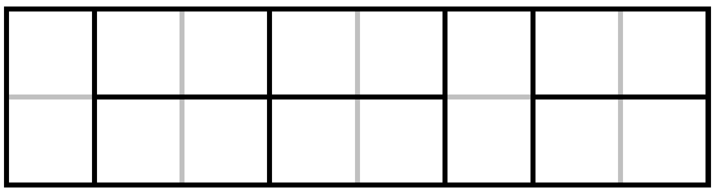
Fibonacci Tilings

Fibonacci numbers (https://www.cut-the-knot.org/arithmetic/Fibonacci.shtml) $\{F_n, n\ge 0\}$ satisfy the recurrence relation

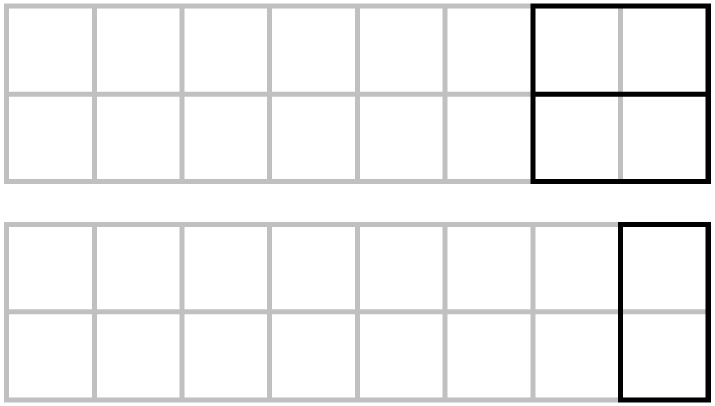
(1) $$F_{n+2}=F_{n+1}+F_n,$$

along with the initial conditions $F_1=1$ and $F_0=0.$

The Fibonacci name has been attached to the sequence $0,1,1,2,3,5,\ldots$ due to the inclusion in his 1202 book *Liber Abaci* of a rabbit reproduction puzzle: under certain constraints the rabbit population at discrete times is given exactly by that sequence. As naturally, the sequence is simulated by counting the tilings with dominoes of a $2\times n$ board:



A tiling of a $2\times n$ board may end with two horizontal dominoes or a single vertical domino:



In the former case, it's an extension of a tiling of a $2 \times (n-2)$ board; in the latter case, it's an extension of a tiling of a $2 \times (n-1)$. If T_n denotes the number of domino tilings of a $2 \times n$ board, then clearly

$$T_n = T_{n-2} + T_{n-1}$$

which is the same recurrence relation that is satisfied by the Fibonacci sequence. By a direct verification, $T_1 = 1$, $T_2 = 2$, $T_3 = 3$, $T_4 = 5$, etc., which shows that $\{T_n\}$ is nothing but a shifted Fibonacci sequence. If we define, $T_0 = 1$, as there is only one way to do nothing; and $T_{-1} = 0$, because there are no boards with negative side lengths, then $F_n = T_{n-1}$, for $n \geq 0$.

The domino tilings are extensively used in *Graham, Knuth, Patashnik* and by *Zeitz*. *Benjamin & Quinn* economize by considering only an upper $1 \times n$ portion of the board (and its tilings). This means tiling a $1 \times n$ board with 1×1 and 1×2 pieces.

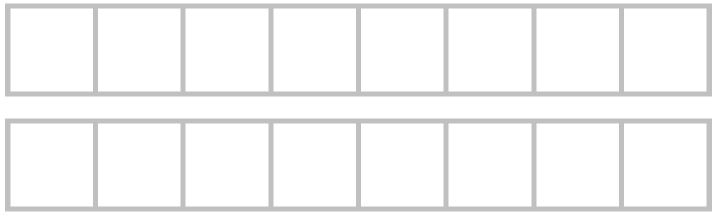
I'll use *Benjamin & Quinn*'s frugal tilings to prove *Cassini's Identity (https://www.cut-the-knot.org/Generalization/CevaPlus.shtml#Cassini)*

$$\boxed{F_{n+1} \cdot F_{n+1} - F_n \cdot F_{n+2} = (-1)^n.}$$

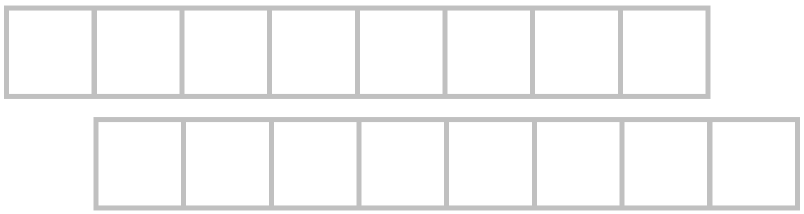
In terms of the tilings, I want to prove that $T_n \cdot T_n - T_{n-1} \cdot T_{n+1} = (-1)^n$.

The meaning of the term $T_n \cdot T_n$ is obvious: this is the number of ways to tile two $1 \times n$ boards where the tilings of the two boards are independent of each other. Similarly, $T_{n-1} T_{n+1}$ is the number of ways to tile two boards: one $1 \times (n-1)$ and one $1 \times (n+1)$. Now, the task is to retrieve the relation between the two numbers annunciated by Cassini's identity.

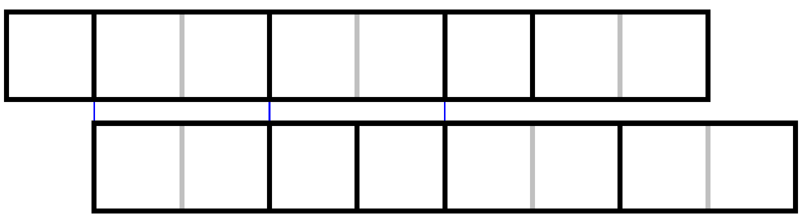
Our setup consists of two $1 \times n$ boards:



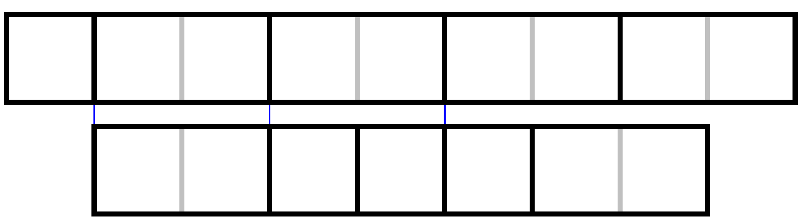
with the bottom board shifted one square to the right:



The tilings of the two boards may or may not have a fault line. A *fault line* is a line on the two boards at which the two tilings are breakable. For example, the tilings below have three fault lines:



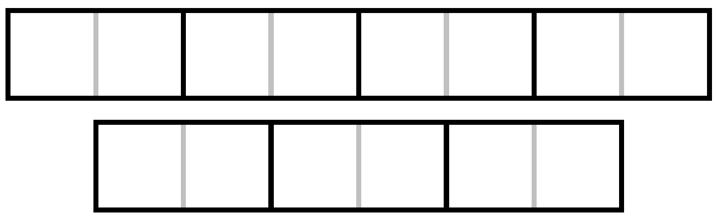
The trick is now to swap tails: the pieces of the two tilings (along with the boards) after the last fault line:



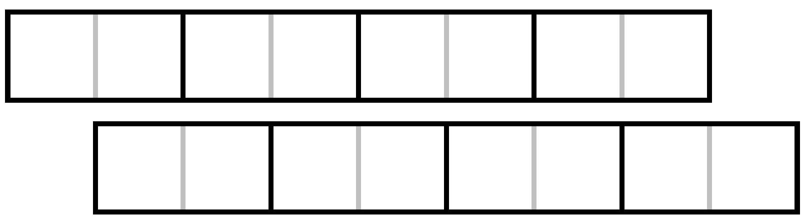
Since the bottom board has been shifted just one square, the swap produces one tiling of a $1 \times (n+1)$ - the top board in the diagram - and one tiling of a $1 \times (n-1)$ board - the bottom board in the diagram. Note that the old faults have been preserved and no new faults have been introduced.

Thus, in the presence of faults, there is a 1-1 correspondence between two n-tilings (T_n) and a pair of $(n-1)$- and $(n+1)$-tilings. The time is to account for the faultless combinations, if any.

But there are. Any 1×1 square induces a fault. This leaves exactly two faultless tilings. If n is odd, both $n-1$ and $n+1$ are even, there is a unique pair of $(n-1)-$ and $(n+1)$-tilings:



If n is even, there is a unique n-tiling that, when shifted, generates no fault lines:



References

1. A. T. Benjamin, J. J. Quinn, *Proofs That Really Count: The Art of Combinatorial Proof (https://www.amazon.com/exec/obidos/ISBN=0883853337/ctksoftwareincA/)*, MAA, 2003
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