2-3 Counting

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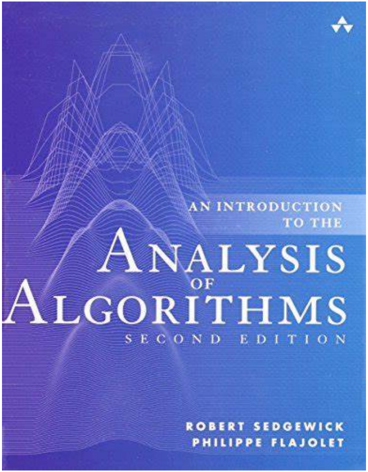
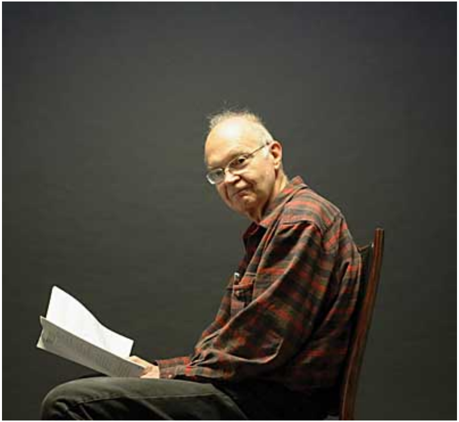


The Analysis of Algorithms

"People who analyze algorithms have double happiness ..."



Donald E. Knuth (1938 $\sim$)

Unfortunately, you have to master some mathematics.

Counting

Sums $\sum$

Binomials $\binom{n}{k}$

Counting

tuples

permutations

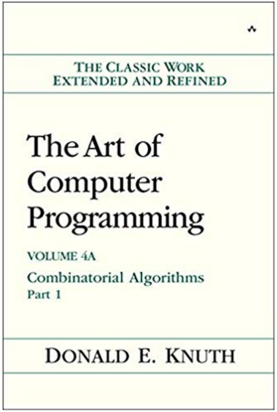
combinations

compositions

partitions

Counting # of functions under (twelve) different restrictions

Counting *vs.* Generating



Generating is about algorithms.

Counting # of functions under (twelve) different restrictions

$$\boxed{f : N \to M \qquad (|N| = n, \quad |M| = m)}$$

$$12 = (2 \times 2) \times 3$$

Elements of N	Elements of M	Any f	Injective f	Surjective f
distinguishable	*distinguishable*			
indistinguishable	*distinguishable*			
distinguishable	*indistinguishable*			
indistinguishable	*indistinguishable*			

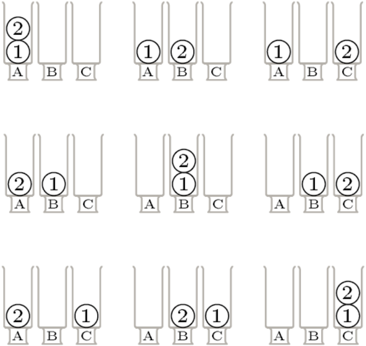
Table: The Twelvefold Way.

Balls	Bins	unrestricted	≤ 1	≥ 1
unlabeled	*unlabeled*			
labeled	*unlabeled*			
unlabeled	*labeled*			
labeled	*labeled*			

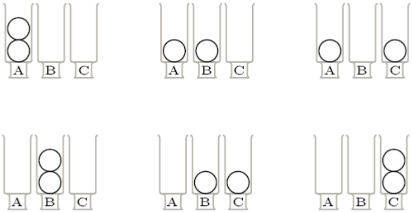
Table: The Twelvefold Way (Balls into Bins Model).

2 balls, 3 bins (unrestricted)

labeled balls into labeled bins



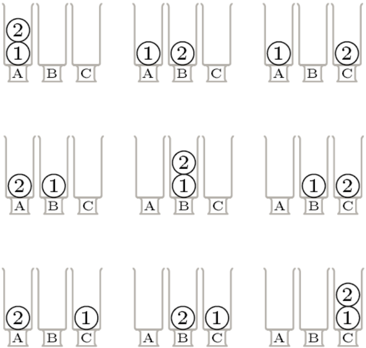
unlabeled balls into labeled bins



Only the # of balls in each bin matters.

2 balls, 3 bins (unrestricted)

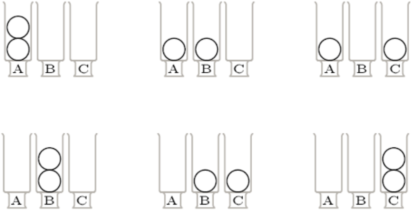
labeled balls into labeled bins



labeled balls into unlabeled bins ...

2 balls, 3 bins (unrestricted)

unlabeled balls into labeled bins



Only the # of balls in each bin matters.

Elements of N	Elements of M	Any f	Injective f	Surjective f
distinguishable	*distinguishable*			
indistinguishable	*distinguishable*			
distinguishable	*indistinguishable*			
indistinguishable	*indistinguishable*			

Table: The Twelvefold Way.

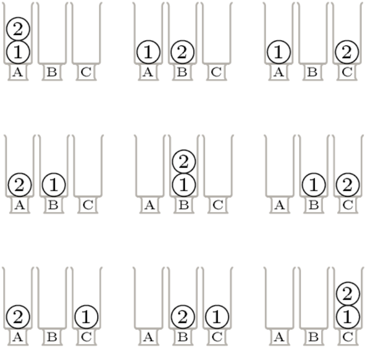
	Any f	Injective f	Surjective f
f			
$f \circ S_n$			
$S_m \circ f$			
$S_m \circ f \circ S_n$			

Table: The Twelvefold Way.

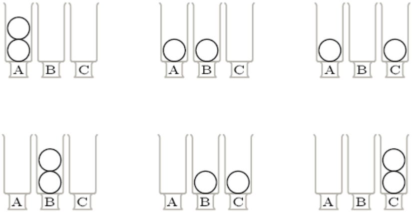
$$S_n = \{f : N \xleftrightarrow[\text{1-1}]{\text{onto}} N\}$$

2 balls, 3 bins (unrestricted)

labeled balls into labeled bins



unlabeled balls into labeled bins



$$f : 1 \mapsto A, \quad 2 \mapsto B$$

$$f' : 2 \mapsto A, \quad 1 \mapsto B$$

$$f' = f \circ (h : 1 \mapsto 2, \quad 2 \mapsto 1)$$

Passing out Apples to Children

k-Permutation $(\text{CS} : 1.2 - 5)$

We need to pass out k distinct apples (pieces of fruit) to n children such that *each child may get at most one apple*.

(a) $k \le n$?

(b) What if $k > n$?

$$n^{\underline{k}} \triangleq n(n-1)\cdots(n-k+1)$$

0

Multisets (CS : 1.5 − 4)

Use multisets to determine the number of ways to pass out k **identical** apples to n children. Assume that a child may get more than one apple.

x_i : the # of apples the i-th child gets

$$x_1 + x_2 + \cdots + x_n = k, \quad x_i \geq 0$$

Integer composition (The order matters!)

$$y_i \triangleq x_i + 1$$

$$y_1 + y_2 + \cdots + y_n = n + k, \qquad y_i \geq 1$$

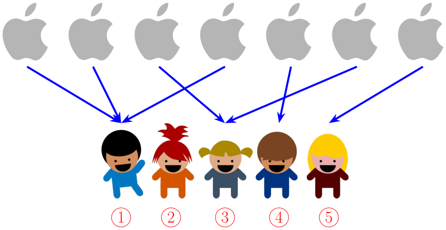
$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

Multisets (CS : 1.5 – 4)

Use **multisets** to determine the number of ways to pass out k identical apples to n children. Assume that a child may get more than one apple.

Q : k-multiset of $[1 \cdots n]$ *vs.* n-multiset of $[1 \cdots k]$

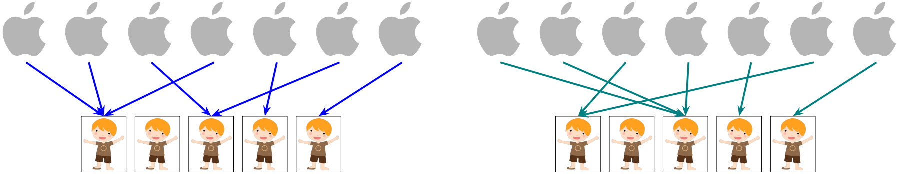
$$k = 7 \qquad n = 5$$



$$\{1, 1, 1, 3, 3, 4, 5\}$$

Integer Partition (CS : 1.5 – 4 Extended)

What is the number of ways to pass out k identical apples to n- .
Assume that a child may get more than one apple.



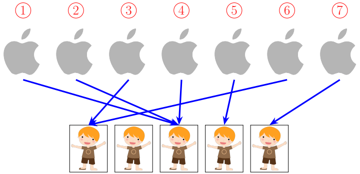
Integer partition of k into $\leq n$ parts (The order does not matter!)

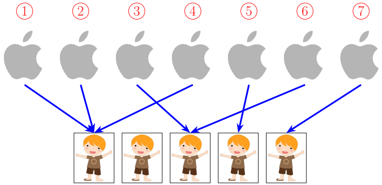
Theorem (G. H. Hardy, Ramanujan (1918))

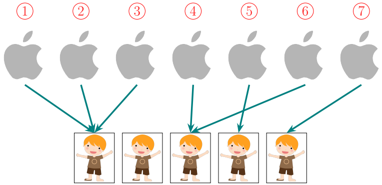
$$p(k) \triangleq \sum_{x=1}^{x=k} p_x(k) \sim \frac{1}{4\sqrt{3}k} \exp\left(\pi\sqrt{\frac{2k}{3}}\right)$$

Set Partition (CS : 1.5 – 4 Extended)

What is the number of ways to pass out k distinct apples to n- . Assume that a child may get more than one apple.







Set partition of $[1 \cdots k]$ into $\leq n$ parts

Set Partition $(\mathrm{CS} : 1.5 - 12)$

$$S(n,k)\ (\left\{ {n \atop k} \right\}) : \#\text{ of set partitions of } [1 \cdots n] \text{ into } k \text{ classes}$$

Stirling number of the second kind

Theorem (Recurrence for $S(n,k)$)

$$S(0,0) = 1, \quad S(n,0) = S(0,n) = 0\ (n > 0)$$

$$S(n,k) = S(n-1,k-1) + kS(n-1,k), \quad n > 0, k > 0$$

Proof.

$$S(n,k) = \underbrace{S(n-1,k-1)}_{n \text{ is alone}} + \underbrace{kS(n-1,k)}_{n \text{ is not alone}}$$

□

$$\text{Bell number: } B_n = \sum_{k=0}^{k=n} \left\{ {n \atop k} \right\}$$

Theorem (Berend & Tassa (2010))

$$B_n < \left(\frac{0.792n}{\ln(n+1)} \right)^n, n \in \mathbb{Z}^+$$

Theorem (de Bruijn (1981))

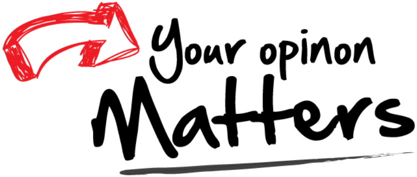
As $n \to \infty$,

$$\frac{\ln B_n}{n} = \ln n - \ln\ln n - 1 + \frac{\ln\ln n}{\ln n} + \frac{1}{\ln n} + \frac{1}{2}\left(\frac{\ln\ln n}{\ln n}\right)^2 + O\left(\frac{\ln\ln n}{(\ln n)^2}\right)$$

THE TWELVEFOLD WAY

balls per urn	unrestricted	≤ 1	≥ 1
n labeled balls, m labeled urns	n-tuples of m things	n-permutations of m things	partitions of $\{1, \ldots, n\}$ into m ordered parts
n unlabeled balls, m labeled urns	n-multicombinations of m things	n-combinations of m things	compositions of n into m parts
n labeled balls, m unlabeled urns	partitions of $\{1, \ldots, n\}$ into $\leq m$ parts	n pigeons into m holes	partitions of $\{1, \ldots, n\}$ into m parts
n unlabeled balls, m unlabeled urns	partitions of n into $\leq m$ parts	n pigeons into m holes	partitions of n into m parts


Thank
You!



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