2-2 The Efficiency of Algorithms

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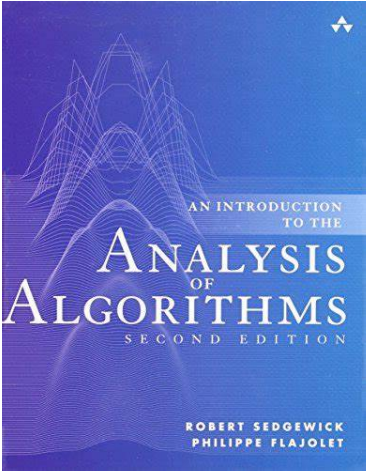
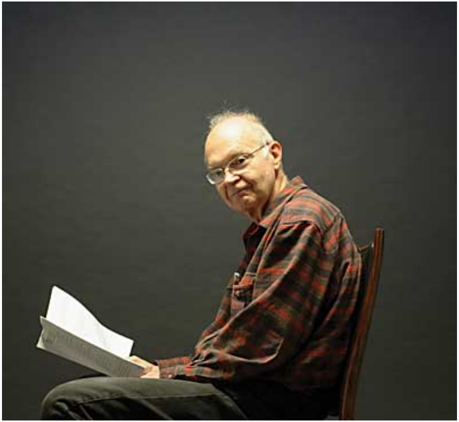


The Analysis of Algorithms



Donald E. Knuth (1938 $\sim$)

Donald E. Knuth (1974)

"For his major contributions to the analysis of algorithms

and the design of programming languages,

and in particular for his contributions to

the "art of computer programming" through

his well-known books in a continuous series by this title."

Fibonacci numbers in the analysis of Euclid's GCD algorithm

H_n in the analysis of FIND-MAX @ Stanford Lecture by Knuth

"People who analyze algorithms have double happiness.

First of all they experience the sheer beauty of elegant mathematical patterns that surround elegant computational procedures.

Then they receive a practical payoff when their theories make it possible to get other jobs done more quickly and more economically."

How Fast is It?

Time (and Space) Complexity of Algorithms

$$O \quad \Omega \quad \Theta$$

$$o \quad \omega$$

Space Complexity of Algorithms

We only care about the extra space caused by the algorithm.

The space for inputs is not part of space complexity of algorithms.

$$\textsc{insertion-sort}(A, n) : O(1) \quad \text{(constant)}$$

Is it the Fastest?

Complexity of Problems

This is much harder and is not our focus today.

Whenever you design an algorithm,

you provide an upper bound for the complexity of the problem.

Whenever you encounter a "hardcore" of the problem,

you obtain a lower bound for all possible algorithms.

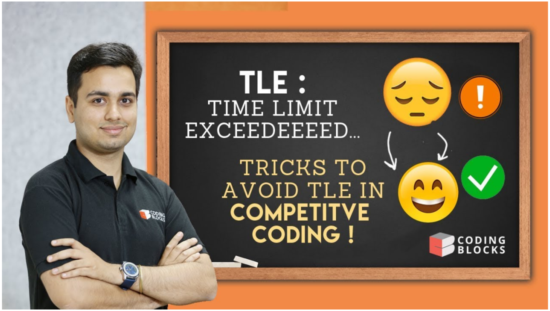
Often, there is an "algorithmic gap" between them.

When the gap is gone, you get the optimal algorithm.

$$\textsf{sorting}(A, n) : \Theta(n \log n) = O(n \log n) \cap \Omega(n \log n)$$

Q : How fast is your algorithm?

A : It runs 3.1415926 seconds.



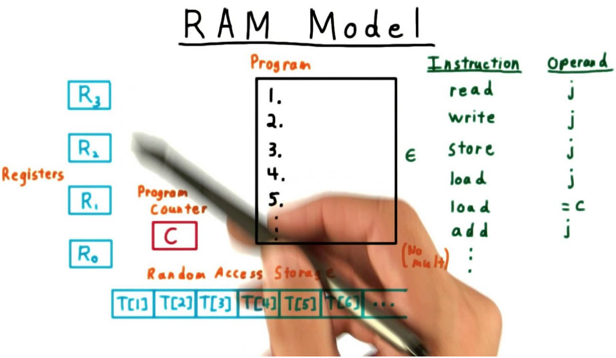
Disadvantages:

- On different machines
- At different time
- On different inputs

No Standards.

We need a uniform model of computation.

The RAM (Random Access Machine) Model of Computation



The RAM (Random Access Machine) Model of Computation

- Each memory access takes constant time.
- Each *"primitive"* operation takes constant time.
- Compound operations should be decomposed.

Counting up the number of time units.

Disadvantages:

- On different machines
- At different time
- On different inputs

Counting up the number of time units

as a function of the input size

in typical cases.

INSERTION-SORT(A)	*cost*	*times*
for $j = 2$ **to** $A.length$	c_1	n
$key = A[j]$	c_2	$n-1$
// Insert $A[j]$ into the sorted sequence $A[1 \,..\, j-1]$.	0	$n-1$
$i = j - 1$	c_4	$n-1$
while $i > 0$ and $A[i] > key$	c_5	$\sum_{j=2}^{n} t_j$
$A[i+1] = A[i]$	c_6	$\sum_{j=2}^{n} (t_j - 1)$
$i = i - 1$	c_7	$\sum_{j=2}^{n} (t_j - 1)$
$A[i+1] = key$	c_8	$n-1$

$$
\begin{aligned}
T(n) = {} & c_1 n + c_2(n-1) + c_4(n-1) \\
& + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8(n-1)
\end{aligned}
$$

… as a function of the input size …

INSERTION-SORT(A)	*cost*	*times*
for $j = 2$ **to** $A.length$	c_1	n
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$$
\begin{aligned}
T(n) = {} & c_1 n + c_2(n-1) + c_4(n-1) \\
& + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8(n-1)
\end{aligned}
$$

$T(n)$: Depends on *which* input of size n

... in typical cases.

Problem P Algorithm A

Inputs: $\mathcal{X}_n$ of size n

$$W(n) = \max_{x \in \mathcal{X}_n} T(x)$$

$$B(n) = \min_{x \in \mathcal{X}_n} T(x)$$

$$A(n) = \boxed{\sum_{x \in \mathcal{X}_n} T(x) \cdot P(x)} = \mathbb{E}[T] = \boxed{\sum_{t \in T(\mathcal{X}_n)} t \cdot P(T = t)}$$

INSERTION-SORT(A)	*cost*	*times*
for $j = 2$ **to** $A.length$	c_1	n
$key = A[j]$	c_2	$n-1$
// Insert $A[j]$ into the sorted sequence $A[1\,..\,j-1]$.	0	$n-1$
$i = j - 1$	c_4	$n-1$
while $i > 0$ and $A[i] > key$	c_5	$\sum_{j=2}^{n} t_j$
$A[i+1] = A[i]$	c_6	$\sum_{j=2}^{n} (t_j - 1)$
$i = i - 1$	c_7	$\sum_{j=2}^{n} (t_j - 1)$
$A[i+1] = key$	c_8	$n-1$

$$B(n) = (c_1 + c_2 + c_4 + c_5 + c_8)n - (c_2 + c_4 + c_5 + c_8)$$

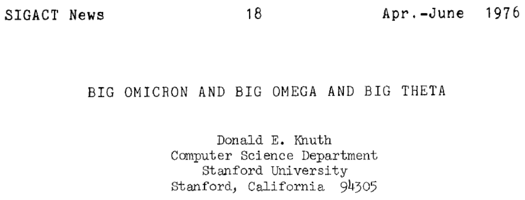
$$W(n) = \frac{c_5 + c_6 + c_7}{2} n^2 + (c_1 + c_2 + c_4 + c_8 - \frac{c_5 + c_6 + c_7}{2})n - (c_2 + c_4 + c_5 + c_8)$$

$$A(n) = 2.25n^2 + 7.75n - 3H_n - 6 \qquad (H_n = \sum_{k=1}^{n} \frac{1}{k} \approx \ln n)$$

Q : How fast is your algorithm?

listen carefully.

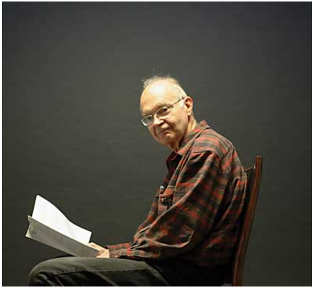
$$W(n) = \frac{c_5 + c_6 + c_7}{2} n^2 + (c_1 + c_2 + c_4 + c_8 - \frac{c_5 + c_6 + c_7}{2}) n - (c_2 + c_4 + c_5 + c_8)$$



SIGACT News 18 Apr.-June 1976

BIG OMICRON AND BIG OMEGA AND BIG THETA

Donald E. Knuth
Computer Science Department
Stanford University
Stanford, California 94305



Reference:

"Big Omicron and Big Omega and Big Theta", Donald E. Knuth, 1976.

Asymptotics

Q : How fast is your algorithm?

$$W(n) = \frac{c_5 + c_6 + c_7}{2} n^2 + (c_1 + c_2 + c_4 + c_8 - \frac{c_5 + c_6 + c_7}{2}) n - (c_2 + c_4 + c_5 + c_8)$$

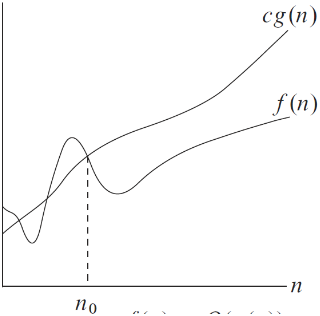
$$\boxed{W(n) = O(n^2)}$$

"Order at most n^2"

"$W(n)$ is a function whose order of magnitude is upper-bounded by a constant times n^2, for all large n."

$$\boxed{f(n) = O(g(n))}$$

"$f(n)$ is a function whose order of magnitude is upper-bounded by a constant times $g(n)$, for all large n."



$$O(g(n)) = \left\{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) \leq cg(n) \right\}$$

$$\boxed{f(n) = O(g(n))}$$

$$O(g(n)) = \Big\{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) \leq cg(n) \Big\}$$

$$\{ \quad \}$$

It is a tradition to write $f(n) = O(g(n))$ instead of $f(n) \in O(g(n))$.

$$42n^2 + 2020n = O(n^2) = O(n^3)$$

$$42n^2 + 2020n \in O(n^2) \subseteq O(n^3)$$

$$O\big(f(n)\big) + O\big(g(n)\big) \triangleq \Big\{h + l \mid h \in O\big(f(n)\big), l \in O\big(g(n)\big)\Big\}$$

$$O\big(f(n)\big)O\big(g(n)\big) \triangleq \Big\{hl \mid h \in O\big(f(n)\big), l \in O\big(g(n)\big)\Big\}$$

$$O\big(f(n)\big) - O\big(g(n)\big) \triangleq$$

$$O(g(n)) = \left\{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) \leq cg(n) \right\}$$

$$42n = O(0.50n^2) \qquad 42n^2 = O(0.50n^2)$$

Q : What does $O(1)$ mean?

A : It means constants.

$$\Omega(g(n)) = \Big\{ f(n) \mid \exists c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq cg(n) \leq f(n) \Big\}$$

$$0.50n^2 = \Omega(42n) \qquad 0.50n^2 = \Omega(42n^2)$$

$$\Theta(g(n)) = \Big\{ f(n) \mid \exists c_1 > 0, \exists c_2 > 0, \exists n_0 > 0, \forall n \geq n_0 : \\ 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \Big\}$$

$$0.50n^2 = \Theta(42n^2)$$

$$o(g(n)) = \left\{ f(n) \mid \forall c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq f(n) < cg(n) \right\}$$

$$42n = o(0.50n^2)$$

$$\omega(g(n)) = \left\{ f(n) \mid \forall c > 0, \exists n_0 > 0, \forall n \geq n_0 : 0 \leq cg(n) < f(n) \right\}$$

$$0.50n^2 = \omega(42n)$$

$$O \quad \Omega \quad \Theta$$

$$o \quad \omega \quad \theta$$

$$f(n) \sim g(n) \iff \lim_{n \to \infty} \frac{f(n)}{g(n)} = 1$$

$$42n^2 + 2020n \sim 42n^2 + 2019n$$

$$f(n) = \Theta(g(n)) \iff f(n) = O(g(n)) \land f(n) = \Omega(g(n))$$

$$f(n) = O(g(n)) \iff g(n) = \Omega(f(n))$$

$$f(n) = o(g(n)) \iff g(n) = \omega(f(n))$$

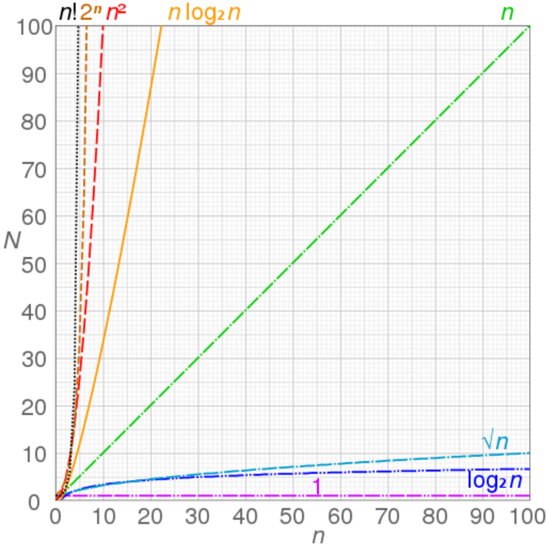
$$O\big(f(n)\big) + O\big(g(n)\big) = O\big(f(n) + g(n)\big)$$

$$O\big(f(n)\big)O\big(g(n)\big) = O\big(f(n)g(n)\big)$$

Q : How to compare functions in terms of $O/\Omega/\Theta$?

$$
\begin{aligned}
O(1) &= O(\log\log n) = O(\log n) = O((\log n)^c) \\
&= O(n^\epsilon) = O(n^c) \\
&= O(n^c \log n) = O(n^{\log n}) = O(c^n) = O(n^n)
\end{aligned}
$$

$$(0 < \epsilon < 1 < c)$$



n! 2^n n^2
n log2 n
n
100
90
80
70
60
N
50
40
30
20
10
0
√n
log2 n
1
0 10 20 30 40 50 60 70 80 90 100
n

Stirling Formula (by *James Stirling*):

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

$$\log(n!) = \Theta(n \log n)$$

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = \Theta(\log n)$$


Learning by
Doing

$$A[0, \ldots n-1] \qquad 1 \le l \le n$$

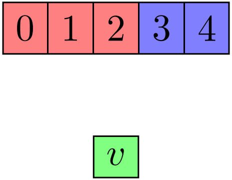
ROTATE(A, n, l) : Rotate A left by l places

0	1	2	3	4

3	4	0	1	2

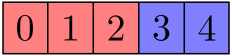
Critical Operation: copy

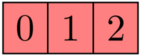
```
procedure ROTATE(A, n, l)
    for i = 1 ... l do
        ROTATE-BY-ONE(A, n)
```



Algorithm	Time	Space
rotate-one-by-one	$nl = O(n^2)$	$O(1)$

```
procedure ROTATE(A, n, l)
    copy A[0 ... l − 1] into v
    move A[l ... n − 1] left l places
    copy v to A[l ... n − 1]
```





Algorithm	Time	Space
rotate-copy	$O(n)$	$l = O(n)$

$n = 5, \quad l = 3$

0	1	2	3	4

3	4	0	1	2

$(0, 2, 4, 1, 3)$

$n = 9, \quad l = 6$

0	1	2	3	4	5	6	7	8

6	7	8	0	1	2	3	4	5

$(0, 3, 6)$ $\quad (1, 7, 4)$ $\quad (2, 8, 5)$

Correctness Proof?

Permutations as Product of Disjoint Cycles

Algorithm	Time	Space
rotate-cyclic	$O(n)$	$O(1)$

$$\boxed{B \cdot A = (A^R \cdot B^R)^R}$$

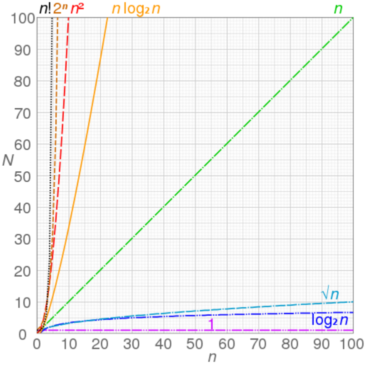
0	1	2	3	4

2	1	0	4	3

3	4	0	1	2

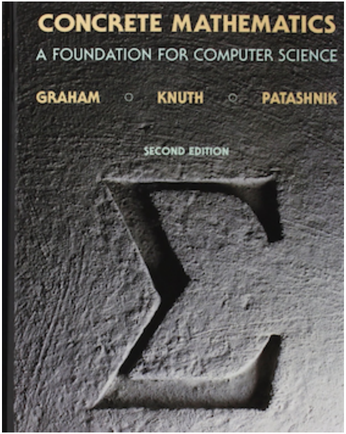
Algorithm	Time	Space
rotate-reverse	$O(n)$	$O(1)$

Algorithm	Time	Space
rotate-one-by-one	$O(n^2)$	$O(1)$
rotate-copy	$O(n)$	$O(n)$
rotate-cyclic	$O(n)$	$O(1)$
rotate-reverse	$O(n)$	$O(1)$



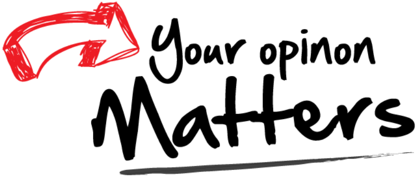
$O \quad \Omega \quad \Theta$

$o \quad \omega$



Chapter 9: Asymptotics


Thank
You!



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