# 3-11 Matchings and Factors

## (Part I: Matchings and Covers)

Hengfeng Wei

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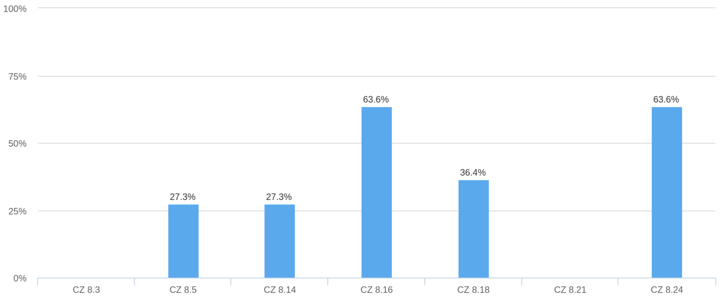
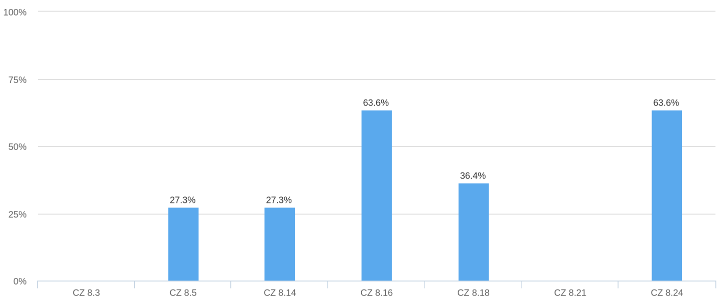
December 10, 2018




8.5 8.14 8.16

8.18 8.24 (The Last Class)




8.5 8.14 8.16 Chinese Postman Problem (The Last Class?)

8.18 8.24 (The Last Class)

比较大的定理（证明比较长的）都不是很理解，想知道期末考什么

点覆盖边覆盖那里只知道有这些性质，了解不是很深

都理解

图的分解的形象意义

无

定理8.3的证明

αβ、α'β'的定义和几个定理推论

为什么中英文书上的定义中\alpha和\beta反了。。

定理8.10的证明看不懂；一些比较几何的构造法证明（比如把顶点排成正多边形，一个点放中间）是怎么保证这些分解不重不漏的?

Kirkman三元系

$$\alpha, \quad \beta, \quad \alpha', \quad \beta'$$

Theorem 8.10 (Tutte's Theorem) (The Last Class)

**Theorem (Hall's Theorem, 1935; Theorem 8.3)**

*Let $G$ be a bipartite graph with partite sets $U$ and $W$ such that $r = |U| \leq |W|$.*
*$G$ contains a matching of cardinality $r \iff G$ satisfies Hall's Condition:*

$$\forall X \subseteq U : \left|N(X)\right| \geq \left|X\right|$$

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TONCAS

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Other TONCAS?

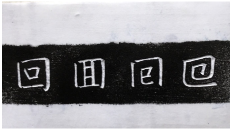
Perfect Matching on Trees (Problem 8.5)

Prove that every tree $T$ has $\leq 1$ perfect matching.

## Perfect Matching on Trees (Problem 8.5)

Prove that every tree $T$ has $\leq 1$ perfect matching.

“这题有四样证法, 你知道吗?”



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2:
3: else                                   ▷ n is even
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**if** $n$ is odd **then**
  $\#$ Perfect Matching $= 0$
**else** ▷ $n$ is even
  Consider $T - r$ ▷ $r$ : root of $G$
  **if** $k_o(T - r) > 1$ **then**

  **else** ▷ $k_o(T - r) = 1$

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        By Induction Hypothesis.

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    Consider a leaf $v$
    $v$ must be matched with its parent $u$

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if T has no perfect matchings then
    # Perfect Matching = 0
else                                        ▷ T has perfect matchings
    Consider a leaf v
    v must be matched with its parent u
    By Induction Hypothesis on each component of G − {u, v}
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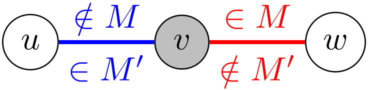

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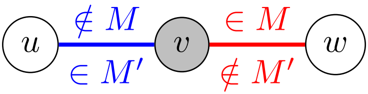


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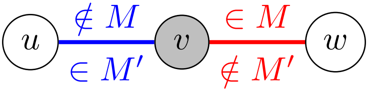


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Contradiction: Cycle

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$$M \Delta M' = (M - M') \cup (M' - M)$$

Consider the subgraph $\mathcal{M}$ with $V(T)$ and $M \Delta M'$.

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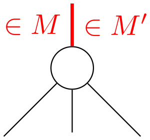
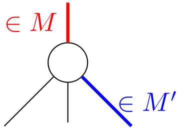



Case I

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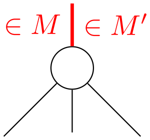


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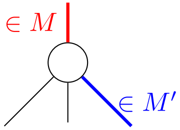


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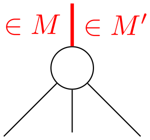


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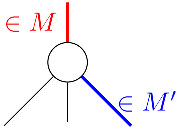


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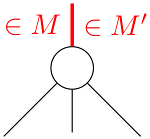


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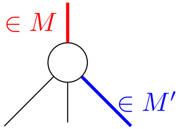


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$$\alpha(G)$$

$$\beta(G)$$

$$\alpha'(G)$$

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| | |
|---|---|
| $\alpha(G)$ | Maximum size of independent set |
| $\beta(G)$ | Minimum size of vertex cover |
| $\alpha'(G)$ | Maximum size of matching |
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**Theorem (Gallai Identities, 1959; Theorem 8.7)**

*If $G$ is graph without isolated vertices, then*

$$\alpha'(G) + \beta'(G) = n(G).$$

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Theorem (König, 1931; Egerváry, 1931)

*If $G$ is a bipartite graph, then*

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Theorem (König, 1931)

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If $G$ is a graph of order $n$, maximum degree $\Delta$ and having no isolated vertices, then

$$\beta(G) \geq \frac{n}{\Delta + 1}.$$

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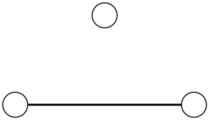

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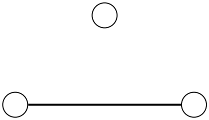
By Contradiction: $\beta < \frac{n}{\Delta+1}$.



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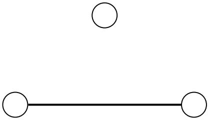

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$$\begin{aligned} \beta \cdot \Delta &< \frac{n\Delta}{\Delta + 1} \\ &= n - \frac{n}{\Delta + 1} \\ &\leq n - 1 \end{aligned}$$

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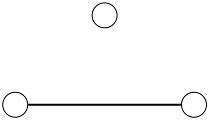

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Contradiction: No isolated vertices.

### Vertex Independence Number (Additional Problem)

If $G$ is a graph of order $n$, maximum degree $\Delta$, then

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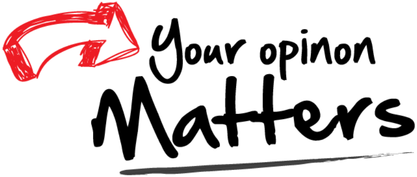
To construct an independent set $S$ with $|S| \geq \frac{n}{\Delta+1}$.

1: **while** $|V(G) > 0|$ **do**
2: Choose $v \in V(G)$
3: $S \leftarrow S \cup \{v\}$
4: $G \leftarrow G - \{v\} - N(v)$


THANK YOU

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