2-8 Probabilistic Analysis

"No Expectation, No Disappointment."

Hengfeng Wei

hfwei@nju.edu.cn

May 16, 2018

Definition (Conditional Expectation on an Event)

$$\mathbb{E}[X \mid E] = \sum_x x \Pr(X = x \mid E)$$

$$\left(\mathbb{E}[X] = \sum_x x \Pr(X = x) \right)$$

Theorem ((CS Theorem 5.23))

Let X be a random variable defined on a sample space Ω.
Let $E_1, E_2, \cdots, E_n$ be a partition *of Ω.*

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X \mid E_i] \Pr(E_i)$$

Theorem (The Law of Total Expectation (CS Theorem 5.23))

Let X be a random variable defined on a sample space Ω.
Let $E_1, E_2, \cdots, E_n$ be a partition of Ω.

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X \mid E_i] \Pr(E_i)$$

Theorem (The Law of Total Expectation (CS Theorem 5.23))

Let X be a random variable defined on a sample space Ω.
Let $E_1, E_2, \cdots, E_n$ be a partition of Ω.

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X \mid E_i] \Pr(E_i)$$

Proof.

By definition. □

Theorem (The Law of Total Expectation (CS Theorem 5.23))

Let X be a random variable defined on a sample space Ω.
Let $E_1, E_2, \cdots, E_n$ be a partition of Ω.

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X \mid E_i] \Pr(E_i)$$

Proof.

By definition.

$$\sum_{x} x \sum_{i=1}^{n} \Pr(X = x, E_i) = \sum_{i=1}^{n} \sum_{x} x \Pr(X = x, E_i)$$

□

($\sharp$) Rational Person Playing a Card Game (CS Problem $5.6 - 4$)

A : \$1.00; Repeat

J : \$2.00; End

K : \$3.00; End

Q : \$4.00; End

($\nexists$) Rational Person Playing a Card Game (CS Problem $5.6-4$)

A : \$1.00; Repeat

J : \$2.00; End

K : \$3.00; End

Q : \$4.00; End

Conditioning on the first draw c

$$\mathbb{E}[X] = \frac{1}{4}\Big(\mathbb{E}[X \mid c = A] + \mathbb{E}[X \mid c = J] + \mathbb{E}[X \mid c = K] + \mathbb{E}[X \mid c = Q]\Big)$$

($\sharp$) Rational Person Playing a Card Game (CS Problem $5.6-4$)

A : \$1.00; Repeat

J : \$2.00; End

K : \$3.00; End

Q : \$4.00; End

Conditioning on the first draw c

$$\mathbb{E}[X] = \frac{1}{4}\Big(\mathbb{E}[X \mid c = A] + \mathbb{E}[X \mid c = J] + \mathbb{E}[X \mid c = K] + \mathbb{E}[X \mid c = Q]\Big)$$

$$\mathbb{E}[X \mid c = A] = \mathbb{E}[X] + 1$$

($\nexists$) Rational Person Playing a Card Game (CS Problem $5.6 - 4$)

A : \$1.00; Repeat

J : \$2.00; End

K : \$3.00; End

Q : \$4.00; End

Conditioning on the first draw c

$$\mathbb{E}[X] = \frac{1}{4}\Big(\mathbb{E}[X \mid c = A] + \mathbb{E}[X \mid c = J] + \mathbb{E}[X \mid c = K] + \mathbb{E}[X \mid c = Q]\Big)$$

$$\mathbb{E}[X \mid c = A] = \mathbb{E}[X] + 1$$

$$4 * A + 1 * Q = \$8.00$$

In-class Exercise: Coin Pattern (Provided by Yifan Pei)

X : # of tosses to get 3 consecutive heads (HHH)

$$\mathbb{E}[X]$$

In-class Exercise: Coin Pattern (Provided by Yifan Pei)

X : # of tosses to get 3 consecutive heads (HHH)

$\mathbb{E}[X]$

Conditioning on the first 3 tosses

In-class Exercise: Coin Pattern (Provided by Yifan Pei)

X : # of tosses to get 3 consecutive heads (HHH)

$$\mathbb{E}[X]$$

Conditioning on the first 3 tosses

$$T, \quad HT, \quad HHT, \quad HHH$$

In-class Exercise: Coin Pattern (Provided by Yifan Pei)

X : # of tosses to get 3 consecutive heads (HHH)

$$\mathbb{E}[X]$$

Conditioning on the first 3 tosses

$$T, \quad HT, \quad HHT, \quad HHH$$

$$\mathbb{E}[X] = \frac{1}{2}(\mathbb{E}[X] + 1) + \frac{1}{4}(\mathbb{E}[X] + 2) + \frac{1}{8}(\mathbb{E}[X] + 3) + \frac{1}{8} \times 3$$

$$\boxed{X : \# \text{ of tosses to get } HHH}$$

$$T, \quad HT, \quad HHT, \quad HHH$$

$$\mathbb{E}[X] = \frac{1}{2}(\mathbb{E}[X] + 1) + \frac{1}{4}(\mathbb{E}[X] + 2) + \frac{1}{8}(\mathbb{E}[X] + 3) + \frac{1}{8} \times 3$$

$$\boxed{X : \# \text{ of tosses to get } HHH}$$

$$T, \quad HT, \quad HHT, \quad HHH$$

$$\mathbb{E}[X] = \frac{1}{2}(\mathbb{E}[X] + 1) + \frac{1}{4}(\mathbb{E}[X] + 2) + \frac{1}{8}(\mathbb{E}[X] + 3) + \frac{1}{8} \times 3$$

$$\boxed{X : \# \text{ of tosses to get } HTH}$$

$$T, \quad HH, \quad HTT, \quad HTH$$

$$\mathbb{E}[X] = \frac{1}{2}(\mathbb{E}[X] + 1) + \frac{1}{4}(\mathbb{E}[X] + 2) + \frac{1}{8}(\mathbb{E}[X] + 3) + \frac{1}{8} \times 3$$

$$\boxed{X : \#\text{ of tosses to get } HHH}$$

$$T, \quad HT, \quad HHT, \quad HHH$$

$$\mathbb{E}[X] = \frac{1}{2}(\mathbb{E}[X] + 1) + \frac{1}{4}(\mathbb{E}[X] + 2) + \frac{1}{8}(\mathbb{E}[X] + 3) + \frac{1}{8} \times 3$$

$$\boxed{X : \#\text{ of tosses to get } HTH}$$

$$T, \quad HH, \quad HTT, \quad HTH$$

$$\mathbb{E}[X] = \frac{1}{2}(\mathbb{E}[X] + 1) + \frac{1}{4}(\mathbb{E}[X] + 2) + \frac{1}{8}(\mathbb{E}[X] + 3) + \frac{1}{8} \times 3$$

$$\boxed{\forall P \in \{H, T\}^3 : \mathbb{E}[X_P] = 14}$$

$$\boxed{X : \#\text{ of tosses to get } HHH}$$

$$T, \quad HT, \quad HHT, \quad HHH$$

$$\mathbb{E}[X] = \frac{1}{2}(\mathbb{E}[X]+1) + \frac{1}{4}(\mathbb{E}[X]+2) + \frac{1}{8}(\mathbb{E}[X]+3) + \frac{1}{8} \times 3$$

$$\boxed{X : \#\text{ of tosses to get } HTH}$$

$$T, \quad HH, \quad HTT, \quad HTH$$

$$\mathbb{E}[X] = \frac{1}{2}(\mathbb{E}[X]+1) + \frac{1}{4}(\mathbb{E}[X]+2) + \frac{1}{8}(\mathbb{E}[X]+3) + \frac{1}{8} \times 3$$

$$\boxed{\forall P \in \{H,T\}^3 : \mathbb{E}[X_P] = 14}$$

$$\boxed{\forall P \in \{H,T\}^n : \mathbb{E}[X_P] = 2(2^n - 1)}$$

$\boxed{X : \# \text{ of tosses to get } HHH}$

$$T, \quad HT, \quad HHT, \quad HHH$$

$$\mathbb{E}[X] = \frac{1}{2}(\mathbb{E}[X] + 1) + \frac{1}{4}(\mathbb{E}[X] + 2) + \frac{1}{8}(\mathbb{E}[X] + 3) + \frac{1}{8} \times 3$$

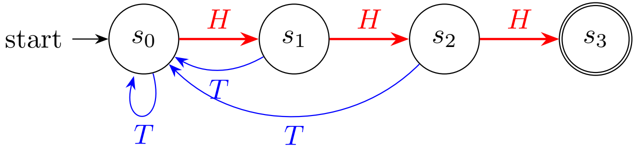
$\boxed{X : \# \text{ of tosses to get } HTH}$

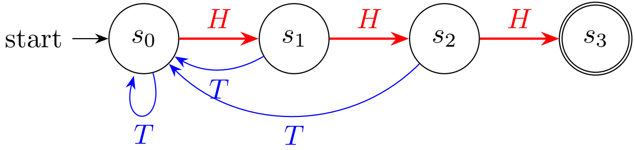
$$T, \quad HH, \quad HTT, \quad HTH$$

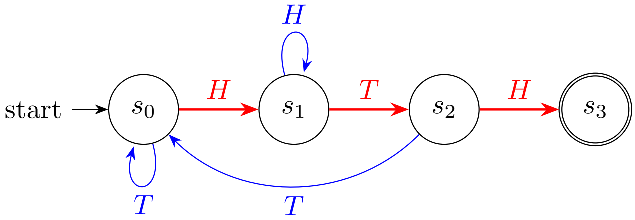
$$\mathbb{E}[X] = \frac{1}{2}(\mathbb{E}[X] + 1) + \frac{1}{4}(\mathbb{E}[X] + 2) + \frac{1}{8}(\mathbb{E}[X] + 3) + \frac{1}{8} \times 3$$

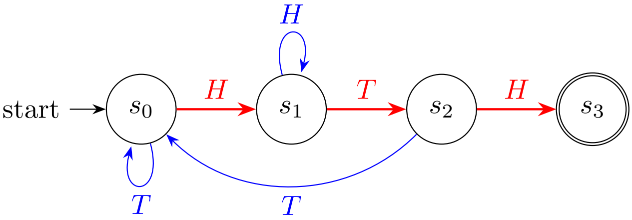
$$\boxed{\forall P \in \{H, T\}^3 : \mathbb{E}[X_P] = 14}$$

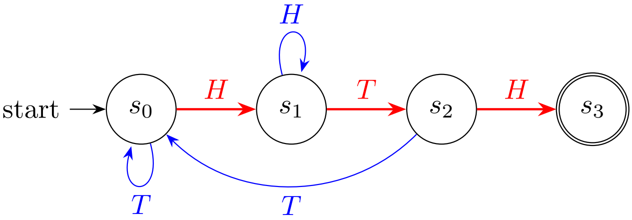
$$\boxed{\forall P \in \{H, T\}^n : \mathbb{E}[X_P] = 2(2^n - 1)}$$


start
s_0
s_1
s_2
s_3
H
H
H
T
T
T

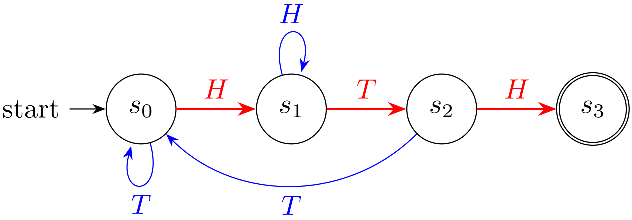

start
s_0
s_1
s_2
s_3
H
H
H
T
T
T


start
s_0
s_1
s_2
s_3
H
H
T
H
T
T


start
s_0
s_1
s_2
s_3
H
T
H
H
T
T



$\boxed{S_i: \text{ Expected number of tosses to reach state } s_i}$



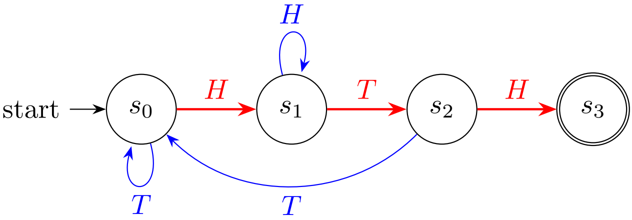
S_i : Expected number of tosses to reach state s_i

$$S_0 = \frac{1}{2}(1 + S_0 + 1 + S_1)$$

$$S_1 = \frac{1}{2}(1 + S_1 + 1 + S_2)$$

$$S_2 = \frac{1}{2}(1 + S_0 + 1 + S_3)$$

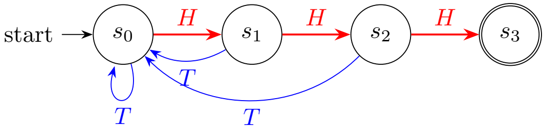
$$S_3 = 0$$

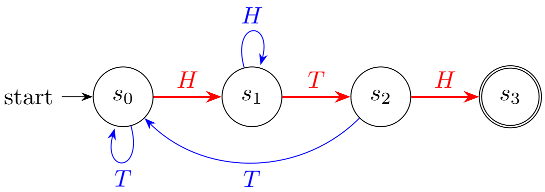


S_i : Expected number of tosses to reach state s_i

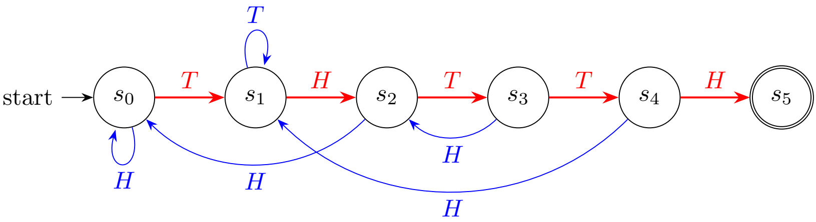
$$S_0 = \frac{1}{2}(1 + S_0 + 1 + S_1)$$
$$S_1 = \frac{1}{2}(1 + S_1 + 1 + S_2)$$
$$S_2 = \frac{1}{2}(1 + S_0 + 1 + S_3)$$
$$S_3 = 0$$

$$S_0 = 10$$
$$S_1 = 8$$
$$S_2 = 6$$
$$S_3 = 0$$

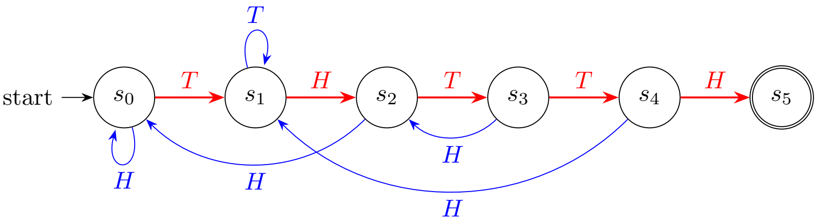




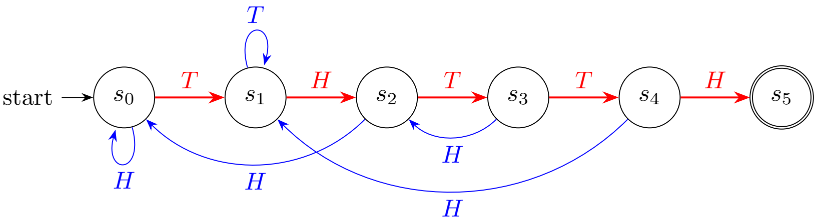
$$\mathbb{E}[X_{HHH}] = 14 > \mathbb{E}[X_{HTH}] = 10$$



start
s_0
s_1
s_2
s_3
s_4
s_5
T
H
T
T
H
T
H
H
H
H



$$S_0 = \frac{1}{2}(1 + S_0 + 1 + S_1)$$
$$S_1 = \frac{1}{2}(1 + S_1 + 1 + S_2)$$
$$S_2 = \frac{1}{2}(1 + S_0 + 1 + S_3)$$
$$S_3 = \frac{1}{2}(1 + S_2 + 1 + S_4)$$
$$S_4 = \frac{1}{2}(1 + S_1 + 1 + S_5)$$
$$S_5 = 0$$



$$S_0 = \frac{1}{2}(1 + S_0 + 1 + S_1)$$

$$S_1 = \frac{1}{2}(1 + S_1 + 1 + S_2)$$

$$S_2 = \frac{1}{2}(1 + S_0 + 1 + S_3)$$

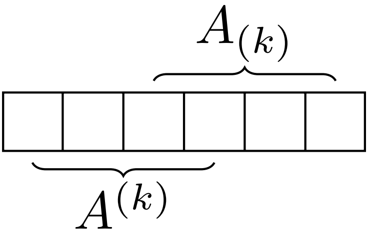
$$S_3 = \frac{1}{2}(1 + S_2 + 1 + S_4)$$

$$S_4 = \frac{1}{2}(1 + S_1 + 1 + S_5)$$

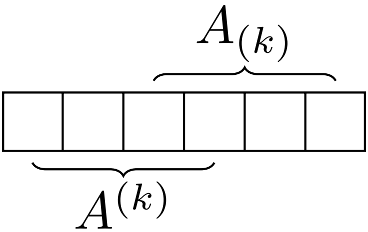
$$S_5 = 0$$

$$A : A = \sum_{k=1}^{n} 2^{k-1} [A^{(k)} = A_{(k)}]$$

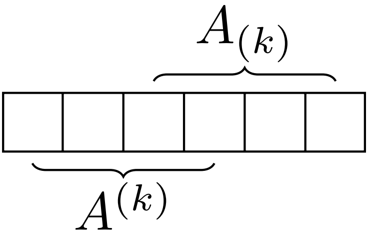
$$A : A = \sum_{k=1}^{n} 2^{k-1} [A^{(k)} = A_{(k)}]$$



$$A : A = \sum_{k=1}^{n} 2^{k-1} [A^{(k)} = A_{(k)}] \qquad \mathbb{E}[X_A] = 2(A : A)$$

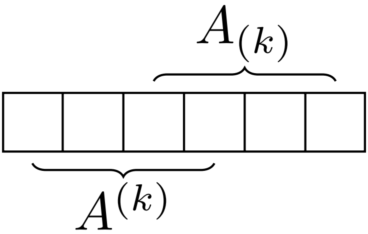


$$A : A = \sum_{k=1}^{n} 2^{k-1} [A^{(k)} = A_{(k)}] \qquad \mathbb{E}[X_A] = 2(A : A)$$



$$A = THTTH$$

$$A : A = \sum_{k=1}^{n} 2^{k-1} [A^{(k)} = A_{(k)}] \qquad \mathbb{E}[X_A] = 2(A : A)$$



$$A = THTTH \qquad \mathbb{E}[X_A] = 2(2^1 + 2^4) = 36$$

$$A = HTHTHHTHTH \qquad \mathbb{E}[X_A] = 2(2^0 + 2^2 + 2^4 + 2^9) = 1066$$

$$\boxed{A : A = \sum_{k=1}^{n} 2^{k-1} [A^{(k)} = A_{(k)}] \qquad \mathbb{E}[X_A] = 2(A : A)}$$

$$\boxed{A : A = \sum_{k=1}^{n} 2^{k-1}[A^{(k)} = A_{(k)}] \qquad \mathbb{E}[X_A] = 2(A : A)}$$

$$\mathbb{E}[X_{H^n}] = 2(2^0 + 2^1 + 2^{n-1}) = 2(2^n - 1)$$

$$\boxed{A : A = \sum_{k=1}^{n} 2^{k-1} [A^{(k)} = A_{(k)}] \qquad \mathbb{E}[X_A] = 2(A : A)}$$

$$\mathbb{E}[X_{H^n}] = 2(2^0 + 2^1 + 2^{n-1}) = 2(2^n - 1)$$

$$\mathbb{E}[X_{H^{\lceil \frac{n}{2} \rceil} T^{\lfloor \frac{n}{2} \rfloor}}] = 2(2^{n-1}) = 2^n$$

Definition (Conditional Expectation on an Event)

$$\mathbb{E}[X \mid E] = \sum_{x} x \Pr(X = x \mid E)$$

Definition (Conditional Expectation on an Event)

$$\mathbb{E}[X \mid E] = \sum_x x \Pr(X = x \mid E)$$

Definition (Conditional Expectation on a Random Variable)

$$\mathbb{E}[X \mid Y = y] = \sum_x x \Pr(X = x \mid Y = y)$$

Definition (Conditional Expectation on an Event)

$$\mathbb{E}[X \mid E] = \sum_x x \Pr(X = x \mid E)$$

Definition (Conditional Expectation on a Random Variable)

$$\mathbb{E}[X \mid Y = y] = \sum_x x \Pr(X = x \mid Y = y)$$

Notation:

$$\mathbb{E}[X \mid Y](y) = \mathbb{E}[X \mid Y = y]$$

Definition (Conditional Expectation on an Event)

$$\mathbb{E}[X \mid E] = \sum_x x \Pr(X = x \mid E)$$

Definition (Conditional Expectation on a Random Variable)

$$\mathbb{E}[X \mid Y = y] = \sum_x x \Pr(X = x \mid Y = y)$$

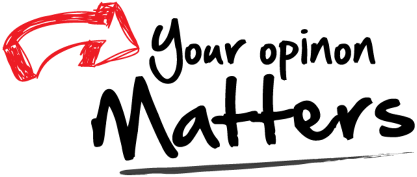
Notation:

$$\mathbb{E}[X \mid Y](y) = \mathbb{E}[X \mid Y = y]$$

Theorem

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid Y]] = \sum_y \mathbb{E}[X \mid Y = y] \Pr(Y = y)$$


Thank
You!



Office 302

Mailbox: H016

hfwei@nju.edu.cn