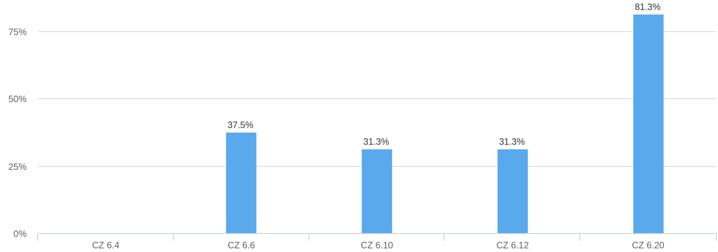
3-10 Traversability

(Part II: Hamiltonian Graphs)

Hengfeng Wei

hfwei@nju.edu.cn

December 10, 2018



CZ 6.20

Theorem (Necessary Condition; Theorem 6.5)

If G is a Hamiltonian graph, then for each nonempty set $S \subset V(G)$,

$$k(G - S) \leq |S|.$$

$\Longrightarrow$ Hamiltonian graphs are 2-connected. (Why?)

Hamiltonian graphs has "good" connectedness.

Graphs with "bad" connectedness are not Hamiltonian.

Theorem (Ore's Theorem, 1960; Theorem 6.6)

Let G be a graph of order $n \geq 3$. If

$$deg(u) + deg(v) \geq n$$

for each pair u, v of nonadjacent vertices of G, then G is Hamiltonian.

Proof.

By Contradiction and Extremality.

By Contradiction:

$\exists G$ satisfying Ore's Condition, but G is not Hamiltonian.

By Extremality:

Consider a critical G: G is not Hamiltonian but $G + uv$ is Hamiltonian.

Contradiction: This critical G is actually Hamiltonian.

□

Theorem (Dirac's Theorem, 1952; Corollary 6.7)

Let G be a graph of order $n \geq 3$. If

$$\forall v \in V(G) : deg(v) \geq n/2,$$

then G is Hamiltonian.

Theorem (Ore's Theorem, 1960; Theorem 6.8)

Let u and v be nonadjacent vertices in a graph G of order n such that

$$deg(u) + deg(v) \geq n.$$

Then $G + uv$ is Hamiltonian $\iff$ G is Hamiltonian.

Definition (Closure $C(G)$)

The closure $C(G)$ of a graph G is the graph obtained from G by iteratively adding edges joining pairs of nonadjacent vertices u and v such that $\deg(u) + \deg(v) \geq n$, until no such pair remains.

Theorem (Bondy-Chavátal Theorem, 1976; Theorem 6.9)

G is Hamiltonian $\iff$ $C(G)$ is Hamiltonian.

Corollary (Corollary 6.10)

If G is a graph of order $n \geq 3$ such that $C(G) = K_n$, then G is Hamiltonian.

Theorem (Lajos Pósa)

Let G be a graph of order $n \geq 3$. If for each integer j with $1 \leq j \leq \frac{n}{2}$, the number of vertices of G with degree at most j is less than j, then G is Hamiltonian.

Theorem (Well-definedness of $C(G)$)

$C(G)$ is well-defined.

$C(G)$ does not depend on the order in which we choose to add edges.

$$\boxed{\Big(G + \langle e_1, \cdots, e_r \rangle = G_1\Big) \land \Big(G + \langle f_1, \cdots, f_s \rangle = G_2\Big) \implies G_1 = G_2}$$

$$G_1 \subseteq G_2 \land G_2 \subseteq G_1$$

$$\forall e_i \in G_1 : e_i \in G_2$$

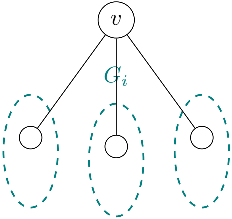
By induction on the order e_i is added to G_1.

Hamiltonian Graphs and 2-Connectedness (Problem 6.20)

Let G be a graph of order $n \geq 3$ having the property that for each $v \in V(G)$, there is a Hamiltonian path with initial vertex v. Show that G is 2-connected but not necessarily Hamiltonian.

2-connected: Connected + No cut-vertex

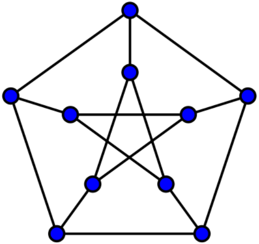
Suppose, by contradiction, v is a cut-vertex of G.



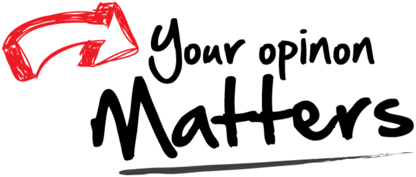
Contradiction: No Hamiltonian path with initial vertex v.

Hamiltonian Graphs and 2-Connectedness (Problem 6.20)

Let G be a graph of order $n \geq 3$ having the property that for each $v \in V(G)$, there is a Hamiltonian path with initial vertex v. Show that G is 2-connected but not necessarily Hamiltonian.




THANK YOU



Office 302

Mailbox: H016

hfwei@nju.edu.cn