1-5 数据与数据结构 (I)

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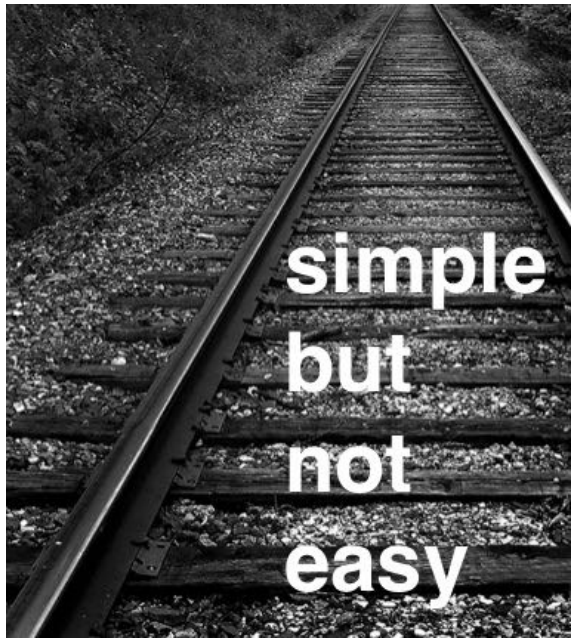
Permutations

Generating All Permutations
Stackable/Queueable Permutations

Generating All Permutations



DH 2.9: # of Permutations

Prove that the number of permutations of n (distinct) elements is $n!$.

"坊间" 证明.

For a_1: We have n choices.

For a_2: We have $n-1$ choices.

For $\cdots$: $\cdots$

Then, # of perms is

$$n \times (n-1) \times \cdots \times 1 = n!$$

□

Prove by mathematical induction on n.

DH 2.9: # of Permutations

Prove that the number of permutations of n (distinct) elements is $n!$.

Prove by mathematical induction on n.

$$P(n): \#\text{ of perms of } n \text{ (distinct) element is } n!$$

B.S. $P(1)$

I.H. $P(n)$

I.S. $P(n) \to P(n+1)$

$$\underbrace{(n+1)}_{\text{1st choice}} \times \underbrace{n!}_{I.H.} = (n+1)!$$

DH 2.11: Generate All Permutations

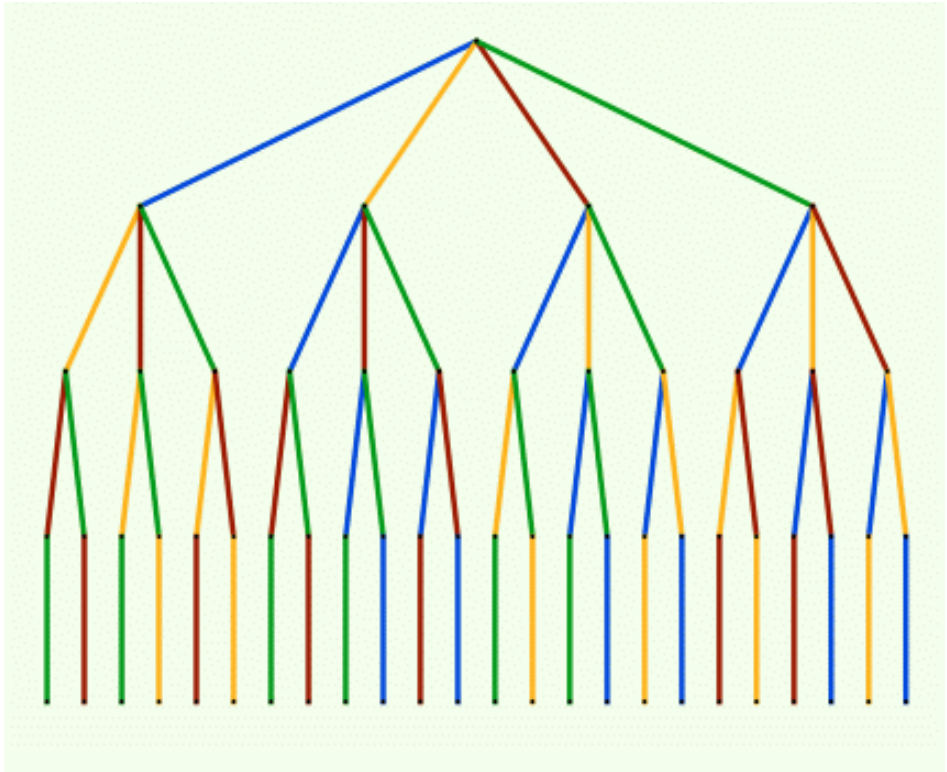
Design an algorithm which, given a positive integer n, generates/prints all the permutations of $[0 \cdots n)$.

```
void perms (A[], n) {
  if (n == 1)
    print ''A[0]''
  else
    for (int i = 0; i < n; ++i)
      print ''A[i]''
      perms(A ← A \ A[i], n - 1)
      print ''\n''
}
```

generate-perms.c

ERROR

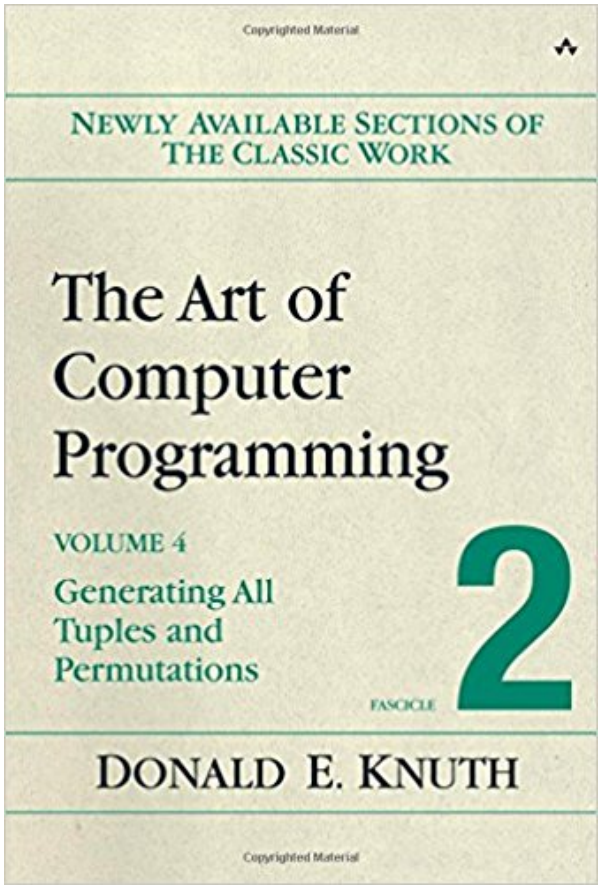
$$A = [0, 1, 2, 3] \qquad n = 4$$



```
void perms (prifix, A[], n) {
  if (n == 1)
    print ''prifix ++ A[0]''
  else
    for (int i = 0; i < n; ++i)
      perms(prefix ← prefix ++ A[i],
          A ← A \ A[i], n - 1) // Space???
      print ''\n''
}
```

```
perms('''', A, n);
```

For more about "Generating All Permutations":



DH 2.10: Permutation Checking

- An integer n
- An array of integers P of length n

To check whether P is a permutation of $1 \cdots n$?

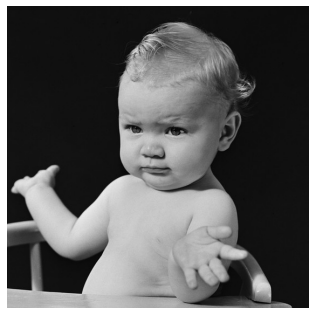
1. Boolean array $[1 \cdots n]$

$\underbrace{O(n)}_{\texttt{time}}$ $\underbrace{O(n)}_{\text{space}}$

2. Sort and then scan

$\underbrace{O(n \log n)}_{\texttt{time}}$ $\underbrace{O(1)}_{\text{space}}$

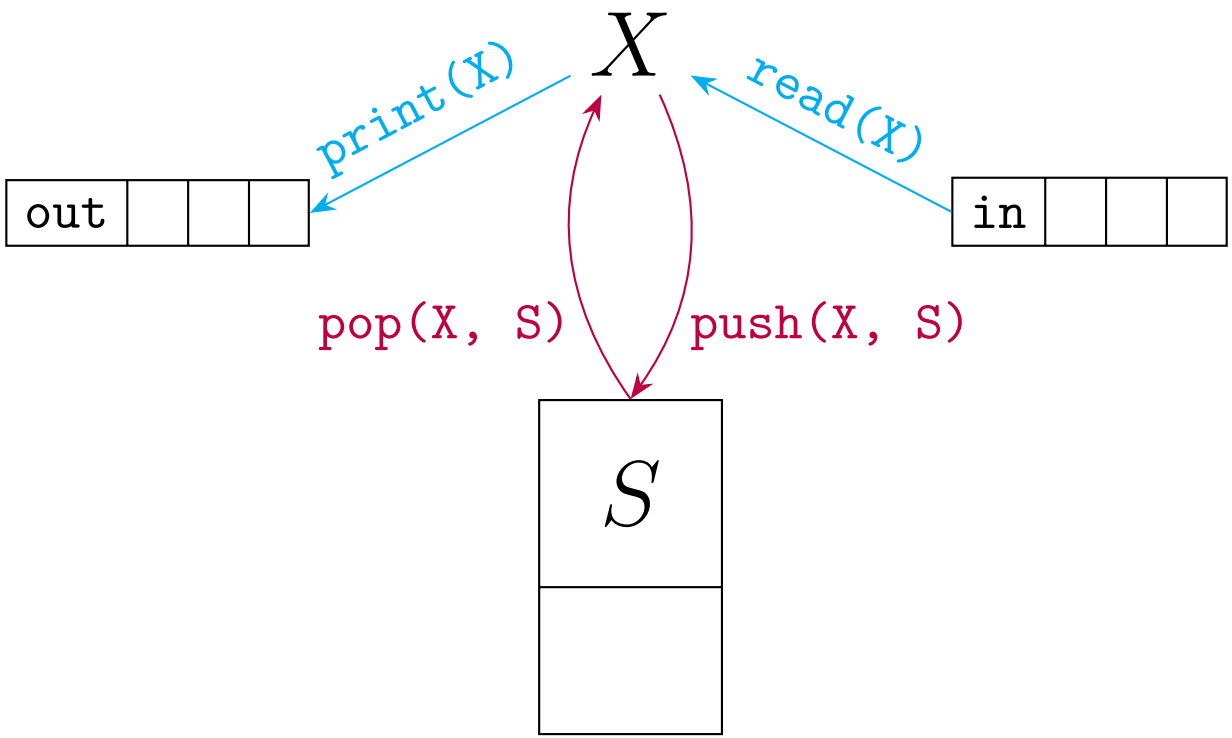
$\underbrace{O(n)}_{\texttt{time}}$ $\underbrace{O(1)}_{\text{space}}$



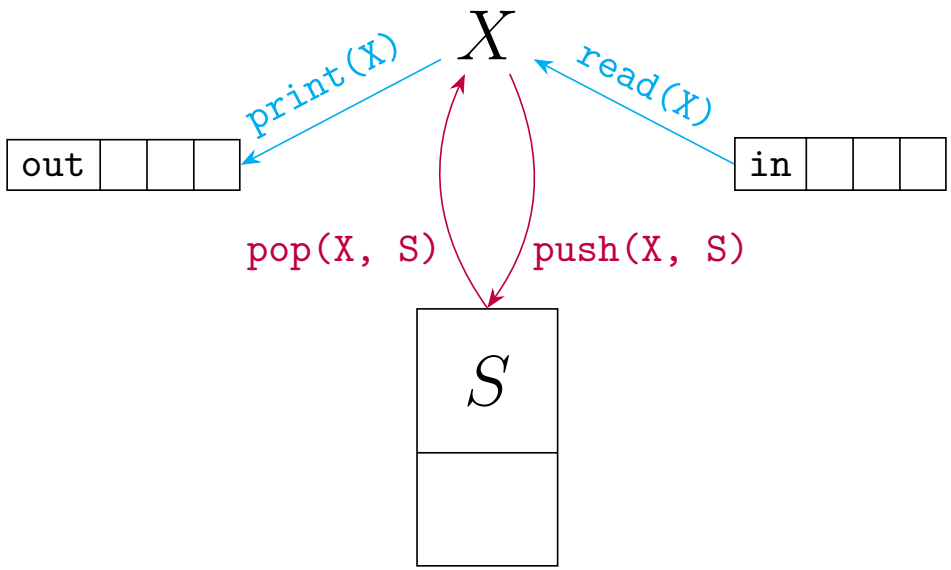
Stackable Permutations

Definition (Stackable Permutations)

$$\texttt{out} = (a_1, \cdots, a_n) \xleftarrow[X=0]{S=\emptyset} \texttt{in} = (1, \cdots, n)$$



Definition (Stackable Permutations)



Q_1 : Meaning of "`read, print, push, pop`"?

Q_2 : Using only "`read, print, push, pop`"?

$$a == X \qquad a > X \ (a < X) \qquad \texttt{top}(S)$$

DH 2.12: Stackable Permutations

(a) **Show** that the following permutations *are* stackable:

(i) $(3, 2, 1)$

(ii) $(3, 4, 2, 1)$

(iii) $(3, 5, 7, 6, 8, 4, 9, 2, 10, 1)$

DH 2.13: Stackable Permutations Checking Algorithm

To check whether a given permutation can be obtained by a stack.

read print push pop is-empty

```
X = 0    S = ∅    in != EOF
```

```
foreach 'a' in out:
  if (! is-empty(S)
       && 'a' == top(S))
    pop(S, X)
    print(X)
    continue
  else ··· // T.B.C
```

```
  else // T.B.C
    while (in != EOF)
      read(X)
      if (X == 'a')
        print(X)
        continue
      else
        push(X, S)
    ERR  // How???
```

DH 2.12: Stackable Permutations

(b) **Prove** that the following permutations are *not* stackable:

(i) $(3, 1, 2)$

(ii) $(4, 5, 3, 7, 2, 1, 6)$

$$(3, 1, 2)$$

$$(4, 5, 3, 7, 2, 1, 6)$$

$$\boxed{\texttt{out} = \cdots a_i \cdots a_j \cdots a_k : i < j < k \wedge a_j < a_k < a_i}$$

312-Pattern

Theorem (Stackable Permutations)

A permutation $(a_1, \cdots, a_n)$ *is stackable* $\iff$ *it is not the case that*

$$312\text{-}\textit{Pattern}: \boxed{\textit{out} = \cdots a_i \cdots a_j \cdots a_k : i < j < k \wedge a_j < a_k < a_i}$$

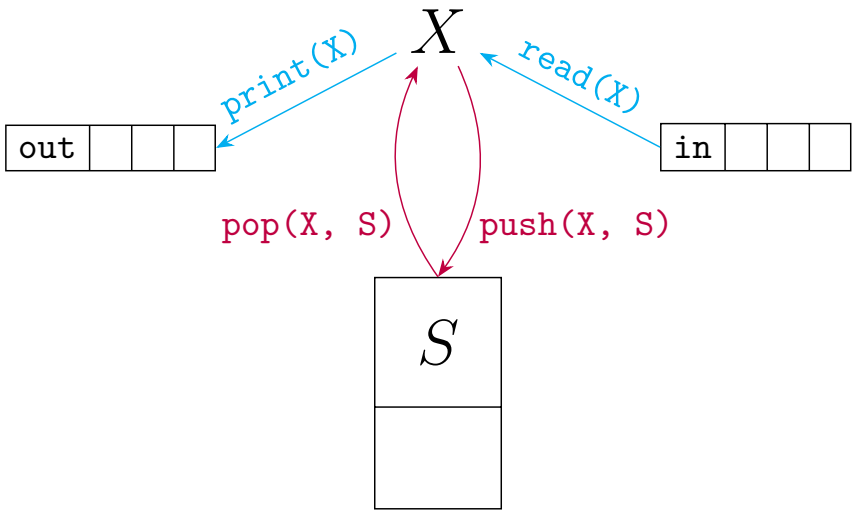
Proof.

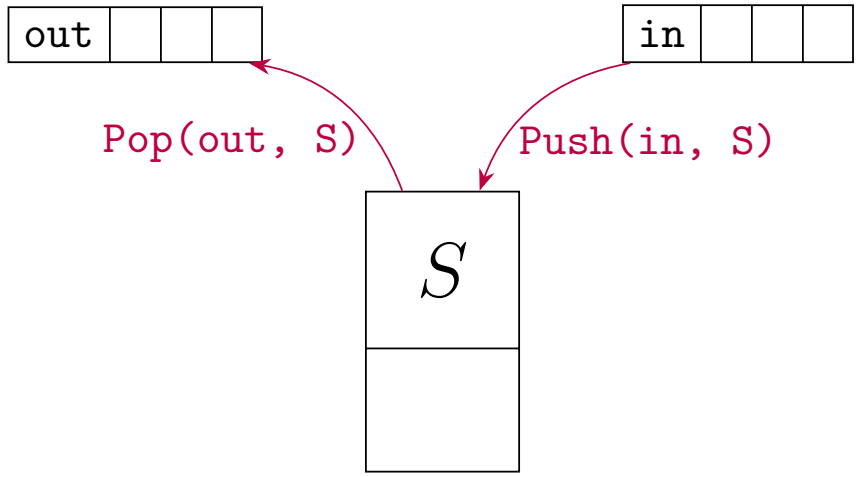
DH 2.12: Stackable Permutations

(c) How many permutations of A_4 *cannot* be obtained by a stack?

$$(1,4,2,3),(2,4,1,3),(3,1,2,4),(3,1,4,2),(3,4,1,2)$$
$$(4,1,2,3),(4,1,3,2),(4,2,1,3),(4,2,3,1),(4,3,1,2)$$

Q : What about A_n?

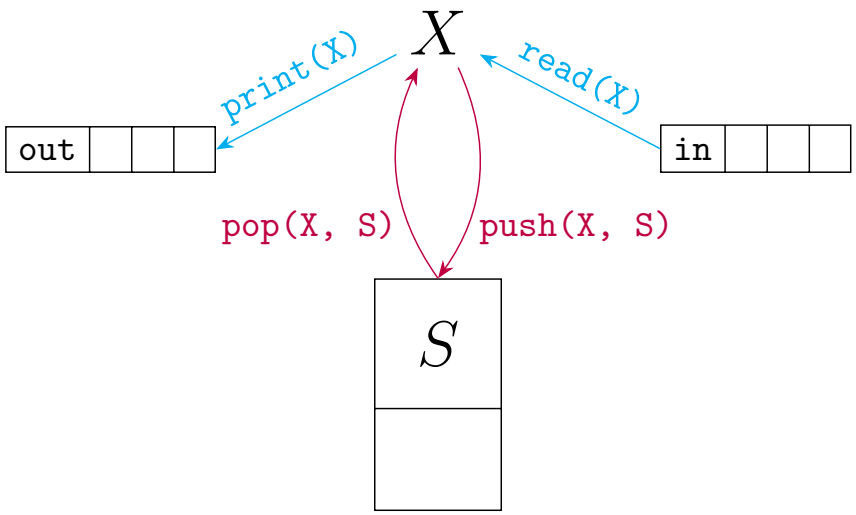


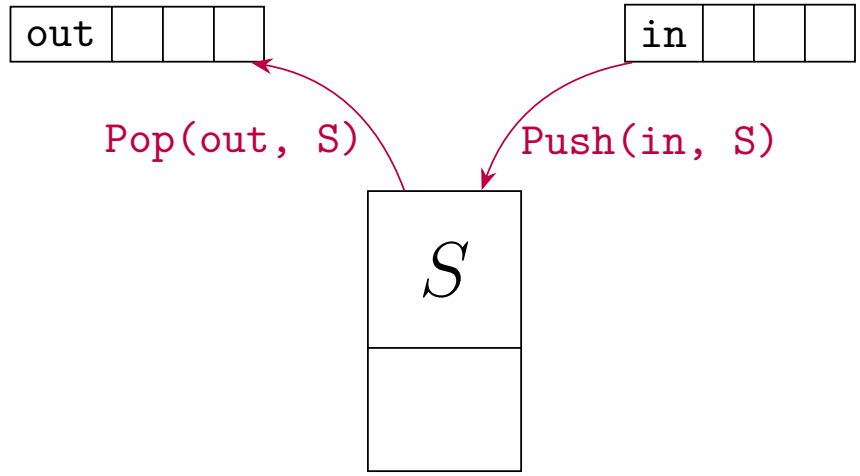


Q : Are $S + X$ and S are **equivalent**?

Producing the same set of permutations.

Accepting the same set of *admissible* operation sequences.





By simulations.

Simulate S by $S + X$:

- Push
- Pop

Simulate $S + X$ by S:

By iterative transformations.

DH 2.12: Stackable Permutations

How many permutations of A_n *cannot* be obtained by a stack?

$$\texttt{Push}: + \qquad \texttt{Pop}: -$$

$$(1,2,3): + - + - + -$$

$$(3,2,1): + + + - - -$$

$$(3,2,5,6,1,4): + + + - - + + - + - - -$$

Definition (Admissible Sequences)

A sequence of "+" and "−" is *admissible* if and only if

1. # of "+" $= n$ # of "−" $= n$
2. $\forall$ prefix : (# of "−") $\leq$ (# of "+")

Theorem

Different admissible sequences correspond to different permutations.

$$+ + + - - + \cdots$$

$$+ + + - - - \cdots$$

Theorem (Reflection Method)

The number of stackable permutations is $\binom{2n}{n} - \binom{2n}{n-1}$.

Proof.

$$\underbrace{\binom{2n}{n}}_{\text{all}} - \underbrace{\binom{2n}{n-1}}_{\text{inadmissible}}$$

$+\,+\,-\,+\,-\,-\,-\,-\,-\,+\,+\,+$

$+\,+\,-\,-\,-\,+$

$-\,-\,+\,-\,+\,+\,+\,-\,-\,+\,+\,+$

$-\,-\,+\,+\,+\,+$

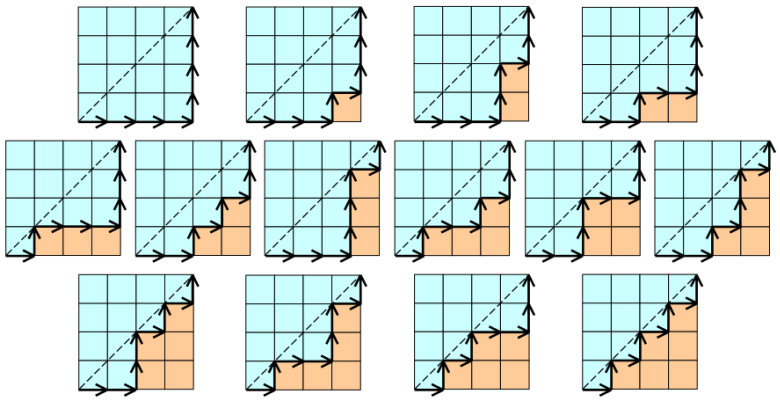
$(\#\text{ of } \text{“}+\text{”}) = (n+1) \qquad (\#\text{ of } \text{“}-\text{”}) = (n-1)$

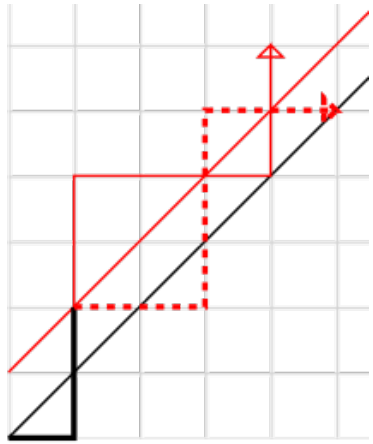
Catalan Number

Parenthesis

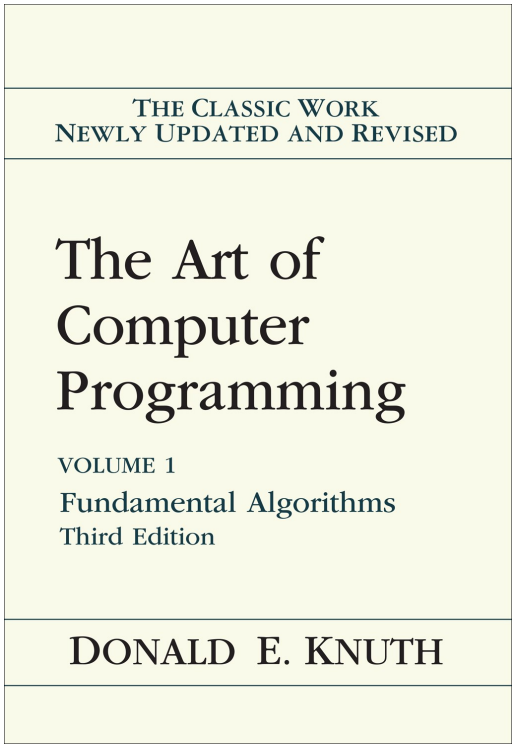
$$(3,2,1):((()))\qquad(1,2,3):()()()$$

Grid Paths Not above the diagonal:





For more about "Stackable Permutations":



Thank
You!