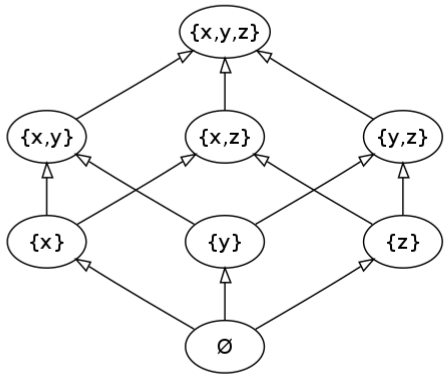
Axiom of power set

In mathematics, the **axiom of power set** is one of the Zermelo–Fraenkel axioms of axiomatic set theory.



The elements of the power set of the set $\{x, y, z\}$ ordered with respect to inclusion.

In the formal language of the Zermelo–Fraenkel axioms, the axiom reads:

$$\forall x\,\exists y\,\forall z\,[z \in y \iff \forall w\,(w \in z \Rightarrow w \in x)]$$

where *y* is the Power set of *x*, $\mathcal{P}(x)$. In English, this says:

> Given any set *x*, there is a set $\mathcal{P}(x)$ such that, given any set *z*, *z* is a member of $\mathcal{P}(x)$ if and only if every element of *z* is also an element of *x*.

More succinctly: *for every set x, there is a set $\mathcal{P}(x)$ consisting precisely of the subsets of x.*

Note the subset relation $\subseteq$ is not used in the formal definition as subset is not a primitive relation in formal set theory; rather, subset is defined in terms of set membership, $\in$. By the axiom of extensionality, the set $\mathcal{P}(x)$ is unique.

The axiom of power set appears in most axiomatizations of set theory. It is generally considered uncontroversial, although constructive set theory prefers a weaker version to resolve concerns about predicativity.

Consequences

The Power Set Axiom allows a simple definition of the Cartesian product of two sets X and Y:

$$X \times Y = \{(x, y) : x \in X \land y \in Y\}.$$

Notice that

$$\begin{array}{l} x, y \in X \cup Y \\ \{x\}, \{x, y\} \in \mathcal{P}(X \cup Y) \\ (x, y) = \{\{x\}, \{x, y\}\} \in \mathcal{P}(\mathcal{P}(X \cup Y)) \end{array}$$

and thus the Cartesian product is a set since

$$X \times Y \subseteq \mathcal{P}(\mathcal{P}(X \cup Y)).$$

One may define the Cartesian product of any finite collection of sets recursively:

$$X_1 \times \cdots \times X_n = (X_1 \times \cdots \times X_{n-1}) \times X_n.$$

Note that the existence of the Cartesian product can be proved without using the power set axiom, as in the case of the Kripke–Platek set theory.

References

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