1-11 Set Theory (IV): Infinity

魏恒峰

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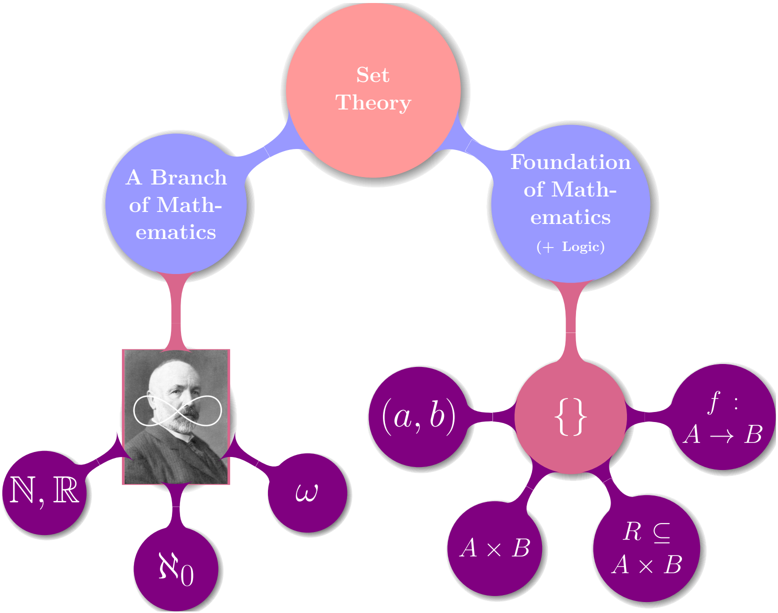
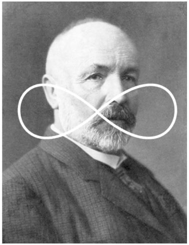
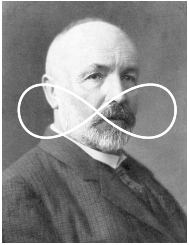
2019 年 12 月 17 日


Set Theory
A Branch of Math-ematics
Foundation of Math-ematics
(+ Logic)
$\mathbb{N}, \mathbb{R}$
$\aleph_0$
ω
(a, b)
$\{\}$
$f : A \to B$
$A \times B$
$R \subseteq A \times B$



Georg Cantor (1845 – 1918)



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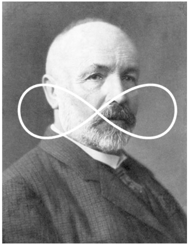

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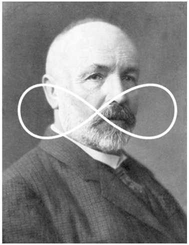

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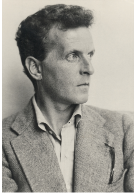
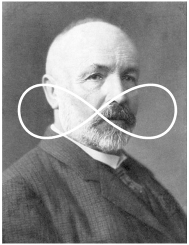


Ludwig Wittgenstein
(1889 – 1951)



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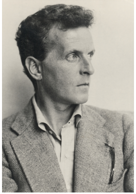

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From his paradise that Cantor with us unfolded, we hold our breath in awe; knowing, we shall not be expelled.

— *David Hilbert*

“没有人能把我们从 *Cantor* 创造的乐园中驱逐出去”

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MATHEMATIKER
BEGRÜNDER DER MENGENLEHRE
1845-1918
DAS WESEN DER MATHEMA
TIK LIEGT IN IHRER FREIHEIT
GEORG CANTOR

"das wesen der mathematik liegt in ihrer freiheit"

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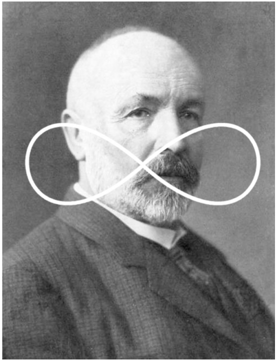
"The essence of mathematics lies in its freedom"

Before Cantor

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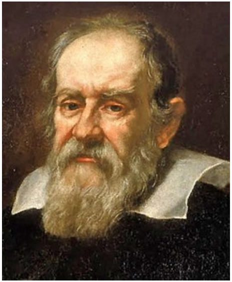
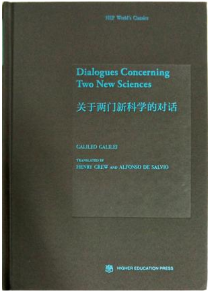
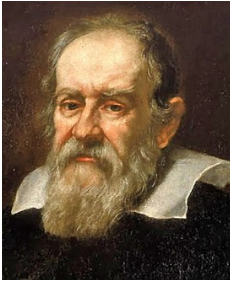
公理: “整体大于部分”

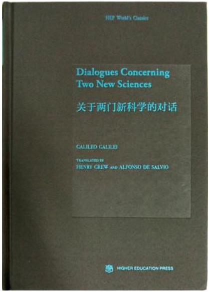


Galileo Galilei (1564 – 1642)



“关于两门新科学的对话” (1638)





Galileo Galilei (1564 – 1642)

“关于两门新科学的对话” (1638)

“用我们有限的心智来讨论无限 · · · ”

$$S_1 = \{1, 2, 3, \cdots, n, \cdots\}$$

$$S_2 = \{1, 4, 9, \cdots, n^2, \cdots\}$$

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无穷数是不可能的。 — *Gottfried Wilhelm Leibniz*

这些证明一开始就期望那些数要具有有穷数的一切性质，或者甚至于把有穷数的性质强加于无穷。

相反，这些无穷数，如果它们能够以任何形式被理解的话，倒是由于它们与有穷数的对应，它们必须具有完全新的数量特征。

这些性质完全依赖于事物的本性，$\cdots$ 而并非来自我们的主观任意性或我们的偏见。

— *Georg Cantor (1885)*

Definition (Dedekind-infinite & Dedekind-finite (Dedekind, 1888))

A set A is *Dedekind-infinite* if there is a bijective function from A onto some proper subset B of A.

A set is *Dedekind-finite* if it is not Dedekind-infinite.

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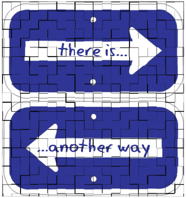

This is a theorem in our theory of infinity.

We have not defined "finite" and "infinite"!

Comparing Sets

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$$= \qquad\qquad\qquad <$$

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$$= \qquad\qquad\qquad <$$

Function

$$=$$

Definition ($|A| = |B|$ ($A \approx B$) (1878))

A and B are *equipotent* if there exists a *bijection* from A to B.

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Abstract from elements: $\{1, 2, 3\} \quad vs. \quad \{a, b, c\}$

Abstract from order: $\{1, 2, 3, \cdots\} \quad vs. \quad \{1, 3, 5, \cdots, 2, 4, 6, \cdots\}$

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Theorem ( )

For any sets A, B, C:

(a) $A \approx B$

(b) $A \approx B \implies B \approx A$

(c) $A \approx B \land B \approx C \implies A \approx C$

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Theorem (The "Equivalence Concept" of Equipotent)

For any sets A, B, C:

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X is finite if

$$\exists n \in \mathbb{N} : |X| = n.$$

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Theorem (UD Theorem 22.6)

Let A be a finite set. There is a unique $n \in \mathbb{N}$ such that $A \approx \{0, 1, \cdots, n-1\}$.

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X is infinite if it is not finite:

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$$g \triangleq f|_{\{0,1,\ldots,n\}} : \{0, 1, \cdots, n\} \to \{0, 1, \cdots, n-1\}$$

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$$\forall n \in \mathbb{N} : |X| \neq n.$$

Theorem (UD Theorem 22.3)

$\mathbb{N}$ *is infinite.* *(So are $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$.)*

By Contradiction.

$$\exists n \in \mathbb{N} : |\mathbb{N}| = n.$$

$$\exists f : \mathbb{N} \xleftrightarrow[\textit{onto}]{1-1} \{0, 1, \cdots, n-1\}$$

$$g \triangleq f|_{\{0,1,\ldots,n\}} : \{0, 1, \cdots, n\} \to \{0, 1, \cdots, n-1\}$$

By the Pigeonhole Principle : g is not 1-1 $\implies$ f is not 1-1

Definition (Infinite)

For any set X,

Countably Infinite

$$|X| = |\mathbb{N}| \triangleq \aleph_0$$

Countable

$$(\text{finite} \lor \text{countably infinite})$$

Uncountable

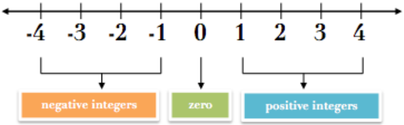
$$(\neg \text{ countable})$$

$$(\text{infinite}) \land \Big(\neg \text{ (countably infinite)}\Big)$$

$$\aleph_0$$

Theorem ($\mathbb{Z}$ is Countable.)

$$|\mathbb{Z}| = |\mathbb{N}|$$



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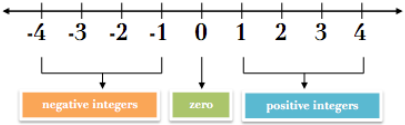

$$0 \quad 1 \quad -1 \quad 2 \quad -2 \quad \cdots$$

Theorem ($\mathbb{Q}$ is Countable. (Cantor 1873-11; Published in 1874))

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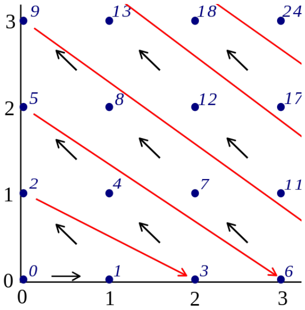
$$q \in \mathbb{Q}^{+} : a/b \; (a, b \in \mathbb{N}^{+})$$

1/1	1/2	1/3	1/4	1/5	1/6	1/7	1/8	⋯
2/1	2/2	2/3	2/4	2/5	2/6	2/7	2/8	⋯
3/1	3/2	3/3	3/4	3/5	3/6	3/7	3/8	⋯
4/1	4/2	4/3	4/4	4/5	4/6	4/7	4/8	⋯
5/1	5/2	5/3	5/4	5/5	5/6	5/7	5/8	⋯
6/1	6/2	6/3	6/4	6/5	6/6	6/7	6/8	⋯
7/1	7/2	7/3	7/4	7/5	7/6	7/7	7/8	⋯
8/1	8/2	8/3	8/4	8/5	8/6	8/7	8/8	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Theorem ($\mathbb{N} \times \mathbb{N}$ is Countable.)

$$|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$$



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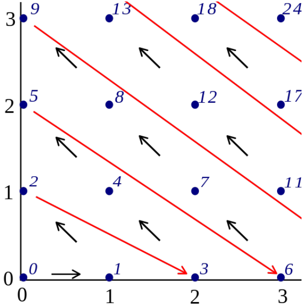

$$\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

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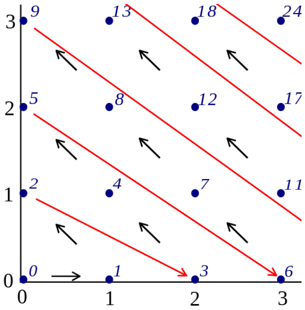

$$\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

$$\pi(k_1, k_2) = \frac{1}{2}(k_1+k_2)(k_1+k_2+1)+k_2$$

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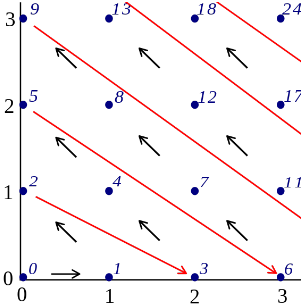

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Cantor Pairing Function

Theorem ($\mathbb{N}^n$ is Countable.)

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Theorem

The Cartesian product of finitely many countable sets is countable.

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$$\mathbb{N}^n \quad \mathit{vs.} \quad \mathbb{N}^{\mathbb{N}}$$

$$\pi^{(n)} : \mathbb{N}^n \to \mathbb{N}$$

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$$\mathbb{N}^n \quad vs. \quad \mathbb{N}^{\mathbb{N}}$$

$$\pi^{(n)} : \mathbb{N}^n \to \mathbb{N}$$

$$\pi^{(n)}(k_1, \ldots, k_{n-1}, k_n) = \pi(\pi^{(n-1)}(k_1, \ldots, k_{n-1}), k_n)$$

Theorem

Any finite union of countable sets is countable.

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$$a_0 \quad b_0 \quad c_0 \quad a_1 \quad b_1 \quad c_1 \cdots$$

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The union of countably many countable sets is countable.

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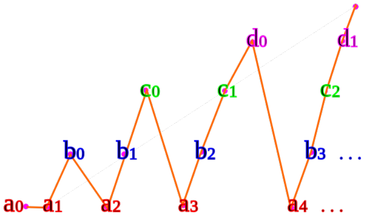

Counting by Diagonals.

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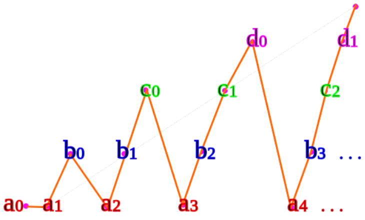

Counting by Diagonals.

We need Axiom of (Countable) Choice!

Beyond

$$\aleph_0$$

Theorem ($\mathbb{R}$ is Uncountable. (Cantor 1873-12; Published in 1874))

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Different "Sizes" of Infinity

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Cantor's Diagonal Argument (1890)

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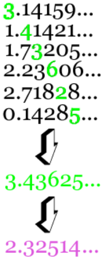

Theorem ($\mathbb{R}$ is Uncountable. (Cantor 1873-12; Published in 1874))

$$|\mathbb{R}| \neq |\mathbb{N}|$$

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3.14159...
1.41421...
1.73205...
2.23606...
2.71828...
0.14285...

3.43625...

2.32514...

By Diagonal Argument.

$$\mathfrak{c} \triangleq |\mathbb{R}|$$

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Was Cantor Surprised?

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Theorem (Brouwer (Topological Invariance of Dimension))

There is no continuous bijections between $\mathbb{R}^m$ and $\mathbb{R}^n$ for $m \neq n$.

Beyond

$$\mathfrak{c}$$

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If $f : A \to \mathcal{P}(A)$, then f is not onto.

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If $f : A \to \mathcal{P}(A)$, then f is not onto.

Proof. Let A be a set and let $f : A \to 2^A$. To show that f is not onto, we must find a $B \in 2^A$ (i.e., $B \subseteq A$) for which there is no $a \in A$ with $f(a) = B$. In other words, B is a set that f "misses." To this end, let

$$B = \{x \in A : x \notin f(x)\}.$$

We claim there is no $a \in A$ with $f(a) = B$.

Suppose, for the sake of contradiction, there is an $a \in A$ such that $f(a) = B$. We ponder: Is $a \in B$?

- If $a \in B$, then, since $B = f(a)$, we have $a \in f(a)$. So, by definition of B, $a \notin f(a)$; that is, $a \notin B$.$\Rightarrow\Leftarrow$
- If $a \notin B = f(a)$, then, by definition of B, $a \in B$.$\Rightarrow\Leftarrow$

Both $a \in B$ and $a \notin B$ lead to contradictions, and hence our supposition [there is an $a \in A$ with $f(a) = B$] is false, and therefore f is not onto. ■

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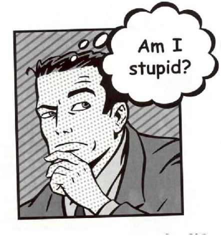

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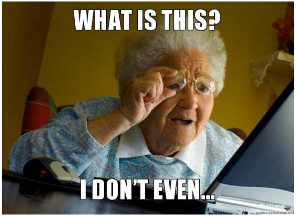

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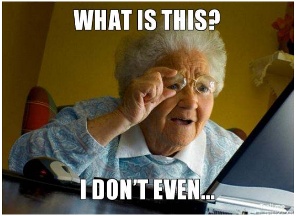

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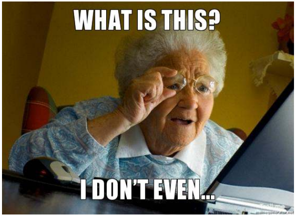

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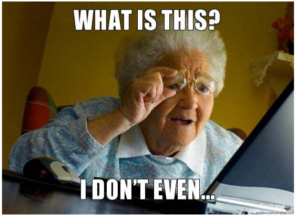

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$$a \in B \iff a \notin B$$

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Diagonal Argument .

a	$f(a)$					
	1	2	3	4	5	$\cdots$
1	1	1	0	0	1	$\cdots$
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$$B = \{0, 1, 1, 0, 1\}$$

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Diagonal Argument (以下仅适用于可数集合 A).

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$$<$$

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$|B| \leq |A|$ (Axiom of Choice)

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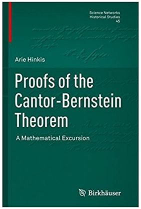

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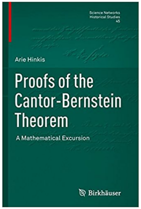

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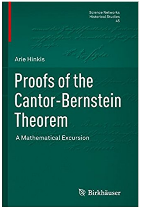

Schröder–Bernstein theorem @ wiki

Q : Is "$\leq$" a total order?

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Theorem (PCC)

Principle of Cardinal Comparability (PCC) $\iff$ *Axiom of Choice*

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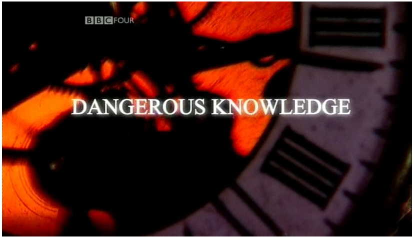
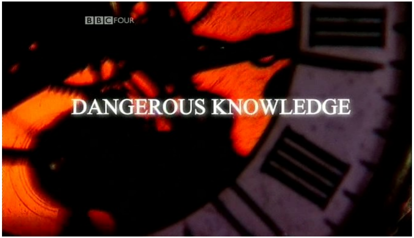
$$\boxed{\mathfrak{c} = 2^{\aleph_0}}$$

Continuum Hypothesis (CH)

$$\boxed{\nexists A : \aleph_0 < |A| < \mathfrak{c}}$$



☞ Dangerous Knowledge (22:20; BBC 2007)



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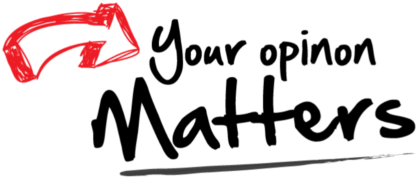
Independence from ZFC:

Kurt Gödel (1940) CH cannot be disproved from ZF.

Paul Cohen (1964) CH cannot be proven from the ZFC axioms.


Thank You!



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