A Theorem on Graphs, with an Application to a Problem of Traffic Control

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QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. J. WALKER, Cornell University, Ithaca, N. Y.

The department of Questions, Discussions, and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

A CORRECTION BY PROFESSOR CANDY

A. L. CANDY, University of Nebraska

For the benefit of those readers of the MONTHLY who have copies of my book *Magic Squares of an Even Order*, I wish to make the following correction: The last four Basic Squares on p. 91, *viz.*, Nos. 5, 6, 7, 8, should be stricken out, because they are duplicates, respectively, of Nos. 2, 1, 4, 3, on the same page, since they give the same "Groups of Four" when rows and columns are permuted so as to retain the same principal diagonals. This reduces the total number E. D. S. (p. 95) from 896 to 880, which agrees with the number obtained by D. H. Lehmer and others. Hence, wherever this number 896 appears elsewhere in this book, it should be changed to 880. See especially pp. 37, 174, and 175.

I am very glad to have this opportunity of making this public acknowledgement of my error.

A CORRECTION BY PROFESSOR MUSSELMAN

J. R. MUSSELMAN, Western Reserve University

I wish to call attention to an error in my article, "On the line of images," this MONTHLY, vol. 45, pp. 421–430. On page 430, line 2 following should read: "of the Kiepert hyperbola and at the point whose line of images as to $A_1A_2A_3$ is the line joining the orthocenter and the symmedian point of $A_1A_2A_3$."

A THEOREM ON GRAPHS, WITH AN APPLICATION TO A PROBLEM OF TRAFFIC CONTROL

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It is the object of this note to answer a question* which is suggested by the problem of traffic control in a modern city: "When may the arcs of a graph be so oriented that one may pass from any vertex to any other, traversing arcs in the positive sense only?" Any graph with this property will be called orientable. The answer is provided by the following theorem:

THEOREM. *A graph is orientable if and only if it remains connected after the removal of any arc.*

Let us suppose that week-day traffic in our city is not particularly heavy, so that all streets are two-way, but that we wish to be able to repair any one street at a time and still detour traffic around it so that any point in the city may be reached from any other point. On week-ends no repairing is done, so

* Proposed by S. Ulam. For a simplification in the proof we are indebted to H. Whitney.

that all streets are available, but due to the heavy traffic (perhaps it is a college town with a noted football team) we wish to make all streets one-way and still be able to get from any point to any other without violating the law. Then the theorem states that if our street-system is suitable for week-day traffic it is also suitable for week-end traffic and conversely. We proceed to give a few definitions and a proof of the theorem.

By a graph G we mean a (finite) set of objects $\langle p, q, \cdots \rangle$ called *vertices*, together with a (finite) set of objects called *arcs*, which join certain pairs of distinct vertices. We shall represent an arc joining p and q by (pq), with the understanding that there may be more than one such arc. G is *connected* if, given any two of its vertices p, q, there exists a chain of arcs joining p and q of the form

$$(1) \qquad (pr_1)(r_1r_2) \cdots (r_sq).$$

G is *oriented* if a direction is assigned to each arc, symbolized by choosing one of the two representations (pq) and (qp) of the unoriented arc, to be called the *admissible* representation. G is *orientable* if it may be oriented in such a way that any two of its vertices may be joined by a chain (1) of arcs, each in its admissible representation. The definitions of *subgraph*, and of the special subgraph obtained from G by removing one of its arcs, are obvious.

Now referring to the theorem, we see that the necessity of the condition is clear, for if G is disconnected by the removal of an arc (pq), then no matter which representation of (pq) we call admissible in an orientation of G, passage either from p to q or from q to p by a chain of admissible arcs will be impossible, so that G is not orientable. It remains to prove the sufficiency of the condition. Suppose G remains connected after the removal of any arc. (Then *a fortiori* G is connected.) Choose a vertex p of G and consider the class of all subgraphs of G which include p and which are orientable. This class is non-void, since the subgraph consisting of p alone satisfies the conditions. Let G' be a sub-graph in this class with maximal number of vertices. We may assume that G' contains all arcs of G whose vertices are in G', since otherwise any such arcs which did not belong to G' could be added, oriented arbitrarily, and G' would remain orientable. Then G' must be identical with G. For suppose p^* is a vertex of G not in G'. Join p^* to p by a chain

$$(2) \qquad (p_1p_2)(p_2p_3) \cdots (p_{n-1}p_n), \qquad [p_1 = p^*;\ p_n = p],$$

and order the arcs of (2) from left to right. Let (p_kp_{k+1}) be the last arc of (2) which is not in G'. Then p_k does not belong to G', while p_{k+1} does. By hypothesis, there exists a chain of arcs

$$(3) \qquad (\bar{p}_1\bar{p}_2)(\bar{p}_2\bar{p}_3) \cdots (\bar{p}_{m-1}\bar{p}_m), \qquad [\bar{p}_1 = p_k;\ \bar{p}_m = p_{k+1}],$$

which does not include (p_kp_{k+1}). Let $(\bar{p}_s\bar{p}_{s+1})$ be the last arc of (3) which does not belong to G'. Then the subgraph G^* consisting of G' plus $(p_{k+1}p_k)$ and the chain

$$(4) \qquad (\bar{p}_1\bar{p}_2)(\bar{p}_2\bar{p}_3) \cdots (\bar{p}_s\bar{p}_{s+1}),$$

where these arcs are to be oriented as written, is clearly orientable, contains p, and has more vertices than G', which is a contradiction and completes the proof.

EVEN PERMUTATIONS AS PRODUCTS OF CYCLES

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THEOREM. *Every even permutation P on n distinct letters can be written as the product of an even number of cycles on these n letters, each cycle being of order m, where $m=2, 3, \cdots$, or n.*

Proof. By definition, any even permutation can be written as the product of an even number of transpositions. It is also known that any even permutaton can be written as the product of cycles of three letters each. The theorem is therefore true for $m=2, 3$. We will show that any even permutation written as the product of an even number of cycles of r letters each, where $2 \leqq r < n$, can be written as the product of the same number of cycles of $r+1$ letters each. The theorem will then follow by induction. Since the permutation group is of degree n, the theorem loses significance for $r \geqq n$.

Suppose that the permutation is written as the product of an even number of cycles each of order r. Let us group the cycles into pairs. Any given pair of cycles will fall into one of the following two types.

I. Pairs of cycles in which all of the letters of the first cycle are repeated in some order in the second.

II. Pairs of cycles in which at least one letter of the first cycle is not repeated in the second.

We will now show that every pair of cycles of type I and type II can be written as a pair of cycles of order $r+1$.

Take any pair of cycles of type I,

$$(a_1 a_2 \cdots a_r)(a_1' a_2' \cdots a_r'). \tag{1}$$

Since the letters of the first cycle are the same as the letters in the second, we may suppose that $a_1 = a_1'$. Since $r < n$, there is an a_{r+1}. If we form

$$(a_1 a_2 \cdots a_r a_{r+1})(a_1 a_{r+1} a_2' \cdots a_r'), \tag{2}$$

it is easy to verify that $(1)=(2)$.

Take any pair of cycles of type II,

$$(a_1 a_2 \cdots a_i \cdots a_r)(a_1' a_2' \cdots a_i' \cdots a_r'). \tag{3}$$

Then let a_i be the last letter in the first cycle which is not found in the second, and let a_j' be the last letter in the second which is not found in the first. If we form

$$(a_1 a_2 \cdots a_j' a_i \cdots a_r)(a_1' a_2' \cdots a_j' a_i \cdots a_r'), \tag{4}$$

it is easy to verify that $(3)=(4)$, and so the theorem follows.