# 2-3 Counting

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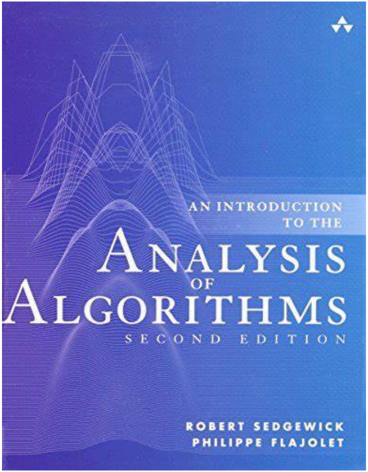
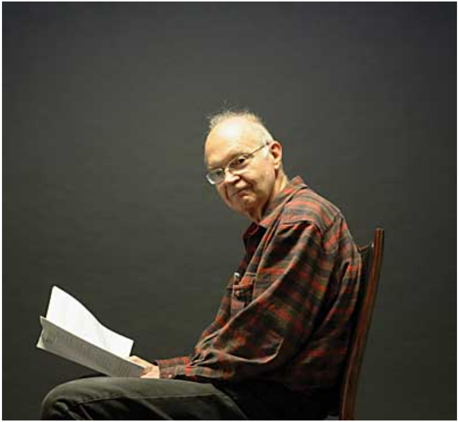
March 12, 2020




The Analysis of Algorithms



Donald E. Knuth (1938 $\sim$)

*"People who analyze algorithms have double happiness.*

*First of all they experience the sheer beauty of elegant mathematical patterns that surround elegant computational procedures.*

*Then they receive a practical payoff when their theories make it possible to get other jobs done more quickly and more economically."*

```
procedure CONUNDRUM(n)
    r ← 0
    for i ← 1 to n do
        for j ← i + 1 to n do
            for k ← i + j − 1 to n do
                r ← r + 1
    return r
```

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=i+j-1}^{n} 1 = \frac{n^2 - n}{2} = \Theta(n^2)$$

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$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=i+j-1}^{n} 1 = \frac{1}{48}\Big(3\big(-1 + (-1)^n\big) + 2n(n+2)(2n-1)\Big) = \Theta(n^3)$$

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=i+j-1}^{n} 1$$

$$= \sum_{i=1}^{n} \sum_{j=i+1}^{n} (n-i-j+2) \left[j \leq n-i+1, i \leq \frac{n}{2}\right]$$

$$= \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=i+1}^{n-i+1} (n-i-j+2)$$

From Zheng (171860658)

## Passing out Apples to Children

width = 0.40figs/apple-children

$k$-Permutation $(\mathrm{CS} : 1.2 - 5)$

We need to pass out $k$ distinct apples (pieces of fruit) to $n$ children such that *each child may get at most one apple*.

(a) $k \le n$?

(b) What if $k > n$?

$$n^{\underline{k}} \triangleq n(n-1)\cdots(n-k+1)$$

0

## Multisets (CS : 1.5 – 4)

Use multisets to determine the number of ways to pass out $k$ **identical** apples to $n$ children. Assume that a child may get more than one apple.

$$x_i : \text{the \# of apples the } i\text{-th child gets}$$

$$x_1 + x_2 + \cdots + x_n = k, \quad x_i \geq 0$$

Integer composition (The order matters!)

$$y_i \triangleq x_i + 1$$

$$y_1 + y_2 + \cdots + y_n = n + k, \qquad y_i \geq 1$$

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

Multisets $(CS : 1.5 - 4)$

Use **multisets** to determine the number of ways to pass out $k$ identical apples to $n$ children. Assume that a child may get more than one apple.

$Q$ : $k$-multiset of $[1 \cdots n]$ *vs.* $n$-multiset of $[1 \cdots k]$

$$k = 7 \qquad n = 5$$

width $=$ 0.50figs/apple-to-children-multiset

$$\{1, 1, 1, 3, 3, 4, 5\}$$

Integer Partition (CS : 1.5 – 4 Extended)

What is the number of ways to pass out $k$ identical apples to $n$- .
Assume that a child may get more than one apple.

width = 0.95figs/apple-to-children-partition

width = 0.95figs/apple-to-children-partition-equiv

Integer partition of $k$ into $\le n$ parts (The order does not matter!)

Theorem (G. H. Hardy, Ramanujan (1918))

$$p(k) \triangleq \sum_{x=1}^{x=k} p_x(k) \sim \frac{1}{4\sqrt{3}k} \exp\left(\pi\sqrt{\frac{2k}{3}}\right)$$

Set Partition $(\text{CS}: 1.5-4 \text{ Extended})$

What is the number of ways to pass out $k$ distinct apples to $n$- .
Assume that a child may get more than one apple.

width = 0.40figs/apple-to-children-set-partition

width = 0.85figs/apple-to-children-set-partition-equiv

width = 0.85figs/apple-to-children-set-partition-nonequiv

Set partition of $[1 \cdots k]$ into $\leq n$ parts

**Set Partition $(\mathrm{CS}: 1.5-12)$**

$$S(n,k) \; (nk) : \# \text{ of set partitions of } [1 \cdots n] \text{ into } k \text{ classes}$$

Stirling number of the second kind

**Theorem (Recurrence for $S(n,k)$)**

$$S(0,0)=1, \quad S(n,0)=S(0,n)=0 \; (n>0)$$

$$S(n,k)=S(n-1,k-1)+kS(n-1,k), \quad n>0, k>0$$

*Proof.*

$$S(n,k)=\underbrace{S(n-1,k-1)}_{n \text{ is alone}}+\underbrace{kS(n-1,k)}_{n \text{ is not alone}}$$

□

$$\text{Bell number: } B_n = \sum_{k=0}^{k=n} nk$$

**Theorem (Berend & Tassa (2010))**

$$B_n < \left(\frac{0.792n}{\ln(n+1)}\right)^n, n \in \mathbb{Z}^+$$

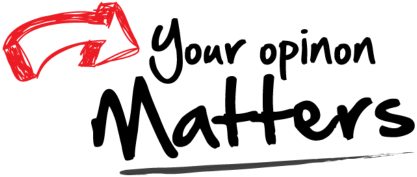
**Theorem (de Bruijn (1981))**

*As* $n \to \infty$,

$$\frac{\ln B_n}{n} = \ln n - \ln\ln n - 1 + \frac{\ln\ln n}{\ln n} + \frac{1}{\ln n} + \frac{1}{2}\left(\frac{\ln\ln n}{\ln n}\right)^2 + O\left(\frac{\ln\ln n}{(\ln n)^2}\right)$$

width = 0.85figs/12-way


Thank
You!

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