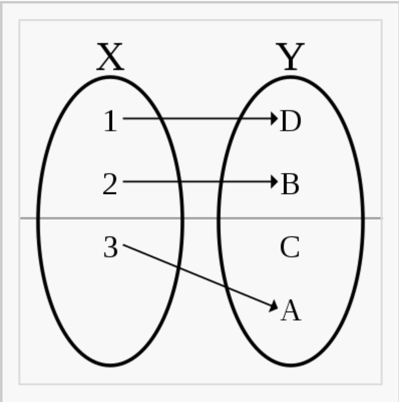
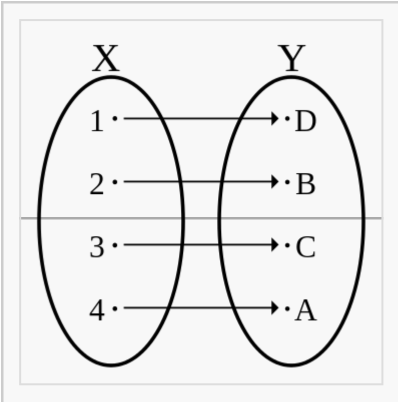
Injective function

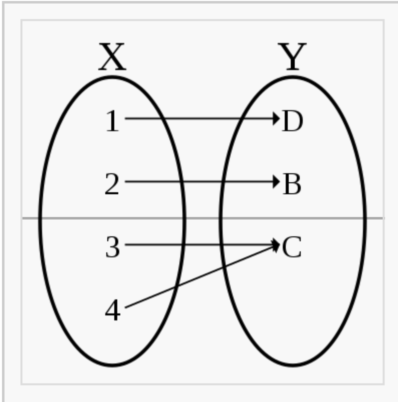
In mathematics, an **injective function** or **injection** or **one-to-one function** is a function that preserves distinctness: it never maps distinct elements of its domain to the same element of its codomain. In other words, every element of the function's codomain is the image of *at most* one element of its domain. The term *one-to-one function* must not be confused with *one-to-one correspondence* (a.k.a. bijective function), which uniquely maps all elements in both domain and codomain to each other (see figures).



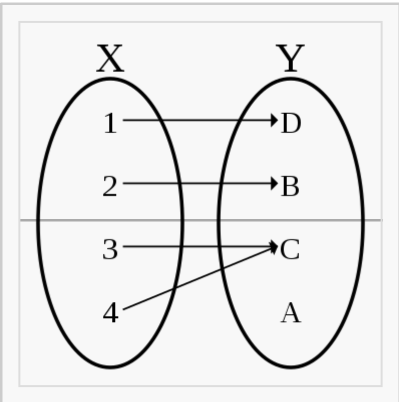
An injective non-surjective function (injection, not a bijection)



An injective surjective function (bijection)



A non-injective surjective function (surjection, not a bijection)



A non-injective non-surjective function (also not a bijection)

Occasionally, an injective function from X to Y is denoted $f\colon X \rightarrowtail Y$, using an arrow with a barbed tail (U+21A3 ↣ RIGHTWARDS ARROW WITH TAIL).[1] The set of injective functions from X to Y may be denoted $Y^{\underline{X}}$ using a notation derived from that used for falling factorial powers, since if X and Y are finite sets with respectively m and n elements, the number of injections from X to Y is $n^{\underline{m}}$ (see the twelvefold way).

A function f that is not injective is sometimes called many-to-one. However, the injective terminology is also sometimes used to mean "single-valued", i.e., each argument is mapped to at most one value.

A monomorphism is a generalization of an injective function in category theory.

Contents

Definition

Let f be a function whose domain is a set X. The function f is said to be **injective** provided that for all a and b in X, whenever $f(a) = f(b)$, then $a = b$; that is, $f(a) = f(b)$ implies $a = b$. Equivalently, if $a \neq b$, then $f(a) \neq f(b)$.

Symbolically,

$$\forall a,b \in X, \quad f(a)=f(b) \Rightarrow a=b$$

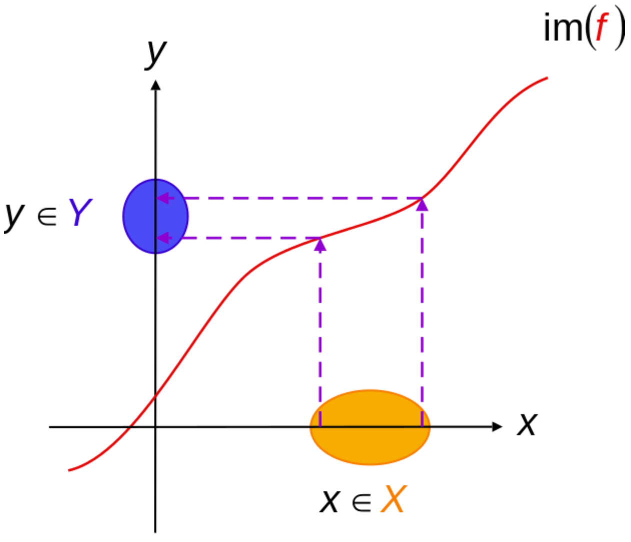
which is logically equivalent to the contrapositive,

$$\forall a,b \in X, \quad a \neq b \Rightarrow f(a) \neq f(b)$$

Examples

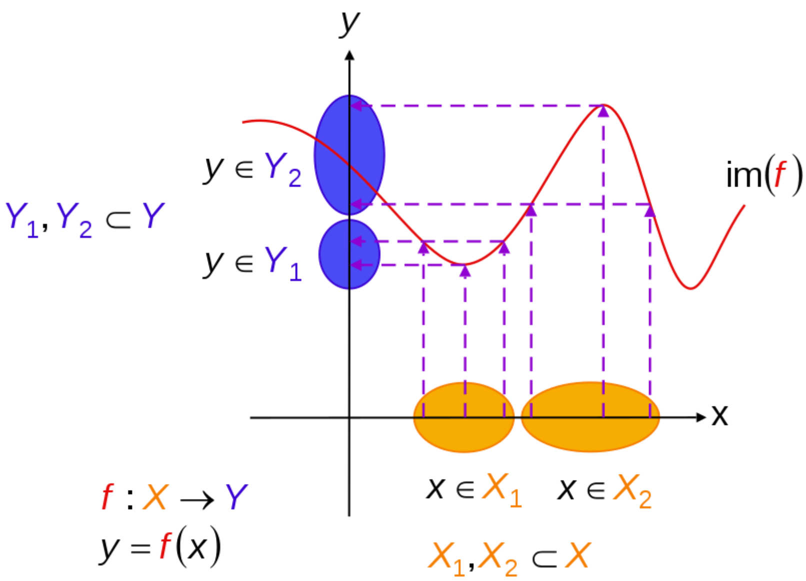
- For any set X and any subset S of X the inclusion map $S \to X$ (which sends any element s of S to itself) is injective. In particular the identity function $X \to X$ is always injective (and in fact bijective).
- If the domain $X = \varnothing$ or X has only one element, the function $X \to Y$ is always injective.
- The function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 2x + 1$ is injective.
- The function $g : \mathbf{R} \to \mathbf{R}$ defined by $g(x) = x^2$ is *not* injective, because (for example) $g(1) = 1 = g(-1)$. However, if g is redefined so that its domain is the non-negative real numbers $[0,+\infty)$, then g is injective.
- The exponential function $\exp : \mathbf{R} \to \mathbf{R}$ defined by $\exp(x) = e^x$ is injective (but not surjective as no real value maps to a negative number).
- The natural logarithm function $\ln : (0, \infty) \to \mathbf{R}$ defined by $x \mapsto \ln x$ is injective.
- The function $g : \mathbf{R} \to \mathbf{R}$ defined by $g(x) = x^n - x$ is not injective, since, for example, $g(0) = g(1) = 0$.

More generally, when X and Y are both the real line $\mathbf{R}$, then an injective function $f : \mathbf{R} \to \mathbf{R}$ is one whose graph is never intersected by any horizontal line more than once. This principle is referred to as the *horizontal line test*.

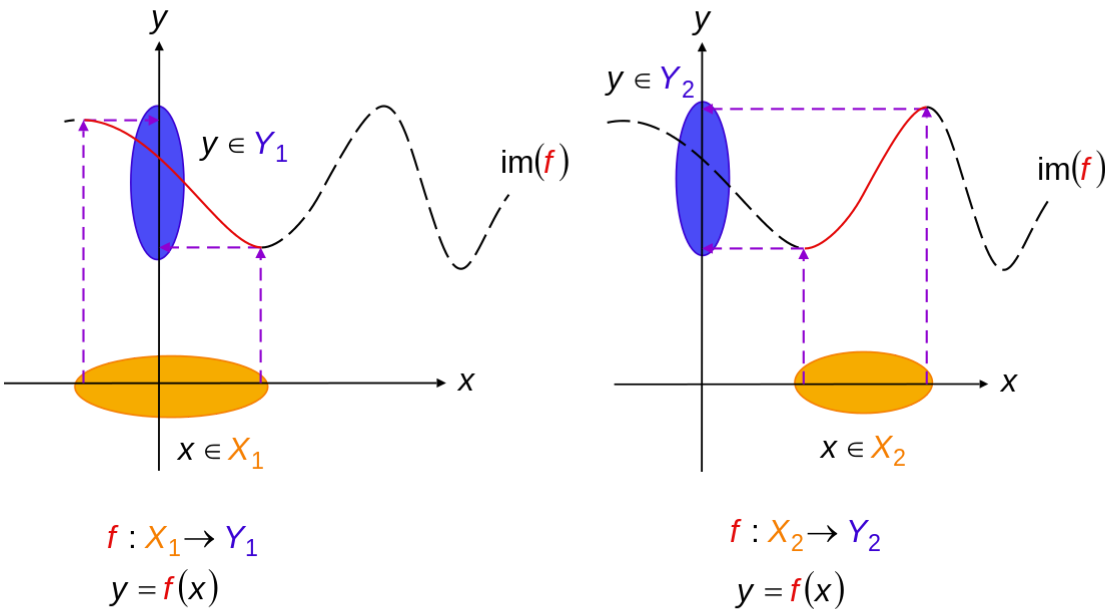


$$f : X \to Y$$
$$y = f(x)$$

Injective functions. Diagramatic interpretation in the Cartesian plane, defined by the mapping $f : X \to Y$, where $y = f(x)$, X = domain of function, Y = range of function, and im(f) denotes image of f. Every one x in X maps to exactly one unique y in Y. The circled parts of the axes represent domain and range sets – in accordance with the standard diagrams above.



Not an injective function. Here X_1 and X_2 are subsets of X, Y_1 and Y_2 are subsets of Y: for two regions where the function is not injective because more than one domain element can map to a single range element. That is, it is possible for *more than one* x in X to map to the *same* y in Y.



Making functions injective. The previous function $f : X \to Y$ can be reduced to one or more injective functions (say) $f : X_1 \to Y_1$ and $f : X_2 \to Y_2$, shown by solid curves (long-dash parts of initial curve are not mapped to anymore). Notice how the rule f has not changed – only the domain and range. X_1 and X_2 are subsets of X, Y_1 and Y_2 are subsets of R: for two regions where the initial function can be made injective so that one domain element can map to a single range element. That is, only one x in X maps to one y in Y.

Injections can be undone

Functions with left inverses are always injections. That is, given $f : X \to Y$, if there is a function $g : Y \to X$ such that, for every $x \in X$,

$$g(f(x)) = x \text{ (} f \text{ can be undone by } g\text{)}$$

then f is injective. In this case, g is called a retraction of f. Conversely, f is called a section of g.

Conversely, every injection f with non-empty domain has a left inverse g (in conventional mathematics[2]). Note that g may not be a complete inverse of f because the composition in the other order, $f \circ g$, may not be the identity on Y. In other words, a function that can be undone or "reversed", such as f, is not necessarily invertible (bijective). Injections are "reversible" but not always invertible.

Although it is impossible to reverse a non-injective (and therefore information-losing) function, one can at least obtain a "quasi-inverse" of it, that is a multiple-valued function.

Injections may be made invertible

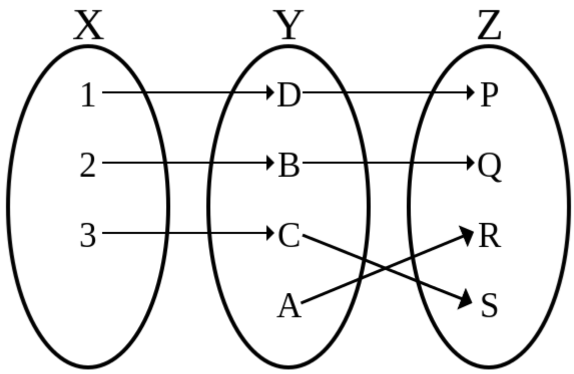
In fact, to turn an injective function $f : X \to Y$ into a bijective (hence invertible) function, it suffices to replace its codomain Y by its actual range $J = f(X)$. That is, let $g : X \to J$ such that $g(x) = f(x)$ for all x in X; then g is bijective. Indeed, f can be factored as $\mathrm{incl}_{J,Y} \circ g$, where $\mathrm{incl}_{J,Y}$ is the inclusion function from J into Y.

More generally, injective partial functions are called partial bijections.

Other properties

- If f and g are both injective, then $f \circ g$ is injective.

- If $g \circ f$ is injective, then f is injective (but g need not be).
- $f : X \to Y$ is injective if and only if, given any functions g, $h : W \to X$ whenever $f \circ g = f \circ h$, then $g = h$. In other words, injective functions are precisely the monomorphisms in the category **Set** of sets.
- If $f : X \to Y$ is injective and A is a subset of X, then $f^{-1}(f(A)) = A$. Thus, A can be recovered from its image $f(A)$.
- If $f : X \to Y$ is injective and A and B are both subsets of X, then $f(A \cap B) = f(A) \cap f(B)$.
- Every function $h : W \to Y$ can be decomposed as $h = f \circ g$ for a suitable injection f and surjection g. This decomposition is unique up to isomorphism, and f may be thought of as the inclusion function of the range $h(W)$ of h as a subset of the codomain Y of h.
- If $f : X \to Y$ is an injective function, then Y has at least as many elements as X, in the sense of cardinal numbers. In particular, if, in addition, there is an injection from Y< to X, then X and Y have the same cardinal number. (This is known as the Cantor–Bernstein–Schroeder theorem.)
- If both X and Y are finite with the same number of elements, then $f : X \to Y$ is injective if and only if f is surjective (in which case f is bijective).
- An injective function which is a homomorphism between two algebraic structures is an embedding.
- Unlike surjectivity, which is a relation between the graph of a function and its codomain, injectivity is a property of the graph of the function alone; that is, whether a function f is injective can be decided by only considering the graph (and not the codomain) of f.



The composition of two injective functions is injective.

Proving that functions are injective

A proof that a function f is injective depends on how the function is presented and what properties the function holds. For functions that are given by some formula there is a basic idea. We use the contrapositive of the definition of injectivity, namely that if $f(x) = f(y)$, then $x = y$.[3]

Here is an example:

$$f = 2x + 3$$

Proof: Let $f : X \to Y$. Suppose $f(x) = f(y)$. So $2x + 3 = 2y + 3 \Rightarrow 2x = 2y \Rightarrow x = y$. Therefore, it follows from the definition that f is injective.

There are multiple other methods of proving that a function is injective. For example, in calculus if f is a differentiable function defined on some interval, then it is sufficient to show that the derivative is always positive or always negative on that interval. In linear algebra, if f is a linear transformation it is sufficient to show that the kernel of f contains only the zero vector. If f is a function with finite domain it is sufficient to look through the list of images of each domain element and check that no image occurs twice on the list.

See also

- Surjective function
- Bijective function
- Partial function
- Injective module
- Bijection, injection and surjection
- Horizontal line test
- Injective metric space

Notes

1. "Unicode" (http://www.unicode.org/charts/PDF/U2190.pdf) (PDF). Retrieved 2013-05-11.
2. This principle is valid in conventional mathematics, but may fail in constructive mathematics. For instance, a left inverse of the inclusion {0,1} → **R** of the two-element set in the reals violates indecomposability by giving a retraction of the real line to the set {0,1}.
3. Williams, Peter. "Proving Functions One-to-One" (http://www.math.csusb.edu/notes/proofs/bpf/node4.html).

References

- Bartle, Robert G. (1976), *The Elements of Real Analysis* (2nd ed.), New York: John Wiley & Sons, ISBN 978-0-471-05464-1, p. 17 *ff*.
- Halmos, Paul R. (1974), *Naive Set Theory*, New York: Springer, ISBN 978-0-387-90092-6, p. 38 *ff*.

External links

- Earliest Uses of Some of the Words of Mathematics: entry on Injection, Surjection and Bijection has the history of Injection and related terms. (http://jeff560.tripod.com/i.html)
- Khan Academy – Surjective (onto) and Injective (one-to-one) functions: Introduction to surjective and injective functions (http://www.khanacademy.org/math/linear-algebra/v/surjective--onto--and-injective--one-to-one--functions)