

1-12 Partial Order and Lattice

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SM Problem 14.44: Consistent Enumerations

Suppose the following are three consistent enumerations of an ordered set $A = \{a, b, c, d\}$:

$A_1 :$ a b c d

$A_2 :$ a c b d

$A_3 :$ a c d b

Assuming the Hasse diagram D of A is **connected**, draw D .

SM Problem 14.44: Consistent Enumerations

Suppose the following are three consistent enumerations of an ordered set $A = \{a, b, c, d\}$:

$$A_1 : \quad a \quad \textcolor{red}{b} \quad \textcolor{red}{c} \quad d$$

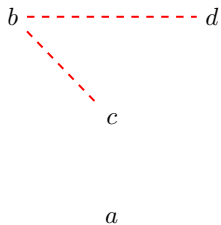
$$A_2 : \quad a \quad c \quad b \quad d$$

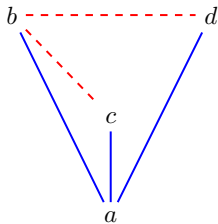
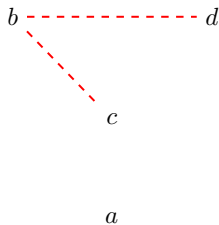
$$A_3 : \quad a \quad c \quad \textcolor{blue}{d} \quad \textcolor{blue}{b}$$

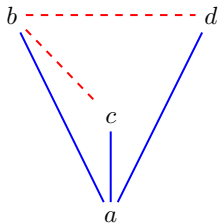
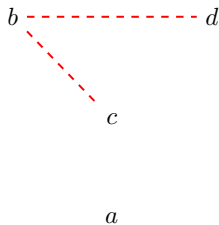
Assuming the Hasse diagram D of A is **connected**, draw D .

$$b \prec_{A_1} c \wedge c \prec_{A_2} b \implies b \parallel_A c$$

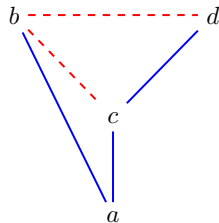
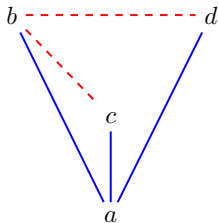
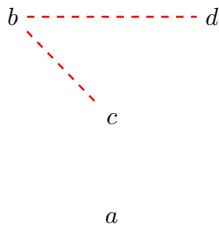
$$d \prec_{A_3} b \wedge b \prec_{A_2} d \implies b \parallel_A d$$



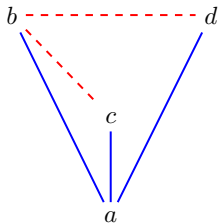
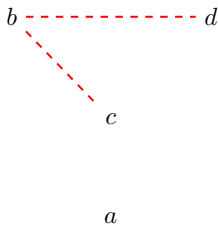




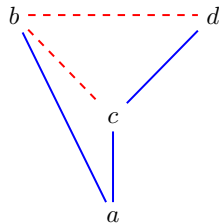
$$\# = 6$$



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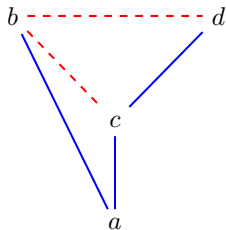
$\# = 3$

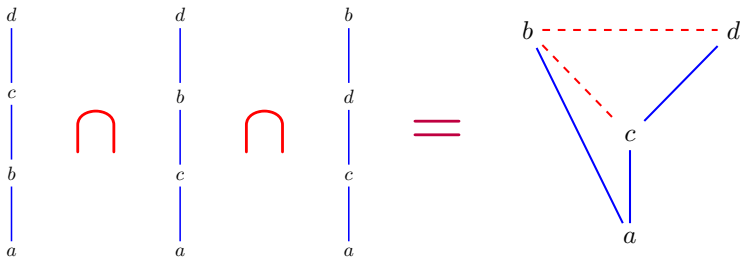
d
|
 c
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 a

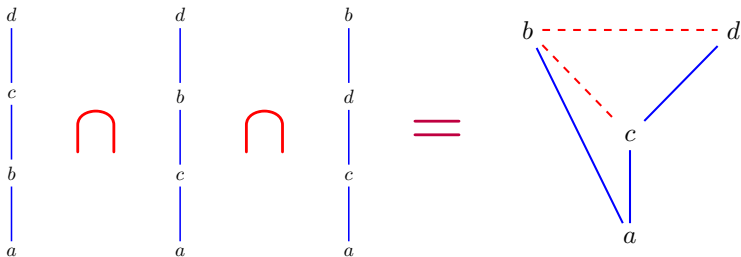
d
|
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 a

b
|
 d
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 c
|
 a

$=$







Theorem

Every partial ordering on a set X is the *intersection* of the total orders on X containing it.

Definition

Lattice A *lattice* is an algebra $\mathcal{L} = (L, \wedge, \vee)$ satisfying,

$$\forall a, b, c \in L,$$

Idempotency:

$$a \wedge a = a \quad a \vee a = a$$

Commutativity:

$$a \wedge b = b \wedge a \quad a \vee b = b \vee a$$

Associativity:

$$(a \wedge b) \wedge c = a \wedge (b \wedge c) \quad (a \vee b) \vee c = a \vee (b \vee c)$$

Absorption:

$$a \wedge (a \vee b) = a \quad a \vee (a \wedge b) = a$$

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(1) Very useful in lattice computations

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(2) The only laws connecting \wedge and \vee

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$$a \leq b \iff a \wedge b = a$$

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SM Problem 14.72: “Weak” Distributive Laws

Prove that for any lattice L :

$$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$$

$$a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$$

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$$a \leq b$$

$$c \leq d$$

$$(a \vee c) \leq (b \vee d)$$

2017-1-final-exam (5): Lattice

假设 (L, \leq) 是格。

如果以下模律 (modular law) 成立, 则称 L 是模格 (modular lattice):

$$\forall x \in L : a \leq b \implies a \vee (x \wedge b) = (a \vee x) \wedge b.$$

以下均假设 L 是模格。

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Distributive Law \implies Modular Law

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$$\forall x \in L : a \leq b \implies a \vee (x \wedge b) = (a \vee x) \wedge b.$$

(1) 请证明模律与以下条件等价:

$$\forall x \in L : a \leq b \implies a \vee (x \wedge b) \geq (a \vee x) \wedge b.$$

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$$a \leq b \implies a \vee (x \wedge b) \leq (a \vee x) \wedge b$$

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$$(a \leq a \vee x) \wedge (a \leq b) \implies a \leq (a \vee x) \wedge b \quad (1)$$

$$(x \leq a \vee x) \wedge b \leq b \implies (x \wedge b) \leq (a \vee x) \wedge b \quad (2)$$

2017-1-final-exam (5): Lattice

$$\forall x \in L : a \leq b \implies a \vee (x \wedge b) = (a \vee x) \wedge b.$$

(2) 请证明: $\forall a, b, c \in L$,

如果 $c \leq a$, $a \wedge b = c \wedge b$, $a \vee b = c \vee b$ 成立, 则 $a = c$.

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Modular Law : $[a \leftarrow c] \quad [b \leftarrow a]$

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2017-1-final-exam (5): Lattice

$$\forall x \in L : a \leq b \implies a \vee (x \wedge b) = (a \vee x) \wedge b.$$

(3) 给定任意元素 $s, t \in L$, 且 $s \leq t$, 构造集合 (称为区间 (interval)):

$$[s, t] \triangleq \{x \in L \mid s \leq x \leq t\}.$$

请证明 $([s, t], \leq)$ 是 L 的子格 (sublattice)。

2017-1-final-exam (5): Lattice

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$$a, b \in [s, t] \implies a \vee b, a \wedge b \in [s, t]$$

5. 格 (Lattice)

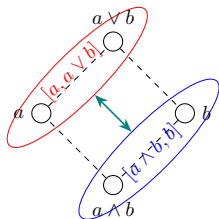
$$\forall x \in L : a \leq b \implies a \vee (x \wedge b) = (a \vee x) \wedge b.$$

(4) 给定任意元素 $a, b \in L$, 定义函数

$$\varphi : [a \wedge b, b] \rightarrow [a, a \vee b] \quad \varphi(x) = x \vee a$$

$$\psi : [a, a \vee b] \rightarrow [a \wedge b, b] \quad \psi(y) = y \wedge b$$

请证明 φ (类似地, ψ) 是从 $[a \wedge b, b]$ 到 $[a, a \vee b]$ 的同构。



Definition (Lattice Isomorphism)

$$(L, \vee_L, \wedge_L) \quad (M, \vee_M, \wedge_M)$$

A *lattice isomorphism* from L to M is a bijection

$$f : L \xrightarrow[\text{onto}]{1-1} M$$

such that $\forall a, b \in L$:

$$f(a \vee_L b) = f(a) \vee_M f(b)$$

$$f(a \wedge_L b) = f(a) \wedge_M f(b)$$

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f preserving \vee and \wedge .

φ preserving \vee and \wedge .

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$$\varphi : [a \wedge b, b] \rightarrow [a, a \vee b] \quad \varphi(x) = x \vee a$$

$$\varphi(x_1 \wedge x_2) = \varphi(x_1) \wedge \varphi(x_2)$$

φ preserving \vee and \wedge .

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$$\varphi(x_1 \wedge x_2) = \varphi(x_1) \wedge \varphi(x_2)$$

$$\varphi(x_1 \wedge x_2) = (x_1 \wedge x_2) \vee a$$

$$\begin{aligned} \varphi(x_1) \wedge \varphi(x_2) &= (x_1 \vee a) \wedge (x_2 \vee a) \\ &= (a \vee x_1) \wedge (x_2 \vee a) \\ &=_{\text{modular law}} a \vee (x_1 \wedge (x_2 \vee a)) \\ &= \dots \end{aligned}$$



$$\varphi : [a \wedge b, b] \rightarrow [a, a \vee b] \quad \varphi(x) = x \vee a$$

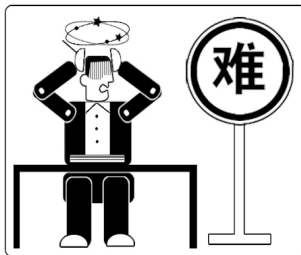
$$\psi : [a, a \vee b] \rightarrow [a \wedge b, b] \quad \psi(y) = y \wedge b$$

φ is bijective.

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φ is bijective.



Theorem (UD Theorem 15.8 (iii))

$$f : A \rightarrow B$$

$$\exists g : B \rightarrow A \left(g \circ f = i_A \wedge f \circ g = i_B \right)$$

$$\implies$$

$$f : A \rightarrow B \text{ is bijective} \wedge g = f^{-1}$$

$$\psi \circ \varphi = id_{[a, a \vee b]} \quad \varphi \circ \psi = id_{[a \wedge b, b]}$$

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$$(\psi \circ \varphi)(y) = \psi(\varphi(y)) = (y \wedge b) \vee a = a \vee (b \wedge y) = (a \vee b) \wedge y = y$$

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$$(\varphi \circ \psi)(x) = \varphi(\psi(x)) = (x \vee a) \wedge b = x \vee (b \wedge a) = x$$

Back to φ preserving \vee and \wedge .

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ψ preserving \wedge :

$$\psi(y_1 \wedge y_2) = y_1 \wedge y_2 \wedge b = (y_1 \wedge b) \wedge (y_2 \wedge b) = \psi(y_1) \wedge \psi(y_2)$$

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$$\psi(\varphi(x_1) \wedge \varphi(x_2)) = \psi(\varphi(x_1)) \wedge \psi(\varphi(x_2)) = x_1 \wedge x_2$$

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Thank
You!