3-11 Matchings and Factors

(Part I: Matchings and Covers)

Hengfeng Wei

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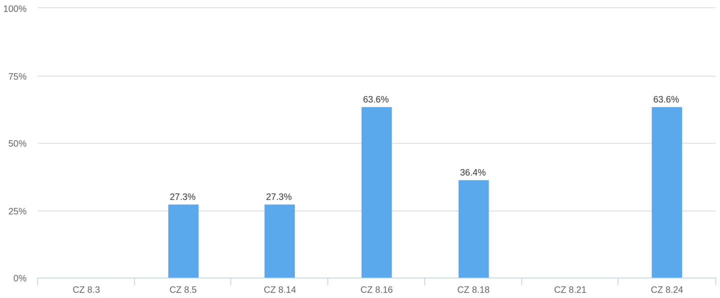
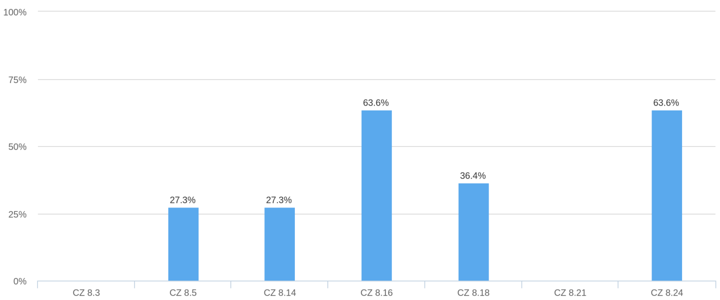
December 10, 2018



8.5 8.14 8.16

8.18 8.24 (The Last Class)



8.5 8.14 8.16 Chinese Postman Problem (The Last Class?)

8.18 8.24 (The Last Class)

比较大的定理（证明比较长的）都不是很理解，想知道期末考什么
点覆盖边覆盖那里只知道有这些性质，了解不是很深
都理解
图的分解的形象意义
无
定理8.3的证明
αβ、α'β'的定义和几个定理推论
为什么中英文书上的定义中\alpha和\beta反了。。
定理8.10的证明看不懂；一些比较几何的构造法证明（比如把顶点排成正多边形，一个点放中间）是怎么保证这些分解不重不漏的?
Kirkman三元系

$$\alpha, \quad \beta, \quad \alpha', \quad \beta'$$

Theorem 8.10 (Tutte's Theorem) (The Last Class)

Theorem (Hall's Theorem, 1935; Theorem 8.3)

Let G be a bipartite graph with partite sets U and W such that $r = |U| \leq |W|$.
G contains a matching of cardinality $r \iff G$ satisfies Hall's Condition:

$$\forall X \subseteq U : |N(X)| \geq |X|$$

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Other TONCAS?

Perfect Matching on Trees (Problem 8.5)

Prove that every tree T has ≤ 1 perfect matching.

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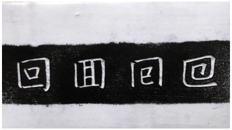
“这题有四样证法, 你知道吗?”



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if n is odd **then**

  $\#$ Perfect Matching $= 0$

else $\triangleright$ n is even

  Consider $T - r$ $\triangleright$ r : root of G

  if $k_o(T - r) > 1$ **then**

  else $\triangleright$ $k_o(T - r) = 1$

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3: **else** ▷ T has perfect matchings

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    v must be matched with its parent u

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    Consider a leaf v
    v must be matched with its parent u
    By Induction Hypothesis on each component of G − {u, v}
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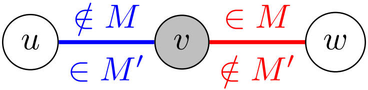

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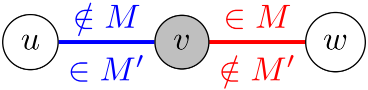

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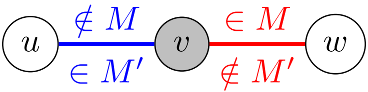

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Contradiction: Cycle

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Consider the subgraph $\mathcal{M}$ with $V(T)$ and $M \Delta M'$.

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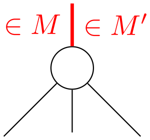

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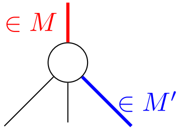

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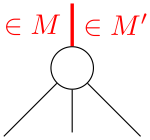

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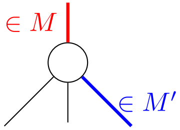

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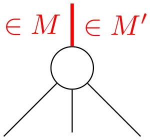

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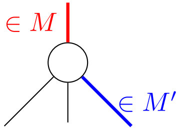

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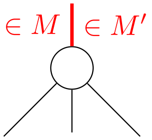

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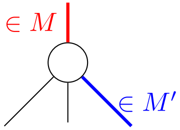

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$\alpha(G)$	Maximum size of independent set
$\beta(G)$	Minimum size of vertex cover
$\alpha'(G)$	Maximum size of matching
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Theorem (Gallai Identities, 1959; Theorem 8.7)

If G is graph without isolated vertices, then

$$\alpha'(G) + \beta'(G) = n(G).$$

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Theorem (König, 1931; Egerváry, 1931)

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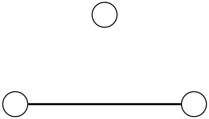

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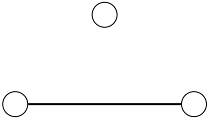

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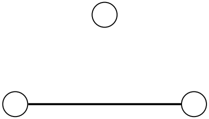

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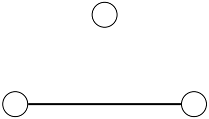

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Vertex Independence Number (Additional Problem)

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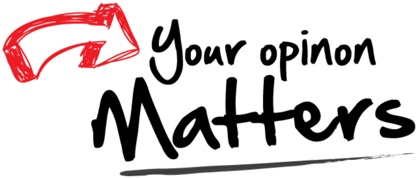
By Construction.

To construct an independent set S with $|S| \geq \frac{n}{\Delta+1}$.

1: **while** $|V(G) > 0|$ **do**
2: Choose $v \in V(G)$
3: $S \leftarrow S \cup \{v\}$
4: $G \leftarrow G - \{v\} - N(v)$


THANK YOU



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