2-8 Probabilistic Analysis

"No Expectation, No Disappointment."

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Searching an Unsorted Array (CLRS Problem $5-2\ (f)$)

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procedure DETERMINISTIC-SEARCH(A[1 ··· n], x)
    i ← 1
    while i ≤ n do
        if A[i] = x then
            return true
        i ← i + 1
    return false
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(f)

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\end{aligned}$$

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\end{aligned}$$

$$k = 1 \implies \mathbb{E}[Y] = \frac{n+1}{2}, \qquad k = n \implies \mathbb{E}[Y] = 1$$

$$\sum_{i=1}^{n-k+1} i\binom{n-i}{k-1} = \binom{n+1}{k+1}$$

Summation by parts (Abel transformation; wiki)

After-class Exercise:

$$\sum_{i=1}^{n-k+1} i\binom{n-i}{k-1} = \binom{n+1}{k+1}$$

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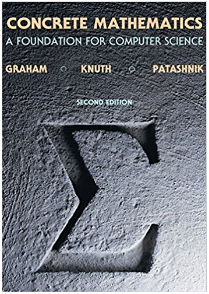

Chapter 5: Binomial Coefficients

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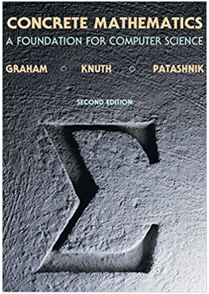

$$r\binom{r-1}{k-1} = k\binom{r}{k}$$

$$\sum_{0\le k\le n}\binom{k}{m} = \binom{n+1}{m+1}$$

Chapter 5: Binomial Coefficients

Indicator Random Variables

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$$\mathbb{E}[Y] = \sum_{i=1}^{n} \Pr\{I_i = 1\} = k \cdot \frac{1}{k} + (n-k) \cdot \frac{1}{k+1} = \frac{n+1}{k+1}$$


Wrong!

$$\Pr\{I_i = 1\} = \begin{cases} \frac{1}{k}, & \text{if } A[i] = x \\ \frac{1}{k+1}, & \text{if } A[i] \neq x \end{cases}$$

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NOT IID

(Independent and Identically Distributed)

$$
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TREASURE
HUNT
on foot


TREASURE
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Hints
&
Tips


Help
Wanted

There are n bins labelled with the numbers $1, 2, \cdots, n$. Balls are placed in these bins one after the other, with the bin into which a ball is placed being independent random variables that assume the value k with probability p_k. Let X be the number of balls placed so that there is at least one ball in every bin.

(a) Assume that $p_k = \frac{1}{n}$. What is the expectation of X?

(b) Assume that $p_k = \frac{1}{n}$. What is the probability distribution of X?

(c) Prove that $\Pr(X > n \ln n + cn) \leq e^{-c}$, $\Pr(X < n \ln n - cn) \leq e^{-c}$.

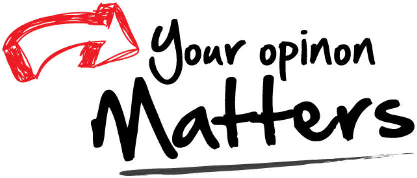
(d) Redo (a) and (b) without the assumption $p_k = \frac{1}{n}$.

(e) Given a deck of n cards, each time you take the top card from the deck, and insert it into the deck at one of the n distinct possible places, each of them with probability $\frac{1}{n}$. What is the expected times for you to perform the procedure above until the bottom card rises to the top?

The Coupon Collector's Problem

Shuffling Cards


Thank
You!



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