2-4 Recurrences

Hengfeng Wei

hfwei@nju.edu.cn

March 24, 2020

Maximum-sum Subarray (mss; Problem $4.1-5$)

$$A[0 \ldots n-1] \qquad \forall 0 \le i \le n-1 : A[i] \in \mathbb{Z}$$

To find (the sum of) a maximum-sum subarray of A

$$A = [-2, 11, -4, 13, -5, -2]$$

$$\text{mss} = 11 + (-4) + 13 = 20$$

$$A = [-2, 1, -3, 4, -1, 2, 1, -5, 4]$$

$$\text{mss} = 4 + (-1) + 2 + 1 = 6$$

$\text{mss-prefix}[i]$: (the sum of) a maximum-sum subarray in $A[1 \cdots i]$

$$\text{mss} = \text{mss-prefix}[n]$$

Q : Is $a_i \in \text{mss-prefix}[i]$?

$$\text{mss-prefix}[i] = \max\{\text{MSS}[i-1], ???\}$$

mss-at$[i]$: (the sum of) a maximum-sum subarray ending with $A[i]$

$$\text{mss} = \max_{0 \le i \le n-1} \text{mss-at}[i]$$

Q : Where does mss-at$[i]$ start?

$$\boxed{\text{mss-at}[i] = \max\{\text{mss-at}[i-1] + A[i], A[i]\}}$$

$$\text{mss-at}[0] = A[0]$$

procedure $\textsc{mss}(A, n)$

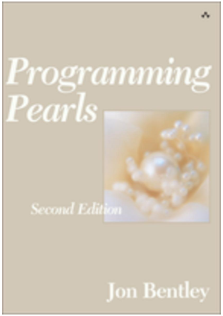
  mss-at$[0] \gets A[0]$

  for all $i \gets 1 \ldots n-1$ **do**

    mss-at$[i] \gets \max\{$mss-at$[i-1] + A[i], 0\}$

  return $\max\limits_{0 \le i \le n-1}$ mss-at$[i]$

procedure $\textsc{mss}(A, n)$
    mss $\leftarrow 0$
    mss-at $\leftarrow A[0]$
    for all $i \leftarrow 1 \ldots n-1$ **do**
        mss-at $\leftarrow \max\{\text{mss-at} + A[i], 0\}$
        mss $\leftarrow \max\{\text{mss}, \text{mss-at}\}$
    return mss



Ulf Grenander	$O(n^3) \implies O(n^2)$
Michael Shamos	$O(n \log n)$, onenight
Jon Bentley	Conjecture: $\Omega(n \log n)$
Michael Shamos	Carnegie Mellon seminar
Jay Kadane	$O(n)$, ≤ 1 minute

Maximum-product Subarray (mps)

$$A[0 \ldots n-1] \qquad \forall 0 \le i \le n-1 : A[i] \in \mathbb{Z}$$

To find (the product of) a maximum-product subarray of A

$$A = [\frac{1}{2}, 4, -2, 5, -\frac{1}{5}, 8$$

$$\textsf{mps} = 4 \times (-2) \times 5 \times (-\frac{1}{5}) \times 8 = 64$$

mps-at$[i]$: (the product of) a maximum-product subarray ending with $A[i]$

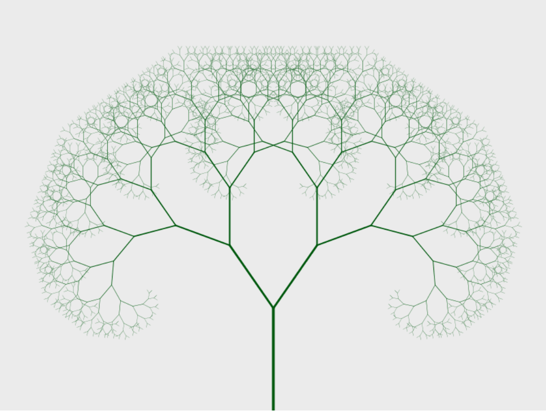
$$\text{mps} = \max_{0 \le i \le n-1} \text{mps-at}[i]$$

Q : Where does mps-at$[i]$ start?

		$\frac{1}{2}$	4	-2	5	$-\frac{1}{5}$	8
MaxP$[i]$	1	$\frac{1}{2}$	4	-2	5	8	64
MinP$[i]$	1	$\frac{1}{2}$	2	-8	-40	-1	-8

$$\text{MaxP}[i] = \max\{\text{MaxP}[i-1] \cdot a_i, \text{MinP}[i-1] \cdot a_i, a_i\}$$
$$\text{MinP}[i] = \min\{\text{MaxP}[i-1] \cdot a_i, \text{MinP}[i-1] \cdot a_i, a_i\}$$

Recurrences



$$T(n) = aT(n/b) + f(n) \quad (a > 0, b > 1)$$

Assume that $T(n)$ is constant for sufficiently small n.

$$\left.\begin{array}{r} f(n) \\ af(\frac{n}{b}) \\ a^2 f(\frac{n}{b^2}) \\ \vdots \\ a^{\log_b n} T(1) = \Theta(n^{\log_b a}) \end{array}\right\} \sum \overset{f(n) \textit{vs. } n^E}{=} \begin{cases} n^{\log_b a}, & f(n) = O(n^{E-\epsilon}) \\ n^{\log_b a} \log n, & f(n) = \Theta(n^E) \\ f(n), & f(n) = \Omega(n^{E+\epsilon}) \end{cases}$$

Solving Recurrences (Problem 2.15)

(1) $\Theta(n^{\log_3 2})$

(2) $\Theta(\log^2 n)$

(3) $\Theta(n)$

(4) $\Theta(n \log n)$

(5) $\Theta(n \log^2 n)$

(6) $\Theta(n^2)$

(7) $\Theta(n^{\frac{3}{2}} \log n)$

(8) $\Theta(n)$

(9) $\Theta(n^{c+1})$

(10) $\Theta(c^{n+1})$

(11) $\cdots$

$$T(n) = T(n/2) + \log n$$

$$T(n) = 2T(n/2) + n \log n$$

Reference:

$$f(n) = \Theta(n^{\log_b a} \log^k n) \implies \Theta(n^{\log_b a} \log^{k+1} n)$$

Gaps in Master Theorem (Problem 2.18)

$$T(n) = 2T(n/2) + \frac{n}{\log n} = \Theta(n \log \log n)$$

Solving Recurrences (Problem 2.15)

(1) $\Theta(n^{\log_3 2})$
(2) $\Theta(\log^2 n)$
(3) $\Theta(n)$
(4) $\Theta(n \log n)$
(5) $\Theta(n \log^2 n)$
(6) $\Theta(n^2)$
(7) $\Theta(n^{\frac{3}{2}} \log n)$
(8) $\Theta(n)$
(9) $\Theta(n^{c+1})$
(10) $\Theta(c^{n+1})$
(11) $\cdots$

$$T(n) = T(n-1) + c^n \quad c > 1$$

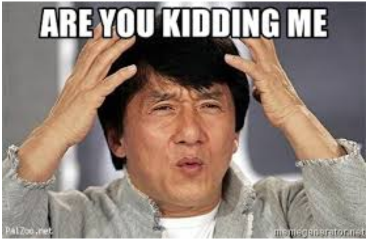
$$T(n) = T(n-1) + n^c \quad c \geq 1$$

$$\int$$

$$(\frac{n}{2}) \cdot (\frac{n}{2})^c \leq T(n) \leq n \cdot n^c$$

Solving Recurrences (Problem 2.15 (11))

$$T(n) = T(n/2) + T(n/4) + T(n/8)$$



Where is $f(n)$?

Solving Recurrences (Problem 2.15 (11))

$$T(n) = T(n/2) + T(n/4) + T(n/8)$$

$$\boxed{T(n) = \Theta(n^{0.879146})}$$

$$\boxed{T(n) = \Theta(n^{\alpha})}$$

$$2^{-\alpha} + 4^{-\alpha} + 8^{-\alpha} = 1$$

```
Solve[2^{-x} + 4^{-x} + 8^{-x} == 1, x] // N
```

Solving Recurrences (Problem 2.15 (11))

$$T(n) = T(n/2) + T(n/4) + T(n/8) + n$$

By recursion-tree.

$$\boxed{T(n) = \Theta(n)}$$

Exercise: Prove it by mathematical induction.

Reference:

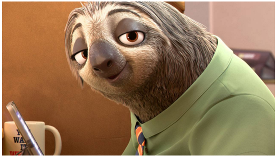
"On the Solution of Linear Recurrence Equations" by Akra & Bazzi, 1996.

$$T(n) = \sum_{i=1}^{k} a_i T(n/b_i) + f(n)$$

Solving Recurrences (Problem 2.17)

$$\begin{aligned}
\mathsf{T}(n) &= \sqrt{n}\ \mathsf{T}(\sqrt{n}) + n \\
&= n^{\frac{1}{2}}\ \mathsf{T}\left(n^{\frac{1}{2}}\right) + n \\
&= n^{\frac{1}{2}} \left(n^{\frac{1}{2^2}}\ \mathsf{T}\left(n^{\frac{1}{2^2}}\right) + n^{\frac{1}{2}}\right) + n \\
&= n^{\frac{1}{2}+\frac{1}{2^2}}\ \mathsf{T}\left(n^{\frac{1}{2^2}}\right) + 2n \\
&= n^{\frac{1}{2}+\frac{1}{2^2}} \left(n^{\frac{1}{2^3}}\ \mathsf{T}\left(n^{\frac{1}{2^3}}\right) + n^{\frac{1}{2^2}}\right) + 2n \\
&= n^{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}}\ \mathsf{T}\left(n^{\frac{1}{2^3}}\right) + 3n \\
&= \cdots \\
&= n^{\sum_{i=1}^{k} \frac{1}{2^i}}\ \mathsf{T}\left(n^{\frac{1}{2^k}}\right) + kn
\end{aligned}$$

$$\mathsf{T}(n) = n^{\sum_{i=1}^{k} \frac{1}{2^i}}\ \mathsf{T}\left(n^{\frac{1}{2^k}}\right) + kn$$



$$n^{\frac{1}{2^k}} = 1$$

$$n^{\frac{1}{2^k}} = 2 \implies k = \log\log n$$

$$\begin{aligned}\mathsf{T}(n) &= n^{\sum_{i=1}^{k} \frac{1}{2^i}}\ \mathsf{T}\left(n^{\frac{1}{2^k}}\right) + kn \\ &= n^{\sum_{i=1}^{\log\log n} \frac{1}{2^i}}\ \mathsf{T}(2) + n\log\log n\end{aligned}$$

$$\sum_{i=1}^{\log\log n} \frac{1}{2^i} < 1 \implies T(n) = \Theta(n\log\log n)$$

Exercise: Prove it by mathematical induction.

$$\boxed{T(n) = \sqrt{n}T(\sqrt{n}) + n}$$

$$\frac{T(n)}{n} = \frac{T(\sqrt{n})}{\sqrt{n}} + 1$$

$$n \leftrightarrow 2^m$$

$$\frac{T(2^m)}{2^m} = \frac{T(2^{m/2})}{2^{m/2}} + 1$$

$$S(m) \leftrightarrow \frac{T(2^m)}{2^m}$$

$$S(m) = S(m/2) + 1 = \Theta(\log m)$$

$$\boxed{T(n) = n \log \log n}$$

Problem (Area-Efficient VLSI Layout)

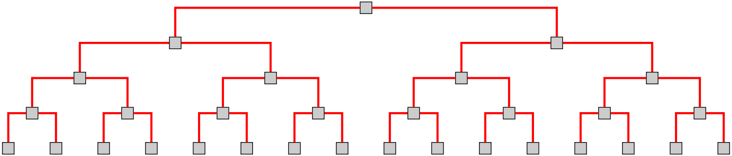
Embed a complete binary tree of n nodes into a grid with minimum area.

- Complete binary tree circuit of

$$\#\text{layer} = 3, 5, 7, \dots$$

- Vertex on grid; no crossing edges
- Area:

$$\underbrace{A(n)}_{\text{area}} = \underbrace{H(n)}_{\text{height}} \times \underbrace{W(n)}_{\text{width}}$$



$$H(n) = H(\frac{n}{2}) + \Theta(1) = \Theta(\log n)$$

$$W(n) = 2W(\frac{n}{2}) + \Theta(1) = \Theta(n)$$

$$\boxed{A(n) = \Theta(n \log n)}$$

$$\boxed{Q : \boxed{H(n)} \times \boxed{W(n)} = n}$$

$$1 \times n$$

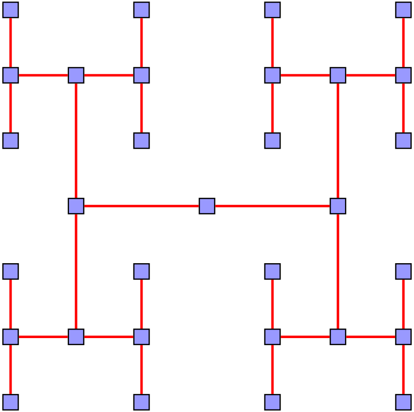
$$\frac{n}{\log n} \times \log n$$

$$\boxed{\sqrt{n} \times \sqrt{n}}$$

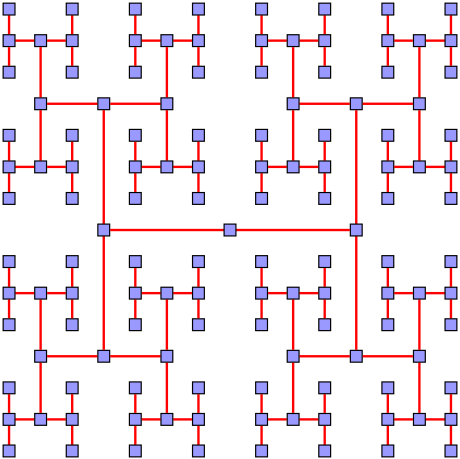
$$H(n) = \Theta(\sqrt{n}),\ W(n) = \Theta(\sqrt{n}),\ A(n) = \Theta(n)$$

$$H(n) = \square H(\frac{n}{\square}) + O(\square)$$

$$\boxed{H(n) = 2H(\frac{n}{4}) + \Theta(1)}$$

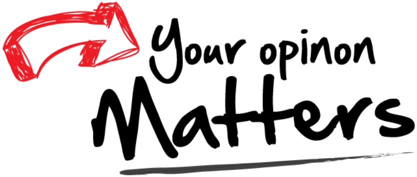


H-layout



"VLSI Theory and Parallel Supercomputing", Charles E. Leiserson, 1989.


Thank You!



Office 302

Mailbox: H016

hfwei@nju.edu.cn