

## 2-8 Probabilistic Analysis

*"No Expectation, No Disappointment."*

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## Definition (Expectation)

$$\mathbb{E}[X] = \sum_x x \Pr(X = x)$$

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Let  $X$  be a discrete random variable that takes on *only nonnegative integer values*  $\mathbb{N}$ .

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## Proof.

$$\sum_{j=1}^{\infty} \sum_{i=1}^j \Pr(X = j) = \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} \Pr(X = j)$$



## Searching an Unsorted Array (CLRS Problem 5 – 2 (f))

---

```
1: procedure DETERMINISTIC-SEARCH( $A[1 \cdots n]$ ,  $x$ )
2:    $i \leftarrow 1$ 
3:   while  $i \leq n$  do
4:     if  $A[i] = x$  then
5:       return true
6:      $i \leftarrow i + 1$ 
7:   return false
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(e)

$$\exists! i : A[i] = x$$

(f)

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$$\sum_{i=1}^{n-k+1} i \binom{n-i}{k-1} = \binom{n+1}{k+1}$$



Summation by parts ([Abel transformation](#); [wiki](#))

How Did I (an ant) Evaluate this Summation:

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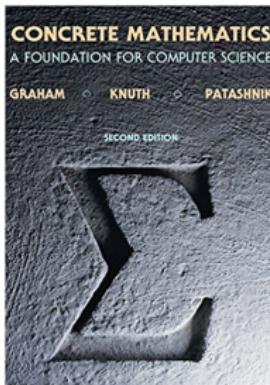
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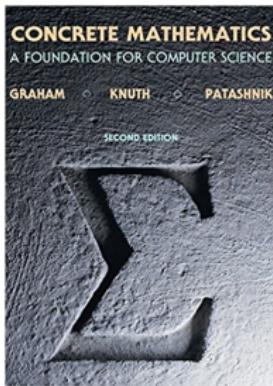
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## Chapter 5: Binomial Coefficients

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$$r \binom{r-1}{k-1} = k \binom{r}{k}$$

$$\sum_{0 \leq k \leq n} \binom{k}{m} = \binom{n+1}{m+1}$$

## Chapter 5: Binomial Coefficients

$$\begin{aligned}
\sum_{i=1}^{n-k+1} i \binom{n-i}{k-1} &= \sum_{i=0}^{n-k} (i+1) \binom{n-i-1}{k-1} \\
&= \sum_{i=0}^{n-k} ((n+1) - (n-i)) \binom{n-i-1}{k-1} \\
&= \sum_{i=0}^{n-k} (n+1) \binom{n-i-1}{k-1} - \sum_{i=0}^{n-k} (n-i) \binom{n-i-1}{k-1} \\
&= (n+1) \sum_{i=0}^{n-k} \binom{n-i-1}{k-1} - k \sum_{i=0}^{n-k} \binom{n-i}{k} \\
&= (n+1) \sum_{m=k-1}^{n-1} \binom{m}{k-1} - k \sum_{m=k}^n \binom{m}{k} \\
&= (n+1) \binom{n}{k} - k \binom{n+1}{k+1} = \binom{n+1}{k+1}
\end{aligned}$$

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$$I_i = \begin{cases} 1, & \text{if } A[i] \text{ is checked} \\ 0, & \text{o.w.} \end{cases}$$

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$$\mathbb{E}[Y] = \sum_{i=1}^n \Pr \{ I_i = 1 \} = k \cdot \frac{1}{k} + (n - k) \cdot \frac{1}{k+1} = \frac{n+1}{k+1}$$





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## Hat-check Problem (CLRS Problem 5.2 – 4)



$X$  : # of customers who get back their own hat

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## Inversions (CLRS Problem 5.2 – 5)

$A[1 \cdots n]$  of  $n$  distinct numbers

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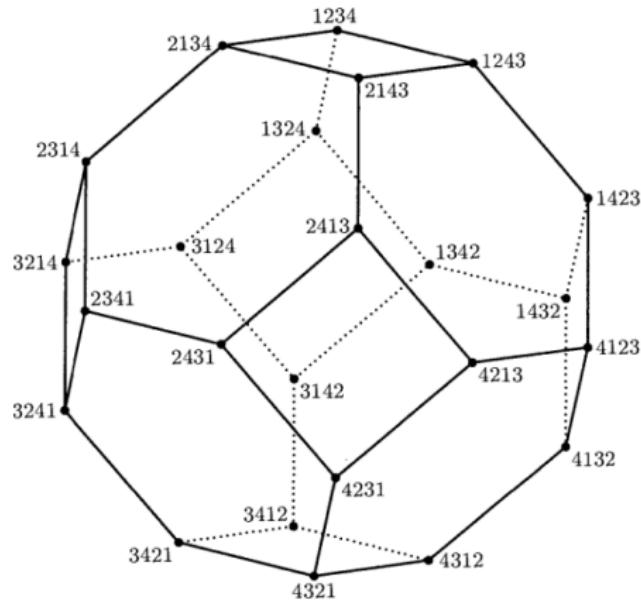
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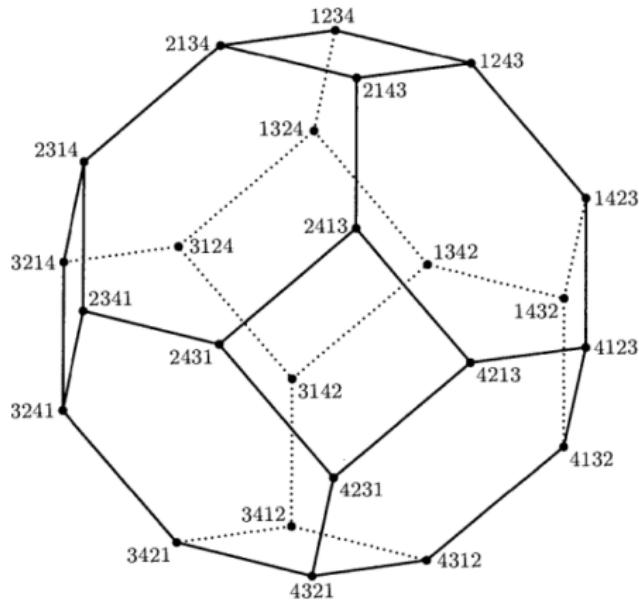
$$X = \sum_{i=1}^{n-1} \sum_{j>i} I_{ij} \quad \mathbb{E}[I_{ij}] = \Pr((i, j) \text{ is an inversion}) = \frac{1}{2}$$

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$$\langle 3214 \rangle \sim \langle 4123 \rangle$$

## Definition (Conditional Expectation on an Event)

$$\mathbb{E}[X | E] = \sum_x x \Pr(X = x | E)$$

$$\left( \mathbb{E}[X] = \sum_x x \Pr(X = x) \right)$$

## Theorem (CS Theorem 5.23))

Let  $X$  be a random variable defined on a sample space  $\Omega$ .

Let  $E_1, E_2, \dots, E_n$  be a **partition** of  $\Omega$ .

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## Theorem (The Law of Total Expectation (CS Theorem 5.23))

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Proof.

By definition.

$$\sum_x x \sum_{i=1}^n \Pr(X = x, E_i) = \sum_{i=1}^n \sum_x x \Pr(X = x, E_i)$$



## (#) Rational Person Playing a Card Game (CS Problem 5.6 – 4)



*A* : \$1.00; Repeat

*J* : \$2.00; End

*K* : \$3.00; End

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Conditioning on the **first** draw  $c$

$$\mathbb{E}[X] = \frac{1}{4} (\mathbb{E}[X \mid c = A] + \mathbb{E}[X \mid c = J] + \mathbb{E}[X \mid c = K] + \mathbb{E}[X \mid c = Q])$$

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$$4 * A + 1 * Q = \$8.00$$

## In-class Exercise: Coin Pattern (Provided by Yifan Pei)



$X$  : # of tosses to get 3 consecutive heads ( $HHH$ )

$$\mathbb{E}[X]$$

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$T$ ,     $HT$ ,     $HHT$ ,     $HHH$

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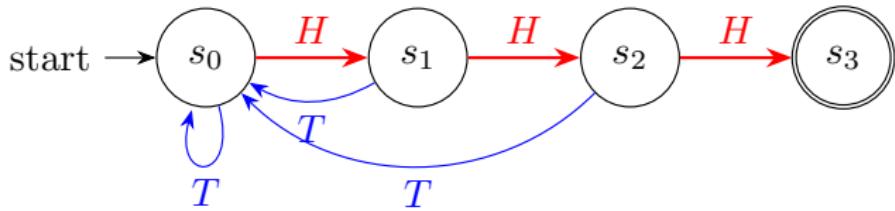
Conditioning on the first 3 tosses

$T, \quad HT, \quad HHT, \quad HHH$

$$\mathbb{E}[X] = \frac{1}{2}(\mathbb{E}[X] + 1) + \frac{1}{4}(\mathbb{E}[X] + 2) + \frac{1}{8}(\mathbb{E}[X] + 3) + \frac{1}{8} \times 3$$

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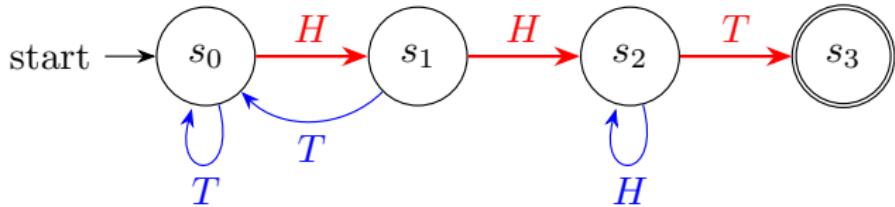
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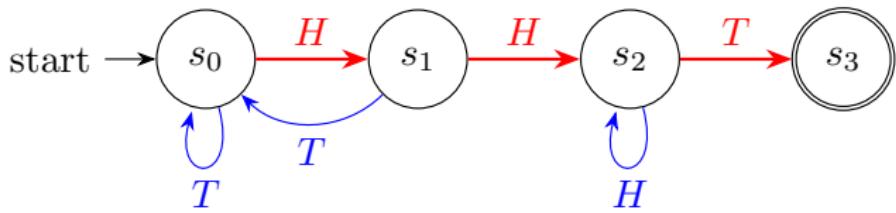
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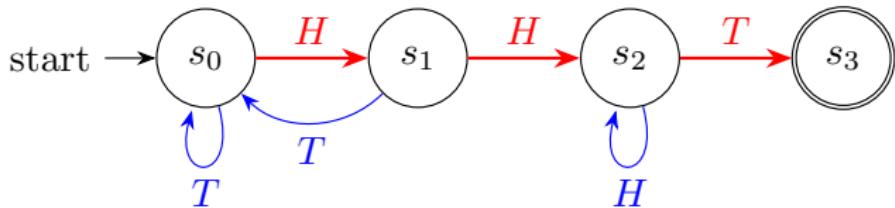
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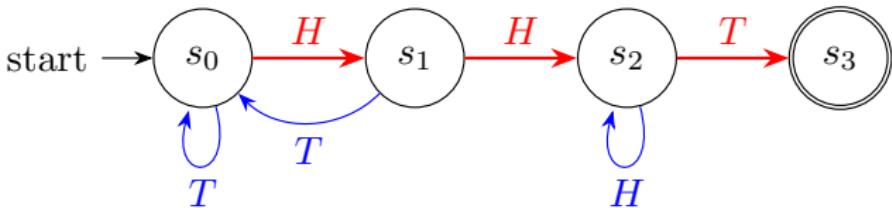
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$S_i$  : Expected number of tosses from state  $s_i$  to reach state  $s_n$



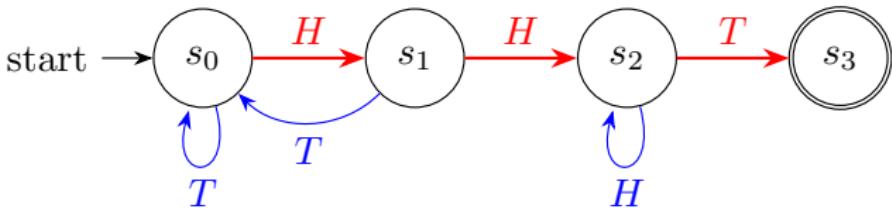
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$$S_0 = \frac{1}{2}(1 + S_0 + 1 + S_1)$$

$$S_1 = \frac{1}{2}(1 + S_0 + 1 + S_2)$$

$$S_2 = \frac{1}{2}(1 + S_2 + 1 + S_3)$$

$$S_3 = 0$$



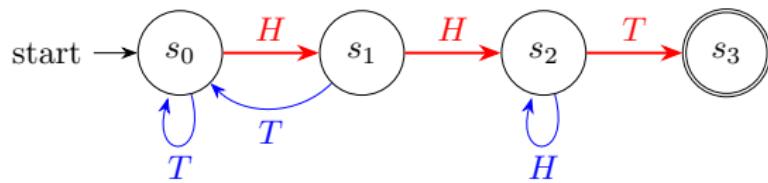
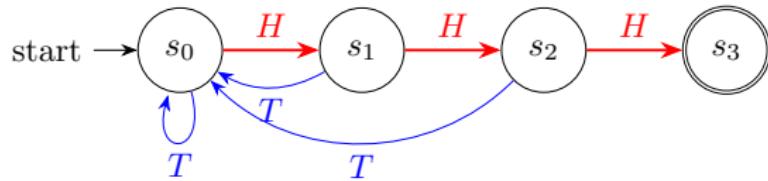
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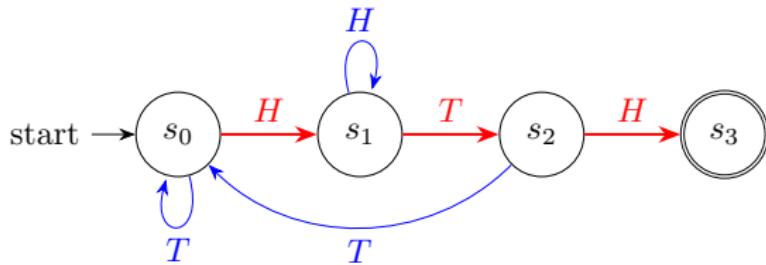
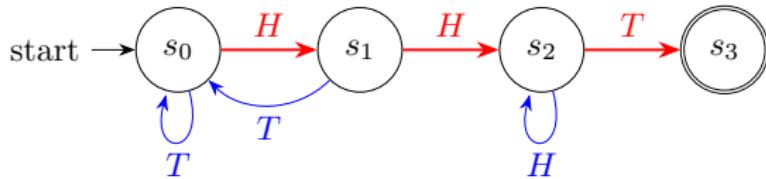
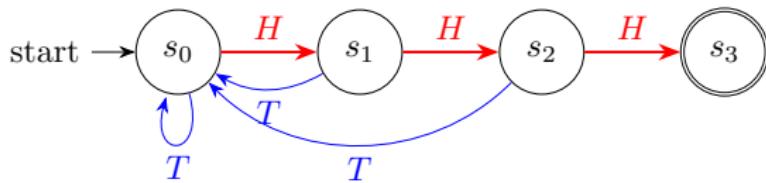
$$S_1 = \frac{1}{2}(1 + S_0 + 1 + S_2) \quad S_0 = 8$$

$$S_2 = \frac{1}{2}(1 + S_2 + 1 + S_3)$$

$$S_3 = 0$$

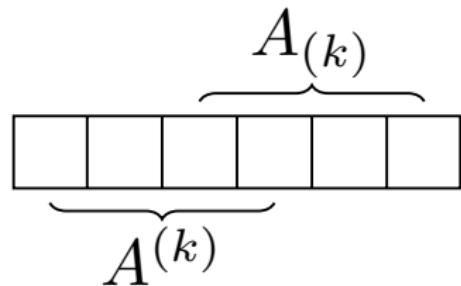


$$\mathbb{E}[X_{HHH}] = 14 > \mathbb{E}[X_{HHT}] = 8$$

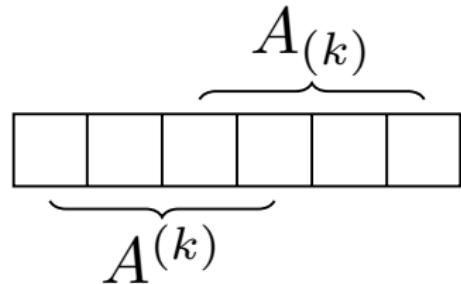


$$A : A = \sum_{k=1}^n 2^{k-1} [A^{(k)} = A_{(k)}] \quad \mathbb{E}[X_A] = 2(A : A)$$

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$$A = THHTTH \quad \mathbb{E}[X_A] = 2(2^1 + 2^4) = 36$$

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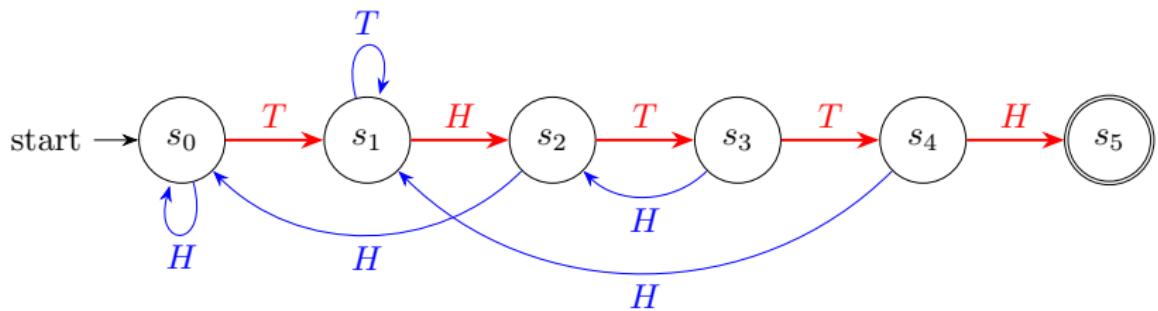
$$\mathbb{E}[X_{H^n}] = 2(2^0 + 2^1 + 2^{n-1}) = 2(2^n - 1)$$

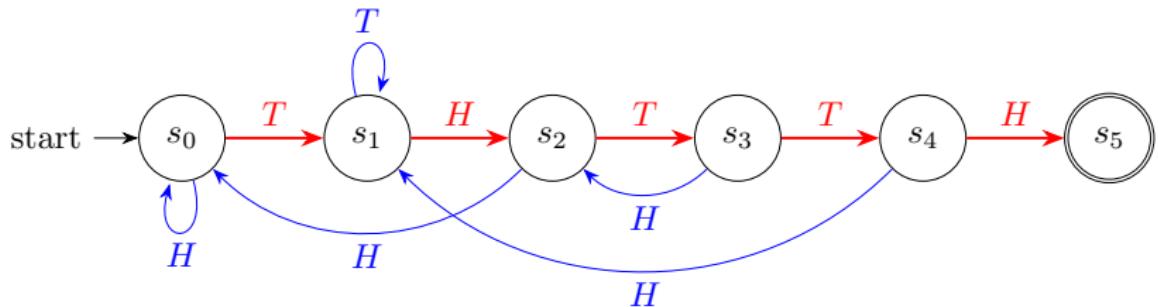
$$A : A = \sum_{k=1}^n 2^{k-1} [A^{(k)} = A_{(k)}] \quad \mathbb{E}[X_A] = 2(A : A)$$

$$A = HHH \quad A = HHT$$

$$\mathbb{E}[X_{H^n}] = 2(2^0 + 2^1 + 2^{n-1}) = 2(2^n - 1)$$

$$\mathbb{E}[X_{H^{n-1}T}] = 2(2^{n-1}) = 2^n$$





$$S_0 = \frac{1}{2}(1 + S_0 + 1 + S_1)$$

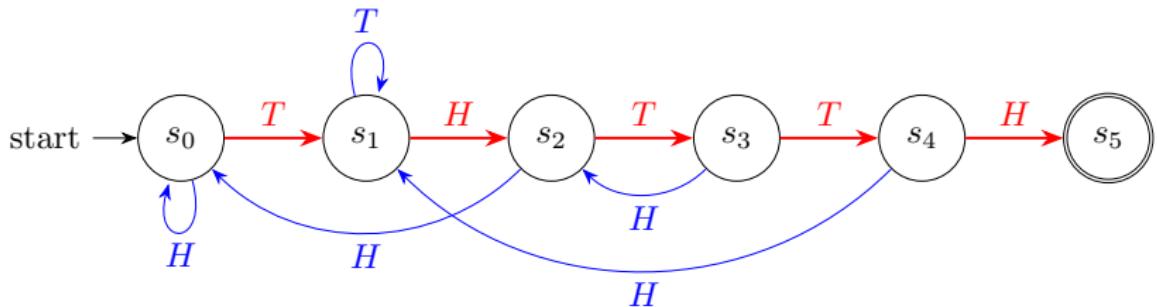
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$$S_0 = \frac{1}{2}(1 + S_0 + 1 + S_1)$$

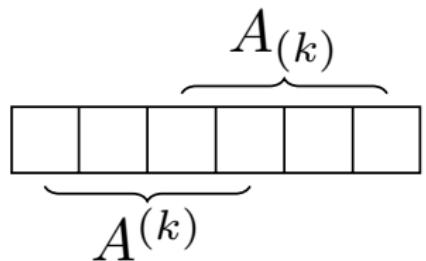
$$S_1 = \frac{1}{2}(1 + S_1 + 1 + S_2)$$

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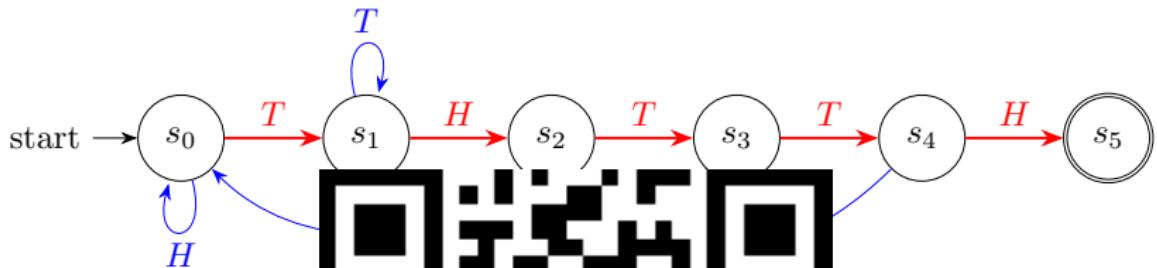
$$S_3 = \frac{1}{2}(1 + S_2 + 1 + S_4)$$

$$S_4 = \frac{1}{2}(1 + S_1 + 1 + S_5)$$

$$S_5 = 0$$



$$2 \sum_{k=1}^n 2^{k-1} [A^{(k)} = A_{(k)}]$$



$$S_0 = \frac{1}{2}(1 + S_1)$$

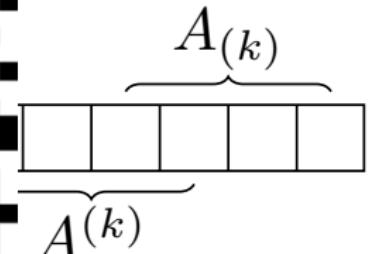
$$S_1 = \frac{1}{2}(1 + S_0 + S_2)$$

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$$S_4 = \frac{1}{2}(1 + S_1 + 1 + S_5)$$

$$S_5 = 0$$



$$2 \sum_{k=1}^n 2^{k-1} [A^{(k)} = A_{(k)}]$$

## Definition (Conditional Expectation on an Event)

$$\mathbb{E}[X | E] = \sum_x x \Pr(X = x | E)$$

## Definition (Conditional Expectation on an Event)

$$\mathbb{E}[X | E] = \sum_x x \Pr(X = x | E)$$

## Definition (Conditional Expectation on a Random Variable)

$$\mathbb{E}[X | Y = y] = \sum_x x \Pr(X = x | Y = y)$$

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Notation:

$$\mathbb{E}[X | Y](y) = \mathbb{E}[X | Y = y]$$

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Theorem

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X | Y]] = \sum_y \mathbb{E}[X | Y = y] \Pr(Y = y)$$





# Hints & Tips



There are  $n$  bins labelled with the numbers  $1, 2, \dots, n$ . Balls are placed in these bins one after the other, with the bin into which a ball is placed being independent random variables that assume the value  $k$  with probability  $p_k$ . Let  $X$  be the number of balls placed so that there is at least one ball in every bin.

- (a) Assume that  $p_k = \frac{1}{n}$ . What is the expectation of  $X$ ?
- (b) Assume that  $p_k = \frac{1}{n}$ . What is the probability distribution of  $X$ ?
- (c) Prove that  $\Pr(X > n \ln n + cn) \leq e^{-c}$ ,  $\Pr(X < n \ln n - cn) \leq e^{-c}$ .
- (d) Redo (a) and (b) without the assumption  $p_k = \frac{1}{n}$ .
- (e) Given a deck of  $n$  cards, each time you take the top card from the deck, and insert it into the deck at one of the  $n$  distinct possible places, each of them with probability  $\frac{1}{n}$ . What is the expected times for you to perform the procedure above until the bottom card rises to the top?

# The Coupon Collector's Problem

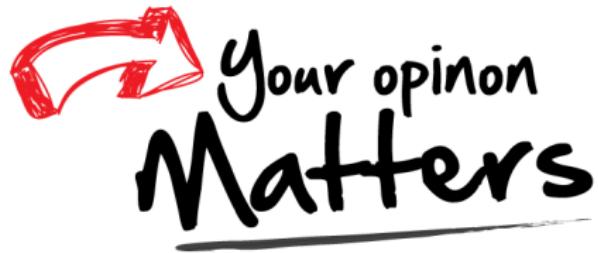


# Shuffling Cards



Chapter “Shuffling Cards” of “Proofs from THE Book”

# Thank You!



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