1-9 关系及其基本性质

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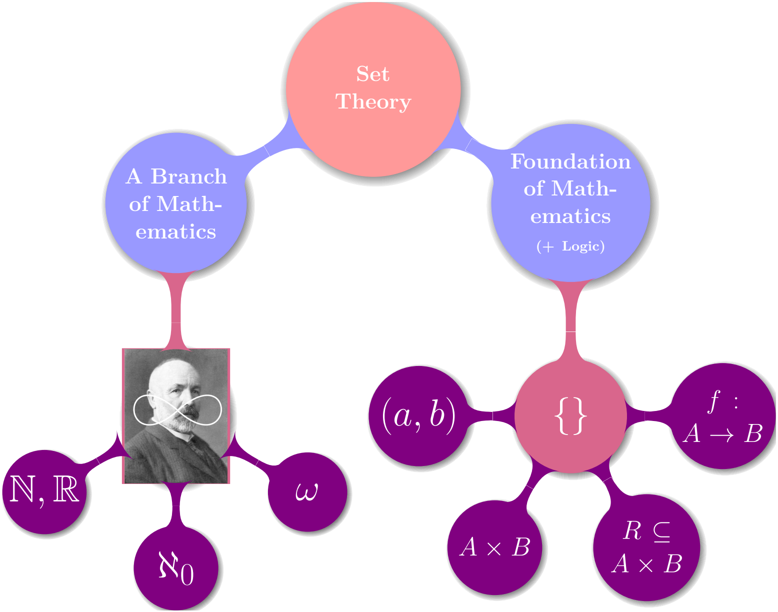
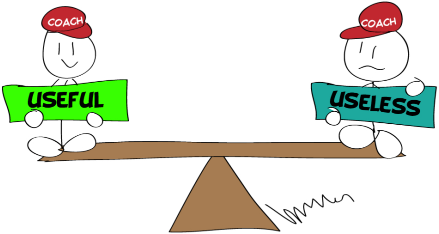
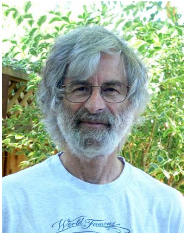


Set Theory
A Branch of Math-ematics
Foundation of Math-ematics
(+ Logic)
$\mathbb{N}, \mathbb{R}$
$\aleph_0$
ω
(a, b)
$\{\}$
$f : A \to B$
$A \times B$
$R \subseteq A \times B$


COACH
USEFUL
COACH
USELESS



Time, Clocks, and the Ordering of Events in a Distributed System

Leslie Lamport
Massachusetts Computer Associates, Inc.

The concept of one event happening before another in a distributed system is examined, and is shown to define a partial ordering of the events. A distributed algorithm is given for synchronizing a system of logical clocks which can be used to totally order the events.

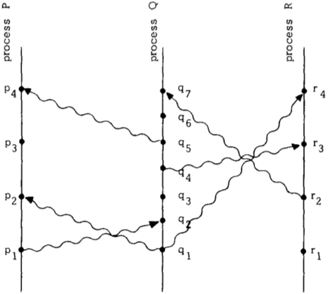


Figure 13. A selection of consistency axioms over an execution $(E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar})$

Auxiliary relations

$\mathsf{sameobj}(e, f) \iff \mathsf{obj}(e) = \mathsf{obj}(f)$

Per-object causality (aka happens-before) order:
$\mathsf{hbo} = ((\mathsf{ro} \cap \mathsf{sameobj}) \cup \mathsf{vis})^+$

Causality (aka happens-before) order: $\mathsf{hb} = (\mathsf{ro} \cup \mathsf{vis})^+$

Axioms

EVENTUAL:
$\forall e \in E.\, \neg(\exists \text{ infinitely many } f \in E.\, \mathsf{sameobj}(e, f) \wedge \neg(e \xrightarrow{\mathsf{vis}} f))$

THINAIR: $\mathsf{ro} \cup \mathsf{vis}$ is acyclic

POCV (Per-Object Causal Visibility): $\mathsf{hbo} \subseteq \mathsf{vis}$

POCA (Per-Object Causal Arbitration): $\mathsf{hbo} \subseteq \mathsf{ar}$

COCV (Cross-Object Causal Visibility): $(\mathsf{hb} \cap \mathsf{sameobj}) \subseteq \mathsf{vis}$

COCA (Cross-Object Causal Arbitration): $\mathsf{hb} \cup \mathsf{ar}$ is acyclic

Figure 17. Optimized state-based multi-value register and its simulation

$$\begin{aligned}
\Sigma &= \mathsf{ReplicaID} \times \mathcal{P}(\mathbb{Z} \times (\mathsf{ReplicaID} \to \mathbb{N}_0))\\
\vec{\sigma}_0 &= \langle r, \emptyset\rangle\\
M &= \mathcal{P}(\mathbb{Z} \times (\mathsf{ReplicaID} \to \mathbb{N}_0))\\
\mathsf{do}(\mathtt{wr}(a), \langle r, V\rangle, t) &=\\
&(\langle r, \{(a, (\lambda s.\, \text{if } s \neq r \text{ then } \max\{v(s) \mid (_, v) \in V\}\\
&\quad \text{else } \max\{v(s) \mid (_, v) \in V\} + 1))\}\rangle, \bot)\\
\mathsf{do}(\mathtt{rd}, \langle r, V\rangle, t) &= (\langle r, V\rangle, \{a \mid (a, _) \in V\})\\
\mathsf{send}(\langle r, V\rangle) &= (\langle r, V\rangle, V)\\
\mathsf{receive}(\langle r, V\rangle, V') &= \langle r, \{(a, v) \in V'' \mid\\
&\quad v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'.\, (a', v') \in V'' \wedge a \neq a'\}\}\rangle,
\end{aligned}$$

where $V'' = \{(a, \bigsqcup\{v' \mid (a, v') \in V \cup V'\}) \mid (a, \_) \in V \cup V'\}$

$\langle s, V\rangle\ [\mathcal{R}_r]\ I \iff (r = s) \wedge (V\ [\mathcal{M}]\ I)$

$$\begin{aligned}
&V\ [\mathcal{M}]\ ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar}), \mathsf{info}) \iff\\
&(\forall (a, v), (a', v') \in V.\, (a = a' \implies v = v')) \wedge\\
&(\forall (a, v) \in V.\, \exists s.\, v(s) > 0) \wedge\\
&(\forall (a, v) \in V.\, v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'.\, (a', v') \in V \wedge a \neq a'\}) \wedge\\
&\exists \text{ distinct } e_{s,k}.\\
&(\{e \in E \mid \exists a.\, \mathsf{oper}(e) = \mathtt{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicaID} \wedge\\
&\quad 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V\}\}) \wedge\\
&(\forall s, j, k.\, (\mathsf{repl}(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}} e_{s,k} \iff j < k)) \wedge\\
&(\forall (a, v) \in V.\, \forall q.\, \{j \mid \mathsf{oper}(e_{q,j}) = \mathtt{wr}(a)\} \cup\\
&\quad \{j \mid \exists s, k.\, e_{q,j} \xrightarrow{\mathsf{vis}} e_{s,k} \wedge \mathsf{oper}(e_{s,k}) = \mathtt{wr}(a)\} =\\
&\quad \{j \mid 1 \leq j \leq v(q)\}) \wedge\\
&(\forall e \in E.\, (\mathsf{oper}(e) = \mathtt{wr}(a) \wedge\\
&\quad \neg\exists f \in E.\, \mathsf{oper}(f) = \mathtt{wr}(_) \wedge e \xrightarrow{\mathsf{vis}} f) \implies (a, _) \in V)
\end{aligned}$$

the former. The only non-trivial obligation is to show that if

$$V\ [\mathcal{M}]\ ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}), \mathsf{info}),$$

then

$$\begin{aligned}
\{a \mid (a, _) \in V\} \subseteq \{a \mid \exists e \in E.\, &\mathsf{oper}(e) = \mathtt{wr}(a) \wedge\\
&\neg\exists f \in E.\, \exists a'.\, \mathsf{oper}(e) = \mathtt{wr}(a') \wedge e \xrightarrow{\mathsf{vis}} f\}
\end{aligned} \tag{13}$$

(the reverse inclusion is straightforwardly implied by $\mathcal{R}_r$).

Take $(a, v) \in V$. We have $\forall (a, v) \in V.\, \exists s.\, v(s) > 0$,

$$v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'.\, (a', v') \in V \wedge a \neq a'\}$$

and

$$\begin{aligned}
\forall (a, v) \in V.\, \forall q.\, &\{j \mid \mathsf{oper}(e_{q,j}) = \mathtt{wr}(a)\} \cup\\
&\{j \mid \exists s, k.\, e_{q,j} \xrightarrow{\mathsf{vis}} e_{s,k} \wedge \mathsf{oper}(e_{s,k}) = \mathtt{wr}(a)\} =\\
&\{j \mid 1 \leq j \leq v(q)\}.
\end{aligned}$$

From this we get that for some $e \in E$

$$\mathsf{oper}(e) = \mathtt{wr}(a) \wedge \neg\exists f \in E.\, \exists a'.\, a' \neq a \wedge \mathsf{oper}(e) = \mathtt{wr}(a') \wedge e \xrightarrow{\mathsf{vis}} f.$$

Since vis is acyclic, this implies that for some $e' \in E$

$$\mathsf{oper}(e') = \mathtt{wr}(a) \wedge \neg\exists f \in E.\, \mathsf{oper}(e') = \mathtt{wr}(\_) \wedge e' \xrightarrow{\mathsf{vis}} f,$$

which establishes (13).

Let us now discharge RECEIVE. Let $\mathsf{receive}(\langle r, V\rangle, V') = \langle r, V'''\rangle$, where

$$\begin{aligned}
V'' &= \{(a, \bigsqcup\{v' \mid (a, v') \in V \cup V'\}) \mid (a, _) \in V \cup V'\};\\
V''' &= \{(a, v) \in V'' \mid v \not\sqsubseteq \bigsqcup\{(a', v') \in V'' \mid a \neq a'\}\}.
\end{aligned}$$

Assume $\langle r, V\rangle\ [\mathcal{R}_r]\ I$, $V'\ [\mathcal{M}]\ J$ and

$$\begin{aligned}
I &= ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar}), \mathsf{info});\\
J &= ((E', \mathsf{repl}', \mathsf{obj}', \mathsf{oper}', \mathsf{rval}', \mathsf{ro}', \mathsf{vis}', \mathsf{ar}'), \mathsf{info}');\\
I \sqcup J &= ((E'', \mathsf{repl}'', \mathsf{obj}'', \mathsf{oper}'', \mathsf{rval}'', \mathsf{ro}'', \mathsf{vis}'', \mathsf{ar}''), \mathsf{info}'').
\end{aligned}$$

By agree we have $I \sqcup J \in \mathsf{IEx}$. Then

$$\begin{aligned}
&(\forall (a, v), (a', v') \in V.\, (a = a' \implies v = v')) \wedge\\
&(\forall (a, v) \in V.\, \exists s.\, v(s) > 0) \wedge\\
&(\forall (a, v) \in V.\, v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'.\, (a', v') \in V \wedge a \neq a'\}) \wedge\\
&\exists \text{ distinct } e_{s,k}.\\
&(\{e \in E \mid \exists a.\, \mathsf{oper}''(e) = \mathtt{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicaID} \wedge\\
&\quad 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V\}\}) \wedge\\
&(\forall s, j, k.\, (\mathsf{repl}''(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}} e_{s,k} \iff j < k)) \wedge\\
&(\forall (a, v) \in V.\, \forall q.\, \{j \mid \mathsf{oper}''(e_{q,j}) = \mathtt{wr}(a)\} \cup\\
&\quad \{j \mid \exists s, k.\, e_{q,j} \xrightarrow{\mathsf{vis}} e_{s,k} \wedge \mathsf{oper}''(e_{s,k}) = \mathtt{wr}(a)\} =\\
&\quad \{j \mid 1 \leq j \leq v(q)\}) \wedge\\
&(\forall e \in E.\, (\mathsf{oper}''(e) = \mathtt{wr}(a) \wedge\\
&\quad \neg\exists f \in E.\, \mathsf{oper}''(f) = \mathtt{wr}(_) \wedge e \xrightarrow{\mathsf{vis}} f) \implies (a, _) \in V)
\end{aligned}$$

and

$$\begin{aligned}
&(\forall (a, v), (a', v') \in V'.\, (a = a' \implies v = v')) \wedge\\
&(\forall (a, v) \in V'.\, \exists s.\, v(s) > 0) \wedge\\
&(\forall (a, v) \in V'.\, v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'.\, (a', v') \in V' \wedge a \neq a'\}) \wedge\\
&\exists \text{ distinct } e_{s,k}.\\
&(\{e \in E' \mid \exists a.\, \mathsf{oper}''(e) = \mathtt{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicaID} \wedge\\
&\quad 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V'\}\}) \wedge\\
&(\forall s, j, k.\, (\mathsf{repl}''(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}'} e_{s,k} \iff j < k)) \wedge\\
&(\forall (a, v) \in V'.\, \forall q.\, \{j \mid \mathsf{oper}''(e_{q,j}) = \mathtt{wr}(a)\} \cup\\
&\quad \{j \mid \exists s, k.\, e_{q,j} \xrightarrow{\mathsf{vis}'} e_{s,k} \wedge \mathsf{oper}''(e_{s,k}) = \mathtt{wr}(a)\} =\\
&\quad \{j \mid 1 \leq j \leq v(q)\}) \wedge\\
&(\forall e \in E'.\, (\mathsf{oper}''(e) = \mathtt{wr}(a) \wedge\\
&\quad \neg\exists f \in E'.\, \mathsf{oper}''(f) = \mathtt{wr}(_) \wedge e \xrightarrow{\mathsf{vis}'} f) \implies (a, _) \in V').
\end{aligned}$$

The agree property also implies

$$\begin{aligned}
\forall s, k.\, 1 \leq k \leq \min\{&\max\{v(s) \mid \exists a.\, (a, v) \in V\},\\
&\max\{v(s) \mid \exists a.\, (a, v) \in V'\}\} \implies e_{s,k} = e'_{s,k}.
\end{aligned}$$

Hence, there exist distinct

$$e''_{s,k} \text{ for } s \in \mathsf{ReplicaID},\ k = 1..(\max\{v(s) \mid \exists a.\, (a, v) \in V'''\}),$$

such that

$$\begin{aligned}
&(\forall s, k.\, 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V\} \implies e''_{s,k} = e_{s,k}) \wedge\\
&(\forall s, k.\, 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V'\} \implies e''_{s,k} = e'_{s,k})
\end{aligned}$$

and

$$\begin{aligned}
&(\{e \in E \cup E' \mid \exists a.\, \mathsf{oper}''(e) = \mathtt{wr}(a)\} =\\
&\{e''_{s,k} \mid s \in \mathsf{ReplicaID} \wedge 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V'''\}\})\\
&\wedge (\forall s, j, k.\, (\mathsf{repl}(e''_{s,k}) = s) \wedge (e''_{s,j} \xrightarrow{\mathsf{ro}''} e''_{s,k} \iff j < k)).
\end{aligned}$$

By the definition of V'' and V''' we have

$$\forall (a, v), (a', v') \in V'''.\, (a = a' \implies v = v').$$

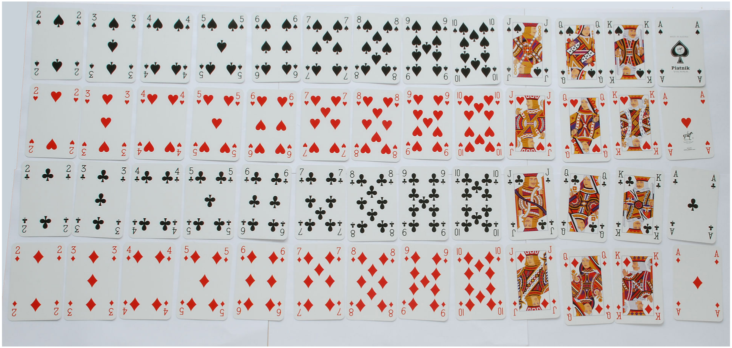
We also straightforwardly get

$$\forall (a, v) \in V'.\, \exists s.\, v(s) > 0$$

and

$$\begin{aligned}
(\forall (a, v) \in V''.\, \forall q.\, &\{j \mid \mathsf{oper}''(e''_{q,j}) = \mathtt{wr}(a)\} \cup\\
&\{j \mid \exists s, k.\, e''_{q,j} \xrightarrow{\mathsf{vis}''} e''_{s,k} \wedge \mathsf{oper}''(e''_{s,k}) = \mathtt{wr}(a)\} =\\
&\{j \mid 1 \leq j \leq v(q)\}).
\end{aligned} \tag{14}$$

Ordered Pair and Cartesian Product



Definitions of (a, b) and $A \times B$ (UD 9.16)

$$(a, b) = \{\{a\}, \{a, b\}\}$$

$$(a, b) = (x, y) \iff a = x \land b = y$$

$$\boxed{\{\{a\}, \{a, b\}\} = \{\{x\}, \{x, y\}\} \implies a = x \land b = y}$$

Q : What is wrong with this proof?

$$(1) \begin{cases} \{a\} & = \{x\} \\ \{a, b\} & = \{x, y\} \end{cases} \implies \begin{cases} a = x \\ b = y \end{cases}$$

$$(2) \begin{cases} \{a\} & = \{x, y\} \\ \{a, b\} & = \{x\} \end{cases} \implies \text{no solution.}$$

Definitions of (a, b) and $A \times B$ (UD 9.16)

$$(a, b) = \{\{a\}, \{a, b\}\}$$

$$(a, b) = (x, y) \iff a = x \land b = y$$

$$\boxed{\{\{a\}, \{a, b\}\} = \{\{x\}, \{x, y\}\} \implies a = x \land b = y}$$

Proof.

CASE $a = b$

$$(a, a) = \{\{a\}\}$$

CASE $a \neq b$

$$\{a\} = \{x\} \quad \{a, b\} = \{x, y\}$$

$\square$

Definitions of (a, b) and $A \times B$ (UD 9.16)

$$(a, b) = \{\{a\}, \{a, b\}\}$$

$$a \in A \land b \in B \implies (a, b) \in \mathcal{P}(\mathcal{P}(A \cup B))$$

$$A \times B = \{x \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid \exists a \in A \, \exists b \in B : x = (a, b)\}$$

$$A \subseteq C \land B \subseteq D \implies A \times B \subseteq C \times D$$

Cartesian Product and "$\subseteq$" (UD 9.13)

$$A \times B \subseteq C \times D \overset{?}{\Longrightarrow} A \subseteq C \land B \subseteq D$$

Disproof.

$$(x, y) \in A \times B \implies (x, y) \in C \times D$$
$$x \in A \land y \in B \implies x \in C \land y \times D$$
$$(x \in A \implies x \in C) \land (y \in B \implies y \in D)$$
$$(A \subseteq C) \land (B \subseteq D)$$

$$A = \emptyset \lor B = \emptyset$$

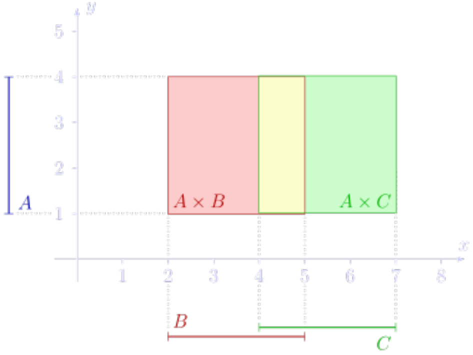
□

$$A \times B \subseteq C \times D \xrightarrow{A,B \neq \emptyset} A \subseteq C \land B \subseteq D$$

By contradiction.

Distributive Laws (UD 9.14)

$$
\begin{aligned}
A \times (B \cup C) &= (A \times B) \cup (A \times C) \\
A \times (B \cap C) &= (A \times B) \cap (A \times C) \\
A \times (B \setminus C) &= (A \times B) \setminus (A \times C)
\end{aligned}
$$



Relation

燕小六: “帮我照顾好我七舅姥爷和我外甥女”

$$G = \{(a,b) : a \text{ 是 } b \text{ 的舅姥爷}\}$$

$$N = \{(a,b) : a \text{ 是 } b \text{ 的外甥女}\}$$

$$G \cup N$$

"B"	Brother	"F"	Father	"O"	Son
"S"	Sister	"M"	Mother	"D"	Dau.

$$G = B \circ M \circ M \qquad N = D \circ S$$

$$G = (B \circ M) \circ M = B \circ (M \circ M)$$

$$R \subseteq X \times Y$$

R is a relation **from** X **to** Y.

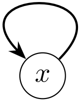
$$R \subseteq X \times X$$

R is a relation **on** X.
(over)

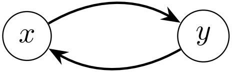
Definition (Equivalence Relation)

R is an equivalence relation on X if R is

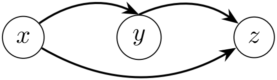
Reflexive: xRx



Symmetric: $xRy \implies yRx$



Transitive: $xRy \wedge yRz \implies xRz$



Definition (Equivalence Class)

$$(X, \sim)$$

The equivalence class of x is a *set*:

$$E_x = \{y \in X : x \sim y\} = [x]_{\sim} = [x]$$

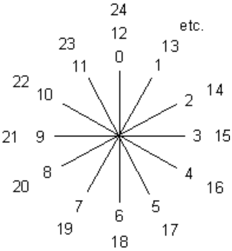
Equivalence Relation (UD 10.5)

$$(X, \sim)$$

Prove that

$$\forall x, y \in X : [x]_{\sim} = [y]_{\sim} \iff x \sim y.$$

Equivalence Relations/Classes as Abstractions



Equivalence Relations/Classes on Polynomials (UD 10.8)

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0 \quad (a_j \in \mathbb{R}, n \in \mathbb{N})$$

(a)

$$p \sim q \iff p(0) = q(0)$$

$$p(x) = x$$

(b)

$$p \sim q \iff \mathsf{deg}(p) = \mathsf{deg}(q)$$

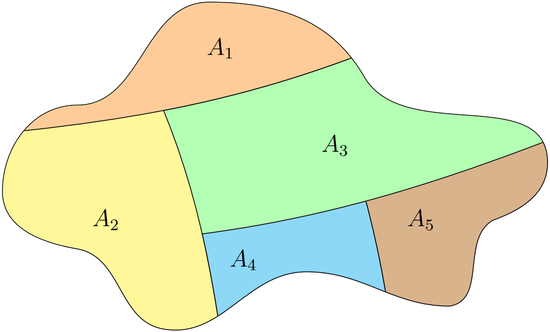
$$p(x) = 3x + 5$$

(c)

$$p \sim q \iff \mathsf{deg}(p) \le \mathsf{deg}(q)$$

$$p(x) = x^2$$

Partition



Definition (Partition)

A family of sets $\{A_\alpha : \alpha \in I\}$ is a *partition* of X if

(i)

$$\forall \alpha \in I : A_\alpha \neq \emptyset$$

$$\forall \alpha \in I\ \exists x \in X : x \in A_\alpha$$

(ii)

$$\bigcup_{\alpha \in I} A_\alpha = X$$

$$\forall x \in X\ \exists \alpha \in I : x \in A_\alpha$$

(iii)

$$\forall \alpha, \beta \in I : A_\alpha \cap A_\beta = \emptyset \vee A_\alpha = A_\beta$$

$$\forall \alpha, \beta \in I : A_\alpha \cap A_\beta \neq \emptyset \implies A_\alpha = A_\beta$$

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商丘
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周至
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襄樊
信阳
达州
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北
万州区
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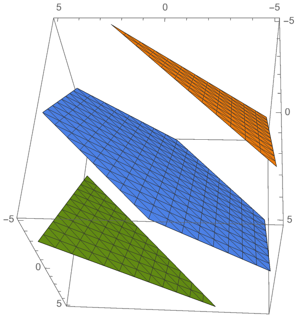
Partitions of $\mathbb{R}^3$ (UD 11.3)

Is $\{A_r : r \in \mathbb{R}\}$ a partition of $\mathbb{R}^3$?

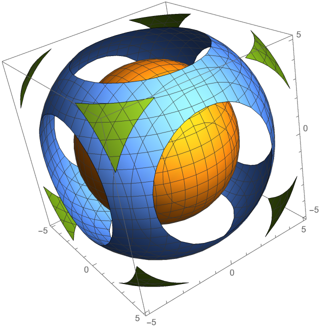
$$A_r = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = r\}$$



Partitions of $\mathbb{R}^3$ (UD 11.3)

Is $\{A_r : r \in \mathbb{R}\}$ a partition of $\mathbb{R}^3$?

$$A_r = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = r^2\}$$



Partitions of the Set of Polynomials (UD 11.7)

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0 \quad (a_j \in \mathbb{R}, n \in \mathbb{N})$$

$$\text{deg}(p = 0) = -\infty$$

(a)

$$A_m = \{p : \text{deg}(p) = m\} \quad m \in \mathbb{N}$$

(c)

$$A_q = \{p : \exists r (p = qr)\} \quad q \in P$$

$$q \in A_q$$

$$p \in A_p$$

$$p \neq q \land r = pq \implies (r \in A_q \cap A_q) \land (A_p \neq A_q)$$

Partitions of the Set of Polynomials (UD 11.7)

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0 \quad (a_j \in \mathbb{R}, n \in \mathbb{N})$$

$$\deg(p = 0) = -\infty$$

(b)

$$A_c = \{p : p(0) = c\} \quad c \in \mathbb{R}$$

(d)

$$A_c = \{p : p(c) = 0\} \quad c \in \mathbb{R}$$

$$p(x) = x^2 + 1$$

Subset and Partition (UD 11.9)

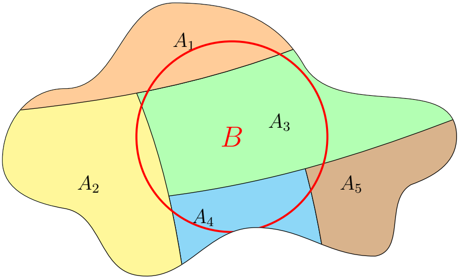
$\{A_\alpha : \alpha \in I\}$ is a partition of $X \neq \emptyset$.

(a)

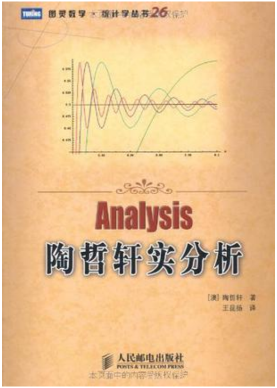
$$B \subseteq X, \quad \forall \alpha \in I : A_\alpha \cap B \neq \emptyset$$

To prove that

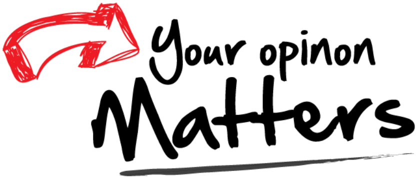
$$\{A_\alpha \cap B : \alpha \in I\} \textit{ is } \text{a partition of } B.$$



Order in the Reals




Thank
You!



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