3-1 Dynamic Programming

(Part I: Examples)

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Printing Neatly (Problem 15-4)

A sequence of n words of lengths $l_1, l_2, \cdots, l_n$

Line width M

$$\textit{extra}[i, j] = M - (j - i) - \sum_{k=i}^{j} l_k$$

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To minimize the sum, over all lines *except the last*, of the cubes of the numbers of extra space characters at the ends of lines.

$$c(n) = \min_{L} \sum_{l_{[i,j]} \in L \,\wedge\, j \neq n} \Big(\textit{extra}[i,j]\Big)^3$$


LaTeX
Beginner's Guide
TEX}

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$C(i)$: $\min$ cost of neatly printing the *first* i words

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$C(i, j)$: $\min$ cost of neatly printing *words* i *through* j

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$$C(i) = \min_{\substack{0 < k \le (n-i+1) \\ \textit{extra}[i, i+k-1] \ge 0}} \Big(\textit{extra}[i, i+k-1]\Big)^3 + C(i+k)$$

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$$O(nW)$$

procedure $\textsc{Printing-Neatly}(n)$

  for $i \leftarrow n \downarrow 1$ **do**

    if $\textit{extra}[i, n] \geq 0$ **then** ▷ put w_i through w_n on a line

      $C[i] \leftarrow 0$

    else

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$$B[1], \quad B[B[1]+1], \quad \cdots$$

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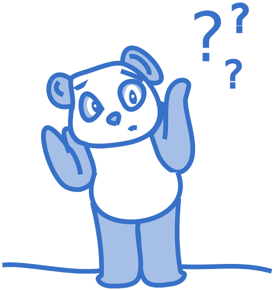

Longest Increasing Subsequence (Problem 15.4-5)

$$A[1 \ldots n]$$

$$5,\ 2,\ 8,\ 6,\ 3,\ 6,\ 9,\ 7$$

Find (the length of) a longest increasing (non-decreasing) subseqence.

$$5,\ \color{red}{2},\ 8,\ 6,\ \color{red}{3},\ \color{red}{6},\ \color{red}{9},\ 7$$

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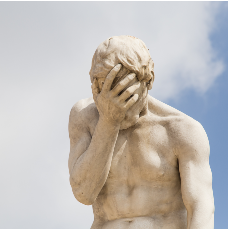

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$$O(n^2) = \Theta(n) \cdot O(n)$$

$$\mathsf{LIS(A)} = \mathsf{LCS}\Big(A, \textsc{Sort}(A)\Big)$$

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$$O(n^2) = O(n \log n) + O(n^2)$$

Longest Increasing Subsequence (Problem 15.4-6)

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E : $E[l] =$ all increasing subsequences of length $l, \quad 1 \leq l \leq n$

$$\max\{k \mid E[k] \neq \emptyset\}$$

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$$\{3 \quad 9 \quad 11 \quad 13\}$$

$$\{4 \quad 6 \quad 10 \quad 15\}$$

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Keep only the *smallest* ending number per distinct length l

$\text{LIS}[i]$:

$E[l] =$ the *smallest ending number* for the increasing subsequence of length l using $A[1 \cdots i]$


THANK YOU