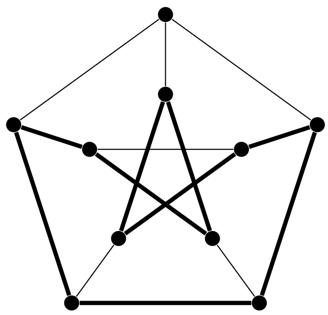
Hamilton cycles

Lecture 6 – Graph Theory 2016 – EPFL – Frank de Zeeuw

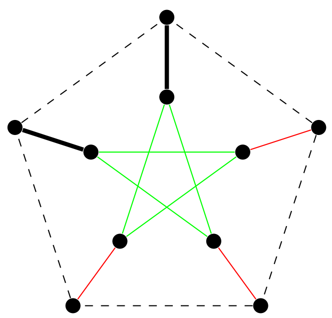
1 Girth and circumference

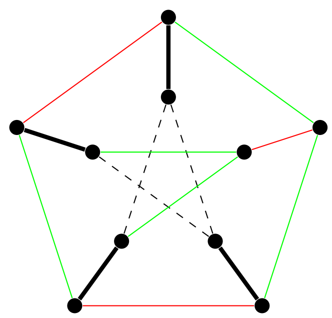
Definition. *The* girth $\mathrm{gir}(G)$ *of a graph* G *is the length of the shortest cycle contained in* G*. The* circumference $\mathrm{circ}(G)$ *is the length of the longest cycle contained in* G*.*

The complete graph K_n for $n \geq 3$ has $\mathrm{gir}(K_n) = 3$ and $\mathrm{circ}(K_n) = n$. For a less obvious example, consider the Petersen graph. It clearly has a 5-cycle, and it is pretty easy to check that it has no 3-cycle or 4-cycle, so it has girth 5. Determining the circumference is more challenging. With a little effort one can find a 9-cycle, like the one depicted below.



Let us prove that it has no 10-cycle, so the circumference is 9. We think of the Petersen graph as an outside 5-cycle and an inside 5-cycle, connected by 5 links. A 10-cycle would have to contain an even number of such links, and not 0 since then we would not get a connected subgraph. Up to isomorphism, this leaves the two cases below: two links or four links (depicted with thick black edges); note that in the case of two links, they must hit adjacent vertices on one of the two 5-cycles, so after possibly swapping the 5-cycles, this is the only case with two links. In each case, we mark edges that cannot be in the cycle with red, and edges that must be in the cycle with green. Whenever a vertex has a red edge, its other two edges must be green or black. And if two edges of a vertex are green or black, then the third edge must be red. In this way we get a contradiction in both cases, either because of a vertex of degree 3 or because of a 5-cycle.



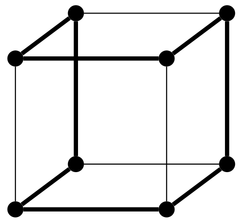


Recall that in the lecture on trees we saw an algorithm for finding the shortest cycle, so we have an algorithm to determine the girth of a graph. The algorithm works by finding, for each edge $xy \in E(G)$, the distance from x to y in $G - xy$, using a breadth-first search tree. The shortest such distance, plus one, is the girth. This works because, given a shortest cycle, if we remove an edge xy from the cycle, then the remaining path must be the shortest path between x and y.

However, determining the circumference is NP-hard, so we do not have a fast algorithm for it. The example of the Petersen graph already illustrates this. In general, a BFS tree gives an easy way to find a shortest path, because it has the special property that a shortest path between two vertices in the BFS tree is also the shortest path in the whole graph. But we do not have a tree or other structure with a similar property for longest paths.

2 Hamilton cycles

Definition. *A* Hamilton cycle *in a graph G is a cycle that contains all vertices of G. A* Hamilton path *in a graph G is a path that contains all vertices of G.*



A graph has a Hamilton cycle if and only if $\mathrm{circ}(G) = |V(G)|$. Just like determining the cicumference, it is NP-hard to find a Hamilton cycle in a graph (or determine that there is none. A related problem is to find a shortest Hamilton cycle in a graph with weighted edges; this is called the *travelling salesman problem* and is one of the most famous NP-hard problems.

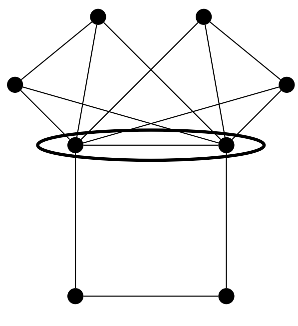
Although we have no general algorithm for finding Hamilton cycles, we can still prove some theorems that are useful in certain situations.

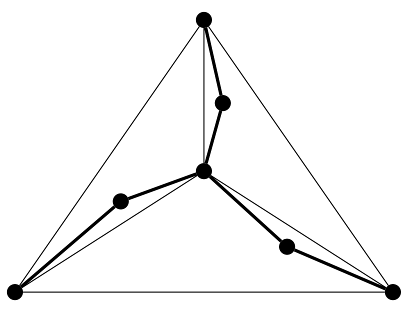
A necessary condition. Given a graph G and a set $S \subset V(G)$ of vertices, we write $G - S$ for the graph obtained by removing the vertices of S from G, along with all the edges that are incident to vertices in S.

Lemma 2.1. *If G has a Hamilton cycle, then for all $S \subset V(G)$, $G - S$ has at most $|S|$ connected components.*

Proof. The Hamilton cycle must visit all the components of $G - S$ (viewed as subgraphs of G), and to get from one component to another the cycle must pass through a vertex of S. Thus every component is connected to S by two edges of the cycle (and possibly by other edges not in the cycle). Since every vertex is incident to two edges of the cycle, we have that twice the number of components is at most twice the number of vertices of S. $\square$

This lemma can be useful to show that a graph does not have a Hamilton cycle. For example, if in the left-hand graph G below we let S consist of the middle two vertices, then $G - S$ has three connected components, so by Lemma 2.1 the graph has no Hamilton cycle.





On the other hand, one can check that the right-hand graph H satisfies the condition that for all $S \subset V(H)$, $H - S$ has at most $|S|$ components. Nevertheless, the graph has no Hamilton cycle. To see this, observe that for the vertices of degree 2, both incident edges would have to be in the cycle; but then the middle vertex would be incident to three edges of the cycle, which is impossible.

3 Sufficient conditions

Next we prove two sufficient conditions for a graph to have a Hamilton cycle. First we show that a graph with many edges must have a Hamilton cycle. However, note that this bound is suprisingly weak, because a graph with this many edges is almost complete. In the proof the following definition will be convenient.

Definition. *The* complement *of a graph G is the graph $\overline{G}$ with vertex set $V(\overline{G}) = V(G)$ and edge set $E(\overline{G}) = \{xy : x, y \in V(G), xy \notin E(G)\}$.*

Theorem 3.1. *If G is a graph with $|E(G)| > \binom{|V(G)|-1}{2} + 1$, then G has a Hamilton cycle.*

Proof. Set $n = |V(G)|$. The statement is clearly true for $n = 1, 2, 3$, so we assume $n > 3$. Note that $\binom{n-1}{2} + 1 = \binom{n}{2} - (n-2)$. Thus the condition of the theorem means that $|E(\overline{G})| < n-2$. Thus $\sum d_{\overline{G}}(v) = 2(n-2) < 2n$, which implies that there must be a vertex v such that $d_{\overline{G}}(v) \leq 1$. Then we have $d_G(v) \geq n-2$. We remove the vertex v from G, and we will apply induction to $G - v$. We distinguish the two cases $d(v) = n-2$ and $d(v) = n-1$.

Suppose $d(v) = n-2$. Then

$$|E(G-v)| = |E(G)| - (n-2) > \binom{n-1}{2} + 1 - (n-2) = \binom{n-2}{2} + 1 = \binom{|V(G-v)| - 1}{2} + 1.$$

Hence, by induction, the graph $G - v$ has a Hamilton cycle C. Since $d(v) = n-2$ and $n > 3$, v must have two neighbors u, w that are adjacent on C. Then we can remove uw from C and replace it by uv and vw, which results in a Hamilton cycle for G.

Suppose $d(v) = n-1$. In this case we only have $|E(G-v)| > \binom{|V(G-v)|-1}{2}$, so we cannot apply induction right away. If $G - v$ is complete, then $G - v$ has a Hamilton cycle, and we can add v as in the previous case. Otherwise, we can add an arbitrary edge e to $G - v$, and apply induction to find a Hamilton cycle C in $G - v + e$. If C does not contain e, then we can again add v as in the previous case. If C does contain e, then removing e from C gives a "Hamilton path" P in $G - v$. Since $d(v) = n-1$, v is connected to all vertices of $G - v$, and in particular to the endpoints u, w of P. Then adding uv and vw to P gives a Hamilton cycle of G. □

The statement in Theorem 3.1 cannot be improved, in the sense that a weaker bound on $|E(G)|$ does not imply a Hamilton cycle. Take for instance the graph G consisting of K_{n-1} and a single vertex connected to a single vertex of K_{n-1}. This graph has $|E(G)| = \binom{n-1}{2} + 1$, but it has no Hamilton cycle, since it has a vertex of degree 1.

As said, the condition in Theorem 3.1 is somewhat weak in the sense that many graphs that have a Hamilton cycle do not satisfy the condition. The following sufficient condition does better by looking at the minimum degree instead of the total number of edges. Its proof is a typical extremal argument that we have seen before, for instance in the proof of a lemma in Lecture 1, which said that a graph must have a cycle of length $\delta(G) + 1$. But note that that lemma by itself is not strong enough to imply a Hamilton cycle.

Theorem 3.2 (Dirac)**.** *Let G be a graph with $|V(G)| \geq 3$. If $\delta(G) \geq \frac{1}{2}|V(G)|$, then G has a Hamilton cycle.*

Proof. First observe that G must be connected, since otherwise each connected component would contain at least $\delta(G) + 1 > \frac{1}{2}|V(G)|$ vertices, which is impossible.

Take a longest path $P = x_1 x_2 \cdots x_k$ in G. By maximality, all neighbors of x_1 and x_k are on the path. Thus $\delta(G) \geq \frac{1}{2}|V(G)|$ gives the following two inequalities:

$$|\{x_i : 1 \leq i \leq k-1, x_i x_k \in E(G)\}| \geq \frac{1}{2}|V(G)|,$$
$$|\{x_i : 1 \leq i \leq k-1, x_{i+1} x_1 \in E(G)\}| \geq \frac{1}{2}|V(G)|.$$

In other words, we have two subsets of size at least $\frac{1}{2}|V(G)|$ that are contained in the set $\{x_1, \ldots, x_{k-1}\}$, which has $k - 1 < |V(G)|$ elements. It follows that the two subsets share an element x_i, which means that we have $x_i x_k \in E(G)$ and $x_{i+1} x_1 \in E(G)$. Then $C = x_i \cdots x_1 x_{i+1} \cdots x_k x_i$ is a cycle.

In fact, C is a Hamilton cycle. Indeed, suppose there is a vertex u not in C. Since G is connected, there is a path from u to (say) x_1. There is a vertex v on this path that is not on C but that is adjacent to some x_j. Then there is a path that goes from v to x_j, then all around the cycle C to a neighbor of x_j. This path contains $k + 1$ vertices, contradicting the maximality of P. $\square$

This theorem is again best possible, in the sense that a weaker bound on the minimum degree would not imply a Hamilton cycle. Take for instance the graph G consisting of two copies of K_k sharing a single vertex. This graph has $n = 2k - 1$ vertices and minimum degree $\delta(G) = k - 1 = \frac{1}{2}|V(G)| - \frac{1}{2}$, but no Hamilton cycle.