1-10 Set Theory (III): Functions

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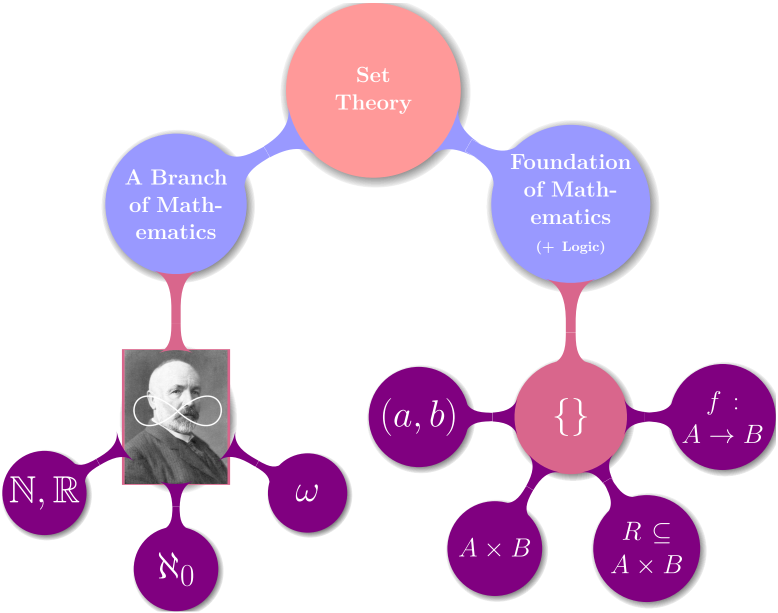
2019 年 12 月 10 日


Set Theory
A Branch of Mathematics
Foundation of Mathematics
(+ Logic)
$\mathbb{N}, \mathbb{R}$
$\aleph_0$
ω
(a, b)
$\{\}$
$f : A \to B$
$A \times B$
$R \subseteq A \times B$

Function

Function

Function

PROOF! PROOF! PROOF!

Definition of Functions

$$R \subseteq A \times B$$

is a *relation* from A to B

Definition (Function)

$R \subseteq A \times B$ is a *function* from A to B if

$$\forall a \in A : \exists! b \in B : (a, b) \in f.$$

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Notations:

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$$dom(f) = A \qquad ran(f) = f(A) \subseteq B \triangleq cod(f)$$

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$$f : a \mapsto b \triangleq f(a)$$

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For Proof:

$$\forall a \in A :$$

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$$\exists! b \in B :$$

$$\forall b, b' \in B : (a, b) \in f \land (a, b') \in f \implies b = b'$$

$$D : \mathbb{R} \to \mathbb{R}$$

$$D(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Dirichlet Function

UD Problem 14.3 (g)

$$f : \mathbb{Q} \to \mathbb{R}$$

$$f(x) = \begin{cases} x + 1 & \text{if } x \in 2\mathbb{Z} \\ x - 1 & \text{if } x \in 3\mathbb{Z} \\ 2 & \text{otherwise} \end{cases}$$

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$$x = 6$$

UD Problem 14.5

$$f : \mathcal{P}(\mathbb{R}) \to \mathbb{Z}$$

$$f(A) = \begin{cases} \min(A \cap \mathbb{N}) & \text{if } A \cap \mathbb{N} \neq \emptyset \\ -1 & \text{if } A \cap \mathbb{N} = \emptyset \end{cases}$$

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By the *Well-Ordering Principle* of $\mathbb{N}$

Axiom (Axiom of Extensionality)

$$\forall A : \forall B : \forall x : (x \in A \iff x \in B) \iff A = B.$$

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Theorem (The Principle of Functional Extensionality)

$$f, g \text{ are functions}:$$
$$f = g \iff dom(f) = dom(g) \wedge \Big(\forall x \in dom(f) : f(x) = g(x)\Big)$$

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It may be that $cod(f) \neq cod(g)$.

Definition (Intersection, Union)

$$f_1, f_2 : A \to B$$

(i) Q : Is $f_1 \cup f_2$ a function from A to B?

(ii) Q : Is $f_1 \cap f_2$ a function from A to B?


Thank You!