2-3 Counting

魏恒峰

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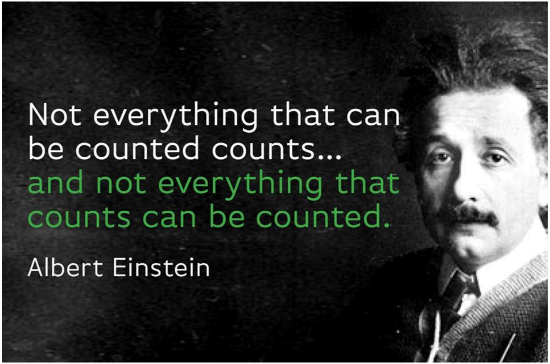
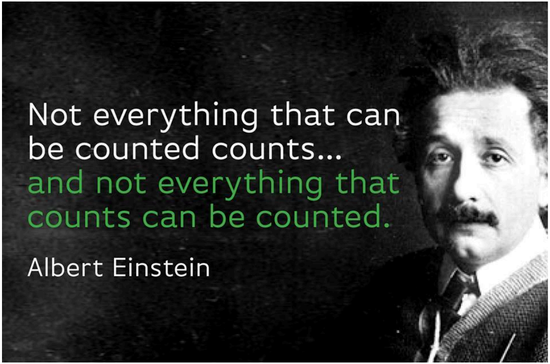

Not everything that can be counted counts...
and not everything that counts can be counted.
Albert Einstein



所以，学好“2-3 组合与计数”是多么重要!

Paring up (CS : $1.2 - 15$)

A tennis club has $2n$ members. We want to pair up the members by twos for singles matches.

(a) In how many ways can we pair up all the members of the club?

(b) Suppose that in addition to specifying who plays whom, we also determine who serves first for each pairing. In how many ways can we specify our pairs?

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$$\binom{2n}{\underbrace{2, 2, \cdots, 2}_{n}} \triangleq \binom{2n}{2}\binom{2n-2}{2}\cdots\binom{4}{2}\binom{2}{2} = \frac{(2n)!}{2^n}$$

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k-Permutation (CS : $1.2 - 5$)

We need to pass out k distinct apples (pieces of fruit) to n children such that *each child may get at most one*.

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$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

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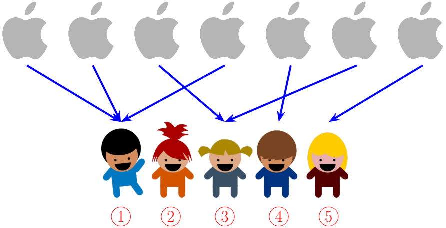

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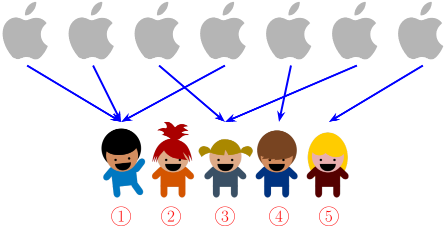

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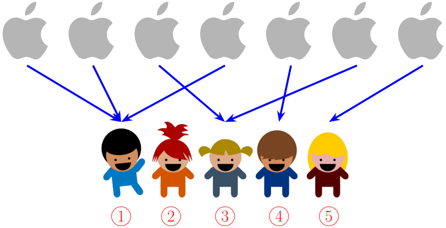

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Partition $(CS : 1.5 - 4$ Extended)

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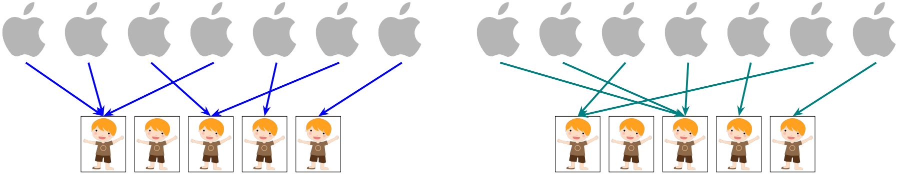

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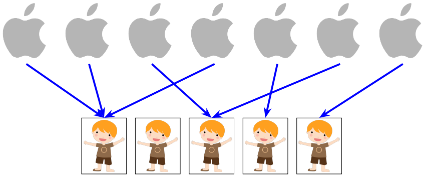

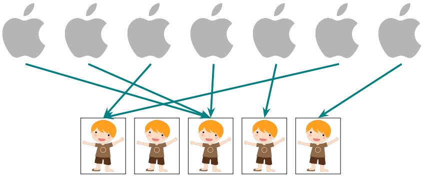

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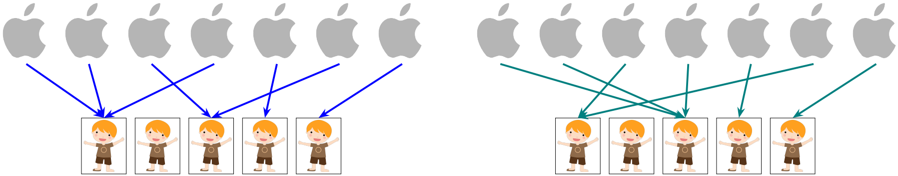

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$$\sum_{x=1}^{x=n} p_x(k)$$

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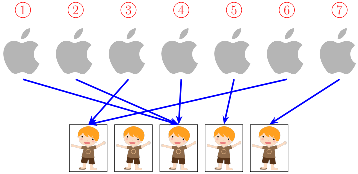

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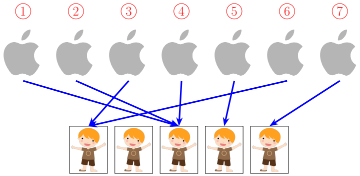

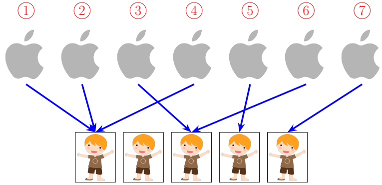

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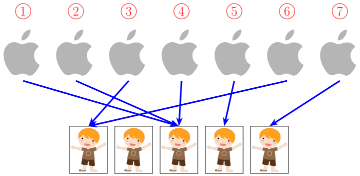

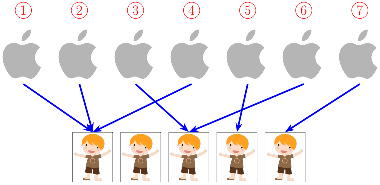

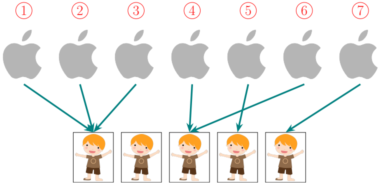

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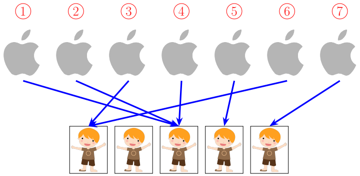

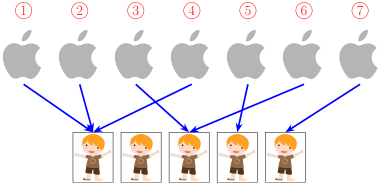

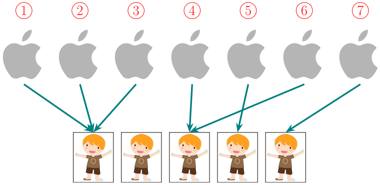

partition of $[1 \cdots k]$ into $\leq n$ parts

Set Partition (CS : $1.5-12$)

$$S(n,k) : \# \text{ of set partitions of } [1 \cdots n] \text{ into } k \text{ classes}$$

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$S(n,k)$: # of set partitions of $[1 \cdots n]$ into k classes

$$S(n,k) = \underbrace{S(n-1,k-1)}_{n \text{ is alone}} + \underbrace{kS(n-1,k)}_{n \text{ is not alone}}, \quad 1 < k < n$$

Computing $\binom{n}{k}$ (CS $1.5:14$)

```
procedure BINOM(n, k)                    ▷ Required: n ≥ k ≥ 0
    if k = 0 ∨ n = k then
        return 1
    return BINOM(n − 1, k) + BINOM(n − 1, k − 1)
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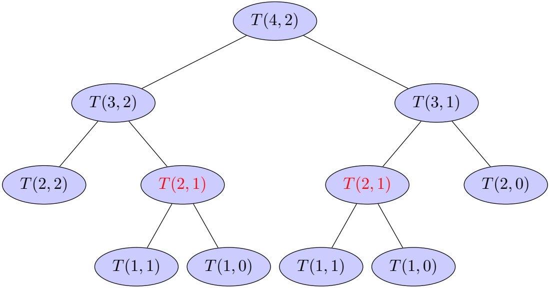

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$$A(n, k) = 1 + A(n-1, k) + A(n-1, k-1)$$

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$$\boxed{T(n,k) = T(n-1,k) + T(n-1,k-1) + c}$$

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

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procedure $\textsc{Binom}(n, k)$ $\quad\triangleright$ Required: $n \geq k \geq 0$

    for $i \leftarrow 0$ **to** n **do**

        $B[i][0] \leftarrow 1$

        $B[i][i] \leftarrow 1$

    for $i \leftarrow 2$ **to** n **do**

        for $j \leftarrow 1$ **to** k **do**

            $B[n][k] \leftarrow B[n-1][k] + B[n-1][k-1]$

    return $B[n][k]$


Thank
You!