3-11 Matchings and Factors

(Part I: Matchings and Covers)

Hengfeng Wei

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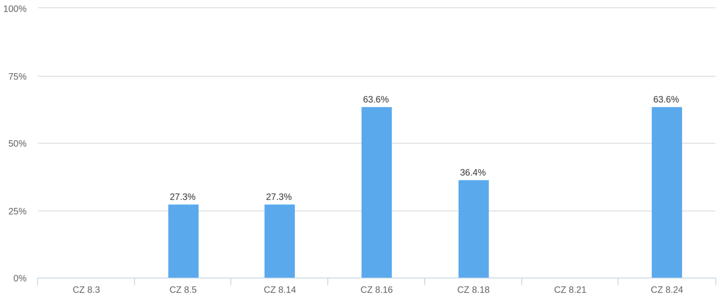
December 10, 2018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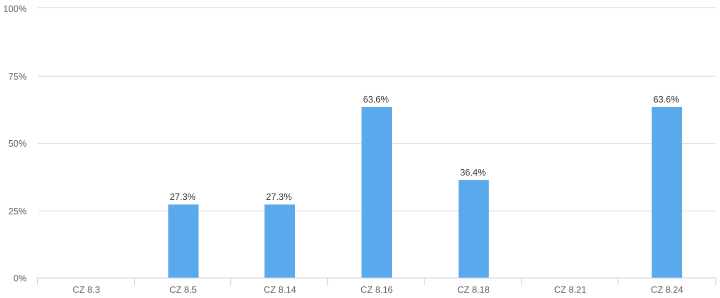
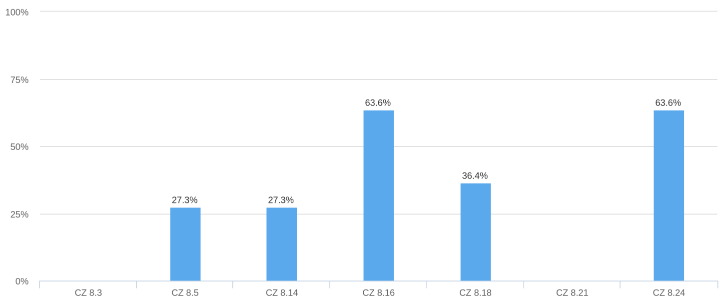


8.5 8.14 8.16

8.18 8.24 (Next Class)



8.5 8.14 8.16 Chinese Postman Problem (Next Class?)

8.18 8.24 (Next Class)

Theorem (Hall's Theorem, 1935; Theorem 8.3)

Let G be a bipartite graph with partite sets U and W such that $r = |U| \leq |W|$.
G contains a matching of cardinality $r \iff G$ satisfies Hall's Condition:

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Other TONCAS?

Perfect Matching on Trees (Problem 8.5)

Prove that every tree T has ≤ 1 perfect matching.

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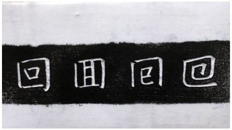

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    Consider T − r                      ▷ r : root of G
    if k_o(T − r) > 1 then

    else                                ▷ k_o(T − r) = 1

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        By Induction Hypothesis.

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    v must be matched with its parent u

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    By Induction Hypothesis on each component of $G - \{u, v\}$

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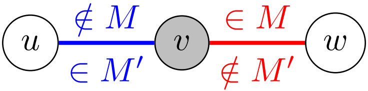

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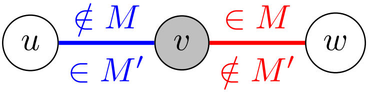

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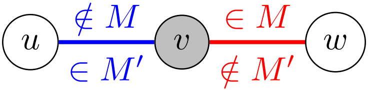

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Contradiction: Cycle

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Consider the subgraph $\mathcal{M}$ with $V(T)$ and $M \Delta M'$.

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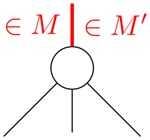
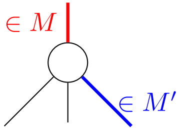


Case I



Case II

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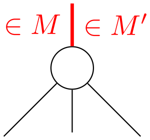

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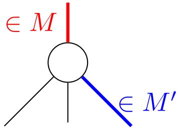

Case II

$$\forall v \in V(\mathcal{M}) :$$

$$\deg(v) = 0 \vee \deg(v) = 2$$

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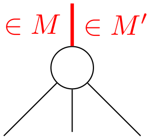

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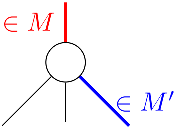

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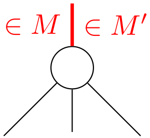

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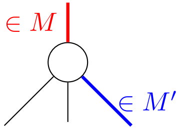

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$$\alpha(G)$$

$$\beta(G)$$

$$\alpha'(G)$$

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$\alpha(G)$	Maximum size of independent set
$\beta(G)$	Minimum size of vertex cover
$\alpha'(G)$	Maximum size of matching
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Theorem (Callai Identities, 1959; Theorem 8.7)

If G is graph without isolated vertices, then

$$\alpha'(G) + \beta'(G) = n(G).$$

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Theorem (Callai Identities, 1959; Theorem 8.8)

If G is graph without isolated vertices, then

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Matching and Edge Cover (Problem 8.14)

A graph G without isolated vertices has a perfect matching if and only if $\alpha'(G) = \beta'(G)$.

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$\alpha(G)$	Maximum size of independent set
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Theorem (König, 1931; Egerváry, 1931)

If G is a bipartite graph, then

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Theorem (König, 1931)

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Vertex Covering Number (Problem 8.16)

If G is a graph of order n, maximum degree Δ and having no isolated vertices, then

$$\beta(G) \geq \frac{n}{\Delta + 1}.$$

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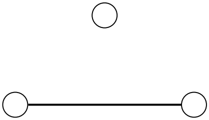

$$n = 3, \Delta = 1, \frac{n}{\Delta + 1} = \frac{3}{2}, \beta = 1$$

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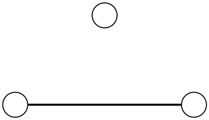

By Contradiction: $\beta < \frac{n}{\Delta+1}$.

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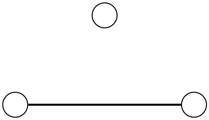

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By Contradiction: $\beta < \frac{n}{\Delta+1}$.

$$\begin{aligned} \beta \cdot \Delta &< \frac{n\Delta}{\Delta + 1} \\ &= n - \frac{n}{\Delta + 1} \\ &\leq n - 1 \end{aligned}$$

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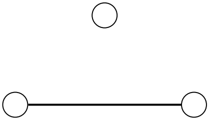

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Contradiction: No isolated vertices.

Vertex Independence Number (Additional Problem)

If G is a graph of order n, maximum degree Δ, then

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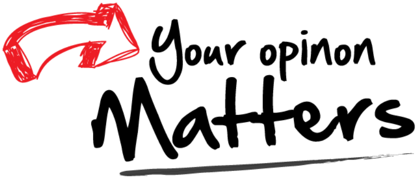
By Construction.

To construct an independent set S with $|S| \geq \frac{n}{\Delta+1}$.

1: **while** $|V(G) > 0|$ **do**
2: Choose $v \in V(G)$
3: $S \leftarrow S \cup \{v\}$
4: $G \leftarrow G - \{v\} - N(v)$


THANK YOU



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