2-11 Heapsort

Hengfeng Wei

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$O, \ \Omega, \ \Theta$

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"What is most efficient way to sort a million 32-bit integers?"

"The bubblesort would be the wrong way to go."

反馈:

O, Ω, Θ 傻傻分不清。

什么时候用哪个?

$6.2-6$ 这道题为什么问的是 Ω, 而不问 O 或 Θ?

Worst-case of MAX-HEAPIFY (TC $6.2-6$)

Show that the worst-case running time of MAX-HEAPIFY on an n-element heap is $\Omega(\log n)$.

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MOVE *vs.* COMPARE

Algorithm $\mathcal{A}$

Inputs $\mathcal{I}$ of size n

	O	Ω	Θ
Best-case			
Worst-case			
Average-case			

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Worst-case			$O = \Omega$
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Worst-case	"power" of $\mathcal{A}$	by example	$O = \Omega$
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Average-case	$\leq$	$\geq$	$O = \Omega$

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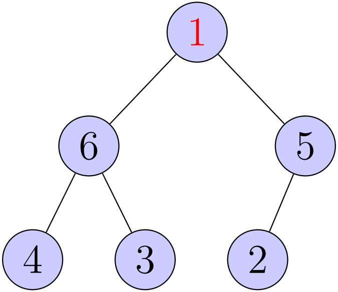

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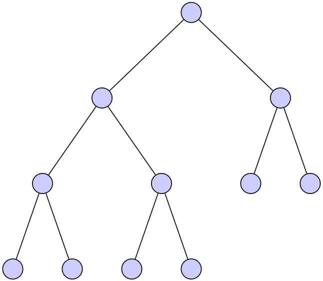

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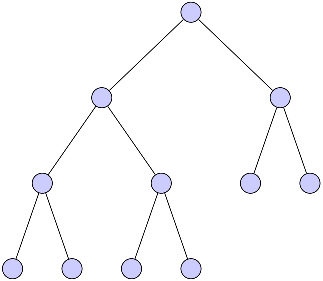

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No Examples Here!

Therefore . . .

Worst-case of MAX-HEAPIFY (TC $6.2 - 6$)

Show that the worst-case running time of MAX-HEAPIFY on an n-element heap is $\Theta(\log n)$.

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Non-proof.

$$\underbrace{\Theta(n)}_{\text{EXTRACT-MAX}} \times \underbrace{\Omega(\log n)}_{\text{MAX-HEAPIFY}} = \Omega(n \log n)$$

□

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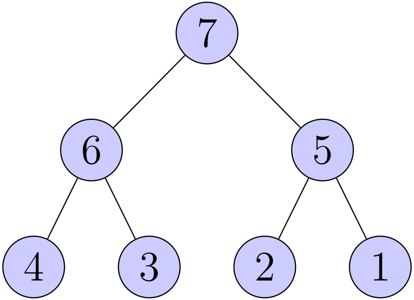

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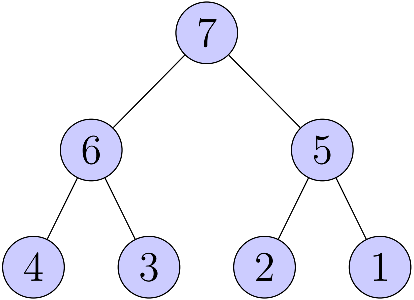
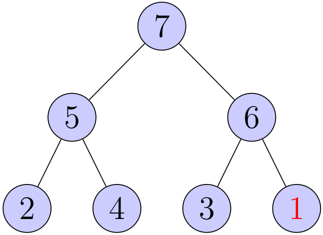
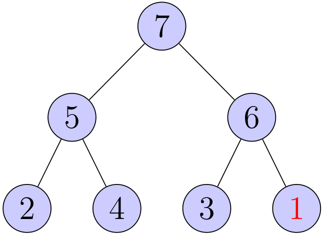


$$T(7) = 2 + 1 + 1 + 1 + 0 + 0 = 5$$


7
5
6
2
4
3
1



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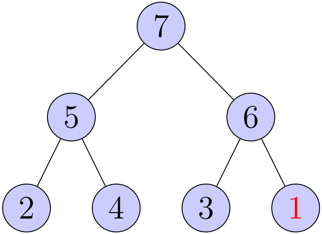

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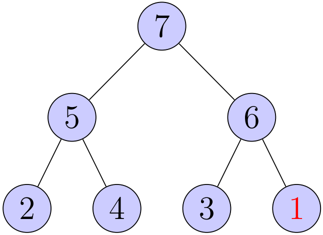

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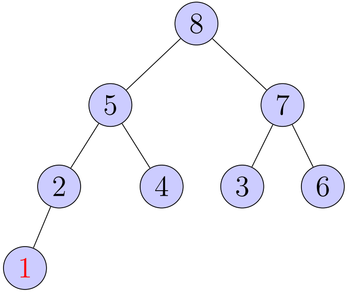

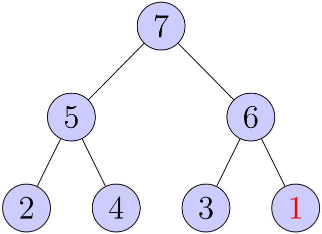

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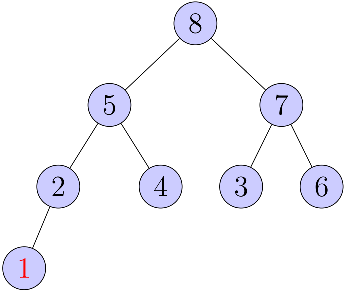

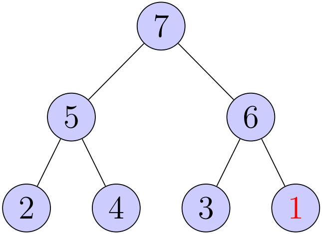

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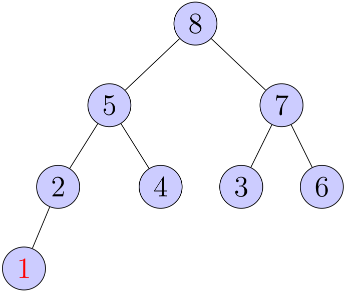

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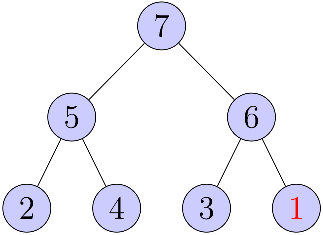

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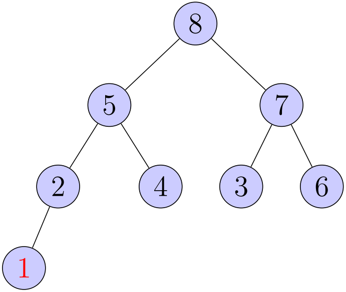

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$$\underbrace{\Theta(n)}_{\text{EXTRACT-MAX}} \times \underbrace{O(\log n)}_{\text{MAX-HEAPIFY}} = O(n \log n)$$

Therefore . . .

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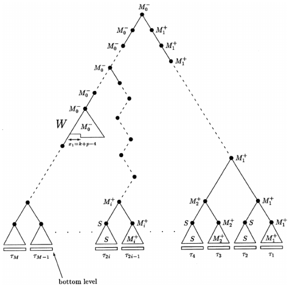

FIG. 2. Initial heap (more detailed).

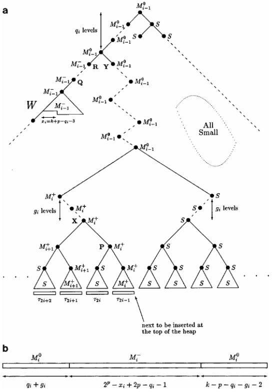

FIG. 3. (a) Odd i; (b) contents of the bottom level of τ_{2i-1}, i odd.

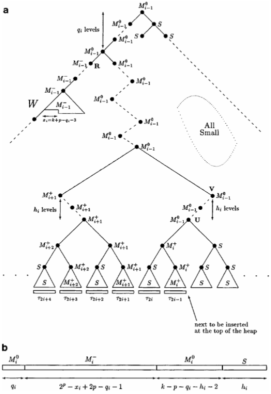

FIG. 4. (a) Even i; (b) contents of the bottom level of τ_{2i-1}, i even.

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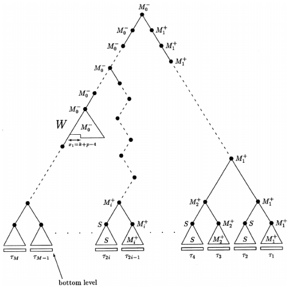

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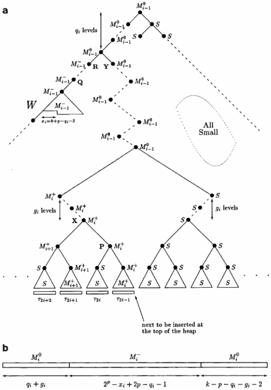

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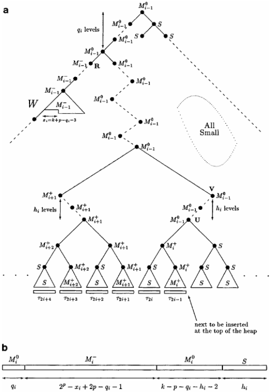

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$$B(n) \le \frac{1}{2} n \log n + O(n \log \log n)$$

Therefore . . .

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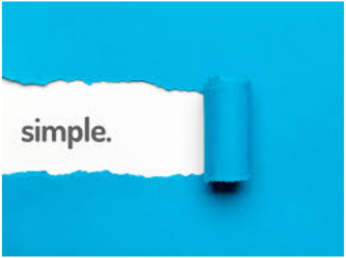

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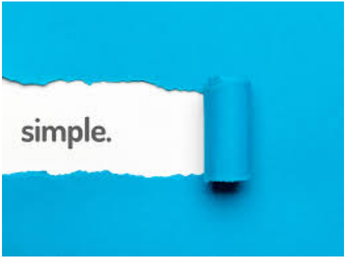

I said simple,
not easy.

“By a surprisingly short counting argument.”

Robert Sedgewick

"By a surprisingly short counting argument."

Robert Sedgewick

D. E. Knuth

"It is elegant.

“By a surprisingly short counting argument.”

Robert Sedgewick

D. E. Knuth

“It is elegant. see exercise 30.”

$$\boxed{f(n) \triangleq \# \text{ of heaps of } n \text{ distinct keys } [1 \cdots n]}$$

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$$f(n) = \binom{n-1}{m} f(m) f(n-1-m)$$

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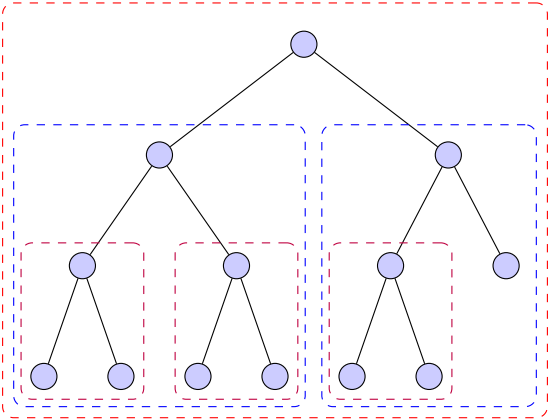
$$\boxed{f(n) = \frac{n!}{\prod_{1 \le i \le n} s_i}}$$

$s_i \triangleq$ size of the subtree rooted at i



$$f(13) = \frac{13!}{13 \cdot 7 \cdot 5 \cdot 3 \cdot 3 \cdot 3} = 506880$$

THE ANALYSIS OF HEAPSORT

Russel Schaffer
Robert Sedgewick

CS-TR-330-91

January 1991

Revised January 1992

On the Best Case of Heapsort*

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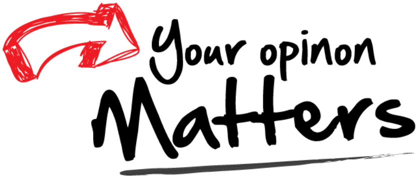
and

A. M. Frieze‡

Department of Mathematics, Carnegie Mellon University, Pittsburgh, Pennsylvania 15213

Received August 14, 1991; revised July 24, 1994


Thank
You!



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