2-14 B-Trees

Hengfeng Wei

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June 02, 2020

Organization and Maintenance of Large Ordered Indexes

R. BAYER and E. MCCREIGHT

Received September 29, 1971

Summary. Organization and maintenance of an index for a dynamic random access file is considered. It is assumed that the index must be kept on some pseudo random access backup store like a disc or a drum. The index organization described allows retrieval, insertion, and deletion of keys in time proportional to $\log_k I$ where I is the size of the index and k is a device dependent natural number such that the performance of the scheme becomes near optimal. Storage utilization is at least 50% but generally much higher. The pages of the index are organized in a special data-structure, so-called B-trees. The scheme is analyzed, performance bounds are obtained, and a near optimal k is computed. Experiments have been performed with indexes up to 100000 keys. An index of size 15000 (100000) can be maintained with an average of 9 (at least 4) transactions per second on an IBM 360/44 with a 2311 disc.

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ID	Name	Gender	Age	...

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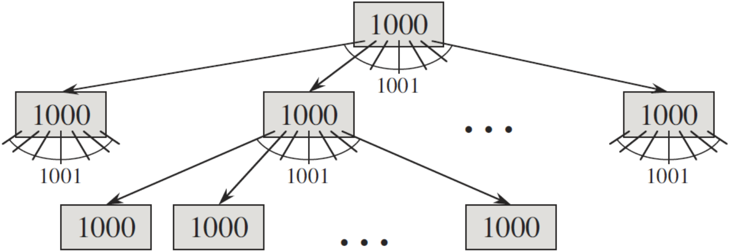


2-way *vs.* multi-way

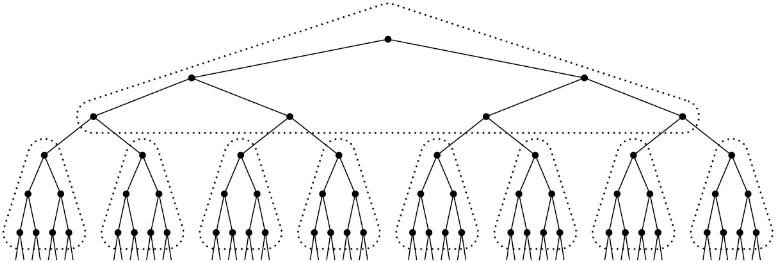
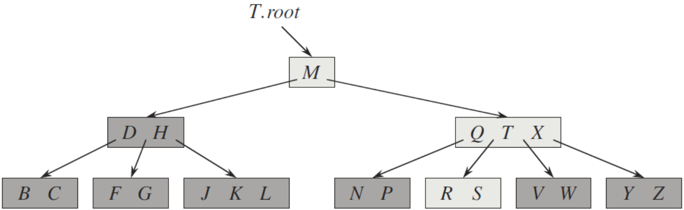


Fig. 29. A large binary search tree can be divided into "pages."

indexes (keys) *vs.* pages

Minimum (TC 18.2-3)

Explain how to find the minimum key stored in a B-tree.



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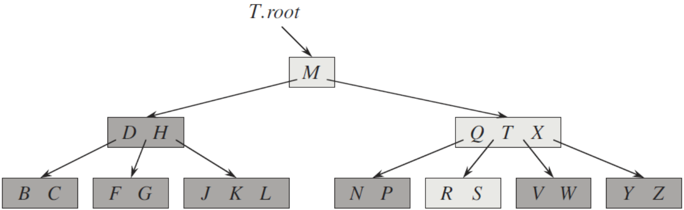

the leftmost key in the leftmost node

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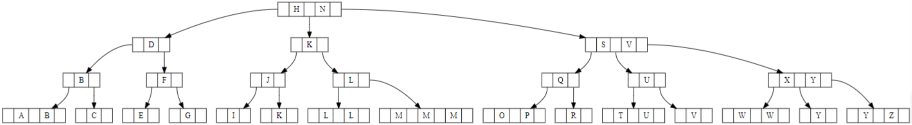

B-tree with 37 characters.

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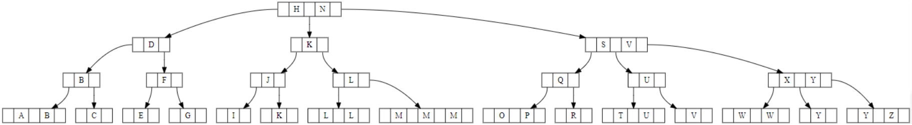

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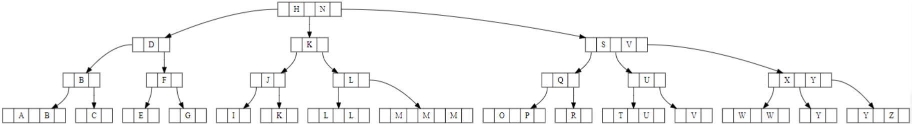

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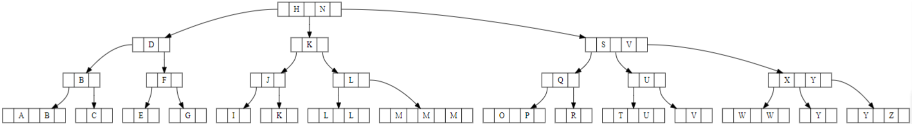


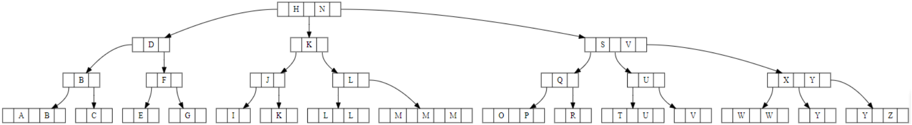
B-tree with 37 characters.

$x.leaf = 0$

$H \quad N \quad S \quad V$

find the rightmost key in $x.c_i$

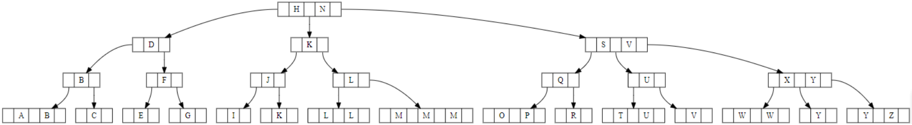
$$(x, i)$$



$$x.leaf = 1$$

$P \quad U \quad T \quad O \quad A$

$$(x, i)$$



$$\color{red}{x.leaf = 1}$$

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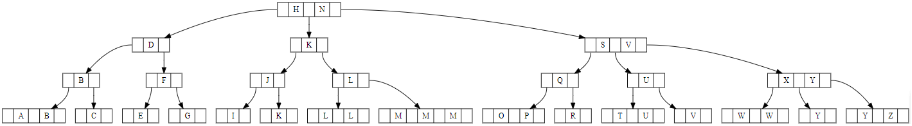
$$i \geq 2 \implies (x, i-1)$$

$i = 1 \implies$ find (y, j) such that x is the leftmost key in $y.c_{j+1}$

$$(x, i)$$



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A is the only key which has no predecessor.

```
procedure B-TREE-PREDECESSOR(T, x, i)        ▷ x.key_i in B-tree T
    if x.leaf = 0 then
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procedure B-TREE-PREDECESSOR(T, x, i) ▷ $x.key_i$ in B-tree T

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  else ▷ $x.leaf = 1 \land i = 1$

    $y \leftarrow x.p$

    while $y \neq T.root \land x = y.c_1$ **do** ▷ exit: $y = T.root \lor x \neq y.c_1$

      $x \leftarrow y$

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            x ← y
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        if x = y.c_1 then                          ▷ y = T.root ∧ x = y.c_1
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        if x = y.c_1 then                       ▷ y = T.root ∧ x = y.c_1
            return "no predecessor"
        else                                    ▷ x ≠ y.c_1
            j ← 2
            while y.c_j ≠ x do
                j ← j + 1
            return (y, j − 1)                   ▷ x = y.c_j
```

Insertion (TC 18.2-4 *)

Suppose that we insert the keys $\{1, 2, \ldots, n\}$ in increasing order into an empty B-tree with minimum degree 2.

How many nodes, denoted X_n, does the final B-tree have?

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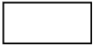

$X_0 = 1$

By Yangjing Dong (June 2018)

https://maxmute.com/TC18.2-4.html

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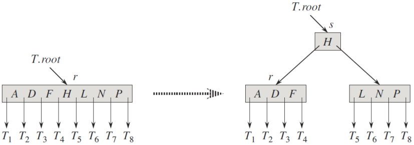

root SPLIT

+2

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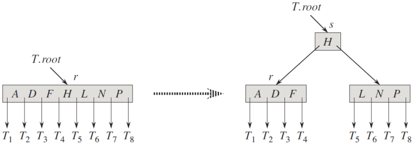

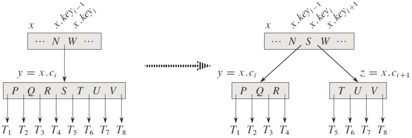

root SPLIT

$+2$

non-root SPLIT

$+1$

(I) Which nodes will SPLIT? S

(II) When does each node $s \in S$ SPLIT? $T_s = \langle s_1, s_2, \ldots \rangle$

(III) How does it SPLIT, as a root or a non-root? $T_s = s_R \uplus s_{NR}$

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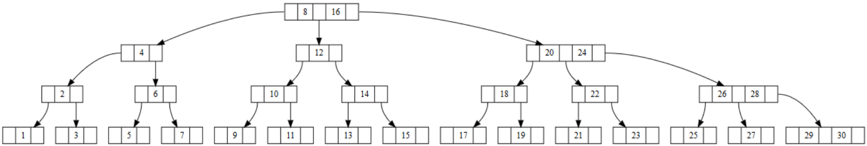

$$1, 2, \ldots, 30$$

$$S = \left\{ \text{the nodes in the rightmost chain} \right\}$$

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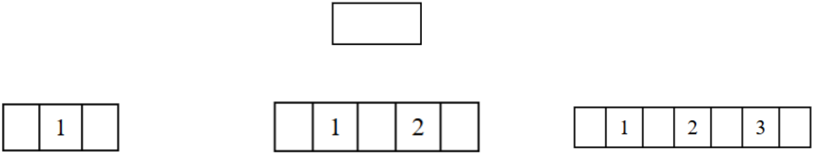
Let's focus the rightmost node first, denoted A.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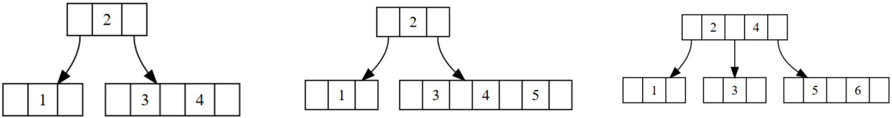
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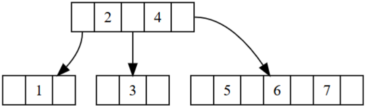
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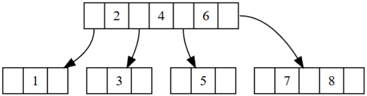
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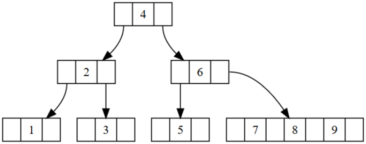
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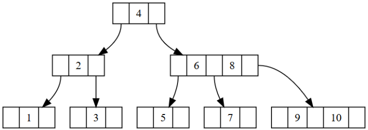
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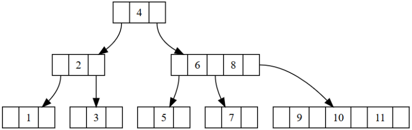
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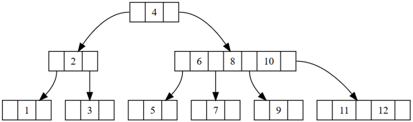
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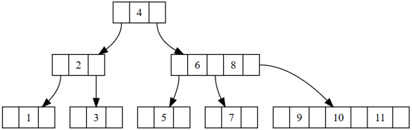

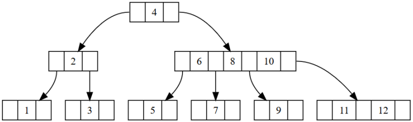

A SPLIT : 4, 6, 8, 10, 12, ...

(II) When does each node SPLIT?

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Let's consider the parent of A, denoted $B \triangleq p(A)$.

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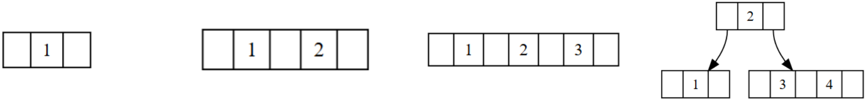
Every time A splits, B obtains a new key.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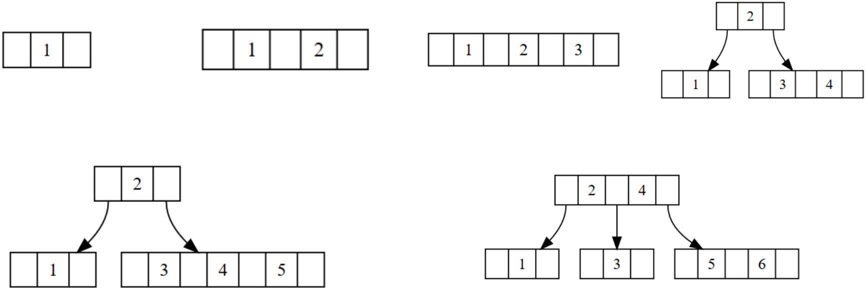
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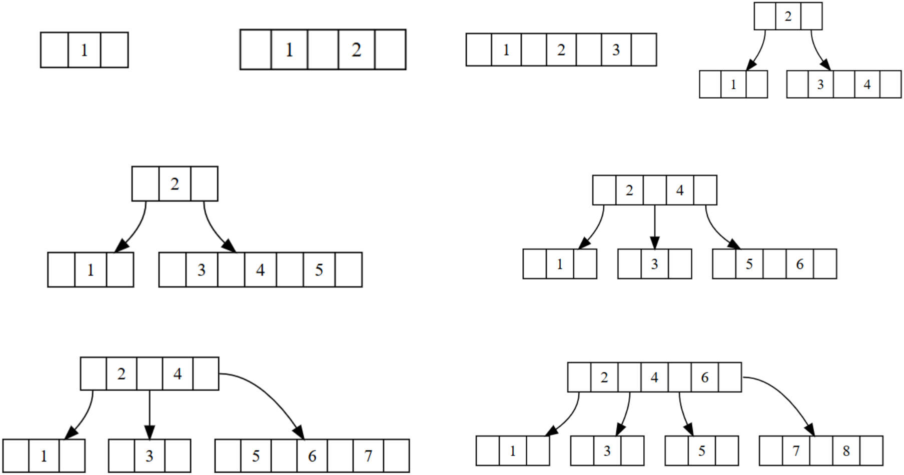
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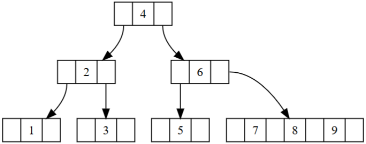
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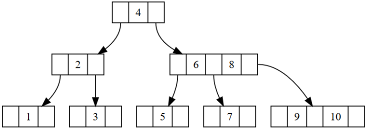
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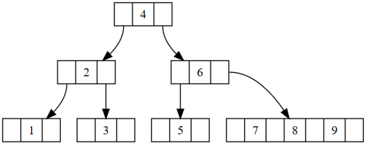
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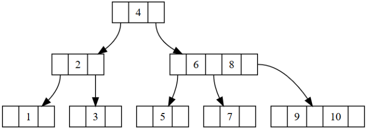
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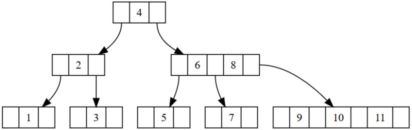
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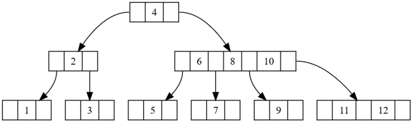
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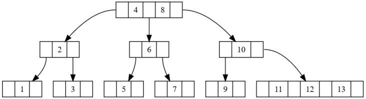
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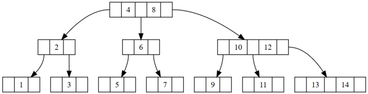
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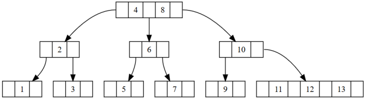
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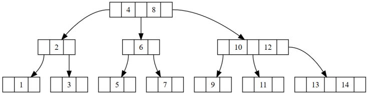
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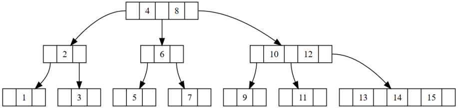
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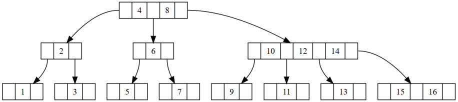
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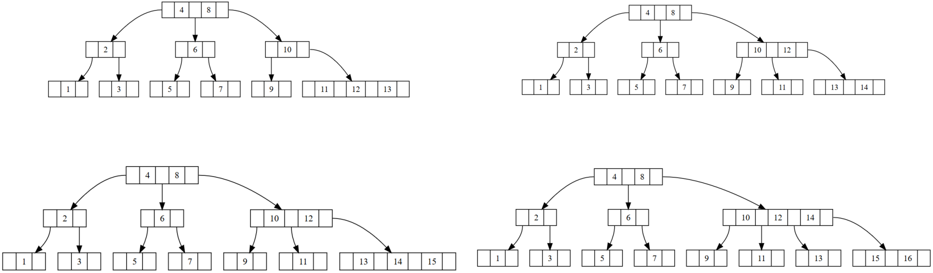
15 16



A SPLIT : 4, 6, 8, 10, 12, ...

B SPLIT : 9, 13, 17, 21, 25, ...

(II) When does each node SPLIT?

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Let's consider the parent of B, denoted $C = p(B)$.

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$$C \text{ SPLIT} : 18, \quad 26, \quad 34, \quad 42, \quad 50, \quad \ldots$$

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$$A : 1 \qquad B : 2 \qquad C : 3$$

T_i : the first time point the i-th node splits

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$$T_i = 2^{i+1} + i - 1$$

$$X_n = 1 + \sum_{s \in S} \Big(2\, |s_R| + |s_{NR}| \Big)$$

$$(T_s = s_R \uplus s_{NR})$$

(III) How does it SPLIT, as a root or a non-root?

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&= 1 + \sum_{i=1}^{\infty} [T_i \le n] \left(\left\lfloor \frac{n - 2^{i+1} - i + 1}{2^i} \right\rfloor \right)
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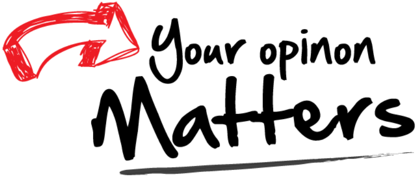
&= 1 + \sum_{i=1}^{\infty} [T_i \le n] \left(\left\lfloor \frac{n - T_i}{2^i} \right\rfloor + 2 \right) \\
&= 1 + \sum_{i=1}^{\infty} [T_i \le n] \left(\left\lfloor \frac{n - 2^{i+1} - i + 1}{2^i} \right\rfloor \right) \\
&= 1 + \sum_{\substack{i \ge 1 \\ 2^{i+1} + i - 1 \le n}} \left\lfloor \frac{n - i + 1}{2^i} \right\rfloor \\
&= 1 + \sum_{\substack{i \ge 0 \\ 2^{i+2} + i \le n}} \left\lfloor \frac{n - i}{2^{i+1}} \right\rfloor
\end{aligned}$$


Thank
You!



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