Subgroup structure of symmetric group:S4

From Groupprops

TAKE A QUIZ ON THIS TOPIC and test the quality of your understanding of it

This article gives specific information, namely, subgroup structure, about a particular group, namely: symmetric group:S4.
View subgroup structure of particular groups | View other specific information about symmetric group:S4

Contents

The symmetric group of degree four has many subgroups.

Note that since S_4 is a complete group, every automorphism is inner, so the classification of subgroups upto conjugacy is equivalent to the classification of subgroups upto automorphism. In other words, every subgroup is an automorph-conjugate subgroup.

Tables for quick information

FACTS TO CHECK AGAINST FOR SUBGROUP STRUCTURE: (finite solvable group)
Lagrange's theorem (order of subgroup times index of subgroup equals order of whole group, so both divide it), |order of quotient group divides order of group (and equals index of corresponding normal subgroup)
Sylow subgroups exist, Sylow implies order-dominating, congruence condition on Sylow numbers|congruence condition on number of subgroups of given prime power order
Hall subgroups exist in finite solvable|Hall implies order-dominating in finite solvable| normal Hall implies permutably complemented, Hall retract implies order-conjugate
MINIMAL, MAXIMAL: minimal normal implies elementary abelian in finite solvable | maximal subgroup has prime power index in finite solvable group

Quick summary

Item	Value
Number of subgroups	30 Compared with $S_n, n = 1,2,3,4,5,\dots$: 1,2,6,**30**,156,1455,11300, 151221
Number of conjugacy classes of subgroups	11 Compared with $S_n, n = 1,2,3,4,5,\dots$: 1,2,4,**11**,19,56,96,296,554,1593
Number of automorphism classes of subgroups	11 Compared with $S_n, n = 1,2,3,4,5,\dots$: 1,2,4,**11**,19,37,96,296,554,1593
Isomorphism classes of Sylow subgroups and the corresponding Sylow numbers and fusion systems	2-Sylow: dihedral group:D8 (order 8), Sylow number is 3, fusion system is non-inner non-simple fusion system for dihedral group:D8 3-Sylow: cyclic group:Z3, Sylow number is 4, fusion system is non-inner fusion system for cyclic group:Z3
Hall subgroups	Given that the order has only two distinct prime factors, the Hall subgroups are the whole group, trivial subgroup, and Sylow subgroups
maximal subgroups	maximal subgroups have order 6 (S3 in S4), 8 (D8 in S4), and 12 (A4 in S4).
normal subgroups	There are four normal subgroups: the whole group, the trivial subgroup, A4 in S4, and normal V4 in S4.

Table classifying subgroups up to automorphisms

TABLE SORTING AND INTERPRETATION: Note that the subgroups in the table below are sorted based on the powers of the prime divisors of the order, first covering the smallest prime in ascending order of powers, then powers of the next prime, then products of powers of the first two primes, then the third prime, then products of powers of the first and third, second and third, and all three primes. The rationale is to cluster together subgroups with similar prime powers in their order. The subgroups are *not* sorted by the magnitude of the order. To sort that way, click the sorting button for the order column. Similarly you can sort by index or by number of subgroups of the automorphism class.

Automorphism class of subgroups	**Representative**	**Isomorphism class**	**Order of subgroups**	**Index of subgroups**	**Number of conjugacy classes (=1 iff automorph-conjugate subgroup)**	**Size of each conjugacy class (=1 iff normal subgroup)**	**Number of subgroups (=1 iff characteristic subgroup)**	**Isomorphism class of quotient (if exists)**	**Subnormal depth (if subnormal)**	**Note**
trivial subgroup	$\{()\}$	trivial group	1	24	1	1	1	symmetric group:S4	1	

S2 in S4	$\{(), (1,2)\}$	cyclic group:Z2	2	12	1	6	6	--	--	
subgroup generated by double transposition in S4	$\{(), (1,2)(3,4)\}$	cyclic group:Z2	2	12	1	3	3	--	2	
Z4 in S4	$\langle (1,2,3,4) \rangle$	cyclic group:Z4	4	6	1	3	3	--	--	
normal Klein four-subgroup of S4	$\{(), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$	Klein four-group	4	6	1	1	1	symmetric group:S3	1	2-core
non-normal Klein four-subgroups of S4	$\langle (1,2), (3,4) \rangle$	Klein four-group	4	6	1	3	3	--	--	
D8 in S4	$\langle (1,2,3,4), (1,3) \rangle$	dihedral group:D8	8	3	1	3	3	--	--	2-Sylow, fusion system is non-inner non-simple fusion system for dihedral group:D8
A3 in S4	$\{(), (1,2,3), (1,3,2)\}$	cyclic group:Z3	3	8	1	4	4	--	--	3-Sylow, fusion system is non-inner fusion system for cyclic group:Z3
S3 in S4	$\langle (1,2,3), (1,2) \rangle$	symmetric group:S3	6	4	1	4	4	--	--	
A4 in S4	$\langle (1,2,3), (1,2)(3,4) \rangle$	alternating group:A4	12	2	1	1	1	cyclic group:Z2	1	
whole group	$\langle (1,2,3,4), (1,2) \rangle$	symmetric group:S4	24	1	1	1	1	trivial group	0	
Total (11 rows)	**--**	**--**	**--**	**--**	**11**	**--**	**30**	**--**	**--**	**--**

Table classifying isomorphism types of subgroups

Group name	Order	Second part of GAP ID (first part is order)	Occurrences as subgroup	Conjugacy classes of occurrence as subgroup	Occurrences as normal subgroup	Occurrences as characteristic subgroup
Trivial group	1	1	1	1	1	1
Cyclic group:Z2	2	1	9	2	0	0
Cyclic group:Z3	3	1	4	1	0	0
Cyclic group:Z4	4	1	3	1	0	0
Klein four-group	4	2	4	2	1	1
Symmetric group:S3	6	1	4	1	0	0
Dihedral group:D8	8	3	3	1	0	0
Alternating group:A4	12	3	1	1	1	1
Symmetric group:S4	24	12	1	1	1	1
Total	**--**	**--**	**30**	**11**	**4**	**4**

Table listing number of subgroups by order

These numbers satisfy the congruence condition on number of subgroups of given prime power order: the number of subgroups of order p^r for a fixed nonnegative integer r is congruent to 1 mod p. For $p = 2$, this means the number is odd, and for $p = 3$, this means the number is congruent to 1 mod 3 (so it is among 1,4,7,...)

Group order	Occurrences as subgroup	Conjugacy classes of occurrence as subgroup	Occurrences as normal subgroup	Occurrences as characteristic subgroup
1	1	1	1	1
2	9	2	0	0
3	4	1	0	0
4	7	3	1	1
6	4	1	0	0
8	3	1	0	0
12	1	1	1	1
24	1	1	1	1
Total	**30**	**11**	**4**	**4**

Table listing numbers of subgroups by group property

Group property	Occurrences as subgroup	Conjugacy classes of occurrence as subgroup	Occurrences as normal subgroup	Occurrences as characteristic subgroup
Cyclic group	17	5	1	1
Abelian group	21	7	2	2
Nilpotent group	24	8	2	2
Solvable group	30	11	4	4

Table listing numbers of subgroups by subgroup property

Subgroup property	Occurences as subgroup	Conjugacy classes of occurrences as subgroup	Automorphism classes of occurrences as subgroup
Subgroup	30	11	11
Normal subgroup	4	4	4
Characteristic subgroup	4	4	4

Subgroup structure viewed as symmetric group

Classification based on partition given by orbit sizes

For any subgroup of S_4, the natural action on $\{1,2,3,4\}$ induces a partition of the set $\{1,2,3\}$ into orbits, which in turn induces an unordered integer partition of the number 4. Below, we classify this information for the subgroups.

Conjugacy class of subgroups	Size of conjugacy class	Induced partition of 4	Direct product of transitive subgroups on each orbit?	Illustration with representative
trivial subgroup	1	1 + 1 + 1 + 1	Yes	The subgroup fixes each point, so the orbits are singleton subsets.
S2 in S4	6	2 + 1 + 1	Yes	$\{(),(1,2)\}$ has orbits $\{1,2\},\{3\},\{4\}$
subgroup generated by double transposition in S4	3	2 + 2	No	$\{(),(1,2)(3,4)\}$ has orbits $\{1,2\},\{3,4\}$
A3 in S4	4	3 + 1	Yes	$\{(),(1,2,3),(1,3,2)\}$ has orbits $\{1,2,3\},\{4\}$
Z4 in S4	3	4	Yes	The action is a transitive group action, so only one orbit.
normal Klein four-subgroup of S4	1	4	Yes	The action is a transitive group action, so only one orbit.
non-normal Klein four-subgroups of S4	3	2 + 2	Yes	$\langle (1,2),(3,4) \rangle$ has orbits $\{1,2\},\{3,4\}$
S3 in S4	4	3 + 1	Yes	$\langle (1,2,3),(1,2) \rangle$ has orbits $\{1,2,3\},\{4\}$
D8 in S4	3	4	Yes	The action is a transitive group action, so only one orbit.
A4 in S4	1	4	Yes	The action is a transitive group action, so only one orbit.
whole group	1	4	Yes	The action is a transitive group action, so only one orbit.

Category: Subgroup structure of particular groups