2-3 Counting

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Weak composition of n with m terms

$$\begin{aligned} 2 &= 2+0+0 \\ &= 1+1+0 \\ &= 1+0+1 \\ &= 0+2+0 \\ &= 0+1+1 \\ &= 0+0+2 \end{aligned}$$

$$\boxed{n = x_1 + x_2 + \ldots + x_m \quad (x_i \geq 0)}$$

$$\binom{n+m-1}{m-1} = \binom{n+m-1}{n}$$

$$\star\ \star\ \star\ \star\ |\ |\ \star\ |\ \star\ \star\ |$$

$7 = 4 + 0 + 1 + 2 + 0$ (Stars and Bars)

Placing $m-1$ bars into $n+(m-1)$ slots.

Composition of n with m terms

$$\boxed{n = x_1 + x_2 + \ldots + x_m \quad (x_i > 0)}$$

$$\star\ \star\ \star\ \star \mid \star \mid \star\ \star$$

$$7 = 4 + 1 + 2 \quad \text{(Stars and Bars)}$$

$$\boxed{\text{Placing } m - 1 \text{ bars into } n - 1 \text{ slots.}}$$

$$\binom{n-1}{m-1}$$

Theorem

The # of *weak composition* *of n with m terms is*

$$\binom{n+m-1}{m-1} = \binom{n+m-1}{n}$$

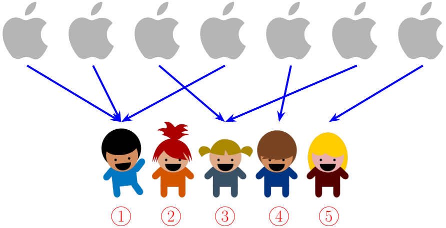
Theorem (CS Theorem 1.8)

The # of n-element *multisets* *chosen from an m-element set is*

$$\binom{n+m-1}{m-1} = \binom{n+m-1}{n}$$

$n = 7 \quad m = 5$ (Apples and Children)

$$\{1, \quad 1, \quad 3, \quad 1, \quad 4, \quad 3, \quad 5\}$$



$$7 = 3 + 0 + 2 + 1 + 1$$

$$
\begin{aligned}
\{1,2,3\} &= \{1,2,3\} \\
&= \{1\} \cup \{2,3\} \\
&= \{2\} \cup \{1,3\} \\
&= \{3\} \cup \{1,2\} \\
&= \{1\} \cup \{2\} \cup \{3\}
\end{aligned}
$$

$\left\{ {n \atop k} \right\}$: # of partitions of N into k parts

Stirling number of the second kind

Set Partition (CS : 1.5 – 12)

$$S(n, k) \ (\left\{ {n \atop k} \right\}) : \# \text{ of set partitions of } [1 \cdots n] \text{ into } k \text{ classes}$$

Theorem (Recurrence for $S(n, k)$)

$$S(0, 0) = 1, \quad S(n, 0) = S(0, n) = 0 \ (n > 0)$$

$$S(n, k) = S(n-1, k-1) + kS(n-1, k), \quad n > 0, k > 0$$

Proof.

$$S(n, k) = \underbrace{S(n-1, k-1)}_{n \text{ is alone}} + \underbrace{kS(n-1, k)}_{n \text{ is not alone}}$$

□

$$\text{Bell number: } B_n = \sum_{k=1}^{k=n} \left\{ {n \atop k} \right\}$$

Theorem (Berend & Tassa (2010))

$$B_n < \left(\frac{0.792n}{\ln(n+1)} \right)^n, n \in \mathbb{Z}^+$$

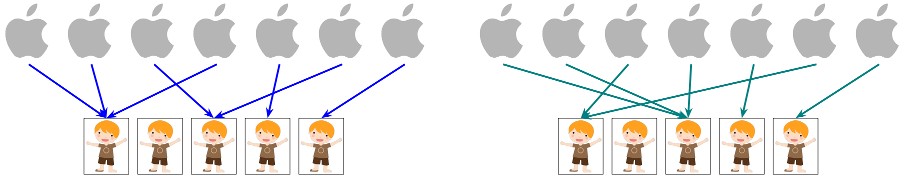
Theorem (de Bruijn (1981))

As $n \to \infty$,

$$\frac{\ln B_n}{n} = \ln n - \ln \ln n - 1 + \frac{\ln \ln n}{\ln n} + \frac{1}{\ln n} + \frac{1}{2} \left(\frac{\ln \ln n}{\ln n} \right)^2 + O\left(\frac{\ln \ln n}{(\ln n)^2} \right)$$

Integer Partition (CS : 1.5 – 4 Extended)

What is the number of ways to pass out k identical apples to n- .
Assume that a child may get more than one apple.

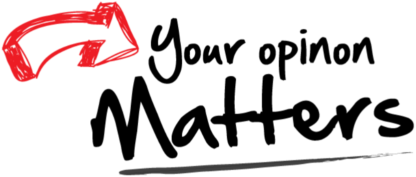


Integer partition of k into $\leq n$ parts (The order does not matter!)

Theorem (G. H. Hardy, Ramanujan (1918))

$$p(k) \triangleq \sum_{x=1}^{x=k} p_x(k) \sim \frac{1}{4\sqrt{3}k} \exp\left(\pi \sqrt{\frac{2k}{3}}\right)$$


Thank
You!



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