2-3 Counting

魏恒峰

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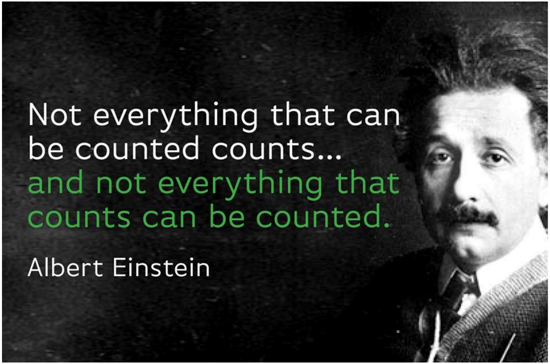
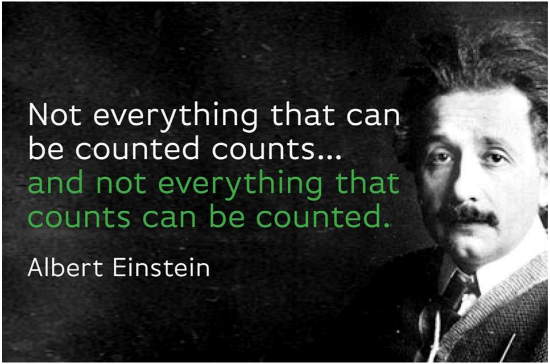

Not everything that can
be counted counts...
and not everything that
counts can be counted.
Albert Einstein



所以，学好 “2-3 组合与计数” 是多么重要!

Paring up (CS : $1.2 - 15$)

A tennis club has $2n$ members. We want to pair up the members by twos for singles matches.

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(b) Suppose that we also determine who serves first for each pairing. In how many ways can we specify our pairs?

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$$\frac{(2n)!}{2^n \cdot n!} \cdot 2^n = \frac{(2n)!}{n!}$$

Passing out Apples to Children

k-Permutation (CS : $1.2 - 5$)

We need to pass out k distinct apples (pieces of fruit) to n children such that *each child may get at most one apple*.

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(b) What if $k > n$?

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$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

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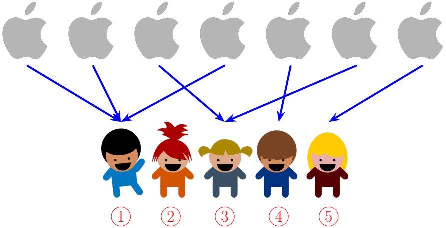

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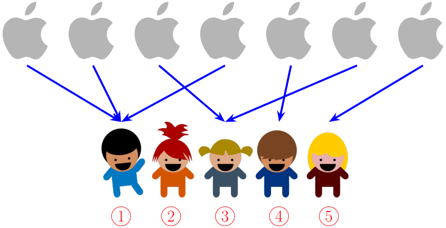

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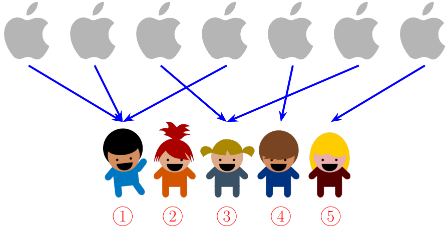

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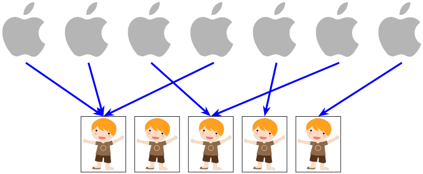

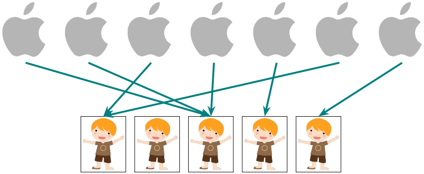

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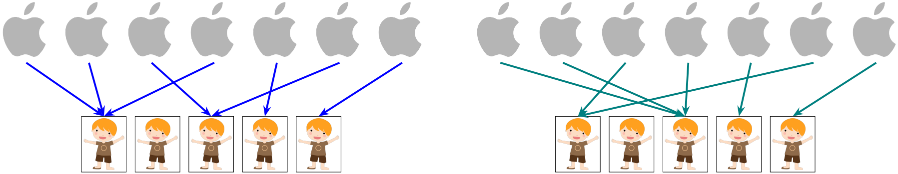

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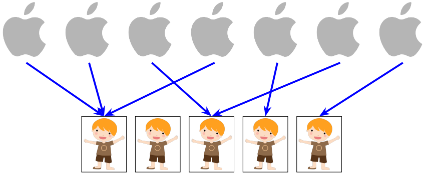

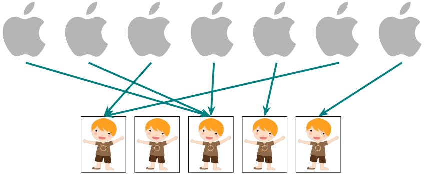

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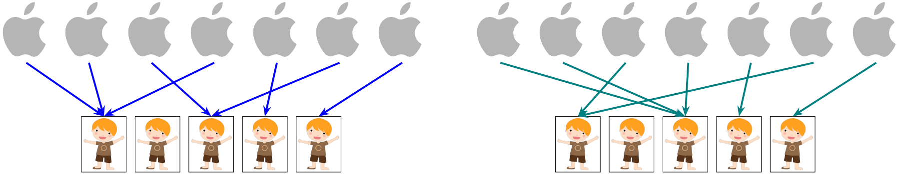

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Theorem ( )

$$p(k) \triangleq \sum_{x=1}^{x=k} p_x(k) \sim \frac{1}{4\sqrt{3}k} \exp\left(\pi\sqrt{\frac{2k}{3}}\right)$$

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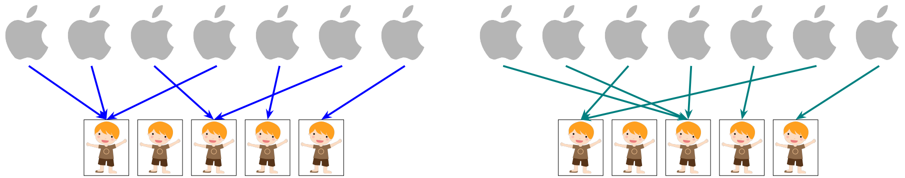

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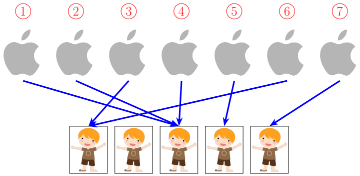

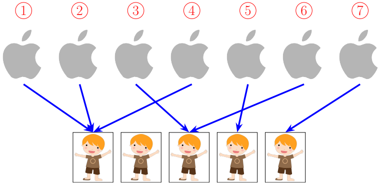

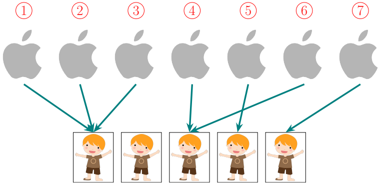

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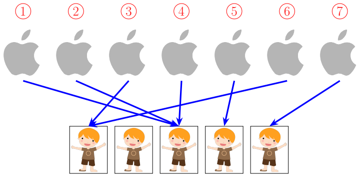

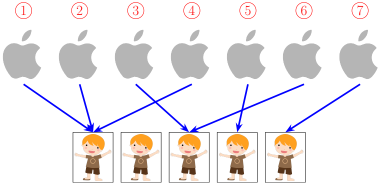

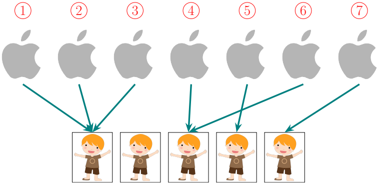

Set partition of $[1 \cdots k]$ into $\leq n$ parts

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Stirling number of the second kind

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Theorem (Recurrence for $S(n,k)$)

$$S(0,0) = 1, \quad S(n,0) = S(0,n) = 0 \ (n > 0)$$

$$S(n,k) = S(n-1,k-1) + kS(n-1,k), \quad n > 0, k > 0$$

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Proof.

$$S(n,k)=\underbrace{S(n-1,k-1)}_{n \text{ is alone}}+\underbrace{kS(n-1,k)}_{n \text{ is not alone}}$$

□

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Theorem (Berend & Tassa (2010))

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Theorem (de Bruijn (1981))

As $n \to \infty$,

$$\frac{\ln B_n}{n} = \ln n - \ln\ln n - 1 + \frac{\ln\ln n}{\ln n} + \frac{1}{\ln n} + \frac{1}{2}\left(\frac{\ln\ln n}{\ln n}\right)^2 + O\left(\frac{\ln\ln n}{(\ln n)^2}\right)$$

THE TWELVEFOLD WAY

balls per urn	unrestricted	≤ 1	≥ 1
n labeled balls, m labeled urns	n-tuples of m things	n-permutations of m things	partitions of $\{1,\ldots,n\}$ into m ordered parts
n unlabeled balls, m labeled urns	n-multicombinations of m things	n-combinations of m things	compositions of n into m parts
n labeled balls, m unlabeled urns	partitions of $\{1,\ldots,n\}$ into $\leq m$ parts	n pigeons into m holes	partitions of $\{1,\ldots,n\}$ into m parts
n unlabeled balls, m unlabeled urns	partitions of n into $\leq m$ parts	n pigeons into m holes	partitions of n into m parts

Computing $\binom{n}{k}$ (CS $1.5 : 14$)

procedure $\textsc{Binom}(n, k)$ ▷ Required: $n \ge k \ge 0$
  if $k = 0 \lor n = k$ **then**
    return 1
  return $\textsc{Binom}(n-1, k) + \textsc{Binom}(n-1, k-1)$

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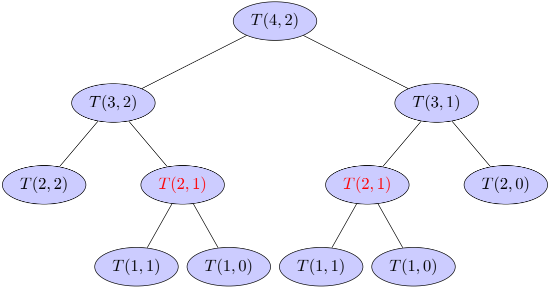

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$$\boxed{T(n,k) = \begin{cases} \\ T(n-1,k) + T(n-1,k-1) + c, \end{cases}}$$

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$$\alpha\binom{n}{0} - c = 0, \quad \alpha\binom{n}{n} - c = 0 \implies \alpha = c$$

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$$T(n,k) = T(n-1,k) + T(n-1,k-1) \implies T(n,k) = \alpha \binom{n}{k}$$

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$$\alpha \binom{n}{k} + \beta = \alpha \binom{n-1}{k} + \beta + \alpha \binom{n-1}{k-1} + \beta + c \implies \beta = -c$$

$$\alpha \binom{n}{0} - c = 0, \quad \alpha \binom{n}{n} - c = 0 \implies \alpha = c$$

$$\boxed{T(n,k) = c \binom{n}{k} - c}$$

7. 斐波那契数列的定义如下：$F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}\ (n \geq 3)$。如果用下面的函数计算斐波那契数列的第 n 项，则其时间复杂度为（　　）。

```
int F(int n)
{
    if (n <= 2)
        return 1;
    else
        return F(n - 1) + F(n - 2);
}
```

A. $O(1)$　　B. $O(n)$　　C. $O(n^2)$　　D. $O(F_n)$

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时间复杂度 of what?

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

Q : How to calculate $\binom{5}{3}$?

```
procedure BINOM(n, k)                          ▷ Required: n ≥ k ≥ 0
    for i ← 0 to n − k do
        B[i][0] ← 1
    for i ← 1 to k do
        B[i][i] ← 1
    for j ← 1 to k do
        for d ← 1 to n − k do
            i ← j + d
            B[i][j] ← B[i − 1][j] + B[i − 1][j − 1]
    return B[n][k]
```

procedure $\textsc{Binom}(n, k)$ ▷ Required: $n \geq k \geq 0$

  for $i \leftarrow 0$ **to** $n - k$ **do**

    $B[i][0] \leftarrow 1$

  for $i \leftarrow 1$ **to** k **do**

    $B[i][i] \leftarrow 1$

  for $j \leftarrow 1$ **to** k **do**

    for $d \leftarrow 1$ **to** $n - k$ **do**

      $i \leftarrow j + d$

      $B[i][j] \leftarrow B[i-1][j] + B[i-1][j-1]$

  return $B[n][k]$

$$(n - k + 1) + (k) + k(n - k) = nk - k^2 + n + 1$$

Thank You!