Bijections from Dyck paths to 321-avoiding permutations revisited

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Abstract

There are (at least) three bijections from Dyck paths to 321-avoiding permutations in the literature, due to Billey-Jockusch-Stanley, Krattenthaler, and Mansour-Deng-Du. How different are they? Denoting them B, K, M respectively, we show that $M = B \circ L = K \circ L'$ where L is the classical Kreweras-Lalanne involution on Dyck paths and L', also an involution, is a sort of derivative of L. Thus $K^{-1} \circ B$, a measure of the difference between B and K, is the product of involutions $L' \circ L$ and turns out to be a very curious bijection: as a permutation on Dyck n-paths it is an nth root of the "reverse path" involution. The proof of this fact boils down to a geometric argument involving pairs of nonintersecting lattice paths.

1 Introduction

Dyck paths and 321-avoiding permutations are two of the many combinatorial manifestations of the Catalan numbers [1, Ex. 6.19]. There are at least three different bijections in the literature from Dyck paths to 321-avoiding permutations, due to Billey-Jockusch-Stanley [2] (1993), Krattenthaler [3] (2001) and Mansour-Deng-Du [4] (2006). We denote them B, K, M respectively. (Krattenthaler actually gave a bijection to 123-avoiding permutations; K is the equivalent bijection to 321-avoiding permutations.) There is also a classical involution L on Dyck paths dating back to 1970 due to Germain Kreweras [5] and discussed by J.C. Lalanne in 1992-93 [6, 7]. In this paper we will show that the following relationships hold between them:

$$M = B \circ L = K \circ L' \tag{1}$$

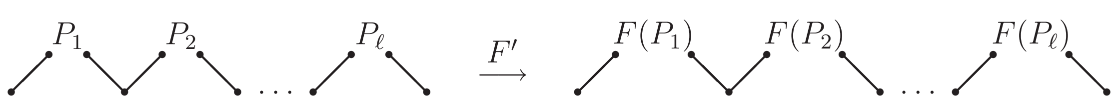
where L' is the "first derivative" (defined below) of L and, like L, is an involution. We will also see that the bijection $K^{-1} \circ B = L' \circ L$, considered as a permutation of Dyck n-paths, has order $2n$, a consequence of the fact that its nth power reverses the path.

The outline of the paper is as follows. In §2, we review Dyck path terminology and introduce the notion of the derivative F' of a mapping F on Dyck paths. Section 3 reviews the ascent-descent code for a Dyck path. Section 4 reviews the left-to-right-maxima and excedance codes for a 321-avoiding permutation. Section 5 describes the involution L. Section 6 translates L to a simpler setting—pairs of nonintersecting lattice paths—and describes L' and $L' \circ L$ in this setting. Section 7 describes the bijections B, K and M. Section 8 then establishes the identities (1) relating B, K and M. Section 9 uses a geometric argument on nonintersecting path pairs to analyze the composition $K^{-1} \circ B = L' \circ L$.

Astrid Reifegerste [8] has also considered bijections involving permutations that avoid a 3-letter pattern and connections between them, and some of our observations regarding "codes" in §3 and §4 can be found in her paper.

2 The derivative of a mapping on Dyck paths

The set $\mathcal{D}$ of Dyck paths is the set of lattice paths consisting of an equal number of upsteps $u = (1, 1)$ and downsteps $d = (1, -1)$ that never dip below ground level, the horizontal line connecting its endpoints. The *size* or *semilength* of a Dyck path is its number of upsteps. A Dyck n-path is one of size n. An *ascent* is a maximal sequence of contiguous upsteps and analogously for a *descent*. A *peak* vertex is one preceded by a u and followed by a d, and a *valley* vertex is defined analogously. An *elevated* Dyck path is a nonempty Dyck path whose only return to ground level occurs at the end. The empty Dyck path is denoted ϵ. Every nonempty Dyck path decomposes uniquely into a concatenation of elevated Dyck paths, called its *components*. For a size-preserving bijection $F : \mathcal{D} \to \mathcal{D}$, its *derivative* F' is defined by applying F to the "elevated" portion of each component (and $F'(\epsilon) := \epsilon$). Schematically,



Clearly, F' is a bijection on $\mathcal{D}$ that preserves not only size but also number of components and their sizes, and if F is an involution, then so is F'.

3 The ascent-descent code of a Dyck n-path

A Dyck path is specified by the lengths of its ascents and descents. For example, the path *uuduuuudddduddudd* has ascent sequence $\mathsf{a} = (\mathsf{a}_i)_{i=1}^k = (2,4,1,1)$ and descent sequence $\mathsf{d} = (\mathsf{d}_i)_{i=1}^k = (1,3,2,2)$ where k is the number of peaks (uds). By definition of Dyck path, each partial sum of the ascent lengths $\mathsf{A}_i := \sum_{j=1}^i \mathsf{a}_j$ is $\geq$ the corresponding partial sum of the descent lengths $\mathsf{D}_i := \sum_{j=1}^i \mathsf{d}_j$. For a Dyck n-path, we necessarily have $\mathsf{A}_k = \mathsf{D}_k = n$ and so the path is determined by the pair $(\mathsf{A}_i)_{i=1}^r,\ (\mathsf{D}_i)_{i=1}^r$ where $r := k-1$ and we call this pair the (truncated) *partial-sum ascent-descent code* of the path. The preceding example has $n = 8,\ k = 4,\ r = 3$, and $\mathsf{A} = (2,6,7), \mathsf{D} = (1,4,6)$. The precise requirements for a valid partial-sum ascent-descent code for a Dyck path of size n are then

$$\begin{gathered} 0 \leq r \leq n-1, \\ 1 \leq \mathsf{A}_1 < \mathsf{A}_2 < \ldots < \mathsf{A}_r \leq n-1, \\ 1 \leq \mathsf{D}_1 < \mathsf{D}_2 < \ldots < \mathsf{D}_r \leq n-1, \\ \mathsf{A}_i \geq \mathsf{D}_i \text{ for } 1 \leq i \leq r. \end{gathered} \tag{2}$$

Note that the "pyramid" path $u^n d^n$, where exponents denote repetition, is the only one with $r = 0$, and its code consists of two empty sequences.

4 Codes for 321-avoiding permutations

A permutation π on $[n]$ has a left-to-right-maxima decomposition as $m_1 L_1 m_2 L_2 \ldots m_k L_k$ where $m_1, m_2, \ldots, m_k$ are the left-to-right maxima of π. For example with $n = 9$,

$$\begin{array}{c|cc|c|cc|c|c|c} {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle 3} & {\scriptstyle 4} & {\scriptstyle 5} & {\scriptstyle 6} & {\scriptstyle 7} & {\scriptstyle 8} & {\scriptstyle 9} \\ 4 & 1 & 3 & 7 & 2 & 5 & 8 & 9 & 6 \\ m_1 & \multicolumn{2}{c|}{L_1} & m_2 & \multicolumn{2}{c|}{L_2} & m_3 & m_4 & L_4 \end{array}$$

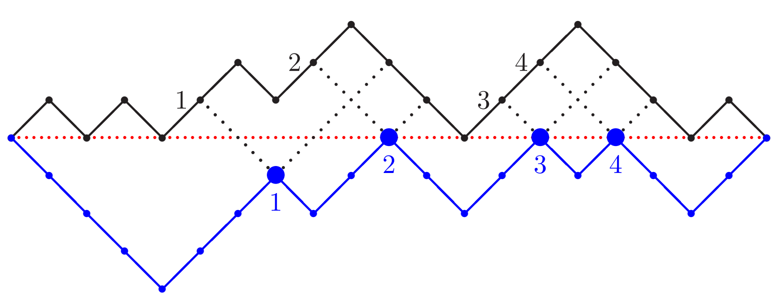
Here, the left-to-right maxima are $4,7,8,9$ and L_3 is empty. Let's call the left-to-right maxima $m = (m_i)_{i=1}^k$ and their positions $p = (p_i)_{i=1}^k$ the LRMax skeleton of π. In the example, $m = (4,7,8,9)$ and $p = (1,4,7,8)$. It is easy to see that a permutation π on $[n]$ is 321-avoiding if and only if the concatenated list $L_1 L_2 \ldots L_r$ is increasing. *Thus a* 321-*avoiding permutation is determined by its LRMax skeleton.* There are two obvious restrictions on the LRMax skeleton: $m_k = n$ and $p_1 = 1$. Delete these entries and, to make things nice, subtract 1 from each remaining entry in p and call the resulting pair, say $(\mathsf{A}_i)_{i=1}^r,\ (\mathsf{D}_i)_{i=1}^r$ where $r := k-1$, the *LRMax code* of the 321-avoiding permutation on $[n]$. Note that, for $1 \le i \le n-1$, $p_{i+1} \le m_i + 1$ (otherwise the first $m_i + 1$ entries would all have to be $\le m_i$, violating the pigeon-hole principle). So, since $\mathsf{A}_i = m_i$ and $\mathsf{D}_i = p_{i+1} - 1$, the requirements for a valid LRMax code for a 321-avoiding permutation on $[n]$ are precisely those in (2). This fact is the basis for Krattenthaler's bijection [3].

An *excedance location* of a permutation π on $[n]$ is an $i \in [n-1]$ for which $\pi(i) > i$ (and $\pi(i)$ is then the corresponding *excedance value*) and a weak excedance refers to an $i \in [n]$ for which $\pi(i) \ge i$. Thus the set of weak excedance locations is the disjoint union of the excedance locations and the fixed points. Now a 321-avoiding permutation on $[n]$ has the following property: if $[n]$ is split into intervals by the fixed points f_i of π so that $[n]$ is the concatenation $I_0\, f_1\, I_1\, f_2\, \ldots\, f_q\, I_q$, then π preserves each interval I_i. For a 321-avoiding permutation π on $[n]$, it follows that the left-to-right-maxima coincide with the weak excedance values and that the permutation is determined just by its (strict) excedance values $v = (v_i)$ and locations $\ell = (\ell_i)$. In other words, in the LRMax skeleton of a 321-avoiding permutation π on $[n]$ the fixed points can safely be omitted at the expense of preserving n and $\pi^{-1}(n)$ (unless n is a fixed point). Since each v_i is ≥ 2, let us again subtract 1 to make things nice and call the result—$\mathsf{A} := v - 1$, $\mathsf{D} := \ell$—the *excedance code* for π. Again, the requirements for a valid excedance code are the same as in (2); this is the basis for the Billey-Jockusch-Stanley bijection [2].

5 The Lalanne-Kreweras involution on Dyck paths

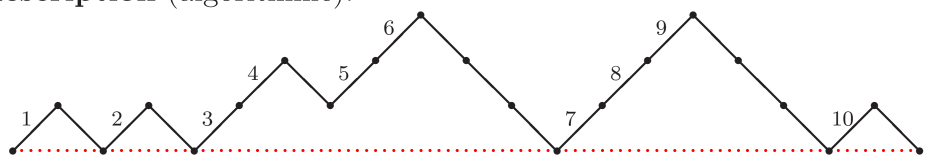
We give two descriptions, illustrated with the same example.

First description [5, 6, 7] (graphical):



Draw a southeast line from the midpoint of each uu and a southwest line from the midpoint of each dd. There will be the same number of each. Mark the point of intersection of the ith southeast and the ith southwest line for each i. Then form the unique (inverted) Dyck path with (inverted) valleys at the marked points, as shown in blue (below ground level) above.

Second description (algorithmic):



(3)

Label the upsteps left to right. Record the label on the first u of each occurrence of uu. The example gives $(3, 5, 7, 8)$. Call this vector D. Do likewise for the downsteps. The example gives $(4, 5, 7, 8)$. Call this vector A. Then form the Dyck path whose partial-sum ascent-descent code is (A, D). (The reader may check that (A, D) satisfy the defining conditions (2) with n the size of the path). The example gives ascent lengths 4,1,2,1,2 and descent lengths 3,2,2,1,2.

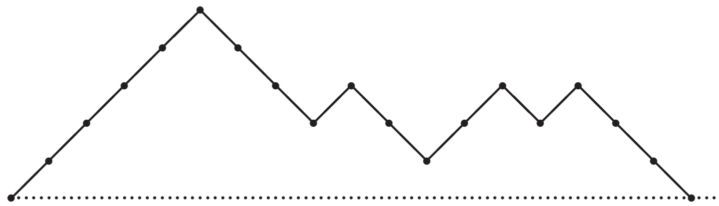
In the next section, following Emeric Deutsch [9], we use a suitable bijection to identify Dyck paths with another manifestation of the Catalan numbers, nonintersecting path pairs (parallelogram polyominoes). In this setting L has perhaps its simplest possible description: flip the path pair in a 45° line. Also, the "reverse path" involution R on Dyck paths translates to "rotate path pair 180°".

6 L, L', $L \circ L'$, and R on Path Pairs

A (*nonintersecting*) *path pair* is an ordered pair (P_1, P_2) of paths of unit steps north, $N = (1, 0)$, and east, $E = (0, 1)$, that intersect only at the initial and terminal points and such that P_1 (the upper path) lies above P_2. The size of a path pair is the number of steps in each path, necessarily the same. The region enclosed by a path pair is known as a parallelogram polyomino.

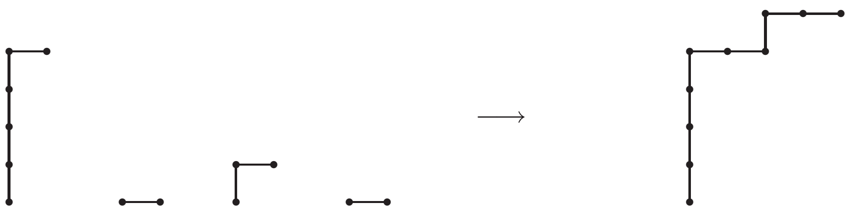
There is a well known bijection [10, p. 182][1, Ex. 6.19(ℓ)] which we will use to identify Dyck paths of size n with path pairs of size $n+1$. An equivalent bijection (up to reversing Dyck paths and rotating path pairs) has been given by Sulanke [11, p. 295]. Here is the bijection (with a slightly simplified description).

Given a Dyck path, first elevate it, that is, prepend u and append d.



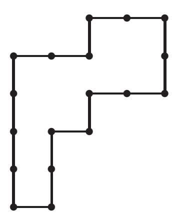
elevated Dyck path

Then extract the elevated path's ascents as N steps except that the last step in each ascent is rendered as an E step, and concatenate:



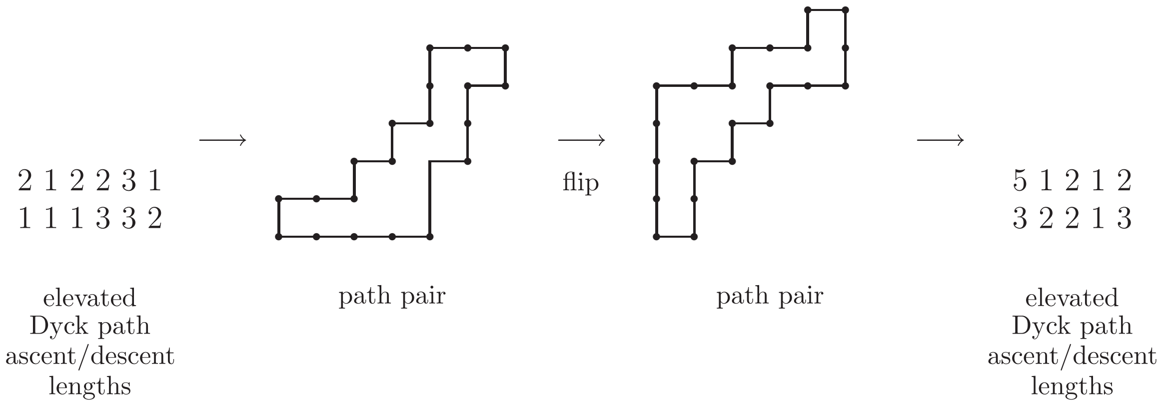
This is the upper path.

Do likewise for the descents to get a path, X say, and then *transfer the last step, necessarily an E, to the start*. This gives the lower path and the resulting path pair is

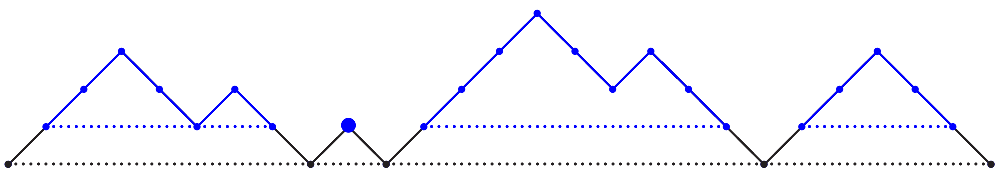


If we let a_i denote the length of the ith ascent in the elevated Dyck path and d_i the length of the ith descent for $1 \le i \le k$, where $k = \#$ peaks ($= \#\,$ascents $= \#\,$descents), then, since the path is elevated, $\sum_{i=1}^{j} \mathsf{a}_i > \sum_{i=1}^{j} \mathsf{d}_i$ for $j = 1, 2, \ldots, k-1$, and hence the ith E step in the upper path lies strictly above the ith E step in the path X for $i < k$. This ensures that the resulting path pair is nonintersecting and the mapping is clearly invertible. Let us call this bijection ϕ.

Using ϕ to identify Dyck paths and nonintersecting path pairs, the Kreweras-Lalanne involution L simplifies to "flip path pair in a 45° line". Again using the Dyck path (3) from §5 to illustrate,



To see the effect of L' on a path pair $P = \phi(D)$ where D is a Dyck path, we need to identify within P the interior of each component of D (in blue below), and this is easy to do.

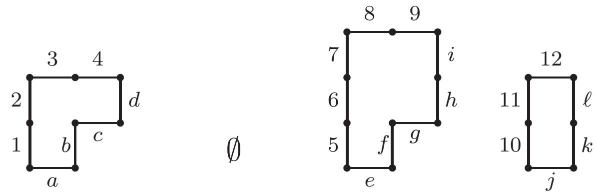


Dyck path D

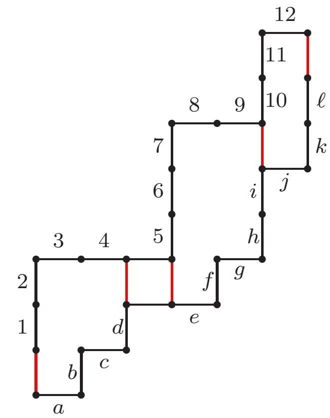
The components of D are determined by the points of contact of D with ground level. These points, including the initial and terminal points, correspond to unit vertical segments (in red in the figure below right) joining a vertex of the upper path to a vertex of the lower path. Furthermore, each hill (ud pair at ground level) in D corresponds to a pair of steps in P that form horizontal sides of a unit square. Keep in mind that ϕ sends a Dyck n-path to a path pair of size $n+1$ *except when* $n = 0$: the empty Dyck path corresponds to the empty path pair. So we can expect a hill in D—a component with empty interior—to exhibit singular behavior under L'. Indeed, the path pair corresponding to the interior of each component of D can be seen in $P = \phi(D)$ as in the illustration, where numerals label steps in each upper path and letters in each lower path, and hills in D show up in P as unlabeled unit squares bounded above by P_1 and below by P_2.

The ascent lengths a and descent lengths d of the components of D are just what is needed to construct the path pairs for their interiors

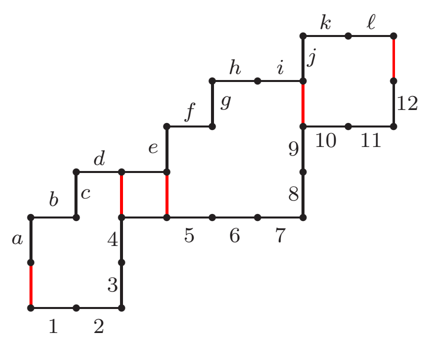
a :	3	1	1	4	1	3
d :	2	2	1	2	3	3



path pair $P = \phi(D)$



Since L flips a path pair, the effect of L' is to flip in a 45° line the pairs of labeled segments separated by red lines while preserving the red lines and unlabeled unit squares. The example yields



$L'(P)$

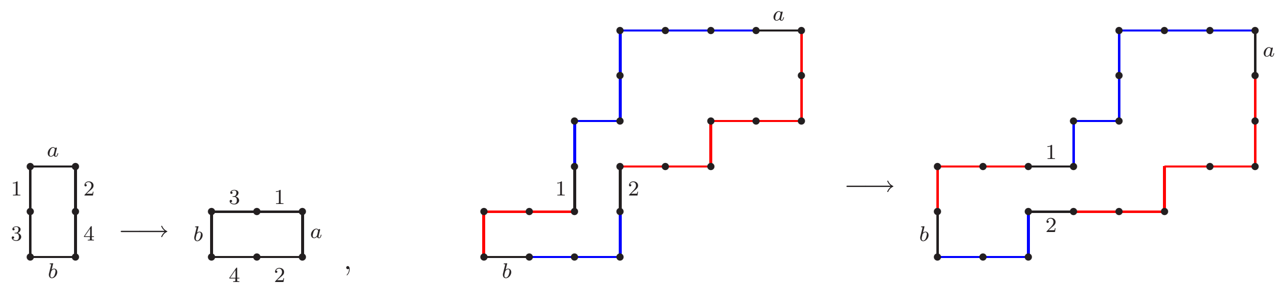
where labels and interior red lines are included for clarity.

Now, since L flips the entire path pair and L' flips a good deal of it, the composition $L' \circ L$ merely tweaks it: given a path pair P, to obtain $L' \circ L(P)$

(i) identify the last (northeasternmost) step in the upper path and the southwesternmost step in the lower path. Both these steps are necessarily flat.

(ii) identify each pair of *vertical* steps that form opposite sides of a unit square.

Then change the two steps in (i) from flat to vertical and all steps in (ii) (if any) from vertical to flat. Two examples are shown below (unchanged steps in color, labels for clarity). Note that step (ii) ensures the resulting path pair is nonintersecting.



effect of $L' \circ L$ on path pairs

Finally, it is clear that reversing a Dyck path, which interchanges the roles of upsteps and downsteps, corresponds under ϕ to rotating a path pair 180°.

7 The bijections B, K and M

The Billey-Jockusch-Stanley bijection B from Dyck paths to 321-avoiding permutations can now be simply described: form the partial-sum ascent-descent code (A, D) of the Dyck path and then use it as the excedance code of a 321-avoiding permutation. For example, the Dyck path (3) of §5 has size $n = 10$, ascent lengths $\mathsf{a} = (1, 1, 2, 2, 3, 1)$ and descent lengths $\mathsf{d} = (1, 1, 1, 3, 3, 1)$ so that $\mathsf{A} = (1, 2, 4, 6, 9)$, $\mathsf{D} = (1, 2, 3, 6, 9)$. With (A, D) as excedance code, $\mathsf{A} + 1$ gives the excedance values and D the excedance locations. We thus immediately have the following partial permutation

$$\begin{array}{cccccccccc} {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle 3} & {\scriptstyle 4} & {\scriptstyle 5} & {\scriptstyle 6} & {\scriptstyle 7} & {\scriptstyle 8} & {\scriptstyle 9} & {\scriptstyle 10} \\ 2 & 3 & 5 & & & 7 & & & 10 & \end{array}$$

and filling in the missing entries in increasing order gives the image permutation: 2 3 5 1 4 7 6 8 10 9.

The Krattenthaler bijection K uses the partial-sum ascent-descent code as the LRMax code of a 321-avoiding permutation. Using the same Dyck path to illustrate, again $\mathsf{A} = (1, 2, 4, 6, 9)$, $\mathsf{D} = (1, 2, 3, 6, 9)$. With (A, D) as LRMax code, the left-to-right-maxima are given by A with n appended, their positions by $\mathsf{D} + 1$ with 1 prepended. Thus we have the partial permutation

$$\begin{array}{cccccccccc} {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle 3} & {\scriptstyle 4} & {\scriptstyle 5} & {\scriptstyle 6} & {\scriptstyle 7} & {\scriptstyle 8} & {\scriptstyle 9} & {\scriptstyle 10} \\ 1 & 2 & 4 & 6 & & & 9 & & & 10 \end{array}$$

and filling in the missing entries in increasing order gives the image permutation: 1 2 4 6 3 5 9 7 8 10.

The Mansour-Deng-Du bijection M is a bit more complicated and here we attempt to simplify its description, referring the reader to [4] for full details of the original description.

First label the upsteps of the Dyck path left to right and record the label on the first u of each uu. Do likewise for the downsteps. For our running example (3), as already noted in §5, the result is $(4,5,7,8)$ for the downsteps and $(3,5,7,8)$ for the upsteps and this pair forms the partial-sum ascent-descent code for $L(P)$. In [4] this pair is denoted (h,t) and is obtained by a different but equivalent process: a graphical construction based on the so-called $(x+y)$-*labelling* of a Dyck path. Next, [4] defines $\sigma_i := s_{h_i} s_{h_i-1} s_{h_i-2} \dots s_{t_i}$ where s_j is the transposition that interchanges j and $j+1$, and goes on to form the image permutation as

$$(1,2,\ \dots,n)\sigma_1\sigma_2\dots\sigma_r,$$

where r is the length of h (and t) and operations are performed left to right. The effect of these operations is simply to displace h_i+1 to the left in the list $(1,2,\ \dots,n)$ so that it is in position t_i, this for each i while leaving all other entries in the same relative order. A little thought shows that this is equivalent to using h and t as the excedance code to produce the image permutation. The example thus yields excedance values $h+1=(5,6,8,9)$ and excedance locations $t=(3,5,7,8)$, and so the image permutation is 1 2 5 3 6 4 8 9 7 10.

8 The identities $M = B \circ L = K \circ L'$

It is now clear that $M = B \circ L$ because, as we have just seen, for a Dyck path P, the excedance code of $M(P)$ is the partial-sum ascent-descent code of $L(P)$ and the bijection B uses the latter code as an excedance code. To see that $M = K \circ L'$, equivalently, $K^{-1} \circ B = L' \circ L$, requires a little more work.

From the descriptions of K and B in the preceding section, we see that the following 4-step process transforms the LRMax code of a 321-avoiding permutation to its excedance code (writing the codes as 2-row matrices with the larger row on top):

1. append n to the top row and and prepend 0 to the bottom row
2. add 1 to each entry of the bottom row
3. delete columns with same top and bottom entry
4. subtract 1 from each entry of the top row.

For example, with $n=13$,

$$\begin{pmatrix} 2 & 3 & 4 & 8 & 9 & 12 \\ 1 & 3 & 4 & 6 & 7 & 10 \end{pmatrix} \xrightarrow{(1)} \begin{pmatrix} 2 & 3 & 4 & 8 & 9 & 12 & 13 \\ 0 & 1 & 3 & 4 & 6 & 7 & 10 \end{pmatrix} \xrightarrow{(2)} \begin{pmatrix} 2 & 3 & 4 & 8 & 9 & 12 & 13 \\ 1 & 2 & 4 & 5 & 7 & 8 & 11 \end{pmatrix}$$

$$\xrightarrow{(3)} \begin{pmatrix} 2 & 3 & 8 & 9 & 12 & 13 \\ 1 & 2 & 5 & 7 & 8 & 11 \end{pmatrix} \xrightarrow{(4)} \begin{pmatrix} 1 & 2 & 7 & 8 & 11 & 12 \\ 1 & 2 & 5 & 7 & 8 & 11 \end{pmatrix} \tag{4}$$

If P is a 321-avoiding permutation and D_1, D_2 are the Dyck paths corresponding to its LRMax and excedance codes respectively, then $K(D_1) = B(D_2)$ and so $D_2 = B^{-1} \circ K(D_1)$. We wish to trace the effects of the these 4 steps on the Dyck path D_1 and show that they produce $L \circ L'(D_1)$; we can then conclude that $B^{-1} \circ K = L \circ L'$ or, taking inverses, that $K^{-1} \circ B = L' \circ L$, as desired.

The trick is to translate Dyck paths to path pairs using ϕ. The composite bijection "partial-sum ascent-descent code $\to$ Dyck path $\to$ path pair" has a simple description as illustrated for the first entry in (4), with $n = 13$:

partial-sum ascent-descent code A,D

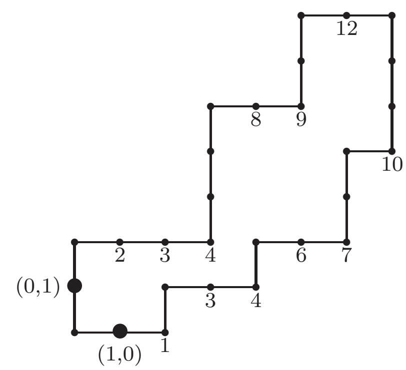
A	2 3 4 8 9 12
D	1 3 4 6 7 10

ascent-descent lengths a,d

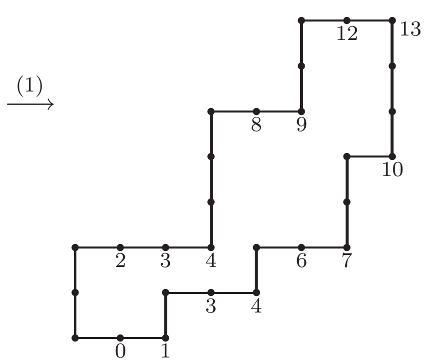
a	2 1 1 4 1 3 1
d	1 2 1 2 1 3 4

A,D show up in path pair as labels on endpoints of interior flat steps counting # steps from (0,1) in upper path, and from (1,0) in lower path

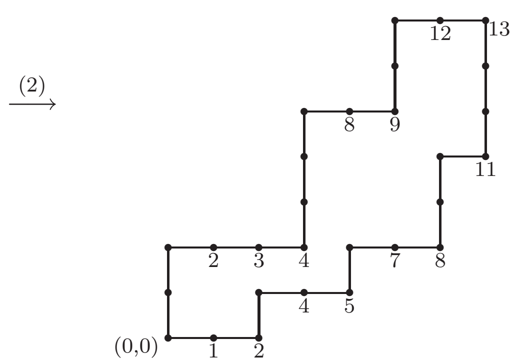
path pair



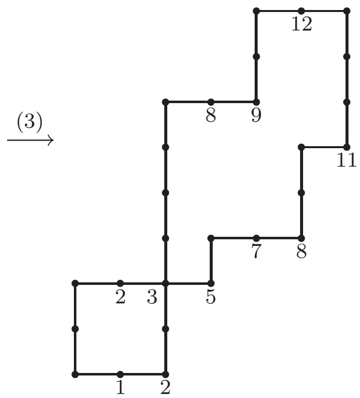
Now we can describe the effect of the 4-step process on the path pair:



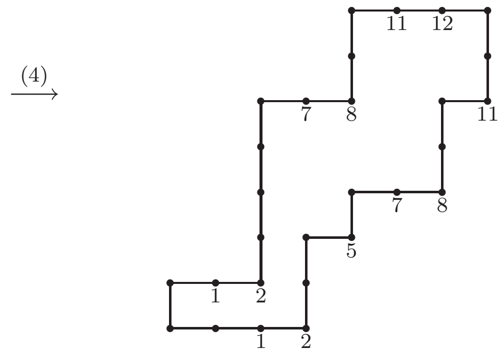
(1) inserts label 0 in lower path, n in upper path



(2) changes labeling on lower path so that steps are counted from the origin $(0, 0)$



(3) swings flat steps forming
sides of unit square
to vertical steps



(4) subtracts 1 from each label in upper path,
and to restore the counting of steps from
(0,1) in the upper path and from (1,0) in
the lower path amounts to rotating the initial
(vertical) step in each path 90° counterclockwise

It is evident that the final result is indeed $L \circ L'$ applied to the initial path pair because the initial path pair is obtained from the final path pair by applying $L' \circ L$ as described in §6. Thus we have shown that $K^{-1} \circ B = L' \circ L$.

9 Analysis of $K^{-1} \circ B = L' \circ L$.

Recall that R is the "reverse path" involution on Dyck paths and is also, under ϕ, the "rotate 180°" involution on path pairs.

Theorem. *On Dyck n-paths,*

$$(L' \circ L)^n = R.$$

Corollary. *For $n \geq 3$, the permutation $L' \circ L$ on Dyck n-paths has order $2n$.*

Proof of Corollary Since R is an involution, the theorem shows that the order of $L' \circ L$ divides $2n$. The assertion can be checked directly for $n = 3$ and for $n \geq 4$, the orbit of the Dyck path $u^{n-1}d^{n-1}ud$ (exponents denote repetition) has size $2n$. □

Proof of Theorem We will consider the effect of repeated application of $L' \circ L$ on a path pair P of size $n+1$. Recall from §6 that $L' \circ L(P)$ is obtained as follows:

(i) identify the last (northeasternmost) step in the upper path and the southwesternmost step in the lower path.

(ii) identify each pair of vertical steps that form opposite sides of a unit square.

Then change the two steps in (i) from flat to vertical and all steps in (ii) (if any) from vertical to flat.

Now consider a path pair as a linkage composed of rods of unit length that must always be aligned either flat or vertical, hinged at the vertices. Applying $L' \circ L$ then simply changes the alignment of some of the rods (steps) but preserves their identity, that is, one may track the progress of a particular step or vertex under repeated applications of $L' \circ L$.

Let us count steps in a path pair clockwise from the origin. Thus the first and $(n+2)$nd steps initiate the upper and lower paths respectively and both are necessarily vertical. If a step S is the ith step in a path pair P, then S becomes step number $i+1 \pmod{2n+2}$ in $L' \circ L(P)$. In particular, under $(L' \circ L)^n$, the initial steps in the upper and lower paths become their terminal steps respectively while every other step passes from its original path to the other one. When it does so, we will say it "turns the corner".

It is clear that, under repeated applications of $L' \circ L$, each vertical step must get flattened before it turns the corner and, once flat, a step stays flat until it turns the corner (when, of course, it becomes vertical). So the crux of the matter is whether or not a step gets flattened *after* it turns the corner. To show that the effect of $(L' \circ L)^n$ is to rotate a path pair 180°, we must show

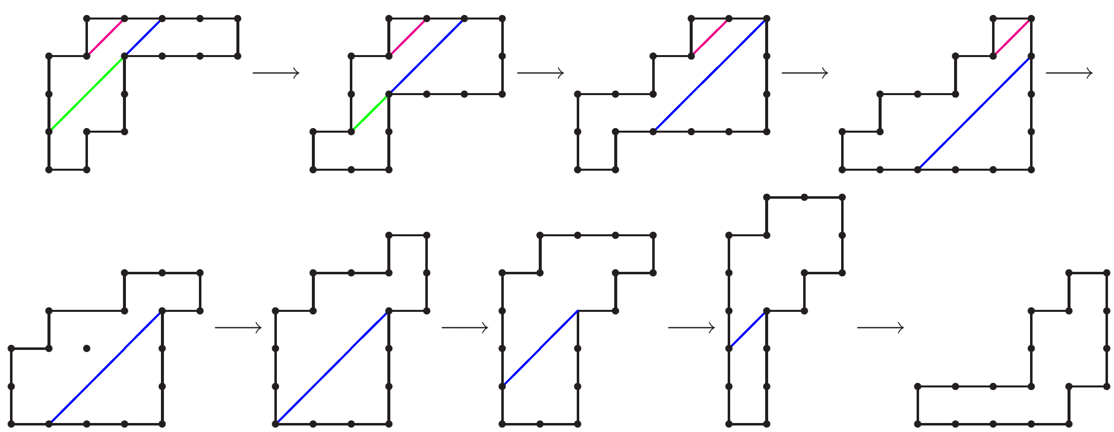
Proposition. *Let P be a path pair of size $n+1$. Under n applications of $L' \circ L$, a step in P gets flattened after it turns the corner if and only if it is immediately preceded by a flat step in the original path pair.*

Proof A *minimal diagonal* in a path pair is a line segment of slope 1 (45°) joining two distinct vertices (either in the same or different paths) and lying strictly inside the path pair except at its endpoints. A simple count shows that there are exactly n minimal diagonals in a path pair of size $n+1$. Given a minimal diagonal, let $V_1 < V_2$ (counting clockwise) denote its endpoints and let S_1, S_2 denote the steps initiated (clockwise) by V_1, V_2 respectively. The key observation is that as V_1, V_2 progress under repeated applications of $L' \circ L$, they remain endpoints of a minimal diagonal until S_1, S_2 form the vertical sides of a unit square. As the next paragraph shows, this will always happen at $(L' \circ L)^i(P)$ for some i, $0 \le i \le n-1$. The next application of $L' \circ L$ will then flatten both S_1 and S_2. Furthermore, by tracing backwards, every instance of a pair of steps forming vertical sides of a unit square in the set $\{(L' \circ L)^i(P)\}_{i=0}^{n-1}$ arises in this way from a minimal diagonal of P.

Applying $L' \circ L$ changes the length of a minimal diagonal by at most 1. Specifically, if V_1, V_2 are interior points of different paths and $\overrightarrow{V_1 V_2}$ points southwest, the length increases by 1; if V_1, V_2 are in the same path and V_2 is not the path's terminal point, the length stays the same; otherwise, the length decreases by 1. It follows that a minimal diagonal can survive at most $n-1$ applications of $L' \circ L$ before being destroyed at the next application.

If (initially) S_1 lies in the upper path and S_2 in the lower path, then both are vertical and get flattened before either turns the corner. If S_1 lies in the lower path and S_2 in the upper, then each is preceded by a flat step and flattening occurs after both S_1 and S_2 have turned the corner. If S_1, S_2 both lie in the same path (either upper or lower), then S_1 is vertical, S_2 is preceded by a flat step and flattening occurs before S_1 turns the corner and after S_2 does so. The Proposition follows.

An example with $n = 8$ is shown along with the progress of 3 of the 8 minimal diagonals, using a different color for each one.



That $K^{-1} \circ B$ turns out to be a product of two "nice" involutions may be somewhat unexpected but recall that every (ordinary) permutation can be expressed as a product of two involutions [12].

References

[1] Richard P. Stanley, *Enumerative Combinatorics* Vol. 2, Cambridge University Press, 1999. Exercise 6.19 and related material on Catalan numbers are available online at http://www-math.mit.edu/~rstan/ec/ .

[2] S. Billey, W. Jockusch and R. Stanley, Some Combinatorial Properties of Schubert Polynomials, *J. Algebraic Combinatorics* **2** (1993), Issue 4, 345–374.

[3] Christian Krattenthaler, Permutations with restricted patterns and Dyck paths, *Advances in Applied Math.* **27** (2001), no. 2-3, 510–530. http://www.mat.univie.ac.at/ kratt/artikel/catperm.html

[4] Toufik Mansour, Eva Y. P. Deng and Rosena R. X. Du, Dyck paths and restricted permutations, *Discrete Applied Math.* **154** (2006), no. 11, 1593–1605. http://www.combinatorics.net.cn/research/Papers_GetFile.aspx?paperID=185

[5] Germain Kreweras, Sur les éventails de segments, *Cahiers du B.U.R.O.* **15** (1970), 3–41.

[6] J.C. Lalanne, Une involution sur les chemins de Dyck, *Europ. J. Combinatorics* **13** (1992), 471–487.

[7] J.C. Lalanne, Sur une involution sur les chemins de Dyck, *Theoretical Comp. Sci.* **117** (1993), 203–215.

[8] Astrid Reifegerste, On the diagram of 132-avoiding permutations, *European J. Combin.* **24** (2003), no. 6, 759–776, http://arxiv.org/abs/math.CO/0208006.

[9] Emeric Deutsch, personal communication, 1999.

[10] Marie-Pierre Delest and Gerard Viennot, Algebraic languages and polyominoes enumeration, *Theoretical Comp. Sci.* **34** (1984) 169–206.

[11] Robert A. Sulanke, A symmetric variation of a distribution of Kreweras and Poupard, *J. Stat. Planning and Inference* **34** (1993) 291–303.

[12] Gap Forum Archive, 1998. http://www.gap-system.org/ForumArchive/Pueschel.1/Markus.1/Re__Fact.7/1.html