

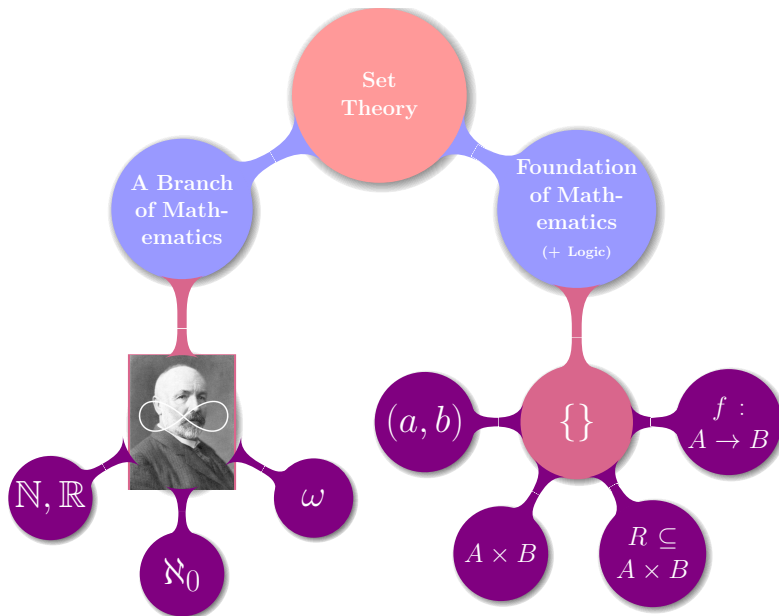
Functions

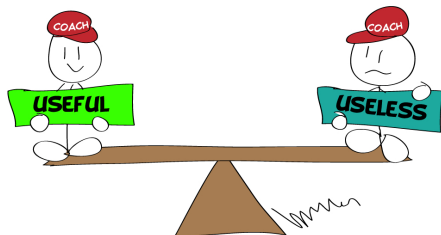
魏恒峰

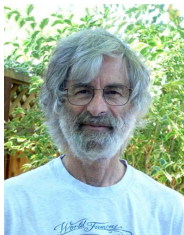
hfwei@nju.edu.cn

2018 年 02 月 26 日









Time, Clocks, and the Ordering of Events in a Distributed System

Leslie Lamport
Massachusetts Computer Associates, Inc.

The concept of one event happening before another in a distributed system is examined, and is shown to define a partial ordering of the events. A distributed algorithm is given for synchronizing a system of logical clocks which can be used to totally order the events.

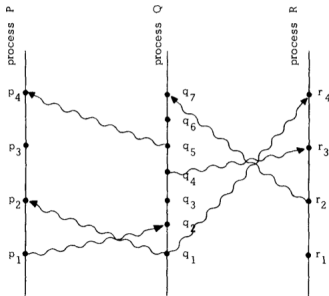


Figure 13. A selection of consistency axioms over an execution $(E, \text{repl}, \text{obj}, \text{oper}, \text{rval}, \text{ro}, \text{vis}, \text{ar})$

Auxiliary relations

$\text{sameobj}(e, f) \iff \text{obj}(e) = \text{obj}(f)$

Per-object causality (aka happens-before) order:

$$\text{hbo} = ((\text{ro} \cap \text{sameobj}) \cup \text{vis})^+$$

Causality (aka happens-before) order: $\text{hb} = (\text{ro} \cup \text{vis})^+$

Axioms

EVENTUAL:

$$\forall e \in E. \neg(\exists \text{ infinitely many } f \in E. \text{sameobj}(e, f) \wedge \neg(e \xrightarrow{\text{vis}} f))$$

THINAIR: $\text{ro} \cup \text{vis}$ is acyclic

POCV (Per-Object Causal Visibility): $\text{hbo} \subseteq \text{vis}$

POCA (Per-Object Causal Arbitration): $\text{hbo} \subseteq \text{ar}$

COCV (Cross-Object Causal Visibility): $(\text{hb} \cap \text{sameobj}) \subseteq \text{vis}$

COCA (Cross-Object Causal Arbitration): $\text{hb} \cup \text{ar}$ is acyclic



Figure 13. A selection of consistency axioms over an execution $(E, \text{repl}, \text{obj}, \text{oper}, \text{rval}, \text{ro}, \text{vis}, \text{ar})$

Auxiliary relations

sameobj(e, f) $\iff \text{obj}(e) = \text{obj}(f)$

Per-object causality (aka happens-before) order:

$\text{hbo} = ((\text{ro} \cap \text{sameobj}) \cup \text{vis})^+$

Causality (aka happens-before) order: $\text{hb} = (\text{ro} \cup \text{vis})^+$

Axioms

EVENTUAL:

$\forall e \in E. \neg(\exists \text{infinitely many } f \in E. \text{sameobj}(e, f) \wedge \neg(e \xrightarrow{\text{vis}} f))$

THINAIR: $\text{ro} \cup \text{vis}$ is acyclic

POCV (Per-Object Causal Visibility): $\text{hbo} \subseteq \text{vis}$

POCA (Per-Object Causal Arbitration): $\text{hbo} \subseteq \text{ar}$

COCV (Cross-Object Causal Visibility): $(\text{hb} \cap \text{sameobj}) \subseteq \text{vis}$

COCA (Cross-Object Causal Arbitration): $\text{hb} \cup \text{ar}$ is acyclic

Figure 17. Optimized state-based multi-value register and its simulation

$$\begin{aligned} \Sigma &= \text{ReplicatedID} \times \mathcal{P}(\mathbb{Z} \times (\text{ReplicatedID} \rightarrow \mathbb{N}_0)) \\ \delta_0 &= (r, \emptyset) \\ M &= \mathcal{P}(\mathbb{Z} \times (\text{ReplicatedID} \rightarrow \mathbb{N}_0)) \\ \text{do}(w(a), (r, V), t) &= \langle (r, \{(a, s) \mid s \neq r \text{ then } \max\{v(s) \mid (a, v) \in V\} \\ &\quad \text{else } \max\{v(s) \mid (a, v) \in V\} + 1\}), \perp \rangle \\ \text{del}(r, (r, V), t) &= ((r, V'), \{(a, v) \mid (a, v) \in V\}) \\ \text{send}((r, V)) &= ((r, V'), V) \\ \text{receive}((r, V), V') &= \langle (r, (a, v) \in V^m) \\ &\quad \vee \bigcup_{j \in \mathbb{Z}} \bigcup_{k \in \mathbb{Z}} \{ \exists a'. (a', v') \in V^m \wedge a \neq a' \} \rangle, \\ &\text{where } V^m = \{(a, \bigcup_{j \in \mathbb{Z}} \{v' \mid (a, v') \in V \cup V^m\} \mid (a, v) \in V \cup V^m\} \\ (a, V) [\mathbb{R}_c] \cdot t \leftrightarrow (a, V') \wedge (V' [M] t) \\ V' [M] \langle (E, \text{repl}, \text{obj}, \text{oper}, \text{rval}, \text{ro}, \text{vis}, \text{ar}), \text{info} \rangle \iff \\ &(\forall(a, v), (a', v') \in V. (a = a' \implies v = v') \wedge \\ &(\forall(a, v) \in V. \exists a, v(s) > 0) \wedge \\ &(\forall(a, v) \in V. \exists a, v(s) > 0) \wedge \\ &\exists \text{distinct } e_{a,k} \\ &(\{e \in E \mid \exists a. \text{oper}(e) = \text{wr}(a)\} = \{e_{a,k} \mid a \in \text{ReplicatedID} \wedge \\ &1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V\}\}) \wedge \\ &(\forall a, j, k. (\text{repl}(e_{a,k}) = a) \wedge (e_{a,j} \xrightarrow{\text{ro}} e_{a,k} \iff j < k)) \wedge \\ &(\forall(a, v) \in V. \forall j. \{j \mid \text{oper}(e_{a,j}) = \text{wr}(a)\} \cup \\ &\{j \mid \exists a, k. e_{a,j} \xrightarrow{\text{ro}} e_{a,k} \wedge \text{oper}(e_{a,k}) = \text{wr}(a)\} = \\ &\{j \mid 1 \leq j \leq v(a)\}) \wedge \\ &(\forall e \in E. (\text{oper}(e) = \text{wr}(a)) \wedge \\ &\neg \exists f \in E. \text{oper}(f) = \text{wr}(a) \wedge e \xrightarrow{\text{ro}} f) \implies (a, v) \in V) \end{aligned}$$

the form. The only non-trivial obligation is to show that if

$$V' [M] \langle (E, \text{repl}, \text{obj}, \text{oper}, \text{rval}, \text{ro}, \text{vis}), \text{info} \rangle,$$

then

$$\{a \mid (a, v) \in V\} \subseteq \{a \mid \exists e \in E. \text{oper}(e) = \text{wr}(a) \wedge \neg \exists f \in E. \exists a'. \text{oper}(e) = \text{wr}(a') \wedge e \xrightarrow{\text{ro}} f\} \quad (13)$$

(the reverse inclusion is straightforwardly implied by \mathbb{R}_c).

$$\text{Take } (a, v) \in V. \text{ We have } \forall(a, v) \in V. \exists a, v(s) > 0. \\ v \in \bigcup_{j \in \mathbb{Z}} \{v' \mid \exists a'. (a', v') \in V \wedge a \neq a'\}$$

and

$$\begin{aligned} \forall(a, v) \in V. \forall j. \{j \mid \text{oper}(e_{a,j}) = \text{wr}(a)\} \cup \\ \{j \mid \exists a, k. e_{a,j} \xrightarrow{\text{ro}} e_{a,k} \wedge \text{oper}(e_{a,k}) = \text{wr}(a)\} = \\ \{j \mid 1 \leq j \leq v(a)\}. \end{aligned}$$

From this we get that for some $e \in E$

$$\begin{aligned} \text{oper}(e) = \text{wr}(a) \wedge \neg \exists f \in E. \exists a'. a' \neq a \wedge \\ \text{oper}(e) = \text{wr}(a') \wedge e \xrightarrow{\text{ro}} f. \end{aligned}$$

Since vis is acyclic, this implies that for some $e' \in E$

$$\text{oper}(e') = \text{wr}(a) \wedge \neg \exists f \in E. \text{oper}(e') = \text{wr}(a') \wedge e' \xrightarrow{\text{ro}} f,$$

which establishes (13).

Let us now discharge **RECEIVE**. Let $\text{receive}((r, V), V') = (r, V'')$, where

$$\begin{aligned} V'' = \{(a, \bigcup_{j \in \mathbb{Z}} \{v' \mid (a, v') \in V \cup V''\}) \mid (a, v) \in V \cup V''\}; \\ V''' = \{(a, v) \in V'' \mid v \in \bigcup_{j \in \mathbb{Z}} \{(a', v') \in V'' \mid a \neq a'\}\}. \end{aligned}$$

Assume $(r, V) [\mathbb{R}_c] f, V' [M] J$ and

$$\begin{aligned} I &= ((E, \text{repl}, \text{obj}, \text{oper}, \text{rval}, \text{ro}, \text{vis}, \text{ar}), \text{info}); \\ J &= ((E', \text{repl}', \text{obj}', \text{oper}', \text{rval}', \text{ro}', \text{vis}', \text{ar}'), \text{info}'); \\ I \sqcup J &= ((E'', \text{repl}'', \text{obj}'', \text{oper}'', \text{rval}'', \text{ro}'', \text{vis}'', \text{ar}''), \text{info}''). \end{aligned}$$

By agree we have $I \sqcup J \in \mathbb{R}_c$. Then

$$\begin{aligned} &(\forall(a, v), (a', v') \in V. (a = a' \implies v = v') \wedge \\ &(\forall(a, v) \in V. \exists a, v(s) > 0)) \wedge \\ &(\forall(a, v) \in V. v \in \bigcup_{j \in \mathbb{Z}} \{v' \mid \exists a'. (a', v') \in V \wedge a \neq a'\}) \wedge \\ &\exists \text{distinct } e_{a,k} \\ &(\{e \in E' \mid \exists a. \text{oper}'(e) = \text{wr}(a)\} = \{e_{a,k} \mid a \in \text{ReplicatedID} \wedge \\ &1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V\}\}) \wedge \\ &(\forall a, j, k. (\text{repl}'(e_{a,k}) = a) \wedge (e_{a,j} \xrightarrow{\text{ro}'} e_{a,k} \iff j < k)) \wedge \\ &(\forall(a, v) \in V. \forall j. \{j \mid \text{oper}'(e_{a,j}) = \text{wr}(a)\} \cup \\ &\{j \mid \exists a, k. e_{a,j} \xrightarrow{\text{ro}'} e_{a,k} \wedge \text{oper}'(e_{a,k}) = \text{wr}(a)\} = \\ &\{j \mid 1 \leq j \leq v(a)\}) \wedge \\ &(\forall e \in E. (\text{oper}'(e) = \text{wr}(a)) \wedge \\ &\neg \exists f \in E. \text{oper}'(f) = \text{wr}(a) \wedge e \xrightarrow{\text{ro}'} f) \implies (a, v) \in V) \end{aligned}$$

and

$$\begin{aligned} &(\forall(a, v), (a', v') \in V'. (a = a' \implies v = v') \wedge \\ &(\forall(a, v) \in V'. \exists a, v(s) > 0) \wedge \\ &(\forall(a, v) \in V'. v \in \bigcup_{j \in \mathbb{Z}} \{v' \mid \exists a'. (a', v') \in V' \wedge a \neq a'\}) \wedge \\ &\exists \text{distinct } e_{a,k} \\ &(\{e \in E' \mid \exists a. \text{oper}'(e) = \text{wr}(a)\} = \{e_{a,k} \mid a \in \text{ReplicatedID} \wedge \\ &1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V'\}\}) \wedge \\ &(\forall a, j, k. (\text{repl}'(e_{a,k}) = a) \wedge (e_{a,j} \xrightarrow{\text{ro}'} e_{a,k} \iff j < k)) \wedge \\ &(\forall(a, v) \in V'. \forall j. \{j \mid \text{oper}'(e_{a,j}) = \text{wr}(a)\} \cup \\ &\{j \mid \exists a, k. e_{a,j} \xrightarrow{\text{ro}'} e_{a,k} \wedge \text{oper}'(e_{a,k}) = \text{wr}(a)\} = \\ &\{j \mid 1 \leq j \leq v(a)\}) \wedge \\ &(\forall e \in E'. (\text{oper}'(e) = \text{wr}(a)) \wedge \\ &\neg \exists f \in E'. \text{oper}'(f) = \text{wr}(a) \wedge e \xrightarrow{\text{ro}'} f) \implies (a, v) \in V'). \end{aligned}$$

The agree property also implies

$$\forall a, k. 1 \leq k \leq \min \{ \max\{v(s) \mid \exists a. (a, v) \in V\}, \max\{v(s) \mid \exists a. (a, v) \in V'\} \} \implies e_{a,k} = e'_{a,k}.$$

Hence, there exist distinct

$$\begin{aligned} e''_{a,s} \text{ for } s \in \text{ReplicatedID}, k = 1..(\max\{v(s) \mid \exists a. (a, v) \in V''\}), \\ \text{such that} \\ &(\forall a, k. 1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V\} \implies e''_{a,k} = e_{a,k}) \wedge \\ &(\forall a, k. 1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V'\} \implies e''_{a,k} = e'_{a,k}) \wedge \\ &\text{and} \\ &(\{e \in E \cup E' \mid \exists a. \text{oper}''(e) = \text{wr}(a)\} = \\ &\{e''_{a,k} \mid s \in \text{ReplicatedID} \wedge 1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V''\}\}) \\ &\wedge (\forall a, j, k. (\text{repl}''(e''_{a,k}) = a) \wedge (e''_{a,j} \xrightarrow{\text{ro}''} e''_{a,k} \iff j < k)). \end{aligned}$$

By the definition of V'' and V''' we have

$$\forall(a, v), (a', v') \in V'''. (a = a' \implies v = v').$$

We also straightforwardly get

$$\forall(a, v) \in V'''. \exists a, v(s) > 0$$

and

$$\begin{aligned} &(\forall(a, v) \in V'''. \forall j. \{j \mid \text{oper}''(e''_{a,j}) = \text{wr}(a)\} \cup \\ &\{j \mid \exists a, k. e''_{a,j} \xrightarrow{\text{ro}''} e''_{a,k} \wedge \text{oper}''(e''_{a,k}) = \text{wr}(a)\} = \\ &\{j \mid 1 \leq j \leq v(a)\}). \end{aligned} \quad (14)$$

Function



Function



(1) from the perspective of set theory

Function



- (1) from the perspective of set theory
- (2) PROOF! PROOF! PROOF!

Definition of Function

Definition (Relation)

Let A and B be sets.

R is a (binary) relation if

$$R \subseteq A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

Definition (Function)

Let A and B be sets.

A **function** f from A to B is a *relation* f from A to B such that

$$\forall a \in A \exists! b \in B (a, b) \in f.$$

Definition (Function)

Let A and B be sets.

A **function** f from A to B is a *relation* f from A to B such that

$$\forall a \in A \exists! b \in B (a, b) \in f.$$

For Proof:

\forall

$$\exists! : \forall b, b' \in B, (a, b) \in f \wedge (a, b') \in f \implies b = b'.$$

Definition (Function)

Let A and B be sets.

A **function** f from A to B is a **relation** f from A to B such that

$$\forall a \in A \exists! b \in B (a, b) \in f.$$

For Proof:

\forall

$$\exists! : \forall b, b' \in B, (a, b) \in f \wedge (a, b') \in f \implies b = b'.$$

$$f : A \rightarrow B, \quad a \mapsto f(a) \quad (b = f(a))$$

$$A : \text{dom}(f) \quad B : \text{cod}(f)$$

Definition (Function)

Let A and B be sets.

A **function** f from A to B is a **relation** f from A to B such that

$$\forall a \in A \exists! b \in B (a, b) \in f.$$

For Proof:

\forall

$$\exists! : \forall b, b' \in B, (a, b) \in f \wedge (a, b') \in f \implies b = b'.$$

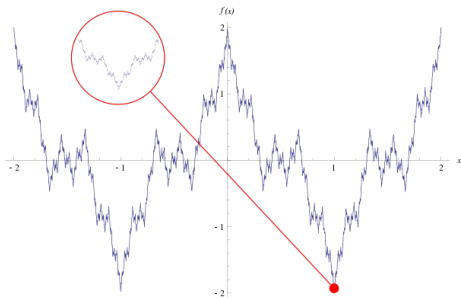
$$f : A \rightarrow B, \quad a \mapsto f(a) \quad (b = f(a))$$

$$A : \text{dom}(f) \quad B : \text{cod}(f)$$

$$\text{ran}(f) = f(A) = \{f(a) \mid a \in A\} \subseteq B$$

$$D(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Dirichlet Function



Weierstrass Function (1872)

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$

$$0 < a < 1, \quad b \in 2\mathbb{N} + 1, \quad ab > 1 + \frac{3}{2}\pi$$

Problem 13.3 (g)

$$f : \mathbb{Q} \rightarrow \mathbb{R}$$

$$f(x) = \begin{cases} x + 1 & \text{if } x \in 2\mathbb{Z} \\ x - 1 & \text{if } x \in 3\mathbb{Z} \\ 2 & \text{otherwise} \end{cases}$$

Problem 13.4

$$f : \mathcal{P}(\mathbb{R}) \rightarrow \mathbb{Z}$$

$$f(A) = \begin{cases} \min(A \cap \mathbb{N}) & \text{if } A \cap \mathbb{N} \neq \emptyset \\ -1 & \text{if } A \cap \mathbb{N} = \emptyset \end{cases}$$

A function $f : A \rightarrow B$ is a set.

$$f \subseteq A \times B$$

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

$$(a, b) = \{\{a\}, \{a, b\}\}$$

Definition (Axiom of Extensionality (集合的外延公理))

$$\forall A \forall B \forall x (x \in A \iff x \in B) \iff A = B.$$

Definition (Axiom of Extensionality (集合的外延公理))

$$\forall A \forall B \forall x (x \in A \iff x \in B) \iff A = B.$$

Definition (函数的外延性原则)

$$f = g \iff \text{dom}(f) = \text{dom}(g) \wedge (\forall x \in \text{dom}(f) : f(x) = g(x))$$

Special Functions (*-jectivity*)

Definition (Injective (one-to-one; 1-1) 单射函数)

$$f : A \rightarrow B$$

$$\forall a_1, a_2 \in A : a_1 \neq a_2 \implies f(a_1) \neq f(a_2)$$

Definition (Injective (one-to-one; 1-1) 单射函数)

$$f : A \rightarrow B \quad f : A \rightarrowtail B$$

$$\forall a_1, a_2 \in A : a_1 \neq a_2 \implies f(a_1) \neq f(a_2)$$

Definition (Injective (one-to-one; 1-1) 单射函数)

$$f : A \rightarrow B \quad f : A \rightarrowtail B$$

$$\forall a_1, a_2 \in A : a_1 \neq a_2 \implies f(a_1) \neq f(a_2)$$

For Proof:

- To prove that f *is* 1-1:

$$\forall a_1, a_2 \in A : f(a_1) = f(a_2) \implies a_1 = a_2$$

Definition (Injective (one-to-one; 1-1) 单射函数)

$$f : A \rightarrow B \quad f : A \rightarrowtail B$$

$$\forall a_1, a_2 \in A : a_1 \neq a_2 \implies f(a_1) \neq f(a_2)$$

For Proof:

- ▶ To prove that f *is* 1-1:

$$\forall a_1, a_2 \in A : f(a_1) = f(a_2) \implies a_1 = a_2$$

- ▶ To show that f *is not* 1-1:

$$\exists a_1, a_2 \in A : a_1 \neq a_2 \wedge f(a_1) = f(a_2)$$

Definition (Surjective (onto) 满射函数)

$$f : A \rightarrow B$$

$$\text{ran}(f) = B$$

Definition (Surjective (onto) 满射函数)

$$f : A \rightarrow B \quad f : A \twoheadrightarrow B$$

$$\text{ran}(f) = B$$

Definition (Surjective (onto) 满射函数)

$$f : A \rightarrow B \quad f : A \twoheadrightarrow B$$

$$\text{ran}(f) = B$$

For Proof:

- To prove that f *is* onto:

$$\forall b \in B \left(\exists a \in A : f(a) = b \right)$$

Definition (Surjective (onto) 满射函数)

$$f : A \rightarrow B \quad f : A \twoheadrightarrow B$$

$$\text{ran}(f) = B$$

For Proof:

- ▶ To prove that f *is* onto:

$$\forall b \in B \left(\exists a \in A : f(a) = b \right)$$

- ▶ To show that f *is not* onto:

$$\exists b \in B \left(\forall a \in A : f(a) \neq b \right)$$

Theorem (Cantor Theorem (ES Theorem 24.4))

Let A be a set.

If $f : A \rightarrow 2^A$, then f is not onto.

Proof.

Proof. Let A be a set and let $f : A \rightarrow 2^A$. To show that f is not onto, we must find a $B \in 2^A$ (i.e., $B \subseteq A$) for which there is no $a \in A$ with $f(a) = B$. In other words, B is a set that f “misses.” To this end, let

$$B = \{x \in A : x \notin f(x)\}.$$

We claim there is no $a \in A$ with $f(a) = B$.

Suppose, for the sake of contradiction, there is an $a \in A$ such that $f(a) = B$. We ponder: Is $a \in B$?

- If $a \in B$, then, since $B = f(a)$, we have $a \in f(a)$. So, by definition of B , $a \notin f(a)$; that is, $a \notin B \Rightarrow \Leftarrow$
- If $a \notin B = f(a)$, then, by definition of B , $a \in B \Rightarrow \Leftarrow$

Both $a \in B$ and $a \notin B$ lead to contradictions, and hence our supposition [there is an $a \in A$ with $f(a) = B$] is false, and therefore f is not onto. ■



Theorem (Cantor Theorem)

Let A be a set.

If $f : A \rightarrow 2^A$, then f is not onto.

Theorem (Cantor Theorem)

Let A be a set.

If $f : A \rightarrow 2^A$, then f is not onto.



Theorem (Cantor Theorem)

Let A be a set.

If $f : A \rightarrow 2^A$, then f is not onto.



Theorem (Cantor Theorem)

Let A be a set.

If $f : A \rightarrow 2^A$, then f is not onto.



Theorem (Cantor Theorem)

Let A be a set.

If $f : A \rightarrow 2^A$, then f is not onto.



Theorem (Cantor Theorem)

Let A be a set.

If $f : A \rightarrow 2^A$, then f is not onto.

Understanding this problem:

$$A = \{1, 2, 3\}$$

Theorem (Cantor Theorem)

Let A be a set.

If $f : A \rightarrow 2^A$, then f is not onto.

Understanding this problem:

$$A = \{1, 2, 3\}$$

$$2^A$$

$$2^A = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Theorem (Cantor Theorem)

Let A be a set.

If $f : A \rightarrow 2^A$, then f is not onto.

Understanding this problem:

$$A = \{1, 2, 3\}$$

$$2^A$$

$$2^A = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Onto

$$\forall B \in 2^A \left(\exists a \in A \ f(a) = B \right).$$

Theorem (Cantor Theorem)

Let A be a set.

If $f : A \rightarrow 2^A$, then f is not onto.

Understanding this problem:

$$A = \{1, 2, 3\}$$

$$2^A$$

$$2^A = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Onto

$$\forall B \in 2^A \left(\exists a \in A \ f(a) = B \right).$$

Not Onto

$$\exists B \in 2^A \left(\forall a \in A \ f(a) \neq B \right).$$

Theorem (Cantor Theorem)

Let A be a set.

If $f : A \rightarrow 2^A$, then f is not onto.

Proof.



Theorem (Cantor Theorem)

Let A be a set.

If $f : A \rightarrow 2^A$, then f is not onto.

Proof.

- ▶ Constructive proof (\exists):

$$B = \{x \in A \mid x \notin f(x)\}.$$



Theorem (Cantor Theorem)

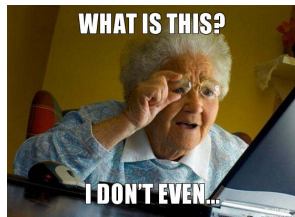
Let A be a set.

If $f : A \rightarrow 2^A$, then f is not onto.

Proof.

- Constructive proof (\exists):

$$B = \{x \in A \mid x \notin f(x)\}.$$



Theorem (Cantor Theorem)

Let A be a set.

If $f : A \rightarrow 2^A$, then f is not onto.

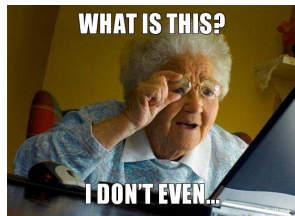
Proof.

- ▶ Constructive proof (\exists):

$$B = \{x \in A \mid x \notin f(x)\}.$$

- ▶ By contradiction (\forall):

$$\exists a \in A : f(a) = B.$$



Theorem (Cantor Theorem)

Let A be a set.

If $f : A \rightarrow 2^A$, then f is not onto.

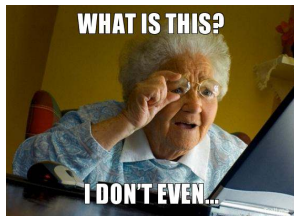
Proof.

- ▶ Constructive proof (\exists):

$$B = \{x \in A \mid x \notin f(x)\}.$$

- ▶ By contradiction (\forall):

$$\exists a \in A : f(a) = B.$$



$$Q : a \in B?$$



Theorem (Cantor Theorem)

Let A be a set.

If $f : A \rightarrow 2^A$, then f is not onto.

对角线论证 (Cantor's diagonal argument) .

Theorem (Cantor Theorem)

Let A be a set.

If $f : A \rightarrow 2^A$, then f is not onto.

对角线论证 (Cantor's diagonal argument) .

a	$f(a)$					
	1	2	3	4	5	...
1	1	1	0	0	1	...
2	0	0	0	0	0	...
3	1	0	0	1	0	...
4	1	1	1	1	1	...
5	0	1	0	1	0	...
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	...



Theorem (Cantor Theorem)

Let A be a set.

If $f : A \rightarrow 2^A$, then f is not onto.

对角线论证 (Cantor's diagonal argument) .

a	$f(a)$					
	1	2	3	4	5	...
1	1	1	0	0	1	...
2	0	0	0	0	0	...
3	1	0	0	1	0	...
4	1	1	1	1	1	...
5	0	1	0	1	0	...
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	...



Theorem (Cantor Theorem)

Let A be a set.

If $f : A \rightarrow 2^A$, then f is not onto.

对角线论证 (Cantor's diagonal argument) .

a	$f(a)$					
	1	2	3	4	5	...
1	1	1	0	0	1	...
2	0	0	0	0	0	...
3	1	0	0	1	0	...
4	1	1	1	1	1	...
5	0	1	0	1	0	...
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	...

$$B = \{0, 1, 1, 0, 1\}$$



Theorem (Cantor Theorem)

Let A be a set.

If $f : A \rightarrow 2^A$, then f is not onto.

对角线论证 (Cantor's diagonal argument) (以下仅适用于可数集合 A).

a	$f(a)$					
	1	2	3	4	5	...
1	1	1	0	0	1	...
2	0	0	0	0	0	...
3	1	0	0	1	0	...
4	1	1	1	1	1	...
5	0	1	0	1	0	...
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	...

$$B = \{0, 1, 1, 0, 1\}$$



Definition (Bijective (one-to-one correspondence) 一一对应)

$$f : A \rightarrow B$$

1-1 & onto

Definition (Bijective (one-to-one correspondence) 一一对应)

$$f : A \rightarrow B \quad f : A \overset{\text{1-1}}{\underset{\text{onto}}{\longleftrightarrow}} B$$

1-1 & onto

Problem 14.12

$$a, b, c, d \in \mathbb{R}, a < b, c < d$$

Define a bijective function:

$$f : [a, b] \xrightarrow[\text{onto}]{1-1} [c, d]$$

$$f : (a, b) \xrightarrow[\text{onto}]{1-1} (c, d)$$

Answer.

$$f(x) = c + \frac{d-c}{b-a}(x-a)$$



Operations on Functions

Operations on Functions

Set

\cup \cap \subseteq

Relation

○ $f^{-1}(a)$ $f(A) \& f^{-1}(B)$

Definition (Intersection, Union)

$$f_1, f_2 : A \rightarrow B$$

- (i) Q : Is $f_1 \cup f_2$ a function from A to B ?
- (ii) Q : Is $f_1 \cap f_2$ a function from A to B ?

Definition (Intersection, Union)

$$f_1, f_2 : A \rightarrow B$$

- (i) Q : Is $f_1 \cup f_2$ a function from A to B ?
- (ii) Q : Is $f_1 \cap f_2$ a function from A to B ?

Definition (Restriction (Problem 15.20))

$$f : A \rightarrow B, A_0 \subseteq A$$

$$f|_{A_0} : A_0 \rightarrow B, \quad f|_{A_0}(a) = f(a), \forall a \in A_0$$

Definition (Composition)

$$f : A \rightarrow B \quad g : C \rightarrow D$$

$$\text{ran}(f) \subseteq C$$

The composition function

$$g \circ f : A \rightarrow D$$

$$(g \circ f)(x) = g(f(x))$$

Definition (Composition)

$$f : A \rightarrow B \quad g : C \rightarrow D$$

$$\text{ran}(f) \subseteq C$$

The composition function

$$g \circ f : A \rightarrow D$$

$$(g \circ f)(x) = g(f(x))$$

Non-commutative:

$$f \circ g \neq g \circ f$$

Theorem (Associative Property for Composition)

$$f : A \rightarrow B \quad g : B \rightarrow C \quad h : C \rightarrow D$$

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Theorem (Associative Property for Composition)

$$f : A \rightarrow B \quad g : B \rightarrow C \quad h : C \rightarrow D$$

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Proof.

Theorem (Associative Property for Composition)

$$f : A \rightarrow B \quad g : B \rightarrow C \quad h : C \rightarrow D$$

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Proof.

(i)

$$\text{dom}(h \circ (g \circ f)) = \text{dom}((h \circ g) \circ f)$$

(ii)

$$(h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x)$$



Theorem (Properties of Composition (UD Theorem 15.7))

$$f : A \rightarrow B \quad g : B \rightarrow C$$

- (i) *If f, g are injective, then $g \circ f$ is injective.*
- (ii) *If f, g are surjective, then $g \circ f$ is surjective.*
- (iii) *If f, g are bijective, then $g \circ f$ is bijective.*

Theorem (Properties of Composition (UD Theorem 15.7))

$$f : A \rightarrow B \quad g : B \rightarrow C$$

- (i) *If f, g are injective, then $g \circ f$ is injective.*
- (ii) *If f, g are surjective, then $g \circ f$ is surjective.*
- (iii) *If f, g are bijective, then $g \circ f$ is bijective.*

Proof for (i).

Theorem (Properties of Composition (UD Theorem 15.7))

$$f : A \rightarrow B \quad g : B \rightarrow C$$

- (i) *If f, g are injective, then $g \circ f$ is injective.*
- (ii) *If f, g are surjective, then $g \circ f$ is surjective.*
- (iii) *If f, g are bijective, then $g \circ f$ is bijective.*

Proof for (i).

$$\forall a_1, a_2 \in A \left((g \circ f)(a_1) = (g \circ f)(a_2) \implies a_1 = a_2 \right)$$



Theorem (Properties of Composition (UD Theorem 15.8))

$$f : A \rightarrow B \quad g : B \rightarrow C$$

- (i) *If $g \circ f$ is injective, then f is injective.*
- (ii) *If $g \circ f$ is surjective, then g is surjective.*
- (iii) *If $g \circ f$ is bijective, then f is injective and g is surjective.*

Theorem (Properties of Composition (UD Theorem 15.8))

$$f : A \rightarrow B \quad g : B \rightarrow C$$

- (i) *If $g \circ f$ is injective, then f is injective.*
- (ii) *If $g \circ f$ is surjective, then g is surjective.*
- (iii) *If $g \circ f$ is bijective, then f is injective and g is surjective.*

Proof.

Left as Exercise (15.9).



Cancellation Property for Composition (Problem 15.11)

$$f : A \rightarrow B \quad g_1, g_2 : B \rightarrow A$$

$$f \circ g_1 = f \circ g_2 \wedge f \text{ is bijective} \implies g_1 = g_2$$

Cancellation Property for Composition (Problem 15.11)

$$f : A \rightarrow B \quad g_1, g_2 : B \rightarrow A$$

$$f \circ g_1 = f \circ g_2 \wedge f \text{ is bijective} \implies g_1 = g_2$$

Remark:

f is one-to-one.

Cancellation Property for Composition (Problem 15.11)

$$f : A \rightarrow B \quad g_1, g_2 : B \rightarrow A$$

$$f \circ g_1 = f \circ g_2 \wedge f \text{ is bijective} \implies g_1 = g_2$$

Remark:

f is one-to-one.

Proof.

Cancellation Property for Composition (Problem 15.11)

$$f : A \rightarrow B \quad g_1, g_2 : B \rightarrow A$$

$$f \circ g_1 = f \circ g_2 \wedge f \text{ is bijective} \implies g_1 = g_2$$

Remark:

f is one-to-one.

Proof.

$$\forall b \in B \left(f \circ g_1(b) = f \circ g_2(b) \implies \dots \right)$$



Definition (Inverse)

Let $f : A \rightarrow B$ be a **bijective** function.

The **inverse** of f is the function $f^{-1} : B \rightarrow A$ defined by

$$f^{-1}(b) = a \iff f(a) = b.$$

Definition (Inverse)

Let $f : A \rightarrow B$ be a **bijjective** function.

The **inverse** of f is the function $f^{-1} : B \rightarrow A$ defined by

$$f^{-1}(b) = a \iff f(a) = b.$$

Q: Why “Bijjective”?

Theorem (UD Theorem 15.4 (ii))

$f : A \rightarrow B$ is bijective $\implies f^{-1}$ is bijective.

Theorem (Solving Equations (UD Theorem 15.4))

$f : A \rightarrow B$ is bijective

(i) $f \circ f^{-1} = i_B$

(ii) $g : B \rightarrow A \wedge f \circ g = i_B \implies g = f^{-1}$

(iii) $f^{-1} \circ f = i_A$

(iv) $g : B \rightarrow A \wedge g \circ f = i_A \implies g = f^{-1}$

Theorem (Solving Equations (UD Theorem 15.4))

$f : A \rightarrow B$ is bijective

(i) $f \circ f^{-1} = i_B$

(ii) $g : B \rightarrow A \wedge f \circ g = i_B \implies g = f^{-1}$

(iii) $f^{-1} \circ f = i_A$

(iv) $g : B \rightarrow A \wedge g \circ f = i_A \implies g = f^{-1}$

Solving the equations:

$$f \circ g = i_B \quad g \circ f = i_A$$

Bijjective \implies Inverse:

$f : A \rightarrow B$ is bijective

\implies

$$\exists g : B \rightarrow A \left(f \circ g = i_B \wedge g \circ f = i_A \right)$$

Bijjective \implies Inverse:

$f : A \rightarrow B$ is bijective

\implies

$$\exists g : B \rightarrow A \left(f \circ g = i_B \wedge g \circ f = i_A \right) \wedge g = f^{-1}$$

Bijjective \implies Inverse:

$f : A \rightarrow B$ is bijective

\implies

$$\exists g : B \rightarrow A \left(f \circ g = i_B \wedge g \circ f = i_A \right) \wedge g = f^{-1}$$

Theorem (Inverse \implies Bijective (UD Theorem 15.8 (iii)))

$$\exists g : B \rightarrow A \left(g \circ f = i_A \wedge f \circ g = i_B \right)$$

\implies

$f : A \rightarrow B$ is bijective

Bijjective \implies Inverse:

$f : A \rightarrow B$ is bijective

\implies

$$\exists g : B \rightarrow A \left(f \circ g = i_B \wedge g \circ f = i_A \right) \wedge g = f^{-1}$$

Theorem (Inverse \implies Bijective (UD Theorem 15.8 (iii)))

$$\exists g : B \rightarrow A \left(g \circ f = i_A \wedge f \circ g = i_B \right)$$

\implies

$$f : A \rightarrow B \text{ is bijective} \wedge g = f^{-1}$$

Theorem (Inverse of Composition (UD Theorem 15.6))

$f : A \rightarrow B, g : B \rightarrow C$ are bijective

(i) $g \circ f$ is bijective

(ii) $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Proof for (ii).

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = i_A$$

$$(g \circ f) \circ (f^{-1} \circ g^{-1}) = i_B$$



$$f : A \rightarrow B, A_0 \subseteq A, B_0 \subseteq B$$

Definition (Image)

The **image** of A_0 under f is the set

$$f(A_0) = \{f(a) \mid a \in A_0\}.$$

Definition (Inverse Image)

The **inverse image** of B_0 under f is the set

$$f^{-1}(B_0) = \{a \in A \mid f(a) \in B_0\}.$$

Theorem (Properties of f and f^{-1} (Theorem 16.7))

$$f : A \rightarrow B, A_0, A_1, A_2 \subseteq A, B_0, B_1, B_2 \subseteq B$$

(i) f , when applied to subsets of A , preserves only " \subseteq " and \cup :

(1) $A_1 \subseteq A_2 \implies f(A_1) \subseteq f(A_2)$

(2) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$

(3) $f(A_1 \cap A_2) \subsetneq f(A_1) \cap f(A_2)$

(4) $f(A \setminus A_0) \neq B \setminus f(A_0)$

(ii) f^{-1} , when applied to subsets of B , preserves \subseteq, \cup, \cap , and \setminus :

(5) $B_1 \subseteq B_2 \implies f^{-1}(B_1) \subseteq f^{-1}(B_2)$

(6) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$

(7) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$

(8) $f^{-1}(B \setminus B_0) = A \setminus f^{-1}(B_0)$

Theorem (Properties of f and f^{-1} (Theorem 16.7))

$$f : A \rightarrow B, A_0 \subseteq A, B_0 \subseteq B$$

(iii) f and f^{-1} :

$$(9) A_0 \subseteq f^{-1}(f(A_0))$$

Q: When is $A_0 = f^{-1}(f(A_0))$?

Theorem (Properties of f and f^{-1} (Theorem 16.7))

$$f : A \rightarrow B, A_0 \subseteq A, B_0 \subseteq B$$

(iii) f and f^{-1} :

$$(9) A_0 \subseteq f^{-1}(f(A_0))$$

Q: When is $A_0 = f^{-1}(f(A_0))$?

$$(10) B_0 \subseteq f(f^{-1}(B_0))$$

Q: When is $B_0 = f(f^{-1}(B_0))$?

Problem 16.20

$$f : A \rightarrow B, \quad A_1, A_2 \subseteq A$$

(i) When is $f(A_1) = f(A_2) \implies A_1 = A_2$?

Problem 16.21

$$f : A \rightarrow B, \quad B_1, B_2 \subseteq B$$

(i) When is $f^{-1}(B_1) = f^{-1}(B_2) \implies B_1 = B_2$?

Thank
You!



Office 302

Mailbox: H016

hfwei@nju.edu.cn