Nowhere continuous function

In mathematics, a **nowhere continuous function**, also called an **everywhere discontinuous function**, is a function that is not continuous at any point of its domain. If f is a function from real numbers to real numbers, then f is nowhere continuous if for each point x there is an $\varepsilon > 0$ such that for each $\delta > 0$ we can find a point y such that $0 < |x - y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon$. Therefore, no matter how close we get to any fixed point, there are even closer points at which the function takes not-nearby values.

More general definitions of this kind of function can be obtained, by replacing the absolute value by the distance function in a metric space, or by using the definition of continuity in a topological space.

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Dirichlet function

One example of such a function is the indicator function of the rational numbers, also known as the **Dirichlet function**, named after German mathematician Peter Gustav Lejeune Dirichlet.[1] This function is written $I_{\mathbf{Q}}$ and has domain and codomain both equal to the real numbers. $I_{\mathbf{Q}}(x)$ equals 1 if x is a rational number and 0 if x is not rational. If we look at this function in the vicinity of some number y, there are two cases:

- If y is rational, then $f(y) = 1$. To show the function is not continuous at y, we need to find an ε such that no matter how small we choose δ, there will be points z within δ of y such that $f(z)$ is not within ε of $f(y) = 1$. In fact, 1/2 is such an ε. Because the irrational numbers are dense in the reals, no matter what δ we choose we can always find an irrational z within δ of y, and $f(z) = 0$ is at least 1/2 away from 1.
- If y is irrational, then $f(y) = 0$. Again, we can take $\varepsilon = 1/2$, and this time, because the rational numbers are dense in the reals, we can pick z to be a rational number as close to y as is required. Again, $f(z) = 1$ is more than 1/2 away from $f(y) = 0$.

In less rigorous terms, between any two irrationals, there is a rational, and vice versa.

The Dirichlet function can be constructed as the double pointwise limit of a sequence of continuous functions, as follows:

$$f(x) = \lim_{k\to\infty}\left(\lim_{j\to\infty}\left(\cos(k!\pi x)\right)^{2j}\right)$$

for integer j and k.

This shows that the Dirichlet function is a Baire class 2 function. It cannot be a Baire class 1 function because a Baire class 1 function can only be discontinuous on a meagre set.[2]

In general, if E is any subset of a topological space X such that both E and the complement of E are dense in X, then the real-valued function which takes the value 1 on E and 0 on the complement of E will be nowhere continuous. Functions of this type were originally investigated by Peter Gustav Lejeune Dirichlet.

Hyperreal characterisation

A real function f is nowhere continuous if its natural hyperreal extension has the property that every x is infinitely close to a y such that the difference $f(x) - f(y)$ is appreciable (i.e., not infinitesimal).

See also

- Thomae's function (also known as the popcorn function) — a function that is continuous at all irrational numbers and discontinuous at all rational numbers.
- Weierstrass function: A function *continuous* everywhere (inside its domain) and *differentiable* nowhere.

References

1. Lejeune Dirichlet, P. G. (1829) "Sur la convergence des séries trigonométriques qui servent à répresenter une fonction arbitraire entre des limites donées" [On the convergence of trigonometric series which serve to represent an arbitrary function between given limits], *Journal für reine und angewandte Mathematik* [Journal for pure and applied mathematics (also known as *Crelle's Journal*)], vol. 4, pages 157–169.
2. Dunham, William (2005). *The Calculus Gallery*. Princeton University Press. p. 197. ISBN 0-691-09565-5.

External links

- Hazewinkel, Michiel, ed. (2001) [1994], "Dirichlet-function" (https://www.encyclopediaofmath.org/index.php?title=p/d032860), *Encyclopedia of Mathematics*, Springer Science+Business Media B.V. / Kluwer Academic Publishers, ISBN 978-1-55608-010-4
- Dirichlet Function — from MathWorld (http://mathworld.wolfram.com/DirichletFunction.html)
- The Modified Dirichlet Function (http://demonstrations.wolfram.com/TheModifiedDirichletFunction/) by George Beck, The Wolfram Demonstrations Project.