1-10 Set Theory (III): Functions

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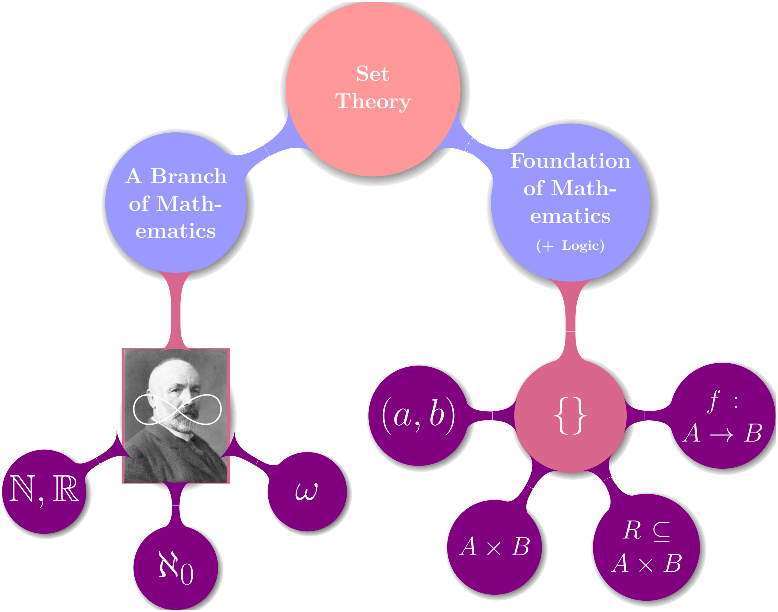

Set Theory
A Branch of Math-ematics
Foundation of Math-ematics
(+ Logic)
$\mathbb{N}, \mathbb{R}$
$\aleph_0$
ω
(a, b)
$\{\}$
$f : A \to B$
$A \times B$
$R \subseteq A \times B$

Functions

PROOF!

Definition of Functions

$$R \subseteq A \times B$$

is a *relation* from A to B

Definition (Function)

$R \subseteq A \times B$ is a *function* from A to B if

$$\forall a \in A : \exists! b \in B : (a, b) \in f.$$

$$f : A \to B$$

$$dom(f) = A \qquad cod(f) = B$$

$$ran(f) = f(A) \subseteq B$$

$$f : a \mapsto b$$

$$f(a) \triangleq b$$

Definition (Function)

$R \subseteq A \times B$ is a *function* from A to B if

$$\forall a \in A : \exists! b \in B : (a, b) \in f.$$

For Proof:

$$\forall a \in A :$$

$$\forall a \in A : \exists b \in B : (a, b) \in f$$

$$\exists! b \in B :$$

$$\forall b, b' \in B : (a, b) \in f \land (a, b') \in f \implies b = b'$$

Definition

The *set* of all functions from X to Y:

$$Y^X = \{f \mid f : X \to Y\}$$

$$Y^X = \{f \in \mathcal{P}(X \times Y) \mid f : X \to Y\}$$

X and Y are finite sets with x and y elements, respectively.

$$|X| = x \quad |Y| = y, \qquad |Y^X| = y^x$$

Definition

The set of all functions from X to Y:

$$Y^X = \{f \mid f : X \to Y\}$$

$$\forall Y : Y^{\emptyset} = \{\emptyset\}$$

$$\emptyset^{\emptyset} = \{\emptyset\}$$

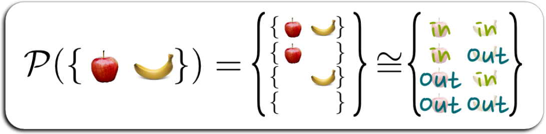
$$\forall X \neq \emptyset : \emptyset^X = \emptyset$$

Definition

The set of all functions from X to Y:

$$Y^X = \{f \mid f : X \to Y\}$$

$$2^X = \{0, 1\}^X \cong \mathcal{P}(X)$$



Definition

The set of all functions from X to Y:

$$Y^X = \{f \mid f : X \to Y\}$$

Q : Is there a set consisting of all functions?

Theorem

There is no set consisting of all functions.

Suppose by contradiction that A is the set of all functions.

For every set X, there exists a function $I_X : \{X\} \to \{X\}$.

$$\bigcup_{I_X \in A} dom(I_X)$$

Functions as Sets

Axiom (Axiom of Extensionality)

$$\forall A\,\forall B\,\forall x : (x \in A \iff x \in B) \iff A = B.$$

Theorem (The Principle of Functional Extensionality)

$$f, g \ \textit{are functions}:$$

$$f = g \iff dom(f) = dom(g) \land \Big(\forall x \in dom(f) : f(x) = g(x)\Big)$$

$$\forall f\,\forall g\,\forall (a,b) : ((a,b) \in f \iff (a,b) \in g) \iff f = g.$$

It may be that $cod(f) \neq cod(g)$.

$$f : A \to B \qquad g : C \to D$$

Q : Is $f \cap g$ a function?

Theorem (Intersection of Functions)

$$f \cap g : (A \cap C) \to (B \cap D)$$

$$f : A \to B \qquad g : C \to D$$

Q : Is $f \cup g$ a function?

Theorem (Union of Functions)

$$f \cup g : (A \cup C) \to (B \cup D) \iff \forall x \in dom(f) \cap dom(g) : f(x) = g(x)$$

UD Problem 14.3 (g)

$$f : \mathbb{Q} \to \mathbb{R}$$

$$f(x) = \begin{cases} x + 1 & \text{if } x \in 2\mathbb{Z} \\ x - 1 & \text{if } x \in 3\mathbb{Z} \\ 2 & \text{otherwise} \end{cases}$$

$$x \in 6\mathbb{Z}$$

$$D : \mathbb{R} \to \mathbb{R}$$

$$D(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Dirichlet Function

Special Functions (*-jectivity*)

Definition (Injective (one-to-one; 1-1) 单射函数)

$$f : A \to B \qquad f : A \rightarrowtail B$$

$$\forall a_1, a_2 \in A : a_1 \neq a_2 \implies f(a_1) \neq f(a_2)$$

For Proof:

- To prove that f *is* 1-1:

$$\forall a_1, a_2 \in A : f(a_1) = f(a_2) \implies a_1 = a_2$$

- To show that f *is not* 1-1:

$$\exists a_1, a_2 \in A : a_1 \neq a_2 \wedge f(a_1) = f(a_2)$$

Definition (Surjective (onto) 满射函数)

$$f : A \to B \qquad f : A \twoheadrightarrow B$$

$$ran(f) = B$$

For Proof:

- To prove that f *is* onto:

$$\forall b \in B \left(\exists a \in A : f(a) = b \right)$$

- To show that f *is not* onto:

$$\exists b \in B \left(\forall a \in A : f(a) \neq b \right)$$

Definition (Bijective (one-to-one correspondence) 一一对应)

$$f : A \to B \qquad f : A \xleftrightarrow[onto]{1-1} B$$

1-1 & onto

Functions as Relations

$f|_X \qquad f(A) \qquad f^{-1}(B) \qquad f^{-1} \qquad f \circ g$

Definition (Restriction)

The *restriction* of a function f to X is the function:

$$f|_X = \{(x, y) \in f \mid x \in X\}$$

$$f : A \to B$$

$$f|_X : A \cap X \to B$$

$$f|_X(x) = f(x), \forall x \in A \cap X$$

Definition (Image)

The *image* of X under a function f is the set

$$f(X) = \{b \mid \exists a \in X : (a, b) \in f\}$$

Definition (Inverse Image)

The *inverse image* of Y under a function f is the set

$$f^{-1}(Y) = \{a \mid \exists b \in Y : (a, b) \in f\}$$

$X \subseteq dom(f),\ Y \subseteq ran(f)$ are not necessary

f may not be invertible in $f^{-1}(Y)$

$$\boxed{y \in f(X) \iff \exists x \in dom(f) \cap X : y = f(x)}$$

$$\boxed{y \in f(X) \iff \exists x \in X : y = f(x)}$$

$$\boxed{x \in f^{-1}(Y) \iff f(x) \in Y}$$

Theorem (Properties of f and f^{-1} (UD Theorem 17.7))

$$f : A \to B \qquad A_1, A_2 \subseteq A,\ B_1, B_2 \subseteq B$$

(i) *f preserves only $\subseteq$ and $\cup$:*

(1) $A_1 \subseteq A_2 \implies f(A_1) \subseteq f(A_2)$

(2) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$

(3) $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$

(4) $f(A_1 \setminus A_2) \subseteq f(A_1) \setminus f(A_2)$

(ii) *f^{-1} preserves $\subseteq, \cup, \cap,$ and $\setminus$:*

(5) $B_1 \subseteq B_2 \implies f^{-1}(B_1) \subseteq f^{-1}(B_2)$

(6) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$

(7) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$

(8) $f^{-1}(B_1 \setminus B_2) = f^{-1}(B_1) \setminus f^{-1}(B_2)$

Theorem (UD Problem 17.5)

$$f : A \to B$$

$$f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$$

$$\begin{aligned}
& b \in f(A_1 \cap A_2) \\
\implies & \exists a \in A_1 \cap A_2 \cap A : b = f(a) \\
\implies & \exists a \in A : a \in A_1 \land a \in A_2 \land b = f(a) \\
\implies & \exists a \in A \cap A_1 : b = f(a) \land \exists a \in A \cap A_2 : b = f(a) \\
\implies & b \in f(A_1) \cap f(A_2)
\end{aligned}$$

Q : When does $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ hold?

f is injective.

Theorem (Properties of f and f^{-1} (UD Theorem 17.7))

$$f : A \to B$$

(iii) *f and f^{-1}:*

(9) $A_0 \subseteq A \implies A_0 \subseteq f^{-1}(f(A_0))$

(10) $B_0 \supseteq f(f^{-1}(B_0))$

Theorem (UD Problem 17.8)

$$A_0 \subseteq A \implies A_0 \subseteq f^{-1}(f(A_0))$$

Theorem

$$f : A \to B$$

$$B_0 \supseteq f(f^{-1}(B_0))$$

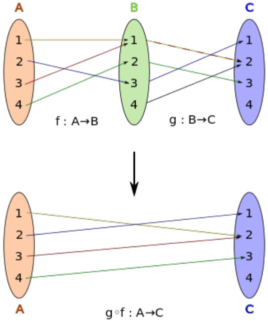
$$\begin{aligned} & b \in f(f^{-1}(B_0)) \\ \Longrightarrow\ & \exists a \in f^{-1}(B_0) : b = f(a) \\ \Longrightarrow\ & \exists a \in A : f(a) \in B_0 \wedge b = f(a) \\ \Longrightarrow\ & b \in B_0 \end{aligned}$$

Q: When does $B_0 = f(f^{-1}(B_0))$ hold?

f is surjective and $B_0 \subseteq B$.

$B_0 \subseteq ran(f)$

Function Composition



Definition (Composition)

$$f : A \to B \qquad g : C \to D$$

$$ran(f) \subseteq C$$

The *composite function* $g \circ f : A \to D$ is defined as

$$(g \circ f)(x) = g(f(x))$$

Why not "$\exists b$" as below?

Definition (Composition)

The *composition* of relations R and S is the relation

$$R \circ S = \{(a, c) \mid \exists b : (a, b) \in S \land (b, c) \in R\}$$

Theorem (Associative Property for Composition)

$$f : A \to B \quad g : B \to C \quad h : C \to D$$

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Proof.

(i)

$$dom(h \circ (g \circ f)) = dom((h \circ g) \circ f)$$

(ii)

$$(h \circ (g \circ f))(x) = ((h \circ g) \circ f)(x)$$

□

Theorem (UD Theorem 16.7)

$$f : A \to B \qquad g : B \to C$$

(i) *If f, g are injective, then $g \circ f$ is injective.*
(ii) *If f, g are surjective, then $g \circ f$ is surjective.*
(iii) *If f, g are bijective, then $g \circ f$ is bijective.*

Proof for (i).

$$\forall a_1, a_2 \in A : \Big((g \circ f)(a_1) = (g \circ f)(a_2) \implies a_1 = a_2 \Big)$$

□

Proof for (ii).

$$\forall c \in C : \Big(\exists a \in A : (g \circ f)(a) = c \Big)$$

□

Theorem (UD Theorem 16.8)

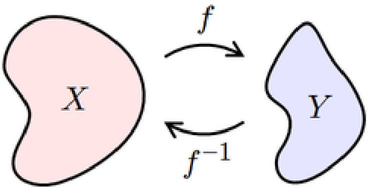
$$f : A \to B \qquad g : B \to C$$

(i) *If* $g \circ f$ *is surjective, then* g *is surjective.*

(ii) *If* $g \circ f$ *is injective, then* f *is injective.*

You can also prove it by contradiction.

Inverse Functions



Definition (Inverse)

Let $f : A \to B$ be a bijective function.

The *inverse* of f is the function $f^{-1}\colon B \to A$ defined by

$$f^{-1}(b) = a \iff f(a) = b.$$

Definition (Invertible)

$f : X \to Y$ is *invertible* if there exists $g : Y \to X$ such that

$$f(x) = y \iff g(y) = x.$$

Theorem

f is invertible $\iff$ f is bijective.

f is invertible $\implies$ f is bijective

g is a function $\implies$ f is injective

$dom(g) = Y \implies f$ is surjective

f is bijective $\implies$ f is invertible

To show that g defined above is indeed a function from Y to X.

Definition (Invertible)

$f : X \to Y$ is *invertible* if there exists $g : Y \to X$ such that

$$f(x) = y \iff g(y) = x.$$

Theorem

$g : Y \to X$ is unique.

By Contradiction

$$f^{-1} \triangleq g$$

$$\boxed{f(x) = y \iff f^{-1}(y) = x}$$

Theorem (UD Theorem 16.4)

$$f : A \to B \textit{ is bijective}$$

(i) $f \circ f^{-1} = I_B$

(ii) $f^{-1} \circ f = I_A$

(iii) f^{-1} *is bijective.*

(iv) $g : B \to A \land f \circ g = I_B \implies g = f^{-1}$

(v) $g : B \to A \land g \circ f = I_A \implies g = f^{-1}$

The ways to find/check f^{-1}.

$$g = f^{-1} \circ (f \circ g) = f^{-1} \circ I_B = f^{-1}$$

Theorem (Inverse of Composition (UD Theorem 16.6))

$$f : A \to B \quad g : B \to C \ \textit{are bijective}$$

(i) $g \circ f$ *is bijective*

(ii) $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Proof for (ii).

It suffices to check either one of the following identities:

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = I_A$$

$$(g \circ f) \circ (f^{-1} \circ g^{-1}) = I_C$$

□

Theorem (UD Theorem 16.8)

$$f : A \to B \quad g : B \to A$$

(iii) $f \circ g = I_B \land g \circ f = I_A \implies g = f^{-1}$

You need to check both identities.

Theorem (UD Theorem 16.8)

$$f : A \to B \qquad g : B \to C$$

(i) *If $g \circ f$ is surjective, then g is surjective.*

(ii) *If $g \circ f$ is injective, then f is injective.*

First show that f is bijective, and then use Theorem 16.4.


Thank
You!