3-5 Minimum Spanning Trees

Hengfeng Wei

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$$\sum_{v \in V(G)} \deg(v) \leq 2(n-1)$$

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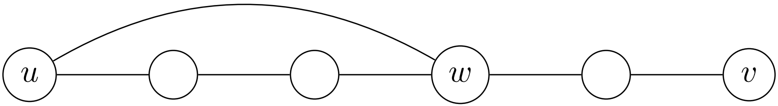

maximal path $P_{u,v}$

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Bridge and Spanning Trees (Problem 4.26)

$$G: \text{ a connected graph}$$

$$e \in E(G) \text{ is a bridge} \iff e \in \forall \text{ST of } G$$

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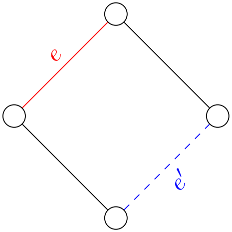

$$T' = \underbrace{T - \{e\}}_{e \in T} + \{e'\}$$

Cut Property

Cut Property (Version I)

X : A part of some MST T of G

$(S, V \setminus S)$: A *cut* such that X does *not* cross $(S, V \setminus S)$

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Correctness of Prim's and Kruskal's algorithms.

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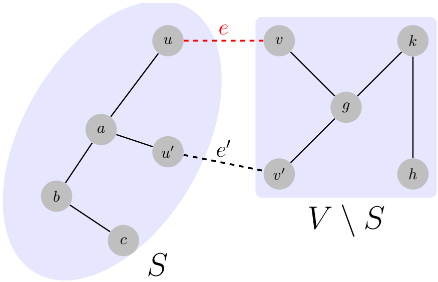

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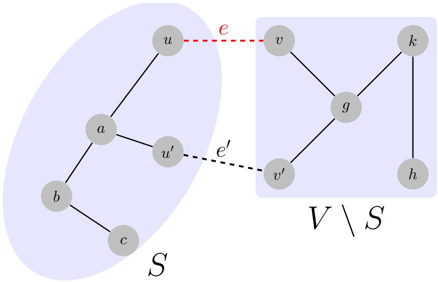

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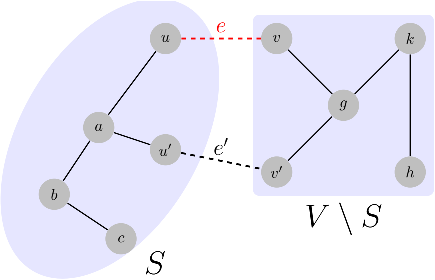

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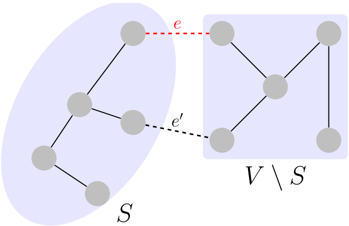
"a" $\to$ "the" $\implies$ "some" $\to$ "all"

Cut Property (Version II)

A cut $(S, V \setminus S)$

Let $e = (u, v)$ be *a* lightest edge across $(S, V \setminus S)$

$\exists$ MST T of $G : e \in T$



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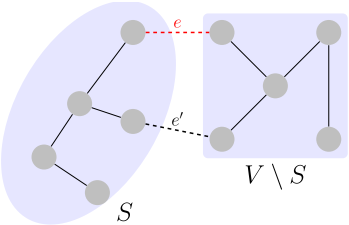

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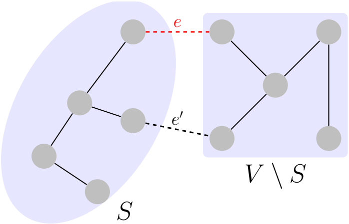

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"a" $\to$ "the" $\Longrightarrow$ "$\exists$" $\to$ "$\forall$"

Application of Cut Property

$$e = (u, v) \in G \text{ is a lightest edge } \implies e \in \exists \text{ MST of } G$$

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Wrong Divide&Conquer Algorithm for MST

$$(V_1, V_2) : \Big| |V_1| - |V_2| \Big| \leq 1$$

$$T_1 + T_2 + \{e\} : e \text{ is a lightest edge across } (V_1, V_2)$$

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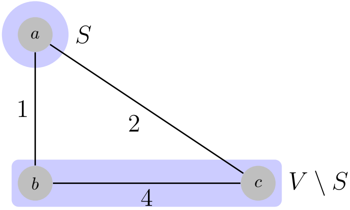
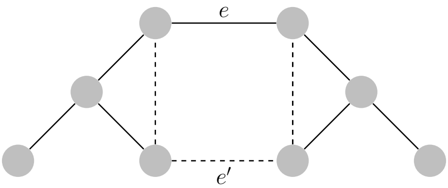


Cycle Property

Cycle Property

- Let C be any cycle in G
- Let $e = (u, v)$ be *a* maximum-weight edge in C

$$\text{Then } \exists \text{ MST } T \text{ of } G : e \notin T.$$



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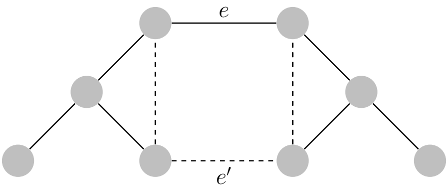

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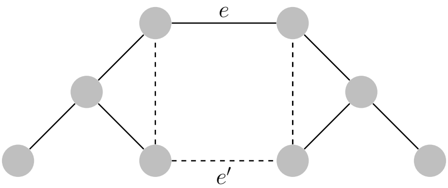

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Reverse-delete algorithm (wiki; clickable)

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"On the Shortest Spanning Subtree of a Graph and the Traveling Salesman Problem"

— Kruskal, 1956.

Application of Cycle Property

$$G = (V, E), \quad |E| > |V| - 1$$

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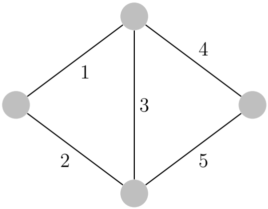

Uniqueness of MST

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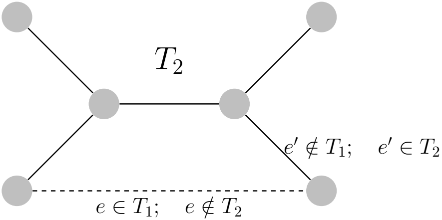
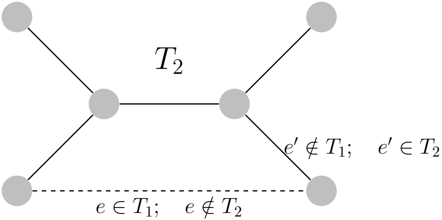

T_2
$e' \notin T_1; \quad e' \in T_2$
$e \in T_1; \quad e \notin T_2$



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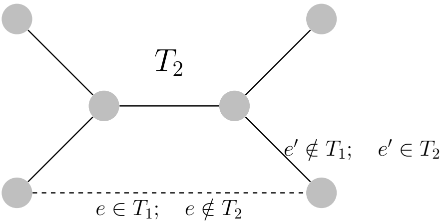

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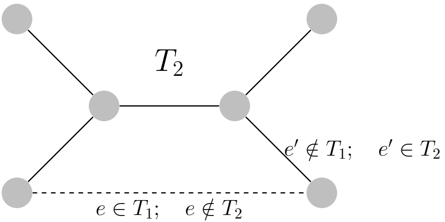

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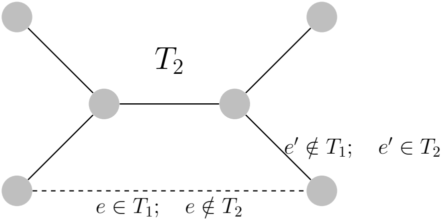

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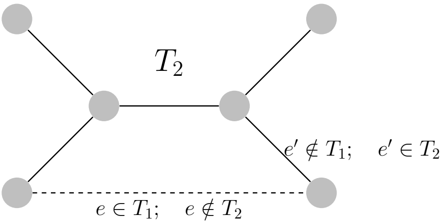

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$$T' = T_2 + \{e\} - \{e'\} \implies w(T') < w(T_2)$$

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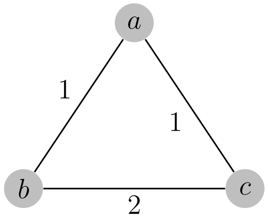

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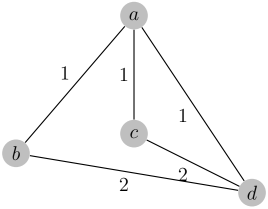

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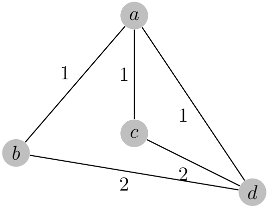

Theorem (After-class Exercise)

Maximum-weight edge in any cycle is unique $\Longrightarrow$ *Unique MST.*

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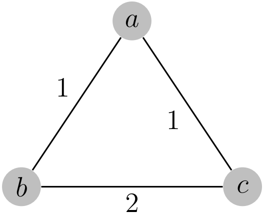

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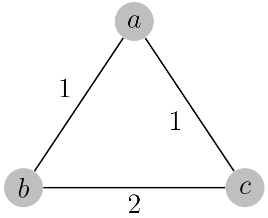

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To decide whether a graph has a unique MST.

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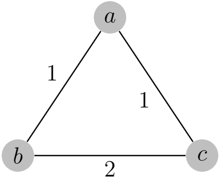

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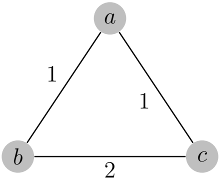

$$\underbrace{\color{red}{T}}_{\text{Any MST}} + \underbrace{\color{red}{\{e\}}, \forall e \notin T}_{\text{Cycle}}$$

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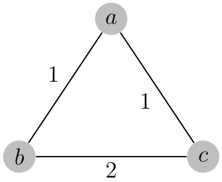

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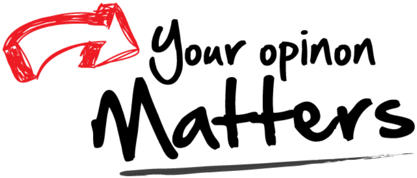
By Kruskal Algorithm.


THANK YOU



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