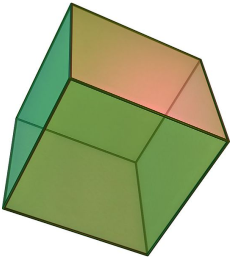
4-5 Polyhedral Groups

Hengfeng Wei

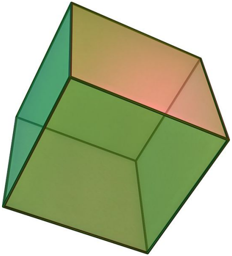
hfwei@nju.edu.cn

April 08, 2019

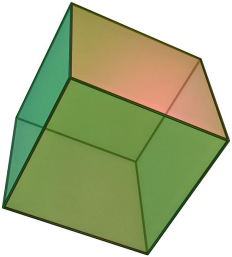

flag永不倒!



$$Sym(C) \cong$$

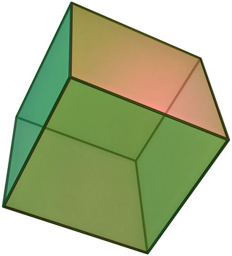


$$Sym(C) \cong S_4$$



$$Sym(C) \cong S_4$$

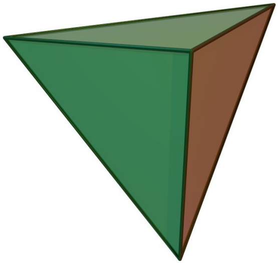
$$\Big|\{H : H \leq Sym(C)\}\Big| =$$



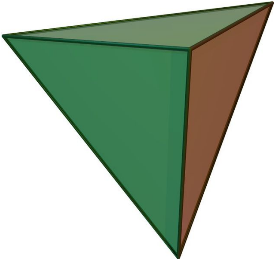
$$Sym(C) \cong S_4$$

$$\Big|\{H : H \leq Sym(C)\}\Big| = 30$$

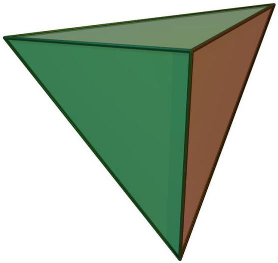

先定一个能达到的小目标



$$Sym(T) \cong$$

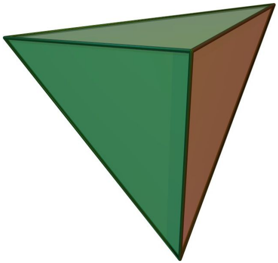


$$Sym(T) \cong A_4$$



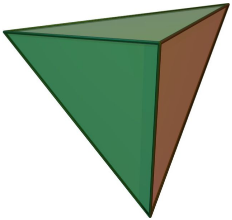
$$Sym(T) \cong A_4$$

$$\Big|\{H : H \leq Sym(T)\}\Big| =$$

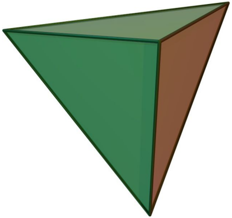


$$Sym(T) \cong A_4$$

$$\Big|\{H : H \leq Sym(T)\}\Big| = 10$$



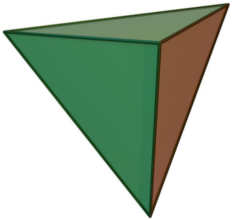
$$Sym(T) \cong A_4$$



$$\color{red}{Sym(T) \cong A_4}$$

Proof.

(1) To find all even perms. in S_4



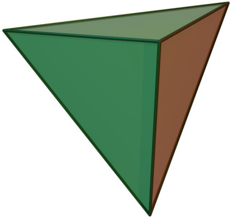
$$Sym(T) \cong A_4$$

Proof.

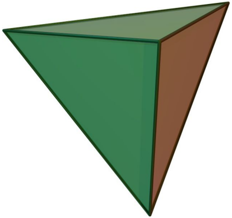
$$(1)\quad \text{To find all even perms. in } S_4$$

$$(2)\quad \text{To show that } \left|Sym(T)\right| < \left|S_4\right|$$

□



$$\left|Sym(T)\right| < \left|S_4\right|$$



$$\left|Sym(T)\right| < \left|S_4\right|$$

$$\because (1\ 2) \notin Sym(T)$$

Rotate through vertices:

$$\text{Fixing } 1: \rho_1 = (2\ 3\ 4) \quad \rho_1^2 = (2\ 4\ 3) \quad \rho_1^3 = 1$$

$$\text{Fixing } 2: \rho_2 = (1\ 3\ 4) \quad \rho_2^2 = (1\ 4\ 3) \quad \rho_2^3 = 1$$

$$\text{Fixing } 3: \rho_3 = (1\ 2\ 4) \quad \rho_3^2 = (1\ 4\ 2) \quad \rho_3^3 = 1$$

$$\text{Fixing } 4: \rho_4 = (1\ 2\ 3) \quad \rho_4^2 = (1\ 3\ 2) \quad \rho_4^3 = 1$$

Rotate through vertices:

$$\text{Fixing } 1: \rho_1 = (2\ 3\ 4) \quad \rho_1^2 = (2\ 4\ 3) \quad \rho_1^3 = 1$$

$$\text{Fixing } 2: \rho_2 = (1\ 3\ 4) \quad \rho_2^2 = (1\ 4\ 3) \quad \rho_2^3 = 1$$

$$\text{Fixing } 3: \rho_3 = (1\ 2\ 4) \quad \rho_3^2 = (1\ 4\ 2) \quad \rho_3^3 = 1$$

$$\text{Fixing } 4: \rho_4 = (1\ 2\ 3) \quad \rho_4^2 = (1\ 3\ 2) \quad \rho_4^3 = 1$$

$$\# = 8 + 1 = 9$$

Rotate through edge-edge:

$$r_1 = (1\ 4)(2\ 3)$$

$$r_2 = (1\ 2)(3\ 4)$$

$$r_3 = (1\ 3)(2\ 4)$$

Rotate through edge-edge:

$$r_1 = (1\ 4)(2\ 3)$$

$$r_2 = (1\ 2)(3\ 4)$$

$$r_3 = (1\ 3)(2\ 4)$$

$$\# = 3$$

$$\begin{aligned} \rho_1 &= (2\ 3\ 4) & \rho_1^2 &= (2\ 4\ 3) \\ \rho_2 &= (1\ 3\ 4) & \rho_2^2 &= (1\ 4\ 3) \\ \rho_3 &= (1\ 2\ 4) & \rho_3^2 &= (1\ 4\ 2) \\ \rho_4 &= (1\ 2\ 3) & \rho_4^2 &= (1\ 3\ 2) \end{aligned}$$

$$\begin{aligned} r_1 &= (1\ 4)(2\ 3) \\ r_2 &= (1\ 2)(3\ 4) \\ r_3 &= (1\ 3)(2\ 4) \end{aligned}$$

$$\boxed{Sym(T) \cong A_4 = \left\{ \text{id}, \quad \underbrace{\text{3-cycle},}_{\#=8} \quad \underbrace{\text{2-2-cycle}}_{\#=3} \right\}}$$

$$\boxed{\left|\{H : H \leq Sym(T)\}\right| = 10}$$

$$\boxed{\Big|\{H : H \le Sym(T)\}\Big| = 10}$$

$$H \le A_4 \Longrightarrow \big|H\big| = 1, 2, 3, 4, 6, 12$$

$$\boxed{\Big|\{H : H \le Sym(T)\}\Big| = 10}$$

$$H \le A_4 \Longrightarrow |H| = 1, 2, 3, 4, 6, 12$$

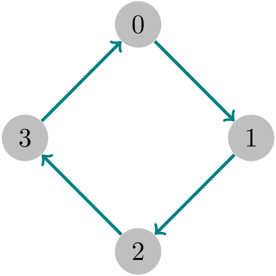
$$|H| = \begin{cases} 1: & \mathrm{id} \quad (\# = 1) \\ 2: & \langle r_1 \rangle, \langle r_2 \rangle, \langle r_3 \rangle \quad (\# = 3) \\ 3: & \langle \rho_1 \rangle, \langle \rho_2 \rangle, \langle \rho_3 \rangle, \langle \rho_4 \rangle \quad (\# = 4) \\ 4: & \{1, r_1, r_2, r_3\} \cong K_4 \quad (\# = 1) \\ 6: & (\# = 0) \\ 12: & A_4 \quad (\# = 1) \end{cases}$$

Theorem (Groups of Order 4)

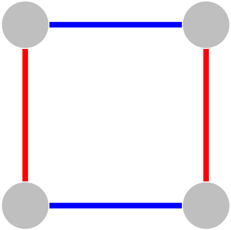
$$|G| = 4 \implies G \cong \mathbb{Z}_4 \lor G \cong K_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

Theorem (Groups of Order 4)

$$|G| = 4 \implies G \cong \mathbb{Z}_4 \vee G \cong K_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$



$\mathbb{Z}_4$



$K_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

Theorem (Groups of Order 4)

$$|G| = 4 \implies G \cong \mathbb{Z}_4 \lor G \cong K_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

Proof.

Theorem (Groups of Order 4)

$$|G| = 4 \implies G \cong \mathbb{Z}_4 \vee G \cong K_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

Proof.

$$|G| = 4, H \leq G \implies |H| = 1, 2, 4$$

Theorem (Groups of Order 4)

$$|G| = 4 \Longrightarrow G \cong \mathbb{Z}_4 \vee G \cong K_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

Proof.

$$|G| = 4, H \leq G \implies |H| = 1, 2, 4$$

$$\exists a \in G : |a| = 4$$

□

Theorem (Groups of Order 4)

$$|G| = 4 \Longrightarrow G \cong \mathbb{Z}_4 \vee G \cong K_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

Proof.

$$|G| = 4, H \leq G \implies |H| = 1, 2, 4$$

$$\forall a \in G : a \neq e \implies |a| = 2$$

$$\exists a \in G : |a| = 4$$

□

Theorem (Groups of Order 4)

$$|G| = 4 \Longrightarrow G \cong \mathbb{Z}_4 \vee G \cong K_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

Proof.

$$|G| = 4, H \leq G \implies |H| = 1, 2, 4$$

$$\forall a \in G : a \neq e \implies |a| = 2$$

$$\exists a \in G : |a| = 4$$

$$G = \langle a \rangle \cong \mathbb{Z}_4$$

□

Theorem (Groups of Order 4)

$$|G| = 4 \Longrightarrow G \cong \mathbb{Z}_4 \vee G \cong K_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

Proof.

$$|G| = 4, H \le G \implies |H| = 1, 2, 4$$

$$\exists a \in G : |a| = 4$$

$$G = \langle a \rangle \cong \mathbb{Z}_4$$

$$\forall a \in G : a \neq e \implies |a| = 2$$

$$H = \{e, a, b, ab\}$$

$$a^2 = b^2 = e, ab = ba$$

□

Theorem (Groups of Order 6)

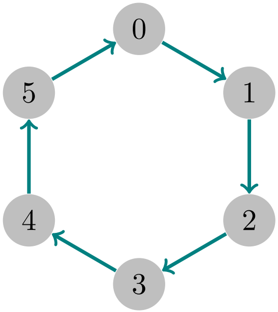
$$|G| = 6 \Longrightarrow$$

Theorem (Groups of Order 6)

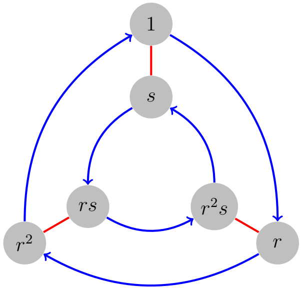
$$|G| = 6 \Longrightarrow G \cong \mathbb{Z}_6 \vee G \cong D_3$$

Theorem (Groups of Order 6)

$$|G| = 6 \Longrightarrow G \cong \mathbb{Z}_6 \vee G \cong D_3$$



$\mathbb{Z}_6$



D_3

Theorem (Groups of Order 6)

$$|G| = 6 \Longrightarrow G \cong \mathbb{Z}_6 \vee G \cong D_3$$

Theorem (Groups of Order 6)

$$|G| = 6 \implies G \cong \mathbb{Z}_6 \vee G \cong D_3$$

$$|G| = 6, H \le G \implies |H| = 1, 2, 3, 6$$

Theorem (Groups of Order 6)

$$\big|G\big| = 6 \Longrightarrow G \cong \mathbb{Z}_6 \vee G \cong D_3$$

$$\big|G\big| = 6, H \leq G \implies \big|H\big| = 1, 2, 3, 6$$

$$(1)\ \exists a \in G, |a| = 6 \implies G = \langle a \rangle \cong \mathbb{Z}_6$$

Theorem (Groups of Order 6)

$$|G| = 6 \implies G \cong \mathbb{Z}_6 \vee G \cong D_3$$

$$|G| = 6, H \leq G \implies |H| = 1, 2, 3, 6$$

$$(1)\ \exists a \in G, |a| = 6 \implies G = \langle a \rangle \cong \mathbb{Z}_6$$

$$(2)\ \forall a \in G, a \neq e \implies |a| = 2 \vee |a| = 3$$

Theorem (Groups of Order 6)

$$|G| = 6 \Longrightarrow G \cong \mathbb{Z}_6 \vee G \cong D_3$$

$$|G| = 6, H \le G \implies |H| = 1, 2, 3, 6$$

$$(1)\ \exists a \in G, |a| = 6 \implies G = \langle a \rangle \cong \mathbb{Z}_6$$

$$(2)\ \forall a \in G, a \ne e \implies |a| = 2 \vee |a| = 3$$

$\exists a \in G : |a| = 2$

$\exists a \in G : |a| = 3$

Theorem (Groups of Order 6)

$$|G| = 6 \Longrightarrow G \cong \mathbb{Z}_6 \vee G \cong D_3$$

$$|G| = 6, H \leq G \implies |H| = 1, 2, 3, 6$$

$$(1)\ \exists a \in G, |a| = 6 \implies G = \langle a \rangle \cong \mathbb{Z}_6$$

$$(2)\ \forall a \in G, a \neq e \implies |a| = 2 \vee |a| = 3$$

$\exists a \in G : |a| = 2$ $\exists a \in G : |a| = 3$

$$\boxed{G = \left\{e, a, a^2, b, ba, ba^2\right\} \quad (a^3 = b^2 = e)}$$

$$\boxed{G = \left\{e, a, a^2, b, ba, ba^2\right\} \quad (a^3 = b^2 = e)}$$

$$\boxed{G = \left\{e, a, a^2, b, ba, ba^2\right\} \quad (a^3 = b^2 = e)}$$

(2.1) $ab = ba$

$$\boxed{G = \left\{e, a, a^2, b, ba, ba^2\right\} \quad (a^3 = b^2 = e)}$$

$$(2.1)\ ab = ba$$

$$G = \langle a, b \mid a^3 = b^2 = e, ab = ba \rangle \cong \mathbb{Z}_6$$

$$\boxed{G = \left\{e, a, a^2, b, ba, ba^2\right\} \quad (a^3 = b^2 = e)}$$

(2.1) $ab = ba$

$$G = \langle a, b \mid a^3 = b^2 = e, ab = ba \rangle \cong \mathbb{Z}_6$$

(2.2) $ab = ba^2$

$$\boxed{G = \left\{e, a, a^2, b, ba, ba^2\right\} \quad (a^3 = b^2 = e)}$$

(2.1) $ab = ba$

$$G = \langle a, b \mid a^3 = b^2 = e, ab = ba \rangle \cong \mathbb{Z}_6$$

(2.2) $ab = ba^2$

$$G = \langle a, b \mid a^3 = b^2 = e, bab^{-1} = a^{-1} \rangle \cong D_3$$

Theorem (Theorem 6.15)

A_4 has no subgroup of order 6.

Theorem (Theorem 6.15)

A_4 has no subgroup of order 6.

By contradiction.

Theorem (Theorem 6.15)

A_4 has no subgroup of order 6.

By contradiction.

Suppose that A_4 has a subgroup H of order 6.

Theorem (Theorem 6.15)

A_4 has no subgroup of order 6.

By contradiction.

Suppose that A_4 has a subgroup H of order 6.

$$H \ncong \mathbb{Z}_6 \implies H \cong D_3$$

Theorem (Theorem 6.15)

A_4 has no subgroup of order 6.

By contradiction.

Suppose that A_4 has a subgroup H of order 6.

$$H \ncong \mathbb{Z}_6 \implies H \cong D_3$$

$$D_3 = \left\{e, a, a^2, b, ba, ba^2\right\} \quad (a^3 = b^2 = e, bab^{-1} = a^{-1})$$

Theorem (Theorem 6.15)

A_4 has no subgroup of order 6.

By contradiction.

Suppose that A_4 has a subgroup H of order 6.

$$H \ncong \mathbb{Z}_6 \implies H \cong D_3$$

$$D_3 = \left\{e, a, a^2, b, ba, ba^2\right\} \quad (a^3 = b^2 = e, bab^{-1} = a^{-1})$$

D_3 contains 3 elements of order 2.

Theorem (Theorem 6.15)

A_4 has no subgroup of order 6.

By contradiction.

Suppose that A_4 has a subgroup H of order 6.

$$H \ncong \mathbb{Z}_6 \implies H \cong D_3$$

$$D_3 = \left\{e, a, a^2, b, ba, ba^2\right\} \quad (a^3 = b^2 = e, bab^{-1} = a^{-1})$$

D_3 contains 3 elements of order 2.

H contains 3 elements of order 2.

Theorem (Theorem 6.15)

A_4 has no subgroup of order 6.

By contradiction.

Suppose that A_4 has a subgroup H of order 6.

$$H \ncong \mathbb{Z}_6 \implies H \cong D_3$$

$$D_3 = \left\{e, a, a^2, b, ba, ba^2\right\} \quad (a^3 = b^2 = e, bab^{-1} = a^{-1})$$

D_3 contains 3 elements of order 2.

H contains 3 elements of order 2.

$$\{1, r_1, r_2, r_3\} \subseteq H$$

Theorem (Theorem 6.15)

A_4 has no subgroup of order 6.

By contradiction.

Suppose that A_4 has a subgroup H of order 6.

$$H \ncong \mathbb{Z}_6 \implies H \cong D_3$$

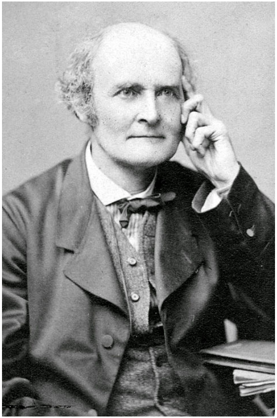
$$D_3 = \left\{e, a, a^2, b, ba, ba^2\right\} \quad (a^3 = b^2 = e, bab^{-1} = a^{-1})$$

D_3 contains 3 elements of order 2.

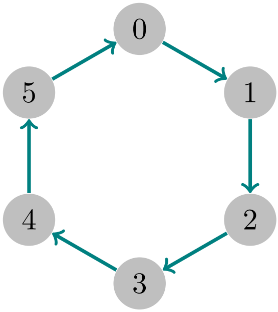
H contains 3 elements of order 2.

$$\{1, r_1, r_2, r_3\} \subseteq H$$

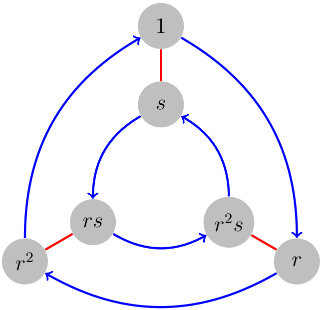
$$K_4 \cong \{1, r_1, r_2, r_3\} \le H \implies 4 \mid 6$$



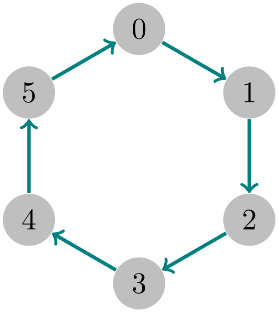
Arthur Cayley (1821 – 1895)



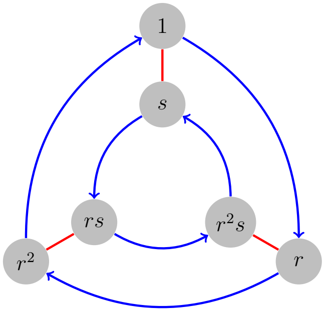
$\mathbb{Z}_6$



D_3



$\mathbb{Z}_6$



D_3

$\Gamma(G, S)$, S is a generating set

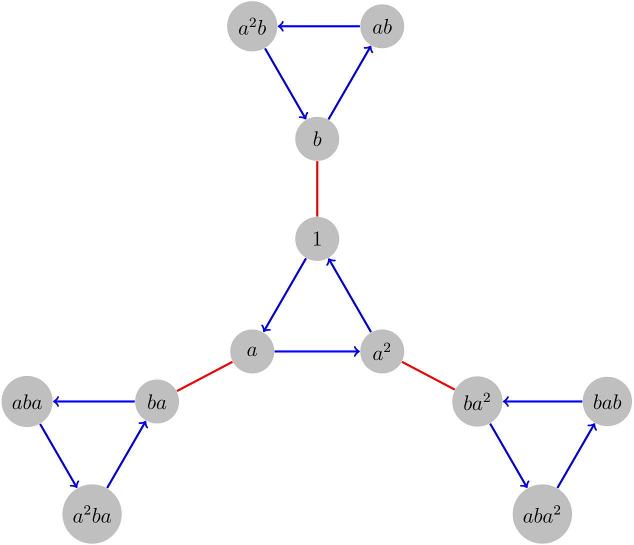
$$\begin{aligned} \rho_1 &= (2\ 3\ 4) & \rho_1^2 &= (2\ 4\ 3) \\ \rho_2 &= (1\ 3\ 4) & \rho_2^2 &= (1\ 4\ 3) \\ \rho_3 &= (1\ 2\ 4) & \rho_3^2 &= (1\ 4\ 2) \\ \rho_4 &= (1\ 2\ 3) & \rho_4^2 &= (1\ 3\ 2) \end{aligned}$$

$$\begin{aligned} r_1 &= (1\ 4)(2\ 3) \\ r_2 &= (1\ 2)(3\ 4) \\ r_3 &= (1\ 3)(2\ 4) \end{aligned}$$

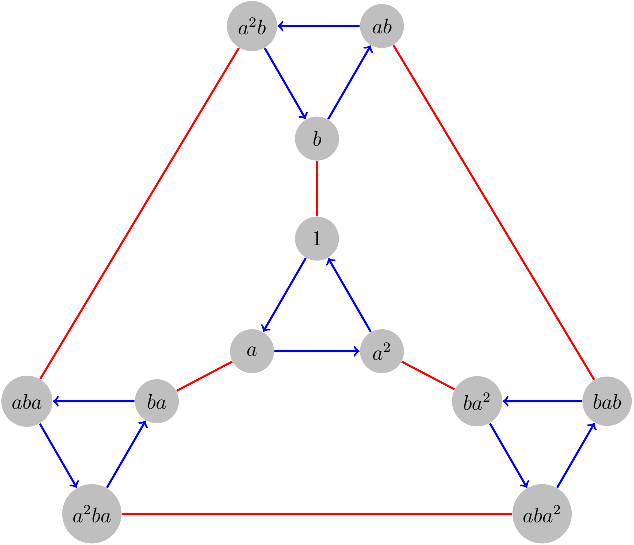
$$\rho_1 = (2\ 3\ 4) \quad \rho_1^2 = (2\ 4\ 3)$$
$$\rho_2 = (1\ 3\ 4) \quad \rho_2^2 = (1\ 4\ 3)$$
$$\rho_3 = (1\ 2\ 4) \quad \rho_3^2 = (1\ 4\ 2)$$
$$\rho_4 = (1\ 2\ 3) \quad \rho_4^2 = (1\ 3\ 2)$$

$$r_1 = (1\ 4)(2\ 3)$$
$$r_2 = (1\ 2)(3\ 4)$$
$$r_3 = (1\ 3)(2\ 4)$$

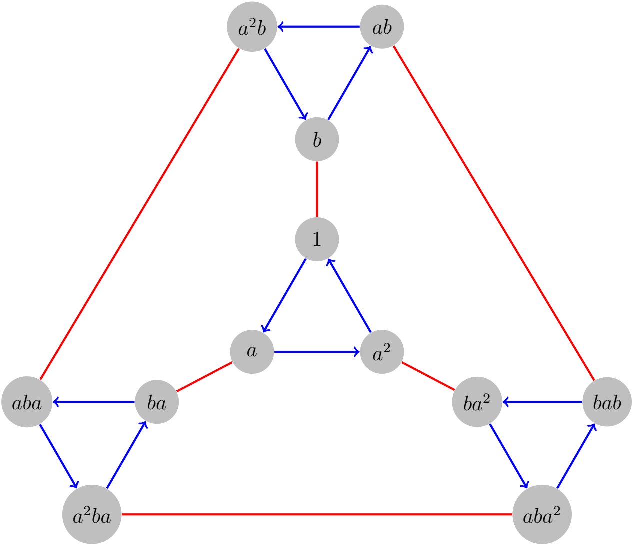
$$a = (1\ 2\ 3) \qquad b = (1\ 2)(3\ 4)$$



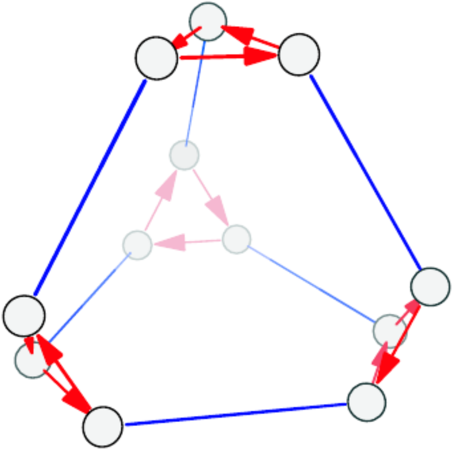
$$a^3 = b^2 = 1$$



$$a^3 = b^2 = 1$$

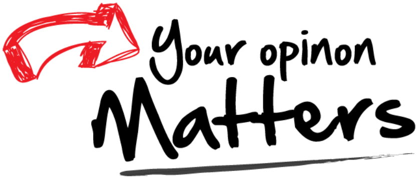


$$a^3 = b^2 = 1 \quad (ba)^3 = 1$$



$Sym(T) \cong A_4$ arranged on a *truncated* tetrahedron


THANK YOU



Office 302

Mailbox: H016

hfwei@nju.edu.cn