1-11 Set Theory (IV): Infinity

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Finite Sets

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"关于有穷，我原以为我是懂的"

Definition (Finite)

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Theorem (Pigeonhole Principle (UD Theorem 22.2))

$$f : \{1, \cdots, m\} \to \{1, \cdots, n\} \; (m, n \in \mathbb{N}^{+}, m > n)$$

f is not one-to-one.

$A \setminus \{a\}$ (UD Problem 22.17)

Let A be a nonempty finite set with $|A| = n$ and let $a \in A$.
Prove that $A \setminus \{a\}$ is finite and $|A \setminus \{a\}| = n - 1$.

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By contradiction and (b).

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$$\sum_{y \in A} f^{-1}(\{y\}) > |A|$$

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Give an example, if possible, of

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$$|A| = n \implies |\mathcal{P}(A)| = 2^n$$

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Is it countable or uncountable?

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Infinite Sequences of 0's and 1's (UD Problem 23.4)

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$$
\begin{array}{|lcl|}
\hline
s_1 & = & \color{red}{0}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \ldots \\
s_2 & = & 1\ \color{red}{1}\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \ldots \\
s_3 & = & 0\ 1\ \color{red}{0}\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0 \ldots \\
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s_5 & = & 1\ 1\ 0\ 1\ \color{red}{0}\ 1\ 1\ 0\ 1\ 0\ 1 \ldots \\
s_6 & = & 0\ 0\ 1\ 1\ 0\ \color{red}{1}\ 1\ 0\ 1\ 1\ 0 \ldots \\
s_7 & = & 1\ 0\ 0\ 0\ 1\ 0\ \color{red}{0}\ 0\ 1\ 0\ 0 \ldots \\
s_8 & = & 0\ 0\ 1\ 1\ 0\ 0\ 1\ \color{red}{1}\ 0\ 0\ 1 \ldots \\
s_9 & = & 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ \color{red}{1}\ 1\ 0 \ldots \\
s_{10} & = & 1\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ \color{red}{0}\ 1 \ldots \\
s_{11} & = & 1\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ \color{red}{0} \ldots \\
\vdots & & \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \ddots \\
\hline
\end{array}
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By Diagonal Argument.

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Prove that

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$$|\mathbb{C}| = |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$$

$$\mathbb{R} \times \mathbb{R} \approx \mathbb{R}$$

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$$(x = 0.a_1 a_2 a_3 \cdots, y = 0.b_1 b_2 b_3 \cdots)$$

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Was Cantor Surprised?

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$$f(x) = x, \ \text{otherwise}$$

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Thank You!