A New Sufficient Condition of Hamiltonian Path

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Abstract:
A Hamiltonian path is a spanning path in a graph i.e. a path through every vertex. In this paper we present an interesting sufficient condition for a graph to possess a Hamiltonian path. In particular we prove that the degree sum of all pairwise nonadjacent vertex-triples is greater than $1/2(3n - 5)$ implies that the graph has a Hamiltonian path, where n is the number of vertices of that graph. Also, the condition is proven to be tight.

Keywords: graphs, Spanning path, Hamiltonian path.

1 Introduction and Previous Works

A Hamiltonian cycle is a spanning cycle in a graph i.e. a cycle through every vertex and a Hamiltonian path is a spanning path. A graph containing a Hamiltonian cycle is said to be Hamiltonian. It is clear that every graph with a Hamiltonian cycle has a Hamiltonian path but the converse is not necessarily true. The study of Hamiltonian cycles and Hamiltonian paths in general and special graphs has been fueled by practical applications and by the issues of complexity. The problem of finding whether a graph G is Hamiltonian is proved to be NP-Complete for general graphs [3]. The problem remains NP-complete [see 3] (1) if G is planar, cubic, 3-connected, and has no face with fewer than 5 edges, (2) if G is bipartite, (3) if G is the square of a graph, (4) if a Hamiltonian path for G is given as part of the instance. On the other hand the problem of finding whether a graph G contains a Hamiltonian path is also proved to be NP-Complete for general graphs [3]. Again it remains NP-complete for conditions (1) and (2) mentioned above. Even the variant, in which either the starting point or the end point or both are specified in the input instance, is also NP-Complete. No easily testable characterization is known for Hamiltonian graphs. Nor there exists any such condition to test whether a graph contains a Hamiltonian path or not. This is why tremendous amount of research has been done in finding the sufficient conditions for the existence of Hamiltonian cycles or Hamiltonian paths in graphs. The existing conditions in the literature exploits many graph parameters among which degree of vertices, especially pairwise nonadjacent vertices, is worth-noting. We first present some of the famous degree related conditions below. But before that we need to introduce and define some notations and definitions we use. Given a graph $G = (V, E)$ and a vertex $u \in V$, we mean by $d(u)$ the degree of u in G. In other words $d(u) = |N_G(u)|$, where $N_G(u)$ denotes the neighbour set of u in the graph G. If $H \subseteq G$ then $d_H(u) = |N_H(u)|$ and $d_{\overline{H}}(u) = \left|N_{G \setminus H}(u)\right|$. By $\delta(G)$ we indicate the degree of a minimum degree vertex in G. $V[G]$ and $E[G]$ are used to denote respectively the vertex set and edge set of G.

Theorem 1.1 (Dirac [2]). *If G is a simple graph with n vertices where $n \geq 3$ and $\delta(G) \geq n/2$, then G is Hamiltonian.* □

Theorem 1.2 (Ore [4]). *Let G be a simple graph with n vertices and u, v be distinct nonadjacent vertices of G with $d(u) + d(v) \geq n$. Then G is Hamiltonian if and only if G + (u, v) is Hamiltonian.*

Theorem 1.3 (Bondy-Chvátal [1]). *If G is a simple graph with n vertices, then G is Hamiltonian if and only if its closure is Hamiltonian..* □

Remark: *The (Hamiltonian) closure of a graph G, denoted C(G), is the supergraph of G on V(G) obtained by iteratively adding edges between pairs of nonadjacent vertices whose degree sum is at least n, until no such pair remains. Fortunately, the closure does not depend on the order in which we choose to add edges when more than one is available i.e. the closure of G is well-defined (For a proof of this statement see [5]).*

Theorem 1.4 (Ore [4]). *If $d(u) + d(v) \geq n$ for every pair of distinct nonadjacent vertices u and v of G, then G is Hamiltonian.* □

Consider the Theorem 1.1 i.e. Dirac's condition. The proof of Dirac's Theorem very cleverly exploits the idea of extremality. The idea was if there is a non-Hamiltonian graph satisfying the hypotheses, then adding edges cannot reduce the minimum degree, so we may restrict our attention to maximal non-Hamiltonian graphs with minimum degree at least $n/2$. By "maximal", we mean that no proper supergraph is also non-Hamiltonian, so $G + (u, v)$ is Hamiltonian whenever u, v are nonadjacent. Note that the maximality of G implies that G has a spanning path from $u = v_1$ to $v = v_n$, i.e. a Hamiltonian path. The rest of the proof tries to find a cross over edge to establish a spanning cycle [see 5 for a complete proof]. The result provided by Ore (Theorem 1.2) is in fact inspired form Dirac's condition. Ore observed that this argument uses $\delta(G) \geq n/2$ only to show that $d(u) + d(v) \geq n$. Therefore, we can weaken the requirement of minimum degree $n/2$ to require only that $d(u) + d(v) \geq n$ whenever u, v are nonadjacent. We also did not need that G was a maximal non-Hamiltonian graph, only that $G + (u, v)$ was Hamiltonian and thereby provided a spanning u, v-path. Theorem 1.4, which is also due to Ore, is in fact an extension of Theorem 1.2. Note carefully that in both Theorems 1.2 and 1.4 the degree sum of pairwise nonadjacent vertices is considered. In this paper we extend the idea presented and adopted in the above-mentioned theorems and present a sufficient condition for a graph to possess a Hamiltonian path imposing conditions on the degree sum of nonadjacent vertex triples. In particular, we here present the following Theorem.

Theorem 1.5. Let $G = (V, E)$ be a connected graph with n vertices. If for all pairwise non-adjacent vertex-triples u, v, and w it holds that $d(u)+d(v)+d(w) \geq \frac{1}{2}(3n-5)$ then G has a Hamiltonian path. □

The paper is organized as follows. In Section 2 we prove the main result of our paper i.e. Theorem 1.5 and establish that the condition of Theorem 1.5 is tight. Finally we conclude in Section 3 with indication to some possible extensions of our research.

2 The Main Result

In this section we first state and prove the following useful Lemma.

Lemma 2.2 *Let G = (V, E) be a connected graph with n vertices and P be a longest path in G. If P is contained in a cycle then P is a Hamiltonian path.*

Proof. Suppose $P \equiv \langle u = u_0, u_1, u_2, \ldots, u_k = v\rangle$ of length k and P is contained in a cycle $C \equiv \langle u = u_0, u_1, u_2, \ldots, u_k = v, u_0 = u\rangle$. Note that $V(C) = V(P)$, since otherwise P would be a part of a longer path, a contradiction. Assume for the sake of contradiction that $k < n - 1$, i.e. P is not Hamiltonian path. Since G is connected, there must be an edge of the form (x, y) such that $x \in V(P) = V(C)$ and $y \in V(G) - V(C)$. Let $x = u_i$. Then there is a path $P' \equiv \langle y, x = u_i, u_{i+1}, \ldots, u_k, u_0, u_1, u_2, \ldots, u_{i-1}\rangle$ with length $k + 1$, which is a contradiction, since P is a longest path in G. □

Now we are ready to prove the main result of our paper i.e. Theorem 1.5 which is restated below for the sake of convenience.

Theorem 1.5. Let $G = (V, E)$ be a connected graph with n vertices. If for all pairwise non-adjacent vertex-triples u, v, and w it holds that $d(u) + d(v) + d(w) \geq \frac{1}{2}(3n - 5)$ then G has a Hamiltonian path.

Proof: Let $P = \langle u_0, u_1, \ldots, u_{p-1}\rangle$ be a longest path in G. And assume for the sake of contradiction that P is not a Hamiltonian path. Now since P is a longest path but not a Hamiltonian path, by the contrapositive of Lemma 2.2, P cannot be contained in a cycle. And since P cannot be contained in a cycle, there cannot be any crossover edge involving u_0 and u_{p-1}. This essentially means that $d(u_0) + d(u_{p-1}) \leq p - 1$. So we must have:

$$\begin{aligned}
& d(u_0) + d(u_{p-1}) + d(w) \geq \frac{1}{2}(3n - 5) \\
\Rightarrow \quad & d(w) \geq \frac{1}{2}(3n - 5) - (d(u_0) + d(u_{p-1})) \\
\Rightarrow \quad & d(w) \geq \frac{1}{2}(3n - 5) - (p - 1) \\
\Rightarrow \quad & d(w) \geq \frac{3}{2}n - p - \frac{3}{2}
\end{aligned}$$

Now we consider $d_P(w)$. We calculate the upper limit of $d_P(w)$ as follows. It is clear that $(w, u_0), (w, u_{p-1}) \notin E$ since otherwise P would not be a longest path in G. Again, Note that w cannot be connected to u_i and u_{i+1}, since in that case we can easily get a path $P'= \langle u_0, u_1, \ldots, u_i, w, u_{i+1}, \ldots, u_{p-1} \rangle$ which is longer than P, leading to a contradiction. So we can write that $d_P(w) \leq \frac{p-2}{2} + 1 = \frac{p}{2}$.

Now we have,

$$\begin{aligned}
d(w) &\geq \frac{3}{2}n - p - \frac{3}{2} \\
\Rightarrow d_P(w) + d_{\overline{P}}(w) &\geq \frac{3}{2}n - p - \frac{3}{2} \\
\Rightarrow \quad d_{\overline{P}}(w) &\geq \frac{3}{2}n - p - \frac{3}{2} - d_P(w) \\
&= \frac{3}{2}n - p - \frac{3}{2} - \frac{p}{2} \\
&= \frac{3}{2}(n - p - 1).
\end{aligned}$$

This leads to a contradiction since $|V(G)\setminus(V(P) \cup \{w\})| = n - p - 1 < \frac{3}{2}(n-p-1)$, which completes the proof. □

In the rest of this section we establish that the condition given in Theorem 1.5 is tight. To establish that we first disprove the following statement..

Statement 2.1.(To Be Disproved) Let $G = (V, E)$ be a connected graph with n vertices. If for all pairwise non-adjacent vertex-triples u, v, and w it holds that $d(u) + d(v) + d(w) \geq \frac{3}{2}(n-2)$ then G has a Hamiltonian path. □

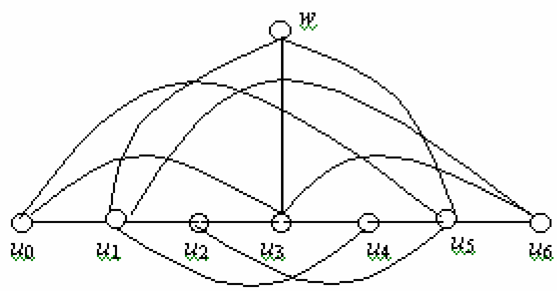

Fig.1. A Graph G with 8 vertices

We disprove Statement 2.1 by presenting a counter example as follows. Consider the graph G in Fig. 1. It can be easily verified that for any nonadjacent vertex-triple u, v, w the condition stated in Statement 2.1 holds i.e. for all vertex triples u, v, and w it holds that $d(u) + d(v) + d(w) \geq \frac{3}{2}(n-2) = 9$ in graph G in Fig. 1. However it can also be verified easily that there exists no Hamiltonian path in G which disproves the Statement 2.1. Now we state the following claim.

Claim 2.2.The condition in Theorem 1.5 is tight.

Proof. The invariant in the condition in Theorem 1.5 is as follows: $d(u) + d(v) + d(w) \geq \frac{1}{2}(3n-5)$. Since the degree sums cannot be fractional numbers, so the next best invariant for the condition would necessarily have to be as follows:

$$\begin{aligned} d(u) + d(v) + d(w) &\geq \frac{1}{2}(3n-5) - 1. \\ &= \frac{1}{2}(3n-5-2) \\ &= \frac{1}{2}(3n-7) \end{aligned}$$

We here prove that this condition can never be achieved. Recall that the invariant in the condition in Statement 2.1 is as follows. $d(u) + d(v) + d(w) \geq \frac{3}{2}(n-2) = \frac{1}{2}(3n-6)$ ($> \frac{1}{2}(3n-7)$). Now since we have disproved Statement 2.1 our claim follows directly.

□

3 Conclusion and Future Works

In this paper we present a new sufficient condition for a graph to possess a Hamiltonian path. Our condition in the form of Theorem 1.5 seems to be significant and interesting. The condition is proven to be tight (Claim 2.2). Also, the sufficient condition we present explores a new idea since the condition is applied on vertex triplets whereas previous existing (degree related) conditions in the literature explores conditions on pair of vertices. The natural extension to this research should be to continue this approach to vertex-quadruples and more and if possible to generalize the cases. Also, since our condition is for Hamiltonian paths only, another direction would be to try for similar conditions for Hamiltonian Cycles.

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