3-10 Traversability

(Part II: Hamiltonian Graphs)

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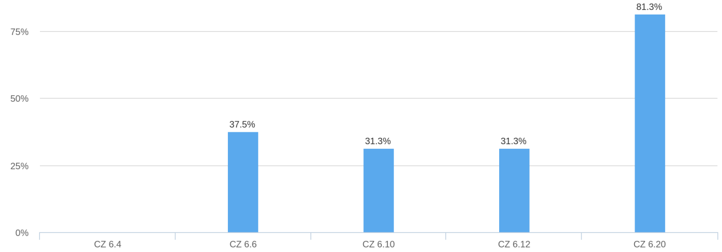

CZ 6.20

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Graphs with "bad" connectedness are not Hamiltonian.

Theorem (Ore's Theorem, 1960; Theorem 6.6)

Let G be a graph of order $n \geq 3$. If

$$deg(u) + deg(v) \geq n$$

for each pair u, v of nonadjacent vertices of G, then G is Hamiltonian.

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Contradiction: This critical G is actually Hamiltonian.

□

Theorem (Dirac's Theorem, 1952; Corollary 6.7)

Let G be a graph of order $n \geq 3$. If

$$\forall v \in V(G) : deg(v) \geq n/2,$$

then G is Hamiltonian.

Theorem (Ore's Theorem, 1960; Theorem 6.8)

Let u and v be nonadjacent vertices in a graph G of order n such that

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Then $G + uv$ is Hamiltonian $\iff$ G is Hamiltonian.

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Definition (Closure $C(G)$)

The closure $C(G)$ of a graph G is the graph obtained from G by iteratively adding edges joining pairs of nonadjacent vertices u and v such that $\deg(u) + \deg(v) \geq n$, until no such pair remains.

Theorem (Bondy-Chavátal Theorem, 1976; Theorem 6.9)

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Corollary (Corollary 6.10)

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Theorem (Lajos Pósa)

Let G be a graph of order $n \geq 3$. If for each integer j with $1 \leq j \leq \frac{n}{2}$, the number of vertices of G with degree at most j is less than j, then G is Hamiltonian.

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By induction on the order e_i is added to G_1.

Hamiltonian Graphs and 2-Connectedness (Problem 6.20)

Let G be a graph of order $n \geq 3$ having the property that for each $v \in V(G)$, there is a Hamiltonian path with initial vertex v.
Show that G is 2-connected but not necessarily Hamiltonian.

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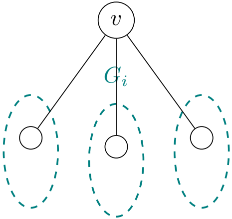

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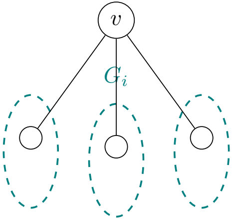

Contradiction: No Hamiltonian path with initial vertex v.

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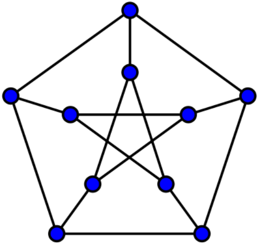


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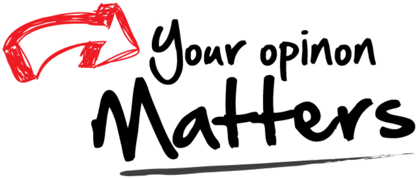

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