2-3 Counting

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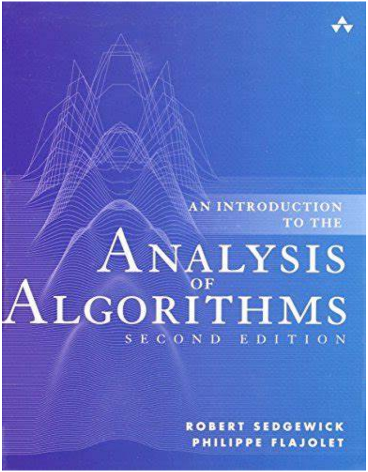
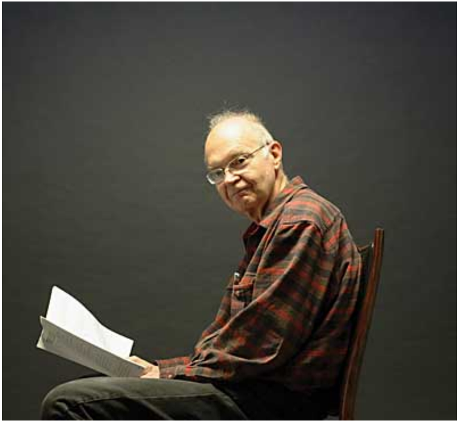


$O \qquad \Omega \qquad \Theta$

$o \qquad \omega$

"People who analyze algorithms have double happiness . . ."



Donald E. Knuth (1938 ∼)

Unfortunately, you have to master some mathematics.

Counting

Sums $\sum$

Binomials $\binom{n}{k}$


PRELIMINARY

Falling and Rising Factorials

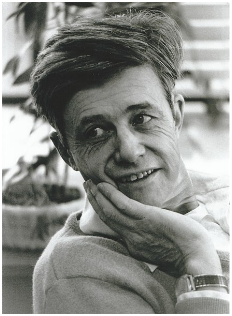
$$m^{\underline{n}} = (m)_n = m(m-1)(m-2)\cdots(m-n+1) = \frac{m!}{(m-n)!}$$

$$m^{\bar{n}} = m^{(n)} = m(m+1)(m+2)\cdots(m+n-1)$$

$$n! = n^{\underline{n}} = 1^{\bar{n}}$$

$$\binom{m}{n} = \frac{m^{\underline{n}}}{n!}$$

Iverson Bracket



Kenneth Eugene Iverson
(1920 $\sim$ 2004)

$$[P] = \begin{cases} 1, & \text{if } P \text{ is true;} \\ 0, & \text{otherwise} \end{cases}$$

$$[n \le m] = \begin{cases} 1, & \text{if } n \le m; \\ 0, & \text{if } n > m \end{cases}$$

Theorem (Sum Principle)

$$S \cap T = \emptyset \implies |S \cup T| = |S| + |T|$$

Theorem (Product Principle)

$$|S \times T| = |S| \times |T|$$

Holds for finite sets S and T.

先学习下加法，$\mathbf{1+1}$，就是

所以$\mathbf{1+1=2}$，这很好理解

那我们趁热打铁学习下一个重要公式吧：

$$\frac{\sum_{w\in W}(-1)^{\det(w)}w(e^{\lambda+\rho})}{e^{\rho}\prod_{\alpha>0}(1-e^{-\alpha})}$$

Counting

tuples

permutations

combinations

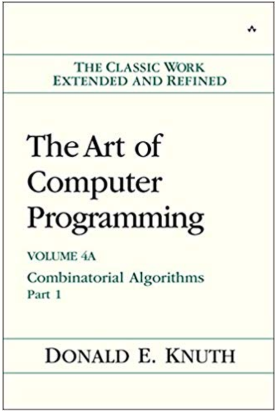
compositions

set partitions

integer partitions

Counting # of functions under (twelve) different restrictions

Counting *vs.* Generating



Generating is more about algorithms.

Counting # of functions under (twelve) different restrictions

$$\boxed{f : N \to M \qquad (|N| = n, \quad |M| = m)}$$

$$12 = (2 \times 2) \times 3$$

Elements of N	Elements of M	Any f	Injective f	Surjective f
distinguishable	*distinguishable*			
indistinguishable	*distinguishable*			
distinguishable	*indistinguishable*			
indistinguishable	*indistinguishable*			

Table: The Twelvefold Way (Functions).

distinguishable *vs.* indistinguishable

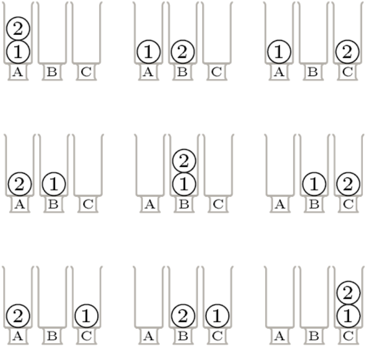
Balls	Bins	unrestricted	≤ 1	≥ 1
unlabeled	*unlabeled*			
labeled	*unlabeled*			
unlabeled	*labeled*			
labeled	*labeled*			

Table: The Twelvefold Way (Balls into Bins Model).

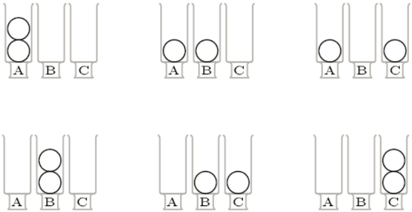
$\boxed{\text{labeled } \textit{vs.} \text{ unlabeled}}$

2 balls, 3 bins (unrestricted)

labeled balls into labeled bins



unlabeled balls into labeled bins

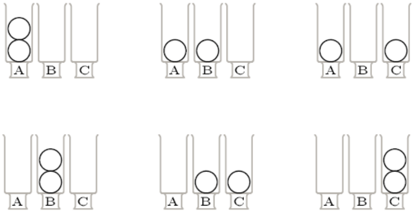


Only # of balls in each bin matters.

2 balls, 3 bins (unrestricted)

$$
\begin{aligned}
2 &= 2 + 0 + 0 \\
&= 1 + 1 + 0 \\
&= 1 + 0 + 1 \\
&= 0 + 2 + 0 \\
&= 0 + 1 + 1 \\
&= 0 + 0 + 2
\end{aligned}
$$

unlabeled balls into labeled bins



$$n = x_1 + x_2 + \ldots + x_m \quad (x_i \geq 0)$$

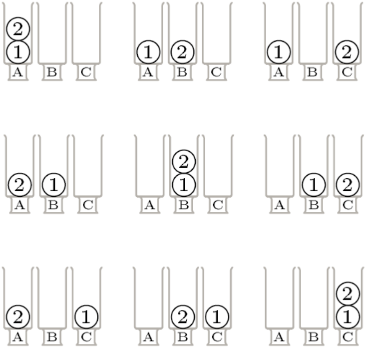
Only # of balls in each bin matters.

weak composition of n with m terms

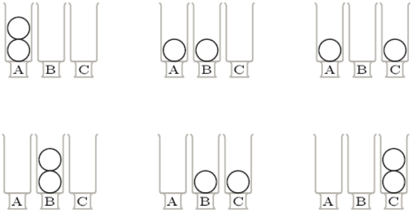
$$\boxed{\left(\!\!\left(n \right)\!\!\right)}$$

2 balls, 3 bins (unrestricted)

labeled balls into labeled bins



unlabeled balls into labeled bins



$$f : 1 \mapsto A, \quad 2 \mapsto B$$

$$f' : 2 \mapsto A, \quad 1 \mapsto B$$

$$f' = f \circ (h : 1 \mapsto 2, \quad 2 \mapsto 1)$$

f class	Any f	Injective f	Surjective f
f	n tuples of m items		
$f \circ S_N$	compositions of n into m parts		

Table: The Twelvefold Way (Functions).

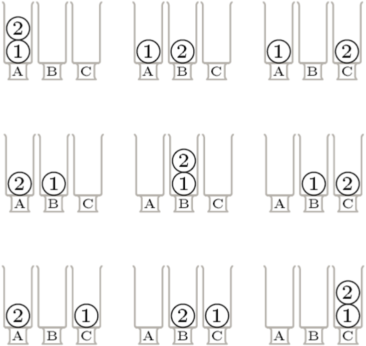
$$S_N = \{f : N \xleftrightarrow[\text{1-1}]{\text{onto}} N\}$$

$$[f] = f \circ S_N = \{f \circ g \mid g \in S_N\}$$

$$\boxed{f, g \text{ are indistinguishable} \iff g \in [f]}$$

2 balls, 3 bins (unrestricted)

labeled balls into labeled bins



labeled balls into unlabeled bins

$$\{1,2\} = \{1,2\}$$

$$= \{1\} \cup \{2\}$$

$\boxed{\text{partition of } N \text{ into } \le m \text{ parts}}$

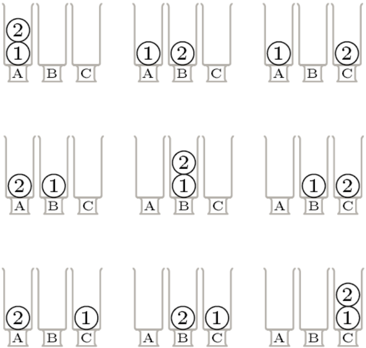
$\left\{ {n \atop k} \right\}$: # of partitions of N into k pa

$$\boxed{\sum_{k=1}^{m} \left\{ {n \atop k} \right\}}$$

$$
\begin{aligned}
\{1,2,3\} &= \{1,2,3\} \\
&= \{1\} \cup \{2,3\} \\
&= \{2\} \cup \{1,3\} \\
&= \{3\} \cup \{1,2\} \\
&= \{1\} \cup \{2\} \cup \{3\}
\end{aligned}
$$

2 balls, 3 bins (unrestricted)

labeled balls into labeled bins



labeled balls into unlabeled bins

$$f : 1 \mapsto A, \quad 2 \mapsto A$$

$$g : 1 \mapsto B, \quad 2 \mapsto B$$

$$h : 1 \mapsto C, \quad 2 \mapsto C$$

$$g = (l : A \mapsto B, \quad B \mapsto A) \circ f$$

f class	Any f	Injective f	Surjective f
f	n tuples of m items		
$f \circ S_N$	compositions of n into m parts		
$S_M \circ f$	partitions of N into $\leq m$ parts		

Table: The Twelvefold Way (Functions).

$$S_M = \{f : M \xleftrightarrow[\text{1-1}]{\text{onto}} M\}$$

$$[f] = S_M \circ f = \{g \circ f \mid g \in S_M\}$$

$$\boxed{f, g \text{ are indistinguishable} \iff g \in [f]}$$

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Table: The Twelvefold Way (Functions).

$$
\begin{aligned}
5 &= 5 \\
&= 41 \\
&= 32 = 311 \\
&= 221 = 21111 \\
&= 11111
\end{aligned}
$$

$$x_1 + x_2 + \cdots + x_k = n$$

$$x_1 \geq x_2 \geq \cdots \geq x_k \geq 1$$

$$\left| \begin{matrix} n \\ k \end{matrix} \right| : \text{\# of (integral) solutions}$$

$$\boxed{\sum_{k=1}^{m} \left| \begin{matrix} n \\ k \end{matrix} \right|}$$

f class	Any f	Injective f	Surjective f
f	n-tuples of m items m^n	n-permutations of m items $m^{\underline{n}}$	???
$f \circ S_N$	weak compositions of n into m parts $\left(\!\binom{n}{m}\!\right)$	n-combinations of m items $\binom{m}{n}$	compositions of n into m parts
$S_M \circ f$	partitions of N into $\leq m$ parts $\sum_{k=1}^{m} \left\{ {n \atop k} \right\}$	n pigeons into m holes $[n \leq m]$	partitions of N into m parts $\left\{ {n \atop m} \right\}$
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Table: The Twelvefold Way (Functions).

$$\left(\!\binom{n}{m}\!\right) \qquad \left\{ {n \atop k} \right\} \qquad \left| {n \atop k} \right|$$

Weak composition of n with m terms

$$
\begin{aligned}
2 &= 2 + 0 + 0 \\
&= 1 + 1 + 0 \\
&= 1 + 0 + 1 \\
&= 0 + 2 + 0 \\
&= 0 + 1 + 1 \\
&= 0 + 0 + 2
\end{aligned}
$$

$$\boxed{n = x_1 + x_2 + \ldots + x_m \quad (x_i \geq 0)}$$

$$\binom{n+m-1}{m-1} = \binom{n+m-1}{n}$$

$$\star \star \star \star \mid \mid \star \mid \star \star \mid$$

$7 = 4 + 0 + 1 + 2 + 0$ (Stars and Bars)

$\boxed{\text{Placing } m - 1 \text{ bars into } n + (m-1) \text{ slots.}}$

Composition of n with m terms

$$\boxed{n = x_1 + x_2 + \ldots + x_m \quad (x_i > 0)}$$

$$\star \star \star \star \mid \star \mid \star \star$$

$$7 = 4 + 1 + 2 \quad \text{(Stars and Bars)}$$

$\boxed{\text{Placing } m-1 \text{ bars into } n-1 \text{ slots.}}$

$$\binom{n-1}{m-1}$$

Theorem

The # of *weak composition* *of n with m terms is*

$$\binom{n+m-1}{m-1} = \binom{n+m-1}{n}$$

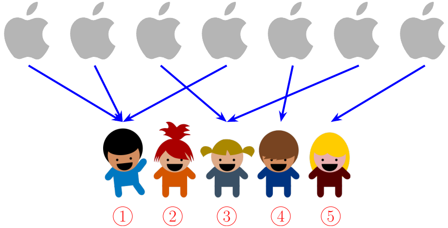
Theorem (CS Theorem 1.8)

The # of n-element *multisets* *chosen from an m-element set is*

$$\binom{n+m-1}{m-1} = \binom{n+m-1}{n}$$

$n = 7 \quad m = 5$ (Apples and Children)

$$\{1, \quad 1, \quad 3, \quad 1, \quad 4, \quad 3, \quad 5\}$$



$$7 = 3 + 0 + 2 + 1 + 1$$

$$\left\{ {n \atop k} \right\}: \text{ \# of partitions of set } N \text{ into } k \text{ parts}$$

$$\begin{aligned}
\{1,2,3\} &= \{1,2,3\} \\
&= \{1\} \cup \{2,3\} \\
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\end{aligned}$$

Stirling number of the second kind

Set Partition $(\mathrm{CS}: 1.5 - 12)$

$$\left\{ n \atop k \right\}: \ \# \text{ of partitions of set } N \text{ into } k \text{ parts}$$

$$\left\{ n \atop k \right\} = \underbrace{\left\{ n-1 \atop k-1 \right\}}_{n \text{ is alone}} + \underbrace{k \left\{ n-1 \atop k \right\}}_{n \text{ is not alone}} \qquad (n > 0, k > 0)$$

$$\text{Bell number: } B_n = \sum_{k=1}^{k=n} \left\{ n \atop k \right\}$$

$$\begin{vmatrix} n \\ k \end{vmatrix} : \# \text{ of partitions of } n \text{ into } k \text{ parts}$$

$$\begin{aligned} 5 &= 5 \\ &= 41 \\ &= 32 = 311 \\ &= 221 = 21111 \\ &= 11111 \end{aligned}$$

$$x_1 + x_2 + \cdots + x_k = n$$

$$x_1 \geq x_2 \geq \cdots \geq x_k \geq 1$$

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Theorem (Recurrence for $\left|{n \atop k}\right|$)

$$\left|{n \atop k}\right| = \left|{n-1 \atop k-1}\right| + \left|{n-k \atop k}\right|$$

Case II : $x_k > 1$

Case I : $x_k = 1$

$$x_1 - 1 \geq x_2 - 1 \geq \cdots \geq x_k - 1 \geq 1$$

$$\boxed{\left|{n-1 \atop k-1}\right|}$$

$$\boxed{\left|{n-k \atop k}\right|}$$

Theorem (G. H. Hardy, Ramanujan (1918))

$$p(n) \triangleq \sum_{x=1}^{x=n} \left| \begin{matrix} n \\ k \end{matrix} \right| \sim \frac{1}{4\sqrt{3}n} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

$p(200) \sim 4,100,251,432,188$

$p(200) = 3,972,999,029,388$

3.203%

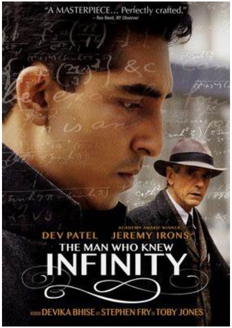

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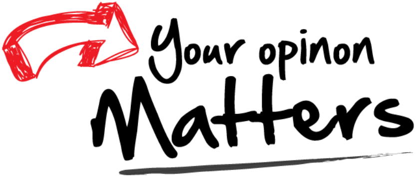
Table: The Twelvefold Way (Functions).

THE TWELVEFOLD WAY

balls per urn	unrestricted	≤ 1	≥ 1
n labeled balls, m labeled urns	n-tuples of m things	n-permutations of m things	partitions of $\{1, \ldots, n\}$ into m ordered parts
n unlabeled balls, m labeled urns	n-multicombinations of m things	n-combinations of m things	compositions of n into m parts
n labeled balls, m unlabeled urns	partitions of $\{1, \ldots, n\}$ into $\leq m$ parts	n pigeons into m holes	partitions of $\{1, \ldots, n\}$ into m parts
n unlabeled balls, m unlabeled urns	partitions of n into $\leq m$ parts	n pigeons into m holes	partitions of n into m parts


Thank You!



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