2-3 Counting

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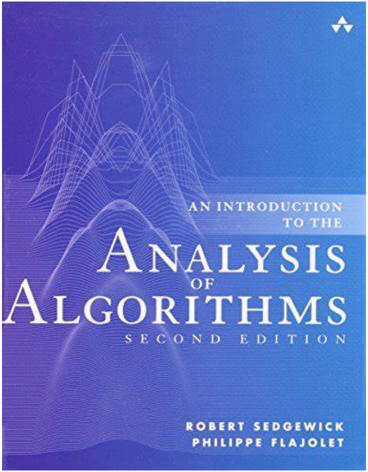
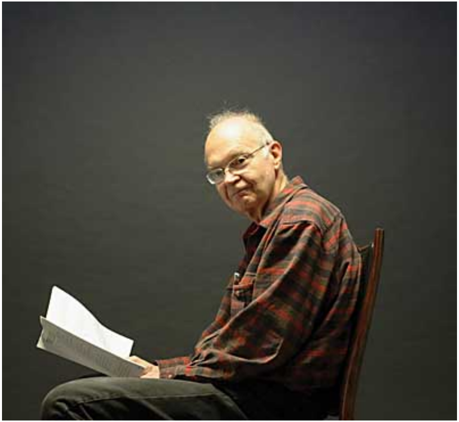


The Analysis of Algorithms



Donald E. Knuth (1938 $\sim$)

"People who analyze algorithms have double happiness.

First of all they experience the sheer beauty of elegant mathematical patterns that surround elegant computational procedures.

Then they receive a practical payoff when their theories make it possible to get other jobs done more quickly and more economically."

```
procedure CONUNDRUM(n)
    r ← 0
    for i ← 1 to n do
        for j ← i + 1 to n do
            for k ← i + j − 1 to n do
                r ← r + 1
    return r
```

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=i+j-1}^{n} 1 = \frac{n^2 - n}{2} = \Theta(n^2)$$

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$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=i+j-1}^{n} 1 = \frac{1}{48}\Big(3\big(-1 + (-1)^n\big) + 2n(n+2)(2n-1)\Big) = \Theta(n^3)$$

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=i+j-1}^{n} 1$$

$$= \sum_{i=1}^{n} \sum_{j=i+1}^{n} (n-i-j+2) \left[j \le n-i+1, i \le \frac{n}{2}\right]$$

$$= \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=i+1}^{n-i+1} (n-i-j+2)$$

$$\sum_{i=1}^{n}\sum_{j=i+1}^{n}\sum_{k=i+j-1}^{n} 1$$

$$= \sum_{i=1}^{n}\sum_{j=i+1}^{n} (n-(i+j-1)+1)\,[i+j-1\le n]$$

$$= \sum_{i=1}^{n}\sum_{j=i+1}^{n} (n-i-j+2)\,[j\le n-i+1] \qquad n-i+1 \ge i+1 \Rightarrow i \le \frac{n}{2}$$

$$= \sum_{i=1}^{\lfloor \frac{n}{2}\rfloor}\sum_{j=i+1}^{n-i+1} (n-i-j+2)$$

$$= \frac{1}{2}\sum_{i=1}^{\lfloor \frac{n}{2}\rfloor} (n-2i+1)(n-2i+2)$$

当n为偶时 $$= \frac{1}{2}\left[\sum_{i=1}^{\frac{n}{2}} (n^2+3n+2) + 4\sum_{i=1}^{\frac{n}{2}} i^2 - \sum_{i=1}^{\frac{n}{2}}(4n+6)i\right]$$

$$= \frac{1}{2}\times\left(\frac{1}{2}(n^3+3n^2+2n) + \frac{n(\frac{n}{2}+1)(n+1)}{3} - \frac{(2n+3)(\frac{n}{2}+1)n}{2}\right)$$

$$= \text{化简} = \frac{2n^3+3n^2-2n}{24} = \frac{1}{48}\left(0+2n(2+n)(2n-1)\right)$$

当n为奇时，$\lfloor \frac{n}{2}\rfloor = \frac{n-1}{2}$，代入，可化得 $= \frac{1}{48}\left(-6+2n(2+n)(2n-1)\right)$

（这个我懒的化了，谁有兴趣化一下，多个常数项）.

通解 $\frac{1}{48}\left(3(-1+(-1)^n)+2n(2+n)(2n-1)\right)$

* $\lfloor \frac{n}{2}\rfloor = \dfrac{n+\frac{(-1)^n+(-1)}{2}}{2}$，代入理应可直接得结果，太繁

From Zheng (171860658)

Passing out Apples to Children

k-Permutation $(\mathrm{CS}: 1.2-5)$

We need to pass out k distinct apples (pieces of fruit) to n children such that *each child may get at most one apple*.

(a) $k \le n$?

(b) What if $k > n$?

$$n^{\underline{k}} \triangleq n(n-1)\cdots(n-k+1)$$

0

Multisets (CS : 1.5 – 4)

Use multisets to determine the number of ways to pass out k **identical** apples to n children. Assume that a child may get more than one apple.

$$x_i : \text{the \# of apples the } i\text{-th child gets}$$

$$x_1 + x_2 + \cdots + x_n = k, \quad x_i \geq 0$$

Integer composition (The order matters!)

$$y_i \triangleq x_i + 1$$

$$y_1 + y_2 + \cdots + y_n = n + k, \qquad y_i \geq 1$$

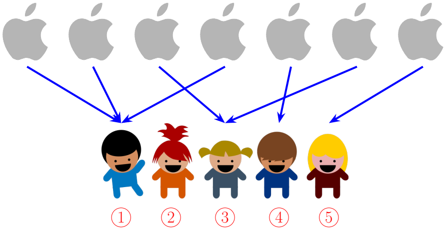
$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

Multisets (CS : 1.5 – 4)

Use **multisets** to determine the number of ways to pass out k identical apples to n children. Assume that a child may get more than one apple.

Q : k-multiset of $[1 \cdots n]$ *vs.* n-multiset of $[1 \cdots k]$

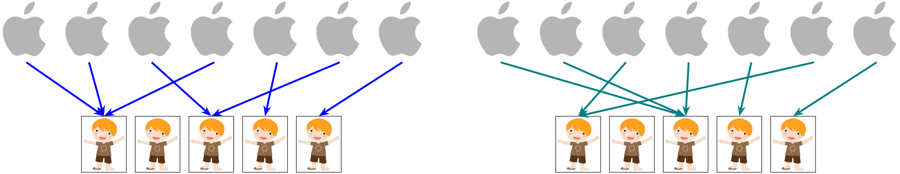
$$k = 7 \qquad n = 5$$



$$\{1, 1, 1, 3, 3, 4, 5\}$$

Integer Partition (CS : 1.5 – 4 Extended)

What is the number of ways to pass out k identical apples to n- .
Assume that a child may get more than one apple.



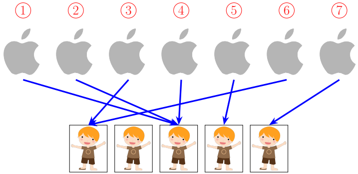
Integer partition of k into $\leq n$ parts (The order does not matter!)

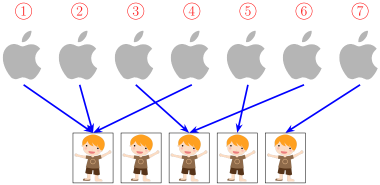
Theorem (G. H. Hardy, Ramanujan (1918))

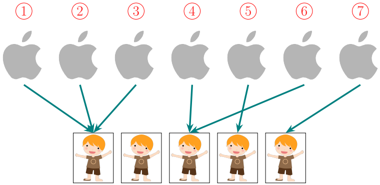
$$p(k) \triangleq \sum_{x=1}^{x=k} p_x(k) \sim \frac{1}{4\sqrt{3}k} \exp\left(\pi\sqrt{\frac{2k}{3}}\right)$$

Set Partition (CS : 1.5 – 4 Extended)

What is the number of ways to pass out k distinct apples to n- .
Assume that a child may get more than one apple.







Set partition of $[1 \cdots k]$ into $\leq n$ parts

Set Partition $(\text{CS} : 1.5 - 12)$

$$S(n,k) \ (\left\{ {n \atop k} \right\}) : \#\text{ of set partitions of } [1 \cdots n] \text{ into } k \text{ classes}$$

Stirling number of the second kind

Theorem (Recurrence for $S(n,k)$)

$$S(0,0) = 1, \quad S(n,0) = S(0,n) = 0 \ (n > 0)$$

$$S(n,k) = S(n-1,k-1) + kS(n-1,k), \quad n > 0, k > 0$$

Proof.

$$S(n,k) = \underbrace{S(n-1,k-1)}_{n \text{ is alone}} + \underbrace{kS(n-1,k)}_{n \text{ is not alone}}$$

□

$$\text{Bell number: } B_n = \sum_{k=0}^{k=n} \left\{ {n \atop k} \right\}$$

Theorem (Berend & Tassa (2010))

$$B_n < \left(\frac{0.792n}{\ln(n+1)} \right)^n, n \in \mathbb{Z}^+$$

Theorem (de Bruijn (1981))

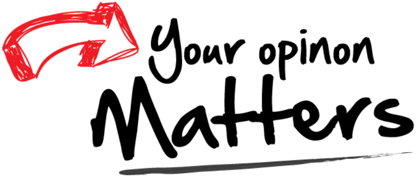
As $n \to \infty$,

$$\frac{\ln B_n}{n} = \ln n - \ln \ln n - 1 + \frac{\ln \ln n}{\ln n} + \frac{1}{\ln n} + \frac{1}{2} \left(\frac{\ln \ln n}{\ln n} \right)^2 + O\left(\frac{\ln \ln n}{(\ln n)^2} \right)$$

THE TWELVEFOLD WAY

balls per urn	unrestricted	≤ 1	≥ 1
n labeled balls, m labeled urns	n-tuples of m things	n-permutations of m things	partitions of $\{1, \ldots, n\}$ into m ordered parts
n unlabeled balls, m labeled urns	n-multicombinations of m things	n-combinations of m things	compositions of n into m parts
n labeled balls, m unlabeled urns	partitions of $\{1, \ldots, n\}$ into $\leq m$ parts	n pigeons into m holes	partitions of $\{1, \ldots, n\}$ into m parts
n unlabeled balls, m unlabeled urns	partitions of n into $\leq m$ parts	n pigeons into m holes	partitions of n into m parts


Thank
You!



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