

2-6 Algorithmic Methods

Hengfeng Wei

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Assume that $T(n)$ is constant for sufficiently small n .

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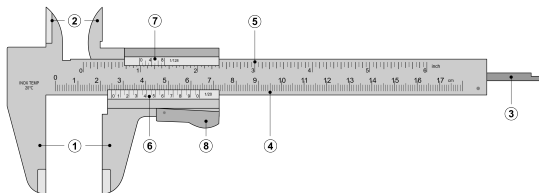
$$T(n) = aT(n/b) + f(n) \quad (a > 0, b > 1)$$

Assume that $T(n)$ is constant for sufficiently small n .

$f(n)$ is asymptotically positive.

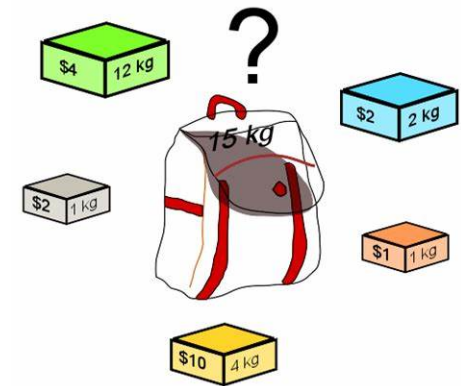
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Convex Polygon Diameter



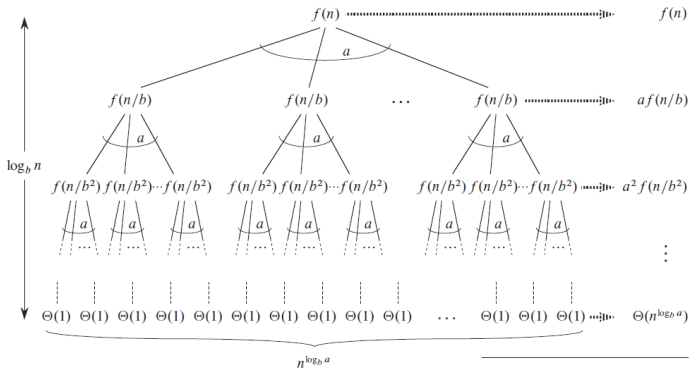
Correctness Proof

Integer Knapsack



Algorithm & Time Complexity

$$T(n) = aT(n/b) + f(n)$$



$$T(n) = n^{\log_b a} T(1) + \sum_{j=0}^{\log_b n - 1} a^j f\left(\frac{n}{b^j}\right)$$

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$$T(n) = n^{\log_b a} \textcolor{red}{T(1)} + \sum_{j=0}^{\log_b n - 1} a^j f\left(\frac{n}{b^j}\right)$$

$f(n) \text{ vs. } n^{\log_b a}$

$$f(n) = O(n^{\log_b a - \epsilon}) \implies T(n) = \Theta(n^{\log_b a})$$

$$f(n) = \Theta(n^{\log_b a}) \implies T(n) = \Theta(n^{\log_b a} \log n)$$

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$f(n)$ is asymptotically positive.

$$T(1) = d = \Theta(1), \quad (d \text{ can be } 0)$$

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$f(n)$ is asymptotically positive.

$$\sum_{j=0}^{\log_b n - 1} a^j f\left(\frac{n}{b^j}\right) = \Omega(n^{\log_b a})$$

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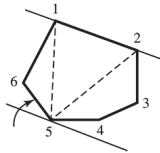
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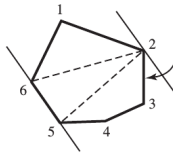
$$T(n) = n^{\log_b a} T(1)$$

$$T(n) = \begin{cases} 0, & T(1) = 0 \\ \Theta(n^{\log_b a}), & T(1) = d \neq 0 \end{cases}$$

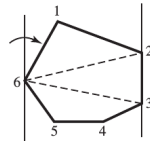
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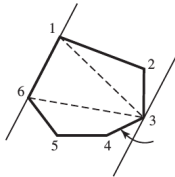
(a)



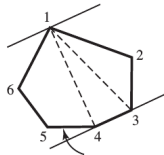
(b)



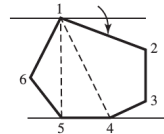
(c)



(d)



(e)



(f)

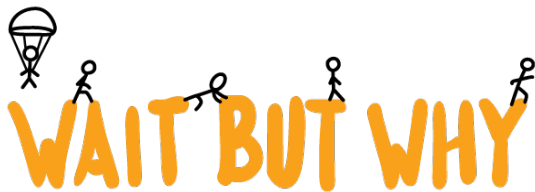


“Computational Geometry”

Ph.D Thesis, Michael Shamos, 1978

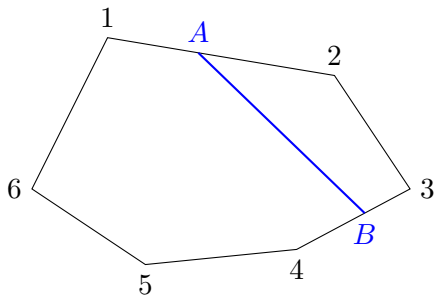
“Solving Geometric Problems with
the Rotating Calipers”, 1983

Correctness



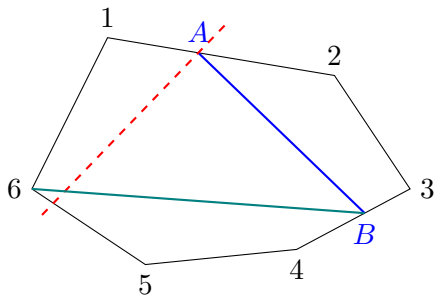
Theorem (DH 4-8)

If AB is a diameter of a convex polygon P , then A and B are vertices.



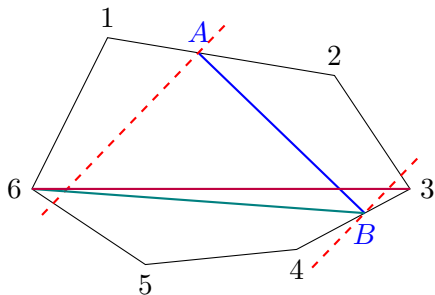
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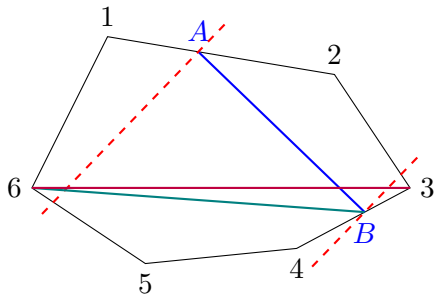
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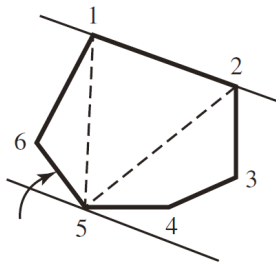
BUT, we have *not* enumerated *all* pairs of vertices.

We have enumerated *all antipodals*.

Definition (Line of Support)

A line L is a *line of support* of a convex polygon P if

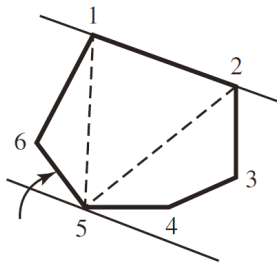
$$L \cap P = \text{a vertex/an edge of } P.$$



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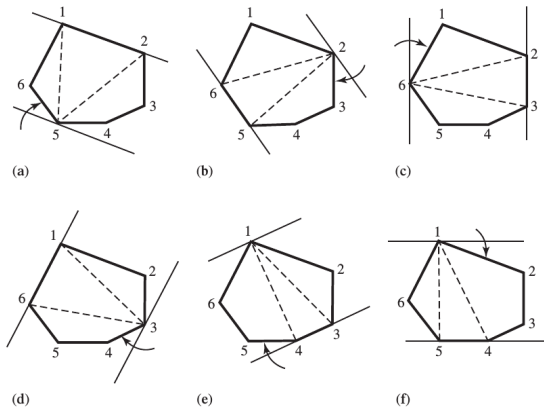
$$L \cap P = \text{a vertex/an edge of } P.$$



$$L \cap P \neq \emptyset \quad P \text{ lies entirely on one side of } L.$$

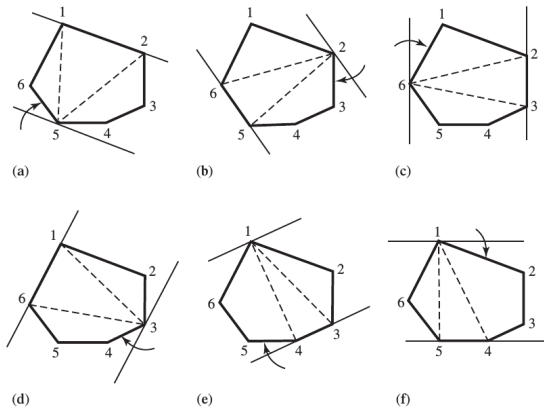
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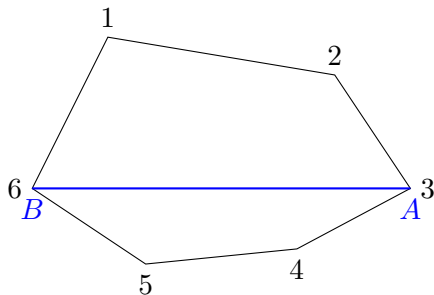
An *antipodal* is a pair of points that admits *parallel supporting lines*.



We have enumerated *all* antipodals by *rotating* through all angles.

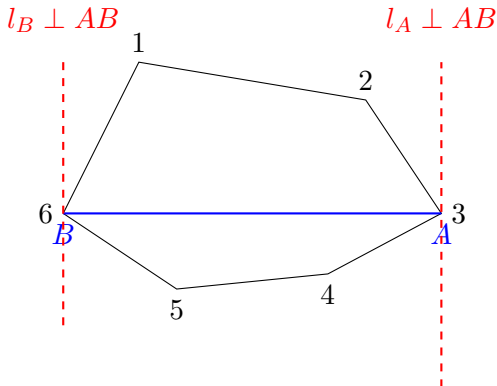
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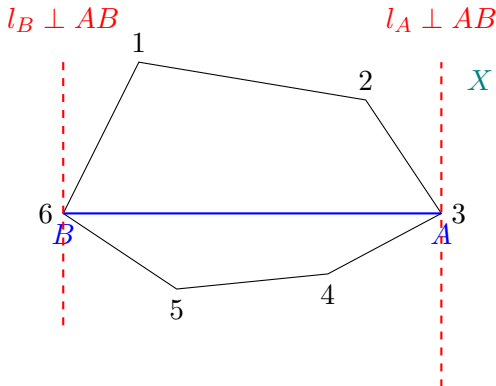
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We claim that l_A and l_B are parallel supporting lines.

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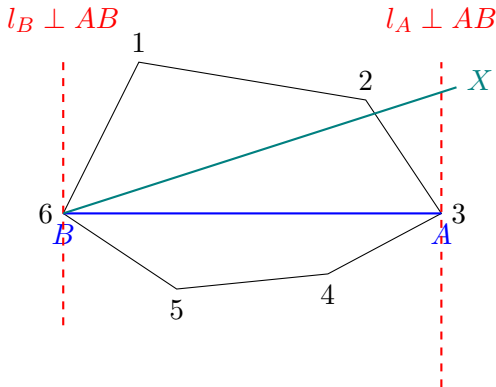


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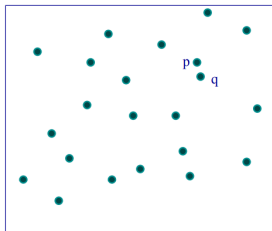
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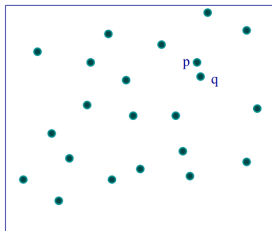
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Finding the Closest Pair of Points (Additional: DH 4-10)



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A Classical and Beautiful Divide-Conquer-Combine Algorithm:



Section 33.4, CLRS

DH 4.13 (Integer Knapsack)

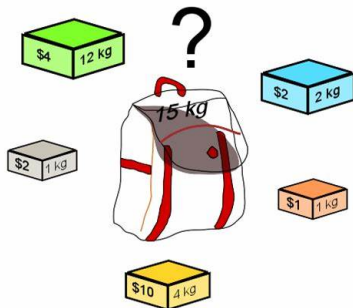
$$N = 5$$

$$Q = [3, 1, 4, 5, 1] \quad (\text{quantity})$$

$$W = [10, 20, 20, 8, 7] \quad (\text{weight})$$

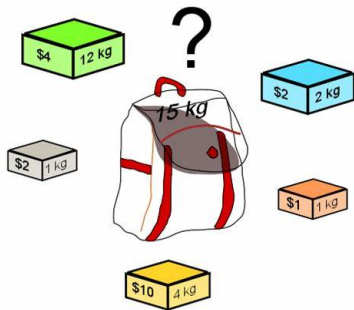
$$P = [17, 42, 35, 16, 15] \quad (\text{profit})$$

$$C = 103$$



0-1 Knapsack

$$\forall i : Q[i] = 1$$



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$$N' = \sum_i Q[i]$$

$$W' = [\dots, \underbrace{W_i}_{\# = Q_i}, \dots]$$

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$K[c, i] :$

The maximal profit obtained using knapsack of capacity c
with items of $x_1 \dots x_i$.

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Using the item x_i or not?

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Using the item x_i or not?

$$K[c, i] = \max \begin{cases} K[c, i - 1], \\ K[c - W[i], i - 1] + P[i], & W[i] \leq c \end{cases}$$

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Time complexity : $\Theta(NC)$

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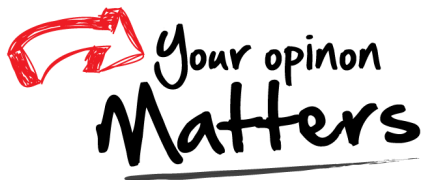
Time complexity : $\Theta(NC)$

Is this a polynomial algorithm?



KEEP
CALM
AND
STAY
TUNED

Thank
You!



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