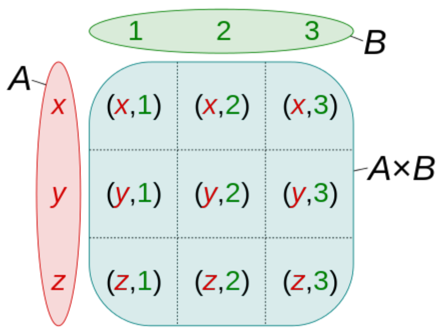
Cartesian product

In set theory (and, usually, in other parts of mathematics), a **Cartesian product** is a mathematical operation that returns a set (or **product set** or simply **product**) from multiple sets. That is, for sets A and B, the Cartesian product $A \times B$ is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$. Products can be specified using set-builder notation, e.g.

$$A \times B = \{\,(a,b) \mid a \in A \ \text{ and } \ b \in B\,\}.$$[1]



Cartesian product $A \times B$ of the sets $A=\{x,y,z\}$ and $B=\{1,2,3\}$

A table can be created by taking the Cartesian product of a set of rows and a set of columns. If the Cartesian product *rows* × *columns* is taken, the cells of the table contain ordered pairs of the form (row value, column value).

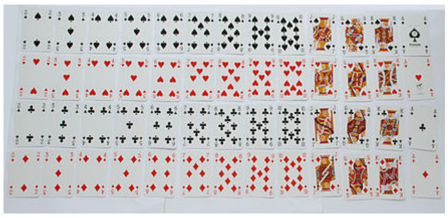
More generally, a Cartesian product of n sets, also known as an ***n*-fold Cartesian product**, can be represented by an array of n dimensions, where each element is an n-tuple. An ordered pair is a 2-tuple or couple.

The Cartesian product is named after René Descartes,[2] whose formulation of analytic geometry gave rise to the concept, which is further generalized in terms of direct product.

Contents

Examples

A deck of cards



Standard 52-card deck

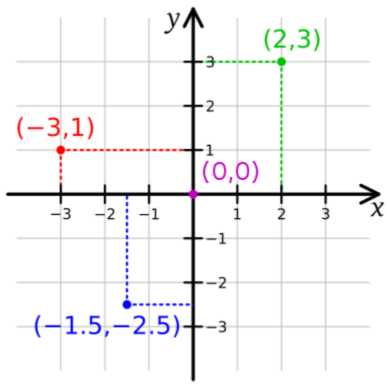
An illustrative example is the standard 52-card deck. The standard playing card ranks {A, K, Q, J, 10, 9, 8, 7, 6, 5, 4, 3, 2} form a 13-element set. The card suits {♠, ♥, ♦, ♣} form a four-element set. The Cartesian product of these sets returns a 52-element set consisting of 52 ordered pairs, which correspond to all 52 possible playing cards.

Ranks × *Suits* returns a set of the form {(A, ♠), (A, ♥), (A, ♦), (A, ♣), (K, ♠), ..., (3, ♣), (2, ♠), (2, ♥), (2, ♦), (2, ♣)}.

Suits × *Ranks* returns a set of the form {(♠, A), (♠, K), (♠, Q), (♠, J), (♠, 10), ..., (♣, 6), (♣, 5), (♣, 4), (♣, 3), (♣, 2)}.

Both sets are distinct, even disjoint.

A two-dimensional coordinate system



Cartesian coordinates of example points

The main historical example is the Cartesian plane in analytic geometry. In order to represent geometrical shapes in a numerical way and extract numerical information from shapes' numerical representations, René Descartes assigned to each point in the plane a pair of real numbers, called its coordinates. Usually, such a pair's first and second components are called its x and y coordinates, respectively; cf. picture. The set of all such pairs (i.e. the Cartesian product $\mathbb{R}\times\mathbb{R}$ with $\mathbb{R}$ denoting the real numbers) is thus assigned to the set of all points in the plane.

Most common implementation (set theory)

A formal definition of the Cartesian product from set-theoretical principles follows from a definition of ordered pair. The most common definition of ordered pairs, the Kuratowski definition, is $(x,y)=\{\{x\},\{x,y\}\}$. Note that, under this definition, $X\times Y\subseteq\mathcal{P}(\mathcal{P}(X\cup Y))$, where $\mathcal{P}$ represents the power set. Therefore, the existence of the Cartesian product of any two sets in ZFC follows from the axioms of pairing, union, power set, and specification. Since functions are usually defined as a special case of relations, and relations are usually defined as subsets of the Cartesian product, the definition of the two-set Cartesian product is necessarily prior to most other definitions.

Non-commutativity and non-associativity

Let A, B, C, and D be sets.

The Cartesian product $A \times B$ is not commutative,

$$A\times B\neq B\times A,$$

because the ordered pairs are reversed unless at least one of the following conditions is satisfied:[3]

- *A* is equal to *B*, or
- *A* or *B* is the empty set.

For example:

A = {1,2}; B = {3,4}

$A \times B$ = {1,2} × {3,4} = {(1,3), (1,4), (2,3), (2,4)}

$B \times A = \{3,4\} \times \{1,2\} = \{(3,1), (3,2), (4,1), (4,2)\}$

$A = B = \{1,2\}$

$A \times B = B \times A = \{1,2\} \times \{1,2\} = \{(1,1), (1,2), (2,1), (2,2)\}$

$A = \{1,2\}$; $B = \varnothing$

$A \times B = \{1,2\} \times \varnothing = \varnothing$
$B \times A = \varnothing \times \{1,2\} = \varnothing$

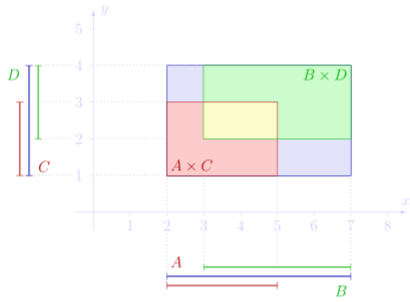
Strictly speaking, the Cartesian product is not associative (unless one of the involved sets is empty).

$$(A \times B) \times C \neq A \times (B \times C)$$

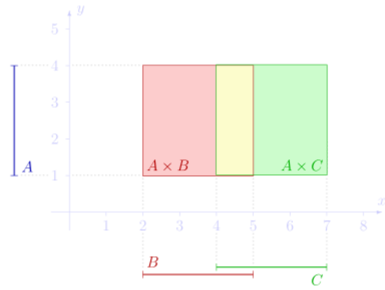
If for example $A = \{1\}$, then $(A \times A) \times A = \{ ((1,1),1) \} \neq \{ (1,(1,1)) \} = A \times (A \times A)$.

Intersections, unions, and subsets

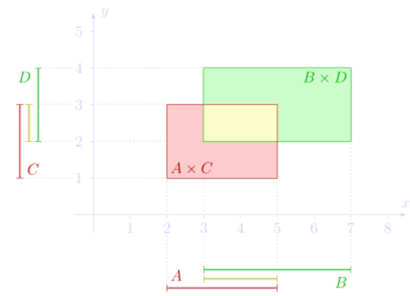
The Cartesian product behaves nicely with respect to intersections, cf. left picture.



$(A \cup B) \times (C \cup D) \neq (A \times C) \cup (B \times D)$ can be seen from the same example.



Example sets $A=\{y\in\mathbb{R}:1\leq y\leq 4\}$, $B=\{x\in\mathbb{R}:2\leq x\leq 5\}$, and $C=\{x\in\mathbb{R}:4\leq x\leq 7\}$, demonstrating $A\times(B\cap C) = (A\times B)\cap(A\times C)$, $A\times(B\cup C) = (A\times B)\cup(A\times C)$, and $A\times(B\setminus C) = (A\times B)\setminus(A\times C)$.



Example sets $A=\{x\in\mathbb{R}:2\leq x\leq 5\}$, $B=\{x\in\mathbb{R}:3\leq x\leq 7\}$, $C=\{y\in\mathbb{R}:1\leq y\leq 3\}$, $D=\{y\in\mathbb{R}:2\leq y\leq 4\}$, demonstrating $(A\cap B)\times(C\cap D) = (A\times C)\cap(B\times D)$.

$$(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$$ [4]

In most cases the above statement is not true if we replace intersection with union, cf. middle picture.

$$(A \cup B) \times (C \cup D) \neq (A \times C) \cup (B \times D)$$

In fact, we have that:

$$(A \times C) \cup (B \times D) = [(A \setminus B) \times C] \cup [(A \cap B) \times (C \cup D)] \cup [(B \setminus A) \times D]$$

For the set difference we also have the following identity:

$$(A \times C) \setminus (B \times D) = [A \times (C \setminus D)] \cup [(A \setminus B) \times C]$$

Here are some rules demonstrating distributivity with other operators (cf. right picture):[3]

$$\begin{aligned}
&A \times (B \cap C) = (A \times B) \cap (A \times C),\\
&A \times (B \cup C) = (A \times B) \cup (A \times C),\\
&A \times (B \setminus C) = (A \times B) \setminus (A \times C),\\
&(A \times B)^\complement = (A^\complement \times B^\complement) \cup (A^\complement \times B) \cup (A \times B^\complement),
\end{aligned}$$ [4]

where $A^\complement$ denotes the absolute complement of A.

Other properties related with subsets are:

$$\begin{aligned}
&\text{if } A \subseteq B \text{ then } A \times C \subseteq B \times C,\\
&\text{if both } A, B \neq \emptyset \text{ then } A \times B \subseteq C \times D \iff A \subseteq C \land B \subseteq D.
\end{aligned}$$ [5]

Cardinality

The cardinality of a set is the number of elements of the set. For example, defining two sets: $A = \{a, b\}$ and $B = \{5, 6\}$. Both set A and set B consist of two elements each. Their Cartesian product, written as $A \times B$, results in a new set which has the following elements:

$$A \times B = \{(a,5), (a,6), (b,5), (b,6)\}.$$

Each element of A is paired with each element of B. Each pair makes up one element of the output set. The number of values in each element of the resulting set is equal to the number of sets whose cartesian product is being taken; 2 in this case. The cardinality of the output set is equal to the product of the cardinalities of all the input sets. That is,

$$|A \times B| = |A| \cdot |B|.$$

Similarly

$$|A \times B \times C| = |A| \cdot |B| \cdot |C|$$

and so on.

The set $A \times B$ is infinite if either A or B is infinite and the other set is not the empty set.[6]

n-ary product

Cartesian power

The **Cartesian square** (or **binary Cartesian product**) of a set X is the Cartesian product $X^2 = X \times X$. An example is the 2-dimensional plane $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ where $\mathbf{R}$ is the set of real numbers: $\mathbf{R}^2$ is the set of all points (x,y) where x and y are real numbers (see the Cartesian coordinate system).

The **cartesian power** of a set X can be defined as:

$$X^n = \underbrace{X \times X \times \cdots \times X}_{n} = \{(x_1, \ldots, x_n) \mid x_i \in X \text{ for all } i = 1, \ldots, n\}.$$

An example of this is $\mathbf{R}^3 = \mathbf{R} \times \mathbf{R} \times \mathbf{R}$, with **R** again the set of real numbers, and more generally $\mathbf{R}^n$.

The n-ary cartesian power of a set X is isomorphic to the space of functions from an n-element set to X. As a special case, the 0-ary cartesian power of X may be taken to be a singleton set, corresponding to the empty function with codomain X.

Finite *n*-ary product

The Cartesian product can be generalized to the ***n*-ary Cartesian product** over n sets $X_1, ..., X_n$:

$$X_1 \times \cdots \times X_n = \{(x_1, \ldots, x_n) : x_i \in X_i\}.$$

It is a set of n-tuples. If tuples are defined as nested ordered pairs, it can be identified to $(X_1 \times ... \times X_{n-1}) \times X_n$.

Infinite products

It is possible to define the Cartesian product of an arbitrary (possibly infinite) indexed family of sets. If I is any index set, and $\{X_i\}_{i\in I}$ is a family of sets indexed by I, then the Cartesian product of the sets in X is defined to be

$$\prod_{i\in I} X_i = \left\{ f : I \to \bigcup_{i\in I} X_i \;\middle|\; (\forall i)(f(i) \in X_i) \right\},$$

that is, the set of all functions defined on the index set such that the value of the function at a particular index i is an element of X_i. Even if each of the X_i is nonempty, the Cartesian product may be empty if the axiom of choice (which is equivalent to the statement that every such product is nonempty) is not assumed.

For each j in I, the function

$$\pi_j : \prod_{i\in I} X_i \to X_j,$$

defined by $\pi_j(f) = f(j)$ is called the ***j*th projection map**.

An important case is when the index set is $\mathbb{N}$, the natural numbers: this Cartesian product is the set of all infinite sequences with the ith term in its corresponding set X_i. For example, each element of

$$\prod_{n=1}^{\infty} \mathbb{R} = \mathbb{R} \times \mathbb{R} \times \cdots$$

can be visualized as a vector with countably infinite real number components. This set is frequently denoted $\mathbb{R}^\omega$, or $\mathbb{R}^{\mathbb{N}}$.

The special case **Cartesian exponentiation** occurs when all the factors X_i involved in the product are the same set X. In this case,

$$\prod_{i\in I} X_i = \prod_{i\in I} X$$

is the set of all functions from I to X, and is frequently denoted X^I. This case is important in the study of cardinal exponentiation.

The definition of finite Cartesian products can be seen as a special case of the definition for infinite products. In this interpretation, an n-tuple can be viewed as a function on $\{1, 2, ..., n\}$ that takes its value at i to be the ith element of the tuple (in some settings, this is taken as the very definition of an n-tuple).

Other forms

Abbreviated form

If several sets are being multiplied together, e.g. $X_1, X_2, X_3, \ldots$, then some authors[7] choose to abbreviate the Cartesian product as simply $\times X_i$.

Cartesian product of functions

If f is a function from A to B and g is a function from X to Y, their Cartesian product $f \times g$ is a function from $A \times X$ to $B \times Y$ with

$$(f \times g)(a, x) = (f(a), g(x)).$$

This can be extended to tuples and infinite collections of functions. Note that this is different from the standard cartesian product of functions considered as sets.

Cylinder

Let A be a set and $B \subseteq A$. Then the *cylinder* of B with respect to A is the Cartesian product $B \times A$ of B and A.

Normally, A is considered to be the universe of the context and is left away. For example, if B is a subset of the natural numbers $\mathbb{N}$, then the cylinder of B is $B \times \mathbb{N}$.

Definitions outside set theory

Category theory

Although the Cartesian product is traditionally applied to sets, category theory provides a more general interpretation of the product of mathematical structures. This is distinct from, although related to, the notion of a Cartesian square in category theory, which is a generalization of the fiber product.

Exponentiation is the right adjoint of the Cartesian product; thus any category with a Cartesian product (and a final object) is a Cartesian closed category.

Graph theory

In graph theory the Cartesian product of two graphs G and H is the graph denoted by $G \times H$ whose vertex set is the (ordinary) Cartesian product $V(G) \times V(H)$ and such that two vertices (u,v) and (u',v') are adjacent in $G \times H$ if and only if $u = u'$ and v is adjacent with v' in H, *or* $v = v'$ and u is adjacent with u' in G. The Cartesian product of graphs is not a product in the sense of category theory. Instead, the categorical product is known as the tensor product of graphs.

See also

- Binary relation
- Concatenation of sets of strings
- Coproduct

- Empty product
- Euclidean space
- Exponential object
- Finitary relation
- Join (SQL) § Cross join
- Orders on the Cartesian product of totally ordered sets
- Product (category theory)
- Product topology
- Product type
- Ultraproduct

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External links

- Cartesian Product at ProvenMath (http://www.apronus.com/provenmath/cartesian.htm)
- Hazewinkel, Michiel, ed. (2001) [1994], "Direct product" (https://www.encyclopediaofmath.org/index.php?title=p/d032730), *Encyclopedia of Mathematics*, Springer Science+Business Media B.V. / Kluwer Academic Publishers, ISBN 978-1-55608-010-4
- How to find the Cartesian Product, Education Portal Academy (http://education-portal.com/academy/lesson/how-to-find-the-cartesian-product.html)