2-8 Probabilistic Analysis

"No Expectation, No Disappointment."

Hengfeng Wei

hfwei@nju.edu.cn

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Definition (Expectation)

$$\mathbb{E}[X] = \sum_x x \Pr(X = x)$$

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Let X be a discrete random variable that takes on only nonnegative integer values $\mathbb{N}$.

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Proof.

$$\sum_{j=1}^{\infty} \sum_{i=1}^{j} \Pr(X = j) = \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} \Pr(X = j)$$

□

Searching an Unsorted Array (CLRS Problem $5-2$ (f))

```
procedure DETERMINISTIC-SEARCH(A[1 ··· n], x)
    i ← 1
    while i ≤ n do
        if A[i] = x then
            return true
        i ← i + 1
    return false
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(e)

$$\exists!\ i : A[i] = x$$

(f)

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\mathbb{E}[Y] &= \sum_{i=1}^{n} i \Pr\{Y = i\} \\
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\end{aligned}$$

$$k = 1 \implies \mathbb{E}[Y] = \frac{n+1}{2}, \qquad k = n \implies \mathbb{E}[Y] = 1$$

$$\sum_{i=1}^{n-k+1} i \binom{n-i}{k-1} = \binom{n+1}{k+1}$$

Summation by parts (Abel transformation; wiki)

How Did I (an ant) Evaluate this Summation:

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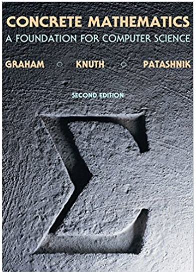

Chapter 5: Binomial Coefficients

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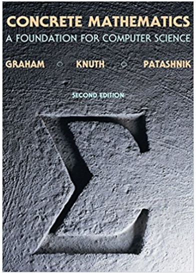

$$r\binom{r-1}{k-1} = k\binom{r}{k}$$

$$\sum_{0 \le k \le n} \binom{k}{m} = \binom{n+1}{m+1}$$

Chapter 5: Binomial Coefficients

$$
\begin{aligned}
\sum_{i=1}^{n-k+1} i\binom{n-i}{k-1} &= \sum_{i=0}^{n-k}(i+1)\binom{n-i-1}{k-1} \\
&= \sum_{i=0}^{n-k}\left((n+1)-(n-i)\right)\binom{n-i-1}{k-1} \\
&= \sum_{i=0}^{n-k}(n+1)\binom{n-i-1}{k-1} - \sum_{i=0}^{n-k}(n-i)\binom{n-i-1}{k-1} \\
&= (n+1)\sum_{i=0}^{n-k}\binom{n-i-1}{k-1} - k\sum_{i=0}^{n-k}\binom{n-i}{k} \\
&= (n+1)\sum_{m=k-1}^{n-1}\binom{m}{k-1} - k\sum_{m=k}^{n}\binom{m}{k} \\
&= (n+1)\binom{n}{k} - k\binom{n+1}{k+1} = \binom{n+1}{k+1}
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$$\Pr\{I_i = 1\} = \begin{cases} \frac{1}{k}, & \text{if } A[i] = x \\ \frac{1}{k+1}, & \text{if } A[i] \neq x \end{cases}$$

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$$\mathbb{E}[Y] = \sum_{i=1}^{n} \Pr\{I_i = 1\} = k \cdot \frac{1}{k} + (n-k) \cdot \frac{1}{k+1} = \frac{n+1}{k+1}$$


Wrong!

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X : $\#$ of customers who get back their own hat $\qquad \mathbb{E}[X]$

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Inversions (CLRS Problem $5.2-5$)

$A[1 \cdots n]$ of n distinct numbers

(i, j) is an inversion of $A : i < j \wedge A[i] > A[j]$

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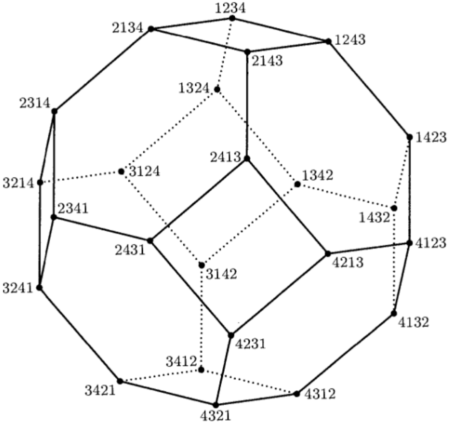

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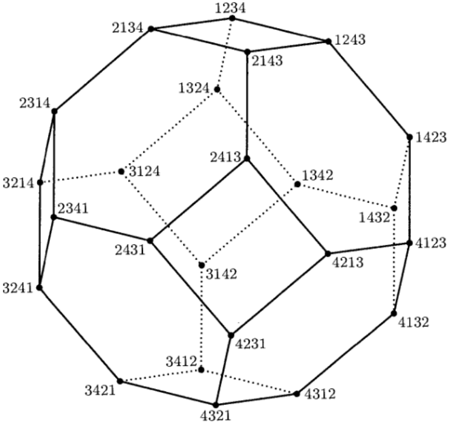

$$\langle 3214 \rangle \sim \langle 4123 \rangle$$

Definition (Conditional Expectation on an Event)

$$\mathbb{E}[X \mid E] = \sum_{x} x \Pr(X = x \mid E)$$

Theorem ( )

Let X be a random variable defined on a sample space Ω.
Let $E_1, E_2, \cdots, E_n$ be a partition of Ω.

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X \mid E_i] \Pr(E_i)$$

Theorem (The Law of Total Expectation)

Let X be a random variable defined on a sample space Ω.
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Proof.

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Definition (Conditional Expectation on a Random Variable)

$$\mathbb{E}[X \mid Y = y] = \sum_x x \Pr(X = x \mid Y = y)$$

Notation:

$$\mathbb{E}[X \mid Y](y) = \mathbb{E}[X \mid Y = y]$$

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$$\mathbb{E}[\mathbb{E}[X \mid Y]] = \sum_{y} \mathbb{E}[X \mid Y = y] \Pr(Y = y)$$


TREASURE
HUNT
on foot


TREASURE
HUNT
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Hints
&
Tips


Help
Wanted

There are n bins labelled with the numbers $1, 2, \cdots, n$. Balls are placed in these bins one after the other, with the bin into which a ball is placed being independent random variables that assume the value k with probability p_k. Let X be the number of balls placed so that there is at least one ball in every bin.

(a) Assume that $p_k = \frac{1}{n}$. What is the expectation of X?

(b) Assume that $p_k = \frac{1}{n}$. What is the probability distribution of X?

(c) Prove that $\Pr(X > n \ln n + cn) \leq e^{-c}$, $\Pr(X < n \ln n - cn) \leq e^{-c}$.

(d) Redo (a) and (b) without the assumption $p_k = \frac{1}{n}$.

(e) Given a deck of n cards, each time you take the top card from the deck, and insert it into the deck at one of the n distinct possible places, each of them with probability $\frac{1}{n}$. What is the expected times for you to perform the procedure above until the bottom card rises to the top?

The Coupon Collector's Problem

Shuffling Cards


Thank You!

Your opinon Matters

Office 302

Mailbox: H016

hfwei@nju.edu.cn