2-9 Sorting and Selection

Hengfeng Wei

hfwei@nju.edu.cn

May 28, 2018

How to Argue?

Show that $\cdots$, Argue that $\cdots$, Explain why $\cdots$

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Show that $\cdots$, Argue that $\cdots$, Explain why $\cdots$

$= $ Prove that $\cdots$

不好，掉坑里了

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“在千山万水人海相遇, 喔, 原来你也在这里”

入坑指南

The Double Dixie Cup Problem

Donald J. Newman
The American Mathematical Monthly
Vol. 67, No. 1 (Jan., 1960), pp. 58-61

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$$1 \to 2 \to m$$

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The Coupon Collector's Problem

Marco Ferrante, Monica Saltalamacchia

In this note we will consider the following problem: how many coupons we have to purchase (on average) to complete a collection. This problem, which takes everybody back to his childhood when this was really "a problem", has been considered by the probabilists since the eighteenth century and nowadays it is still possible to derive some new results, probably original or at least never published. We will present some classic results, some new formulas, some alternative approaches to obtain known results and a couple of amazing expressions.

"兄弟同心，其利断金"

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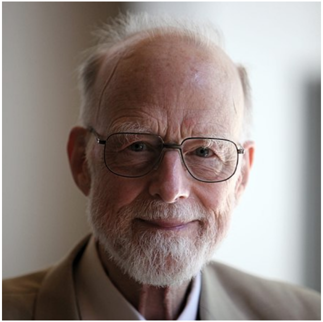
"兄弟同心，其利断金"

$$n \log n + (m-1) n \log \log n + n C_m + o(n), \quad n \to \infty, m \text{ fixed}$$

QUICKSORT (Tony Hoare, 1959/1960)



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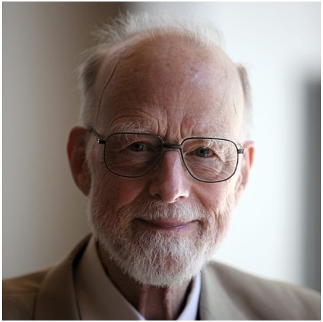

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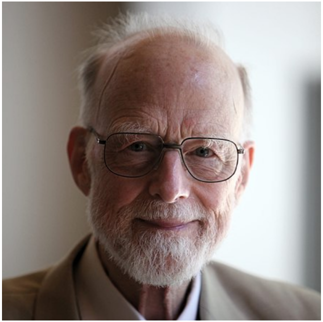

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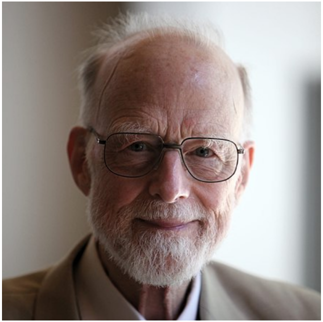

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"I call it my billion-dollar mistake."

Best-Case Complexity of QUICKSORT $(7.4-2)$

Show that QUICKSORT's *best-case* running time is $\Omega(n \log n)$.

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By substitution.

Sorting Almost-Sorted Inputs (Problem $7.2 - 4$)

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$$\textsc{Insertion-sort} : \Theta(n)$$

$$\textsc{Quicksort} : \Omega(n \log n)$$

Median-of-3 PARTITION (Problem $7-5$)

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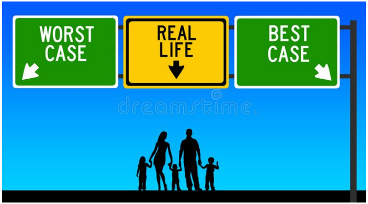

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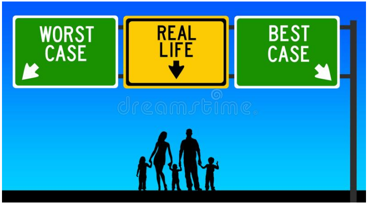

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The Analysis of Quicksort Programs*

Robert Sedgewick

Received January 19, 1976

Summary. The Quicksort sorting algorithm and its best variants are presented and analyzed. Results are derived which make it possible to obtain exact formulas describing the total expected running time of particular implementations on real computers of Quicksort and an improvement called the median-of-three modification. Detailed analysis of the effect of an implementation technique called loop unwrapping is presented. The paper is intended not only to present results of direct practical utility, but also to illustrate the intriguing mathematics which arises in the complete analysis of this important algorithm.

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$$B_N = \frac{12}{35}(N+1)(H_{N+1} - H_{M+2}) + \frac{37}{245}(N+1) - \frac{12}{7}\frac{N+1}{M+2} + 1 \quad \text{exchanges}$$

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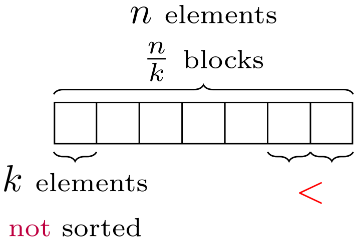
Robert Sedgewick

Sorts a $\frac{n}{k}$-sorted Array (Problem $8.1 - 4$)



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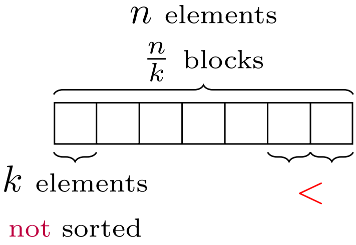

$\Omega(n \log k)$

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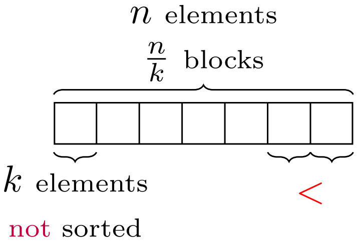

$\Omega(n \log k)$ $\qquad$ $O(n \log k)$

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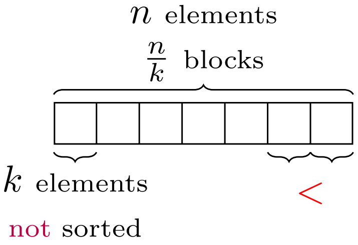

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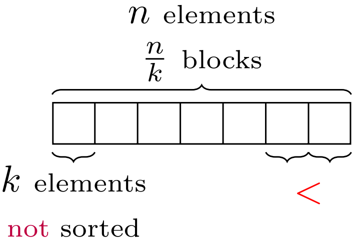

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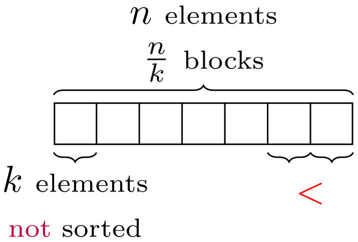
$$(k!)^{\frac{n}{k}} \leq L \leq 2^H$$

$\frac{n}{k}$-sorts an arbitrary array



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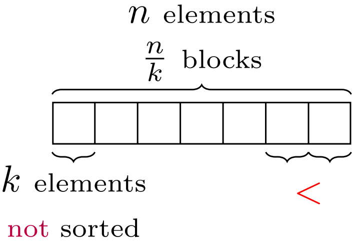

$O(?)$

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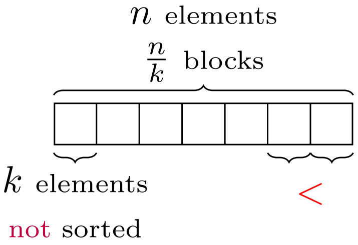

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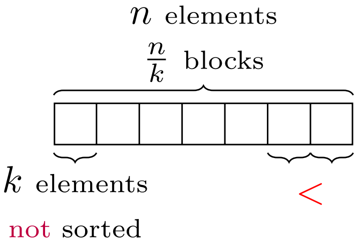

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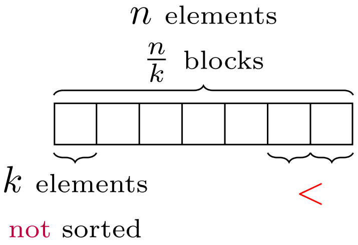

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$$L \geq \binom{n}{\underbrace{k, \ldots, k}_{\frac{n}{k}}} = \frac{n!}{(k!)^{\frac{n}{k}}} \implies \Omega(n \log(n/k))$$

Sorting $[0, n^3 - 1]$ (Problem $8.3 - 4$)

Sort n integers in $[0, n^3 - 1]$ in $O(n)$ time.

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$$3n \cdot T(\frac{\Box}{n})$$

Sorting in Place in Linear Time (Problem $8-2$ (e))

Suppose that the n records have keys in the range $[0, k]$.
Modify COUNTING-SORT to sort them in place ($O(k)$) in $O(n+k)$ time.

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	1	2	3	4	5	6	7	8
A :	2	5	3	0	2	3	0	3

	0	1	2	3	4	5
C :	2	0	2	3	0	1
C :	2	2	4	7	7	8

	1	2	3	4	5	6	7	8
B :	0	0	2	2	3	3	3	5

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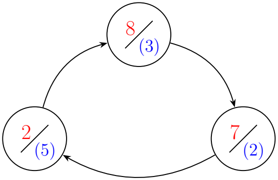

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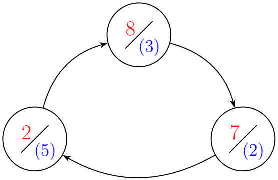
`for` $(i \leftarrow n$ `to` $1)$:



Finding the 2nd Smallest Element (Problem $9.1-1$)

Show that the 2nd smallest of n elements can be found with $n + \lceil \log n \rceil - 2$ comparisons in the worst case.

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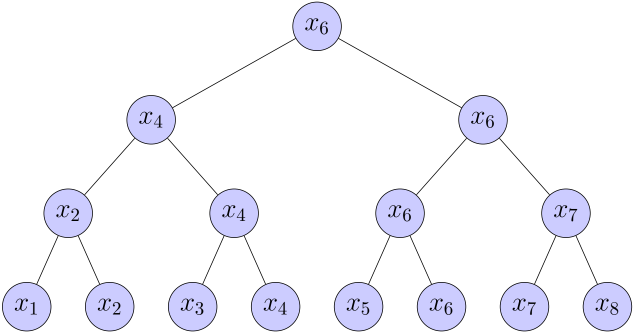

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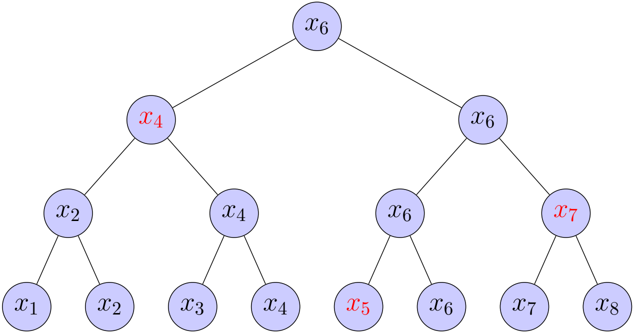

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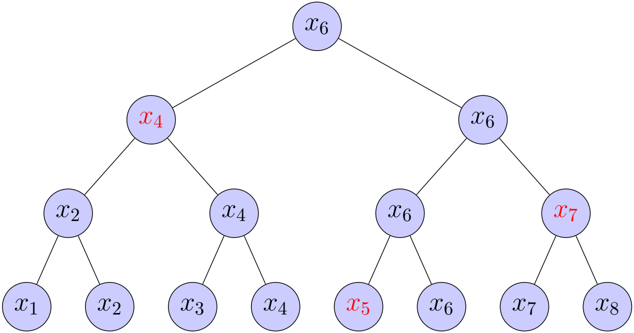

#Potential 2nd smallest elements $\leq \lceil \log n \rceil$

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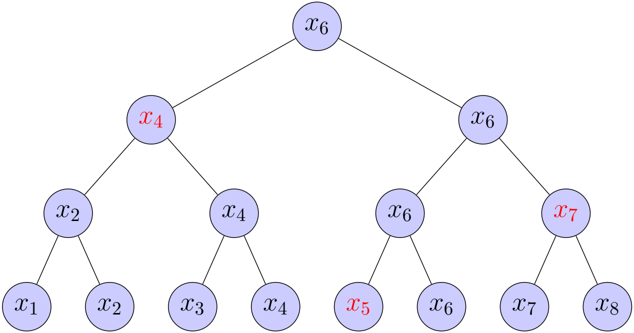

\#Potential 2nd smallest elements $\leq \lceil \log n \rceil$

Q : Can we do even better?

$$\Omega = n + \lceil \log n \rceil - 2$$

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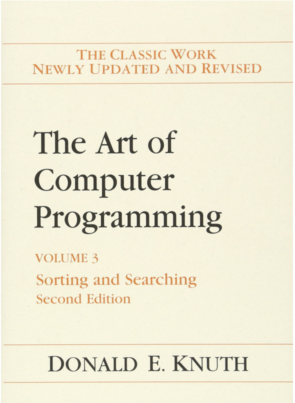

TAOCP Vol 3 (Page 209, Section 5.3.3)

k Numbers Closest to the Median (Problem $9.3 - 7$)

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$$\frac{n}{2} - \frac{k}{2} \sim \frac{n}{2} + \frac{k}{2} \qquad \times$$

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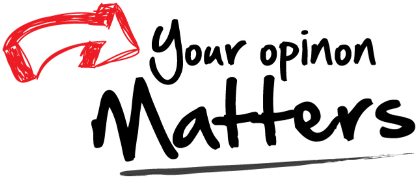
median + subtraction + $(k+1)$-th smallest + partition + add back


Thank You!



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