1-12 Partial Order and Lattice

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SM Problem 14.44: Consistent Enumerations

Suppose the following are three consistent enumerations of an ordered set $A = \{a, b, c, d\}$:

$$
\begin{array}{lcccc}
A_1 : & a & b & c & d \\
A_2 : & a & c & b & d \\
A_3 : & a & c & d & b
\end{array}
$$

Assuming the Hasse diagram D of A is connected, draw D.

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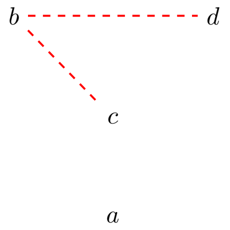
Assuming the Hasse diagram D of A is connected, draw D.

$$b \prec_{A_1} c \land c \prec_{A_2} b \implies b \parallel_A c$$

$$d \prec_{A_3} b \land b \prec_{A_2} d \implies b \parallel_A d$$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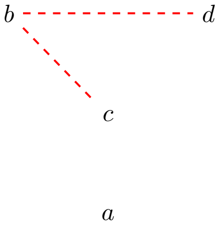
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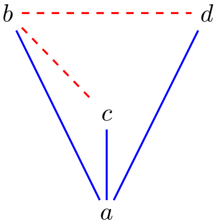
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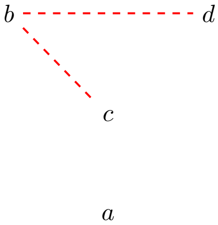

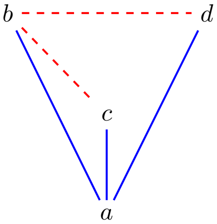

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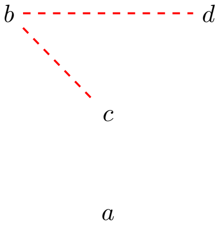

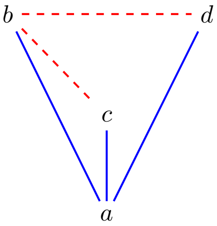

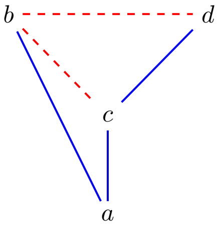

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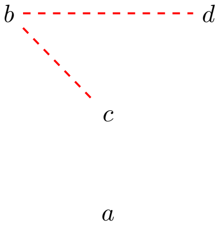

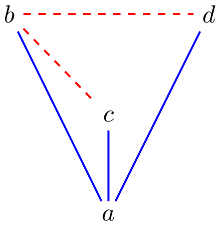

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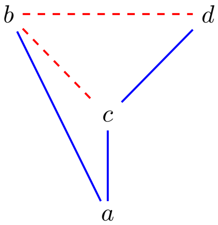
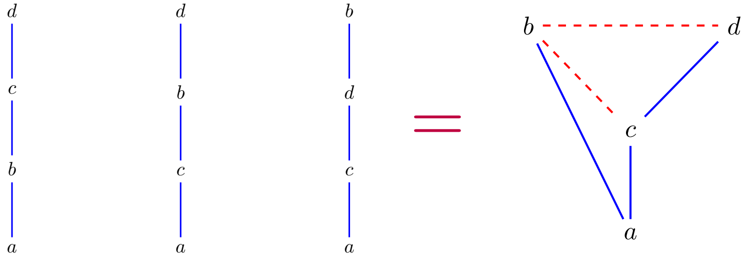


$\# = 3$



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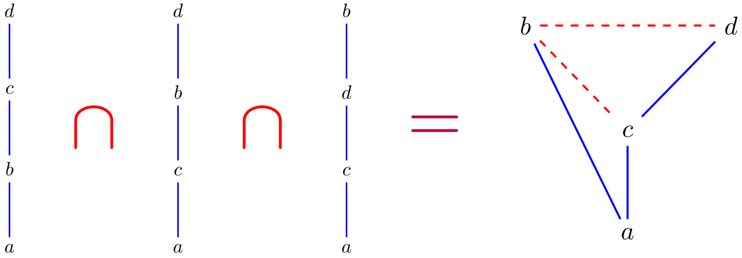
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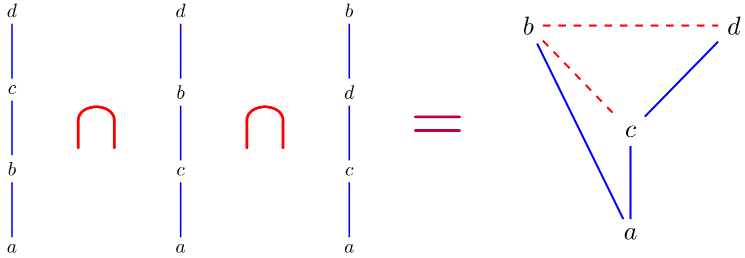

Theorem

Every partial ordering on a set X is the intersection of the total orders on X containing it.

Definition

Lattice A *lattice* is an algebra $\mathcal{L} = (L, \wedge, \vee)$ satisfying,

$$\forall a, b, c \in L,$$

Idempotency:

$$a \wedge a = a \quad a \vee a = a$$

Commutativity:

$$a \wedge b = b \wedge a \quad a \vee b = b \vee a$$

Associativity:

$$(a \wedge b) \wedge c = a \wedge (b \wedge c) \quad (a \vee b) \vee c = a \vee (b \vee c)$$

Absorption:

$$a \wedge (a \vee b) = a \quad a \vee (a \wedge b) = a$$

$$\boxed{a \wedge (a \vee b) = a \quad a \vee (a \wedge b) = a}$$

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(1) Very useful in lattice computations

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$$a \wedge a = a \wedge (a \vee (a \wedge b)) = a$$

(2) The only laws connecting $\wedge$ and $\vee$

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$$a \leq b \iff a \wedge b = a$$
$$a \leq b \iff a \vee b = b$$

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SM Problem 14.72: "Weak" Distributive Laws

Prove that for any lattice L:

$$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$$
$$a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$$

SM Problem 14.72: "Weak" Distributive Laws

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$$a \vee (b \wedge c) \leq a \vee b$$
$$a \vee (b \wedge c) \leq a \vee c$$

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$$a \vee (b \wedge c) \leq a \vee b$$
$$a \vee (b \wedge c) \leq a \vee c$$

$$\frac{a \leq b \qquad c \leq d}{(a \vee c) \leq (b \vee d)}$$

2017-1-final-exam (5): Lattice

假设 $(L, \leq)$ 是格。
如果以下模律 (modular law) 成立, 则称 L 是模格 (modular lattice):

$$\forall x \in L : a \leq b \implies a \vee (x \wedge b) = (a \vee x) \wedge b.$$

以下均假设 L 是模格。

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$$\textit{vs.}\ \ a \vee (x \wedge b) = (a \vee x) \wedge (a \vee b)$$

Distributive Law $\implies$ Modular Law

2017-1-final-exam (5): Lattice

$$\forall x \in L : a \le b \implies a \vee (x \wedge b) = (a \vee x) \wedge b.$$

(1) 请证明模律与以下条件等价:

$$\forall x \in L : a \le b \implies a \vee (x \wedge b) \ge (a \vee x) \wedge b.$$

2017-1-final-exam (5): Lattice

$$\forall x \in L : a \le b \implies a \lor (x \land b) = (a \lor x) \land b.$$

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$$\implies$$

$$\Big(\big(a \lor (x \land b) = (a \lor x) \land b\big) \iff \big(a \lor (x \land b) \ge (a \lor x) \land b\big) \Big).$$

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$$\boxed{a \le b \implies a \vee (x \wedge b) \le (a \vee x) \wedge b}$$

$$\boxed{a \leq b \implies a \lor (x \land b) \leq (a \lor x) \land b}$$

$$(a \leq a \lor x) \land (a \leq b) \implies a \leq (a \lor x) \land b \tag{1}$$

$$(x \leq a \lor x) \land b \leq b \implies (x \land b) \leq (a \lor x) \land b \tag{2}$$

2017-1-final-exam (5): Lattice

$$\forall x \in L : a \le b \implies a \vee (x \wedge b) = (a \vee x) \wedge b.$$

(2) 请证明: $\forall a, b, c \in L$,
如果 $c \le a$, $a \wedge b = c \wedge b$, $a \vee b = c \vee b$ 成立, 则 $a = c$.

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$$\text{Modular Law} : [a \leftarrow c] \quad [b \leftarrow a]$$

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$$[x := b]$$

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$$\begin{aligned} c \le a \implies c \lor (b \land a) &= (c \lor b) \land a \\ c \lor (c \land b) &= (a \lor b) \land a \end{aligned}$$

2017-1-final-exam (5): Lattice

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2017-1-final-exam (5): Lattice

$$\forall x \in L : a \le b \implies a \vee (x \wedge b) = (a \vee x) \wedge b.$$

(3) 给定任意元素 $s, t \in L$, 且 $s \le t$, 构造集合 (称为区间 (interval)):

$$[s, t] \triangleq \{x \in L \mid s \le x \le t\}.$$

请证明 $([s, t], \le)$ 是 L 的子格 (sublattice)。

2017-1-final-exam (5): Lattice

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请证明 $([s, t], \leq)$ 是 L 的子格 (sublattice)。

$$a, b \in [s, t] \implies a \vee b, a \wedge b \in [s, t]$$

5. 格 (Lattice)

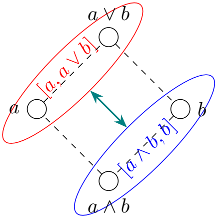
$$\forall x \in L : a \le b \implies a \vee (x \wedge b) = (a \vee x) \wedge b.$$

(4) 给定任意元素 $a, b \in L$, 定义函数

$$\varphi : [a \wedge b, b] \to [a, a \vee b] \quad \varphi(x) = x \vee a$$

$$\psi : [a, a \vee b] \to [a \wedge b, b] \quad \psi(y) = y \wedge b$$

请证明 φ (类似地, ψ) 是从 $[a \wedge b, b]$ 到 $[a, a \vee b]$ 的同构。



Definition (Lattice Isomorphism)

$$(L, \vee_L, \wedge_L) \qquad (M, \vee_M, \wedge_M)$$

A *lattice isomorphism* from L to M is a bijection

$$f : L \xleftrightarrow[onto]{1-1} M$$

such that $\forall a, b \in L$:

$$f(a \vee_L b) = f(a) \vee_M f(b)$$
$$f(a \wedge_L b) = f(a) \wedge_M f(b)$$

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f preserving $\vee$ and $\wedge$.

φ preserving $\vee$ and $\wedge$.

$$\varphi : [a \wedge b, b] \to [a, a \vee b] \quad \varphi(x) = x \vee a$$

φ preserving $\vee$ and $\wedge$.

$$\varphi : [a \wedge b, b] \to [a, a \vee b] \quad \varphi(x) = x \vee a$$

$$\varphi(x_1 \wedge x_2) = \varphi(x_1) \wedge \varphi(x_2)$$

φ preserving $\vee$ and $\wedge$.

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$$\varphi(x_1 \wedge x_2) = (x_1 \wedge x_2) \vee a$$

$$\begin{aligned}
\varphi(x_1) \wedge \varphi(x_2) &= (x_1 \vee a) \wedge (x_2 \vee a) \\
&= (a \vee x_1) \wedge (x_2 \vee a) \\
&=_{\text{modular law}} a \vee (x_1 \wedge (x_2 \vee a)) \\
&= \cdots
\end{aligned}$$


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$$\varphi : [a \wedge b, b] \to [a, a \vee b] \quad \varphi(x) = x \vee a$$

$$\psi : [a, a \vee b] \to [a \wedge b, b] \quad \psi(y) = y \wedge b$$

φ is bijective.

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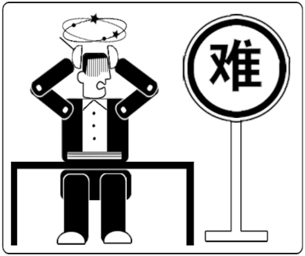

Theorem (UD Theorem 15.8 (iii))

$$f : A \to B$$

$$\exists g : B \to A \left(g \circ f = i_A \land f \circ g = i_B \right)$$

$$\Longrightarrow$$

$$f : A \to B \textit{ is bijective} \land g = f^{-1}$$

$$\psi \circ \varphi = id_{[a, a \vee b]} \qquad \varphi \circ \psi = id_{[a \wedge b, b]}$$

$$\psi \circ \varphi = id_{[a, a \vee b]} \qquad \varphi \circ \psi = id_{[a \wedge b, b]}$$

$$(\psi \circ \varphi)(y) = \psi(\varphi(y)) = (y \wedge b) \vee a = a \vee (b \wedge y) = (a \vee b) \wedge y = y$$

$$\psi \circ \varphi = id_{[a, a \vee b]} \qquad \varphi \circ \psi = id_{[a \wedge b, b]}$$

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$$(\varphi \circ \psi)(x) = \varphi(\psi(x)) = (x \vee a) \wedge b = x \vee (b \wedge a) = x$$

Back to φ preserving $\vee$ and $\wedge$.

Back to φ preserving $\vee$ and $\wedge$.

ψ preserving $\wedge$:

$$\psi(y_1 \wedge y_2) = y_1 \wedge y_2 \wedge b = (y_1 \wedge b) \wedge (y_2 \wedge b) = \psi(y_1) \wedge \psi(y_2)$$

Back to φ preserving $\vee$ and $\wedge$.

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$$\psi(y_1 \wedge y_2) = y_1 \wedge y_2 \wedge b = (y_1 \wedge b) \wedge (y_2 \wedge b) = \psi(y_1) \wedge \psi(y_2)$$

$$\psi\big(\varphi(x_1) \wedge \varphi(x_2)\big) = \psi\big(\varphi(x_1)\big) \wedge \psi\big(\varphi(x_2)\big) = x_1 \wedge x_2$$

Back to φ preserving $\vee$ and $\wedge$.

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$$\psi(y_1 \wedge y_2) = y_1 \wedge y_2 \wedge b = (y_1 \wedge b) \wedge (y_2 \wedge b) = \psi(y_1) \wedge \psi(y_2)$$

$$\psi\big(\varphi(x_1) \wedge \varphi(x_2)\big) = \psi\big(\varphi(x_1)\big) \wedge \psi\big(\varphi(x_2)\big) = x_1 \wedge x_2$$

$$\varphi(x_1 \wedge x_2) = \varphi\Big(\psi\big(\varphi(x_1) \wedge \varphi(x_2)\big)\Big) = \varphi(x_1) \wedge \varphi(x_2)$$

Thank You!