1-9 Set Theory (II): Relations

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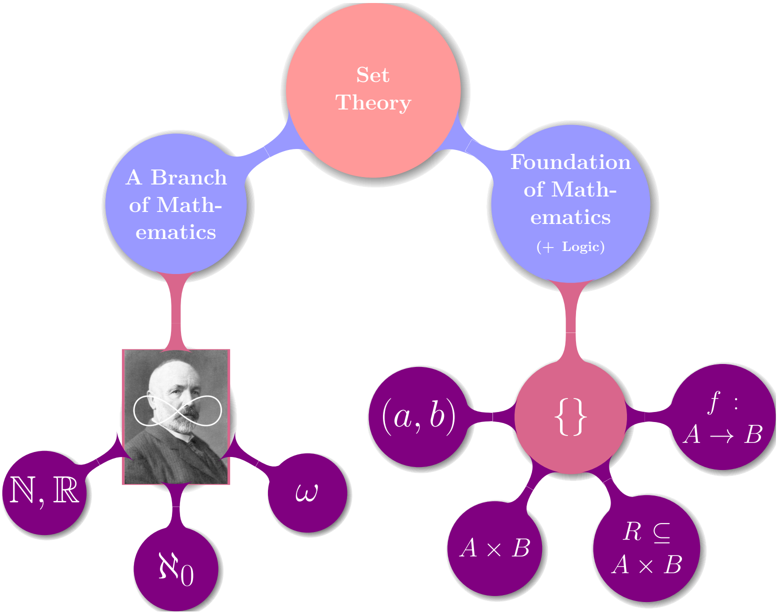
Set Theory
A Branch of Mathematics
Foundation of Mathematics
(+ Logic)
$\mathbb{N}, \mathbb{R}$
$\aleph_0$
ω
(a, b)
$\{\}$
$f : A \to B$
$A \times B$
$R \subseteq A \times B$

Figure 13. A selection of consistency axioms over an execution $(E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar})$

Auxiliary relations

$\mathsf{sameobj}(e, f) \iff \mathsf{obj}(e) = \mathsf{obj}(f)$

Per-object causality (aka happens-before) order:
$\mathsf{hbo} = ((\mathsf{ro} \cap \mathsf{sameobj}) \cup \mathsf{vis})^+$

Causality (aka happens-before) order: $\mathsf{hb} = (\mathsf{ro} \cup \mathsf{vis})^+$

Axioms

EVENTUAL:
$\forall e \in E.\, \neg(\exists \text{ infinitely many } f \in E.\, \mathsf{sameobj}(e, f) \wedge \neg(e \xrightarrow{\mathsf{vis}} f))$

THINAIR: $\mathsf{ro} \cup \mathsf{vis}$ is acyclic

POCV (Per-Object Causal Visibility): $\mathsf{hbo} \subseteq \mathsf{vis}$

POCA (Per-Object Causal Arbitration): $\mathsf{hbo} \subseteq \mathsf{ar}$

COCV (Cross-Object Causal Visibility): $(\mathsf{hb} \cap \mathsf{sameobj}) \subseteq \mathsf{vis}$

COCA (Cross-Object Causal Arbitration): $\mathsf{hb} \cup \mathsf{ar}$ is acyclic

Figure 17. Optimized state-based multi-value register and its simulation

$$
\begin{aligned}
\Sigma &= \mathsf{ReplicaID} \times \mathcal{P}(\mathbb{Z} \times (\mathsf{ReplicaID} \to \mathbb{N}_0)) \\
\vec{\sigma}_0 &= \langle r, \emptyset \rangle \\
M &= \mathcal{P}(\mathbb{Z} \times (\mathsf{ReplicaID} \to \mathbb{N}_0)) \\
\mathsf{do}(\mathtt{wr}(a), \langle r, V\rangle, t) &= \\
&(\langle r, \{(a, (\lambda s.\, \text{if } s \neq r \text{ then } \max\{v(s) \mid (\_, v) \in V\} \\
&\qquad \text{else } \max\{v(s) \mid (\_, v) \in V\} + 1))\}\rangle, \bot) \\
\mathsf{do}(\mathtt{rd}, \langle r, V\rangle, t) &= (\langle r, V\rangle, \{a \mid (a, \_) \in V\}) \\
\mathsf{send}(\langle r, V\rangle) &= (\langle r, V\rangle, V) \\
\mathsf{receive}(\langle r, V\rangle, V') &= \langle r, \{(a, v) \in V'' \mid \\
&\qquad v \not\sqsubseteq \textstyle\bigsqcup\{v' \mid \exists a'.\, (a', v') \in V'' \wedge a \neq a'\}\}\rangle,
\end{aligned}
$$

where $V'' = \{(a, \bigsqcup\{v' \mid (a, v') \in V \cup V'\}) \mid (a, \_) \in V \cup V'\}$

$\langle s, V\rangle\ [\mathcal{R}_r]\ I \iff (r = s) \wedge (V\ [\mathcal{M}]\ I)$

$$
\begin{aligned}
&V\ [\mathcal{M}]\ ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar}), \mathsf{info}) \iff \\
&\quad (\forall (a, v), (a', v') \in V.\, (a = a' \implies v = v')) \wedge \\
&\quad (\forall (a, v) \in V.\, \exists s.\, v(s) > 0) \wedge \\
&\quad (\forall (a, v) \in V.\, v \not\sqsubseteq \textstyle\bigsqcup\{v' \mid \exists a'.\, (a', v') \in V \wedge a \neq a'\}) \wedge \\
&\quad \exists \text{ distinct } e_{s,k}. \\
&\quad (\{e \in E \mid \exists a.\, \mathsf{oper}(e) = \mathtt{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicaID} \wedge \\
&\qquad 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V\}\}) \wedge \\
&\quad (\forall s, j, k.\, (\mathsf{repl}(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}} e_{s,k} \iff j < k)) \wedge \\
&\quad (\forall (a, v) \in V.\, \forall q.\, \{j \mid \mathsf{oper}(e_{q,j}) = \mathtt{wr}(a)\} \cup \\
&\qquad \{j \mid \exists s, k.\, e_{q,j} \xrightarrow{\mathsf{vis}} e_{s,k} \wedge \mathsf{oper}(e_{s,k}) = \mathtt{wr}(a)\} = \\
&\qquad \{j \mid 1 \leq j \leq v(q)\}) \wedge \\
&\quad (\forall e \in E.\, (\mathsf{oper}(e) = \mathtt{wr}(a) \wedge \\
&\qquad \neg\exists f \in E.\, \mathsf{oper}(f) = \mathtt{wr}(\_) \wedge e \xrightarrow{\mathsf{vis}} f) \implies (a, \_) \in V)
\end{aligned}
$$

the former. The only non-trivial obligation is to show that if

$$V\ [\mathcal{M}]\ ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}), \mathsf{info}),$$

then

$$
\begin{aligned}
\{a \mid (a, \_) \in V\} \subseteq \{a \mid \exists e \in E.\, &\mathsf{oper}(e) = \mathtt{wr}(a) \wedge \\
&\neg\exists f \in E.\, \exists a'.\, \mathsf{oper}(e) = \mathtt{wr}(a') \wedge e \xrightarrow{\mathsf{vis}} f\}
\end{aligned}
\tag{13}
$$

(the reverse inclusion is straightforwardly implied by $\mathcal{R}_r$).

Take $(a, v) \in V$. We have $\forall (a, v) \in V.\, \exists s.\, v(s) > 0$,

$$v \not\sqsubseteq \textstyle\bigsqcup\{v' \mid \exists a'.\, (a', v') \in V \wedge a \neq a'\}$$

and

$$
\begin{aligned}
&\forall (a, v) \in V.\, \forall q.\, \{j \mid \mathsf{oper}(e_{q,j}) = \mathtt{wr}(a)\} \cup \\
&\qquad \{j \mid \exists s, k.\, e_{q,j} \xrightarrow{\mathsf{vis}} e_{s,k} \wedge \mathsf{oper}(e_{s,k}) = \mathtt{wr}(a)\} = \\
&\qquad \{j \mid 1 \leq j \leq v(q)\}.
\end{aligned}
$$

From this we get that for some $e \in E$

$$
\begin{aligned}
&\mathsf{oper}(e) = \mathtt{wr}(a) \wedge \neg\exists f \in E.\, \exists a'.\, a' \neq a \wedge \\
&\qquad\qquad \mathsf{oper}(e) = \mathtt{wr}(a') \wedge e \xrightarrow{\mathsf{vis}} f.
\end{aligned}
$$

Since vis is acyclic, this implies that for some $e' \in E$

$$\mathsf{oper}(e') = \mathtt{wr}(a) \wedge \neg\exists f \in E.\, \mathsf{oper}(e') = \mathtt{wr}(\_) \wedge e' \xrightarrow{\mathsf{vis}} f,$$

which establishes (13).

Let us now discharge RECEIVE. Let $\mathsf{receive}(\langle r, V\rangle, V') = \langle r, V'''\rangle$, where

$$
\begin{aligned}
V'' &= \{(a, \textstyle\bigsqcup\{v' \mid (a, v') \in V \cup V'\}) \mid (a, \_) \in V \cup V'\}; \\
V''' &= \{(a, v) \in V'' \mid v \not\sqsubseteq \textstyle\bigsqcup\{(a', v') \in V'' \mid a \neq a'\}\}.
\end{aligned}
$$

Assume $\langle r, V\rangle\ [\mathcal{R}_r]\ I$, $V'\ [\mathcal{M}]\ J$ and

$$
\begin{aligned}
I &= ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar}), \mathsf{info}); \\
J &= ((E', \mathsf{repl}', \mathsf{obj}', \mathsf{oper}', \mathsf{rval}', \mathsf{ro}', \mathsf{vis}', \mathsf{ar}'), \mathsf{info}'); \\
I \sqcup J &= ((E'', \mathsf{repl}'', \mathsf{obj}'', \mathsf{oper}'', \mathsf{rval}'', \mathsf{ro}'', \mathsf{vis}'', \mathsf{ar}''), \mathsf{info}'').
\end{aligned}
$$

By agree we have $I \sqcup J \in \mathsf{IEx}$. Then

$$
\begin{aligned}
&(\forall (a, v), (a', v') \in V.\, (a = a' \implies v = v')) \wedge \\
&(\forall (a, v) \in V.\, \exists s.\, v(s) > 0) \wedge \\
&(\forall (a, v) \in V.\, v \not\sqsubseteq \textstyle\bigsqcup\{v' \mid \exists a'.\, (a', v') \in V \wedge a \neq a'\}) \wedge \\
&\exists \text{ distinct } e_{s,k}. \\
&(\{e \in E \mid \exists a.\, \mathsf{oper}''(e) = \mathtt{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicaID} \wedge \\
&\quad 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V\}\}) \wedge \\
&(\forall s, j, k.\, (\mathsf{repl}''(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}} e_{s,k} \iff j < k)) \wedge \\
&(\forall (a, v) \in V.\, \forall q.\, \{j \mid \mathsf{oper}''(e_{q,j}) = \mathtt{wr}(a)\} \cup \\
&\quad \{j \mid \exists s, k.\, e_{q,j} \xrightarrow{\mathsf{vis}} e_{s,k} \wedge \mathsf{oper}''(e_{s,k}) = \mathtt{wr}(a)\} = \\
&\quad \{j \mid 1 \leq j \leq v(q)\}) \wedge \\
&(\forall e \in E.\, (\mathsf{oper}''(e) = \mathtt{wr}(a) \wedge \\
&\quad \neg\exists f \in E.\, \mathsf{oper}''(f) = \mathtt{wr}(\_) \wedge e \xrightarrow{\mathsf{vis}} f) \implies (a, \_) \in V)
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&(\forall e \in E'.\, (\mathsf{oper}''(e) = \mathtt{wr}(a) \wedge \\
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\end{aligned}
$$

The agree property also implies

$$
\begin{aligned}
\forall s, k.\, 1 \leq k \leq \min\{&\max\{v(s) \mid \exists a.\, (a, v) \in V\}, \\
&\max\{v(s) \mid \exists a.\, (a, v) \in V'\}\} \implies e_{s,k} = e'_{s,k}.
\end{aligned}
$$

Hence, there exist distinct

$$e''_{s,k} \text{ for } s \in \mathsf{ReplicaID},\ k = 1..(\max\{v(s) \mid \exists a.\, (a, v) \in V'''\}),$$

such that

$$
\begin{aligned}
&(\forall s, k.\, 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V\} \implies e''_{s,k} = e_{s,k}) \wedge \\
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\end{aligned}
$$

and

$$
\begin{aligned}
&(\{e \in E \cup E' \mid \exists a.\, \mathsf{oper}''(e) = \mathtt{wr}(a)\} = \\
&\{e''_{s,k} \mid s \in \mathsf{ReplicaID} \wedge 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V'''\}\}) \\
&\wedge (\forall s, j, k.\, (\mathsf{repl}(e''_{s,k}) = s) \wedge (e''_{s,j} \xrightarrow{\mathsf{ro}''} e''_{s,k} \iff j < k)).
\end{aligned}
$$

By the definition of V'' and V''' we have

$$\forall (a, v), (a', v') \in V'''.\, (a = a' \implies v = v').$$

We also straightforwardly get

$$\forall (a, v) \in V'.\, \exists s.\, v(s) > 0$$

and

$$
\begin{aligned}
&(\forall (a, v) \in V''.\, \forall q.\, \{j \mid \mathsf{oper}''(e''_{q,j}) = \mathtt{wr}(a)\} \cup \\
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\end{aligned}
\tag{14}
$$

Figure 13. A selection of consistency axioms over an execution $(E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar})$

Auxiliary relations

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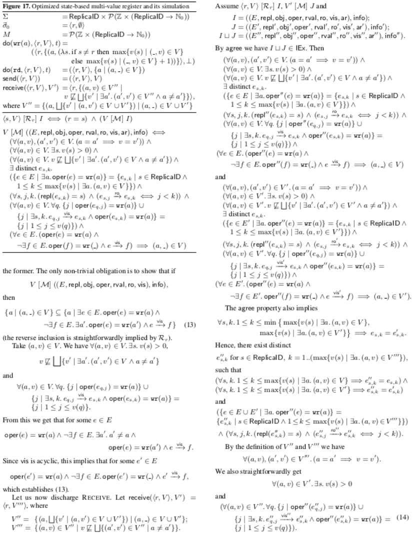

Figure 17. Optimized state-based multi-value register and its simulation

$$\begin{aligned}
\Sigma &= \mathsf{ReplicaID} \times \mathcal{P}(\mathbb{Z} \times (\mathsf{ReplicaID} \to \mathbb{N}_0))\\
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&\quad v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'. (a', v') \in V'' \wedge a \neq a'\}\}\rangle,
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where $V'' = \{(a, \bigsqcup\{v' \mid (a, v') \in V \cup V'\}) \mid (a, \_) \in V \cup V'\}$

$\langle s, V\rangle \; [\mathcal{R}_r] \; I \iff (r = s) \wedge (V \; [\mathcal{M}] \; I)$

$$\begin{aligned}
&V \; [\mathcal{M}] \; ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar}), \mathsf{info}) \iff\\
&\quad (\forall (a, v), (a', v') \in V. (a = a' \implies v = v')) \wedge\\
&\quad (\forall (a, v) \in V. \exists s. v(s) > 0) \wedge\\
&\quad (\forall (a, v) \in V. v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'. (a', v') \in V \wedge a \neq a'\}) \wedge\\
&\quad \exists \text{ distinct } e_{s,k}.\\
&\quad (\{e \in E \mid \exists a. \mathsf{oper}(e) = \mathtt{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicaID} \wedge\\
&\qquad 1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V\}\}) \wedge\\
&\quad (\forall s, j, k. (\mathsf{repl}(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}} e_{s,k} \iff j < k)) \wedge\\
&\quad (\forall (a, v) \in V. \forall q. \{j \mid \mathsf{oper}(e_{q,j}) = \mathtt{wr}(a)\} \cup\\
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&\quad (\forall e \in E. (\mathsf{oper}(e) = \mathtt{wr}(a) \wedge\\
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the former. The only non-trivial obligation is to show that if

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then

$$\begin{aligned}
\{a \mid (a, _) \in V\} \subseteq \{a \mid \exists e \in E. \mathsf{oper}(e) = \mathtt{wr}(a) \wedge\\
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\end{aligned} \tag{13}$$

(the reverse inclusion is straightforwardly implied by $\mathcal{R}_r$).

Take $(a, v) \in V$. We have $\forall (a, v) \in V. \exists s. v(s) > 0$,

$$v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'. (a', v') \in V \wedge a \neq a'\}$$

and

$$\begin{aligned}
&\forall (a, v) \in V. \forall q. \{j \mid \mathsf{oper}(e_{q,j}) = \mathtt{wr}(a)\} \cup\\
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&\quad \{j \mid 1 \leq j \leq v(q)\}.
\end{aligned}$$

From this we get that for some $e \in E$

$$\mathsf{oper}(e) = \mathtt{wr}(a) \wedge \neg\exists f \in E. \exists a'. a' \neq a \wedge \mathsf{oper}(e) = \mathtt{wr}(a') \wedge e \xrightarrow{\mathsf{vis}} f.$$

Since vis is acyclic, this implies that for some $e' \in E$

$$\mathsf{oper}(e') = \mathtt{wr}(a) \wedge \neg\exists f \in E. \mathsf{oper}(e') = \mathtt{wr}(\_) \wedge e' \xrightarrow{\mathsf{vis}} f,$$

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Let us now discharge RECEIVE. Let $\mathsf{receive}(\langle r, V\rangle, V') = \langle r, V'''\rangle$, where

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V'' &= \{(a, \bigsqcup\{v' \mid (a, v') \in V \cup V'\}) \mid (a, _) \in V \cup V'\};\\
V''' &= \{(a, v) \in V'' \mid v \not\sqsubseteq \bigsqcup\{(a', v') \in V'' \mid a \neq a'\}\}.
\end{aligned}$$

Assume $\langle r, V\rangle \; [\mathcal{R}_r] \; I$, $V' \; [\mathcal{M}] \; J$ and

$$\begin{aligned}
I &= ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar}), \mathsf{info});\\
J &= ((E', \mathsf{repl}', \mathsf{obj}', \mathsf{oper}', \mathsf{rval}', \mathsf{ro}', \mathsf{vis}', \mathsf{ar}'), \mathsf{info}');\\
I \sqcup J &= ((E'', \mathsf{repl}'', \mathsf{obj}'', \mathsf{oper}'', \mathsf{rval}'', \mathsf{ro}'', \mathsf{vis}'', \mathsf{ar}''), \mathsf{info}'').
\end{aligned}$$

By agree we have $I \sqcup J \in \mathsf{IEx}$. Then

$$\begin{aligned}
&(\forall (a, v), (a', v') \in V. (a = a' \implies v = v')) \wedge\\
&(\forall (a, v) \in V. \exists s. v(s) > 0) \wedge\\
&(\forall (a, v) \in V. v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'. (a', v') \in V \wedge a \neq a'\}) \wedge\\
&\exists \text{ distinct } e_{s,k}.\\
&(\{e \in E \mid \exists a. \mathsf{oper}''(e) = \mathtt{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicaID} \wedge\\
&\quad 1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V\}\}) \wedge\\
&(\forall s, j, k. (\mathsf{repl}''(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}} e_{s,k} \iff j < k)) \wedge\\
&(\forall (a, v) \in V. \forall q. \{j \mid \mathsf{oper}''(e_{q,j}) = \mathtt{wr}(a)\} \cup\\
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&(\forall e \in E. (\mathsf{oper}''(e) = \mathtt{wr}(a) \wedge\\
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&(\forall (a, v), (a', v') \in V'. (a = a' \implies v = v')) \wedge\\
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The agree property also implies

$$\begin{aligned}
\forall s, k. 1 \leq k \leq \min\{&\max\{v(s) \mid \exists a. (a, v) \in V\},\\
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Hence, there exist distinct

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$$\begin{aligned}
&(\forall s, k. 1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V\} \implies e''_{s,k} = e_{s,k}) \wedge\\
&(\forall s, k. 1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V'\} \implies e''_{s,k} = e'_{s,k})
\end{aligned}$$

and

$$\begin{aligned}
&(\{e \in E \cup E' \mid \exists a. \mathsf{oper}''(e) = \mathtt{wr}(a)\} =\\
&\{e''_{s,k} \mid s \in \mathsf{ReplicaID} \wedge 1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V'''\}\})\\
&\wedge (\forall s, j, k. (\mathsf{repl}(e''_{s,k}) = s) \wedge (e''_{s,j} \xrightarrow{\mathsf{ro}''} e''_{s,k} \iff j < k)).
\end{aligned}$$

By the definition of V'' and V''' we have

$$\forall (a, v), (a', v') \in V'''. (a = a' \implies v = v').$$

We also straightforwardly get

$$\forall (a, v) \in V'. \exists s. v(s) > 0$$

and

$$\begin{aligned}
&(\forall (a, v) \in V''. \forall q. \{j \mid \mathsf{oper}''(e''_{q,j}) = \mathtt{wr}(a)\} \cup\\
&\quad \{j \mid \exists s, k. e''_{q,j} \xrightarrow{\mathsf{vis}''} e''_{s,k} \wedge \mathsf{oper}''(e''_{s,k}) = \mathtt{wr}(a)\} =\\
&\quad \{j \mid 1 \leq j \leq v(q)\}).
\end{aligned} \tag{14}$$


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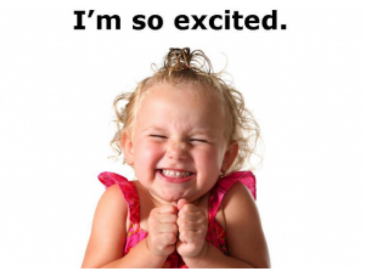
I'm so excited.

Definition (Relations)

A *relation* R from A to B is a subset of $A \times B$:

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$$A \times B \triangleq \{(a, b) \mid a \in A \land b \in B\}$$

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$$(a, b) = (c, d) \iff a = c \land b = d$$

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Q : Are you satisfied with the definitions above?

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Proof.

$$\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$$

CASE I : $a = b$

CASE II : $a \neq b$

□

Definition (Ordered Pairs (Norbert Wiener; 1914))

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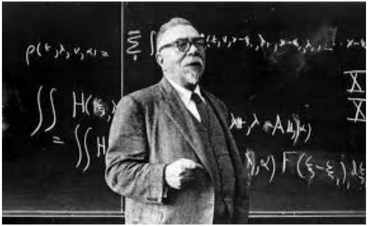

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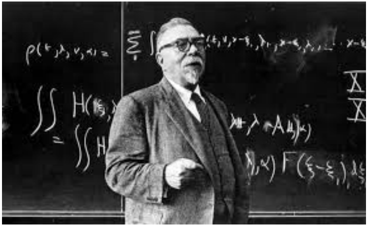

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$$\{\{a\}, \{a, b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B))$$

$\square$

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Definition (Notations)

$$(a, b) \in R \qquad R(a, b) \qquad aRb$$

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Examples

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- Both $A \times B$ and $\emptyset$ are relations from A to B.

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- Both $A \times B$ and $\emptyset$ are relations from A to B.
- $$< = \{(a, b) \in \mathbb{R} \times \mathbb{R} \mid a \text{ is less than } b\}$$
- $$D = \{(a, b) \in \mathbb{N} \times \mathbb{N} \mid \exists q \in \mathbb{N} : a \cdot q = b\}$$
- P : the set of people

$$M = \{(a, b) \in P \times P \mid a \text{ is the mother of } b\}$$

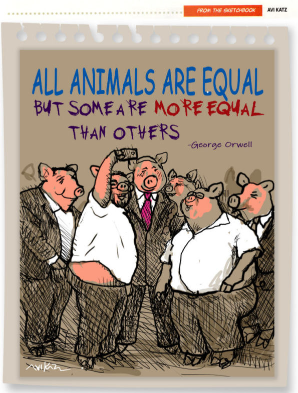
$$B = \{(a, b) \in P \times P \mid a \text{ is the brother of } b\}$$

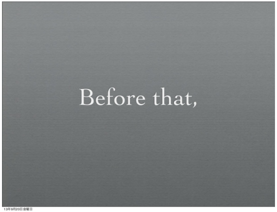
Important Relations:

Equivalence Relations (1-9)

Functions (1-10)

Ordering Relations (1-12)





3 Definitions

5 Operations

7 Properties

$$R = \{(0,1),(0,2),(0,3),(1,1),(1,2),(1,3),(2,3)\}$$

3 Definitions

Definition (Domain)

$$\mathrm{dom}(R) = \{a \mid \exists b : (a, b) \in R\}$$

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Theorem

ran(R) *is a set.*

$$\operatorname{ran}(R) = \{b \in \bigcup\bigcup R \mid \exists a : (a, b) \in R\}$$

Definition (Field)

$$\operatorname{fld}(R) = \operatorname{dom}(R) \cup \operatorname{ran}(R)$$

5 Operations

Definition (Inverse)

The *inverse* of R is the relation

$$R^{-1} = \{(a, b) \mid (b, a) \in R\}$$

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Theorem

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Definition (Restriction)

The *restriction* of R to X is the relation

$$R|_X = \{(a, b) \in R \mid a \in X\}$$

Definition (Image)

The *image* of X under R is the set

$$R[X] = \{b \in \mathrm{ran}(R) \mid \exists a \in X : (a, b) \in R\}$$

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The *inverse image* of Y under R is the set

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$$R \subseteq A \times B \qquad X \subseteq A \qquad Y \subseteq B$$

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$$R^{-1}[R[X]] \ ? \ X$$

$$R[R^{-1}[Y]] \ ? \ Y$$

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$$R^{-1}[R[X]] \; ? \; X$$

$$R[R^{-1}[Y]] \; ? \; Y$$

Theorem

$$R[X_1 \cup X_2] = R[X_1] \cup R[X_2]$$

$$R[X_1 \cap X_2] \subseteq R[X_1] \cap R[X_2]$$

$$R[X_1 \setminus X_2] \supseteq R[X_1] \setminus R[X_2]$$

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$$b \in R[X_1 \cup X_2]$$

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\iff & \exists a \in X_1 : (a, b) \in R \lor \exists a \in X_2 : (a, b) \in R
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Definition (Composition)

The *composition* of relations R and S is the relation

$$R \circ S = \{(a, c) \mid \exists b : (a, b) \in S \wedge (b, c) \in R\}$$

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$$R \circ R = \{\cdots\}$$

$$\leq \circ \leq \; = \; \leq$$

$$\leq \circ \geq \; = \; \mathbb{R} \times \mathbb{R}$$

Theorem

$$(R \circ S)^{-1} = S^{-1} \circ R^{-1}$$

Theorem

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$$(a, b) \in (R \circ S)^{-1} \iff \cdots$$

Theorem

$$(R \circ S) \circ T = R \circ (S \circ T)$$

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\begin{aligned}
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$$
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& (a, b) \in (R \circ S) \circ T \\
\iff & \exists c : (a, c) \in T \land (c, b) \in R \circ S \\
\iff & \exists c : (a, c) \in T \land (\exists d : (c, d) \in S \land (d, b) \in R)
\end{aligned}
$$

$$
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\iff & \exists d : (a, d) \in S \circ T \land (d, b) \in R \\
\iff & (a, b) \in R \circ (S \circ T)
\end{aligned}
$$

燕小六: “帮我照顾好我七舅姥爷和我外甥女”

“舅姥爷”：姥姥的兄弟

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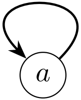
7 Properties

$$R \subseteq X \times X$$

Definition (Reflexive)

$$\forall a \in X : (a, a) \in R$$



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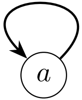

Definition (Irreflexive)

$$\forall a \in X : (a, a) \notin R$$

$$A = \{1, 2, 3\}, R \subseteq A \times A$$

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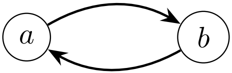
Definition (Symmetric)

$$\forall a, b \in X : aRb \implies bRa$$



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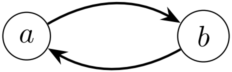

Definition (AntiSymmetric)

$$\forall a, b \in X : (aRb \wedge bRa) \implies a = b$$

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$$\begin{tikzcd} a \arrow[r, bend left] & b \arrow[l, bend left] \end{tikzcd}$$

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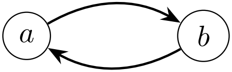

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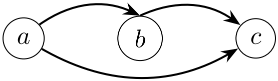
Definition (Transitive)

$$\forall a, b, c \in X : aRb \wedge bRc \implies aRc$$



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$$\emptyset$$

$$R \subseteq X \times X$$

Definition (Connex)

$$\forall a, b \in X : aRb \vee bRa$$

$$R \subseteq X \times X$$

Definition (Connex)

$$\forall a, b \in X : aRb \lor bRa$$

Definition (Trichotomous)

$$\forall a, b \in X : \text{ exactly one of } aRb, bRa, \text{ or } a = b \text{ holds}$$

Theorem

$$R \textit{ is reflexive } \iff I \subseteq R$$

$$I = \{(a, a) \in A \times A \mid a \in A\}$$

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$$(1, 2), (2, 3), (1, 3), (4, 4)$$

Equivalence Relations

Definition (Equivalence Relation)

R is an *equivalence relation* on X iff R is

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$$a \sim b \iff a \,\%\, 12 = b \,\%\, 12$$

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Why are equivalence relations important?

Equivalence Relations as Abstractions

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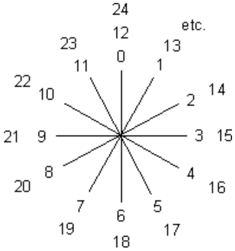

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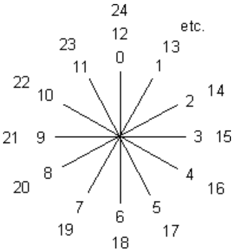
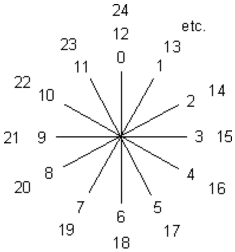


"全国人民代表大会各省代表团"

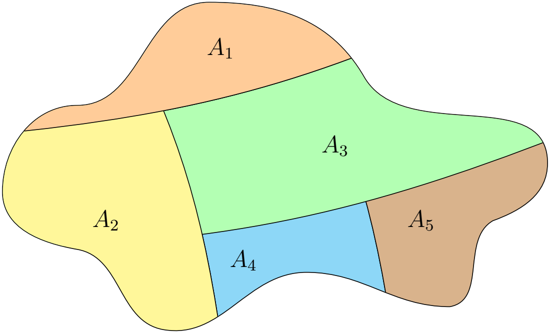
Equivalence Relations as Abstractions



“全国人民代表大会各省代表团”

Equivalence Relation $\Longleftrightarrow$ Partition

Partition



“不空、不漏、不重”

Definition (Partition)

A family of sets $\{A_\alpha : \alpha \in I\}$ is a *partition* of X if

(i)
$$\forall \alpha \in I : A_\alpha \neq \emptyset$$

(ii)
$$\bigcup_{\alpha \in I} A_\alpha = X$$

(iii)
$$\forall \alpha, \beta \in I : A_\alpha \cap A_\beta = \emptyset \lor A_\alpha = A_\beta$$

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Equivalence Relation $R \subseteq X \times X \implies$ Partition Π of X

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Definition (Quotient Set)

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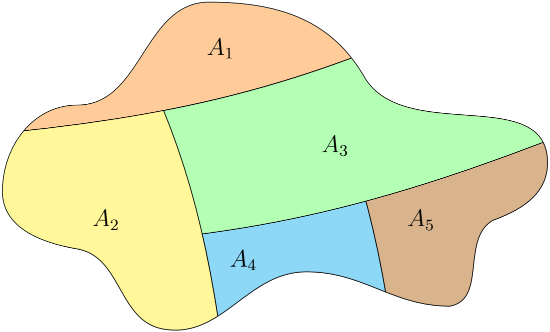
$$\forall x, y, z \in X : xRy \land yRz \implies xRz$$



Equivalence Relation $\Longleftrightarrow$ Partition

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Definition ($\mathbb{Z}$)

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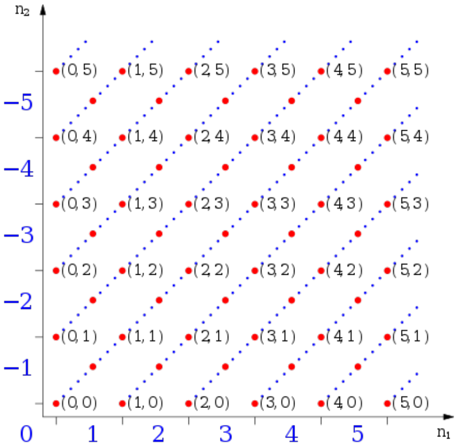
$$[(1, 3)]_{\sim} = \{(0, 2), (1, 3), (2, 4), (3, 5), \cdots\} \triangleq -2 \in \mathbb{Z}$$



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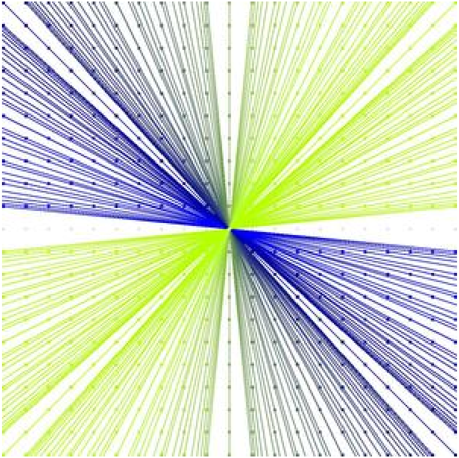

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How to define $\mathbb{R}$ as equivalence classes of ordered pairs of $\mathbb{Q}$?

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Thank You!