Proof that the set of all possible curves is of cardinality $\aleph_2$?

I am reading George Gamow's book *One Two Three... Infinity* and a certain assertion Gamow makes seems a little startling considering what I already know of the subject of infinite cardinality.

> But the number of all geometrical points, though larger than the number of integer and fractional numbers, is not the largest one known to mathematicians. In fact it was found the *variety of all possible curves, including those of the most unusual shapes, has a larger membership than the collection of all geometric points, and thus has to be described by the third number in the infinite sequence.* [Gamow, 22]

He later does explicitly say the number of all possible curves is $\aleph_2$. But nowhere else can I find reference to this being true, nor does he provide any sort of reasoning to its validity.

My question is how can you prove the set of all possible curves is of cardinality $\aleph_2$ and not $\aleph_1$ or $\aleph_0$?

(set-theory) (cardinals) (infinity)

edited Dec 31 '16 at 4:33
bof
41.6k ● 3 ■ 43 ▲ 103

asked Dec 31 '16 at 3:50
Cxyn
68 ▲ 9

2 It would be clearer what sort of answer you're looking for if you elaborated on "what [you] already know of the subject of infinite cardinality" and what exactly you find startling about the statement. (By the way, the statement is incorrect, so you are right to be skeptical.) – Eric Wofsey Dec 31 '16 at 3:57

3 The short answer is that Gamow was a physicist, not a mathematician or set theorist, and he didn't know what he was talking about in that passage. By the way, I think it would help if you provided a bit more context, by quoting the part just before, where Gamow says that $\aleph_1$ is the number of geometrical points. – bof Dec 31 '16 at 4:07

3 The most charitable explanation I can think of is that Gamow was assuming the generalized continuum hypothesis, so that $\aleph_2 = \beth_2 = 2^{2^{\aleph_0}}$, and that he considered the graph of an ***arbitrary*** function $f: \mathbb{R} \to \mathbb{R}$ to be one of his "possible curves, including those of the most unusual shapes". Of course, if you assume $2^{\aleph_0} = \aleph_2$, then $\aleph_2$ ***is*** the cardinality of the set of ***continuous*** curves in the plane, but I'm sure that isn't what Gamow had in mind. – bof Dec 31 '16 at 4:12

2 Answers

This assertion is incorrect and even if corrected is misleading. Here is what I'm guessing Gamow had in mind. You can describe a (parametrized) "curve" in the plane as a function $f: \mathbb{R} \to \mathbb{R}^2$. The sets $\mathbb{R}$ and $\mathbb{R}^2$ each have cardinality $2^{\aleph_0}$, so the set of all such functions has cardinality

$$(2^{\aleph_0})^{2^{\aleph_0}} = 2^{\aleph_0 \cdot 2^{\aleph_0}} = 2^{2^{\aleph_0}}.$$

By Cantor's theorem, this is strictly larger than the cardinality of the plane itself, which is $2^{\aleph_0}$.

However, it is not correct that $2^{2^{\aleph_0}} = \aleph_2$. The standard axioms of set theory actually are not strong enough to determine whether $2^{2^{\aleph_0}} = \aleph_2$: it might be true, or $2^{2^{\aleph_0}}$ could be larger than $\aleph_2$ (possibly much much much larger). Relatedly, it is not correct to say that $2^{\aleph_0} = \aleph_1$ either: it is possible that this is true, but you cannot prove that it is true from the standard axioms.

The statement is further misleading because it is not particularly reasonable to refer to an arbitrary function $\mathbb{R} \to \mathbb{R}^2$ as a "curve". Indeed, by this definition a "curve" is (roughly speaking) just a completely arbitrary collection of points in the plane. Normally when mathematicians use the word "curve", they mean something much more restricted, such as a *continuous* function $\mathbb{R} \to \mathbb{R}^2$. And in fact, there are only $2^{\aleph_0}$ such continuous functions, so with that definition is is not true that there are more curves than there are points in the plane.

answered Dec 31 '16 at 4:16
Eric Wofsey
125k ● 7 ■ 142 ▲ 246

It seems a bit harsh to call $2^{2^{\aleph_0}} = \aleph_2$ "not correct". Unsurprisingly, Gamow did not discuss set-theoretic axioms in his book. In 1947 I don't think it was established consensus even among mathematicians that ZFC was "***the*** standard axioms" of set theory; it would be perfectly reasonable then (as it would today) to take ZFC+GCH as your background theory, especially if you're a physicist rather than a specialist in set-theory. (On the other hand, his conception of "curves" seems remarkably expansive for a physicist.) – bof Dec 31 '16 at 4:39

Gamow seems to have a few things confused. The cardinality of the set of all curves is $2^{\aleph_0}$, which can be seen by considering the fact that the rationals are dense in the reals and curves are continuous. This is equal to the cardinality of the set of all geometric points, not greater than it. Also, $2^{\aleph_0}$ is not the same as $\aleph_1$. The statement $2^{\aleph_0} = \aleph_1$ (the continuum hypothesis) is not provable in ZFC.

answered Dec 31 '16 at 4:16

eyeballfrog
4,039 ■ 4 ▲ 20