How to find a particular solution for a non-homogeneous recurrence relation

Consider a non-homogeneous linear recurrence relation of order k with constant coefficients:

$$c_0 a_n + c_1 a_{n-1} + \ldots + c_k a_{n-k} = f(n). \tag{1}$$

Let

$$a_n^{(p)} \text{ be a particular solution of the recurrence relation (1)}$$

and let

$$a_n^{(h)} \text{ be the general solution of the corrsponding homogeneous recurrence relation}$$

$$c_0 a_n + c_1 a_{n-1} + \ldots + c_k a_{n-k} = 0. \tag{2}$$

Note that $a_n^{(h)}$ involves k parameters. Then

$$a_n = a_n^{(h)} + a_n^{(p)} \tag{3}$$

is the general solution of the non-homogeneous recurrence relation (1). *(Prove this!)* Note that a_n also involves k parameters. And conversely: the general solution of the non-homogeneous recurrence relation (1) is of the form (3) for any fixed particular solution $a_n^{(p)}$. *(Prove this, too!)*

How do we find a particular solution $a_n^{(p)}$*?*

1. Find an appropriate trial set-up for $a_n^{(p)}$ in the table below.
2. Adjust your set-up using the *additional guidelines*. Your $a_n^{(p)}$ will involve a number of parameters.
3. Find the values for these parameters by observing that $a_n^{(p)}$ should be a solution of the *non-homogeneous* recurrence relation (1).

Trial set-up for $a_n^{(p)}$

$f(n)$	Trial set-up for $a_n^{(p)}$
$b_0 n^t + b_1 n^{t-1} + \ldots + b_{t-1} n + b_t$	$B_0 n^t + B_1 n^{t-1} + \ldots + B_{t-1} n + B_t$
r^n	Ar^n
$n^t r^n$	$r^n(B_0 n^t + B_1 n^{t-1} + \ldots + B_{t-1} n + B_t)$

Additional guidelines:

- If $f(n) = f_1(n) + f_2(n)$, then $a_n^{(p)}$ should be the sum of particular solutions corresponding to $f_1(n)$ and $f_2(n)$.
- If $f(n)$ (or a term in $f(n)$) is a solution of the corresponding homogeneous recurrence relation, then multiply the trial $a_n^{(p)}$ by n^s for the smallest integer s such that $n^s a_n^{(p)}$ (and every summand thereof) is NOT a solution of the homogeneous recurrence relation.

Example 1. Consider the second-order non-homogeneous linear recurrence relation with constant coefficients $a_n = aa_{n-1} + ba_{n-2} + f(n)$ with characteristic roots $r_1 = 2$ and $r_2 = 5$. Depending on $f(n)$, we seek a particular solution in the following form:

$f(n)$	Set-up for $a_n^{(p)}$
3	A
$5n^2 - 3n + 2$	$An^2 + Bn + C$
$(6n^2 + 1) \cdot 3^n$	$(An^2 + Bn + C) \cdot 3^n$
$5 \cdot 2^n$	$A2^n \cdot n$
$5 \cdot 2^n + 6 \cdot 3^n$	$An2^n + B \cdot 3^n$
$5 \cdot 2^n + 6 \cdot 5^n$	$An2^n + Bn5^n$

Example 2. Consider the second-order non-homogeneous linear recurrence relation with constant coefficients $a_n = aa_{n-1} + ba_{n-2} + f(n)$ with a double characteristic root $r_1 = r_2 = -2$. Depending on $f(n)$, we seek a particular solution in the following form:

$f(n)$	Set-up for $a_n^{(p)}$
3	A
$5n^2 - 3n + 2$	$An^2 + Bn + C$
$6 \cdot 3^n$	$A \cdot 3^n$
$5 \cdot (-2)^n$	$A(-2)^n \cdot n^2$
$5n \cdot (-2)^n$	$(An + B)(-2)^n \cdot n^2$
$5n^2 \cdot (-2)^n$	$(An^2 + Bn + C)(-2)^n \cdot n^2$