Formal (natural) deduction in propositional logic

Lila Kari

University of Waterloo

'I know what you're thinking about,' said Tweedledum; 'but it isn't so, nohow.'

'Contrariwise,' continued Tweedledee, 'if it was so, it might be; and if it were so, it would be; but as it isn't, it ain't. That's logic.'

(Lewis Caroll, "Alice in Wonderland")

Formal deducibility

- We have seen how to prove arguments valid by using truth tables and other semantic methods ($\models$).
- We now want to replace this approach by a purely syntactic one, that is, we give formal rules for deduction which are purely **syntactical**.
- We want to define a relation called formal deducibility ($\vdash$) to allow us to **mechanically/syntactically** check the correctness of a proof that an argument is valid.
- The intuitive meaning of $\vdash$ is similar to the meaning of $\models$, in that it signifies argument validity. However, the method of proving validity is different.
- The significance of the word "formal" will be explained later. The important point is that formal deducibility is concerned with the syntactic structure of formulas and proofs of argument validity can be checked mechanically.

Gentzen (Jaskowski, independently) - 1934

"My starting point was this: The formalization of logical deduction, especially as it has been developed by Frege, Russell, and Hilbert, is rather far removed from the forms of deduction used in practice in mathematical proofs. Considerable formal advantages are achieved in return. In contrast, I intended first to set up a formal system which comes as close as possible to actual reasoning. The result was a calculus of natural deduction' (Gentzen, 1934)

NOTE: In this course, we will use the terms natural deduction and formal deduction interchangeably.

Notational conventions

- Suppose $\Sigma = \{A_1, A_2, A_3, \dots\}$ is a set of logic formulas. For convenience, Σ may be written as a sequence $A_1, A_2, A_3, \dots$.
- The set $\Sigma \cup \{A\}$, where A is a formula, may be written as Σ, A
- If Σ, Σ' are two sets of formulas, then $\Sigma \cup \Sigma'$ may be written as Σ, Σ'
- We use the symbol "$\vdash$" to denote the relation of formal deducibility and write

$$\Sigma \ \vdash \ A$$

  to mean that A is formally deducible (provable) from Σ.
- Formal deducibility is a relation between Σ (a set of formulas which are the premises of the argument) and A (a formula which is the conclusion of the argument).

11 rules of formal deduction

Note: Hereafter, we may call a valid argument also a theorem.

(1)	(Ref)	$A \vdash A$
(2)	$(+)$	**If** $\Sigma \vdash A$ is a theorem **then** $\Sigma, \Sigma' \vdash A$ is a theorem.
(3)	$(\neg -)$	**If** $\Sigma, \neg A \vdash B$ is a theorem, and $\Sigma, \neg A \vdash \neg B$ is a theorem **then** $\Sigma \vdash A$ is a theorem.
(4)	$(\rightarrow -)$	**If** $\Sigma \vdash A \rightarrow B$ is a theorem, and $\Sigma \vdash A$ is a theorem **then** $\Sigma \vdash B$ is a theorem.
(5)	$(\rightarrow +)$	**If** $\Sigma, A \vdash B$ is a theorem **then** $\Sigma \vdash A \rightarrow B$ is a theorem.

11 rules of formal deduction

(6)	$(\wedge -)$	**If** $\Sigma \vdash A \wedge B$ is a theorem **then** $\Sigma \vdash A$ is a theorem, and $\Sigma \vdash B$ is a theorem.
(7)	$(\wedge +)$	**If** $\Sigma \vdash A$ is a theorem, and $\Sigma \vdash B$ is a theorem **then** $\Sigma \vdash A \wedge B$ is a theorem.
(8)	$(\vee -)$	**If** $\Sigma, A \vdash C$ is a theorem, and $\Sigma, B \vdash C$ is a theorem **then** $\Sigma, A \vee B \vdash C$ is a theorem.
(9)	$(\vee +)$	**If** $\Sigma \vdash A$ is a theorem **then** $\Sigma \vdash A \vee B$ is a theorem, and $\Sigma \vdash B \vee A$ is a theorem.

11 rules of formal deduction

(10)	$(\leftrightarrow -)$	**If** $\Sigma \vdash A \leftrightarrow B$ is a theorem, and $\Sigma \vdash A$ is a theorem, **then** $\Sigma \vdash B$ is a theorem.
		If $\Sigma \vdash A \leftrightarrow B$ is a theorem, and $\Sigma \vdash B$ is a theorem **then** $\Sigma \vdash A$ is a theorem.
(11)	$(\leftrightarrow +)$	**If** $\Sigma, A \vdash B$ is a theorem, and $\Sigma, B \vdash A$ is a theorem **then** $\Sigma \vdash A \leftrightarrow B$ is a theorem.

Example

Show that

$(\in)$ If $A \in \Sigma$
then $\Sigma \vdash A$.

Proof: Suppose $A \in \Sigma$ and $\Sigma' = \Sigma - \{A\}$.

(1) $A \vdash A$ $\quad (by(Ref))$.
(2) $A, \Sigma' \vdash A$ $\quad (by(+), (1))$

(That is, $\Sigma \vdash A$.)

- Step (1) is generated directly by the rule (Ref).
- Step (2) is generated by the rule $(+)$, which is applied to Step (1).
- At each of the steps, the rule appplied, and the preceding steps concerned (if any), form a justification for this step, and are written on the right.
- These steps are said to form a formal proof of the last line, $\Sigma \vdash A$.

Example

Prove that $A \to B, B \to C \vdash A \to C$

The following sequence
(1)$A \to B, B \to C, A \vdash A \to B \quad (by(\in))$
(2)$A \to B, B \to C, A \vdash A \quad (by(\in))$
(3)$A \to B, B \to C, A \vdash B \quad (by(\to -), (1), (2))$
(4)$A \to B, B \to C, A \vdash B \to C \quad (by(\in))$
(5)$A \to B, B \to C, A \vdash C \quad (by(\to -), (4), (3))$
(6)$A \to B, B \to C \vdash A \to C \quad (by(\to +), (5))$
consists of six steps.

At each step, one of the 11 rules of formal deduction, or $(\in)$ (which has just been proved) is applied.

On the right are written justifications for the steps (in blue).

These six steps form a formal proof of $A \to B, B \to C \vdash A \to C$, which is generated at the last step.

Comments

- A demonstrated $\Sigma \vdash A$ (that is, for which we have a formal proof) is called a scheme of formal deducibility, or a theorem.
- Rules of formal deduction are purely syntactical.
  For instance, from

$$(*) \quad \Sigma, \neg A \vdash B$$

$$(**) \quad \Sigma, \neg A \vdash \neg B$$

  we can generate $(***) \quad \Sigma \vdash A$, by applying $(\neg -)$.
  - The premise Σ from $(***)$ is exactly the same Σ as in the premises from $(*)$ and $(**)$.
  - The B of $(*), (**)$ is the same formula.
  - The conclusion A of $(***)$ results by deleting the leftmost $\neg$ of $\neg A$ in the premises of (*) and (**).
- Therefore it can be checked mechanically (by a computer) whether the rules are used correctly.

Intuitive meaning of rules

- The elimination (introduction) of a connective means that one occurrence of this connective is eliminated (introduced) in the conclusion of the scheme of formal deducibility generated by the rule.
- Remark: The rules $(\vee-)$ and $(\neg-)$ are somewhat different in this respect.
- $(\neg\ -)$ expresses the method of indirect proof or proof by contradiction: if a contradiction (denoted by B and $\neg B$) follows from certain premises (denoted by Σ) with an additional supposition that a certain proposition does not hold (denoted by $\neg A$), then this proposition is deducible from the premises (denoted by $\Sigma \ \vdash \ A$.)

Intuitive meaning of rules

- $(\vee -)$ expresses the method of proof by cases. If C follows from A and B separately, then C follows from "A or B".
- $(\rightarrow +)$ expresses that to prove "If A then B" from certain premises (denoted by $\Sigma \vdash A \rightarrow B$), it is sufficient to prove B from the premises together with A (denoted by $\Sigma, A \vdash B$).
- These rules are similar to the rules of natural deduction from [Huth & Ryan], but with a different notation: $(\wedge +)$ and $(\wedge -)$ correspond to the rules for conjunction, $(\vee +)$ and $(\vee -)$ to the rules for disjunction, the rule $(\rightarrow -)$ to the "rule for eliminating implication", the rule $(\rightarrow +)$ to the rule "implies introduction", $(\neg -)$ to the rule for negation, etc.

The 11 rules - Summary

(1)	(Ref)	$A \vdash A$	(Reflexivity)
(2)	$(+)$	If $\Sigma \vdash A$, then $\Sigma, \Sigma' \vdash A$.	(Addition of premises)
(3)	$(\neg -)$	If $\Sigma, \neg A \vdash B$, $\Sigma, \neg A \vdash \neg B$, then $\Sigma \vdash A$.	($\neg$ elimination)
(4)	$(\rightarrow -)$	If $\Sigma \vdash A \rightarrow B$, $\Sigma \vdash A$, then $\Sigma \vdash B$.	($\rightarrow$ elimination)
(5)	$(\rightarrow +)$	If $\Sigma, A \vdash B$, then $\Sigma \vdash A \rightarrow B$.	($\rightarrow$ introduction)

The 11 rules - Summary

(6)	$(\wedge -)$	If $\Sigma \vdash A \wedge B$, then $\Sigma \vdash A$, $\Sigma \vdash B$.	$(\wedge$ elimination$)$
(7)	$(\wedge +)$	If $\Sigma \vdash A$, $\Sigma \vdash B$, then $\Sigma \vdash A \wedge B$.	$(\wedge$ introduction$)$
(8)	$(\vee -)$	If $\Sigma, A \vdash C$, $\Sigma, B \vdash C$, then $\Sigma, A \vee B \vdash C$.	$(\vee$ elimination$)$
(9)	$(\vee +)$	If $\Sigma \vdash A$, then $\Sigma \vdash A \vee B$, $\Sigma \vdash B \vee A$.	$(\vee$ introduction$)$

The 11 rules - Summary

(10)	$(\leftrightarrow -)$	If $\Sigma \vdash A \leftrightarrow B$, $\Sigma \vdash A$, then $\Sigma \vdash B$.	($\leftrightarrow$ elimination)
		If $\Sigma \vdash A \leftrightarrow B$, $\Sigma \vdash B$, then $\Sigma \vdash A$.	($\leftrightarrow$ elimination)
(11)	$(\leftrightarrow +)$	If $\Sigma, A \vdash B$, $\Sigma, B \vdash A$, then $\Sigma \vdash A \leftrightarrow B$.	($\leftrightarrow$ introduction)

Observation: Each of these rules is not a single rule, but a scheme of rules, because Σ is *any set* of formulas, and A, B, C are *any* formulas.

Formal deducibility: Finding a proof

- Formula A is formally deducible from the set of formulas Σ, written as $\Sigma \vdash A$, iff $\Sigma \vdash A$ is generated by a (finite number of applications of) the rules of formal deduction.
- The sequence of rules generating $\Sigma \vdash A$ is called a formal proof.
- A "scheme of formal deducibility" (theorem) may have various formal proofs. Perhaps one may not know how to construct a formal proof for it.
- It is significant however that any proposed formal proof for a theorem can be checked mechanically to decide whether it is indeed a formal proof of this theorem.

Formal deducibility: Checking a proof

• To check whether a sequence of steps is indeed a formal proof of a "scheme of formal deducibility" (theorem), we :

- check whether the rules of formal deduction are correctly applied at each step, and
- check whether the last term of the formal proof is identical with the desired scheme of formal deducibility (theorem).

• In this sense, rules of formal deduction and formal proofs serve to clarify the concepts of inference and proofs from informal reasoning.

Observations

- Tautological consequence ($\Sigma \models A$) and formal deducibility ($\Sigma \vdash A$) are different matters.
- The former ($\Sigma \models A$) belongs to semantics, while the latter ($\Sigma \vdash A$) belongs to syntax.
- Both tautological consequence and formal deducibility are studied in the metalanguage, by means of reasoning which is informal.
- "$\models$" and "$\vdash$" are not symbols in $\mathcal{L}^p$. They should not be confused with "$\rightarrow$", which is a symbol in $\mathcal{L}^p$, a connective used for forming formulas.
- The connection between $\models$ and $\rightarrow$ is that "$A \models B$ iff $A \rightarrow B$ is a tautology."
- The connection between $\vdash$ and $\rightarrow$ is that "$A \vdash B$ iff $\emptyset \vdash A \rightarrow B$" (exercise)

Formal deducibility: Complete definition

Definition (Formal deducibility). A formula A is formally deducible from Σ, written as $\Sigma \vdash A$, iff $\Sigma \vdash A$ is generated by (a finite number of applications of) the rules of formal deduction.

By the above definition, $\Sigma \vdash A$ holds iff there is a finite sequence

$$(1)\ \Sigma_1 \vdash A_1$$

$$\dots$$

$$(n)\ \Sigma_n \vdash A_n$$

such that

- each term $\Sigma_k \vdash A_k$ $(k = 1, \dots, n)$ is generated by one rule of formal deduction, and
- $\Sigma_n \vdash A_n$ is $\Sigma \vdash A$ (that is, $\Sigma_n = \Sigma$ and $A_n = A$)

Comments on the definition

To say that $\Sigma_k \vdash A_k$ is generated by a rule of formal deduction, say $(\neg -)$, means that in the subsequence

$$\Sigma_1 \vdash A_1$$

$$\ldots$$

$$\Sigma_{k-1} \vdash A_{k-1}$$

which precedes $\Sigma_k \vdash A_k$, there are two terms

$$\Sigma_k, \neg A_k \vdash B,$$

$$\Sigma_k, \neg A_k \vdash \neg B$$

where B is an arbitrary formula.

Comments on the definition

- The sequence

$$(1)\ \Sigma_1 \vdash A_1$$

$$\dots$$

$$(n)\ \Sigma_n \vdash A_n$$

  is called a formal proof. It is a formal proof of its last term $\Sigma_n \vdash A_n$.
- Now the significance of the word "formal" has been explained in full.
- The definition of formal deducibility is an inductive one. We may compare this definition with the definition of Form$(\mathcal{L}^p)$ to see that schemes of deducibility correspond to formulas, rules of formal deduction to formation rules.

Proving statements about formal deduction

Statements concerning formal deducibility can be proved by induction on its complexity (structural induction).

- The basis of induction is to prove that $A \vdash A$, which is generated directly by rule (Ref), has a certain property.
- The induction step is to prove that the other ten rules preserve this property.

For instance, in the case of $(\vee -)$, we suppose

$$\Sigma, A \vdash C$$

$$\Sigma, B \vdash C$$

have the required property and show that

$$\Sigma, A \vee B \vdash C$$

also has this property.

Finiteness of premise set

Theorem. If $\Sigma \vdash A$, then there is some finite $\Sigma^0 \subseteq \Sigma$ such that $\Sigma^0 \vdash A$.

Proof. By induction on the complexity of $\Sigma \vdash A$.

Basis: The premise A of $A \vdash A$ generated by (Ref) is itself finite.

Induction Step: We distinguish ten cases. For each case, assume that the premises have the property and show that the corresponding conclusion has the property.

Example: Case of $(\to -)$: "If $\Sigma \vdash A \to B$, $\Sigma \vdash A$, then $\Sigma \vdash B$."
By induction hypothesis, there exist finite subsets Σ_1 and Σ_2 of Σ such that $\Sigma_1 \vdash A \to B$ and $\Sigma_2 \vdash A$. By $(+)$ we have $\Sigma_1, \Sigma_2 \vdash A \to B$, as well as $\Sigma_1, \Sigma_2 \vdash A$.
Then, by $(\to -)$, we have $\Sigma_1, \Sigma_2 \vdash B$, where $\Sigma_1 \cup \Sigma_2$ is a finite subset of Σ.
The proof of the other cases is left as exercise.

Finitness of premise set

This theorem captures the intuition that, in a proof involving only finitely many steps, we can only use finitely many formulas in Σ.

Transitivity of deducibility

Theorem.
Let $\Sigma \subseteq$ Form$(\mathcal{L}^p)$ and $A_1, A_2, \ldots, A_n$ be formulas in Form$(\mathcal{L}^p)$. If $\Sigma \vdash A_i$ for all $i = 1, \ldots n$ and $A_1, A_2, \ldots, A_n \vdash A$, then $\Sigma \vdash A$.

Proof.

(1)	$A_1, \ldots A_n \vdash A$	by supposition
(2)	$A_1, \ldots, A_{n-1} \vdash A_n \to A$	by $(\to +), (1)$
$\ldots$		
(3)	$\emptyset \vdash A_1 \to (\ldots(A_n \to A)\ldots)$	similar to obtaining (2)
(4)	$\Sigma \vdash A_1 \to (\ldots(A_n \to A)\ldots)$	by $(+), (3)$
(5)	$\Sigma \vdash A_1$	by supposition
(6)	$\Sigma \vdash A_2 \to (\ldots(A_n \to A)\ldots)$	by $(\to -), (4), (5)$
$\ldots$		
(7)	$\Sigma \vdash A_n \to A$	similar to obtaining (6)
(8)	$\Sigma \vdash A_n$	by supposition
(9)	$\Sigma \vdash A$	by$(\to -), (7), (8)$

Remarks

- The theorem of transitivity of deducibility is denoted by (Tr).
- The sequence $A_1, \dots A_n$ in (Tr) is finite. Otherwise (3) in the above proof would be obtained by applying $(\rightarrow\ +)$ infinitely many times, contradicting the finiteness of a formal proof.
- The conclusion of a scheme of formal deducibility consists of one formula. When a number of schemes of formal deducibility have the same premises, we may write $\Sigma \vdash A_1, \dots, A_n$ for $\Sigma \vdash A_1, \dots, \Sigma \vdash A_n$.
- Thus, (Tr) may be written as:
  If $\Sigma \vdash A_1, \dots, A_n$,
  and $A_1, \dots, A_n \vdash A$,
  then $\Sigma \vdash A$.

Natural deduction

- Since the rules of formal deduction (for propositional logic) express naturally and intuitively the rules of informal reasoning, the formal deduction based upon these rules is also called natural deduction.
- A type of natural deduction was introduced by Gentzen in 1934.
- There are other types of formal deduction, one of which will be introduced later.

Some useful theorems

Theorem. $\neg\neg A \vdash A$.

Proof.

$$\begin{array}{lll}
(1) & \neg\neg A, \neg A \vdash \neg A & \text{by } (\in) \\
(2) & \neg\neg A, \neg A \vdash \neg\neg A & \text{by } (\in) \\
(3) & \neg\neg A \vdash A & \text{by } (\neg-), (1), (2).
\end{array}$$

Some useful theorems

Theorem (Reductio ad absurdum, $(\neg+)$)
If $\Sigma, A \vdash B$ and $\Sigma, A \vdash \neg B$, then $\Sigma \vdash \neg A$.

Proof

(1)	$\Sigma, A \vdash B$	by supposition
(2)	$\Sigma^0, A \vdash B$	take finite $\Sigma^0 \subseteq \Sigma$
(3)	$\Sigma, \neg\neg A \vdash \Sigma^0$	by $(\in)$
(4)	$\neg\neg A \vdash A$	by (a)
(5)	$\Sigma, \neg\neg A \vdash A$	by $(+), (4)$
(6)	$\Sigma, \neg\neg A \vdash B$	by (Tr) $, (3), (5), (2)$
(7)	$\Sigma, \neg\neg A \vdash \neg B$	analogous to(6)
(8)	$\Sigma \vdash \neg A$	by $(\neg-), (6), (7)$.

Comments

- In (2) and (3) of the preceding proof, we used a finite subset Σ^0 to replace Σ because Σ may be infinite and accordingly not available in (Tr).
- Suppose $\Sigma^0 = C_1, \ldots C_n$. Then (3) consists of n steps

$$\Sigma, \neg\neg A \vdash C_1,$$

$$\ldots$$

$$\Sigma, \neg\neg A \vdash C_n.$$

  These can be written in one step because they are generated by the same rule ($\in$).
- The theorem of reductio ad absurdum is denoted by ($\neg+$) and sometimes called ($\neg$ introduction).

$(\neg-)$ and $(\neg+)$

- $(\neg+)$ and $(\neg-)$ are similar in shape but different in strength.
- $(\neg-)$ is stronger than $(\neg+)$.
- $(\neg+)$ has just been proved.
- However, if $(\neg-)$ is replaced by $(\neg+)$ in the 11 rules, then $(\neg-)$ cannot be proved.

Syntactical equivalence

For two formulas A and B we write

$$A \dashv\vdash B$$

for $A \vdash B$ and $B \vdash A$.

A and B are said to be syntactically equivalent iff $A \dashv\vdash B$ holds.

We write $\dashv$ to denote the converse of $\vdash$.

Lemma If $A \dashv\vdash A'$ and $B \dashv\vdash B'$ then

(1)$\neg A \dashv\vdash \neg A'$.
(2)$A \wedge B \dashv\vdash A' \wedge B'$.
(3)$A \vee B \dashv\vdash A' \vee B'$.
(4)$A \rightarrow B \dashv\vdash A' \rightarrow B'$.
(5)$A \leftrightarrow B \dashv\vdash A' \leftrightarrow B'$.

Note the resemblance to analogous results about tautological equivalences $\models\!\!\dashv$.

Replaceability of syntactically equivalent formulas in formal deduction, and other theorems

Theorem (Replacement of syntactically equivalent formulas) (Repl.)
If $B \vdash\dashv C$ and A' results from A by replacing some (not necessarily all) occurrences of B in A by C, then $A \vdash\dashv A'$.

Theorem (exercise)
$A_1, A_2, \ldots A_n \vdash A$ iff $\emptyset \vdash A_1 \wedge \ldots \wedge A_n \to A$.

Theorem (exercise)
$A_1, \ldots, A_n \vdash A$ iff $\emptyset \vdash A_1 \to (\ldots (A_n \to A) \ldots)$.

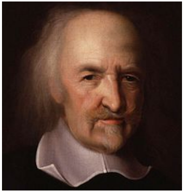
Special case of formal deduction - empty premise set

- When the premise is empty we have the special case $\emptyset \vdash A$ of formal deducibility.
- Obviously, $\emptyset \vdash A$ iff $\Sigma \vdash A$ for any Σ.
- It has been mentioned before that A is said to be formally provable from Σ when $\Sigma \vdash A$ holds.
- When $\emptyset \vdash A$ holds, A is said to be formally provable (from nothing).
- The laws of non-contradiction $\neg(A \land \neg A)$ and excluded middle $A \lor \neg A$ are instances of formally provable formulas, that is,
- $\emptyset \vdash \neg(A \land \neg A)$ and
- $\emptyset \vdash A \lor \neg A$

Why formal deduction?

- What is it that makes mathematics and computer science different from other academic disciplines?
- What is it that distinguishes a mathematician/computer scientist from a poet, a linguist, a biologist, or a civil engineer?
- One of the things that sets mathematics/computer science apart is the insistence upon proof.

A famous story: " [Thomas Hobbes] was 40 years old before he looked on Geometry; which happened accidentally. Being in a Gentleman's Library, Euclid's *Elements* lay open, and 'twas at the 47 El.libri 1 [the Pythagorean Theorem]. He read the Proposition. By G–, sayd he (he would now and then sweare an emphaticall Oath by way of emphasis) this is impossible! So he reads the Demonstration of it, which referred him back to such a Proposition; which proposition he read. That referred him back to another, which he also read. Et sic deinceps [and so on] that at last he was demonstratively convinced of that trueth. This made him in love with Geometry." (John Aubrey, *Brief Lives*)



Thomas Hobbes, English philosopher, 1588-1679

Why formal deduction?

- (Tauto)logical consequence, $\models$, which is defined in terms of value assignments and truth values corresponds to informal deducibility and involves semantics.
- Formal deducibility in general, which is defined by a finite number of rules of formal deduction, is concerned with the syntactical structures of formulas and involves syntax.
- Our goal was to define something called formal (natural) deduction, based on 11 axioms, from which we could **prove formally everything that is correct semantically**.

What makes a certain system of formal deduction “good”?

- For a certain system of formal deducibility to be “good” is has to be connected to informal reasoning in the following sense:
  - **(I) We should not be able to prove incorrect statements!** (soundness)
  - **(II) We should be able to prove, from first principles (in our case, the 11 rules), every correct statement!** (completeness)

(I) Soundness of a formal deduction system

Suppose that the statement **"If $\Sigma \vdash A$ then $\Sigma \models A$"** is true for any Σ and A.

This means that what formal deducibility expresses about premises and conclusions, also holds in informal reasoning.

In other words, it means that we cannot prove incorrect statements.

If this property holds for a given system of formal deducibility, that system is called sound.

Soundness Theorem.
The system of natural deduction, based on the 11 (eleven) given rules of formal deduction, is sound.

Proof of the Soundness Theorem

Theorem (Soundness Theorem) If $\Sigma \vdash A$ then $\Sigma \models A$, where $\vdash$ means the formal deduction based on the 11 given rules.

Proof Induction on the complexity of $\Sigma \vdash A$.

That is, we prove that each of the 11 rules of formal deduction preserves the property: "The statement obtained by replacing $\vdash$ by $\models$ in each rule holds."

We prove the cases of (Ref), $(\neg-)$ and $(\vee-)$.

Case of (Ref). $A \models A$. Obvious.

Proof of soundness, the case of $(\neg-)$

Case of $(\neg-)$.

We shall prove that

> If $\Sigma, \neg A \models B$ and
> $\Sigma, \neg A \models \neg B$,
> then $\Sigma \models A$.

Suppose $\Sigma \not\models A$, that is, there is a value assignment v such that $v(\Sigma) = 1$ and $v(A) = 0$.

Then, $v(\neg A) = 1$.

Since we are given that $\Sigma, \neg A \models B$ and $\Sigma, \neg A \models \neg B$, we have $v(B) = 1$ and $v(\neg B) = 1$, which is a contradiction.

Hence $\Sigma \models A$, and the proof of this case is complete.

Proof of soundness, the case of $(\vee-)$

Case of $(\vee-)$.

We shall prove that

> If $\Sigma, A \models C$ and
> $\Sigma, B \models C$,
> then $\Sigma, A \vee B \models C$.

Suppose v is an arbitrary value assignment such that $v(\Sigma) = 1$ and $v(A \vee B) = 1$. Then $v(A) = 1$ or $v(B) = 1$.

Subcase 1: If $v(A) = 1$, then, by $\Sigma, A \models C$, we have that $v(C) = 1$.

Subcase 2: If $v(B) = 1$, then, by $\Sigma, B \models C$, we have that $v(C) = 1$.

Hence $v(C) = 1$ and, accordingly, $\Sigma, A \vee B \models C$.

The cases of the other rules of formal deduction are proved similarly.

q.e.d.

(II) Completeness of a formal deduction system

Suppose that the statement **“If $\Sigma \models A$ then $\Sigma \vdash A$”** is true for any Σ and A.

This means that what holds in informal reasoning can be proved using formal deducibility.

In other words, it means that whatever is correct can be formally proved.

If this property holds for a system of formal deducibility, that system is called complete.

Completeness Theorem.
The system of natural deduction based on the 11 (eleven) given rules of formal deduction is complete.

Proof of the Completeness Theorem

Theorem (Completeness Theorem) If $\Sigma \models A$ then $\Sigma \vdash A$, where $\vdash$ means the formal deduction based on the 11 given rules.

Proof in three steps:

1. If $A_1, A_2, \ldots, A_n \models A$ then $\emptyset \models (A_1 \to (A_2 \to ...(A_n \to A) \ldots)$
2. If $\emptyset \models A$ then $\emptyset \vdash A$ (every tautology is formally provable)
3. If $\emptyset \vdash (A_1 \to (A_2 \to \ldots (A_n \to A) \ldots)$ then $A_1, A_2, \ldots, A_n \vdash A$.

The idea is to prove the required statement in the case when $\Sigma = \emptyset$ (that is, prove that every tautology is formally provable, Step (2)), and then "convert" from a general Σ to $\emptyset$ and viceversa.

We first prove that the "conversion" works, i.e., prove (1) and (3).

Steps (1), (3) of Completeness Theorem

(1) If $A_1, A_2, \ldots, A_n \models A$ then $\emptyset \models (A_1 \to (A_2 \to ...(A_n \to A) \ldots)$
Proof by contradiction. Assume there exists a value assignment v such that $v((A_1 \to (A_2 \to ...(A_n \to A) \ldots)) = 0$.
This formula being structured as a series of nested implications, this implies that $v(A_1) = v(A_2) = \cdots = v(A_n) = 1$ and $v(A) = 0$.
This contradicts our hypothesis that $A_1, A_2, \ldots, A_n \models A$.

(3) If $\emptyset \vdash (A_1 \to (A_2 \to \ldots (A_n \to A) \ldots)$ then $A_1, A_2, \ldots, A_n \vdash A$.
Since we have $\emptyset \vdash (A_1 \to (A_2 \to \ldots (A_n \to A) \ldots)$, we deduce that $A_1, \ldots A_n \vdash (A_1 \to (A_2 \to \ldots (A_n \to A) \ldots)$, by rule $(+)$.
Then, n applications of the rules $(\in)$ and $(\to -)$ lead to $A_1, A_2 \ldots A_n \vdash A$.
* *Technically, both steps should be by induction on n.*

Step (2) of Completeness Theorem

(2) If $\emptyset \models A$ then $\emptyset \vdash A$ (every tautology is formally provable)
Proof idea: Assume that A is a tautology, and A has n atoms. Construct 2^n subproofs for A, one for each value assignment, and use the Law of Excluded Middle, $\emptyset \vdash p \vee \neg p$, the rule $(\vee-)$, and (Transitivity), to put them together. More precisely:

Let the n atoms be $p_1, p_2, \ldots, p_n$, and let v be a value assignment. Define (relative to v):

$$p_i' = \begin{cases} p_i & \text{if } v(p_i) = 1 \\ \neg p_i & \text{if } v(p_i) = 0 \end{cases}$$

Lemma

Lemma. Let A be a formula with propositional variables $p_1, p_2, \dots, p_n$, and let v be a value assignment. Then

- if $v(A) = 1$ then $p'_1, p'_2 \dots, p'_n \vdash A$, and
- if $v(A) = 0$ then $p'_1, p'_2 \dots, p'_n \vdash \neg A$

Proof. Structural induction on the formula A.

Base Case: If A is an atom, $A = p$. If $v(p) = 1$, then $p' = p$ and $p \vdash A$; if $v(p) = 0$ then $p' = \neg p$ and $\neg p \vdash \neg A$, by (Ref).

Inductive Step. We have to consider the case of all five formation rules for formulas.

All cases are similar, let us prove the case $A = B \to C$, where we assume the property holds for B and for C (ind. hypothesis, I.H.).

Inductive step of Lemma, case $A = B \to C$

Let $b_1, b_2, \ldots b_k$ be the propositional atoms for B and $c_1, c_2, \ldots c_l$ be the propositional atoms for C.

Subcase (1): $v(A) = v(B \to C) = 0$.

We have to prove that $p'_1, p'_2, \ldots p'_n \vdash \neg(B \to C)$.

Since $v(B \to C) = 0$, we have that $v(B) = 1$ and $v(C) = 0$.

By I.H. we have $b'_1, b'_2, \ldots b'_k \vdash B$ and $c'_1, c'_2, \ldots c'_l \vdash \neg C$.

We apply $(+)$ to obtain $p'_1, p'_2, \ldots p'_n \vdash B$ and $p'_1, p'_2, \ldots p'_n \vdash \neg C$.

Then, by $(\wedge+)$ we obtain $p'_1, p'_2, \ldots p'_n \vdash B \wedge \neg C$.

All we have to do now is to prove that $B \wedge \neg C \vdash \neg(B \to C)$, and then use $(Tr.)$ to obtain the required proof.

Prove that $B \wedge \neg C \vdash \neg(B \to C)$

1.	$B \wedge \neg C, B \to C$	$\vdash$	$B \wedge \neg C$	$(\in)$
2.	$B \wedge \neg C, B \to C$	$\vdash$	B	$(1, \wedge-)$
3.	$B \wedge \neg C, B \to C$	$\vdash$	$B \to C$	$(\in)$
4.	$B \wedge \neg C, B \to C$	$\vdash$	C	$(2, 3 \to -)$
5.	$B \wedge \neg C, B \to C$	$\vdash$	$\neg C$	$(1, \wedge-)$
6.	$B \wedge \neg C$	$\vdash$	$\neg(B \to C)$	$(4, 5, \neg+)$

Thus, $p'_1, p'_2, \ldots p'_n \vdash B \wedge \neg C \vdash \neg(B \to C)$ - this case is done

The (other) subcases of $A = B \to C$

- $v(A) = v(B \to C) = 1$
- In this case we have to prove that $p'_1, p'_2, \dots p'_n \vdash (B \to C)$.
- We have three subcases:
  - $v(B) = 1,\ v(C) = 1$, need to prove $B \wedge C \vdash B \to C$
  - $v(B) = 0,\ v(C) = 1$, need $\neg B \wedge C \vdash B \to C$
  - $v(B) = 0,\ v(C) = 0$, need $\neg B \wedge \neg C \vdash B \to C$

All subcase proofs are similar, and the proofs for the other main cases for $A = \neg B$, $A = B \wedge C$, $A = B \vee C$, $A = B \leftrightarrow C$ (with their own subcases) are also similar.

Q.E.D.

Step (2) of Completeness Theorem proof: $\emptyset \models A$ implies $\emptyset \vdash A$

- Corollary: Every tautology A is formally provable, that is, $\emptyset \models A$ implies $\emptyset \vdash A$.
  Proof. Since all 2^n value assignments make a tautology A true, for every possible choice for $p'_1, p'_2, \ldots p'_n$, we can find a formal proof $p'_1, p'_2, \ldots p'_n \vdash A$; such 2^n proofs exist, by the Lemma.
- Each proof shows that the n premises p'_i corresponding to that truth table row (p_i if $p_i = 1$, and $\neg p_i$ if $p_i = 0$), entail A.
- We then use the rule $(\vee-)$ to combine all the 2^n subproofs for $p'_1, p'_2, \ldots, p'_n \vdash A$, into one "big proof" for A, that has as premises all of $(p_i \vee \neg p_i)$, $1 \leq i \leq n$.
- Lastly, we use the Law of Excluded Middle, $\emptyset \vdash p_i \vee \neg p_i$, for all variables p_i, together with the "big proof," and (Transitivity), to obtain $\emptyset \vdash A$.

Example

Find a formal proof for the tautology $A = \neg q \wedge (p \to q) \to \neg p$.

1.	p, q	$\vdash$	A	(separately) (proof for row 1)
2.	$p, \neg q$	$\vdash$	A	(separately) (proof for row 2)
3.	$\neg p, q$	$\vdash$	A	(separately) (proof for row 3)
4.	$\neg p, \neg q$	$\vdash$	A	(separately) (proof for row 4)
5.	$p, q \vee \neg q$	$\vdash$	A	(1, 2 $\vee-$)
6.	$\neg p, q \vee \neg q$	$\vdash$	A	(3, 4 $\vee-$)
7.	$p \vee \neg p, q \vee \neg q$	$\vdash$	A	(5, 6 $\vee-$) ("big proof" for A)
8.	$\emptyset$	$\vdash$	$p \vee \neg p$	(proved) (Excl.Middle for p)
9.	$\emptyset$	$\vdash$	$q \vee \neg q$	(proved) (Excl.Middle for q)
10.	$\emptyset$	$\vdash$	A	(8, 9, 7, Tr) (Transitivity)

The formal proof for A corresponding to row 1 of the truth table, $\mathbf{v(p) = v(q) = 1}$

Prove that $p, q \vdash \neg q \wedge (p \rightarrow q) \rightarrow \neg p$

1.	$p, q, \neg q \wedge (p \rightarrow q)$	$\vdash$	q	$(\in)$
2.	$p, q, \neg q \wedge (p \rightarrow q)$	$\vdash$	$\neg q \wedge (p \rightarrow q)$	$(\in)$
3.	$p, q, \neg q \wedge (p \rightarrow q)$	$\vdash$	$\neg q$	$(2, \wedge -)$
4.	$q, \neg q \wedge (p \rightarrow q)$	$\vdash$	$\neg p$	$(1, 3, \neg +)$
5.	$p, q, \neg q \wedge (p \rightarrow q)$	$\vdash$	$\neg p$	$(4, +)$
6.	p, q	$\vdash$	$\neg q \wedge (p \rightarrow q) \rightarrow \neg p$	$(5, \rightarrow +)$

The other subcases (corresponding to the other rows of the truth table) are proved similarly, and the previous Lemma guarantees that these proofs always exist.

Q.E.D.

Connection between syntax and semantics

- The Soundness and Completeness Theorems associate the syntactic notion of formal deduction with the semantic notion of (tauto)logical consequence, and establishes the equivalence between them.
- The Soundness and Completeness Theorems say that with natural deduction (as defined by the 11 rules) we can prove

  The truth,
  the whole truth, (completeness),
  and nothing but the truth. (soundness)

"Do you swear to tell the truth, the whole truth, and nothing but the truth in the most entertaining way possible?"

Note: Formal deduction cannot be used to prove that an argument is invalid

- The fallacy (invalid argument) of denying the antecedent:

  $$\begin{array}{l} p \to q \\ \neg p \\ \hline \neg q \end{array}$$

- Recall Descartes, who famously said "I think, therefore I am". The joke was that Descartes goes into a bar and the bartender asks him if he wants another drink. "I think not," says Descartes, and vanishes in a puff of smoke.
- What was wrong with this joke? One cannot conclude $\neg q$ from $p \to q$ and $\neg p$. The argument is invalid.
- However, while we can prove that $p \to q, \neg p \not\models \neg q$, we cannot prove invalidity of this argument using formal deduction, $\vdash$.

Another logical fallacy

"Why are you standing here on this street corner, wildly waiving your hands and shouting?"
"I am keeping away the elephants."
"But there aren't any elephants here."
"You bet: that's because I'm here."
What's wrong with this joke?

- The fallacy (invalid argument) of affirming the consequent:

$$\frac{p \to q \qquad q}{p}$$

Another system of formal deduction

(1)	$A, B \vdash A \wedge B$	Law of combination
(2)	$A \wedge B \vdash B$	Law of simplification
(3)	$A \wedge B \vdash A$	Var. of law of simplification
(4)	$A \vdash A \vee B$	Law of addition
(5)	$B \vdash A \vee B$	Var. of law of addition
(6)	$A, A \rightarrow B \vdash B$	Modus ponens
(7)	$\neg B, A \rightarrow B \vdash \neg A$	Modus tollens
(8)	$A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$	Hypothetical syllogism
(9)	$A \vee B, \neg A \vdash B$	Disjunctive syllogism
(10)	$A \vee B, \neg B \vdash A$	Var. of disjunctive syllogism
(11)	$A \rightarrow B, \neg A \rightarrow B \vdash B$	Law of cases
(12)	$A \leftrightarrow B \vdash A \rightarrow B$	Equivalence elimination
(13)	$A \leftrightarrow B \vdash B \rightarrow A$	Var. of equivalence elimination
(14)	$A \rightarrow B, B \rightarrow A \vdash A \leftrightarrow B$	Equivalence introduction
(15)	$A, \neg A \vdash B$	Inconsistency law

Exercise

Exercise

Assuming that we are in the system of natural deduction, prove the laws (1) - (15) on the preceding slide.

Completeness of the new system of formal deduction (based on 15 rules)

The new system we have defined is not complete.

To see this, consider the law of excluded middle, which is $p \vee \neg p$.

This law holds without premises, that is, $\emptyset \models p \vee \neg p$. It is easy to see that, without premises, none of the rules of the new system can be used.

Consequently, $p \vee \neg p$ cannot be derived if the rules of deduction are restricted those listed to rules (1) - (15) from the new system.

Deduction Law

Deduction Law.

If

$$B, A_1, A_2, A_3, \ldots \vdash C$$

then

$$A_1, A_2, A_3, \ldots \vdash B \to C.$$

Together with this law, the new system is complete.

Formal deduction exercises

Use only the 11 axioms (rules) of formal deduction and the theorems proved in class to show that:

1. $A \vdash \neg\neg A$
2. $A \vee B, \neg A \vdash B$ (disjunctive syllogism)
3. $\neg B, A \rightarrow B \vdash \neg A$ (modus tollens)
4. $A \rightarrow B \vdash\dashv \neg B \rightarrow \neg A$ (contrapositive)
5. If $A \vdash B$ then $\neg B \vdash \neg A$
6. $\neg(A \vee B) \vdash\dashv \neg A \wedge \neg B$ (de Morgan)
7. $\neg(A \wedge B) \vdash\dashv \neg A \vee \neg B$ (de Morgan)
8. $A \vee B \vdash\dashv \neg A \rightarrow B$
9. $\emptyset \vdash \neg(A \wedge \neg A)$ and $\emptyset \vdash A \vee \neg A$
10. $A \rightarrow B, \neg A \rightarrow B \vdash B$ (law of cases)
11. $A, \neg A \vdash B$ (inconsistency law)

Formal deduction exercises contd.

Use only the 11 axioms (rules) of formal deduction and the theorems proved in class to show that:

1. $A \to (B \vee C), A \to \neg B, C \to \neg D \vdash A \to \neg D$
2. $A \to (B \to C), C \to \neg D, \neg E \to D, A \wedge B \vdash E$
3. $\neg A \to C \vee D, B \to E \wedge F, E \to D, \neg D \vdash (A \to B) \to C$
4. $\neg(A \vee B) \to (C \to D), \neg A \wedge \neg D \vdash \neg B \to \neg C$
5. $B \vee A, B \to A \vdash \neg(A \to \neg A)$
6. $A \to B, A \vee (A \wedge C) \vdash A \wedge B$
7. $(A \wedge B) \vee (A \wedge C) \vdash A \wedge (B \vee C)$
8. $A \vee (D \wedge C), A \to (B \wedge C) \vdash D \vee C$
9. $A \to (B \wedge C) \vdash (A \to B) \wedge (A \to C)$

Formal deduction proof strategies (i)

- If the conclusion is an implication, that is, we have to prove $\Sigma \vdash A \to B$, then try adding A to the set of premises and try to prove B. In other words, prove $\Sigma, A \vdash B$ first. If this is proved, then one application of $(\to +)$ will result in $\Sigma \vdash A \to B$.
- If the premises contain a disjunction, that is, if we have to prove $\Sigma, A \vee B \vdash C$, then try to use "proof by cases" $(\vee -)$. In other words, prove separately $\Sigma, A \vdash C$ (Case 1), then $\Sigma, B \vdash C$ (Case 2), and then put these two proofs together with one application of $(\vee -)$ to obtain $\Sigma, A \vee B \vdash C$.
- If we have to prove $A \vdash B$ and the direct proof does not work, try "proving the contrapositive", that is, try to prove $\neg B \vdash \neg A$, and then use the theorem "$A \to B \vdash\dashv \neg B \to \neg A$" (you have to prove it, to be allowed to use it!)

Formal deduction proof strategies (ii)

- If everything else fails, try "proof by contradiction", that is, apply $(\neg+)$ or $(\neg-)$, as follows:
- $(\neg-)$: If we want to prove $\Sigma \vdash B$ and we do not know how, start with the modified premises $\Sigma, \neg B$ (we added a new premise, $\neg B$, to the initial set), and try to reach a contradiction:
  - Prove that $\Sigma, \neg B \vdash C$, for some formula C
  - Prove that $\Sigma, \neg B \vdash \neg C$, for the same C (any C would do!)
  - If we succeed proving both facts, we reached a contradiction (we proved both C and $\neg C$).
  - This means that our new assumption/premise, $\neg B$, was incorrect, and its opposite (that is, B) holds.
  - Formally, from $\Sigma, \neg B \vdash C$ and $\Sigma, \neg B \vdash \neg C$, one application of $(\neg-)$ yields $\Sigma \vdash B$.
- $(\neg+)$: Used for cases in which we have difficulties proving $\Sigma \vdash \neg B$. In that case we prove separately that $\Sigma, B \vdash C$, and $\Sigma, B \vdash \neg C$, then apply $(\neg+)$ to obtain $\Sigma \vdash \neg B$.

Formal deduction proof strategies (iii)

- These are only general proof strategies (not algorithms!), and there is no guarantee that they will be the best strategy for all formal deduction proofs.