GROUP THEORY

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I'm Yu Tsumura.

Example of an Infinite Group Whose Elements Have Finite Orders

BY YU · 10/27/2017

Problem 594

Is it possible that each element of an infinite group has a finite order?
If so, give an example. Otherwise, prove the non-existence of such a group.

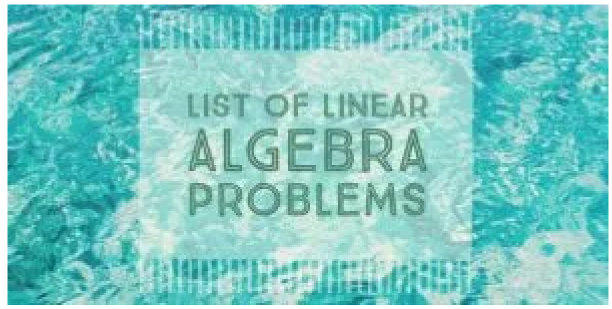
♡ Add to solve later

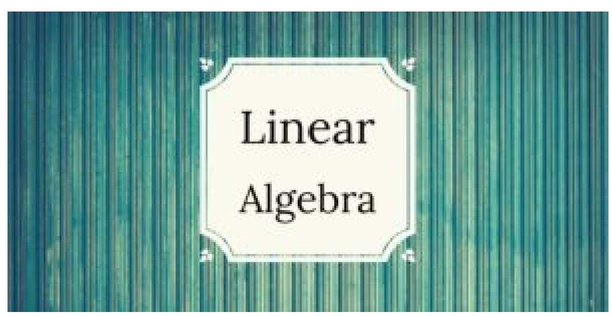
Solution.

We give an example of a group of infinite order each of whose elements has a finite order.
Consider the group of rational numbers $\mathbb{Q}$ and its subgroup $\mathbb{Z}$.
The quotient group $\mathbb{Q}/\mathbb{Z}$ will serve as an example as we verify below.

LINEAR ALGEBRA PROBLEMS BY TOPICS

The list of linear algebra problems is available here.





Note that each element of $\mathbb{Q}/\mathbb{Z}$ is of the form

$$\frac{m}{n}+\mathbb{Z},$$

where m and n are integers.

This implies that the representatives of $\mathbb{Q}/\mathbb{Z}$ are rational numbers in the interval $[0, 1)$.
There are infinitely many rational numbers in $[0, 1)$, and hence the order of the group $\mathbb{Q}/\mathbb{Z}$ is infinite.

On the other hand, as each element of $\mathbb{Q}/\mathbb{Z}$ is of the form $\frac{m}{n}+\mathbb{Z}$ for $m, n \in \mathbb{Z}$, we have

$$n\cdot\left(\frac{m}{n}+\mathbb{Z}\right)=m+\mathbb{Z}=0+\mathbb{Z}$$

because $m \in \mathbb{Z}$.
Thus the order of the element $\frac{m}{n}+\mathbb{Z}$ is at most n.
Hence the order of each element of $\mathbb{Q}/\mathbb{Z}$ is finite.

Therefore, $\mathbb{Q}/\mathbb{Z}$ is an infinite group whose elements have finite orders.

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Group of Order 18 is Solvable

Let G be a finite group of order 18. Show that the group G is solvable. Definition Recall that a group G is said to be solvable if G has a subnormal series

$$\{e\} = G_0 \triangleleft G_1$$

such [...]

MATHJAX

Mathematical equations are created by MathJax. See How to use MathJax in WordPress if you want to write a mathematical blog.

RECENT POSTS

The Additive Group of Rational Numbers and The Multiplicative Group of Positive Rational Numbers are Not Isomorphic

Let $(\mathbb{Q}, +)$ be the additive group of rational numbers and let $(\mathbb{Q}_{>0}, \times)$ be the multiplicative group of positive rational numbers. Prove that $(\mathbb{Q}, +)$ and $(\mathbb{Q}_{>0}, \times)$ are not isomorphic as groups. Proof. Suppose, towards a [...]

The Group of Rational Numbers is Not Finitely Generated

(a) Prove that the additive group $\mathbb{Q} = (\mathbb{Q}, +)$ of rational numbers is not finitely generated. (b) Prove that the multiplicative group $\mathbb{Q}^* = (\mathbb{Q} \setminus \{0\},$ of nonzero rational numbers is not finitely generated. Proof. (a) Prove that the additive [...]

GROUP THEORY

If the Order of a Group is Even, then the Number of Elements of Order 2 is Odd

LINEAR ALGEBRA

Diagonalize the 3 by 3 Matrix if it is Diagonalizable

GROUP THEORY

If Generators x, y Satisfy the Relation $xy^2 = y^3x$, $yx^2 = x^3y$, then the Group is Trivial

LINEAR ALGEBRA

How to Find Eigenvalues of a Specific Matrix.

LINEAR ALGEBRA

Characteristic Polynomial, Eigenvalues, Diagonalization Problem (Princeton University Exam)

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4. Eigenvalues of Real Skew-Symmetric Matrix are Zero or Purely Imaginary and the Rank is Even
5. How to Find a Basis for the Nullspace, Row Space, and Range of a Matrix
6. The Intersection of Two Subspaces is also a Subspace

Every Finitely Generated Subgroup of Additive Group $\mathbb{Q}$ of Rational Numbers is Cyclic Let $\mathbb{Q}=(\mathbb{Q}, +)$ be the additive group of rational numbers. (a) Prove that every finitely generated subgroup of $(\mathbb{Q}, +)$ is cyclic. (b) Prove that $\mathbb{Q}$ and $\mathbb{Q} \times \mathbb{Q}$ are not isomorphic as groups. Proof. (a) Prove that every finitely generated […]

Commutator Subgroup and Abelian Quotient Group Let G be a group and let $D(G)=[G,G]$ be the commutator subgroup of G. Let N be a subgroup of G. Prove that the subgroup N is normal in G and G/N is an abelian group if and only if $N \supset D(G)$. Definitions. Recall that for any $a, b \in G$, the […]

7 Determine Whether Each Set is a Basis for $\mathbb{R}^3$

8 Find all Values of x such that the Given Matrix is Invertible

9 Express a Vector as a Linear Combination of Other Vectors

10 Positive definite Real Symmetric Matrix and its Eigenvalues

SITE MAP & INDEX

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Normal Subgroups, Isomorphic Quotients, But Not Isomorphic

Let G be a group. Suppose that H_1, H_2, N_1, N_2 are all normal subgroup of G, $H_1 \lhd N_2$, and $H_2 \lhd N_2$. Suppose also that N_1/H_1 is isomorphic to N_2/H_2. Then prove or disprove that N_1 is isomorphic to N_2.   Proof. We give a [...]

Torsion Subgroup of an Abelian Group, Quotient is a Torsion-Free Abelian Group

Let A be an abelian group and let $T(A)$ denote the set of elements of A that have finite order. (a) Prove that $T(A)$ is a subgroup of A. (The subgroup $T(A)$ is called the torsion subgroup of the abelian group A and elements of $T(A)$ are called torsion [...]

Group of p-Power Roots of 1 is Isomorphic to a Proper Quotient of Itself Let p be a prime number. Let

$$G = \{z \in \mathbb{C} \mid z$$

be the group of p-power roots of 1 in $\mathbb{C}$. Show that the map $\Psi : G \to G$ mapping z to z^p is a surjective homomorphism. Also deduce from this that G is isomorphic to a proper quotient of G […]

Tags: finite group group of rational numbers group theory infinite group quotient group

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2 RESPONSES

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Lalitha 12/15/2017 at 1:40 AM

Please explain what are representatives in Q/Z ? You mean elements of Q/Z?
"This implies that the representatives of Q/Z are rational numbers in the interval [0,1)."

From this sentence please explain the remaining part once again :
"On the other hand, as each element of Q/Z is of the form mn+Z for m,n∈Z, we have…"
Why m=0? and why is the order of any element in Q/Z is atmost n ?

Thanks

Reply

Yu 12/15/2017 at 2:04 AM

Note that each element in the group $\mathbb{Q}/\mathbb{Z}$ is of the form $q + \mathbb{Z}$, where q ∈\Q . *Wesaythat*q *isarepresentativefortheelement*q+\Z$.

For the second part, m is not necessarily 0. But $m + \mathbb{Z}$ is equal to $0 + \mathbb{Z}$.

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