2-4 Recurrences

魏恒峰

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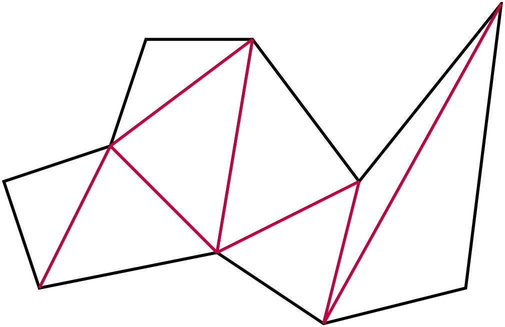
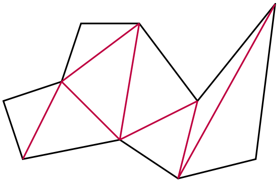
Triangulating Polygons



Ear Lemma (Problem $4.1 - 16$)

$\#$ of triangles (Problem $4.1 - 17$)



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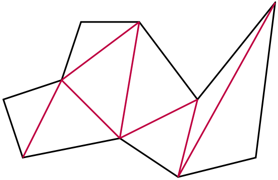

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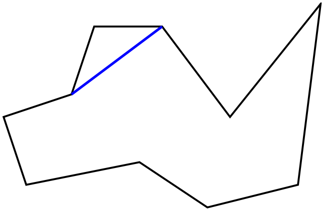

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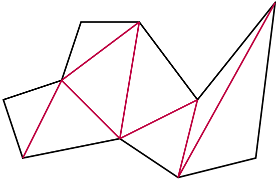

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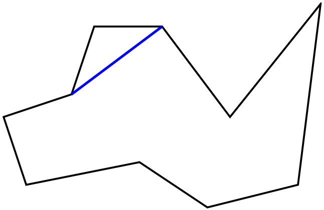

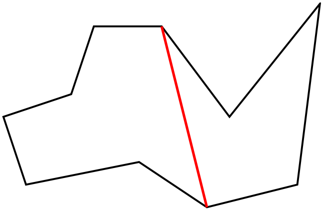

Lemma (Ear Lemma)

A triangle has 3 ears, and a larger triangulated polygon has at least 2 non-adjacent ears.

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Q : Can every polygon be triangulated?

Theorem (Existence of Triangulation)

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> *They break the interior of the polygon into a number of triangles, because any larger polygon can be split by adding a diagonal."*

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Theorem (Existence of Diagonal)

Every polygon with $n > 3$ *has a diagonal.*

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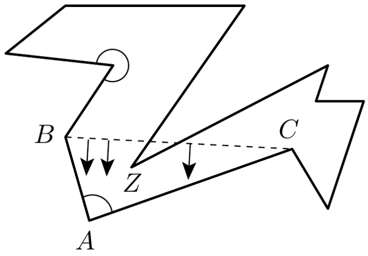
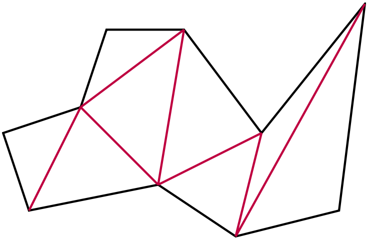


Theorem (Coloring)

Any triangulated polygon polygon is 3*-colorable.*



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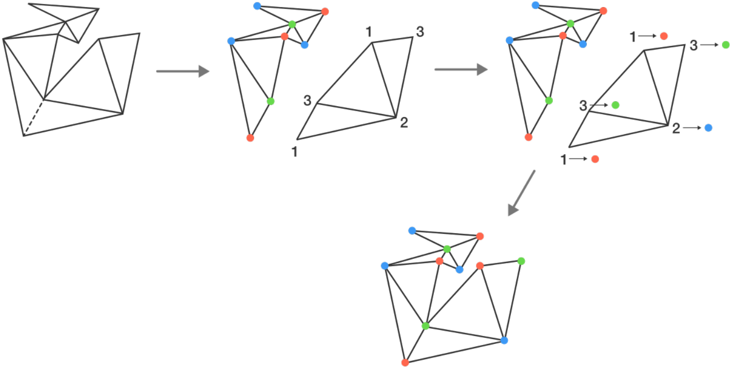

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Q : How many "BIG BROs" to hire?

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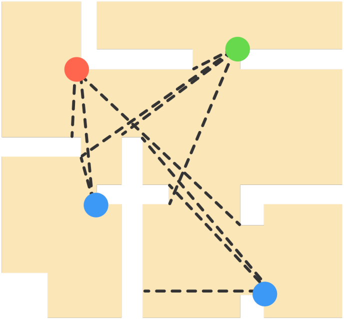

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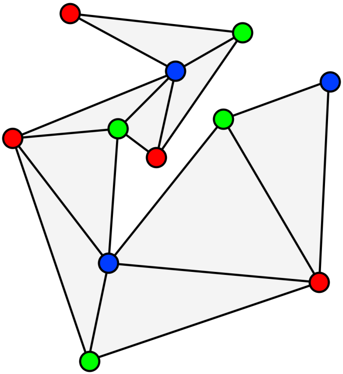

Theorem (The Art Gallery Theorem (O))

For any art gallery with n walls, $\lfloor \frac{n}{3} \rfloor$ "BIG BROs" suffice.

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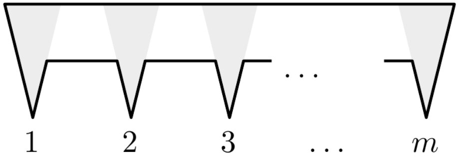
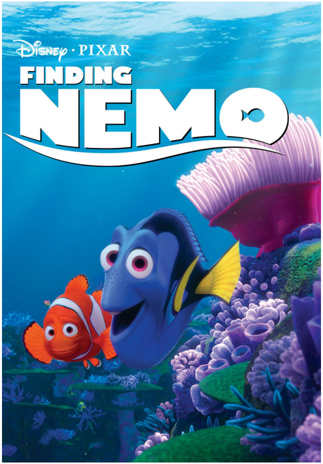


$$n = 3m$$

Fish Recurrence



Fish Recurrence (Problem $4.2-8$)

At the end of each year, a state fish hatchery puts 2000 fish into a lake.

The number of fish in the lake at the beginning of the year doubles by the end of the year due to reproduction.

Give a recurrence for the number of fish in the lake after n years, and solve the recurrence.

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Solving Recurrence

Why again?

base cases

First-order Linear Recurrence (CS $4.2-11$)

$$T(n)=\begin{cases} 1 & n=0 \\ 2T(n-1)+n2^n & n>0 \end{cases}$$

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$$\boxed{T(n) = r^n a + \sum_{i=1}^{n} r^{n-i} g(i)}$$

Theorem (First-order Linear Recurrences)

$$T(n) = x_n T(n-1) + y_n \quad \textit{for } n > 0 \textit{ with } T(0) = 0$$

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Proof.

$$\underbrace{\frac{T(n)}{x_n x_{n-1} \cdots x_1}}_{\text{summation factor}} = \frac{T(n-1)}{x_{n-1} \cdots x_1} + \frac{y_n}{x_n x_{n-1} \cdots x_1}$$

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$$S(n) \triangleq \frac{T(n)}{x_n x_{n-1} \cdots x_1}$$

□

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$$\boxed{T(n) = 2(n+1)(H_{n+1} - \frac{3}{2})}$$

After-class Exercise

$$T(n) = T(n-1) - \frac{2T(n-1)}{n} + 2\left(1 - \frac{2T(n-1)}{n}\right), n > 0 \text{ with } T(0) = 0$$

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Theorem (Linear Recurrences with Constant Coefficients)

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_t a_{n-t} \quad \textit{for } n \geq t$$

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$$\beta^{n-t} \left((n-t)\, q(\beta) + \beta q'(\beta) \right) = 0$$

□

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$$\boxed{a_n = n2^{n-1}}$$

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$$\boxed{a_n = c_1 2^n + c_2 i^n + c_3 (-i)^n}$$

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$$1,\ 0,\ -1,\ 0,\ 1,\ 0,\ -1,\ 0,\ 1,\ 0,\ -1,\ 0 \cdots$$

After-class Exercise

Given initial conditions a_0, a_1, and a_2 for which the growth rate of the solution to

$$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3},\ n > 2$$

is

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First-order Linear Non-homogeneous Recurrences

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_t a_{n-t} + r \quad \text{for } n \geq t$$

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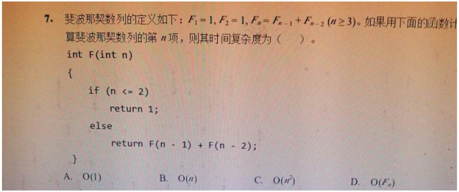

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int F(int n)
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    else
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A. $O(1)$　　B. $O(n)$　　C. $O(n^2)$　　D. $O(F_n)$

$$F(n) = F(n-1) + F(n-2) + 2, \ n \geq 3 \ (F(1) = F(2) = 0)$$

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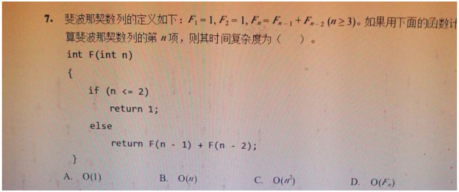

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$$\boxed{T(n,k) = \begin{cases} 0, & k = 0 \lor n = k \\ T(n-1,k) + T(n-1,k-1) + c, & \text{o.w.} \end{cases}}$$

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$$a_0 = 0 = c_0 + c_1 + 1$$
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$$\boxed{a_n = 2 \cdot 3^n - 3 \cdot 2^n + 1}$$

More Issues about Linear Recurrences

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_t a_{n-t} + g(n) \quad \text{for } n \geq t$$

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Generating Functions and Asymptotic Analysis

$$a_n = f_1(n)a_{n-1} + f_2(n)a_{n-2} + \cdots + f_t(n)a_{n-t} \quad \text{for } n \geq t$$

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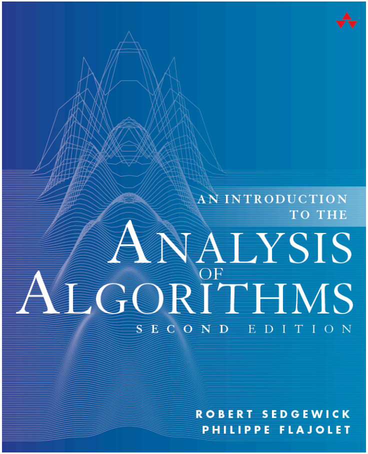
Genearating Functions


AN INTRODUCTION
TO THE
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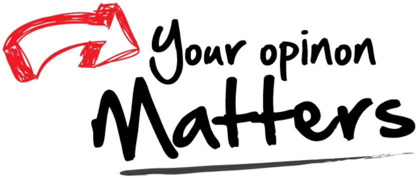
OF
ALGORITHMS
SECOND EDITION
ROBERT SEDGEWICK
PHILIPPE FLAJOLET


Thank
You!



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