Geometric Programming

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Outline

- Standard GP and Convex GP
- Convexity of LogSumExp
- Generalized GP
- Dual GP
- Example

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Monomials and Posynomials

Monomial as a function $f : \mathbf{R}_+^n \to \mathbf{R}$:

$$f(x) = d x_1^{a^{(1)}} x_2^{a^{(2)}} \dots x_n^{a^{(n)}}$$

where the multiplicative constant $d \geq 0$ and the exponential constants $a^{(j)} \in \mathbf{R}, j = 1, 2, \dots, n$

Sum of monomials is called a posynomial:

$$f(x) = \sum_{k=1}^{K} d_k x_1^{a_k^{(1)}} x_2^{a_k^{(2)}} \dots x_n^{a_k^{(n)}}.$$

where $d_k \geq 0,\ k = 1, 2, \dots, K$, and $a_k^{(j)} \in \mathbf{R},\ j = 1, 2, \dots, n, k = 1, 2, \dots, K$

Example: $\sqrt{2} x^{-0.5} y^{\pi} z$ is a monomial, $x - y$ is not a posynomial

Standard GP & Convex GP

- GP standard form with variables x:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 1, \quad i = 1, 2, \ldots, m, \\ & h_l(x) = 1, \quad l = 1, 2, \ldots, M \end{array}$$

where $f_i, i = 0, 1, \ldots, m$ are posynomials and $h_l, l = 1, 2, \ldots, M$ are monomials

Log transformation: $y_j = \log x_j, b_{ik} = \log d_{ik}, b_l = \log d_l$

- GP convex form with variables y:

$$\begin{array}{ll} \text{minimize} & p_0(y) = \log \sum_{k=1}^{K_0} \exp(a_{0k}^T y + b_{0k}) \\ \text{subject to} & p_i(y) = \log \sum_{k=1}^{K_i} \exp(a_{ik}^T y + b_{ik}) \leq 0, \quad i = 1, 2, \ldots, m, \\ & q_l(y) = a_l^T y + b_l = 0, \quad l = 1, 2, \ldots, M \end{array}$$

In convex form, GP with only monomials reduces to LP

Convexity of LogSumExp

Log sum inequality (readily proved by the convexity of $f(t) = t \log t, t \geq 0$):

$$\sum_{i=1}^{n} a_i \log \frac{a_i}{b_i} \geq \left(\sum_{i=1}^{n} a_i \right) \log \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}$$

where $a_i, b_i \in \mathbf{R}_+, i = 1, 2, \ldots, n$

Let $\hat{b}_i = \log b_i$ and $\sum_{i=1}^{n} a_i = 1$:

$$\log \left(\sum_{i=1}^{n} e^{\hat{b}_i} \right) \geq a^T \hat{b} - \sum_{i=1}^{n} a_i \log a_i$$

So LogSumExp is the conjugate function of negative entropy

Since all conjugate functions are convex, LogSumExp is convex

GP and Convexity

- The following problem can be turned into an equivalent standard GP:

$$
\begin{array}{ll}
\text{maximize} & x/y \\
\text{subject to} & 2 \leq x \leq 3 \\
& x^2 + 3y/z \leq \sqrt{y} \\
& x/y = z^2
\end{array}
$$

$$
\begin{array}{ll}
\text{minimize} & x^{-1}y \\
\text{subject to} & 2x^{-1} \leq 1, \quad (1/3)x \leq 1 \\
& x^2 y^{-1/2} + 3y^{1/2} z^{-1} \leq 1 \\
& xy^{-1}z^{-2} = 1
\end{array}
$$

- Let p, q be posynomials and r monomial

$$
\begin{array}{ll}
\text{minimize} & p(x)/(r(x) - q(x)) \\
\text{subject to} & r(x) > q(x)
\end{array}
$$

which is equivalent to

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & p(x) \leq t(r(x) - q(x)) \\ & (q(x)/r(x)) < 1 \end{array}$$

which is in turn equivalent to

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & (p(x)/t + q(x))/r(x) \leq 1 \\ & (q(x)/r(x)) < 1 \end{array}$$

- Generalized posynomials: f is a generalized posynomial if it can be formed using addition, multiplication, positive power, and maximum, starting from posynomials. Composition of posynomials.

- Generalized GP: minimize generalized posynomials over upper bound inequality constraints on other generalized posynomials
  Generalized GP can be turned into equivalent standard GP

Generalized GP

- Rule 1: Composing posynomials $\{f_{ij}(\mathbf{x})\}$ with a posynomial with nonnegative exponents $\{a_{ij}\}$ is a generalized posynomial

- Rule 2: The maximum of a finite number of posynomials is also a generalized posynomial

- Rule 3: f_1 and f_2 are posynomials and h is a monomial: $F_3(\mathbf{x}) = \frac{f_1(\mathbf{x})}{h(\mathbf{x})-f_2(\mathbf{x})}$

Example:

$$\begin{array}{ll} \text{maximize} & \max\{(x_1 + x_2^{-1})^{0.5},\, x_1 x_3\} + (x_2 + x_3^{-2.9})^{1.5} \\ \text{subject to} & \frac{(x_2 x_3 + x_2/x_1)^{\pi}}{x_1 x_2 - \max\{x_1^2 x_3^3, x_1 + x_3\}} \leq 10, \\ \text{variables} & x_1, x_2, x_3. \end{array}$$

Freely available software: Stanford CVX & GGPLAB
(`http://www.stanford.edu/~boyd/ggplab`)

Unconstrained GP

minimize

$$f(x) = \log\left(\sum_{i=1}^{m} \exp(a_i^T x + b_i)\right)$$

Optimality condition has no analytic solution:

$$\nabla f(x^*) = \frac{1}{\sum_{j=1}^{m} \exp(a_j^T x^* + b_j)} \sum_{i=1}^{m} \exp(a_i^T x^* + b_i) a_i = 0$$

Dual GP

Primal problem: unconstrained GP in variables y

$$\text{minimize} \quad \log \sum_{i=1}^{N} \exp(a_i^T y + b_i).$$

Lagrange dual in variables ν:

$$\begin{array}{ll} \text{maximize} & b^T\nu - \sum_{i=1}^N \nu_i \log \nu_i \\ \text{subject to} & \mathbf{1}^T\nu = 1, \\ & \nu \succeq 0, \\ & A^T\nu = 0 \end{array}$$

Dual GP

Primal problem: General GP in variables y

$$
\begin{array}{ll}
\text{minimize} & \log \sum_{j=1}^{k_0} \exp(a_{0j}^T y + b_{0j}) \\
\text{subject to} & \log \sum_{j=1}^{k_i} \exp(a_{ij}^T y + b_{ij}) \leq 0, \quad i = 1, \ldots, m,
\end{array}
$$

Lagrange dual problem:

$$
\begin{array}{ll}
\text{maximize} & b_0^T \nu_0 - \sum_{j=1}^{k_0} \nu_{0j} \log \nu_{0j} + \sum_{i=1}^{m} \left(b_i^T \nu_i - \sum_{j=1}^{k_i} \nu_{ij} \log \frac{\nu_{ij}}{\mathbf{1}^T \nu_i} \right) \\
\text{subject to} & \nu_i \succeq 0, \quad i = 0, \ldots, m, \\
& \mathbf{1}^T \nu_0 = 1, \\
& \sum_{i=0}^{m} A_i^T \nu_i = 0
\end{array}
$$

where variables are $\nu_i, \quad i = 0, 1, \ldots, m$

A_0 is the matrix of the exponential constants in the objective function, and $A_i, \quad i = 1, 2, \ldots, m$ are the matrices of the exponential constants in the constraint functions

Example: DMC Capacity Problem

Discrete Memoryless Channel (DMC): $x \in \mathbf{R}^n$ is distribution of input; $y \in \mathbf{R}^m$ is distribution of output;

$P \in \mathbf{R}^{m \times n}$ gives conditional probabilities: $y = Px$

Primal Channel Capacity Problem:

$$\begin{array}{ll} \text{maximize} & -c^T x - \sum_{i=1}^m y_i \log y_i \\ \text{subject to} & x \geq 0,\ \mathbf{1}^T x = 1,\ y = Px, \end{array}$$

where $c_j = -\sum_{i=1}^m p_{ij} \log p_{ij}$

Dual Channel Capacity Problem is a simple GP:

$$\begin{array}{ll} \text{minimize} & \log \sum_{i=1}^m e^{u_i} \\ \text{subject to} & c + P^T u \geq 0, \end{array}$$

Properties of GP

- Nonlinear nonconvex problem can be turned into nonlinear convex problem

- Linearly constrained dual problem

- Theoretical structures: global optimality, zero duality gap, KKT condition, sensitivity analysis

- Numerical efficiency: interior-point, robust

- Surprisingly wide range of applications

Summary

- Nonlinearity: Posynomial or LogSumExp
- Standard GP, Convex GP and Dual GP
- A variety of problems: Structural design in mechanical engineering, Growth modeling in economics (1960s-1970s), Analog and digital circuit design (late 1990s), Communication system problems (early 2000s) and many other applications in practice and analysis

Reading assignment: Sections 4.5, 5.7 of textbook.

- S. Boyd, S.-J. Kim, L. Vandenberghe, and A. Hassibi, "A Tutorial on Geometric Programming," Optimization and Engineering, 8(1):67-127, 2007.
- M. Chiang, "Geometric programming for communication systems," Foundations and Trends in Communications and Information Theory, 2005.