3-1 Dynamic Programming

(Part II: "Theory")

Hengfeng Wei

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September 27, 2018

Definition (Optimal Substructure)

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How to prove "YES"?

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How to show "NO"?

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How to prove "YES"?

How to use "Cut-and-Paste"?

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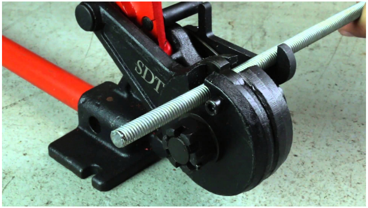
$\boxed{\textit{Relative to Subproblems}}$

Rod Cutting



Optimal Substructure of Rod-Cutting (Problem 15.3-5)

Limit l_i : # of pieces of length $i, \quad 1 \leq i \leq n$

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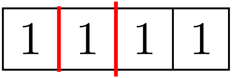

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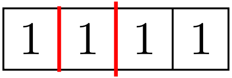

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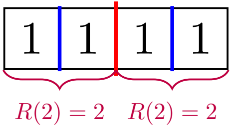


Well done

"Show that the optimal-substructure property described in Section 15.1 *no longer holds."*

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Where is the leftmost cut?

$$R(i, L) = \max_{1 \le j \le i} \Big(p_j + R(i - j, L[j \mapsto L_j - 1]) \Big)$$

Currency Exchange (Problem 15.3-6)

$1, 2, \cdots, n$ currencies

$$i \xrightarrow{r_{ij}} j$$

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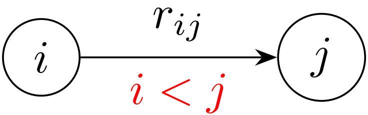
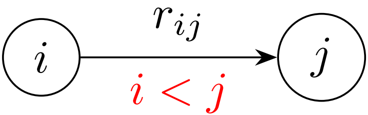
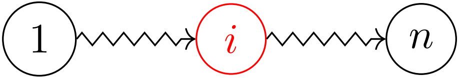
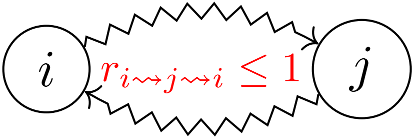
c_k : Commission charged for k trades

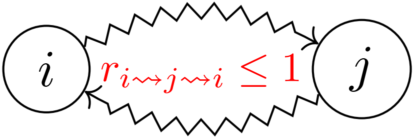

i
r_{ij}
j
$i < j$



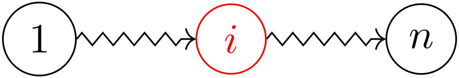
An optimal sequence of trades from 1 to n through i:

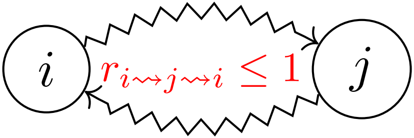



i
$r_{i \rightsquigarrow j \rightsquigarrow i} \le 1$
j



An *optimal* *sequence of trades from* 1 *to* n *through* i*:*





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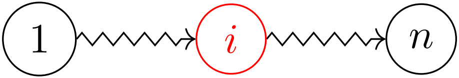
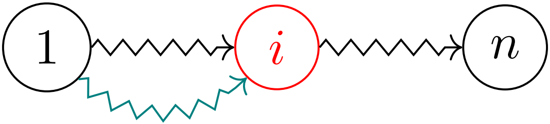
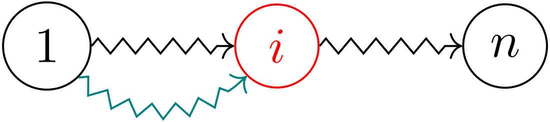


By Contradiction.



$$\text{CASE I} : s_{1 \rightsquigarrow i} \cap s_{i \rightsquigarrow n} = \emptyset$$



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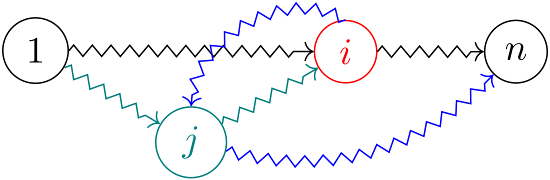

$\textsc{Case II} : j \in s_{1 \leadsto i} \cap s_{i \leadsto n}$

$$
\begin{aligned}
& 1 \leadsto j \leadsto n \\
\geq\ & 1 \leadsto j \leadsto i \leadsto j \leadsto n \\
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Longest Path Problem

To find a *simple* path of maximum length from s to t in a graph.

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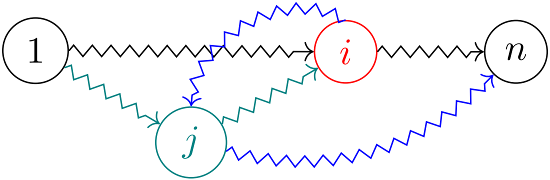

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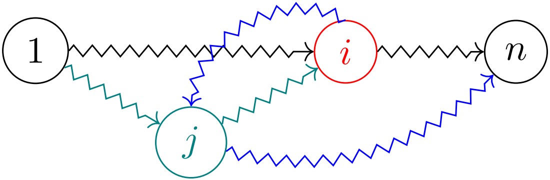

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Does the longest path problem really have no optimal substructure?

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WAIT WAIT...
DON'T TELL ME!®
FROM NPR® & WBEZ® CHICAGO

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through exactly the intermediate vertices in I

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What is the next vertex from s*?*

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$$L(t, t, I) = 0, \qquad L(s, t, \emptyset) = \begin{cases} 1, & \text{if } (s, t) \in E \\ 0, & \text{otherwise} \end{cases}$$

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DP does not necessarily lead to efficient (polynomial) algorithms.

The (decision version of the) longest path problem is NP-hard!

The Knapsack Problem

The Change-making Problem

Coins values: $x_1, x_2, \cdots, x_n$

Amount: v

Is it possible to make change for v?

The Change-making Problem (a.k.a Subset Sum)

Without repetition: $0/1$

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$C[i, w]$: Make change for w using only values of $x_1 \ldots x_i$?

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$$O(nv)$$

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Unbounded repetition: ∞

$$C[i, w] \text{ vs. } C[w]$$

$$C[i, w] = C[i-1, w] \lor (C[i, w - x_i] \land w \geq x_i)$$

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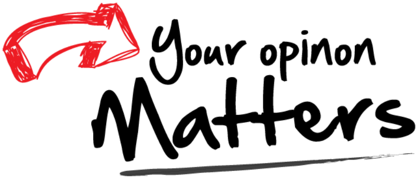
Problem (Hungry-Lion Game)

- *A sheep in danger*
- *Hungry lions in a strict hierarchy*


THANK YOU



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