

## 4-4 Direct Products

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What do you mean by “是一回事”?



## Theorem

If  $G = H \times K$ ,  
then  $\exists H' \cong H, K' \cong K$ ,  
such that  $G$  is the internal direct product of  $H$  and  $K$ .



## Theorem ((Theorem 9.27))

If  $G$  is the internal direct product of  $H$  and  $K$ ,  
then  $G \cong H \times K$ .

$$G = H \times K$$

### Definition

$$H' = H \times \{e_K\} = \{(h', e_K) : h' \in H\} \cong H$$

$$K' = \{e_H\} \times K = \{(e_H, k') : k' \in K\} \cong K$$

$$H' \leq G, \quad K' \leq G$$

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$$G = H'K'$$

$$H' \cap K' = \{(e_H, e_K)\} = \{e\}$$

$H'$  and  $K'$  commute.

## Theorem

*If  $G = H \times K$ ,  
then  $\exists H' \cong H, K' \cong K$ ,  
such that  $G$  is the internal direct product of  $H'$  and  $K'$ .*

## Definition (Internal Direct Product)

Let  $G$  be a group with subgroups  $H$  and  $K$  satisfying

$$G = HK$$

$$H \cap K = \{e\}$$

$H$  and  $K$  commute.

Then,  $G$  is the internal direct product of  $H$  and  $K$ .

$$G = H \times K$$

## Definition

$$H' = H \times \{e_K\} = \{(h', e_K) : h' \in H\} \cong H$$

$$K' = \{e_H\} \times K = \{(e_H, k') : k' \in K\} \cong K$$

$$H' \leq G, \quad K' \leq GH' \triangleleft G, \quad K' \triangleleft G$$

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$$G = H'K'$$

$$H' \cap K' = \{e\}$$

$H'$  and  $K'$  commute.

### Definition (Internal Direct Product)

Let  $G$  be a group with subgroups  $H$  and  $K$  satisfying

$$G = HK$$

$$H \cap K = \{e\}$$

$H$  and  $K$  commute

Then,  $G$  is the internal direct product of  $H$  and  $K$ .

### Definition (Internal Direct Product (Equivalent))

Let  $G$  be a group with *normal* subgroups  $H$  and  $K$  satisfying

$$G = HK$$

$$H \cap K = \{e\}$$

Then,  $G$  is the internal direct product of  $H$  and  $K$ .

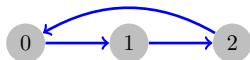


$$\mathbb{Z}_2 \times \mathbb{Z}_3 = G \cong \mathbb{Z}_6$$

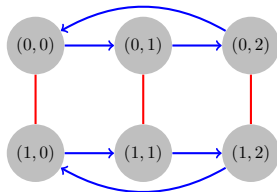




$\mathbb{Z}_2$



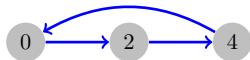
$\mathbb{Z}_3$



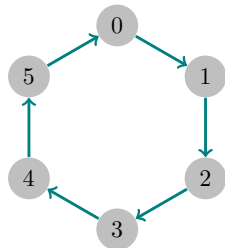
$\mathbb{Z}_2 \times \mathbb{Z}_3$



$\mathbb{Z}_2 \cong \{0, 3\} \leq \mathbb{Z}_6$

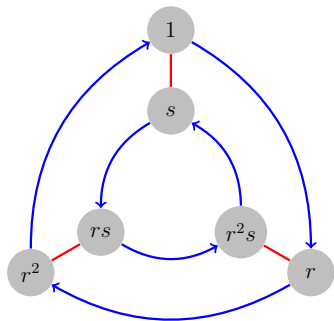


$\mathbb{Z}_3 \cong \{0, 2, 4\} \leq \mathbb{Z}_6$



$\mathbb{Z}_6$

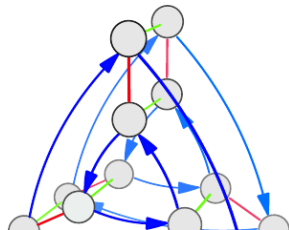
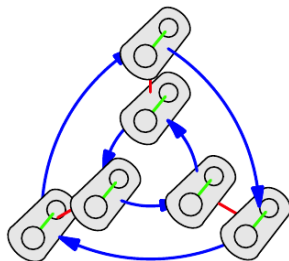
$$D_6 \cong D_3 \times \mathbb{Z}_2$$



$D_3$

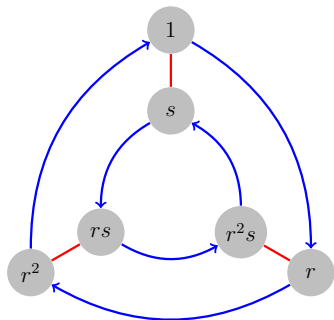


$\mathbb{Z}_2$



$$D_6 \cong D_3 \times \mathbb{Z}_2$$

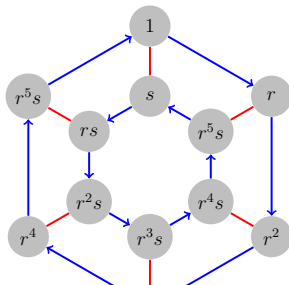
$$D_6 = D'_3 \mathbb{Z}'_2 \quad (D'_3 \triangleleft D_6, \mathbb{Z}'_2 \triangleleft D_6)$$



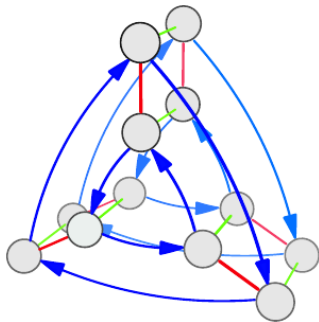
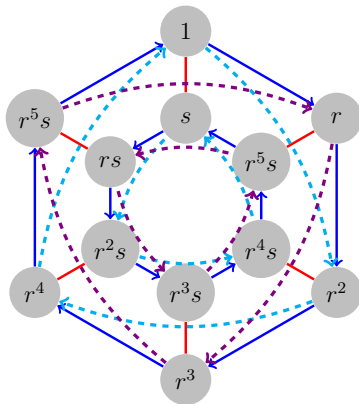
$D_3$



$\mathbb{Z}_2$



$$D_6 \cong D_3 \times \mathbb{Z}_2$$


 $D_3 \times \mathbb{Z}_2$ 

 $D_6$

$$D_{2n} \cong D_n \times \mathbb{Z}_2 \quad (n \text{ is odd})$$

$$\mathbb{Z}_2 \cong (\mathbb{Z}'_2 \triangleq \{1, r^n\}) \triangleleft D_{2n}$$

$$D_n \cong (D'_n \triangleq \langle r^2, s \rangle) \triangleleft D_{2n}$$

$D_n$  is the internal direct product of  $\mathbb{Z}'_2$  and  $D'_n$ .



能不能告诉我另一件事?

## Definition (Internal Direct Product)

Let  $G$  be a group with **normal** subgroups  $H$  and  $K$  satisfying

$$G = HK$$

$$H \cap K = \{e\}$$

Then,  $G$  is the internal direct product of  $H$  and  $K$ .



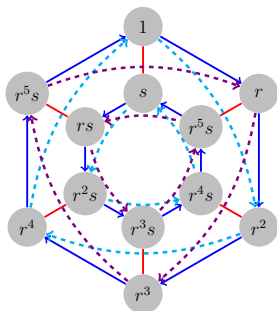
Theorem (The Second Isomorphism Theorem (**Diamond Theorem**))

$$H \leq G, N \triangleleft G \implies H/(H \cap N) \cong HN/N.$$

## Theorem

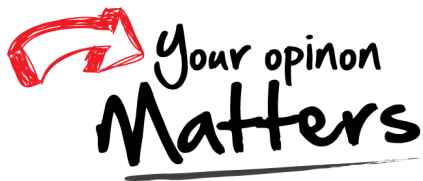
*If  $G$  is the internal direct product of its normal subgroups  $H$  and  $K$ ,  
then  $G/H \cong K$ ,  $G/K \cong H$ .*

$$D_6 = \langle r^2, s \rangle \{1, r^3\} \implies D_6 / \langle r^2, s \rangle \cong \{1, r^3\} \cong \mathbb{Z}_2$$









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