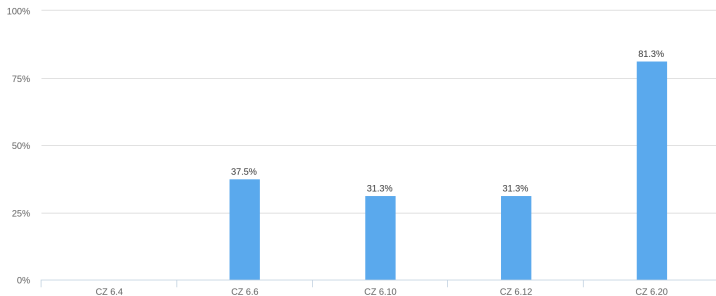# 3-10 Traversability

(Part I: Eulerian Graphs)

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CZ 6.20 (Next Class)

这次习题相对简单【因为是必选的所以随机选了一个】，希望老师可以多回顾一下课本内容，比如哈密尔顿图的各种充分条件和证明，对哈密尔顿图和欧拉图的用法做一些拓展

欧拉图和汉密尔顿图的联系 就是在建模时如何确定图的节点和边

g-cage 对于不同大小而言都是唯一的吗？（书上只给到n=8） Kirkman triple system 和 factorization 的等价推导 Theorem 8.10的证明

6.3，不想看书，看自闭了，希望老师可以直接讲一下

如何打印欧拉闭路/欧拉迹/哈密顿闭路

定理6.5

无

暂无

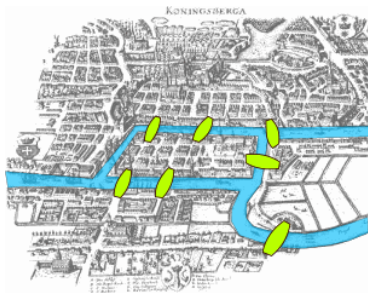无

陶老师上课讲的中国邮递员问题没明白

暂无，我就来抽个奖

定理6.12的证明

嗝(^·ェ·^)

希望能讲一下fluery算法 ●

可以总结一下证明的方法，其实每次都可以这样，不一定要堂上讲，可以整理之后做成讲义课后发，比如怎么么证明有欧拉回路等等

Fleury (Hierholzer)

Chinese Postman Problem (Next Class)
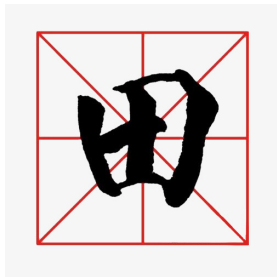
6.3 Exploration & 6.4 Excursion (Not Required)

Leonhard Euler (1707 – 1783)

Graph Theory   Topology

Theorem (Leonhard Euler 1735)

A connected graph G is Eulerian if and only if the degree of each vertex of G is even.

"$\Longleftarrow$" (Carl Hierholzer 1873); (Proofs on PPT from Tao & on CZ).

By strong mathematical induction on the number m of edges of G.

$$\text{Inductive Step} : m = k + 1$$

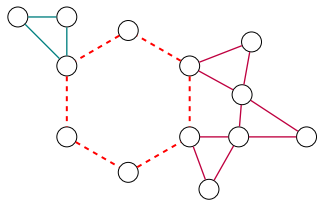Let $C = u \sim v$ be any trail in G of the maximum length.

$$v = u \implies C = u \sim u$$

$$H = G - E(C) = \bigcup H_i$$

$$H = G - E(C) = \bigcup H_i$$

(I) $\forall v \in H : \deg(v)$ is even

(II) $\forall i : \left| E(H_i) \right| < m$

By I.H., each H_i has an Eulerian circuit C_i.

$$\forall i : V(H_i) \cap V(C) \neq \emptyset$$

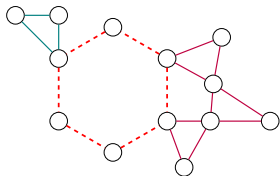Glue together each C_i with C to get an Eulerian circuit of G.

```
1:  procedure HIERHOLZER(G)
2:      u ∈ V(G)
3:      C ←  any circuit u ∼ u in G

4:      while ∃v ∈ C : deg(v) > 0 do
5:          H ← G − E(C)
6:          v ←  any vertex in V(C) such that deg(v) > 0
7:          C' ←  any circuit v ∼ v in H
8:          C ← C ⊗ C'                    ▷ Glue C' = v ∼ v with C via v

9:      return C
```


Q : Time Complexity?

Q : Data Structures?

$O(m)$: Using doubly linked list

Fleury's Algorithm (1883)

(I) $v_0 \in V(G);$ $P_0 = v_0$

(II) Suppose $C_i = v_0 e_1 v_1 e_2 \cdots e_i v_i$.

   Choose e_{i+1} from $E(G) - \{e_1, e_2, \cdots, e_i\}$ as follows:

   (a) e_{i+1} is incident with v_i

   (b) Unless there is no alternative, e_{i+1} is not a bridge of
      $G - \{e_1, e_2, \cdots, e_i\}$

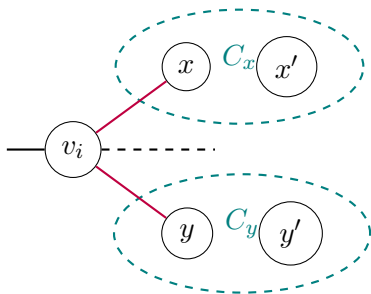(III) Stop when step (II) can no longer be implemented

Is that ok?

PROOF

Theorem (Bridges in Fleury's Algorithm)

At any stage, v_i is incident with ≤ 1 bridge in $E(G) - \{e_1, e_2, \cdots, e_i\}$.

<div align="center">By Contradiction.</div>

Suppose that v_i is incident with ≥ 2 bridges in $E(G) - \{e_1, e_2, \cdots, e_i\}$.
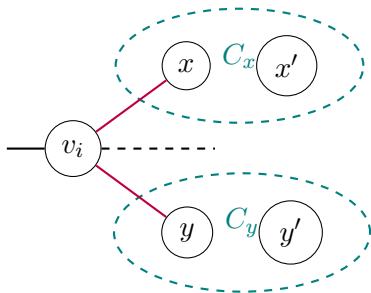

$\exists x' \in C_x : \deg(x)$ is odd

$\exists y' \in C_y : \deg(y)$ is odd

We have found 2 odd vertices.

Q : What is the contradiction?

Is $\deg(v_i)$ odd or even?

We have found 2 odd vertices.

Is $\deg(v_i)$ odd or even?

CASE I: $\deg(v_i)$ is odd.

Contradiction:

Only v_0 and v_i can have odd degrees!

CASE II: $\deg(v_i)$ is even.

$$v_i = v_0$$

Contradiction: No odd vertices!

1: **procedure** FLEURY(G)
2:     $v_0 \in V(G)$ ▷ Choose any starting vertex
3:     $C \leftarrow v_0$ ▷ Keep track of the circuit
4:     $i \leftarrow 0$, $V_0 \leftarrow V(G)$, $E_0 \leftarrow E(G)$

5:     **while** $\deg(v_i) > 0$ in E_i **do**   ▷ Stop otherwise
6:         **if** $\deg(v_i) = 1$ in E_i **then**   ▷ No alternative: go the bridge
7:             $e_{i+1} \triangleq v_i v_{i+1}$
8:             $V_{i+1} \leftarrow V_i - \{v_i\}$ ▷ Delete the isolated vertex v_i
9:         **else**   ▷ Have alternatives: don't go the bridge
10:             Choose $e_{i+1} \triangleq v_i v_{i+1}$ that is not a bridge of (V_i, E_i)
11:             $V_{i+1} \leftarrow V_i$ ▷ No isolated vertex produced
12:         $C \leftarrow C e_{i+1} v_{i+1}$
13:         $E_{i+1} \leftarrow E_i - \{e_{i+1}\}$
14:         $i \leftarrow i + 1$
15:     **return** C

PROOF

We need to prove that C returned by FLEURY is an Eulerian circuit.

Definition (Eulerian Circuit)

A connected graph is Eulerian if there exists a closed trail that includes every edge of G.
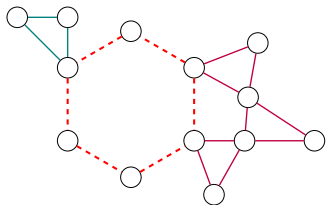
A trail is a walk in which all the edges are distinct.

Closed	Trail	Include every edge of G
∵ even degrees	∵ used edges are deleted	By Contradiction.

Include every edge of G

By Contradiction.

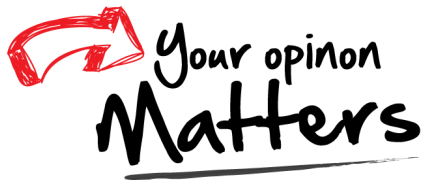We know that $C : v_0 \sim v_0$ $\qquad E' \triangleq E(G) - E(C) \neq \emptyset$

$\deg(v_0) = 0$ (Otherwise, FLEURY is not terminated.)

$G|_{E'}$ is disconnected from v_0

Impossible:

(I) Don't go the bridge unless there is no alternative.

(II) Delete the isolated vertex left by going the bridge.

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