Surjectivity implies injectivity

Let S be a finite set.Let F be a surjective function from S to S.

How do I prove that it is injective?

(functions) (elementary-set-theory)

asked Sep 9 '11 at 10:20
Mohan
5,409 ● 8 ■ 38 ▲ 93

Have you tried counting elements yet? – Sebastian Sep 9 '11 at 10:32

Suppose $x \neq y \in S$ and that $f(x) = f(y)$. Let $|S| = n$. How many distinct elements can lie in the image of f? – m_t_ Sep 9 '11 at 10:36

2 Answers

Let S be a *finite* set, and $f : S \to S$ a function. Then the following are equivalent:

- f is injective.
- f is surjective.
- f is bijective.

This is really just a counting argument. First, suppose f is injective. If S has n elements, by our assumption, this means the image of f has at least n elements. But the image of f is contained in S, so it has at most n elements; so the image of f contains exactly n elements and is therefore the whole of S, i.e. f is surjective.

Next, suppose f is surjective. So, for each y in S, there is an x in S such that $y = f(x)$; we choose one such x for each y and define a function $g : S \to S$ so that $g(y) = x$. By construction, $f(g(y)) = y$, so g must be injective, and hence, must be surjective by the above argument. So g is a bijection, and f is a left inverse for g. But a left inverse for a bijection is also a right inverse, so this implies f is a bijection, and *a fortiori* an injection.

Notice that the very first part of the argument fails when S is not finite. For example, let us consider the function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) = x + 1$. This function is certainly injective but is not surjective. Similarly, the function $g : \mathbb{N} \to \mathbb{N}$ defined by $f(0) = 0$ and $f(x + 1) = x$ is surjective, but not injective.

answered Sep 9 '11 at 10:37
Zhen Lin
57k ● 4 ■ 90 ▲ 185

Why is the function g injective? – Mohan Sep 9 '11 at 11:06

@user774025: Because we send y to its x such that $f(x) = y$. Since f is a function there can only be one element as $f(x)$. – Asaf Karagila Sep 9 '11 at 11:46

3 Though technically correct, the claim that "the image of [an injective] f has at least n elements" is odd and misleading. It follows from the definition of a function that the image of any function has *at most* n elements when its domain has n elements. So proving the first part really just amounts to noticing that injectivity implies the image of f has exactly n elements, i.e., it coincides with S. – pash Jul 26 '13 at 18:10

Suppose that f is an injective function and not surjective, i.e. there is point $y \in S$ such that there is no point $x \in S$ with $f(x) = y$. Since f is a function, every $x \in S$ must work as abscissa in the relation f. Hence we must have some $x_1 \neq x_2$ with $f(x_1) = f(x_2)$, which gives a contradiction. Therefore f must be onto.

edited May 5 '16 at 2:04
Szmagpie
1,329 ■ 6 ▲ 17

answered Sep 21 '13 at 20:43
Taha Guma el turki
11 ▲ 1