1!1!2 3!3!4

# 1-9 Set Theory (II): Relations

# 魏恒峰

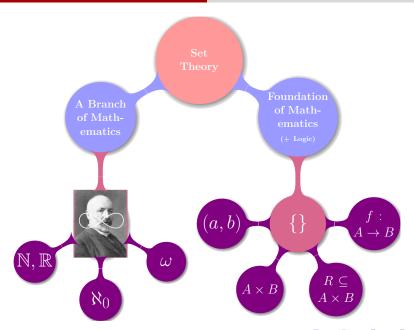
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# Time, Clocks, and the Ordering of Events in a Distributed System

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The concept of one event happening before another in a distributed system is examined, and is shown to define a partial ordering of the events. A distributed algorithm is given for synchronizing a system of logical clocks which can be used to totally order the events.

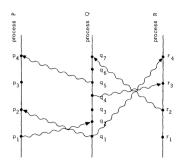


Figure 13. A selection of consistency axioms over an execution (E. repl. obi, oper, rval, ro, vis, ar)

Auxiliary relations

 $sameobi(e, f) \iff obi(e) = obi(f)$ Per-object causality (aka happens-before) order:

 $hbo = ((ro \cap sameobi) \cup vis)^+$ 

Causality (aka happens-before) order: hb = (ro ∪ vis)+

Axioms

EVENTUAL:

 $\forall e \in E. \neg (\exists \text{ infinitely many } f \in E. \text{ sameobj}(e, f) \land \neg (e \xrightarrow{\text{vis}} f))$ 

THINAIR: ro ∪ vis is acvelic

POCV (Per-Object Causal Visibility): hbo ⊆ vis POCA (Per-Object Causal Arbitration): hbo ⊆ ar

COCV (Cross-Object Causal Visibility): (hb ∩ sameobi) ⊂ vis

COCA (Cross-Object Causal Arbitration): hb ∪ ar is acyclic

Figure 17. Optimized state-based multi-value register and its simulation = ReplicalD  $\times P(\mathbb{Z} \times (ReplicalD \rightarrow \mathbb{N}_0))$ =(r,0) $= \mathcal{P}(\mathbb{Z} \times (\mathsf{ReplicalD} \to \mathbb{N}_0))$ 

do(wr(a), (r, V), t) = $(\langle r, \{(a, (\lambda s, \text{if } s \neq r \text{ then } \max\{v(s) \mid (\cdot, v) \in V\}$ else  $\max\{v(s) \mid (-v) \in V3 + 1)(3), \bot \}$ 

 $do(xd, (r, V), t) = ((r, V), \{a \mid (a, .) \in V\})$ send((r, V)) $\operatorname{receive}(\langle r, V \rangle, V') = \langle r, \{(a, v) \in V'' \mid$ 

 $v \not\sqsubseteq \bigcup \{v' \mid \exists a'.(a',v') \in V'' \land a \neq a'\}\}$ , where  $V'' = \{(a, \lfloor |\{v' \mid (a, v') \in V \cup V'\}) \mid (a, .) \in V \cup V'\}$  $(s, V) [R_r] I \iff (r = s) \land (V [M] I)$ 

 $V[M]((E, repl. obj. oper, rval, ro, vis, ar), info) \iff$  $(\forall (a, v), (a', v') \in V, (a = a', \Longrightarrow v = v')) \land$  $(\forall (a, v) \in V, \exists s, v(s) > 0) \land$ 

 $(\forall (a,v) \in V. v \not\sqsubseteq \bigsqcup \{v' \mid \exists a'. (a',v') \in V \land a \neq a'\}) \land$ ∃ distinct e<sub>s,b</sub>  $\{\ell e \in E \mid \exists a. oper(e) = vr(a)\} = \{e, \downarrow \mid s \in ReplicalD \land \}$  $1 \le k \le \max\{v(s) \mid \exists a, (a, v) \in V\}\}) \land$  $(\forall s, j, k, (repl(e_{s,k}) = s) \land (e_{s,i} \xrightarrow{s_s} e_{s,k} \iff j < k)) \land$ 

 $(\forall (a, v) \in V. \forall q. \{j \mid oper(e_{g,j}) = wr(a)\} \cup$  $\{i \mid \exists s, k, e_{s,i} \xrightarrow{\forall i} e_{s,k} \land oper(e_{s,k}) = wr(a)\} =$ 

 $\{j \mid 1 \le j \le v(q)\}\} \land$  $(\forall e \in E. (\mathsf{oper}(e) = \mathsf{wz}(a) \land$ 

 $\neg\exists f \in E \text{ oper}(f) = \text{wr}(\cdot) \land e \xrightarrow{\text{w}} f) \implies (a \cdot ) \in V$ 

the former. The only non-trivial obligation is to show that if  $V[\mathcal{M}]$  ((E. repl. obi. oper. rval. ro. vis), info).

 $\{a \mid (a, \downarrow) \in V\} \subseteq \{a \mid \exists e \in E, oper(e) = wr(a) \land A\}$  $\neg \exists f \in E. \exists a'. oper(e) = wr(a') \land e \xrightarrow{\forall a} f$  (13) (the reverse inclusion is straightforwardly implied by  $R_r$ ).

Take  $(a, v) \in V$ . We have  $\forall (a, v) \in V$ .  $\exists s. v(s) > 0$ ,

 $\forall (a, v) \in V, \forall a, \{i \mid \mathsf{oper}(e_{v,i}) = \mathsf{wr}(a)\} \cup$  $\{j \mid \exists s, k. \ e_{g,j} \xrightarrow{\operatorname{sig}} e_{s,k} \wedge \operatorname{oper}(e_{s,k}) = \operatorname{wr}(a)\} =$  $\{i \mid 1 \le i \le v(a)\}.$ 

From this we get that for some  $e \in E$  $oper(a) = wr(a) \land \neg \exists f \in E, \exists a', a' \neq a \land$  $oper(e) \equiv wr(e') \wedge e \xrightarrow{\forall e} f$ .

Since vis is acyclic, this implies that for some  $e' \in E$ 

 $oper(e') = wx(a) \land \neg \exists f \in E, oper(e') = wx(a) \land e' \xrightarrow{\vee a} f$ . which establishes (13) Let us now discharge RECEIVE. Let  $receive(\langle r, V \rangle, V') =$ (r, V"), where

 $V'' = \{(a, | |\{v' \mid (a, v') \in V \cup V'\}) \mid (a, ...) \in V \cup V'\}:$  $V^{\prime\prime\prime} = \{(a, v) \in V^{\prime\prime} \mid v \not\subseteq | | \{(a', v') \in V^{\prime\prime} \mid a \neq a'\} \}.$ 

Assume (r, V)  $[R_r]$  I, V' [M] J and

I = ((E, repl, obj, oper, rval, ro, vis, ar), info);J = ((E', repl', obi', oper', rval', ro', vis', ar'), info'):  $I \sqcup J = ((E'', repl'', obj'', oper'', rval'', ro'', vis'', ar''), info").$ 

By agree we have  $I \sqcup J \in \mathbb{R}$ . Then

 $(\forall (a, v), (a', v') \in V. (a = a' \Longrightarrow v = v')) \land$  $(\forall (a, v) \in V, \exists s, v(s) > 0) \land$  $(\forall (a, v) \in V. v \not\subseteq | |\{v' \mid \exists a'. (a', v') \in V \land a \neq a'\}) \land$  $\exists$  distinct  $e_{s,k}$ .

 $(\{e \in E \mid \exists a. \mathsf{oper}^e(e) = \mathsf{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicalD} \land A$  $1 \le k \le \max\{v(s) \mid \exists a, (a, v) \in V\}\}) \land$  $(\forall s, j, k. (repl''(e_{s,k}) = s) \land (e_{s,j} \xrightarrow{ra} e_{s,k} \iff j < k)) \land$  $(\forall (a, v) \in V. \forall q. \{j \mid \mathsf{oper}^{\pi}(c_{g,i}) = \mathsf{wr}(a)\} \cup$ 

 $\{j \mid \exists s, k. c_{g,j} \xrightarrow{\forall s} c_{s,k} \land oper''(c_{s,k}) = wr(a)\} =$  $\{j\mid 1\leq j\leq v(q)\})\wedge$  $(\forall e \in E. (\mathsf{oper''}(e) = \mathsf{wr}(a) \land$ 

 $\neg \exists f \in E. \mathsf{oper}''(f) = \mathsf{wr}(.) \land e \xrightarrow{\mathsf{vis}} f) \Longrightarrow (a,..) \in V$ 

 $(\forall (a,v),(a',v') \in V'.(a=a' \implies v=v')) \land$  $(\forall (a, v) \in V', \exists s, v(s) > 0) \land$  $(\forall (a, v) \in V'. v \not\subseteq \bigsqcup \{v' \mid \exists a'. (a', v') \in V' \land a \neq a'\}) \land$  $\exists$  distinct  $e_{++-}$  $\{le \in E' \mid \exists a. oper''(e) \equiv yrr(a)\} \equiv le_{+k} \mid s \in Replical D \land$ 

 $1 \le k \le \max\{v(s) \mid \exists a. (a, v) \in V'\}\} \land$  $(\forall s, i, k, (repl''(e_{+k}) = s) \land (e_{+i} \xrightarrow{nc'} e_{+k} \iff i < k)) \land$  $(\forall (a,v) \in V', \forall q, \{j \mid oper''(e_{q,j}) = wx(a)\} \cup$  $\{j \mid \exists s, k, e_{n,i} \xrightarrow{\text{vis'}} e_{s,k} \land \text{oper''}(e_{s,k}) = \text{wr}(a)\} =$ 

 $\{i \mid 1 \le i \le v(a)\}\} \land$  $(\forall e \in E', (\mathsf{oper}^n(e) = \mathsf{wr}(a) \land$  $\neg \exists f \in E', oper''(f) = vr(\bot) \land c \xrightarrow{viv} f) \Longrightarrow (a, \bot) \in V'),$ 

The agree property also implies  $\forall s, k, 1 \le k \le \min \{ \max\{v(s) \mid \exists a, (a, v) \in V \},\$ 

 $\max\{v(s) \mid \exists a. (a, v) \in V'\}\} \implies e_{s,k} = e'_{s,k}.$ Hence, there exist distinct  $e''_{s,k}$  for  $s \in \text{Replical D}$ ,  $k = 1..(\max\{v(s) \mid \exists a. (a, v) \in V^m\})$ ,

 $(\forall s, k, 1 \le k \le \max\{v(s) \mid \exists a, (a, v) \in V\} \Longrightarrow c'', = c, \iota) \land$  $(\forall s, k, 1 \le k \le \max\{v(s) \mid \exists a, (a, v) \in V'\} \Longrightarrow e''_{s,k} = e'_{s,k})$ 

 $(\{e \in E \cup E' \mid \exists a. oper''(e) = wr(a)\} =$  $\{e_{s,k}^{"} \mid s \in \text{Replical D} \land 1 \le k \le \max\{v(s) \mid \exists a. (a, v) \in V^{**}\}\}\$ 

 $\wedge (\forall s, j, k. (repl(e''_{s,k}) = s) \wedge (e''_{s,i} \xrightarrow{so''} e''_{s,k} \iff j < k)).$ By the definition of V'' and V''' we have  $\forall (a, v), (a', v') \in V''', (a = a' \Longrightarrow v = v').$ 

We also straightforwardly get  $\forall (a, v) \in V'$ ,  $\exists s. v(s) > 0$ 

 $(\forall (a, v) \in V'', \forall a, \{j \mid \mathsf{oper}''(e_s'', i) = \mathsf{wr}(a)\} \cup$ 

 $\{j \mid \exists s, k. e_{s,i}^{\prime\prime} \xrightarrow{\text{wist}^{\prime\prime}} e_{s,k}^{\prime\prime} \land \text{oper}^{\prime\prime}(e_{s,k}^{\prime\prime}) = \text{wr}(a)\} = (14)$  $\{j \mid 1 \le j \le v(q)\}\}$ .

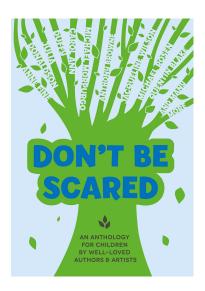


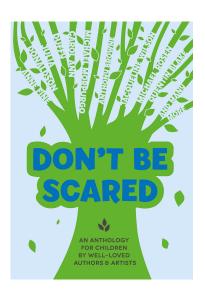
Figure 13. A selection of consistency axioms over an execution (E, repl, obj, oper, rval, ro, vis, ar)Auxiliary relations

```
sameobj(e,f) \iff obj(e) = obj(f)
Per-object causality (aka happens-before) order:
hbo = ((ro \cap sameobj) \cup vis)^+
Causality (aka happens-before) order: hb = (ro \cup vis)^+
Axioms

EVENTUAL:
ve \in E - (\exists infinitely many f \in E. sameobj(e, f) \land \neg (e \xrightarrow{vis} f))
THINAIR: ro \cup vis is acyclic
POCV (Per-Object Causal Visibility): hbo \subseteq vis
POCA (Per-Object Causal Visibility): (hb \cap sameobj) \subseteq vis
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Figure 17. Optimized state-based multi-value register and its simulation
                                                                                                             Assume (r, V) [R_r] I, V' [M] J and
                               = ReplicalD \times P(\mathbb{Z} \times (ReplicalD \rightarrow \mathbb{N}_0))
                                                                                                                       I = ((E, repl, obj, oper, rval, ro, vis, ar), info);
                               =(r,0)
                                                                                                                       J = ((E', repl', obi', oper', rval', ro', vis', ar'), info'):
                              = \mathcal{P}(\mathbb{Z} \times (\mathsf{ReplicalD} \to \mathbb{N}_0))
                                                                                                               I \sqcup J = ((E'', repl'', obj'', oper'', rval'', ro'', vis'', ar''), info").
do(wr(a), (r, V), t) =
                                                                                                             By agree we have I \sqcup J \in \mathbb{R}. Then
             (\langle r, \{(a, (\lambda s, \text{if } s \neq r \text{ then } \max\{v(s) \mid (\cdot, v) \in V\}
                                                                                                                 (\forall (a, v), (a', v') \in V. (a = a' \implies v = v')) \land
                                   else \max\{v(s) \mid (-v) \in V\} + 1)(1), \bot)
do(xd, (r, V), t) = ((r, V), \{a \mid (a, .) \in V\})
                                                                                                                 (\forall (a, v) \in V, \exists s, v(s) > 0) \land
                                                                                                                 (\forall (a, v) \in V. v \not\subseteq | |\{v' \mid \exists a'. (a', v') \in V \land a \neq a'\}) \land
send((r, V))
                                                                                                                 \exists distinct e_{s,k}.
\operatorname{receive}(\langle r, V \rangle, V') = \langle r, \{(a, v) \in V'' \mid
                                v \not\sqsubseteq \bigcup \{v' \mid \exists a'.(a',v') \in V'' \land a \neq a'\}\},
                                                                                                                 (\{e \in E \mid \exists a. \mathsf{oper}^e(e) = \mathsf{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicalD} \land A
where V'' = \{(a, \lfloor |\{v' \mid (a, v') \in V \cup V'\}) \mid (a, .) \in V \cup V'\}
                                                                                                                     1 \le k \le \max\{v(s) \mid \exists a, (a, v) \in V\}\}) \land
                                                                                                                 (\forall s, j, k. (\mathsf{repl}^{tr}(e_{s,k}) = s) \land (e_{s,j} \xrightarrow{ra} e_{s,k} \iff j < k)) \land
(s, V) [R_r] I \iff (r = s) \land (V [M] I)
                                                                                                                 (\forall (a, v) \in V. \forall q. \{j \mid \mathsf{oper}^{\pi}(c_{g,i}) = \mathsf{wr}(a)\} \cup
V[M]((E, repl. obj. oper, rval, ro, vis, ar), info) \iff
                                                                                                                      \{j \mid \exists s, k. c_{g,j} \xrightarrow{\forall s} c_{s,k} \land oper''(c_{s,k}) = wr(a)\} =
   (\forall (a, v), (a', v') \in V, (a = a', \Longrightarrow v = v')) \land
                                                                                                                      \{j\mid 1\leq j\leq v(q)\})\wedge
   (\forall (a, v) \in V, \exists s, v(s) > 0) \land
                                                                                                              (\forall e \in E. (\mathsf{oper''}(e) = \mathsf{wr}(a) \land
   (\forall (a,v) \in V. v \not\sqsubseteq \bigsqcup \{v' \mid \exists a'. (a',v') \in V \land a \neq a'\}) \land
                                                                                                                      \neg \exists f \in E. \mathsf{oper}''(f) = \mathsf{wr}(.) \land e \xrightarrow{\mathsf{vis}} f) \Longrightarrow (a,..) \in V
    ∃ distinct e<sub>s,b</sub>
   \{\ell_e \in E \mid \exists a. oper(e) = vr(a)\} = \{\ell_{e-k} \mid s \in ReplicalD \land ... \}
      1 \le k \le \max\{v(s) \mid \exists a, (a, v) \in V\}\} \land
                                                                                                                (\forall (a,v),(a',v') \in V'.(a=a' \implies v=v')) \land
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                                                                                                                (\forall (a, v) \in V', \exists s, v(s) > 0) \land
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                                                                                                                 \exists distinct e_{++-}
       \{j \mid \exists s, k. \, e_{q,j} \xrightarrow{\forall i} e_{a,k} \land \mathsf{oper}(e_{a,k}) = \mathsf{wr}(a)\} =
                                                                                                                \{e \in E' \mid \exists a. \text{ oper}''(e) = wr(a)\} = \{e_{s,k} \mid s \in \text{Replical D} \land a
       \{j \mid 1 \le j \le v(q)\}\} \land
                                                                                                                   1 \le k \le \max\{v(s) \mid \exists a. (a, v) \in V'\}\} \land
   (\forall e \in E, (oper(e) = wx(a) \land
                                                                                                                 (\forall s, i, k, (repl''(e_{+k}) = s) \land (e_{+i} \xrightarrow{no'} e_{+k} \iff i < k)) \land
       \neg\exists f \in E \text{ oper}(f) = \text{wr}(\cdot) \land e \xrightarrow{\text{w}} f) \implies (e \cdot \cdot) \in V
                                                                                                                (\forall (a,v) \in V', \forall q, \{j \mid oper''(e_{q,j}) = wx(a)\} \cup
                                                                                                                      \{j \mid \exists s, k, e_{a,i} \xrightarrow{\forall a'} e_{a,k} \land oper''(e_{a,k}) = wr(a)\} =
the former. The only non-trivial obligation is to show that if
                                                                                                                      \{i \mid 1 \le i \le v(a)\}\} \land
             V[M] ((E, repl. obi, oper, rval, ro, vis), info).
                                                                                                             (\forall e \in E', (\mathsf{oper}^n(e) = \mathsf{wr}(a) \land
                                                                                                                      \neg \exists f \in E', oper''(f) = vr(\bot) \land c \xrightarrow{viv} f) \Longrightarrow (a, \bot) \in V'),
                                                                                                                   The agree property also implies
 \{a \mid (a, \downarrow) \in V\} \subseteq \{a \mid \exists e \in E. oper(e) = wr(a) \land A\}
                                                                                                              \forall s, k, 1 \le k \le \min \{ \max\{v(s) \mid \exists a, (a, v) \in V \}, 
                     \neg \exists f \in E, \exists a', oper(e) = wr(a') \land e \xrightarrow{\forall a} f (13)
                                                                                                                                     \max\{v(s) \mid \exists a. (a, v) \in V'\}\} \implies e_{s,k} = e'_{s,k}.
(the reverse inclusion is straightforwardly implied by R_r).
    Take (a, v) \in V. We have \forall (a, v) \in V. \exists s. v(s) > 0,
                                                                                                             Hence, there exist distinct
                                                                                                             e''_{s,k} for s \in \text{Replical D}, k = 1..(\max\{v(s) \mid \exists a. (a, v) \in V^m\}),
                  v \not\sqsubseteq | \{v' \mid \exists a'. (a', v') \in V \land a \neq a'\}
                                                                                                             (\forall s, k, 1 \le k \le \max\{v(s) \mid \exists a, (a, v) \in V\} \Longrightarrow c'', = c, \iota) \land
       \forall (a, v) \in V, \forall a, \{i \mid \mathsf{oper}(e_{v,i}) = \mathsf{wr}(a)\} \cup
                                                                                                             (\forall s, k, 1 \le k \le \max\{v(s) \mid \exists a, (a, v) \in V'\} \Longrightarrow e''_{s,k} = e'_{s,k})
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                                                                                                             (\{e \in E \cup E' \mid \exists a. oper''(e) = wr(a)\} =
From this we get that for some e \in E
                                                                                                             \{e_{s,k}^{"} \mid s \in \text{Replical D} \land 1 \le k \le \max\{v(s) \mid \exists a. (a, v) \in V^{**}\}\}\
                                                                                                               \land (\forall s, j, k. (repl(e''_{s,k}) = s) \land (e''_{s,j} \xrightarrow{so''} e''_{s,k} \iff j < k)).
 oper(e) = wr(a) \land \neg \exists f \in E, \exists a', a' \neq a \land
                                                                                                                 By the definition of V'' and V''' we have
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                                                                                                                            \forall (a, v), (a', v') \in V''', (a = a' \Longrightarrow v = v').
Since vis is acyclic, this implies that for some e' \in E
                                                                                                             We also straightforwardly get
    oper(e') = wr(a) \land \neg \exists f \in E, oper(e') = wr(a) \land e' \xrightarrow{\forall a} f
                                                                                                                                          \forall (a, v) \in V', \exists s, v(s) > 0
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    Let us now discharge RECEIVE. Let receive(\langle r, V \rangle, V') =
(r, V"), where
                                                                                                               (\forall (a, v) \in V'', \forall a, \{j \mid \mathsf{oper}''(e_s'', i) = \mathsf{wr}(a)\} \cup
    V'' = \{(a, | |\{v' \mid (a, v') \in V \cup V'\}) \mid (a, ...) \in V \cup V'\}:
                                                                                                                        \{j \mid \exists s, k. e''_{s,t} \xrightarrow{\text{wid}'} e''_{s,k} \land \text{oper}''(e''_{s,k}) = \text{wr}(a)\} = (14)
    V^{\prime\prime\prime} = \{(a, v) \in V^{\prime\prime} \mid v \not\subseteq | | \{(a', v') \in V^{\prime\prime} \mid a \neq a'\} \}.
                                                                                                                        \{j \mid 1 \le j \le v(q)\}\}.
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R

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# Definition (Cartesian Products)

The Cartesian product  $A \times B$  of A and B is defined as

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$$(a,b) = (c,d) \iff a = c \land b = d$$

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Q: Are you satisfied with the definitions above?



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Definition (Ordered Pairs (Kazimierz Kuratowski; 1921))

$$(a,b) \triangleq \big\{ \{a\}, \{a,b\} \big\}$$



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$$\big\{\{a\},\{a,b\}\big\} = \big\{\{c\},\{c,d\}\big\}$$

$$(a,b) \triangleq \{\{a\},\{a,b\}\}$$

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$$\left\{ \{a\}, \{a, b\} \right\} = \left\{ \{c\}, \{c, d\} \right\}$$
 
$$\left\{ a\} \in \left\{ \{c\}, \{c, d\} \right\} \land \{a, b\} \in \left\{ \{c\}, \{c, d\} \right\}$$

$$(a,b) \triangleq \{\{a\},\{a,b\}\}$$

$$(a,b) = (c,d) \iff a = c \land b = d$$

$$\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$$
$$\{a\} \in \{\{c\}, \{c, d\}\} \land \{a, b\} \in \{\{c\}, \{c, d\}\}$$
$$(\{a\} = \{c\} \lor \{a\} = \{c, d\}) \land (\{a, b\} = \{c\} \lor \{a, b\} = \{c, d\})$$

$$(a,b) \triangleq \{\{a\},\{a,b\}\}$$

$$(a,b) = (c,d) \iff a = c \land b = d$$

$$\{\{a\}, \{a,b\}\} = \{\{c\}, \{c,d\}\}$$
 
$$\{a\} \in \{\{c\}, \{c,d\}\} \land \{a,b\} \in \{\{c\}, \{c,d\}\}$$
 
$$(\{a\} = \{c\} \lor \{a\} = \{c,d\}) \land (\{a,b\} = \{c\} \lor \{a,b\} = \{c,d\})$$
 
$$(\{a\} = \{c\} \land \{a,b\} = \{c\}) \lor (\{a\} = \{c\} \land \{a,b\} = \{c,d\}) \lor (\{a\} = \{c,d\})$$



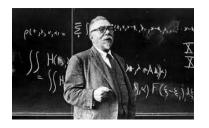
# Definition (Ordered Pairs (Norbert Wiener; 1914))

$$(a,b) \triangleq \Big\{ \big\{ \{a\},\emptyset \big\}, \big\{ \{b\} \big\} \Big\}$$



# Definition (Ordered Pairs (Norbert Wiener; 1914))

$$(a,b) \triangleq \Big\{ \big\{ \big\{ a \big\}, \emptyset \big\}, \big\{ \big\{ b \big\} \big\} \Big\}$$



$$(a,b) = (c,d) \iff a = c \land b = d$$



Definition (Cartesian Products)

The Cartesian product  $A \times B$  of A and B is defined as

$$A\times B\triangleq$$

## Definition (Cartesian Products)

The Cartesian product  $A \times B$  of A and B is defined as

$$A\times B\triangleq \{(a,b)\mid a\in A\wedge b\in B\}$$

#### Theorem

 $A \times B$  is a set.

# Thank You!