

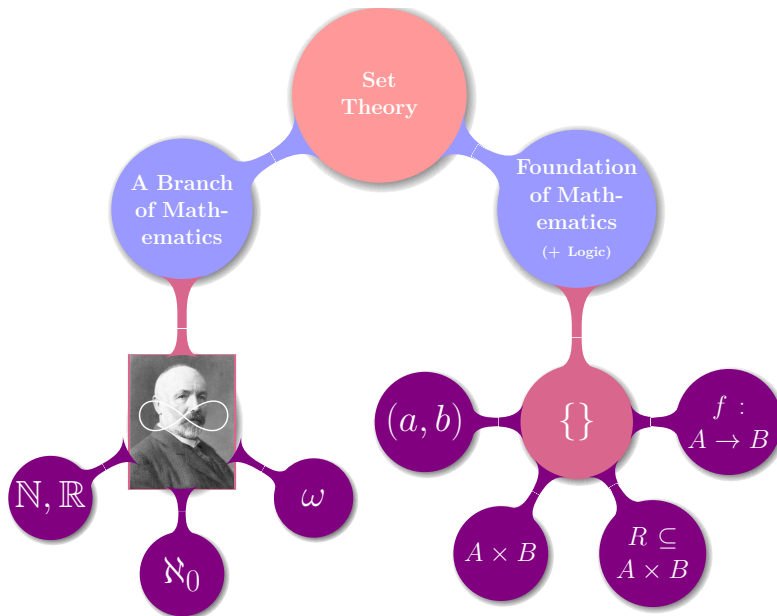
1-11 Set Theory (IV): Infinity

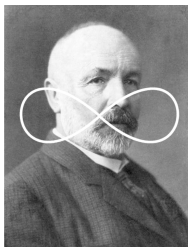
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Georg Cantor (1845 – 1918)



David Hilbert (1862 – 1943)



Leopold Kronecker
(1823 – 1891)



Henri Poincaré
(1854 – 1912)



Ludwig Wittgenstein
(1889 – 1951)

*From his paradise that Cantor with us unfolded, we hold our
breath in awe; knowing, we shall not be expelled.*

— *David Hilbert*

“没有人能把我们从 Cantor 创造的乐园中驱逐出去”

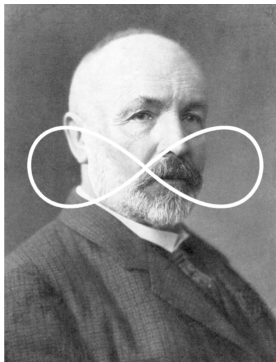




“das wesen der mathematik liegt in ihrer freiheit”

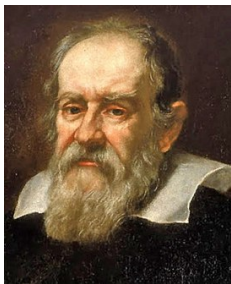
“The essence of mathematics lies in its freedom”

Before Cantor





公理: “整体大于部分”



Galileo Galilei (1564 – 1642)

“关于两门新科学的对话” (1638)

“用我们有限的心智来讨论无限...”

$$S_1 = \{1, 2, 3, \dots, n, \dots\}$$

$$S_2 = \{1, 4, 9, \dots, n^2, \dots\}$$

$$|S_1| = |S_2| \quad S_2 \subset S_1$$

“部分等于全体”



吓得我吃了一鲸

说到底，“等于”、“大于”和“小于”诸性质不能用于无限，而只能用于有限的数量。
— Galileo Galilei

无穷数是不可能的。
— Gottfried Wilhelm Leibniz

这些证明一开始就期望那些数要具有有穷数的一切性质，或者甚至于把有穷数的性质强加于无穷。

相反，这些无穷数，如果它们能够以任何形式被理解的话，倒是由于它们与有穷数的对应，它们必须具有完全新的数量特征。

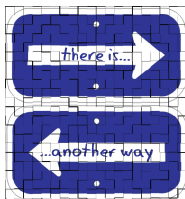
这些性质完全依赖于事物的本性，… 而并非来自我们的主观任意性或我们的偏见。

— Georg Cantor (1885)

Definition (Dedekind-infinite & Dedekind-finite (Dedekind, 1888))

A set A is *Dedekind-infinite* if there is a bijective function from A onto some proper subset B of A .

A set is *Dedekind-finite* if it is not Dedekind-infinite.



This is a **theorem** in our theory of infinity.



We have not defined “finite” and “infinite”!

Comparing Sets



Function



Definition ($|A| = |B|$ ($A \approx B$) (1878))

A and B are *equipotent* if there exists a *bijection* from A to B .

$\overline{\overline{A}}$ (two *abstractions*)

Abstract from elements: $\{1, 2, 3\}$ vs. $\{a, b, c\}$

Abstract from order: $\{1, 2, 3, \dots\}$ vs. $\{1, 3, 5, \dots, 2, 4, 6, \dots\}$

Definition ($|A| = |B|$ ($A \approx B$) (1878))

A and B are *equipotent* if there exists a *bijection* from A to B .

Q : Is “ \approx ” an equivalence relation?

Theorem (The “Equivalence Concept” of Equipotent)

For any sets A, B, C :

- (a) $A \approx B$
- (b) $A \approx B \implies B \approx A$
- (c) $A \approx B \wedge B \approx C \implies A \approx C$

Definition (Finite)

X is finite if

$$\exists n \in \mathbb{N} : |X| = n.$$

$$|X| = |\{0, 1, \dots, n-1\}|$$

Theorem (UD Theorem 22.6)

*Let A be a finite set. There is a **unique** $n \in \mathbb{N}$ such that $A \approx \{0, 1, \dots, n-1\}$.*

Definition (Infinite)

X is infinite if it is not finite:

$$\forall n \in \mathbb{N} : |X| \neq n.$$

Theorem (UD Theorem 22.3)

\mathbb{N} is infinite. (So are \mathbb{Z} , \mathbb{Q} , \mathbb{R} .)

By Contradiction.

$$\exists n \in \mathbb{N} : |\mathbb{N}| = n.$$

$$\exists f : \mathbb{N} \xrightarrow[\text{onto}]{1-1} \{0, 1, \dots, n-1\}$$

$$g \triangleq f|_{\{0,1,\dots,n\}} : \{0, 1, \dots, n\} \rightarrow \{0, 1, \dots, n-1\}$$

$$g \text{ is not 1-1} \implies f \text{ is not 1-1}$$

Definition (Infinite)

For any set X ,

Countably Infinite

$$|X| = |\mathbb{N}| \triangleq \aleph_0$$

Countable

(finite \vee countably infinite)

Uncountable

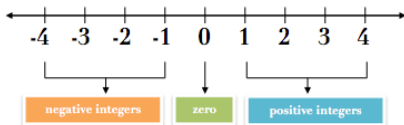
(\neg countable)

(infinite) \wedge (\neg (countably infinite))

\aleph_0

Theorem (\mathbb{Z} is Countable.)

$$|\mathbb{Z}| = |\mathbb{N}|$$



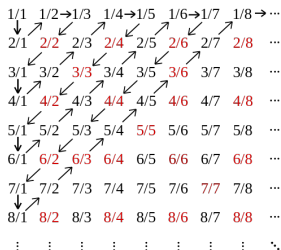
0 1 -1 2 -2 ...

Theorem (\mathbb{Q} is Countable. (Cantor 1873-11; Published in 1874))

$$|\mathbb{Q}| = |\mathbb{N}|$$

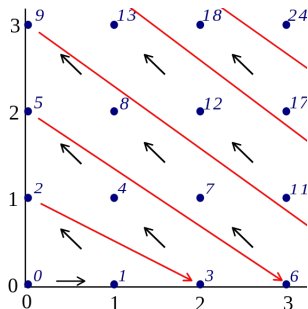
$|\mathbb{Q}| = |\mathbb{N}|$ (UD Problem 23.12)

$$q \in \mathbb{Q}^+ : a/b \ (a, b \in \mathbb{N}^+)$$



Theorem ($\mathbb{N} \times \mathbb{N}$ is Countable.)

$$|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$$



$$\pi : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$$

$$\pi(k_1, k_2) = \frac{1}{2}(k_1 + k_2)(k_1 + k_2 + 1) + k_2$$

Cantor Pairing Function

Theorem (\mathbb{N}^n is Countable.)

$$|\mathbb{N}^n| = |\mathbb{N}|$$

Theorem

*The Cartesian product of **finitely many** countable sets is countable.*

$$\mathbb{N}^n \quad \text{vs.} \quad \mathbb{N}^{\mathbb{N}}$$

$$\pi^{(n)} : \mathbb{N}^n \rightarrow \mathbb{N}$$

$$\pi^{(n)}(k_1, \dots, k_{n-1}, k_n) = \pi(\pi^{(n-1)}(k_1, \dots, k_{n-1}), k_n)$$

Theorem

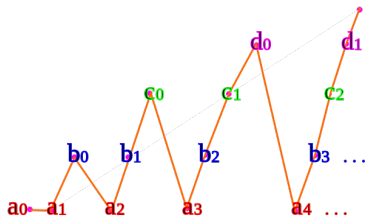
Any *finite* union of countable sets is countable.

$$A = \{a_n \mid n \in \mathbb{N}\} \quad B = \{b_n \mid n \in \mathbb{N}\} \quad C = \{c_n \mid n \in \mathbb{N}\}$$

$$a_0 \quad b_0 \quad c_0 \quad a_1 \quad b_1 \quad c_1 \cdots$$

Theorem

The union of *countably many* countable sets is countable.



Counting by Diagonals.

We need Axiom of (Countable) Choice!

Beyond

\aleph_0

Theorem (\mathbb{R} is Uncountable. (Cantor 1873-12; Published in 1874))

$$|\mathbb{R}| \neq |\mathbb{N}|$$



Different “Sizes” of Infinity

Cantor’s Diagonal Argument (1890)

Theorem (\mathbb{R} is Uncountable. (Cantor 1873-12; Published in 1874))

$$|\mathbb{R}| \neq |\mathbb{N}|$$

By Contradiction.

$$f : \mathbb{R} \xrightarrow[\text{onto}]{1-1} \mathbb{N}$$

3.14159...
1.41421...
1.73205...
2.23606...
2.71828...
0.14285...



3.43625...



2.32514...

By Diagonal Argument.

$$\mathfrak{c} \triangleq |\mathbb{R}|$$

Theorem ($|\mathbb{R}|$ (Cantor 1877))

$$|(0, 1)| = |\mathbb{R}| = |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}^{n \in \mathbb{N}}|$$

Proof.

$$f(x) = \tan \frac{(2x - 1)\pi}{2}$$

$$|(0, 1)| = |(-\frac{\pi}{2}, \frac{\pi}{2})| = |\mathbb{R}|$$

$$(x = 0.a_1a_2a_3\cdots, y = 0.b_1b_2b_3\cdots) \mapsto 0.a_1b_1a_2b_2a_3b_3\cdots$$



Theorem ($|\mathbb{R}|$ (Cantor 1877))

$$|(0, 1)| = |\mathbb{R}| = |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}^n|$$

“Je le vois, mais je ne le crois pas !”

“I see it, but I don't believe it !”

— Cantor's letter to Dedekind (1877).

Q : Then, what is “dimension”?

Theorem (Brouwer (Topological Invariance of Dimension))

*There is no **continuous** bijections between \mathbb{R}^m and \mathbb{R}^n for $m \neq n$.*

Beyond



Theorem (Cantor's Theorem (1891))

$$|A| \neq |\mathcal{P}(A)|$$

Theorem (Cantor Theorem (ES Theorem 24.4))

If $f : A \rightarrow \mathcal{P}(A)$, then f is not onto.

Proof. Let A be a set and let $f : A \rightarrow 2^A$. To show that f is not onto, we must find a $B \in 2^A$ (i.e., $B \subseteq A$) for which there is no $a \in A$ with $f(a) = B$. In other words, B is a set that f “misses.” To this end, let

$$B = \{x \in A : x \notin f(x)\}.$$

We claim there is no $a \in A$ with $f(a) = B$.

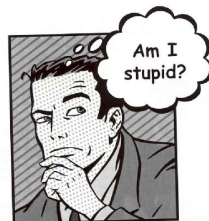
Suppose, for the sake of contradiction, there is an $a \in A$ such that $f(a) = B$. We ponder: Is $a \in B$?

- If $a \in B$, then, since $B = f(a)$, we have $a \in f(a)$. So, by definition of B , $a \notin f(a)$; that is, $a \notin B. \Rightarrow \Leftarrow$
- If $a \notin B = f(a)$, then, by definition of B , $a \in B. \Rightarrow \Leftarrow$

Both $a \in B$ and $a \notin B$ lead to contradictions, and hence our supposition [there is an $a \in A$ with $f(a) = B$] is false, and therefore f is not onto. ■

Theorem (Cantor Theorem)

If $f : A \rightarrow \mathcal{P}(A)$, then f is not onto.



Theorem (Cantor Theorem)

If $f : A \rightarrow \mathcal{P}(A)$, then f is not onto.

Understanding this problem:

$$A = \{1, 2, 3\}$$

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

Onto

$$\forall B \in 2^A : (\exists a \in A : f(a) = B)$$

Not Onto

$$\exists B \in 2^A : (\forall a \in A : f(a) \neq B)$$

Theorem (Cantor Theorem)

If $f : A \rightarrow \mathcal{P}(A)$, then f is not onto.

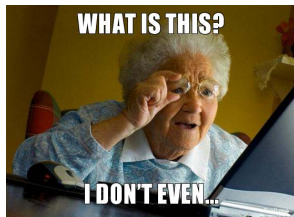
$$\exists B \in \mathcal{P}(A) : \left(\forall a \in A : f(a) \neq B \right)$$

- Constructive proof (\exists):

$$B = \{a \in A \mid a \notin f(a)\}$$

- By contradiction (\forall):

$$\exists a \in A : f(a) = B.$$



$$Q : a \in B?$$

$$a \in B \iff a \notin B$$

Theorem (Cantor Theorem)

If $f : A \rightarrow \mathcal{P}(A)$, then f is not onto.

Diagonal Argument (以下仅适用于可数集合 A).

a	$f(a)$					
	1	2	3	4	5	...
1	1	1	0	0	1	...
2	0	0	0	0	0	...
3	1	0	0	1	0	...
4	1	1	1	1	1	...
5	0	1	0	1	0	...
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	...

$$B = \{0, 1, 1, 0, 1\}$$

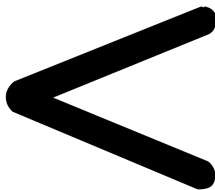


Theorem (Cantor Theorem)

$$|A| < |\mathcal{P}(A)|$$

$$A \quad \mathcal{P}(A) \quad \mathcal{P}(\mathcal{P}(A)) \quad \dots$$

There is no largest infinity.



Definition ($|A| \leq |B|$)

$|A| \leq |B|$ if there exists an *one-to-one* function f from A into B .

Q : What about onto function $f : A \rightarrow B$?

$|B| \leq |A|$ (Axiom of Choice)

Definition ($|A| < |B|$)

$$|A| < |B| \iff |A| \leq |B| \wedge |A| \neq |B|$$

$$|\mathbb{N}| < |\mathbb{R}|$$

$$|X| < |2^X|$$

$$|\mathbb{N}| < |2^{\mathbb{N}}|$$

Definition (Countable Revisited)

X is countable:

$$(\exists n \in \mathbb{N} : |X| = n) \vee |X| = |\mathbb{N}|$$

Theorem (Proof for Countable)

X is countable iff

$$|X| \leq |\mathbb{N}|.$$

X is countable iff there exists a *one-to-one* function

$$f : X \rightarrow \mathbb{N}.$$

Subsets of Countable Set (UD Corollary 23.4)

Every subset B of a countable set A is countable.

$$f : A \xrightarrow[\text{onto}]{1-1} \mathbb{N} \quad g = f|_B$$

Slope (UD Problem 23.3 (a))

(a) The set of all lines with rational slopes

$$(\mathbb{Q}, \mathbb{R})$$

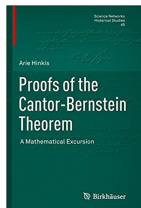
$$|\mathbb{R}| \leq |\mathbb{Q} \times \mathbb{R}| \leq |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$$

Q : Is “ \leq ” a partial order?

Theorem (Cantor-Schröder-Bernstein (1887))

$$|X| \leq |Y| \wedge |Y| \leq |X| \implies |X| = |Y|$$

$$\exists \text{ one-to-one } f : A \rightarrow B \wedge g : B \rightarrow A \implies \exists \text{ bijection } h : A \rightarrow B$$



Schröder-Bernstein
theorem @ wiki

Q : Is “ \leq ” a total order?

Theorem (PCC)

Principle of Cardinal Comparability (PCC) \iff Axiom of Choice

Theorem (UD Theorem 24.11)

$$|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$$

$$|\mathbb{R}| \leq |\mathcal{P}(\mathbb{N})| \quad |\mathcal{P}(\mathbb{N})| \leq |\mathbb{R}|$$

$$\mathfrak{c} \triangleq |\mathbb{R}| = |\mathcal{P}(\mathbb{N})| = |2^{\mathbb{N}}| \triangleq 2^{\aleph_0}$$

$$\boxed{\mathfrak{c} = 2^{\aleph_0}}$$

Continuum Hypothesis (CH)

$$\nexists A : \aleph_0 < |A| < \mathfrak{c}$$



👉 Dangerous Knowledge (22:20; BBC 2007)

Independence from ZFC:

Kurt Gödel (1940) CH cannot be disproved from ZF.

Paul Cohen (1964) CH cannot be proven from the ZFC axioms.

Thank
You!



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