3-9 Connectivity

Hengfeng Wei

hfwei@nju.edu.cn

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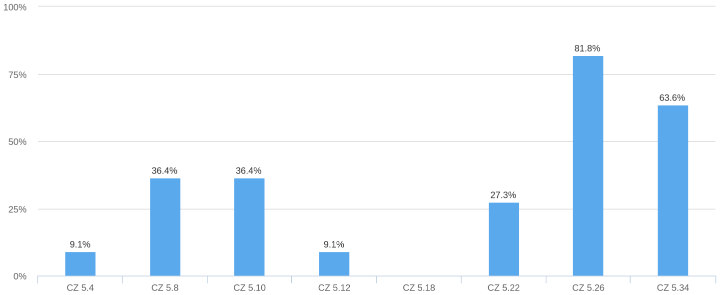

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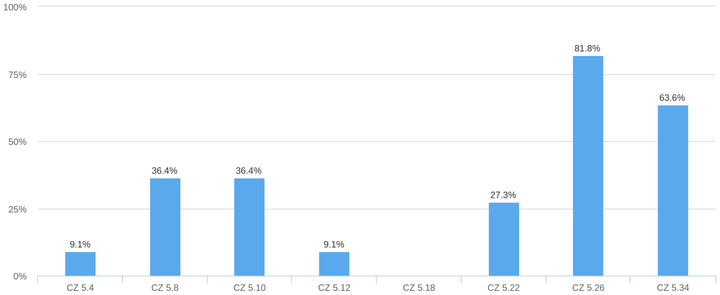
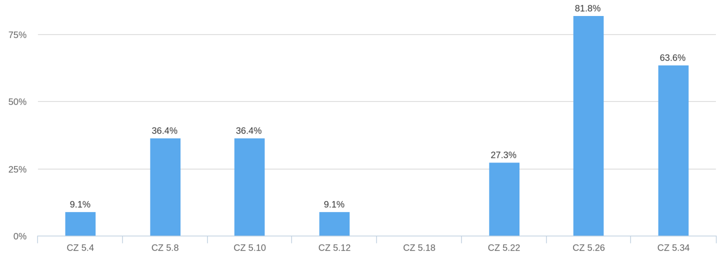


$$5.10 \qquad 5.34 \qquad 5.22 \qquad 5.26$$



5.10 5.34 5.22 5.26

Menger's Theorem (Theorem 5.16; Theorem 5.21)

2-Connectivity (Problem 5.10)

A connected graph G with $m \geq 2$ is *nonseparable*

$$\Longleftrightarrow$$

any two *adjacent* edges of G lie on a common cycle of G.

2-Connectivity (Problem 5.10)

A connected graph G with $m \geq 2$ is *nonseparable*

$$\iff$$

any two *adjacent* edges of G lie on a common cycle of G.

Proof.

“ $\implies$ ”

2-Connectivity (Problem 5.10)

A connected graph G with $m \geq 2$ is *nonseparable*

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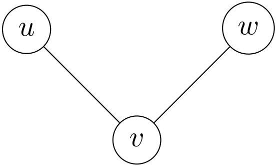
Proof.

" $\Longrightarrow$ "

G is nonseparable

$\Longrightarrow$ u, w lie on a common cycle



□

2-Connectivity (Problem 5.10)

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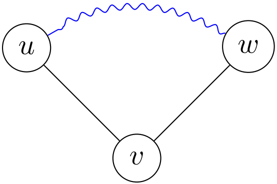
"$\implies$"



G is nonseparable

$\implies u, w$ lie on a common cycle

$\implies \exists$ path $u \sim w$

$\implies \exists$ cycle $u - v - w \sim u$

□

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A connected graph G with $m \geq 2$ is *nonseparable*

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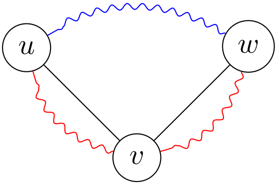


G is nonseparable

$\Longrightarrow$ u, w lie on a common cycle

$\Longrightarrow$ $\exists$ path $u \sim w$

$\Longrightarrow$ $\exists$ cycle $u - v - w \sim u$

$\square$

2-Connectivity (Problem 5.10)

A connected graph G with $m \geq 2$ is *nonseparable*

$$\iff$$

any two *adjacent* edges of G lie on a common cycle of G.

Proof.

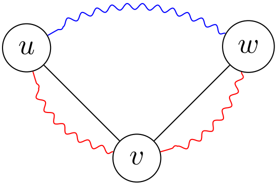
“ $\Longrightarrow$ ”

G is nonseparable

$\Longrightarrow$ u, w lie on a common cycle

$\Longrightarrow$ $\exists$ path $u \sim w$ that does not contain v

$\Longrightarrow$ $\exists$ cycle $u - v - w \sim u$



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Proof.

"$\Longleftarrow$"

By Contradiction.

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Proof.

"$\Longleftarrow$"

By Contradiction.

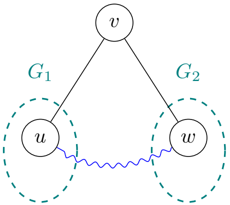
Suppose v is a cut-vertex of G

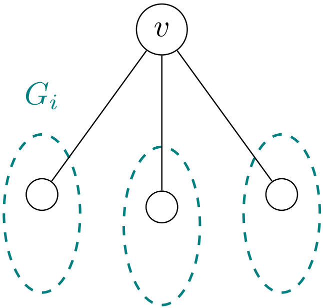
$\implies$ $G - v$ contains ≥ 2 comps $G_1, G_2, \cdots$

$\implies$ $\exists u \in G_1, w \in G_2 : v - u \land v - w$

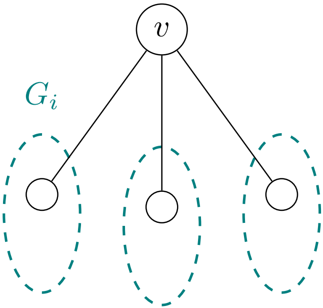
$\implies$ $v - u, v - w$ lie on a common cycle

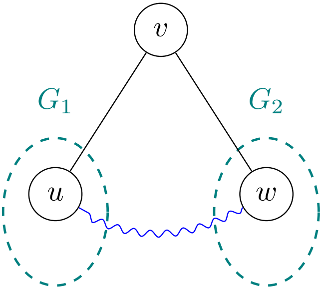
$\implies$ $\exists$ path $u \sim w$ that does not contain v



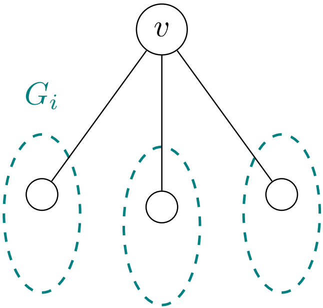


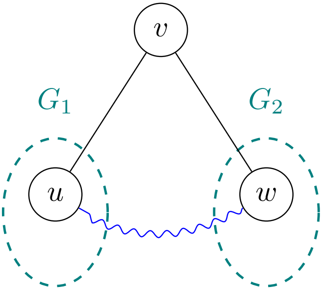
$$\forall G_i \ \exists v_i \in G_i \ v - v_i$$





$\forall G_i \; \exists v_i \in G_i \; v - v_i$





$\forall G_i \; \exists v_i \in G_i \; v - v_i$

$\forall v \in S \; \forall G_i \; \exists v_i \in G_i \; v - v_i$

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A connected graph G with $m \geq 2$ is *nonseparable*

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2-Connectivity (Extended Problem)

A connected graph G with $m \geq 2$ is *nonseparable*

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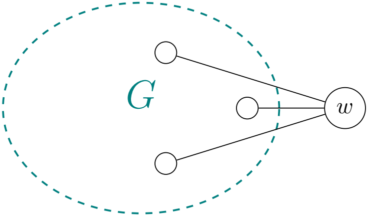
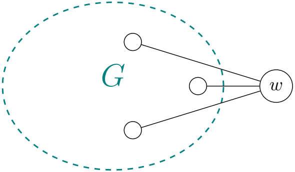
any two edges of G lie on a common cycle of G.

Expansion Lemma (Problem 5.34; Theorem 5.18)

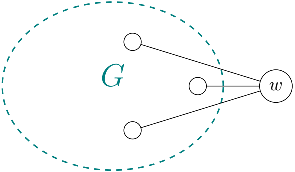
Let G be a k-connected graph and let S be any set of k vertices.

If a graph H is obtained from G by adding a new vertex w and joining w to the vertices of S, then H is also k-connected.





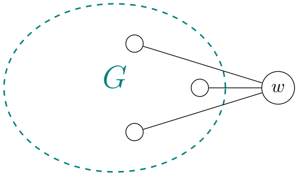
To prove $\kappa(H) \geq k$



To prove $\kappa(H) \geq k$

Let U be a vertex-cut of H.

We prove that $|U| \geq k$.



To prove $\kappa(H) \geq k$

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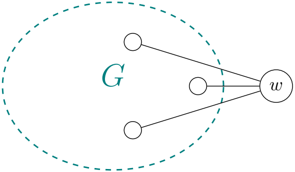
Case I: U is a vertex-cut of G

Case II: U is not a vertex-cut of G



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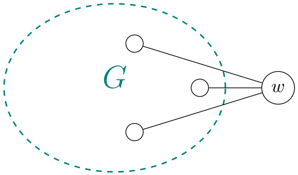
$|U| \geq k$



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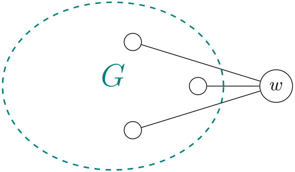
CASE II: U is not a vertex-cut of G

$w \in U$



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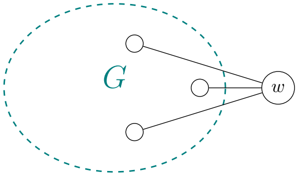
$w \in U$

$U - w$ is a vertex-cut of G



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CASE II: U is not a vertex-cut of G

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$U - w$ is a vertex-cut of G

$$|U| \geq k + 1$$

2-Connectivity (Extended Problem)

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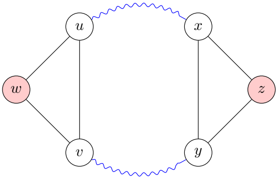


2-Connectivity (Extended Problem)

A connected graph G with $m \geq 2$ is *nonseparable*

$$\iff$$

any two edges of G lie on a common cycle of G.

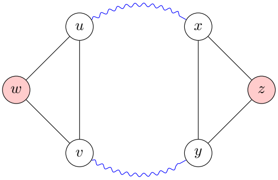
$$\Longrightarrow$$

Consider two edges uv and xy.

Add w, z

Add $wu, wv; zx, zy$

w and z lie on a common cycle



Effects of Removing an Edge on Connectivity (Problem 5.22 (a))

(a) If G is k-connected and $e = uv \in E(G)$, then $G - e$ is $(k-1)$-connected.

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$$\text{To prove } \kappa(G) \geq k \implies \kappa(G-e) \geq k-1$$

Effects of Removing an Edge on Connectivity (Problem 5.22 (a))

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$$\text{To prove } \kappa(G) \geq k \implies \kappa(G-e) \geq k-1$$

Choose any $U \subseteq V(G)$ with $|U| < k-1$.

We prove that $G - e - U$ is connected.

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$$G \text{ is } k\text{-connected} \implies G - U \text{ is connected}$$

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Suppose, by contradiction, that $G - e - U$ is not connected.

Choose any $U \subseteq V(G)$ with $|U| < k - 1$.

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G is k-connected $\implies$ $G - U$ is connected

Suppose, by contradiction, that $G - e - U$ is not connected.

$e = uv$ is a bridge of $G - U$

Choose any $U \subseteq V(G)$ with $|U| < k - 1$.

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$e = uv$ is a bridge of $G - U$

$U \cup \{u\}$ is a vertex-cut of G

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But $\Big| U \cup \{u\} \Big| < k$

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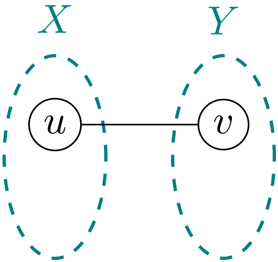
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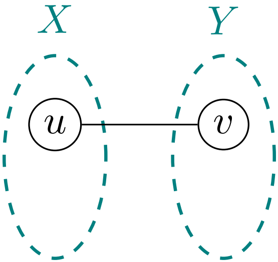
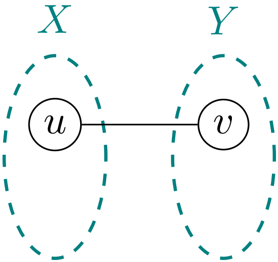


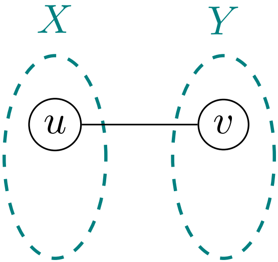
CASE I : $|X| \geq 2 \vee |Y| \geq 2$



CASE I : $|X| \geq 2 \vee |Y| \geq 2$

$U \cup \{u\}$ is a vertex-cut of G

But $\left| U \cup \{u\} \right| < k$



CASE II : $|X| = |Y| = 1$

CASE I : $|X| \geq 2 \vee |Y| \geq 2$

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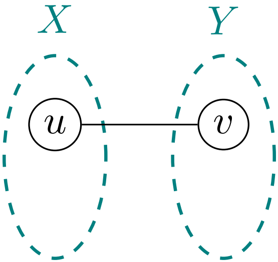
But $\left| U \cup \{u\} \right| < k$



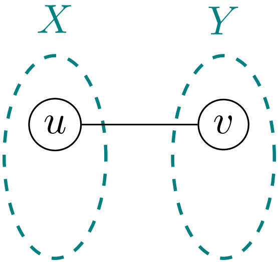
$\textsc{Case I}: |X| \geq 2 \lor |Y| \geq 2$

$U \cup \{u\}$ is a vertex-cut of G

But $\Big|U \cup \{u\}\Big| < k$

$\textsc{Case II}: |X| = |Y| = 1$

$|U| = n - 2 < k - 1$



$\textsc{Case I} : |X| \geq 2 \vee |Y| \geq 2$

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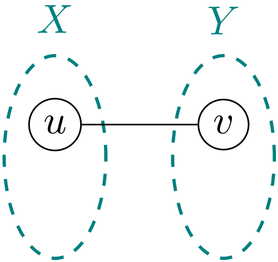
But $\left| U \cup \{u\} \right| < k$

$\textsc{Case II} : |X| = |Y| = 1$

$|U| = n - 2 < k - 1$

$\kappa(G) \geq k > n - 1$



$\textsc{Case I}: |X| \geq 2 \vee |Y| \geq 2$

$U \cup \{u\}$ is a vertex-cut of G

But $\left| U \cup \{u\} \right| < k$

$\textsc{Case II}: |X| = |Y| = 1$

$|U| = n - 2 < k - 1$

$\kappa(G) \geq k > n - 1$

But $0 \leq \kappa(G) \leq n - 1$

Effects of Removing an Edge on Connectivity (Problem 5.22 (b))

(b) If G is k-edge-connected and $e = uv \in E(G)$, then $G - e$ is $(k-1)$-edge-connected.

$$\lambda(G) \geq k \implies \lambda(G - e) \geq k - 1$$

Effects of Removing an Edge on Connectivity (Problem 5.22 (b))

(b) If G is k-edge-connected and $e = uv \in E(G)$, then $G - e$ is $(k-1)$-edge-connected.

$$\lambda(G) \geq k \implies \lambda(G - e) \geq k - 1$$

Choose any $X \subseteq E(G)$ with $|X| < k - 1$.

We prove that $G - e - X$ is connected.

Effects of Removing an Edge on Connectivity (Problem 5.22 (b))

(b) If G is k-edge-connected and $e = uv \in E(G)$, then $G - e$ is $(k-1)$-edge-connected.

$$\lambda(G) \ge k \implies \lambda(G - e) \ge k - 1$$

Choose any $X \subseteq E(G)$ with $|X| < k - 1$.

We prove that $G - e - X$ is connected.

$$G - e - X = G - (e + E) \text{ is connected}$$

Effects of Removing an Edge on Connectivity (Problem 5.22 (b))

(b) If G is k-edge-connected and $e = uv \in E(G)$, then $G - e$ is $(k-1)$-edge-connected.

$$\lambda(G) \geq k \implies \lambda(G - e) \geq k - 1$$

Choose any $X \subseteq E(G)$ with $|X| < k - 1$.

We prove that $G - e - X$ is connected.

$G - e - X = G - (e + E)$ is connected $(\because \lambda(G) \geq k)$

$$\kappa(G - e) \leq \kappa(G)$$

$$\kappa(G - e) \leq \kappa(G)$$

Effects of Removing a Vertex on Connectivity (Extended Problem)

$$\text{Is } \kappa(G - v) \leq \kappa(G)?$$

$$\text{Is } \lambda(G - v) \leq \lambda(G)?$$

$$\kappa(G - e) \leq \kappa(G)$$

Effects of Removing a Vertex on Connectivity (Extended Problem)

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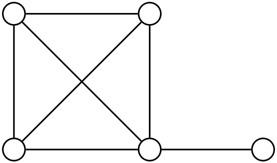
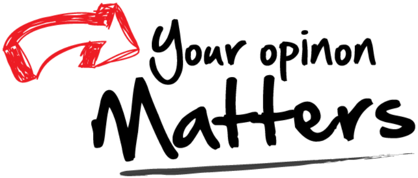
$$\text{Is } \lambda(G - v) \leq \lambda(G)?$$



Degree Condition for $\lambda(G) = \delta(G)$ (Problem 5.26)

If G is graph of order n such that $\delta(G) \geq (n-1)/2$, then $\lambda(G) = \delta(G)$.


THANK YOU



Office 302

Mailbox: H016

hfwei@nju.edu.cn