Direct Products and Quotient Groups

Hengfeng Wei

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What do you mean by “是一回事”?

Theorem

$$\textit{If } G = H \times K,$$
$$\textit{then } \exists H' \cong H, K' \cong K,$$
$$\textit{such that } G \textit{ is the internal direct product of } H \textit{ and } K.$$

Theorem

$$\textit{If } G \textit{ is the internal direct product of } H \textit{ and } K,$$
$$\textit{then } G \cong H \times K.$$

$$\mathbb{Z}_2 \times \mathbb{Z}_3 = G \cong \mathbb{Z}_6$$

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$\mathbb{Z}_2$

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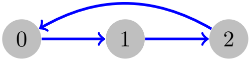

$\mathbb{Z}_3$

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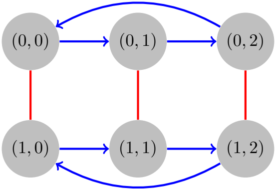
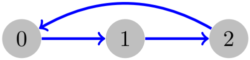


$\mathbb{Z}_2 \times \mathbb{Z}_3$



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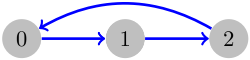
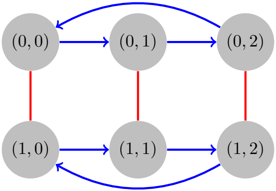


$\mathbb{Z}_3$



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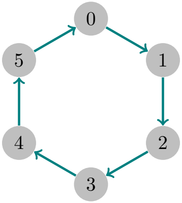

$\mathbb{Z}_6$

$$G = H \times K$$

Definition

$$H' = H \times \{e_K\} = \{(h', e_K) : h' \in H)\}$$

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H' and K' commute.

Theorem

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such that G is the internal direct product of H' and K'.

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such that G *is the internal direct product of* H' *and* K'.

Definition (Internal Direct Product)

Let G be a group with subgroups H and K satisfying

$$G = HK$$

$$H \cap K = \{e\}$$

H and K commute.

Then, G is the internal direct product of H and K.

$$G = H \times K$$

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Definition (Internal Direct Product (Equivalent))

Let G be a group with normal subgroups H and K satisfying

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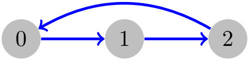
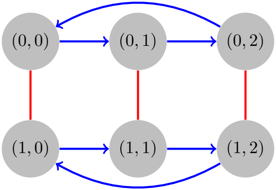
Then, G is the internal direct product of H and K.

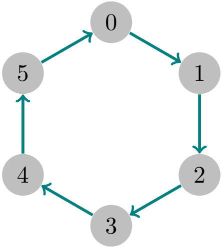
$\mathbb{Z}_2$



$\mathbb{Z}_3$

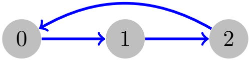


$\mathbb{Z}_2 \times \mathbb{Z}_3$



$\mathbb{Z}_6$

$\mathbb{Z}_2$



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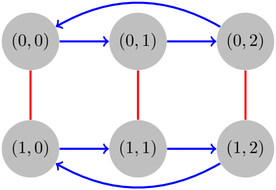
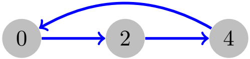
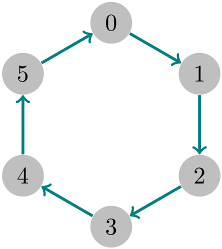


$\mathbb{Z}_2 \times \mathbb{Z}_3$

$\mathbb{Z}_2 \cong \{0, 3\} \leq \mathbb{Z}_6$

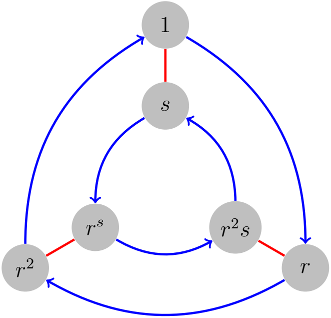


$\mathbb{Z}_3 \cong \{0, 2, 4\} \leq \mathbb{Z}_6$

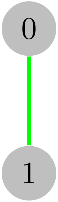


$\mathbb{Z}_6$

$$D_6 \cong D_3 \times \mathbb{Z}_2$$



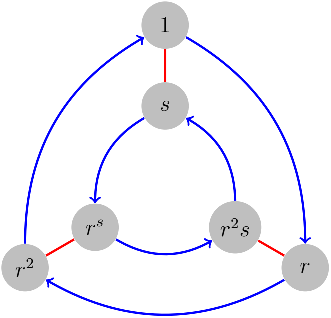
D_3



$\mathbb{Z}_2$

$D_3 \times \mathbb{Z}_2$

$$D_6 \cong D_3 \times \mathbb{Z}_2$$



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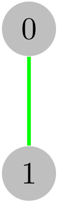

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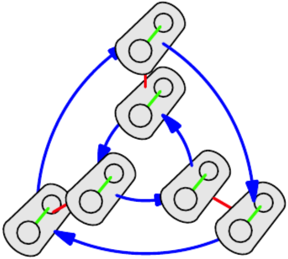

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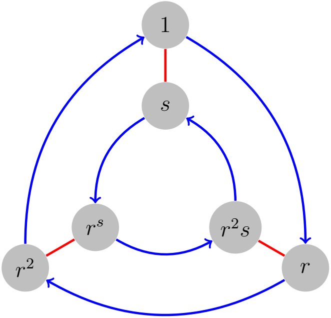

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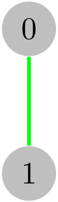

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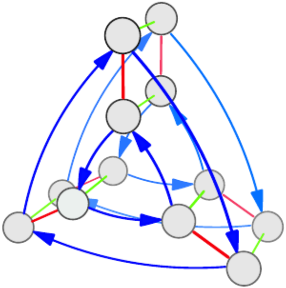

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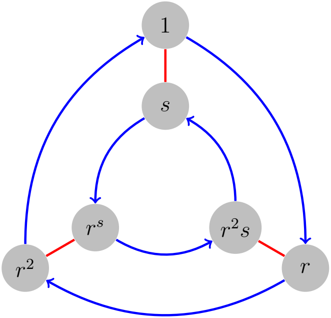

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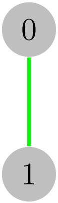

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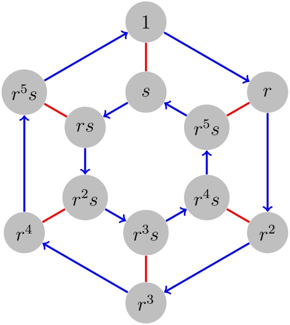
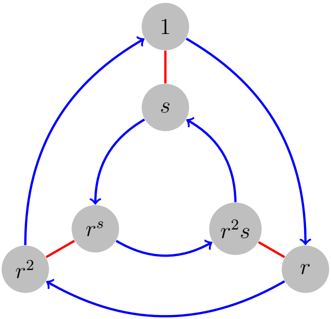


D_6

$$D_6 \cong D_3 \times \mathbb{Z}_2$$

$$D_6 = D_3' \mathbb{Z}_2' \quad (D_3' \lhd D_6,\ \mathbb{Z}_2' \lhd D_6)$$



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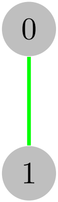

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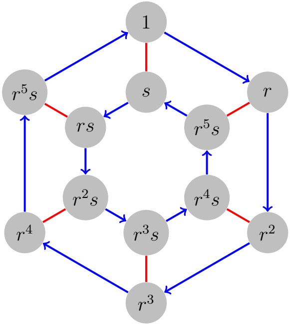

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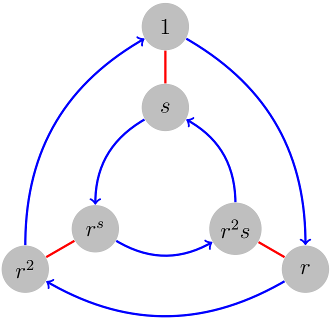

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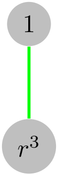
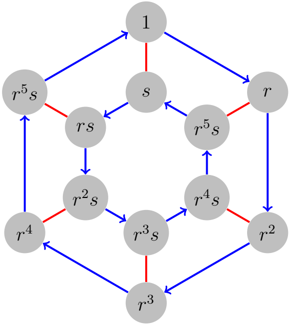


$$\mathbb{Z}_2 \cong \left\{1, r^3\right\} \lhd D_6$$



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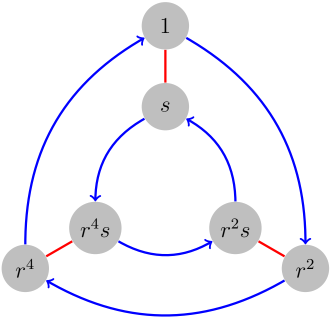
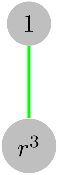


$$D_3 \cong \left\{1, r^2, r^4, s, r^2 s, r^4 s\right\} \lhd D_6$$



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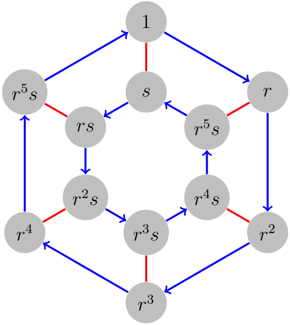
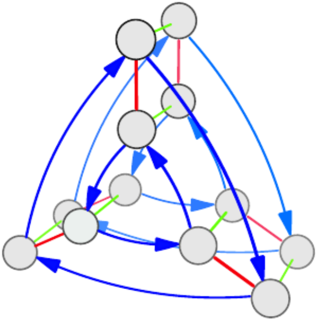


D_6

$D_6 \cong D_3 \times \mathbb{Z}_2$



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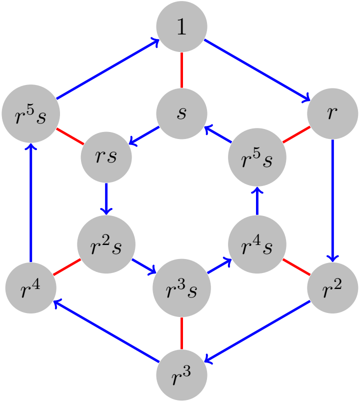

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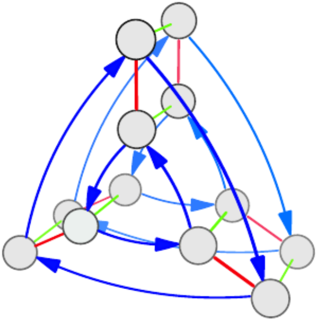

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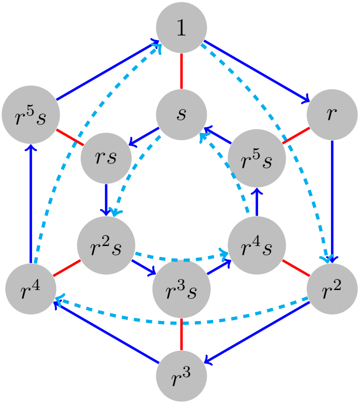

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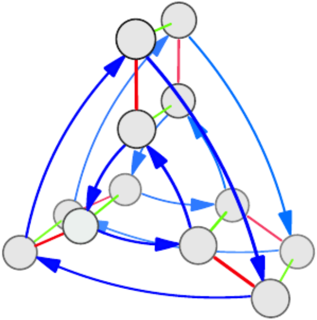

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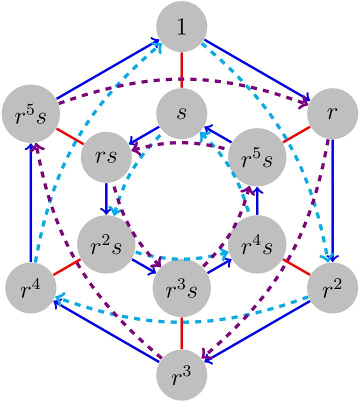

D_6

$$D_{2n} \cong D_n \times \mathbb{Z}_2 \quad (n \text{ is odd})$$

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$$\mathbb{Z}_2 \cong (\mathbb{Z}_2' \triangleq \{1, r^n\}) \lhd D_{2n}$$

$$D_{2n} \cong D_n \times \mathbb{Z}_2 \quad (n \text{ is odd})$$

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$$D_n \cong (D_n' \triangleq \langle r^2, s \rangle) \lhd D_{2n}$$

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$$D_n \cong (D_n' \triangleq \langle r^2, s \rangle) \lhd D_{2n}$$

D_n is the internal direct product of $\mathbb{Z}_2'$ and D_n'.

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Let G be a group with normal subgroups H and K satisfying

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Theorem (The Second Isomorphism Theorem)

$$H \leq G, N \lhd G \Longrightarrow H/(H \cap N) \cong HN/N.$$

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Theorem (The Second Isomorphism Theorem (Diamond Theorem))

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Theorem

If G is the internal direct product of its normal subgroups H and K, then $G/H \cong K, \quad G/K \cong H$.

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$$D_6 = \langle r^2, s \rangle \left\{1, r^3\right\}$$

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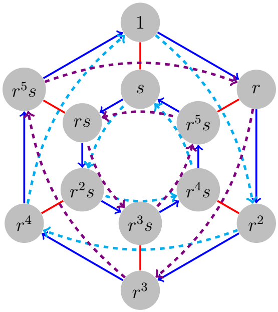
If G is the internal direct product of its normal subgroups H and K, then $G/H \cong K, \quad G/K \cong H$.

$$D_6 = \langle r^2, s \rangle \left\{1, r^3\right\} \implies D_6/\langle r^2, s \rangle \cong \left\{1, r^3\right\} \cong \mathbb{Z}_2$$



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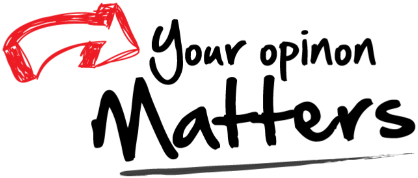
Theorem

$$\text{If } G \cong H \times K$$

$$\text{then } G/H \times 1 \cong K, \quad G/K \times 1 \cong H.$$


THANK YOU



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