# 1-11 有穷与无穷

魏恒峰

hfwei@nju.edu.cn

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*"das wesen der mathematik liegt in ihrer freiheit"*

*"The essence of mathematics lies in its freedom"*

# Dangerous Knowledge (BBC 2007)


$$c = \aleph_1$$

# Comparing Sets

$=$ $\qquad\qquad$ $<$

Function

$$=$$

**Definition ($|A| = |B|$ ($A \approx B$) (1878))**

Two sets of $A$ and $B$ are *equipotent* if there exists a *bijection* from $A$ to $B$.

"$=$" is an equivalence relation.

$\overline{\overline{A}}$ (two abstractions)

$$\{1, 2, 3\} \quad \textit{vs.} \quad \{a, b, c\}$$

$$\{1, 2, 3, \cdots\} \quad \textit{vs.} \quad \{1, 3, 5, \cdots, 2, 4, 6, \cdots\}$$

**Definition (Finite and Infinite)**

For any set $X$,

Finite

$$\exists n \in \mathbb{N} : |X| = n \qquad (0 \in \mathbb{N})$$

Infinite ($\neg$ finite)

$$\forall n \in \mathbb{N} : |X| \neq n$$

**Definition (Finite and Infinite)**

For any set $X$,

Countably Infinite

$$|X| = |\mathbb{N}| \triangleq \aleph_0$$

Countable

(finite $\lor$ countably infinite)

Uncountably Infinite

$$(\neg \text{ finite}) \land \Big(\neg \text{ (countably infinite)}\Big)$$

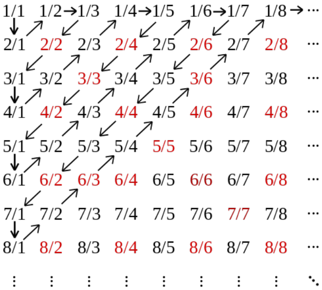
$$(\neg \text{ countable})$$

## Theorem ($\aleph_0$ (1874))

$$|\mathbb{Q}| = |\mathbb{N}|$$

## $|\mathbb{Q}| = |\mathbb{N}|$ (UD 22.9)

$$q \in \mathbb{Q}^+ : a/b \ (a, b \in \mathbb{N}^+)$$



$$|\mathbb{N}| = |\mathbb{Z}|$$

$$|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$$

**Theorem ($\mathbb{R}$ is uncountably infinite (1874).)**

$$|\mathbb{R}| \neq |\mathbb{N}| \qquad (|\mathbb{R}| > |\mathbb{N}|)$$

Different "Sizes" of Infinity

Cantor's Diagonal Argument (1890)

**Theorem (Cantor's Theorem (1891))**

$$|X| \neq |2^X| \qquad (|X| < |2^X|)$$

## Infinite Sequences of $0$'s and $1$'s (UD $22.3$)

Is the set of all infinite sequences of $0$'s and $1$'s finite, countably infinite, or uncountable?

### Proof.

By Cantor's diagonal argument $\implies$ uncountable. $\square$

### Nonproof.

$$f : \{\{0,1\}^*\} \to \mathbb{N}$$

$$f(x_0 x_1 \cdots) = \sum_{i=0}^{\infty} x_i 2^i$$

$\square$

Theorem ($|\mathbb{R}|$ (1877))

$$(0,1) = |\mathbb{R}| = |\mathbb{R}| \times |\mathbb{R}| = |\mathbb{R}|^{n \in \mathbb{N}}$$

*"Je le vois, mais je ne le crois pas !"*

*"I see it, but I don't believe it !"*

*— Cantor's letter to Dedekind (1877).*

$Q$ : *Then, what is "dimension"?*

$$<$$

**Definition ($|A| \le |B|$)**

$|A| \le |B|$ if there exists an *one-to-one* function $f$ from $A$ into $B$.

$$\text{bijection } f : A \to f(A) \; (\subseteq B)$$

*$Q$ : What about onto function $f : A \to B$?*

$|B| \le |A|$ (Axiom of Choice)

**Definition** ($|A| < |B|$)

$|A| < |B| \iff |A| \le |B| \land |A| \neq |B|$

$$|\mathbb{N}| < |\mathbb{R}|$$

$$|X| < |2^X|$$

### Definition (Countable Revisited)

$X$ is countable:

$$\exists n \in \mathbb{N} : |X| = n \vee |X| = |\mathbb{N}|$$

### Theorem (Proof for Countable (UD Exercise 22.5))

*$X$ is countable iff there exists a one-to-one function*

$$f : A \to \mathbb{N}.$$

*$X$ is countable iff*

$$|X| \leq |\mathbb{N}|.$$

### Subsets of Countable Set (UD 22.6; UD Corollary 22.4)

Every subset $B$ of a countable set $A$ is countable.

### Set Union (UD 22.1)

Give an example, if possible, of

(c) a countably infinite collection of *pairwise disjoint* nonempty sets whose union is finite.

(b) a countably infinite collection of nonempty sets whose union is finite.

$$\Big(\{A_i : i \in R\} \quad A_i = \{1\}\Big) = \{\{1\}\}$$

$$|A| = n \implies |2^A| = 2^n$$

Slope (UD $22.2$ $(e)$)

(e) the set of all lines with rational slopes

$$(\mathbb{Q},\ \mathbb{R})$$

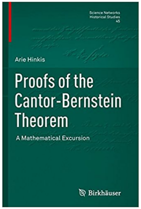
$$|\mathbb{R}| \le |\mathbb{Q} \times \mathbb{R}| \le |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$$

$Q$ : *Is "$\leq$" a partial order?*

Theorem (Cantor-Schröder–Bernstein (1887))

$$|X| \leq |Y| \wedge |Y| \leq |X| \implies |X| = |Y|$$

$\exists$ *one-to-one* $f : A \to B \wedge g : B \to A \implies \exists$ *bijection* $h : A \to B$




$Q$ : *Is "$\leq$" a total order?*

**Theorem (PCC)**

*Principle of Cardinal Comparability (PCC)* $\iff$ *Axiom of Choice*

# Finite Sets

“关于有穷，我原以为我是懂的”

### Definition (Finite)

$X$ is finite if

$$\exists n \in \mathbb{N} : |X| = n$$

### Theorem (Pigeonhole Principle (UD Theorem 21.2))

$$f : \{1, \cdots, m\} \to \{1, \cdots, n\} \ (m, n \in \mathbb{N}^+, m > n)$$

*$f$ is not one-to-one.*

### $A \setminus \{a\}$ (UD 21.15)

Let $A$ be a nonempty finite set with $|A| = n$ and let $a \in A$.
Prove that $A \setminus \{a\}$ is finite and $|A \setminus \{a\}| = n - 1$.

$$f : A \to \{1, \cdots, n\}$$

$$f|_{A \setminus \{a\}} : A \setminus \{a\} \to \{1, \cdots, n\} \setminus \{f(a)\} \to \{1, \cdots, n-1\}$$

$|A| \leq |B|$ (UD $21.17$)

$A$ and $B$ are finite sets and $f : A \to B$ is one-to-one.
Show that $|A| \leq |B|$.

By contradiction and the pigeonhole principle.

(UD $21.16$)

(a) $A$ is a finite set and $B \subseteq A$. We showed that $B$ is finite (Corollary $20.11$). Show that $|B| \leq |A|$.

one-to-one $f : B \to A$

(b) $A$ is a finite set and $B \subseteq A$. Show that if $B \neq A$, then $|B| < |A|$.

$\exists a : a \in A \land a \notin B \qquad f : B \to A \setminus \{a\} \qquad |B| \leq |A \setminus \{a\}|$

(c) If two finite sets $A$ and $B$ satisfy $B \subseteq A$ and $|A| \leq |B|$, then $A = B$.

By contradiction and (b).

**Cardinality of $|\text{ran}(f)|$ (UD $21.18$)**

Let $A$ and $B$ be sets with $A$ finite.

$$f : A \to B$$

Prove that $|\text{ran}(f)| \le |A|$.

one-to-one $g : \text{ran}(f) \to A$

(No Axiom of Choice Here)

## $f : A \to A$ (UD 21.19)

Let $A$ be a finite set.

$$f : A \to A$$

Prove that

$$f \text{ is one-to-one} \iff f \text{ is onto.}$$

$$\Longrightarrow$$

By contradiction.

$$f' : A \to A \setminus \{a\}$$

$$\Longleftarrow$$

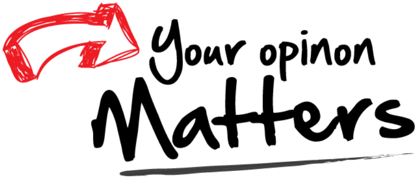
$$\forall y \in A\ \exists x \in A : y = f(x)$$

$\forall y$, choose $x : (g : g(y) = x)$

$g$ is bijective.

$$f(g(y)) = f(x) = y \implies f = g^{-1}$$


Thank
You!




Office 302

Mailbox: H016

hfwei@nju.edu.cn