1-10 Set Theory (III): Functions

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$$y \in f(X) \iff \exists x \in X : y = f(x)$$

$$x \in f^{-1}(Y) \iff f(x) \in Y$$

$$a \in A \implies f(a) \in f(A)$$

$$f(A_1 \cup A_2) = f(A_1) \cup f(A_2) \qquad f(\bigcup_{\alpha \in I} A_\alpha) = \bigcup \{f(A_\alpha) \mid \alpha \in I\}$$

$$f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2) \qquad f(\bigcap_{\alpha \in I} A_\alpha) \subseteq \bigcap \{f(A_\alpha) \mid \alpha \in I\}$$

UD Problem 16.14

$$f : A \to B \qquad g_1, g_2 : B \to A$$

$$f \circ g_1 = f \circ g_2$$

(a) Show that if f is bijective, then $g_1 = g_2$.

(b) If $g_1 \circ f = g_2 \circ f$ and f is bijective, must $g_1 = g_2$?

$$f^{-1} \circ (f \circ g_1) = f^{-1} \circ (f \circ g_2)$$

$$(g_1 \circ f) \circ f^{-1} = (g_2 \circ f) \circ f^{-1}$$

Theorem

$$f \circ g_1 = f \circ g_2 \land f \textit{ is injective} \implies g_1 = g_2.$$

$$\begin{aligned}
& (f \circ g_1)(x) = (f \circ g_2)(x) \\
\implies & f(g_1(x)) = f(g_2(x)) \\
\implies & g_1(x) = g_2(x) \quad (f \text{ is injective})
\end{aligned}$$

Theorem

$$g_1 \circ f = g_2 \circ f \wedge f \textit{ is surjective} \implies g_1 = g_2.$$

$$\forall b \in B : \exists a \in A : b = f(a)$$

$$g_1(b) = g_1(f(a)) = g_2(f(a)) = g_2(b)$$

(Composition) UD Problem 16.22

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that f is onto and $f \circ f \circ f = f$. Prove that f is bijective.

$$
\begin{aligned}
& f \text{ is onto} \\
\implies & \forall y \in \mathbb{R} : \exists x \in \mathbb{R} : f(x) = y \\
\implies & \forall y \in \mathbb{R} : \exists x \in \mathbb{R} : f^3(x) = f^2(y) \\
\implies & \forall y \in \mathbb{R} : \exists x \in \mathbb{R} : f(x) = f^2(y) \\
\implies & \forall y \in \mathbb{R} : \exists x \in \mathbb{R} : y = f^2(y) \\
\implies & \forall y \in \mathbb{R} : y = f^2(y) \\
\implies & f^2 = Id_{\mathbb{R}}
\end{aligned}
$$

Theorem

If $f : X \to X$, *then*

$$f^2 = Id_X \implies f \text{ is injective.}$$

$$\begin{aligned} & f(x_1) = f(x_2) \\ \implies & f^2(x_1) = f^2(x_2) \\ \implies & Id_X(x_1) = Id_X(x_2) \\ \implies & x_1 = x_2 \end{aligned}$$

Image (UD Problem 17.22)

$$f : A \to B, \quad A_1, A_2 \subseteq A$$

(i) If $f(A_1) = f(A_2)$, must $A_1 = A_2$?

(ii) When is $f(A_1) = f(A_2) \implies A_1 = A_2$?

$$\begin{aligned}
& a_1 \in A_1 \\
\implies & f(a_1) \in f(A_1) \\
\implies & f(a_1) \in f(A_2) \\
\implies & \exists a_2 \in A_2 : f(a_2) = f(a_1) \\
\implies & a_1 = a_2 \quad \text{(if } f \text{ is injective)} \\
\implies & a_1 \in A_2
\end{aligned}$$

(Inverse Image) UD Problem 17.23

$$f : A \to B, \quad B_1, B_2 \subseteq B$$

(i) If $f^{-1}(B_1) = f^{-1}(B_2)$, must $B_1 = B_2$?

(ii) When is $f^{-1}(B_1) = f^{-1}(B_2) \implies B_1 = B_2$?

$$\begin{aligned}
& b_1 \in B_1 \\
\implies & \exists a_1 \in A : f(a_1) = b_1 \in B_1 \quad \text{(if } f \text{ is surjective)} \\
\implies & \exists a_1 \in A : a_1 \in f^{-1}(B_1) \\
\implies & \exists a_1 \in A : a_1 \in f^{-1}(B_2) \\
\implies & \exists a_1 \in A : f(a_1) \in B_2 \\
\implies & b_1 \in B_2 \quad (f(a_1) = b_1, \text{ since } f \text{ is a function})
\end{aligned}$$

Monotonicity

Assume that $F : \mathcal{P}(A) \to \mathcal{P}(A)$ and that F has the monotonicity property:

$$X \subseteq Y \subseteq A \implies F(X) \subseteq F(Y).$$

Define

$$B = \bigcap \{X \subseteq A \mid F(X) \subseteq X\}$$

$$C = \bigcup \{X \subseteq A \mid X \subseteq F(X)\}.$$

(a) Show that $F(B) = B$ and $F(C) = C$.

(b) Show that if $F(X) = X$, then $B \subseteq X \subseteq C$.

$$F(X) = X \implies F(X) \subseteq X \land X \subseteq F(X) \implies B \subseteq X \subseteq C$$

$$B = \bigcap \{X \subseteq A \mid F(X) \subseteq X\}$$

$$F(B) \subseteq B$$

$$\begin{aligned}
F(B) &= F(\bigcap \{X \subseteq A \mid F(X) \subseteq X\}) \\
&\subseteq \bigcap \{F(X) : X \subseteq A \land F(X) \subseteq X\} \\
&\subseteq \bigcap \{X : X \subseteq A \land F(X) \subseteq X\} \\
&= B
\end{aligned}$$

$$B = \bigcap \{X \subseteq A \mid F(X) \subseteq X\}$$

$$B \subseteq F(B)$$

$$F(B) \subseteq B \implies F(F(B)) \subseteq F(B) \implies B \subseteq F(B)$$

$$C = \bigcup \{X \subseteq A \mid X \subseteq F(X)\}.$$

$$C \subseteq F(C)$$

$$\begin{aligned}
C &= \bigcup \{X \subseteq A \mid X \subseteq F(X)\} \\
&\subseteq \bigcup \{F(X) \subseteq A \mid X \subseteq F(X)\} \\
&= F(\bigcup \{X \subseteq A \mid X \subseteq F(X)\}) \\
&= F(C)
\end{aligned}$$

$$C = \bigcup\{X \subseteq A \mid X \subseteq F(X)\}.$$

$$F(C) \subseteq C$$

$$C \subseteq F(C) \implies F(C) \subseteq F(F(C)) \implies F(C) \subseteq C$$


Thank
You!