# Pigeonhole Principle/Solutions

These are the solutions to the problems related to the **Pigeonhole Principle**.

## Contents



## Introductory

### I1

The Martian must pull 5 socks out of the drawer to guarantee he has a pair. In this case the pigeons are the socks he pulls out and the holes are the colors. Thus, if he pulls out 5 socks, the Pigeonhole Principle states that some two of them have the same color. Also, note that it is possible to pull out 4 socks without obtaining a pair.

### I2

Consider the residues of the elements of $S$, modulo $n$. By the Pigeonhole Principle, there exist distinct $a, b \in S$ such that $a \equiv b \pmod{n}$, as desired.

## Intermediate

### M1

The maximum number of friends one person in the group can have is n-1, and the minimum is 0. If all of the members have at least one friend, then each individual can have somewhere between $1$ to $n-1$ friends; as there are $n$ individuals, by pigeonhole there must be at least two with the same number of friends. If one individual has no friends, then the remaining friends must have from $1$ to $n-2$ friends for the remaining friends not to also have no friends. By pigeonhole again, this leaves at least $1$ other person with $0$ friends.

### M2

For the difference to be a multiple of 5, the two integers must have the same remainder when divided by 5. Since there are 5 possible remainders (0-4), by the pigeonhole principle, at least two of the integers must share the same remainder. Thus, the answer is 1 (E).

### M3

Multiplying both sides by $q$, we have

$$|xq-p|<\frac{1}{n}.$$

Now, we wish to find a $q$ between 1 and $n$ such that $xq$ is within $\frac{1}{n}$ of some integer. Let $\{a\}$ denote the fractional part of $a$. Now, we sort the pigeons $\{x\},\{2x\},\ldots,\{nx\}$ into the holes $(0,1/n)),(1/n,2/n),\ldots,((n-1)/n,1)$. If any pigeon falls into the first or last hole, we are done. Therefore assume otherwise; then some two pigeons $\{ix\},\{jx\}\in(k/n,(k+1)/n)$ for $1\leq k<n-1$. Assume, without loss of generality, that $j-i>0$. Then we have that $\{(j-i)x\}$ must fall into the first or last hole, contradiction.

## Olympiad

### O1

By the Triangle Inequality Theorem, a side of a triangle must be less than the sum of the other two sides. This is equivalent to say that every side must be greater than the subtraction of the other two sides.

Consider the following statements:

1. If we have 2 line segments of lengths $a,b$ (so that $a\geq b$ and $a-b<1$); then if we have at least a line segment of length $c$ left to check (so that $b\geq c$), we will get that $a,b$ and $c$ are sides of a triangle. This is true because $c\geq 1$. This means if we check, for example, $5.5$ and $5$ then any number less than or equal to $5$ will satisfy the condition of them being sides of a triangle.

2. If we have 3 line segments of lengths $a,b,c$ to check inside an interval with a form $[k,2k[$, so that $k\geq 1$ we will find that they are sides of a triangle. This is true because $|a-b|<k$, $|a-c|<k$, $|b-c|<k$ and $a\geq b\geq c\geq k$.

3. If we have 2 line segments of lengths $a,b$ so that $a\geq b$, then $a,b,c$ are sides of a triangle if $b\geq c$ and $c>a-b$. This is a generalization of 1.

Now let's consider the intervals $[1,2[$ , $[2,4[$ and $[4,10]$

If we have 3 lines segment of lengths $a,b,c\in[1,2[$, then they are sides of a triangle because of 2.

If we have 3 lines segment of lengths $a,b,c\in[2,4[$, then they are sides of a triangle because of 2.

To analyze the case of having 3 lines segment of lengths $a,b,c\in[4,10]$ we can have two subcases. In both of them we will assume that 2 line segments' lengths are in $[1,2[$ and 2 line segments' lengths are in $[2,4[$ (otherwise it wouldn't be necessary to check because we would have 3 lenghts in the same interval). Also, that the difference between the two lengths in $[2,4[$ is greater than or equal to 1. (So that we can't apply 1.)

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A. If we can find a, b ∈ [4, 10] so that |a − b| < 3, we are done because there is a c ∈ [2, 4[ so that c ≥ 3 (Using 3. and the assumptions)
B. If we can't find a, b ∈ [4, 10] so that |a − b| < 3, that means the lengths are 4, 7, 10. But if we look at them, they are 3 sides of a triangle.
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Finally, as we wish to distribute 7 lengths in 3 intervals, the Pigeonhole Principle can be used to guarantee that at least 3 line segments' lengths belong to a same interval, and therefore to satisfy the condition.

### O2

For the difference to be a multiple of 7, the integers must have equal modulo 7 residues. To avoid having 15 with the same residue, 14 numbers with different modulo 7 residues can be picked ( $14*7=98$). Thus, two numbers are left over and have to share a modulo 7 residue with the other numbers under the pigeonhole principle.

### O3

Label the numbers in the set $x_1, \ldots, x_{100}$, consider the 100 subsets $\{x_1\}, \{x_1, x_2\}, \ldots, \{x_1, \ldots, x_{100}\}$ and for each of these subsets, compute its sum. If none of these sums are divisible by $100$, then there are $100$ sums and $99$ residue classes mod $100$ (excluding $0$). Therefore two of these sums are the same mod $100$, say $x_1 + \cdots + x_i \equiv x_1 + \cdots + x_j \pmod{100}$ (with $i < j$). Then $x_{i+1} + \cdots + x_j \equiv 0 \pmod{100}$, and the subset $\{x_{i+1}, \ldots, x_j\}$ suffices.

### O4

Inscribe a regular $9$-gon in a circle, and it will divide the circle into $9$ equal arcs. The length of the side of this $9$-gon is $\simeq 1.71$, and this is an upper bound on the distance of any two points on the arc. From the pigeonhole principle one of the arcs contains at least two of the points.

### O5

The pigeonhole principle is used in these solutions (PDF) (http://usamts.org/Solutions/Solution4_1_18.pdf).

### O6

In the worst case, consider that senator $S$ hates a set of 3 senators, while he himself is hated by a completely different set of 3 other senators. Thus, given one senator, there may be a maximum of 6 other senators whom he cannot work with. If we have a minimum of 7 committees, there should be at least one committee suitable for the senator $S$ after the assignment of the 6 conflicting senators.