A one-to-one function from a finite set to itself is onto - how to prove by induction?

I'm not sure if I can do this without knowing what f actually is?

> Let X be a finite set with n elements and $f : X \to X$ a one-to-one function. Prove by induction that f is an onto function.

Any pointers? I don't even know how to make a base case for this.

(elementary-set-theory) (induction)

edited Apr 19 '13 at 4:39
Zev Chonoles
105k ● 16 ■ 202 ▲ 378

asked Apr 19 '13 at 4:37
K. Barresi
139 ● 1 ■ 2 ▲ 6

THe base case is pretty pretty much always with $n = 1$. In this case X would be a finite set with 1 element. – Eleven-Eleven Apr 19 '13 at 4:41

1 From there, Let X is a finite set with k elements and f:X->X a one to one function and assume that f is an onto function. What does that mean? Apply that assumption to the set with n+1 elements... – Eleven-Eleven Apr 19 '13 at 4:44

Hm..I wonder if I can actually use Pidgeon Hole for this...thanks for the help getting started Christopher! – K. Barresi Apr 19 '13 at 4:49

no pigeonhole here. You have to use definitions of 1-1 and onto functions. Pigeonhole implies a counting problem. This is not a counting problem. This is an induction argument using known definitions. You anchor your base case and you make an assumption on k and then try to show it holds for k+1 effectively causing a "chain reaction" proof on all values that n can take on. – Eleven-Eleven Apr 19 '13 at 4:56

2 Answers

Induct on $|X|$. The base case $|X| = 1$ is obvious since then there is only one function $X \to X$.

Now, suppose inductively that $|E| \leq n$ implies that every injective $E \to E$ is surjective. Suppose $|X| = n + 1$ and let $f : X \to X$ be injective. Seeking a contradiction, suppose f is not surjective so $|f(X)| \leq n$. Then $g : f(X) \to f(X)$ given by $g(t) = f(t)$ is injective and the inductive hypothesis implies g is surjective. That is, $g(f(X)) = f(X)$ so for every $y \in X$ there exists an $x \in X$ such that $f(f(x)) = f(y) \Rightarrow f(x) = y$. Thus f is surjective, a contradiction. Hence f is surjective and this closes the induction.

edited May 9 '13 at 12:28

answered Apr 19 '13 at 4:44
Brian Fitzpatrick
20.2k ● 4 ■ 28 ▲ 55

Thanks Brian. The E threw me off for a second, but I read through a few times and that cleared it up. – K. Barresi Apr 19 '13 at 4:53

1 Of course the base case $|X| = 0$ would be "even more obvious". ;) – Hagen von Eitzen Apr 19 '13 at 5:21

I altered the proof to make it a bit nicer. Hagen--I agree! – Brian Fitzpatrick Apr 19 '13 at 5:32

The last line is kind of hard to understand, you get a contradiction because f is surjective therefore f is surjective... – shinzou Jan 14 '15 at 20:45

Is this true for infinite set also? – blue boy Aug 25 at 23:22

An alternative, non-inductive approach. Makes use of the definition of Dedekind-infinite/finite.

Suppose we have injective (1-1) function $f : X \to X$

Using proof by contrapositive, suppose that f is *not* surjective (onto).

Let $X' = f(X)$. Show X' is a proper subset of X.

Construct $f' : X' \to X$, the inverse of f on X'.

Show f' is both injective and surjective. By definition, X would then be infinite.

Taking the contrapositive, if X is finite then f is surjective (onto).

edited Apr 22 '13 at 3:24

answered Apr 21 '13 at 19:27

Dan Christensen
7,407 ● 1 ■ 17 ▲ 30