

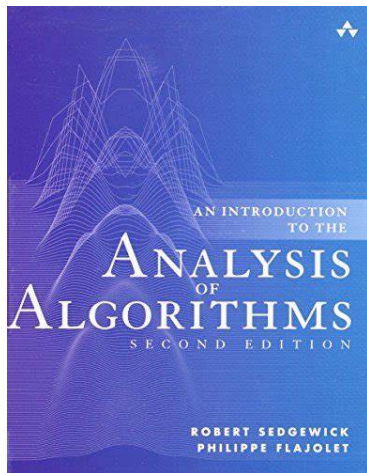
2-3 Counting

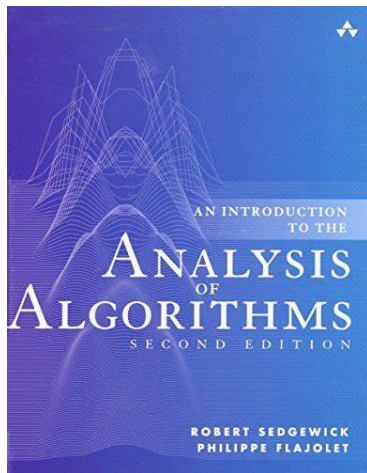
Hengfeng Wei

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March 12, 2020



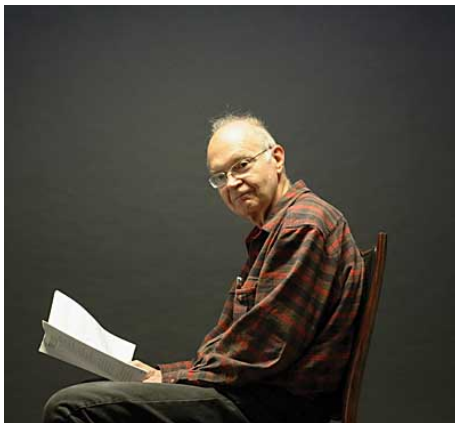




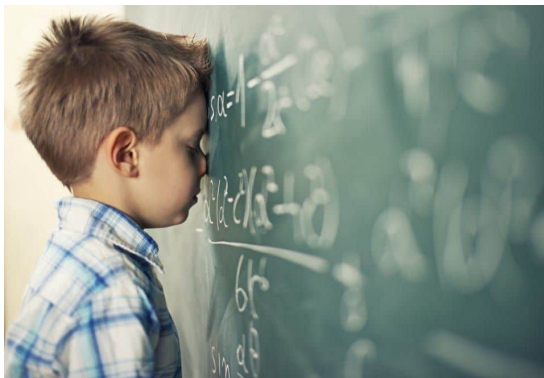
O Ω Θ

o ω

*“People who **analyze algorithms** have **double happiness** ...”*



Donald E. Knuth (1938 ~)



Unfortunately, you have to master some **mathematics**.



Counting



Counting

Sums Σ



Counting

Sums Σ

Binomials $\binom{n}{k}$

PRELIMINARY

Falling and Rising Factorials

$$m^{\underline{n}} = (m)_n = m(m-1)(m-2)\cdots(m-n+1) = \frac{m!}{(m-n)!}$$

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Falling and Rising Factorials

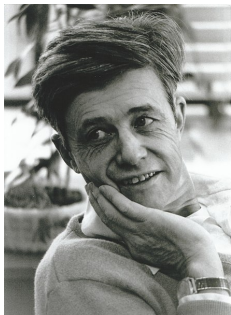
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$$\binom{m}{n} = \frac{m^{\underline{n}}}{n!}$$

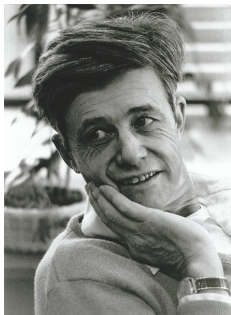
Iverson Bracket



$$[P] = \begin{cases} 1, & \text{if } P \text{ is true;} \\ 0, & \text{otherwise} \end{cases}$$

Kenneth Eugene Iverson
(1920 ~ 2004)

Iverson Bracket



$$[P] = \begin{cases} 1, & \text{if } P \text{ is true;} \\ 0, & \text{otherwise} \end{cases}$$

$$[n \leq m] = \begin{cases} 1, & \text{if } n \leq m; \\ 0, & \text{if } n > m \end{cases}$$

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Theorem (Sum Principle)

$$S \cap T = \emptyset \implies |S \cup T| = |S| + |T|$$

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$$|S \times T| = |S| \times |T|$$

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Theorem (Product Principle)

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Holds for **finite** sets S and T .

先学习下加法， $1 + 1$ ，就是



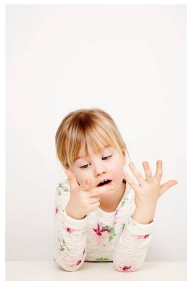
所以 $1 + 1 = 2$ ，这很好理解

那我们趁热打铁学习下一个重要公式吧：

$$\frac{\sum_{w \in W} (-1)^{\det(w)} w(e^{\lambda + \rho})}{e^{\rho} \prod_{\alpha > 0} (1 - e^{-\alpha})}$$



Counting



Counting



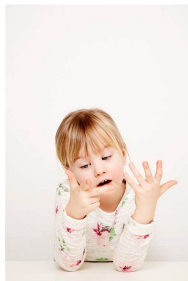
Counting # of functions under (twelve) different restrictions

Counting

tuples

permutations

combinations



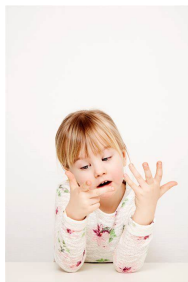
Counting # of functions under (twelve) different restrictions

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compositions

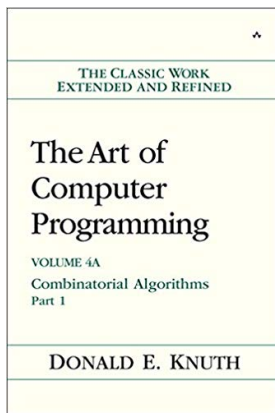
set partitions

integer partitions

Counting # of functions under (twelve) different restrictions

Counting *vs.* Generating

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Generating is more about algorithms.

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Elements of N	Elements of M	Any f	Injective f	Surjective f
<i>distinguishable</i>	<i>distinguishable</i>			
<i>indistinguishable</i>	<i>distinguishable</i>			
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Table: The Twelfold Way (Functions).

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Table: The Twelfold Way (Functions).

distinguishable *vs.* indistinguishable

Balls	Bins	unrestricted	≤ 1	≥ 1
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<i>labeled</i>	<i>unlabeled</i>			
<i>unlabeled</i>	<i>labeled</i>			
<i>labeled</i>	<i>labeled</i>			

Table: The Twelfold Way (Balls into Bins Model).

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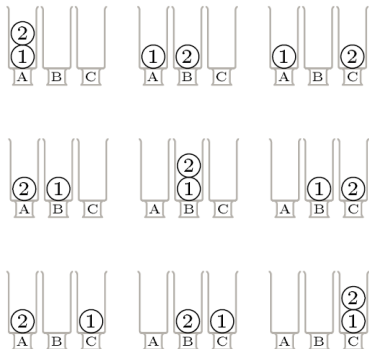
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labeled *vs.* unlabeled

2 balls, 3 bins (unrestricted)

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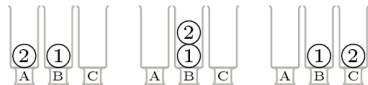
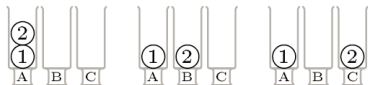
labeled balls into labeled bins



2 balls, 3 bins

(unrestricted)

labeled balls into labeled bins

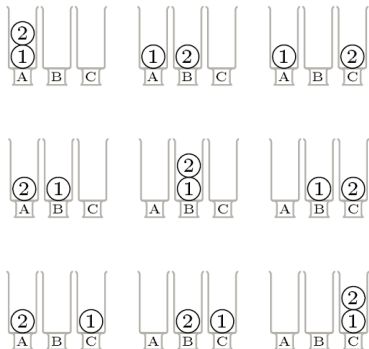


unlabeled balls into labeled bins

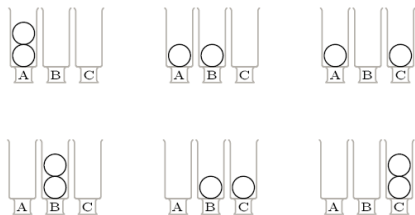
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(unrestricted)

labeled balls into labeled bins



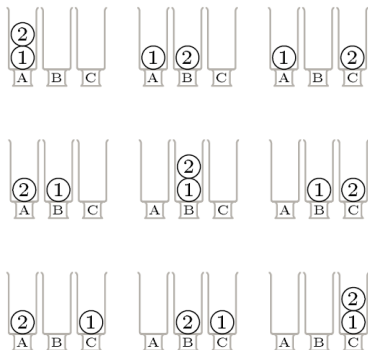
unlabeled balls into labeled bins



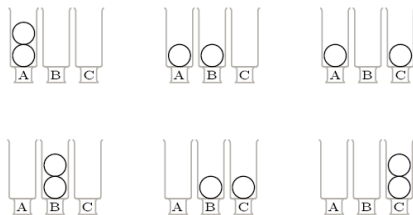
2 balls, 3 bins

(unrestricted)

labeled balls into labeled bins



unlabeled balls into labeled bins

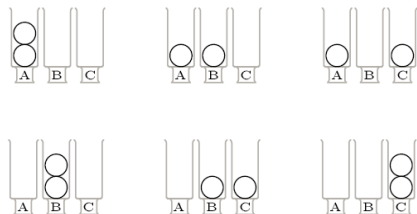


Only # of balls in each bin matters.

2 balls, 3 bins

(unrestricted)

unlabeled balls into labeled bins



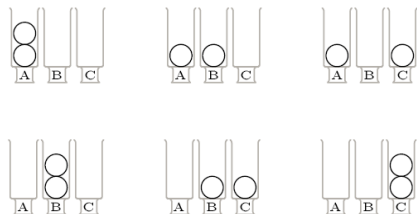
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2 balls, 3 bins

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$$2 = 2 + 0 + 0$$

unlabeled balls into labeled bins



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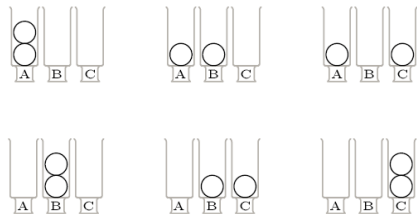
2 balls, 3 bins

(unrestricted)

$$2 = 2 + 0 + 0$$

$$= 1 + 1 + 0$$

unlabeled balls into labeled bins

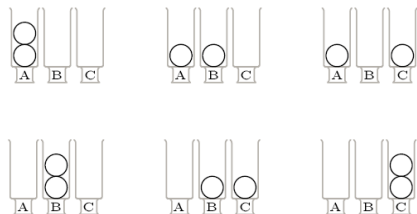


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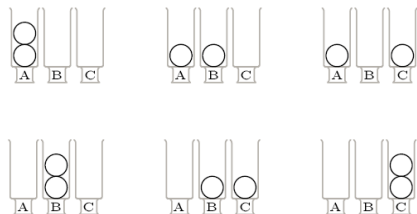
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$$= 0 + 0 + 2$$

2 balls, 3 bins

(unrestricted)

unlabeled balls into labeled bins



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$$n = x_1 + x_2 + \dots + x_m \quad (x_i \geq 0)$$

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2 balls, 3 bins

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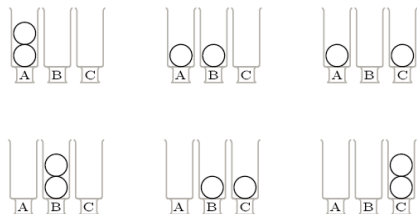
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unlabeled balls into labeled bins



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Only # of balls in each bin matters. weak composition of n with m terms

(unrestricted)

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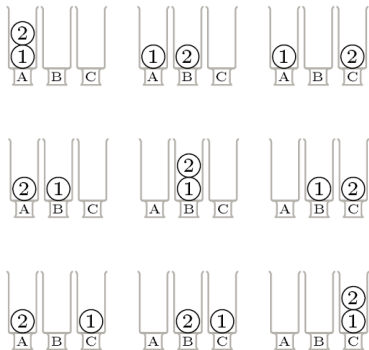
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$$\left(\binom{n}{k} \right)$$

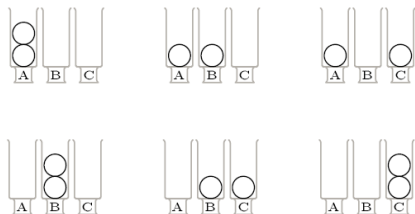
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(unrestricted)

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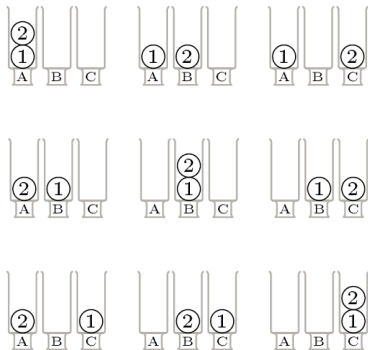
unlabeled balls into labeled bins



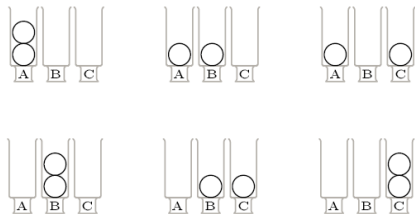
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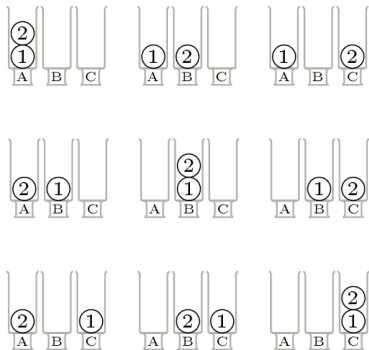
$$f : 1 \mapsto A, \quad 2 \mapsto B$$

$$f' : 2 \mapsto A, \quad 1 \mapsto B$$

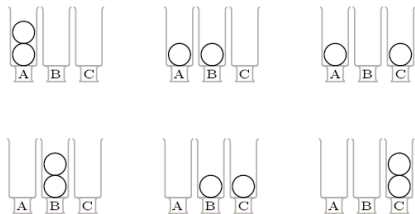
2 balls, 3 bins

(unrestricted)

labeled balls into labeled bins



unlabeled balls into labeled bins



$$f : 1 \mapsto A, \quad 2 \mapsto B$$

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$$f' = f \circ (h : 1 \mapsto 2, \quad 2 \mapsto 1)$$

f class	Any f	Injective f	Surjective f
f	n tuples of m items		
$f \circ S_N$	compositions of n into m parts		

Table: The Twelfold Way (Functions).

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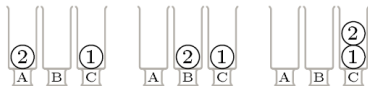
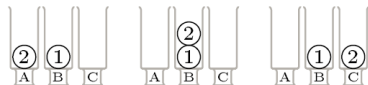
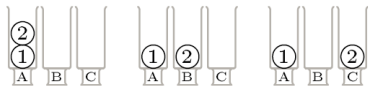
$$f, g \text{ are indistinguishable} \iff g \in [f]$$

2 balls, 3 bins (unrestricted)

2 balls, 3 bins

(unrestricted)

labeled balls into labeled bins

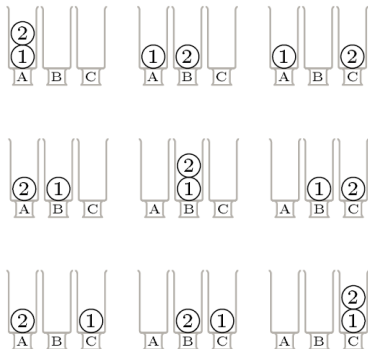


2 balls, 3 bins

(unrestricted)

labeled balls into unlabeled bins

labeled balls into labeled bins



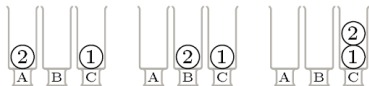
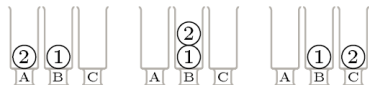
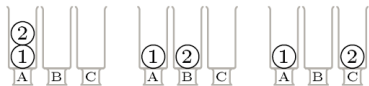
2 balls, 3 bins

(unrestricted)

labeled balls into unlabeled bins

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$$\begin{aligned}\{1, 2\} &= \{1, 2\} \\ &= \{1\} \cup \{2\}\end{aligned}$$

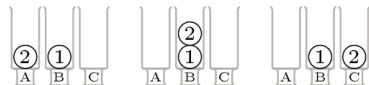
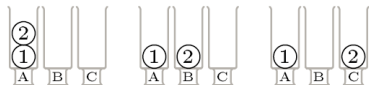


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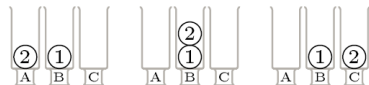
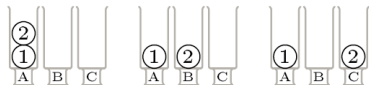
partition of N into $\leq m$ parts

2 balls, 3 bins

(unrestricted)

labeled balls into unlabeled bins

labeled balls into labeled bins



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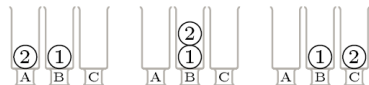
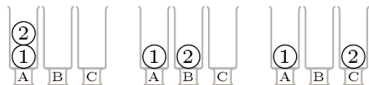
$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$: # of partitions of N into k parts

2 balls, 3 bins

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partition of N into $\leq m$ parts

$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$: # of partitions of N into k parts

$$\sum_{k=1}^m \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$$

$$\begin{aligned}\{1, 2, 3\} &= \{1, 2, 3\} \\ &= \{1\} \cup \{2, 3\} \\ &= \{2\} \cup \{1, 3\} \\ &= \{3\} \cup \{1, 2\} \\ &= \{1\} \cup \{2\} \cup \{3\}\end{aligned}$$

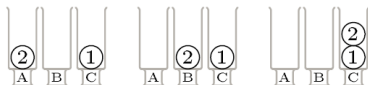
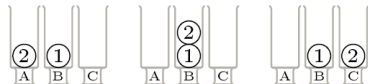
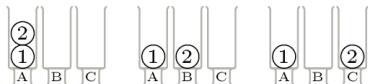
2 balls, 3 bins (unrestricted)

2 balls, 3 bins

(unrestricted)

labeled balls into labeled bins

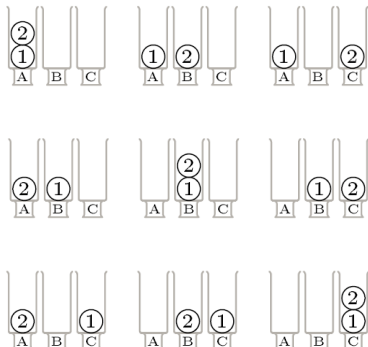
labeled balls into unlabeled bins



2 balls, 3 bins

(unrestricted)

labeled balls into labeled bins



labeled balls into unlabeled bins

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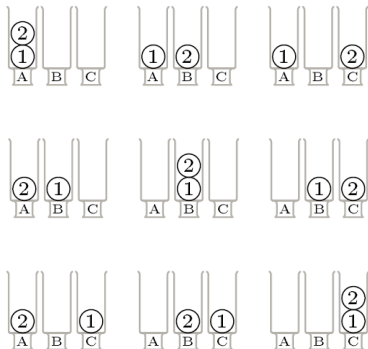
$$g : 1 \mapsto B, \quad 2 \mapsto B$$

$$h : 1 \mapsto C, \quad 2 \mapsto C$$

2 balls, 3 bins

(unrestricted)

labeled balls into labeled bins



labeled balls into unlabeled bins

$$f : 1 \mapsto A, \quad 2 \mapsto A$$

$$g : 1 \mapsto B, \quad 2 \mapsto B$$

$$h : 1 \mapsto C, \quad 2 \mapsto C$$

$$g = (l : A \mapsto B, \quad B \mapsto A) \circ f$$

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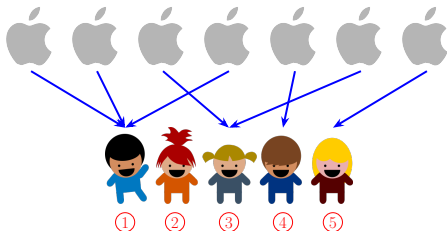
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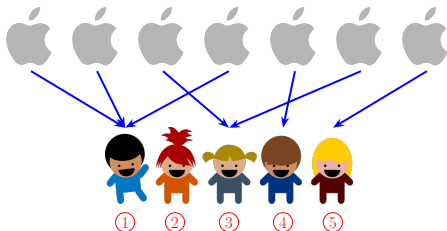
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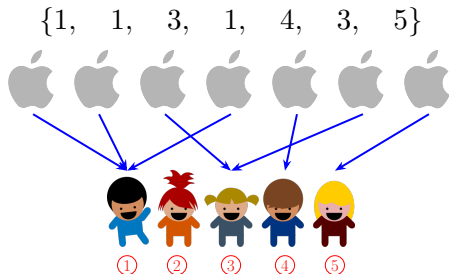
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$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$: # of partitions of set N into k parts

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Stirling number of the second kind

Set Partition (CS : 1.5 – 12)

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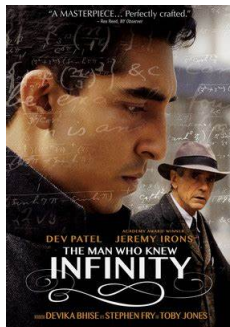
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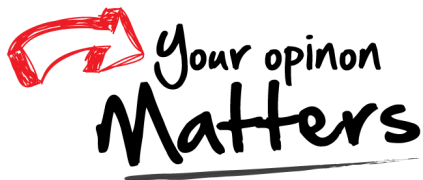
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f	n -tuples of m items m^n	n -permutations of m items m^n	???
$f \circ S_N$	weak compositions of n into m parts $\left(\binom{n}{m}\right) = \binom{n+m-1}{m-1} = \binom{n+m-1}{n}$	n -combinations of m items $\binom{m}{n}$	compositions of n into m parts $\binom{n-1}{m-1}$
$S_M \circ f$	partitions of N into $\leq m$ parts $\sum_{k=1}^m \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$	n pigeons into m holes $[n \leq m]$	partitions of N into m parts $\left\{ \begin{matrix} n \\ m \end{matrix} \right\}$
$S_M \circ f \circ S_N$	partitions of n into $\leq m$ parts $\sum_{k=1}^m \left \begin{matrix} n \\ k \end{matrix} \right $	n pigeons into m holes $[n \leq m]$	partitions of n into m parts $\left \begin{matrix} n \\ m \end{matrix} \right $

Table: The Twelfold Way (Functions).

THE TWELVEFOLD WAY

<i>balls per urn</i>	unrestricted	≤ 1	≥ 1
n labeled balls, m labeled urns	n -tuples of m things	n -permutations of m things	partitions of $\{1, \dots, n\}$ into m ordered parts
n unlabeled balls, m labeled urns	n -multicombinations of m things	n -combinations of m things	compositions of n into m parts
n labeled balls, m unlabeled urns	partitions of $\{1, \dots, n\}$ into $\leq m$ parts	n pigeons into m holes	partitions of $\{1, \dots, n\}$ into m parts
n unlabeled balls, m unlabeled urns	partitions of n into $\leq m$ parts	n pigeons into m holes	partitions of n into m parts

Thank
You!



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