What We Talk About When We Talk About Isomorphism Theorems

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Q : Do isomorphic groups behave exactly the same?

$$H \lhd G,\ K \lhd G,\ H \cong K \Longrightarrow G/H \cong G/K.$$

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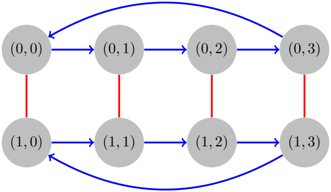

$$K = \{(0,0),(0,2)\}$$

$$G = Z_2 \times Z_4$$

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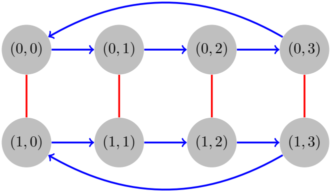

$K = \{(0,0),(0,2)\}$

$$G = Z_2 \times Z_4$$

$$G = \mathbb{Z},\ H = 2\mathbb{Z},\ K = 3\mathbb{Z}$$

Problem 9.3-23

$$G \times H \cong H \times K \implies G \cong K$$

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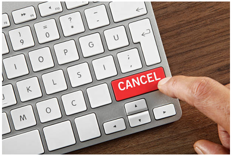

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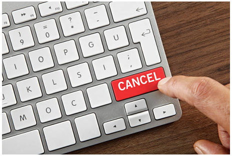

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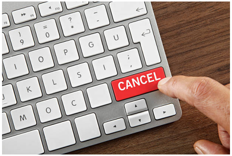

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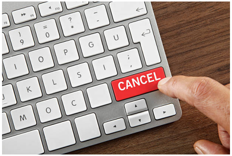

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"On Cancellation in Groups" by R. Hirshon, 1969

$$G \times H \cong H \times K, \ |K| < \infty \Longrightarrow G \cong K$$

Problem 11.4-17

$\phi : G_1 \to G_2$ is a surjective group homomorphism.

$$H_1 \lhd G_1, \quad \phi(H_1) = H_2 \Longrightarrow G_1/H_1 \cong G_2/H_2$$

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$$G_1 = \mathbb{Z}_2, \quad G_2 = \{e\}, \quad H_1 = \{0\}, \quad H_2 = \{e\}$$

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Find all homomorphisms from $\mathbb{Z}_{24}$ to $\mathbb{Z}_{18}$.

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$$\phi(1) = 0, 9, 6, 12, 3, 15$$

Emmy Noether ($1882 \sim 1935$)

Theorem (The First Isomorphism Theorem)

$$\psi : G \to H \implies \frac{G}{Ker\ \psi} \cong \psi(G)$$

Theorem (The Second Isomorphism Theorem)

$$H \le G, N \lhd G \implies \frac{H}{H \cap N} \cong \frac{HN}{N}$$

Theorem (The Third Isomorphism Theorem)

$$H \lhd G, N \lhd G, N \subseteq H \implies \frac{G}{H} \cong \frac{G/N}{H/N}$$

Theorem (The Fourth Isomorphism Theorem (Correspondence))

$$N \lhd G \implies$$

$$\{(normal)\ subgroups\ of\ G\ containing\ N\} \leftrightarrow \{(normal)\ subgroups\ of\ G/N\}$$

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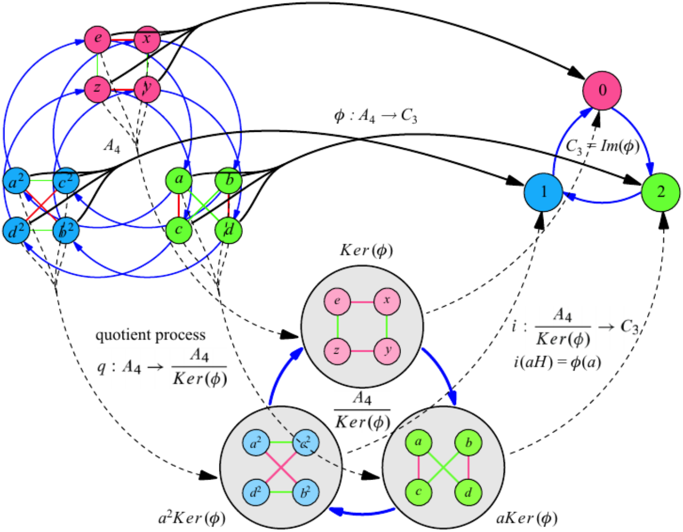
$$\boxed{\frac{G}{\text{Ker } \psi} : \text{Quotient } G \text{ out by Ker } \psi}$$

$$\begin{aligned} \rho_1 &= (2\ 3\ 4) & \rho_1^2 &= (2\ 4\ 3) \\ \rho_2 &= (1\ 3\ 4) & \rho_2^2 &= (1\ 4\ 3) \\ \rho_3 &= (1\ 2\ 4) & \rho_3^2 &= (1\ 4\ 2) \\ \rho_4 &= (1\ 2\ 3) & \rho_4^2 &= (1\ 3\ 2) \end{aligned}$$

$$\begin{aligned} r_1 &= (1\ 4)(2\ 3) \\ r_2 &= (1\ 2)(3\ 4) \\ r_3 &= (1\ 3)(2\ 4) \end{aligned}$$

$$\boxed{Sym(T) \cong A_4 = \left\{ \text{id}, \quad \underbrace{\text{3-cycle},}_{\#=8} \quad \underbrace{\text{2-2-cycle}}_{\#=3} \right\}}$$



$$\phi : A_4 \to C_3 \qquad (\text{Ker } \phi = K_4)$$

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$$\text{To show } \frac{G_1}{N} \cong G_2.$$

$$\frac{\mathbb{Z} \times \mathbb{Z}}{\langle (1,1) \rangle} \cong \mathbb{Z}$$

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$$\text{Ker } f = \langle (1,1) \rangle$$

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$$h \in H \cap N \implies hN = N$$

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$$G = \mathbb{Z}_{24}, \quad H = \langle 4 \rangle, \quad N = \langle 6 \rangle$$

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$$ab = \gcd(a, b) \cdot \operatorname{lcm}(a, b)$$

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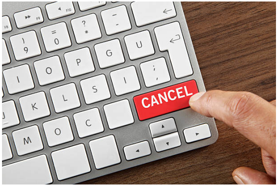

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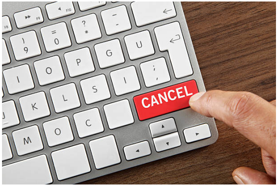

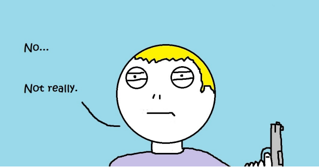

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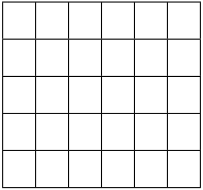

View G and H from the point of view of N

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Absorption!!!

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$n \mid m$

$m\mathbb{Z} \lhd n\mathbb{Z} \lhd \mathbb{Z}$

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$$\{0, 1\} \cong \frac{\{0, 1, 2, \cdots, 9\}}{}$$

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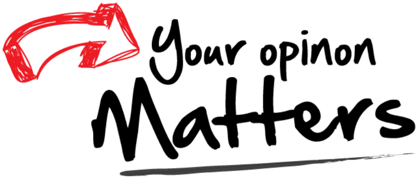
THANK YOU



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