

score: 400

500

3-2 Greedy Algorithms

(How to justify your greed?)

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Proof: Inductive Exchange Argument

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Theorem (Greedy-Choice Property)

There is an optimal solution that makes the greedy choice.

Theorem (Optimal Substructure Property)

*A problem exhibits **optimal substructure** if an optimal solution to the problem contains within it optimal solutions to subproblems.*

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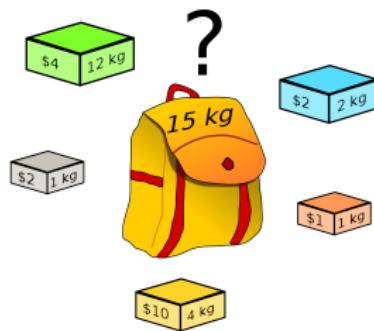
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Two key ingredients in induction.

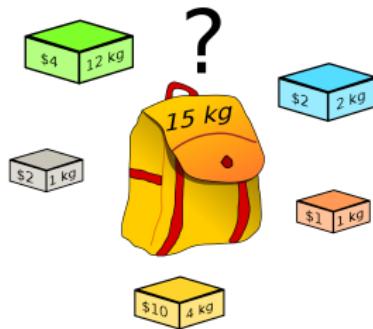
Fractional Knapsack Problem (Problem 16.2-1)

Prove that the fractional knapsack problem has the greedy-choice property.



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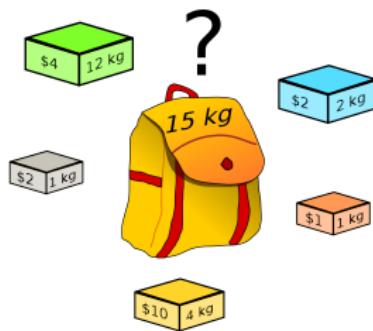
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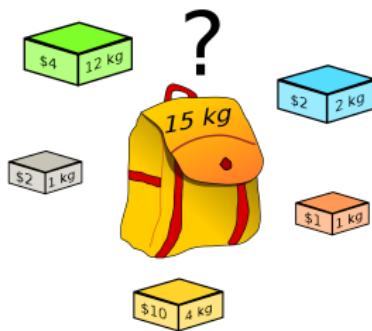


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$$\frac{v_1}{w_1} \geq \frac{v_2}{w_2} \geq \dots \geq \frac{v_n}{w_n}$$

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As most item of $\frac{v_1}{w_1}$ as possible.

Change-Making Problem (Problem 16-1 (a))



Making change for v cents using the fewest number of coins.

Change-Making Problem (Problem 16-1 (a))



25



10



5



1

Making change for v cents using the fewest number of coins.

$$\text{OPT} : v = 25 \times A^* + 10 \times B^* + 5 \times C^* + 1 \times D^*$$

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(A^*, B^*, C^*, D^*) vs. (A, B, C, D)

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(A^*, B^*, C^*, D^*) vs. (A, B, C, D)

$$A^* = A$$

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Lemma (Properties of OPT)

- (I) $B^* \leq 2$
- (II) $C^* \leq 1$
- (III) $D^* \leq 4$
- (IV) $\neg(B^* = 2 \wedge C^* = 1)$

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Theorem

$$10 \times B^* + 5 \times C^* + 1 \times D^* \leq 24$$

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Change-Making Problem (Problem 16-1 (c))



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$$\{1, 3, 4\}, \quad v = 6$$

Change-Making Problem (Problem 16-1 (c))



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G : 4, 1, 1 **vs.** OPT : 3, 3

Change-Making Problem (Problem 16-1 (c))



$$\{1, 3, 4\}, \quad v = 6$$

$$G : 4, 1, 1 \quad vs. \quad OPT : 3, 3$$

Why does the previous proof fail here?

Change-Making Problem (Problem 16-1 (b))

$$c^0, c^1, c^2, \dots, c^k, \quad c > 1, k \geq 1$$

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$$\color{red}{a_i^*} = \color{blue}{a_i}$$

Canonical Coin Systems



Live with Your Deadlines (Additional Problem)

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Live with Your Deadlines (Additional Problem)

$HW_i : (Length : t_i, \ Deadline : d_i)$



4

$OT : 2$

$PS : 5$

$OJ : 6$

$PA : 8$

Live with Your Deadlines (Additional Problem)

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$Finish\ time : f_i = s_i + t_i$

$\textcolor{blue}{Lateness} : l_i = f_i - d_i$

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$$\min (L \triangleq \max_i l_i)$$

What are your greedy strategies?

**“I love deadlines.
I like the
whooshing
sound they
make as they
fly by. ”**

Douglas Adams

Shortest Job First: $\min t_i$

Urgentest Job First: $\min(d_i - t_i)$

Earliest Deadline First



Proof: Exchange Argument.

$OPT: J_1^*, J_2^*, \dots, J_n^*$

$G: J_1, J_2, \dots, J_n, \quad (d_i \leq d_j, \forall i \leq j)$

Theorem

The greedy solution has no idle time and no inversions.

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Theorem

All schedules with no idle time and no inversions have the same maximum lateness.

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Immediate Inversion

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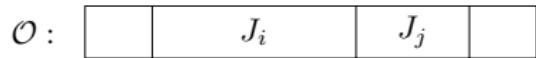
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Immediate Inversion

$$d_j < d_i$$



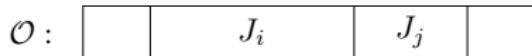
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$$l'_j \leq l_j$$



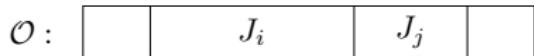
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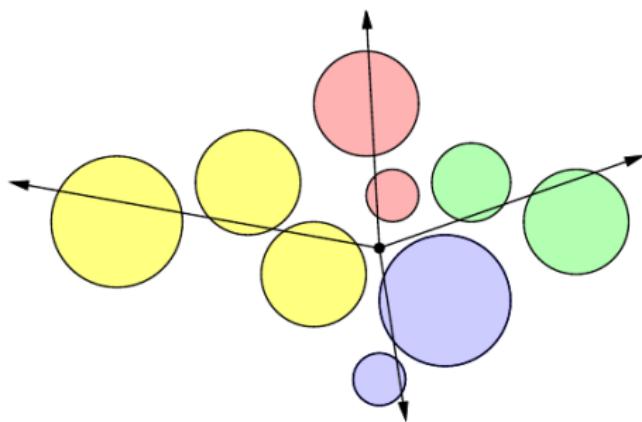
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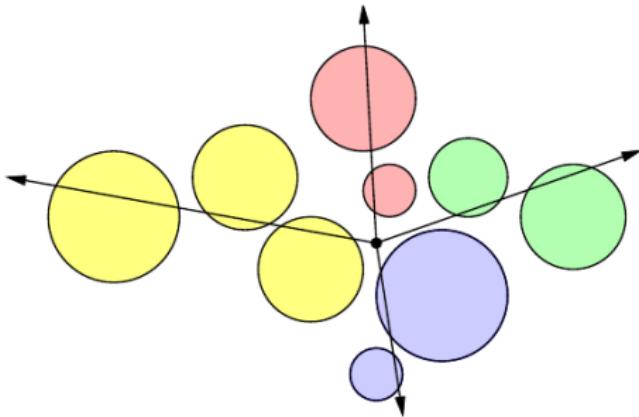
$$l'_i = f_j - d_i < f_j - d_j = l_j$$



Minimum Shots Problem (Additional Problem)



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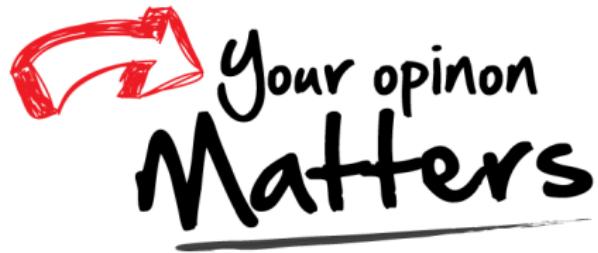


Assumption:

Possible to shoot a ray that does not intersect any balloons.







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