1-13 Boolean Algebra

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Definition (Boolean Algebra)

A *boolean algebra* $\mathcal{B} = (B, \wedge, \vee, ', 0, 1)$ is a bounded, complemented, and distributive lattice.

$$\forall a, b, c \in B,$$

Idempotency:

Commutativity:

Associativity:

Absorption:

Complements:

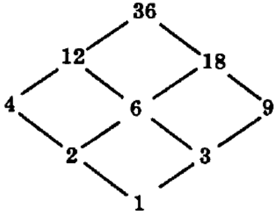
$$a \wedge a' = 0 \quad a \vee a' = 1$$

Distributivity:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \qquad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

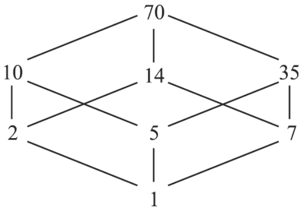
Problem 2: D_n

D_n is a boolean algebra if and only if $n = p_1 p_2 \cdots p_k$ for some k, where all p_i are distinct primes.



D_{36}

$36 = 2^2 \times 3^2$



D_{70}

$70 = 2 \times 5 \times 7$

D_n is a boolean algebra

$$\implies D_n \text{ is a complemented distributive lattice}$$

$$\implies \forall x \in D_n : x \text{ has a complement}$$

$$\implies \forall x \in D_n : \exists y \in D_n : (x \wedge y = 0) \wedge (x \vee y = 1)$$

$$\implies \forall x \in D_n : \exists y \in D_n : \texttt{gcd}(x, y) = 1 \wedge \texttt{lcm}(x, y) = n$$

$$\implies \forall x \in D_n : \exists y \in D_n : xy = n \wedge (x, y) = 1$$

$$\implies \forall x \in D_n : (x, n/x) = 1$$

$$\implies n = p_1 p_2 p_k \wedge \text{all } p_i \text{ are unique primes}$$

Problem 3: Atom

Let $\mathcal{B} = (B, \leq)$ is a Boolean algebra.

$$\forall a \in B : \mathsf{Atom}(a) = \{x \leq a \mid x \text{ is an atom}\}$$

Suppose $\mathcal{B}$ is finite. To prove:

$$\forall a \in B : a \neq 0 \implies \mathsf{Atom}(a) \neq \emptyset.$$

Atoms: those elements which *immediately* succeed 0

$$\forall a \in B : a \neq 0 \implies \mathsf{Atom}(a) \neq \emptyset.$$

By contradiction.

$$a \neq 0 \land \mathsf{Atom}(a) = \emptyset$$

$$\implies a \text{ is not an atom} \quad (o.w.,\ a \in \mathsf{Atom}(a))$$

$$\implies \exists x_1 : 0 < x_1 < a \quad (a \neq 0)$$

$$\implies x_1 \text{ is not an atom} \quad (o.w.,\ x_1 \in \mathsf{Atom}(a))$$

$$\implies \exists x_2 : 0 < x_2 < x_1 \quad (x_1 \neq 0)$$

$$\implies \cdots$$

$$\boxed{\cdots < x_2 < x_1 < a}$$

Problem 4: Isomorphic

All finite Boolean algebras of the same cardinality are isomorphic.

Theorem (Representation Theorem for Finite Boolean Algebras)

Every finite Boolean algebra is isomorphic to a Boolean algebra $\mathcal{P}(X)$ for some finite set X.

Additional Problem: Isomorphic

Is every Boolean algebra isomorphic to $\mathcal{P}(X)$ for some set X?

Finite-Cofinite Algebra

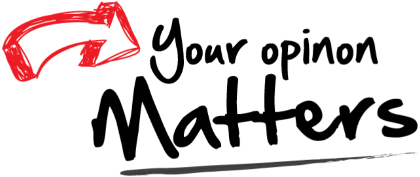
$$\boxed{F(\mathbb{N}) = \{X \subseteq \mathbb{N} \mid X \text{ is finite} \lor \mathbb{N} \setminus X \text{ is finite}\}}$$

$$|F(\mathbb{N})| = \aleph_0$$

If $F(\mathbb{N})$ is isomorphic to $\mathcal{P}(X)$ for some X:

$$|F(\mathbb{N})| \geq 2^{\aleph_0}$$


Thank
You!



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