2-8 Probabilistic Analysis

"No Expectation, No Disappointment."

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Expectation

Definition (Expectation)

$$\mathbb{E}[X] = \sum_{x} x \Pr(X = x)$$

Expectation

Theorem (Computing Expectation)

Let X be a discrete random variable that takes on only nonnegative integer values.

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} i \Pr(X = i) = \sum_{i=1}^{\infty} \Pr(X \leq i)$$

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Let X be a discrete random variable that takes on only nonnegative integer values.

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Proof.

$\square$

Searching an Unsorted Array (CLRS Problem $5-2\ (f)$)

```
procedure DETERMINISTIC-SEARCH(A[1···n], x)
    i ← 1
    while i ≤ n do
        if A[i] = x then
            return true
        i ← i + 1
    return false
```

Searching an Unsorted Array (CLRS Problem $5-2$ (f))

procedure $\textsc{Deterministic-Search}(A[1\cdots n], x)$
  $i \leftarrow 1$
  while $i \le n$ **do**
    if $A[i] = x$ **then**
      return *true*
    $i \leftarrow i + 1$
  return *false*

(e)

$$\exists!\ i : A[i] = x$$

(f)

$$\exists!_k\ i : A[i] = x$$

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\mathbb{E}[Y] &= \sum_{i=1}^{n} i \Pr\{Y = i\} \\
&= \sum_{i=1}^{n} i \Pr\{A[i] = x\} \\
&= \frac{1}{n} \sum_{i=1}^{n} i
\end{aligned}$$

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&= \sum_{i=1}^{n} i \Pr\{A[i] = x\} \\
&= \frac{1}{n} \sum_{i=1}^{n} i \\
&= \frac{n+1}{2}
\end{aligned}$$

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\mathbb{E}[Y] &= \sum_{i=1}^{n-k+1} i \Pr\{Y = i\} \\
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&= \sum_{i=1}^{n-k+1} i \frac{\binom{n-i}{k-1}}{\binom{n}{k}} = \frac{1}{\binom{n}{k}} \sum_{i=1}^{n-k+1} i \binom{n-i}{k-1}
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\end{aligned}$$

$$k = 1 \implies \mathbb{E}[Y] = \frac{n+1}{2}, \qquad k = n \implies \mathbb{E}[Y] = 1$$

$$\sum_{i=1}^{n-k+1} i \binom{n-i}{k-1} = \binom{n+1}{k+1}$$

Summation by parts (Abel transformation; wiki)

After-class Exercise:

$$\sum_{i=1}^{n-k+1} i\binom{n-i}{k-1} = \binom{n+1}{k+1}$$

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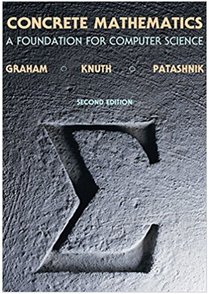

Chapter 5: Binomial Coefficients

After-class Exercise:

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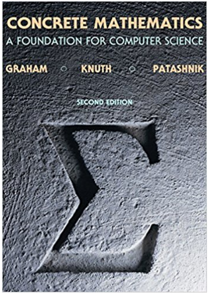

$$r\binom{r-1}{k-1} = k\binom{r}{k}$$

$$\sum_{0 \le k \le n} \binom{k}{m} = \binom{n+1}{m+1}$$

Chapter 5: Binomial Coefficients

Indicator Random Variables

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$$\Pr\{I_i = 1\} = \begin{cases} \frac{1}{k}, & \text{if } A[i] = x \\ \frac{1}{k+1}, & \text{if } A[i] \neq x \end{cases}$$

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$$\mathbb{E}[Y] = \sum_{i=1}^{n} \Pr\{I_i = 1\} = k \cdot \frac{1}{k} + (n-k) \cdot \frac{1}{k+1} = \frac{n+1}{k+1}$$

$$
\begin{aligned}
\mathbb{E}[Y] &= \sum_{i=1}^{n-k+1} i \Pr\{Y = i\} \\
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&= \frac{1}{\binom{n}{k}} \binom{n+1}{k+1} = \frac{n+1}{k+1}
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Wrong!

$$\Pr\{I_i = 1\} = \begin{cases} \frac{1}{k}, & \text{if } A[i] = x \\ \frac{1}{k+1}, & \text{if } A[i] \neq x \end{cases}$$

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$$i = 1 \implies \Pr\{I_1 = 1\} = 1$$

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NOT IID

(Independent and Identically Distributed)

Hat-check Problem (CLRS Problem $5.2 - 4$)

Hat-check Problem (CLRS Problem $5.2-4$)

X : # of customers who get back their own hat

$$I_i = \begin{cases} 1 & \text{customer } i \text{ gets back his/her hat} \\ 0 & \text{o.w.} \end{cases}$$

Inversions (CLRS Problem $5.2-5$)

Conditional Expectation

Definition (Conditional Expectation on an Event)

$$\mathbb{E}[X \mid E] = \sum_x x \Pr(X = x \mid E)$$

Theorem ( )

Let X be a random variable defined on a sample space Ω.
Let $E_1, E_2, \cdots, E_n$ be a partition of Ω.

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X \mid E_i] \Pr(E_i)$$

Theorem (The Law of Total Expectation)

Let X be a random variable defined on a sample space Ω.
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Proof.

□

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Definition (Conditional Expectation on a Random Variable)

$$\mathbb{E}[X \mid Y = y] = \sum_x x \Pr(X = x \mid Y = y)$$

Notation:

$$\mathbb{E}[X \mid Y](y) = \mathbb{E}[X \mid Y = y]$$

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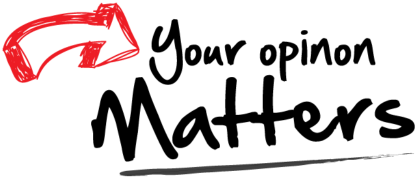
$$\mathbb{E}[\mathbb{E}[X \mid Y]] = \sum_{y} \mathbb{E}[X \mid Y = y] \Pr(Y = y)$$


Thank
You!



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