

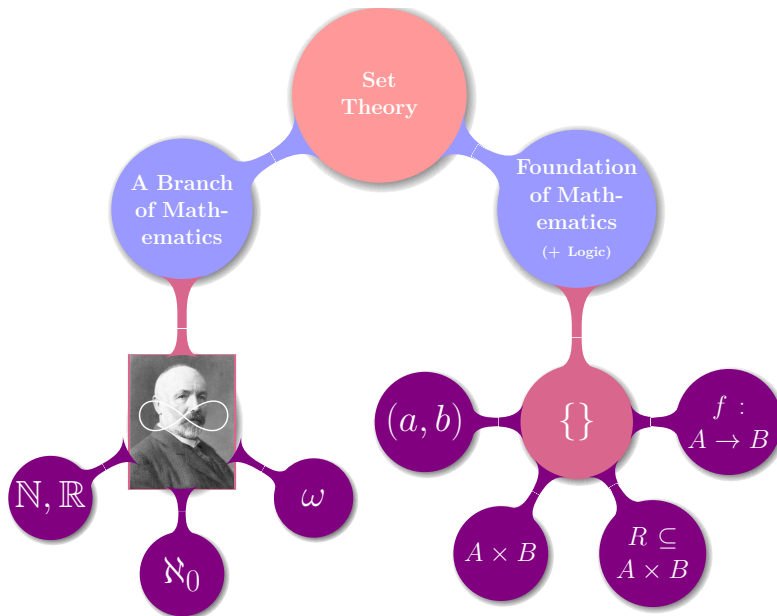
1-11 Set Theory (IV): Infinity

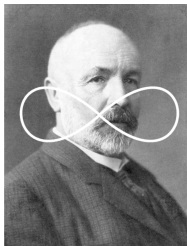
魏恒峰

hfwei@nju.edu.cn

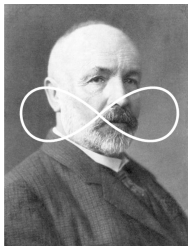
2019 年 12 月 17 日







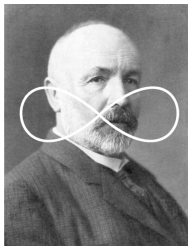
Georg Cantor (1845 – 1918)



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Leopold Kronecker
(1823 – 1891)



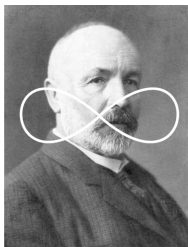
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Henri Poincaré
(1854 – 1912)



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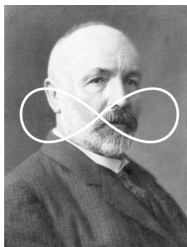
Leopold Kronecker
(1823 – 1891)



Henri Poincaré
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Ludwig Wittgenstein
(1889 – 1951)



Georg Cantor (1845 – 1918)



David Hilbert (1862 – 1943)



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*From his paradise that Cantor with us unfolded, we hold our
breath in awe; knowing, we shall not be expelled.*

— *David Hilbert*

“没有人能把我们从 Cantor 创造的乐园中驱逐出去”

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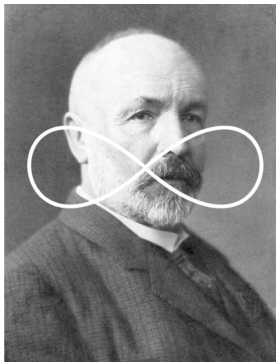
“das wesen der mathematik liegt in ihrer freiheit”



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“The essence of mathematics lies in its freedom”

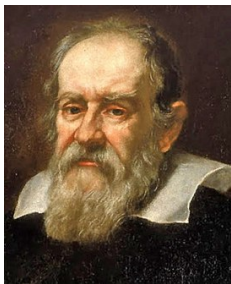
Before Cantor







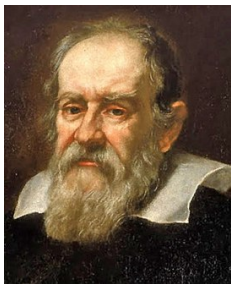
公理: “整体大于部分”



Galileo Galilei (1564 – 1642)



“关于两门新科学的对话” (1638)



Galileo Galilei (1564 – 1642)

“关于两门新科学的对话” (1638)

“用我们有限的心智来讨论无限...”

$$S_1 = \{1, 2, 3, \dots, n, \dots\}$$

$$S_2 = \{1, 4, 9, \dots, n^2, \dots\}$$

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“部分等于全体”

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说到底，“等于”、“大于”和“小于”诸性质不能用于无限，而只能用于有限的数量。
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无穷数是不可能的。
— Gottfried Wilhelm Leibniz

这些证明一开始就期望那些数要具有有穷数的一切性质，或者甚至于把有穷数的性质强加于无穷。

相反，这些无穷数，如果它们能够以任何形式被理解的话，倒是由于它们与有穷数的对应，它们必须具有完全新的数量特征。

这些性质完全依赖于事物的本性，… 而并非来自我们的主观任意性或我们的偏见。

— Georg Cantor (1885)

Definition (Dedekind-infinite & Dedekind-finite (Dedekind, 1888))

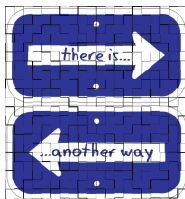
A set A is *Dedekind-infinite* if there is a bijective function from A onto some proper subset B of A .

A set is *Dedekind-finite* if it is not Dedekind-infinite.

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This is a **theorem** in our theory of infinity.



We have not defined “finite” and “infinite”!

Comparing Sets

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Function



Definition ($|A| = |B|$ ($A \approx B$) (1878))

A and B are *equipotent* if there exists a *bijection* from A to B .

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Abstract from order: $\{1, 2, 3, \dots\}$ vs. $\{1, 3, 5, \dots, 2, 4, 6, \dots\}$

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Theorem ()

For any sets A, B, C :

- (a) $A \approx B$
- (b) $A \approx B \implies B \approx A$
- (c) $A \approx B \wedge B \approx C \implies A \approx C$

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Theorem (The “Equivalence Concept” of Equipotent)

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Definition (Finite)

X is finite if

$$\exists n \in \mathbb{N} : |X| = n.$$

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Theorem (UD Theorem 22.6)

*Let A be a finite set. There is a **unique** $n \in \mathbb{N}$ such that $A \approx \{0, 1, \dots, n-1\}$.*

Definition (Infinite)

X is infinite if it is not finite:

$$\forall n \in \mathbb{N} : |X| \neq n.$$

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\mathbb{N} is infinite. (So are \mathbb{Z} , \mathbb{Q} , \mathbb{R} .)

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Definition (Infinite)

For any set X ,

Countably Infinite

$$|X| = |\mathbb{N}| \triangleq \aleph_0$$

Countable

(finite \vee countably infinite)

Uncountable

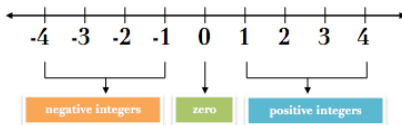
(\neg countable)

(infinite) \wedge (\neg (countably infinite))

\aleph_0

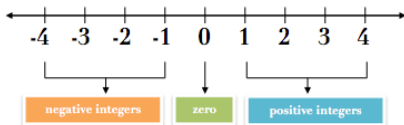
Theorem (\mathbb{Z} is Countable.)

$$|\mathbb{Z}| = |\mathbb{N}|$$



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0 1 -1 2 -2 ...

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$|\mathbb{Q}| = |\mathbb{N}|$ (UD Problem 23.12)

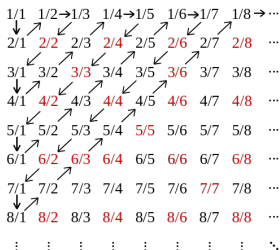
$$q \in \mathbb{Q}^+ : a/b \ (a, b \in \mathbb{N}^+)$$

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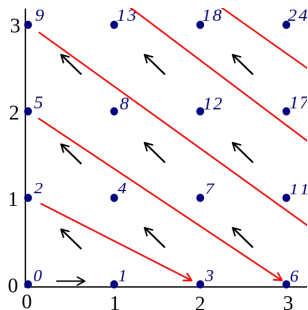
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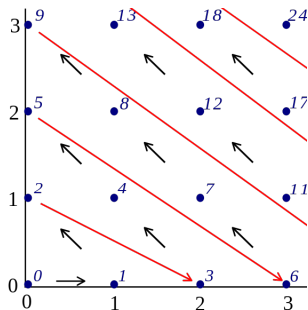
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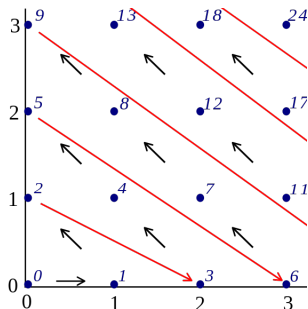
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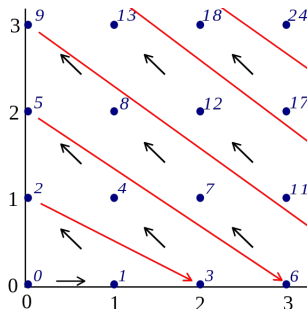


$$\pi : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$$

$$\pi(k_1, k_2) = \frac{1}{2}(k_1 + k_2)(k_1 + k_2 + 1) + k_2$$

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Cantor Pairing Function

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$$\mathbb{N}^n \quad \text{vs.} \quad \mathbb{N}^{\mathbb{N}}$$

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$$\pi^{(n)}(k_1, \dots, k_{n-1}, k_n) = \pi(\pi^{(n-1)}(k_1, \dots, k_{n-1}), k_n)$$

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*Any **finite** union of countable sets is countable.*

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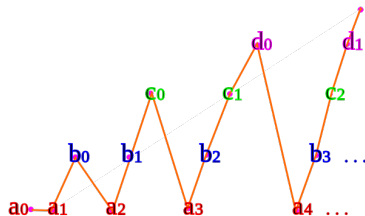
$$a_0 \quad b_0 \quad c_0 \quad a_1 \quad b_1 \quad c_1 \cdots$$

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*The union of **countably many** countable sets is countable.*

Theorem

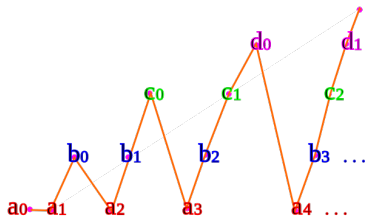
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Counting by Diagonals.

Theorem

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Counting by Diagonals.

We need Axiom of (Countable) Choice!

Beyond

\aleph_0

Theorem (\mathbb{R} is Uncountable. (Cantor 1873-12; Published in 1874))

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**VERY
IMPORTANT**

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Different “Sizes” of Infinity

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Different “Sizes” of Infinity

Cantor’s Diagonal Argument (1890)

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1.41421...
1.73205...
2.23606...
2.71828...
0.14285...



3.43625...



2.32514...

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By Diagonal Argument.

$$\mathfrak{c} \triangleq |\mathbb{R}|$$

Theorem ($|\mathbb{R}|$ (Cantor 1877))

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Theorem ($|\mathbb{R}|$ (Cantor 1877))

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“Je le vois, mais je ne le crois pas !”

“I see it, but I don't believe it !”

— Cantor's letter to Dedekind (1877).

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Q : Then, what is “dimension”?

Theorem (Brouwer (Topological Invariance of Dimension))

*There is no **continuous** bijections between \mathbb{R}^m and \mathbb{R}^n for $m \neq n$.*

Beyond



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If $f : A \rightarrow \mathcal{P}(A)$, then f is not onto.

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Theorem (Cantor Theorem (ES Theorem 24.4))

If $f : A \rightarrow \mathcal{P}(A)$, then f is not onto.

Proof. Let A be a set and let $f : A \rightarrow 2^A$. To show that f is not onto, we must find a $B \in 2^A$ (i.e., $B \subseteq A$) for which there is no $a \in A$ with $f(a) = B$. In other words, B is a set that f “misses.” To this end, let

$$B = \{x \in A : x \notin f(x)\}.$$

We claim there is no $a \in A$ with $f(a) = B$.

Suppose, for the sake of contradiction, there is an $a \in A$ such that $f(a) = B$. We ponder: Is $a \in B$?

- If $a \in B$, then, since $B = f(a)$, we have $a \in f(a)$. So, by definition of B , $a \notin f(a)$; that is, $a \notin B. \Rightarrow \Leftarrow$
- If $a \notin B = f(a)$, then, by definition of B , $a \in B. \Rightarrow \Leftarrow$

Both $a \in B$ and $a \notin B$ lead to contradictions, and hence our supposition [there is an $a \in A$ with $f(a) = B$] is false, and therefore f is not onto. ■

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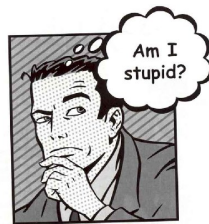
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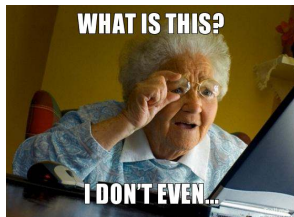
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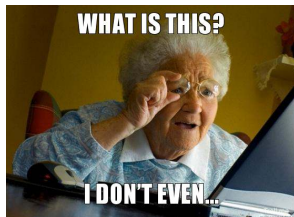
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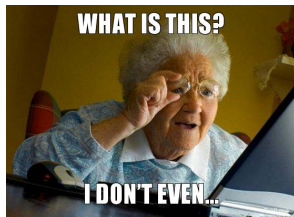
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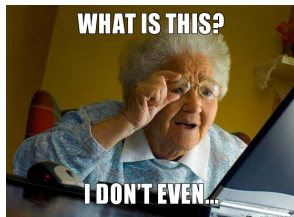
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Diagonal Argument (以下仅适用于可数集合 A).

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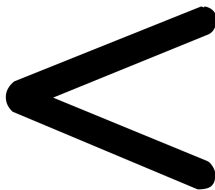
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There is no largest infinity.



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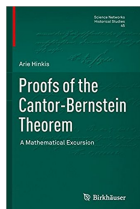
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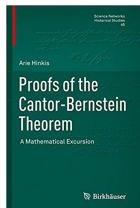


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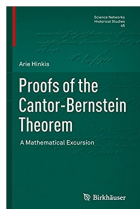


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Schröder-Bernstein
theorem @ wiki

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Theorem (PCC)

Principle of Cardinal Comparability (PCC) \iff Axiom of Choice

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Continuum Hypothesis (CH)

$$\nexists A : \aleph_0 < |A| < \mathfrak{c}$$



👉 Dangerous Knowledge (22:20; BBC 2007)



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Independence from ZFC:

Kurt Gödel (1940) CH cannot be disproved from ZF.

Paul Cohen (1964) CH cannot be proven from the ZFC axioms.

Thank
You!



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