Lecture 4 supplement: detailed proof

Here are the details of the proof we gave today that if $|A| \leq |B|$ and if $|B| \leq |A|$ that $|A| = |B|$. This is called the Cantor-Schröder-Bernstein Theorem.

See Wikipedia for another writeup.

Definitions

First a reminder of some relevant definitions:

- A function $f : A \to B$ is one-to-one if for all x_1 and $x_2 \in A, f(x_1) \neq f(x_2)$ unless $x_1 = x_2$. We will use the contrapositive definition:

  f is **one-to-one** if, for all x_1 and x_2, if $f(x_1) = f(x_2)$ then $x_1 = x_2$.

- A function f is **onto** if, for every y in the codomain, there is some x in the domain with $f(x) = y$.
- $|A| \leq |B|$ if there exists a function $f : A \to B$ that is one-to-one.
- $|A| = |B|$ if there exists a function $f : A \to B$ that is both one-to-one and onto.

The proof

We will do a direct proof. Assume that $|A| \leq |B|$ and $|B| \leq |A|$. By definition, this means that there exists functions $f : A \to B$ and $g : B \to A$ that are both one-to-one.

Our goal is to piece these together to form a function $h : A \to B$ which is both one-to-one *and* onto.

Chains

To build the function h, we need to give its output on every input. To define it, we need to consider *chains* of elements that are formed by repeatedly applying f and g.

The chain of an element $x \in A$ contains $x, f(x), g(f(x)), f(g(f(x))), g(f(g(f(x))))$ and so on. It also contains any elements that can be reached by going *backwards* along the chain. That is, if there happens to be some y such that $g(y) = x$, then y is in the chain.

There need not be such a y because g is not onto. However, if there is a y, it must be unique, because g is one-to-one. If such a y exists, we will call it $g^{-1}(y)$. This discussion shows that g^{-1} is a partial function.

Similarly, the chain of x will include $f^{-1}(g^{-1}(x))$, $g^{-1}(f^{-1}(g^{-1}(x)))$ and so on.

We want to distinguish between various types of chains, based on what happens as you walk backwards along them (that is, if we consider x, $g^{-1}(x), f^{-1}(g^{-1}(x))$, ... as defined above). There are 4 types:

1. The chain forms a loop
2. Chains that go "backwards" forever without repeating.
3. Chains that stop in A. That is, they end on some x with $g^{-1}(x)$ undefined.
4. Chains that stop in B.

Note that every element of both A and B is part of exactly one chain.

Constructing h

We define h as follows. If x is in a chain of type 1, 2, or 3, then we define $h(x) = f(x)$. If x is in a chain of type 4, then we define $h(x) = g^{-1}(x)$. $g^{-1}(x)$ is defined, because if it wasn't, then x would be in a chain of type 3.

What's left is to show that h is one-to-one and onto.

Proof that h is one-to-one

We must show that whenever $h(x_1) = h(x_2)$, that $x_1 = x_2$. We will prove this directly: assume that $h(x_1) = h(x_2)$. Notice that $h(x)$ is always part of the same chain as x. Therefore, x_1 and x_2 must be in the same chain.

Let's consider the possible types of chains:

- If the chain of x_1 and x_2 is of type 1, 2, or 3, then $h(x_1) = f(x_1)$ and $h(x_2) = f(x_2)$. Therefore,

  $$f(x_1) = h(x_1) = h(x_2) = f(x_2)$$

  Since f is one-to-one, this implies that $x_1 = x_2$ as required.

- If the chain is of type 4, then we have that $h(x_1) = y_1$ with $g(y_1) = x_1$, and $h(x_2) = y_2$ with $g(y_2) = x_2$. Since $h(x_1) = h(x_2)$, we have $y_1 = y_2$, so

  $$x_1 = g(y_1) = g(y_2) = x_2$$

  as required.

In any case, we have shown that $x_1 = x_2$, so we conclude that h is one-to-one.

Proof that h is onto

Given an arbitrary $y \in B$, we must find some $x \in A$ with $h(x) = y$. We consider the chain containing y.

- If that chain is of type 1, 2, or 3, then we know there is some x such that $f(x) = y$. Since x and y are in the same chain, we have that x's chain is of type 1, 2 or 3, so $h(x) = f(x) = y$.
- If the chain is of type 4, then we know that $g(y)$ is also in a chain of type 4. That means that $h(g(y)) = y$. Therefore there is some x (namely $g(y)$) that maps to y.

In either case, we have found an element that maps to y, so h is onto.

Conclusion

We have defined a function $h : A \to B$ and shown that it is both one-to-one and onto. Therefore (by definition) $|A| = |B|$.