3-7 Relax! We are SSSP Algorithms.

Hengfeng Wei

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Definition (Shortest Path)

$$G = (V, E, w): \text{ weighted digraph}$$

$$\delta(u, v) = \begin{cases} \min\{w(p) : u \rightsquigarrow^p v\} & \text{if } u \rightsquigarrow v \\ \infty & \text{o.w.} \end{cases}$$

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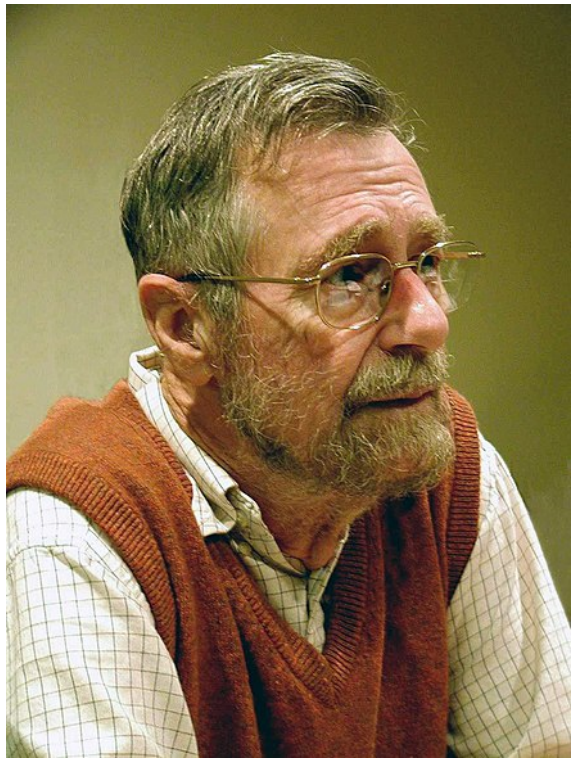
Path *vs.* Simple path



For fundamental contributions to programming as a high, intellectual challenge;

for eloquent insistence and practical demonstration that programs should be composed correctly, not just debugged into correctness;

for illuminating perception of problems at the foundations of program design.

— *Turing Award*, 1972

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Min-heap: $O(E \log V)$

Fib-heap: $O(V \log V + E)$

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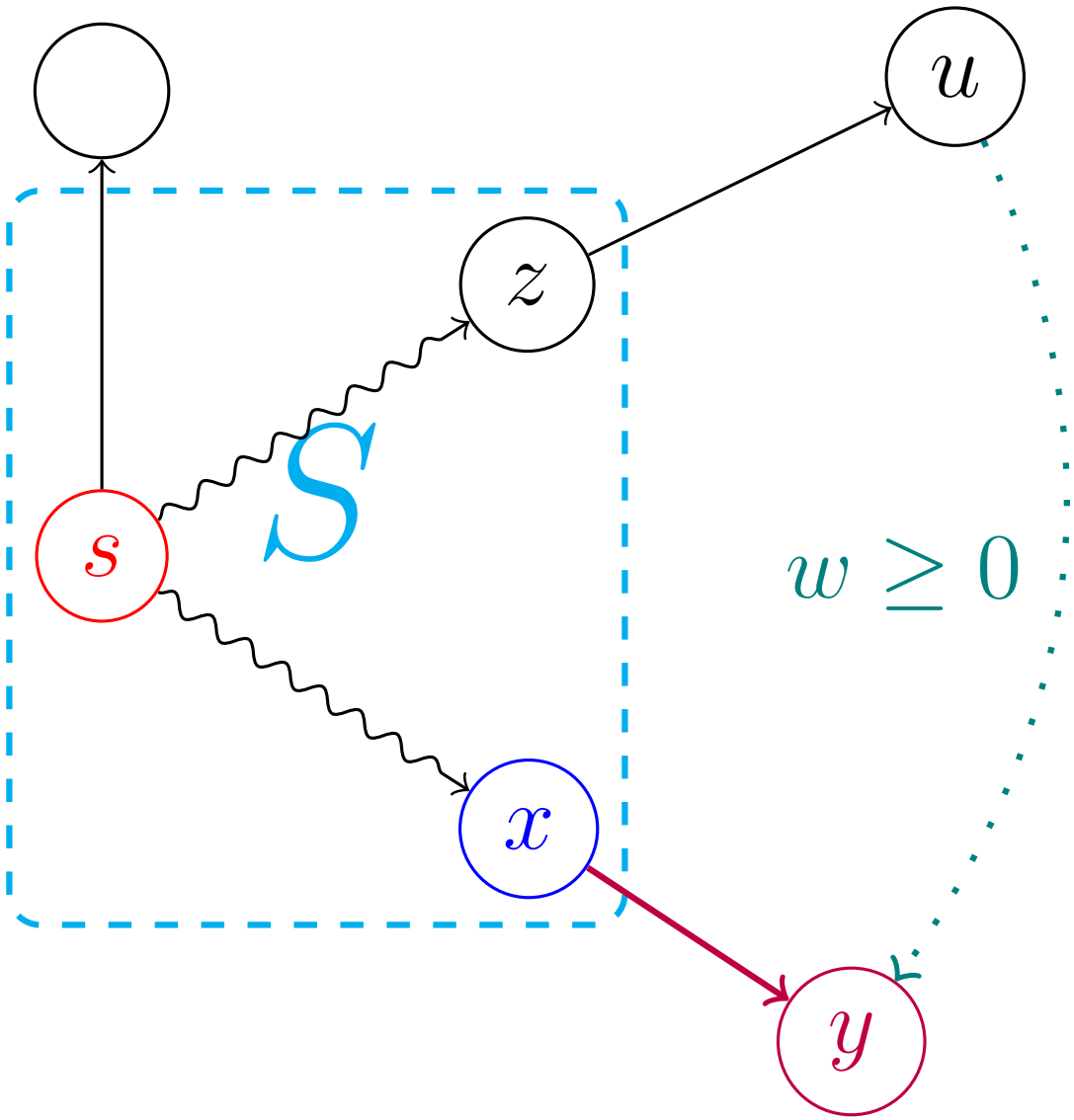
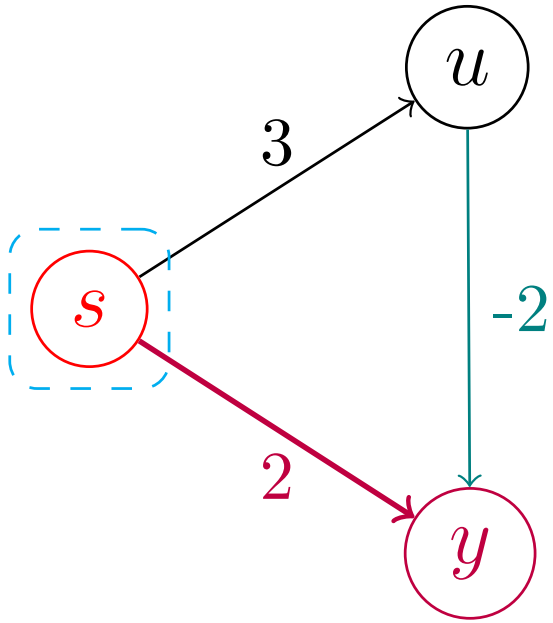

u
z
S
s
w ≥ 0
x
y

Negative-weight Edges for Dijkstra's Algorithm (Problem 24.3-2)



Negative-weight Edges for Dijkstra's Algorithm (Additional Problem 24.3-10)

- All negative-weight egdes are from s
- No negative-weight cycles

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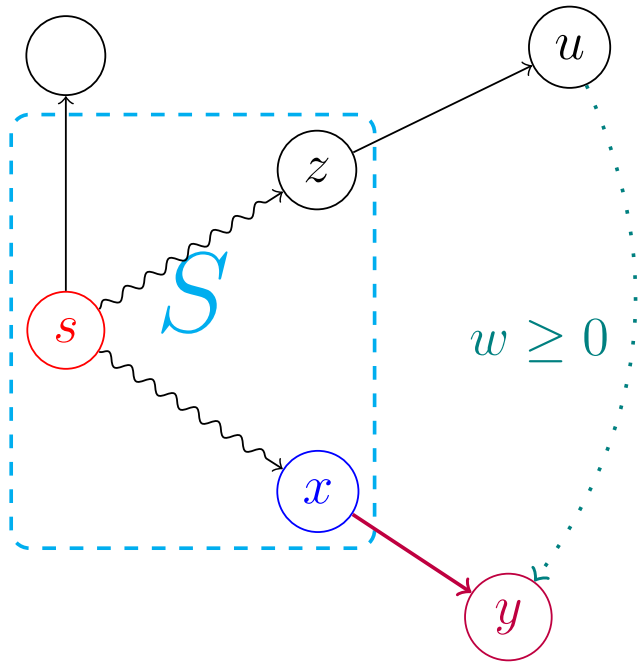

Checking Output of Dijkstra's Algorithm (Problem 24.3-4)

$$\forall v \in V : v.\pi, v.d$$

To check whether π and d match some shortest-paths tree?

$$O(V + E)$$

(1) π forms a tree

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(2) $s.d = 0$

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$$\boxed{u \triangleq v.\pi}$$

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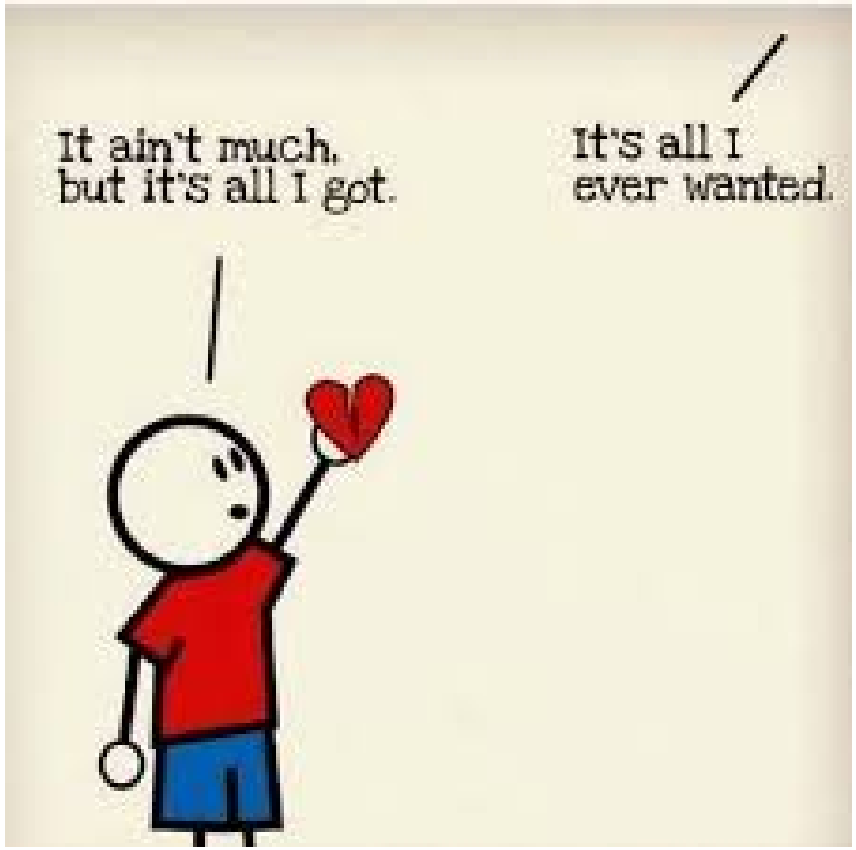
(4) $\forall (v', v) \in E : v'.d + w(v', v) \geq v.d$


It ain't much,
but it's all I got.
It's all I
ever wanted.

$$\forall v \in V : v.d = \delta(s, v)$$

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$$v.d = u.d + w(u, v) > \delta(s, v)$$

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$$\begin{cases} u.d = \delta(s, u) & \boxed{v.\pi} \\ \boxed{u.d > \delta(s, u)} \end{cases}$$

Lawler's Algorithm on DAG

$\Longleftarrow$

Dijkstra's Algorithm on Digraph with Nonnegative-weight Edges

$\Longrightarrow$

Bellman-Ford Algorithm on Digraph with Negative-weight Edges

```
1: procedure DAG-SSSP(G, w, s)
2:     Init-Single-Source(G, s)
3:     Topo-Sort(G)
4:     for u ∈ V in topo. order do
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$$\Theta(V + E)$$

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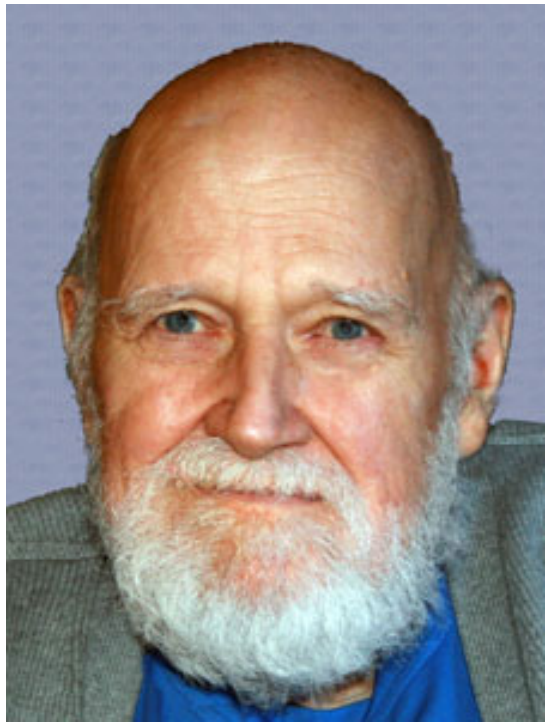
Q : Why is $\delta(s, u)$ determined right now?

Little Modification to DAG-SSSP (Problem 24.2-2)

```
1: procedure DAG-SSSP(G, w, s)
2:     Init-Single-Source(G, s)
3:     Topo-Sort(G)
4:     for the first |V| − 1 vertices u ∈ V in topo. order do
5:         for v ∈ G.Adj[u] do
6:             Relax(u, v, w)
```

Richard Bellman (1920—1984)



Lester Randolph Ford Jr. (1927—2017)

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1: procedure BELLMAN-FORD(G, w, s)
2:     INIT-SINGLE-SOURCE(G, s)
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4:         for (u, v) ∈ E do
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6:     for (u, v) ∈ E do
7:         if v.d > u.d + w(u, v) then
8:             return FALSE
9:     return TRUE
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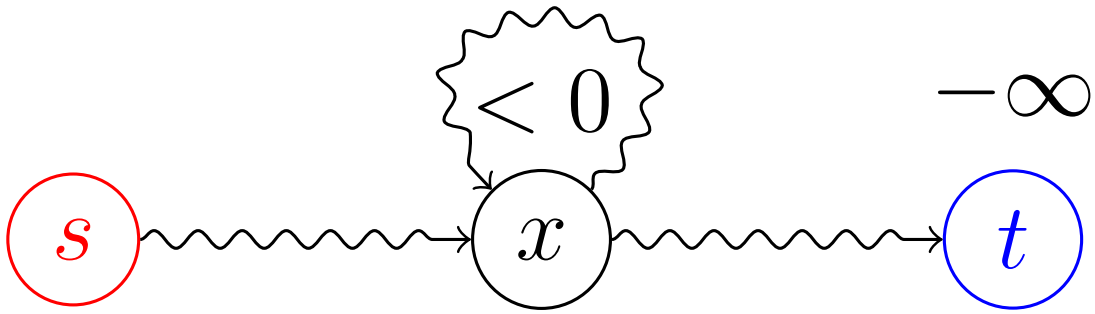
Deal with Negative-weight Cycles (Problem 24.1-4)



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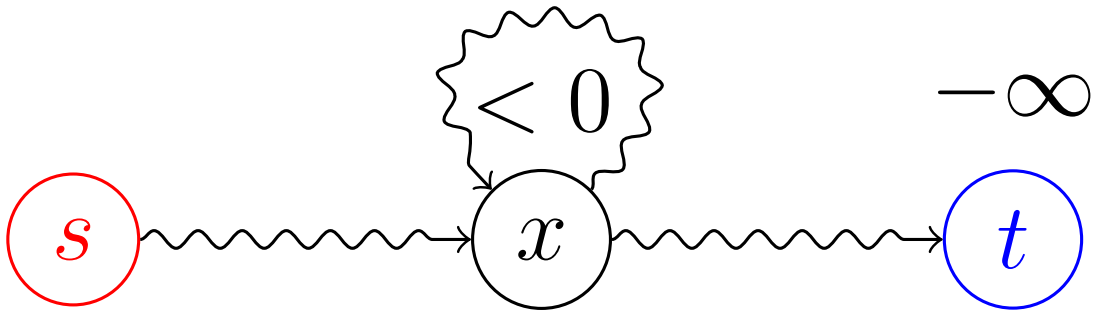

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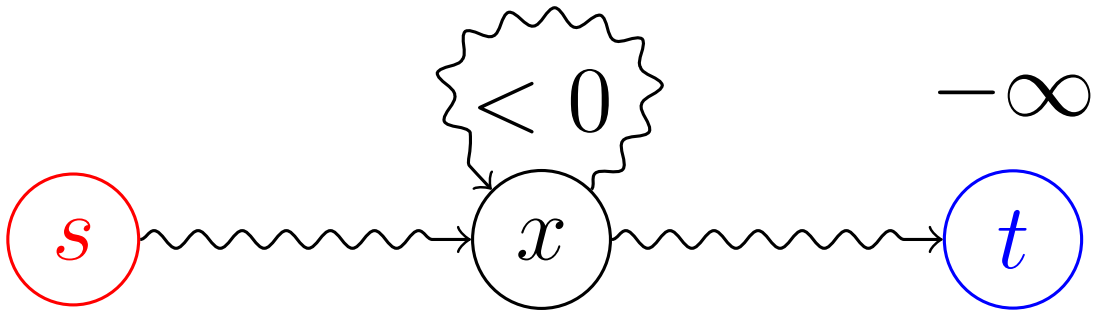

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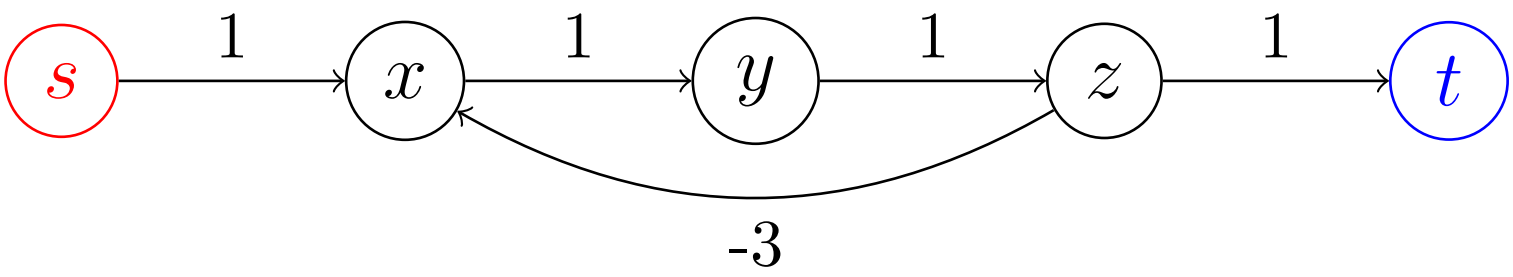
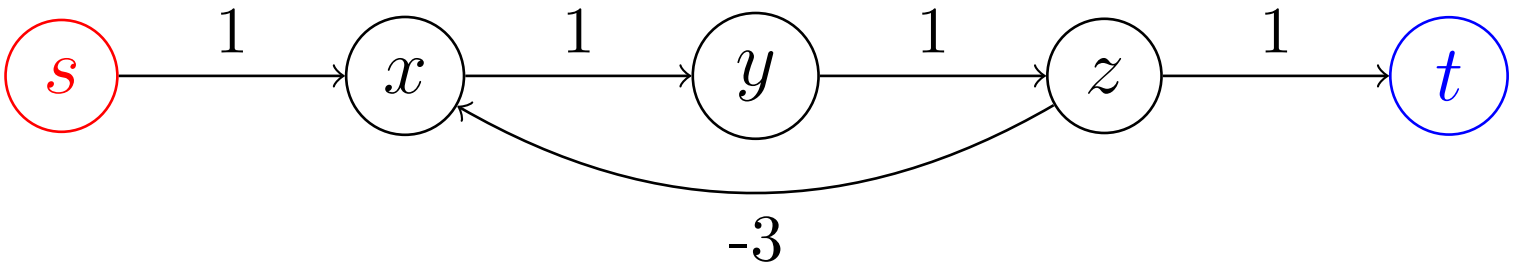

s
1
x
1
y
1
z
1
t
-3



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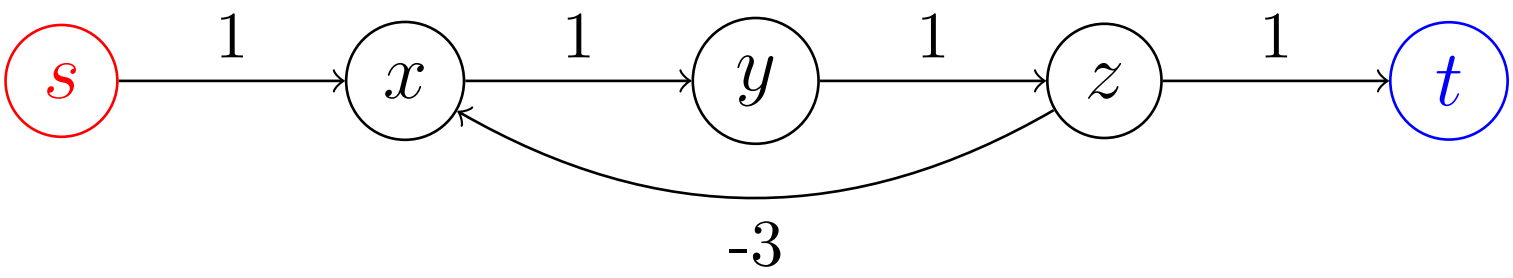

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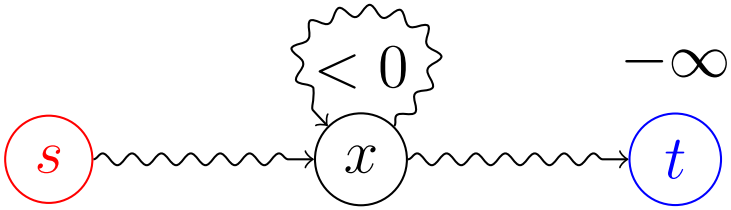

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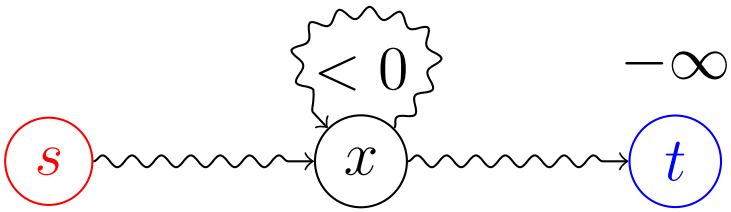

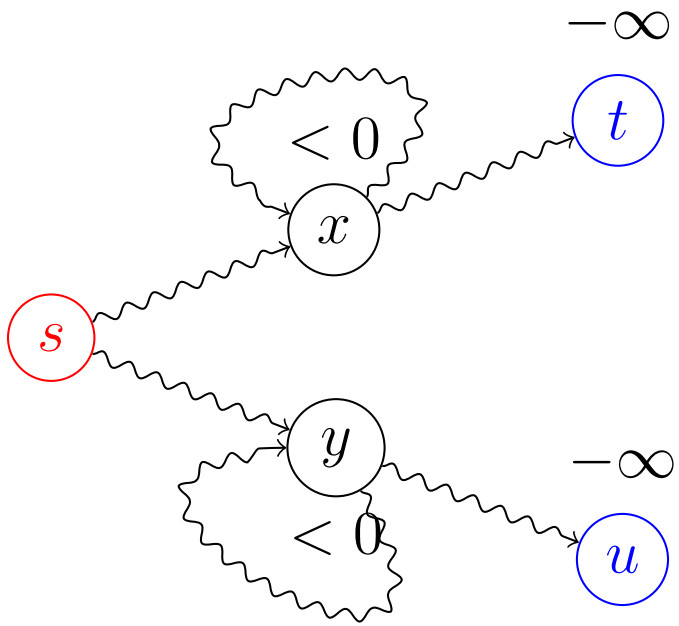

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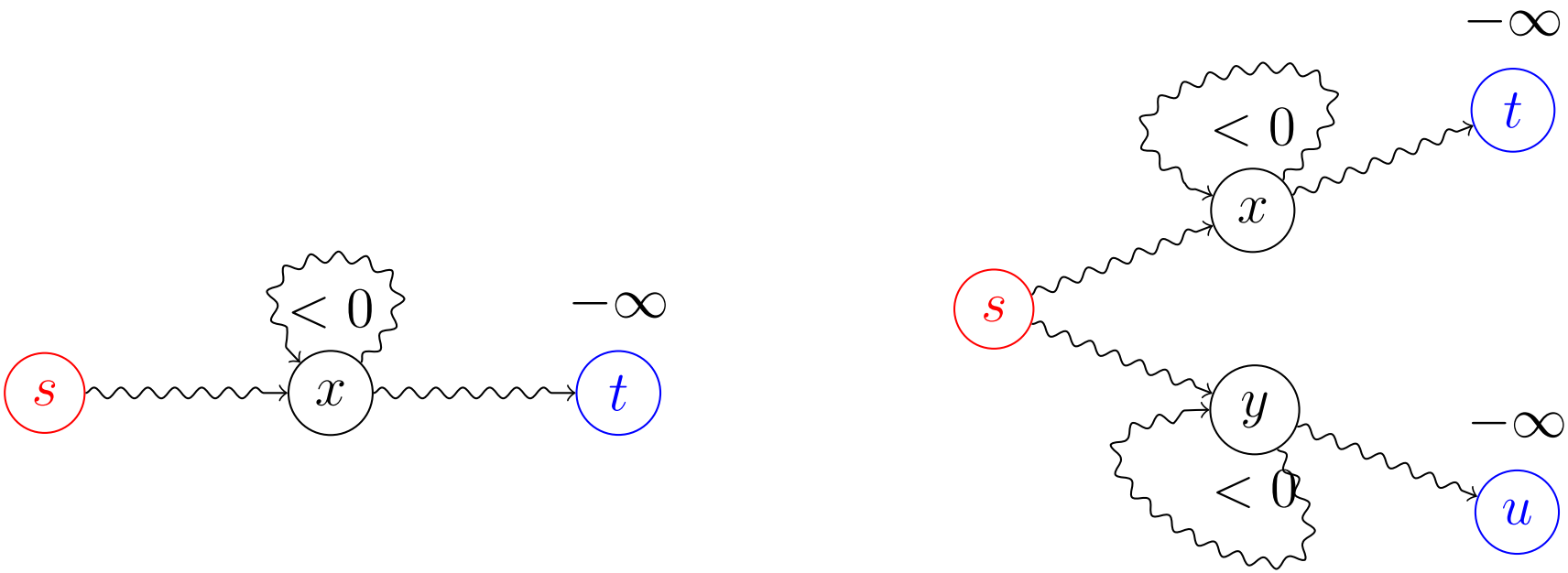

Theorem (The $|V|$-th Pass of Bellman-Ford Algorithm)

For every reachable negative-weight cycle, at least one edge of it has been relaxed in the $|V|$-th pass.

Terminate Early in Bellman-Ford Algorithm (Problem 24.1-3)

$$G = (V, E) \text{ without negative-weight cycles}$$

$$m \triangleq \min_{v \in V} \left\{ \text{Len}(\delta(s, v)) \right\} \text{ (Unknown!)}$$

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9:         if f = FALSE then
10:            return
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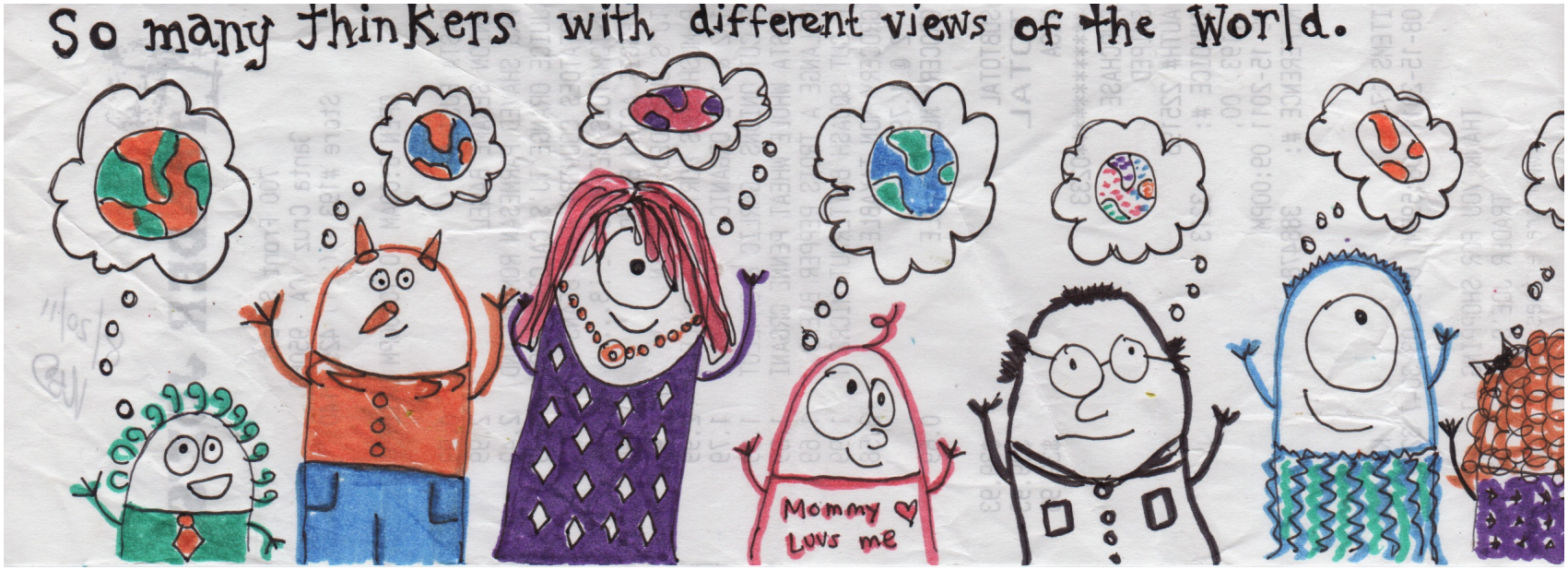
Two Different Views of Bellman-Ford Algorithms



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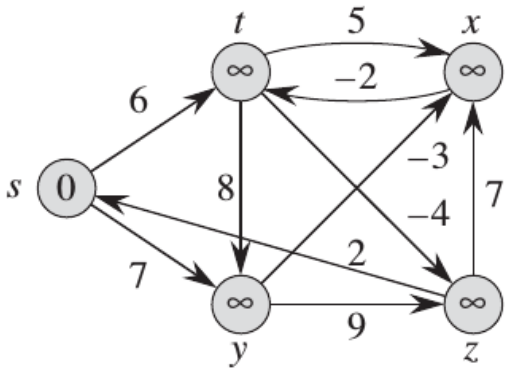
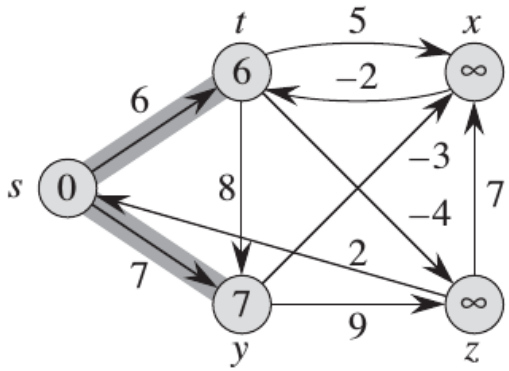
Bellman-Ford Algorithm ≡ Dijkstra's Algorithm with QUEUE

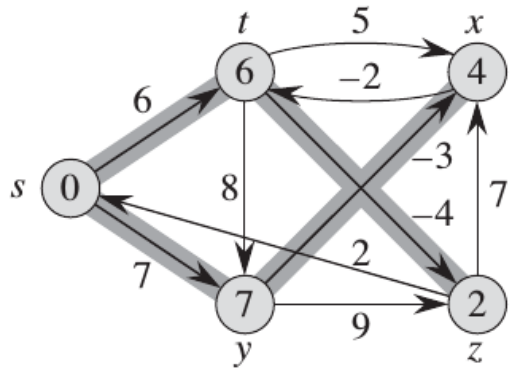
Bellman-Ford Algorithm $\equiv$ Dijkstra's Algorithm with QUEUE



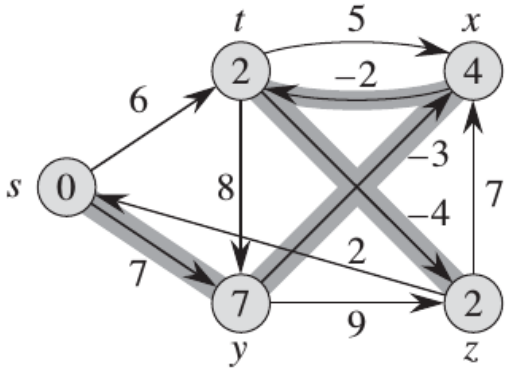
(a)



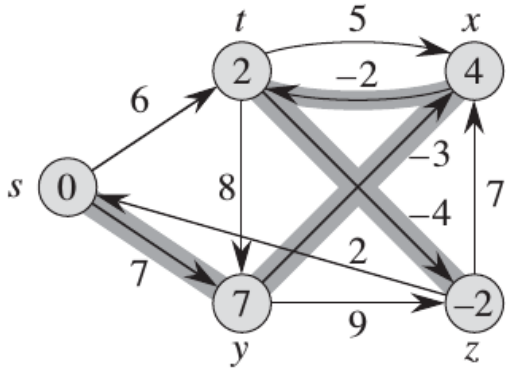
(b)



(c)



(d)



(e)

Bellman-Ford Algorithm is a DP Algorithm.

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$d[i, v]$: the length of the shortest path $s \rightsquigarrow v$ consisting of $\leq i$ edges

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$$d[i,v] = \begin{cases} 0 & i = 0 \wedge v = s \\ \infty & i = 0 \wedge v \neq s \\ \min\left\{d[i-1,v], \min\limits_{(u,v)\in E}\left\{d[i-1,u] + w(u,v)\right\}\right\} & \text{o.w.} \end{cases}$$

```
1:  procedure BELLMAN-FORD-DP(G, w, s)
2:      d[0, s] ← 0
3:      for (v ≠ s) ∈ V do
4:          d[0, v] ← ∞
5:      for i ← 1 to |V| − 1 do
6:          for v ∈ V do
7:              d[i, v] = d[i − 1, v]
8:              for (u, v) ∈ E do
9:                  if d[i − 1, v] > d[i − 1, u] + w(u, v) then
10:                     d[i, v] = d[i − 1, u] + w(u, v)
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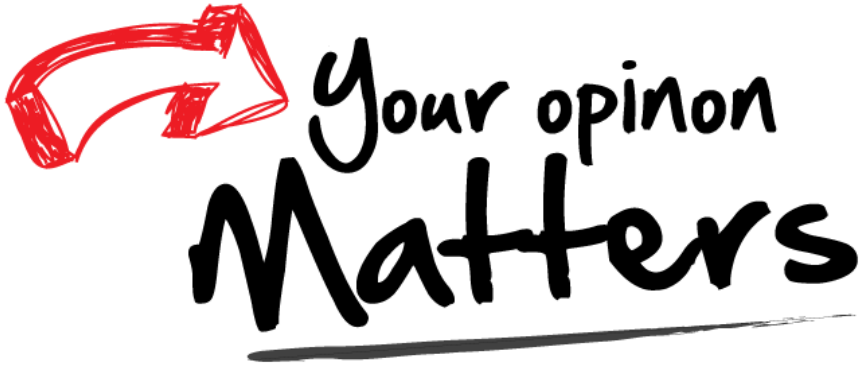
```

$$Q: d[i,v] \implies d[v]?$$


THANK YOU



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