2-5 Recursion

魏恒峰

hfwei@nju.edu.cn

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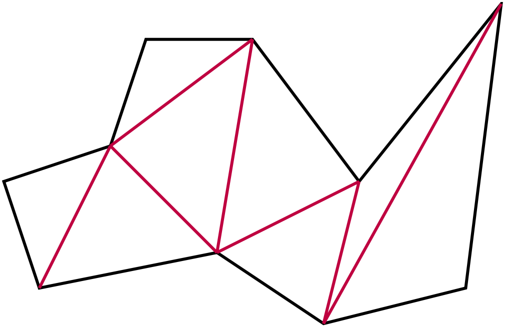
Triangulating Polygons



The Art Gallery Problem

Q : How many "BIG BROs" to hire?

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A triangulated polygon is either a triangle with three ears or has at least two ears (which are *not necessarily non-adjacent*).

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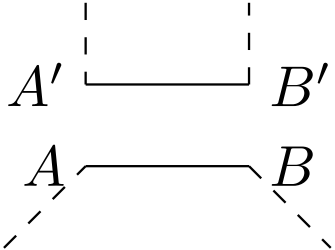

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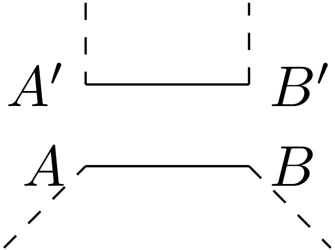
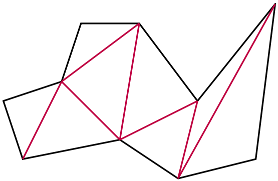


Q : *Adjacent ears?*

of triangles (Problem $4.1 - 17$)



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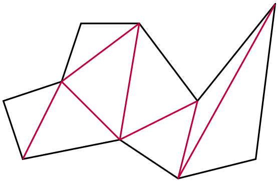

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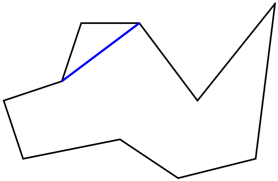

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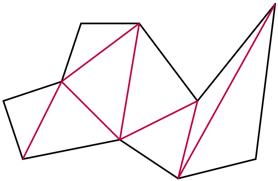

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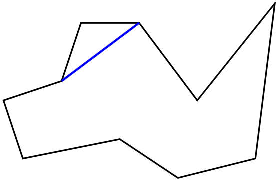

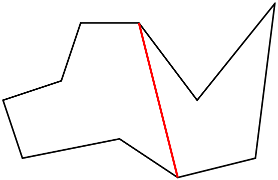

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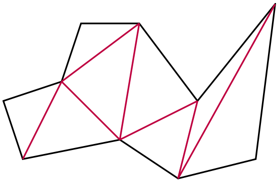

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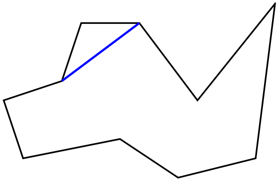

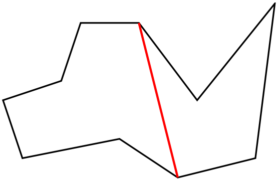

Q : Existence of diagonals?

Lemma (Ear Lemma)

A triangle has 3 ears, and a larger triangulated polygon has at least 2 non-adjacent ears.

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Q : Can every polygon be triangulated?

Theorem (Existence of Triangulation)

Any polygon can be triangulated.

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Theorem (Existence of Diagonal)

Every polygon with $n > 3$ has a diagonal.

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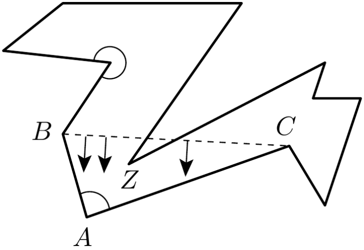
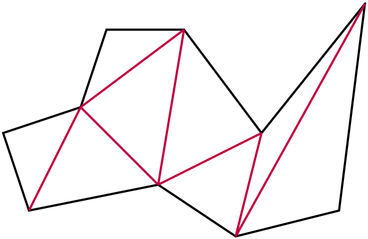


Theorem (Coloring)

Any triangulated polygon polygon is 3*-colorable.*



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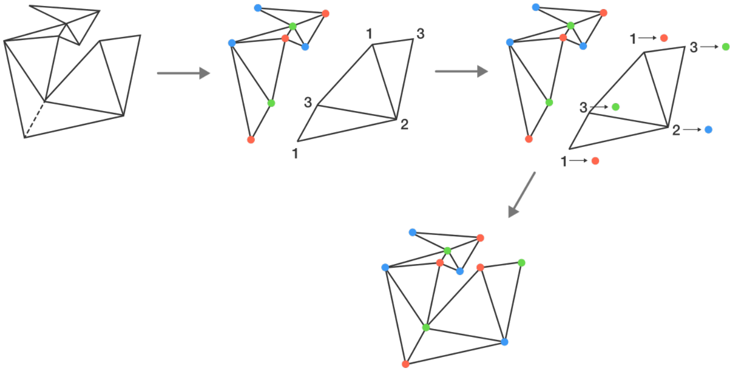

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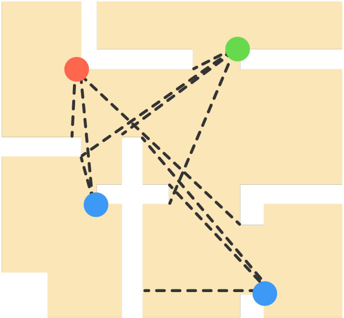

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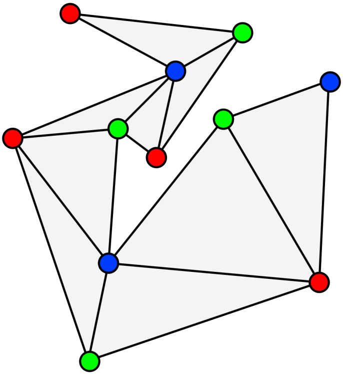

Theorem (The Art Gallery Theorem (O))

For any art gallery with n walls, $\lfloor \frac{n}{3} \rfloor$ "BIG BROs" suffice.

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There exists an art gallery with n walls such that $\lfloor \frac{n}{3} \rfloor$ "BIG BROs" are necessary.

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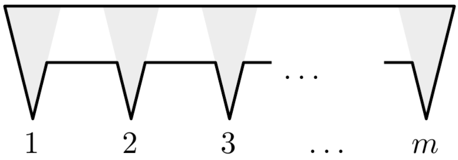

$$n = 3m$$

"How to Guard a Museum?"

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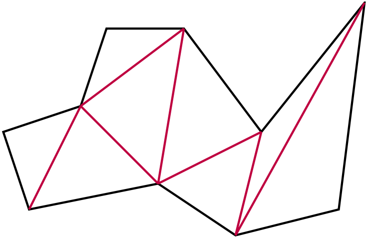

$$O(n \log n)$$

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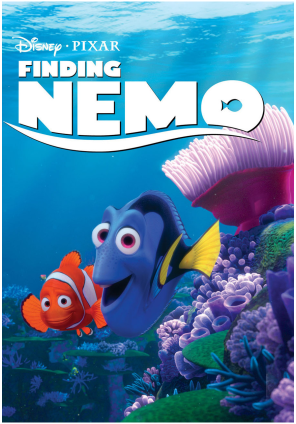
$$O(n)$$

Fish Recurrence



Fish Recurrence (Problem $4.2 - 8$)

At the end of each year, a state fish hatchery puts 2000 fish into a lake.

The number of fish in the lake at the beginning of the year doubles by the end of the year due to reproduction.

Give a recurrence for the number of fish in the lake after n years, and solve the recurrence.

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Solving Recurrence

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Not Exactly!


走馬看花

First-order Linear Recurrence (CS $4.2 - 11$)

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$$\boxed{T(n) = r^n a + \sum_{i=1}^{n} r^{n-i} g(i)}$$

Theorem (First-order Linear Recurrences)

$$T(n) = x_n T(n-1) + y_n \quad \textit{for } n > 0 \textit{ with } T(0) = 0$$

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Proof.

$$\underbrace{\frac{T(n)}{x_n x_{n-1} \cdots x_1}}_{\text{summation factor}} = \frac{T(n-1)}{x_{n-1} \cdots x_1} + \frac{y_n}{x_n x_{n-1} \cdots x_1}$$

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$$S(n) \triangleq \frac{T(n)}{x_n x_{n-1} \cdots x_1}$$

□

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$$\boxed{T(n) = 2(n+1)(H_{n+1} - \frac{3}{2})}$$

After-class Exercise

$$T(n) = T(n-1) - \frac{2T(n-1)}{n} + 2\left(1 - \frac{2T(n-1)}{n}\right), n > 0 \text{ with } T(0) = 0$$

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Theorem (Linear Recurrences with Constant Coefficients)

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_t a_{n-t} \quad \textit{for } n \geq t$$

$$a_0, a_1, \cdots, a_{t-1}$$

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$$\beta_1\ (m_1), \beta_2\ (m_2), \cdots, \beta_i\ (m_i), \cdots, \beta_k\ (m_k)$$

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$$\boxed{a_n = \sum_{0 \leq j < m_1} c_{1j} n^j \beta_1^n + \sum_{0 \leq j < m_2} c_{2j} n^j \beta_2^n + \cdots + \sum_{0 \leq j < m_k} c_{kj} n^j \beta_k^n}$$

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$$\beta^{n-t} \left((n-t)\, q(\beta) + \beta q'(\beta) \right) = 0$$

$\square$

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$$1,\ 0,\ -1,\ 0,\ 1,\ 0,\ -1,\ 0,\ 1,\ 0,\ -1,\ 0 \cdots$$

After-class Exercise

To give initial conditions a_0, a_1, and a_2 such that the growth rate of the solution to

$$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3},\ n > 2$$

is *(1) constant*; *(2) exponential*; *(3) fluctuating in sign*.

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First-order Linear Non-homogeneous Recurrences

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_t a_{n-t} + r \quad \text{for } n \geq t$$

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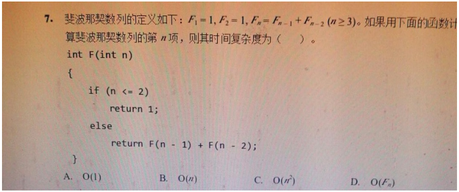

7. 斐波那契数列的定义如下：$F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}\ (n \geq 3)$。如果用下面的函数计算斐波那契数列的第 n 项，则其时间复杂度为（　）。

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int F(int n)
{
    if (n <= 2)
        return 1;
    else
        return F(n - 1) + F(n - 2);
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A. $O(1)$　B. $O(n)$　C. $O(n^2)$　D. $O(F_n)$

$$F(n) = F(n-1) + F(n-2) + 2,\ n \geq 3\ (F(1) = F(2) = 0)$$

First-order Linear Non-homogeneous Recurrences

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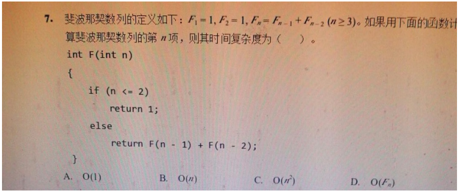

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$$F(n) = F(n-1) + F(n-2) + 2,\ n \geq 3\ (F(1) = F(2) = 0)$$

$$\boxed{T(n,k) = \begin{cases} 0, & k = 0 \vee n = k \\ T(n-1,k) + T(n-1,k-1) + c, & \text{o.w.} \end{cases}}$$

$$a_n = 5a_{n-1} - 6a_{n-2} + 2,\ n \geq 2\ (a_0 = 0, a_1 = 1)$$

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$$\begin{aligned} a_0 &= 0 = c_0 + c_1 + 1 \\ a_1 &= 1 = 3c_0 + 2c_1 + 1 \end{aligned}$$

$$\boxed{a_n = 2 \cdot 3^n - 3 \cdot 2^n + 1}$$

More Issues about Linear Recurrences

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_t a_{n-t} + g(n) \quad \text{for } n \geq t$$

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$$a_n = a_n^g + a_n^p$$

How to Find a Particular Solution for a Non-homogeneous Recurrence Relation?

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_t a_{n-t} \quad \text{for } n \geq t$$

$$t \geq 5$$

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Generating Functions and Asymptotic Analysis

$$a_n = f_1(n)a_{n-1} + f_2(n)a_{n-2} + \cdots + f_t(n)a_{n-t} \quad \text{for } n \geq t$$

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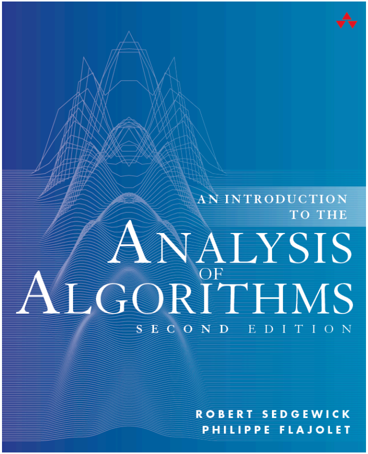
Genearating Functions


AN INTRODUCTION TO THE
ANALYSIS OF ALGORITHMS
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ROBERT SEDGEWICK
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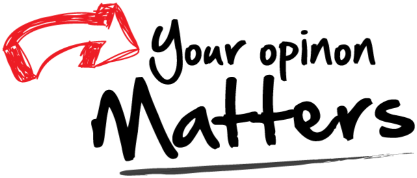
Thank
You!



Office 302

Mailbox: H016

hfwei@nju.edu.cn