

## 2-2 The Efficiency of Algorithms

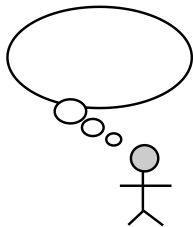
魏恒峰

hfwei@nju.edu.cn

2018 年 04 月 02 日



- (1) Diameter of Convex Polygon:  $\Theta(n)$
- (2) Lower Bound for Sorting:  $\Omega(n \lg n)$
- (3) Traversal over Trees: DFS/BFS ( $\Theta(n)$ )

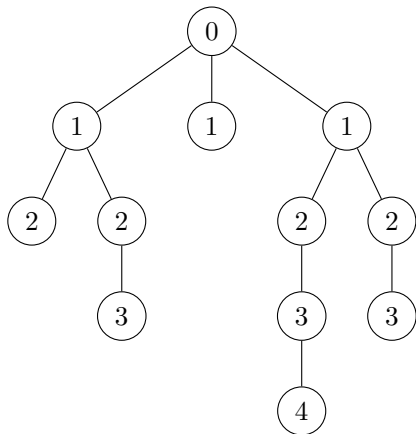


- (1) Diameter of Convex Polygon:  $\Theta(n)$
- (2) Lower Bound for Sorting:  $\Omega(n \lg n)$
- (3) Traversal over Trees: DFS/BFS ( $\Theta(n)$ )

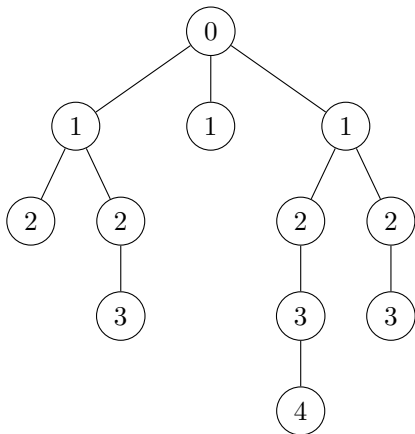
I have thought that ...



## DH 4.2 (a): Sum of Depths

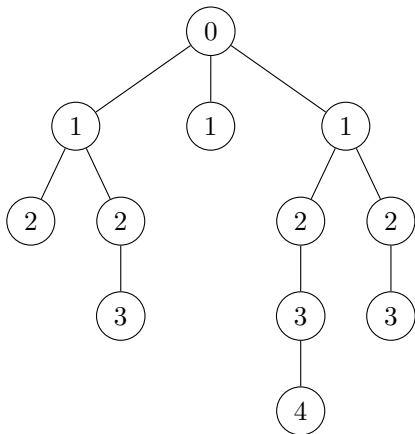


## DH 4.2 (a): Sum of Depths



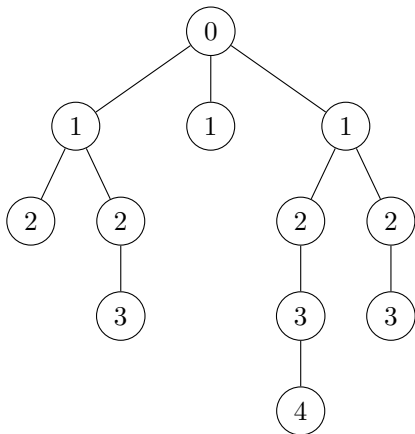
$$\text{sum-of-depths}(r) = \left\{ \begin{array}{l} \sum_{v: \text{child of } r} \text{sum-of-depths}(v) + \text{depth of } r, \end{array} \right.$$

## DH 4.2 (a): Sum of Depths



$$\text{sum-of-depths}(r) = \begin{cases} \text{depth of } r, & r \text{ is a leaf} \\ \sum_{v: \text{child of } r} \text{sum-of-depths}(v) + \text{depth of } r, & \text{o.w.} \end{cases}$$

## DH 4.2 (a): Sum of Depths



$$\text{sum-of-depths}(r, d) = \begin{cases} d, & r \text{ is a leaf} \\ \sum_{v: \text{child of } r} \text{sum-of-depths}(v, d+1) + d, & \text{o.w.} \end{cases}$$



---

**Algorithm 1** Calculate the sum of depths of all nodes of a tree  $T$ .

---

```
1: procedure SUM-OF-DEPTHS()  
2:   return SUM-OF-DEPTHS( $T, 0$ )  
  
3: procedure SUM-OF-DEPTHS( $r, depth$ )                                ▷  $r$ : root of a tree  
4:   if  $T$  is a leaf then  
5:     return  $depth$   
6:   for all child vertex  $v$  of  $r$  do  
7:      $depth \leftarrow depth + \text{SUM-OF-DEPTHS}(v, depth + 1)$   
8:   return  $depth$ 
```

---

$$\text{sum-of-depths}(r, d) = \begin{cases} d, & r \text{ is a leaf} \\ \sum_{v:\text{child of } r} \text{sum-of-depths}(v, d+1) + d, & \text{o.w.} \end{cases}$$

$$\text{sum-of-depths}(r, d) = \begin{cases} d, & r \text{ is a leaf} \\ \sum_{v:\text{child of } r} \text{sum-of-depths}(v, d+1) + d, & \text{o.w.} \end{cases}$$

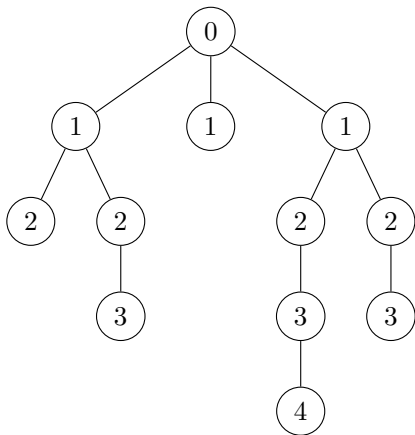
Master Theorem?

$$\text{sum-of-depths}(r, d) = \begin{cases} d, & r \text{ is a leaf} \\ \sum_{v:\text{child of } r} \text{sum-of-depths}(v, d+1) + d, & \text{o.w.} \end{cases}$$

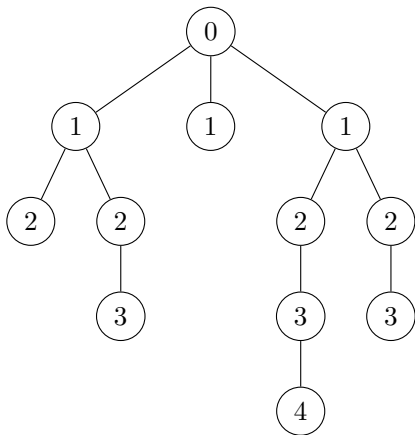
Master Theorem?

$$\Theta(m + n) = \Theta(n)$$

## DH 4.2 (b): Number of Nodes at Depth $K$

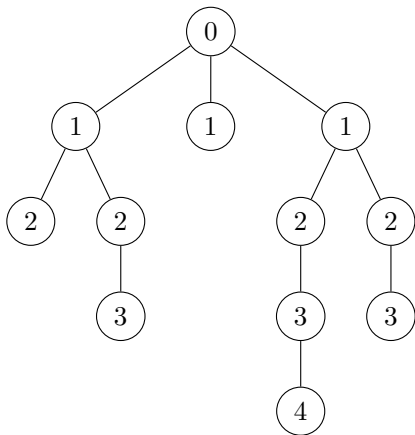


## DH 4.2 (b): Number of Nodes at Depth $K$



$$\text{nodes-at-depth}(r, k) = \left\{ \begin{array}{l} \sum_{v: \text{child of } r} \text{nodes-at-depth}(v, k-1), \end{array} \right.$$

## DH 4.2 (b): Number of Nodes at Depth $K$



$$\text{nodes-at-depth}(r, k) = \begin{cases} 1, & k = 0 \\ 0, & k > 0 \wedge r \text{ is a leaf} \\ \sum_{v: \text{child of } r} \text{nodes-at-depth}(v, k-1), & \text{o.w.} \end{cases}$$

---

**Algorithm 2** Count the number of nodes in  $T$  at depth  $K$ .

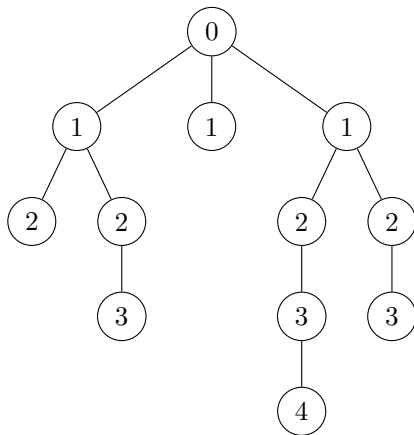
---

```
1: procedure NODES-AT-DEPTH()  
2:   return NODES-AT-DEPTH( $T, K$ )  
  
3: procedure NODES-AT-DEPTH( $r, k$ )                                ▷  $r$ : root of a tree  
4:   if  $k = 0$  then  
5:     return 1  
6:   if  $r$  is a leaf then  
7:     return 0  
8:    $num \leftarrow 0$   
9:   for all child vertex  $v$  of  $r$  do  
10:     $num \leftarrow num + \text{NODES-AT-DEPTH}(v, k - 1)$   
11:   return  $num$ 
```

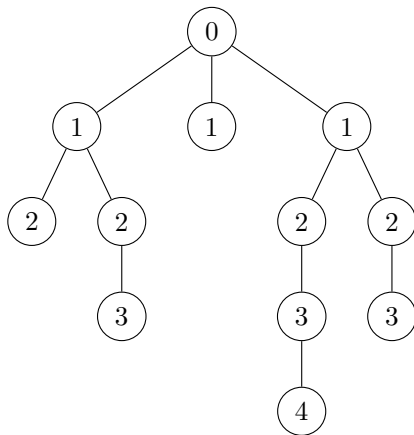
---



## DH 4.2 (c): Any Leaf at an Even Depth?

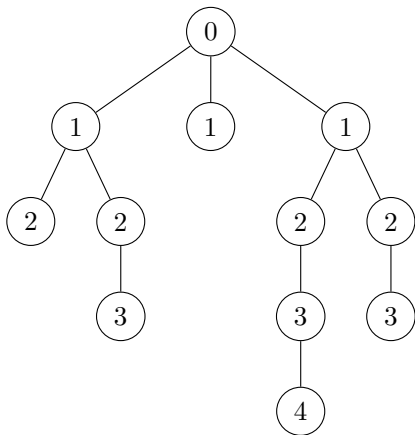


## DH 4.2 (c): Any Leaf at an Even Depth?



$$\text{leaf-at-depth}(r, \textit{parity}) = \begin{cases} \sum_{v: \text{child of } r} (v, 1 - \textit{parity}), \end{cases}$$

## DH 4.2 (c): Any Leaf at an Even Depth?



$$\text{leaf-at-depth}(r, \textit{parity}) = \begin{cases} 1 - \textit{parity}, & r \text{ is a leaf} \\ \sum_{v: \text{child of } r} (v, 1 - \textit{parity}), & \text{o.w.} \end{cases}$$

---

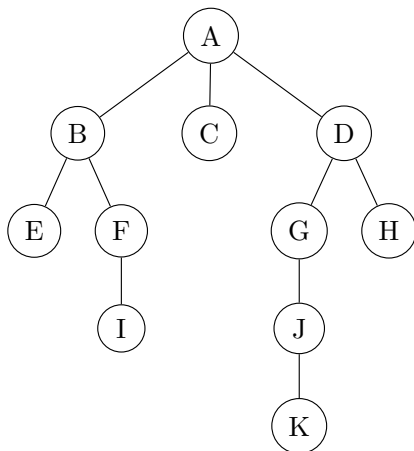
**Algorithm 3** Check whether a tree  $T$  has any leaf at an even depth.

---

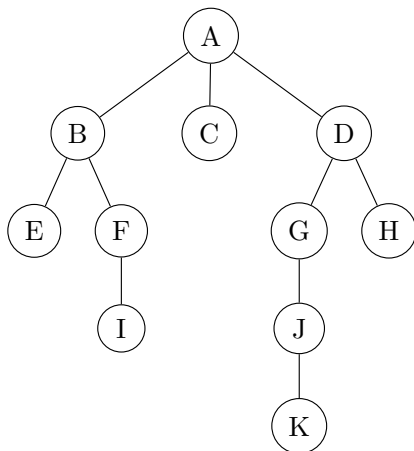
```
1: procedure LEAF-AT-EVEN-DEPTH()  
2:   return LEAF-AT-DEPTH( $T, even = 0$ )  
  
3: procedure LEAF-AT-DEPTH( $r, parity$ )                                ▷  $r$ : root of a tree  
4:   if  $r$  is a leaf then  
5:     return  $1 - parity$   
  
6:    $result \leftarrow 0$   
7:   for all child vertex  $v$  of  $r$  do  
8:      $result \leftarrow result \vee \text{LEAF-AT-DEPTH}(v, 1 - parity)$   
9:   return  $result$ 
```

---

### DH 4.3 (a): Sum of Contents at Each Depth



### DH 4.3 (a): Sum of Contents at Each Depth



---

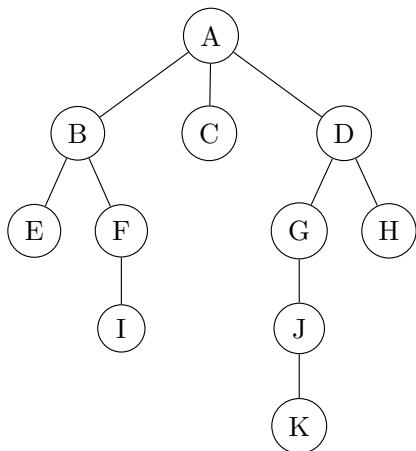
**Algorithm 4** Calculate the sum of contents of nodes of a tree  $T$  at each depth.

---

```
1: procedure SUM-AT-DEPTH( $r$ )                                ▷  $r$ : root of the tree  $T$ 
2:    $r.depth \leftarrow 0$ 
3:    $Q \leftarrow \emptyset$ 
4:   ENQUEUE( $Q, r$ )
5:   while  $Q \neq \emptyset$  do
6:      $u \leftarrow$  DEQUEUE( $Q$ )
7:      $sumAtDepth[u.depth] \leftarrow sumAtDepth[u.depth] + u.content$ 
8:     for all child vertex  $v$  of  $u$  do
9:        $v.depth \leftarrow u.depth + 1$ 
10:      ENQUEUE( $Q, v$ )
```

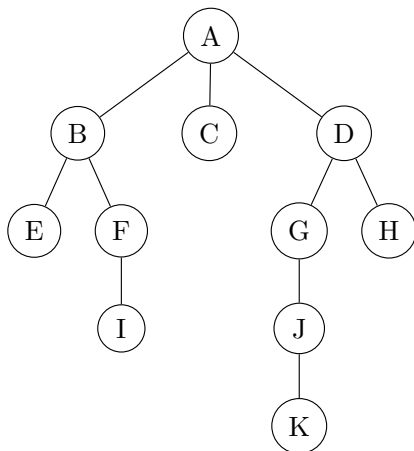
---

### DH 4.3 (b): Depth $K$ with the Maximum Number of Nodes





### DH 4.3 (b): Depth $K$ with the Maximum Number of Nodes



---

**Algorithm 5** Count the number of nodes of a tree  $T$  at each depth.

---

```
1: procedure NODES-AT-DEPTH( $r$ )                                ▷  $r$ : root of the tree  $T$ 
2:    $r.depth \leftarrow 0$ 
3:    $Q \leftarrow \emptyset$ 
4:   ENQUEUE( $Q, r$ )
5:   while  $Q \neq \emptyset$  do
6:      $u \leftarrow$  DEQUEUE( $Q$ )
7:      $nodesAtDepth[u.depth] += 1$ 
8:     for all child vertex  $v$  of  $u$  do
9:        $v.depth \leftarrow u.depth + 1$ 
10:      ENQUEUE( $Q, v$ )
```

---

# Lower Bound for Comparison-based Sorting

## Lower Bound for Comparison-based Sorting



## Lower Bound for Comparison-based Sorting (DH 6.13)

Prove a lower bound of  $O(n \lg n)$  on the time complexity of any comparison-based sorting algorithm.

## Lower Bound for Comparison-based Sorting (DH 6.13)

Prove a lower bound of  $O(n \lg n)$  on the time complexity of any comparison-based sorting algorithm.



WRONG

## Lower Bound for Comparison-based Sorting (DH 6.13)

Prove a lower bound of  $O(n \lg n)$  on the time complexity of any comparison-based sorting algorithm.



WRONG

## Lower Bound for Comparison-based Sorting (DH 6.13)

Prove a lower bound of  $\Omega(n \lg n)$  on the time complexity of any comparison-based sorting algorithm.

## Lower Bound for Comparison-based Sorting (DH 6.13)

Prove a lower bound of  $O(n \lg n)$  on the time complexity of any comparison-based sorting algorithm.



WRONG

## Lower Bound for Comparison-based Sorting (DH 6.13)

Prove a lower bound of  $\Omega(n \lg n)$  on the time complexity of any comparison-based sorting algorithm on inputs of size  $n$ .



## Lower Bound for Comparison-based Sorting (DH 6.13)

Prove a lower bound of  $\Omega(n \lg n)$  on the time complexity of any **comparison-based** sorting algorithm on inputs of size  $n$ .

## Lower Bound for Comparison-based Sorting (DH 6.13)

Prove a lower bound of  $\Omega(n \lg n)$  on the time complexity of any **comparison-based** sorting algorithm on inputs of size  $n$ .

### Computational Model:

*the only way to gain order info.*

## Lower Bound for Comparison-based Sorting (DH 6.13)

Prove a lower bound of  $\Omega(n \lg n)$  on the time complexity of any **comparison-based** sorting algorithm on inputs of size  $n$ .

### Computational Model:

*the only way to gain order info.*

$$x \in [1 \cdots 99]$$

$$x/10$$

## Lower Bound for Comparison-based Sorting (DH 6.13)

Prove a lower bound of  $\Omega(n \lg n)$  on the time complexity of any **comparison-based** sorting algorithm on inputs of size  $n$ .

Cost Model:

*the critical operations to count*

Computational Model:

*the only way to gain order info.*

$$x \in [1 \cdots 99]$$

$$x/10$$

## Lower Bound for Comparison-based Sorting (DH 6.13)

Prove a lower bound of  $\Omega(n \lg n)$  on the time complexity of any **comparison-based** sorting algorithm on inputs of size  $n$ .

Cost Model:

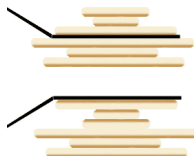
*the critical operations to count*

Computational Model:

*the only way to gain order info.*

$$x \in [1 \cdots 99]$$

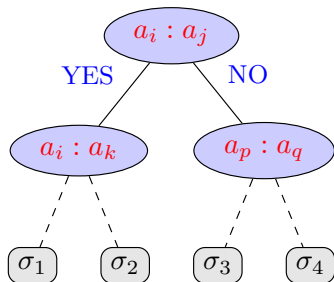
$$x/10$$



“Bounds For Sorting By Prefix Reversal”, 1979

# Decision Tree Model

# Decision Tree Model



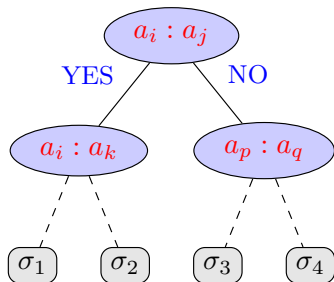
Nodes: comparisons  $a_i : a_j$

$<, \leq, =, \geq, >$

Edges: two-way decisions (Y/N)

Leaves: possible permutations

# Decision Tree Model



Nodes: comparisons  $a_i : a_j$

$<, \leq, =, \geq, >$

Edges: two-way decisions (Y/N)

Leaves: possible permutations

Assumption (By aware of any assumptions !!!):

All the input elements are **distinct**.

$$a_i < a_j$$



# Decision Tree Model

Any Comparison-based Sorting Algorithm  $\xRightarrow{\text{modeled by}}$  A Decision Tree

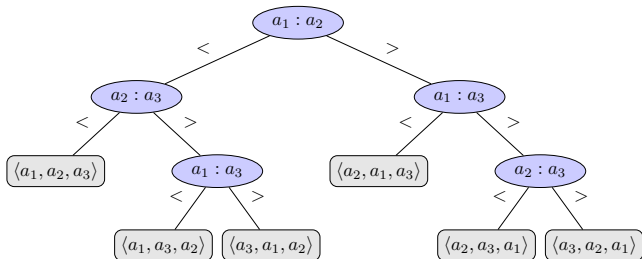
# Decision Tree Model

Any Comparison-based Sorting Algorithm  $\xrightarrow{\text{modeled by}}$  A Decision Tree



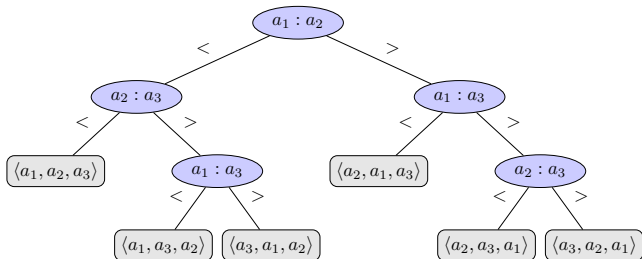
# Decision Tree Model

Any Comparison-based Sorting Algorithm  $\xRightarrow{\text{modeled by}}$  A Decision Tree



# Decision Tree Model

Any Comparison-based Sorting Algorithm  $\xrightarrow{\text{modeled by}}$  A Decision Tree



The decision tree for **insertion sort** on three elements.

# Decision Tree Model

Any Comparison-based Sorting Algorithm modeled by A Decision Tree

# Decision Tree Model

Any Comparison-based Sorting Algorithm modeled by A Decision Tree

---

```
1: procedure           -SORT( $A, n$ )
2:   for  $i \leftarrow 1$  to  $n - 1$  do
3:     for  $j \leftarrow i + 1$  to  $n$  do
4:       if  $A[j] < A[i]$  then
5:         SWAP( $A[j], A[i]$ )
```

---

## Decision Tree Model

Any Comparison-based Sorting Algorithm  $\xrightarrow{\text{modeled by}}$  A Decision Tree

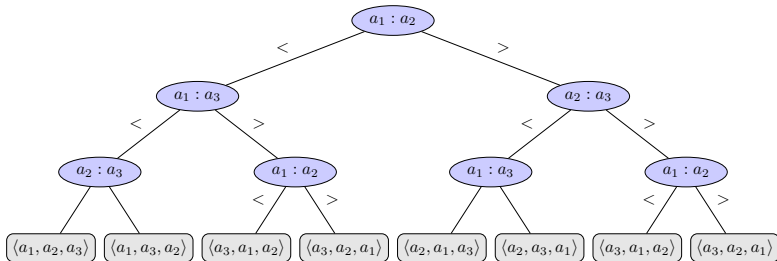
---

```
1: procedure SELECTION-SORT( $A, n$ )
2:   for  $i \leftarrow 1$  to  $n - 1$  do
3:     for  $j \leftarrow i + 1$  to  $n$  do
4:       if  $A[j] < A[i]$  then
5:         SWAP( $A[j], A[i]$ )
```

---

# Decision Tree Model

Any Comparison-based Sorting Algorithm  $\xRightarrow{\text{modeled by}}$  A Decision Tree



The decision tree for **selection sort** on three elements.



# Decision Tree Model

Any Comparison-based Sorting Algorithm  $\mathcal{A}$   $\xRightarrow{\text{modeled by}}$  A Decision Tree  $\mathcal{T}$

# Decision Tree Model

Any Comparison-based Sorting Algorithm  $\mathcal{A}$   $\xRightarrow{\text{modeled by}}$  A Decision Tree  $\mathcal{T}$

Algorithm  $\mathcal{A}$  on a specific input of size  $n$   $\xRightarrow{\text{modeled by}}$  A path through  $\mathcal{T}$

# Decision Tree Model

Any Comparison-based Sorting Algorithm  $\mathcal{A}$   $\xRightarrow{\text{modeled by}}$  A Decision Tree  $\mathcal{T}$

Algorithm  $\mathcal{A}$  on a specific input of size  $n$   $\xRightarrow{\text{modeled by}}$  A path through  $\mathcal{T}$

Worst-case time complexity of  $\mathcal{A}$   $\xRightarrow{\text{modeled by}}$  The height of  $\mathcal{T}$

# Decision Tree Model

Any Comparison-based Sorting Algorithm  $\mathcal{A}$   $\xRightarrow{\text{modeled by}}$  A Decision Tree  $\mathcal{T}$

Algorithm  $\mathcal{A}$  on a specific input of size  $n$   $\xRightarrow{\text{modeled by}}$  A path through  $\mathcal{T}$

Worst-case time complexity of  $\mathcal{A}$   $\xRightarrow{\text{modeled by}}$  The height of  $\mathcal{T}$

Worst-case Lower Bound of Comparison-based Sorting on inputs of size  $n$   
 $\xRightarrow{\text{modeled by}}$   
The Minimum Height of All  $\mathcal{T}$ s

# Decision Tree Model

Worst-case Lower Bound of Comparison-based Sorting on inputs of size  $n$

modeled by

The Minimum Height of All  $\mathcal{T}_s$

# Decision Tree Model

Worst-case Lower Bound of Comparison-based Sorting on inputs of size  $n$

modeled by

The Minimum Height of All  $\mathcal{T}$ s

To be a full binary tree:

$$\# \text{ of leaves} \leq 2^h$$

# Decision Tree Model

Worst-case Lower Bound of Comparison-based Sorting on inputs of size  $n$

modeled by 

The Minimum Height of All  $\mathcal{T}$ s

To be a full binary tree:

$$\# \text{ of leaves} \leq 2^h$$

To be a correct sorting algorithm:

$$\# \text{ of leaves} \geq n!$$

# Lower Bound for Comparison-based Sorting

$$n! \leq \# \text{ of leaves} \leq 2^h$$



# Lower Bound for Comparison-based Sorting

$$n! \leq \# \text{ of leaves} \leq 2^h$$

$$h \geq \lg n! = \Omega(n \lg n)$$

## Lower Bound for Comparison-based Sorting

$$n! \leq \# \text{ of leaves} \leq 2^h$$

$$h \geq \lg n! = \Omega(n \lg n)$$

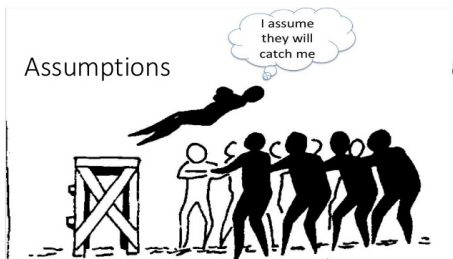
Stirling Formula (by *James Stirling*):

$$n! = \Theta\left(\sqrt{2\pi n} \left(\frac{n}{e}\right)^n\right)$$

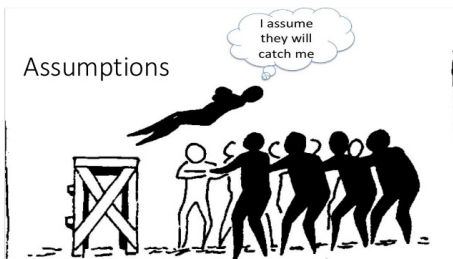


# Looking Back

# Looking Back



# Looking Back



Assumptions (By aware of any assumptions !!!):

- (a) **Comparison-based** sorting algorithms.
- (b) All the input elements are **distinct**.

## The $k$ -sorted Problem

An array  $A[1 \cdots n]$  is  **$k$ -sorted** if it can be divided into  $k$  blocks, each of size  $n/k$  (we assume that  $n/k \in \mathbb{N}$ ), such that the elements in each block are larger than the elements in earlier blocks and smaller than elements in later blocks. The elements within each block need **not** be sorted.

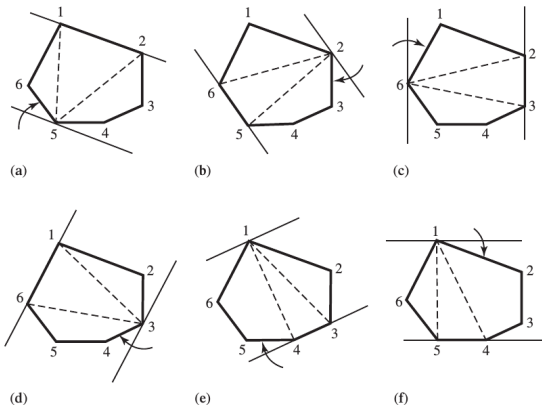
- (a) Describe an algorithm that  **$k$ -sorts an arbitrary array** in  $O(n \log k)$  time.
- (b) Prove that any comparison-based  $k$ -sorting algorithm requires  $\Omega(n \log k)$  comparisons in the worst case.
- (c) Describe an algorithm that **completely sorts an already  $k$ -sorted array** in  $O(n \log(n/k))$  time.
- (d) Prove that any comparison-based algorithm to completely sort a  $k$ -sorted array requires  $\Omega(n \log(n/k))$  comparisons in the worst case.

## Convex Polygon Diameter (DH 6.8)

Show that the “Convex Polygon Diameter” algorithm is of linear-time complexity.

## Convex Polygon Diameter (DH 6.8)

Show that the “Convex Polygon Diameter” algorithm is of linear-time complexity.





## Convex Polygon Diameter (DH 6.8)

Show that the “Convex Polygon Diameter” algorithm is of **linear-time** complexity.

*Q* : Linear-time of **WHAT**?

## Convex Polygon Diameter (DH 6.8)

Show that the “Convex Polygon Diameter” algorithm is of **linear-time** complexity.

*Q* : Linear-time of **WHAT**?

*A* : Linear-time of **the size of input**

## Convex Polygon Diameter (DH 6.8)

Show that the “Convex Polygon Diameter” algorithm is of **linear-time** complexity.

*Q* : Linear-time of **WHAT**?

*A* : Linear-time of **the size of input**

*Q* : What is the input?

## Convex Polygon Diameter (DH 6.8)

Show that the “Convex Polygon Diameter” algorithm is of **linear-time** complexity.

*Q* : Linear-time of **WHAT**?

*A* : Linear-time of **the size of input**

*Q* : What is the input?

*A* : A convex polygon

## Convex Polygon Diameter (DH 6.8)

Show that the “Convex Polygon Diameter” algorithm is of **linear-time** complexity.

*Q* : Linear-time of **WHAT**?

*A* : Linear-time of **the size of input**

*Q* : What is the input?

*A* : A convex polygon **represented by  $n$  vertices**

## Convex Polygon Diameter (DH 6.8)

Show that the “Convex Polygon Diameter” algorithm is of **linear-time** complexity.

*Q* : Linear-time of **WHAT**?

*A* : Linear-time of **the size of input**

*Q* : What is the input?

*A* : A convex polygon **represented by  $n$  vertices**

*Q* : What are the critical operations?

## Convex Polygon Diameter (DH 6.8)

Show that the “Convex Polygon Diameter” algorithm is of **linear-time** complexity.

*Q* : Linear-time of **WHAT**?

*A* : Linear-time of **the size of input**

*Q* : What is the input?

*A* : A convex polygon **represented by  $n$  vertices**

*Q* : What are the critical operations?

*A* :  $d(p_1, p_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

## Convex Polygon Diameter (DH 6.8)

Show that the “Convex Polygon Diameter” algorithm is of **linear-time** complexity.

*Q* : Linear-time of **WHAT**?

*A* : Linear-time of **the size of input**

*Q* : What is the input?

*A* : A convex polygon **represented by  $n$  vertices**

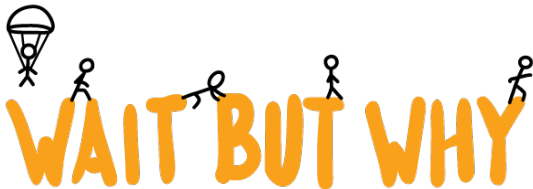
*Q* : What are the critical operations?

*A* :  $d(p_1, p_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$$\Theta(c \cdot n) = \Theta(n)$$



# Correctness

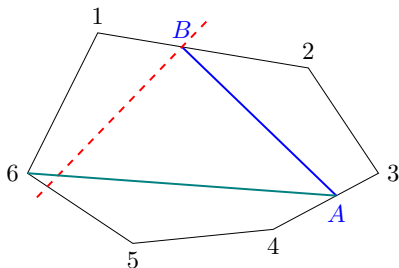


## Theorem

*For a convex polygon, a pair of vertices determine the diameter.*

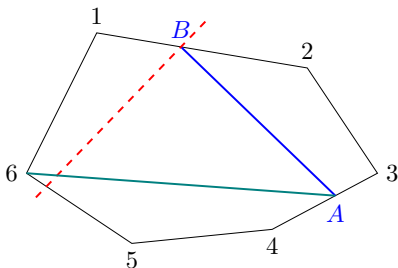
## Theorem

*For a convex polygon, a pair of vertices determine the diameter.*



## Theorem

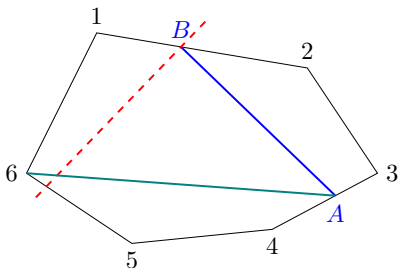
*For a convex polygon, a pair of vertices determine the diameter.*



BUT, we have *not* enumerated *all* pairs of vertices.

## Theorem

*For a convex polygon, a pair of vertices determine the diameter.*

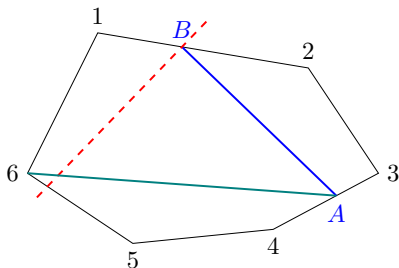


BUT, we have *not* enumerated *all* pairs of vertices.

We have enumerated *all* pairs of vertices

## Theorem

*For a convex polygon, a pair of vertices determine the diameter.*



BUT, we have *not* enumerated *all* pairs of vertices.

We have enumerated *all* pairs of vertices  
that *admits parallel supporting lines*.

## Definition (Line of Support)

A line  $L$  is a *line of support* of a convex polygon  $P$  if

$$L \cap P = \text{a vertex/an edge of } P.$$

## Definition (Line of Support)

A line  $L$  is a *line of support* of a convex polygon  $P$  if

$$L \cap P = \text{a vertex/an edge of } P.$$

$L \cap P \neq \emptyset$  and  $P$  lies entirely on one side of  $L$ .



### Definition (Line of Support)

A line  $L$  is a *line of support* of a convex polygon  $P$  if

$$L \cap P = \text{a vertex/an edge of } P.$$

$$L \cap P \neq \emptyset \text{ and } P \text{ lies entirely on one side of } L.$$

### Definition (Antipodal)

An *antipodal* is a pair of points that admits parallel supporting lines.

### Definition (Line of Support)

A line  $L$  is a *line of support* of a convex polygon  $P$  if

$$L \cap P = \text{a vertex/an edge of } P.$$

$$L \cap P \neq \emptyset \text{ and } P \text{ lies entirely on one side of } L.$$

### Definition (Antipodal)

An *antipodal* is a pair of points that admits parallel supporting lines.

We have enumerated *all* antipodals.

## Theorem

*If  $AB$  is a diameter of a convex polygon  $P$ , then  $AB$  is an antipodal.*

## Theorem

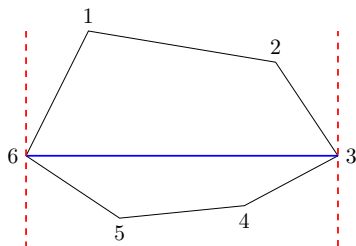
*If  $AB$  is a diameter of a convex polygon  $P$ , then  $AB$  is an antipodal.*

Proof.

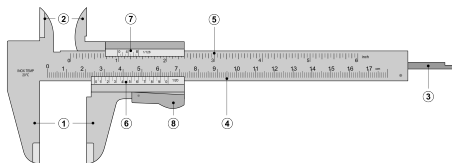
## Theorem

*If  $AB$  is a diameter of a convex polygon  $P$ , then  $AB$  is an antipodal.*

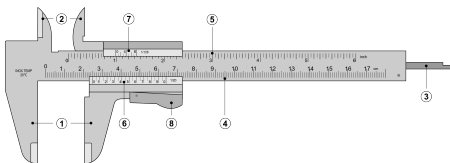
Proof.



# Rotating Caliper



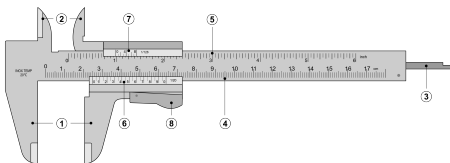
# Rotating Caliper



“Computational Geometry”

Ph.D Thesis, Michael Shamos, 1978

# Rotating Caliper



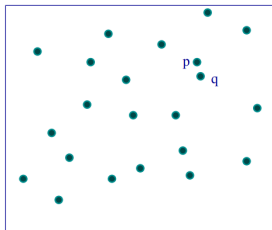
“Computational Geometry”  
Ph.D Thesis, Michael Shamos, 1978



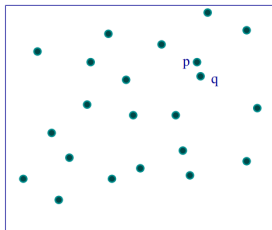
“Solving Geometric Problems with  
the Rotating Calipers”, 1983



## Finding the Closest Pair of Points

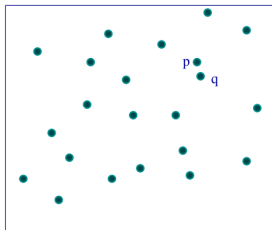


## Finding the Closest Pair of Points



A Classic and Beautiful Divide-Conquer Algorithm:

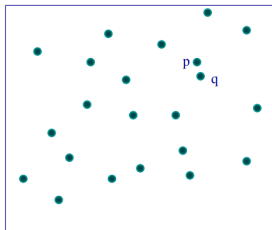
## Finding the Closest Pair of Points



A Classic and Beautiful Divide-Conquer Algorithm:



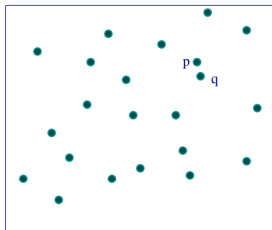
## Finding the Closest Pair of Points



A Classic and Beautiful Divide-Conquer-Combine Algorithm:



## Finding the Closest Pair of Points



A Classic and Beautiful Divide-Conquer-Combine Algorithm:



### Section 33.4, CLRS





- (a) Given an array  $A[0 \cdots n - 1]$ , to determine whether there is a value that *occurs more than  $\lfloor n/k \rfloor$  times* in  $\Theta(n \lg k)$  time and  $\Theta(k)$  extra space.
- (b) Prove that the *lower bound* of this problem is  $\Theta(n \lg k)$ .



- (a) Given an array  $A[0 \cdots n - 1]$ , to determine whether there is a value that *occurs more than  $\lfloor n/k \rfloor$  times* in  $\Theta(n \lg k)$  time and  $\Theta(k)$  extra space.
- (b) Prove that the *lower bound* of this problem is  $\Theta(n \lg k)$ .

## TIP#1

Take  $k = 2$ .

$\Theta(n)$  &  $\Theta(1)$

Thank  
You!



Office 302

Mailbox: H016

hfwei@nju.edu.cn