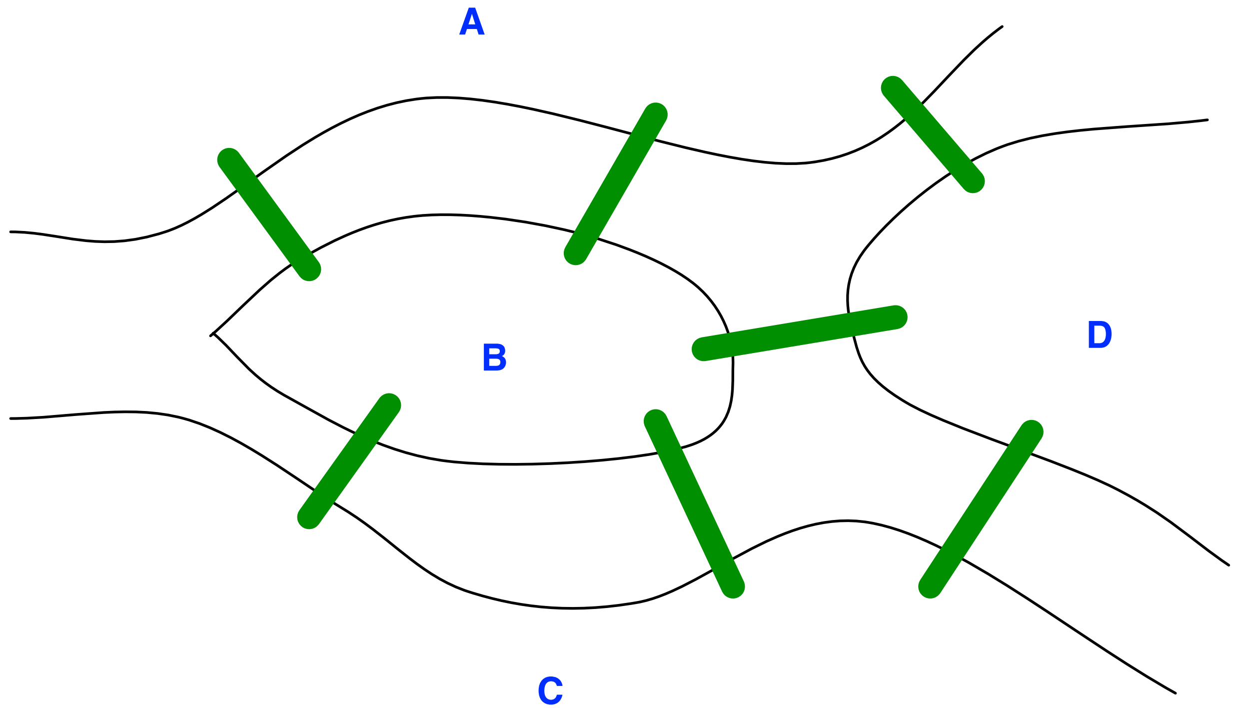
Euler tours and Hamilton cycles

Euler tours

Konigsberg Bridge Problem : Can you pass through each of the bridges and return to the starting point.


A
B
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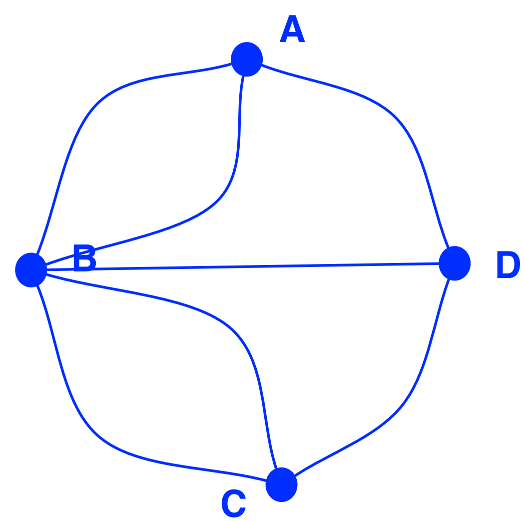
A
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C
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Definition 1 *An Eulerian circuit (Eulerian trail) in a graph is a circuit (trail) which contains all edges. A graph is called Eulerian if it has an Eulerian circuit.*

Definition 2 *A vertex is even (odd) if its degree is even (odd). A graph is called even if every vertex has an even degree.*

Lemma 1 *If every vertex of a graph G has degree at least 2 then G contains a cycle.*

Theorem 2 *Let G be a connected graphs. Then G is Eulerian if and only if it is even.*

Proof.

- Any passage of an Eulerian circuit through a vertex v contributes two to its degree.

- Induction on the number of edges. For the inductive step, let C be a cycle in G. Apply induction to each component of $G - C$ and paste Eulerian circuits of the components when traversing C.

Fleury's Algorithm

Procedure Fleury(G- Eulerian graph)

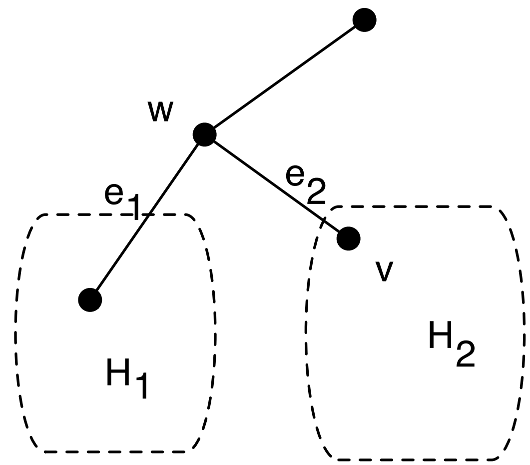
1. Let $v \in V(G)$, $T := v$, $w := v$
2. while $E(G) \neq \emptyset$ do:
3. Select edge $e = wu$ incident to w which is not a cut edge unless there is no alternative.
4. $T := Teu$, $w := u$
5. Delete e from G.

Theorem 3 *Fleury's algorithm finds an Eulerian circuit in an Eulerian graph G.*

Proof.

- During the execution of the algorithm there is at most one bridge incident to vertex w, and this edge will be used last allowing to use all edges incident to every vertex of G.

- Indeed, suppose w is incident to two bridges e_1, e_2 and let $H_1, \ldots, H_l$ be the components of $G - w$. Suppose $v \notin H_1$. Then every vertex in $V(H_1)$ has an even degree in G when w is examined but then H_1 has exactly one vertex of an odd degree.

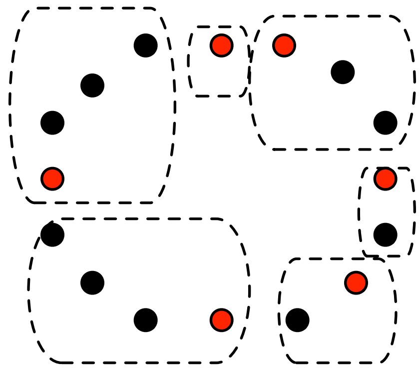


Hamiltion cycles

A Hamiltonian cycle in a graph G is a spanning subgraph of G which is isomorphic to a cycle.

Definition 3 *Graph is called Hamiltonian if it has a Hamiltonian cycle.*

Proposition 4 *If G has a Hamiltonian cycle, then for each nonempty set $S \subseteq V$, the graph $G - S$ has at most $|S|$ components.*



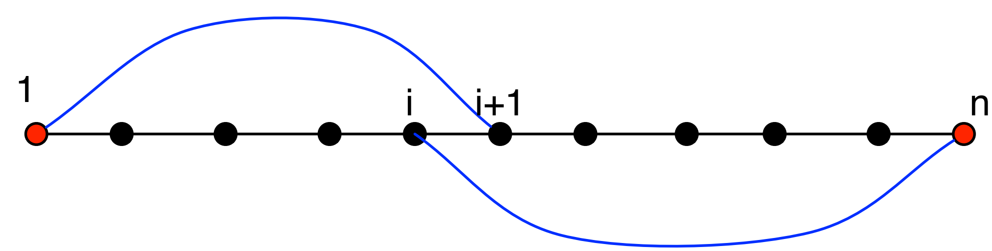
Theorem 5 (Dirac's Theorem) *If G is a graph on at least three vertices with minimum degree $\delta(G) \geq n(G)/2$ then G has a Hamiltonian cycle.*

Proof.

- Consider a maximal counterexample G.

- Then G has a spanning path $v_1, \ldots, v_n$.

- If there is an index $i = 1, \ldots, n-1$ such that $v_i v_n \in E(G)$ and $v_{i+1} v_1 \in E(G)$. Then $v_1 v_{i+1} v_{i+2} \ldots v_n v_i v_{i-1} \ldots, v_2$ is a Hamiltonian cycle.



- Let $S = \{i | v_1 v_{i+1} \in E\}$ and $T = \{i | v_n v_i \in S\}$ where $i =$

$1, \ldots, n-1$. We have

$$|S \cup T| + |S \cap T| = |S| + |T| = d(u) + d(v) \geq n.$$

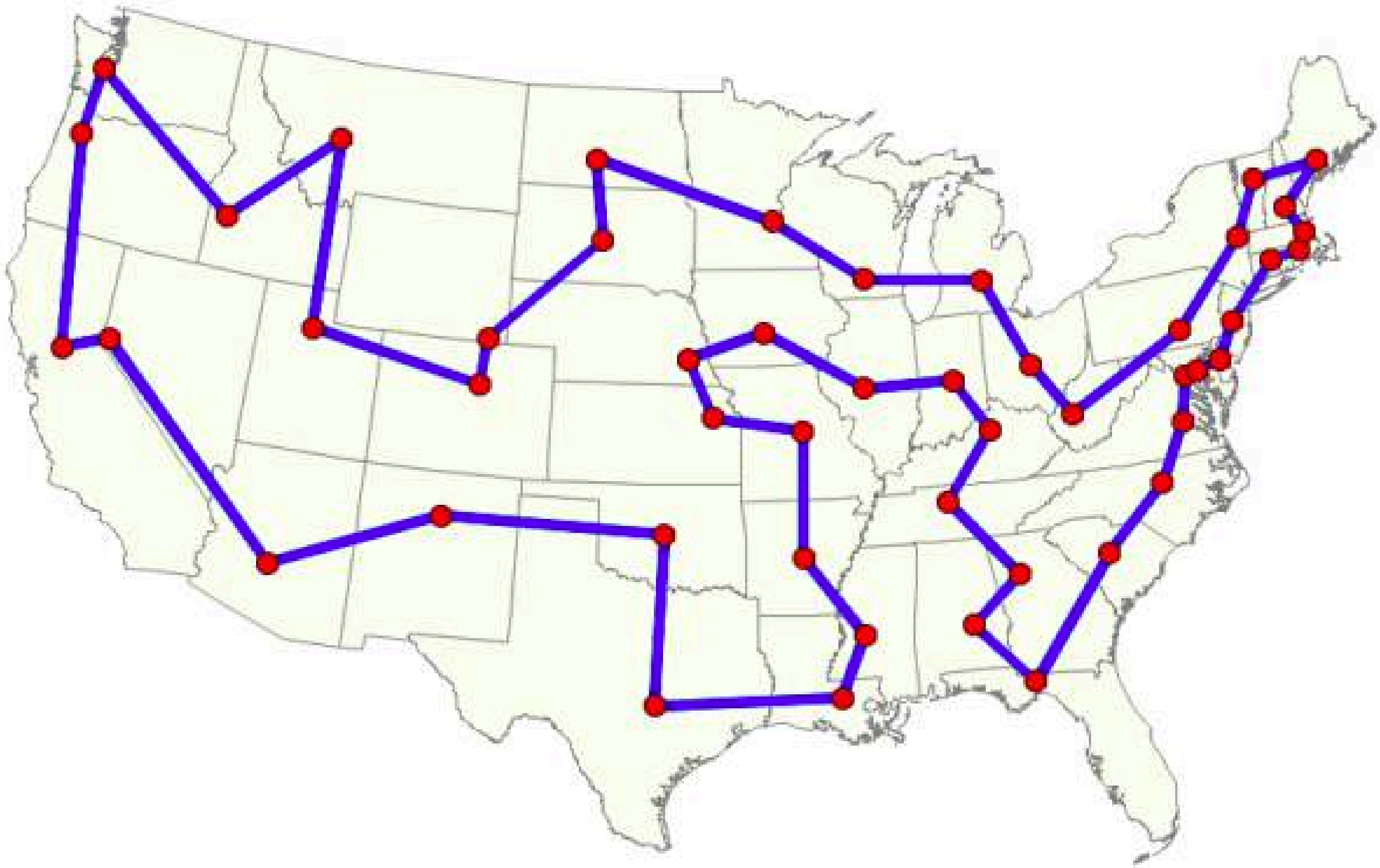
and $|S \cup T| \leq n-1$ gives $|S \cap T| \geq 1$. Consequently there is an i such that $uv_{i+1} \in E$ and $vv_i \in E$.

The Traveling Salesman Problem

Given a complete graph G with positive weights on the edges, find a minimum weight spanning cycle in G.

9.4 Proc OptNet: TSP

Total distance = 10,627.75 miles



Remarks

- The problem is NP-complete.

- 2-opt heuristic: If $C := v_1, \ldots, v_n$ is the current solution, consider two edges $e = v_i v_{i+1}$, $e' = v_j v_{j+1}$ and check if $C' := C - e - e' + f + f'$ where $f = v_i v_j$, $f' = v_{i+1} v_{j+1}$ has a smaller weight.

- Lin-Kernighan heuristic considers switches of more than two edges.

- Even finding a constant approximation of the TSP efficiently is not possible if $P \neq NP$.

Theorem 6 *In the class of the graphs which satisfy the triangle inequality, there is an efficient algorithm that finds a spanning cycle of weight at most two times the optimal.*

Proof.

- The triangle inequality: $w_{i,j} + w_{j,k} \geq w_{i,k}$.

- We find a solution of weight at most $2M$ where M is the weight of a minimum weight spanning tree.

- Double all edges of a minimum-weight spanning tree and find an Eulerian circuit (the weight of which is $2M$).

- Transform circuit into a cycle: Suppose v_j is traversed more than once, say $v_i \to v_j \to v_k$ and $v_p \to v_i \to v_q$. Delete v_iv_j, v_jv_k and add v_iv_k which by the triangle inequality has a smaller weight.