2-7 Discrete Probability

Hengfeng Wei

hfwei@nju.edu.cn

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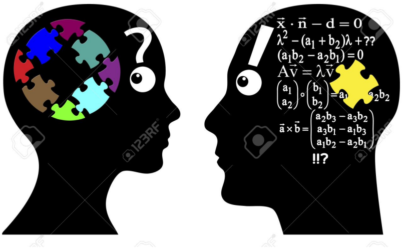
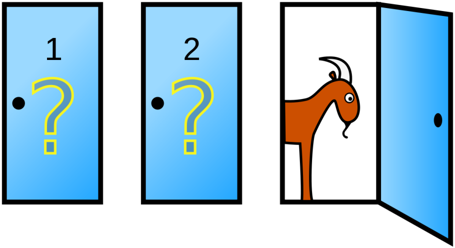
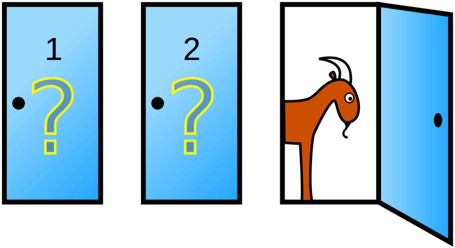
Let us calculate [calculemus].



Intuition vs. Calculation

The Monty-Hall Problem





You: Randomly pick a door (No. 1)

I: Open a door which has a goat (No. 3)

(I know what's behind the doors)

Q : Do you want to switch to door 2?

C_i : The car is behind door i $(i = 1, 2, 3)$

$$\boxed{\Pr\{C_i\} = \frac{1}{3}}$$

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Y_1 : you initially pick door 1

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$$\boxed{\Pr\{X_1\} = \frac{1}{3}}$$

I_3 : I open door 3

$$\boxed{\Pr\{C_2 \mid I_3, Y_1\}}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{C_2, I_3, Y_1\}}{\Pr\{I_3, Y_1\}} = \frac{\Pr\{I_3, Y_1 \mid C_2\} \Pr\{C_2\}}{\Pr\{I_3 \mid Y_1\} \Pr\{Y_1\}}$$

$$= \frac{\Pr\{I_3, Y_1 \mid C_2\}}{\Pr\{I_3 \mid Y_1\}}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{C_2, I_3, Y_1\}}{\Pr\{I_3, Y_1\}} = \frac{\Pr\{I_3, Y_1 \mid C_2\}\Pr\{C_2\}}{\Pr\{I_3 \mid Y_1\}\Pr\{Y_1\}}$$
$$= \frac{\Pr\{I_3, Y_1 \mid C_2\}}{\Pr\{I_3 \mid Y_1\}}$$

$$\Pr\{I_3, Y_1 \mid C_2\} = \frac{\Pr\{I_3, Y_1, C_2\}}{\Pr\{C_2\}} = \frac{\Pr\{I_3 \mid C_2, Y_1\}\Pr\{C_2, Y_1\}}{\Pr\{C_2\}}$$
$$= \frac{\Pr\{I_3 \mid C_2, Y_1\}\Pr\{Y_1 \mid C_2\}\Pr\{C_2\}}{\Pr\{C_2\}}$$
$$= \frac{1}{3}\Pr\{I_3 \mid C_2, Y_1\}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{3\Pr\{I_3 \mid Y_1\}}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{3 \Pr\{I_3 \mid Y_1\}}$$

$$\begin{aligned}
\Pr\{I_3 \mid Y_1\} &= \Pr\{I_3 \mid C_1, Y_1\} \Pr\{C_1 \mid Y_1\} \\
&+ \Pr\{I_3 \mid C_2, Y_1\} \Pr\{C_2 \mid Y_1\} \\
&+ \Pr\{I_3 \mid C_3, Y_1\} \Pr\{C_3 \mid Y_1\} \\
&= \frac{1}{3}\Big(\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\} \Big)
\end{aligned}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}}$$

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$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}}$$

It depends on how I choose the door to open!

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}}$$

It depends on how I choose the door to open!

$$\Pr\{I_3 \mid C_1, Y_1\} = \frac{1}{2}$$

$$\Pr\{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr\{I_3 \mid C_3, Y_1\} = 0$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}}$$

It depends on how I choose the door to open!

$$\Pr\{I_3 \mid C_1, Y_1\} = \frac{1}{2}$$

$$\Pr\{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr\{I_3 \mid C_3, Y_1\} = 0$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{2}{3}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}}$$

$$\boxed{\Pr\{I_3 \mid C_3, Y_1\} = 0}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}}$$

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$$\Pr\{C_1 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_1, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\}}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\} + \Pr\{I_3 \mid C_3, Y_1\}}$$

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$$\frac{\Pr\{C_2 \mid I_3, Y_1\}}{\Pr\{C_1 \mid I_3, Y_1\}} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\}}$$

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$$\frac{\Pr\{C_2 \mid I_3, Y_1\}}{\Pr\{C_1 \mid I_3, Y_1\}} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\}}$$

$$\boxed{\Pr\{C_2 \mid I_3, Y_1\} > \Pr\{C_1 \mid I_3, Y_1\} \iff \Pr\{I_3 \mid C_2, Y_1\} > \Pr\{I_3 \mid C_1, Y_1\}}$$

$$\boxed{\Pr\{C_2 \mid I_3, Y_1\} > \Pr\{C_1 \mid I_3, Y_1\} \iff \Pr\{I_3 \mid C_2, Y_1\} > \Pr\{I_3 \mid C_1, Y_1\}}$$

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$$\Pr\{I_3 \mid C_1, Y_1\} = \frac{1}{2}$$

$$\Pr\{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{2}{3}$$

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Q : Switching *vs.* Choosing between the two remaining doors randomly?

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\}}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\}}$$

$$\Pr\{I_3 \mid C_1, Y_1\} = q$$

$$\Pr\{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\}}$$

$$\Pr\{I_3 \mid C_1, Y_1\} = q$$

$$\Pr\{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{1}{1+q}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\}}$$

$$\Pr\{I_3 \mid C_1, Y_1\} = q$$

$$\Pr\{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{1}{1+q} \in [\frac{1}{2}, 1]$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\}}$$

$$\Pr\{I_3 \mid C_1, Y_1\} = q$$

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$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{1}{1+q} \in [\frac{1}{2}, 1]$$

$$\Pr\{C_1 \mid I_3, Y_1\} \in [0, \frac{1}{2}]$$

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$$\Pr\{C_1 \mid I_3, Y_1\} \in [0, \frac{1}{2}]$$

Always Switch!

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\}}$$

$$\Pr\{I_3 \mid C_1, Y_1\} = q$$

$$\Pr\{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{1}{1+q} \in [\frac{1}{2}, 1]$$

$$\Pr\{C_1 \mid I_3, Y_1\} \in [0, \frac{1}{2}]$$

Always Switch!

$$\boxed{\Pr\{C_2 \mid I_3, Y_1\} > \Pr\{C_1 \mid I_3, Y_1\} \iff \Pr\{I_3 \mid C_2, Y_1\} > \Pr\{I_3 \mid C_1, Y_1\}}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\}}$$

$$\boxed{\Pr\{C_2 \mid I_3, Y_1\} > \Pr\{C_1 \mid I_3, Y_1\} \iff \Pr\{I_3 \mid C_2, Y_1\} > \Pr\{I_3 \mid C_1, Y_1\}}$$

$$\Pr\{I_3 \mid C_1, Y_1\} = \frac{1}{2}$$

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$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{2}{3}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\}}$$

$$\boxed{\Pr\{C_2 \mid I_3, Y_1\} > \Pr\{C_1 \mid I_3, Y_1\} \iff \Pr\{I_3 \mid C_2, Y_1\} > \Pr\{I_3 \mid C_1, Y_1\}}$$

I know what's behind the doors.

$$\Pr\{I_3 \mid C_1, Y_1\} = \frac{1}{2}$$

$$\Pr\{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{2}{3}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{\Pr\{I_3 \mid C_2, Y_1\}}{\Pr\{I_3 \mid C_1, Y_1\} + \Pr\{I_3 \mid C_2, Y_1\}}$$

$$\boxed{\Pr\{C_2 \mid I_3, Y_1\} > \Pr\{C_1 \mid I_3, Y_1\} \iff \Pr\{I_3 \mid C_2, Y_1\} > \Pr\{I_3 \mid C_1, Y_1\}}$$

I know what's behind the doors.

$$\Pr\{I_3 \mid C_1, Y_1\} = \frac{1}{2}$$

$$\Pr\{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{2}{3}$$

I do not know what's behind the doors, and opens one randomly that happens not to reveal the car.

$$\Pr\{I_3 \mid C_1, Y_1\} = \frac{1}{2}$$

$$\Pr\{I_3 \mid C_2, Y_1\} = \frac{1}{2}$$

$$\Pr\{C_2 \mid I_3, Y_1\} = \frac{1}{2}$$

The Boy/Girl Puzzle

Both Girls (CS Problem $5.3 - 12$)

Mr. and Mrs. Smith have two children of different ages, what is the probability that they has two girls

(a) given that one of the children is a girl?

(b) given that the older child is a girl?

G_1 : the older child is a girl

G_2 : the younger child is a girl

$$G_1 : \text{the older child is a girl}$$

$$G_2 : \text{the younger child is a girl}$$

$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\}$

$$G_1 : \text{the older child is a girl}$$

$$G_2 : \text{the younger child is a girl}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1 \vee G_2\}}$$

G_1 : the older child is a girl

G_2 : the younger child is a girl

$$
\begin{aligned}
\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} &= \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1 \vee G_2\}} \\
&= \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1\} + \Pr\{G_2\} - \Pr\{G_1 \wedge G_2\}}
\end{aligned}
$$

$$G_1 : \text{the older child is a girl}$$

$$G_2 : \text{the younger child is a girl}$$

$$\begin{aligned}
\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} &= \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1 \vee G_2\}} \\
&= \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1\} + \Pr\{G_2\} - \Pr\{G_1 \wedge G_2\}} \\
&= \frac{1/4}{3/4} = \frac{1}{3}
\end{aligned}$$

G_1 : the older child is a girl

G_2 : the younger child is a girl

$$\begin{aligned}\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} &= \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1 \vee G_2\}} \\ &= \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1\} + \Pr\{G_2\} - \Pr\{G_1 \wedge G_2\}} \\ &= \frac{1/4}{3/4} = \frac{1}{3}\end{aligned}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1\}$$

G_1 : the older child is a girl

G_2 : the younger child is a girl

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1 \vee G_2\}} = \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1\} + \Pr\{G_2\} - \Pr\{G_1 \wedge G_2\}} = \frac{1/4}{3/4} = \frac{1}{3}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1\} = \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1\}}$$

$$G_1 : \text{the older child is a girl}$$

$$G_2 : \text{the younger child is a girl}$$

$$\begin{aligned}\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} &= \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1 \vee G_2\}} \\ &= \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1\} + \Pr\{G_2\} - \Pr\{G_1 \wedge G_2\}} \\ &= \frac{1/4}{3/4} = \frac{1}{3}\end{aligned}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1\} = \frac{\Pr\{G_1 \wedge G_2\}}{\Pr\{G_1\}} = \frac{1/4}{1/2} = \frac{1}{2}$$

$$\boxed{(G_1, G_2), (G_1, B_2), (B_1, G_2), (B_1, B_2)}$$

$$\boxed{(G_1, G_2), (G_1, B_2), (B_1, G_2), (B_1, B_2)}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{1}{3}$$

$$\boxed{(G_1, G_2), (G_1, B_2), (B_1, G_2), (B_1, B_2)}$$

$$\Pr\{G_1 \land G_2 \mid G_1 \lor G_2\} = \frac{1}{3}$$

$$(G_1, G_2), (G_1, B_2), (B_1, G_2), (B_1, B_2)$$

$$\boxed{(G_1, G_2), (G_1, B_2), (B_1, G_2), (B_1, B_2)}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{1}{3}$$

$$(G_1, G_2), (G_1, B_2), (B_1, G_2), (B_1, B_2)$$

$$\Pr\{G_1 \wedge G_2 \mid G_1\} = \frac{1}{2}$$

$$\boxed{(G_1, G_2), (G_1, B_2), (B_1, G_2), (B_1, B_2)}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{1}{3}$$

$$(G_1, G_2), (G_1, B_2), (B_1, G_2), (B_1, B_2)$$

$$\Pr\{G_1 \wedge G_2 \mid G_1\} = \frac{1}{2}$$

$$(G_1, G_2), (G_1, B_2), (B_1, G_2), (B_1, B_2)$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{1}{3}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1\} = \frac{1}{2}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{1}{3}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1\} = \frac{1}{2}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \boxed{\frac{2}{3}} \Pr\{G_1 \wedge G_2 \mid G_1\}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{1}{3}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1\} = \frac{1}{2}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \boxed{\frac{2}{3}} \Pr\{G_1 \wedge G_2 \mid G_1\}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \boxed{\Pr\{G_1 \mid G_1 \vee G_2\}} \Pr\{G_1 \wedge G_2 \mid G_1\}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{1}{3}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1\} = \frac{1}{2}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \boxed{\frac{2}{3}} \Pr\{G_1 \wedge G_2 \mid G_1\}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \boxed{\Pr\{G_1 \mid G_1 \vee G_2\}} \Pr\{G_1 \wedge G_2 \mid G_1\}$$

$$\Pr\{G_1 \mid G_1 \vee G_2\} = \frac{\Pr\{G_1 \wedge (G_1 \vee G_2)\}}{\Pr\{G_1 \vee G_2\}}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{1}{3}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1\} = \frac{1}{2}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \boxed{\frac{2}{3}} \Pr\{G_1 \wedge G_2 \mid G_1\}$$

$$\Pr\{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \boxed{\Pr\{G_1 \mid G_1 \vee G_2\}} \Pr\{G_1 \wedge G_2 \mid G_1\}$$

$$\Pr\{G_1 \mid G_1 \vee G_2\} = \frac{\Pr\{G_1 \wedge (G_1 \vee G_2)\}}{\Pr\{G_1 \vee G_2\}} = \frac{\Pr\{G_1\}}{\Pr\{G_1 \wedge G_2\}} = \frac{2}{3}$$


Easy!


Easy!


KEEP
CALM
AND
WE ARE NOT
FINISHED YET

Both Girls (CS Problem $5.3 - 12$)

Mr. and Mrs. Smith have two children of different ages, what is the probability that they has two girls

(a) given that one of the children is a girl?

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Q : **How** do you know that "one of the children is a girl"?

(I) From all families with two children, at least one of whom is a girl, a (Smith's) family is chosen at random.

(II) From all families with two children, one child (of Smith) is selected at random that happens to be a girl.

Q : **How** do you know that "one of the children is a girl"?

(II) g : From all families with two children, one child (of Smith) is selected at random that happens to be a girl.

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Q : **How** do you know that "one of the children is a girl"?

(II) g : From all families with two children, one child (of Smith) is selected at random that happens to be a girl.

$$\Pr\{G_1 \land G_2 \mid g\} = \frac{\{G_1 \land G_2 \land g\}}{\Pr\{g\}} = \frac{1/4}{1/2} = \frac{1}{2}$$

After-class Exercise

A new couple, known to have two children, has just moved into town. Suppose that the mother is encountered walking with one of her children. If this child is a girl, what is the probability that both children are girls?

Searching an Unsorted Array (CLRS Problem $5-2$ (f))

```
procedure DETERMINISTIC-SEARCH(A[1···n], x)
    i ← 1
    while i ≤ n do
        if A[i] = x then
            return true
        i ← i + 1
    return false
```

Searching an Unsorted Array (CLRS Problem $5-2$ (f))

procedure DETERMINISTIC-SEARCH($A[1 \cdots n], x$)
    $i \leftarrow 1$
    while $i \leq n$ **do**
        if $A[i] = x$ **then**
            return *true*
        $i \leftarrow i + 1$
    return *false*

(e)

$$\exists!\ i : A[i] = x$$

(f)

$$\exists!_k\ i : A[i] = x$$

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Y : # of comparisons

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$$\mathbb{E}[Y] = \sum_{i=1}^{n} i \Pr\{Y = i\}$$

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Y : # of comparisons

$$\begin{aligned}\mathbb{E}[Y] &= \sum_{i=1}^{n} i \Pr\{Y = i\} \\ &= \sum_{i=1}^{n} i \Pr\{A[i] = x\}\end{aligned}$$

$$\exists!\ i : A[i] = x$$

Y : $\#$ of comparisons

$$\begin{aligned}
\mathbb{E}[Y] &= \sum_{i=1}^{n} i \Pr\{Y = i\} \\
&= \sum_{i=1}^{n} i \Pr\{A[i] = x\} \\
&= \frac{1}{n} \sum_{i=1}^{n} i
\end{aligned}$$

$$\exists!\ i : A[i] = x$$

Y : # of comparisons

$$\begin{aligned}\mathbb{E}[Y] &= \sum_{i=1}^{n} i \Pr\{Y = i\} \\ &= \sum_{i=1}^{n} i \Pr\{A[i] = x\} \\ &= \frac{1}{n} \sum_{i=1}^{n} i \\ &= \frac{n+1}{2}\end{aligned}$$

$$\exists!_k \; i : A[i] = x$$

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$$\mathbb{E}[Y] = \sum_{i=1}^{n-k+1} i \Pr\{Y = i\}$$

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$$\begin{aligned}\mathbb{E}[Y] &= \sum_{i=1}^{n-k+1} i \Pr\{Y = i\} \\ &= \sum_{i=1}^{n-k+1} i \Pr\{i \text{ is the first index among } k \text{ indices } \textit{s.t. } A[i] = x\}\end{aligned}$$

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&= \sum_{i=1}^{n-k+1} i \frac{\binom{n-i}{k-1}}{\binom{n}{k}}
\end{aligned}$$

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&= \sum_{i=1}^{n-k+1} i \frac{\binom{n-i}{k-1}}{\binom{n}{k}} = \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} i \binom{n-i}{k-1}
\end{aligned}$$

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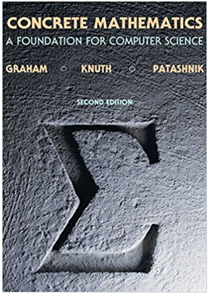
&= \frac{1}{\binom{n}{k}} \binom{n+1}{k+1} = \frac{n+1}{k+1}
\end{aligned}$$

$$k = 1 \implies \mathbb{E}[Y] = \frac{n+1}{2}, \qquad k = n \implies \mathbb{E}[Y] = 1$$

After-class Exercise

$$\sum_{i=1}^{n-k+1} i\binom{n-i}{k-1} = \binom{n+1}{k+1}$$



$$r\binom{r-1}{k-1} = k\binom{r}{k}$$

$$\sum_{0\le k\le n}\binom{k}{m} = \binom{n+1}{m+1}$$

Chapter 5: Binomial Coefficients

$$
\begin{aligned}
\sum_{i=1}^{n-k+1} i\binom{n-i}{k-1} &= \sum_{i=0}^{n-k}(i+1)\binom{n-i-1}{k-1} \\
&= \sum_{i=0}^{n-k}\left((n+1)-(n-i)\right)\binom{n-i-1}{k-1} \\
&= \sum_{i=0}^{n-k}(n+1)\binom{n-i-1}{k-1} - \sum_{i=0}^{n-k}(n-i)\binom{n-i-1}{k-1} \\
&= (n+1)\sum_{i=0}^{n-k}\binom{n-i-1}{k-1} - k\sum_{i=0}^{n-k}\binom{n-i}{k} \\
&= (n+1)\sum_{m=k-1}^{n-1}\binom{m}{k-1} - k\sum_{m=k}^{n}\binom{m}{k} \\
&= (n+1)\binom{n}{k} - k\binom{n+1}{k+1} = \binom{n+1}{k+1}
\end{aligned}
$$

Indicator Random Variables

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$$I_i = \begin{cases} 1, & \text{if } A[i] \text{ is checked} \\ 0, & \text{o.w.} \end{cases}$$

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$$\mathbb{E}[Y] = \mathbb{E}\left[\sum_{i=1}^{n} I_i\right] = \sum_{i=1}^{n} \mathbb{E}[I_i] = \sum_{i=1}^{n} \Pr\{I_i = 1\}$$

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$$\Pr\{I_i = 1\} = \begin{cases} \frac{1}{k}, & \text{if } A[i] = x \\ \frac{1}{k+1}, & \text{if } A[i] \neq x \end{cases}$$

Indicator Random Variables

$$I_i = \begin{cases} 1, & \text{if } A[i] \text{ is checked} \\ 0, & \text{o.w.} \end{cases}$$

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$$\Pr\{I_i = 1\} = \begin{cases} \frac{1}{k}, & \text{if } A[i] = x \\ \frac{1}{k+1}, & \text{if } A[i] \neq x \end{cases}$$

$$\mathbb{E}[Y] = \sum_{i=1}^{n} \Pr\{I_i = 1\} = k \cdot \frac{1}{k} + (n-k) \cdot \frac{1}{k+1} = \frac{n+1}{k+1}$$


Wrong!

$$\Pr\{I_i = 1\} = \begin{cases} \frac{1}{k}, & \text{if } A[i] = x \\ \frac{1}{k+1}, & \text{if } A[i] \neq x \end{cases}$$

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$$i = 1 \implies \Pr\{I_1 = 1\} = 1$$

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NOT IID

(Independent and Identically Distributed)

$$\mathbb{E}[Y] = \sum_{i=1}^{n-k+1} \Pr\{Y \geq i\}$$

$$\begin{aligned}\mathbb{E}[Y] &= \sum_{i=1}^{n-k+1} \Pr\{Y \geq i\} \\ &= \sum_{i=1}^{n-k+1} \frac{\binom{n-i+1}{k}}{\binom{n}{k}}\end{aligned}$$

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&= \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} \binom{n-i+1}{k}
\end{aligned}
$$

$$
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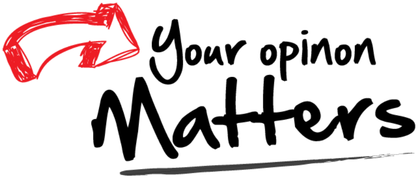
\end{aligned}
$$

order statistics?
balls-into-bins?


Thank
You!



Office 302

Mailbox: H016

hfwei@nju.edu.cn