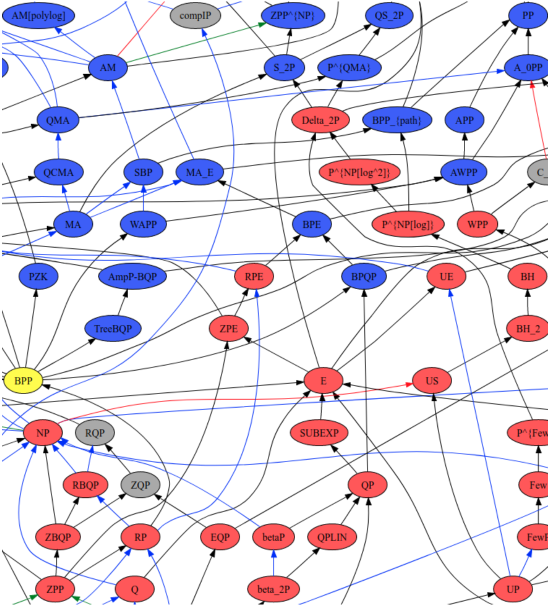
# 4-13 Randomized Algorithms

Hengfeng Wei

hfwei@nju.edu.cn

June 10, 2019


1/2


AM[polylog]
compIP
ZPP^{NP}
QS_2P
PP
AM
S_2P
P^{QMA}
A_0PP
QMA
Delta_2P
BPP_{path}
APP
QCMA
SBP
MA_E
P^{NP[log^2]}
AWPP
MA
WAPP
BPE
P^{NP[log]}
WPP
PZK
AmpP-BQP
RPE
BPQP
UE
BH
TreeBQP
ZPE
BH_2
BPP
E
US
NP
RQP
SUBEXP
RBQP
ZQP
QP
ZBQP
RP
EQP
betaP
QPLIN
ZPP
Q
beta_2P
UP

$$P \subseteq ZPP \subseteq RP \subseteq BPP \subseteq PP$$

Exercise 5.2.2.9

Definition ($ZPP$: Zero-error Probabilistic Polynomial Time)

$$L \in ZPP$$

$$\iff$$

$$\exists A \ (\textit{probabilistic polynomial-time algorithm}):$$

$$Pr\Big(A(x) = L(x)\Big) \geq \frac{1}{2}$$

$$Prob\Big(A(x) = ?\Big) = 1 - Pr\Big(A(x) = L(x)\Big) \leq \frac{1}{2}$$

$Q$ : Why 1/2?

$$ZPP_\delta : ZPP_{1/3} = ZPP_{1/2} = ZPP_{2/3}$$

$$L \in ZPP_{\delta}$$

$A^{(k)}$ : Repeat $A$ $k$ times independently

Output the non-"?" value if any; Otherwise, output "?"

$L \in ZPP_{\alpha}$ for some $\alpha$

$$Pr\Big(A^{(k)}(x) = L(x)\Big) = 1 - Pr\Big(A^{(k)}(x) = ?\Big) \geq 1 - (1-\delta)^k$$

$$L \in ZPP_{1-(1-\delta)^k}$$

Definition ($RP$: Randomized Polynomial time (One-Sided Error))

$$L \in RP$$

$$\iff$$

$$\exists A \ (\textit{probabilistic polynomial-time algorithm}):$$

$$x \in L \implies Pr\Big(A(x) = 1\Big) \geq \frac{1}{2}$$

$$x \notin L \implies Pr\Big(A(x) = 0\Big) = 1$$

$Q$ : Why 1/2?

$$RP_\delta : RP_{1/3} = RP_{1/2} = RP_{2/3}$$

$$L \in RP_\delta$$

$A^{(k)}$ : Repeat $A$ $k$ times independently

Accept $x$ iff any of the $k$ runs accepts

$$L \in RP_\alpha \text{ for some } \alpha$$

$$Pr\Big(x \in L \wedge A^{(k)}(x) = 1\Big) = 1 - Pr\Big(x \in L \wedge A^{(k)}(x) = 0\Big) \geq 1 - (1-\delta)^k$$

$$L \in RP_{1-(1-\delta)^k}$$

Definition ($BPP$: Bounded-error Probabilistic Polynomial time (Two-Sided Error))

$$L \in BPP$$

$$\iff$$

$$\exists A \ (\textit{probabilistic polynomial-time algorithm}):$$

$$\exists \epsilon, 0 < \epsilon \leq 1/2 : Pr\Big(A(x) = L(x)\Big) \geq \frac{1}{2} + \epsilon$$

$Q$ : Why $1/2$?

$Q$ : Why $\epsilon$?

$$L \in BPP_{p \triangleq (\frac{1}{2}+\delta)}$$

$$A^{(k)} : \text{Repeat } A \ k \text{ times independently}$$

Output the "majority" ($\# \geq \lceil k/2 \rceil$) value

$$L \in BPP_{\alpha} \text{ for some } \alpha$$

$$Pr\Big(A^{(k)}(x) = L(x)\Big) \geq 1 - \sum_{i=0}^{\lfloor k/2 \rfloor} \binom{t}{i} p^i (1-p)^{k-i} > 1 - \frac{1}{2}(1-4\delta^2)^{k/2}$$

$$L \in BPP_{1-\epsilon} \implies k \geq \frac{2 \ln 2\epsilon}{\ln(1-4\delta^2)}$$

Definition ($BPP$: Bounded-error Probabilistic Polynomial time (Two-Sided Error))

$$L \in BPP$$

$$\iff$$

$$\exists A \; (\textit{probabilistic polynomial-time algorithm}):$$

$$\exists \epsilon, 0 < \epsilon \leq 1/2 : Pr\Big(A(x) = L(x)\Big) \geq \frac{1}{2} + \epsilon$$

$Q$ : Why $\epsilon$?

$Q$ : What about $Pr\Big(A(x) = L(x)\Big) > \frac{1}{2}$?

$Q$ : What about $Pr\Big(A(x) = L(x)\Big) \geq \frac{1}{2} + n^{-c}$ for some constant $c$?

$$Pr\Big(A(x) = L(x)\Big) \geq \frac{1}{2} + n^{-c} \text{ for some constant } c$$

$$L \in BPP_{p \triangleq (\frac{1}{2} + n^{-c})}$$

$$A^{(k)} : \text{Repeat } A \; k \text{ times independently}$$

Output the "majority" ($\# \geq \lceil k/2 \rceil$) of $x_1, x_2, \ldots, x_k$

$$L \in BPP_{\alpha} \text{ for some } \alpha$$

## Indicator random variables

$$X_i = \begin{cases} 1, & x_i = L(x) \\ 0, & \text{otherwise} \end{cases}$$

$$X = \sum_{i=1}^{k} X_i$$

$$Pr\Big(X \geq \frac{1}{2}k\Big) \geq \cdots$$

$$\boxed{\forall 0 < \delta < 1 : Pr\Big(X < (1-\delta)pk\Big) < e^{-\frac{\delta^2}{3}pk}}$$

$$\text{Fix } \delta = 1 - \frac{1}{2p}$$

$$Pr\Big(X \geq \frac{1}{2}k\Big) \geq 1 - e^{\frac{k}{3n^c}}$$

$$Pr\left(X \geq \frac{1}{2}k\right) \geq 1 - e^{\frac{k}{3n^c}}$$

Choose $k = 3n^{c+d}$ for some constant $d$

$$Pr\left(X \geq \frac{1}{2}k\right) \geq 1 - e^{-n^d}$$

$$L \in BPP_{1-e^{-n^d}}$$

$$\boxed{\forall \text{ constant } c, d > 0 : BPP_{\frac{1}{2}+\frac{1}{n^c}} = BPP_{1-\frac{1}{e^{n^d}}}}$$

Definition ($PP$: Probabilistic Polynomial time (Unbounded Error))

$$L \in BPP$$

$$\Longleftrightarrow$$

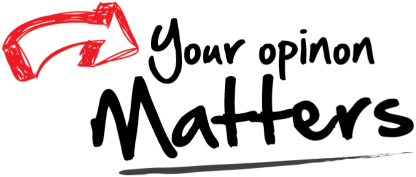
$$\exists A \; (\textit{probabilistic polynomial-time algorithm}):$$

$$Pr\Big(A(x) = L(x)\Big) > \frac{1}{2}$$

$$Pr\Big(A^{(k)}(x) = L(x)\Big) \geq 1 - \delta$$

$k$ may be exponential of $n$


THANK YOU




Office 302

Mailbox: H016

hfwei@nju.edu.cn