3-8 Cool? We are APSP Algorithms.

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Definition (Shortest Path)

$$G = (V, E, w): \text{ weighted digraph}$$

$$\delta(u,v) = \begin{cases} \min\left\{w(p) : u \overset{p}{\rightsquigarrow} v\right\} & \text{if } u \rightsquigarrow v \\ \infty & \text{o.w.} \end{cases}$$

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Path *vs.* Simple path

Robert W. Floyd (1936–1901)

For having a clear influence on methodologies for the creation of efficient and reliable software, and for helping to found the following important subfields of computer science:

the theory of parsing, the semantics of programming languages, automatic program verification, automatic program synthesis, and analysis of algorithms

— *Turing Award*, 1978

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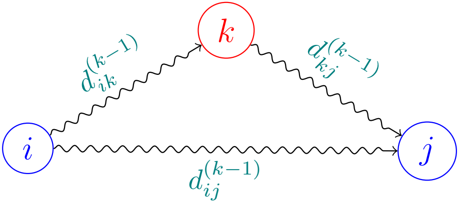
$$\boxed{D^{(n)} \triangleq \left(d_{ij}^{(n)}\right)}$$

$$k \in \mathrm{SP}_{ij}^{(k)}?$$



$$d_{ij}^{(k)} = \begin{cases} w_{ij} & k = 0 \\ \min\left\{d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right\} & k \geq 1 \end{cases}$$

procedure FLOYD-WARSHALL(W)

  $D^{(0)} = W$

  for $k \leftarrow 1$ **to** n **do**

    $D^{(k)} \triangleq \left(d_{ij}^{(k)}\right) \leftarrow$ a new $n \times n$ matrix

    for $i \leftarrow 1$ **to** n **do**

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$$\text{Space} : \Theta(n^3) \implies \Theta(n^2)$$

Floyd-Warshall Made Simple (Problem 25.2-4)

```
procedure FLOYD-WARSHALL-SIMPLIFIED(W)
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$$D^{(k-1)} \Longrightarrow D^{(k)} \qquad D \Longrightarrow D$$

"Decrease" does no harm.

Negative-weight Cycle Detection (Problem 25.2-6)

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Proof. This is trivial.

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Proof.

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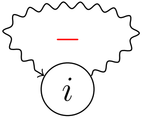
" $\impliedby$ "



A Cycle

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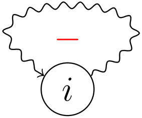

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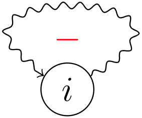

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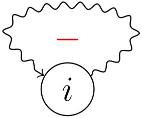

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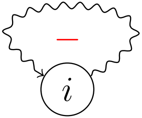
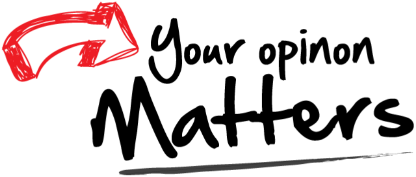


A Simple Cycle

$$d_{ii}^{(n)} < 0$$

□


THANK YOU



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