

4-5 Polyhedral Groups (II)

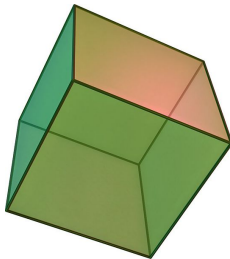
(Cube)

Hengfeng Wei

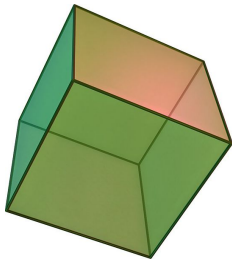
hfwei@nju.edu.cn

April 08, 2019





$$\text{Sym}(C) \cong S_4$$

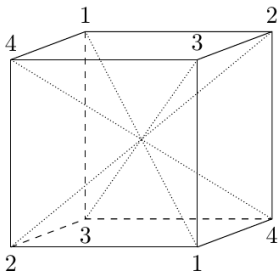


$$\text{Sym}(C) \cong S_4$$

$$\left| \{H : H \leq \text{Sym}(C)\} \right| = 30$$

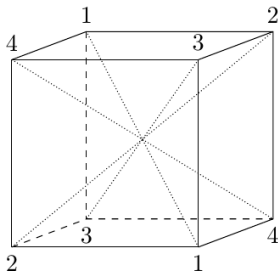
$$|\mathrm{Sym}(C)| \leq 24$$

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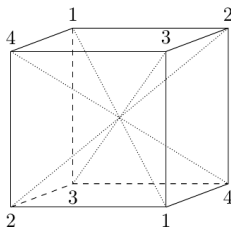
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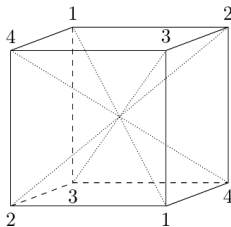
$$\text{Sym}(C) \leq S_4$$

$$|\text{Sym}(C)| = \underbrace{6}_{\text{Facing Upward}} \times \underbrace{4}_{\text{Rotation}}$$

Order of 1: id ($\# = 1$)

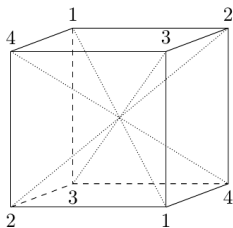


Order of 1: id ($\# = 1$)



Order of 4: face-to-face ($\# = 9$)

$$\begin{aligned}
 f_{td} &= (1\ 2\ 3\ 4) & f_{td}^2 &= (1\ 3)(2\ 4) & f_{td}^3 &= (1\ 4\ 3\ 2) \\
 f_{lr} &= (1\ 3\ 2\ 4) & f_{lr}^2 &= (1\ 2)(3\ 4) & f_{lr}^3 &= (1\ 4\ 2\ 3) \\
 f_{fb} &= (1\ 2\ 4\ 3) & f_{fb}^2 &= (1\ 4)(2\ 3) & f_{fb}^3 &= (1\ 3\ 4\ 2)
 \end{aligned}$$



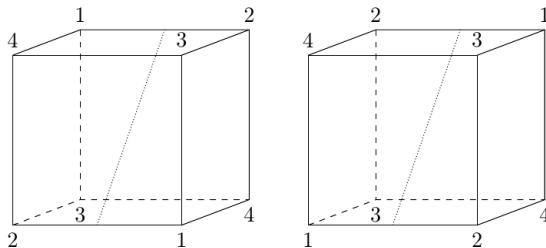
Order of 3: vertex-to-vertex ($\# = 8$)

$$v_1 = (2\ 3\ 4) \quad v_1^2 = (2\ 4\ 3)$$

$$v_2 = (1\ 4\ 3) \quad v_2^2 = (1\ 3\ 4)$$

$$v_3 = (1\ 2\ 4) \quad v_3^2 = (1\ 4\ 2)$$

$$v_4 = (1\ 2\ 3) \quad v_4^2 = (1\ 3\ 2)$$



Order of 2: edge-to-edge ($\# = 6$)

$$e_{12} = (1\ 2) \quad e_{13} = (1\ 3) \quad e_{14} = (1\ 4)$$

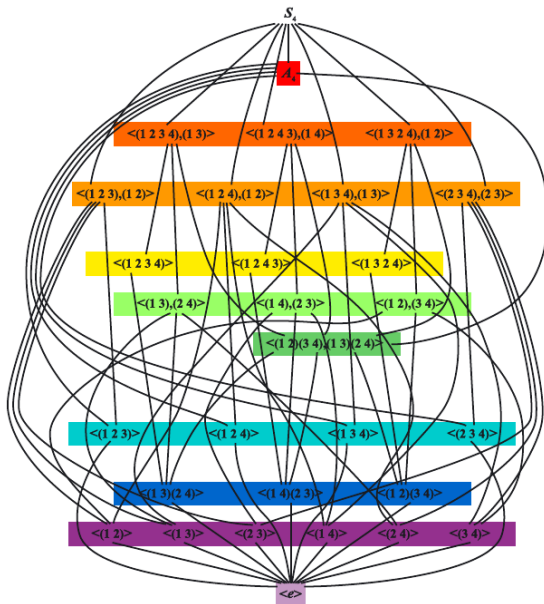
$$e_{23} = (2\ 3) \quad e_{24} = (2\ 4) \quad e_{34} = (3\ 4)$$

flag永不倒!



$$\left| \{H : H \leq \text{Sym}(C)\} \right| = 30$$

order of subgroup	conjugacy classes	<div> <div>A_1</div> <div>D_1</div> <div>S_1</div> </div> <div> <div>Z_4</div> <div>V</div> <div>V</div> </div> <div> <div>Z_3</div> <div>Z_2</div> <div>Z_1</div> </div>
24	S_4	
12	A_4	
8	$\langle(1\ 2\ 3\ 4), (1\ 3)\rangle$ $\langle(1\ 2\ 4\ 3), (1\ 4)\rangle$ $\langle(1\ 3\ 2\ 4), (1\ 2)\rangle$	
6	$\langle(1\ 2\ 3), (1\ 2)\rangle$ $\langle(1\ 2\ 4), (1\ 2)\rangle$ $\langle(1\ 3\ 4), (1\ 3)\rangle$ $\langle(2\ 3\ 4), (2\ 3)\rangle$	
4	$\langle(1\ 2\ 3\ 4)\rangle$ $\langle(1\ 2\ 4\ 3)\rangle$ $\langle(1\ 3\ 2\ 4)\rangle$ $\langle(1\ 3), (2\ 4)\rangle$ $\langle(1\ 4), (2\ 3)\rangle$ $\langle(1\ 2), (3\ 4)\rangle$ $\langle(1\ 2)(3\ 4), (1\ 3)(2\ 4)\rangle$	
3	$\langle(1\ 2\ 3)\rangle$ $\langle(1\ 2\ 4)\rangle$ $\langle(1\ 3\ 4)\rangle$ $\langle(2\ 3\ 4)\rangle$	
2	$\langle(1\ 3)(2\ 4)\rangle$ $\langle(1\ 4)(2\ 3)\rangle$ $\langle(1\ 2)(3\ 4)\rangle$ $\langle(1\ 2)\rangle$ $\langle(1\ 3)\rangle$ $\langle(2\ 3)\rangle$ $\langle(1\ 4)\rangle$ $\langle(2\ 4)\rangle$ $\langle(3\ 4)\rangle$	
1	$\langle e \rangle$	



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Order of 2: edge-to-edge ($\# = 6$)

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$$H \leq S_4 \implies |H| = 1, \quad 2, \quad 3, \quad 4, \quad 6, \quad 8, \quad 12, \quad 24$$

$$|H| = \begin{cases} 1 : & \text{id} \quad (\# = 1) \\ 2 : & (\# = 6 + 3 = 9) \\ 3 : & v_1, v_2, v_3, v_4 \quad (\# = 4) \\ 4 : & (\# = 7) \\ 6 : & (\# = 4) \\ 8 : & (\# = 3) \\ 12 : & A_4 \quad (\# = 1) \\ 24 : & S_4 \quad (\# = 1) \end{cases}$$

$$|G| = 4 \implies G \cong \mathbb{Z}_4 \vee G \cong K_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

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$$\textcolor{brown}{v}_1 = (2\ 3\ 4) \quad v_1^2 = (2\ 4\ 3)$$

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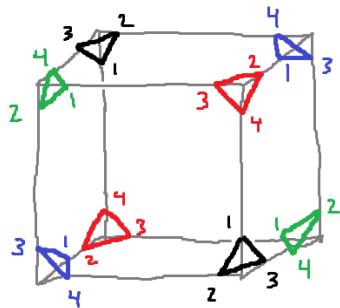
$$\textcolor{red}{v}_2 = (1\ 4\ 3) \quad v_2^2 = (1\ 3\ 4)$$

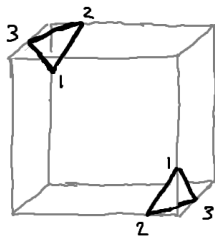
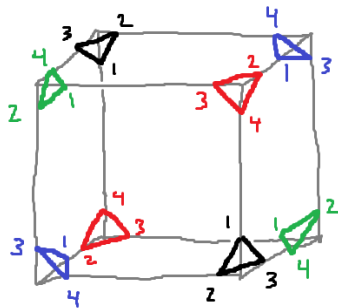
$$\textcolor{red}{v}_3 = (1\ 2\ 4) \quad v_3^2 = (1\ 4\ 2)$$

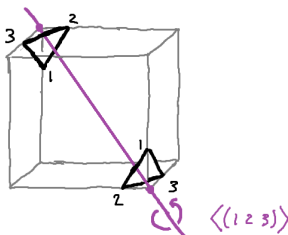
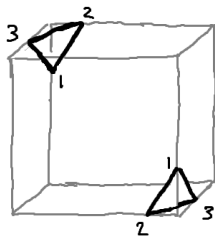
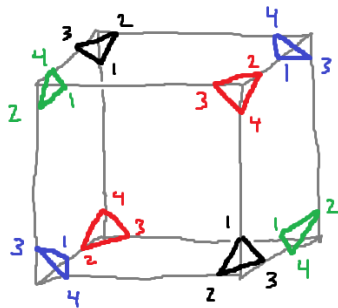
$$\textcolor{red}{v}_4 = (1\ 2\ 3) \quad v_4^2 = (1\ 3\ 2)$$

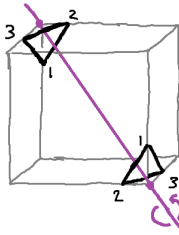
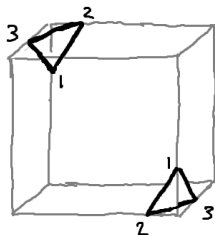
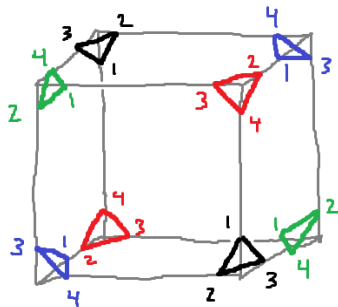
Theorem

There are only 4 subgroups $\cong \textcolor{red}{D}_3$ in S_4 .

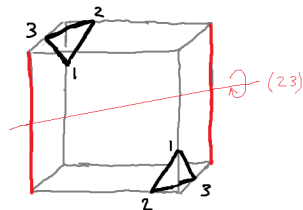








$\langle (123) \rangle$



(23)

$$|G| = 8 \implies G \cong \mathbb{Z}_8, \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \mathbb{Z}_4 \times \mathbb{Z}_2, \quad D_4, \quad Q_8$$

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$$Q_8 = \{\pm 1, \pm I, \pm J, \pm K\} \quad (\text{Example 3.15})$$

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$$H \not\cong Q_8 \implies |H| \geq 9$$

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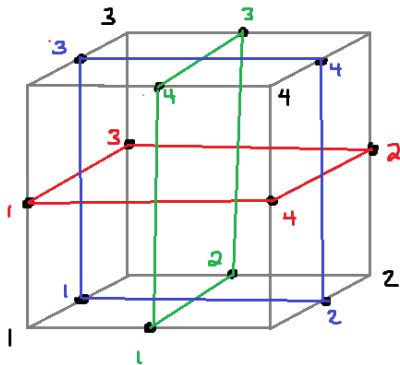
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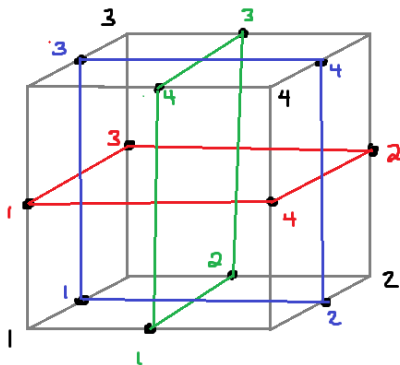
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Theorem

There are only 3 subgroups $\cong D_4$ of S_4 .





$$H_{\text{green}} = \langle (1\ 2\ 3\ 4), (1\ 3) \rangle$$

$$H_{\text{blue}} = \langle (1\ 2\ 4\ 3), (1\ 4) \rangle$$

$$H_{\text{red}} = \langle (1\ 3\ 2\ 4), (1\ 2) \rangle$$

$$|G| = 12 \implies G \cong \mathbb{Z}_{12}, \quad \mathbb{Z}_6 \times \mathbb{Z}_2, \quad D_6, \quad A_4, \quad \text{Dic}_{12}$$

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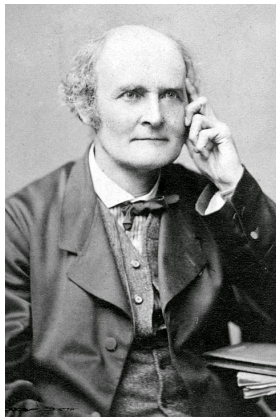
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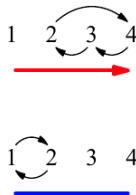
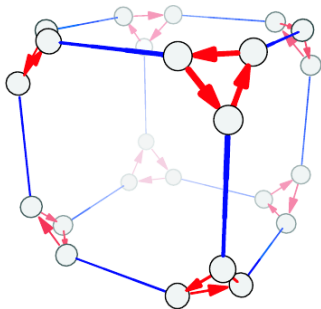
$$H \cong A_4$$

Theorem

There is only one subgroup of order 12 in S_4 .



Arthur Cayley (1821 – 1895)



$\text{Sym}(C) \cong S_4$ arranged on a *truncated* cube







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