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Communication

Short proof of Menger's Theorem

F. Göring

Department of Mathematics, Technical University of Ilmenau, D-98684 Ilmenau, Germany

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Abstract

A short proof of the classical theorem of Menger concerning the number of disjoint AB-paths of a finite digraph for two subsets A and B of its vertex set is given. © 2000 Elsevier Science B.V. All rights reserved.

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Proofs of Menger's Theorem are given in [1,3–6]. T. Böhme, J. Harant and I gave another short proof in [2, update edition]; here I shall give a yet shorter proof.

For terminology and notation not defined here we refer to [2]. A graph with no edges is denoted by its vertex set. Let D be a finite digraph (loops and multiple edges being allowed). For (possibly empty) sets of vertices A and B of D let an *AB-separator* be a set of vertices of D such that the graph obtained from D by deleting these vertices contains no path from A to B. Note that a single vertex of $A \cap B$ is considered a path from A to B, too. An *AB-connector* is a subgraph of D such that each of its components is a path from A to B having no inner vertex in common with A or B (in particular the empty graph is also an AB-connector).

Theorem (Menger [5]). *Let D be a finite digraph, A and B sets of vertices of D, and s the minimum number of vertices forming an AB-separator. Then there is an AB-connector C with $|C \cap A| = s$.*

Proof. If D is edgeless then set $C = A \cap B$. Hence we may assume: D contains an edge e from x to y, the theorem holds for $D' = D - e$, and D' has an AB-separator S with $|S| < s$. Then $P = S \cup \{x\}$ and $Q = S \cup \{y\}$ are AB-separators of D. Thus $|P| = |Q| = |S| + 1 = s$. An AP-separator (as well as an QB-separator) of D' is an AB-separator of D. Consequently,

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D' has an AP-connector X containing P and a QB-connector Y containing Q. Since $X \cap Y = S$ one can set $C = (X \cup Y) + e$. $\square$

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