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 - Zero error

Definition: RP

The complexity class RP is the class of all languages L for which there exists a polynomial PTM M such that

$$x \in L \implies Prob[M(x) = 1] \ge \frac{1}{2}$$

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Let PTM M decide a language $L \in RP$

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- If M(x) = 1, then $x \in L$.
- If M(x) = 0, then $x \in L$ or $x \notin L$, we can't really tell.

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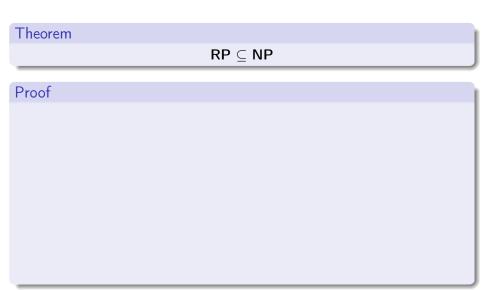
$$x \in L \implies Prob[M(x) = 1] = 1$$

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Theorem

 $\mathsf{RP}\subseteq \mathsf{NP}$



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Proof

 \bullet Let L be a language in $\ensuremath{\mathsf{RP}}$. Let M be a polynomial probabilistic Turing Machine that decides L.

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- So *L* ∈ **NP**

Boosting for $\ensuremath{\mathsf{RP}}$

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- $\Pr[\text{algorithm makes a mistake } t \text{ times}] \leq \frac{1}{2^t}$
- Thus, we can make the error exponentially small by polynomial number of repetitions.

Examples of RP and coRP

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Example of RP

Primes, the set of all prime numbers, is in $\ensuremath{\mathsf{RP}}$. This was shown by Adelman and Huang (1992), who gave a primality testing algorithm that always rejected composite numbers, and accepted primes with probability at least 1/2

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Example of coRP

Primes, the set of all prime numbers, is in coRP also.

Miller and Rabin gave a primality test that always accepts prime numbers, but rejects composites with probability at least 1/2

Bounded-error Probabilistic Polynomial time

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Primes, the set of all prime numbers, is in BPP .

BPP is closed under complement.

BPP = coBPP

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The Chernoff Bound

Suppose X_1, \ldots, X_t are t independent random variables with values in $\{0,1\}$ and for every i, $\Pr[X_i=1]=p$.

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The Chernoff Bound

Suppose X_1, \ldots, X_t are t independent random variables with values in $\{0,1\}$ and for every i, $\Pr[X_i=1]=p$. Then

$$Pr\left[\frac{1}{t}\sum_{i=1}^{t}X_{i}-p>\epsilon\right]<\mathrm{e}^{-\epsilon^{2}\cdot\frac{t}{2p(1-p)}}$$

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 for $t = (\ln 2^n)/c$

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$$= e^{-ct}, \text{ where } c \text{ is some constant}$$

$$< \frac{1}{27} \text{ for } t = (\ln 2^{n})/c$$

• So by running the algorithm O(n) times, reduce probability exponentially

$RP \subseteq BPP$

RP

- $x \in L \Rightarrow Prob[M(x) = 1] \ge \frac{1}{2}$
- $x \notin L \Rightarrow Prob[M(x) = 1] = 0$

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- Can boost RP probability by running twice:

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- Can boost RP probability by running twice:

$$x \in L \Rightarrow Prob[M(x) = 1] \ge \frac{3}{4}$$

$$x \notin L \Rightarrow Prob[M(x) = 1] = 0$$

Clearly the above definition satisfies BPP

Randomized Computation

Nabil Mustafa

Computational Complexity

Definition: **ZPP**

The complexity class \mathbf{ZPP} is the class of all languages L for which there exists a polynomial \mathbf{PTM} M such that

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$$\forall x, \ Prob[M(x) = 'Don't \ know'] \le 1/2$$

Whenever M answers with a 0 or a 1, it answers correctly. If M is not sure, it'll output a 'Don't know'. On any input x, it outputs 'Don't know' with probability at most 1/2.

Theorem

 $ZPP = RP \cap coRP$

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Claim: $ZPP \subseteq RP \cap coRP$

 \bullet Show that $\mathsf{ZPP} \subseteq \mathsf{RP}\$ and $\mathsf{ZPP} \subseteq \mathsf{coRP}\$

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- ullet Show that $\mathsf{ZPP} \subseteq \mathsf{RP}$ and $\mathsf{ZPP} \subseteq \mathsf{coRP}$
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- ullet We prove $\mathsf{ZPP}\subseteq\mathsf{coRP}$, the other proof is similar
- Let L ∈ ZPP . Then ∃ PTM M that, for all x, either correctly decides x ∈ L or outputs 'Don't know'.
- Let M' be the Turing Machine that on input x, returns 1 if M(x) = `Don't know' and otherwise returns M(x).
 - \times $x \in L$: M(x) = 1 or M(x) = 'Don't know', so M'(x) = 1.
 - ▶ $x \notin L$: M(x) = 0, which is correct, or M(x) = 'Don't know' where M'(x) returns incorrectly with probability $\leq 1/2$.
- So $L \in \mathbf{coRP}$, and $\mathbf{ZPP} \subseteq \mathbf{coRP}$

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 - ▶ The case for $x \notin L$ similar.