3-8 Cool? We are APSP Algorithms.

Hengfeng Wei

hfwei@nju.edu.cn

November 19, 2018

Please Help Me Out Here.

Definition (Shortest Path)

$$G = (V, E, w) : \text{ weighted digraph}$$

$$\delta(u, v) = \begin{cases} \min \left\{ w(p) : u \overset{p}{\rightsquigarrow} v \right\} & \text{if } u \rightsquigarrow v \\ \infty & \text{o.w.} \end{cases}$$

Path *vs.* Simple path

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Shortest-path Problem *vs.* Longest-path Problem

Digraph *vs.* Undirected Graph

Single Source Digraph

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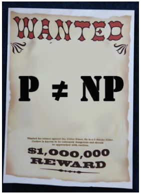

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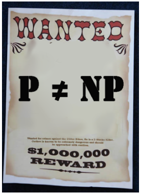

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NP-hard

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Single-source $s \rightsquigarrow$ Single-target t

Shortest Path Algorithms

Luis Goddyn, Math 408

Given an edge weighted graph (G, d), $d : E(G) \to \mathbb{Q}$ and two vertices $s, t \in V(G)$, the *Shortest Path Problem* is to find an s, t-path P whose total weight is as small as possible. Here, G may be either directed or undirected. A path in a graph is a sequence $v_0 e_1, v_1, \ldots, v_k$ of vertices and edges such that no vertex or edge appears twice, and e_i joins v_{i-1} to v_i. If G is directed, then e_i should be oriented from v_{i-1} to v_i.

Minimum-weight Perfect Matching

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And Errors.

INTERESTED?

let's talk.

Robert W. Floyd (1936–1901)

For having a clear influence on methodologies for the creation of efficient and reliable software, and for helping to found the following important subfields of computer science:

the theory of parsing, the semantics of programming languages, automatic program verification, automatic program synthesis, and analysis of algorithms

— *Turing Award*, 1978

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the weight of a shortest path from i to j

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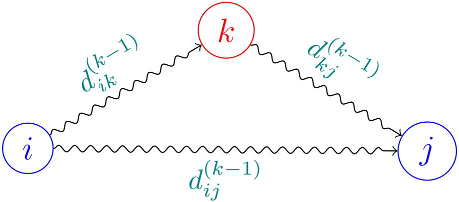
$$\boxed{D^{(n)} \triangleq \left(d_{ij}^{(n)}\right)}$$

$$k \in \mathrm{SP}_{ij}^{(k)}?$$



$$d_{ij}^{(k)} = \begin{cases} w_{ij} & k = 0 \\ \min\left\{d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right\} & k \geq 1 \end{cases}$$

procedure FLOYD-WARSHALL(W)

$D^{(0)} = W$

for $k \gets 1$ **to** n **do**

$D^{(k)} \triangleq \left(d_{ij}^{(k)}\right) \gets$ a new $n \times n$ matrix

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$$\text{Space} : \Theta(n^3)$$

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$$\text{Space}: \Theta(n^3) \implies \Theta(n^2)$$

Floyd-Warshall Made Simple (Problem 25.2-4)

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$$D^{(k-1)} \Longrightarrow D^{(k)} \qquad D \Longrightarrow D$$

"Decrease" does no harm.

Negative-weight Cycle Detection (Problem 25.2-6)

To detect negative-weight cycle (NC) using FLOYD-WARSHALL.

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“ $\implies$ ”

Proof. This is trivial.

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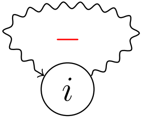
Proof.

" $\implies$ "

" $\impliedby$ "



A Cycle

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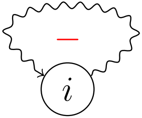

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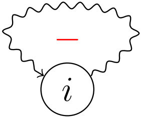

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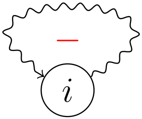

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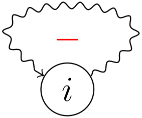

A Simple Cycle

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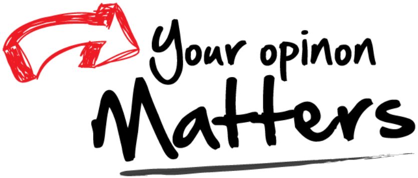
THANK YOU



Office 302

Mailbox: H016

hfwei@nju.edu.cn