2-2 The Efficiency of Algorithms

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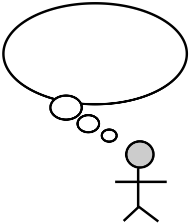
(1) Diameter of Convex Polygon: $\Theta(n)$

(2) Lower Bound for Sorting: $\Omega(n \lg n)$

(3) Traversal over Trees: DFS/BFS ($\Theta(n)$)



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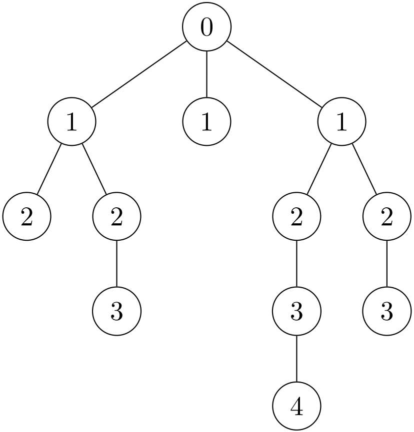
I have thought that $\cdots$

DH 4.2 (a): Sum of Depths



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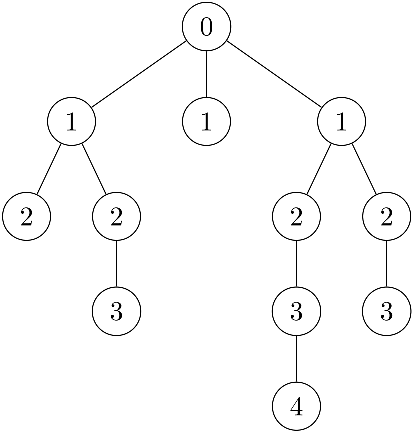

$$\text{sum-of-depths}(r) = \begin{cases} & \\ \displaystyle\sum_{v:\text{child of } r} \text{sum-of-depths}(v) + \text{depth of } r, & \end{cases}$$

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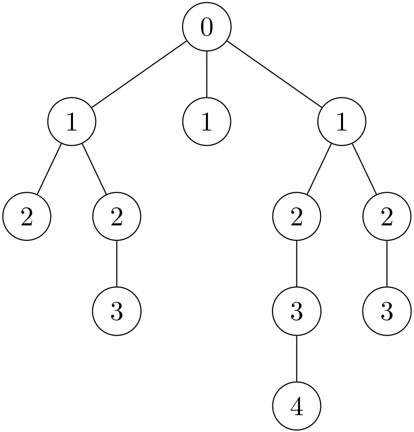

$$\text{sum-of-depths}(r) = \begin{cases} \text{depth of } r, & r \text{ is a leaf} \\ \displaystyle\sum_{v:\text{child of } r} \text{sum-of-depths}(v) + \text{depth of } r, & \text{o.w.} \end{cases}$$

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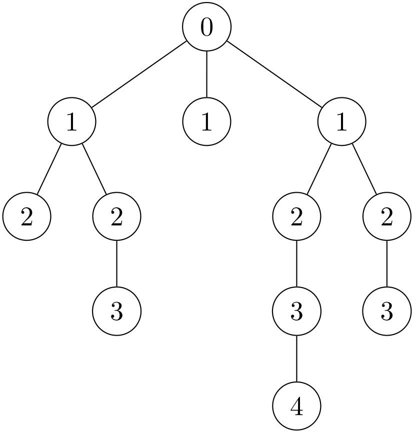

$$\text{sum-of-depths}(r, d) = \begin{cases} d, & r \text{ is a leaf} \\ \sum\limits_{v\text{:child of } r} \text{sum-of-depths}(v, d+1) + d, & \text{o.w.} \end{cases}$$

Algorithm 1 Calculate the sum of depths of all nodes of a tree T.

procedure $\textsc{Sum-of-Depths}()$
    return $\textsc{Sum-of-Depths}(T, 0)$

procedure $\textsc{Sum-of-Depths}(r, \mathit{depth})$ ▷ r: root of a tree
    if T is a leaf **then**
        return depth
    for all child vertex v of r **do**
        $\mathit{depth} \leftarrow \mathit{depth} + \textsc{Sum-of-Depths}(v, \mathit{depth} + 1)$
    return depth

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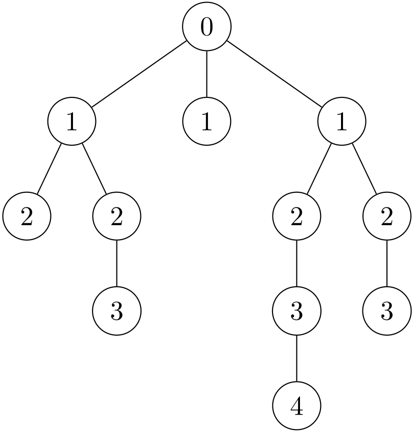
Master Theorem?

$$\text{sum-of-depths}(r, \color{red}{d}) = \begin{cases} d, & r \text{ is a leaf} \\ \displaystyle\sum_{v\text{:child of } r} \text{sum-of-depths}(v, \color{red}{d+1}) + d, & \text{o.w.} \end{cases}$$

Master Theorem?

$$\Theta(m+n) = \Theta(n)$$

DH 4.2 (b): Number of Nodes at Depth K



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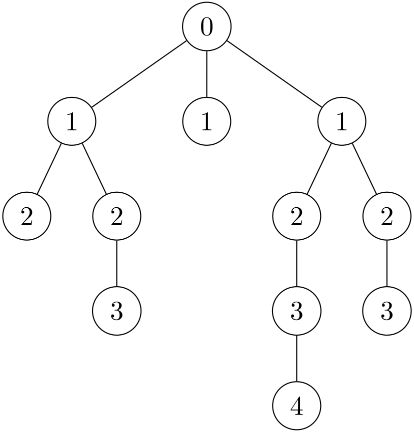

$$\text{nodes-at-depth}(r, k) = \begin{cases} \\ \displaystyle\sum_{v:\text{child of } r} \text{nodes-at-depth}(v, k-1), \end{cases}$$

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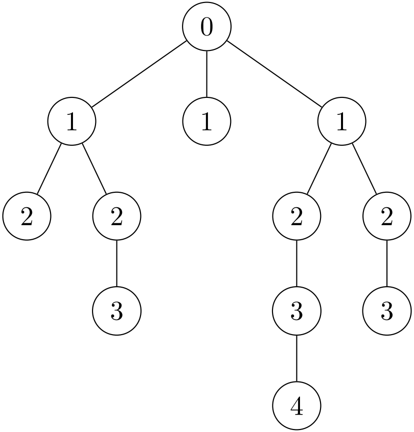


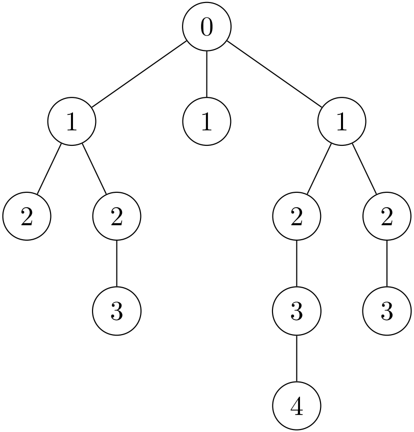
$$\text{nodes-at-depth}(r, k) = \begin{cases} 1, & k = 0 \\ 0, & k > 0 \land r \text{ is a leaf} \\ \displaystyle\sum_{v:\text{child of } r} \text{nodes-at-depth}(v, k-1), & \text{o.w.} \end{cases}$$

Algorithm 2 Count the number of nodes in T at depth K.

```
procedure NODES-AT-DEPTH()
    return NODES-AT-DEPTH(T, K)

procedure NODES-AT-DEPTH(r, k)                    ▷ r: root of a tree
    if k = 0 then
        return 1
    if r is a leaf then
        return 0
    num ← 0
    for all child vertex v of r do
        num ← num + NODES-AT-DEPTH(v, k − 1)
    return num
```

DH 4.2 (c): Any Leaf at an Even Depth?



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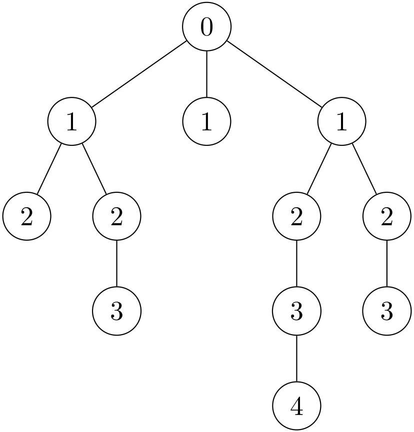

$$\text{leaf-at-depth}(r, \color{red}{parity}) = \begin{cases} \\ \displaystyle\sum_{v\text{:child of } r} (v, \color{red}{1 - parity}), \end{cases}$$

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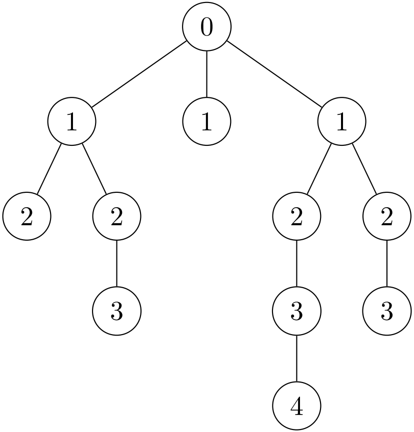


$$\text{leaf-at-depth}(r, \color{red}{parity}) = \begin{cases} 1 - parity, & r \text{ is a leaf} \\ \displaystyle\sum_{v\text{:child of } r} (v, \color{red}{1 - parity}), & \text{o.w.} \end{cases}$$

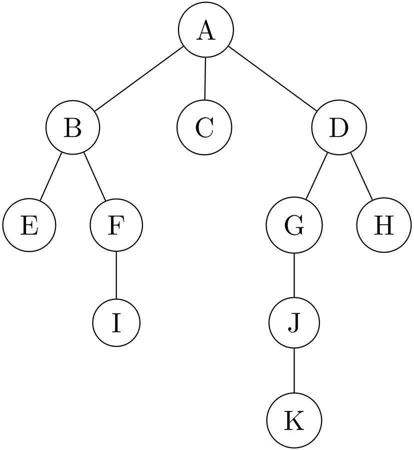
Algorithm 3 Check whether a tree T has any leaf at an even depth.

procedure $\textsc{Leaf-at-Even-Depth}()$
    return $\textsc{Leaf-at-Depth}(T, even = 0)$

procedure $\textsc{Leaf-at-Depth}(r, parity)$ $\triangleright$ r: root of a tree
    if r is a leaf **then**
        return $1 - parity$

    $result \leftarrow 0$
    for all child vertex v of r **do**
        $result \leftarrow result \lor \textsc{Leaf-at-Depth}(v, 1 - parity)$
    return $result$

DH 4.3 (a): Sum of Contents at Each Depth



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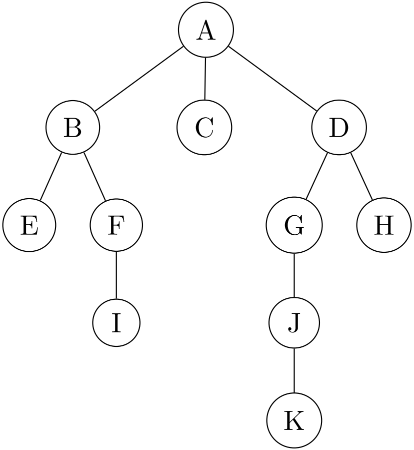
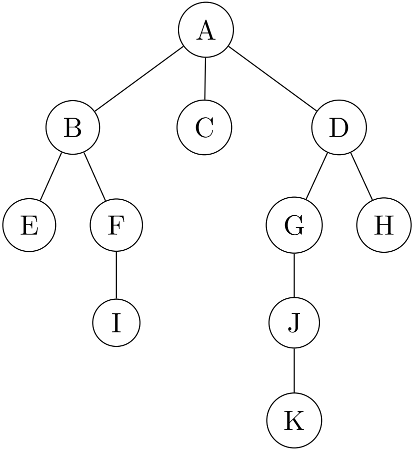


Algorithm 4 Calculate the sum of contents of nodes of a tree T at each depth.

```
procedure SUM-AT-DEPTH(r)                          ▷ r: root of the tree T
    r.depth ← 0
    Q ← ∅
    ENQUEUE(Q, r)
    while Q ≠ ∅ do
        u ← DEQUEUE(Q)
        sumAtDepth[u.depth] += u.content
        for all child vertex v of u do
            v.depth ← u.depth + 1
            ENQUEUE(Q, v)
```

DH 4.3 (b): Depth K with the Maximum Number of Nodes



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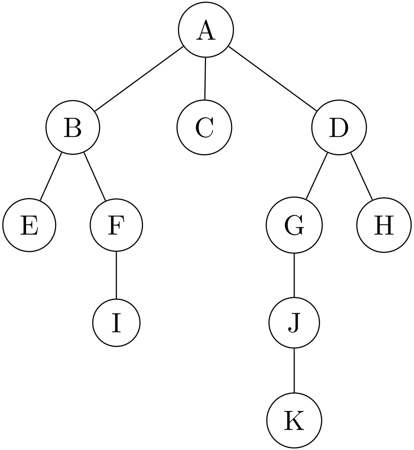

Algorithm 5 Count the number of nodes of a tree T at each depth.

procedure $\textsc{Nodes-at-Depth}(r)$ ▷ r: root of the tree T

  $r.depth \leftarrow 0$

  $Q \leftarrow \emptyset$

  $\textsc{Enqueue}(Q, r)$

  while $Q \neq \emptyset$ **do**

    $u \leftarrow \textsc{Dequeue}(Q)$

    $nodesAtDepth[u.depth] \mathrel{+}= 1$

    for all child vertex v of u **do**

      $v.depth \leftarrow u.depth + 1$

      $\textsc{Enqueue}(Q, v)$

Lower Bound for Comparion-based Sorting

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Lower Bound for Comparison-based Sorting (DH 6.13)

Prove a lower bound of $O(n \lg n)$ on the time complexity of any comparison-based sorting algorithm.

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Computational Model:

the only way to gain order info.

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$$x \in [1 \cdots 99]$$

$$x/10$$

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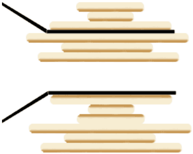

"Bounds For Sorting By Prefix Reversal", 1979

An argument from a student:

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- Therefore, *the total number of comparisons* is at least

$$\sum_{i=1}^{n-1} \lg i = \Omega(n \lg n).$$

COMPUTER **SCIENCE** Questions Tags

Is this lower bound proof for the comparison-based sorting problem correct?

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"This makes claims without justification." — *D.W.*

Decision Tree Model

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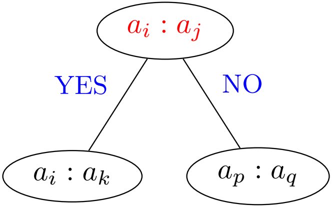

Nodes: comparisions $a_i : a_j$

$$<, \leq, =, \geq, >$$

Edges: two-way decisions (Y/N)

Leaves: possible permutations

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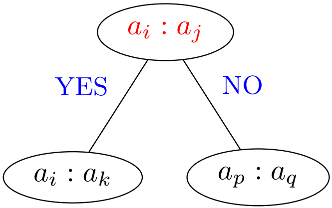

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Assumption (By aware of any assumptions !!!):

All the input elements are **distinct**.

$$a_i < a_j$$

Decision Tree Model

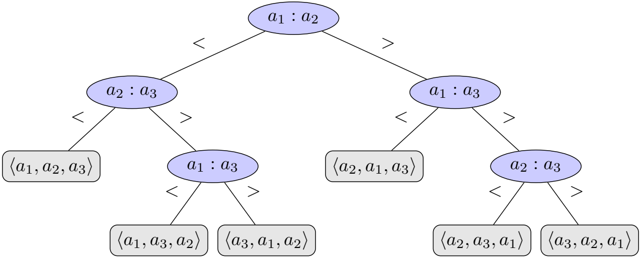
$$\boxed{\text{Any Comparison-based Sorting Algorithm} \xRightarrow{\text{modeled by}} \text{A Decision Tree}}$$

Decision Tree Model

$$\boxed{\text{Any Comparison-based Sorting Algorithm} \xrightarrow{\text{modeled by}} \text{A Decision Tree}}$$

Decision Tree Model

Any Comparison-based Sorting Algorithm $\xrightarrow[\text{- - - - - - - -}]{\text{modeled by}}$ A Decision Tree



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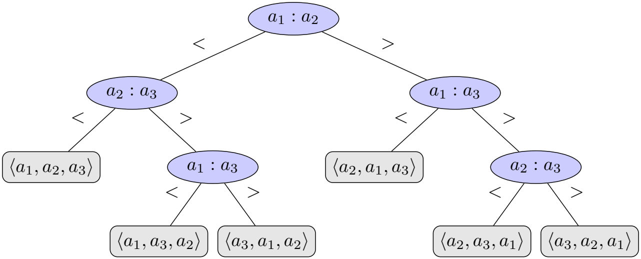

The decision tree for insertion sort on three elements.

Decision Tree Model

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Any Comparison-based Sorting Algorithm $\xRightarrow{\text{modeled by}}$ A Decision Tree

procedure $\text{-}\textsc{Sort}(A, n)$
  for $i \leftarrow 1$ **to** $n - 1$ **do**
    for $j \leftarrow i + 1$ **to** n **do**
      if $A[j] < A[i]$ **then**
        $\textsc{Swap}(A[j], A[i])$

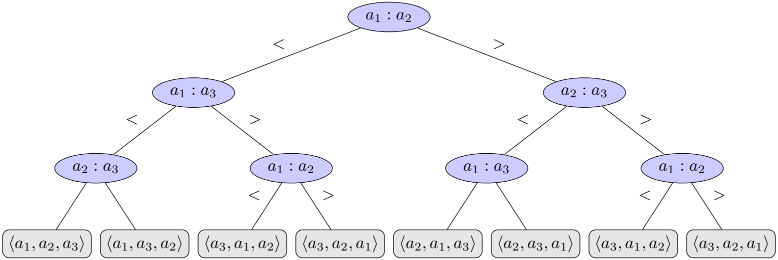
Decision Tree Model

Any Comparison-based Sorting Algorithm $\xRightarrow{\text{modeled by}}$ A Decision Tree

```
procedure SELECTION-SORT(A, n)
    for i ← 1 to n − 1 do
        for j ← i + 1 to n do
            if A[j] < A[i] then
                SWAP(A[j], A[i])
```

Decision Tree Model

Any Comparison-based Sorting Algorithm $\xRightarrow{\text{modeled by}}$ A Decision Tree



The decision tree for selection sort on three elements.

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Algorithm $\mathcal{A}$ on a specific input of size $n \xRightarrow{\text{modeled by}}$ A path through $\mathcal{T}$

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$$\boxed{\begin{array}{c}\text{Worst-case Lower Bound of Comparison-based Sorting on inputs of size } n \\ \xRightarrow{\text{modeled by}} \\ \text{The Minimum Height of All } \mathcal{T}\text{s}\end{array}}$$

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The Minimum Height of All $\mathcal{T}$s

To be a full binary tree:

$$\#\text{ of leaves } \leq 2^h$$

Decision Tree Model

Worst-case Lower Bound of Comparison-based Sorting on inputs of size n

$$\xRightarrow{\text{modeled by}}$$

The Minimum Height of All $\mathcal{T}$s

To be a full binary tree:

$$\# \text{ of leaves } \le 2^h$$

To be a correct sorting algorithm:

$$\# \text{ of leaves } \ge n!$$

Lower Bound for Comparison-based Sorting

$$\boxed{n! \le \# \text{ of leaves} \le 2^h}$$

Lower Bound for Comparison-based Sorting

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Lower Bound for Comparison-based Sorting

$$\boxed{n! \le \# \text{ of leaves} \le 2^h}$$

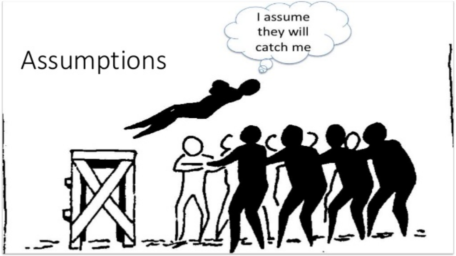
$$h \ge \lg n! = \Omega(n \lg n)$$

Stirling Formula (by *James Stirling*):

$$n! = \Theta\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right)$$

Looking Back

Looking Back



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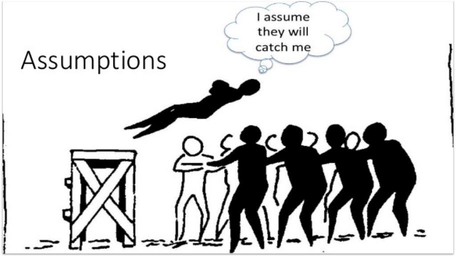

Assumptions (By aware of any assumptions !!!):

(a) Comparison-based sorting algorithms.

(b) All the input elements are **distinct**.

The k-sorted Problem

An array $A[1 \cdots n]$ is k-sorted if it can be divided into k blocks, each of size n/k (we assume that $n/k \in \mathbb{N}$), such that the elements in each block are larger than the elements in earlier blocks and smaller than elements in later blocks. The elements within each block need *not* be sorted.

(a) Describe an algorithm that k-sorts an arbitrary array in $O(n \log k)$ time.

(b) Prove that any comparison-based k-sorting algorithm requires $\Omega(n \log k)$ comparisons in the worst case.

(c) Describe an algorithm that completely sorts an already k-sorted array in $O(n \log(n/k))$ time.

(d) Prove that any comparison-based algorithm to completely sort a k-sorted array requires $\Omega(n \log(n/k))$ comparisons in the worst case.

Convex Polygon Diameter (DH 6.8)

Show that the “Convex Polygon Diameter” algorithm is of linear-time complexity.

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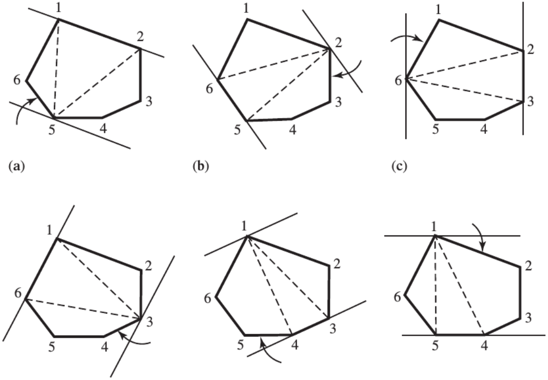

(a) (b) (c)

(d) (e) (f)

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Q : Linear-time of **WHAT**?

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A : A convex polygon

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A : $d(p_1, p_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

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$$\Theta(c \cdot n) = \Theta(n)$$

Correctness

Theorem (A Simple Observation)

For a convex polygon, a pair of vertices determine the diameter.

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Proof.

□

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BUT, we have *not* enumerated *all* pairs of vertices.

Theorem (A Simple Observation)

For a convex polygon, a pair of vertices determine the diameter.

Proof.

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For a convex polygon, a pair of vertices determine the diameter.

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BUT, we have *not* enumerated *all* pairs of vertices.

We have enumerated *all* pairs of vertices
which *admits parallel supporting lines*.

Definition (Line of Support)

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$$L \cap P = \text{ a vertex/an edge of } P.$$

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Definition (Antipodal)

An *antipodal* is a pair of points that admits parallel supporting lines.

We have enumerated *all* antipodals.

Theorem

If AB is a diameter of a convex polygon P, then AB is an antipodal.

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Proof.

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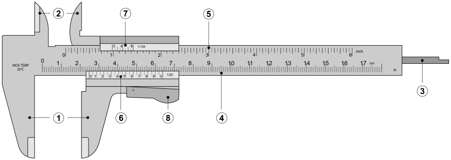
Proof.

$\square$

Rotating Caliper



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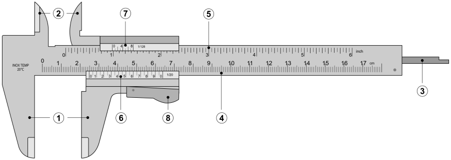
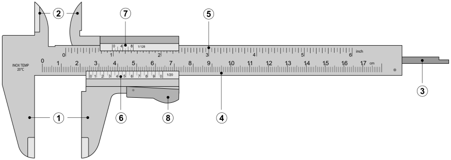


"Computational Geometry"

Ph.D Thesis, Michael Shamos, 1978

Rotating Caliper



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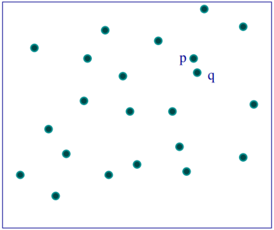
"Solving Geometric Problems with the Rotating Calipers", 1983

Finding the Closest Pair of Points



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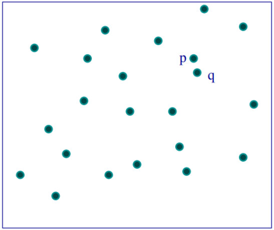
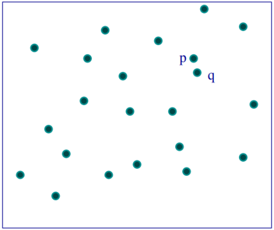


A Classic and Beautiful Divide-Conquer Algorithm:

Finding the Closest Pair of Points



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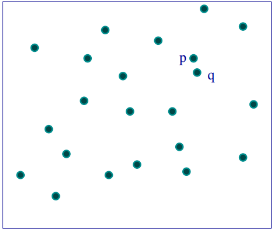
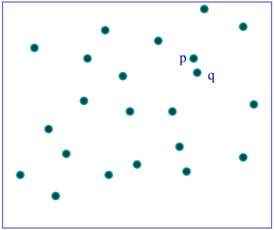


A Classic and Beautiful Divide-Conquer-Combine Algorithm:

Finding the Closest Pair of Points



A Classic and Beautiful Divide-Conquer-Combine Algorithm:

Section 33.4, CLRS


TREASURE
HUNT
on foot


TREASURE
HUNT
on foot


Hints
&
Tips


Help
Wanted

Repeated Elements Problem


Thank You!