4-1 LP overview. (George Dantzig 1947)

- LP & "Simplex" Method (ProblemOverflow Q&A) Why Simplex?
  (29-1:) Linear-inequality Feasibility ⟺ LP)
- Duality. (duality of LPs in General form)
  29.4-2

TODAY

- Applications of LP & Duality.
  - SSSP 29.2-2 29.2-3
  - max-flow $\boxed{29.2\text{-}4 \quad 29.4\text{-}3}$
  - maximum-bipartite-matching. 29.2-6

Scribe Note

Duality.

(通[illegible].)

Duality

Primal $\underline{\underline{P}}$

$\max\ c^T x$

s.t.

$Ax \le b$

$x \ge 0$

Dual $\underline{\underline{D}}$

$\min\ b^T y$

$A^T y \ge c$

$y \ge 0$

Example

$\max\ 3x_1 + x_2 + 2x_3$

s.t.

$x_1 + x_2 + 3x_3 \le 30$ ① y_1

$2x_1 + 2x_2 + 5x_3 \le 24$ ② y_2

$4x_1 + x_2 + 2x_3 \le 36$ ③ y_3

$x_1,\ x_2,\ x_3 \ge 0.$

Lagrange Multipliers ⇓

$x^* = (8, 4, 0),\ v^* = 28$ 寻找证据

Goal of primal: Find proof for "28".

$\max\ 3x_1 + x_2 + 2x_3$

How to show that $x^* = (8,4,0)$ is optimal? certificate

~~$\frac{1}{2}\times① + \frac{1}{2}\times②$: $3x_1 + 3x_2 + 8x_3$~~

~~$2x_1$~~

~~$\frac{1}{2}\times②$: $x_1 + x_2 + \frac{5}{2}x_3$~~

①+②: $3x_1 + 3x_2 + 8x_3 \le 54$

$3x_1 + x_2 + 2x_3 \le$

①$+\frac{1}{2}\times$②: $3x_1 + \frac{3}{2}x_2 + 4x_3 \le 48$

①$+\frac{1}{2}\times$②: $2x_1 + 2x_2 + \frac{11}{2}x_3$ ×

$0\times$①$+\frac{1}{6}\times$②$+\frac{2}{3}\times$③: $3x_1 + x_2 + \frac{13}{6}x_3 \le 28$

Example.

$\min\ 30y_1 + 24y_2 + 36y_3$

s.t.

$y_1 + 2y_2 + 4y_3 \ge 3$

$y_1 + 2y_2 + y_3 \ge 1$

$3y_1 + 5y_2 + 2y_3 \ge 2$

$y_1,\ y_2,\ y_3 \ge 0.$

~~$(y^* = (0, \frac{1}{6}, \frac{2}{3})\ \ y^* = 28.$~~

Goal: To provide upper bound for "$3x_1 + x_2 + 2x_3$" as min as possible

$3x_1 + x_2 + 2x_3$

$\le y_1 \times ① + y_2 \times ② + y_3 \times ③$

$= (y_1 + 2y_2 + 4y_3)\,x_1$ ≥3 determined by the variable types ($\ge 0, \le 0, \gtrless 0$)

$+ (y_1 + 2y_2 + y_3) \times x_2$

$+ (3y_1 + 5y_2 + 2y_3)\,x_3$

$\le 30y_1 + 24y_2 + 36y_3$

To establish the upper bound for P

$y_1 \ge 0 \quad y_2 \ge 0 \quad y_3 \ge 0.$

由不等式 determined by constraint types. ($\ge, \le, =$)

EX 29.4-2 Dual programs for general linear programs.

max. $3x_1 + x_2 + 2x_3$

s.t.

$x_1 + x_2 + 3x_3 \geq 30$

$2x_1 + 2x_2 + 5x_3 \leq 24$

$4x_1 + x_2 + 2x_3 = 36$

$x_1 \geq 0$

$x_2 \leq 0$

$x_3 \gtrless 0$.

min. $30y_1 + 24y_2 + 36y_3$

s.t.

$y_1 + 2y_2 + 4y_3 \geq 3$

$y_1 + 2y_2 + y_3 \leq 1$

$3y_1 + 5y_2 + 2y_3 = 2$

$y_1 \leq 0$

$y_2 \geq 0$

$y_3 \gtrless 0$.

Now, you should be able to give dual to primal P of general form in symbolic way.

~~Ex~~:

Weak Duality: (Thm 29.8).

$$c^T x \leq b^T y.$$

Pf. $Ax \leq b$ $\quad A^T y \geq c$

$y^T A \geq c^T$

$$c^T x \leq \boxed{y^T} A x \leq y^T b = b^T y.$$

primal

multiplier.

Dual.

Corollary 29.9.

Let x^* be a feasible solution to P

y^* be a feasible solution to D.

If $c^T x^* = b^T y^* \Rightarrow$ x^* is optimal to P

y^* is optimal to D.

Thm 29.10 (LP Strong Duality)

If P has a (bounded) optimal solution x^*,

then its dual D has a (bounded) optimal solution y^*

and $c^T x^* = b^T y^*$.

P \ D	Finite optimum	Unbounded	Infeasible
FO	1	×	×
Un.	×	×	3
Inf.	×	3	2

Case 2: Inf.

$$\max x_1$$
$$x_1+x_2 \le -1$$
$$-x_1-x_2 \le -1$$
$$x_1, x_2 \text{ free}$$
$$(x_1, x_2 \geq 0)$$

(Inf.)

$$\min -y_1-y_2$$
$$y_1-y_2=1$$
$$y_1-y_2=0$$
$$y_1 \geq 0$$
$$y_2 \geq 0.$$

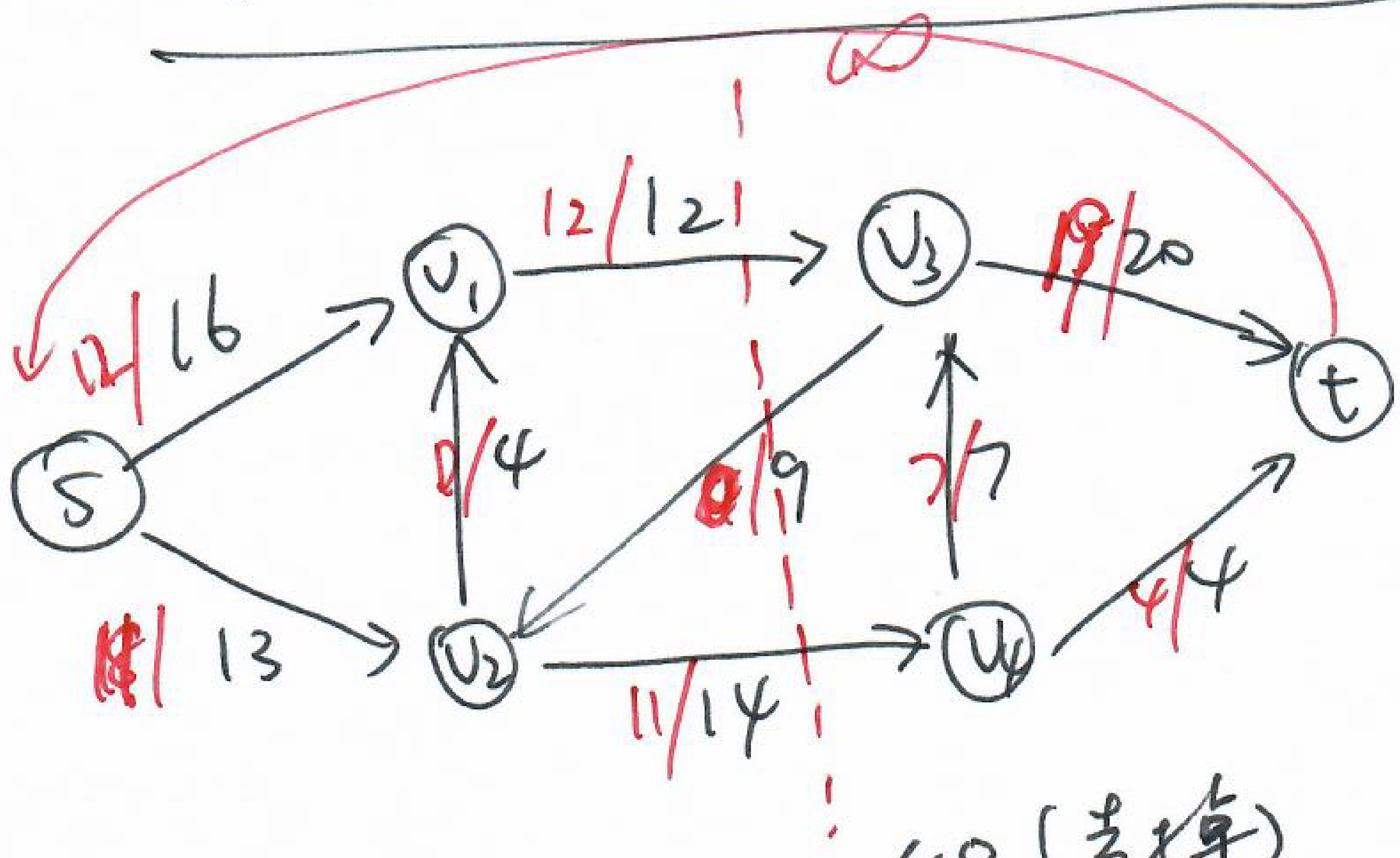
Case 3: (Inf.)

$$\max x_1$$
$$x_1+x_2 \le -1$$
$$-x_1-x_2 \le -1$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

(Unbounded)

$$\min -y_1-y_2$$
$$y_1-y_2 \geq 1$$
$$y_1-y_2 \geq 0$$
$$y_1 \geq 0$$
$$y_2 \geq 0$$

LP for the Max-Flow Problem



$$\max \sum_{v\in V} f_{sv} \boxed{-\sum_{v\in V} f_{vs}}$$

≤0 (去掉). WHY?

$f = \begin{pmatrix} \vdots \\ f_{uv} \\ \vdots \end{pmatrix}$ $c = \begin{pmatrix} c_{uv} \end{pmatrix}$

$\left\{ \begin{array}{l} f_{uv}: \text{flows on edge } (u,v)\in E. \\ c_{uv}: \text{capacity of edge } (u,v)\in E. \end{array} \right.$

f_{ts}, $c_{ts}=\infty$

s.t.

$f_{uv} \le \cancel{c(u,v)}\ c_{uv}$ $\quad \cancel{\forall u,v\in V}\ \forall (u,v)\in E$ $\quad |E|\ \cancel{|V|^2} \Rightarrow |E|+1$

$\sum_{v\in V} f_{vu} = \sum_{v\in V} f_{uv}$ $\quad \forall u\in V\setminus\{s,t\}$. $\quad |V|-2 \Rightarrow |V|$

$f_{uv} \ge 0$. $\quad \cancel{\forall u,v\in V}\ \forall (u,v)\in E$ $\quad |E|\ \cancel{|V|^2} \Rightarrow |E|+1$.

$\Downarrow$

$\cancel{\boxed{\sum_{v\in V} f_{uv} - \sum_{v\in V} f_{vu} = 0}}$.

(Example: For $u = v_1$
$-f_{v_1v_3} + f_{sv_1} + f_{v_2v_1} = 0$.

In Matrix Form: $\underset{\Uparrow\ (\text{Incoming})}{\sum_{v\in V} f_{vu}} - \underset{\Uparrow\ (\text{outgoing})}{\sum_{v\in V} f_{uv}} = 0$. Edge E $\boxed{A}\begin{pmatrix}\vdots\end{pmatrix}$

node-edge incidence matrix.

max. $\sum_{v\in V} f_{sv}$. max f_{ts}

s.t.

$Af = 0$

$f \le c$ $\quad |E|$

$f \ge 0$ $\quad |E|$

node-edge incidence matrix (关联矩阵)

Example.

$$A: \begin{array}{c|ccccccccc} & sv_1 & sv_2 & v_1v_3 & v_2v_1 & v_2v_4 & v_3v_2 & v_3t & v_4v_3 & v_4t \\ \hline s & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ & & & & 0 & 0 & 0 & 0 & 0 & 0 \\ v_1 & +1 & 0 & -1 & -1 & & & & & \\ v_2 & & 1 & & -1 & -1 & 1 & & & \\ v_3 & & & 1 & & & -1 & -1 & 1 & \\ v_4 & & & & & 1 & & & -1 & -1 \\ t & & & & & & & 1 & & 1 \end{array} \begin{pmatrix} f_{sv_1} \\ f_{sv_2} \\ \vdots \\ f_{v_4t} \end{pmatrix} = 0.$$

A row: for each node. (indegree; outdegree)
A col for each edge (+1, −1).

max-flow LP: P

$\max \ f_{ts} = (0,0,\dots,1)\cdot f$

s.t. $|E|+1$

p $\quad |V| \cdot \boxed{A} f = 0 \qquad |V|$

d. $\quad I \cdot f \le c \qquad |E|$

$\boxed{f \geq 0} \qquad |E|$

$f: |E|+1,$

Dual program D.

~~relax~~

$\Rightarrow p_u$ for each u $\quad p_u$.

$\Rightarrow d_{uv}$ for each $uv \in E$.

$\min.\ c^T d. \quad \sum_{uv \in E} c_{uv} \cdot d_{uv}$

s.t. $(A^T \ I)\begin{pmatrix} p \\ d \end{pmatrix} \geq (0,0,\dots 1) \qquad \boxed{\forall (u,v) \in E}$

$d \geq 0 \quad |E|$

p is free.

Example:

edge. $v_1 v_3$

$\begin{pmatrix} A \\ I \end{pmatrix} f \qquad (A^T \ I)\begin{pmatrix} p \\ d \end{pmatrix} \geq (0, 0, 0, \dots 1).$

min

$0 \cdot f_{v_1 v_3} \qquad \forall (u,v) \in E. \quad p_u$

$-p_{v_1} + p_{v_3} + d_{v_1 v_3} \geq 0.$

$-p_v + p_u + d_{uv} \geq 0. \qquad \forall (u,v) \in E.$

$-p_t + p_s + d_{ts} \geq 1 \qquad (t s) \in E.$

$d_{uv} \geq 0.$ without proof $d_{uv} \in \{0,1\}$

$p_u \geq 0. \quad p_u \in \{0,1\}.$

Exercise:

Run the Simplex Method on P & D!!!

(1) $d_{ts} = 0.$

(2) $p_s - p_t \geq 1$

$\Rightarrow p_s = 1, \ p_t = 0.$

(3) $S = \{ v \in V \mid p_v = 1 \}$ $\quad S, \bar{S}$ cut.

$\bar{S} = \{ v \in V \mid p_u = 0 \}.$

capacity of a cut

(4) min $\sum_{uv \in E} c_{uv} \cdot d_{uv}.$

We hope that $d_{uv} = 1 \Leftrightarrow u \in S, \ v \in \bar{S}, \ uv \in E.$ $\quad p_u = 1, \ p_v = 0$

(5) $-p_v + p_u + d_{uv} \geq 0 \qquad \forall uv \in E.$

0	0	0
0	1	0
1	0	1
1	1	0

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