3-9 Connectivity

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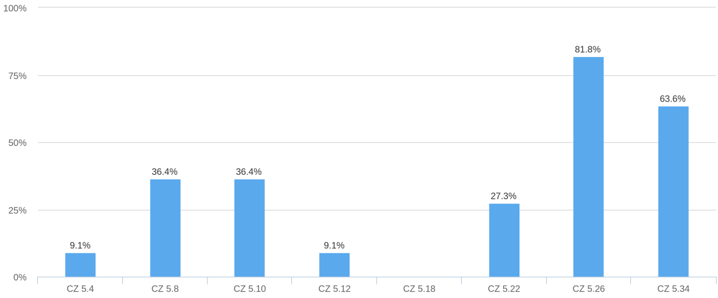


5.10 5.34 5.22 5.26

如果两个割点相连，那么联通块怎么划分
联通快呢）

menger定理吧

暂无

好像没有......

Menger定理的证明看不懂

menger定理的证明

不。。不记得了

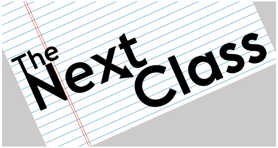
还好理解，只是都不怎么容易理解

menger定理的证明没太理解 老师辛苦了！

点割集，边割集

Menger's Theorem

(Theorem 5.16; Theorem 5.21)



2-Connectivity (Problem 5.10)

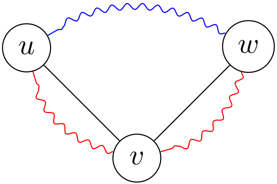
A connected graph G with $m \geq 2$ is *nonseparable*

$$\iff$$

any two *adjacent* edges of G lie on a common cycle of G.

Proof.

"$\implies$"



G is nonseparable

$\implies$ u, w lie on a common cycle

$\implies$ $\exists$ path $u \sim w$ that does not contain v

$\implies$ $\exists$ cycle $u - v - w \sim u$

$\square$

2-Connectivity (Problem 5.10)

A connected graph G with $m \geq 2$ is *nonseparable*

$$\iff$$

any two *adjacent* edges of G lie on a common cycle of G.

Proof.

"$\Longleftarrow$"

By Contradiction.

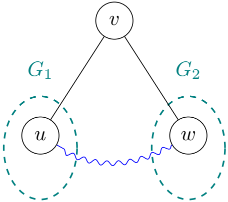
Suppose v is a cut-vertex of G

$\implies$ $G - v$ contains ≥ 2 comps $G_1, G_2, \cdots$

$\implies$ $\exists u \in G_1, w \in G_2 : v - u \wedge v - w$

$\implies$ $v - u, v - w$ lie on a common cycle

$\implies$ $\exists$ path $u \sim w$ that does not contain v



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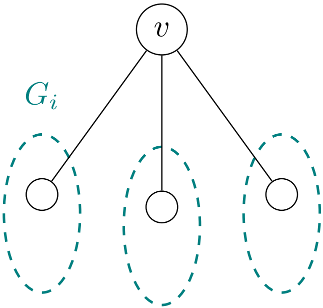
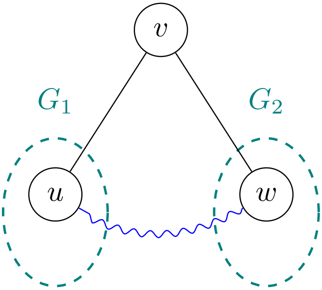


$\forall G_i \; \exists v_i \in G_i \; v - v_i$



$\forall v \in S \; \forall G_i \; \exists v_i \in G_i \; v - v_i$

2-Connectivity (Problem 5.10)

A connected graph G with $m \geq 2$ is *nonseparable*

$$\Longleftrightarrow$$

any two *adjacent* edges of G lie on a common cycle of G.

2-Connectivity (Extended Problem)

A connected graph G with $m \geq 2$ is *nonseparable*

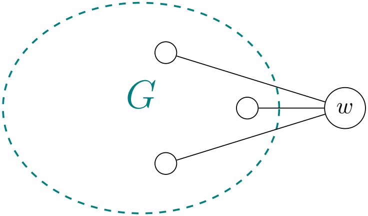
$$\Longleftrightarrow$$

any two edges of G lie on a common cycle of G.

Expansion Lemma (Problem 5.34; Theorem 5.18)

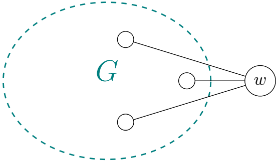
Let G be a k-connected graph and let S be any set of k vertices.

If a graph H is obtained from G by adding a new vertex w and joining w to the vertices of S, then H is also k-connected.



We need to prove that

$$\forall v \in V(G): \text{ there exist } k \text{ internally disjoint } v - w \text{ paths}$$

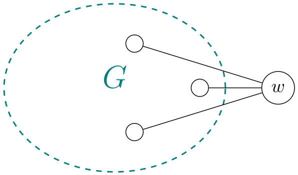


This holds because

$$\forall v \in V(G): \text{ there exist internally disjoint } v - s_i \ (\forall s_i \in S) \text{ paths}$$

Corollary (5.19; Proved using Theorem 5.18)

If G is a k-connected graph and $u, v_1, v_2, \cdots, v_k$ are $k+1$ distinct vertices of G, then there exist internally disjoint $u - v_i$ paths $(1 \leq i \leq k)$ in G.



To prove $\kappa(H) \geq k$

Let U be a vertex-cut of H.

We prove that $|U| \geq k$.

Case I: U is a vertex-cut of G

$$|U| \geq k$$

Case II: U is not a vertex-cut of G

$$w \in U$$

$U - w$ is a vertex-cut of G

$$|U| \geq k + 1$$

2-Connectivity (Extended Problem)

A connected graph G with $m \geq 2$ is *nonseparable*

$$\iff$$

any two edges of G lie on a common cycle of G.

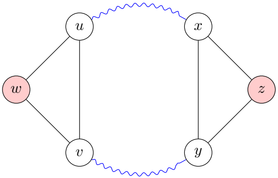
$$\Longrightarrow$$

Consider two edges uv and xy.

Add w, z

Add $wu, wv; zx, zy$

w and z lie on a common cycle



Effects of Removing an Edge on Connectivity (Problem 5.22 (a))

(a) If G is k-connected and $e = uv \in E(G)$, then $G - e$ is $(k-1)$-connected.

$$\text{To prove } \kappa(G) \geq k \implies \kappa(G - e) \geq k - 1$$

Choose any $U \subseteq V(G)$ with $|U| < k - 1$.

We prove that $G - e - U$ is connected.

Choose any $U \subseteq V(G)$ with $|U| < k - 1$.

We prove that $G - e - U$ is connected.

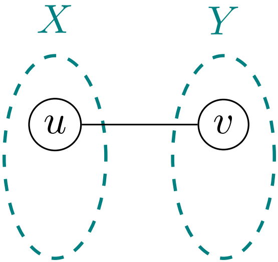
G is k-connected $\implies$ $G - U$ is connected

Suppose, by contradiction, that $G - e - U$ is not connected.

$e = uv$ is a bridge of $G - U$

$U \cup \{u\}$ is a vertex-cut of G

But $\Big| U \cup \{u\} \Big| < k$



CASE I : $|X| \geq 2 \vee |Y| \geq 2$

$U \cup \{u\}$ is a vertex-cut of G

But $\Big|U \cup \{u\}\Big| < k$

CASE II : $|X| = |Y| = 1$

$|U| = n - 2 < k - 1$

$\kappa(G) \geq k > n - 1$

But $0 \leq \kappa(G) \leq n - 1$

Effects of Removing an Edge on Connectivity (Problem 5.22 (b))

(b) If G is k-edge-connected and $e = uv \in E(G)$, then $G - e$ is $(k-1)$-edge-connected.

$$\lambda(G) \geq k \implies \lambda(G - e) \geq k - 1$$

Choose any $X \subseteq E(G)$ with $|X| < k - 1$.

We prove that $G - e - X$ is connected.

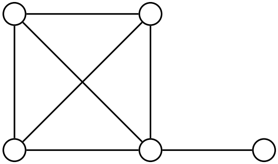
$G - e - X = G - (e + E)$ is connected $(\because \lambda(G) \geq k)$

$$\kappa(G - e) \le \kappa(G)$$

Effects of Removing a Vertex on Connectivity (Extended Problem)

$$\text{Is } \kappa(G - v) \le \kappa(G)?$$

$$\text{Is } \lambda(G - v) \le \lambda(G)?$$



Effects of Removing a Vertex on Connectivity (After-class Exercise)

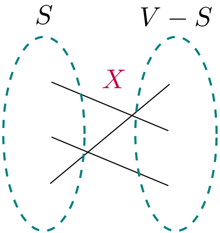
$$\text{Is } \kappa(G) \ge k \implies \kappa(G - v) \ge k - 1?$$

Degree Condition for $\lambda(G) = \delta(G)$ (Problem 5.26)

If G is graph of order n such that $\delta(G) \geq (n-1)/2$, then $\lambda(G) = \delta(G)$.

$$\lambda(G) \leq \delta(G)$$

We prove that $\lambda(G) \geq \delta(G)$.



$$\lambda(G) = |X|$$

$$1 \leq |S| = k \leq n/2, \quad |V - S| = n - k$$

$$\lambda \geq k\Big(\delta - (k-1)\Big) \geq \delta$$

Decision	Author(s)	Year	Complexity	Comments
Edge Connectivity				
$\lambda = 2$ or $\lambda = 3$	Tarjan [26]	1972	$O(m + n)$	uses Depth First Search
λ	Even and Tarjan [6]	1975	$O(mn \times \min\{m^{1/2}, n^{2/3}\})$	n calls to max-flow
λ (digraphs)	Schnorr [25]	1979	$O(\lambda mn)$	n calls to max-flow
λ	Esfahanian & Hakimi [3]	1984	$O(\lambda mn)$	$\leq n/2$ calls to max-flow
λ (digraphs)	Esfahanian & Hakimi [3]	1984	$O(\lambda mn)$	$\leq n/2$ calls to max-flow
λ	Matula [23]	1987	$O(mn)$	uses dominating sets
$\lambda = k$	Matula [23]	1987	$O(kn^2)$	
λ (digraphs)	Mansour & Schieber [22]	1989	$O(mn)$	
$\lambda = k$	Gabow [9]	1991	$O(m+k^2 n\log(n/k))$	uses matroids
Vertex Connectivity				
$\kappa = 2$	Tarjan [26]	1972	$O(m + n)$	uses Depth First Search
$\kappa = 3$	Hopcroft & Tarjan [18]	1973	$O(m + n)$	uses triconnected components
κ	Even & Trajan [6]	1975	$O((\kappa(n - \delta - 1)mn^{2/3})$	max-flow based
$\kappa = k$	Even [4]	1975	$O(kn^3)$	max-flow based
κ	Galil [12]	1980	$O(\min\{\kappa, n^{2/3}\}mn)$	max-flow based
$\kappa = k$	Galil [12]	1980	$O(\min\{k, n^{1/2}\}kmn)$	max-flow based
κ	Esfahanian & Hakimi [3]	1984	$O((n - \delta - 1 + \delta(\delta - 1)/2)mn^{2/3})$	max-flow based
$\kappa = 4$	Kanevsky & Ramachandran [20]	1991	$O(n^2)$	
κ	Henzinger & Rao [17]	1996	$O(\kappa mn\log n)$	randomised algorithm

Table 1: A chronology of connectivity algorithms

Theorem (Menger's Theorem (Theorem 5.16))

Let u and v be nonadjacent vertices in a graph G.

The minimum number of vertices in a $u - v$ separating set equals the maximum number of internally disjoint $u - v$ paths in G.

How do CASE 1, CASE 2, and CASE 3 cover all possibilities?

Are CASE 1 and CASE 2 mutually exclusive?

What is the key to use the induction hypothesis in CASE 2?

Are CASE 1 and CASE 3 mutually exclusive?

What will fail if we do not exclude CASE 1 from CASE 3?

Can you restate these three cases in terms of $N(u)$ and $N(v)$?

Can you rearrange these three cases to make them (hopefully) easier to understand?

Case I: There exists a minimum $u - v$ separating set W in G containing a vertex x that is adjacent to both u and v.

$$\exists W : \exists x \in W : x - u \wedge x - v$$

Case II: There exists a minimum $u - v$ separating set W in G containing a vertex in W that is not adjacent to u and a vertex in W that is not adjacent to v.

$$\begin{aligned} \exists W : \exists x \in W : x \not- u \\ \wedge \exists y \in W : y \not- v \end{aligned}$$

Case III: For each minimum $u - v$ separating set W in G, either every vertex of W is adjacent to u and not adjacent to v or every vertex of W is adjacent to v and not adjacent to u.

$$\begin{aligned} \forall W : \forall x \in W : x - u \wedge x \not- v \\ \vee \forall x \in W : x - v \wedge x \not- u \end{aligned}$$

$$\mathrm{I} : \exists W : \exists x \in W : x - u \land x - v \qquad \mathrm{I}' : \forall W : \forall x \in W : x \not- u \lor x \not- v$$

$$\begin{aligned} \mathrm{II} : \exists W : \exists x \in W &: x \not- u \\ \land \exists y \in W &: y \not- v \end{aligned}$$

$$\begin{aligned} \mathrm{II}' : \forall W : \forall x \in W &: x - u \\ \lor \forall y \in W &: y - v \end{aligned}$$

$$\begin{aligned} \mathrm{III} : \forall W : \forall x \in W &: x - u \land x \not- v \\ \lor \forall x \in W &: x - v \land x \not- u \end{aligned}$$

$$\mathrm{III} \equiv \mathrm{II}' \land \mathrm{I}'$$

II

II′

I

III

II

$$
\begin{aligned}
\mathrm{II}: \exists W: \exists x \in W: x \not- u \\
\land\ \exists y \in W: y \not- v
\end{aligned}
$$

II$'$

$$
\mathrm{I}: \exists W: \exists x \in W: x - u \land x - v
$$

$$
\begin{aligned}
\mathrm{III}: \forall W: \forall x \in W: x - u \land x \not- v \\
\lor\ \forall x \in W: x - v \land x \not- u
\end{aligned}
$$

II

$$
\begin{aligned}
\exists W: W \nsubseteq N(u) \\
\land\ W \nsubseteq N(v)
\end{aligned}
$$

II$'$

$$
\exists W: \exists x \in W: x \in N(u) \cap N(v)
$$

$$
\begin{aligned}
\forall W: W \subseteq N(u) \land W \cap N(v) = \emptyset \\
\lor\ W \subseteq N(v) \land W \cap N(u) = \emptyset
\end{aligned}
$$

II

$$\text{II}: \exists W : W \not\subseteq N(u) \\ \land W \not\subseteq N(v)$$

Q : What is the key to use the induction hypothesis in CASE II?

II′

$$\text{I}: \exists W : \exists x \in W : x \in N(u) \cap N(v)$$

$$\text{III}: \forall W : W \subseteq N(u) \land W \cap N(v) = \emptyset \\ \lor W \subseteq N(v) \land W \cap N(u) = \emptyset$$

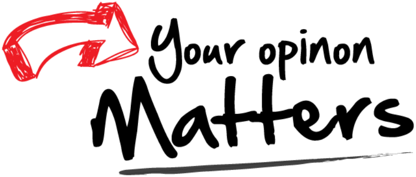
Q : What will fail if we do not exclude CASE I from CASE III?

Theorem (Menger's Theorem for Edge-Connectivity (Theorem 5.21))

For distinct vertices u and v in a graph G,

the minimum number of edges of G that separate u and v equals the maximum number of pairwise edge-disjoint $u-v$ paths in G.


THANK YOU



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