

2-8 Probabilistic Analysis

"No Expectation, No Disappointment."

Hengfeng Wei

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May 16, 2018



Definition (Expectation)

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Let X be a discrete random variable that takes on *only nonnegative integer values* \mathbb{N} .

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Proof.

$$\sum_{j=1}^{\infty} \sum_{i=1}^j \Pr(X = j) = \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} \Pr(X = j)$$



Searching an Unsorted Array (CLRS Problem 5 – 2 (f))

```
1: procedure DETERMINISTIC-SEARCH( $A[1 \cdots n]$ ,  $x$ )
2:    $i \leftarrow 1$ 
3:   while  $i \leq n$  do
4:     if  $A[i] = x$  then
5:       return true
6:      $i \leftarrow i + 1$ 
7:   return false
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(e)

$$\exists! i : A[i] = x$$

(f)

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$$\sum_{i=1}^{n-k+1} i \binom{n-i}{k-1} = \binom{n+1}{k+1}$$



Summation by parts ([Abel transformation](#); [wiki](#))

How Did I (an ant) Evaluate this Summation:

$$\sum_{i=1}^{n-k+1} i \binom{n-i}{k-1} = \binom{n+1}{k+1}$$

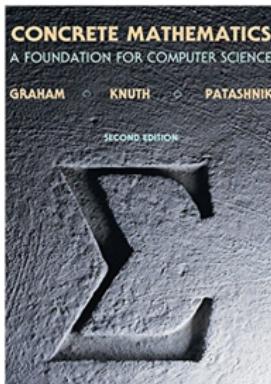
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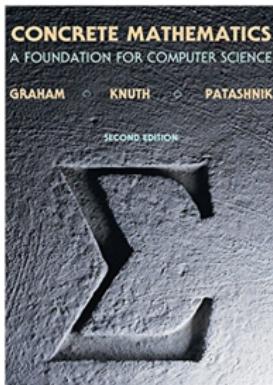
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Chapter 5: Binomial Coefficients

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$$r \binom{r-1}{k-1} = k \binom{r}{k}$$

$$\sum_{0 \leq k \leq n} \binom{k}{m} = \binom{n+1}{m+1}$$

Chapter 5: Binomial Coefficients

$$\begin{aligned}
\sum_{i=1}^{n-k+1} i \binom{n-i}{k-1} &= \sum_{i=0}^{n-k} (i+1) \binom{n-i-1}{k-1} \\
&= \sum_{i=0}^{n-k} ((n+1) - (n-i)) \binom{n-i-1}{k-1} \\
&= \sum_{i=0}^{n-k} (n+1) \binom{n-i-1}{k-1} - \sum_{i=0}^{n-k} (n-i) \binom{n-i-1}{k-1} \\
&= (n+1) \sum_{i=0}^{n-k} \binom{n-i-1}{k-1} - k \sum_{i=0}^{n-k} \binom{n-i}{k} \\
&= (n+1) \sum_{m=k-1}^{n-1} \binom{m}{k-1} - k \sum_{m=k}^n \binom{m}{k} \\
&= (n+1) \binom{n}{k} - k \binom{n+1}{k+1} = \binom{n+1}{k+1}
\end{aligned}$$

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$$I_i = \begin{cases} 1, & \text{if } A[i] \text{ is checked} \\ 0, & \text{o.w.} \end{cases}$$

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$$\mathbb{E}[Y] = \sum_{i=1}^n \Pr \{ I_i = 1 \} = k \cdot \frac{1}{k} + (n - k) \cdot \frac{1}{k+1} = \frac{n+1}{k+1}$$





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Hat-check Problem (CLRS Problem 5.2 – 4)



X : # of customers who get back their own hat

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Inversions (CLRS Problem 5.2 – 5)

$A[1 \cdots n]$ of n distinct numbers

(i, j) is an **inversion** of $A : i < j \wedge A[i] > A[j]$

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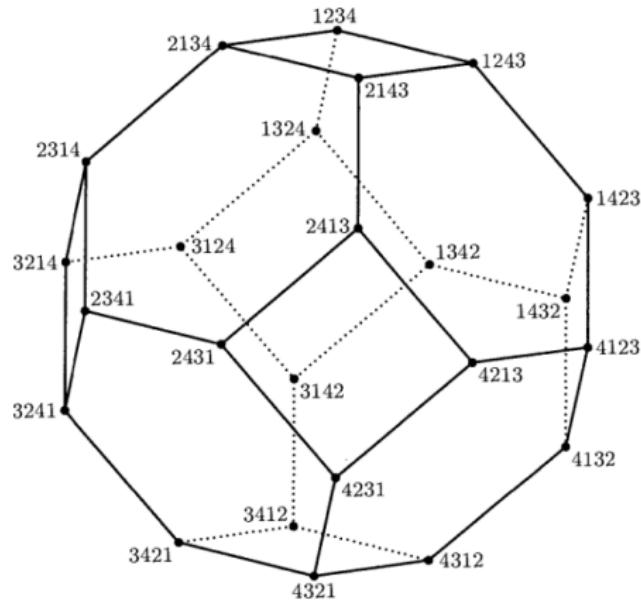
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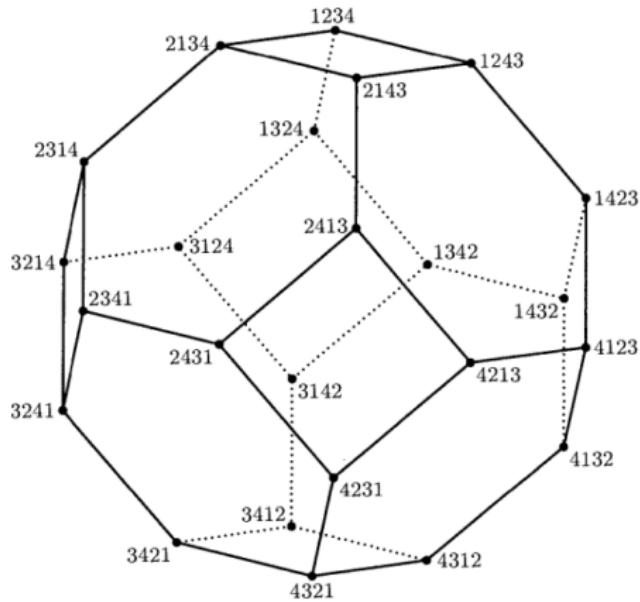
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Q : Average # of swaps (comparisons) of INSERTION-SORT?

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$$\langle 3214 \rangle \sim \langle 4123 \rangle$$

Definition (Conditional Expectation on an Event)

$$\mathbb{E}[X | E] = \sum_x x \Pr(X = x | E)$$

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Theorem (CS Theorem 5.23))

Let X be a random variable defined on a sample space Ω .

Let E_1, E_2, \dots, E_n be a **partition** of Ω .

$$\mathbb{E}[X] = \sum_{i=1}^n \mathbb{E}[X \mid E_i] \Pr(E_i)$$

Theorem (The Law of Total Expectation (CS Theorem 5.23))

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Proof.

By definition.



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Proof.

By definition.

$$\sum_x x \sum_{i=1}^n \Pr(X = x, E_i) = \sum_{i=1}^n \sum_x x \Pr(X = x, E_i)$$



(#) Rational Person Playing a Card Game (CS Problem 5.6 – 4)



A : \$1.00; Repeat

J : \$2.00; End

K : \$3.00; End

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Conditioning on the **first** draw c

$$\mathbb{E}[X] = \frac{1}{4} (\mathbb{E}[X \mid c = A] + \mathbb{E}[X \mid c = J] + \mathbb{E}[X \mid c = K] + \mathbb{E}[X \mid c = Q])$$

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$$4 * A + 1 * Q = \$8.00$$

In-class Exercise: Coin Pattern (Provided by Yifan Pei)



X : # of tosses to get 3 consecutive heads (HHH)

$$\mathbb{E}[X]$$

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T , HT , HHT , HHH

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Conditioning on the first 3 tosses

$T, \quad HT, \quad HHT, \quad HHH$

$$\mathbb{E}[X] = \frac{1}{2}(\mathbb{E}[X] + 1) + \frac{1}{4}(\mathbb{E}[X] + 2) + \frac{1}{8}(\mathbb{E}[X] + 3) + \frac{1}{8} \times 3$$

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Definition (Conditional Expectation on a Random Variable)

$$\mathbb{E}[X | Y = y] = \sum_x x \Pr(X = x | Y = y)$$

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$$\mathbb{E}[X | Y](y) = \mathbb{E}[X | Y = y]$$

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Theorem

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X | Y]] = \sum_y \mathbb{E}[X | Y = y] \Pr(Y = y)$$





Hints & Tips



There are n bins labelled with the numbers $1, 2, \dots, n$. Balls are placed in these bins one after the other, with the bin into which a ball is placed being independent random variables that assume the value k with probability p_k . Let X be the number of balls placed so that there is at least one ball in every bin.

- (a) Assume that $p_k = \frac{1}{n}$. What is the expectation of X ?
- (b) Assume that $p_k = \frac{1}{n}$. What is the probability distribution of X ?
- (c) Prove that $\Pr(X > n \ln n + cn) \leq e^{-c}$, $\Pr(X < n \ln n - cn) \leq e^{-c}$.
- (d) Redo (a) and (b) without the assumption $p_k = \frac{1}{n}$.
- (e) Given a deck of n cards, each time you take the top card from the deck, and insert it into the deck at one of the n distinct possible places, each of them with probability $\frac{1}{n}$. What is the expected times for you to perform the procedure above until the bottom card rises to the top?

The Coupon Collector's Problem

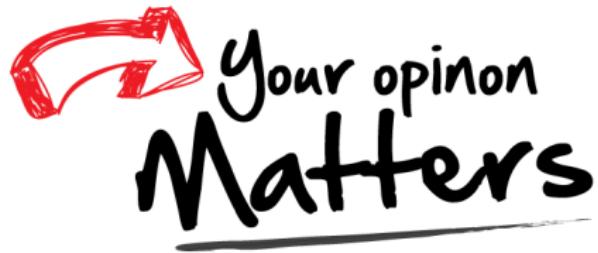


Shuffling Cards



Chapter “Shuffling Cards” of “Proofs from THE Book”

Thank You!



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