Solutions of Linear Nonhomogeneous Recurrence Relations

A Case for Thought

We already mentioned that finding a particular solution for a nonhomogeneous problem can be more involved than those exemplified in the previous lecture. Let us first highlight our point with the following example.

Example

1. Solve $a_{n+2}+a_{n+1}-6a_n=2^n$ for $n \geq 0$.

   Solution First we observe that the homogeneous problem
   $$u_{n+2} + u_{n+1} -6u_n=0$$
   has the general solution $u_n=A\ 2^n +B(-3)^n$ for $n \geq 0$ because the associated characteristic equation $\lambda^2+ \lambda -6 =0$ has 2 distinct roots $\lambda_1=2$ and $\lambda_2=-3$. Since the r.h.s. of the nonhomogeneous recurrence relation is 2^n, if we formally follow the strategy in the previous lecture we would try $v_n=C2^n$ for a particular solution. But there is a difficulty: $C2^n$ fits into the format of u_n which is a solution of the homogeneous problem. In other words it can't be a particular solution of the *nonhomogeneous* problem. This is really because ``2" happens to be one of the 2 roots λ_1 and λ_2. However, we suspect that a particular solution would still have to have 2^n as a factor, so we try $v_n=Cn2^n$. Substituting it to $v_{n+2}+v_{n+1} -6v_n=2^n$, we obtain
   $$C(n+2)2^{n+2}+C(n+1)2^{n+1}-6Cn2^n = 2^n ,$$
   i.e. $10C2^n=2^n$ or $C=1/10$. Hence a particular solution is $v_n=(n/10)2^n$ and the general solution of our nonhomogeneous recurrence relation is
   $$a_n=A2^n+B(-3)^n + \frac{n}{10} 2^n , \quad n \geq 0 .$$

In general, it is important that a correct form, often termed **ansatz** in physics, for a particular solution is used before we fix up the unknown constants in the solution ansatz. The choice of the form of a particular solution, covering the cases in this current lecture as well as the previous one, can be summarized below.

Method of Undertermined Coefficients

Consider a linear, constant coefficient recurrence relation of the form
$$c_m a_{n+m}+ \ldots + c_1 a_{n+1}+c_0 a_n=g(n) , \quad c_0 c_m \neq 0 , \quad n \geq 0. \qquad (*)$$
Suppose function $g(n)$, the **nonhomogeneous part** of the recurrence relation, is of the following form
$$g(n)= \mu^n(b_0+ b_1 n + \ldots + b_k n^k) , \qquad (**)$$
where $k \in \mathbb{N}$, μ, $b_0, \ldots, b_k$ are constants, and μ is a root of multiplicity M of the associated characteristic equation
$$c_m \lambda^m + \ldots + c_1 \lambda +c_0 =0 .$$
Then a particular solution v_n of (*) should be sought in the form

v_n

$$= \left[\sum_{i=0}^{k} B_i n^i\right] \mu^n\, n^M$$

$$= \mu^n\,(B_0 + B_1 n + \ldots + B_{k-1} n^{k-1} + B_k n^k)\, n^M, \qquad (***)$$

where constants $B_0, \ldots, B_k$ are to be determined from the requirement that $a_n = v_n$ should satisfy the recurrence relation (*). Obviously the v_n in (***) is composed of two parts: one is the $\mu^n(B_0+B_1 n+\ldots+B_k n^k)$ which is of the same *form* of $g(n)$ in (**), the other is the n^M which is a necessary adjustment for the case when μ, appearing in $g(n)$ in (**), is *also* a root (of multiplicity M) of the characteristic equation of the associated homogeneous recurrence relation.

Note

1. If μ is not a root of the characteristic equation then just choose $M=0$, implying alternatively that μ is a ``root" of 0 multiplicity.
2. We can also try $\tilde{v}_n = \mu^n \left(\sum_{j=0}^{k+M} A_j n^j\right)$. If we rewrite $\tilde{v}_n$ as $\mu^n \sum_{j=M}^{k+M} A_j n^j + \mu^n \sum_{j=0}^{M-1} A_j n^j$, then the 1st part is essentially (***) while the 2nd part is just a solution of the *homogeneous* problem. It is however obvious that v_n in (***) is simpler $\tilde{v}_n$.
3. We briefly hint why v_n is chosen in the form of (***). Let Δ, $f(\lambda)$ and $P_s(\lambda)$ be defined in the same way as we did in the derivation hints of the theorem in the lecture just before the previous one, and we'll also make use of some intermediate results there. Recall that (*) can be written as $f(\Delta)a_n = g(n)$ and (**) implies $g(n) \in P_k(\mu)$.

   (i) If $f(\mu) \neq 0$, then $f(\Delta)\, P_k(\mu) \subseteq P_k(\mu)$. Hence if we try $v_n = (B_0 + B_1 n + \ldots + B_k n^k)\, \mu^n \in P_k(\mu)$, then we can derive a set of exactly $(k+1)$ linear equations in $B_0, \ldots, B_k$, which can be used to determine these B_i's.

   (ii) If μ is a root of $f(\lambda)=0$ with multiplicity $M \geq 1$, then
   $$f(\Delta)P_{M-1}(\mu) \subseteq \{0\},\quad f(\Delta)P_{M+k}(\mu) \subseteq P_k(\mu).$$
   Hence if we try $v_n = n^M(B_0 + \ldots + B_k n^k)\, \mu^n \in P_{M+k}(\mu)$, we'll again have a set of exactly $(k+1)$ linear equations as the coefficients of the terms $\mu^n, \mu^n n, \ldots, \mu^n n^k$. The $(k+1)$ constants $B_0, \ldots, B_k$ can thus be determined from these linear equations.

Examples

2. Find the general solution of $f(n+2)-6f(n+1)+9f(n)=5\times 3^n,\quad n \geq 0$.

   Solution Let $f(n)=u_n+v_n$, with u_n being the general solution of the homogeneous problem and v_n a particular solution.

   (a) Find u_n: The associated characteristic equation $\lambda^2 - 6\lambda + 9 = 0$ has a repeated root $\lambda = 3$ with multiplicity 2. Hence the general solution of the homogeneous problem
   $$u_{n+2} - 6u_{n+1} + 9u_n = 0,\quad n \geq 0$$
   is $u_n = (A+Bn)3^n$.

   (b) Find v_n: Since the r.h.s. of the recurrence relation, the nonhomogeneous part, is 5×3^n and 3 is a root of multiplicity 2 of the characteristic equation (i.e. $\mu = 3$, $k=0$, $M=2$), we try due to (***) $v_n = B_0\, \mu^n \times n^M \equiv Cn^2 3^n$: we just need to observe that $C3^n$ is of the form 5×3^n and that the extra factor n^2 is due to $\mu=3$ being a double root of the characteristic equation. Thus
   $$\begin{aligned} 5\times 3^n &= v_{n+2} - 6v_{n+1} + 9v_n \\ &= C(n+2)^2 3^{n+2} - 6C(n+1)^2 3^{n+1} + 9Cn^2 3^n \\ &= 18C3^n. \end{aligned}$$

Hence $C=5/18$ and $v_n=(5/18)n^2 3^n$. Therefore our general solution reads

$$f(n) = \left(A + Bn + \frac{5}{18} n^2 \right) 3^n , \quad n \geq 0 .$$

3. Find the particular solution of

$$a_{n+4}-5a_{n+3}+9a_{n+2}-7a_{n+1} +2a_n=3 , \; n \geq 0$$

satisfying the initial conditions $a_0 = 2$, $a_1 = -1/2$, $a_2 = -5$, $a_3 = -31/2$.

Solution We first find the general solution u_n for the homogeneous problem. We then find a particular solution v_n for the nonhomogeneous problem without considering the initial conditions. Then $a_n=u_n+v_n$ would be the general solution of the nonhomogeneous problem. We finally make use of the initial conditions to determine the arbitrary constants in the general solution so as to arrive at our required particular solution.

(a) Find u_n: Since the associated characteristic equation

$$\lambda^4-5\lambda^3 +9\lambda^2 -7\lambda + 2 =0$$

has the sum of all the coefficients being zero, i.e. $1-5+9-7+2=0$, it must have a root $\lambda=1$. After factorising out $(\lambda-1)$ via $\lambda^4-5\lambda^3+9\lambda^2-7\lambda + 2 = (\lambda-1)(\lambda^3-4\lambda^2+5\lambda -2)$, the rest of the roots will come from $\lambda^3-4\lambda^2+5\lambda -2 =0$. Notice that $\lambda^3-4\lambda^2+5\lambda-2=0$ can again be factorised by a factor $(\lambda -1)$ because $1-4+5-2=0$. This way we can derive in the end that the roots are

$\lambda_1 =1$ with multiplicity $m_1=3$, and

$\lambda_2=2$ with multiplicity $m_2=1$.

Thus the general solutions for the homogeneous problem is

$$u_n = (A+Bn+Cn^2)1^n+D2^n ,$$

or simply $u_n=A+Bn+Cn^2+D2^n$ because $1^n \equiv 1$.

(b) Find v_n: Notice that the nonhomogeneous part is a constant 3 which can be written as 3×1^n when cast into the form of (**), and that 1 is in fact a root of multiplicity 3. In other words, we have in (***) $\mu =1$, $k=0$ and $M=3$. Hence we try a particular solution $v_n=En^3\cdot 1^n =En^3$. The substitution of v_n into the nonhomogeneous recurrence equations then gives, using a formula in the subsection *Binomial Expansions* in the *Preliminary Mathematics* at the beginning of these notes,

$$\begin{aligned} 3 &= v_{n+4}-5v_{n+3}+9v_{n+2}-7v_{n+1}+2v_n \\ &= E(n+4)^3-5E(n+3)^3+9E(n+2)^3-7E(n+1)^3+2En^3 \\ &= E(n^3 + 3n^2\times4+3n\times4^2+4^3) -5E(n^3 + 3n^2\times3+3n\times3^2+3^3) \\ &\quad + 9E(n^3 + 3n^2\times2+3n\times2^2+2^3) -7E(n^3 + 3n^2\times1+3n\times1^2+1^3) + 2En^3 \\ &= -6E , \end{aligned}$$

i.e. $E= -1/2$. Hence $v_n= -n^3/2$.

Note Should you find it very tedious to perform the expansions in the above, you could just substitute, say, $n=0$ into

$$3 = E(n+4)^3-5E(n+3)^3+9E(n+2)^3-7E(n+1)^3+2En^3$$

to obtain readily $3=E4^3-5E3^3+9E2^3-7E=-6E$. Incidentally you don't have to substitute $n=0$; you can in fact substitute any value for n because the equation is valid for all n. Obviously this *alternative* technique also applys even if there are more than 1 unknowns in the equation; we just need to substitute sufficiently many distinct values for n to collect enough equations to determine the unknowns. The drawback of this technique is that you have to make sure that the form you have proposed for v_n is absolutely correct through the use of the proper theory, otherwise an error in the form for v_n will go undetected in this alternative approach.

(c) The general solution of the nonhomogenous problem thus read

$$a_n = u_n + v_n = A + Bn + Cn^2 + D2^n - n^3/2.$$

(d) We now ask the solution in (c) to comply with the initial conditions.

Initial Conditions	Induced Equations	Solutions
$a_0 = 2$	$A+D = 2$	$A = 3$
$a_1 = -1/2$	$A+B+C+2D = 0$	$B = -2$
$a_2 = -5$	$A+2B+4C+4D = -1$	$C = 1$
$a_3 = -31/2$	$A+3B+9C+8D = -2$	$D = -1$

Hence our required particular solution takes the following final form

$$a_n = 3 - 2n + n^2 - n^3/2 - 2^n, \quad n \geq 0 .$$

4. Find the general solution of $a_{n+1}-a_n=n2^n+1$ for $n \geq 0$.

Solution

(a) The general solution for homogeneous problem is $u_n=A$ because the only root of the characteristic equation is $\lambda_1=1$.

(b) Since $n2^n+1 = 2^n \times n + 1^n$ is of the form $\mu_1{}^n(b_1n+b_0)+ \mu^n{}_2\, c_0$ and $\mu_2=1$ is a simple root of the characteristic equation, we try the similar form $v_n = 2^n(B+Cn) + Dn$ for a particular solution. Substiting v_n into the recurrence relation, we have

$$n2^n+1 = v_{n+1}-v_n = 2^{n+1}(B+C(n+1)) + D(n+1) - 2^n(B+Cn) - Dn$$
$$= 2^n(Cn+B+2C) + D ,$$

i.e.

$$2^n ((C-1)n + (B+2C)) + (D-1) = 0 .$$

In order the above equation be identically 0 for all $n \geq 0$, we require all its coefficients to be 0, i.e.

$$C-1=0 , \quad B+2C=0 , \quad D-1=0 .$$

Hence $B=-2$, $C=1$ and $D=1$ and the particular solution $v_n =2^n(n-2)+n$.

(c) The general solution is u_n+v_n and thus reads

$$a_n = 2^n(n-2)+n+A , \; n \geq 0 .$$

5. Let $m \in \mathbb{N}$ and $S(n)= \sum_{i=0}^{n} i^m$ for $n \in \mathbb{N}$. Convert the problem of finding $S(n)$ to a problem of solving a recurrence relation.

Solution We first observe

$$S(n+1)= (n+1)^m + \sum_{i=0}^{n} i^m = S(n) + (n+1)^m .$$

Since the general solution will contain just 1 arbitrary constant, 1 initial condition should suffice. Hence $S(n)$ is the solution of

$$S(n+1)-S(n) = (n+1)^m, \quad \forall n \in \mathbb{N}$$
$$S(0) = 0 .$$

Note A similar procedure for solving linear, constant coefficient nonhomogeneous recurrence relations can be found in the book by Stephen B Maurer *et al*, *Discrete Algorithmic Mathematics*, Addison-Wesley, 1991.