1-10 Set Theory (III): Functions

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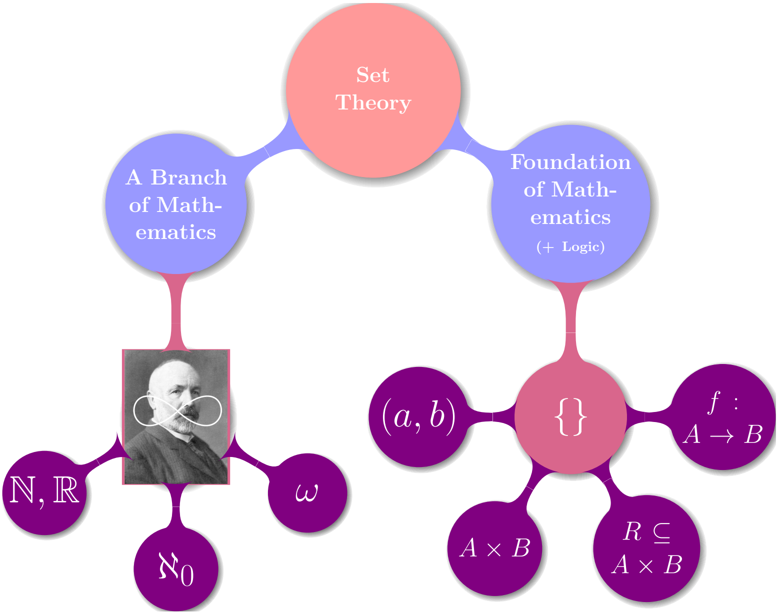
Set Theory
A Branch of Mathematics
Foundation of Mathematics
(+ Logic)
$\mathbb{N}, \mathbb{R}$
$\aleph_0$
ω
(a, b)
$\{\}$
$f : A \to B$
$A \times B$
$R \subseteq A \times B$

Functions

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PROOF! PROOF! PROOF!

Definition of Functions

$$R \subseteq A \times B$$

is a *relation* from A to B

Definition (Function)

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$$\forall a \in A : \exists! b \in B : (a, b) \in f.$$

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$$dom(f) = A \qquad cod(f) = B$$

$$ran(f) = f(A) \subseteq B$$

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$$f : a \mapsto b$$

$$f(a) \triangleq b$$

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$$\exists! b \in B :$$

$$\forall b, b' \in B : (a, b) \in f \land (a, b') \in f \implies b = b'$$

$$D : \mathbb{R} \to \mathbb{R}$$

$$D(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Dirichlet Function

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The *set* of all functions from X to Y:

$$Y^X = \{f \mid f : X \to Y\}$$

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Definition

The *set* of all functions from X to Y:

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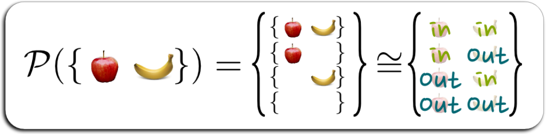
$$2^X = \{0, 1\}^X \cong \mathcal{P}(X)$$



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For every set X, there exists a function $I_X : \{X\} \to \{X\}$.

$$\bigcup_{I_X \in A} dom(I_X)$$

Functions as Sets

Axiom (Axiom of Extensionality)

$$\forall A : \forall B : \forall x : (x \in A \iff x \in B) \iff A = B.$$

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Theorem (The Principle of Functional Extensionality)

$$f, g \ \textit{are functions}:$$

$$f = g \iff dom(f) = dom(g) \land \Big(\forall x \in dom(f) : f(x) = g(x)\Big)$$

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$$f, g \ \textit{are functions}:$$

$$f = g \iff \text{dom}(f) = \text{dom}(g) \wedge \Big(\forall x \in \text{dom}(f) : f(x) = g(x)\Big)$$

It may be that $cod(f) \neq cod(g)$.

$$f : A \to B \qquad g : C \to D$$

$$Q : \text{Is } f \cap g \text{ a function?}$$

$$f : A \to B \qquad g : C \to D$$

Q : Is $f \cap g$ a function?

Theorem (Intersection of Functions)

$$f \cap g : (A \cap C) \to (B \cap D)$$

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Theorem (Union of Functions)

$$f \cup g : (A \cup C) \to (B \cup D) \iff \forall x \in dom(f) \cap dom(g) : f(x) = g(x)$$

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UD Problem 14.3 (g)

$$f : \mathbb{Q} \to \mathbb{R}$$

$$f(x) = \begin{cases} x + 1 & \text{if } x \in 2\mathbb{Z} \\ x - 1 & \text{if } x \in 3\mathbb{Z} \\ 2 & \text{otherwise} \end{cases}$$

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$$x \in 6\mathbb{Z}$$

UD Problem 14.5

$$f : \mathcal{P}(\mathbb{R}) \to \mathbb{Z}$$

$$f(A) = \begin{cases} \min(A \cap \mathbb{N}) & \text{if } A \cap \mathbb{N} \neq \emptyset \\ -1 & \text{if } A \cap \mathbb{N} = \emptyset \end{cases}$$

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By the *Well-Ordering Principle* of $\mathbb{N}$

Special Functions (*-jectivity*)

Definition (Injective (one-to-one; 1-1) 单射函数)

$$f : A \to B$$

$$\forall a_1, a_2 \in A : a_1 \neq a_2 \implies f(a_1) \neq f(a_2)$$

Definition (Injective (one-to-one; 1-1) 单射函数)

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For Proof:

- To prove that f *is* 1-1:

$$\forall a_1, a_2 \in A : f(a_1) = f(a_2) \implies a_1 = a_2$$

- To show that f *is not* 1-1:

$$\exists a_1, a_2 \in A : a_1 \neq a_2 \land f(a_1) = f(a_2)$$

Definition (Surjective (onto) 满射函数)

$$f : A \to B$$

$$ran(f) = B$$

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If $f : A \to 2^A$, then f is not onto.

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Proof. Let A be a set and let $f : A \to 2^A$. To show that f is not onto, we must find a $B \in 2^A$ (i.e., $B \subseteq A$) for which there is no $a \in A$ with $f(a) = B$. In other words, B is a set that f "misses." To this end, let

$$B = \{x \in A : x \notin f(x)\}.$$

We claim there is no $a \in A$ with $f(a) = B$.

Suppose, for the sake of contradiction, there is an $a \in A$ such that $f(a) = B$. We ponder: Is $a \in B$?

- If $a \in B$, then, since $B = f(a)$, we have $a \in f(a)$. So, by definition of B, $a \notin f(a)$; that is, $a \notin B$. $\Rightarrow\Leftarrow$
- If $a \notin B = f(a)$, then, by definition of B, $a \in B$. $\Rightarrow\Leftarrow$

Both $a \in B$ and $a \notin B$ lead to contradictions, and hence our supposition [there is an $a \in A$ with $f(a) = B$] is false, and therefore f is not onto. ■

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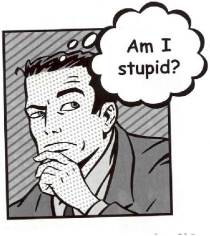

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$$\forall B \in 2^A : \Big(\exists a \in A : f(a) = B\Big)$$

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Not Onto

$$\exists B \in 2^A : \Big(\forall a \in A : f(a) \neq B\Big)$$

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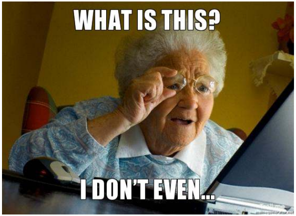

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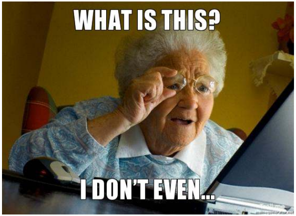

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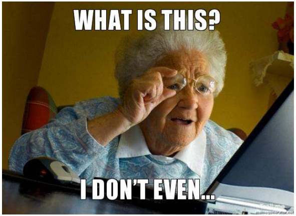

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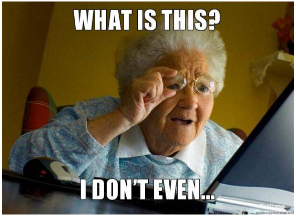

$$Q : a \in B\,?$$

$$a \in B \iff a \notin B$$

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a	$f(a)$					
	1	2	3	4	5	$\cdots$
1	1	1	0	0	1	$\cdots$
2	0	0	0	0	0	$\cdots$
3	1	0	0	1	0	$\cdots$
4	1	1	1	1	1	$\cdots$
5	0	1	0	1	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$

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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$

$$B = \{0, 1, 1, 0, 1\}$$

$\square$

Theorem (Cantor Theorem)

If $f : A \to 2^A$, then f is not onto.

对角线论证 (Cantor's diagonal argument) (以下仅适用于可数集合 A).

a	$f(a)$					
	1	2	3	4	5	$\cdots$
1	1	1	0	0	1	$\cdots$
2	0	0	0	0	0	$\cdots$
3	1	0	0	1	0	$\cdots$
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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$

$$B = \{0, 1, 1, 0, 1\}$$

$\square$


Thank You!