JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 22, 96–111 (1968)

Applications of Menger's Graph Theorem

HAZEL PERFECT

Department of Pure Mathematics, The University, Sheffield, England

Submitted by Richard Bellman

In their book "Flows in Networks" [4], Ford and Fulkerson devote an interesting section (Chapter II, Section 10) to the discussion of a number of combinatorial theorems on representatives of sets, and they deduce these results from the fundamental max-flow min-cut theorem. This latter theorem may not unreasonably be regarded as a quantified version of Menger's graph theorem, and in the present paper we derive similar combinatorial theorems from Menger's theorem itself. In a final section, Menger's theorem is brought to bear on the study of a notion of independence which generalizes that of transversal independence [3, 16]. Our account is largely expository in the sense that the techniques we use are, for the most part, implicit in the literature of combinatorial analysis. It seems, nevertheless, appropriate to present a unified derivation of some of the consequences of Menger's theorem.

I am grateful to Professor R. Rado for describing to me a very simple deduction of Hall's criterion for systems of distinct representatives from Menger's theorem and arousing my interest in the possibility of other similar deductions. Also, in the course of preparing this paper, I have been able to discuss both content and presentation with Dr. L. Mirsky, and I am grateful for the valuable help he has given me.

1. MENGER'S THEOREM

Throughout, we shall use the word "graph" to mean a directed graph which may generally be either finite or infinite. (An undirected graph would serve equally well in some of the arguments.) Let $G = [N; \mathscr{E}]$ be a graph, N being its nonempty set of nodes and $\mathscr{E}$ its set of edges. By a *path* in G we shall understand a finite sequence

$$a_1, (a_1, a_2), a_2, \ldots, (a_{m-1}, a_m), a_m \tag{1.1}$$

of distinct nodes a_i $(1 \leqslant i \leqslant m)$ and distinct edges (a_{j-1}, a_j) $(2 \leqslant j \leqslant m)$. Here (a_{j-1}, a_j) is to be distinguished from (a_j, a_{j-1}), since the graph is directed; (a_{j-1}, a_j) denotes the edge directed *from* a_{j-1} *to* a_j. The nodes

a_1, a_m are called respectively the *initial* and *terminal* nodes of the path (1.1); the other nodes are called *intermediate*.

A set of paths in G, no two of which have a node in common will be said to be *pairwise node-disjoint*, abbreviated to *pnd*. If L, $M \subseteq N$ and $a_1 \in L$, $a_m \in M$, we shall speak of (1.1) as a *path from L to M* in G. Again, if $K \subseteq N$, we shall understand by $G - K$ the graph obtained from G by the deletion of each node in K and also each edge which has at least one end in K.

Now let L, $M \subseteq N$, $L \cap M = \phi$. A subset $K \subseteq N$ is said to *separate M from L* in G if there is no path from L to M in the graph $G - K$. It is at once clear that if there are n *pnd* paths from L to M in G, where n is a positive integer, then there must be at least n nodes in K. The following fundamental theorem, which asserts the converse of this statement, was proved for finite graphs by Menger [12] and extended to infinite graphs by Erdös; see for example [10, p. 247].

THEOREM 1.1. *If n is a positive integer, and no set of fewer than n nodes separates M from L in G, then there are (at least) n pnd paths from L to M in G.*

Possibly the simplest proof of Menger's theorem is that due to Dirac [2]. Erdös's extension to infinite graphs is entirely straight-forward. Alternatively, Menger's theorem may be deduced from the max-flow min-cut theorem of Ford and Fulkerson; see for example [4, p. 55].

It may be remarked that Theorem 1.1 remains true if the positive integer n in the statement is replaced by an infinite cardinal; see [1]. However, this extension would seem to be of limited interest, and we shall not need to appeal to it in the following sections.

2. REPRESENTATIVES OF FAMILIES OF SETS

Let $\mathfrak{A} = (A_i : i \in I)$ be a family[1] of subsets of a set E. A family $(e_k : k \in K)$ of elements of E is called a *system of representatives* of $\mathfrak{A}$ if, for some bijection $\theta : K \to I$, the relations $e_k \in A_{\theta(k)} (k \in K)$ are valid. If, in addition, the e_k are distinct, then $(e_k : k \in K)$ is called a *system of distinct representatives* (SDR) of $\mathfrak{A}$. A *transversal* of $\mathfrak{A}$ is the range of a SDR of $\mathfrak{A}$, and a *partial transversal* (PT) of $\mathfrak{A}$ is a transversal of a subfamily of $\mathfrak{A}$. The number of elements in a PT is called its *length*. Finally, if d is a nonnegative integer, a transversal of a subfamily of $\mathfrak{A}$ consisting of all but d of the sets A_i is called a *transversal of $\mathfrak{A}$ of defect d*.

[1] A "family" is a mapping by definition, and consequently we need not have $A_i \neq A_j$ when $i \neq j$.

Now let $\mathfrak{B} = (B_j : j \in J)$ be a second family of subsets of E. A family of elements of E which is a system of representatives of $\mathfrak{A}$ and also of $\mathfrak{B}$ is called a *system of common representatives* (SCR) of $\mathfrak{A}$ and $\mathfrak{B}$. A subset of E which is a transversal of $\mathfrak{A}$ and of $\mathfrak{B}$ is called a *common transversal* (CT) of $\mathfrak{A}$ and $\mathfrak{B}$; and a *common partial transversal* (CPT) of $\mathfrak{A}$ and $\mathfrak{B}$ is a CT of a subfamily of $\mathfrak{A}$ and a subfamily of $\mathfrak{B}$. *Length* is defined as for PT's of a single family.

Unless otherwise stated, I, J will be assumed throughout to be finite sets, though generally E may be finite or infinite. It is, of course, nothing more than a matter of convenience to retain both terms "length" and "defect" when speaking of a PT of a finite family.

A celebrated theorem of P. Hall [6] gives necessary and sufficient conditions for the existence of a transversal of a (finite) family $\mathfrak{A} = (A_i : i \in I)$; and we shall begin by describing the very simple deduction of this result from Menger's theorem.[2] Probably the easiest ad hoc proof of Hall's theorem is the one due to Halmos and Vaughan [7]. Hall's theorem appears as a special case of later results in this section, but we give an independent deduction in view of its extreme simplicity. In fact, we state and prove a slightly more general result, namely the "defect" form of Hall's theorem which was noted by Ore [18].

As is usual, we denote by $|X|$ the cardinal number of the set X. Further, for each $I' \subseteq I$, we denote by $A(I')$ the union $\cup (A_i : i \in I')$ of members of the subfamily of $\mathfrak{A}$ indexed by I'; and a similar notation is adopted for other families.

THEOREM 2.1. *Suppose* $0 \leqslant d < |I|$. *The family* $\mathfrak{A} = (A_i : i \in I)$ *of subsets of* E *has a transversal of defect* d *if and only if*

$$|A(I')| \geqslant |I'| - d \tag{2.1}$$

whenever $I' \subseteq I$.

Hall's theorem corresponds to the case $d = 0$.

The necessity of the conditions is clear. In order to deduce their sufficiency from Menger's theorem,[3] we first describe the appropriate graph $G = [N; \mathscr{E}]$. There is no essential loss of generality in supposing that $I \cap E = \phi$, and we shall repeatedly make this kind of assumption. We take

$$N = I \cup E, \qquad \mathscr{E} = \{(i, e) \in I \times E : e \in A_i\}.$$

If the set of nodes $\tilde{I} \cup \tilde{E}$, where $\tilde{I} \subseteq I$, $\tilde{E} \subseteq E$, separates E from I in G, then there can be no nodes in $I - \tilde{I}$ joined to nodes in $E - \tilde{E}$, and so

$$A(I - \tilde{I}) \subseteq \tilde{E}.$$

[2] M. Hall's infinite analogue [5] of P. Hall's theorem is not deducible by the methods of the present paper.

[3] Here, and often elsewhere, we mean Erdös's extension of Menger's theorem, since E may be infinite.

Therefore, from (2.1) with $I' = I - \tilde{I}$,

$$| I - \tilde{I} | - d \leqslant | \tilde{E} |;$$

i.e.,

$$| \tilde{I} | + | \tilde{E} | \geqslant | I | - d.$$

From Menger's theorem, therefore, there are at least $| I | - d$ *pnd* paths (which in this case are simply edges) from I to E in G, and this clearly means that $\mathfrak{A}$ has a transversal of defect d.

Next, we apply Menger's theorem to the deduction of certain results on CT's.

THEOREM 2.2. *Let p be a positive integer. The families $\mathfrak{A} = (A_i : i \in I)$, $\mathfrak{B} = (B_j : j \in J)$ of subsets of E have a CPT of length p if and only if*

$$| A(I') \cap B(J') | \geqslant | I' | + | J' | - | I | - | J | + p,$$

whenever $I' \subseteq I$, $J' \subseteq J$.

In the case $| I | = | J | = p$, this theorem gives necessary and sufficient conditions for the existence of a CT of the families $\mathfrak{A}$ and $\mathfrak{B}$, and is a special case of a very general result of Ford and Fulkerson on SCR's of two families; see [4, p. 74].

Instead of proving Theorem 2.2 directly, we consider a generalization in a somewhat different direction and then deduce Theorem 2.2. All the families considered in Theorem 2.3 are finite.

THEOREM 2.3. *Let p be a positive integer. Let $\mathfrak{A} = (A_i : i \in I)$, $\mathfrak{B} = (B_j : j \in J)$ be families of subsets of E, and let $\mathfrak{P} = (P_k : k \in K)$, $\mathfrak{Q} = (Q_\ell : \ell \in L)$ be families of subsets of I, J, respectively. Then there exist subfamilies of $\mathfrak{A}$ and $\mathfrak{B}$, whose index sets are PT's of length p of $\mathfrak{P}$, $\mathfrak{Q}$, respectively, and which possess a CT, if and only if*

$$\begin{aligned} &| A(I') \cap B(J') | + | P(K') | + | Q(L') | \\ &\geqslant | I' | + | J' | + | K' | + | L' | - | K | - | L | + p, \end{aligned} \tag{2.2}$$

whenever

$$K' \subseteq K, \qquad L' \subseteq L, \qquad I' \subseteq P(K'), \qquad J' \subseteq Q(L'). \tag{2.3}$$

This result was stated and proved by different means in [19]. We obtain Theorem 2.2 from Theorem 2.3 by taking $P_k = I$ for all $k \in K$ and $Q_\ell = J$ for all $\ell \in L$.

Consider the graph $G = [N; \mathscr{E}]$ defined as follows. We suppose that the sets, I, J, K, L, E are pairwise disjoint, and we take

$$N = K \cup I \cup E \cup J \cup L, \qquad \mathscr{E} = \mathscr{E}_1 \cup \mathscr{E}_2 \cup \mathscr{E}_3 \cup \mathscr{E}_4,$$

where

$$\mathscr{E}_1 = \{(k, i) \in K \times I : i \in P_k\}, \qquad \mathscr{E}_2 = \{(i, e) \in I \times E : e \in A_i\},$$
$$\mathscr{E}_3 = \{(e, j) \in E \times J : e \in B_j\}, \qquad \mathscr{E}_4 = \{(j, \ell) \in J \times L : j \in Q_\ell\}.$$

Now subfamilies of $\mathfrak{A}$ and $\mathfrak{B}$, whose index sets are PT's of length p of $\mathfrak{P}$ and $\mathfrak{Q}$, and which have a CT, will exists if and only if there are p *pnd* paths from K to L in G.

It is readily seen that the set of nodes

$$V = \tilde{K} \cup \tilde{I} \cup \tilde{E} \cup \tilde{J} \cup \tilde{L},$$

where

$$\tilde{K} = K - K', \qquad \tilde{L} = L - L', \qquad \tilde{I} = P(K') - I', \qquad \tilde{J} = Q(L') - J',$$
$$\tilde{E} = A(I') \cap B(J'),$$

and I', J', K', L' are any sets satisfying (2.3), separates L from K in G. But, if I', J', K', L' can be so chosen that (2.2) is false, then, for such a choice,

$$|\tilde{E}| + |\tilde{I}| + |\tilde{J}| < -|\tilde{K}| - |\tilde{L}| + p;$$

i.e.,

$$|V| < p.$$

Therefore, there cannot be as many as p *pnd* paths from K to L in G. This proves the necessity of the conditions.

Now suppose the conditions of the theorem hold, and let

$$V = \tilde{K} \cup \tilde{I} \cup \tilde{E} \cup \tilde{J} \cup \tilde{L},$$

where $\tilde{K} \subseteq K$, $\tilde{I} \subseteq I$, etc., be *any* set of nodes separating L from K in G. Then

$$A(P(K - \tilde{K}) - \tilde{I}) \cap B(Q(L - \tilde{L}) - \tilde{J}) \subseteq \tilde{E}.$$

For, otherwise, there is a point $e \in E - \tilde{E}$ such that $e \in A_i$ for some $i \in P(K - \tilde{K}) - \tilde{I}$ and $e \in B_j$ for some $j \in Q(L - \tilde{L}) - \tilde{J}$, i.e., there is a point $e \in E - \tilde{E}$ such that $e \in A_i$ for some $i \notin \tilde{I}$, $i \in P_k$ for some $k \in K - \tilde{K}$ and $e \in B_j$ for some $j \notin \tilde{J}$, $j \in Q_\ell$ for some $\ell \in L - \tilde{L}$; and this contradicts the assumption that V separates L from K in G. Therefore, from (2.2) with $I' = P(K - \tilde{K}) - \tilde{I}$, $J' = Q(L - \tilde{L}) - \tilde{J}$, $K' = K - \tilde{K}$, $L' = L - \tilde{L}$,

$$\begin{aligned} |\tilde{E}| \geqslant & -|P(K - \tilde{K})| - |Q(L - \tilde{L})| + |P(K - \tilde{K})| - \tilde{I}| \\ & + |Q(L - \tilde{L}) - \tilde{J}| + |K - \tilde{K}| + |L - \tilde{L}| - |K| - |L| + p \\ \geqslant & \; p - (|\tilde{I}| + |\tilde{J}| + |\tilde{K}| + |\tilde{L}|); \end{aligned}$$

i.e.,

$$|V| \geqslant p.$$

By Menger's theorem, therefore, there are at least p *pnd* paths from K to L in G; and so the proof is complete.

3. Marginal Elements for a Single Family

We turn now to a theorem (Theorem 3.1 below) equivalent to one of Hoffman and Kuhn [8] which gives necessary and sufficient conditions for the existence of a transversal of a (finite) family $\mathfrak{A} = (A_i : i \in I)$ of subsets of E containing a set $M \subseteq E$ of prescribed (marginal) elements. Hoffman and Kuhn's proof of their result makes use of the techniques of linear programming, but they also remarked that a straightforward argument by induction, along the lines of Halmos and Vaughan's proof of Hall's theorem, could be given. Here, we shall first deduce Theorem 3.1 from Menger's theorem and then indicate a particularly easy deduction on similar lines from Hall's theorem.

The simple device used in the proof of Theorem 3.1 is used again in Section 5 below, and so it is convenient to state it first in more general terms. It allows us to apply Menger's theorem to find conditions for the existence of *pnd* paths whose set of initial nodes contains a prescribed set.

Let $G = [N; \mathscr{E}]$ be a graph and L, M disjoint subsets of N with L finite. Let $L' \subseteq L$, and let k be an integer such that $|L'| \leqslant k \leqslant |L|$. We construct a new graph G' whose set of nodes is $N \cup S$, where S is any set satisfying the conditions $N \cap S = \phi$, $|S| = |L| - k$, and whose set of edges is

$$\mathscr{E} \cup \{(x, s) : x \in L - L', s \in S\}.$$

Lemma. *If there exist $|L|$ pnd paths from L to $M \cup S$ in G', then there exist k pnd paths from L to M in G, whose set of initial nodes contains the set L'.*

The proof is immediate. It is also easily seen that the result is false if L is infinite.

Theorem 3.1. *Let $M \subseteq E$. The family $\mathfrak{A} = (A_i : i \in I)$ of subsets of E has a transversal with M as a subset if and only if*

(1) $|A(I')| \geqslant |I'|$

and

(2) $|A(I') \cap M| \geqslant |I'| + |M| - |I|$,

whenever $I' \subseteq I$.

Hoffman and Kuhn stated the necessary and sufficient conditions in a different form, but it is not difficult to see that Theorem 3.1 is equivalent to their result.

The necessity of the conditions is easily verified. We shall prove their sufficiency now, first under the assumption that E is a finite set.

It is clear from (1) that $|E| \geqslant |I|$ and from (2) that $|I| \geqslant |M|$. Assume that $I \cap E = \phi$, and consider the graph $G = [N; \mathscr{E}]$ where

$$N = E \cup I, \qquad \mathscr{E} = \{(e, i) \in E \times I : e \in A_i\}.$$

Let G' be the graph whose set of nodes is $N \cup S$, where $N \cap S = \phi$, $|S| = |E| - |I|$, and whose set of edges is $\mathscr{E} \cup \{(e, s) : e \in E - M, s \in S\}$.

Now let $V = \tilde{E} \cup \tilde{I} \cup \tilde{S}$, where $\tilde{E} \subseteq E$, etc., be a set of nodes separating $I \cup S$ from E in G'. First, the only nodes in $E - M$ to which nodes in $S - \tilde{S}$ can be joined must lie in $\tilde{E}$; so either $\tilde{S} = S$ or $\tilde{E} \supseteq E - M$. Next, clearly

$$A(I - \tilde{I}) \subseteq \tilde{E};$$

and so

$$A(I - \tilde{I}) \cap M \subseteq \tilde{E} \cap M.$$

Therefore, from (1) with $I' = I - \tilde{I}$,

$$|\tilde{E}| \geqslant |I - \tilde{I}| = |I| - |\tilde{I}|;$$

and, from (2) with $I' = I - \tilde{I}$,

$$|\tilde{E} \cap M| \geqslant |I - \tilde{I}| + |M| - |I| = |M| - |\tilde{I}|.$$

Hence, if $\tilde{S} = S$,

$$\begin{aligned} |V| &= |\tilde{E}| + |\tilde{I}| + |\tilde{S}| \\ &= |\tilde{E}| + |\tilde{I}| + |E| - |I| \\ &\geqslant |E|; \end{aligned}$$

and, if $\tilde{E} \supseteq E - M$,

$$\begin{aligned} |V| &= |\tilde{E}| + |\tilde{I}| + |\tilde{S}| \\ &\geqslant |\tilde{E}| + |\tilde{I}| \\ &= |E - M| + |\tilde{E} \cap M| + |\tilde{I}| \\ &= |E - M| + |M| = |E|. \end{aligned}$$

Therefore, in both cases, $|V| \geqslant |E|$, and hence, by Menger's theorem, there are at least $|E|$ *pnd* paths from E to $I \cup S$ in G'. It follows from the lemma that there are $|I|$ *pnd* paths from E to I in G whose set of initial nodes contains the set M. This clearly implies that $\mathfrak{A}$ has a transversal with M as a subset, and the proof is complete in the case that E is finite.

Now suppose that E is infinite and that (1), (2) hold. If A_i is finite, write $\bar{A}_i = A_i$ and, if A_i is infinite, write $\bar{A}_i = (A_i \cap M) \cup X_i$, where X_i is any set with $|X_i| = |I|$, $X_i \subseteq A_i \cap (E - M)$. It follows easily from (1), (2) that, whenever $I' \subseteq I$,

$$|\bar{A}(I')| \geqslant |I'|$$

and

$$|\bar{A}(I') \cap M| \geqslant |I'| + |M| - |I|.$$

Therefore, by the above proof, $(\bar{A}_i : i \in I)$ possesses a transversal containing M, and so *a fortiori* does $\mathfrak{A}$.

The construction of G' from G in the proof of Theorem 3.1 is essentially equivalent to the formation of a new family $\mathfrak{A}'$ of subsets of E consisting of the sets A_i of $\mathfrak{A}$ together with $|E|-|I|$ copies of the set $E-M$. The given family $\mathfrak{A}$ has a transversal with M as a subset if and only if the family $\mathfrak{A}'$ has a transversal; and a straightforward deduction from Hall's theorem leads to another proof of Theorem 3.1.

Another approach to the theorem of Hoffman and Kuhn was described in [19]. This makes use of similar ideas to those treated in Section 5 below. For a discussion of the infinite analogue of Theorem 3.1, see [16]. This infinite version also appears in [13] as a corollary of a more general result on systems of representatives with repetition.

4. Marginal Elements for Two Families

We consider the (finite) families $\mathfrak{A} = (A_i : i \in I)$, $\mathfrak{B} = (B_j : j \in J)$ of subsets of E, where $|I| = |J|$, and we begin with the statement and proof of an analogue of Theorem 3.1.

Theorem 4.1. *Let $M \subseteq E$, and let $|I| = |J| = n$. The families $\mathfrak{A} = (A_i : i \in I)$, $\mathfrak{B} = (B_j : j \in J)$ of subsets of E have a CT containing M as a subset if and only if*

$$|A(I') \cap M| + |B(J') \cap M| + |A(I') \cap B(J') \cap (E - M)| \geqslant |I'| + |J'| + |M| - n, \tag{4.1}$$

whenever $I' \subseteq I$, $J' \subseteq J$.

The special case $B_j = E (j \in J)$ yields Theorem 3.1 again, but the proof which we describe below does not reduce in this case to the proof of Theorem 3.1 given in Section 3. In some respect it appears to be a simpler application of Menger's theorem, though the initial construction is less obvious. Nowhere do we need the assumption that E is finite.

We observe that the conditions (4.1) imply that $|M| \leqslant n$, and certainly, if $\mathfrak{A}$ and $\mathfrak{B}$ have a CT containing M, then again $|M| \leqslant n$.

Now consider the graphs G, G^* defined in the following way. Assume I, E, J are pairwise disjoint, and let $G = [N; \mathscr{E}]$, where

$$N = I \cup E \cup J, \qquad \mathscr{E} = \mathscr{E}_1 \cup \mathscr{E}_2;$$

and

$$\mathscr{E}_1 = \{(i, e) \in I \times E : e \in A_i\}, \qquad \mathscr{E}_2 = \{(e, j) \in E \times J : e \in B_j\}.$$

Let $M^{(1)}$, $M^{(2)}$ be sets with $|M^{(1)}| = |M^{(2)}| = |M|$, disjoint from each other and from $N - M$, and consider bijections of M into $M^{(1)}$ and $M^{(2)}$. For each $e \in M$, denote by $e^{(1)}$, $e^{(2)}$ its images in $M^{(1)}$, $M^{(2)}$ and, for each $X \subseteq M$, denote by $X^{(1)}$, $X^{(2)}$ its images in $M^{(1)}$, $M^{(2)}$. Take $G^* = [N^* : \mathscr{E}^*]$, where

$$N^* = I \cup M^{(2)} \cup (E - M) \cup M^{(1)} \cup J, \qquad \mathscr{E}^* = \mathscr{E}^*_{11} \cup \mathscr{E}^*_{12} \cup \mathscr{E}^*_{21} \cup \mathscr{E}^*_{22}$$

and

$$\mathscr{E}^*_{11} = \{(i, e^{(1)}) \in I \times M^{(1)} : e \in A_i\}, \quad \mathscr{E}^*_{12} = \{(i, e) \in I \times (E - M) : e \in A_i\},$$

$$\mathscr{E}^*_{21} = \{(e^{(2)}, j) \in M^{(2)} \times J : e \in B_j\}, \quad \mathscr{E}^*_{22} = \{(e, j) \in (E - M) \times J : e \in B_j\}.$$

Now, clearly, $\mathfrak{A}$ and $\mathfrak{B}$ have a CT containing M if and only if there are n *pnd* paths from I to J in G whose set of intermediate nodes contains M. The crux of the argument is the observation that this is the case if and only if there are $n + |M|$ *pnd* paths from $I \cup M^{(2)}$ to $M^{(1)} \cup J$ in G^*.

It is readily seen that the set of nodes

$$V = \tilde{I} \cup \tilde{M}^{(1)} \cup (\widetilde{E-M}) \cup \tilde{M}^{(2)} \cup \tilde{J},$$

where

$$\tilde{I} = I - I'; \quad \tilde{J} = J - J', \quad \tilde{M}^{(1)} = (A(I') \cap M)^{(1)}, \quad \tilde{M}^{(2)} = (B(J') \cap M)^{(2)},$$

$$\widetilde{E-M} = A(I') \cap B(J') \cap (E-M),$$

and I', J' are any subsets of I, J, separates $M^{(1)} \cup J$ from $I \cup M^{(2)}$ in G^*. But, if I', J' can be so chosen that (4.1) is false, then, for such a choice,

$$|\tilde{M}^{(1)}| + |\tilde{M}^{(2)}| + |\widetilde{E-M}| < -|\tilde{I}| - |\tilde{J}| + |M| + n;$$

i.e.

$$|V| < n + |M|;$$

and so there cannot be as many as $n + |M|$ *pnd* paths from $I \cup M^{(2)}$ to $M^{(1)} \cup J$ in G^*. This proves the necessity of the conditions.

Now suppose the conditions of the theorem hold, and let

$$V = \tilde{I} \cup \tilde{M}^{(1)} \cup (\widetilde{E-M}) \cup \tilde{M}^{(2)} \cup \tilde{J},$$

where $\tilde{I} \subseteq I$, $\tilde{M}^{(1)} \subseteq M^{(1)}$, etc., be *any* set of nodes separating $M^{(1)} \cup J$ from $I \cup M^{(2)}$ in G^*. The only nodes in $M^{(1)}$ to which nodes in $I - \tilde{I}$ can be joined must lie in $\tilde{M}^{(1)}$, and so

$$(A(I - \tilde{I}) \cap M)^{(1)} \subseteq \tilde{M}^{(1)};$$

and similarly

$$(B(J - \tilde{J}) \cap M)^{(2)} \subseteq \tilde{M}^{(2)}.$$

Further, the only nodes in $E–M$ joined to nodes in both $I–\tilde{I}$ and $J–\tilde{J}$ must lie in $\widetilde{E–M}$, and so

$$A(I–\tilde{I}) \cap B(J–\tilde{J}) \cap (E–M) \subseteq \widetilde{E–M}.$$

Therefore, from (4.1) with $I' = I - \tilde{I}$, $J' = J - \tilde{J}$, it follows that

$$| \tilde{M}^{(1)} | + | \tilde{M}^{(2)} | + | \widetilde{E–M} | \geqslant | I–\tilde{I} | + | J–\tilde{J} | + | M | - n;$$

i.e.,

$$| V | \geqslant n + | M | .$$

Hence, by Menger's theorem, there are $n + | M |$ *pnd* paths from $I \cup M^{(2)}$ to $M^{(1)} \cup J$ in G^*.

The proof of the theorem is now complete.

We have made use of a device in this proof which allows Menger's theorem to be applied to find conditions for the existence of *pnd* paths whose set of intermediate nodes contains a prescribed set. It might be interesting to investigate a more general formulation of this technique.

Instead of applying Menger's theorem to prove Theorem 4.1, we could use Theorem 2.2. Let us define two new families $\mathfrak{A}^*$, $\mathfrak{B}^*$ in terms of $\mathfrak{A}$, $\mathfrak{B}$: $\mathfrak{A}^*$ consists of the sets $(A_i \cap M)^{(1)} \cup \{A_i \cap (E\text{-}M)\}$ $(i \in I)$ together with $| M |$ copies of the set $M^{(2)}$, and $\mathfrak{B}^*$ consists of the sets $(B_j \cap M)^{(2)} \cup \{B_j \cap (E\text{-}M)\}$ $(j \in J)$ together with $| M |$ copies of the set $M^{(1)}$. (The construction of the families $\mathfrak{A}^*$, $\mathfrak{B}^*$ from $\mathfrak{A}$, $\mathfrak{B}$ is, of course, suggested by the construction of G^* from G in the earlier proof.) It is readily verified that $\mathfrak{A}$, $\mathfrak{B}$ have a CT with M as a subset if and only if $\mathfrak{A}^*$, $\mathfrak{B}^*$ have a CT (of length $n + | M |$). An application of Theorem 2.2 (in the case $p = | I | = | J |$) yields conditions which can be shown to be equivalent to (4.1). The details of the argument are described in [17].

It is interesting to observe, further, that the theorems on marginal elements (Theorems 3.1, 4.1) can be used to prove much more general results on systems of representatives, and systems of common representatives, with repetitions. In this connexion, we refer to [17] for a deduction from Theorem 4.1 of a result of Ford and Fulkerson [4, p. 74] on SCR's. A similar procedure can evidently be carried through in the simpler situation concerning representatives of a single family.

5. Independence Spaces

We shall use the notation $\{x_1 ,..., x_k\}_{\neq}$ to indicate that the set $\{x_1 ,..., x_k\}$ consists of the *distinct* elements $x_1 ,..., x_k$.

A collection $\mathscr{X}$ of subsets of a non-empty set X is called an *independence structure* on X if it satisfies the following axioms.

I(1) $\phi \in \mathscr{X}$.

I(2) If $A \in \mathscr{X}$ and $B \subseteq A$, then $B \in \mathscr{X}$.

I(3) If the sets $\{a_1, ..., a_m\}_{\neq}$, $\{b_1, ..., b_{m+1}\}_{\neq}$ belong to $\mathscr{X}$ then, for some k satisfying $1 \leqslant k \leqslant m+1$, the set $\{a_1, ..., a_m, b_k\}_{\neq}$ belongs to $\mathscr{X}$.

I(4) $\mathscr{X}$ is of 'finite character'; i.e., if every finite subset of a set $A \subseteq X$ belongs to $\mathscr{X}$, then A belongs to $\mathscr{X}$.

An ordered pair $(X, \mathscr{X})$ in which the first component is a nonempty set X and the second component is an independence structure on X is called an *independence space*. The sets belonging to $\mathscr{X}$ are called *independent sets*.

Finite independence spaces were first studied in detail (under the name of "matroids") by H. Whitney, whose fundamental paper on the subject [25] appeared in 1935. More recently much work has been done by R. Rado and others on independence spaces which may be finite or infinite; see for example [21-24].

The most familiar (nontrivial) example of an independence structure is that defined by linear independence in a vector space; and the proof of the existence of a basis in a vector space can easily be carried over to yield the result that maximal independent subsets exist in an arbitrary independence space $(X, \mathscr{X})$. Further, any two maximal independent subsets have the same cardinal number. For finite spaces, this is an immediate consequence of the "replacement" axiom I(3). For infinite spaces, the result was first proved by Rado [22]; an alternative argument can be given which follows H. Löwig's proof for bases of a vector space (see [9, p. 240]).

Now let $\mathfrak{A} = (A_i : i \in I)$ be a family of subsets of the (nonempty) set E. Here I may be infinite. The collection of PT's of $\mathfrak{A}$ trivially satisfies I(1), I(2). Further, if no element of E belongs to more than a finite number of the A_i,[4] then the collection can be shown to satisfy I(4); see for example [16]. It has recently been shown [3, 16] that the collection of PT's also has the replacement property I(3). Proofs of Theorems 2.2, 3.1 can be given which make use of the theory of "transversal independence." Generalizations of this theory have been studied in [19] and a proof of Theorem 2.3 has been given in these lines.

It is of interest to observe that the replacement property I(3) of transversal independence can also be established quite simply from Menger's theorem. In Theorem 5.1 below we prove a more general result which has close connections with the theory developed in [19].

---

[4] We shall refer to such an assumption as a condition of "local finiteness."

Let $G = [N; \mathscr{E}]$ be a graph, and let L, M be disjoint subsets of N. Suppose, further, that no path from L to M in G has intermediate nodes in L. We shall say that a subset K of L is *$G(L, M)$-independent* if either $K = \phi$ or if $K \neq \phi$ and there exist *pnd* paths from K to M in G, one through each point of K. The collection of $G(L, M)$-independent subsets of L will be denoted by $\mathscr{L}$.

THEOREM 5.1. *If, for each point $x \in L$, there is only a finite number of paths from x to M in G, then $(L, \mathscr{L})$ is an independence space.*

Theorem 5.1, in particular, yields the result that the PT's of $\mathfrak{A} = (A_i : i \in I)$ form an independence structure on E provided no element of E belongs to infinitely many A's. To see this, we simply take G to to be the graph $[N; \mathscr{E}]$, where $N = E \cup I$ $(E \cap I = \phi)$, $\mathscr{E} = \{(e, i) : e \in A_i\}$. Another simple corollary of Theorem 5.1 is the result that, if $\mathfrak{B} = (B_j : j \in J)$ is also a family of subsets of E, then the collection of subsets I' of I with the property that $(A_i : i \in I')$ has a CT with a subfamily of $\mathfrak{B}$ is an independence structure on I (under appropriate conditions of local finiteness). For other similar results, see [19].

Only I(3), I(4) require any proof, and only for I(4) is any assumption of local finiteness needed. Our main interest is in I(3) which we establish here by means of Menger's theorem. We prove I(4) by a routine argument depending on a powerful selection principle of R. Rado [22]; see also [15, Theorem 4.7].

Let m be a positive integer, and let K_1, K_2 with $|K_1| = m$, $|K_2| = m + 1$, be $G(L, M)$-independent subsets of L. Write $N = K_1 \cup K_2 \cup C$, where $C \cap (K_1 \cup K_2) = \phi$. Take any set S disjoint from N and such that $|S| = |K_1 \cup K_2| - (m + 1)$. We consider a new graph G' whose nodes form the set $N \cup S$ and whose set of edges is

$$\mathscr{E} \cup \{(x, s) : x \in (K_1 \cup K_2) - K_1, s \in S\}.$$

The lemma in Section 3 yields the result that, if there are $|K_1 \cup K_2|$ *pnd* paths from $K_1 \cup K_2$ to $M \cup S$ in G', then there is a $G(L, M)$-independent subset K of L satisfying $|K| = m + 1$, $K_1 \subseteq K \subseteq K_1 \cup K_2$.

Now let $V = \tilde{K} \cup \tilde{C} \cup \tilde{S}$, where $\tilde{K} \subseteq K_1 \cup K_2$, $\tilde{C} \subseteq C$, $\tilde{S} \subseteq S$, be a set of nodes separating $M \cup S$ from $K_1 \cup K_2$ in G'. Since there are m *pnd* paths from K_1 to M in G with no intermediate nodes in K_2

(i) $|\tilde{K} \cap K_1| + |\tilde{C}| \geqslant m$,

and, since there are $m + 1$ *pnd* paths from K_2 to M in G,

(ii) $|\tilde{K}| + |\tilde{C}| \geqslant m + 1$.

Also, clearly, either $\tilde{S} = S$ or $\tilde{K} \supseteq (K_1 \cup K_2) - K_1$. If $\tilde{S} = S$ then, from (ii)

$$\begin{aligned} |V| &= |\tilde{K}| + |\tilde{C}| + |K_1 \cup K_2| - (m + 1) \\ &\geqslant |K_1 \cup K_2|. \end{aligned}$$

If $\tilde{K} \supseteq (K_1 \cup K_2) - K_1$ then, from (i),

$$\begin{aligned} |V| &= |K_1 \cup K_2| - m + |\tilde{K} \cap K_1| + |\tilde{C}| + |\tilde{S}| \\ &\geqslant |K_1 \cup K_2| . \end{aligned}$$

Therefore, in each case, $|V| \geqslant |K_1 \cup K_2|$, and hence, by Menger's theorem, there are $|K_1 \cup K_2|$ *pnd* paths from $K_1 \cup K_2$ to $M \cup S$ in G'. From the observation above, the proof of I(3) is complete.

Now let P be the set of paths in G from L to M. Let $\mathscr{P}$ be a collection of subsets of P with the property that $P' \in \mathscr{P}$ if and only if either $P' = \phi$ or the paths of P' are *pnd*. It is at once clear that $\mathscr{P}$ satisfies the conditions I(1), I(2), I(4) of an independence structure (but not in general I(3)). We define, for each $x \in L$, A_x to be the set of paths from x to M in G, and consider the family $\mathfrak{A} = (A_x : x \in L)$ of finite (disjoint) subsets of P. Let K be a subset of L with the property that, for every finite $K' \subseteq K$, $K' \in \mathscr{L}$. This means that, for every finite $K' \subseteq K$, there is an (injective) choice function $\theta_{K'} : K' \to P$ of $(A_x : x \in K')$ such that $\theta_{K'}(K') \in \mathscr{P}$. By Rado's selection principle, therefore, there is a choice function θ of $(A_x : x \in K)$ such that, for each finite $K' \subseteq K$, there exists a finite $K'' \subseteq K$ with $K' \subseteq K''$ and $\theta(x) = \theta_{K''}(x)$ for every $x \in K'$. Now every finite subset P' of $\theta(K)$ is the image under θ of a finite subset K' of K. Therefore

$$P' = \theta(K') = \theta_{K''}(K')$$

for some finite subset K'' of K with $K'' \supseteq K'$. Hence, by the property I(2) of $\mathscr{P}$,

$$P' \in \mathscr{P}$$

since $\theta_{K''}(K'') \in \mathscr{P}$. So, by the property I(4) of $\mathscr{P}$, $\theta(K) \in \mathscr{P}$. This means that there is a set of *pnd* paths from K to M in G, one through each point of K, and so $K \in \mathscr{L}$.

The proof of the theorem is therefore complete.

Our last application of Menger's theorem generalizes a result of L. Mirsky [14]. This states that, *if* $\mathfrak{A} = (A_i : i \in I)$, $\mathfrak{B} = (B_j : j \in J)$ *are (finite) families of subsets of E and if* $\mathfrak{A}' \subseteq \mathfrak{A}$, $\mathfrak{B}' \subseteq \mathfrak{B}$, *then the following statements are equivalent.*

(a) *There exist families* $\mathfrak{A}_0$, $\mathfrak{B}_0$ *with* $\mathfrak{A}' \subseteq \mathfrak{A}_0 \subseteq \mathfrak{A}$, $\mathfrak{B}' \subseteq \mathfrak{B}_0 \subseteq \mathfrak{B}$ *which have a CT.*

(b) $\mathfrak{A}'$ *and a subfamily of* $\mathfrak{B}$ *have a CT, and* $\mathfrak{B}'$ *and a subfamily of* $\mathfrak{A}$ *have a CT.*

Again, let $G = [N; \mathscr{E}]$ be a graph, and let L, M be disjoint subsets of N. Suppose, further, that no path from L to M in G has intermediate nodes in L or M. We define $G(L, M)$-independent subsets of L as above, and now we make the further definition that a subset R of M is to be called $G(L, M)$-independent if either $R = \phi$, or if $R \neq \phi$ and there exist *pnd* paths from L to R in G one through each point of R.[5] The collection, $\mathscr{M}$ say, of $G(L, M)$-independent subsets of M is clearly also an independence structure on M provided that, for each $x \in R$, there is only a finite number of paths from L to x in G.

THEOREM 5.2. *Suppose at least one of L, M is finite, and let $L' \in \mathscr{L}$, $M' \in \mathscr{M}$. Then there exist sets L_0, M_0 satisfying $L' \subseteq L_0 \subseteq L$, $M' \subseteq M_0 \subseteq M$ and which are the sets of initial and terminal nodes of a set of pnd paths from L to M in G.*

The proof of Theorem 5.2 which we give below makes essential use of the finiteness of one of the sets L and M.[6]

Mirsky's theorem referred to above is an immediate consequence of Theorem 5.2. We take G to be the graph whose set of nodes is $I \cup E \cup J$ and whose set of edges is $\{(i, e) : e \in A_i\} \cup \{(e, j) : e \in B_j\}$; and we take I, J in the roles of L, M.

We turn, finally, to the proof of Theorem 5.2. Let L_0 be any maximal $G(L, M)$-independent subset of L containing L', and let M_0 be any maximal $G(L, M)$-independent subset of M containing M'. It is clear that L_0, M_0 are finite. Also, since all maximal $G(L, M)$-independent subsets of M have the same cardinal, $|L_0| \leqslant |M_0|$. Similarly, $|M_0| \leqslant |L_0|$, and so

$$|L_0| = |M_0| = p \text{ (say).}$$

Let P be a particular set of p *pnd* paths from L_0 to M in G, and let $\tilde{P}$ be a particular set of p *pnd* paths from L to M_0 in G. Since no set of *pnd* paths from L to M in G contains more than p members, it follows from Menger's theorem that there is a set of p nodes of G, S say, which separates M from L in G. In view of this separation property, there must be a node of S on each of the paths of P (and so exactly one on each since p is finite) and similarly (exactly) one node of S on each path of $\tilde{P}$.

Now consider p paths from L_0 to M_0 formed from those parts of the paths of P from L_0 to S followed by those parts of the paths of $\tilde{P}$ from S to M_0.

[5] It should be noted that $G(L, M)$-independent subsets of M are to be distinguished from $G(M, L)$-independent subsets of M.

[6] Since writing this paper I have learned that Dr. J. S. Pym has now obtained a more general version of Theorem 5.2 which applies to infinite sets. Dr. Pym's work will be published in due course. For the case of a bipartite graph the result for infinite sets is already known and is a generalization of the Schröder-Bernstein theorem; a statement and proof of this generalization are given in [20]. Theorems on representatives of sets which may be deduced from this latter result include the infinite analogues of theorems of Hoffman and Kuhn (on marginal elements) and Mendelsohn and Dulmage [11] (see also [20]).

We show that these new paths are *pnd*. For, otherwise, there must be two of these paths with a node in common, say the paths whose consecutive nodes are

$$x_1, x_2, \ldots, x_t = s, x_{t+1}, \ldots, x_m$$

and

$$x_1', x_2', \ldots, x_u' = s', x_{u+1}', \ldots, x_n',$$

where $s, s' \in S$. Clearly, the only possibility for a common node is $x_i = x_j'$ where $t + 1 \leqslant i < m$, $1 < j \leqslant u - 1$ or $1 < i \leqslant t - 1$, $u + 1 \leqslant j < n$. Take the first case for definiteness. Then

$$x_1', x_2', \ldots, x_j' = x_i, x_{i+1}, \ldots, x_m$$

is the sequence of nodes of a path from L_0 to M_0 containing no point of S. This contradicts the assumption that S separates M from L in G; and the proof is complete.

References

1. G. A. Dirac. Extensions of Menger's theorem. *J. London Math. Soc.* (3) **38** (1963), 148–161.
2. G. A. Dirac. Short proof of Menger's graph theorem. *Mathematika* **13** (1966), 42–44.
3. J. Edmonds and D. R. Fulkerson. Transversals and matroid partition. *J. Res. Nat. Bur. Standards* **69B** (1965), 147–153.
4. L. R. Ford, Jr. and D. R. Fulkerson. "Flows in Networks." Princeton Univ. Press, Princeton, New Jersey, 1962.
5. M. Hall, Jr. Distinct representatives of subsets. *Bull. Amer. Math. Soc.* **54** (1948), 922–926.
6. P. Hall. On representatives of subsets. *J. London Math. Soc.* **10** (1935), 26–30.
7. P. R. Halmos and H. E. Vaughan. The marriage problem. *Amer. J. Math.* **72** (1950), 214–215.
8. A. J. Hoffman and H. W. Kuhn. Systems of distinct representatives and linear programming. *Amer. Math. Monthly* **63** (1956), 455–460.
9. N. Jacobson. "Lectures in Abstract Algebra," Vol. II. New York, 1953.
10. D. König. "Theorie der endlichen und unendlichen Graphen." Leipzig, 1936.
11. N. S. Mendelsohn and A. L. Dulmage. Some generalizations of the problem of distinct representatives. *Canad. J. Math.* **10** (1958), 230–241.
12. K. Menger. Zur allgemeinen Kurventheorie. *Fund. Math.* **10** (1927), 96–115.
13. L. Mirsky. Systems of representatives with repetition. *Proc. Camb. Phil. Soc.* **63** (1967), 1135–1140.
14. L. Mirsky. A theorem on common transversals. *Math. Ann.* (to appear).
15. L. Mirsky and H. Perfect. Systems of representatives. *J. Math. Anal. Appl.* **15** (1966), 520–568.

16. L. Mirsky and H. Perfect. Applications of the notion of independence to problems of combinatorial analysis. *J. Comb. Theory* **2** (1967), 327–357.
17. L. Mirsky and H. Perfect. Comments on certain combinatorial theorems of Ford and Fulkerson. *Archiv. der Math.* (to appear).
18. O. Ore. Graphs and matching theorems. *Duke Math. J.* **22** (1955), 625–639.
19. H. Perfect. Independence spaces and combinatorial problems. *Proc. London Math. Soc.* (to appear).
20. H. Perfect and J. S. Pym. An extension of Banach's mapping theorem with applications to problems concerning common representatives. *Proc. Camb. Phil. Soc.* **62** (1966), 187–192.
21. R. Rado. A theorem on independence relations. *Quart. J. Math.* (Oxford) **13** (1942), 83–89.
22. R. Rado. Axiomatic treatment of rank in infinite sets. *Canad. J. Math.* **1** (1949), 337–343.
23. R. Rado. Note on independence functions. *Proc. London Math. Soc.* **7** (1957), 300–320.
24. R. Rado. Abstract linear dependence. *Colloq. Math.* **14** (1966), 257–264.
25. H. Whitney. On the abstract properties of linear dependence. *Amer. J. Math.* **57** (1935), 509–533.