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Proof method: Multidimensional induction

For the last couple of posts we have been dealing with mathematical induction. This post shall be no different. However, until now we have been limited to one variable, so this post will deal with induction on multiple variables – or in other words multidimensional induction as the headline suggests. I will treat the two dimensional case in this post, but the expansion to n dimensions should be relatively easy. So for now we will assume that we have a statement P(m,n) which we need to prove.

If we have the statement P(m,n) then we need to prove three things:

1. **Basecase: We need to prove a base case P(a,b) where a is the smallest value for which m is valid and b is the smallest value where n is valid.**
2. **Induction over m: We assume that P(k,b) is valid for some positive integer k. Then we need to prove that P(k+1,b) is valid.**
3. **Induction over n: We assume that P(h,k) is valid for some positive integers h,k. Then we need to prove that P(h,k+1) is valid.**

If we can prove these three things then we have proved that P(m,n) is true for all $m \geq a$ and $n \geq b$. point 1 and 2 ensures that the statement is true for all P(m,a) just as in the in the single variable case. Once we have proven that then adding 3 ensures that it holds for all n as well. But let us take an example.

Example

Proposition: Let f(m,n) be a function with f(1,1) = 2 and

$$f(m+1,n) = f(m,n) + 2(m+n)$$

$$f(m,n+1) = f(m,n) + 2(m+n-1)$$

For all $m, n \in \mathbb{N}$ prove that $f(m,n) = (m+n)^2 - (m+n) - 2n + 2$

Proof:

Basecase: We need to check that $(m+n)^2 - (m+n) - 2n + 2 = 2$ for m,n = 1. So let us plug it into the formula. $f(1,1) = (1+1)^2 - (1+1) - 2 + 2 = 4 - 2 - 2 + 2 = 2$. So the base case holds.

Induction on m: Assuming that the statement f(k,1) is true we need to prove that f(k+1,1) is true as well. For this we will use a direct proof.

$$f(k+1,1) = f(k,1) + 2(k+1)$$

The first part is $f(k,1) = (k+1)^2 - (k+1) - 2 + 2 = k^2 + 2k + 1 - k - 1 = k^2 + k$ by assumption which gives us

$$f(k+1,1) = (k^2 + k) + 2(k+1) =$$

$$k^2 + 3k + 2 =$$

$$k^2 + 3k + 2 + k - j + 2 - 2 =$$

$$(k^2 + 4k + 4) - (k + 2)$$

So far I rearranged the formula a bit and added k-k and 2-2 which I am allowed to, since you can always add 0. The first parantheses can be rewritten as a quadratic term since $(k + 2)^2 = (k^2 + 4k + 4)$.

$$(k + 2)^2 - (k + 2) =$$

$$((k + 1) + 1)^2 - ((k+1) + 1) \quad + 2(1) - 2$$

Which proves that it holds for f(k+1,1).

Induction on n: Now we assume that the proposition holds for f(h,k) and we need to prove that it holds for f(h,k+1) as well. Once again we will use a direct proof to show this

$$f(h,k+1) = f(h,k) + 2(h+k - 1) =$$

$$(h+k)^2 - (h+k) - 2k + 2 + 2(h + k - 1) =$$

$$(h+k)^2 - (h+k) - 2k + 2 + 2(h + k) - 2$$

Since = $(h+k)^2 + 2(h+k) + 1 = (h+k+1)^2$ we can rewrite the equation to

$$(h+(k+1))^2 - (h+k) - 2k - 1 =$$

$$(h+(k+1))^2 - (h+k+1) - 2(k+1) + 2$$

which is equal to the proposition that $f(h,k+1) = (h+(k+1))^2 - (h+k+1) - 2(k+1) + 2$.

From these three cases it follows that $f(m,n) = (m+n)^2 - (m+n) - 2n + 2$ for all all natural number m,n.

Wrapping up

There is another method for conducting inductive proofs in the multidimensional case, it is covered nicely in this note. It builds on assuming that for any m+n = k the proposition holds and we then need to show that it holds for m+n = k+1.

The metaphor of dominoes also gave rise to the chosen post photo, which was kindly shared under the creative common license by Malkav. My crop of the photo is of course shared under the same conditions.

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Proof method: Strong Induction
As I promised in the Proof by induction post, I would follow up to elaborate on the proof by induction topic. Here is part of the follow up, known as the proof by strong induction. What I covered last time, is sometimes also known as weak induction. In weak

Proof methods: Proof by mathematical induction
It has been a while since I last posted something about proof methods, but lets dig that up again and take a look at a fourth method. The first three were direct proof, proof by contradiction and contrapositive proofs. Proof by induction is a

Proof Method: Contrapositive proof
On my journey to improve my mathematical rigour I have covered direct proofs and Proof by Contradiction. In this post I will cover the third method for proving theorems. Reading up on different methods for proving things I realized that a

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Induction for statements with more than one variable. - MathHub

March 17, 2016

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How to prove that a dynamic programming algorithm is a monotonic function - MathHub

April 5, 2016

[…] found some help regarding multidimensional induction in a blogpost, but I don't know how to approach the […]

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April 5, 2016

[…] found this blog post with a good example of 2D induction. It seems in this example, 2D induction is needed […]

rohan

August 6, 2017

i liked this blog an i need help as I want to prove that sum of n sqauare numbers is not equal sum of other n square numbers I have no idea how to do it .i will be very glad if you help me.

Ryan Jackson

October 8, 2019

In the 'Induction on m' section, I think that the character 'j' is inserted instead of 'k' the line:
k^2 + 3k + 2 + k – j +2 -2

I think it should be
k^2 + 3k + 2 + k – j +2 -2

since you are adding k-k and 2-2 to k^2 + 3k + 2

Ryan Jackson

October 8, 2019

Sorry, I meant that I think it should be
k^2 + 3k + 2 + k – k +2 -2
Lol

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