Why are modular lattices important?

A lattice $(L, \leq)$ is said to be modular when

$$(\forall x, a, b \in L) \quad x \leq b \implies x \vee (a \wedge b) = (x \vee a) \wedge b,$$

where $\vee$ is the join operation, and $\wedge$ is the meet operation. (Join and meet.)

The ideals of a ring form a modular lattice. So do submodules of a module. These facts are easy to prove, but I have never seen any striking examples of their utility. Actually, in a seminar I took part in, the speaker said the modularity condition wasn't very natural and that there was an ongoing search for better ones (this was in the context of the Gabriel dimension and its generalization to lattices -- unfortunately, I didn't understand much of that).

I would like to see some motivation for this notion. That is, I would like to know when it is useful, and if it is natural. At the moment, it doesn't look any more natural to me than any random condition in the language of lattices. If you could shed some light on the opinion I quote in the previous paragraph, it would be very helpful as well. I would be especially interested in algebraic motivation, as I know *very* little about other areas if mathematics.

(abstract-algebra) (soft-question) (lattice-orders) (big-picture) (motivation)

edited Oct 22 '16 at 11:07
Alexey
606 4 20

asked May 7 '12 at 14:14
user23211

2 I think it would be nice if you added the definition of modular lattice to the question. – Tara B May 7 '12 at 15:30

@TaraB Thanks! I've added the definition. – user23211 May 7 '12 at 15:39

1 @ymar: There is a general "spectrum" construction one can apply to a distributive lattice to obtain a topological space. The spectrum of the lattice of ideals of a commutative ring in this sense is the same thing as its Zariski spectrum. Oddly enough, topological spaces – or rather their lattice of open sets – are themselves examples of special distributive lattices, called frames. – Zhen Lin May 10 '12 at 10:46

@ZhenLin It sounds very interesting! Are there any books or papers I could read to introduce myself to the subject? A google search just gives me loads of physics papers. – user23211 May 10 '12 at 11:22

4 If you search for "the many lives of lattice theory", links like ams.org/notices/199711/comm-rota.pdf (an article by G. C. Rota containing some high level background on modular lattices) and dpt-info.u-strasbg.fr/~cronse/lt.html (a summary containing concrete examples for "Rota's thesis") come up. (Also searching for "combinatorics the rota way" gives interesting results.) – Thomas Klimpel May 13 '12 at 12:26

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4 Answers

For reference:

Modularity: $x \leq b \implies x \vee (a \wedge b) = (x \vee a) \wedge b$

Algebraically it is a relaxed associativity condition for the meet and join operations. Graphically, it means that the forbidden Pentagon diagram will have one or more of its sides crushed. I can't think of any more proof for their naturality other than the fact the submodules of a module and the set of normal subgroups of a group are all modular lattices. Groups and modules are very natural!

Similarly its cousin the *distributive lattice* is algebraically distributivity of meet and join over each other. Modules do not normally have a distributive lattice of submodules.

Distributive lattices are natural because their prototype is the lattice of subsets of a given set with intersection and union operations. It is known that every distributive lattice is lattice isomorphic to such a set lattice.

P.S.: I didn't know this before, but I found that von Neumann apparently made use of complemented modular lattices in his book *Continuous Geometry*, so I would also look there for inspiration.

Added: You were requesting some places where modularity was explicitly used. When Ward and Dilworth went about abstracting the study of ideals in a ring to "multiplicative lattices", they managed to do primary decomposition in what they called *Noether lattices*. These were of course supposed to generalize the lattice of ideals of a Noetherian ring, and general multiplicative lattices are far too wild, so they needed to make some additional natural requirements for the multiplicative lattice. Among these assumptions were the ACC (to make it Noetherian), the property that every element should be a join of principal elements, and finally *modularity* of the lattice. I'm not an expert in the topic but I think modularity was probably crucial in their proofs using *residuals*.

edited May 13 '12 at 1:20

answered May 7 '12 at 15:41
rschwieb
91.7k 10 79 202

Thank you for the answer. Could you explain what you mean by the Pentagon diagram having its sides crushed? And why is it forbidden? – user23211 May 7 '12 at 19:21

@ymar I think this is known most places as *Dedekind's modularity criterion*: a lattice is modular iff it does not contain a pentagon like this. There is a counterpart for distributive lattices that I thought was attributed to Birkhoff but I can't seem to find a reference. For any pentagon you draw like that in a modular lattice, one of the sides *must* collapse to a point (or else you have a real pentagon!). – rschwieb May 7 '12 at 19:34

@ymar On the pentagon, put b over x on the leftmost points and a on the rightmost point, then try to compute the modular law we had above. – rschwieb May 7 '12 at 19:39

Thank you, I see. I've heard about the result that all distributive lattices are (up to isomorphism) sublattices of power sets, but I've never seen a proof. Could you recommend some book that has proofs of this and other facts you've mentioned in it? – user23211 May 7 '12 at 21:33

1 While I'm very grateful for this answer because it made some things clearer for me, the motivation you provide isn't satisfactory for me. I realize there may not be anything better, but I want to give this question another chance by starting a bounty. This is why I haven't accepted your answer yet. – user23211 May 10 '12 at 9:58

The definition of modularity looks more natural to me if I think of it as follows (rather than as a modified associativity or a weakened distributivity). Given any element a of a lattice L, there is a rather obvious way to map any element $x \in L$ to a "nearest" element $\geq a$, namely send x to $a \vee x$. Think of this map as "projecting " elements into the part of L above a. There is, of course, a dual notion of projecting elements into the part of L below a given element b, namely $x \mapsto b \wedge x$. If $a \leq b$, then we can combine these ideas to "project" any x into the interval $[a, b] = \{z \in L : a \leq z \leq b\}$, namely first project x above a and then project the result below b. (One needs to check that the second projection doesn't ruin what the first achieved; the final result is still above a as well as below b.) Again, duality provides another way to project x into $[a, b]$, namely first project it below b and then project the result above a. So in general we have two competing, equally natural notions of projecting an element x into an interval $[a, b]$. Modularity eliminates the competition; it says precisely that these two projections always agree.

answered Jul 15 '13 at 4:58
Andreas Blass
44.5k ● 3 ■ 47 ▲ 100

2 Thank you for an excellent view that is at the same time intuitive and motivated :D – Musa Al-hassy Jun 23 '14 at 16:27

2 I wish I could give more points for this wonderful answer! Thanks a lot! – Isomorphism Aug 27 '14 at 13:45

Thank you very much for the excellent answer! – Fawzy Hegab Oct 26 '15 at 8:35

2 So the modular law $x \leq b \Rightarrow x \vee (a \wedge b) = (x \vee a) \wedge b$ should better be written as $a \leq b \Rightarrow a \vee (x \wedge b) = (a \vee x) \wedge b$. Finally this makes sense to me. – Martin Brandenburg Feb 7 '16 at 16:33

I'd like to complete Andreas's answer by the following consequence of the modularity property:

There is a natural isomorphism between $[a, a \vee b]$ and $[a \wedge b, b]$. In group or module theory, this is the natural isomorphism $(A + B)/A = B/(A \cap B)$. Any argument that uses this (2nd? 3rd? 1st?) isomorphism theorem applies as well to the lattice of submodules, normal subgroups, etc., and more generally any modular lattice.

answered Oct 17 '14 at 17:37
user46358
201 ■ 2 ▲ 3

How to conclude $(A + B)/A = B/(A \cap B)$ from $[a, a \vee b] \cong [a \wedge b, b]$? – user153312 Dec 6 '15 at 14:43

It looks like a formal analogy: replace interval by / , ∨ by + and ∧ by ∩ – Noix07 Mar 25 '16 at 11:35

Let H be a Hilbert space, write $L(H)$ for the poset of *closed* subspaces of H. Note that $L(H)$ is a lattice: the meet is the intesection of subspaces, the join is the closure of the sum of subspaces.

Then H is finite-dimensional if and only if $L(H)$ is modular.

edited Dec 4 '16 at 10:44
Cloudscape
1,942 ■ 7 ▲ 22

answered Oct 27 '16 at 19:05
Gejza Jenča
1,093 ■ 8 ▲ 20