

2-7 Discrete Probability

"Probability is too important to be left to the experts."

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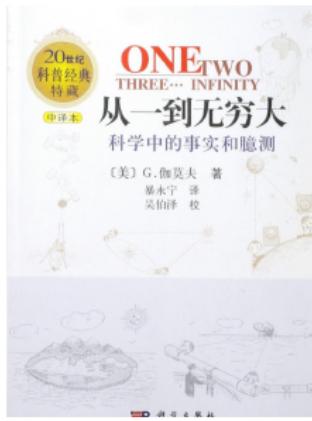


Q : What is Probability?

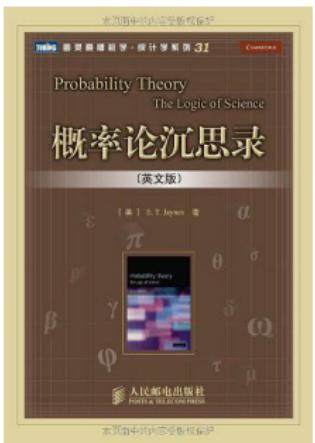


Q : What is Probability?

(Objective? Subjective? Neutral?)

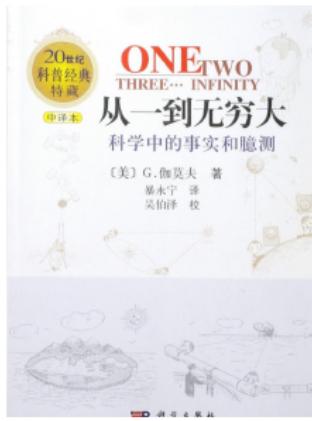


MY
INTUITION
TOLD
ME

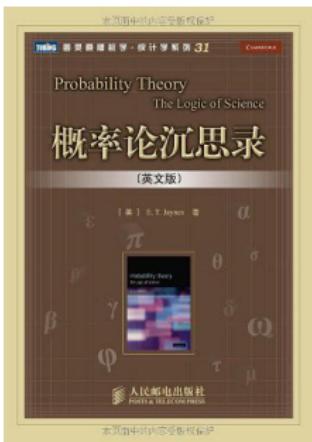


“... and the many paradoxes show clearly that we, as humans, lack a well grounded intuition in this matter.”

— “*The Art of Probability*”, Richard W. Hamming



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“... and the many paradoxes show clearly that we, as humans, lack a well grounded intuition in this matter.”

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“When called upon to judge probability, people actually judge something else and believe they have judged probability.”

— “*Thinking, Fast and Slow*”, Daniel Kahneman



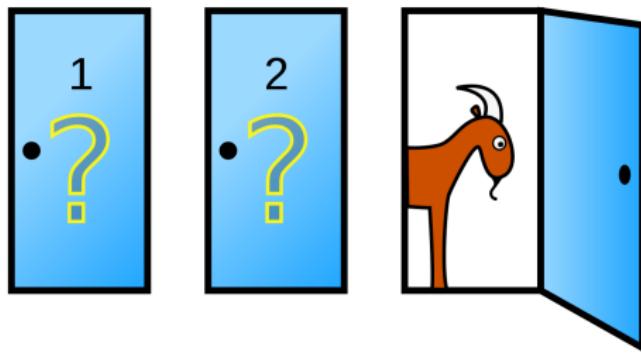


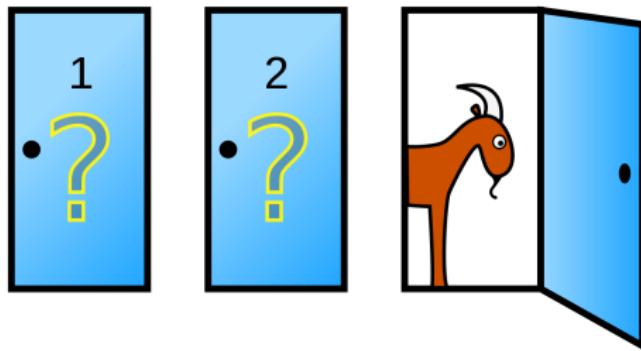
Let us calculate [calculemus].



- (a) Monty Hall problem
- (b) Boy or Girl paradox
- (c) Searching unsorted array

The Monty-Hall Problem





You: Randomly pick a door (No. 1)

I: Open a door which has a goat (No. 3)
(I know what's behind the doors)

Q : Do you want to switch to door 2?

C_i : The car is behind door i ($i = 1, 2, 3$)

$$\Pr \{C_i\} = \frac{1}{3}$$

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Y_1 : you initially pick door 1

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$$\Pr \{C_i\} = \frac{1}{3}$$

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$$\Pr \{X_1\} = \frac{1}{3}$$

I_3 : I open door 3

$$\Pr \{C_2 | I_3, Y_1\}$$

$$\begin{aligned}\Pr \{C_2 | I_3, Y_1\} &= \frac{\Pr \{C_2, I_3, Y_1\}}{\Pr \{I_3, Y_1\}} = \frac{\Pr \{I_3, Y_1 | C_2\} \Pr \{C_2\}}{\Pr \{I_3 | Y_1\} \Pr \{Y_1\}} \\ &= \frac{\Pr \{I_3, Y_1 | C_2\}}{\Pr \{I_3 | Y_1\}}\end{aligned}$$

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$$\begin{aligned}\Pr \{I_3, Y_1 | C_2\} &= \frac{\Pr \{I_3, Y_1, C_2\}}{\Pr \{C_2\}} = \frac{\Pr \{I_3 | C_2, Y_1\} \Pr \{C_2, Y_1\}}{\Pr \{C_2\}} \\ &= \frac{\Pr \{I_3 | C_2, Y_1\} \Pr \{Y_1 | C_2\} \Pr \{C_2\}}{\Pr \{C_2\}} \\ &= \frac{1}{3} \Pr \{I_3 | C_2, Y_1\}\end{aligned}$$

$$\Pr \{C_2 | I_3, Y_1\} = \frac{\Pr \{I_3 | C_2, Y_1\}}{3 \Pr \{I_3 | Y_1\}}$$

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$$\begin{aligned}\Pr \{I_3 | Y_1\} &= \Pr \{I_3 | C_1, Y_1\} \Pr \{C_1 | Y_1\} \\&\quad + \Pr \{I_3 | C_2, Y_1\} \Pr \{C_2 | Y_1\} \\&\quad + \Pr \{I_3 | C_3, Y_1\} \Pr \{C_3 | Y_1\} \\&= \frac{1}{3} \left(\Pr \{I_3 | C_1, Y_1\} + \Pr \{I_3 | C_2, Y_1\} + \Pr \{I_3 | C_3, Y_1\} \right)\end{aligned}$$

$$\Pr \{C_2 | I_3, Y_1\} = \frac{\Pr \{I_3 | C_2, Y_1\}}{\Pr \{I_3 | C_1, Y_1\} + \Pr \{I_3 | C_2, Y_1\} + \Pr \{I_3 | C_3, Y_1\}}$$

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It depends on how I choose the door to open!

$$\Pr \{C_2 | I_3, Y_1\} = \frac{\Pr \{I_3 | C_2, Y_1\}}{\Pr \{I_3 | C_1, Y_1\} + \Pr \{I_3 | C_2, Y_1\} + \Pr \{I_3 | C_3, Y_1\}}$$

It depends on how I choose the door to open!

$$\Pr \{I_3 | C_1, Y_1\} = \frac{1}{2}$$

$$\Pr \{I_3 | C_2, Y_1\} = 1$$

$$\Pr \{I_3 | C_3, Y_1\} = 0$$

$$\Pr \{C_2 | I_3, Y_1\} = \frac{\Pr \{I_3 | C_2, Y_1\}}{\Pr \{I_3 | C_1, Y_1\} + \Pr \{I_3 | C_2, Y_1\} + \Pr \{I_3 | C_3, Y_1\}}$$

It depends on how I choose the door to open!

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$$\Pr \{C_2 | I_3, Y_1\} = \frac{2}{3}$$

$$\Pr \{C_2 | I_3, Y_1\} = \frac{\Pr \{I_3 | C_2, Y_1\}}{\Pr \{I_3 | C_1, Y_1\} + \Pr \{I_3 | C_2, Y_1\} + \Pr \{I_3 | C_3, Y_1\}}$$

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{\Pr \{I_3 \mid C_2, Y_1\}}{\Pr \{I_3 \mid C_1, Y_1\} + \Pr \{I_3 \mid C_2, Y_1\} + \Pr \{I_3 \mid C_3, Y_1\}}$$

$$\boxed{\Pr \{I_3 \mid C_3, Y_1\} = 0}$$

$$\Pr \{C_2 | I_3, Y_1\} = \frac{\Pr \{I_3 | C_2, Y_1\}}{\Pr \{I_3 | C_1, Y_1\} + \Pr \{I_3 | C_2, Y_1\} + \Pr \{I_3 | C_3, Y_1\}}$$

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$$\Pr \{C_1 | I_3, Y_1\} = \frac{\Pr \{I_3 | C_1, Y_1\}}{\Pr \{I_3 | C_1, Y_1\} + \Pr \{I_3 | C_2, Y_1\}}$$

$$\Pr \{C_2 | I_3, Y_1\} = \frac{\Pr \{I_3 | C_2, Y_1\}}{\Pr \{I_3 | C_1, Y_1\} + \Pr \{I_3 | C_2, Y_1\} + \Pr \{I_3 | C_3, Y_1\}}$$

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$$\frac{\Pr \{C_2 | I_3, Y_1\}}{\Pr \{C_1 | I_3, Y_1\}} = \frac{\Pr \{I_3 | C_2, Y_1\}}{\Pr \{I_3 | C_1, Y_1\}}$$

$$\Pr \{C_2 | I_3, Y_1\} > \Pr \{C_1 | I_3, Y_1\} \iff \Pr \{I_3 | C_2, Y_1\} > \Pr \{I_3 | C_1, Y_1\}$$

$$\Pr \{C_2 \mid I_3, Y_1\} > \Pr \{C_1 \mid I_3, Y_1\} \iff \Pr \{I_3 \mid C_2, Y_1\} > \Pr \{I_3 \mid C_1, Y_1\}$$

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$$\Pr \{I_3 \mid C_2, Y_1\} = 1$$

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Q : Switching vs. Choosing between the two remaining doors randomly?

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{\Pr \{I_3 \mid C_2, Y_1\}}{\Pr \{I_3 \mid C_1, Y_1\} + \Pr \{I_3 \mid C_2, Y_1\}}$$

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{\Pr \{I_3 \mid C_2, Y_1\}}{\Pr \{I_3 \mid C_1, Y_1\} + \Pr \{I_3 \mid C_2, Y_1\}}$$

$$\Pr \{I_3 \mid C_1, Y_1\} = q$$

$$\Pr \{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{\Pr \{I_3 \mid C_2, Y_1\}}{\Pr \{I_3 \mid C_1, Y_1\} + \Pr \{I_3 \mid C_2, Y_1\}}$$

$$\Pr \{I_3 \mid C_1, Y_1\} = q$$

$$\Pr \{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{1}{1+q}$$

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{\Pr \{I_3 \mid C_2, Y_1\}}{\Pr \{I_3 \mid C_1, Y_1\} + \Pr \{I_3 \mid C_2, Y_1\}}$$

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$$\Pr \{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{1}{1+q} \in [\frac{1}{2}, 1]$$

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{\Pr \{I_3 \mid C_2, Y_1\}}{\Pr \{I_3 \mid C_1, Y_1\} + \Pr \{I_3 \mid C_2, Y_1\}}$$

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Always Switch!

$$\Pr \{C_2 | I_3, Y_1\} = \frac{\Pr \{I_3 | C_2, Y_1\}}{\Pr \{I_3 | C_1, Y_1\} + \Pr \{I_3 | C_2, Y_1\}}$$

$$\Pr \{I_3 | C_1, Y_1\} = q$$

$$\Pr \{I_3 | C_2, Y_1\} = 1$$

$$\Pr \{C_2 | I_3, Y_1\} = \frac{1}{1+q} \in [\frac{1}{2}, 1]$$

$$\Pr \{C_1 | I_3, Y_1\} \in [0, \frac{1}{2}]$$

Always Switch!

$$\Pr \{C_2 | I_3, Y_1\} > \Pr \{C_1 | I_3, Y_1\} \iff \Pr \{I_3 | C_2, Y_1\} > \Pr \{I_3 | C_1, Y_1\}$$

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{\Pr \{I_3 \mid C_2, Y_1\}}{\Pr \{I_3 \mid C_1, Y_1\} + \Pr \{I_3 \mid C_2, Y_1\}}$$

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$$\Pr \{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{2}{3}$$

$$\Pr \{C_2 | I_3, Y_1\} = \frac{\Pr \{I_3 | C_2, Y_1\}}{\Pr \{I_3 | C_1, Y_1\} + \Pr \{I_3 | C_2, Y_1\}}$$

$$\boxed{\Pr \{C_2 | I_3, Y_1\} > \Pr \{C_1 | I_3, Y_1\} \iff \Pr \{I_3 | C_2, Y_1\} > \Pr \{I_3 | C_1, Y_1\}}$$

I know what's behind the doors.

$$\Pr \{I_3 | C_1, Y_1\} = \frac{1}{2}$$

$$\Pr \{I_3 | C_2, Y_1\} = 1$$

$$\Pr \{C_2 | I_3, Y_1\} = \frac{2}{3}$$

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I know what's behind the doors.

I do not know what's behind the doors,
and opens one randomly that happens
not to reveal the car.

$$\Pr \{I_3 | C_1, Y_1\} = \frac{1}{2}$$

$$\Pr \{I_3 | C_1, Y_1\} = \frac{1}{2}$$

$$\Pr \{I_3 | C_2, Y_1\} = 1$$

$$\Pr \{I_3 | C_2, Y_1\} = \frac{1}{2}$$

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$$\Pr \{C_2 | I_3, Y_1\} = \frac{1}{2}$$

The Boy/Girl Puzzle



Both Girls (CS Problem 5.3 – 12)

Mr. and Mrs. Smith have two children of different ages,
what is the probability that they has two girls

- (a) given that **one of the children** is a girl?
- (b) given that **the older child** is a girl?



G_1 : the older child is a girl

G_2 : the younger child is a girl

G_1 : the older child is a girl

G_2 : the younger child is a girl

$$\Pr \{G_1 \wedge G_2 \mid G_1 \vee G_2\}$$

G_1 : the older child is a girl

G_2 : the younger child is a girl

$$\Pr \{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{\Pr \{G_1 \wedge G_2\}}{\Pr \{G_1 \vee G_2\}}$$

G_1 : the older child is a girl

G_2 : the younger child is a girl

$$\begin{aligned}\Pr \{G_1 \wedge G_2 \mid G_1 \vee G_2\} &= \frac{\Pr \{G_1 \wedge G_2\}}{\Pr \{G_1 \vee G_2\}} \\ &= \frac{\Pr \{G_1 \wedge G_2\}}{\Pr \{G_1\} + \Pr \{G_2\} - \Pr \{G_1 \wedge G_2\}}\end{aligned}$$

G_1 : the older child is a girl

G_2 : the younger child is a girl

$$\begin{aligned}\Pr \{G_1 \wedge G_2 \mid G_1 \vee G_2\} &= \frac{\Pr \{G_1 \wedge G_2\}}{\Pr \{G_1 \vee G_2\}} \\&= \frac{\Pr \{G_1 \wedge G_2\}}{\Pr \{G_1\} + \Pr \{G_2\} - \Pr \{G_1 \wedge G_2\}} \\&= \frac{1/4}{3/4} = \frac{1}{3}\end{aligned}$$

G_1 : the older child is a girl

G_2 : the younger child is a girl

$$\begin{aligned}\Pr \{G_1 \wedge G_2 \mid G_1 \vee G_2\} &= \frac{\Pr \{G_1 \wedge G_2\}}{\Pr \{G_1 \vee G_2\}} \\&= \frac{\Pr \{G_1 \wedge G_2\}}{\Pr \{G_1\} + \Pr \{G_2\} - \Pr \{G_1 \wedge G_2\}} \\&= \frac{1/4}{3/4} = \frac{1}{3}\end{aligned}$$

$$\Pr \{G_1 \wedge G_2 \mid G_1\}$$

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$$\Pr \{G_1 \wedge G_2 \mid G_1\} = \frac{\Pr \{G_1 \wedge G_2\}}{\Pr \{G_1\}} = \frac{1/4}{1/2} = \frac{1}{2}$$

$(G_1, G_2), (G_1, B_2), (B_1, G_2), (B_1, B_2)$

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$$\Pr \{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{1}{3}$$

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$(G_1, G_2), (G_1, B_2), (B_1, G_2), (B_1, B_2)$

$$\Pr \{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{1}{3}$$

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$$\Pr \{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{1}{3}$$

$$\Pr \{G_1 \wedge G_2 \mid G_1\} = \frac{1}{2}$$

$$\Pr \{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \boxed{\frac{2}{3}} \Pr \{G_1 \wedge G_2 \mid G_1\}$$

$$\Pr \{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{1}{3}$$

$$\Pr \{G_1 \wedge G_2 \mid G_1\} = \frac{1}{2}$$

$$\Pr \{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \boxed{\frac{2}{3}} \Pr \{G_1 \wedge G_2 \mid G_1\}$$

$$\Pr \{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \boxed{\Pr \{G_1 \mid G_1 \vee G_2\}} \Pr \{G_1 \wedge G_2 \mid G_1\}$$

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$$\Pr \{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \boxed{\Pr \{G_1 \mid G_1 \vee G_2\}} \Pr \{G_1 \wedge G_2 \mid G_1\}$$

$$\Pr \{G_1 \mid G_1 \vee G_2\} = \frac{\Pr \{G_1 \wedge (G_1 \vee G_2)\}}{\Pr \{G_1 \vee G_2\}}$$

$$\Pr \{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{1}{3}$$

$$\Pr \{G_1 \wedge G_2 \mid G_1\} = \frac{1}{2}$$

$$\Pr \{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \boxed{\frac{2}{3}} \Pr \{G_1 \wedge G_2 \mid G_1\}$$

$$\Pr \{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \boxed{\Pr \{G_1 \mid G_1 \vee G_2\}} \Pr \{G_1 \wedge G_2 \mid G_1\}$$

$$\Pr \{G_1 \mid G_1 \vee G_2\} = \frac{\Pr \{G_1 \wedge (G_1 \vee G_2)\}}{\Pr \{G_1 \vee G_2\}} = \frac{\Pr \{G_1\}}{\Pr \{G_1 \wedge G_2\}} = \frac{2}{3}$$





Both Girls (CS Problem 5.3 – 12)

Mr. and Mrs. Smith have two children of different ages,
what is the probability that they has two girls

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- (I) From all families with two children, at least one of whom is a girl, a (Smith's) family is chosen at random.
- (II) From all families with two children, one child (of Smith) is selected at random that happens to be a girl.

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$$\Pr \{G_1 \wedge G_2 \mid g\} = \frac{\{G_1 \wedge G_2 \wedge g\}}{\Pr \{g\}} = \frac{1/4}{1/2} = \frac{1}{2}$$

After-class Exercise

A new couple, known to have two children, has just moved into town. Suppose that the mother is encountered walking with one of her children. If this child is a girl, what is the probability that both children are girls?



Searching an Unsorted Array (CLRS Problem 5 – 2 (f))

```
1: procedure DETERMINISTIC-SEARCH( $A[1 \cdots n]$ ,  $x$ )
2:    $i \leftarrow 1$ 
3:   while  $i \leq n$  do
4:     if  $A[i] = x$  then
5:       return true
6:      $i \leftarrow i + 1$ 
7:   return false
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(e)

$$\exists! i : A[i] = x$$

(f)

$$\exists!_k i : A[i] = x$$

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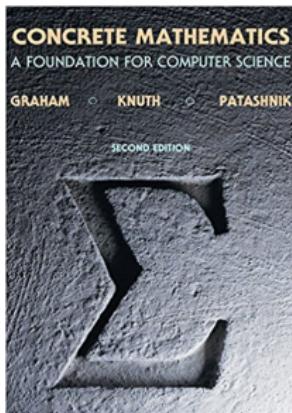
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After-class Exercise

$$\sum_{i=1}^{n-k+1} i \binom{n-i}{k-1} = \binom{n+1}{k+1}$$



$$r \binom{r-1}{k-1} = k \binom{r}{k}$$

$$\sum_{0 \leq k \leq n} \binom{k}{m} = \binom{n+1}{m+1}$$

Chapter 5: Binomial Coefficients

$$\begin{aligned}
\sum_{i=1}^{n-k+1} i \binom{n-i}{k-1} &= \sum_{i=0}^{n-k} (i+1) \binom{n-i-1}{k-1} \\
&= \sum_{i=0}^{n-k} ((n+1) - (n-i)) \binom{n-i-1}{k-1} \\
&= \sum_{i=0}^{n-k} (n+1) \binom{n-i-1}{k-1} - \sum_{i=0}^{n-k} (n-i) \binom{n-i-1}{k-1} \\
&= (n+1) \sum_{i=0}^{n-k} \binom{n-i-1}{k-1} - k \sum_{i=0}^{n-k} \binom{n-i}{k} \\
&= (n+1) \sum_{m=k-1}^{n-1} \binom{m}{k-1} - k \sum_{m=k}^n \binom{m}{k} \\
&= (n+1) \binom{n}{k} - k \binom{n+1}{k+1} = \binom{n+1}{k+1}
\end{aligned}$$

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$$\mathbb{E}[Y] = \sum_{i=1}^n \Pr \{ I_i = 1 \} = k \cdot \frac{1}{k} + (n - k) \cdot \frac{1}{k+1} = \frac{n+1}{k+1}$$





$$\Pr \{I_i = 1\} = \begin{cases} \frac{1}{k}, & \text{if } A[i] = x \\ \frac{1}{k+1}, & \text{if } A[i] \neq x \end{cases}$$



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NOT IID
(Independent and Identically Distributed)

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\end{aligned}$$

order statistics?
balls-into-bins?





- (a) Given an array $A[0 \cdots n - 1]$, to determine whether there is a value that *occurs more than $\lfloor n/k \rfloor$ times* in $\Theta(n \lg k)$ time and $\Theta(k)$ extra space.
- (b) Prove that the *lower bound* of this problem is $\Theta(n \lg k)$.

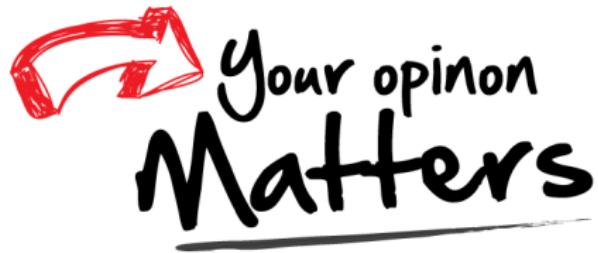
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TIP #1

Take $k = 2$.

$\Theta(n)$ time & $\Theta(1)$ space

Thank You!



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