a personal view of the theory of computation

I see it, but I don't believe it

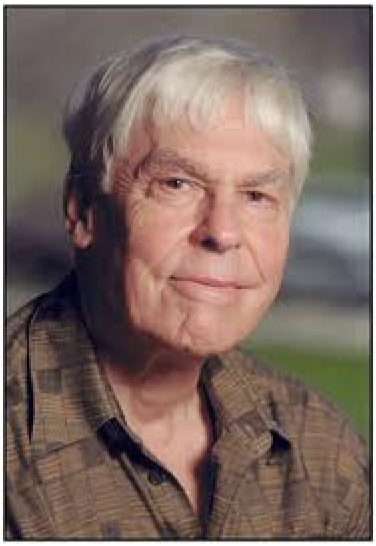
APRIL 4, 2011

tags: eversion*,* exotic*,* paradox*,* spheres

by rjlipton

Shocking results about spheres

(https://rjlipton.files.wordpress.com/2011/04/images.jpeg)



Steven Smale is an eminent mathematician who has made seminal contributions to many areas of mathematics. He solved the Poincaré Conjecture for dimension $d \geq 5$; he made fundamental contributions to the theory of dynamical systems; he, with Lenore Blum and Michael Shub, created the **real number (http://en.wikipedia.org/wiki/Blum–Shub–Smale_machine)** $\mathsf{P} = \mathsf{NP}$ question. He is a Fields Medalist, standing alongside Michael Atiyah, Paul Cohen, and Alexander Grothendieck in 1966. His medal was awarded especially for his brilliant work on the Poincaré Conjecture—his was the first non-trivial result on it.

Today I would like to discuss mathematical shocks, which Smale and others have created.

There has been much discussion here and elsewhere on whether we tend to guess right or not. I think that there are plenty of surprises, which are results that we did not predict. See this discussion for my previous thoughts about **surprises (https://rjlipton.wordpress.com/2009/09/27/surprises-in-mathematics-and-theory/)**. I would characterize a **shock** as something that is even more than a surprise. A shock is a result that leaves us speechless, that takes us completely off-guard, that at first seems impossible.

Smale created one of the great shocks in mathematics, but he also has done some "shocking" things. Perhaps the most shocking was this statement that caused great discomfort within NSF:

> *He once said that his best work had been done "on the beaches of Rio."*

This statement, which was probably true, and which should have sent more top mathematicians to sit on the beaches somewhere, was not understood by politicians. The result was a ban, for a while, on Smale's funding.

Some Shocks

I will give four results that I think fall into the category of shocking. Each result will be about a shocking property of the sphere—that is a ball, the kind you can hold in your hand, the kind that we use to play games such as soccer or baseball or basketball. Sometimes the balls have to be in higher dimensions than three, so you cannot quite hold them in your hand.

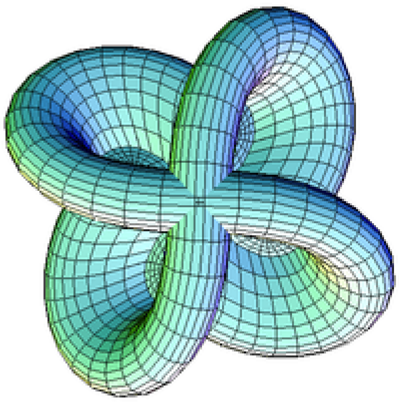
(https://rjlipton.files.wordpress.com/2011/04/balls.png)

● **Sphere Eversion:** This shock is Smale's result that it is possible to turn a sphere inside out, without leaving a crease or hole. Silvio Levy **here (http://www.math.sunysb.edu/CDproject/OvUM/cool-links/www.geom.umn.edu/docs/outreach/oi/history.html)** gives a detailed history of this result.

My high-level understanding of the history is that in 1957 Smale proved a general abstract theorem about embeddings of spheres into Euclidean spaces. Apparently he did not notice at first that his abstract theorem implied that there had to be a way to continuously move a sphere in three space so that it turned inside out. Of course the motion must have the sphere pass through itself, but otherwise not cause any tear or crease. This cannot be done in two-dimensions, but magically can be done in three.

This was so shocking that Raoul Bott, one of the founders of differential topology, told Smale that his abstract theorem must be wrong. He had a technical point that he thought must be violated, but he was eventually convinced that Smale was correct. Still neither Smale nor Bott nor others could "see" how to actually turn a sphere inside out. Finally Bernard Morin discovered how the construction worked. One of the final shocking parts of this story is that Morin is blind, yet he was one of the first to understand and "see" how to turn a sphere inside out. A tremendous indication of Morin's mathematical ability.

Check out this for a series of **pictures (http://www.cs.berkeley.edu/~sequin/SCULPTS/SnowSculpt04/MORIN/VISUALIZATION/visualiz.htm)** that give insight into how eversion works, or watch the movie **here (http://video.google.com/videoplay?docid=-6626464599825291409#)**. The "half-way" point of the construction, which is called the Morin surface looks like this:



(https://rjlipton.files.wordpress.com/2011/04/morin.png)

Clearly just from this picture you can see that the construction is quite complex, and requires the sphere to be manipulated in a manner that is non-trivial.

● **Sphere Cardinality:** This shock is not Georg Cantor's famous diagonalization theorem: that the reals are uncountable. That did not shock him. The shock was his proof that the interval $[0, 1]$ and the unit square $[0, 1] \times [0, 1]$ have the same cardinality. Okay it really is not exactly about spheres, but it can be viewed as showing that the unit circle and the unit disk have the same cardinality.

When he first proved this he wrote to Richard Dedekind:

> *"Je le vois, mais je ne le crois pas!" ("I see it, but I don't believe it!")*

That a line and a square could be put in a one-to-one correspondence with each other seems to have really surprised Cantor. The proof raises an interesting point too.

The way that Cantor first defined the map from square to line was by,

$$f(x,y) = x_1, y_1, x_2, y_2, \dots$$

He interleaved the digits that defined the two numbers. This is a natural idea, but it has a problem that was pointed out to him by Dedekind: the mapping is not one-to-one. The problem is caused by the usual ambiguity of the decimal representation of reals:

$$0.99999\cdots = 1.00000\dots$$

Today we could easily argue around this issue, since we could use Cantor-Bernstein's Theorem to make the interleaving argument work. But Cantor used another trick; he used that real numbers have a unique representation as continued fractions, provided they are not rational:

> ***Theorem:*** *Let x be a number in the interval $[0,1]$. Then it has a continued fraction representation of the form:*
>
> $$x = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + \dots}}}$$
>
> *where $a, b, c, \dots$ are all positive integers. Further the representation is **unique if x is irrational.***

This is a neat representation theorem, and the key part is that for irrational numbers it is unique. So there is no ambiguity issue, the problem that can make arguments about real sets messy. Of course for rationals it is easy to see that it is not in general unique.

Cantor used this theorem to do his interleaving, and since the representation is unique he could make his argument rigorous. Fernando Gouvêa has a wonderful **article (http://www.maa.org/pubs/AMM-March11_Cantor.pdf)** in this March's Math Monthly on exactly this issue

● **Sphere Measure:** This shock is that a solid sphere can be divided into a finite number of pieces, and these pieces can be reassembled to form another solid sphere that is twice the size. This is called the Banach-Tarski Paradox—I guess a true statement that is called a paradox might be considered a shock. Truth in blogging: this is the only result here that uses the axiom of choice, so one can, and some do, argue that the shock is due to the axiom of choice alone.

The result is due to Stefan Banach and Alfred Tarski, who proved it in 1924.

There is a very readable **book (http://www.amazon.com/Pea-Sun-Mathematical-Paradox/dp/1568812132)**, "The Pea and the Sun: A Mathematical Paradox" by Leonard Wapner on this result. He does a great job making the result understandable, and I enjoyed reading it a while ago very much.

● **Sphere With Exotic Structure:** This shock is that there are *exotic spheres*. An exotic sphere is a differentiable manifold that is the same as the standard Euclidean n-sphere with respect to continuous bijections, but not with respect to differential ones. The discovery of such spheres in 1956, in dimension 7, is due to the **work (http://en.wikipedia.org/wiki/Exotic_sphere)** of John Milnor, who just won the Abel prize, for this and many other great results.

Since then a great deal of work has been done on this topic, although one major open problem remains: it is unknown whether such spheres exist in four dimensions.

Michael Freedman, Robert Gompf, Scott Morrison, and Kevin Walker say that it is believed to be false, that is that such spheres do not exist. In their 2010 **paper (http://arxiv.org/abs/0906.5177)** they discuss approaches to resolve the conjecture that will make heavy use of computation. Take a look at the paper, but here is a taste of what they are doing:

> ***(https://rjlipton.files.wordpress.com/2011/04/amd.png)***
> *Computing the two-variable polynomial for K2 took approximately 4 weeks on a dual core AMD Opteron 285 with 32 gb of RAM. At this point, we haven't been able to do the calculation for K3 . With the current version of the program, after about two weeks the program runs out of memory and aborts.*

It would be cool if high-performance computation helped resolve this long standing open problem.

Open Problems

What is your favorite shock? Do you agree that these are really shocks? See **this (http://motls.blogspot.com/2009/05/shocks-in-mathematics-and-physics.html)** for another view of shocks.

from → History*,* People*,* Proofs

17 Comments leave one →

1. **martin PERMALINK**
   April 4, 2011 8:15 pm
   My favourite shock also invoves spheres and is the Banach-Tarski Paradox. (http://en.wikipedia.org/wiki/Banach_Tarski)

   REPLY
2. **Harman PERMALINK**
   April 4, 2011 8:51 pm
   Beautiful results. Thanks for sharing. The video for the sphere inversion is an achievement in education.

   REPLY
3. **rrtucci PERMALINK**
   April 5, 2011 12:26 am
   Has sphere inversion been observed nature, say in biology or chemistry?

   REPLY
4. **rrtucci PERMALINK**
   April 5, 2011 12:27 am
   Sorry. I meant "observed in nature"

   REPLY
5. **proaonuiq PERMALINK**
   April 5, 2011 3:55 am
   Favourite surprises…and favourite shock:

   https://rjlipton.wordpress.com/2009/11/22/new-streaming-algorithms-for-old-problems/

   REPLY

6. **mathanon PERMALINK**
   April 5, 2011 2:03 pm
   Cantor's result probably wouldn't have astonished Duns Scotus although Scotus took such interesting bijections as arguments against the existence of points (indivisibles).

   http://books.google.com/books?id=l821N_9GSusC&pg=PA579

   REPLY
7. **fnord PERMALINK**
   April 5, 2011 11:58 pm
   I was pretty shocked many years ago that the Axiom of Choice and Zermelo's Theorem implied each other. I've never really trusted the Axiom of Choice for infinities bigger than countable ones.

   I read some contemporary math journals from the time after Zermelo published his paper, and opinion was pretty evenly divided between tossing both the Axiom of Choice and Zermelo's paper, and keeping them.

   Someone once opined that while it is obvious that the Axiom of Choice works when choosing one each from a large collection of pairs of shoes, it is less obvious that it also works with socks.

   (What's yellow and implies the Axiom of Choice? Zorn's Lemon.)

   REPLY
8. **John Sidles PERMALINK**
   April 6, 2011 7:34 am
   The midpoint of a minimum-energy sphere eversions is a one-sided immersion of the projective plane known as Boy's Surface. Thus surface can be seen in one of the beautiful of the sphere eversion animations, *The Optiverse*, by John Sullivan, George Francis and Stuart Levy.

   Our QSE Group uses Boy's Surface as a training exercise for quantum systems engineering students, by requiring them to specify natural complex coordinates upon it, and then numerically locate closed geodesic trajectories … this exercise is a warm-up for integrating quantum dynamical simulation trajectories on large-dimensional projective manifolds.

   The most amazing result of this exercise is that KAM theory works … a subset of the closed geodesic trajectories on Boy's Surface *are* stable against small perturbations … and yet, generically the geodesic trajectories ergodically cover the surface (a sample graphic is here).

   The stability of the Solar System is thought to depend upon the same delicate balance, of KAM-stable versus ergodically unstable dynamical trajectories, upon a state-space that is richly endowed with an elegant symplectic structure.

   What's amazing is that this delicate dynamical balance endures robustly for billions of years … well … except maybe for (1) the Asteroid Belt, and (2) the Rings of Saturn.

   But heck … as far as our own planet *Earth* is concerned, KAM theory has worked fine for billions of cycles … "We all see it every day, but we shouldn't believe it" (to paraphrase Dick's post) … at least, not without a sense of amazement.

   REPLY
9. **MattF PERMALINK**
   April 6, 2011 8:01 am
   It may be obvious, but it's worth emphasizing that the Banach-Tarski dissects a sphere into pieces that are *not measurable*, i.e., there is no way to define their volume.

My own favorite I-don’t-believe-it assertion is Goodstein’s Theorem. I mean– c’mon people, how can that be true?

REPLY

10. **Sasho PERMALINK**
April 6, 2011 8:48 am
I was quite shocked to learn that the volume of the n-dimensional unit ball goes exponentially fast to 0 with n. Or equivalently, the n-dim ball inscribed in the n-dim cube takes an exponentially small fraction of the cube’s volume. The sphere is just way too round in high dimension. While the fact is elementary to prove, it is a good example of how high dimensional geometry and intuition don’t go together, at all.

To rtucci: the “rules” of the eversion game are a little too strange for nature. Maybe there is an indirect application to physics where continuous transformations of the sphere model something (and john sidles probably knows about it). But an explicit eversion in nature would involve materials that can stretch all they want, can pass through themselves, but by no means should tear or even crease.

REPLY

11. **Woett PERMALINK**
April 6, 2011 9:43 am
Last week I learned that $P = BPP$ under very plausible assumptions.. Maybe it says a lot about me, but initially I was completely stunned to learn that ‘randomness doesn’t add anything’

REPLY

12. **chazisop PERMALINK**
April 6, 2011 7:51 pm
I am 2 months away from my bachelor in CS. So here’s some of the shocks I have received so far:

1. The first is from high school: complex numbers. It got me wondering what else our teachers were hiding from us and how I was supposed to find 100 solutions for x^100 = 1, when I could figure out 1. I had to wait until a university course to learn the technique.

2. A Turing Machine can compute everything the most powerful supercomputer can.

3. The world is digital: https://rjlipton.wordpress.com/2009/10/04/the-world-is-digital/

4. A recent unveiling: I wondered if there are problems that are unrecognizable and complementary unrecognizable. By asking on the TCS.SE website, I was directed to the arithmetical hierarchy, which of I knew only by name. So it turns out not only they do, but there is an infinite hierarchy of them that has been proven sound to collapsing!

REPLY

13. **Don Blazys PERMALINK**
April 7, 2011 6:13 am
Here’s a result that is not merely shocking, but absolutely horrifying. It was discovered by yours truly in January of 1999.

First, let us recall that the equality relation is always defined such that things that are equal have all and only the same properties. Now, if you think that the two sides of the equation:

(T/T)*c^3 = (c/c)*c^3

have “essentially” the same properties, then you had better think again, because we can’t even derive an identity such as:

(T/T)*c^3 = T*(c/T)^((3*ln(c)/(ln(T))-1)/(ln(c)/(ln(T))-1)) from its so called “equal”

(c/c)*c^3 ,

which means that the two sides have radically different properties, especially in how they relate to the properties of logarithms!

Don.

REPLY

14. **Gil Kalai PERMALINK**
April 8, 2011 11:04 am
Smale’s result which leads to the sphere inversion “shock” as well as Nash theorem from 54 on C^1 embeddings are examples of a very general phenomenon diecovered by Gromov in the 70s known as the h-principle. There is a 2002 book about it by Eliashberg and Mishachev in addition to Gromov’s book “Partial differential relations”. Somehow as mentioned in the post the special cases of the general phenomenon are more “shocking”.

    There are quite a few “shocks” even if we restrict our attention to spheres. A little shok is that the basketball (to the left of the baseball and football in the post,) is combinatorially isomorphic to the octahedron.

    REPLY
15. **Gilbert Bernstein PERMALINK**
April 13, 2011 3:56 am
I’m surprised that you advocated HPC as a resolution to their efficiency problems. In my (admittedly brief) experience with writing programs to prove mathematical properties, some hefty constant factors can be gained by optimizing a first implementation: algorithmically (using more efficient or tailored data structures) or by rewriting in a lower level language (e.g. translating from Python/SAGE to C) or with some bit-level hackery. (e.g. use bit vectors to track large arrays mod 2, or avoiding big number arithmetic by working mod p)

    REPLY
16. **Raphael PERMALINK**
April 16, 2011 7:51 am
Historically, the fact that there are indeed algorithmically undecidable problems was a shock. Many a mathematician did not believe it at fist, considering Turing’s machine model faulty in some way.

    Gödel’s incompleteness theorems shocked the community, too.

    REPLY

Trackbacks

1. Strange Places To Prove Theorems « Gödel’s Lost Letter and P=NP