3-11 Matchings and Factors

(Part II: Perfect Matchings)

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Chinese Postman Problem (CPP)

(Postman Tour Problem, Route Inspection Problem)

管梅谷(1934-)

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奇偶点图上作业法*

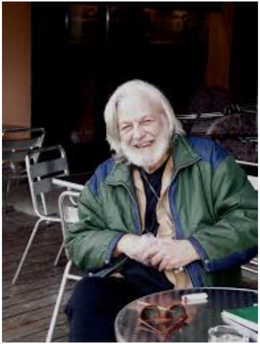
管梅谷
(山东师范学院)

§1. 問題的提出

在邮局搞綫性規划时，发現了下述問題："一个投递員每次上班，要走遍他負責送信的段[1]，然后回到邮局. 問应該怎样走才能使所走的路程最短."

《奇偶点图上作业法》，1960

Translated into English in 1962



Jack Edmonds (1934-)

MATCHING, EULER TOURS AND THE CHINESE POSTMAN

Jack EDMONDS
University of Waterloo, Waterloo, Ontario, Canada

and

Ellis L. JOHNSON
IBM Watson Research Center, Yorktown Heights, New York, U.S.A.

Received 20 May 1972
Revised manuscript received 3 April 1973

The solution of the Chinese postman problem using matching theory is given. The convex hull of integer solutions is described as a linear programming polyhedron. This polyhedron is used to show that a good algorithm gives an optimum solution. The algorithm is a specialization of the more general *b*-matching blossom algorithm. Algorithms for finding Euler tours and related problems are also discussed.

"Matching, Euler Tours and the Chinese Postman", 1973(1965)

Definition (Chinese Postman Problem)

Given an undirected weighted graph G with $w(e) > 0$,
to find the shortest tour such that each edge is traversed at least once.

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Q : What is the relation between Postman Tour and Eulerian Tour?

P: A postman tour of G

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Let $1 + x_e$ $(x_e \in \mathbb{N})$ be the number of times edge e is in P.

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Let $1 + x_e$ $(x_e \in \mathbb{N})$ be the number of times edge e is in P.

Construct $G' = G + e \cdot x_e$

P: A postman tour of G

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Let $1 + x_e$ ($x_e \in \mathbb{N}$) be the number of times edge e is in P.

Construct $G' = G + e \cdot x_e$

$\boxed{P \text{ is an Eulerian tour of } G'.}$

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Definition (Chinese Postman Problem)

Given an undirected weighted graph G with $w(e) > 0$,
to find $x_e \in \mathbb{N}$ for each edge e of G

$$\text{to minimize } \sum_e w(e) x_e,$$

such that $G' = G + e \cdot x_e$ is an Eulerian graph.

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Q : What are the possible values of each x_e?

Definition (Chinese Postman Problem)

Given an undirected weighted graph G with $w(e) > 0$,
to find $x_e \in \{0, 1\}$ for each edge e of G

$$\text{to minimize } \sum_e w(e)x_e,$$

such that $G' = G + e \cdot x_e$ is an Eulerian graph.

$V_o \leftarrow \{v \in V(G) : \deg(v) \text{ is odd}\}$

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Construct a complete weighted graph G_p with vertices V_o:

for $u, v \in V_o$ **do**

    $w(u, v) \leftarrow$ the length of the shortest path between u and v

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Construct a complete weighted graph G_p with vertices V_o:
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Find a minimum-weighted perfect matching M of G_p

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for $(u, v) \in M$ **do**

$\quad p \leftarrow$ the shortest path between u and v

$\quad \forall e \in p : x_e \leftarrow 1$

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Find a minimum-weighted perfect matching M of G_p

for (u, v) ∈ M do
    p ← the shortest path between u and v
    ∀e ∈ p : x_e ← 1
```

Q : What if some edge $e \in E(G)$ is in two shortest paths corresponding to (two) matching edges of G_p?

Theorem (Edge-disjointness of Shortest Paths)

No edge $e \in E(G)$ is in two shortest paths corresponding to (two) matching edges of G_p.

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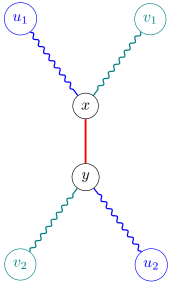

Suppose that

$$\exists e \in E(G) : e \in u_1 \rightsquigarrow u_2 \land e \in v_1 \rightsquigarrow v_2$$

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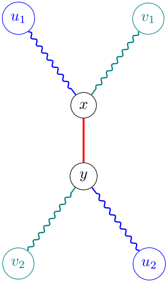

Suppose that

$$\exists e \in E(G) : e \in u_1 \rightsquigarrow u_2 \wedge e \in v_1 \rightsquigarrow v_2$$

Contradiction:

$$u_1 \rightsquigarrow v_1, u_2 \rightsquigarrow v_2 \implies \text{smaller perfect matching}$$

□

Theorem (Property of CHINESE-POSTMAN)

The edges with $x_e = 1$ obtained by CHINESE-POSTMAN *is a minimum collection of edge-disjoint paths connecting pairs of odd vertices.*

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To prove that these $x_e = 1$ obtained by Chinese-Postman satisfies:

Definition (Chinese Postman Problem)

Given an undirected weighted graph G with $w(e) > 0$,
to find $x_e \in \{0, 1\}$ for each edge e of G

$$\text{to minimize } \sum_e w(e)x_e,$$

such that $G' = G + e \cdot x_e$ is an Eulerian graph.

Lemma (Chinese-Postman Gives a Postman Tour)

$G' = G + e \cdot x_e$ *is an Eulerian graph.*

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Proof.

A collection of edge-disjoint paths connecting pairs of odd vertices. □

Lemma (Chinese-Postman Gives an Optimal Postman Tour)

$$\sum_e w(e)x_e \text{ is minimized.}$$

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Theorem (Property of Chinese-Postman)

The edges with $x_e = 1$ obtained by Chinese-Postman *is a minimum collection of edge-disjoint simple paths connecting pairs of odd vertices.*

Lemma (CHINESE-POSTMAN Gives an Optimal Postman Tour)

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P: An optimal postman tour of G

Let $1 + x_e$ ($x_e \in \mathbb{N}$) be the number of times edge e is in P.

We show that the edges with $x_e = 1$ is a collection of edge-disjoint simple paths connecting pairs of odd vertices.

Lemma (Property of Optimal Postman Tours)

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Let $1 + x_e$ ($x_e \in \mathbb{N}$) be the number of times edge e is in P.

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Proof.

By Construction.

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The edges with $x_e = 1$ is a collection of edge-disjoint simple paths connecting pairs of odd vertices.

Proof.

By Construction.

An odd number of edges e with $x_e = 1$ meet odd nodes.

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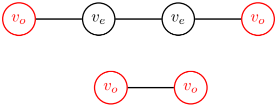
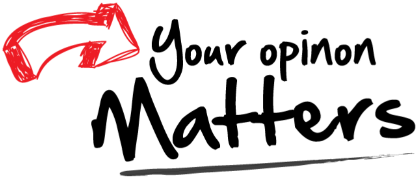


□


THANK YOU



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