1-5 Data Structures

魏恒峰

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2019 年 11 月 21 日

Pseudocode

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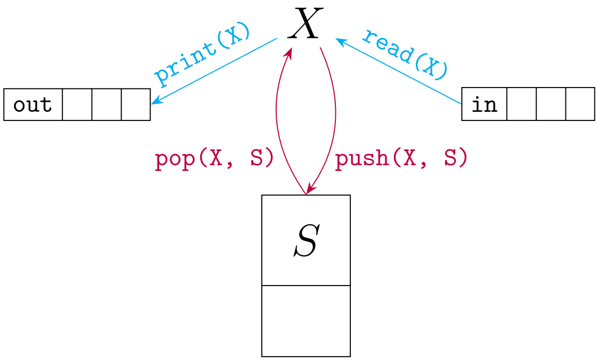
Make everything as
SIMPLE
as possible,
but not simpler.
- ALBERT EINSTEIN (1879-1955)
QUOTESEVERLASTING.COM

Pseudocode

"Executable" at an abstract level.

Stackable Permutations

Definition (Stackable Permutations)



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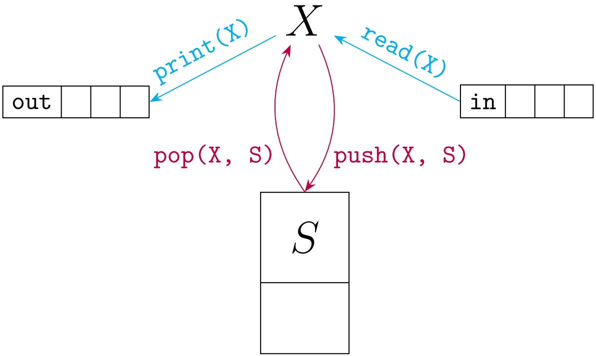
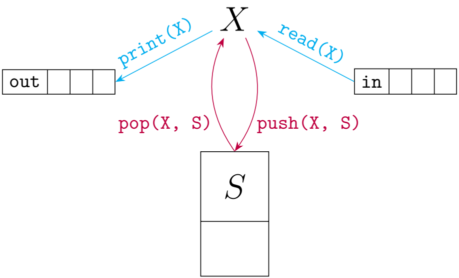
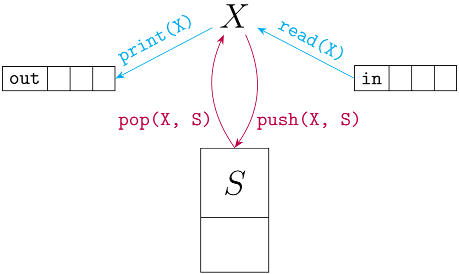
$$\boxed{\texttt{out} = (a_1, \cdots, a_n) \xleftarrow[X = \perp]{S = \emptyset} \texttt{in} = (1, \cdots, n)}$$





We can assume that X is always blank.



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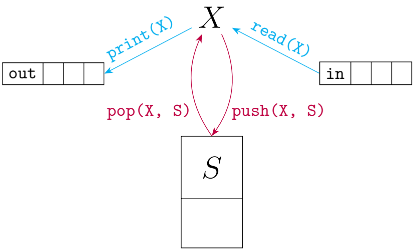
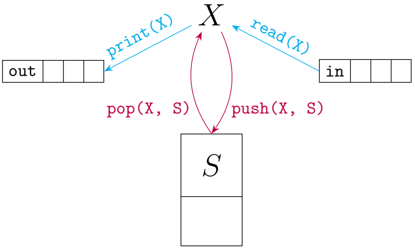
`read` + `push` `read` + `print`

`pop` + `print` `pop` + `push`


X
print(X)
read(X)
out
in
pop(X, S)
push(X, S)
S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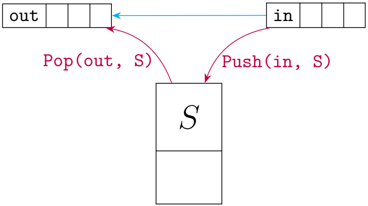
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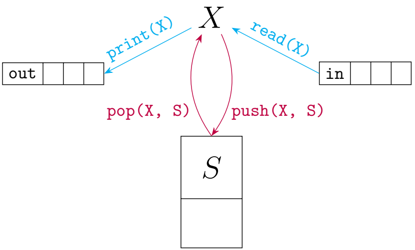
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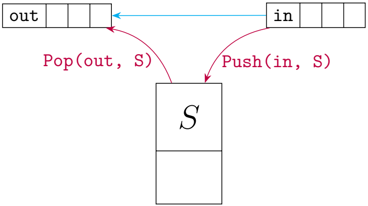
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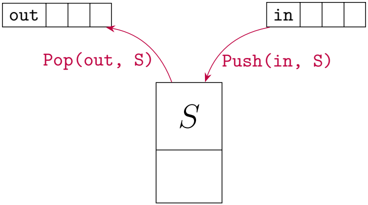
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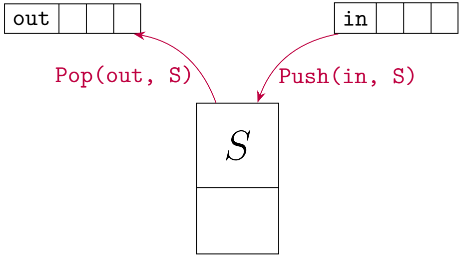

DH 2.12: Stackable Permutations

(a) Show that the following permutations *are* stackable:

- (i) $(3, 2, 1)$
- (ii) $(3, 4, 2, 1)$
- (iii) $(3, 5, 7, 6, 8, 4, 9, 2, 10, 1)$

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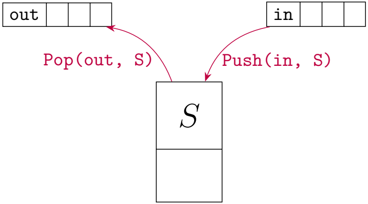
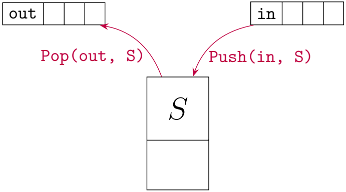


DH 2.13: Stackable Permutations Checking Algorithm

To check whether a given permutation can be obtained by a stack.



procedure STACKABLE(out)
  for all $a_j \in out$ **do**
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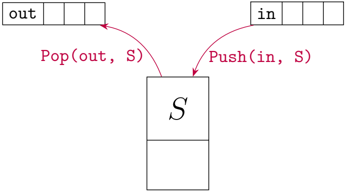


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procedure STACKABLE(out)
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Q : What is wrong with STACKABLE*?*

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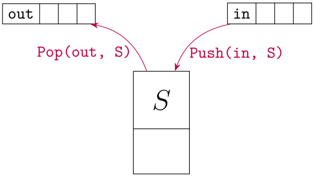

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(b) **Prove** that the following permutations are *not* stackable:

  (i) $(3, 1, 2)$
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312-Pattern

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A permutation $(a_1, \cdots, a_n)$ *is stackable* $\iff$ *it is not the case that*

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stackable $\Longrightarrow$ $\nexists$ 312-Pattern

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$\exists$ algorithm

$\square$

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312-Pattern $\implies$ non-stackable.

$i < j \land a_j < a_i$: $\mathsf{Push}_j \quad \mathsf{Push}_i \quad \mathsf{Pop}_i \quad \mathsf{Pop}_j$

$j < k \land a_j < a_k$: $\mathsf{Push}_j \quad \mathsf{Pop}_j \quad \mathsf{Push}_k \quad \mathsf{Pop}_k$

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procedure STACKABLE(out)
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$\exists i : i < j \land a_k < a_i$

DH 2.12: Stackable Permutations

(c) How many permutations of A_4 *cannot* be obtained by a stack?

$$(1,4,2,3),(2,4,1,3),(3,1,2,4),(3,1,4,2),(3,4,1,2)$$
$$(4,1,2,3),(4,1,3,2),(4,2,1,3),(4,2,3,1),(4,3,1,2)$$

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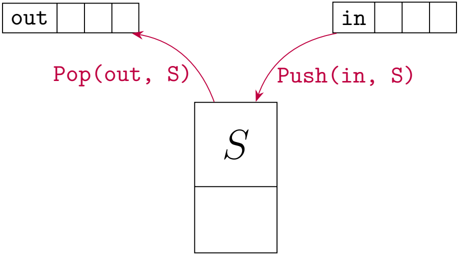
Q : What about A_n?

DH 2.12: Stackable Permutations

How many permutations of $\{1 \cdots n\}$ are stackable?



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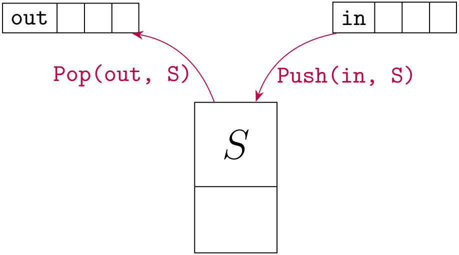

Q : How many *admissible* operation sequences of "Push" and "Pop"?

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Why is f bijective (1-1)?

Theorem

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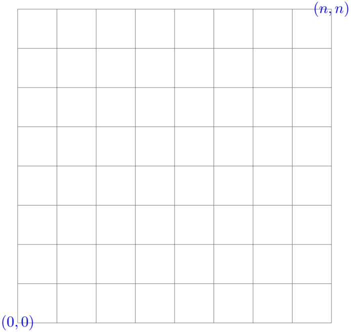

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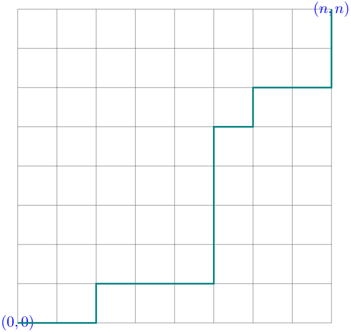

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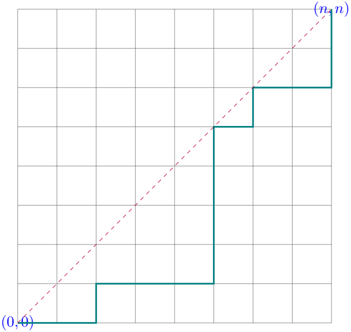

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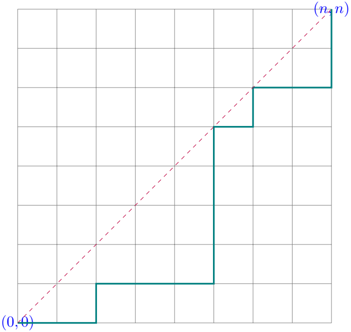
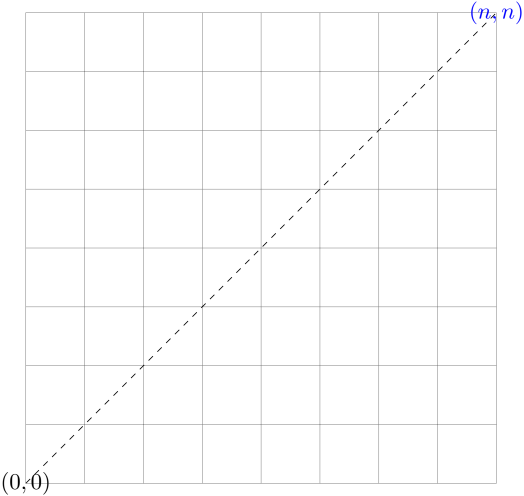
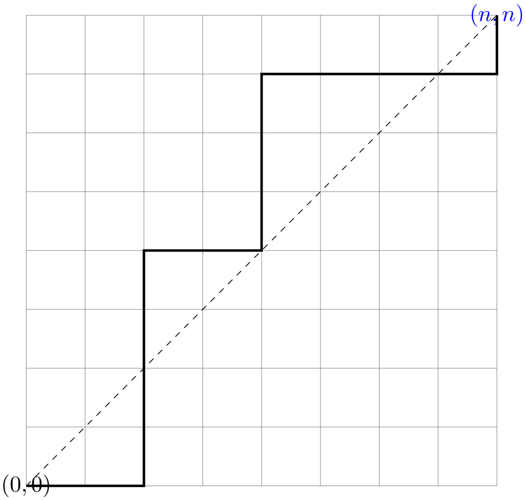
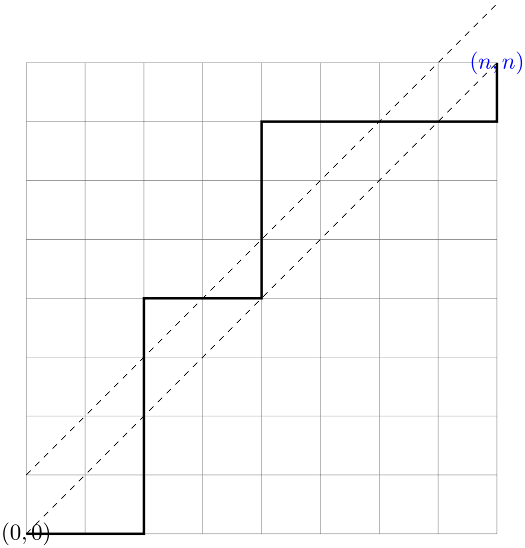


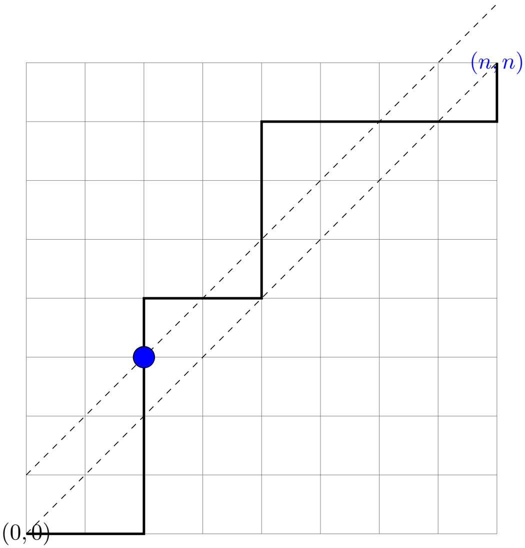
$$\underbrace{\binom{2n}{n}}_{\text{all}} - \underbrace{\binom{2n}{n-1}}_{\text{inadmissible}}$$


(n, n)
$(0, 0)$

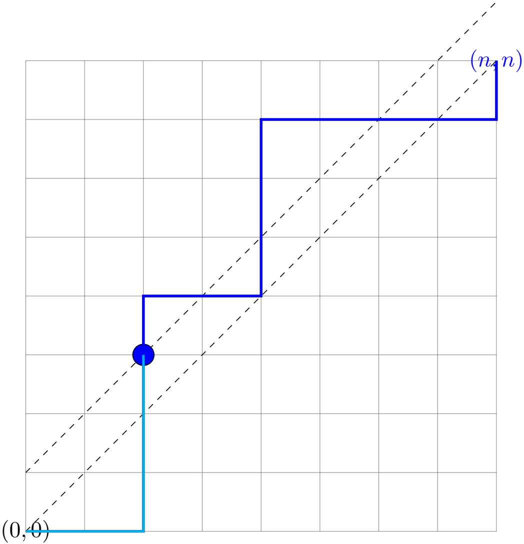

(n, n)
(0, 0)



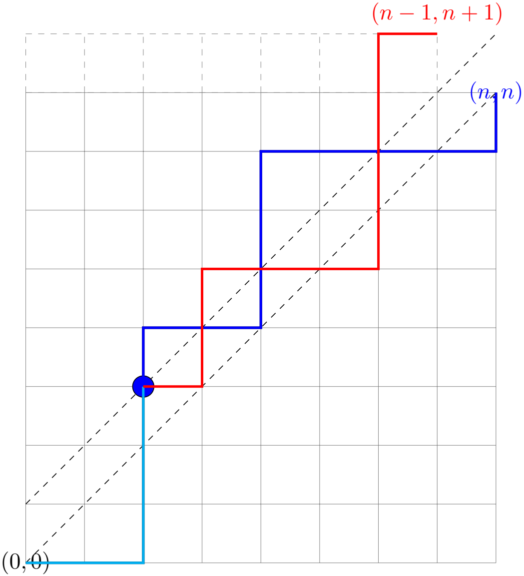
(n, n)
$(0, 0)$



(n, n)
(0,0)



(n,n)
$(0,0)$


$(n-1, n+1)$
(n, n)
$(0, 0)$

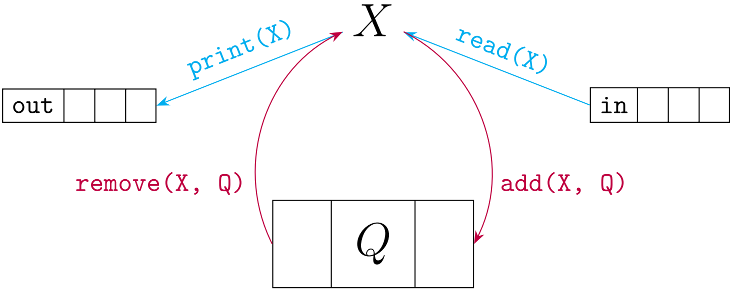
$$\binom{2n}{n} - \binom{2n}{n-1}$$

Catalan Number

$$(3,2,1) : ((())) \qquad (1,2,3) : ()()()$$

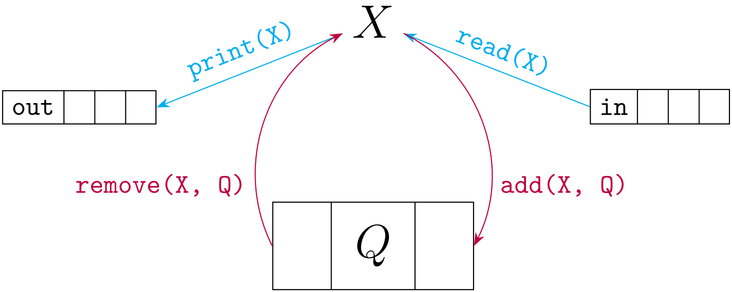
Queueable Permutations

DH 2.14: Queueable Permutations



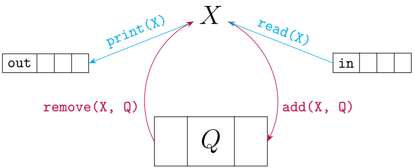
DH 2.14: Queueable Permutations

$$\boxed{\texttt{out} = (a_1, \cdots, a_n) \xleftarrow[X = \perp]{Q = \emptyset} \texttt{in} = (1, \cdots, n)}$$



DH 2.14: Queueable Permutations

(b) Prove that every permutation are queueable.



DH 2.14: Queueable Permutations

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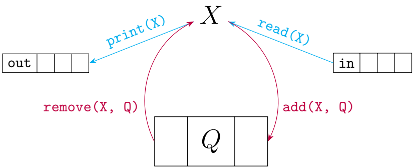
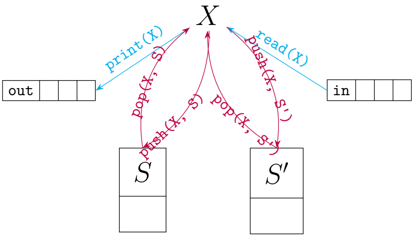


procedure QUEUEABLE(out)
  for all $a \in in$ **do**
    $read(X)$
    $add(X, Q)$
  for all $a \in out$ **do**
    while $Head(Q) \neq a$ **do**
      $remove(X, Q)$
      $add(X, Q)$
    $remove(X, Q)$
    $print(X)$

DH 2.14: Queueable Permutations

(c) Prove that every permutation can be obtained by two stacks.



DH 2.14: Queueable Permutations

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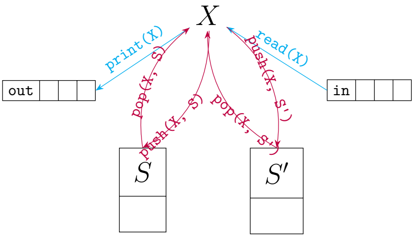
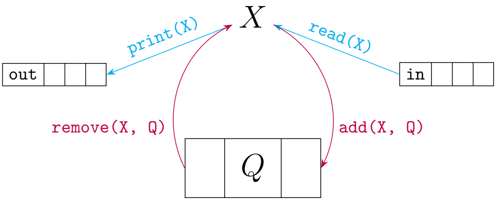
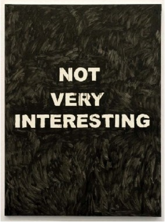
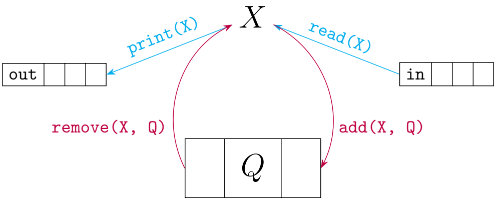



```
procedure DoubleStackable(out)
    for all a ∈ in do
        read(X)
        push(X, S)
    for all a ∈ out do
        while top(S) ≠ a do
            pop(X, S)
            push(X, S')
        pop(X, S)
        print(X)
        while S' ≠ ∅ do
            pop(X, S')
            push(X, S)
```

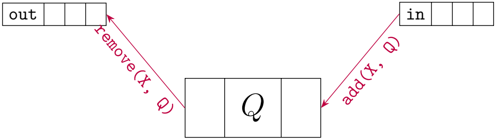


All are queueable.

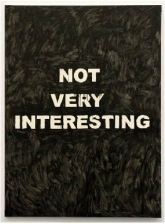


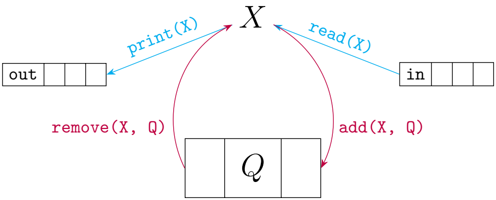


All are queueable.



Only one is queueable.





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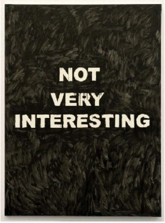

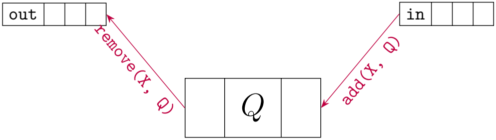

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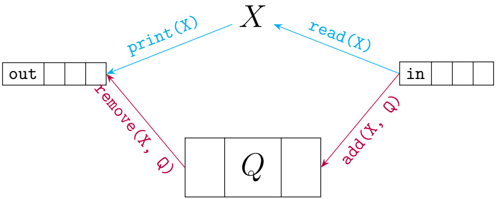
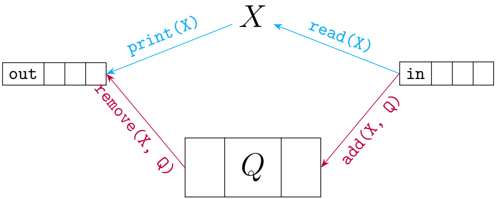

X
print(X)
read(X)
out
in
remove(X, Q)
add(X, Q)
Q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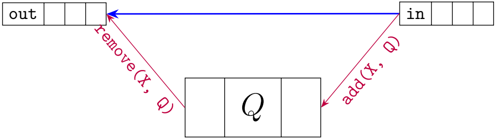
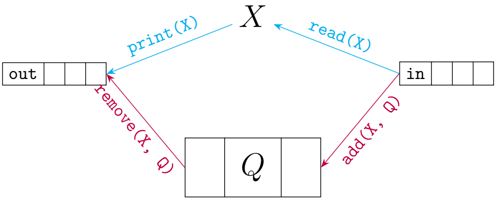
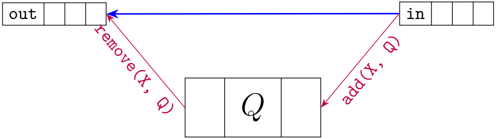
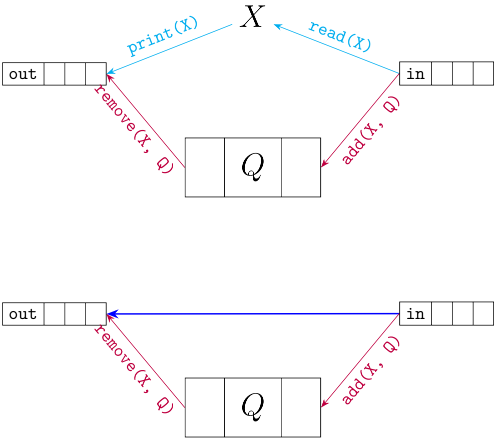

out
in
remove(X, Q)
add(X, Q)
Q





3 2 1 *is not queueable*



$3\ 2\ 1$ *is not queueable*

Theorem (Queueable Permutations)

A permutation $(a_1, \cdots, a_n)$ *is* *queueable* $\iff$ *it is not the case that*

$$321\text{-}Pattern : \boxed{out = \cdots a_i \cdots a_j \cdots a_k : i < j < k \land a_i > a_j > a_k}$$

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Proof.

□

Theorem (# of Queueable Permutations)

The number of queueable permutations of $[1 \cdots n]$ *is* $\binom{2n}{n} - \binom{2n}{n-1}$.

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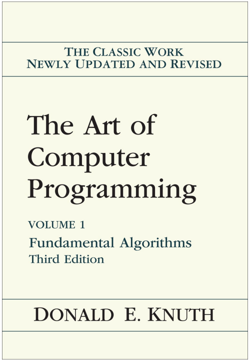
Proof.

□

For more about "Stackable/Queueable Permutations" (Section 2.2.1)



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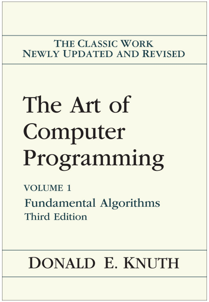
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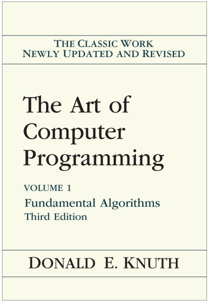
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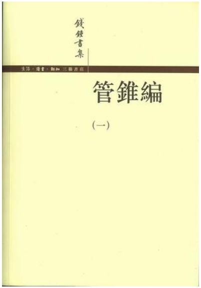
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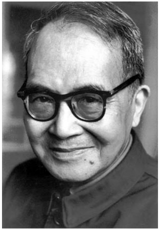


Thank You!