4. Fleury's Algorithm

There are several algorithms for constructing Euler Walks in (multi)graphs. One we mention here is called Fleury's Algorithm.

In using Fleury's algorithm, we assume that the graph $G = (V, e, f)$ is connected, that the order of G is at least two, and that all the verticies are of even degree or else there are exactly two verticies of odd degree. We also use the term *cut edge* (isthmus); an edge e of a connected graph $G = (V, E, f)$ is a *cut edge* (isthmus) of G if the graph $(V, E - \{e\}, f|(E - \{e\}))$ is not connected. The following result is also used. You saw this proof in Math 161 last year.

Theorem 1. *An edge e of a graph $G = (V, E, f)$ is a cut edge (isthmus) of G if and only if e is not contained in any cycle in G.*

Proof. The first part of the proof is by contradiction. Suppose that there is a cut edge (isthmus) e of F that is contained in a cycle C. Since $G - \{e\}$ is not connected, there exist verticies u and v of G that are connected in G but not in $G - \{e\}$. Thus, there is a (u, v)-path P in G, and R must belong to P. Suppose that $e = \{a, b\}$ and a precedes b in P. Then a section of P forms a (u, a)-path in G and a section of P forms a (b, v)-path on G. Since e is a member of the cycle C, $C - e$ is an (a, b)-path. Hence u and v are connected in $G - \{e\}$. This is a contradiction. We have proved that, if e is a cut edge (isthmus) of G, then e is not contained in any cycle in G.

Now suppose that e is not a cut edge (isthmus) of G. If e is a loop, then the loop is a cycle. Suppose that e is not a loop, and let $f(e) = \{u, v\}$, where $u \neq v$. Since e is not a cut edge (isthmus) of G, $G - \{e\}$ is connected. Therefor there is a path P in $G - \{e\}$ from u to v. Then $P \cup \{e\}$ is a cycle in G and e is contained in thus cycle. Thus we have proved that, if e is not contained in any cycle in G, then e is a cut edge (isthmus) of G. $\square$

Fleury's algorithm

1. Choose a vertex u_0 of odd degree if there is one. Otherwise choose an arbitary vertex u_0, set $W_0 = u_0$, $V_0 = V$, $E_0 = E$, and $i = 0$.

2. If there is no edge in E_i incident with u_i, stop.

3. If there is exactly one edge in E_i incident with u_i, let $e_{i+1} = u_i u_{i+1}$ denote this edge. (If e_{i+1} is a loop then $u_{i+1} = u_i$, otherwise $u_{i+1} \neq u_i$).

   Let $W_{i+1} = W_i e_{i+1} u_{i+1}$, $E_{i+1} = E_i - \{e_{i+1}\}$, $V_{i+1} = V_i - \{u_i\}$, replace i by $i+1$, and go to step 2.

4. If there is more than one edge in E_i incident with u_i, choose an edge e_{i+1} that is not a cut edge (isthmus) of the graph $(V_i, E_i, f|E_i)$, let u_{i+1} denote the other vertex incident with e_{i+1}, let $W_{i+1} = W_i e_{i+1} u_{i+1}$, $E_{i+1} = E_i - \{e_{i+1}\}$, $V_{i+1} = V_i$, replace i by $i+1$ and go to step 2.

We give an example to illustrate Fleury's algorithm.

Example

Use Fleury's algorithm to construct an Euler trail/walk in the graph shown in Figure **??**.

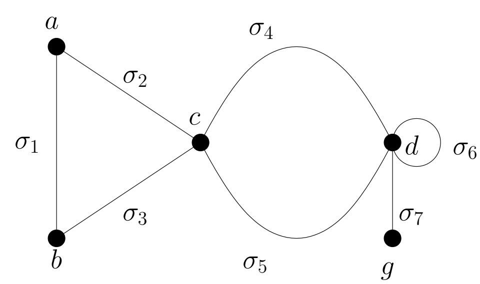

Figure 1: Move along

Analysis

We suggest that you draw a picture of the graph that remains each time we remove an edge or an edge and a vertex. We proceed with Fleury's algorithm as follows.

1. **1** There are two verticies d and g of odd degree. Let $u_0 = d$, set $W_0 = u_0$, $V_0 = V$, $E_0 = E$ and $i = 0$.

2. **4** There are four edges σ_4, σ_5, σ_6 and σ_7 in E_0 incident with u_0, and σ_7 is a cut edge (isthmus). Let $e_1 = \sigma_4$. Then $u_1 = c$, $W_1 = u_0 e_1 u_1$, $E_1 = E_0 - \{e_1\}$, $V_1 = V_0$, and i becomes 1.

3. **4** There are three edges σ_2, σ_3, and σ_5 in E_1 incident with u_1, and σ_5 is a cut edge (isthmus). Let $e_2 = \sigma_2$. Let $u_2 = a$, $W_2 = u_0 e_1 u_1 e_2 u_2$, $E_2 = E_1 - \{e_2\}$, $V_2 = V_1$, and i becomes 2.

4. **3** There is one edge σ_1 in E_2 that is incident with u_2. Let $e_3 = \sigma_1$. Then $u_3 = b$, $W_3 = u_0e_1u_1e_1u_2e_2u_3$, $E_3 = E_2 - \{e_3\}$, $V_3 = V_2 - \{u_2\}$, and i becomes 3.

5. **3** There is one edge σ_3 in E_3 incident with u_3. Let $e_4 = \sigma_3$. Then $u_4 = u_1$, $W_4 = u_0e_1u_1e_1u_2e_2u_3e_3u_4$, $E_4 = E_3 - \{e_4\}$, $V_4 = V_3 - \{u_3\}$, and i becomes 4.

6. **3** There is one edge σ_5 in E_4 incident with u_4. Let $e_5 = \sigma_5$. Then $u_5 = u_0$, $W_5 = u_0e_1u_1e_1u_2e_2u_3e_3u_4e_5u_5$, $E_5 = E_4 - \{e_5\}$, $V_5 = V_4 - \{u_4\}$, and i becomes 5.

7. **4** There are two edges σ_6 and σ_7 in E_5 incident with u_5, and σ_7 is a cut edge (isthmus). Let $e_6 = \sigma_6$. Then $u_6 = u_5$, $W_6 = u_0e_1u_1e_1u_2e_2u_3e_3u_4e_5u_5e_6u_6$, $E_6 = E_5 - \{e_6\}$, $V_6 = V_5$, and i becomes 6.

8. **3** There is one edge σ_7 in E_6 incident with u_6. Let $e_7 = \sigma_7$. Then $u_7 = g$, $W_7 = u_0e_1u_1e_1u_2e_2u_3e_3u_4e_5u_5e_6u_6e_7u_7$, $E_7 = E_6 - \{e_7\}$, $V_7 = V_6 - \{u_6\}$, and i becomes 7.

9. **2** There are no edges in E_7, so we stop.

Notice that W_7 is an Euler trail.

The proof the Fleury's algorithm is correct will be done in class.

It is more difficult to establish the complexity of Fleury's algorithm than some of the other algorithms. One time-consuming step is to decide whether an edge is an isthmus. As a matter of information, if n denotes the number of verticies of the graph and ϵ denotes the number of edges, then the complexity is $O(\epsilon n^2)$. Since in a simple graph $\epsilon \leq n(n-1)/2$, the complexity of Fleury's algorithm when applied to a simple graph is $O(n^4)$. The proof given in class also gives an algorithm (as mentioned) which gives a faster algorithm called Hierholzer's Algorithm.

We mention that *counting* the number of Euler Walks is has hard as counting the number of Hamilton Cycles, so this is a case where existence is fast to decide, but enumerating is very hard.

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