4-5 Polyhedral Groups (II)

(Cube)

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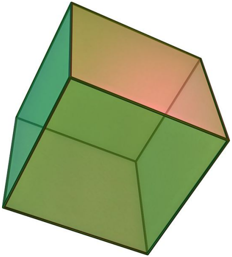
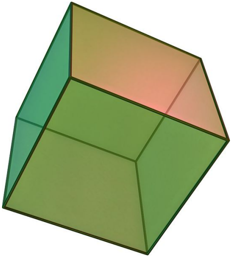


$$Sym(C) \cong S_4$$



$$Sym(C) \cong S_4$$

$$\Big|\{H : H \leq Sym(C)\}\Big| = 30$$

$$\left|Sym(C)\right| \leq 24$$

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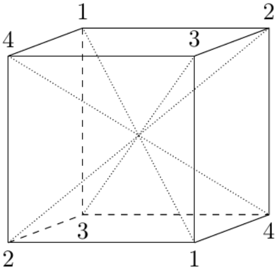
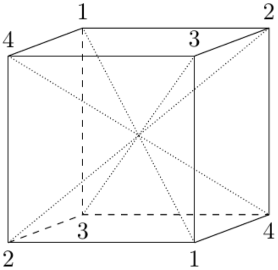


$$Sym(C) \leq S_4$$

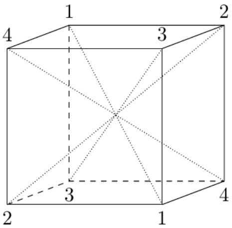
$$\left|Sym(C)\right| \leq 24$$



$$Sym(C) \leq S_4$$

$$\left|Sym(C)\right| = \underbrace{6}_{\text{Facing Upward}} \times \underbrace{4}_{\text{Rotation}}$$

Order of 1: id $(\# = 1)$



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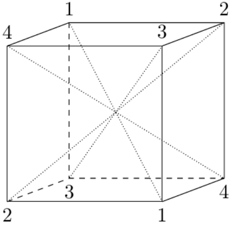
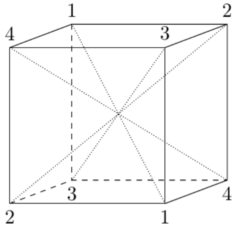


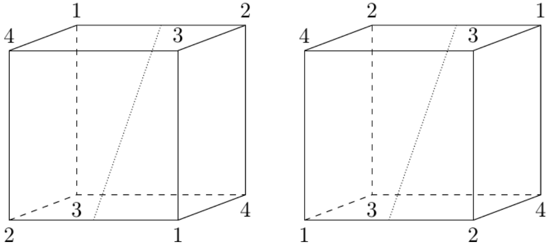
Order of 4: face-to-face $(\# = 9)$

$$
\begin{array}{lll}
f_{td} = (1\ 2\ 3\ 4) & f_{td}^2 = (1\ 3)(2\ 4) & f_{td}^3 = (1\ 4\ 3\ 2) \\
f_{lr} = (1\ 3\ 2\ 4) & f_{lr}^2 = (1\ 2)(3\ 4) & f_{lr}^3 = (1\ 4\ 2\ 3) \\
f_{fb} = (1\ 2\ 4\ 3) & f_{fb}^2 = (1\ 4)(2\ 3) & f_{fb}^3 = (1\ 3\ 4\ 2)
\end{array}
$$



Order of 3: vertex-to-vertex $(\# = 8)$

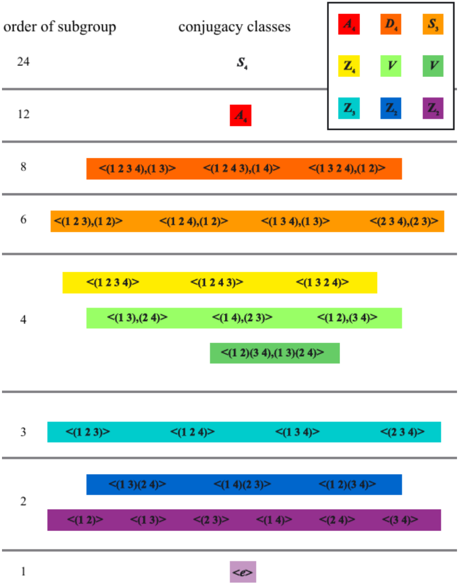
$$
\begin{aligned}
v_1 &= (2\ 3\ 4) & v_1^2 &= (2\ 4\ 3) \\
v_2 &= (1\ 4\ 3) & v_2^2 &= (1\ 3\ 4) \\
v_3 &= (1\ 2\ 4) & v_3^2 &= (1\ 4\ 2) \\
v_4 &= (1\ 2\ 3) & v_4^2 &= (1\ 3\ 2)
\end{aligned}
$$



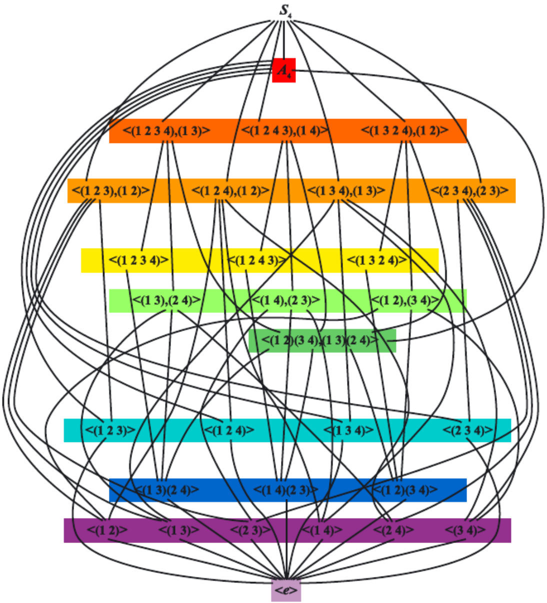
Order of 2: edge-to-edge $(\# = 6)$

$$e_{12} = (1\ 2) \quad e_{13} = (1\ 3) \quad e_{14} = (1\ 4)$$
$$e_{23} = (2\ 3) \quad e_{24} = (2\ 4) \quad e_{34} = (3\ 4)$$

$$\left|\{H : H \leq Sym(C)\}\right| = 30$$



order of subgroup
conjugacy classes
A_4 D_4 S_3
Z_4 V V
Z_3 Z_2 Z_2
24
S_4
12
A_4
8
<(1 2 3 4),(1 3)> <(1 2 4 3),(1 4)> <(1 3 2 4),(1 2)>
6
<(1 2 3),(1 2)> <(1 2 4),(1 2)> <(1 3 4),(1 3)> <(2 3 4),(2 3)>
4
<(1 2 3 4)> <(1 2 4 3)> <(1 3 2 4)>
<(1 3),(2 4)> <(1 4),(2 3)> <(1 2),(3 4)>
<(1 2)(3 4),(1 3)(2 4)>
3
<(1 2 3)> <(1 2 4)> <(1 3 4)> <(2 3 4)>
2
<(1 3)(2 4)> <(1 4)(2 3)> <(1 2)(3 4)>
<(1 2)> <(1 3)> <(2 3)> <(1 4)> <(2 4)> <(3 4)>
1
<e>


S4
A4
<(1 2 3 4),(1 3)>
<(1 2 4 3),(1 4)>
<(1 3 2 4),(1 2)>
<(1 2 3),(1 2)>
<(1 2 4),(1 2)>
<(1 3 4),(1 3)>
<(2 3 4),(2 3)>
<(1 2 3 4)>
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<(1 4)>
<(2 4)>
<(3 4)>
<e>

Order of 1: id $(\# = 1)$

Order of 4: face-to-face $(\# = 9)$

$$f_{td} = (1\ 2\ 3\ 4) \quad f_{td}^2 = (1\ 3)(2\ 4) \quad f_{td}^3 = (1\ 4\ 3\ 2)$$

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Order of 3: vertex-to-vertex $(\# = 8)$

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$$e_{23} = (2\ 3) \quad e_{24} = (2\ 4) \quad e_{34} = (3\ 4)$$

$$H \leq S_4 \Longrightarrow |H| = 1, \quad 2, \quad 3, \quad 4, \quad 6, \quad 8, \quad 12, \quad 24$$

$$|H| = \begin{cases} 1: & \text{id} \quad (\# = 1) \\ 2: & (\# = 6 + 3 = 9) \\ 3: & v_1, v_2, v_3, v_4 \quad (\# = 4) \\ 4: & (\# = 7) \\ 6: & (\# = 4) \\ 8: & (\# = 3) \\ 12: & A_4 \quad (\# = 1) \\ 24: & S_4 \quad (\# = 1) \end{cases}$$

$$\boxed{|G| = 4 \Longrightarrow G \cong \mathbb{Z}_4 \vee G \cong K_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2}$$

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$H \cong \mathbb{Z}_4$:

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$$H \cong \mathbb{Z}_4 : f_{fd}, f_{lr}, f_{fb} \quad (\# = 3)$$

$$\boxed{|G| = 4 \implies G \cong \mathbb{Z}_4 \lor G \cong K_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2}$$

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$$H \cong K_4 = \{e, a, b, ab\} \quad (a^2 = b^2 = e, ab = ba)$$

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$$\{(1), (1\ 2), (3\ 4), (1\ 2)(3\ 4)\}$$

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$$\boxed{|G| = 6 \Longrightarrow G \cong \mathbb{Z}_6 \vee G \cong D_3}$$

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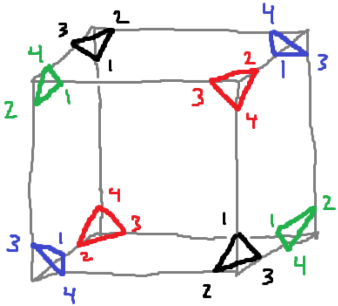
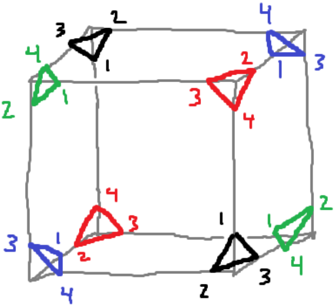
$$v_4 = (1\ 2\ 3) \quad v_4^2 = (1\ 3\ 2)$$

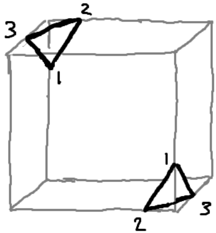
Theorem

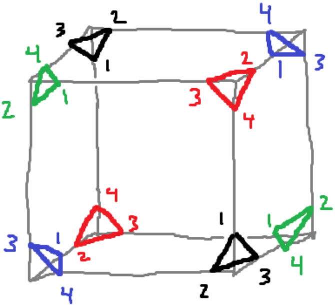
There are only 4 subgroups $\cong D_3$ in S_4.



3 2 1
4 3 1
4 2 1
2 3 4
4 3 2
3 1 4
1 2 3
1 2 4


1
2
3
4


1
2
3



3
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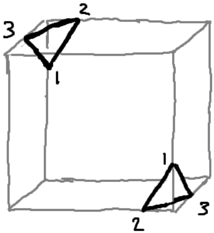

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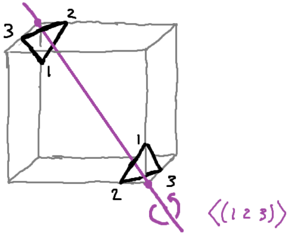

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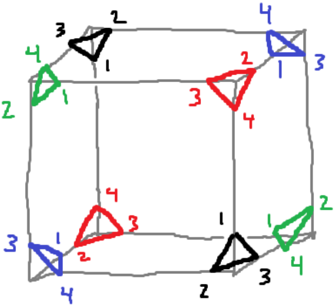
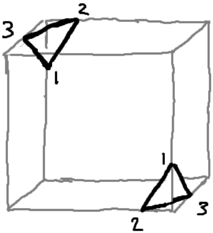
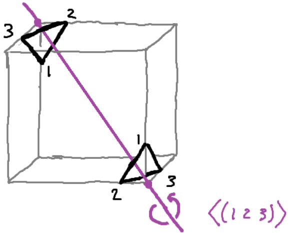
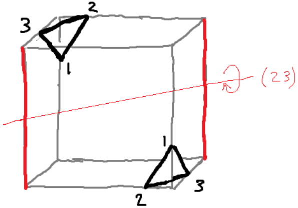
⟨(1 2 3)⟩






⟨(1 2 3)⟩


(2 3)

$$\boxed{|G| = 8 \Longrightarrow G \cong \mathbb{Z}_8, \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \quad \mathbb{Z}_4 \times \mathbb{Z}_2, \quad D_4, \quad Q_8}$$

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$Q_8 = \{\pm 1, \pm I, \pm J, \pm K\}$ (Example 3.15)

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$$Q_8 = \{\pm 1, \pm I, \pm J, \pm K\} \quad \text{(Example 3.15)}$$

$$H \ncong \mathbb{Z}_8$$

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$$H \ncong \mathbb{Z}_8$$

$$H \ncong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

$$H \ncong \mathbb{Z}_4 \times \mathbb{Z}_2$$

$$H \ncong Q_8 \implies |H| \geq 9$$

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$$H \cong D_4 = \left\{1, r, r^2, r^3, s, rs, r^2 s, r^3 s\right\} \quad (r^4 = 1, s^2 = 1, srs = r^{-1})$$

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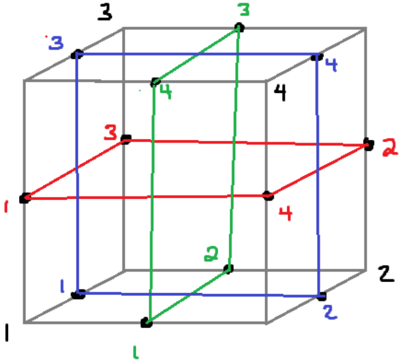
Theorem

There are only 3 *subgroups* $\cong D_4$ *of* S_4.



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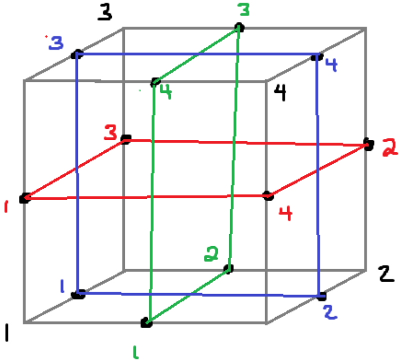

$$H_{\text{green}} = \langle (1\ 2\ 3\ 4), (1\ 3) \rangle$$
$$H_{\text{blue}} = \langle (1\ 2\ 4\ 3), (1\ 4) \rangle$$
$$H_{\text{red}} = \langle (1\ 3\ 2\ 4), (1\ 2) \rangle$$

$$\boxed{|G| = 12 \Longrightarrow G \cong \mathbb{Z}_{12}, \quad \mathbb{Z}_6 \times \mathbb{Z}_2, \quad D_6, \quad A_4, \quad \mathrm{Dic}_{12}}$$

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$$H \cong A_4$$

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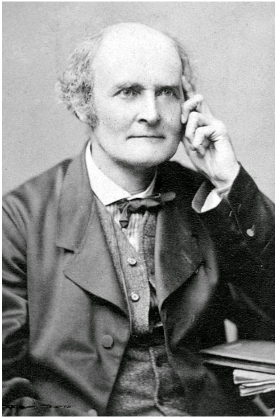
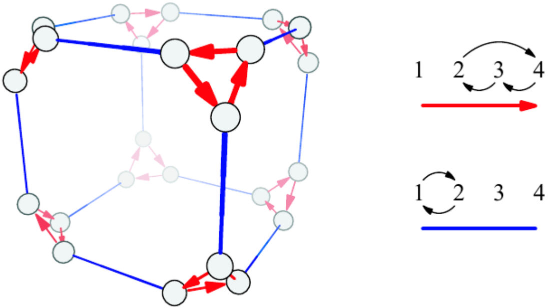
$$H \cong A_4$$

Theorem

There is only one subgroup of order 12 *in* S_4.



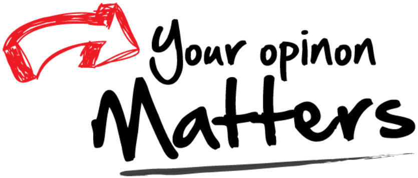
Arthur Cayley (1821 – 1895)



$Sym(C) \cong S_4$ arranged on a *truncated* cube


FLAG


THANK YOU



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