3-8 Cool? We are APSP Algorithms.

Hengfeng Wei

hfwei@nju.edu.cn

November 19, 2018

Please Help Me Out Here.

Definition (Shortest Path)

$$G = (V, E, w): \text{ weighted digraph}$$

$$\delta(u, v) = \begin{cases} \min \left\{ w(p) : u \overset{p}{\leadsto} v \right\} & \text{if } u \leadsto v \\ \infty & \text{o.w.} \end{cases}$$

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Digraph *vs.* Undirected Graph

Single Source Digraph

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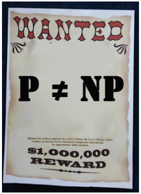

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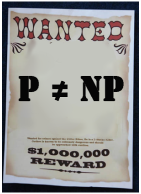

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Single-source $s \rightsquigarrow$ Single-target t

Shortest Path Algorithms

Luis Goddyn, Math 408

Given an edge weighted graph (G, d), $d : E(G) \to \mathbb{Q}$ and two vertices $s, t \in V(G)$, the *Shortest Path Problem* is to find an s,t-path P whose total weight is as small as possible. Here, G may be either directed or undirected. A path in a graph is a sequence $v_0 e_1, v_1, \ldots, v_k$ of vertices and edges such that no vertex or edge appears twice, and e_i joins v_{i-1} to v_i. If G is directed, then e_i should be oriented from v_{i-1} to v_i.

Minimum-weight Perfect Matching

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And Errors.

INTERESTED?

Let's talk.

Robert W. Floyd (1936–1901)

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the theory of parsing, the semantics of programming languages, automatic program verification, automatic program synthesis, and analysis of algorithms

— *Turing Award*, 1978

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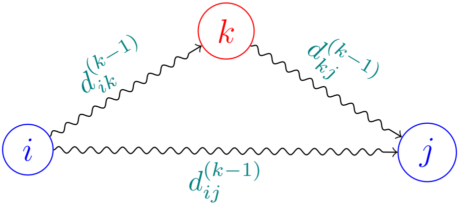
$$\boxed{D^{(n)} \triangleq \left(d_{ij}^{(n)}\right)}$$

$$k \in \mathrm{SP}_{ij}^{(k)}?$$



$$d_{ij}^{(k)} = \begin{cases} w_{ij} & k = 0 \\ \min\left\{ d_{ij}^{(k-1)}, \underbrace{d_{ik}^{(k-1)} + d_{kj}^{(k-1)}} \right\} & k \geq 1 \end{cases}$$

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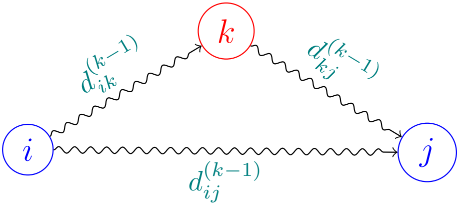

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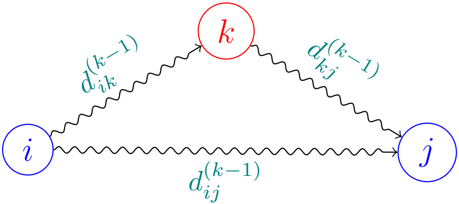

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$\cdots$, but we assume that there are no negative-weight cycles.

— Section 25.2 of CLRS

procedure FLOYD-WARSHALL(W)

$D^{(0)} = W$

for $k \leftarrow 1$ **to** n **do**

$D^{(k)} \triangleq \left(d_{ij}^{(k)}\right) \leftarrow$ a new $n \times n$ matrix

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$$\text{Space} : \Theta(n^3) \implies \Theta(n^2)$$

Floyd-Warshall Made Simple (Problem 25.2-4)

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"Decrease" does no harm to the correctness.

Negative-weight Cycle Detection (Problem 25.2-6)

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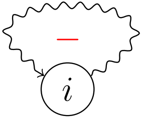

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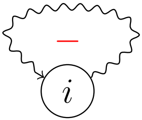

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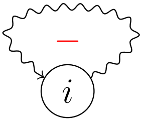

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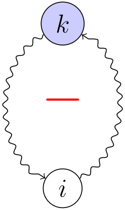

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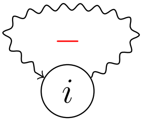

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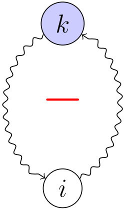

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$$
\begin{aligned}
L^{(1)} &= \langle w_{sv} \rangle_{v \in V} \\
L^{(2)} &= L^{(1)} \cdot W \\
\cdots &= \cdots \\
L^{(n-1)} &= \Big(\Big(\Big(L^{(1)} \cdot W \Big) \cdot W \Big) \cdot \ldots \cdot W \Big)
\end{aligned}
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$$l_v^{(m)} = \min_{1 \le u \le n} \left\{ l_u^{(i-1)} + w_{uv} \right\}, \quad m \ge 2$$

$$d_v^{(i)} = \min_{u \to v} \left\{ d_u^{(i-1)} + w_{uv} \right\}, \quad i \ge 1$$

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$$d_v^{(0)} = \begin{cases} 0 & v = s \\ \infty & v \ne s \end{cases}$$

procedure BELLMAN-FORD-DP(G, w, s)

    $d[0, s] \leftarrow 0$
    for $(v \neq s) \in V$ **do**
        $d[0, v] \leftarrow \infty$

    for $i \leftarrow 1$ **to** $|V| - 1$ **do**
        for $v \in V$ **do**
            $d[i, v] = \infty$
            for $(u, v) \in E$ **do**
                if $d[i, v] > d[i-1, u] + w(u, v)$ **then** $\triangleright$
                    $d[i, v] = d[i-1, u] + w(u, v)$

```
procedure Bellman-Ford-DP(G, w, s)
    d[0, s] ← 0
    for (v ≠ s) ∈ V do
        d[0, v] ← ∞

    for i ← 1 to |V| − 1 do
        for v ∈ V do
            d[i, v] = ∞
            for (u, v) ∈ E do
                if d[i, v] > d[i − 1, u] + w(u, v) then        ▷ Simplify?
                    d[i, v] = d[i − 1, u] + w(u, v)
```

procedure BELLMAN-FORD-DP-SIMPLIFIED(G, w, s)
    $d[s] \leftarrow 0$
    for $(v \neq s) \in V$ **do**
        $d[v] \leftarrow \infty$

    for $i \leftarrow 1$ **to** $|V| - 1$ **do**
        for $v \in V$ **do**
            for $(u, v) \in E$ **do**
                if $d[v] > d[u] + w(u, v)$ **then** $\triangleright$
                    $d[v] = d[u] + w(u, v)$

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        for $v \in V$ **do**
            for $(u, v) \in E$ **do**
                if $d[v] > d[u] + w(u, v)$ **then** $\triangleright$ Relax!
                    $d[v] = d[u] + w(u, v)$

```
procedure BELLMAN-FORD-DP-SIMPLIFIED(G, w, s)
    d[s] ← 0
    for (v ≠ s) ∈ V do
        d[v] ← ∞

    for i ← 1 to |V| − 1 do
        for v ∈ V do
            for (u, v) ∈ E do
                if d[v] > d[u] + w(u, v) then                ▷ Relax!
                    d[v] = d[u] + w(u, v)
```

```
procedure BELLMAN-FORD-WITHOUT-NE(G, w, s)
    INIT-SINGLE-SOURCE(G, s)

    for i ← 1 to |V| − 1 do
        for (u, v) ∈ E do
            RELAX(u, v, w)
```

Bellman-Ford: $L \cdot W$

SLOW-ALL-PAIRS-SHORTEST-PATHS: $W \cdot W$

Bellman-Ford: $L \cdot W$

SLOW-ALL-PAIRS-SHORTEST-PATHS: $W \cdot W$

SLOW-ALL-PAIRS-SHORTEST-PATHS is n instances of Bellman-Ford, one for each source.

SSSP as a Product of Matrices and a Vector (Problem 25.1-5)

To express SSSP as a product of matrices and a vector.

$$L^{(n-1)} = \left(\left(\left(L^{(1)} \cdot W \right) \cdot W \right) \cdot \ldots \cdot W \right)$$

SSSP as a Product of Matrices and a Vector (Problem 25.1-5)

To express SSSP as a product of matrices and a vector.

$$L^{(n-1)} = \left(\left(\left(L^{(1)} \cdot W\right) \cdot W\right) \cdot \ldots \cdot W\right)$$

Q : Associative? Repeated Squaring?

Negative-weight Cycle Detection (Problem 25.1-9)

To detect negative-weight cycle (NC) using FASTER-ALL-PAIRS-SHORTEST-PATHS.

Minimum-length Negative-weight Cycle (Problem 25.1-10)

To find the length of a minimum-length negative-weight cycle (NC).


THANK YOU

Your opinon Matters

Office 302

Mailbox: H016

hfwei@nju.edu.cn