A Simple Proof of Menger's Theorem

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ABSTRACT

A proof of Menger's theorem is presented.

We use the notation and terminology of Bondy and Murty [1].

Let D be a directed graph. If $\{u\}$, $\{v\}$, and S are disjoint subsets of $V(D)$ and u and v are nonadjacent, then S separates u and v if every (u, v)-path has a vertex in S.

Proofs of Menger's theorem are given in [2–14].

Menger's Theorem. If no set of fewer than n vertices separates nonadjacent vertices u and v in a directed graph D, then there are n internally disjoint (u, v)-paths.

Proof. The proof uses induction on n. The theorem is trivial for $n = 1$. Suppose u and v are separated by no set of less than $n + 1$ vertices ($n \geq 1$). By the induction hypothesis there are n internally disjoint (u, v)-paths $P_1, \ldots, P_n$. Since the set of second vertices of $P_1, \ldots, P_n$ does not separate u and v, there is a (u, v)-path P whose initial arc is not on P_i, $i = 1, \ldots, n$. Let x be the first vertex on P after u which is also on some P_i, $1 \leq i \leq n$. Let P_{n+1} be the (u, x)-section of P. Assume $P_1, \ldots, P_n, P_{n+1}$ have been chosen so that the distance in $D - \{u\}$ from x to v is a minimum. If $x = v$ we are done, so assume not.

In $D - \{x\}$ there are n internally disjoint (u, v)-paths $Q_1, \ldots, Q_n$, again by the induction hypothesis. Assume $Q_1, \ldots, Q_n$ have been chosen so that a minimum number of arcs in $B = A(D) - \bigcup_{i=1}^{n+1} A(P_i)$ are used. Let H be the directed graph consisting of the vertices and arcs of Q_1,

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..., Q_n together with the vertex x. Choose some P_k, $1 \leq k \leq n + 1$, whose initial arc is not in $A(H)$. Let y be the first vertex on P_k after u which is in $V(H)$. If $y = v$ we are done, so assume not.

If $y = x$ then let R be the shortest (x, v)-path in $D - \{u\}$. Let z be the first vertex of R on some Q_j, $1 \leq j \leq n$. Then the distance in $D - \{u\}$ from z to v is less than the distance from x to v. This contradicts the choice of $P_1, \ldots, P_n, P_{n+1}$.

If y is on some Q_i, $1 \leq i \leq n$, then the (u, y)-section of Q_i has an arc in B. Otherwise, two paths in $\{P_1, \ldots, P_n, P_{n+1}\}$ intersect at a vertex other than u, v, or x. Now if we replace the (u, y)-section of Q_i by the (u, y)-section of P_k we get n internally disjoint (u, v)-paths in $D - \{x\}$ using less arcs in B than $Q_1, \ldots, Q_n$. This is a contradiction. ∎

A similar proof can be used for the undirected version of Menger's theorem.

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