1-12 Partial Order and Lattice Theory

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SM Problem 14.44

Suppose the following are three consistent enumerations of an ordered set $A = \{a, b, c, d\}$:

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\begin{array}{lcccc}
A_1: & a & b & c & d \\
A_2: & a & c & b & d \\
A_3: & a & c & d & b
\end{array}
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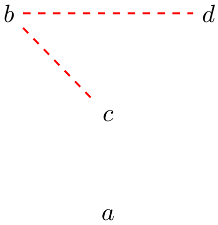
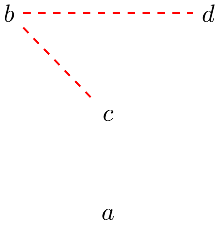
$$b \prec_{A_1} c \land c \prec_{A_2} b \implies b \parallel_A c$$

$$d \prec_{A_3} b \land b \prec_{A_2} d \implies b \parallel_A d$$


b
d
c
a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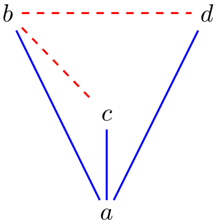
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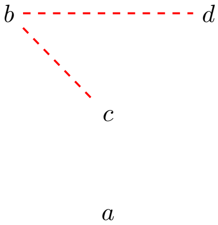
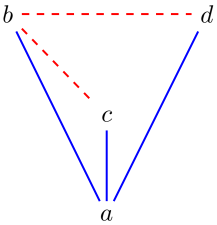
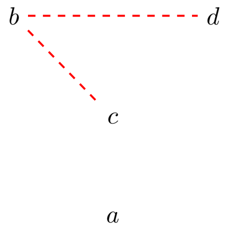
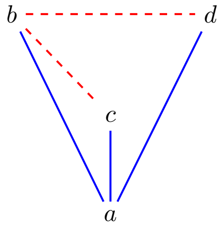
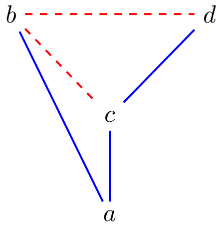




$\# = 6$







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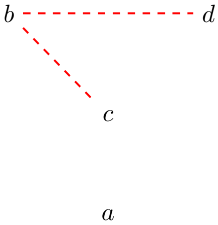

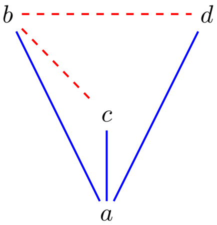

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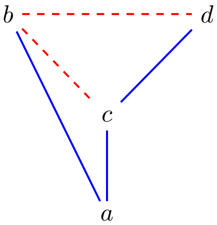

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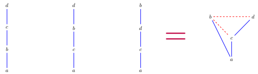
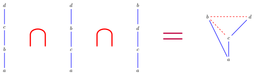


Theorem



Theorem

Let A be a nonempty finite set with $|A| = n$ and let $a \in A$.
Prove that $A \setminus \{a\}$ is finite and $|A \setminus \{a\}| = n - 1$.

Set Union (UD Problem 23.1)

Give an example, if possible, of

(a) A countably infinite collection of pairwise disjoint finite sets whose union is countably infinite.

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$$|A| = n \implies |\mathcal{P}(A)| = 2^n$$

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By Diagonal Argument.

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Was Cantor Surprised?

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Thank
You!