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Negative probability

The probability of the outcome of an experiment is never negative, but quasiprobability distributions can be defined that allow a **negative probability**, or **quasiprobability** for some events. These distributions may apply to unobservable events or conditional probabilities.

Contents

Physics and mathematics

In 1942, Paul Dirac wrote a paper "The Physical Interpretation of Quantum Mechanics"[1] where he introduced the concept of negative energies and negative probabilities:

> "Negative energies and probabilities should not be considered as nonsense. They are well-defined concepts mathematically, like a negative of money."

The idea of negative probabilities later received increased attention in physics and particularly in quantum mechanics. Richard Feynman argued[2] that no one objects to using negative numbers in calculations: although "minus three apples" is not a valid concept in real life, negative money is valid. Similarly he argued how negative probabilities as well as probabilities above unity possibly could be useful in probability calculations.

Mark Burgin gives another example:

> "Let us consider the situation when an attentive person A with the high knowledge of English writes some text T. We may ask what the probability is for the word “texxt” or “wrod” to appear in his text T. Conventional probability theory gives 0 as the answer. However, we all know that there are usually misprints. So, due to such a misprint this word may appear but then it would be corrected. In terms of extended probability, a negative value (say, −0.1) of the probability for the word “texxt” to appear in his text T means that this word may appear due to a misprint but then it’ll be corrected and will not be present in the text T."
>
> — Mark Burgin, Burgin, Mark (2010). "Interpretations of Negative Probabilities". arXiv:1008.1287 (https://arxiv.org/abs/1008.1287) [physics.data-an (https://arxiv.org/archive/physics.data-an)].

Negative probabilities have later been suggested to solve several problems and paradoxes.[3] *Half-coins* provide simple examples for negative probabilities. These strange coins were introduced in 2005 by Gábor J. Székely.[4] Half-coins have infinitely many sides numbered with 0,1,2,... and the positive even numbers are taken with negative probabilities. Two half-coins make a complete coin in the sense that if we flip two half-coins then the sum of the outcomes is 0 or 1 with probability 1/2 as if we simply flipped a fair coin.

In *Convolution quotients of nonnegative definite functions*[5] and *Algebraic Probability Theory* [6] Imre Z. Ruzsa and Gábor J. Székely proved that if a random variable X has a signed or quasi distribution where some of the probabilities are negative then one can always find two random variables, Y and Z, with ordinary (not signed / not quasi) distributions such that X, Y are independent and X + Y = Z in distribution. Thus X can always be interpreted as the "difference" of two ordinary random variables, Z and Y. If Y is interpreted as a measurement error of X and the observed value is Z then the negative regions of the distribution of X are masked / shielded by the error Y.

Another example known as the Wigner distribution in phase space, introduced by Eugene Wigner in 1932 to study quantum corrections, often leads to negative probabilities.[7] For this reason, it has later been better known as the Wigner quasiprobability distribution. In 1945, M. S. Bartlett worked out the mathematical and logical consistency of such negative valuedness.[8] The Wigner distribution function is routinely used in physics nowadays, and provides the cornerstone of phase-space quantization. Its negative features are an asset to the formalism, and often indicate quantum interference. The negative regions of the distribution are shielded from direct observation by the quantum uncertainty principle: typically, the moments of such a non-positive-semidefinite quasiprobability distribution are highly constrained, and prevent *direct measurability* of the negative regions of the distribution. But these regions contribute negatively and crucially to the expected values of observable quantities computed through such distributions, nevertheless.

An example: the double slit experiment

Consider a double slit experiment with photons. The two waves exiting each slit can be written as:

$$f_1(x)=\sqrt{\frac{dN/dt}{2\pi/d}}\frac{1}{\sqrt{d^2+(x+a/2)^2}}\exp\left[i(h/\lambda)\sqrt{d^2+(x+a/2)^2}\right],$$

and

$$f_2(x)=\sqrt{\frac{dN/dt}{2\pi/d}}\frac{1}{\sqrt{d^2+(x-a/2)^2}}\exp\left[i(h/\lambda)\sqrt{d^2+(x-a/2)^2}\right],$$

where d is the distance to the detection screen, a is the separation between the two slits, x the distance to the center of the screen, λ the wavelength and dN/dt is the number of photons emitted per unit time at the source. The amplitude of measuring a photon at distance x from the center of the screen is the sum of these two amplitudes coming out of each hole, and therefore the probability that a photon is detected at position x will be given by the square of this sum:

$$I(x)=|f_1(x)+f_2(x)|^2=|f_1(x)|^2+|f_2(x)|^2+[f_1^*(x)f_2(x)+f_1(x)f_2^*(x)],$$

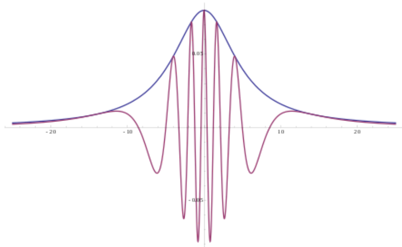
This should strike you as the well-known probability rule:

$$\begin{aligned}P(\texttt{photon reaches x going through either slit}) &= P(\texttt{photon reaches x going through slit 1})\\ &\quad + P(\texttt{photon reaches x going through slit 2})\\ &\quad - P(\texttt{photon reaches x going through both slits})\\ \\ &= P(\texttt{photon reaches x} \mid \texttt{went through slit 1})\, P(\texttt{going through slit 1})\\ &\quad + P(\texttt{photon reaches x} \mid \texttt{went through slit 2})\, P(\texttt{going through slit 2})\\ &\quad - P(\texttt{photon reaches x going through both slits})\\ \\ &= P(\texttt{photon reaches x} \mid \texttt{went through slit 1})\,\frac{1}{2}\\ &\quad + P(\texttt{photon reaches x} \mid \texttt{went through slit 2})\,\frac{1}{2}\\ &\quad - P(\texttt{photon reaches x going through both slits})\end{aligned}$$

whatever the last term means. Indeed, if one closes either one of the holes forcing the photon to go through the other slit, the two corresponding intensities are

$$I_1(x)=|f_1(x)|^2=\frac{1}{2}\frac{dN}{dt}\frac{d/\pi}{d^2+(x+a/2)^2} \quad \text{and}$$

$$I_2(x)=|f_2(x)|^2=\frac{1}{2}\frac{dN}{dt}\frac{d/\pi}{d^2+(x-a/2)^2}.$$



In blue, the sum of the probabilities of going through holes 1 and 2; in red, minus the joint probability of going through "both holes". The interference pattern is obtained by adding the two curves.

But now, if one does interpret each of these terms in this way, the joint probability takes negative values roughly every $\lambda \frac{d}{a}$!

$$\begin{aligned} I_{12}(x) &= [f_1^*(x) f_2(x) + f_1(x) f_2^*(x)] \\ &= \frac{1}{2} \frac{dN}{dt} \frac{d/\pi}{\sqrt{d^2 + (x - a/2)^2} \sqrt{d^2 + (x + a/2)^2}} \sin\left[(h/\lambda)(\sqrt{d^2 + (x + a/2)^2} - \sqrt{d^2 + (x - a/2)^2})\right] \end{aligned}$$

However, these negative probabilities are never observed as one can't isolate the cases in which the photon "goes through both slits", but can hint at the existence of anti-particles

Finance

Negative probabilities have more recently been applied to mathematical finance. In quantitative finance most probabilities are not real probabilities but pseudo probabilities, often what is known as risk neutral probabilities. These are not real probabilities, but theoretical "probabilities" under a series of assumptions that helps simplify calculations by allowing such pseudo probabilities to be negative in certain cases as first pointed out by Espen Gaarder Haug in 2004.[9]

A rigorous mathematical definition of negative probabilities and their properties was recently derived by Mark Burgin and Gunter Meissner (2011). The authors also show how negative probabilities can be applied to financial option pricing.[10]

Engineering

The concept of negative probabilities have also been proposed for reliable facility location models where facilities are subject to negatively correlated disruption risks when facility locations, customer allocation, and backup service plans are determined simultaneously.[11][12] Li et al.[13] proposed a virtual station structure that transforms a facility network with positively correlated disruptions into an equivalent one with added virtual supporting stations, and these virtual stations were subject to independent disruptions. This approach reduces a problem from one with correlated disruptions to one without. Xie et al.[14] later showed how negatively correlated disruptions can also be addressed by the same modeling framework, except that a supporting station now may be disrupted with a “failure propensity” which

“... inherits all mathematical characteristics and properties of a failure probability except that we allow it to be larger than 1...”

This finding paves ways for using compact mixed-integer mathematical programs to optimally design reliable location of service facilities under site-dependent and positive/negative/mixed disruption correlations.[15]

The proposed “propensity” concept in Xie et al.[14] turns out to be what Feynman and others referred to as “quasi-probability.” Note that when a quasi-probability is larger than 1, then 1 minus this value gives a negative probability. The truly physically verifiable observation is the facility disruption states, and there is no direct information on the station states or their corresponding probabilities. Hence the failure probability of the stations, interpreted as “probabilities of imagined intermediary states,” could exceed unity.

See also

- Existence of states of negative norm (or fields with the wrong sign of the kinetic term, such as Pauli–Villars ghosts) allows the probabilities to be negative. See Ghosts (physics).
- Signed measure
- Wigner quasiprobability distribution

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