(十二) 图论：匹配与网络流
(Matching and Network Flow)

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3 Theorems + 1 Algorithm

To maximize the size of a mathematical structure $\mathcal{S}$ in G

To maximize the size of a mathematical structure $\mathcal{S}$ in G

Theorem

$\mathcal{S}$ is maximum iff G does not contain $\mathcal{S}$-augmenting objects.

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Algorithm

Repeatedly finding $\mathcal{S}$-augmenting objects until no more ones exist.

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Theorem (Weak Duality Theorem)

The size of a maximum $\mathcal{S}$ $\leq$ *The size of a minimum* $\mathcal{S}'$

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Theorem (Weak Duality Theorem)

The size of a maximum $\mathcal{S} \leq$ *The size of a minimum* $\mathcal{S}'$

Theorem (Strong Duality Theorem)

The size of a maximum $\mathcal{S} =$ *The size of a minimum* $\mathcal{S}'$


let's get
married
today

The Marriage Problem (Philip Hall, 1935)

If there is a finite set of girls, each of whom knows several boys, under what conditions can all the girls marry boys in such a way that each girl marries a boy that she knows?

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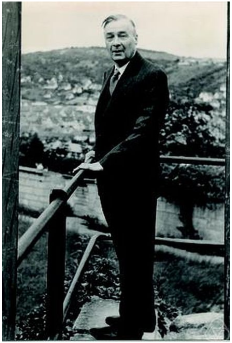
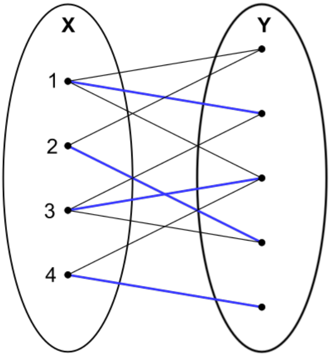


Philip Hall ($1904 \sim 1982$)


X
Y
1
2
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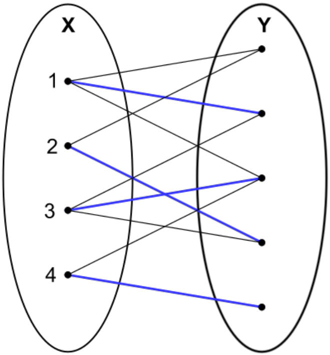
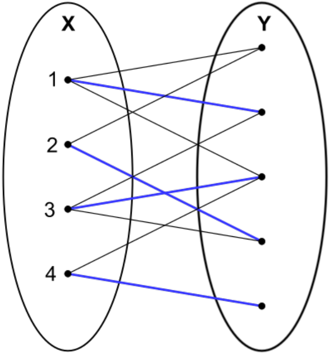
4



Definition (Matching (匹配))

A matching in a graph G is a set of edges with no shared endpoints.

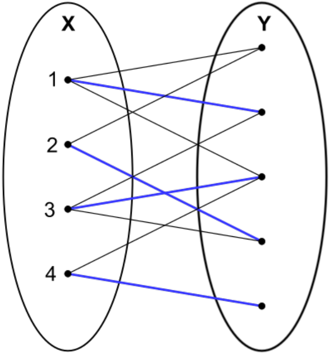


Definition (X-Perfect Matching (X-Saturating Matching))

Let $G = (X, Y, E)$ be a bipartite graph.

An X-perfect matching of G is a matching which covers each vertex in X.

$$|X| \leq |Y|$$



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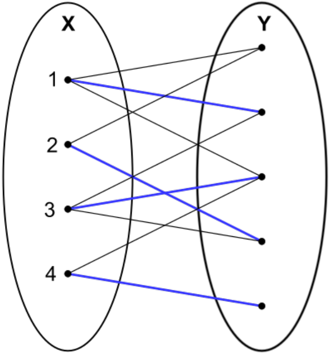

$$\forall W \subseteq X.\ |W| \le |N(W)|$$

Theorem (Hall Theorem; 1935)

Let $G = (X, Y, E)$ be a bipartite graph. There is an X-perfect matching of G iff

$$\forall W \subseteq X. \left|W\right| \le \left|N_G(W)\right|$$

By induction on the number $\left|X\right|$ of vertices in X.

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Basis Step: $\left|X\right| = 1$. $\left|X\right| \leq \left|N_G(X)\right|$.

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Induction Hypothesis: Suppose that it holds if $\left|X\right| < m$.

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- CASE II: There is a set of $k < m$ girls in X who know k boys in Y.

Theorem (Hall Theorem; 1935)

Let $G = (X, Y, E)$ be a bipartite graph. There is a X-perfect matching of G iff

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Definition (M-alternating Paths)

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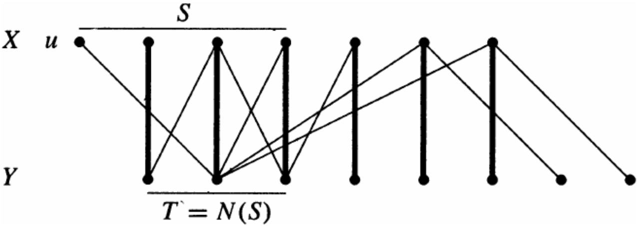

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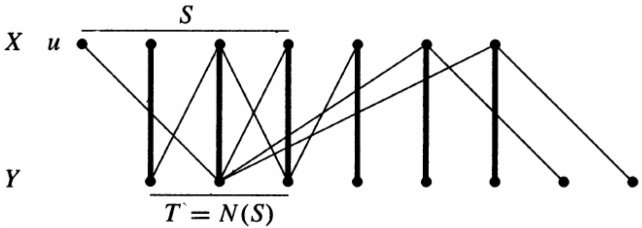

Let M be a *maximum* matching.

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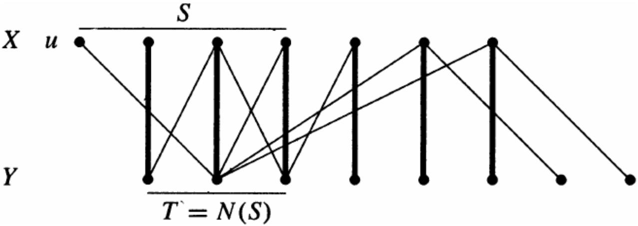

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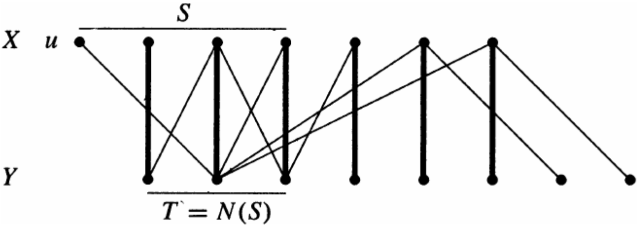

Let M be a *maximum* matching.

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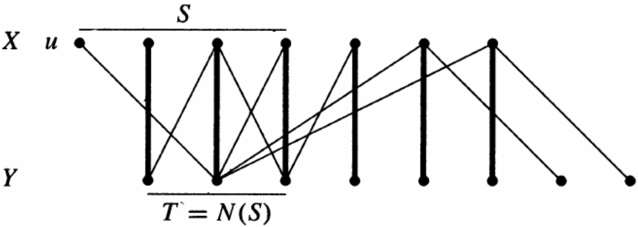
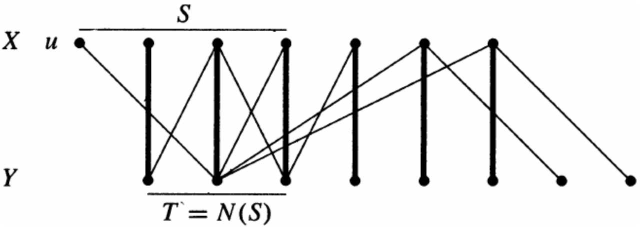
Consider all *M-alternating paths* starting from u.



$T \triangleq$ the set of vertices in Y reachable from u by M-alternating paths.



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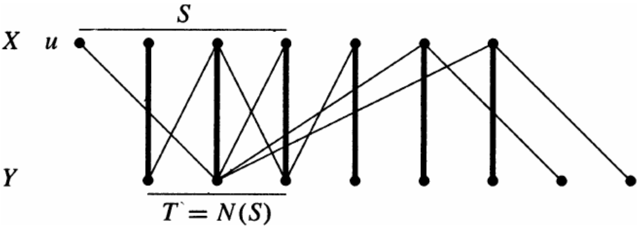

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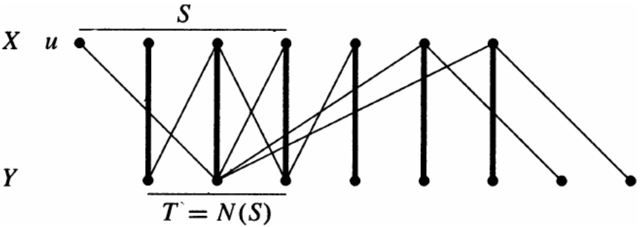
We will show that

$$T = N(S) \wedge \left|T\right| = \left|S - \{u\}\right|$$



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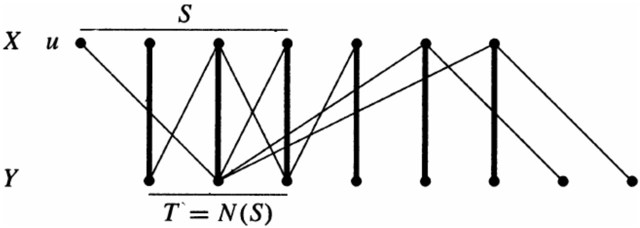

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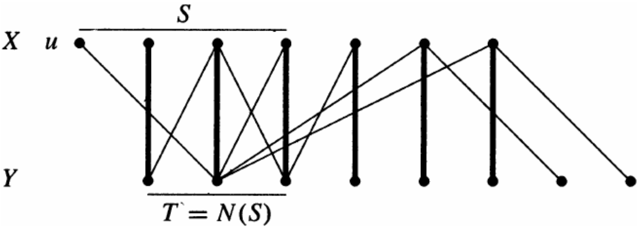

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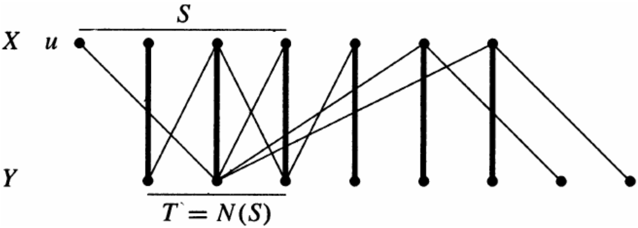

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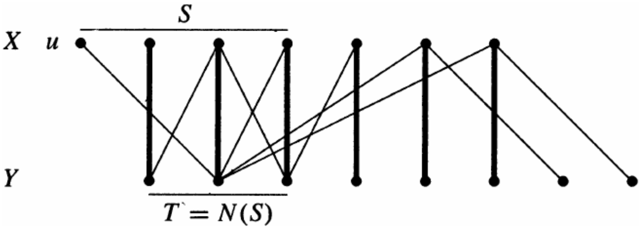

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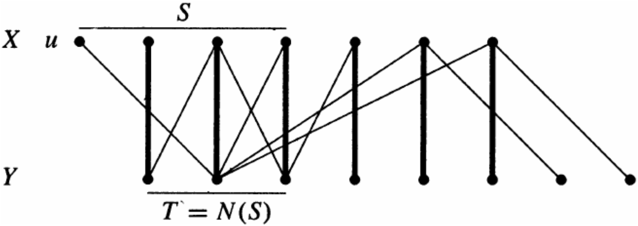

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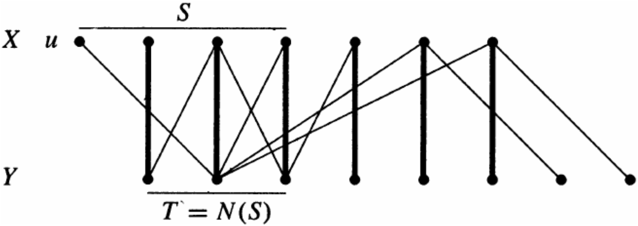

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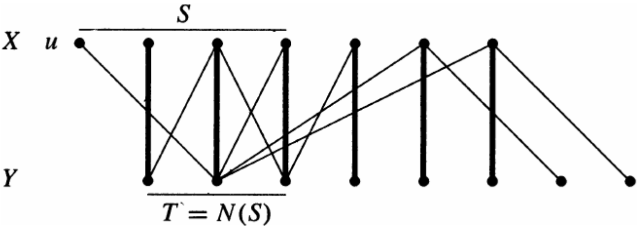

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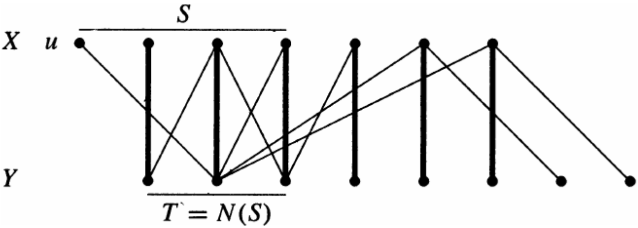

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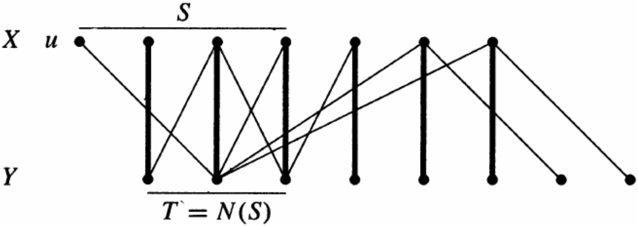

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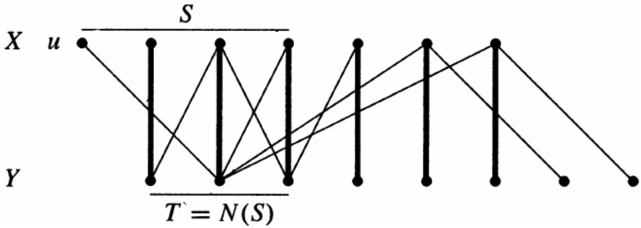

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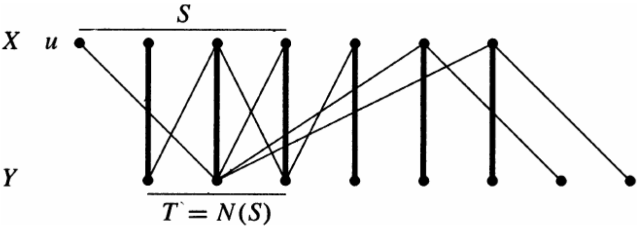

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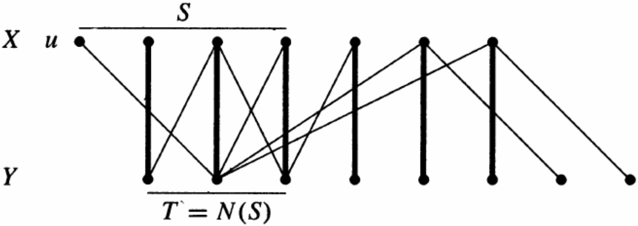

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Theorem (Hall Theorem; 1935)

Let $G = (X, Y, E)$ be a bipartite graph. There is a X-perfect matching of G iff

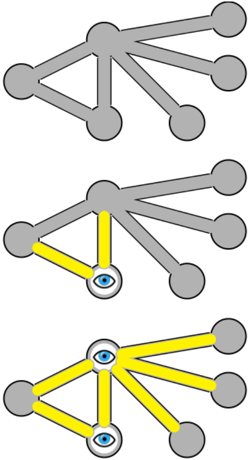
$$\forall W \subseteq X. \left|W\right| \leq \left|N_G(W)\right|$$

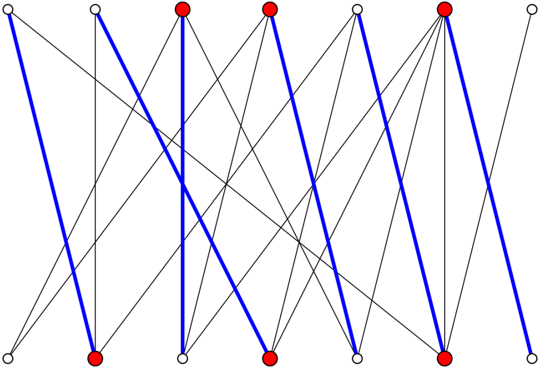
algorithm

Definitions (Vertex Cover (点覆盖))

A vertex cover of a graph G is a set $Q \subseteq V(G)$ that covers all edges.

$$\forall e \in E(G).\ e \cap Q \neq \emptyset$$

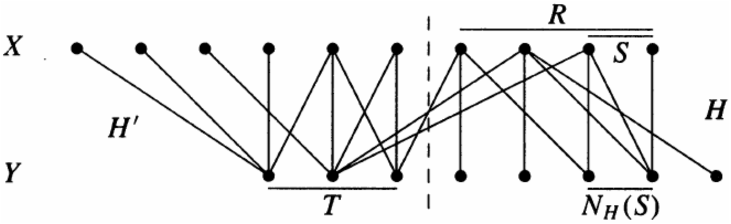


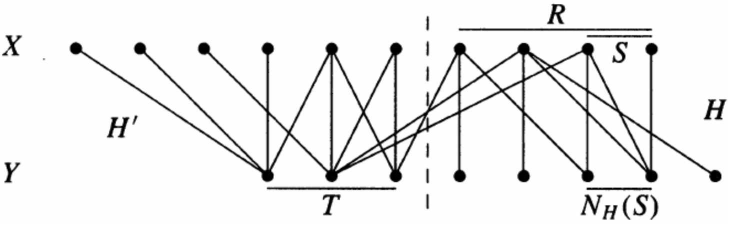


examples

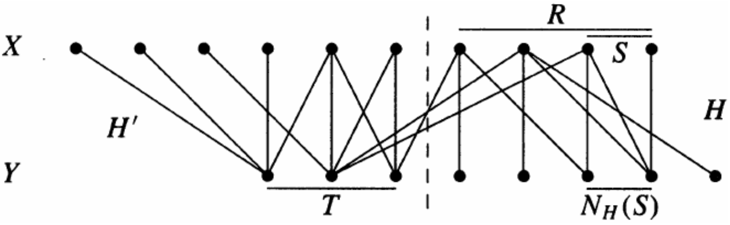
Theorem (König (1931), Egerváry (1931))

If G is a bipartite graph, then the maximum size of a mathching in G equals the minimum size of a vertex cover of G


R
X
S
H'
H
Y
T
$N_H(S)$

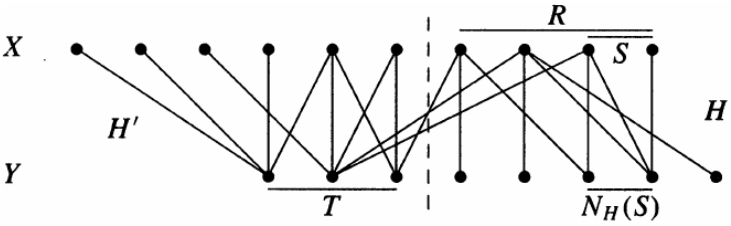


$$R = Q \cap X \qquad T = Q \cap Y$$



$$R = Q \cap X \qquad T = Q \cap Y$$

$$H \triangleq (R \cup (Y - T))\text{-induced subgraph of } G$$
$$H' \triangleq (T \cup (X - R))\text{-induced subgraph of } G$$



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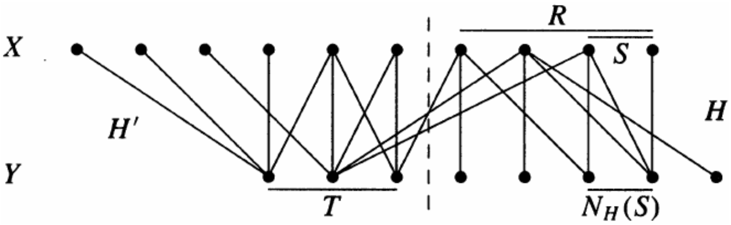
G has no edges from $X - R$ to $Y - T$.



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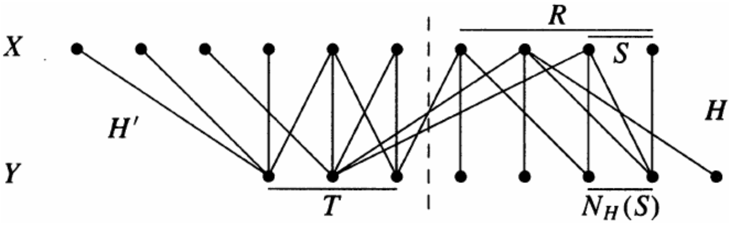

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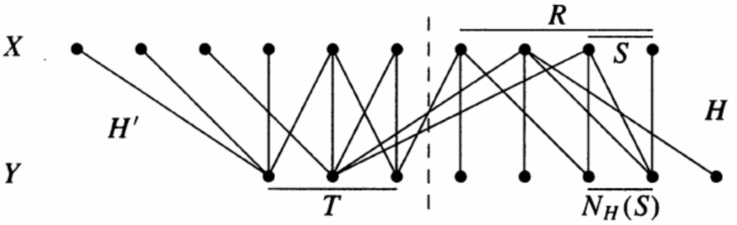

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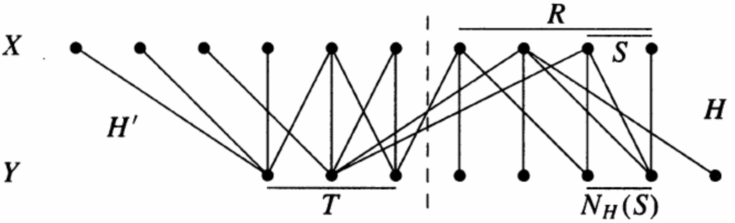

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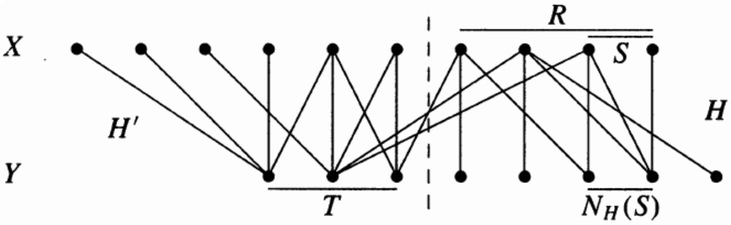

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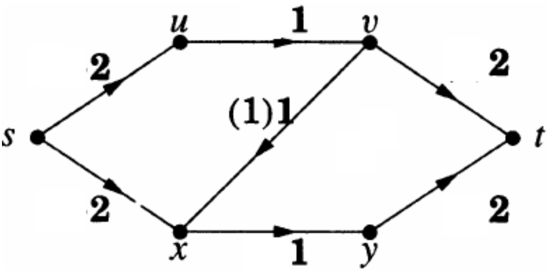
$$T \cup (R - S + N_H(S)) \text{ is a smaller vertex cover than } Q$$

independent sets and covers

Definition (Network (网络))

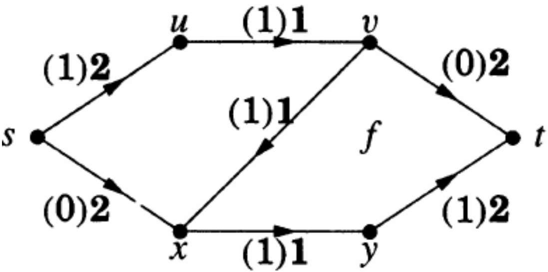
A network is a digraph with

- a distinguished source vertex s,
- a distinguished sink vertex t,
- a capacity $c(e) \geq 0$ on each edge e



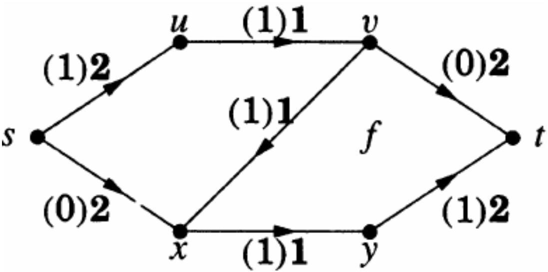
Definition (Flow (流))

A flow f is a function that assigns a value $f(e)$ to each edge e.



Definition (Flow (流))

A flow f is a function that assigns a value $f(e)$ to each edge e.



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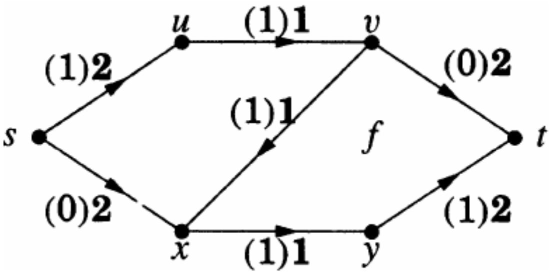


$$f^+(v) = \sum_{vw \in E} f(vw) \qquad f^-(v) = \sum_{uv \in E} f(uv)$$

Definition (Feasible)

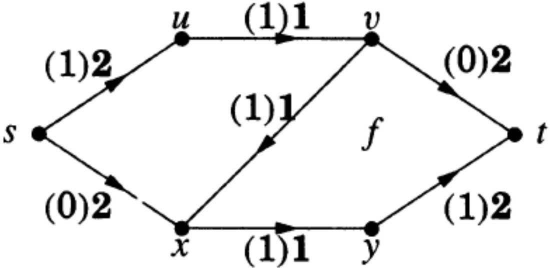
A flow f is feasible if it satisfies

Capacity Constraints:

$$\forall e \in E(G).\ 0 \leq f(e) \leq c(e)$$

Conservation Constraints:

$$\forall v \in V(G) - \{s, t\}.\ f^{+}(v) = f^{-}(v)$$



Definition (Value (值))

The value $\mathsf{val}(f)$ of a flow f is

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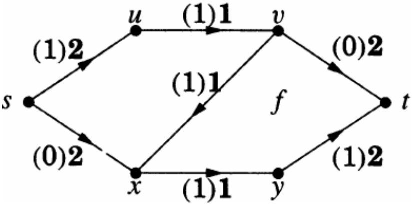

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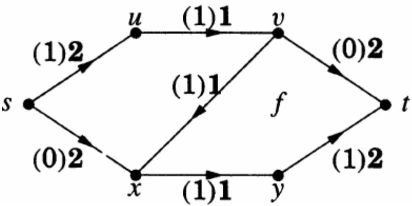

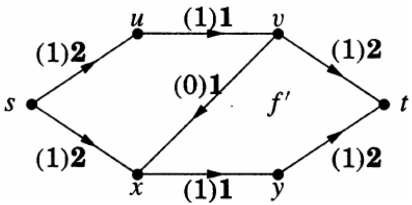

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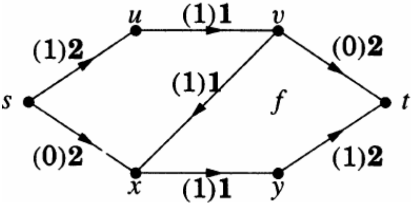
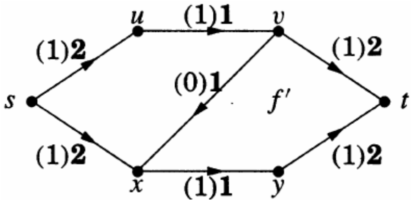
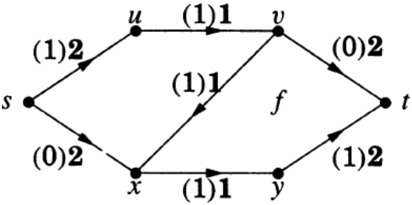




Definition (Maximum Flow (最大流))

A maximum flow is a feasible flow of maximum value.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f
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x
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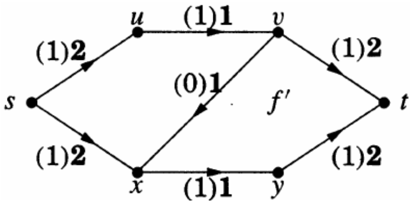
y


u
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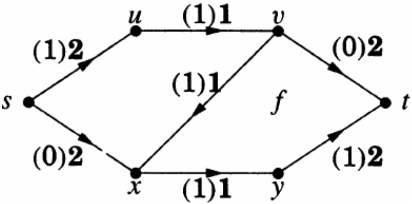
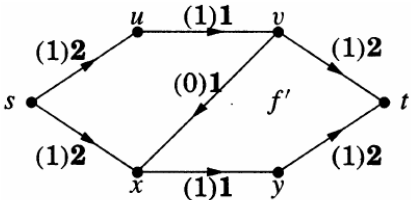
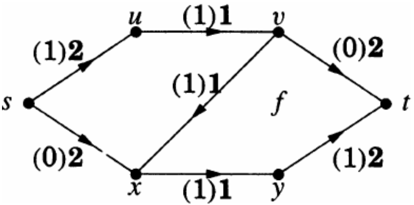
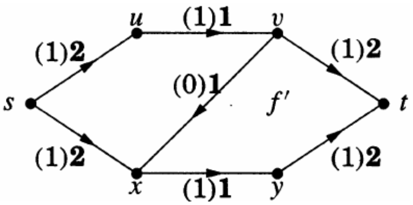




$$s \to x \textcolor{red}{\to} v \to t$$





$$s \to x \to v \to t$$

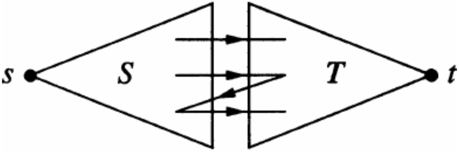
Definition (f-augmenting Paths (增广路径))

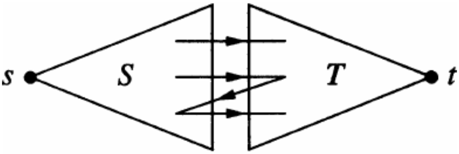
$$\min_{e \in E(P)} \epsilon(e)$$

Definition (Source/Sink Cut (割))

In a network, a source/sink cut $[S, T]$ consists of the edges from a source set S to a sink set T, where

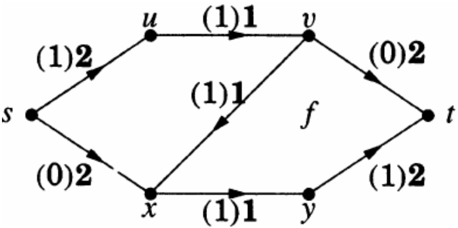
$$V = S \uplus T \wedge s \in S \wedge t \in T$$





Definition (Capacity of Cut (割的容量))

$$\mathsf{cap}(S,T) = \sum_{u \in S, v \in T, uv \in E} c(uv)$$



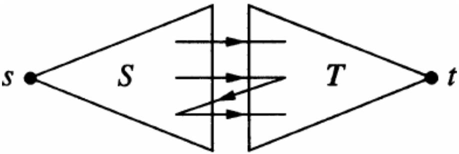
Definition (Minimum Cut (最小割))

A minimum cut is a cut of minimum value.

Theorem (Weak Duality (弱对偶定理))

Let f be any feasible flow and $[S, T]$ be any source/sink cut.

$$\mathit{val}(f) \leq \mathit{cap}(S, T).$$



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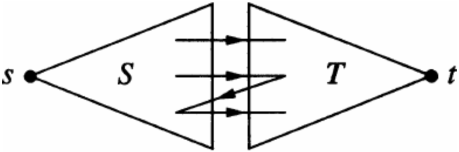

$$\mathrm{val}(f) = f^{+}(S) - f^{-}(S)$$

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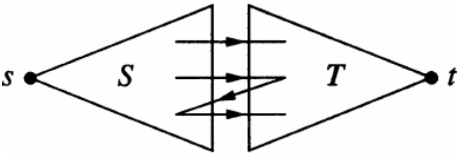

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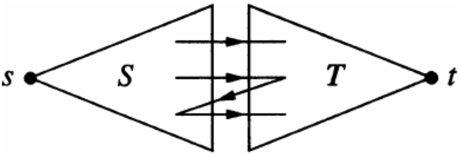
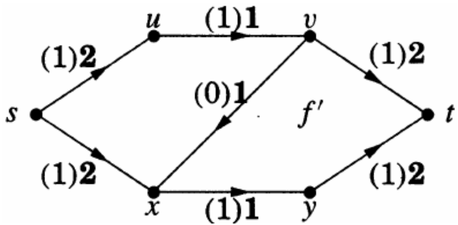


$$\mathsf{val}(f) = f^+(S) - f^-(S) \leq f^+(S) \leq \mathsf{cap}(S, T)$$

Lemma

$$\max_{f} \mathit{val}(f) \leq \min_{[S,T]} \mathit{cap}(S,T)$$



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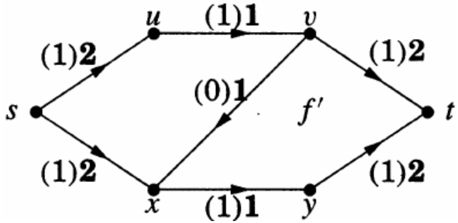


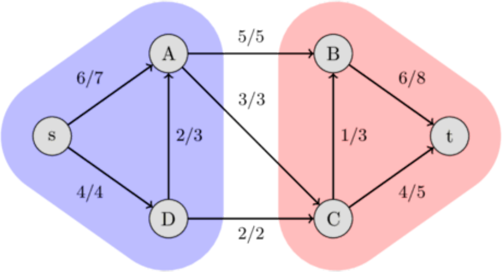
Theorem (Max-flow Min-cut Theorem (Ford and Fulkerson; 1956))

$$\max_{f} val(f) = \min_{[S,T]} cap(S,T)$$

(Strong Duality)

L. R. Ford Jr. ($1927 \sim 2017$)

D. R. Fulkerson ($1924 \sim 1976$)


5/5
A
B
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s
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4/5
D
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2/2

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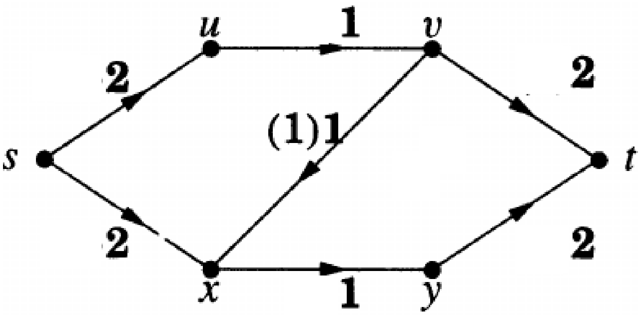

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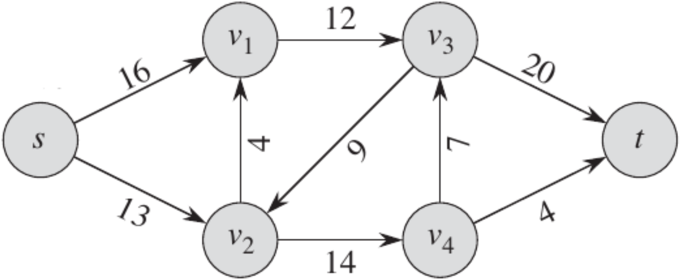

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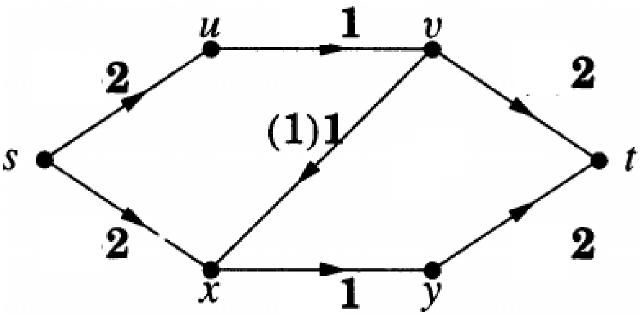

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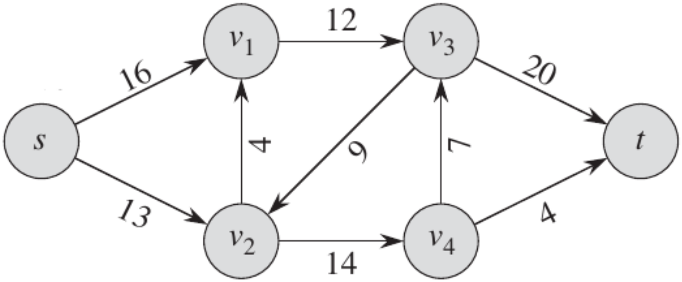

Theorem (Hall Theorem; 1935)

There is an X-perfect matching of G iff

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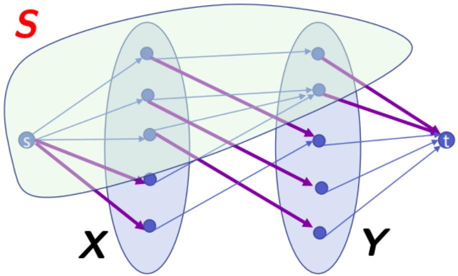

$\forall x \in X.\ c(s,x) = 1 \quad \forall y \in Y.\ c(y,t) = 1 \quad \forall x \in X, y \in Y.\ c(x,y) = \infty$

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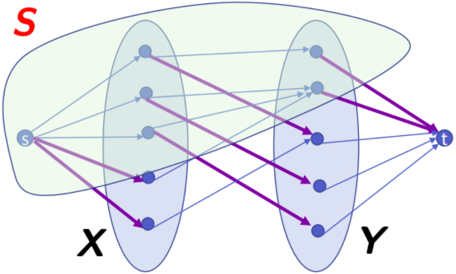

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We need to show that $\max_{f} \mathsf{val}(f) = \left|X\right|$.

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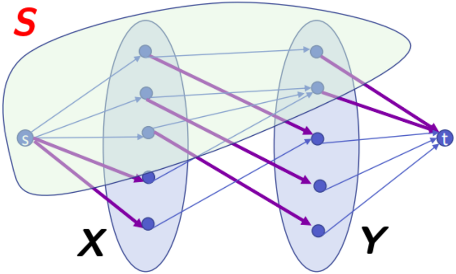

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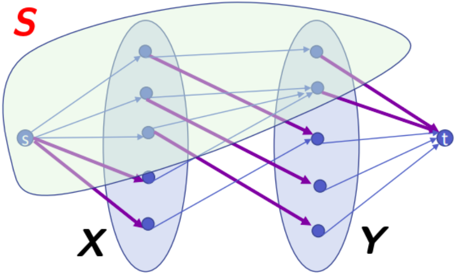

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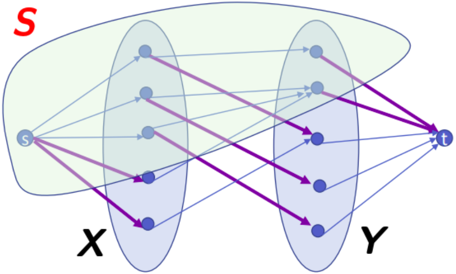
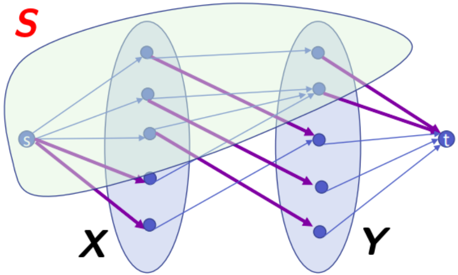


$$\min_{[S,\overline{S}]} \mathsf{cap}(S, \overline{S}) \le \left|X\right|$$

Therefore, we need to show that $\min_{[S,\overline{S}]} \mathsf{cap}(S, \overline{S}) \ge \left|X\right|$.

Let $[S, \overline{S}]$ be a minimum cut. We need to show that $\mathsf{cap}(S, \overline{S}) = \left|X\right|$.



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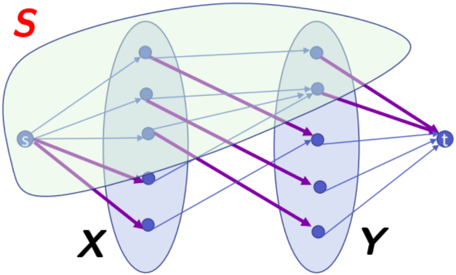

$$\boxed{N(S \cap X) \subseteq (S \cap Y)}$$

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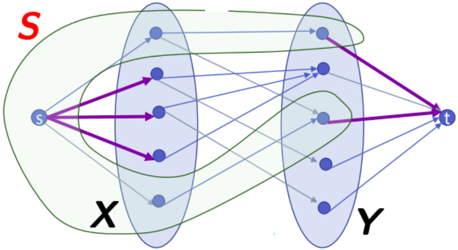

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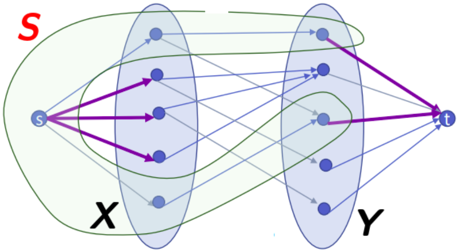
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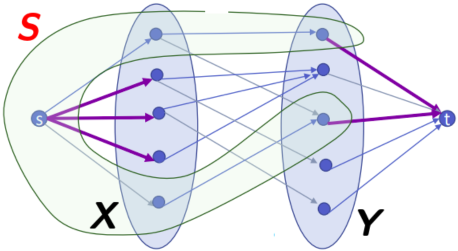
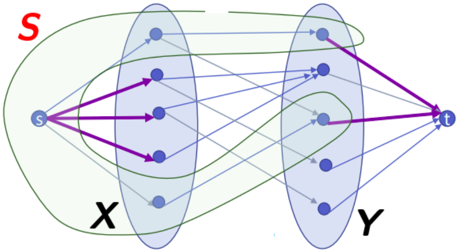


$$\mathrm{cap}(S,\overline{S}) = \sum_{u\in S, v\in \overline{S}} c(x,y)$$



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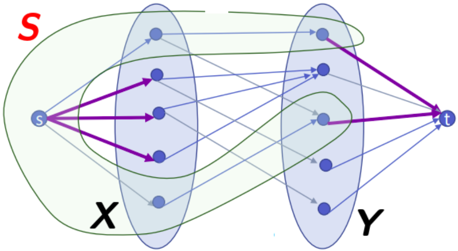
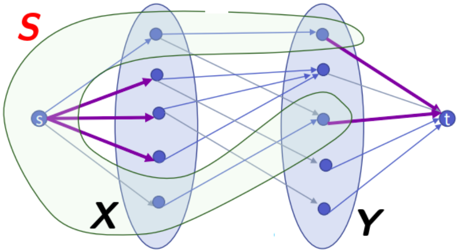
$$= \sum_{v \in \overline{S} \cap X} c(s, v) + \sum_{u \in S \cap Y} c(u, t)$$



$$
\begin{aligned}
\mathsf{cap}(S,\overline{S}) &= \sum_{u\in S, v\in\overline{S}} c(x,y) \\
&= \sum_{v\in\overline{S}\cap X} c({\color{red}s},v) + \sum_{u\in S\cap Y} c(u,{\color{red}t}) \\
&= \left|X\right| - \left|S\cap X\right| + \left|S\cap Y\right|
\end{aligned}
$$



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&{\color{red}\geq} \left|X\right| - \left|S\cap X\right| + {\color{red}\left|N(S\cap X)\right|}
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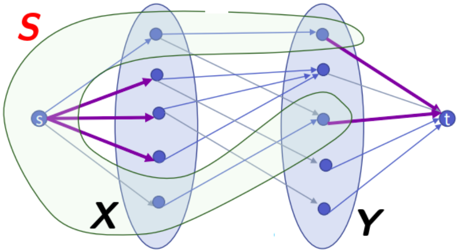

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&{\color{red}\geq} \left|X\right| - \left|S \cap X\right| + {\color{red}\left|N(S \cap X)\right|} \\
&{\color{blue}\geq} \left|X\right|
\end{aligned}
$$

Theorem (König (1931), Egerváry (1931))

If G is a bipartite graph, then the maximum size of a mathching in G equals the minimum size of a vertex cover of G

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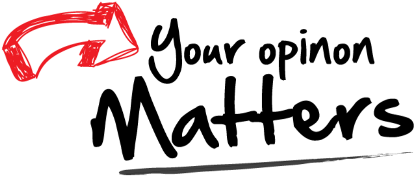
Thank You!



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