2-3 Counting

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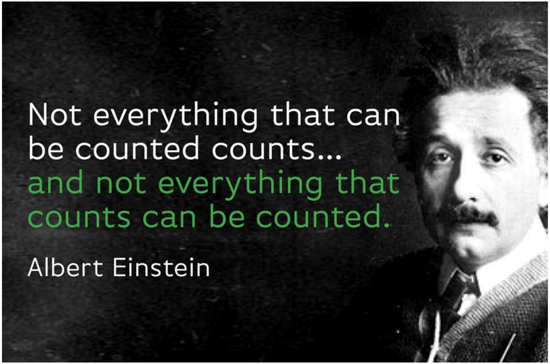
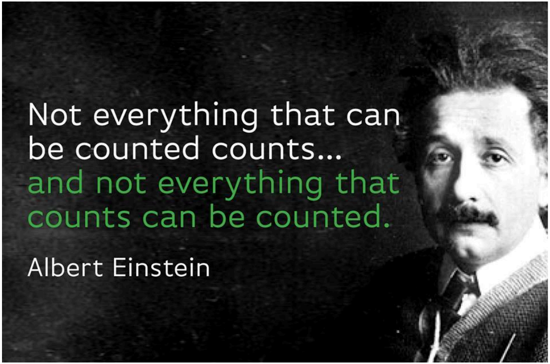
2018 年 03 月 12 日


Not everything that can
be counted counts...
and not everything that
counts can be counted.
Albert Einstein



所以，学好“2-3 组合与计数”是多么重要!

Paring up (CS : $1.2 - 15$)

A tennis club has $2n$ members. We want to pair up the members by twos for singles matches.

(a) In how many ways can we pair up all the members of the club?

(b) Suppose that in addition to specifying who plays whom, we also determine who serves first for each pairing. In how many ways can we specify our pairs?

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$$\binom{2n}{\underbrace{2, 2, \cdots, 2}_{n}} \triangleq \binom{2n}{2}\binom{2n-2}{2}\cdots\binom{4}{2}\binom{2}{2} = \frac{(2n)!}{2^n}$$

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$$\frac{(2n)!}{2^n} \cdot 2^n = (2n)!$$

k-Permutation (CS : $1.2 - 5$)

We need to pass out k distinct pieces of fruit to n children such that *each child may get at most one*.

(a) $k \le n$?

(b) What if $k > n$?

$$n^{\underline{k}}$$

$$0$$

Multisets (CS : $1.5-4$)

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$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

Multisets (CS : $1.5 - 4$)

Use **multisets** to determine the number of ways to pass out k identical apples to n children. Assume that a child may get more than one apple.

$$k = 5 \qquad n = 4 : [1 \cdots 4]$$

balls into bins

$$\{1, 1, 1, 2, 3, 3\}$$

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k-multiset of $[1 \cdots n]$

n-multiset of $[1 \cdots k]$

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Partition (CS : $1.5 - 4$)

Use multisets to determine the number of ways to pass out k identical apples to n-胞胎. Assume that a child may get more than one apple.

Computing $\binom{n}{k}$ (CS $1.5:14$)

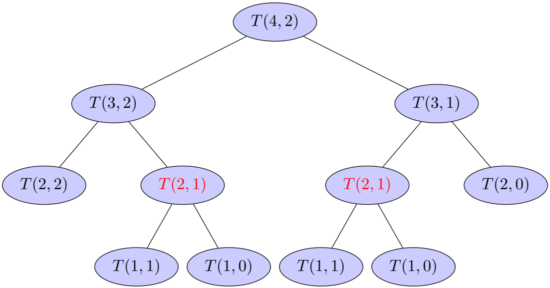
Algorithm 1 Computing $\binom{n}{k}$.

```
procedure BINOM(n, k)                       ▷ Required: n ≥ k ≥ 0
    if k = 0 ∨ n = k then
        return 1
    return BINOM(n − 1, k) + BINOM(n − 1, k − 1)
```

Computing $\binom{n}{k}$ (CS $1.5:14$)

Algorithm 2 Computing $\binom{n}{k}$.

procedure $\textsc{Binom}(n, k)$ ▷ Required: $n \ge k \ge 0$
  if $k = 0 \lor n = k$ **then**
    return 1
  return $\textsc{Binom}(n-1, k) + \textsc{Binom}(n-1, k-1)$



Algorithm 3 Computing $\binom{n}{k}$.

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procedure Binom(n, k)                          ▷ Required: n ≥ k ≥ 0
    if k = 0 ∨ n = k then
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```

Algorithm 4 Computing $\binom{n}{k}$.

procedure $\textsc{Binom}(n, k)$ ▷ Required: $n \ge k \ge 0$

  if $k = 0 \lor n = k$ **then**

    return 1

  return $\textsc{Binom}(n-1, k) + \textsc{Binom}(n-1, k-1)$

(i) $\#$ of "$+$":

Algorithm 5 Computing $\binom{n}{k}$.

```
procedure BINOM(n, k)                          ▷ Required: n ≥ k ≥ 0
    if k = 0 ∨ n = k then
        return 1
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```

(i) # of "+":

$$A(n,k) = 1 + A(n-1,k) + A(n-1,k-1)$$

Algorithm 6 Computing $\binom{n}{k}$.

```
procedure Binom(n, k)                              ▷ Required: n ≥ k ≥ 0
    if k = 0 ∨ n = k then
        return 1
    return Binom(n − 1, k) + Binom(n − 1, k − 1)
```

(i) # of "+":

$$A(n,k) = 1 + A(n-1,k) + A(n-1,k-1)$$

(ii) # of recursive calls of Binom:

Algorithm 7 Computing $\binom{n}{k}$.

```
procedure BINOM(n, k)                    ▷ Required: n ≥ k ≥ 0
    if k = 0 ∨ n = k then
        return 1
    return BINOM(n − 1, k) + BINOM(n − 1, k − 1)
```

(i) # of "+":

$$A(n,k) = 1 + A(n-1,k) + A(n-1,k-1)$$

(ii) # of recursive calls of BINOM:

$$R(n,k) = 2 + R(n-1,k) + R(n-1,k-1)$$

Algorithm 8 Computing $\binom{n}{k}$.

```
procedure BINOM(n, k)                          ▷ Required: n ≥ k ≥ 0
    if k = 0 ∨ n = k then
        return 1
    return BINOM(n − 1, k) + BINOM(n − 1, k − 1)
```

(i) # of "+":

$$A(n,k) = 1 + A(n-1,k) + A(n-1,k-1)$$

(ii) # of recursive calls of $\textsc{Binom}$:

$$R(n,k) = 2 + R(n-1,k) + R(n-1,k-1)$$

$$\boxed{T(n,k) = T(n-1,k) + T(n-1,k-1) + c}$$

$$
\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}
$$

Algorithm 9 Computing $\binom{n}{k}$.

procedure $\textsc{Binom}(n, k)$ ▷ Required: $n \geq k \geq 0$

  for $i \leftarrow 0$ **to** n **do**

    $B[i][0] \leftarrow 1$

    $B[i][i] \leftarrow 1$

  for $i \leftarrow 2$ **to** n **do**

    for $j \leftarrow 1$ **to** k **do**

      $B[n][k] \leftarrow B[n-1][k] + B[n-1][k-1]$

  return $B[n][k]$


Thank You!