3-11 Matchings and Factors

(Part I: Matchings and Covers)

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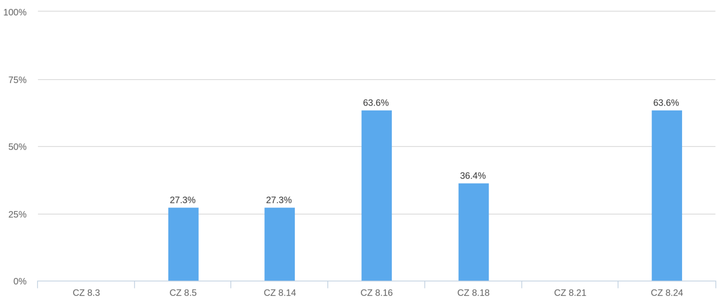
December 10, 2018



8.5 8.14 8.16 Chinese Postman Problem (The Last Class?)

8.18 8.24 (The Last Class)

比较大的定理（证明比较长的）都不是很理解，想知道期末考什么
点覆盖边覆盖那里只知道有这些性质，了解不是很深
都理解
图的分解的形象意义
无
定理8.3的证明
αβ、α'β'的定义和几个定理推论
为什么中英文书上的定义中\alpha和\beta反了。。
定理8.10的证明看不懂；一些比较几何的构造法证明（比如把顶点排成正多边形，一个点放中间）是怎么保证这些分解不重不漏的?
Kirkman三元系

$$\alpha, \quad \beta, \quad \alpha', \quad \beta'$$

Theorem 8.10 (Tutte's Theorem)   (The Last Class)

Theorem (Hall's Theorem, 1935; Theorem 8.3)

Let G be a bipartite graph with partite sets U and W such that $r = |U| \leq |W|$.
G contains a matching of cardinality $r \iff G$ satisfies Hall's Condition:

$$\forall X \subseteq U : \big|N(X)\big| \geq \big|X\big|$$

TONCAS

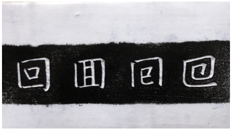
(The Obvious Necessary Conditions are Also Sufficient)

Other TONCAS?

Perfect Matching on Trees (Problem 8.5)

Prove that every tree T has ≤ 1 perfect matching.

“这题有四样证法, 你知道吗?”



Perfect Matching on Trees (Problem 8.5)

Prove that every tree T has ≤ 1 perfect matching.

By strong mathematical induction on the order n of trees.

Inductive step: Consider a tree of order n.

if n is odd **then**
  $\#$ Perfect Matching $= 0$
else $\triangleright$ n is even
  Consider $T - r$ $\triangleright$ r : root of G
  if $k_o(T - r) > 1$ **then**
    $\#$ Perfect Matching $= 0$
  else $\triangleright$ $k_o(T - r) = 1$
    By Induction Hypothesis.

Perfect Matching on Trees (Problem 8.5)

Prove that every tree T has ≤ 1 perfect matching.

By strong mathematical induction on the order n of trees.

Inductive step: Consider a tree of order n.

```
if T has no perfect matchings then
    # Perfect Matching = 0
else                                        ▷ T has perfect matchings
    Consider a leaf v
    v must be matched with its parent u
    By Induction Hypothesis on each component of G − {u, v}
```

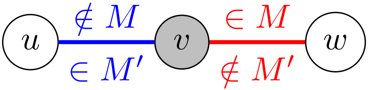
Perfect Matching on Trees (Problem 8.5)

Prove that every tree T has ≤ 1 perfect matching.

By Contradiction.

Suppose that there are two different perfect matchings M and M' on T.

$\exists v : v$ is matched with different vertices in M and M'.



Q : What about u and w?

Contradiction: Cycle

Perfect Matching on Trees (Problem 8.5)

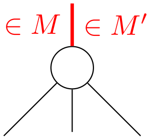
Prove that every tree T has ≤ 1 perfect matching.

By Contradiction.

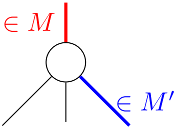
Suppose that there are two different perfect matchings M and M' on T.

$$M \Delta M' = (M - M') \cup (M' - M)$$

Consider the subgraph $\mathcal{M}$ with $V(T)$ and $M \Delta M'$.



CASE I



CASE II

$$\forall v \in V(\mathcal{M}):$$

$$\deg(v) = 0 \vee \deg(v) = 2$$

$$T \text{ is a tree} \implies \deg(v) = 0$$

$$\deg(v) = 0 \implies \text{CASE I}$$

Theorem (Gallai Identities, 1959; Theorem 8.7)

If G is graph without isolated vertices, then

$$\alpha'(G) + \beta'(G) = n(G).$$

$\alpha(G)$	Maximum size of independent set
$\beta(G)$	Minimum size of vertex cover
$\alpha'(G)$	Maximum size of matching
$\beta'(G)$	Minimum size of edge cover

Theorem (Gallai Identities, 1959; Theorem 8.8)

If G is graph without isolated vertices, then

$$\alpha(G) + \beta(G) = n(G).$$

Matching and Edge Cover (Problem 8.14)

A graph G without isolated vertices has a perfect matching if and only if $\alpha'(G) = \beta'(G)$.

" $\Longrightarrow$ "

G has a perfect matching

$\Longrightarrow$ n is even $\land \alpha'(G) = n/2$

$\Longrightarrow$ $\beta'(G) = n/2$

" $\Longleftarrow$ "

$\alpha'(G) = \beta'(G)$

$\Longrightarrow$ $\alpha'(G) = n/2 \land n$ is even

$\Longrightarrow$ G has a perfect matching

Theorem (König, 1931; Egerváry, 1931)

If G is a bipartite graph, then

$$\alpha'(G) = \beta(G).$$

$\alpha(G)$	Maximum size of independent set
$\beta(G)$	Minimum size of vertex cover
$\alpha'(G)$	Maximum size of matching
$\beta'(G)$	Minimum size of edge cover

Theorem (König, 1931)

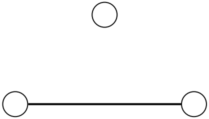
If G is a bipartite graph, then

$$\alpha(G) = \beta'(G).$$

Vertex Covering Number (Problem 8.16)

If G is a graph of order n, maximum degree Δ and having no isolated vertices, then

$$\beta(G) \geq \frac{n}{\Delta + 1}.$$



$$n = 3, \Delta = 1, \frac{n}{\Delta + 1} = \frac{3}{2}, \beta = 1$$

By Contradiction: $\beta < \frac{n}{\Delta+1}$.

$$\begin{aligned} \beta \cdot \Delta &< \frac{n\Delta}{\Delta + 1} \\ &= n - \frac{n}{\Delta + 1} \\ &\leq n - 1 \end{aligned}$$

Vertex Covering Number (Problem 8.16)

If G is a graph of order n, maximum degree Δ and having no isolated vertices, then

$$\beta(G) \geq \frac{n}{\Delta + 1}.$$

Double Counting:

What is the number of neighbors of a vertex cover C?

$$N(C) \leq |C| \cdot \Delta$$

$$N(C) = n - |C|$$

$$n - |C| \leq |C| \cdot \Delta \implies |C| \geq \frac{n}{\Delta + 1}$$

Vertex Independence Number (Additional Problem)

If G is a graph of order n, maximum degree Δ, then

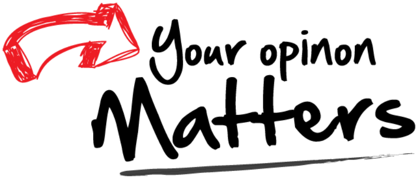
$$\alpha(G) \geq \frac{n}{\Delta + 1}.$$

By Construction.

To construct an independent set S with $|S| \geq \frac{n}{\Delta+1}$.

1: **while** $|V(G) > 0|$ **do**
2: Choose $v \in V(G)$
3: $S \leftarrow S \cup \{v\}$
4: $G \leftarrow G - \{v\} - N(v)$


THANK YOU



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