

Network flows and Menger's theorem

Recall...

Theorem (max flow, min cut – strong duality). *Let G be a network. The maximum value of a flow equals the minimum capacity of a cut.*

We prove this strong duality result via the Ford-Fulkerson algorithm.

Algorithmic proof of Max flow, min cut thm. Apply Ford-Fulkerson algorithm to obtain a feasible flow f^* such that there is no s, t path in the residual network.

(Does algorithm terminate?) If capacities are rational, then each augmentation increases the flow by a multiple of $\frac{1}{z}$ where z is the least common multiple of the denominators, and since the flow value is bounded above by any $s - t$ cut capacity, we force termination in finite time.

(Do we obtain a flow whose value is equal to some s, t -cut?)

Define S to be the set of vertices reachable by a walk (or path) from s in the last residual network. Then $s \in S$ and $t \notin S$ since no s, t path exists. We have

$$\begin{aligned} 0 &= \sum_{e \in [S, \overline{S}]} r(e) \\ &= \sum_{e \in [S, \overline{S}]} [c(e) - f(e) + f(e^{-1})] \\ &= - \sum_{e \in [S, \overline{S}]} [f(e) - f(e^{-1})] + \sum_{e \in [S, \overline{S}]} c(e) \\ &= -(\text{flow value of } f^*) + \text{cap}[S, \overline{S}] \quad \text{by previous lemma} \\ \implies \text{flow value of } f^* &= \text{cap}[S, \overline{S}] \end{aligned}$$

Therefore, by weak duality, f^* must be a maximum flow (and $[S, \overline{S}]$ is a minimum cut). ■

Note that this proof only holds for networks with rational capacities. In fact, the Ford-Fulkerson algorithm may yield augmenting paths forever if there are irrational capacities in the network.

Edmonds and Karp later modified the Ford-Fulkerson algorithm to work for all real capacities (by searching for shortest augmenting paths).

Corollary (integrality theorem). *If all capacities of a network are integers, then there is a maximum flow assigning integral flow to each edge. Furthermore, some maximum flow can be partitioned into flows of unit value along paths from source to sink.*

We want to use the max flow, min cut thm to prove Menger's thms. For each version of Menger's thm, we encode the path problem using network flows with integer capacities. See Remark 4.3.15 for such transformations.

Theorem (Menger - edge, directed, local version). *Let G be a digraph and let $s, t \in V(G)$. The maximum number of edge-disjoint directed paths from s to t equals the minimum number of edges whose removal destroys all s, t -directed paths.*

Proof. Make G into a network by assigning each arc capacity 1 and letting s be the source vertex and t the sink vertex.

By assigning cap 1 to each edge, we ensure that units of flow from s to t correspond to pairwise edge-disjoint directed s, t -paths.

Let f be a maximum flow and let $[S, \overline{S}]$ be a minimum cut. (So $s \in S$ and $t \in \overline{S}$.) Since every arc has capacity 1, the $\text{cap}[S, \overline{S}]$ is precisely the # of edges in the edge cut, so $\text{cap}[S, \overline{S}]$ gives # of edges whose removal destroys all s, t -directed paths. Thus,

$$\min \# \text{ of edges that destroy all } s, t\text{-directed paths} \leq \text{cap}[S, \overline{S}].$$

Furthermore, by the integrality thm, the flow f yields a set of directed paths from s to t that are pairwise edge-disjoint (since any edge has at most one unit of flow assigned to it and thereby appears in at most one directed s, t -path). So

$$\text{flow value of } f \leq \max \# \text{ of edge-disjoint } s, t\text{-directed paths}.$$

The other direction of the inequality is clear since we must remove at least one edge from each of the edge-disjoint s, t -directed paths and (no edge can destroy two edge-disjoint directed paths). ■

Theorem (Menger - edge, undirected, local version). *Let G be an undirected graph. Let s, t be vertices of G . Then the maximum number of s, t -paths that are pairwise edge-disjoint equals the minimum number of edges that destroy all s, t -paths.*

Proof. Make G into a digraph D by replacing each edge $uv \in E(G)$ with a pair of arcs: (u, v) and (v, u) . Assign each arc capacity 1 and find a maximum flow from s to t and a minimum $s - t$ cut $[S, \overline{S}]$.

Suppose there are a pair of edges (a, b) and (b, a) both with flow 1. (While these two edges are disjoint in the digraph, they are the same edge in the undirected graph.) Change the flow on both to flow 0. This does not change the value of the flow. Now, each original undirected edge ab has either flow 0 or 1 (but not 2), and so, as in the previous proof,

$$\max_{\text{edge-disjoint } s, t\text{-paths}} \# \text{ of pairwise} \geq \max \text{ flow value}.$$

Alternate perspective:

In other words, if P_1, P_2 are $s - t$ paths that go through arcs (a, b) and (b, a) , i.e.,

$$P_1 = (s, v_1) \cdots (v_j, a)(a, b)(b, v_{j+1}), \cdots (v_m, t)$$

$$P_2 = (s, u_1) \cdots (u_k, b)(b, a)(a, u_{k+1}), \cdots (u_n, t),$$

then we can replace these paths with two paths that use neither (a, b) nor (b, a) , namely

$$\widetilde{P}_1 = (s, v_1) \cdots (v_j, a)(a, u_{k+1}), \cdots (u_n, t)$$

$$\widetilde{P}_2 = (s, u_1) \cdots (u_k, b)(b, v_{j+1}), \cdots (v_m, t).$$

Furthermore, since all arcs have capacity 1, the size of the edge cut $[S, \overline{S}]$ is exactly $\text{cap}[S, \overline{S}]$. Since this set of edges disconnects the vertices s and t in the digraph D , it certainly is a set of $s - t$ disconnecting edges in the original, undirected graph G . This implies that

$$\text{min cut capacity} \geq \min \# \text{ of edges that destroy all } s, t\text{-paths.}$$

Using the max flow-min cut theorem, we have that

$$\max \# \text{ of pairwise edge-disjoint } s, t\text{-paths} \geq \text{max flow value} = \text{min cut capacity} \geq \min \# \text{ of edges that destroy all } s, t\text{-paths.}$$

As in the previous proof, the other direction of the inequality is clear, as at least one edge must be removed from each of the edge-disjoint s, t -paths in order to destroy all s, t -paths. ■

Now original edge version of Menger's theorem follows as a corollary.

Theorem (Menger - edge, undirected, global version). *A graph G is k -edge-connected if and only if every pair of vertices are joined by k pairwise edge disjoint paths.*

Vertex colorings

Problem A company manufactures n chemicals C_1, C_2, \dots, C_n . Certain pairs of these chemicals are incompatible and would cause explosions if brought into contact with each other. As a precautionary measure the company wishes to partition its warehouse into compartments, and store incompatible chemicals in different compartments. What is the least number of compartments into which the warehouse should be partitioned?

Define a graph with

- a vertex representing each chemical,
- and two vertices are adjacent if and only if their corresponding chemicals are incompatible.

The least # of compartments into which the warehouse should be partitioned is the least # of colors needed to properly color the graph.

Definition A *k -coloring* of a graph G is a function $f: V(G) \rightarrow \{1, 2, \dots, k\}$. It is *proper* if $u \sim v \Rightarrow f(u) \neq f(v)$. G is said to be *k -colorable* if G has a proper k -coloring.

Definition The minimum k such that G has a proper k -coloring is called the *chromatic number* of G and is denoted $\chi(G)$.

Remarks

- We are only considering proper colorings in our study, so we refer to a proper k -coloring simply as a k -coloring.
- We refer to the labels $\{1, 2, \dots, k\}$ as *colors*.
- G is k -colorable $\iff \chi(G) \leq k$.

- k is the bound on the size of the color palette – not a requirement on using all those colors.
- Graphs with loops cannot be properly colored and multiple edges are irrelevant when considering colorability, so we consider simple graphs only.

Example $\chi(K_n) = n$

Definition The *clique number* of a graph G , denoted $\omega(G)$, is the maximum size of a set of pairwise adjacent vertices in G . (Note that $\omega(G) = \alpha(\overline{G})$.)

Proposition. For any graph G , $\chi(G) \geq \omega(G)$.

Proof. Vertices of a clique require distinct colors. ■

Is a large clique always what determines the value of the chromatic number? Are there graphs G for which $\chi(G) > \omega(G)$?

Examples

- $\chi(C_5) = 3$
- Consider graph below:



Alternative view of coloring Vertices of one color form a *color class*. Each color class is an independent set so a k -coloring of a graph G can be viewed as a partition of the vertices of G into (at most) k independent sets.

Proposition. For any graph G with n vertices, $\chi(G) \geq \frac{n}{\alpha(G)}$.

Proof. Each color class of an optimal coloring of G is independent set, so

$$n = \sum_{i=1}^{\chi(G)} (\# \text{ of vertices in } i\text{th color class}) \leq \chi(G)\alpha(G).$$

■

Problem 5.1.12 Prove or disprove: Every k -chromatic graph G has a proper k -coloring in which some color class has $\alpha(G)$ vertices.

Problem 5.1.14 Prove or disprove: For every graph G with n vtc's,

$$\chi(G) \leq n - \alpha(G) + 1.$$

Characterizations of k -colorable graphs

- $\chi(G) = 1 \iff G$ is edgeless.
- $\chi(G) = 2 \iff G$ is bipartite $\iff G$ has no odd cycles.
- $\chi(G) \leq 3 \iff ??$

Definition A graph G is *k -critical* if $\chi(G) = k$ and for all proper subgraphs H of G , $\chi(H) < k$.

Characterizations of k -critical graphs

- G is 2-critical $\iff G = K_2$.
- G is 3-critical $\iff G$ is an odd cycle.

- G is 4-critical \iff ??

Remark No good characterization of 4-critical graphs or test for 3-colorability is known.