Countably infinite product of countably infinite sets has cardinality of the continuum

How to prove that the countably infinite product of countably infinite sets has cardinality of the continuum?

I know that it is uncountable thus the only thing to prove is the existence of a one-one function from the set $\prod_{n\in\mathbb{N}} \mathbb{N}$ to $\mathbb{R}$.

Thanks

(set-theory)

asked Dec 1 '15 at 18:56
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It would be enough to prove that the cardinality of the product is not greater than the cardinality of $\mathbb{R}$, assuming the continuum hypothesis is true. – Peter Dec 1 '15 at 19:02

@Peter Not, it wouldn't. The continuum hypothesis plays no role here. – Andrés E. Caicedo Dec 1 '15 at 19:16

1 A standard approach is to note that you can identify the members of $\prod_n \mathbb{N}$ with the irrational numbers (in $(0,1)$, say), for instance, via continued fractions. This shows the result, since it is easy to see that $\mathbb{R}$ has the same size as the set of irrationals. – Andrés E. Caicedo Dec 1 '15 at 19:19

Please explain why the proof that the cardinality of the product is not greater than the cardinality of $\mathbb{R}$ would not be enough ? – Peter Dec 1 '15 at 19:22

1 @Peter I think he is talking about assuming the CH. Your proof would not show the result holds in normal set theory, which is probably what OP needs to show. – Trevor Norton Dec 1 '15 at 19:38

1 Answer

There are two things which need to be proved:

- **$\mathbb{N}^\mathbb{N}$ has size at least that of $\mathbb{R}$.** To each real in $(0,1)$, we can associate an infinite string of zeroes and ones - its binary expansion. This (ignoring the expansions which are eventually all "1"s) yields an injection from $(0,1)$ into $\mathbb{N}^\mathbb{N}$. *Note that it is not enough to merely observe that $\mathbb{N}^\mathbb{N}$ is uncountable - it is consistent that there are uncountable sets of size strictly less than that of $\mathbb{R}$.*
- **$\mathbb{N}^\mathbb{N}$ injects into $\mathbb{R}$.** This is slightly more complicated. If you understand why $\mathbb{N}^\mathbb{N}$ and $2^\mathbb{N}$ have the same cardinality, it's enough to observe that the map defined above had range $2^\mathbb{N}$; if you haven't seen that yet, then here's a straightforward (if somewhat unnatural) injection: given $\alpha = (a_i)_{i\in\mathbb{N}} \in \mathbb{N}^\mathbb{N}$, let $f(\alpha)$ be the real with binary expansion

$$0.0...010...010...01...$$

where the ith block of zeroes has length $a_i + 1$.

answered Dec 8 '15 at 4:48
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