1-8 Set Theory: Axioms and Operations

魏恒峰

hfwei@nju.edu.cn

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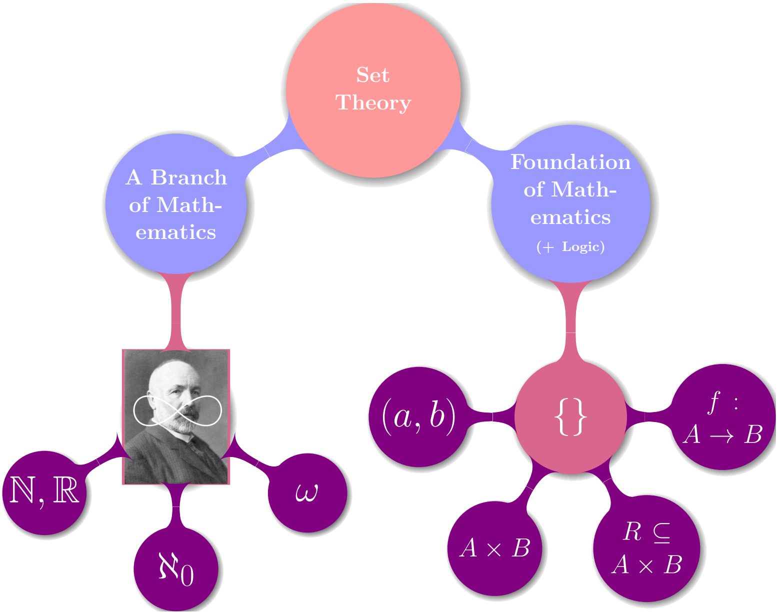
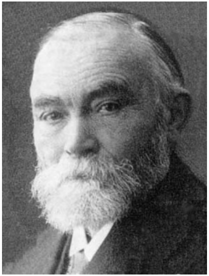
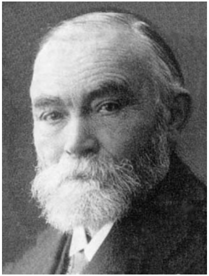

Set Theory
A Branch of Mathematics
Foundation of Mathematics
(+ Logic)
$\mathbb{N}, \mathbb{R}$
$\aleph_0$
ω
(a, b)
$\{\}$
$f : A \to B$
$A \times B$
$R \subseteq A \times B$



Gottlob Frege (1848–1925)



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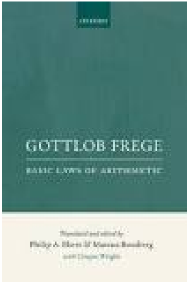
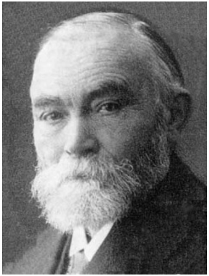


“Basic Laws of Arithmetic”



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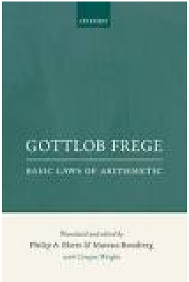
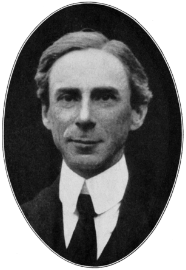
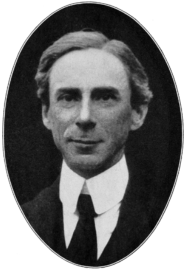


“Basic Laws of Arithmetic”

对于一个科学工作者来说，最不幸的事情莫过于：当他的工作接近完成时，却发现那大厦的基础已经动摇。 — 《附录二》，*1902*



Bertrand Russell (1872–1970)



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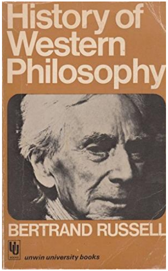

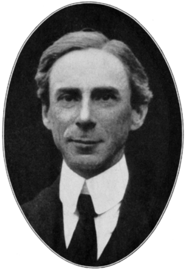

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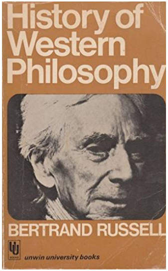
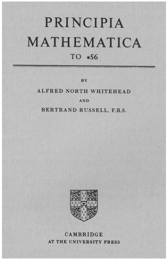
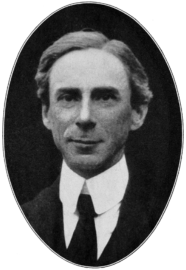



PRINCIPIA MATHEMATICA
TO *56

BY
ALFRED NORTH WHITEHEAD
AND
BERTRAND RUSSELL, F.R.S.

CAMBRIDGE
AT THE UNIVERSITY PRESS



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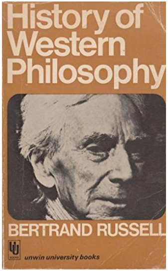
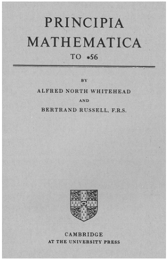
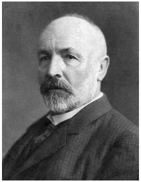




我们将集合理解为任何将我们思想中那些确定而彼此独立的对象放在一起而形成的聚合。

— *Georg Cantor*《超穷数理论基础》



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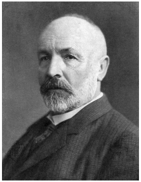

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For any predicate $\psi(x)$, *there is a set* X:

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$$R = \{x \mid x \notin x\}$$

$$Q : R \in R\,?$$

Q : 既然朴素集合论存在悖论，你是如何做作业的?


YOU HAVE
TO FIX IT!

Theorem (Russell's Paradox)

$$\{x \mid x \notin x\} \textit{ is not a set.}$$

Axiomatic Set Theory (ZFC)

Ernst Zermelo (1871–1953)

Abraham Fraenkel (1891–1965)

First-order Language

Parentheses: $(,)$

Variables: $x, y, z, \cdots$

Connectives: $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$

Quantifiers: $\forall$, $\exists$

Equality: $=$

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Quantifiers: $\forall$, $\exists$

Equality: $=$

Constants: $a, b, c, \cdots$

Functions: $f, g, h, \cdots$

Predicates: $R, P, Q, \cdots$

First-order Language for Sets $\mathcal{L}_{Set}$

First-order Language for Sets $\mathcal{L}_{Set} = \{\in\}$

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Quantifiers: $\forall, \exists$

Equality: $=$

Constants:

Functions:

Predicates: $\in$

Everything we consider in $\mathcal{L}_{Set}$ is a set.

Q : What is "$\in$"?

Q : What are "sets"?

We don't define them directly.

We only describe their properties in an axiomatic way.

(1) To draw a straight line from any point to any point.

(2) To extend a finite straight line continuously in a straight line.

(3) To describe a circle with any center and radius.

(4) That all right angles are equal to one another.

(5) The parallel postulate.

$$E, E; P, U, R, P; I, C; F$$

Definition ($\notin$)

$$x \notin A \triangleq \neg(x \in A).$$

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Definition ($\subseteq, \subset$)

$$A \subseteq B \triangleq \forall x (x \in A \implies x \in B)$$

$$A \subset B \triangleq A \subseteq B \land A \neq B$$

Axiom (Paring Axiom)

For any sets x and y, there is a set having as members just x and y:

$$\forall x\, \forall y\, \exists B\, \big(\forall z(z \in B \iff z = x \lor z = y)\big).$$

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Definition ("$\{x, y\}$")

$$\{x, y\} \triangleq \text{ the unique set obtained by paring } x \text{ and } y.$$

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$$\{x, y\} = \{y, x\}.$$

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Definition ("$\{x\}$")

$$\{x\} \triangleq \{x, x\}.$$

Axiom (Union Axiom (Simplified Version))

For any sets x and y, there is a set whose members are the elements belonging either to x or to y (or both):

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Definition ("$x \cup y$")

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Definition (“$\{x, y, z\}$”)

$$\{x, y, z\} \triangleq \{x, y\} \cup \{z\}.$$

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We can use pairing and union together to form finite sets.

Axiom (Union Axiom (Extended Version))

For any set A, there is a set B such that:

$$\forall x\ (x \in B \iff x \textit{ belongs to some member of } A).$$

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Theorem

$$\bigcup \emptyset = \emptyset.$$

Axiom (Replacement Axioms (Simplified Version; Subset Axioms; Separation Axioms))

Let $\psi(u)$ be a predicate. For any set u, there is a set B which is a subset of u such that each element x of B satisfies $\psi(x)$:

$$\forall u\ \exists B\ (\forall x(x \in B \iff x \in u \land \psi(x))).$$

Definition ("$\{x \in u \mid \psi(x)\}$")

$$\{x \in u \mid \psi(x)\} \triangleq \text{ the unique set obtained by separating } u \text{ with } \psi.$$

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Definition ("$u \cap v$")

$$u \cap v \triangleq \{x \in u \mid x \in v\}.$$

Theorem ("$\bigcap A$" (Arbitrary Intersection))

For any nonempty set A, there is a unique set B such that

$$\forall x\, (x \in B \iff x \text{ belongs to every member of } A).$$

$$\forall x\, \big(x \in B \iff \forall y \in A(x \in y)\big).$$

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Proof.

Let c be a fixed member of A.

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□

"$\bigcap \emptyset$"

$\bigcap \emptyset$ is *not* a set.

Theorem (No Universal Set)

There is no universal set.

$$\nexists B(\forall x(x \in B)).$$

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$$B \in B \iff B \in A \land B \notin B$$

$$B \in A \implies (B \in B \iff B \notin B)$$

□

Theorem (Russell's Paradox)

$$\{x \mid x \notin x\} \text{ is } \textit{not} \text{ a set.}$$

Theorem (Russell's Paradox)

$\{x \mid x \notin x\}$ *is* *not* *a set.*

$$B = \{x \in A \mid x \notin x\}.$$

does *not* lead to contradiction.

Definition ("$u \setminus v$" (Relative Complement))

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□

We can never look for objects "not in B" unless we know where to start looking. — *UD (Chapter 6; Page 64)*

Axiom (Power Set Axiom)

For any set A, there is a set whose members are the subsets of A:

$$\forall A\ \exists B\ \forall x(x \in B \iff x \subseteq A).$$

Definition (“$\mathcal{P}(A)$”)

$$\mathcal{P}(A) \triangleq \text{ the unique power set of } A.$$

The is *not* correct!

$$\mathcal{P}(A) \triangleq \{x \mid x \subseteq A\}$$


Thank
You!