2-3 Counting

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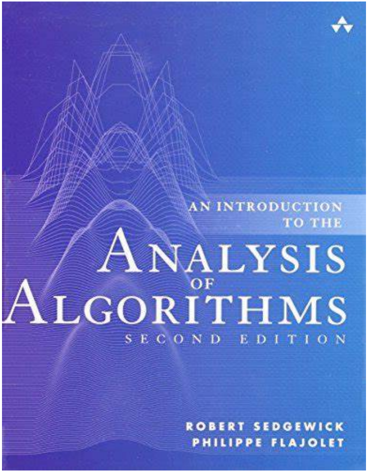
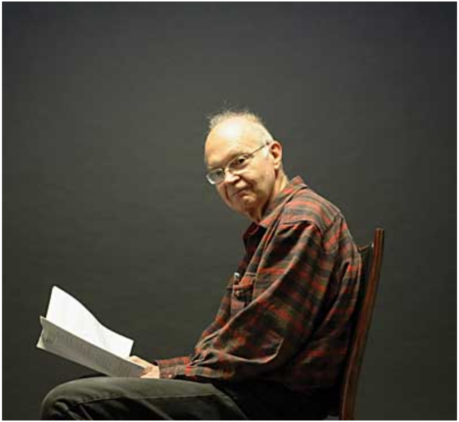


The Analysis of Algorithms

"People who analyze algorithms have double happiness ..."



Donald E. Knuth (1938 $\sim$)

Unfortunately, you have to master some mathematics.

Counting

Sums $\sum$

Binomials $\binom{n}{k}$

Counting

tuples

permutations

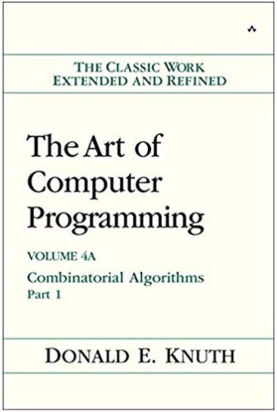
combinations

compositions

partitions

Counting # of functions under (twelve) different restrictions

Counting *vs.* Generating



Generating is about algorithms.

Falling and Rising Factorials

$$m^{\underline{n}} = (m)_n = m(m-1)(m-2)\cdots(m-n+1) = \frac{m!}{(m-n)!}$$

$$m^{\bar{n}} = m^{(n)} = m(m+1)(m+2)\cdots(m+n-1)$$

$$n! = n^{\underline{n}} = 1^{\bar{n}}$$

$$\binom{m}{n} = \frac{m^{\underline{n}}}{n!}$$

Iverson Bracket

$$[P] = \begin{cases} 1, & \text{if } P \text{ is true;} \\ 0, & \text{otherwise} \end{cases}$$

$$[n \leq m] = \begin{cases} 1, & \text{if } n \leq m; \\ 0, & \text{if } n > m \end{cases}$$

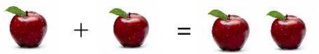
Theorem

Sum Rule

Theorem

Product Rule

先学习下加法，$1+1$，就是



所以$1+1=2$，这很好理解

那我们趁热打铁学习下一个重要公式吧：

$$\frac{\sum_{w \in W}(-1)^{\det(w)} w(e^{\lambda+\rho})}{e^{\rho}\prod_{\alpha>0}(1-e^{-\alpha})}$$

Counting # of functions under (twelve) different restrictions

$$\boxed{f : N \to M \qquad (|N| = n, \quad |M| = m)}$$

$$12 = (2 \times 2) \times 3$$

Elements of N	Elements of M	Any f	Injective f	Surjective f
distinguishable	*distinguishable*			
indistinguishable	*distinguishable*			
distinguishable	*indistinguishable*			
indistinguishable	*indistinguishable*			

Table: The Twelvefold Way (Functions).

distinguishable *vs.* indistinguishable

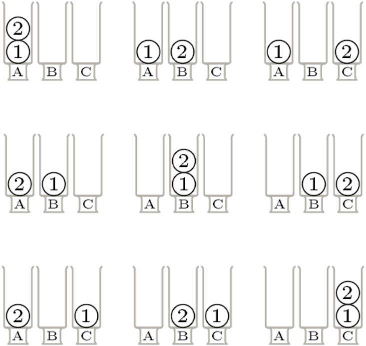
Balls	Bins	unrestricted	≤ 1	≥ 1
unlabeled	*unlabeled*			
labeled	*unlabeled*			
unlabeled	*labeled*			
labeled	*labeled*			

Table: The Twelvefold Way (Balls into Bins Model).

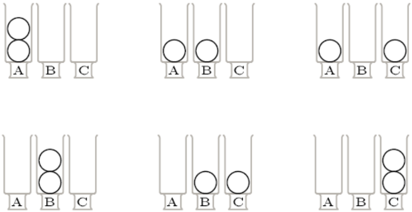
labeled *vs.* unlabeled

2 balls, 3 bins (unrestricted)

labeled balls into labeled bins



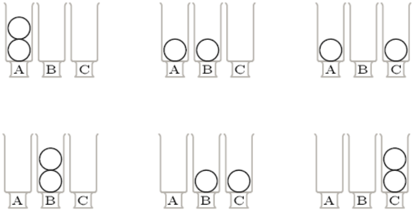
unlabeled balls into labeled bins



Only # of balls in each bin matters.

2 balls, 3 bins (unrestricted)

unlabeled balls into labeled bins



$$
\begin{aligned}
2 &= 2+0+0 \\
&= 1+1+0 \\
&= 1+0+1 \\
&= 0+2+0 \\
&= 0+1+1 \\
&= 0+0+2
\end{aligned}
$$

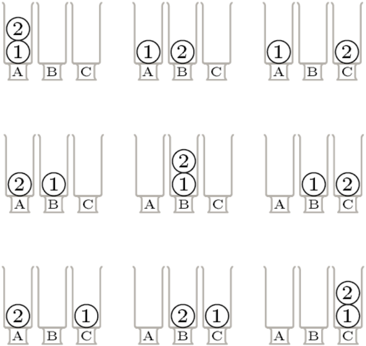
Only # of balls in each bin matters.

$$n = n_1 + n_2 + \ldots + n_m \quad (n_i \geq 0)$$

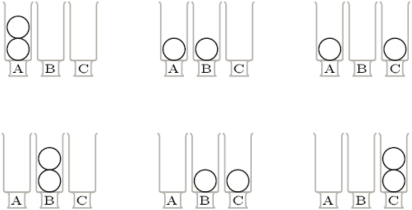
weak composition of n with m terms

2 balls, 3 bins (unrestricted)

labeled balls into labeled bins



unlabeled balls into labeled bins



$$f : 1 \mapsto A, \quad 2 \mapsto B$$

$$f' : 2 \mapsto A, \quad 1 \mapsto B$$

$$f' = f \circ (h : 1 \mapsto 2, \quad 2 \mapsto 1)$$

f class	Any f	Injective f	Surjective f
f	n tuples of m items		
$f \circ S_N$	compositions of n into m parts		

Table: The Twelvefold Way (Functions).

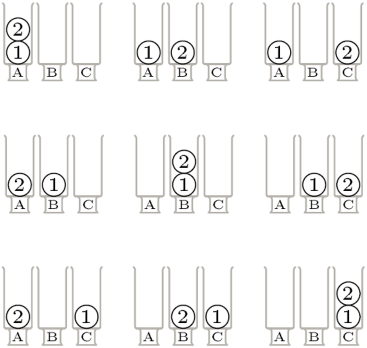
$$S_N = \{f : N \xleftrightarrow[\text{1-1}]{\text{onto}} N\}$$

$$[f] = f \circ S_N = \{f \circ g \mid g \in S_N\}$$

$$\boxed{f, g \text{ are indistinguishable} \iff g \in [f]}$$

2 balls, 3 bins (unrestricted)

labeled balls into labeled bins



labeled balls into unlabeled bins

$$\{1,2\} = \{1,2\}$$

$$= \{1\} \cup \{2\}$$

$\boxed{\text{partition of } N \text{ into } \le m \text{ parts}}$

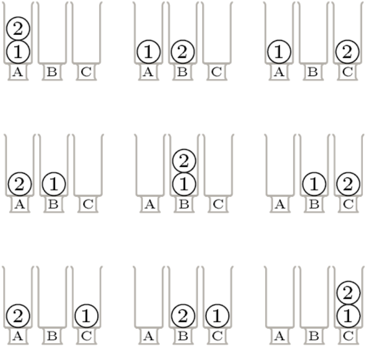
$\left\{ {n \atop k} \right\}$: # of partitions of N into k pa

$$\boxed{\sum_{k=1}^{m} \left\{ {n \atop k} \right\}}$$

$$
\begin{aligned}
\{1,2,3\} &= \{1,2,3\} \\
&= \{1\} \cup \{2,3\} \\
&= \{2\} \cup \{1,3\} \\
&= \{3\} \cup \{1,2\} \\
&= \{1\} \cup \{2\} \cup \{3\}
\end{aligned}
$$

2 balls, 3 bins (unrestricted)

labeled balls into labeled bins



labeled balls into unlabeled bins

$$f: 1 \mapsto A, \quad 2 \mapsto A$$

$$g: 1 \mapsto B, \quad 2 \mapsto B$$

$$h: 1 \mapsto C, \quad 2 \mapsto C$$

$$g = (l : A \mapsto B, \quad B \mapsto A) \circ f$$

f class	Any f	Injective f	Surjective f
f	n tuples of m items		
$f \circ S_N$	compositions of n into m parts		
$S_M \circ f$	partitions of N into $\le m$ parts		

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$$S_M = \{f : M \xleftrightarrow[\text{1-1}]{\text{onto}} M\}$$

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$S_M \circ f \circ S_N$	partitions of n into $\le m$ parts		

Table: The Twelvefold Way (Functions).

$$
\begin{aligned}
5 &= 5 \\
&= 41 \\
&= 32 = 311 \\
&= 221 = 21111 \\
&= 11111
\end{aligned}
$$

$$x_1 + x_2 + \cdots + x_k = n$$

$$x_1 \ge x_2 \ge \cdots \ge x_k \ge 1$$

$\left| \begin{matrix} n \\ k \end{matrix} \right|$: # of (integral) solutions

$$\sum_{k=1}^{m} \left| \begin{matrix} n \\ k \end{matrix} \right|$$

f class	Any f	Injective f	Surjective f
f	n-tuples of m items m^n	n-permutations of m items $m^{\underline{n}}$	???
$f \circ S_N$	weak compositions of n into m parts	n-combinations of m items $\binom{m}{n}$	compositions of n into m parts
$S_M \circ f$	partitions of N into $\le m$ parts $\sum_{k=1}^{m} \left\| {n \atop k} \right\|$	n pigeons into m holes $[n \le m]$	partitions of N into m parts $\left\| {n \atop m} \right\|$
$S_M \circ f \circ S_N$	partitions of n into $\le m$ parts $\sum_{k=1}^{m} \left\{ {n \atop k} \right\}$	n pigeons into m holes $[n \le m]$	partitions of n into m parts $\left\{ {n \atop m} \right\}$

Table: The Twelvefold Way (Functions).

Multisets (CS : 1.5 – 4)

Use multisets to determine the number of ways to pass out k **identical** apples to n children. Assume that a child may get more than one apple.

x_i : the # of apples the i-th child gets

$$x_1 + x_2 + \cdots + x_n = k, \quad x_i \geq 0$$

Integer composition (The order matters!)

$$y_i \triangleq x_i + 1$$

$$y_1 + y_2 + \cdots + y_n = n + k, \qquad y_i \geq 1$$

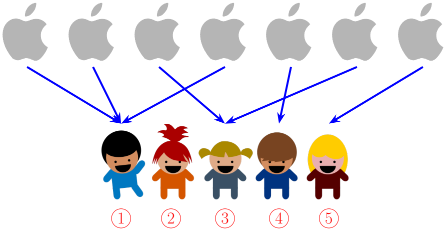
$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

Multisets (CS : 1.5 – 4)

Use **multisets** to determine the number of ways to pass out k identical apples to n children. Assume that a child may get more than one apple.

Q : k-multiset of $[1 \cdots n]$ *vs.* n-multiset of $[1 \cdots k]$

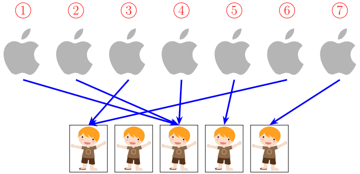
$$k = 7 \qquad n = 5$$

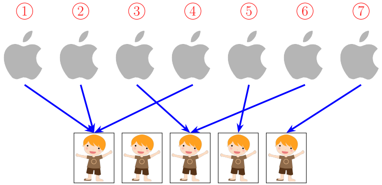


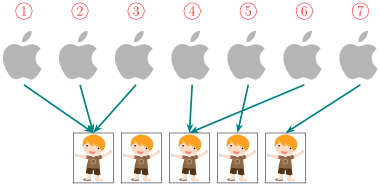
$$\{1, 1, 1, 3, 3, 4, 5\}$$

Set Partition (CS : 1.5 – 4 Extended)

What is the number of ways to pass out k distinct apples to n- .
Assume that a child may get more than one apple.







Set partition of $[1 \cdots k]$ into $\leq n$ parts

Set Partition $(\mathrm{CS}: 1.5-12)$

$$S(n,k)\ (\left\{ n \atop k \right\}) : \#\text{ of set partitions of } [1 \cdots n] \text{ into } k \text{ classes}$$

Stirling number of the second kind

Theorem (Recurrence for $S(n,k)$)

$$S(0,0)=1, \quad S(n,0)=S(0,n)=0\ (n>0)$$

$$S(n,k)=S(n-1,k-1)+kS(n-1,k), \quad n>0, k>0$$

Proof.

$$S(n,k)=\underbrace{S(n-1,k-1)}_{n \text{ is alone}}+\underbrace{kS(n-1,k)}_{n \text{ is not alone}}$$

□

$$\text{Bell number: } B_n = \sum_{k=0}^{k=n} \left\{ n \atop k \right\}$$

Theorem (Berend & Tassa (2010))

$$B_n < \left(\frac{0.792n}{\ln(n+1)} \right)^n, n \in \mathbb{Z}^+$$

Theorem (de Bruijn (1981))

As $n \to \infty$,

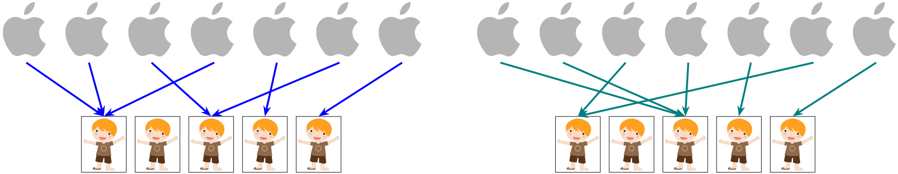
$$\frac{\ln B_n}{n} = \ln n - \ln\ln n - 1 + \frac{\ln\ln n}{\ln n} + \frac{1}{\ln n} + \frac{1}{2}\left(\frac{\ln\ln n}{\ln n}\right)^2 + O\left(\frac{\ln\ln n}{(\ln n)^2}\right)$$

THE TWELVEFOLD WAY

balls per urn	unrestricted	≤ 1	≥ 1
n labeled balls, m labeled urns	n-tuples of m things	n-permutations of m things	partitions of $\{1, \ldots, n\}$ into m ordered parts
n unlabeled balls, m labeled urns	n-multicombinations of m things	n-combinations of m things	compositions of n into m parts
n labeled balls, m unlabeled urns	partitions of $\{1, \ldots, n\}$ into $\leq m$ parts	n pigeons into m holes	partitions of $\{1, \ldots, n\}$ into m parts
n unlabeled balls, m unlabeled urns	partitions of n into $\leq m$ parts	n pigeons into m holes	partitions of n into m parts

Integer Partition (CS : 1.5 – 4 Extended)

What is the number of ways to pass out k identical apples to n- .
Assume that a child may get more than one apple.

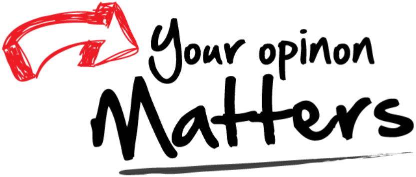


Integer partition of k into $\leq n$ parts (The order does not matter!)

Theorem (G. H. Hardy, Ramanujan (1918))

$$p(k) \triangleq \sum_{x=1}^{x=k} p_x(k) \sim \frac{1}{4\sqrt{3}k} \exp\left(\pi\sqrt{\frac{2k}{3}}\right)$$


Thank
You!



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