# Natural Deduction for Predicate Logic

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# Outline

1. Predicate logic as a formal language
   - Terms
   - Formulae
   - Free and bound variables
   - Substitution
2. Proof theory of predicate logic
3. Quantifier equivalences

# Syntax

- In our examples, there are two sorts of things:
  - $B(x)$, $M(x,y)$, $B(x) \wedge \neg F(x)$ are formulae. They denote truth values;
  - $y$, $paul$, $m(x)$ are terms. They denote objects.
- Hence a predicate vocabulary has three sets.
- $\mathcal{P}$ is a set of predicate symbols ($B(x), M(x,y)$ etc).
- $\mathcal{F}$ is a set of function symbols ($m(x)$ etc).
- $\mathcal{C}$ is a set of constant symbols ($andy, paul$ etc).
- A function symbol $f \in \mathcal{F}$ with arity $n$ (or $n$-arity) takes $n$ arguments.
- Observe that a 0-arity (or nullary) function is in fact a constant.
- Hence $\mathcal{C} \subseteq \mathcal{F}$. We can ignore $\mathcal{C}$ for convenience.

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# Terms

## Definition

Terms are defined as follows.

- Any variable is a term;
- If $c \in \mathcal{F}$ is a nullary function symbol, $c$ is a term;
- If $t_1, t_2, \ldots, t_n$ are terms and $f \in \mathcal{F}$ has arity $n > 0$, then $f(t_1, t_2, \ldots, t_n)$ is a term;
- Nothing else is a term.

- In Backus Naur form, we have

$$t \ ::= \ x \mid c \mid f(t, \ldots, t)$$

where $x \in \text{var}$ is a variable, $c \in \mathcal{F}$ a nullary function symbol, and $f \in \mathcal{F}$ a function symbol with arity $> 0$.

# Terms

- Let $n, f, g \in \mathcal{F}$ be function symbols with arity 0, 1, and 2 respectively.
- $g(f(n), n)$, $f(f(n))$, $f(g(n, g(f(n), n)))$ are terms.
- $g(n)$, $f(n, n)$, $n(g)$ are not terms.
- Let $0, 1, \ldots$ be nullary function symbols, and $+, -, \times$ binary function symbols.
- $+(\times(3, x), 1)$, $+(\times(x, x), +(\times(2, \times(x, y))), \times(y, y))$ are terms.
- In infix notation, they are $(3 \times x) + 1$, $(x \times x) + ((2 \times (x \times y)) + (y \times y))$.

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# Formulae

## Definition

Formulae are defined as follows.

- If $P \in \mathcal{P}$ is a predicate symbol with arity $n \geq 1$, and $t_1, t_2, \ldots t_n$ are terms over $\mathcal{F}$, then $P(t_1, t_2, \ldots, t_n)$ is a formula;
- If $\phi$ is a formula, so is $(\neg\phi)$;
- If $\phi$ and $\psi$ are formulae, so are $(\phi \wedge \psi)$, $(\phi \vee \psi)$, and $(\phi \implies \psi)$.
- If $\phi$ is a formula and $x$ is a variable, then $(\forall x \phi)$ and $(\exists x \phi)$ are formulae;
- Nothing else is a formula.

- In Backus Naur form, we have

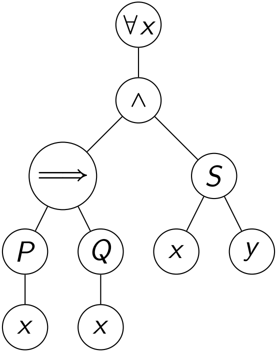
$$\phi ::= P(t_1, \ldots, t_n) \mid (\neg\phi) \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \implies \phi) \mid (\forall x \phi) \mid (\exists x \phi)$$

where $P \in \mathcal{P}$ is a predicate symbol of arity $n$, $t_1, \ldots, t_n$ terms over $\mathcal{F}$, and $x \in \text{var}$ a variable.

# Convention

- It is very tedious to write parentheses.
- We will assume the following binding priorities.
  - $\neg$, $\forall x$ and $\exists x$ (tightest)
  - $\vee$, $\wedge$
  - $\implies$ (right-associative and loosest)

# Parse Tree




- A predicate logic formula can be represented as a parse tree.
  - $\forall x$, $\exists y$ are nodes;
  - arguments of function symbols are also nodes.
- The above figure gives the parse tree of $\forall x((P(x) \implies Q(x)) \wedge S(x, y))$.

# Example

## Example

Write "every son of my father is my brother" in predicate logic.

## Proof.

Let *me* denote 'me', $S(x, y)$ ($x$ is a son of $y$), $F(x, y)$ ($x$ is the father of $y$), and $B(x, y)$ ($x$ is a brother of $y$) be predicate symbols of arity 2. Consider

$$\forall x \forall y (F(x, me) \land S(y, x) \implies B(y, me)).$$

Alternatively, let $f$ ($f(x)$ is the father of $x$) be a unary function symbol. Consider

$$\forall x (S(x, f(me)) \implies B(x, me)).$$

□

- Translating an English sentence into predicate logic can be tricky.
- Can you identify problem(s) in the example?

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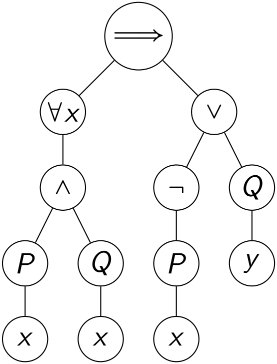
# Constants and Variables

- Let $c, d$ be constants (nullary functions).
- Consider $\forall x(P(x) \implies Q(x)) \wedge P(c) \implies Q(c)$.
  - If $P(x)$ implies $Q(x)$ for all $x$ and $P(c)$ is true, then $Q(c)$ is true.
- Intuitively, $\forall y(P(y) \implies Q(y)) \wedge P(c) \implies Q(c)$ should have the same meaning.
- $\forall y(P(y) \implies Q(y)) \wedge P(d) \implies Q(d)$ is different.
  - We do not know if $Q(c)$ is true.
- Things can get very complicated when there are several variables.
  - $\forall x((P(x) \implies Q(x)) \wedge S(x, y))$
  - $\forall z((P(z) \implies Q(z)) \wedge S(z, y))$
  - $\forall y((P(y) \implies Q(y)) \wedge S(y, x))$

# Free and Bound Variables

## Definition

Let $\phi$ be a predicate logic formula. An occurrence of $x$ in $\phi$ is free in $\phi$ if it is a leaf node without ancestor nodes $\forall x$ or $\exists x$ in the parse tree of $\phi$. Otherwise, the occurrence of $x$ is bound. The scope of $\forall x$ in $\forall x \phi$ is the formula $\phi$ minus any subformula in $\phi$ of the form $\forall x \psi$ or $\exists x \psi$.




$$(\forall x (P(x) \wedge Q(x))) \Longrightarrow (\neg P(x) \vee Q(y))$$

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# Subsitution

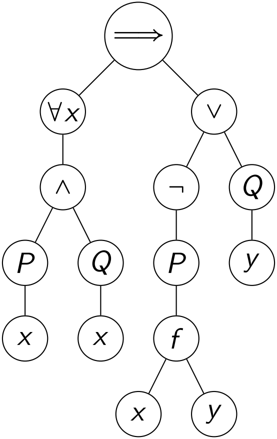
- Variables denote objects in predicate logic.
- Hence variables can be replaced by terms (but not formulae).
  - Replace $x$ in $x \neq x + 1$ by 2 to get $2 \neq 2 + 1$.
  - What if we replace $x$ by $2 = 2$?
- However, bound variables should not be replaced.
- The variables $x$ and $y$ in $\forall x \phi$ and $\exists y \psi$ denote <u>all</u> or <u>some</u> objects respectively.
  - What if we replace $x$ in $\exists x (x = 0)$ by 1?

### Definition

Given a variable $x$, a term $t$ and a formula $\phi$, define $\phi[t/x]$ to be the formula obtained by replacing each free occurrence of $x$ in $\phi$ with $t$.

# Example

- Let $\phi = (\forall x(P(x) \wedge Q(x))) \implies (\neg P(x) \vee Q(y))$. Consider $\phi[f(x,y)/x]$.




$$(\forall x(P(x) \wedge Q(x))) \implies (\neg P(x) \vee Q(y))[f(x,y)/x]$$

# Variable Capture in Substitution

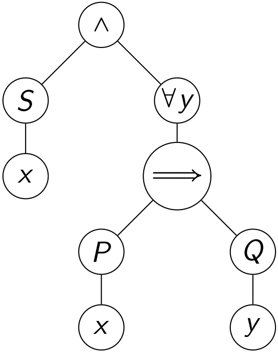
- Let $\phi = \exists y(y < x)$ and $\psi = \exists z(z < x)$.
  - Since $\phi$ and $\psi$ only differ in bound variables, they should have the same meaning.
- Consider $\phi[(y-1)/x] = \exists y(y < y-1)$.
- The variable $y$ in $y-1$ is caught by the bound variable in $\phi$.
- Consider $\psi[(y-1)/x] = \exists z(z < y-1)$.
- The variable $y$ in $y-1$ is not caught in the substitution $\psi[(y-1)/x]$.

**Definition**

Let $t$ be a term, $x$ a variable, and $\phi$ a formula. $t$ is <u>free for</u> $x$ in $\phi$ if no free $x$ leaf in $\phi$ occurs in the scope of $\forall y$ or $\exists y$ for any variable $y$ occurring in $t$.

- Examples: $y-1$ is free for $x$ in $\exists z(z < x)$; $y-1$ is not free for $x$ in $\exists y(y < x)$.

# Example




- Consider $\phi = S(x) \wedge \forall y (P(x) \implies Q(y))$ and $t = f(y, y)$.
- The two occurrences of $x$ in $\phi$ are free.
- The right occurrence of $x$ in $\phi$ is in the scope of $\forall y$ and $y$ occurs in $t$.
- $t$ is not free for $x$ in $\phi$.

# Substitution and Variable Capture

- When $t$ is not free for $x$ in $\phi$, the substitution $\phi[t/x]$ is not desirable.
- However, we can always rename bound variables for substitution.
- When we write $\phi[t/x]$, we mean all bound variables in $\phi$ are renamed so that $t$ is free for $x$ in $\phi$.
- Examples.
  - $\phi = \exists y(y < x)$ and $t = y - 1$. $t$ is not free for $x$ in $\phi$. Rename the bound variable $y$ to $z$ and obtain $\psi = \exists z(z < x)$. $t$ is free for $x$ in $\psi$.
  - $\phi = S(x) \land \forall y(P(x) \implies Q(y))$ and $t = f(y, y)$. $t$ is not free for $x$ in $\phi$. Rename the bound variable $y$ to $z$ and obtain $\psi = S(x) \land \forall z(P(x) \implies Q(z))$. $t$ is free for $x$ in $\psi$.

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# Natural Deduction for Predicate Logic

- Similar to propositional logic, predicate logic has its natural deduction proof system.
- Naturally, the natural deduction proof rules for contradiction ($\perp$), negation ($\neg$), and Boolean connectives ($\vee$, $\wedge$, $\implies$) are the same as those in propositional logic.
- Additionally, there are proof rules for equality ($=$) and quantification ($\forall$ and $\exists$).
- Again, these additional rules have two types: introduction and elimination rules.

# Equality

- Let $s$ and $t$ be terms.
- What do we mean by $s = t$?
- Shall we say $2 + 1 = 2 + 1$?
- What about $2^{61} - 1 = 2305843009213693951$?
- Apparently, if two terms are syntactically equal, they are equal.
  - This is called <u>intensional equality</u>.
- In practice, if two terms denote the same object, they are equal.
  - This is called <u>extensional equality</u>.

# Natural Deduction Proof Rules for Equality

- The introduction rule for equality is as follows.

$$\frac{}{t = t} = i$$

- The elimination rule for equality is as follows.

$$\frac{t_1 = t_2 \quad \phi[t_1/x]}{\phi[t_2/x]} = e$$

  ($t_1$ and $t_2$ are free for $x$ in $\phi$).
  - The requirement "$t_1$ and $t_2$ are free for $x$ in $\phi$" is called the side condition of the proof rule.
- By convention, we assume the side condition holds in all substitutions.

# Example

## Example

Show

$x+1=1+x, (x+1)>1 \implies (x+1)>0 \vdash (1+x)>1 \implies (1+x)>0.$

## Proof.

| | | |
|---|---|---|
| 1 | $x+1=1+x$ | premise |
| 2 | $(x+1)>1 \implies (x+1)>0$ | premise |
| 3 | $(1+x)>1 \implies (1+x)>0$ | $=$e 1, 2 |

In step 3, take $\phi = x>1 \implies x>0$, $t_1 = x+1$, and $t_2 = 1+x$. Then
$\phi[t_1/x] = (x+1)>1 \implies (x+1)>0$,
$\phi[t_2/x] = (1+x)>0 \implies (1+x)>0$. □

# Symmetry of Equality

## Example

Show $t_1 = t_2 \vdash t_2 = t_1$.

## Proof.

$$\begin{array}{lll} 1 & t_1 = t_2 & \text{premise} \\ 2 & t_1 = t_1 & =\text{i} \\ 3 & t_2 = t_1 & =\text{e}, 1, 2 \end{array}$$

Take $\phi = (x = t_1)$. $\phi[t_1/x] = (t_1 = t_1)$ and $\phi[t_2/x] = (t_2 = t_1)$. $\square$

# Transitivity of Equality

## Example

Show $t_1 = t_2, t_2 = t_3 \vdash t_1 = t_3$.

## Proof.

| | | |
|---|---|---|
| 1 | $t_2 = t_3$ | premise |
| 2 | $t_1 = t_2$ | premise |
| 3 | $t_1 = t_3$ | $=$e, 1, 2 |

Take $\phi = (t_1 = x)$. $\phi[t_2/x] = (t_1 = t_2)$ and $\phi[t_3/x] = (t_1 = t_3)$. □

- Thus, the rules $=$i and $=$e give us the reflexity, symmetry, and transitivity of equality.

# Natural Deduction Proof Rules for Universal Quantification

- The elimination rule for universal quantification is the following:

$$\frac{\forall x \phi}{\phi[t/x]} \; \forall x e$$

  when $t$ is free for $x$ in $\phi$.

- To see why $t$ must be free for $x$ in $\phi$, let $\phi$ be $\exists y(x < y)$. For natural numbers, $\forall x \exists y(x < y)$ is clearly true ("for any number, there is a larger number"). But if we take $t = y$, $\phi[t/x] = \exists y(y < y)$. This is wrong. Hence $t$ must be free for $x$ in $\phi$.
  - If we really need to replace $x$ by $y$ in this case, we should rewrite $\exists y(x < y)$ to $\exists z(x < z)$ and obtain $\exists z(x < z)[x/y] = \exists z(y < z)$.

# Natural Deduction Proof Rules for Universal Quantification

- The introduction rule for universal quantification opens a new box for a fresh variable $x_0$:

$$\dfrac{\boxed{\begin{array}{l} x_0 \\ \qquad \vdots \\ \qquad \phi[x_0/x] \end{array}}}{\forall x \phi} \ \forall x i$$

  (By "fresh," we mean $x_0$ does not occur outside of the box.)
- Informally, the rule $\forall x i$ says "if we can establish $\phi[x_0/x]$ for a fresh $x_0$, then we can derive $\forall x \phi$."
  - Intuitively, $x_0$ can be an arbitrary term since it is fresh and assumes nothing. If we can show $\phi[x_0/x]$, we have $\forall x \phi$.
  - Another way to see this is to replace $x_0$ by a term $t$ in the box. We would have a proof for $\phi[t/x]$. That is, we have shown $\forall x \phi$.

# Example

## Example

Show $\forall x(P(x) \implies Q(x)), \forall x P(x) \vdash \forall x Q(x)$.

## Proof.

| | | | |
|---|---|---|---|
| 1 | | $\forall x(P(x) \implies Q(x))$ | premise |
| 2 | | $\forall x P(x)$ | premise |
| 3 | $x_0$ | $P(x_0) \implies Q(x_0)$ | $\forall x$e 1 |
| 4 | | $P(x_0)$ | $\forall x$e 2 |
| 5 | | $Q(x_0)$ | $\implies$e 4, 3 |
| 6 | | $\forall x Q(x)$ | $\forall x$i 3–5 |

$\square$

# Example

## Example

Show $P(t), \forall x(P(x) \implies \neg Q(x)) \vdash \neg Q(t)$ for any term $t$.

## Proof.

| | | |
|---|---|---|
| 1 | $P(t)$ | premise |
| 2 | $\forall x(P(x) \implies \neg Q(x))$ | premise |
| 3 | $P(t) \implies \neg Q(t)$ | $\forall x$e 2 |
| 4 | $\neg Q(t)$ | $\implies$ e 1, 3 |

$\square$

- In step 3, we apply $\forall x$e by replacing $x$ with $t$. We could apply the same rule with a different term, say, $a$. Hence the rule $\forall x$e is in fact a <u>scheme</u> of rules; one for each term $t$ (free of $x$ in $\phi$).
- Also, we have different introduction and elimination rule. for different variables. That is, we have $\forall x$i, $\forall x$e, $\forall y$i, $\forall y$e, and so on. We will simply write $\forall$i and $\forall$ e when bound variables are clear.

# Universal Quantification and Conjunction

- It is helpful to compare proof rules for universal quantification and conjunction.
- Introduction rules:
  - To establish $\forall x \phi$, we need to show $\phi[t/x]$ for any term $t$. This is accomplished by proving $\phi[x_0/x]$ with the box for a fresh variable $x_0$;
  - To establish $\phi \wedge \psi$, we need to show $\phi$ and $\psi$.
- Elimination rules:
  - To eliminate $\forall x \phi$, we pick a term (free for $x$ in $\phi$) and deduce $\phi[t/x]$;
  - To eliminate $\phi \wedge \psi$, we deduce $\phi$ (or $\psi$).

# Natural Deduction Proof Rule for Existential Quantification

- The introduction rule for existential quantification is as follows.

$$\frac{\phi[t/x]}{\exists x \phi}\ \exists x i$$

  when $t$ is free for $x$ in $\phi$.
- To see why $t$ must be free for $x$ in $\phi$, consider $\exists x \forall y (x = y)$. This is clearly wrong for, say, natural numbers. Let $\phi = \forall y (x = y)$ and $t = y$. Since $\phi[t/x] = \forall y (y = y)$ is deducible (=i, $\forall y$i), we would have $\exists x \forall y (x = y)$.
- Recall the elimination rule for universal quantification:

$$\frac{\forall x \phi}{\phi[t/x]}\ \forall x e$$

  when $t$ is free for $x$ in $\phi$.
- $\forall x$e is the "dual" of $\exists x$i.
  - Recall the duality of $\wedge$e and $\vee$i.

# Natural Deduction Proof Rule for Existential Quantification

- The elimination rule for existential quantification is as follows.

$$\dfrac{\exists x \phi \qquad \boxed{\begin{array}{ll} x_0 & \\ & \phi[x_0/x] \\ & \vdots \\ & \chi \end{array}}}{\chi}\ \exists x e$$

- Informally, the rule $\exists x e$ says: to show $\chi$ from $\exists x \phi$, we show $\chi$ by assuming $\phi[x_0/x]$ for a fresh variable $x_0$.
  - Intuitively, $x_0$ stands for an unknown term $t$ such that $\phi[t/x]$ holds. If we can deduce $\chi$ by assuming $\phi[t/x]$, then $\chi$ is deducible from $\exists x \phi$.
- Note that $x_0$ must not occur in $\chi$.

# Existential Quantification and Disjunction

- It is helpful to compare the elimination rules for existential quantification and disjunction.
- Recall

$$\frac{\phi \vee \psi \qquad \boxed{\begin{matrix}\phi \\ \vdots \\ \chi\end{matrix}} \qquad \boxed{\begin{matrix}\psi \\ \vdots \\ \chi\end{matrix}}}{\chi} \vee e$$

- To eliminate $\phi \vee \psi$, we show that $\chi$ is deducible by assuming $\phi$ or assuming $\psi$.
- To eliminate $\exists x \phi$, we show that $\chi$ is deducible by assuming $\phi[x_0/x]$.

# Subformula Property I

- An elimination rule has <u>subformula property</u> if it must conclude with a subformula of the eliminated formula.
- For example, both $\wedge e_1$ and $\neg e$ have the subformula property.

$$\frac{\phi \wedge \psi}{\phi} \wedge e_1 \quad \frac{\neg\neg\phi}{\phi} \neg\neg e$$

- Since the conclusion of $\forall x e$ has the same logical structure as the eliminated formula, we also say $\forall x e$ has the subformula property.

$$\frac{\forall x \phi}{\phi[t/x]} \forall x e$$

  - Strictly speaking, $\phi[t/x]$ may not be a subformula of $\forall x \phi$.

# Subformula Property II

- The subformula property helps proof search.
  - We need not invent a formula for rules with the property.
  - Such rules are good for automated proof search.
- $\lor e$ and $\exists x e$ however do not have the subformula property.

$$\frac{\phi \lor \psi \qquad \boxed{\begin{array}{c}\phi\\ \vdots\\ \chi\end{array}} \qquad \boxed{\begin{array}{c}\psi\\ \vdots\\ \chi\end{array}}}{\chi}\ \lor e \qquad\qquad \frac{\exists x \phi \qquad \boxed{\begin{array}{ll} x_0 & \\ & \phi[x_0/x] \\ & \vdots \\ & \chi \end{array}}}{\chi}\ \exists x e$$

  - The conclusion $\chi$ must be chosen carefully.

# Examples I

**Example**

Show $\forall x \phi \vdash \exists x \phi$.

**Proof.**

| | | |
|---|---|---|
| 1 | $\forall x \phi$ | premise |
| 2 | $\phi[x/x]$ | $\forall x$e 1 |
| 3 | $\exists x \phi$ | $\exists x$i 2 |

(Is $x$ free for $x$ in $\phi[x/x]$?) $\square$

- Is it correct?

# Examples II

**Example**

Show $\forall x(P(x) \implies Q(x)), \exists x P(x) \vdash \exists x Q(x)$.

*Proof.*

| | | | |
|---|---|---|---|
| 1 | | $\forall x(P(x) \implies Q(x))$ | premise |
| 2 | | $\exists x P(x)$ | premise |
| 3 | $x_0$ | $P(x_0)$ | assumption |
| 4 | | $P(x_0) \implies Q(x_0)$ | $\forall x$e 1 |
| 5 | | $Q(x_0)$ | $\implies$e 3, 4 |
| 6 | | $\exists x Q(x)$ | $\exists x$i 5 |
| 7 | | $\exists x Q(x)$ | $\exists x$e 2, 3–6 |

(Can we close the box at line 5 instead of 6? Why not?) □

# Examples III

**Example**

Show $\exists x P(x), \forall x \forall y (P(x) \implies Q(y)) \vdash \forall y Q(y)$.

**Proof.**

| | | | |
|---|---|---|---|
| 1 | | $\exists x P(x)$ | premise |
| 2 | | $\forall x \forall y (P(x) \implies Q(y))$ | premise |
| 3 | $y_0$ | | |
| 4 | $x_0$ | $P(x_0)$ | assumption |
| 5 | | $\forall y (P(x_0) \implies Q(y))$ | $\forall x$e 2 |
| 6 | | $P(x_0) \implies Q(y_0)$ | $\forall y$e 5 |
| 7 | | $Q(y_0)$ | $\implies$e 4, 6 |
| 8 | | $Q(y_0)$ | $\exists x$e 1, 4–7 |
| 9 | | $\forall y Q(y)$ | $\forall y$i 3–8 |

$\square$

- Fresh variables in box must not appear outside!
- If not, we could show $\exists x P(x), \forall x(P(x) \implies Q(x)) \vdash \forall y Q(y)$!

| | | | |
|---|---|---|---|
| 1 | | $\exists x P(x)$ | premise |
| 2 | | $\forall x(P(x) \implies Q(x))$ | premise |
| 3 | $x_0$ | | |
| 4 | $x_0$ | $P(x_0)$ | assumption |
| 5 | | $P(x_0) \implies Q(x_0)$ | $\forall x$e 2 |
| 6 | | $Q(x_0)$ | $\implies$ e 4, 5 |
| 7 | | $Q(x_0)$ | $\exists x$e 1, 4–6 |
| 8 | | $\forall y Q(y)$ | $\forall y$i 3–7 |

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# Equivalent Predicate Logic Formulae I

- Let $\phi$ and $\psi$ be predicate logic formulae.
- $\phi \dashv\vdash \psi$ denotes tha $\phi \vdash \psi$ and $\psi \vdash \phi$.

# Equivalent Predicate Logic Formulae II

## Theorem

*Let $\phi$ and $\psi$ be predicate logic formulae. We have*

1. (a) $\neg\forall x\phi \dashv\vdash \exists x\neg\phi$; (b) $\neg\exists x\phi \dashv\vdash \forall x\neg\phi$.
2. *When $x$ is not free in $\psi$:*
   (a) $\forall x\phi \wedge \psi \dashv\vdash \forall x(\phi \wedge \psi)$; (b) $\forall x\phi \vee \psi \dashv\vdash \forall x(\phi \vee \psi)$;
   (c) $\exists x\phi \wedge \psi \dashv\vdash \exists x(\phi \wedge \psi)$; (d) $\exists x\phi \vee \psi \dashv\vdash \exists x(\phi \vee \psi)$;
   (e) $\forall x(\psi \implies \phi) \dashv\vdash \psi \implies \forall x\phi$;
   (f) $\exists x(\phi \implies \psi) \dashv\vdash \forall x\phi \implies \psi$;
   (g) $\forall x(\phi \implies \psi) \dashv\vdash \exists x\phi \implies \psi$;
   (h) $\exists x(\psi \implies \phi) \dashv\vdash \psi \implies \exists x\phi$ .
3. (a) $\forall x\phi \wedge \forall x\psi \dashv\vdash \forall x(\phi \wedge \psi)$; (b) $\exists x\phi \vee \exists x\psi \dashv\vdash \exists(\phi \vee \psi)$.
4. (a) $\forall x\forall y\phi \dashv\vdash \forall y\forall x\phi$ (b) $\exists x\exists y\phi \dashv\vdash \exists y\exists x\phi$.

# $\exists x \neg \phi \vdash \neg \forall x \phi$

| | | | |
|---|---|---|---|
| 1 | | $\exists x \neg \phi$ | premise |
| 2 | | $\forall x \phi$ | assumption |
| 3 | $x_0$ | $\neg \phi[x_0/x]$ | assumption |
| 4 | | $\phi[x_0/x]$ | $\forall$e 2 |
| 5 | | $\bot$ | $\neg$e 4, 3 |
| 6 | | $\bot$ | $\exists x$e 1, 3–5 |
| 7 | | $\neg \forall x \phi$ | $\neg$i 2–6 |

# $\neg\forall x\phi \vdash \exists x\neg\phi$

| | | | |
|---|---|---|---|
| 1 | | $\neg\forall x\phi$ | premise |
| 2 | | $\neg\exists x\neg\phi$ | assumption |
| 3 | $x_0$ | | |
| 4 | | $\neg\phi[x_0/x]$ | assumption |
| 5 | | $\exists x\neg\phi$ | $\exists x$i 4 |
| 6 | | $\bot$ | $\neg$e 5, 2 |
| 7 | | $\phi[x_0/x]$ | RAA 4–6 |
| 8 | | $\forall x\phi$ | $\forall x$i 3–7 |
| 9 | | $\bot$ | $\neg$e 8, 1 |
| 10 | | $\exists x\neg\phi$ | RAA 2–9 |

- The proof structure is similar to $\neg(p_1 \wedge p_2) \vdash \neg p_1 \vee \neg p_2$.

# $\neg(p_1 \wedge p_2) \vdash \neg p_1 \vee \neg p_2$

| | | |
|---|---|---|
| 1 | $\neg(p_1 \wedge p_2)$ | premise |
| 2 | $\neg(\neg p_1 \vee \neg p_2)$ | assumption |
| 3 | $\neg p_1$ | assumption |
| 4 | $\neg p_1 \vee \neg p_2$ | $\vee i_1$ 3 |
| 5 | $\perp$ | $\neg e$ 4, 2 |
| 6 | $p_1$ | RAA 3–5 |
| 3' | $\neg p_2$ | assumption |
| 4' | $\neg p_1 \vee \neg p_2$ | $\vee i_2$ 3' |
| 5' | $\perp$ | $\neg e$ 4', 2' |
| 6' | $p_2$ | RAA 3'–5' |
| 7 | $p_1 \wedge p_2$ | $\wedge i$ 6, 6' |
| 8 | $\perp$ | $\neg e$ 7, 1 |
| 9 | $\neg p_1 \vee \neg p_2$ | RAA 2–8. |

# $\forall x \phi \wedge \psi \vdash \forall x(\phi \wedge \psi)$ and $\forall x(\phi \wedge \psi) \vdash \forall x \phi \wedge \psi$ ($x$ not free in $\psi$)

| | | | |
|---|---|---|---|
| 1 | | $(\forall x \phi) \wedge \psi$ | premise |
| 2 | | $\forall x \phi$ | $\wedge e_1$ 1 |
| 3 | | $\psi$ | $\wedge e_2$ 2 |
| 4 | $x_0$ | | |
| 5 | | $\phi[x_0/x]$ | $\forall x e$ 2 |
| 6 | | $\phi[x_0/x] \wedge \psi$ | $\wedge i$ 5, 3 |
| 7 | | $(\phi \wedge \psi)[x_0/x]$ | $x$ not free in $\psi$ |
| 8 | | $\forall x(\phi \wedge \psi)$ | $\forall x i$ 4–7 |
| 1 | | $\forall x(\phi \wedge \psi)$ | premise |
| 2 | $x_0$ | | |
| 3 | | $(\phi \wedge \psi)[x_0/x]$ | $\forall x e$ 1 |
| 4 | | $\phi[x_0/x] \wedge \psi$ | $x$ not free in $\psi$ |
| 5 | | $\psi$ | $\wedge e_2$ 4 |
| 6 | | $\phi[x_0/x]$ | $\wedge e_1$ 4 |
| 7 | | $\forall x \phi$ | $\forall x i$ 2–6 |
| 8 | | $\forall x \phi \wedge \psi$ | $\wedge i$ 7, 5 |

# $(\exists x \phi) \vee (\exists x \psi) \vdash \exists x(\phi \vee \psi)$

| | | | |
|---|---|---|---|
| 1 | | $(\exists x \phi) \vee (\exists x \psi)$ | premise |
| 2 | | $\exists x \phi$ | assumption |
| 3 | $x_0$ | $\phi[x_0/x]$ | assumption |
| 4 | | $\phi[x_0/x] \vee \psi[x_0/x]$ | $\vee$i$_1$ 3 |
| 5 | | $(\phi \vee \psi)[x_0/x]$ | same as 4 |
| 6 | | $\exists x(\phi \vee \psi)$ | $\exists$xi 5 |
| 7 | | $\exists x(\phi \vee \psi)$ | $\exists$xe 2, 3–6 |
| 2' | | $\exists x \psi$ | assumption |
| 3' | $y_0$ | $\psi[y_0/x]$ | assumption |
| 4' | | $\phi[y_0/x] \vee \psi[y_0/x]$ | $\vee$i$_2$ 3' |
| 5' | | $(\phi \vee \psi)[y_0/x]$ | same as 4' |
| 6' | | $\exists x(\phi \vee \psi)$ | $\exists$xi 5' |
| 7' | | $\exists x(\phi \vee \psi)$ | $\exists$xe 2', 3'–6' |
| 8 | | $\exists x(\phi \vee \psi)$ | $\vee$e 1, 2–7, 2'–7' |

# $\exists x \exists y \phi \vdash \exists y \exists x \phi$

| | | | |
|---|---|---|---|
| 1 | | $\exists x \exists y \phi$ | premise |
| 2 | $x_0$ | $(\exists y \phi)[x_0/x]$ | assumption |
| 3 | | $\exists y (\phi[x_0/x])$ | $x$ and $y$ different |
| 4 | $y_0$ | $\phi[x_0/x][y_0/y]$ | assumption |
| 5 | | $\phi[y_0/y][x_0/x]$ | $x$, $y$, $x_0$, $y_0$ different |
| 6 | | $\exists x \phi[y_0/y]$ | $\exists x$i 5 |
| 7 | | $\exists y \exists x \phi$ | $\exists y$i 6 |
| 8 | | $\exists y \exists x \phi$ | $\exists y$e 3, 4–7 |
| 9 | | $\exists y \exists x \phi$ | $\exists x$e 1, 2–8 |