3-5 Minimum Spanning Trees

Hengfeng Wei

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$$\sum_{v \in V(G)} \deg(v) \leq 2(n-1)$$

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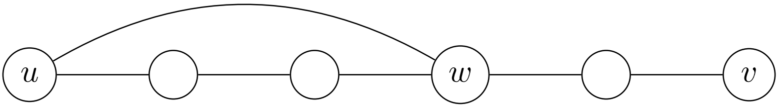

maximal path $P_{u,v}$

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$$G: \text{ a connected graph}$$

$$e \in E(G) \text{ is a bridge} \iff e \in \forall \text{ST of } G$$

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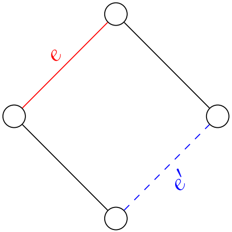

$$T' = \underbrace{T - \{e\}}_{e \in T} + \{e'\}$$

Cut Property

Cut Property (Version I)

$$X: \text{A part of some MST } T \text{ of } G$$

$$(S, V \setminus S): \text{A } \textit{cut} \text{ such that } X \text{ does } \textit{not} \text{ cross } (S, V \setminus S)$$

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Correctness of Prim's and Kruskal's algorithms.

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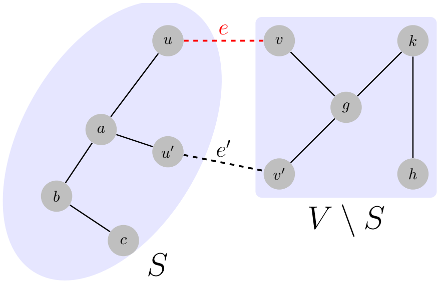

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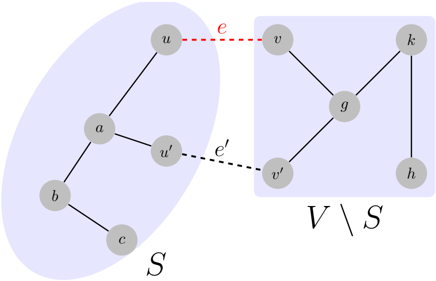

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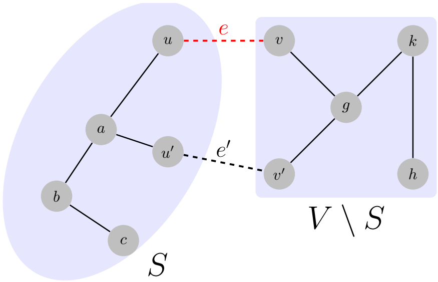

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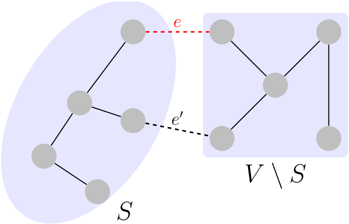
"a" $\rightarrow$ "the" $\Longrightarrow$ "some" $\rightarrow$ "all"

Cut Property (Version II)

A cut $(S, V \setminus S)$

Let $e = (u, v)$ be *a* lightest edge across $(S, V \setminus S)$

$\exists$ MST T of $G : e \in T$



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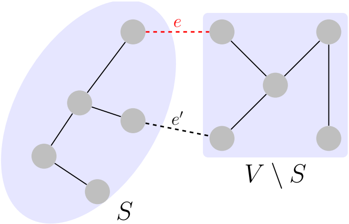

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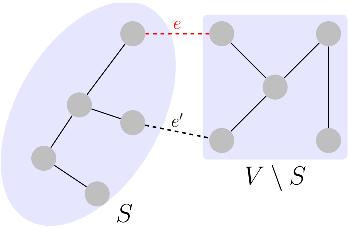

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Wrong Divide&Conquer Algorithm for MST

$$(V_1, V_2) : \Big| |V_1| - |V_2| \Big| \le 1$$

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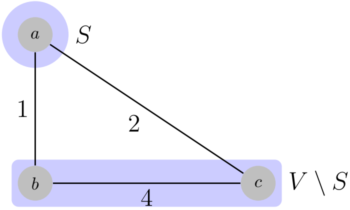

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- Let C be any cycle in G
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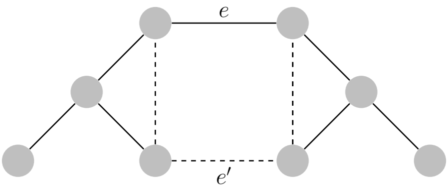

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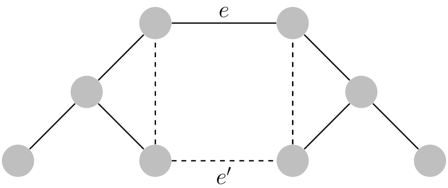

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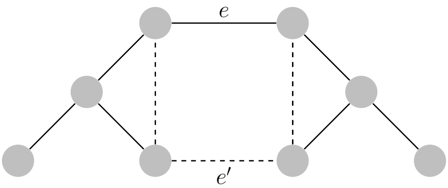

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Reverse-delete algorithm (wiki; clickable)

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"On the Shortest Spanning Subtree of a Graph and the Traveling Salesman Problem"

— Kruskal, 1956.

Application of Cycle Property (Problem 4.30)

T : a MST of a connected weight graph G

T is a unique MST of G

$$\Longleftrightarrow$$

$$\forall e \in G \setminus T : w(e) > w(\text{every other edge on the cycle in } T + e)$$

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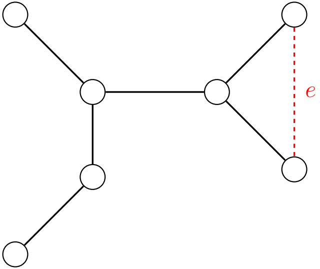

Uniqueness of MST

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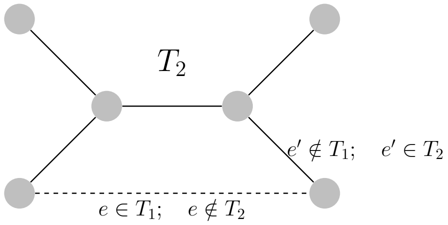
T_2
$e' \notin T_1; \quad e' \in T_2$
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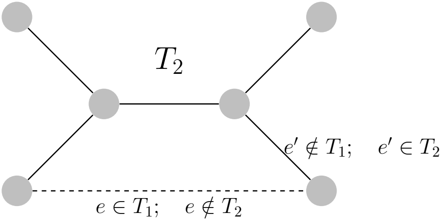

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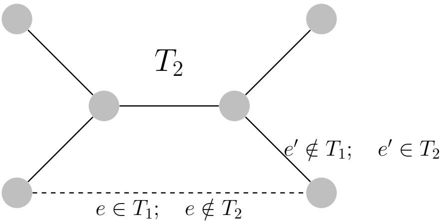

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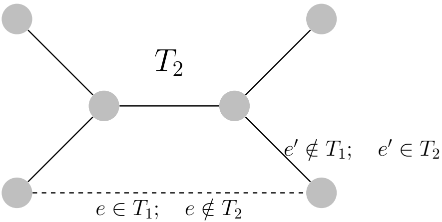

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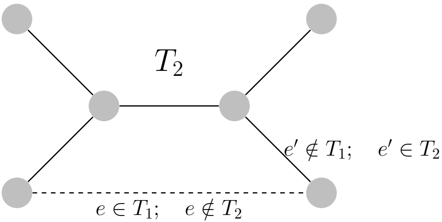

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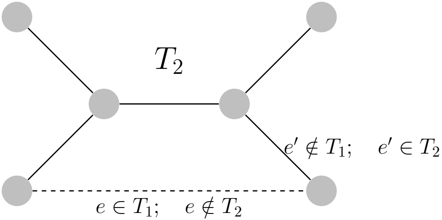

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$$T' = T_2 + \{e\} - \{e'\} \implies w(T') < w(T_2)$$

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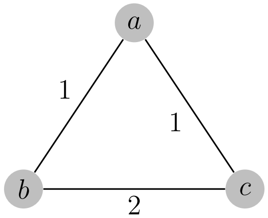

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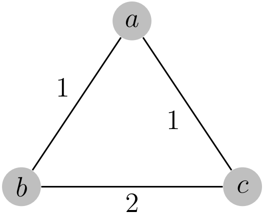

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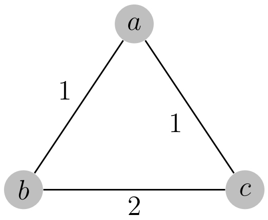

Theorem (After-class Exercise)

Minimum-weight edge across any cut is unique $\Longrightarrow$ *Unique MST.*

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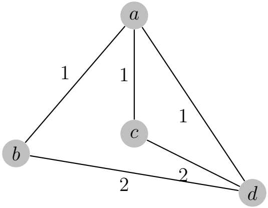

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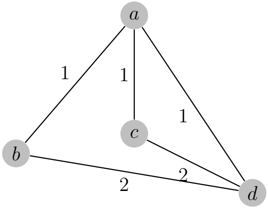

Theorem (After-class Exercise)

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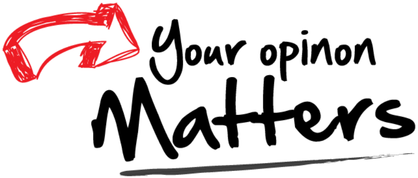
THANK YOU



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