

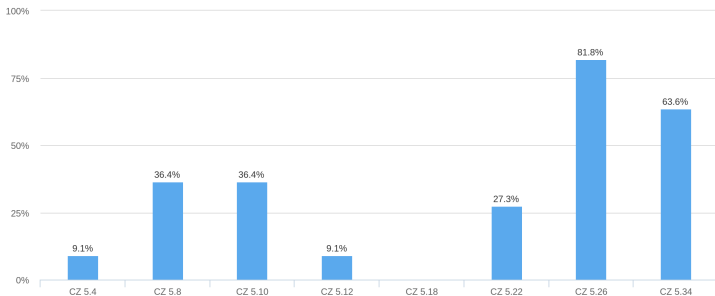
3-9 Connectivity

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5.10

5.34

5.22

5.26

如果两个割点相连，那么联通块怎么划分！
联通快呢）

menger定理吧

暂无

好像没有.....

Menger定理的证明看不懂

menger定理的证明

不。。不记得了

还好理解，只是都不怎么容易理解

menger定理的证明没太理解 老师辛苦了！

点割集，边割集

Menger's Theorem (Theorem 5.16; Theorem 5.21)



2-Connectivity (Problem 5.10)

A connected graph G with $m \geq 2$ is *nonseparable*



any two *adjacent* edges of G lie on a common cycle of G .

Proof.

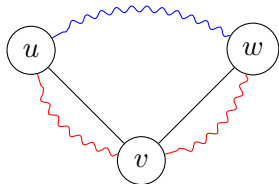
“ \implies ”

G is nonseparable

$\implies u, w$ lie on a common cycle

$\implies \exists$ path $u \sim w$ that does not contain v

$\implies \exists$ cycle $u - v - w \sim u$



2-Connectivity (Problem 5.10)

A connected graph G with $m \geq 2$ is *nonseparable*



any two *adjacent* edges of G lie on a common cycle of G .

Proof.

“ \Leftarrow ”

By Contradiction.

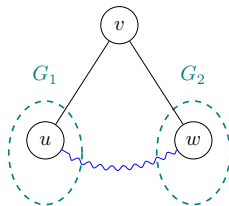
Suppose v is a cut-vertex of G

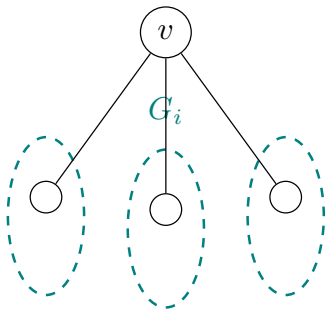
$\implies G - v$ contains ≥ 2 comps G_1, G_2, \dots

$\implies \exists u \in G_1, w \in G_2 : v - u \wedge v - w$

$\implies v - u, v - w$ lie on a common cycle

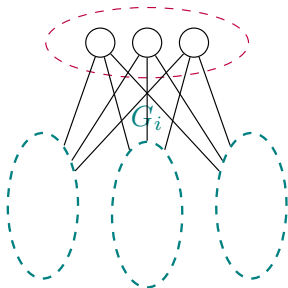
$\implies \exists$ path $u \sim w$ that does not contain v





$$\forall G_i \exists v_i \in G_i : v - v_i$$

S : Minimum Vertex Cut



$$\forall v \in S \forall G_i \exists v_i \in G_i : v - v_i$$

2-Connectivity (Problem 5.10)

A connected graph G with $m \geq 2$ is *nonseparable*



any two *adjacent* edges of G lie on a common cycle of G .

2-Connectivity (Extended Problem)

A connected graph G with $m \geq 2$ is *nonseparable*

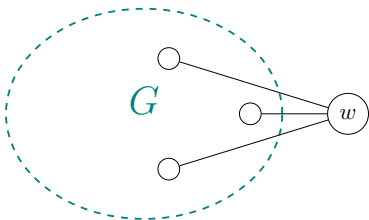


any two edges of G lie on a common cycle of G .

Expansion Lemma (Problem 5.34; Theorem 5.18)

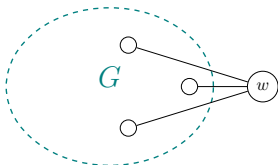
Let G be a k -connected graph and let S be any set of k vertices.

If a graph H is obtained from G by adding a new vertex w and joining w to the vertices of S , then H is also k -connected.



We prove that

$\forall v \in V(G) : \text{there exist } k \text{ internally disjoint } v - w \text{ paths}$

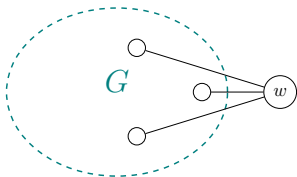


This holds because

$\forall v \in V(G) : \text{there exist internally disjoint } v - s_i \ (\forall s_i \in S) \text{ paths}$

Corollary (5.19; Proved using Theorem 5.18)

If G is a k -connected graph and u, v_1, v_2, \dots, v_k are $k + 1$ distinct vertices of G , then there exist internally disjoint $u - v_i$ paths ($1 \leq i \leq k$) in G .



To prove $\kappa(H) \geq k$

Let U be a vertex-cut of H .

We prove that $|U| \geq k$.

CASE I: U is a
vertex-cut of G

$$|U| \geq k$$

CASE II: U is not a vertex-cut of G

$$w \in U$$

$U - w$ is a vertex-cut of G

($\because U$ is a vertex-cut of H)

$$|U| \geq k + 1$$

$$w \notin U$$

$$\implies U \subseteq G$$

$$\implies S \subseteq U \subseteq G$$

($\because U$ is a vertex-cut of H)

$$\implies |U| \geq k$$

2-Connectivity (Extended Problem)

A connected graph G with $m \geq 2$ is *nonseparable*



any two edges of G lie on a common cycle of G .

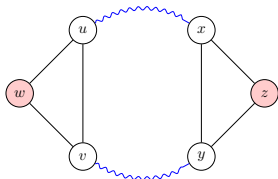


Consider two edges uv and xy .

Add w, z

Add $wu, wv; zx, zy$

w and z lie on a common cycle



Effects of Removing an Edge on Connectivity (Problem 5.22 (a))

- (a) If G is k -connected and $e = uv \in E(G)$, then $G - e$ is $(k - 1)$ -connected.

To prove $\kappa(G) \geq k \implies \kappa(G - e) \geq k - 1$

Choose any $U \subseteq V(G)$ with $|U| < k - 1$.

We prove that $G - e - U$ is connected.

Choose any $U \subseteq V(G)$ with $|U| < k - 1$.

We prove that $G - e - U$ is connected.

G is k -connected $\implies G - U$ is connected

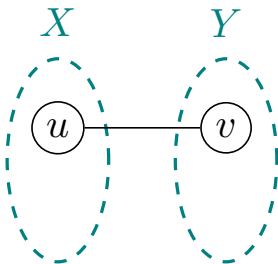
Suppose, by contradiction, that $G - e - U$ is not connected.

$e = uv$ is a bridge of $G - U$

$U \cup \{u\}$ is a vertex-cut of G

But $|U \cup \{u\}| < k$





CASE I : $|X| \geq 2 \vee |Y| \geq 2$

$U \cup \{u\}$ is a vertex-cut of G

But $|U \cup \{u\}| < k$

CASE II : $|X| = |Y| = 1$

$|U| = n - 2 < k - 1$

$\kappa(G) \geq k > n - 1$

But $0 \leq \kappa(G) \leq n - 1$

Effects of Removing an Edge on Connectivity (Problem 5.22 (b))

- (b) If G is k -edge-connected and $e = uv \in E(G)$, then $G - e$ is $(k - 1)$ -edge-connected.

$$\lambda(G) \geq k \implies \lambda(G - e) \geq k - 1$$

Choose any $X \subseteq E(G)$ with $|X| < k - 1$.

We prove that $G - e - X$ is connected.

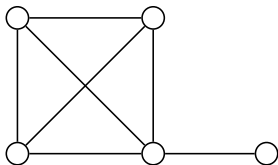
$G - e - X = G - (e + X)$ is connected ($\because \lambda(G) \geq k$)

$$\kappa(G - e) \leq \kappa(G)$$

Effects of Removing a Vertex on Connectivity (Extended Problem)

Is $\kappa(G - v) \leq \kappa(G)$?

Is $\lambda(G - v) \leq \lambda(G)$?



Effects of Removing a Vertex on Connectivity (After-class Exercise)

Is $\kappa(G) \geq k \implies \kappa(G - v) \geq k - 1$?

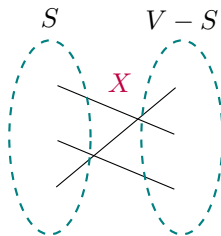
$$\text{Is } \lambda(G) \geq 1 \implies \lambda(G - v) \geq 1?$$

Degree Condition for $\lambda(G) = \delta(G)$ (Problem 5.26)

If G is graph of order n such that $\delta(G) \geq (n-1)/2$, then $\lambda(G) = \delta(G)$.

$$\lambda(G) \leq \delta(G)$$

We prove that $\lambda(G) \geq \delta(G)$.



$$\lambda(G) = |X|$$

$$1 \leq |S| = k \leq n/2, \quad |V-S| = n-k$$

$$\lambda \geq k(\delta - (k-1)) \geq \delta$$

Decision	Author(s)	Year	Complexity	Comments
<i>Edge Connectivity</i>				
$\lambda = 2$ or $\lambda = 3$	Tarjan [26]	1972	$O(m + n)$	uses Depth First Search
λ	Even and Tarjan [6]	1975	$O(mn \times \min\{m^{1/2}, n^{2/3}\})$	n calls to max-flow
λ (digraphs)	Schnorr [25]	1979	$O(\lambda mn)$	n calls to max-flow
λ	Esfahanian & Hakimi [3]	1984	$O(\lambda mn)$	$\leq n/2$ calls to max-flow
λ (digraphs)	Esfahanian & Hakimi [3]	1984	$O(\lambda mn)$	$\leq n/2$ calls to max-flow
λ	Matula [23]	1987	$O(mn)$	uses dominating sets
$\lambda = k$	Matula [23]	1987	$O(kn^2)$	
λ (digraphs)	Mansour & Schieber [22]	1989	$O(mn)$	
$\lambda = k$	Gabow [9]	1991	$O(m+k^2n\log(n/k))$	uses matroids
<i>Vertex Connectivity</i>				
$\kappa = 2$	Tarjan [26]	1972	$O(m + n)$	uses Depth First Search
$\kappa = 3$	Hopcroft & Tarjan [18]	1973	$O(m + n)$	uses triconnected components
κ	Even & Trajan [6]	1975	$O((\kappa(n - \delta - 1)mn^{2/3}))$	max-flow based
$\kappa = k$	Even [4]	1975	$O(kn^3)$	max-flow based
κ	Galil [12]	1980	$O(\min\{\kappa, n^{2/3}\}mn)$	max-flow based
$\kappa = k$	Galil [12]	1980	$O(\min\{k, n^{1/2}\}kmn)$	max-flow based
κ	Esfahanian & Hakimi [3]	1984	$O((n - \delta - 1 + \delta(\delta - 1)/2)mn^{2/3})$	max-flow based
$\kappa = 4$	Kanevsky & Ramachandran [20]	1991	$O(n^2)$	
κ	Henzinger & Rao [17]	1996	$O(\kappa mn \log n)$	randomised algorithm

Table 1: A chronology of connectivity algorithms





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