3-1 Dynamic Programming

(Part II: "Theory")

Hengfeng Wei

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September 27, 2018

Definition (Optimal Substructure)

A problem exhibits ***optimal substructure*** if an optimal solution to the problem contains within it optimal solutions to subproblems.

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How to prove "YES"?

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How to show "NO"?

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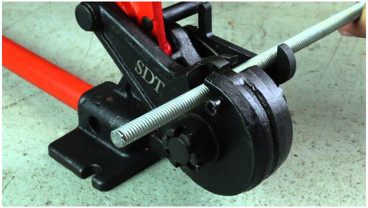
$\boxed{\textit{Relative to Subproblems}}$

Rod Cutting



Optimal Substructure of Rod-Cutting (Problem 15.3-5)

Limit l_i : # of pieces of length i, $\quad 1 \le i \le n$

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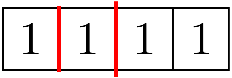

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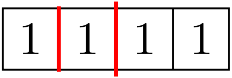

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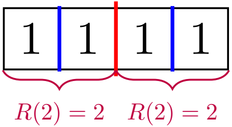


Well done

"Show that the optimal-substructure property described in Section 15.1 *no longer holds."*

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$R(i, L) : \max$ *revenue obtainable by cutting up a rod of length* i

with the length limit array L

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$$R(i, L) = \max_{1 \le j \le i} \Big(p_j + R(i - j, L[j \mapsto L_j - 1]) \Big)$$

Currency Exchange (Problem 15.3-6)

$1, 2, \cdots, n$ currencies

$$i \xrightarrow{r_{ij}} j$$

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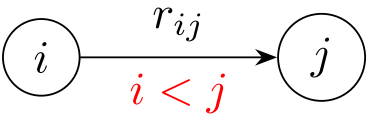
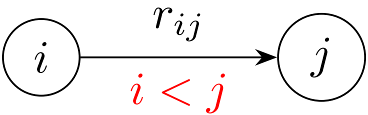
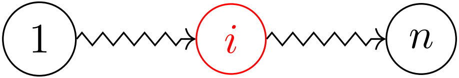
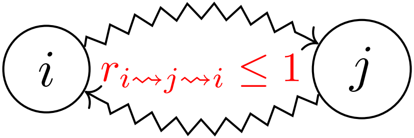
c_k : Commission charged for k trades

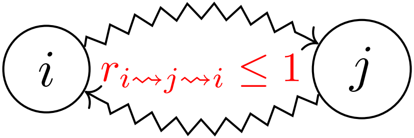

r_{ij}
i
j
$i < j$



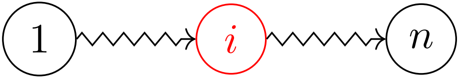
An optimal sequence of trades from 1 to n through i:

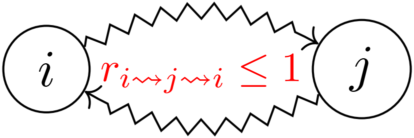



i
$r_{i \rightsquigarrow j \rightsquigarrow i} \leq 1$
j



An optimal sequence of trades from 1 *to* n *through* i*:*





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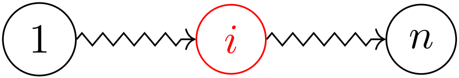
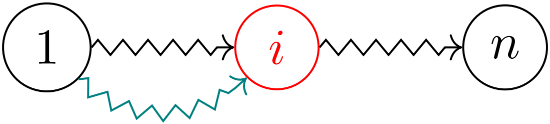
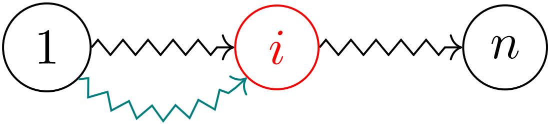


By Contradiction.



$$\text{Case I}: s_{1 \rightsquigarrow i} \cap s_{i \rightsquigarrow n} = \emptyset$$



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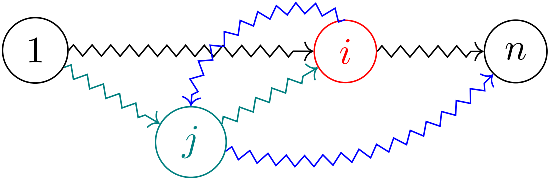

$\textsc{Case II} : j \in s_{1 \rightsquigarrow i} \cap s_{i \rightsquigarrow n}$

$$
\begin{aligned}
& 1 \rightsquigarrow j \rightsquigarrow n \\
\geq\ & 1 \rightsquigarrow j \rightsquigarrow i \rightsquigarrow j \rightsquigarrow n \\
=\ & 1 \rightsquigarrow j \rightsquigarrow i \rightsquigarrow n \\
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\end{aligned}
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Longest Path Problem

To find a *simple* path of maximum length from s to t in a graph.

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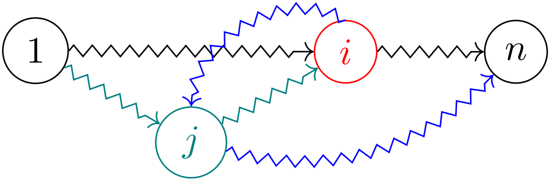

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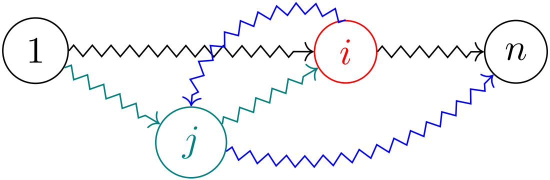

$$
\begin{aligned}
& 1 \leadsto j \leadsto n \\
& \geq 1 \leadsto j \leadsto i \leadsto j \leadsto n \\
& = 1 \leadsto j \leadsto i \leadsto n \\
& > 1 \leadsto i \leadsto n
\end{aligned}
$$

Does the longest path problem really have no optimal substructure?

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WAIT WAIT...
DON'T TELL ME!®
FROM NPR® & WBEZ® CHICAGO

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through exactly the intermediate vertices in I

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$$L(s, t) =$$

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through *exactly* *the intermediate vertices in I*

$$L(s, t) = \max_{I \subseteq V} L(s, t, I)$$

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What is the next vertex from s?

$L(s,t,A)$: *the longest path from s to t*

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What is the next vertex from s?

$$L(s,t,I) = 1 + \max_{\substack{(s,x) \in E \\ x \in I}} l(x,t,I \setminus \{x\})$$

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$$L(t, t, I) = 0, \qquad L(s, t, \emptyset) = \begin{cases} 1, & \text{if } (s, t) \in E \\ 0, & \text{otherwise} \end{cases}$$

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Cycle $\implies$ *No Order on Vertices*

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DP does not necessarily lead to efficient (polynomial) algorithms.

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Cycle $\implies$ *No Order on Vertices*

DP does not necessarily lead to efficient (polynomial) algorithms.

The (decision version of the) longest path problem is NP-hard!

The Change-making Problem

Coins values: $x_1, x_2, \cdots, x_n$

Amount: v

Is it possible to make change for v?

The Change-making Problem

Without repetition: $0/1$

The Change-making Problem

Without repetition: $0/1$

$C[i, w]$: Make change for w using only values of $x_1 \ldots x_i$?

$$C[n, v]$$

Without repetition: $0/1$

$C[i, w]$: Make change for w using only values of $x_1 \dots x_i$?

$$C[n, v]$$

$\boxed{\textit{Using value } x_i \textit{ or not?}}$

$$C[i, w] = \underbrace{C[i-1, w]}_{\notin} \vee \big(\underbrace{C[i-1, w - x_i] \wedge w \geq x_i}_{\in} \big)$$

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The $0/1$ Knapsack Problem (Problem 16.2-2)

Values : v_i Weights : w_i Capacity : W

Taking as valuable a load as possible

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Taking as valuable a load as possible

$C[i, w]$: Knapsack with capacity w using only items of values $v_1 \ldots v_i$

$\boxed{\textit{Using value } v_i \textit{ or not?}}$

$$C[i, w] = \max\Big(\underbrace{C[i-1, w]}_{\notin}, \big(\underbrace{w \geq x_i \implies C[i-1, w-w_i] + v_i}_{\in}\big)\Big)$$

The Change-making Problem

Unbounded repetition: ∞

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$$C[w] = \bigvee_{i:\, x_i \le w} C[w - x_i]$$

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$$O(nv)$$

The Change-making Problem

Unbounded repetition: ∞

$C[i, w]$ *vs.* $C[w]$

$$C[i, w] = C[i-1, w] \vee \left(C[i, w - x_i] \wedge w \geq x_i\right)$$

$$C[w] = \bigvee_{i:\, x_i \leq w} C[w - x_i]$$

The Change-making Problem (Problem $16\text{-}1$ (d))

Unbounded repetition: ∞

Using the fewest number of coins

The Change-making Problem (Problem 16-1 (d))

Unbounded repetition: ∞

Using the fewest number of coins

$\boxed{\textit{Using value } x_i \textit{ or not?}}$

$$C[i,w] = \min\Big(\underbrace{C[i-1,w]}_{\notin}, \big(\underbrace{w \ge x_i \implies C[i, w-x_i] + 1}_{\in}\big)\Big)$$

The Change-making Problem (Problem 16-1 (d))

Unbounded repetition: ∞

Using the fewest number of coins

$\boxed{\textit{What is the first coin to use?}}$

$$C[w] = \min_{i:\, x_i \le w} C[w - x_i] + 1$$

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Unbounded repetition with $\leq k$ coins overall

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Problem (Hungry-Lion Game)

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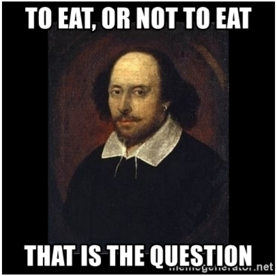
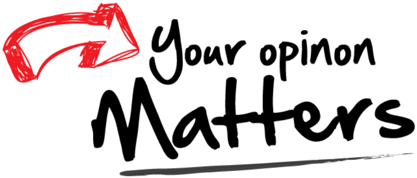



THANK YOU



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