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Correspondence theorem (group theory)

In the area of mathematics known as group theory, the **correspondence theorem**,[1][2][3][4][5][6][7][8] sometimes referred to as the **fourth isomorphism theorem**[6][9][note 1][note 2] or the **lattice theorem**,[10] states that if N is a normal subgroup of a group G, then there exists a bijection from the set of all subgroups A of G containing N, onto the set of all subgroups of the quotient group G/N. The structure of the subgroups of G/N is exactly the same as the structure of the subgroups of G containing N, with N collapsed to the identity element.

Specifically, if

G is a group,
N is a normal subgroup of G,
$\mathcal{G}$ is the set of all subgroups A of G such that $N \subseteq A \subseteq G$, and
$\mathcal{N}$ is the set of all subgroups of G/N,

then there is a bijective map $\phi : \mathcal{G} \to \mathcal{N}$ such that

$\phi(A) = A/N$ for all $A \in \mathcal{G}$.

One further has that if A and B are in $\mathcal{G}$, and $A' = A/N$ and $B' = B/N$, then

- $A \subseteq B$ if and only if $A' \subseteq B'$;
- if $A \subseteq B$ then $|B : A| = |B' : A'|$, where $|B : A|$ is the index of A in B (the number of cosets bA of A in B);
- $\langle A, B\rangle/N = \langle A', B'\rangle$, where $\langle A, B\rangle$ is the subgroup of G generated by $A \cup B$;
- $(A \cap B)/N = A' \cap B'$, and
- A is a normal subgroup of G if and only if A' is a normal subgroup of G/N.

This list is far from exhaustive. In fact, most properties of subgroups are preserved in their images under the bijection onto subgroups of a quotient group.

More generally, there is a monotone Galois connection (f^*, f_*) between the lattice of subgroups of G (not necessarily containing N) and the lattice of subgroups of G/N: the lower adjoint of a subgroup H of G is given by $f^*(H) = HN/N$ and the upper adjoint of a subgroup K/N of G/N is a given by $f_*(K/N) = K$. The associated closure operator on subgroups of G is $\bar{H} = HN$; the associated kernel operator on subgroups of G/N is the identity.

Similar results hold for rings, modules, vector spaces, and algebras.

See also

- Modular lattice

Notes

1. Some authors use "fourth isomorphism theorem" to designate the Zassenhaus lemma; see for example by Alperin & Bell (p. 13) or Robert Wilson (2009). *The Finite Simple Groups*. Springer. p. 7. ISBN 978-1-84800-988-2.
2. Depending how one counts the isomorphism theorems, the correspondence theorem can also be called the 3rd isomorphism theorem; see for instance H.E. Rose (2009), p. 78.

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