2-4 Recurrences

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Maximal Sum Subarray (Problem $4.1 - 5$)

- array $A[1 \cdots n], a_i >=< 0$
- to find (the sum of) an MS in A

$$A[-2, 1, -3, \boxed{4, -1, 2, 1}, -5, 4]$$

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Trial and error.

- try subproblem MSS$[i]$: the sum of the MS (MS[i]) in $A[1 \cdots i]$
- goal: mss = MSS$[n]$

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- try subproblem MSS$[i]$: the sum of the MS (MS[i]) in $A[1 \cdots i]$
- goal: mss $=$ MSS$[n]$
- question: Is $a_i \in$ MS$[i]$?
- recurrence:

$$\text{MSS}[i] = \max\{\text{MSS}[i-1], ???\}$$

Solution.

- subproblem MSS$[i]$: the sum of the MS *ending with* a_i or 0
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$$\boxed{\text{mss} = \max_{1 \le i \le n} \text{MSS}[i]}$$

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- question: where does the $\text{MS}[i]$ start?
- recurrence:

$$\boxed{\text{MSS}[i] = \max\left\{\text{MSS}[i-1] + a_i, 0\right\}} \quad \text{(prove it!)}$$

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- initialization: $\text{MSS}[0] = 0$

procedure $\mathrm{MSS}(A[1 \cdots n])$

  MSS$[0] \leftarrow 0$

  for $i \leftarrow 1$ **to** n **do**

    MSS$[i] \leftarrow \max\{$MSS$[i-1] + A[i], 0\}$

  return $\max\limits_{1 \le i \le n}$ MSS$[i]$

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procedure MSS(A[1 ··· n])
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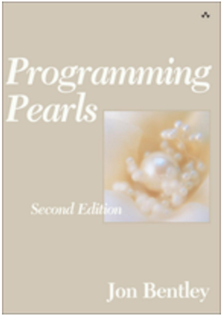
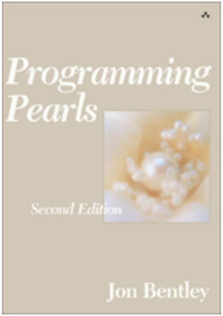
    return mss
```



Ulf Grenander $O(n^3) \implies O(n^2)$



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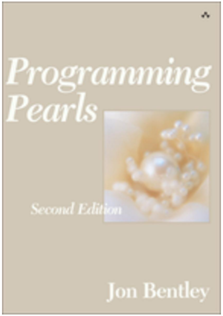

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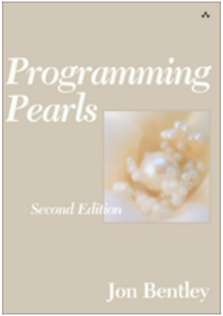

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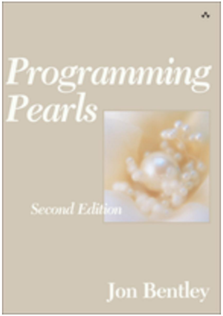

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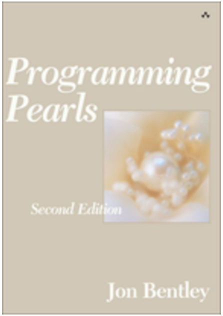

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Maximum-product subarray

Maximum-product subarray (Problem 7.4)

- Array $A[1 \ldots n]$
- Find maximum-product subarray of A

(1) $a_i \in \mathbb{N}$

(2) $a_i \in \mathbb{Z}$

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sum *vs.* product

Maximum-product subarray

Subproblem: $\mathsf{MaxP}[i], \mathsf{MinP}[i]$

		$\frac{1}{2}$	4	-2	5	$-\frac{1}{5}$	8
$\mathsf{MaxP}[i]$	1	$\frac{1}{2}$	4	-2	5	8	64
$\mathsf{MinP}[i]$	1	$\frac{1}{2}$	2	-8	-40	-1	-8

$$\mathsf{MaxP}[i] = \max\{\mathsf{MaxP}[i-1] \cdot a_i, \mathsf{MinP}[i-1] \cdot a_i, a_i\}$$
$$\mathsf{MinP}[i] = \min\{\mathsf{MaxP}[i-1] \cdot a_i, \mathsf{MinP}[i-1] \cdot a_i, a_i\}$$

2d

Problem (Area-Efficient VLSI Layout)

Embedding a complete binary tree into a grid with minimum area.

- Complete binary tree circuit of

$$\#\text{layer} = 3, 5, 7, \ldots$$

- Vertex on grid; no crossing edges
- Area:

$$\text{area} = \text{width} \times \text{height}$$

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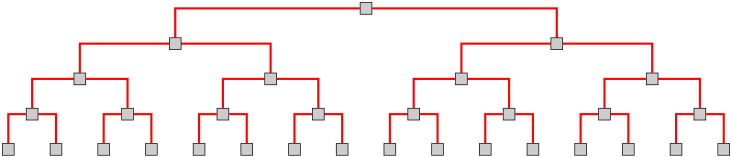

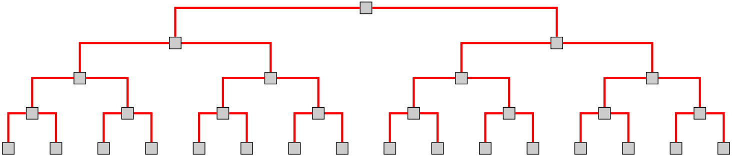

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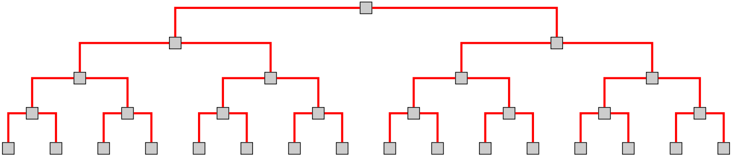

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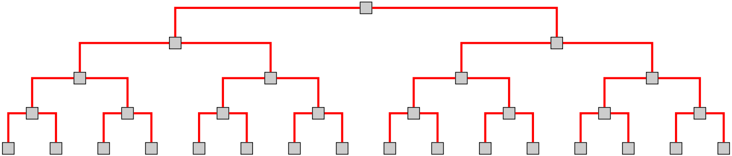

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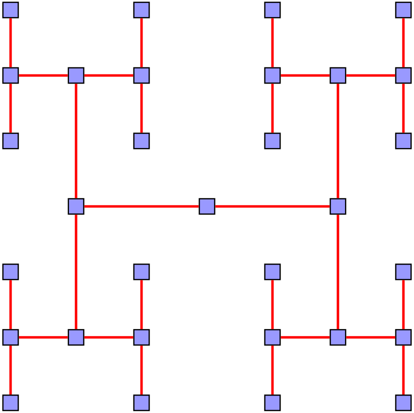
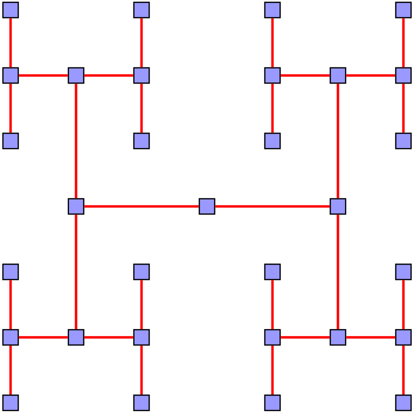
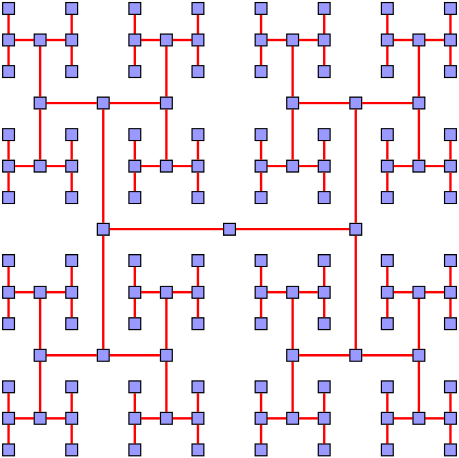
$$\boxed{H(n) = 2H(\frac{n}{4}) + \Theta(1)}$$





H-layout



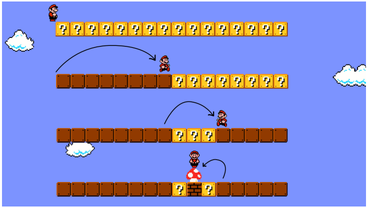
"VLSI Theory and Parallel Supercomputing", Charles E. Leiserson, 1989.

Binary Search (CLRS $4.5-3$)

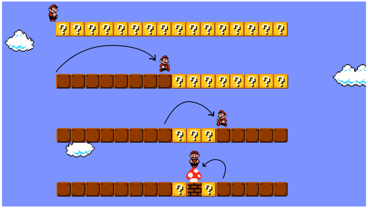
$$T(n) = 2T(n/2) + \Theta(1)$$

procedure $\textsc{BinarySearch}(A, L, R, x)$
  if $R < L$ **then**
    return -1
  $m \leftarrow L + (R-1)/2$
  if $A[m] = x$ **then**
    return m
  else if $A[m] > x$ **then**
    return $\textsc{BinarySearch}(A, L, m-1, x)$
  else
    return $\textsc{BinarySearch}(A, m+1, R, x)$

$$\boxed{T(n) = \Theta(n \log n)}$$



$$T(n) = \begin{cases} \max\left\{T(\lfloor \frac{n-1}{2} \rfloor), T(\lceil \frac{n-1}{2} \rceil)\right\} + 1, & n > 2 \\ 1, & n = 1 \end{cases}$$



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Theorem

The worst case time complexity (# of comparisons) of BinarySearch *on an input size of* n $=$ *# of bits in the binary representation of* n.


TREASURE
HUNT
on foot


Hints
&
Tips


Help
Wanted

Analysis of Mergesort in CLRS (# of Comparisions; $a_i : \infty$ not Counted)

(a) Analyze the worst case $W(n)$ and the best case $B(n)$ time complexity of mergesort *as accurately as possible*.
Explore the relation between them and the binary representations of numbers.
Plot $W(n)$ and $B(n)$ and explain what you observe.

(b) Analyze the average case $A(n)$ time complexity of mergesort.
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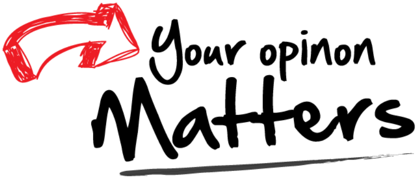
$W(n)$: Consider $W(n+1)$

$$\boxed{W(n) = W(\lfloor \frac{n}{2} \rfloor) + W(\lceil \frac{n}{2} \rceil) + (n-1)}$$

Thank You!



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