Functions

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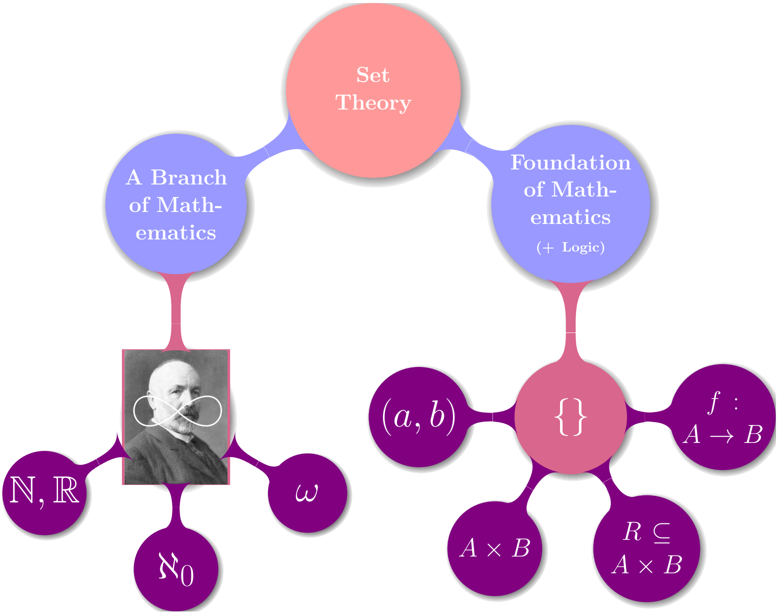
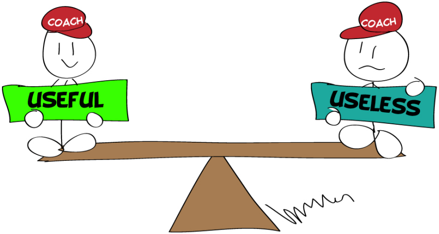

Set Theory
A Branch of Mathematics
Foundation of Mathematics
(+ Logic)
$\mathbb{N}, \mathbb{R}$
$\aleph_0$
ω
(a, b)
$\{\}$
$f : A \to B$
$A \times B$
$R \subseteq A \times B$


COACH
USEFUL
COACH
USELESS

Function

PROOF! PROOF! PROOF!

Definition of Function

Definition (Relation)

Let A and B be sets.

R is a (binary) relation if

$$R \subseteq A \times B = \{(a, b) \mid a \in A \land b \in B\}$$

$$(a, b) = \{\{a\}, \{a, b\}\} \quad \text{(UD Problem 9.16)}$$

Definition (Function)

A **function** f from A to B is a *relation* f from A to B such that

$$\forall a \in A \ \exists! b \in B \ (a, b) \in f.$$

For Proof:

$$\forall$$

$$\exists! : \forall b, b' \in B, (a, b) \in f \land (a, b') \in f \implies b = b'.$$

Notations:

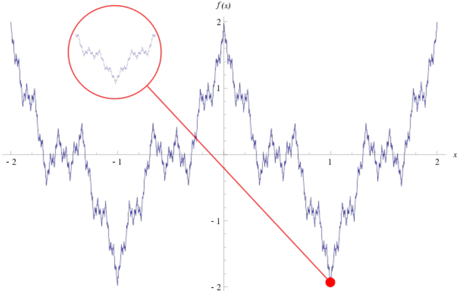
$$f : A \to B, \quad a \mapsto b = f(a)$$

$$A : \mathrm{dom}(f) \qquad B : \mathrm{cod}(f)$$

$$\mathrm{ran}(f) = f(A) = \{f(a) \mid a \in A\} \subseteq B$$

$$D(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Dirichlet Function



Weierstrass Function (1872)

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$

$$0 < a < 1,\ b \in 2\mathbb{N} + 1,\ ab > 1 + \frac{3}{2}\pi$$

UD Problem $13.3\ (g)$

$$f : \mathbb{Q} \to \mathbb{R}$$

$$f(x) = \begin{cases} x + 1 & \text{if } x \in 2\mathbb{Z} \\ x - 1 & \text{if } x \in 3\mathbb{Z} \\ 2 & \text{otherwise} \end{cases}$$

UD Problem 13.4

$$f : \mathcal{P}(\mathbb{R}) \to \mathbb{Z}$$

$$f(A) = \begin{cases} \min(A \cap \mathbb{N}) & \text{if } A \cap \mathbb{N} \neq \emptyset \\ -1 & \text{if } A \cap \mathbb{N} = \emptyset \end{cases}$$

Definition (Axiom of Extensionality (集合的外延公理))

$$\forall A \forall B \forall x (x \in A \iff x \in B) \iff A = B.$$

Definition (函数的外延性原则)

$$f = g \iff \mathrm{dom}(f) = \mathrm{dom}(g) \land \Big(\forall x \in \mathrm{dom}(f) : f(x) = g(x) \Big)$$

Special Functions (*-jectivity*)

Definition (Injective (one-to-one; 1-1) 单射函数)

$$f : A \to B \qquad f : A \rightarrowtail B$$

$$\forall a_1, a_2 \in A : a_1 \neq a_2 \implies f(a_1) \neq f(a_2)$$

For Proof:

- To prove that f *is* 1-1:

$$\forall a_1, a_2 \in A : f(a_1) = f(a_2) \implies a_1 = a_2$$

- To show that f *is not* 1-1:

$$\exists a_1, a_2 \in A : a_1 \neq a_2 \wedge f(a_1) = f(a_2)$$

Definition (Surjective (onto) 满射函数)

$$f : A \to B \qquad f : A \twoheadrightarrow B$$

$$\text{ran}(f) = B$$

For Proof:

- To prove that f *is* onto:

$$\forall b \in B \Big(\exists a \in A : f(a) = b \Big)$$

- To show that f *is not* onto:

$$\exists b \in B \Big(\forall a \in A : f(a) \neq b \Big)$$

Theorem (Cantor Theorem (ES Theorem 24.4))

Let A be a set.
If $f : A \to 2^A$, then f is not onto.

Proof.

Proof. Let A be a set and let $f : A \to 2^A$. To show that f is not onto, we must find a $B \in 2^A$ (i.e., $B \subseteq A$) for which there is no $a \in A$ with $f(a) = B$. In other words, B is a set that f "misses." To this end, let

$$B = \{x \in A : x \notin f(x)\}.$$

We claim there is no $a \in A$ with $f(a) = B$.

Suppose, for the sake of contradiction, there is an $a \in A$ such that $f(a) = B$. We ponder: Is $a \in B$?

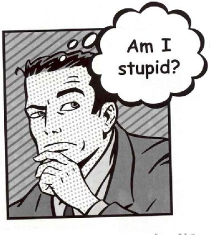
- If $a \in B$, then, since $B = f(a)$, we have $a \in f(a)$. So, by definition of B, $a \notin f(a)$; that is, $a \notin B$. $\Rightarrow\Leftarrow$
- If $a \notin B = f(a)$, then, by definition of B, $a \in B$. $\Rightarrow\Leftarrow$

Both $a \in B$ and $a \notin B$ lead to contradictions, and hence our supposition [there is an $a \in A$ with $f(a) = B$] is false, and therefore f is not onto. ■

□

Theorem (Cantor Theorem)

Let A be a set.
If $f : A \to 2^A$, then f is not onto.



Theorem (Cantor Theorem)

Let A be a set.
If $f : A \to 2^A$, then f is not onto.

Understanding this problem:

$$A = \{1, 2, 3\}$$

2^A

$$2^A = \Big\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\Big\}$$

Onto

$$\forall B \in 2^A \ \Big(\exists a \in A \ f(a) = B\Big).$$

Not Onto

$$\exists B \in 2^A \ \Big(\forall a \in A \ f(a) \neq B\Big).$$

Theorem (Cantor Theorem)

Let A be a set.
If $f : A \to 2^A$, then f is not onto.

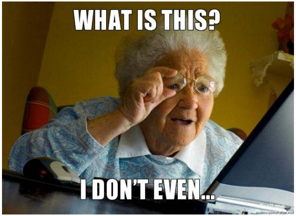
Proof.

- Constructive proof ($\exists$):

$$B = \{x \in A \mid x \notin f(x)\}.$$

- By contradiction ($\forall$):

$$\exists a \in A : f(a) = B.$$



$$Q : a \in B?$$

$\square$

Theorem (Cantor Theorem)

Let A be a set.
If $f : A \to 2^A$, then f is not onto.

对角线论证 (Cantor's diagonal argument) (以下仅适用于可数集合 A).

a	$f(a)$					
	1	2	3	4	5	$\cdots$
1	1	1	0	0	1	$\cdots$
2	0	0	0	0	0	$\cdots$
3	1	0	0	1	0	$\cdots$
4	1	1	1	1	1	$\cdots$
5	0	1	0	1	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$

$$B = \{0, 1, 1, 0, 1\}$$

Definition (Bijective (one-to-one correspondence) 一一对应)

$$f : A \to B \qquad f : A \xleftrightarrow[onto]{1-1} B$$

1-1 & onto

UD Problem 14.12

$$a, b, c, d \in \mathbb{R},\ a < b,\ c < d$$

Define a bijective function:

$$f : [a, b] \xleftrightarrow[onto]{1-1} [c, d]$$

$$f : (a, b) \xleftrightarrow[onto]{1-1} (c, d)$$

Answer.

$$f(x) = c + \frac{d - c}{b - a}(x - a)$$

$\square$

Operations on Functions

Set

$\cup \quad \cap \quad \subseteq$

Relation

$\circ \quad f^{-1}(a) \quad f(A) \& f^{-1}(B)$

Definition (Intersection, Union)

$$f_1, f_2 : A \to B$$

(i) Q : Is $f_1 \cup f_2$ a function from A to B?

(ii) Q : Is $f_1 \cap f_2$ a function from A to B?

Definition (Restriction (UD Problem 15.20))

$$f : A \to B,\ A_0 \subseteq A$$

$$f \mid_{A_0} : A_0 \to B, \qquad f \mid_{A_0} (a) = f(a), \forall a \in A_0$$

Definition (Composition)

$$f : A \to B \qquad g : C \to D$$

$$\mathrm{ran}(f) \subseteq C$$

The composition function

$$g \circ f : A \to D$$

$$(g \circ f)(x) = g(f(x))$$

Non-commutative:

$$f \circ g \neq g \circ f$$

Theorem (Associative Property for Composition)

$$f : A \to B \quad g : B \to C \quad h : C \to D$$

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Proof.

(i)

$$\mathsf{dom}(h \circ (g \circ f)) = \mathsf{dom}((h \circ g) \circ f)$$

(ii)

$$(h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x)$$

□

Theorem (Properties of Composition (UD Theorem 15.7))

$$f : A \to B \qquad g : B \to C$$

(i) *If f, g are injective, then $g \circ f$ is injective.*

(ii) *If f, g are surjective, then $g \circ f$ is surjective.*

(iii) *If f, g are bijective, then $g \circ f$ is bijective.*

Proof for (i).

$$\forall a_1, a_2 \in A \Big((g \circ f)(a_1) = (g \circ f)(a_2) \implies a_1 = a_2 \Big)$$

□

Theorem (Properties of Composition (UD Theorem 15.8))

$$f : A \to B \qquad g : B \to C$$

(i) *If $g \circ f$ is injective, then f is injective.*
(ii) *If $g \circ f$ is surjective, then g is surjective.*
(iii) *If $g \circ f$ is bijective, then f is injective and g is surjective.*

Proof.

Left as an Exercise (15.9).

□

Cancellation Property for Composition (UD Problem 15.11)

$$f : A \to B \qquad g_1, g_2 : B \to A$$

$$f \circ g_1 = f \circ g_2 \land f \text{ is bijective} \implies g_1 = g_2$$

Remark:

f is one-to-one.

Proof.

$$\forall b \in B \Big(f \circ g_1(b) = f \circ g_2(b) \implies \cdots \Big)$$

□

Definition (Inverse)

Let $f : A \to B$ be a bijective function.
The inverse of f is the function $f^{-1} : B \to A$ defined by

$$f^{-1}(b) = a \iff f(a) = b.$$

Q: Why "Bijective"?

Theorem (UD Theorem 15.4 (ii))

$$f : A \to B \text{ is bijective} \implies f^{-1} \text{ is bijective.}$$

Theorem (Solving Equations (UD Theorem 15.4))

$$f : A \to B \textit{ is bijective}$$

(i) $f \circ f^{-1} = i_B$

(ii) $g : B \to A \land f \circ g = i_B \implies g = f^{-1}$

(iii) $f^{-1} \circ f = i_A$

(iv) $g : B \to A \land g \circ f = i_A \implies g = f^{-1}$

Solving the equations:

$$f \circ g = i_B \qquad g \circ f = i_A$$

Bijective $\implies$ Inverse:

$$f : A \to B \text{ is bijective}$$

$$\implies$$

$$\exists g : B \to A \left(f \circ g = i_B \land g \circ f = i_A\right) \land g = f^{-1}$$

Theorem (Inverse $\implies$ Bijective (UD Theorem 15.8 (iii)))

$$\exists g : B \to A \left(g \circ f = i_A \land f \circ g = i_B\right)$$

$$\implies$$

$$f : A \to B \textit{ is bijective} \land g = f^{-1}$$

Theorem (Inverse of Composition (UD Theorem 15.6))

$$f : A \to B, g : B \to C \text{ are bijective}$$

(i) $g \circ f$ *is bijective*

(ii) $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Proof for (ii).

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = i_A$$

$$(g \circ f) \circ (f^{-1} \circ g^{-1}) = i_B$$

□

$$f : A \to B,\ A_0 \subseteq A,\ B_0 \subseteq B$$

Definition (Image)

The image of A_0 under f is the set

$$f(A_0) = \{f(a) \mid a \in A_0\}.$$

Definition (Inverse Image)

The inverse image of B_0 under f is the set

$$f^{-1}(B_0) = \{a \in A \mid f(a) \in B_0\}.$$

Theorem (Properties of f and f^{-1} (UD Theorem 16.7))

$$f : A \to B,\ A_0, A_1, A_2 \subseteq A,\ B_0, B_1, B_2 \subseteq B$$

(i) *f, when applied to subsets of A, preserves only "$\subseteq$" and $\cup$:*

(1) $A_1 \subseteq A_2 \implies f(A_1) \subseteq f(A_2)$

(2) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$

(3) $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$

(4) $f(A \setminus A_0) \neq B \setminus f(A_0)$

(ii) *f^{-1}, when applied to subsets of B, preserves $\subseteq, \cup, \cap$, and $\setminus$:*

(5) $B_1 \subseteq B_2 \implies f^{-1}(B_1) \subseteq f^{-1}(B_2)$

(6) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$

(7) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$

(8) $f^{-1}(B \setminus B_0) = A \setminus f^{-1}(B_0)$

Theorem (Properties of f and f^{-1} (UD Theorem 16.7))

$$f : A \to B,\ A_0 \subseteq A,\ B_0 \subseteq B$$

(iii) *f and f^{-1}:*

(9) $A_0 \subseteq f^{-1}(f(A_0))$

Q: *When is $A_0 = f^{-1}(f(A_0))$?*

(10) $B_0 \subseteq f(f^{-1}(B_0))$

Q: *When is $B_0 = f(f^{-1}(B_0))$?*

UD Problem 16.20

$$f : A \to B, \quad A_1, A_2 \subseteq A$$

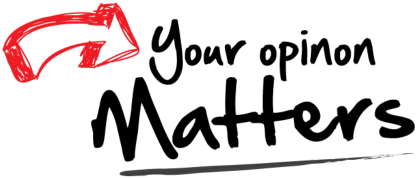
(i) When is $f(A_1) = f(A_2) \implies A_1 = A_2$?

UD Problem 16.21

$$f : A \to B, \quad B_1, B_2 \subseteq B$$

(i) When is $f^{-1}(B_1) = f^{-1}(B_2) \implies B_1 = B_2$?


Thank
You!



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