APPLYING QUOTIENT GROUPS TO AN UNSOLVED PROBLEM IN ART

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The Dutch artist M. C. Escher was always interested in repeating patterns. His work on this theme initially had patterns repeating in the plane, such as *Lizard* (see Figure 1). Later he created a repeating pattern in a bounded region: in *Circle Limit I* (see Figure 2), copies of the central figures appear again, at a smaller scale, as we move closer to the boundary of the circle.

FIGURE 1. Escher's *Lizard*, 1942.

In 1956 Escher created *Print Gallery* (see Figure 3). In it there is a person in a gallery who sees the roof of the very gallery he is standing in. This spiral symmetry is flawed: the central region of the picture is blank and Escher signed his name there. How should the pattern in the picture extend into the blank spot? This question would be unanswered for over 40 years.

In 2000, Hendrik Lenstra saw a copy of *Print Gallery* and realized that the pattern Escher probably had in mind, but could not execute, is a *rotated* analogue of the repeating pattern on boxes of cocoa made by the Dutch chocolate company Droste (see left box in Figure 4): a nurse holds a box of Droste cocoa containing a picture of a nurse holding a box, and so on. The repetition of a pattern inside itself at a smaller scale is called the Droste effect. Droste says on its website [2] (click the "1863–1918" link there) that this design was developed around 1900 and was inspired by a Swiss drawing *La Belle Chocolatière*, but that seems to be false: look at the nurse on cocoa boxes of the Russian company Einem from 1897 in the right box in Figure 4.

The connection of the Droste effect to math is quotient groups of $\mathbf{C}^\times$. In Figure 5, we plot 2 and its integral powers along the positive real axis, labeling 2 as q. Every element of $\mathbf{C}^\times$ can be

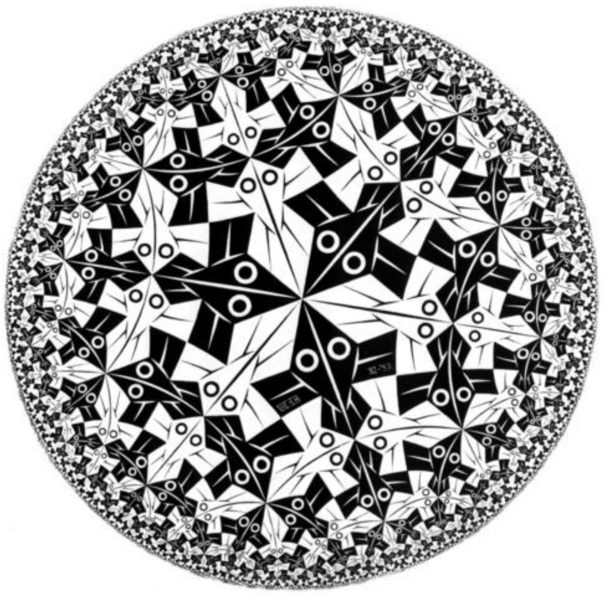

FIGURE 2. Escher's *Circle Limit I*, 1958.

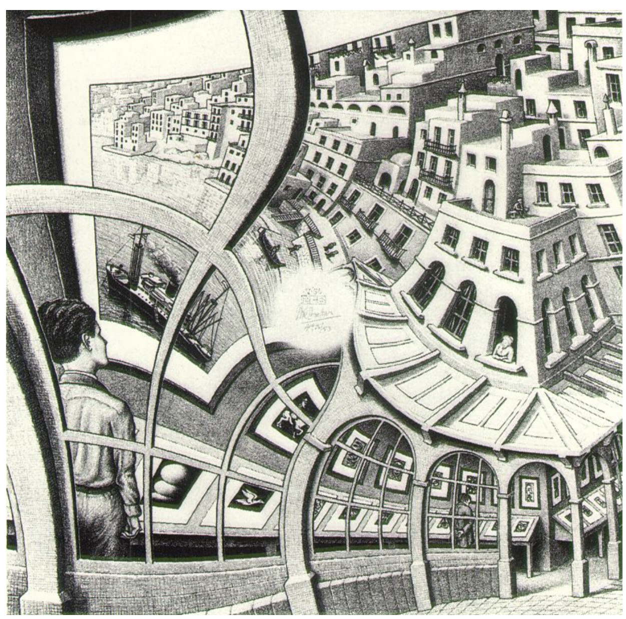

FIGURE 3. Escher's *Print Gallery*, 1956.

carried into the annulus $A = \{z : 1 \leq z < 2\}$ by multiplication or division by a power of 2, *i.e.*, the annulus A provides representatives of the quotient group $\mathbf{C}^\times/2^{\mathbf{Z}}$.[1] The blue diamonds are all copies of the blue diamond in A, which is multiplied by 2, 4, and so on, or by 1/2, 1/4, and so on

[1]The outer boundary of A is related by a factor of 2 to the inner boundary, so they match in $\mathbf{C}^\times/2^{\mathbf{Z}}$, which is a torus (doughnut).

FIGURE 4. Droste cocoa box and earlier Einem cocoa box.

to get the other blue diamonds. Two points z and w in A are marked, along with other points in the cosets $z2^{\mathbf{Z}}$ and $w2^{\mathbf{Z}}$. Every coset lies on a straight line out of the origin, because multiplying by 2, or any power of 2, does not change the angle.

Lenstra's key insight about *Print Gallery* is that the pattern Escher wanted corresponds to $\mathbf{C}^\times/q^{\mathbf{Z}}$ where q is a *nonreal* complex number with $|q| \neq 1$. Look at Figure 6, where $q = 2i$. Like Figure 5, the cosets in $\mathbf{C}^\times/(2i)^{\mathbf{Z}}$ are represented by $A = \{z : 1 \leq |z| < 2\}$, but the way points not in A correspond to points in A is different than when $q = 2$: a coset in $\mathbf{C}^\times/(2i)^{\mathbf{Z}}$ is not on a straight line out of the origin. Look at $zq^{\mathbf{Z}}$ and $wq^{\mathbf{Z}}$ and the blue diamonds. Multiplying by $q = 2i$ doesn't just double the distance from the origin, but also rotates around the origin by 90 degrees counterclockwise, while dividing by $2i$ halves the distance from the origin and rotates by 90 degrees clockwise. This is why, when we multiply z or w or the blue diamond in A by integral powers of $2i$, the new copies of them are moving *around* the origin, not just towards or away from the origin. Figures 7 and 8 are pictures for $\mathbf{C}^\times/q^{\mathbf{Z}}$ when $q = \sqrt{2}(1+i)$ and $q = 2(\cos 1 + i \sin 1)$. These q have absolute value 2, so A represents the cosets again, but the relation of points and figures in A and in q-power copies of A is different than before (everything rotates at new angles).

The curves in Figures 6, 7, and 8 are plots of the real powers of q, which are all subgroups of $\mathbf{C}^\times$ isomorphic to $\mathbf{R}$. The corresponding curve in Figure 5, where $q = 2$, is $2^{\mathbf{R}} = \mathbf{R}_{>0}$, which does not wrap around the origin just as the blue diamonds in Figure 5 don't wrap around the origin. The choice $q = 2$ corresponds to the traditional Droste effect, and the choice of nonreal q is like *Print Gallery*. Do you see the analogy with the way the blue diamonds are repeating?

Lenstra and colleagues in Leiden determined the value of q for *Print Gallery* to be $-20.883 + 8.596i$,[2], and they filled in the empty part of the picture without the origin in 2002. The result is in Figure 9. See [1] and [4] for mathematical details, or watch [3] for a general audience lecture on this phenomenon.

Can you articulate the pattern in Escher's work in Figure 10 using the group $\mathbf{C}^\times$?

[2]The exact value is $\exp(2\pi i(\log 256)/(2\pi i + \log 256))$.

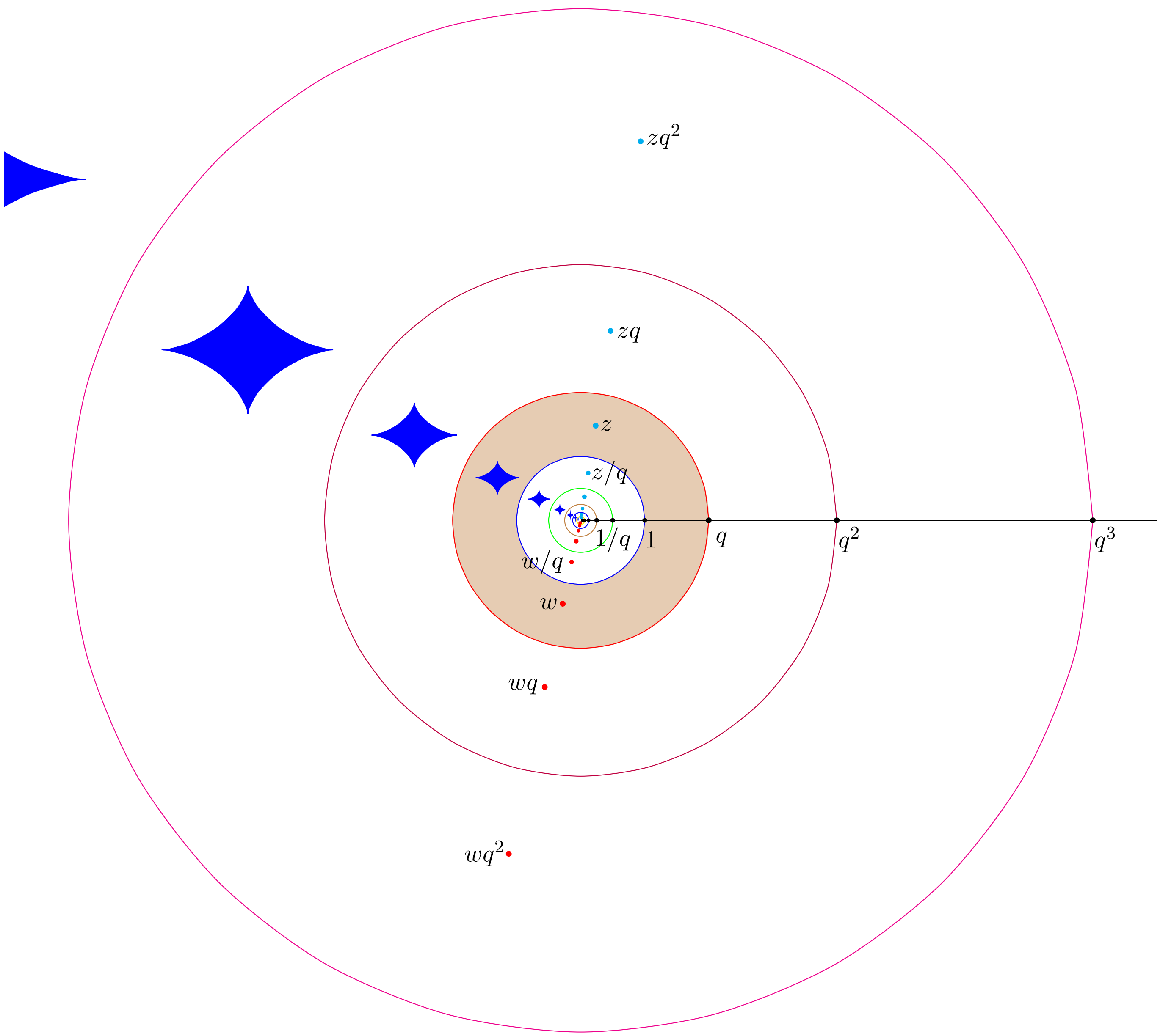

FIGURE 5. $\mathbf{C}^\times/q^{\mathbf{Z}}$ when $q = 2$.

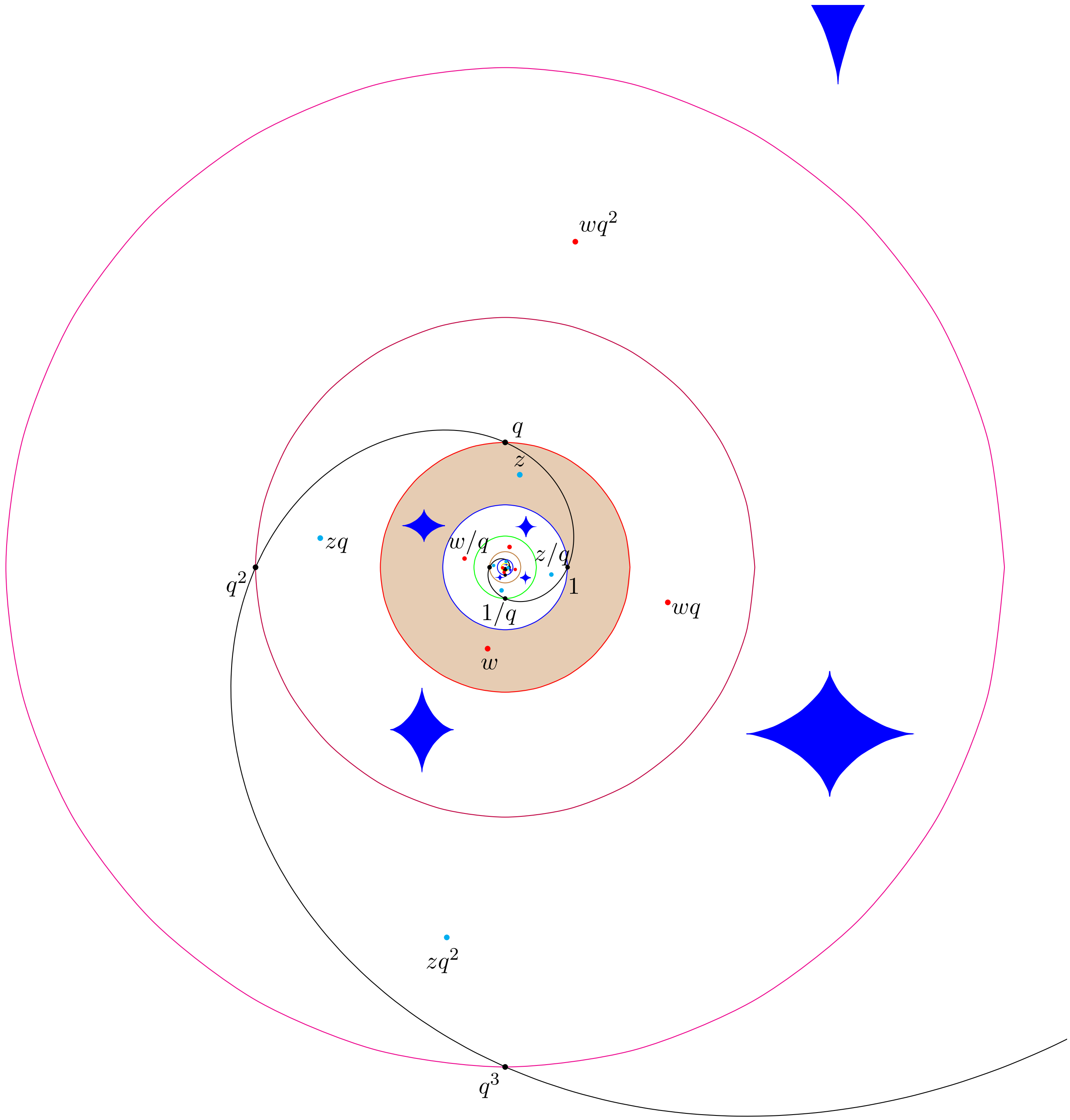

FIGURE 6. $\mathbf{C}^\times/q^{\mathbf{Z}}$ when $q = 2i$.

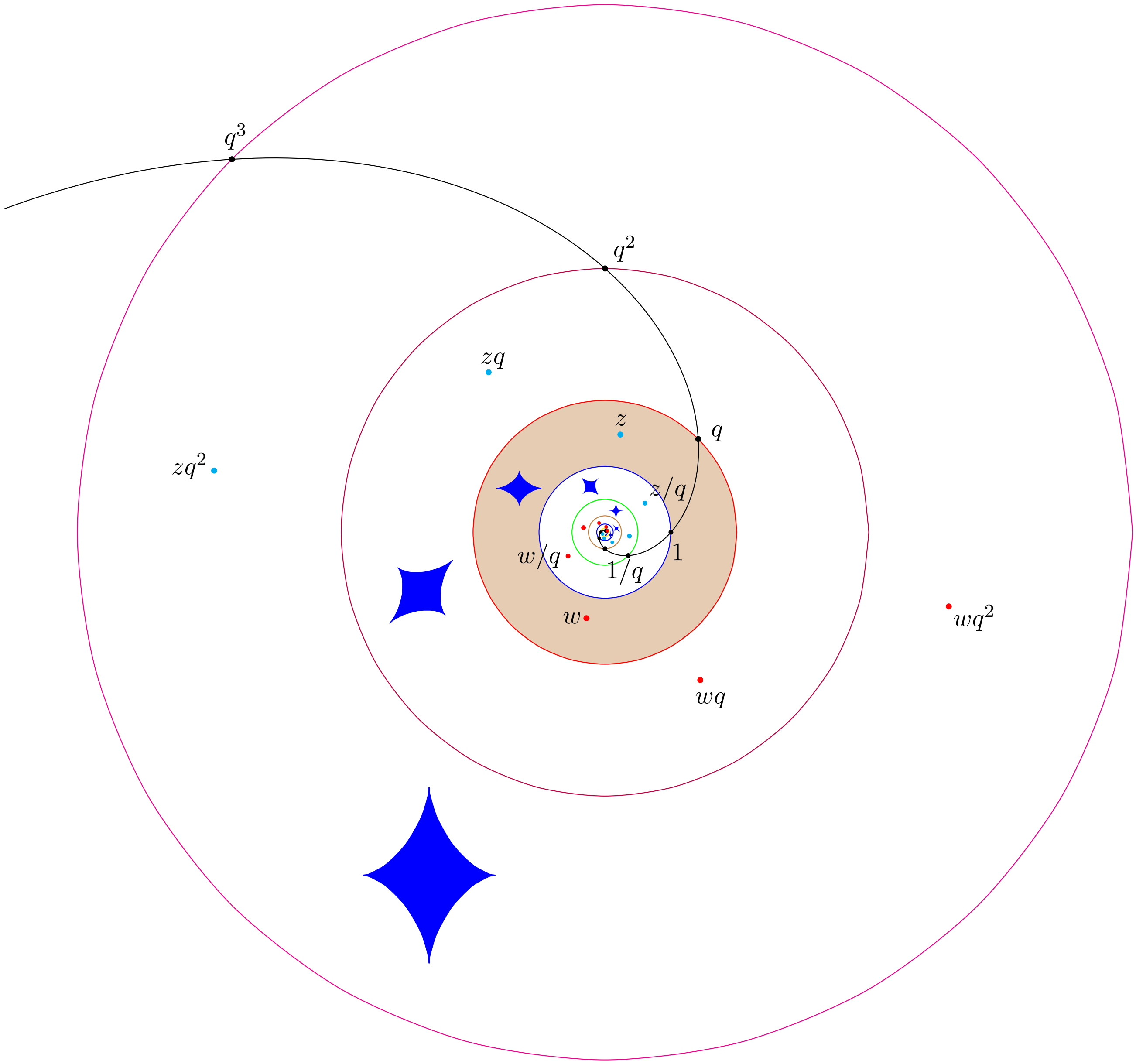

FIGURE 7. $\mathbf{C}^\times/q^{\mathbf{Z}}$ when $q = \sqrt{2}(1+i)$.

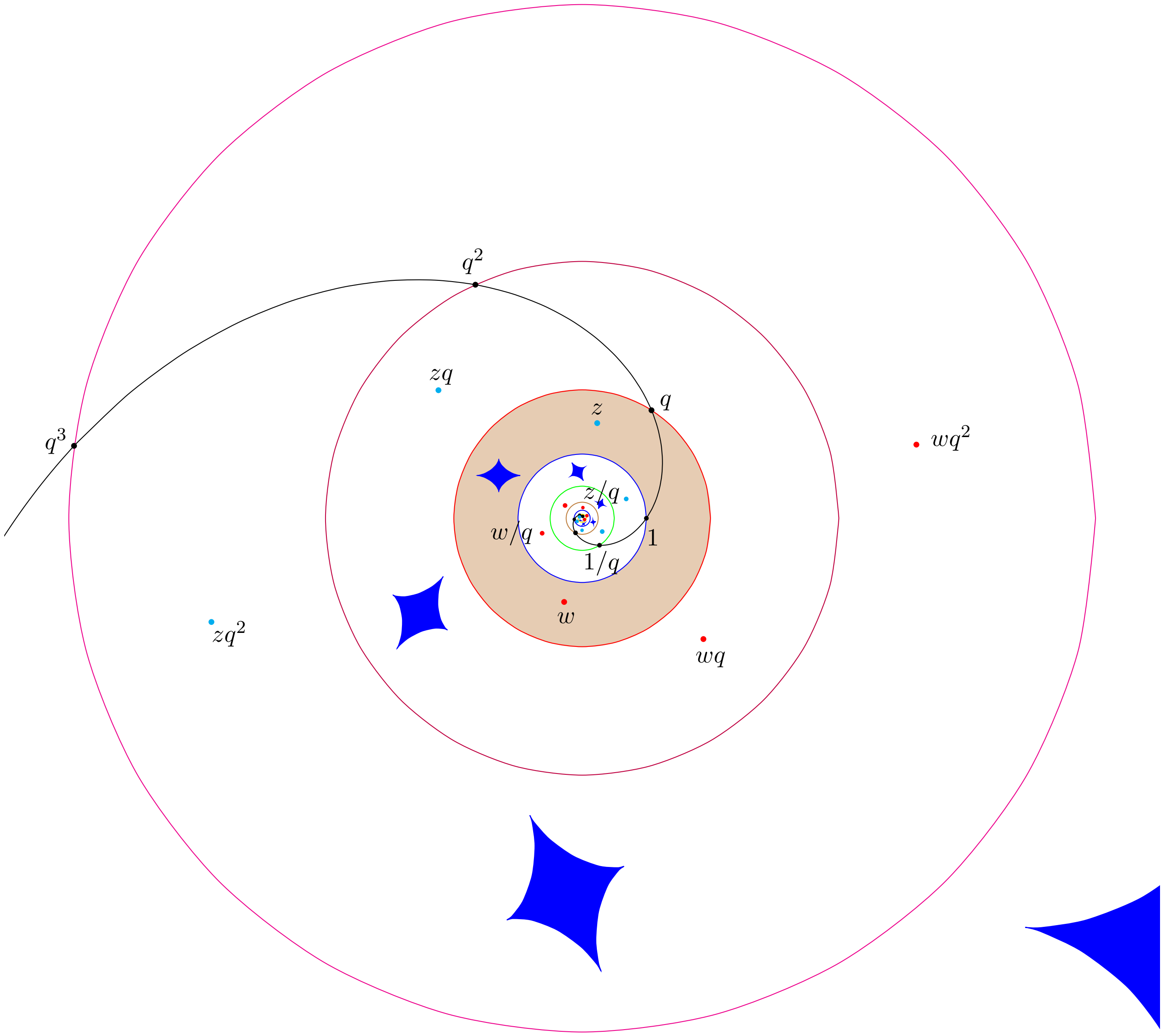

FIGURE 8. $\mathbf{C}^\times/q^{\mathbf{Z}}$ when $q = 2(\cos 1 + i \sin 1)$.

FIGURE 9. *Print Gallery* filled in, 2002.

FIGURE 10. Escher at work.

Appendix A. The group structure of $\mathbf{C}^\times/q^{\mathbf{Z}}$ for $|q| \neq 1$.

The pictures of $\mathbf{C}^\times/q^{\mathbf{Z}}$ suggest that this quotient group resembles a torus geometrically. Let's show how this works algebraically. A torus is $\mathbf{R}^2/\mathbf{Z}^2 \cong (\mathbf{R}/\mathbf{Z})^2$, so our task is to see how $\mathbf{C}^\times/q^{\mathbf{Z}}$ is isomorphic to $(\mathbf{R}/\mathbf{Z})^2$ when $|q| \neq 1$.

We can write any $z \in \mathbf{C}^\times$ in polar coordinates as $r(\cos\theta + i\sin\theta)$, where $r > 0$ and θ is a suitable angle. The second factor $u := \cos\theta + i\sin\theta$ has absolute value 1. (Explicitly, $r = |z|$ and $u = z/|z|$.) Both $\mathbf{R}_{>0}$ and $U = \{u \in \mathbf{C}^\times : |u| = 1\}$ are subgroups of $\mathbf{C}^\times$ and the decomposition $z = ru$ provides an isomorphism of groups $\mathbf{C}^\times \cong \mathbf{R}_{>0} \times U$. The group $\mathbf{R}/\mathbf{Z}$ is isomorphic to U using trigonometric functions: the function $x \bmod \mathbf{Z} \mapsto (\cos 2\pi x, \sin 2\pi x)$ is an isomorphism from $\mathbf{R}/\mathbf{Z}$ to U.

To study $\mathbf{C}^\times/q^{\mathbf{Z}}$ when $q \in \mathbf{C}^\times$ and $|q| \neq 1$, we first consider $q > 0$ and then the general case.

Case 1: $q > 0$ and $q \neq 1$. Then $q^{\mathbf{Z}}$ is a subgroup of $\mathbf{R}_{>0}$, so $\mathbf{C}^\times/q^{\mathbf{Z}} \cong (\mathbf{R}_{>0} \times U)/(q^{\mathbf{Z}} \times \{1\}) \cong (\mathbf{R}_{>0}/q^{\mathbf{Z}}) \times U$. The base q logarithm provides an isomorphism $\log_q \colon \mathbf{R}_{>0} \to \mathbf{R}$ that turns $q^{\mathbf{Z}}$ into $\mathbf{Z}$, so $\mathbf{R}_{>0}/q^{\mathbf{Z}} \cong \mathbf{R}/\mathbf{Z}$. Thus $\mathbf{C}^\times/q^{\mathbf{Z}} \cong \mathbf{R}/\mathbf{Z} \times U \cong \mathbf{R}/\mathbf{Z} \times \mathbf{R}/\mathbf{Z}$, which is a torus.

Case 2: any q with $|q| \neq 1$. We will show $\mathbf{C}^\times/q^{\mathbf{Z}} \cong \mathbf{C}^\times/|q|^{\mathbf{Z}}$, and therefore by the previous case $\mathbf{C}^\times/q^{\mathbf{Z}}$ is a torus. Here we will assume familiarity with the complex exponential function e^z, which is a surjective homomorphism $\mathbf{C} \to \mathbf{C}^\times$ (with kernel $2\pi i\mathbf{Z}$).

We can write q as e^{a+bi} for some real a and b. Then $|q| = e^a$, so $a \neq 0$. Define $f \colon \mathbf{C}^\times \to \mathbf{C}^\times$ by $f(z) = z|z|^{-(b/a)i}$. This unusual choice is a homomorphism of $\mathbf{C}^\times$ to itself and it has the crucial property that it sends q to $|q|$:

$$f(q) = q|q|^{-(b/a)i} = q(e^a)^{-(b/a)i} = qe^{-bi} = e^{a+bi}e^{-bi} = e^a = |q|.$$

This function is not just a homomorphism, but an isomorphism. To see that f is injective we check its kernel is trivial: if $f(z) = 1$ then $z = |z|^{(b/a)i}$, and taking absolute values of both sides gives us $|z| = 1$, so $z = 1^{(b/a)i} = 1$. To see that f is surjective, pick $w \in \mathbf{C}^\times$. We want to find $z \in \mathbf{C}^\times$ such that $z|z|^{-(b/a)i} = w$. Write w in polar form as $re^{i\theta}$ and write the unknown z in polar form as se^{it}. Then $z|z|^{-(b/a)i} = (se^{it})s^{-(b/a)i} = se^{i(t-(b/a)(\log s))}$. In order to solve $z|z|^{-(b/a)i} = w$ we want to solve $se^{i(t-(b/a)(\log s))} = re^{i\theta}$ for some s and t, so we must use $s = r$ and therefore we need $e^{i(t-(b/a)\log r)} = e^{i\theta}$; use $t = (b/a)\log r + \theta$.

Since $f(q) = |q|$, we have $f(q^{\mathbf{Z}}) = f(q)^{\mathbf{Z}} = |q|^{\mathbf{Z}}$. Therefore the isomorphism $f \colon \mathbf{C}^\times \to \mathbf{C}^\times$ induces an isomorphism $\mathbf{C}^\times/q^{\mathbf{Z}} \to \mathbf{C}^\times/f(q)^{\mathbf{Z}} = \mathbf{C}^\times/|q|^{\mathbf{Z}}$.

References

[1] B. de Smit and H. W. Lenstra, "The Mathematical Structure of Escher's Print Gallery," *Notices of the Amer. Math. Soc.* **50** (2003), 446–451.

[2] Droste History, http://www.droste.nl/cms-export/english/about_droste/history/.

[3] H. W. Lenstra, "Escher and the Droste effect," BIRS Public Lecture May 8, 2018, at http://www.birs.ca/publiclectures/2018/18pl001.

[4] Escher and the Droste effect, at http://escherdroste.math.leidenuniv.nl/.