

score: 400

500

## 3-2 Greedy Algorithms

(How to justify your greed?)

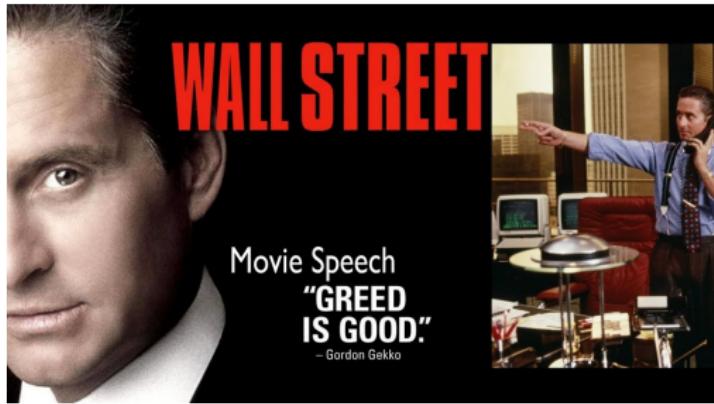
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*“GREED IS GOOD.”*



*“BUT”*

*“Greedy algorithms without **proofs** are considered wrong.”*

## Proof: Inductive Exchange Argument

Theorem (Greedy-Choice Property)

*There is an optimal solution that makes the greedy choice.*

Theorem (Optimal Substructure Property)

*A problem exhibits optimal substructure if an optimal solution to the problem contains within it optimal solutions to subproblems.*

## Live with Your Deadlines (Additional Problem)

$HW_i : (Length : t_i, \ Deadline : d_i)$



4

$OT : 2$

$PS : 5$

$OJ : 6$

$PA : 8$

$Finish\ time : f_i = s_i + t_i$

$\textcolor{blue}{Lateness} : l_i = f_i - d_i \quad (0 \text{ if } f_i \leq d_i)$

$$\min (L \triangleq \max_i l_i)$$

# *What are your greedy strategies?*

**“I love deadlines.  
I like the  
whooshing  
sound they  
make as they  
fly by. ”**

Douglas Adams

Shortest Job First:  $\min t_i$

$$t_1 = 1, d_1 = 100$$

$$t_2 = 10, d_2 = 10$$

Urgentest Job First:  $\min(d_i - t_i)$

$$t_1 = 1, d_1 = 2$$

$$t_2 = 10, d_2 = 10$$

## *Earliest Deadline First*



## Proof: Inductive Exchange Argument.

$OPT : J_1^*, J_2^*, \dots, J_n^*$

$G : J_1, J_2, \dots, J_n, \quad (d_i \leq d_j, \forall i \leq j)$

$\exists J_i : d_{J_i} < d_{J_1^*}$

$$d_{J_i} < d_{J_1^*}$$

$J_1^*$		$J_i^*$		$J_i$		
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$J_i$		$J_i^*$		$J_1^*$		
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$$l'_{J_1^*} = f'_{J_1^*} - d_{J_1^*} = f_{J_i} - d_{J_1^*} < f_{J_i} - d_{J_i} = l_{J_i}$$

$$l'_{J_i^*} = f'_{J_i^*} - d_{J_i^*} < f_{J_i} - d_{J_i^*} \leq f_{J_i} - d_{J_i} = l_{J_i}$$

## Theorem

*All schedules with no idle time and no inversions have the same maximum lateness.*

## Theorem (Optimal Substructure Property)

A problem exhibits *optimal substructure* if an optimal solution to the problem contains within it optimal solutions to subproblems.

$$L(s, S)$$

The optimal lateness obtainable from scheduling the jobs in  $S$

which are available to start at the common start time  $s$ .

What is the first job to schedule?

$$L(s, S) = \min_{1 \leq i \leq n} \left( \max (L(0, \{J_i\}), L(t_i, J \setminus \{J_i\})) \right)$$

$$J = \{J_1, J_2, J_3\}$$

$$t_1 = 6, t_2 = 2, t_3 = 2$$

$$d_1 = 2, d_2 = 7, d_3 = 8$$

$$\mathcal{O} : J_1, J_3, J_2 \quad \quad \quad \mathcal{O}' : J_2, J_3$$

$$L(0, \{J_1, J_2, J_3\}) = 4 \quad \quad \quad L(6, \{J_2, J_3\}) = 2$$

$$\mathcal{O}' \not\subset \mathcal{O}$$

$$\exists \text{ } \mathcal{O} \triangleq J_1, J_2, J_3 : \mathcal{O}' \subset \mathcal{O}$$

## Theorem

*The greedy solution has no idle time and no inversions.*

## Theorem

*There is an optimal schedule with no idle time.*

## Theorem

*There is an optimal schedule with no idle time and no inversions.*

*By Exchange Argument.*

## Theorem

*All schedules with no idle time and no inversions have the same maximum lateness.*

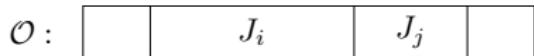
## Theorem

*There is an optimal schedule with no idle time and no inversions.*

### Immediate Inversion

$$d_j < d_i$$

$$l'_j \leq l_j$$

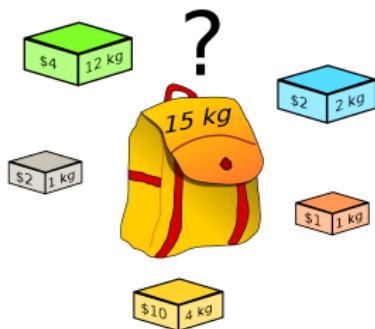


$$l'_i = f_j - d_i < f_j - d_j = l_j$$



## Fractional Knapsack Problem (Problem 16.2-1)

Prove that the fractional knapsack problem has the greedy-choice property.



$\exists$  an optimal solution which contains the greedy choice.

$$\frac{v_1}{w_1} \geq \frac{v_2}{w_2} \geq \dots \geq \frac{v_n}{w_n}$$

As most item of  $\frac{v_1}{w_1}$  as possible.

## Change-Making Problem (Problem 16-1 (a))



25



10



5



1

*Making change for  $v$  cents using the fewest number of coins.*

$$\text{OPT} : v = 25 \times A^* + 10 \times B^* + 5 \times C^* + 1 \times D^*$$

$$\mathsf{G} : v = 25 \times A + 10 \times B + 5 \times C + 1 \times D$$

$(A^*, B^*, C^*, D^*)$  vs.  $(A, B, C, D)$

$$A^* = A$$

$$\text{OPT} : v = 25 \times A^* + 10 \times B^* + 5 \times C^* + 1 \times D^*$$

### Lemma (Properties of OPT)

- (I)  $B^* \leq 2$
- (II)  $C^* \leq 1$
- (III)  $D^* \leq 4$
- (IV)  $\neg(B^* = 2 \wedge C^* = 1)$

### Theorem

$$10 \times B^* + 5 \times C^* + 1 \times D^* \leq 24$$

$$5 \times C^* + 1 \times D^* \leq 9$$

$$1 \times D^* \leq 4$$

$$(A^*, B^*, C^*, D^*) = (A, B, C, D)$$

## Change-Making Problem (Problem 16-1 (c))



$$\{1, 3, 4\}, \quad v = 6$$

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$$G : 4, 1, 1 \quad vs. \quad OPT : 3, 3$$

*Why does the previous proof fail here?*

## Change-Making Problem (Problem 16-1 (b))

$$c^0, c^1, c^2, \dots, c^k, \quad c > 1, k \geq 1$$

$$\text{OPT} : v = \sum_{i=0}^{i=k} c^i \color{red}{a_i^*}$$

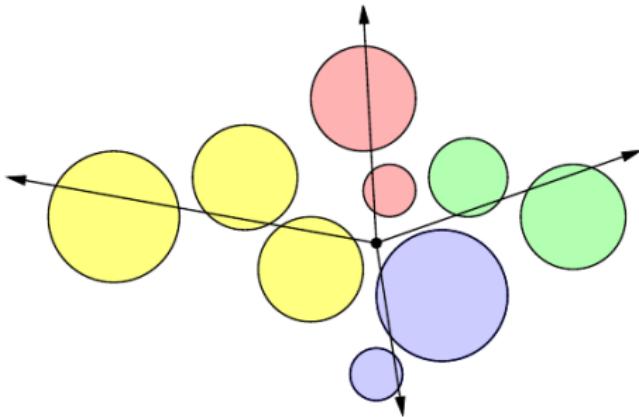
$$\text{G} : v = \sum_{i=0}^{i=k} c^i \color{blue}{a_i}$$

$$\color{red}{a_i^*} = \color{blue}{a_i}$$

# Canonical Coin Systems



## Minimum Shots Problem (Additional Problem)



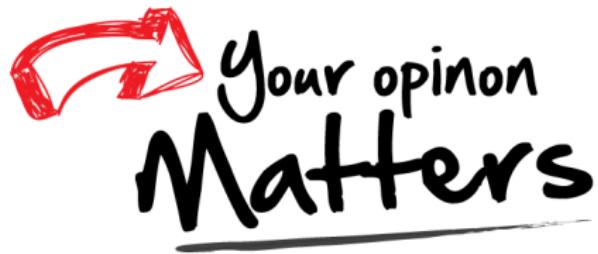
### **Assumption:**

*Possible to shoot a ray that does not intersect any balloons.*



KEEP  
CALM  
MAKE  
NO  
ASSUMPTION





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