1-9 Set Theory (II): Relations

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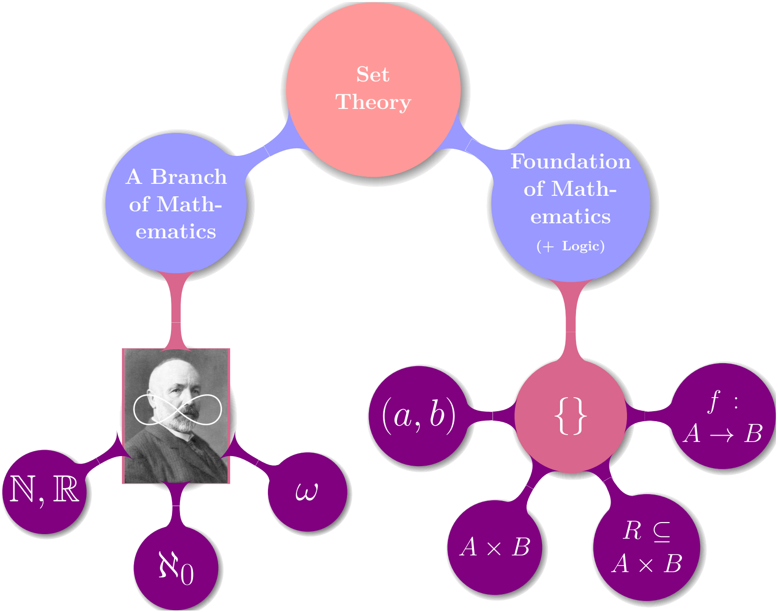
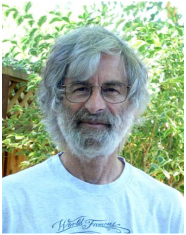

Set Theory
A Branch of Mathematics
Foundation of Mathematics
(+ Logic)
$\mathbb{N}, \mathbb{R}$
$\aleph_0$
ω
(a, b)
$\{\}$
$f : A \to B$
$A \times B$
$R \subseteq A \times B$



Time, Clocks, and the Ordering of Events in a Distributed System

Leslie Lamport
Massachusetts Computer Associates, Inc.

The concept of one event happening before another in a distributed system is examined, and is shown to define a partial ordering of the events. A distributed algorithm is given for synchronizing a system of logical clocks which can be used to totally order the events.

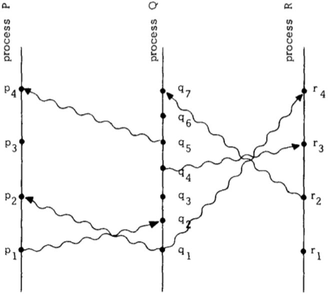


Figure 13. A selection of consistency axioms over an execution $(E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar})$

Auxiliary relations

$\mathsf{sameobj}(e, f) \iff \mathsf{obj}(e) = \mathsf{obj}(f)$

Per-object causality (aka happens-before) order:

$\mathsf{hbo} = ((\mathsf{ro} \cap \mathsf{sameobj}) \cup \mathsf{vis})^+$

Causality (aka happens-before) order: $\mathsf{hb} = (\mathsf{ro} \cup \mathsf{vis})^+$

Axioms

EVENTUAL:

$\forall e \in E.\, \neg(\exists \text{ infinitely many } f \in E.\, \mathsf{sameobj}(e, f) \wedge \neg(e \xrightarrow{\mathsf{vis}} f))$

THINAIR: $\mathsf{ro} \cup \mathsf{vis}$ is acyclic

POCV (Per-Object Causal Visibility): $\mathsf{hbo} \subseteq \mathsf{vis}$

POCA (Per-Object Causal Arbitration): $\mathsf{hbo} \subseteq \mathsf{ar}$

COCV (Cross-Object Causal Visibility): $(\mathsf{hb} \cap \mathsf{sameobj}) \subseteq \mathsf{vis}$

COCA (Cross-Object Causal Arbitration): $\mathsf{hb} \cup \mathsf{ar}$ is acyclic

Figure 17. Optimized state-based multi-value register and its simulation

$$
\begin{aligned}
\Sigma &= \mathsf{ReplicaID} \times \mathcal{P}(\mathbb{Z} \times (\mathsf{ReplicaID} \to \mathbb{N}_0)) \\
\vec{\sigma}_0 &= \langle r, \emptyset \rangle \\
M &= \mathcal{P}(\mathbb{Z} \times (\mathsf{ReplicaID} \to \mathbb{N}_0)) \\
\mathsf{do}(\mathtt{wr}(a), \langle r, V \rangle, t) &= \\
&(\langle r, \{(a, (\lambda s.\, \text{if } s \neq r \text{ then } \max\{v(s) \mid (\_, v) \in V\} \\
&\qquad \text{else } \max\{v(s) \mid (\_, v) \in V\} + 1))\}\rangle, \bot) \\
\mathsf{do}(\mathtt{rd}, \langle r, V \rangle, t) &= (\langle r, V \rangle, \{a \mid (a, \_) \in V\}) \\
\mathsf{send}(\langle r, V \rangle) &= (\langle r, V \rangle, V) \\
\mathsf{receive}(\langle r, V \rangle, V') &= \langle r, \{(a, v) \in V'' \mid \\
&\quad v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'.\, (a', v') \in V'' \wedge a \neq a'\}\}\rangle,
\end{aligned}
$$

where $V'' = \{(a, \bigsqcup\{v' \mid (a, v') \in V \cup V'\}) \mid (a, \_) \in V \cup V'\}$

$\langle s, V \rangle\ [\mathcal{R}_r]\ I \iff (r = s) \wedge (V\ [\mathcal{M}]\ I)$

$$
\begin{aligned}
&V\ [\mathcal{M}]\ ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar}), \mathsf{info}) \iff \\
&\quad (\forall (a, v), (a', v') \in V.\, (a = a' \implies v = v')) \wedge \\
&\quad (\forall (a, v) \in V.\, \exists s.\, v(s) > 0) \wedge \\
&\quad (\forall (a, v) \in V.\, v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'.\, (a', v') \in V \wedge a \neq a'\}) \wedge \\
&\quad \exists \text{ distinct } e_{s,k}. \\
&\quad (\{e \in E \mid \exists a.\, \mathsf{oper}(e) = \mathtt{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicaID} \wedge \\
&\qquad 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V\}\}) \wedge \\
&\quad (\forall s, j, k.\, (\mathsf{repl}(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}} e_{s,k} \iff j < k)) \wedge \\
&\quad (\forall (a, v) \in V.\, \forall q.\, \{j \mid \mathsf{oper}(e_{q,j}) = \mathtt{wr}(a)\} \cup \\
&\qquad \{j \mid \exists s, k.\, e_{q,j} \xrightarrow{\mathsf{vis}} e_{s,k} \wedge \mathsf{oper}(e_{s,k}) = \mathtt{wr}(a)\} = \\
&\qquad \{j \mid 1 \leq j \leq v(q)\}) \wedge \\
&\quad (\forall e \in E.\, (\mathsf{oper}(e) = \mathtt{wr}(a) \wedge \\
&\qquad \neg\exists f \in E.\, \mathsf{oper}(f) = \mathtt{wr}(\_) \wedge e \xrightarrow{\mathsf{vis}} f) \implies (a, \_) \in V)
\end{aligned}
$$

the former. The only non-trivial obligation is to show that if

$$V\ [\mathcal{M}]\ ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}), \mathsf{info}),$$

then

$$
\begin{aligned}
\{a \mid (a, \_) \in V\} \subseteq \{a \mid \exists e \in E.\, &\mathsf{oper}(e) = \mathtt{wr}(a) \wedge \\
&\neg\exists f \in E.\, \exists a'.\, \mathsf{oper}(e) = \mathtt{wr}(a') \wedge e \xrightarrow{\mathsf{vis}} f\}
\end{aligned} \tag{13}
$$

(the reverse inclusion is straightforwardly implied by $\mathcal{R}_r$).

Take $(a, v) \in V$. We have $\forall (a, v) \in V.\, \exists s.\, v(s) > 0$,

$$v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'.\, (a', v') \in V \wedge a \neq a'\}$$

and

$$
\begin{aligned}
&\forall (a, v) \in V.\, \forall q.\, \{j \mid \mathsf{oper}(e_{q,j}) = \mathtt{wr}(a)\} \cup \\
&\qquad \{j \mid \exists s, k.\, e_{q,j} \xrightarrow{\mathsf{vis}} e_{s,k} \wedge \mathsf{oper}(e_{s,k}) = \mathtt{wr}(a)\} = \\
&\qquad \{j \mid 1 \leq j \leq v(q)\}.
\end{aligned}
$$

From this we get that for some $e \in E$

$$\mathsf{oper}(e) = \mathtt{wr}(a) \wedge \neg\exists f \in E.\, \exists a'.\, a' \neq a \wedge \mathsf{oper}(e) = \mathtt{wr}(a') \wedge e \xrightarrow{\mathsf{vis}} f.$$

Since vis is acyclic, this implies that for some $e' \in E$

$$\mathsf{oper}(e') = \mathtt{wr}(a) \wedge \neg\exists f \in E.\, \mathsf{oper}(e') = \mathtt{wr}(\_) \wedge e' \xrightarrow{\mathsf{vis}} f,$$

which establishes (13).

Let us now discharge RECEIVE. Let $\mathsf{receive}(\langle r, V \rangle, V') = \langle r, V''' \rangle$, where

$$
\begin{aligned}
V'' &= \{(a, \bigsqcup\{v' \mid (a, v') \in V \cup V'\}) \mid (a, \_) \in V \cup V'\}; \\
V''' &= \{(a, v) \in V'' \mid v \not\sqsubseteq \bigsqcup\{(a', v') \in V'' \mid a \neq a'\}\}.
\end{aligned}
$$

Assume $\langle r, V \rangle\ [\mathcal{R}_r]\ I$, $V'\ [\mathcal{M}]\ J$ and

$$
\begin{aligned}
I &= ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar}), \mathsf{info}); \\
J &= ((E', \mathsf{repl}', \mathsf{obj}', \mathsf{oper}', \mathsf{rval}', \mathsf{ro}', \mathsf{vis}', \mathsf{ar}'), \mathsf{info}'); \\
I \sqcup J &= ((E'', \mathsf{repl}'', \mathsf{obj}'', \mathsf{oper}'', \mathsf{rval}'', \mathsf{ro}'', \mathsf{vis}'', \mathsf{ar}''), \mathsf{info}'').
\end{aligned}
$$

By agree we have $I \sqcup J \in \mathsf{IEx}$. Then

$$
\begin{aligned}
&(\forall (a, v), (a', v') \in V.\, (a = a' \implies v = v')) \wedge \\
&(\forall (a, v) \in V.\, \exists s.\, v(s) > 0) \wedge \\
&(\forall (a, v) \in V.\, v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'.\, (a', v') \in V \wedge a \neq a'\}) \wedge \\
&\exists \text{ distinct } e_{s,k}. \\
&(\{e \in E \mid \exists a.\, \mathsf{oper}''(e) = \mathtt{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicaID} \wedge \\
&\quad 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V\}\}) \wedge \\
&(\forall s, j, k.\, (\mathsf{repl}''(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}} e_{s,k} \iff j < k)) \wedge \\
&(\forall (a, v) \in V.\, \forall q.\, \{j \mid \mathsf{oper}''(e_{q,j}) = \mathtt{wr}(a)\} \cup \\
&\quad \{j \mid \exists s, k.\, e_{q,j} \xrightarrow{\mathsf{vis}} e_{s,k} \wedge \mathsf{oper}''(e_{s,k}) = \mathtt{wr}(a)\} = \\
&\quad \{j \mid 1 \leq j \leq v(q)\}) \wedge \\
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&\quad \neg\exists f \in E'.\, \mathsf{oper}''(f) = \mathtt{wr}(\_) \wedge e \xrightarrow{\mathsf{vis}'} f) \implies (a, \_) \in V').
\end{aligned}
$$

The agree property also implies

$$
\begin{aligned}
\forall s, k.\, 1 \leq k \leq \min\{&\max\{v(s) \mid \exists a.\, (a, v) \in V\}, \\
&\max\{v(s) \mid \exists a.\, (a, v) \in V'\}\} \implies e_{s,k} = e'_{s,k}.
\end{aligned}
$$

Hence, there exist distinct

$$e''_{s,k} \text{ for } s \in \mathsf{ReplicaID},\ k = 1..(\max\{v(s) \mid \exists a.\, (a, v) \in V'''\}),$$

such that

$$
\begin{aligned}
&(\forall s, k.\, 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V\} \implies e''_{s,k} = e_{s,k}) \wedge \\
&(\forall s, k.\, 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V'\} \implies e''_{s,k} = e'_{s,k})
\end{aligned}
$$

and

$$
\begin{aligned}
&(\{e \in E \cup E' \mid \exists a.\, \mathsf{oper}''(e) = \mathtt{wr}(a)\} = \\
&\{e''_{s,k} \mid s \in \mathsf{ReplicaID} \wedge 1 \leq k \leq \max\{v(s) \mid \exists a.\, (a, v) \in V'''\}\}) \\
&\wedge (\forall s, j, k.\, (\mathsf{repl}(e''_{s,k}) = s) \wedge (e''_{s,j} \xrightarrow{\mathsf{ro}''} e''_{s,k} \iff j < k)).
\end{aligned}
$$

By the definition of V'' and V''' we have

$$\forall (a, v), (a', v') \in V'''.\, (a = a' \implies v = v').$$

We also straightforwardly get

$$\forall (a, v) \in V'.\, \exists s.\, v(s) > 0$$

and

$$
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&V\ [\mathcal{M}]\ ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar}), \mathsf{info}) \iff\\
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&\quad (\forall s, j, k. (\mathsf{repl}(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}} e_{s,k} \iff j < k)) \wedge\\
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\{a \mid (a, \_) \in V\} \subseteq \{a \mid \exists e \in E. \mathsf{oper}(e) = \mathtt{wr}(a) \wedge\\
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(the reverse inclusion is straightforwardly implied by $\mathcal{R}_r$).

Take $(a, v) \in V$. We have $\forall (a, v) \in V. \exists s. v(s) > 0$,

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\end{aligned}
$$

Since vis is acyclic, this implies that for some $e' \in E$

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Let us now discharge RECEIVE. Let $\mathsf{receive}(\langle r, V\rangle, V') = \langle r, V'''\rangle$, where

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\end{aligned}
$$

Assume $\langle r, V\rangle\ [\mathcal{R}_r]\ I$, $V'\ [\mathcal{M}]\ J$ and

$$
\begin{aligned}
I &= ((E, \mathsf{repl}, \mathsf{obj}, \mathsf{oper}, \mathsf{rval}, \mathsf{ro}, \mathsf{vis}, \mathsf{ar}), \mathsf{info});\\
J &= ((E', \mathsf{repl}', \mathsf{obj}', \mathsf{oper}', \mathsf{rval}', \mathsf{ro}', \mathsf{vis}', \mathsf{ar}'), \mathsf{info}');\\
I \sqcup J &= ((E'', \mathsf{repl}'', \mathsf{obj}'', \mathsf{oper}'', \mathsf{rval}'', \mathsf{ro}'', \mathsf{vis}'', \mathsf{ar}''), \mathsf{info}'').
\end{aligned}
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By agree we have $I \sqcup J \in \mathsf{IEx}$. Then

$$
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&(\forall (a, v), (a', v') \in V. (a = a' \implies v = v')) \wedge\\
&(\forall (a, v) \in V. \exists s. v(s) > 0) \wedge\\
&(\forall (a, v) \in V. v \not\sqsubseteq \bigsqcup\{v' \mid \exists a'. (a', v') \in V \wedge a \neq a'\}) \wedge\\
&\exists \text{ distinct } e_{s,k}.\\
&(\{e \in E \mid \exists a. \mathsf{oper}''(e) = \mathtt{wr}(a)\} = \{e_{s,k} \mid s \in \mathsf{ReplicaID} \wedge\\
&\quad 1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V\}\}) \wedge\\
&(\forall s, j, k. (\mathsf{repl}''(e_{s,k}) = s) \wedge (e_{s,j} \xrightarrow{\mathsf{ro}} e_{s,k} \iff j < k)) \wedge\\
&(\forall (a, v) \in V. \forall q. \{j \mid \mathsf{oper}''(e_{q,j}) = \mathtt{wr}(a)\} \cup\\
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The agree property also implies

$$
\begin{aligned}
&\forall s, k. 1 \leq k \leq \min\{\max\{v(s) \mid \exists a. (a, v) \in V\},\\
&\qquad \max\{v(s) \mid \exists a. (a, v) \in V'\}\} \implies e_{s,k} = e'_{s,k}.
\end{aligned}
$$

Hence, there exist distinct

$$e''_{s,k} \text{ for } s \in \mathsf{ReplicaID},\ k = 1..(\max\{v(s) \mid \exists a. (a, v) \in V'''\}),$$

such that

$$
\begin{aligned}
&(\forall s, k. 1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V\} \implies e''_{s,k} = e_{s,k}) \wedge\\
&(\forall s, k. 1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V'\} \implies e''_{s,k} = e'_{s,k})
\end{aligned}
$$

and

$$
\begin{aligned}
&(\{e \in E \cup E' \mid \exists a. \mathsf{oper}''(e) = \mathtt{wr}(a)\} =\\
&\{e''_{s,k} \mid s \in \mathsf{ReplicaID} \wedge 1 \leq k \leq \max\{v(s) \mid \exists a. (a, v) \in V'''\}\})\\
&\wedge (\forall s, j, k. (\mathsf{repl}(e''_{s,k}) = s) \wedge (e''_{s,j} \xrightarrow{\mathsf{ro}''} e''_{s,k} \iff j < k)).
\end{aligned}
$$

By the definition of V'' and V''' we have

$$\forall (a, v), (a', v') \in V'''. (a = a' \implies v = v').$$

We also straightforwardly get

$$\forall (a, v) \in V'. \exists s. v(s) > 0$$

and

$$
\begin{aligned}
&(\forall (a, v) \in V''. \forall q. \{j \mid \mathsf{oper}''(e''_{q,j}) = \mathtt{wr}(a)\} \cup\\
&\quad \{j \mid \exists s, k. e''_{q,j} \xrightarrow{\mathsf{vis}''} e''_{s,k} \wedge \mathsf{oper}''(e''_{s,k}) = \mathtt{wr}(a)\} =\\
&\quad \{j \mid 1 \leq j \leq v(q)\}).
\end{aligned} \tag{14}
$$


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Q : Are you satisfied with the definitions above?

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$$(\{a\} = \{c\} \lor \{a\} = \{c, d\}) \land (\{a, b\} = \{c\} \lor \{a, b\} = \{c, d\})$$

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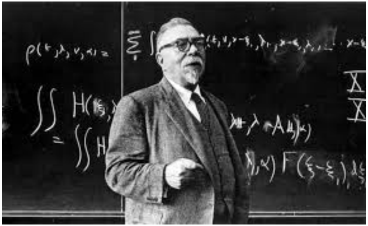
Definition (Ordered Pairs (Norbert Wiener; 1914))

$$(a, b) \triangleq \Big\{\{\{a\}, \emptyset\}, \{\{b\}\}\Big\}$$



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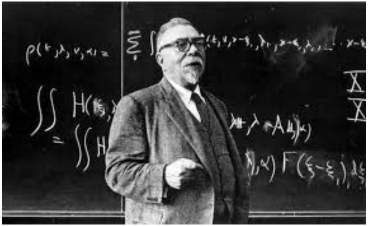

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Theorem

$A \times B$ is a set.


Thank
You!