1-11 Set Theory (IV): Infinity

魏恒峰

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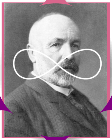
2019 年 12 月 17 日

Set Theory

A Branch of Mathematics



$\mathbb{N}, \mathbb{R}$

$\aleph_0$

ω

Foundation of Mathematics

(+ Logic)

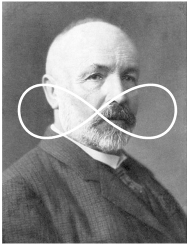
$\{\}$

(a, b)

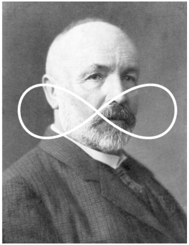
$f : A \to B$

$A \times B$

$R \subseteq A \times B$



Georg Cantor (1845 – 1918)



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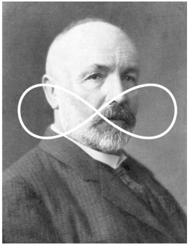

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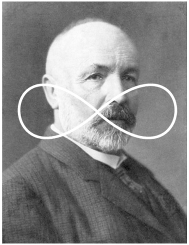

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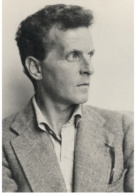
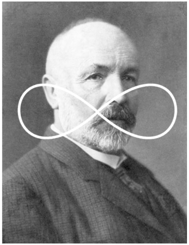


Ludwig Wittgenstein
(1889 – 1951)



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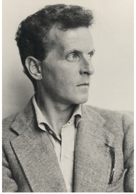

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From his paradise that Cantor with us unfolded, we hold our breath in awe; knowing, we shall not be expelled.

— *David Hilbert*

"没有人能把我们从 *Cantor* 创造的乐园中驱逐出去"

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MATHEMATIKER
BEGRÜNDER DER MENGENLEHRE
1845-1918
DAS WESEN DER MATHEMA
TIK LIEGT IN IHRER FREIHEIT
GEORG CANTOR

"das wesen der mathematik liegt in ihrer freiheit"

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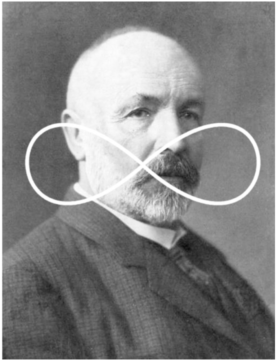
"The essence of mathematics lies in its freedom"

Before Cantor

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决定经典
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一部高度展示人类的逻辑理性与逻辑思维能力的体系教本
全新修订版
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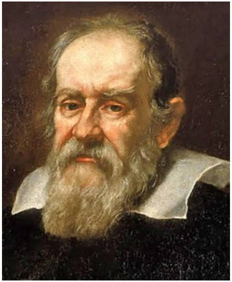
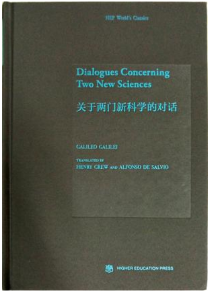
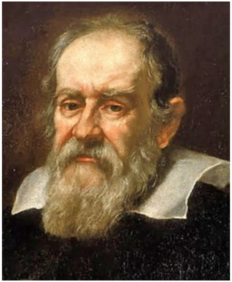
公理: “整体大于部分”



Galileo Galilei (1564 – 1642)



“关于两门新科学的对话” (1638)



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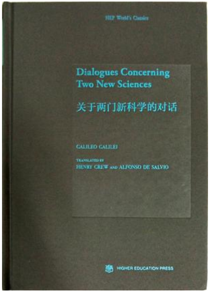

“关于两门新科学的对话” (1638)

“用我们有限的心智来讨论无限 $\cdots$”

$$S_1 = \{1, 2, 3, \cdots, n, \cdots\}$$
$$S_2 = \{1, 4, 9, \cdots, n^2, \cdots\}$$

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说到底，“等于”、“大于”和“小于”诸性质不能用于无限，而只能用于有限的数量。 — *Galileo Galilei*

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无穷数是不可能的。 — *Gottfried Wilhelm Leibniz*

这些证明一开始就期望那些数要具有有穷数的一切性质，或者甚至于把有穷数的性质强加于无穷。

相反，这些无穷数，如果它们能够以任何形式被理解的话，倒是由于它们与有穷数的对应，它们必须具有完全全新的数量特征。

这些性质完全依赖于事物的本性，$\cdots$ 而并非来自我们的主观任意性或我们的偏见。

— *Georg Cantor (1885)*

Definition (Dedekind-infinite & Dedekind-finite (Dedekind, 1888))

A set A is *Dedekind-infinite* if there is a bijective function from A onto some proper subset B of A.

A set is *Dedekind-finite* if it is not Dedekind-infinite.

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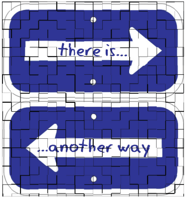

We will prove this as a theorem in our theory of infinity.

We have not defined "finite" and "infinite"!!!

Comparing Sets

Comparing Sets

$$=$$

$$<$$

Comparing Sets

$$= \qquad\qquad <$$

Function

$$=$$

Definition ($|A| = |B|$ $(A \approx B)$ (1878))

A and B are *equipotent* if there exists a *bijection* from A to B.

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Abstract from order: $\{1, 2, 3, \cdots\} \quad vs. \quad \{1, 3, 5, \cdots, 2, 4, 6, \cdots\}$

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Theorem ( )

For any sets A, B, C:

(a) $A \approx B$

(b) $A \approx B \implies B \approx A$

(c) $A \approx B \land B \approx C \implies A \approx C$

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Theorem (The "Equivalence Concept" of Equipotent)

For any sets A, B, C:

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Definition (Finite)

X is finite if

$$\exists n \in \mathbb{N} : |X| = n.$$

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Theorem (UD Theorem 22.6)

Let A be a finite set. There is a unique $n \in \mathbb{N}$ such that $A \approx \{0, 1, \cdots, n-1\}$.

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X is infinite if it is not finite:

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$\mathbb{N}$ is infinite. (So are $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$.)

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Definition (Infinite)

For any set X,

Countably Infinite

$$|X| = |\mathbb{N}| \triangleq \aleph_0$$

Countable

$$(\text{finite} \lor \text{countably infinite})$$

Uncountable

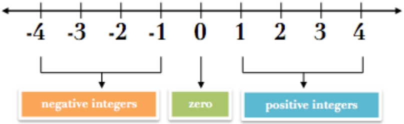
$$(\neg\ \text{countable})$$

$$(\text{infinite}) \land \Big(\neg\ (\text{countably infinite})\Big)$$

$$\aleph_0$$

Theorem ($\mathbb{Z}$ is Countable.)

$$|\mathbb{Z}| = |\mathbb{N}|$$



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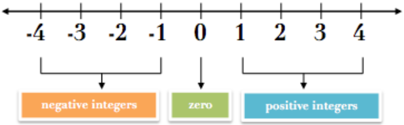

$$0 \quad 1 \quad -1 \quad 2 \quad -2 \quad \cdots$$

Theorem ($\mathbb{Q}$ is Countable. (Cantor 1873-11; Published in 1874))

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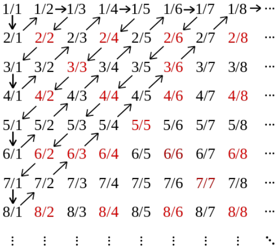

Theorem ($\mathbb{N} \times \mathbb{N}$ is Countable.)

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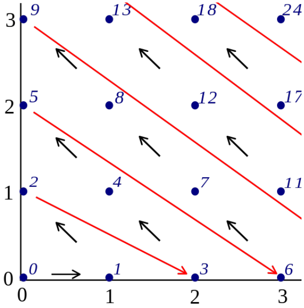

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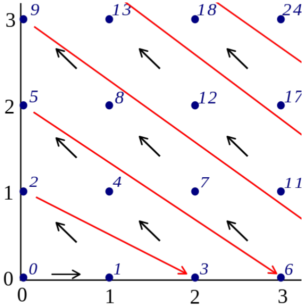

$$\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

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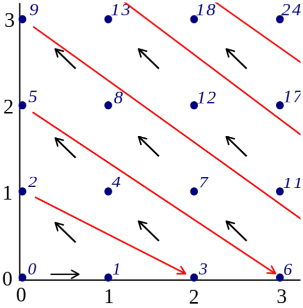

$$\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

$$\pi(k_1, k_2) = \frac{1}{2}(k_1+k_2)(k_1+k_2+1)+k_2$$

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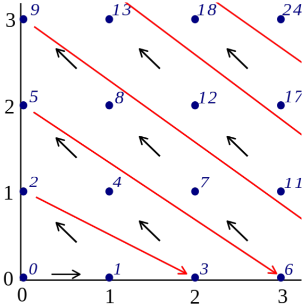

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Cantor Pairing Function

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$$\pi^{(n)}(k_1, \ldots, k_{n-1}, k_n) = \pi(\pi^{(n-1)}(k_1, \ldots, k_{n-1}), k_n)$$

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$$a_0 \quad b_0 \quad c_0 \quad a_1 \quad b_1 \quad c_1 \cdots$$

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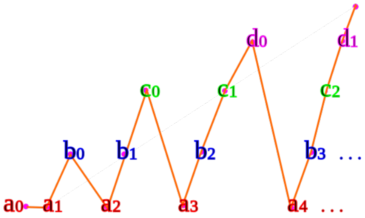

Counting by Diagonals.

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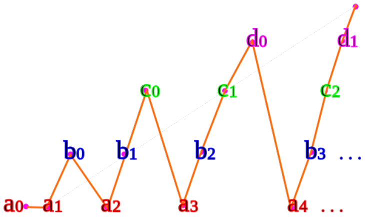

Counting by Diagonals.

We need Axiom of (Countable) Choice!

$$\aleph_0$$

Theorem ($\mathbb{R}$ is Uncountable. (Cantor 1873-12; Published in 1874))

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Cantor's Diagonal Argument (1890)

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3.14159...
1.41421...
1.73205...
2.23606...
2.71828...
0.14285...

3.43625...

2.32514...

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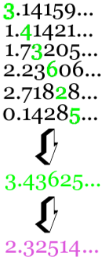

By Diagonal Argument.

Theorem (Cantor's Theorem (1891))

$$|A| \neq |2^A|$$

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Proof. Let A be a set and let $f : A \to 2^A$. To show that f is not onto, we must find a $B \in 2^A$ (i.e., $B \subseteq A$) for which there is no $a \in A$ with $f(a) = B$. In other words, B is a set that f "misses." To this end, let

$$B = \{x \in A : x \notin f(x)\}.$$

We claim there is no $a \in A$ with $f(a) = B$.

Suppose, for the sake of contradiction, there is an $a \in A$ such that $f(a) = B$. We ponder: Is $a \in B$?

- If $a \in B$, then, since $B = f(a)$, we have $a \in f(a)$. So, by definition of B, $a \notin f(a)$; that is, $a \notin B$. $\Rightarrow\Leftarrow$
- If $a \notin B = f(a)$, then, by definition of B, $a \in B$. $\Rightarrow\Leftarrow$

Both $a \in B$ and $a \notin B$ lead to contradictions, and hence our supposition [there is an $a \in A$ with $f(a) = B$] is false, and therefore f is not onto. ■

Theorem (Cantor Theorem)

If $f : A \to 2^A$*, then* f *is not onto.*

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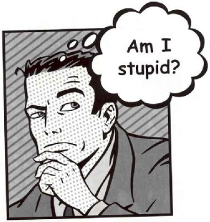

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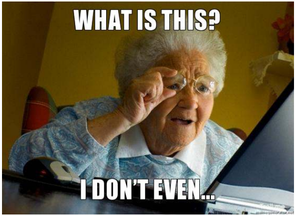

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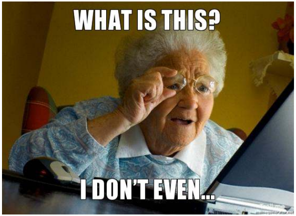

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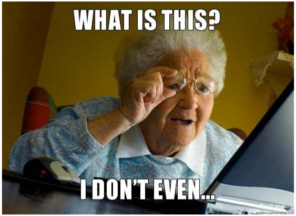

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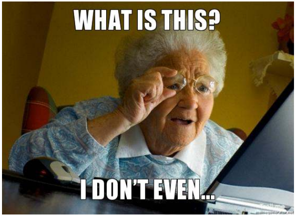

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Diagonal Argument .

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a	$f(a)$					
	1	2	3	4	5	$\cdots$
1	1	1	0	0	1	$\cdots$
2	0	0	0	0	0	$\cdots$
3	1	0	0	1	0	$\cdots$
4	1	1	1	1	1	$\cdots$
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Diagonal Argument (以下仅适用于可数集合 A).

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Infinite Sequences of 0's and 1's (UD Problem 23.4)

Is the set of all infinite sequences of 0's and 1's finite, countably infinite, or uncountable?

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$$
\begin{array}{ll}
s_1 & = \color{red}{0}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \ldots \\
s_2 & = 1\ \color{red}{1}\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \ldots \\
s_3 & = 0\ 1\ \color{red}{0}\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0 \ldots \\
s_4 & = 1\ 0\ 1\ \color{red}{0}\ 1\ 0\ 1\ 0\ 1\ 0\ 1 \ldots \\
s_5 & = 1\ 1\ 0\ 1\ \color{red}{0}\ 1\ 1\ 0\ 1\ 0\ 1 \ldots \\
s_6 & = 0\ 0\ 1\ 1\ 0\ \color{red}{1}\ 1\ 0\ 1\ 1\ 0 \ldots \\
s_7 & = 1\ 0\ 0\ 0\ 1\ 0\ \color{red}{0}\ 0\ 1\ 0\ 0 \ldots \\
s_8 & = 0\ 0\ 1\ 1\ 0\ 0\ 1\ \color{red}{1}\ 0\ 0\ 1 \ldots \\
s_9 & = 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ \color{red}{1}\ 1\ 0 \ldots \\
s_{10} & = 1\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ \color{red}{0}\ 1 \ldots \\
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\vdots & \quad \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \ddots
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$$

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s = \color{blue}{1\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1} \ldots
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Theorem (Brouwer (Topological Invariance of Dimension))

There is no continuous bijections between $\mathbb{R}^m$ and $\mathbb{R}^n$ for $m \neq n$.

$$<$$

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$|B| \leq |A|$ (Axiom of Choice)

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$$|A| < |B| \iff |A| \leq |B| \land |A| \neq |B|$$

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$$|\mathbb{N}| < |\mathbb{R}|$$

$$|X| < |2^X|$$

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X is countable:

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Subsets of Countable Set (UD Problem 22.6; UD Corollary 22.4)

Every subset B of a countable set A is countable.

Set Union (UD Problem 22.1)

Give an example, if possible, of

(c) a countably infinite collection of *pairwise disjoint* nonempty sets whose union is finite.

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$$|A| = n \implies |2^A| = 2^n$$

Slope (UD Problem 22.2 (e))

(e) the set of all lines with rational slopes

Slope (UD Problem 22.2 (*e*))

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$$(\mathbb{Q}, \mathbb{R})$$

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$$|\mathbb{R}| \le |\mathbb{Q} \times \mathbb{R}| \le |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$$

Q : Is "$\leq$" a partial order?

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Theorem (Cantor-Schröder–Bernstein (1887))

$$|X| \leq |Y| \land |Y| \leq |X| \implies |X| = |Y|$$

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$$|X| \leq |Y| \wedge |Y| \leq |X| \implies |X| = |Y|$$

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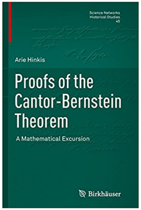

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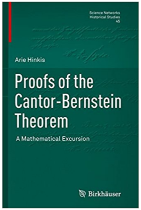

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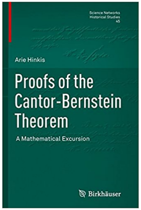

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Theorem (PCC)

Principle of Cardinal Comparability (PCC) $\iff$ *Axiom of Choice*

Finite Sets

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"关于有穷，我原以为我是懂的"

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Theorem (Pigeonhole Principle (UD Theorem 21.2))

$$f : \{1, \cdots, m\} \to \{1, \cdots, n\} \ (m, n \in \mathbb{N}^+, m > n)$$

f *is not one-to-one.*

$A \setminus \{a\}$ (UD Problem 21.15)

Let A be a nonempty finite set with $|A| = n$ and let $a \in A$.
Prove that $A \setminus \{a\}$ is finite and $|A \setminus \{a\}| = n - 1$.

$A \setminus \{a\}$ (UD Problem 21.15)

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$$f : A \xleftrightarrow[onto]{1-1} \{1, \cdots, n\}$$

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$|A| \le |B|$ (UD Problem 21.17)

A and B are finite sets and $f : A \to B$ is one-to-one.
Show that $|A| \le |B|$.

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By contradiction and the pigeonhole principle.

(UD Problem 21.16)

(a) A is a finite set and $B \subseteq A$. We showed that B is finite (Corollary 20.11). Show that $|B| \le |A|$.

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one-to-one $f : B \to A$

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By contradiction and (b).

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(No Axiom of Choice Here)

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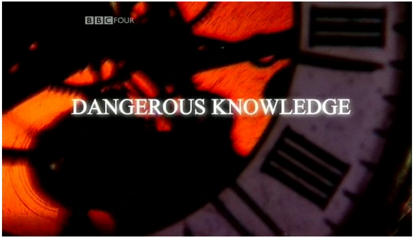
$$g \text{ is bijective.}$$

$$f(g(y)) = f(x) = y \implies f = g^{-1}$$

Dangerous Knowledge (BBC 2007)



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Independence from ZFC:

Kurt Gödel (1940) CH cannot be disproved from ZF.

Paul Cohen (1964) CH cannot be proven from the ZFC axioms.


Thank You!

Your opinon Matters

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