1-8 Set Theory: Axioms and Operations

魏恒峰

hfwei@nju.edu.cn

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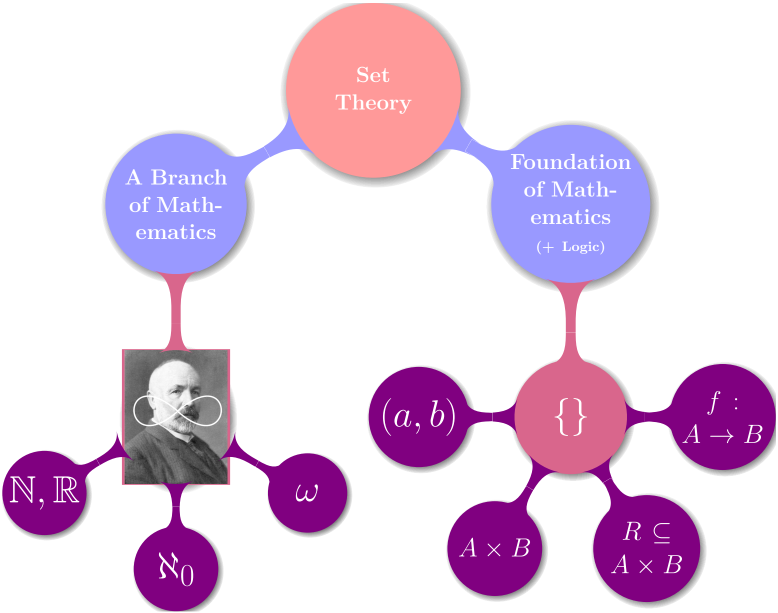
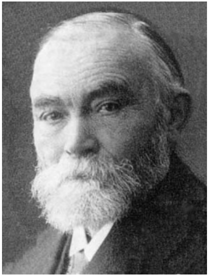
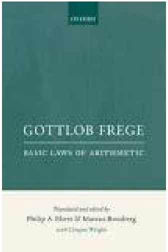

Set Theory
A Branch of Math- ematics
Foundation of Math- ematics
(+ Logic)
$\mathbb{N}, \mathbb{R}$
$\aleph_0$
ω
(a, b)
$\{\}$
$f : A \to B$
$A \times B$
$R \subseteq A \times B$



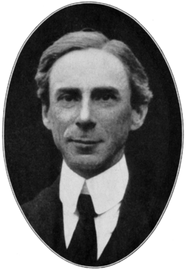
Gottlob Frege (1848–1925)



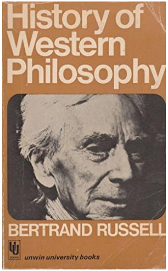
"Basic Laws of Arithmetic"
(1893 & 1903)

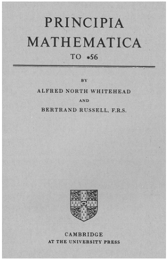
对于一个科学工作者来说，最不幸的事情莫过于：当他的工作接近完成时，却发现那大厦的基础已经动摇。

— 《附录二》，*1902*



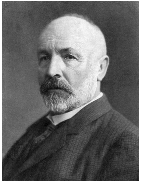
Bertrand Russell (1872–1970)





我们将集合理解为任何将我们思想中那些确定而彼此独立的对象放在一起而形成的聚合。

— *Georg Cantor*《超穷数理论基础》



Georg Cantor (1845–1918)

Theorem (概括原则)

For any predicate $\psi(x)$, there is a set X:

$$X = \{x \mid \psi(x)\}$$

Theorem (概括原则)

For any predicate $\psi(x)$*, there is a set* X*:*

$$X = \{x \mid \psi(x)\}.$$

Definition (Russell's Paradox)

$$\psi(x) \triangleq \text{“}x \notin x\text{”}$$

$$R = \{x \mid x \notin x\}$$

$$Q : R \in R\,?$$

Q : 既然朴素集合论存在悖论，你是如何做作业的?

Theorem (Russell's Paradox)

$$\{x \mid x \notin x\} \textit{ is not a set.}$$

Axiomatic Set Theory (ZFC)

Ernst Zermelo (1871–1953)

Abraham Fraenkel (1891–1965)

First-order Language for Sets $\mathcal{L}_{Set} = \{\in\}$

Parentheses: $(,)$

Variables: $x, y, z, \cdots$

Connectives: $\wedge, \vee, \neg, \to, \leftrightarrow$

Quantifiers: $\forall, \exists$

Equality: $=$

Constants:

Functions:

Predicates: $\in$

Everything we consider in $\mathcal{L}_{Set}$ is a set.

Q : What is "$\in$"?

Q : What are "sets"?

We don't define them directly.

We only describe their properties in an axiomatic way.

(1) To draw a straight line from any point to any point.

(2) To extend a finite straight line continuously in a straight line.

(3) To describe a circle with any center and radius.

(4) That all right angles are equal to one another.

(5) The parallel postulate.

Definition ($\notin$)

$$x \notin A \triangleq \neg(x \in A).$$

Definition ($\subseteq$)

$$A \subseteq B \triangleq \forall x (x \in A \implies x \in B)$$

Axiom (Axiom of Extensionality)

If two sets have exactly the same members, then they are equal.

$$\forall A \, \forall B \, (\forall x (x \in A \iff x \in B) \implies A = B).$$

$$\forall A \, \forall B \, (A \subseteq B \land B \subseteq A \implies A = B).$$

$$\forall A \, \forall B \, (A \subseteq B \land B \subseteq A \iff A = B).$$

Axiom (Empty Set Axiom)

There is a set having no members:

$$\exists B \, \forall x (x \notin B).$$

Theorem (Uniqueness of Empty Set)

There is only one empty set.

Definition ("$\emptyset$")

$$\emptyset \triangleq \text{ the unique unique empty set.}$$

Axiom (Paring Axiom)

For any sets x and y, there is a set having as members just x and y:

$$\forall x \, \forall y \, \exists B \, \big(\forall z (z \in B \iff z = x \lor z = y)\big).$$

Definition ("$\{x, y\}$")

$$\{x, y\} \triangleq \text{ the unique set obtained by paring } x \text{ and } y.$$

Theorem

$$\{x, y\} = \{y, x\}.$$

Definition ("$\{x\}$")

$$\{x\} \triangleq \{x, x\}.$$

Axiom (Union Axiom (Simplified Version))

For any sets x and y, there is a set whose members are the elements belonging either to x or to y (or both):

$$\forall x \, \forall y \, \exists B \, \big(\forall z (z \in B \iff z \in x \lor z \in y)\big).$$

Definition ("$x \cup y$")

$$x \cup y \triangleq \text{ the unique set obtained by unioning } x \text{ and } y.$$

Definition ("$\{x, y\}$")

$$\{x, y\} \triangleq \text{ the unique set obtained by paring } x \text{ and } y.$$

Definition ("$\{x\}$")

$$\{x\} \triangleq \{x, x\}.$$

Definition ("$\{x, y, z\}$")

$$\{x, y, z\} \triangleq \{x, y\} \cup \{z\}.$$

We can use pairing and union together to form finite sets.

Axiom (Union Axiom (Extended Version))

For any set A, there is a set B such that:

$$\forall x \, (x \in B \iff x \text{ belongs to some member of } A).$$

$$\forall x \big(x \in B \iff \exists y \in A (x \in y)\big).$$

Definition ("$\bigcup A$" (Arbitrary Union))

$$\bigcup A \triangleq \text{ the unique set obtained by unioning } A.$$

Theorem

$$\bigcup \{x, y\} = x \cup y.$$

Theorem

$$\bigcup \emptyset = \emptyset.$$

Axiom (Replacement Axioms (Simplified Version: Subset Axioms; Separation Axioms))

Let ψ be a predicate. For any set u, there is a set B which is a subset of u such that each element x of B satisfies $\psi(x)$:

$$\forall u \, \exists B \, (\forall x (x \in B \iff x \in u \wedge \psi(x))).$$

Definition ("$\{x \in u \mid \psi(x)\}$")

$\{x \in u \mid \psi(x)\} \triangleq$ the unique set obtained by separating from u with ψ.

Definition ("$u \cap v$")

$$u \cap v \triangleq \{x \in u \mid x \in v\}.$$

Theorem ("$\bigcap A$" (Arbitrary Intersection))

For any nonempty set A, there is a unique set B such that

$$\forall x\, (x \in B \iff x \textit{ belongs to every member of } A).$$

$$\forall x\, \big(x \in B \iff \forall y \in A (x \in y)\big).$$

Proof.

Let c be a fixed member of A.

$$\bigcap A \triangleq \{x \in c \mid x \text{ belongs to every other member of } A\}.$$

$\square$

"$\bigcap \emptyset$"

$\bigcap \emptyset$ is *not* a set.

Theorem (No Universal Set)

There is no universal set.

$$\nexists B (\forall x (x \in B)).$$

Proof.

For any set A, we construct a set not in A.

$$B = \{x \in A \mid x \notin x\}$$

$$B \in B \iff B \in A \land B \notin B$$

$$\boxed{B \notin A}$$

$$B \in A \implies (B \in B \iff B \notin B)$$

□

Definition ("$u \setminus v$" (Relative Complement))

$$u \setminus v \triangleq \{x \in u \mid x \notin v\}.$$

Theorem (No "Absolute Complement")

For any set B, the following is not a set:

$$\{x \mid x \notin B\}.$$

Proof.

By Contradiction.

$\{x \mid x \notin B\} \cup B$ would be a universal set.

□

We can never look for objects "not in B" unless we know where to start looking. — *UD (Chapter 6; Page 64)*

Axiom (Power Set Axiom)

For any set A, there is a set whose members are the subsets of A:

$$\forall A \, \exists B \, \forall x (x \in B \iff x \subseteq A).$$

Definition ("$\mathcal{P}(A)$")

$$\mathcal{P}(A) \triangleq \text{ the unique power set of } A.$$

The is *not* correct!

$$\mathcal{P}(A) \triangleq \{x \mid x \subseteq A\}$$

Set Operations (I)

$\cap \qquad \cup \qquad \setminus$

Theorem 7.4. *Let X denote a set, and A, B, and C denote subsets of X. Then*

1. $\emptyset \subseteq A$ *and* $A \subseteq A$.
2. $(A^c)^c = A$.
3. $A \cup \emptyset = A$.
4. $A \cap \emptyset = \emptyset$.
5. $A \cap A = A$.
6. $A \cup A = A$.
7. $A \cap B = B \cap A$. *(Commutative property)*
8. $A \cup B = B \cup A$. *(Commutative property)*
9. $(A \cup B) \cup C = A \cup (B \cup C)$. *(Associative property)*
10. $(A \cap B) \cap C = A \cap (B \cap C)$. *(Associative property)*
11. $A \cap B \subseteq A$.
12. $A \subseteq A \cup B$.
13. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. *(Distributive property)*
14. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. *(Distributive property)*
15. $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$. *(DeMorgan's law)*
    (When X is the universe we also write $(A \cup B)^c = A^c \cap B^c$.)
16. $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$. *(DeMorgan's law)*
    (When X is the universe we also write $(A \cap B)^c = A^c \cup B^c$.)
17. $A \setminus B = A \cap B^c$.
18. $A \subseteq B$ *if and only if* $(X \setminus B) \subseteq (X \setminus A)$.
    (When X is the universe we also write $A \subseteq B$ if and only if $B^c \subseteq A^c$.)
19. $A \subseteq C$ *and* $B \subseteq C$ *if and only if* $A \cup B \subseteq C$.
20. $C \subseteq A$ *and* $C \subseteq B$ *if and only if* $C \subseteq A \cap B$.
21. $A \cup B = A$ *if and only if* $B \subseteq A$.
22. $A \cap B = B$ *if and only if* $B \subseteq A$.

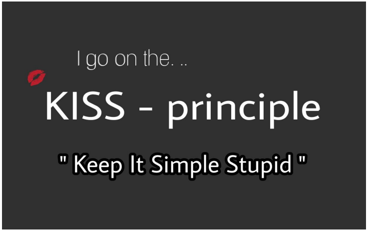
Theorem (Distributive Property (Theorem 7.4 (13)))

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Proof.
If $x \in A \cup (B \cap C)$, then $x \in A$ or $x \in B \cap C$. Suppose first that $x \in A$. Then $x \in A \cup B$ and $x \in A \cup C$. In this first case, we see that $x \in (A \cup B) \cap (A \cup C)$. Now suppose that $x \in B \cap C$. Then $x \in B$ and $x \in C$. Since $x \in B$, we see that $x \in A \cup B$. Since we also have $x \in C$, we see that $x \in A \cup C$. Therefore, $x \in (A \cup B) \cap (A \cup C)$ in this case as well. In either case $x \in (A \cup B) \cap (A \cup C)$ and we may conclude that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

To complete the proof, we must now show that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. So if $x \in (A \cup B) \cap (A \cup C)$, then $x \in A \cup B$ and $x \in A \cup C$. It is, once again, helpful to break this into two cases, since we know that either $x \in A$ or $x \notin A$. Now if $x \in A$, then $x \in A \cup (B \cap C)$. If $x \notin A$, then the fact that $x \in A \cup B$ implies that x must be in B. Similarly, the fact that $x \in A \cup C$ implies that x must be in C. Therefore, $x \in B \cap C$. Hence $x \in A \cup (B \cap C)$. In either case $x \in A \cup (B \cap C)$ and we may conclude that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

Since we proved containment in both directions we may conclude that the two sets are equal. ■



Theorem (DeMorgan's Law (Theorem 7.4 (15)))

Let X denote a set, and $A, B \subseteq X$.

$$X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$$

$$Q : A, B \subseteq X?$$

Theorem (DeMorgan's Law)

Let A, B, C be three sets.

$$C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$$

Set Operations (II)

$\bigcap \qquad \bigcup$

$$\bigcup_{j=1}^{n} A_j \triangleq A_1 \cup A_2 \cup \cdots \cup A_n \qquad \bigcap_{j=1}^{n} A_j \triangleq A_1 \cap A_2 \cap \cdots \cap A_n$$

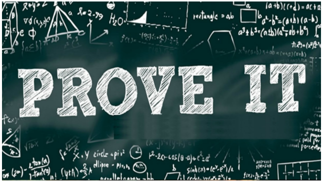
$$\bigcup_{j=1}^{\infty} A_j \triangleq A_1 \cup A_2 \cup \cdots \qquad \bigcap_{j=1}^{\infty} A_j \triangleq A_1 \cap A_2 \cap \cdots$$

$$\bigcup_{\alpha \in I} A_\alpha \triangleq \left\{ x \mid \exists \alpha \in I : x \in A_\alpha \right\} \qquad \bigcap_{\alpha \in I} A_\alpha \triangleq \left\{ x \mid \forall \alpha \in I : x \in A_\alpha \right\}$$

Theorem (DeMorgan's Law (UD Exercise 8.9))

$$X \setminus \bigcup_{\alpha \in I} A_\alpha = \bigcap_{\alpha \in I} (X \setminus A_\alpha)$$

$$X \setminus \bigcap_{\alpha \in I} A_\alpha = \bigcup_{\alpha \in I} (X \setminus A_\alpha)$$



DeMorgan's Law (UD Problem 8.14)

$$A = \mathbb{R} \setminus \bigcap_{n \in \mathbb{Z}^+} \left(\mathbb{R} \setminus \{-n, -n+1, \cdots, 0, \cdots, n-1, n\}\right)$$

$$X_n = \{-n, -n+1, \cdots, 0, \cdots, n-1, n\}$$

$$\begin{aligned} A &= \mathbb{R} \setminus \bigcap_{n \in \mathbb{Z}^+} (\mathbb{R} \setminus X_n) \\ &= \mathbb{R} \setminus \Big(\mathbb{R} \setminus \bigcup_{n \in \mathbb{Z}^+} X_n\Big) \\ &= \mathbb{R} \setminus \Big(\mathbb{R} \setminus \mathbb{Z}\Big) \\ &= \mathbb{Z} \end{aligned}$$

Set Operations (III)

$\mathcal{P}(X)$

$$\boxed{S \in \mathcal{P}(X) \iff S \subseteq X}$$

Prove that for any set S:

$$\{\emptyset, \{\emptyset\}\} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(S)))$$

Proof.

$$\{\emptyset, \{\emptyset\}\} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(S))) \iff \{\emptyset, \{\emptyset\}\} \subseteq \mathcal{P}(\mathcal{P}(S))$$

$$\{\emptyset\} \in \mathcal{P}(\mathcal{P}(S))$$

$$\emptyset \in \mathcal{P}(\mathcal{P}(S)) \qquad \iff \{\emptyset\} \subseteq \mathcal{P}(S)$$

$$\iff \emptyset \subseteq \mathcal{P}(S) \qquad \iff \emptyset \in \mathcal{P}(S)$$

$$\iff \emptyset \subseteq S$$

□

UD Exercise 9.4

$$\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$$

Proof.

$$\begin{aligned}
& x \in \mathcal{P}(A) \cap \mathcal{P}(B) \\
\iff & x \in \mathcal{P}(A) \land x \in \mathcal{P}(B) \\
\iff & x \subseteq A \land x \subseteq B \\
\iff & x \subseteq A \cap B \\
\iff & x \in \mathcal{P}(A \cap B)
\end{aligned}$$

□

UD Problem 9.9

$$\bigcap_{\alpha\in I} \mathcal{P}(A_\alpha) = \mathcal{P}(\bigcap_{\alpha\in I} A_\alpha)$$

Proof.

$$\begin{aligned} & x \in \bigcap_{\alpha\in I} \mathcal{P}(A_\alpha) \\ \iff & \forall \alpha \in I : x \in \mathcal{P}(A_\alpha) \\ \iff & \forall \alpha \in I : x \subseteq A_\alpha \\ \iff & x \subseteq \bigcap_{\alpha\in I} A_\alpha \\ \iff & x \in \mathcal{P}(\bigcap_{\alpha\in I} A_\alpha) \end{aligned}$$

□

Video:

Message To Future Generations — Bertrand Russell


Thank
You!