ISOMETRIES OF THE PLANE AND LINEAR ALGEBRA

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1. Introduction

An *isometry* of $\mathbf{R}^2$ is a function $h\colon \mathbf{R}^2 \to \mathbf{R}^2$ that preserves the distance between vectors:

$$||h(v) - h(w)|| = ||v - w||$$

for all v and w in $\mathbf{R}^2$, where $||(x, y)|| = \sqrt{x^2 + y^2}$.

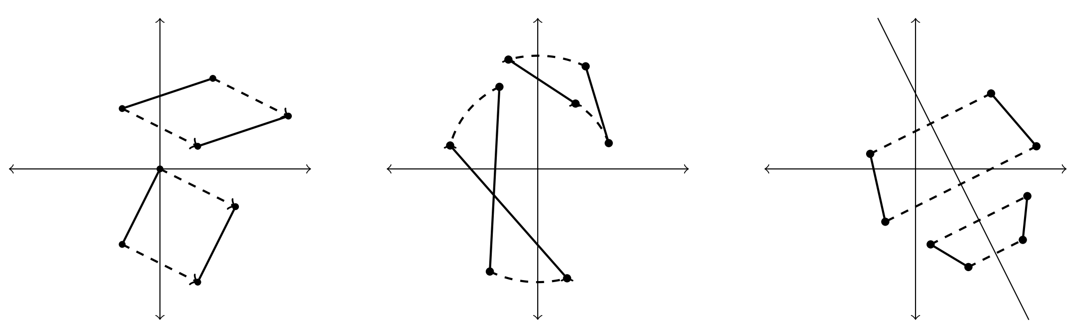
Example 1.1. The identity transformation: $\mathrm{id}(v) = v$ for all $v \in \mathbf{R}^2$.

Example 1.2. Negation: $-\mathrm{id}(v) = -v$ for all $v \in \mathbf{R}^2$.

Example 1.3. Translation: fixing $u \in \mathbf{R}^2$, let $t_u(v) = v + u$. Easily $||t_u(v) - t_u(w)|| = ||v - w||$.

Example 1.4. Rotations around points and reflections across lines in the plane are isometries of $\mathbf{R}^2$. Formulas for these isometries will be given in Example 3.2 and Section 5.

The effects of a translation, rotation (around the origin) and reflection across a line in $\mathbf{R}^2$ are pictured below on sample line segments.



The composition of two isometries of $\mathbf{R}^2$ is an isometry and *if* an isometry is invertible, its inverse is also an isometry. The three kinds of isometries pictured above (translations, rotations, reflections) are each invertible (translate by the negative vector, rotate by the opposite angle, reflect a second time across the same line). A general isometry of $\mathbf{R}^2$ is invertible, but to prove this will require some work.

In Section 2, we will see how to study isometries using dot products instead of distances. The dot product is more convenient to use than distance because of its algebraic properties. Section 3 introduces the matrix transformations on $\mathbf{R}^2$, called orthogonal matrices, that are isometries. In Section 4 we will see that all isometries of $\mathbf{R}^2$ can be expressed in terms of translations and orthogonal matrix transformations. In particular, this will imply that every isometry of $\mathbf{R}^2$ is invertible. Section 5 discusses the isometries of $\mathbf{R}^2$.

2. Isometries and dot products

Using translations, we can reduce the study of isometries of $\mathbf{R}^2$ to the case of isometries fixing $\mathbf{0}$.

Theorem 2.1. *Every isometry of $\mathbf{R}^2$ can be uniquely written as the composition $t \circ k$ where t is a translation and k is an isometry fixing the origin.*

Proof. Let $h\colon \mathbf{R}^2 \to \mathbf{R}^2$ be an isometry. If $h = t_w \circ k$, where t_w is translation by a vector w and k is an isometry fixing $\mathbf{0}$, then for all v in $\mathbf{R}^2$ we have $h(v) = t_w(k(v)) = k(v)+w$. Setting $v = \mathbf{0}$ we get $w = h(\mathbf{0})$, so w is determined by h. Then $k(v) = h(v)-w = h(v)-h(\mathbf{0})$, so k is determined by h. Turning this around, if we define $t(v) = v + h(\mathbf{0})$ and $k(v) = h(v) - h(\mathbf{0})$, then t is a translation, k is an isometry fixing $\mathbf{0}$, and $h(v) = k(v) + h(\mathbf{0}) = t_w \circ k$, where $w = h(\mathbf{0})$. $\square$

Theorem 2.2. *For a function $h\colon \mathbf{R}^2 \to \mathbf{R}^2$, the following are equivalent:*

(1) *h is an isometry and $h(\mathbf{0}) = \mathbf{0}$,*
(2) *h preserves dot products: $h(v) \cdot h(w) = v \cdot w$ for all $v, w \in \mathbf{R}^2$.*

Proof. The link between length and dot product is the formula

$$||v||^2 = v \cdot v.$$

Suppose h satisfies (1). Then for any vectors v and w in $\mathbf{R}^2$,

$$||h(v) - h(w)|| = ||v - w||. \tag{2.1}$$

As a special case, when $w = \mathbf{0}$ in (2.1) we get $||h(v)|| = ||v||$ for all $v \in \mathbf{R}^2$. Squaring both sides of (2.1) and writing the result in terms of dot products makes it

$$(h(v) - h(w)) \cdot (h(v) - h(w)) = (v - w) \cdot (v - w).$$

Carrying out the multiplication,

$$h(v) \cdot h(v) - 2h(v) \cdot h(w) + h(w) \cdot h(w) = v \cdot v - 2v \cdot w + w \cdot w. \tag{2.2}$$

The first term on the left side of (2.2) equals $||h(v)||^2 = ||v||^2 = v \cdot v$ and the last term on the left side of (2.2) equals $||h(w)||^2 = ||w||^2 = w \cdot w$. Canceling equal terms on both sides of (2.2), we obtain $-2h(v) \cdot h(w) = -2v \cdot w$, so $h(v) \cdot h(w) = v \cdot w$.

Now assume h satisfies (2), so

$$h(v) \cdot h(w) = v \cdot w \tag{2.3}$$

for all v and w in $\mathbf{R}^2$. Therefore

$$\begin{aligned} ||h(v) - h(w)||^2 &= (h(v) - h(w)) \cdot (h(v) - h(w)) \\ &= h(v) \cdot h(v) - 2h(v) \cdot h(w) + h(w) \cdot h(w) \\ &= v \cdot v - 2v \cdot w + w \cdot w \quad \text{by (2.3)} \\ &= (v - w) \cdot (v - w) \\ &= ||v - w||^2, \end{aligned}$$

so $||h(v) - h(w)|| = ||v - w||$. Thus h is an isometry. Setting $v = w = \mathbf{0}$ in (2.3), we get $||h(\mathbf{0})||^2 = 0$, so $h(\mathbf{0}) = \mathbf{0}$. $\square$

Corollary 2.3. *The only isometry of $\mathbf{R}^2$ fixing $\mathbf{0}$ and the standard basis is the identity.*

Proof. Let $h\colon \mathbf{R}^2 \to \mathbf{R}^2$ be an isometry that satisfies

$$h(\mathbf{0}) = \mathbf{0},\ h\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix},\ h\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}.$$

Theorem 2.2 says

$$h(v)\cdot h(w) = v \cdot w$$

for all v and w in $\mathbf{R}^2$. Fix $v \in \mathbf{R}^2$ and let w run over the standard basis vectors $e_1 = \binom{1}{0}$ and $e_2 = \binom{0}{1}$, so we see

$$h(v)\cdot h(e_i) = v \cdot e_i.$$

Since h fixes each e_i,

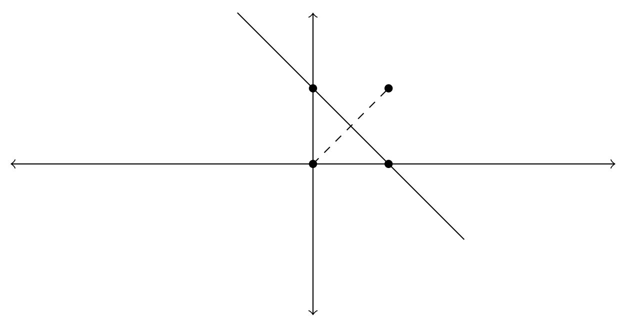
$$h(v)\cdot e_i = v \cdot e_i.$$

Writing $v = c_1e_1 + c_2e_2$, we get

$$h(v)\cdot e_i = c_i$$

for $i = 1, 2$, so $h(v) = c_1e_1 + c_2e_2 = v$. As v was arbitrary, h is the identity on $\mathbf{R}^2$. □

It is essential in Corollary 2.3 that the isometry fixes $\mathbf{0}$. An isometry of $\mathbf{R}^2$ fixing the standard basis *without* fixing $\mathbf{0}$ need not be the identity! For example, reflection across the line $x + y = 1$ in $\mathbf{R}^2$ is an isometry of $\mathbf{R}^2$ fixing $(1, 0)$ and $(0, 1)$ but not $\mathbf{0} = (0, 0)$. See below.



If we knew all isometries of $\mathbf{R}^2$ were invertible, then Corollary 2.3 would imply that two isometries f and g taking the same values at $\mathbf{0}$ and the standard basis are equal: apply Corollary 2.3 to the isometry $f^{-1}\circ g$ to see this composite is the identity, so $f = g$. However, we do not yet know that all isometries are invertible; that is one of our main tasks.

3. Orthogonal matrices

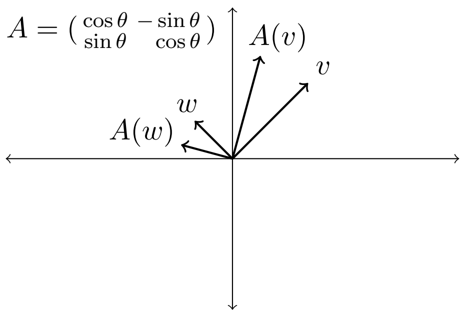
A large supply of isometries of $\mathbf{R}^2$ that fix $\mathbf{0}$ come from special types of matrices.

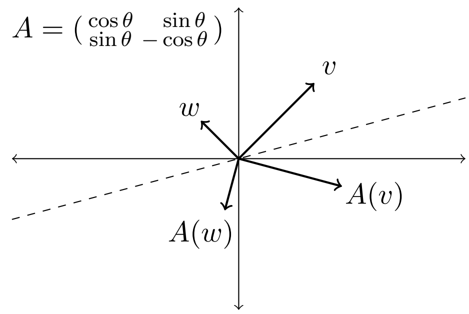
Definition 3.1. A 2×2 matrix A is called *orthogonal* if $AA^\top = I_2$, or equivalently if $A^\top A = I_2$.

A matrix is orthogonal when its transpose is its inverse. Since $\det(A^\top) = \det A$, any orthogonal matrix A satisfies $(\det A)^2 = 1$, so $\det A = \pm 1$.

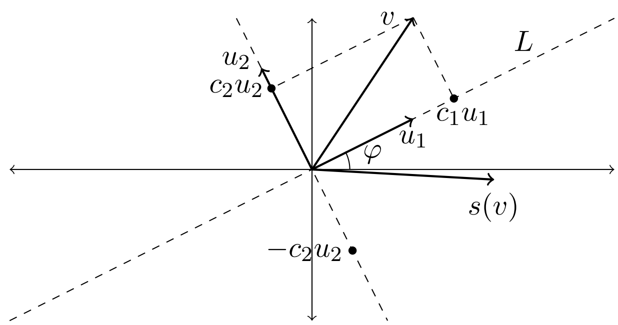
Example 3.2. By algebra, $AA^\top = I_2$ if and only if $A = \left(\begin{smallmatrix} a & -\varepsilon b\\ b & \varepsilon a\end{smallmatrix}\right)$, where $a^2 + b^2 = 1$ and $\varepsilon = \pm 1$. Writing $a = \cos\theta$ and $b = \sin\theta$, we get the matrices $\left(\begin{smallmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{smallmatrix}\right)$. Algebraically, these types of matrices are distinguished by their determinants: the first type has determinant 1 and the second type has determinant -1.

Geometrically, the effect of these matrices is pictured below. On the left, $\left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$ is a counterclockwise rotation by angle θ around the origin. On the right, $\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{smallmatrix}\right)$ is a reflection across the line through the origin at angle $\theta/2$ with respect to the positive x-axis. (Check $\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{smallmatrix}\right)$ squares to the identity, as any reflection should.)





Let's explain why $\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{smallmatrix}\right)$ is a reflection at angle $\theta/2$. See the figure below. Pick a line L through the origin, say at an angle φ with respect to the positive x-axis. To find a formula for reflection across L, we'll use a basis of $\mathbf{R}^2$ with one vector **on** L and the other vector **perpendicular** to L. The unit vector $u_1 = \binom{\cos\varphi}{\sin\varphi}$ lies on L and the unit vector $u_2 = \binom{-\sin\varphi}{\cos\varphi}$ is perpendicular to L. For any $v \in \mathbf{R}^2$, write $v = c_1u_1 + c_2u_2$ with $c_1, c_2 \in \mathbf{R}$.



The reflection of v across L is $s(v) = c_1u_1 - c_2u_2$. Writing $a = \cos\varphi$ and $b = \sin\varphi$ (so $a^2 + b^2 = 1$), in standard coordinates

$$v = c_1u_1 + c_2u_2 = c_1\begin{pmatrix} a \\ b \end{pmatrix} + c_2\begin{pmatrix} -b \\ a \end{pmatrix} = \begin{pmatrix} c_1a - c_2b \\ c_1b + c_2a \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{3.1}$$

and

$$\begin{aligned} s(v) &= c_1u_1 - c_2u_2 \\ &= \begin{pmatrix} a & b \\ b & -a \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ b & -a \end{pmatrix}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}^{-1} v \quad \text{by (3.1)} \\ &= \begin{pmatrix} a & b \\ b & -a \end{pmatrix}\begin{pmatrix} a & b \\ -b & a \end{pmatrix} v \\ &= \begin{pmatrix} a^2 - b^2 & 2ab \\ 2ab & -(a^2 - b^2) \end{pmatrix} v. \end{aligned}$$

By the sine and cosine duplication formulas, the last matrix is $\left(\begin{smallmatrix} \cos(2\varphi) & \sin(2\varphi) \\ \sin(2\varphi) & -\cos(2\varphi) \end{smallmatrix}\right)$. Therefore $\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{smallmatrix}\right)$ is a reflection across the line through the origin at angle $\theta/2$.

The geometric meaning of the condition $A^\top A = I_2$ is that the columns of A are mutually perpendicular unit vectors (check!). From this we see how to create orthogonal matrices: starting with an orthonormal basis of $\mathbf{R}^2$, a 2×2 matrix having this basis as its columns (in any order) is an orthogonal matrix, and all 2×2 orthogonal matrices arise in this way.

Let $\mathrm{O}_2(\mathbf{R})$ denote the set of 2×2 orthogonal matrices:

$$\mathrm{O}_2(\mathbf{R}) = \{A \in \mathrm{GL}_2(\mathbf{R}) : AA^\top = I_2\}. \tag{3.2}$$

Theorem 3.3. *The set $\mathrm{O}_2(\mathbf{R})$ is a group under matrix multiplication.*

Proof. Clearly $I_2 \in \mathrm{O}_2(\mathbf{R})$. For $A \in \mathrm{O}_2(\mathbf{R})$, the inverse of A^{-1} is $(A^{-1})^\top$ since

$$(A^{-1})^\top = (A^\top)^\top = A.$$

Therefore $A^{-1} \in \mathrm{O}_2(\mathbf{R})$. If A and B are in $\mathrm{O}_2(\mathbf{R})$, then

$$(AB)(AB)^\top = ABB^\top A^\top = AA^\top = I_2,$$

so $AB \in \mathrm{O}_2(\mathbf{R})$. $\square$

Theorem 3.4. *If $A \in \mathrm{O}_2(\mathbf{R})$, then the transformation $h_A \colon \mathbf{R}^2 \to \mathbf{R}^2$ given by $h_A(v) = Av$ is an isometry of $\mathbf{R}^2$ that fixes $\mathbf{0}$.*

Proof. Trivially the function h_A fixes $\mathbf{0}$. To show h_A is an isometry, by Theorem 2.2 it suffices to show

$$Av \cdot Aw = v \cdot w \tag{3.3}$$

for all $v, w \in \mathbf{R}^2$.

The fundamental link between the dot product and transposes is

$$v \cdot Aw = A^\top v \cdot w \tag{3.4}$$

for *any* 2×2 matrix A and $v, w \in \mathbf{R}^2$. Replacing v with Av in (3.4),

$$Av \cdot Aw = A^\top(Av) \cdot w = (A^\top A)v \cdot w.$$

This is equal to $v \cdot w$ for all v and w precisely when $A^\top A = I_2$. $\square$

Example 3.5. Negation on $\mathbf{R}^2$ comes from the matrix $-I_2$, which is orthogonal: $-\mathrm{id} = h_{-I_2}$.

The proof of Theorem 3.4 gives us a more geometric description of $\mathrm{O}_2(\mathbf{R})$ than (3.2):

$$\mathrm{O}_2(\mathbf{R}) = \{A \in \mathrm{GL}_2(\mathbf{R}) : Av \cdot Aw = v \cdot w \text{ for all } v, w \in \mathbf{R}^2\}. \tag{3.5}$$

The label "orthogonal matrix" suggests it should just be a matrix that preserves orthogonality of vectors:

$$v \cdot w = 0 \Longrightarrow Av \cdot Aw = 0 \tag{3.6}$$

for all v and w in $\mathbf{R}^2$. While orthogonal matrices do satisfy (3.6), since (3.6) is a special case of the condition $Av \cdot Aw = v \cdot w$ in (3.5), equation (3.6) is *not* a characterization of orthogonal matrices. That is, orthogonal matrices (which preserve *all* dot products) are not the only matrices that preserve orthogonality of vectors (dot products equal to 0). A simple example of a nonorthogonal matrix satisfying (3.6) is a scalar matrix cI_2, where $c \neq \pm 1$. Since $(cv) \cdot (cw) = c^2(v \cdot w)$, cI_2 does not preserve dot products in general but it does preserve dot products equal to 0. It's natural to ask what matrices besides orthogonal matrices preserve orthogonality. Here is the answer.

Theorem 3.6. *A 2×2 real matrix A satisfies (3.6) if and only if A is a scalar multiple of an orthogonal matrix.*

Proof. If $A = cA'$ where A' is orthogonal, then $Av \cdot Aw = c^2(A'v \cdot A'w) = c^2(v \cdot w)$, so if $v \cdot w = 0$ then $Av \cdot Aw = 0$.

Now assume A satisfies (3.6). Then the vectors Ae_1, Ae_2 are mutually perpendicular, so the columns of A are perpendicular to each other. We want to show that they have the same length.

Note that $e_1+e_2 \perp e_1-e_2$, so by (3.6) and linearity $Ae_1+Ae_2 \perp Ae_1-Ae_2$. Writing this in the form $(Ae_1+Ae_2)\cdot(Ae_1-Ae_2) = 0$ and expanding, we are left with $Ae_1 \cdot Ae_1 = Ae_2 \cdot Ae_2$, so $||Ae_1|| = ||Ae_2||$. Therefore the columns of A are mutually perpendicular vectors with the same length. Call this common length c. If $c = 0$ then $A = O = 0 \cdot I_2$. If $c \neq 0$ then the matrix $(1/c)A$ has an orthonormal basis as its columns, so it is an orthogonal matrix. Therefore $A = c((1/c)A)$ is a scalar multiple of an orthogonal matrix. □

4. Isometries of $\mathbf{R}^2$ form a group

We now establish the converse to Theorem 3.4, and in particular establish that isometries fixing $\mathbf{0}$ are invertible linear maps.

Theorem 4.1. *Any isometry $h\colon \mathbf{R}^2 \to \mathbf{R}^2$ fixing $\mathbf{0}$ has the form $h(v) = Av$ for some $A \in \mathrm{O}_2(\mathbf{R})$. In particular, h is linear and invertible.*

Proof. By Theorem 2.2,

$$h(v) \cdot h(w) = v \cdot w$$

for all $v, w \in \mathbf{R}^2$. What does this say about the effect of h on the standard basis? Taking $v = w = e_i$,

$$h(e_i) \cdot h(e_i) = e_i \cdot e_i,$$

so $||h(e_i)||^2 = 1$. Therefore $h(e_i)$ is a unit vector. Taking $v = e_1$ and $w = e_2$, we get

$$h(e_1) \cdot h(e_2) = e_1 \cdot e_2 = 0.$$

Therefore the vectors $h(e_1), h(e_2)$ are mutually perpendicular unit vectors (an orthonormal basis of $\mathbf{R}^2$).

Let A be the 2×2 matrix with i-th column equal to $h(e_i)$. Since the columns are mutually perpendicular unit vectors, $A^\top A$ equals I_2, so A is an orthogonal matrix and thus acts as an isometry of $\mathbf{R}^2$ by Theorem 3.4. By the definition of A, $A(e_i) = h(e_i)$ for all i. Therefore A and h are isometries with the same values at the standard basis. Moreover, we know A is invertible since it is an orthogonal matrix.

Consider now the isometry $A^{-1} \circ h$. It fixes $\mathbf{0}$ as well as the standard basis. By Corollary 2.3, $A^{-1} \circ h$ is the identity, so $h(v) = Av$ for all $v \in \mathbf{R}^2$: h is given by an orthogonal matrix. □

Theorem 4.2. *For $A \in \mathrm{O}_2(\mathbf{R})$ and $w \in \mathbf{R}^2$, the function $h_{A,w}\colon \mathbf{R}^2 \to \mathbf{R}^2$ given by*

$$h_{A,w}(v) = Av + w = (t_w A)(v)$$

is an isometry. Moreover, every isometry of $\mathbf{R}^2$ has this form for unique w and A.

Proof. The indicated formula always gives an isometry, since it is the composition of a translation and orthogonal transformation, which are both isometries.

To show any isometry of $\mathbf{R}^2$ has the form $h_{A,w}$ for some A and w, let $h\colon \mathbf{R}^2 \to \mathbf{R}^2$ be an isometry. By Theorem 2.1, $h = k(v) + h(\mathbf{0})$ where k is an isometry of $\mathbf{R}^2$ fixing $\mathbf{0}$. Theorem 4.1 tells us there is an $A \in \mathrm{O}_2(\mathbf{R})$ such that $k(v) = Av$ for all $v \in \mathbf{R}^2$, so

$$h(v) = Av + h(\mathbf{0}) = h_{A,w}(v)$$

where $w = h(\mathbf{0})$.

If $h_{A,w} = h_{A',w'}$ as functions on $\mathbf{R}^2$, then evaluating both sides at $\mathbf{0}$ gives $w = w'$. Therefore $Av + w = A'v + w$ for all v, so $Av = A'v$ for all v, which implies $A = A'$. $\square$

Theorem 4.3. *The set* $\mathrm{Iso}(\mathbf{R}^2)$ *of isometries of* $\mathbf{R}^2$ *is a group under composition.*

Proof. The only property that has to be checked is invertibility. By Theorem 4.2, we can write any isometry h as $h(v) = Av + w$ where $A \in \mathrm{O}_2(\mathbf{R})$. Its inverse is $g(v) = A^{-1}v - A^{-1}w$. $\square$

Let's look at composition in $\mathrm{Iso}(\mathbf{R}^2)$ when we write isometries as $h_{A,w}$ from Theorem 4.2. We have

$$\begin{aligned} h_{A,w}(h_{A',w'}(v)) &= A(A'v + w') + w \\ &= AA'v + Aw' + w \\ &= h_{AA',Aw'+w}(v). \end{aligned}$$

This is similar to the multiplication law in the $ax + b$ group:

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a' & b' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} aa' & ab' + b \\ 0 & 1 \end{pmatrix}.$$

In fact, if we write an isometry $h_{A,w} \in \mathrm{Iso}(\mathbf{R}^2)$ as an $(n+1) \times (n+1)$ matrix $\left(\begin{smallmatrix} A & w \\ 0 & 1 \end{smallmatrix}\right)$, where the 0 in the bottom is a row vector of n zeros, then the composition law in $\mathrm{Iso}(\mathbf{R}^2)$ is multiplication of the corresponding $(n+1) \times (n+1)$ matrices, so $\mathrm{Iso}(\mathbf{R}^2)$ can be viewed as a subgroup of $\mathrm{GL}_{n+1}(\mathbf{R})$, acting on $\mathbf{R}^2$ as the column vectors $\binom{v}{1}$ in $\mathbf{R}^{n+1}$ (not a subspace!).

Corollary 4.4. *Two isometries of* $\mathbf{R}^2$ *that are equal at* $\mathbf{0}$ *and at a basis of* $\mathbf{R}^2$ *are the same.*

This strengthens Corollary 2.3 since we allow any basis, not just the standard basis.

Proof. Let h and h' be isometries of $\mathbf{R}^2$ such that $h = h'$ at $\mathbf{0}$ and at a basis of $\mathbf{R}^2$, say v_1, v_2. Then $h^{-1}h'$ is an isometry of $\mathbf{R}^2$ fixing $\mathbf{0}$ and each v_i. By Theorem 4.1, $h^{-1}h'$ is in $\mathrm{O}_2(\mathbf{R})$, so the fact that it fixes each v_i implies it fixes every *linear combination* of the v_i's, which exhausts $\mathbf{R}^2$. Thus $h^{-1}h'$ is the identity on $\mathbf{R}^2$, so $h' = h$. $\square$

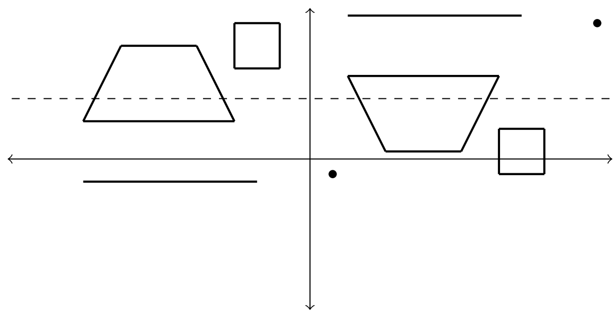
5. Geometric description of isometries of $\mathbf{R}^2$

We know from Theorem 4.2 what all the isometries of $\mathbf{R}^2$ look like by formulas. In this section we describe what they are like geometrically.

The isometries of $\mathbf{R}$ are the functions $h(x) = x + c$ and $h(x) = -x + c$ for $c \in \mathbf{R}$. (Of course, this case can be worked out easily from scratch without all the earlier preliminary material.)

Now consider isometries of $\mathbf{R}^2$. Write an isometry $h \in \mathrm{Iso}(\mathbf{R}^2)$ in the form $h(v) = Av + w$ with $A \in \mathrm{O}_2(\mathbf{R})$. By Example 3.2, A is a rotation or reflection, depending on the determinant.

There turn out to be four possibilities for h: translations, rotations, reflections, and glide reflections. A *glide reflection* is the composition of a reflection and a nonzero translation in a direction parallel to the line of reflection. A picture of a glide reflection is in the figure below, where the (horizontal) line of reflection is dashed and the translation is a movement to the right.



The image above, which includes "before" and "after" states, suggests a physical interpretation of a glide reflection: it is the result of turning the plane in space like a half-turn of a screw. A more picturesque image, suggested to me by Michiel Vermeulen, is the effect of successive steps with a left foot and then a right foot in the sand or snow (if your feet are mirror reflections).

The possibilities for isometries of f are collected in Table 1 below. It describes how the type of an isometry h is determined by $\det A$ and the geometry of the set of fixed points of h (solutions to $h(v) = v$), which is empty, a point, a line, or the plane. (The only isometry belonging to more than one of the four possibilities is the identity, which is both a translation and a rotation, so we make the identity its own row in the table.) The table also shows that a description of the fixed points can be obtained algebraically from A and w.

Isometry	Condition	Fixed pts
Identity	$A = I_2, w = 0$	$\mathbf{R}^2$
Nonzero Translation	$A = I_2, w \neq 0$	$\emptyset$
Nonzero Rotation	$\det A = 1, A \neq I_2$	$(I_2 - A)^{-1}w$
Reflection	$\det A = -1,\ Aw = -w$	$w/2 + \ker(A - I_2)$
Glide Reflection	$\det A = -1,\ Aw \neq -w$	$\emptyset$

TABLE 1. Isometries of $\mathbf{R}^2$: $h(v) = Av + w$, $A \in \mathrm{O}_2(\mathbf{R})$.

To justify the information in the table we move down the middle column. The first two rows are obvious, so we start with the third row.

Row 3: Suppose $\det A = 1$ and $A \neq I_2$, so $A = \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$ for some θ and $\cos\theta \neq 1$. We want to show h is a rotation. First of all, h has a unique fixed point: $v = Av + w$ precisely when $w = (I_2 - A)v$. We have $\det(I_2 - A) = 2(1 - \cos\theta) \neq 0$, so $I_2 - A$ is invertible and $p = (I_2 - A)^{-1}w$ is the fixed point of h. Then $w = (I_2 - A)p = p - Ap$, so

$$h(v) = Av + (p - Ap) = A(v - p) + p. \tag{5.1}$$

Since A is a rotation by θ around the origin, (5.1) shows h is a rotation by θ around P.

Rows 4, 5: Suppose $\det A = -1$, so $A = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{smallmatrix}\right)$ for some θ and $A^2 = I_2$. We again look at fixed points of h. As before, $h(v) = v$ for some v if and only if $w = (I_2 - A)v$.

But unlike the previous case, now $\det(I_2 - A) = 0$ (check!), so $I_2 - A$ is not invertible and therefore w may or may not be in the image of $I_2 - A$. When w is in the image of $I_2 - A$, we will see that h is a reflection. When w is not in the image of $I_2 - A$, we will see that h is a glide reflection.

Suppose the isometry $h(v) = Av + w$ with $\det A = -1$ has a fixed point. Then $w/2$ must be a fixed point. Indeed, let p be any fixed point, so $p = Ap + w$. Since $A^2 = I_2$,

$$Aw = A(p - Ap) = Ap - p = -w,$$

so

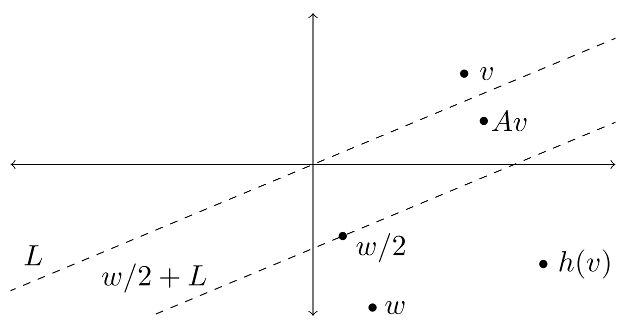
$$h\left(\frac{w}{2}\right) = A\left(\frac{w}{2}\right) + w = \frac{1}{2}Aw + w = \frac{w}{2}.$$

Conversely, if $h(w/2) = w/2$ then $A(w/2) + w = w/2$,, so $Aw = -w$.

Thus h has a fixed point if and only if $Aw = -w$, in which case

$$h(v) = Av + w = A\left(v - \frac{w}{2}\right) + \frac{w}{2}. \tag{5.2}$$

Since A is a reflection across some line L through 0, (5.2) says h is a reflection across the parallel line $w/2 + L$ passing through $w/2$. (Algebraically, $L = \{v : Av = v\} = \ker(A - I_2)$. Since $A - I_2$ is not invertible and not identically 0, its kernel really is 1-dimensional.)



Now assume h has no fixed point, so $Aw \neq -w$. We will show h is a glide reflection. (The formula $h = Av+w$ shows h is the composition of a reflection and a nonzero translation, but w need not be parallel to the line of reflection of A, which is $\ker(A - I_2)$, so this formula for h does *not* show directly that h is a glide reflection.) We will now take stronger advantage of the fact that $A^2 = I_2$.

Since $O = A^2 - I_2 = (A - I_2)(A + I_2)$ and $A \neq \pm I_2$ (after all, $\det A = -1$), $A + I_2$ and $A - I_2$ are not invertible. Therefore the subspaces

$$W_1 = \ker(A - I_2), \quad W_2 = \ker(A + I_2)$$

are both nonzero, and neither is the whole plane, so W_1 and W_2 are both one-dimensional. We already noted that W_1 is the line of reflection of A (fixed points of A form the kernel of $A - I_2$). It turns out that W_2 is the line perpendicular to W_1. To see why, pick $w_1 \in W_1$ and $w_2 \in W_2$, so

$$Aw_1 = w_1, \quad Aw_2 = -w_2.$$

Then, since $Aw_1 \cdot Aw_2 = w_1 \cdot w_2$ by orthogonality of A, we have

$$w_1 \cdot (-w_2) = w_1 \cdot w_2.$$

Thus $w_1 \cdot w_2 = 0$, so $w_1 \perp w_2$.

Now we are ready to show h is a glide reflection. Pick nonzero vectors $w_i \in W_i$ for $i = 1, 2$, and use $\{w_1, w_2\}$ as a basis of $\mathbf{R}^2$. Write $w = h(\mathbf{0})$ in terms of this basis: $w = c_1w_1 + c_2w_2$.

To say there are no fixed points for h is the same as $Aw \neq -w$, so $w \notin W_2$. That is, $c_1 \neq 0$. Then

$$h(v) = Av + w = (Av + c_2w_2) + c_1w_1. \tag{5.3}$$

Since $A(c_2w_2) = -c_2w_2$, our previous discussion shows $v \mapsto Av + c_2w_2$ is a reflection across the line $c_2w_2/2 + W_1$. Since c_1w_1 is a nonzero vector in W_1, (5.3) exhibits h as the composition of a reflection across the line $c_2w_2/2 + W_1$ and a nonzero translation by c_1w_1, whose direction is parallel to the line of reflection, so h is a glide reflection.

We have now justified the information in Table 1. Each row describes a different kind of isometry. Using fixed points it is easy to distinguish the first four rows from each other and to distinguish glide reflections from any isometry besides translations. A glide reflection can't be a translation since any isometry of $\mathbf{R}^2$ is uniquely of the form $h_{A,w}$, and translations have $A = I_2$ while glide reflections have $\det A = -1$.

Lemma 5.1. *A composition of two reflections of* $\mathbf{R}^2$ *is a translation or a rotation.*

Proof. The product of two matrices with determinant -1 has determinant 1, so the composition of two reflections has the form $v \mapsto Av + w$ where $\det A = 1$. Such isometries are translations or rotations by Table 1 (consider the identity to be a trivial translation or rotation). □

Theorem 5.2. *Each isometry of* $\mathbf{R}^2$ *is a composition of at most* 2 *reflections except for glide reflections, which are a composition of* 3 (*and no fewer*) *reflections.*

Proof. We check the theorem for each type of isometry in Table 1 besides reflections, for which the theorem is obvious.

The identity is the square of any reflection.

For a translation $t(v) = v + w$, let A be the matrix representing the reflection across the line $w^\perp$. Then $Aw = -w$. Set $s_1(v) = Av + w$ and $s_2(v) = Av$. Both s_1 and s_2 are reflections, and $(s_1 \circ s_2)(v) = A(Av) + w = v + w$ since $A^2 = I_2$.

Now consider a rotation, say $h(v) = A(v - p) + p$ for some $A \in \mathrm{O}_2(\mathbf{R})$ with $\det A = 1$ and $p \in \mathbf{R}^2$. We have $h = t \circ r \circ t^{-1}$, where t is translation by p and $r(v) = Av$ is a rotation around the origin. Let A' be any reflection matrix (*e.g.*, $A' = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$). Set $s_1(v) = AA'v$ and $s_2(v) = A'v$. Both s_1 and s_2 are reflections and $r = s_1 \circ s_2$ (check). Therefore

$$h = t \circ r \circ t^{-1} = (t \circ s_1 \circ t^{-1}) \circ (t \circ s_2 \circ t^{-1}). \tag{5.4}$$

The conjugate of a reflection by a translation (or by any isometry, for that matter) is another reflection, as an explicit calculation using Table 1 shows. Thus, (5.4) expresses the rotation h as a composition of 2 reflections.

Finally we consider glide reflections. Since this is the composition of a translation and a reflection, it is a composition of 3 reflections. We can't use fewer reflections to get a glide reflection, since a composition of two reflections is either a translation or a rotation by Lemma 5.1 and we know that a glide reflection is not a translation or rotation (or reflection). □

In Table 2 we record the minimal number of reflections whose composition can equal a particular type of isometry of $\mathbf{R}^2$.

That each isometry of $\mathbf{R}^2$ is a composition of at most 3 reflections can be proved geometrically, without recourse to a prior classification of all isometries of the plane. We will give a rough sketch of the argument. We will take for granted (!) that an isometry that fixes at

Isometry	Min. Num. Reflections	dim(fixed set)
Identity	0	2
Nonzero Translation	2	0
Nonzero Rotation	2	0
Reflection	1	1
Glide Reflection	3	0

Table 2. Counting Reflections in an Isometry

least two points is a reflection across the line through those points or is the identity. (This is related to Corollary 2.3 when $n = 2$.) Pick any isometry h of $\mathbf{R}^2$. We may suppose h is not a reflection or the identity (the identity is the square of any reflection), so h has at most one fixed point. If h has one fixed point, say P, choose $Q \neq P$. Then $h(Q) \neq Q$ and the points Q and $h(Q)$ lie on a common circle centered at P (because $h(P) = P$). Let s be the reflection across the line through P that is perpendicular to the line connecting Q and $h(Q)$. Then $s \circ h$ fixes P and Q, so $s \circ h$ is the identity or is a reflection. Thus $h = s \circ (s \circ h)$ is a reflection or a composition of two reflections. If h has no fixed points, pick any point P. Let s be the reflection across the perpendicular bisector of the line connecting P and $h(P)$, so $s \circ h$ fixes P. Thus $s \circ h$ has a fixed point, so our previous argument shows $s \circ h$ is either the identity, a reflection, or the composition of two reflections, so h is the composition of at most 3 reflections.

A byproduct of this argument, which did not use the classification of isometries, is another proof that all isometries of $\mathbf{R}^2$ are invertible: any isometry is a composition of reflections and reflections are invertible.