Symmetry Axes

The symmetry axes of an object are lines about which it can be rotated through some angle which brings the object to a new orientation which appears identical to its starting position. An axis is called n-fold if the smallest angle which brings the object back to its original appearance is $360/n$ degrees. For example, a plane pentagon has a single 5-fold (72 degree) axis of symmetry and five 2-fold (180 degree) axes of symmetry. The 5-fold axis is orthogonal to its plane, while the five 2-fold axes each lie in the plane and pass through one of the vertices and the opposite edge midpoint. All the symmetry axes of a polyhedron necessarily intersect at a common point at the center of the object.

Part of the beauty of polyhedra lies in their many axes of symmetry. Every uniform polyhdron (regular or semiregular) falls into one of the following five classes according to where its symmetry axes lie. Be sure to examine the models listed here and identify all their symmetry axes. You can create your own models of axes of symmetry with the cylinder intersecter. A great toy to help one understand symmetry axes is the Zometool.

The Thirteen Axes of the Cube and/or Octahedron

The cube contains three different types of symmetry axes:

- three 4-fold axes, each of which passes through the centers of two opposite faces,
- four 3-fold axes, each of which passes through two opposite vertices, and
- six 2-fold axes, each of which passes through the midpoints of two opposite edges.

Here is a model of the cube displaying all of its symmetry axes. The model also includes the octahedron which is dual to the cube, and so shows that the cube and the octahedron possess the same symmetry. In the case of the octahedron the same thirteen axes appear as:

- three 4-fold axes, each of which passes through two opposite vertices,
- four 3-fold axes, each of which passes through the centers of two opposite faces, and
- six 2-fold axes, each of which passes through the midpoints of two opposite edges.

Some other polyhedra with the same thirteen symmetry axes are the cuboctahedron (in which the 2-fold axes pass through the vertices) and the rhombicuboctahedron (in which each axis goes through the center of two of the 26 faces).Yet another is the snub cube. (Find its axes!) However, the snub cube is chiral, lacking any planes of mirror symmetry. So the axes of symmetry are not enough to completely characterize the symmetry of a polyhedron; they do not distinguish between chiral and reflexible polyhedra.

The Thirty One Axes of the Icosahedron and/or Dodecahedron

The icosahedron and dodecahedron, being mutually dual, share their symmetry axes, so one model can show all 31 axes of either. Again there are three kinds of axes. For the icosahedron these are:

- ten 3-fold axes, each of which passes through the centers of two opposite faces,
- six 5-fold axes, each of which passes through two opposite vertices, and
- fifteen 2-fold axes, each of which passes through the midpoints of two opposite edges.

In the case of the dodecahedron the same thirty-one axes appear as:

- ten 3-fold axes, each of which passes through two opposite vertices,
- six 5-fold axes, each of which passes through the centers of two opposite faces, and

- fifteen 2-fold axes, each of which passes through the midpoints of two opposite edges.

If we place a cube inside a dodecahedron or an octahedron inside a dodecahedron, we can see how the symmetry axes of the cube/octahedron relate to the symmetry axes of the dodecahedron:

- the four 3-fold axes of the cube/octahedron are four of the ten 3-fold axes of the dodecahedron,
- the three 4-fold axes of the cube/octahedron are merely 2-fold axes in the dodecahedron,
- the 2-fold axes of the cube/octahedron are not axes of the dodecahedron (and vice versa).

The Seven Axes of the Tetrahedron

The tetrahedron has only two different types of axes, as this model of the tetrahedron and its seven axes of symmetry displays:

- four 3-fold axes, each of which passes through one vertex and the center of the opposite face,
- three 2-fold axes, each of which passes through the midpoints of two opposite edges.

Because the tetrahedron (unlike the other polyhedra above) has faces opposite vertices (rather than faces opposite faces and vertices opposite vertices), it does not have three kinds of symmetry axes.

If we place a tetrahedron inside a cube, we see how the symmetry axes of the tetrahedron relate to the symmetry axes of the cube:

- The four 3-fold axes of the cube are the same as the four 3-fold axes of the tetrahedron,
- The three 4-fold axes of the cube show up as merely 2-fold axes in the tetrahedron.
- The 2-fold axes of the cube are not axes of symmetry of the tetrahedron.

We can also place a tetrahedron inside a dodecahedron and see that every symmetry axis of the tetrahedron is an axis of the same type for the dodecahedron.

The *n*+1 Axes of a Prism

An *n*-gonal prism or antiprism also has only two kinds of axis of symmetry as this pentagonal prism model shows:

- one *n*-fold axis perpendicular to the *n*-gons, and
- *n* 2-fold axes

Prism symmetry is called *dihedral* symmetry. If n is odd each 2-fold axis goes through one face and one opposite edge. If n is even, half the 2-fold axes go through two face centers and the other half go through 2 edge midpoints.

Exercise: Which of the four symmetry types listed above, if any, does the pentagonal pyramid fall under ?

Answer: None of the above; unlike the uniform polyhedra, pyramids have no 2-fold axes of symmetry. The pentagonal pyramid has only a single 5-fold axis. This is called *cyclic* symmetry.

Exercise: Examine all of the Platonic solids, Kepler-Poinsot solids, Archimedean solids, Archimedean duals, prisms, antiprisms, dipyramids, and trapezohedra, and for each polyhedron, determine which of the above four classes of symmetry it falls under.
