1-11 Set Theory (IV): Infinity

魏恒峰

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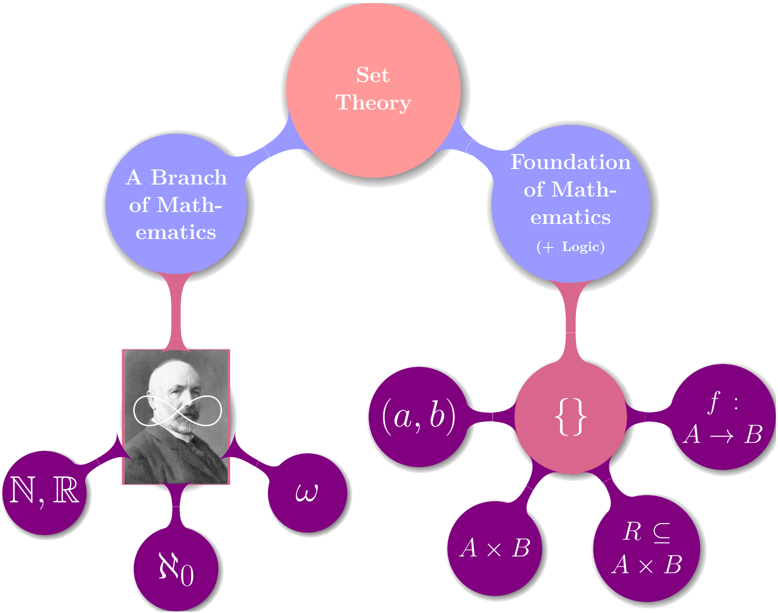
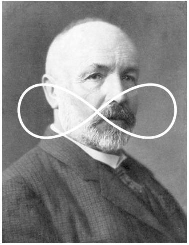
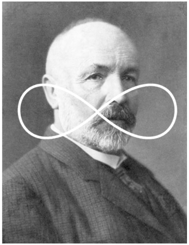

Set Theory
A Branch of Mathematics
Foundation of Mathematics
(+ Logic)
$\mathbb{N}, \mathbb{R}$
$\aleph_0$
ω
(a, b)
$\{\}$
$f : A \to B$
$A \times B$
$R \subseteq A \times B$



Georg Cantor (1845 – 1918)



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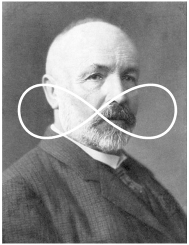
Leopold Kronecker
(1823 – 1891)



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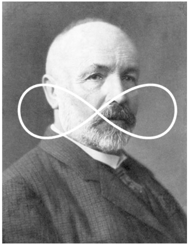
Henri Poincaré
(1854 – 1912)



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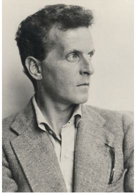
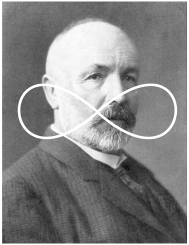


Ludwig Wittgenstein
(1889 – 1951)



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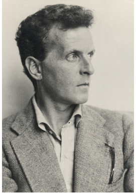

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From his paradise that Cantor with us unfolded, we hold our breath in awe; knowing, we shall not be expelled.

— *David Hilbert*

“没有人能把我们从 *Cantor* 创造的乐园中驱逐出去”

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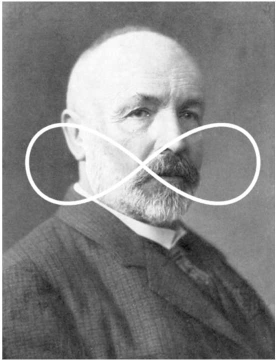
GEORG CANTOR
MATHEMATIKER
BEGRÜNDER DER MENGENLEHRE
1845-1918
DAS WESEN DER MATHEMA
TIK LIEGT IN IHRER FREIHEIT
GEORG CANTOR

"das wesen der mathematik liegt in ihrer freiheit"

"das wesen der mathematik liegt in ihrer freiheit"

"The essence of mathematics lies in its freedom"

Before Cantor

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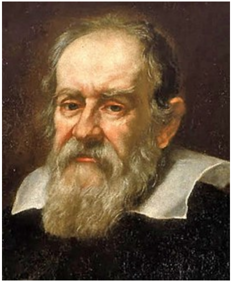
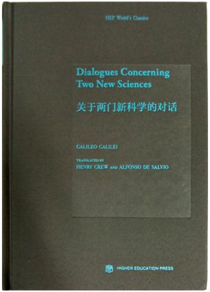
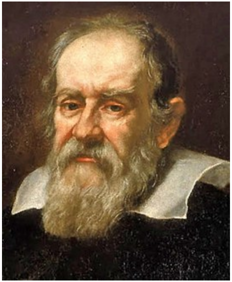
公理: “整体大于部分”



Galileo Galilei (1564 – 1642)



"关于两门新科学的对话" (1638)



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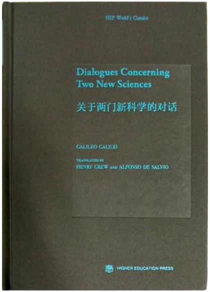

“关于两门新科学的对话” (1638)

“用我们有限的心智来讨论无限 · · · ”

$$S_1 = \{1, 2, 3, \cdots, n, \cdots\}$$
$$S_2 = \{1, 4, 9, \cdots, n^2, \cdots\}$$

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"部分等于全体"

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说到底，“等于”、“大于”和“小于”诸性质不能用于无限，而只能用于有限的数量。 — *Galileo Galilei*

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无穷数是不可能的。 — *Gottfried Wilhelm Leibniz*

这些证明一开始就期望那些数要具有有穷数的一切性质，或者甚至于把有穷数的性质强加于无穷。

相反，这些无穷数，如果它们能够以任何形式被理解的话，倒是由于它们与有穷数的对应，它们必须具有完全新的数量特征。

这些性质完全依赖于事物的本性，⋯ 而并非来自我们的主观任意性或我们的偏见。

— *Georg Cantor (1885)*

Definition (Dedekind-infinite & Dedekind-finite (Dedekind, 1888))

A set A is *Dedekind-infinite* if there is a bijective function from A onto some proper subset B of A.

A set is *Dedekind-finite* if it is not Dedekind-infinite.

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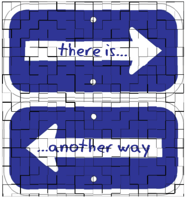

We will prove this as a theorem in our theory of infinity.

We have not defined “finite” and “infinite”!!!

Comparing Sets

Comparing Sets

$$= \qquad\qquad\qquad <$$

Comparing Sets

$=$ $<$

Function

$$=$$

Definition ($|A| = |B|$ $(A \approx B)$ (1878))

A and B are *equipotent* if there exists a *bijection* from A to B.

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Abstract from elements: $\{1, 2, 3\} \quad vs. \quad \{a, b, c\}$

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Abstract from elements: $\{1, 2, 3\} \quad vs. \quad \{a, b, c\}$

Abstract from order: $\{1, 2, 3, \cdots\} \quad vs. \quad \{1, 3, 5, \cdots, 2, 4, 6, \cdots\}$

Definition ($|A| = |B|$ ($A \approx B$) (1878))

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Q : Is "$\approx$" an equivalence relation?

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Theorem ( )

For any sets A, B, C:

(a) $A \approx B$

(b) $A \approx B \implies B \approx A$

(c) $A \approx B \land B \approx C \implies A \approx C$

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Theorem (The "Equivalence Concept" of Equipotent)

For any sets A, B, C:

(a) $A \approx B$

(b) $A \approx B \implies B \approx A$

(c) $A \approx B \land B \approx C \implies A \approx C$

Definition (Finite and Infinite)

For any set X,

Finite

$$\exists n \in \mathbb{N} : |X| = n \qquad (0 \in \mathbb{N})$$

Infinite $(\neg$ finite)

$$\forall n \in \mathbb{N} : |X| \neq n$$

Definition (Finite and Infinite)

For any set X,

Countably Infinite

$$|X| = |\mathbb{N}| \triangleq \aleph_0$$

Countable

$$(\text{finite} \lor \text{countably infinite})$$

Uncountably Infinite

$$(\neg \text{ finite}) \land \Big(\neg \text{ (countably infinite)}\Big)$$

$$(\neg \text{ countable})$$

Theorem ($\mathbb{Z}$ is Countable.)

$$|\mathbb{Z}| = |\mathbb{N}|$$

Theorem ($\mathbb{Q}$ is Countable. (Cantor 1873-11))

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Theorem ($\mathbb{Q}$ is Countable. (Cantor 1873-11))

$$|\mathbb{Q}| = |\mathbb{N}|$$

$|\mathbb{Q}| = |\mathbb{N}|$ (UD Problem 23.12)

$$q \in \mathbb{Q}^{+} : a/b \; (a, b \in \mathbb{N}^{+})$$

1/1	1/2	1/3	1/4	1/5	1/6	1/7	1/8	⋯
2/1	2/2	2/3	2/4	2/5	2/6	2/7	2/8	⋯
3/1	3/2	3/3	3/4	3/5	3/6	3/7	3/8	⋯
4/1	4/2	4/3	4/4	4/5	4/6	4/7	4/8	⋯
5/1	5/2	5/3	5/4	5/5	5/6	5/7	5/8	⋯
6/1	6/2	6/3	6/4	6/5	6/6	6/7	6/8	⋯
7/1	7/2	7/3	7/4	7/5	7/6	7/7	7/8	⋯
8/1	8/2	8/3	8/4	8/5	8/6	8/7	8/8	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Theorem ($\mathbb{N} \times \mathbb{N}$ is Countable.)

$$|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$$

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$$|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$$

$$f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

Theorem ($\mathbb{N} \times \mathbb{N}$ is Countable.)

$$|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$$

$$f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

$$f(m,n) = n + \frac{(m+n)(m+n+1)}{2}$$

Theorem ($\mathbb{R}$ is Uncountable. (Cantor 1873-12))

$$|\mathbb{R}| \neq |\mathbb{N}|$$

Theorem (ℝ is Uncountable. (Cantor 1873-12))

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Different "Sizes" of Infinity

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Different "Sizes" of Infinity

Cantor's Diagonal Argument (1890)

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$$f : \mathbb{R} \xleftrightarrow[onto]{1-1} \mathbb{N}$$

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$$f : \mathbb{R} \xleftrightarrow[onto]{1-1} \mathbb{N}$$

3.14159...
1.41421...
1.73205...
2.23606...
2.71828...
0.14285...

3.43625...

2.32514...

Theorem (ℝ is Uncountable. (Cantor 1873-12))

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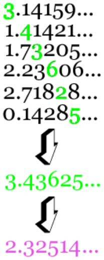

By Diagonal Argument.

Theorem (Cantor's Theorem (1891))

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Theorem (Cantor Theorem (ES Theorem 24.4))

If $f : A \to 2^A$, then f is not onto.

Proof. Let A be a set and let $f : A \to 2^A$. To show that f is not onto, we must find a $B \in 2^A$ (i.e., $B \subseteq A$) for which there is no $a \in A$ with $f(a) = B$. In other words, B is a set that f "misses." To this end, let

$$B = \{x \in A : x \notin f(x)\}.$$

We claim there is no $a \in A$ with $f(a) = B$.

Suppose, for the sake of contradiction, there is an $a \in A$ such that $f(a) = B$. We ponder: Is $a \in B$?

- If $a \in B$, then, since $B = f(a)$, we have $a \in f(a)$. So, by definition of B, $a \notin f(a)$; that is, $a \notin B$.$\Rightarrow\Leftarrow$
- If $a \notin B = f(a)$, then, by definition of B, $a \in B$.$\Rightarrow\Leftarrow$

Both $a \in B$ and $a \notin B$ lead to contradictions, and hence our supposition [there is an $a \in A$ with $f(a) = B$] is false, and therefore f is not onto. ■

Theorem (Cantor Theorem)

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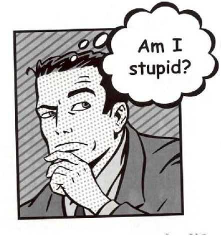

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$$2^A = \mathcal{P}(A) = \Big\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\Big\}$$

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Onto

$$\forall B \in 2^A : \Big(\exists a \in A : f(a) = B\Big)$$

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Not Onto

$$\exists B \in 2^A : \Big(\forall a \in A : f(a) \neq B\Big)$$

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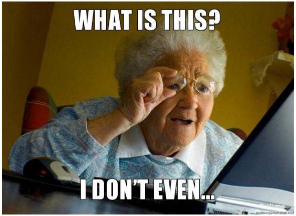

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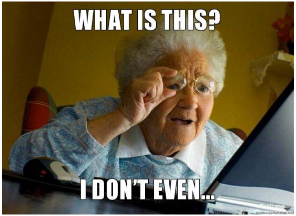

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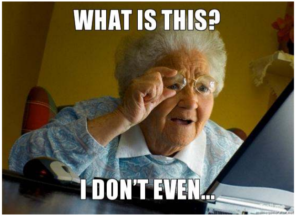

$$Q : a \in B\,?$$

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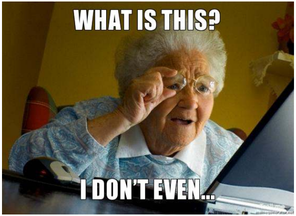

$$Q : a \in B\,?$$

$$a \in B \iff a \notin B$$

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Diagonal Argument .

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Diagonal Argument .

a	$f(a)$					
	1	2	3	4	5	$\cdots$
1	1	1	0	0	1	$\cdots$
2	0	0	0	0	0	$\cdots$
3	1	0	0	1	0	$\cdots$
4	1	1	1	1	1	$\cdots$
5	0	1	0	1	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$

□

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4	1	1	1	1	1	$\cdots$
5	0	1	0	1	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$

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3	1	0	0	1	0	$\cdots$
4	1	1	1	1	1	$\cdots$
5	0	1	0	1	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$

$$B = \{0, 1, 1, 0, 1\}$$

□

Theorem (Cantor Theorem)

If $f : A \to 2^A$, then f is not onto.

Diagonal Argument (以下仅适用于可数集合 A).

a	$f(a)$					
	1	2	3	4	5	$\cdots$
1	1	1	0	0	1	$\cdots$
2	0	0	0	0	0	$\cdots$
3	1	0	0	1	0	$\cdots$
4	1	1	1	1	1	$\cdots$
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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$

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Infinite Sequences of 0's and 1's (UD Problem 23.4)

Is the set of all infinite sequences of 0's and 1's finite, countably infinite, or uncountable?

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$$
\begin{array}{lcl}
s_1 & = & \color{red}{0}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \ldots \\
s_2 & = & 1\ \color{red}{1}\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \ldots \\
s_3 & = & 0\ 1\ \color{red}{0}\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0 \ldots \\
s_4 & = & 1\ 0\ 1\ \color{red}{0}\ 1\ 0\ 1\ 0\ 1\ 0\ 1 \ldots \\
s_5 & = & 1\ 1\ 0\ 1\ \color{red}{0}\ 1\ 1\ 0\ 1\ 0\ 1 \ldots \\
s_6 & = & 0\ 0\ 1\ 1\ 0\ \color{red}{1}\ 1\ 0\ 1\ 1\ 0 \ldots \\
s_7 & = & 1\ 0\ 0\ 0\ 1\ 0\ \color{red}{0}\ 0\ 1\ 0\ 0 \ldots \\
s_8 & = & 0\ 0\ 1\ 1\ 0\ 0\ 1\ \color{red}{1}\ 0\ 0\ 1 \ldots \\
s_9 & = & 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ \color{red}{1}\ 1\ 0 \ldots \\
s_{10} & = & 1\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ \color{red}{0}\ 1 \ldots \\
s_{11} & = & 1\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ \color{red}{0} \ldots \\
\vdots & & \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \ddots
\end{array}
$$

$$
s = \color{blue}{1\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1} \ldots
$$

By Diagonal Argument.

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$\square$

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Q : Then, what is "dimension"?

$$<$$

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$|B| \le |A|$ (Axiom of Choice)

Definition ($|A| < |B|$)

$$|A| < |B| \iff |A| \leq |B| \land |A| \neq |B|$$

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$$|A| < |B| \iff |A| \le |B| \land |A| \ne |B|$$

$$|\mathbb{N}| < |\mathbb{R}|$$

$$|X| < |2^X|$$

Definition (Countable Revisited)

X is countable:

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Subsets of Countable Set (UD Problem 22.6; UD Corollary 22.4)

Every subset B of a countable set A is countable.

Set Union (UD Problem 22.1)

Give an example, if possible, of

(c) a countably infinite collection of *pairwise disjoint* nonempty sets whose union is finite.

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$$|A| = n \implies |2^A| = 2^n$$

Slope (UD Problem 22.2 (e))

(e) the set of all lines with rational slopes

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Q : Is "$\leq$" a partial order?

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Theorem (Cantor-Schröder–Bernstein (1887))

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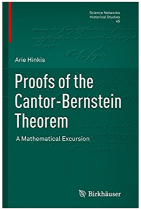

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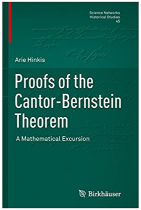

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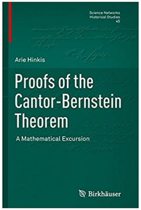

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Theorem (PCC)

Principle of Cardinal Comparability (PCC) $\iff$ *Axiom of Choice*

Finite Sets

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"关于有穷，我原以为我是懂的"

Definition (Finite)

X is finite if

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Theorem (Pigeonhole Principle (UD Theorem 21.2))

$$f : \{1, \cdots, m\} \to \{1, \cdots, n\} \ (m, n \in \mathbb{N}^+, m > n)$$

f is not one-to-one.

$A \setminus \{a\}$ (UD Problem 21.15)

Let A be a nonempty finite set with $|A| = n$ and let $a \in A$.
Prove that $A \setminus \{a\}$ is finite and $|A \setminus \{a\}| = n - 1$.

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A and B are finite sets and $f : A \to B$ is one-to-one.
Show that $|A| \leq |B|$.

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By contradiction and the pigeonhole principle.

(UD Problem 21.16)

(a) A is a finite set and $B \subseteq A$. We showed that B is finite (Corollary 20.11). Show that $|B| \leq |A|$.

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By contradiction and (b).

Cardinality of $|\text{ran}(f)|$ (UD Problem 21.18)

Let A and B be sets with A finite.

$$f : A \to B$$

Prove that $|\text{ran}(f)| \leq |A|$.

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(No Axiom of Choice Here)

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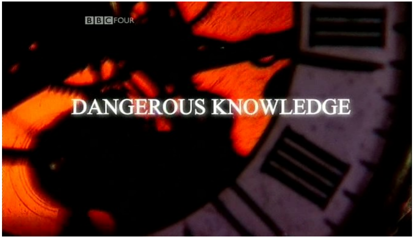
$$f(g(y)) = f(x) = y \implies f = g^{-1}$$

Dangerous Knowledge (BBC 2007)



Continuum Hypothesis (CH):

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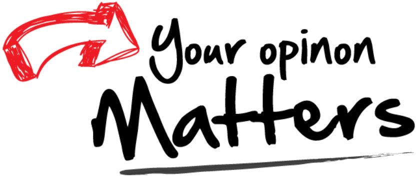
☞ Dangerous Knowledge (22:20)

Independence from ZFC:

Kurt Gödel (1940) CH cannot be disproved from ZF.

Paul Cohen (1964) CH cannot be proven from the ZFC axioms.


Thank
You!



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