A4 in S4

From Groupprops

> This article is about a particular subgroup in a group, up to equivalence of subgroups (i.e., an isomorphism of groups that induces the corresponding isomorphism of subgroups). The subgroup is (up to isomorphism) alternating group:A4 and the group is (up to isomorphism) symmetric group:S4 (see subgroup structure of symmetric group:S4).
> The subgroup is a normal subgroup and the quotient group is isomorphic to cyclic group:Z2.
> **VIEW**: Group-subgroup pairs with the same subgroup part | Group-subgroup pairs with the same group part| Group-subgroup pairs with the same quotient part | All pages on particular subgroups in groups

This article describes the subgroup H in the group G, where G is the symmetric group of degree four, acting on the set $\{1,2,3,4\}$ (for concreteness) and H is the alternating group of degree four, i.e., the subset of G comprising the even permutations.

H is a subgroup of index two, and its unique other coset (which is both a left coset and right coset) is the set of odd permutations.

Explicitly:

$$H = \{(), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3), (1,2,3), (1,3,2), (1,3,4), (1,4,3), (1,2,4), (1,4,2), (2,3,4), (2,4,3)\}$$

See also element structure of symmetric group:S4 (to understand more about the elements, and which of them are even and which are odd permutations) and subgroup structure of symmetric group:S4.

Contents

- 1 Cosets
- 2 Complements
  - 2.1 Properties related to complementation
- 3 Subgroup-defining functions
- 4 Description in alternative interpretations of the whole group
- 5 Subgroup properties
  - 5.1 Invariance under automorphisms and endomorphisms: properties
  - 5.2 Advanced properties related to invariance/resemblance
- 6 GAP implementation
  - 6.1 Finding this subgroup inside the group as a black box
  - 6.2 Constructing the group and the subgroup

Cosets

H is a subgroup of index two, hence a normal subgroup. It has exactly two cosets: the subgroup itself and the rest of the group. Each of these is both a left coset and a right coset. The subgroup is the set of even permutations, and the other coset is the set of odd permutations. Explicitly:

$$H = \{(), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3), (1,2,3), (1,3,2), (1,3,4), (1,4,3), (1,2,4), (1,4,2), (2,3,4), (2,4,3)\}$$

$$G \setminus H = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4), (1,2,3,4), (1,4,3,2), (1,2,4,3), (1,3,4,2), (1,4,2,3), (1,3,2,4)\}$$

For more on the element structure and interaction with conjugacy class structure, see element structure of symmetric group:S4#Interpretation as symmetric group.

Complements

> **COMPLEMENTS TO NORMAL SUBGROUP: TERMS/FACTS TO CHECK AGAINST**:
> **TERMS**: permutable complements | permutably complemented subgroup | lattice-complemented subgroup | complemented normal subgroup (normal subgroup that has permutable complement, equivalently, that has lattice complement) | retract (subgroup having a normal complement)
> **FACTS**: complement to normal subgroup is isomorphic to quotient | complements to abelian normal subgroup are automorphic | complements to normal subgroup need not be automorphic | Schur-Zassenhaus theorem (two parts: normal Hall implies permutably complemented and Hall retract implies order-conjugate)

There are *six* different candidates for a permutable complement to H in G. Since H is a normal subgroup of G, these are also precisely the lattice complements of H in G. Each of these is isomorphic to the quotient group cyclic group:Z2 and in fact, in this case, they are all conjugate subgroups in G:

$$\{(),(1,2)\},\{(),(1,3)\},\{(),(1,4)\},\{(),(2,3)\},\{(),(2,4)\},\{(),(3,4)\}$$

Properties related to complementation

Property	Meaning	Satisfied?	Explanation	Related examples
complemented normal subgroup	normal subgroup with permutable complement	Yes		Find all subgroups of the same big group satisfying the property | Find all subgroups of the same big group dissatisfying the property| Find all occurrences of the subgroup in other big groups satisfying the property | Find all subgroups of the same big group dissatisfying the property
complemented characteristic subgroup	characteristic subgroup with permutable complement	Yes		Find all subgroups of the same big group satisfying the property | Find all subgroups of the same big group dissatisfying the property| Find all occurrences of the subgroup in other big groups satisfying the property | Find all subgroups of the same big group dissatisfying the property
complemented fully invariant subgroup	fully invariant subgroup with permutable complement	Yes		Find all subgroups of the same big group satisfying the property | Find all subgroups of the same big group dissatisfying the property| Find all occurrences of the subgroup in other big groups satisfying the property | Find all subgroups of the same big group dissatisfying the property
permutably complemented subgroup	has a permutable complement	Yes		Find all subgroups of the same big group satisfying the property | Find all subgroups of the same big group dissatisfying the property| Find all occurrences of the subgroup in other big groups satisfying the property | Find all subgroups of the same big group dissatisfying the property
lattice-complemented subgroup	has a lattice complement	Yes		Find all subgroups of the same big group satisfying the property | Find all subgroups of the same big group dissatisfying the property| Find all occurrences of the subgroup in other big groups satisfying the property | Find all subgroups of the same big group dissatisfying the property
retract	has a normal complement	No		Find all subgroups of the same big group satisfying the property | Find all subgroups of the same big group dissatisfying the property| Find all occurrences of the subgroup in other big groups satisfying the property | Find all subgroups of the same big group dissatisfying the property
direct factor	normal subgroup with normal complement	No		Find all subgroups of the same big group satisfying the property | Find all subgroups of the same big group dissatisfying the property| Find all occurrences of the subgroup in other big groups satisfying the property | Find all subgroups of the same big group dissatisfying the property

Subgroup-defining functions

The subgroup is a characteristic subgroup and arises as a result of many common subgroup-defining functions on the whole group. Some of these are given below:

Subgroup-defining function	Meaning in general	Why it takes this value
derived subgroup (also called commutator subgroup)	subgroup generated by commutators	All the even permutations are commutators. In general, the alternating group is the derived subgroup of the corresponding symmetric group.
subgroup generated by squares	subgroup generated by squares	The squares are precisely the elements in the subgroup. The identity element is its own square, each 3-cycle is the square of its inverse, and each double transposition is the square of a 4-cycle.
hypocenter	stable member of lower central series	H is the derived subgroup of G, and $[G, H] = H$.
Jacobson radical	intersection of all maximal normal subgroups	It is the unique maximal normal subgroup. The whole group is a one-headed group and this subgroup is the head.
3-Sylow closure	normal closure of any 3-Sylow subgroup, or join of all the 3-Sylow subgroups	This is the subgroup generated by all the 3-cycles.

Description in alternative interpretations of the whole group

Interpretation of G	Corresponding interpretation of H
As the symmetric group of degree four.	alternating group of degree four.
As the projective general linear group of degree two over field:F3	projecitve special linear group of degree two over field:F3
As the full tetrahedral group, i.e., the full group of symmetries of the regular tetrahedron	Orientation-preserving symmetries of the regular tetrahedron.
As the group of orientation-preserving symmetries of the cube or octahedron.	?
The triangle group with parameters $(3,3,2)$	The corresponding von Dyck group with parameters $(3,3,2)$
The von Dyck group with parameters $(4,3,2)$	?

Subgroup properties

Invariance under automorphisms and endomorphisms: properties

Property	Meaning	Satisfied?	Explanation	Comment
normal subgroup	equals all conjugate subgroups	Yes	index two implies normal	
characteristic subgroup	invariant under all automorphisms	Yes		
fully invariant subgroup	invariant under all endomorphisms	Yes		
coprime automorphism-invariant subgroup	invariant under automorphisms of order coprime to the group	Yes		
cofactorial automorphism-invariant subgroup	invariant under automorphisms whose order has no prime factors other than those in the order of the group	Yes		
subgroup-cofactorial automorphism-invariant subgroup	invariant under automorphisms whose order has no prime factors other than those in the order of the subgroup	Yes		

Advanced properties related to invariance/resemblance

Property	Meaning	Satisfied?	Explanation	Comment
verbal subgroup		Yes		
isomorph-free subgroup	no other isomorphic subgroup	Yes		
isomorph-containing subgroup	contains any isomorphic subgroup	Yes		
order-unique subgroup, index-unique subgroup	no other subgroup of that order, respectively index. Order-unique and index-unique are equivalent properties in finite groups.	Yes		
normal subgroup having no nontrivial homomorphism to its quotient group		Yes		
homomorph-containing subgroup	contains any homomorphic image of it in the whole group	Yes		
variety-containing subgroup		No		

GAP implementation

Finding this subgroup inside the group as a black box

Here, a group G that we know to be isomorphic to the symmetric group of degree four is given, and we need to locate in that the alternating group of degree four. Different ways of constructing/locating this subgroup are given below.

Description	Functions used
`DerivedSubgroup(G)`	DerivedSubgroup

To assign H to any of these, do `H :=` followed by that. For instance:

```
H := DerivedSubgroup(G);
```

Constructing the group and the subgroup

Because of GAP's native implementation of symmetric groups, this can be easily achieved using SymmetricGroup and AlternatingGroup:

```
gap> G := SymmetricGroup(4);;
gap> H := AlternatingGroup(4);;
```

Note that double semicolons have been used to suppress confirmatory output, but you may prefer to use single semicolons.

Category: Particular subgroups