

1-10 Set Theory (III): Functions

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$$y \in f(X) \iff \exists x \in X : y = f(x)$$

$$x \in f^{-1}(Y) \iff f(x) \in Y$$

$$a \in A \implies f(a) \in f(A)$$

$$f(A_1 \cup A_2) = f(A_1) \cup f(A_2) \qquad f\left(\bigcup_{\alpha \in I} A_\alpha\right) = \bigcup \{f(A_\alpha) \mid \alpha \in I\}$$

$$f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2) \qquad f\left(\bigcap_{\alpha \in I} A_\alpha\right) \subseteq \bigcap \{f(A_\alpha) \mid \alpha \in I\}$$

UD Problem 16.14

$$f : A \rightarrow B \quad g_1, g_2 : B \rightarrow A$$

$$f \circ g_1 = f \circ g_2$$

- (a) Show that if f is bijective, then $g_1 = g_2$.
- (b) If $g_1 \circ f = g_2 \circ f$ and f is bijective, must $g_1 = g_2$?

$$f^{-1} \circ (f \circ g_1) = f^{-1} \circ (f \circ g_2)$$

$$(g_1 \circ f) \circ f^{-1} = (g_2 \circ f) \circ f^{-1}$$

Theorem

$$f \circ g_1 = f \circ g_2 \wedge f \text{ is injective} \implies g_1 = g_2.$$

$$\begin{aligned} (f \circ g_1)(x) &= (f \circ g_2)(x) \\ \implies f(g_1(x)) &= f(g_2(x)) \\ \implies g_1(x) &= g_2(x) \quad (f \text{ is injective}) \end{aligned}$$

Theorem

$$g_1 \circ f = g_2 \circ f \wedge f \text{ is surjective} \implies g_1 = g_2.$$

$$\forall b \in B : \exists a \in A : b = f(a)$$

$$g_1(b) = g_1(f(a)) = g_2(f(a)) = g_2(b)$$

(Composition) UD Problem 16.22

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function such that f is onto and $f \circ f \circ f = f$.
Prove that f is bijective.

f is onto

$$\implies \forall y \in \mathbb{R} : \exists x \in \mathbb{R} : f(x) = y$$

$$\implies \forall y \in \mathbb{R} : \exists x \in \mathbb{R} : f^3(x) = f^2(y)$$

$$\implies \forall y \in \mathbb{R} : \exists x \in \mathbb{R} : f(x) = f^2(y)$$

$$\implies \forall y \in \mathbb{R} : \exists x \in \mathbb{R} : y = f^2(y)$$

$$\implies \forall y \in \mathbb{R} : y = f^2(y)$$

$$\implies f^2 = Id_{\mathbb{R}}$$

Theorem

If $f : X \rightarrow X$, then

$$f^2 = Id_X \implies f \text{ is injective.}$$

$$\begin{aligned} f(x_1) &= f(x_2) \\ \implies f^2(x_1) &= f^2(x_2) \\ \implies Id_X(x_1) &= Id_X(x_2) \\ \implies x_1 &= x_2 \end{aligned}$$

Image (UD Problem 17.22)

$$f : A \rightarrow B, \quad A_1, A_2 \subseteq A$$

- (i) If $f(A_1) = f(A_2)$, must $A_1 = A_2$?
- (ii) When is $f(A_1) = f(A_2) \implies A_1 = A_2$?

$$a_1 \in A_1$$

$$\implies f(a_1) \in f(A_1)$$

$$\implies f(a_1) \in f(A_2)$$

$$\implies \exists a_2 \in A_2 : f(a_2) = f(a_1)$$

$$\implies a_1 = a_2 \quad (\text{if } f \text{ is injective})$$

$$\implies a_1 \in A_2$$

(Inverse Image) UD Problem 17.23

$$f : A \rightarrow B, \quad B_1, B_2 \subseteq B$$

- (i) If $f^{-1}(B_1) = f^{-1}(B_2)$, must $B_1 = B_2$?
- (ii) When is $f^{-1}(B_1) = f^{-1}(B_2) \implies B_1 = B_2$?

$$b_1 \in B_1$$

$$\implies \exists a_1 \in A : f(a_1) = b_1 \in B_1 \quad (\text{if } f \text{ is surjective})$$

$$\implies \exists a_1 \in A : a_1 \in f^{-1}(B_1)$$

$$\implies \exists a_1 \in A : a_1 \in f^{-1}(B_2)$$

$$\implies \exists a_1 \in A : f(a_1) \in B_2$$

$$\implies b_1 \in B_2 \quad (f(a_1) = b_1, \text{ since } f \text{ is a function})$$

Monotonicity

Assume that $F : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ and that F has the monotonicity property:

$$X \subseteq Y \subseteq A \implies F(X) \subseteq F(Y).$$

Define

$$B = \bigcap \{X \subseteq A \mid F(X) \subseteq X\}$$

$$C = \bigcup \{X \subseteq A \mid X \subseteq F(X)\}.$$

- (a) Show that $F(B) = B$ and $F(C) = C$.
- (b) Show that if $F(X) = X$, then $B \subseteq X \subseteq C$.

$$F(X) = X \implies F(X) \subseteq X \wedge X \subseteq F(X) \implies B \subseteq X \subseteq C$$

$$B = \bigcap \{X \subseteq A \mid F(X) \subseteq X\}$$

$$F(B) \subseteq B$$

$$\begin{aligned} F(B) &= F\left(\bigcap \{X \subseteq A \mid F(X) \subseteq X\}\right) \\ &\subseteq \bigcap \{F(X) : X \subseteq A \wedge F(X) \subseteq X\} \\ &\subseteq \bigcap \{X : X \subseteq A \wedge F(X) \subseteq X\} \\ &= B \end{aligned}$$

$$B = \bigcap \{X \subseteq A \mid F(X) \subseteq X\}$$

$$B \subseteq F(B)$$

$$F(B) \subseteq B \implies F(F(B)) \subseteq F(B) \implies B \subseteq F(B)$$

$$C = \bigcup \{X \subseteq A \mid X \subseteq F(X)\}.$$

$$C \subseteq F(C)$$

$$\begin{aligned} C &= \bigcup \{X \subseteq A \mid X \subseteq F(X)\} \\ &\subseteq \bigcup \{F(X) \subseteq A \mid X \subseteq F(X)\} \\ &= F\left(\bigcup \{X \subseteq A \mid X \subseteq F(X)\}\right) \\ &= F(C) \end{aligned}$$

$$C = \bigcup \{X \subseteq A \mid X \subseteq F(X)\}.$$

$$F(C) \subseteq C$$

$$C \subseteq F(C) \implies F(C) \subseteq F(F(C)) \implies F(C) \subseteq C$$

Thank
You!