What We Talk About When We Talk About Isomorphism Theorems

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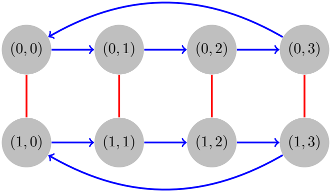
Q : Do isomorphic groups behave exactly the same?

$$H \lhd G,\ K \lhd G,\ H \cong K \Longrightarrow G/H \cong G/K.$$

$$H = \{(0,0),(1,0)\}$$



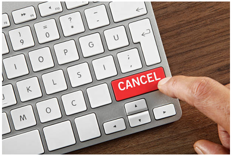
$$K = \{(0,0),(0,2)\}$$

$$G = Z_2 \times Z_4$$

$$G = \mathbb{Z},\ H = 2\mathbb{Z},\ K = 3\mathbb{Z}$$

Problem 9.3-23

$$G \times H \cong H \times K \Longrightarrow G \cong K$$



$$G = \mathbb{Z}, \quad H = \{e\}, \quad K = \prod_{n \in \mathbb{N}} \mathbb{Z}$$

"On Cancellation in Groups" by R. Hirshon, 1969

$$G \times H \cong H \times K, \; |K| < \infty \Longrightarrow G \cong K$$

Problem 11.4-17

$$\phi : G_1 \to G_2 \text{ is a surjective group homomorphism.}$$

$$H_1 \lhd G_1, \quad \phi(H_1) = H_2 \implies G_1/H_1 \cong G_2/H_2$$

$$G_1 = \mathbb{Z}_2, \quad G_2 = \{e\}, \quad H_1 = \{0\}, \quad H_2 = \{e\}$$

Problem 11.4-5

Find all homomorphisms from $\mathbb{Z}_{24}$ to $\mathbb{Z}_{18}$.

$$\phi : \mathbb{Z}_{24} \to \mathbb{Z}_{18}$$

$$\phi(1) = a? \quad \phi(x) = xa \pmod{18}$$

$$\phi(1) = a \implies \mathrm{ord}(a) \mid \mathrm{ord}(1)$$

Theorem

$$\boxed{ord(\phi(x)) \mid ord(x)}$$

$$\mathrm{ord}(a) \mid \gcd(24, 18) = 6$$

$$\phi(1) = 0, 9, 6, 12, 3, 15$$

Emmy Noether ($1882 \sim 1935$)

Theorem (The First Isomorphism Theorem)

$$\psi : G \to H \implies \frac{G}{Ker\ \psi} \cong \psi(G)$$

Theorem (The Second Isomorphism Theorem)

$$H \leq G, N \lhd G \implies \frac{H}{H \cap N} \cong \frac{HN}{N}$$

Theorem (The Third Isomorphism Theorem)

$$H \lhd G, N \lhd G, N \subseteq H \implies \frac{G}{H} \cong \frac{G/N}{H/N}$$

Theorem (The Fourth Isomorphism Theorem (Correspondence))

$$N \lhd G \implies$$

$$\{\textit{(normal) subgroups of } G \textit{ containing } N\} \leftrightarrow \{\textit{(normal) subgroups of } G/N$$

Theorem (The First Isomorphism Theorem)

$$\psi : G \to H \Longrightarrow \frac{G}{Ker\ \psi} \cong \psi(G)$$

$$Q: \ What\ if\ \psi\ is\ injective?$$

$$G \cong \psi(G)$$

$$Q: \ How\ to\ decide\ whether\ \psi\ is\ injective\ or\ not?$$

Theorem (Ker ψ and Injectivity)

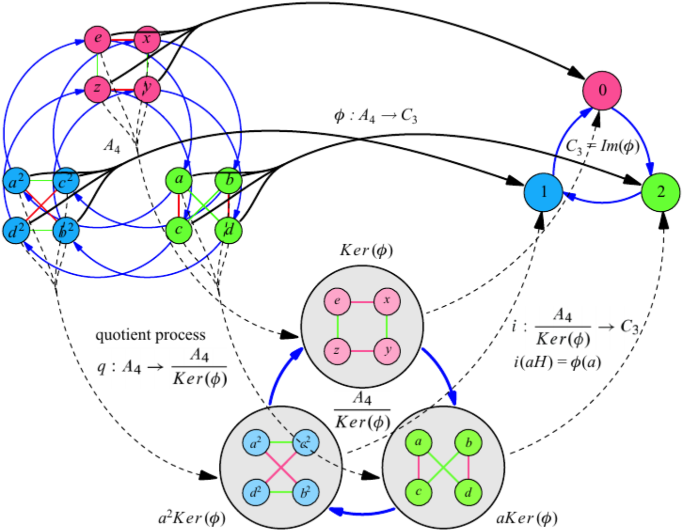
$$\psi : G \to H \ \ is\ injective\ \iff Ker\ \psi = \{e_G\}$$

$$\boxed{\frac{G}{\text{Ker } \psi} : \text{Quotient } G \text{ out by Ker } \psi}$$

$$
\begin{aligned}
\rho_1 &= (2\ 3\ 4) & \rho_1^2 &= (2\ 4\ 3) \\
\rho_2 &= (1\ 3\ 4) & \rho_2^2 &= (1\ 4\ 3) \\
\rho_3 &= (1\ 2\ 4) & \rho_3^2 &= (1\ 4\ 2) \\
\rho_4 &= (1\ 2\ 3) & \rho_4^2 &= (1\ 3\ 2)
\end{aligned}
$$

$$
\begin{aligned}
r_1 &= (1\ 4)(2\ 3) \\
r_2 &= (1\ 2)(3\ 4) \\
r_3 &= (1\ 3)(2\ 4)
\end{aligned}
$$

$$\boxed{Sym(T) \cong A_4 = \left\{ \text{id}, \quad \underbrace{\text{3-cycle},}_{\#=8} \quad \underbrace{\text{2-2-cycle}}_{\#=3} \right\}}$$



$$\phi : A_4 \to C_3 \qquad (\text{Ker } \phi = K_4)$$

Theorem (The First Isomorphism Theorem)

$$\psi : G \to H \Longrightarrow \frac{G}{Ker\ \psi} \cong \psi(G)$$

$$\text{To show } \frac{G_1}{N} \cong G_2.$$

$$\frac{\mathbb{Z} \times \mathbb{Z}}{\langle (1,1) \rangle} \cong \mathbb{Z}$$

$$f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$$

$$f(m,n) = m - n$$

$$\text{Ker } f = \langle (1,1) \rangle$$

Theorem (The Second Isomorphism Theorem)

$$H \leq G, N \lhd G \Longrightarrow \frac{H}{H \cap N} \cong \frac{HN}{N}$$

What if $H \cap N = \{e\}$?

$$H \cong \frac{HN}{N}$$

$$h \in H \leftrightarrow hN \subseteq HN$$

What if $h \in H \cap N$ $(h \neq e)$?

$$h \in H \cap N \implies hN = N$$

Theorem (The Second Isomorphism Theorem)

$$H \le G, N \lhd G \Longrightarrow \frac{H}{H \cap N} \cong \frac{HN}{N}$$

Problem 11.4-7

$$G = \mathbb{Z}_{24}, \quad H = \langle 4 \rangle, \quad N = \langle 6 \rangle$$

$$H \cap N = \langle 12 \rangle$$

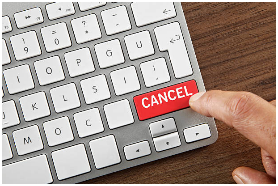
$$HN = \langle 2 \rangle = \bigcup_{h \in H} hN$$

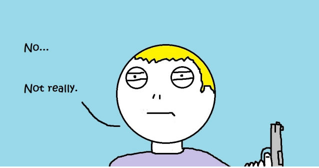
$$\frac{H}{H \cap N} \cong \frac{HN}{N} \Longrightarrow \frac{\langle 4 \rangle}{\langle 12 \rangle} \cong \frac{\langle 2 \rangle}{\langle 6 \rangle}$$

$$ab = \gcd(a, b) \cdot \operatorname{lcm}(a, b)$$

Theorem (The Third Isomorphism Theorem)

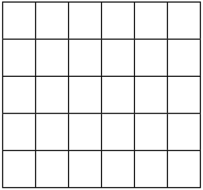
$$H \lhd G, N \lhd G, N \subseteq H \Longrightarrow \frac{G}{H} \cong \frac{G/N}{H/N}$$





Theorem (The Third Isomorphism Theorem)

$$H \lhd G, N \lhd G, N \subseteq H \Longrightarrow \frac{G}{H} \cong \frac{G/N}{H/N}$$



View G and H from the point of view of N

Theorem (The Third Isomorphism Theorem)

$$H \lhd G, N \lhd G, N \subseteq H \Longrightarrow \frac{G}{H} \cong \frac{G/N}{H/N}$$

$Q:$ *What do the elements in* $\frac{G}{H}$ *look like?*

$$gH \in \frac{G}{H}$$

$Q:$ *What do the elements in* $\frac{G/N}{H/N}$ *look like?*

$$gN \cdot (H/N)$$

$$\boxed{gN \cdot (H/N) \mapsto gH}$$

Absorption!!!

Theorem (The Third Isomorphism Theorem)

$$H \lhd G, N \lhd G, N \subseteq H \Longrightarrow \frac{G}{H} \cong \frac{G/N}{H/N}$$

$$n \mid m$$

$$m\mathbb{Z} \lhd n\mathbb{Z} \lhd \mathbb{Z}$$

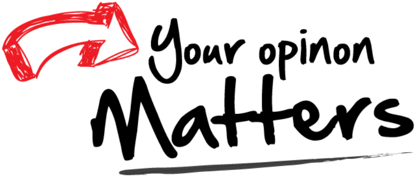
$$\frac{\mathbb{Z}}{n\mathbb{Z}} \cong \frac{\mathbb{Z}/m\mathbb{Z}}{n\mathbb{Z}/m\mathbb{Z}}$$

$$10\mathbb{Z} \lhd 2\mathbb{Z} \lhd \mathbb{Z}$$

$$\frac{\mathbb{Z}}{2\mathbb{Z}} \cong \frac{\mathbb{Z}/10\mathbb{Z}}{2\mathbb{Z}/10\mathbb{Z}}$$

$$\{0, 1\} \cong \frac{\{0, 1, 2, \cdots, 9\}}{\{0, 2, 4, 6, 8\}}$$


THANK YOU



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