1-11 有穷与无穷

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GEORG CANTOR
MATHEMATIKER
BEGRÜNDER DER MENGENLEHRE
1845-1918
DAS WESEN DER MATHEMA
TIK LIEGT IN IHRER FREIHEIT
GEORG CANTOR

"das wesen der mathematik liegt in ihrer freiheit"

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"The essence of mathematics lies in its freedom"

Dangerous Knowledge (BBC 2007)

$$c = \aleph_1$$

Comparing Sets

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$=$ $<$

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Function

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$$\{1, 2, 3, \cdots\} \quad \textit{vs.} \quad \{1, 3, 5, \cdots, 2, 4, 6, \cdots\}$$

Definition (Finite and Infinite)

For any set X,

Finite

$$\exists n \in \mathbb{N} : |X| = n \qquad (0 \in \mathbb{N})$$

Infinite $(\neg$ finite)

$$\forall n \in \mathbb{N} : |X| \neq n$$

Definition (Finite and Infinite)

For any set X,

Countably Infinite

$$|X| = |\mathbb{N}| \triangleq \aleph_0$$

Countable

(finite $\vee$ countably infinite)

Uncountably Infinite

$$(\neg \text{ finite}) \wedge \Big(\neg \text{ (countably infinite)}\Big)$$

$$(\neg \text{ countable})$$

Theorem ($\aleph_0$ (1874))

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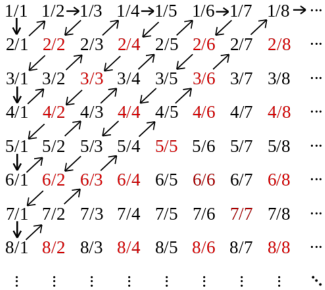
$|\mathbb{Q}| = |\mathbb{N}|$ (UD 22.9)

$$q \in \mathbb{Q}^+ : a/b \; (a, b \in \mathbb{N}^+)$$



$$|\mathbb{N}| = |\mathbb{Z}|$$

$$|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$$

Theorem ($\mathbb{R}$ is uncountably infinite (1874).)

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Different "Sizes" of Infinity

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Theorem (Cantor's Theorem (1891))

$$|X| \neq |2^X| \qquad (|X| < |2^X|)$$

Infinite Sequences of 0's and 1's (UD 22.3)

Is the set of all infinite sequences of 0's and 1's finite, countably infinite, or uncountable?

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By Cantor's diagonal argument $\implies$ uncountable. □

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Nonproof.

$$f : \{\{0,1\}^*\} \to \mathbb{N}$$

$$f(x_0 x_1 \cdots) = \sum_{i=0}^{\infty} x_i 2^i$$

□

Theorem ($|\mathbb{R}|$ (1877))

$$(0,1) = |\mathbb{R}| = |\mathbb{R}| \times |\mathbb{R}| = |\mathbb{R}|^{n\in\mathbb{N}}$$

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Q : *Then, what is "dimension"?*

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$|B| \le |A|$ (Axiom of Choice)

Definition ($|A| < |B|$)

$$|A| < |B| \iff |A| \le |B| \land |A| \neq |B|$$

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$$|\mathbb{N}| < |\mathbb{R}|$$

$$|X| < |2^X|$$

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X is countable:

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Subsets of Countable Set (UD 22.6; UD Corollary 22.4)

Every subset B of a countable set A is countable.

Set Union (UD 22.1)

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$$|A| = n \implies |2^A| = 2^n$$

Slope (UD 22.2 (e))

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$$|\mathbb{R}| \le |\mathbb{Q} \times \mathbb{R}| \le |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$$

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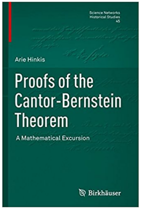

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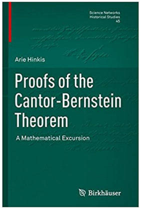

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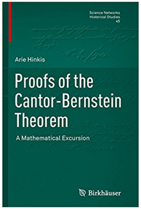

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Theorem (PCC)

Principle of Cardinal Comparability (PCC) $\iff$ Axiom of Choice

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Theorem (Pigeonhole Principle (UD Theorem 21.2))

$$f : \{1, \cdots, m\} \to \{1, \cdots, n\} \ (m, n \in \mathbb{N}^+, m > n)$$

f is not one-to-one.

$A \setminus \{a\}$ (UD 21.15)

Let A be a nonempty finite set with $|A| = n$ and let $a \in A$.
Prove that $A \setminus \{a\}$ is finite and $|A \setminus \{a\}| = n - 1$.

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A and B are finite sets and $f : A \to B$ is one-to-one.
Show that $|A| \leq |B|$.

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By contradiction and the pigeonhole principle.

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By contradiction and (b).

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(No Axiom of Choice Here)

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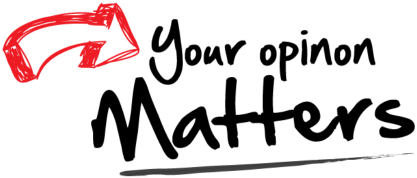
$$f(g(y)) = f(x) = y \implies f = g^{-1}$$


Thank You!



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