2-3 Counting

Hengfeng Wei

hfwei@nju.edu.cn

March 26, 2020

Solving Recurrences

Don't we already know how to use the "Master Theorem"?

You need to learn more to handle with more complex recurrences which may arise in average-case time complexity analysis.

Is it hard?

No.

Q : Recurrences without Base Cases?

"Boundary conditions represent another class of details that we typically ignore."

"We shall generally omit statements of the boundary conditions of recurrences and assume that $T(n)$ is constant for small n."

— "Technicalities in recurrences", Chapter 4, CLRS

Not Exactly!

First-order Linear Recurrence (CS 4.2 – 11)

$$T(n) = \begin{cases} 1 & n = 0 \\ 2T(n-1) + n2^n & n > 0 \end{cases}$$

Theorem (First-order Linear Recurrences with *Constant Coefficients* (CS Theorem 4.5))

$$T(n) = \begin{cases} a & n = 0 \\ rT(n-1) + g(n) & n > 0 \end{cases}$$

$$\boxed{T(n) = r^n a + \sum_{i=1}^{n} r^{n-i} g(i)}$$

Theorem (First-order Linear Recurrences)

$$T(n) = x_n T(n-1) + y_n \quad \textit{for } n > 0 \textit{ with } T(0) = 0$$

$$\boxed{T(n) = y_0 + \sum_{1 \le j < n} y_j x_{j+1} x_{j+2} \cdots x_n}$$

Proof.

$$\underbrace{\frac{T(n)}{x_n x_{n-1} \cdots x_1}}_{\text{summation factor}} = \frac{T(n-1)}{x_{n-1} \cdots x_1} + \frac{y_n}{x_n x_{n-1} \cdots x_1}$$

$$S(n) \triangleq \frac{T(n)}{x_n x_{n-1} \cdots x_1}$$

□

$$T(n) = (1 + \frac{1}{n})T(n-1) + 2 \quad \text{for } n > 1 \text{ with } T(1) = 0$$

$$x_n = 1 + \frac{1}{n} \implies x_n x_{n-1} \cdots x_1 = n + 1$$

$$\frac{T(n)}{n+1} = \frac{T(n-1)}{n} + \frac{2}{n+1} \quad \text{for } n > 1$$

$$\frac{T(n)}{n+1} = \frac{T(1)}{2} + 2 \sum_{3 \le k \le n+1} \frac{1}{k}$$

$$\boxed{T(n) = 2(n+1)(H_{n+1} - \frac{3}{2})}$$

After-class Exercise

$$T(n) = T(n-1) - \frac{2T(n-1)}{n} + 2\left(1 - \frac{2T(n-1)}{n}\right), n > 0 \text{ with } T(0) = 0$$

Theorem (Linear Recurrences with Constant Coefficients)

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_t a_{n-t} \quad \textit{for } n \geq t$$

$$a_0, a_1, \cdots, a_{t-1}$$

$$\boxed{q(x) \equiv x^t - r_1 x^{t-1} - r_2 x^{t-2} - \cdots - r_t}$$

$$\beta_1 \ (m_1), \beta_2 \ (m_2), \cdots, \beta_i \ (m_i), \cdots, \beta_k \ (m_k)$$

$$m_1 + m_2 + \cdots + m_k = t$$

$$\boxed{a_n = \sum_{0 \leq j < m_1} c_{1j} n^j \beta_1^n + \sum_{0 \leq j < m_2} c_{2j} n^j \beta_2^n + \cdots + \sum_{0 \leq j < m_k} c_{kj} n^j \beta_k^n}$$

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_t a_{n-t} \quad \text{for } n \geq t$$

$$\boxed{q(x) \equiv x^t - r_1 x^{t-1} - r_2 x^{t-2} - \cdots - r_t}$$

Proof.

$$\beta \ (m = 2)$$

$$\beta^n = r_1 \beta^{n-1} + r_2 \beta^{n-2} + \cdots + r_t \beta^{n-t} \quad \text{for } n \geq t$$

$$\beta^{n-t} q(\beta) = 0$$

$$n\beta^n = r_1 (n-1) \beta^{n-1} + r_2 (n-2) \beta^{n-2} + \cdots + r_t (n-t) \beta^{n-t} \quad \text{for } n \geq t$$

$$\beta^{n-t} \left((n-t)\, q(\beta) + \beta q'(\beta) \right) = 0$$

$\square$

$$a_n = 5a_{n-1} - 6a_{n-2},\ n \geq 2\ (a_0 = 0, a_1 = 1)$$

$$\boxed{x^2 - 5x + 6 = (x-2)(x-3) = 0} \implies x = 2, 3$$

$$\boxed{a_n = c_0 3^n + c_1 2^n}$$

$$a_0 = 0 = c_0 + c_1$$
$$a_1 = 1 = 3c_0 + 2c_1$$

$$\boxed{a_n = 3^n - 2^n}$$

$$a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3}, n \geq 3 \; (a_0 = 0, a_1 = 1, a_2 = 4)$$

$$\boxed{x^3 - 5x^2 + 8x - 4 = 0}$$

$$(x-1)(x-2)^2 = 0 \implies x_1 = 1, x_2 = 2, x_2' = 2$$

$$a_n = c_1 \cdot 1^n + c_2 \cdot 2^n + c_2' \cdot n2^n$$

$$a_0 = 0 = c_1 + c_2$$
$$a_1 = 1 = c_1 + 2c_2 + 2c_2'$$
$$a_2 = 4 = c_1 + 4c_2 + 8c_2'$$

$$\boxed{a_n = n2^{n-1}}$$

$$a_n = 2a_{n-1} - a_{n-2} + 2a_{n-3},\ n \ge 3\ (a_0 = 1, a_1 = 0, a_2 = -1)$$

$$\boxed{x^3 - 2x^2 + x - 2} = (x^2+1)(x-2) = 0 \implies x = 2, i, -i$$

$$\boxed{a_n = c_1 2^n + c_2 i^n + c_3 (-i)^n}$$

$$\begin{aligned} a_0 &= 1 = c_1 + c_2 + c_3 \\ a_1 &= 0 = 2c_1 + c_2 i - c_3 i \\ a_2 &= -1 = 4c_1 - c_2 - c_3 \end{aligned}$$

$$\boxed{a_n = \frac{1}{2} i^n \left(1 + (-1)^n\right)}$$

$$1,\ 0,\ -1,\ 0,\ 1,\ 0,\ -1,\ 0,\ 1,\ 0,\ -1,\ 0 \cdots$$

After-class Exercise

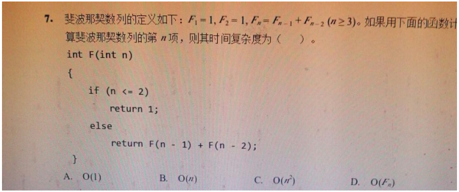
To give initial conditions a_0, a_1, and a_2 such that the growth rate of the solution to

$$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}, \ n > 2$$

is *(1) constant*; *(2) exponential*; *(3) fluctuating in sign.*

First-order Linear Non-homogeneous Recurrences

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_t a_{n-t} + r \quad \text{for } n \geq t$$



7. 斐波那契数列的定义如下：$F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}\ (n \geq 3)$。如果用下面的函数计算斐波那契数列的第 n 项，则其时间复杂度为（　　）。

```
int F(int n)
{
    if (n <= 2)
        return 1;
    else
        return F(n - 1) + F(n - 2);
}
```

A. $O(1)$　B. $O(n)$　C. $O(n^2)$　D. $O(F_n)$

$$F(n) = F(n-1) + F(n-2) + 2,\ n \geq 3\ (F(1) = F(2) = 0)$$

$$\boxed{T(n,k) = \begin{cases} 0, & k = 0 \vee n = k \\ T(n-1,k) + T(n-1,k-1) + c, & o.w. \end{cases}}$$

$$a_n = 5a_{n-1} - 6a_{n-2} + 2,\ n \geq 2\ (a_0 = 0, a_1 = 1)$$

$$\boxed{a_n = c_0 3^n - c_1 2^n + c_2}$$

$$\boxed{c_2 = 5c_2 - 6c_2 + 2} \implies c_2 = 1$$

$$a_0 = 0 = c_0 + c_1 + 1$$
$$a_1 = 1 = 3c_0 + 2c_1 + 1$$

$$\boxed{a_n = 2 \cdot 3^n - 3 \cdot 2^n + 1}$$

More Issues about Linear Recurrences

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_t a_{n-t} + g(n) \quad \text{for } n \geq t$$

$$a_n = a_n^h + a_n^p$$

How to Find a Particular Solution for a Non-homogeneous Recurrence Relation?

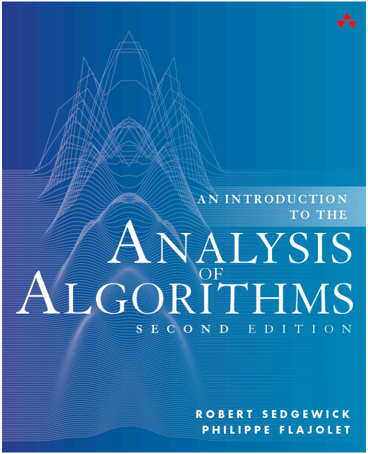
$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_t a_{n-t} \quad \text{for } n \geq t$$

$$t \geq 5$$

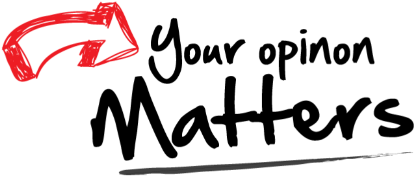
Generating Functions and Asymptotic Analysis

$$a_n = f_1(n)a_{n-1} + f_2(n)a_{n-2} + \cdots + f_t(n)a_{n-t} \quad \text{for } n \geq t$$

Generating Functions


AN INTRODUCTION
TO THE
ANALYSIS
OF
ALGORITHMS
SECOND EDITION
ROBERT SEDGEWICK
PHILIPPE FLAJOLET


Thank
You!



Office 302

Mailbox: H016

hfwei@nju.edu.cn