Homework

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SM Problem 14.62

Suppose A and B are *well-ordered* *isomorphic* sets.
Show that there is only one *isomorphic mapping* $f : A \to B$.

Definition (Well-ordered Set (SM Definition 14.1))

An ordered set S is said to be *well-ordered* if every non-empty subset of S has a first element.

Definition (Isomorphic)

Two ordered sets A and B are said to be *isomorphic*, written $A \simeq B$, if $\exists f : A \xleftrightarrow[onto]{1-1} B$ which preserves the order relations

$$\forall a, a' \in A : a \prec a' \iff f(a) \prec f(a')$$

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Remark: What if "similarity mapping"?

Definition (Similarity Mapping)

A function $f : A \xrightarrow{1-1} B$ is called a *similarity mapping* from A to B if f preserves the order relations

$$\forall a, a' \in A : a \prec a' \iff f(a) \prec f(a')$$

Counterexample for "similarity mapping":

$$A = B = \mathbb{N} \qquad f : a \mapsto a \qquad f' : a \mapsto a + 1$$

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Lemma

X is a well-ordered set. $f : X \to X$ is a similarity mapping.
Then $\forall x \in X : f(x) \geq x$.

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X is a well-ordered set.
Then X has only one automorphism, i.e., id_X.

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Then $\forall x \in X : f(x) \geq x$.

Proof.

By contradiction.

$$Y = \{x \in X \mid f(x) < x\} \neq \emptyset$$

Well-ordered: $x = \min Y \implies f(x) < x$

Similarity: $f(f(x)) < f(x) \implies f(x) \in X$

$\square$

Lemma

X is a well-ordered set.
Then X has only one automorphism, i.e., id_X.

Proof.

Suppose $f : X \to X$ is an automorphism.

f is a similarity from X to $X \implies \forall x \in X : f(x) \geq x$

f^{-1} is a similarity from X to $X \implies \forall x \in X : f^{-1}(x) \geq x$

f is a similarity from X to $X \implies x = f(f^{-1}(x)) \geq f(x)$

$$id_X : f(x) = x$$

□

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Show that there is only one *isomorphic mapping* $f : A \to B$.

Proof.

By contradiction.

$$\exists \text{ isomorphic mappings } f, g : A \to B$$

$$h = g^{-1} \circ f \text{ is an automorphism on } A \implies h = id_A$$

$$f = g \circ h = g \circ id_A = g$$

□

Boolean Algebra

(i) 等势且有穷的布尔代数均同构。

Proof.

$$|B_1| = |B_2| = m \qquad f : B_1 \xleftrightarrow[onto]{1-1} B_2$$

Theorem (Representation Theorem)

B is a finite Boolean algebra. A_B is the set of atoms of B.

$$f : B \xleftrightarrow[onto]{1-1} \mathcal{P}(A_B)$$

$$x = a_1 + a_2 + \cdots + a_r \mapsto \{a_1, a_2, \cdots, a_r\} \text{ is an isomorphism.}$$

$$|A_{B_1}| = |A_{B_2}| = n = \log_2 m \quad A_{B_1} = \{b_1, \cdots, b_n\} \quad A_{B_2} = \{b_2, \cdots, b_n\}$$

□

Boolean Algebra

(ii) 举例说明等势但无穷的布尔代数不一定同构。

Not Exactly an Answer.

并非任何 Boolean Algebra 皆同构于某幂集代数。

有穷-余有穷 (finite co-finite) 代数 $F(\mathbb{N})$

$$F(\mathbb{N}) = \{X \in \mathcal{P}(\mathbb{N}) \mid X \text{ is finite} \lor \mathbb{N} \setminus X \text{ is finite}\}$$

$$|F(\mathbb{N})| = \aleph_0$$

$$f : F(\mathbb{N}) \xrightarrow{1-1} \mathbb{N} \qquad \{a_1, a_2, \cdots, a_n\} = p_{a_1} p_{a_2} \cdots p_{a_n}$$

□

Theorem (Representation Theorem)

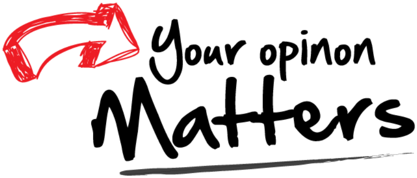
(i) 任何有穷 *Boolean Algebra* 同构于某幂集代数。

(ii) 有穷 *Boolean Algebra* 之势呈形 2^n。

(iii) 两个等势的有穷 *Boolean Algebra* 是同构的。

(iv) 并非任何 *Boolean Algebra* 皆同构于某幂集代数。


Thank
You!



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