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Cayley Graphs of Groups and Their Applications

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CAYLEY GRAPHS OF GROUPS AND THEIR APPLICATIONS

A Masters Thesis

Presented to

The Graduate College of

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In Partial Fulfillment

Of the Requirements for the Degree

Master of Science, Mathematics

By

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ABSTRACT

Cayley graphs are graphs associated to a group and a set of generators for that group (there is also an associated directed graph). The purpose of this study was to examine multiple examples of Cayley graphs through group theory, graph theory, and applications. We gave background material on groups and graphs and gave numerous examples of Cayley graphs and digraphs. This helped investigate the conjecture that the Cayley graph of any group (except Z_2) is hamiltonian. We found the conjecture to still be open. We found Cayley graphs and hamiltonian cycles could be applied to campanology (in particular, to the change ringing of bells).

KEYWORDS: abstract algebra, group theory, Cayley's Theorem, Cayley graphs, campagnology

This abstract is approved as to form and content

Dr. Les Reid
Chairperson, Advisory Committee
Missouri State University

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1. INTRODUCTION

Cayley graphs geometrically display the actions of a group. To examine this thoroughly, we will discuss the necessary background information on group theory and graph theory. Cayley graphs are dependent on a specific set of generators. Examples will be provided to show that the same group can have a different Cayley graph. The related concept of a Schreier coset graph is also introduced. Cayley graphs have undirected edges while Cayley digraphs have directed edges. We will focus primarily on Cayley graphs.

If graphs have a path that starts from one vertex, connects all of the other vertices, only hits every vertex once, and returns to the original vertex, then the graph is hamiltonian. We will prove that the product of any two or more hamiltonian graphs is hamiltonian. There is a well-known conjecture that every connected Cayley graph is hamiltonian. We will discuss this for certain types of Cayley graphs and groups. The Factor Group Lemma says if we find a hamiltonian cycle in the digraph of a quotient group, then under certain conditions, the digraph of the group is hamiltonian.

Lastly, an application of Cayley graphs will be shown through change-ringing. In this specific type of bell ringing, there are certain methods and principles that ringers use that are based off of permutations. These methods and principles correspond to hamiltonian cycles on Cayley graphs. In the case of four bells, that Cayley graph is a truncated octahedron with diagonals on its square faces.

2. PRELIMINARIES

2.1 GROUP THEORY

DEFINITION 2.1: (Dummit, Foote, 2004) A *group* is an ordered pair $(G, *)$ where G is a set and $*$ is a binary operation on G satisfying the following axioms:

(**i**) $(a * b) * c = a * (b * c)$, for all $a, b, c \in G$

(**ii**) $\exists\ e \in G$ such that for all $a \in G$ we have $a * e = e * a = a$

(**iii**) for each $a \in G\ \exists\ a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$

EXAMPLE 2.2: The dihedral group of order $2n$, denoted D_{2n}, is the group of symmetries of an n-gon. These symmetries include rotations, denoted r, and reflections, denoted s. The rotation of 0 degrees is the identity, id. Multiple rotations give powers of r and multiple rotations are powers of s. When there are n rotations, we return to the identity. Similarly, when there are 2 reflections, we return to the identity. Thus $D_{2n} = \langle r, s \mid r^n = s^2 = 1 \text{ and } srs = r^{-1} \rangle$.

EXAMPLE 2.3: Z_n is the group of rotations on an n-gon. The elements of Z_n start with the identity and $x \in Z_n$ creates the rest of the elements of Z_n by repeatedly applying the operation of x. These elements are powers of x. Hence $Z_n = \{1, x, x^2, ..., x^{n-1}\}$ where $x^n = 1$.

DEFINITION 2.4: A group is called *abelian* if $a * b = b * a$ for all $a, b \in G$.

EXAMPLE 2.5: Z_n is an abelian group.

DEFINITION 2.6: Let G be a group. The subset H of G is a *subgroup* of G if H is nonempty and H is closed under products and inverses.

DEFINITION 2.7: Let S be a subset of a group G. Then the subgroup generated by S, $\langle S \rangle$, is the smallest subgroup of G containing every element of S.

DEFINITION 2.8: A group H is *cyclic* if H can be generated by a single element. More specifically, there is some element $x \in H$ such that $H=\{x^n \mid n \in \mathbb{Z}\}$. In other words, $H=\langle x\rangle$.

EXAMPLE 2.9: $Z_n = \langle x\rangle$ is a cyclic group.

DEFINITION 2.10: Let $\Omega = \{1, 2, 3, ..., n\}$ and let S_n be the set of all bijections from Ω to itself. The *symmetric group of degree* n, S_n, is a group under function composition. A *cycle* is a string of integers which represents the elements of S_n which cyclically permute these integers (and fixes all other integers). The cycle $(a_1a_2, ...a_m)$ is the permutation that sends a_i to a_{i+1}, $1 \leq i \leq m-1$ and sends a_m to a_1. Every permutation is a product of disjoint cycles.

EXAMPLE 2.11: Consider the product (246135)(123456). This product is computed by starting with a number from a the right cycle. Start with 1 from the right cycle. $1 \rightarrow 2 \rightarrow 4$. Since 1 goes to 2 in the right cycle and 2 goes to 4 in the left cycle, our end product says that 1 goes to 4, (14...). Let us finish the product. $4 \rightarrow 5 \rightarrow 2$, $2 \rightarrow 3 \rightarrow 5$, and $5 \rightarrow 6 \rightarrow 1$. So we have (1425) thus far. Since 5 goes to 1, this closes the cycle. Now let's see where 3 goes. $3 \rightarrow 4 \rightarrow 6$ and $6 \rightarrow 1 \rightarrow 3$. Thus we have the cycle (36). So the overall product of (246135)(123456) = (1425)(36).

DEFINITION 2.12: For any subgroup N of a group G and any $g \in G$, $gN = \{gn \mid n \in N\}$ is a left coset of N in G.

DEFINITION 2.13: A subgroup N of a group G is called *normal* if $gNg^{-1} = N$ for all $g \in G$.

DEFINITION 2.14: Let N be a normal subgroup of a group G. The *quotient group* G/N is the set of all left cosets of N. In other words, $G/N = \{gN \mid g \in G\}$. The multiplication of elements is $gNhN = ghN$ for $g, h \in G$. Intuitively, G/N is the group which essentially "collapses" N.

Now we discuss generators and relations to prepare us for Cayley's theorem;

a theorem which allows groups to be represented as symmetric groups. Groups can be represented by generators from a specific set. This allows every element of the group to be represented as a product of powers of some of those generators.

EXAMPLE 2.15: Consider $D_8 = \{1, r, r^2, r^3, s, rs, r^2s, r^3s\}$. Note that all rotations are some power of r, so we see that r generates those four elements. Next, s generates itself. However, if we combine powers of r and s together, we obtain the remaining elements. Thus this group is generated by two elements, $D_8 = \langle r, s\rangle$ and the generating set is $S = \{r, s\}$.

THEOREM 2.16: Every group is isomorphic to a subgroup of some symmetric group. If G is a group of order n, then G is isomorphic to a subgroup of S_n.

EXAMPLE 2.17: $D_8 = \langle r, s\rangle = \langle r, s \mid r^4 = s^2 = 1, sr = r^{-1}s\rangle$. The elements are $\{1, r, r^2, r^3, s, rs, r^2s, r^3s\}$. Assign the numbers $1, 2, ..., 8$ to these elements, so that the set is in the order 12345678. Left multiply every element in the set by r. This permutes the elements of D_8. Some of the calculations are $r * 1 = r$, $r * r = r^2$, and $r * s = rs$. The resulting set is $\{r, r^2, r^3, 1, rs, r^2s, r^3s, s\}$. Compare the two sets to see the first element of D_8 is sent to r, which we identified as 2. So this cycle is now 23416785. Compare the two orderings, 12345678 and 23416785, and their positions to find the permutation (1234)(5678). Now multiply every element of D_8 by s to obtain the set $\{s, sr, sr^2, sr^3, s^2, srs, sr^2s, sr^3s\}$. Recall that D_8 is non abelian, so we cannot simply reorder the multiplication of elements. We instead use the properties above which tells us that $r^4 = s^2 = 1, sr = r^{-1}s$. Hence, our left multiplication of s gives us the set $\{s, r^3s, r^2s, rs, 1, r^3, r^2, r\}$ which is 58761432. Again compare the positions of these numbers. 12345678 and 58761432 give the permutation (15)(28)(37)(46). This shows D_8 is isomorphic to a subgroup of S_8, per Cayley's Theorem.

2.2 GRAPH THEORY

DEFINITION 2.18: (Brualdi, 2010) A *graph*, G, is a pair (V, E) where V is a set and E is a set of 2-element subsets of V. The set V is almost always finite and its elements are called *vertices*. The elements of E are called *edges*. Hence, V is the vertex set of G and E is the edge set of G. Edges are often written $\{x, y\}$, but we denote them as xy. Furthermore, the edge $xy \in E$ is the same edge as $yx \in E$. Finally, *neighbors* are vertices that are adjacent.

DEFINITION 2.19: A sequence $(x_1, x_2, ..., x_n)$ of vertices in a graph G=(V, E) is called a *walk* when $x_i x_{i+1}$ is an edge for each $i = 1, 2, ..., n-1$. A *path* is a walk whose vertices are distinct.

DEFINITION 2.20: When $n \geq 3$, a path $(x_1, x_2, ..., x_n)$ of n distinct vertices is called a *cycle* such that $x_1 x_n$ is also an edge in the graph G.

DEFINITION 2.21: A graph $G = (V, E)$ is said to be *hamiltonian* if there exists a cycle $(x_1 x_2 ... x_n)$ so that every vertex of G appears exactly once in the sequence.

DEFINITION 2.22: Let Γ_1 and Γ_2 be two graphs with respective vertex sets $V(\Gamma_1)$ and $V(\Gamma_2)$. The *vertex set of the product* $\Gamma_1 \times \Gamma_2$ is defined as $V(\Gamma_1) \times V(\Gamma_2)$. Let $v_1, w_1 \in V(\Gamma_1)$. Let $v_2, w_2 \in V(\Gamma_2)$. We connect (v_1, v_2) and (w_1, w_2) if $v_1 = w_1$ and v_2, w_2 are adjacent in Γ_2 or if v_1, w_1 are adjacent in Γ_1 and $v_2 = w_2$.

2.3 CAYLEY GRAPHS

Cayley graphs represent groups geometrically. We can read many of the abstract group actions from these diagrams.

DEFINITION 2.23: Given a group G and a subset S of G, the *Cayley graph*, $Cay(G : S)$, is the undirected graph with vertex set G and edge set containing an edge from g to sg and from g to $s^{-1}g$ whenever $g \in G$ and $s \in S$. If $|g| = 2$, the edge from $g \to sg$ and $g \to s^{-1}g$ are the same. This results in one edge.

REMARK 2.24: $Cay(G : S)$ is connected if and only if S is a generating set of G.

The Cayley graph of a group depends on the set of generators. This will be further discussed in later examples.

DEFINITION 2.25: The *Cayley digraph*, $DiCay(G : S)$, is the Cayley graph with directed edges. g to sg for $g \in G$ and $s \in S$ is two edges with direction.

For an element of order 2, the two directed edges of a Cayley graph, as seen in Fig. 1, are shown as two curved edges. In Cayley digraphs, there are as many directed edges as there are elements in the group. Note that in a Cayley graph, this will also be true for edges except elements of order 2. There will be half as many since the two edges collapse into one. Let us view some examples of Cayley graphs on the proceeding page.

EXAMPLE 2.26: Find the Cayley graph for $D_8 = \langle r, s \rangle$.

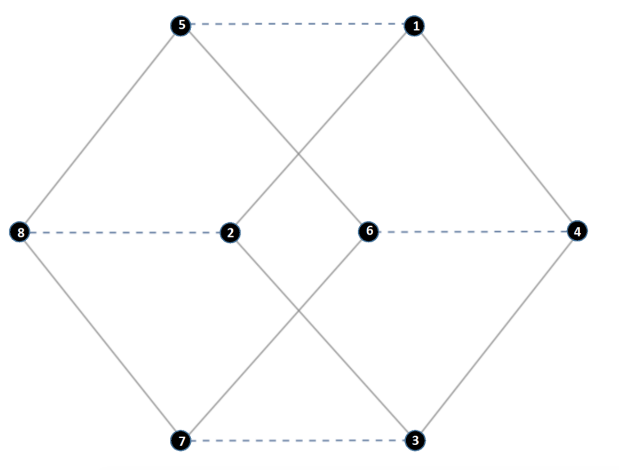

Figure 1: $Cay(D_8 : \{r, s, \})$.

Fig. 1 allows us to visualize the abstract structure of the group D_8. For example, since the group action of left multiplying r by s gives us sr and then left multiplying by s again gives us r. This means s is bi-directional. We choose to collapse it as one edge and this gives us a Cayley graph. Multiplication by s is shown in the graph as dotted lines.

Now we note that there are different Cayley graphs depending on the generating set.

EXAMPLE 2.27: Now consider $D_8 = \langle s, rs \rangle$.

We have the same properties that $r^4 = s^2 = 1$ and $sr = r^{-1}s$. But the Cayley graph will be different from the last example. We will still have our inner four vertices for $1, r, r^2, r^3$ and outer four vertices for s, sr, sr^2, and sr^3. However when we left multiply rs by each vertex, we obtain a different graph than before.

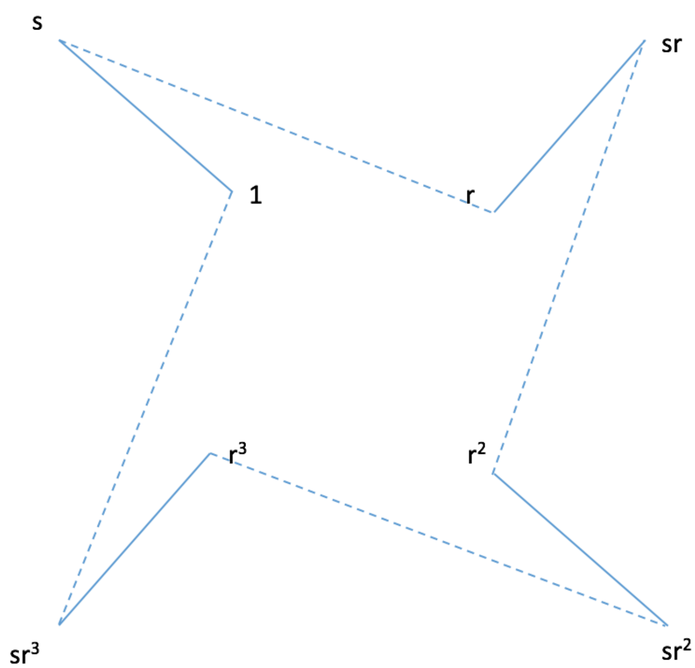

Figure 2: $Cay(D_8, \{s, rs\})$.

In Fig. 2, the dotted line represents left multiplication by rs and the solid line represents left multiplication by s. Not shown on the diagram, each line is bidirectional. Note that 1 times s is s. If we multiply that result by s, we get back to 1. Similarly, left multiplying s by rs, $rs * s$, we obtain r. And again $rs * r \rightarrow sr^{-1}r \rightarrow s$. Thus we see that every multiplication is bi-directional.

EXAMPLE 2.28: $Q_8 = \langle i, j \rangle$. Assign the elements $Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$

as the numbers 12345678 per their position. Left multiply all the elements of Q_8 by i so that $i * 1 = 1$, $i * j = k$, $j * k = i$, and $k * i = j$. The resulting set is $i * Q_8 = \{i, -i, -1, 1, k, -k, -j, j\}$ or 34217865. Now we compare the two sets. Recall the assignment of numbering for the elements. Similar to our previous example, we obtain the permutation (1324)(5768). Left multiply all of Q_8 by j so that the resulting set is $j * Q_8 = \{j, -j, -k, k, -1, 1, i, -i\}$ or 56872134. This permutation is (1526)(3847). The following image is the Cayley graph representation of these group actions via permutations.

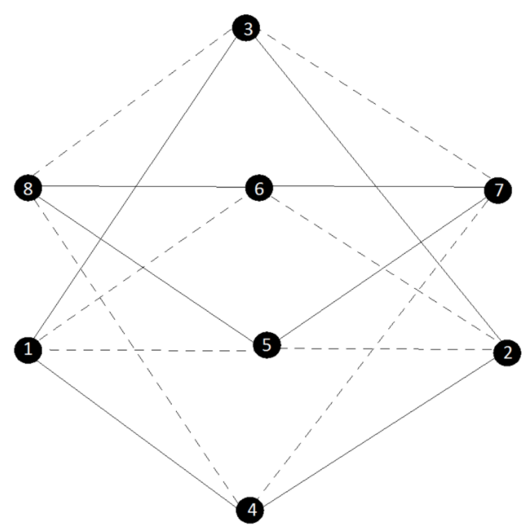

Figure 3: $Cay(Q_8 : \{i, j\})$.

Fig. 3 shows us a cube with diagonals inside as squares. If we follow the graph according to the permutations, we see that the red lines represent left multiplication by i. Similarly, the blue lines represent left multiplication by j.

EXAMPLE 2.29: Cuboctahedron Cayley graph.

Fig. 4 is the Cayley graph of a cuboctahedron that is generated by {(123), (1234)}.

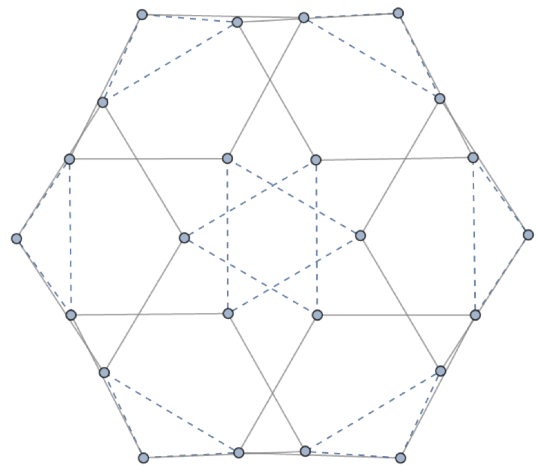

Figure 4: $Cay(S_3 : \{(123), (1234)\})$.

Now let's examine the Schreier coset graph. We are collapsing the Cayley graph into a smaller graph. We are able to view a handful of our group cosets as one single element. However, this means we may now see mulitple edges.

DEFINITION 2.30: Let G be a finite group generated by S. Let H be a subgroup of G. The *Schreier left coset graph* $Sch(G/H : S)$ has the left cosets of H in G as its vertices. Also, two vertices are adajacent if and only if left multiplication of $x \in S$ implies one coset is taken to another coset.

The Schreier coset graph is drawn without direction.

EXAMPLE 2.31: Create the Schreier coset graph for $Z_4 = \langle(1234)\rangle$. Above we described the Schreier coset graph for quotient groups. For this example, our quotient group is S_4/Z_4 and we are looking at the cosets of Z_4. Note that before we can try to draw this graph, we must find the cosets of Z_4. These are listed in Tbl. 1. Next we need to do left multiplication of $(12)(34), (12), (34), and (23)$ by all of the cosets to see which coset we are taken to. Note, $a = (12)(34)$, $b = (12)$, $c = (34)$, and $d = (23)$. This work for one coset is displayed in the Tbl. 2. The results in Tbl. 2 and left multiplicaton on the other cosets will provide us with the graph in Fig. 5. In Fig. 5, left multiplication by a is represented by a solid line, b is

Table 1: Cosets of Z_4

Coset	Elements of Coset
Z_4	id $(1234), (13)(24), (1432)\}$
$(12)Z_4$	$\{(12), (234), (1324), (143)\}$
$(23)Z_4$	$\{(23), (134), (1243), (142)\}$
$(34)Z_4$	$\{(34), (124), (1423), (132)\}$
$(12)(34)Z_4$	$\{(12)(34), (24), (14)(23), (13)\}$
$(123)Z_4$	$\{(123), (1342), (243), (14)\}$

Table 2: Left multiplication of $(12)Z_4$

Left mulitplication on a $(12)Z_4$	Coset Result
$a * (12)Z_4$	$(34)Z_4$
$b * (12)Z_4$	Z_4
$c * (12)Z_4$	$(12)(34)Z_4$
$d * (12)Z_4$	$(34)Z_4$

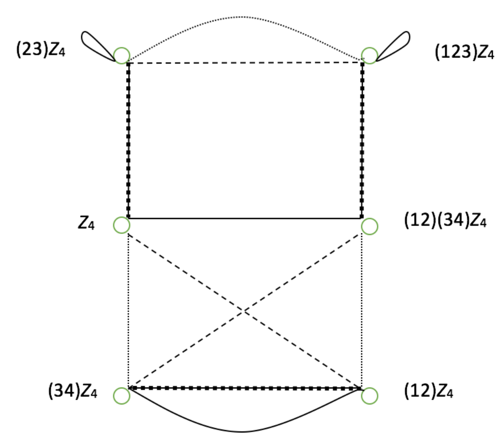

Figure 5: $Sch(Z_4 : \{(1234)\})$.

the dashed line, c is the dotted line, and d is the solid line with dashes. Notice two cosets have a loop. This is added to show that a multiplied by the cosets produces the same coset.

3. FREE GROUPS

DEFINITION 3.1: Let S be a set of elements with no relations. $F(S)$ is the *free group* generated by S.

EXAMPLE 3.2: Let S= $\{a, b\}$. The $F(S)$ is generated by a and b which are of the form a, aa, ab, $abab$, bab, a^{-1}, $b^{-1}a^{-1}$ etc. All of these elements, called *words*, are considered distinct. These words can be concatenated to create additional elements. Since there are no relations in this set, these additional elements are unique.

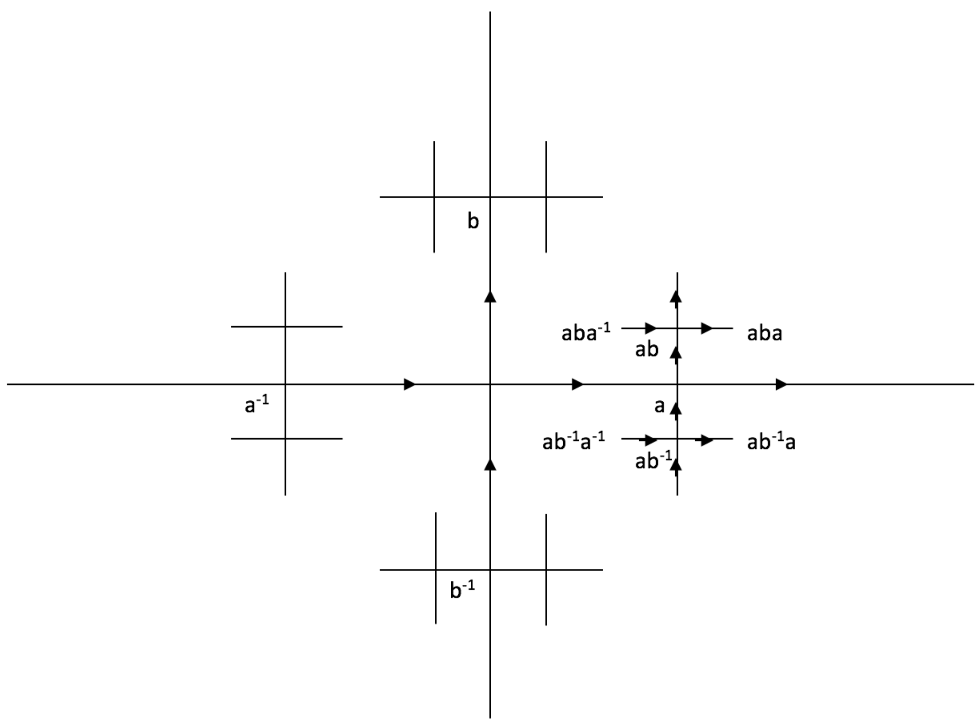

Figure 6: $Cay(F(S) : \{a, b\})$.

In the above Fig. 6, we see the graph of the free group on our set S with no relations. Consider the a - axis. The powers of a are given as $\{..., a^{-2}, a^{-1}, 1, a, a^2, ...\}$.

The b - axis is similar. On this image, the labels are extended for multiplications on a. When we move one unit right on the a - axis, we obtain a. Multiplying a by a power of b gives us our next two elements that send us up and down respectively, ab and ab^{-1}. The rest of the elements listed are found in the same manner. Note that since there are no relations on this set, ab and ba are not the same element. That means when we start to extend from a multiplying by a power of b, we are never going to cross any line from the b - axis. Hence, the graph becomes a fractal. A small portion of that idea is shown in the figure and you may assume the image continues to extend.

Free Groups and Cayley graphs are used in the proof of the Banach-Tarski paradox. This paradox shows that a solid sphere in 3-dimensions can be dissected into a finite number of disjoint sets, then recombined to obtain two separate but identical copies of the original sphere.

EXAMPLE 3.3: Now let's consider a quick example of a group that is not free. Imagine a group whose set is $\mathbb{Z}^2$. This gives us our normal coordinate plane for the a and b axes. Note that this is an abelian group. Thus this set has a relation. Hence the group generated by a and b on $\mathbb{Z}^2$ is not a free group. The Cayley graph is in Fig. 7.

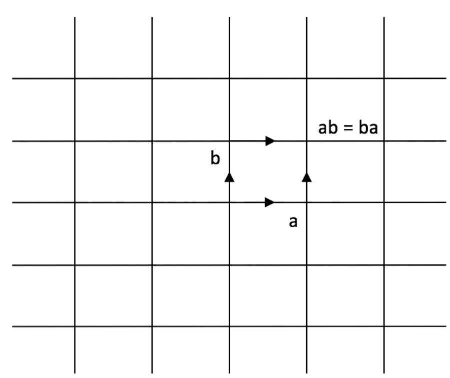

Figure 7: $Cay(\mathbb{Z}^2 : \{a, b\})$.

4. HAMILTONIAN GRAPHS

LEMMA 4.1: If Γ_1 and Γ_2 are hamiltonian, then $\Gamma_1 \times \Gamma_2$ is hamiltonian.

Proof. We divide the proof into two cases. The first discusses when we have two graphs that are either even and even or even and odd. The second case discusses two graphs that are both odd.

Case 1: $|\Gamma_1|$ or $|\Gamma_2|$ is even.

Without loss of generality, suppose $|\Gamma_1|$ is even. Then Γ_1 has an even number of vertices. Let Γ_1 have the respective hamlitonian cycle $(v_1, v_2, ..., v_{2n})$. Similarly, let Γ_2 have the hamiltonian cycle $(w_1, w_2, ..., w_m)$. Note that Γ_2 may have either an even number or odd number of vertices. We now construct a hamiltonian cycle for $\Gamma_1 \times \Gamma_2$ starting with (v_1, w_1).

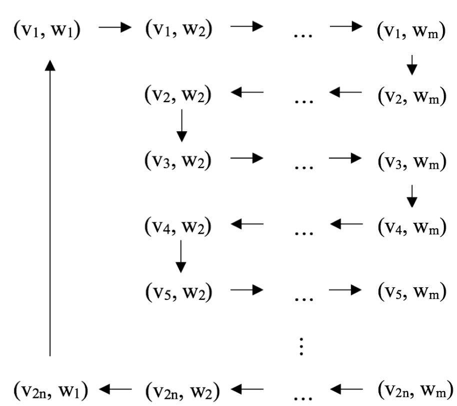

Figure 8: A hamiltonian cycle when $|\Gamma_1|$ is even.

Now that we have shown that a hamiltonian exists, by Fig. 8, we note that no matter $|\Gamma_2|$, we are able to still find a hamiltonian cycle. That is because the elements of Γ_2 dictate the number of columns in the image. We see by our route that the number of columns does not effect the result. Thus, by the above demonstration for any Γ_1 with an even number of vertices and a hamiltonian cycle, and for any Γ_2, $\Gamma_1 \times \Gamma_2$ has a hamiltonian cycle.

Case 2: Both $|\Gamma_1|$ and $|\Gamma_2|$ are odd.

Now we suppose that Γ_1 has a hamiltonian cycle $(v_1, v_2, ..., v_{2n+1})$ and that Γ_2 has a hamiltonian cycle $(w_1, w_2, ...2_{2m+1})$. We construct a hamiltonian cycle for $\Gamma_1 \times \Gamma_2$ in the following figure.

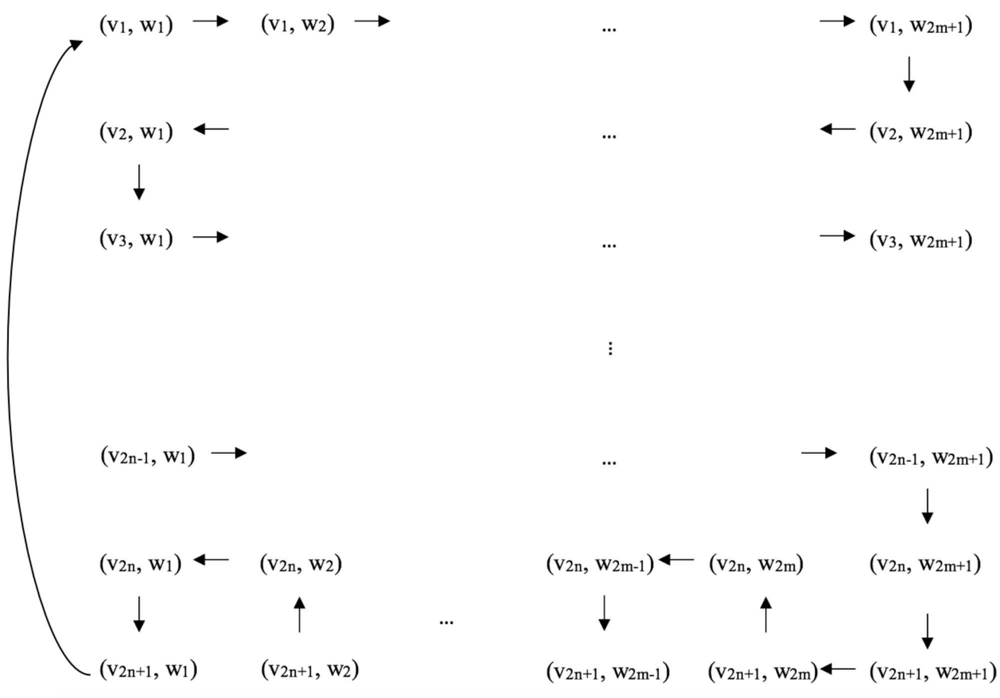

Figure 9: A hamiltonian cycle when $|\Gamma_1|$ and $|\Gamma_2|$ are odd.

By Fig. 9, $\Gamma_1 \times \Gamma_2$ is hamiltonian when $|\Gamma_1|$ and $|\Gamma_2|$ are odd. Therefore, these two cases show that for any two graphs who have hamiltonian cycles, their cartesian product also has a hamiltonian cycle. □

THEOREM 4.2: Let $\Gamma_1, \Gamma_2, ..., \Gamma_n$ be hamiltonian. Then $\Gamma_1 \times \Gamma_2 \times .... \times \Gamma_n$ is hamiltonian.

Proof. We prove this theorem by induction.

Case $n = 1$. This is the trivial case since we already assume that Γ_1 has a hamiltonian cycle.

Case $n = k$. Suppose that $\Gamma_1, \Gamma_2, ..., \Gamma_k$ each have a hamiltonian cycle. We assume $\Gamma_1 \times \Gamma_2 \times ... \times \Gamma_k$ has a hamiltonian cycle.

Case $n = k + 1$.We now desire to show that $\Gamma_1 \times \Gamma_2 \times ... \times \Gamma_{k+1}$ has a hamiltonian cycle. First note that both $\Gamma_1 \times \Gamma_2 \times ... \times \Gamma_k$ and Γ_{k+1} have a hamiltonian cycle. By the previous lemma, their product also has a hamiltonian cycle. So, $\Gamma_1 \times \Gamma_2 \times ... \times \Gamma_{k+1}$ also has a hamiltonian cycle by induction. $\square$

Let us now consider an example applying the theorem for two graphs.

EXAMPLE 4.3: Consider the graphs $\Gamma_1 = \{1, 2, 3, 4\}$ and $\Gamma_2 = \{a, b, c\}$. We wish to find a hamiltonian cycle on $\Gamma_1 \times \Gamma_2$. First view Fig. 10 displaying Γ_1 and Γ_2. Fig. 11 shows the cross product of the two graphs.

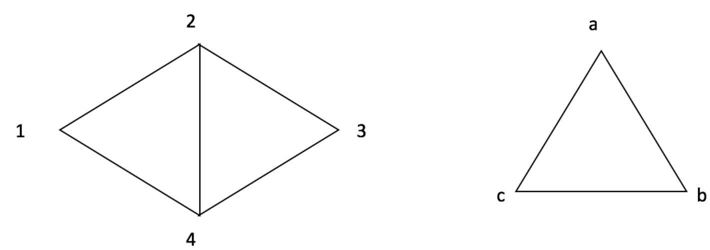

Figure 10: Γ_1 and Γ_2.

EXAMPLE 4.4: Consider $Cay(Z_n : \{t^2, t^3\})$. If n is odd, then t^2 generates a hamiltonian cycle. If n is not a multiple of 3, t^3 generates a hamiltonian cycle.

EXAMPLE 4.5: Find a hamiltonian path on $Cay(Z_{12} : \{t^2, t^3\})$.

Note that Z_{12} is isomorphic to $(\mathbb{Z}_{12}, t)$, so we can consider this Cayley graph of 12 vertices.

Fig. 12 shows multiplication by t^2 through solid lines and multiplication through t^3 with dashed lines. We can see a hamiltonian path starting with the

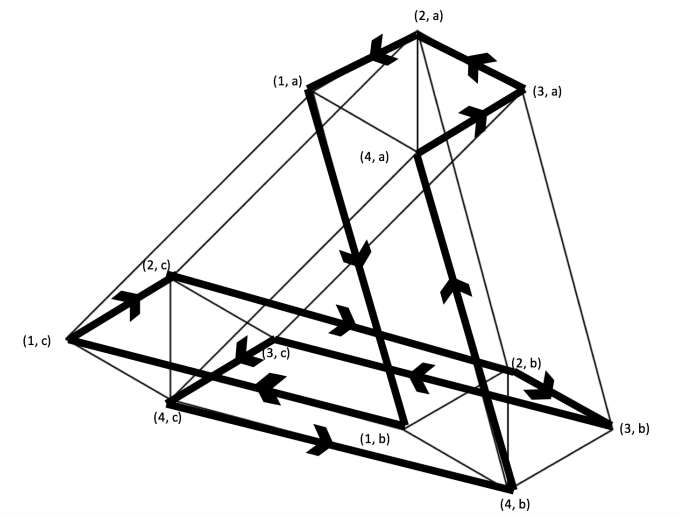

Figure 11: A hamiltonian cycle in $\Gamma_1 \times \Gamma_2$.

north most vertex and continuing as such: $0 \to 3 \to 6 \to 9 \to 11 \to 2 \to 5 \to 8 \to 10 \to 1 \to 4 \to 7$.

EXAMPLE 4.6: Find $Cay(S_4 : \{(123), (1234)\})$. To create this Cayley graph, we begin left multiplication from the identity by (1234). The identity is denoted "id" in the graph. These vertices are found $(1234) * id = (1234)$, $(1234) * (1234) = (13)(24)$, $(1234) * (13)(24) = (1432)$, and $(1234) * (1432) = id$. Since this left multiplication was done four times, we are able to create a square from the operation. Fig. 13 displays this Cayley graph. The dotted line represents left mulitplication

by (1234). The solid line is left multiplication by (123). To obtain the other vertices, we continue to left multiply each vertex by (123). In the same manner, we obtain (123)$*$id = (123), $(123) * (123) = (132)$, and $(123) * (132) =$ id. This time three operations develops a triangle shape. The rest of the diagram is obtained in this manner, where we make sure that every possible vertex is multiplied by both generators. This Cayley graph is a truncated octahedron in the third dimension. Note, there exists a hamiltonian cycle in this graph.

LEMMA 4.7: Factor Group Lemma.

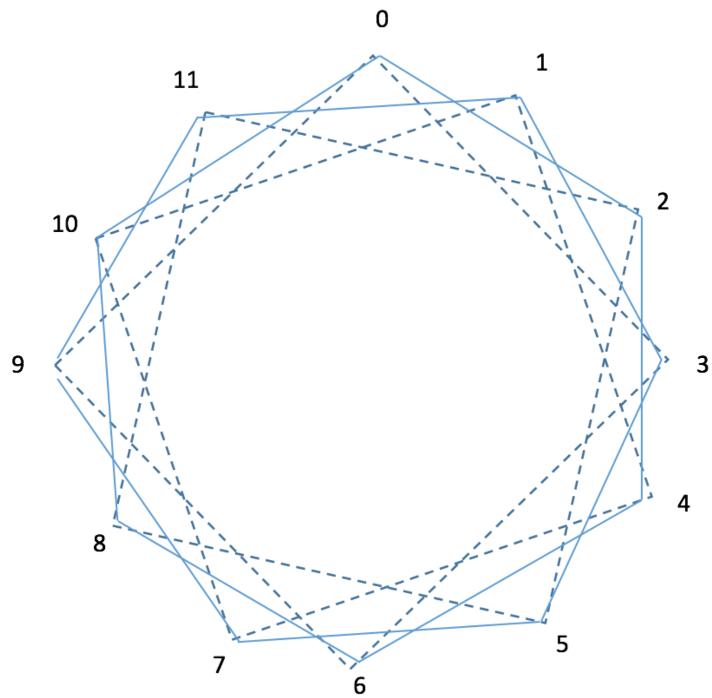

Figure 12: $Cay(Z_{12} : \{t^2, t^3\})$.

Let H be a cyclic, normal subgroup of index n in a finite group G and let S be a generating set of G. If $(s_iH : 1 \leq i \leq n)$ is a hamiltonian cycle in the Cayley digraph of G/H where $s_1, s_2, ..., s_n \in S$ and $s_1s_2...s_n$ is a generator of H, then $|H| *$ $(s_i : 1 \leq i \leq n)$ is hamiltonian cycle in the $DiCay(G : S)$.

Let us demonstrate the Factor Group Lemma by the following example.

EXAMPLE 4.8: Let $G = D_8 = \langle s, rs \rangle$. Let $H = \langle r \rangle$. Take G/H and by Lagrange, we know there are two elements. G/H is isomorphic to Z_2. So by the Factor Group Lemma, we need to find a hamiltonian cycle in $DiCay(Z_2 : \{sH\})$. Since we are looking at the digraph, we will have two curved edges between two vertices. If we go from H to sH and back by multiplying by sH, we have a hamiltonian cycle. However, $s_1s_2 = s^2 = 1$ does not generate H, so the lemma does not apply. We note that $sH = rsH$ and with this representation $s_1s_2 = srs = r^{-1}$ which does generate H. By the Factor Group Lemma, we obtain a hamiltonian cycle on $DiCay(D_8$

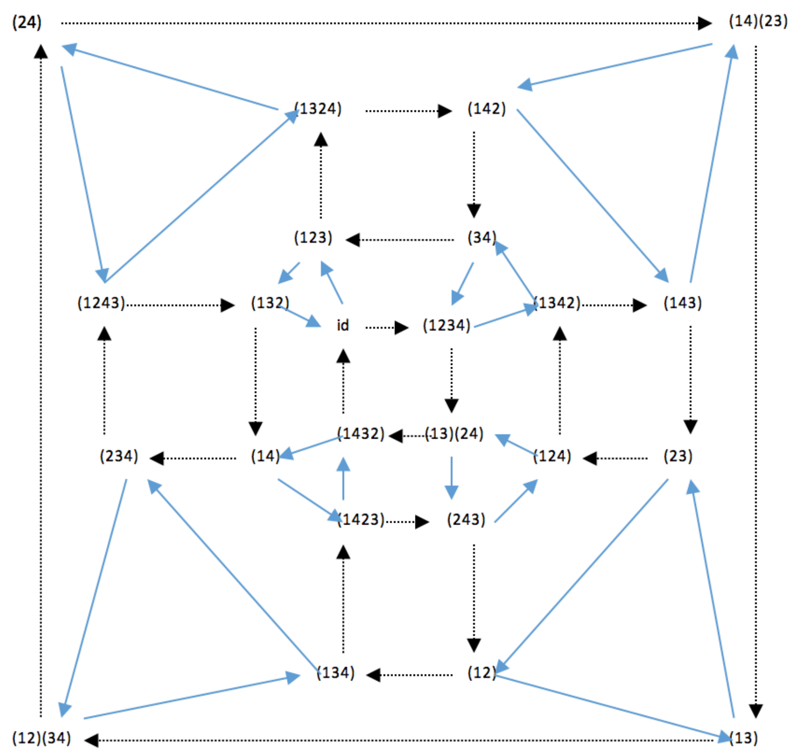

Figure 13: $Cay(S_4 : \{(123), (1234)\})$.

$: \{s, rs\})$ which, in fact, is the cycle in Fig. 14.

THEOREM 4.9: There is a hamiltonian path on every Cayley digraph on an abelian group.

Proof. We prove this theorem by induction.

Let G be an abelian group. Let S be our generating set.

Case: If $|S| = 1$, then the group is Z_n and has one generator. Hence, there already exists a hamiltonian path.

Case: If $|S| = k$, we assume there exists a hamiltonian path.

Case: Suppose $|S| = k+1$. Suppose $S = \{s_0, s_1, ..., s_k\}$. Choose $s_0 \in S$. $G/\langle s_0 \rangle$ is a subgroup of G. Furthermore, it is a normal subgroup since G is abelian. Now our generating set has one fewer elements. So our generating set was once S and is now $\bar{S} = \{\bar{s_1}, ..., \bar{s_k}\}$. So, since there is one fewer generator, $Cay(G/\langle s_0 \rangle : \bar{S})$ has a hamiltonian path. But how do we find a hamiltonian path for G? Let $s_0 \in S$. De-

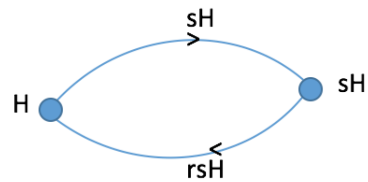

Figure 14: Cayley graph of Z_2 generated by cosets.

fine $d = |s_0|$ - 1. Then $(s_0 * d, s_1, s_0 * d, ..., s_k, s_0 * d)$ is a hamiltonian path in $Cay(G : S)$. This means, we do s_0 d times, then s_1 once, and so on. We are essentially almost finishing a cycle, but we skip the last step to close the path and move onto a different generator of S. Then we again continue the cycle of s_0 with one less step. Eventually, we have a hamiltonian path. $\square$

CONJECTURE 4.10: (Folklore, Alspach, 1985) Any connected Cayley graph is hamiltonian.

EXAMPLE 4.11: Consider $Cay(Z_4 : x)$. Then we have a cyclic group with elements $\{1, x, x^2, x^3\}$. Fig. 15 displays a hamiltonian path and a hamiltonian cycle.

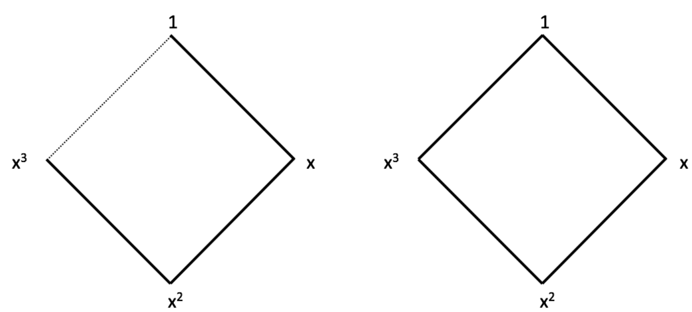

Figure 15: A hamiltonian path and cycle on $Cay(Z_4 : x)$.

CONJECTURE 4.12: (Marusic, 1983) Every connected Cayley graph of an abelian group of order at least three is hamiltonian.

The Cayley graph of Z_2 is a line, so it cannot have a hamiltonian cycle. The proof of this conjecture invokes ideas similar to that of Lemma 4.1.

The question remains, can we find a hamiltonian cycle on every Cayley graph of a nonabelian group?

THEOREM 4.13: $Cay(D_{2n} : \{r, s\})$ is hamiltonian.

Proof. For any dihedral group with generator r, there will be an n-gon generated by r. Furthermore, since our generators are $\langle r, s\rangle$, left multiplying each vertex by s extends the vertices of the n-gon. Each of these lines connects through multiplication by r. This is displayed in Fig. 16.

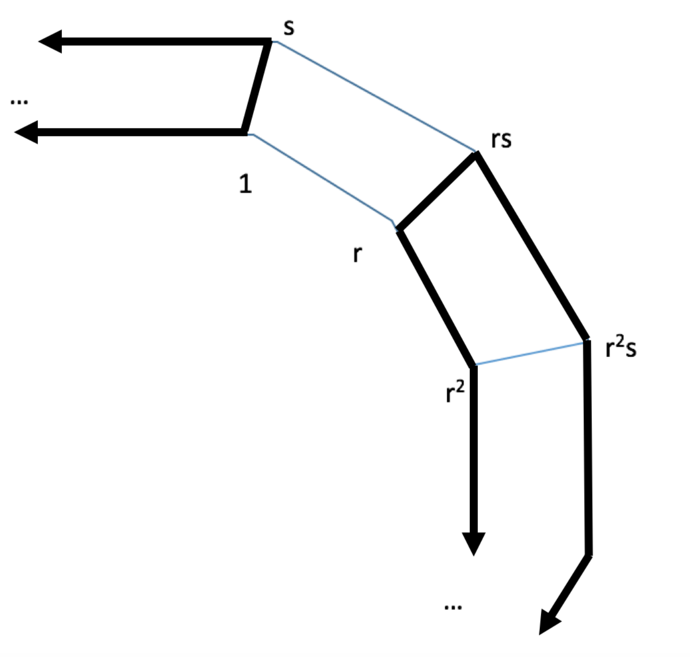

Figure 16: A hamiltonian cycle on D_{2n}.

In this image, we see for all n, D_{2n} will have two n-gons. One on the inside and the other on the outside. The vertices are connected respectively. A hamiltonian cycle, in the thick line, is constructed by starting at vertex 1, moving clockwise among the inner n-gon, stoping at the vertex r^{n-1} to left multiply by s, such that we now move counterclockwise on the outer n-gon until we get to vertex s, and move back on the connecting line to 1. $\square$

Now we discuss the Cayley graphs of semi-direct products and cartesian products.

DEFINITION 4.14: The *semi-direct product* between two cyclic groups is $Z_m \rtimes_k Z_n = \langle a, b \mid a^m = b^n = 1 \text{ with } bab^{-1} = a^k \rangle$ and $k^n \equiv 1 \bmod m$.

THEOREM 4.15: (Curran, Gallian, 1996) The graph $Cay(Z_m \rtimes_k Z_n : \{a, b\})$ is hamiltonian when $mn > 2$.

EXAMPLE 4.16: What is $Cay(Z_7 \rtimes_k Z_3 : \{a, b\})$?

First we write out $Z_7 \rtimes_k Z_3 = < a, b \mid a^7 = b^3 = 1 \text{ with } bab^{-1} = a^k >$ and $k^3 \equiv 1 \bmod 7$. Now we find a k that satisfies the congruency. $k = 2$ satisfies the congruency. Let us view the images generated by a, b, and b^2.

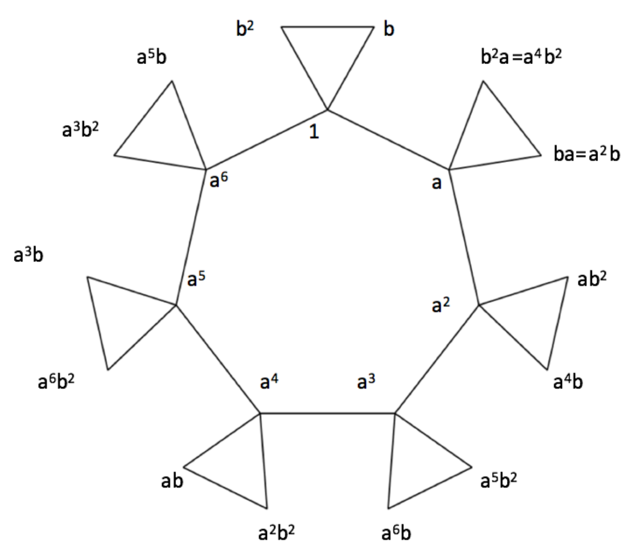

Figure 17: The graph generated by a and b at every vertex of a.

Fig. 17 we see that the generator a creates a heptagon. At this moment, we only have vertices with a and the property that $a^7 = 1$. So we no longer multiply by a. The next step is to consider what happens to each vertex upon left multiplication by b. Looking at each vertex of the heptagon, we see that left multiplication by b creates a triangle with each vertex. Notice two vertices of the upper right most triangle have two notations. When we left multiply a by b, we obtain ba. However, we choose to rewrite this vertex with the property that $bab^{-1} = a^2$. Hence, $ba = a^2b$ and $b^2a = a^4b^2$. All of the other vertices are written in that respective manner. Notice we have 21 total vertices and these will be all of them for our resulting graph. However, we must consider, what happens when we multiply one of the outer vertices of each triangle by a? Let us find out below.

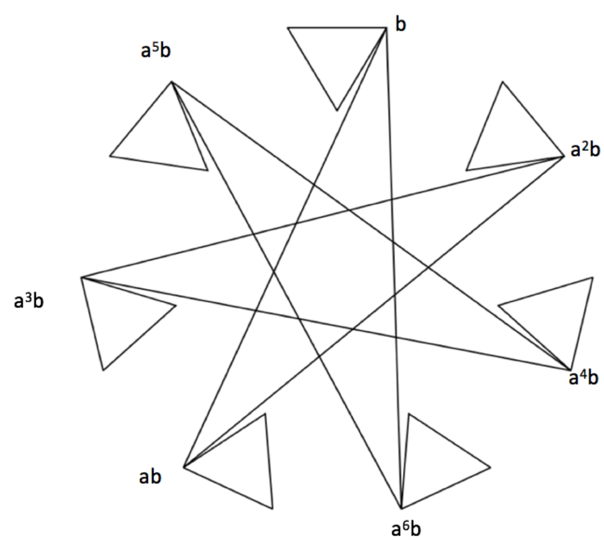

Figure 18: Left multiply the vertex b by a.

As we continue to left multiply b by a in Fig. 18, we connect the vertices b, ab, a^2b, and so on until we obtain a^7b which is just b. The triangles are left in the image as a reference to the image before. Similarly, let us now examine left multiplication of a by b^2 in the next image.

The next image, Fig. 19, left multiplying b^2 by a continuously connects the labeled 7 vertices as displayed. Now let us conjoin the three images to see the overall Cayley graph for this example.

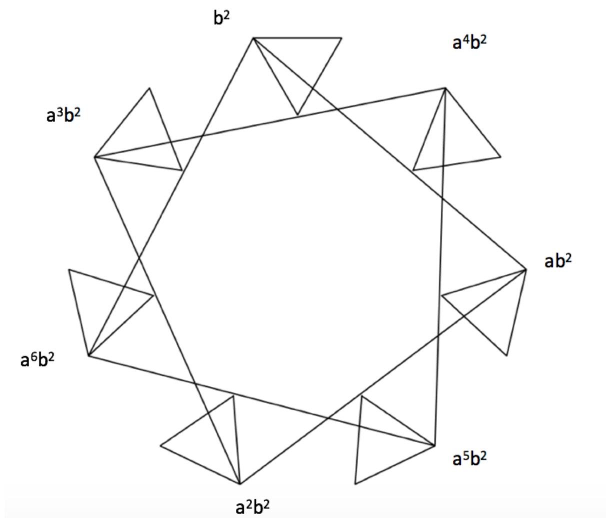

Figure 19: Left multipying vertex b^2 by a.

The Cayley graph of $Z_7 \rtimes_2 Z_3$ is given by the Fig. 20.

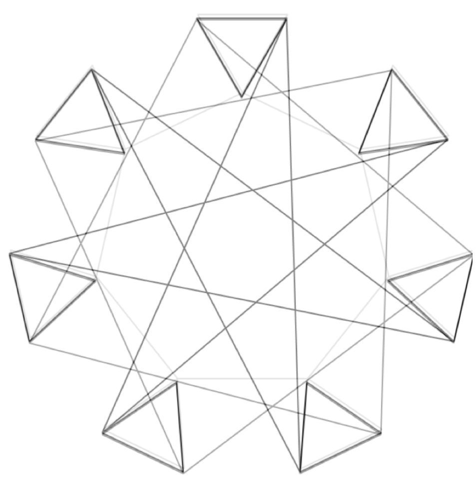

Figure 20: $Cay(Z_7 \rtimes_2 Z_3 : \{a, b\}$.

EXAMPLE 4.17: Find a hamiltonian cycle on $Z_7 \rtimes_2 Z_3$.

Consider the following image.

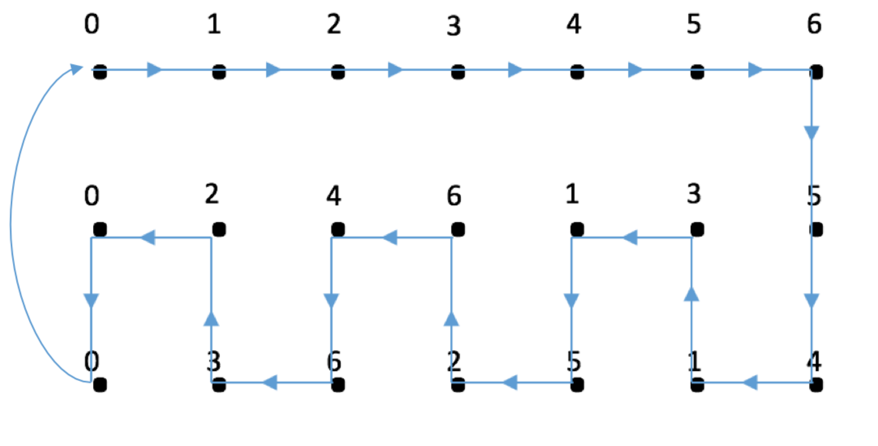

Figure 21: A hamiltonian cycle in $Z_7 \rtimes_2 Z_3$.

In Fig. 21, reorganize the vertices of $Z_7 \rtimes_2 Z_3$ in the above manner. The first row are the vertices of the heptagon. The second row are the vertices of the wider star. The last row are the vertices of the thinner star. Lastly, each column represents the respective triangle of that vertex. We are able to find a hamiltonian cycle quickly. If we were to examine the Cayley graph, it would be difficult to find a hamiltonian cycle. The method in this figure was to follow the vertices of the heptagon, do not close, move to the respective triangle, move one edge of a star, back to the next triangle, and so on. Graphically, we see a similar image to the proof of two graphs having a hamiltonian cycle.

THEOREM 4.18: $(Cay(G : S) \times Cay(H : T)) = Cay(G \times H : [(S \times \{1\}) \cup (\{1\} \times T)])$.

EXAMPLE 4.19: Consider $Cay(Z_3 : s) \times Cay(Z_4 : t)$.

The Cayley graph of Z_3 is a triangle and is a square for Z_4. Thus when multiply these graphs together, we obtain $Cay(Z_3 \times Z_4 : \{st\})$, which is a dodecagon.

The analogous conjecture is false for Cayley digraphs with the following example.

EXAMPLE 4.20: (Rankin, 1948) The $DiCay(A_6 : \{(24653), (163)(245)\}$ is not hamiltonian.

5. CAMPANOLOGY AND CAYLEY GRAPHS

Campanology is the study of bell-ringing. This research focuses on change-ringing, the act of ringing bells in a controlled and methodical manner. First, a brief discussion of campanology is necessary.

The bells played will be numbered by $1, 2, ..., n$ for n bells. Bell 1 represents the treble and bell n the tenor. The operation of ringing from row to row is calculated through permutation and will be denoted as a *change*. For a function $f : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$, the domain represents the position of a bell's ring and the range denotes the new position of a bell's ring. Change-ringing is represented through permutations on S_n. An *extent* is when a full ringing has occured. More particularly, an extent occurs when all the possible cyles of S_n have been rung once with the additional row of the identity. This means we expect to have $n! + 1$ rows. Throughout these changes, conditions must follow to be a change-ringing:

(i) The first and last change are both the identity, which is called *rounds*.

(ii) No other change is repeated.

(iii) From one change to the next, no bell moves places more than one position away from its previous spot.

These three conditions must occur in order to have a change-ringing. There are additional conditions that are optional to the ringers. In this research, we do not consider them. Among change-ringing, specific patterns are followed called methods and principles. *Methods* are when all bells have the same work except one bell plain hunts which means that one bell is fixed to a specific pattern in ringing. *Principles* are when all bells work the same and no bell is fixed. We will examine some of these patterns.

Before we begin, we are allowed so many position changes, called *switches*, depending on our n. Switches will be represented by cycles. First consider the case of 3 bells. Since we have 3 positions, we may exchange the first position with the second or the second with the third. Recall that our third condition disables us to move from the first position to the third. Hence for 3 bells, our switch options are (12) and (23). Similarly, for 4 bells, our options are $(12), (23), (34)$, and switching (12)(34) simultaneously. There is a pattern here, as described in the following remark.

REMARK 5.1: (White, 1987) Let $F(n)$ represent the nth Fibonacci number. There exist $F(n) - 1$ possible switches for n bells.

As discussed before, there are only 2 switches on 3 bells. This is $F(3)$-1, which is 2. For 4 bells, $F(4)$-1 is 4, which matches the switches described above. Now we can find how many switches exist for 5 bells, without figuring them out. $F(5)$-1 = 7 implies there are 7 possible switches for 5 bells, which are $(12), (23)$,

$$(34), (45), (12)(34), (23)(45), (12)(45), (12)(23)(34)(45).$$

Earlier, in the definition of Schreier coset, we defined these switches via letters. We will continue to use that representation: $a = (12)(34)$, $b = (12)$, $c = (34)$, and $d = (23)$.

EXAMPLE 5.2: Start with rounds for S_4 and do the following switches from left to right: a, d, a, d, a, d. In future examples, the commas and spaces will be omitted and repetitions will be denoted in exponential format. So we perform the switches $adadad$ or $(ad)^3$. Following the designated switches, we obtain $(1234) \rightarrow (2143) \rightarrow (2413) \rightarrow (4231) \rightarrow (4321) \rightarrow (3412)$. This happens to be the first column of a method called Plain Bob Minimus.

Our research began with viewing the elements of S_4 on a flattened truncated octahedron and trying to find hamiltonian cycles. We only considered switches b, c, and d. This is the diagram we studied.

In Fig. 22, we always began with the upper left vertex labeled 1234 for rounds and tried to find as many cycles as we could. After we found a cycle, we would

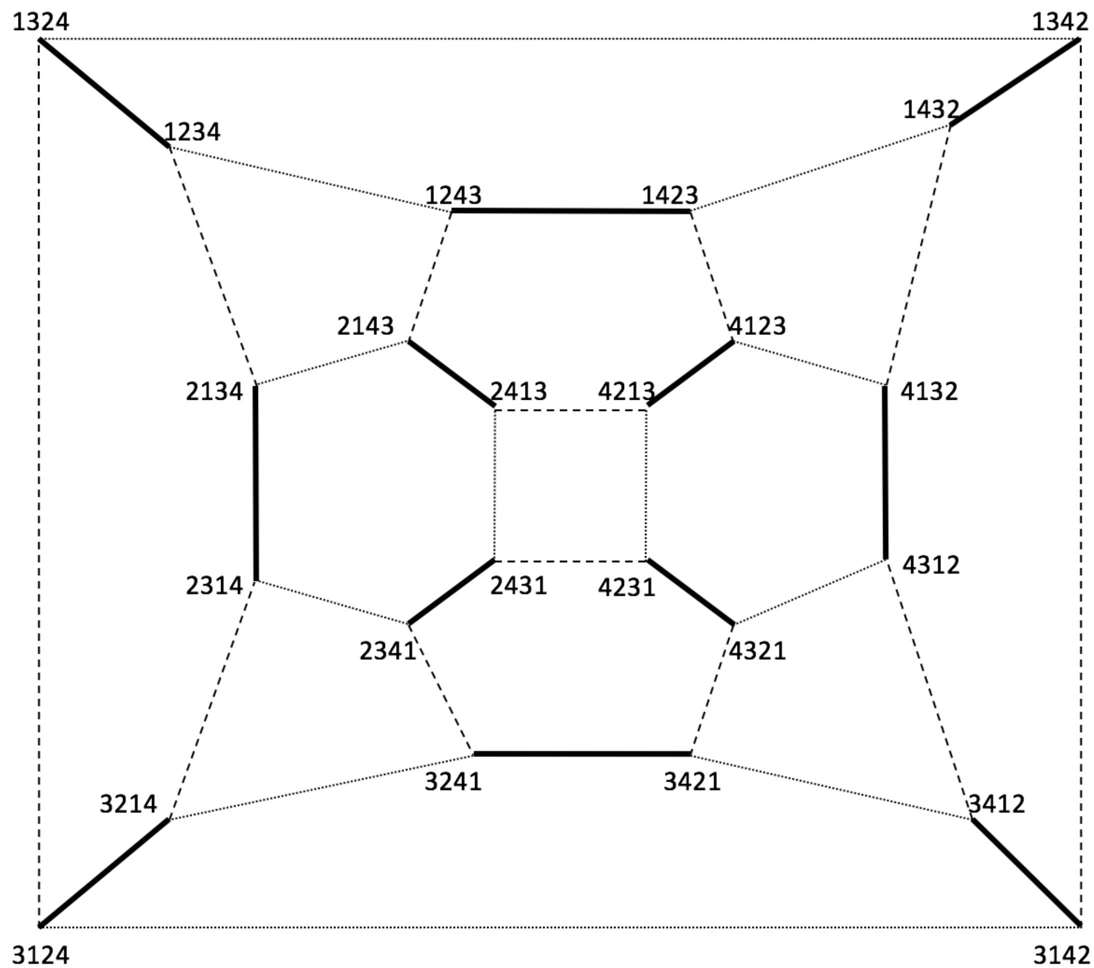

Figure 22: Cayley diagram for 4 bells, without simultaneous switches.

write down the switches accordingly. Then we compared them to the principles and methods. Without the switch a, we missed the majority of documented change-ringings. However in our research, we were able to match our words with Double Court and Double Canterbury. The following figure displays the same graph in a different perspective.

Each method and principle has a Cayley graph that lays on a truncated octahedron. The truncated octahedron is given in the Fig. 23.

THEOREM 5.3: (White, 1989) Let Δ be the set of all switches on S_n such that the product of cycles are disjoint. An extent on n bells can be composed if and

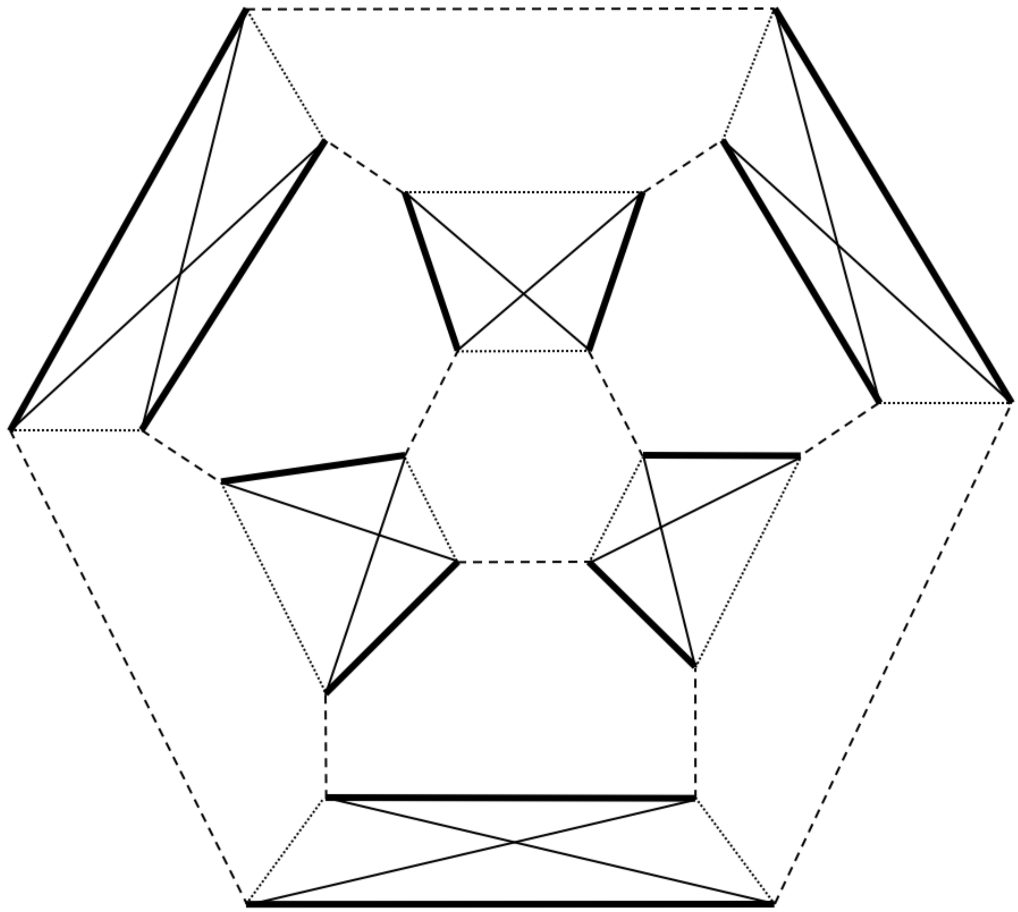

Figure 23: Cayley Graph for 4 bells

only if $Cay(S_n, \Delta)$ is hamiltonian.

From the theorem, we expect that for every Cayley graph of a change-ringing, there is a hamiltonian cycle. Now for every extent, not including the last rounds, there will be n rows. We divide the extent into n columns. The first row of each column is called a lead and we will call it the sequence of numbers a word denoted w. Note that w is obtained through switches from Δ in S_n. Also, $w^m = e$ such that w is an m-cycle and e is the identity (rounds) of S_n. Let us consider an example to determine the words.

EXAMPLE 5.4: Consider the Plain Bob Minimus extent listed below. Our word

1234	1342	1423
2143	3124	4132
2413	3214	4312
4231	2341	3421
4321	2431	3241
3412	4213	2314
3142	4123	2134
1324	1432	1243

for Plain Bob Minimus is $w = (ad)^3ac$. Note here that $(ab)^3$ is presumed to be identified as one switch. So w is length 3. This means $w^3 = (1234)$. This makes sense because if we expanded w, we would see that it is a total of 8 switches, or 8 changes. The order of S_4 is 24 and in change ringing, we need all of the permutations. So w must occur 3 times in order to have all of our changes, called a full extent. Recall that each extent is rung with an extra change of rounds at the end. So the actual last ringing is (1234). Since change ringing requires all of these ringings to be different, it is easy to see a hamiltonian cycle exists. Now let us actually confirm that w is a 3-cycle. First, recall the definitions of a, b, c, d. Now let us compute $ad = (1243)$. Then $(ad)^3 = (1243)^3 = (1342)$. Then calculate $ac = (12)$. So, $(ad)^3ac$ $= (1342)(12) = (234)$. So $w = (234)$. The order of a a 3-cycle is 3, so $w^3 = (1234)$.

We see that our word takes effect at the beginning of every column. A hamiltonian circuit on Schreier graph of left cosets gives us a word which tells us row by row (or letter by letter) how to change. This gives us our full extent.

EXAMPLE 5.5: Recall the example about Schreier coset graph on $Z_4 = \langle(1234)\rangle$. There are 8 hamiltonian cycles that exist on that graph and they are listed below.

Fig. 24, 25 displays 8 hamiltonian cycles. The words beneath each cycle describe the switches need to create the cycle. Since this is a coset graph and all the elements of our group have essentially 'collapsed' into cosets, we can return to the original Cayley graph and determine their are 8 hamiltonian cycles for Z_4. Now we

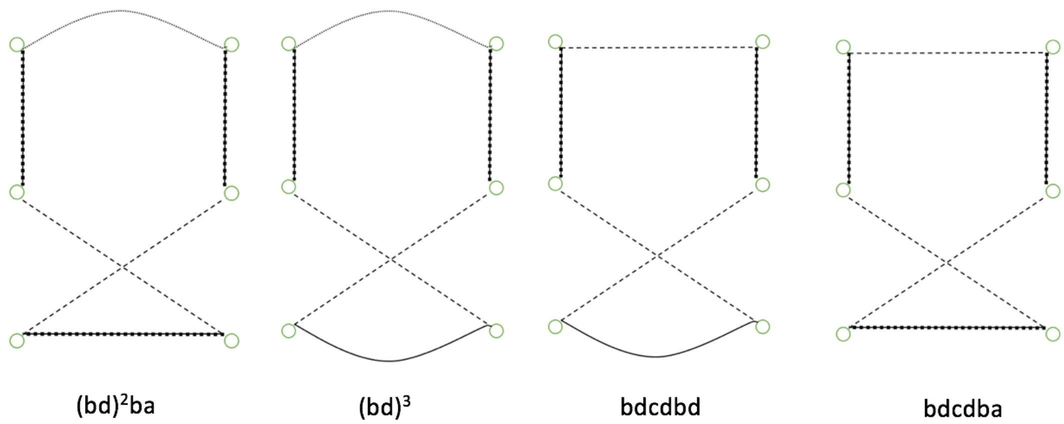

Figure 24: 4 hamiltonian cycles of $Sch(Z_4 : (1234))$.

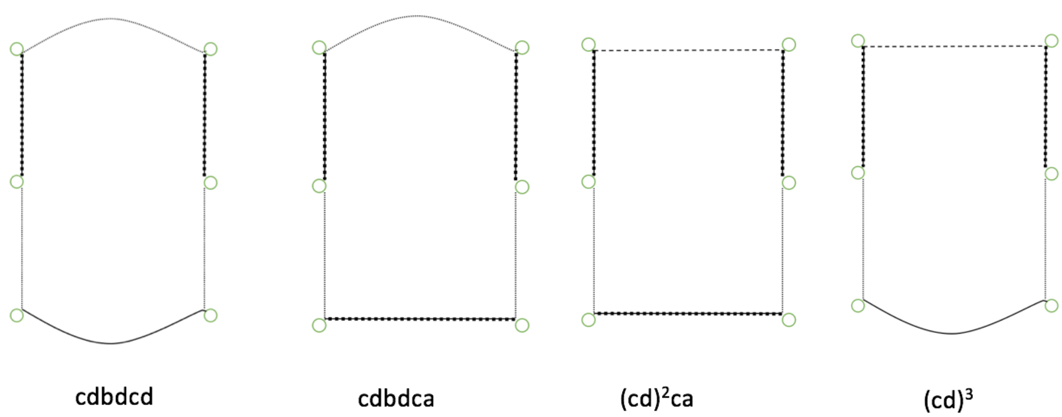

Figure 25: 4 hamiltonian cycles of $Sch(Z_4 : (1234))$.

need to determine which cycles give a full extent. Each word is calculated in the below table, Tbl. 3, and the last column determines the order.

Each word has a calculated cycle and order. The words are calculated in order of upper most hamiltonian cycle from Figures 24 and 25. From Tbl. 3, the order of each word tells us which words provide full extents. We are able to determine that numbers 1, 4, 6, and 7 all provide 24 changes and hence a full extent. These four words happen to represent four of the principles. They are listed in re-

Table 3: Coset words and their order

Word	Cycle	Order
1. $(bd)^2ba$	(1342)	4
2. $(bd)^3$	id	1
3. $bdcdbd$	(14)(23)	2
4. $bdcdba$	(1243)	4
5. $cdbdcd$	(14)(23)	2
6. $cdbdca$	(1243)	4
7. $(cd)^2ca$	(1342)	4
8. $(cd)^3$	id	1

spective order of the numbering: Erin, Stanton, Reverse Stanton, and Reverse Erin.

Aside from Plain Bob Minimus and the above principles, there exist other patterns that change-ringers follow. In total, there are 11 methods and 4 principles. Tbl. 4 displays the methods, principles, and their words for 4 bells.

Table 4: 11 methods and 4 principles

Name	Word
Plain Bob	$(ad)^3ac$
Reverse Bob	$adab(ad)^2$
Double Bob	$adabadac$
Canterbury	$adcbcdad$
Reverse Canterbury	$bd(ad)^2bc$
Double Canterbury	$bdcbcdbc$
Single Court	$bd(ad)^2bd$
Reverse Court	$ad(cd)^2ad$
Double Court	$bd(cd)^2bd$
St. Nicholas	$bdabadbc$
Reverse St. Nicholas	$adcbcdac$
Erin	$(bd)^2ba$
Reverse Erin	$(cd)^2ca$
Stanton	$bdcdba$
Reverse Stanton	$cdbdca$

The following eleven images in Fig. 26-29 show the hamiltonian cycle of all of the methods and principles that overlay this graph (Polster, 2003).

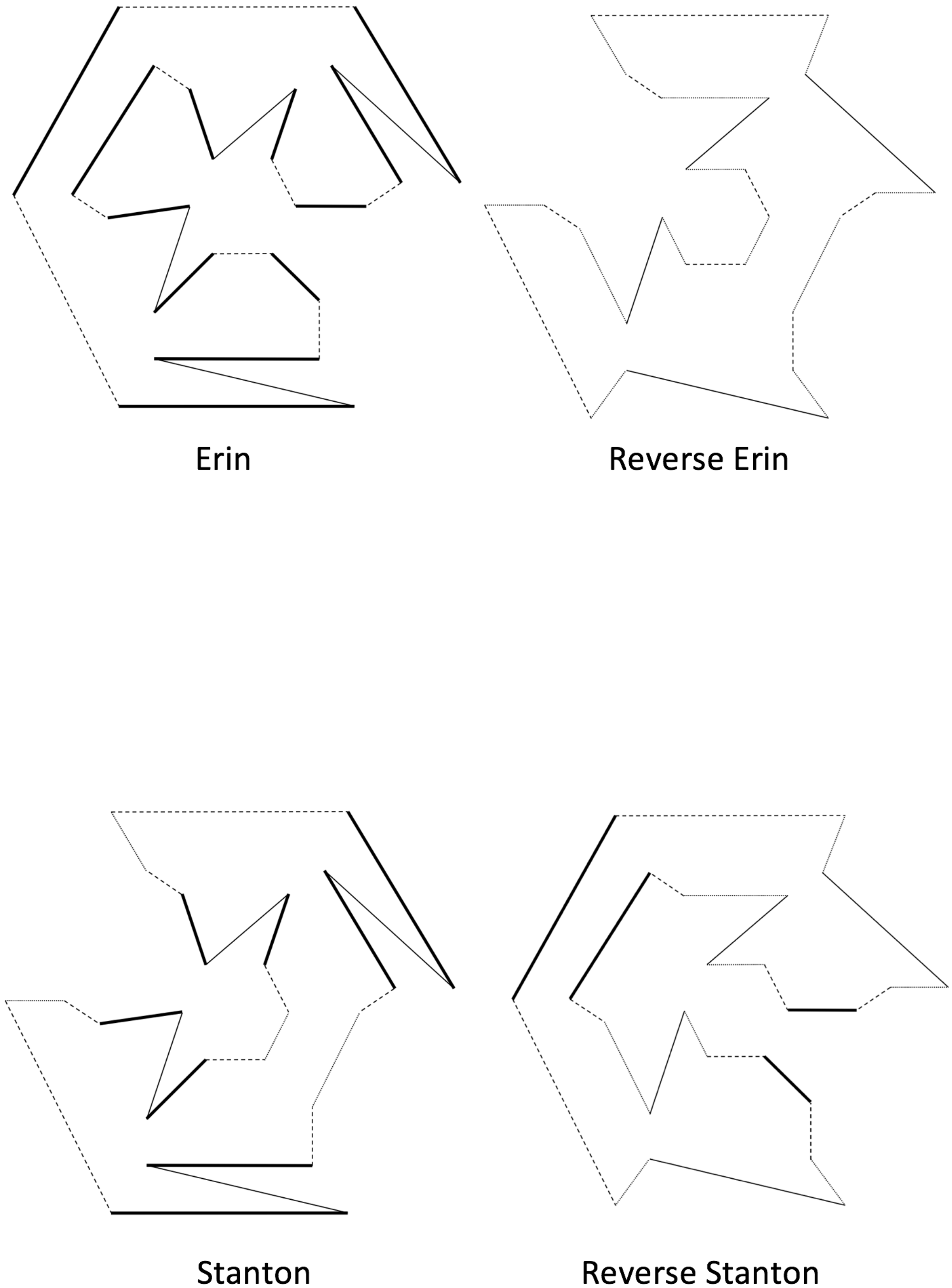

Figure 26: The four principles of 4 bells

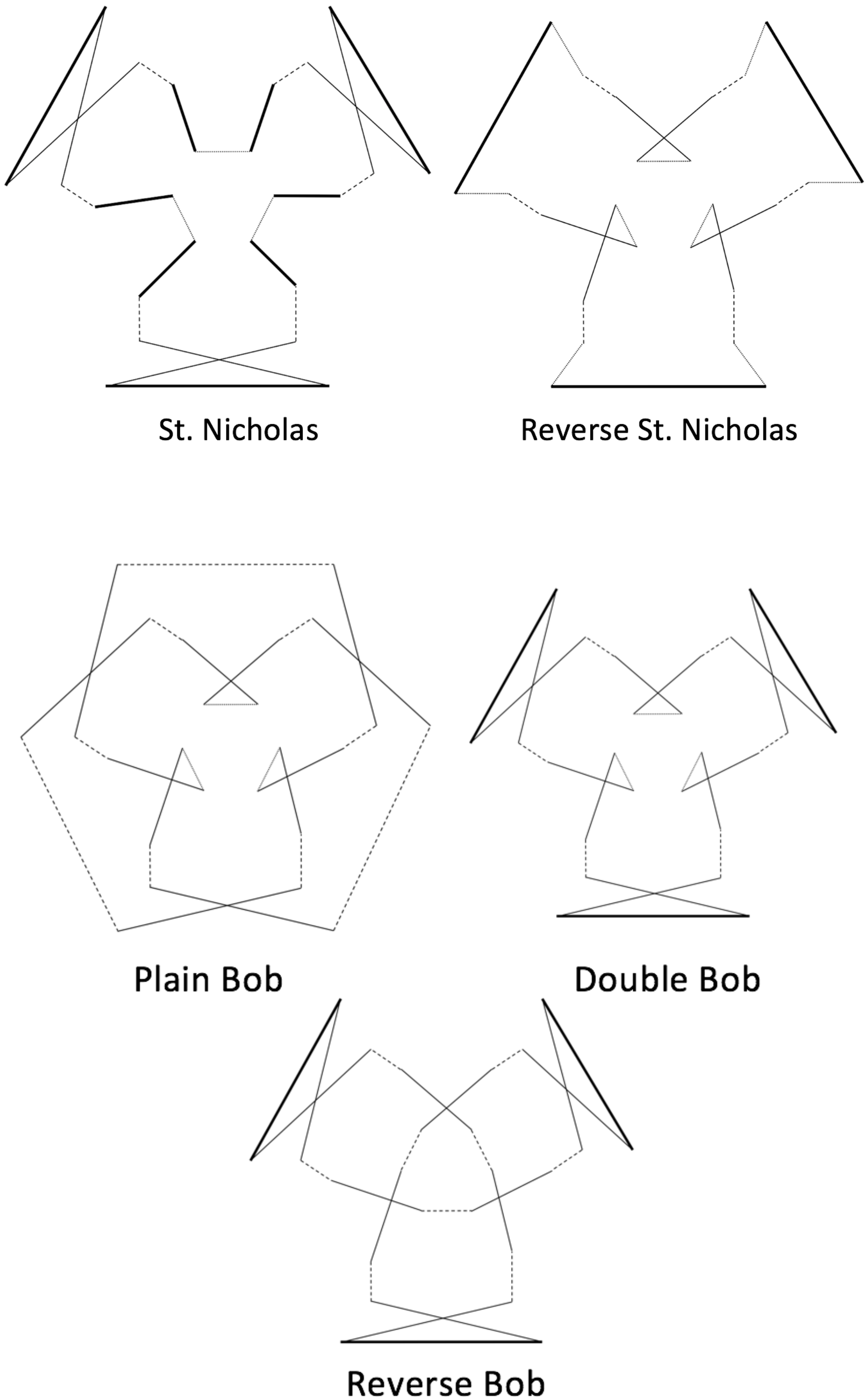

Figure 27: 5 methods of 4 bells

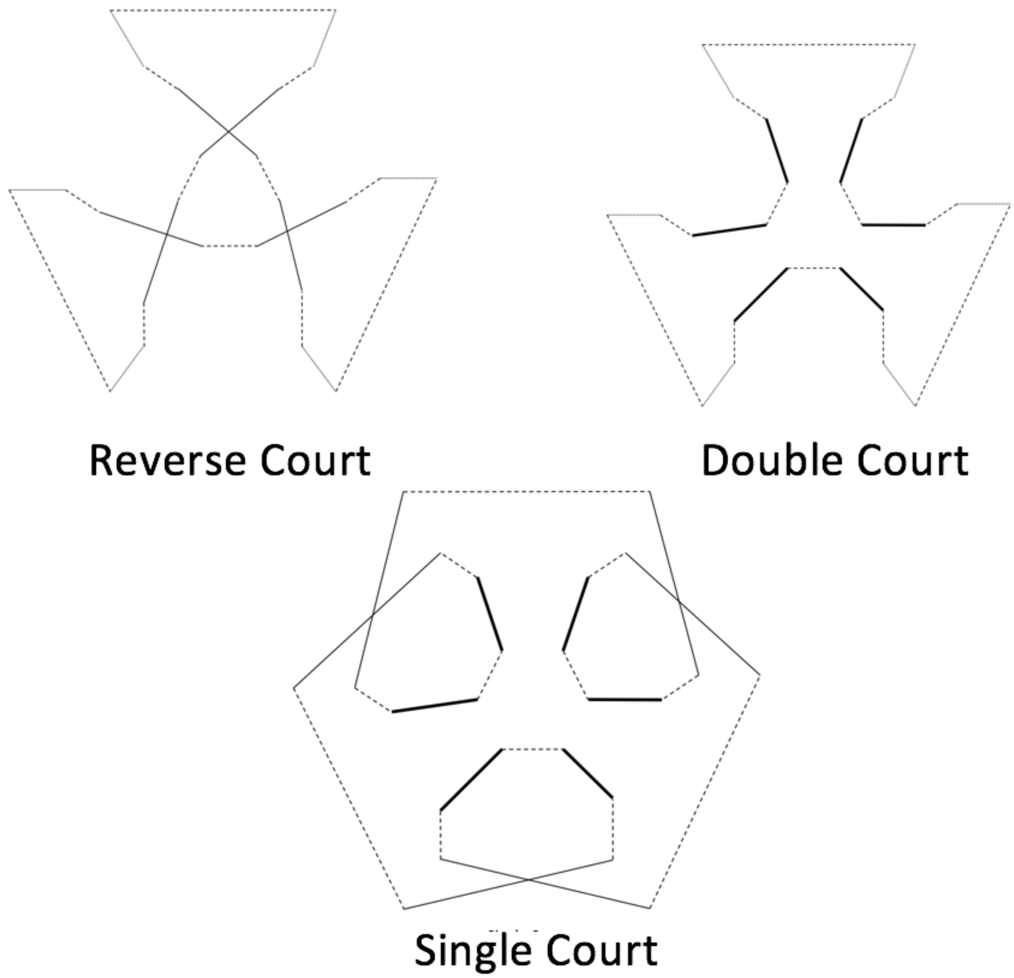

Figure 28: 3 methods of 4 bells

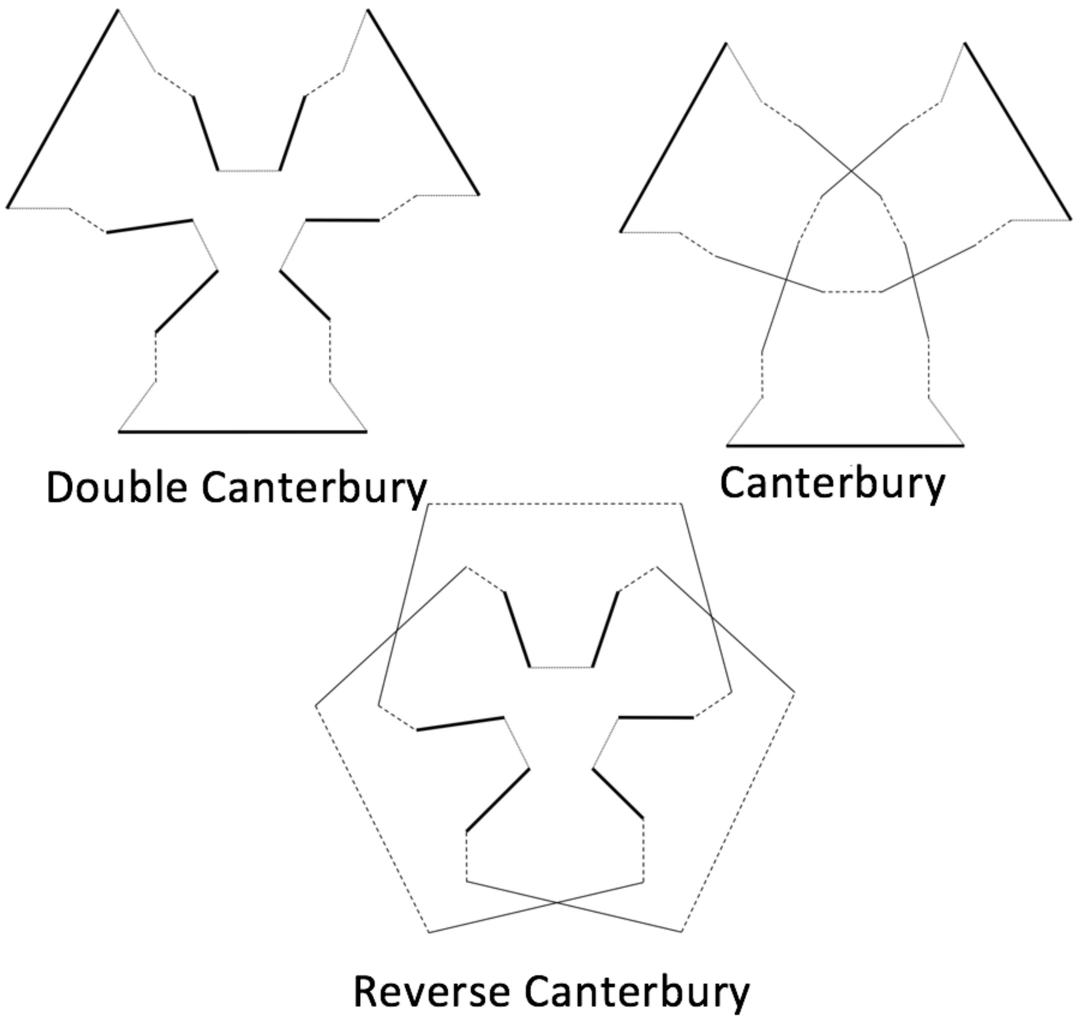

Figure 29: 3 methods of 4 bells

6. CONCLUSION

Cayley graphs are very useful in explaining and visualing groups and their actions. Depending on the generating set, there are different graphs for different groups. The Schreier left coset graph helps visual the understanding of cosets.

We showed that for any hamiltonian graphs, their product is hamiltonian. We applied this to an arbitrary example and then discussed results with Cayley graphs. In addition to Schreier coset graph, the Factor Group Lemma displayed that if a coset generated Cayley digraph (from a quotient group) has a hamiltonian cycle, then under certain conditions so does the Cayley digraph of the group.

The conjecture that all connected Cayley graphs have hamiltonian cycles was dissected into parts. We proved that every Cayley digraph of an abelian group has a hamiltonian path. In fact, we discussed that all connected Cayley graphs of abelian groups are hamiltonian, for groups of order greater than 2. We furthered this conjecture by showing that all dihedral groups generated by $\{r, s\}$ are hamiltonian. The question still remains if there exists a hamiltonian graph for other non-abelian groups.

Lastly, Cayley graphs display the permutations of a bell ringing from change to change. The Cayley graphs of each method and principle display a hamiltonian cycle which expresses the full extent of the method or principle.

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