Natural Deduction for Propositional Logic

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Outline

1. Natural Deduction

2. Propositional logic as a formal language

3. Semantics of propositional logic
   - The meaning of logical connectives
   - Soundness of Propositional Logic
   - Completeness of Propositional Logic

Natural Deduction

- In our examples, we (informally) infer new sentences.
- In natural deduction, we have a collection of proof rules.
  - These proof rules allow us to infer new sentences logically followed from existing ones.
- Supose we have a set of sentences: $\phi_1, \phi_2, \ldots, \phi_n$ (called premises), and another sentence ψ (called a conclusion).
- The notation

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$$

  is called a sequent.
- A sequent is valid if a proof (built by the proof rules) can be found.
- We will try to build a proof for our examples. Namely,

$$p \wedge \neg q \implies r, \neg r, p \vdash q.$$

Proof Rules for Natural Deduction – Conjunction

- Suppose we want to prove a conclusion $\phi \wedge \psi$. What do we do?
  - Of course, we need to prove both ϕ and ψ so that we can conclude $\phi \wedge \psi$.
- Hence the proof rule for conjunction is

$$\frac{\phi \quad \psi}{\phi \wedge \psi} \wedge i$$

  - Note that premises are shown above the line and the conclusion is below. Also, $\wedge i$ is the name of the proof rule.
  - This proof rule is called "conjunction-introduction" since we introduce a conjunction ($\wedge$) in the conclusion.

Proof Rules for Natural Deduction – Conjunction

- For each connective, we have introduction proof rule(s) and also elimination proof rule(s).
- Suppose we want to prove a conclusion ϕ from the premise $\phi \wedge \psi$. What do we do?
  - We don't do any thing since we know ϕ already!
- Here are the elimination proof rules:

$$\frac{\phi \wedge \psi}{\phi} \wedge e_1 \qquad\qquad \frac{\phi \wedge \psi}{\psi} \wedge e_2$$

- The rule $\wedge e_1$ says: if you have a proof for $\phi \wedge \psi$, then you have a proof for ϕ by applying this proof rule.
- Why do we need two rules?
  - Because we want to manipulate syntax only.

Examples

Example

Prove $p \wedge q, r \vdash q \wedge r$.

Proof.

We are looking for a proof of the form:

$$\begin{array}{c} p \wedge q \quad r \\ \vdots \\ q \wedge r \end{array}$$

□

Example

Prove $p \wedge q, r \vdash q \wedge r$.

Proof.

We are looking for a proof of the form:

$$\dfrac{\dfrac{p \wedge q}{q}\ \wedge e_2 \qquad r}{q \wedge r}\ \wedge i$$

We will write proofs in lines:

1	$p \wedge q$	premise
2	r	premise
3	q	$\wedge e_2$ 1
4	$q \wedge r$	$\wedge i$ 3, 2

□

Proof Rules for Natural Deduction – Double Negation

- Suppose we want to prove ϕ from a proof for $\neg\neg\phi$. What do we do?
  - There is no difference between ϕ and $\neg\neg\phi$. The same proof suffices!
- Hence we have the following proof rules:

$$\frac{\phi}{\neg\neg\phi}\ \neg\neg i \qquad\qquad \frac{\neg\neg\phi}{\phi}\ \neg\neg e$$

Examples

Example

Prove $p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$.

Proof.

We are looking for a proof like:

$$\begin{array}{c} p \quad \neg\neg(q \wedge r) \\ \vdots \\ \neg\neg p \wedge r \end{array}$$

$\square$

Examples

Example

Prove $p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$.

Proof.

We are looking for a proof like:

$$\dfrac{\dfrac{p}{\neg\neg p}\ \neg\neg i \qquad \dfrac{\dfrac{\neg\neg(q \wedge r)}{q \wedge r}\ \neg\neg e}{r}\ \wedge e_2}{\neg\neg p \wedge r}\ \wedge i$$

□

Examples

Example

Prove $p, \neg\neg(q \wedge r) \vdash \neg\neg p \wedge r$.

Proof.

We are looking for a proof like:

1	p	premise
2	$\neg\neg(q \wedge r)$	premise
3	$\neg\neg p$	$\neg\neg i$ 1
4	$q \wedge r$	$\neg\neg e$ 2
5	r	$\wedge e_2$ 4
6	$\neg\neg p \wedge r$	$\wedge i$ 3, 5

$\square$

Proof Rules for Natural Deduction – Implication

- Suppose we want to prove ψ from proofs for ϕ and $\phi \implies \psi$. What do we do?
  - We just put the two proofs for ϕ and $\phi \implies \psi$ together.
- Here is the proof rule:

$$\frac{\phi \quad \phi \implies \psi}{\psi} \implies e$$

- This proof rule is also called *modus ponens*.
- Here is another proof rule related to implication:

$$\frac{\phi \implies \psi \quad \neg\psi}{\neg\phi} \; MT$$

- This proof rule is called *modus tollens*.

Example

Example

Prove $p \implies (q \implies r), p, \neg r \vdash \neg q$.

Proof.

1	$p \implies (q \implies r)$	premise
2	p	premise
3	$\neg r$	premise
4	$q \implies r$	$\implies e$ 2, 1
5	$\neg q$	MT 4, 3

$\square$

Proof Rules for Natural Deduction – Implication

- Suppose we want to prove $\phi \implies \psi$. What do we do?
  - We assume ϕ to prove ψ. If succeed, we conclude $\phi \implies \psi$ without any assumption.
  - Note that ϕ is added as an assumption and then removed so that $\phi \implies \psi$ does not depend on ϕ.
- We use "box" to simulate this strategy.
- Here is the proof rule:

$$\frac{\boxed{\begin{array}{c}\phi \\ \vdots \\ \psi\end{array}}}{\phi \implies \psi} \implies i$$

- At any point in a box, you can only use a sentence ϕ before that point. Moreover, no box enclosing the occurrence of ϕ has been closed.

Example

Example

Prove $\neg q \implies \neg p \vdash p \implies \neg\neg q$.

Proof.

$$\boxed{\dfrac{\neg q \implies \neg p \quad \dfrac{p}{\neg\neg p}\,\neg\neg i}{\neg\neg q}\,MT}$$
$$\overline{p \implies \neg\neg q} \implies i$$

1	$\neg q \implies \neg p$	premise
2	p	assumption
3	$\neg\neg p$	$\neg\neg i$ 2
4	$\neg\neg q$	MT 1, 3
5	$p \implies \neg\neg q$	$\implies i$ 2-4

□

Theorems

Example

Prove $\vdash p \implies p$.

Proof.

1	p	assumption
2	$p \implies p$	$\implies i$ 1 - 1

□

In the box, we have $\phi \equiv \psi \equiv p$.

Definition

A sentence ϕ such that $\vdash \phi$ is called a <u>theorem</u>.

Examples

Example

Prove $p \wedge q \implies r \vdash p \implies (q \implies r)$.

Proof.

1	$p \wedge q \implies r$	premise	
2	p	assumption	$\rceil$
3	q	assumption $\rceil$	$\mid$
4	$p \wedge q$	$\wedge i$ 2, 3 $\mid$	$\mid$
5	r	$\implies e$ 4, 1 $\rfloor$	$\mid$
6	$q \implies r$	$\implies i$ 3-5	$\rfloor$
7	$p \implies (q \implies r)$	$\implies i$ 2-6	

□

Proof Rules for Natural Deduction – Disjunction

- Suppose we want to prove $\phi \lor \psi$. What do we do?
  - We can either prove ϕ or ψ.
- Here are the proof rules:

$$\frac{\phi}{\phi \lor \psi} \lor i_1 \qquad\qquad \frac{\psi}{\phi \lor \psi} \lor i_2$$

  - Note the symmetry with $\land e_1$ and $\land e_2$.

$$\frac{\phi \land \psi}{\phi} \land e_1 \qquad\qquad \frac{\phi \land \psi}{\psi} \land e_2$$

  - Can we have a corresponding symmetric elimination rule for disjunction? Recall

$$\frac{\phi \quad \psi}{\phi \land \psi} \land i$$

Proof Rules for Natural Deduction – Disjunction

- Suppose we want to prove χ from $\phi \vee \psi$. What do we do?
  - We assume ϕ to prove χ and then assume ψ to prove χ.
  - If both succeed, χ is proved from $\phi \vee \psi$ without assuming ϕ and ψ.
- Here is the proof rule:

$$\frac{\phi \vee \psi \qquad \boxed{\begin{array}{c}\phi\\ \vdots\\ \chi\end{array}} \qquad \boxed{\begin{array}{c}\psi\\ \vdots\\ \chi\end{array}}}{\chi}\ \vee e$$

- In addition to nested boxes, we may have parallel boxes in our proofs.

Example

Recall that our syntax does not admit commutativity.

Example

Prove $p \vee q \vdash q \vee p$.

Proof.

$$\dfrac{p \vee q \quad \boxed{\dfrac{p}{q \vee p}\ \vee i_2} \quad \boxed{\dfrac{q}{q \vee p}\ \vee i_1}}{q \vee p}\ \vee e$$

1	$p \vee q$	premise	
2	p	assumption	$\rceil$
3	$q \vee p$	$\vee i_2$ 2	$\rfloor$
4	q	assumption	$\rceil$
5	$q \vee p$	$\vee i_1$ 4	$\rfloor$
6	$q \vee p$	$\vee e$ 1, 2-3, 4-5	

□

Example

Example

Prove $q \implies r \vdash p \lor q \implies p \lor r$.

Proof.

1	$q \implies r$	premise	
2	$p \lor q$	assumption	
3	p	assumption	
4	$p \lor r$	$\lor i_1$ 3	
5	q	assumption	
6	r	$\implies e$ 5, 1	
7	$p \lor r$	$\lor i_2$ 6	
8	$p \lor r$	$\lor e$ 2, 3-4, 5-7	
9	$p \lor q \implies p \lor r$	$\implies i$ 2-8	

$\square$

Example

Example

Prove $p \wedge (q \vee r) \vdash (p \wedge q) \vee (p \wedge r)$.

Proof.

1	$p \wedge (q \vee r)$	premise
2	p	$\wedge e_1$ 1
3	$q \vee r$	$\wedge e_2$ 1
4	q	assumption
5	$p \wedge q$	$\wedge i$ 2, 4
6	$(p \wedge q) \vee (p \wedge r)$	$\vee i_1$ 5
7	r	assumption
8	$p \wedge r$	$\wedge i$ 2, 7
9	$(p \wedge q) \vee (p \wedge r)$	$\vee i_2$ 8
10	$(p \wedge q) \vee (p \wedge r)$	$\vee e$ 3, 4-6, 7-9

□

Example

Example

Prove $(p \wedge q) \vee (p \wedge r) \vdash p \wedge (q \vee r)$.

Proof.

1	$(p \wedge q) \vee (p \wedge r)$	premise
2	$p \wedge q$	assumption
3	p	$\wedge e_1$ 2
4	q	$\wedge e_2$ 2
5	$q \vee r$	$\vee i_1$ 4
6	$p \wedge (q \vee r)$	$\wedge i$ 3, 5
7	$p \wedge r$	assumption
8	p	$\wedge e_1$ 7
9	r	$\wedge e_2$ 7
10	$q \vee r$	$\vee i_2$ 9
11	$p \wedge (q \vee r)$	$\wedge i$ 8, 10
12	$p \wedge (q \vee r)$	$\vee e$ 1, 2-6, 7-11

Contradiction

Definition

Contradictions are sentences of the form $\phi \wedge \neg\phi$ or $\neg\phi \wedge \phi$.

- Examples:
  - $p \wedge \neg p$, $\neg(p \vee q \implies r) \wedge (p \vee q \implies r)$.
- Logically, any sentence can be proved from a contradiction.
  - If $0 = 1$, then $100 \neq 100$.
- Particularly, if ϕ and ψ are contradictions, we have $\phi \dashv\vdash \psi$.
  - $\phi \dashv\vdash \psi$ means $\phi \vdash \psi$ and $\psi \vdash \phi$ (called provably equivalent).
- Since all contradictions are equivalent, we will use the symbol $\perp$ (called "bottom") for them.
- We are now ready to discuss proof rules for negation.

Proof Rules for Natural Deduction – Negation

- Since any sentence can be proved from a contradiction, we have

$$\frac{\bot}{\phi}\ \bot e$$

- When both ϕ and $\neg\phi$ are proved, we have a contradiction.

$$\frac{\phi \quad \neg\phi}{\bot}\ \neg e$$

  - The proof rule could be called $\bot i$. We use $\neg e$ because it eliminates a negation.

Example

Example

Prove $\neg p \vee q \vdash p \implies q$.

Proof.

1	$\neg p \vee q$	premise	
2	$\neg p$	assumption	
3	p	assumption	
4	$\perp$	$\neg e$ 3, 2	
5	q	$\perp e$ 4	
6	$p \implies q$	$\implies i$ 3-5	
7	q	assumption	
8	p	assumption	
9	q	copy 7	
10	$p \implies q$	$\implies i$ 8-9	
11	$p \implies q$	$\vee e$ 1, 2-6, 7-10	

□

Proof Rules for Natural Deduction – Negation

- Suppose we want to prove $\neg\phi$. What do we do?
  - We assume ϕ and try to prove a contradiction. If succeed, we prove $\neg\phi$.
- Here is the proof rule:

$$\frac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \bot\end{array}}}{\neg\phi}\ \neg i$$

Example

Example

Prove $p \implies q, p \implies \neg q \vdash \neg p$.

Proof.

1	$p \implies q$	premise	
2	$p \implies \neg q$	premise	
3	p	assumption	⌉
4	q	$\implies e$ 3, 1	|
5	$\neg q$	$\implies e$ 3, 2	|
6	$\perp$	$\neg e$ 4, 5	⌋
7	$\neg p$	$\neg i$ 3-6	

$\square$

Example

Example

Prove $p \land \neg q \implies r, \neg r, p \vdash q$.

Proof.

1	$p \land \neg q \implies r$	premise	
2	$\neg r$	premise	
3	p	premise	
4	$\neg q$	assumption	$\rceil$
5	$p \land \neg q$	$\land i$ 3, 4	$\vert$
6	r	$\implies e$ 5, 1	$\vert$
7	$\bot$	$\neg e$ 6, 2	$\rfloor$
8	$\neg\neg q$	$\neg i$ 4-7	
9	q	$\neg\neg e$ 8	

$\square$

Derived Rules

- Some rules can actually be derived from others.

Examples

Prove $p \implies q, \neg q \vdash \neg p$ (modus tollens).

Proof.

1	$p \implies q$	premise
2	$\neg q$	premise
3	p	assumption
4	q	$\implies e$ 3, 1
5	$\bot$	$\neg e$ 4, 2
6	$\neg p$	$\neg i$ 3-5

□

Derived Rules

Examples

Prove $p \vdash \neg\neg p$ ($\neg\neg i$)

Proof.

1	p	premise	
2	$\neg p$	assumption	$\rceil$
3	$\bot$	$\neg e$ 1, 2	$\rfloor$
4	$\neg\neg p$	$\neg i$ 2-3	

$\square$

- These rules can be replaced by their proofs and are not necessary.
  - They are just macros to help us write shorter proofs.

Reductio ad absurdum (RAA)

Example

Prove $\neg p \implies \bot \vdash p$ (RAA).

Proof.

1	$\neg p \implies \bot$	premise	
2	$\neg p$	assumption	⌉
3	$\bot$	$\implies e$ 2, 1	⌋
4	$\neg\neg p$	$\neg i$ 2-3	
5	p	$\neg\neg e$ 4	

$\square$

Tertium non datur, Law of the Excluded Middle (LEM)

Example

Prove $\vdash p \vee \neg p$.

Proof.

1	$\neg(p \vee \neg p)$	assumption
2	p	assumption
3	$p \vee \neg p$	$\vee i_1$ 2
4	$\bot$	$\neg e$ 3, 1
5	$\neg p$	$\neg i$ 2-4
6	$p \vee \neg p$	$\vee i_2$ 5
7	$\bot$	$\neg e$ 6, 1
8	$\neg\neg(p \vee \neg p)$	$\neg i$ 1-7
9	$p \vee \neg p$	$\neg\neg e$ 8

$\square$

Proof Rules for Natural Deduction (Summary)

Conjunction ($\wedge$)

$$\frac{\phi \quad \psi}{\phi \wedge \psi} \wedge i \qquad \frac{\phi \wedge \psi}{\phi} \wedge e_1 \qquad \frac{\phi \wedge \psi}{\psi} \wedge e_2$$

Disjunction ($\vee$)

$$\frac{\phi}{\phi \vee \psi} \vee i_1 \qquad \frac{\psi}{\phi \vee \psi} \vee i_2 \qquad \frac{\phi \vee \psi \quad \boxed{\begin{array}{c}\phi \\ \vdots \\ \chi\end{array}} \quad \boxed{\begin{array}{c}\psi \\ \vdots \\ \chi\end{array}}}{\chi} \vee e$$

Implication ($\Longrightarrow$)

$$\frac{\boxed{\begin{array}{c}\phi \\ \vdots \\ \psi\end{array}}}{\phi \Longrightarrow \psi} \Longrightarrow i \qquad \frac{\phi \quad \phi \Longrightarrow \psi}{\psi} \Longrightarrow e$$

Proof Rules for Natural Deduction (Summary)

Negation ($\neg$)

$$\frac{\boxed{\begin{array}{c}\phi \\ \vdots \\ \bot\end{array}}}{\neg\phi}\ \neg i \qquad\qquad \frac{\phi \quad \neg\phi}{\bot}\ \neg e$$

Contradiction ($\bot$)

(no introduction rule)

$$\frac{\bot}{\phi}\ \bot e$$

Double negation ($\neg\neg$)

(no introduction rule)

$$\frac{\neg\neg\phi}{\phi}\ \neg\neg e$$

Useful Derived Proof Rules

$$\frac{\phi \implies \psi \quad \neg\psi}{\neg\phi}\ MT \qquad \frac{\phi}{\neg\neg\phi}\ \neg\neg i$$

$$\frac{\boxed{\begin{array}{c}\neg\phi\\ \vdots\\ \bot\end{array}}}{\phi}\ RAA \qquad \frac{}{\phi \vee \neg\phi}\ LEM$$

Provable Equivalence

- Recall $p \dashv\vdash q$ means $p \vdash q$ and $q \vdash p$.
- Here are some provably equivalent sentences:

$$
\begin{array}{rcl}
\neg(p \land q) & \dashv\vdash & \neg q \lor \neg p \\
\neg(p \lor q) & \dashv\vdash & \neg q \land \neg p \\
p \implies q & \dashv\vdash & \neg q \implies \neg p \\
p \implies q & \dashv\vdash & \neg p \lor q \\
p \land q \implies p & \dashv\vdash & r \lor \neg r \\
p \land q \implies r & \dashv\vdash & p \implies (q \implies r)
\end{array}
$$

- Try to prove them.

Proof by Contradiction

- Although it is very useful, the proof rule RAA is a bit puzzling.

$$\frac{\boxed{\begin{array}{c}\neg\phi \\ \vdots \\ \bot\end{array}}}{\phi}\ RAA$$

- Instead of proving ϕ directly, the proof rule allows indirect proofs.
  - If $\neg\phi$ leads to a contradiction, then ϕ must hold.
- Note that indirect proofs are not "constructive."
  - We do not show why ϕ holds; we only know $\neg\phi$ is impossible.
- In early 20th century, some logicians and mathematicians chose not to prove indirectly. They are intuitionistic logicians or mathematicians.
- For the same reason, intuitionists also reject

$$\frac{}{\phi \vee \neg\phi}\ LEM \qquad\qquad \frac{\neg\neg\phi}{\phi}\ \neg\neg e$$

Proof by Contradiction

Theorem

There are $a, b \in \mathbb{R} \setminus \mathbb{Q}$ *such that* $a^b \in \mathbb{Q}$.

Proof.

Let $b = \sqrt{2}$. There are two cases:

- If $b^b \in \mathbb{Q}$, we are done since $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$.
- If $b^b \notin \mathbb{Q}$, choose $a = b^b = \sqrt{2}^{\sqrt{2}}$. Then $a^b = (b^b)^b = \sqrt{2}^{\sqrt{2} \cdot \sqrt{2}} = \sqrt{2}^2 = 2$. Since $\sqrt{2}^{\sqrt{2}}, \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$, we are done. □

- An intuitionist would criticize the proof since it does not tell us what a, b give $a^b \in \mathbb{Q}$.
  - We know (a, b) is either $(\sqrt{2}, \sqrt{2})$ or $(\sqrt{2}^{\sqrt{2}}, \sqrt{2})$.

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Well-Formedness

Definition

A well-formed formula is constructed by applying the following rules finitely many times:

- atom: Every propositional atom $p, q, r, \ldots$ is a well-formed formula;
- $\neg$: If ϕ is a well-formed formula, so is $(\neg\phi)$;
- $\wedge$: If ϕ and ψ are well-formed formulae, so is $(\phi \wedge \psi)$;
- $\vee$: If ϕ and ψ are well-formed formulae, so is $(\phi \vee \psi)$;
- $\implies$: If ϕ and ψ are well-formed formulae, so is $(\phi \implies \psi)$.

- More compactly, well-formed formulae are defined by the following grammar in Backus Naur form (BNF):

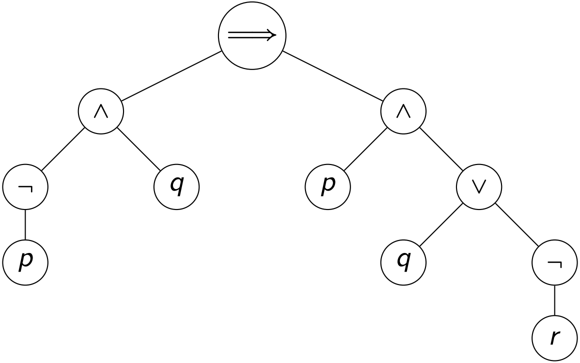
$$\phi ::= p \mid (\neg\phi) \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \implies \phi)$$

Inversion Principle

- How do we check if $(((\neg p) \wedge q) \implies (p \wedge (q \vee (\neg r))))$ is well-formed?
- Although a well-formed formula needs five grammar rules to construct, the construction process can always be inverted.
  - This is called inversion principle.
- To show $(((\neg p) \wedge q) \implies (p \wedge (q \vee (\neg r))))$ is well-formed, we need to show both $((\neg p) \wedge q)$ and $(p \wedge (q \vee (\neg r)))$ are well-formed.
- To show $((\neg p) \wedge q)$ is well-formed, we need to show both $(\neg p)$ and q are well-formed.
  - q is well-formed since it is an atom.
- To show $(\neg p)$ is well-formed, we need to show p is well-formed.
  - p is well-formed since it is an atom.
- Similarly, we can show $(p \wedge (q \vee (\neg r)))$ is well-formed.

Parse Tree

- The easiest way to decide whether a formula is well-formed is perhaps by drawing its parse tree.



Subformulae

- Given a well-formed formula, its subformulae are the well-formed formulae corresponding to its parse tree.
- For instance, the subformulae of the well-formed formulae $(((\neg p) \land q) \implies (p \land (q \lor (\neg r))))$ are

$$
\begin{array}{l}
p \\
q \\
r \\
(\neg p) \\
(\neg r) \\
((\neg p) \land q) \\
(q \lor (\neg r)) \\
(p \land (q \lor (\neg r))) \\
(((\neg p) \land q) \implies (p \land (q \lor (\neg r))))
\end{array}
$$

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From $\vdash$ to $\vDash$

- We have developed a calculus to determine whether $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid.
  - That is, from the premises $\phi_1, \phi_2, \ldots, \phi_n$, we can conclude ψ.
  - Our calculus is syntactic. It depends on the syntactic structures of $\phi_1, \phi_2, \ldots, \phi_n$, and ψ.
- We will introduce another relation between premises $\phi_1, \phi_2, \ldots, \phi_n$ and a conclusion ψ.

$$\phi_1, \phi_2, \ldots, \phi_n \vDash \psi.$$

  - The new relation is defined by 'truth values' of atomic formulae and the semantics of logical connectives.

Truth Values and Models

Definition

The set of truth values is $\{F, T\}$ where F represents 'false' and T represents 'true.'

Definition

A valuation or model of a formula ϕ is an assignment from each proposition atom in ϕ to a truth value.

Truth Values of Formulae

Definition

Given a valuation of a formula ϕ, the truth value of ϕ is defined inductively by the following truth tables:

ϕ	ψ	$\phi \land \psi$
F	F	F
F	T	F
T	F	F
T	T	T

ϕ	ψ	$\phi \lor \psi$
F	F	F
F	T	T
T	F	T
T	T	T

ϕ	ψ	$\phi \implies \psi$
F	F	T
F	T	T
T	F	F
T	T	T

ϕ	$\neg\phi$
F	T
T	F

$\top$
T

$\bot$
F

Example

- $\phi \wedge \psi$ is T when ϕ and ψ are T.
- $\phi \vee \psi$ is T when ϕ or ψ is T.
- $\bot$ is always F; $\top$ is always T.
- $\phi \implies \psi$ is T when ϕ "implies" ψ.

Example

Consider the valuation $\{q \mapsto \mathrm{T}, p \mapsto \mathrm{F}, r \mapsto \mathrm{F}\}$ of $(q \wedge p) \implies r$. What is the truth value of $(q \wedge p) \implies r$?

Proof.

Since the truth values of q and p are T and F respectively, the truth value of $q \wedge p$ is F. Moreover, the truth value of r is F. The truth value of $(q \wedge p) \implies r$ is T. □

Truth Tables for Formulae

- Given a formula ϕ with propositional atoms $p_1, p_2, \ldots, p_n$, we can construct a truth table for ϕ by listing 2^n valuations of ϕ.

Example

Find the truth table for $(p \implies \neg q) \implies (q \vee \neg p)$.

Proof.

p	q	$\neg p$	$\neg q$	$p \implies \neg q$	$q \vee \neg p$	$(p \implies \neg q) \implies (q \vee \neg p)$
F	F	T	T	T	T	T
F	T	T	F	T	T	T
T	F	F	T	T	F	F
T	T	F	F	F	T	T

$\square$

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   - Completeness of Propositional Logic

Validity of Sequent Revisited

- Informally $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid if we can derive ψ with assumptions $\phi_1, \phi_2, \ldots, \phi_n$.
  - We have formalized "deriving ψ with assumptions $\phi_1, \phi_2, \ldots, \phi_n$" by "constructing a proof in a formal calculus."
- We can give another interpretation by valuations and truth values.
- Consider a valuation ν over all propositional atoms in $\phi_1, \phi_2, \ldots, \phi_n, \psi$.
  - By "assumptions $\phi_1, \phi_2, \ldots, \phi_n$," we mean "$\phi_1, \phi_2, \ldots, \phi_n$ are T under the valuation ν.
  - By "deriving ψ,", we mean ψ is also T under the valuation ν.
- Hence, "we can derive ψ with assumptions $\phi_1, \phi_2, \ldots, \phi_n$" actually means "if $\phi_1, \phi_2, \ldots, \phi_n$ are T under a valuation, then ψ must be T under the same valuation.

Semantic Entailment

Definition

We say

$$\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$$

holds if for every valuations where $\phi_1, \phi_2, \ldots, \phi_n$ are T, ψ is also T. In this case, we also say $\phi_1, \phi_2, \ldots, \phi_n$ semantically entail ψ.

- Examples
  - $p \wedge q \vDash p$. For every valuation where $p \wedge q$ is T, p must be T. Hence $p \wedge q \vDash p$.
  - $p \vee q \nvDash q$. Consider the valuation $\{p \mapsto \mathrm{T}, q \mapsto \mathrm{F}\}$. We have $p \vee q$ is T but q is F. Hence $p \vee q \nvDash q$.
  - $\neg p, p \vee q \vDash q$. Consider any valuation where $\neg p$ and $p \vee q$ are T. Since $\neg p$ is T, p must be F under the valuation. Since p is F and $p \vee q$ is T, q must be T under the valuation. Hence $\neg p, p \vee q \vDash q$.
- The validity of $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is defined by syntactic calculus. $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ is defined by truth tables. Do these two relations coincide?

Soundness Theorem for Propositional Logic

Theorem (Soundness)

Let $\phi_1, \phi_2, \ldots, \phi_n$ and ψ be propositional logic formulae. If $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid, then $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds.

Proof.

Consider the assertion $M(k)$:

"For all sequents $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi (n \geq 0)$ that have a proof of length k, then $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds."

$k = 1$. The only possible proof is of the form

1	ϕ	premise

This is the proof of $\phi \vdash \phi$. For every valuation such that ϕ is T, ϕ must be T. That is, $\phi \vDash \phi$.

Soundness Theorem for Propositional Logic

Proof (cont'd).

Assume $M(i)$ for $i < k$. Consider a proof of the form

$$
\begin{array}{lll}
1 & \phi_1 & \text{premise} \\
2 & \phi_2 & \text{premise} \\
 & \vdots & \\
\text{n} & \phi_n & \text{premise} \\
 & \vdots & \\
\text{k} & \psi & \text{justification}
\end{array}
$$

We have the following possible cases for justification:

i $\wedge i$. Then ψ is $\psi_1 \wedge \psi_2$. In order to apply $\wedge i$, ψ_1 and ψ_2 must appear in the proof. That is, we have $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi_1$ and $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi_2$. By inductive hypothesis, $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi_1$ and $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi_2$. Hence $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi_1 \wedge \psi_2$ (Why?).

Proof (cont'd).

ii $\lor e$. Recall the proof rule for $\lor e$:

$$\dfrac{\eta_1 \lor \eta_2 \qquad \boxed{\begin{array}{c}\eta_1\\ \vdots \\ \psi\end{array}} \qquad \boxed{\begin{array}{c}\eta_2\\ \vdots \\ \psi\end{array}}}{\psi}\ \lor e$$

In order to apply $\lor e$, $\eta_1 \lor \eta_2$ must appear in the proof. We have $\phi_1, \phi_2, \ldots, \phi_n \vdash \eta_1 \lor \eta_2$. By turning "assumptions" η_1 and η_2 to "premises," we obtain proofs for $\phi_1, \phi_2, \ldots, \phi_n, \eta_1 \vdash \psi$ and $\phi_1, \phi_2, \ldots, \phi_n, \eta_2 \vdash \psi$. By inductive hypothesis, $\phi_1, \phi_2, \ldots, \phi_n \vDash \eta_1 \lor \eta_2$, $\phi_1, \phi_2, \ldots, \phi_n, \eta_1 \vDash \psi$, and $\phi_1, \phi_2, \ldots, \phi_n, \eta_2 \vDash \psi$. Consider any valuation such that $\phi_1, \phi_2, \ldots, \phi_n$ evaluates to T. $\eta_1 \lor \eta_2$ must be T. If η_1 is T under the valuation, ψ is also T (Why?). Similarly for η_2 is T. Thus $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$.

Soundness Theorem for Propositional Logic

Proof (cont'd).

iii Other cases are similar. Prove the case of $\implies e$ to see if you understand the proof. $\square$

- The soundness theorem shows that our calculus does not go wrong.
- If there is a proof of a sequent, then the conclusion must be true for all valuations where all premises are true.
- The theorem also allows us to show the non-existence of proofs.
- Given a sequent $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$, how do we prove there is no proof for the sequent?
  - Try to find a valuation where $\phi_1, \phi_2, \ldots, \phi_n$ are T but ψ is F.

Outline

1. Natural Deduction
2. Propositional logic as a formal language
3. Semantics of propositional logic
   - The meaning of logical connectives
   - Soundness of Propositional Logic
   - Completeness of Propositional Logic

Completeness Theorem for Propositional Logic

- "$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid" and "$\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds" are very different.
  - "$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid" requires proof search (syntax);
  - "$\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds" requires a truth table (semantics).
- If "$\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds" implies "$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid," then our natural deduction proof system is <u>complete</u>.
- The natural deduction proof system is both sound and complete. That is

  $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid iff $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds.

Completeness Theorem for Propositional Logic

- We will show the natural deduction proof system is complete.
- That is, if $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds, then there is a natural deduction proof for the sequent $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$.
- Assume $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$. We proceed in three steps:
  1. $\vDash \phi_1 \implies (\phi_2 \implies (\ldots(\phi_n \implies \psi)))$ holds;
  2. $\vdash \phi_1 \implies (\phi_2 \implies (\ldots(\phi_n \implies \psi)))$ is valid;
  3. $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid.

Completeness Theorem for Propositional Logic (Step 1)

Lemma

If $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ *holds, then* $\vDash \phi_1 \implies (\phi_2 \implies (\ldots(\phi_n \implies \psi)))$ *holds.*

Proof.

Suppose $\vDash \phi_1 \implies (\phi_2 \implies (\ldots(\phi_n \implies \psi)))$ does not hold. Then there is valuation where $\phi_1, \phi_2, \ldots, \phi_n$ is T but ψ is F. A contradiction to $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$. $\square$

Definition

Let ϕ be a propositional logic formula. We say ϕ is a <u>tautology</u> if $\vDash \phi$.

- A tautology is a propositional logic formula that evaluates to T for all of its valuations.

Completeness Theorem for Propositional Logic (Step 2)

- Our goal is to show the following theorem:

Theorem

If $\vDash \eta$ holds, then $\vdash \eta$ is valid.

- Similar to tautologies, we introduce the following definition:

Definition

Let ϕ be a propositional logic formula. We say ϕ is a <u>theorem</u> if $\vdash \phi$.

- Two types of theorems:
  - If $\vdash \phi$, ϕ is a theorem proved by the natural deduction proof system.
  - The soundness theorem for propositional logic is another type of theorem proved by mathematical reasoning (less formally).

Completeness Theorem for Propositional Logic (Step 2)

Proposition

Let ϕ be a formula with propositional atoms $p_1, p_2, \ldots, p_n$. Let l be a line in ϕ's truth table. For all $1 \leq i \leq n$, let $\hat{p}_i$ be p_i if p_i is T in l; otherwise $\hat{p}_i$ is $\neg p_i$. Then

1. *$\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \phi$ is valid if the entry for ϕ at l is T;*
2. *$\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \neg\phi$ is valid if the entry for ϕ at l is F.*

Proof.

We prove by induction on the height of the parse tree of ϕ.

- ϕ is a propositional atom p. Then $p \vdash p$ or $\neg p \vdash \neg p$ have one-line proof.
- ϕ is $\neg\phi_1$.
  - If ϕ is T at l. Then ϕ_1 is F. By IH, $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \neg\phi_1 (\equiv \phi)$.
  - If ϕ is F at l. Then ϕ_1 is T. By IH, $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \phi_1$. Using $\neg\neg i$, we have $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \neg\neg\phi_1 (\equiv \neg\phi)$.

Completeness Theorem for Propositional Logic (Step 2)

Proof (cont'd).

- ϕ is $\phi_1 \implies \phi_2$.
  - If ϕ is F at l, then ϕ_1 is T and ϕ_2 is F at l. By IH, $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \phi_1$ and $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \neg\phi_2$. Consider

1	$\phi_1 \implies \phi_2$	assumption
	$\vdots$	
i	ϕ_1	IH
i + 1	ϕ_2	$\implies e$ i, 1
	$\vdots$	
j	$\neg\phi_2$	IH
j + 1	$\perp$	$\neg$ e i+1, j
j + 2	$\neg(\phi_1 \implies \phi_2)$	$\neg$ i 1-(j+1)

Completeness Theorem for Propositional Logic (Step 2)

Proof (cont'd).

- ϕ is $\phi_1 \implies \phi_2$.
  - If ϕ is T at l, we have three subcases. Consider the case where ϕ_1 and ϕ_2 are F at l. Then

1	ϕ_1	assumption
	$\vdots$	
i	$\neg\phi_1$	IH
i + 1	$\bot$	$\neg$ e 1, i
i + 2	ϕ_2	$\bot$ e (i+1)
i + 3	$\phi_1 \implies \phi_2$	$\implies$ i 1-(i+2)

The other two subcases are simple exercises.

Completeness Theorem for Propositional Logic (Step 2)

Proof (cont'd).

- ϕ is $\phi_1 \land \phi_2$.
  - If ϕ is T at l, then ϕ_1 and ϕ_2 are T at l. By IH, we have $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \phi_1$ and $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \phi_2$. Using $\land$ i, we have $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n \vdash \phi_1 \land \phi_2$.
  - If ϕ is F at l, there are three subcases. Consider the subcase where ϕ_1 and ϕ_2 are F at l. Then

1	$\phi_1 \land \phi_2$	assumption
2	ϕ_1	$\land$ e$_1$ 1
	$\vdots$	
i	$\neg\phi_1$	IH
i + 1	$\bot$	$\neg$ e 2, i
i + 2	$\neg(\phi_1 \land \phi_2)$	$\neg$ i 1-(i+1)

The other two subcases are simple exercises.

Completeness Theorem for Propositional Logic (Step 2)

Proof.

- ϕ is $\phi_1 \vee \phi_2$.
  - If ϕ is F at l, then ϕ_1 and ϕ_2 are F at l. Then

1	$\phi_1 \vee \phi_2$	assumption
2	ϕ_1	assumption
	$\vdots$	
i	$\neg\phi_1$	IH
i + 1	$\bot$	$\neg$ e 2, i
i + 2	ϕ_2	assumption
	$\vdots$	
j	$\neg\phi_2$	IH
j + 1	$\bot$	$\neg$ e i+2, j
j + 2	$\bot$	$\vee$ e 2-(i+1), (i+2)-(j+1)
j + 3	$\neg(\phi_1 \vee \phi_2)$	$\neg$i 1-(j+2)

  - If ϕ is T at l, there are three subcases. All of them are simple exercises.

□

Completeness Theorem for Propositional Logic (Step 2)

Theorem

If ϕ is a tautology, then ϕ is a theorem.

Proof.

Let ϕ have propositional atoms $p_1, p_2, \ldots, p_n$. Since ϕ is a tautology, each line in ϕ's truth table is T. By the above proposition, we have the following 2^n proofs for ϕ:

$$\begin{array}{rcl} \neg p_1, \neg p_2, \ldots, \neg p_n & \vdash & \phi \\ p_1, \neg p_2, \ldots, \neg p_n & \vdash & \phi \\ \neg p_1, p_2, \ldots, \neg p_n & \vdash & \phi \\ & \vdots & \\ p_1, p_2, \ldots, p_n & \vdash & \phi \end{array}$$

We apply the rule LEM and the $\vee$e rule to obtain a proof for $\vdash \phi$. (See the following example.) $\square$

Completeness Theorem for Propositional Logic (Step 2)

Example

Observe that $\vDash p \implies (q \implies p)$. Prove $\vdash p \implies (q \implies p)$.

Proof.

1	$p \vee \neg p$	LEM
2	p	assumption
3	$q \vee \neg q$	LEM
4	q	assumption
	$\vdots$	
i	$p \implies (q \implies p)$	$p, q \vdash p \implies (q \implies p)$
i + 1	$\neg q$	assumption
	$\vdots$	
j	$p \implies (q \implies p)$	$p, \neg q \vdash p \implies (q \implies p)$
j + 1	$p \implies (q \implies p)$	$\vee$e 3, 4-i, (i+1)-j
j + 2	$\neg p$	assumption
j + 3	$q \vee \neg q$	LEM
j + 4	q	assumption
	$\vdots$	
k	$p \implies (q \implies p)$	$\neg p, q \vdash p \implies (q \implies p)$
k + 1	$\neg q$	assumption
	$\vdots$	
l	$p \implies (q \implies p)$	$\neg p, \neg q \vdash p \implies (q \implies p)$
l + 1	$p \implies (q \implies p)$	$\vee$e (j+3), (j+4)-k, (k+1)-l
l + 2	$p \implies (q \implies p)$	$\vee$e 1, 2-(j+1), (j+2)-(l+1)

□

Lemma

If $\phi_1 \implies (\phi_2 \implies (\cdots(\phi_n \implies \psi)))$ *is a theorem, then* $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ *is valid.*

Proof.

Consider

1	ϕ_1	premise
2	ϕ_2	premise
	$\vdots$	
n	ϕ_n	premise
	$\vdots$	
i	$\phi_1 \implies (\phi_2 \implies (\cdots(\phi_n \implies \psi)))$	theorem
i + 1	$\phi_2 \implies (\cdots(\phi_n \implies \psi))$	$\implies$e 1, i
i + 2	$\phi_3 \implies (\cdots(\phi_n \implies \psi))$	$\implies$e 2, (i+1)
	$\vdots$	
i + n - 1	$\phi_n \implies \psi$	$\implies$e (n-1), (i+n-2)
i + n	ψ	$\implies$e n, (i+n-1)

□