Finite and Infinite

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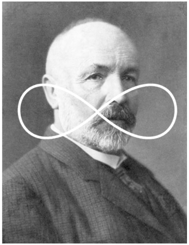
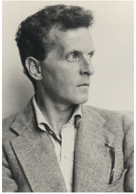


Georg Cantor (1845 – 1918)

David Hilbert (1862 – 1943)

Leopold Kronecker
(1823 – 1891)

Henri Poincaré
(1854 – 1912)



Ludwig Wittgenstein
(1889 – 1951)

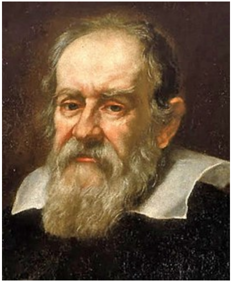
From his paradise that Cantor with us unfolded, we hold our breath in awe; knowing, we shall not be expelled.

— David Hilbert

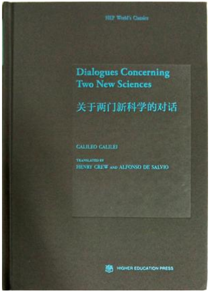
"没有人能把我们从 *Cantor* 创造的乐园中驱逐出去"

"das wesen der mathematik liegt in ihrer freiheit"

"The essence of mathematics lies in its freedom"



Galileo Galilei (1564 – 1642)



《关于两门新科学的对话》(1638)

"用我们有限的心智来讨论无限 $\cdots$"

$$S_1 = \{1, 2, 3, \cdots, n, \cdots\}$$
$$S_2 = \{1, 4, 9, \cdots, n^2, \cdots\}$$

$|S_1| = |S_2|, S_1 \subsetneqq S_2$

"部分等于全体"

说到底，"等于"、"大于"和"小于"诸性质不能用于无限，而只能用于有限的数量。 — *Galileo Galilei*

无穷数是不可能的。 — *Gottfried Wilhelm Leibniz*

这些证明一开始就期望那些数要具有有穷数的一切性质，或者甚至于把有穷数的性质强加于无穷。

相反，这些无穷数，如果它们能够以任何形式被理解的话，倒是由于它们与有穷数的对应，它们必须具有完全新的数量特征。

这些性质完全依赖于事物的本性，⋯ 而并非来自我们的主观任意性或我们的偏见。

— *Georg Cantor (1885)*

Definition (Dedekind-infinite & Dedekind-finite (Dedekind, 1888))

A set A is Dedekind-infinite if there is a bijective function from A onto some proper subset B of A.

A set is Dedekind-finite if it is not Dedekind-infinite.

Comparing Sets

$=$ $<$

Function

$$=$$

Definition ($|A| = |B|$ ($A \approx B$) (1878))

Two sets of A and B are *equipotent* if there exists a *bijection* from A to B.

"$\approx$" is an equivalence relation.

$\overline{\overline{A}}$ (two abstractions)

$\{1, 2, 3\}$ *vs.* $\{a, b, c\}$

$\{1, 2, 3, \cdots\}$ *vs.* $\{1, 3, 5, \cdots, 2, 4, 6, \cdots\}$

Definition (Finite and Infinite)

For any set X,

Finite

$$\exists n \in \mathbb{N} : |X| = n \qquad (0 \in \mathbb{N})$$

Infinite ($\neg$ finite)

$$\forall n \in \mathbb{N} : |X| \neq n$$

Definition (Finite and Infinite)

For any set X,

Countably Infinite

$$|X| = |\mathbb{N}| \triangleq \aleph_0$$

Countable

(finite $\lor$ countably infinite)

Uncountably Infinite

$$(\neg \text{ finite}) \land \Big(\neg \text{ (countably infinite)}\Big)$$

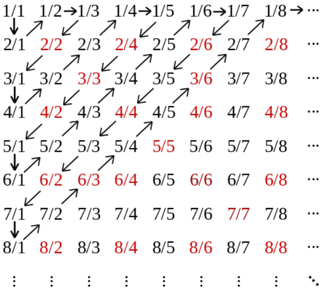
$$(\neg \text{ countable})$$

Theorem ($\aleph_0$ (1874))

$$|\mathbb{Q}| = |\mathbb{N}|$$

$|\mathbb{Q}| = |\mathbb{N}|$ (UD Problem 22.9)

$$q \in \mathbb{Q}^+ : a/b \; (a, b \in \mathbb{N}^+)$$



$$|\mathbb{N}| = |\mathbb{Z}|$$

$$|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$$

$$f : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \qquad f(m, n) = n + \frac{(m+n)(m+n+1)}{2}$$

Theorem ($\mathbb{R}$ is uncountably infinite (1874) .)

$$|\mathbb{R}| \neq |\mathbb{N}| \qquad (|\mathbb{R}| > |\mathbb{N}|)$$

Different "Sizes" of Infinity

Cantor's Diagonal Argument (1890)

Proof.

See UD Theorem 22.12. $\quad f : \mathbb{N} \xleftrightarrow[onto]{1-1} (0, 1).$ □

Theorem (Cantor's Theorem (1891))

$$|X| \neq |2^X| \qquad (|X| < |2^X|)$$

Infinite Sequences of 0's and 1's (UD Problem 22.3)

Is the set of all infinite sequences of 0's and 1's finite, countably infinite, or uncountable?

Proof.

By Cantor's diagonal argument $\implies$ uncountable. □

Nonproof.

$$f : \{\{0,1\}^*\} \to \mathbb{N}$$

$$f(x_0 x_1 \cdots) = \sum_{i=0}^{\infty} x_i 2^i$$

□

Theorem ($|\mathbb{R}|$ (1877))

$$|(0,1)| = |\mathbb{R}| = |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}^{n \in \mathbb{N}}|$$

Proof.

$$|(0,1)| = |(-\frac{\pi}{2}, \frac{\pi}{2})| = |\mathbb{R}|$$

$$(x = 0.a_1 a_2 a_3 \cdots, y = 0.b_1 b_2 b_3 \cdots) \mapsto 0.a_1 b_1 a_2 b_2 a_3 b_3 \cdots$$

□

"Je le vois, mais je ne le crois pas !"

"I see it, but I don't believe it !"

— Cantor's letter to Dedekind (1877).

Q : Then, what is "dimension"?

$$<$$

Definition ($|A| \leq |B|$)

$|A| \leq |B|$ if there exists an *one-to-one* function f from A into B.

bijection $f : A \to f(A)$ ($\subseteq B$)

Q : What about onto function $f : A \to B$?

$|B| \leq |A|$ (Axiom of Choice)

Definition ($|A| < |B|$)

$|A| < |B| \iff |A| \le |B| \land |A| \ne |B|$

$$|\mathbb{N}| < |\mathbb{R}|$$

$$|X| < |2^X|$$

Definition (Countable Revisited)

X is countable:

$$\exists n \in \mathbb{N} : |X| = n \lor |X| = |\mathbb{N}|$$

Theorem (Proof for Countable (UD Exercise 22.5))

X is countable iff there exists a one-to-one function

$$f : X \to \mathbb{N}.$$

X is countable iff

$$|X| \leq |\mathbb{N}|.$$

Subsets of Countable Set (UD Problem 22.6; UD Corollary 22.4)

Every subset B of a countable set A is countable.

Set Union (UD Problem 22.1)

Give an example, if possible, of

(c) a countably infinite collection of *pairwise disjoint* nonempty sets whose union is finite.

(b) a countably infinite collection of nonempty sets whose union is finite.

$$\Big(\{A_i : i \in R\} \quad A_i = \{1\}\Big) = \{\{1\}\}$$

$$|A| = n \implies |2^A| = 2^n$$

Slope (UD Problem 22.2 (e))

(e) the set of all lines with rational slopes

$$(\mathbb{Q}, \mathbb{R})$$

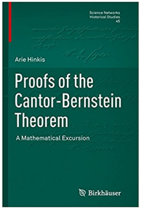
$$|\mathbb{R}| \le |\mathbb{Q} \times \mathbb{R}| \le |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$$

Q : *Is "$\leq$" a partial order?*

Theorem (Cantor-Schröder–Bernstein (1887))

$$|X| \leq |Y| \wedge |Y| \leq |X| \implies |X| = |Y|$$

$$\exists \text{ one-to-one } f : A \to B \wedge g : B \to A \implies \exists \text{ bijection } h : A \to B$$



Q : Is "$\leq$" a total order?

Theorem (PCC)

Principle of Cardinal Comparability (PCC) $\iff$ Axiom of Choice

Finite Sets

"关于有穷，我原以为我是懂的"

Definition (Finite)

X is finite if

$$\exists n \in \mathbb{N} : |X| = n$$

Theorem (Pigeonhole Principle (UD Theorem 21.2))

$$f : \{1, \cdots, m\} \to \{1, \cdots, n\} \ (m, n \in \mathbb{N}^+, m > n)$$

f is not one-to-one.

$A \setminus \{a\}$ (UD Problem 21.15)

Let A be a nonempty finite set with $|A| = n$ and let $a \in A$.
Prove that $A \setminus \{a\}$ is finite and $|A \setminus \{a\}| = n - 1$.

$$f : A \xleftrightarrow[onto]{1-1} \{1, \cdots, n\}$$

$$f|_{A \setminus \{a\}} : A \setminus \{a\} \xleftrightarrow[onto]{1-1} \{1, \cdots, n\} \setminus \{f(a)\} \xleftrightarrow[onto]{1-1} \{1, \cdots, n-1\}$$

$|A| \leq |B|$ (UD Problem 21.17)

A and B are finite sets and $f : A \to B$ is one-to-one.
Show that $|A| \leq |B|$.

By contradiction and the pigeonhole principle.

(UD Problem 21.16)

(a) A is a finite set and $B \subseteq A$. We showed that B is finite (Corollary 20.11). Show that $|B| \leq |A|$.

one-to-one $f : B \to A$

(b) A is a finite set and $B \subseteq A$. Show that if $B \neq A$, then $|B| < |A|$.

$\exists a : a \in A \land a \notin B \qquad f : B \to A \setminus \{a\} \qquad |B| \leq |A \setminus \{a\}|$

(c) If two finite sets A and B satisfy $B \subseteq A$ and $|A| \leq |B|$, then $A = B$.

By contradiction and (b).

Cardinality of $|\text{ran}(f)|$ (UD Problem 21.18)

Let A and B be sets with A finite.

$$f : A \to B$$

Prove that $|\text{ran}(f)| \leq |A|$.

one-to-one $g : \text{ran}(f) \to A$

(No Axiom of Choice Here)

$f : A \to A$ (UD Problem 21.19)

Let A be a finite set.

$$f : A \to A$$

Prove that

$$f \text{ is one-to-one} \iff f \text{ is onto.}$$

$$\Longleftarrow$$

$$\Longrightarrow$$

$$\forall y \in A \; \exists x \in A : y = f(x)$$

By contradiction.

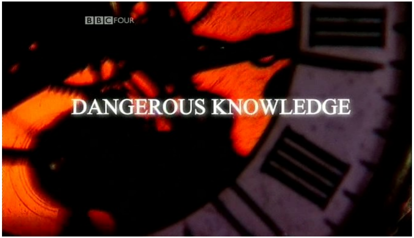
$\forall y$, choose $x : (g : g(y) = x)$

$$f' : A \to A \setminus \{a\}$$

g is bijective.

$$f(g(y)) = f(x) = y \implies f = g^{-1}$$

Dangerous Knowledge (BBC 2007)



Continuum Hypothesis (CH):

$$\boxed{c = \aleph_1}$$

$$c = 2^{\aleph_0} = \aleph_1$$

☞ Dangerous Knowledge (22:20)

Independence from ZFC:

Kurt Gödel (1940) CH cannot be disproved from ZF.

Paul Cohen (1964) CH cannot be proven from the ZFC axioms.


Thank You!

Your opinon Matters

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