Direct Products and Quotient Groups

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What do you mean by “是一回事”?

Theorem

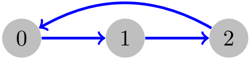
$$\textit{If } G = H \times K,$$
$$\textit{then } \exists H' \cong H, K' \cong K,$$
$$\textit{such that } G \textit{ is the internal direct product of } H \textit{ and } K.$$

Theorem

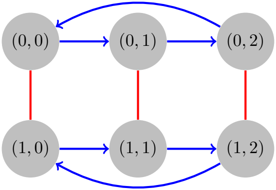
$$\textit{If } G \textit{ is the internal direct product of } H \textit{ and } K,$$
$$\textit{then } G \cong H \times K.$$

$$\mathbb{Z}_2 \times \mathbb{Z}_3 = G \cong \mathbb{Z}_6$$

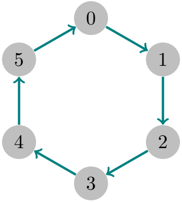
$\mathbb{Z}_2$



$\mathbb{Z}_3$



$\mathbb{Z}_2 \times \mathbb{Z}_3$



$\mathbb{Z}_6$

$$G = H \times K$$

Definition

$$H' = H \times \{e_K\} = \{(h', e_K) : h' \in H)\} \cong H$$
$$K' = \{e_H\} \times K = \{(e_H, k') : k' \in K)\} \cong K$$

$$H' \leq G, \quad K' \leq G$$

$$G = H'K'$$

$$H' \cap K' = \{(e_H, e_K)\} = \{e\}$$

H' and K' commute.

Theorem

$$\textit{If } G = H \times K,$$

$$\textit{then } \exists H' \cong H, K' \cong K,$$

such that G is the internal direct product of H' and K'.

Definition (Internal Direct Product)

Let G be a group with subgroups H and K satisfying

$$G = HK$$

$$H \cap K = \{e\}$$

$$H \text{ and } K \text{ commute.}$$

Then, G is the internal direct product of H and K.

$$G = H \times K$$

Definition

$$H' = H \times \{e_K\} = \{(h', e_K) : h' \in H)\} \cong H$$

$$K' = \{e_H\} \times K = \{(e_H, k') : k' \in K)\} \cong K$$

$$H' \leq G, \quad K' \leq G H' \lhd G, \quad K' \lhd G$$

$$G = H'K'$$

$$H' \cap K' = \{e\}$$

H' and K' commute.

Definition (Internal Direct Product)

Let G be a group with subgroups H and K satisfying

$$G = HK$$

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$$H \text{ and } K \text{ commute}$$

Then, G is the internal direct product of H and K.

Definition (Internal Direct Product (Equivalent))

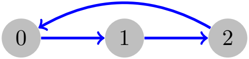
Let G be a group with normal subgroups H and K satisfying

$$G = HK$$

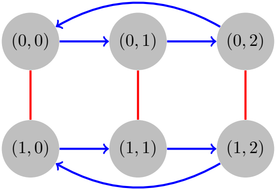
$$H \cap K = \{e\}$$

Then, G is the internal direct product of H and K.

$\mathbb{Z}_2$

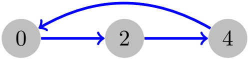


$\mathbb{Z}_3$

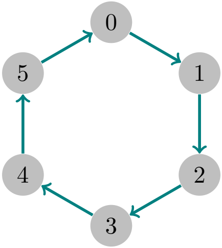


$\mathbb{Z}_2 \times \mathbb{Z}_3$

$\mathbb{Z}_2 \cong \{0, 3\} \leq \mathbb{Z}_6$

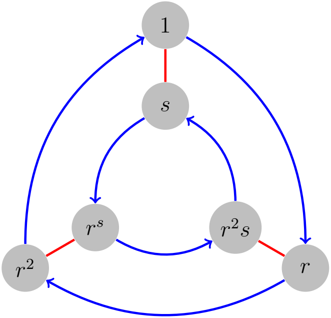


$\mathbb{Z}_3 \cong \{0, 2, 4\} \leq \mathbb{Z}_6$

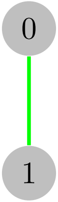


$\mathbb{Z}_6$

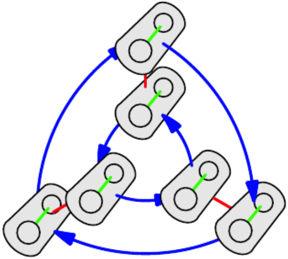
$$D_6 \cong D_3 \times \mathbb{Z}_2$$

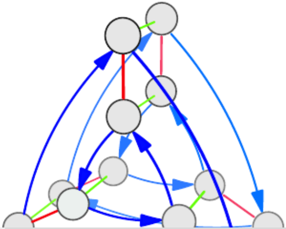


D_3



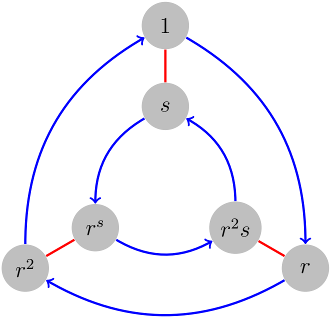
$\mathbb{Z}_2$



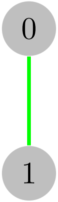


$$D_6 \cong D_3 \times \mathbb{Z}_2$$

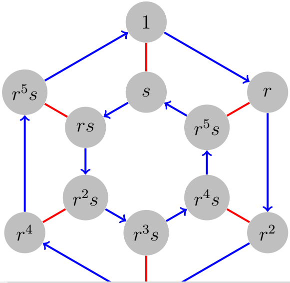
$$D_6 = D_3' \mathbb{Z}_2' \quad (D_3' \lhd D_6,\ \mathbb{Z}_2' \lhd D_6)$$

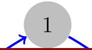


D_3

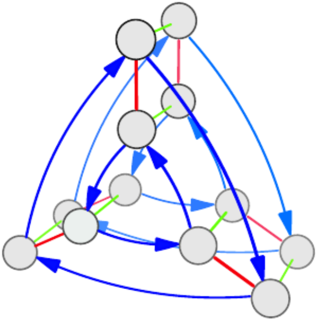


$\mathbb{Z}_2$

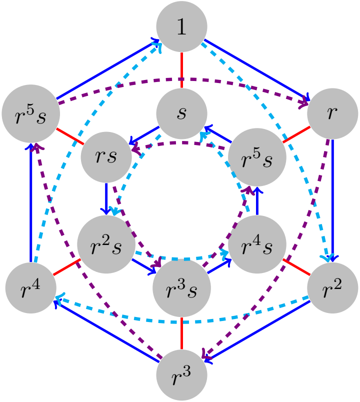




$$D_6 \cong D_3 \times \mathbb{Z}_2$$



$D_3 \times \mathbb{Z}_2$



D_6

$$D_{2n} \cong D_n \times \mathbb{Z}_2 \quad (n \text{ is odd})$$

$$\mathbb{Z}_2 \cong (\mathbb{Z}_2' \triangleq \{1, r^n\}) \lhd D_{2n}$$

$$D_n \cong (D_n' \triangleq \langle r^2, s \rangle) \lhd D_{2n}$$

D_n is the internal direct product of $\mathbb{Z}_2'$ and D_n'.

Definition (Internal Direct Product (Equivalent))

Let G be a group with normal subgroups H and K satisfying

$$G = HK$$

$$H \cap K = \{e\}$$

Then, G is the internal direct product of H and K.

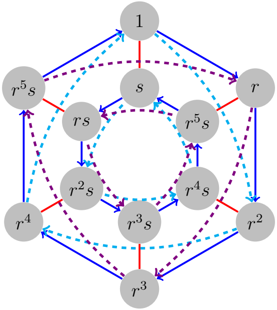
Theorem (The Second Isomorphism Theorem (Diamond Theorem))

$$H \le G, N \lhd G \Longrightarrow H/(H \cap N) \cong HN/N.$$

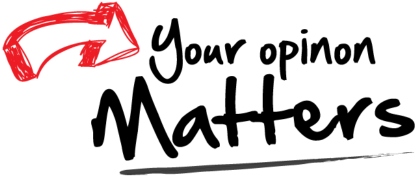
Theorem

If G is the internal direct product of its normal subgroups H and K, then $G/H \cong K, \quad G/K \cong H$.

$$D_6 = \langle r^2, s\rangle \left\{1, r^3\right\} \implies D_6/\langle r^2, s\rangle \cong \left\{1, r^3\right\} \cong \mathbb{Z}_2$$




THANK YOU



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