1-5 数据与数据结构 (I)

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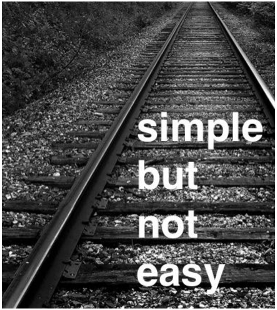
Permutations

Permutations

Generating All Permutations

Stackable/Queueable Permutations

Generating All Permutations



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$$\underbrace{(n+1)}_{\text{1st choice}} \times \underbrace{n!}_{I.H.} = (n+1)!$$

$\square$

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Design an algorithm which, given a positive integer n, generates/prints all the permutations of $[0 \cdots n)$.

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```
void perms (A[], n) {
  if (n == 1)
    print ''A[0]''
  else
    for (int i = 0; i < n; ++i)
      print ''A[i]''
      perms(A ← A \ A[i], n - 1)
      print ''\n''
}
```

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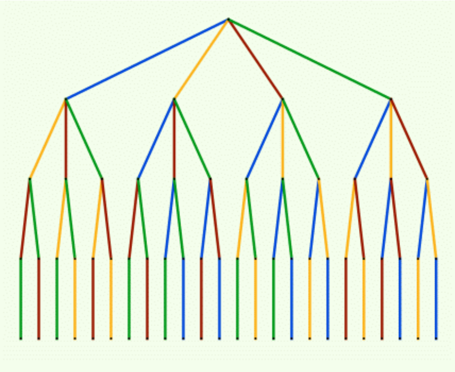
```

generate-perms.c


ERROR


ERROR

$$A = [0, 1, 2, 3] \qquad n = 4$$



```
void perms (prifix, A[], n) {
  if (n == 1)
    print ''prifix ++ A[0]''
  else
    for (int i = 0; i < n; ++i)
      perms(prefix ← prefix ++ A[i],
            A ← A \ A[i], n - 1)
      print ''\n''
}
```

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void perms (prefix, A[], n) {
  if (n == 1)
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    for (int i = 0; i < n; ++i)
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}
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```
perms('''', A, n);
```

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void perms (prefix, A[], n) {
  if (n == 1)
    print ''prefix ++ A[0]''
  else
    for (int i = 0; i < n; ++i)
      perms(prefix ← prefix ++ A[i],
            A ← A \ A[i], n - 1) // Space???
      print ''\n''
}
```

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perms('''', A, n);
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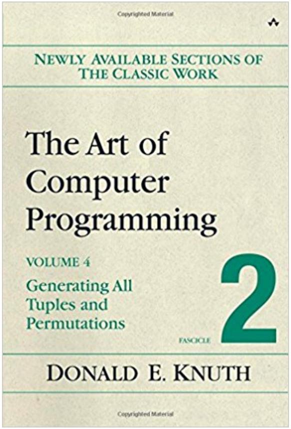
For more about "Generating All Permutations":



DH 2.10: Permutation Checking

- An integer n
- An array of integers P of length n

To check whether P is a permutation of $1 \cdots n$?

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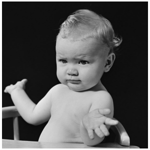

Stackable Permutations

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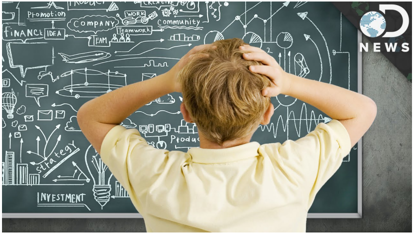
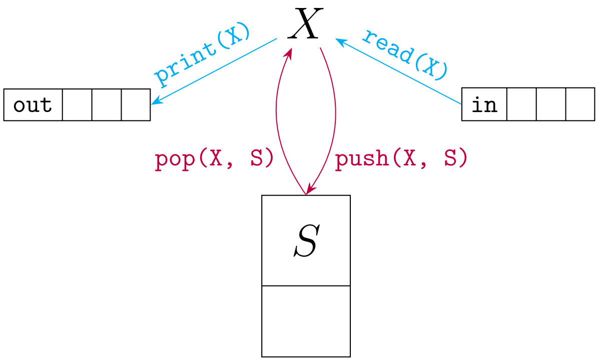
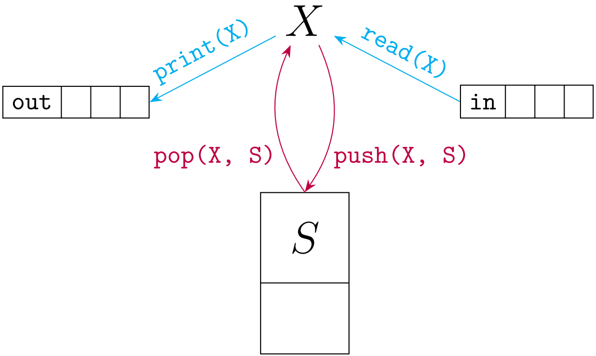


Definition (Stackable Permutations)



Definition (Stackable Permutations)

$$\boxed{\texttt{out} = (a_1, \cdots, a_n) \xleftarrow[X=0]{S=\emptyset} \texttt{in} = (1, \cdots, n)}$$



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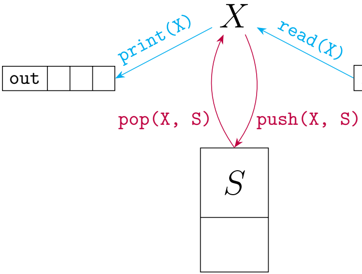

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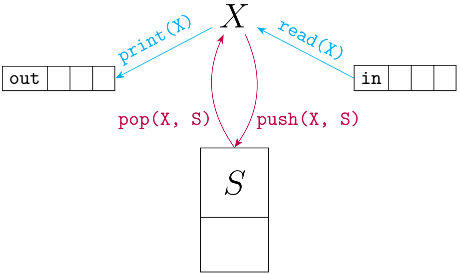

Q_1 : Meaning of "`read, print, push, pop`"?

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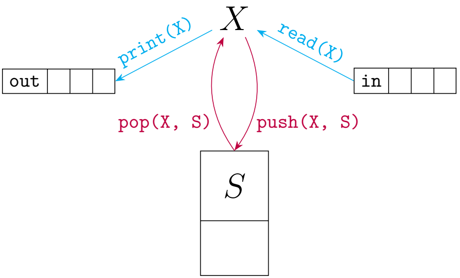
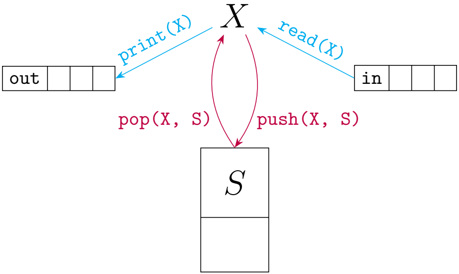


Q_1 : Meaning of "`read`, `print`, `push`, `pop`"?

Q_2 : Using only "`read`, `print`, `push`, `pop`"?

$$a == X$$

Definition (Stackable Permutations)



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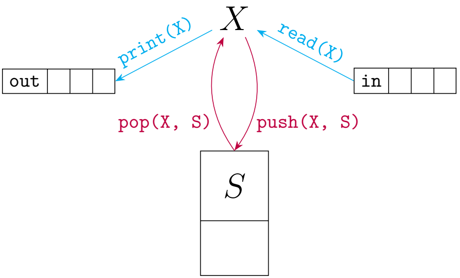

Q_1 : Meaning of "`read, print, push, pop`"?

Q_2 : Using only "`read, print, push, pop`"?

$$a == X \qquad a > X \ (a < X) \qquad \texttt{top}(S)$$

DH 2.12: Stackable Permutations

(a) **Show** that the following permutations *are* stackable:

(i) $(3, 2, 1)$

(ii) $(3, 4, 2, 1)$

(iii) $(3, 5, 7, 6, 8, 4, 9, 2, 10, 1)$

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DH 2.13: Stackable Permutations Checking Algorithm

To check whether a given permutation can be obtained by a stack.

`read` `print` `push` `pop` `is-empty`

```
X = 0    S = ∅    in != EOF
```

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X = 0     S = ∅     in != EOF
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```
foreach 'a' in out:
  if (! is-empty(S)
      && 'a' == top(S))
    pop(S, X)
    print(X)
  else ··· // T.B.C
```

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read print push pop is-empty

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```
else // T.B.C
  while (in != EOF)
    read(X)
    if (X == 'a')
      print(X)
      break
    else
      push(X, S)
  ERR
```

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  ERR   // How???
```

DH 2.12: Stackable Permutations

(b) **Prove** that the following permutations are *not* stackable:

(i) $(3, 1, 2)$

(ii) $(4, 5, 3, 7, 2, 1, 6)$

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$$(3, 1, 2)$$

$$(4, 5, 3, 7, 2, 1, 6)$$

$$\boxed{\texttt{out} = \cdots a_i \cdots a_j \cdots a_k : i < j < k \land a_j < a_k < a_i}$$

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(b) **Prove** that the following permutations are *not* stackable:

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$$(3, 1, 2)$$

$$(4, 5, 3, 7, 2, 1, 6)$$

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312-Pattern

Theorem (Stackable Permutations)

A permutation $(a_1, \cdots, a_n)$ *is stackable* $\iff$ *it is not the case that*

$$312\text{-}\textit{Pattern}: \boxed{\textit{out} = \cdots a_i \cdots a_j \cdots a_k : i < j < k \land a_j < a_k < a_i}$$

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Proof.

NO PROOF WARRANTY

$\square$

DH 2.12: Stackable Permutations

(c) How many permutations of A_4 *cannot* be obtained by a stack?

$$(1,4,2,3),(2,4,1,3),(3,1,2,4),(3,1,4,2),(3,4,1,2)$$
$$(4,1,2,3),(4,1,3,2),(4,2,1,3),(4,2,3,1),(4,3,1,2)$$

DH 2.12: Stackable Permutations

(c) How many permutations of A_4 *cannot* be obtained by a stack?

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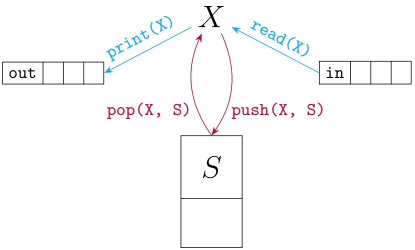
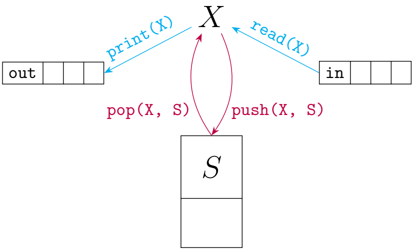
$$(4, 1, 2, 3), (4, 1, 3, 2), (4, 2, 1, 3), (4, 2, 3, 1), (4, 3, 1, 2)$$

Q : What about A_n?


X
print(X)
read(X)
out
in
pop(X, S)
push(X, S)
S



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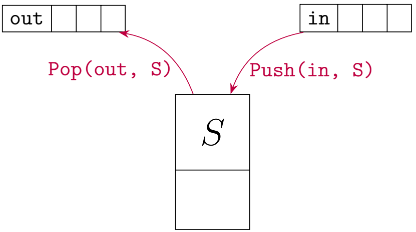

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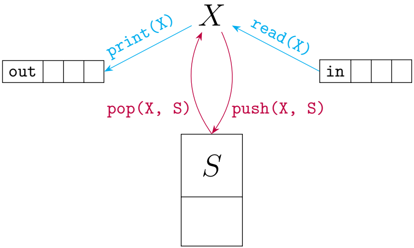
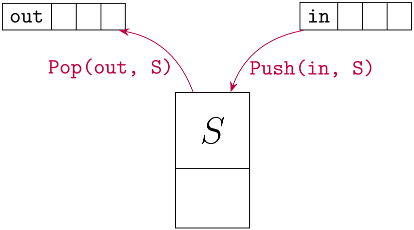
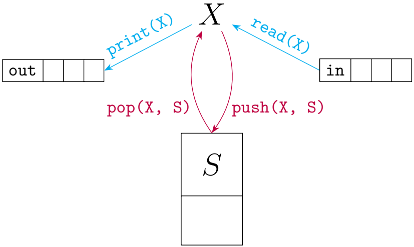
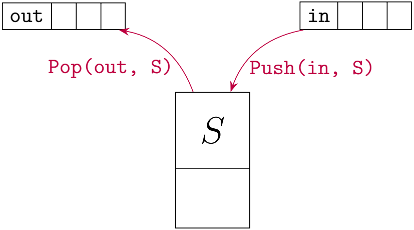




Q : Are $S + X$ and S are **equivalent**?





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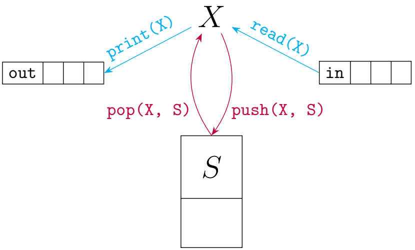

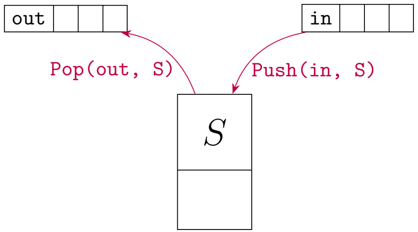

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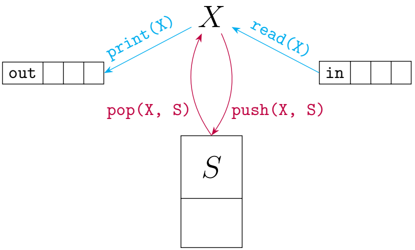
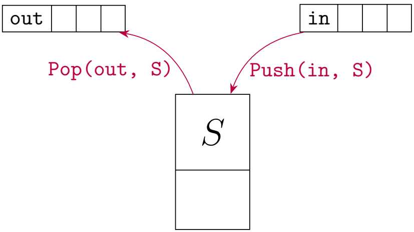
Producing the same set of permutations.

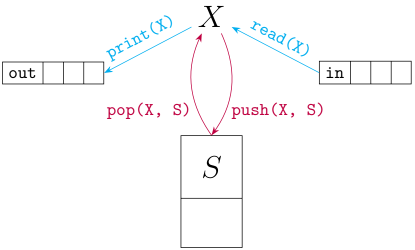




Q : Are $S + X$ and S are **equivalent**?

Producing the same set of permutations.

Accepting the same set of *admissible* operation sequences.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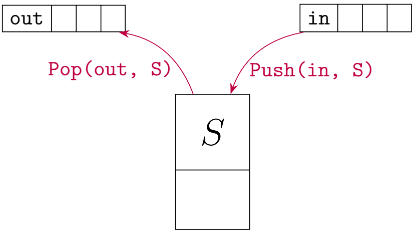
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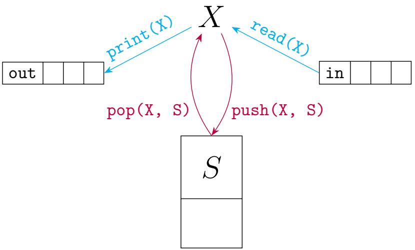
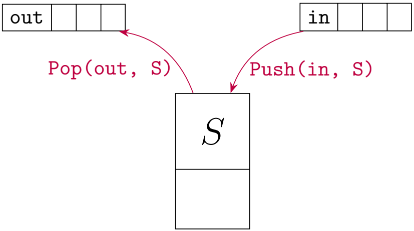
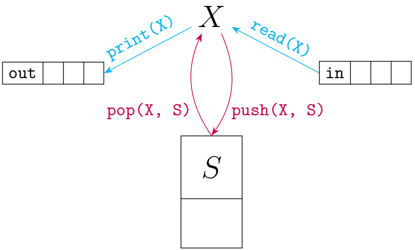
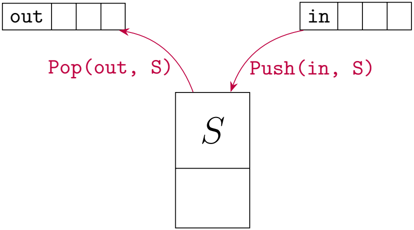




By simulations.

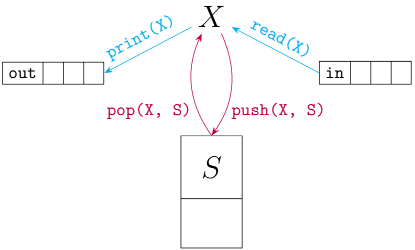


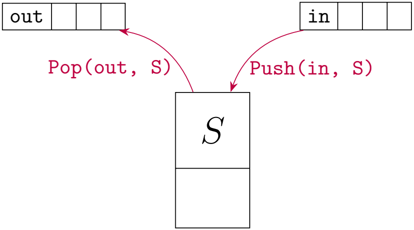


By simulations.

Simulate S by $S + X$:

- Push
- Pop





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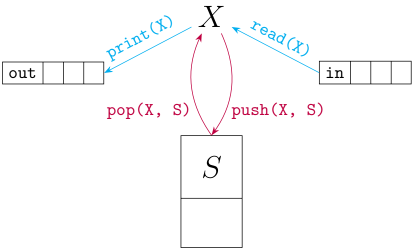
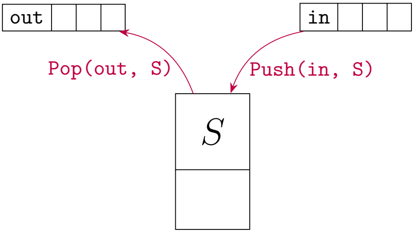
Simulate $S + X$ by S:





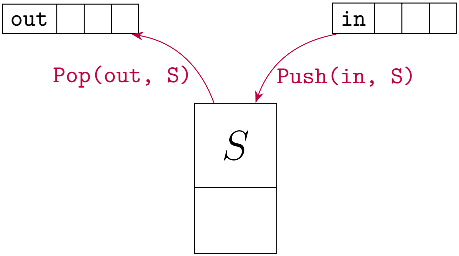
By simulations.

Simulate S by $S + X$:

- Push
- Pop

Simulate $S + X$ by S:

By iterative transformations.

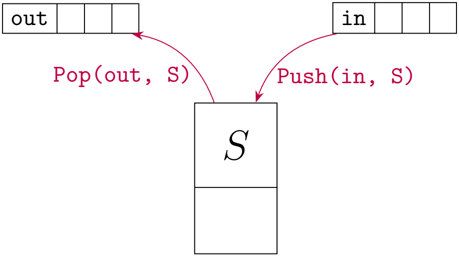


$(1, 2, 3)$: Push Pop Push Pop Push Pop

$(3, 2, 1)$: Push Push Push Pop Pop Pop

DH 2.12: Stackable Permutations

How many permutations of $\{1 \cdots n\}$ are stackable on the model S?



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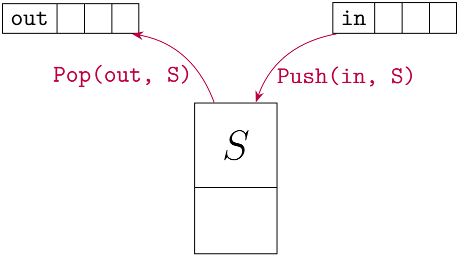

Q : How many *admissible* operation sequences of "Push" and "Pop"?

Definition (Admissible Operation Sequences)

An operation sequence of "`Push`" and "`Pop`" is *admissible* if and only if

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Theorem

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Proof.

$$\texttt{Push Push Push Pop Pop } \textcolor{red}{\texttt{Push}} \cdots$$

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□

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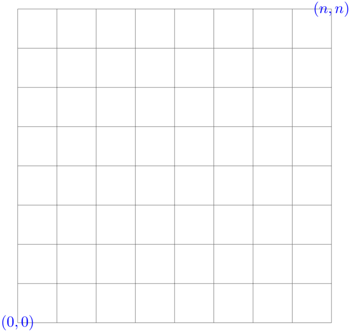

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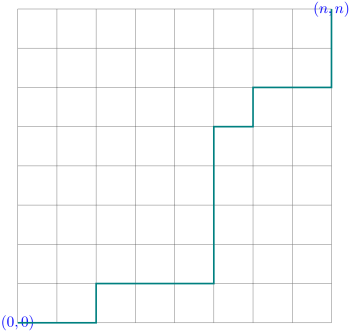

$\square$

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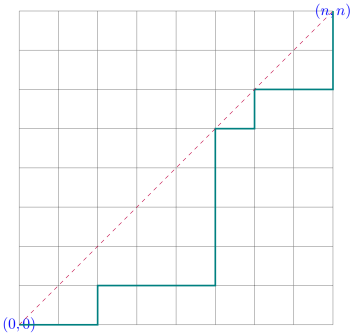

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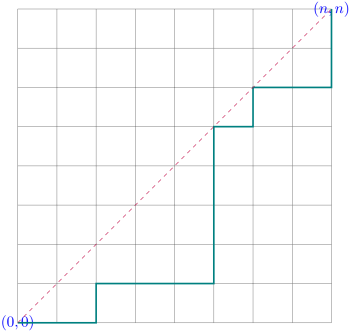
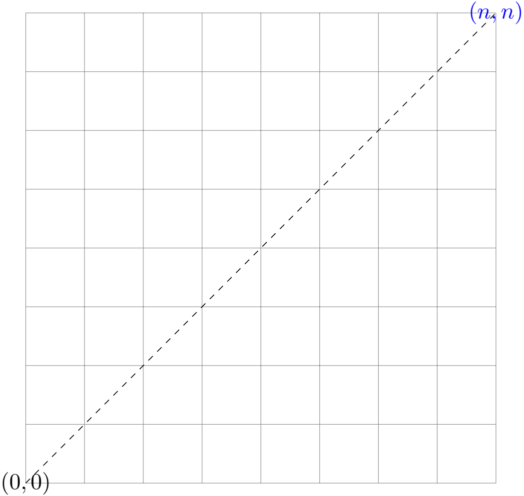
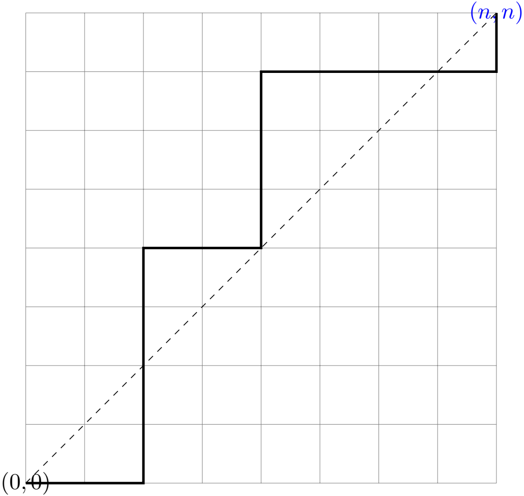


$$\underbrace{\binom{2n}{n}}_{\text{all}} - \underbrace{\binom{2n}{n-1}}_{\text{inadmissible}}$$

□


(n, n)
$(0, 0)$



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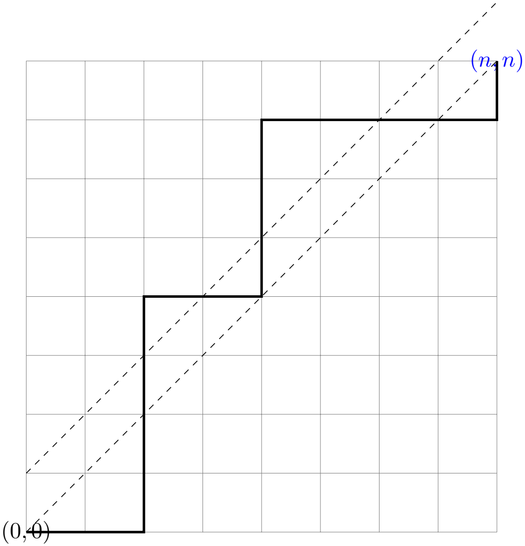

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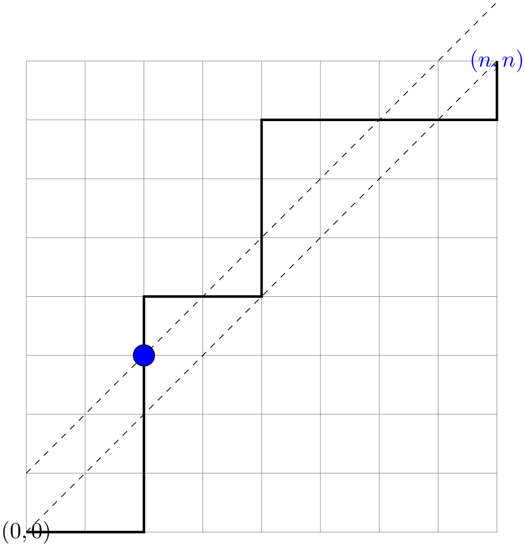
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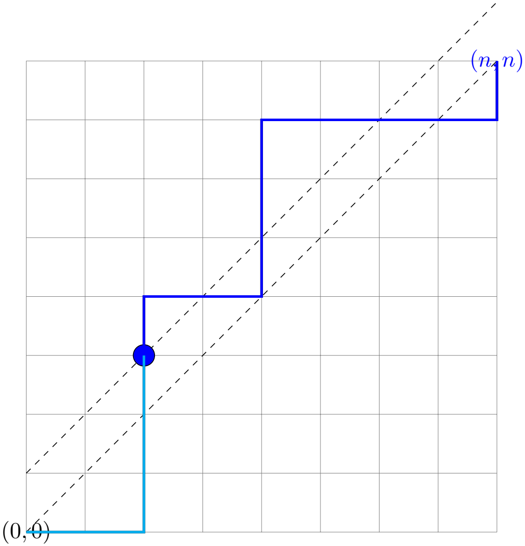

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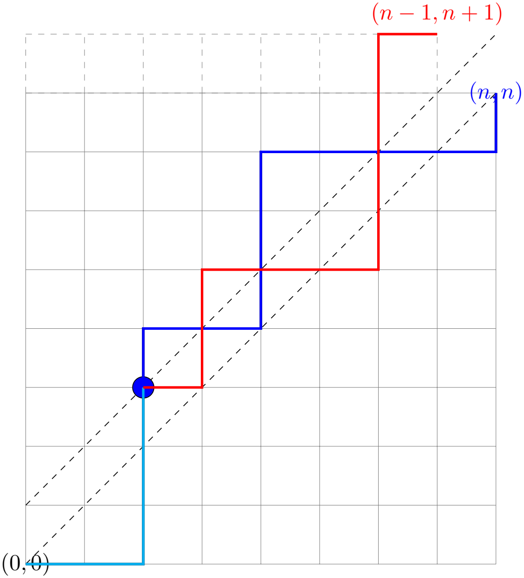
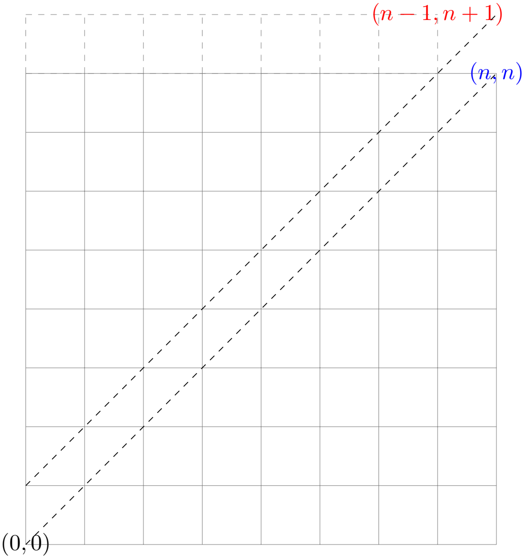

$(n-1, n+1)$
(n, n)
$(0, 0)$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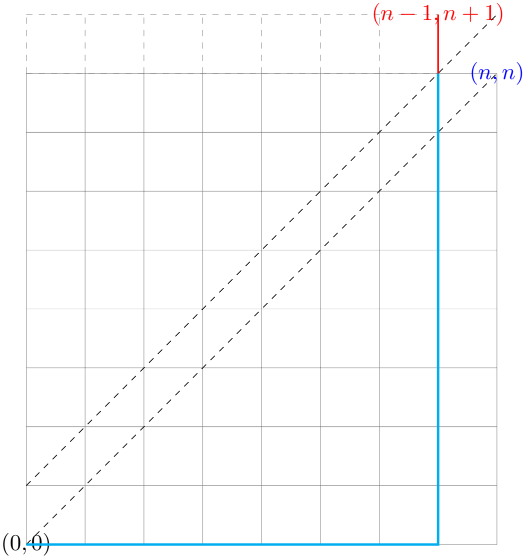
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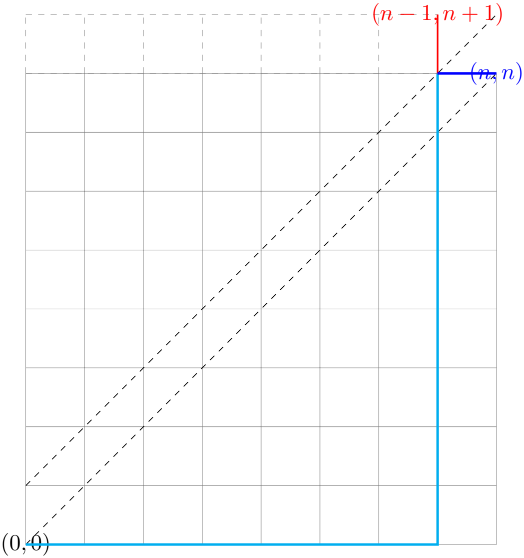
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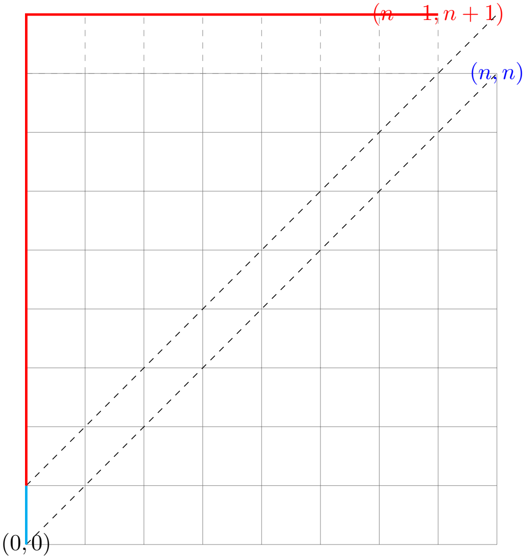
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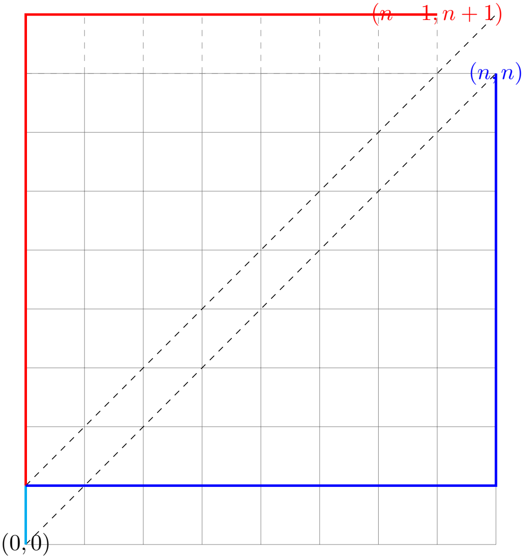
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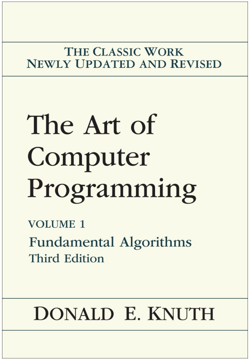
Catalan Number

$$(3, 2, 1) : ((())) \qquad (1, 2, 3) : ()()()$$

For more about "Stackable Permutations" (Section $2.2.1$):




Thank You!