2-4 Recurrences

魏恒峰

hfwei@nju.edu.cn

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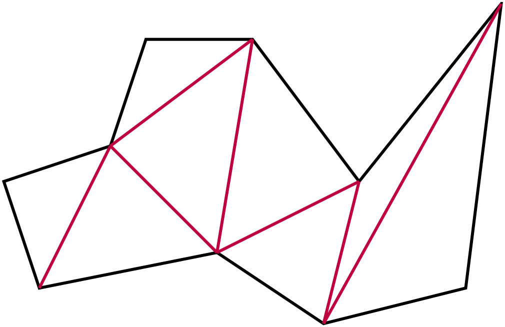
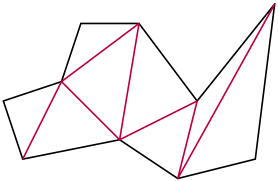
Triangulating Polygons



Ear Lemma (Problem $4.1 - 16$)

of triangles (Problem $4.1 - 17$)



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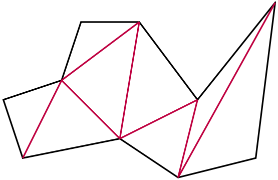

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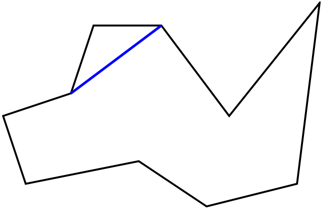

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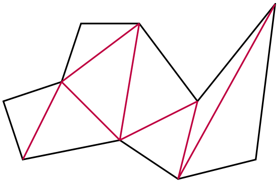

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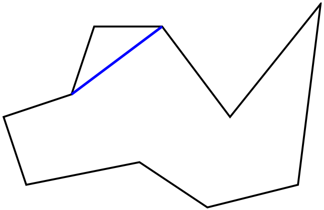

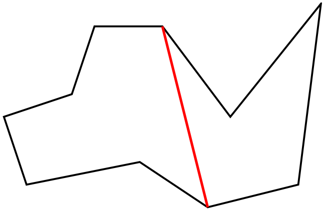

Lemma (Ear Lemma)

A triangle has 3 ears, and a larger ***triangulated*** *polygon has at least 2 non-adjacent ears.*

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Q : Can every polygon be triangulated?

Theorem (Existence of Triangulation)

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Every polygon with $n > 3$ has a diagonal.

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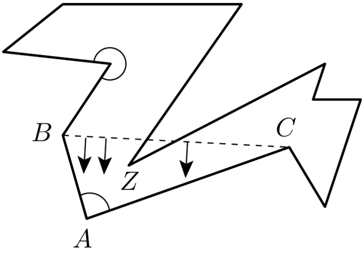
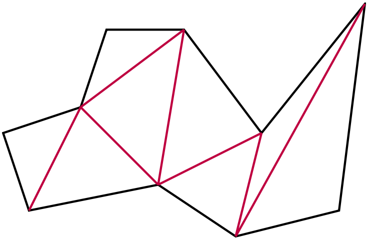


Theorem (Coloring)

Any triangulated polygon polygon is 3-colorable.



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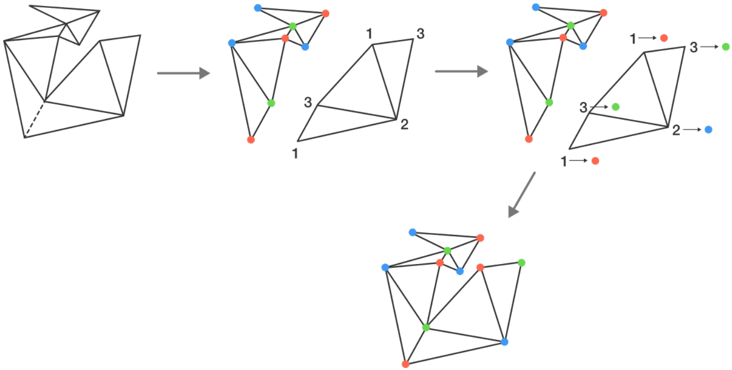

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Q : How many "BIG BROs" to hire?

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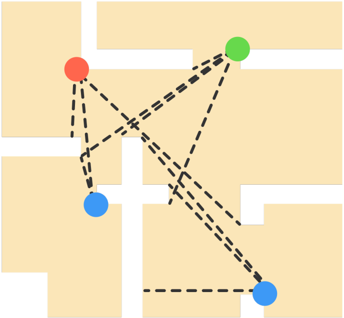

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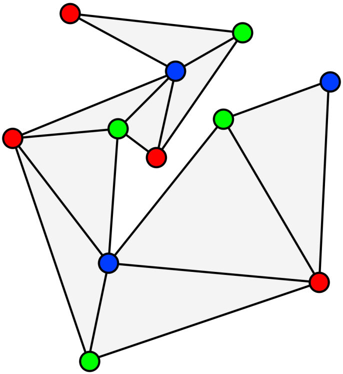

Theorem (The Art Gallery Theorem (O))

For any art gallery with n walls, $\lfloor \frac{n}{3} \rfloor$ "BIG BROs" suffice.

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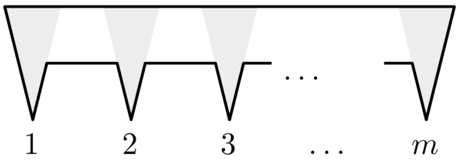
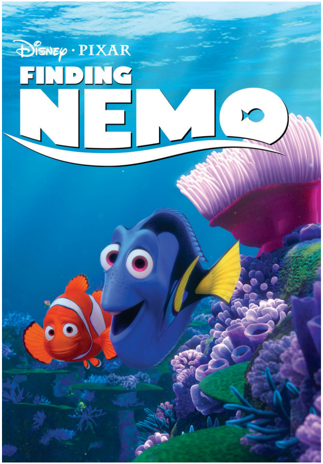


$$n = 3m$$

Fish Recurrence



Fish Recurrence (Problem $4.2-8$)

At the end of each year, a state fish hatchery puts 2000 fish into a lake.

The number of fish in the lake at the beginning of the year doubles by the end of the year due to reproduction.

Give a recurrence for the number of fish in the lake after n years, and solve the recurrence.

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$$T(0) = 2000$$

Solving Recurrence

Why again?

base cases

First-order Linear Recurrence (CS $4.2 - 11$)

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$$\boxed{T(n) = r^n a + \sum_{i=1}^{n} r^{n-i} g(i)}$$

Theorem (First-order Linear Recurrences)

$$T(n) = x_n T(n-1) + y_n \quad \textit{for } n > 0 \textit{ with } T(0) = 0$$

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$$\underbrace{\frac{T(n)}{x_n x_{n-1} \cdots x_1}}_{\text{summation factor}} = \frac{T(n-1)}{x_{n-1} \cdots x_1} + \frac{y_n}{x_n x_{n-1} \cdots x_1}$$

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$$S(n) \triangleq \frac{T(n)}{x_n x_{n-1} \cdots x_1}$$

□

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$$\boxed{T(n) = 2(n+1)(H_{n+1} - \frac{3}{2})}$$

After-class Exercise

$$T(n) = T(n-1) - \frac{2T(n-1)}{n} + 2\left(1 - \frac{2T(n-1)}{n}\right), n > 0 \text{ with } T(0) = 0$$

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$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_t a_{n-t} \quad \textit{for } n \geq t$$

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$$\beta^{n-t} \left((n-t)\, q(\beta) + \beta q'(\beta) \right) = 0$$

□

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$$\boxed{a_n = n2^{n-1}}$$

$$a_n = 2a_{n-1} - a_{n-2} + 2a_{n-3}, n \geq 3 \; (a_0 = 1, a_1 = 0, a_2 = -1)$$

$$a_n = \frac{1}{2} i^n \left(1 + (-1)^n\right)$$

After-class Exercise

$$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}, n > 3$$

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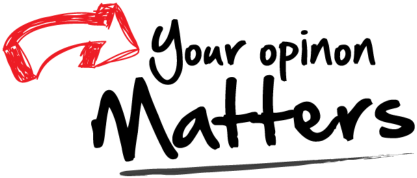
First-order Linear Non-homogeneous Recurrences

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_t a_{n-t} + r \quad \text{for } n \geq t$$


Thank You!



Office 302

Mailbox: H016

hfwei@nju.edu.cn