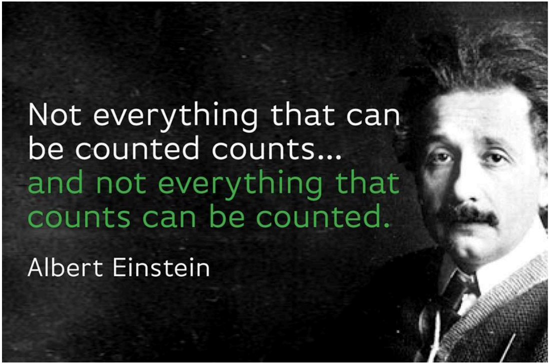
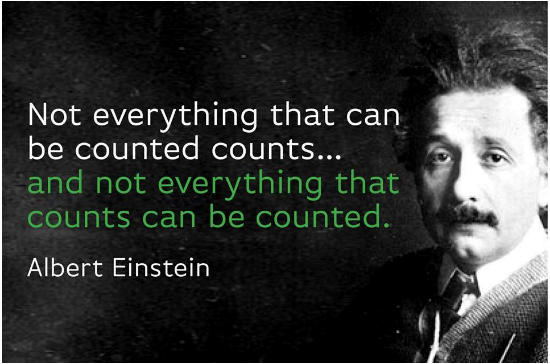
2-3 Counting

魏恒峰

hfwei@nju.edu.cn

2018 年 04 月 11 日


Not everything that can be counted counts...
and not everything that counts can be counted.
Albert Einstein



所以，学好“2-3 组合与计数”是多么重要!

Paring up (CS : $1.2 - 15$)

A tennis club has $2n$ members. We want to pair up the members by twos for singles matches.

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(b) Suppose that we also determine who serves first for each pairing. In how many ways can we specify our pairs?

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$$\frac{(2n)!}{2^n \cdot n!} \cdot 2^n = \frac{(2n)!}{n!}$$

Passing out Apples to Children

k-Permutation $(\text{CS} : 1.2 - 5)$

We need to pass out k distinct apples (pieces of fruit) to n children such that *each child may get at most one apple*.

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(b) What if $k > n$?

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$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

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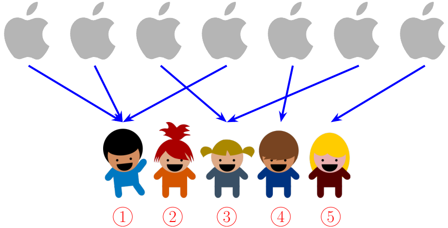

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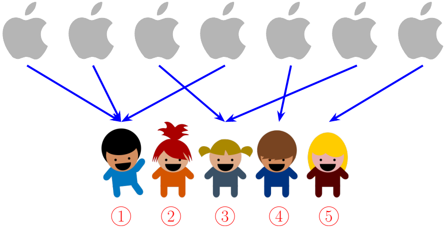

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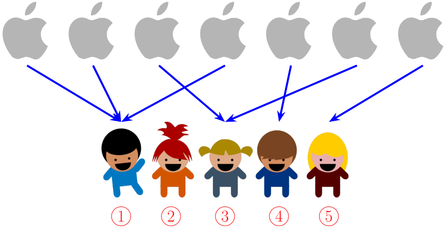

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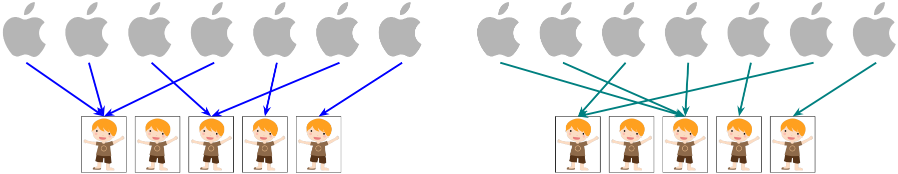

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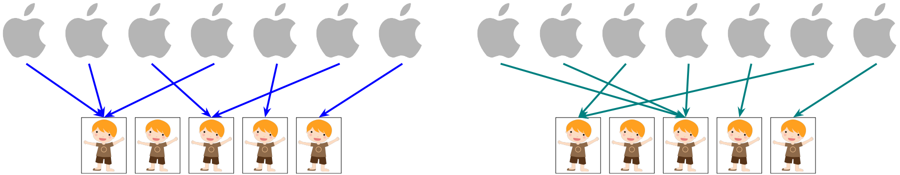

Integer partition of k into $\leq n$ parts

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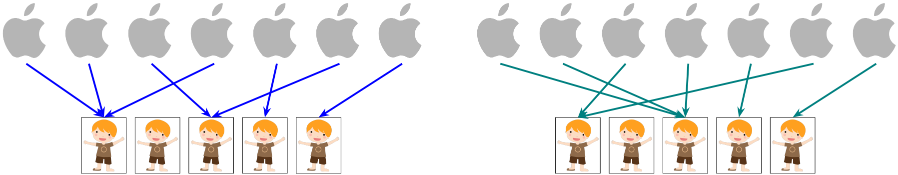

Integer partition of k into $\leq n$ parts

Theorem (G. H. Hardy, Ramanujan (1918))

$$p(k) \triangleq \sum_{x=1}^{x=n} p_x(k) \sim \frac{1}{4\sqrt{3}n} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

$p_n(k)$: # of partitions of k into n parts

Theorem (Recurrence for $p_n(k)$)

$$p_n(k) = p_{n-1}(k-1) + p_n(k-n)$$

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Proof.

$$1 \le x_1 \le x_2 \le \cdots \le x_n$$

Case $x_1 > 1$

$$1 < x_1 \le x_2 \le \cdots \le x_n$$

Case $x_1 = 1$

$$1 = x_1 \le x_2 \le \cdots \le x_n$$

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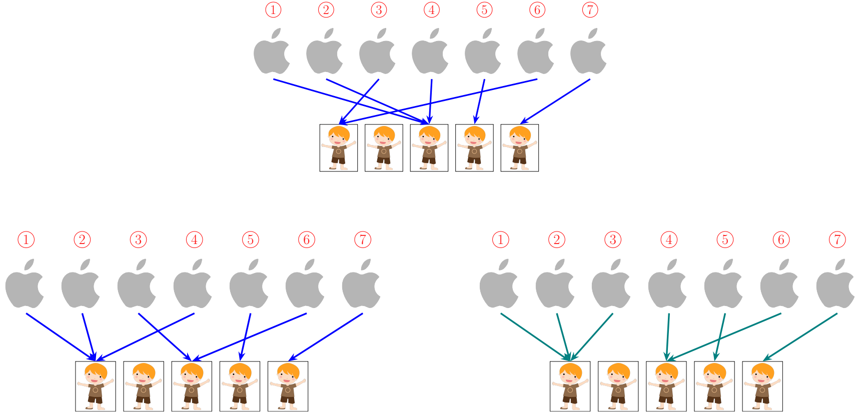

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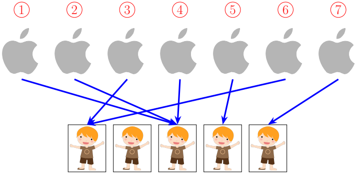

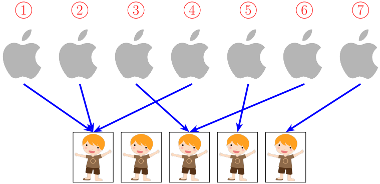

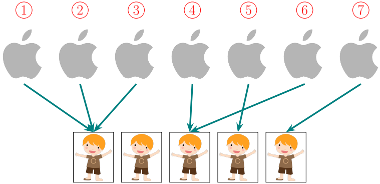

Set partition of $[1 \cdots k]$ into $\leq n$ parts

Set Partition (CS : $1.5-12$)

$$S(n,k)\ \left(\left\{ n \atop k \right\}\right) : \#\text{ of set partitions of } [1\cdots n] \text{ into } k \text{ classes}$$

Set Partition (CS : $1.5 - 12$)

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Stirling number of the second kind

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Theorem (Recurrence for $S(n,k)$)

$$S(0,0) = 1, \quad S(n,0) = S(0,n) = 0 \ (n > 0)$$

$$S(n,k) = S(n-1,k-1) + kS(n-1,k), \quad n > 0, k > 0$$

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Proof.

$$S(n,k) = \underbrace{S(n-1,k-1)}_{n \text{ is alone}} + \underbrace{kS(n-1,k)}_{n \text{ is not alone}}$$

□

Computing $\binom{n}{k}$ (CS $1.5 : 14$)

```
procedure BINOM(n, k)                          ▷ Required: n ≥ k ≥ 0
    if k = 0 ∨ n = k then
        return 1
    return BINOM(n − 1, k) + BINOM(n − 1, k − 1)
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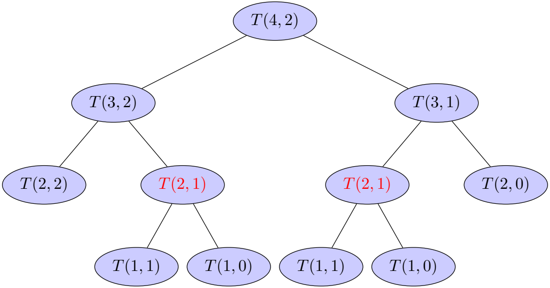

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$$A(n,k) = 1 + A(n-1,k) + A(n-1,k-1)$$

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(ii) # of recursive calls of $\textsc{Binom}$:

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$$\boxed{T(n,k) = T(n-1,k) + T(n-1,k-1) + c}$$

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

```
procedure BINOM(n, k)                          ▷ Required: n ≥ k ≥ 0
    for i ← 0 to n do
        B[i][0] ← 1
        B[i][i] ← 1
    for i ← 2 to n do
        for j ← 1 to k do
            B[n][k] ← B[n − 1][k] + B[n − 1][k − 1]
    return B[n][k]
```


Thank
You!