2-4 Recurrences

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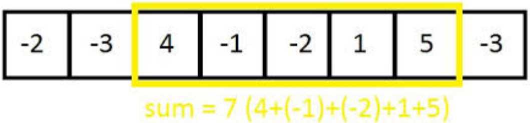


$$O(n^3) \implies O(n^2) \implies O(n \log n) \implies O(n)$$

$$T(n) = aT(n/b) + f(n)$$

Master Theorem

$$T(n) = 4T(n/2) + n^2 \log n$$

$$T(n) = 2T(n/2) + \frac{n}{\log n}$$

$$T(n) = \sqrt{n}T(\sqrt{n}) + n$$

$$T(n) = T(n-1) + T(n/2) + n$$

Maximum-sum Subarray (mss; Problem 4.1-5)

$$A[0 \ldots n-1] \qquad \forall\, 0 \le i \le n-1 : A[i] \in \mathbb{Z}$$

To find (the sum of) a maximum-sum subarray of A

$$A = [-2, 11, -4, 13, -5, -2]$$

$$\textsf{mss} = 11 + (-4) + 13 = 20$$

$$\forall\, 0 \le i \le n-1 : A[i] < 0$$

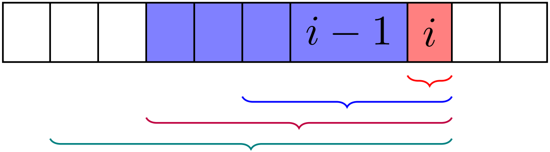
$$\textsf{mss} = 0 \text{ } vs. \text{ } \textsf{mss} = \max_{0 \le i \le n-1} A[i]$$

$\textsf{mss-at}[i]$: (the sum of) a maximum-sum subarray ending with $A[i]$

$$\textsf{mss} = \max_{0 \le i \le n-1} \textsf{mss-at}[i]$$

What is the relation between $\textsf{mss-at}[i-1]$ and $\textsf{mss-at}[i]$?

Q : Where does $\textsf{mss-at}[i]$ start?



$$\boxed{\textsf{mss-at}[i] = \max\{\textsf{mss-at}[i-1] + A[i], A[i]\}}$$

procedure $\textsc{mss}(A, n)$

  $\textsf{mss-at}[0] \gets A[0]$

  for $i \gets 1 \ldots n-1$ **do**

    $\textsf{mss-at}[i] \gets \max\{\textsf{mss-at}[i-1] + A[i], A[i]\}$

  return $\max\limits_{0 \le i \le n-1} \textsf{mss-at}[i]$

time	*space*
$O(n)$	$O(n)$

procedure MSS(A, n)

  mss $\gets -\infty$

  mss-at $\gets A[0]$

  for $i \gets 1 \ldots n-1$ **do**

    mss-at $\gets \max\{$mss-at $+ A[i], A[i]\}$

    mss $\gets \max\{$mss, mss-at$\}$

  return mss

time	*space*
$O(n)$	$O(1)$

Maximum-product Subarray (mps)

$$A[0 \ldots n-1] \qquad \forall 0 \le i \le n-1 : A[i] \in \mathbb{Z}$$

To find (the product of) a maximum-product subarray of A

$$A = [\frac{1}{2}, 4, -2, 5, -\frac{1}{5}, 8]$$

$$\textsf{mps} = 4 \times (-2) \times 5 \times (-\frac{1}{5}) \times 8 = 64$$

线性时间内求解最大积子数组问题

0
0
0 views

asked 1 second ago in tutorial by ant-hengxin (30 points)

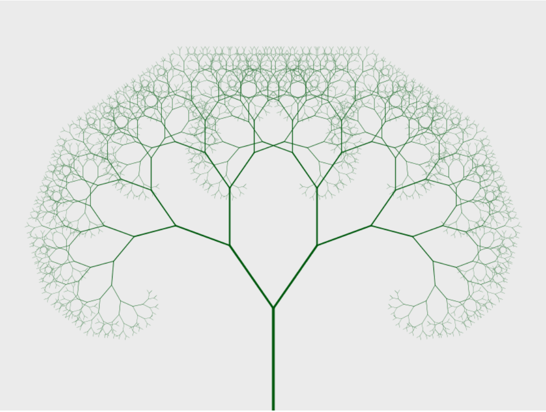
在习题课上，我们已经知道了（这里是一般将来过去完成时态）如何在 $O(n)$ 时间内解决最大和子数组问题。

那么，如何在 $O(n)$ 时间内解决**最大积**子数组问题？请给出算法。

tutorial algorithm

answer comment

Recurrences



$$T(n) = aT(n/b) + f(n) \quad (a > 0,\ b > 1)$$

Assume that $T(n)$ is constant for sufficiently small n.

$$\left.\begin{array}{r} f(n) \\ af(\frac{n}{b}) \\ a^2 f(\frac{n}{b^2}) \\ \vdots \\ a^{\log_b n} T(1) = \Theta(n^{\log_b a}) \end{array}\right\} \sum \overset{f(n)\ \underset{=}{vs.}\ n^E}{} \begin{cases} n^{\log_b a}, & f(n) = O(n^{E-\epsilon}) \\ n^{\log_b a} \log n, & f(n) = \Theta(n^E) \\ f(n), & f(n) = \Omega(n^{E+\epsilon}) \end{cases}$$

$\boxed{E \triangleq \log_b a}$ *(critical exponent)*

TC 4.5-4: Gap in Mater Theorem

$$T(n) = 4T(n/2) + n^2 \log n = \Theta(n^2 \log^2 n)$$

$$E \triangleq \log_b a = 2$$

$$f(n) = O(n^{E-\epsilon}) \qquad f(n) = \Theta(n^E) \qquad f(n) = \Omega(n^{E+\epsilon})$$

$$n^2 \log n = o(n^{2+\epsilon})$$

$$\boxed{f(n) = \Theta(n^{\log_b a} \log^k n) \implies \Theta(n^{\log_b a} \log^{k+1} n) \qquad (k \geq 0)}$$

TC Problem 4-3 (e): Gaps in Master Theorem

$$T(n) = 2T(n/2) + \frac{n}{\log n}$$

$$E \triangleq \log_b a = 1$$

$$f(n) = O(n^{E-\epsilon}) \qquad f(n) = \Theta(n^E) \qquad f(n) = \Omega(n^{E+\epsilon})$$

$$\frac{n}{\log n} = \omega(n^{1-\epsilon})$$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$$T(n) = 2T(n/2) + \frac{n}{\log n}$$

$$\boxed{T(n) = \Theta(n)}$$

$T(n) = \Omega(n)$ $\quad T(n) \geq cn$

$$\begin{aligned} T(n) &= 2T(n/2) + \frac{n}{\log n} \\ &\geq 2c \cdot \frac{n}{2} + \frac{n}{\log n} \\ &= cn + \frac{n}{\log n} \end{aligned}$$

$c = 1$

$T(n) = O(n)$ $\quad T(n) \leq cn - d$

$$\begin{aligned} T(n) &= 2T(n/2) + \frac{n}{\log n} \\ &\leq 2(c \cdot \frac{n}{2} - d) + \frac{n}{\log n} \\ &= cn + \frac{n}{\log n} - 2d \end{aligned}$$

$$\frac{n}{\log n} \leq d$$

$$T(n) = 2T(n/2) + \frac{n}{\log n}$$

$$L(n) = 2L(n/2) + 1 = \Theta(n) \qquad H(n) = 2H(n/2) + n = \Theta(n \log n)$$

$$\boxed{T(n) = \Theta(n \log \log n)}$$

数学归纳法证明 $T(n) = 2T(n/2) + \frac{n}{\log n} = \Theta(n \log\log n)$

0

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1 view

asked 32 minutes ago in homework by ant-hengxin (30 points)
recategorized 4 minutes ago by ant-hengxin

我们在习题课上已经知道（这里是一般将来过去完成时态）如下递归式的解：

$$T(n) = 2T(n/2) + \frac{n}{\log n} = \Theta(n \log\log n)$$

请问如何用数学归纳法证明？

recurrence homework

answer comment

$$
\begin{aligned}
T(n) &= 2T(n/2) + \frac{n}{\log n} \\
&= 2(2T(\frac{n}{4}) + \frac{\frac{n}{2}}{\log \frac{n}{2}}) + \frac{n}{\log n} \\
&= 2^2 T(\frac{n}{2^2}) + \frac{n}{\log n - 1} + \frac{n}{\log n} \\
&= \dots \\
&= 2^k T(\frac{n}{2^k}) + \sum_{i=0}^{k-1} \frac{n}{\log n - i}
\end{aligned}
$$

$$\frac{n}{2^k} = 1 \implies k = \log n$$

$$\frac{n}{2^k} = 1 \implies k = \log n$$

$$\begin{aligned}
T(n) &= 2^k T(\frac{n}{2^k}) + \sum_{i=0}^{k-1} \frac{n}{\log n - i} \\
&= \Theta(n) + n \sum_{i=0}^{\log n - 1} \frac{1}{\log n - i} \\
&= \Theta(n) + nH_{\log n} \\
&= \Theta(n \log\log n)
\end{aligned}$$

$$T(n) = 2T(n/2) + \frac{n}{\log n}$$

$$\boxed{n = 2^k}$$

$$T(2^k) = 2T(2^{k-1}) + \frac{2^k}{k}$$

$$\frac{T(2^k)}{2^k} = \frac{T(2k-1)}{2^{k-1}} + \frac{1}{k}$$

$$\boxed{S(k) \triangleq \frac{T(2^k)}{2^k}}$$

$$S(k) = S(k-1) + \frac{1}{k} = H_k = \Theta(\log k)$$

$$T(n) = \Theta(n \log \log n)$$

$$\boxed{f(n) = \Theta(n^{\log_b a} \log^{k} n) \implies \Theta(n^{\log_b a} \log^{k+1} n) \qquad (k \geq 0)}$$

Master theorem @ wiki

$$f(n) = \Theta(n^{\log_b a} \log^{k} n)$$

$$k > -1 \implies T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$$

$$k = -1 \implies T(n) = \Theta(n^{\log_b a} \log \log n)$$

$$k < -1 \implies T(n) = \Theta(n^{\log_b a})$$

TC Problem 4-3 (j)

$$
\begin{aligned}
\mathrm{T}(n) &= \sqrt{n}\,\mathrm{T}(\sqrt{n}) + n \\
&= n^{\frac{1}{2}}\,\mathrm{T}\left(n^{\frac{1}{2}}\right) + n \\
&= n^{\frac{1}{2}}\left(n^{\frac{1}{2^2}}\,\mathrm{T}\left(n^{\frac{1}{2^2}}\right) + n^{\frac{1}{2}}\right) + n \\
&= n^{\frac{1}{2}+\frac{1}{2^2}}\,\mathrm{T}\left(n^{\frac{1}{2^2}}\right) + 2n \\
&= n^{\frac{1}{2}+\frac{1}{2^2}}\left(n^{\frac{1}{2^3}}\,\mathrm{T}\left(n^{\frac{1}{2^3}}\right) + n^{\frac{1}{2^2}}\right) + 2n \\
&= n^{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}}\,\mathrm{T}\left(n^{\frac{1}{2^3}}\right) + 3n \\
&= \cdots \\
&= n^{\sum_{i=1}^{k}\frac{1}{2^i}}\,\mathrm{T}\left(n^{\frac{1}{2^k}}\right) + kn
\end{aligned}
$$

$$\mathrm{T}(n) = n^{\sum_{i=1}^{k} \frac{1}{2^i}} \mathrm{T}\left(n^{\frac{1}{2^k}}\right) + kn$$

$$n^{\frac{1}{2^k}} = 1$$

$$n^{\frac{1}{2^k}} = 2 \implies k = \log\log n$$

$$\begin{aligned} \mathrm{T}(n) &= n^{\sum_{i=1}^{k} \frac{1}{2^i}} \mathrm{T}\left(n^{\frac{1}{2^k}}\right) + kn \\ &= n^{\sum_{i=1}^{\log\log n} \frac{1}{2^i}} \mathrm{T}(2) + n \log\log n \end{aligned}$$

$$\sum_{i=1}^{\log\log n} \frac{1}{2^i} < 1 \implies T(n) = \Theta(n \log\log n)$$

Exercise: Prove it by mathematical induction.

$$\boxed{T(n) = \sqrt{n}T(\sqrt{n}) + n}$$

$$\frac{T(n)}{n} = \frac{T(\sqrt{n})}{\sqrt{n}} + 1$$

$$n \leftrightarrow 2^m$$

$$\frac{T(2^m)}{2^m} = \frac{T(2^{m/2})}{2^{m/2}} + 1$$

$$S(m) \leftrightarrow \frac{T(2^m)}{2^m}$$

$$S(m) = S(m/2) + 1 = \Theta(\log m)$$

$$\boxed{T(n) = n \log \log n}$$

TC 4.4-5

To determine a good asymptotic upper bound.

$$T(n) = T(n-1) + T(n/2) + n$$


鄢振宇(1015198808) 2020/3/23 10:20:14

题目 3 (TC 4.4-5)
Use a recursion tree to determine a good asymptoti
T(n)=T(n-1)+T(n/2)+n. Use the substitution met

这题给出O(2^n)算对么

蚂蚁蚂蚁(245552163) 2020/3/23 10:21:33

也可以

鄢振宇(1015198808) 2020/3/23 10:21:42

好叭

```
>>> a[50001]/a[50000]
1.0002484609013023
>>> a[90001]/a[90000]
1.0001465270347825
```

· 拿Python跑了一下……总感觉很奇怪

· 不像是多项式，但是指数的话，底数又好像非常小

蚂蚁蚂蚁(245552163) 2020/3/23 10:27:30

是的。很奇怪的一个递归式。不是多项式，所以给出一个指数的上界也可以。我也不知道精确的界，查过一点资料，说既不是多项式，也不是指数的。

exponential. It seems that $\log T(n) \sim (\log n)^2/(2\log 2)$ and one can probably check that, for every positive ε, the property

$$\exp((\log n)^{2-\varepsilon}) \leqslant T(n) \leqslant \exp((\log n)^{2+\varepsilon})$$

solution @ math.stackexchange

$$T(n) = T(n-1) + T(n/2) + n$$

$$T(n) = -2(n+2)$$


Thank
You!

Your opinon Matters

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