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A generalization of Menger's Theorem

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ABSTRACT

This paper generalizes one of the celebrated results in Graph Theory due to Karl. A. Menger (1927), which plays a crucial role in many areas of flow and network theory. This paper also introduces and characterizes strength reducing sets of nodes and arcs in weighted graphs.

1. Introduction

Weighted graph theory has numerous applications in various fields like clustering analysis, operations research, database theory, network analysis, information theory, etc. Connectivity concepts play a key role in applications related with graphs and weighted graphs. Several authors including Bondy and Fan [1–3], Broersma et al. [4], Enomoto [5], Mathew and Sunitha [6–9] introduced many connectivity concepts in weighted graphs following the works of Dirac [10], Erdos, Gallai and Grotschel [11,12].

In this article, we introduce some new connectivity concepts in weighted graphs. In a weighted graph model, for example, in an information network or an electric circuit, the reduction of flow between pairs of nodes is more relevant and may frequently occur than the total disruption of the flow or the disconnection of entire networks [8,9]. This concept is our motivation. As weighted graphs are generalized structures of graphs, the concepts introduced in this article also generalize the classic connectivity concepts.

A *weighted graph* G is a graph in which every arc e is assigned a nonnegative number $w(e)$, called the *weight* of e. The set of all the neighbors of a vertex v in G is denoted by $N_G(v)$ or simply $N(v)$, and its cardinality by $d_G(v)$ or $d(v)$ [13]. The *weighted degree* of v is defined as $d_G^w(v) = \sum_{x \in N(v)} w(vx)$. When no confusion occurs, we denote $d_G^w(v)$ by $d^w(v)$. The *weight of a cycle* is defined as the sum of the weights of its edges. An unweighted graph can be regarded as a weighted graph in which every edge e is assigned weight $w(e) = 1$. Thus, in an unweighted graph, $d^w(v) = d(v)$ for every vertex v, and the weight of a cycle is simply the length of the cycle. An *optimal cycle* is a cycle which has a maximum weight [1].

2. Strength reducing sets

In a weighted graph G, we can associate to each pair of nodes in G, a real number called strength of connectedness. It is evaluated using strengths of different paths joining the given pair of nodes. We have a set of new definitions which are given below.

Definition 1 (*[8]*). Let G be a weighted graph. The strength of a path P (respectively, strength of a cycle C) of n edges e_i, for $1 \le i \le n$, denoted by $s(P)$ (respectively, $s(C)$), is equal to $s(P) = \min_{1 \le i \le n}\{w(e_i)\}$.

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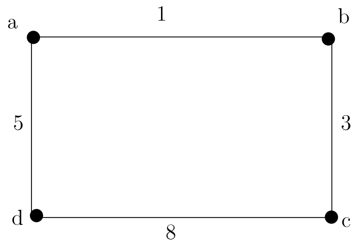

Fig. 1. Strength of connectedness.

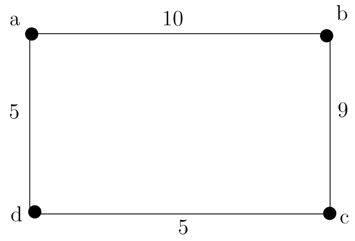

Fig. 2. Partial cutnodes and strongest paths.

Definition 2 ([8]). Let G be a weighted graph. The strength of connectedness of a pair of nodes $u, v \in V(G)$, denoted by $CONN_G(u, v)$ is defined as $CONN_G(u, v) = \text{Max}\{s(P) : P \text{ is a } u\text{–}v \text{ path in } G\}$. If u and v are in different components of G, then $CONN_G(u, v) = 0$.

Example 1. Let $G(V, E)$ be a weighted graph with $V = \{a, b, c, d\}$ and $E = \{e_1 = (a, b), e_2 = (b, c), e_3 = (c, d), e_4 = (d, a)\}$ with $w(e_1) = 1$, $w(e_2) = 3$, $w(e_3) = 8$, $w(e_4) = 5$. (Fig. 1)

Here, $CONN_G(a, b) = 3$, $CONN_G(a, c) = 5$, $CONN_G(a, d) = 5$, $CONN_G(b, c) = 3$, $CONN_G(b, d) = 3$, $CONN_G(c, d) = 8$. Next we have an obvious result.

Proposition 1 ([8]). *Let G be a weighted graph and H, a weighted subgraph of G. Then for any pair of nodes $u, v \in G$, we have $CONN_H(u, v) \leq CONN_G(u, v)$.*

Definition 3 ([8]). A u–v path in a weighted graph G is said to be strongest u–v path if $s(P) = CONN_G(u, v)$.

Definition 4 ([8]). Let G be a weighted graph. A node w is said to be a partial cutnode (p-cutnode for short) of G if there exists a pair of nodes u, v in G such that $u \neq v \neq w$ and $CONN_{G-w}(u, v) < CONN_G(u, v)$.

It is proved that a node w in a weighted graph G is a p-cutnode if and only if w is an internal node of every maximum spanning tree [8].

Example 2. Let $G(V, E)$ be a weighted graph (Fig. 2) with $V = \{a, b, c, d\}$ and $E = \{e_1 = (a, b), e_2 = (b, c), e_3 = (c, d), e_4 = (d, a)\}$ with $w(e_1) = 10$, $w(e_2) = 9$, $w(e_3) = 5$, $w(e_4) = 5$.

Node b is a partial cutnode since $CONN_{G-b}(a, c) = 3 < 9 = CONN_G(u, v)$. Also note that the path abc is the unique strongest a–b path in G.

Definition 5 ([6,8]). Let G be a weighted graph. An arc $e = (u, v)$ is said to be a partial bridge (p-bridge for short) if $CONN_{G-e}(u, v) < CONN_G(u, v)$. A p-bridge is said to be a partial bond (p-bond for short) if $CONN_{G-e}(x, y) < CONN_G(x, y)$ with at least one of x or y is different from both u and v and is said to be a partial cutbond (p-cutbond for short) if both x and y are different from u and v.

Partial bridges are characterized in [8] and partial bonds and cutbonds in [6].

Example 3. Let $G(V, E)$ be a weighted graph with $V = \{a, b, c, d\}$ and $E = \{e_1 = (a, b), e_2 = (b, c), e_3 = (c, d), e_4 = (d, a)\}$ with $w(e_1) = 10$, $w(e_2) = 9$, $w(e_3) = 8$, $w(e_4) = 5$. (Fig. 2 with $w(c, d) = 8$.) Here all arcs except arc (a, d) are partial bonds. In particular, arc (b, c) is a partial cutbond since $CONN_{G-(b,c)}(a, d) = 5 < 8 = CONN_G(a, d)$.

Now we define the concept of strength reducing set as follows.

Definition 6. A strength reducing set (srs) of nodes in a weighted graph G is a set of nodes $S \subseteq V(G)$ with the property that either $CONN_{G-S}(u, v) < CONN_G(u, v)$ for some pair of nodes $u, v \in V(G) - S$ or $G - S$ is trivial. If S contains a single node w, then w is a partial cutnode.

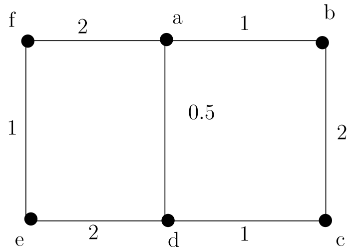

Fig. 3. Strength reducing sets.

Definition 7. A strength reducing set of arcs in a weighted graph G is a set of arcs $F \subseteq E(G)$ with the property that $CONN_{G-F}(u, v) < CONN_G(u, v)$ for some pair of nodes $u, v \in V(G)$ with at least one of u and v different from the end nodes of arcs in F, where $G - F$ is the graph obtained by deleting all arcs in F from G.

If F contains a single arc e, then e is a partial bond.

Example 4. Let $G(V, E)$ be a weighted graph with $V = \{a, b, c, d, e, f\}$ and $E = \{e_1 = (a, b), e_2 = (b, c), e_3 = (c, d), e_4 = (d, e), e_5 = (e, f), e_6 = (f, a), e_7 = (a, d)\}$ with $w(e_1) = w(e_3) = w(e_5) = 1$, $w(e_2) = w(e_4) = w(e_6) = 2$, $w(e_7) = 0.5$ (Fig. 3).

$S = \{b, f\}$ is a strength reducing set of nodes since $CONN_{G-S}(a, d) = 0.5 < 1 = CONN_G(a, d)$. Also $F = \{(a, b), (e, f)\}$ is a strength reducing set of arcs.

Now we can consider a particular pair u, v of nodes and obtain sets of nodes or arcs whose removal from G reduces $CONN_G(u, v)$.

Definition 8. Let u, v be any two nodes of a weighted graph G. A u–v strength reducing set of nodes in G is a set of nodes $S \subseteq V(G)$ such that $CONN_{G-S}(u, v) < CONN_G(u, v)$. If S contains a single node w, then w is a partial cutnode.

Note that a u–v strength reducing set of nodes do not always exist. In Example 4, no b–c strength reducing set of nodes exists as the arc (b, c) itself is a strongest b–c path.

Definition 9. Let u, v be any two nodes of a weighted graph G. A u–v strength reducing set of arcs in G is a set of arcs $F \subseteq E(G)$ such that $CONN_{G-F}(u, v) < CONN_G(u, v)$. If F contains a single arc e, then e is a partial bridge.

Definition 10. A u–v strength reducing set of nodes(arcs) with n elements is said to be a minimum u–v strength reducing set of nodes (arcs) if there exists no u–v strength reducing set of nodes (arcs) with less than n elements.

Definition 11. Let G be a weighted graph. Then an arc $e = (x, y)$ is said to be strong if its weight is at least equal to the strength of connectedness between its end nodes in G. An arc $e = (x, y) \in E$ is said to be α-strong if $CONN_{G-e}(x, y) < w(e)$, β-strong if $CONN_{G-e}(x, y) = w(e)$ and a δ-arc if $CONN_{G-e} > w(e)$.

Clearly, an arc e is strong if it is either α-strong or β-strong. If (x, y) is a strong arc, then x and y are said to be strong neighbors to each other.

Next we have an important, but obvious result.

Theorem 1. *Let G be a connected weighted graph and let u, v be two nodes in G such that (u, v) is not a strong arc. A set $S \subseteq V(G)$ is a u–v strength reducing set in G if and only if every strongest u–v path in G contains at least one node from S.*

A similar result for a strength reducing set of arcs is given below.

Theorem 2. *Let G be a connected weighted graph and let u, v be two nodes in G. Then a set F of arcs in G is a u–v strength reducing set of arcs if and only if every strongest path from u to v contains at least one arc of F.*

The weighted degree of a weighted graph is discussed in [1]. We define a new type of degree in weighted graphs called strong degree as follows.

Definition 12. Let G be a weighted graph. The *strong degree* of a node $v \in V(G)$ is defined as the sum of weights of all strong arcs incident at v. It is denoted by $d_s(v)$. The minimum strong degree of G is denoted by $\delta_s(G)$ and maximum strong degree $\Delta_s(G)$.

Also if $N_s(v)$ denote the set of all strong neighbors of v, then $d_s(v) = \sum_{u \in N_s(v)} w((u, v))$.

Definition 13. The number of strong arcs in a weighted graph G is said to be the strong size of G. It is denoted by $ss(G)$.

Example 5. Let $G(V, E)$ be a weighted graph with $V = \{a, b, c, d\}$ and $E = \{e_1 = (a, b), e_2 = (b, c), e_3 = (c, d), e_4 = (d, a), e_5 = (b, d), e_6 = (a, c)\}$ with $w(e_1) = w(e_3) = 1, w(e_2) = w(e_4) = 2, w(e_5) = 7, w(e_6) = 0.1$. In this graph, (b, c), (a, d) and (b, d) are the only strong arcs. Hence, $ss(G) = 3$. So $d_s(a) = 2$ whereas $d_w(a) = 3.1$. Also $d_s(b) = d_s(d) = 10, d_s(b) = d_s(d) = 9, d_w(c) = 3.1, d_s(c) = 2$. Thus $\delta_s(G) = 2$ and $\Delta_s(G) = 9$.

Now we have a lemma which is to be used in the following result.

Lemma 1. *Let G be a weighted graph. if x is a node in G, with at least one α-strong arc incident on x, then $ss(G - x) < ss(G)$.*

Proof. Let G be a weighted graph with a node x as given in the statement of the theorem. Let (w, x) be an α-strong arc incident on x. We know that an arc (x, y) is strong if $CONN_G(x, y) \leq w(x, y)$ and is α-strong if $CONN_{G-(x,y)}(x, y) \leq w(x, y)$. Since $G - (x, y)$ is a subgraph of G, by Proposition 1, $CONN_{G-(x,y)}(x, y) \leq CONN_G(x, y)$. Thus if (x, y) is α-strong,

$$w(x, y) \geq CONN_{G-(x,y)} \geq CONN_G(x, y)$$

which implies that (x, y) is strong. As the removal of the node x from G also removes the arc (x, w) from G, it follows that the number of strong arcs of $G - x$ will be less than that of G, that is $ss(G - x) < ss(G)$. □

Also note that if E is a set of strong arcs with cardinality n in G, the removal of E from G reduces the strong size of G by n. That is $ss(G - E) = ss(G) - n$.

Next we present a generalization of one of the celebrated results in Graph Theory due to Karl. A. Menger (1927).

Theorem 3 (*Generalization of the node version of Menger's Theorem*)**.** *Let G be a weighted graph. For any two nodes $u, v \in V(G)$ such that (u, v) is not strong, the maximum number of internally disjoint strongest u–v paths in G is equal to the number of nodes in a minimal u–v strength reducing set.*

Proof. We shall prove the result by induction on the strong size $ss(G)$ of G. When $ss(G) = 0$, the only possibility is that the graph G is empty, so that between any pair of nodes u and v, there do not exist a path in G and both parameters given in the statement of the theorem reduce to zero and hence the result is trivially true for any pair of nodes $u, v \in V(G)$.

Assume that the theorem is true for all weighted graphs G with strong size less than m where $m \geq 1$. Let G be a weighted graph of strong size m. Let $u, v \in V(G)$ such that (u, v) is not strong. If u and v are in different components of G, the theorem is obviously true. So assume that u and v belong to the same component of G. Then either (u, v) is not adjacent or (u, v) is a δ-arc. In both cases u–v strength reducing sets of nodes exist in G. (If (u, v) is strong, then reduction of any number of nodes will not reduce the strength of connectivity between u and v and hence no strength reducing set of nodes exist.)

Now suppose that $S_G(u, v)$ is a minimal strength reducing set of nodes in G with $|S_G(u, v)| = k \geq 1$. By Theorem 1, each strongest u–v path must contain at least one member from $S_G(u, v)$. Hence, any u–v strength reducing set must contain at least as many nodes as the number of internally disjoint strongest $u - v$ paths. In other words, there exists at most k internally disjoint strongest u–v paths. We show that G contains exactly k internally disjoint strongest u–v paths.

If $k = 1$, then $| S_G(u, v) |= 1$. Let $S_G(u, v) = \{w\}$. Then $CONN_{G-\{w\}}(u, v) < CONN_G(u, v)$. That is w is a partial cutnode of G. So every strongest u–v path must pass through w. Hence, the number of internally disjoint u–v paths is one and the result is true. So assume that $k \geq 2$.

Case I: G has a minimal u–v strength reducing set of nodes containing a node x such that both (u, x) and (x, v) are α-strong arcs.

Let $S_G(u, v)$ be the minimal u–v strength reducing set of nodes with the above mentioned property. Then $S_G(u, v) - \{x\}$ is a minimal u–v strength reducing set in $G - \{x\}$ having $k - 1$ nodes. Since both (u, x) and (x, v) are α-strong, they are clearly strong and hence $ss(G - \{x\}) < ss(G)$. By induction, it follows that $G - \{x\}$ contains $k - 1$ internally disjoint strongest u–v paths. Since (u, x) and (x, v) are α-strong, $P = (u, x, v)$ is a strongest u–v path. Thus, we have k internally disjoint strongest u–v paths in G.

Case II: For every minimal u–v strength reducing set $S_G(u, v)$ in G, either every node in $S_G(u, v)$ is an α- strong neighbor of u (that is if w is a node in $S_G(u, v)$, then, (u, w) is an arc which is the unique strongest u–w path.) but not of v or every node in $S_G(u, v)$ is an α-strong neighbor of v but not of u.

Suppose that every node in $S_G(u, v)$ is an α-strong neighbor of u but not of v. Consider a strongest u–v path P in G. Let x be the first node of P which is in $S_G(u, v)$. Then (u, x) is α-strong and since (x, v) is not α-strong, there exists at least one node say y other than u and v such that (x, y) is β-strong. Denote the arc (x, y) by e.

Claim. *Every u–v strength reducing set in $G - \{e\}$ has exactly k nodes.*

If possible suppose that there exists a minimal u–v strength reducing set in $G - \{e\}$ with $k - 1$ nodes say $Z = \{z_1, z_2, \ldots, z_{k-1}\}$. Then $Z \cup \{x\}$ is a minimal u–v strength reducing set in G. Note that every $z_i, i = 1, 2, \ldots, k - 1$ and x are α-strong neighbors of u. Since $Z \cup \{y\}$ also is a minimal u–v strength reducing set in G, it follows that y is an α-strong neighbor of u contradicting the fact that arc (x, y) is β-strong (The arcs (u, x), (u, y) and (x, y) form a triangle with arc (x, y) as the weakest arc. The unique weakest arc of a cycle is a δ-arc). Thus k is the minimum number of nodes in a u–v strength reducing set in $G - \{e\}$. Since $ss(G - \{e\}) < ss(G)$, it follows by induction that there are k internally disjoint u–v paths in $G - \{e\}$ and hence in G.

Case III: There exists a minimum u–v strength reducing set W in G such that no member of W is an α-strong neighbor of both u and v and W contains at least one node which is not an α-strong neighbor of u and at least one node which is not an α-strong neighbor of v.

Let W be a minimal u–v strength reducing set with k elements having the above properties. Let $W = \{w_1, w_2, \ldots, w_k\}$. Consider all strongest paths from u to v. Since W is minimal, $w_i, i = 1, 2, \ldots, k$ must belong to at least one such path. Let G_u be the subgraph of G consisting of all u–w_i subpaths of all strongest u–v paths in which $w_i \in W$ is the only node of the path belonging to W. Note that if $CONN_G(u, v) = t$, then $\mu(x, y) \geq t$ for all arc (x, y) in these paths. let G'_u be the graph constructed from G_u by adding a new node v' and joining v' to each node w_i for $i = 1, 2, \ldots, k$. Let $\sigma(v') = 1$ and $\mu(w_i, v') = \sigma(w_i)$ for every $i = 1, 2, \ldots, k$. The graphs G_v and G'_v are defined similarly.

Since W contains a node that is not an α-strong neighbor of u and a node that is not an α-strong neighbor of v (Note that all newly introduced arcs are strong), we have $ss(G'_u) < ss(G)$ and $ss(G'_v) < ss(G)$.

Clearly, $S_{G'_u}(u, v') = k$ and $S_{G'_v}(u', v) = k$. So by induction G'_u contains k internally disjoint u–v' paths say $A_i, i = 1, 2, \ldots, k$ where A_i contains w_i. Also G'_v contains k internally disjoint $u' - v$ paths say $B_i, i = 1, 2, \ldots, k$ where B_i contains w_i. Let A'_i be the u–w_i subpaths of A_i and B'_i be the w_i–v subpath of B_i for $1 \leq i \leq k$. Now k internally disjoint strongest u–v paths can be constructed by joining A_i and B_i for $i = 1, 2, \ldots, k$ and the theorem is proved by induction. □

Toward the end, we state the arc version of Theorem 3 without proof. The proof is similar to that of Theorem 3.

Theorem 4 (*Generalization of the Arc Version of Menger's Theorem*)**.** *Let $G : (\sigma, \mu)$ be a connected weighted graph and let $u, v \in V(G)$. Then the maximum number of arc disjoint strongest u–v paths in G is equal to the number of arcs in a minimal u–v strength reducing set.*

We conclude this article with the following remarks.

3. Concluding remarks

Connectivity concepts are the key in graph clustering and network problems. The classical parameters are dealing with the disconnection of the graph. In practical applications, the reduction in the flow is more frequent than the disconnection. The authors made an attempt to redefine connectivity concepts in weighted graphs. One of the major theorems in Graph Theory due to Menger is generalized.

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