Cardinality Part I

Original Notes adopted from December 3, 2001 (Week 13)

S and T have the same **cardinality** if there exists $f\colon S \to T$ one-to-one onto (i.e. a "pairing") or **one-to-one correspondence.**
We showed that $|\mathbb{N}| = |E| = |\mathbb{Q}^+|$
$|S| = |\mathbb{N}|$ iff S is an infinite set whose elements can be listed. We call such sets "countably infinite", or say they have cardinality $\aleph_0$.
$|S| = \aleph_0$ means $|S| = |\mathbb{N}|$.
$|[0,1]| \neq \aleph_0$
Proof We'll show no list can contain all numbers in [0,1].
$a_{ij} \in \{0, 1, 2, 3, 4, ...9\}$
Suppose we have a list $c_1, c_2, c_3, \cdots$, write them as
$c_1 = .a_{11}a_{12}a_{13}a_{14}a_{15}\cdots$
$c_2 = .a_{21}a_{22}a_{23}a_{24}a_{25}\cdots$
$c_3 = .a_{31}a_{32}a_{33}a_{34}a_{35}\cdots$
$\cdots\cdots$

In ambiguous cases, pick representation with all 9's. e.g. $.34999\cdots = .3500000$.
Let $x = .b_1b_2b_3b_4\cdots$ where b_j any digit other than 0, 9 or a_{jj}

Then x isn't among numbers listed for it differs from the kth number listed in its kth place.
Therefore $|[0,1]| \neq \aleph_0$
We say [0,1] has the cardinality of the continuum, or $|[0,1]| = c$

Definition. $|S| \leqslant |T|$ ("The cardinality of S is less than or equal to the cardinality of T") if there exists $T_0 \subset T$ such that $|S| = |T_0|$.

We say $|S| < |T|$ if $|S| \leqslant |T|$ and $|S| \neq |T|$

Claim: $|\mathbb{N}| < |[0,1]|$. We just proved $|\mathbb{N}| \neq |[0,1]|$.
$\{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, .010, \cdots\}$
This is an easier way:
Let $T_0 = \{1, 1/2, 1/3, 1/4, ...\} \subset [0,1]$
Let $f : N \to T_0$ by $f(n) = 1/n$.
Since f is one-to-one onto and onto, $|\mathbb{N}| = |T_0|$.
Therefore $|\mathbb{N}| \leqslant |T| = |[0,1]$, or $|$ on $\aleph_0 < c$.
We defined $|S| \leqslant |T|$ to mean $|S| = |T_0|$ for some $T_0 \subset T$.
Suppose that also $|T| \leqslant |S|$. Must $|S| = |T|$?

$|S| \leqslant |T|$ means there exists $f\colon S \to T$, f one-to-one (not necessarily onto)
$|T| \leqslant |S|$ means there exists $g\colon T \to S, g$ one-to-one
$|S| = |T|$ means there exists $h\colon S \to T, h$ one-to-one and onto

Theorem. (**Schroeder-Bernstein or Cantor-Bernstein Theorem**)
If $|S| \leqslant T$ and $|T| \leqslant |S|$, then $|S| = |T|$.

Theorem. If $a < b$ and $c < d$, then $|[a,b]| = |[b,d]|$ and $|(a,b)| = |(c,d)|$

Proof: Let $f(x) = c(\frac{x-b}{a-b}) + d(\frac{x-a}{b-a})$. Then
$f : [a,b] \Rightarrow [c,d]$ one-to-one and onto
$f : (a,b) \Rightarrow (c,d)$ one-to-one and onto

Eg. $(\pi, \frac{3\pi}{2})$, [0,1].
$|(\pi, \frac{3\pi}{2})| \leqslant |[\pi, \frac{3\pi}{2}]| = |[\pi + 0.1, \frac{3\pi}{2} - 0.1| \leqslant |(\pi, \frac{3\pi}{2})|$
S-B $\Rightarrow |(\pi, \frac{3\pi}{2})| = |[\pi, \frac{3\pi}{2}]|$

Corollary. If $a < b$ and $c < d$, then $|[a,b]| = |(c,d)| = |[c,d)| = |(c,d]| = |[c,d]|$. The cardinalities of any intervals (closed or not) are equal.

Eg. $f(x) = \tan x$
$f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$, one-to-one and onto
Therefore $|(-\frac{\pi}{2}, \frac{\pi}{2})| = |\mathbb{R}|$
Therefore $|\mathbb{R}| = |[0,1]| = c$.

$[0,1] \times [0,1] = \{(x,y) : x \in [0,1], y \in [0,1]\}$
Let $S = [0,1] \times [0,1]$ be the unit square.
To see $|[0,1]| \leqslant |S|$
Let $S_0 = \{(x,y) \in S : y = 0\}$.
Let $f : [0,1] \to S_0$ by $f(x) = (x,0)$.
Therefore $|[0,1]| = |S_0| \Rightarrow |[0,1]| \leqslant |S|$.

Is $|S| = |[0,1]|$?
Represent points in S as infinite decimals:
$(x,y) = (.a_1a_2a_3\cdots, .b_1b_2b_3\cdots)$
Choose all 9's in ambiguous cases.
Let $f : S \Rightarrow [0,1]$ by $f(.a_1a_2a_3\cdots, .b_1b_2b_3\cdots) = .a_1b_1a_2b_2a_3b_3a_4b_4\cdots$
f is one-to-one (but not onto).
For example, $.1707070707070\cdots$ is not in the range of f; it would have to come from ($.10000\cdots, .777\cdots$), but this is written as $(.0999\cdots, .777\cdots)$.
Since f is one-to-one, $|S| \leqslant |[0,1]|$.
Schroeder-Berstein $\Rightarrow |S| = |(0,1]| = c$.

Theorem. If $|S_i| = c$ for $i = 1,2,3,\cdots$, then $|\bigcup_{i=1}^{\infty} S_i| = c$.
$(\bigcup_{i=1}^{\infty} S_i = S_1 \bigcup S_2 \bigcup S_3 \cdots)$

Proof: Clearly $|\bigcup_{i=1}^{\infty} S_i| \geqslant c$, since $S_i \subset \bigcup_{i=1}^{\infty} S_i$.
Write $\bigcup_{i=1}^{\infty} S_i = S_1 \bigcup (S_2 \backslash S_1) \bigcup (S_3 \backslash (S_1 \bigcup S_2)) \bigcup (S_4 \backslash (S_1 \bigcup S_2 \bigcup S_3)) \cdots$ as a disjoint union.

Can construct $f : \bigcup_{i=1}^{\infty} S_i \to \mathbb{R}$ as follows: let f on S_1 be any one-to-one function from S_1 to (0,1); f on $S_2 \backslash S_1$ is any one-to-one function from $S_2 \backslash S_1$ onto (1,2), etc.
Then $f : \bigcup_{i=1}^{\infty} S_i \to \mathbb{R}$ is one-to-one.
Therefore $|\bigcup_{i=1}^{\infty} S_i| \leqslant |\mathbb{R}|$, S-B $\Rightarrow |\bigcup_{i=1}^{\infty} S_i| = |\mathbb{R}| = c$

In words: a countable union of sets of cardinality c has cardinality c.
Countable numbers of squares of unit sides covers $\mathbb{R}^2$, so $|\mathbb{R}^2| = c$.

Theorem. Let S = set of all sets of real numbers (ie. the collection of subsets of $\mathbb{R}$). Then $|S| > c$ (ie $|S| > |\mathbb{R}|$).

Proof: First, $|\mathbb{R}| \leqslant |S|$.
For each $x \in \mathbb{R}$, let $f(x) = \{x\}$ (singleton subset of $\mathbb{R}$)
If $S_0 = \{$all singleton subsets of $\mathbb{R}\}$, $f : \mathbb{R} \Rightarrow S_0$ one-to-one and onto.
$|\mathbb{R}| \leqslant |S|$ by definition.
Must show: $|S| \neq |\mathbb{R}|$

Suppose there exists $g : \mathbb{R} \to S$, and show g can't be onto.
For $x \in \mathbb{R}$, $g(x)$ is a subset of $\mathbb{R}$.
Let $T = \{x \in \mathbb{R} : x \notin g(x)\}$.

Claim: there is no $y \in \mathbb{R}$ such that $g(y) = T$.
For if $g(y) = T$, is $y \in g(y)$ or not?
If $y \in g(y)$, then $y \notin T$ (=$g(y)$). Contradiction.
Therefore $y \notin g(y)$.
But if $y \notin g(y)$, $y \in T$, so $y \in g(y)$. Another contradiction.
Therefore there is no such y, and so g is not onto.
The cardinality of S, we call 2^c. Therefore $2^c > c$.