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Akra–Bazzi method

In computer science, the **Akra–Bazzi method**, or **Akra–Bazzi theorem**, is used to analyze the asymptotic behavior of the mathematical recurrences that appear in the analysis of divide and conquer algorithms where the sub-problems have substantially different sizes. It is a generalization of the master theorem for divide-and-conquer recurrences, which assumes that the sub-problems have equal size. It is named after mathematicians Mohamad Akra and Louay Bazzi.

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Formulation

The Akra–Bazzi method applies to recurrence formulas of the form

$$T(x) = g(x) + \sum_{i=1}^{k} a_i T(b_i x + h_i(x)) \qquad \text{for } x \geq x_0.$$

The conditions for usage are:

- sufficient base cases are provided
- a_i and b_i are constants for all i
- $a_i > 0$ for all i
- $0 < b_i < 1$ for all i
- $|g(x)| \in O(x^c)$, where c is a constant and O notates Big O notation
- $|h_i(x)| \in O\left(\frac{x}{(\log x)^2}\right)$ for all i
- x_0 is a constant

The asymptotic behavior of $T(x)$ is found by determining the value of p for which $\sum_{i=1}^{k} a_i b_i^p = 1$ and plugging that value into the equation

$$T(x) \in \Theta\left(x^p\left(1 + \int_1^x \frac{g(u)}{u^{p+1}} du\right)\right)$$

(see Θ). Intuitively, $h_i(x)$ represents a small perturbation in the index of T. By noting that $\lfloor b_i x \rfloor = b_i x + (\lfloor b_i x \rfloor - b_i x)$ and that the absolute value of $\lfloor b_i x \rfloor - b_i x$ is always between 0 and 1, $h_i(x)$ can be used to ignore the floor function in the index. Similarly, one can also ignore the ceiling function. For example, $T(n) = n + T\left(\frac{1}{2}n\right)$ and $T(n) = n + T\left(\left\lfloor \frac{1}{2}n \right\rfloor\right)$ will, as per the Akra–Bazzi theorem, have the same asymptotic behavior.

Example

Suppose $T(n)$ is defined as 1 for integers $0 \leq n \leq 3$ and $n^2 + \frac{7}{4}T\left(\left\lfloor \frac{1}{2}n \right\rfloor\right) + T\left(\left\lceil \frac{3}{4}n \right\rceil\right)$ for integers $n > 3$. In applying the Akra–Bazzi method, the first step is to find the value of p for which $\frac{7}{4}\left(\frac{1}{2}\right)^p + \left(\frac{3}{4}\right)^p = 1$. In this example, $p = 2$. Then, using the formula, the asymptotic behavior can be determined as follows:

$$\begin{aligned}
T(x) &\in \Theta\left(x^p\left(1 + \int_1^x \frac{g(u)}{u^{p+1}}\,du\right)\right) \\
&= \Theta\left(x^2\left(1 + \int_1^x \frac{u^2}{u^3}\,du\right)\right) \\
&= \Theta(x^2(1 + \ln x)) \\
&= \Theta(x^2 \log x).
\end{aligned}$$

Significance

The Akra–Bazzi method is more useful than most other techniques for determining asymptotic behavior because it covers such a wide variety of cases. Its primary application is the approximation of the runtime of many divide-and-conquer algorithms. For example, in the merge sort, the number of comparisons required in the worst case, which is roughly proportional to its runtime, is given recursively as $T(1) = 0$ and

$$T(n) = T\left(\left\lfloor \frac{1}{2}n \right\rfloor\right) + T\left(\left\lceil \frac{1}{2}n \right\rceil\right) + n - 1$$

for integers $n > 0$, and can thus be computed using the Akra–Bazzi method to be $\Theta(n \log n)$.

References

- Mohamad Akra, Louay Bazzi: On the solution of linear recurrence equations. *Computational Optimization and Applications* **10**(2):195–210, 1998.
- Tom Leighton: Notes on Better Master Theorems for Divide-and-Conquer Recurrences (http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.39.1636), Manuscript. Massachusetts Institute of Technology, 1996, 9 pages.
- Proof and application on few examples (http://www.mpi-inf.mpg.de/~mehlhorn/DatAlg2008/NewMasterTheorem.pdf)