1-9 Set Theory (II): Relations

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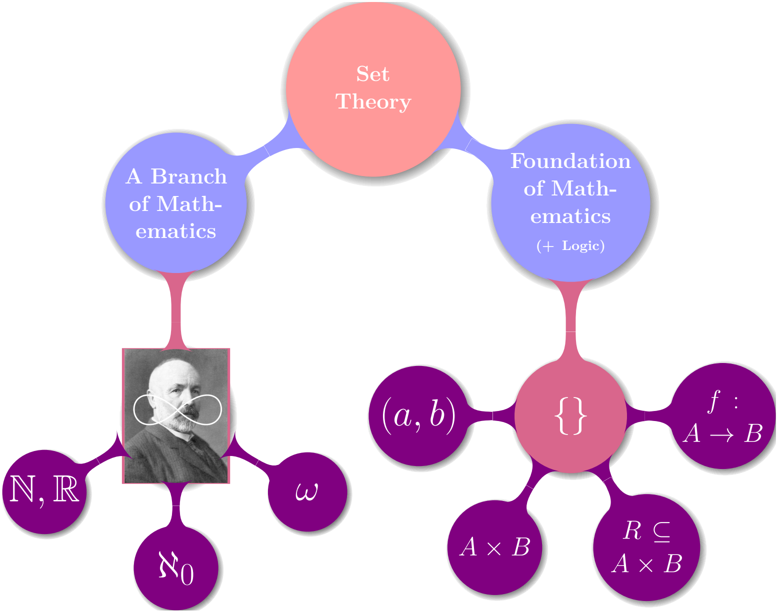
Set Theory
A Branch of Mathematics
Foundation of Mathematics
(+ Logic)
$\mathbb{N}, \mathbb{R}$
$\aleph_0$
ω
(a, b)
$\{\}$
$f : A \to B$
$A \times B$
$R \subseteq A \times B$

Figure 13. A selection of consistency axioms over an execution $(E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar})$

Auxiliary relations

$\mathsf{sameobj}(e, f) \iff \mathsf{obj}(e) = \mathsf{obj}(f)$

Per-object causality (aka happens-before) order:

$\mathsf{hbo} = ((\mathsf{ro} \cap \mathsf{sameobj}) \cup \mathsf{vis})^+$

Causality (aka happens-before) order: $\mathsf{hb} = (\mathsf{ro} \cup \mathsf{vis})^+$

Axioms

EVENTUAL:

$\forall e \in E. \neg(\exists \text{ infinitely many } f \in E. \mathsf{sameobj}(e, f) \wedge \neg(e \xrightarrow{\mathsf{vis}} f))$

THINAIR: $\mathsf{ro} \cup \mathsf{vis}$ is acyclic

POCV (Per-Object Causal Visibility): $\mathsf{hbo} \subseteq \mathsf{vis}$

POCA (Per-Object Causal Arbitration): $\mathsf{hbo} \subseteq \mathsf{ar}$

COCV (Cross-Object Causal Visibility): $(\mathsf{hb} \cap \mathsf{sameobj}) \subseteq \mathsf{vis}$

COCA (Cross-Object Causal Arbitration): $\mathsf{hb} \cup \mathsf{ar}$ is acyclic

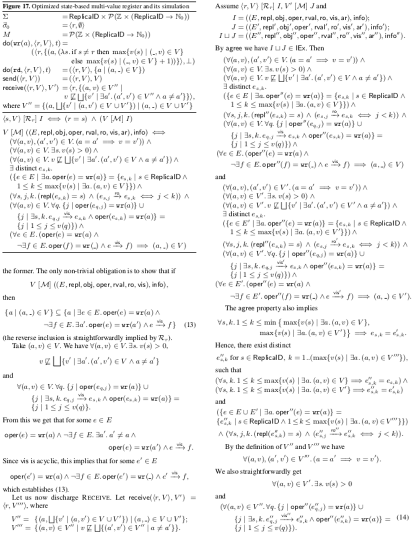

Figure 17. Optimized state-based multi-value register and its simulation

$$
\begin{aligned}
\Sigma &= \mathsf{ReplicaID} \times \mathcal{P}(\mathbb{Z} \times (\mathsf{ReplicaID} \to \mathbb{N}_0))\\
\vec{\sigma}_0 &= \langle r, \emptyset \rangle\\
M &= \mathcal{P}(\mathbb{Z} \times (\mathsf{ReplicaID} \to \mathbb{N}_0))\\
\mathsf{do}(\mathtt{wr}(a), \langle r, V\rangle, t) &= \\
&(\langle r, \{(a, (\lambda s.\, \text{if } s \neq r \text{ then } \max\{v(s) \mid (\_, v) \in V\}\\
&\qquad \text{else } \max\{v(s) \mid (\_, v) \in V\} + 1))\}\rangle, \bot)\\
\mathsf{do}(\mathtt{rd}, \langle r, V\rangle, t) &= (\langle r, V\rangle, \{a \mid (a, \_) \in V\})\\
\mathsf{send}(\langle r, V\rangle) &= (\langle r, V\rangle, V)\\
\mathsf{receive}(\langle r, V\rangle, V') &= \langle r, \{(a, v) \in V'' \mid\\
&\quad v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'. (a', v') \in V'' \wedge a \neq a'\}\}\rangle,\\
\text{where } V'' &= \{(a, \bigsqcup\{v' \mid (a, v') \in V \cup V'\}) \mid (a, \_) \in V \cup V'\}
\end{aligned}
$$

$\langle s, V\rangle\ [\mathcal{R}_r]\ I \iff (r = s) \wedge (V\ [\mathcal{M}]\ I)$

$$
\begin{aligned}
&V\ [\mathcal{M}]\ ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar}), \mathsf{info}) \iff\\
&\quad (\forall (a, v), (a', v') \in V. (a = a' \implies v = v')) \wedge\\
&\quad (\forall (a, v) \in V. \exists s. v(s) > 0) \wedge\\
&\quad (\forall (a, v) \in V. v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'. (a', v') \in V \wedge a \neq a'\}) \wedge\\
&\quad \exists \text{ distinct } e_{s,k}.\\
&\quad (\{e \in E \mid \exists a. \mathsf{oper}(e) = \mathtt{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicaID} \wedge\\
&\qquad 1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V\}\}) \wedge\\
&\quad (\forall s, j, k. (\mathsf{repl}(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}} e_{s,k} \iff j < k)) \wedge\\
&\quad (\forall (a, v) \in V. \forall q. \{j \mid \mathsf{oper}(e_{q,j}) = \mathtt{wr}(a)\} \cup\\
&\qquad \{j \mid \exists s, k. e_{q,j} \xrightarrow{\mathsf{vis}} e_{s,k} \wedge \mathsf{oper}(e_{s,k}) = \mathtt{wr}(a)\} =\\
&\qquad \{j \mid 1 \leq j \leq v(q)\}) \wedge\\
&\quad (\forall e \in E. (\mathsf{oper}(e) = \mathtt{wr}(a) \wedge\\
&\qquad \neg\exists f \in E. \mathsf{oper}(f) = \mathtt{wr}(\_) \wedge e \xrightarrow{\mathsf{vis}} f) \implies (a, \_) \in V)
\end{aligned}
$$

the former. The only non-trivial obligation is to show that if

$$V\ [\mathcal{M}]\ ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}), \mathsf{info}),$$

then

$$
\begin{aligned}
\{a \mid (a, \_) \in V\} \subseteq \{a \mid \exists e \in E. \mathsf{oper}(e) = \mathtt{wr}(a) \wedge&\\
\neg\exists f \in E. \exists a'. \mathsf{oper}(e) = \mathtt{wr}(a') \wedge e \xrightarrow{\mathsf{vis}} f\}& \quad (13)
\end{aligned}
$$

(the reverse inclusion is straightforwardly implied by $\mathcal{R}_r$).

Take $(a, v) \in V$. We have $\forall (a, v) \in V. \exists s. v(s) > 0$,

$$v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'. (a', v') \in V \wedge a \neq a'\}$$

and

$$
\begin{aligned}
&\forall (a, v) \in V. \forall q. \{j \mid \mathsf{oper}(e_{q,j}) = \mathtt{wr}(a)\} \cup\\
&\qquad \{j \mid \exists s, k. e_{q,j} \xrightarrow{\mathsf{vis}} e_{s,k} \wedge \mathsf{oper}(e_{s,k}) = \mathtt{wr}(a)\} =\\
&\qquad \{j \mid 1 \leq j \leq v(q)\}.
\end{aligned}
$$

From this we get that for some $e \in E$

$$\mathsf{oper}(e) = \mathtt{wr}(a) \wedge \neg\exists f \in E. \exists a'. a' \neq a \wedge \mathsf{oper}(e) = \mathtt{wr}(a') \wedge e \xrightarrow{\mathsf{vis}} f.$$

Since vis is acyclic, this implies that for some $e' \in E$

$$\mathsf{oper}(e') = \mathtt{wr}(a) \wedge \neg\exists f \in E. \mathsf{oper}(e') = \mathtt{wr}(\_) \wedge e' \xrightarrow{\mathsf{vis}} f,$$

which establishes (13).

Let us now discharge RECEIVE. Let $\mathsf{receive}(\langle r, V\rangle, V') = \langle r, V'''\rangle$, where

$$
\begin{aligned}
V'' &= \{(a, \bigsqcup\{v' \mid (a, v') \in V \cup V'\}) \mid (a, \_) \in V \cup V'\};\\
V''' &= \{(a, v) \in V'' \mid v \not\sqsubseteq \bigsqcup\{(a', v') \in V'' \mid a \neq a'\}\}.
\end{aligned}
$$

Assume $\langle r, V\rangle\ [\mathcal{R}_r]\ I, V'\ [\mathcal{M}]\ J$ and

$$
\begin{aligned}
I &= ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar}), \mathsf{info});\\
J &= ((E', \mathsf{repl}', \mathsf{obj}', \mathsf{oper}', \mathsf{rval}', \mathsf{ro}', \mathsf{vis}', \mathsf{ar}'), \mathsf{info}');\\
I \sqcup J &= ((E'', \mathsf{repl}'', \mathsf{obj}'', \mathsf{oper}'', \mathsf{rval}'', \mathsf{ro}'', \mathsf{vis}'', \mathsf{ar}''), \mathsf{info}'').
\end{aligned}
$$

By agree we have $I \sqcup J \in \mathsf{IEx}$. Then

$$
\begin{aligned}
&(\forall (a, v), (a', v') \in V. (a = a' \implies v = v')) \wedge\\
&(\forall (a, v) \in V. \exists s. v(s) > 0) \wedge\\
&(\forall (a, v) \in V. v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'. (a', v') \in V \wedge a \neq a'\}) \wedge\\
&\exists \text{ distinct } e_{s,k}.\\
&(\{e \in E \mid \exists a. \mathsf{oper}''(e) = \mathtt{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicaID} \wedge\\
&\quad 1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V\}\}) \wedge\\
&(\forall s, j, k. (\mathsf{repl}''(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}} e_{s,k} \iff j < k)) \wedge\\
&(\forall (a, v) \in V. \forall q. \{j \mid \mathsf{oper}''(e_{q,j}) = \mathtt{wr}(a)\} \cup\\
&\quad \{j \mid \exists s, k. e_{q,j} \xrightarrow{\mathsf{vis}} e_{s,k} \wedge \mathsf{oper}''(e_{s,k}) = \mathtt{wr}(a)\} =\\
&\quad \{j \mid 1 \leq j \leq v(q)\}) \wedge\\
&(\forall e \in E. (\mathsf{oper}''(e) = \mathtt{wr}(a) \wedge\\
&\quad \neg\exists f \in E. \mathsf{oper}''(f) = \mathtt{wr}(\_) \wedge e \xrightarrow{\mathsf{vis}} f) \implies (a, \_) \in V)
\end{aligned}
$$

and

$$
\begin{aligned}
&(\forall (a, v), (a', v') \in V'. (a = a' \implies v = v')) \wedge\\
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&(\forall e \in E'. (\mathsf{oper}''(e) = \mathtt{wr}(a) \wedge\\
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\end{aligned}
$$

The agree property also implies

$$
\begin{aligned}
\forall s, k. 1 \leq k \leq \min\{&\max\{v(s) \mid \exists a. (a, v) \in V\},\\
&\max\{v(s) \mid \exists a. (a, v) \in V'\}\} \implies e_{s,k} = e'_{s,k}.
\end{aligned}
$$

Hence, there exist distinct

$$e''_{s,k} \text{ for } s \in \mathsf{ReplicaID},\ k = 1..(\max\{v(s) \mid \exists a. (a, v) \in V'''\}),$$

such that

$$
\begin{aligned}
&(\forall s, k. 1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V\} \implies e''_{s,k} = e_{s,k}) \wedge\\
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\end{aligned}
$$

and

$$
\begin{aligned}
&(\{e \in E \cup E' \mid \exists a. \mathsf{oper}''(e) = \mathtt{wr}(a)\} =\\
&\{e''_{s,k} \mid s \in \mathsf{ReplicaID} \wedge 1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V'''\}\})\\
&\wedge (\forall s, j, k. (\mathsf{repl}(e''_{s,k}) = s) \wedge (e''_{s,j} \xrightarrow{\mathsf{ro}''} e''_{s,k} \iff j < k)).
\end{aligned}
$$

By the definition of V'' and V''' we have

$$\forall (a, v), (a', v') \in V'''. (a = a' \implies v = v').$$

We also straightforwardly get

$$\forall (a, v) \in V'. \exists s. v(s) > 0$$

and

$$
\begin{aligned}
&(\forall (a, v) \in V''. \forall q. \{j \mid \mathsf{oper}''(e''_{q,j}) = \mathtt{wr}(a)\} \cup\\
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Figure 13. A selection of consistency axioms over an execution $(E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar})$

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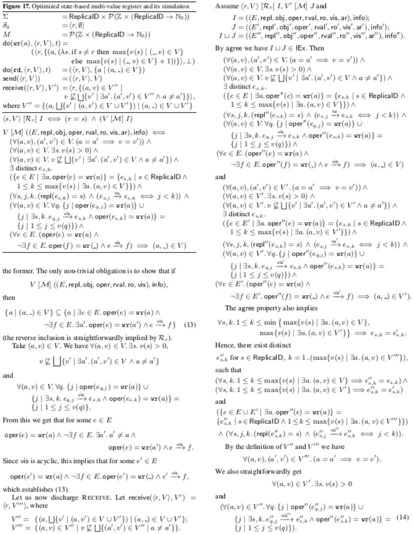

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$$\langle s, V \rangle\ [\mathcal{R}_r]\ I \iff (r = s) \wedge (V\ [\mathcal{M}]\ I)$$

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\begin{aligned}
&V\ [\mathcal{M}]\ ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar}), \mathsf{info}) \iff \\
&\quad (\forall (a, v), (a', v') \in V. (a = a' \implies v = v')) \wedge \\
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&\quad (\forall s, j, k. (\mathsf{repl}(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}} e_{s,k} \iff j < k)) \wedge \\
&\quad (\forall (a, v) \in V. \forall q. \{j \mid \mathsf{oper}(e_{q,j}) = \mathtt{wr}(a)\} \cup \\
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&\quad (\forall e \in E. (\mathsf{oper}(e) = \mathtt{wr}(a) \wedge \\
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then

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\{a \mid (a, \_) \in V\} \subseteq \{a \mid \exists e \in E. \mathsf{oper}(e) = \mathtt{wr}(a) \wedge \\
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\end{aligned}
\tag{13}
$$

(the reverse inclusion is straightforwardly implied by $\mathcal{R}_r$).

Take $(a, v) \in V$. We have $\forall (a, v) \in V. \exists s. v(s) > 0$,

$$v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'. (a', v') \in V \wedge a \neq a'\}$$

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\end{aligned}
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From this we get that for some $e \in E$

$$
\begin{aligned}
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&\qquad \mathsf{oper}(e) = \mathtt{wr}(a') \wedge e \xrightarrow{\mathsf{vis}} f.
\end{aligned}
$$

Since vis is acyclic, this implies that for some $e' \in E$

$$\mathsf{oper}(e') = \mathtt{wr}(a) \wedge \neg\exists f \in E. \mathsf{oper}(e') = \mathtt{wr}(\_) \wedge e' \xrightarrow{\mathsf{vis}} f,$$

which establishes (13).

Let us now discharge RECEIVE. Let $\mathsf{receive}(\langle r, V \rangle, V') = \langle r, V''' \rangle$, where

$$
\begin{aligned}
V'' &= \{(a, \bigsqcup\{v' \mid (a, v') \in V \cup V'\}) \mid (a, \_) \in V \cup V'\}; \\
V''' &= \{(a, v) \in V'' \mid v \not\sqsubseteq \bigsqcup\{(a', v') \in V'' \mid a \neq a'\}\}.
\end{aligned}
$$

Assume $\langle r, V \rangle\ [\mathcal{R}_r]\ I$, $V'\ [\mathcal{M}]\ J$ and

$$
\begin{aligned}
I &= ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar}), \mathsf{info}); \\
J &= ((E', \mathsf{repl}', \mathsf{obj}', \mathsf{oper}', \mathsf{rval}', \mathsf{ro}', \mathsf{vis}', \mathsf{ar}'), \mathsf{info}'); \\
I \sqcup J &= ((E'', \mathsf{repl}'', \mathsf{obj}'', \mathsf{oper}'', \mathsf{rval}'', \mathsf{ro}'', \mathsf{vis}'', \mathsf{ar}''), \mathsf{info}'').
\end{aligned}
$$

By agree we have $I \sqcup J \in \mathsf{IEx}$. Then

$$
\begin{aligned}
&(\forall (a, v), (a', v') \in V. (a = a' \implies v = v')) \wedge \\
&(\forall (a, v) \in V. \exists s. v(s) > 0) \wedge \\
&(\forall (a, v) \in V. v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'. (a', v') \in V \wedge a \neq a'\}) \wedge \\
&\exists \text{ distinct } e_{s,k}. \\
&(\{e \in E \mid \exists a. \mathsf{oper}''(e) = \mathtt{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicaID} \wedge \\
&\quad 1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V\}\}) \wedge \\
&(\forall s, j, k. (\mathsf{repl}''(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}} e_{s,k} \iff j < k)) \wedge \\
&(\forall (a, v) \in V. \forall q. \{j \mid \mathsf{oper}''(e_{q,j}) = \mathtt{wr}(a)\} \cup \\
&\quad \{j \mid \exists s, k. e_{q,j} \xrightarrow{\mathsf{vis}} e_{s,k} \wedge \mathsf{oper}''(e_{s,k}) = \mathtt{wr}(a)\} = \\
&\quad \{j \mid 1 \leq j \leq v(q)\}) \wedge \\
&(\forall e \in E. (\mathsf{oper}''(e) = \mathtt{wr}(a) \wedge \\
&\quad \neg\exists f \in E. \mathsf{oper}''(f) = \mathtt{wr}(\_) \wedge e \xrightarrow{\mathsf{vis}} f) \implies (a, \_) \in V)
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and

$$
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&(\forall (a, v), (a', v') \in V'. (a = a' \implies v = v')) \wedge \\
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\end{aligned}
$$

The agree property also implies

$$
\begin{aligned}
&\forall s, k. 1 \leq k \leq \min\{\max\{v(s) \mid \exists a. (a, v) \in V\}, \\
&\qquad \max\{v(s) \mid \exists a. (a, v) \in V'\}\} \implies e_{s,k} = e'_{s,k}.
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Hence, there exist distinct

$$e''_{s,k} \text{ for } s \in \mathsf{ReplicaID},\ k = 1..(\max\{v(s) \mid \exists a. (a, v) \in V'''\}),$$

such that

$$
\begin{aligned}
&(\forall s, k. 1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V\} \implies e''_{s,k} = e_{s,k}) \wedge \\
&(\forall s, k. 1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V'\} \implies e''_{s,k} = e'_{s,k})
\end{aligned}
$$

and

$$
\begin{aligned}
&(\{e \in E \cup E' \mid \exists a. \mathsf{oper}''(e) = \mathtt{wr}(a)\} = \\
&\{e''_{s,k} \mid s \in \mathsf{ReplicaID} \wedge 1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V'''\}\}) \\
&\wedge (\forall s, j, k. (\mathsf{repl}(e''_{s,k}) = s) \wedge (e''_{s,j} \xrightarrow{\mathsf{ro}''} e''_{s,k} \iff j < k)).
\end{aligned}
$$

By the definition of V'' and V''' we have

$$\forall (a, v), (a', v') \in V'''. (a = a' \implies v = v').$$

We also straightforwardly get

$$\forall (a, v) \in V'. \exists s. v(s) > 0$$

and

$$
\begin{aligned}
&(\forall (a, v) \in V''. \forall q. \{j \mid \mathsf{oper}''(e''_{q,j}) = \mathtt{wr}(a)\} \cup \\
&\quad \{j \mid \exists s, k. e''_{q,j} \xrightarrow{\mathsf{vis}''} e''_{s,k} \wedge \mathsf{oper}''(e''_{s,k}) = \mathtt{wr}(a)\} = \\
&\quad \{j \mid 1 \leq j \leq v(q)\}).
\end{aligned}
\tag{14}
$$


ANNE FINE
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AND MANY MORE
DON'T BE SCARED
AN ANTHOLOGY FOR CHILDREN BY WELL-LOVED AUTHORS & ARTISTS


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I'm so excited.

Definition (Relations)

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$$R \subseteq A \times B$$

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The *Cartesian product* $A \times B$ of A and B is defined as

$$A \times B \triangleq \{(a, b) \mid a \in A \wedge b \in B\}$$

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$$(a, b) = (c, d) \iff a = c \land b = d$$

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Q : Are you satisfied with the definitions above?

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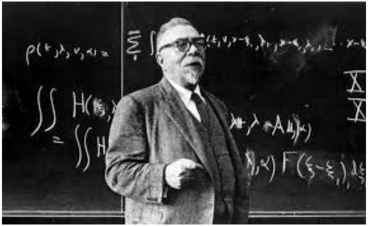
Case I : $a = b$

Case II : $a \neq b$

□

Definition (Ordered Pairs (Norbert Wiener; 1914))

$$(a, b) \triangleq \Big\{\{\{a\}, \emptyset\}, \{\{b\}\}\Big\}$$



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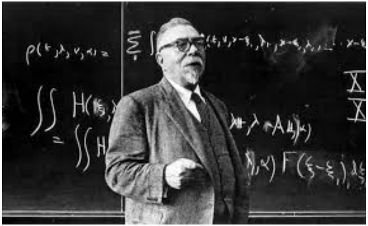

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$$\{\{a\},\{a,b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B))$$

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Definition (Notations)

$$(a, b) \in R \qquad R(a, b) \qquad aRb$$

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Examples

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- $$D = \{(a, b) \in \mathbb{N} \times \mathbb{N} \mid \exists q \in \mathbb{N} : a \cdot q = b\}$$
- P : the set of people

$$M = \{(a, b) \in P \times P \mid a \text{ is the mother of } b\}$$

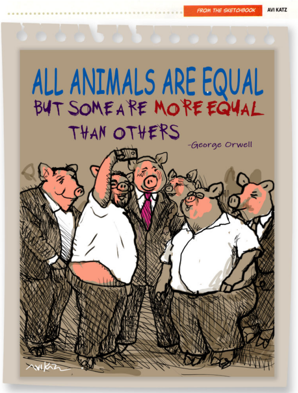
$$B = \{(a, b) \in P \times P \mid a \text{ is the brother of } b\}$$

Important Relations:

Equivalence Relations (1-9)

Functions (1-10)

Ordering Relations (1-12)



3 Definitions

5 Operations

7 Properties

$$R = \{(0,1),(0,2),(0,3),(1,1),(1,2),(1,3),(2,3)\}$$

3 Definitions

Definition (Domain)

$$\operatorname{dom}(R) = \{a \mid \exists b : (a, b) \in R\}$$

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Theorem

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ran(R) is a set.

$$\operatorname{ran}(R) = \{b \in \bigcup\bigcup R \mid \exists a : (a, b) \in R\}$$

Definition (Field)

$$\operatorname{fld}(R) = \operatorname{dom}(R) \cup \operatorname{ran}(R)$$

5 Operations

Definition (Inverse)

The *inverse* of R is the relation

$$R^{-1} = \{(a, b) \mid (b, a) \in R\}$$

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Theorem

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Definition (Restriction)

The *restriction* of R to X is the relation

$$R|_X = \{(a, b) \in R \mid a \in X\}$$

Definition (Image)

The *image* of X under R is the set

$$R[X] = \{b \in \mathrm{ran}(R) \mid \exists a \in X : (a, b) \in R\}$$

Definition (Image)

The *image* of X under R is the set

$$R[X] = \{b \in \mathrm{ran}(R) \mid \exists a \in X : (a, b) \in R\} = \mathrm{ran}(R|_X)$$

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Definition (Inverse Image)

The *inverse image* of Y under R is the set

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$$R^{-1}[Y] = \{b \in \mathrm{dom}(R) \mid \exists b \in Y : (a, b) \in R\} = \mathrm{ran}(R^{-1}|_Y)$$

$$R \subseteq A \times B \qquad X \subseteq A \qquad Y \subseteq B$$

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$$R^{-1}[R[X]] \ ? \ X$$

$$R[R^{-1}[Y]] \ ? \ Y$$

$$R \subseteq A \times B \qquad X \subseteq A \qquad Y \subseteq B$$

$$R^{-1}[R[X]] \; ? \; X$$

$$R[R^{-1}[Y]] \; ? \; Y$$

Theorem

$$R[X_1 \cup X_2] = R[X_1] \cup R[X_2]$$

$$R[X_1 \cap X_2] \subseteq R[X_1] \cap R[X_2]$$

$$R[X_1 \setminus X_2] \supseteq R[X_1] \setminus R[X_2]$$

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$$b \in R[X_1 \cup X_2]$$

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\iff & b \in R[X_1] \lor b \in R[X_2]
\end{aligned}$$

Definition (Composition)

The *composition* of relations R and S is the relation

$$R \circ S = \{(a, c) \mid \exists b : (a, b) \in S \wedge (b, c) \in R\}$$

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$$R \circ R = \{\cdots\}$$

$$\leq \circ \leq \; = \; \leq$$

$$\leq \circ \geq \; = \; \mathbb{R} \times \mathbb{R}$$

Theorem

$$(R \circ S)^{-1} = S^{-1} \circ R^{-1}$$

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$$(a, b) \in (R \circ S)^{-1} \iff \cdots$$

Theorem

$$(R \circ S) \circ T = R \circ (S \circ T)$$

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$$
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\iff\ & \exists c : (a,c) \in T \land (\exists d : (c,d) \in S \land (d,b) \in R)
\end{aligned}
$$

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\iff\ & \exists d : \exists c : (a, c) \in T \land (c, d) \in S \land (d, b) \in R \\
\iff\ & \exists d : (\exists c : (a, c) \in T \land (c, d) \in S) \land (d, b) \in R \\
\iff\ & \exists d : (a, d) \in S \circ T \land (d, b) \in R \\
\iff\ & (a, b) \in R \circ (S \circ T)
\end{aligned}
$$

燕小六: “帮我照顾好我七舅姥爷和我外甥女”

“舅姥爷”：姥姥的兄弟

"舅姥爷": 姥姥的兄弟

$$G = \{(a, b) : a \text{ 是 } b \text{ 的舅姥爷}\}$$

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$$M = \{(a, b) \mid a \text{ is the mother of } b\}$$

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“舅姥爷”：姥姥的兄弟

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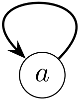
7 Properties

$$R \subseteq X \times X$$

Definition (Reflexive)

$$\forall a \in X : (a, a) \in R$$



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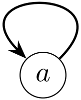

Definition (Irreflexive)

$$\forall a \in X : (a, a) \notin R$$

$$A = \{1, 2, 3\}, R \subseteq A \times A$$

$$\{(1,1), (1,3), (2,1), (2,2), (3,3)\}$$

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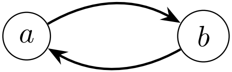
Definition (Symmetric)

$$\forall a, b \in X : aRb \implies bRa$$



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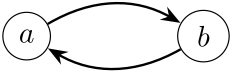

Definition (AntiSymmetric)

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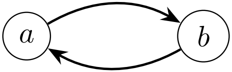

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$$>$$

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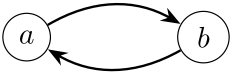

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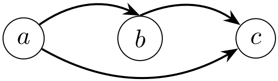
$$R \subseteq X \times X$$

Definition (Transitive)

$$\forall a, b, c \in X : aRb \wedge bRc \implies aRc$$



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$$\emptyset$$

$$R \subseteq X \times X$$

Definition (Connex)

$$\forall a, b \in X : aRb \vee bRa$$

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$$\forall a, b \in X : aRb \vee bRa$$

Definition (Trichotomous)

$$\forall a, b \in X : \text{ exactly one of } aRb, bRa, \text{ or } a = b \text{ holds}$$

Theorem

$$R \textit{ is reflexive} \iff I \subseteq R$$

$$I = \{(a,a) \in A \times A \mid a \in A\}$$

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$$(1,2),(2,3),(1,3),(4,4)$$

Equivalence Relations

Definition (Equivalence Relation)

R is an *equivalence relation* on X iff R is

- reflexive
- symmetric
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$$a \sim b \iff a \,\%\, 12 = b \,\%\, 12$$

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Why are equivalence relations important?

Equivalence Relations as Abstractions

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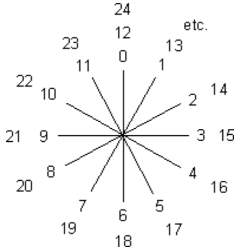
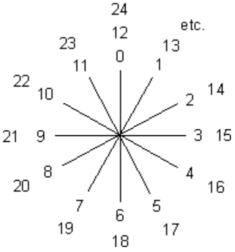
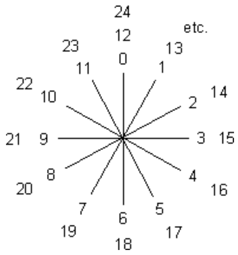


Equivalence Relations as Abstractions



“全国人民代表大会各省代表团”

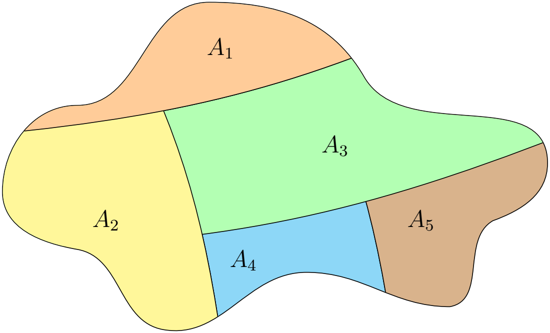
Equivalence Relations as Abstractions



“全国人民代表大会各省代表团”

Equivalence Relation $\Longleftrightarrow$ Partition

Partition



“不空、不漏、不重”

Definition (Partition)

A family of sets $\{A_\alpha : \alpha \in I\}$ is a *partition* of X if

(i)
$$\forall \alpha \in I : A_\alpha \neq \emptyset$$

(ii)
$$\bigcup_{\alpha \in I} A_\alpha = X$$

(iii)
$$\forall \alpha, \beta \in I : A_\alpha \cap A_\beta = \emptyset \lor A_\alpha = A_\beta$$

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Equivalence Relation $R \subseteq X \times X \implies$ Partition Π of X

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The *equivalence class* of *a modulo R* is a set:

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Definition (Quotient Set)

The *quotient set* is a set:

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Theorem

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Partition Π of $X \implies$ Equivalence Relation $R \subseteq X \times X$

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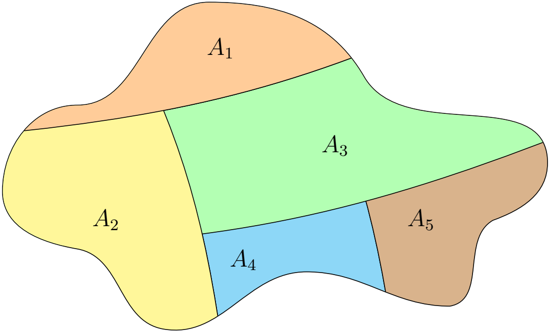
$$\forall x, y, z \in X : xRy \land yRz \implies xRz$$



Equivalence Relation $\Longleftrightarrow$ Partition

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Definition ($\mathbb{Z}$)

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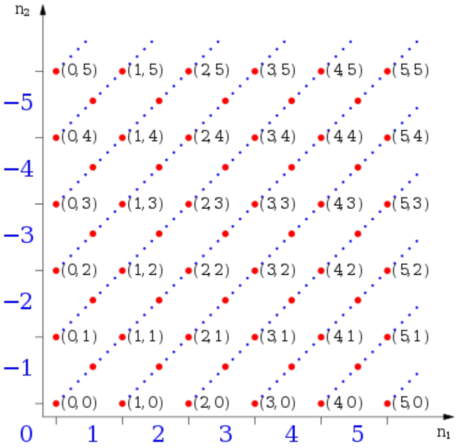
Definition ($\mathbb{Z}$)

$$\mathbb{Z} \triangleq \mathbb{N} \times \mathbb{N} / \sim$$

$$[(1,3)]_{\sim} = \{(0,2), (1,3), (2,4), (3,5), \cdots\} \triangleq -2 \in \mathbb{Z}$$



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Definition ($+_{\mathbb{Z}}$)

$$[(m_1, n_1)] +_{\mathbb{Z}} [(m_2, n_2)] = [m_1 +_{\mathbb{N}} m2, n_1 +_{\mathbb{N}} n_2]$$

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Definition

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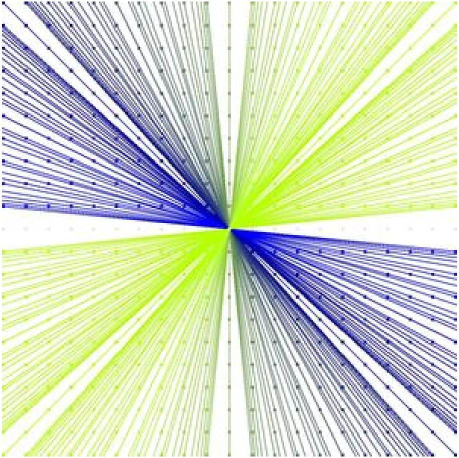

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How to define $\mathbb{R}$ as equivalence classes of ordered pairs of $\mathbb{Q}$?

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Thank
You!