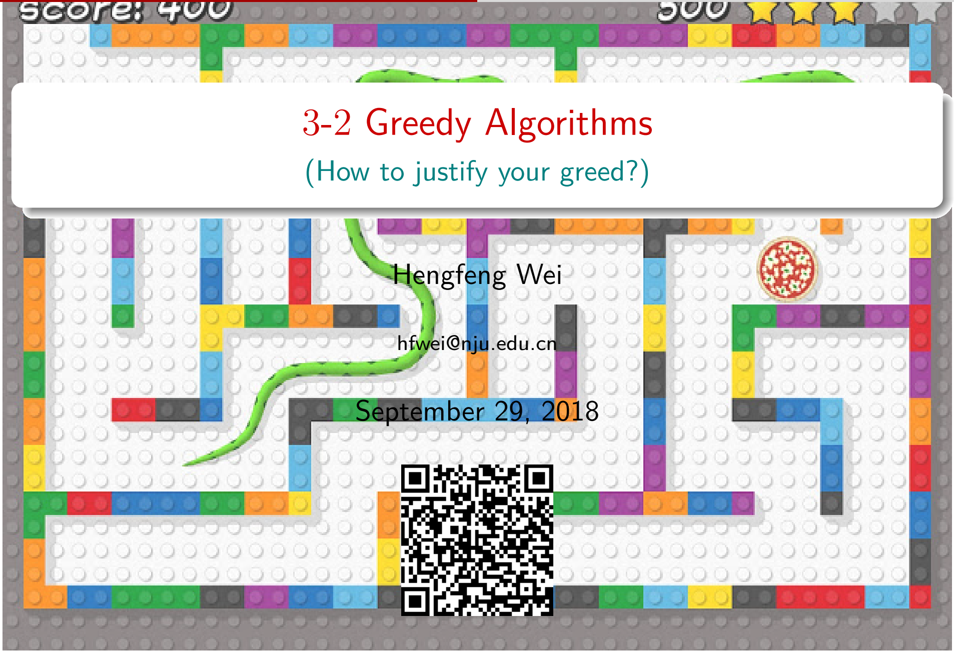

3-2 Greedy Algorithms

(How to justify your greed?)

Hengfeng Wei

hfwei@nju.edu.cn

September 29, 2018

"GREED IS GOOD."

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"BUT"

"Greedy algorithms without proofs are considered wrong."

Proof: Inductive Exchange Argument

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Theorem (Greedy-Choice Property)

There is an optimal solution that makes the greedy choice.

Theorem (Optimal Substructure Property)

A problem exhibits optimal substructure if an optimal solution to the problem contains within it optimal solutions to subproblems.

Live with Your Deadlines (Additional Problem)

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$$HW_i : (\mathit{Length} : t_i, \ \mathit{Deadline} : d_i)$$

4

$OT : 2$

$PS : 5$

$OJ : 6$

$PA : 8$

Live with Your Deadlines (Additional Problem)

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$$\textit{Finish time} : f_i = s_i + t_i$$

$$\textit{Lateness} : l_i = f_i - d_i \quad (0 \text{ if } f_i \le d_i)$$

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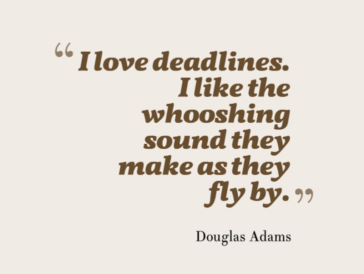
$$\min \left(L \triangleq \max_i l_i \right)$$

What are your greedy strategies?



Shortest Job First: $\min t_i$

Shortest Job First: $\min t_i$

Urgentest Job First: $\min(d_i - t_i)$

$$t_1 = 1,\ d_1 = 100$$

$$t_2 = 10,\ d_2 = 10$$

Shortest Job First: $\min t_i$

Urgentest Job First: $\min(d_i - t_i)$

$$t_1 = 1,\ d_1 = 100$$

$$t_2 = 10,\ d_2 = 10$$

$$t_1 = 1,\ d_1 = 2$$

$$t_2 = 10,\ d_2 = 10$$

Earliest Deadline First

Proof: Inductive Exchange Argument.

$$OPT : J_1^*,\ J_2^*,\ \cdots,\ J_n^*$$

$$G : J_1,\ J_2,\ \cdots,\ J_n, \quad (d_i \le d_j,\ \forall i \le j)$$

$$\exists J_i : d_{J_i} < d_{J_1^*}$$

$$d_{J_i} < d_{J_1^*}$$

$\mathcal{O}$:	J_1^*		J_i^*		J_i		

$\mathcal{O}'$:	J_i		J_i^*		J_1^*		

$$d_{J_i} < d_{J_1^*}$$

$\mathcal{O}:$	J_1^*		J_i^*		J_i		

$\mathcal{O}':$	J_i		J_i^*		J_1^*		

$$l'_{J_1^*} = f'_{J_1^*} - d_{J_1^*} = f_{J_i} - d_{J_1^*} < f_{J_i} - d_{J_i} = l_{J_i}$$

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Theorem

All schedules with no idle time and no inversions have the same maximum lateness.

Theorem (Optimal Substructure Property)

A problem exhibits optimal substructure if an optimal solution to the problem contains within it optimal solutions to subproblems.

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$$L(s, S)$$

The optimal lateness obtainable from scheduling the jobs in S

which are available to start at the common start time s.

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$$L(s, S)$$

The optimal lateness obtainable from scheduling the jobs in S

which are available to start at the common start time s.

What is the first job to schedule?

$$L(s, S) = \min_{1 \le i \le n} \Big(\max \left(L(0, \{J_i\}), L(t_i, J \setminus \{J_i\}) \right) \Big)$$

$$J = \{J_1, J_2, J_3\}$$

$$t_1 = 6, t_2 = 2, t_3 = 2$$

$$d_1 = 2, d_2 = 7, d_3 = 8$$

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$\mathcal{O} : J_1, J_3, J_2$

$L(0, \{J_1, J_2, J_3\}) = 4$

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$$\mathcal{O}' \not\subset \mathcal{O}$$

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$$\mathcal{O}' \not\subset \mathcal{O}$$

$$\exists\ \mathcal{O} \triangleq J_1, J_2, J_3 : \mathcal{O}' \subset \mathcal{O}$$

Theorem

The greedy solution has no idle time and no inversions.

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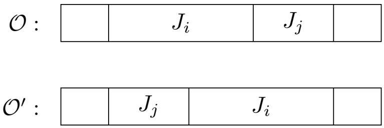
Immediate Inversion

$$d_j < d_i$$



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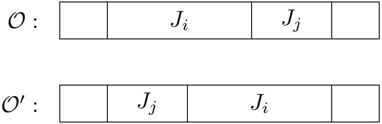

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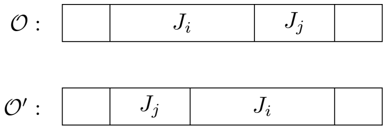
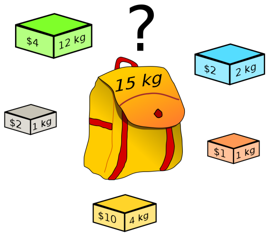


$$l'_j \le l_j$$

$$l'_i = f_j - d_i < f_j - d_j = l_j$$

Fractional Knapsack Problem (Problem 16.2-1)

Prove that the fractional knapsack problem has the greedy-choice property.



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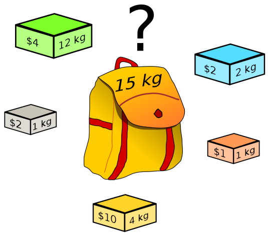

$\exists$ *an optimal solution which contains the greedy choice.*

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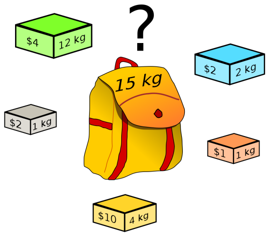

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$$\frac{v_1}{w_1} \geq \frac{v_2}{w_2} \geq \cdots \geq \frac{v_n}{w_n}$$

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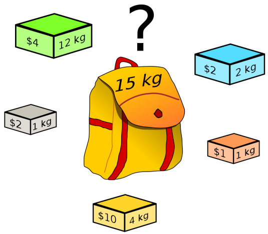

$\exists$ *an optimal solution which contains the greedy choice.*

$$\frac{v_1}{w_1} \geq \frac{v_2}{w_2} \geq \cdots \geq \frac{v_n}{w_n}$$

As most item of $\dfrac{v_1}{w_1}$ as possible.

Change-Making Problem (Problem 16-1 (a))

Making change for v cents using the fewest number of coins.

Change-Making Problem (Problem 16-1 (a))

25

10

5

1

Making change for v cents using the fewest number of coins.

$$\text{OPT} : v = 25 \times A^* + 10 \times B^* + 5 \times C^* + 1 \times D^*$$

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$$\text{G}: v = 25 \times A + 10 \times B + 5 \times C + 1 \times D$$

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$$(A^*, B^*, C^*, D^*) \textit{ vs. } (A, B, C, D)$$

$$\text{OPT}: v = 25 \times A^* + 10 \times B^* + 5 \times C^* + 1 \times D^*$$

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$$(A^*, B^*, C^*, D^*) \text{ vs. } (A, B, C, D)$$

$$A^* = A$$

$$\text{OPT}: v = 25 \times A^* + 10 \times B^* + 5 \times C^* + 1 \times D^*$$

$$\mathsf{OPT} : v = 25 \times A^* + 10 \times B^* + 5 \times C^* + 1 \times D^*$$

Lemma (Properties of OPT)

(I) $B^* \leq 2$

(II) $C^* \leq 1$

(III) $D^* \leq 4$

(IV) $\neg(B^* = 2 \land C^* = 1)$

$$\text{OPT}: v = 25 \times A^* + 10 \times B^* + 5 \times C^* + 1 \times D^*$$

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Theorem

$$10 \times B^* + 5 \times C^* + 1 \times D^* \leq 24$$

$$5 \times C^* + 1 \times D^* \leq 9$$

$$1 \times D^* \leq 4$$

$$\text{OPT} : v = 25 \times A^* + 10 \times B^* + 5 \times C^* + 1 \times D^*$$

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(I) $B^* \leq 2$

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$$(A^*, B^*, C^*, D^*) = (A, B, C, D)$$

Change-Making Problem (Problem 16-1 (c))

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$$\{1, 3, 4\}, \qquad v = 6$$

Change-Making Problem (Problem 16-1 (c))

$$\{1, 3, 4\}, \qquad v = 6$$

$$\text{G} : 4, 1, 1 \quad \textit{vs.} \quad \text{OPT} : 3, 3$$

Change-Making Problem (Problem 16-1 (c))

$$\{1, 3, 4\}, \qquad v = 6$$

$$\mathsf{G} : 4, 1, 1 \quad \textit{vs.} \quad \mathsf{OPT} : 3, 3$$

Why does the previous proof fail here?

Change-Making Problem (Problem 16-1 (b))

$$c^0, c^1, c^2, \cdots, c^k, \quad c > 1, k \geq 1$$

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Change-Making Problem (Problem 16-1 (b))

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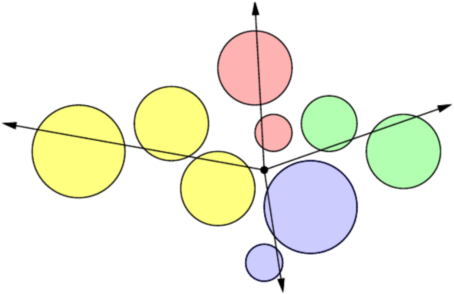
$$a_i^* = a_i$$

Canonical Coin Systems

Minimum Shots Problem (Additional Problem)



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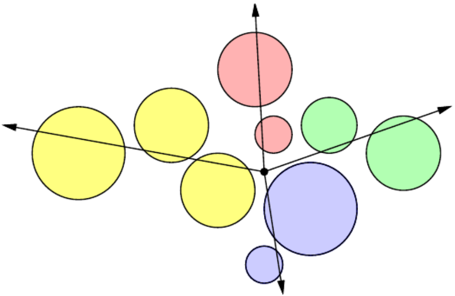
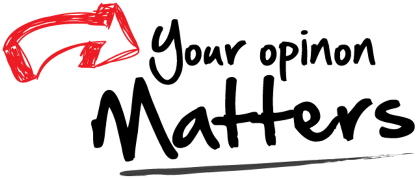


Assumption:

Possible to shoot a ray that does not intersect any balloons.


KEEP
CALM
MAKE
NO
ASSUMPTION


THANK YOU



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