1-13 Boolean Algebra

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Definition (Boolean Algebra)

A *boolean algebra* $\mathcal{B} = (B, \wedge, \vee, ', 0, 1)$ is a bounded, complemented, and distributive lattice.

$$\forall a, b, c \in B,$$

Idempotency:

Commutativity:

Associativity:

Absorption:

Complements:

$$a \wedge a' = 0 \quad a \vee a' = 1$$

Distributivity:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \qquad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

Problem 2: D_n

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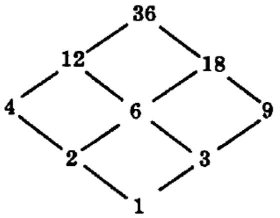
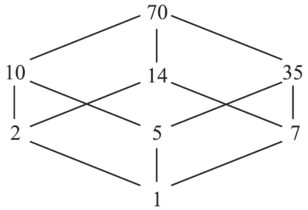


D_{36}

$36 = 2^2 \times 3^2$



D_{70}

$70 = 2 \times 5 \times 7$

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$\implies$ $n = p_1 p_2 p_k \wedge$ all p_i are unique primes

Problem 3: Atom

Let $\mathcal{B} = (B, \leq)$ is a Boolean algebra.

$$\forall a \in B : \mathsf{Atom}(a) = \{x \leq a \mid x \text{ is an atom}\}$$

Suppose $\mathcal{B}$ is finite. To prove:

$$\forall a \in B : a \neq 0 \implies \mathsf{Atom}(a) \neq \emptyset.$$

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Atoms: those elements which *immediately* succeed 0

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Problem 4: Isomorphic

All finite Boolean algebras of the same cardinality are isomorphic.

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Theorem (Representation Theorem for Finite Boolean Algebras)

Every finite Boolean algebra is isomorphic to a Boolean algebra $\mathcal{P}(X)$ for some finite set X.

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Finite-Cofinite Algebra

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If $F(\mathbb{N})$ is isomorphic to $\mathcal{P}(X)$ for some X:

$$|F(\mathbb{N})|$$

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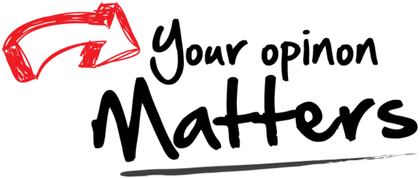
If $F(\mathbb{N})$ is isomorphic to $\mathcal{P}(X)$ for some X:

$$|F(\mathbb{N})| \geq 2^{\aleph_0}$$

Thank You!



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