3-11 Matchings and Factors

(Part I: Matchings and Covers)

Hengfeng Wei

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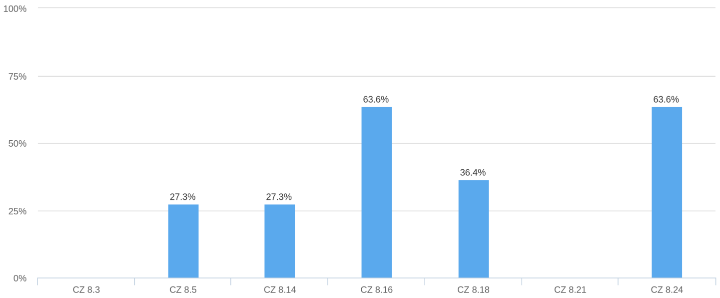
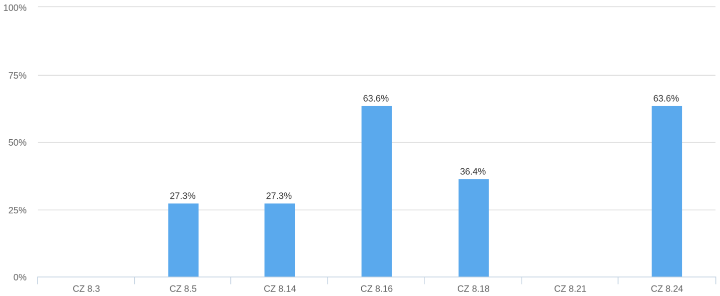
December 10, 2018



8.5 8.14 8.16

8.18 8.24 (The Last Class)



8.5 8.14 8.16 Chinese Postman Problem (The Last Class?)

8.18 8.24 (The Last Class)

比较大的定理（证明比较长的）都不是很理解，想知道期末考什么

点覆盖边覆盖那里只知道有这些性质，了解不是很深

都理解

图的分解的形象意义

无

定理8.3的证明

αβ、α'β'的定义和几个定理推论

为什么中英文书上的定义中\alpha和\beta反了。。

定理8.10的证明看不懂； 一些比较几何的构造法证明（比如把顶点排成正多边形，一个点放中间）是怎么保证这些分解不重不漏的?

Kirkman三元系

$$\alpha, \quad \beta, \quad \alpha', \quad \beta'$$

Theorem 8.10 (Tutte's Theorem) (The Last Class)

Theorem (Hall's Theorem, 1935; Theorem 8.3)

Let G be a *bipartite graph* *with partite sets U and W such that $r = |U| \le |W|$.*
G contains a matching of cardinality $r \iff G$ satisfies *Hall's Condition:*

$$\forall X \subseteq U : \big|N(X)\big| \ge \big|X\big|$$

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(The Obvious Necessary Conditions are Also Sufficient)

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Other TONCAS?

Perfect Matching on Trees (Problem 8.5)

Prove that every tree T has ≤ 1 perfect matching.

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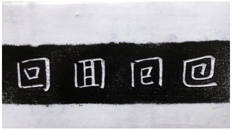
“这题有四样证法, 你知道吗?”



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else ▷ n is even

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if n is odd **then**
    $\#$ Perfect Matching $= 0$
else $\triangleright$ n is even
    Consider $T - r$ $\triangleright$ r : root of G
    if $k_o(T - r) > 1$ **then**
    else $\triangleright$ $k_o(T - r) = 1$

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    Consider T − r                    ▷ r : root of G
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        By Induction Hypothesis.
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1: **if** T has no perfect matchings **then**
2: $\quad \#$ Perfect Matching $= 0$
3: **else** $\triangleright$ T has perfect matchings

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  Consider a leaf v
  v must be matched with its parent u

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    Consider a leaf v
    v must be matched with its parent u
    By Induction Hypothesis on each component of $G - \{u, v\}$

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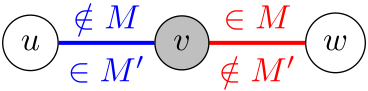

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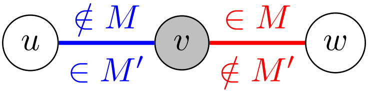

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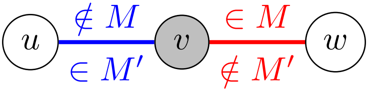

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Contradiction: Cycle

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Consider the subgraph $\mathcal{M}$ with $V(T)$ and $M \Delta M'$.

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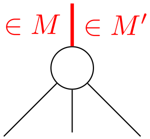
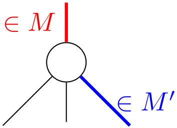


CASE I



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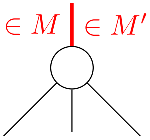

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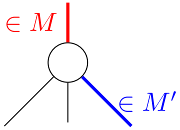

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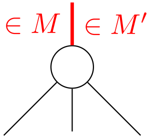

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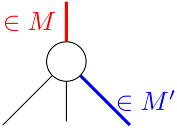

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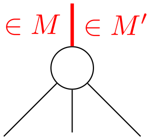

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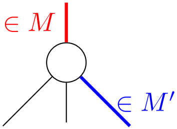

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$$\beta(G)$$

$$\alpha'(G)$$

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$\alpha(G)$	Maximum size of independent set
$\beta(G)$	Minimum size of vertex cover
$\alpha'(G)$	Maximum size of matching
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Theorem (Gallai Identities, 1959; Theorem 8.7)

If G is graph without isolated vertices, then

$$\alpha'(G) + \beta'(G) = n(G).$$

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A graph G without isolated vertices has a perfect matching if and only if $\alpha'(G) = \beta'(G)$.

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Theorem (König, 1931; Egerváry, 1931)

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If G is a graph of order n, maximum degree Δ and having no isolated vertices, then

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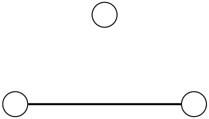

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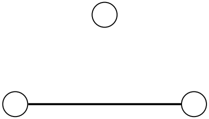

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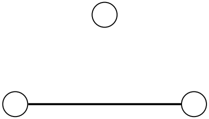

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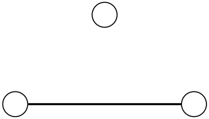

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Contradiction: $\exists$ isolated vertices in the other $n - \beta$ vertices.

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Vertex Independence Number (Additional Problem)

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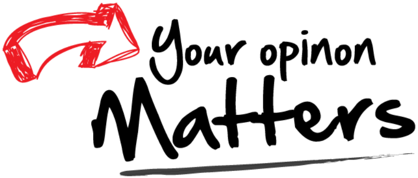
By Construction.

To construct an independent set S with $|S| \geq \frac{n}{\Delta+1}$.

while $|V(G) > 0|$ **do**
    Choose $v \in V(G)$
    $S \leftarrow S \cup \{v\}$
    $G \leftarrow G - \{v\} - N(v)$


THANK YOU



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