

1-9 Set Theory (II): Relations

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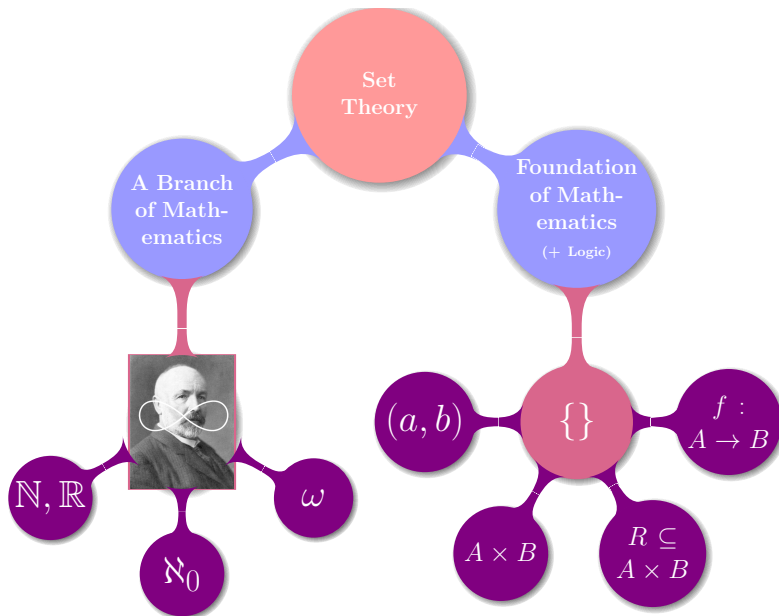




Figure 13. A selection of consistency axioms over an execution $(E, \text{repl}, \text{obj}, \text{oper}, \text{rval}, \text{ro}, \text{vis}, \text{ar})$

Auxiliary relations

sameobj(e, f) \iff obj(e) = obj(f)

Per-object causality (aka happens-before) order:

hbo = $((\text{ro} \cap \text{sameobj}) \cup \text{vis})^+$

Causality (aka happens-before) order: hb = $(\text{ro} \cup \text{vis})^+$

Axioms

EVENTUAL:

$\forall e \in E. \neg(\exists \text{infinitely many } f \in E. \text{sameobj}(e, f) \wedge \neg(e \xrightarrow{\text{vis}} f))$

THINAIR: $\text{ro} \cup \text{vis}$ is acyclic

POCV (Per-Object Causal Visibility): $\text{hbo} \subseteq \text{vis}$

POCA (Per-Object Causal Arbitration): $\text{hbo} \subseteq \text{ar}$

COCV (Cross-Object Causal Visibility): $(\text{hb} \cap \text{sameobj}) \subseteq \text{vis}$

COCA (Cross-Object Causal Arbitration): $\text{hb} \cup \text{ar}$ is acyclic

Figure 17. Optimised state-based multi-value register and its simulation

Σ	$\equiv \text{Replicated} \times \mathcal{P}(\mathbb{Z} \times (\text{Replicated} \rightarrow \mathbb{N}_0))$
\bar{a}_0	$\equiv (r, \emptyset)$
M	$\equiv \mathcal{P}(\mathbb{Z} \times (\text{Replicated} \rightarrow \mathbb{N}_0))$
$\text{do}(\text{wr}(a), (r, V), t) \equiv$	$\langle (r, \{a, k \mid a \neq r \text{ then } \max\{r(s) \mid (s, v) \in V\} \mid \perp, v\} \in V \mid$ $\quad \text{else } \max\{r(a) \mid (s, v) \in V\} + 1\}), \perp \rangle$
$\text{do}(\text{rd}, (r, V), t) \equiv$	$\langle (r, V), \{a \mid (a, v) \in V\} \rangle$
$\text{send}(\text{rd}, (r, V), V) \equiv$	$\langle (r, V), V \rangle$
$\text{receive}((r, V), V) \equiv$	$\langle r, \{a, v \in V \mid$ $\quad \exists \bar{a} \in \bar{a}_0 \text{ s.t. } \exists \bar{a}' \in V \text{ s.t. } \bar{a}' \in V'' \wedge a \neq \bar{a}'\} \rangle$
where $V'' = \{(a, v) \mid (a, v) \in V \cup V'\} \mid (a, v) \in V \cup V'\}$	
$(a, V) \text{ [R]} \cdot f \iff (r = a) \wedge (V \mid M) \cdot f$	
$V \text{ [M]} \{ (E, \text{repl}, \text{obj}, \text{oper}, \text{rval}, \text{ro}, \text{vis}, \text{ar}), \text{info} \} \iff$	
$(\forall(a, v), (a', v') \in V. (a = a' \implies v = v')) \wedge$	
$(\forall(a, v) \in V. \exists a. v(s) > 0) \wedge$	
$(\forall(a, v) \in V. v \not\subseteq \bigcup\{v' \mid \exists \bar{a}', (a', v') \in V \wedge a \neq \bar{a}'\}) \wedge$	
$\exists \text{distinct } e_{a,k}$	
$\{(e \in E \mid \exists a. \text{oper}(e) = \text{wr}(a)) = \{e_{a,k} \mid a \in \text{Replicated} \wedge$	
$1 \leq k \leq \max\{r(s) \mid \exists a. (a, v) \in V\}\}) \wedge$	
$(\forall a, j, k. (\text{repl}(e_{a,k}) = a) \wedge (e_{a,j} \xrightarrow{\text{ro}} e_{a,k} \iff j < k)) \wedge$	
$(\forall(a, v) \in V. \forall q. \{j \mid \text{oper}(e_{a,j}) = \text{wr}(a)\} \cup$	
$\{j \mid \exists a, k. e_{a,j} \xrightarrow{\text{ro}} e_{a,k} \wedge \text{oper}(e_{a,k}) = \text{wr}(a)\} =$	
$\{j \mid 1 \leq j \leq v(q)\}) \wedge$	
$(\forall e \in E. (\text{oper}(e) = \text{wr}(a)) \wedge$	
$\neg \exists f \in E. (\text{oper}(f) = \text{wr}(\perp) \wedge e \xrightarrow{\text{ro}} f) \implies (a, \perp) \in V)$	

the former. The only non-trivial obligation is to show that if

$$V \text{ [M]} \{ (E, \text{repl}, \text{obj}, \text{oper}, \text{rval}, \text{ro}, \text{vis}), \text{info} \},$$

then

$$\{a \mid (a, \perp) \in V\} \subseteq \{a \mid \exists e \in E. \text{oper}(e) = \text{wr}(a) \wedge$$

$$\neg \exists f \in E. \exists \bar{a}'. \text{oper}(e) = \text{wr}(\bar{a}') \wedge e \xrightarrow{\text{ro}} f\} \quad (13)$$

(the reverse inclusion is straightforwardly implied by $R_{a,k}$).

Take $(a, v) \in V$. We have $\forall(a, v) \in V. \exists a. v(s) > 0$,

$$v \not\subseteq \bigcup\{v' \mid \exists \bar{a}', (a', v') \in V \wedge a \neq \bar{a}'\}$$

and

$$\forall(a, v) \in V. \forall q. \{j \mid \text{oper}(e_{a,j}) = \text{wr}(a)\} \cup$$

$$\{j \mid \exists a, k. e_{a,j} \xrightarrow{\text{ro}} e_{a,k} \wedge \text{oper}(e_{a,k}) = \text{wr}(a)\} =$$

$$\{j \mid 1 \leq j \leq v(q)\}$$

From this we get that for some $e \in E$

$$\text{oper}(e) = \text{wr}(a) \wedge \neg \exists f \in E. \exists \bar{a}'. \bar{a}' \neq a \wedge$$

$$\text{oper}(e) = \text{wr}(\bar{a}') \wedge e \xrightarrow{\text{ro}} f.$$

Since vis is acyclic, this implies that for some $e' \in E$

$$\text{oper}(e') = \text{wr}(a) \wedge \neg \exists f \in E. \text{oper}(e') = \text{wr}(\perp) \wedge e' \xrightarrow{\text{ro}} f,$$

which establishes (13).

Let us now discharge RECEIVE. Let $\text{receive}((r, V), V) \equiv$

(r, V'') , where

$$V'' = \{(a, v) \mid \exists \bar{a}' \in V \text{ s.t. } \bar{a}' \in V'' \wedge (a, v') \in V \cup V''\} \mid (a, v) \in V \cup V'';$$

$$V'' = \{(a, v) \in V \mid v \not\subseteq \bigcup\{(a', v') \in V'' \mid a \neq a'\}\}.$$

Assume $(r, V) \text{ [R]} \cdot f, V' \text{ [M]} \cdot J$ and

$$I = ((E, \text{repl}, \text{obj}, \text{oper}, \text{rval}, \text{ro}, \text{vis}, \text{ar}), \text{info});$$

$$J = ((E', \text{repl}', \text{obj}', \text{oper}', \text{rval}', \text{ro}', \text{vis}', \text{ar}'), \text{info}');$$

$$I \sqcup J = ((E'', \text{repl}'', \text{obj}'', \text{oper}'', \text{rval}'', \text{ro}'', \text{vis}'', \text{ar}''), \text{info}'').$$

By agree we have $I \sqcup J \in \mathcal{R} \text{Exec}$. Then

$$(\forall(a, v), (a', v') \in V. (a = a' \implies v = v')) \wedge$$

$$(\forall(a, v) \in V. \exists a. v(s) > 0) \wedge$$

$$(\forall(a, v) \in V. v \not\subseteq \bigcup\{v' \mid \exists \bar{a}', (a', v') \in V \wedge a \neq \bar{a}'\}) \wedge$$

$$\exists \text{distinct } e_{a,k}$$

$$\{(e \in E \mid \exists a. \text{oper}'(e) = \text{wr}(a)) = \{e_{a,k} \mid a \in \text{Replicated} \wedge$$

$$1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V\}\}) \wedge$$

$$(\forall a, j, k. (\text{repl}'(e_{a,k}) = a) \wedge (e_{a,j} \xrightarrow{\text{ro}} e_{a,k} \iff j < k)) \wedge$$

$$(\forall(a, v) \in V. \forall q. \{j \mid \text{oper}'(e_{a,j}) = \text{wr}(a)\} \cup$$

$$\{j \mid \exists a, k. e_{a,j} \xrightarrow{\text{ro}} e_{a,k} \wedge \text{oper}'(e_{a,k}) = \text{wr}(a)\} =$$

$$\{j \mid 1 \leq j \leq v(q)\}) \wedge$$

$$(\forall e \in E'. (\text{oper}'(e) = \text{wr}(a)) \wedge$$

$$\neg \exists f \in E'. \text{oper}'(f) = \text{wr}(\perp) \wedge e \xrightarrow{\text{ro}} f) \implies (a, \perp) \in V)$$

and

$$(\forall(a, v), (a', v') \in V'. (a = a' \implies v = v')) \wedge$$

$$(\forall(a, v) \in V'. \exists a. v(s) > 0) \wedge$$

$$(\forall(a, v) \in V'. v \not\subseteq \bigcup\{v' \mid \exists \bar{a}', (a', v') \in V' \wedge a \neq \bar{a}'\}) \wedge$$

$$\exists \text{distinct } e_{a,k}$$

$$\{(e \in E' \mid \exists a. \text{oper}''(e) = \text{wr}(a)) = \{e_{a,k} \mid a \in \text{Replicated} \wedge$$

$$1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V'\}\}) \wedge$$

$$(\forall a, j, k. (\text{repl}''(e_{a,k}) = a) \wedge (e_{a,j} \xrightarrow{\text{ro}} e_{a,k} \iff j < k)) \wedge$$

$$(\forall(a, v) \in V'. \forall q. \{j \mid \text{oper}''(e_{a,j}) = \text{wr}(a)\} \cup$$

$$\{j \mid \exists a, k. e_{a,j} \xrightarrow{\text{ro}} e_{a,k} \wedge \text{oper}''(e_{a,k}) = \text{wr}(a)\} =$$

$$\{j \mid 1 \leq j \leq v(q)\}) \wedge$$

$$(\forall e \in E''. (\text{oper}''(e) = \text{wr}(a)) \wedge$$

$$\neg \exists f \in E''. \text{oper}''(f) = \text{wr}(\perp) \wedge e \xrightarrow{\text{ro}} f) \implies (a, \perp) \in V').$$

The agree property also implies

$$\forall a, k. 1 \leq k \leq \min\{\max\{v(s) \mid \exists a. (a, v) \in V\},$$

$$\max\{v(s) \mid \exists a. (a, v) \in V'\}\} \implies e_{a,k} = e'_{a,k}.$$

Hence, these exist distinct

$$e''_{a,k} \text{ for } a \in \text{Replicated}, k = 1..(\max\{v(s) \mid \exists a. (a, v) \in V''\}),$$

such that

$$(\forall a, k. 1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V\} \implies e''_{a,k} = e_{a,k}) \wedge$$

$$(\forall a, k. 1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V'\} \implies e''_{a,k} = e'_{a,k})$$

and

$$\{(e \in E' \mid \exists a. \text{oper}''(e) = \text{wr}(a)) =$$

$$\{e''_{a,k} \mid a \in \text{Replicated} \wedge 1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V''\}\})$$

$$\wedge (\forall a, j, k. (\text{repl}''(e''_{a,k}) = a) \wedge (e''_{a,j} \xrightarrow{\text{ro}} e''_{a,k} \iff j < k)).$$

By the definition of V'' and V''' we have

$$(\forall(a, v), (a', v') \in V'''. (a = a' \implies v = v')).$$

We also straightforwardly get

$$(\forall(a, v) \in V'''. \exists a. v(s) > 0)$$

and

$$(\forall(a, v) \in V'''. \forall q. \{j \mid \text{oper}''(e''_{a,j}) = \text{wr}(a)\} \cup$$

$$\{j \mid \exists a, k. e''_{a,j} \xrightarrow{\text{ro}} e''_{a,k} \wedge \text{oper}''(e''_{a,k}) = \text{wr}(a)\} = \quad (14)$$

$$\{j \mid 1 \leq j \leq v(q)\}).$$



I'm so excited.



Definition (Relations)

A *relation* R from A to B is a subset of $A \times B$:

$$R \subseteq A \times B$$

Definition (Cartesian Products)

The *Cartesian product* $A \times B$ of A and B is defined as

$$A \times B \triangleq \{(a, b) \mid a \in A \wedge b \in B\}$$

Axiom (Ordered Pairs)

$$(a, b) = (c, d) \iff a = c \wedge b = d$$

Q : Are you satisfied with the definitions above?

Axiom (Ordered Pairs)

$$(a, b) = (c, d) \iff a = c \wedge b = d$$



Definition (Ordered Pairs (Kazimierz Kuratowski; 1921))

$$(a, b) \triangleq \{\{a\}, \{a, b\}\}$$

Definition (Ordered Pairs (Kazimierz Kuratowski; 1921))

$$(a, b) \triangleq \{\{a\}, \{a, b\}\}$$

Theorem

$$(a, b) = (c, d) \iff a = c \wedge b = d$$

Proof.

$$\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$$

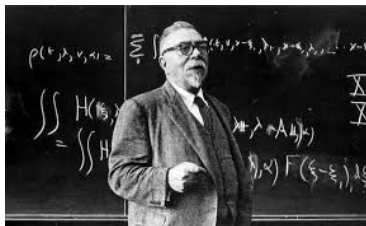
CASE I : $a = b$

CASE II : $a \neq b$



Definition (Ordered Pairs (Norbert Wiener; 1914))

$$(a, b) \triangleq \left\{ \left\{ \{a\}, \emptyset \right\}, \left\{ \{b\} \right\} \right\}$$



Theorem

$$(a, b) = (c, d) \iff a = c \wedge b = d$$

Definition (Cartesian Products)

The *Cartesian product* $A \times B$ of A and B is defined as

$$A \times B \triangleq \{(a, b) \mid a \in A \wedge b \in B\}$$

$$X^2 \triangleq X \times X$$

Theorem

$A \times B$ *is* a set.

Proof.

$$A \times B \triangleq \{(a, b) \in ? \mid a \in A \wedge b \in B\}$$

$$\{\{a\}, \{a, b\}\} \in ?\mathcal{P}(\mathcal{P}(A \cup B))$$



Definition (Relations)

A *relation* R from A to B is a subset of $A \times B$:

$$R \subseteq A \times B$$

If $A = B$, R is called a relation *on* A .

Definition (Notations)

$$(a, b) \in R \quad R(a, b) \quad aRb$$

Definition (Relations)

A *relation* R from A to B is a subset of $A \times B$:

$$R \subseteq A \times B$$

Examples

- ▶ Both $A \times B$ and \emptyset are relations from A to B .



$$< = \{(a, b) \in \mathbb{R} \times \mathbb{R} \mid a \text{ is less than } b\}$$



$$D = \{(a, b) \in \mathbb{N} \times \mathbb{N} \mid \exists q \in \mathbb{N} : a \cdot q = b\}$$

- ▶ P : the set of people

$$M = \{(a, b) \in P \times P \mid a \text{ is the mother of } b\}$$

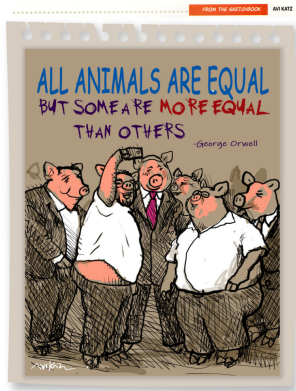
$$B = \{(a, b) \in P \times P \mid a \text{ is the brother of } b\}$$

Important Relations:

Equivalence Relations (1-9)

Functions (1-10)

Ordering Relations (1-12)



Before that,

3 Definitions

5 Operations

7 Properties

$$R = \{(0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3), (2, 3)\}$$

3 Definitions

Definition (Domain)

$$\text{dom}(R) = \{a \mid \exists b : (a, b) \in R\}$$

Theorem

$\text{dom}(R)$ *is* a set.

$$\text{dom}(R) = \{a \in \bigcup \bigcup R \mid \exists b : (a, b) \in R\}$$

$$(a, b) = \{\{a\}, \{a, b\}\} \in R$$

$$\{a, b\} \in \bigcup R$$

$$a \in \bigcup \bigcup R$$

Definition (Range)

$$\text{ran}(R) = \{b \mid \exists a : (a, b) \in R\}$$

Theorem

$\text{ran}(R)$ *is* a set.

$$\text{ran}(R) = \{b \in \bigcup \bigcup R \mid \exists a : (a, b) \in R\}$$

Definition (Field)

$$\text{fld}(R) = \text{dom}(R) \cup \text{ran}(R)$$

5 Operations

Definition (Inverse)

The *inverse* of R is the **relation**

$$R^{-1} = \{(a, b) \mid (b, a) \in R\}$$

Theorem

$$(R^{-1})^{-1} = R$$

Definition (Restriction)

The *restriction* of R to X is the **relation**

$$R|_X = \{(a, b) \in R \mid \mathbf{a} \in \mathbf{X}\}$$

Definition (Image)

The *image* of X under R is the set

$$R[X] = \{b \in \text{ran}(R) \mid \exists a \in X : (a, b) \in R\} = \text{ran}(R|_X)$$

Definition (Inverse Image)

The *inverse image* of Y under R is the set

$$R^{-1}[Y] = \{b \in \text{dom}(R) \mid \exists a \in Y : (a, b) \in R\} = \text{ran}(R^{-1}|_Y)$$

$$R \subseteq A \times B \quad X \subseteq A \quad Y \subseteq B$$

$$R^{-1}[R[X]] \stackrel{?}{=} X$$

$$R[R^{-1}[Y]] \stackrel{?}{=} Y$$



Theorem

$$R[X_1 \cup X_2] = R[X_1] \cup R[X_2]$$

$$R[X_1 \cap X_2] \subseteq R[X_1] \cap R[X_2]$$

$$R[X_1 \setminus X_2] \supseteq R[X_1] \setminus R[X_2]$$

$$b \in R[X_1 \cup X_2]$$

$$\iff \exists a \in X_1 \cup X_2 : (a, b) \in R$$

$$\iff \exists a \in X_1 : (a, b) \in R \vee \exists a \in X_2 : (a, b) \in R$$

$$\iff b \in R[X_1] \vee b \in R[X_2]$$

Definition (Composition)

The *composition* of relations R and S is the **relation**

$$R \circ S = \{(a, c) \mid \exists b : (a, b) \in S \wedge (b, c) \in R\}$$

$$R = \{(0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3), (2, 3)\}$$

$$R \circ R = \{\dots\}$$

$$\leq \circ \leq = \leq$$

$$\leq \circ \geq = \mathbb{R} \times \mathbb{R}$$

Theorem

$$(R \circ S)^{-1} = S^{-1} \circ R^{-1}$$

$$(a, b) \in (R \circ S)^{-1} \iff \dots$$

Theorem

$$(R \circ S) \circ T = R \circ (S \circ T)$$

$$(a, b) \in (R \circ S) \circ T \iff \dots$$

$$(a, b) \in (R \circ S) \circ T$$

$$\iff \exists c : (a, c) \in T \wedge (c, b) \in R \circ S$$

$$\iff \exists c : (a, c) \in T \wedge (\exists d : (c, d) \in S \wedge (d, b) \in R)$$

$$\iff \exists d : \exists c : (a, c) \in T \wedge (c, d) \in S \wedge (d, b) \in R$$

$$\iff \exists d : (\exists c : (a, c) \in T \wedge (c, d) \in S) \wedge (d, b) \in R$$

$$\iff \exists d : (a, d) \in S \circ T \wedge (d, b) \in R$$

$$\iff (a, b) \in R \circ (S \circ T)$$



燕小六：“帮我照顾好我七舅姥爷和我外甥女”

“舅姥爷”: 姥姥的兄弟

$$G = \{(a, b) : a \text{ 是 } b \text{ 的舅姥爷}\}$$

$$M = \{(a, b) \mid a \text{ is the mother of } b\}$$

$$B = \{(a, b) \mid a \text{ is the brother of } b\}$$

$$G = B \circ (M \circ M)$$

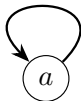
$$G = B \circ (M \circ M) = (B \circ M) \circ M$$

7 Properties

$$R \subseteq X \times X$$

Definition (Reflexive)

$$\forall a \in X : (a, a) \in R$$



Definition (Irreflexive)

$$\forall a \in X : (a, a) \notin R$$

$$A = \{1, 2, 3\}, R \subseteq A \times A$$

$$\{(1, 1), (1, 3), (2, 1), (2, 2), (3, 3)\}$$

$$\{(1, 2), (2, 3), (3, 1)\}$$

$$\{(1, 2), (2, 2), (2, 3), (3, 1)\}$$

$$R \subseteq X \times X$$

Definition (Symmetric)

$$\forall a, b \in X : aRb \implies bRa$$



Definition (AntiSymmetric)

$$\forall a, b \in X : (aRb \wedge bRa) \implies a = b$$

> *is* antisymmetric.

$$A = \{1, 2, 3\}, R \subseteq A \times A$$

$$\{(1, 1), (1, 2), (1, 3), (2, 1), (3, 1), (3, 3)\}$$

$$\{(1, 2), (2, 3), (2, 2), (3, 1)\}$$

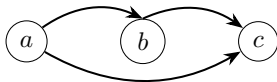
$$\{(1, 1), (2, 2), (3, 3)\}$$

$$\{(1, 2), (2, 1), (2, 3)\}$$

$$R \subseteq X \times X$$

Definition (Transitive)

$$\forall a, b, c \in X : aRb \wedge bRc \implies aRc$$



$$A = \{1, 2, 3\}, R \subseteq A \times A$$

$$\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}$$

$$\{(1, 2), (2, 3), (3, 1)\}$$

$$\{(1, 3)\}$$

$$\emptyset$$

$$R \subseteq X \times X$$

Definition (Connex)

$$\forall a, b \in X : aRb \vee bRa$$

Definition (Trichotomous)

$$\forall a, b \in X : \text{exactly one of } aRb, bRa, \text{ or } a = b \text{ holds}$$

Theorem

$$R \text{ is reflexive} \iff I \subseteq R$$

$$I = \{(a, a) \in A \times A \mid a \in A\}$$

Theorem

$$R \text{ is symmetric} \iff R^{-1} = R$$

Theorem

$$R \text{ is transitive} \iff R \circ R \subseteq R$$

$$(1, 2), (2, 3), (1, 3), (4, 4)$$

Equivalence Relations

Definition (Equivalence Relation)

R is an *equivalence relation* on X iff R is

- ▶ reflexive
- ▶ symmetric
- ▶ transitive

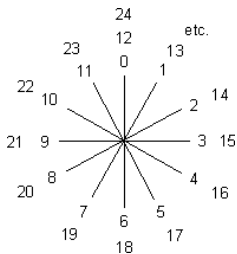
$$= \in \mathbb{R} \times \mathbb{R}$$

$$\parallel \in \mathbb{L} \times \mathbb{L}$$

$$a \sim b \iff a \% 12 = b \% 12$$

Why are equivalence relations important?

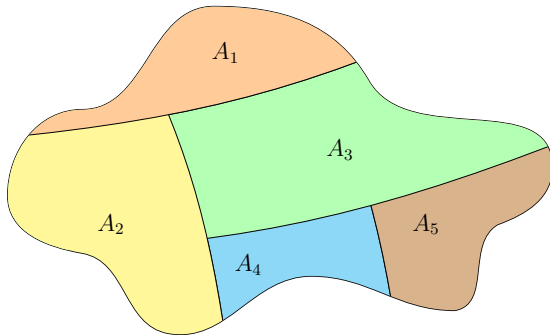
Equivalence Relations as Abstractions



“全国人民代表大会各省代表团”

Equivalence Relation \iff Partition

Partition



“不空、不漏、不重”

Definition (Partition)

A family of sets $\{A_\alpha : \alpha \in I\}$ is a *partition* of X if

(i)

$$\begin{aligned} &\forall \alpha \in I : A_\alpha \neq \emptyset \\ &(\forall \alpha \in I \exists x \in X : x \in A_\alpha) \end{aligned}$$

(ii)

$$\begin{aligned} &\bigcup_{\alpha \in I} A_\alpha = X \\ &(\forall x \in X \exists \alpha \in I : x \in A_\alpha) \end{aligned}$$

(iii)

$$\begin{aligned} &\forall \alpha, \beta \in I : A_\alpha \cap A_\beta = \emptyset \vee A_\alpha = A_\beta \\ &(\forall \alpha, \beta \in I : A_\alpha \cap A_\beta \neq \emptyset \implies A_\alpha = A_\beta) \end{aligned}$$

Equivalence Relation $R \subseteq X \times X \implies$ Partition Π of X

Definition (Equivalence Class)

The *equivalence class* of a modulo R is a **set**:

$$[a]_R = \{b \in X : aRb\}$$

Definition (Quotient Set)

The *quotient set* is a **set**:

$$X/R = \{[a]_R \mid a \in X\}$$

Theorem

$X/R = \{[a]_R \mid a \in X\}$ is a partition of X .

$$\forall a \in X : [a]_R \neq \emptyset$$

$$\forall a \in X : \exists b \in X : a \in [b]_R$$

Theorem

$$\forall a \in X, b \in X : [a]_R \cap [b]_R = \emptyset \vee [a]_R = [b]_R$$

$$\forall a \in X, b \in X : [a]_R \cap [b]_R \neq \emptyset \implies [a]_R = [b]_R$$

Partition Π of $X \implies$ Equivalence Relation $R \subseteq X \times X$

Definition

$$(a, b) \in R \iff \exists S \in \Pi : a \in S \wedge b \in S$$

$$R = \{(a, b) \in X \times X \mid \exists S \in \Pi : a \in S \wedge b \in S\}$$

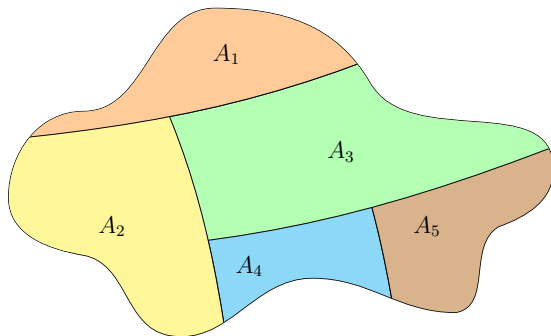
Theorem

R is an equivalence relation on X .

$$\forall x \in X : xRx$$

$$\forall x, y \in X : xRy \implies yRx$$

$$\forall x, y, z \in X : xRy \wedge yRz \implies xRz$$



Equivalence Relation \iff Partition

Definition

$$\sim \subseteq \mathbb{N}^2 \times \mathbb{N}^2$$

$$(a, b) \sim (c, d) \iff a +_{\mathbb{N}} d = b +_{\mathbb{N}} c$$

Theorem

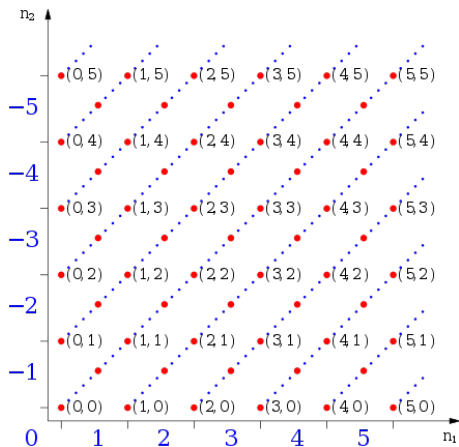
\sim is an equivalence relation.

Q : What is $\mathbb{N} \times \mathbb{N} / \sim$?

Definition (\mathbb{Z})

$$\mathbb{Z} \triangleq \mathbb{N} \times \mathbb{N} / \sim$$

$$[(1, 3)]_{\sim} = \{(0, 2), (1, 3), (2, 4), (3, 5), \dots\} \triangleq -2 \in \mathbb{Z}$$



$$\mathbb{Z} \triangleq \mathbb{N} \times \mathbb{N} / \sim$$

Definition $(+_{\mathbb{Z}})$

$$[(m_1, n_1)] +_{\mathbb{Z}} [(m_2, n_2)] = [m_1 +_{\mathbb{N}} m_2, n_1 +_{\mathbb{N}} n_2]$$

Definition $(\cdot_{\mathbb{Z}})$

$$\begin{aligned} & [(m_1, n_1)] \cdot_{\mathbb{Z}} [(m_2, n_2)] \\ &= [m_1 \cdot_{\mathbb{N}} m_2 +_{\mathbb{N}} n_1 \cdot_{\mathbb{N}} n_2, m_1 \cdot_{\mathbb{N}} n_2 +_{\mathbb{N}} n_1 \cdot_{\mathbb{N}} m_2] \end{aligned}$$

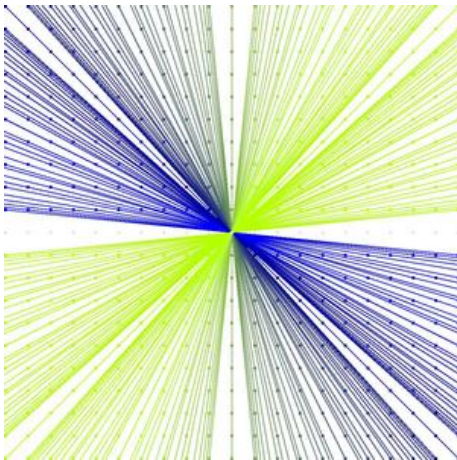
Definition

$$\sim \subseteq (\mathbb{Z} \times \mathbb{Z} \setminus \{0_{\mathbb{Z}}\})^2$$

$$(a, b) \sim (c, d) \iff a \cdot_{\mathbb{Z}} d = b \cdot_{\mathbb{Z}} c$$

Definition (\mathbb{Q})

$$\mathbb{Q} \triangleq \mathbb{Z} \times \mathbb{Z} / \sim$$



$$\mathbb{Q} \triangleq \mathbb{Z} \times \mathbb{Z} / \sim$$

How to define \mathbb{R} as equivalence classes of ordered pairs of \mathbb{Q} ?



Thank
You!