3-11 Matchings and Factors

(Part I: Matchings and Covers)

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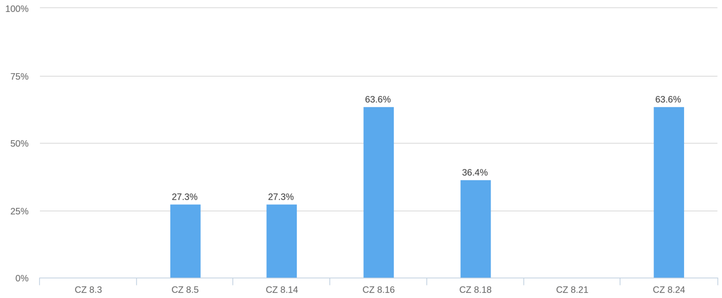
December 10, 2018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CZ 8.3
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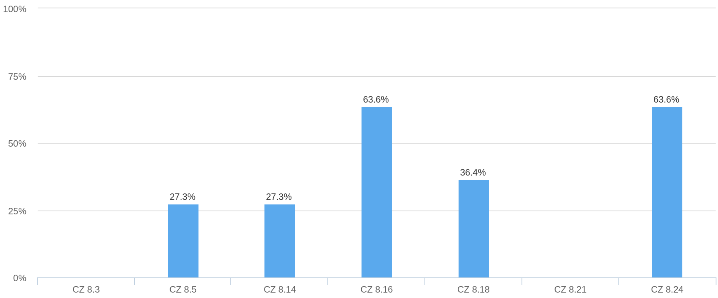
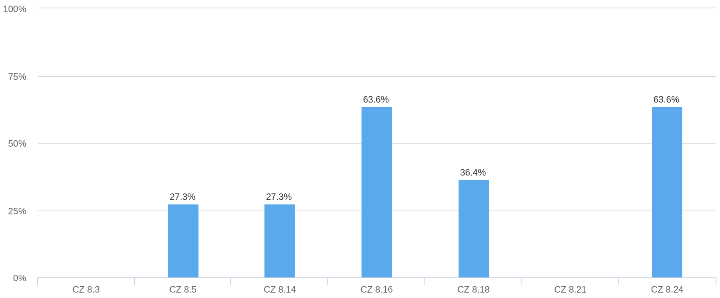


8.5 8.14 8.16

8.18 8.24 (Next Class)



8.5 8.14 8.16 Chinese Postman Problem

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Theorem (Hall's Theorem, 1935; Theorem 8.3)

Let G be a bipartite graph with partite sets U and W such that $r = |U| \leq |W|$.
G contains a matching of cardinality $r \iff G$ satisfies Hall's Condition:

$$\forall X \subseteq U : |N(X)| \geq |X|$$

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Other TONCAS?

Perfect Matching on Trees (Problem 8.5)

Prove that every tree has ≤ 1 perfect matching.

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$\alpha(G)$	Maximum size of independent set
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Theorem (Callai Identities, 1959; Theorem 8.7)

If G is graph without isolated vertices, then

$$\alpha'(G) + \beta'(G) = n(G).$$

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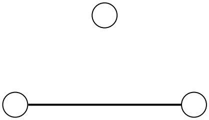

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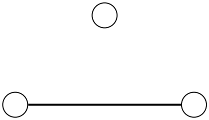

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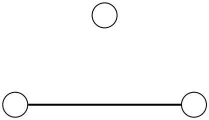

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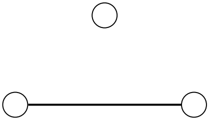

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Contradiction: No isolated vertices.

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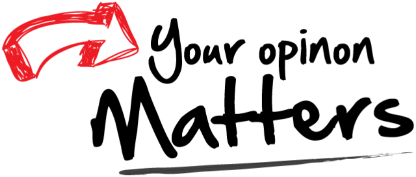
while $|V(G) > 0|$ **do**
  Choose $v \in V(G)$
  $S \leftarrow S \cup \{v\}$
  $G \leftarrow G - \{v\} - N(v)$


THANK YOU



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