1-8 集合及其运算

魏恒峰

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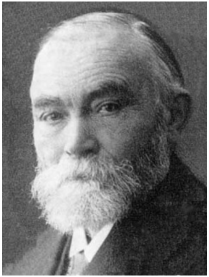
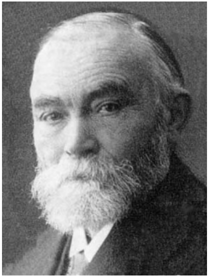


Gottlob Frege (1848–1925)



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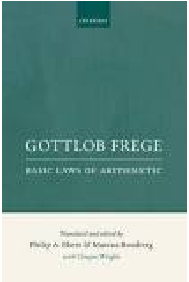
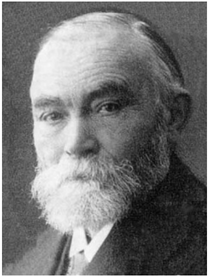
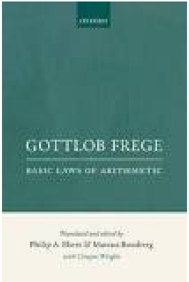


“Basic Laws of Arithmetic”

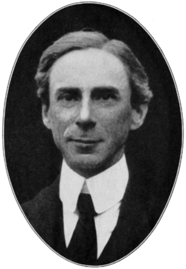




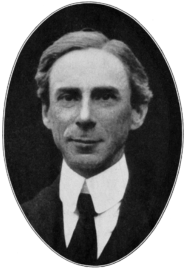
Gottlob Frege (1848–1925)

“Basic Laws of Arithmetic”

对于一个科学工作者来说，最不幸的事情莫过于:
当他的工作接近完成时，却发现那大厦的基础已经动摇。



Bertrand Russell (1872–1970)



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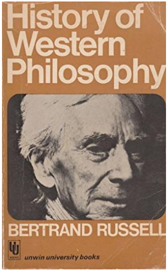

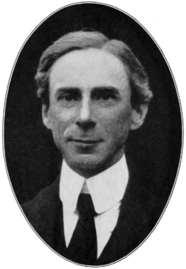

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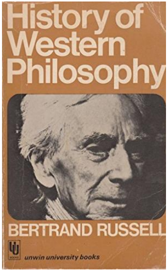
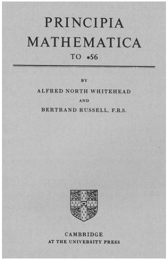
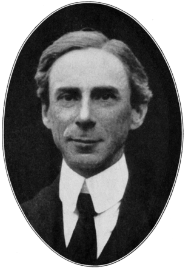




PRINCIPIA
MATHEMATICA
TO *56

BY
ALFRED NORTH WHITEHEAD
AND
BERTRAND RUSSELL, F.R.S.

CAMBRIDGE
AT THE UNIVERSITY PRESS



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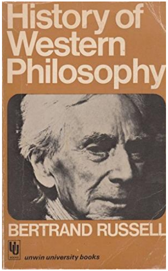

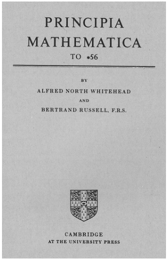

我们将集合理解为任何将我们思想中那些确定而彼此独立的对象放在一起而形成的聚合。

– *Cantor*《超穷数理论基础》

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Theorem (概括原则)

$$\forall \psi(x) \exists X : X = \{x \mid \psi(x)\}.$$

Definition (Russell's Paradox)

$$\psi(x) = \text{“}x \notin x\text{”}$$

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$$R = \{x \mid x \notin x\}$$

$$Q : R \in R\,?$$

Q： 既然朴素集合论存在悖论，你是如何做作业的？


YOU HAVE
TO FIX IT!

$\{x \mid x \notin x\}$ **does not exist!**

A Little Axiomatic Set Theory (ZFC)

Ernst Zermelo (1871–1953)

Abraham Fraenkel (1891–1965)

Definition (Axiom Schema of Separation)

$$\forall \psi(x) : \Big(\forall X \exists Y : Y = \{x \in X \mid \psi(x)\}\Big).$$

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$$R = \{x \mid x \notin x\}$$

Theorem

$\{x \mid x \notin x\}$ **is not a set.**

$$\forall X : R_X = \{x \in X \mid x \notin x\}$$

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$$Q : R_X \in R_X?$$

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There is no universe set.

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Proof.

By contradiction.

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□

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Theorem

There is no universe set. *(It is too "big" to be a set!)*

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Proof.

By contradiction.

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□

Definition ("$\cap$")

$$\begin{aligned} A \cap B &= \{x \in A \mid x \in B\} \\ &= \{x \mid x \in A \wedge x \in B\} \end{aligned}$$

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A \setminus B &= \{x \in A \mid x \notin B\} \\
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We can never look for objects “not in B” unless we know where to start looking. So we use A to tell us where to look for elements not in B. *– UD (Chapter 6)*

Definition (Axiom of Extensionality)

$$\forall A \forall B \forall x (x \in A \iff x \in B) \iff A = B.$$

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$$\{a, a\} = \{a\}$$

Set Operations

$\cap \qquad \cup \qquad \setminus$

UD 7.1 (b): Distributive Property

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

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$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Theorem (Distributive Property (Theorem 7.1))

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Proof. □

UD 7.1 (c): DeMorgan's Law

Let X denote a set, and $A, B \subseteq X$.

$$X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$$

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UD $7.1(d)$

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$Q : A, B \subseteq X?$ $\quad$ $(\text{“} \Leftarrow: X = \emptyset \text{”})$

UD 7.8

Consider the following sets:

(i) $(A \cap B) \setminus (A \cap B \cap C)$

(ii) $A \cap B \setminus (A \cap B \cap C)$

(iii) $A \cap B \cap C^c$

(iv) $(A \cap B) \setminus C$

(v) $(A \setminus C) \cap (B \setminus C)$

(a) Which of the sets above are written ambiguously, if any?

(c) Prove that $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$.

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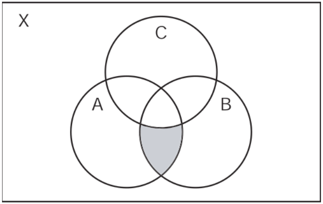

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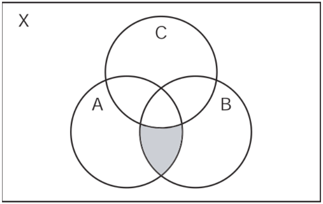

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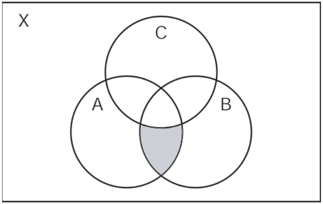

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$$A \setminus C = A \cap C^c$$

$$A \setminus C = \{x \mid x \in A \land x \notin C\}$$

UD 7.9

Prove that the union of two sets can be rewritten as the union of two disjoint sets.

(a) Prove that $(A \setminus B) \cap B = \emptyset$

(b) Prove that $A \cup B = (A \setminus B) \cup B$

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“太容易了，一时没反应过来”

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By contradiction.

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By contraction.

$(A \setminus B) \cup B = \cdots$

“太容易了，一时没反应过来”

Set Family $\{A_\alpha : \alpha \in I\}$

$$\bigcap \qquad \bigcup$$

$$\bigcup_{j=1}^{n} A_j = A_1 \cup A_2 \cup \cdots \cup A_n \qquad \bigcap_{j=1}^{n} A_j = A_1 \cap A_2 \cap \cdots \cap A_n$$

$$\bigcup_{j=1}^{n} A_j = A_1 \cup A_2 \cup \cdots \cup A_n \qquad \bigcap_{j=1}^{n} A_j = A_1 \cap A_2 \cap \cdots \cap A_n$$

$$\bigcup_{j=1}^{\infty} A_j = A_1 \cup A_2 \cup \cdots \qquad \bigcap_{j=1}^{\infty} A_j = A_1 \cap A_2 \cap \cdots$$

$$\bigcup_{\alpha \in I} A_\alpha = \{x \in X \mid \exists \alpha \in I : x \in A_\alpha\}$$

$$\bigcap_{\alpha \in I} A_\alpha = \{x \in X \mid \forall \alpha \in I : x \in A_\alpha\}$$

$$\bigcup_{\alpha \in I} A_\alpha = \{x \in X \mid \exists \alpha \in I : x \in A_\alpha\}$$

$$\bigcap_{\alpha \in I} A_\alpha = \{x \in X \mid \forall \alpha \in I : x \in A_\alpha\}$$

$Q: \ I \neq \emptyset$ for $\bigcap_{\alpha \in I} A_\alpha$ (UD P_{91})

$$\bigcup_{\alpha \in I} A_\alpha = \{x \in X \mid \exists \alpha \in I : x \in A_\alpha\}$$

$$\bigcap_{\alpha \in I} A_\alpha = \{x \in X \mid \forall \alpha \in I : x \in A_\alpha\}$$

$$Q: \; I \neq \emptyset \text{ for } \bigcap_{\alpha \in I} A_\alpha \quad (\text{UD } P_{91})$$

$$Q: \; I \neq \emptyset \text{ for } \bigcup_{\alpha \in I} A_\alpha$$

UD Exercise 8.9

$$X \setminus \bigcup_{\alpha \in I} A_\alpha = \bigcap_{\alpha \in I} (X \setminus A_\alpha)$$

$$X \setminus \bigcap_{\alpha \in I} A_\alpha = \bigcup_{\alpha \in I} (X \setminus A_\alpha)$$

UD 8.8

$$A = \mathbb{R} \setminus \bigcap_{n \in \mathbb{Z}^+} (\mathbb{R} \setminus \{-n, -n+1, \cdots, 0, \cdots, n-1, n\})$$

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$$\begin{aligned}
A &= \mathbb{R} \setminus \bigcap_{n \in \mathbb{Z}^+} (\mathbb{R} \setminus \{-n, -n+1, \cdots, 0, \cdots, n-1, n\}) \\
&= \mathbb{R} \setminus \Big(\mathbb{R} \setminus \bigcup_{n \in \mathbb{Z}^+} \{-n, -n+1, \cdots, 0, \cdots, n-1, n\}\Big) \\
&= \bigcup_{n \in \mathbb{Z}^+} \{-n, -n+1, \cdots, 0, \cdots, n-1, n\} \\
&= \mathbb{Z}
\end{aligned}$$

UD 8.9

$$A = \mathbb{Q} \setminus \bigcap_{n \in \mathbb{Z}} (\mathbb{R} \setminus \{2n\})$$

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UD 8.1

Consider the intervals of real numbers given by

$$A_n = [0, 1/n), B_n = [0, 1/n], C_n = (0, 1/n)$$

(b) Find $\bigcap_{n=1}^{\infty} A_n, \quad \bigcap_{n=1}^{\infty} B_n, \quad \bigcap_{n=1}^{\infty} C_n.$

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UD 8.4

$$\{A_n : n \in \mathbb{Z}^+\}, \quad \{B_n : n \in \mathbb{Z}^+\}$$

UD 8.4

Definition (Axiom of Power Set)

$$\forall X \exists Y \forall u (u \in Y \iff u \subseteq X)$$

UD $9.2(a)$

$$\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$$

UD 9.4

$$A \subseteq B \iff \mathcal{P}(A) \subseteq \mathcal{P}(B)$$

UD 9.12

UD 9.13

$$A \times B \subseteq C \times D \implies (?) A \subseteq C \land B \subseteq D$$

UD 9.14

$$
A \times (B \cup C) = (A \times B) \cup (A \times C)
$$
$$
A \times (B \cap C) = (A \times B) \cap (A \times C)
$$

UD 9.14

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$
$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

$$A \times (B \setminus C) = (A \times B) \setminus (A \times C)$$

UD 9.16


Thank
You!