3-9 Connectivity

Hengfeng Wei

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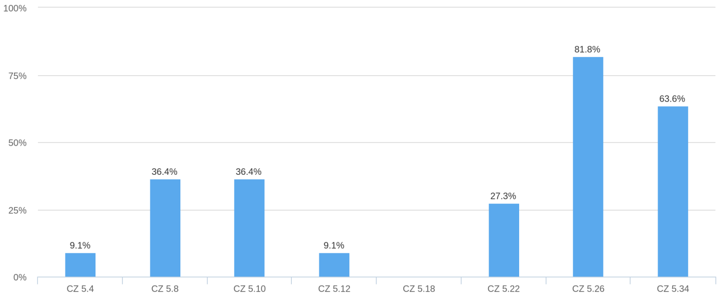
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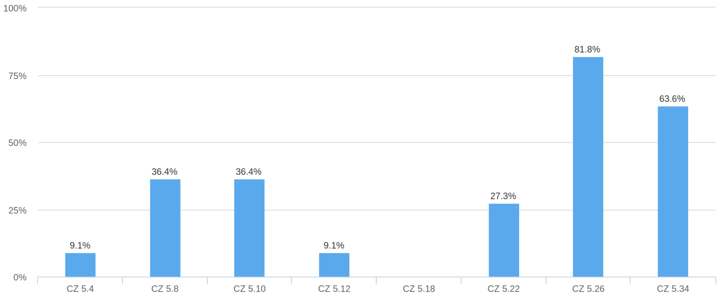
CZ 5.34



5.10 5.34 5.22 5.26

如果两个割点相连，那么联通块怎么划分
联通快呢）

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点割集，边割集

Menger's Theorem

(Theorem 5.16; Theorem 5.21)

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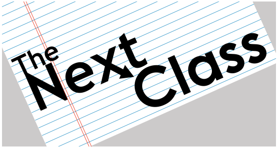

2-Connectivity (Problem 5.10)

A connected graph G with $m \geq 2$ is *nonseparable*

$$\Longleftrightarrow$$

any two *adjacent* edges of G lie on a common cycle of G.

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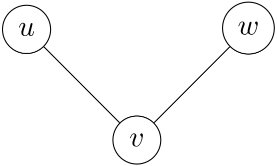
"$\Longrightarrow$"

G is nonseparable

$\Longrightarrow$ u, w lie on a common cycle



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A connected graph G with $m \geq 2$ is *nonseparable*

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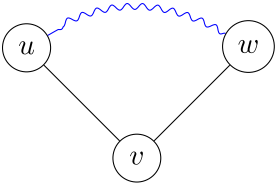

G is nonseparable

$\Longrightarrow$ u, w lie on a common cycle

$\Longrightarrow$ $\exists$ path $u \sim w$

$\Longrightarrow$ $\exists$ cycle $u - v - w \sim u$

$\square$

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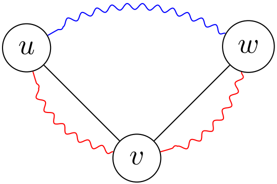

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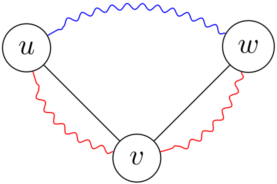
$\implies$ u, w lie on a common cycle

$\implies$ $\exists$ path $u \sim w$ that does not contain v

$\implies$ $\exists$ cycle $u - v - w \sim u$



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By Contradiction.

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By Contradiction.

Suppose v is a cut-vertex of G

$\implies$ $G - v$ contains ≥ 2 comps $G_1, G_2, \cdots$

$\implies$ $\exists u \in G_1, w \in G_2 : v - u \wedge v - w$

$\implies$ $v - u, v - w$ lie on a common cycle

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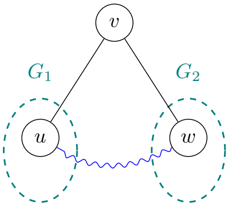
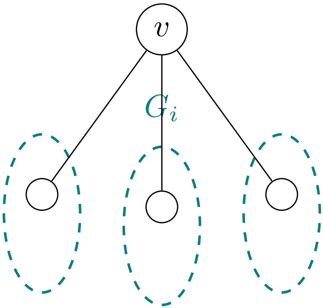
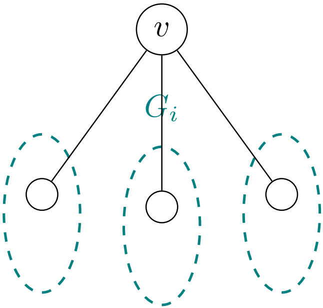
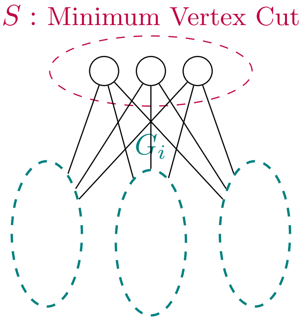




$$\forall G_i \ \exists v_i \in G_i \ : v - v_i$$





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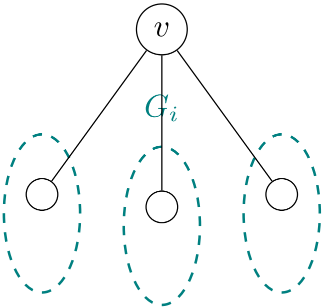

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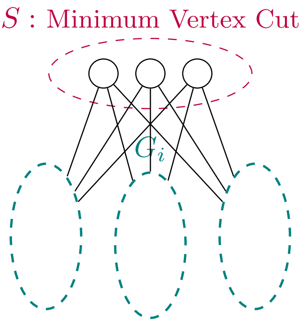

$\forall v \in S \; \forall G_i \; \exists v_i \in G_i \;\; : v - v_i$

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2-Connectivity (Extended Problem)

A connected graph G with $m \geq 2$ is *nonseparable*

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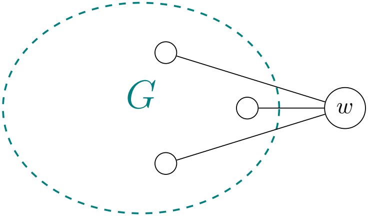
Expansion Lemma (Problem 5.34; Theorem 5.18)

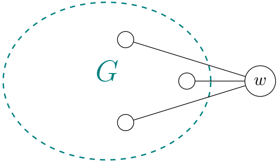
Let G be a k-connected graph and let S be any set of k vertices.

If a graph H is obtained from G by adding a new vertex w and joining w to the vertices of S, then H is also k-connected.



We prove that

$$\forall v \in V(G) : \text{ there exist } k \text{ internally disjoint } v - w \text{ paths}$$



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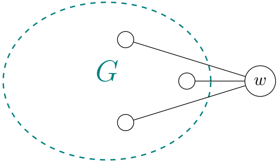

This holds because

$$\forall v \in V(G): \text{ there exist internally disjoint } v - s_i \ (\forall s_i \in S) \text{ paths}$$

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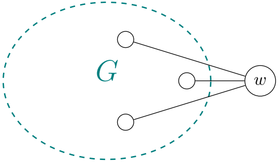

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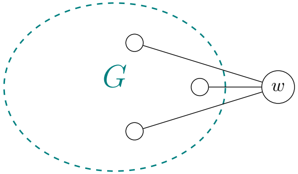
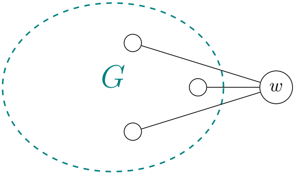
Corollary (5.19; Proved using Theorem 5.18)

If G is a k-connected graph and $u, v_1, v_2, \cdots, v_k$ are $k+1$ distinct vertices of G, then there exist internally disjoint $u - v_i$ paths $(1 \leq i \leq k)$ in G.



To prove $\kappa(H) \geq k$



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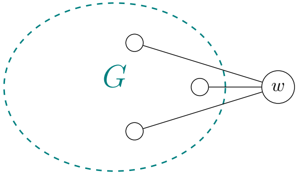
Let U be a vertex-cut of H.

We prove that $|U| \geq k$.



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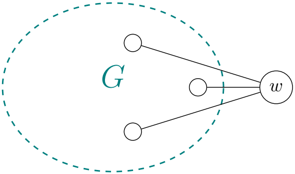

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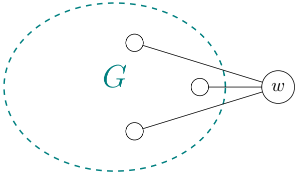

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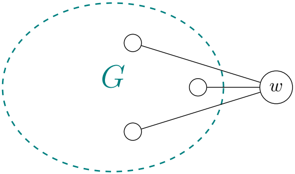

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$w \in U$

$U - w$ is a vertex-cut of G

$(\because U$ is a vertex-cut of $H)$

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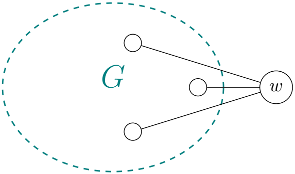

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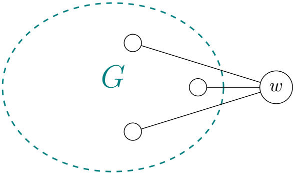
($\because U$ is a vertex-cut of H)

$|U| \geq k + 1$



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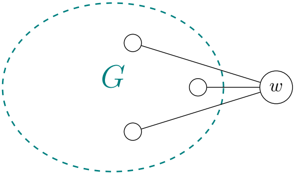

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$\implies U \subseteq G$

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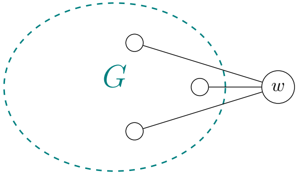

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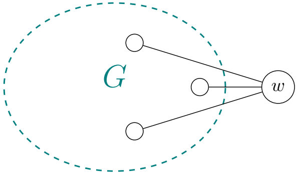

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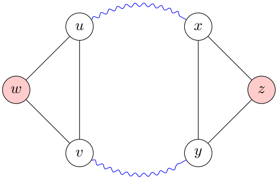

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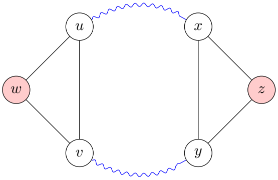
Consider two edges uv and xy.

Add w, z

Add $wu, wv; zx, zy$

w and z lie on a common cycle



Effects of Removing an Edge on Connectivity (Problem 5.22 (a))

(a) If G is k-connected and $e = uv \in E(G)$, then $G - e$ is $(k-1)$-connected.

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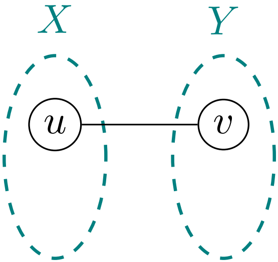
X
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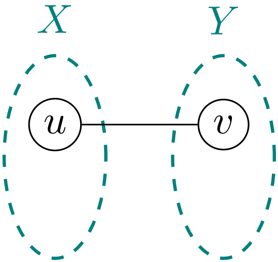
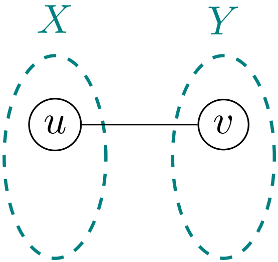


CASE I : $|X| \geq 2 \vee |Y| \geq 2$



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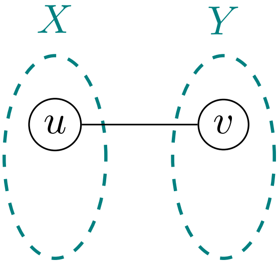

Case II : $|X| = |Y| = 1$

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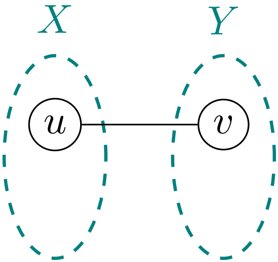

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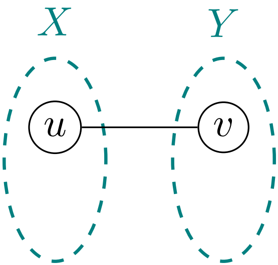

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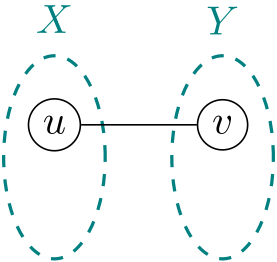

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But $\left| U \cup \{u\} \right| < k$

CASE II : $|X| = |Y| = 1$

$|U| = n - 2 < k - 1$

$\kappa(G) \geq k > n - 1$

But $0 \leq \kappa(G) \leq n - 1$

Effects of Removing an Edge on Connectivity (Problem 5.22 (b))

(b) If G is k-edge-connected and $e = uv \in E(G)$, then $G - e$ is $(k-1)$-edge-connected.

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$$\text{Is } \kappa(G - v) \le \kappa(G)?$$

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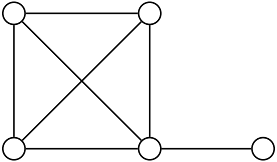

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$$\text{Is } \kappa(G - v) \le \kappa(G)?$$

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Effects of Removing a Vertex on Connectivity (After-class Exercise)

$$\text{Is } \kappa(G) \ge k \implies \kappa(G - v) \ge k - 1?$$

$$\text{Is } \lambda(G) \ge k \implies \lambda(G - v) \ge k - 1?$$

Degree Condition for $\lambda(G) = \delta(G)$ (Problem 5.26)

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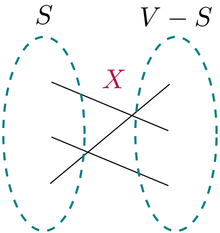

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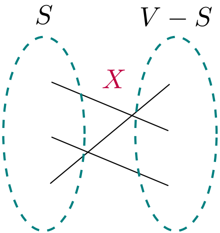

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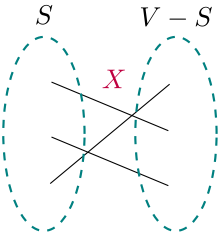

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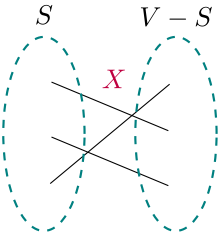

$$\lambda(G) = |X|$$

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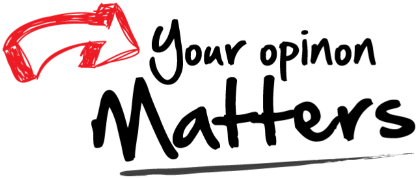
$$\lambda \geq k\Big(\delta - (k-1)\Big) \geq \delta$$

Decision	Author(s)	Year	Complexity	Comments
Edge Connectivity				
$\lambda = 2$ or $\lambda = 3$	Tarjan [26]	1972	$O(m + n)$	uses Depth First Search
λ	Even and Tarjan [6]	1975	$O(mn \times \min\{m^{1/2}, n^{2/3}\})$	n calls to max-flow
λ (digraphs)	Schnorr [25]	1979	$O(\lambda mn)$	n calls to max-flow
λ	Esfahanian & Hakimi [3]	1984	$O(\lambda mn)$	$\leq n/2$ calls to max-flow
λ (digraphs)	Esfahanian & Hakimi [3]	1984	$O(\lambda mn)$	$\leq n/2$ calls to max-flow
λ	Matula [23]	1987	$O(mn)$	uses dominating sets
$\lambda = k$	Matula [23]	1987	$O(kn^2)$	
λ (digraphs)	Mansour & Schieber [22]	1989	$O(mn)$	
$\lambda = k$	Gabow [9]	1991	$O(m+k^2 n\log(n/k))$	uses matroids
Vertex Connectivity				
$\kappa = 2$	Tarjan [26]	1972	$O(m + n)$	uses Depth First Search
$\kappa = 3$	Hopcroft & Tarjan [18]	1973	$O(m + n)$	uses triconnected components
κ	Even & Trajan [6]	1975	$O((\kappa(n - \delta - 1)mn^{2/3})$	max-flow based
$\kappa = k$	Even [4]	1975	$O(kn^3)$	max-flow based
κ	Galil [12]	1980	$O(\min\{\kappa, n^{2/3}\}mn)$	max-flow based
$\kappa = k$	Galil [12]	1980	$O(\min\{k, n^{1/2}\}kmn)$	max-flow based
κ	Esfahanian & Hakimi [3]	1984	$O((n - \delta - 1 + \delta(\delta - 1)/2)mn^{2/3})$	max-flow based
$\kappa = 4$	Kanevsky & Ramachandran [20]	1991	$O(n^2)$	
κ	Henzinger & Rao [17]	1996	$O(\kappa mn\log n)$	randomised algorithm

Table 1: A chronology of connectivity algorithms


THANK YOU



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