2-14 B-Trees

Hengfeng Wei

hfwei@nju.edu.cn

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Organization and Maintenance of Large Ordered Indexes

R. BAYER and E. MCCREIGHT

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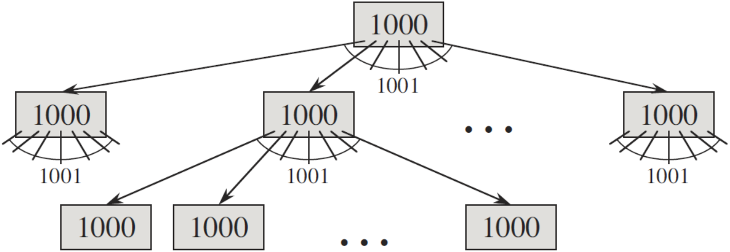
Summary. Organization and maintenance of an index for a dynamic random access file is considered. It is assumed that the index must be kept on some pseudo random access backup store like a disc or a drum. The index organization described allows retrieval, insertion, and deletion of keys in time proportional to $\log_k I$ where I is the size of the index and k is a device dependent natural number such that the performance of the scheme becomes near optimal. Storage utilization is at least 50% but generally much higher. The pages of the index are organized in a special data-structure, so-called B-trees. The scheme is analyzed, performance bounds are obtained, and a near optimal k is computed. Experiments have been performed with indexes up to 100000 keys. An index of size 15000 (100000) can be maintained with an average of 9 (at least 4) transactions per second on an IBM 360/44 with a 2311 disc.

"Bayer and McCreight introduced B-trees in 1972;

they did not explain their choice of name."



2-way *vs.* multi-way

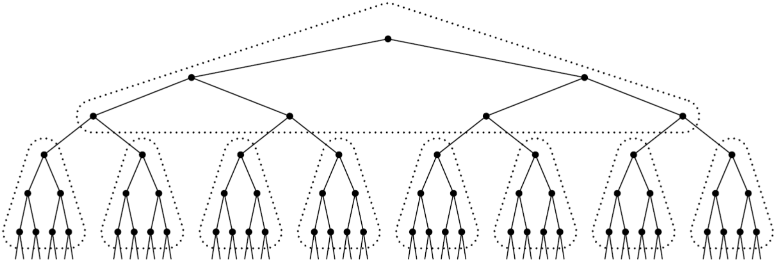
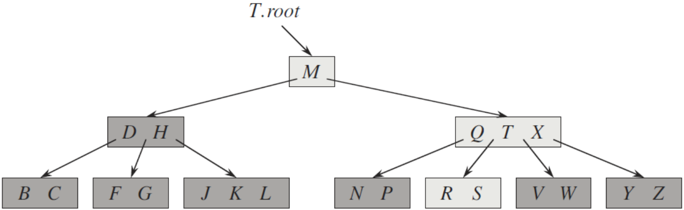


Fig. 29. A large binary search tree can be divided into "pages."

node *vs.* pages

Minimum (TC 18.2-3)

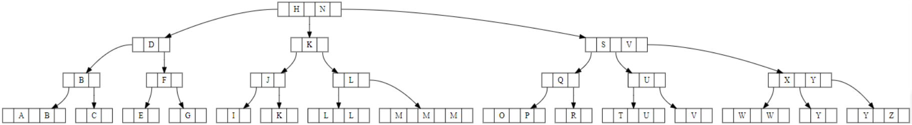
Explain how to find the minimum key stored in a B-tree.



the leftmost key in the leftmost node

Predecessor (TC 18.2-3)

Explain how to find the predecessor of a given key (x, i) stored in a B-tree.



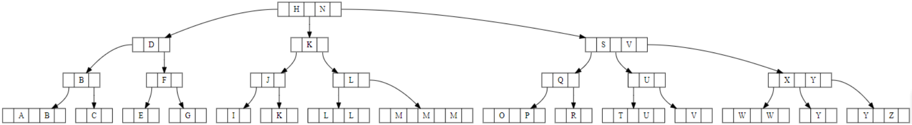
B-tree with 37 characters.

$$x.leaf = 0$$

$H \quad N \quad S \quad V \quad Q$

find the rightmost key in $x.c_i$

$$(x, i)$$



$$x.leaf = 1$$

$P \quad U \quad T \quad O \quad A$

$$i \geq 2 \implies (x, i-1)$$

$i = 1 \implies$ find (y, j) such that x is the leftmost key in $y.c_{j+1}$

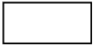
A is the only key which has no predecessor.

procedure B-TREE-PREDECESSOR(T, x, i) ▷ $x.key_i$ in B-tree T
    if $x.leaf = 0$ **then**
        return the rightmost key in $x.c_i$
    else if $i \geq 2$ **then** ▷ $x.leaf = 1$
        return $(x, i-1)$
    else ▷ $x.leaf = 1 \land i = 1$
        $y \leftarrow x.p$
        while $y \neq T.root \land x = y.c_1$ **do** ▷ exit: $y = T.root \lor x \neq y.c_1$
            $x \leftarrow y$
            $y \leftarrow y.p$
        if $x = y.c_1$ **then** ▷ $y = T.root \land x = y.c_1$
            return "no predecessor"
        else ▷ $x \neq y.c_1$
            $j \leftarrow 2$
            while $y.c_j \neq x$ **do**
                $j \leftarrow j+1$
            return $(y, j-1)$ ▷ $x = y.c_j$

Insertion (TC 18.2-4 *)

Suppose that we insert the keys $\{1, 2, \ldots, n\}$ in increasing order into an empty B-tree with minimum degree 2.

How many nodes, denoted X_n, does the final B-tree have?

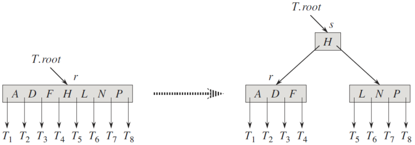


$$X_0 = 1$$

By Yangjing Dong (June 2018)

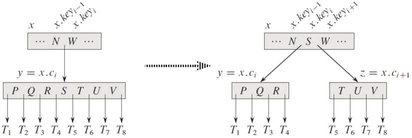
https://maxmute.com/TC18.2-4.html

Only SPLIT can create new nodes.



root SPLIT

+2



non-root SPLIT

+1

(I) Which nodes will SPLIT? S

(II) When does each node $s \in S$ SPLIT? $T_s = \{s_1, s_2, \dots\}$

(III) How does it SPLIT, as a root or a non-root? $T_s = s_R \uplus s_{NR}$

$$X_n = 1 + \sum_{s \in S} \Big(2\, |s_R| + |s_{NR}| \Big)$$

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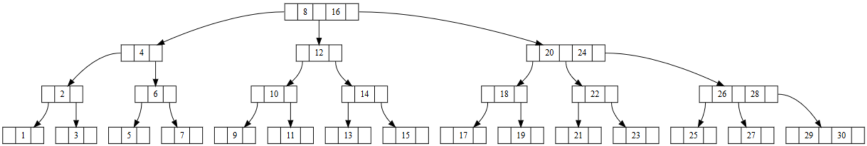
(I) Which nodes will SPLIT?

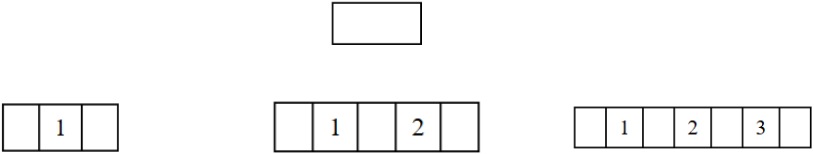


$$1, 2, \ldots, 30$$

$$S = \left\{\text{the nodes in the rightmost chain}\right\}$$

(II) When does each node SPLIT?

Let's focus the rightmost node first, denoted A.


1
1
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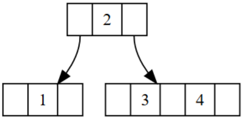
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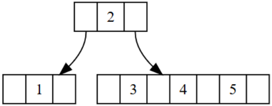
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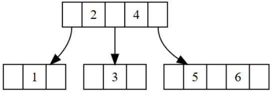
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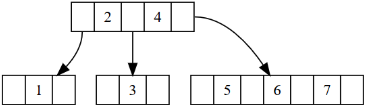
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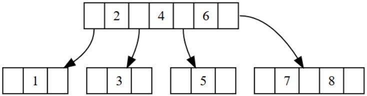
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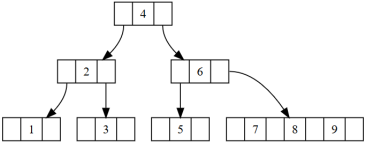
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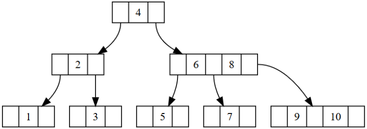
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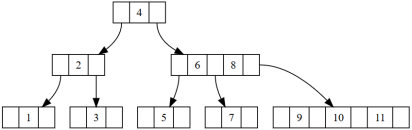
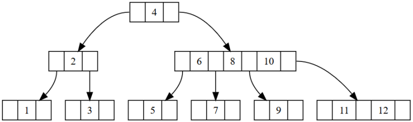


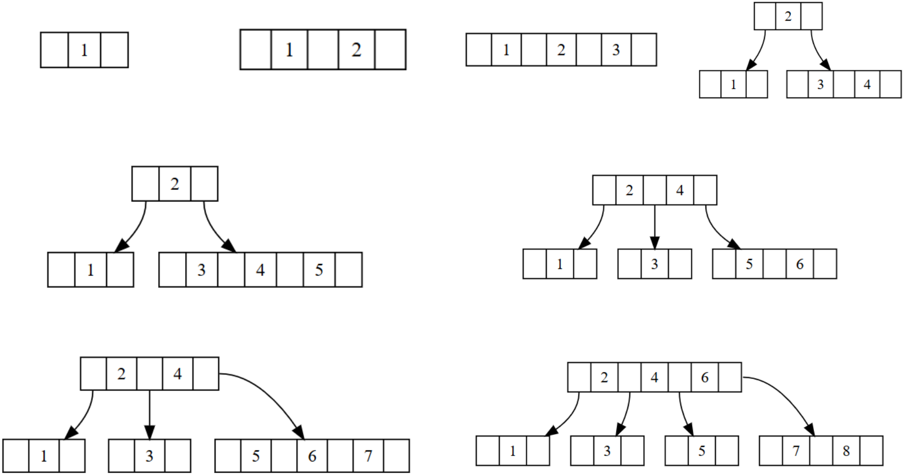


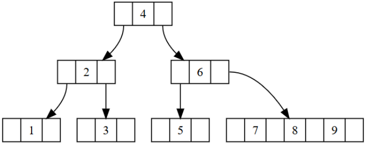
A SPLIT : 4, 6, 8, 10, 12, . . .

(II) When does each node SPLIT?

Let's consider the parent of A, denoted $B \triangleq p(A)$.

Every time A splits, B obtains a new key.

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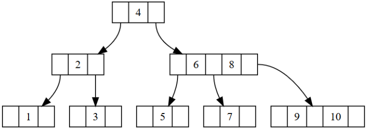
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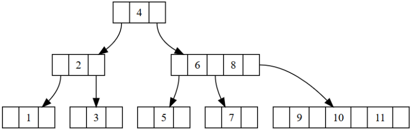
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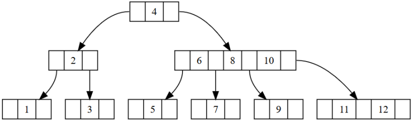
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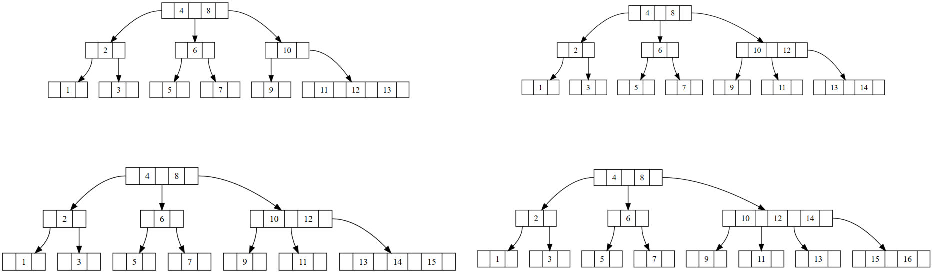

B SPLIT : $9, \quad 13, \quad 17, \quad 21, \quad 25, \quad \ldots$

$$A \text{ SPLIT}: 4, \quad 6, \quad 8, \quad 10, \quad 12, \quad \ldots$$

$$B \text{ SPLIT}: 9, \quad 13, \quad 17, \quad 21, \quad 25, \quad \ldots$$

(II) When does each node SPLIT?

Let's consider the parent of B, denoted $C = p(B)$.

$$A \text{ SPLIT}: 4, \quad 6, \quad 8, \quad 10, \quad 12, \quad \ldots$$

$$B \text{ SPLIT}: 9, \quad 13, \quad 17, \quad 21, \quad 25, \quad \ldots$$

$$C \text{ SPLIT}: 18, \quad 26, \quad 34, \quad 42, \quad 50, \quad \ldots$$

$$A:1 \qquad B:2 \qquad C:3$$

$T_i:$ the first time point the i-th node splits

$$T_1 = 4$$

$$T_i = \underbrace{T_{i-1}}_{\text{its right child first split}} + \underbrace{2 \times 2^{i-1}}_{\text{its right child split twice more}} + \underbrace{1}_{\text{insert one more}}$$

$$T_i = 2^{i+1} + i - 1$$

$$X_n = 1 + \sum_{s \in S} \Big(2 \; |s_R| + |s_{NR}| \Big)$$

$$(T_s = s_R \uplus s_{NR})$$

(III) How does it SPLIT, as a root or a non-root?

$$s_R = \{s_1\} \qquad s_{NR} = \{s_2, s_3, \ldots\}$$

$$|s_R| = 1 \qquad |s_{NR}| = |T_s| - 1$$

$$X_n = 1 + \sum_{s \in S} \Big(2 + |T_s| - 1 \Big) = 1 + \sum_{s \in S} \Big(|T_s| + 1 \Big)$$

$$X_n = 1 + \sum_{s \in S} \Big(|T_s| + 1 \Big) \qquad T_i = 2^{i+1} + i - 1$$

$$\begin{aligned}
X_n &= 1 + \sum_{i=1}^{\infty} [T_i \le n] \left(\left(\left\lfloor \frac{n - T_i}{2^i} \right\rfloor + 1 \right) + 1 \right) \\
&= 1 + \sum_{i=1}^{\infty} [T_i \le n] \left(\left\lfloor \frac{n - T_i}{2^i} \right\rfloor + 2 \right) \\
&= 1 + \sum_{i=1}^{\infty} [T_i \le n] \left(\left\lfloor \frac{n - 2^{i+1} - i + 1}{2^i} \right\rfloor \right) \\
&= 1 + \sum_{\substack{i \ge 1 \\ 2^{i+1} + i - 1 \le n}} \left\lfloor \frac{n - i + 1}{2^i} \right\rfloor \\
&= 1 + \sum_{\substack{i \ge 0 \\ 2^{i+2} + i \le n}} \left\lfloor \frac{n - i}{2^{i+1}} \right\rfloor
\end{aligned}$$


Thank
You!

Your opinon Matters

Office 302

Mailbox: H016

hfwei@nju.edu.cn