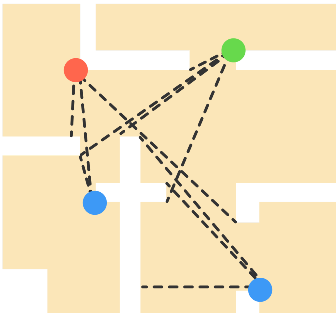
Art Gallery Problem

Given the layout of a museum, what is the minimum number of guards needed to guard every point in the museum? This often called the **Art Gallery Problem**, is an example of a problem at the intersection of several areas, including geometry math, and optimization. By working through this problem, one can explore ideas from different areas of mathematics, and these ideas can be combined to solve real-world problems!

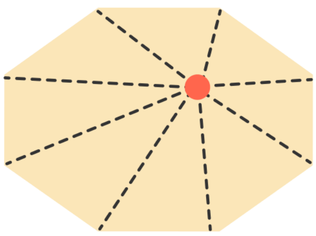


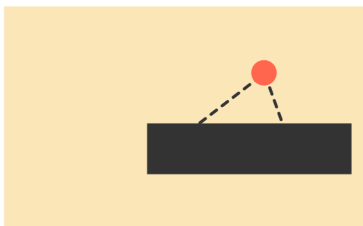
Placing four guards at the given points will guard the entire museum.

Contents

Guards and layouts

First, let's define mathematically what we mean to guard a museum. In this problem, guards must stay at fixed positions to see every angle from their position by rotating. To represent this mathematically, a point P in the museum is *visible* to the line segment from the guard to point P lies within the museum or along the boundary.





The guard in the museum on the left can see all points from his position. The guard in the museum on the right cannot se wall (shaded in black), so the lower right corner of the museum is unguarded.

We will assume our museums have straight walls, so the floor plan of the museum is a polygon in the plane (this analysis apply to museums such as Frank Gehry's Guggenheim Museum Bilbao). A polygon is **convex** if the entire line segment j two points in the polygon lies in the polygon. Let's start by thinking about a triangle, which is the simplest convex polygon

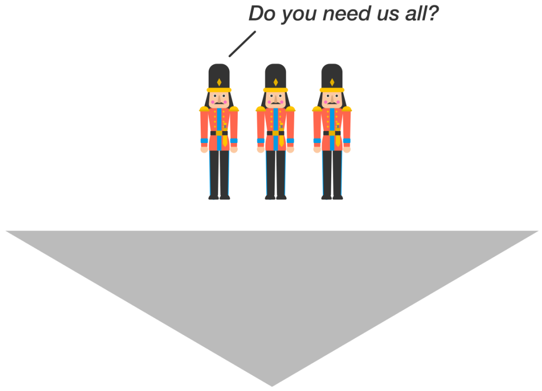
TRY IT YOURSELF

How many guards does it take to guard a museum in the shape of a triangle?

- 1
- 2
- 3

Details and Assumptions: A point P in the museum is visible to a guard if the line segment from the guard to P lies within the triangle (or along the boundary).

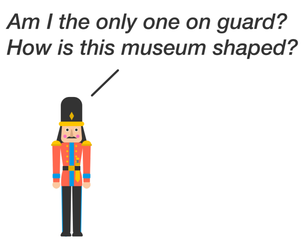
See Guarding a Museum for more details on guarding a museum.



1 if the triangle is acu
triangle is obtuse

Since a triangle is convex, by positing a guard anywhere in the museum, the line segment between the line and any othe
the museum lies in the museum. This holds for any convex shape, so any convex polygon can be guarded by a single gu
opposite direction also true?

TRY IT YOURSELF



Suppose a museum in the shape of a polygon can be guarded by a single guard. Does this imply the museum is convex?

Details: A point P in the museum is visible to a guard if the line segment from the guard to P lies within the triangle (or along the boundary).

See Guarding a Museum for details on guarding a museum.

- Not enough informat
- No, the museum is n convex.
- The museum is conv be a triangle.
- Yes, the museum is a convex.

Finding the Number of Guards

What is the minimum number of guards needed to guard a museum whose floor plan is a polygon with n walls? Note tha
answer this question, we need to show

- there exist positions for the guards such that every point in the museum is guarded
- no fewer guards can guard every point in the museum.

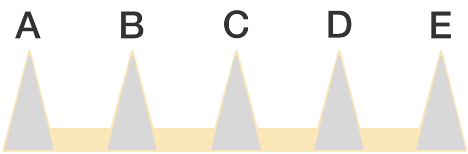
First, we define the floor function $\lfloor x \rfloor$ for any real number x to be the largest integer less than or equal to x. Think of the
function as rounding down to the nearest integer. For example,

$$\lfloor 3.4 \rfloor = 3, \lfloor 5.9 \rfloor = 5, \lfloor 10 \rfloor = 10.$$

THEOREM

Art Gallery Theorem: Any museum with n walls can be guarded by at most $\lfloor \frac{n}{3} \rfloor$ guards.

This problem was first solved by Vasek Chvatal in 1975 and below, we will give the beautiful proof due to Steve Fisk in 19
Fisk's proof of this theorem is **constructive**, giving an algorithm (or sequence of steps) that tells us exactly where to plac
guards. To show that bound in the theorem is tight, consider the museum with 15 walls in the shape of a comb.



Then the guard for point A must be stationed within the shaded triangle with vertex A, the guard for point B must be sta the shaded triangle with vertex B, etc. Since these triangles do not overlap, at least 5 guards are needed. But by the Art Theorem, $\lfloor \frac{15}{3} \rfloor = 5$ guards are also sufficient, which we can observe by placing the guards at the lower left corner of ea triangle. In general, the comb museum layout gives an example of a museum with $3n$ walls that requires exactly $\lfloor \frac{3n}{3} \rfloor =$ which shows that the bound in the theorem is best possible.

Generalizations

It seems that the worst case example of a comb museum occurs because there are very sharp "corners" that restrict the of guards. What if we consider museums whose walls meet at right angles, creating 90° corners? These floor plans corr orthogonal polygons, and three proofs given by Kahn-Klawe-Kleitman, Lubiw, and Sack-Toussaint show that there is alw configuration of $\lfloor \frac{n}{4} \rfloor$ guards that will guard the entire museum.

Applications

Problems from computational geometry also naturally arise in video game programming, where it is often necessary to p computations based on a virtual world to create a realistic user experience. Think about a video game you have played i virtual world and how the game must solve challenges such as detecting when objects collide, representing the surface c the virtual world, detecting motion from your input to the game, and determining the appearance/visibility of objects as yo through the world. All of these problems involve elements of computational geometry, computer graphics, computer scier algorithms.

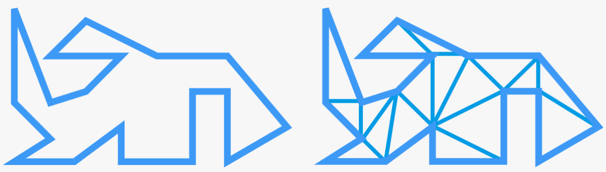
Other applications of computational geometry include

- route planning for GPS: determining location, speed, and direction
- integrated circuit design
- designing and building objects such as cars, ships, and aircraft
- computer vision, to determine lines of sight and designing special effects in movies
- robotics, to plan motion and visibility

Proof of Art Gallery Theorem

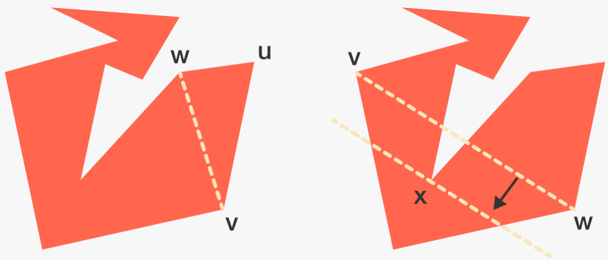
PROOF

We will prove this theorem through a sequence of claims. First, a *triangulation* of a polygon is a decomposition of the p into triangles by drawing non-intersecting diagonals between pairs of vertices.



Claim 1: Any polygon P can be triangulated.

We prove this claim by induction on the number n of vertices. For $n = 3$, the polygon P is a triangle, which is already triangulated. For $n \geq 4$, we will find a single diagonal (i.e., a line segment that lies within P connecting a pair of vertic splits the polygon into two smaller parts. Then the triangulation of the entire polygon can be obtained by pasting togeth triangulation of the two different parts. Since the sum of the interior angles of P is $(n-2)180^{\circ}$, there is a vertex u of interior angle less than 180°. Let v, w be the neighboring vertices of u in the polygon. If the line segment v, w lies with polygon, then this line segment is the desired diagonal. Otherwise, triangle $\triangle uvw$ contains other vertices. Move the li segment v, w towards u until it hits the final vertex x in triangle $\triangle uvw$. Then line segment ux lies within the polygon a be taken as the desired diagonal, proving the claim.

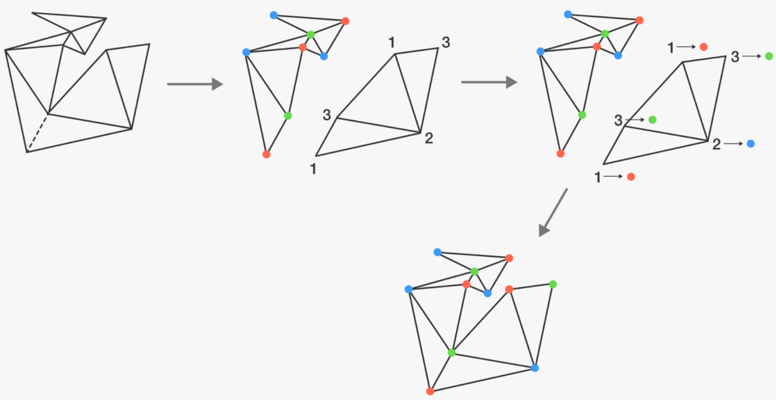


Our second claim involves coloring the vertices of the triangulated polygon in the following way: given a triangulated pc *3-coloring* is a coloring of the vertices such that every triangle has 3 different colored vertices.

Claim 2: Any triangulated polygon is 3-colorable.

We will proceed by induction on the number of vertices in the polygon. For $n = 3$, the polygon is a triangle and we car three different colors for the three vertices. Now, consider any triangulated polygon with $n > 3$ vertices. Pick any two v and v that are connected by a diagonal (i.e., connected by an edge in the triangulation but not in the original polygon). split the polygon into two triangulated polygons using edge (u, v) and by induction, the two triangulated polygons are 3 colorable. Let (red, blue, green) be the colors in the first triangulation T_1 and let $(1, 2, 3)$ be the colors for the second triangulation T_2. Then identify the color of u in the first triangulation to the number labeling u in the second triangulatio identify the color of v in the first triangulation to the number labeling v in the second triangulation. Finally, identify the la remaining colors in both triangulations with each other.

Then we obtain a coloring for the entire triangulation by preserving the color of all vertices in the first triangulation and u colors identified with $(1, 2, 3)$ for the vertices in the second triangulation. This gives a 3-coloring of the entire triangulat polygon and proves Claim 2.



Claim 3: For any 3-coloring of a triangulation, there exists a color such that the number of vertices of this color is $\leq \lfloor$ placing guards on these vertices will guard the entire museum.

Without loss of generality, suppose the colors of the vertices are red, green, blue such that the number r of red vertices than or equal to the number g of green vertices, which is less than or equal to the number b of blue vertices. The total r vertices is n, so $r + g + b = n$. If $r > \lfloor \frac{n}{3} \rfloor$, then g and b would also be strictly greater than $\lfloor \frac{n}{3} \rfloor$, and the identity $r + g + b = n$ cannot be satisfied. Therefore, $r \leq \lfloor \frac{n}{3} \rfloor$. Now, by placing a guard at every vertex colored by red, obse every triangle in the triangulation has exactly one red node and thus, exactly one guard. Also, any point P in the muse

contained in a triangle in the triangulated polygon, and P is visible from the vertex of the triangle colored by red. This g
placement of at most $\lfloor \frac{n}{3} \rfloor$ guards that guard the entire museum, proving Claim 3.

Claims 1, 2, and 3 together give a proof of the Art Gallery Theorem. $\square$

Since there are many different possible triangulations and 3-colorings, there may be multiple ways to place the guards, b
choice will result in at most $\lfloor \frac{n}{3} \rfloor$ guards to guard the entire museum.