3-6 Decompositions of Graphs

(DFS, Cycle, DAG, Toposort, SCC)

Hengfeng Wei

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October 29, 2018

Robert Tarjan

John Hopcroft

"For fundamental achievements
in the design and analysis of algorithms and data structures."

— Turing Award, 1986

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DEPTH-FIRST SEARCH AND LINEAR GRAPH ALGORITHMS*

ROBERT TARJAN†

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Key words. Algorithm, backtracking, biconnectivity, connectivity, depth-first, graph, search, spanning tree, strong-connectivity.

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"DFS is a powerful technique with many applications."

The Hammer of DFS

Power of DFS:

Graph Traversal $\Longrightarrow$ Graph Decomposition

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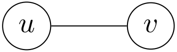
Structure! Structure! Structure!

Graph *structure* induced by DFS:

states of 

types of 

Graph *structure* induced by DFS:

states of v

types of u — v

life time of v:

v : d[v], f[v]

d[v]: Bicomp

f[v]: Toposort, SCC

Definition (Classification of Edges)

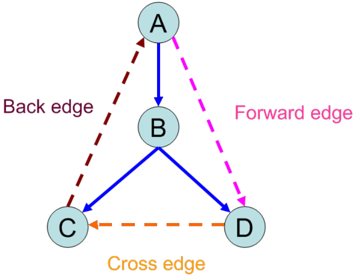
We can define four edge edges in terms of the depth-first forest G_π produced by a DFS on G:

Tree edge: edge in G_π

Back edge: $\rightarrow$ ancestor

Forward edge: $\rightarrow$ descendant (nontree edge)

Cross edge: $\rightarrow$ $(\neg\text{ancestor}) \wedge (\neg\text{descendant})$



DFS on Undirected Graphs (Problem 22.3-6)

Classifying an edge (u, v) as a tree edge or a back edge according to whether (u, v) or (v, u) is encountered first during the depth-first search

is equivalent to

classifying it according to the ordering of the four types in the classification scheme.

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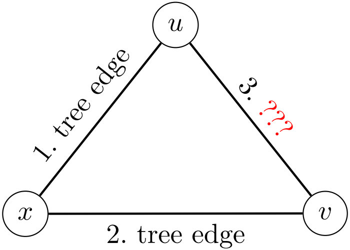

Theorem (Theorem 22.10)

In a depth-first search of an undirected graph G, every edge of G is either a tree edge or a back edge.

Proof.

Let (u, v) be an arbitrary edge of G, and suppose without loss of generality that $u.d < v.d$. Then the search must discover and finish v before it finishes u (while u is gray), since v is on u's adjacency list.

If the first time that the search explores edge (u, v), it is in the direction from u to v, then v is undiscovered (white) until that time, for otherwise the search would have explored this edge already in the direction from v to u. Thus, (u, v) becomes a tree edge.

If the search explores (u, v) first in the direction from v to u, then (u, v) is a back edge, since u is still gray at the time the edge is first explored. □

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"First Types"	*vs.*	"First Time"
tree edge	$\iff$	tree edge
back edge	$\iff$	back edge

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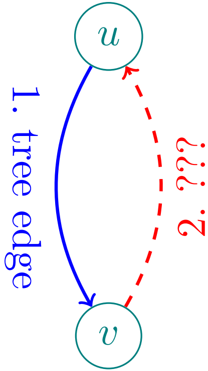

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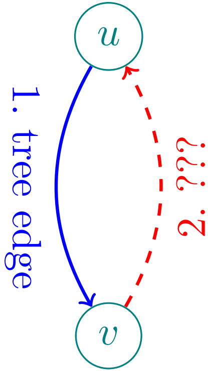

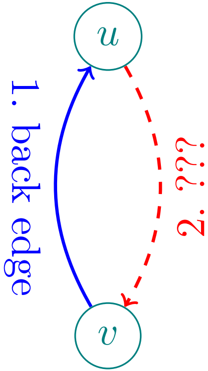

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Theorem (Disjoint or Contained)

$$\forall u, v : [_u\ ]_u \cap [_v\ ]_v = \emptyset \bigvee \left([_u\ ]_u \subset [_v\ ]_v \vee [_v\ ]_v \subset [_u\ ]_u \right)$$

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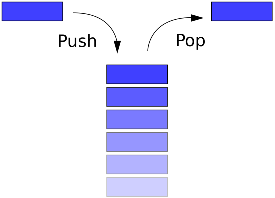

□

Edge Types and Life Time of Vertices in DFS

$$\forall u \to v :$$

- tree/forward edge: $[_u \ [_v \ ]_v \ ]_u$
- back edge: $[_v \ [_u \ ]_u \ ]_v$
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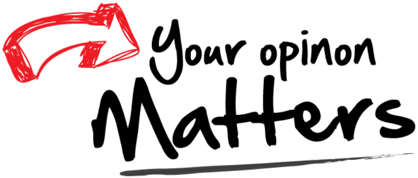
$$\nexists \text{ cycle} \implies \boxed{u \to v \iff \text{f}[v] < \text{f}[u]}$$


THANK YOU



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