

3-11 Matchings and Factors

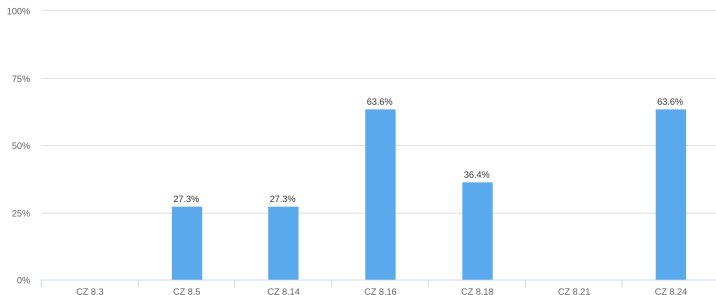
(Part I: Matchings and Covers)

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8.5 8.14 8.16 Chinese Postman Problem (The Last Class?)
8.18 8.24 (The Last Class)

比较大的定理（证明比较长的）都不是很理解，想知道期末考什么

点覆盖边覆盖那里只知道有这些性质，了解不是很深

都理解

图的分解的形象意义

无

定理8.3的证明

$\alpha\beta$ 、 $\alpha'\beta'$ 的定义和几个定理推论

为什么中英文书上的定义中 α 和 β 反了。。

定理8.10的证明看不懂；一些比较几何的构造法证明（比如把顶点排成正多边形，一个点放中间）是怎么保证这些分解不重不漏的？

Kirkman三元系

$$\alpha, \beta, \alpha', \beta'$$

Theorem 8.10 (Tutte's Theorem) (The Last Class)

Theorem (Hall's Theorem, 1935; Theorem 8.3)

Let G be a *bipartite graph* with partite sets U and W such that $r = |U| \leq |W|$.

G contains a matching of cardinality $r \iff G$ satisfies *Hall's Condition*:

$$\forall X \subseteq U : |N(X)| \geq |X|$$

TONCAS

(The Obvious Necessary Conditions are Also Sufficient)

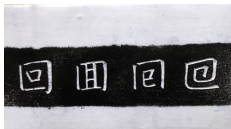


Other TONCAS?

Perfect Matching on Trees (Problem 8.5)

Prove that every tree T has ≤ 1 perfect matching.

“这题有四样证法, 你知道吗?”



Perfect Matching on Trees (Problem 8.5)

Prove that every tree T has ≤ 1 perfect matching.

By strong mathematical induction on the order n of trees.

Inductive step: Consider a tree of order n .

```
1: if  $n$  is odd then
2:   # Perfect Matching = 0
3: else                                ▷  $n$  is even
4:   Consider  $T - r$                   ▷  $r$  : root of  $G$ 
5:   if  $k_o(T - r) > 1$  then
6:     # Perfect Matching = 0
7:   else                              ▷  $k_o(T - r) = 1$ 
8:     By Induction Hypothesis.
```

Perfect Matching on Trees (Problem 8.5)

Prove that every tree T has ≤ 1 perfect matching.

By strong mathematical induction on the order n of trees.

Inductive step: Consider a tree of order n .

-
-
- 1: **if** T has no perfect matchings **then**
 - 2: $\# \text{ Perfect Matching} = 0$
 - 3: **else** $\triangleright T$ has perfect matchings
 - 4: Consider a leaf v
 - 5: v **must** be matched with its parent u
 - 6: **By Induction Hypothesis** on each component of $G - \{u, v\}$
-

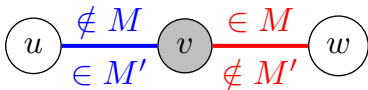
Perfect Matching on Trees (Problem 8.5)

Prove that every tree T has ≤ 1 perfect matching.

By Contradiction.

Suppose that there are two different perfect matchings M and M' on T .

$\exists v : v$ is matched with different vertices in M and M' .



Q : What about u and w ?

Contradiction: Cycle

Perfect Matching on Trees (Problem 8.5)

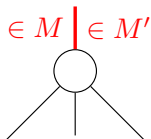
Prove that every tree T has ≤ 1 perfect matching.

By Contradiction.

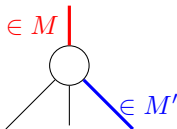
Suppose that there are two different perfect matchings M and M' on T .

$$M \Delta M' = (M - M') \cup (M' - M)$$

Consider the subgraph \mathcal{M} with $V(T)$ and $M \Delta M'$.



CASE I



CASE II

$$\forall v \in V(\mathcal{M}) :$$

$$\deg(v) = 0 \vee \deg(v) = 2$$

$$T \text{ is a tree} \implies \deg(v) = 0$$

$$\deg(v) = 0 \implies \text{CASE I}$$

Theorem (Gallai Identities, 1959; Theorem 8.7)

If G is graph without isolated vertices, then

$$\alpha'(G) + \beta'(G) = n(G).$$

$\alpha(G)$ Maximum size of independent set

$\beta(G)$ Minimum size of vertex cover

$\alpha'(G)$ Maximum size of matching

$\beta'(G)$ Minimum size of edge cover

Theorem (Gallai Identities, 1959; Theorem 8.8)

If G is graph without isolated vertices, then

$$\alpha(G) + \beta(G) = n(G).$$

Matching and Edge Cover (Problem 8.14)

A graph G without isolated vertices has a perfect matching if and only if $\alpha'(G) = \beta'(G)$.

“ \implies ”

“ \impliedby ”

G has a perfect matching

$$\implies n \text{ is even} \wedge \alpha'(G) = n/2$$

$$\implies \beta'(G) = n/2$$

$$\alpha'(G) = \beta'(G)$$

$$\implies \alpha'(G) = n/2 \wedge n \text{ is even}$$

$$\implies G \text{ has a perfect matching}$$

Theorem (König, 1931; Egerváry, 1931)

If G is a *bipartite graph*, then

$$\alpha'(G) = \beta(G).$$

$\alpha(G)$ Maximum size of independent set

$\beta(G)$ Minimum size of vertex cover

$\alpha'(G)$ Maximum size of matching

$\beta'(G)$ Minimum size of edge cover

Theorem (König, 1931)

If G is a *bipartite graph*, then

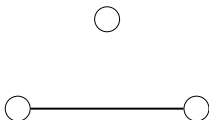
$$\alpha(G) = \beta'(G).$$

Vertex Covering Number (Problem 8.16)

If G is a graph of order n , maximum degree Δ and **having no isolated vertices**, then

$$\beta(G) \geq \frac{n}{\Delta + 1}.$$

By Contradiction: $\beta < \frac{n}{\Delta + 1}$.



$$n = 3, \Delta = 1, \frac{n}{\Delta + 1} = \frac{3}{2}, \beta = 1$$

$$\begin{aligned} \beta \cdot \Delta &< \frac{n\Delta}{\Delta + 1} \\ &= n - \frac{n}{\Delta + 1} \\ &\leq n - 1 \end{aligned}$$



Vertex Covering Number (Problem 8.16; Solution Provided by Dai)

If G is a graph of order n , maximum degree Δ and having no isolated vertices, then

$$\beta(G) \geq \frac{n}{\Delta + 1}.$$

By Contradiction: $\beta < \frac{n}{\Delta + 1}$.

$$\begin{aligned}\beta \cdot \Delta &< \frac{n\Delta}{\Delta + 1} \\ &= n - \frac{n}{\Delta + 1} \\ &< n - \beta\end{aligned}$$

Contradiction: \exists isolated vertices in the other $n - \beta$ vertices.

Vertex Covering Number (Problem 8.16)

If G is a graph of order n , maximum degree Δ and having no isolated vertices, then

$$\beta(G) \geq \frac{n}{\Delta + 1}.$$

Double Counting:

What is the number of **neighbors** of a vertex cover C ?

$$N(C) \leq |C| \cdot \Delta$$

$$N(C) = n - |C|$$

$$n - |C| \leq |C| \cdot \Delta \implies |C| \geq \frac{n}{\Delta + 1}$$

Vertex Independence Number (Additional Problem)

If G is a graph of order n , maximum degree Δ , then

$$\alpha(G) \geq \frac{n}{\Delta + 1}.$$

By Construction.

To construct an independent set S with $|S| \geq \frac{n}{\Delta+1}$.

```
1: while  $|V(G)| > 0$  do
2:   Choose  $v \in V(G)$ 
3:    $S \leftarrow S \cup \{v\}$ 
4:    $G \leftarrow G - \{v\} - N(v)$ 
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