2-3 Counting

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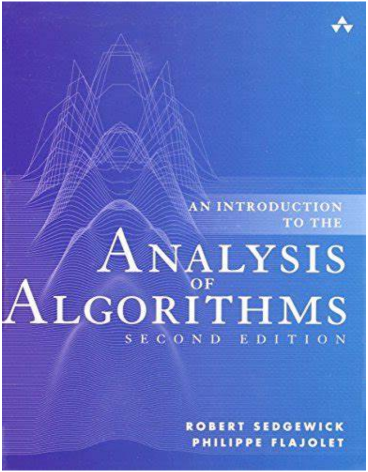
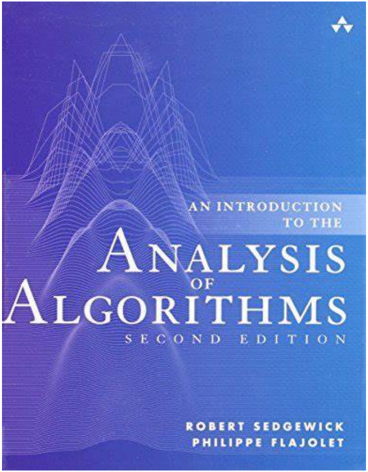
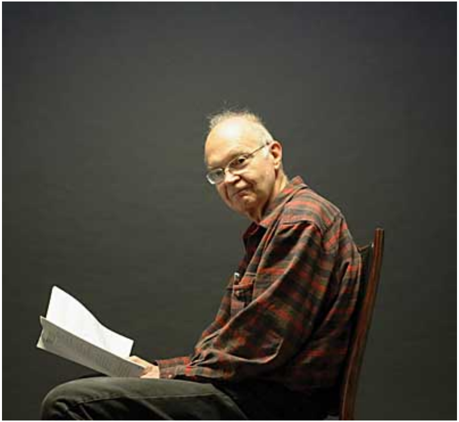

AN INTRODUCTION
TO THE
ANALYSIS
OF
ALGORITHMS
SECOND EDITION
ROBERT SEDGEWICK
PHILIPPE FLAJOLET



$O \qquad \Omega \qquad \Theta$

$o \qquad \omega$

"People who analyze algorithms have double happiness ..."



Donald E. Knuth (1938 $\sim$)

Unfortunately, you have to master some mathematics.

Counting

Counting

Sums $\sum$

Counting

Sums $\sum$

Binomials $\binom{n}{k}$


PRELIMINARY

Falling and Rising Factorials

$$m^{\underline{n}} = (m)_n = m(m-1)(m-2)\cdots(m-n+1) = \frac{m!}{(m-n)!}$$

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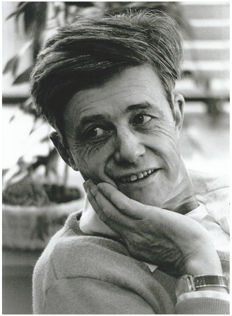
$$n! = n^{\underline{n}} = 1^{\bar{n}}$$

$$\binom{m}{n} = \frac{m^{\underline{n}}}{n!}$$

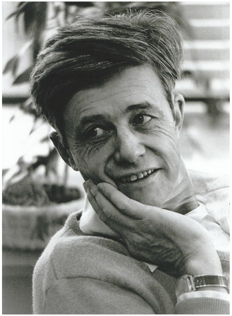
Iverson Bracket



Kenneth Eugene Iverson
(1920 $\sim$ 2004)

$$[P] = \begin{cases} 1, & \text{if } P \text{ is true;} \\ 0, & \text{otherwise} \end{cases}$$

Iverson Bracket



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$$[P] = \begin{cases} 1, & \text{if } P \text{ is true;} \\ 0, & \text{otherwise} \end{cases}$$

$$[n \le m] = \begin{cases} 1, & \text{if } n \le m; \\ 0, & \text{if } n > m \end{cases}$$

Theorem (Sum Principle)

$$S \cap T = \emptyset \implies |S \cup T| = |S| + |T|$$

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Theorem (Product Principle)

$$|S \times T| = |S| \times |T|$$

Holds for finite sets S and T.

先学习下加法，$1+1$，就是

所以$1+1=2$，这很好理解

那我们趁热打铁学习下一个重要公式吧：

$$\frac{\sum_{w \in W} (-1)^{\det(w)} w(e^{\lambda+\rho})}{e^{\rho} \prod_{\alpha>0} (1 - e^{-\alpha})}$$

Counting

Counting

Counting # of functions under (twelve) different restrictions

Counting

tuples

permutations

combinations

Counting # of functions under (twelve) different restrictions

Counting

tuples

permutations

combinations

compositions

set partitions

integer partitions

Counting # of functions under (twelve) different restrictions

Counting *vs.* Generating

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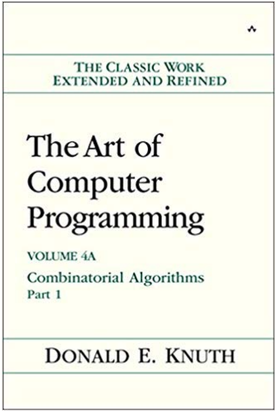

Generating is more about algorithms.

Counting # of functions under (twelve) different restrictions

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$$\boxed{f : N \to M \qquad (|N| = n, \quad |M| = m)}$$

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Elements of N	Elements of M	Any f	Injective f	Surjective f
distinguishable	*distinguishable*			
indistinguishable	*distinguishable*			
distinguishable	*indistinguishable*			
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Table: The Twelvefold Way (Functions).

Counting # of functions under (twelve) different restrictions

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Table: The Twelvefold Way (Functions).

$\boxed{\text{distinguishable } \textit{vs.} \text{ indistinguishable}}$

Balls	Bins	unrestricted	≤ 1	≥ 1
unlabeled	*unlabeled*			
labeled	*unlabeled*			
unlabeled	*labeled*			
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Table: The Twelvefold Way (Balls into Bins Model).

Balls	Bins	unrestricted	≤ 1	≥ 1
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unlabeled	*labeled*			
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Table: The Twelvefold Way (Balls into Bins Model).

labeled *vs.* unlabeled

2 balls, 3 bins (unrestricted)

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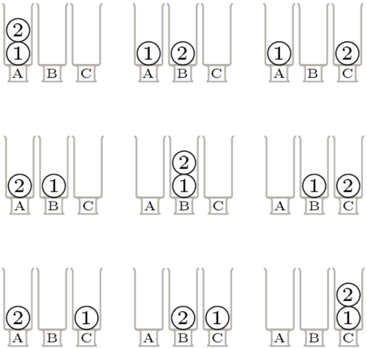
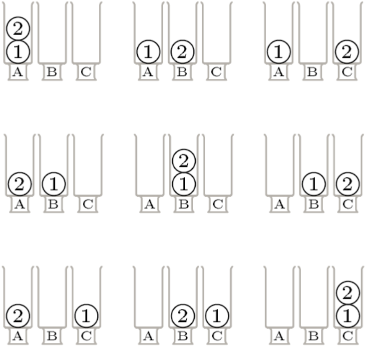
labeled balls into labeled bins



2 balls, 3 bins (unrestricted)

labeled balls into labeled bins



unlabeled balls into labeled bins

2 balls, 3 bins (unrestricted)

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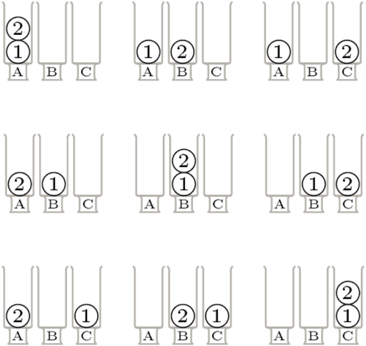

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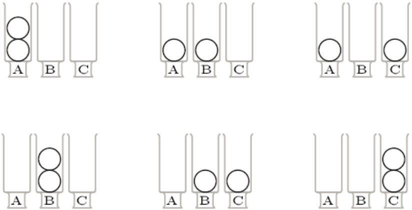
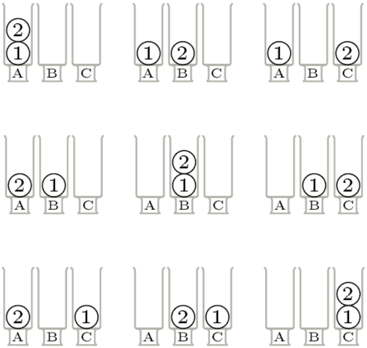


2 balls, 3 bins (unrestricted)

labeled balls into labeled bins



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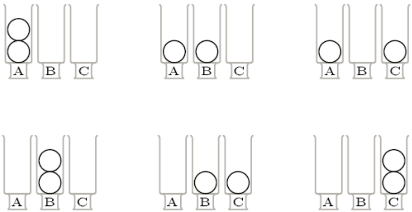
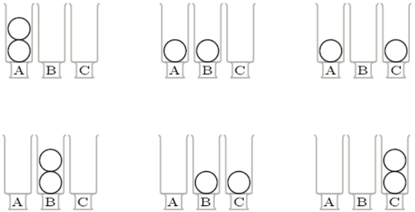


Only # of balls in each bin matters.

2 balls, 3 bins (unrestricted)

unlabeled balls into labeled bins



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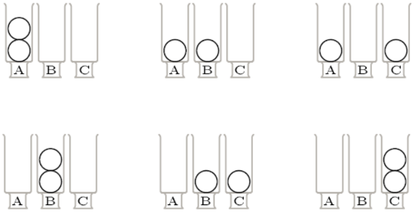
2 balls, 3 bins (unrestricted)

$$2 = 2 + 0 + 0$$

unlabeled balls into labeled bins



Only # of balls in each bin matters.

2 balls, 3 bins (unrestricted)

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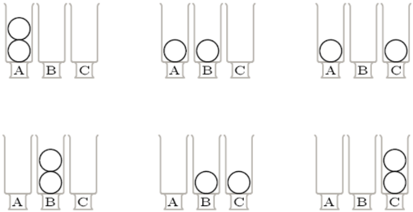
\begin{aligned}
2 &= 2 + 0 + 0 \\
&= 1 + 1 + 0
\end{aligned}
$$

unlabeled balls into labeled bins



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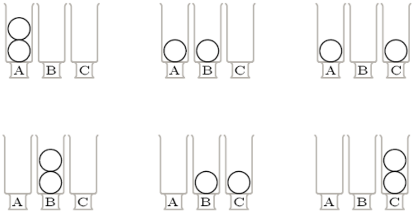

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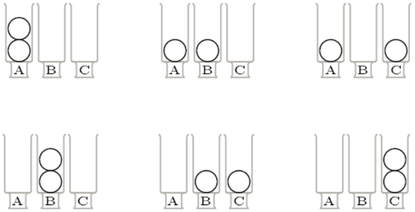

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$$

$$n = x_1 + x_2 + \ldots + x_m \quad (x_i \geq 0)$$

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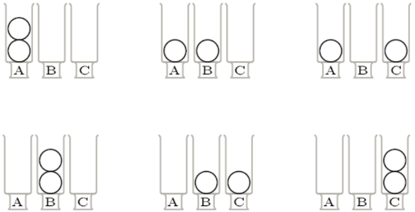
unlabeled balls into labeled bins



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weak composition of n with m terms

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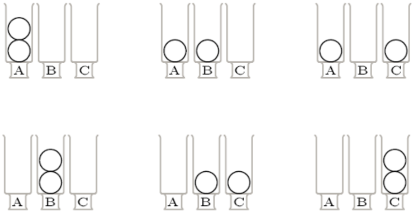
unlabeled balls into labeled bins



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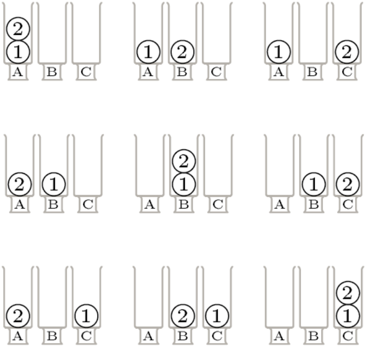
weak composition of n with m terms

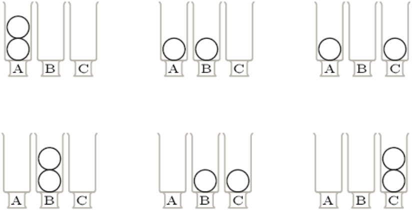
$$\boxed{\left(\!\!\binom{n}{m}\!\!\right)}$$

2 balls, 3 bins (unrestricted)

labeled balls into labeled bins



unlabeled balls into labeled bins



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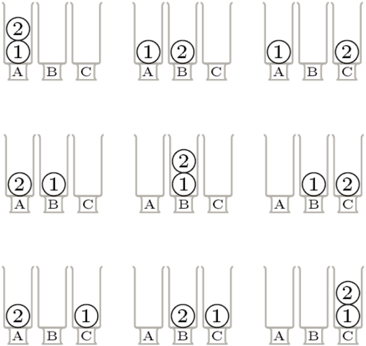

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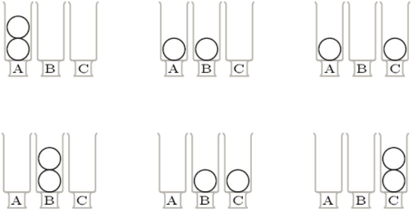
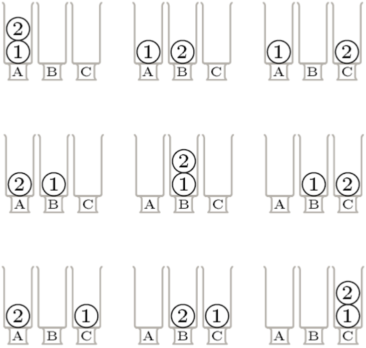


$$f : 1 \mapsto A, \quad 2 \mapsto B$$

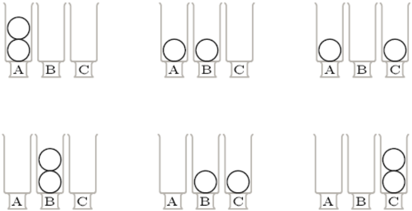
$$f' : 2 \mapsto A, \quad 1 \mapsto B$$

2 balls, 3 bins (unrestricted)

labeled balls into labeled bins



unlabeled balls into labeled bins



$$f : 1 \mapsto A, \quad 2 \mapsto B$$

$$f' : 2 \mapsto A, \quad 1 \mapsto B$$

$$f' = f \circ (h : 1 \mapsto 2, \quad 2 \mapsto 1)$$

f class	Any f	Injective f	Surjective f
f	n tuples of m items		
$f \circ S_N$	compositions of n into m parts		

Table: The Twelvefold Way (Functions).

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$$\boxed{f, g \text{ are indistinguishable} \iff g \in [f]}$$

2 balls, 3 bins (unrestricted)

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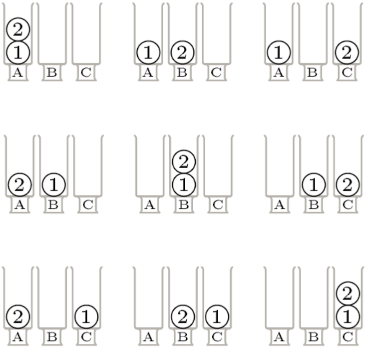
labeled balls into labeled bins



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labeled balls into unlabeled bins

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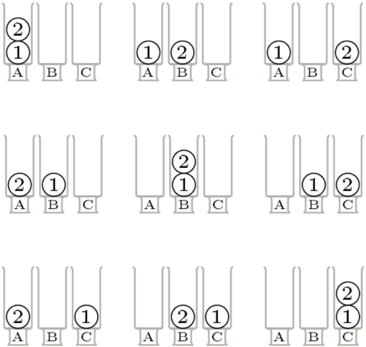

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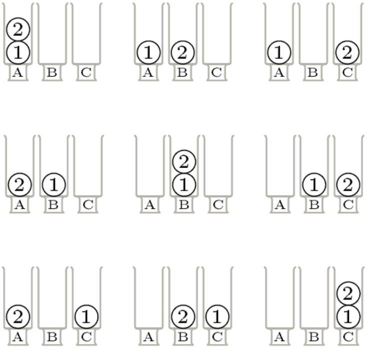

labeled balls into unlabeled bins

$$\{1, 2\} = \{1, 2\}$$

$$= \{1\} \cup \{2\}$$

2 balls, 3 bins (unrestricted)

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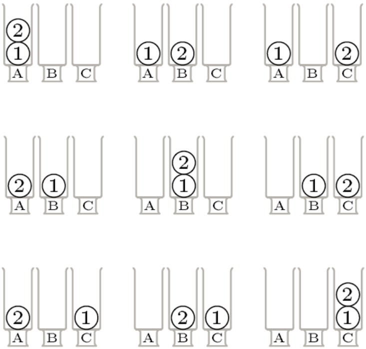

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$$\{1,2\} = \{1,2\}$$
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$\boxed{\text{partition of } N \text{ into } \leq m \text{ parts}}$

2 balls, 3 bins (unrestricted)

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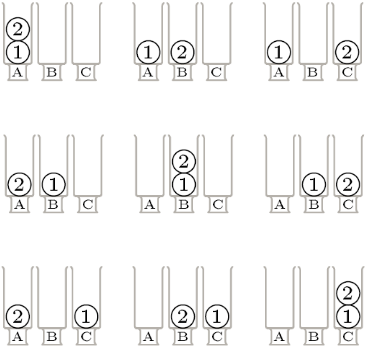

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$\left\{ n \atop k \right\}$: # of partitions of N into k pa

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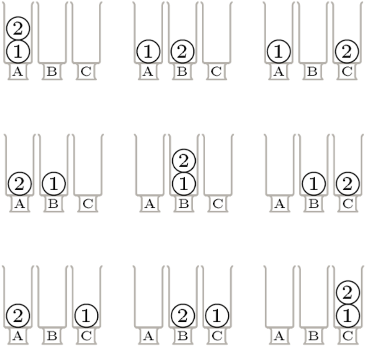

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$\left\{ {n \atop k} \right\}$: # of partitions of N into k pa

$$\boxed{\sum_{k=1}^{m} \left\{ {n \atop k} \right\}}$$

$$
\begin{aligned}
\{1,2,3\} &= \{1,2,3\} \\
&= \{1\} \cup \{2,3\} \\
&= \{2\} \cup \{1,3\} \\
&= \{3\} \cup \{1,2\} \\
&= \{1\} \cup \{2\} \cup \{3\}
\end{aligned}
$$

2 balls, 3 bins (unrestricted)

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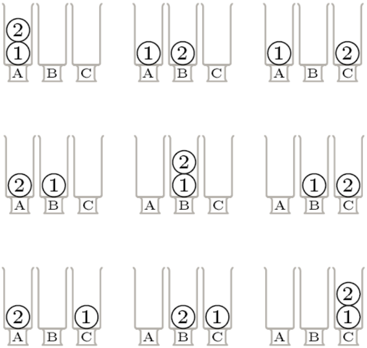
labeled balls into labeled bins



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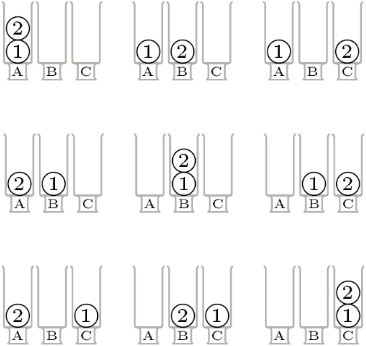


labeled balls into unlabeled bins

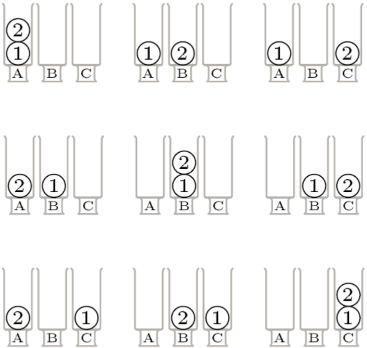
$$f : 1 \mapsto A, \quad 2 \mapsto A$$

$$g : 1 \mapsto B, \quad 2 \mapsto B$$

$$h : 1 \mapsto C, \quad 2 \mapsto C$$

2 balls, 3 bins (unrestricted)

labeled balls into labeled bins



labeled balls into unlabeled bins

$$f : 1 \mapsto A, \quad 2 \mapsto A$$

$$g : 1 \mapsto B, \quad 2 \mapsto B$$

$$h : 1 \mapsto C, \quad 2 \mapsto C$$

$$g = (l : A \mapsto B, \quad B \mapsto A) \circ f$$

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$f \circ S_N$	compositions of n into m parts		
$S_M \circ f$	partitions of N into $\leq m$ parts		

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$S_M \circ f \circ S_N$			

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Table: The Twelvefold Way (Functions).

$$
\begin{aligned}
5 &= 5 \\
&= 41 \\
&= 32 = 311 \\
&= 221 = 21111 \\
&= 11111
\end{aligned}
$$

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Weak composition of n with m terms

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$\boxed{\text{Placing } m-1 \text{ bars into } n+(m-1) \text{ slots.}}$

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Theorem

The # of *weak composition* *of* n *with* m *terms is*

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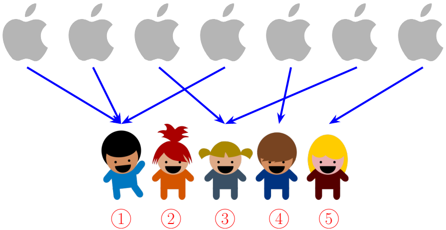

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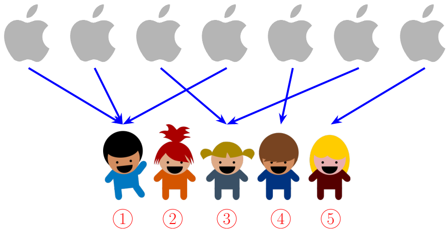

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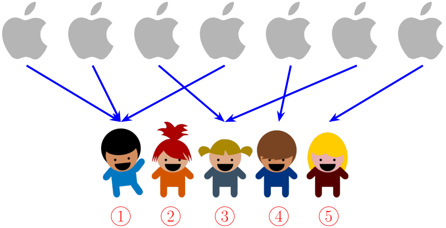
$$\{1, \quad 1, \quad 3, \quad 1, \quad 4, \quad 3, \quad 5\}$$



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$\left\{ n \atop k \right\}$: # of partitions of set N into k parts

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Stirling number of the second kind

Set Partition $(\mathrm{CS}: 1.5 - 12)$

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$$\text{Bell number: } B_n = \sum_{k=1}^{k=n} \left\{ n \atop k \right\}$$

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$$p(n) \triangleq \sum_{k=1}^{k=n} \left| \begin{matrix} n \\ k \end{matrix} \right| \sim \frac{1}{4\sqrt{3}n} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

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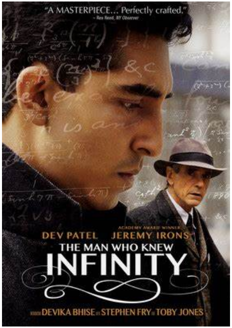

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Table: The Twelvefold Way (Functions).

THE TWELVEFOLD WAY

balls per urn	unrestricted	≤ 1	≥ 1
n labeled balls, m labeled urns	n-tuples of m things	n-permutations of m things	partitions of $\{1, \ldots, n\}$ into m ordered parts
n unlabeled balls, m labeled urns	n-multicombinations of m things	n-combinations of m things	compositions of n into m parts
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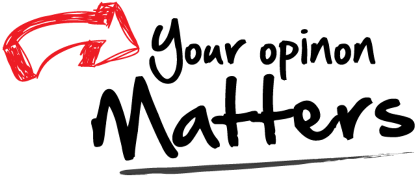
Thank You!



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