# The Axiom of Infinity and The Natural Numbers

Bernd Schröder

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2. But they do not guarantee the existence of infinite sets.
3. In fact, the superstructure over the empty set is a model that satisfies all the axioms so far and which does not contain any infinite sets. (Remember that the superstructure itself is not a set in the model.)

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The superstructure over $I$ is a model that satisfies all axioms introduced so far.

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1. *There is a special element in* $\mathbb{N}$*, which we denote by* $1$.
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4. **Principle of Induction**. *If* $S \subseteq \mathbb{N}$ *is such that* $1 \in S$ *and for each* $n \in S$ *we also have* $n' \in S$, *then* $S = \mathbb{N}$.

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5. *For all* $m, n \in \mathbb{N}$ *if* $m' = n'$*, then* $m = n$.

*The above properties are also called the* **Peano Axioms** *for the natural numbers.*

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**Proof.** *(Defining $\mathbb{N}$.)*
Let $I$ be the set from the Axiom of Infinity. Let $1 := \{\emptyset\} = \emptyset \cup \{\emptyset\} \in I$. For each $n \in I$, let $n' := n \cup \{n\}$. Call a subset $S \subseteq I$ a **successor set** iff $\emptyset \notin S$, $1 \in S$ and for all $n \in S$ we have that $n' \in S$.

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**Proof.** *(Defining $\mathbb{N}$.)*
Let $I$ be the set from the Axiom of Infinity. Let $1 := \{\emptyset\} = \emptyset \cup \{\emptyset\} \in I$. For each $n \in I$, let $n' := n \cup \{n\}$. Call a subset $S \subseteq I$ a **successor set** iff $\emptyset \notin S$, $1 \in S$ and for all $n \in S$ we have that $n' \in S$. Then $I \setminus \{\emptyset\}$ is a successor set. Moreover, all successor sets are subsets of $I$. Define $\mathbb{N} := \bigcap \mathscr{S}$ to be the intersection of the set $\mathscr{S}$ of all successor sets.

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Every successor set contains $1$. Therefore $1 \in \bigcap \mathscr{S} = \mathbb{N}$, as was to be proved.

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Let $n \in \mathbb{N}$. Because $n \in \mathbb{N} = \bigcap \mathscr{S}$, we conclude that $n \in S$ for all $S \in \mathscr{S}$.

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Let $n \in \mathbb{N}$. Because $n \in \mathbb{N} = \bigcap \mathscr{S}$, we conclude that $n \in S$ for all $S \in \mathscr{S}$. By definition of successor sets, $n' = n \cup \{n\} \in S$ for all $S \in \mathscr{S}$.

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Let $S \subseteq \mathbb{N}$ be so that $1 \in S$ and so that for every $n \in S$ we have that $n' \in S$. Because $S$ is a successor set, by definition of $\mathbb{N}$ we conclude $\mathbb{N} \subseteq S$.

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Now let $m,n \in \mathbb{N}$ with $m' = n'$ be arbitrary but fixed. Then $m \cup \{m\} = m' = n' = n \cup \{n\}$. Suppose for a contradiction that $m \neq n$. Then $\{n\} \neq \{m\}$, which implies $n \in m$ and $m \in n$. By the above, $n \subseteq m$ and $m \subseteq n$, that is, $m = n$, contradiction.

**Proof of part 5.** *(For all $m, n \in \mathbb{N}$ if $m' = n'$, then $m = n$.)* We first use part 4 to prove that every element of $n \in \mathbb{N}$ is a subset of $n$. Let $S := \left\{ n \in \mathbb{N} : [\forall m \in n : m \subseteq n] \right\}$. Trivially, $\{\emptyset\} \in S$, that is, $1 \in S$. For $n \in S$ we have $n' = n \cup \{n\}$. Hence, if $m \in n'$, then $m = n \subseteq n'$ or $m \in n$, which means $m \subseteq n \subseteq n'$.

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9. For example, to start, the engine must turn over. The handcrank from the really old movies has been replaced with an electric motor that cranks as we turn the key. Knowing that we need the engine to turn over is helpful when starting a car with electrical problems.