

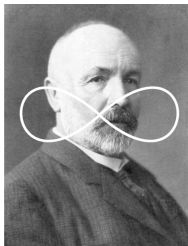
Finite and Infinite

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Georg Cantor (1845 – 1918)



David Hilbert (1862 – 1943)



Leopold Kronecker
(1823 – 1891)



Henri Poincaré
(1854 – 1912)



Ludwig Wittgenstein
(1889 – 1951)

*From his paradise that Cantor with us unfolded, we hold our
breath in awe; knowing, we shall not be expelled.*

— *David Hilbert*

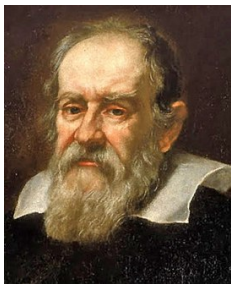
“没有人能把我们从 Cantor 创造的乐园中驱逐出去”





“das wesen der mathematik liegt in ihrer freiheit”

“The essence of mathematics lies in its freedom”



Galileo Galilei (1564 – 1642)



《关于两门新科学的对话》(1638)

“用我们有限的心智来讨论无限 …”

$$S_1 = \{1, 2, 3, \dots, n, \dots\}$$

$$S_2 = \{1, 4, 9, \dots, n^2, \dots\}$$

$$|S_1| = |S_2|, S_1 \subsetneq S_2$$

“部分等于全体”



吓得我吃了一鲸

说到底，“等于”、“大于”和“小于”诸性质不能用于无限，而
只能用于有限的数量。
— Galileo Galilei

无穷数是不可能的。
— Gottfried Wilhelm Leibniz

这些证明一开始就期望那些数要具有有穷数的一切性质，或者甚至于把有穷数的性质强加于无穷。

相反，这些无穷数，如果它们能够以任何形式被理解的话，倒是由于它们与有穷数的对应，它们必须具有完全新的数量特征。

这些性质完全依赖于事物的本性， \cdots 而并非来自我们的主观任意性或我们的偏见。

— Georg Cantor (1885)

Definition (Dedekind-infinite & Dedekind-finite (Dedekind, 1888))

A set A is **Dedekind-infinite** if there is a bijective function from A onto some proper subset B of A .

A set is **Dedekind-finite** if it is not Dedekind-infinite.



Comparing Sets



Function



Definition ($|A| = |B|$ ($A \approx B$) (1878))

Two sets of A and B are *equipotent* if there exists a *bijection* from A to B .

“ \approx ” is an equivalence relation.

\overline{A} (two *abstractions*)

$\{1, 2, 3\}$ vs. $\{a, b, c\}$

$\{1, 2, 3, \dots\}$ vs. $\{1, 3, 5, \dots, 2, 4, 6, \dots\}$

Definition (Finite and Infinite)

For any set X ,

Finite

$$\exists n \in \mathbb{N} : |X| = n \quad (0 \in \mathbb{N})$$

Infinite (\neg finite)

$$\forall n \in \mathbb{N} : |X| \neq n$$

Definition (Finite and Infinite)

For any set X ,

Countably Infinite

$$|X| = |\mathbb{N}| \triangleq \aleph_0$$

Countable

(finite \vee countably infinite)

Uncountably Infinite

$$(\neg \text{finite}) \wedge (\neg (\text{countably infinite}))$$

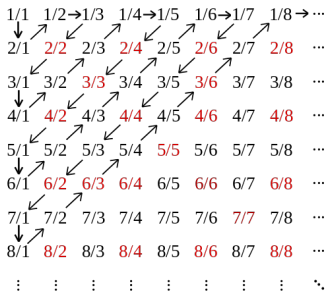
$$(\neg \text{countable})$$

Theorem (\aleph_0 (1874))

$$|Q| = |N|$$

$$|\mathbb{Q}| = |\mathbb{N}| \text{ (UD 22.9)}$$

$$q \in \mathbb{Q}^+ : a/b \ (a, b \in \mathbb{N}^+)$$



$$|\mathbb{N}| = |\mathbb{Z}|$$

$$|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$$

$$f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \quad f(m, n) = n + \frac{(m+n)(m+n+1)}{2}$$

Theorem (\mathbb{R} is uncountably infinite (1874) .)

$$|\mathbb{R}| \neq |\mathbb{N}| \quad (|\mathbb{R}| > |\mathbb{N}|)$$

Proof.

See UD Theorem 22.12. $f : \mathbb{N} \xrightarrow[\text{onto}]{1-1} (0, 1)$.



Different “Sizes” of Infinity

Cantor’s Diagonal Argument (1890)

Theorem (Cantor’s Theorem (1891))

$$|X| \neq |2^X| \quad (|X| < |2^X|)$$

Infinite Sequences of 0's and 1's (UD 22.3)

Is the set of all infinite sequences of 0's and 1's finite, countably infinite, or uncountable?

Proof.

By Cantor's diagonal argument \implies uncountable. □

Nonproof.

$$f : \{\{0, 1\}^*\} \rightarrow \mathbb{N}$$

$$f(x_0x_1\cdots) = \sum_{i=0}^{\infty} x_i 2^i$$
□

Theorem ($|\mathbb{R}|$ (1877))

$$|(0, 1)| = |\mathbb{R}| = |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}^{n \in \mathbb{N}}|$$

Proof.

$$|(0, 1)| = |(-\frac{\pi}{2}, \frac{\pi}{2})| = |\mathbb{R}|$$

$$(x = 0.a_1a_2a_3\cdots, y = 0.b_1b_2b_3\cdots) \mapsto 0.a_1b_1a_2b_2a_3b_3\cdots$$

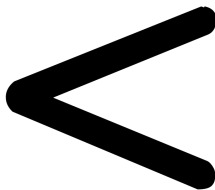


“Je le vois, mais je ne le crois pas !”

“I see it, but I don't believe it !”

— Cantor's letter to Dedekind (1877).

Q : Then, what is “dimension”?



Definition ($|A| \leq |B|$)

$|A| \leq |B|$ if there exists an *one-to-one* function f from A into B .

bijection $f : A \rightarrow f(A) (\subseteq B)$

Q : What about onto function $f : A \rightarrow B$?

$|B| \leq |A|$ (Axiom of Choice)

Definition ($|A| < |B|$)

$$|A| < |B| \iff |A| \leq |B| \wedge |A| \neq |B|$$

$$|\mathbb{N}| < |\mathbb{R}|$$

$$|X| < |2^X|$$

Definition (Countable Revisited)

X is countable:

$$\exists n \in \mathbb{N} : |X| = n \vee |X| = |\mathbb{N}|$$

Theorem (Proof for Countable (UD Exercise 22.5))

X is countable iff there exists a *one-to-one* function

$$f : X \rightarrow \mathbb{N}.$$

X is countable iff

$$|X| \leq |\mathbb{N}|.$$

Subsets of Countable Set (UD 22.6; UD Corollary 22.4)

Every subset B of a countable set A is countable.

Set Union (UD 22.1)

Give an example, if possible, of

- (c) a countably infinite collection of *pairwise disjoint* nonempty sets whose union is finite.
- (b) a countably infinite collection of nonempty sets whose union is finite.

$$\left(\{A_i : i \in \mathbb{R}\} \quad A_i = \{1\} \right) = \{\{1\}\}$$

$$|A| = n \implies |2^A| = 2^n$$

Slope (UD 22.2 (e))

(e) the set of all lines with rational slopes

$$(\mathbb{Q}, \mathbb{R})$$

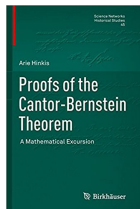
$$|\mathbb{R}| \leq |\mathbb{Q} \times \mathbb{R}| \leq |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$$

Q : Is “ \leq ” a partial order?

Theorem (Cantor-Schröder-Bernstein (1887))

$$|X| \leq |Y| \wedge |Y| \leq |X| \implies |X| = |Y|$$

$$\exists \text{ one-to-one } f : A \rightarrow B \wedge g : B \rightarrow A \implies \exists \text{ bijection } h : A \rightarrow B$$



Q : Is " \leq " a total order?

Theorem (PCC)

Principle of Cardinal Comparability (PCC) \iff Axiom of Choice

Finite Sets



“关于有穷，我原以为我是懂的”

Definition (Finite)

X is finite if

$$\exists n \in \mathbb{N} : |X| = n$$

Theorem (Pigeonhole Principle (UD Theorem 21.2))

$$f : \{1, \dots, m\} \rightarrow \{1, \dots, n\} \quad (m, n \in \mathbb{N}^+, m > n)$$

f is not one-to-one.

$A \setminus \{a\}$ (UD Problem 21.15)

Let A be a nonempty finite set with $|A| = n$ and let $a \in A$.

Prove that $A \setminus \{a\}$ is finite and $|A \setminus \{a\}| = n - 1$.

$$f : A \xrightarrow[\text{onto}]{1-1} \{1, \dots, n\}$$

$$f|_{A \setminus \{a\}} : A \setminus \{a\} \xrightarrow[\text{onto}]{1-1} \{1, \dots, n\} \setminus \{f(a)\} \xrightarrow[\text{onto}]{1-1} \{1, \dots, n-1\}$$

$|A| \leq |B|$ (UD Problem 21.17)

A and B are finite sets and $f : A \rightarrow B$ is one-to-one.
Show that $|A| \leq |B|$.



By contradiction and the pigeonhole principle.

(UD Problem 21.16)

- (a) A is a finite set and $B \subseteq A$. We showed that B is finite (Corollary 20.11). Show that $|B| \leq |A|$.

one-to-one $f : B \rightarrow A$

- (b) A is a finite set and $B \subseteq A$. Show that if $B \neq A$, then $|B| < |A|$.

$$\exists a : a \in A \wedge a \notin B \quad f : B \rightarrow A \setminus \{a\} \quad |B| \leq |A \setminus \{a\}|$$

- (c) If two finite sets A and B satisfy $B \subseteq A$ and $|A| \leq |B|$, then $A = B$.

By contradiction and (b).

Cardinality of $|\text{ran}(f)|$ (UD Problem 21.18)

Let A and B be sets with A finite.

$$f : A \rightarrow B$$

Prove that $|\text{ran}(f)| \leq |A|$.

one-to-one $g : \text{ran}(f) \rightarrow A$

(No Axiom of Choice Here)

$f : A \rightarrow A$ (UD Problem 21.19)

Let A be a finite set.

$$f : A \rightarrow A$$

Prove that

$$f \text{ is one-to-one} \iff f \text{ is onto.}$$

\Leftarrow

\Rightarrow

By contradiction.

$$f' : A \rightarrow A \setminus \{a\}$$

$$\forall y \in A \exists x \in A : y = f(x)$$

$$\forall y, \text{ choose } x : (g : g(y) = x)$$

g is bijective.

$$f(g(y)) = f(x) = y \implies f = g^{-1}$$

Dangerous Knowledge (BBC 2007)



Continuum Hypothesis (CH):

$$\boxed{c = \aleph_1}$$

$$c = 2^{\aleph_0} = \aleph_1$$



Dangerous Knowledge (22:20)

Independence from ZFC:

Kurt Gödel (1940) CH cannot be disproved from ZF.

Paul Cohen (1964) CH cannot be proven from the ZFC axioms.

Thank
You!



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