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Schröder–Bernstein theorem

In set theory, the **Schröder–Bernstein theorem** states that, if there exist injective functions $f : A \to B$ and $g : B \to A$ between the sets A and B, then there exists a bijective function $h : A \to B$. In terms of the cardinality of the two sets, this means that if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$; that is, A and B are equipollent. This is a useful feature in the ordering of cardinal numbers.

This theorem does not rely on the axiom of choice. However, its various proofs are non-constructive, as they depend on the law of excluded middle, and are therefore rejected by intuitionists.[1]

The theorem is named after Felix Bernstein and Ernst Schröder. It is also known as **Cantor–Bernstein theorem**, or **Cantor–Schröder–Bernstein**, after Georg Cantor who first published it without proof.

Contents

Proof

The following proof is attributed to Julius König.[2]

Assume without loss of generality that A and B are disjoint. For any a in A or b in B we can form a unique two-sided sequence of elements that are alternately in A and B, by repeatedly applying f and g to go right and g^{-1} and f^{-1} to go left (where defined).

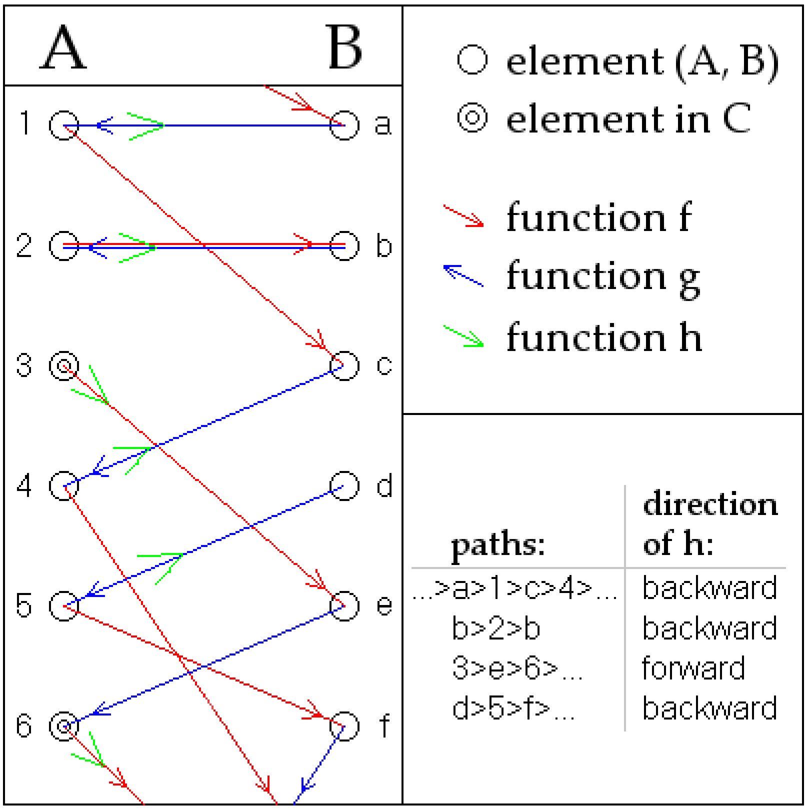
$$\cdots \to f^{-1}(g^{-1}(a)) \to g^{-1}(a) \to a \to f(a) \to g(f(a)) \to \cdots$$

For any particular a, this sequence may terminate to the left or not, at a point where f^{-1} or g^{-1} is not defined.

By the fact that f and g are injective functions, each a in A and b in B is in exactly one such sequence to within identity: if an element occurs in two sequences, all elements to the left and to the right must be the same in both, by the definition of the sequences. Therefore, the sequences form a partition of the (disjoint) union of A and B. Hence it suffices to produce a bijection between the elements of A and B in each of the sequences separately, as follows:

Call a sequence an *A-stopper* if it stops at an element of A, or a *B-stopper* if it stops at an element of B. Otherwise, call it *doubly infinite* if all the elements are distinct or *cyclic* if it repeats. See the picture for examples.

- For an *A-stopper*, the function f is a bijection between its elements in A and its elements in B.
- For a *B-stopper*, the function g is a bijection between its elements in B and its elements in A.
- For a *doubly infinite* sequence or a *cyclic* sequence, either f or g will do (g is used in the picture).



König's definition of a bijection $h{:}A \to B$ from given example injections $f{:}A \to B$ and $g{:}B \to A$. An element in A and B is denoted by a number and a letter, respectively. The sequence 3 → e → 6 → ... is an A-stopper, leading to the definitions $h(3) = f(3) = e$, $h(6) = f(6)$, ... The sequence $d \to 5 \to f \to$ ... is a B-stopper, leading to $h(5) = g^{-1}(5) = d$, ... The sequence ... $\to a \to 1 \to c \to 4 \to$ ... is doubly infinite, leading to $h(1) = g^{-1}(1) = a$, $h(4) = g^{-1}(4) = c$, ... The sequence $b \to 2 \to b$ is cyclic, leading to $h(2) = g^{-1}(2) = b$.

Original proof

An earlier proof by Cantor relied, in effect, on the axiom of choice by inferring the result as a corollary of the well-ordering theorem.[3] The argument given above shows that the result can be proved without using the axiom of choice. Note however that the principle of excluded middle is used to do the analysis into cases, so this proof does not work in non-classical logic.

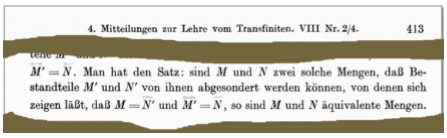
There is also a proof which uses Tarski's fixed point theorem.[4]

History

The traditional name "Schröder–Bernstein" is based on two proofs published independently in 1898. Cantor is often added because he first stated the theorem in 1887, while Schröder's name is often omitted because his proof turned out to be flawed while the name of Richard Dedekind, who first proved it, is not connected with the theorem. According to Bernstein, Cantor had suggested the name *equivalence theorem* (Äquivalenzsatz).[5]

- **1887 Cantor** publishes the theorem, however without proof.[6][5]
- **1887** On July 11, **Dedekind** proves the theorem (not relying on the axiom of choice)[7] but neither publishes his proof nor tells Cantor about it. Ernst Zermelo discovered Dedekind's proof and in 1908[8] he publishes his own proof based on the *chain theory* from Dedekind's paper *Was sind und was sollen die Zahlen?*[5][9]

- **1895 Cantor** states the theorem in his first paper on set theory and transfinite numbers. He obtains it as an easy consequence of the linear order of cardinal numbers.[10][11] However, he couldn't prove the latter theorem, which is shown in 1915 to be equivalent to the axiom of choice by Friedrich Moritz Hartogs.[5][12]
- **1896 Schröder** announces a proof (as a corollary of a theorem by Jevons).[13]
- **1896 Schröder** publishes a proof sketch[14] which, however, is shown to be faulty by Alwin Reinhold Korselt in 1911[15] (confirmed by Schröder).[5][16]



4. Mitteilungen zur Lehre vom Transfiniten. VIII Nr. 2/4. 413

$\overline{\overline{M'}} = \overline{\overline{N}}$. Man hat den Satz: sind M und N zwei solche Mengen, daß Bestandteile M' und N' von ihnen abgesondert werden können, von denen sich zeigen läßt, daß $\overline{\overline{M}} = \overline{\overline{N'}}$ und $\overline{\overline{M'}} = \overline{\overline{N}}$, so sind M und N äquivalente Mengen.

Cantor's first statement of the theorem (1887)[6]

- **1897 Bernstein**, a 19 years old student in Cantor's Seminar, presents his proof.[17][18]
- **1897** Almost simultaneously, but independently, **Schröder** finds a proof.[17][18]
- **1897** After a visit by Bernstein, **Dedekind** independently proves the theorem a second time.
- **1898 Bernstein's** proof (not relying on the axiom of choice) is published by Émile Borel in his book on functions.[19] (Communicated by Cantor at the 1897 International Congress of Mathematicians in Zürich.) In the same year, the proof also appears in **Bernstein's** dissertation.[20][5]

Both proofs of Dedekind are based on his famous memoir *Was sind und was sollen die Zahlen?* and derive it as a corollary of a proposition equivalent to statement C in Cantor's paper,[10] which reads $A \subseteq B \subseteq C$ and $|A|=|C|$ implies $|A|=|B|=|C|$. Cantor observed this property as early as 1882/83 during his studies in set theory and transfinite numbers and was therefore (implicitly) relying on the Axiom of Choice.

See also

- Myhill isomorphism theorem
- Schröder–Bernstein theorem for measurable spaces
- Schröder–Bernstein theorems for operator algebras
- Schröder–Bernstein property

Notes

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External links

- Weisstein, Eric W. "Schröder-Bernstein Theorem" (http://mathworld.wolfram.com/Schroeder-BernsteinTheorem.html). *MathWorld*.
- Cantor-Schroeder-Bernstein theorem (https://ncatlab.org/nlab/show/Cantor-Schroeder-Bernstein+theorem) in *nLab*
- Cantor-Bernstein's Theorem in a Semiring (https://link.springer.com/content/pdf/10.1007%2Fs00283-011-9242-3.pdf) by Marcel Crabbé.
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