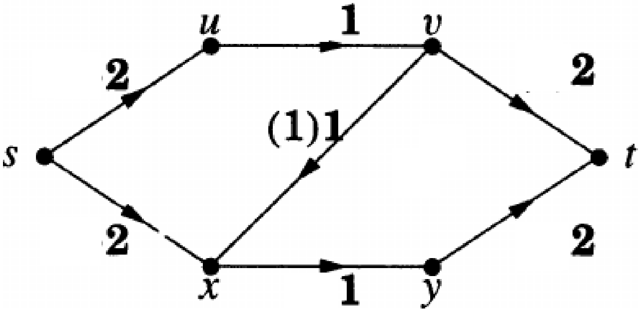

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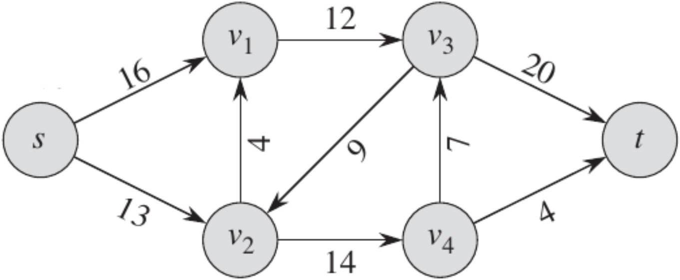

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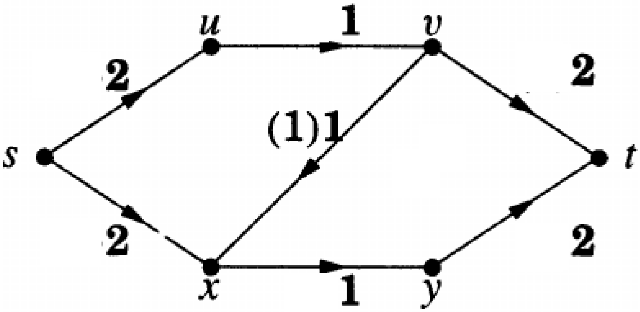

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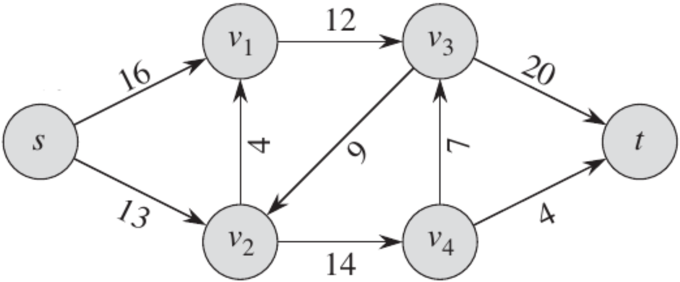

Theorem (Hall Theorem; 1935)

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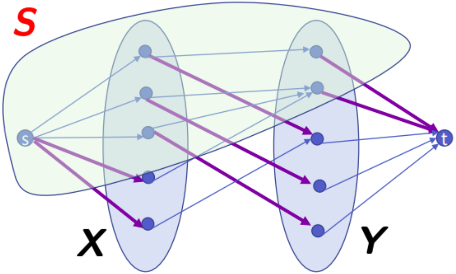

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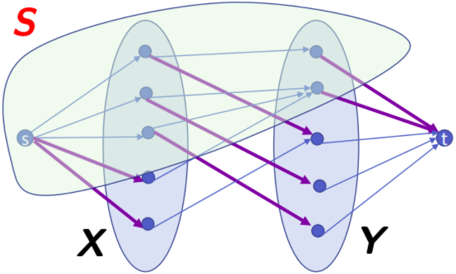

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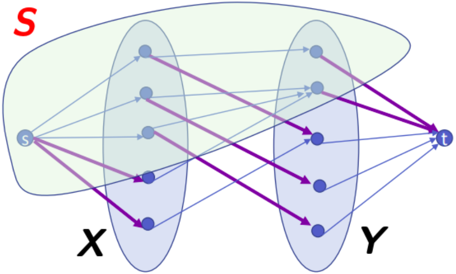

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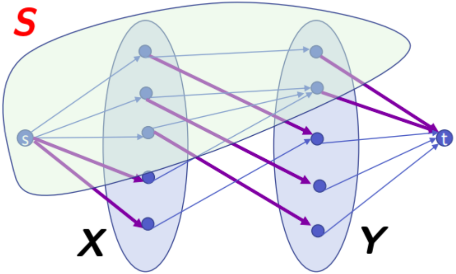

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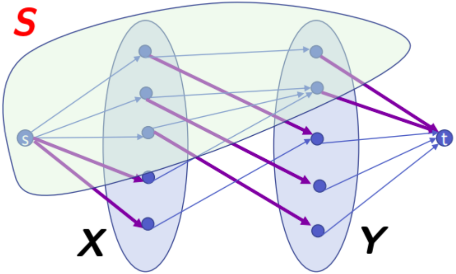

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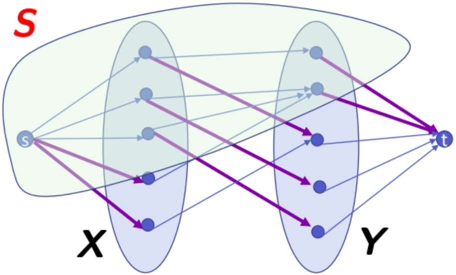
Therefore, we need to show that $\min_{[S,\overline{S}]} \mathsf{cap}(S, \overline{S}) \geq \left| X \right|$.

Let $[S, \overline{S}]$ be a minimum cut. We need to show that $\mathsf{cap}(S, \overline{S}) = \left|X\right|$.



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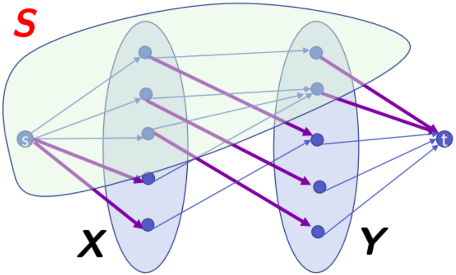

$$\boxed{N(S \cap X) \subseteq (S \cap Y)}$$

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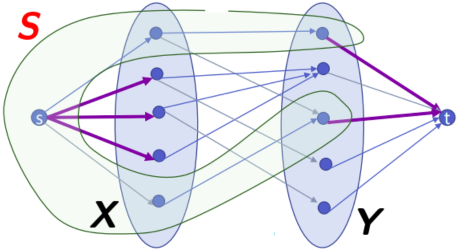

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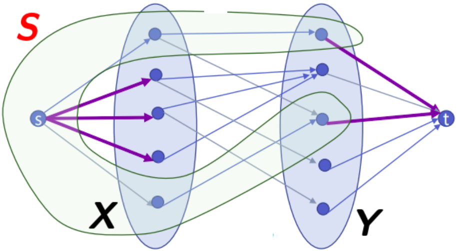
S
s
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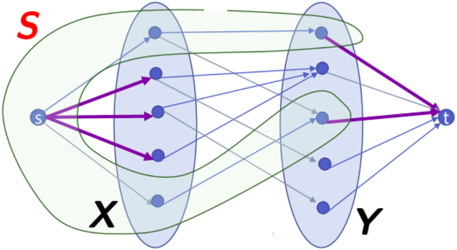
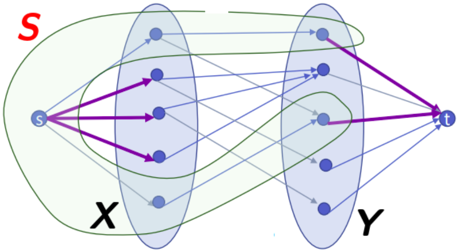
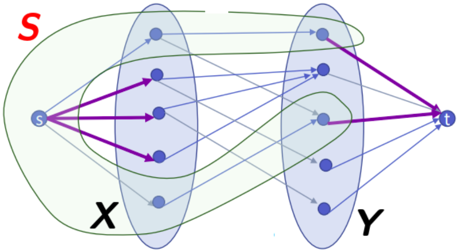


$$\mathrm{cap}(S, \overline{S}) = \sum_{u \in S, v \in \overline{S}} c(x, y)$$



$$\text{cap}(S, \overline{S}) = \sum_{u \in S, v \in \overline{S}} c(x, y)$$

$$= \sum_{v \in \overline{S} \cap X} c(s, v) + \sum_{u \in S \cap Y} c(u, t)$$



$$
\begin{aligned}
\mathsf{cap}(S,\overline{S}) &= \sum_{u\in S, v\in \overline{S}} c(x,y) \\
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\end{aligned}
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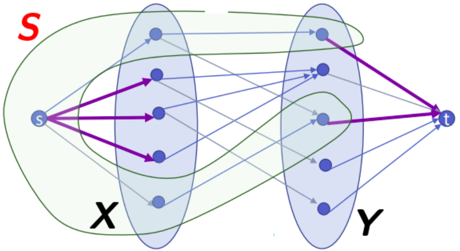

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\begin{aligned}
\mathsf{cap}(S,\overline{S}) &= \sum_{u\in S, v\in\overline{S}} c(x,y) \\
&= \sum_{v\in\overline{S}\cap X} c({\color{red}s},v) + \sum_{u\in S\cap Y} c(u,{\color{red}t}) \\
&= \left|X\right| - \left|S\cap X\right| + \left|S\cap Y\right| \\
&{\color{red}\geq} \left|X\right| - \left|S\cap X\right| + {\color{red}\left|N(S\cap X)\right|}
\end{aligned}
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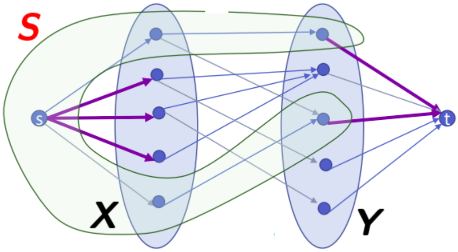

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&= \sum_{v\in \overline{S}\cap X} c({\color{red}s},v) + \sum_{u\in S\cap Y} c(u,{\color{red}t}) \\
&= \left|X\right| - \left|S\cap X\right| + \left|S\cap Y\right| \\
&{\color{red}\geq} \left|X\right| - \left|S\cap X\right| + {\color{red}\left|N(S\cap X)\right|} \\
&{\color{blue}\geq} \left|X\right|
\end{aligned}$$

Theorem (König (1931), Egerváry (1931))

If G is a bipartite graph, then the maximum size of a mathching in G equals the minimum size of a vertex cover of G

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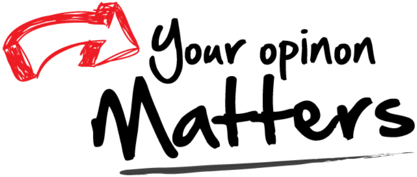
Thank
You!



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