2-11 Heapsort

Hengfeng Wei

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Obama in a job interview at Google

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"What is most efficient way to sort a million 32-bit integers?"

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Best case Worst case Average case

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Best case　　Worst case　　Average case

Worst-case of MAX-HEAPIFY (TC 6.2-6)

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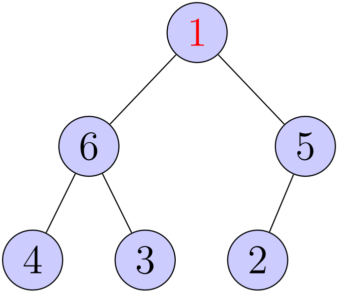

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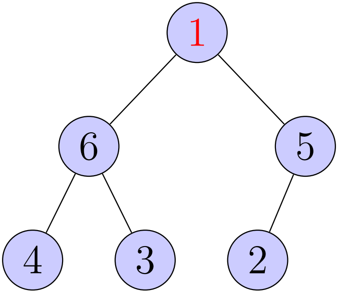

COMPARE *vs.* EXCHANGE

Worst-case of MAX-HEAPIFY (Section 6.2 of CLRS)

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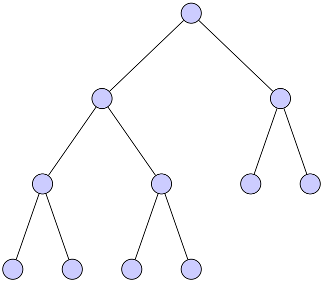

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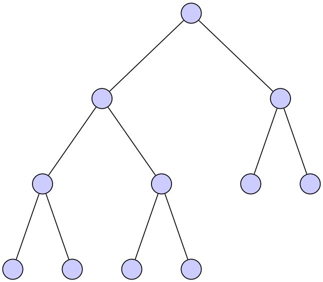

$$W(n) \leq H(n)$$

No Examples Here!

Therefore . . .

Worst-case of MAX-HEAPIFY

Show that the worst-case running time of MAX-HEAPIFY on an n-element heap is $\Theta(\log n)$.

	O	Ω	Θ
Worst-case	"power" of $\mathcal{A}$	by example	$O = \Omega$

Worst-case of HEAPSORT (TC 6.4-4)

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Non-proof.

$$\underbrace{\Theta(n)}_{\text{EXTRACT-MAX}} \times \underbrace{\Omega(\log n)}_{\text{MAX-HEAPIFY}} = \Omega(n \log n)$$

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What is wrong?

$\square$

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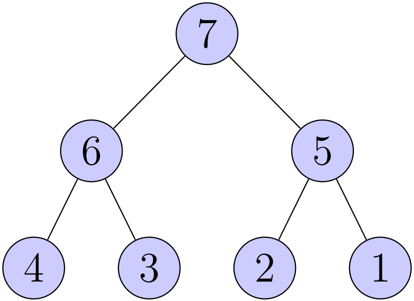
Heap in decreasing order?



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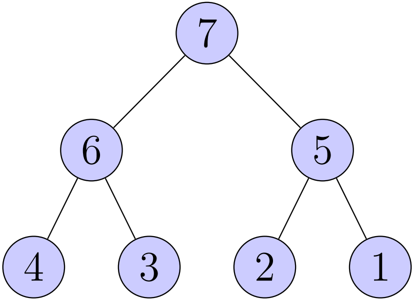
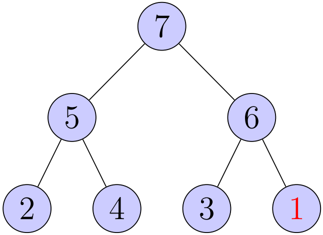
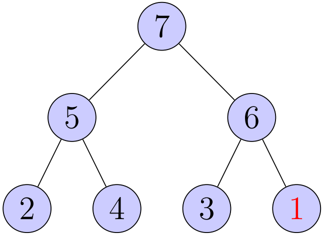
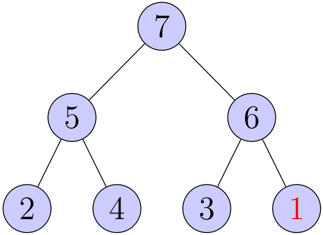


$$T(7) = 2 + 1 + 1 + 1 + 0 + 0 = 5$$


7
5
6
2
4
3
1



$$T(7) = 2 + 2 + 2 + 1 + 1 + 0 = 8$$



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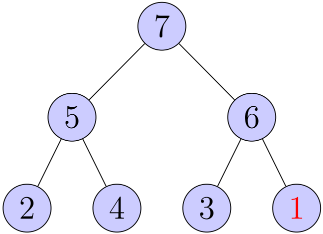

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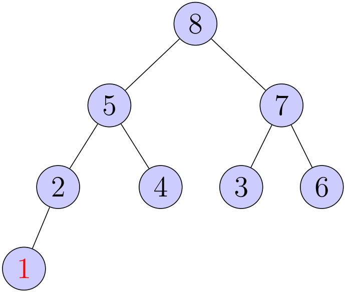

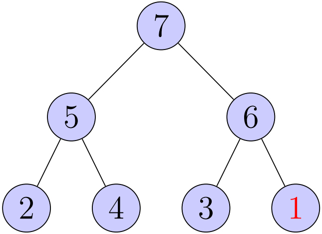

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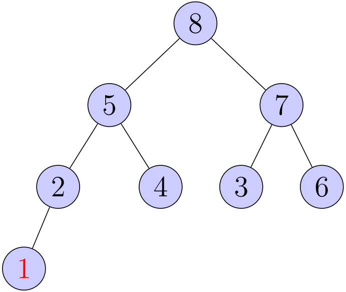
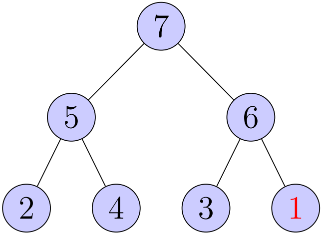


(Ex. 23, Section 5.2.3, TAOCP Vol 3)

$$\sum_{r=1}^{n-1} \lfloor \log r \rfloor$$



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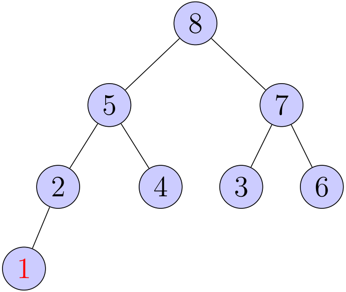

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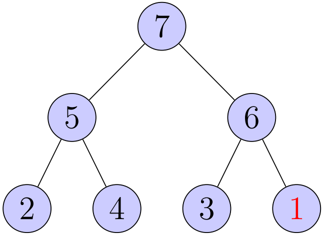
$$\sum_{r=1}^{n-1} \lfloor \log r \rfloor = n \lfloor \log n \rfloor - 2^{\lfloor \log n \rfloor + 1} + 2$$



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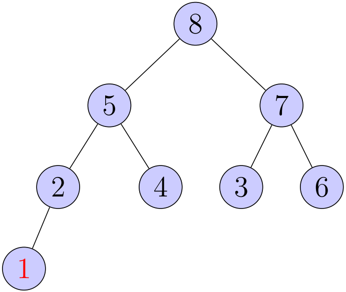

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$$\sum_{r=1}^{n-1} \lfloor \log r \rfloor = n \lfloor \log n \rfloor - 2^{\lfloor \log n \rfloor + 1} + 2 = \Omega(n \log n)$$

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No Examples Here!

$$\underbrace{\Theta(n)}_{\text{EXTRACT-MAX}} \times \underbrace{O(\log n)}_{\text{MAX-HEAPIFY}} = O(n \log n)$$

Therefore . . .

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Show that the worst-case running time of HEAPSORT is $\Theta(n \log n)$.

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Algorithm $\mathcal{A}$

Inputs $\mathcal{I}$ of size n

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Best-case			
Worst-case			

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Best-case of HEAPSORT (Ex. 32, Section 5.2.3, TAOCP Vol 3)

Prove that the number of heapsort promotions, B, is always at least $\frac{1}{2}N \log N + O(N)$, if the keys being sorted are distinct.

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The largest m elements form a subtree.

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$$B(n) \geq \frac{1}{4} n \log n + O(n) + B(\lfloor n/2 \rfloor)$$

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$$\sum_{k=1}^{\lfloor m/2 \rfloor} \lfloor \log k \rfloor = \frac{1}{2} m \log m + O(m)$$

$$B(n) \geq \frac{1}{4} n \log n + O(n) + B(\lfloor n/2 \rfloor) \implies B(n) \geq \frac{1}{2} n \log n + O(n)$$

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	O	Ω	Θ
Best-case	?	$\sim \frac{1}{2} n \log n + O(n)$	$O = \Omega$
Worst-case	$\sim n \log n$	$\sim n \log n$	$O = \Omega$

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$$\frac{1}{2} n \log n + O(n) \leq ? \leq n \log n$$

	O	Ω	Θ
Best-case	?	$\sim \frac{1}{2} n \log n + O(n)$	$O = \Omega$
Worst-case	$\sim n \log n$	$\sim n \log n$	$O = \Omega$

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$$B(n) \leq \frac{1}{2} n \log n + O(n \log \log n).$$

By Example.

“On the Best Case of Heapsort” (1994)

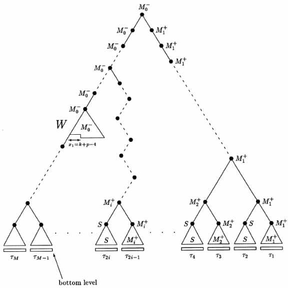

FIG. 2. Initial heap (more detailed).

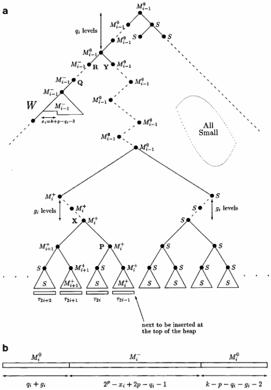

FIG. 3. (a) Odd i; (b) contents of the bottom level of τ_{2i-1}, i odd.

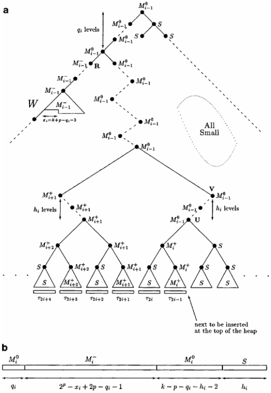

FIG. 4. (a) Even i; (b) contents of the bottom level of τ_{2i-1}, i even.

Therefore . . .

Best-case of HEAPSORT

Show that when all elements are distinct, the best-case running time of HEAPSORT is $\Theta(n \log n)$.

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Inputs $\mathcal{I}$ of size n

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Best-case			
Worst-case			
Average-case			

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Average-case			

Algorithm $\mathcal{A}$

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Average-case	$\leq$	$\geq$	

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Best-case	by example	"weakness" of $\mathcal{A}$	$O = \Omega$
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Average-case of HEAPSORT

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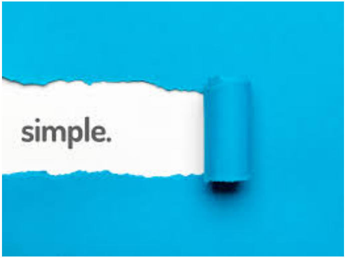

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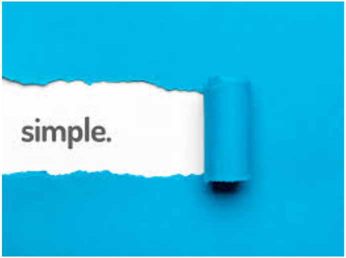

I said simple, not easy.

“By a surprisingly short counting argument.”

“The Analysis of Heapsort” (Sedgewick ; 1992)

Robert Sedgewick

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Robert Sedgewick

D. E. Knuth

“It is elegant.

“By a surprisingly short counting argument.”

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Robert Sedgewick

D. E. Knuth

“It is elegant. see exercise 30.”

Heap Identity (Additional)

$$\forall h \geq 1 : \lceil \log(\lfloor \frac{1}{2} h \rfloor + 1) \rceil + 1 = \lceil \log(h+1) \rceil$$

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$$\lfloor \log \lfloor \frac{1}{2}h \rfloor \rfloor + 1 = \lceil \log(\lfloor \frac{1}{2}h \rfloor + 1) \rceil = \lceil \log(h+1) \rceil - 1 = \lfloor \log h \rfloor$$

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(Depth of the parent of h) + 1 = Depth of h

k-way Merging (TC 6.5-9)

Give an $O(n \log k)$-time algorithm to merge k sorted lists with n elements in total into one sorted list.

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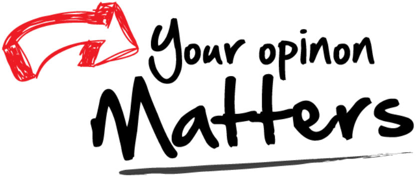
$$k = 2 \implies O(n)$$

Always maintain a min-heap of size k

whose root contains the next smallest element.


Thank
You!



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