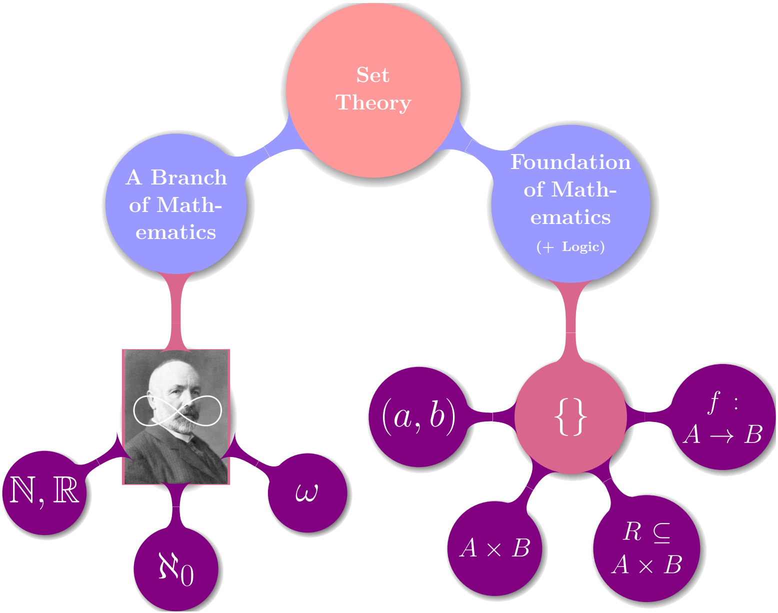
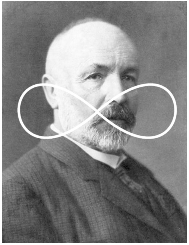
1-11 Set Theory (IV): Infinity

魏恒峰

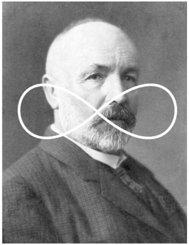
hfwei@nju.edu.cn

2019 年 12 月 17 日


Set Theory
A Branch of Mathematics
Foundation of Mathematics
(+ Logic)
$\mathbb{N}, \mathbb{R}$
$\aleph_0$
ω
(a, b)
$\{\}$
$f : A \to B$
$A \times B$
$R \subseteq A \times B$

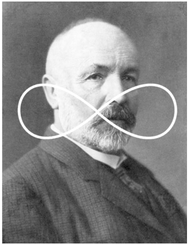


Georg Cantor (1845 – 1918)



Georg Cantor (1845 – 1918)

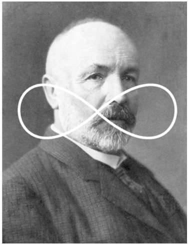
Leopold Kronecker
(1823 – 1891)



Georg Cantor (1845 – 1918)

Leopold Kronecker
(1823 – 1891)

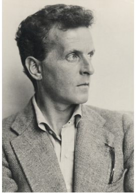
Henri Poincaré
(1854 – 1912)



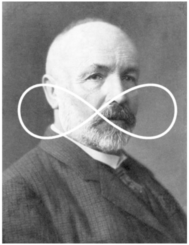
Georg Cantor (1845 – 1918)

Leopold Kronecker
(1823 – 1891)

Henri Poincaré
(1854 – 1912)



Ludwig Wittgenstein
(1889 – 1951)

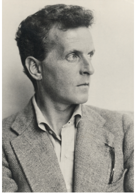


Georg Cantor (1845 – 1918)

David Hilbert (1862 – 1943)

Leopold Kronecker
(1823 – 1891)

Henri Poincaré
(1854 – 1912)



Ludwig Wittgenstein
(1889 – 1951)

From his paradise that Cantor with us unfolded, we hold our breath in awe; knowing, we shall not be expelled.

— *David Hilbert*

"没有人能把我们从 *Cantor* 创造的乐园中驱逐出去"

From his paradise that Cantor with us unfolded, we hold our breath in awe; knowing, we shall not be expelled.

— *David Hilbert*

"没有人能把我们从 *Cantor* 创造的乐园中驱逐出去"

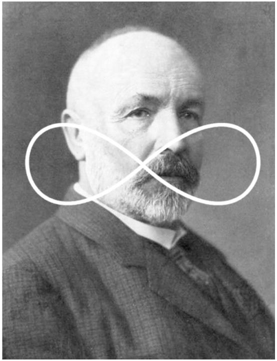

GEORG CANTOR
MATHEMATIKER
BEGRÜNDER DER MENGENLEHRE
1845 - 1918
DAS WESEN DER MATHEMA
TIK LIEGT IN IHRER FREIHEIT
GEORG CANTOR

"das wesen der mathematik liegt in ihrer freiheit"

"das wesen der mathematik liegt in ihrer freiheit"

"The essence of mathematics lies in its freedom"

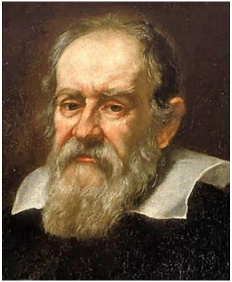
Before Cantor



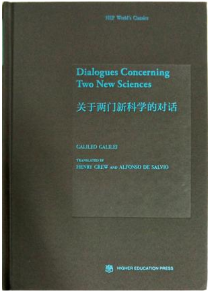

决定经典
001
全新修订版
[古希腊] 欧几里得 / 著
几何原本
建立空间秩序最久远最权威的逻辑推演语系
Euclid's Elements
13卷图释全本


决定经典
001
全新修订版
[古希腊] 欧几里得 / 著
几何原本
建立空间秩序最久远最权威的逻辑推演语系
Euclid's Elements
13卷图释全本

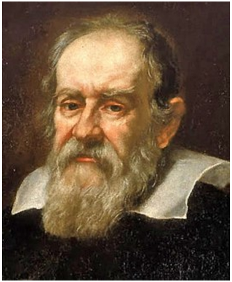
公理: “整体大于部分”

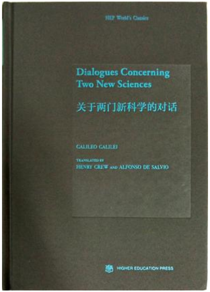


Galileo Galilei (1564 – 1642)



“关于两门新科学的对话” (1638)





Galileo Galilei (1564 – 1642)

“关于两门新科学的对话” (1638)

“用我们有限的心智来讨论无限 · · · ”

$$S_1 = \{1, 2, 3, \cdots, n, \cdots\}$$
$$S_2 = \{1, 4, 9, \cdots, n^2, \cdots\}$$

$$S_1 = \{1, 2, 3, \cdots, n, \cdots\}$$
$$S_2 = \{1, 4, 9, \cdots, n^2, \cdots\}$$

$|S_1| = |S_2| \qquad S_2 \subset S_1$

$$S_1 = \{1, 2, 3, \cdots, n, \cdots\}$$
$$S_2 = \{1, 4, 9, \cdots, n^2, \cdots\}$$

$|S_1| = |S_2| \qquad S_2 \subset S_1$

"部分等于全体"

$$S_1 = \{1, 2, 3, \cdots, n, \cdots\}$$

$$S_2 = \{1, 4, 9, \cdots, n^2, \cdots\}$$

$|S_1| = |S_2| \qquad S_2 \subset S_1$

“部分等于全体”

$$S_1 = \{1, 2, 3, \cdots, n, \cdots\}$$
$$S_2 = \{1, 4, 9, \cdots, n^2, \cdots\}$$

$|S_1| = |S_2| \qquad S_2 \subset S_1$

"部分等于全体"

说到底，"等于"、"大于"和"小于"诸性质不能用于无限，而只能用于有限的数量。 — *Galileo Galilei*

$$S_1 = \{1, 2, 3, \cdots, n, \cdots\}$$

$$S_2 = \{1, 4, 9, \cdots, n^2, \cdots\}$$

$|S_1| = |S_2| \qquad S_2 \subset S_1$

“部分等于全体”

说到底，“等于”、“大于”和“小于”诸性质不能用于无限，而只能用于有限的数量。 — *Galileo Galilei*

无穷数是不可能的。 — *Gottfried Wilhelm Leibniz*

这些证明一开始就期望那些数要具有有穷数的一切性质，或者甚至于把有穷数的性质强加于无穷。

相反，这些无穷数，如果它们能够以任何形式被理解的话，倒是由于它们与有穷数的对应，它们必须具有完全新的数量特征。

这些性质完全依赖于事物的本性，$\cdots$ 而并非来自我们的主观任意性或我们的偏见。

— *Georg Cantor (1885)*

Definition (Dedekind-infinite & Dedekind-finite (Dedekind, 1888))

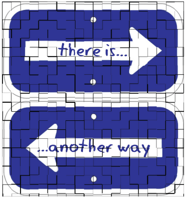
A set A is *Dedekind-infinite* if there is a bijective function from A onto some proper subset B of A.

A set is *Dedekind-finite* if it is not Dedekind-infinite.

Definition (Dedekind-infinite & Dedekind-finite (Dedekind, 1888))

A set A is *Dedekind-infinite* if there is a bijective function from A onto some proper subset B of A.

A set is *Dedekind-finite* if it is not Dedekind-infinite.



This is a theorem in our theory of infinity.

We have not defined "finite" and "infinite"!

Comparing Sets

Comparing Sets

$$= \qquad\qquad\qquad <$$

Comparing Sets

$$= \qquad\qquad <$$

Function

$$=$$

Definition ($|A| = |B|$ ($A \approx B$) (1878))

A and B are *equipotent* if there exists a *bijection* from A to B.

Definition ($|A| = |B|$ ($A \approx B$) (1878))
A and B are *equipotent* if there exists a *bijection* from A to B.

$$\overline{\overline{A}} \quad \text{(two abstractions)}$$

Definition ($|A| = |B|$ ($A \approx B$) (1878))

A and B are *equipotent* if there exists a *bijection* from A to B.

$$\overline{\overline{A}} \quad \text{(two abstractions)}$$

Abstract from elements: $\{1, 2, 3\} \quad vs. \quad \{a, b, c\}$

Definition ($|A| = |B|$ ($A \approx B$) (1878))

A and B are *equipotent* if there exists a *bijection* from A to B.

$$\overline{\overline{A}} \quad \text{(two abstractions)}$$

Abstract from elements: $\{1, 2, 3\} \quad vs. \quad \{a, b, c\}$

Abstract from order: $\{1, 2, 3, \cdots\} \quad vs. \quad \{1, 3, 5, \cdots, 2, 4, 6, \cdots\}$

Definition ($|A| = |B|$ ($A \approx B$) (1878))

A and B are *equipotent* if there exists a *bijection* from A to B.

Q : Is "$\approx$" an equivalence relation?

Definition ($|A| = |B|$ ($A \approx B$) (1878))

A and B are *equipotent* if there exists a *bijection* from A to B.

Q : Is "$\approx$" an equivalence relation?

Theorem ( )

For any sets A, B, C:

(a) $A \approx B$

(b) $A \approx B \implies B \approx A$

(c) $A \approx B \land B \approx C \implies A \approx C$

Definition ($|A| = |B|$ ($A \approx B$) (1878))

A and B are *equipotent* if there exists a *bijection* from A to B.

Q : Is "$\approx$" an equivalence relation?

Theorem (The "Equivalence Concept" of Equipotent)

For any sets A, B, C:

(a) $A \approx B$

(b) $A \approx B \implies B \approx A$

(c) $A \approx B \land B \approx C \implies A \approx C$

Definition (Finite)

X is finite if

$$\exists n \in \mathbb{N} : |X| = n.$$

Definition (Finite)

X is finite if

$$\exists n \in \mathbb{N} : |X| = n.$$

$$|X| = |\{0, 1, \cdots, n-1\}|$$

Definition (Finite)

X is finite if

$$\exists n \in \mathbb{N} : |X| = n.$$

$$|X| = |\{0, 1, \cdots, n-1\}|$$

Theorem (UD Theorem 22.6)

Let A be a finite set. There is a unique $n \in \mathbb{N}$ such that $A \approx \{0, 1, \cdots, n-1\}$.

Definition (Infinite)

X is infinite if it is not finite:

$$\forall n \in \mathbb{N} : |X| \neq n.$$

Definition (Infinite)

X is infinite if it is not finite:

$$\forall n \in \mathbb{N} : |X| \neq n.$$

Theorem (UD Theorem 22.3)

$\mathbb{N}$ *is infinite.*

Definition (Infinite)

X is infinite if it is not finite:

$$\forall n \in \mathbb{N} : |X| \neq n.$$

Theorem (UD Theorem 22.3)

$\mathbb{N}$ *is infinite.*

By Contradiction.

Definition (Infinite)

X is infinite if it is not finite:

$$\forall n \in \mathbb{N} : |X| \neq n.$$

Theorem (UD Theorem 22.3)

$\mathbb{N}$ *is infinite.*

By Contradiction.

$$\exists n \in \mathbb{N} : |\mathbb{N}| = n.$$

Definition (Infinite)

X is infinite if it is not finite:

$$\forall n \in \mathbb{N} : |X| \neq n.$$

Theorem (UD Theorem 22.3)

$\mathbb{N}$ *is infinite.*

By Contradiction.

$$\exists n \in \mathbb{N} : |\mathbb{N}| = n.$$

$$\exists f : \mathbb{N} \xleftrightarrow[onto]{1-1} \{0, 1, \cdots, n-1\}$$

Definition (Infinite)

X is infinite if it is not finite:

$$\forall n \in \mathbb{N} : |X| \neq n.$$

Theorem (UD Theorem 22.3)

$\mathbb{N}$ *is infinite.*

By Contradiction.

$$\exists n \in \mathbb{N} : |\mathbb{N}| = n.$$

$$\exists f : \mathbb{N} \xleftrightarrow[onto]{1-1} \{0, 1, \cdots, n-1\}$$

$$g \triangleq f|_{\{0,1,\ldots,n\}} : \{0, 1, \cdots, n\} \to \{0, 1, \cdots, n-1\}$$

Definition (Infinite)

X is infinite if it is not finite:

$$\forall n \in \mathbb{N} : |X| \neq n.$$

Theorem (UD Theorem 22.3)

$\mathbb{N}$ is infinite.

By Contradiction.

$$\exists n \in \mathbb{N} : |\mathbb{N}| = n.$$

$$\exists f : \mathbb{N} \xleftrightarrow[onto]{1-1} \{0, 1, \cdots, n-1\}$$

$$g \triangleq f|_{\{0,1,\ldots,n\}} : \{0, 1, \cdots, n\} \to \{0, 1, \cdots, n-1\}$$

By the Pigeonhole Principle : g is not 1-1

Definition (Infinite)

X is infinite if it is not finite:

$$\forall n \in \mathbb{N} : |X| \neq n.$$

Theorem (UD Theorem 22.3)

$\mathbb{N}$ *is infinite.* *(So are $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$.)*

By Contradiction.

$$\exists n \in \mathbb{N} : |\mathbb{N}| = n.$$

$$\exists f : \mathbb{N} \xleftrightarrow[onto]{1-1} \{0, 1, \cdots, n-1\}$$

$$g \triangleq f|_{\{0,1,\ldots,n\}} : \{0, 1, \cdots, n\} \to \{0, 1, \cdots, n-1\}$$

By the Pigeonhole Principle : g is not 1-1 $\implies$ f is not 1-1

Definition (Infinite)

For any set X,

Countably Infinite

$$|X| = |\mathbb{N}| \triangleq \aleph_0$$

Countable

$$(\text{finite} \lor \text{countably infinite})$$

Uncountable

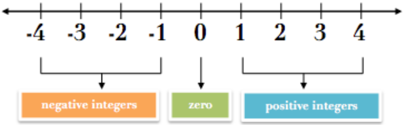
$$(\neg \text{ countable})$$

$$(\text{infinite}) \land \Big(\neg \text{ (countably infinite)}\Big)$$

$$\aleph_0$$

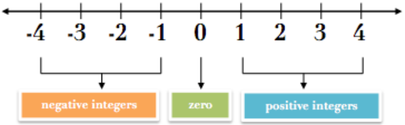
Theorem ($\mathbb{Z}$ is Countable.)

$$|\mathbb{Z}| = |\mathbb{N}|$$



Theorem ($\mathbb{Z}$ is Countable.)

$$|\mathbb{Z}| = |\mathbb{N}|$$



$$0 \quad 1 \quad -1 \quad 2 \quad -2 \quad \cdots$$

Theorem ($\mathbb{Q}$ is Countable. (Cantor 1873-11; Published in 1874))

$$|\mathbb{Q}| = |\mathbb{N}|$$

Theorem ($\mathbb{Q}$ is Countable. (Cantor 1873-11; Published in 1874))

$$|\mathbb{Q}| = |\mathbb{N}|$$

$|\mathbb{Q}| = |\mathbb{N}|$ (UD Problem 23.12)

$$q \in \mathbb{Q}^{+} : a/b \; (a, b \in \mathbb{N}^{+})$$

Theorem ($\mathbb{Q}$ is Countable. (Cantor 1873-11; Published in 1874))

$$|\mathbb{Q}| = |\mathbb{N}|$$

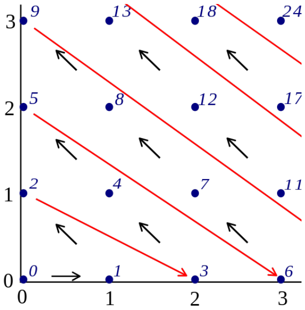
$|\mathbb{Q}| = |\mathbb{N}|$ (UD Problem 23.12)

$$q \in \mathbb{Q}^{+} : a/b \; (a, b \in \mathbb{N}^{+})$$

1/1	1/2	1/3	1/4	1/5	1/6	1/7	1/8	⋯
2/1	2/2	2/3	2/4	2/5	2/6	2/7	2/8	⋯
3/1	3/2	3/3	3/4	3/5	3/6	3/7	3/8	⋯
4/1	4/2	4/3	4/4	4/5	4/6	4/7	4/8	⋯
5/1	5/2	5/3	5/4	5/5	5/6	5/7	5/8	⋯
6/1	6/2	6/3	6/4	6/5	6/6	6/7	6/8	⋯
7/1	7/2	7/3	7/4	7/5	7/6	7/7	7/8	⋯
8/1	8/2	8/3	8/4	8/5	8/6	8/7	8/8	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

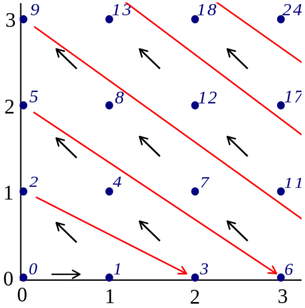
Theorem ($\mathbb{N} \times \mathbb{N}$ is Countable.)

$$|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$$



Theorem ($\mathbb{N} \times \mathbb{N}$ is Countable.)

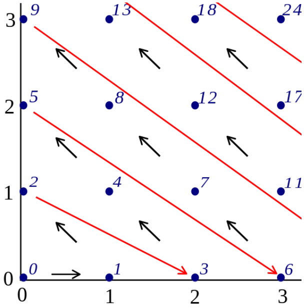
$$|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$$



$$\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

Theorem ($\mathbb{N} \times \mathbb{N}$ is Countable.)

$$|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$$

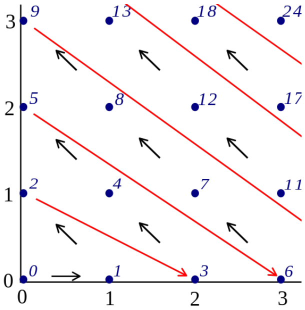


$$\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

$$\pi(k_1, k_2) = \frac{1}{2}(k_1+k_2)(k_1+k_2+1)+k_2$$

Theorem ($\mathbb{N} \times \mathbb{N}$ is Countable.)

$$|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$$



$$\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

$$\pi(k_1, k_2) = \frac{1}{2}(k_1+k_2)(k_1+k_2+1)+k_2$$

Cantor Pairing Function

Theorem ($\mathbb{N}^n$ is Countable.)

$$|\mathbb{N}^n| = |\mathbb{N}|$$

Theorem ($\mathbb{N}^n$ is Countable.)

$$|\mathbb{N}^n| = |\mathbb{N}|$$

Theorem

The Cartesian product of finitely many countable sets is countable.

Theorem ($\mathbb{N}^n$ is Countable.)

$$|\mathbb{N}^n| = |\mathbb{N}|$$

Theorem

The Cartesian product of finitely many countable sets is countable.

$$\mathbb{N}^n \quad vs. \quad \mathbb{N}^{\mathbb{N}}$$

Theorem ($\mathbb{N}^n$ is Countable.)

$$|\mathbb{N}^n| = |\mathbb{N}|$$

Theorem

The Cartesian product of finitely many countable sets is countable.

$$\mathbb{N}^n \quad vs. \quad \mathbb{N}^{\mathbb{N}}$$

$$\pi^{(n)} : \mathbb{N}^n \to \mathbb{N}$$

Theorem ($\mathbb{N}^n$ is Countable.)

$$|\mathbb{N}^n| = |\mathbb{N}|$$

Theorem

The Cartesian product of finitely many countable sets is countable.

$$\mathbb{N}^n \quad vs. \quad \mathbb{N}^{\mathbb{N}}$$

$$\pi^{(n)} : \mathbb{N}^n \to \mathbb{N}$$

$$\pi^{(n)}(k_1, \ldots, k_{n-1}, k_n) = \pi(\pi^{(n-1)}(k_1, \ldots, k_{n-1}), k_n)$$

Theorem

Any finite union of countable sets is countable.

Theorem

Any finite union of countable sets is countable.

$$A = \{a_n \mid n \in \mathbb{N}\} \quad B = \{b_n \mid n \in \mathbb{N}\} \quad C = \{c_n \mid n \in \mathbb{N}\}$$

Theorem

Any finite union of countable sets is countable.

$$A = \{a_n \mid n \in \mathbb{N}\} \quad B = \{b_n \mid n \in \mathbb{N}\} \quad C = \{c_n \mid n \in \mathbb{N}\}$$

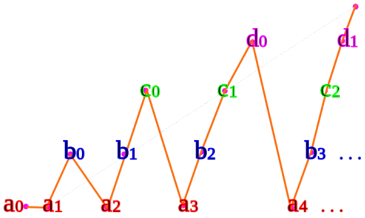
$$a_0 \quad b_0 \quad c_0 \quad a_1 \quad b_1 \quad c_1 \cdots$$

Theorem

The union of countably many countable sets is countable.

Theorem

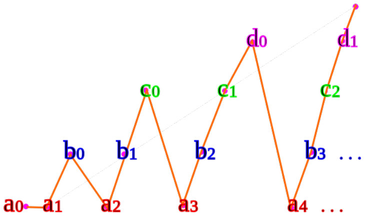
The union of countably many countable sets is countable.



Counting by Diagonals.

Theorem

The union of countably many countable sets is countable.



Counting by Diagonals.

We need Axiom of (Countable) Choice!

Beyond

$$\aleph_0$$

Theorem ($\mathbb{R}$ is Uncountable. (Cantor 1873-12; Published in 1874))

$$|\mathbb{R}| \neq |\mathbb{N}|$$

Theorem ($\mathbb{R}$ is Uncountable. (Cantor 1873-12; Published in 1874))

$$|\mathbb{R}| \neq |\mathbb{N}|$$

Theorem (ℝ is Uncountable. (Cantor 1873-12; Published in 1874))

$$|\mathbb{R}| \neq |\mathbb{N}|$$

Different "Sizes" of Infinity

Theorem ($\mathbb{R}$ is Uncountable. (Cantor 1873-12; Published in 1874))

$$|\mathbb{R}| \neq |\mathbb{N}|$$

Different "Sizes" of Infinity

Cantor's Diagonal Argument (1890)

Theorem ($\mathbb{R}$ is Uncountable. (Cantor 1873-12; Published in 1874))

$$|\mathbb{R}| \neq |\mathbb{N}|$$

Theorem ($\mathbb{R}$ is Uncountable. (Cantor 1873-12; Published in 1874))

$$|\mathbb{R}| \neq |\mathbb{N}|$$

By Contradiction.

Theorem ($\mathbb{R}$ is Uncountable. (Cantor 1873-12; Published in 1874))

$$|\mathbb{R}| \neq |\mathbb{N}|$$

By Contradiction.

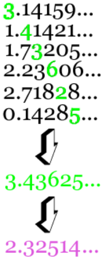
$$f : \mathbb{R} \xleftrightarrow[onto]{1-1} \mathbb{N}$$

Theorem (ℝ is Uncountable. (Cantor 1873-12; Published in 1874))

$$|\mathbb{R}| \neq |\mathbb{N}|$$

By Contradiction.

$$f : \mathbb{R} \xleftrightarrow[onto]{1-1} \mathbb{N}$$



Theorem (ℝ is Uncountable. (Cantor 1873-12; Published in 1874))

$$|\mathbb{R}| \neq |\mathbb{N}|$$

By Contradiction.

$$f : \mathbb{R} \xleftrightarrow[onto]{1-1} \mathbb{N}$$

3.14159...
1.41421...
1.73205...
2.23606...
2.71828...
0.14285...

$\Downarrow$

3.43625...

$\Downarrow$

2.32514...

By Diagonal Argument.

$$\mathfrak{c} \triangleq |\mathbb{R}|$$

Theorem ($|\mathbb{R}|$ (Cantor 1877))

$$|(0,1)| = |\mathbb{R}| = |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}^{n \in \mathbb{N}}|$$

Theorem ($|\mathbb{R}|$ (Cantor 1877))

$$|(0,1)| = |\mathbb{R}| = |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}^{n \in \mathbb{N}}|$$

Proof.

Theorem ($|\mathbb{R}|$ (Cantor 1877))

$$|(0,1)| = |\mathbb{R}| = |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}^{n \in \mathbb{N}}|$$

Proof.

$$f(x) = \tan \frac{(2x-1)\pi}{2}$$

$$|(0,1)| = |(-\frac{\pi}{2}, \frac{\pi}{2})| = |\mathbb{R}|$$

Theorem ($|\mathbb{R}|$ (Cantor 1877))

$$|(0,1)| = |\mathbb{R}| = |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}^{n \in \mathbb{N}}|$$

Proof.

$$f(x) = \tan \frac{(2x-1)\pi}{2}$$

$$|(0,1)| = |(-\frac{\pi}{2}, \frac{\pi}{2})| = |\mathbb{R}|$$

$$(x = 0.a_1 a_2 a_3 \cdots, y = 0.b_1 b_2 b_3 \cdots)$$

Theorem ($|\mathbb{R}|$ (Cantor 1877))

$$|(0,1)| = |\mathbb{R}| = |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}^{n \in \mathbb{N}}|$$

Proof.

$$f(x) = \tan \frac{(2x-1)\pi}{2}$$

$$|(0,1)| = |(-\frac{\pi}{2}, \frac{\pi}{2})| = |\mathbb{R}|$$

$$(x = 0.a_1 a_2 a_3 \cdots, y = 0.b_1 b_2 b_3 \cdots) \mapsto 0.a_1 b_1 a_2 b_2 a_3 b_3 \cdots$$

$\square$

Theorem ($|\mathbb{R}|$ (Cantor 1877))

$$|(0,1)| = |\mathbb{R}| = |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}^n|$$

"Je le vois, mais je ne le crois pas !"

"I see it, but I don't believe it !"

— *Cantor's letter to Dedekind (1877).*

Theorem ($|\mathbb{R}|$ (Cantor 1877))

$$|(0,1)| = |\mathbb{R}| = |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}^n|$$

"Je le vois, mais je ne le crois pas !"

"I see it, but I don't believe it !"

— *Cantor's letter to Dedekind (1877).*

Q : *Then, what is "dimension"?*

Theorem ($|\mathbb{R}|$ (Cantor 1877))

$$|(0,1)| = |\mathbb{R}| = |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}^n|$$

"Je le vois, mais je ne le crois pas !"

"I see it, but I don't believe it !"

— *Cantor's letter to Dedekind (1877).*

Q : Then, what is "dimension"?

Theorem (Brouwer (Topological Invariance of Dimension))

There is no continuous bijections between $\mathbb{R}^m$ and $\mathbb{R}^n$ for $m \neq n$.

Beyond

$$\mathfrak{c}$$

Theorem (Cantor's Theorem (1891))

$$|A| \neq |\mathcal{P}(A)|$$

Theorem (Cantor's Theorem (1891))

$$|A| \neq |\mathcal{P}(A)|$$

Theorem (Cantor Theorem (ES Theorem 24.4))

If $f : A \to \mathcal{P}(A)$, then f is not onto.

Theorem (Cantor's Theorem (1891))

$$|A| \neq |\mathcal{P}(A)|$$

Theorem (Cantor Theorem (ES Theorem 24.4))

If $f : A \to \mathcal{P}(A)$, then f is not onto.

Proof. Let A be a set and let $f : A \to 2^A$. To show that f is not onto, we must find a $B \in 2^A$ (i.e., $B \subseteq A$) for which there is no $a \in A$ with $f(a) = B$. In other words, B is a set that f "misses." To this end, let

$$B = \{x \in A : x \notin f(x)\}.$$

We claim there is no $a \in A$ with $f(a) = B$.

Suppose, for the sake of contradiction, there is an $a \in A$ such that $f(a) = B$. We ponder: Is $a \in B$?

- If $a \in B$, then, since $B = f(a)$, we have $a \in f(a)$. So, by definition of B, $a \notin f(a)$; that is, $a \notin B$.$\Rightarrow\Leftarrow$
- If $a \notin B = f(a)$, then, by definition of B, $a \in B$.$\Rightarrow\Leftarrow$

Both $a \in B$ and $a \notin B$ lead to contradictions, and hence our supposition [there is an $a \in A$ with $f(a) = B$] is false, and therefore f is not onto. ■

Theorem (Cantor Theorem)

If $f : A \to \mathcal{P}(A)$*, then* f *is not onto.*

Theorem (Cantor Theorem)

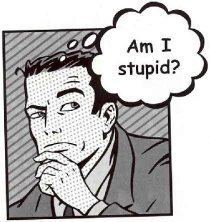
If $f : A \to \mathcal{P}(A)$, *then* f *is not onto.*

Theorem (Cantor Theorem)

If $f : A \to \mathcal{P}(A)$, then f is not onto.

Theorem (Cantor Theorem)

If $f : A \to \mathcal{P}(A)$, *then* f *is not onto.*



Theorem (Cantor Theorem)

If $f : A \to \mathcal{P}(A)$, then f is not onto.

Understanding this problem:

$$A = \{1, 2, 3\}$$

Theorem (Cantor Theorem)

If $f : A \to \mathcal{P}(A)$, then f is not onto.

Understanding this problem:

$$A = \{1, 2, 3\}$$

$$\mathcal{P}(A) = \Big\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\Big\}$$

Theorem (Cantor Theorem)

If $f : A \to \mathcal{P}(A)$, then f is not onto.

Understanding this problem:

$$A = \{1, 2, 3\}$$

$$\mathcal{P}(A) = \Big\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\Big\}$$

Onto

$$\forall B \in \mathcal{P}(A) : \Big(\exists a \in A : f(a) = B\Big)$$

Theorem (Cantor Theorem)

If $f : A \to \mathcal{P}(A)$, then f is not onto.

Understanding this problem:

$$A = \{1, 2, 3\}$$

$$\mathcal{P}(A) = \Big\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\Big\}$$

Onto

$$\forall B \in \mathcal{P}(A) : \Big(\exists a \in A : f(a) = B\Big)$$

Not Onto

$$\exists B \in \mathcal{P}(A) : \Big(\forall a \in A : f(a) \neq B\Big)$$

Theorem (Cantor Theorem)

If $f : A \to \mathcal{P}(A)$, then f is not onto.

$$\exists B \in \mathcal{P}(A) : \Big(\forall a \in A : f(a) \neq B \Big)$$

Theorem (Cantor Theorem)

If $f : A \to \mathcal{P}(A)$, then f is not onto.

$$\exists B \in \mathcal{P}(A) : \Big(\forall a \in A : f(a) \neq B\Big)$$

- Constructive proof ($\exists$):

$$B = \{a \in A \mid a \notin f(a)\}$$

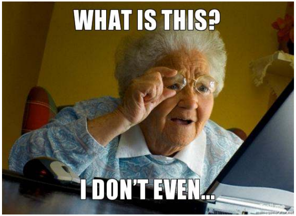
Theorem (Cantor Theorem)

If $f : A \to \mathcal{P}(A)$, then f is not onto.

$$\exists B \in \mathcal{P}(A) : \Big(\forall a \in A : f(a) \neq B\Big)$$

- Constructive proof ($\exists$):

$$B = \{a \in A \mid a \notin f(a)\}$$



Theorem (Cantor Theorem)

If $f : A \to \mathcal{P}(A)$, then f is not onto.

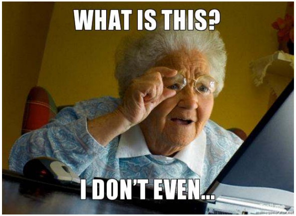
$$\exists B \in \mathcal{P}(A) : \Big(\forall a \in A : f(a) \neq B\Big)$$

- Constructive proof ($\exists$):

$$B = \{a \in A \mid a \notin f(a)\}$$

- By contradiction ($\forall$):

$$\exists a \in A : f(a) = B.$$



Theorem (Cantor Theorem)

If $f : A \to \mathcal{P}(A)$, then f is not onto.

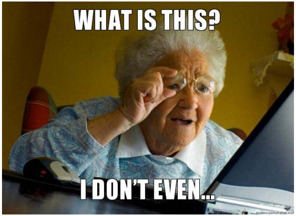
$$\exists B \in \mathcal{P}(A) : \Big(\forall a \in A : f(a) \neq B \Big)$$

- Constructive proof ($\exists$):

$$B = \{a \in A \mid a \notin f(a)\}$$

- By contradiction ($\forall$):

$$\exists a \in A : f(a) = B.$$



$$Q : a \in B\,?$$

Theorem (Cantor Theorem)

If $f : A \to \mathcal{P}(A)$, then f is not onto.

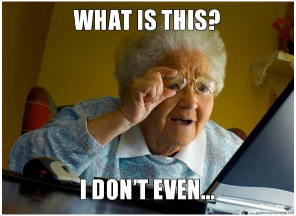
$$\exists B \in \mathcal{P}(A) : \Big(\forall a \in A : f(a) \neq B\Big)$$

- Constructive proof ($\exists$):

$$B = \{a \in A \mid a \notin f(a)\}$$

- By contradiction ($\forall$):

$$\exists a \in A : f(a) = B.$$



$$Q : a \in B\,?$$

$$a \in B \iff a \notin B$$

Theorem (Cantor Theorem)

If $f : A \to \mathcal{P}(A)$*, then* f *is not onto.*

Diagonal Argument .

Theorem (Cantor Theorem)

If $f : A \to \mathcal{P}(A)$, then f is not onto.

Diagonal Argument .

a	$f(a)$					
	1	2	3	4	5	$\cdots$
1	1	1	0	0	1	$\cdots$
2	0	0	0	0	0	$\cdots$
3	1	0	0	1	0	$\cdots$
4	1	1	1	1	1	$\cdots$
5	0	1	0	1	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$

$\square$

Theorem (Cantor Theorem)

If $f : A \to \mathcal{P}(A)$, then f is not onto.

Diagonal Argument .

a	$f(a)$					
	1	2	3	4	5	$\cdots$
1	1	1	0	0	1	$\cdots$
2	0	0	0	0	0	$\cdots$
3	1	0	0	1	0	$\cdots$
4	1	1	1	1	1	$\cdots$
5	0	1	0	1	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$

$\square$

Theorem (Cantor Theorem)

If $f : A \to \mathcal{P}(A)$, then f is not onto.

Diagonal Argument .

a	$f(a)$					
	1	2	3	4	5	$\cdots$
1	1	1	0	0	1	$\cdots$
2	0	0	0	0	0	$\cdots$
3	1	0	0	1	0	$\cdots$
4	1	1	1	1	1	$\cdots$
5	0	1	0	1	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$

$$B = \{0, 1, 1, 0, 1\}$$

□

Theorem (Cantor Theorem)

If $f : A \to \mathcal{P}(A)$, then f is not onto.

Diagonal Argument (以下仅适用于可数集合 A).

a	$f(a)$					
	1	2	3	4	5	$\cdots$
1	1	1	0	0	1	$\cdots$
2	0	0	0	0	0	$\cdots$
3	1	0	0	1	0	$\cdots$
4	1	1	1	1	1	$\cdots$
5	0	1	0	1	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$

$$B = \{0, 1, 1, 0, 1\}$$

$\square$

Theorem (Cantor Theorem)

$$|A| < |\mathcal{P}(A)|$$

Theorem (Cantor Theorem)

$$|A| < |\mathcal{P}(A)|$$

$$A \qquad \mathcal{P}(A) \qquad \mathcal{P}(\mathcal{P}(A)) \qquad \dots$$

Theorem (Cantor Theorem)

$$|A| < |\mathcal{P}(A)|$$

$$A \qquad \mathcal{P}(A) \qquad \mathcal{P}(\mathcal{P}(A)) \qquad \ldots$$

There is no largest infinity.

$$<$$

Definition ($|A| \leq |B|$)

$|A| \leq |B|$ if there exists an *one-to-one* function f from A into B.

Definition ($|A| \leq |B|$)

$|A| \leq |B|$ if there exists an *one-to-one* function f from A into B.

Q : What about onto function $f : A \to B$?

Definition ($|A| \leq |B|$)

$|A| \leq |B|$ if there exists an *one-to-one* function f from A into B.

Q : What about onto function $f : A \to B$?

$|B| \leq |A|$

Definition ($|A| \le |B|$)

$|A| \le |B|$ if there exists an *one-to-one* function f from A into B.

Q : What about onto function $f : A \to B$?

$|B| \le |A|$ (Axiom of Choice)

Definition ($|A| < |B|$)

$$|A| < |B| \iff |A| \leq |B| \land |A| \neq |B|$$

Definition ($|A| < |B|$)

$|A| < |B| \iff |A| \le |B| \land |A| \ne |B|$

$$|\mathbb{N}| < |\mathbb{R}|$$

$$|X| < |2^X|$$

$$|\mathbb{N}| < |2^{\mathbb{N}}|$$

Definition (Countable Revisited)

X is countable:

$$(\exists n \in \mathbb{N} : |X| = n) \lor |X| = |\mathbb{N}|$$

Definition (Countable Revisited)

X is countable:

$$(\exists n \in \mathbb{N} : |X| = n) \lor |X| = |\mathbb{N}|$$

Theorem (Proof for Countable)

X is countable iff

$$|X| \le |\mathbb{N}|.$$

Definition (Countable Revisited)

X is countable:

$$(\exists n \in \mathbb{N} : |X| = n) \lor |X| = |\mathbb{N}|$$

Theorem (Proof for Countable)

X is countable iff

$$|X| \leq |\mathbb{N}|.$$

X is countable iff there exists a one-to-one function

$$f : X \to \mathbb{N}.$$

Definition (Countable Revisited)

X is countable:

$$(\exists n \in \mathbb{N} : |X| = n) \lor |X| = |\mathbb{N}|$$

Theorem (Proof for Countable)

X is countable iff

$$|X| \leq |\mathbb{N}|.$$

X is countable iff there exists a one-to-one function

$$f : X \to \mathbb{N}.$$

Subsets of Countable Set (UD Corollary 23.4)

Every subset B of a countable set A is countable.

Definition (Countable Revisited)

X is countable:

$$(\exists n \in \mathbb{N} : |X| = n) \vee |X| = |\mathbb{N}|$$

Theorem (Proof for Countable)

X is countable iff

$$|X| \le |\mathbb{N}|.$$

X is countable iff there exists a one-to-one function

$$f : X \to \mathbb{N}.$$

Subsets of Countable Set (UD Corollary 23.4)

Every subset B of a countable set A is countable.

$$f : A \xrightarrow{1-1} \mathbb{N}$$

Definition (Countable Revisited)

X is countable:

$$(\exists n \in \mathbb{N} : |X| = n) \lor |X| = |\mathbb{N}|$$

Theorem (Proof for Countable)

X is countable iff

$$|X| \le |\mathbb{N}|.$$

X is countable iff there exists a one-to-one function

$$f : X \to \mathbb{N}.$$

Subsets of Countable Set (UD Corollary 23.4)

Every subset B of a countable set A is countable.

$$f : A \xrightarrow{1-1} \mathbb{N} \qquad g = f|_B$$

Slope (UD Problem 23.3 (a))

(a) The set of all lines with rational slopes

Slope (UD Problem 23.3 (a))

(a) The set of all lines with rational slopes

$$(\mathbb{Q}, \mathbb{R})$$

Slope (UD Problem 23.3 (a))

(a) The set of all lines with rational slopes

$$(\mathbb{Q}, \mathbb{R})$$

$$|\mathbb{R}| \le |\mathbb{Q} \times \mathbb{R}| \le |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$$

Q : Is "$\leq$" a partial order?

Q : Is "≤" a partial order?

Theorem (Cantor-Schröder–Bernstein (1887))

$$|X| \le |Y| \land |Y| \le |X| \implies |X| = |Y|$$

$$Q: \textit{Is } \text{“}\leq\text{” } \textit{a partial order?}$$

Theorem (Cantor-Schröder–Bernstein (1887))

$$|X| \leq |Y| \wedge |Y| \leq |X| \implies |X| = |Y|$$

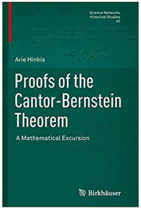
$$\exists \textit{ one-to-one } f : X \to Y \wedge g : Y \to X \implies \exists \textit{ bijection } h : X \to Y$$

Q : Is "$\leq$" a partial order?

Theorem (Cantor-Schröder–Bernstein (1887))

$$|X| \leq |Y| \wedge |Y| \leq |X| \implies |X| = |Y|$$

$$\exists \textit{ one-to-one } f : X \to Y \wedge g : Y \to X \implies \exists \textit{ bijection } h : X \to Y$$

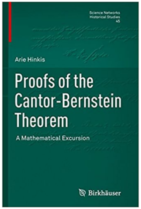


Q : *Is "$\leq$" a partial order?*

Theorem (Cantor-Schröder–Bernstein (1887))

$$|X| \leq |Y| \land |Y| \leq |X| \implies |X| = |Y|$$

$$\exists \textit{ one-to-one } f : X \to Y \land g : Y \to X \implies \exists \textit{ bijection } h : X \to Y$$

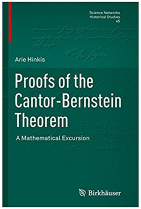


Q : Is "$\leq$" a partial order?

Theorem (Cantor-Schröder–Bernstein (1887))

$$|X| \leq |Y| \wedge |Y| \leq |X| \implies |X| = |Y|$$

$$\exists \text{ *one-to-one* } f : X \to Y \wedge g : Y \to X \implies \exists \text{ *bijection* } h : X \to Y$$



Schröder–Bernstein theorem @ wiki

Q : Is "$\leq$" a total order?

$Q: \textit{Is } \text{“}\leq\text{”} \textit{ a total order?}$

Theorem (PCC)

Principle of Cardinal Comparability (PCC) $\iff$ *Axiom of Choice*

Theorem (UD Theorem 24.11)

$$|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$$

Theorem (UD Theorem 24.11)

$$|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$$

$$|\mathbb{R}| \le |\mathcal{P}(\mathbb{N})| \qquad |\mathcal{P}(\mathbb{N})| \le |\mathbb{R}|$$

Theorem (UD Theorem 24.11)

$$|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$$

$$|\mathbb{R}| \leq |\mathcal{P}(\mathbb{N})| \qquad |\mathcal{P}(\mathbb{N})| \leq |\mathbb{R}|$$

$$\mathfrak{c} \triangleq |\mathbb{R}| = |\mathcal{P}(\mathbb{N})| = |2^{\mathbb{N}}| \triangleq 2^{\aleph_0}$$

Theorem (UD Theorem 24.11)

$$|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$$

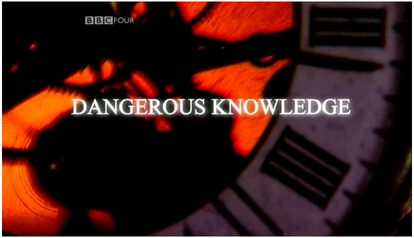
$$|\mathbb{R}| \le |\mathcal{P}(\mathbb{N})| \qquad |\mathcal{P}(\mathbb{N})| \le |\mathbb{R}|$$

$$\mathfrak{c} \triangleq |\mathbb{R}| = |\mathcal{P}(\mathbb{N})| = |2^{\mathbb{N}}| \triangleq 2^{\aleph_0}$$

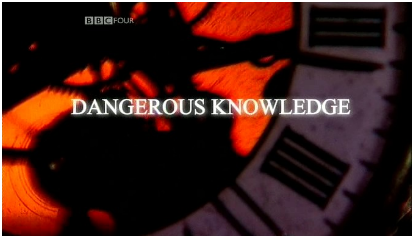
$$\boxed{\mathfrak{c} = 2^{\aleph_0}}$$

Continuum Hypothesis (CH)

$$\boxed{\nexists A : \aleph_0 < |A| < \mathfrak{c}}$$



☞ Dangerous Knowledge (22:20; BBC 2007)



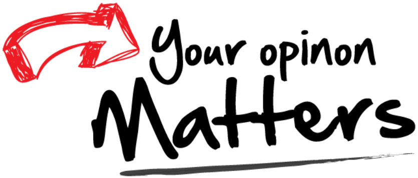
☞ Dangerous Knowledge (22:20; BBC 2007)

Independence from ZFC:

Kurt Gödel (1940) CH cannot be disproved from ZF.

Paul Cohen (1964) CH cannot be proven from the ZFC axioms.


Thank You!



Office 302

Mailbox: H016

hfwei@nju.edu.cn