3-9 Connectivity

(Part II: Menger's Theorem)

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如果两个割点相连，那么联通块怎么划分
联通快呢）

menger定理吧

暂无

好像没有……

Menger定理的证明看不懂

menger定理的证明

不。。不记得了

还好理解，只是都不怎么容易理解

menger定理的证明没太理解 老师辛苦了！

点割集，边割集

Menger's Theorem

(Theorem 5.16; Theorem 5.21)

Theorem (Menger's Theorem (Theorem 5.16))

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The minimum number of vertices in a $u-v$ separating set equals the maximum number of internally disjoint $u-v$ paths in G.

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Can you rearrange these three cases to make them (hopefully) easier to understand?

By induction on the number m of edges of G.

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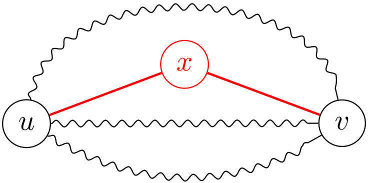
Case I: There exists a minimum $u - v$ separating set W in G containing a vertex x that is adjacent to both u and v.



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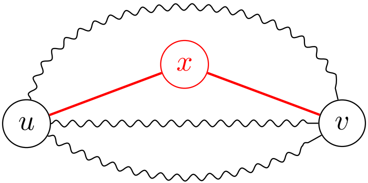
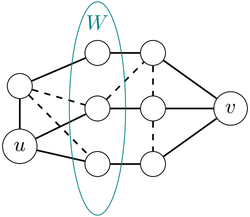


$W - \{x\}$ is a minimum $u - v$ separating set in $G - x$

By induction on the number m of edges of G.

CASE II: There exists a minimum $u - v$ separating set W in G containing a vertex in W that is not adjacent to u and a vertex in W that is not adjacent to v.



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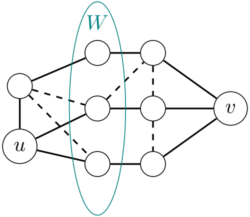

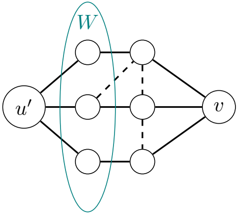

G'_v

By induction on the number m of edges of G.

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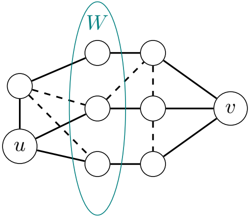
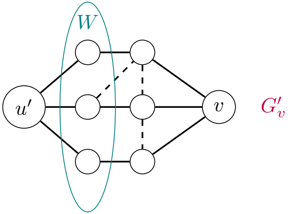
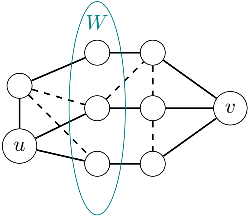


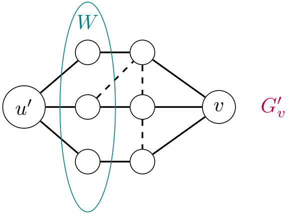


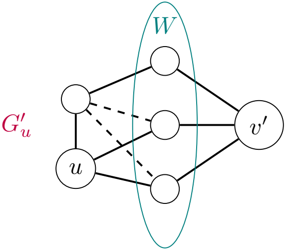
$$\boxed{m(G'_v) < m(G)}$$

By induction on the number m of edges of G.

CASE II: There exists a minimum $u-v$ separating set W in G containing a vertex in W that is not adjacent to u and a vertex in W that is not adjacent to v.





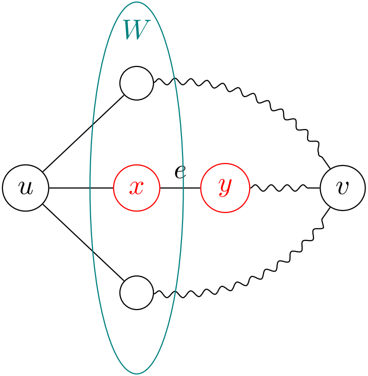


$\boxed{m(G'_v) < m(G)}$

$\boxed{m(G'_u) < m(G)}$

By induction on the number m of edges of G.

CASE III: For each minimum $u - v$ separating set W in G, either every vertex of W is adjacent to u and not adjacent to v or every vertex of W is adjacent to v and not adjacent to u.



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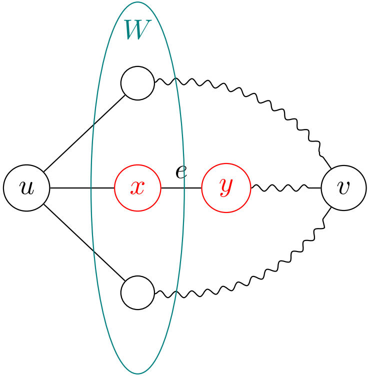

$$\boxed{P = u, x, y, \ldots, v}$$

A $u - v$ shortest simple path in G

By induction on the number m of edges of G.

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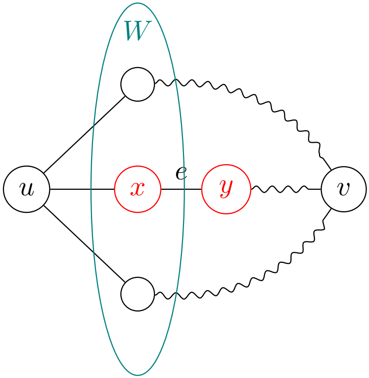

$$\boxed{P = u, x, y, \ldots, v}$$

A $u-v$ shortest simple path in G

$$m(G-e) < m(G)$$

A minimum $u-v$ separating set in $G-e$ contains k vertices.

CASE I: There exists a minimum $u-v$ separating set W in G containing a vertex x that is adjacent to both u and v.

CASE II: There exists a minimum $u-v$ separating set W in G containing a vertex in W that is not adjacent to u and a vertex in W that is not adjacent to v.

CASE III: For each minimum $u-v$ separating set W in G, either every vertex of W is adjacent to u and not adjacent to v or every vertex of W is adjacent to v and not adjacent to u.

CASE I: There exists a minimum $u - v$ separating set W in G containing a vertex x that is adjacent to both u and v.

$$\exists W : \exists x \in W : x - u \land x - v$$

CASE II: There exists a minimum $u - v$ separating set W in G containing a vertex in W that is not adjacent to u and a vertex in W that is not adjacent to v.

CASE III: For each minimum $u - v$ separating set W in G, either every vertex of W is adjacent to u and not adjacent to v or every vertex of W is adjacent to v and not adjacent to u.

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$$\exists W : \exists x \in W : x - u \wedge x - v$$

CASE II: There exists a minimum $u-v$ separating set W in G containing a vertex in W that is not adjacent to u and a vertex in W that is not adjacent to v.

$$\begin{aligned} \exists W : \exists x \in W : x \not- u \\ \wedge \exists y \in W : y \not- v \end{aligned}$$

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$$\forall W : \forall x \in W : x - u \land x \not- v$$
$$\lor\ \forall x \in W : x - v \land x \not- u$$

$$\mathrm{I} : \exists W : \exists x \in W : x - u \land x - v$$

$$\begin{aligned} \mathrm{II} : \exists W : \exists x \in W &: x \not- u \\ \land \exists y \in W &: y \not- v \end{aligned}$$

$$\begin{aligned} \mathrm{III} : \forall W : \forall x \in W &: x - u \land x \not- v \\ \lor \forall x \in W &: x - v \land x \not- u \end{aligned}$$

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$$\mathrm{III} \equiv \mathrm{II}' \wedge \mathrm{I}'$$

$$\mathrm{I} : \exists W : \exists x \in W : x - u \land x - v$$

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II

II′

I

III

II

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\begin{aligned}
\mathrm{II}: \exists W : \exists x \in W : x \not- u \\
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\end{aligned}
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II′

$$
\mathrm{I}: \exists W : \exists x \in W : x - u \wedge x - v
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$$
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II

$$\exists W : W \not\subseteq N(u) \\ \land W \not\subseteq N(v)$$

II′

II

$$
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\end{aligned}
$$

II′

$$
\mathrm{I}: \exists W : \exists x \in W : x - u \land x - v
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II′

$$
\exists W : \exists x \in W : x \in N(u) \cap N(v)
$$

$$
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$$\forall W : W \subseteq N(u) \land W \cap N(v) = \emptyset \\ \lor W \subseteq N(v) \land W \cap N(u) = \emptyset$$

II

$$\text{II}: \exists W: W \nsubseteq N(u) \land W \nsubseteq N(v)$$

II′

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For distinct vertices u and v in a graph G,

the minimum number of edges of G that separate u and v equals the maximum number of pairwise edge-disjoint $u-v$ paths in G.

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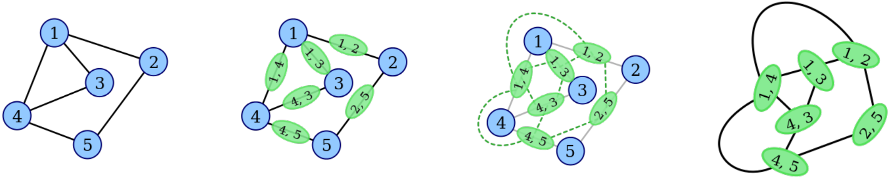
Line Graph



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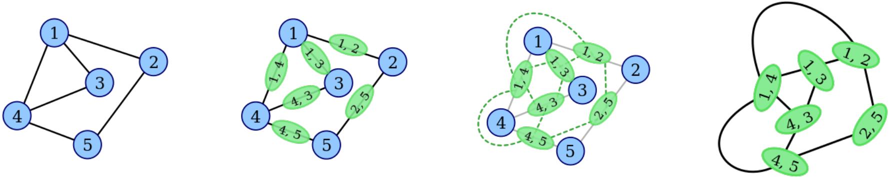
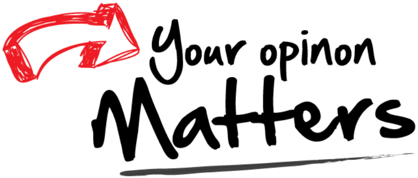


Definition 4.2.18 & Theorem 4.2.19
of "Introduction to Graph Theory" by Douglas B. West


THANK YOU



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