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# Some topics in Lattice theory

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# SOME TOPICS IN LATTICE THEORY

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# CHAPTER I

## INTRODUCTION

The phenomenal development in Algebra which has occurred in recent years has been largely the result of a changed point of view toward the subject, displacement of formalism by generalization and abstraction. The most striking characteristic of this point of view is the deduction of the theoretical properties of such formal systems as groups, rings, fields, linear algebras, linear spaces, and lattices. Perhaps, from the point of application, lattices are the most general of all algebraic systems.

In order to appreciate lattice theory, one must not only be familiar with the above concepts, but should be reasonably conversant with the fundamental notions of topology, logic, and abstract algebra. A familiarity with these notions also makes lattice theory much more vivid, and enables one to make correlations between the ideas of lattice theory and ideas with which every mathematician is acquainted.

It is the purpose of this thesis to give a simple and self-contained exposition of the theory of some topics found in lattice theory, to develop some of its contacts with other mathematical systems, and to give concrete examples of special kinds of lattices. Rather than an exhaustive treatment, it is the aim of this thesis to develop the nucleus of the theory.

It is the hope that those readers who are not familiar with this relatively new field will find herein sufficient inspiration to delve into some of the deeper aspects of lattice theory.

Lattice theory may be thought of as a generalization of Boolean Algebra. [8]. It is a generalization of this system in that some but not all of the defining postulates of a Boolean Algebra are required of a lattice. For instance, the distributive laws of a Boolean Algebra are not among the postulates characterizing a lattice. Further, not every lattice need contain the "zero" and the "universal" elements of Boolean Algebra, and hence not every lattice is "complemented" in George Boole's sense. This lifting of certain restrictions obviously makes possible more concrete examples of the unrestricted system than of the restricted system - Boolean Algebra. Of course, some examples may satisfy the distributive laws, and some may satisfy the complementarity property. If a lattice satisfies the distributive laws, it is called a "distributive lattice"; if it is complemented, it is a "complemented lattice". In particular, a Boolean Algebra is a complemented, distributive lattice.

Mathematics in general employs very few undefined terms. This is especially true of lattice theory which essentially contains, besides general logical concepts and undefined elements, the single undefined relation "includes." Its properties are contained in the following definition of a partially ordered system - an intrinsic part of the definition of a lattice.

1. Partially ordered system. By a partially ordered system is meant a system X in which a relation $x \geq y$ (read x "includes" y) is defined and satisfies

P1: For all x, $x \geq x$ (Reflexive property of $\geq$)

P2: If $x \geq y$ and $y \geq x$, then $x = y$ (Anti-symmetric property of $\geq$)

P3: If $x \geq y$ and $y \geq z$, then $x \geq z$ (Transitive property of $\geq$).

We shall sometimes write $x \leq y$ to mean $y \geq x$, and $x > y$ (or $y < x$) to mean $x \geq y$ although $x \neq y$. This notation is familiar and standard.

Duality. By the "Converse" of a relation $\rho$ is meant the relation $\gamma$ such that x is in the relation $\gamma$ to y if and only if y is in the relation $\rho$ to x. This gives rise to the principle of duality in lattice theory:

Any theorem which is true in a partially ordered system remains true if the symbols $\leq$ and $\geq$ are interchanged throughout the statement of the theorem. [2; 327]. Throughout this thesis only one of two dual theorems will be proved.

2. Minimal and maximal elements. By a "least" element of a subset X of a partially ordered system P, we mean an element $a \in X$ such that $a \leq x$ for all $x \in X$; by a "minimal" element, we mean one, say a, such that $a \geq x$ for no $x \in X$. By P2 a least element is minimal, but the Converse is not true.

Dual to the notions of least and minimal elements, we may define "greatest" and "maximal" elements. A greatest element is maximal, but the converse need not be true. In fact, by P2, X can have at most one least and one greatest element, whereas it can have many minimal and maximal elements.

Theorem 1. 1: Any finite subset X of a partially ordered system has minimal and maximal numbers.

Proof: Let the elements of X be $x_1$, $x_2$, ..., $x_n$. Let $m_1 = x_1$; select $m_k$ as $x_k$ if $x_k < m_{k-1}$ and otherwise as $m_{k-1}$. Then $m_n$ will be a minimal. Dually, X will have a maximal number.

Not all partially ordered systems have a least element, and not all have a greatest element. To illustrate this, consider the set I of all integers such that if P is an element of I then P is greater than zero. The three postulates for partial ordering are easily verified if we define $a \geq b$ to mean a is a multiple of b.

$a \geq a$, that is to say every element divides itself. If $a \geq b$ and $b \geq a$, then $a = b$, for a and b are greater than zero. If $a \geq b$ and $b \geq c$, then $a \geq c$, for $a \geq b$ implies that there exists an integer $k_1$ of the set such that $a = k_1 b$ and $b \geq c$ implies the existence of an integer $k_2$ such that $b = k_2 c$. By the associative law, we have $a = k_1(k_2 c) = (k_1 k_2)c$ and therefore $a \geq c$.

There does not exist an integer p of I such that for all $x \in I$ it is true that $x \leq p$. Hence the set does not contain a greatest element.

If we consider the set I of all integers such that if P is an element of I then P is greater than 1, the set does not contain a least element.

We shall use the symbols 0 and I to denote respectively the (unique) least and greatest elements of a partially ordered system P, whenever they exist.

We shall say an element x is "between" a and b if and only if $a \geq x \geq b$ or $a \leq x \leq b$. And a subset X of a partially ordered system P will be called "convex" if and only if it contains with any a and b, every element "between" a and b.

By "a covers b", it is meant that $a > b$, while no x satisfies $a > x > b$ (that is, no x exists between a and b). [5;9].

3. <u>Definition and examples of a lattice</u>. By an upper bound to a subset X of a partially ordered system P is meant an element of P which contains every x in X. A least upper bound is an upper bound contained in every upper bound. Dually, by a lower bound to a subset X of a partially ordered system P is meant an element of P, say a, satisfying $a \leq x$ for all x in X. A greatest lower bound is a lower bound including every lower bound. Obviously, greatest lower bound and least upper bound are unique if they exist.

We may now define a "lattice" as a partially ordered system any two of whose elements x and y have a greatest lower bound or "meet", $x \cap y$, and a least upper bound or "join", $x \cup y$. [5;16]

<u>Examples</u>. (1) Let $X_i$ $(i = 1,2,\ldots)$ be sets of points in the Euclidean plane. Then $X_j \cap X_k$ is the set product of the subsets $X_j$ and $X_k$. Also $X_j \cup X_k$ is the set sum of the subsets

$X_j$ and $X_k$. (2) Consider the set of positive integers $I(1,2,3,\ldots)$. By $x_i \leq x_j$ ($x_j \in I$, $x_i \in I$) is meant $x_j$ is a multiple of $x_i$. Then $x_i \cap x_j$ will mean the greatest common divisor of $x_i$ and $x_j$ and $x_i \cup x_j$ will mean the least common multiple of $x_i$ and $x_j$. For instance, $9 \cap 6 = 3$ and $9 \cup 6 = 18$.

By a "sublattice" of a lattice L is meant a subset which contains with any two elements their join and their meet.

3. <u>Lattices as abstract algebras</u>. The operations already defined and those which follow have various algebraic properties, many of which are identical with the familiar laws of arithmetic. They are in effect paraphrases of the universally accepted principles of logic which are involved in every piece of deductive argument. [2;313]. The operations of meet and join are respectively analogous to ordinary addition and multiplication of real numbers by virtue of the following laws. We shall prove those that do not carry over directly to arithmetic.

L1: (Idempotent) $x \cap x = x$ and $x \cup x = x$

Proof: By definition of greatest lower bound $x \cap y = z$ means $x \geq z$ and $y \geq z$, and if $x \geq w$ and $y \geq w$, then $z \geq w$. By definition of least upper bound $x \cup y = z$ means $z \geq x$ and $z \geq y$, and if $w \geq x$ and $w \geq y$, the $w \geq z$. It follows that $x \cap x = x$, for $x \geq x$ by the reflexive property of $\geq$, and by the tautological property of logic $x \geq w$ implies $x \geq w$. The proof of the second part is the dual of the first.

L2: (Commutative) $x \cap y = y \cap x$ and $x \cup y = y \cup x$.

L3: (Associative) $x \cap (y \cap z) = (x \cap y) \cap z$ and $x \cup (y \cup z)$ $(x \cup y) \cup z$.

L4: (Absorbtive) $x \cap (x \cup y) = x$ and $x \cup (x \cap y) = x$.

Proof:[1] It is sufficient to show that $x \geq x \cap (x \cup y)$ and $x \cap (x \cup y) \geq x$. $x \geq x \cap (x \cup y)$ by definition of greatest lower bound. By the one sided modular law $x \cap (y \cup x) = (x \cap y) \cup x \geq x$ by the definition of least upper bound.

The reader will find easy illustrations of the above laws from Venn's diagram in figure 1. [13;8]. In this type of diagram, classes are represented by circles or other areas, and the diagram is so drawn as always to represent the universal element I, for whatever terms may be in question. Thus for two terms

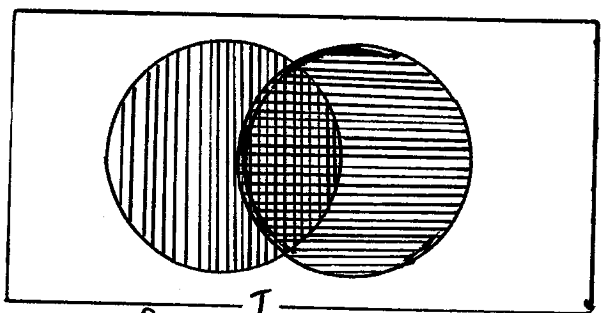


fig.I

a and b, one draws overlapping circles, as in the figure. Here the left-hand circle is x; the right-hand circle is y. For clarity, x is shaded with lines parallel to the margin of the paper and y is shaded with lines perpendicular to the margin of the paper. $x \cap y$, therefore, is the cross hatched area, for in that area are all the points contained in x and y. $x \cup y$ is all the shaded area, for in that area are all the points contained in x or y.

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[1]To prove this property we shall have to use the universally true "one-sided modular law" - If $x \geq z$, then $x \cap (y \cup z) = (x \cap y) \cup z$. (We discuss this law somewhat in detail in chapter II).

4. <u>The combination of lattices</u>. The "sum" $X + Y$ of two partially ordered systems $X$ and $Y$ is the system $Z$ whose elements are the elements $x \in X$ plus the elements $y \in Y$. Order is preserved within $X$ and $Y$, while $x \geq y$ and $x \leq y$ are assumed never to hold.

The "product" $XY$ of $X$ and $Y$ is the system $W$ whose elements are the couples $[x,y]$ with $x \in X, y \in Y$, $[x,y]$ $[x',y']$ meaning that $x \geq x'$ and $y \geq y'$. And finally, the "power" $Y^X$ of $Y$ to exponent $X$ is the system $V$ whose elements are the functions $y = f(x)$ with domain $X$ and range $Y$ for which $x \geq x'$ implies $f(x) \geq f(x')$ which is to say, the monotonic functions $X \to Y$. In $Y^X$, $f \geq g$ means that $f(x) \geq g(x)$ for all $x$ in $X$.

In the terminology of the above paragraph, the "sum" of two lattices is never a lattice: for if $x$ and $y$ come from different summands, they have no upper bound.

By the "direct union" $XY$ of two algebras $X$ and $Y$ having the same operations, will be meant the system itself having these operations, whose elements are the couples $[x,y][x \in X, y \in Y]$, in which algebraic combination is performed component-by-component.

Theorem 1.2: The "product" of two lattices is the direct union of the lattices regarded as abstract algebras. Hence it is always a lattice.

Proof: From the above, $[x \cup x', y \cup y']$ is not only an upper bound to $[x,y]$ and $[x',y']$, but it is contained in every other upper bound - hence it is a least upper bound. The proof is completed by duality.

## CHAPTER II

## MODULAR LATTICES

1. Lattice polynomial. We shall define a "lattice polynomial" as any polynomial function of variables $x_1$, $x_2$, ..., $x_n$, which is either a monomial $x_i$, or a join or meet of polynomials already defined.

Theorem 2.1: Let $f(x_1,x_2,\dots,x_n)$ be any lattice polynomial, and suppose $x_i \le y_i$ for all i. Then $f(x_1,x_2,\dots,x_n) \le f(y_1,y_2,\dots,y_n)$: Lattice polynomials are monotonic functions.

Proof: By induction, it suffices to show that $x \le x_1$ implies $x \cap y \le x_1 \cap y$ and $x \cup y \le x_1 \cup y$. But if $x \le x_1$, then $x \cap y = (x \cap x_1) \cap y = x \cap (x_1 \cap y) \le x_1 \cap y$ and similarly $x_1 \cup y = (x_1 \cup y = (x_1 \cup x) \cup y = x_1 \cup (x \cup y) \ge x \cup y$.

Corollary 1: We have the one sided distributive laws $x \cap (y \cup z) \ge (x \cap y) \cup (x \cap z)$ and $x \cup (y \cap z) \le (x \cup y) \cap (x \cup z)$.

Proof: By theorem 2.1, $x \cap (y \cup z)$ is an upper bound to both $x \cap y$ and $x \cap z$; hence it contains their join. The second law follows by duality.

Assuming $x \ge z$ in corollary 1, we get the self-dual one-sided modular law,

Corollary 2: If $x \ge z$, then $x \cap (y \cup z) \ge (x \cap y) \cup z$.

Corollary 3: Always $(x \cap y) \cup (u \cap v) \le (x \cup u) \cap (y \cup v)$.

Proof: By theorem 2.1, $(x \cap y) \le (x \cup u) \cap (y \cup v)$; similarly, $(u \cap v) \le (x \cup u) \cap (y \cup v)$. The conclusion is now obvious.

2. Definition of a modular lattice. Just as the elements of some groups (but not all) satisfy $a.a =$ the identity of the

group and just as the elements of many groups (but not all) satisfy $ab=ba$, so the elements of many lattices (but not all) satisfy the following so-called modular law (an obvious strengthening of the one-sided modular law).

L5. If $x\leq z$, then $x\cup(y\cap z)=(x\cup y)\cap z$. [9:34]

A lattice will be called "modular" if and only if its elements satisfy the modular identity.

Theorem 2.2: Each of the following identities implies L5 and both of the others.

L5': $x<z<x\cup y$ implies $z=x\cup(y\cap z)$
L5": $[x\cap(y\cup z)]\cup(y\cap z)=[x\cup(y\cap z)]\cap(y\cup z)$
L5"': $[x\cap(y\cup z)]\cup(y\cap(x\cup z)]=(y\cup z)\cap(z\cup x)\cap(x\cup y)$

Proof: The structure of the proof is as follows ($\rightarrow$ shall read implies)

$$\begin{array}{l} L5\rightarrow L5'\rightarrow L5''\rightarrow L5 \\ \updownarrow \\ L5''' \end{array}$$

(1) To show that L5$\rightarrow$L5', assume L5 and the hypothesis of L5'. The conclusion of L5' becomes $Z=x\cup(y\cap z)=(x\cup y)\cap z=Z$. (2) To show that L5'$\rightarrow$L5", assume L5'. By L5' and the definitions of least upper bound and greatest lower bound, the left-hand member of L5", $[x\cap(y\cup z)]\cup(y\cap z)=x\cup(y\cap z)$. Also the right hand member of L5", $[x\cup(y\cap z)]\cap(y\cup z)=x\cup(y\cap z)$. Hence $[x\cap(y\cup z)]\cup(y\cap z)=[x\cup(y\cap z)]\cap(y\cup z)$. (3) To show that L5"$\rightarrow$L5, rewrite L5 as follows:

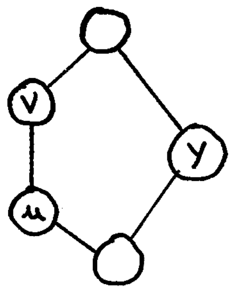
If $z\leq y$, then $Z\cup(x\cap y)=(z\cup x)\cap y$. This is permissible for we have only interchanged x, y, and z. Then L5" becomes $(x\cap y)\cup z=(x\cup z)\cap y$ which is L5 rearranged. (4) To show that L5$\rightarrow$ L5''', consider L5"'. $[x\cap(y\cup z)]\cup[y\cap(x\cup z)]=y\cup z\cap(z\cup x)\cap(x\cup y)$. $[x\cap(y\cup z)]\leq[y\cap(x\cup z)]$ hence by L5, the left hand member of L5"' =

$\{[x\cap(y\cup z)]\cup y\}\cap(x\cup z)$. Since $y\leq y\cup z$, we may again apply L5 and show that $\{[x\cap(y\cup z)\cup y\}=(y\cup x)\cap(y\cup z)$. Then $[x\cap(y\cup z)]\cup[y\cap(x\cup z)]=(y\cup z)\cap(z\cup x)\cap(x\cup y)$. 5 To show that L5'''$\to$L5, apply the hypothesis of L5 to L5' and by the definition of least upper bound and the absorbtive law get $x\cup(y\cap z)=(x\cup y)\cap z$, which is L5.

We have shown, therefore, that L5$\to$L5' by 1, L5'$\to$L5" by 2, L5"$\to$L5 by 3, L5$\to$L5''' by 4 and L5'''$\to$L5 by 5 completing the proof.

Theorem 2.3: A lattice is non modular if and only if it contains the lattice of figure 2 as a sublattice.

Fig. 2




Note: See the appendix for the explanation of this figure.

Proof: The lattice graphed fails to satisfy L5. Conversely, unless L5 holds in a lattice, by the one-sided modular law we have

$$u=x\cup(y\cap z)<(x\cup y)\cap z=v$$

But in this case the five elements u, v, y, $u\cup y$, and $v\cap y$ form a sublattice isomorphic with the one in figure 2. For $u\cup y\leq v\cup y\leq (x\cup y)\cup y=x\cup y=x\cup(y\cap z)\cup y=u\cup y$, whence $v\cup y=u\cup y$. Dually, $u\cap y=v\cap y$ and now it is clear by the absorbtive law that we have a sublattice.

Corollary 1: In a modular lattice, $v>u$ is incompatible with $y\cup u=y\cup v$ and $y\cap u=y\cap v$; moreover this condition conversely implies modularity.

Corollary 2: In a modular lattice, ( ') if x and y cover a, and $x\neq y$, then $x\cup y$ covers x and y, and dually ( ") if a covers

$x$ and $y$, and $x \neq y$, then $x$ and $y$ cover $x \cap y$.

Proof: Unless $x \cup y$ covered (say $x$,) the sublattice generated by $x$, $y$, and any element between $x$ and $x$ $y$ would be isomorphic with the lattice graphed in F2.

3. <u>Example of a modular lattice</u>. The importance of modular lattices is shown in the following theorem.

Theorem 2.4: The normal subgroups of any group form a modular lattice.

Proof: In any lattice, the one-sided modular law holds. Suppose $X$, $Y$, and $Z$ are normal subgroups of a group, and that $X \leq Z$. Then $X \cup Y$ certainly contains all products $xy[x \in X, y \in Y]$; while conversely, since $xy(x'y') = (xx')(x'^{-1}yx'y')$ and $(xy)^{-1} = y^{-1}x^{-1} = (y^{-1}xy)^{-1}y^{-1}$, the $xy$ form a group containing $X$ and $V$ - whence $X \cup Y$ is the set of the $xy$. Consequently, if $u \in (X \cup Y) \cap Z$, then $u = xy = z[z \in Z]$, and $y = x^{-1}z \in Z$ (since $x \in X \leq Z$ which implies $u = xy \in X \cup (Y \cap Z)$. We conclude $X(Y \cap Z) \geq (X \cup Y) \cap Z$ which with P2 and the one-sided modular law implies L5.

4. <u>Complemented modular lattices</u>. A (modular) lattice is "complemented" if and only if it has a 0 and I and

L6: Every $x$ has a "complement" $x'$, such that $x \cap x' = 0$ and $x \cup x' = I$.

Theorem 2.5: Each of the following conditions is necessary and sufficient that a modular lattice of finite dimensions be complemented, and so is the dual of each condition.

L6': Given $a \leq x \leq b$, then $x$ has a "relative complement" $y$ satisfying $x \cap y = a$, $x \cup y = b$.

L6'': Every element is a join of atoms.[1]

L6''': I is the join of atoms.

Proof: We shall prove the implications L6$\rightarrow$L6'$\rightarrow$L6''$\rightarrow$ L6; because of finite - dimensionality, we can assume the existence of 0 and I. Given $a \leq x \leq b$,

$(a \cup x') \cap b \cap x = (a \cup x') \cap x = a \cup (x' \cap x) = a$,

$x \cup a \cup (x' \cap b) = x \cup (x' \cap b) = (x \cup x') \cap b = b$,

and so $(a \cup x') \cap b = a \cup (x' \cap b)$ by the relative complement we want. Again if $<0<x<a$ and L6' holds, then a is the join of x and its relative complement y in a/0; hence by induction a is the join of atoms, proving L6''. The implication L6''$\rightarrow$L6''' is trivial when I exists, and so it remains to prove that L6''' implies L6.

But setting $x = x_0$, we can construct a chain between $x_0$ and I as follows. If $x_k < I$, then by L6''' there exists a $p_{k+1}$ not contained in $x_k$; set $x_{k+1} = x_k \cup p_{k+1}$. Some $x_k$ will be I; define $x' = p_1 \cup p2 \cup \ldots p_n$. Evidently $x \cup x' = x \cup p_1 u \ldots \cup p_n = I$.

5. <u>Modular functionals</u>. By the dimension d [x] of an element x of a partially ordered system P, is meant the maximum "length" d of a chain $x_0 < x_1 < \ldots < x_d = x$ having x for its greatest element. Similarly, by d [P] is meant the maximum length of a chain in P.

It is readily seen that d [P] is the maximum of d [x]; it is also readily seen that in determining dimensions one need consider only "connected" relative chains. The notion of dimension is especially important if P has a 0 and satisfies

---

[1]An "atom" is an element which covers 0. This corresponds to Euclids' definition of a point as "that which has no parts," and of an atom as something "indivisible." [5;9]

the

Jordan-Dedekind chain condition: All connected chains between fixed end-points have the same length. [5;11]

Under these circumstances, represent each $x \in P$ by a vertex on a horizontal plane d [x] units above 0. If x covers y, then there is a connected chain $x > y > y_1 > \ldots > O$ of length d [y] + 1 from x to 0, and conversely. Hence x covers y if and only if $x > y$ and $d[x] = d[y] + 1$.

Measure, probability, and dimension are familiar instances of functionals (real-valued functions) defined on lattices. In order to analyze the properties of such functionals, we shall need the definition:

A functional m [x] defined on a lattice is called "modular" if and only if

M1: $m[x]+m[y]=m[x \cup y]+m[x \cap y]$.

It is called "positive", if and only if

M2: $x \geq y$ implies $m[x] > m[y]$.

It is called "of bounded variation," if and only if the sums

$\sum_{i=1}^{n} \left\{ m[x_i]-m[x_{i-1}] \right\}$ $(x_0 < x_1 < \ldots < x_m)$ are bounded. [5;40]

Probability and measure are always positive and modular; dimension is positive and often modular. The above definition of a functional defined on a lattice contains as special cases the usual definitions[1] of bounded variation, both for ordinary

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[1]Fréchet defined a "metric space" as a class of elements (points), together with a definition of the "distance" $\delta(x,y)$ between all pairs of points in a plane, the latter being assumed to satisfy four conditions: ① $\delta(x,x)=0$, ② if $x \neq y$, then $\delta(x,y) > 0$, ③ $\delta(x,y)=\delta(y,x)$, ④ the triangle inequality $\delta(x,y)+\delta(y,z) \geq \delta(x,z)$. [5;1]

real functions and for functions of sets. Also, any positive functional on a lattice with O and I is of bounded variation (since $\sum_{i=1}^{n}\{m[x_i]-m[x_{i-1}]\}$ is always bounded by $m[I]-m[O]$).

6. <u>Metric lattices</u>. By a "quasi-metric lattice," is meant a lattice L with a positive modular functional $m[x]$. We shall call $m[x\cup y]-m[x\cap y]=m[x\cup y\ x\cap y]$ the "quasi-distance" between x and y, and shall denote it by $\delta(x,y)$.

If M2': $x>y$ implies $m[x]>m[y]$, then we shall call L a "metric lattice," $m[x]$ will be called "sharply positive, and quasi-distance will be called "distance."

Through these definitions, we can apply the metric ideas of Fréchet and his followers to lattice theory.

Theorem 2.6: $\delta(a\cup x,a\cup y)+\delta(a\cap x,a\cap y)\leq\delta(x,y)$.

Proof: $\delta(a\cup x,a\cup y)+\delta(a\cap x,a\cap y)$ is by definition $m[a\cup x\cup y]-m[(a\cup x)\cap(a\cup y)]+m[(a\cap x)\cup(a\cap y)]-m[a\cap x\cap y]$ which is, by the one sided distributive law, at most

$$m[a\cup x\cup y]-m[a\cup(x\cap y)]+m[a\cap(x\cup y)]-m[a\cap x\cap y].$$

Transposing the middle terms, and using M , this becomes

$$m[a]+m[x\cup y]-m[a]-m[x\cap y]=\delta(x,y).$$

7. <u>Modularity of metric lattices</u>.

Theorem 2.7: Any metric lattice is modular [5;42]

Proof: In any lattice, $x\leq z$ implies $x\cup(y\cap z)\leq(x\cup y)\cap z$ (one-sided modular law), hence it suffices to exclude the case $x\cup(y\cap z)<(x\cup y)\cap z$ by proving $m[x\cup(y\cap z)]-m[(x\cup y)\cap z]=0$. But by condition M1, this difference is

$$m[x]+m[y\cap z]-m[x\cap y\cap z]-m[x\cup y]-m[z]+m[x\cup y\cup z].$$ And since $x\leq z$, this reduces after rearrangement to

$(m[y \cap z]+m[y \cup z]-m[z])-(m[x \cup y]+m[x \cap y]-m[x]$, which is by M1 simply $m[y]-m[y]=0$.

Corollary. A lattice of finite dimensions is modular if and only if its dimension function is modular.

## CHAPTER III

## DISTRIBUTIVE LATTICES

1. Definition. A lattice will be called "distributive" if and only if it satisfies identically

L6: $(x\cap y)\cup(y\cap z)\cup(z\cap x)=(x\cup y)\cap(y\cup z)\cap(z\cup x)$.

L6': $x\cap(y\cup z)=(x\cap y)\cup(x\cap z)$.

L6": $x\cup(y\cap z)=(x\cup y)\cap(x\cup z)$.

Theorem 3.1: Each of the identities L6, L6' and L6" implies L5 (modular identity) and both of the others.[1]

Proof: Assuming L6 and $x\geq z$, we get by direct substitution $(x\cap y)\cup z=(y\cup z)\cap x$, which is L5 rearranged. Similarly, assuming L6' and $x\geq z$, we get $x\cap(y\cup z)=(x\cap y)\cup z$. Finally, assuming $x\leq z$ in L6", we get $x\cup(y\cap z)=(x\cup y)\cap z$. Hence each of the conditions L6, L6', and L6" implies L5.

Now assuming L6" we get by the expansion,

$[(x\cap y)\cup(y\cap z)]\cup(z\cap x)$

$=[(x\cap y)\cup(y\cap z)\cup z]\cap[(x\cap y)\cup(y\cap z)\cup x]$

$=[(x\cap y)\cup z]\cap[(y\cap z)\cup x]$ by L3-L4

$=(x\cup z)\cap(y\cup z)\cap(y\cup x)\cap(z\cup x)$ by L6" again

$=(x\cup y)\cap(y\cup z)\cap(z\cup x)$ by L2-L3

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[1]Birkhoff thinks it "curious" that C. S. Pierce should have thought that every lattice was distributive [5;74]. This error has been demonstrated many times. The writer also gives an example (page 18) which shows that not every lattice satisfies the distributive property.

which is L6. Conversely, abbreviating L6 to the form $u=v$, we get from the equality $x\cup u=x\cup v, x\cup(y\cap z)$ on the left-hand side, and on the right-hand side, using L5,

$$x\cup\{[y\cup z]\cap[(x\cup y)\cap(x\cup z)]\}=(x\cup y\cup z)\cap(x\cup y)\cap(x\cup z).$$

But by L3-L4, this is $(x\cup y)\cap(x\cup z)$, the right-hand side of L6". Thus L6" and L6 are equivalent; dually L6' and L6 are equivalent, completing the proof.

Corollary. Any distributive lattice is modular.

Remark. It can easily be shown that the converse is not true; for if we define greatest lower bound as the greatest common divisor of two real numbers and least upper bound as the least common multiple of the numbers, the modular identity is satisfied and the distributive law is not. We prove this as follows.[1]

Denote the greatest common divisor of a and b by (a,b) and the least common multiple of a and b by [a,b]. The modular law now becomes:

If C is a multiple of a, then [a,(b,c)] ([a,b],c).
Let $c=Ka$. Then (b,c) bx+cy bx+kay ....1

"The product of two numbers is equal to the product of their least common multiple and greatest common divisor." Then, if $a_1$ is the greatest common divisor of a and bx + kay and L is their least common multiple,

$$a_1 L=abx + Ka^2y \text{ ................2}$$

$$L=\frac{a}{a_1}bx + \frac{a^2}{a_1}Ky$$

---

[1] The writer is responsible to Mrs. Georgia C. Smith for this proof.

Now the greatest common divisor of a and bx is the greatest common divisor of a and b. For suppose a number divides bx and a, but not b. Looking at 1, anything that divides bx and a must divide (b,c), which contradicts. Then $[a,b]a_1 = ab$, $[a,b] = \frac{ab}{a_1}$

$$L = [a_1,b]x + k\frac{a}{a_1}ay \text{ from } 2$$

$$[a,b]x + cy\frac{a}{a_1} = ([a,b],c).$$

$\frac{a}{a_1}$ has not factor common to [a,b]x. Also x, y, and (b,c) have no factor common to all three by 1. Suppose C had a factor which would divide x and y but not b. This contradicts 1. Then there is no number which divides x, y, and the right-hand side of the last equation and therefore it is the greatest common divisor.

Using the same notation, L6' becomes :

$$(a,[b,c]\ [(a,b),(a,c)].$$

If we use the particular example a=4, b=5, c=16, we see that L6' is not satisfied, for

$$(4,80) \neq [1,1]$$

Theorem 3.2: A lattice which is not distributive contains one of the examples of figure 3 as a sublattice. [5;75].

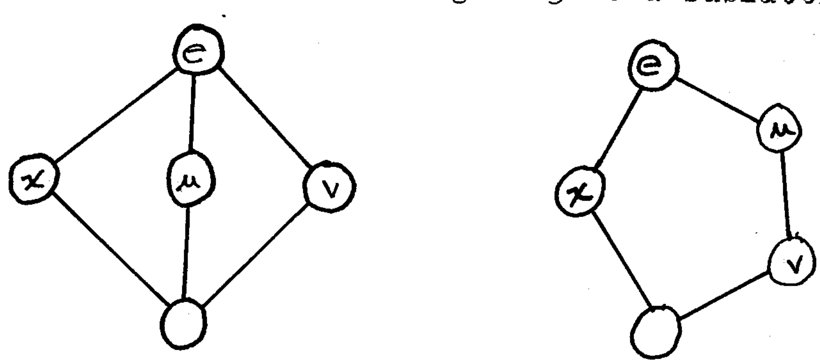


fig. 3

Corollary. A lattice is distributive if and only if relative complements in it are uniquely determined. [5;75]. This means that, given $a \leq x \leq b$, at most one y exists satisfying $x \cap y = a$ and $x \cup y = b$.

Illustration:

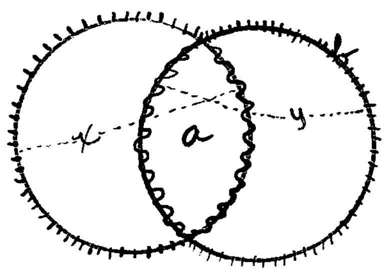


fig. 4

Proof: In a distributive lattice, $x \cap u = x \cap v$ and $x \cup u = x \cup v$ imply $u = u \cap (x \cup v) = (u \cap x) \cup (u \cap v)$

$$(v \cap x) \cup (u \cap v) = v \cap (u \cup v) = v$$

Conversely, in a non-distributive lattice, the element x in either example of fig. 3 has two relative complements.

Theorem 3.3: In a distributive lattice, complementation is unique and orthocomplementation.[1]

Proof: If $a \cup x = I$ and $a \cap y = o$, then

$$x = o \cup x = (a \cap y) \cup x = (a \cup x) \cap (y \cup x) = I \cap (y \cup x) = y \cup x.$$

If also $a \cap x = o$ and $a \cup y = I$, then similarly $y = y \cup x$, whence $x = y$ proving unicity.

By the commutative law, the relation of complementarity is symmetric, and so $(a')' = a$. Again, if $a \leq b$, then $a \cup a' = I$ and $a \cap b' \leq b \cap b' = o$. Hence $a' = b' \cup a'$ and $b' \leq a'$. That is the

---

[1]Lattices are "orthocomplemented" in the sense that complementation is defined satisfying L6 and $(x \cap y)' = x' \cup y'$, $(x \cup y)' = x' \cap y'$, and $(x')' = x$.

correspondence $a \to a'$ is a dual automorphism, completing the proof.

We are now in a position to prove one of the profound theorems of algebra. This is M. H. Stone's theorem[1] which brings up to date, so to say, Boolean Algebra. This is accomplished by showing that every Boolean Algebra can be transformed into the modern algebraic concept of a ring. To be sure, the ring is a highly restricted one, but is a ring nevertheless. If we remember that a Boolean Algebra is a "highly restricted" lattice (distributive and complemented), we should not be surprised that it makes contact with a very special kind of another algebraic system. Although the proof that every Boolean Algebra can be transformed into a ring is well known, the following proof (Theorem 34) is the author's.

2. Ring. A ring R is a mathematical system composed of elements (a,b,c,...) and two single valued operations, addition + and multiplication x , relative to which the system is closed, and

R1: The elements constitute an abelian group relative to the operation +.

R2: Ax(bxc)=(axb)xc for all a, b, and c of R.

R3: ax(b+c)=(axb)+(axc) and

(b+c)xa=(bxa)+cxa) for all a, b, and c of R.

R4: If axb=bxa is true for every pair of elements of R, then R is a "commutative ring." A ring is called a "ring with

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[1] M. H. Stone "Subsumption of Boolean Algebras Under the Theory of Rings," Proc. Nat. Academy Science, 1935, vol. 21, pp. 103-5.

a unit" if there exists an element i such that $a \times i = i \times a = a$ for all a of R.

3. <u>Boolean Ring</u>. A "Boolean ring" is a commutative ring each of whose elements is idempotent. That is to say $xx = x$ for all x.

Theorem 3.4: Every Boolean Algebra can be transformed into a Boolean ring.

Proof: Definition 3.1. Define $a \times b$ to be $a \cap b$ and $a+b$ to be $(a \cap b') \cup (a' \cap b)$.

We shall show along with R2, R3, and R4,

R1' For every a and b of R there exists a unique C of R such that $a+b=c$.

R" $a+(b+c)=(a+b)+c$ for all a, b, and C of R.

R"' There exists a unique element, o, of R such that $a+o=o+a=a$ for all a of R.

$R1^{IV}$ For every element a there exists an a - inverse (-a) such that $a+(-a)=(-a)+a=o$.

The above four posulates are Dickson's definition of a group [11;47]. It is obvious that under the definition of + the group is Abelian.

R1' transforms under definition 3.1 to $(a \cap b') \cup (a' \cap b)$ which is unique and is an element of the lattice.

R1" transforms to

$$\{[(a \cap b') \cup (a' \cap b)] \cap c'\} \cup \{[(a \cap b') \cup (a' \cap b)]' \cap c\} = \{a \cap [(b \cap c') \cup (b' \cap c)]'\} \cup \{a' \cap [(b \cap c') \cup (b8 \cap c)]\}.$$

$$(a \cap b8 \cap c8) \cup \{(a' \cap b \cap c') \cup [(a' \cup b) \cap (a \cup b') \cap c]\} = \{[a \cap (b' \cup c) \cap (b \cup c')] \cup (a' \cap b \cap c')\} \cup (a' \cap b' \cap c).$$

$(a\cap b'\cap c')\cup\{(a\cup b)\cap(a\cup b'\cup c')\cap(c\cup a')\cap(c\cup b)\}=\{(a\cup b)\cap(a\cup c')\cap$
$(b'\cup c\cup a')\cap(b\cup c')\}$ $(a'\cup b'\cap c)$.

$(a'\cup b\cup c')\cap(a\cup b'\cup c')\cap(a'\cup b'\cup c)\cap(a\cup b\cup c)=(a'\cup b\cup c')\cap(a\cup b'\cup c')$

$(a'\cup b\cup c')\cap(a\cup b'\cup c')\cap(a'\cup b'\cup c)\cap(a\cup b\cup c)=(a'\cup b\cup c')\cap(a\cup b'\cup c')\cap$
$(a'\cup b'\cup c)\cap(a\cup b\cup c)$.

The identity element is 0, for R1''' transforms to $(a\cap 0')\cup(a'\cap 0)=a\cup 0=a$. To prove 0 is unique, assume a second identity, say $0_1$. Then $0+0_1=0$, using $0_1$ as the identity. Also $0_1+0=0_1$ using 0 as the identity. From the two equations and commutativity $0+0_1\equiv 0_1+0$ and therefore $0\equiv 0_1$.

To show RI$^{IV}$ consider

$a+a\rightarrow(a\cap a')\cup(a'\cap a)=0$, hence every element is its own inverse.

RI$^{V}$ follows easily from definition 3.1 and the definition of greatest lower bound.

R2 transforms to $a\cap(b\cap c)=(a\cap b)\cap c$ which is L3 (associative law of a lattice).

R3 becomes by definition 3.1

$a\cap[(b'\cap c)\cup(b\cap c')]=[(a\cap b)'\cap(a\cap c)]\cup[(a\cap b)\cap(a\cap c)']$

$(a\cap b'\cap c)\cup(a\cap b\cap c')=\{(a\cap b)'\cup[(a\cap b)\cap(a\cap c)']\}\cap\{(a\cap c)\cup[(a\cap b)\cap(a\cap c)']\}$

$(a\cap b'\cap c)\cup(a\cap b\cap c')=\{(a\cap b)'\cup[(a\cap b)\cap(a\cap c)']\}\cap\{(a\cap c)\cup[(a\cap b)\cap(a\cap c)']\}$

$=(a\cap b)'\cup(a\cap b)\cap[(a\cap b'\cup(a\cap c)']\cap[(a\cap c)\cup(a\cap b)]\cap$
$(a\cap c)\cup(a\cap c)'$.

$$\begin{aligned}(a\cap b'\cap c)\cup(a\cap b\cap c')&=[(a'\cup b')\cup(a'\cup c')]\cap[a\cap(c\cup b)].\\&=[a\cap(c\cup b)\cap a']\cup[a\cup(c\cup b)\cap(b'\cup c')]\\&=[(a\cap c)\cup(a\cap b)]\cap(b'\cup c')\\&=[(b'\cup c')\cap(a\cap c)]\cup[(b'\cup c')\cap(a\cup b)]\\&=[(b'\cup c')\cap(a\cap c)]\cup[(a\cap b)\cap c']\\&=(a\cap b'\cap c)\cup(a\cap b\cap c').\end{aligned}$$

R4 transforms to a b b a which is L3 (Commutative law of a lattice).

Thus it has been established that under definition 3.1, a Boolean Algebra is a Boolean ring.

Theorem 3.3: Every Boolean Ring can be transformed into a Boolean Algebra. [5;96]

Proof: Given a Boolean ring R with unit 1 and $x \geq y$ defined to mean $xy=y$. Then (1) $1 \geq x \geq 0$ for $x1=x$ and $x0=0$, (2) $x \geq x$ for $xx=x$ by idempotence, (3) If $x \geq y$ and $y \geq x$, then $x=yx=xy=y$ by commutativity, (4) If $x \geq y$ and $y \geq z$, then $x \geq z$ for $xy=y$, $yz=z$, $(xy)z=z$, $x(yz)=x$, $xz=z$, then $x \geq z$, (5) $x \geq xy$, $x(xy=(xx)y=xy$, (6) $y \geq xy$, $y(xy)=(yy)x=yx$, (7) If $x \geq z$ and $y \geq z$, then $xy \geq z$ for $x(yz)=x(yz)=xz=z$, (8) The correspondence $x \rightleftarrows 1-x$ is a dual automorphism of period two. Since $xy=y$ implies $(1-y)(1-x)=1-y-x+xy=(1-x)$, it inverts inclusion and hence $x'=1-x$.

Hence our definition makes R into a partially ordered system with 0 and I by (1) - (4), in which $x \cap y$ exists and is $xy$ by (5) - (7), and by (8) $x \cup y$ exists and is $1-(1-x)(1-y)=x+y-xy$. Also if $x'=1-x$, $x \cap x'=x(1-x)=0$ and $x \cup x'=x+(1-x)-x(1-x)=I$. Hence our definition makes R into a complemented lattice. Finally, $x \cap (y \cup z)=x(y+z-yz)=xy+xz-xyz=xy+xz-xyz=xy+xz-xyxz=(x \cap y) \cup (x \cap z)$ and so R is a Boolean Algebra.

Theorem 3.4: Each of the following identities is equivalent to distributivity

(a) $x \cup (y \cap z) \geq (x \cup y) \cap z$,

(b) $(y \cup z) \cap (z \cup x) \cap (x \cup y) = (y \cup z) \cap [x \cup (y \cap z)$

Proof: The structure of our proof is as follows (The arrow is read "implies") $D \Leftrightarrow a \Leftrightarrow b$.

To show that D$\rightarrow$a, consider D: $x\cup(y\cap z)=(x\cup y)\cap(x\cup z)$. Since $z\leq x\cup z$, we have from a $x\cup(y\cap z)\geq(x\cup y)\cap z$.

To show that a$\rightarrow$D, again consider D. Always $x\cup(y\cap z)\leq(x\cup y)\cap(x\cup z)$. By a, $(x\cup y)\cap(x\cup z)\leq x\cup[y\cap(x\cup z)]\leq x\cup[x\cup(z\cap y)]$. Applying a $x\cup[x\cup(z\cap y)]\leq x\cup x\cup(z\cap y)=x\cup(z\cap y)$. Therefore $(x\cup y)\cap(x\cup z)\leq x\cup(z\cap y)$ but $(x\cup y)\cap(x\cup z)\geq x\cup(y\cap z)$ from the one-sided modular law and therefore $(x\cup y)\cap(x\cup z)=x\cup(y\cap z)$. To show that a$\rightarrow$b consider $(x\cup y)\cap z\cup x)\leq x\cup[y\cap(z\cup x)]$ which is obtained by applying a. Also $x\cup[y\cap(z\cup x)]\leq x\cup[x\cup(z\cap y)]$ by a and the commutative law. $x\cup[x\cup(z\cap y)=x\cup(z\cap y)$ by the associative law. Therefore $(y\cup z)\cap(z\cup x)\cap(x\cup y)\leq(y\cup z)\cap[x\cup(z\cap y)]$. But by the one-sided distributive law: $x\cup(z\cap y)\leq(x\cup z)\cap(x\cup y)$, $(y\cup z)\cap[x\cup(z\cap y)]\leq(y\cup z)\cap(x\cup z)\cap(x\cup y)$ by the associate law and therefore $(y\cup z)\cap(z\cup x)\cap(x\cup y)=(y\cup z)\cap[x\cup(y\cap z)]$.

To show that b$\rightarrow$a, we delete $x\cup z$ from the right-hand member of b and get $x\cup(y\cap z)\geq(y\cup z)\cap(z\cup x)\cap(x\cup y)$. By deleting x from $(y\cup z)\cap(z\cup x)\cap(x\cup y)$ we get $x\cup(y\cap z)\geq(y\cup z)\cap z\cap(x\cup y)=z\cap(x\cup y)$ by the absorbtive law. Therefore $x\cup(y\cap z)\geq z\cap(x\cup y)$ which is a.

# CHAPTER IV

## APPLICATION TO LOGIC

This thesis would be incomplete without a word relative to the lattice-theoretic aspect of the logic of propositions. Traditional logic (law of excluded middle) involves Boolean algebras, and modifications of it involve other lattices. In order to show just how propositions constitute a Boolean algebra, we shall consider a set of propositions (x, y, z, ...). Denote the propositions "x and y," "either x or y," and "not x," by $x \cup y$, $x \cap y$, and $x'$ respectively. Under this notation, the identities of Boolean algebra relate logically equivalent statements. Thus (L2) "John sleeps and Henry walks" is true or false according as "Henry walks and John sleeps" is true or false. Denote the compound proposition "x implies y" ("if x, then y") by $x \rightarrow y$. Clearly $x \rightarrow y$ is true or false according as "y or not x" is true or false. Similarly, we denote "x is equivalent to y" by $x \sim y$, and identify it with "x implies y and y implies x" - i. e. $(x \rightarrow y) \cup (y \rightarrow x)$.

Also, 0 denotes the proposition asserting nothing, and I the proposition which asserts everything. Incidentally, $x \sim y$ is the symmetric difference $x+y$ and $x \rightarrow y$ is $y-x$ (the part of y not included in x). Our notation is dual to the usual one.

We can also show that many compound propositions are "tautologies," that is, true merely in virtue of their logical structure. This amounts algebraically to saying that they are equal to zero. The simplest of these is $x \cap x'$("x or not x"). It is obvious that in a Boolean algebra the following are also tautologies: (1) $I \to x$, (2) $x \to o$, (3) $x \to x$, (4) $x \sim x$.

Before proving that under the above definitions of "meet" and "join" propositions constitute a Boolean algebra it is necessary that some of the rules of traditional logic be reviewed. [15;56]. The operations "and," "or," and "not" are symbolized respectively by $\cdot$, v, and $-$. Also $\to$ is read "implies" and $\equiv$ is read "logical equivalent." The rules will be listed and briefly explained.

1. $p \to q \equiv -q \to -p$; i. e., p implies q is logically equivalent to not - p implies not - q.

2. $p \to q \equiv -p \vee q$; i. e., p implies q is logically equivalent to not - p or q.

3. $(p \vee q) \equiv (q \vee p)$

4. $(p \cdot q) \equiv (q \cdot p)$

These say that "and" and "or" are commutative.

5. $-(p \cdot q) \equiv -p \vee -q$; i. e., not-(p and q) is logically equivalent to not p or not q.

5' $-(p \vee q) \equiv (-p \cdot -q)$; i. e., not -(p or q) is logically equivalent to not p and not q.

5 and 5' represent a duality principle in logic, that is to say, a principle permitting one logical symbol to be exchanged for another provided that all signs are reversed.

6. $p \vee (q \vee r) \equiv (p \vee q) \vee p$

7. $p \cdot (q \vee r) \equiv (p \cdot q) \vee r$

These say that "and" and "or" are associative.

8. $pv(q\cdot r)\equiv(pvq)\cdot(pvr)$

9. $p\cdot(qvr)\equiv(p\cdot q)v(p\cdot r)$ Distributive property

10. $pv-p$ This is the law of the "excluded middle" and asserts that either p is true or -p is true.

10'. $p\supset p$

10". $-(p\cdot-p)$

Formulas 10, 10' and 10" serve to verify a group of universally valid propositions - true by virtue of their structure. They are known in logic as tautologies. 10' and 10" are algebraically equivalent to zero.

It is now obvious that is "v" serves as the symbol for "greatest lower bound," "." for "least upper bound," and "-" for "complement of" then the propositional calculus of the traditional logic constitutes a Boolean algebra. This must be regarded as a significant statement. It is significant from the point of view that it connects traditional logic with Boolean rings, a modern algebraic concept. This means, among other things, that the calculus of traditional logic is being further explored whenever contributions are made to the theory of Boolean rings. It must be remembered also that it is the broad generality of lattice theory which encourages the search for the kind of relationship as the above.

It should be remembered that just as traditional algebra has been modernized by men such as Albert, Wedderburn, Stone, Artin, etc., so has traditional logic been brought up to date by men such as Tarski, Church, Rosser, Carnap, Göfel, and Brower. Brower's school negates the principles of excluded middle. Thus De Morgan's laws 5' and 5" are disturbed and hence the property of complementarity in Boolean algebra. But is must be remembered that Boolean algebra is a very special lattice, and perhaps even Brower's logic has a lattice - theoretic aspect. That this is the case is evidenced by Birkhoff's characterization of a "Browerian logic" in terms of lattices - theoretic concepts [5;128]

## CHAPTER V

## CONCLUSION

As forestated, it has been the purpose of the writer to give a simple exposition of the theory of some topics found in lattice theory, to develop some of its contacts with other mathematical systems, and to give examples of special kinds of lattices. Considerable attention has been paid to the arrangement of material in accordance with the best pedagogical methods. For example, several definitions, operations, and theorems concerned with fundamental notions of lattice theory are given in the earlier chapters which gives the reader a general idea of what is to follow.

Chapter I is introductory in character. It gives a short historical sketch of lattice theory making reference to George Boole. Further, definitions of a partially ordered system, minimal and maximal elements are lattices are presented. Four universal laws have been introduced, two with proof. The other two are analagous to ordinarly laws of arithmetic.

The writer believes that the only method of gaining an understanding of the notions of lattice theory is by studying a number of examples and illustrations of different kinds of lattices. Hence the theory is illustrated by a large number of examples drawn from different authors. In particular, an appropriate example (Venn's diagram, fig. 1) is given for the idempotent, associative, commutative, and absorbtive laws.

Chapter II deals exclusively with modular and complemented modular lattices while Chapter III deals with distributive and complemented distributive lattices. The treatment of the two chapters is almost the same. The idea of metric lattices is introduced in Chapter II which discusses modular lattices. The contact that a lattice makes with a Boolean ring is shown in Chapter III.

Chapter IV gives the reader a somewhat general idea of the lattice-theoretic aspects of theoretical logic. The writer has not tried to give a complete survey of mathematical logic. This difficult and paradoxical field obviously lies outside the scope of this thesis.

## APPENDIX

In terms of the definition of "covers," it is easy to define the graph of any finite partially ordered system X, as the graph whose verticees are the different elements x, y, z, ... of X, in which x and y are joined by a line segment if and only if x covers y or y covers x. If the graph is so drawn that whenever x covers y, the vertex x is "higher" than the vertex y, it is called a "Hasse diagram" of X.

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