

# 4-5 Polyhedral Groups (II)

## (Cube)

Hengfeng Wei

hfwei@nju.edu.cn

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$$|C| \leq 24$$

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1. facing upward
2. 24 **oriented** edges

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$$|C| = 24 \Leftarrow 4 \text{ main diagonals}$$

- ▶ Order of 1: id ( $\# = 1$ )
- ▶ Order of 4: face-to-face

$$\begin{array}{lll}
 f_{td} = (1\ 2\ 3\ 4) & f_{td}^2 = (1\ 3)(2\ 4) & f_{td}^3 = (1\ 4\ 3\ 2) \\
 f_{lr} = (1\ 2\ 4\ 3) & f_{lr}^2 = (1\ 4)(2\ 3) & f_{lr}^3 = (1\ 3\ 4\ 2) \\
 f_{fb} = (1\ 4\ 2\ 3) & f_{fb}^2 = (1\ 2)(3\ 4) & f_{fb}^3 = (1\ 3\ 2\ 4)
 \end{array}$$

$$C \cong S_4$$

- ▶ Order of 3: vertex-to-vertex

$$v_1 = (2\ 3\ 4) \quad v_1^2 = (2\ 4\ 3)$$

$$v_2 = (1\ 4\ 3) \quad v_2^2 = (1\ 3\ 4)$$

$$v_3 = (1\ 2\ 4) \quad v_3^2 = (1\ 4\ 2)$$

$$v_4 = (1\ 2\ 3) \quad v_4^2 = (1\ 3\ 2)$$

- ▶ Order of 2: edge-to-edge

$$e_{12} = (1\ 2) \quad e_{13} = (1\ 3) \quad e_{14} = (1\ 4)$$

$$e_{23} = (2\ 3) \quad e_{24} = (2\ 4) \quad e_{34} = (3\ 4)$$

# Subgroups of $S_4$

Possible orders: 1   2   3   4   6   8   12   24

- ▶  $|H| = 1$ :  $\# = 1$
- ▶  $|H| = 24$ :  $\# = 1$
- ▶  $|H| = 2$ :  $\# = 6 + 3 = 9$
- ▶  $|H| = 3$ :  $\# = 4$

# Subgroups of order 4

- ▶  $H \cong \mathbb{Z}_4$ :  $\# = 3$
- ▶  $H \cong K_4 = \{e, a, b, c\} (a^2 = b^2 = (ab)^2 = e)$

$$\{(1), (1\ 2), (3\ 4), (1\ 2)(3\ 4)\}$$

$$\{(1), (1\ 3), (2\ 4), (1\ 3)(2\ 4)\}$$

$$\{(1), (1\ 4), (2\ 3), (1\ 4)(2\ 3)\}$$

$$\{(1), (1\ 2)(1\ 3), (2\ 4), (1\ 4)(2\ 3)\}$$

$$\# = 3 + 4 = 7$$



# Subgroups of order 6

$$H \not\cong \mathbb{Z}_6$$

$$H \cong S_3 = \{1, r, r^2, s, rs, r^2s\} \quad (r^3 = 1, s^2 = 1, srs = r^{-1})$$

Figure here.

## Theorem

*There are only 4 subgroups of order 6 in  $S_4$ .*

$$r = (1\ 3\ 2), \quad s = (1\ 3)$$

What does  $srs = r^{-1}$  mean?

## Subgroups of order 8

$$H \not\cong \mathbb{Z}_8$$

$$H \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

$$H \not\cong \mathbb{Z}_4 \times \mathbb{Z}_2$$

$$H \not\cong Q_8 : \implies |H| \geq 9$$

$$H \cong D_4 = \{1, r, r^2, r^3, s, rs, r^2s, r^3s\} \quad (r^4 = 1, s^2 = 1, srs = r^{-1})$$

Figure here.

### Theorem

*There are only 3 subgroups of order 8 of  $S_4$ .*

# Subgroups of order 12

$$H \cong \mathbb{Z}_{12}, \mathbb{Z}_6 \times \mathbb{Z}_2, D_6, A_4, Dic_{12}$$

$$H \cong A_4$$

Figure here.

## Theorem

*There is only one subgroup of order 12 in  $S_4$ .*

## Proof.







Office 302

Mailbox: H016

hfwei@nju.edu.cn