

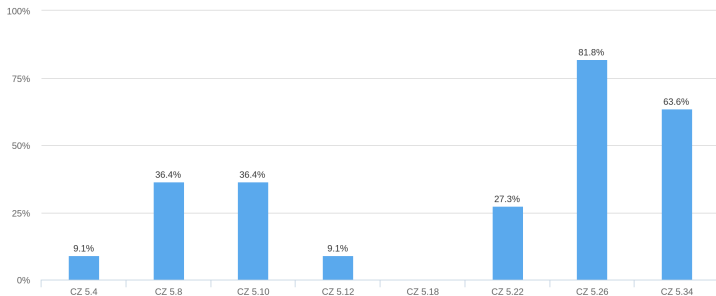
## 3-9 Connectivity

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5.10

5.34

5.22

5.26

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如果两个割点相连，那么联通块怎么划分！  
联通快呢）

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menger定理吧

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暂无

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好像没有.....

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Menger定理的证明看不懂

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menger定理的证明

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不。。不记得了

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还好理解，只是都不怎么容易理解

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menger定理的证明没太理解 老师辛苦了！

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点割集，边割集

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## Menger's Theorem (Theorem 5.16; Theorem 5.21)



## 2-Connectivity (Problem 5.10)

A connected graph  $G$  with  $m \geq 2$  is *nonseparable*



any two *adjacent* edges of  $G$  lie on a common cycle of  $G$ .

Proof.

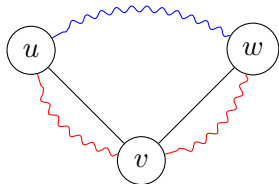
“  $\implies$  ”

$G$  is nonseparable

$\implies u, w$  lie on a common cycle

$\implies \exists$  path  $u \sim w$  that does not contain  $v$

$\implies \exists$  cycle  $u - v - w \sim u$



## 2-Connectivity (Problem 5.10)

A connected graph  $G$  with  $m \geq 2$  is *nonseparable*



any two *adjacent* edges of  $G$  lie on a common cycle of  $G$ .

Proof.

“ $\Leftarrow$ ”

By Contradiction.

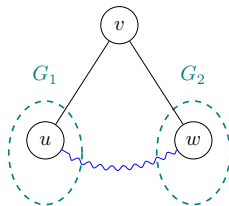
Suppose  $v$  is a cut-vertex of  $G$

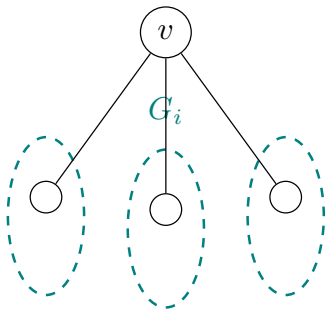
$\implies G - v$  contains  $\geq 2$  comps  $G_1, G_2, \dots$

$\implies \exists u \in G_1, w \in G_2 : v - u \wedge v - w$

$\implies v - u, v - w$  lie on a common cycle

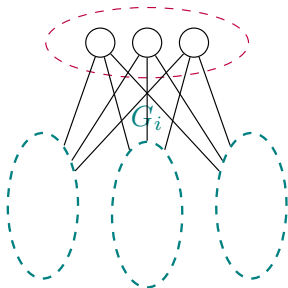
$\implies \exists$  path  $u \sim w$  that does not contain  $v$





$$\forall G_i \exists v_i \in G_i : v - v_i$$

$S$  : Minimum Vertex Cut



$$\forall v \in S \forall G_i \exists v_i \in G_i : v - v_i$$

## 2-Connectivity (Problem 5.10)

A connected graph  $G$  with  $m \geq 2$  is *nonseparable*



any two *adjacent* edges of  $G$  lie on a common cycle of  $G$ .

## 2-Connectivity (Extended Problem)

A connected graph  $G$  with  $m \geq 2$  is *nonseparable*

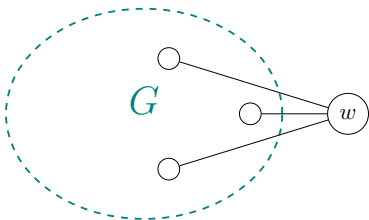


any two edges of  $G$  lie on a common cycle of  $G$ .

## Expansion Lemma (Problem 5.34; Theorem 5.18)

Let  $G$  be a  $k$ -connected graph and let  $S$  be any set of  $k$  vertices.

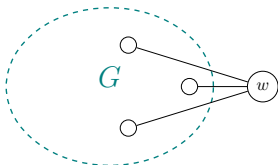
If a graph  $H$  is obtained from  $G$  by adding a new vertex  $w$  and joining  $w$  to the vertices of  $S$ , then  $H$  is also  $k$ -connected.





We prove that

$\forall v \in V(G) : \text{there exist } k \text{ internally disjoint } v - w \text{ paths}$

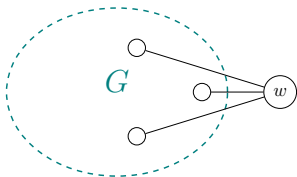


This holds because

$\forall v \in V(G) : \text{there exist internally disjoint } v - s_i \ (\forall s_i \in S) \text{ paths}$

**Corollary (5.19; Proved using Theorem 5.18)**

*If  $G$  is a  $k$ -connected graph and  $u, v_1, v_2, \dots, v_k$  are  $k + 1$  distinct vertices of  $G$ , then there exist internally disjoint  $u - v_i$  paths ( $1 \leq i \leq k$ ) in  $G$ .*



To prove  $\kappa(H) \geq k$

Let  $U$  be a vertex-cut of  $H$ .

We prove that  $|U| \geq k$ .

CASE I:  $U$  is a vertex-cut of  $G$

$$|U| \geq k$$

CASE II:  $U$  is not a vertex-cut of  $G$

$$w \in U$$

$U - w$  is a vertex-cut of  $G$

( $\because U$  is a vertex-cut of  $H$ )

$$|U| \geq k + 1$$

$$w \notin U$$

$$\implies U \subseteq G$$

$$\implies S \subseteq U \subseteq G$$

( $\because U$  is a vertex-cut of  $H$ )

$$\implies |U| \geq k$$

## 2-Connectivity (Extended Problem)

A connected graph  $G$  with  $m \geq 2$  is *nonseparable*



any two edges of  $G$  lie on a common cycle of  $G$ .

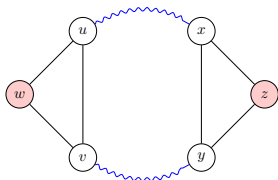


Consider two edges  $uv$  and  $xy$ .

Add  $w, z$

Add  $wu, wv; zx, zy$

$w$  and  $z$  lie on a common cycle



## Effects of Removing an Edge on Connectivity (Problem 5.22 (a))

- (a) If  $G$  is  $k$ -connected and  $e = uv \in E(G)$ , then  $G - e$  is  $(k - 1)$ -connected.

To prove  $\kappa(G) \geq k \implies \kappa(G - e) \geq k - 1$

Choose any  $U \subseteq V(G)$  with  $|U| < k - 1$ .

We prove that  $G - e - U$  is connected.

Choose any  $U \subseteq V(G)$  with  $|U| < k - 1$ .

We prove that  $G - e - U$  is connected.

$G$  is  $k$ -connected  $\implies G - U$  is connected

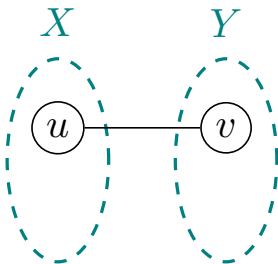
Suppose, by contradiction, that  $G - e - U$  is not connected.

$e = uv$  is a bridge of  $G - U$

$U \cup \{u\}$  is a vertex-cut of  $G$

But  $|U \cup \{u\}| < k$





CASE I :  $|X| \geq 2 \vee |Y| \geq 2$

$U \cup \{u\}$  is a vertex-cut of  $G$

But  $|U \cup \{u\}| < k$

CASE II :  $|X| = |Y| = 1$

$|U| = n - 2 < k - 1$

$\kappa(G) \geq k > n - 1$

But  $0 \leq \kappa(G) \leq n - 1$

## Effects of Removing an Edge on Connectivity (Problem 5.22 (b))

- (b) If  $G$  is  $k$ -edge-connected and  $e = uv \in E(G)$ , then  $G - e$  is  $(k - 1)$ -edge-connected.

$$\lambda(G) \geq k \implies \lambda(G - e) \geq k - 1$$

Choose any  $X \subseteq E(G)$  with  $|X| < k - 1$ .

We prove that  $G - e - X$  is connected.

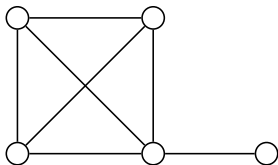
$G - e - X = G - (e + X)$  is connected ( $\because \lambda(G) \geq k$ )

$$\kappa(G - e) \leq \kappa(G)$$

## Effects of Removing a Vertex on Connectivity (Extended Problem)

Is  $\kappa(G - v) \leq \kappa(G)$ ?

Is  $\lambda(G - v) \leq \lambda(G)$ ?



## Effects of Removing a Vertex on Connectivity (After-class Exercise)

Is  $\kappa(G) \geq k \implies \kappa(G - v) \geq k - 1$ ?

$$\text{Is } \lambda(G) \geq 1 \implies \lambda(G - v) \geq 1 \quad 12$$

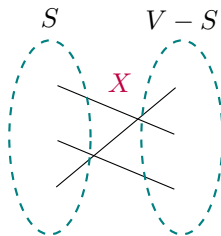


## Degree Condition for $\lambda(G) = \delta(G)$ (Problem 5.26)

If  $G$  is graph of order  $n$  such that  $\delta(G) \geq (n-1)/2$ , then  $\lambda(G) = \delta(G)$ .

$$\lambda(G) \leq \delta(G)$$

We prove that  $\lambda(G) \geq \delta(G)$ .



$$\lambda(G) = |X|$$

$$1 \leq |S| = k \leq n/2, \quad |V - S| = n - k$$

$$\lambda \geq k(\delta - (k - 1)) \geq \delta$$

Decision	Author(s)	Year	Complexity	Comments
<i>Edge Connectivity</i>				
$\lambda = 2$ or $\lambda = 3$	Tarjan [26]	1972	$O(m + n)$	uses Depth First Search
$\lambda$	Even and Tarjan [6]	1975	$O(mn \times \min\{m^{1/2}, n^{2/3}\})$	$n$ calls to max-flow
$\lambda$ (digraphs)	Schnorr [25]	1979	$O(\lambda mn)$	$n$ calls to max-flow
$\lambda$	Esfahanian & Hakimi [3]	1984	$O(\lambda mn)$	$\leq n/2$ calls to max-flow
$\lambda$ (digraphs)	Esfahanian & Hakimi [3]	1984	$O(\lambda mn)$	$\leq n/2$ calls to max-flow
$\lambda$	Matula [23]	1987	$O(mn)$	uses dominating sets
$\lambda = k$	Matula [23]	1987	$O(kn^2)$	
$\lambda$ (digraphs)	Mansour & Schieber [22]	1989	$O(mn)$	
$\lambda = k$	Gabow [9]	1991	$O(m+k^2n\log(n/k))$	uses matroids
<i>Vertex Connectivity</i>				
$\kappa = 2$	Tarjan [26]	1972	$O(m + n)$	uses Depth First Search
$\kappa = 3$	Hopcroft & Tarjan [18]	1973	$O(m + n)$	uses triconnected components
$\kappa$	Even & Trajan [6]	1975	$O((\kappa(n - \delta - 1)mn^{2/3}))$	max-flow based
$\kappa = k$	Even [4]	1975	$O(kn^3)$	max-flow based
$\kappa$	Galil [12]	1980	$O(\min\{\kappa, n^{2/3}\}mn)$	max-flow based
$\kappa = k$	Galil [12]	1980	$O(\min\{k, n^{1/2}\}kmn)$	max-flow based
$\kappa$	Esfahanian & Hakimi [3]	1984	$O((n - \delta - 1 + \delta(\delta - 1)/2)mn^{2/3})$	max-flow based
$\kappa = 4$	Kanevsky & Ramachandran [20]	1991	$O(n^2)$	
$\kappa$	Henzinger & Rao [17]	1996	$O(\kappa mn \log n)$	randomised algorithm

**Table 1:** A chronology of connectivity algorithms

## Theorem (Menger's Theorem (Theorem 5.16))

Let  $u$  and  $v$  be *nonadjacent* vertices in a graph  $G$ .

The *minimum number of vertices in a  $u - v$  separating set* equals the *maximum number of internally disjoint  $u - v$  paths in  $G$ .*

How do CASE 1, CASE 2, and CASE 3 cover all possibilities?

Are CASE 1 and CASE 2 mutually exclusive?

What is the key to use the induction hypothesis in CASE 2?

Are CASE 1 and CASE 3 mutually exclusive?

What will fail if we do not exclude CASE 1 from CASE 3?

Can you restate these three cases in terms of  $N(u)$  and  $N(v)$ ?

Can you rearrange these three cases to make them (hopefully) easier to understand?

**CASE I:** There exists a minimum  $u - v$  separating set  $W$  in  $G$  containing a vertex  $x$  that is adjacent to both  $u$  and  $v$ .

$$\exists W : \exists x \in W : x - u \wedge x - v$$

**CASE II:** There exists a minimum  $u - v$  separating set  $W$  in  $G$  containing a vertex in  $W$  that is not adjacent to  $u$  and a vertex in  $W$  that is not adjacent to  $v$ .

$$\begin{aligned} \exists W : \exists x \in W : x \not- u \\ \wedge \exists y \in W : y \not- v \end{aligned}$$

**CASE III:** For each minimum  $u - v$  separating set  $W$  in  $G$ , either every vertex of  $W$  is adjacent to  $u$  and not adjacent to  $v$  or every vertex of  $W$  is adjacent to  $v$  and not adjacent to  $u$ .

$$\begin{aligned} \forall W : \forall x \in W : x - u \wedge x \not- v \\ \vee \forall x \in W : x - v \wedge x \not- u \end{aligned}$$

$$\text{I : } \exists W : \exists x \in W : x - u \wedge x - v \quad \text{I' : } \forall W : \forall x \in W : x \not- u \vee x \not- v$$

$$\text{II : } \exists W : \exists x \in W : x \not- u \\ \wedge \exists y \in W : y \not- v$$

$$\text{II' : } \forall W : \forall x \in W : x - u \\ \vee \forall y \in W : y - v$$

$$\text{III : } \forall W : \forall x \in W : x - u \wedge x \not- v \\ \vee \forall x \in W : x - v \wedge x \not- u$$

$$\text{III} \equiv \text{II'} \wedge \text{I'}$$

II

II'

I

III

II

$$\text{II} : \exists W : \exists x \in W : x \not\sim u \\ \wedge \exists y \in W : y \not\sim v$$

II'

$$\text{I} : \exists W : \exists x \in W : x \sim u \wedge x \sim v$$

$$\text{III} : \forall W : \forall x \in W : x \sim u \wedge x \not\sim v \\ \vee \forall x \in W : x \sim v \wedge x \not\sim u$$

II

$$\exists W : W \not\subseteq N(u) \\ \wedge W \not\subseteq N(v)$$

II'

$$\exists W : \exists x \in W : x \in N(u) \cap N(v)$$

$$\forall W : W \subseteq N(u) \wedge W \cap N(v) = \emptyset \\ \vee W \subseteq N(v) \wedge W \cap N(u) = \emptyset$$

## II

$$\text{II} : \exists W : W \not\subseteq N(u) \\ \wedge W \not\subseteq N(v)$$

$Q$  : What is the key to use the induction hypothesis in CASE II?

## II'

$$\text{I} : \exists W : \exists x \in W : x \in N(u) \cap N(v)$$

$$\text{III} : \forall W : W \subseteq N(u) \wedge W \cap N(v) = \emptyset \\ \vee W \subseteq N(v) \wedge W \cap N(u) = \emptyset$$

$Q$  : What will fail if we do not exclude CASE I from CASE III?

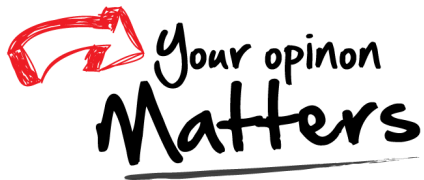
Theorem (Menger's Theorem for Edge-Connectivity (Theorem 5.21))

*For distinct vertices  $u$  and  $v$  in a graph  $G$ ,*

*the minimum number of edges of  $G$  that separate  $u$  and  $v$   
equals the maximum number of pairwise edge-disjoint  $u - v$  paths in  $G$ .*







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