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Geometry and Groups

The Dihedral Group: Consider a regular n-gon. Then rotating it by a multiple of $2\pi/n$ leaves it unchanged, as does a reflection through any one of its axes of symmetry. Thus if a represents rotation by $2\pi/n$ and c represents reflection through one of its axes of symmetry, then all the symmetry-preserving rotations and reflections (alternatively, reflections can be replaced by rotations in 3D) can be generated using a, c, with defining relations $a^n = c^2 = (ac)^2 = 1$. The last relation can be seen by realizing $ac = ca^{-1}$. This group is called the 'dihedral group ' of order $2n$.

The Tetrahedral Group: Consider a tetradhedron that is free to rotate about its center. Any one of the four vertices can be brought to the position of any other, and then there are three configurations the other vertices can take. Thus there are $4 \times 3 = 12$ operations. Note if one vertex is fixed, the other three can only be rotated cyclically, thus the tetrahedral group contains all possible 3-cycles, hence it contains A_4. But since its order is the same as that of A_4, the tetradhedral group must be A_4.

The Octahedral/Hexahedral Group: Note the centers of the faces of an octahedron can be thought of as the vertices of a cube, and conversely.

For a cube, we may rotate any given vertex to the position of any of the eight vertices, and then choose one of three rotations (the edges that the given vertex belong to can take one of three positions), hence there are 24 operations in all. Now consider the four diagonals of the cube, which are permuted amongst themselves. Note that two distinct rotations of the octahedral group correspond to two distinct permutations of the four diagonals because no rotation except the identiy can map all four diagonals into themselves, thus the octahedral group is precisely S_4.

The Icosahedral/Dodecahedral Group: Again, the centers of the faces of one of these solids can be viewed as the vertices of the other.

Given a dodecahedron, we can rotate any one of its vertices to the position of any one of the twenty vertices, and once there, we can choose among three rotations, so there are 60 distinct rotations.

In Euclid’s Elements (Book XIII, Proposition 17) a dodecahedron is derived from a cube such that each of the twelve edges of the cube is a diagonal in one of the faces of the dodecahedron. Conversely, starting with a given diagonal of a dodecahedron, a unique cube can be constructed with its edges being the diagonals of the dodecahedron, with one of the edges being the chosen diagonal. Each face has five diagonals, so there are exactly five cubes that can be constructed in this manner. Now two distinct permutations of these five cubes correspond to distinct rotations, because a little thought shows that only the identity will leave the five cubes in place, thus the dodecahedral group is isomorphic to some subgroup of S_5. But we shall see the only subgroup of S_5 of order 60 is A_5, so this must be the answer.

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Notes
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