3-1 Dynamic Programming

(Part II: “Theory”)

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Definition (Optimal Substructure)

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How to show "NO"?

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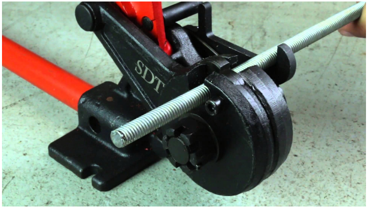
Relative to Subproblems

Rod Cutting



Optimal Substructure of Rod-Cutting (Problem 15.3-5)

Limit l_i : $\#$ of pieces of length i, $\quad 1 \le i \le n$

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Limit l_i : # of pieces of length i, $1 \le i \le n$

$n = 4$

length i	1	2	3	4
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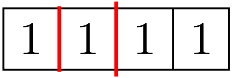

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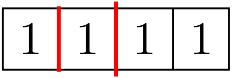

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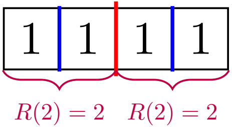


Well done

"Show that the optimal-substructure property described in Section 15.1 *no longer holds."*

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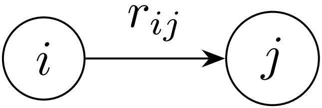
$$R(i, L) = \max_{1 \le j \le i} \Big(p_j + R(i - j, L[j \mapsto L_j - 1]) \Big)$$

Currency Exchange (Problem 15.3-6)

$1, 2, \cdots, n$ currencies

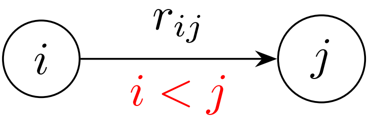


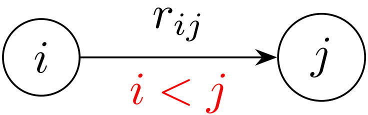
Currency Exchange (Problem 15.3-6)

$1, 2, \cdots, n$ currencies

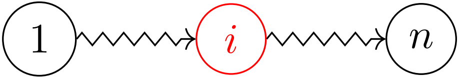
$$i \xrightarrow{r_{ij}} j$$

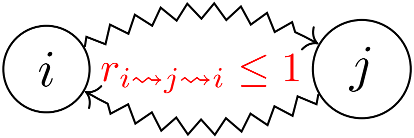
c_k : Commission charged for k trades

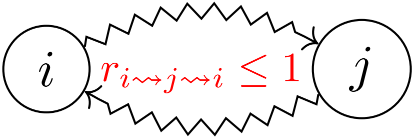

r_{ij}
i
j
$i < j$



An optimal sequence of trades from 1 *to* n *through* i*:*




i
$r_{i \rightsquigarrow j \rightsquigarrow i} \leq 1$
j



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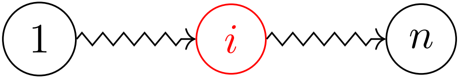
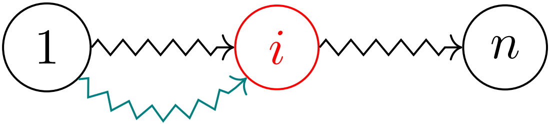


$$i \rightleftarrows j \qquad r_{i \rightsquigarrow j \rightsquigarrow i} \leq 1$$

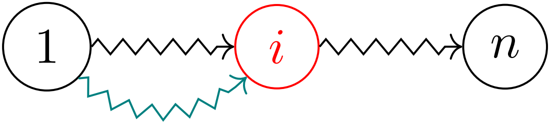
An optimal sequence of trades from 1 to n through i:

$$1 \rightsquigarrow i \rightsquigarrow n$$

By Contradiction.



$\textsc{Case I}: s_{1 \rightsquigarrow i} \cap s_{i \rightsquigarrow n} = \emptyset$



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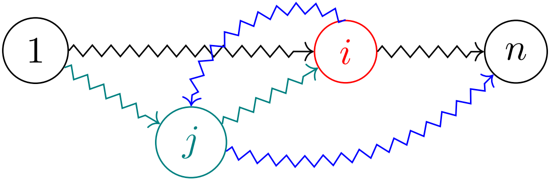

$$\textsc{Case II} : j \in s_{1 \rightsquigarrow i} \cap s_{i \rightsquigarrow n}$$

$$\begin{aligned} & 1 \rightsquigarrow j \rightsquigarrow n \\ \geq\ & 1 \rightsquigarrow j \rightsquigarrow i \rightsquigarrow j \rightsquigarrow n \\ =\ & 1 \rightsquigarrow j \rightsquigarrow i \rightsquigarrow n \\ >\ & 1 \rightsquigarrow i \rightsquigarrow n \end{aligned}$$

Longest Path Problem

To find a *simple* path of maximum length from s to t in a graph.

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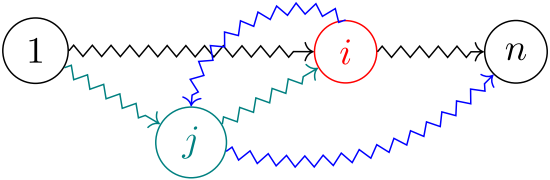

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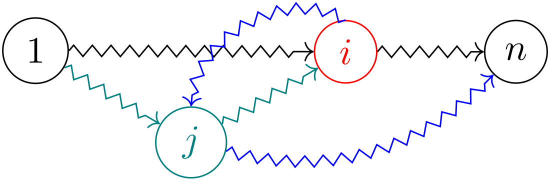

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$$> 1 \rightsquigarrow i \rightsquigarrow n$$

Does the longest path problem really have no optimal substructure?

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WAIT WAIT...
DON'T TELL ME!®
FROM NPR® & WBEZ® CHICAGO

$l(s, t, A)$: *the longest path from* s *to* t *avoiding the vertices in* A

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$$l(s,t,A) = 1 + \max_{x \notin A \cup \{s\} : (s,x) \in E} l(x,t,A \cup \{s\}) \quad (s \neq t)$$

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$$O(n2^n)$$

efficiency $\cdots$

The Knapsack Problem

The Change-making Problem (Problem 14.13)

- Coins values: $x_1 \dots x_n$
- Amount: v
- Is it possible to make change for v?

The Change-making Problem (Problem 14.13 (2), Problem 14.2 (Subset sum))

(2) Without repetition $(0/1)$

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Subproblem: $C[i, w]$: Make change for w using only values of $x_1 \dots x_i$?

Goal: $C[n, v]$

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Make choice: Using value x_i or not?

Recurrence:

$$C[i, w] = C[i-1, w] \vee (C[i-1, w-x_i] \wedge w \geq x_i)$$

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Init:

$$C[i, 0] = \text{true } \forall i = 0 \ldots n$$
$$C[0, w] = \text{false, if } w > 0$$
$$C[0, 0] = \text{true}$$

Time: $O(nv)$

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(3) Unbounded repetition with $\leq k$ coins

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Subproblem: $C[i, w, l]$: Possible to make change for w with $\leq l$ coins of values of $x_1 \ldots x_i$?

Goal: $C[n, v, k]$

The Change-making Problem (Problem 14.13 (3))

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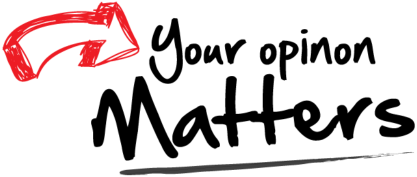
THANK YOU



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