1-8 Set Theory: Axioms and Operations

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Set Operations (I)

$\cap \qquad \cup \qquad \setminus$

UD Problem 7.1 (d)

Let $A, B \subseteq X$.

$$A \subseteq B \iff (X \setminus B) \subseteq (X \setminus A)$$

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UD Problem 7.1 (f)

$$A \cap B = B \iff B \subseteq A$$

UD Problem 7.2

Let $A, B \subseteq X$.

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$Q : A, B \subseteq X?$

We need only $B \subseteq X$.

UD Problem 7.14

Prove that the union of two sets can be rewritten as the union of two disjoint sets.

(a) Prove that $(A \setminus B) \cap B = \emptyset$

(b) Prove that $A \cup B = (A \setminus B) \cup B$

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By contradiction.

$(A \setminus B) \cup B = \cdots$

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UD Problem 7.19

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$$A \cap B = \emptyset \iff B \subseteq (X \setminus A)$$

UD Problem 7.20

$$(A \cup B) \setminus (C \cup D) = (A \setminus (C \cup D)) \cup (B \setminus (C \cup D))$$

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$$E \triangleq C \cup D$$

Set Operations (II)

$\bigcap \qquad \bigcup$

UD Problem 8.1

$$A_n = [0, 1/n) \quad B_n = [0, 1/n] \quad C_n = (0, 1/n)$$

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(b) Find $\bigcap_{n=1}^{\infty} A_n = \{0\} \quad \bigcap_{n=1}^{\infty} B_n = \{0\} \quad \bigcap_{n=1}^{\infty} C_n = \emptyset$

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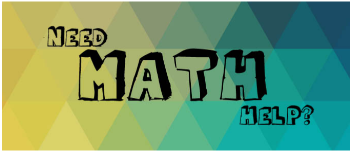

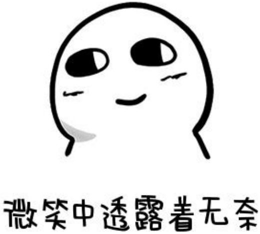

UD Problem 8.1

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Theorem (The Nested Interval Theorem (Cantor))

设 $\{[a_n, b_n]\}$ 为递降闭区间套序列，即

$$[a_1, b_1] \supset [a_2, b_2] \supset \cdots \supset [a_n, b_n] \supset \cdots.$$

如果 $\lim_{n\to\infty}(b_n - a_n) = 0$，则存在唯一的点 c，使得 $c \in [a_n, b_n], \forall n \geq 1$.

UD Problem 8.6

$$\forall n \in \mathbb{Z}^+ : A_n \subset B_n \not\Rightarrow \bigcap_{n=1}^{\infty} A_n \subset \bigcap_{n=1}^{\infty} B_n$$

$$A_n = [0, 1/n) \quad B_n = [0, 1/n]$$

UD Problem 8.6

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$$A_n = [0, 1/n) \quad B_n = [0, 1/n]$$

UD Problem 8.14

$$A = \mathbb{R} \setminus \bigcap_{n \in \mathbb{Z}^+} \left(\mathbb{R} \setminus \{-n, -n+1, \cdots, 0, \cdots, n-1, n\} \right)$$

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$$X_n = \{-n, -n+1, \cdots, 0, \cdots, n-1, n\}$$

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$$\begin{aligned} A &= \mathbb{R} \setminus \bigcap_{n \in \mathbb{Z}^+} (\mathbb{R} \setminus X_n) \\ &= \mathbb{R} \setminus \Big(\mathbb{R} \setminus \bigcup_{n \in \mathbb{Z}^+} X_n\Big) \end{aligned}$$

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UD Problem 8.15

$$A = \mathbb{Q} \setminus \bigcap_{n \in \mathbb{Z}} (\mathbb{R} \setminus \{2n\})$$

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$$\begin{aligned} A &= \mathbb{Q} \setminus \bigcap_{n \in \mathbb{Z}} (\mathbb{R} \setminus \{2n\}) \\ &= \mathbb{Q} \setminus \Big(\mathbb{R} \setminus \bigcup_{n \in \mathbb{Z}} \{2n\}\Big) \\ &= \mathbb{Q} \setminus \Big(\bigcup_{n \in \mathbb{Z}} \{2n\}\Big)^c \\ &= \mathbb{Q} \cap \bigcup_{n \in \mathbb{Z}} \{2n\} \\ &= \{2n : n \in \mathbb{Z}\} \end{aligned}$$

Set Operations (III)

$\mathcal{P}(X)$

UD Problem 9.8

$$A \subseteq B \iff \mathcal{P}(A) \subseteq \mathcal{P}(B)$$

UD Problem 9.9

$$\bigcup_{\alpha \in I} \mathcal{P}(A_\alpha) \subseteq \mathcal{P}(\bigcup_{\alpha \in I} A_\alpha)$$

UD Problem 9.10

$$\bigcap_{\alpha \in I} \mathcal{P}(A_\alpha) = \mathcal{P}(\bigcap_{\alpha \in I} A_\alpha)$$

Proof. $\square$

UD Problem 9.10

$$\bigcap_{\alpha \in I} \mathcal{P}(A_\alpha) = \mathcal{P}(\bigcap_{\alpha \in I} A_\alpha)$$

Proof.

$$x \in \bigcap_{\alpha \in I} \mathcal{P}(A_\alpha)$$

□

UD Problem 9.10

$$\bigcap_{\alpha \in I} \mathcal{P}(A_\alpha) = \mathcal{P}(\bigcap_{\alpha \in I} A_\alpha)$$

Proof.

$$\begin{aligned} & x \in \bigcap_{\alpha \in I} \mathcal{P}(A_\alpha) \\ \iff & \forall \alpha \in I : x \in \mathcal{P}(A_\alpha) \end{aligned}$$

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Proof.

$$\begin{aligned} & x \in \bigcap_{\alpha \in I} \mathcal{P}(A_\alpha) \\ \iff & \forall \alpha \in I : x \in \mathcal{P}(A_\alpha) \\ \iff & \forall \alpha \in I : x \subseteq A_\alpha \end{aligned}$$

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$$\bigcap_{\alpha \in I} \mathcal{P}(A_\alpha) = \mathcal{P}(\bigcap_{\alpha \in I} A_\alpha)$$

Proof.

$$\begin{aligned} & x \in \bigcap_{\alpha \in I} \mathcal{P}(A_\alpha) \\ \iff & \forall \alpha \in I : x \in \mathcal{P}(A_\alpha) \\ \iff & \forall \alpha \in I : x \subseteq A_\alpha \\ \iff & x \subseteq \bigcap_{\alpha \in I} A_\alpha \end{aligned}$$

□

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Proof.

$$\begin{aligned}
& x \in \bigcap_{\alpha \in I} \mathcal{P}(A_\alpha) \\
\iff & \forall \alpha \in I : x \in \mathcal{P}(A_\alpha) \\
\iff & \forall \alpha \in I : x \subseteq A_\alpha \\
\iff & x \subseteq \bigcap_{\alpha \in I} A_\alpha \\
\iff & x \in \mathcal{P}(\bigcap_{\alpha \in I} A_\alpha)
\end{aligned}$$

□

UD Problem 9.19

$$A \times (B \setminus C) = (A \times B) \setminus (A \times C)$$


Thank You!