surjective map and cardinality

I work in ZF(without the axiom of choice). Let A, B be sets such that $|A|$ and $|B|$ are both defined and let $f: A \to B$ a surjective function. Can I prove that $|A| \geq |B|$? Or it can't be provable?

(set-theory) (axiom-of-choice)

edited Mar 25 '14 at 22:38
mle
1,596 ● 1 ■ 10 ▲ 27

asked Mar 25 '14 at 20:04
andreasvr
357 ■ 1 ▲ 11

2 Answers

If A is well-orderable, then the answer is yes.

One classic example is in models where $\aleph_1 \nleq 2^{\aleph_0}$. But we can still prove in ZF that there is a surjection from $\mathcal{P}(\omega)$ onto ω_1.

Another classic example is when we have an infinite set without a countably infinite subset. In that case we can prove that there is such set which can be mapped onto ω; but by definition there is no injection back.

The assertion that if $f: A \to B$ is surjective then there is $g: B \to A$ injective is known as **The Partition Principle**. It is clearly implied by the axiom of choice, and we can show quite easily that it is not provable in ZF itself (it has quite a lot of consequences which we know are consistent).

However the question whether or not the partition principle implies the axiom of choice is the oldest [still] open question in set theory.

edited Mar 25 '14 at 21:48

answered Mar 25 '14 at 20:31
Asaf Karagila
275k ● 31 ■ 374 ▲ 694

How I can prove that the Partition Principle isn't provable in ZF? Thank you. – andreasvr Mar 25 '14 at 20:36

Exhibiting the consistency of either one of the examples with ZF would suffice. (For the former try the Feferman-Levy model, for the latter any model with an infinite Dedekind-finite set, e.g. Cohen's first model.) – Asaf Karagila Mar 25 '14 at 20:41

@Andres: Yes. Sleep deprivation does that. Thank you. – Asaf Karagila Mar 25 '14 at 21:49

If A and B are finite, they yes, via the Pigeonhole Principle.

For infinite sets, see Dedekind Infinite Sets in ZF and the summary at the top of that article.

answered Mar 25 '14 at 20:19
Eric Towers
24.1k ● 2 ■ 17 ▲ 51

Sorry, but in your link I don't find the answer at my question. Could you be more precise? – andreasvr Mar 25 '14 at 20:31