3-1 Dynamic Programming

(Part II: "Theory")

Hengfeng Wei

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September 27, 2018

Definition (Optimal Substructure)

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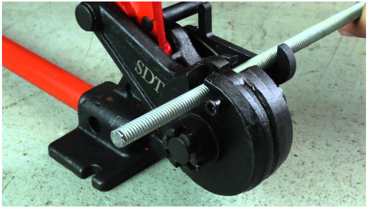
$\boxed{\textit{Relative to Subproblems}}$

Rod Cutting



Optimal Substructure of Rod-Cutting (Problem 15.3-5)

Limit l_i : $\#$ of pieces of length i, $\quad 1 \le i \le n$

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length i	1	2	3	4
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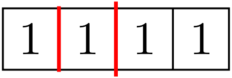

$$R(4) = 3$$

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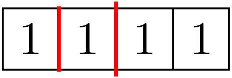

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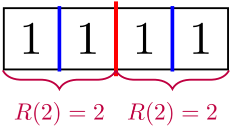


Well done

"Show that the optimal-substructure property described in Section 15.1 *no longer holds."*

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with the length limit array L

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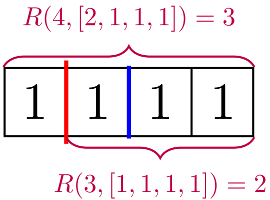

Currency Exchange (Problem 15.3-6)

$1, 2, \cdots, n$ currencies

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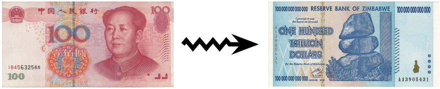

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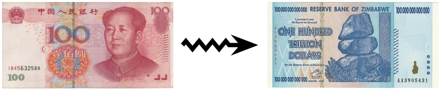

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c_k : Commission charged for k trades

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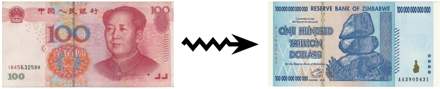

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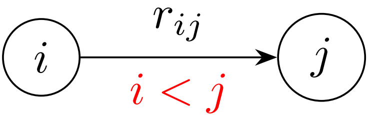
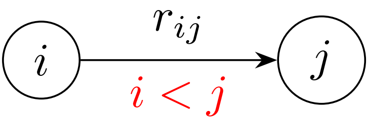
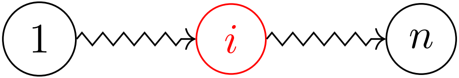
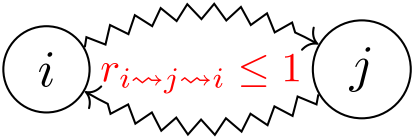
$$c_k = 0$$


i
r_{ij}
j
$i < j$

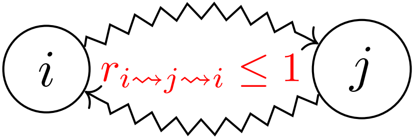


An *optimal* *sequence of trades from* 1 *to* n *through* i*:*





i
$r_{i \rightsquigarrow j \rightsquigarrow i} \leq 1$
j



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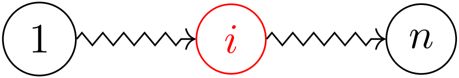

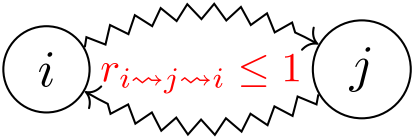

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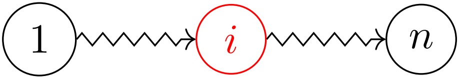
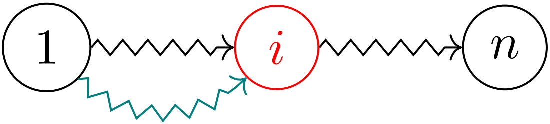
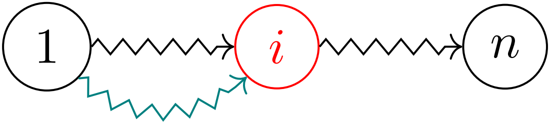


By Contradiction.



$\textsc{Case I}: s_{1 \rightsquigarrow i} \cap s_{i \rightsquigarrow n} = \emptyset$



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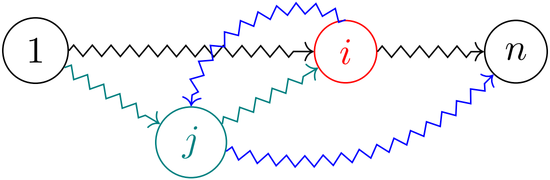

Case II : $j \in s_{1 \rightsquigarrow i} \cap s_{i \rightsquigarrow n}$

$$
\begin{aligned}
& 1 \rightsquigarrow j \rightsquigarrow n \\
\geq\ & 1 \rightsquigarrow j \rightsquigarrow i \rightsquigarrow j \rightsquigarrow n \\
=\ & 1 \rightsquigarrow j \rightsquigarrow i \rightsquigarrow n \\
>\ & 1 \rightsquigarrow i \rightsquigarrow n
\end{aligned}
$$

Longest Path Problem

To find a *simple* path of maximum length from s to t in a graph.

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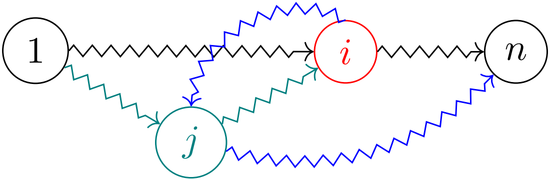

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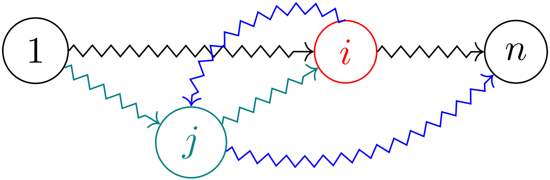

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Does the longest path problem really have no optimal substructure?

Does the longest path problem really have no optimal substructure?

WAIT WAIT...
DON'T TELL ME!®
FROM NPR® & WBEZ® CHICAGO

$L(s, t, A)$: *the longest path from* s *to* t

through exactly the intermediate vertices in I

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$$L(s,t,I) = 1 + \max_{\substack{(s,x) \in E \\ x \in I}} l(x,t,I \setminus \{x\})$$

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What is the next vertex from s?

$$L(s, t, I) = 1 + \max_{\substack{(s,x) \in E \\ x \in I}} l(x, t, I \setminus \{x\})$$

$$L(t, t, I) = 0, \qquad L(s, t, \emptyset) = \begin{cases} 1, & \text{if } (s, t) \in E \\ 0, & \text{otherwise} \end{cases}$$

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Cycle $\implies$ *No Order on Vertices*

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DP does not necessarily lead to efficient (polynomial) algorithms.

The (decision version of the) longest path problem is NP-hard!

The Change-Making Problem

Coins values: $x_1, x_2, \cdots, x_n$

Amount: v

Is it possible to make change for v?

The Change-Making Problem

Without repetition: $0/1$

The Change-Making Problem

Without repetition: $0/1$

$C[i, w]$: Make change for w using only values of $x_1 \ldots x_i$?

$$C[n, v]$$

Without repetition: $0/1$

$C[i, w]$: Make change for w using only values of $x_1 \dots x_i$?

$$C[n, v]$$

$\boxed{\textit{Using value } x_i \textit{ or not?}}$

$$C[i, w] = \underbrace{C[i-1, w]}_{\notin} \lor \big(\underbrace{C[i-1, w - x_i] \land w \ge x_i}_{\in}\big)$$

Without repetition: $0/1$

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The $0/1$ Knapsack Problem (Problem 16.2-2)

Values : v_i Weights : w_i Capacity : W

Taking as valuable a load as possible

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$C[i, w]$: Knapsack with capacity w using only items of values $v_1 \ldots v_i$

$\boxed{\textit{Using value } v_i \textit{ or not?}}$

$$C[i, w] = \max\Big(\underbrace{C[i-1, w]}_{\notin}, \big(\underbrace{w \geq x_i \implies C[i-1, w-w_i] + v_i}_{\in}\big)\Big)$$

The Change-Making Problem

Unbounded repetition: ∞

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$$C[w] = \bigvee_{i:\ x_i \le w} C[w - x_i]$$

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$$O(nv)$$

The Change-Making Problem

Unbounded repetition: ∞

$C[i, w]$ *vs.* $C[w]$

$$C[i, w] = C[i-1, w] \vee \big(C[i, w - x_i] \wedge w \ge x_i\big)$$

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The Change-Making Problem (Problem $16\text{-}1$ (d))

Unbounded repetition: ∞

Using the fewest number of coins

The Change-Making Problem (Problem 16-1 (d))

Unbounded repetition: ∞

Using the fewest number of coins

Using value x_i or not?

$$C[i, w] = \min \Big(\underbrace{C[i-1, w]}_{\notin}, \big(\underbrace{w \ge x_i \implies C[i, w - x_i] + 1}_{\in} \big) \Big)$$

The Change-Making Problem (Problem 16-1 (d))

Unbounded repetition: ∞

Using the fewest number of coins

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Unbounded repetition with $\leq k$ coins overall

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Problem (Hungry-Lion Game)

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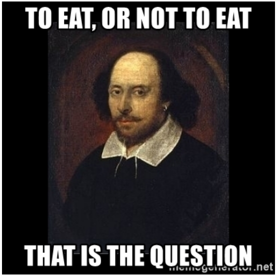


THANK YOU

Your opinon
Matters

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