Question about the proof of Cantor's Theorem

Cantor's Theorem can be stated as follows:

Theorem: Let A be a set. Then $|P(A)| > |A|$, where $P(A)$ is the power-set of A.

Proof: Suppose there exists a surjection $\phi : A \to P(A)$. Let $B = \{a \in A | a \notin \phi(a)\}$. Then since ϕ is a surjection, $B = \phi(b)$ for some $b \in A$.

Either $b \in B$ or $b \notin B$. If $b \in B$, we get a contradiction and if $b \notin B$, we also get a contradiction. QED

My question about the proof is how do we know that there exists such a set B as defined in the proof? I ask this question because it is known that a similarly-defined set $C = \{a \in A | a \notin a\}$ is known not to exist: If $C \in C$, we get a contradiction. If $C \notin C$, we also get a contradiction. This is Russell's paradox. Cantor's proof falls apart if we cannot justify the existence of set B in Cantor's proof.

(elementary-set-theory)

asked Sep 18 '15 at 3:08
Craig Feinstein
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1 The set B does exist by the axiom schema of specification. – Shahab Sep 18 '15 at 3:17

And, by the same token, the set C exists as well. – Andrés E. Caicedo Sep 18 '15 at 3:29

The set C DOES exist by the Comprehension Axiom Schema. Russell's Paradox comes from assuming $\{a | a \notin a\}$ exists. The set B also exists by Comprehension. The idea is that, given a set A, we can obtain a set of all ,and only, those members of A that have some stated property.But we cannot always omit the "A" from the sentence without allowing paradoxes. – DanielWainfleet Sep 18 '15 at 4:51

1 Answer

The set B exists by the Axiom of Specification. For the same reason, the set $C = \{a \in A | a \notin a\}$ ***does*** exist. The contradiction you speak of only arises if you assume that $C \in A$; this contradiction proves that $C \notin A$. If we assume the Axiom of Foundation, then $C = A$. What does ***not*** exist is $\{a | a \notin a\}$; note the difference.

edited Sep 18 '15 at 3:32

answered Sep 18 '15 at 3:26
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