4-13 Randomized Algorithms

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1/2

Definition (ZPP: Zero-error Probabilistic Polynomial Time)

$$L \in ZPP$$

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$$\exists A \; (probabilistic\ polynomial\text{-}time\ algorithm):$$

$$Pr\Big(A(x) = L(x)\Big) \geq \frac{1}{2}$$

$$Prob\Big(A(x) = ?\Big) = 1 - Pr\Big(A(x) = L(x)\Big) \leq \frac{1}{2}$$

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Definition (RP: Randomized Polynomial time (One-Sided Error))

$$L \in RP$$

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$$x \in L \implies Pr\Big(A(x) = 1\Big) \geq \frac{1}{2}$$

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$$\exists A \; (\textit{probabilistic polynomial-time algorithm}):$$

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$$L \in BPP_{1-\epsilon} \implies k \geq \frac{2\ln 2\epsilon}{\ln(1-4\delta^2)}$$

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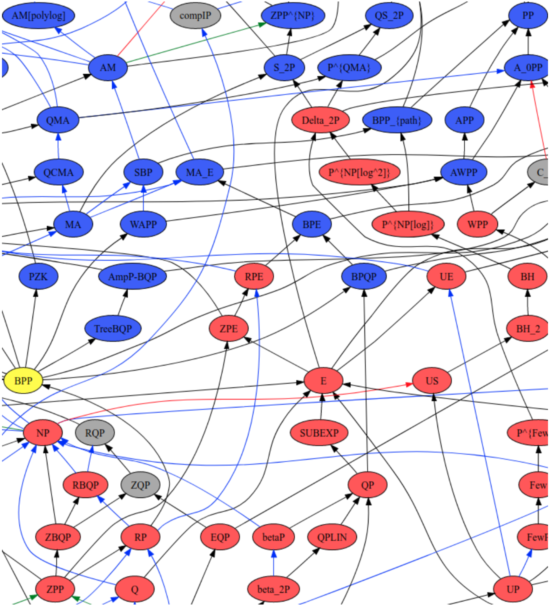
$$Pr\Big(A(x) = L(x)\Big) \geq \frac{1}{2} + \frac{1}{2^{n^c}} \text{ for some constant } c$$


AM[polylog]
compIP
ZPP^{NP}
QS_2P
PP
AM
S_2P
P^{QMA}
A_0PP
QMA
Delta_2P
BPP_{path}
APP
QCMA
SBP
MA_E
P^{NP[log^2]}
AWPP
MA
WAPP
BPE
P^{NP[log]}
WPP
PZK
AmpP-BQP
RPE
BPQP
UE
BH
TreeBQP
ZPE
BH_2
BPP
E
US
NP
RQP
SUBEXP
RBQP
ZQP
QP
ZBQP
RP
EQP
betaP
QPLIN
ZPP
Q
beta_2P
UP

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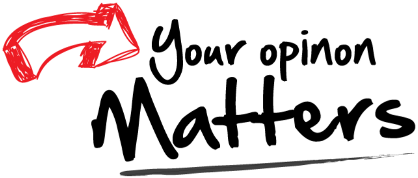
Exercise 5.2.2.9


THANK YOU



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