Functions

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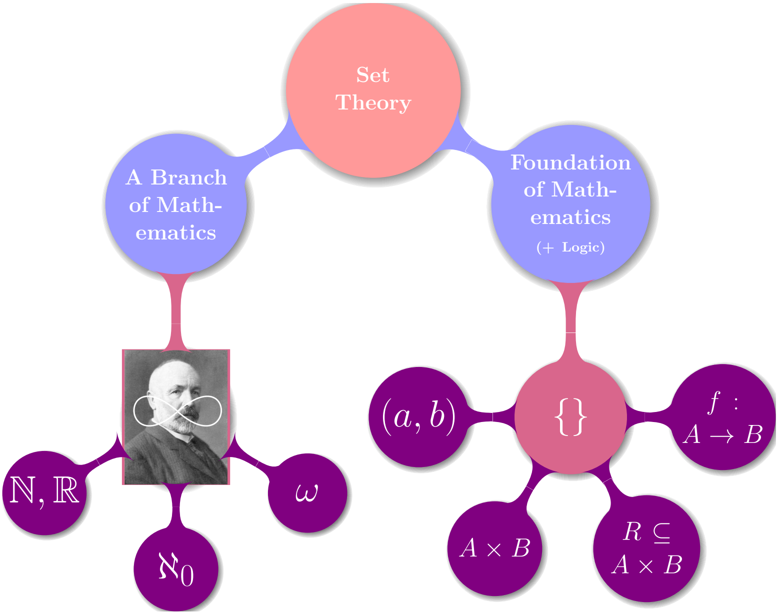
Set Theory
A Branch of Mathematics
Foundation of Mathematics
(+ Logic)
$\mathbb{N}, \mathbb{R}$
$\aleph_0$
ω
(a, b)
$\{\}$
$f : A \to B$
$A \times B$
$R \subseteq A \times B$

Definition of Function

Definition (Function)

Let A and B be sets.
A **function** f from A to B is a *relation* f from A to B such that

$$\forall a \in A \; \exists! b \in B \; (a, b) \in f.$$

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$$f : A \to B, \quad a \mapsto f(a) \qquad \Big(b = f(a)\Big)$$

$$A : \mathrm{dom}(f) \qquad B : \mathrm{cod}(f)$$

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$$\mathrm{ran}(f) = f(A) = \{f(a) \mid a \in A\} \subseteq B$$

A function $f : A \to B$ is a set.

$$f \subseteq A \times B$$

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

$$(a, b) = \{\{a\}, \{a, b\}\}$$

Definition (Axiom of Extensionality (集合的外延公理))

$$\forall A \forall B \forall x (x \in A \iff x \in B) \iff A = B.$$

Intensionality (内涵) *vs.* Extensionality (外延)

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Intensionality (内涵) *vs.* Extensionality (外延)

Definition (函数的外延性原则)

$$f = g \iff \text{dom}(f) = \text{dom}(g) \land (\forall x \in \text{dom}(f) : f(x) = g(x))$$

Properties of Functions

Definition (Injective (one-to-one; 1-1) 单射函数)

$$f : A \to B$$

$$\forall a_1, a_2 \in A : a_1 \neq a_2 \implies f(a_1) \neq f(a_2)$$

Definition (Injective (one-to-one; 1-1) 单射函数)

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- To show that f *is not* 1-1:

$$\exists a_1, a_2 \in A : a_1 \neq a_2 \land f(a_1) = f(a_2)$$

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$$f : A \to B$$

$$\mathrm{ran}(f) = B$$

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- To prove that f *is* onto:

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- To show that f *is not* onto:

$$\exists b \in B \left(\forall a \in A : f(a) \neq b \right)$$

Theorem (Cantor Theorem (ES Theorem 24.4))

Let A be a set.
If $f : A \to 2^A$, then f is not onto.

Proof.

Proof. Let A be a set and let $f : A \to 2^A$. To show that f is not onto, we must find a $B \in 2^A$ (i.e., $B \subseteq A$) for which there is no $a \in A$ with $f(a) = B$. In other words, B is a set that f "misses." To this end, let

$$B = \{x \in A : x \notin f(x)\}.$$

We claim there is no $a \in A$ with $f(a) = B$.

Suppose, for the sake of contradiction, there is an $a \in A$ such that $f(a) = B$. We ponder: Is $a \in B$?

- If $a \in B$, then, since $B = f(a)$, we have $a \in f(a)$. So, by definition of B, $a \notin f(a)$; that is, $a \notin B$. $\Rightarrow\Leftarrow$
- If $a \notin B = f(a)$, then, by definition of B, $a \in B$. $\Rightarrow\Leftarrow$

Both $a \in B$ and $a \notin B$ lead to contradictions, and hence our supposition [there is an $a \in A$ with $f(a) = B$] is false, and therefore f is not onto. ■

□

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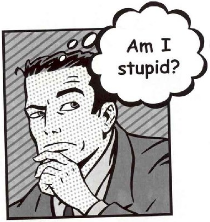

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$$\forall B \in 2^A \; \Big(\exists a \in A \; f(a) = B\Big).$$

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$$\exists B \in 2^A \; \Big(\forall a \in A \; f(a) \neq B\Big).$$

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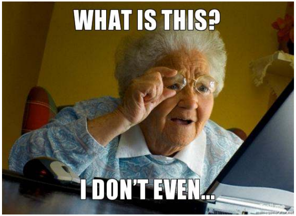

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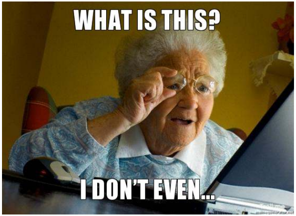

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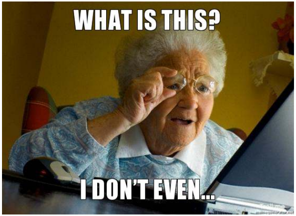

$$Q : a \in B?$$

$\square$

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a	$f(a)$					
	1	2	3	4	5	$\cdots$
1	1	1	0	0	1	$\cdots$
2	0	0	0	0	0	$\cdots$
3	1	0	0	1	0	$\cdots$
4	1	1	1	1	1	$\cdots$
5	0	1	0	1	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$

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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$

$$B = \{0, 1, 1, 0, 1\}$$

□

Theorem (Cantor Theorem)

Let A be a set.
If $f : A \to 2^A$, then f is not onto.

对角线论证 (Cantor's diagonal argument) (以下仅适用于可数集合 A).

a	$f(a)$					
	1	2	3	4	5	$\cdots$
1	1	1	0	0	1	$\cdots$
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$$B = \{0, 1, 1, 0, 1\}$$

$\square$

Definition (Bijective (one-to-one correspondence) 一一对应)

$$f : A \to B \qquad f : A \quad B$$

1-1 & onto

proof examples

Operations on Functions

Definition (Intersection, Union)

$$f_1, f_2 : A \to B$$

(i) Q : Is $f_1 \cup f_2$ a function from A to B?

(ii) Q : Is $f_1 \cap f_2$ a function from A to B?

Definition (Composition)

$$f : A \to B \qquad g : C \to D$$

$$\mathrm{ran}(f) \subseteq C$$

The composition function

$$g \circ f : A \to D$$

$$(g \circ f)(x) = g(f(x))$$

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$$(g \circ f)(x) = g(f(x))$$

Non-commutative:

$$f \circ g \neq g \circ f$$

Theorem (Associative Property for Composition)

$$f : A \to B \quad g : B \to C \quad h : C \to D$$

$$h \circ (g \circ f) = (h \circ g) \circ f$$

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$$f : A \to B \quad g : B \to C \quad h : C \to D$$

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Proof.

(i)

$$\text{dom}(h \circ (g \circ f)) = \text{dom}((h \circ g) \circ f)$$

(ii)

$$(h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x)$$

□

Theorem (Properties of Composition (UD Theorem 15.7))

$$f : A \to B \qquad g : B \to C$$

(i) *If f, g are injective, then $g \circ f$ is injective.*

(ii) *If f, g are surjective, then $g \circ f$ is surjective.*

(iii) *If f, g are bijective, then $g \circ f$ is bijective.*

Theorem (Properties of Composition (UD Theorem 15.7))

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(iii) *If f, g are bijective, then $g \circ f$ is bijective.*

Proof for (i).

$$\forall a_1, a_2 \in A \left((g \circ f)(a_1) = (g \circ f)(a_2) \implies a_1 = a_2 \right)$$

□

Theorem (Properties of Composition (UD Theorem 15.8))

$$f : A \to B \qquad g : B \to C$$

(i) *If $g \circ f$ is injective, then f is injective.*

(ii) *If $g \circ f$ is surjective, then g is surjective.*

Cancellation Property for Composition (Problem 15.11)

Definition (Inverse)

$$f : A \to B$$

$$f : X \to Y \quad A \subseteq X \quad B \subseteq Y$$

Definition (Image)

The image of A under f is the set

$$f(A) = \{f(a) \mid a \in A\}.$$

Definition (Inverse Image)

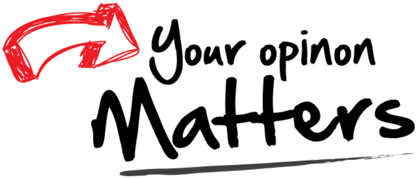
The inverse image of B under f is the set

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

$$Q_1 : A \text{ vs. } f^{-1}(f(A))$$

$$Q_2 : B \text{ vs. } f(f^{-1}(B))$$


Thank
You!



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