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(Some of) the many uses of Eulerian graphs in graph theory (plus some applications)

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Abstract

In this survey type article, various connections between eulerian graphs and other graph properties such as being hamiltonian, nowhere-zero flows, the cycle-plus-triangles problem and problems derived from it, are demonstrated. It is also shown how compatible cycle decompositions can be used to construct loopless 4-regular graphs having precisely one hamiltonian cycle, or to prove the equivalence between the Chinese Postman Problem and the Minimum Cycle Covering Problem in the planar bridgeless case.

0. Introduction and preliminaries

Concepts not defined in this paper can be found in [15,16]. There and in this article as well, we call a graph *eulerian* if all of its vertices have even degree; i.e., in this context an eulerian graph need not be connected (others call such graphs *even* graphs). Also, we call a *cycle* what many others call a *circuit*, and call a *closed trail* what others call a *cycle*. A subgraph G' of the graph G is called *dominating* if every edge of G is incident with a vertex of G'. Call a graph *essentially* (*cyclically*, *resp.*) *k-edge-connected* if it cannot be separated into two nontrivial (cyclic, resp.) components by the removal of less than k edges. Unless stated otherwise, a graph may have loops and/or multiple edges.

In what follows we demonstrate various connections between properties (of, in some instances, special classes) of eulerian graphs and other graph properties such as being hamiltonian, colorability, nowhere-zero flows, etc. Compatible cycle decompositions are used to construct loopless 2k-regular graphs having precisely one hamiltonian cycle and to show the equivalence between the Chinese Postman Problem and The Minimum Cycle Covering Problem in the planar bridgeless case. Corresponding to the three

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lectures given by the author at the Paul Catlin Memorial Conference, this article is divided into three parts which altogether demonstrate the usefulness of dealing with eulerian graphs when studying certain other, seemingly unrelated, problems. However, these connections are by no means complete. The interested reader is referred to the books quoted above.

1. Part 1: The Euler–Hamilton connection

In 1936, D. König published his book Theorie der endlichen und unendlichen Graphen [34]; it is generally considered to be the first book on graph theory. Let us have a look at p. 26 of the original edition [p. 87 (33) of the *English translation*]. Comparing the existence problem of eulerian trails with that of hamiltonian cycles he says:

> Die Ähnlichkeit zwischen diesen zwei Fragestellungen beruht bloß auf einer rein äußerlichen Analogie, und die zweite Frage führt auf schwierige und ungelöste Probleme: über die Bedingung der Existenz einer Hamiltonschen Linie ist im allgemeinen nichts bekannt. Hier müßten viel tiefer liegende Eigenschaften der Graphen herangezogen werden als für die Frage der Eulerschen Linie.
>
> [*The similarity between these two problems rests on a purely external analogy, but the second problem is difficult and unsolved. Nothing in general is known concerning questions of existence of a Hamiltonian cycle. Here much deeper properties of graphs must be involved than in the question of an Euler trail.*]

This statement demonstrates great insight by someone who at the time could not have possibly had a notion of NP-complete problems as opposed to polynomially solvable problems. On the other hand, as we shall see below, there are NP-complete problems when dealing with special types of eulerian trails, for example (which, however, were not known at D. König's time). In any case, *I decided to make a living on the above remark.*

As a student (1966) I tried the harder of the two problems. So, how do I find in general a hamiltonian cycle (in particular, in graphs having few edges only; e.g., in 3-regular graphs)? Having read D. König's book as the only available source at the time (Berge's book was in the hands of another student), I had the following

1.1. General idea

In a given graph $\boldsymbol{G}$, try to find a set $\boldsymbol{S}$ of cycles of $\boldsymbol{G}, \boldsymbol{S} = \{\boldsymbol{C}_1, \ldots, \boldsymbol{C}_r\}$, such that each vertex of $\boldsymbol{G}$ belongs to at least one $\boldsymbol{C}_i$, and $\boldsymbol{C}_i$ and $\boldsymbol{C}_j$ have a $\boldsymbol{K}_2$ or nothing in common for different i and j.

Then $\boldsymbol{G}$ is hamiltonian if and only if $\boldsymbol{I}(\boldsymbol{S})$ is a tree (where $I(S)$ is defined by $V(I(S)) = S$, and $C_iC_j \in E(I(S))$ if and only if $E(C_i) \cap E(C_j) \neq \emptyset$).

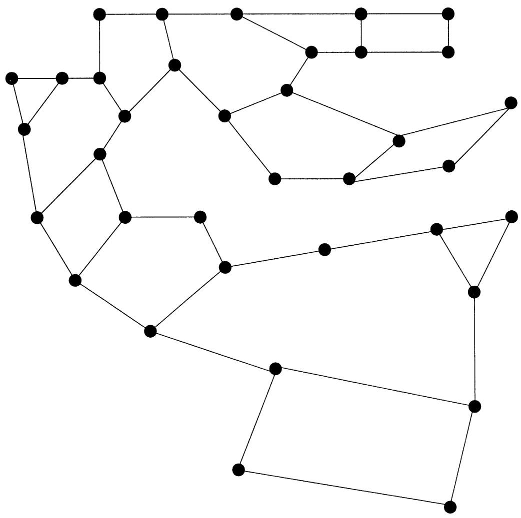


Fig. 1.

As a demonstration of this idea, we look at the graph of Fig. 1 and view it as a 2-connected, spanning, outerplane subgraph of some other graph (which need not be planar at all); the face boundaries of the bounded faces are the elements of $\boldsymbol{S}$, and $\boldsymbol{I}(\boldsymbol{S})$ is nothing but the weak dual of this subgraph. Its unique hamiltonian cycle is the face boundary of the outer face, or equivalently (and in a less topological phrasing), the set of edges belonging to precisely one element of $\boldsymbol{S}$.

In other words, we are looking for a 2-connected spanning outerplanar subgraph $\boldsymbol{G}^*$ of $\boldsymbol{G}$ such that $\boldsymbol{G}^*$ has no vertex of degree greater than 3. In fact, this idea is general enough to go ahead and prove that *the prism over a planar cubic 2-connected graph is hamiltonian* (where the prism of G is formed by labelling the vertices of G, making a second copy of the labelled graph, and joining vertices with the same label by an edge). But it is not general enough to prove that the square of every 2-connected graph is hamiltonian. So, we add a *bit* to the above statement to obtain

1.2. A bit more general idea

In a given graph $\boldsymbol{G}$, try to find a set $\boldsymbol{S}$ of cycles of $\boldsymbol{G}, \boldsymbol{S} = \{\boldsymbol{C}_1, \ldots, \boldsymbol{C}_r\}$, such that each vertex of $\boldsymbol{G}$ belongs to at least one $\boldsymbol{C}_i$, and $\boldsymbol{C}_i$ and $\boldsymbol{C}_j$ have a $\boldsymbol{K}_2$ or nothing *or precisely one vertex* $\boldsymbol{x}$ in common for different i and j. In the latter case, $\boldsymbol{S}$ contains a $\boldsymbol{C}_k$ such that $\boldsymbol{C}_k$ and $\boldsymbol{C}_i$ have a common edge incident with $\boldsymbol{x}$, and so do $\boldsymbol{C}_k$ and $\boldsymbol{C}_j$. Then $\boldsymbol{G}$ is hamiltonian if and only if $\boldsymbol{I}(\boldsymbol{S})$ is a tree.

Fig. 2 demonstrates this idea much the same way Fig. 1 did with respect to the preceding idea, with the unique hamiltonian cycle being defined again by the edges belonging to precisely one element of $\boldsymbol{S}$. Note, however, that in this case $\boldsymbol{G}^*$ may

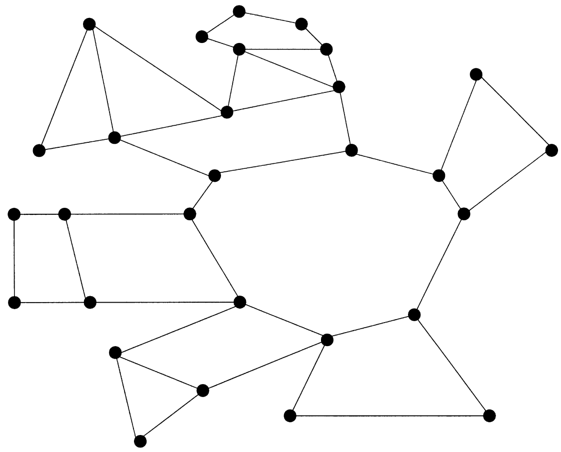


Fig. 2.

have 4-valent vertices as well, while in the preceding case the spanning subgraph has 2- and 3-valent vertices only.

Note that also in this more general case the subgraph $\boldsymbol{G}^*$ defined by the edges of the elements of $\boldsymbol{S}$ is spanning, 2-connected and outerplanar. Indeed, this more general idea suffices to prove that *the square of every 2-connected graph is hamiltonian*. How come? For that matter, consider a connected eulerian DT-graph $\boldsymbol{E}$ (DT means that every edge is incident with a vertex of degree at most two). Consider a fixed eulerian trail $\boldsymbol{T}$ of $\boldsymbol{E}$ and let us consider the run through $\boldsymbol{T}$ like driving in a car. So, reaching a vertex of degree greater than two is like reaching a CROSSING where one usually has TRAFFIC LIGHTS which tell us how to behave in constructing a hamiltonian cycle in $\boldsymbol{E}^2$. In fact, if v is such crossing, consider a transition x, xv, v, vy, y through v by $\boldsymbol{T}$. x and y are 2-valent because we drive along a DT-graph. If this transition does not pass v for the last time, then x is a red traffic light telling us that we have to use the edge xy which is definitely an edge of $\boldsymbol{E}^2$; otherwise, x is a green traffic light in which case we use the two edges of the above transition to pick up v before reaching y. Thus, we construct a hamiltonian cycle in the square of a connected eulerian DT-graph.

This construction also works if we ‘hook up’ several disjoint connected eulerian DT-graphs by paths in such a way that we obtain a connected graph in which no vertex belongs to more than two path edges, and such that the path edges are the bridges of this new graph (of course, we still need to remain in the class of DT-graphs if we want to construct a hamiltonian cycle in the square of such a graph). Disregarding now the DT-property, we have altogether a connected graph $\boldsymbol{S}$ (not to be confused with the above set of cycles) which can be written in the form

$$\boldsymbol{S} = \boldsymbol{E} \cup \boldsymbol{P}$$

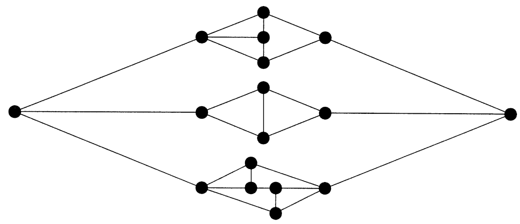


Fig. 3.

and where $\boldsymbol{E}$ is an eulerian graph, $\boldsymbol{P}$ is a path-forest (= linear forest), and $\boldsymbol{E}$ and $\boldsymbol{P}$ are edge disjoint, and the edges of $\boldsymbol{P}$ are the bridges of $\boldsymbol{S}$ (this can be achieved by deleting edges of $\boldsymbol{P}$ if necessary). Call such an $\boldsymbol{S}$ an *EPS-graph* of a given graph $\boldsymbol{G}$ if $\boldsymbol{S}$ spans $\boldsymbol{G}$.

Which graphs have an EPS-graph? In 1970 I proved the following

Theorem 1 (Fleischner [10]). *Let v and w be distinct vertices of the connected bridgeless graph $\boldsymbol{G}$. Then $\boldsymbol{G}$ has an EPS-graph $\boldsymbol{S}$ such that both v and w belong to $\boldsymbol{E}$ and $\boldsymbol{d}(v;\boldsymbol{P}) = \boldsymbol{0}$ and $\boldsymbol{d}(w;\boldsymbol{P}) < 2$.*

This theorem and the preceding construction yield the following

Corollary 2. *The total graph of a connected bridgeless graph is hamiltonian.*

In fact, in 1973/74 Art Hobbs and I proved that *the total graph of any given graph G is hamiltonian if and only if G has an EPS-graph*, and we determined the most general block-cutpoint-structure a graph may have such that it contains an EPS-graph [19].

In proving that the square of a 2-connected graph is Hamiltonian, one cannot immediately apply the above theorem. Instead, one can assume first that such graph is edge-critical with respect to 2-connectedness, and then one finds a DT-graph hanging in there (see [11]).

In 1990, a much shorter proof was published by Ríha (Brno, Czech Republic), [39], but in view of the relevance of EPS-graphs for the hamiltonicity of total graphs (iff !), I do not know whether Ríha's method will embrace as many results as the theory of EPS-graphs.

Now, EPS-graphs are the best you can hope for in the case of arbitrary connected bridgeless graphs. The latter will not have connected spanning eulerian subgraphs, in general, simply because of the possible existence of 2-valent vertices. However, even if you restrict yourself to *minimum valency* $> \mathbf{2}$, *you can't always get what you want*, as can be seen in Fig. 3.

However, as an application of the *Splitting Lemma* (see, e.g., [15, Lemma III.26] and *Petersen's Theorem* on the existence of a 1-factor in a cubic bridgeless graph, you can prove the next result [17].

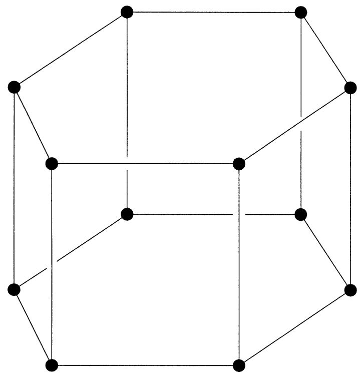


Fig. 4.

Theorem 3. *Let* G *be a bridgeless graph without* 2*-valent vertices. Then* G *has a spanning eulerian subgraph without isolated vertices.*

To guarantee that such spanning eulerian subgraph is also connected, you need additional properties (e.g., 4-edge-connectedness, or that every edge belongs to a cycle of length 3 or 4, etc.). I do not elaborate further in this direction, because the question of which graphs have a connected spanning eulerian subgraph (an NP-complete problem in this general form), was a central theme in the work of our friend and colleague Paul Catlin, and others write about it as their contribution to the proceedings of this conference ([30,36,6], and also see [5,7]). Instead, let us see what else we can do with EPS-graphs. For that matter, consider a prism over an even cycle. It has many hamiltonian cycles; however, it has just two hamiltonian cycles using all 'pillars' of the prism (i.e., all edges not belonging to the even cycle 'on the bottom' nor its copy 'on the top'). Such a hamiltonian cycle cannot exist if we consider the prism over an odd cycle (Fig. 4). On the other hand, the prism over a path has precisely one hamiltonian cycle; it uses all edges of the given path and all edges of its copy, plus the two outer 'pillars'.

By proceeding analogously as before we may 'hook up' various such even cycles by paths to produce graphs whose prisms are hamiltonian. In this case, however, we require that vertices belonging to such even cycles appear at most as end-vertices of the path-forest.

In a way, one may get confused if one wants to decide whether this construction results from the '*general idea*' or from the '*a bit more general idea*'. It just depends on how you look at the construction: if you do the whole construction at once, then the second idea prevails; if, however, you first construct hamiltonian cycles in the prisms over even cycles, and then 'hook up' the hamiltonian cycles in the prisms over paths, then the first idea prevails. I had chosen the latter approach when I dealt with hamiltonian cycles in prisms over cubic graphs. However, if one proceeds inductively in building up ever larger cycles, then the more general idea reduces to the first idea.

Now, we have formed prisms over … what type of graphs? We have even cycles and paths, the whole thing is connected, and the path edges are the bridges of these graphs. So we have an EPS-graph of a special type, namely: the eulerian part is bipartite (simply because it is a set of totally disjoint even cycles), and since the path edges are the bridges of the EPS-graph, the latter is also bipartite. Thus, we call such EPS-graph a *BEPS-graph*. Analogous to what Art Hobbs and I proved regarding the equivalence of EPS-graphs and hamiltonian cycles in total graphs, one has the following

Theorem 4 (Fleischner [14]). *A cubic graph has a hamiltonian prism if and only if it contains a* (*spanning*) *BEPS-graph.*

Assuming the validity of the 4CT (in its equivalent form on the existence of a Tait coloring in planar cubic bridgeless graphs), it follows easily that planar 2-connected cubic graphs have a BEPS-graph in which the path forest is a matching. Just take the even cycles defined by the blue and red perfect matching, and hook them up by some yellow edges. So, *the existence of a BEPS-graph in a connected planar*, *bridgeless cubic graph is a necessary condition for the validity of the* 4*CT*. In fact, in [14] I could prove the existence of BEPS-graphs (having additional properties) in such graphs, *without using the* 4*CT*. The proof is long and tedious due to complications which arise from the possible existence of triangles. This is all the more astonishing as triangles are 'peanuts' in the case of a proof of the 4CT (for planar cubic graphs).

So, we see that there is some relation between hamiltonian cycles in certain 'derived' graphs and eulerian parts in spanning subgraphs of a special type in the 'base' graphs. However, there is a much closer relationship between a yet unsolved hamiltonian problem and another unsolved eulerian problem.

Call an eulerian trail $\boldsymbol{T}$ of a plane connected eulerian graph $\boldsymbol{G}$ an ***A****-trail* if consecutive edges of $\boldsymbol{T}$ are consecutive in a face boundary of $\boldsymbol{G}$. The following two conjectures are equivalent indeed (see, e.g., [15, pp. VI.108–VI.114]).

Conjecture 5 (Barnette and Tutte). Every planar, 3-connected, cubic, bipartite graph has a hamiltonian cycle.

Conjecture 6. Every simple eulerian plane triangulation has an A-trail.

Regner in her Ph.D. thesis [38] implicitly showed that the existence of A-trails in arbitrary planar 3-connected eulerian graphs is an NP-complete problem. Starting from the dual of a planar cubic 3-connected graph, one can construct a 3-connected planar eulerian graph (having triangular and quadrangular face boundaries only) which has an A-trail if and only if the original cubic graph has a hamiltonian cycle. Andersen and I improved on her work, showing (as a consequence) that the existence of spanning hypertrees in hypergraphs is an NP-complete problem [3]. However, I put forward the following (see [15, Conjecture VI.86]).

Conjecture 7. Every planar 4-connected eulerian graph has an A-trail.

Observe that in the case of simple 2-connected outerplane eulerian graphs, the existence of such an A-trail has been proved by Regner as well ([15, Theorem VI.63]). Improving on one of her results in this direction, I was able to develop a polynomial algorithm for producing such an A-trail [15, pp. VI.161–VI.163]).

Going back to the original conjecture of Barnette and Tutte which lies at the heart of the development of the theory of A-trails, I wonder whether it would be possible to prove it in a way similar to the proof of the 4CT. That is, with the help of a discharge procedure find first a certain set of unavoidable configurations in a 3-connected planar cubic bipartite graph, and then find reductions such that a hamiltonian cycle in the smaller (cubic bipartite) graph extends to a hamiltonian cycle in the original graph.

Finally, let us consider some seemingly unrelated conjectures.

Conjecture A. Every cyclically 4-edge-connected 3-regular graph has a dominating cycle.

Conjecture B. Every essentially 4-edge-connected graph has a dominating connected eulerian subgraph.

Conjecture C (Thomassen [48]). Every 4-connected line graph is hamiltonian.

Conjecture D (Mathews and Sumner [37]). Every 4-connected claw-free graph is hamiltonian.

Bill Jackson and I had shown in the 1980s that Conjectures A, B, C are equivalent indeed [22]. At a workshop in Enschede (The Netherlands) in November 1995, Ryjácek (Pilsen, Czech Republic [40]) showed the equivalence of Conjectures C and D. We observe, on the other hand, that the equivalences shown in [22] rest on a classical result of Harary and Nash-Williams whereby connected dominating eulerian subgraphs of $\boldsymbol{G}$ correspond to hamiltonian cycles in the line graph $\boldsymbol{L}(\boldsymbol{G})$, and that G being essentially k-edge-connected implies $\boldsymbol{L}(\boldsymbol{G})$ being k-connected.

2. Part 2: The Erdős–Seymour–Tutte connection

Consider a 4-regular graph $\boldsymbol{G}$ which can be decomposed into a hamiltonian cycle and n triangles (phrased differently, $\boldsymbol{G}$ has a 2-factorization such that one 2-factor is a hamiltonian cycle and the other consists of triangles). Du and Hsu [8] had conjectured that such graphs always have an independent set of n vertices (there cannot be more since every triangle contributes at most one vertex to an independent set). In fact, Paul Erdős went further by conjecturing that such graphs are even 3-colorable. This conjecture (having become known as the *Cycle-plus-Triangles Problem*) was proved in 1991 by Stiebitz and myself [24]. Our proof rests on the following result of Alon and Tarsi [1].

Theorem 8. *Let $\boldsymbol{D}$ be an orientation of a given graph $\boldsymbol{G}$ such that the number of arc-induced eulerian subdigraphs of $\boldsymbol{D}$ having an even number of arcs differs from the corresponding number of subdigraphs with an odd number of arcs. Then the list-chromatic number (and the more so the chromatic number) of $\boldsymbol{G}$ is at most $\Delta^+(\boldsymbol{D})+\mathbf{1}$.*

Observe that such an orientation always exists; e.g., considering an acyclic orientation, the empty digraph is the only eulerian subdigraph, hence the numbers considered in the theorem are different indeed, for this orientation.

Considering a 4-regular graph $\boldsymbol{G}$ as described above, we were able to show that the number of *all* arc-induced eulerian subdigraphs in a given eulerian orientation of $\boldsymbol{G}$ is of the form $\mathbf{4s+2}$. In fact, this number is independent of the actual eulerian orientation, and since $\boldsymbol{G}$ has an even number of edges, $\mathbf{4s+2}$ implies that the numbers quoted in the theorem are different (simply observe that for an arc subset inducing an eulerian subdigraph, its complement also defines an eulerian subdigraph and has the same parity since we consider an eulerian orientation and the total number of edges is even). Invoking the above theorem, the list-chromatic number and the more so the chromatic number of $\boldsymbol{G}$ is 3 (it cannot be smaller because of the existence of triangles).

However, there is a 1–1-correspondence between the eulerian orientations of $\boldsymbol{G}$ and the eulerian subdigraphs of an eulerian orientation $\boldsymbol{D}$ of $\boldsymbol{G}$. (Note that the reversal of the orientation of the arcs of an eulerian subdigraph of $\boldsymbol{D}$ yields another eulerian orientation of $\boldsymbol{G}$.) Hence $\mathbf{4s+2}$ is the number of eulerian orientations of $\boldsymbol{G}$, and an inclusion–exclusion argument guarantees the existence of an eulerian orientation $\boldsymbol{D}'$ in which the 'inscribed' triangles of $\boldsymbol{G}$ (viewed as a set of diagonals inscribed in the hamiltonian cycle) are transitively oriented (see [25]; this fact had already been observed, however, by Alon and Tarsi). Thus, $\boldsymbol{D}'$ induces in each of these triangles a source and a sink. Split away, for each of these triangles, their paths of length 2 and suppress the 2-valent vertices thus obtained. We arrive at a digraph $\boldsymbol{D}^*$ whose underlying graph $\boldsymbol{G}^*$ is bipartite; its vertex bipartition is defined by the sets of sources and sinks in the triangles of $\boldsymbol{D}'$, respectively. These two classes, however, are independent sets of $\boldsymbol{G}$ as well, and each has size n; therefore, the original conjecture of Du and Hsu follows in a stronger form even without using the above theorem of Alon and Tarsi. The remaining n vertices of $\boldsymbol{G}$, however, need not form an independent set; therefore we could not prove Paul Erdös' conjecture without using the Alon–Tarsi result. Such proof (entirely constructive) has been given by Sachs [41].

We apply the observations of the preceding paragraph to the theory of cubic graphs and nowhere-zero (integer) flows as follows. The (inscribed) triangles of $\boldsymbol{G}$ correspond to digons of G^*; thus, deleting just one edge in each of these digons we arrive at a (hamiltonian) bipartite cubic graph H^*. However, by a simple construction, we may reach H^* directly from G. To see this, replace every (inscribed) triangle $\Delta = \langle a, b, c \rangle$ of G by a claw having a, b, c as its end-vertices and a new vertex $z(\Delta)$ as its center. This yields the cubic graph H in which the hamiltonian cycle of G appears as a dominating cycle C containing all but the new vertices $z(\Delta)$ (note that in this construction, C is chordless). The transition from H to H^* is then guaranteed by the above D' and the above splitting operation. The following corollary is slightly more general since it includes all cubic graphs having a dominating cycle (even if the dominating cycle is a hamiltonian cycle; however, if we replace one end of every chord of the dominating cycle by a triangle, then we arrive at a cubic graph with a chordless dominating cycle).

Corollary 9. *Let H be a cubic graph with dominating cycle C. There is a non-empty matching $M \subset E(H) - E(C)$ such that the cubic graph H^* homeomorphic to $H - M$ is hamiltonian and bipartite.*

Consequently, we call a matching M with the properties as described in this corollary, a *bipartizing matching*. Now let us consider an arbitrary cubic non-bipartite hamiltonian graph H, and let C be a hamiltonian cycle of H. Consider the Tait coloring of H in which the edges of C are alternately colored b and r, whereas the chords of C are colored y. In order to show that the chords can be partitioned into two non-empty bipartizing matchings M_1 and M_2, we consider the 2-factors Q_1 and Q_2, defined by the edges colored r and y, b and y, respectively, as sets of oriented cycles. Those chords which have different orientations in Q_1 and Q_2 form the set M_1, the other chords form M_2. To see that these two disjoint sets of chords are bipartizing matchings indeed, consider first the cubic graph H_1 homeomorphic to $H - M_1$ together with the orientation induced by the cycles of Q_1 and Q_2. Reversing now the orientation of the elements of M_2 (these are the chords of the hamiltonian cycle C_1 of H_1 induced by C), we see that every vertex of H_1 is either a source or a sink; i.e., H_1 is bipartite. Next, reversing the orientation of the cycles of Q_2 and arguing as above, we see that M_1 and M_2 interchange their roles; hence H_2, the cubic graph homeomorphic to $H - M_2$, is bipartite. Now we make another observation. For $i = 1, 2$, consider a Tait coloring of H_i as above, and let $Q_{i,1}$ and $Q_{i,2}$ be the corresponding bicolored 2-factors. Consider the set of cyclic orientations of $Q_{i,j}$ (analogous to what we did above) for fixed $j \in \{1, 2\}$. Since H_i is bipartite, there is a cyclic orientation such that the orientation of the edges in $Q_{i,j} \cap C_i$ induces a cyclic orientation of the hamiltonian cycle C_i of H_i induced by C; and vice versa, a cyclic orientation of C_i induces a cyclic orientation in each of the two $Q_{i,j}$.

The connection of the preceding considerations to the proof of Paul Erdös' conjecture is evident. Therefore, the question arises where the connection lies between these considerations on the one hand, and *Seymour's 6-Flow-Theorem*, [44], and *Tutte's 5-Flow-Conjecture* on the other hand. In what follows, we consider a *nowhere-zero k-flow* of an arbitrary graph G as a labelling of the edges of G taken from the set of integers $\{1,2,\ldots,k-1\}$, together with an orientation $\boldsymbol{D}$ of $\boldsymbol{G}$ such that at every vertex υ, the sum of the labels of the arcs ending at υ equals the sum of the labels of the arcs starting at υ. We adopt the analogous understanding for the concept of *partial k-flows* in which, in addition, some of the edges are allowed to have *zero* as a label. Now we are in a position to prove the following.

Theorem 10. *Suppose the cubic graph* $\boldsymbol{G}_3$ *has a dominating cycle* $\boldsymbol{C}$. *Then it has a nowhere-zero 6-flow such that*

(i) *all edges with a label exceeding* 3 *lie in* $\boldsymbol{C}$; *and*
(ii) $\boldsymbol{C}$ *is cyclically oriented.*

Proof. We use the notation and considerations of the preceding paragraph. Suppose first that $\boldsymbol{C}$ is a hamiltonian cycle. If $\boldsymbol{G}_3$ is bipartite, then we obtain via $\boldsymbol{Q}_1$ a nowhere-zero 3-flow (just start with a cyclic orientation of $\boldsymbol{C}$ which induces a cyclic orientation of the elements of $\boldsymbol{Q}_1$, and define the edge labelling by the number of elements of $\boldsymbol{C}$ and $\boldsymbol{Q}_1$ an edge belongs to. If, however, $\boldsymbol{G}_3$ is not bipartite, then we partition the set of chords of $\boldsymbol{C}$ into two bipartizing matchings $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$. Now, a fixed cyclic orientation of $\boldsymbol{C}$ will induce a cyclic orientation of the cycles of $\boldsymbol{Q}_{1,j}$ and $\boldsymbol{Q}_{2,k}$ for any choice of $j,k \in \{1,2\}$. Extending these sets of cycles to sets of cycles in $\boldsymbol{G}_3$ and using $\boldsymbol{C}$ as above, we readily obtain a nowhere-zero 4-flow of $\boldsymbol{G}_3$. Since an m-flow is also an n-flow for $m<n$, we obtain a nowhere-zero 6-flow as required by the theorem if $\boldsymbol{G}_3$ is hamiltonian. Whence we assume that $\boldsymbol{C}$ is not a hamiltonian cycle. (We note in passing that the above 4-flow can be obtained in a simpler way; however, due to the subsequent discussion we chose the procedure described above.)

By the preceding corollary, $\boldsymbol{G}_3$ has a bipartizing matching $\boldsymbol{M}$. Let $\boldsymbol{M}'$ be a matching of $\boldsymbol{G}_3-\boldsymbol{E}(\boldsymbol{C})$ such that $\boldsymbol{G}_3-\boldsymbol{M}'$ is homeomorphic to a hamiltonian cubic graph $\boldsymbol{G}_3'$, and $\boldsymbol{M}'\cap\boldsymbol{M}=\emptyset$ (in fact, there are at least 2^r such matchings, where r is the number of vertices not contained in $\boldsymbol{C}$).

We assume first that $\boldsymbol{M}'$ is not a bipartizing matching; i.e., $\boldsymbol{G}_3'$ is not bipartite. By the above, $\boldsymbol{G}_3'$ has a nowhere-zero 4-flow such that a corresponding orientation of $\boldsymbol{G}_3'$ yields a cyclic orientation of $\boldsymbol{C}'$ — the hamiltonian cycle of $\boldsymbol{G}_3'$ induced by $\boldsymbol{C}$. This yields a nowhere-zero 4-flow and an orientation of $\boldsymbol{G}_3-\boldsymbol{M}'$ such that $\boldsymbol{C}$ is cyclically oriented. This in turn yields a partial 4-flow and a partial orientation of $\boldsymbol{G}_3$ such that all edges not on $\boldsymbol{C}$ have label 1 and are oriented unless they belong to $\boldsymbol{M}'$ in which case they carry the label 0 and are not oriented (or can be oriented either way); and these are the only edges with label 0. However, the edges of $\boldsymbol{M}'$ are in $\boldsymbol{G}_3-\boldsymbol{M}$ since $\boldsymbol{M}$ and $\boldsymbol{M}'$ are disjoint, and $\boldsymbol{G}_3^*$ — the hamiltonian cubic graph homeomorphic to $\boldsymbol{G}_3-\boldsymbol{M}$ — is bipartite. Moreover, the cyclic orientation of $\boldsymbol{C}$ induces a cyclic orientation of $\boldsymbol{Q}_1^*$ and $\boldsymbol{Q}_2^*$ (see the respective consideration with respect to $\boldsymbol{H}_i$, $\boldsymbol{Q}_{i,1}$, $\boldsymbol{Q}_{i,2}$ above). Choose $\boldsymbol{Q}_j^*$ for fixed $j\in\{1,2\}$; it corresponds to a set $\boldsymbol{S}^*$ of oriented cycles in $\boldsymbol{G}_3$ (note that the edges in $\boldsymbol{M}'$ belong to elements of $\boldsymbol{S}^*$).

We now turn the partial 4-flow into a nowhere-zero 6-flow in $\boldsymbol{G}_3$ as follows. The edges not belonging to any cycle of $\boldsymbol{S}^*$ inherit their orientation and label from the partial 4-flow, whereas the other edges are given the orientation they carry in the respective cycle of S^*. Therefore, the orientation of C is cyclic. As for the label of an edge of the latter type, we increase it by 2 if it belongs to M' or if it has the same orientation in the partial 4-flow as in the corresponding cycle of S^*; otherwise we label it with 1 (this corresponds to the subtraction $2-1$ if we interpret S^* as a partial 3-flow of G_3 where the edges of the cycles of S^* are given the label 2 and the other edges are given the label 0). In any case, we have constructed an integer flow. Because of the properties of the original partial 4-flow and because of the orientation of the elements of S^* and their interpretation as a partial 3-flow, it follows that no label can be larger than 5, that edges not in C carry either of the labels 1,2,3, and that this flow is nowhere-zero on C (because the partial 4-flow had already this property). Thus we have a nowhere-zero 6-flow as required indeed.

Finally, assume that M' is a bipartizing matching; whence G_3' is bipartite and thus has a nowhere-zero 3-flow with a corresponding orientation such that C' is cyclically oriented. We now use the same arguments as in the preceding case to obtain even a nowhere-zero 5-flow which, of course, is also a nowhere-zero 6-flow. This finishes the proof of the theorem. □

The first and the last part of the preceding proof yield the following.

Corollary 11. *Let G_3 be a cubic graph having a dominating cycle C. If G_3 has two disjoint bipartizing matchings, then it has a nowhere-zero 5-flow with the additional properties stated in the preceding theorem.*

We note in passing that Alon and Tarsi were already aware of the preceding theorem and corollary before the latter had been stated in [25]. However, I extracted the corollary from the proof of the preceding theorem for good reasons. For, any minimal counter-example to Tutte's 5-flow Conjecture must be a snark (having additional properties; see, e.g., [32]). On the other hand, Bill Jackson and I had put forward the following conjecture, [22], which is the same as Conjecture A in Part 1 of the present paper.

DOMINATING CYCLE CONJECTURE (DCC). *Every cyclically 4-edge-connected cubic graph has a dominating cycle.*

Earlier I had restricted this conjecture to the class of snarks, [12]. Observe that the DCC is true in the planar case (it follows from Tutte's famous work on bridges in planar graphs which also implies that every 4-connected planar graph is hamiltonian). On the other hand, the validity of the DCC plus the preceding theorem would yield a new proof of Seymour's 6-Flow Theorem.

Stiebitz and I tried to find snarks with a dominating cycle, having no two disjoint bipartizing matchings; we didn't succeed. All we know is that a greedy approach won't work; i.e., one cannot start from a given bipartizing matching (one that exists by our theorem), and construct another one disjoint from it. We do not even know what the algorithmic complexity is to decide whether such pair of matchings exists or, if it exists, to find it. Nevertheless, we put forward the following conjecture.

BIPARTIZING MATCHINGS CONJECTURE (BMC). *Every snark having a dominating cycle has two disjoint bipartizing matchings.*

Clearly, since a minimal counter-example to Tutte's 5-Flow Conjecture must be a snark, the validity of both the DCC and the BMC together with the preceding theorem would yield a proof of the first conjecture. In this context, one might guess that *if a cyclically* 4-*edge-connected cubic graph has a chordless dominating cycle*, *then it is* 3-*edge-colorable* (and thus has a nowhere-zero 4-flow) (due to Luis Goddyn (oral communication)). However, a short while ago, this conjecture was disproved by A. Huck; this was communicated to me by T.R. Jensen who suggested that the conjecture might be true if one replaces 4 with 5.

As for 3-edge-colorability of cubic graphs in general and a connection to eulerian subdigraphs, there is an interesting feature discovered by F. Jaeger. As we have noted before, in an eulerian orientation D of an eulerian graph G the number of arc-induced eulerian subdigraphs is independent of the actual eulerian orientation and is, in fact, the number of eulerian orientations of G. However, there is another invariant. Denoting by $ee(D)$ and $eo(D)$ the numbers of arc-induced eulerian subdigraphs having an even number, odd number of arcs, respectively, the *invariant* is $|ee(D)-eo(D)|$. As F. Jaeger observed in [33], if G_3 is a planar 2-connected cubic graph and $G = L(G_3)$ is its line graph (which is 4-regular and thus eulerian), then *this invariant is the number of Tait colorings* of G_3. That is (see the Theorem by Alon and Tarsi above), $ee(D) \neq eo(D)$ *is a necessary and sufficient condition for* 3-*edge-colorability of* G_3.

Another interesting feature regarding eulerian orientations of 4-regular graphs had been discovered by N. Alon (private communication) namely: *If the number of eulerian orientations of the* 4-*regular graph* G *is not of the form* $3t+1$, *then* G *has a* 3-*regular subgraph*. This opens chances for a new proof of the famous *Berge-Sauer-Conjecture* (see, e.g., [4], p. 246, problem 3) which was proved independently by Tashkinov [46] and Zhang Limin [50] in 1982.

What about generalizations of the Cycle-plus-Triangles Problem? *What happens if we replace the triangles with larger cycles or with complete graphs*?

Well, if we 'inscribe' cycles arbitrarily into the hamiltonian cycle, all we are getting is the class of hamiltonian 4-regular graphs; among them are infinitely many 4-chromatic graphs for which every color class in every 4-coloring has the same size, [26]. So let's start with inscribing pairwise disjoint complete graphs into the hamiltonian cycle. That is, we consider a graph G having a decomposition into a hamiltonian cycle and disjoint complete graphs, and let $k > 3$ be an integer. If each of the complete subgraphs in this decomposition has at most k vertices, then G is k-colorable (see [25] which also contains a nice application to coloring the integers). This is best possible not only in the sense that if a complete subgraph on k vertices exists, then $\boldsymbol{G}$ cannot be $(k-1)$-colorable, but the conclusion no longer holds for $k=3$; this is exhibited in [25] by a simple example due to H. Sachs.

Finally, let us consider 4-regular graphs decomposable into a hamiltonian cycle $\boldsymbol{C}$ and a 2-factor consisting of r cycles whose lengths are denoted by $k_1,\dots,k_r$. However, because of what was said at the beginning of the preceding paragraph, we make an assumption on how these r cycles are being inscribed into $\boldsymbol{C}$. It is the natural one: namely, that in each of these cycles the vertices are cyclically ordered in accordance with the cyclic ordering of the vertices on $\boldsymbol{C}$ (phrased differently, each of the inscribed cycles does not intersect itself if $\boldsymbol{C}$ is visualized as a regular polygon in the plane). We say that these *cycles are conformly inscribed*. In a forthcoming paper G. Sabidussi and I prove the following

Theorem 12. *If $\boldsymbol{G}$ is a 4-regular graph of order n, decomposable into a hamiltonian cycle $\boldsymbol{C}$ and a 2-factor consisting of r cycles conformly inscribed into $\boldsymbol{C}$, then $\boldsymbol{G}$ has at least $(n-r)/3$ independent vertices.*

One way of proving this theorem rests on a simple application of the aforementioned Theorem of Alon and Tarsi. Starting with a given vertex, run through $\boldsymbol{C}$ and mark in each of the inscribed cycles the first vertex reached. Denote this set of first vertices by $\boldsymbol{V}_f$. Then it is possible to find an acyclic orientation of $\boldsymbol{G}-\boldsymbol{V}_f$ with maximum out-degree 2. By the Theorem of Alon and Tarsi, $\boldsymbol{G}-\boldsymbol{V}_f$ is 3-colorable. Since it has $n-r$ vertices, one of the color classes in any 3-coloring must have size at least $(n-r)/3$. However, there is a more direct way of proving this result by first deleting in a run through $\boldsymbol{C}$ the last edge of each of the inscribed cycles and of $\boldsymbol{C}$ as well. Then one constructs inductively a 3-coloring in this spanning subgraph starting at the given vertex, as one walks along $\boldsymbol{C}$. This induces a 3-coloring in $\boldsymbol{G}-\boldsymbol{V}_f$.

Sabidussi and I turned to the consideration of such graphs because Sabidussi thought that in the Cycle-plus-Triangles Problem it is the number 3 which is essential. So we thought that if each k_i is a multiple of $3, i=1,\dots,r$, then we might be able to find an independent set of size $n/3$ even if such graphs are not 3-colorable. We soon found examples where this was not the case. So we restricted ourselves to such graphs where all the inscribed cycles have *uniform cycle lengths modulo 3*, i.e., where $k_i=3k$ for $i=1,\dots,r$, and some integer $k>1$. But even in this case we found examples showing that our expectations were too high. In fact, the following is true.

Theorem 13. *For every integer $k>1$, there are infinitely many 4-regular graphs $\boldsymbol{G}$ of order $n=3kr$ decomposable into a hamiltonian cycle $\boldsymbol{C}$ and a 2-factor $\boldsymbol{Q}$ of r conformly inscribed cycles of uniform cycle length $3k$, such that $\boldsymbol{G}$ has no more than $(7n-3r)/21$ independent vertices.*

This is a far cry indeed from any hope to extend the conjecture of Du and Hsu to a wider range of 4-regular graphs; in fact, this result is closer to the previous theorem.

Also, we show in [26] that even in the restricted class of 4-regular graphs defined just before stating Theorem 13, deciding 3-colorability is an NP-complete problem. Thus, if $\boldsymbol{G}$ is a 4-regular graph of order n and decomposable into a hamiltonian cycle and a 2-factor having r conformly inscribed cycles of uniform cycle length $3k$, $k > 1$ (and thus $n = 3kr$), all we can say regarding the independence number $\alpha(\boldsymbol{G})$, is that it can be written in the form

$$\alpha(\boldsymbol{G}) = (n - cr)/\mathbf{3}$$

for some rational c satisfying $0 \leqslant c \leqslant 1$; and that there are infinitely many such $\boldsymbol{G}$ for which $c = 3/7$. Define $\alpha_n := \min\{\alpha(\boldsymbol{G})\}$ where the minimum is taken over all $\boldsymbol{G}$ of the above type having order n. Then the preceding two theorems imply the following

Corollary 14. *There are infinitely many* $n = 3kr$, *where* $k > 1$ *and* $r > 1$ *are integers, such that* $\alpha_n = (n - c_n r)/3$ *for some rational* c_n *satisfying* $3/7 \leqslant c_n$.

Consequently, it makes sense to study the sequence $\{c_n\colon n{=}3kr; k > 1, r > 1\}$ (which we have not done yet). *What is its least upper bound, what is its largest lower bound? Is it convergent?* We cannot answer these questions but we think that 1/2 might be a good guess for the least upper bound. That is, for every $\boldsymbol{G}$ of the above type, we conjecture that $\alpha(\boldsymbol{G}) \geqslant (2n{-}r)/6$, and that this inequality is close to being best possible.

3. Part 3: Some bits and pieces of the international compatibility connection

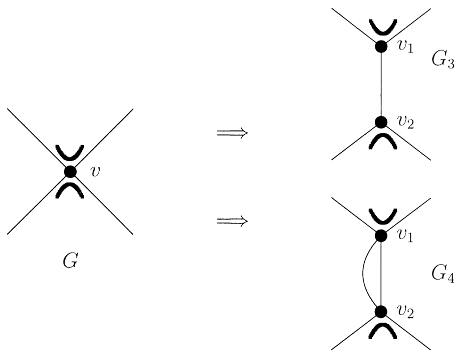
Once upon a time... there were three questions: Q1, Q2, and Q3.

Q1. *Given an eulerian trail* $\boldsymbol{T}$ *in an eulerian graph* $\boldsymbol{G}$ *without* 2*-valent vertices, does there exist a cycle decomposition* $\boldsymbol{S}$ *of* $\boldsymbol{G}$ *such that consecutive edges in* $\boldsymbol{T}$ *always belong to different elements of* $\boldsymbol{S}$? (*G. Sabidussi*, 1975; see, e.g., [27])

Q2. *Given a cycle* $\boldsymbol{C}$ *in a* 3*-regular graph* $\boldsymbol{G}_3$, *is there another cycle* $\boldsymbol{C}'$ *in* $\boldsymbol{G}_3$ *such that* $\boldsymbol{C}'$ *contains all the vertices of* $\boldsymbol{C}$? (*P.D. Seymour*, 1990; workshop on cycles in graphs, Barbados)

Q3. *Does there exist a* 4*-regular simple graph* $\boldsymbol{G}_4$ *having precisely one hamiltonian cycle*? (*J. Sheehan* [45])

Let us consider Q1 first. With the help of the Splitting Lemma it is easy to show (and was proved implicitly already by Kotzig [35]) that the converse of Q1 is true. That is, given a cycle decomposition of $\boldsymbol{G}$, there is an eulerian trail with the property stated in Q1. Thus we call a cycle decomposition and an eulerian trail *compatible* if they have this property. Note that an eulerian trail and the elements of a cycle decomposition as well define a pairing of the half-edges incident to v, for every vertex v of $\boldsymbol{G}$ (we have to speak of half-edges since we allow loops and multiple edges). Conversely, an arbitrary such pairing defines a trail decomposition of $\boldsymbol{G}$. In what follows we call such pairing a transition system $\boldsymbol{X}(\boldsymbol{G})$; if it arises from an eulerian trail $\boldsymbol{T}$, we denote it by $\boldsymbol{X}_T$, and likewise by $\boldsymbol{X}_S$ if it arises from a cycle decomposition $\boldsymbol{S}$. As a generalization,



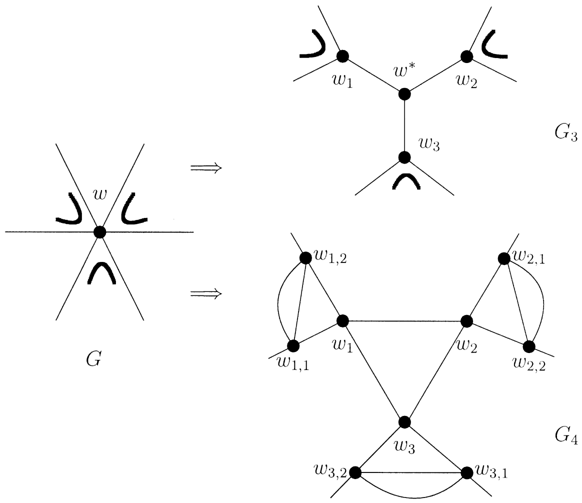


Fig. 5. The little arcs mark the transitions of $\boldsymbol{T}$ in $\boldsymbol{G}$ and the transformation of $\boldsymbol{G}$ into $\boldsymbol{G}_3$ and $\boldsymbol{G}_4$.

we call *two arbitrary transition systems of* $\boldsymbol{G}$ *compatible* if they have no transition in common (where a *transition* is simply an element of a transition system, or just a pair of adjacent half-edges, depending on the context). Similarly, we call a set $\boldsymbol{S}_0$ of cycles of $\boldsymbol{G}$ compatible with a given $\boldsymbol{X}(\boldsymbol{G})$ if no transition defined by an element of $\boldsymbol{S}_0$ belongs to $\boldsymbol{X}(\boldsymbol{G})$. As for the original Q1, the following had been shown already in [12]:

It suffices to restrict oneself to $\boldsymbol{G}$ *having* 4- *and* 6-*valent vertices only.* $(*)$

Because of $(*)$ and in view of the above three questions, there are certain constructions relating respective graphs $\boldsymbol{G}, \boldsymbol{G}_3, \boldsymbol{G}_4$ to each other (see Fig. 5).

Observe:

The eulerian trail $\boldsymbol{T}$ of $\boldsymbol{G}$ corresponds to a dominating cycle $\boldsymbol{C}$ in $\boldsymbol{G}_3$. $(**)$

The eulerian trail $\boldsymbol{T}$ of $\boldsymbol{G}$ corresponds to a hamiltonian cycle $\boldsymbol{H}$ in $\boldsymbol{G}_4$. $(***)$

In fact, we have the following (here $\boldsymbol{E}(\boldsymbol{S}_0)$ means the set of edges belonging to the union of the edge sets of the elements of $\boldsymbol{S}_0$).

Theorem 15 (Fleischner [18]). *Let $\boldsymbol{G}, \boldsymbol{G}_3, \boldsymbol{G}_4$, and $\boldsymbol{T}, \boldsymbol{C}, \boldsymbol{H}$ be as above. Any two of the following statements are equivalent.*

(1) *$\boldsymbol{G}$ contains a set $\boldsymbol{S}_0 \neq \phi$ of vertex disjoint cycles compatible with $\boldsymbol{T}$ such that $\boldsymbol{T}$ induces an eulerian trail in $\boldsymbol{G}_0 := \boldsymbol{G} - \boldsymbol{E}(\boldsymbol{S}_0)$. (That is, $\boldsymbol{G}_0$ contains an eulerian trail whose transition system contains all transitions of $\boldsymbol{T}$ which are not destroyed by the removal of the edges of the elements of $\boldsymbol{S}_0$.)*
(2) *$\boldsymbol{G}_3$ has a dominating cycle $\boldsymbol{C}'$ different from $\boldsymbol{C}$ and containing all vertices of $\boldsymbol{C}$.*
(3) *$\boldsymbol{G}_4$ contains more than one hamiltonian cycle.*

In view of statement (1) of the theorem it seems natural to adopt the following strategy:

Search in $\boldsymbol{G}$ for a set $\boldsymbol{S}_0 \neq \phi$ of vertex disjoint circles compatible with $\boldsymbol{T}$ such that $\boldsymbol{G}_0 := \boldsymbol{G} - \boldsymbol{E}(\boldsymbol{S}_0)$ has an eulerian trail $\boldsymbol{T}_0$ induced by $\boldsymbol{T}$. If this is possible, apply induction and thus you have constructed a cycle decomposition compatible with $\boldsymbol{T}$.

In fact, I proved in [12] that such $\boldsymbol{S}_0$ exists if $\boldsymbol{G}$ contains at most one 6-valent vertex and all other vertices are 4-valent. The proof rests on Smith's Theorem whereby *in a cubic graph, every edge belongs to an even number of hamiltonian cycles*. In the same paper, however, I presented a graph satisfying $(*)$ and having precisely two 6-valent vertices such that an $\boldsymbol{S}_0$ as described above does not exist; it has precisely one cycle decomposition compatible with the given eulerian trail. Due to the preceding theorem and the transformation exhibited in Fig. 5 (see also $(**)$ and $(***)$), I was then able to construct an infinite family of 3-regular graphs for which the answer to Seymour's question Q2 is negative (although his question seemed quite natural in view of Smith's Theorem above). Similarly, I gave a partial positive answer to Sheehan's question Q3 by constructing infinitely many loopless 4-regular graphs having precisely one hamiltonian cycle, [18] (from them one easily constructs loopless $2k$-regular graphs having precisely one hamiltonian cycle, for any positive integer k). Unfortunately, they all have multiple edges. In order to 'wipe out' these multiple edges one basically would have to replace them by 4-regular graphs constituting a positive answer to either of the following two questions.

Does there exist a simple 4-regular graph containing two edges such that there is precisely one hamiltonian cycle containing them?

Does there exist a simple 4-regular graph containing two edges such that there is precisely one hamiltonian cycle containing one of them and avoiding the other?

In view of a paper by Thomason [47], it seems that answering any one of these two questions might take a nontrivial effort.

We note in passing that the *Dominating Cycle Conjecture* (DCC) (see above) might play an important role also in the context of the *Cycle Double Cover Conjecture* (CDCC). The latter states that

every bridgeless graph can be covered by a system of cycles such that every edge belongs to precisely two cycles of the system.

It is well known that in proving the CDCC one can restrict oneself to the consideration of snarks. However, if the DCC is true and Sabidussi's question Q1 has an affirmative answer (and the handful of compatibilists tend to think it does), then — via the construction exhibited in Fig. 5 and $(**)$ — a compatible cycle decomposition in $\boldsymbol{G}$ would yield a cycle double cover in $\boldsymbol{G}_3$ containing the corresponding dominating cycle. This observation lends support to the *Strong Cycle Double Cover Conjecture* (SCDCC) which states that

every bridgeless graph has a cycle double cover containing any given cycle.

Regarding compatible cycle decompositions, there is another conjecture by Bill Jackson and myself which, in its less general form, says the following.

Conjecture 16. Let $\boldsymbol{G}$ be a cyclically 6-edge-connected 4-regular graph, and let $\boldsymbol{X}(\boldsymbol{G})$ be an arbitrary system of transitions. Then there is a cycle decomposition of $\boldsymbol{G}$ compatible with $\boldsymbol{X}(\boldsymbol{G})$.

Jaeger [31] observed that the truth of this conjecture immediately implies the truth of the CDCC: just consider the line graph of a snark and define the system of transitions in the line graph via the triangles corresponding to the vertices of the snark. Then a compatible cycle decomposition of the line graph corresponds to a cycle double cover of the snark, and vice versa.

On the other hand, in the planar case we do not have as strong a connectivity condition as in the above conjecture. All one needs is an obvious restriction; namely, that a given $\boldsymbol{X}(\boldsymbol{G})$ does not contain *separating* transitions, where a transition is separating if its two half-edges define an edge-cut of size 2, or if they belong to the same loop. In this case we have the following

Theorem 17 (Fleischner [27]). *Let there be a given a system of transitions* $\boldsymbol{X}(\boldsymbol{G})$ *without separating transitions, where* $\boldsymbol{G}$ *is an arbitrary planar eulerian graph. Then there is a cycle decomposition of* $\boldsymbol{G}$ *compatible with* $\boldsymbol{X}(\boldsymbol{G})$.

In fact, this theorem has several applications and generalizations in the theory of planar graphs such as proving the existence of removable cycles and the SCDCC [13,20,21]. It also serves as the basis for a more general theorem on the existence of a cycle decomposition compatible with a given 'partition system' (whereby at every vertex the set of incident edges is partitioned in such a way that a certain cut-condition is fulfilled) [23]. This, in turn, implies Seymour's (Integer) Sums-of-Circuits Theorem (the cut-conditions of these two theorems are closely interrelated) and his Even Cycle Decomposition Theorem [42,43].

However, the above theorem can also be used to show that in a connected planar bridgeless graph, any solution of the Chinese Postman Problem yields a solution of the Minimum Weight Cycle Covering Problem [28]. In fact, all one has to do is the following: double the edges traversed twice by a postman tour, to obtain a planar eulerian graph; considering the problem for every block of this eulerian graph, use the doubled edges for an initial set of transitions and extend it to a system of transitions by pairing the remaining edges at every vertex. Since the original graph is bridgeless, no block of the derived eulerian graph is a cycle stemming from doubling an edge.

All of the above results were proved in the first half of the 1980s, and it was already clear at that time that the Petersen graph and graphs derived from it constituted counter-examples for the nonplanar case (see, e.g., [12,28]). In fact, in more recent developments most of these results have been extended to graphs without a Petersen minor or a K_5-minor, respectively (see, e.g., [2,9,29,49]). However, some of these proofs require far more effort than the respective proofs in the planar case.

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