Transfinite number

In mathematics, **transfinite numbers** are numbers that are "infinite" in the sense that they are larger than all finite numbers, yet not necessarily absolutely infinite. These include the **transfinite cardinals**, which are cardinal numbers used to quantify the size of infinite sets, and the **transfinite ordinals**, which are ordinal numbers used to provide an ordering of infinite sets.[1][2][3] The term *transfinite* was coined by Georg Cantor in 1915,[4] who wished to avoid some of the implications of the word *infinite* in connection with these objects, which were, nevertheless, not *finite*. Few contemporary writers share these qualms; it is now accepted usage to refer to transfinite cardinals and ordinals as "infinite". Nevertheless, the term "transfinite" also remains in use.

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Definition

Any finite number can be used in at least two ways: as an ordinal and as a cardinal. Cardinal numbers specify the size of sets (e.g., a bag of five marbles), whereas ordinal numbers specify the order of a member within an ordered set[5] (e.g., "the third man from the left" or "the twenty-seventh day of January"). When extended to transfinite numbers, these two concepts become distinct. A transfinite cardinal number is used to describe the size of an infinitely large set,[3] while a transfinite ordinal is used to describe the location within an infinitely large set that is ordered.[5] The most notable ordinal and cardinal numbers are, respectively:

- ω (Omega): the lowest transfinite ordinal number. It is also the order type of the natural numbers under their usual linear ordering.
- $\aleph_0$ (Aleph-null): the first transfinite cardinal number. It is also the cardinality of the infinite set of the natural numbers. If the axiom of choice holds, the next higher cardinal number is aleph-one, $\aleph_1$. If not, there may be other cardinals which are incomparable with aleph-one and larger than aleph-nought. Either way, there are no cardinals between aleph-nought and aleph-one.

The continuum hypothesis is the proposition that there are no intermediate cardinal numbers between $\aleph_0$ and the cardinality of the continuum (the cardinality of the set of real numbers):[3] or equivalently that $\aleph_1$ is the cardinality of the set of real numbers. In Zermelo–Fraenkel set theory, neither the continuum hypothesis nor its negation can be proven without violating consistency.

Some authors, including P. Suppes and J. Rubin, use the term *transfinite cardinal* to refer to the cardinality of a Dedekind-infinite set in contexts where this may not be equivalent to "infinite cardinal"; that is, in contexts where the axiom of countable choice is not assumed or is not known to hold. Given this definition, the following are all equivalent:

- $\mathfrak{m}$ is a transfinite cardinal. That is, there is a Dedekind infinite set A such that the cardinality of A is $\mathfrak{m}$.
- $\mathfrak{m} + 1 = \mathfrak{m}$.
- $\aleph_0 \leq \mathfrak{m}$.
- There is a cardinal $\mathfrak{n}$ such that $\aleph_0 + \mathfrak{n} = \mathfrak{m}$.

See also

- Absolutely infinite
- Actual infinity
- Aleph number
- Beth number
- Cardinal number
- Epsilon numbers (mathematics)
- Infinity plus one
- Infinitesimal
- Ordinal number

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