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Dedekind-infinite set

In mathematics, a set *A* is **Dedekind-infinite** (named after the German mathematician Richard Dedekind) if some proper subset *B* of *A* is equinumerous to *A*. Explicitly, this means that there is a bijective function from *A* onto some proper subset *B* of *A*. A set is **Dedekind-finite** if it is not Dedekind-infinite. Proposed by Dedekind in 1888, Dedekind-infiniteness was the first definition of "infinite" that did not rely on the definition of the natural numbers.[1]

Until the foundational crisis of mathematics showed the need for a more careful treatment of set theory most mathematicians assumed that a set is infinite if and only if it is Dedekind-infinite. In the early twentieth century, Zermelo–Fraenkel set theory, today the most commonly used form of axiomatic set theory, was proposed as an axiomatic system to formulate a theory of sets free of paradoxes such as Russell's paradox. Using the axioms of Zermelo–Fraenkel set theory with the originally highly controversial axiom of choice included (**ZFC**) one can show that a set is Dedekind-finite if and only if it is finite in the sense of having a finite number of elements. However, there exists a model of Zermelo–Fraenkel set theory without the axiom of choice (**ZF**) in which there exists an infinite, Dedekind-finite set, showing that the axioms of **ZF** are not strong enough to prove that every set that is Dedekind-finite has a finite number of elements.[2][1] There are definitions of finiteness and infiniteness of sets besides the one given by Dedekind that do not depend on the axiom of choice.

A vaguely related notion is that of a **Dedekind-finite ring**. A ring is said to be a Dedekind-finite ring if $ab = 1$ implies $ba = 1$ for any two ring elements *a* and *b*. These rings have also been called **directly finite** rings.

Contents

Comparison with the usual definition of infinite set

This definition of "infinite set" should be compared with the usual definition: a set *A* is infinite when it cannot be put in bijection with a finite ordinal, namely a set of the form $\{0, 1, 2, ..., n-1\}$ for some natural number *n* – an infinite set is one that is literally "not finite", in the sense of bijection.

During the latter half of the 19th century, most mathematicians simply assumed that a set is infinite if and only if it is Dedekind-infinite. However, this equivalence cannot be proved with the axioms of Zermelo–Fraenkel set theory without the axiom of choice (AC) (usually denoted "**ZF**"). The full strength of AC is not needed to prove the equivalence; in fact, the equivalence of the two definitions is strictly weaker than the axiom of countable choice (CC). (See the references below.)

Dedekind-infinite sets in ZF

The following conditions are equivalent in **ZF**. In particular, note that all these conditions can be proved to be equivalent without using the AC.

- A is **Dedekind-infinite**.
- There is a function $f : A \to A$ that is injective but not surjective.
- There is an injective function $f : \mathbf{N} \to A$, where **N** denotes the set of all natural numbers.
- A has a countably infinite subset.

Every Dedekind-infinite set A also satisfies the following condition:

- There is a function $f : A \to A$ that is surjective but not injective.

This is sometimes written as "A is **dually Dedekind-infinite**". It is not provable (in **ZF** without the AC) that dual Dedekind-infinity implies that A is Dedekind-infinite. (For example, if B is an infinite but Dedekind-finite set, and A is the set of finite one-to-one sequences from B, then "drop the last element" is a surjective but not injective function from A to A, yet A is Dedekind finite.)

It can be proved in ZF that every dually Dedekind infinite set satisfies the following (equivalent) conditions:

- There exists a surjective map from A onto a countably infinite set.
- The powerset of A is Dedekind infinite

(Sets satisfying these properties are sometimes called **weakly Dedekind infinite**.)

It can be shown in ZF that weakly Dedekind infinite sets are infinite.

ZF also shows that every well-ordered infinite set is Dedekind infinite.

History

The term is named after the German mathematician Richard Dedekind, who first explicitly introduced the definition. It is notable that this definition was the first definition of "infinite" that did not rely on the definition of the natural numbers (unless one follows Poincaré and regards the notion of number as prior to even the notion of set). Although such a definition was known to Bernard Bolzano, he was prevented from publishing his work in any but the most obscure journals by the terms of his political exile from the University of Prague in 1819. Moreover, Bolzano's definition was more accurately a relation that held between two infinite sets, rather than a definition of an infinite set *per se*.

For a long time, many mathematicians did not even entertain the thought that there might be a distinction between the notions of infinite set and Dedekind-infinite set. In fact, the distinction was not really realised until after Ernst Zermelo formulated the AC explicitly. The existence of infinite, Dedekind-finite sets was studied by Bertrand Russell and Alfred North Whitehead in 1912; these sets were at first called *mediate cardinals* or *Dedekind cardinals*.

With the general acceptance of the axiom of choice among the mathematical community, these issues relating to infinite and Dedekind-infinite sets have become less central to most mathematicians. However, the study of Dedekind-infinite sets played an important role in the attempt to clarify the boundary between the finite and the infinite, and also an important role in the history of the AC.

Relation to the axiom of choice

Since every infinite well-ordered set is Dedekind-infinite, and since the AC is equivalent to the well-ordering theorem stating that every set can be well-ordered, clearly the general AC implies that every infinite set is Dedekind-infinite. However, the equivalence of the two definitions is much weaker than the full strength of AC.

In particular, there exists a model of **ZF** in which there exists an infinite set with no denumerable subset. Hence, in this model, there exists an infinite, Dedekind-finite set. By the above, such a set cannot be well-ordered in this model.

If we assume the axiom CC (i. e., AC_ω), then it follows that every infinite set is Dedekind-infinite. However, the equivalence of these two definitions is in fact strictly weaker than even the CC. Explicitly, there exists a model of **ZF** in which every infinite set is Dedekind-infinite, yet the CC fails (assuming consistency of **ZF**).

Proof of equivalence to infinity, assuming axiom of countable choice

That every Dedekind-infinite set is infinite can be easily proven in ZF: every finite set has by definition a bijection with some finite ordinal n, and one can prove by induction on n that this is not Dedekind-infinite.

By using the axiom of countable choice (denotation: axiom CC) one can prove the converse, namely that every infinite set X is Dedekind-infinite, as follows:

First, define a function over the natural numbers (that is, over the finite ordinals) $f : \mathbf{N} \to \text{Power}(\text{Power}(X))$, so that for every natural number n, $f(n)$ is the set of finite subsets of X of size n (i.e. that have a bijection with the finite ordinal n). $f(n)$ is never empty, or otherwise X would be finite (as can be proven by induction on n).

The image of f is the countable set $\{f(n) | n \in \mathbf{N}\}$, whose members are themselves infinite (and possibly uncountable) sets. By using the axiom of countable choice we may choose one member from each of these sets, and this member is itself a finite subset of X. More precisely, according to the axiom of countable choice, a (countable) set exists, $G = \{g(n) | n \in \mathbf{N}\}$, so that for every natural number n, $g(n)$ is a member of $f(n)$ and is therefore a finite subset of X of size n.

Now, we define U as the union of the members of G. U is an infinite countable subset of X, and a bijection from the natural numbers to U, $h : \mathbf{N} \to U$, can be easily defined. We may now define a bijection $B : X \to X \setminus h(0)$ that takes every member not in U to itself, and takes $h(n)$ for every natural number to $h(n + 1)$. Hence, X is Dedekind-infinite, and we are done.

Generalizations

Expressed in category-theoretical terms, a set A is Dedekind-finite if in the category of sets, every monomorphism $f : A \to A$ is an isomorphism. A von Neumann regular ring R has the analogous property in the category of (left or right) R-modules if and only if in R, $xy = 1$ implies $yx = 1$. More generally, a *Dedekind-finite ring* is any ring that satisfies the latter condition. Beware that a ring may be Dedekind-finite even if its underlying set is Dedekind-infinite, e.g. the integers.

Notes

1. Moore, Gregory H. (2013) [unabridged republication of the work originally published in 1982 as Volume 8 in the series "Studies in the History of Mathematics and Physical Sciences" by Springer-Verlag, New York]. *Zermelo's Axiom of Choice: Its Origins, Development & Influence*. Dover Publications. ISBN 978-0-486-48841-7.
2. Herrlich, Horst (2006). *Axiom of Choice*. Lecture Notes in Mathematics 1876. Springer-Verlag. ISBN 978-3540309895.

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