1-11 有穷与无穷

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"das wesen der mathematik liegt in ihrer freiheit"

"The essence of mathematics lies in its freedom"

Dangerous Knowledge (BBC 2007)

$$c = \aleph_1$$

Comparing Sets

$$= \qquad\qquad <$$

Function

$$=$$

Definition ($|A| = |B|$ ($A \approx B$) (1878))
Two sets of A and B are *equipotent* if there exists a *bijection* from A to B.

"$=$" is an equivalence relation.

$\overline{\overline{A}}$ (two abstractions)

$$\{1, 2, 3\} \quad \textit{vs.} \quad \{a, b, c\}$$

$$\{1, 2, 3, \cdots\} \quad \textit{vs.} \quad \{1, 3, 5, \cdots, 2, 4, 6, \cdots\}$$

Definition (Finite and Infinite)

For any set X,

Finite

$$\exists n \in \mathbb{N} : |X| = n \qquad (0 \in \mathbb{N})$$

Infinite $(\neg$ finite)

$$\forall n \in \mathbb{N} : |X| \neq n$$

Q : How to prove that a set is infinite?

By contradiction.

Definition (Finite and Infinite)

For any set X,

Countably Infinite

$$|X| = |\mathbb{N}| \triangleq \aleph_0$$

Countable

(finite $\lor$ countably infinite)

Uncountably Infinite

$$(\neg \text{ finite}) \land \Big(\neg \text{ (countably infinite)}\Big)$$

$$(\neg \text{ countable})$$

Theorem ($\aleph_0$ (1874))

$$|\mathbb{Q}| = |\mathbb{N}|$$

$|\mathbb{Q}| = |\mathbb{N}|$ (UD 22.9)

$$q \in \mathbb{Q}^+ : a/b \; (a, b \in \mathbb{N}^+)$$

1/1	1/2	1/3	1/4	1/5	1/6	1/7	1/8	⋯
2/1	2/2	2/3	2/4	2/5	2/6	2/7	2/8	⋯
3/1	3/2	3/3	3/4	3/5	3/6	3/7	3/8	⋯
4/1	4/2	4/3	4/4	4/5	4/6	4/7	4/8	⋯
5/1	5/2	5/3	5/4	5/5	5/6	5/7	5/8	⋯
6/1	6/2	6/3	6/4	6/5	6/6	6/7	6/8	⋯
7/1	7/2	7/3	7/4	7/5	7/6	7/7	7/8	⋯
8/1	8/2	8/3	8/4	8/5	8/6	8/7	8/8	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

$$|\mathbb{N}| = |\mathbb{Z}|$$

$$|\mathbb{N}| = |\mathbb{N}| \times |\mathbb{N}|$$

Theorem ($\mathbb{R}$ is uncountably infinite (1874).)

$$|\mathbb{R}| \neq |\mathbb{N}| \qquad (|\mathbb{R}| > |\mathbb{N}|)$$

Different "Sizes" of Infinity

Cantor's Diagonal Argument (1890)

Theorem (Cantor's Theorem (1891))

$$|X| \neq |2^X| \qquad (|X| < |2^X|)$$

Theorem ($|\mathbb{R}|$ (1877))

$$(0,1) = |\mathbb{R}| = |\mathbb{R}| \times |\mathbb{R}| = |\mathbb{R}|^{n \in \mathbb{N}}$$

"Je le vois, mais je ne le crois pas !"

"I see it, but I don't believe it !"

— Cantor's letter to Dedekind (1877).

Q : *What is "dimension"?*

$$<$$

Definition ($|A| \le |B|$)

$|A| \le |B|$ if there exists an *one-to-one* function f from A into B.

bijection $f : A \to f(A)$ $(\subseteq B)$

Q : What about onto function $f : A \to B$?

$|B| \le |A|$ (Axiom of Choice)

Definition ($|A| < |B|$)

$|A| < |B| \iff |A| \le |B| \land |A| \ne |B|$

$$|\mathbb{N}| < |\mathbb{R}|$$

$$|X| < |2^X|$$

Definition (Countable Revisited)

X is countable:

$$\exists n \in \mathbb{N} : |X| = n \vee |X| = |\mathbb{N}|$$

Proof for Countable (UD Exercise 22.5)

X is countable iff there exists a one-to-one function

$$f : A \to \mathbb{N}.$$

X is countable iff

$$|X| \leq |\mathbb{N}|.$$

Subsets of Countable Set (UD 22.6; UD Corollary 22.4)

Every subset B of a countable set A is countable.

Set Union (UD 22.1)

Give an example, if possible, of

(c) a countably infinite collection of *pairwise disjoint* nonempty sets whose union is finite.

(b) a countably infinite collection of nonempty sets whose union is finite.

$$\Big(\{A_i : i \in R\} \quad A_i = \{1\}\Big) = \{\{1\}\}$$

$$|A| = n \implies |2^A| = 2^n$$

Slope (UD 22.2 (e))

(e) the set of all lines with rational slopes

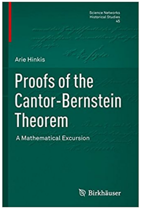
$$(\mathbb{Q},\ \mathbb{R})$$

Q : *Is "$\leq$" a partial order?*

Theorem (Cantor-Schröder–Bernstein (1887))

$$|X| \leq |Y| \wedge |Y| \leq |X| \implies |X| = |Y|$$

$$\exists \textit{ one-to-one } f : A \to B \wedge g : B \to A \implies \exists \textit{ bijection } h : A \to B$$



Q : *Is "$\leq$" a total order?*

Theorem (PCC)

Principle of Cardinal Comparability (PCC) $\iff$ *Axiom of Choice*

Finite Sets

"关于有穷，我原以为我是懂的"

学生反馈 (改编版)

"明明很显然的事情，为什么要那么繁琐的证明？

依靠直觉不可以吗？"

Definition (Finite)

X is finite if

$$\exists n \in \mathbb{N} : |X| = n$$

Theorem (Pigeonhole Principle (UD Theorem 21.2))

$$f : \{1, \cdots, m\} \to \{1, \cdots, n\} \ (m, n \in \mathbb{N}^+, m > n)$$

f is not one-to-one.

$A \setminus \{a\}$ (UD 21.15)

Let A be a nonempty finite set with $|A| = n$ and let $a \in A$.
Prove that $A \setminus \{a\}$ is finite and $|A \setminus \{a\}| = n - 1$.

$$f : A \to \{1, \cdots, n\}$$

$$f|_{A \setminus \{a\}} : A \setminus \{a\} \to \{1, \cdots, n\} \setminus \{f(a)\}$$

$|A| \leq |B|$ (UD 21.17)

A and B are finite sets and $f : A \to B$ is one-to-one.
Show that $|A| \leq |B|$.

By contradiction and the pigeonhole principle.

(UD 21.16)

(a) A is a finite set and $B \subseteq A$. We showed that B is finite (Corollary 20.11). Show that $|B| \leq |A|$.

one-to-one $f : B \to A$

(b) A is a finite set and $B \subseteq A$. Show that if $B \neq A$, then $|B| < |A|$.

$\exists a : a \in A \land a \notin B \qquad f : B \to A \setminus \{a\} \qquad |B| \leq |A \setminus \{a\}|$

(c) If two finite sets A and B satisfy $B \subseteq A$ and $|A| \leq |B|$, then $A = B$.

By contradiction and (b).

Cardinality of $|\mathrm{ran}(f)|$ (UD 21.18)

Let A and B be sets with A finite.

$$f : A \to B$$

Prove that $|\mathrm{ran}(f)| \le |A|$.

one-to-one $g : \mathrm{ran}(f) \to A$

(No Axiom of Choice Here)

$f : A \to A$ (UD 21.19)

Let A be a finite set.

$$f : A \to A$$

Prove that

$$f \text{ is one-to-one} \iff f \text{ is onto.}$$

$$\Longrightarrow$$

By contradiction.

$$f' : A \to A \setminus \{a\}$$

$$\Longleftarrow$$

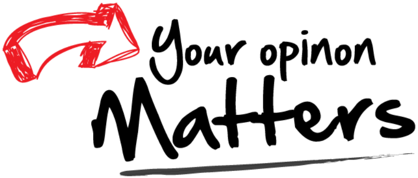
$$\forall y \in A \; \exists x \in A : y = f(x)$$

$$\forall y, \text{choose } x : (g : g(y) = x)$$

g is bijective.

$$f(g(y)) = f(x) = y \implies f = g^{-1}$$


Thank
You!



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