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Abstract. In this article we introduce one of the most frequently encountered sequence in mathematics, the sequence of Catalan numbers. The numerous and varied occurrences and applications in combinatorial problems as well as their relations with other famous numerical sequences make Catalan numbers very suitable for creative teaching of mathematics at all levels.

1. Introduction

We give a brief historical overview of the exciting mathematical story of Catalan numbers. The interested reader may refer to the article [**5**]. The recent discover by Luo Jianjin in 1988 addressed the first appearance of Catalan numbers to Chinese mathematician Ming Antu (c.1692-c.1763) who wrote a book in 1731 which included some trigonometric expansions involving Catalan numbers [**4**]. Leonard Euler (1707-1783) in his letter from 1751 to Christian Goldbach (1690-1764) defines Catalan numbers C_n as the numbers of triangulations of $(n+2)$-gon and finds the generating function for these numbers. Another correspondent of Euler was Johann Segner (1704-1777) who found the recurrence relation for Catalan numbers. In 1838 Eugene Catalan (1814-1894) studied the problem of different parenthesizations of n factors. Arthur Cayley (1821-1895) counted the plane trees in 1859 by using the generating function method. Despite theirs omnipresence the Catalan numbers are named recently and the wide adoption of the name begins in early 1970's.

Richard Stanley, the leading modern combinatorist, in his famous book *Enumerative Combinatorics* [**6**] provide a list of more than 200 combinatorial interpretations of Catalan numbers. This collection is recently published in a separate monograph [**7**]. In a 2008 interview [**1**] he confessed that Catalan numbers are his favorite number sequence. Another extensive monograph on Catalan numbers is [**3**].

The Catalan numbers are determined by

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$$C_n = \frac{1}{n+1}\binom{2n}{n}, \ n = 0, 1, 2, \dots. \tag{1.1}$$

The first members of the sequence $\{C_n\}_{n\geqslant 0}$ are

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4869, 16796, 58786, 208012, \dots$$

The Catalan sequence is numbered by A000108 in the Neil Sloan's On-Line Encyclopedia of Integer Sequences (OEIS) as probably its the longest entry.

In this paper we present the major combinatorial interpretations of Catalan numbers. The choice is arbitrary and depends on space and preference of the author. The paper is only invitation to studies of the subject and may serve as starting point for own investigation. In section 2 we count binary trees and derive the major number theoretic properties. We show the connections of Catalan numbers with Fibonacci sequence and Chebyshev polynomials. In section 3 we count binary terms and introduce Tamari order on the set of binary trees. We introduce the special convex polytope called associahedron whose vertices are Catalan objects. The section 4 includes more examples on combinatorial counting: triangulations of convex polygon, upper-diagonal walking and ballot problem and pattern avoiding problems for permutations.

2. Binary trees and the number theoretic properties

In graph theory the tree is a simple graph with no cycles. The nodes of a tree which are incident to a unique edge are called leaves, the remaining nodes are called internal nodes. The rooted tree is a tree with the distinguished node called the root. Any rooted tree gives a partial order on the set of nodes by $u \leqslant v$ if u lies on the unique path from v to the root. A tree is called full binary tree if all internal nodes included the root have exactly two successors. A binary tree is the plane tree if it is embedded in the plane, so the ordering of the successors of the internal nodes are given from left to right. Let $\mathcal{T}$ be the class of full plane rooted binary trees graded by the number of internal nodes $\mathcal{T} = \sqcup_{n\in\mathbb{N}}\mathcal{T}_n$. For the reason of shortness we will simply call the elements $T \in \mathcal{T}$ the binary trees. There is a binary operation $\bullet$ on the set $\mathcal{T}$ that associates to any trees T_1 and T_2, the rooted tree $\bullet(T_1, T_2)$ obtained by grafting the roots of T_1 and T_2 on a common new root. The grafting operation $\bullet$ is not commutative since it depends on the order of factors. The class $\mathcal{T}$ may be defined recursively by

(i) $\mathcal{T}_0 = \{\bullet\}$

(ii) If $T_1 \in \mathcal{T}_{n_1}$ and $T_2 \in \mathcal{T}_{n_2}$ then $\bullet(T_1, T_2) \in \mathcal{T}_{n_1+n_2+1}$.

Define the generating function

$$C(x) = \sum_{T\in\mathcal{T}} x^{|T|}. \tag{2.1}$$

Since for $n \geqslant 1$ any binary tree $T \in \mathcal{T}_n$ is of the form $T = \bullet(T_1, T_2)$ we have

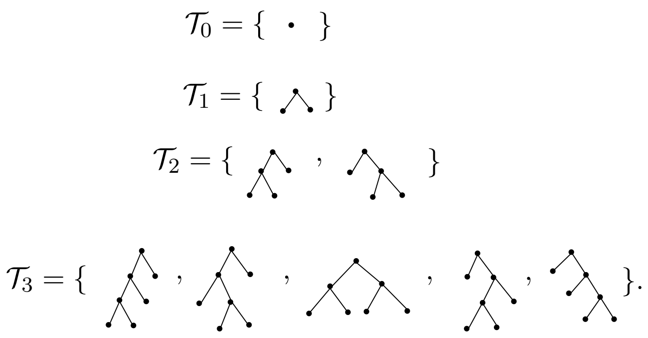


FIGURE 1. The binary trees with $n \leqslant 3$ internal nodes

$$\sum_{T \in \mathcal{T}} x^{|T|} = 1 + \sum_{T_1, T_2 \in \mathcal{T}} x^{|\bullet(T_1, T_2)|} = 1 + x \sum_{T_1 \in \mathcal{T}} x^{|T_1|} \sum_{T_2 \in \mathcal{T}} x^{|T_2|}.$$

Therefore the function $C(x)$ satisfies the functional relation

$$C(x) = 1 + xC(x)^2. \tag{2.2}$$

Consequently, since $C(0) = 0$, we obtain

$$C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

The expansion of the square root term into a power series gives

$$C(x) = \frac{1}{2x}[1 - \sum_{n=0}^{\infty} \binom{1/2}{n} (-4x)^n],$$

which by a simple calculation leads to

$$C(x) = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^n.$$

According to (2.1) we obtain that the numbers $C_n = |\mathcal{T}_n|$ of binary trees $T \in \mathcal{T}_n$ are Catalan numbers (1.1).

From the functional relation (2.2) and Caushy product formula for power series we obtain

$$\sum_{n=0}^{\infty} C_n x^n = 1 + \sum_{n=0}^{\infty} (\sum_{i+j=n} C_i C_j) x^{n+1},$$

which implies the recurrence relation for Catalan numbers

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1}, \; C_0 = C_1 = 1. \tag{2.3}$$

Let L_n be the path graph on n nodes, see Figure 2. Define the adjacency matrix $A_n = (a_{i,j})_{n \times n}$ by $a_{i,j} = 1$ if (i,j) is the edge of L_n and $a_{i,j} = 0$ otherwise. Let $q_n(u)$ be the characteristic polynomial

$$q_n(u) = \det(uE - A).$$

By expanding the determinant on elements of the first row we obtain

$$q_{n+1}(u) = uq_n(u) - q_{n-1}(u), q_0(u) = 1, q_1(u) = u.$$

The substitution $U_n(u) = q_n(2u)$ leads to

$$U_{n+1}(u) = 2uU_n(u) - U_{n-1}(u), U_0(u) = 1, U_1(u) = 2u,$$

which is exactly the recurrence relation satisfied by Chebyshev polynomials of the second kind, whose defining relation is

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}.$$

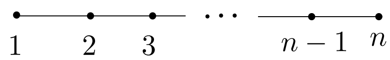


FIGURE 2. The path graph L_n

Define the generating function $G(u,v) = \sum_{n=0}^{\infty} q_n(u)v^n$. By using the recurrence relation for $q_n(u)$ and summing up we obtain

$$G(u,v) = \frac{1}{1 - uv + v^2}.$$

On the other hand, rewrite the equation (2.2) in the form

$$C(x) = \frac{1}{1 - xC(x)},$$

and iterate this identity to get the continued fraction expansion

$$C(x) = \frac{1}{1 - \frac{x}{1 - \frac{x}{1 - \frac{x}{1 - \cdots}}}}.$$

The convergents to this continued fraction are defined recursively

$$P_1(x) = 1, P_n(x) = \frac{1}{1 - xP_{n-1}(x)}, n \geqslant 1.$$

Suppose we have $P_n(x) = \frac{p_{n-1}(x)}{p_n(x)}$ for some sequence of polynomials $p_n(x)$, which therefore satisfies

$$p_{n+1}(x) = p_n(x) - xp_{n-1}(x), p_0(x) = p_1(x) = 1.$$

Specially for $x = -1$ we obtain that $F_n = p_n(-1)$ is the Fibonacci sequence and that $\frac{F_{n-1}}{F_n}$ are convergents to continued fraction expansion of the golden ratio $\phi = C(-1) = \frac{1+\sqrt{5}}{2}$. Summing up the generating function $F(x,y) = \sum_{n=0}^{\infty} p_n(x)y^n$ by using the recurrence relation for $p_n(x)$ we get

$$F(x,y) = \frac{1}{1-y+xy^2}.$$

The substitution of variables $x = 1/u^2, y = uv$ gives $F(1/u^2, uv) = G(u,v)$, which implies

$$q_n(u) = u^n p_n(\frac{1}{u^2}).$$

This identity relates Catalan numbers and Chebyshev polynomials.

3. Binary terms and Associahedron

The set of binary terms $\mathbf{T}$ on a set of variables $X = \{x_1, x_2, x_3, \ldots\}$ and a binary function symbol $\cdot$ is recursively defined:

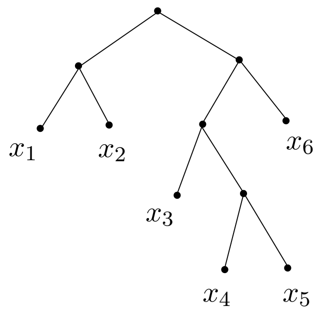
(i) every variable is a term $X \subset \mathbf{T}$
(ii) if $t_1, t_2 \in \mathbf{T}$ then $(t_1 \cdot t_2) \in \mathbf{T}$.

The length of a term $t \in \mathbf{T}_n$ is the number n of appearance of the functional symbol in the expression of t. There is a well known bijection between binary trees with n internal nodes and binary terms of the length n obtained by parenthesizes the string $x_1 \cdot x_2 \cdots x_n \cdot x_{n+1}$. To each internal node associate the binary function symbol. The $n+1$ leaves of T are ordered rightward, so to the i^{th} leaf associate the variable x_i for any $i = 1, 2, \ldots, n+1$, see Figure 3.

A grupoid is a set G with a binary operation $* : G \times G \to G$. Giving the values to variables $e : X \to G$ gives rice to the valuation map on terms $\widetilde{e} : \mathbf{T} \to G$. If the operation $*$ is associative $(a * b) * c = a * (b * c)$, for any $a, b, c \in G$ then all terms corresponding to the trees of the same size have the same value.

There is a natural partial order on the set $\mathbf{T}_n$ induced by the relation $(x \cdot y) \cdot z \leqslant x \cdot (y \cdot z)$. We define $t_1 \leqslant t_2$ if and only if the term t_2 may be obtained from the term t_1 by only rightward application of associativity law. Note that this is also an ordering of binary trees by correspondence with binary terms. At Figure 4 is presented the partial ordered set of terms on four variables.

Polytopes are high dimensional analogues of polyhedra. A *convex polytope* P in $\mathbb{R}^n$ is the convex hull $P = \mathrm{Conv}\{a_1, \ldots, a_m\}$ of a finite set of points $a_1, \ldots, a_m \in \mathbb{R}^n$. The dimension of P is its affine dimension. The supporting hyperplane of the polytope P is the affine hyperplane H such that $H \cap P$ is nonempty and P is contained in one of the half-space determined by H. The intersections of P with supporting hyperplanes are called faces of P. The set of faces is ordered



$$(x_1 \cdot x_2) \cdot ((x_3 \cdot (x_4 \cdot x_5)) \cdot x_6)$$

FIGURE 3. The binary tree and the corresponding binary term

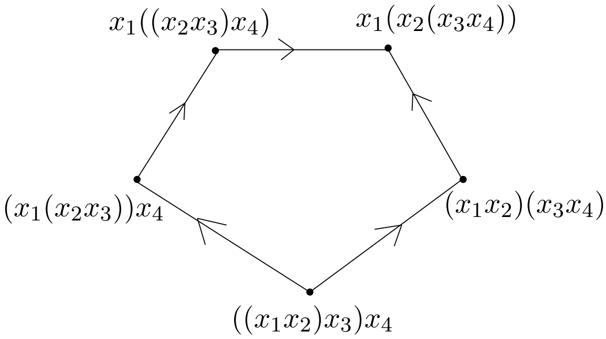


FIGURE 4. Tamari lattice $\mathbf{T}_3$

by inclusion. Each face of P is itself a polytope of the less dimension. Vertices are 0-dimensional faces, edges are 1-dimensional faces and faces of codimension one are called facets. The graph $G(P)$ of polytope is its 1-dimensional skeleton, i.e. the union of vertices and edges of P. The simplest polytopes are the simplex Δ^{n-1} and the cube I^{n-1}. The simplex is realized as the convex hull $\Delta^{n-1} = \mathrm{Conv}\{e_1, e_2, \dots, e_n\}$, where e_i is the i^{th} coordinate vector in $\mathbb{R}^n$ and the cube I^{n-1} is the convex hull of the indicator vectors $e_S \in \{0,1\}^{n-1}$ of subsets $S \subset \{1, 2, \dots, n-1\}$. Hence the numbers of vertices are $f_0(\Delta^{n-1}) = n$ and $f_0(I^{n-1}) = 2^{n-1}$. Any face of the simplex $F_S \subset \Delta^{n-1}$ is encoded by the subset $S \subset \{e_1, \dots, e_n\}$. Thus the face lattice of the simplex Δ^{n-1} is the Boolean algebra B_n on the n-elements set.

The story of Catalan numbers would not be complete without the associahedron, which is a convex polytope whose vertices enumerate Catalan objects. Thefirst appearance of the associahedron was in 1963. when James Stasheff, while studying the homotopy of loop spaces, constructed a cell complex whose vertices correspond to the binary terms of the length $n-1$. This cell complex turns out to be a boundary complex of a convex polytope As^{n-1} which is called Stasheff's polytope or the associahedron. Since the vertices of As^{n-1} correspond to the binary terms, i.e. to the binary trees, it follows that the number of vertices $f_0(As^{n-1}) = C_n$ is the Catalan number.

The easiest way to realize the associahedron As^{n-1} is by truncations of faces of the simplex Δ^{n-1}. We start with the path graph L_n on n nodes and write sets S of vertices such that the induced subgraphs are connected. Then perform truncations in a direct order on faces of Δ^{n-1} which correspond to complements S^c. For example if $n = 4$ and L_4 is the path with edges $\{12, 23, 34\}$ we perform truncations on faces $\{1, 4, 12, 14, 34, 123, 124, 134, 234\}$, see Figure 5.

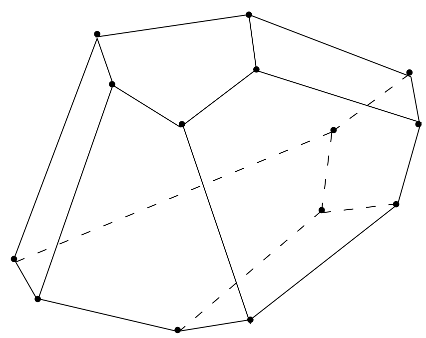


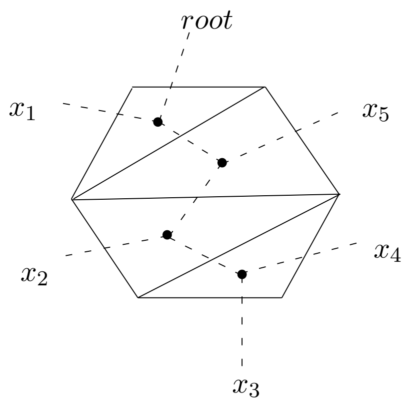
FIGURE 5. The associahedron As^3

4. More combinatorial enumeration

In this section we present three the most known enumeration problems connected with Catalan numbers.

4.1. Triangulations of the convex polygon. This is Euler's problem from 1751 and it is the first occurrence of Catalan numbers. Given a convex $(n+2)$-gon, find all different ways to divide it into triangles by nonintersecting diagonals.

To any such triangulation we can associate in a unique way the plane binary tree such that triangles correspond to internal vertices and edges of triangulations correspond to edges of the three. It remains to choose the distinguished edge of polygon which determines the root. This also uniquely associates the binary term on $(n+1)$ variables to the triangulation of $(n+2)$-gon.



$$x_1((x_2(x_3x_4))x_5)$$

FIGURE 6. A triangulation of the hexagon and the corresponding tree

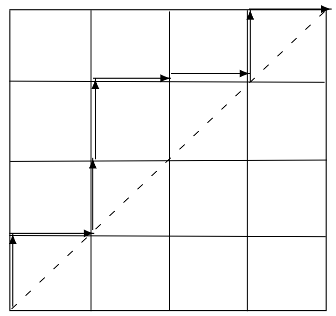
4.2. Upper-diagonal walking or Ballot problem. Given a $n \times n$-tableau on which we can walk only rightwards and upwards. What is the number of allowed paths from the initial position $(0,0)$ to the final position (n,n). The problem is also known as the ballot problem introduced in 1887 by Joseph Bertrand (1822-1900). In an election where two candidates receive the same number of votes what is the number of different possible voting so that the first candidate has never had fewer votes.

The translation of the problem is obvious. To each allowed path, equivalently to each voting order we can associate the ward in two letters, say A and B. These wards are called Dyck words and may be interpreted as possible ways of correctly matching pairs of parentheses if $A =($ and $B =)$, see Figure 7.

Let us construct the set $S = \cup_{n\geqslant 0} S_n$ of Dyck words recursively, where S_n contains words of the length n:

- ◇ $S_0 = \{\}$
- ◇ If $x \in S_{n-1}$ then $(x) \in S_n$
- ◇ If $x \in S_i$ and $y \in S_j$ then $(xy) \in S_{i+j+1}$.

Let $|S_n|$ be the number of words of the length n. By definition is obvious that $|S_n|$ satisfies the recurrence relation (2.3) for Catalan numbers.



$$ABAABBAB = ()(())()$$

FIGURE 7. The allowed path and the corresponding Dyck word

4.3. Avoiding patterns of permutations. It was Percy MacMahon (1854-1929) who first prove a result in pattern avoidance problems for permutations. His two volume *Combinatorial Analysis* (1915/6) was the one of the first monographs in Combinatorics.

A permutation ω on n letters is said to be (123)-avoiding if there are no $i < j < k$ such that $\omega(i) < \omega(j) < \omega(k)$. Let $A_n(123)$ be the set of such permutations. Count the number $|A_n(123)|$ of its elements. For example $A_3(123) = \{132, 213, 231, 312, 321\}$ and $A_4(123) = \{1432, 2143, 2413, 2431, 3142, 3214, 3241, 3412, 3421, 4132, 4213, 4231, 4312, 4321\}$. It is immediately seen that $\omega \in A_n(123)$ if and only if ω can be divided into two decreasing subsequences.

We construct a bijection between (123)-avoiding permutations and Dyck words. Given $\omega \in A_n(123)$, we say that $\omega(i)$ is a right-to-left maximum if $\omega(i) > \omega(j)$ for any $j > i$. For example, the right-to-left maxima of 58327641 are $1, 4, 6, 7, 8$. Then for the right-to-left maxima $m_1, \ldots, m_s$ we have the presentation

$$\omega = w_s m_s w_{s-1} m_{s-1} \cdots w_1 m_1,$$

where w_i is the subword (possible empty) of ω between m_{i+1} and m_i. The sequence $w_s w_{s-1} \ldots w_1$ is decreasing since ω is (123)-avoiding. Read the decomposition of ω from right to left. To each m_i we associate subword $A^{m_i - m_{i-1}}$ (with $m_0 = 0$) and to each w_i we associate $B^{|w_i|+1}$. The obtained word is a Dyck word. For example

$$58327641 \longrightarrow ABA^3BA^2BAB^3AB^2.$$

To the upper-diagonal walk on Figure 7 the associated permutation is 4231.

The reader can try to prove that $|A_n(\omega)| = C_n$ regardless to $\omega \in S_3$. The proof may be found in the famous Donald Knuth's monograph [**2**].

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