2-6 Algorithmic Methods

Hengfeng Wei

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$$T(n) = aT(n/b) + f(n) \quad (a > 0, b > 1)$$

Assume that $T(n)$ is constant for sufficiently small n.

$$T(n) = aT(n/b) + f(n) \quad (a > 0, b > 1)$$

Assume that $T(n)$ is constant for sufficiently small n.

$$\boxed{T(1) = 0 \; vs. \; T(1) = d \neq 0}$$

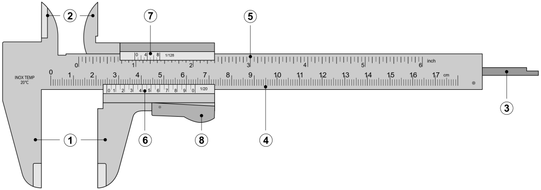
$$T(n) = aT(n/b) + f(n) \quad (a > 0, b > 1)$$

Assume that $T(n)$ is constant for sufficiently small n.

$$f(n) > 0$$

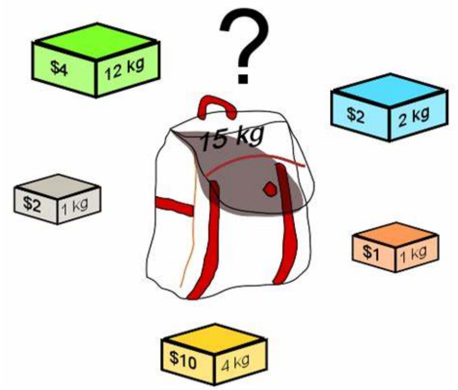
$$\boxed{T(1) = 0 \ vs. \ T(1) = d \neq 0}$$

Convex Polygon Diameter



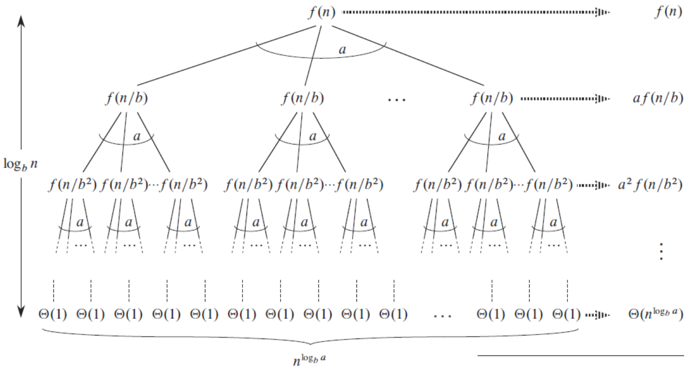
Correctness Proof

Integer Knapsack



Algorithm & Time Complexity

$$T(n) = aT(n/b) + f(n)$$



$$T(n) = n^{\log_b a} T(1) + \sum_{j=0}^{\log_b n - 1} a^j f\left(\frac{n}{b^j}\right)$$

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$$\boxed{f(n) \textit{ vs. } n^{\log_b a}}$$

$$f(n) = O(n^{\log_b a - \epsilon}) \implies T(n) = \Theta(n^{\log_b a})$$

$$f(n) = \Theta(n^{\log_b a}) \implies T(n) = \Theta(n^{\log_b a} \log n)$$

$$f(n) = \Omega(n^{\log_b a + \epsilon}) \implies T(n) = \Theta(f(n))$$

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$$a^{\log_b n - 1} f\left(\frac{n}{b^{\log_b n - 1}}\right) = a^{\log_b n - 1} f(b) = a^{\log_b n} \frac{f(b)}{a}$$

$$T(n) = n^{\log_b a} T(1) + \sum_{j=0}^{\log_b n - 1} a^j f\left(\frac{n}{b^j}\right)$$

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$$\begin{aligned} T(n) &= n^{\log_b a} T(1) + \sum_{j=0}^{\log_b n - 1} a^j f\left(\frac{n}{b^j}\right) \\ &= n^{\log_b a} \left(\frac{f(b^{\log_b n})}{a^{\log_b n}} + \ldots + \frac{f(b^2)}{a^2} + \frac{f(b)}{a} + T(1) \right) \end{aligned}$$

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$$f(n) > 0$$

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$$f(n) > 0$$

$$T(1) = d = \Theta(1), \quad (d \text{ can be } 0)$$

$$T(n) = n^{\log_b a} T(1) + \sum_{j=0}^{\log_b n - 1} a^j f\left(\frac{n}{b^j}\right)$$

$$f(n) = O(n^{\log_b a - \epsilon}) \implies T(n) = \Theta(n^{\log_b a})$$

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$$f(n) > 0$$

$$\sum_{j=0}^{\log_b n - 1} a^j f\left(\frac{n}{b^j}\right) = \Omega(n^{\log_b a})$$

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What if $f(n) = 0$?

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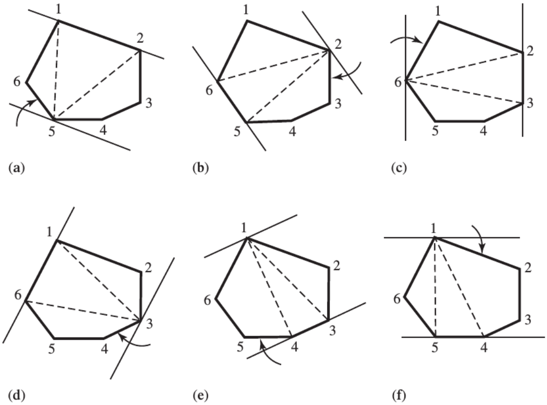
What if $f(n) = 0$?

$$T(n) = n^{\log_b a} T(1)$$

$$T(n) = \begin{cases} 0, & T(1) = 0 \\ \Theta(n^{\log_b a}), & T(1) = d \neq 0 \end{cases}$$

Convex Polygon Diameter



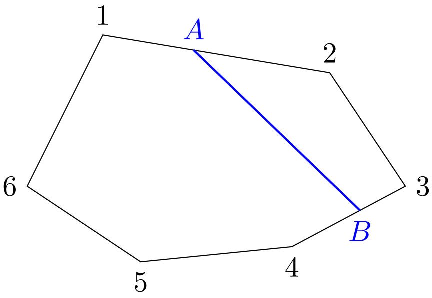
“Computational Geometry”
Ph.D Thesis, Michael Shamos, 1978

“Solving Geometric Problems with the Rotating Calipers”, 1983

Correctness

Theorem (DH 4-8)

If AB is a diameter of a convex polygon P, then A and B are vertices.



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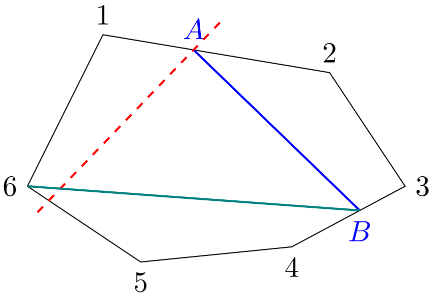

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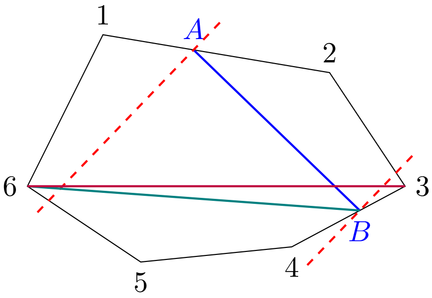

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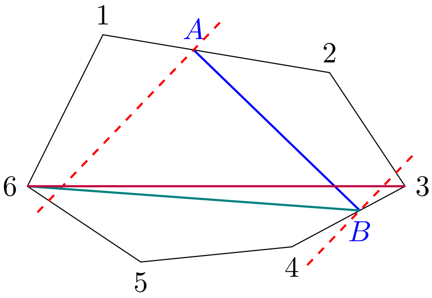


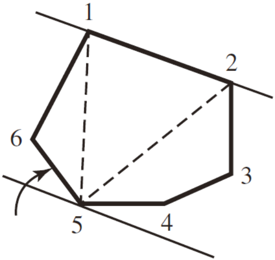
BUT, we have *not* enumerated *all* pairs of vertices.

We have enumerated *all antipodals*.

Definition (Line of Support)

A line L is a *line of support* of a convex polygon P if

$$L \cap P = \text{ a vertex/an edge of } P.$$



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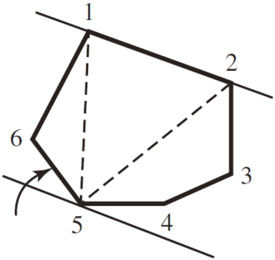
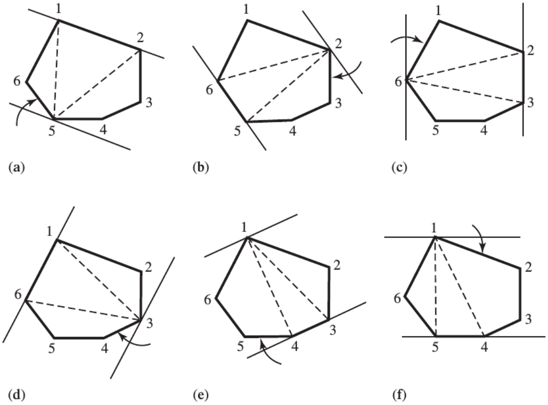


$L \cap P \neq \emptyset$ $\qquad$ P lies entirely on one side of L.

Definition (Antipodal)

An *antipodal* is a pair of points that admits *parallel supporting lines*.



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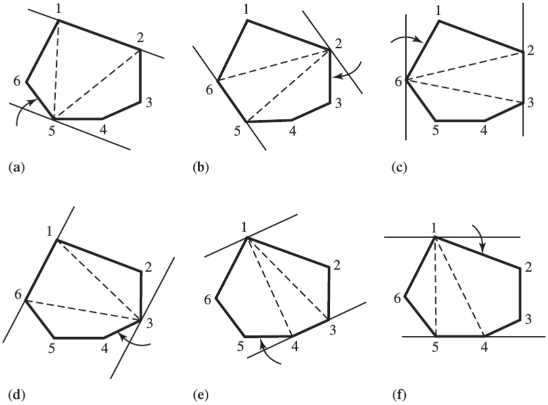
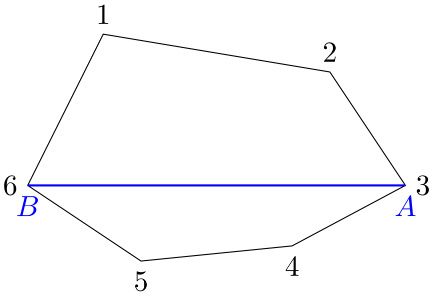


We have enumerated *all* antipodals by *rotating* through all angles.

Theorem (We Won't Miss the Diameter)

If AB is a diameter of a convex polygon P, then AB is an antipodal.



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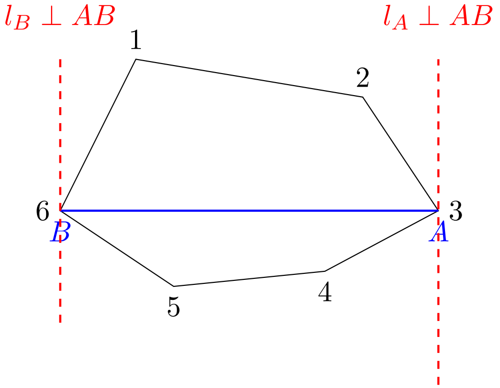

We claim that l_A and l_B are parallel supporting lines.

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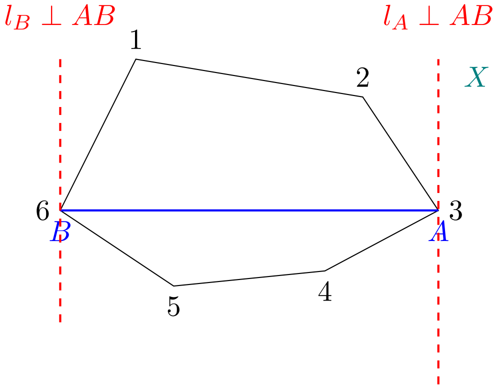

We claim that l_A and l_B are parallel supporting lines.

$l_A \cap P \neq \emptyset$ P lies entirely on one side of l_A.

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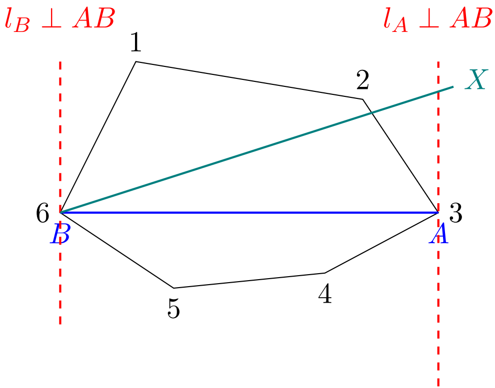

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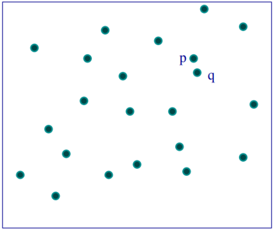
Finding the Closest Pair of Points (Additional: DH 4-10)



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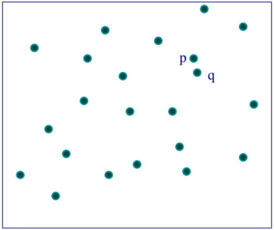


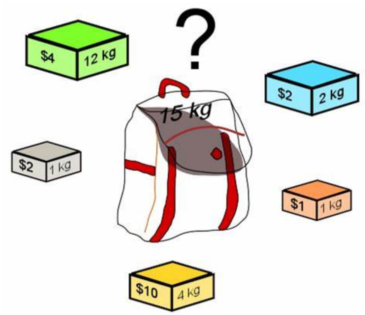
A Classical and Beautiful Divide-Conquer-Combine Algorithm:

Section 33.4, CLRS

DH 4.13 (Integer Knapsack)

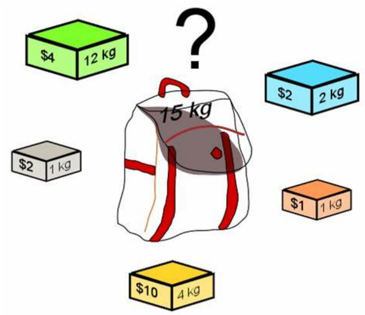
$$N = 5$$
$$Q = [3, 1, 4, 5, 1] \quad (quantity)$$
$$W = [10, 20, 20, 8, 7] \quad (weight)$$
$$P = [17, 42, 35, 16, 15] \quad (profit)$$

$$C = 103$$



0-1 Knapsack

$$\forall i : Q[i] = 1$$



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$$N' = \sum_i Q[i]$$
$$W' = [\dots, \underbrace{W_i}_{\# = Q_i}, \dots]$$
$$P' = [\dots, \underbrace{P_i}_{\# = Q_i}, \dots]$$

$K[c, i]$:

The maximal profit obtained using knapsack of capacity c with items of $x_1 \dots x_i$.

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$K[C, N]$

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$K[C, N]$

$\boxed{\text{Using the item } x_i \text{ or not?}}$

$K[c, i]$:

The maximal profit obtained using knapsack of capacity c with items of $x_1 \ldots x_i$.

$K[C, N]$

$\boxed{\text{Using the item } x_i \text{ or not?}}$

$$K[c, i] = \max \begin{cases} K[c, i-1], \\ K[c - W[i], i-1] + P[i], \quad W[i] \leq c \end{cases}$$

$K[c, i]$:

The maximal profit obtained using knapsack of capacity c with items of $x_1 \dots x_i$.

$K[C, N]$

Using the item x_i or not?

$$K[c, i] = \max \begin{cases} K[c, i-1], \\ K[c - W[i], i-1] + P[i], & W[i] \leq c \end{cases}$$

Time complexity : $\Theta(NC)$

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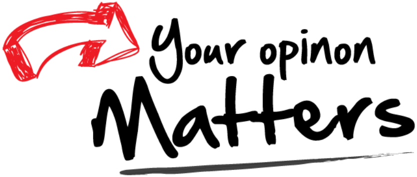
Time complexity : $\Theta(NC)$

Is this a polynomial algorithm?


KEEP
CALM
AND
STAY
TUNED


Thank You!



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