2-5 Linear Recurrences

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Don't we already know how to use the "Master Theorem"?

$$T(n) = aT(n/b) + f(n)$$

Linear recurrences which may arise in average-case analysis.

$$\boxed{a_n = f(a_{n-1}, a_{n-2}, \ldots, a_{n-t}) + g(n)}$$

recurrence type	typical example
first-order	
linear	$a_n = na_{n-1} - 1$
nonlinear	$a_n = 1/(1 + a_{n-1})$
second-order	
linear	$a_n = a_{n-1} + 2a_{n-2}$
nonlinear	$a_n = a_{n-1}a_{n-2} + \sqrt{a_{n-2}}$
variable coefficients	$a_n = na_{n-1} + (n-1)a_{n-2} + 1$
tth order	$a_n = f(a_{n-1}, a_{n-2}, \ldots, a_{n-t})$
full-history	$a_n = n + a_{n-1} + a_{n-2} \ldots + a_1$
divide-and-conquer	$a_n = a_{\lfloor n/2 \rfloor} + a_{\lceil n/2 \rceil} + n$

Table 2.1 Classification of recurrences

Theorem (First-order Linear Recurrences with Constant Coefficients)

$$T(n) = rT(n-1) + g(n) \quad for\ n > 0\ with\ T(0) = a$$

$$\boxed{T(n) = r^n a + \sum_{i=1}^{n} r^{n-i} g(i)}$$

Theorem (First-order Linear Recurrences)

$$T(n) = x_n T(n-1) + y_n \quad \textit{for } n > 0 \textit{ with } T(0) = 0$$

$$\boxed{T(n) = y_n + \sum_{1 \le j < n} y_j x_{j+1} x_{j+2} \cdots x_n}$$

$$\begin{aligned}
T(n) &= x_n T(n-1) + y_n \\
&= x_n \left(x_{n-1} T(n-2) + y_{n-1} \right) + y_n \\
&= x_n x_{n-1} T(n-2) + x_n y_{n-1} + y_n \\
&= x_n x_{n-1} \left(x_{n-2} T(n-3) + y_{n-2} \right) + x_n y_{n-1} + y_n \\
&= x_n x_{n-1} x_{n-2} T(n-3) + x_n x_{n-1} y_{n-2} + x_n y_{n-1} + y_n \\
&= \dots
\end{aligned}$$

Theorem (First-order Linear Recurrences)

$$T(n) = x_n T(n-1) + y_n \quad \textit{for } n > 0 \textit{ with } T(0) = 0$$

$$\boxed{T(n) = y_n + \sum_{1 \le j < n} y_j x_{j+1} x_{j+2} \cdots x_n}$$

$$\underbrace{\frac{T(n)}{x_n x_{n-1} \cdots x_1}}_{\text{summation factor}} = \frac{T(n-1)}{x_{n-1} \cdots x_1} + \frac{y_n}{x_n x_{n-1} \cdots x_1}$$

$$S(n) \triangleq \frac{T(n)}{x_n x_{n-1} \cdots x_1}$$

$$S(n) = S(n-1) + \frac{y_n}{x_n x_{n-1} \cdots x_1}$$

$$\boxed{T(n) = (1 + \frac{1}{n})T(n-1) + 2 \quad \text{for } n > 1 \text{ with } T(1) = 0}$$

$$x_n = 1 + \frac{1}{n} = \frac{n+1}{n} \implies x_n x_{n-1} \cdots x_1 = n + 1$$

$$\frac{T(n)}{n+1} = \frac{T(n-1)}{n} + \frac{2}{n+1} \quad \text{for } n > 1$$

$$\frac{T(n)}{n+1} = \frac{T(1)}{2} + 2 \sum_{3 \le k \le n+1} \frac{1}{k}$$

$$\boxed{T(n) = 2(n+1)(H_{n+1} - \frac{3}{2}) \approx 2n \ln n - 1.846n}$$

$$\boxed{T(n) = (1 + \frac{1}{n})T(n-1) + 2 \quad \text{for } n > 1 \text{ with } T(1) = 0}$$

$$\boxed{T(n) = 2(n+1)(H_{n+1} - \frac{3}{2}) \approx 2n \ln n - 1.846n}$$

$$T(n) = (n+1) + \frac{1}{n} \sum_{1 \le i \le n} \Big(T(i-1) + T(n-i) \Big)$$

$$\text{for } n > 1 \text{ with } T(0) = T(1) = 0$$

average number of comparisons of QUICKSORT

After-class Exercise

$$T(n) = T(n-1) - \frac{2T(n-1)}{n} + 2\left(1 - \frac{2T(n-1)}{n}\right), n > 0 \text{ with } T(0) = 0$$

"On Random 2-3 Trees", Andrew C Yao, 1978

Higher-order Linear Recurrences

$$a_n = x_1 a_{n-1} + x_2 a_{n-2} + \cdots + x_t a_{n-t} + g_n \quad \text{for } n \geq t$$

$$\text{with } a_0, a_1, \cdots, a_{t-1}$$

Theorem (Linear Homogeneous Recurrences with Constant Coefficients)

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_t a_{n-t} \quad \textit{for } n \geq t$$

$$\textit{with } a_0, a_1, \cdots, a_{t-1}$$

"Characteristic polynomial": $\boxed{q(x) \equiv x^t - r_1 x^{t-1} - r_2 x^{t-2} - \cdots - r_t}$

$$\beta_1\ (m_1), \beta_2\ (m_2), \cdots, \beta_i\ (m_i), \cdots, \beta_k\ (m_k)$$

$$\beta_i \textit{ is a root with multiplicity } m_i \textit{ and } m_1 + m_2 + \cdots + m_k = t$$

$$\boxed{a_n = \sum_{0 \leq j < m_1} c_{1j} n^j \beta_1^n + \sum_{0 \leq j < m_2} c_{2j} n^j \beta_2^n + \cdots + \sum_{0 \leq j < m_k} c_{kj} n^j \beta_k^n}$$

a_n *is a linear combination of* $n^j \beta^n$ *(called "particular solutions")*

$$F_n = F_{n-1} + F_{n-2} \quad \text{for } n > 1 \text{ with } F_0 = 0,\ F_1 = 1$$

$$\boxed{x^2 - x - 1 = 0} \implies \phi = \frac{1+\sqrt{5}}{2},\ \hat{\phi} = \frac{1-\sqrt{5}}{2}$$

$$\boxed{F_n = c_1\phi^n + c_2\hat{\phi}^n}$$

$$\begin{aligned} F_0 &= 0 = c_1 + c_2 \\ F_1 &= 1 = c_1\phi + c_2\hat{\phi} \end{aligned}$$

$$\boxed{F_n = \frac{1}{\sqrt{5}}(\phi^n - \hat{\phi}^n)}$$

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_t a_{n-t} \quad \text{for } n \geq t$$

$$\boxed{q(x) \equiv x^t - r_1 x^{t-1} - r_2 x^{t-2} - \cdots - r_t}$$

$$\boxed{a_n = \sum_{0 \leq j < m_1} c_{1j} n^j \beta_1^n + \sum_{0 \leq j < m_2} c_{2j} n^j \beta_2^n + \cdots + \sum_{0 \leq j < m_k} c_{kj} n^j \beta_k^n}$$

Proof.

Take β $(m = 2)$ for example.

$$\beta^n = r_1 \beta^{n-1} + r_2 \beta^{n-2} + \cdots + r_t \beta^{n-t} \quad \text{for } n \geq t$$

$$\beta^{n-t} q(\beta) = 0$$

$$n\beta^n = r_1 (n-1) \beta^{n-1} + r_2 (n-2) \beta^{n-2} + \cdots + r_t (n-t) \beta^{n-t} \quad \text{for } n \geq t$$

$$\beta^{n-t} \left((n-t)\, q(\beta) + \beta q'(\beta) \right) = 0$$

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_t a_{n-t} \quad \text{for } n \ge t$$

$$\boxed{q(x) \equiv x^t - r_1 x^{t-1} - r_2 x^{t-2} - \cdots - r_t}$$

$$\boxed{a_n = \sum_{0 \le j < m_1} c_{1j} n^j \beta_1^n + \sum_{0 \le j < m_2} c_{2j} n^j \beta_2^n + \cdots + \sum_{0 \le j < m_k} c_{kj} n^j \beta_k^n}$$

Proof Cont.

a_n is a linear combination of $n^j \beta^n$

We also need to prove that there are *no* other solutions.

$$a_n = 5a_{n-1} - 6a_{n-2} \quad \text{for } n \geq 2 \text{ with } a_0 = 0, a_1 = 1$$

$$\boxed{x^2 - 5x + 6 = (x-2)(x-3) = 0} \implies x = 2,\ 3$$

$$\boxed{a_n = c_1 2^n + c_2 3^n}$$

$$a_0 = 0 = c_1 + c_2$$
$$a_1 = 1 = 2c_1 + 3c_2$$

$$\boxed{a_n = 3^n - 2^n}$$

$$a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3} \quad \text{for } n \geq 3 \text{ with } a_0 = 0,\ a_1 = 1,\ a_2 = 4$$

$$\boxed{x^3 - 5x^2 + 8x - 4 = 0}$$

$$(x-1)(x-2)^2 = 0 \implies x_1 = 1,\ x_2 = 2,\ x_2' = 2$$

$$a_n = c_1 \cdot 1^n + c_2 \cdot 2^n + c_2' \cdot n2^n$$

$$\begin{aligned} a_0 &= 0 = c_1 + c_2 \\ a_1 &= 1 = c_1 + 2c_2 + 2c_2' \\ a_2 &= 4 = c_1 + 4c_2 + 8c_2' \end{aligned}$$

$$\boxed{a_n = n2^{n-1}}$$

$$a_n = 2a_{n-1} - a_{n-2} + 2a_{n-3} \quad \text{for } n \geq 3 \text{ with } a_0 = 1,\ a_1 = 0,\ a_2 = -1$$

$$\boxed{x^3 - 2x^2 + x - 2} = (x^2+1)(x-2) = 0 \implies x = 2,\ i,\ -i$$

$$\boxed{a_n = c_1 2^n + c_2 i^n + c_3 (-i)^n}$$

$$a_0 = 1 = c_1 + c_2 + c_3$$
$$a_1 = 0 = 2c_1 + c_2 i - c_3 i$$
$$a_2 = -1 = 4c_1 - c_2 - c_3$$

$$\boxed{a_n = \frac{1}{2} i^n \left(1 + (-1)^n\right)}$$

$$1,\ 0,\ -1,\ 0,\ 1,\ 0,\ -1,\ 0,\ 1,\ 0,\ -1,\ 0 \cdots$$

$$a_n = 2a_{n-1} - a_{n-2} + 2a_{n-3} \quad \text{for } n \geq 3 \text{ with } a_0,\ a_1,\ a_2$$

$$\boxed{a_n = c_1 2^n + c_2 i^n + c_3 (-i)^n}$$

$$\begin{aligned} a_0 &= c_1 + c_2 + c_3 \\ a_1 &= 2c_1 + c_2 i - c_3 i \\ a_2 &= 4c_1 - c_2 - c_3 \end{aligned}$$

$$a_0 = 1,\ a_1 = 0,\ a_2 = -1 \implies a_n = \frac{1}{2} i^n \left(1 + (-1)^n\right)$$

$$a_0 = 1,\ a_1 = 2,\ a_2 = 4 \implies a_n = 2^n$$

Pay attention to initial conditions in linear recurrences!

Additional Problem

To give initial conditions a_0, a_1, and a_2 such that the growth rate of the solution to

$$a_n = 2a_{n-1} - a_{n-2} + 2a_{n-3}, \ n > 2$$

is *(1) constant*; *(2) exponential*; *(3) fluctuating in sign.*

First-order Linear Non-homogeneous Recurrences

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_t a_{n-t} + r \quad \text{for } n \geq t$$

$$a_n = 5a_{n-1} - 6a_{n-2} + 2 \quad \text{for } n \geq 2 \text{ with } a_0 = 0,\ a_1 = 1$$

$$a_n - 5a_{n-1} + 6a_{n-2} - 2 = 0 \quad \text{for } n \geq 2 \text{ with } a_0 = 0,\ a_1 = 1$$

$$a_{n-1} - 5a_{n-2} + 6a_{n-3} - 2 = 0 \quad \text{for } n \geq 3 \text{ with } a_2 = 7$$

$$a_n - 6a_{n-1} + 11a_{n-2} - 6a_{n-3} = 0$$

$$\boxed{x^3 - 6x^2 + 11x - 6 = (x-1)(x-2)(x-3) \implies x_1 = 1,\ x_2 = 2,\ x_3 = 3}$$

$$\boxed{a_n = c_1 1^n + c_2 2^n + c_3 3^n}$$

$$\boxed{a_n = 2 \cdot 3^n - 3 \cdot 2^n + 1}$$

More Issues about Linear Recurrences

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_t a_{n-t} + g(n) \quad \text{for } n \geq t$$

$$a_n = a_n^h + a_n^p$$

How to Find a Particular Solution for a Non-homogeneous Recurrence?

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_t a_{n-t} \quad \text{for } n \geq t$$

$$\boxed{q(x) \equiv x^t - r_1 x^{t-1} - r_2 x^{t-2} - \cdots - r_t}$$

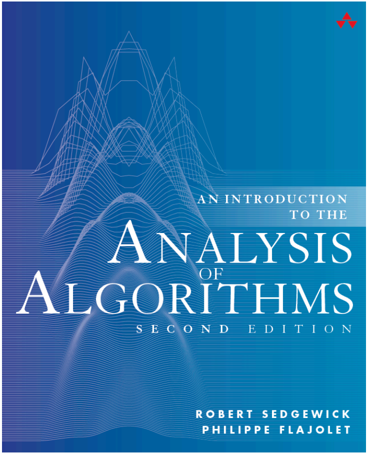
$$\boxed{a_n = \sum_{0 \leq j < m_1} c_{1j} n^j \beta_1^n + \sum_{0 \leq j < m_2} c_{2j} n^j \beta_2^n + \cdots + \sum_{0 \leq j < m_k} c_{kj} n^j \beta_k^n}$$

$$t \geq 5$$

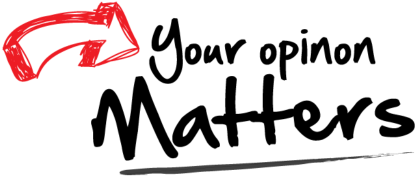
Generating Functions and Asymptotic Analysis

$$a_n = f_1(n)a_{n-1} + f_2(n)a_{n-2} + \cdots + f_t(n)a_{n-t} \quad \text{for } n \geq t$$

Generating Functions


AN INTRODUCTION
TO THE
ANALYSIS
OF
ALGORITHMS
SECOND EDITION
ROBERT SEDGEWICK
PHILIPPE FLAJOLET


Thank
You!



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