3-9 Connectivity

Hengfeng Wei

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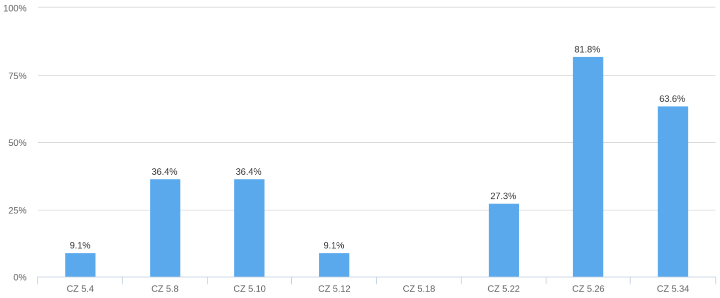
November 26, 2018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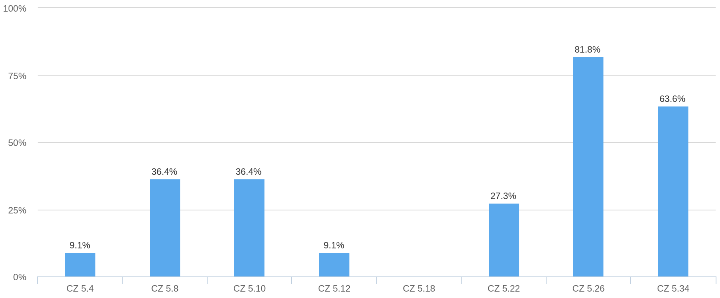

5.10 5.34 5.22 5.26

如果两个割点相连，那么联通块怎么划分
联通快呢）

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点割集，边割集

Menger's Theorem

(Theorem 5.16; Theorem 5.21)

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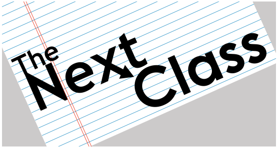

2-Connectivity (Problem 5.10)

A connected graph G with $m \geq 2$ is *nonseparable*

$$\iff$$

any two *adjacent* edges of G lie on a common cycle of G.

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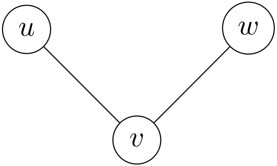
G is nonseparable

$\implies$ u, w lie on a common cycle



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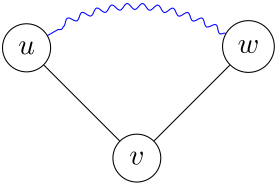
$\implies$ $\exists$ path $u \sim w$

$\implies$ $\exists$ cycle $u - v - w \sim u$



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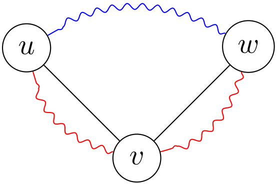

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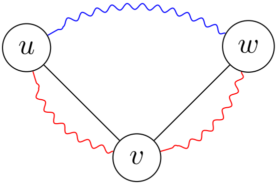

G is nonseparable

$\implies$ u, w lie on a common cycle

$\implies$ $\exists$ path $u \sim w$ that does not contain v

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By Contradiction.

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Proof.

"$\Longleftarrow$"

By Contradiction.

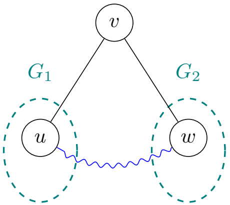
Suppose v is a cut-vertex of G

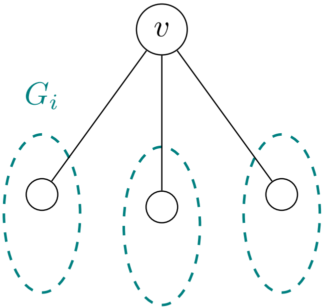
$\implies$ $G - v$ contains ≥ 2 comps $G_1, G_2, \cdots$

$\implies$ $\exists u \in G_1, w \in G_2 : v - u \land v - w$

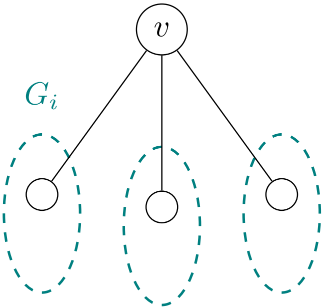
$\implies$ $v - u, v - w$ lie on a common cycle

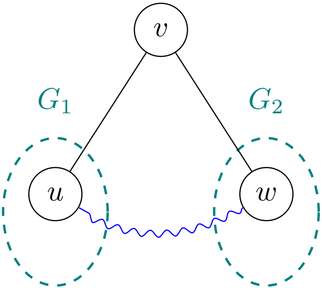
$\implies$ $\exists$ path $u \sim w$ that does not contain v





$$\forall G_i \; \exists v_i \in G_i \; v - v_i$$





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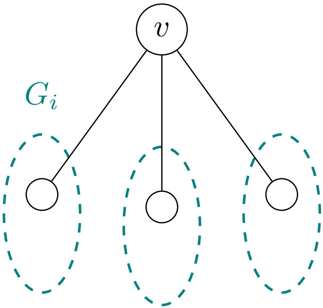

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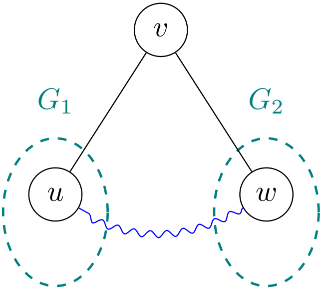

$$\color{red}{\forall v \in S} \; \forall G_i \; \exists v_i \in G_i \; v - v_i$$

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2-Connectivity (Extended Problem)

A connected graph G with $m \geq 2$ is *nonseparable*

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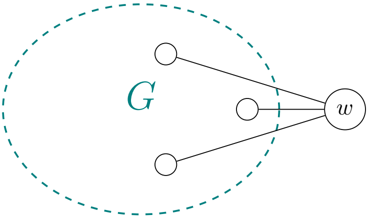
any two edges of G lie on a common cycle of G.

Expansion Lemma (Problem 5.34; Theorem 5.18)

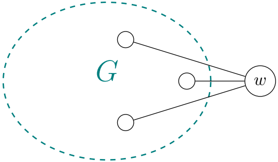
Let G be a k-connected graph and let S be any set of k vertices.

If a graph H is obtained from G by adding a new vertex w and joining w to the vertices of S, then H is also k-connected.



We need to prove that

$$\forall v \in V(G): \text{ there exist } k \text{ internally disjoint } v - w \text{ paths}$$



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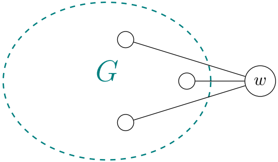

This holds because

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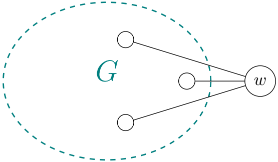

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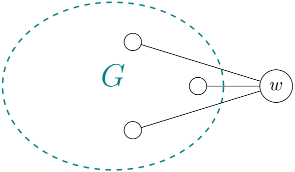
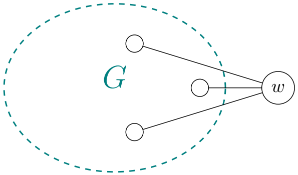
Corollary (5.19; Proved using Theorem 5.18)

If G is a k-connected graph and $u, v_1, v_2, \cdots, v_k$ are $k+1$ distinct vertices of G, then there exist internally disjoint $u - v_i$ paths $(1 \leq i \leq k)$ in G.



To prove $\kappa(H) \geq k$



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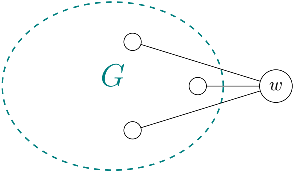
Let U be a vertex-cut of H.

We prove that $|U| \geq k$.



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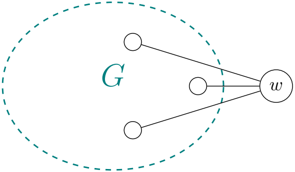

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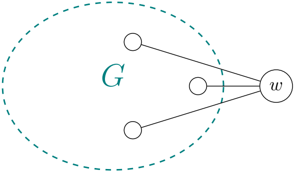

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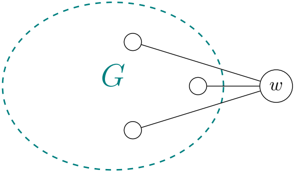

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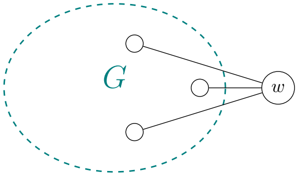
$U - w$ is a vertex-cut of G



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$$|U| \geq k + 1$$

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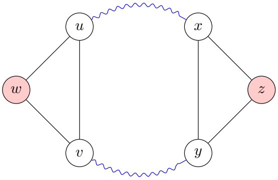

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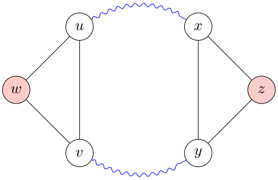
Consider two edges uv and xy.

Add w, z

Add $wu, wv; zx, zy$

w and z lie on a common cycle



Effects of Removing an Edge on Connectivity (Problem 5.22 (a))

(a) If G is k-connected and $e = uv \in E(G)$, then $G - e$ is $(k-1)$-connected.

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We prove that $G - e - U$ is connected.

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$$\text{But } \Big| U \cup \{u\} \Big| < k$$

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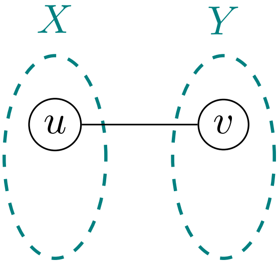
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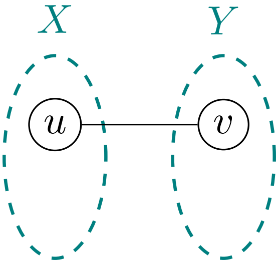
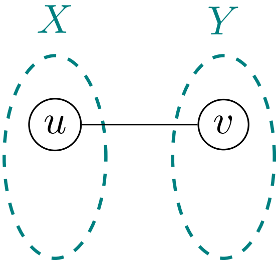


CASE I : $|X| \geq 2 \vee |Y| \geq 2$



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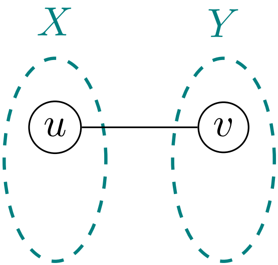

CASE II : $|X| = |Y| = 1$

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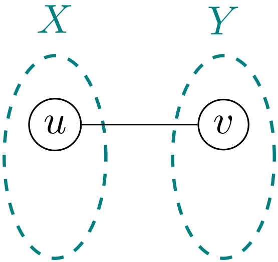

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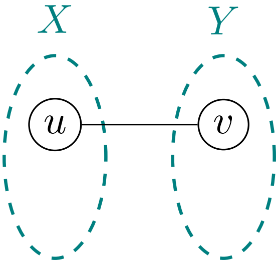

Case II : $|X| = |Y| = 1$

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$$\kappa(G) \geq k > n - 1$$

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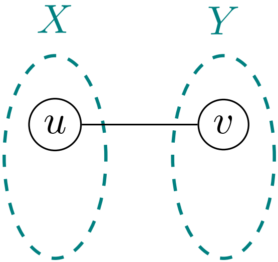

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$\kappa(G) \geq k > n - 1$

But $0 \leq \kappa(G) \leq n - 1$

Effects of Removing an Edge on Connectivity (Problem 5.22 (b))

(b) If G is k-edge-connected and $e = uv \in E(G)$, then $G - e$ is $(k-1)$-edge-connected.

$$\lambda(G) \geq k \implies \lambda(G - e) \geq k - 1$$

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We prove that $G - e - X$ is connected.

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$$G - e - X = G - (e + E) \text{ is connected}$$

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We prove that $G - e - X$ is connected.

$G - e - X = G - (e + E)$ is connected $(\because \lambda(G) \geq k)$

$$\kappa(G - e) \leq \kappa(G)$$

$$\kappa(G - e) \le \kappa(G)$$

Effects of Removing a Vertex on Connectivity (Extended Problem)

$$\text{Is } \kappa(G - v) \le \kappa(G)?$$

$$\text{Is } \lambda(G - v) \le \lambda(G)?$$

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Effects of Removing a Vertex on Connectivity (Extended Problem)

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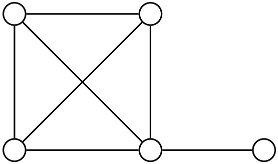

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Effects of Removing a Vertex on Connectivity (Extended Problem)

$$\text{Is } \kappa(G - v) \leq \kappa(G)?$$

$$\text{Is } \lambda(G - v) \leq \lambda(G)?$$

Effects of Removing a Vertex on Connectivity (After-class Exercise)

$$\text{Is } \kappa(G) \geq k \implies \kappa(G - v) \geq k - 1?$$

$$\text{Is } \lambda(G) \geq k \implies \lambda(G - v) \geq k - 1?$$

Degree Condition for $\lambda(G) = \delta(G)$ (Problem 5.26)

If G is graph of order n such that $\delta(G) \geq (n-1)/2$, then $\lambda(G) = \delta(G)$.

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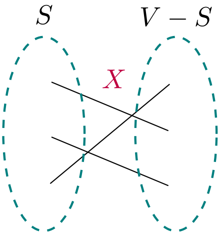

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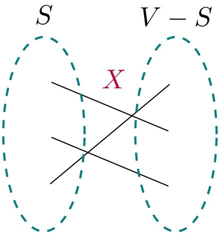

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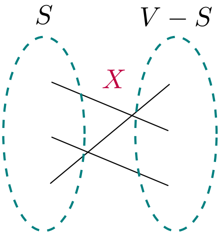

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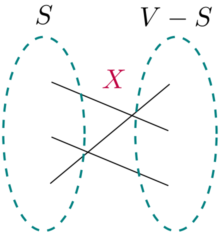

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Decision	Author(s)	Year	Complexity	Comments
		Edge Connectivity		
$\lambda = 2$ or $\lambda = 3$	Tarjan [26]	1972	$O(m + n)$	uses Depth First Search
λ	Even and Tarjan [6]	1975	$O(mn \times \min\{m^{1/2}, n^{2/3}\})$	n calls to max-flow
λ (digraphs)	Schnorr [25]	1979	$O(\lambda mn)$	n calls to max-flow
λ	Esfahanian & Hakimi [3]	1984	$O(\lambda mn)$	$\leq n/2$ calls to max-flow
λ (digraphs)	Esfahanian & Hakimi [3]	1984	$O(\lambda mn)$	$\leq n/2$ calls to max-flow
λ	Matula [23]	1987	$O(mn)$	uses dominating sets
$\lambda = k$	Matula [23]	1987	$O(kn^2)$	
λ (digraphs)	Mansour & Schieber [22]	1989	$O(mn)$	
$\lambda = k$	Gabow [9]	1991	$O(m+k^2 n\log(n/k))$	uses matroids
		Vertex Connectivity		
$\kappa = 2$	Tarjan [26]	1972	$O(m + n)$	uses Depth First Search
$\kappa = 3$	Hopcroft & Tarjan [18]	1973	$O(m + n)$	uses triconnected components
κ	Even & Trajan [6]	1975	$O((\kappa(n - \delta - 1)mn^{2/3})$	max-flow based
$\kappa = k$	Even [4]	1975	$O(kn^3)$	max-flow based
κ	Galil [12]	1980	$O(\min\{\kappa, n^{2/3}\}mn)$	max-flow based
$\kappa = k$	Galil [12]	1980	$O(\min\{k, n^{1/2}\}kmn)$	max-flow based
κ	Esfahanian & Hakimi [3]	1984	$O((n - \delta - 1 + \delta(\delta - 1)/2)mn^{2/3})$	max-flow based
$\kappa = 4$	Kanevsky & Ramachandran [20]	1991	$O(n^2)$	
κ	Henzinger & Rao [17]	1996	$O(\kappa mn\log n)$	randomised algorithm

Table 1: A chronology of connectivity algorithms

Theorem (Menger's Theorem (Theorem 5.16))

Let u and v be nonadjacent vertices in a graph G.

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Can you rearrange these three cases to make them (hopefully) easier to understand?

Case I: There exists a minimum $u-v$ separating set W in G containing a vertex x that is adjacent to both u and v.

Case II: There exists a minimum $u-v$ separating set W in G containing a vertex in W that is not adjacent to u and a vertex in W that is not adjacent to v.

Case III: For each minimum $u-v$ separating set W in G, either every vertex of W is adjacent to u and not adjacent to v or every vertex of W is adjacent to v and not adjacent to u.

Case I: There exists a minimum $u - v$ separating set W in G containing a vertex x that is adjacent to both u and v.

$$\exists W : \exists x \in W : x - u \wedge x - v$$

Case II: There exists a minimum $u - v$ separating set W in G containing a vertex in W that is not adjacent to u and a vertex in W that is not adjacent to v.

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$$\begin{aligned} \exists W : \exists x \in W : x \not- u \\ \wedge \exists y \in W : y \not- v \end{aligned}$$

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$$\begin{aligned} \forall W : \forall x \in W &: x - u \wedge x \not- v \\ \vee\ \forall x \in W &: x - v \wedge x \not- u \end{aligned}$$

$$\mathrm{I}: \exists W : \exists x \in W : x - u \land x - v$$

$$\begin{aligned} \mathrm{II}: \exists W : \exists x \in W &: x \not- u \\ \land \exists y \in W &: y \not- v \end{aligned}$$

$$\begin{aligned} \mathrm{III}: \forall W : \forall x \in W &: x - u \land x \not- v \\ \lor \forall x \in W &: x - v \land x \not- u \end{aligned}$$

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$$\begin{aligned} \mathrm{II}' : \forall W : \forall x \in W &: x - u \\ \vee\ \forall y \in W &: y - v \end{aligned}$$

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$$\mathrm{I}: \exists W : \exists x \in W : x - u \land x - v \qquad \mathrm{I}' : \forall W : \forall x \in W : x \not- u \lor x \not- v$$

$$\begin{aligned} \mathrm{II} : \exists W &: \exists x \in W : x \not- u \\ &\land \exists y \in W : y \not- v \end{aligned} \qquad \begin{aligned} \mathrm{II}' : \forall W &: \forall x \in W : x - u \\ &\lor \forall y \in W : y - v \end{aligned}$$

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$$\mathrm{III} \equiv \mathrm{II}' \wedge \mathrm{I}'$$

$$\mathrm{I}: \exists W: \exists x \in W: x - u \wedge x - v$$

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$$\begin{aligned}\mathrm{III}: \forall W: \forall x \in W&: x - u \wedge x \not- v \\ \vee\ \forall x \in W&: x - v \wedge x \not- u\end{aligned}$$

$$\mathrm{III} \equiv \mathrm{II}' \wedge \mathrm{I}'$$

II

II′

I

III

II

$$
\begin{aligned}
\text{II}: \exists W : \exists x \in W : x \not- u \\
\wedge\ \exists y \in W : y \not- v
\end{aligned}
$$

II′

$$\text{I}: \exists W : \exists x \in W : x - u \wedge x - v$$

$$
\begin{aligned}
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\end{aligned}
$$

II

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II′

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II

$$\begin{aligned} \exists W : W \nsubseteq N(u) \\ \land W \nsubseteq N(v) \end{aligned}$$

II′

II

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II'

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II

$$\exists W : W \not\subseteq N(u) \land W \not\subseteq N(v)$$

II'

$$\exists W : \exists x \in W : x \in N(u) \cap N(v)$$

II

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II

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$$\mathrm{I}: \exists W : \exists x \in W : x - u \land x - v$$

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$$\mathrm{III}: \forall W : \forall x \in W : x - u \land x \not- v \lor \forall x \in W : x - v \land x \not- u$$

$$\forall W : W \subseteq N(u) \land W \cap N(v) = \emptyset \lor W \subseteq N(v) \land W \cap N(u) = \emptyset$$

II

$$\mathrm{II}: \exists W : W \nsubseteq N(u) \\ \land W \nsubseteq N(v)$$

II′

$$\mathrm{I}: \exists W : \exists x \in W : x \in N(u) \cap N(v)$$

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II

$$\begin{aligned}\text{II} : \exists W : W &\not\subseteq N(u) \\ \land\, W &\not\subseteq N(v)\end{aligned}$$

Q : What is the key to use the induction hypothesis in Case II?

II′

$$\text{I} : \exists W : \exists x \in W : x \in N(u) \cap N(v)$$

$$\begin{aligned}\text{III} : \forall W : W &\subseteq N(u) \land W \cap N(v) = \emptyset \\ \lor\, W &\subseteq N(v) \land W \cap N(u) = \emptyset\end{aligned}$$

II

$$\text{II} : \exists W : W \not\subseteq N(u) \\ \land W \not\subseteq N(v)$$

Q : What is the key to use the induction hypothesis in CASE II?

II′

$$\text{I} : \exists W : \exists x \in W : x \in N(u) \cap N(v)$$

$$\text{III} : \forall W : W \subseteq N(u) \land W \cap N(v) = \emptyset \\ \lor W \subseteq N(v) \land W \cap N(u) = \emptyset$$

Q : What will fail if we do not exclude CASE I from CASE III?

Theorem (Menger's Theorem for Edge-Connectivity (Theorem 5.21))

For distinct vertices u and v in a graph G,

the minimum number of edges of G that separate u and v equals the maximum number of pairwise edge-disjoint $u - v$ paths in G.


THANK YOU

Your opinon Matters

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