

2-7 Discrete Probability

“Life is a school of probability — Walter Bagehot.”

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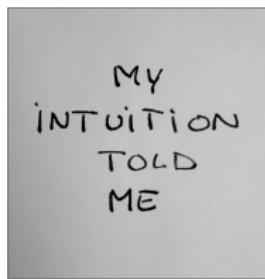


Q : What is Probability?



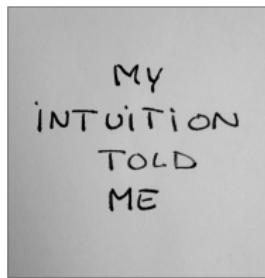
Q : What is Probability?

(Objective? Subjective? Neutral?)



*"...and the many **paradoxes** show clearly that we, as humans, lack a well grounded intuition in this matter."*

— “*The Art of Probability*”, Richard W. Hamming



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*“When called upon to judge probability, people actually judge something else and **believe** they have judged probability.”*

— “*Thinking, Fast and Slow*”, Daniel Kahneman



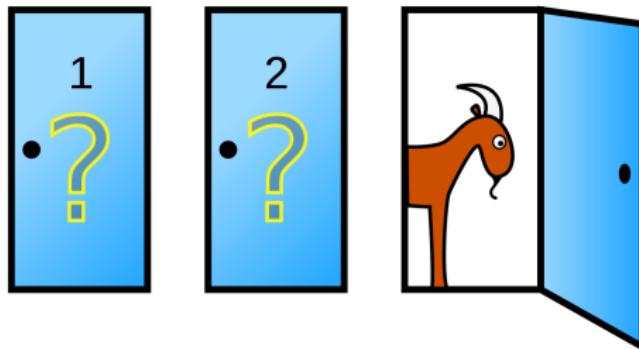


Let us calculate [calculemus].

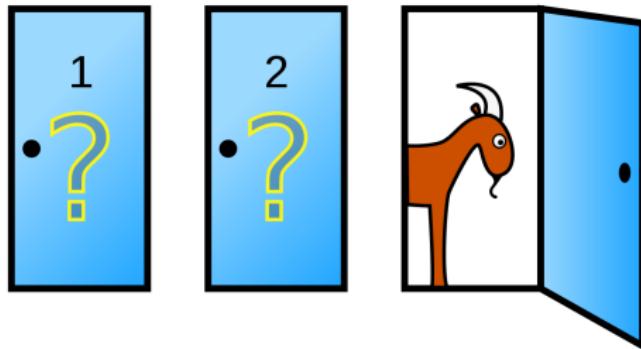


- (a) Monty Hall problem
- (b) Boy or Girl paradox
- (c) Searching unsorted array

The Monty-Hall Problem



The Monty-Hall Problem



You: Randomly pick a door (No. 1)

I: Open a door which has a goat (No. 3)
(I know what's behind the doors)

Q : Do you want to switch to door 2?

C_i : The car is behind door i ($i = 1, 2, 3$)

$$\Pr \{C_i\} = \frac{1}{3}$$

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ASSUMPTION: The car is initially hidden randomly behind the doors.

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ASSUMPTION: The car is initially hidden randomly behind the doors.

Y_1 : you initially pick door 1

$$\Pr \{Y_1\} = \frac{1}{3}$$

C_i : The car is behind door i ($i = 1, 2, 3$)

$$\Pr \{C_i\} = \frac{1}{3}$$

ASSUMPTION: The car is initially hidden randomly behind the doors.

Y_1 : you initially pick door 1

$$\Pr \{Y_1\} = \frac{1}{3}$$

ASSUMPTION: Your initial choice is random.

I_3 : I open door 3

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ASSUMPTION: I know what's behind the doors.

$$\Pr \{C_2 \mid I_3, Y_1\}$$

I_3 : I open door 3

ASSUMPTION: I know what's behind the doors.

ASSUMPTION: If you initially pick the car, then I open a door randomly.

$$\Pr \{C_2 \mid I_3, Y_1\}$$

I_3 : I open door 3

ASSUMPTION: I know what's behind the doors.

ASSUMPTION: If you initially pick the car, then I open a door randomly.

ASSUMPTION: I always open a door to reveal a goat and never the car.

$$\Pr \{C_2 \mid I_3, Y_1\}$$

$$\Pr \{C_2 | I_3, Y_1\} = \frac{\Pr \{C_2, I_3, Y_1\}}{\Pr \{I_3, Y_1\}}$$

$$\Pr \{C_2 | I_3, Y_1\} = \frac{\Pr \{C_2, I_3, Y_1\}}{\Pr \{I_3, Y_1\}} = \frac{\Pr \{I_3, Y_1 | C_2\} \Pr \{C_2\}}{\Pr \{I_3 | Y_1\} \Pr \{Y_1\}}$$

$$\begin{aligned}\Pr \{C_2 \mid I_3, Y_1\} &= \frac{\Pr \{C_2, I_3, Y_1\}}{\Pr \{I_3, Y_1\}} = \frac{\Pr \{I_3, Y_1 \mid C_2\} \Pr \{C_2\}}{\Pr \{I_3 \mid Y_1\} \Pr \{Y_1\}} \\ &= \frac{\Pr \{I_3, Y_1 \mid C_2\}}{\Pr \{I_3 \mid Y_1\}}\end{aligned}$$

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$$\Pr \{I_3, Y_1 \mid C_2\} = \Pr \{I_3 \mid C_2, Y_1\} \Pr \{Y_1 \mid C_2\}$$

$$\begin{aligned}\Pr \{C_2 | I_3, Y_1\} &= \frac{\Pr \{C_2, I_3, Y_1\}}{\Pr \{I_3, Y_1\}} = \frac{\Pr \{I_3, Y_1 | C_2\} \Pr \{C_2\}}{\Pr \{I_3 | Y_1\} \Pr \{Y_1\}} \\ &= \frac{\Pr \{I_3, Y_1 | C_2\}}{\Pr \{I_3 | Y_1\}}\end{aligned}$$

$$\begin{aligned}\Pr \{I_3, Y_1 | C_2\} &= \Pr \{I_3 | C_2, Y_1\} \Pr \{Y_1 | C_2\} \\ &= \frac{1}{3} \Pr \{I_3 | C_2, Y_1\}\end{aligned}$$

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$$\Pr \{C_2 | I_3, Y_1\} = \frac{\Pr \{I_3 | C_2, Y_1\}}{3 \Pr \{I_3 | Y_1\}}$$

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$$\Pr \{I_3 \mid Y_1\}$$

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{\Pr \{I_3 \mid C_2, Y_1\}}{3 \Pr \{I_3 \mid Y_1\}}$$

$$\begin{aligned}\Pr \{I_3 \mid Y_1\} &= \Pr \{I_3 \mid C_1, Y_1\} \Pr \{C_1 \mid Y_1\} \\&\quad + \Pr \{I_3 \mid C_2, Y_1\} \Pr \{C_2 \mid Y_1\} \\&\quad + \Pr \{I_3 \mid C_3, Y_1\} \Pr \{C_3 \mid Y_1\}\end{aligned}$$

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$$\Pr \{C_2 | I_3, Y_1\} = \frac{\Pr \{I_3 | C_2, Y_1\}}{3 \Pr \{I_3 | Y_1\}}$$

$$\begin{aligned}\Pr \{I_3 | Y_1\} &= \Pr \{I_3 | C_1, Y_1\} \Pr \{C_1 | Y_1\} \\&\quad + \Pr \{I_3 | C_2, Y_1\} \Pr \{C_2 | Y_1\} \\&\quad + \Pr \{I_3 | C_3, Y_1\} \Pr \{C_3 | Y_1\} \\&= \frac{1}{3} \left(\Pr \{I_3 | C_1, Y_1\} + \Pr \{I_3 | C_2, Y_1\} + \Pr \{I_3 | C_3, Y_1\} \right)\end{aligned}$$

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$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{\Pr \{I_3 \mid C_2, Y_1\}}{\Pr \{I_3 \mid C_1, Y_1\} + \Pr \{I_3 \mid C_2, Y_1\} + \Pr \{I_3 \mid C_3, Y_1\}}$$

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It depends on how I choose the door to open!

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{\Pr \{I_3 \mid C_2, Y_1\}}{\Pr \{I_3 \mid C_1, Y_1\} + \Pr \{I_3 \mid C_2, Y_1\} + \Pr \{I_3 \mid C_3, Y_1\}}$$

It depends on how I choose the door to open!

$$\Pr \{I_3 \mid C_1, Y_1\} = \frac{1}{2}$$

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{\Pr \{I_3 \mid C_2, Y_1\}}{\Pr \{I_3 \mid C_1, Y_1\} + \Pr \{I_3 \mid C_2, Y_1\} + \Pr \{I_3 \mid C_3, Y_1\}}$$

It depends on how I choose the door to open!

$$\Pr \{I_3 \mid C_1, Y_1\} = \frac{1}{2}$$

$$\Pr \{I_3 \mid C_2, Y_1\} = 1$$

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It depends on how I choose the door to open!

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$$\Pr \{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr \{I_3 \mid C_3, Y_1\} = 0$$

$$\boxed{\Pr \{C_2 \mid I_3, Y_1\} = \frac{2}{3}}$$

$$\Pr \{C_2 | I_3, Y_1\} = \frac{\Pr \{I_3 | C_2, Y_1\}}{\Pr \{I_3 | C_1, Y_1\} + \Pr \{I_3 | C_2, Y_1\} + \Pr \{I_3 | C_3, Y_1\}}$$

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{\Pr \{I_3 \mid C_2, Y_1\}}{\Pr \{I_3 \mid C_1, Y_1\} + \Pr \{I_3 \mid C_2, Y_1\} + \Pr \{I_3 \mid C_3, Y_1\}}$$

$$\boxed{\Pr \{I_3 \mid C_3, Y_1\} = 0}$$

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$$\Pr \{C_1 | I_3, Y_1\} = \frac{\Pr \{I_3 | C_1, Y_1\}}{\Pr \{I_3 | C_1, Y_1\} + \Pr \{I_3 | C_2, Y_1\}}$$

$$\Pr \{C_2 | I_3, Y_1\} = \frac{\Pr \{I_3 | C_2, Y_1\}}{\Pr \{I_3 | C_1, Y_1\} + \Pr \{I_3 | C_2, Y_1\} + \Pr \{I_3 | C_3, Y_1\}}$$

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$$\frac{\Pr \{C_2 | I_3, Y_1\}}{\Pr \{C_1 | I_3, Y_1\}} = \frac{\Pr \{I_3 | C_2, Y_1\}}{\Pr \{I_3 | C_1, Y_1\}}$$

$$\Pr \{C_2 | I_3, Y_1\} > \Pr \{C_1 | I_3, Y_1\} \iff \Pr \{I_3 | C_2, Y_1\} > \Pr \{I_3 | C_1, Y_1\}$$

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$$\Pr \{I_3 | C_2, Y_1\} = 1$$

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$$\Pr \{C_2 | I_3, Y_1\} = \frac{\Pr \{I_3 | C_2, Y_1\}}{\Pr \{I_3 | C_1, Y_1\} + \Pr \{I_3 | C_2, Y_1\} + \Pr \{I_3 | C_3, Y_1\}}$$

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$$\Pr \{I_3 | C_1, Y_1\} = \frac{1}{2}$$

$$\Pr \{I_3 | C_2, Y_1\} = 1$$

$$\Pr \{C_2 | I_3, Y_1\} = \frac{2}{3}$$

Q : Switching vs. Choosing between the two remaining doors randomly?

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{\Pr \{I_3 \mid C_2, Y_1\}}{\Pr \{I_3 \mid C_1, Y_1\} + \Pr \{I_3 \mid C_2, Y_1\}}$$

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{\Pr \{I_3 \mid C_2, Y_1\}}{\Pr \{I_3 \mid C_1, Y_1\} + \Pr \{I_3 \mid C_2, Y_1\}}$$

$$\Pr \{I_3 \mid C_1, Y_1\} = q$$

$$\Pr \{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr \{I_3 \mid C_3, Y_1\} = 0$$

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{\Pr \{I_3 \mid C_2, Y_1\}}{\Pr \{I_3 \mid C_1, Y_1\} + \Pr \{I_3 \mid C_2, Y_1\}}$$

$$\Pr \{I_3 \mid C_1, Y_1\} = q$$

$$\Pr \{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr \{I_3 \mid C_3, Y_1\} = 0$$

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{1}{1+q}$$

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{\Pr \{I_3 \mid C_2, Y_1\}}{\Pr \{I_3 \mid C_1, Y_1\} + \Pr \{I_3 \mid C_2, Y_1\}}$$

$$\Pr \{I_3 \mid C_1, Y_1\} = q$$

$$\Pr \{I_3 \mid C_2, Y_1\} = 1$$

$$\Pr \{I_3 \mid C_3, Y_1\} = 0$$

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{1}{1+q} \in [\frac{1}{2}, 1]$$

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{\Pr \{I_3 \mid C_2, Y_1\}}{\Pr \{I_3 \mid C_1, Y_1\} + \Pr \{I_3 \mid C_2, Y_1\}}$$

$$\Pr \{I_3 \mid C_1, Y_1\} = q$$

$$\Pr \{I_3 \mid C_2, Y_1\} = 1$$

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$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{1}{1+q} \in [\frac{1}{2}, 1]$$

$$\Pr \{C_1 \mid I_3, Y_1\} \in [0, \frac{1}{2}]$$

$$\Pr \{C_2 | I_3, Y_1\} = \frac{\Pr \{I_3 | C_2, Y_1\}}{\Pr \{I_3 | C_1, Y_1\} + \Pr \{I_3 | C_2, Y_1\}}$$

$$\Pr \{I_3 | C_1, Y_1\} = q$$

$$\Pr \{I_3 | C_2, Y_1\} = 1$$

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$$\Pr \{C_2 | I_3, Y_1\} = \frac{1}{1+q} \in [\frac{1}{2}, 1]$$

$$\Pr \{C_1 | I_3, Y_1\} \in [0, \frac{1}{2}]$$

$$\boxed{\Pr \{C_2 | I_3, Y_1\} > \Pr \{C_1 | I_3, Y_1\} \iff \Pr \{I_3 | C_2, Y_1\} > \Pr \{I_3 | C_1, Y_1\}}$$

$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{\Pr \{I_3 \mid C_2, Y_1\}}{\Pr \{I_3 \mid C_1, Y_1\} + \Pr \{I_3 \mid C_2, Y_1\}}$$

$$\Pr \{I_3 \mid C_1, Y_1\} = q$$

$$\Pr \{I_3 \mid C_2, Y_1\} = 1$$

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$$\Pr \{C_2 \mid I_3, Y_1\} = \frac{1}{1+q} \in [\frac{1}{2}, 1]$$

$$\Pr \{C_1 \mid I_3, Y_1\} \in [0, \frac{1}{2}]$$

$$\Pr \{C_2 \mid I_3, Y_1\} > \Pr \{C_1 \mid I_3, Y_1\} \iff \Pr \{I_3 \mid C_2, Y_1\} > \Pr \{I_3 \mid C_1, Y_1\}$$

Always Switch!

$$\Pr \{C_2 | I_3, Y_1\} = \frac{\Pr \{I_3 | C_2, Y_1\}}{\Pr \{I_3 | C_1, Y_1\} + \Pr \{I_3 | C_2, Y_1\}}$$

$$\Pr \{C_2 | I_3, Y_1\} > \Pr \{C_1 | I_3, Y_1\} \iff \Pr \{I_3 | C_2, Y_1\} > \Pr \{I_3 | C_1, Y_1\}$$

ASSUMPTION: I KNOW.

$$\Pr \{I_3 | C_1, Y_1\} = \frac{1}{2}$$

$$\Pr \{I_3 | C_2, Y_1\} = 1$$

$$\Pr \{C_2 | I_3, Y_1\} = \frac{2}{3}$$

$$\Pr \{C_2 | I_3, Y_1\} = \frac{\Pr \{I_3 | C_2, Y_1\}}{\Pr \{I_3 | C_1, Y_1\} + \Pr \{I_3 | C_2, Y_1\}}$$

$$\Pr \{C_2 | I_3, Y_1\} > \Pr \{C_1 | I_3, Y_1\} \iff \Pr \{I_3 | C_2, Y_1\} > \Pr \{I_3 | C_1, Y_1\}$$

ASSUMPTION: I DON'T KNOW EITHER.

ASSUMPTION: I KNOW.

Opens one randomly that happens not to reveal the car.

$$\Pr \{I_3 | C_1, Y_1\} = \frac{1}{2}$$

$$\Pr \{I_3 | C_2, Y_1\} = 1$$

$$\Pr \{C_2 | I_3, Y_1\} = \frac{2}{3}$$

$$\Pr \{C_2 | I_3, Y_1\} = \frac{\Pr \{I_3 | C_2, Y_1\}}{\Pr \{I_3 | C_1, Y_1\} + \Pr \{I_3 | C_2, Y_1\}}$$

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ASSUMPTION: I DON'T KNOW EITHER.

ASSUMPTION: I KNOW.

$$\Pr \{I_3 | C_1, Y_1\} = \frac{1}{2}$$

$$\Pr \{I_3 | C_2, Y_1\} = 1$$

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Opens one randomly that happens not to reveal the car.

$$\Pr \{I_3 | C_1, Y_1\} = \frac{1}{2}$$

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$$\Pr \{C_2 | I_3, Y_1\} = \frac{1}{2}$$



Monty Hall problem (wiki)

The Boy/Girl Puzzle



Both Girls (CS Problem 5.3 – 12)

Mr. and Mrs. Smith have two children of different ages,
what is the probability that they has **two girls**,

- (a) given that **one of the children** is a girl?
- (b) given that **the older child** is a girl?



G_1 : the older child is a girl

G_2 : the younger child is a girl

G_1 : the older child is a girl

G_2 : the younger child is a girl

$$\Pr \{G_1 \wedge G_2 \mid G_1 \vee G_2\}$$

G_1 : the older child is a girl

G_2 : the younger child is a girl

$$\Pr \{G_1 \wedge G_2 \mid G_1 \vee G_2\} = \frac{\Pr \{G_1 \wedge G_2\}}{\Pr \{G_1 \vee G_2\}}$$

G_1 : the older child is a girl

G_2 : the younger child is a girl

$$\begin{aligned}\Pr \{G_1 \wedge G_2 \mid G_1 \vee G_2\} &= \frac{\Pr \{G_1 \wedge G_2\}}{\Pr \{G_1 \vee G_2\}} \\ &= \frac{\Pr \{G_1 \wedge G_2\}}{\Pr \{G_1\} + \Pr \{G_2\} - \Pr \{G_1 \wedge G_2\}}\end{aligned}$$

G_1 : the older child is a girl

G_2 : the younger child is a girl

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$(G_1, G_2), (G_1, B_2), (B_1, G_2), (B_1, B_2)$

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Both Girls (CS Problem 5.3 – 12)

Mr. and Mrs. Smith have two children of different ages,
what is the probability that they has **two girls**,

- (a) given that **one of the children** is a girl?

Both Girls (CS Problem 5.3 – 12)

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Q : How do you know that “one of the children is a girl”?

Q : How do you know that “one of the children is a girl”?

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Q : How do you know that “one of the children is a girl”?



- (I) From all families with two children, at least one of whom is a girl, a (Smith's) family is chosen at random.
- (II) From all families with two children, one child (of Smith) is selected at random that happens to be a girl.

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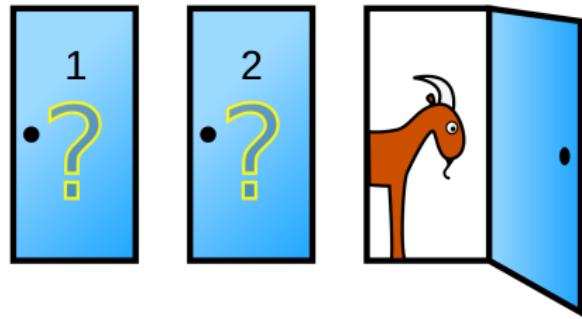
- (I) I **KNOW** them well and I tell you that “one of the children is a girl”.

Q : How do you know that “one of the children is a girl”?



- (I) I **KNOW** them well and I tell you that “one of the children is a girl”.
- (II) I **DON'T KNOW** them. I just open a room door and see a girl.

The Monty-Hall Problem Comes Back



Q : **How** do you know that “one of the children is a girl”?

- (II) *g* : From all families with two children, **one child (of Smith)** is selected at random that happens to be a girl.

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After-class Exercise:

A new couple, known to have two children, has just moved into town. Suppose that the mother is encountered walking with one of her children. If this child is a girl, what is the probability that both children are girls?





Boy or Girl paradox (wiki)

Searching an Unsorted Array (CLRS Problem 5 – 2 (f))

```
1: procedure DETERMINISTIC-SEARCH( $A[1 \cdots n]$ ,  $x$ )
2:    $i \leftarrow 1$ 
3:   while  $i \leq n$  do
4:     if  $A[i] = x$  then
5:       return true
6:      $i \leftarrow i + 1$ 
7:   return false
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(e)

$$\exists! i : A[i] = x$$

(f)

$$\exists!_k i : A[i] = x$$

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$$\sum_{i=1}^{n-k+1} i \binom{n-i}{k-1} = \binom{n+1}{k+1}$$



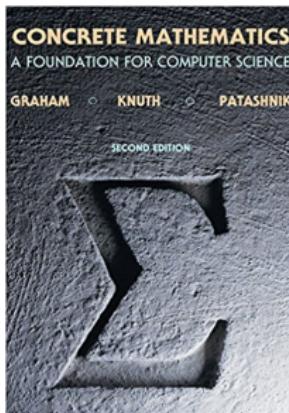
Summation by parts (Abel transformation; wiki)

After-class Exercise:

$$\sum_{i=1}^{n-k+1} i \binom{n-i}{k-1} = \binom{n+1}{k+1}$$

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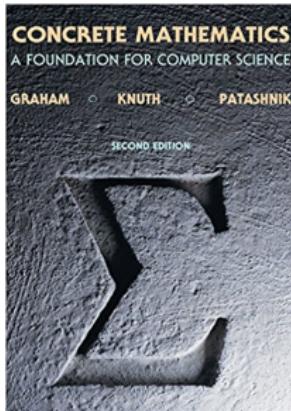
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Chapter 5: Binomial Coefficients

After-class Exercise:

$$\sum_{i=1}^{n-k+1} i \binom{n-i}{k-1} = \binom{n+1}{k+1}$$



$$r \binom{r-1}{k-1} = k \binom{r}{k}$$

$$\sum_{0 \leq k \leq n} \binom{k}{m} = \binom{n+1}{m+1}$$

Chapter 5: Binomial Coefficients

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$$\Pr \{ I_i = 1 \} = \begin{cases} \frac{1}{k}, & \text{if } A[i] = x \\ \frac{1}{k+1}, & \text{if } A[i] \neq x \end{cases}$$

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$$\mathbb{E}[Y] = \sum_{i=1}^n \Pr \{ I_i = 1 \} = k \cdot \frac{1}{k} + (n - k) \cdot \frac{1}{k+1} = \frac{n+1}{k+1}$$





$$\Pr \{I_i = 1\} = \begin{cases} \frac{1}{k}, & \text{if } A[i] = x \\ \frac{1}{k+1}, & \text{if } A[i] \neq x \end{cases}$$



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NOT IID
(Independent and Identically Distributed)

$$\begin{aligned}\mathbb{E}[Y] &= \sum_{i=1}^{n-k+1} i \Pr\{Y = i\} \\ &= \sum_{i=1}^{n-k+1} \Pr\{Y \geq i\}\end{aligned}$$

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&= \frac{1}{\binom{n}{k}} \binom{n+1}{k+1} = \frac{n+1}{k+1}
\end{aligned}$$





Hints & Tips



There are n bins labelled with the numbers $1, 2, \dots, n$. Balls are placed in these bins one after the other, with the bin into which a ball is placed being independent random variables that assume the value k with probability p_k . Let X be the number of balls placed so that there is at least one ball in every bin.

- (a) Assume that $p_k = \frac{1}{n}$. What is the expectation of X ?
- (b) Assume that $p_k = \frac{1}{n}$. What is the probability distribution of X ?
- (c) Prove that $\Pr(X > n \ln n + cn) \leq e^{-c}$, $\Pr(X < n \ln n - cn) \leq e^{-c}$.
- (d) Redo (a) and (b) without the assumption $p_k = \frac{1}{n}$.
- (e) Given a deck of n cards, each time you take the top card from the deck, and insert it into the deck at one of the n distinct possible places, each of them with probability $\frac{1}{n}$. What is the expected times for you to perform the procedure above until the bottom card rises to the top?

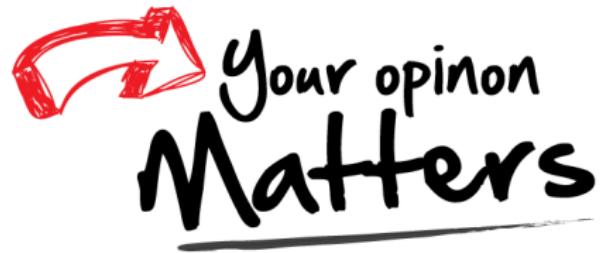
The Coupon Collector's Problem



Shuffling Cards



Thank You!



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