Images and Definitions for the Concept of Function
Author(s): Shlomo Vinner and Tommy Dreyfus
Source: *Journal for Research in Mathematics Education*, Vol. 20, No. 4 (Jul., 1989), pp. 356-366
Published by: National Council of Teachers of Mathematics
Stable URL: http://www.jstor.org/stable/749441

Journal for Research in Mathematics Education
1989, Vol. 20, No. 4, 356–366

IMAGES AND DEFINITIONS FOR THE CONCEPT OF FUNCTION

SHLOMO VINNER, *Hebrew University of Jerusalem*
TOMMY DREYFUS, *Center for Technological Education*

Images held by 271 college students and 36 junior high school teachers for the concept of a mathematical function were compared to the definitions they gave for the concept. A questionnaire was designed to exhibit the cognitive schemes for the function concept that become active in identification and construction problems and to make possible the comparison of these schemes with the definition. Many of the definitions and even more of the images were primitive among all but the mathematics majors and the teachers. Discrepancies between image and definition were frequent for all subjects who gave the Dirichlet-Bourbaki definition.

This study examined some aspects of the images and definitions that college students and junior high school teachers have for the concept of function. Concept images and concept definitions (henceforth called *images* and *definitions*) were discussed in detail in several papers (Tall & Vinner, 1981; Vinner, 1983; Vinner & Hershkowitz, 1980). Therefore we will introduce them here very briefly. All mathematical concepts except the primitive ones have formal definitions. Many of these definitions are introduced to high school or college students at one time or another. The student, on the other hand, does not necessarily use the definition when deciding whether a given mathematical object is an example or nonexample of the concept. In most cases, he or she decides on the basis of a *concept image,* that is, the set of all the mental pictures associated in the student's mind with the concept name, together with all the properties characterizing them. (By mental picture we mean any kind of representation—picture, symbolic form, diagram, graph, etc.) The student's image is a result of his or her experience with examples and nonexamples of the concept. Hence, the set of mathematical objects considered by the student to be examples of the concept is not necessarily the same as the set of mathematical objects determined by the definition. If these two sets are not the same, the student's behavior may differ from what the teacher expects. To improve communication, we need to understand why it fails; therefore, it is important to explore students' images of various mathematical concepts.

Some mathematical concepts have strong graphical aspects; others do not. For concepts not having strong graphical aspects, the image includes mainly symbolic representations or formulas, as well as the set of all properties associated with the

---

This research was done while the authors were research fellows at the Weizmann Institute of Science, Rehovot, Israel. A partial report of the study appeared in Dreyfus and Vinner, 1982. The authors would like to thank Ein-Yah Gurah for help with the analysis of the questionnaires.

concept. The concept of function discussed in this paper has both strong graphical and strong nongraphical aspects.

The modern concept of function, which can be called the Dirichlet-Bourbaki concept, is that of a correspondence between two nonempty sets that assigns to every element in the first set (the domain) exactly one element in the second set (the codomain). To avoid the term *correspondence,* one may talk about a set of ordered pairs that satisfies a certain condition. We will, however, not use this extreme formality.

The Dirichlet-Bourbaki approach defined as functions many correspondences that were not recognized as functions by previous generations of mathematicians (Malik, 1981). Among these correspondences are discontinuous functions, functions defined on split domains (i.e., by different rules on different subdomains), functions with a finite number of exceptional points, and functions defined by means of a graph. Although the Dirichlet-Bourbaki approach is frequently presented in textbooks and curricula, the examples used to illustrate and work with the concept are usually, sometimes exclusively, functions whose rule of correspondence is given by a formula. This practice may lead to students' images being based on the appearance of a formula, even though their definition may well be of the Dirichlet-Bourbaki type. Thus, when asked about the function definition, a student may well come up with the Dirichlet-Bourbaki formulation, but when working on identification or construction tasks, his or her behavior might be based on the formula conception.

This inconsistent behavior is a specific case of the compartmentalization phenomenon mentioned in Vinner, Hershkowitz, and Bruckheimer (1981). This phenomenon occurs when a person has two different, potentially conflicting schemes in his or her cognitive structure. Certain situations stimulate one scheme, and other situations stimulate the other. Inconsistent behavior is not the only indication of compartmentalization. Sometimes, a given situation does not stimulate the scheme that is the most relevant to the situation. Instead, a less relevant scheme is activated. For instance, respondents can give the Dirichlet-Bourbaki definition and even accept that a certain discontinuous correspondence is a function; when asked to justify this, however, they do not use the definition but rather say that it is a discontinuous function.

This study investigated the following research questions:

1. What are the common definitions of the function concept given by college students before they start their calculus course?

2. What are the main images of the function concept that these students use in identification and construction tasks?

3. Are there statistically significant differences between groups of students with different majors in the way they conceive functions?

4. How frequently do students compartmentalize their formal definition of function and their image of the function concept?

METHOD

Sample

Our sample consisted of several groups of first-year college students in two Israeli institutions and some junior high school mathematics teachers who had not majored in mathematics. The college students had studied functions several years earlier in high school, but the concept had not yet been reintroduced to them in their college mathematics courses.

The students in our sample majored in mathematics, physics, chemistry, biology, economics, agriculture, technological education, or industrial design. For the purpose of the analysis they were split into four groups, determined by the level of mathematics courses required for their majors: low level, 33 students majoring in industrial design; intermediate level, 67 students majoring in economics or agriculture; high level, 113 students majoring in chemistry, biology, or technological education; and mathematics level, 58 students majoring in physics or mathematics. The sample included a fifth group, 36 junior high school mathematics teachers who participated in an in-service training course.

Originally, our sample consisted of 511 respondents, but in the final analysis we included only the 307 respondents whose questionnaires contained enough data to make a judgement (see below). This procedure might have caused some bias in the results: They might present a better picture than the reality. The percentage of students we dropped decreased with increasing level. We speculate that if we had taken these students into account, the differences between the groups may have been more pronounced.

The Questionnaire

The questionnaire in Figure 1 was administered, in Hebrew, to all subjects in the sample.

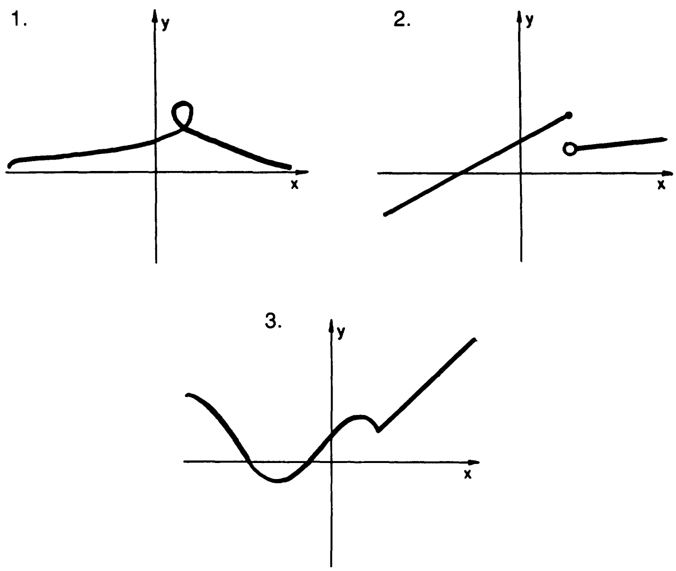
Questions 1 through 6 were designed to examine some aspects of the function image of the respondents, whereas Question 7 was designed to examine their definitions. These definitions were not necessarily the definitions given to them by their teachers or textbooks. The respondents could reflect impressions that were based on personal experience with examples and nonexamples of the concept.

In Questions 1 through 6 the respondents had to circle one of *Yes, No,* or *I do not know*. They were also asked to explain their answers. We included in the final sample only those respondents who gave a function definition (Question 7) and at least one explanation to another question, or, if the definition was missing, explanations to at least three questions. Thus the responses of the 271 students and 36 teachers described above were left for analysis.

Procedure

The questionnaire was administered to the students in their classes. They were not asked to fill in their names, only their background information. It took them at most 20 minutes to complete the questionnaire.

Does there exist a function whose graph is:



4. Does there exist a function which assigns to every number different from 0 its square and to 0 it assigns 1?
5. Does there exist a function all of whose values are equal to each other?
6. Does there exist a function whose values for integral numbers are non-integral and whose values for non-integral numbers are integral?
7. What is a function, in your opinion?

Figure 1. The questionnaire.

About 50 randomly chosen questionnaires were analyzed in detail by the authors, who consulted additional experts when faced with problematic cases. On the basis of this analysis, the definition categories and the image categories were determined. The remainder of the questionnaires were analyzed by a research assistant who was trained for this purpose. Each problematic case was discussed with the authors until full agreement about the classification was reached.

RESULTS

Definition Categories

We categorized the students' definitions of a function into six categories, a refinement of the categorization by Vinner (1983). We illustrate each category by some sample responses.

I. *Correspondence:* A function is any correspondence between two sets that assigns to every element in the first set exactly one element in the second set (the Dirichlet-Bourbaki definition).

"A correspondence between two sets of elements."

"For every element in A there is one and only one element in B."

II. *Dependence Relation:* A function is a dependence relation between two variables (y depends on x).

"One factor depending on the other one."

"A dependence between two variables."

"A connection between two magnitudes."

III. *Rule:* A function is a rule. A rule is expected to have some regularity, whereas a correspondence may be "arbitrary." The domain and the codomain were usually not mentioned here, contrary to Category I, where they were.

"Something that connects the value of x with the value of y."

"The result of a certain rule applied to a varying number."

"A relation between x and y is a function."

IV. *Operation:* A function is an operation or a manipulation (one acts on a given number, generally by means of algebraic operations, in order to get its image).

"An operation."

"An operation done on certain values of x that assigns to every value of x a value of $y = f(x)$."

"Transmitting values to other values according to certain conditions."

V. *Formula:* A function is a formula, an algebraic expression, or an equation.

"It is an equation expressing a certain relation between two objects."

"A mathematical expression that gives a connection between two factors."

"An equation connecting two factors."

VI. *Representation:* The function is identified, in a possibly meaningless way, with one of its graphical or symbolic representations.

"A graph that can be described mathematically."

"A collection of numbers in a certain order which can be expressed in a graph."

"$y = f(x)$."

"$y(F) = x$."

Table 1 shows the distribution of the concept definition categories for the five groups.

There were significant differences between the five groups in the number of students giving the Dirichlet-Bourbaki definition, $\chi^2(24, N = 307) = 82.9, p < .0001$. An examination of the data indicates that the percentage of students giving some version of this definition increased with the level of the mathematics course the students were taking.

Table 1
Distribution of Concept Definitions in the Five Groups

Definition category	Low (N = 33)	Intermediate (N = 67)	High (N = 113)	Mathematics (N = 58)	Teachers (N = 36)	Entire sample (N = 307)
Correspondence	2	12	17	26	25	82
Dependence relation	12	18	36	12	3	81
Rule	4	9	9	7	3	32
Operation	2	1	7	3	1	14
Formula	6	13	8	3	0	30
Representation	4	6	11	3	1	25
Others	3	8	25	4	3	43

Concept Images

Various aspects of the function concept, as conceived by the students, were expressed in their answers to Questions 1 through 6. Some of the major aspects of the function concept that played a crucial role in the explanations given by the respondents are as follows:

1. *One-valuedness:* If a correspondence assigns exactly one value to every element in its domain, then it is a function. If not, then it is not a function.

2. *Discontinuity:* The graph has a gap. The correspondence is discontinuous at one point in its domain.

3. *Split Domain:* The domain of the correspondence splits into two subdomains, in each of which a different rule of correspondence holds. As a consequence, the graph may change its character from one subdomain to the other.

4. *Exceptional Point:* There is a point of exception for a given correspondence, that is, a point for which the general rule of correspondence does not hold.

Sometimes, a certain aspect was for some students the reason for rejecting the given relation as a function, whereas for other students it was the reason for accepting it. For instance, some explanations of a negative answer to Question 2 mentioned that the graph was discontinuous and therefore could not be the graph of a function; and some explanations of a positive answer to the same question stated that discontinuous functions were legitimate functions. Similarly, the split-domain argument and the exceptional-point argument were used both for accepting and for rejecting certain examples as functions.

In the following discussion, we deal only with explanations to the correct answers, because that is where systematic differences between the groups were found. As far as justifications of wrong answers are concerned, no differences were found between the five groups of our sample. The higher level respondents justified more of their correct answers, justified them better, and used a wider spectrum of explanations than did the lower level respondents. However, when the higher level respondents gave a wrong answer, they did not behave appreciably differently from their lower level colleagues. The distribution of the correct answers to Questions 1 through 6 is given in Table 2; significant differences between the groups were found for each question, $\chi^2(5, N = 307), p < .01$.

We now turn to the justifications given by the students for their correct answers. Questions 1 through 4 may be considered as identification problems: The moment a question is understood, one knows immediately whether the correspondence under consideration is a function or not. Questions 5 and 6 are construction problems: One has to define a function in order to justify the answer. Table 3 deals with Questions 1 through 4. It gives the distribution of the explanations used by the students to justify their correct answers.

We would like to make some remarks concerning Table 3. In Question 2, some of the students were not familiar with the convention that an empty circle denotes an omitted point. Because of that, they thought that at the point where the graph is "torn," two values were given. Therefore, they claimed that there was no function with the given graph and justified their claim with a one-valuedness argument. These answers were considered as correct answers. It is reasonable to assume that those respondents to Question 2 who used the one-valuedness argument were also aware of the discontinuity argument and the split-domain argument. We believe this because the discontinuity and the split domain are so obvious when one looks at the graph. However, these respondents realized that what really matters when deciding whether a given correspondence is a function or not is the one-valuedness requirement. The one-valuedness argument is a kind of superargument to which the other arguments are subordinate.

Table 2
Distribution of Correct Answers to Questions 1–6 in the Five Groups and the Entire Sample

	Group					
Question	Low (N = 33)	Intermediate (N = 67)	High (N = 113)	Mathematics (N = 58)	Teachers (N = 36)	Entire sample (N = 307)
1	18	44	72	43	35	212
2	9	32	76	50	34	201
3	12	27	60	42	34	175
4	3	15	56	35	27	136
5	13	31	61	39	24	168
6	4	10	26	10	12	65

Table 3
Distribution of Explanations Among Respondents With Correct Answers to Questions 1–4 (Entire Sample)

	Question			
Argument	1 (N = 212)	2 (N = 201)	3 (N = 175)	4 (N = 136)
One-valuedness	126	66	40	14
Discontinuity		37		
Split domain		33	50	
Exceptional point				48
Other	53	37	26	41
No explanation	33	28	59	33

More precisely, an ordering of explanations can be observed, with the highest being the one-valuedness argument; the second, the discontinuity argument; the third, the split-domain argument; and the fourth, observed in some lower level students, an argument that we labeled the "mathematical convention" argument. A typical instance of this last argument is "Because mathematicians have decided that this is a function." Higher level students tended to choose higher level explanations, and vice versa. On the other hand, of 87 respondents who incorrectly answered that there is no function whose graph is the one in Question 2, 40% justified their claim by the discontinuity argument (for example, "A graph of a function must be continuous."), and 41% justified it by the split-domain argument ("A graph of a function cannot have two rules of correspondence. It cannot change its character.").

Seventy-seven percent of the respondents rejecting the graph in Question 3 gave the split-domain argument. Some respondents who rejected the graph in Question 3 accepted the one in Question 2, explaining that in Question 2 the character of the graph was the same throughout its entire domain (straight lines), whereas in Question 3 the character changed at the passage from a straight line to a curve. The idea that a graph of a function has to have a stable character throughout its entire domain plays a crucial role for these respondents. Among the respondents above, there were some who expressed their belief that for the graph in Question 2, it was possible to find a single formula. Their argumentation went as follows:

> In Question 2 we are concerned with straight lines only; all straight lines are of the form $y = ax + b$; on the other hand, it is impossible to find a single formula for the graph in Question 3, because a straight line and a curved line are involved. Two parts of a graph which have different characters cannot be described by a single formula.

Here, the idea that a function should be given by a single formula plays a major role. Of course, the reasoning is erroneous in the framework of the Dirichlet-Bourbaki definition.

It should be mentioned here that other respondents accepted the graph in Question 3 and rejected the one in Question 2. Their arguments were based on continuity: "A graph is the graph of a function only if it is continuous."

In Question 4, the one-valuedness argument was not as popular as in previous questions. This is probably so because the exceptional point of the given correspondence drew the main attention of the respondents and stimulated them to relate to it. However, among the teachers the one-valuedness argument was still the dominant one.

Forty-five percent of those who rejected the function in Question 4 did it with the exceptional-point argument, whereas most of the remaining 55% did not explain their answer.

Some of the respondents to Question 4 replied *Yes*, but for wrong reasons. They tried to give a formula for the function that took care of the exception. A typical answer was the following: "Yes. The function is $y = x^2 + 1$." Such answers were scored as incorrect.

Questions 5 and 6 were construction questions. Hence, we are discussing them separately. As shown in Table 2, only 55% of the entire sample answered Question 5 correctly. Of these, 64% justified it by a general expression, such as $y = c$, or by a specific example, such as $y = 5$. Fifteen percent of those with the correct answer suggested a formula containing x, such as $y = x/x$ or $y = x^0$. Here, we believe, the operational aspect of function was expressed: One has to do something to x in order to obtain the corresponding y.

A typical wrong answer to Question 5 was "Yes, $y = x$." This is, of course, a function whose values are equal to their arguments and not to each other. Nevertheless, we consider this answer as symptomatic rather than accidental. Students usually pay less attention to the conceptual aspects of a given notion and more attention to its computational or operational aspects. Hence, many students are unfamiliar with the terminology that relates to the conceptual aspects of the mathematical notion.

As to Question 6, only 21% of the entire sample answered it in a way that could not be immediately classified as wrong. Of these, only 37% also justified their answer by providing a correct example, such as the following:

$$f(x) = \begin{cases} \tfrac{1}{2}, & x \text{ integral} \\ 5, & x \text{ nonintegral} \end{cases}$$

Many students answered the question with *yes* but then tried to suggest one algebraic rule that was supposed to define the function. A typical answer (22% of the entire sample) was "Yes, $y = 1/x$." This mistake (discussed in Vinner, 1983) was considered symptomatic rather than accidental. It was probably caused by the following line of thought: For integral values of x, the expression $1/x$ is nonintegral (this is true only if $|x| > 1$, but $x = \pm 1$ was probably ignored). The expression $1/x$, for integral x, brought to the respondents' attention nonintegral numbers of the form $1/n$, $n = 2, 3, 4, \ldots$. Having this type of nonintegral numbers in mind, and ignoring all other types, the respondents substituted it in the formula $1/x$. The result was n, an integral number. This made them believe that $y = 1/x$ is the rule for the required function.

Only a correct example was considered as an explanation for an affirmative answer to Question 6. Only 8% of the sample gave a correct example.

Compartmentalization is perhaps the most interesting aspect of this study, because it is related to the question of how often inconsistent or noncoherent behaviors occur in mathematics learning. Table 4 provides this information about our sample for the case of the function concept. It shows all cases in which respondents

Table 4
Distribution of Compartmentalization Cases in Respondents With the Dirichlet-Bourbaki Definition

	Group					
	Low (N = 2)	Intermediate (N = 12)	High (N = 17)	Mathematics (N = 26)	Teachers (N = 25)	Entire Sample (N = 82)
Number	2	12	12	6	14	46
Percentage	100	100	71	23	56	56

gave the Dirichlet-Bourbaki definition for the function concept but did not use the definition when answering Questions 1 through 6, sometimes ending up with inconsistent behavior.

DISCUSSION

One of the goals of this study was to expose some common images of the function concept held by college students at the beginning of their calculus courses. This has direct implications for teaching. If one wants to teach functions to a group similar to one of the groups in the study, it is important to know the starting point of its members (Dreyfus & Eisenberg, 1982). Taking into account the difficulties mentioned in this study and also in Malik (1981) and Vinner (1983), at least a doubt should be raised whether the Dirichlet-Bourbaki approach to the function concept should be taught in courses where it is not intensively needed. If discontinuous functions, functions with split domains, functions with exceptional points, or other strange functions are needed, we think that they should be introduced as cases extending the students' previous experience. The formal definition should be only a conclusion of the various examples introduced to the students.

In the study we also examined the differences between some groups. These differences might seem trivial if our questionnaire is considered as a common achievement test. However, this is not the case. In our questionnaire, some components of mathematical thought are examined, such as the ability to reason and the ability to apply a definition in a coherent way. It turns out that the course level of a student is a good indication of the student's reasoning patterns. As indicated in Table 4, there are fewer cases of compartmentalization by students at higher level courses. We speculate that compartmentalization is quite rare in mathematically oriented students, since they are more aware of their thought processes and since they are more likely to reflect on them (see Burton, 1984; Schoenfeld, 1985; Skemp, 1979).

We would like to conclude this paper with another comment that has direct implications for teaching. One must remember that a concept is not acquired in one step. Several stages precede the complete acquisition and mastery of a complex concept. In these intermediate stages, some peculiar behaviors are likely to occur. Several cognitive schemes, some even conflicting with each other, may act in the same person in different situations that are closely related in time (see also Davis, 1980). The knowledge of these particular cognitive schemes may make the teacher more sensitive to students' reactions and thus improve communication.

REFERENCES

Burton, L. (1984). Mathematical thinking: The struggle for meaning. *Journal for Research in Mathematics Education, 15,* 35–49.

Davis, R. (1980). The postulation of certain specific explicit, commonly-shared frames. *Journal of Mathematical Behavior, 3,* 167–201.

Dreyfus, T., & Eisenberg, T. (1982). Intuitive functional concepts: A baseline study on intuitions. *Journal for Research in Mathematics Education, 13,* 360–380.

Dreyfus, T., & Vinner, S. (1982). Some aspects of the function concept in college students and junior high school teachers. In A. Vermandel (Ed.), *Proceedings of the Sixth International Conference for the Psychology of Mathematics Education* (pp. 12–17). Antwerp, Belgium: Universitaire Instelling.

Malik, M. A. (1981). Historical and pedagogical aspects of the definition of function. *International Journal of Mathematics Education in Science and Technology, 11,* 489–492.

Schoenfeld, A. (1985). Metacognitive and epistemological issues in mathematical understanding. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 361–380). Hillsdale, NJ: Erlbaum.

Skemp, R. (1979). *Intelligence, learning, and action.* Chichester, England: Wiley.

Tall, D., & Vinner, S. (1981). Concept images and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12,* 151–169.

Vinner, S. (1983). Concept definition, concept image and the notion of function. *International Journal of Mathematical Education in Science and Technology, 14,* 293–305.

Vinner, S., & Hershkowitz, R. (1980). Concept images and common cognitive paths in the development of some simple geometrical concepts. In R. Karplus (Ed.), *Proceedings of the Fourth International Conference for the Psychology of Mathematics Education* (pp. 177–184). Berkeley: University of California, Lawrence Hall of Science.

Vinner, S., Hershkowitz, R., & Bruckheimer, M. (1981). Some cognitive factors as causes of mistakes in addition of fractions. *Journal for Research in Mathematics Education, 12,* 70–76.

AUTHORS

SHLOMO VINNER, Professor, Department of Science Teaching, The Hebrew University, Jerusalem, Israel

TOMMY DREYFUS, Center for Technological Education, P.O. Box 305, Holon, Israel