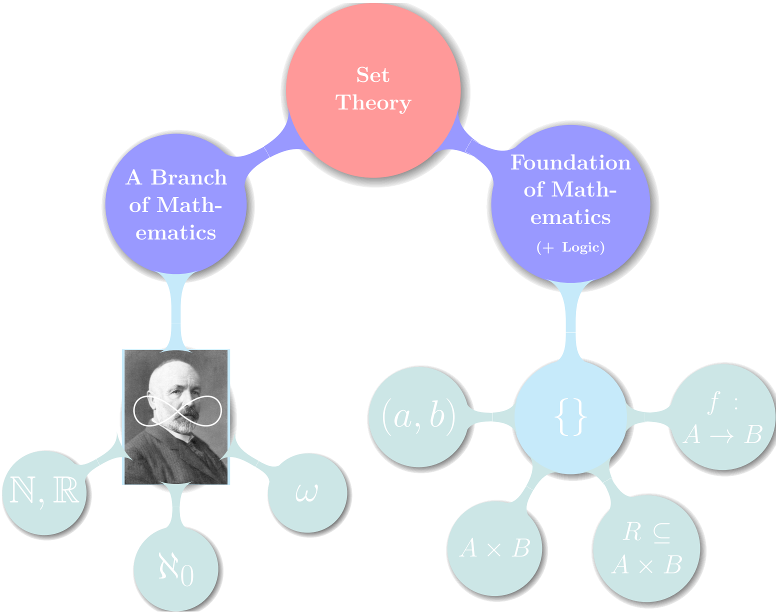
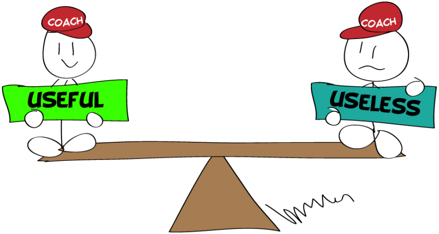
1-9 关系及其基本性质

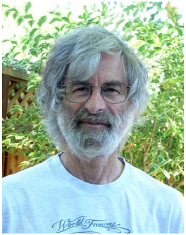
魏恒峰

hfwei@nju.edu.cn

2017 年 12 月 11 日


Set Theory
A Branch of Math-ematics
Foundation of Math-ematics
(+ Logic)
$\mathbb{N}, \mathbb{R}$
$\aleph_0$
ω
(a, b)
$\{\}$
$f : A \to B$
$A \times B$
$R \subseteq A \times B$


COACH
USEFUL
COACH
USELESS



Time, Clocks, and the Ordering of Events in a Distributed System

Leslie Lamport
Massachusetts Computer Associates, Inc.

The concept of one event happening before another in a distributed system is examined, and is shown to define a partial ordering of the events. A distributed algorithm is given for synchronizing a system of logical clocks which can be used to totally order the events.

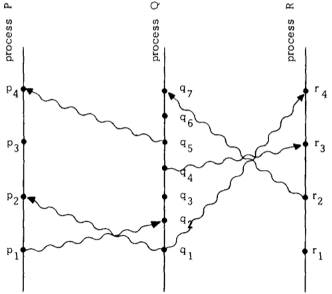


Figure 13. A selection of consistency axioms over an execution $(E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar})$

Auxiliary relations

$\mathsf{sameobj}(e, f) \iff \mathsf{obj}(e) = \mathsf{obj}(f)$

Per-object causality (aka happens-before) order:

$\mathsf{hbo} = ((\mathsf{ro} \cap \mathsf{sameobj}) \cup \mathsf{vis})^+$

Causality (aka happens-before) order: $\mathsf{hb} = (\mathsf{ro} \cup \mathsf{vis})^+$

Axioms

EVENTUAL:

$\forall e \in E.\, \neg(\exists \text{ infinitely many } f \in E.\, \mathsf{sameobj}(e, f) \wedge \neg(e \xrightarrow{\mathsf{vis}} f))$

THINAIR: $\mathsf{ro} \cup \mathsf{vis}$ is acyclic

POCV (Per-Object Causal Visibility): $\mathsf{hbo} \subseteq \mathsf{vis}$

POCA (Per-Object Causal Arbitration): $\mathsf{hbo} \subseteq \mathsf{ar}$

COCV (Cross-Object Causal Visibility): $(\mathsf{hb} \cap \mathsf{sameobj}) \subseteq \mathsf{vis}$

COCA (Cross-Object Causal Arbitration): $\mathsf{hb} \cup \mathsf{ar}$ is acyclic

Figure 17. Optimized state-based multi-value register and its simulation

$$
\begin{aligned}
\Sigma &= \mathsf{ReplicaID} \times \mathcal{P}(\mathbb{Z} \times (\mathsf{ReplicaID} \to \mathbb{N}_0)) \\
\vec{\sigma}_0 &= \langle r, \emptyset \rangle \\
M &= \mathcal{P}(\mathbb{Z} \times (\mathsf{ReplicaID} \to \mathbb{N}_0)) \\
\mathsf{do}(\mathtt{wr}(a), \langle r, V\rangle, t) &= \\
&(\langle r, \{(a, (\lambda s.\, \text{if } s \neq r \text{ then } \max\{v(s) \mid (\_, v) \in V\} \\
&\qquad \text{else } \max\{v(s) \mid (\_, v) \in V\} + 1))\}\rangle, \bot) \\
\mathsf{do}(\mathtt{rd}, \langle r, V\rangle, t) &= (\langle r, V\rangle, \{a \mid (a, \_) \in V\}) \\
\mathsf{send}(\langle r, V\rangle) &= (\langle r, V\rangle, V) \\
\mathsf{receive}(\langle r, V\rangle, V') &= \langle r, \{(a, v) \in V'' \mid \\
&\qquad v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'.\, (a', v') \in V'' \wedge a \neq a'\}\}\rangle, \\
\text{where } V'' &= \{(a, \bigsqcup\{v' \mid (a, v') \in V \cup V'\}) \mid (a, \_) \in V \cup V'\}
\end{aligned}
$$

$$\langle s, V\rangle\ [\mathcal{R}_r]\ I \iff (r = s) \wedge (V\ [\mathcal{M}]\ I)$$

$$
\begin{aligned}
&V\ [\mathcal{M}]\ ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar}), \mathsf{info}) \iff \\
&\quad (\forall (a, v), (a', v') \in V.\, (a = a' \implies v = v')) \wedge \\
&\quad (\forall (a, v) \in V.\, \exists s.\, v(s) > 0) \wedge \\
&\quad (\forall (a, v) \in V.\, v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'.\, (a', v') \in V \wedge a \neq a'\}) \wedge \\
&\quad \exists \text{ distinct } e_{s,k}. \\
&\quad (\{e \in E \mid \exists a.\, \mathsf{oper}(e) = \mathtt{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicaID} \wedge \\
&\qquad 1 \le k \le \max\{v(s) \mid \exists a.\, (a, v) \in V\}\}) \wedge \\
&\quad (\forall s, j, k.\, (\mathsf{repl}(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}} e_{s,k} \iff j < k)) \wedge \\
&\quad (\forall (a, v) \in V.\, \forall q.\, \{j \mid \mathsf{oper}(e_{q,j}) = \mathtt{wr}(a)\} \cup \\
&\qquad \{j \mid \exists s, k.\, e_{q,j} \xrightarrow{\mathsf{vis}} e_{s,k} \wedge \mathsf{oper}(e_{s,k}) = \mathtt{wr}(a)\} = \\
&\qquad \{j \mid 1 \le j \le v(q)\}) \wedge \\
&\quad (\forall e \in E.\, (\mathsf{oper}(e) = \mathtt{wr}(a) \wedge \\
&\qquad \neg\exists f \in E.\, \mathsf{oper}(f) = \mathtt{wr}(\_) \wedge e \xrightarrow{\mathsf{vis}} f) \implies (a, \_) \in V)
\end{aligned}
$$

the former. The only non-trivial obligation is to show that if

$$V\ [\mathcal{M}]\ ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}), \mathsf{info}),$$

then

$$
\begin{aligned}
\{a \mid (a, \_) \in V\} \subseteq \{a \mid \exists e \in E.\, &\mathsf{oper}(e) = \mathtt{wr}(a) \wedge \\
&\neg\exists f \in E.\, \exists a'.\, \mathsf{oper}(e) = \mathtt{wr}(a') \wedge e \xrightarrow{\mathsf{vis}} f\}
\end{aligned} \tag{13}
$$

(the reverse inclusion is straightforwardly implied by $\mathcal{R}_r$).

Take $(a, v) \in V$. We have $\forall (a, v) \in V.\, \exists s.\, v(s) > 0$,

$$v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'.\, (a', v') \in V \wedge a \neq a'\}$$

and

$$
\begin{aligned}
&\forall (a, v) \in V.\, \forall q.\, \{j \mid \mathsf{oper}(e_{q,j}) = \mathtt{wr}(a)\} \cup \\
&\quad \{j \mid \exists s, k.\, e_{q,j} \xrightarrow{\mathsf{vis}} e_{s,k} \wedge \mathsf{oper}(e_{s,k}) = \mathtt{wr}(a)\} = \\
&\quad \{j \mid 1 \le j \le v(q)\}.
\end{aligned}
$$

From this we get that for some $e \in E$

$$\mathsf{oper}(e) = \mathtt{wr}(a) \wedge \neg\exists f \in E.\, \exists a'.\, a' \neq a \wedge \mathsf{oper}(e) = \mathtt{wr}(a') \wedge e \xrightarrow{\mathsf{vis}} f.$$

Since vis is acyclic, this implies that for some $e' \in E$

$$\mathsf{oper}(e') = \mathtt{wr}(a) \wedge \neg\exists f \in E.\, \mathsf{oper}(e') = \mathtt{wr}(\_) \wedge e' \xrightarrow{\mathsf{vis}} f,$$

which establishes (13).

Let us now discharge RECEIVE. Let $\mathsf{receive}(\langle r, V\rangle, V') = \langle r, V'''\rangle$, where

$$
\begin{aligned}
V'' &= \{(a, \bigsqcup\{v' \mid (a, v') \in V \cup V'\}) \mid (a, \_) \in V \cup V'\}; \\
V''' &= \{(a, v) \in V'' \mid v \not\sqsubseteq \bigsqcup\{(a', v') \in V'' \mid a \neq a'\}\}.
\end{aligned}
$$

Assume $\langle r, V\rangle\ [\mathcal{R}_r]\ I$, $V'\ [\mathcal{M}]\ J$ and

$$
\begin{aligned}
I &= ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar}), \mathsf{info}); \\
J &= ((E', \mathsf{repl}', \mathsf{obj}', \mathsf{oper}', \mathsf{rval}', \mathsf{ro}', \mathsf{vis}', \mathsf{ar}'), \mathsf{info}'); \\
I \sqcup J &= ((E'', \mathsf{repl}'', \mathsf{obj}'', \mathsf{oper}'', \mathsf{rval}'', \mathsf{ro}'', \mathsf{vis}'', \mathsf{ar}''), \mathsf{info}'').
\end{aligned}
$$

By agree we have $I \sqcup J \in \mathsf{IEx}$. Then

$$
\begin{aligned}
&(\forall (a, v), (a', v') \in V.\, (a = a' \implies v = v')) \wedge \\
&(\forall (a, v) \in V.\, \exists s.\, v(s) > 0) \wedge \\
&(\forall (a, v) \in V.\, v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'.\, (a', v') \in V \wedge a \neq a'\}) \wedge \\
&\exists \text{ distinct } e_{s,k}. \\
&(\{e \in E \mid \exists a.\, \mathsf{oper}''(e) = \mathtt{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicaID} \wedge \\
&\quad 1 \le k \le \max\{v(s) \mid \exists a.\, (a, v) \in V\}\}) \wedge \\
&(\forall s, j, k.\, (\mathsf{repl}''(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}} e_{s,k} \iff j < k)) \wedge \\
&(\forall (a, v) \in V.\, \forall q.\, \{j \mid \mathsf{oper}''(e_{q,j}) = \mathtt{wr}(a)\} \cup \\
&\quad \{j \mid \exists s, k.\, e_{q,j} \xrightarrow{\mathsf{vis}} e_{s,k} \wedge \mathsf{oper}''(e_{s,k}) = \mathtt{wr}(a)\} = \\
&\quad \{j \mid 1 \le j \le v(q)\}) \wedge \\
&(\forall e \in E.\, (\mathsf{oper}''(e) = \mathtt{wr}(a) \wedge \\
&\quad \neg\exists f \in E.\, \mathsf{oper}''(f) = \mathtt{wr}(\_) \wedge e \xrightarrow{\mathsf{vis}} f) \implies (a, \_) \in V)
\end{aligned}
$$

and

$$
\begin{aligned}
&(\forall (a, v), (a', v') \in V'.\, (a = a' \implies v = v')) \wedge \\
&(\forall (a, v) \in V'.\, \exists s.\, v(s) > 0) \wedge \\
&(\forall (a, v) \in V'.\, v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'.\, (a', v') \in V' \wedge a \neq a'\}) \wedge \\
&\exists \text{ distinct } e_{s,k}. \\
&(\{e \in E' \mid \exists a.\, \mathsf{oper}''(e) = \mathtt{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicaID} \wedge \\
&\quad 1 \le k \le \max\{v(s) \mid \exists a.\, (a, v) \in V'\}\}) \wedge \\
&(\forall s, j, k.\, (\mathsf{repl}''(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}'} e_{s,k} \iff j < k)) \wedge \\
&(\forall (a, v) \in V'.\, \forall q.\, \{j \mid \mathsf{oper}''(e_{q,j}) = \mathtt{wr}(a)\} \cup \\
&\quad \{j \mid \exists s, k.\, e_{q,j} \xrightarrow{\mathsf{vis}'} e_{s,k} \wedge \mathsf{oper}''(e_{s,k}) = \mathtt{wr}(a)\} = \\
&\quad \{j \mid 1 \le j \le v(q)\}) \wedge \\
&(\forall e \in E'.\, (\mathsf{oper}''(e) = \mathtt{wr}(a) \wedge \\
&\quad \neg\exists f \in E'.\, \mathsf{oper}''(f) = \mathtt{wr}(\_) \wedge e \xrightarrow{\mathsf{vis}'} f) \implies (a, \_) \in V').
\end{aligned}
$$

The agree property also implies

$$
\begin{aligned}
\forall s, k.\, 1 \le k \le \min\{&\max\{v(s) \mid \exists a.\, (a, v) \in V\}, \\
&\max\{v(s) \mid \exists a.\, (a, v) \in V'\}\} \implies e_{s,k} = e'_{s,k}.
\end{aligned}
$$

Hence, there exist distinct

$$e''_{s,k} \text{ for } s \in \mathsf{ReplicaID},\ k = 1..(\max\{v(s) \mid \exists a.\, (a, v) \in V'''\}),$$

such that

$$
\begin{aligned}
&(\forall s, k.\, 1 \le k \le \max\{v(s) \mid \exists a.\, (a, v) \in V\} \implies e''_{s,k} = e_{s,k}) \wedge \\
&(\forall s, k.\, 1 \le k \le \max\{v(s) \mid \exists a.\, (a, v) \in V'\} \implies e''_{s,k} = e'_{s,k})
\end{aligned}
$$

and

$$
\begin{aligned}
&(\{e \in E \cup E' \mid \exists a.\, \mathsf{oper}''(e) = \mathtt{wr}(a)\} = \\
&\{e''_{s,k} \mid s \in \mathsf{ReplicaID} \wedge 1 \le k \le \max\{v(s) \mid \exists a.\, (a, v) \in V'''\}\}) \\
&\wedge (\forall s, j, k.\, (\mathsf{repl}(e''_{s,k}) = s) \wedge (e''_{s,j} \xrightarrow{\mathsf{ro}''} e''_{s,k} \iff j < k)).
\end{aligned}
$$

By the definition of V'' and V''' we have

$$\forall (a, v), (a', v') \in V'''.\, (a = a' \implies v = v').$$

We also straightforwardly get

$$\forall (a, v) \in V'.\, \exists s.\, v(s) > 0$$

and

$$
\begin{aligned}
&(\forall (a, v) \in V''.\, \forall q.\, \{j \mid \mathsf{oper}''(e''_{q,j}) = \mathtt{wr}(a)\} \cup \\
&\quad \{j \mid \exists s, k.\, e''_{q,j} \xrightarrow{\mathsf{vis}''} e''_{s,k} \wedge \mathsf{oper}''(e''_{s,k}) = \mathtt{wr}(a)\} = \\
&\quad \{j \mid 1 \le j \le v(q)\}).
\end{aligned} \tag{14}
$$

Power Set

{a,b,c}

{ {},
{a}, {b}, {c},
{a,b}, {a,c}, {b,c},
{a,b,c} }

Definition (Axiom of Power Set)

$$\forall X \exists Y \forall u (u \in Y \iff u \subseteq X)$$

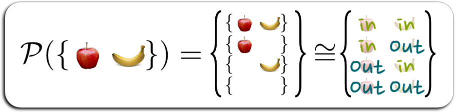
$$\mathcal{P}(X)$$

Definition (Axiom of Power Set)

$$\forall X \exists Y \forall u (u \in Y \iff u \subseteq X)$$

$$\mathcal{P}(X)$$

$$2^X = \{0, 1\}^X$$



$$\boxed{S \in \mathcal{P}(X) \iff S \subseteq X}$$

“$\subseteq$” (UD 9.2)

$$\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$$

"$\subseteq$" (UD 9.2)

$$\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$$

Proof.

$$\forall x\Big(x \in \mathcal{P}(A) \cup \mathcal{P}(B) \implies x \in \mathcal{P}(A \cup B)\Big)$$

$\square$

“$\subseteq$” (UD 9.2)

$$\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$$

Proof.

$$\forall x \Big(x \in \mathcal{P}(A) \cup \mathcal{P}(B) \implies x \in \mathcal{P}(A \cup B) \Big)$$

□

$$\mathcal{P}(A) \cup \mathcal{P}(B) \neq \mathcal{P}(A \cup B)$$

“$\subseteq$” (UD 9.2)

$$\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$$

Proof.

$$\forall x \Big(x \in \mathcal{P}(A) \cup \mathcal{P}(B) \implies x \in \mathcal{P}(A \cup B) \Big)$$

□

$$\mathcal{P}(A) \cup \mathcal{P}(B) \neq \mathcal{P}(A \cup B)$$

UD Exercise 9.3

$$\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$$

“$\subseteq$” (UD 9.4)

$$A \subseteq B \iff \mathcal{P}(A) \subseteq \mathcal{P}(B)$$

The “element-chasing” method.

“$\subseteq$” (UD 9.4)

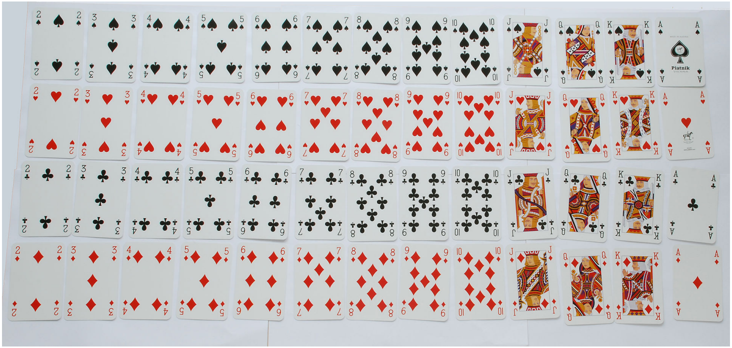
$$A \subseteq B \iff \mathcal{P}(A) \subseteq \mathcal{P}(B)$$

The “element-chasing” method.

A proof using the following equation:

$$\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$$

Ordered Pair and Cartesian Product



Definitions of (a, b) and $A \times B$ (UD 9.16)

$$(a, b) = \{\{a\}, \{a, b\}\}$$

$$(a, b) = (x, y) \iff a = x \land b = y$$

Definitions of (a, b) and $A \times B$ (UD 9.16)

$$(a, b) = \{\{a\}, \{a, b\}\}$$

$$(a, b) = (x, y) \iff a = x \land b = y$$

$$\boxed{\{\{a\}, \{a, b\}\} = \{\{x\}, \{x, y\}\} \implies a = x \land b = y}$$

Definitions of (a, b) and $A \times B$ (UD 9.16)

$$(a, b) = \{\{a\}, \{a, b\}\}$$

$$(a, b) = (x, y) \iff a = x \land b = y$$

$$\boxed{\{\{a\}, \{a, b\}\} = \{\{x\}, \{x, y\}\} \implies a = x \land b = y}$$

What are the flaws in the following proof:

$$\begin{cases} \{a\} & = \{x\} \\ \{a, b\} & = \{x, y\} \end{cases} \implies \begin{cases} a = x \\ b = y \end{cases} \qquad \begin{cases} \{a\} & = \{x, y\} \\ \{a, b\} & = \{x\} \end{cases} \implies \text{no solution.}$$

Definitions of (a, b) and $A \times B$ (UD 9.16)

$$(a, b) = \{\{a\}, \{a, b\}\}$$

$$(a, b) = (x, y) \iff a = x \land b = y$$

$$\boxed{\{\{a\}, \{a, b\}\} = \{\{x\}, \{x, y\}\} \implies a = x \land b = y}$$

Proof.

CASE $a = b$

CASE $a \neq b$

□

Definitions of (a, b) and $A \times B$ (UD 9.16)

$$(a, b) = \{\{a\}, \{a, b\}\}$$

$$a \in A \land b \in B \implies (a, b) \in \mathcal{P}(\mathcal{P}(A \cup B))$$

Definitions of (a, b) and $A \times B$ (UD 9.16)

$$(a, b) = \{\{a\}, \{a, b\}\}$$

$$a \in A \land b \in B \implies (a, b) \in \mathcal{P}(\mathcal{P}(A \cup B))$$

$$A \times B = \{x \in \mathcal{P}(\mathcal{P}(A \cup B)) \mid \exists a \in A \, \exists b \in B : x = (a, b)\}$$

$$A \subseteq C \land B \subseteq D \implies A \times B \subseteq C \times D$$

(UD 9.13)

$$A \times B \subseteq C \times D \overset{?}{\Longrightarrow} A \subseteq C \land B \subseteq D$$

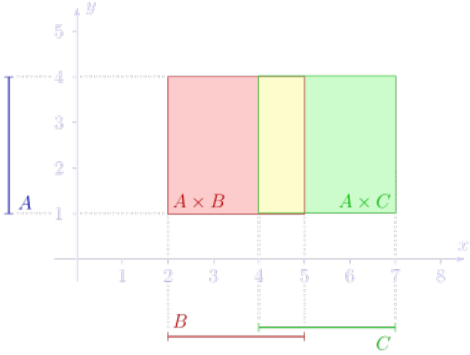
$$A = \emptyset$$

$$A \times B \subseteq C \times D \xRightarrow{A,B\neq\emptyset} A \subseteq C \land B \subseteq D$$

By contradiction.

Distributive Laws (UD 9.14)

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$
$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$
$$A \times (B \setminus C) = (A \times B) \setminus (A \times C)$$



Relation

燕小六: “帮我照顾好我七舅姥爷和我外甥女”

$$G = \{(a, b) : a \text{ 是 } b \text{ 的舅姥爷}\}$$

$$N = \{(a, b) : a \text{ 是 } b \text{ 的外甥女}\}$$

$$G \cup N$$

$$G = \{(a, b) : a \text{ 是 } b \text{ 的舅姥爷}\}$$

$$N = \{(a, b) : a \text{ 是 } b \text{ 的外甥女}\}$$

$$G \cup N$$

"B"	Brother	"F"	Father	"O"	Son
"S"	Sister	"M"	Mather	"D"	Dau.

$$G = \{(a,b) : a \text{ 是 } b \text{ 的舅姥爷}\}$$

$$N = \{(a,b) : a \text{ 是 } b \text{ 的外甥女}\}$$

$$G \cup N$$

"B"	Brother	"F"	Father	"O"	Son
"S"	Sister	"M"	Mather	"D"	Dau.

$$G = B \circ M \circ M$$

$$G = \{(a,b) : a \text{ 是 } b \text{ 的舅姥爷}\}$$

$$N = \{(a,b) : a \text{ 是 } b \text{ 的外甥女}\}$$

$$G \cup N$$

"B"	Brother	"F"	Father	"O"	Son
"S"	Sister	"M"	Mather	"D"	Dau.

$$G = B \circ M \circ M \qquad N = D \circ S$$

$$G = \{(a,b) : a \text{ 是 } b \text{ 的舅姥爷}\}$$

$$N = \{(a,b) : a \text{ 是 } b \text{ 的外甥女}\}$$

$$G \cup N$$

"B"	Brother	"F"	Father	"O"	Son
"S"	Sister	"M"	Mather	"D"	Dau.

$$G = B \circ M \circ M \qquad N = D \circ S$$

$$G = (B \circ M) \circ M = B \circ (M \circ M)$$

$$R \subseteq X \times Y$$

R is a relation **from** X **to** Y.

$$R \subseteq X \times X$$

R is a relation **on** X.

Definition (Equivalence Relation)

R is an equivalence relation on $X \times X$ if

Reflexive: (fig here)

Symmetric:

Transitive:

Definition (Equivalence Relation)

R is an equivalence relation on $X \times X$ if

- Reflexive: (fig here)
- Symmetric:
- Transitive:

Definition (Equivalence Class)

$$(X, \sim)$$

$$E_x = \{y \in X : x \sim y\} = [x]_{\sim}$$

Equivalence Relation (UD 10.5)

$$(X, \sim)$$

Prove that

$$\forall x, y \in X : [x]_{\sim} = [y]_{\sim} \iff x \sim y.$$

Thank You!