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THE EVOLUTION OF . . .

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Function: Part I

N. Luzin

What follows is the first part of a translation by Abe Shenitzer (in two parts) of an article by N. Luzin that appeared (in the 1930s) in the first edition of *The Great Soviet Encyclopedia*, Vol. 59, pp. 314–334. The article describes the evolution of the function concept.

In its most general form the term "function" denotes a connection between variable quantities. If a quantity x can take on arbitrary values and there is given a rule by means of which it is possible to associate with these values definite values of a quantity y, then we say that y is a function of x and denote this by symbolic notations such as $y = f(x)$, or $y = F(x)$, or $y = \varphi(x)$, and so on. We call the quantity x the independent variable, or argument, and y the dependent variable. However, this definition of the term function is somewhat vague and must be sharpened as follows: (1) concerning the variation of the independent variable x we must decide on an interval of variation $a < x < b$ and on whether x is to take on all values from a to b (the case of a continuous independent variable) or only some of them, say, only integral values; (2) we must make precise the nature of the rule that tells us how one is to associate a value y to a particular value x; (3) concerning the nature of the argument x it must be decided whether x is real or complex, and so on.

The function concept is one of the most fundamental concepts of modern mathematics. It did not arise suddenly. It arose more than two hundred years ago out of the famous debate on the vibrating string and underwent profound changes in the very course of that heated polemic. From that time on this concept has deepened and evolved continuously, and this twin process continues to this very day. That is why no single formal definition can include the full content of the function concept. This content can be understood only by a study of the main lines of the development that is extremely closely linked with the development of science in general and of mathematical physics in particular.

The fundamental vibrations of a mass system

Consider any mass system (say, a bridge) in a state of equilibrium. If it is slightly perturbed, then, in an effort to return to the state of equilibrium, the system begins to vibrate. A vibration is called *fundamental* if all points of the system pass through their respective positions of equilibrium at the same time. The study of the motion of a system with a single degree of freedom was essentially completed in the 17th century, and in the 18th century there began the study of the motions of systems with many degrees of freedom. The first steps in this direction were taken by the great Johann Bernoulli (1727). In order to study the motion of a vibrating string he performed a thought experiment in which he placed n equal and equally spaced weights on a stretched horizontal weightless string. He gives the periods of the fundamental vibrations when the number of weights is less than 8 and states the important principle that in a fundamental vibration the force acting on a material particle is always proportional to the distance of that particle from its equilibrium position. Using this principle he shows that the ratio $(y_{k+1} - 2y_k + y_{k-1})/y_k$ must be independent of k; here y_k is the distance of the k-th weight from the weightless thread when the latter is in the equilibrium position. It is assumed that at all times the amplitudes of the vibrating particles are infinitesimal. By means of this approach Bernoulli obtained a finite difference equation for y_k. The constants entering these finite difference equations are determined from an n-th degree algebraic equation. To each root of this equation there corresponds a definite fundamental vibration of the whole system. Bernoulli was unable to show that the roots of this equation are real and simple.

Somewhat later (1732–36), J. Bernoulli's son, Daniel, and his friend Euler tackled the analogous problem of the determination of the fundamental vibrations of a vertical weightless string attached at its upper end, fitted with n weights and free to swing in the air. Daniel Bernoulli was a superb experimenter. He first gave an experimental solution for $n = 2$ and 3 and then supplied a theoretical justification. Euler, who was just as superb a mathematician, treated the general case and showed that in a fundamental vibration the sides of a vibrating polygon intersect the vertical position of the string in fixed points. Then both of them began to investigate other systems, for example, a plate submerged in a liquid and swaying in it, the swinging of a heavy stick suspended at one end and, finally, a pendulum.

In all these problems Bernoulli and Euler investigated only fundamental vibrations. Whenever the force depended only on the location of the particle, the fundamental vibrations were harmonic, that is, the displacement of the k-th particle was given by the formula $y_k = f_k \cos at$, where f_k was specific for each particle and all particles had the same period $T = 2\pi/a$. D. Bernoulli explicitly formulated in the general case the existence of fundamental vibrations but was not able to show that the roots of the auxiliary equation are real and distinct. What is of greatest importance is the fundamental fact that, at the time, neither of them was able to express an arbitrary motion of the system in terms of fundamental vibrations alone. Much earlier (Rameau, 1726), music theorists pointed out that in addition to fundamental tones musical instruments also produce overtones. It is important to note that preoccupation with fundamental vibrations derived from the following error: beginning with the investigation of the great Taylor (1713), mathematicians clung to the erroneous view, shared by D. Bernoulli, that every composite vibration tends very rapidly to *status uniformis*, that is, to a fundamental vibration. To some extent this is true of physical situations in which friction, air resistance, and so on, cause dispersion of energy and thus give prominence to a fundamental component. The trouble was, however, that this conclusion was tacitly carried over to the mathematical apparatus, that is, to solutions of differential equations that are completely free of this side effect.

Passing to the limit from discrete to continuous systems. D. Bernoulli and Euler passed without hesitation from finite systems of particles to continuous systems by thinking of the latter as composed of a great many, or of infinitely many, particles. The boldness of 18th-century mathematicians is well known. Except for Varignon, N. Bernoulli, and d'Alembert, none of them appreciated the difficulties involved in passing to the limit. They regarded it as obvious that a proposition that holds for every finite value of n continues to hold as n goes to infinity. They had a blurred notion of the difference between "very large" and "infinitely large," and of the difference between results of limited accuracy and results whose accuracy can be indefinitely improved. They used finite differences instead of differentials and sums instead of integrals and ignored distinctions in either case. Usually, the transfer of conclusions from the finite to the infinite case was made either in the finished formulas or at the very beginning of an investigation. A very important example of the first approach is found in D. Bernoulli's paper dealing with the oscillation of a heavy homogeneous flexible string suspended at the top. He begins with a weightless string loaded by means of n weights, solves this problem, and, by assuming n in the answer to be infinitely large, obtains the solution of the problem of the vibrations of a heavy flexible string in the form $y = \cos(t/T)f(x)$, where x is the abscissa of a point of the string, y is the deviation from the equilibrium position, and $f(x) = 1 - x/a + (x/2!\,a)^2 - (x/3!\,a)^3 \ldots$. Here a is determined by the condition $f(l) = 0$, where l is the length of the string.

On the basis of an earlier result involving weights Bernoulli concludes that the equation $f(l) = 0$ has infinitely many roots $a_0, a_1, a_2, a_3, \ldots,$ and that the root a_k corresponds to a fundamental vibration in which the heavy string has k fixed points in addition to the suspension point.

Direct determination of the fundamental vibrations from the differential equation. A very important example of passing to the limit at the very beginning of an investigation is the replacing of a system of n ordinary differential equations

$$\frac{d^2 y_k}{dt^2} = f(y_{k+1}, y_k, y_{k-1}) \tag{1}$$

by the single partial differential equation

$$\frac{\partial^2 y}{\partial t^2} = F\left(y, \frac{\partial y}{\partial x}, \frac{\partial^2 y}{\partial x^2}\right). \tag{2}$$

This is done by putting $y_k = y_{k-1} + \Delta y_{k-1}$ and $y_{k+1} = y_{k-1} + 2\Delta y_{k-1} + \Delta^2 y_{k-1}$ and replacing finite differences by differentials. In this approach, in place of a system of algebraic equations linking the initial values of the y_k in a fundamental vibration—and thus in place of a single finite difference equation that combines all of these equations—comes a single differential equation for the initial shape. The latter equation is also obtained by putting $y = Y\cos at$ in equation (2) for the determination of the fundamental vibration and by requiring Y to depend on x alone (rather than on x and t). Also, one must take into consideration the special conditions that apply to the first and last points of the system and express them by means of particular equations (boundary conditions). The first to investigate a vibrating string in this way was Taylor (1713). He showed that the form of a vibrating string is that of a curve whose radii of curvature are to one another as the ordinates. In other words, he obtained the differential equation $y'' = -n^2 y$. After two integrations this equation yielded a quantity proportional to the sine of the argument, which, in turn, is proportional to the abscissa. Taylor did not write his solution explicitly because at that time the symbol "sin" for the sine function had not yet been introduced. That is why he could not formulate the question of the uniqueness of the integration constants satisfying all conditions. Here is where Taylor made his famous error to the effect that there is a unique fundamental vibration and that, for an arbitrary initial motion, every other motion of a vibrating string tends to the fundamental vibration found by him. J. Hermann and D. Bernoulli repeated Taylor's error. When he obtained Taylor's solution by means of his own approach, D. Bernoulli said that the form of a vibrating string is *socia* *trochoidis* (this was before the introduction of the term sine curve). Neither of these authors (1716 and 1728) suspected the possibility of the existence of other motions of the vibrating string. D. Bernoulli first anticipated the existence of many other fundamental vibrations when he began to treat the problem of the vibrations of a freely suspended heavy elastic string (1732 and 1739), which he regarded as an analog of a vibrating string. In this connection he experimented with a vibrating string and noted that it did not reject pieces of paper placed at its nodes. At that time (1734) Euler still mentioned only fundamental vibrations. It was only in 1744 that Euler, in the course of the investigation of the fundamental vibrations of an elastic plate fastened at one edge, showed that the auxiliary equation, with roots corresponding to the fundamental vibrations, has infinitely many solutions, which he tried to approximate.

The debate about the vibrating string

The paper of d'Alembert. While D. Bernoulli and Euler hinted in their papers at a multiplicity of fundamental vibrations of a vibrating string, it was d'Alembert who gave an almost exhaustive solution of this problem in his famous paper of 1747. He states directly that the aim of his paper is to prove that the problem of the shape of the vibrating string has infinitely many solutions other than the "companion of the cycloid." D'Alembert's method is as follows. He begins with the differential equation $\partial^2 y/\partial t^2 = \partial^2 y/\partial x^2$ and uses the identities $d(\partial y/\partial x) = (\partial^2 y/\partial x^2)\,dx + (\partial^2 y/\partial x\,\partial t)\,dt$ and $d(\partial y/\partial t) = (\partial^2 y/\partial x\,\partial t)\,dx + (\partial^2 y/\partial t^2)\,dt$ to obtain as consequences the relations

$$d\left(\frac{\partial y}{\partial t} + \frac{\partial y}{\partial x}\right) = \left(\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2 y}{\partial x\,\partial t}\right)(dt + dx), \quad \text{and}$$

$$d\left(\frac{\partial y}{\partial t} - \frac{\partial y}{\partial x}\right) = \left(\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x\,\partial t}\right)(dt - dx).$$

From this he directly concludes that $\partial y/\partial t + \partial y/\partial x$ depends only on $t + x$, and $\partial y/\partial t - \partial y/\partial x$ depends only on $t - x$, that is, $\partial y/\partial t + \partial y/\partial x = \Phi(t + x)$ and $\partial y/\partial t - \partial y/\partial x = \Delta(t - x)$. Hence $dy = (\partial y/\partial t)\,dt + (\partial y/\partial x)\,dx = \frac{1}{2}\Phi(t + x)d(t + x) + \frac{1}{2}\Delta(t - x)d(t - x)$.

Integrating the latter expression, d'Alembert obtains the final solution $y = \psi(t + x) + \delta(t - x)$, which he unhesitatingly calls the "general solution." In the case when the string, attached at the points $x = 0$ and $x = l$ of the $0X$-axis, passes through the equilibrium position (the $0X$-axis) at time $t = 0$, this solution becomes $y = \psi(x + t) - \psi(x - t)$, where ψ is an even periodic function with period $2l$. In the case when the form of the string at the initial moment $t = 0$ is given by $y = \Sigma(x)$ and the velocity of its particles at that moment is given by the formula $\partial y/\partial t = \sigma(x)$, the solution takes the form $y = \psi(t + x) - \psi(t - x)$, where ψ is a periodic function with period $2l$ determined from the supplementary conditions $\psi(x) - \psi(-x) = \Sigma(x)$ and $\psi(+x) + \psi(-x) = \int\sigma(x)\,dx$. This, essentially, completes the paper.

Euler's solution. In 1748, Euler, following d'Alembert, tackles the same problem. He notes that his solution differs inessentially from that of d'Alembert but stresses that it is *truly the general solution*. Euler assumes that the initial velocity (at $t = 0$) of the particles of the string is zero and that the initial form of the string (at $t = 0$) is $y = f(x)$. Under these conditions Euler's solution is $y = \frac{1}{2}f(x + t) + \frac{1}{2}f(x - t)$.

Also, Euler is the first to note that the period of the vibration of the string is independent of the initial form as long as the latter cannot be subdivided into identical aliquot parts. At first sight it might seem that, apart from minor points, the solutions of Euler and d'Alembert are identical. But this is not at all the case. Both men use the same terminology but use the same words to denote different things. They agree on one thing, namely, that the term "equation" means equality of two analytic expressions (without entering on a discussion of what constitutes an analytic expression). Also, both agree that if two analytic expressions take on the same values at all points of an interval, they must be identical. But d'Alembert and Euler differ fundamentally in the meaning they assign to the word "function": *d'Alembert meant by it any analytic expression while Euler meant by it any curve drawn with a free hand.*

The debate between d'Alembert and Euler. That Euler and d'Alembert subscribed to diametrically opposite views became clear during the lively debate that sharpened ideas and gave them exact formulations. D'Alembert was the first to look for contradictions in Euler's interpretation of the word "function." He writes: "One cannot imagine a more general expression for a quantity y than that of supposing it to be a function of x and t; in which case the problem of the vibrating string has a solution only if the different forms of that string are contained in the same equation." D'Alembert concludes that his own and Euler's solution make sense only if the given function $f(x)$ is *periodic*. Euler's objection takes the form of a question: "If the obtained solution is to be regarded as deficient in those special cases when the form of the string cannot be contained by a single equation, what is one to mean by a solution in such cases?" He insists that his "geometric construction is always correct, regardless of the initial form of the string," that "the different parts of the initial curve are not connected at all by an equation but are connected simply by their description," and that "knowledge of a geometric curve is entirely sufficient for the knowledge of the motion without recourse to computations." D'Alembert's reply was not long in coming. He insists on his interpretation of what is a solution and notes the frequently neglected fact that the very differential equation $\partial^2 y/\partial t^2 = \partial^2 y/\partial x^2$ demands that the ratio $\partial^2 y/\partial x^2$ have a definite (finite) value, that is, that the curve have a definite curvature at each point. In particular, this applies to the endpoints of the string where, in view of the equality $\partial^2 y/\partial x^2 = 0$, the radius of curvature must be infinitely large. Also, the presence of points such as corners, artificially linking curves of different nature, makes the force indeterminate there and consequently makes motion impossible: "here nature itself blocks computations." "We shall leave it to physics to worry" about the question of the motion of such a composite string. Euler declined to continue the debate but noted that it is possible to develop a theory of differential equations containing such "improper" or "mixed" functions. In response to d'Alembert's criticism he points out that his solution, employing such "improper" functions, confirms, for example, the propagation of shocks along a string—a fact observed by D. Bernoulli. D'Alembert insists on the validity of his viewpoint and repeats that the presence of a corner on the string makes a solution impossible.

The ideas of D. Bernoulli. D. Bernoulli approached the problem in an altogether different manner. He had already acquired a measure of experience in the study of acoustical problems, and it dawned on him that a vibrating string has infinitely many fundamental vibrations. On the basis of his study of discrete systems he concluded that the most general motion of a string can be obtained by the composition of fundamental vibrations.

D. Bernoulli's ideas matured in 1753, and he concluded that the equation

$$y = \alpha \sin x \cos t + \beta \sin 2x \cos 2t + \gamma \sin 3x \cos 3t + \cdots$$

encompasses the solution of d'Alembert as well as that of Euler. Thus D. Bernoulli discovered an extremely important principle of mathematical physics and deserves to be honored not only for having formulated it but also for having clearly understood its far-reaching consequences. But while D. Bernoulli understood the importance and meaning of his principle of composition of vibrations, he was unable to justify it mathematically and thus provoked the strongest criticism of both d'Alembert and Euler. Euler pointed out that D. Bernoulli fails to note the totally unacceptable consequence implicit in his ideas, which is that an entirely arbitrary function of a variable x is representable by means of a series of sines of multiple arcs. Euler thought that such a function must be odd and periodic. We see that Euler again makes implicit use of the principle that if two analytic expressions have the same numerical values on some interval they must be identical everywhere. D. Bernoulli responded by pointing out that his formula contains infinitely many indeterminate coefficients that can be used to free the curve to pass through an arbitrarily large number of points of the given curve and thus to obtain an arbitrarily close approximation. Regarding the possibility of leaving out of this process one or another particular point, D. Bernoulli refers to d'Alembert's earlier criticism of Euler. To this Euler replied that it is extremely difficult, if not impossible, to choose the coefficients in the manner required by D. Bernoulli. As for d'Alembert, he stated that he fully agrees with Euler's criticism of D. Bernoulli and that he goes beyond it, for he is of the opinion that not every (analytic) periodic function can be represented by means of a sine series, that every function represented by a sine series must possess continuous curvature, and that the coincidence of two curves at infinitely many points does not necessarily make them identical. The nature of the controversy between d'Alembert and D. Bernoulli readily shows that, in modern terms, the former was an "arithmetizer" of mathematical analysis and the latter was a physicist who viewed things from a physicist's point of view.

The entrance of Lagrange. At the time when the most eminent mathematicians argued about the mathematical principles associated with the problem of the vibrating string, a young unknown, Lagrange, appeared on the stage. He immediately attracted attention by his "deft" computations (1759). Lagrange investigated the condition of the problem of the vibrating string with utmost care and adopted a definite position in the controversy by completely siding with Euler and opposing d'Alembert as well as D. Bernoulli. In an effort to prove Euler's correctness Lagrange put first and foremost the *interpolation problem*. He takes one of Euler's "improper" functions, that is, a graphically given curve, composed in general of pieces of completely different curves, and subdivides the axis of abscissas into small equal segments. Then he erects at the division points perpendiculars, thereby determining a sequence of points on the graphical curve, and seeks an interpolation curve passing through these points. Lagrange works with *linear trigonometric interpolation* with a bounded number of terms. This made his interpolation curves "laws" even for d'Alembert, since they were given by simple analytic expressions. Having thus solved the interpolation problem, Lagrange looks for a solution of the problem of a vibrating string for the *interpolation curve*. By passing to the limit a number of times he obtains in the end Euler's formula for the form of a vibrating string. It is worth noting that Lagrange passed by a colossal discovery without noticing it. En route to the final derivation of Euler's formulas Lagrange obtains *Fourier's trigonometric series*. Had he merely interchanged limits, Lagrange would have discovered the formation law of Fourier coefficients, and this would have ended all debates. But Lagrange's efforts were aimed in a different direction, and, while virtually brushing against the discovery, he was so completely unaware of it that he aimed at D. Bernoulli the phrase: "It is a pity that so clever a theory is untenable." Ironically, it was precisely the ideas of D. Bernoulli, as eventually handled by Fourier, that essentially ended the controversy. In another paper (1760) Lagrange returns to the problem of the string, follows d'Alembert's method, obtains the latter's solution, and is certain that he "made no use of any continuity" ("continuity" in the sense of Euler, that is, in modern terms, "analytic continuability"). This was not quite the case, for, as is well known, Lagrange was fully convinced that every continuous (in the modern sense) function is infinitely differentiable and can be expanded in a Taylor series with the possible exception of isolated points. This being so, it is extremely difficult to decide where in Lagrange's arguments Euler-continuity does or does not enter. His critics objected only to certain special points but did not go into the fundamental side of his investigations and admitted that his computations were, by and large, "singularly deft." D'Alembert attacked, above all, Lagrange's frequent passing to the limit. His sharp mind began to fully appreciate the difficulties associated with this operation. D'Alembert also objected to Lagrange's use of divergent series. Lagrange's response was to the effect that "so far, no one has made a mistake by replacing the series $1 + x + x^2 + x^3 + \cdots$ by the formula $1/(1 - x)$." In defending himself against d'Alembert's criticism (which the latter had also directed against Euler) to the effect that a vibrating string must have curvature at its points, Lagrange says that "nature cannot dwell on computations, for, in physical terms, there are no corners on a string; there is always a certain roundness to the stiffness of the string." In subsequent correspondence d'Alembert forces Lagrange to admit that his solution tacitly assumes the existence and finiteness of all derivatives. And since d'Alembert and Lagrange shared the prevalent contemporary conviction that the existence of derivatives of all orders implied that a function could be expanded in a Taylor series, Lagrange was forced to admit that he implicitly introduced Euler-continuity, that is, the representation of a function by means of equations—a point always insisted upon by d'Alembert.

Later Lagrange made one more attempt to bolster his considerations whose truth he was deeply convinced of. In this new exposition he passes through a definite number of points a curve that is a solution of the problem of the string and consists of m sine curves. It is important to note that these points are no longer on the given curve but near it. Lagrange calls the curve he introduced a *generatrix*. He notes that when m is very large, the generatrix deviates very little from the initial form of the string, so that the initial form may be regarded as a piece of the generatrix. He then poses the following question: does this not imply that the initial form of the string consists of sine curves? His answer is that when it comes to "geometric" identity such an assumption is unavoidable, but that in all other cases the initial curve is a kind of asymptote indefinitely approached by the generatrix without the two curves ever becoming one. And from the coefficients of his interpolation formula Lagrange deduces the conclusion that one can ignore the deviation of the generatrix only if the initial form has derivatives of all orders—a property that must be preserved throughout the time of the motion of the string. Motion of the string is possible under these conditions alone. Lagrange does not tell the reader that this assertion represents a complete refusal to defend Euler's viewpoint (which was his initial objective) and an acceptance of d'Alembert's position. The latter stubbornly insisted that the use of divergent series is inadmissible. It is worth pointing out that he quotes the function $\sqrt[3]{\sin x}$ as an example that an everywhere finite function need not have a Taylor series. It does not escape his penetrating vision that this very example goes against him. Indeed, on the one hand, we have an "equation," so that this form of a string admits a solution. On the other hand, we no longer have the finiteness of all derivatives. To help matters, d'Alembert says that infinitely large values of derivatives are admissible as long as there are no jumps. The debates lasted another 20 years without a final solution.

Fourier's discovery

At present the concept of function is not as finally crystalized and undeniably established as it seemed to be at one time at the end of the 19th century. It is no exaggeration to say that at present the function concept is still evolving and that the controversy about the vibrating string continues, except for the obvious fact that the scientific circumstances, the personalities involved, and the terminology are different. If we now look back at the 18th-century debate, what is especially striking is the remarkable perspicacity and intuitive power of the debating thinkers and the tremendous richness of deep analytic ideas connected with this controversy and largely generated by it. In this sense the debate was a motley tangle of profound and extremely difficult questions pertaining to the possibility of passing to the limit and the interchanging of limits; the conditions under which one may use diverging series; the convergence of the Taylor series of an infinitely differentiable function; the difference between a function and its analytic representation; the analytic continuability of a function; the concept of arbitrariness; infinite determinants; curves without curvature and curves consisting of just corner points; interpolation; discontinuities of functions and, in particular, of trigonometric series. The latter loomed so large during and after the debate that they have later been justly called "the axis of rotation of all of mathematical analysis." Even in the light of modern mathematical analysis it is not easy to get to know the particulars of the clash of all these ideas. What makes matters even more difficult is that we are not quite certain that we correctly understand the viewpoint of each of the debating thinkers. For example, as early as 1744 Euler communicated to Goldbach the formula $\Sigma_1^\infty (\sin nx)/n = (\pi - x)/2$ without at all concluding that *two analytic expressions that coincide on an interval need not coincide everywhere*. At that time such a conclusion would have been thought monstrous, and Euler, in possession of a fact confirming this conclusion, failed to take note of it for some, to us, inexplicable reason. In general, in the light of modern mathematical analysis, matters could, apparently, be described as follows. The key question of the debate pertained to the relation between an analytic definition of a function and a definition that is to some extent physical; does there exist a formula that gives the exact initial position of a string deflected from its position of equilibrium *in an arbitrary manner*? Neither the sophisticated analytic mind of d'Alembert, nor the creative efforts of Euler, D. Bernoulli, and Lagrange sufficed to solve this problem. The person fated to accomplish this task was Fourier. In 1807, to everybody's amazement, Fourier gave the rule for the coefficients a_n and b_n of the trigonometric series

$$a_0/2 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

representing an "arbitrarily given" function $f(x)$. The formulas

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\alpha)\cos n\alpha\, d\alpha \quad \text{and} \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(\alpha)\sin n\alpha\, d\alpha,$$

now known as the Fourier formulas, categorically decided the controversy in favor of D. Bernoulli, the main objection to whose view was precisely the absence of a rule for the computation of the coefficients of a trigonometric series representing an "arbitrarily" given function $f(x)$. True, there remained the objection to Fourier's results fed by the fact that it was not known whether or not his series converged. An immediate argument in favor of Fourier was the extreme simplicity of his rule for the computation of the coefficients of the trigonometric series. A final argument in his favor was a succession of papers by Lejeune Dirichlet (1805–1859) in which he proved the convergence of the Fourier series of any function $f(x)$ with a finite number of minima and maxima [on an interval]. Fourier's discovery produced tremendous bewilderment and confusion among all mathematicians. It toppled all concepts. Up until that time everyone, including Euler and d'Alembert, thought that every analytic expression represents only curves whose successive parts depend on one another. Euler introduced his term "continuous function" to express this mutual dependence of the parts of a function (the modern meaning of this term is completely different). Under the influence of Euler's view of continuity Lagrange tried to show in his theory of analytic functions (1797) that every continuous function can be expanded in a Taylor series: already at that time one sensed that there was a connection between the different parts of a function that can be expanded in a Taylor series, for one was aware that knowledge of a small arc of the curve implied knowledge of the whole curve. Now Fourier showed that such claims are futile and impossible, for a physicist who draws a curve in an arbitrary manner is free to change its course at his whim; but once the curve has been drawn it can be represented by means of a single analytic expression. This suggested the paradoxical result that there is no organic connection between different parts of the same straight line or between different arcs of the same circle, since Fourier's discovery showed that one can subsume under a single analytic formula, a single equation, a continuous curve consisting of segments of different straight lines or arcs of different circles. True, some timid voices noted that the equation of a single straight line or a single circle looked "simpler" than a Fourier expansion. But it soon became clear that this criterion of "simplicity" was completely useless, since it compelled one to use only algebraic functions and banned the use of infinite expansions, compromised by Fourier's discovery, whose importance and usefulness grew from day to day.