3-5 Minimum Spanning Trees

Hengfeng Wei

hfwei@nju.edu.cn

October 22, 2018



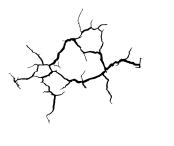


Be Careful with Your Proofs!!!

Theorem (A "Real" Theorem)

Theorem (A "Faked" Theorem)

Proof.



Proof.



Cut Property

Cut Property (Version I)

X: A part of some MST T_1 of G

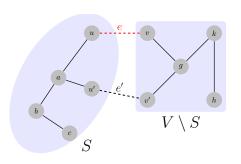
 $(S, V \setminus S)$: A cut such that X does not cross $(S, V \setminus S)$

e: A lightest edge across $(S, V \setminus S)$

Then $X \cup \{e\}$ is a part of some MST T_2 of G.

Correctness of Prim's and Kruskal's algorithms.

By Exchange Argument.



$$T' = \underbrace{T}_{X \subseteq T} + \{e\} - \{e'\}$$
if $e \notin T$

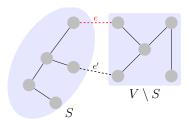
"a" \rightarrow "the" \Longrightarrow "some" \rightarrow "all"

Cut Property (Version II)

A cut
$$(S, V \setminus S)$$

Let e = (u, v) be a lightest edge across $(S, V \setminus S)$

\exists MST T of $G: e \in T$



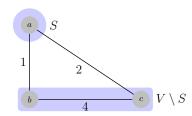
$$T' = \underbrace{T + \{e\}}_{\text{if } e \notin T} - \{e'\}$$

"a"
$$\rightarrow$$
 "the" \Longrightarrow " \exists " \rightarrow " \forall "

A Wrong Divide&Conquer Algorithm for MST

$$(V_1, V_2) : ||V_1| - |V_2|| \le 1$$

 $T_1 + T_2 + \{e\}$: e is a lightest edge across (V_1, V_2)

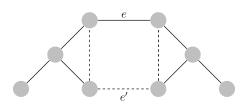


Cycle Property

Cycle Property

- \blacktriangleright Let C be any cycle in G
- ▶ Let e = (u, v) be a maximum-weight edge in C

Then \exists MST T of $G: e \notin T$.



$$T' = \underbrace{T - \{e\}}_{\text{if } e \in T} + \{e'\}$$

"a" \rightarrow "the" \Longrightarrow " \exists " \rightarrow " \forall "

Anti-Kruskal Algorithm

Reverse-delete algorithm (wiki; clickable)

$$O(m \log n ((\log \log n)^3))$$

Proof.

Cycle Property

$$T \subseteq F \implies \exists T' : T' \subseteq F - \{e\}$$

"On the Shortest Spanning Subtree of a Graph and the Traveling Salesman Problem"

— Kruskal, 1956.

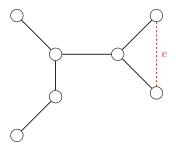
Application of Cycle Property (Problem 4.30)

T: a MST of a connected weight graph G

T is a unique MST of G



 $\forall e \in G \setminus T : w(e) > w(\text{every other edge on the cycle in } T + e)$



Uniqueness of MST

Uniqueness of MST (Problem 4.29)

Distinct weights \implies Unique MST

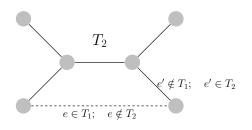
By Contradiction.

$$\exists$$
 MSTs $T_1 \neq T_2$

$$\Delta E = \{ e \mid e \in T_1 \setminus T_2 \lor e \in T_2 \setminus T_1 \}$$

$$e = \min \Delta E$$

$$e \in T_1 \setminus T_2 \ (w.l.o.g)$$



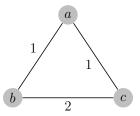
$$T_2 + \{e\} \implies C$$

$$\exists (e' \in C) \notin T_1 \implies e' \in T_2 \setminus T_1 \implies e' \in \Delta E \implies w(e') > w(e)$$

$$T' = T_2 + \{e\} - \{e'\} \implies w(T') < w(T_2)$$

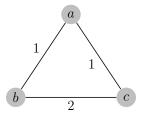
Condition for Uniqueness of MST

Unique MST \implies Distinct weights



Unique MST

Unique MST \implies Minimum-weight edge across any cut is unique

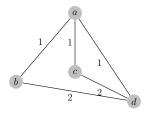


Theorem (After-class Exercise)

Minimum-weight edge across any cut is unique \implies $Unique\ MST$

Unique MST

Unique MST \implies Maximum-weight edge in any cycle is unique



Theorem (After-class Exercise)

Maximum-weight edge in any cycle is unique \implies Unique MST







Existence of Cycle (Problem 4.8)

$$\forall v \in V(G) : \deg(v) \geq 2 \implies G \text{ contains a cycle}$$

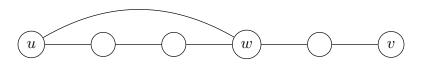
By Contradiction.

$$m = n - k(G) \le n - 1$$

$$\sum_{v \in V(G)} \deg(v) \le 2(n-1)$$

Existence of Cycle (Problem 4.8)

$$\forall v \in V(G) : \deg(v) \geq 2 \implies G \text{ contains a cycle}$$



maximal path $P_{u,v}$

Existence of Cycle (Problem 4.8)

$$\forall v \in V(G) : \deg(v) \geq 2 \implies G \text{ contains a cycle}$$

$$\forall v \in V(G) : \deg(v) \ge 2$$

$$\implies \nexists v \in V(G) : \deg(v) = 1$$

 $\xrightarrow{\text{Theorem } 4.3} G$ is not a tree

Consider each component G' of G

 $\xrightarrow{\text{Theorem } 4.2} \exists u, v \in V(G') : u, v \text{ are connected by } \geq 2 \text{ paths}$

- $\implies G'$ contains a cycle
- \implies G contains a cycle

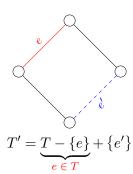
Bridge and Spanning Trees (Problem 4.26)

G: a connected graph

 $e \in E(G)$ is a bridge $\iff e \in \forall ST \text{ of } G$

By Contradiction.

ST of G - e





MST from the point of view of greedy-algorithm

MIT 6.046J: "Design and Analysis of Algorithms", Spring 2015





Office 302

Mailbox: H016

hfwei@nju.edu.cn