1-11 有穷与无穷

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GEORG CANTOR
MATHEMATIKER
BEGRÜNDER DER MENGENLEHRE
1845-1918
DAS WESEN DER MATHEMA
TIK LIEGT IN IHRER FREIHEIT
GEORG CANTOR

"das wesen der mathematik liegt in ihrer freiheit"

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"The essence of mathematics lies in its freedom"

Dangerous Knowledge (BBC 2007)

$$c = \aleph_1$$

Comparing Sets

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$=$ $<$

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Function

$$=$$

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$\{1, 2, 3, \cdots\}$ *vs.* $\{1, 3, 5, \cdots, 2, 4, 6, \cdots\}$

Definition (Finite and Infinite)

For any set X,

Finite

$$\exists n \in \mathbb{N} : |X| = n \qquad (0 \in \mathbb{N})$$

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By contradiction.

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Countably Infinite

$$|X| = |\mathbb{N}| \triangleq \aleph_0$$

Countable

(finite $\lor$ countably infinite)

Uncountably Infinite

$$(\neg \text{ finite}) \land \Big(\neg \text{ (countably infinite)}\Big)$$

$$(\neg \text{ countable})$$

Theorem ($\aleph_0$ (1874))

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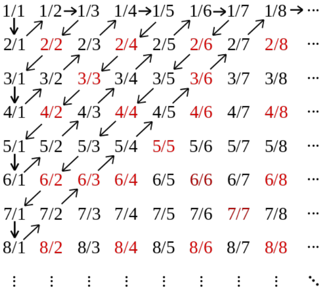
$|\mathbb{Q}| = |\mathbb{N}|$ (UD 22.9)

$$q \in \mathbb{Q}^+ : a/b \; (a, b \in \mathbb{N}^+)$$



$$|\mathbb{N}| = |\mathbb{Z}|$$

$$|\mathbb{N}| = |\mathbb{N}| \times |\mathbb{N}|$$

Theorem ($\mathbb{R}$ is uncountably infinite (1874).)

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Theorem (Cantor's Theorem (1891))

$$|X| \neq |2^X| \qquad (|X| < |2^X|)$$

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Q : *What is "dimension"?*

$$<$$

Definition ($|A| \le |B|$)

$|A| \le |B|$ if there exists an *one-to-one* function f from A into B.

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$|B| \leq |A|$ (Axiom of Choice)

Definition ($|A| < |B|$)

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$$|\mathbb{N}| < |\mathbb{R}|$$

$$|X| < |2^X|$$

Definition (Countable Revisited)

X is countable:

$$\exists n \in \mathbb{N} : |X| = n \lor |X| = |\mathbb{N}|$$

Proof for Countable (UD Exercise 22.5)

X is countable iff there exists a one-to-one function

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Subsets of Countable Set (UD 22.6; UD Corollary 22.4)

Every subset B of a countable set A is countable.

Set Union (UD 22.1)

Give an example, if possible, of

(c) a countably infinite collection of *pairwise disjoint* nonempty sets whose union is finite.

(b) a countably infinite collection of nonempty sets whose union is finite.

$$\left(\{A_i : i \in R\} \quad A_i = \{1\}\right) = \{\{1\}\}$$

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$$|A| = n \implies |2^A| = 2^n$$

Slope (UD 22.2 (e))

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$$(\mathbb{Q},\ \mathbb{R})$$

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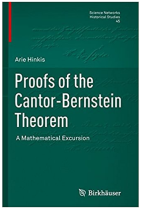

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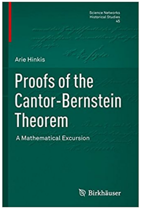

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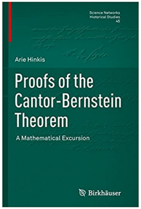

Q : Is "$\leq$" a total order?

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Theorem (PCC)

Principle of Cardinal Comparability (PCC) $\iff$ *Axiom of Choice*

Finite Sets

Finite Sets

“关于有穷，我原以为我是懂的”

学生反馈 (改编版)

"明明很显然的事情，为什么要那么繁琐的证明？

依靠直觉不可以吗？"

Definition (Finite)

X is finite if

$$\exists n \in \mathbb{N} : |X| = n$$

Theorem (Pigeonhole Principle (UD Theorem 21.2))

$$f : \{1, \cdots, m\} \to \{1, \cdots, n\} \; (m, n \in \mathbb{N}^+, m > n)$$

f is not one-to-one.

$A \setminus \{a\}$ (UD 21.15)

Let A be a nonempty finite set with $|A| = n$ and let $a \in A$.
Prove that $A \setminus \{a\}$ is finite and $|A \setminus \{a\}| = n - 1$.

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$$f|_{A \setminus \{a\}} : A \setminus \{a\} \to \{1, \cdots, n\} \setminus \{f(a)\}$$

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Show that $|A| \leq |B|$.

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By contradiction and the pigeonhole principle.

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By contradiction and (b).

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(No Axiom of Choice Here)

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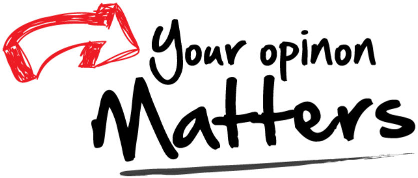
$$f(g(y)) = f(x) = y \implies f = g^{-1}$$


Thank You!



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