1-11 Set Theory (IV): Infinity

魏恒峰

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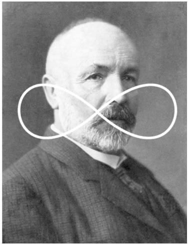
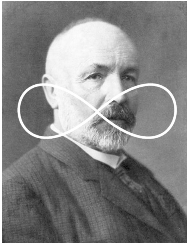


Georg Cantor (1845 – 1918)



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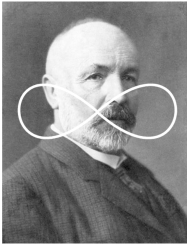

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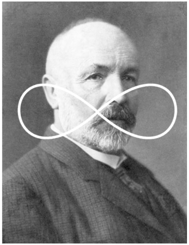

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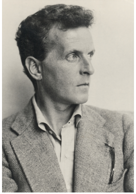
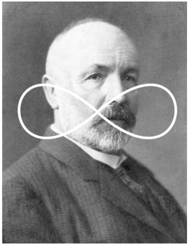


Ludwig Wittgenstein
(1889 – 1951)



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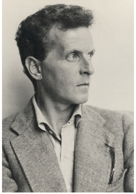

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"没有人能把我们从 *Cantor* 创造的乐园中驱逐出去"

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MATHEMATIKER
BEGRÜNDER DER MENGENLEHRE
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TIK LIEGT IN IHRER FREIHEIT
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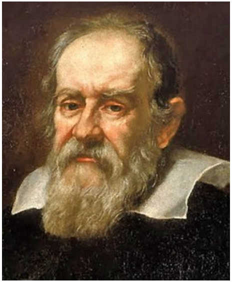
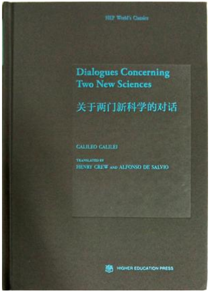
"The essence of mathematics lies in its freedom"



Galileo Galilei (1564 – 1642)



《关于两门新科学的对话》(1638)

"用我们有限的心智来讨论无限 ⋯"

$$S_1 = \{1, 2, 3, \cdots, n, \cdots\}$$

$$S_2 = \{1, 4, 9, \cdots, n^2, \cdots\}$$

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无穷数是不可能的。 — *Gottfried Wilhelm Leibniz*

这些证明一开始就期望那些数要具有有穷数的一切性质，或者甚至于把有穷数的性质强加于无穷。

相反，这些无穷数，如果它们能够以任何形式被理解的话，倒是由于它们与有穷数的对应，它们必须具有完全新的数量特征。

这些性质完全依赖于事物的本性，$\cdots$ 而并非来自我们的主观任意性或我们的偏见。

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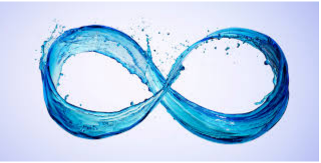
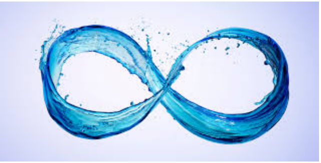
Definition (Dedekind-infinite & Dedekind-finite (Dedekind, 1888))

A set A is Dedekind-infinite if there is a bijective function from A onto some proper subset B of A.

A set is Dedekind-finite if it is not Dedekind-infinite.



Comparing Sets



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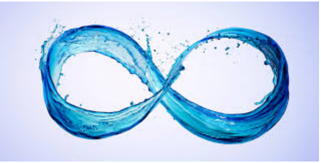

Comparing Sets

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Function

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Two sets of A and B are *equipotent* if there exists a *bijection* from A to B.

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$$\{1, 2, 3, \cdots\} \quad vs. \quad \{1, 3, 5, \cdots, 2, 4, 6, \cdots\}$$

Definition (Finite and Infinite)

For any set X,

Finite

$$\exists n \in \mathbb{N} : |X| = n \qquad (0 \in \mathbb{N})$$

Infinite ($\neg$ finite)

$$\forall n \in \mathbb{N} : |X| \neq n$$

Definition (Finite and Infinite)

For any set X,

Countably Infinite

$$|X| = |\mathbb{N}| \triangleq \aleph_0$$

Countable

$$(\text{finite} \lor \text{countably infinite})$$

Uncountably Infinite

$$(\neg \text{ finite}) \land \big(\neg \text{ (countably infinite)}\big)$$

$$(\neg \text{ countable})$$

Theorem ($\aleph_0$ (1874))

$$|\mathbb{Q}| = |\mathbb{N}|$$

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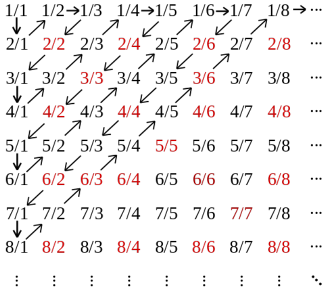
$|\mathbb{Q}| = |\mathbb{N}|$ (UD Problem 22.9)

$$q \in \mathbb{Q}^+ : a/b \ (a, b \in \mathbb{N}^+)$$



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$$|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$$

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$$|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$$

$$f : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \qquad f(m, n) = n + \frac{(m+n)(m+n+1)}{2}$$

Theorem ($\mathbb{R}$ is uncountably infinite (1874) .)

$$|\mathbb{R}| \neq |\mathbb{N}| \qquad (|\mathbb{R}| > |\mathbb{N}|)$$

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Proof.

See UD Theorem 22.12. $\quad f : \mathbb{N} \xleftrightarrow[onto]{1-1} (0, 1).$

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Theorem (Cantor's Theorem (1891))

$$|X| \neq |2^X| \qquad (|X| < |2^X|)$$

Infinite Sequences of 0's and 1's (UD Problem 22.3)

Is the set of all infinite sequences of 0's and 1's finite, countably infinite, or uncountable?

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Nonproof.

$$f : \{\{0,1\}^*\} \to \mathbb{N}$$

$$f(x_0 x_1 \cdots) = \sum_{i=0}^{\infty} x_i 2^i$$

□

Theorem ($|\mathbb{R}|$ (1877))

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"Je le vois, mais je ne le crois pas !"

"I see it, but I don't believe it !"

— *Cantor's letter to Dedekind (1877).*

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Q : Then, what is "dimension"?

$$<$$

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$|B| \leq |A|$ (Axiom of Choice)

Definition ($|A| < |B|$)

$$|A| < |B| \iff |A| \leq |B| \land |A| \neq |B|$$

Definition ($|A| < |B|$)

$|A| < |B| \iff |A| \le |B| \land |A| \ne |B|$

$$|\mathbb{N}| < |\mathbb{R}|$$

$$|X| < |2^X|$$

Definition (Countable Revisited)

X is countable:

$$\exists n \in \mathbb{N} : |X| = n \vee |X| = |\mathbb{N}|$$

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X is countable iff there exists a one-to-one function

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Subsets of Countable Set (UD Problem 22.6; UD Corollary 22.4)

Every subset B of a countable set A is countable.

Set Union (UD Problem 22.1)

Give an example, if possible, of

(c) a countably infinite collection of *pairwise disjoint* nonempty sets whose union is finite.

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$$|A| = n \implies |2^A| = 2^n$$

Slope (UD Problem 22.2 (e))

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Q : *Is "$\leq$" a partial order?*

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Theorem (Cantor-Schröder–Bernstein (1887))

$$|X| \leq |Y| \land |Y| \leq |X| \implies |X| = |Y|$$

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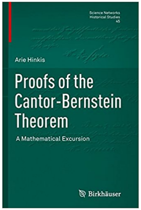

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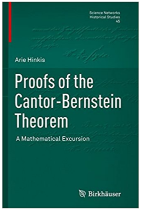

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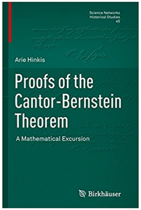

Q : Is "$\leq$" a total order?

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Theorem (PCC)

Principle of Cardinal Comparability (PCC) $\iff$ *Axiom of Choice*

Finite Sets

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“关于有穷，我原以为我是懂的”

Definition (Finite)

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Theorem (Pigeonhole Principle (UD Theorem 21.2))

$$f : \{1, \cdots, m\} \to \{1, \cdots, n\} \ (m, n \in \mathbb{N}^{+}, m > n)$$

f is not one-to-one.

$A \setminus \{a\}$ (UD Problem 21.15)

Let A be a nonempty finite set with $|A| = n$ and let $a \in A$.
Prove that $A \setminus \{a\}$ is finite and $|A \setminus \{a\}| = n - 1$.

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$$f|_{A \setminus \{a\}} : A \setminus \{a\} \xleftrightarrow[onto]{1-1} \{1, \cdots, n\} \setminus \{f(a)\}$$

$A \setminus \{a\}$ (UD Problem 21.15)

Let A be a nonempty finite set with $|A| = n$ and let $a \in A$.
Prove that $A \setminus \{a\}$ is finite and $|A \setminus \{a\}| = n - 1$.

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$|A| \leq |B|$ (UD Problem 21.17)

A and B are finite sets and $f : A \to B$ is one-to-one.
Show that $|A| \leq |B|$.

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By contradiction and the pigeonhole principle.

(UD Problem 21.16)

(a) A is a finite set and $B \subseteq A$. We showed that B is finite (Corollary 20.11). Show that $|B| \leq |A|$.

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By contradiction and (b).

Cardinality of $|\mathrm{ran}(f)|$ (UD Problem 21.18)

Let A and B be sets with A finite.

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(No Axiom of Choice Here)

$f : A \to A$ (UD Problem 21.19)

Let A be a finite set.

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$$f \text{ is one-to-one} \iff f \text{ is onto.}$$

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$$\forall y \in A \; \exists x \in A : y = f(x)$$

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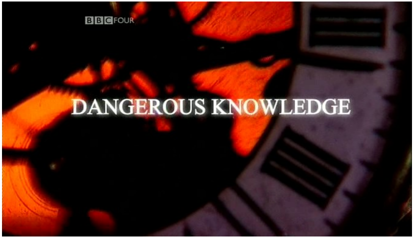
g is bijective.

$$f(g(y)) = f(x) = y \implies f = g^{-1}$$

Dangerous Knowledge (BBC 2007)



Continuum Hypothesis (CH):

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☞ Dangerous Knowledge (22:20)

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☞ Dangerous Knowledge (22:20)

Independence from ZFC:

Kurt Gödel (1940) CH cannot be disproved from ZF.

Paul Cohen (1964) CH cannot be proven from the ZFC axioms.


Thank
You!

Your opinon Matters

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