

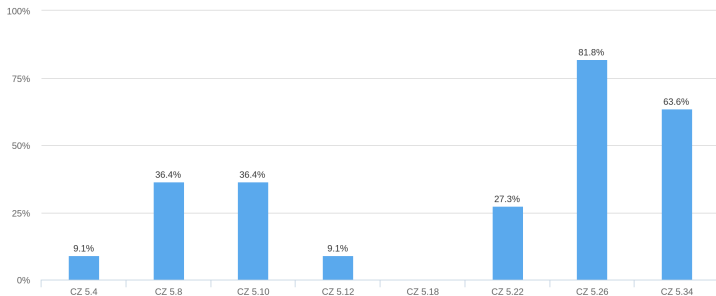
## 3-9 Connectivity

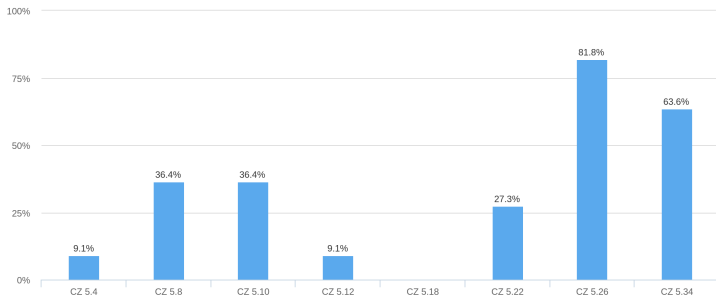
Hengfeng Wei

hfwei@nju.edu.cn

November 26, 2018







5.10

5.34

5.22

5.26

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如果两个割点相连，那么联通块怎么划分！  
(联通快呢)

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menger定理吧

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Menger定理的证明看不懂

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## Menger's Theorem (Theorem 5.16; Theorem 5.21)

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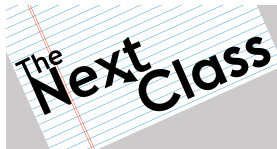
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## Menger's Theorem (Theorem 5.16; Theorem 5.21)



## 2-Connectivity (Problem 5.10)

A connected graph  $G$  with  $m \geq 2$  is *nonseparable*



any two *adjacent* edges of  $G$  lie on a common cycle of  $G$ .

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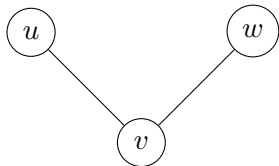
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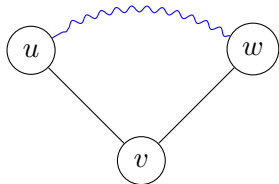
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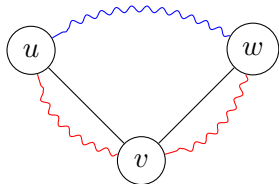
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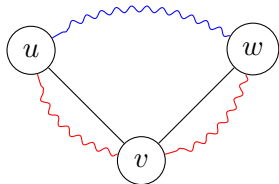
“  $\implies$  ”

$G$  is nonseparable

$\implies u, w$  lie on a common cycle

$\implies \exists$  path  $u \sim w$  that does not contain  $v$

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By Contradiction.

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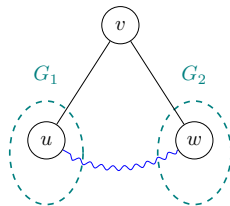
Suppose  $v$  is a cut-vertex of  $G$

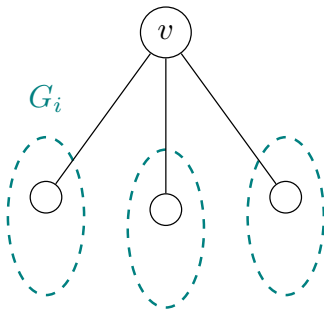
$\implies G - v$  contains  $\geq 2$  comps  $G_1, G_2, \dots$

$\implies \exists u \in G_1, w \in G_2 : v - u \wedge v - w$

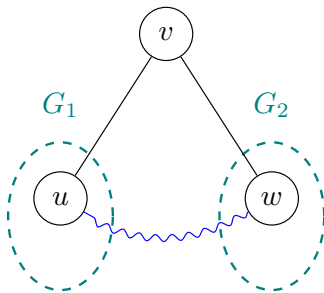
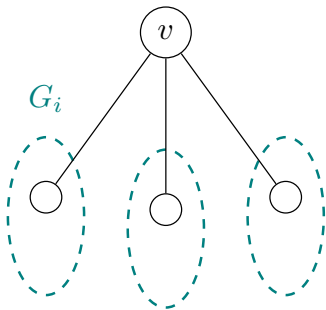
$\implies v - u, v - w$  lie on a common cycle

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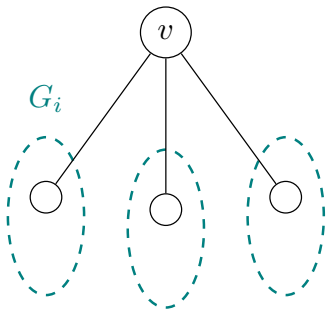




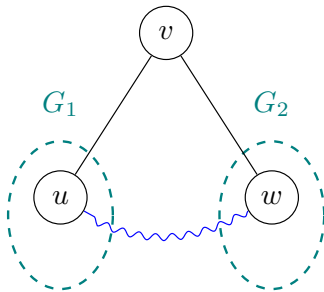
$$\forall G_i \exists v_i \in G_i \ v - v_i$$



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$$\forall v \in S \ \forall G_i \exists v_i \in G_i \ v - v_i$$



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## 2-Connectivity (Extended Problem)

A connected graph  $G$  with  $m \geq 2$  is *nonseparable*

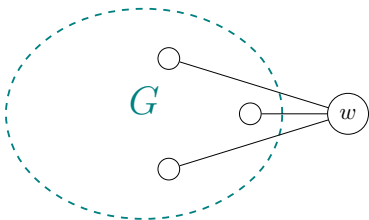


any two edges of  $G$  lie on a common cycle of  $G$ .

## Expansion Lemma (Problem 5.34; Theorem 5.18)

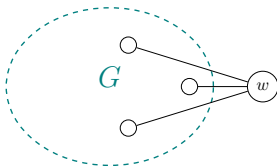
Let  $G$  be a  $k$ -connected graph and let  $S$  be any set of  $k$  vertices.

If a graph  $H$  is obtained from  $G$  by adding a new vertex  $w$  and joining  $w$  to the vertices of  $S$ , then  $H$  is also  $k$ -connected.



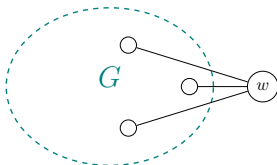
We need to prove that

$\forall v \in V(G) : \text{there exist } k \text{ internally disjoint } v - w \text{ paths}$



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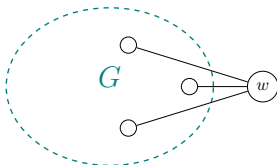


This holds because

$\forall v \in V(G) : \text{there exist internally disjoint } v - s_i \text{ } (\forall s_i \in S) \text{ paths}$

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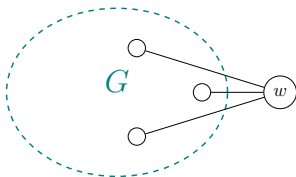


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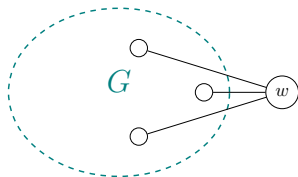
$\forall v \in V(G) : \text{there exist internally disjoint } v - s_i \ (\forall s_i \in S) \text{ paths}$

**Corollary (5.19; Proved using Theorem 5.18)**

*If  $G$  is a  $k$ -connected graph and  $u, v_1, v_2, \dots, v_k$  are  $k + 1$  distinct vertices of  $G$ , then there exist internally disjoint  $u - v_i$  paths ( $1 \leq i \leq k$ ) in  $G$ .*



To prove  $\kappa(H) \geq k$

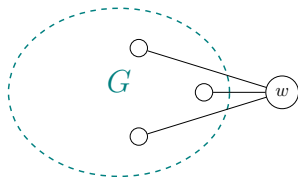


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Let  $U$  be a vertex-cut of  $H$ .

We prove that  $|U| \geq k$ .





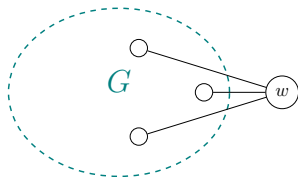
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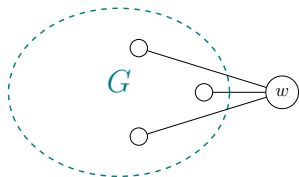
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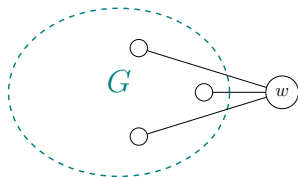
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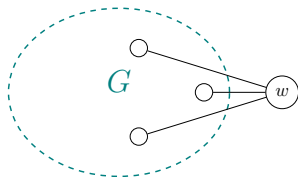
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$$|U| \geq k + 1$$

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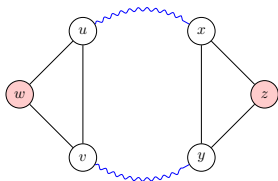
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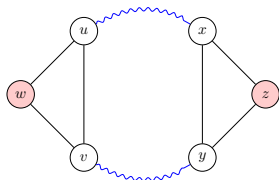


Consider two edges  $uv$  and  $xy$ .

Add  $w, z$

Add  $wu, wv; zx, zy$

$w$  and  $z$  lie on a common cycle



## Effects of Removing an Edge on Connectivity (Problem 5.22 (a))

- (a) If  $G$  is  $k$ -connected and  $e = uv \in E(G)$ , then  $G - e$  is  $(k - 1)$ -connected.

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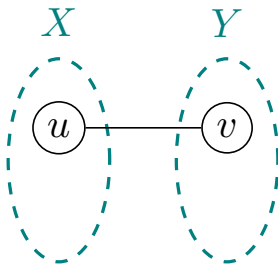
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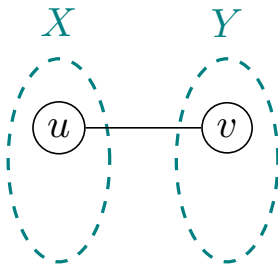
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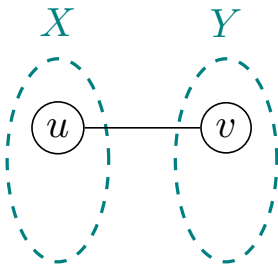
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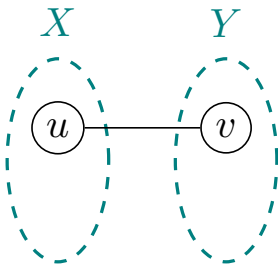
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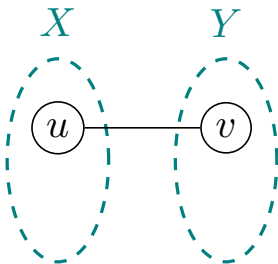
CASE II :  $|X| = |Y| = 1$

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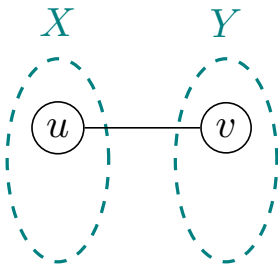
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$|U| = n - 2 < k - 1$



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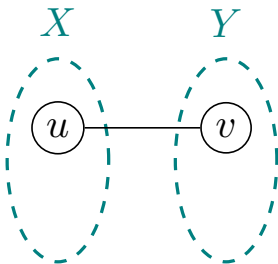
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$\kappa(G) \geq k > n - 1$



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$\kappa(G) \geq k > n - 1$

But  $0 \leq \kappa(G) \leq n - 1$

## Effects of Removing an Edge on Connectivity (Problem 5.22 (b))

- (b) If  $G$  is  $k$ -edge-connected and  $e = uv \in E(G)$ , then  $G - e$  is  $(k - 1)$ -edge-connected.

$$\lambda(G) \geq k \implies \lambda(G - e) \geq k - 1$$

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$$G - e - X = G - (e + X) \text{ is connected}$$

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We prove that  $G - e - X$  is connected.

$G - e - X = G - (e + X)$  is connected ( $\because \lambda(G) \geq k$ )

$$\kappa(G - e) \leq \kappa(G)$$



$$\kappa(G - e) \leq \kappa(G)$$

## Effects of Removing a Vertex on Connectivity (Extended Problem)

Is  $\kappa(G - v) \leq \kappa(G)$ ?

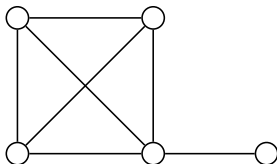
Is  $\lambda(G - v) \leq \lambda(G)$ ?

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### Effects of Removing a Vertex on Connectivity (After-class Exercise)

$$\text{Is } \kappa(G) \geq k \implies \kappa(G - v) \geq k - 1?$$

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## Degree Condition for $\lambda(G) = \delta(G)$ (Problem 5.26)

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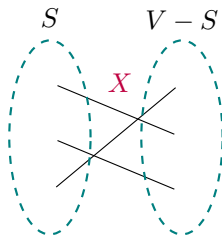
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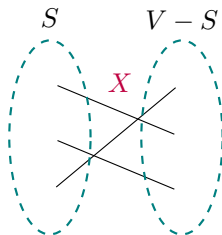
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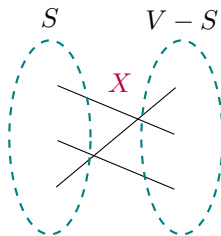
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$$\lambda \geq k(\delta - (k - 1))$$

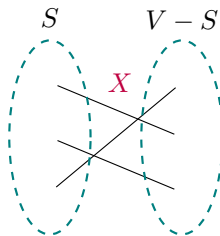


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$$\lambda \geq k(\delta - (k-1)) \geq \delta$$

Decision	Author(s)	Year	Complexity	Comments
<i>Edge Connectivity</i>				
$\lambda = 2$ or $\lambda = 3$	Tarjan [26]	1972	$O(m + n)$	uses Depth First Search
$\lambda$	Even and Tarjan [6]	1975	$O(mn \times \min\{m^{1/2}, n^{2/3}\})$	$n$ calls to max-flow
$\lambda$ (digraphs)	Schnorr [25]	1979	$O(\lambda mn)$	$n$ calls to max-flow
$\lambda$	Esfahanian & Hakimi [3]	1984	$O(\lambda mn)$	$\leq n/2$ calls to max-flow
$\lambda$ (digraphs)	Esfahanian & Hakimi [3]	1984	$O(\lambda mn)$	$\leq n/2$ calls to max-flow
$\lambda$	Matula [23]	1987	$O(mn)$	uses dominating sets
$\lambda = k$	Matula [23]	1987	$O(kn^2)$	
$\lambda$ (digraphs)	Mansour & Schieber [22]	1989	$O(mn)$	
$\lambda = k$	Gabow [9]	1991	$O(m+k^2n\log(n/k))$	uses matroids
<i>Vertex Connectivity</i>				
$\kappa = 2$	Tarjan [26]	1972	$O(m + n)$	uses Depth First Search
$\kappa = 3$	Hopcroft & Tarjan [18]	1973	$O(m + n)$	uses triconnected components
$\kappa$	Even & Trajan [6]	1975	$O((\kappa(n - \delta - 1)mn^{2/3}))$	max-flow based
$\kappa = k$	Even [4]	1975	$O(kn^3)$	max-flow based
$\kappa$	Galil [12]	1980	$O(\min\{\kappa, n^{2/3}\}mn)$	max-flow based
$\kappa = k$	Galil [12]	1980	$O(\min\{k, n^{1/2}\}kmn)$	max-flow based
$\kappa$	Esfahanian & Hakimi [3]	1984	$O((n - \delta - 1 + \delta(\delta - 1)/2)mn^{2/3})$	max-flow based
$\kappa = 4$	Kanevsky & Ramachandran [20]	1991	$O(n^2)$	
$\kappa$	Henzinger & Rao [17]	1996	$O(\kappa mn \log n)$	randomised algorithm

**Table 1:** A chronology of connectivity algorithms





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