

1-1: The Counterfeit Coin Problem

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You have *eight* similar coins and a beam balance.

At most one coin is counterfeit and hence underweight.

How can you detect whether there is an underweight coin, and if so, which one, using the balance only *twice*?

— E.D. Schell, 1945 (*American Mathematical Monthly*)

— *Problem 1.8 of UD*

The **minimum** number of weighings ...
In the worst-case scenario
Decision tree
“min-max”

Put equal numbers of coins on opposite sides of the balance.
Same?

What is the First Step?

$$L = x \quad R = x$$

Possible outcomes:

Balanced

L Rises

R Rises

Balanced: The “Standard Coin” Variation

Key point: G

L Rises: The “Labelled Coin” Variation

Key point: PH & PL & G

A Special Case of the “Labelled Coin” Variation

The counterfeit coin is known to be light.

Recursive algorithm:

$$\frac{1}{3}$$

Lower bound:

a single weighing of any sort cannot do better than trisection

The “Labelled Coin” Variation

Recursive algorithm:

*Whenever we place coins on the scale, we must be sure to put **equal** number of PL (therefore PH) coins on the two sides.*

Lower bound:

cannot do better than in the “Light Coin” variation

The “Labelled Coin” Variation in the 12 Coins Example

The “Standard Coin” Variation

$$M(n) = (3^n - 1)/2$$

The Weighing Algorithm (0)

Compute n , the minimum number of weighings:

$$(3^{n-1} - 3)/2 < |S| \leq (3^n - 3)/2$$

The Weighing Algorithm (1)

$$S = S_1 \cup S_2 \cup S_3$$

$$|S_1| = |S_2| \quad |S_1 \cup S_2| \leq 3^{n-1} - 1$$

$$|S_3| \leq (3^{n-1} - 1)/2$$

The Weighing Algorithm (2: Balanced)

$$S_3 \cup \{\text{a standard coin}\}$$

$$S_3 = S'_1 \cup S'_2 \cup S'_3$$

$$|S_3| \leq (3^{n-1} - 1)/2$$

$$|S'_1| = |S'_2| + 1 \text{ (the standard coin)} \quad |S_1 \cup S_2| \dots$$

$$|S'_3| \leq (3^{n-2} - 1)/2$$

The Weighing Algorithm (2: Not Balanced)

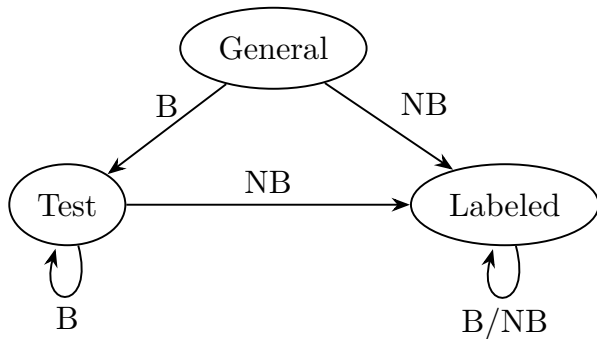
$$|S_1 \cup S_2| \leq 3^{n-1} - 1$$

$$|S_1 \cup S_2| = S'_1 \cup S'_2 \cup S'_3$$

$$|S'_1| = |S'_2| \leq 3^{n-2} \quad |S'_1|_{PH} = |S'_2|_{PL}$$

$$|S'_3| \leq 3^{n-2}$$

The Weighing Algorithm (3: Recurse)







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