# 1-11 有限与无限

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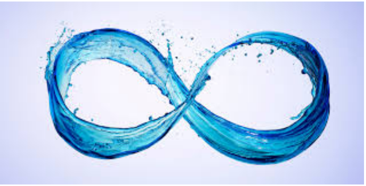

Comparing Sets

Finite vs. Infinity

# Comparing Sets

$=$ $\qquad$ $<$

[redacted]

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$\{1, 2, 3, \cdots\} \qquad \{1, 3, 5, \cdots, 2, 4, 6, \cdots\}$

Definition (Finite and Infinite)

For any set $X$,

Finite

$$\exists n \in \mathbb{N} : |X| = n \qquad (0 \in \mathbb{N})$$

Infinite $(\neg$ finite)

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Infinite (¬ finite)

Countably Infinite

$$|X| = |\mathbb{N}| \triangleq \aleph_0$$

Countable (finite ∨ countably infinite)

Uncountably Infinite (¬ countable)

Theorem ($\aleph_0$ (1874))

$$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}|$$

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## $|\mathbb{Q}| = |\mathbb{N}|$ (UD 22.9)

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1/1 | 1/2 → | 1/3 | 1/4 → | 1/5 | 1/6 → | 1/7 | 1/8 → | ⋯ |
| 2/1 | 2/2 | 2/3 | 2/4 | 2/5 | 2/6 | 2/7 | 2/8 | ⋯ |
| 3/1 | 3/2 | 3/3 | 3/4 | 3/5 | 3/6 | 3/7 | 3/8 | ⋯ |
| 4/1 | 4/2 | 4/3 | 4/4 | 4/5 | 4/6 | 4/7 | 4/8 | ⋯ |
| 5/1 | 5/2 | 5/3 | 5/4 | 5/5 | 5/6 | 5/7 | 5/8 | ⋯ |
| 6/1 | 6/2 | 6/3 | 6/4 | 6/5 | 6/6 | 6/7 | 6/8 | ⋯ |
| 7/1 | 7/2 | 7/3 | 7/4 | 7/5 | 7/6 | 7/7 | 7/8 | ⋯ |
| 8/1 | 8/2 | 8/3 | 8/4 | 8/5 | 8/6 | 8/7 | 8/8 | ⋯ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋱ |

Theorem ($\mathbb{R}$ is uncountably infinite (1874).)

$$|\mathbb{R}| \neq |\mathbb{N}| \qquad (|\mathbb{R}| > |\mathbb{N}|)$$

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Different "Sizes" of Infinity

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Cantor's Diagonal Argument

Different "Sizes" of Infinity

**Theorem (Cantor's Theorem (1891))**

$$|X| \neq |2^X| \qquad (|X| < |2^X|)$$

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$Q$ : What is "dimension"?

$$<$$

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$$|\mathbb{N}| < |\mathbb{R}|$$

$$|X| < |2^X|$$

**Definition (Countable Revisited)**

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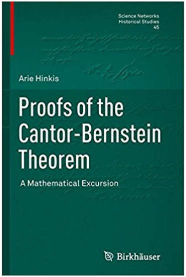

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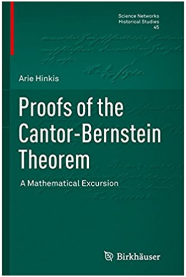
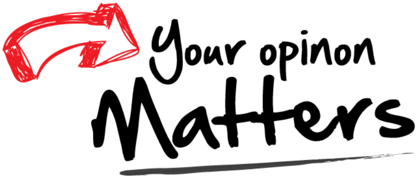


**Thank You!**

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