(六) 集合: 函数 (Functions)

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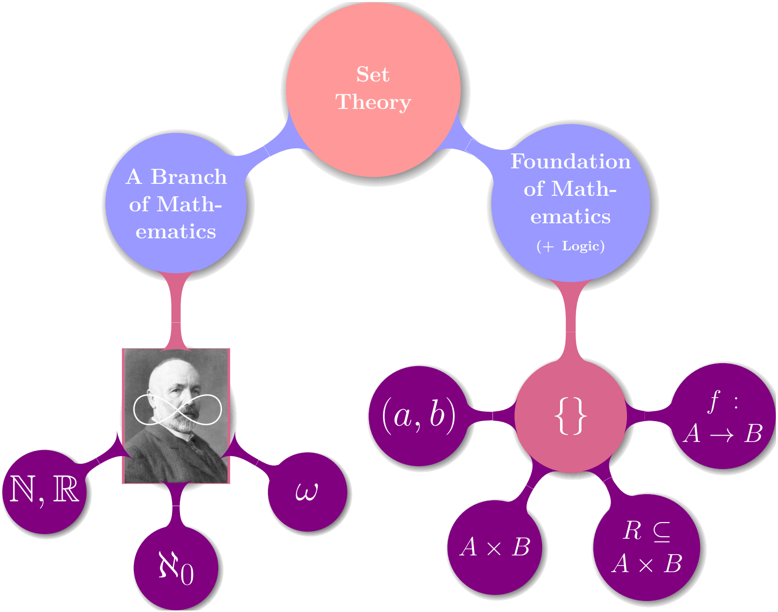
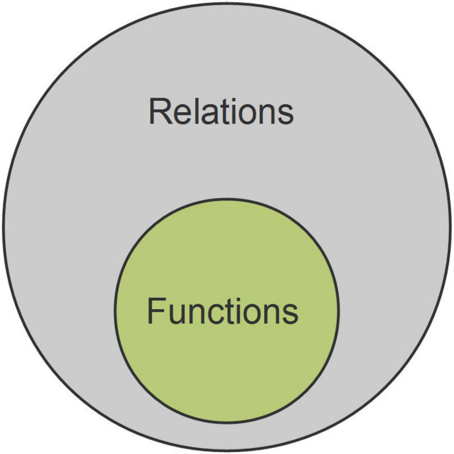

Set Theory
A Branch of Math-ematics
Foundation of Math-ematics
(+ Logic)
$\mathbb{N}, \mathbb{R}$
$\aleph_0$
ω
(a, b)
$\{\}$
$f : A \to B$
$A \times B$
$R \subseteq A \times B$

从“关系”的角度理解“函数”

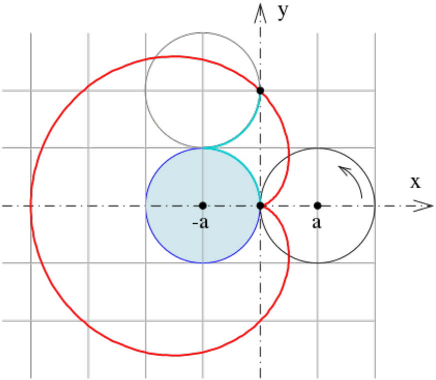


$$f(x) = 2x + 1$$

“函数” 也是 “关系”

$$\{\dots, (-2, -3), (-1, -1), (0, 1), (1, 3), \dots\}$$

$$(x^2+y^2)^2+4ax(x^2+y^2)-4a^2y^2=0$$



“函数”不允许“一对多”

Functions

Functions

PROOF!

Definition of Functions

$$R \subseteq A \times B$$

is a *relation* from A to B

Definition (Function)

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$$\forall a \in A.\ \exists!\ b \in B.\ (a, b) \in f.$$

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$$\mathrm{dom}(f) = A \qquad \mathrm{cod}(f) = B$$

$$\mathrm{ran}(f) = f(A) \subseteq B$$

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$$f : a \mapsto b$$

$$f(a) \triangleq b$$

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$$\forall a \in A.$$

$$\forall a \in A.\ \exists b \in B. (a, b) \in f$$

$$\exists! b \ \in B.$$

$$\forall b, b' \in B.\ (a, b) \in f \land (a, b') \in f \implies b = b'$$

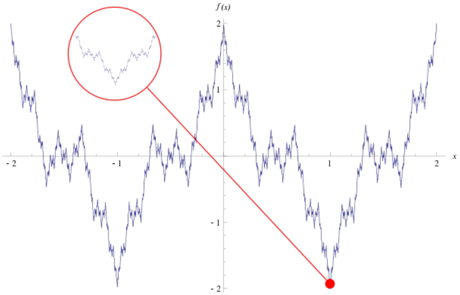
$$I_X : X \to X$$

X 上的恒等函数

$$\forall x \in X.\ I_X(x) = x$$

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$

$$0 < a < 1,\ b \text{ is a positive odd integer},\ ab > 1 + \frac{3}{2}\pi$$



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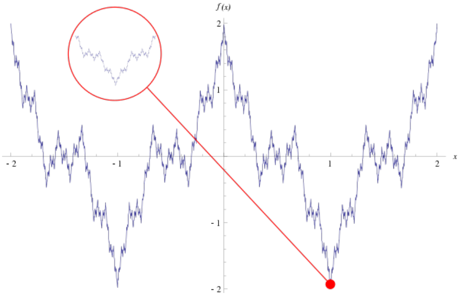

Weierstrass Function (1872)

“处处连续, 但处处不可导”

Definition (Y^X)

The *set* of all functions from X to Y:

$$Y^X = \{f \mid f : X \to Y\}$$

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There is no set consisting of all functions.

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$$\bigcup_{I_X \in A} \mathrm{dom}(I_X)$$

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For every set X, there exists a function $I_X : \{X\} \to \{X\}$.

$\bigcup_{I_X \in A} \mathrm{dom}(I_X)$ would be the universe that does not exist!

Functions as Sets

Theorem (函数的外延性原理 (The Principle of Functional Extensionality))

$$f, g \ \textit{are functions}:$$

$$f = g \iff dom(f) = dom(g) \wedge \big(\forall x \in dom(f).\ f(x) = g(x)\big)$$

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It may be that $\text{cod}(f) \neq \text{cod}(g)$.

$$f : A \to B \qquad g : C \to D$$

Q : Is $f \cap g$ a function?

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Theorem (Intersection of Functions)

$$f \cap g : (A \cap C) \to (B \cap D)$$

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Q : Is $f \cup g$ a function?

Theorem (Union of Functions)

$$f \cup g : (A \cup C) \to (B \cup D) \iff \forall x \in dom(f) \cap dom(g).\ f(x) = g(x)$$

$$f : A \to B \qquad g : C \to D$$

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Theorem (Union of Functions)

$$f \cup g : (A \cup C) \to (B \cup D) \iff \forall x \in dom(f) \cap dom(g).\ f(x) = g(x)$$

$$f : \mathbb{Q} \to \mathbb{R}$$

$$f(x) = \begin{cases} x + 1, & \text{if } x \in 2\mathbb{Z} \\ x - 1, & \text{if } x \in 3\mathbb{Z} \\ 2, & \text{otherwise} \end{cases}$$

$$f : \mathcal{P}(\mathbb{R}) \to \mathbb{Z}$$

$$f(A) = \begin{cases} \min(A \cap \mathbb{N}) & \text{if } A \cap \mathbb{N} \neq \emptyset \\ -1 & \text{if } A \cap \mathbb{N} = \emptyset \end{cases}$$

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$$\mathrm{dom}(f) \cap \mathrm{dom}(g) = \emptyset$$

By the *Well-Ordering Principle* of $\mathbb{N}$

$$D : \mathbb{R} \to \mathbb{R}$$

$$D(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Dirichlet Function

"处处不连续"

Special Functions (*-jectivity*)

Definition (Injective (one-to-one; 1-1) 单射函数)

$$f : A \to B$$

$$\forall a_1, a_2 \in A.\ a_1 \neq a_2 \to f(a_1) \neq f(a_2)$$

Definition (Injective (one-to-one; 1-1) 单射函数)

$$f : A \to B \qquad f : A \rightarrowtail B$$

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For Proof:

- To prove that f *is* 1-1:

$$\forall a_1, a_2 \in A.\ f(a_1) = f(a_2) \to a_1 = a_2$$

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For Proof:

- To prove that f *is* 1-1:

$$\forall a_1, a_2 \in A.\ f(a_1) = f(a_2) \to a_1 = a_2$$

- To show that f *is not* 1-1:

$$\exists a_1, a_2 \in A.\ a_1 \neq a_2 \land f(a_1) = f(a_2)$$

Definition (Surjective (onto) 满射函数)

$$f : A \to B$$

$$\mathrm{ran}(f) = B$$

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$$f : A \to B \qquad f : A \twoheadrightarrow B$$

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- To prove that f *is* onto:

$$\forall b \in B.\ (\exists a \in A.\ f(a) = b)$$

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$$f : A \to B \qquad f : A \twoheadrightarrow B$$

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$$\forall b \in B.\ (\exists a \in A.\ f(a) = b)$$

- To show that f *is not* onto:

$$\exists b \in B.\ (\forall a \in A.\ f(a) \neq b)$$

Definition (Bijective (one-to-one correspondence) 双射; 一一对应)

$$f : A \to B$$

1-1 & onto

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$$f : A \to B \qquad f : A \xleftrightarrow[onto]{1-1} B$$

1-1 & onto

$$f : \mathbb{Z} \to \mathbb{N}, \qquad f(x) = x^2 + 1$$

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$$f : \mathbb{N} \to \mathbb{Q}, \qquad f(x) = \frac{1}{x}$$

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$$f : \mathbb{N} \to \mathbb{N}, \qquad f(x) = 2^x$$

$$f : \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}, \qquad f(z, n) = \frac{z}{n+1}$$

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$$f : \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}, \qquad f(z, n) = \frac{z}{n+1}$$

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \qquad f(x, y) = (x + 1, y + 1)$$

Theorem (Cantor Theorem)

If $f : A \to 2^A$, *then* f *is not onto.*

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If $f : A \to 2^A$, then f is not onto.

Proof. Let A be a set and let $f : A \to 2^A$. To show that f is not onto, we must find a $B \in 2^A$ (i.e., $B \subseteq A$) for which there is no $a \in A$ with $f(a) = B$. In other words, B is a set that f "misses." To this end, let

$$B = \{x \in A : x \notin f(x)\}.$$

We claim there is no $a \in A$ with $f(a) = B$.

Suppose, for the sake of contradiction, there is an $a \in A$ such that $f(a) = B$. We ponder: Is $a \in B$?

- If $a \in B$, then, since $B = f(a)$, we have $a \in f(a)$. So, by definition of B, $a \notin f(a)$; that is, $a \notin B$.$\Rightarrow\Leftarrow$
- If $a \notin B = f(a)$, then, by definition of B, $a \in B$.$\Rightarrow\Leftarrow$

Both $a \in B$ and $a \notin B$ lead to contradictions, and hence our supposition [there is an $a \in A$ with $f(a) = B$] is false, and therefore f is not onto. ■

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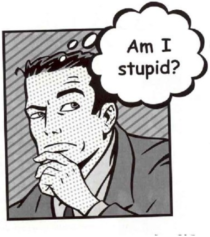

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Understanding this problem:

$$A = \{1, 2, 3\}$$

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$$2^A = \mathcal{P}(A) = \Big\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\Big\}$$

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Onto

$$\forall B \in 2^A. \Big(\exists a \in A.\ f(a) = B\Big)$$

Not Onto

$$\exists B \in 2^A. \Big(\forall a \in A.\ f(a) \neq B\Big)$$

$$f(1) = \{1, 2\}$$
$$f(2) = \{1, 3\}$$
$$f(3) = \emptyset$$

$$f(1) = \{1, 2\}$$
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$$B = \{2, 3\}$$

$$
\begin{aligned}
f(1) &= \{1, 2\} \\
f(2) &= \{1, 3\} \\
f(3) &= \emptyset
\end{aligned}
$$

$$B = \{2, 3\}$$

$$B = \{x \in \{1, 2, 3\} \mid x \notin f(x)\} = \{2, 3\}$$

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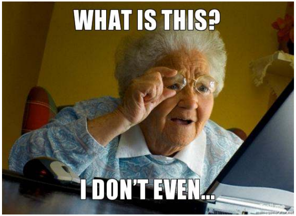

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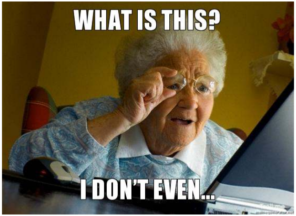

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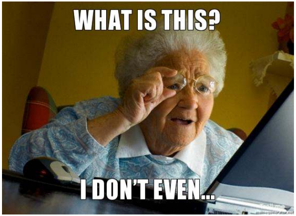

$$Q : a \in B?$$

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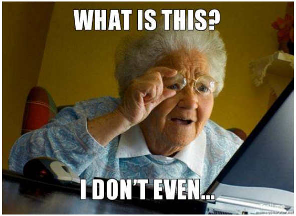

$$Q : a \in B?$$

$$a \in B \iff a \notin B$$

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a	$f(a)$					
	1	2	3	4	5	$\cdots$
1	1	1	0	0	1	$\cdots$
2	0	0	0	0	0	$\cdots$
3	1	0	0	1	0	$\cdots$
4	1	1	1	1	1	$\cdots$
5	0	1	0	1	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$

□

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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$

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3	1	0	0	1	0	$\cdots$
4	1	1	1	1	1	$\cdots$
5	0	1	0	1	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$

$$B = \{0, 1, 1, 0, 1\}$$

$\square$

Theorem (Cantor Theorem)

If $f : A \to 2^A$, then f is not onto.

对角线论证 (Cantor's diagonal argument) (以下仅适用于可数集合 A).

a	$f(a)$					
	1	2	3	4	5	$\cdots$
1	1	1	0	0	1	$\cdots$
2	0	0	0	0	0	$\cdots$
3	1	0	0	1	0	$\cdots$
4	1	1	1	1	1	$\cdots$
5	0	1	0	1	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$

$$B = \{0, 1, 1, 0, 1\}$$

□

Functions as Relations

$f|_X \qquad f(A) \qquad f^{-1}(B) \qquad f^{-1} \qquad f \circ g$

Definition (Restriction)

The *restriction* of a function f to X is the function:

$$f|_X = \{(x, y) \in f \mid x \in X\}$$

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$$f|_X = \{(x,y) \in f \mid x \in X\}$$

$$f : A \to B$$

$$f|_X : A \cap X \to B$$

$$f|_X(x) = f(x), \forall x \in A \cap X$$

Definition (Image)

The *image* of X under a function f is the set

$$f(X) = \{b \mid \exists a \in X : (a, b) \in f\}$$

Definition (Inverse Image)

The *inverse image* of Y under a function f is the set

$$f^{-1}(Y) = \{a \mid \exists b \in Y : (a, b) \in f\}$$

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The *inverse image* of Y under a function f is the set

$$f^{-1}(Y) = \{a \mid \exists b \in Y : (a, b) \in f\}$$

$X \subseteq \mathrm{dom} f$, $Y \subseteq \mathrm{ran} f$ are not necessary

f may not be invertible in $f^{-1}(Y)$

$$\boxed{y \in f(X) \iff \exists x \in \operatorname{dom} f \cap X : y = f(x)}$$

$$\boxed{y \in f(X) \iff \exists x \in X : y = f(x)}$$

$$\boxed{x \in f^{-1}(Y) \iff f(x) \in Y}$$

Theorem (Properties of f and f^{-1} (UD Theorem 17.7))

$$f : A \to B \qquad A_1, A_2 \subseteq A,\ B_1, B_2 \subseteq B$$

(i) *f preserves only $\subseteq$ and $\cup$:*

(1) $A_1 \subseteq A_2 \implies f(A_1) \subseteq f(A_2)$

(2) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$

(3) $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$

(4) $f(A_1 \setminus A_2) \supseteq f(A_1) \setminus f(A_2)$

(ii) *f^{-1} preserves $\subseteq, \cup, \cap,$ and $\setminus$:*

(5) $B_1 \subseteq B_2 \implies f^{-1}(B_1) \subseteq f^{-1}(B_2)$

(6) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$

(7) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$

(8) $f^{-1}(B_1 \setminus B_2) = f^{-1}(B_1) \setminus f^{-1}(B_2)$

Theorem (UD Problem 17.5)

$$f : A \to B$$

$$f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$$

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$b \in f(A_1 \cap A_2)$

Theorem (UD Problem 17.5)

$$f : A \to B$$

$$f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$$

$$\begin{aligned} & b \in f(A_1 \cap A_2) \\ \implies & \exists a \in A_1 \cap A_2 \cap A : b = f(a) \end{aligned}$$

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\implies & \exists a \in A : a \in A_1 \land a \in A_2 \land b = f(a) \\
\implies & \exists a \in A \cap A_1 : b = f(a) \land \exists a \in A \cap A_2 : b = f(a)
\end{aligned}$$

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\implies & b \in f(A_1) \cap f(A_2)
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Q : When does $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ hold?

Theorem (UD Problem 17.5)

$$f : A \to B$$

$$f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$$

$$\begin{aligned}
& b \in f(A_1 \cap A_2) \\
\implies & \exists a \in A_1 \cap A_2 \cap A : b = f(a) \\
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\end{aligned}$$

Q : When does $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ hold?

f is injective.

Theorem (Properties of f and f^{-1} (UD Theorem 17.7))

$$f : A \to B$$

(iii) f *and* f^{-1}:

(9) $A_0 \subseteq A \implies A_0 \subseteq f^{-1}(f(A_0))$

(10) $B_0 \supseteq f(f^{-1}(B_0))$

Theorem (Properties of f and f^{-1} (UD Theorem 17.7))

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Theorem (UD Problem 17.8)

$$A_0 \subseteq A \implies A_0 \subseteq f^{-1}(f(A_0))$$

Theorem

$$f : A \to B$$

$$B_0 \supseteq f(f^{-1}(B_0))$$

Theorem

$$f : A \to B$$

$$B_0 \supseteq f(f^{-1}(B_0))$$

$$b \in f(f^{-1}(B_0))$$

Theorem

$$f : A \to B$$

$$B_0 \supseteq f(f^{-1}(B_0))$$

$$\begin{aligned} & b \in f(f^{-1}(B_0)) \\ \implies & \exists a \in f^{-1}(B_0) : b = f(a) \end{aligned}$$

Theorem

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$$B_0 \supseteq f(f^{-1}(B_0))$$

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& b \in f(f^{-1}(B_0)) \\
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\implies & \exists a \in A : f(a) \in B_0 \land b = f(a) \\
\implies & b \in B_0
\end{aligned}$$

Theorem

$$f : A \to B$$

$$B_0 \supseteq f(f^{-1}(B_0))$$

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& b \in f(f^{-1}(B_0)) \\
\implies & \exists a \in f^{-1}(B_0) : b = f(a) \\
\implies & \exists a \in A : f(a) \in B_0 \wedge b = f(a) \\
\implies & b \in B_0
\end{aligned}$$

Q: When does $B_0 = f(f^{-1}(B_0))$ hold?

Theorem

$$f : A \to B$$

$$B_0 \supseteq f(f^{-1}(B_0))$$

$$\begin{aligned} & b \in f(f^{-1}(B_0)) \\ \Longrightarrow\ & \exists a \in f^{-1}(B_0) : b = f(a) \\ \Longrightarrow\ & \exists a \in A : f(a) \in B_0 \land b = f(a) \\ \Longrightarrow\ & b \in B_0 \end{aligned}$$

Q: When does $B_0 = f(f^{-1}(B_0))$ hold?

f is surjective and $B_0 \subseteq B$.

Theorem

$$f : A \to B$$

$$B_0 \supseteq f(f^{-1}(B_0))$$

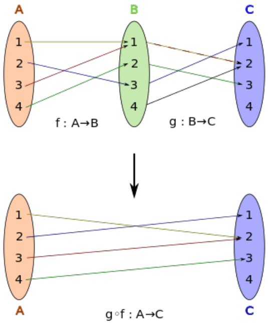
$$\begin{aligned} & b \in f(f^{-1}(B_0)) \\ \implies & \exists a \in f^{-1}(B_0) : b = f(a) \\ \implies & \exists a \in A : f(a) \in B_0 \land b = f(a) \\ \implies & b \in B_0 \end{aligned}$$

Q: When does $B_0 = f(f^{-1}(B_0))$ hold?

f is surjective and $B_0 \subseteq B$.

$B_0 \subseteq \operatorname{ran} f$

Function Composition



Definition (Composition)

$$f : A \to B \qquad g : C \to D$$

$$\text{ran} f \subseteq C$$

The *composite function* $g \circ f : A \to D$ is defined as

$$(g \circ f)(x) = g(f(x))$$

Definition (Composition)

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$$\operatorname{ran} f \subseteq C$$

The *composite function* $g \circ f : A \to D$ is defined as

$$(g \circ f)(x) = g(f(x))$$

Why not "$\exists b$" as below?

Definition (Composition)

The *composition* of relations R and S is the relation

$$R \circ S = \{(a, c) \mid \exists b : (a, b) \in S \land (b, c) \in R\}$$

Theorem (Associative Property for Composition)

$$f : A \to B \quad g : B \to C \quad h : C \to D$$

$$h \circ (g \circ f) = (h \circ g) \circ f$$

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Proof.

Theorem (Associative Property for Composition)

$$f : A \to B \quad g : B \to C \quad h : C \to D$$

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Proof.

(i)

$$\mathrm{dom} h \circ (g \circ f) = \mathrm{dom} (h \circ g) \circ f$$

(ii)

$$(h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x)$$

$\square$

Theorem (UD Theorem 16.7)

$$f : A \to B \qquad g : B \to C$$

(i) *If* f, g *are injective, then* $g \circ f$ *is injective.*

(ii) *If* f, g *are surjective, then* $g \circ f$ *is surjective.*

(iii) *If* f, g *are bijective, then* $g \circ f$ *is bijective.*

Theorem (UD Theorem 16.7)

$$f : A \to B \qquad g : B \to C$$

(i) *If f, g are injective, then $g \circ f$ is injective.*

(ii) *If f, g are surjective, then $g \circ f$ is surjective.*

(iii) *If f, g are bijective, then $g \circ f$ is bijective.*

Proof for (i).

$$\forall a_1, a_2 \in A : \Big((g \circ f)(a_1) = (g \circ f)(a_2) \implies a_1 = a_2 \Big)$$

□

Theorem (UD Theorem 16.7)

$$f : A \to B \qquad g : B \to C$$

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Proof for (i).

$$\forall a_1, a_2 \in A : \Big((g \circ f)(a_1) = (g \circ f)(a_2) \implies a_1 = a_2 \Big)$$

□

Proof for (ii).

$$\forall c \in C : \Big(\exists a \in A : (g \circ f)(a) = c \Big)$$

□

Theorem (UD Theorem 16.8)

$$f : A \to B \qquad g : B \to C$$

(i) *If* $g \circ f$ *is surjective, then* g *is surjective.*

(ii) *If* $g \circ f$ *is injective, then* f *is injective.*

Theorem (UD Theorem 16.8)

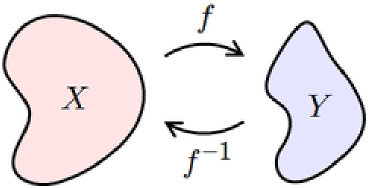
$$f : A \to B \qquad g : B \to C$$

(i) *If $g \circ f$ is surjective, then g is surjective.*

(ii) *If $g \circ f$ is injective, then f is injective.*

You can also prove it by contradiction.

Inverse Functions



Definition (Inverse)

Let $f : A \to B$ be a bijective function.

The *inverse* of f is the function $f^{-1}\colon B \to A$ defined by

$$f^{-1}(b) = a \iff f(a) = b.$$

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Definition (Invertible)

$f : X \to Y$ is *invertible* if there exists $g : Y \to X$ such that

$$f(x) = y \iff g(y) = x.$$

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Theorem

f is invertible $\iff$ f is bijective.

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f is invertible $\iff$ f is bijective.

f is invertible $\implies$ f is bijective

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Theorem

f is invertible $\iff$ f is bijective.

f is invertible $\implies$ f is bijective

g is a function $\implies$ f is injective

$\text{dom}\, g = Y \implies f$ is surjective

Definition (Invertible)

$f : X \to Y$ is *invertible* if there exists $g : Y \to X$ such that

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Theorem

f is invertible $\iff$ f is bijective.

f is invertible $\implies$ f is bijective

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$\text{dom} g = Y \implies f$ is surjective

f is bijective $\implies$ f is invertible

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f is invertible $\implies$ f is bijective

g is a function $\implies$ f is injective

$\mathrm{dom}\, g = Y \implies f$ is surjective

f is bijective $\implies$ f is invertible

To show that g defined above is indeed a function from Y to X.

Definition (Invertible)

$f : X \to Y$ is *invertible* if there exists $g : Y \to X$ such that

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Definition (Invertible)

$f : X \to Y$ is *invertible* if there exists $g : Y \to X$ such that

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Theorem

$g : Y \to X$ is unique.

Definition (Invertible)

$f : X \to Y$ is *invertible* if there exists $g : Y \to X$ such that

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$g : Y \to X$ is unique.

By Contradiction

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$f : X \to Y$ is *invertible* if there exists $g : Y \to X$ such that

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By Contradiction

$$f^{-1} \triangleq g$$

Definition (Invertible)

$f : X \to Y$ is *invertible* if there exists $g : Y \to X$ such that

$$f(x) = y \iff g(y) = x.$$

Theorem

$g : Y \to X$ is unique.

By Contradiction

$$f^{-1} \triangleq g$$

$$\boxed{f(x) = y \iff f^{-1}(y) = x}$$

Theorem (UD Theorem 16.4)

$$f : A \to B \textit{ is bijective}$$

(i) $f \circ f^{-1} = I_B$

(ii) $f^{-1} \circ f = I_A$

(iii) f^{-1} *is bijective.*

(iv) $g : B \to A \land f \circ g = I_B \implies g = f^{-1}$

(v) $g : B \to A \land g \circ f = I_A \implies g = f^{-1}$

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The ways to find/check f^{-1}.

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(v) $g : B \to A \land g \circ f = I_A \implies g = f^{-1}$

The ways to find/check f^{-1}.

$$g = f^{-1} \circ (f \circ g) = f^{-1} \circ I_B = f^{-1}$$

Theorem (Inverse of Composition (UD Theorem 16.6))

$$f : A \to B \quad g : B \to C \ \textit{are bijective}$$

(i) $g \circ f$ *is bijective*

(ii) $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Proof for (ii).

It suffices to check either one of the following identities:

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = I_A$$

$$(g \circ f) \circ (f^{-1} \circ g^{-1}) = I_C$$

□

Theorem (UD Theorem 16.8)

$$f : A \to B \quad g : B \to A$$

(iii) $f \circ g = I_B \land g \circ f = I_A \implies g = f^{-1}$

Theorem (UD Theorem 16.8)

$$f : A \to B \quad g : B \to A$$

(iii) $f \circ g = I_B \land g \circ f = I_A \implies g = f^{-1}$

You need to check both identities.

Theorem (UD Theorem 16.8)

$$f : A \to B \quad g : B \to A$$

(iii) $f \circ g = I_B \land g \circ f = I_A \implies g = f^{-1}$

You need to check both identities.

Theorem (UD Theorem 16.8)

$$f : A \to B \qquad g : B \to C$$

(i) *If $g \circ f$ is surjective, then g is surjective.*

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Theorem (UD Theorem 16.8)

$$f : A \to B \quad g : B \to A$$

(iii) $f \circ g = I_B \land g \circ f = I_A \implies g = f^{-1}$

You need to check both identities.

Theorem (UD Theorem 16.8)

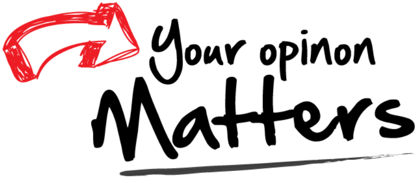
$$f : A \to B \qquad g : B \to C$$

(i) *If $g \circ f$ is surjective, then g is surjective.*

(ii) *If $g \circ f$ is injective, then f is injective.*

First show that f is bijective, and then use Theorem 16.4.


Thank
You!



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