The Counterfeit Coin Problem
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THE COUNTERFEIT COIN PROBLEM.

By C. A. B. Smith.

There is a problem which is much in favour at present of which one version is this : a man has 12 pennies among which there may be a counterfeit coin, which can only be told apart by its weight being different from the others. How can one tell in not more than three weighings whether there is a counterfeit penny, if so which one it is, and whether it is heavier or lighter than a normal penny? (*Math. Gaz.*, XXIX (1945), pp. 227–229; XXX (1946), pp. 231–234.) This puzzle seems to have originated in America. The purpose of the present note is to make clear what is the best possible solution in the most general case of any version of the problem. No originality is claimed for these solutions, as no doubt many of them have been obtained independently many times before : but I am not aware that any complete and systematic account has yet been published.

The general problem and its various versions.

More generally we will suppose that the man has been given a set S of pennies. This set has been classified as follows : for some of the pennies it is known that they do not weigh less than the correct amount, but possibly more. For brevity these will be called " Above-weight " or " A " coins. Others will be known to weigh not more than the correct amount : these will be " Below-weight " or " B " coins. The third set are " Correct " or " C " coins. The fourth are " Doubtful " or " D " : these may be of any weight. In the set S we shall suppose that there are $a(S)$ coins classified as A, $b(S)$ classified as B, $c(S)$ C's and $d(S)$ D's.

In what we shall call the " weak version no. 1 " of the problem the man is now told that there is not more than one counterfeit coin, and he is allowed m weighings on an ordinary balance (using coins only, no weights) to find out (i) if there is a counterfeit, (ii) which it is, (iii) whether it is heavier or lighter than normal. Clearly at each weighing we must put an equal number of pennies on the two scales of the balance, for otherwise if the counterfeit was only slightly different in weight it would go undetected. Thus if there is no counterfeit the scales will balance at each weighing. So that in order to detect that a given penny is counterfeit it is necessary for it to be weighed at least once—*i.e.* no penny is left off the scales in every weighing.

Version 2 differs from version 1 in that it is now certain that there is an incorrect coin. We can then solve the problem in m weighings, even if we add a further coin of sort A or sort B which is never put on the scales.

In version 3 it is again certain that there is an incorrect coin, but it is only required to find which it is, not whether it is heavier or lighter than normal. In this case we can again have one more coin than version 1, and it may be any sort of coin. In view of these remarks it will clearly be sufficient to solve the version 1, since the solutions for the other versions follow immediately.

In the " strong " form of the problem (in any version) we make the additional restriction that it is necessary to state in advance exactly which coins are to be put on the scales at each weighing, the choice being uninfluenced by the results of the previous weighings. (For example, the man might have to leave the weighings to his small son, who was too young to follow any complicated instructions.) Clearly when there is a solution of the strong form there is also a solution of the weak form : surprisingly enough, the converse is also true except for a very small number of cases.

One obvious remark : to any version of the problem for which we have a solution we may add an unlimited number of C (correct) coins. These will not be weighed, and will not affect the solution in any way.

We shall use the following notation.

For any real number x, $[x+]$ denotes the smallest integer $\geqq x$, and $[x-]$ the greatest integer $\leqq x$. Thus $[3+]=[3-]=[2\frac{1}{2}+]=[3\frac{1}{2}-]=3$. If u is any integer, we can always write $u=2v+w$, where v, w are integers $\leqq [\frac{1}{3}u+]$.

Pennies of sorts A, B, and D will be called "questionable pennies".

We shall write, for example,

$A+C$ for "an A coin + a C coin" : $A+A=2A$: $O=$ the set of no coins.

Clearly it is not possible to solve the problem for the combinations D, $2D$, $D+A$, $D+B$, $A+B$. These will be called the "insoluble combinations", and we shall suppose that they do not occur. In all other cases we shall show that the problem is soluble in a finite number m of weighings, and determine the least value of m. Clearly also there is effective symmetry between the letters A and B, and so we shall tacitly assume throughout that any solution or result in which the letters A, B (or a, b) appear stands also for the corresponding result in which these two letters are interchanged.

Solution for weak form (version 1).

The idea of this solution is this: we use the information gained at each weighing to reclassify the coins, so that, for instance, a coin which was known to be D before the weighing might now be narrowed down to A (above-weight) or even C (correct). In that way, after m weighings we hope to whittle down the possibilities to a single penny, known to be counterfeit, or to show that all the coins are C.

It is most convenient to introduce for simplicity an extra coin E, which is never put on the scales: the idea is that we then always whittle down the possibilities to a single coin, but if the coin is E, then we know that in fact there is no counterfeit coin. In any set of coins S we define $e(S)$ to be the number of E coins in S, so that $e(S)=0$ or 1.

Now in any weighing we must do this: we must take the set S_A of above-weight pennies and split it up into three parts (some possibly empty), $S_A(R)$ which is put on the right-hand scale pan, $S_A(L)$ on the left-hand pan, and $S_A(N)$ on neither. Similarly for S_B, S_C, S_D, and S_E. If $a(R)$ denotes the number of coins in $S_A(R)$, *i.e.* the number of A pennies on the right-hand pan, and similarly for other cases, we must have:

$$a(R)+a(N)+a(L)=a(S) \text{ (similarly for } b, c, d), \qquad (1)$$

$$e(R)=e(L)=0 \text{ ; } e(N)=e(S) \text{ (by definition of } E), \qquad (2)$$

$$a(R)+b(R)+c(R)+d(R)=a(L)+b(L)+c(L)+d(L). \qquad (3)$$

There are now three possibilities. Firstly, we may suppose that the right-hand pan is heavier (which we may denote by $R>L$). This may be due to either an above-weight penny, A or D, on the right-hand pan, or to a below-weight penny, B or D, on the left-hand pan. Accordingly we may now readjust our classification thus:

A pennies = those in $S_A(R)+S_D(R)$,
B pennies = those in $S_B(L)+S_D(L)$,
C pennies = all others.

If we write $a(R>L)$ for the number of pennies now classified A, we see

$$a(R>L)=a(R)+d(R),$$
$$b(R>L)=b(L)+d(L),$$
$$d(R>L)=e(R>L)=0. \qquad (4)$$

Secondly, however, we might find that it is the left-hand pan that is heavier $(R<L)$. We then have by symmetry :

$$a(R<L)=a(L)+d(L),$$
$$b(R<L)=b(R)+d(R),$$
$$d(R<L)=e(R<L)=0. \qquad \text{.....(5)}$$

The third possible case is that the scales balance : then we know that all the coins on the scales are good ones, while those not on the scales retain their classification. Denoting this case by $R=L$, we have

$$a(R=L)=a(N), \text{ and similarly for } b, d, e. \qquad \text{.....(6)}$$

Suppose now we write $\sigma(S)=a(S)+b(S)+2d(S)+e(S)$, then from equations (4), (5) and (6) we get

$$\sigma(R>L)+\sigma(R=L)+\sigma(R<L)=\sigma(S), \qquad \text{.....(7)}$$

where the symbols have the obvious meanings. But the number σ has the following properties. Except for the uninteresting case in which we know that all the coins are C, we must have $\sigma>0$. In addition, when $\sigma=1$ we can see the solution of the problem at once without any weighings, while if $\sigma>1$ we can not. Thus the solution of the problem may be considered as the reduction of the value of σ to 1 in as small a number of moves as possible. Now since we cannot tell in advance whether $R>$, $=$, or $<L$, it follows that the greatest reduction in the value of σ that we can be certain of getting in one weighing is from $\sigma(S)$ to

$$\mu=\max[\sigma(R>L),\ \sigma(R=L),\ \sigma(R<L)]. \qquad \text{.....(8}$$

Now from (7) we see that $\mu+\mu+\mu\geqq\sigma(S)$, so that $\mu\geqq[\frac{1}{3}\sigma(S)+]$. By induction it follows that in m weighings we cannot be certain of reducing the value of σ to less than $[3^{-m}\sigma(S)+]$. Thus if we can get σ down to 1 in m weighings we must have $\sigma(S)\leqslant 3^m$. This, however, is for the set containing the extra coin E : for the original set of coins (Z, say, so that $S=Z+E$) before the addition of E we must have :

Theorem 1. If the problem can be solved for a set Z in m weighings, then $\sigma(Z)\leqq 3^m-1$.

Conversely, if $\sigma(Z)\leqq 3^m-1$, then $\sigma(S)\leqq 3^m$, and we can certainly solve the problem in m or fewer weighings if we can be certain of reducing σ to a value not exceeding 3^{m-r} in r weighings. And that in turn will certainly be possible if we can reduce σ from its value, σ', say, before any weighing, to $[\frac{1}{3}\sigma'+]$ after that weighing. We must accordingly consider for what sets such a reduction is possible. Here is a list of some such sets, with the correct method of weighing, where a symbol such as U/V or W/X means that we weigh the set U on the left-hand pan, leave the set V or the set W off the balance, and put the set X on the right-hand pan.

List 1.

O/O or A or E/O.
A/O or B or $2B$ or $(B+E)$ or D or E/A.
A/O or B or E/C.
$2A/B$ or $(B+E)$ or D or $E/2A$.
$(A+B)/(A+B)$ or $(A+E)$ or $E/2D$.
C/O or B or E/D.
$A/2A$ or $(A+E)$ or B or $2B$ or $(B+E)$ or E/D.
$3A/(B+D)$ or $(D+E)/3A$.

D/O or A or $2A$ or $(A+B)$ or $(A+E)$ or E/D.
$2D/(A+3B)$ or $(A+2B+E)$ or $(3A+E)/2D$.
$4A/(B+D+E)/4A$.
$(A+B)/E$ or $(D+E)/(A+B)$.
$(A+B)/E/2C$.
$(A+2B)/(D+E)/(A+2B)$.
$(A+D)/A$ or $2A$ or $(2A+E)$ or $E/(A+D)$.
$(A+D)/D$ or $(D+E)/(B+D)$.

From these simple examples we can get an unlimited number of more complicated ones by adding any set of coins to one scale pan, and the same set to the other scale pan and also to the set of coins left off the balance—thus from the trivial case $O/E/O$ we may get the non-trivial one

$$6A+3D/6A+3D+E/6A+3D.$$

This operation does not alter the values of a, b, c, d, or e modulo 3, so that we may examine all possible cases, and we find that practically all of them are covered by the above list. Naturally the insoluble ones are not, and the other exceptional ones are the following, in which we can reduce σ from σ' to $[\frac{1}{3}\sigma'+1+]=\mu$, but not to $[\frac{1}{3}\sigma'+]$.

List 2.

$\sigma=3$, $\mu=2$: $A+B+C+E$.
$\sigma=5$, $\mu=3$: $3A+B+E$.
$\sigma=6$, $\mu=3$: $(3A+B+D)$, $(3A+D+E)$, $(2A+B+D+E)$.
$\sigma=9$, $\mu=4$: $5A+B+D+E$.
$\sigma=6r+2$, $\mu=2r+2$: $(A+3rD+E)$, $(3r+1)D$.
$\sigma=6r+3$, $\mu=2r+2$: $A+(3r+1)D$, $(3r+1)D+E$.

For example,

the correct way to weigh $A+B+C+E$ to give $\mu=2$ is $A/(C+E)/B$:

but because the coin E must always be left off the balance we can find no weighing which will reduce σ to 1.

Now, with the exception of $A+B+C+E$, none of these sets contains a C coin : and the set $A+B+C+E$ cannot result from any of the above weighings. (For, if it had resulted, since it contains the E coin, it must have been the set left off the balance : and there cannot have been any coins put on either of the two pans, or we would have had at least two C coins afterwards. But one would never weigh $O/A+B+C+E/O$, since $A/B+E/C$ would be better.)

Thus we see that, with the sole exception $A+B+C+E$, all sets containing one or more C coins may have their σ reduced to $\mu=[\frac{1}{3}\sigma+]$ in one weighing, and similarly in all subsequent weighings, and therefore if $\sigma\leqq 3^m$ they can be solved in at most m weighings. But after one weighing we certainly must have at least one C coin, as may be seen on examining List 1 (except for the insoluble sets, and certain sets soluble in one weighing). Hence in order that we may solve the problem in m weighings, it is necessary and sufficient that σ should be reduced to not more than 3^{m-1} in one weighing. And if $\sigma(S)\leqq 3^m$, that can certainly be done, except possibly for the sets of List 2. In fact, on examining that list we find the following failures, which are not soluble in m weighings, but are in $m+1$.

$$A+\tfrac{1}{2}(3^m-3)D+E,\quad \tfrac{1}{2}(3^m-1)D,\quad \tfrac{1}{2}(3^m-1)D+E,\quad A+\tfrac{1}{2}(3^m-1)D,$$

and when $m=1$, $A+B+C+E$, when $m=2$, $5A+B+D+E$.

Remembering that we started with a set Z, where $S = Z + E$, we obtain :

Theorem 2. The problem can be solved in m weighings for all sets Z for which $\sigma(Z) \leqslant 3^m - 1$, except for the insoluble sets and the following sets which need $m+1$ weighings :

$$A + \tfrac{1}{2}(3^m - 3)D, \qquad \tfrac{1}{2}(3^m - 1)D,$$
$$\text{when } m = 1,\ A + B + C, \qquad \text{when } m = 2,\ 5A + B + D.$$

This theorem accordingly provides a complete solution for the weak form of the problem, in version 1. In accordance with the remarks we made earlier, we can readily deduce from this the complete solution for the other versions.

Solution for strong form (*Version* 1).

In this form of the problem we have to imagine we have a scheme stating beforehand exactly which coins are to be placed on each pan at each weighing. Now we can conveniently represent such a scheme in the following way. Firstly, we number the pennies off as $P_1, P_2, \ldots$, up to P_n (say). Then with each coin P_r we associate a vector $v(P_r)$ with m components,

$$v(P_r) = (v_1, v_2, \ldots v_m)$$

where the sth component v_s is defined to be 1, -1, or 0, according as the penny P_r is placed respectively in the right-hand pan, the left-hand pan, or neither. (For convenience we shall use an inverted figure I to denote -1, as in reverse notation.)

Because there must be an equal number of coins on the two scale pans at each weighing, we see that these vectors must satisfy

$$\sum_r v(P_r) = \underset{\sim}{0} = (000\ldots0). \qquad \text{(9)}$$

Furthermore, the only information we have to show which penny is incorrect is the results of the weighings. If, for example, we find that the right-hand pan goes down in the first weighing, the left-hand one in the second, and neither in the third, we see that this must be due either to an above-weight coin which is placed on the right-hand pan in the first weighing, the left in the second, and neither in the third, and so has vector (1, I, 0, ...), or else it may be due to a below-weight penny with vector (I, 1, 0, ...).

Thus, in order that the problem may be soluble, we must have the further conditions :

(i) If $v(P_r) = v(P_s)$, $r \neq s$, then either P_r and P_s are one A and one B, or at least one of them is C.

(ii) If $v(P_r) = -v(P_s)$, $r \neq s$, then either P_r and P_s are both A or both B, or at least one of them is C.

(iii) We do not introduce an extra coin E which is never weighed, so that no coin has vector $\underset{\sim}{0}$. ......(10)

Conversely, any set of vectors $v(P_r)$ satisfying the conditions (9) and (10) provides a solution of the problem in m weighings.

In order to help us find such solutions we notice that the $(3^m - 1)$ non-zero m-vectors may be divided into pairs of vectors $\pm v$ of opposite sign. We shall call these simply " pairs of vectors ", and the number $\tfrac{1}{2}(3^m - 1)$ of such pairs we shall denote by M. From condition (10) we see that in any set of vectors forming a solution we may choose from each pair either :

(i) one vector, associated with any coin, either A, B, or D (we do not introduce an extra E coin in the strong form), or

(ii) both vectors of the pair $\pm v$, associated with two A coins, or two B coins, or

(iii) one vector of the pair v, repeated twice; once associated with an A coin, and once with a B coin. In addition, we may associate with C coins any of the vectors without restriction, including the zero vector.

Suppose, then, that we can find a set of vectors $v_1, v_2, \dots v_r$ (say), not more than one being chosen from each pair, whose sum is zero. Then we can easily construct from them a solution in this way: with each of the vectors v_1, $v_2, \dots v_r$ we associate a coin (of any sort, but most usefully of sort D). With the remaining pairs $\pm v_{r+1}, \pm v_{r+2}, \dots \pm v_M$ we may associate if we wish pairs of A coins or pairs of B coins, for such pairs of vectors have zero sum. We see that so far as solutions of this sort are possible, since we may have up to r D coins, and then up to $(2M-2r)\,A$ and B coins, we must have

$$\sigma = 2d + a + b \leqslant 2r + (2M - 2r) = 2M = 3^m - 1,$$

the same condition as for the weak form. Thus we may expect roughly the same set of solutions for the strong as for the weak form: more exactly we have:

Theorem 3. For $r=0$ and $3 \leqq r \leqq M-1$, and for no other values of r, it is possible to find a set of r m-vectors $v_1, v_2, \dots v_r$ satisfying the conditions:

(i) every component of every vector is 1, 0, or $\overline{1}$,

(ii) no $v_s = \underset{\sim}{0}$,

(iii) if $s \neq t$, then $v_s \neq \pm v_t$,

(iv) $\Sigma v_s = \underset{\sim}{0}$.

Proof. For $m = 1, 2, 3$, the theorem is readily verified directly.
Suppose, then, that it has been proved for $m = m' \geqq 3$. We wish to show that it follows for $m = m' + 1$. To do this we divide the range of r into 4 parts.

(I) If $3 \leqq r \leqq \frac{1}{2}(3^{m'} - 3)$, we simply take the solution for $m = m'$ and add on new component, identically 0.

(II) If $\frac{1}{2}(3^{m'} - 3) < r \leqq \frac{1}{2}(3^{m'+1} - 9)$, we write $r = 2g + h$, where g and $h \leqq \frac{1}{2}(3^{m'} - 3)$.

We take the solution for g m'-vectors, and add to it firstly an extra component 1, and secondly an extra component $\overline{1}$, thus obtaining a set of $2g(m'+1)$-vectors with zero sum. In addition, we take the solution for h m'-vectors and add to it the extra component 0. In all we have thus a set of $(2g+h) = r(m'+1)$-vectors, which may be seen to form a solution under the above conditions.

(III) If $r = \frac{1}{2}(3^{m'+1} - 7)$ or $\frac{1}{2}(3^{m'+1} - 5)$, we write $r = 2g + h + 4$ where

$$g = \tfrac{1}{2}(3^{m'} - 5) \text{ and } h \leqq \tfrac{1}{2}(3^{m'} - 3).$$

Now we can find a solution for g m'-vectors: it will leave two pairs of m'-vectors unused, say $\pm\xi$ and $\pm\eta$. Then we can build up a solution thus we take:

(the solution for g m'-vectors with extra component 1);
(the same solution with extra component $\overline{1}$);
(the solution for h m'-vectors with extra component 0);
(the two vectors $\pm\xi$, with extra component 1);
(the two vectors $\pm\eta$, with extra component $\overline{1}$).

(IV) If $r=\frac{1}{2}(3^{m'+1}-3)$ we write $r=3g+3$ where $g=\frac{1}{2}(3^{m'}-3)$, we have a solution of g m'-vectors leaving out some pair, $\pm\xi$, say. We may then take as solution :

(the solution for g m'-vectors with extra component 1) ;
(the same solution with extra component 0) ;
(the same solution with extra component $\bar{1}$) ;
(the vector ξ, with extra component 1) ;
(the vector $-\xi$, with extra component 0) ;
(the vector $\underset{\sim}{0}$, with extra component $\bar{1}$).

This completes the induction.

We can see, moreover, that there can be no solution when $r=1, 2$; while when $r=\frac{1}{2}(3^m-1)=M$ there is no solution, because if we take any set of vectors obtained by choosing one from each of the M pairs, the sum of all these vectors must have all its components odd, and therefore cannot vanish.

The solution we obtain for $M-1$ vectors is particularly interesting : it was first sent to me by Mr. R. L. Brooks. It is this : we choose one vector from every pair except the pair $\pm(1\ 1\ 1\ \ldots\ 1)$ according to the rules :

(i) if all components preceding the first zero component are equal they are all 1's ;
(ii) otherwise the first component is $\bar{1}$.

This type of solution is not, however, the only possible one. If $m \geqslant 3$, then for all values of r from 3 to M (but not for $r=1$ or 2) we can find solutions of the equation

$$v_1+v_2+\ldots+v_{r-1}+2v_r=\underset{\sim}{0},$$

where no two vectors belong to the same pair. This result may be proved in much the same way as Theorem 3 for $r=3$ to $M-1$, while we get the solution for $r=M$ from Brooks's solution of $\Sigma v_s=\underset{\sim}{0}$ for $(M-1)$-vectors by changing the vectors $(\bar{1}\bar{1}11\ \ldots\ 1)$, $(\bar{1}1\bar{1}\bar{1}\ \ldots\ \bar{1})$, and $(\bar{1}111\ \ldots\ 1)$ to $(11\bar{1}\bar{1}\ \ldots\ \bar{1})$, $(1\bar{1}11\ \ldots\ 1)$, and $2(\bar{1}111\ \ldots\ 1)$ and adding the vector $(\bar{1}\bar{1}\bar{1}\bar{1}\ \ldots\ \bar{1})$—the resulting set of vectors still having zero sum. We may now associate the vectors $v_1, v_2, \ldots v_{r-1}$ with any $(r-1)$ coins whatever, the vector v_r with one A and also one B coin, and we may again add up to $(M-r)$ pairs of A and B coins.

Again, the equation $v_1+v_2+2v_3+2v_4=\underset{\sim}{0}$ is solved by $v_1=(110\ \ldots)$, $v_2=(1\bar{1}0\ \ldots)$, $v_3=(\bar{1}01\ \ldots)$, $v_4=(00\bar{1}\ \ldots)$ when $m \geqslant 3$, and so provides a solution for such arrangements as $D+D+(A+B)+(A+B)+2\lambda A+2\mu B$.

We will now show how these arrangements give us the general solution. The cases $m=1, 2$ can easily be disposed of separately, so that we can confine ourselves to cases in which m, the number of weighings, is not less than 3. Consider first the situation when $d(Z)$, the number of D coins, is at least 2. We then see that we can solve the problem by assignments of vectors such as the following, where each A or D standing separately is given one vector from some pair, while in bracketed sums like $(A+B)$, A and B are both assigned the same vector.

(i) If $a(Z)$, $b(Z)$ are both even, solutions are

$$\begin{array}{ll} qD & 3 \leqslant q \leqslant M-1, \\ D+D+(A+B)+(A+B), & \\ D+D+A+A, & \\ 0, & \end{array}$$

to any of which we may add pairs of vectors corresponding to $(A+A)$ or $(A+C)$ or $(C+D)$, provided that we do not use vectors from more than M pairs altogether.

(ii) If $a(Z)$ is odd and $b(Z)$ is even, solutions are

$$qD + A, \qquad 2 \leqslant q \leqslant M - 2$$

again with pairs $(A + A)$, etc., as before, and

$$(M - 1)D + (A + C).$$

(iii) If $a(Z)$, $b(Z)$ are both odd, solutions are

$$qD + (A + B), \qquad 2 \leqslant q \leqslant M - 1,$$

again with pairs.

Since $\sigma(Z) = a(Z) + b(Z) + 2d(Z)$, we see that these arrangements solve the problem in all cases for which $d(Z) \geqslant 2$, $\sigma(Z) \leqslant 2M = 3^m - 1$, except for the following ones :

List 3.

$$\left.\begin{array}{l} MD = \tfrac{1}{2}(3^m - 1)D \\ (M - 1)D + A \end{array}\right\} \text{ (as in weak form),}$$

$$2D + 2(M - 2)A,$$

$$2D \text{ (insoluble).}$$

Clearly all of these are insoluble in m weighings—although all except the last may be done in $(m + 1)$. For example, a solution for MD would involve choosing M vectors, one from each pair, to sum to zero : and that we have seen to be impossible. (Theorem 3.)

If we investigate the other cases in which $d(Z) = 0$ or 1, we find that, besides the insoluble ones, the following cases with $\sigma(Z) \leqslant 3^m - 1$ cannot be solved in m moves.

List 4.

$$d(Z) = 1, \quad c(Z) = 0, \quad a(Z) + b(Z) = 3^m - 3\ ;$$

$$d(Z) = 0, \quad c(Z) = 0, \quad a(Z) + b(Z) = 3^m - 2\ ;$$

$$(3^m - 4)A + D\ ;$$

$$(3^m - 2)A + B\ ;$$

$$(3^m - 2)A + B + C\ ;$$

and also when $m = 2$ the following cases :

$$A + 5B + D \text{ (as in weak form) ;}$$

$$3A + D\ ;$$

$$2A + 2D\ ;$$

$$2A + 2B + 2D\ ;$$

$$A + B + 3D.$$

So we see that there are a small number of cases for which $\sigma(Z) \leqslant 3^m - 1$ which can be solved in the weak form but not in the strong form : but that in general the two forms agree : and, moreover, if we are given at least two correct coins, then all arrangements with $\sigma(Z) \leqslant 3^m - 1$ are soluble in both forms in m weighings.

The cases in which all coins are of the same sort.

The most interesting cases are those in which all the coins are of the same sort, *i.e.* all A's, all B's, or all D's. Remembering what we said about the relations between the different versions of the problem, we see that if all the coins are of sort A (*i.e.* none below normal weight) or all of sort B (*i.e.* not above normal weight), then we can solve the problem in the weak form in m weighings,—

in version 1 (in which there need not be a dud) for any number of coins from 2 to $3^m - 1$, and

in versions 2 *and* 3 (in which there certainly is a dud) for any number of coins up to 3^m.

The strong form differs only in that in version 1 $(3^m - 2)$ coins are not soluble in m weighings.

If all the coins are D, *i.e.* of unknown weight, then in either form we can solve the problem in m weighings,—

in versions 1 *and* 2 (in which we must say if the dud is heavy or light) for any number of coins from 3 to $\frac{1}{2}(3^m - 3)$, and

in version 3 (in which we merely find the dud) for any number of coins up to $\frac{1}{2}(3^m - 1)$, except 2. But if we are given in addition at least one correct coin, then we can solve the problem in versions 1 and 2 for any number of D's up to $\frac{1}{2}(3^m - 1)$, and in version 3 up to $\frac{1}{2}(3^m + 1)$.

The problem with weights.

In the above discussion we supposed that we were provided with a balance, but no weights. If we are provided with weights (so that we can determine the difference in weight between the two scale pans), and we know the weight of a correct penny, then the effect on the problem is similar to that of having an unlimited number of correct coins, except that we can now determine the exact weight of the counterfeit coin, and not merely whether it is heavier or lighter than normal.

Thus we can now solve the problem in m weighings in version 1 for *all* arrangements with $\sigma(Z) \leqslant 3^m - 1$, and corresponding results hold for versions 2 and 3.

If, however, we are allowed weights but do not know the weight of a correct coin, then the problem becomes very much more complicated, and I haven't yet obtained a general solution. However, one doesn't necessarily lose anything by not knowing the correct weight, as is shown by the fact that the following problem is soluble (I leave the solution to the reader):

"A man has 13 coins, not more than one of which is counterfeit. He marks them with the letters $a, b, \ldots m$, and hands them to his assistant to make three weighings, telling him exactly which coins to place on each pan in each weighing. From the three weights obtained he is able to tell the weight of a correct penny, whether there is an incorrect one, and if so which it is and how much it weighs. How can he do that?"

In conclusion, I would like to express my thanks to Dr. A. H. Stone for advice and suggestions regarding the paper.

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1520. THE FOUR-DIMENSIONAL SPACE-TIME CONTINUUM. . . . This is the Space that is at this moment only present before our Eye, the only Space that was, or that will be, from Everlasting to Everlasting. This Moment Exhibits infinite Space, but there is a Space also wherein all Moments are infinitely Exhibited, and the Everlasting Duration of infinite Space is another Region and Room of Joys. Wherein all Ages appear together, all Occurrences stand up at once, and the innumerable and endless Myriads of yeers that were before the Creation, and will be after the World is ended, are objected as a clear and stable Object, whose several parts extended out at length, give an inward Infinity to this moment, and compose an Eternitie that is seen by all Comprehensors and Enjoyers.—(From the *Felicities of Thomas Traherne*, pub. Dobell, 1934, p. 104. Written about the middle of the 17th C.) [Per Mr. A. R. Pargeter.]

1521. Time really is nothing but a huge circle. You divide a circle of three hundred and sixty degrees into twenty-four hours, and you get fifteen degrees of arc that is the equivalent of each hour.—E. S. Gardner, *The case of the buried clock*, p. 82. [Per Mr. C. D. T. Owen.]