1-9 Set Theory (II): Relations

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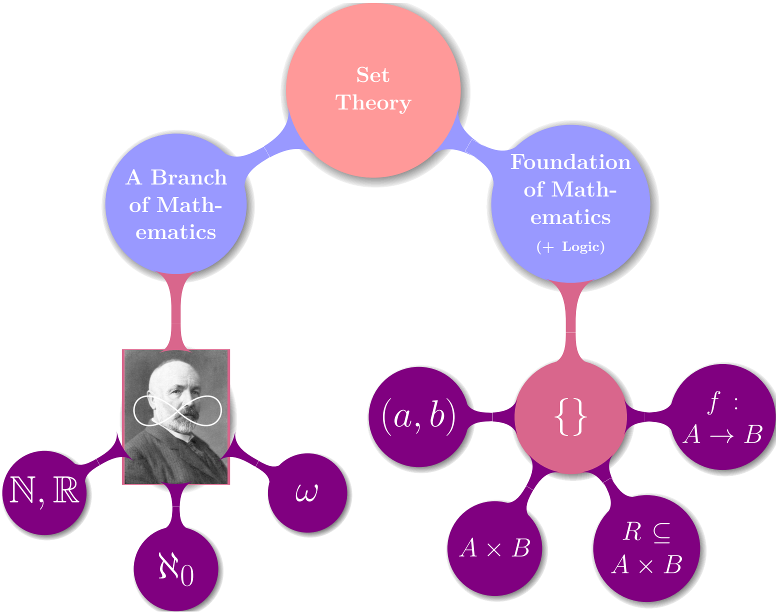

Set Theory
A Branch of Mathematics
Foundation of Mathematics
(+ Logic)
$\mathbb{N}, \mathbb{R}$
$\aleph_0$
ω
(a, b)
$\{\}$
$f : A \to B$
$A \times B$
$R \subseteq A \times B$

Figure 13. A selection of consistency axioms over an execution $(E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar})$

Auxiliary relations

$\mathsf{sameobj}(e, f) \iff \mathsf{obj}(e) = \mathsf{obj}(f)$

Per-object causality (aka happens-before) order:
$\mathsf{hbo} = ((\mathsf{ro} \cap \mathsf{sameobj}) \cup \mathsf{vis})^+$

Causality (aka happens-before) order: $\mathsf{hb} = (\mathsf{ro} \cup \mathsf{vis})^+$

Axioms

EVENTUAL:
$\forall e \in E.\ \neg(\exists \text{ infinitely many } f \in E.\ \mathsf{sameobj}(e, f) \wedge \neg(e \xrightarrow{\mathsf{vis}} f))$

THINAIR: $\mathsf{ro} \cup \mathsf{vis}$ is acyclic

POCV (Per-Object Causal Visibility): $\mathsf{hbo} \subseteq \mathsf{vis}$

POCA (Per-Object Causal Arbitration): $\mathsf{hbo} \subseteq \mathsf{ar}$

COCV (Cross-Object Causal Visibility): $(\mathsf{hb} \cap \mathsf{sameobj}) \subseteq \mathsf{vis}$

COCA (Cross-Object Causal Arbitration): $\mathsf{hb} \cup \mathsf{ar}$ is acyclic

Figure 17. Optimized state-based multi-value register and its simulation

$\Sigma = \mathsf{ReplicaID} \times \mathcal{P}(\mathbb{Z} \times (\mathsf{ReplicaID} \to \mathbb{N}_0))$

$\bar{\sigma}_0 = \langle r, \emptyset \rangle$

$M = \mathcal{P}(\mathbb{Z} \times (\mathsf{ReplicaID} \to \mathbb{N}_0))$

$\mathsf{do}(\mathtt{wr}(a), \langle r, V\rangle, t) = (\langle r, \{(a, (\lambda s.\ \text{if } s \neq r \text{ then } \max\{v(s) \mid (\_, v) \in V\} \text{ else } \max\{v(s) \mid (\_, v) \in V\} + 1))\}\rangle, \bot)$

$\mathsf{do}(\mathtt{rd}, \langle r, V\rangle, t) = (\langle r, V\rangle, \{a \mid (a, \_) \in V\})$

$\mathsf{send}(\langle r, V\rangle) = (\langle r, V\rangle, V)$

$\mathsf{receive}(\langle r, V\rangle, V') = \langle r, \{(a, v) \in V'' \mid v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'. (a', v') \in V'' \wedge a \neq a'\}\}\rangle$,
where $V'' = \{(a, \bigsqcup\{v' \mid (a, v') \in V \cup V'\}) \mid (a, \_) \in V \cup V'\}$

$\langle s, V\rangle\ [\mathcal{R}_r]\ I \iff (r = s) \wedge (V\ [\mathcal{M}]\ I)$

$V\ [\mathcal{M}]\ ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar}), \mathsf{info}) \iff$
$(\forall (a, v), (a', v') \in V.\ (a = a' \implies v = v')) \wedge$
$(\forall (a, v) \in V.\ \exists s.\ v(s) > 0) \wedge$
$(\forall (a, v) \in V.\ v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'. (a', v') \in V \wedge a \neq a'\}) \wedge$
$\exists$ distinct $e_{s,k}$.
$(\{e \in E \mid \exists a.\ \mathsf{oper}(e) = \mathtt{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicaID} \wedge 1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V\}\}) \wedge$
$(\forall s, j, k.\ (\mathsf{repl}(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}} e_{s,k} \iff j < k)) \wedge$
$(\forall (a, v) \in V.\ \forall q.\ \{j \mid \mathsf{oper}(e_{q,j}) = \mathtt{wr}(a)\} \cup \{j \mid \exists s, k.\ e_{q,j} \xrightarrow{\mathsf{vis}} e_{s,k} \wedge \mathsf{oper}(e_{s,k}) = \mathtt{wr}(a)\} = \{j \mid 1 \leq j \leq v(q)\}) \wedge$
$(\forall e \in E.\ (\mathsf{oper}(e) = \mathtt{wr}(a) \wedge \neg\exists f \in E.\ \mathsf{oper}(f) = \mathtt{wr}(\_) \wedge e \xrightarrow{\mathsf{vis}} f) \implies (a, \_) \in V)$

the former. The only non-trivial obligation is to show that if

$$V\ [\mathcal{M}]\ ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}), \mathsf{info}),$$

then

$$\{a \mid (a, \_) \in V\} \subseteq \{a \mid \exists e \in E.\ \mathsf{oper}(e) = \mathtt{wr}(a) \wedge \neg\exists f \in E.\ \exists a'.\ \mathsf{oper}(e) = \mathtt{wr}(a') \wedge e \xrightarrow{\mathsf{vis}} f\} \quad (13)$$

(the reverse inclusion is straightforwardly implied by $\mathcal{R}_r$).
Take $(a, v) \in V$. We have $\forall (a, v) \in V.\ \exists s.\ v(s) > 0$,

$$v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'. (a', v') \in V \wedge a \neq a'\}$$

and

$$\forall (a, v) \in V.\ \forall q.\ \{j \mid \mathsf{oper}(e_{q,j}) = \mathtt{wr}(a)\} \cup \{j \mid \exists s, k.\ e_{q,j} \xrightarrow{\mathsf{vis}} e_{s,k} \wedge \mathsf{oper}(e_{s,k}) = \mathtt{wr}(a)\} = \{j \mid 1 \leq j \leq v(q)\}.$$

From this we get that for some $e \in E$

$$\mathsf{oper}(e) = \mathtt{wr}(a) \wedge \neg\exists f \in E.\ \exists a'.\ a' \neq a \wedge \mathsf{oper}(e) = \mathtt{wr}(a') \wedge e \xrightarrow{\mathsf{vis}} f.$$

Since vis is acyclic, this implies that for some $e' \in E$

$$\mathsf{oper}(e') = \mathtt{wr}(a) \wedge \neg\exists f \in E.\ \mathsf{oper}(e') = \mathtt{wr}(\_) \wedge e' \xrightarrow{\mathsf{vis}} f,$$

which establishes (13).
Let us now discharge RECEIVE. Let $\mathsf{receive}(\langle r, V\rangle, V') = \langle r, V'''\rangle$, where

$$V'' = \{(a, \bigsqcup\{v' \mid (a, v') \in V \cup V'\}) \mid (a, \_) \in V \cup V'\};$$
$$V''' = \{(a, v) \in V'' \mid v \not\sqsubseteq \bigsqcup\{(a', v') \in V'' \mid a \neq a'\}\}.$$

Assume $\langle r, V\rangle\ [\mathcal{R}_r]\ I$, $V'\ [\mathcal{M}]\ J$ and

$I = ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar}), \mathsf{info})$;
$J = ((E', \mathsf{repl}', \mathsf{obj}', \mathsf{oper}', \mathsf{rval}', \mathsf{ro}', \mathsf{vis}', \mathsf{ar}'), \mathsf{info}')$;
$I \sqcup J = ((E'', \mathsf{repl}'', \mathsf{obj}'', \mathsf{oper}'', \mathsf{rval}'', \mathsf{ro}'', \mathsf{vis}'', \mathsf{ar}''), \mathsf{info}'')$.

By agree we have $I \sqcup J \in \mathsf{IEx}$. Then

$(\forall (a, v), (a', v') \in V.\ (a = a' \implies v = v')) \wedge$
$(\forall (a, v) \in V.\ \exists s.\ v(s) > 0) \wedge$
$(\forall (a, v) \in V.\ v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'. (a', v') \in V \wedge a \neq a'\}) \wedge$
$\exists$ distinct $e_{s,k}$.
$(\{e \in E \mid \exists a.\ \mathsf{oper}''(e) = \mathtt{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicaID} \wedge 1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V\}\}) \wedge$
$(\forall s, j, k.\ (\mathsf{repl}''(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}} e_{s,k} \iff j < k)) \wedge$
$(\forall (a, v) \in V.\ \forall q.\ \{j \mid \mathsf{oper}''(e_{q,j}) = \mathtt{wr}(a)\} \cup \{j \mid \exists s, k.\ e_{q,j} \xrightarrow{\mathsf{vis}} e_{s,k} \wedge \mathsf{oper}''(e_{s,k}) = \mathtt{wr}(a)\} = \{j \mid 1 \leq j \leq v(q)\}) \wedge$
$(\forall e \in E.\ (\mathsf{oper}''(e) = \mathtt{wr}(a) \wedge \neg\exists f \in E.\ \mathsf{oper}''(f) = \mathtt{wr}(\_) \wedge e \xrightarrow{\mathsf{vis}} f) \implies (a, \_) \in V)$

and

$(\forall (a, v), (a', v') \in V'.\ (a = a' \implies v = v')) \wedge$
$(\forall (a, v) \in V'.\ \exists s.\ v(s) > 0) \wedge$
$(\forall (a, v) \in V'.\ v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'. (a', v') \in V' \wedge a \neq a'\}) \wedge$
$\exists$ distinct $e_{s,k}$.
$(\{e \in E' \mid \exists a.\ \mathsf{oper}''(e) = \mathtt{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicaID} \wedge 1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V'\}\}) \wedge$
$(\forall s, j, k.\ (\mathsf{repl}''(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}'} e_{s,k} \iff j < k)) \wedge$
$(\forall (a, v) \in V'.\ \forall q.\ \{j \mid \mathsf{oper}''(e_{q,j}) = \mathtt{wr}(a)\} \cup \{j \mid \exists s, k.\ e_{q,j} \xrightarrow{\mathsf{vis}'} e_{s,k} \wedge \mathsf{oper}''(e_{s,k}) = \mathtt{wr}(a)\} = \{j \mid 1 \leq j \leq v(q)\}) \wedge$
$(\forall e \in E'.\ (\mathsf{oper}''(e) = \mathtt{wr}(a) \wedge \neg\exists f \in E'.\ \mathsf{oper}''(f) = \mathtt{wr}(\_) \wedge e \xrightarrow{\mathsf{vis}'} f) \implies (a, \_) \in V')$.

The agree property also implies

$$\forall s, k.\ 1 \leq k \leq \min\{\max\{v(s) \mid \exists a. (a, v) \in V\}, \max\{v(s) \mid \exists a. (a, v) \in V'\}\} \implies e_{s,k} = e'_{s,k}.$$

Hence, there exist distinct

$$e''_{s,k} \text{ for } s \in \mathsf{ReplicaID},\ k = 1..(\max\{v(s) \mid \exists a. (a, v) \in V'''\}),$$

such that

$(\forall s, k.\ 1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V\} \implies e''_{s,k} = e_{s,k}) \wedge$
$(\forall s, k.\ 1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V'\} \implies e''_{s,k} = e'_{s,k})$

and

$(\{e \in E \cup E' \mid \exists a.\ \mathsf{oper}''(e) = \mathtt{wr}(a)\} = \{e''_{s,k} \mid s \in \mathsf{ReplicaID} \wedge 1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V'''\}\})$
$\wedge\ (\forall s, j, k.\ (\mathsf{repl}(e''_{s,k}) = s) \wedge (e''_{s,j} \xrightarrow{\mathsf{ro}''} e''_{s,k} \iff j < k))$.

By the definition of V'' and V''' we have

$$\forall (a, v), (a', v') \in V'''.\ (a = a' \implies v = v').$$

We also straightforwardly get

$$\forall (a, v) \in V'.\ \exists s.\ v(s) > 0$$

and

$$(\forall (a, v) \in V''.\ \forall q.\ \{j \mid \mathsf{oper}''(e''_{q,j}) = \mathtt{wr}(a)\} \cup \{j \mid \exists s, k.\ e''_{q,j} \xrightarrow{\mathsf{vis}''} e''_{s,k} \wedge \mathsf{oper}''(e''_{s,k}) = \mathtt{wr}(a)\} = \{j \mid 1 \leq j \leq v(q)\}). \quad (14)$$


ANNE FINE
JULIA DONALDSON
CAROL ANN DUFFY
MICHAEL MORPURGO
ANTHONY BROWNE
JACQUELINE WILSON
MICHAEL ROSEN
QUENTIN BLAKE
AND MANY MORE
DON'T BE SCARED
AN ANTHOLOGY FOR CHILDREN BY WELL-LOVED AUTHORS & ARTISTS


I'm so excited.

Definition (Relations)

A *relation* R from A to B is a subset of $A \times B$:

$$R \subseteq A \times B$$

Definition (Cartesian Products)

The *Cartesian product* $A \times B$ of A and B is defined as

$$A \times B \triangleq \{(a, b) \mid a \in A \wedge b \in B\}$$

Axiom (Ordered Pairs)

$$(a, b) = (c, d) \iff a = c \wedge b = d$$

Q : Are you satisfied with the definitions above?

Axiom (Ordered Pairs)

$$(a, b) = (c, d) \iff a = c \land b = d$$

Definition (Ordered Pairs (Kazimierz Kuratowski; 1921))

$$(a, b) \triangleq \{\{a\}, \{a, b\}\}$$

Definition (Ordered Pairs (Kazimierz Kuratowski; 1921))

$$(a, b) \triangleq \{\{a\}, \{a, b\}\}$$

Theorem

$$(a, b) = (c, d) \iff a = c \land b = d$$

Proof.

$$\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$$

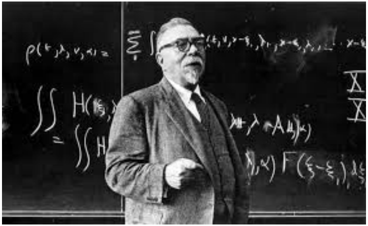
CASE I : $a = b$

CASE II : $a \neq b$

$\square$

Definition (Ordered Pairs (Norbert Wiener; 1914))

$$(a, b) \triangleq \Big\{\{\{a\}, \emptyset\}, \{\{b\}\}\Big\}$$



Theorem

$$(a, b) = (c, d) \iff a = c \land b = d$$

Definition (Cartesian Products)

The *Cartesian product* $A \times B$ of A and B is defined as

$$A \times B \triangleq \{(a, b) \mid a \in A \land b \in B\}$$

$$X^2 \triangleq X \times X$$

Theorem

$A \times B$ is a set.

Proof.

$$A \times B \triangleq \{(a, b) \in ? \mid a \in A \land b \in B\}$$

$$\{\{a\}, \{a, b\}\} \in ?\mathcal{P}(\mathcal{P}(A \cup B))$$

□

Definition (Relations)

A *relation* R from A to B is a subset of $A \times B$:

$$R \subseteq A \times B$$

If $A = B$, R is called a relation on A.

Definition (Notations)

$$(a, b) \in R \qquad R(a, b) \qquad aRb$$

Definition (Relations)

A *relation* R from A to B is a subset of $A \times B$:

$$R \subseteq A \times B$$

Examples

- Both $A \times B$ and $\emptyset$ are relations from A to B.
- $$< \; = \{(a, b) \in \mathbb{R} \times \mathbb{R} \mid a \text{ is less than } b\}$$
- $$D = \{(a, b) \in \mathbb{N} \times \mathbb{N} \mid \exists q \in \mathbb{N} : a \cdot q = b\}$$
- P : the set of people

$$M = \{(a, b) \in P \times P \mid a \text{ is the mother of } b\}$$

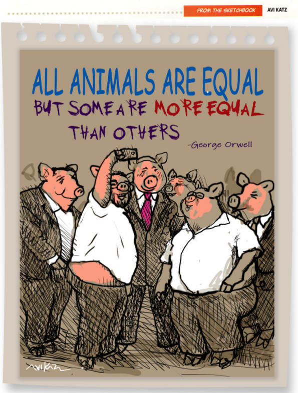
$$B = \{(a, b) \in P \times P \mid a \text{ is the brother of } b\}$$

Important Relations:

Equivalence Relations (1-9)

Functions (1-10)

Ordering Relations (1-12)



3 Definitions

5 Operations

7 Properties

$$R = \{(0,1),(0,2),(0,3),(1,1),(1,2),(1,3),(2,3)\}$$

3 Definitions

Definition (Domain)

$$\mathrm{dom}(R) = \{a \mid \exists b : (a, b) \in R\}$$

Theorem

$dom(R)$ is a set.

$$\mathrm{dom}(R) = \{a \in ?\bigcup\bigcup R \mid \exists b : (a, b) \in R\}$$

$$(a, b) = \{\{a\}, \{a, b\}\} \in R$$

$$\{a, b\} \in \bigcup R$$

$$a \in \bigcup\bigcup R$$

Definition (Range)

$$\operatorname{ran}(R) = \{b \mid \exists a : (a, b) \in R\}$$

Theorem

$ran(R)$ is a set.

$$\operatorname{ran}(R) = \{b \in \bigcup\bigcup R \mid \exists a : (a, b) \in R\}$$

Definition (Field)

$$\operatorname{fld}(R) = \operatorname{dom}(R) \cup \operatorname{ran}(R)$$

5 Operations

Definition (Inverse)

The *inverse* of R is the relation

$$R^{-1} = \{(a, b) \mid (b, a) \in R\}$$

Theorem

$$(R^{-1})^{-1} = R$$

Definition (Restriction)

The *restriction* of R to X is the relation

$$R|_X = \{(a, b) \in R \mid a \in X\}$$

Definition (Image)

The *image* of X under R is the set

$$R[X] = \{b \in \text{ran}(R) \mid \exists a \in X : (a, b) \in R\} = \text{ran}(R|_X)$$

Definition (Inverse Image)

The *inverse image* of Y under R is the set

$$R^{-1}[Y] = \{b \in \text{dom}(R) \mid \exists b \in Y : (a, b) \in R\} = \text{ran}(R^{-1}|_Y)$$

$$R \subseteq A \times B \qquad X \subseteq A \qquad Y \subseteq B$$

$$R^{-1}[R[X]] \; ? \; X$$

$$R[R^{-1}[Y]] \; ? \; Y$$

Theorem

$$R[X_1 \cup X_2] = R[X_1] \cup R[X_2]$$

$$R[X_1 \cap X_2] \subseteq R[X_1] \cap R[X_2]$$

$$R[X_1 \setminus X_2] \supseteq R[X_1] \setminus R[X_2]$$

$$\begin{aligned}
& b \in R[X_1 \cup X_2] \\
\iff & \exists a \in X_1 \cup X_2 : (a, b) \in R \\
\iff & \exists a \in X_1 : (a, b) \in R \lor \exists a \in X_2 : (a, b) \in R \\
\iff & b \in R[X_1] \lor b \in R[X_2]
\end{aligned}$$

Definition (Composition)

The *composition* of relations R and S is the relation

$$R \circ S = \{(a, c) \mid \exists b : (a, b) \in S \land (b, c) \in R\}$$

$$R = \{(0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3), (2, 3)\}$$
$$R \circ R = \{\cdots\}$$

$$\leq \circ \leq \ = \ \leq$$

$$\leq \circ \geq \ = \mathbb{R} \times \mathbb{R}$$

Theorem

$$(R \circ S)^{-1} = S^{-1} \circ R^{-1}$$

$$(a, b) \in (R \circ S)^{-1} \iff \cdots$$

Theorem

$$(R \circ S) \circ T = R \circ (S \circ T)$$

$$(a, b) \in (R \circ S) \circ T \iff \cdots$$

$$
\begin{aligned}
& \color{red}{(a,b)} \in (R \circ S) \circ T \\
\iff\ & \exists c : (a,c) \in T \wedge (c,b) \in R \circ S \\
\iff\ & \exists c : (a,c) \in T \wedge (\exists d : (c,d) \in S \wedge (d,b) \in R) \\
\iff\ & \exists d : \exists c : (a,c) \in T \wedge (c,d) \in S \wedge (d,b) \in R \\
\iff\ & \exists d : (\exists c : (a,c) \in T \wedge (c,d) \in S) \wedge (d,b) \in R \\
\iff\ & \exists d : (a,d) \in S \circ T \wedge (d,b) \in R \\
\iff\ & (a,b) \in R \circ (S \circ T)
\end{aligned}
$$

燕小六: "帮我照顾好我七舅姥爷和我外甥女"

"舅姥爷": 姥姥的兄弟

$$G = \{(a, b) : a \text{ 是 } b \text{ 的舅姥爷}\}$$

$$M = \{(a, b) \mid a \text{ is the mother of } b\}$$

$$B = \{(a, b) \mid a \text{ is the brother of } b\}$$

$$G = B \circ (M \circ M)$$

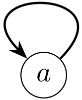
$$G = B \circ (M \circ M) = (B \circ M) \circ M$$

7 Properties

$$R \subseteq X \times X$$

Definition (Reflexive)

$$\forall a \in X : (a, a) \in R$$



Definition (Irreflexive)

$$\forall a \in X : (a, a) \notin R$$

$$A = \{1, 2, 3\}, R \subseteq A \times A$$

$$\{(1,1), (1,3), (2,1), (2,2), (3,3)\}$$

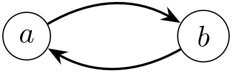
$$\{(1,2), (2,3), (3,1)\}$$

$$\{(1,2), (2,2), (2,3), (3,1)\}$$

$$R \subseteq X \times X$$

Definition (Symmetric)

$$\forall a, b \in X : aRb \implies bRa$$



Definition (AntiSymmetric)

$$\forall a, b \in X : (aRb \land bRa) \implies a = b$$

$>$ *is* antisymmetric.

$$A = \{1, 2, 3\}, R \subseteq A \times A$$

$$\{(1,1), (1,2), (1,3), (2,1), (3,1), (3,3)\}$$

$$\{(1,2), (2,3), (2,2), (3,1)\}$$

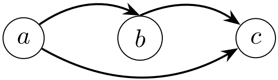
$$\{(1,1), (2,2), (3,3)\}$$

$$\{(1,2), (2,1), (2,3)\}$$

$$R \subseteq X \times X$$

Definition (Transitive)

$$\forall a, b, c \in X : aRb \land bRc \implies aRc$$



$$A = \{1, 2, 3\}, R \subseteq A \times A$$

$$\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\}$$

$$\{(1,2),(2,3),(3,1)\}$$

$$\{(1,3)\}$$

$$\emptyset$$

$$R \subseteq X \times X$$

Definition (Connex)

$$\forall a, b \in X : aRb \lor bRa$$

Definition (Trichotomous)

$$\forall a, b \in X : \text{ exactly one of } \; aRb, bRa, \; \text{or } a = b \text{ holds}$$

Theorem

$$R \text{ is reflexive} \iff I \subseteq R$$

$$I = \{(a,a) \in A \times A \mid a \in A\}$$

Theorem

$$R \text{ is symmetric} \iff R^{-1} = R$$

Theorem

$$R \text{ is transitive} \iff R \circ R \subseteq R$$

$$(1,2),(2,3),(1,3),(4,4)$$

Equivalence Relations

Definition (Equivalence Relation)

R is an *equivalence relation* on X iff R is

- reflexive
- symmetric
- transitive

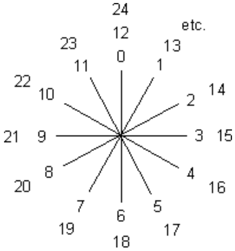
$$= \in \mathbb{R} \times \mathbb{R}$$

$$\parallel \in \mathbb{L} \times \mathbb{L}$$

$$a \sim b \iff a \,\%\, 12 = b \,\%\, 12$$

Why are equivalence relations important?

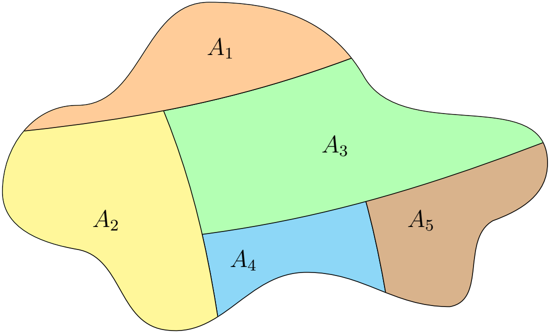
Equivalence Relations as Abstractions



“全国人民代表大会各省代表团”

Equivalence Relation $\Longleftrightarrow$ Partition

Partition



“不空、不漏、不重”

Definition (Partition)

A family of sets $\{A_\alpha : \alpha \in I\}$ is a *partition* of X if

(i)

$$\forall \alpha \in I : A_\alpha \neq \emptyset$$

$$(\forall \alpha \in I \; \exists x \in X : x \in A_\alpha)$$

(ii)

$$\bigcup_{\alpha \in I} A_\alpha = X$$

$$(\forall x \in X \; \exists \alpha \in I : x \in A_\alpha)$$

(iii)

$$\forall \alpha, \beta \in I : A_\alpha \cap A_\beta = \emptyset \lor A_\alpha = A_\beta$$

$$(\forall \alpha, \beta \in I : A_\alpha \cap A_\beta \neq \emptyset \implies A_\alpha = A_\beta)$$

Equivalence Relation $R \subseteq X \times X \implies$ Partition Π of X

Definition (Equivalence Class)

The *equivalence class* of *a modulo* R is a set:

$$[a]_R = \{b \in X : aRb\}$$

Definition (Quotient Set)

The *quotient set* is a set:

$$X/R = \{[a]_R \mid a \in X\}$$

Theorem

$$X/R = \{[a]_R \mid a \in X\} \textit{ is a partition of } X.$$

$$\forall a \in X : [a]_R \neq \emptyset$$

$$\forall a \in X : \exists b \in X : a \in [b]_R$$

Theorem

$$\forall a \in X, b \in X : [a]_R \cap [b]_R = \emptyset \lor [a]_R = [b]_R$$

$$\forall a \in X, b \in X : [a]_R \cap [b]_R \neq \emptyset \implies [a]_R = [b]_R$$

Partition Π of $X \implies$ Equivalence Relation $R \subseteq X \times X$

Definition

$$(a,b) \in R \iff \exists S \in \Pi : a \in S \land b \in S$$

$$R = \{(a,b) \in X \times X \mid \exists S \in \Pi : a \in S \land b \in S\}$$

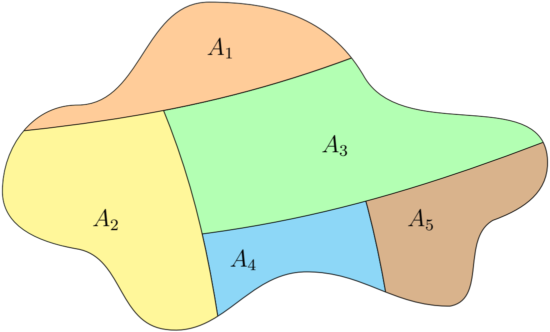
Theorem

R is an equivalence relation on X.

$$\forall x \in X : xRx$$

$$\forall x, y \in X : xRy \implies yRx$$

$$\forall x, y, z \in X : xRy \land yRz \implies xRz$$



Equivalence Relation $\Longleftrightarrow$ Partition

Definition

$$\sim \subseteq \mathbb{N} \times \mathbb{N}$$

$$(a, b) \sim (c, d) \iff a +_{\mathbb{N}} d = b +_{\mathbb{N}} c$$

Theorem

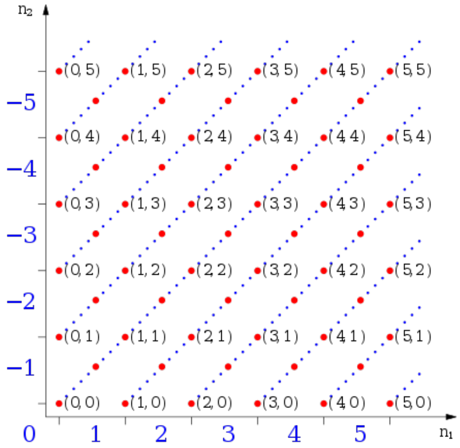
$\sim$ is an equivalence relation.

Q : What is $\mathbb{N} \times \mathbb{N} / \sim$?

Definition ($\mathbb{Z}$)

$$\mathbb{Z} \triangleq \mathbb{N} \times \mathbb{N} / \sim$$

$$[(1, 3)]_{\sim} = \{(0, 2), (1, 3), (2, 4), (3, 5), \cdots\} \triangleq -2 \in \mathbb{Z}$$



$$\mathbb{Z} \triangleq \mathbb{N} \times \mathbb{N} / \sim$$

Definition ($+_{\mathbb{Z}}$)

$$[(m_1, n_1)] +_{\mathbb{Z}} [(m_2, n_2)] = [m_1 +_{\mathbb{N}} m2, n_1 +_{\mathbb{N}} n_2]$$

Definition ($\cdot_{\mathbb{Z}}$)

$$\begin{aligned} &[(m_1, n_1)] \cdot_{\mathbb{Z}} [(m_2, n_2)] \\ = &[m_1 \cdot_{\mathbb{N}} m_2 +_{\mathbb{N}} n_1 \cdot_{\mathbb{N}} n_2, m_1 \cdot_{\mathbb{N}} n_2 +_{\mathbb{N}} n_1 \cdot_{\mathbb{N}} m_2] \end{aligned}$$

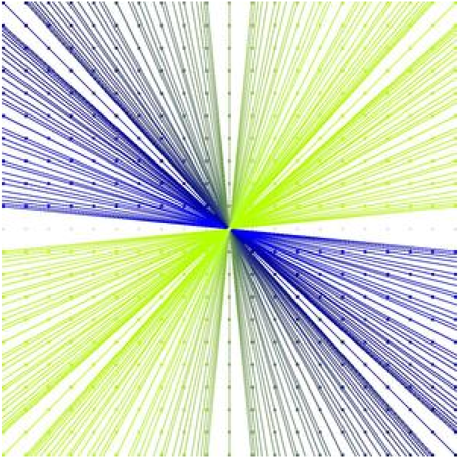
Definition

$$\sim \subseteq \mathbb{Z} \times \mathbb{Z} \setminus \{0_{\mathbb{Z}}\}$$

$$(a, b) \sim (c, d) \iff a \cdot_{\mathbb{Z}} d = b \cdot_{\mathbb{Z}} c$$

Definition ($\mathbb{Q}$)

$$\mathbb{Q} \triangleq \mathbb{Z} \times \mathbb{Z} / \sim$$



$$\mathbb{Q} \triangleq \mathbb{Z} \times \mathbb{Z} / \sim$$

How to define $\mathbb{R}$ as equivalence classes of ordered pairs of $\mathbb{Q}$?


Thank
You!