Dimension theory

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The part of topology in which for every compactum, and subsequently also for more general classes of topological spaces, there is defined in some natural way a numerical topological invariant, the dimension, which coincides if X is a polyhedron (in particular, a manifold) with the number of its coordinates in the sense of elementary or differential geometry. The first general definition of dimension was given by L.E.J. Brouwer (1913) for compacta and even for the wider class of complete metric spaces. The definition is constructed inductively as follows. The empty set is assigned dimension -1. Assuming that the spaces of dimension $\leq n$, and hence their subsets, have been defined, one says that a space X has dimension $\leq n+1$ if between any two disjoint closed sets A and B of X there is a partition Φ of dimension $\leq n$ (here a partition between two sets A and B in a space X is a closed subset Φ of this space such that the complement $X \setminus \Phi$ is the sum of two disjoint open sets C and D, one of which contains A and the other one B). In 1921, P.S. Urysohn and K. Menger, independently of Brouwer and each other, arrived at a definition, equivalent in the case of compacta and even of arbitrary separable metric spaces, which differs from Brouwer's definition in that one of the two closed sets A, B is supposed to consist of one point. The definitions of dimension in the senses of Brouwer and Urysohn–Menger can be formulated for arbitrary Hausdorff spaces. The topological invariants that they define are called the large and the small inductive dimension, respectively, and are denoted by $\operatorname{Ind} X$ and $\operatorname{ind} X$. Always $\operatorname{ind} X \leq \operatorname{Ind} X$.

A completely different approach to the concept of dimension originated from H. Lebesgue, who stated the following theorem: An n-dimensional cube Q^n in the sense of elementary geometry can be covered by a finite number of closed sets (even cubes) of diameter $< \epsilon$, for any positive number ϵ, in such a way that the multiplicity (or order) of this covering is $n+1$, whereas, for a sufficiently small $\epsilon > 0$, there is no covering of Q^n with multiplicity $< n+1$ and consisting of closed sets of diameter $< \epsilon$ (here the multiplicity of some (finite) collection of sets is the largest integer m for which in the given collection there are m sets with non-empty intersection). Nowadays one can reformulate Lebesgue's theorem as follows: The number of coordinates of the cube Q^n is the smallest integer n for which there is an arbitrarily fine (that is, consisting of elements with arbitrarily small diameter) covering of multiplicity $n+1$ by closed sets. This theorem, proved for the first time by Brouwer, leads to the following definition. The (covering) dimension $\dim X$ of a compactum X is the smallest number n such that for any $\epsilon > 0$ the compactum X has a covering of multiplicity $n+1$ consisting of closed sets of diameter $\leq \epsilon$. Without changing the content of this definition, one may replace closed sets by open sets in its formulation.

When defining the dimension of a compactum X, one uses the notion of the diameter of a set, which is related to the metric, rather than the topology, of X. However, it can be proved that the number $\dim X$ thus defined is nevertheless a topological invariant of X, that is, two homeomorphic compacta have the same dimension. This fact can be established directly, but it is also easily deducible from the fact that $\dim X$ can also be given by a direct topological definition, relying only on the topology of X.

A covering of a given topological space is any finite collection of open subsets whose union is the whole space. A covering α' is finer than a covering α if α' is inscribed in α, that is, if each element of α' is a subset of at least one element of α. It turns out that the dimension $\dim X$ can be defined as follows: The number $\dim X$ is the smallest integer n such that for every covering of X there is a covering of multiplicity $n+1$ inscribed in it. However, this definition can obviously be formulated not only for compacta, but for arbitrary topological spaces, and makes it possible to define a dimension for them. The dimension $\dim X$ defined in this way for topological spaces enables one to construct a significant theory that is rich in mathematical facts, at least if one stays in the class of normal spaces (and hence, in particular, also metrizable spaces).

One of the main problems of dimension theory is the determination of the widest conditions under which the fundamental Urysohn identity holds, namely

$$\operatorname{ind} X = \operatorname{Ind} X = \dim X .$$

It turns out that it holds for all separable metrizable spaces, that is, for all normal spaces with a countable base, and also for the spaces of locally compact Hausdorff groups (Pasynkov's theorem). Without assuming separability, for metrizable spaces one can assert only the validity of the Katětov formula

$$\operatorname{ind} X \leq \operatorname{Ind} X = \dim X,$$

and for Hausdorff compacta the Aleksandrov formula

$$\dim X \leq \operatorname{ind} X \leq \operatorname{Ind} X .$$

There are, however, Hausdorff compacta X for which

$$\dim X \neq \operatorname{ind} X$$

(the Lunts–Lokutsievskii example) and Hausdorff compacta X for which

$$\operatorname{ind} X \neq \operatorname{Ind} X$$

(Filippov's example).

The following Nöbeling–Pontryagin theorem is very informative: A necessary and sufficient condition for a topological space X to be homeomorphic to a subspace of a Euclidean space of finite dimension is that X should be a normal space of finite dimension with a countable base. This makes it possible to regard finite-dimensional compacta and, in general, finite-dimensional normal spaces with a countable base as subspaces of Euclidean spaces.

In this connection, the so-called theorem on ϵ-shifts is of particular interest: A compactum X in some Euclidean space $\mathbf{R}^m$ has dimension $\dim X \leq n$ if and only if for any $\epsilon > 0$ it can be transformed into a polyhedron of dimension $\leq n$ by means of an ϵ-shift in the space $\mathbf{R}^m$ (here an ϵ-shift of a subspace X of a Euclidean space $\mathbf{R}^m$ is a continuous mapping f of X into the Euclidean space $\mathbf{R}^m$ containing it, under which the distance $\rho(x, fx)$ of any point $x \in X$ from its image fx is less than ϵ). The intuitive meaning of this theorem resides in the fact that every compactum of a given finite dimension n, regarded as a set in some Euclidean space $\mathbf{R}^m$, can by an arbitrarily small modification (which consists of the ϵ-shift) be transformed into a polyhedron of the same, but not of smaller, dimension. This theorem, as well as the definition of the dimension $\dim X$ for compacta, can be restated in purely topological terms, where again "arbitrarily small" numbers $\epsilon > 0$ are replaced by "arbitrarily fine" coverings ω. This makes it possible to formulate an analogous theorem for any normal space, and to arrive at the conclusion that in some (still visualizable) geometrical sense every n-dimensional normal space is "similar" and even "differs arbitrarily little from" an n-dimensional polyhedron.

One of the main theorems in dimension theory is the so-called theorem on essential mappings, which lies at the foundation of a considerable part of this theory. Let f be a continuous mapping from a (normal) space X onto an n-dimensional ball Q^n with boundary S^{n-1}. Let $\Phi \subset X$ be the pre-image of the sphere S^{n-1} under this mapping, $\Phi = f^{-1} S^{n-1}$. A mapping $f: X \rightarrow Q^n$ is called essential if every continuous mapping $g: X \rightarrow Q^n$ that coincides with f at all points $x \in \Phi$ is a mapping onto the whole ball Q^n. The celebrated Aleksandrov theorem states that a normal space X has dimension $\dim X \geq n$ if and only if X can be essentially mapped onto an n-dimensional ball. From this theorem one can deduce the sum theorem (proved for compacta by Urysohn and Menger already at the very beginning of the development of dimension theory): If a (normal) space X of dimension $\dim X = n$ is the union of a finite or countable number of closed subsets Φ_k, then for at least one of these Φ_k one has $\dim \Phi_k = n$.

The theorem on essential mappings lies at the base of so-called homological dimension theory, which makes it possible to apply methods of algebraic topology to the study of dimension under rather general assumptions. The concept of homological dimension of a space is connected with the concepts of a cycle and homology, and hence assumes that alongside with the topological space X one is also given a commutative group $\mathfrak{G}$, called the coefficient group. Then one can speak of cycles of the compactum X with this coefficient group, of their supports $\Phi \subset X$, and, in particular, of cycles homologous to zero in X for the coefficient domain $\mathfrak{G}$, where these concepts can be equivalently understood both in the sense of the Aleksandrov–Čech homology theory, and in the sense of the Vietoris homology theory.

After this, one can define the homological dimension of a compactum X with coefficient group $\mathfrak{G}$ as the largest integer n for which X has an $(n-1)$-dimensional cycle z^{n-1} homologous to zero in X, but not homologous to zero on some support Φ of it. It turns out that if $\dim X < \infty$, then $\dim X$ is the homological dimension for the group κ which is the quotient group of the group of all real numbers by the subgroup of integers, and is the largest of all homological dimensions.

If from cycles and homology one moves to cocycles and cohomology, then one obtains the cohomological dimension. Moreover, the cohomological dimension of a compactum for a given discrete group $\mathfrak{A}$ is the homological dimension for the compact Hausdorff group $\mathfrak{B}$ dual to $\mathfrak{A}$ in the sense of Pontryagin's theory of characters. Hence it follows that if $\dim X < \infty$, then $\dim X$ coincides with the cohomological dimension for the group of integers.

For references see Dimension.

Comments

The small and large inductive dimensions are usually defined for the classes of regular and normal spaces, respectively. Outside these classes they show pathological behaviour.

Katětov's formula was obtained independently by K. Morita.

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