3-5 Minimum Spanning Trees

Hengfeng Wei

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October 22, 2018

Uniqueness of MST (Problem 4.30)

$$T \text{ is a unique MST of } G$$

$$\Longleftrightarrow$$

$$\forall e \in G \setminus T : w(e) > w(\text{every other edge on the cycle in } T + e)$$

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$$\text{Distinct weights} \implies \text{Unique MST}$$

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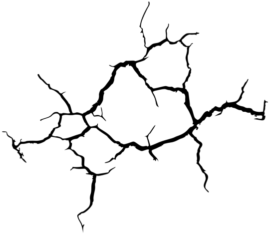
Cut Property

Cycle Property

Be Careful with Your Proofs!!!

Theorem (A "Real" Theorem)

Proof.



$\square$

Theorem (A "Real" Theorem)

Theorem (A "Faked" Theorem)

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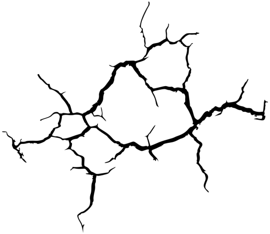

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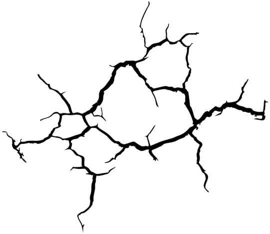

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Cut Property

Cut Property (Version I)

$$\begin{aligned} X &: \text{A part of some MST } T_1 \text{ of } G \\ (S, V \setminus S) &: \text{A } \textit{cut} \text{ such that } X \text{ does } \textit{not} \text{ cross } (S, V \setminus S) \\ e &: \text{A lightest edge across } (S, V \setminus S) \end{aligned}$$

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Correctness of Prim's and Kruskal's algorithms.

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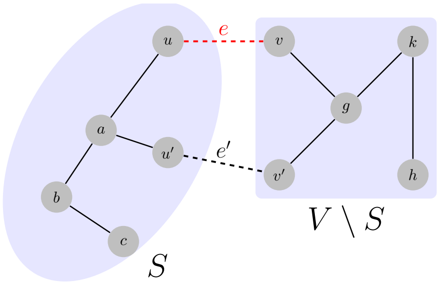

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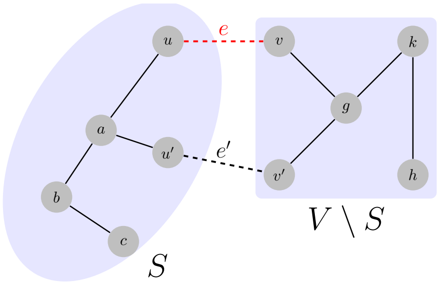

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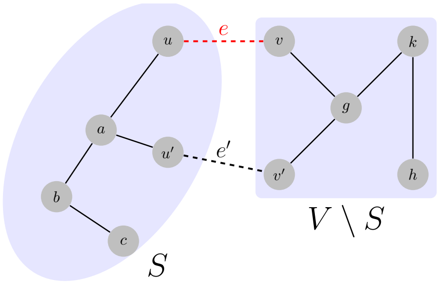

$$T' = \underbrace{\underbrace{T}_{\color{blue}X \subseteq T} + \{e\}}_{\color{red}\text{if } e \notin T} - \{e'\}$$

"a" $\rightarrow$ "the" $\color{red}\Longrightarrow$ "some" $\rightarrow$ "all"

Cut Property (Version II)

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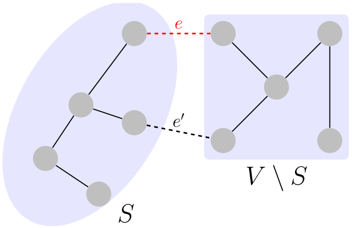
$\exists$ MST T of $G : e \in T$



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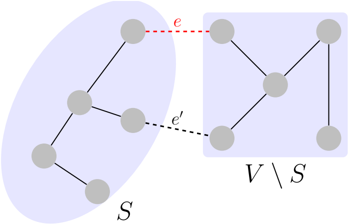

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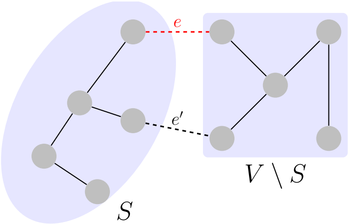

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A Wrong Divide&Conquer Algorithm for MST

$$(V_1, V_2) : \Big| |V_1| - |V_2| \Big| \leq 1$$

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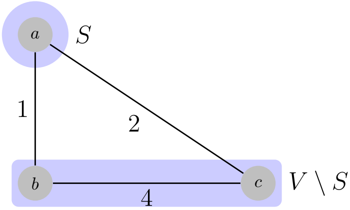

Cycle Property

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- Let C be any cycle in G
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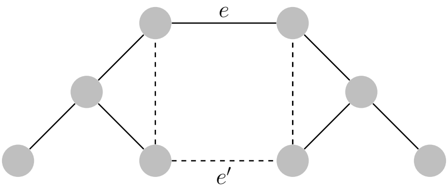

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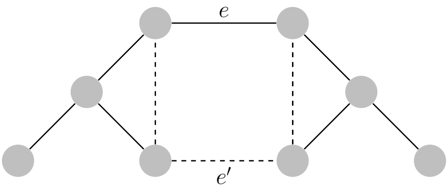

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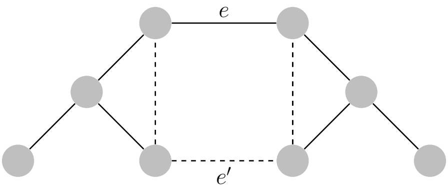

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Reverse-delete algorithm (wiki; clickable)

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"On the Shortest Spanning Subtree of a Graph and the Traveling Salesman Problem"

— Kruskal, 1956.

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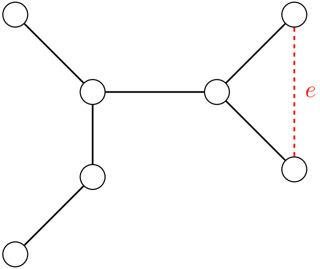

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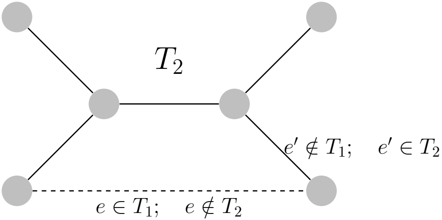
T_2
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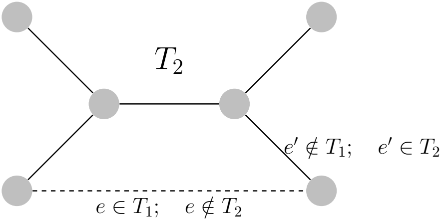

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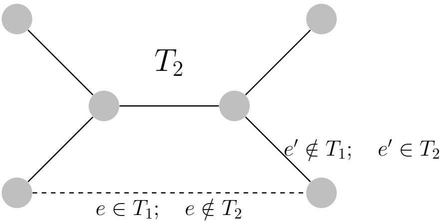

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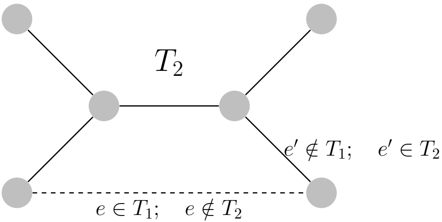

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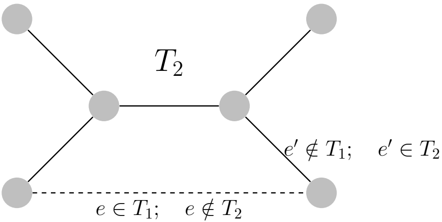

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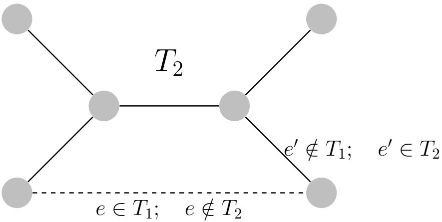

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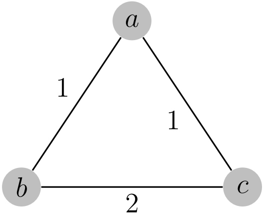

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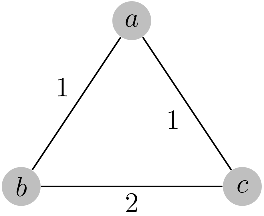

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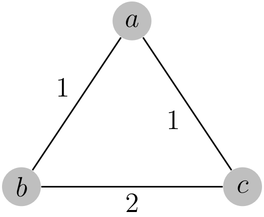

Theorem (After-class Exercise)

Minimum-weight edge across any cut is unique $\Longrightarrow$ *Unique MST*

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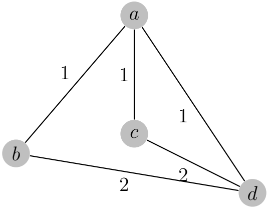

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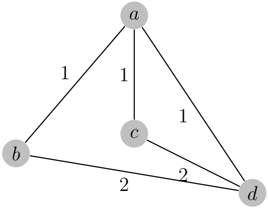

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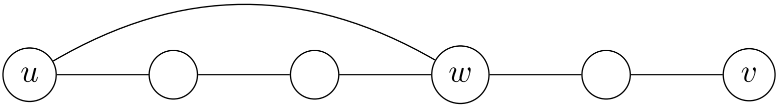

maximal path $P_{u,v}$

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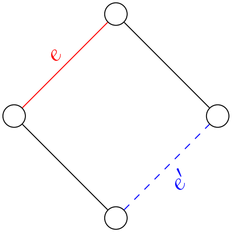

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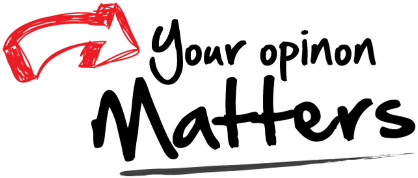
MST from the point of view of greedy-algorithm

MIT 6.046J: *"Design and Analysis of Algorithms"*, Spring 2015


THANK YOU



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