MATHEMATICS

Good examples of double induction

Asked 10 years, 2 months ago Active 4 years, 6 months ago Viewed 13k times

I'm looking for good examples where double induction is necessary. What I mean by double induction is induction on ω^2. These are intended as examples in an "Automatas and Formal Languages" course.

One standard example is the following: in order to cut an $n \times m$ chocolate bar into its constituents, we need $nm - 1$ cuts. However, there is a much better proof without using induction.

Another example: the upper bound \mathchoice$\Bigg(((\binom{a+b}{a}$\mathchoice$\Bigg)))$ on Ramsey numbers. The problem with this example is that it can be recast as induction on $a + b$, while I want something which is inherently inducting on ω^2.

Lukewarm example: Ackermann's function, which seems to be pulled out of the hat (unless we know about the primitive recursive hierarchy).

Better examples: the proof of other theorems in Ramsey theory (e.g. Van der Waerden or Hales-Jewett). While these can possibly be recast as induction on ω, it's less obvious, and so intuitively we really think of these proofs as double induction.

Another example: cut elimination in the sequent calculus. In this case induction on ω^2 might actually be necessary (although I'm not sure about that).

The problem with my positive examples is that they are all quite technical and complicated. So I'm looking for a simple, non-contrived example where induction on ω^2 cannot be easily replaced with regular induction (or with an altogether simpler argument). Any suggestions?

induction examples-counterexamples big-list

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edited Sep 16 '16 at 5:21
Martin Sleziak
52.1k 16 155 311

asked Jan 21 '11 at 18:42
Yuval Filmus
53.9k 5 84 155

4 Maybe I am misunderstanding you, but can't *every* proof of a statement by induction on two natural number parameters a, b be recast as a proof by induction on a+b? – Qiaochu Yuan Jan 21 '11 at 19:05

2 It can be that a decreases but b increases, for example. – Yuval Filmus Jan 21 '11 at 19:13

1 I'm not sure I understand what you mean by that. How would that affect a proof by *strong* induction on a+b? – Qiaochu Yuan Jan 21 '11 at 19:32

4 It could happen that a decreases by 1 and b increases by 2, so that in total $a+b$ increases. You can fix that whenever you have an upper bound on the increase of b which depends only on a. – Yuval Filmus Jan 21 '11 at 19:44

Many proofs in low-dimensional topology require induction over small ordinals, such as the proof of Haken finiteness. (Basically you need to show an m-tuplet of complexity functions decreases lexicographically.) However these are obviously outside the scope of an intro course. – Cheerful Parsnip Jan 21 '11 at 20:22

I see. So you are looking for a natural example where you do *not* have such a bound? This seems difficult. Most objects we induct on in elementary mathematics are basically controlled by a single "size" parameter. – Qiaochu Yuan Jan 21 '11 at 20:49

5 Qiaochu, the Ackerman function provably cannot be organized as a recursion on $a+b$, since this would place the function within the class of primitive recursive functions (which are closed under that kind of recursion), but it is known not to be primitive recursive. – JDH Jan 22 '11 at 1:00

@JDH: yes, I misunderstood the question. My apologies. – Qiaochu Yuan Jan 22 '11 at 1:31

9 Answers

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A nice example arises by relativizing Goodstein's Theorem from $\epsilon_0 = \omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}$ down to ω^2 .

ω^2 **Goodstein's Theorem** Given naturals $\rm\ a,\ b,\ c\ $ and an arbitrary increasing "base-bumping" function $\rm\ g(n)\ $ on $\mathbb N$ the following iteration eventually reaches 0 (i.e. $\rm\ a = c = 0$).

$$\rm a\ b + c\ \ \to\ \ \ \ \ a\ \ \ g(b)\ +\ \ c\ \ -\ 1\ \ \ if\ \ \ \ c > 0$$

$$\rm \to\ \ (a-1)\ g(b)\ +\ g(b) - 1\ \ \ if\ \ \ \ c = 0$$

Note: The above iteration is really on triples $\rm\ (a,b,c)\ $ but I chose the above notation in order to emphasize the relationship with radix notation and with Cantor Normal form for ordinals $< \epsilon_0$. For more on Goodstein's Theorem see the link in Andres's post or see my 1995\12\11 sci.math post with Cantor Normal form for ordinals $< \epsilon_0$. For more on Goodstein's Theorem see the link in Andres's post or see my 1995 (12 (11 sci.math post.

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edited Apr 13 '17 at 12:20
Community ♦
1

answered Jan 22 '11 at 0:25
Bill Dubuque
246k 32 245 780

14

Let me begin with an example of an induction of length ϵ_0: The proof that Goodstein sequences terminate. I mention this because when I decided to understand this result, I began to compute the length of these sequences and eventually came to a conjecture for a general formula (!) for the length of the sequence. It turned out that proving the conjecture was easy, because the proof organized itself as an induction of length ϵ_0. I was both very amused and very intrigued by this. The little paper that came out of this adventure is here.

Now, I also found once a natural example of an induction of length ω^2 when studying a "Ramsey type" problem: the size of min-homogeneous sets for regressive functions on pairs. What I liked about this example is that Ackermann's function injected itself into the picture and ended up providing me with the right rates of growth. The details are in a paper here.

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answered Jan 22 '11 at 0:07
Andrés E. Caicedo
73k 8 192 305

Sigh, I didn't notice your similar answer get posted while I was composing mine since I compose my answers on Mathoverflow (to get the instant MathJaxification), so I don't see new post notifications while I'm editing. But it's quite refreshing to find here someone who thinks so similarly. Never did that occur in many years on sci.math! – Bill Dubuque Jan 22 '11 at 0:49

@Bill: "it's quite refreshing to find here someone who thinks so similarly." Yes indeed. :-) – Andrés E. Caicedo Jan 22 '11 at 3:49

@Andres: That's actually a really nice "little paper"! – Desiato Mar 22 '12 at 21:25

@Desiato: Thanks! – Andrés E. Caicedo Mar 22 '12 at 22:43

8

Though you've already dismissed it as 'lukewarm', Ackermann's function (proving totality I think is what is wanted) is your most accessible option (I think it is a great option).

It's not contrived/unnatural because it is motivated by very different concepts. If you want to construct another example (prove $f(x, y) = g(x, y)$), you'd probably want to have x and y very much asymmetric (in the sense that they should be used in syntactically very different ways in the computations). And Ackermann's function does just that.

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answered Jan 21 '11 at 23:07
Mitch

Mitch
7,837 2 31 65

5

One could concoct a simple example, like proving that every sequence of the following moves on pairs of natural numbers eventually terminates:

$(i, j) \mapsto (i-1, N)$ for any natural number N.

$(i, j) \mapsto (i, j-1)$

(Edited to make it an inherently ω^2 problem.)

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edited Jan 21 '11 at 20:23

answered Jan 21 '11 at 19:48
Cheerful Parsnip
24.2k 2 46 110

This is both artificial and provable by induction on $3i + j$. The same line of thought leads to Paris-Harrington, but I'm not sure we can really go that far... – Yuval Filmus Jan 21 '11 at 20:11

How about (i,j) maps to (i-1,N) for any N. I realize it's artificial. – Cheerful Parsnip Jan 21 '11 at 20:19

1 There must be some combinatorial game theory example along these lines. Anyone? – Yuval Filmus Jan 21 '11 at 22:35

Unfortunately, all the CGT examples I know for this sort of thing really just amount (as this example does) to the definition of ω^2 in disguise... – Steven Stadnicki May 18 '11 at 23:44

4

Let [n] denote the ordered set (0, ..., n). Show that there are precisely $\backslash\mathrm{mathchoice}\left(\left(\left(\binom{n+m+1}{n+1}\backslash\mathrm{mathchoice}\right)\right)\right)$ order-preserving maps $[n] \to [m]$.

Also note that the collection of objects [n], together with order-preserving functions, forms a category... one can show by double induction that every map has an epi-monic factorization. I haven't actually tried doing these without double induction... it just seemed more natural that way, so I don't really know whether this is a good example, or I was just being silly.

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answered Jan 22 '11 at 1:25
Dylan Wilson
5,380 20 31

1 This is a pretty well-known combinatorial problem, with a neat combinatorial solution. Also, the double induction proof can probably be rephrased in terms of induction on $n + m$. – Yuval Filmus Jan 22 '11 at 7:53

What about prove $m + n = n + m$ for $m, n \in \mathbb{N}$? In particular, see this (site talking about double induction).

3

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answered Jan 21 '11 at 19:26

PrimeNumber
12.6k 7 51 72

4 I find such proofs artificial - we all know that $m+n=n+m$. It's not a course on the axiomatic method, so restricting yourself to use only the Peano axioms doesn't seem justified. – Yuval Filmus Jan 21 '11 at 19:47

3

I suggest the following proof for Bezout's identity. It is a double induction, in a sense, since it proves a proposition for all pairs a, b of natural numbers, by using "regular" induction on $\min(a, b)$. (I propose this as a general way to prove a statement on multiple natural variables: choose a function $f(n_1, \ldots, n_k) \mapsto N$ and then show that for all n, if $f(n_1, \ldots, n_k) = n$ then $n_1, \ldots, n_k$ satisfy the statement. This is then proved by induction on n).

Theorem: Let a and b be natural numbers and let $d = GCD(a, b)$ be their greatest common divisor. Then there exist integer x and y such that $d = ax + by$.

Proof: We use induction on $b = \min\{a, b\}$. Base case: If $b = 1$ then $d = 1$, and, for $x = 0$, $y = 1$ it is $1 = ax + by$. Inductive step: Assume that for each pair a, b, with $a \geq b$ and $b \in \{1, 2, \ldots, n-1\}$ there are integers x, y such that $GCD(a, b) = ax + by$. Consider a pair a, b with $a \geq b$ and $b = n$, and let q and r be the quotient and remainder in the division of a by b. If $r = 0$ then $GCD(a, b) = b = ax + by$ for $x = 0$ and $y = 1$. Otherwise, being $d = GCD(a, b) = GCD(b, r)$, from the inductive hypothesis there are integers x', y' such that $d = bx' + ry'$. By replacing $r = a - qb$ we get $d = bx' + (a - qb)y' = ay' + b(x' - qy') = ax + by$ for $x = y'$ e $y = x' - qy'$. \QED

Giuseppe Lancia (giulan@gmail.com)

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answered Nov 7 '12 at 16:36

Giuseppe Lancia
39 1

1 Induction on $\min(a, b)$ is the same as induction on $a + b$, it is just induction on ω. Besides, your proof is really an analysis of the extended Euclidean algorithm, and is perhaps better presented as such. – Yuval Filmus Nov 17 '12 at 22:26

1

There is a double induction in the recent paper David G Glynn, "A condition for arcs and MDS codes", Des. Codes Cryptogr. (2011) 58:215-218. See Lemma 2.4. It is about an identity involving subdeterminants of a general matrix and appears to need a double induction.

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answered Mar 29 '11 at 3:52

user8825

I was looking for something elementary - these students probably don't know what a subdeterminant is. – Yuval Filmus Apr 3 '11 at 3:03

1

For equations of parabolic or hyperbolic type in two independent variables the integration process is essentially a double induction. To find the values of the dependent variables at time $t + \Delta t$ one integrates with respect to x from one boundary to the other by utilizing the data at time t as if they were coefficients which contribute to defining the problem of this integration.

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answered Jul 14 '12 at 2:12

sameerg

11 3