2-7 Discrete Probability

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OPEN
TOPICS/2018

Searching an Unsorted Array (CLRS Problem $5-2\ (f)$)

```
procedure DETERMINISTIC-SEARCH(A[1···n], x)
    i ← 1
    while i ≤ n do
        if A[i] = x then
            return true
        i ← i + 1
    return false
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(e)

$$\exists!\ i : A[i] = x$$

(f)

$$\exists!_k\ i : A[i] = x$$

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\mathbb{E}[Y] &= \sum_{i=1}^{n} i \Pr\{Y = i\} \\
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\mathbb{E}[Y] &= \sum_{i=1}^{n} i \Pr\{Y = i\} \\
&= \sum_{i=1}^{n} i \Pr\{A[i] = x\} \\
&= \frac{1}{n} \sum_{i=1}^{n} i
\end{aligned}$$

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&= \sum_{i=1}^{n} i \Pr\{A[i] = x\} \\
&= \frac{1}{n} \sum_{i=1}^{n} i \\
&= \frac{n+1}{2}
\end{aligned}$$

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$$\begin{aligned} \mathbb{E}[Y] &= \sum_{i=1}^{n-k+1} i \Pr\{Y = i\} \\ &= \sum_{i=1}^{n-k+1} i \Pr\{i \text{ is the first index among } k \text{ indices } \textit{s.t. } A[i] = x\} \end{aligned}$$

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&= \sum_{i=1}^{n-k+1} i \frac{\binom{n-i}{k-1}}{\binom{n}{k}}
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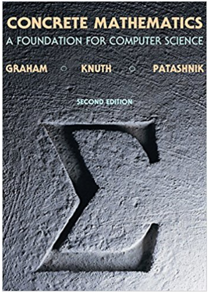
\end{aligned}$$

$$k = 1 \implies \mathbb{E}[Y] = \frac{n+1}{2}, \qquad k = n \implies \mathbb{E}[Y] = 1$$

After-class Exercise

$$\sum_{i=1}^{n-k+1} i\binom{n-i}{k-1} = \binom{n+1}{k+1}$$



$$r\binom{r-1}{k-1} = k\binom{r}{k}$$

$$\sum_{0\le k\le n}\binom{k}{m} = \binom{n+1}{m+1}$$

Chapter 5: Binomial Coefficients

$$
\begin{aligned}
\sum_{i=1}^{n-k+1} i\binom{n-i}{k-1} &= \sum_{i=0}^{n-k}(i+1)\binom{n-i-1}{k-1} \\
&= \sum_{i=0}^{n-k}\left((n+1)-(n-i)\right)\binom{n-i-1}{k-1} \\
&= \sum_{i=0}^{n-k}(n+1)\binom{n-i-1}{k-1} - \sum_{i=0}^{n-k}(n-i)\binom{n-i-1}{k-1} \\
&= (n+1)\sum_{i=0}^{n-k}\binom{n-i-1}{k-1} - k\sum_{i=0}^{n-k}\binom{n-i}{k} \\
&= (n+1)\sum_{m=k-1}^{n-1}\binom{m}{k-1} - k\sum_{m=k}^{n}\binom{m}{k} \\
&= (n+1)\binom{n}{k} - k\binom{n+1}{k+1} = \binom{n+1}{k+1}
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Indicator Random Variables

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$$I_i = \begin{cases} 1, & \text{if } A[i] \text{ is checked} \\ 0, & \text{o.w.} \end{cases}$$

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$$\mathbb{E}[Y] = \mathbb{E}\left[\sum_{i=1}^{n} I_i\right] = \sum_{i=1}^{n} \mathbb{E}[I_i] = \sum_{i=1}^{n} \Pr\{I_i = 1\}$$

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$$\Pr\{I_i = 1\} = \begin{cases} \frac{1}{k}, & \text{if } A[i] = x \\ \frac{1}{k+1}, & \text{if } A[i] \neq x \end{cases}$$

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$$\mathbb{E}[Y] = \sum_{i=1}^{n} \Pr\{I_i = 1\} = k \cdot \frac{1}{k} + (n-k) \cdot \frac{1}{k+1} = \frac{n+1}{k+1}$$


Wrong!

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$$i = n \implies \Pr\{I_n = 1\} = 0$$

NOT IID

(Independent and Identically Distributed)

$$\mathbb{E}[Y] = \sum_{i=1}^{n-k+1} \Pr\{Y \geq i\}$$

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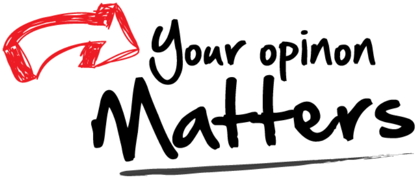
$$

order statistics?
balls-into-bins?


Thank
You!



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