1-11 Set Theory (IV): Infinity

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Finite Sets

"关于有穷，我原以为我是懂的"

Definition (Finite)

X is finite if

$$\exists n \in \mathbb{N} : |X| = n$$

Theorem (Pigeonhole Principle (UD Theorem 22.2))

$$f : \{1, \cdots, m\} \to \{1, \cdots, n\} \; (m, n \in \mathbb{N}^+, m > n)$$

f is not one-to-one.

$A \setminus \{a\}$ (UD Problem 22.17)

Let A be a nonempty finite set with $|A| = n$ and let $a \in A$.
Prove that $A \setminus \{a\}$ is finite and $|A \setminus \{a\}| = n - 1$.

$$f : A \xleftrightarrow[onto]{1-1} \{1, \cdots, n\}$$

$$f|_{A \setminus \{a\}} : A \setminus \{a\} \xleftrightarrow[onto]{1-1} \{1, \cdots, n\} \setminus \{f(a)\} \xleftrightarrow[onto]{1-1} \{1, \cdots, n-1\}$$

(UD Problem 22.18)

(a) A is a finite set and $B \subseteq A$. We showed that B is finite (Corollary 21.10). Show that $|B| \le |A|$.

one-to-one $f : B \to A$

(b) A is a finite set and $B \subseteq A$. Show that if $B \neq A$, then $|B| < |A|$.

$\exists a : a \in A \land a \notin B$ $\quad$ $f : B \to A \setminus \{a\}$ $\quad$ $|B| \le |A \setminus \{a\}|$

(c) If two finite sets A and B satisfy $B \subseteq A$ and $|A| \le |B|$, then $A = B$.

By contradiction and (b).

$f : A \to A$ (UD Problem 22.21)

Let A be a finite set.

$$f : A \to A$$

Prove that

$$f \text{ is one-to-one} \iff f \text{ is onto.}$$

$$\Longrightarrow$$

By Contradiction.

$$f : A \to A \setminus \{a\}$$

By Pigeonhole Principle.

$$\Longleftarrow$$

By Contradiction.

$$\sum_{y \in A} f^{-1}(\{y\}) > |A|$$

Set Union (UD Problem 23.1)

Give an example, if possible, of

(a) A countably infinite collection of pairwise disjoint finite sets whose union is countably infinite.

$$\forall n \in \mathbb{N} : A_n = \{n\} \qquad \bigcup_{n \in \mathbb{N}} A_n = \mathbb{N}$$

(b) A countably infinite collection of nonempty sets whose union is finite.

$$\forall n \in \mathbb{N} : A_n = \{1\} \qquad \bigcup_{n \in \mathbb{N}} A_n = \{1\}$$

(c) A countably infinite collection of pairwise disjoint nonempty sets whose union is finite.

$$|A| = n \implies |\mathcal{P}(A)| = 2^n$$

UD Problem 23.3 (d)

Is it countable or uncountable?

$$A = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x + y = 1\}$$

$$f : \mathbb{R} \xleftrightarrow[onto]{1-1} A$$

$$f(x) = (x, 1 - x)$$

Infinite Sequences of 0's and 1's (UD Problem 23.4)

Is the set of all infinite sequences of 0's and 1's finite, countably infinite, or uncountable?

$$
\begin{array}{|ll|}
\hline
s_1 & = \color{red}{0}\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\ldots \\
s_2 & = 1\,\color{red}{1}\,1\,1\,1\,1\,1\,1\,1\,1\,1\ldots \\
s_3 & = 0\,1\,\color{red}{0}\,1\,0\,1\,0\,1\,0\,1\,0\ldots \\
s_4 & = 1\,0\,1\,\color{red}{0}\,1\,0\,1\,0\,1\,0\,1\ldots \\
s_5 & = 1\,1\,0\,1\,\color{red}{0}\,1\,1\,0\,1\,0\,1\ldots \\
s_6 & = 0\,0\,1\,1\,0\,\color{red}{1}\,1\,0\,1\,1\,0\ldots \\
s_7 & = 1\,0\,0\,0\,1\,0\,\color{red}{0}\,0\,1\,0\,0\ldots \\
s_8 & = 0\,0\,1\,1\,0\,0\,1\,\color{red}{1}\,0\,0\,1\ldots \\
s_9 & = 1\,1\,0\,0\,1\,1\,0\,0\,\color{red}{1}\,1\,0\ldots \\
s_{10} & = 1\,1\,0\,1\,1\,1\,0\,0\,1\,\color{red}{0}\,1\ldots \\
s_{11} & = 1\,1\,0\,1\,0\,1\,0\,0\,1\,0\,\color{red}{0}\ldots \\
\vdots & \quad \vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\vdots\,\ddots \\
\hline
\end{array}
$$

$$
\boxed{s \;= \color{blue}{1\,0\,1\,1\,1\,0\,1\,0\,0\,1\,1}\ldots}
$$

By Diagonal Argument.

Infinite Sequences of 0's and 1's (UD Problem 23.4)

Is the set of all infinite sequences of 0's and 1's finite, countably infinite, or uncountable?

$$f : \{\{0,1\}^*\} \to \mathbb{N}$$

$$f(x_0 x_1 \cdots) = \sum_{i=0}^{\infty} x_i 2^i$$

Complex Numbers (UD Problem 24.16)

Prove that

$$|\mathbb{R}| = |\mathbb{C}|, \quad \mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

$$|\mathbb{C}| = |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$$

$$\mathbb{R} \times \mathbb{R} \approx \mathbb{R}$$

$$(x = 0.a_1 a_2 a_3 \cdots, y = 0.b_1 b_2 b_3 \cdots) \mapsto 0.a_1 b_1 a_2 b_2 a_3 b_3 \cdots$$

Was Cantor Surprised?

UD Problem 24.15

$$(0, 1) \approx (0, 1) \times (0, 1)$$

Theorem (Cantor-Schröder–Bernstein (1887))

$$|X| \leq |Y| \land |Y| \leq |X| \implies |X| = |Y|$$

$$\exists \textit{ one-to-one } f : X \to Y \land g : Y \to X \implies \exists \textit{ bijection } h : X \to Y$$

$$f : (0, 1) \to (0, 1) \times (0, 1)$$

$$f(x) = (x, 0.5)$$

$$g : (0, 1) \times (0, 1) \times (0, 1)$$

$$(x = 0.a_1 a_2 a_3 \cdots, y = 0.b_1 b_2 b_3 \cdots) \mapsto 0.a_1 b_1 a_2 b_2 a_3 b_3 \cdots$$

Bijections (UD Problem 21.21)

$$[0,1] \approx (0,1)$$

$$0, 1, \quad \frac{1}{2}, \frac{1}{3}, \quad \frac{1}{4}, \frac{1}{5} \cdots$$

$$f(0) = \frac{1}{2} \quad f(1) = \frac{1}{3}$$

$$\forall n \geq 4 : f(\frac{1}{n-2}) = \frac{1}{n}$$

$$f(x) = x, \text{ otherwise}$$

$$(-\infty, \infty) \approx (0, \infty)$$

$$f(x) = e^x$$

$$(0, \infty) \approx (0, 1)$$

$$f(x) = \frac{x}{x+1}$$

$$[0, 1] \approx (0, 1]$$

$$f(0) = \frac{1}{2} \quad f(\frac{1}{2}) = \frac{2}{3} \quad f(\frac{2}{3}) = \frac{3}{4} \quad \cdots \quad f(x) = x$$


Thank
You!