3-6 Decompositions of Graphs

(DFS, DAG, Toposort, Cycle, SCC)

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October 29, 2018

Robert Tarjan

John Hopcroft

"For fundamental achievements
in the design and analysis of algorithms and data structures."

— Turing Award, 1986

SIAM J. COMPUT.
Vol. 1, No. 2, June 1972

DEPTH-FIRST SEARCH AND LINEAR GRAPH ALGORITHMS*

ROBERT TARJAN†

Abstract. The value of depth-first search or "backtracking" as a technique for solving problems is illustrated by two examples. An improved version of an algorithm for finding the strongly connected components of a directed graph and an algorithm for finding the biconnected components of an undirect graph are presented. The space and time requirements of both algorithms are bounded by $k_1V + k_2E + k_3$ for some constants k_1, k_2, and k_3, where V is the number of vertices and E is the number of edges of the graph being examined.

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"DFS is a powerful technique with many applications."

The Hammer of DFS

Power of DFS:

Graph Traversal $\Longrightarrow$ Graph Decomposition

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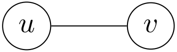
Structure! Structure! Structure!

Graph *structure* induced by DFS:

states of v

types of



Graph *structure* induced by DFS:

states of v

types of u — v

lifetime of v:

v : $\text{d}[v], \text{f}[v]$

$\text{f}[v]$: TOPOSORT, SCC

$\text{d}[v]$: BICOMP (Problem 22-2)

Definition (Classification of Edges)

We can define four edge types in terms of the depth-first forest G_π produced by a DFS on G:

Tree edge: edge in G_π

Back edge: $\rightarrow$ ancestor

Forward edge: $\rightarrow$ descendant (*nontree* edge)

Cross edge: $\rightarrow (\neg\text{ancestor}) \land (\neg\text{descendant})$

DFS on Undirected Graphs (Problem 22.3-6)

Classifying an edge (u, v) as a tree edge or a back edge according to whether (u, v) or (v, u) is encountered first during the depth-first search

is equivalent to

classifying it according to the ordering of the four types in the classification scheme.

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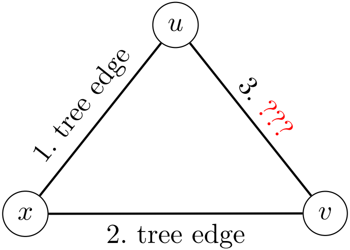

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answered 16 hours ago

A Apass.Jack

3,173 2 25

Thanks. However, I am still confused. I have added an example to explain my confusion. Could you please have a look at it? – hengxin 3 hours ago

I am checking ... It looks like the answer is clear to me. – Apass.Jack 3 hours ago

I will let you try following the procedure in the book step by step for the next few minutes. Or tell me if you have already tried. (Hopefully, I will visit your university...) (this comment will be removed later.) – Apass.Jack 3 hours ago

I am going to update my answer now. It may take 5 minutes to half an hour. – Apass.Jack 2 hours ago

:) I am waiting (both on the Internet and in my university). – hengxin 2 hours ago

add a comment

Theorem (Theorem 22.10)

In a depth-first search of an undirected graph G, every edge of G is either a tree edge or a back edge.

Proof.

Let (u, v) be an arbitrary edge of G, and suppose without loss of generality that $u.d < v.d$. Then the search must discover and finish v before it finishes u (while u is gray), since v is on u's adjacency list.

If the first time that the search explores edge (u, v), it is in the direction from u to v, then v is undiscovered (white) until that time, for otherwise the search would have explored this edge already in the direction from v to u. Thus, (u, v) becomes a tree edge.

If the search explores (u, v) first in the direction from v to u, then (u, v) is a back edge, since u is still gray at the time the edge is first explored. $\square$

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"First Type"	*vs.*	"First Time"
tree edge	$\iff$	tree edge
back edge	$\iff$	back edge

$$
\begin{array}{ccc}
\text{“First Type”} & \Longleftarrow & \text{“First Time”} \\
\text{tree edge} & \Longleftarrow & \text{tree edge} \\
\text{back edge} & \Longleftarrow & \text{back edge}
\end{array}
$$

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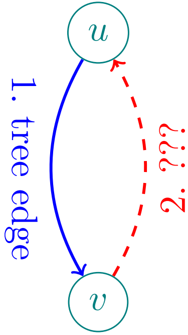
back edge	$\Longleftarrow$	back edge



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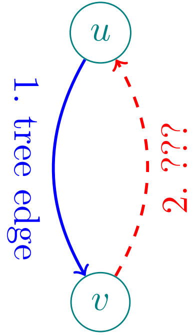

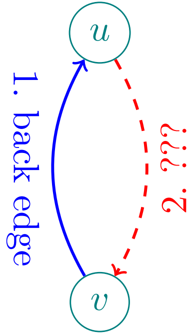

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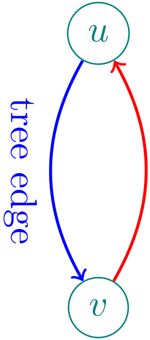

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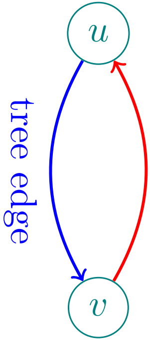
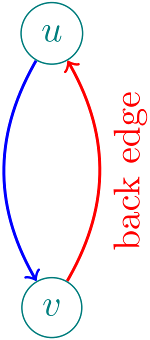
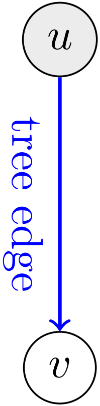
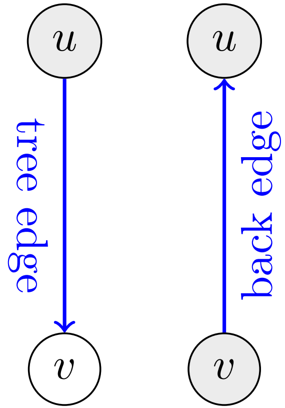





u
tree edge
v


u
v
tree edge
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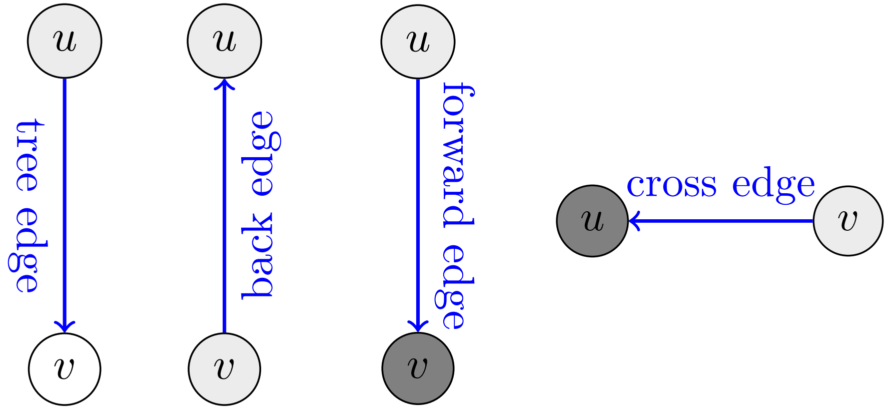
u
v
tree edge
u
v
back edge
u
v
forward edge
u
v
cross edge

Edge Types and Lifetime of Vertices in DFS

$$\forall u \to v :$$

- tree/forward edge: $[_u\ [_v\ ]_v\ ]_u$
- back edge: $[_v\ [_u\ ]_u\ ]_v$
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On digraphs:

$$\nexists \text{ back edge} \iff \text{DAG} \iff \exists \text{ topo. ordering}$$

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Sort vertices in *decreasing* order of their *finish* times.

Cycle Detection (Problem 22.4-3)

Whether an undirected graph G contains a cycle?

$$O(|V|)$$

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$$O(|V|)$$

$$\text{tree: } |E| = |V| - 1 \implies \text{ check } |E| \geq |V|$$

Cycle Detection

	Digraph	Undirected graph
DFS		
BFS		

Cycle Detection

	Digraph	Undirected graph
DFS	back edge $\iff$ cycle	
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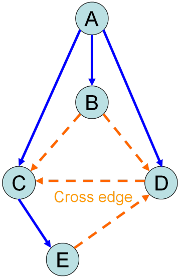

Theorem (Digraph as DAG)

Every digraph is a dag of its SCCs.

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Two tiered structure of digraphs:

digraph $\equiv$ a dag of SCCs

SCC: equivalence class over reachability

$\boxed{\text{digraph} \equiv \text{a dag of SCCs}}$

Kosaraju's SCC algorithm, 1978

"SCCs can be topo-sorted

in decreasing order of their highest finish time."

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Semiconnected Digraph (Problem 4.14)

$$\forall u, v \in V : u \rightsquigarrow v \lor v \rightsquigarrow u$$

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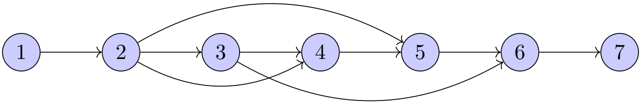

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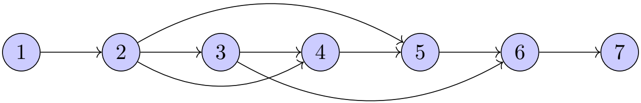

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Tarjan's TOPOSORT + Check edges (v_i, v_{i+1})

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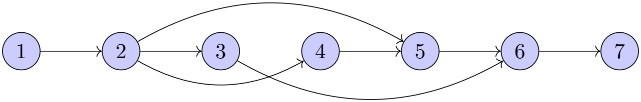


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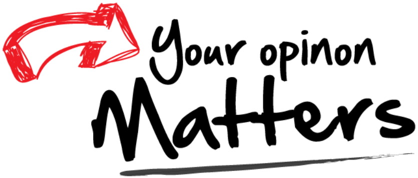

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