

## 2-9 Sorting and Selection

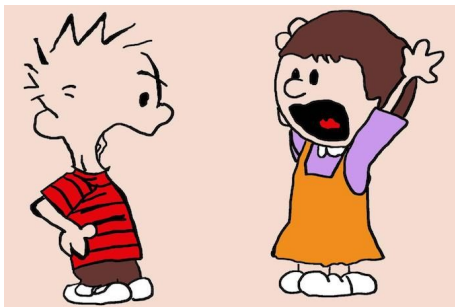
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May 28, 2018



## How to Argue?



Show that  $\dots$ , Argue that  $\dots$ , Explain why  $\dots$   
= Prove that  $\dots$

不好，掉坑里了



“在千山万水人海相遇，喔，原来你也在这里”

# 入坑指南 (Coupon Collector Problem)

## The Double Dixie Cup Problem

Donald J. Newman

*The American Mathematical Monthly*

Vol. 67, No. 1 (Jan., 1960), pp. 58-61

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Topics: [Mathematical theorems](#)

## The Coupon Collector's Problem

Marco Ferrante, Monica Saltalamacchia

In this note we will consider the following problem: how many coupons we have to purchase (on average) to complete a collection. This problem, which takes everybody back to his childhood when this was really “a problem”, has been considered by the probabilists since the eighteenth century and nowadays it is still possible to derive some new results, probably original or at least never published. We will present some classic results, some new formulas, some alternative approaches to obtain known results and a couple of amazing expressions.

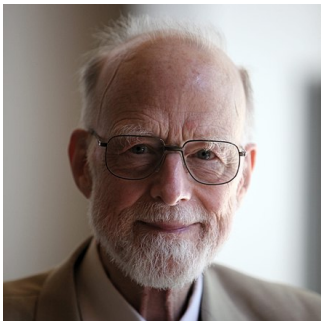


“兄弟同心，其利断金”版本

$$1 \rightarrow 2 \rightarrow m$$

$$n \log n + (m-1)n \log \log n + nC_m + o(n), \quad n \rightarrow \infty, m \text{ fixed}$$

## QUICKSORT (Tony Hoare, 1959/1960)



Hoare Logic:  $\{P\} S \{Q\}$

null pointer

*"I call it my billion-dollar mistake."*

## Best-Case Complexity of QUICKSORT (7.4 – 2)

Show that QUICKSORT's *best-case* running time is  $\Omega(n \log n)$ .

$$T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n)$$

$$T(n) = \underbrace{n}_{\text{PARTITION}} \underbrace{\log n}_{\text{Height}} \quad (\text{Recursion Tree})$$

$$T(n) = \min_{0 \leq q \leq n-1} \left( T(q) + T(n - q - 1) \right) + \Theta(n)$$

$$T(n) = \Omega(n \log n)$$

By substitution. (Section 7.4.1, CLRS)

## Sorting Almost-Sorted Inputs (Problem 7.2 – 4)

Argue that INSERTION-SORT would tend to beat QUICKSORT on *almost-sorted* inputs.

$$\# \text{ inversions} = \Theta(n)$$

$$\text{INSERTION-SORT} : \Theta(n)$$

$$\text{QUICKSORT} : \Omega(n \log n)$$

## Median-of-3 PARTITION (Problem 7 – 5)

Argue that in the  $\Omega(n \log n)$  running time of QUICKSORT, the *median-of-3* method affects only the constant factor.



$$T(n) = \min_{0 \leq q \leq n-1} \left( T(q) + T(n - q - 1) \right) + \Theta(n)$$

$$T(n) = \Omega(n \log n)$$



# The Analysis of Quicksort Programs\*

Robert Sedgewick

Received January 19, 1976

*Summary.* The Quicksort sorting algorithm and its best variants are presented and analyzed. Results are derived which make it possible to obtain exact formulas describing the total expected running time of particular implementations on real computers of Quicksort and an improvement called the median-of-three modification. Detailed analysis of the effect of an implementation technique called loop unwrapping is presented. The paper is intended not only to present results of direct practical utility, but also to illustrate the intriguing mathematics which arises in the complete analysis of this important algorithm.



Robert Sedgewick

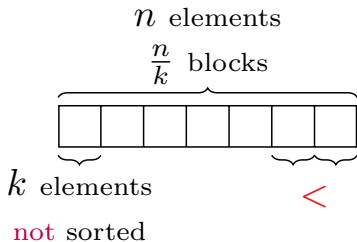
$$B_N = \frac{12}{35} (N+1) (H_{N+1} - H_{M+2}) + \frac{37}{245} (N+1) - \frac{12}{7} \frac{N+1}{M+2} + 1 \quad \text{exchanges}$$

$$C_N = \frac{12}{7} (N+1) (H_{N+1} - H_{M+2}) + \frac{37}{49} (N+1) - \frac{24}{7} \frac{N+1}{M+2} + 2 \quad \text{comparisons}$$

$$B_N = (N+1) \left( \frac{1}{3} H_{N+1} - \frac{1}{3} H_{M+2} + \frac{1}{6} - \frac{1}{M+2} \right) + \frac{1}{2} \quad \text{exchanges}$$

$$C_N = (N+1) (2H_{N+1} - 2H_{M+2} + 1) \quad \text{comparisons,}$$

## Sorts a $\frac{n}{k}$ -sorted Array (Problem 8.1 – 4)

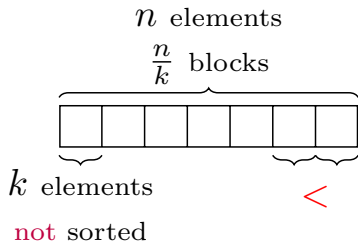


$$\Omega(n \log k) \quad O(n \log k)$$

$$\Omega : \frac{n}{k} (k \log k)$$

$$(k!)^{\frac{n}{k}} \leq L \leq 2^H$$

$\frac{n}{k}$ -sorts an arbitrary array



$O(?)$      $\Omega(?)$

$$L \geq \binom{n}{\underbrace{k, \dots, k}_{\frac{n}{k}}} = \frac{n!}{(k!)^{\frac{n}{k}}} \implies \Omega(n \log(n/k))$$

$$O(n \log k)$$

反馈: 这是什么意思? 我们并没学过多变量的渐近符号的定义。

### Problem 3.1 – 8

When  $n$  and  $m$  go to  $\infty$  independently at different rates:

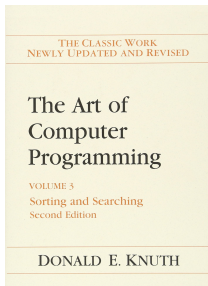
$$O(g(n, m)) = \{f(n, m) \mid \exists c > 0, \exists n_0 > 0, \exists m_0 > 0 : \\ \forall n \geq n_0 \forall m \geq m_0, 0 \leq f(n, m) \leq cg(n, m)\}$$

$$k = O(1) \quad k = \Theta(n); \quad n \rightarrow \infty$$

$$\Theta(1 + \alpha), \quad \alpha = \frac{n}{m}$$

## Two ways of understanding

$$1 + \alpha$$



$$\Theta(1 + \alpha)$$

$$n \rightarrow \infty, m \rightarrow \infty$$

$$n = f(m), m \rightarrow \infty$$

## Sorting $[0, n^3 - 1]$ (Problem 8.3 – 4)

Sort  $n$  integers in  $[0, n^3 - 1]$  in  $O(n)$  time.

$n$ -ary  $d = 3$

$n = 5 : [15, 39, 20, 123, 98] = \{030, 124, 040, 443, 343\}$

$$\Theta\left(d(\underbrace{n}_{\text{blue}} + \underbrace{n}_{\text{red}})\right) = \Theta(n)$$

Any other costs?

$$3n \cdot T(\square \div n)$$

## Sorting in Place in Linear Time (Problem 8 – 2 (e))

Suppose that the  $n$  records have keys in the range  $[0, k]$ .

Modify COUNTING-SORT to sort them **in place** ( $O(k)$ ) in  $O(n + k)$  time.

	1	2	3	4	5	6	7	8
A:	2	5	3	0	2	3	0	3

	0	1	2	3	4	5
C:	2	0	2	3	0	1

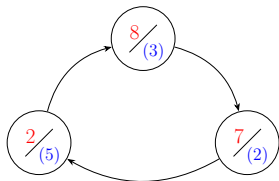
C:	2	2	4	7	7	8
----	---	---	---	---	---	---

	1	2	3	4	5	6	7	8
B:	0	0	2	2	3	3	3	5

	1	2	3	4	5	6	7	8
A:	2	0	3	0	2	3	3	5

	0	1	2	3	4	5
C:	1	2	4	7	7	8

for ( $i \leftarrow n$  to 1):



## Assigning elements according to $C$

	1	2	3	4	5	6	7	8
$A :$	2	5	3	0	2	3	0	3

	0	1	2	3	4	5
$C :$	2	0	2	3	0	1

	1	2	3	4	5	6	7	8
$A :$	0	0	2	2	3	3	3	5

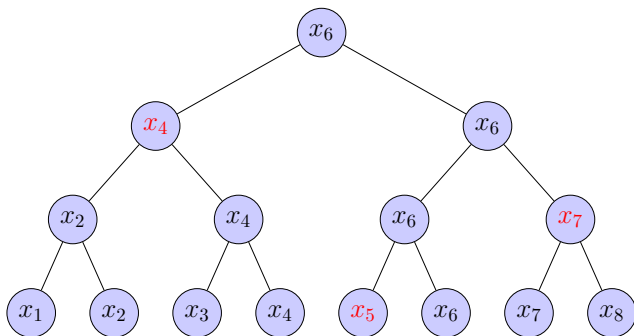
$Q$  : What is the possible problem?



## Finding the 2nd Smallest Element (Problem 9.1 – 1)

Show that the 2nd smallest of  $n$  elements can be found with  $n + \lceil \log n \rceil - 2$  comparisons in the worst case.

$$(n - 1) + (n - 1 - 1) = 2n - 3$$



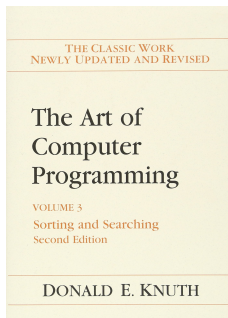
# Potential 2nd smallest elements  $\leq \lceil \log n \rceil$

$$n + \lceil \log n \rceil - 2 = (n - 1) + (\lceil \log n \rceil - 1)$$

*Q* : Can we do even better?

## Adversary Argument

$$\Omega = n + \lceil \log n \rceil - 2 = (n - 1) + (\lceil \log n \rceil - 1)$$

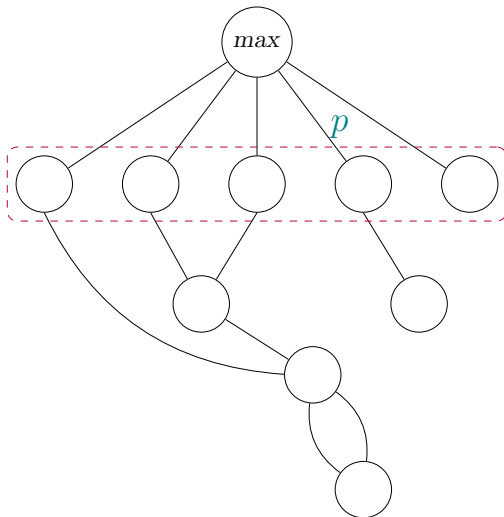


TAOCP, Vol. 3 (Page 209, Section 5.3.3)

## Theorem

*Any comparison-based algorithm to find secondLargest in  $n$  keys must do at least  $n + \lceil \log n \rceil - 2$  comparisons in the worst case.*

$$n + \lceil \log n \rceil - 2 = (n - 1) + (\lceil \log n \rceil - 1)$$



$$p \geq \lceil \log n \rceil$$

Adversary  $\mathcal{A}$ :

$$x > y$$

$$x < y$$



Algorithm  $\mathbb{A}$ :

COMPARE( $x, y$ )

$\exists \mathcal{A} \forall \mathbb{A} \left( \geq \lceil \log n \rceil \text{ distinct keys lost directly to } \max \text{ in } \mathbb{A} \right)$



$$\forall x : w(x) = 1$$

COMPARE( $x, y$ )

CASE	REPLY	(LOCAL) ACTION
$w(x) > w(y)$	$x > y$	$w'(x) \leftarrow w(x) + w(y); w'(y) \leftarrow 0$
$w(x) = w(y) > 0$	$x > y$	$w'(x) \leftarrow w(x) + w(y); w'(y) \leftarrow 0$
$w(x) = w(y) = 0$	No Conflict	NOP
$w(y) > w(x)$	$y > x$	$w'(y) \leftarrow w(x) + w(y); w'(x) \leftarrow 0$

贏者通吃; 敗者永不得翻身



$$\sum_x w(x) = n$$

$$w(x) = 0 \iff x \text{ has lost}$$

$$w(x) > 0 \iff x \text{ has not yet lost}$$

### Theorem

For any algorithm  $\mathbb{A}$  that finds *secondLargest*, when it stops,

$$\exists! x : w(x) > 0.$$

$$t = \text{max}, \quad w(t) = n$$

## Theorem

*$t$  has directly won against  $\geq \lceil \log n \rceil$  distinct keys.*

## Proof.

$K$  : # of comparisons  $t$  wins against **previously undefeated** keys

$$n = w_K$$

$w_k$  :  $w(t)$  after  $t$  winning against  $k$  **previously undefeated** keys

$$w_k \leq 2w_{k-1}$$

$$n = w_K \leq 2^K w_0 = 2^K \implies K \geq \log n \implies K \geq \lceil \log n \rceil$$



## *max&min* (Problem 9.1-2)

Prove the lower bound of  $\lceil 3n/2 \rceil - 2$  comparisons in the worst case to find both the *max* and the *min* of  $n$  numbers.

$$\mathbb{A} : \lceil 3n/2 \rceil - 2$$

Adversary  $\mathcal{A}$ :

$$n/2 + (n - 2)$$



Algorithm  $\mathbb{A}$ :

$$2n - 2$$

units of WIN/LOSE info.

$$2n - 2 = 2 \times (n/2) + 1 \times (n - 2)$$

## “01” Pattern

To check whether an array  $A[1 \cdots n]$  of  $n$  bits contains the “01” pattern.  
Do we have to check every bit?

No, if  $n$  is odd;

Yes, if  $n$  is even.

1 2 3 4 5 6 7 8 9

--	--	--	--	--	--	--	--	--

First checking  $A[2, 4, \dots, n - 1]$

	0		0		0		0	
--	---	--	---	--	---	--	---	--

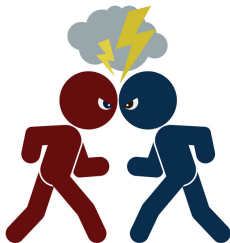
	1		1		1		1	
--	---	--	---	--	---	--	---	--

			0				1	
--	--	--	---	--	--	--	---	--

	1		1		0		0	
--	---	--	---	--	---	--	---	--

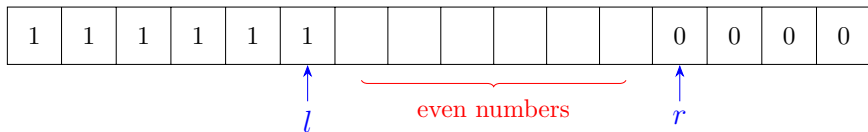
Adversary  $\mathcal{A}$ :

0/1



Algorithm  $\mathcal{A}$ :

CHECK( $i$ )



## $k$ Numbers Closest to the Median (Problem 9.3 – 7)

$S : n$  distinct numbers       $k \leq n$

$$\frac{n}{2} - \frac{k}{2} \sim \frac{n}{2} + \frac{k}{2} \quad (\text{By rank})$$

$$S = \{800, 6, 900, \textcolor{red}{50}, 7\}, \quad k = 2 \implies \{6, 7\} \quad (\text{By distance})$$

$$S - 50 = \{750, -44, 850, \textcolor{red}{0}, -43\}$$

median + subtraction +  $(k + 1)$ -th smallest + partition + add back





# The Coupon Collector's Problem

Marco Ferrante, Monica Saltalamacchia

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Thank  
You!



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