3-5 Minimum Spanning Trees

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[Problem: 4.8]

$$\forall v \in V(G) : \deg(v) \geq 2 \implies G \text{ contains a cycle}$$

Cut Property

$$G = (V, E, w)$$

Cut Property (I)

$$
\begin{aligned}
X &: \text{ A part of some MST } T \text{ of } G \\
(S, V \setminus S) &: \text{ A \textit{cut} such that } X \text{ does \textit{not} cross } (S, V \setminus S) \\
e &: \text{ \textit{A} lightest edge across } (S, V \setminus S)
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$(S, V \setminus S)$: A *cut* such that X does *not* cross $(S, V \setminus S)$

e : *A* lightest edge across $(S, V \setminus S)$

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Correctness of Prim's and Kruskal's algorithms.

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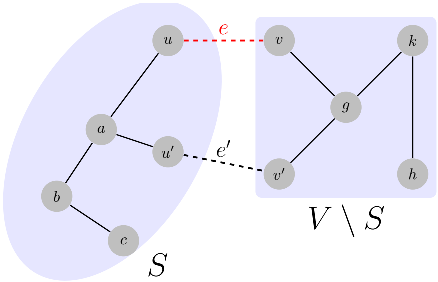

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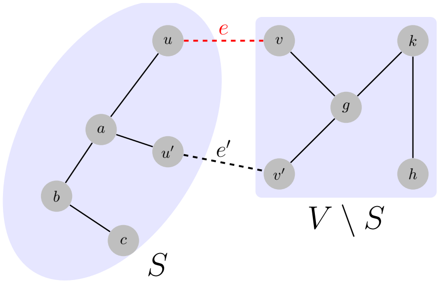

$$T' = \underbrace{T + \{e\}}_{\text{if } e \notin T} - \{e'\}$$

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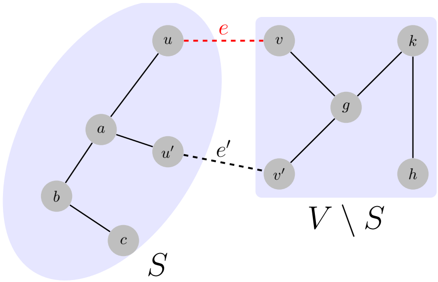

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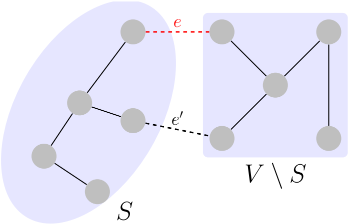
"a" $\to$ "the" $\Longrightarrow$ "some" $\to$ "all"

Cut Property (II)

A cut $(S, V \setminus S)$

Let $e = (u, v)$ be *a* lightest edge across $(S, V \setminus S)$

$\exists$ MST T of $G : e \in T$



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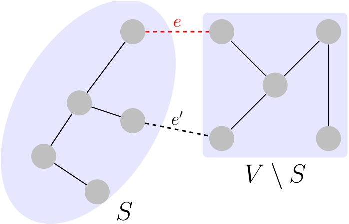

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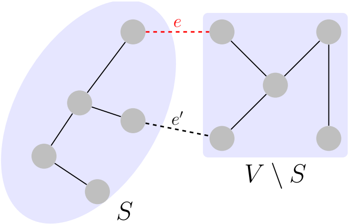

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"a" $\to$ "the" $\Longrightarrow$ "$\exists$" $\to$ "$\forall$"

Application of Cut Property [Problem: 10.15 (3)]

$$e = (u, v) \in G \text{ is a lightest edge } \implies e \in \exists \text{ MST of } G$$

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Wrong Divide&Conquer Algorithm for MST [Problem: 10.21]

$$(V_1, V_2) : \Big| |V_1| - |V_2| \Big| \le 1$$

$$T_1 + T_2 + \{e\} : e \text{ is a lightest edge across } (V_1, V_2)$$

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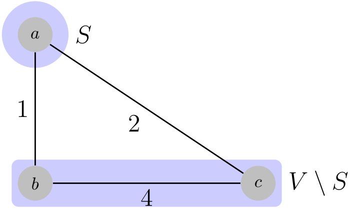
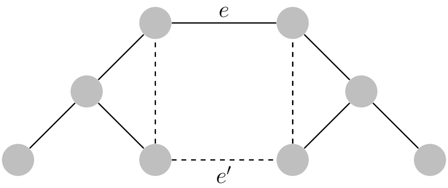


Cycle Property

Cycle Property [Problem: 10.19(b)]

- Let C be any cycle in G
- Let $e = (u, v)$ be *a* maximum-weight edge in C

$$\text{Then } \exists \text{ MST } T \text{ of } G : e \notin T.$$



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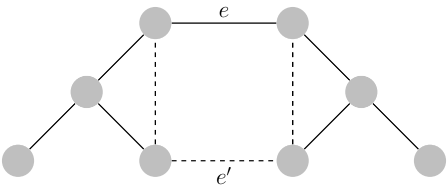

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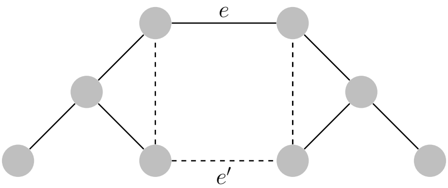

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"On the Shortest Spanning Subtree of a Graph and the Traveling Salesman Problem"

— Kruskal, 1956.

Application of Cycle Property [Problem: 10.15 (1)]

$$G = (V, E), \quad |E| > |V| - 1$$

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$$C \subseteq G, \quad e \in C$$

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Application of Cycle Property [Problem: 10.15 (2)]

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Application of Cycle Property [Problem: 10.15 (5)]

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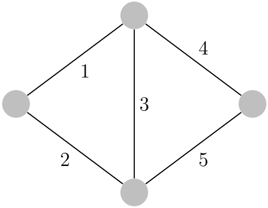

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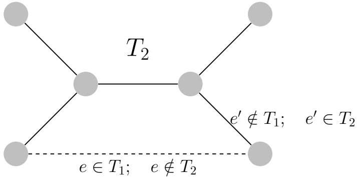

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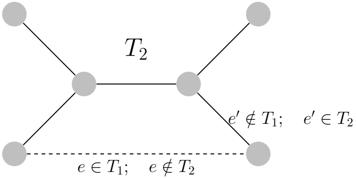

$$T_2 + \{e\} \implies C$$

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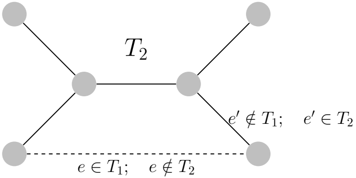

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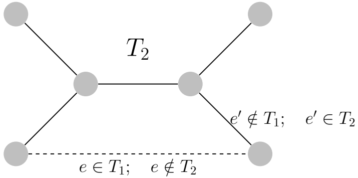

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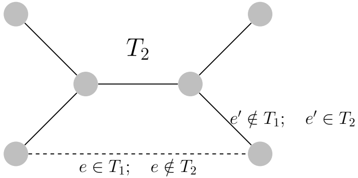

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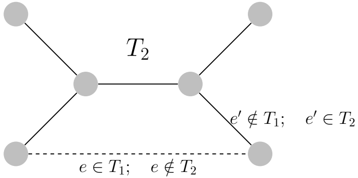

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$$T' = T_2 + \{e\} - \{e'\} \implies w(T') < w(T_2)$$

Condition for Uniqueness of MST [Problem: 10.18 (2)]

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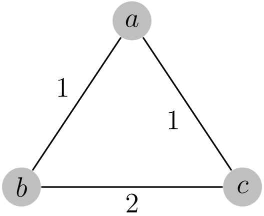

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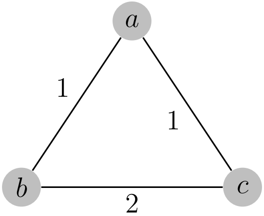

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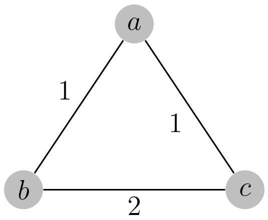

Theorem (After-class Exercise)

Minimum-weight edge across any cut is unique $\Longrightarrow$ *Unique MST.*

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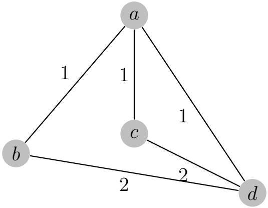

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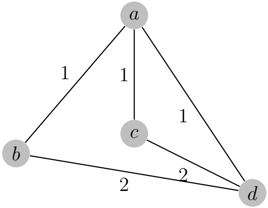

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To decide whether a graph has a unique MST.

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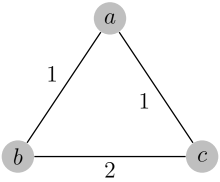

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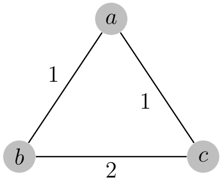

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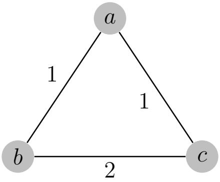

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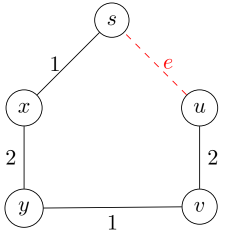

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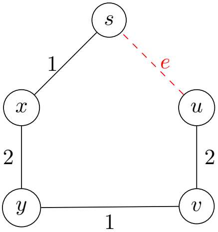

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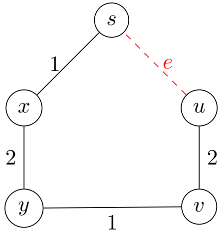

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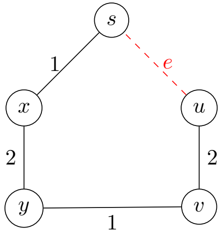

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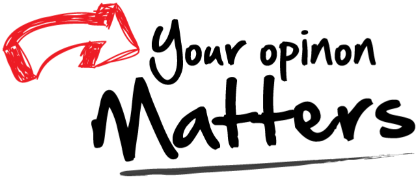

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