1-12 Partial Order and Lattice

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SM Problem 14.44: Consistent Enumerations

Suppose the following are three consistent enumerations of an ordered set $A = \{a, b, c, d\}$:

$$\begin{array}{lcccc} A_1 : & a & b & c & d \\ A_2 : & a & c & b & d \\ A_3 : & a & c & d & b \end{array}$$

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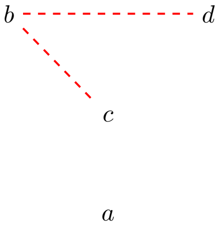
$$b \prec_{A_1} c \land c \prec_{A_2} b \implies b \parallel_A c$$

$$d \prec_{A_3} b \land b \prec_{A_2} d \implies b \parallel_A d$$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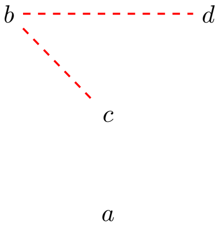
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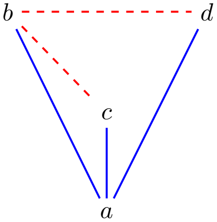
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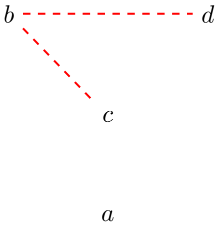
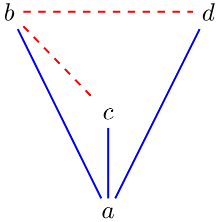
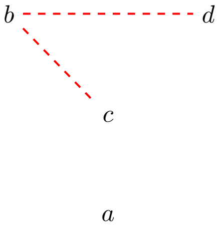
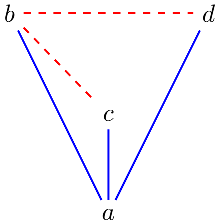




$\# = 6$





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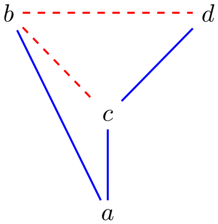

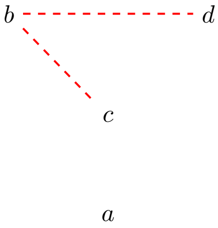

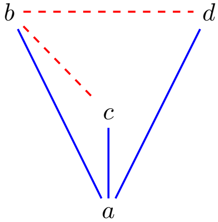

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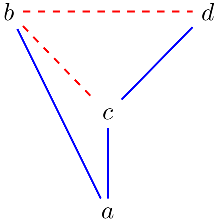

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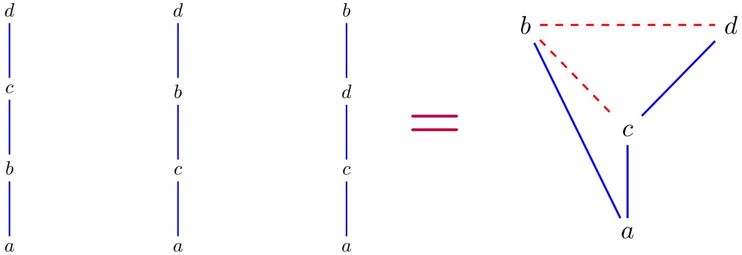

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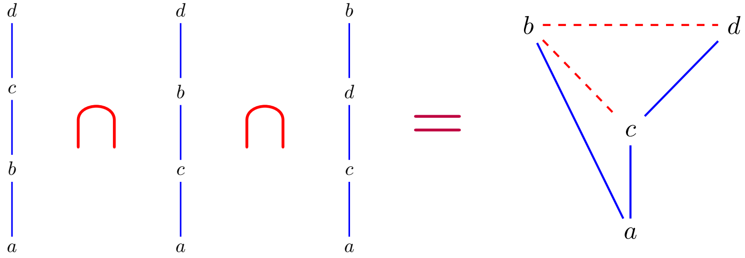
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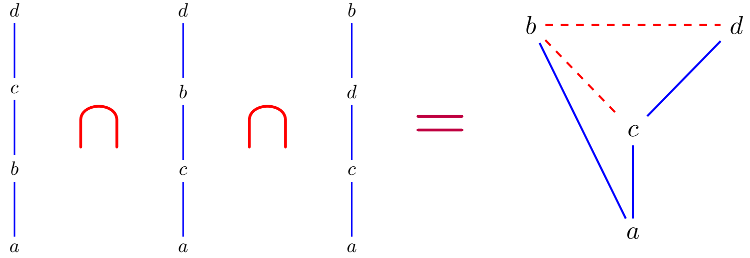

Theorem

Every partial ordering on a set X is the intersection of the total orders on X containing it.

SM Problem 14.62: Isomorphic Well-Ordered Sets

Suppose A and B are well-ordered isomorphic sets. Show that there is only one isomorphic mapping $f : A \to B$.

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Q : What about "totally-ordered" isomorphic sets?

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$$\boxed{f(x) = \min\Big(B \setminus f(\{a \in A : a < x\})\Big)}$$

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$$f : A \xrightarrow[onto]{1-1} B$$

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For any isomorphic mapping $g : A \to B$, we show that $g = f$.

$$\boxed{f(x) = \min\left(B \setminus f(\{a \in A : a < x\})\right)}$$

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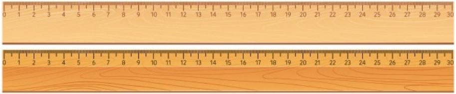

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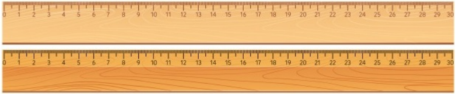

Theorem (Mathematical Induction for Well-Ordered Sets)

Let $\mathcal{S} = (S, <)$ be a well-ordered set. If $P(x)$ is a predicate such that

1. $P(\min S)$ *holds,*
2. $(\forall y < x : P(y)) \implies P(x)$,

then $\forall x \in S : P(x)$.

$$\boxed{f(x) = \min\Big(B \setminus f(\{a \in A : a < x\})\Big)}$$

We need to prove $\forall x \in A : g(x) = f(x)$.

By induction on the structure of A.

$$\boxed{f(x) = \min\Big(B \setminus f(\{a \in A : a < x\})\Big)}$$

Base Case: Consider $a \leftarrow \min A$.

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Definition (Lattice)

A *lattice* is an algebra $\mathcal{L} = (L, \wedge, \vee)$ satisfying,

$$\forall a, b, c \in L,$$

Idempotency:

$$a \wedge a = a \quad a \vee a = a$$

Commutativity:

$$a \wedge b = b \wedge a \quad a \vee b = b \vee a$$

Associativity:

$$(a \wedge b) \wedge c = a \wedge (b \wedge c) \quad (a \vee b) \vee c = a \vee (b \vee c)$$

Absorption:

$$a \wedge (a \vee b) = a \quad a \vee (a \wedge b) = a$$

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$$a \leq b \iff a \wedge b = a$$
$$a \leq b \iff a \vee b = b$$

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SM Problem 14.72: "Weak" Distributive Laws

Prove that for any lattice L:

$$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$$
$$a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$$

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$$a \vee (b \wedge c) \leq a \vee b$$
$$a \vee (b \wedge c) \leq a \vee c$$

$$\frac{a \leq b \qquad c \leq d}{(a \vee c) \leq (b \vee d)}$$


Thank
You!