2-3 Counting

魏恒峰

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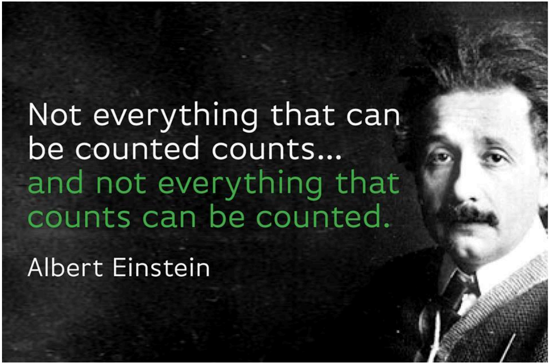
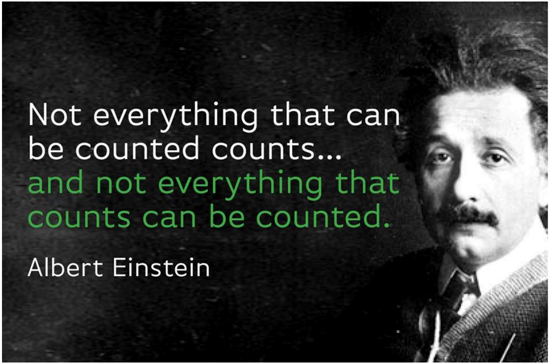

Not everything that can
be counted counts...
and not everything that
counts can be counted.
Albert Einstein



所以，学好“2-3 组合与计数”是多么重要!

Paring up (CS : $1.2 - 15$)

A tennis club has $2n$ members. We want to pair up the members by twos for singles matches.

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(b) Suppose that we also determine who serves first for each pairing. In how many ways can we specify our pairs?

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$$\frac{(2n)!}{2^n \cdot n!} \cdot 2^n = \frac{(2n)!}{n!}$$

Passing out Apples to Children

k-Permutation $(\text{CS} : 1.2 - 5)$

We need to pass out k distinct apples (pieces of fruit) to n children such that *each child may get at most one apple*.

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(b) What if $k > n$?

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$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

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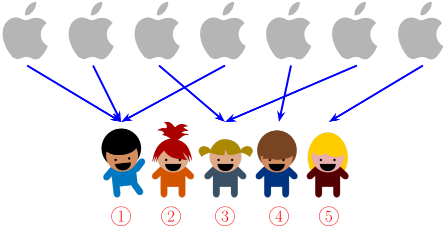

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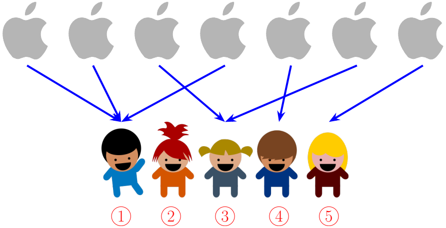

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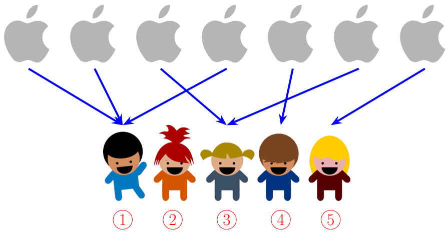

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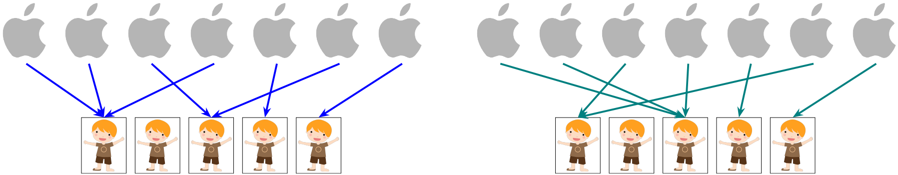

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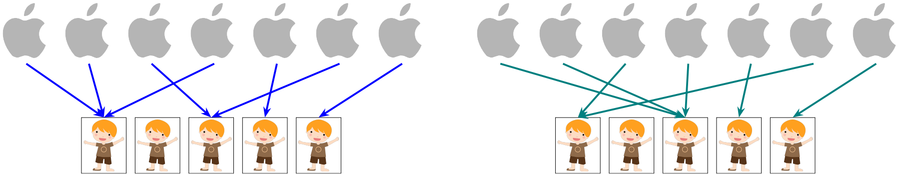

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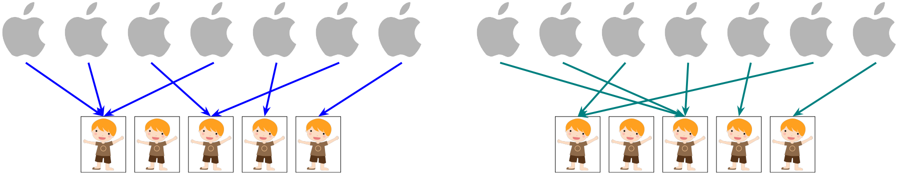

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Theorem ( )

$$p(k) \triangleq \sum_{x=1}^{x=n} p_x(k) \sim \frac{1}{4\sqrt{3}n} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

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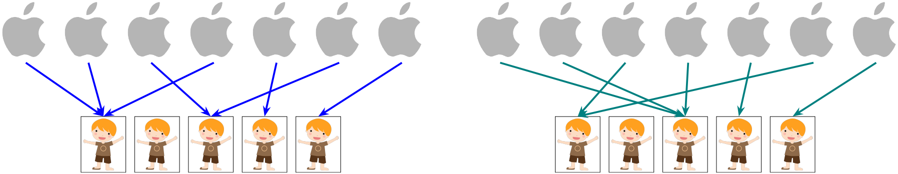

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Theorem (Recurrence for $p_n(k)$)

$$p_n(k) = p_{n-1}(k-1) + p_n(k-n)$$

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Proof.

$$1 \le x_1 \le x_2 \le \cdots \le x_n$$

Case $x_1 > 1$

$$1 < x_1 \le x_2 \le \cdots \le x_n$$

Case $x_1 = 1$

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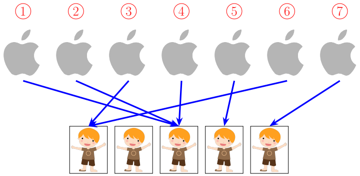

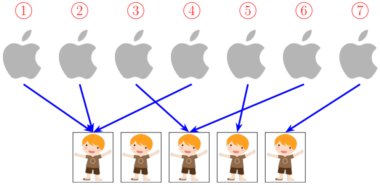

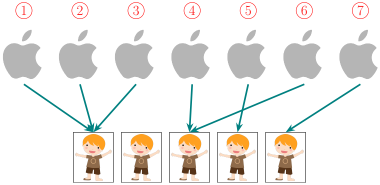

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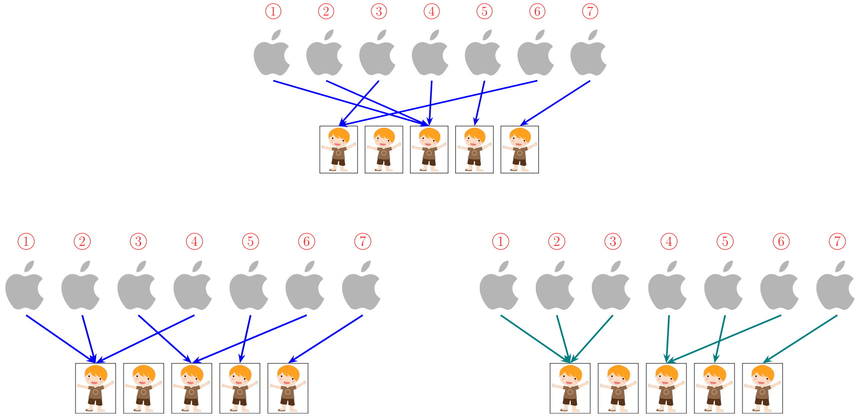

Set partition of $[1 \cdots k]$ into $\le n$ parts

Set Partition (CS : $1.5 - 12$)

$$S(n,k) \ (\left\{ {n \atop k} \right\}) : \# \text{ of set partitions of } [1 \cdots n] \text{ into } k \text{ classes}$$

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$$S(0,0) = 1, \quad S(n,0) = S(0,n) = 0 \; (n > 0)$$

$$S(n,k) = S(n-1,k-1) + kS(n-1,k), \quad n > 0, k > 0$$

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Proof.

$$S(n,k) = \underbrace{S(n-1,k-1)}_{n \text{ is alone}} + \underbrace{kS(n-1,k)}_{n \text{ is not alone}}$$

□

Bell number: $B_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}$

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Theorem (de Bruijn (1981))

As $n \to \infty$,

$$\frac{\ln B_n}{n} = \ln n - \ln\ln n - 1 + \frac{\ln\ln n}{\ln n} + \frac{1}{\ln n} + \frac{1}{2}\left(\frac{\ln\ln n}{\ln n}\right)^2 + O\left(\frac{\ln\ln n}{(\ln n)^2}\right)$$

Computing $\binom{n}{k}$ (CS $1.5:14$)

```
procedure BINOM(n, k)                          ▷ Required: n ≥ k ≥ 0
    if k = 0 ∨ n = k then
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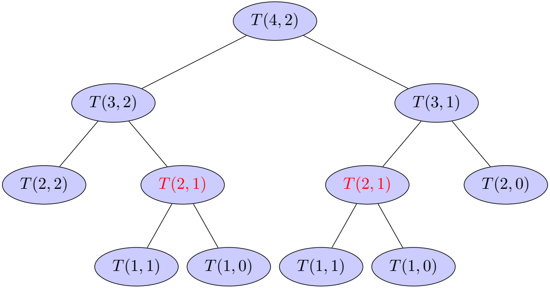

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$$\alpha\binom{n}{0} - c = 0, \quad \alpha\binom{n}{n} - c = 0 \implies \alpha = c$$

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$$\boxed{T(n,k) = c\binom{n}{k} - c}$$

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

$$
\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}
$$

Q : How to calculate $\binom{5}{3}$?

```
procedure BINOM(n, k)                                    ▷ Required: n ≥ k ≥ 0
    for i ← 0 to n − k do
        B[i][0] ← 1
    for i ← 1 to k do
        B[i][i] ← 1
    for j ← 1 to k do
        for d ← 1 to n − k do
            i ← j + d
            B[i][j] ← B[i − 1][j] + B[i − 1][j − 1]
    return B[n][k]
```

```
procedure BINOM(n, k)                         ▷ Required: n ≥ k ≥ 0
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    return B[n][k]
```

$$(n-k+1)+(k)+k(n-k)=nk-k^2+n+1$$


Thank You!