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Georg Cantor's first set theory article

Georg Cantor published his first set theory article in 1874, and it contains the first theorems of transfinite set theory, which studies infinite sets and their properties.[1] One of these theorems is "Cantor's revolutionary discovery" that the set of all real numbers is uncountably, rather than countably, infinite.[2] This theorem is proved using **Cantor's first uncountability proof**, which differs from the more familiar proof using his diagonal argument. The title of the article, "**On a Property of the Collection of All Real Algebraic Numbers**" ("*Ueber eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen*"), refers to its first theorem: the set of real algebraic numbers is countable. In 1879, Cantor modified his uncountability proof by using the topological notion of a set being dense in an interval.

Georg Cantor, c. 1870

Cantor's 1874 article also contains a proof of the existence of transcendental numbers. As early as 1930, mathematicians have disagreed on whether this proof is constructive or non-constructive.[3] Books as recent as 2014 and 2015 indicate that this disagreement has not been resolved.[4] Since Cantor's proof either constructs transcendental numbers or does not, an analysis of his article can determine whether his proof is constructive or non-constructive.[5] Cantor's correspondence with Richard Dedekind shows the development of his ideas and reveals that he had a choice between two proofs, one that uses the uncountability of the real numbers and one that does not.

Historians of mathematics have examined Cantor's article and the circumstances in which it was written. For example, they have discovered that Cantor was advised to leave out his uncountability theorem in the article he submitted; he added it during proofreading. They have traced this and other facts about the article to the influence of Karl Weierstrass and Leopold Kronecker. Historians have also studied Dedekind's contributions to the article, including his contributions to the theorem on the countability of the real algebraic numbers. In addition, they have looked at the article's legacy, which includes the impact that the uncountability theorem and the concept of countability have had on mathematics.

Contents

The article

Cantor's article is short, less than four and a half pages.[A] It begins with a discussion of the real algebraic numbers and a statement of his first theorem: The set of real algebraic numbers can be put into one-to-one correspondence with the set of positive integers.[6] Cantor restates this theorem in terms more familiar to mathematicians of his time: The set of real algebraic numbers can be written as an infinite sequence in which each number appears only once.[7]

Cantor's second theorem works with a closed interval $[a, b]$, which is the set of real numbers $\geq a$ and $\leq b$. The theorem states: Given any sequence of real numbers $x_1, x_2, x_3, \ldots$ and any interval $[a, b]$, there is a number in $[a, b]$ that is not contained in the given sequence. Hence, there are infinitely many such numbers.[8]

Cantor observes that combining his two theorems yields a new proof of Liouville's theorem that every interval $[a, b]$ contains infinitely many transcendental numbers.[8]

Cantor then remarks that his second theorem is:

> the reason why collections of real numbers forming a so-called continuum (such as, all real numbers which are ≥ 0 and ≤ 1) cannot correspond one-to-one with the collection (ν) [the collection of all positive integers]; thus I have found the clear difference between a so-called continuum and a collection like the totality of real algebraic numbers.[9]

This remark contains Cantor's uncountability theorem, which only states that an interval $[a, b]$ cannot be put into one-to-one correspondence with the set of positive integers. It does not state that this interval is an infinite set of larger cardinality than the set of positive integers. Cardinality is defined in Cantor's next article, which was published in 1878.[10]

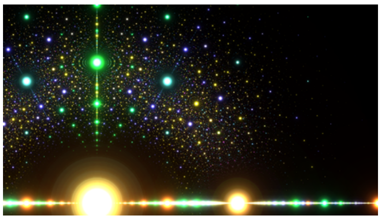
Proof of Cantor's uncountability theorem
Cantor does not explicitly prove his uncountability theorem, which follows easily from his second theorem. To prove it, we use proof by contradiction. Assume that the interval $[a, b]$ can be put into one-to-one correspondence with the set of positive integers, or equivalently: The real numbers in $[a, b]$ can be written as a sequence in which each real number appears only once. Applying Cantor's second theorem to this sequence and $[a, b]$ produces a real number in $[a, b]$ that does not belong to the sequence. This contradicts the original assumption, and proves the uncountability theorem.[11]

Cantor only states his uncountability theorem. He does not use it in any proofs.[6]

The proofs

First theorem

To prove that the set of real algebraic numbers is countable, define the *height* of a polynomial of degree n with integer coefficients as: $n - 1 + |a_0| + |a_1| + \ldots + |a_n|$, where $a_0, a_1, \ldots, a_n$ are the coefficients of the polynomial. Order the polynomials by their height, and order the real roots of polynomials of the same height by numeric order. Since there are only a finite number of roots of polynomials of a given height, these orderings put the real algebraic numbers into a sequence. Cantor went a step further and produced a sequence in which each real algebraic number appears just once. He did this by only using polynomials that are irreducible over the integers. The following table contains the beginning of Cantor's enumeration.[12]



Algebraic numbers on the complex plane colored by polynomial degree. (red = 1, green = 2, blue = 3, yellow = 4). Points become smaller as the integer polynomial coefficients become larger.

Cantor's enumeration of the real algebraic numbers		
Real algebraic number	Polynomial	Height of polynomial
$x_1 = 0$	x	1
$x_2 = -1$	$x + 1$	2
$x_3 = 1$	$x - 1$	2
$x_4 = -2$	$x + 2$	3
$x_5 = -\frac{1}{2}$	$2x + 1$	3
$x_6 = \frac{1}{2}$	$2x - 1$	3
$x_7 = 2$	$x - 2$	3
$x_8 = -3$	$x + 3$	4
$x_9 = \frac{-1 - \sqrt{5}}{2}$	$x^2 + x - 1$	4
$x_{10} = -\sqrt{2}$	$x^2 - 2$	4
$x_{11} = -\frac{1}{\sqrt{2}}$	$2x^2 - 1$	4
$x_{12} = \frac{1 - \sqrt{5}}{2}$	$x^2 - x - 1$	4
$x_{13} = -\frac{1}{3}$	$3x + 1$	4
$x_{14} = \frac{1}{3}$	$3x - 1$	4
$x_{15} = \frac{-1 + \sqrt{5}}{2}$	$x^2 + x - 1$	4
$x_{16} = \frac{1}{\sqrt{2}}$	$2x^2 - 1$	4
$x_{17} = \sqrt{2}$	$x^2 - 2$	4
$x_{18} = \frac{1 + \sqrt{5}}{2}$	$x^2 - x - 1$	4
$x_{19} = 3$	$x - 3$	4

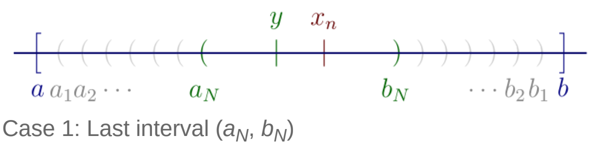
Second theorem

Only the first part of Cantor's second theorem needs to be proved. It states: Given any sequence of real numbers $x_1, x_2, x_3, \ldots$ and any interval $[a, b]$, there is a number in $[a, b]$ that is not contained in the given sequence.[B]

To find a number in $[a, b]$ that is not contained in the given sequence, construct two sequences of real numbers as follows: Find the first two numbers of the given sequence that are in (a, b). Denote the smaller of these two numbers by a_1 and the larger by b_1. Similarly, find the first two numbers of the given sequence that are in (a_1, b_1). Denote the smaller by a_2 and the larger by b_2. Continuing this procedure generates a sequence of intervals $(a_1, b_1), (a_2, b_2), (a_3, b_3), \ldots$ such that each interval in the sequence contains all succeeding intervals—that is, it generates a sequence of nested intervals. This implies that the sequence $a_1, a_2, a_3, \ldots$ is increasing and the sequence $b_1, b_2, b_3, \ldots$ is decreasing.[13]

Either the number of intervals generated is finite or infinite. If finite, let (a_N, b_N) be the last interval. If infinite, take the limits $a_\infty = \lim_{n \to \infty} a_n$ and $b_\infty = \lim_{n \to \infty} b_n$. Since $a_n < b_n$ for all n, either $a_\infty = b_\infty$ or $a_\infty < b_\infty$. Thus, there are three cases to consider:

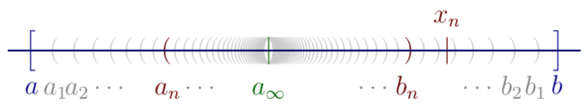
Case 1: There is a last interval (a_N, b_N). Since at most one x_n can be in this interval, every y in this interval except x_n (if it exists) is not contained in the given sequence.



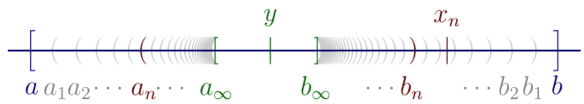
Case 1: Last interval (a_N, b_N)

Case 2: $a_\infty = b_\infty$. Then a_∞ is not contained in the given sequence since for all n: a_∞ belongs to the interval (a_n, b_n), but as Cantor observes, x_n does not.

Proof that for all n: $x_n \notin (a_n, b_n)$
This is implied by the stronger result: For all n, (a_n, b_n) excludes $x_1, \ldots, x_{2n}$, which is proved by induction. Basis step: If $a_1 = x_j$, $b_1 = x_k$, and $E_1 = \max(j, k)$, then (a_1, b_1) excludes $x_1, \ldots, x_{E_1}$. Also, $E_1 \geq 2$ since the interval (a_1, b_1) excludes its two endpoints. Inductive step: Assume that (a_n, b_n) excludes $x_1, \ldots, x_{E_n}$ and $E_n \geq 2n$. If $a_{n+1} = x_j$, $b_{n+1} = x_k$, and $E_{n+1} = \max(j, k)$, then (a_{n+1}, b_{n+1}) excludes $x_1, \ldots, x_{E_{n+1}}$. Also, $E_{n+1} \geq E_n + 2 \geq 2n + 2 = 2(n + 1)$ since the interval (a_{n+1}, b_{n+1}) excludes the numbers that (a_n, b_n) excludes plus the two endpoints a_{n+1} and b_{n+1}. Therefore, for all n: (a_n, b_n) excludes $x_1, \ldots, x_{2n}$. (This proof is similar to a proof that Cantor published in 1879. The main difference is that Cantor's proof is embedded in a larger proof and uses notation from it. Our proof does not depend on this notation.[proof 1])



Case 2: $a_\infty = b_\infty$



Case 3: $a_\infty < b_\infty$

Case 3: $a_\infty < b_\infty$. Then every y in $[a_\infty, b_\infty]$ is not contained in the given sequence since for all n: y belongs to (a_n, b_n) but x_n does not.[14]

The proof is complete since, in all cases, at least one real number in $[a, b]$ has been found that is not contained in the given sequence.[C]

Cantor's proofs are constructive and have been used to write a computer program that generates the digits of a transcendental number. This program applies Cantor's construction to a sequence containing all the real algebraic numbers between 0 and 1. The article that discusses this program gives some of its output, which shows how the construction generates a transcendental.[15]

Example of Cantor's construction

An example illustrates how Cantor's construction works. Consider the sequence: $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots$ This sequence is obtained by ordering the rational numbers in (0, 1) by increasing denominators, ordering those with the same denominator by increasing numerators, and omitting reducible fractions. The table below shows the first five steps of the construction. The table's first column contains the intervals (a_n, b_n). The second column lists the terms visited during the search for the first two terms in (a_n, b_n). These two terms are in red.[16]

Generating a number using Cantor's construction

Interval	Finding the next interval	Interval (decimal)
$\left(\frac{1}{3}, \frac{1}{2}\right)$	$\frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}$	$(0.3333\ldots, 0.5000\ldots)$
$\left(\frac{2}{5}, \frac{3}{7}\right)$	$\frac{4}{7}, \ldots, \frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \ldots, \frac{6}{17}, \frac{7}{17}$	$(0.4000\ldots, 0.4285\ldots)$
$\left(\frac{7}{17}, \frac{5}{12}\right)$	$\frac{8}{17}, \ldots, \frac{11}{29}, \frac{12}{29}, \frac{13}{29}, \ldots, \frac{16}{41}, \frac{17}{41}$	$(0.4117\ldots, 0.4166\ldots)$
$\left(\frac{12}{29}, \frac{17}{41}\right)$	$\frac{18}{41}, \ldots, \frac{27}{70}, \frac{29}{70}, \frac{31}{70}, \ldots, \frac{40}{99}, \frac{41}{99}$	$(0.4137\ldots, 0.4146\ldots)$
$\left(\frac{41}{99}, \frac{29}{70}\right)$	$\frac{43}{99}, \ldots, \frac{69}{169}, \frac{70}{169}, \frac{71}{169}, \ldots, \frac{98}{239}, \frac{99}{239}$	$(0.4141\ldots, 0.4142\ldots)$

Since the sequence contains all the rational numbers in (0, 1), the construction generates an irrational number, which turns out to be $\sqrt{2} - 1$.[17]

Proof that the number generated is √2 − 1

The proof uses Farey sequences and continued fractions. The Farey sequence F_n is the increasing sequence of completely reduced fractions whose denominators are $\leq n.$ If $\frac{a}{b}$ and $\frac{c}{d}$ are adjacent in a Farey sequence, the lowest denominator fraction between them is their mediant $\frac{a+c}{b+d}$. This mediant is adjacent to both $\frac{a}{b}$ and $\frac{c}{d}$ in the Farey sequence F_{b+d}.[18]

Cantor's construction produces mediants because the rational numbers were sequenced by increasing denominator. The first interval in the table is $(\frac{1}{3}, \frac{1}{2})$. Since $\frac{1}{3}$ and $\frac{1}{2}$ are adjacent in $F_3,$ their mediant $\frac{2}{5}$ is the first fraction in the sequence between $\frac{1}{3}$ and $\frac{1}{2}$. Hence, $\frac{1}{3} < \frac{2}{5} < \frac{1}{2}$. In this inequality, $\frac{1}{2}$ has the smallest denominator, so the second fraction is the mediant of $\frac{2}{5}$ and $\frac{1}{2}$, which equals $\frac{3}{7}$. This implies: $\frac{1}{3} < \frac{2}{5} < \frac{3}{7} < \frac{1}{2}$. Therefore, the next interval is $(\frac{2}{5}, \frac{3}{7})$.

We will prove that the endpoints of the intervals converge to the continued fraction $[0; 2, 2, \ldots]$. This continued fraction is the limit of its convergents:

$$\frac{p_n}{q_n} = [0; 2, \ldots, 2] \ \ (n \text{ 2's}).$$

The p_n and q_n sequences satisfy the equations:[19]

$$\begin{array}{lll} p_0 = 0 & p_1 = 1 & p_{n+1} = 2p_n + p_{n-1} \text{ for } n \geq 1 \\ q_0 = 1 & q_1 = 2 & q_{n+1} = 2q_n + q_{n-1} \text{ for } n \geq 1 \end{array}$$

First, we prove by induction that for odd *n*, the *n*-th interval in the table is:

$$\left(\frac{p_n + p_{n-1}}{q_n + q_{n-1}}, \frac{p_n}{q_n}\right),$$

and for even *n*, the interval's endpoints are reversed: $\left(\frac{p_n}{q_n}, \frac{p_n + p_{n-1}}{q_n + q_{n-1}}\right)$.

This is true for the first interval since:

$$\left(\frac{p_1 + p_0}{q_1 + q_0}, \frac{p_1}{q_1}\right) = \left(\frac{1}{3}, \frac{1}{2}\right).$$

Assume that the inductive hypothesis is true for the *k*-th interval. If *k* is odd, this interval is:

$$\left(\frac{p_k + p_{k-1}}{q_k + q_{k-1}}, \frac{p_k}{q_k}\right).$$

The mediant of its endpoints $\frac{2p_k + p_{k-1}}{2q_k + q_{k-1}} = \frac{p_{k+1}}{q_{k+1}}$ is the first fraction in the sequence between these endpoints.

Hence, $\frac{p_k + p_{k-1}}{q_k + q_{k-1}} < \frac{p_{k+1}}{q_{k+1}} < \frac{p_k}{q_k}$.

In this inequality, $\frac{p_k}{q_k}$ has the smallest denominator, so the second fraction is the mediant of $\frac{p_{k+1}}{q_{k+1}}$ and $\frac{p_k}{q_k}$, which equals $\frac{p_{k+1} + p_k}{p_{k+1} + q_k}$.

This implies: $\frac{p_k + p_{k-1}}{q_k + q_{k-1}} < \frac{p_{k+1}}{q_{k+1}} < \frac{p_{k+1} + p_k}{p_{k+1} + q_k} < \frac{p_k}{q_k}$.

Therefore, the (*k* + 1)-st interval is $\left(\frac{p_{k+1}}{q_{k+1}}, \frac{p_{k+1} + p_k}{p_{k+1} + q_k}\right)$.

This is the desired interval; $\frac{p_{k+1}}{q_{k+1}}$ is the left endpoint because *k* + 1 is even. Thus, the inductive hypothesis is true for the (*k* + 1)-st interval. For even *k*, the proof is similar. This completes the inductive proof.

Since the right endpoints of the intervals are decreasing and every other endpoint is $\frac{p_{2n-1}}{q_{2n-1}}$, their limit equals $\lim_{n\to\infty} \frac{p_n}{q_n}$. The left endpoints have the same limit because they are increasing and every other endpoint is $\frac{p_{2n}}{q_{2n}}$. As mentioned above, this limit is the continued fraction $[0; 2, 2, \ldots]$, which equals $\sqrt{2} - 1$.[20]

Cantor's 1879 uncountability proof

Everywhere dense

In 1879, Cantor published a new uncountability proof that modifies his 1874 proof. He first defines the topological notion of a point set P being "everywhere dense in an interval" (which is quite often shortened to "dense in an interval"):[D]

> If P lies partially or completely in the interval [α, β], then the remarkable case can happen that *all* intervals [γ, δ] contained in [α, β], *no matter how small,* contain points of P. In such a case, we will say that P is *everywhere dense in the interval* [α, β].[E]

We will use a, b, c, d rather than α, β, γ, δ. Cantor assumes that an interval $[c, d]$ satisfies $c < d$.

Since our discussion of Cantor's 1874 proof was simplified by using open intervals rather than closed intervals, the same simplification is used here. This requires an equivalent definition of everywhere dense: A set P is everywhere dense in the interval $[a, b]$ if and only if every subinterval (c, d) of $[a, b]$ contains at least one point of P.[21]

Cantor did not specify how many points of P a subinterval (c, d) must contain. He did not need to specify this because assuming that every subinterval contains at least one point of P implies that they contain infinitely many points of P. This is proved by generating a sequence of points belonging to both P and (c, d). Since P is dense in $[a, b]$, the subinterval (c, d) contains at least one point x_1 of P. Now consider the subinterval (x_1, d). It contains at least one point x_2 of P, which satisfies $x_2 > x_1$. In general, after generating x_n, the subinterval (x_n, d) is used to obtain the point x_{n+1}, which satisfies $x_{n+1} > x_n$. The points x_n are all unique and belong to both P and (c, d).

Cantor's 1879 proof

Cantor's 1879 proof is the same as his 1874 proof except for a new proof of the first part of his second theorem: Given any sequence P of real numbers $x_1, x_2, x_3, \ldots$ and any interval $[a, b]$, there is a number in $[a, b]$ that is not contained in the sequence P. The new proof has only two cases.[proof 1]

In the first case, P is not dense in $[a, b]$. By definition, P is dense if and only if for all $(c, d) \subseteq [a, b]$, there is an $x \in P$ such that $x \in (c, d)$. Taking the negation of each side of the "if and only if" produces: P is not dense in $[a, b]$ if and only if there exists a $(c, d) \subseteq [a, b]$ such that for all $x \in P$, we have $x \notin (c, d)$. Thus, every number in (c, d) is not contained in the sequence P.[proof 1] This case handles cases 1 and 3 of Cantor's 1874 proof.

In the second case, P is dense in $[a, b]$. The denseness of P is used to recursively define a nested sequence of intervals that excludes all elements of P. The definition begins with $a_1 = a$ and $b_1 = b$. The definition's inductive case starts with the interval (a_n, b_n), which because of the denseness of P contains infinitely many elements of P. From these elements of P, we take the two with smallest indices and denote the least of these two numbers by a_{n+1} and the greatest by b_{n+1}. Cantor proved that for all n: $x_n \notin (a_n, b_n)$.[proof 1] We proved this in a previous section.

The sequence a_n is increasing and bounded above by b, so it has a limit A, which satisfies $a_n < A$. The sequence b_n is decreasing and bounded below by a, so it has a limit B, which satisfies $B < b_n$. Also, $a_n < b_n$ implies $A \le B$. Therefore, $a_n < A \le B < b_n$. If $A < B$, then for every n: $x_n \notin (A, B)$ because x_n is not in the larger interval (a_n, b_n). This contradicts P being dense in $[a, b]$. Therefore, $A = B$. Since for all n: $A \in (a_n, b_n)$ but $x_n \notin (a_n, b_n)$, the limit A is a real number that is not contained in the sequence P.[proof 1] This case handles case 2 of Cantor's 1874 proof.

Cantor's new proof first takes care of the easy case of the sequence P not being dense in the interval. Then it deals with the more difficult case of P being dense. This division into cases not only indicates which sequences are most difficult to handle, but it also reveals the important role denseness plays in the proof.[proof 1]

In the Example of Cantor's construction, each successive nested interval excludes rational numbers for two different reasons. It will exclude the finitely many rationals visited in the search for the first two rationals within the interval (these two rationals will have the least indices). These rationals are then used to form an interval that excludes the rationals visited in the search along with infinitely many more rationals. However, it still contains infinitely many rationals since our sequence of rationals is dense in [0, 1]. Forming this interval from the two rationals with the least indices guarantees that this interval excludes an initial segment of our sequence that contains at least two more elements than the preceding initial segment. Since the denseness of our sequence guarantees that this process never ends, all rationals will be excluded.[proof 1] Because of the ordering of the rationals in our sequence, the intersection of the nested intervals is the set $\{\sqrt{2} - 1\}$.

The development of Cantor's ideas

The development leading to Cantor's 1874 article appears in the correspondence between Cantor and Richard Dedekind. On November 29, 1873, Cantor asked Dedekind whether the collection of positive integers and the collection of positive real numbers "can be corresponded so that each individual of one collection corresponds to one and only one individual of the other?" Cantor added that collections having such a correspondence include the collection of positive rational numbers, and collections of the form $(a_{n_1, n_2, \ldots, n_\nu})$ where $n_1, n_2, \ldots, n_\nu$, and ν are positive integers.[22]

Dedekind replied that he was unable to answer Cantor's question, and said that it "did not deserve too much effort because it has no particular practical interest." Dedekind also sent Cantor a proof that the set of algebraic numbers is countable.[23]

On December 2, Cantor responded that his question does have interest: "It would be nice if it could be answered; for example, provided that it could be answered *no*, one would have a new proof of Liouville's theorem that there are transcendental numbers."[24]

On December 7, Cantor sent Dedekind a proof by contradiction that the set of real numbers is uncountable. Cantor starts by assuming that the real numbers in $[0,1]$ can be written as a sequence. Then, he applies a construction to this sequence to produce a number in $[0,1]$ that is not in the sequence, thus contradicting his assumption.[25] Together, the letters of December 2 and 7 provide a non-constructive proof of the existence of transcendental numbers.[26] Also, the proof in Cantor's December 7th letter shows some of the reasoning that led to his discovery that the real numbers form an uncountable set.[27]

Cantor's December 7, 1873 proof

The proof is by contradiction and starts by assuming that the real numbers in $[0,1]$ can be written as a sequence:

$$\text{(I)}\ \ \omega_1, \omega_2, \omega_3, \ldots, \omega_n, \ldots$$

An increasing sequence is extracted from this sequence by letting $\omega_1^1 =$ the first term, $\omega_1^2 =$ the next largest term following ω_1^1, $\omega_1^3 =$ the next largest term following ω_1^2, and so forth. The same procedure is applied to the remaining members of the original sequence to extract another increasing sequence. By continuing this process of extracting sequences, one sees that the sequence (I) can be decomposed into the infinitely many sequences:[25]

$$\begin{array}{l}(1)\ \ \omega_1^1, \omega_1^2, \omega_1^3, \ldots, \omega_1^n, \ldots \\ (2)\ \ \omega_2^1, \omega_2^2, \omega_2^3, \ldots, \omega_2^n, \ldots \\ (3)\ \ \omega_3^1, \omega_3^2, \omega_3^3, \ldots, \omega_3^n, \ldots \\ \quad \vdots \end{array}$$

Let $[p,q]$ be an interval such that no term of sequence (1) lies in it. For example, let p and q satisfy $\omega_1^1 < p < q < \omega_1^2$. Then $\omega_1^1 < p < q < \omega_1^n$ for $n \geq 2$, so no term of sequence (1) lies in $[p,q]$.[25]

Now consider whether the terms of the other sequences lie outside $[p,q]$. All terms of some of these sequences may lie outside of $[p,q]$; however, there must be some sequence such that not all its terms lie outside $[p,q]$. Otherwise, the numbers in $[p,q]$ would not be contained in sequence (I), contrary to the initial hypothesis. Let sequence (k) be the first sequence that contains a term in $[p,q]$ and let ω_k^n be the first term. Since $p < \omega_k^n < q$, let p_1 and q_1 satisfy $p < p_1 < q_1 < \omega_k^n < q$. Then $[p,q] \supsetneq [p_1, q_1]$ and the terms of sequences $(1), (2), \ldots, (k-1)$ lie outside of $[p_1, q_1]$.[25]

Repeat the above argument starting with $[p_1, q_1]$: Let sequence (k_1) be the first sequence containing a term in $[p_1, q_1]$ and let $\omega_{k_1}^n$ be the first term. Since $p_1 < \omega_{k_1}^n < q_1$, let p_2 and q_2 satisfy $p_1 < p_2 < q_2 < \omega_{k_1}^n < q_1$. Then $[p_1, q_1] \supsetneq [p_2, q_2]$ and the terms of sequences $(k_1), \ldots, (k_2 - 1)$ lie outside of $[p_2, q_2]$.[25]

One sees that it is possible to form an infinite sequence of intervals $[p,q] \supsetneq [p_1, q_1] \supsetneq [p_2, q_2] \supsetneq \ldots$ such that:

the members of the $1^{st}, 2^{nd}, \ldots, (k-1)^{st}$ sequence lie outside $[p,q]$;
the members of the $k^{th}, \ldots, (k_1 - 1)^{st}$ sequence lie outside $[p_1, q_1]$;
the members of the $(k_1)^{th}, \ldots, (k_2 - 1)^{st}$ sequence lie outside $[p_2, q_2]$;
$\ldots$.[25]

Since p_n and q_n are bounded monotonic sequences, the limits $\lim\limits_{n\to\infty} p_n$ and $\lim\limits_{n\to\infty} q_n$ exist. Also, $p_n < q_n$ for all n implies $\lim\limits_{n\to\infty} p_n \leq \lim\limits_{n\to\infty} q_n$. Hence, there is at least one number η in $(0,1)$ that lies in all the intervals $[p,q]$ and $[p_n, q_n]$. Namely, η can be any number in $[\lim\limits_{n\to\infty} p_n, \lim\limits_{n\to\infty} q_n]$. This implies that η lies outside all the sequences $(1), (2), (3), \ldots,$ contradicting the initial hypothesis that sequence (I) contains all the real numbers in $[0,1]$. Therefore, the set of all real numbers is uncountable.[25]

Dedekind received Cantor's proof on December 8. On that same day, Dedekind simplified the proof and mailed his proof to Cantor. Cantor used Dedekind's proof in his article.[28] The letter containing Cantor's December 7th proof was not published until 1937.[29]

On December 9, Cantor announced the theorem that allowed him to construct transcendental numbers as well as prove the uncountability of the set of real numbers:

> I show directly that if I start with a sequence
> (I) $\omega_1, \omega_2, \ldots, \omega_n, \ldots$
> I can determine, in *every* given interval $[\alpha, \beta]$, a number η that is not included in (I).[30]

This is the second theorem in Cantor's article. It comes from realizing that his construction can be applied to any sequence, not just to sequences that supposedly enumerate the real numbers. So Cantor had a choice between two proofs that demonstrate the existence of transcendental numbers: one proof is constructive, but the other is not. We now compare the proofs assuming that we have a sequence consisting of all the real algebraic numbers.

The constructive proof applies Cantor's construction to this sequence and the interval [*a*, *b*] to produce a transcendental number in this interval.[8]

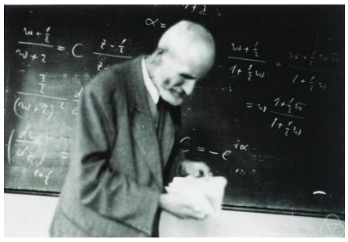
The non-constructive proof uses two proofs by contradiction:

1. The proof by contradiction used to prove the uncountability theorem (see Proof of Cantor's uncountability theorem).
2. The proof by contradiction used to prove the existence of transcendental numbers from the countability of the real algebraic numbers and the uncountability of real numbers. Cantor's December 2nd letter mentions this existence proof but does not contain it. Here is a proof: Assume that there are no transcendental numbers in [*a*, *b*]. Then all the numbers in [*a*, *b*] are algebraic. This implies that they form a subsequence of the sequence of all real algebraic numbers, which contradicts Cantor's uncountability theorem. Thus, the assumption that there are no transcendental numbers in [*a*, *b*] is false. Therefore, there is a transcendental number in [*a*, *b*].[F]

Cantor chose to publish the constructive proof, which not only produces a transcendental number but is also shorter and avoids two proofs by contradiction. The non-constructive proof from Cantor's correspondence is simpler than the one above because it works with all the real numbers rather than the interval [*a*, *b*]. This eliminates the subsequence step and all occurrences of [*a*, *b*] in the second proof by contradiction.[8]

The disagreement about Cantor's existence proof

The correspondence containing Cantor's non-constructive reasoning was published in 1937. By then, other mathematicians had rediscovered its non-constructive proof. As early as 1921, this proof was attributed to Cantor and criticized for not producing any transcendental numbers.[32] In that year, Oskar Perron stated: "… Cantor's proof for the existence of transcendental numbers has, along with its simplicity and elegance, the great disadvantage that it is only an existence proof; it does not enable us to actually specify even a single transcendental number."[33]


Oskar Perron

Some mathematicians have attempted to correct this misunderstanding of Cantor's work. In 1930, the set theorist Abraham Fraenkel stated that Cantor's method is "… a method that incidentally, contrary to a widespread interpretation, is fundamentally constructive and not merely existential."[34] In 1972, Irving Kaplansky wrote: "It is often said that Cantor's proof is not 'constructive,' and so does not yield a tangible transcendental number. This remark is not justified. If we set up a definite listing of all algebraic numbers … and then apply the diagonal procedure …, we get a perfectly definite transcendental number (it could be computed to any number of decimal places)."[35]


Abraham Fraenkel

Cantor's diagonal argument has often replaced his 1874 construction in expositions of his proof. The diagonal argument is constructive and produces a more efficient computer program than his 1874 construction. Using it, a computer program has been written that computes the digits of a transcendental number in polynomial time. The program that uses Cantor's 1874 construction requires at least sub-exponential time.[G]

The disagreement about Cantor's proof occurs because two groups of mathematicians are talking about different proofs: the constructive one that Cantor published and the non-constructive one that was later rediscovered. The opinion that Cantor's proof is non-constructive appears in some books that were quite successful as measured by the length of time new editions or reprints appeared—for example: Eric Temple Bell's *Men of Mathematics* (1937; still being reprinted), Godfrey Hardy and E. M. Wright's *An Introduction to the Theory of Numbers* (1938; 2008 6th edition), Garrett Birkhoff and Saunders Mac Lane's *A Survey of Modern Algebra* (1941; 1997 5th edition), and Michael Spivak's *Calculus* (1967; 2008 4th edition).[36] Since these books view Cantor's proof as non-constructive, they do not mention his constructive proof. On the other hand, the quotations above from Fraenkel and Kaplansky show that they knew Cantor's work can be used non-constructively. The disagreement about Cantor's proof shows no sign of being resolved: since 2014, at least two books have appeared stating that Cantor's proof is constructive, and at least four have appeared stating that his proof does not construct any (or a single) transcendental.[37]

Asserting that Cantor gave a non-constructive proof can lead to erroneous statements about the history of mathematics. In *A Survey of Modern Algebra*, Birkhoff and Mac Lane state: "Cantor's argument for this result [Not every real number is algebraic] was at first rejected by many mathematicians, since it did not exhibit any specific transcendental number." Birkhoff and Mac Lane are talking about the non-constructive proof.[38] Cantor's proof produces transcendental numbers, and there appears to be no evidence that his argument was rejected.[7] Even Leopold Kronecker, who had strict views on what is acceptable in mathematics and who could have delayed publication of Cantor's article, did not delay it.[39] In fact, applying Cantor's construction to the sequence of real algebraic numbers produces a limiting process that Kronecker accepted—namely, it determines a number to any required degree of accuracy.[40][H]

The influence of Weierstrass and Kronecker on Cantor's article

Historians of mathematics have discovered the following facts about Cantor's article "On a Property of the Collection of All Real Algebraic Numbers":

- Cantor's uncountability theorem was left out of the article he submitted. He added it during proofreading.[44]
- The article's title refers to the set of real algebraic numbers. The main topic in Cantor's correspondence was the set of real numbers.[45]
- The proof of Cantor's second theorem came from Dedekind. However, it omits Dedekind's explanation of why the limits a_∞ and b_∞ exist.[46]
- Cantor restricted his first theorem to the set of real algebraic numbers. The proof he was using demonstrates the countability of the set of all algebraic numbers.[23]

Karl Weierstrass

To explain these facts, historians have pointed to the influence of Cantor's former professors, Karl Weierstrass and Leopold Kronecker. Cantor discussed his results with Weierstrass on December 23, 1873.[47] Weierstrass was first amazed by the concept of countability, but then found the countability of the set of real algebraic numbers useful.[48] Cantor did not want to publish yet, but Weierstrass felt that he must publish at least his results concerning the algebraic numbers.[47]

Leopold Kronecker, 1865

From his correspondence, it appears that Cantor only discussed his article with Weierstrass. However, Cantor told Dedekind: "The restriction which I have imposed on the published version of my investigations is caused in part by local circumstances …"[47] Cantor biographer Joseph Dauben believes that "local circumstances" refers to Kronecker who, as a member of the editorial board of *Crelle's Journal*, had delayed publication of an 1870 article by Eduard Heine, one of Cantor's colleagues. Cantor would submit his article to *Crelle's Journal*.[49]

Weierstrass advised Cantor to leave his uncountability theorem out of the article he submitted, but Weierstrass also told Cantor that he could add it as a marginal note during proofreading, which he did.[44] It appears in a remark at the end of the article's introduction.[50] The opinions of Kronecker and Weierstrass both played a role here. Kronecker did not accept infinite sets, and it seems that Weierstrass did not accept that two infinite sets could be so different, with one being countable and the other not.[51] Weierstrass changed his opinion later.[52] Without the uncountability theorem, the article needed a title that did not refer to this theorem. Cantor chose *Ueber eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen* ("On a Property of the Collection of All Real Algebraic Numbers"), which refers to the countability of the set of real algebraic numbers, the result that Weierstrass found useful.[53]

Kronecker's influence appears in the proof of Cantor's second theorem. Cantor used Dedekind's version of the proof except he left out why the limits $a_\infty = \lim_{n \to \infty} a_n$ and $b_\infty = \lim_{n \to \infty} b_n$ exist. Dedekind had used his "principle of continuity" to prove they exist. This principle (which is equivalent to the least upper bound property of the real numbers) comes from Dedekind's construction of the real numbers, a construction Kronecker did not accept.[54]

Cantor restricted his first theorem to the set of real algebraic numbers even though Dedekind had sent him a proof that handled all algebraic numbers.[23] Cantor did this for expository reasons and because of "local circumstances."[55] This restriction simplifies the article because the second theorem works with real sequences. Hence, the construction in the second theorem can be applied directly to the enumeration of the real algebraic numbers to produce "an effective procedure for the calculation of transcendental numbers." This procedure would be acceptable to Weierstrass.[56]

Dedekind's contributions to Cantor's article

Richard Dedekind, c. 1870

Since 1856, Dedekind had developed theories involving infinitely many infinite sets—for example: ideals, which he used in algebraic number theory, and Dedekind cuts, which he used to construct the real numbers. This work enabled him to understand and contribute to Cantor's work.[57]

Dedekind's first contribution concerns the theorem that the set of real algebraic numbers is countable. Cantor is usually given credit for this theorem, but the mathematical historian José Ferreirós calls it "Dedekind's theorem."[58] Their correspondence reveals what each mathematician contributed to the theorem.

In his letter introducing the concept of countability, Cantor stated without proof that the set of positive rational numbers is countable, as are sets of the form $(a_{n_1,\ n_2,\ \ldots,\ n_\nu})$ where $n_1, n_2, \ldots, n_\nu$, and ν are positive integers.[59] Cantor's second result uses indexed numbers: a set of the form $(a_{n_1,\ n_2,\ \ldots,\ n_\nu})$ is the range of a function from the ν indices to the set of real numbers. His second result implies his first: let $\nu = 2$ and $a_{n_1,\ n_2} = \frac{n_1}{n_2}$. The function can be quite general—for example, $a_{n_1,\ n_2,\ n_3,\ n_4,\ n_5} = (\frac{n_1}{n_2})^{\frac{1}{n_3}} + \tan(\frac{n_4}{n_5})$.

Dedekind replied with a proof of the theorem that the set of all algebraic numbers is countable.[23]

In his reply, Cantor did not claim to have proved Dedekind's result. He did indicate how he proved his theorem about indexed numbers: "Your proof that (n) [the set of positive integers] can be correlated one-to-one with the field of all algebraic numbers is approximately the same as the way I prove my contention in the last letter. I take $n_1{}^2 + n_2{}^2 + \cdots + n_\nu{}^2 = \mathfrak{N}$ and order the elements accordingly."[60] Cantor's ordering cannot handle indices that are 0.[61]

Dedekind's second contribution is his proof of Cantor's second theorem. Dedekind sent this proof in reply to Cantor's letter that contained the uncountability theorem, which Cantor proved using infinitely many sequences. Cantor next wrote that he had found a simpler proof that did not use infinitely many sequences.[62] So Cantor had a choice of proofs and chose to publish Dedekind's.[63]

Cantor thanked Dedekind privately for his help: "… your comments (which I value highly) and your manner of putting some of the points were of great assistance to me."[47] However, he did not mention Dedekind's help in his article. In previous articles, he had acknowledged help received from Kronecker, Weierstrass, Heine, and Hermann Schwarz. Cantor's failure to mention Dedekind's contributions damaged his relationship with Dedekind. Dedekind stopped replying to his letters and did not resume the correspondence until October 1876.[64][I]

The legacy of Cantor's article

Cantor's article introduced the uncountability theorem and the concept of countability. Both would lead to significant developments in mathematics. The uncountability theorem demonstrated that one-to-one correspondences can be used to analyze infinite sets. In 1878, Cantor used them to define and compare cardinalities. He also constructed one-to-one correspondences to prove that the n-dimensional spaces $\mathbf{R}^n$ (where $\mathbf{R}$ is the set of real numbers) and the set of irrational numbers have the same cardinality as $\mathbf{R}$.[65][J]

In 1883, Cantor extended the natural numbers with his infinite ordinals. This extension was necessary for his work on the Cantor–Bendixson theorem. Cantor discovered other uses for the ordinals—for example, he used sets of ordinals to produce an infinity of sets having different infinite cardinalities.[67] His work on infinite sets together with Dedekind's set-theoretical work created set theory.[68]

The concept of countability led to countable operations and objects that are used in various areas of mathematics. For example, in 1878, Cantor introduced countable unions of sets.[69] In the 1890s, Émile Borel used countable unions in his theory of measure, and René Baire used countable ordinals to define his classes of functions.[70] Building on the work of Borel and Baire, Henri Lebesgue created his theories of measure and integration, which were published from 1899 to 1901.[71]

Countable models are used in set theory. In 1922, Thoralf Skolem proved that if conventional axioms of set theory are consistent, then they have a countable model. Since this model is countable, its set of real numbers is countable. This consequence is called Skolem's paradox, and Skolem explained why it does not contradict Cantor's uncountability theorem: although there is a one-to-one correspondence between this set and the set of positive integers, no such one-to-one correspondence is a member of the model. Thus the model considers its set of real numbers to be uncountable, or more precisely, the first-order sentence that says the set of real numbers is uncountable is true within the model.[72] In 1963, Paul Cohen used countable models to prove his independence theorems.[73]

See also

- Cantor's theorem

Notes

A. In letter to Dedekind dated December 25, 1873, Cantor states that he has written and submitted "a short paper" titled *On a Property of the Set of All Real Algebraic Numbers*. (Noether & Cavaillès 1937, p. 17; English translation: Ewald 1996, p. 847.)

B. This implies the rest of the theorem — namely, there are infinitely many numbers in [*a*, *b*] that are not contained in the given sequence. For example, let $[0,1]$ be the interval and consider its pairwise disjoint subintervals $[0,\frac{1}{2}]$, $[\frac{3}{4},\frac{7}{8}]$, $[\frac{15}{16},\frac{31}{32}]$, $\ldots$. Applying the first part of the theorem to each subinterval produces infinitely many numbers in $[0,1]$ that are not contained in the given sequence. In general, for the interval $[a,b]$, apply the first part of the theorem to the subintervals $[a,a+\frac{1}{2}(b-a)]$, $[a+\frac{3}{4}(b-a),a+\frac{7}{8}(b-a)]$, $[a+\frac{15}{16}(b-a),a+\frac{31}{32}(b-a)]$, $\ldots$.

C. The main difference between Cantor's proof and the above proof is that he generates the sequence of closed intervals [a_n, b_n]. To find a_{n+1} and b_{n+1}, he uses the interior of the interval [a_n, b_n], which is the open interval (a_n, b_n). Generating open intervals combines Cantor's use of closed intervals and their interiors, which allows the case diagrams to depict all the details of the proof.

D. Cantor was not the first to define "everywhere dense" but his terminology was adopted with or without the "everywhere" (everywhere dense: Arkhangel'skii & Fedorchuk 1990, p. 15; dense: Kelley 1991, p. 49). In 1870, Hermann Hankel had defined this concept using different terminology: "a multitude of points … *fill the segment* if no interval, however small, can be given within the segment in which one does not find at least one point of that multitude" (Ferreirós 2007, pp. 155). Hankel was building on Peter Gustav Lejeune Dirichlet's 1829 article that contains the Dirichlet function, a non-(Riemann) integrable function whose value is 0 for rational numbers and 1 for irrational numbers. (Ferreirós 2007, p. 149.)

E. The original German text from Cantor 1879, p. 2 (Cantor's closed set notation (α . . . β) is translated to [α, β]): Liegt *P* theilweise oder ganz im Intervalle (α . . . β), so kann der bemerkenswerthe Fall eintreten, dass *jedes noch so kleine* in (α . . . β) enthaltene Intervall (γ . . . δ) Punkte von *P* enthält. In einem solchen Falle wollen wir sagen, dass *P im Intervalle* (α . . . β) *überall-dicht* sei.

F. The beginning of our proof is derived from the proof below by restricting the numbers in this proof to the interval [*a*, *b*]. However, we derive the contradiction by using a subsequence because Cantor was using sequences in his 1873 work on countability.
German text: Satz 68. Es gibt transzendente Zahlen.
Gäbe es nämlich keine transzendenten Zahlen, so wären alle Zahlen algebraisch, das Kontinuum also identisch mit der Menge aller algebraischen Zahlen. Das ist aber unmöglich, weil die Menge aller algebraischen Zahlen abzählbar ist, das Kontinuum aber nicht.[31]
Translation: Theorem 68. There are transcendental numbers.
If there were no transcendental numbers, then all numbers would be algebraic. Hence, the continuum would be identical to the set of all algebraic numbers. However, this is impossible because the set of all algebraic numbers is countable, but the continuum is not.

G. The program using the diagonal method produces n digits in $O(n^2 \log^2 n \log\log n)$ steps, while the program using the 1874 method requires at least $O(2^{\sqrt[3]{n}})$ steps to produce n digits. (Gray 1994, pp. 822–823.)

H. Kronecker's opinion was: "Definitions must contain the means of reaching a decision in a finite number of steps, and existence proofs must be conducted so that the quantity in question can be calculated with any required degree of accuracy."[41] So Kronecker would accept Cantor's argument as a valid existence proof, but he would not accept its conclusion that transcendental numbers exist. For Kronecker, they do not exist because their definition contains no means for deciding in a finite number of steps whether or not a given number is transcendental.[42] To prove that Cantor's construction calculates numbers to any required degree of accuracy, we need to prove: Given a k, an n can be computed such that $b_n - a_n \leq \frac{1}{k}$ where (a_n, b_n) is the n-th interval of Cantor's construction. An example of how to prove this is given in Gray 1994, p. 822. Cantor's diagonal argument provides an accuracy of 10^{-n} after n real algebraic numbers have been calculated because each of these numbers generates one digit of the transcendental number.[43]

I. José Ferreirós analyzed the relations between Cantor and Dedekind, which began in 1872 and ended in 1899. (Ferreirós 1993, pp. 344, 357.) For example, he analyzed why "Relations between both mathematicians were difficult after 1874, when they underwent an interruption…" (Ferreirós 1993, pp. 344, 348–352.)

J. Cantor's method of constructing a one-to-one correspondence between the set of irrational numbers and **R** can be used to construct one between the set of transcendental numbers and **R**.[66] The construction begins with the set of transcendental numbers T and removes a countable subset $\{t_n\}$ (for example, $t_n = \frac{e}{n}$). Let this set be T_0. Then $T = T_0 \cup \{t_n\} = T_0 \cup \{t_{2n-1}\} \cup \{t_{2n}\}$, and $\mathbf{R} = T \cup \{a_n\} = T_0 \cup \{t_n\} \cup \{a_n\}$ where a_n is the sequence of real algebraic numbers. So both T and **R** are the union of three pairwise disjoint sets: T_0 and two countable sets. A one-to-one correspondence between T and **R** is given by the function: $g(t) = t$ if $t \in T_0$, $g(t_{2n-1}) = t_n$, and $g(t_{2n}) = a_n$.

Note: Cantor's 1879 proof

1. Since Cantor's proof has not been published in English, an English translation is given alongside the original German text, which is from Cantor 1879, pp. 5–7. The translation starts one sentence before the proof because this sentence mentions Cantor's 1874 proof. Cantor states it was printed in Borchardt's Journal. Crelle's Journal was also called Borchardt's Journal from 1856-1880 when Carl Wilhelm Borchardt edited the journal (Audin 2011, p. 80). Square brackets are used to identify this mention of Cantor's earlier proof, to clarify the translation, and to provide page numbers. Also, "Mannichfaltigkeit" (manifold) has been translated to "set" and Cantor's notation for closed sets (α . . . β) has been translated to [α, β]. Cantor changed his terminology from Mannichfaltigkeit to Menge (set) in his 1883 article, which introduced sets of ordinal numbers (Kanamori 2012, p. 5). Currently in mathematics, a manifold is type of topological space.

English translation	German text
[Page 5] . . . But this contradicts a very general theorem, which we have proved with full rigor in Borchardt's Journal, Vol. 77, page 260; namely, the following theorem: *"If one has a simply [countably] infinite sequence* $\omega_1, \omega_2, \ldots, \omega_\nu, \ldots$ *of real, unequal numbers that proceed according to some rule, then in every given interval [α, β] a number η (and thus infinitely many of them) can be specified that does not occur in this sequence (as a member of it)."* In view of the great interest in this theorem, not only in the present discussion, but also in many other arithmetical as well as analytical relations, it might not be superfluous if we develop the argument followed there [Cantor's 1874 proof] more clearly here by using simplifying modifications. Starting with the sequence: $\omega_1, \omega_2, \ldots, \omega_\nu, \ldots$ (which we give [denote by] the symbol (ω)) and an arbitrary interval [α, β], where α < β, we will now demonstrate that in this interval a real number η can be found that does *not* occur in (ω). I. We first notice that if our set (ω) is *not everywhere dense* in the interval [α, β], then within this interval another interval [γ, δ] must be present, all of whose numbers do not belong to (ω). From the interval [γ, δ], one can then choose any number for η. It lies in the interval [α, β] and definitely does *not* occur in our sequence (ω). Thus, this case presents no special considerations and we can move on to the *more difficult* case. II. Let the set (ω) be *everywhere dense* in the interval [α, β]. In this case, every interval [γ,δ] located in [α,β], however small, contains numbers of our sequence (ω). To show that, *nevertheless,* numbers η in the interval [α, β] exist that do not occur in (ω), we employ the following observation. Since some numbers in our sequence: $\omega_1, \omega_2, \ldots, \omega_\nu, \ldots$	[Seite 5] . . . Dem widerspricht aber ein sehr allgemeiner Satz, welchen wir in Borchardt's Journal, Bd. 77, pag. 260, mit aller Strenge bewiesen haben, nämlich der folgende Satz: *"Hat man eine einfach unendliche Reihe* $\omega_1, \omega_2, \ldots, \omega_\nu, \ldots$ *von reellen, ungleichen Zahlen, die nach irgend einem Gesetz fortschreiten, so lässt sich in jedem vorgegebenen, Intervalle (α . . . β) eine Zahl η (und folglich lassen sich deren unendlich viele) angeben, welche nicht in jener Reihe (als Glied derselben) vorkommt."* In Anbetracht des grossen Interesses, welches sich an diesen Satz, nicht blos bei der gegenwärtigen Erörterung, sondern auch in vielen anderen sowohl arithmetischen, wie analytischen Beziehungen, knüpft, dürfte es nicht überflüssig sein, wenn wir die dort befolgte Beweisführung [Cantors 1874 Beweis], unter Anwendung vereinfachender Modificationen, hier deutlicher entwickeln. Unter Zugrundelegung der Reihe: $\omega_1, \omega_2, \ldots, \omega_\nu, \ldots$ (welcher wir das Zeichen (ω) beilegen) und eines beliebigen Intervalles (α . . . β), wo α < β ist, soll also nun gezeigt werden, dass in diesem Intervalle eine reelle Zahl η gefunden werden kann, welche in (ω) *nicht* vorkommt. I. Wir bemerken zunächst, dass wenn unsre Mannichfaltigkeit (ω) in dem Intervall (α . . . β) *nicht überall-dicht* ist, innerhalb dieses Intervalles ein anderes (γ . . . δ) vorhanden sein muss, dessen Zahlen sämmtlich nicht zu (ω) gehören; man kann alsdann für η irgend eine Zahl des Intervalls (γ . . . δ) wählen, sie liegt im Intervalle (α . . . β) und kommt sicher in unsrer Reihe (ω) *nicht* vor. Dieser Fall bietet daher keinerlei besondere Umstände; und wir können zu dem *schwierigeren* übergehen. II. Die Mannichfaltigkeit (ω) sei im Intervalle (α . . . β) *überall-dicht*. In diesem Falle enthält jedes, noch so kleine in (α . . . β) gelegene Intervall (γ . . . δ) Zahlen unserer Reihe (ω). Um zu zeigen, dass *nichtsdestoweniger* Zahlen η im Intervalle (α . . . β) existiren, welche in (ω) nicht vorkommen, stellen wir die folgende Betrachtung an. Da in unserer Reihe: $\omega_1, \omega_2, \ldots, \omega_\nu, \ldots$
[Page 6]	[Seite 6]

definitely occur *within* the interval [α, β], one of these numbers must have the *least index,* let it be ω_{κ_1}, and another: ω_{κ_2} with the next larger index.

Let the smaller of the two numbers ω_{κ_1}, ω_{κ_2} be denoted by α', the larger by β'. (Their equality is impossible because we assumed that our sequence consists of nothing but unequal numbers.)

Then according to the definition:

$$\alpha < \alpha' < \beta' < \beta,$$

furthermore:

$$\kappa_1 < \kappa_2;$$

and all numbers ω_μ of our sequence, for which $\mu \le \kappa_2$, do *not* lie in the interior of the interval [α', β'], as is immediately clear from the definition of the numbers κ_1, κ_2. Similarly, let ω_{κ_3} and ω_{κ_4} be the two numbers of our sequence with smallest indices that fall in the *interior* of the interval [α', β'] and let the smaller of the numbers ω_{κ_3}, ω_{κ_4} be denoted by α'', the larger by β''.

Then one has:

$$\alpha' < \alpha'' < \beta'' < \beta',$$

$$\kappa_2 < \kappa_3 < \kappa_4;$$

and one sees that all numbers ω_μ of our sequence, for which $\mu \le \kappa_4$, do *not* fall into the *interior* of the interval [α'', β''].

After one has followed this rule to reach an interval $[\alpha^{(\nu - 1)}, \beta^{(\nu - 1)}]$, the next interval is produced by selecting the first two (i. e. with lowest indices) numbers of our sequence (ω) (let them be $\omega_{\kappa_{2\nu - 1}}$ and $\omega_{\kappa_{2\nu}}$) that fall into the *interior* of $[\alpha^{(\nu - 1)}, \beta^{(\nu - 1)}]$. Let the smaller of these two numbers be denoted by $\alpha^{(\nu)}$, the larger by $\beta^{(\nu)}$.

The interval $[\alpha^{(\nu)}, \beta^{(\nu)}]$ then lies in the *interior* of all preceding intervals and has the *specific* relation with our sequence (ω) that all numbers ω_μ, for which $\mu \le \kappa_{2\nu}$, *definitely do not lie in its interior*. Since obviously:

$$\kappa_1 < \kappa_2 < \kappa_3 < \ldots, \omega_{\kappa_{2\nu-2}} < \omega_{\kappa_{2\nu-1}} < \omega_{\kappa_{2\nu}}, \ldots$$

and these numbers, as indices, are *whole* numbers, so:

$$\kappa_{2\nu} \ge 2\nu,$$

and hence:

$$\nu < \kappa_{2\nu};$$

thus, we can certainly say (and this is sufficient for the following):

That if ν is an arbitrary whole number, the [real] quantity ω_ν lies outside the interval $[\alpha^{(\nu)} \ldots \beta^{(\nu)}]$.

sicher Zahlen *innerhalb* des Intervalls (α . . . β) vorkommen, so muss eine von diesen Zahlen den *kleinsten Index* haben, sie sei ω_{κ_1}, und eine andere: ω_{κ_2} mit dem nächst grösseren Index behaftet sein.

Die kleinere der beiden Zahlen ω_{κ_1}, ω_{κ_2} werde mit α', die grössere mit β' bezeichnet. (Ihre Gleichheit ist ausgeschlossen, weil wir voraussetzten, dass unsere Reihe aus lauter ungleichen Zahlen besteht.)

Es ist alsdann der Definition nach:

$$\alpha < \alpha' < \beta' < \beta,$$

ferner:

$$\kappa_1 < \kappa_2;$$

und ausserdem ist zu bemerken, dass alle Zahlen ω_μ unserer Reihe, für welche $\mu \le \kappa_2$, *nicht* im Innern des Intervalls (α' . . . β') liegen, wie aus der Bestimmung der Zahlen κ_1, κ_2 sofort erhellt. Ganz ebenso mögen ω_{κ_3}, ω_{κ_4} die beiden mit den kleinsten Indices versehenen Zahlen unserer Reihen [see note 1 below] sein, welche in das *Innere* des Intervalls (α' . . . β') fallen und die kleinere der Zahlen ω_{κ_3}, ω_{κ_4} werde mit α'', die grössere mit β'' bezeichnet.

Man hat alsdann:

$$\alpha' < \alpha'' < \beta'' < \beta',$$

$$\kappa_2 < \kappa_3 < \kappa_4;$$

und man erkennt, dass alle Zahlen ω_μ unserer Reihe, für welche $\mu \le \kappa_4$ *nicht* in das *Innere* des Intervalls (α'' . . . β'') fallen.

Nachdem man unter Befolgung des gleichen Gesetzes zu einem Intervall $(\alpha^{(\nu - 1)}, \ldots \beta^{(\nu - 1)})$ gelangt ist, ergiebt sich das folgende Intervall dadurch aus demselben, dass man die beiden ersten (d. h. mit niedrigsten Indices versehenen) Zahlen unserer Reihe (ω) aufstellt (sie seien $\omega_{\kappa_{2\nu-1}}$ und $\omega_{\kappa_{2\nu}}$), welche in das *Innere* von $(\alpha^{(\nu-1)} \ldots \beta^{(\nu-1)})$ fallen; die kleinere dieser beiden Zahlen werde mit $\alpha^{(\nu)}$, die grössere mit $\beta^{(\nu)}$ bezeichnet.

Das Intervall $(\alpha^{(\nu)} \ldots \beta^{(\nu)})$ liegt alsdann im *Innern* aller vorangegangenen Intervalle und hat zu unserer Reihe (ω) die *eigenthümliche* Beziehung, dass alle Zahlen ω_μ, für welche $\mu \le \kappa_{2\nu}$ *sicher nicht in seinem Innern* liegen. Da offenbar:

$$\kappa_1 < \kappa_2 < \kappa_3 < \ldots, \omega_{\kappa_{2\nu-2}} < \omega_{\kappa_{2\nu-1}} < \omega_{\kappa_{2\nu}}, \ldots$$

und diese Zahlen, als Indices, *ganze* Zahlen sind, so ist:

$$\kappa_{2\nu} \ge 2\nu,$$

und daher:

$$\nu < \kappa_{2\nu};$$

wir können daher, und dies ist für das Folgende ausreichend, gewiss sagen:

	Dass, wenn ν eine beliebige ganze Zahl ist, die Grösse ω_ν ausserhalb des Intervalls ($\alpha^{(\nu)}$ . . . $\beta^{(\nu)}$) liegt.
[Page 7] Since the numbers α', α'', α''', . . ., $\alpha^{(\nu)}$, . . . are continually increasing by value while simultaneously being enclosed in the interval [α, β], they have, by a well-known fundamental theorem of the theory of magnitudes [see note 2 below], a limit that we denote by A, so that: $A = \text{Lim}\ \alpha^{(\nu)}$ for $\nu = \infty$. The same applies to the numbers β', β'', β''', . . ., $\beta^{(\nu)}$, . . ., which are continually decreasing and likewise lying in the interval [α, β]. We call their limit B, so that: $B = \text{Lim}\ \beta^{(\nu)}$ for $\nu = \infty$. Obviously, one has: $\alpha^{(\nu)} < A \leq B < \beta^{(\nu)}$. But it is easy to see that the case A < B can *not* occur here since otherwise every number ω_ν of our sequence would lie *outside* of the interval [A, B] by lying outside the interval [$\alpha^{(\nu)}$, $\beta^{(\nu)}$]. So our sequence (ω) would *not* be *everywhere dense* in the interval [α, β], contrary to the assumption. Thus, there only remains the case A = B and now it is demonstrated that the number: $\eta = A = B$ does *not* occur in our sequence (ω). If it were a member of our sequence, such as the ν^{th}, then one would have: $\eta = \omega_\nu$. But the latter equation is not possible for any value of ν because η is in the *interior* of the interval [$\alpha^{(\nu)}$, $\beta^{(\nu)}$], but ω_ν lies *outside* of it.	[Seite 7] Da die Zahlen α', α'', α''',, . . ., $\alpha^{(\nu)}$,, . . . ihrer Grösse nach fortwährend wachsen, dabei jedoch im Intervalle (α . . . β) eingeschlossen sind, so haben sie, nach einem bekannten Fundamentalsatze der Grössenlehre, eine Grenze, die wir mit A bezeichnen, so dass: $A = \text{Lim}\ \alpha^{(\nu)}$ für $\nu = \infty$. Ein Gleiches gilt für die Zahlen β', β'', β''', . . ., $\beta^{(\nu)}$, . . . welche fortwährend abnehmen und dabei ebenfalls im Intervalle (α . . . β) liegen; wir nennen ihre Grenze B, so dass: $B = \text{Lim}\ \beta^{(\nu)}$ für $\nu = \infty$. Man hat offenbar: $\alpha^{(\nu)} < A \leq B < \beta^{(\nu)}$. Es ist aber leicht zu sehen, dass der Fall A < B hier *nicht* vorkommen kann; da sonst jede Zahl ω_ν, unserer Reihe *ausserhalb* des Intervalles (A . . . B) liegen würde, indem ω_ν, ausserhalb des Intervalls ($\alpha^{(\nu)}$ . . . $\beta^{(\nu)}$) gelegen ist; unsere Reihe (ω) wäre im Intervall (α . . . β) *nicht überalldicht,* gegen die Voraussetzung. Es bleibt daher nur der Fall A = B übrig und es zeigt sich nun, dass die Zahl: $\eta = A = B$ in unserer Reihe (ω) *nicht* vorkommt. Denn, würde sie ein Glied unserer Reihe sein, etwa das ν^{te}, so hätte man: $\eta = \omega_\nu$. Die letztere Gleichung ist aber für keinen Werth von ν möglich, weil η im *Innern* des Intervalls [$\alpha^{(\nu)}$, $\beta^{(\nu)}$], ω_ν aber *ausserhalb* desselben liegt.
Note 1: This is the only occurrence of "unserer Reihen" ("our sequences") in the proof. There is only one sequence involved in Cantor's proof and everywhere else "Reihe" ("sequence") is used, so it is most likely a typographical error and should be "unserer Reihe" ("our sequence"), which is how it has been translated. Note 2: Grössenlehre, which has been translated as "the theory of magnitudes", is a term used by 19th century German mathematicians that refers to the theory of discrete and continuous magnitudes. (Ferreirós 2007, pp. 41–42, 202.)	

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1. Ferreirós 2007, p. 171.
2. Dauben 1993, p. 4.
3. "[Cantor's method is] a method that incidentally, contrary to a widespread interpretation, is fundamentally constructive and not merely existential." (Fraenkel 1930, p. 237; English translation: Gray 1994, p. 823.)

4. "Cantor's proof of the existence of transcendental numbers is not just an existence proof. It can, at least in principle, be used to construct an explicit transcendental number." (Sheppard 2014, p. 131.) "Meanwhile Georg Cantor, in 1874, had produced a revolutionary proof of the existence of transcendental numbers, without actually constructing any." (Stewart 2015, p. 285.)
5. Gray 1994, pp. 819–821.
6. Cantor 1874. English translation: Ewald 1996, pp. 840–843.
7. Gray 1994, p. 828.
8. Cantor 1874, p. 259. English translation: Ewald 1996, pp. 840–841.
9. Cantor 1874, p. 259. English translation: Gray 1994, p. 820.
10. Cantor 1878, p. 242.
11. Gray 1994, p. 820.
12. Cantor 1874, pp. 259–260. English translation: Ewald 1996, p. 841.
13. Cantor 1874, pp. 260–261. English translation: Ewald 1996, pp. 841–842.
14. Cantor 1874, p. 261. English translation: Ewald 1996, p. 842.
15. Gray 1994, p. 822.
16. Havil 2012, pp. 208–209.
17. Havil 2012, p. 209.
18. LeVeque 1956, pp. 154–155.
19. LeVeque 1956, p. 174.
20. Weisstein 2003, p. 541.
21. Arkhangel'skii & Fedorchuk 1990, p. 16.
22. Noether & Cavaillès 1937, pp. 12–13. English translation: Gray 1994, p. 827; Ewald 1996, p. 844.
23. Noether & Cavaillès 1937, p. 18. English translation: Ewald 1996, p. 848.
24. Noether & Cavaillès 1937, p. 13. English translation: Gray 1994, p. 827.
25. Noether & Cavaillès 1937, pp. 14–15. English translation: Ewald 1996, pp. 845–846.
26. Gray 1994, p. 827
27. Dauben 1979, p. 51.
28. Noether & Cavaillès 1937, p. 19. English translation: Ewald 1996, p. 849.
29. Ewald 1996, p. 843.
30. Noether & Cavaillès 1937, p. 16. English translation: Gray 1994, p. 827.
31. Perron 1921, p. 162.
32. Gray 1994, pp. 827–828.
33. Perron 1921, p. 162. English translation: Gray 1994, p. 828.
34. Fraenkel 1930, p. 237. English translation: Gray 1994, p. 823.
35. Kaplansky 1972, p. 25.
36. Bell 1937, pp. 568–569; Hardy & Wright 1938, p. 159 (6th ed., pp. 205–206); Birkhoff & Mac Lane 1941, p. 392, (5th ed., pp. 436–437); Spivak 1967, pp. 369–370 (4th ed., pp. 448–449).
37. Proof is constructive: Dasgupta 2014, p. 107; Sheppard 2014, pp. 131–132. Proof is non-constructive: Jarvis 2014, p. 18; Chowdhary 2015, p. 19; Stewart 2015, p. 285; Stewart & Tall 2015, p. 333.
38. Birkhoff & Mac Lane 1941, p. 392, (5th ed., pp. 436–437).
39. Edwards 1989; Gray 1994, p. 828.
40. Edwards 1989, pp. 74–75.
41. Burton 1995, p. 595.
42. Dauben 1979, p. 69.
43. Gray 1994, p. 824.
44. Ferreirós 2007, p. 184.
45. Noether & Cavaillès 1937, pp. 12–16. English translation: Ewald 1996, pp. 843–846.
46. Dauben 1979, p. 67.
47. Noether & Cavaillès 1937, pp. 16–17. English translation: Ewald 1996, p. 847.
48. Grattan-Guinness 1971, p. 124.
49. Dauben 1979, pp. 67, 308–309.
50. See "The article" section. Also: Cantor 1874, p. 259; English translation: Ewald 1996, p. 841.
51. Ferreirós 2007, pp. 184–185, 245.
52. "It is unclear when his attitude changed, but there is evidence that by the mid-1880s he was accepting the conclusion that infinite sets are of different powers [cardinalities]." (Ferreirós 2007, p. 185.)
53. Ferreirós 2007, p. 177.
54. Dauben 1979, pp. 67–68.

55. Ferreirós 2007, p. 183.
56. Ferreirós 2007, p. 185.
57. Ferreirós 2007, pp. 109–111, 172–174.
58. Ferreirós 1993, p. 350.
59. Noether & Cavaillès 1937, pp. 12–13. English translation: Ewald 1996, p. 844.
60. Noether & Cavaillès 1937, p. 13. English translation: Ewald 1996, p. 845.
61. Ferreirós 2007, p. 179.
62. Noether & Cavaillès 1937, pp. 14–16, 19. English translation: Ewald 1996, pp. 845–847, 849.
63. Ferreirós 1993, pp. 358–359.
64. Ferreirós 1993, p. 350.
65. Cantor 1878, pp. 245–254.
66. Cantor 1879, p. 4.
67. Ferreirós 2007, pp. 267–273.
68. Ferreirós 2007, pp. xvi, 320–321, 324.
69. Cantor 1878, p. 243.
70. Hawkins 1970, pp. 103–106, 127.
71. Hawkins 1970, pp. 118, 120–124, 127.
72. Ferreirós 2007, pp. 362–363.
73. Cohen 1963, pp. 1143–1144.

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