2-11 Heapsort

Hengfeng Wei

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Obama in a job interview at Google

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"What is most efficient way to sort a million 32-bit integers?"

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Best case   Worst case   Average case

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Best case Worst case Average case

Worst-case of Max-Heapify (TC 6.2-6)

Show that the worst-case running time of Max-Heapify on an n-element heap is $\Omega(\log n)$.

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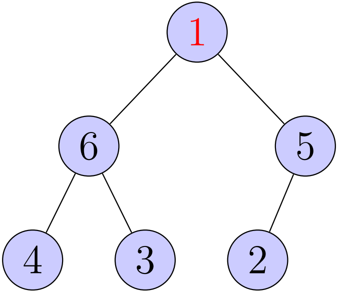

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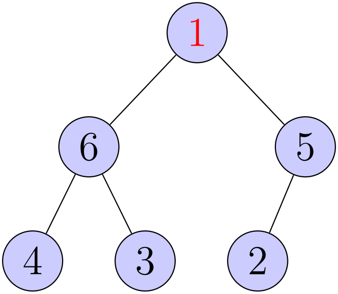

Compare *vs.* Exchange

Worst-case of Max-Heapify (Section 6.2 of CLRS)

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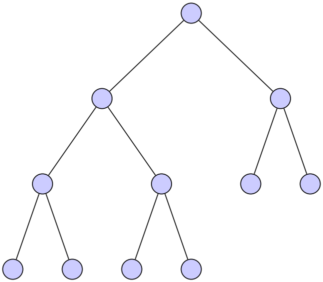

$$W(n) \le H(n)$$

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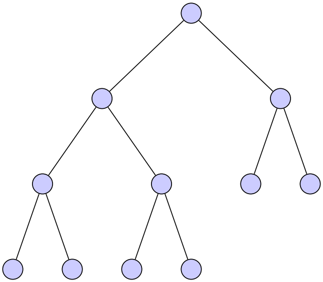

$$W(n) \leq H(n)$$

No Examples Here!

Therefore . . .

Worst-case of MAX-HEAPIFY

Show that the worst-case running time of MAX-HEAPIFY on an n-element heap is $\Theta(\log n)$.

	O	Ω	Θ
Worst-case	"power" of $\mathcal{A}$	by example	$O = \Omega$

Worst-case of HEAPSORT (TC 6.4-4)

Show that the worst-case running time of HEAPSORT is $\Omega(n \log n)$.

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Non-proof.

$$\underbrace{\Theta(n)}_{\text{EXTRACT-MAX}} \times \underbrace{\Omega(\log n)}_{\text{MAX-HEAPIFY}} = \Omega(n \log n)$$

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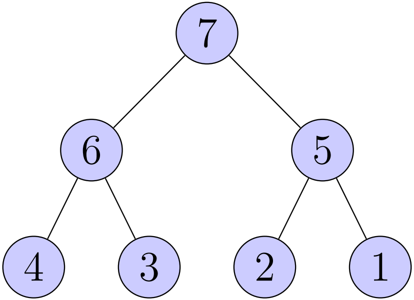
What is wrong?

Worst-case of HEAPSORT (TC 6.4 – 4)

Show that the worst-case running time of HEAPSORT is $\Omega(n \log n)$.

Heap in decreasing order?



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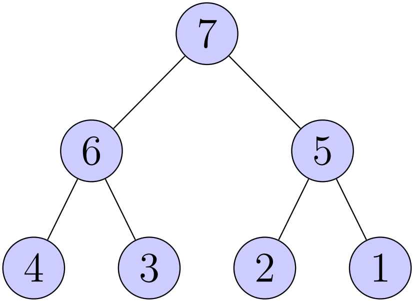
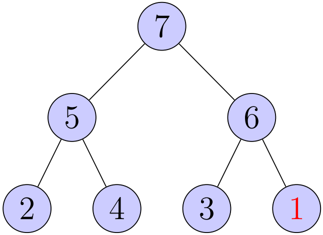
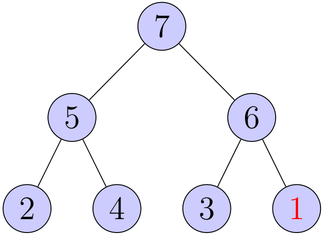
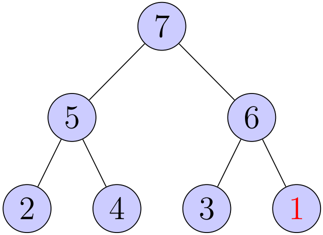


$$T(7) = 2 + 1 + 1 + 1 + 0 + 0 = 5$$


7
5
6
2
4
3
1



$$T(7) = 2 + 2 + 2 + 1 + 1 + 0 = 8$$



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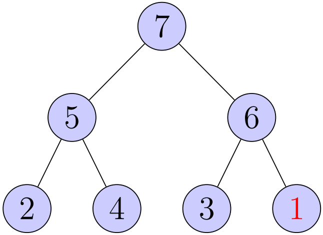

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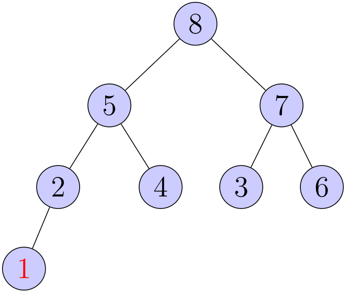

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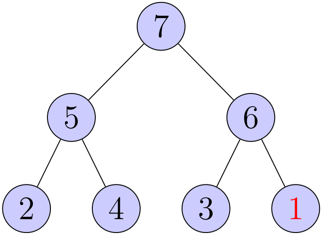

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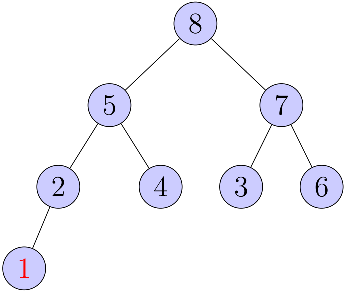
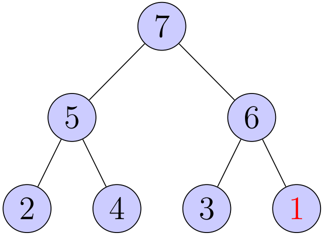
(Ex. 23, Section 5.2.3, TAOCP Vol 3)

$$\sum_{r=1}^{n-1} \lfloor \log r \rfloor$$





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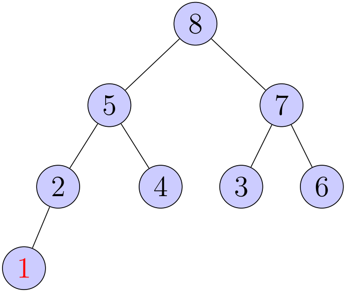
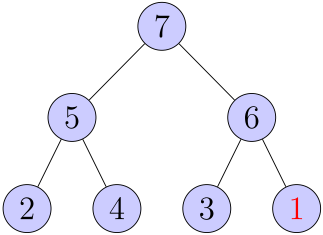


(Ex. 23, Section 5.2.3, TAOCP Vol 3)

$$\sum_{r=1}^{n-1} \lfloor \log r \rfloor = n \lfloor \log n \rfloor - 2^{\lfloor \log n \rfloor + 1} + 2$$



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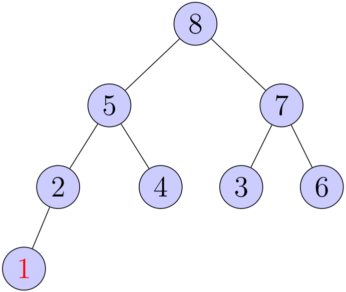

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$$\sum_{r=1}^{n-1} \lfloor \log r \rfloor = n \lfloor \log n \rfloor - 2^{\lfloor \log n \rfloor + 1} + 2 = \Omega(n \log n)$$

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$$\underbrace{\Theta(n)}_{\text{EXTRACT-MAX}} \times \underbrace{O(\log n)}_{\text{MAX-HEAPIFY}} = O(n \log n)$$

Therefore . . .

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Worst-case	"power" of $\mathcal{A}$	by example	$O = \Omega$

Algorithm $\mathcal{A}$

Inputs $\mathcal{I}$ of size n

	O	Ω	Θ
Best-case			
Worst-case			

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Best-case of Heapsort (Ex. 32, Section 5.2.3, TAOCP Vol 3)

Prove that the number of heapsort promotions, B, is always at least $\frac{1}{2} N \log N + O(N)$, if the keys being sorted are distinct.

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The largest m elements form a subtree.

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$$B(n) \geq \frac{1}{4} n \log n + O(n) + B(\lfloor n/2 \rfloor) \implies B(n) \geq \frac{1}{2} n \log n + O(n)$$

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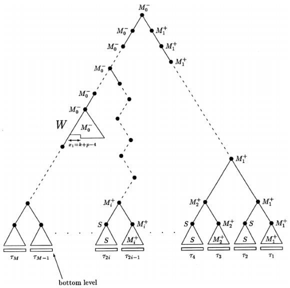


FIG. 2. Initial heap (more detailed).

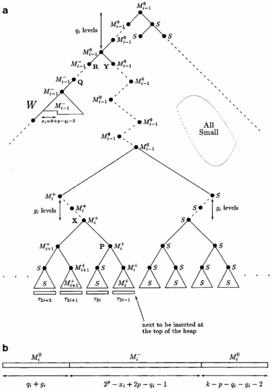


FIG. 3. (a) Odd i; (b) contents of the bottom level of τ_{2i-1}, i odd.

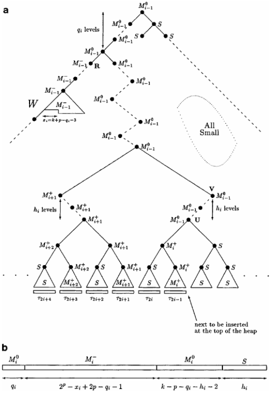
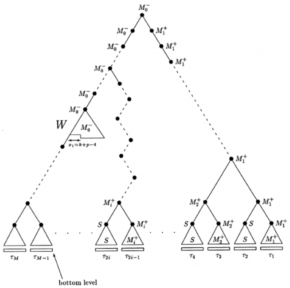


FIG. 4. (a) Even i; (b) contents of the bottom level of τ_{2i-1}, i even.

“On the Best Case of Heapsort” (1994)



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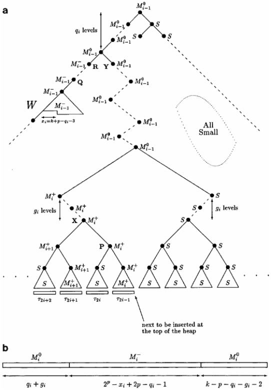

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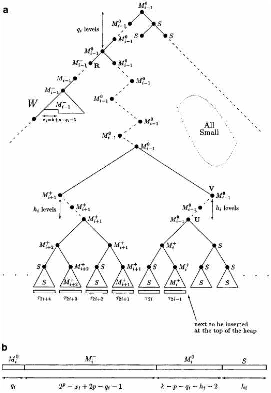

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$$B(n) \leq \frac{1}{2} n \log n + O(n \log \log n)$$

Therefore . . .

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Best-case			
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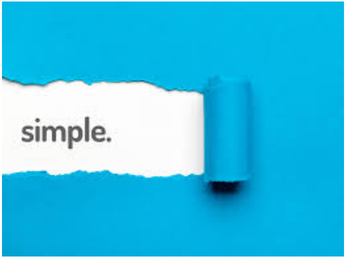

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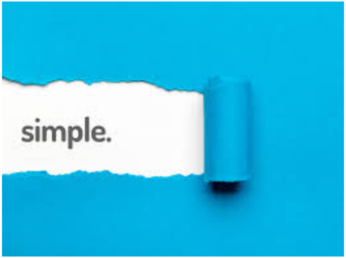

I said simple,
not easy.

“By a surprisingly short counting argument.”

Robert Sedgewick

"By a surprisingly short counting argument."

Robert Sedgewick

D. E. Knuth

"It is elegant.

"By a surprisingly short counting argument."

Robert Sedgewick

D. E. Knuth

"It is elegant. see exercise 30."

Heap Identity (Additional)

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$$\lfloor \log \lfloor \frac{1}{2}h \rfloor \rfloor + 1 = \lceil \log(\lfloor \frac{1}{2}h \rfloor + 1) \rceil = \lceil \log(h+1) \rceil - 1 = \lfloor \log h \rfloor$$

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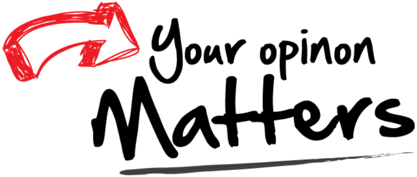
Depth of h = (Depth of the parent of h) + 1


Thank
You!



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