2-7 Discrete Probability

"Life is a school of probability — Walter Bagehot"

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Two Extra Tasks

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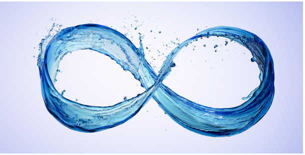
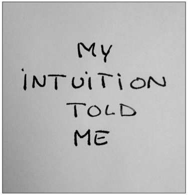
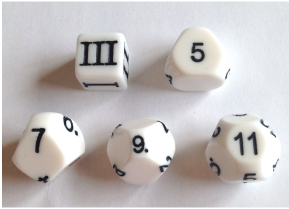
Q : What is probability?

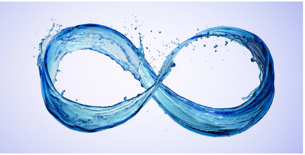
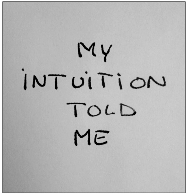


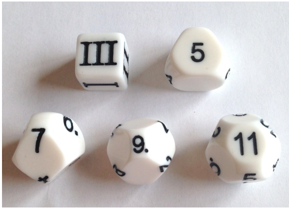




"... and the many *paradoxes* *show clearly that we, as humans,* *lack a well grounded intuition* *in this matter."*

— *"The Art of Probability", Richard W. Hamming*



"...and the many paradoxes show clearly that we, as humans, lack a well grounded intuition in this matter."

— "The Art of Probability", Richard W. Hamming

"When called upon to judge probability, people actually judge something else and **believe** *they have judged probability."*

— "Thinking, Fast and Slow", Daniel Kahneman

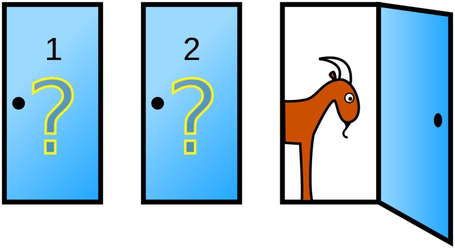
Let us calculate [calculemus].

(a) Monty Hall problem

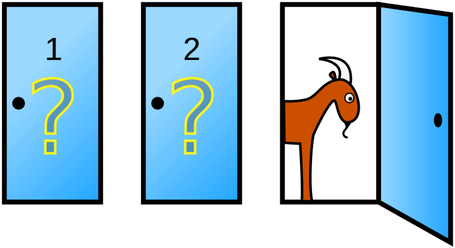
(b) Boy or Girl paradox

(c) Searching unsorted array

The Monty-Hall Problem



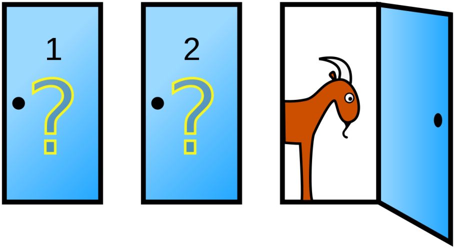
The Monty-Hall Problem



You: Randomly pick a door (No. 1)

I: Open a door which has a goat (No. 3)

The Monty-Hall Problem



You: Randomly pick a door (No. 1)

I: Open a door which has a goat (No. 3)

Q : Do you want to switch to door 2?

C_i : The car is behind door i $(i = 1, 2, 3)$

$$\boxed{\Pr\{C_i\} = \frac{1}{3}}$$

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C_i : The car is behind door i $(i = 1, 2, 3)$

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ASSUMPTION: The car is initially hidden randomly behind the doors.

Y_1 : you initially pick door 1

$$\boxed{\Pr\{Y_1\} = \frac{1}{3}}$$

ASSUMPTION: Your initial choice is random.

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ASSUMPTION: If you initially pick the car, then I open a door randomly.

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ASSUMPTION: If you initially pick the car, then I open a door randomly.

ASSUMPTION: I always open a door to reveal a goat and never the car.

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ASSUMPTION: I know what's behind the doors.

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ASSUMPTION: I always open a door to reveal a goat and never the car.

$$\boxed{\Pr\{C_2 \mid I_3, Y_1\}}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{C_2, I_3, Y_1\}}{\Pr\{I_3, Y_1\}}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{C_2, I_3, Y_1\}}{\Pr\{I_3, Y_1\}} = \frac{\Pr\{I_3, Y_1 \mid C_2\}\Pr\{C_2\}}{\Pr\{I_3 \mid Y_1\}\Pr\{Y_1\}}$$

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$$\Pr\{I_3, Y_1 \mid C_2\} = \Pr\{I_3 \mid C_2, Y_1\}\Pr\{Y_1 \mid C_2\}$$

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$$= \frac{\Pr\{I_3, Y_1 \mid C_2\}}{\Pr\{I_3 \mid Y_1\}}$$

$$\Pr\{I_3, Y_1 \mid C_2\} = \Pr\{I_3 \mid C_2, Y_1\} \Pr\{Y_1 \mid C_2\}$$
$$= \frac{1}{3} \Pr\{I_3 \mid C_2, Y_1\}$$

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$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{3 \Pr\{I_3 \mid Y_1\}}$$

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$\Pr\{I_3 \mid Y_1\}$

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$$\begin{aligned}
\Pr\{I_3 \mid Y_1\} &= \Pr\{I_3 \mid C_1, Y_1\}\Pr\{C_1 \mid Y_1\} \\
&+ \Pr\{I_3 \mid C_2, Y_1\}\Pr\{C_2 \mid Y_1\} \\
&+ \Pr\{I_3 \mid C_3, Y_1\}\Pr\{C_3 \mid Y_1\}
\end{aligned}$$

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$$\begin{aligned}
\Pr\{I_3 \mid Y_1\} &= \Pr\{I_3 \mid C_1, Y_1\}\Pr\{C_1 \mid Y_1\} \\
&\quad + \Pr\{I_3 \mid C_2, Y_1\}\Pr\{C_2 \mid Y_1\} \\
&\quad + \Pr\{I_3 \mid C_3, Y_1\}\Pr\{C_3 \mid Y_1\} \\
&= \frac{1}{3}\Big(\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}\Big)
\end{aligned}$$

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It depends on how I choose the door to open!

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}}$$

It depends on how I choose the door to open!

$$\Pr\{I_3 \mid C_1, Y_1\} = \frac{1}{2}$$

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$$\boxed{\Pr\{C_2 \mid I_3, Y_1\} = \frac{2}{3}}$$

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$$\Pr\{C_1 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_1, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\}}$$

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$$\boxed{\Pr\{C_2 \mid I_3, Y_1\} > \Pr\{C_1 \mid I_3, Y_1\} \iff \Pr\{I_3 \mid C_2, Y_1\} > \Pr\{I_3 \mid C_1, Y_1\}}$$

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$$\Pr\{I_3 \mid C_1, Y_1\} = \frac{1}{2}$$

$$\Pr\{I_3 \mid C_2, Y_1\} = 1$$

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Q : Switching *vs.* Choosing between the two remaining doors randomly?

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\}}$$

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$$\Pr\{I_3 \mid C_1, Y_1\} = q$$

$$\Pr\{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr\{I_3 \mid C_3, Y_1\} = 0$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\}}$$

$$\Pr\{I_3 \mid C_1, Y_1\} = q$$

$$\Pr\{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr\{I_3 \mid C_3, Y_1\} = 0$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{1}{1+q}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\}}$$

$$\Pr\{I_3 \mid C_1, Y_1\} = q$$

$$\Pr\{I_3 \mid C_2, Y_1\} = 1$$

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$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{1}{1+q} \in [\frac{1}{2}, 1]$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\}}$$

$$\Pr\{I_3 \mid C_1, Y_1\} = q$$

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$$\Pr\{C_1 \mid I_3, Y_1\} \in [0, \frac{1}{2}]$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\}}$$

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$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\}}$$

$$\Pr\{I_3 \mid C_1, Y_1\} = q$$

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$\boxed{\textit{Always Switch!}}$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\}}$$

$$\boxed{\Pr\{C_2 \mid I_3, Y_1\} > \Pr\{C_1 \mid I_3, Y_1\} \iff \Pr\{I_3 \mid C_2, Y_1\} > \Pr\{I_3 \mid C_1, Y_1\}}$$

Assumption: I know.

$$\Pr\{I_3 \mid C_1, Y_1\} = \frac{1}{2}$$

$$\Pr\{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{2}{3}$$

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Assumption: I know.

$$\Pr\{I_3 \mid C_1, Y_1\} = \frac{1}{2}$$

$$\Pr\{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{2}{3}$$

Assumption: I don't know either.

Opens one randomly that happens not to reveal the car.

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\}}$$

$$\boxed{\Pr\{C_2 \mid I_3, Y_1\} > \Pr\{C_1 \mid I_3, Y_1\} \iff \Pr\{I_3 \mid C_2, Y_1\} > \Pr\{I_3 \mid C_1, Y_1\}}$$

Assumption: I know.

$$\Pr\{I_3 \mid C_1, Y_1\} = \frac{1}{2}$$

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$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{1}{2}$$

Monty Hall problem (wiki)

The Boy/Girl Puzzle

Both Girls (CS Problem $5.3-12$)

Mr. and Mrs. Smith have two children of different ages, what is the probability that they has two girls,

(a) given that one of the children is a girl?

(b) given that the older child is a girl?

$$G_1 : \text{the older child is a girl}$$

$$G_2 : \text{the younger child is a girl}$$

$$G_1 : \text{the older child is a girl}$$

$$G_2 : \text{the younger child is a girl}$$

$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\}$

$$G_1 : \text{the older child is a girl}$$

$$G_2 : \text{the younger child is a girl}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1 \vee G_2\}}$$

$$G_1 : \text{the older child is a girl}$$

$$G_2 : \text{the younger child is a girl}$$

$$\begin{aligned}
\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} &= \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1 \vee G_2\}} \\
&= \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1\} + \Pr\{G_2\} - \Pr\{G_1 \wedge G_2\}}
\end{aligned}$$

G_1 : the older child is a girl

G_2 : the younger child is a girl

$$
\begin{aligned}
\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} &= \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1 \vee G_2\}} \\
&= \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1\} + \Pr\{G_2\} - \Pr\{G_1 \wedge G_2\}} \\
&= \frac{1/4}{3/4} = \frac{1}{3}
\end{aligned}
$$

G_1 : the older child is a girl

G_2 : the younger child is a girl

$$\begin{aligned}
\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} &= \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1 \vee G_2\}} \\
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&= \frac{1/4}{3/4} = \frac{1}{3}
\end{aligned}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1\}$$

G_1 : the older child is a girl

G_2 : the younger child is a girl

$$\begin{aligned}\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} &= \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1 \vee G_2\}} \\ &= \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1\} + \Pr\{G_2\} - \Pr\{G_1 \wedge G_2\}} \\ &= \frac{1/4}{3/4} = \frac{1}{3}\end{aligned}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1\} = \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1\}}$$

G_1 : the older child is a girl

G_2 : the younger child is a girl

$$\begin{aligned}
\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} &= \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1 \vee G_2\}} \\
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&= \frac{1/4}{3/4} = \frac{1}{3}
\end{aligned}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1\} = \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1\}} = \frac{1/2}{1/2} = \frac{1}{2}$$

$$\boxed{(G_1, G_2), (G_1, B_2), (B_1, G_2), (B_1, B_2)}$$

$$\boxed{(G_1, G_2), (G_1, B_2), (B_1, G_2), (B_1, B_2)}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{1}{3}$$

$$\boxed{(G_1, G_2), (G_1, B_2), (B_1, G_2), (B_1, B_2)}$$

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$$\Pr\{G_1 \wedge G_2 \mid G_1\} = \frac{1}{2}$$

$$(G_1, G_2), (G_1, B_2), (B_1, G_2), (B_1, B_2)$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{1}{3}$$

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$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{1}{3}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1\} = \frac{1}{2}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \boxed{\frac{2}{3}} \Pr\{G_1 \wedge G_2 \mid G_1\}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{1}{3}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1\} = \frac{1}{2}$$

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$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \boxed{\Pr\{G_1 \mid G_1 \vee G_2\}} \Pr\{G_1 \wedge G_2 \mid G_1\}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{1}{3}$$

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$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \boxed{\frac{2}{3}} \Pr\{G_1 \wedge G_2 \mid G_1\}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \boxed{\Pr\{G_1 \mid G_1 \vee G_2\}} \Pr\{G_1 \wedge G_2 \mid G_1\}$$

$$\Pr\{G_1 \mid G_1 \vee G_2\} = \frac{\Pr\{G_1 \wedge (G_1 \vee G_2)\}}{\Pr\{G_1 \vee G_2\}}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{1}{3}$$

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$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \boxed{\Pr\{G_1 \mid G_1 \vee G_2\}} \Pr\{G_1 \wedge G_2 \mid G_1\}$$

$$\Pr\{G_1 \mid G_1 \vee G_2\} = \frac{\Pr\{G_1 \wedge (G_1 \vee G_2)\}}{\Pr\{G_1 \vee G_2\}} = \frac{\Pr\{G_1\}}{\Pr\{G_1 \wedge G_2\}} = \frac{2}{3}$$


Easy!


Easy!


KEEP
CALM
AND
WE ARE NOT
FINISHED YET

Both Girls (CS Problem $5.3 - 12$)

Mr. and Mrs. Smith have two children of different ages, what is the probability that they has two girls,

(a) given that one of the children is a girl?

Both Girls (CS Problem $5.3-12$)

Mr. and Mrs. Smith have two children of different ages, what is the probability that they has two girls,

(a) given that one of the children is a girl?

Q : **How** do you know that "one of the children is a girl"?

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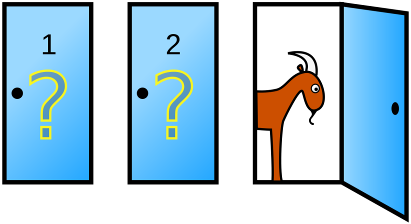
(I) I KNOW them well and I tell you that "one of the children is a girl".

Q : **How** do you know that "one of the children is a girl"?

(I) I KNOW them well and I tell you that "one of the children is a girl".

(II) I DON'T KNOW them. I just open a room door and see a girl.

The Monty-Hall Problem Comes Back



Q : **How** do you know that "one of the children is a girl"?

(II) g : I DON'T KNOW them. I just open a room door and see a girl.

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(II) g : I DON'T KNOW them. I just open a room door and see a girl.

$$\Pr\{G_1 \wedge G_2 \mid g\} = \frac{\{G_1 \wedge G_2 \wedge g\}}{\Pr\{g\}}$$

Q : **How** do you know that "one of the children is a girl"?

(II) g : I DON'T KNOW them. I just open a room door and see a girl.

$$\Pr\{G_1 \land G_2 \mid g\} = \frac{\{G_1 \land G_2 \land g\}}{\Pr\{g\}} = \frac{1/4}{1/2} = \frac{1}{2}$$

After-class Exercise:

A new couple, known to have two children, has just moved into town. Suppose that the mother is encountered walking with one of her children. If this child is a girl, what is the probability that both children are girls?

Boy or Girl paradox (wiki)

Q : What is probability?

Q : What is probability?

Objective: Frequentist

Subjective: Bayesians

Probability interpretations (wiki)

Probability interpretations (wiki)

Kolmogorov axioms

The INDEPENDENT

$$\Pr(EF) = \Pr(E) \times \Pr(F) \quad (E \perp F)$$

The Independent

$$\Pr(EF) = \Pr(E) \times \Pr(F) \quad {\color{red}(E \perp F)}$$

$$\Pr(E) = \Pr(E \mid F)$$

The Independent

$$\Pr(EF) = \Pr(E) \times \Pr(F) \quad (E \perp F)$$

$$\Pr(E) = \Pr(E \mid F)$$

$$Q : E \perp F \implies F \perp E\,?$$

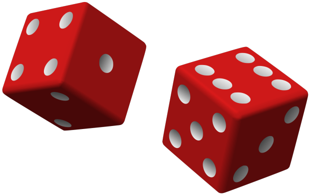
The INDEPENDENT

$$\Pr(EF) = \Pr(E) \times \Pr(F) \quad (E \perp F)$$

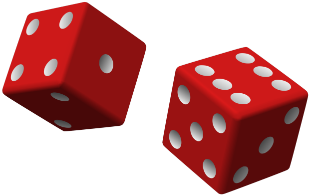
$$\Pr(E) = \Pr(E \mid F)$$

$$Q : E \perp F \implies F \perp E\,?$$

$$Q : E \perp F \implies E \perp F^C\,?$$

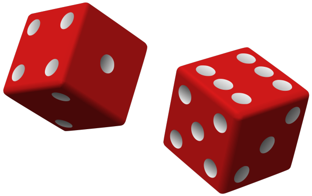


$$(d_1, d_2)$$



$$(d_1, d_2)$$

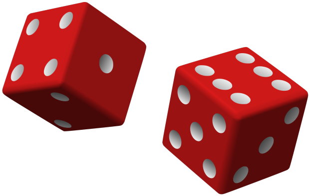
$$E : d_1 + d_2 = 6 \qquad F : d_1 = 4$$



$$(d_1, d_2)$$

$$E : d_1 + d_2 = 6 \qquad F : d_1 = 4$$

$$Q : E \perp F\,?$$



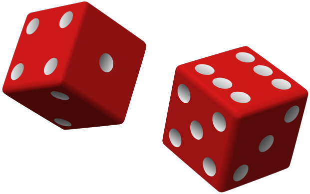
$$(d_1, d_2)$$

$$E : d_1 + d_2 = 7 \qquad F : d_1 = 4$$

$$Q : E \perp F\,?$$

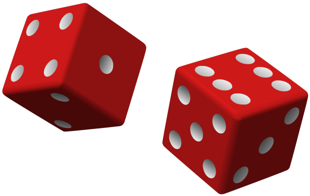
$$Q: E \perp F \land E \perp G \implies E \perp (FG)?$$

$$Q: E \perp F \wedge E \perp G \implies E \perp (FG)\,?$$



$$(d_1, d_2)$$

$$Q: E \perp F \wedge E \perp G \implies E \perp (FG)\,?$$

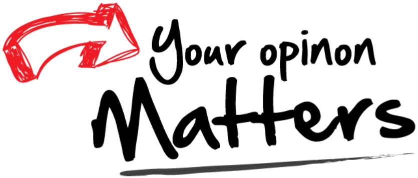


$$(d_1, d_2)$$

$$E : d_1 + d_2 = 7$$

$$F : d_1 = 4 \qquad G : d_2 = 3$$


Thank You!



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