1-5 Data Structures

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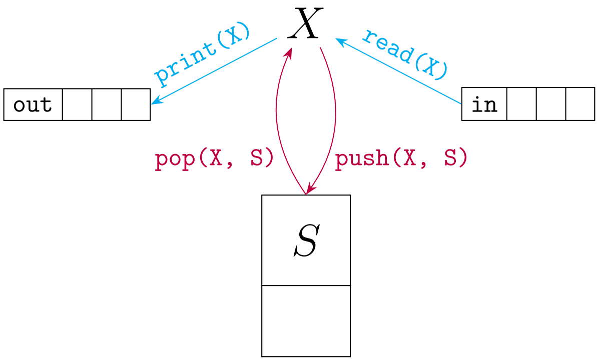
Pseudocode

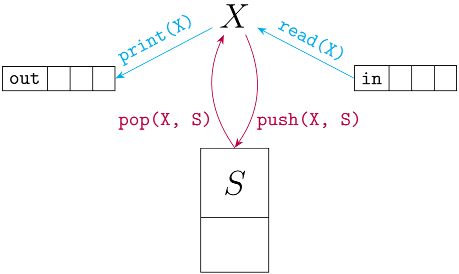
"Executable" at an abstract level.

Stackable Permutations

Definition (Stackable Permutations)

$$\boxed{\texttt{out} = (a_1, \cdots, a_n) \xleftarrow[X = \perp]{S = \emptyset} \texttt{in} = (1, \cdots, n)}$$

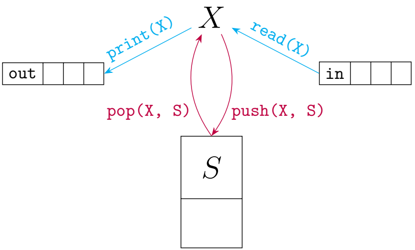


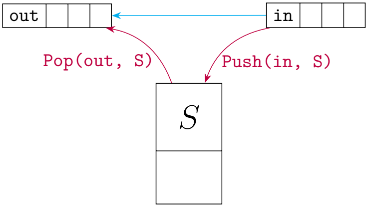


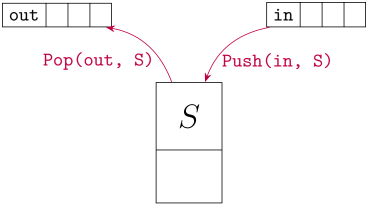
We can assume that X is always blank.

read + push   read + print

pop + print   pop + push

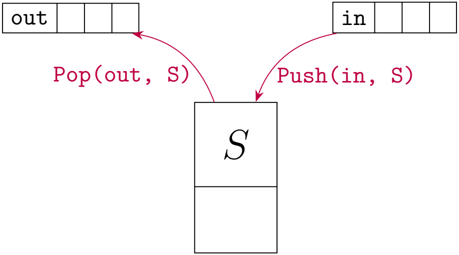

X
print(X)
read(X)
out
in
pop(X, S)
push(X, S)
S


out
in
Pop(out, S)
Push(in, S)
S


out
in
Pop(out, S)
Push(in, S)
S

Definition (Stackable Permutations)

$$\boxed{\texttt{out} = (a_1, \cdots, a_n) \xleftarrow{S = \emptyset} \texttt{in} = (1, \cdots, n)}$$



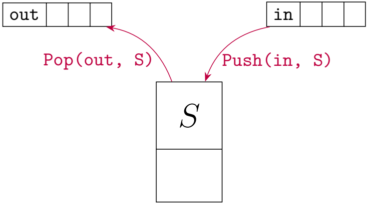
DH 2.12: Stackable Permutations

(a) Show that the following permutations *are* stackable:

(i) $(3, 2, 1)$

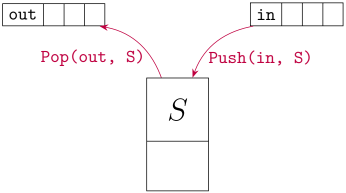
(ii) $(3, 4, 2, 1)$

(iii) $(3, 5, 7, 6, 8, 4, 9, 2, 10, 1)$



DH 2.13: Stackable Permutations Checking Algorithm

To check whether a given permutation can be obtained by a stack.

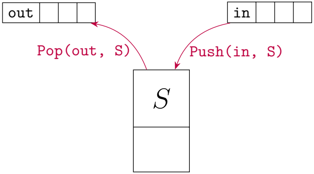


procedure STACKABLE(out)
  for all $a_j \in out$ **do**
    while top$(S) \neq a_j$ **do**
      Push(in, S)
    Pop(out, S)

Q : What is wrong with STACKABLE*?*

DH 2.13: Stackable Permutations Checking Algorithm

To check whether a given permutation can be obtained by a stack.



procedure STACKABLE(out)
    for all $a_j \in out$ **do**
        while $\mathsf{top}(S) \neq a_j \land in \neq \emptyset$ **do**
            $\mathsf{Push}(in, S)$
        if $\mathsf{top}(S) = a_j$ **then**
            $\mathsf{Pop}(out, S)$
        else $\triangleright$ $\mathsf{top}(S) \neq a_j \land in = \emptyset$
            return F
    return T

DH 2.12: Stackable Permutations

(b) **Prove** that the following permutations are *not* stackable:

- (i) $(3, 1, 2)$
- (ii) $(4, 5, 3, 7, 2, 1, 6)$

$$(3, 1, 2)$$

$$(4, 5, 3, 7, 2, 1, 6)$$

$$\boxed{\texttt{out} = \cdots a_i \cdots a_j \cdots a_k : i < j < k \land a_j < a_k < a_i}$$

312-Pattern

Theorem (Stackable Permutations)

A permutation $(a_1, \cdots, a_n)$ *is stackable* $\iff$ *it is not the case that*

$$312\text{-}Pattern: \boxed{out = \cdots a_i \cdots a_j \cdots a_k : i < j < k \land a_j < a_k < a_i}$$

Proof.

stackable $\Longrightarrow \nexists$ 312-Pattern

$\nexists$ 312-Pattern $\Longrightarrow$ stackable

312-Pattern $\Longrightarrow$ non-stackable

$\exists$ algorithm

$\square$

Theorem (Stackable Permutations)

A permutation $(a_1, \cdots, a_n)$ *is stackable* $\iff$ *it is not the case that*

$$312\text{-}Pattern : \boxed{out = \cdots a_i \cdots a_j \cdots a_k : i < j < k \land a_j < a_k < a_i}$$

312-Pattern $\implies$ non-stackable.

$i < j \land a_j < a_i$: $\mathsf{Push}_j \quad \mathsf{Push}_i \quad \mathsf{Pop}_i \quad \mathsf{Pop}_j$

$j < k \land a_j < a_k$: $\mathsf{Push}_j \quad \mathsf{Pop}_j \quad \mathsf{Push}_k \quad \mathsf{Pop}_k$

$i < k \land a_k < a_i$: $\mathsf{Push}_k \quad \mathsf{Push}_i \quad \mathsf{Pop}_i \quad \mathsf{Pop}_k$

$\square$

Theorem (Stackable Permutations)

A permutation $(a_1, \cdots, a_n)$ *is stackable* $\iff$ *it is not the case that*

$$\text{312-}\mathit{Pattern}: \boxed{out = \cdots a_i \cdots a_j \cdots a_k : i < j < k \land a_j < a_k < a_i}$$

$\nexists$ 312-Pattern $\implies$ Obtainable by STACKABLE.

STACKABLE fails $\implies$ $\exists$ 312-Pattern.

```
procedure STACKABLE(out)
    for all a_j ∈ out do
        while top(S) ≠ a_j ∧ in ≠ ∅ do
            Push(in, S)

        if top(S) = a_j then
            Pop(out, S)
        else        ▷ top(S) ≠ a_j ∧ in = ∅
            return F

    return T
```

$a_j \neq \mathsf{top}(S) \land in = \emptyset$

a_j is covered by some a_k in S

$\exists k : j < k \land a_j < a_k$

Why is a_k in S?

$\exists i : i < j \land a_k < a_i$

DH 2.12: Stackable Permutations

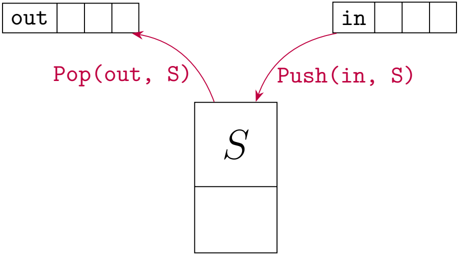
(c) How many permutations of A_4 *cannot* be obtained by a stack?

$$(1,4,2,3),(2,4,1,3),(3,1,2,4),(3,1,4,2),(3,4,1,2)$$
$$(4,1,2,3),(4,1,3,2),(4,2,1,3),(4,2,3,1),(4,3,1,2)$$

Q : What about A_n?

DH 2.12: Stackable Permutations

How many permutations of $\{1 \cdots n\}$ are stackable?



Q : How many *admissible* operation sequences of "`Push`" and "`Pop`"?

Definition (Admissible Operation Sequences)

An operation sequence of "`Push`" and "`Pop`" is *admissible* if and only if

(i) # of "`Push`" $= n$ # of "`Pop`" $= n$

(ii) $\forall$ prefix : (# of "`Pop`") $\leq$ (# of "`Push`")

$$\boxed{\text{\# of admissible operation sequences} = \text{\# of stackable perms}}$$

$$\{\text{admissible operation sequences}\} \xrightarrow{\exists f : 1-1} \{\text{stackable perms}\}$$

$$f(s) \triangleq \textit{Execute} \text{ this admissible operation sequence } s$$

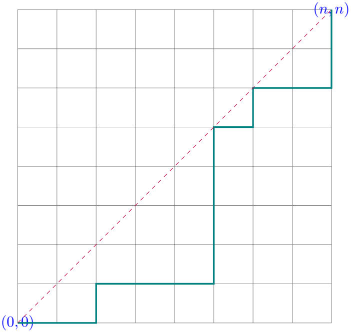
Why is f bijective (1-1)?

Theorem

The number of admissible operation sequences of "Push" and "Pop" is $\binom{2n}{n} - \binom{2n}{n-1}$.

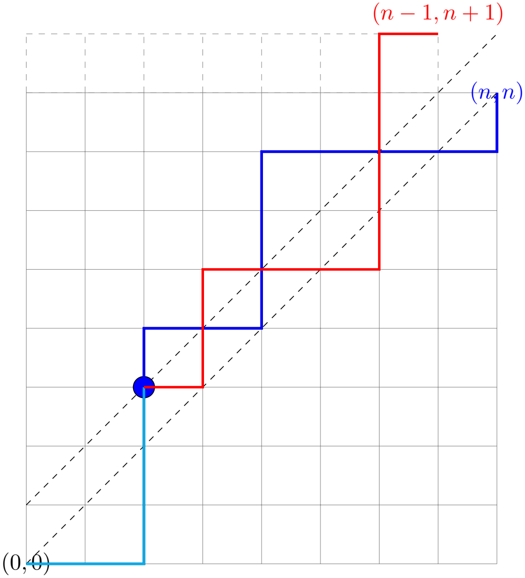
Proof: The Reflection Method.

$$\texttt{Push} : \rightarrow \qquad \texttt{Pop} : \uparrow$$



$$\underbrace{\binom{2n}{n}}_{\text{all}} - \underbrace{\binom{2n}{n-1}}_{\text{inadmissible}}$$

$\square$


$(n-1, n+1)$
(n, n)
$(0, 0)$

$$\binom{2n}{n} - \binom{2n}{n-1}$$

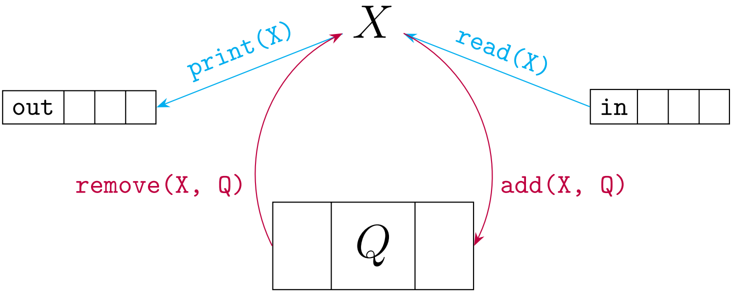
Catalan Number

$$(3,2,1) : ((())) \qquad (1,2,3) : ()()()$$

Queueable Permutations

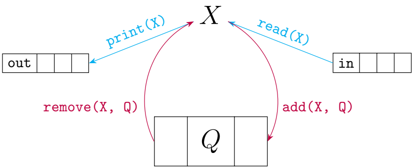
DH 2.14: Queueable Permutations

$$\boxed{\texttt{out} = (a_1, \cdots, a_n) \xleftarrow[X = \perp]{Q = \emptyset} \texttt{in} = (1, \cdots, n)}$$



DH 2.14: Queueable Permutations

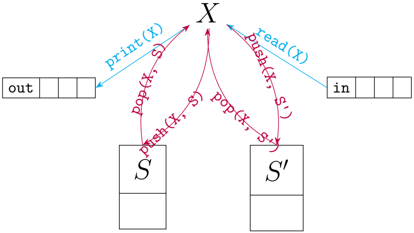
(b) Prove that every permutation are queueable.



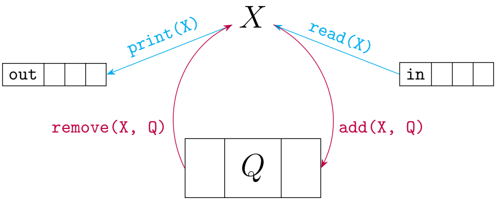

```
procedure QUEUEABLE(out)
    for all a ∈ in do
        read(X)
        add(X, Q)
    for all a ∈ out do
        while Head(Q) ≠ a do
            remove(X, Q)
            add(X, Q)
        remove(X, Q)
        print(X)
```

DH 2.14: Queueable Permutations

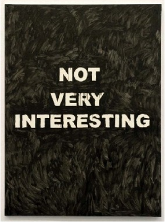
(c) Prove that every permutation can be obtained by two stacks.

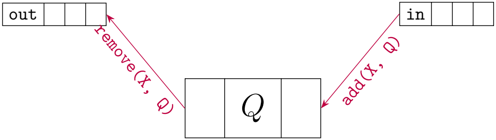



```
procedure DoubleStackable(out)
    for all a ∈ in do
        read(X)
        push(X, S)
    for all a ∈ out do
        while top(S) ≠ a do
            pop(X, S)
            push(X, S')
        pop(X, S)
        print(X)
        while S' ≠ ∅ do
            pop(X, S')
            push(X, S)
```

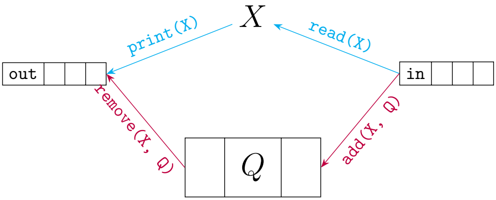


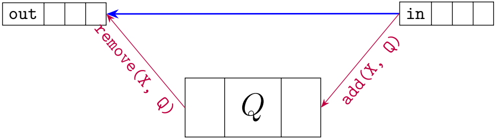
All are queueable.





Only one is queueable.





3 2 1 *is not queueable*

Theorem (Queueable Permutations)

A permutation $(a_1, \cdots, a_n)$ *is* *queueable* $\iff$ *it is not the case that*

$$321\text{-}Pattern : \boxed{out = \cdots a_i \cdots a_j \cdots a_k : i < j < k \land a_i > a_j > a_k}$$

Proof.

□

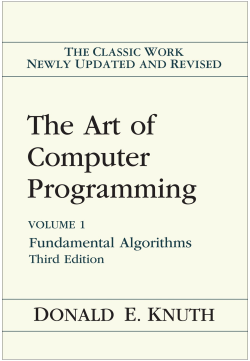
Theorem (# of Queueable Permutations)

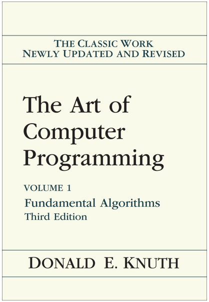
The number of queueable permutations of $[1 \cdots n]$ *is* $\binom{2n}{n} - \binom{2n}{n-1}$.

Proof.

$\square$

For more about “Stackable/Queueable Permutations”
(Section 2.2.1)

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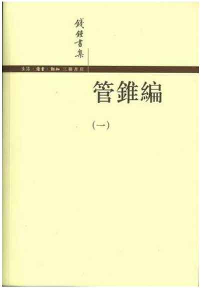
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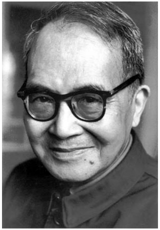


Thank
You!