2-4 Recurrences

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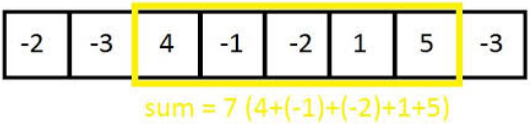


$$O(n^3) \implies O(n^2) \implies O(n \log n) \implies O(n)$$

$$T(n) = aT(n/b) + f(n)$$

Solving it

Algorithm Analysis

Algorithm Design

Maximum-sum Subarray (mss; Problem 4.1-5)

$$A[0 \ldots n-1] \qquad \forall\, 0 \le i \le n-1 : A[i] \in \mathbb{Z}$$

To find (the sum of) a maximum-sum subarray of A

$$A = [-2, 11, -4, 13, -5, -2]$$

$$\mathsf{mss} = 11 + (-4) + 13 = 20$$

$$\forall\, 0 \le i \le n-1 : A[i] < 0$$

$$\mathsf{mss} = 0 \textit{ vs. } \mathsf{mss} = \max_{0 \le i \le n-1} A[i]$$

$\mathsf{mss\text{-}at}[i]$: (the sum of) a maximum-sum subarray ending with $A[i]$

$$\mathsf{mss} = \max_{0 \le i \le n-1} \mathsf{mss\text{-}at}[i]$$

Q : Where does $\mathsf{mss\text{-}at}[i]$ start?

$$\boxed{\mathsf{mss\text{-}at}[i] = \max\{\mathsf{mss\text{-}at}[i-1] + A[i], A[i]\}}$$

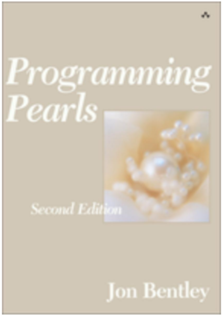
$$\mathsf{mss\text{-}at}[0] = A[0]$$

```
procedure MSS(A, n)
    mss-at[0] ← A[0]
    for i ← 1 ... n − 1 do
        mss-at[i] ← max{mss-at[i − 1] + A[i], A[i]}
    return max_{0≤i≤n−1} mss-at[i]
```

time	*space*
$O(n)$	$O(n)$

```
procedure MSS(A, n)
    mss ← −∞
    mss-at ← A[0]
    for i ← 1 … n − 1 do
        mss-at ← max{mss-at + A[i], A[i]}
        mss ← max{mss, mss-at}
    return mss
```

time	*space*
$O(n)$	$O(1)$



Ulf Grenander $O(n^3) \implies O(n^2)$

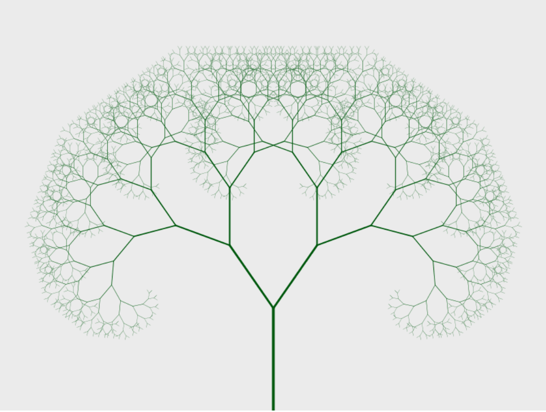
Michael Shamos $O(n \log n)$, one night

Jon Bentley Conjecture: $\Omega(n \log n)$

Michael Shamos Carnegie Mellon seminar

Jay Kadane $O(n)$, ≤ 1 minute

Recurrences



$$T(n) = aT(n/b) + f(n) \quad (a > 0,\ b > 1)$$

Assume that $T(n)$ is constant for sufficiently small n.

$$\left.\begin{array}{r} f(n) \\ af(\frac{n}{b}) \\ a^2 f(\frac{n}{b^2}) \\ \vdots \\ a^{\log_b n} T(1) = \Theta(n^{\log_b a}) \end{array}\right\} \sum \overset{f(n)\ vs.\ n^E}{=} \begin{cases} n^{\log_b a}, & f(n) = O(n^{E-\epsilon}) \\ n^{\log_b a} \log n, & f(n) = \Theta(n^E) \\ f(n), & f(n) = \Omega(n^{E+\epsilon}) \end{cases}$$

$$E \triangleq \log_b a \qquad \textit{(critical exponent)}$$

TC 4.5-4: Gap in Mater Theorem

$$T(n) = 4T(n/2) + n^2 \log n = \Theta(n^2 \log^2 n)$$

$$E \triangleq \log_b a = 2$$

$$f(n) = O(n^{E-\epsilon}) \qquad f(n) = \Theta(n^E) \qquad f(n) = \Omega(n^{E+\epsilon})$$

$$n^2 \log n = o(n^{2+\epsilon})$$

$$\boxed{f(n) = \Theta(n^{\log_b a} \log^k n) \implies \Theta(n^{\log_b a} \log^{k+1} n) \qquad (k \geq 0)}$$

TC 4-3 (e): Gaps in Master Theorem

$$T(n) = 2T(n/2) + \frac{n}{\log n}$$

$$E \triangleq \log_b a = 1$$

$$f(n) = O(n^{E-\epsilon}) \qquad f(n) = \Theta(n^E) \qquad f(n) = \Omega(n^{E+\epsilon})$$

$$\frac{n}{\log n} = \omega(n^{1-\epsilon})$$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$$T(n) = 2T(n/2) + \frac{n}{\log n}$$

$$\boxed{T(n) = \Theta(n)}$$

$$T(n) = \Omega(n) \qquad T(n) \geq cn$$

$$T(n) = 2T(n/2) + \frac{n}{\log n}$$
$$\geq 2c \cdot \frac{n}{2} + \frac{n}{\log n}$$
$$= cn + \frac{n}{\log n}$$

$$c = 1$$

$$T(n) = O(n) \qquad T(n) \leq cn - d$$

$$T(n) = 2T(n/2) + \frac{n}{\log n}$$
$$\leq 2(c \cdot \frac{n}{2} - d) + \frac{n}{\log n}$$
$$= cn + \frac{n}{\log n} - 2d$$

$$\frac{n}{\log n} \leq d$$

$$
\begin{aligned}
T(n) &= 2T(n/2) + \frac{n}{\log n} \\
&= 2(2T(\frac{n}{4}) + \frac{\frac{n}{2}}{\log \frac{n}{2}}) + \frac{n}{\log n} \\
&= 2^2 T(\frac{n}{2^2}) + \frac{n}{\log n - 1} + \frac{n}{\log n} \\
&= \dots \\
&= 2^k T(\frac{n}{2^k}) + \sum_{i=0}^{k-1} \frac{n}{\log n - i}
\end{aligned}
$$

$$\frac{n}{2^k} = 1 \implies k = \log n$$

$$
\begin{aligned}
T(n) &= 2^k T(\frac{n}{2^k}) + \sum_{i=0}^{k-1} \frac{n}{\log n - i} \\
&= \Theta(n) + n \sum_{i=0}^{\log n - 1} \frac{1}{\log n - i} \\
&= \Theta(n) + n H_{\log n} \\
&= \Theta(n \log \log n)
\end{aligned}
$$

$$T(n) = 2T(n/2) + \frac{n}{\log n}$$

$$\boxed{n = 2^k}$$

$$T(2^k) = 2T(2^{k-1}) + \frac{2^k}{k}$$

$$\frac{T(2^k)}{2^k} = \frac{T(2k-1)}{2^{k-1}} + \frac{1}{k}$$

$$\boxed{S(k) \triangleq \frac{T(2^k)}{2^k}}$$

$$S(k) = S(k-1) + \frac{1}{k} = H_k = \Theta(\log k)$$

$$T(n) = \Theta(n \log\log n)$$

$$\boxed{f(n) = \Theta(n^{\log_b a} \log^{k} n) \implies \Theta(n^{\log_b a} \log^{k+1} n) \qquad (k \ge 0)}$$

Master theorem @ wiki

$$f(n) = \Theta(n^{\log_b a} \log^{k} n)$$

$$k > -1 \implies T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$$

$$k = -1 \implies T(n) = \Theta(n^{\log_b a} \log \log n)$$

$$k < -1 \implies T(n) = \Theta(n^{\log_b a})$$

TC Problem 4-3 (i)

$$T(n) = T(n-2) + \frac{1}{\log n}$$

Problem (Area-Efficient VLSI Layout)

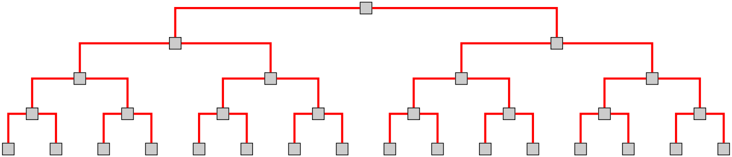
Embed a complete binary tree of n nodes into a grid with minimum area.

- Complete binary tree circuit of

$$\#\text{layer} = 3, 5, 7, \ldots$$

- Vertex on grid; no crossing edges
- Area:

$$\underbrace{A(n)}_{\text{area}} = \underbrace{H(n)}_{\text{height}} \times \underbrace{W(n)}_{\text{width}}$$



$$H(n) = H(\frac{n}{2}) + \Theta(1) = \Theta(\log n)$$

$$W(n) = 2W(\frac{n}{2}) + \Theta(1) = \Theta(n)$$

$$\boxed{A(n) = \Theta(n \log n)}$$

$$\boxed{Q : \boxed{H(n)} \times \boxed{W(n)} = n}$$

$$1 \times n$$

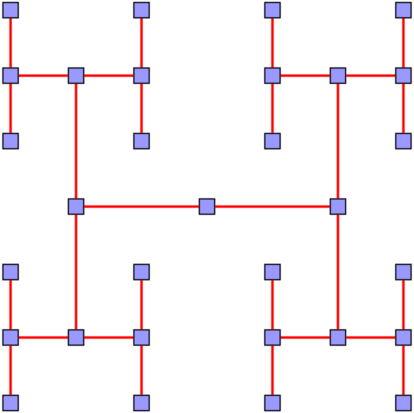
$$\frac{n}{\log n} \times \log n$$

$$\boxed{\sqrt{n} \times \sqrt{n}}$$

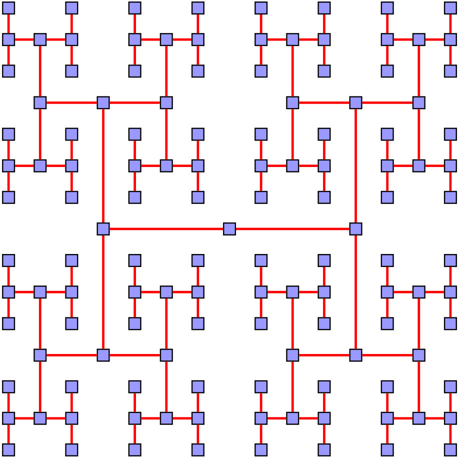
$$H(n) = \Theta(\sqrt{n}),\ W(n) = \Theta(\sqrt{n}),\ A(n) = \Theta(n)$$

$$H(n) = \square H(\frac{n}{\square}) + O(\square)$$

$$\boxed{H(n) = 2H(\frac{n}{4}) + \Theta(1)}$$

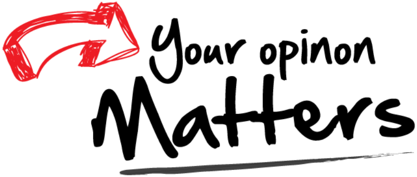


H-layout



“VLSI Theory and Parallel Supercomputing”, Charles E. Leiserson, 1989.

Thank You!



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