3-5 Minimum Spanning Trees

Hengfeng Wei

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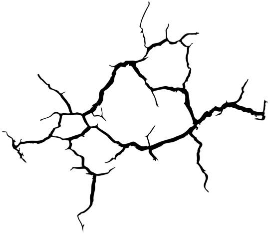
October 22, 2018

Be Careful with Your Proofs!!!

Theorem (A "Real" Theorem)

Proof.



$\square$

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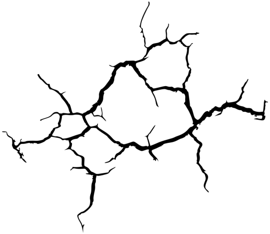

□

Theorem (A "Faked" Theorem)

Proof.

□

Cut Property

Cut Property (Version I)

$$
\begin{aligned}
X &: \text{ A part of some MST } T_1 \text{ of } G \\
(S, V \setminus S) &: \text{ A } \textit{cut} \text{ such that } X \text{ does } \textit{not} \text{ cross } (S, V \setminus S) \\
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$(S, V \setminus S)$: A *cut* such that X does *not* cross $(S, V \setminus S)$

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Then $X \cup \{e\}$ is a part of some MST T_2 of G.

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Correctness of Prim's and Kruskal's algorithms.

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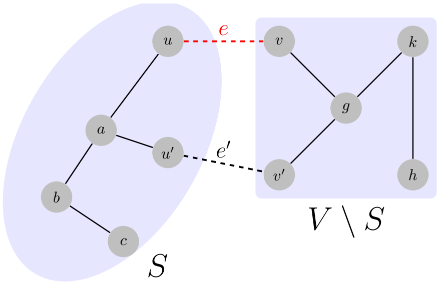

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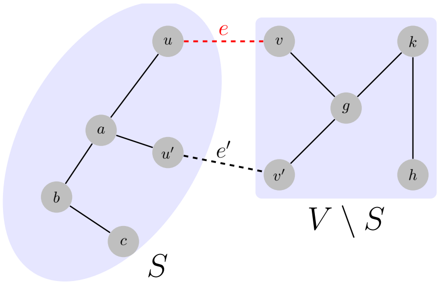

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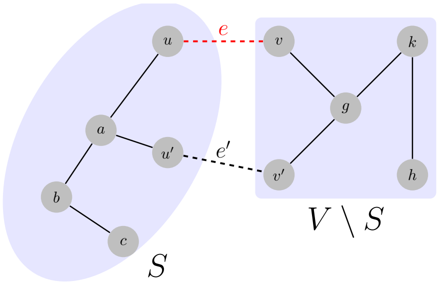


$$T' = \underbrace{\underbrace{T}_{\color{blue}{X \subseteq T}} + \{e\} - \{e'\}}_{\color{red}{\text{if } e \notin T}}$$

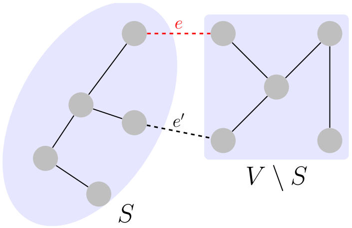
"a" $\to$ "the" $\color{red}{\Longrightarrow}$ "some" $\to$ "all"

Cut Property (Version II)

A cut $(S, V \setminus S)$

Let $e = (u, v)$ be *a* lightest edge across $(S, V \setminus S)$

$\exists$ MST T of $G : e \in T$



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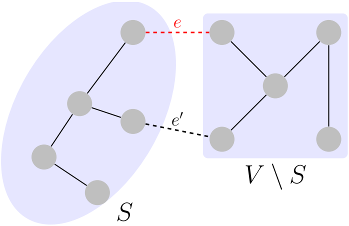

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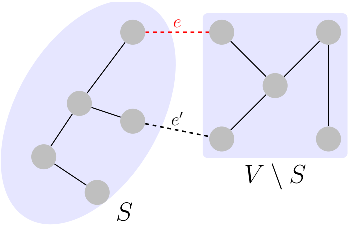

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"a" $\to$ "the" $\Longrightarrow$ "$\exists$" $\to$ "$\forall$"

A Wrong Divide&Conquer Algorithm for MST

$$(V_1, V_2) : \Big| |V_1| - |V_2| \Big| \le 1$$

$$T_1 + T_2 + \{e\} : e \text{ is a lightest edge across } (V_1, V_2)$$

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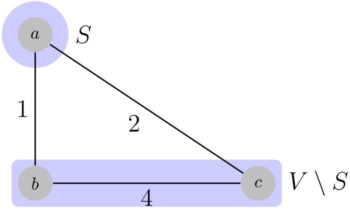

Cycle Property

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- Let C be any cycle in G
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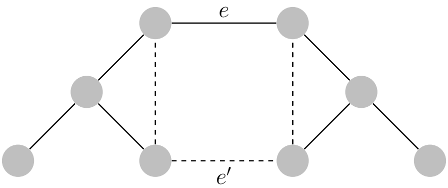

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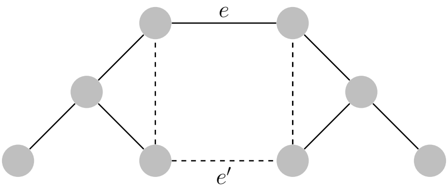

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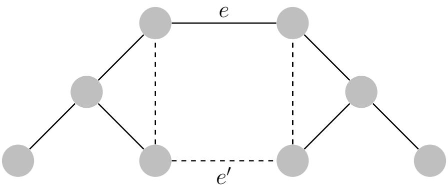

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$$\text{“a”} \to \text{“the”} \implies \text{“}\exists\text{”} \to \text{“}\forall\text{”}$$

Anti-Kruskal Algorithm

Reverse-delete algorithm (wiki; clickable)

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"On the Shortest Spanning Subtree of a Graph and the Traveling Salesman Problem"

— Kruskal, 1956.

Application of Cycle Property (Problem 4.30)

$$T: \text{ a MST of a connected weight graph } G$$

$$T \text{ is a } \text{unique} \text{ MST of } G$$

$$\iff$$

$$\forall e \in G \setminus T : w(e) > w(\text{every other edge on the cycle in } T + e)$$

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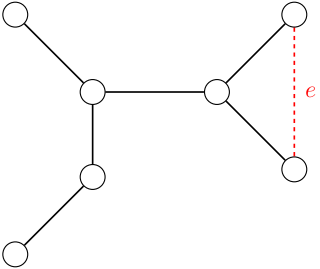

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$$\text{Distinct weights} \implies \text{Unique MST}$$

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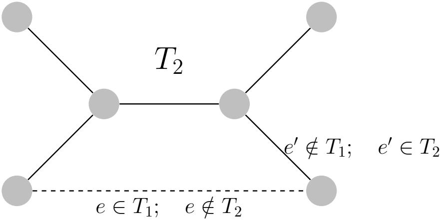
$$e \in T_1 \setminus T_2 \ (w.l.o.g)$$


T_2
$e' \notin T_1; \quad e' \in T_2$
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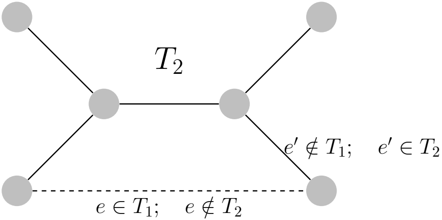
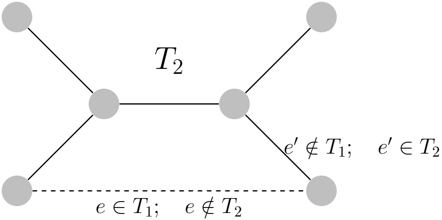


$$T_2 + \{e\} \implies C$$



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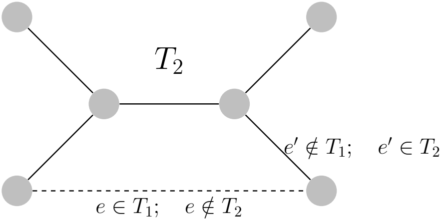

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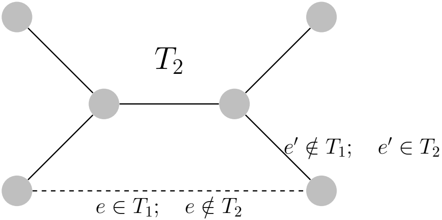

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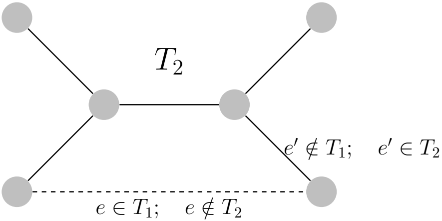

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Condition for Uniqueness of MST

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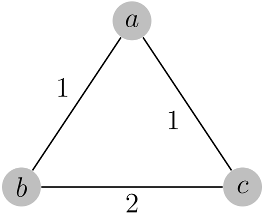

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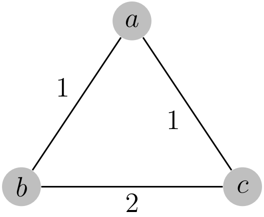

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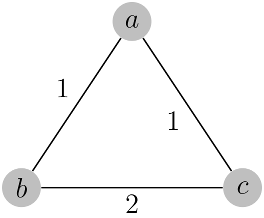

Theorem (After-class Exercise)

Minimum-weight edge across any cut is unique $\Longrightarrow$ *Unique MST*

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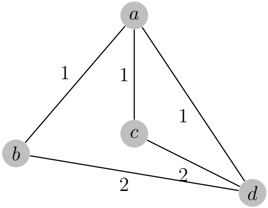

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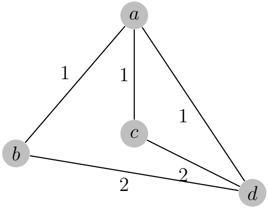

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$$\sum_{v \in V(G)} \deg(v) \leq 2(n-1)$$

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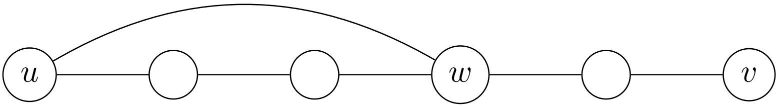

maximal path $P_{u,v}$

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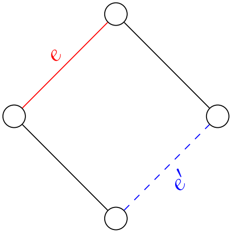

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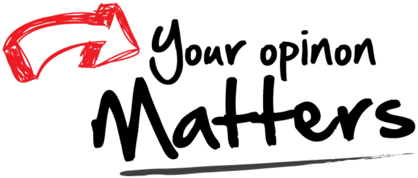
MST from the point of view of greedy-algorithm

MIT 6.046J: *"Design and Analysis of Algorithms"*, Spring 2015


THANK YOU



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