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Binary relation

In mathematics, a **binary relation** on a set A is a collection of ordered pairs of elements of A. In other words, it is a subset of the Cartesian product $A^2 = A \times A$. More generally, a binary relation between two sets A and B is a subset of $A \times B$. The terms **correspondence**, **dyadic relation** and **2-place relation** are synonyms for binary relation.

An example is the "divides" relation between the set of prime numbers **P** and the set of integers **Z**, in which every prime p is associated with every integer z that is a multiple of p (but with no integer that is not a multiple of p). In this relation, for instance, the prime 2 is associated with numbers that include −4, 0, 6, 10, but not 1 or 9; and the prime 3 is associated with numbers that include 0, 6, and 9, but not 4 or 13.

Binary relations are used in many branches of mathematics to model concepts like "is greater than", "is equal to", and "divides" in arithmetic, "is congruent to" in geometry, "is adjacent to" in graph theory, "is orthogonal to" in linear algebra and many more. The concept of function is defined as a special kind of binary relation. Binary relations are also heavily used in computer science.

A binary relation is the special case $n = 2$ of an n-ary relation $R \subseteq A_1 \times \ldots \times A_n$, that is, a set of n-tuples where the jth component of each n-tuple is taken from the jth domain A_j of the relation. An example for a ternary relation on **Z**×**Z**×**Z** is " ... lies between ... and ...", containing e.g. the triples (5,2,8), (5,8,2), and (−4,9,−7).

In some systems of axiomatic set theory, relations are extended to classes, which are generalizations of sets. This extension is needed for, among other things, modeling the concepts of "is an element of" or "is a subset of" in set theory, without running into logical inconsistencies such as Russell's paradox.

Contents

Formal definition

A **binary relation** R between arbitrary sets (or classes) X (the **set of departure**) and Y (the **set of destination** or **codomain**) is specified by its **graph** G, which is a subset of the Cartesian product $X \times Y$. The binary relation R itself is usually identified with its graph G, but some authors define it as an ordered triple (X, Y, G), which is otherwise referred to as a **correspondence**.[1]

The statement $(x, y) \in G$ is read "x **is R-related to** y", and is denoted by xRy or $R(x, y)$. The latter notation corresponds to viewing R as the characteristic function of the subset G of $X \times Y$, i.e. $R(x, y)$ equals to 1 (true), if $(x, y) \in G$, and 0 (false) otherwise.

The order of the elements in each pair of G is important: if $a \neq b$, then aRb and bRa can be true or false, independently of each other. Resuming the above example, the prime 3 divides the integer 9, but 9 doesn't divide 3.

The **domain** of R is the set of all x such that xRy for at least one y. The **range** of R is the set of all y such that xRy for at least one x. The **field** of R is the union of its domain and its range.[2][3][4]

Is a relation more than its graph?

According to the definition above, two relations with identical graphs but different domains or different codomains are considered different. For example, if $G = \{(1,2),(1,3),(2,7)\}$, then $(\mathbb{Z},\mathbb{Z},G)$, $(\mathbb{R},\mathbb{N},G)$, and $(\mathbb{N},\mathbb{R},G)$ are three distinct relations, where $\mathbb{Z}$ is the set of integers, $\mathbb{R}$ is the set of real numbers and $\mathbb{N}$ is the set of natural numbers.

Especially in set theory, binary relations are often defined as sets of ordered pairs, identifying binary relations with their graphs. The domain of a binary relation R is then defined as the set of all x such that there exists at least one y such that $(x, y) \in R$, the **range** of R is defined as the set of all y such that there exists at least one x such that $(x, y) \in R$, and the **field** of R is the union of its domain and its range.[2][3][4]

A special case of this difference in points of view applies to the notion of function. Many authors insist on distinguishing between a function's codomain and its range. Thus, a single "rule," like mapping every real number x to x^2, can lead to distinct functions $f: \mathbb{R} \to \mathbb{R}$ and $f: \mathbb{R} \to \mathbb{R}^+$, depending on whether the images under that rule are understood to be reals or, more restrictively, non-negative reals. But others view functions as simply sets of ordered pairs with unique first components. This difference in perspectives does raise some nontrivial issues. As an example, the former camp considers surjectivity—or being onto—as a property of functions, while the latter sees it as a relationship that functions may bear to sets.

Either approach is adequate for most uses, provided that one attends to the necessary changes in language, notation, and the definitions of concepts like restrictions, composition, inverse relation, and so on. The choice between the two definitions usually matters only in very formal contexts, like category theory.

Example

Example: Suppose there are four objects {ball, car, doll, gun} and four persons {John, Mary, Ian, Venus}. Suppose that John owns the ball, Mary owns the doll, and Venus owns the car. Nobody owns the gun and Ian owns nothing. Then the binary relation "is owned by" is given as

1st example relation

	ball	car	doll	gun
John	+	−	−	−
Mary	−	−	+	−
Ian	−	−	−	−
Venus	−	+	−	−

2nd example relation

	ball	car	doll	gun
John	+	−	−	−
Mary	−	−	+	−
Venus	−	+	−	−

R = ({ball, car, doll, gun}, {John, Mary, Ian, Venus}, {(ball, John), (doll, Mary), (car, Venus)}).

Thus the first element of R is the set of objects, the second is the set of persons, and the last element is a set of ordered pairs of the form (object, owner).

The pair (ball, John), denoted by ${}_{\text{ball}}R_{\text{John}}$ means that the ball is owned by John.

Two different relations could have the same graph. For example: the relation

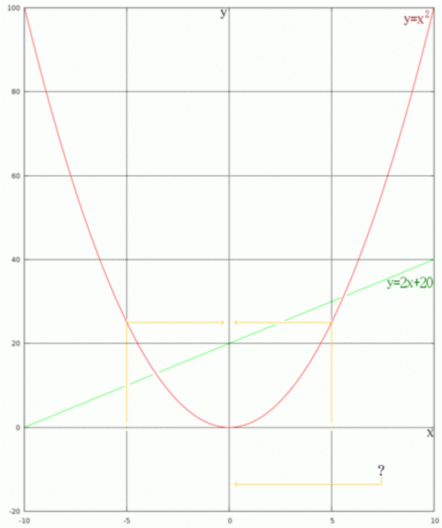
({ball, car, doll, gun}, {John, Mary, Venus}, {(ball, John), (doll, Mary), (car, Venus)})

is different from the previous one as everyone is an owner. But the graphs of the two relations are the same.

Nevertheless, R is usually identified or even defined as G(R) and "an ordered pair $(x, y) \in$ G(R)" is usually denoted as "$(x, y) \in R$".[5]

Special types of binary relations

Some important types of binary relations R between two sets X and Y are listed below. To emphasize that X and Y can be different sets, some authors call such binary relations **heterogeneous**.[6][7]



Example relations between real numbers. **Red:** $y=x^2$. **Green:** $y=2x+20$.

Uniqueness properties:

- **injective** (also called **left-unique**[8]): for all x and z in X and y in Y it holds that if xRy and zRy then $x = z$. For example, the green relation in the diagram is injective, but the red relation is not, as it relates e.g. both $x = -5$ and $z = +5$ to $y = 25$.
- **functional** (also called **univalent**[9] or **right-unique**[8] or **right-definite**[10]): for all x in X, and y and z in Y it holds that if xRy and xRz then $y = z$; such a binary relation is called a **partial function**. Both relations in the picture are functional. An example for a non-functional relation can be obtained by rotating the red graph clockwise by 90 degrees, i.e. by considering the relation $x=y^2$ which relates e.g. $x=25$ to both $y=-5$ and $z=+5$.
- **one-to-one** (also written **1-to-1**): injective and functional. The green relation is one-to-one, but the red is not.

Totality properties (only definable if the sets of departure X resp. destination Y are specified; not to be confused with a total relation):

- **left-total**:[8] for all x in X there exists a y in Y such that xRy. For example, R is left-total when it is a function or a multivalued function. Note that this property, although sometimes also referred to as *total*, is different from the definition of *total* in the next section. Both relations in the picture are left-total. The relation $x=y^2$, obtained from the above rotation, is not left-total, as it doesn't relate, e.g., $x = -14$ to any real number y.
- **surjective** (also called **right-total**[8] or **onto**): for all y in Y there exists an x in X such that xRy. The green relation is surjective, but the red relation is not, as it doesn't relate any real number x to e.g. $y = -14$.

Uniqueness and totality properties:

- A **function**: a relation that is functional and left-total. Both the green and the red relation are functions.
- An **injective function** or **injection**: a relation that is injective, functional, and left-total.
- A **surjective function** or **surjection**: a relation that is functional, left-total, and right-total.
- A **bijection**: a surjective one-to-one or surjective injective function is said to be **bijective**, also known as **one-to-one correspondence**.[11] The green relation is bijective, but the red is not.

Difunctional

Less commonly encountered is the notion of **difunctional** (or **regular**) relation, defined as a relation R such that $R=RR^{-1}R$.[12]

To understand this notion better, it helps to consider a relation as mapping every element $x \in X$ to a set $xR = \{ y \in Y \mid xRy \}$.[12] This set is sometimes called the **successor neighborhood** of x in R; one can define the **predecessor neighborhood** analogously.[13] Synonymous terms for these notions are **afterset** and respectively **foreset**.[6]

A difunctional relation can then be equivalently characterized as a relation R such that wherever x_1R and x_2R have a non-empty intersection, then these two sets coincide; formally $x_1R \cap x_2R \neq \emptyset$ implies $x_1R = x_2R$.[12]

As examples, any function or any functional (right-unique) relation is difunctional; the converse doesn't hold. If one considers a relation R from set to itself ($X = Y$), then if R is both transitive and symmetric (i.e. a partial equivalence relation), then it is also difunctional.[14] The converse of this latter statement also doesn't hold.

A characterization of difunctional relations, which also explains their name, is to consider two functions $f: A \to C$ and $g: B \to C$ and then define the following set which generalizes the kernel of a single function as joint kernel: $\ker(f, g) = \{ (a, b) \in A \times B \mid f(a) = g(b) \}$. Every difunctional relation $R \subseteq A \times B$ arises as the joint kernel of two functions $f: A \to C$ and $g: B \to C$ for some set C.[15]

In automata theory, the term **rectangular relation** has also been used to denote a difunctional relation. This terminology is justified by the fact that when represented as a boolean matrix, the columns and rows of a difunctional relation can be arranged in such a way as to present rectangular blocks of true on the (asymmetric) main diagonal.[16] Other authors however use the term "rectangular" to denote any heterogeneous relation whatsoever.[7]

Relations over a set

If $X = Y$ then we simply say that the binary relation is over X, or that it is an **endorelation** over X.[17] In computer science, such a relation is also called a **homogeneous** (binary) relation.[7][17][18] Some types of endorelations are widely studied in graph theory, where they are known as simple directed graphs permitting loops.

The set of all binary relations ***Rel***(X) on a set X is the set $2^{X \times X}$ which is a Boolean algebra augmented with the involution of mapping of a relation to its inverse relation. For the theoretical explanation see Relation algebra.

Some important properties that a binary relation R over a set X may have are:

- **reflexive**: for all x in X it holds that xRx. For example, "greater than or equal to" (≥) is a reflexive relation but "greater than" (>) is not.
- **irreflexive** (or **strict**): for all x in X it holds that **not** xRx. For example, > is an irreflexive relation, but ≥ is not.
- **coreflexive relation**: for all x and y in X it holds that if xRy then $x = y$.[19] An example of a coreflexive relation is the relation on integers in which each odd number is related to itself and there are no other relations. The equality relation is the only example of a both reflexive and coreflexive relation, and any coreflexive relation is a subset of the identity relation.

> The previous 3 alternatives are far from being exhaustive; e.g. the red relation $y=x^2$ from the above picture is neither irreflexive, nor coreflexive, nor reflexive, since it contains the pair (0,0), and (2,4), but not (2,2), respectively.

- **symmetric**: for all x and y in X it holds that if xRy then yRx. "Is a blood relative of" is a symmetric relation, because x is a blood relative of y if and only if y is a blood relative of x.
- **antisymmetric**: for all x and y in X, if xRy and yRx then $x = y$. For example, ≥ is anti-symmetric; so is >, but vacuously (the condition in the definition is always false).[20]
- **asymmetric**: for all x and y in X, if xRy then **not** yRx. A relation is asymmetric if and only if it is both anti-symmetric and irreflexive.[21] For example, > is asymmetric, but ≥ is not.
- **transitive**: for all x, y and z in X it holds that if xRy and yRz then xRz. For example, "is ancestor of" is transitive, while "is parent of" is not. A transitive relation is irreflexive if and only if it is asymmetric.[22]
- **total**: for all x and y in X it holds that xRy or yRx (or both). This definition for *total* is different from *left total* in the previous section. For example, ≥ is a total relation.
- **trichotomous**: for all x and y in X exactly one of xRy, yRx or $x = y$ holds. For example, > is a trichotomous relation, while the relation "divides" on natural numbers is not.[23]
- **Right Euclidean**: for all x, y and z in X it holds that if xRy and xRz, then yRz.
- **Left Euclidean**: for all x, y and z in X it holds that if yRx and zRx, then yRz.
- **Euclidean**: A Euclidean relation is both left and right Euclidean. Equality is a Euclidean relation because if $x=y$ and $x=z$, then $y=z$.
- **serial**: for all x in X, there exists y in X such that xRy. "*Is greater than*" is a serial relation on the integers. But it is not a serial relation on the positive integers, because there is no y in the positive integers such that $1>y$.[24] However, "*is less than*" is a serial relation on the positive integers, the rational numbers and the real numbers. Every reflexive relation is serial: for a given x, choose $y=x$. A serial relation can be equivalently characterized as

every element having a non-empty successor neighborhood (see the previous section for the definition of this notion). Similarly an **inverse serial** relation is a relation in which every element has non-empty predecessor neighborhood.[13]

- **set-like** (or **local**): for every x in X, the class of all y such that yRx is a set. (This makes sense only if relations on proper classes are allowed.) The usual ordering < on the class of ordinal numbers is set-like, while its inverse > is not.

A relation that is reflexive, symmetric, and transitive is called an equivalence relation. A relation that is symmetric, transitive, and serial is also reflexive. A relation that is only symmetric and transitive (without necessarily being reflexive) is called a partial equivalence relation.

A relation that is reflexive, antisymmetric, and transitive is called a partial order. A partial order that is total is called a total order, *simple order*, linear order, or a chain.[25] A linear order where every nonempty subset has a least element is called a well-order.

Binary endorelations by property

	reflexivity	symmetry	transitivity	symbol	example
directed graph				→	
undirected graph	irreflexive	symmetric			
tournament	irreflexive	antisymmetric			pecking order
dependency	reflexive	symmetric			
strict weak order	irreflexive	antisymmetric	yes	<	
total preorder	reflexive		yes	≤	
preorder	reflexive		yes	≤	preference
partial order	reflexive	antisymmetric	yes	≤	subset
partial equivalence		symmetric	yes		
equivalence relation	reflexive	symmetric	yes	∼, ≅, ≈, ≡	equality
strict partial order	irreflexive	antisymmetric	yes	<	proper subset

Operations on binary relations

If R, S are binary relations over X and Y, then each of the following is a binary relation over X and Y:

- **Union**: $R \cup S \subseteq X \times Y$, defined as $R \cup S = \{ (x, y) \mid (x, y) \in R \text{ or } (x, y) \in S \}$. For example, ≥ is the union of > and =.
- **Intersection**: $R \cap S \subseteq X \times Y$, defined as $R \cap S = \{ (x, y) \mid (x, y) \in R \text{ and } (x, y) \in S \}$.

If R is a binary relation over X and Y, and S is a binary relation over Y and Z, then the following is a binary relation over X and Z: (see main article *composition of relations*)

- **Composition**: $S \circ R$, also denoted $R\,;S$ (or $R \circ S$), defined as $S \circ R = \{ (x, z) \mid \text{there exists } y \in Y, \text{ such that } (x, y) \in R \text{ and } (y, z) \in S \}$. The order of R and S in the notation $S \circ R$, used here agrees with the standard notational order for composition of functions. For example, the composition "is mother of" ∘ "is parent of" yields "is maternal grandparent of", while the composition "is parent of" ∘ "is mother of" yields "is grandmother of".

A relation R on sets X and Y is said to be **contained** in a relation S on X and Y if R is a subset of S, that is, if $x\,R\,y$ always implies $x\,S\,y$. In this case, if R and S disagree, R is also said to be **smaller** than S. For example, > is contained in ≥.

If R is a binary relation over X and Y, then the following is a binary relation over Y and X:

- **Inverse** or **converse**: R^{-1}, defined as $R^{-1} = \{ (y, x) \mid (x, y) \in R \}$. A binary relation over a set is equal to its inverse if and only if it is symmetric. See also duality (order theory). For example, "is less than" (<) is the inverse of "is greater than" (>).

If R is a binary relation over X, then each of the following is a binary relation over X:

- **Reflexive closure**: $R^=$, defined as $R^= = \{ (x, x) \mid x \in X \} \cup R$ or the smallest reflexive relation over X containing R. This can be proven to be equal to the intersection of all reflexive relations containing R.
- **Reflexive reduction**: $R^{\neq}$, defined as $R^{\neq} = R \setminus \{ (x, x) \mid x \in X \}$ or the largest irreflexive relation over X contained in R.
- **Transitive closure**: R^+, defined as the smallest transitive relation over X containing R. This can be seen to be equal to the intersection of all transitive relations containing R.
- **Reflexive transitive closure**: R^*, defined as $R^* = (R^+)^=$, the smallest preorder containing R.
- **Reflexive transitive symmetric closure**: $R^{\equiv}$, defined as the smallest equivalence relation over X containing R.

Complement

If R is a binary relation over X and Y, then the following too:

- The **complement** S is defined as $x\ S\ y$ if not $x\ R\ y$. For example, on real numbers, ≤ is the complement of >.

The complement of the inverse is the inverse of the complement.

If $X = Y$, the complement has the following properties:

- If a relation is symmetric, the complement is too.
- The complement of a reflexive relation is irreflexive and vice versa.
- The complement of a strict weak order is a total preorder and vice versa.

The complement of the inverse has these same properties.

Restriction

The restriction of a binary relation on a set X to a subset S is the set of all pairs (x, y) in the relation for which x and y are in S.

If a relation is reflexive, irreflexive, symmetric, antisymmetric, asymmetric, transitive, total, trichotomous, a partial order, total order, strict weak order, total preorder (weak order), or an equivalence relation, its restrictions are too.

However, the transitive closure of a restriction is a subset of the restriction of the transitive closure, i.e., in general not equal. For example, restricting the relation "x is parent of y" to females yields the relation "x is mother of the woman y"; its transitive closure doesn't relate a woman with her paternal grandmother. On the other hand, the transitive closure of "is parent of" is "is ancestor of"; its restriction to females does relate a woman with her paternal grandmother.

Also, the various concepts of completeness (not to be confused with being "total") do not carry over to restrictions. For example, on the set of real numbers a property of the relation "≤" is that every non-empty subset S of $\mathbf{R}$ with an upper bound in $\mathbf{R}$ has a least upper bound (also called supremum) in $\mathbf{R}$. However, for a set of rational numbers this supremum is not necessarily rational, so the same property does not hold on the restriction of the relation "≤" to the set of rational numbers.

The *left-restriction* (*right-restriction*, respectively) of a binary relation between X and Y to a subset S of its domain (codomain) is the set of all pairs (x, y) in the relation for which x (y) is an element of S.

Algebras, categories, and rewriting systems

Various operations on binary endorelations can be treated as giving rise to an algebraic structure, known as relation algebra. It should not be confused with relation*al* algebra which deals in finitary relations (and in practice also finite and many-sorted).

For heterogenous binary relations, a category of relations arises.[7]

Despite their simplicity, binary relations are at the core of an abstract computation model known as an abstract rewriting system.

Sets versus classes

Certain mathematical "relations", such as "equal to", "member of", and "subset of", cannot be understood to be binary relations as defined above, because their domains and codomains cannot be taken to be sets in the usual systems of axiomatic set theory. For example, if we try to model the general concept of "equality" as a binary relation =, we must take the domain and codomain to be the "class of all sets", which is not a set in the usual set theory.

In most mathematical contexts, references to the relations of equality, membership and subset are harmless because they can be understood implicitly to be restricted to some set in the context. The usual work-around to this problem is to select a "large enough" set A, that contains all the objects of interest, and work with the restriction $=_A$ instead of =. Similarly, the "subset of" relation $\subseteq$ needs to be restricted to have domain and codomain $P(A)$ (the power set of a specific set A): the resulting set relation can be denoted $\subseteq_A$. Also, the "member of" relation needs to be restricted to have domain A and codomain $P(A)$ to obtain a binary relation $\in_A$ that is a set. Bertrand Russell has shown that assuming $\in$ to be defined on all sets leads to a contradiction in naive set theory.

Another solution to this problem is to use a set theory with proper classes, such as NBG or Morse–Kelley set theory, and allow the domain and codomain (and so the graph) to be proper classes: in such a theory, equality, membership, and subset are binary relations without special comment. (A minor modification needs to be made to the concept of the ordered triple (X, Y, G), as normally a proper class cannot be a member of an ordered tuple; or of course one can identify the function with its graph in this context.)[26] With this definition one can for instance define a function relation between every set and its power set.

The number of binary relations

The number of distinct binary relations on an n-element set is 2^{n^2} (sequence A002416 in the OEIS):

Number of n-element binary relations of different types

n	all	transitive	reflexive	preorder	partial order	total preorder	total order	equivalence relation
0	1	1	1	1	1	1	1	1
1	2	2	1	1	1	1	1	1
2	16	13	4	4	3	3	2	2
3	512	171	64	29	19	13	6	5
4	65536	3994	4096	355	219	75	24	15
n	2^{n^2}		2^{n^2-n}			$\Sigma_{k=0}^{n} k! S(n, k)$	$n!$	$\Sigma_{k=0}^{n} S(n, k)$
OEIS	**A002416**	**A006905**	**A053763**	**A000798**	**A001035**	**A000670**	**A000142**	**A000110**

Notes:

- The number of irreflexive relations is the same as that of reflexive relations.
- The number of strict partial orders (irreflexive transitive relations) is the same as that of partial orders.
- The number of strict weak orders is the same as that of total preorders.
- The total orders are the partial orders that are also total preorders. The number of preorders that are neither a partial order nor a total preorder is, therefore, the number of preorders, minus the number of partial orders, minus the number of total preorders, plus the number of total orders: 0, 0, 0, 3, and 85, respectively.
- the number of equivalence relations is the number of partitions, which is the Bell number.

The binary relations can be grouped into pairs (relation, complement), except that for $n = 0$ the relation is its own complement. The non-symmetric ones can be grouped into quadruples (relation, complement, inverse, inverse complement).

Examples of common binary relations

- order relations, including strict orders:
  - greater than

- greater than or equal to
- less than
- less than or equal to
- divides (evenly)
- is a subset of
- equivalence relations:
  - equality
  - is parallel to (for affine spaces)
  - is in bijection with
  - isomorphy
- dependency relation, a finite, symmetric, reflexive relation.
- independency relation, a symmetric, irreflexive relation which is the complement of some dependency relation.

See also

- Confluence (term rewriting)
- Hasse diagram
- Incidence structure
- Logic of relatives
- Order theory
- Triadic relation

Notes

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