1-11 Set Theory (IV): Infinity

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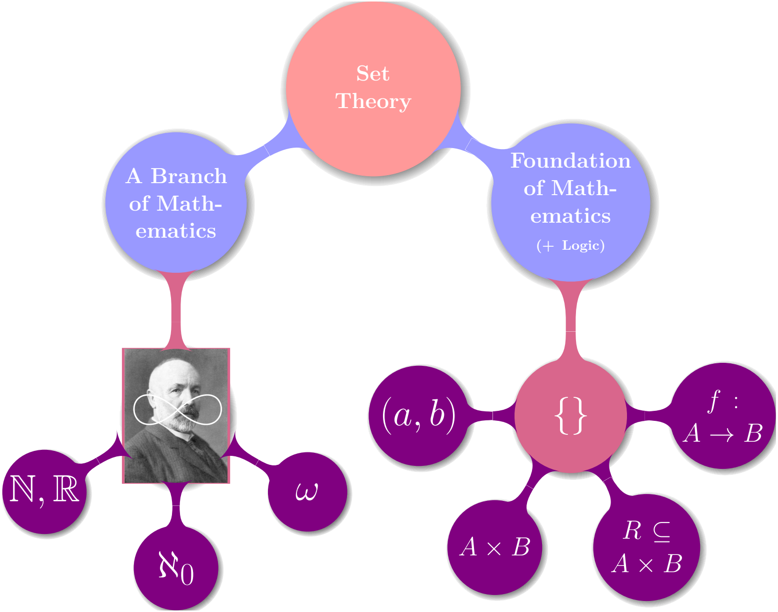
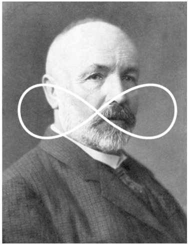

Set Theory
A Branch of Math-ematics
Foundation of Math-ematics
(+ Logic)
$\mathbb{N}, \mathbb{R}$
$\aleph_0$
ω
(a, b)
$\{\}$
$f : A \to B$
$A \times B$
$R \subseteq A \times B$

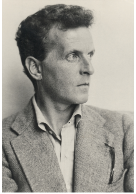


Georg Cantor (1845 – 1918)

David Hilbert (1862 – 1943)

Leopold Kronecker
(1823 – 1891)

Henri Poincaré
(1854 – 1912)



Ludwig Wittgenstein
(1889 – 1951)

From his paradise that Cantor with us unfolded, we hold our breath in awe; knowing, we shall not be expelled.

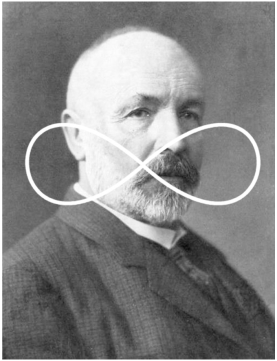
— David Hilbert

“没有人能把我们从 *Cantor* 创造的乐园中驱逐出去”

"das wesen der mathematik liegt in ihrer freiheit"

"The essence of mathematics lies in its freedom"

Before Cantor




决定经典
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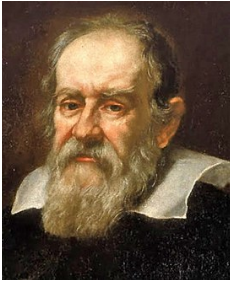
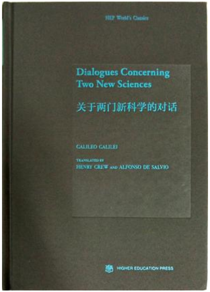
13卷图释全本

公理: “整体大于部分”



Galileo Galilei (1564 – 1642)



“关于两门新科学的对话” (1638)

“用我们有限的心智来讨论无限 · · · ”

$$S_1 = \{1, 2, 3, \cdots, n, \cdots\}$$
$$S_2 = \{1, 4, 9, \cdots, n^2, \cdots\}$$

$|S_1| = |S_2|$ $S_2 \subset S_1$

“部分等于全体”

说到底，“等于”、“大于”和“小于”诸性质不能用于无限，而只能用于有限的数量。 — *Galileo Galilei*

无穷数是不可能的。 — *Gottfried Wilhelm Leibniz*

这些证明一开始就期望那些数要具有有穷数的一切性质，或者甚至于把有穷数的性质强加于无穷。

相反，这些无穷数，如果它们能够以任何形式被理解的话，倒是由于它们与有穷数的对应，它们必须具有完全新的数量特征。

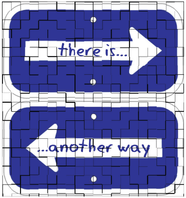
这些性质完全依赖于事物的本性，$\cdots$ 而并非来自我们的主观任意性或我们的偏见。

— *Georg Cantor (1885)*

Definition (Dedekind-infinite & Dedekind-finite (Dedekind, 1888))

A set A is *Dedekind-infinite* if there is a bijective function from A onto some proper subset B of A.

A set is *Dedekind-finite* if it is not Dedekind-infinite.



This is a theorem in our theory of infinity.

We have not defined “finite” and “infinite”!

Comparing Sets

$$= \qquad\qquad <$$

Function

$$=$$

Definition ($|A| = |B|$ ($A \approx B$) (1878))

A and B are *equipotent* if there exists a *bijection* from A to B.

$\overline{\overline{A}}$ (two abstractions)

Abstract from elements: $\{1, 2, 3\}$ *vs.* $\{a, b, c\}$

Abstract from order: $\{1, 2, 3, \cdots\}$ *vs.* $\{1, 3, 5, \cdots, 2, 4, 6, \cdots\}$

Definition ($|A| = |B|$ ($A \approx B$) (1878))

A and B are *equipotent* if there exists a *bijection* from A to B.

Q : Is "$\approx$" an equivalence relation?

Theorem (The "Equivalence Concept" of Equipotent)

For any sets A, B, C:

(a) $A \approx B$

(b) $A \approx B \implies B \approx A$

(c) $A \approx B \land B \approx C \implies A \approx C$

Definition (Finite)

X is finite if

$$\exists n \in \mathbb{N} : |X| = n.$$

$$|X| = |\{0, 1, \cdots, n-1\}|$$

Theorem (UD Theorem 22.6)

Let A be a finite set. There is a unique $n \in \mathbb{N}$ such that $A \approx \{0, 1, \cdots, n-1\}$.

Definition (Infinite)

X is infinite if it is not finite:

$$\forall n \in \mathbb{N} : |X| \neq n.$$

Theorem (UD Theorem 22.3)

$\mathbb{N}$ *is infinite. (So are $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$.)*

By Contradiction.

$$\exists n \in \mathbb{N} : |\mathbb{N}| = n.$$

$$\exists f : \mathbb{N} \xleftrightarrow[onto]{1-1} \{0, 1, \cdots, n-1\}$$

$$g \triangleq f|_{\{0,1,\ldots,n\}} : \{0, 1, \cdots, n\} \to \{0, 1, \cdots, n-1\}$$

By the Pigeonhole Principle : g is not 1-1 $\implies$ f is not 1-1

Definition (Infinite)

For any set X,

Countably Infinite

$$|X| = |\mathbb{N}| \triangleq \aleph_0$$

Countable

$$(\text{finite} \lor \text{countably infinite})$$

Uncountable

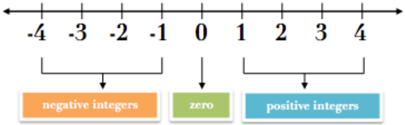
$$(\lnot \text{ countable})$$

$$(\text{infinite}) \land \Big(\lnot \text{ (countably infinite)}\Big)$$

$$\aleph_0$$

Theorem ($\mathbb{Z}$ is Countable.)

$$|\mathbb{Z}| = |\mathbb{N}|$$



$$0 \quad 1 \quad -1 \quad 2 \quad -2 \quad \cdots$$

Theorem ($\mathbb{Q}$ is Countable. (Cantor 1873-11; Published in 1874))

$$|\mathbb{Q}| = |\mathbb{N}|$$

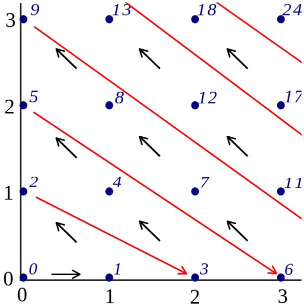
$|\mathbb{Q}| = |\mathbb{N}|$ (UD Problem 23.12)

$$q \in \mathbb{Q}^{+} : a/b \; (a, b \in \mathbb{N}^{+})$$

1/1	1/2 →	1/3	1/4 →	1/5	1/6 →	1/7	1/8 →	⋯
2/1	2/2	2/3	2/4	2/5	2/6	2/7	2/8	⋯
3/1	3/2	3/3	3/4	3/5	3/6	3/7	3/8	⋯
4/1	4/2	4/3	4/4	4/5	4/6	4/7	4/8	⋯
5/1	5/2	5/3	5/4	5/5	5/6	5/7	5/8	⋯
6/1	6/2	6/3	6/4	6/5	6/6	6/7	6/8	⋯
7/1	7/2	7/3	7/4	7/5	7/6	7/7	7/8	⋯
8/1	8/2	8/3	8/4	8/5	8/6	8/7	8/8	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Theorem ($\mathbb{N} \times \mathbb{N}$ is Countable.)

$$|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$$



$$\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

$$\pi(k_1, k_2) = \frac{1}{2}(k_1+k_2)(k_1+k_2+1)+k_2$$

Cantor Pairing Function

Theorem ($\mathbb{N}^n$ is Countable.)

$$|\mathbb{N}^n| = |\mathbb{N}|$$

Theorem

The Cartesian product of finitely many countable sets is countable.

$$\mathbb{N}^n \quad vs. \quad \mathbb{N}^{\mathbb{N}}$$

$$\pi^{(n)} : \mathbb{N}^n \to \mathbb{N}$$

$$\pi^{(n)}(k_1, \ldots, k_{n-1}, k_n) = \pi(\pi^{(n-1)}(k_1, \ldots, k_{n-1}), k_n)$$

Theorem

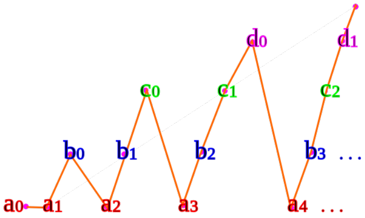
Any finite union of countable sets is countable.

$$A = \{a_n \mid n \in \mathbb{N}\} \quad B = \{b_n \mid n \in \mathbb{N}\} \quad C = \{c_n \mid n \in \mathbb{N}\}$$

$$a_0 \quad b_0 \quad c_0 \quad a_1 \quad b_1 \quad c_1 \cdots$$

Theorem

The union of countably many countable sets is countable.



Counting by Diagonals.

We need Axiom of (Countable) Choice!

Beyond

$$\aleph_0$$

Theorem ($\mathbb{R}$ is Uncountable. (Cantor 1873-12; Published in 1874))

$$|\mathbb{R}| \neq |\mathbb{N}|$$

Different "Sizes" of Infinity

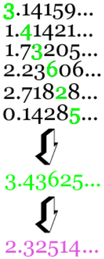
Cantor's Diagonal Argument (1890)

Theorem ($\mathbb{R}$ is Uncountable. (Cantor 1873-12; Published in 1874))

$$|\mathbb{R}| \neq |\mathbb{N}|$$

By Contradiction.

$$f : \mathbb{R} \xleftrightarrow[onto]{1-1} \mathbb{N}$$



By Diagonal Argument.

$$\mathfrak{c} \triangleq |\mathbb{R}|$$

Theorem ($|\mathbb{R}|$ (Cantor 1877))

$$|(0,1)| = |\mathbb{R}| = |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}^{n \in \mathbb{N}}|$$

Proof.

$$f(x) = \tan \frac{(2x-1)\pi}{2}$$

$$|(0,1)| = |(-\frac{\pi}{2}, \frac{\pi}{2})| = |\mathbb{R}|$$

$$(x = 0.a_1 a_2 a_3 \cdots, y = 0.b_1 b_2 b_3 \cdots) \mapsto 0.a_1 b_1 a_2 b_2 a_3 b_3 \cdots$$

$\square$

Theorem ($|\mathbb{R}|$ (Cantor 1877))

$$|(0,1)| = |\mathbb{R}| = |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}^n|$$

"Je le vois, mais je ne le crois pas !"

"I see it, but I don't believe it !"

— *Cantor's letter to Dedekind (1877).*

Q : Then, what is "dimension"?

Theorem (Brouwer (Topological Invariance of Dimension))

There is no continuous bijections between $\mathbb{R}^m$ and $\mathbb{R}^n$ for $m \neq n$.

Beyond

$$\mathfrak{c}$$

Theorem (Cantor's Theorem (1891))

$$|A| \neq |\mathcal{P}(A)|$$

Theorem (Cantor Theorem (ES Theorem 24.4))

If $f : A \to \mathcal{P}(A)$, then f is not onto.

Proof. Let A be a set and let $f : A \to 2^A$. To show that f is not onto, we must find a $B \in 2^A$ (i.e., $B \subseteq A$) for which there is no $a \in A$ with $f(a) = B$. In other words, B is a set that f "misses." To this end, let

$$B = \{x \in A : x \notin f(x)\}.$$

We claim there is no $a \in A$ with $f(a) = B$.

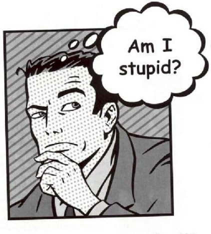
Suppose, for the sake of contradiction, there is an $a \in A$ such that $f(a) = B$. We ponder: Is $a \in B$?

- If $a \in B$, then, since $B = f(a)$, we have $a \in f(a)$. So, by definition of B, $a \notin f(a)$; that is, $a \notin B$.$\Rightarrow\Leftarrow$
- If $a \notin B = f(a)$, then, by definition of B, $a \in B$.$\Rightarrow\Leftarrow$

Both $a \in B$ and $a \notin B$ lead to contradictions, and hence our supposition [there is an $a \in A$ with $f(a) = B$] is false, and therefore f is not onto. ■

Theorem (Cantor Theorem)

If $f : A \to \mathcal{P}(A)$, then f is not onto.



Theorem (Cantor Theorem)

If $f : A \to \mathcal{P}(A)$, then f is not onto.

Understanding this problem:

$$A = \{1, 2, 3\}$$

$$\mathcal{P}(A) = \Big\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\Big\}$$

Onto

$$\forall B \in \mathcal{P}(A) : \Big(\exists a \in A : f(a) = B\Big)$$

Not Onto

$$\exists B \in \mathcal{P}(A) : \Big(\forall a \in A : f(a) \neq B\Big)$$

Theorem (Cantor Theorem)

If $f : A \to \mathcal{P}(A)$, then f is not onto.

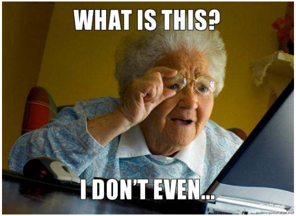
$$\exists B \in \mathcal{P}(A) : \Big(\forall a \in A : f(a) \neq B\Big)$$

- Constructive proof ($\exists$):

$$B = \{a \in A \mid a \notin f(a)\}$$

- By contradiction ($\forall$):

$$\exists a \in A : f(a) = B.$$



$$Q : a \in B\,?$$

$$a \in B \iff a \notin B$$

Theorem (Cantor Theorem)

If $f : A \to \mathcal{P}(A)$, then f is not onto.

Diagonal Argument (以下仅适用于可数集合 A).

a	$f(a)$					
	1	2	3	4	5	$\cdots$
1	1	1	0	0	1	$\cdots$
2	0	0	0	0	0	$\cdots$
3	1	0	0	1	0	$\cdots$
4	1	1	1	1	1	$\cdots$
5	0	1	0	1	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$

$$B = \{0, 1, 1, 0, 1\}$$

□

Theorem (Cantor Theorem)

$$|A| < |\mathcal{P}(A)|$$

$$A \qquad \mathcal{P}(A) \qquad \mathcal{P}(\mathcal{P}(A)) \qquad \ldots$$

There is no largest infinity.

$$<$$

Definition ($|A| \leq |B|$)

$|A| \leq |B|$ if there exists an *one-to-one* function f from A into B.

Q : What about onto function $f : A \to B$?

$|B| \leq |A|$ (Axiom of Choice)

Definition ($|A| < |B|$)

$|A| < |B| \iff |A| \le |B| \land |A| \ne |B|$

$$|\mathbb{N}| < |\mathbb{R}|$$

$$|X| < |2^X|$$

$$|\mathbb{N}| < |2^{\mathbb{N}}|$$

Definition (Countable Revisited)

X is countable:

$$(\exists n \in \mathbb{N} : |X| = n) \lor |X| = |\mathbb{N}|$$

Theorem (Proof for Countable)

X is countable iff

$$|X| \le |\mathbb{N}|.$$

X is countable iff there exists a one-to-one function

$$f : X \to \mathbb{N}.$$

Subsets of Countable Set (UD Corollary 23.4)

Every subset B of a countable set A is countable.

$$f : A \xrightarrow{1-1} \mathbb{N} \qquad g = f|_B$$

Slope (UD Problem 23.3 (a))

(a) The set of all lines with rational slopes

$$(\mathbb{Q}, \mathbb{R})$$

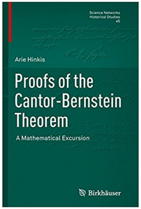
$$|\mathbb{R}| \leq |\mathbb{Q} \times \mathbb{R}| \leq |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$$

Q : Is "≤" a partial order?

Theorem (Cantor-Schröder–Bernstein (1887))

$$|X| \le |Y| \wedge |Y| \le |X| \implies |X| = |Y|$$

$$\exists \textit{ one-to-one } f : X \to Y \wedge g : Y \to X \implies \exists \textit{ bijection } h : X \to Y$$



Schröder–Bernstein theorem @ wiki

Q : Is "$\leq$" a total order?

Theorem (PCC)

Principle of Cardinal Comparability (PCC) $\iff$ *Axiom of Choice*

Theorem (UD Theorem 24.11)

$$|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$$

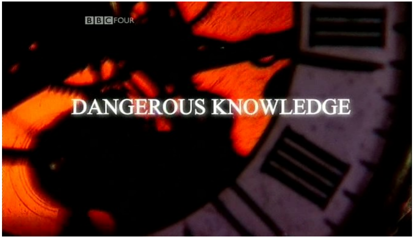
$$|\mathbb{R}| \le |\mathcal{P}(\mathbb{N})| \qquad |\mathcal{P}(\mathbb{N})| \le |\mathbb{R}|$$

$$\mathfrak{c} \triangleq |\mathbb{R}| = |\mathcal{P}(\mathbb{N})| = |2^{\mathbb{N}}| \triangleq 2^{\aleph_0}$$

$$\boxed{\mathfrak{c} = 2^{\aleph_0}}$$

Continuum Hypothesis (CH)

$$\boxed{\nexists A : \aleph_0 < |A| < \mathfrak{c}}$$



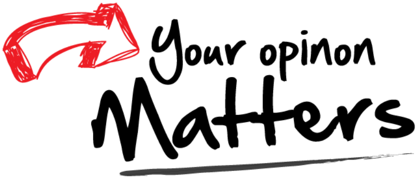
☞ Dangerous Knowledge (22:20; BBC 2007)

Independence from ZFC:

Kurt Gödel (1940) CH cannot be disproved from ZF.

Paul Cohen (1964) CH cannot be proven from the ZFC axioms.

Thank You!



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