1-3 Proof

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Theorem (First Principle of Mathematical Induction (Theorem 18.1))

For an integer n, let $P(n)$ denote an assertion. Suppose that

(i) *$P(1)$ is true, and*

(ii) *for all positive integers n, if $P(n)$ is true, then $P(n+1)$ is true.*

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Let us calculate [calculemus].

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$$P(n) \triangleq Q(1) \land \cdots \land Q(n)$$

说好的数学归纳法呢?

PMI(I) $\to$ PMI(II) (“标准” 证明示例)

$$P(n) \triangleq Q(1) \wedge \cdots \wedge Q(n)$$

用第一数学归纳法证明 $\forall n \in \mathbb{N}^+ : P(n)$。

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Proof.

By mathematical induction on $\mathbb{N}^+$.

Basis Step: $P(1)$

Inductive Hypothesis: $P(n)$

Inductive Step: $P(n) \rightarrow P(n+1)$

Therefore, $P(n)$ holds for all positive integers. $\square$

Theorem (Second Principle of Mathematical Induction)

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Theorem (Well-ordering Principle of $\mathbb{N}$)

Every non-empty subset of the natural numbers contains a minimum.

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$$Q(n) \triangleq n \notin S$$

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Inductive Step: $P(n) \to P(n+1)$

- $A' \leftarrow A \setminus a$
- $x \leftarrow \min A'$
- Compare x with a

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Inductive Step: $P(n) \to P(n+1)$

$$\forall n \in \mathbb{N} : P(n) \quad vs. \quad P(\infty)$$

Numbers

Suppose $A \subseteq \{1, 2, \cdots, 2n\}$ with $|A| = n + 1$. Please prove that:

(1) There are two numbers in A which are relatively prime.

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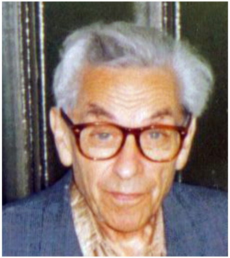
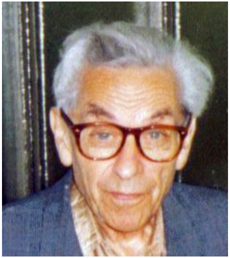
There must be two numbers in A with the same odd part.



Paul Erdős (1913 – 1996)



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Paul Erdős with Terence Tao

Theorem (Erdős-Szekeres Theorem)

Let n be a positive integer.
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$$n = 3$$

$$7, 8, 9, 4, 5, 6, 1, 2, 3$$

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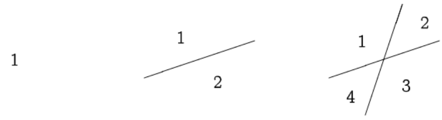
Theorem (Primes 1 (Mod 4) Theorem)

There are infinitely many primes that are congruent to 1 modulo 4.

Lines in the Plane

(1) What is the maximum number L_n of regions determined by n straight lines in the plane?



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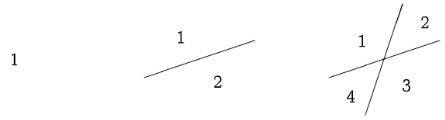
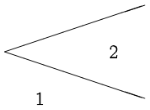


$L_0 = 1$ $L_1 = 2$ $L_2 = 4$

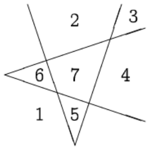
$$L_n = L_{n-1} + n = \frac{1}{2}n(n+1) + 1$$

Lines in the Plane

(2) What is the maximum number Z_n of regions determined by n bent lines, each containing one "zig", in the plane?



$Z_1 = 2$



$Z_2 = 7$

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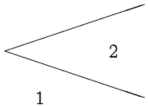

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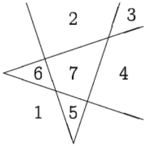

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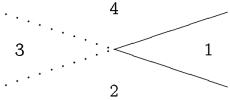

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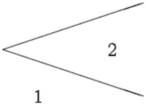

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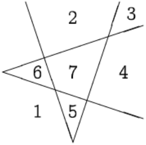

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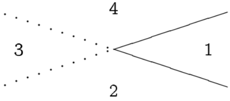
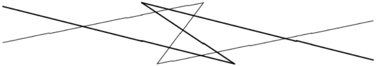


$$Z_n = L_{2n} - 2n = 2n^2 - n + 1$$

Lines in the Plane

(3) What's the maximum number ZZ_n of regions determined by n "zig-zag" lines in the plane?



$ZZ_2 = 12$

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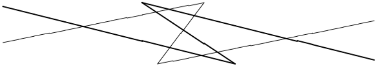

$$ZZ_n = ZZ_{n-1} + 9n - 8 = \frac{9}{2}n^2 - \frac{7}{2}n + 1$$

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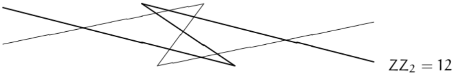

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$$9n - 8 = 9(n-1) + 1$$


Thank You!