Recap

In the last lecture, we had studied the time derivatives of the position vector (i.e. velocity and acceleration) of a particle P w.r.t different coordinate systems of a ref. frame.

In kinematics, it is interesting to relate the motion of the same body (modelled as a particle or an RB) in two different reference frames.

velocity, acceleration, etc.

Since the time-derivative of vectors is reference-frame-dependent we will discuss the relationship between the time derivatives of the same vector $\underline{A}$ in different reference frames 'F' and 'm'

You will see that the derivatives are related through:

$$\left.\frac{d\underline{A}}{dt}\right|_F = \left.\frac{d\underline{A}}{dt}\right|_m + \underline{\omega}_{m|F} \times \underline{A}$$

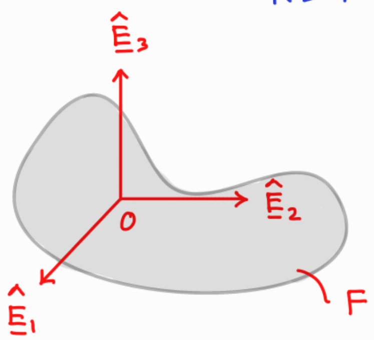
angular velocity of moving ref. 'm' w.r.t ref 'F'

will define it formally next

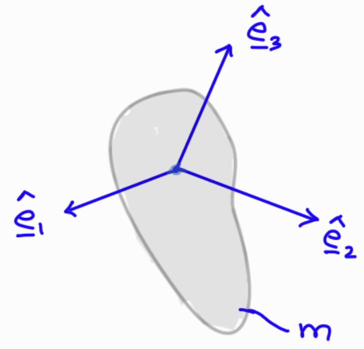
You will see that the time derivatives in the two different ref. frames differ only if they are rotating relative to each other.

Angular velocity vector of a rotating ref. frame 'm' w.r.t fixed frame F

Consider a $\underbrace{\text{fixed frame}}_{\text{RB `F'}}$ and a $\overbrace{\text{rotating frame}}^{\text{RB `m'}}$ of reference:



Fixed frame (F)



Rotating frame (m)

$$\left.\frac{d\hat{\underline{E}}_i}{dt}\right|_F = \underline{0}$$

w.r.t

$$\left.\frac{d\hat{\underline{e}}_i}{dt}\right|_m = \underline{0}$$

w.r.t

$\underline{\omega}_{m|F}$: angular velocity vector of rigid body (RB) 'm' w.r.t RB 'F' → measures the rate of change of orientation of RB 'm' w.r.t. RB 'F'

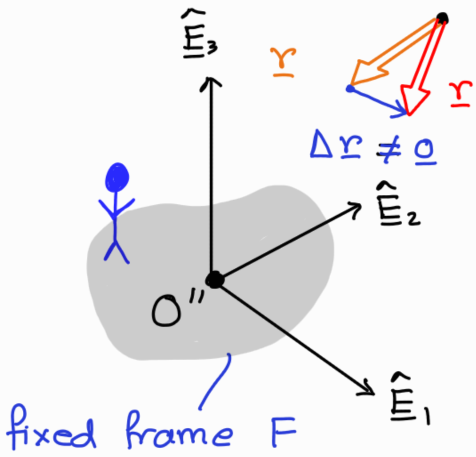
How do we measure orientation?

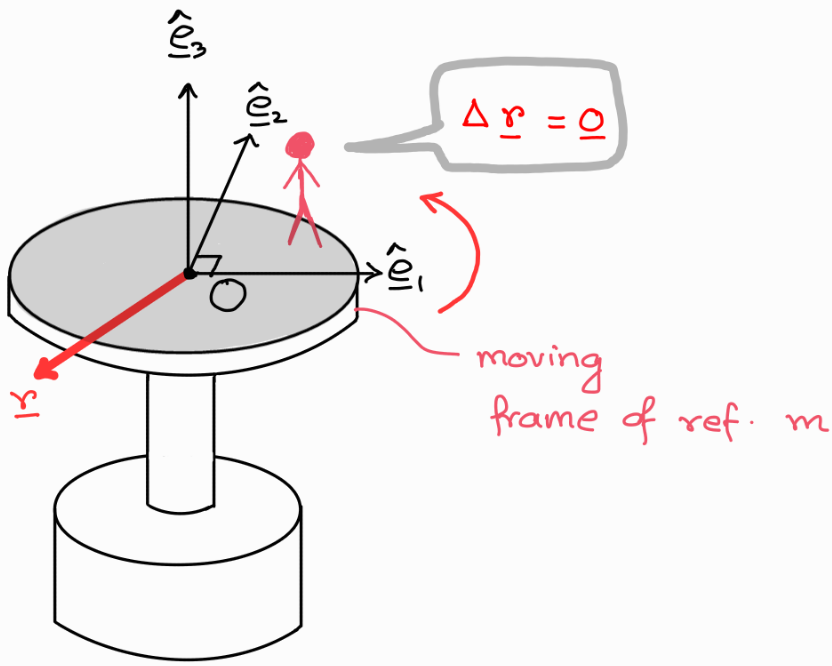
⇓

Let $\hat{\underline{e}}_1, \hat{\underline{e}}_2, \hat{\underline{e}}_3$ be three mutually perpendicular vectors embedded in RB 'm', and $\hat{\underline{E}}_1, \hat{\underline{E}}_2, \hat{\underline{E}}_3$ are three m.p vectors embedded in RB 'F', then if the orientation of the triad $\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3$ appears to change to an observer tied to RB 'F', then

$$\underline{\omega}_{m|F} \neq \underline{0}$$

Here is a more physical example





Note that $\left.\dfrac{d\hat{E}_1}{dt}\right|_F = \left.\dfrac{d\hat{E}_2}{dt}\right|_F = \left.\dfrac{d\hat{E}_3}{dt}\right|_F = \underline{0}$

Also, $\left.\dfrac{d\hat{e}_1}{dt}\right|_m = \left.\dfrac{d\hat{e}_2}{dt}\right|_m = \left.\dfrac{d\hat{e}_3}{dt}\right|_m = \underline{0}$

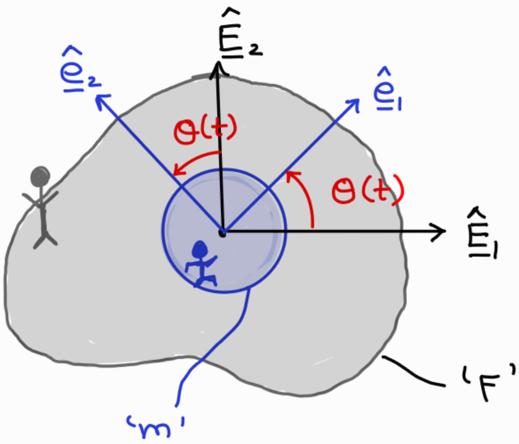
but if any one of the following is non-zero,

$$\left.\frac{d\hat{e}_1}{dt}\right|_F, \quad \left.\frac{d\hat{e}_2}{dt}\right|_F, \quad \left.\frac{d\hat{e}_3}{dt}\right|_F \neq \underline{0}$$

then $\underline{\omega}_{m|F} \neq \underline{0}$

Let us consider a simple case of planar motion of rotating frame 'm' w.r.t. fixed ref frame 'F'

Every point/particle fixed in 'm' moves in a plane, hence called planar motion



$$\underline{\hat{e}}_1 = \cos\theta\ \underline{\hat{E}}_1 + \sin\theta\ \underline{\hat{E}}_2$$

$$\underline{\hat{e}}_2 = -\sin\theta\ \underline{\hat{E}}_1 + \cos\theta\ \underline{\hat{E}}_2$$

$$\left.\frac{d\underline{\hat{e}}_1}{dt}\right|_F = \left.\frac{d}{dt}\left(\cos\theta\ \underline{\hat{E}}_1 + \sin\theta\ \underline{\hat{E}}_2\right)\right|_F$$

$$= (-\sin\theta)\,\dot{\theta}\ \underline{\hat{E}}_1 + (\cos\theta)\,\dot{\theta}\ \underline{\hat{E}}_2$$

(Note that $\left.\frac{d\underline{\hat{E}}_i}{dt}\right|_F = 0$)

manipulate using cross product

$$= \dot{\theta}\ \underline{\hat{E}}_3 \times \left(\sin\theta\ \underline{\hat{E}}_2 + \cos\theta\ \underline{\hat{E}}_1\right)$$

$\downarrow$

Recognize that this equation can be expressed as

$$\left.\frac{d\underline{\hat{e}}_1}{dt}\right|_F = \underline{\omega}_{m|F} \times \underline{\hat{e}}_1, \quad \text{where } \underline{\omega}_{m|F} = \dot{\theta}\ \underline{\hat{E}}_3$$

Similarly, we can show that

$$\left.\frac{d\underline{\hat{e}}_2}{dt}\right|_F = \underline{\omega}_{m|F} \times \underline{\hat{e}}_2, \quad \text{and} \quad \left.\frac{d\underline{\hat{e}}_3}{dt}\right|_F = \underline{\omega}_{m|F} \times \underline{\hat{e}}_3$$

$$\therefore \boxed{\left.\frac{d\underline{\hat{e}}_i}{dt}\right|_F = \underline{\omega}_{m|F} \times \underline{\hat{e}}_i} \quad \text{— I}$$

We can show that for any vector $\underline{r}$, whose orientation changes but the magnitude does not change, the time derivative w.r.t fixed frame 'F' is:

simple rotation

$$\boxed{\left.\frac{d\underline{r}}{dt}\right|_F = \underline{\omega}_{m|F} \times \underline{r}} \quad \text{— II}$$

Proof: $\quad \underline{r} = r_1 \hat{\underline{e}}_1 + r_2 \hat{\underline{e}}_2 + r_3 \hat{\underline{e}}_3$ where r_1, r_2, r_3 are constant values

$$\left.\frac{d\underline{r}}{dt}\right|_F = \cancelto{0}{\frac{dr_1}{dt}}\, \hat{\underline{e}}_1 + r_1 \left.\frac{d\hat{\underline{e}}_1}{dt}\right|_F + \cancelto{0}{\frac{dr_2}{dt}}\, \hat{\underline{e}}_2 + r_2 \left.\frac{d\hat{\underline{e}}_2}{dt}\right|_F$$

$$+ \cancelto{0}{\frac{dr_3}{dt}}\, \hat{\underline{e}}_3 + r_3 \left.\frac{d\hat{\underline{e}}_3}{dt}\right|_F$$

Use (I)

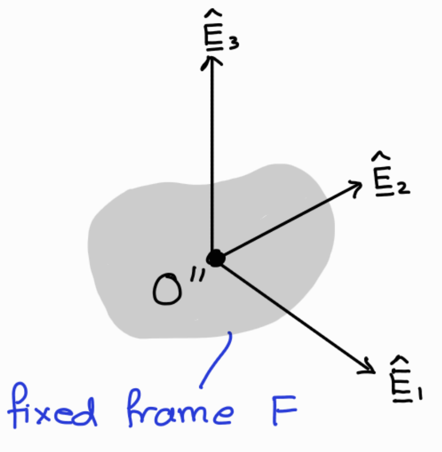
$$= r_1 \left(\underline{\omega}_{m|F} \times \hat{\underline{e}}_1\right) + r_2 \left(\underline{\omega}_{m|F} \times \hat{\underline{e}}_2\right) + r_3 \left(\underline{\omega}_{m|F} \times \hat{\underline{e}}_3\right)$$

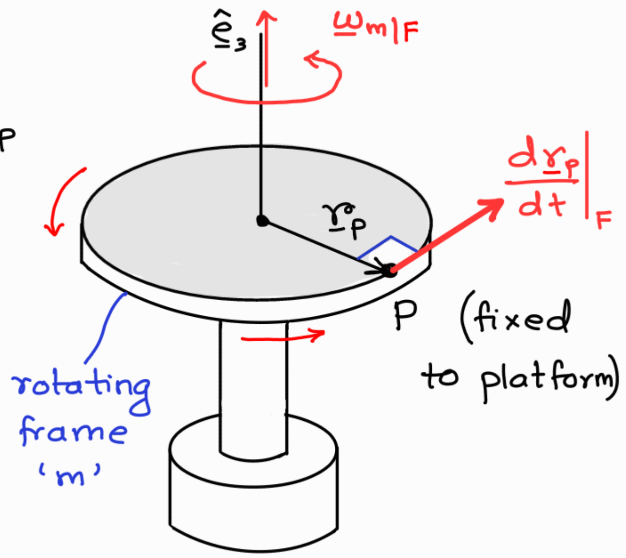
$$= \underline{\omega}_{m|F} \times \left(r_1 \hat{\underline{e}}_1 + r_2 \hat{\underline{e}}_2 + r_3 \hat{\underline{e}}_3\right)$$

$$= \underline{\omega}_{m|F} \times \underline{r}$$

In regard to kinematics of a particle (or point), we have understood that derivative of the position vector of a pt P fixed to a moving frame 'm' that rotates relative to 'F' is given by

Velocity vector of P fixed in 'm' $\; = \left.\frac{d\underline{r}_P}{dt}\right|_F = \underline{\omega}_{m|F} \times \underline{r}_P$





Relationship of velocity and acceleration vectors of a point w.r.t rotating frame 'm' and fixed frame 'F'

We will establish the relationship for a general time-varying vector, say $\underline{A}(t)$, that is no longer fixed to the moving ref. frame 'm', but also changing w.r.t 'm'

$$\text{i.e.} \quad \left.\frac{d\underline{A}}{dt}\right|_m \neq 0$$

① If 'A' is a scalar $\Rightarrow \left.\frac{dA}{dt}\right|_F = \left.\frac{dA}{dt}\right|_m$

(absolute $\Leftrightarrow$ ref. frame independent)

② If $\underline{A}$ is a vector $\Rightarrow \underbrace{\left.\frac{d\underline{A}}{dt}\right|_F \neq \left.\frac{d\underline{A}}{dt}\right|_m}_{\text{Let's see why!}}$

$$\underline{A} = A_i \, \underline{\hat{E}}_i \quad (\text{frame } F)$$

$$\underline{A} = a_i \, \underline{\hat{e}}_i \quad (\text{frame } m)$$

representation of the same vector $\underline{A}$ using two different sets of triads of unit vectors

$$\left.\frac{d\underline{A}}{dt}\right|_F = \left.\dot{\underline{A}}\right|_F = \frac{d}{dt}\left(A_1\underline{\hat{E}}_1 + A_2\underline{\hat{E}}_2 + A_3\underline{\hat{E}}_3\right)\Big|_F$$

$$= \dot{A}_1\underline{\hat{E}}_1 + \dot{A}_2\underline{\hat{E}}_2 + \dot{A}_3\underline{\hat{E}}_3 \Big|_F$$

$$\left.\frac{d\underline{A}}{dt}\right|_m = \left.\dot{\underline{A}}\right|_m = \frac{d}{dt}\left(a_i\,\underline{\hat{e}}_i\right)\Big|_m = \dot{a}_i\,\underline{\hat{e}}_i\Big|_m \leftarrow (\text{ESC})$$

$$\dot{\underline{A}}\Big|_F = \frac{d}{dt}\left(a_1\,\hat{\underline{e}}_1 + a_2\,\hat{\underline{e}}_2 + a_3\,\hat{\underline{e}}_3\right)\Big|_F$$

$$= \underbrace{\dot{a}_1\,\hat{\underline{e}}_1 + \dot{a}_2\,\hat{\underline{e}}_2 + \dot{a}_3\,\hat{\underline{e}}_3}_{\dot{\underline{A}}|_m} + \underbrace{a_1\,\dot{\hat{\underline{e}}}_1 + a_2\,\dot{\hat{\underline{e}}}_2 + a_3\,\dot{\hat{\underline{e}}}_3}$$

We can sub in the values of $\dot{\hat{\underline{e}}}_i = \underline{\omega}_{m|F} \times \hat{\underline{e}}_i$ from (I)

$$a_1\,\dot{\hat{\underline{e}}}_1 + a_2\,\dot{\hat{\underline{e}}}_2 + a_3\,\dot{\hat{\underline{e}}}_3$$

$$= a_1(\underline{\omega}_{m|F} \times \hat{\underline{e}}_1) + a_2(\underline{\omega}_{m|F} \times \hat{\underline{e}}_2) + a_3(\underline{\omega}_{m|F} \times \hat{\underline{e}}_3)$$

$$= \underline{\omega}_{m|F} \times \underbrace{(a_1\,\hat{\underline{e}}_1 + a_2\,\hat{\underline{e}}_2 + a_3\,\hat{\underline{e}}_3)}_{\underline{A}}$$

$$= \underline{\omega}_{m|F} \times \underline{A}$$

$$\boxed{\dot{\underline{A}}\Big|_F = \dot{\underline{A}}\Big|_m + \left(\underline{\omega}_{m|F} \times \underline{A}\right)}$$

Even if the fixed frame 'F' was also a moving frame, the above equation would still hold.

Also, think that NO frame of reference is absolutely fixed!

e.g. Earth spins & revolves around the sun

If the components of vector $\underline{A}$ remain constant w.r.t. moving frame 'm'

$\dot{\underline{A}}\big|_m = 0$, then the rate of change of $\underline{A}$ is the cross-product of $\underline{\omega}_{m|F}$ and $\underline{A}$ $\Rightarrow$ $\dot{\underline{A}}\big|_F = \underline{\omega}_{m|F} \times \underline{A}$ (same as (II))

Note: $\dot{\underline{A}}\big|_F \neq \dot{\underline{A}}\big|_m$, unless $\underline{\omega}_{m|F} = 0$ (or)

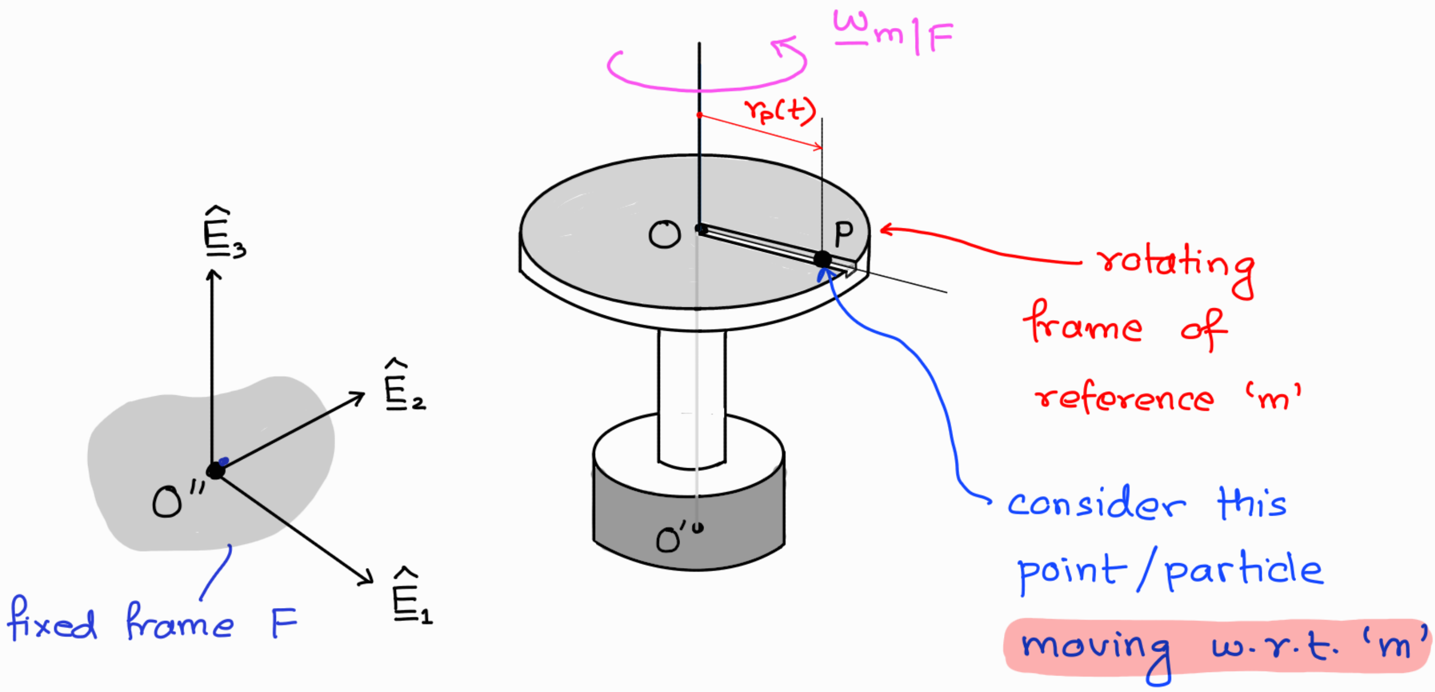
$\underline{\omega}_{m|F}$ is parallel to $\underline{A}$

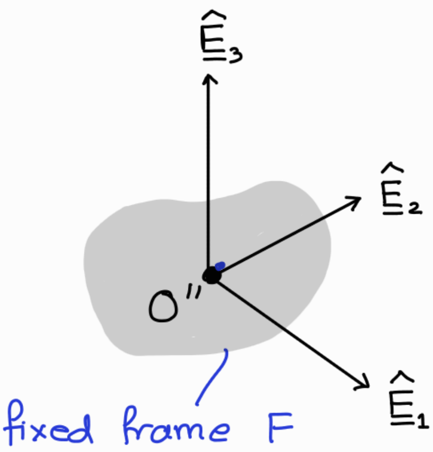
Relation of velocity vector of P wrt 'm' and 'F'

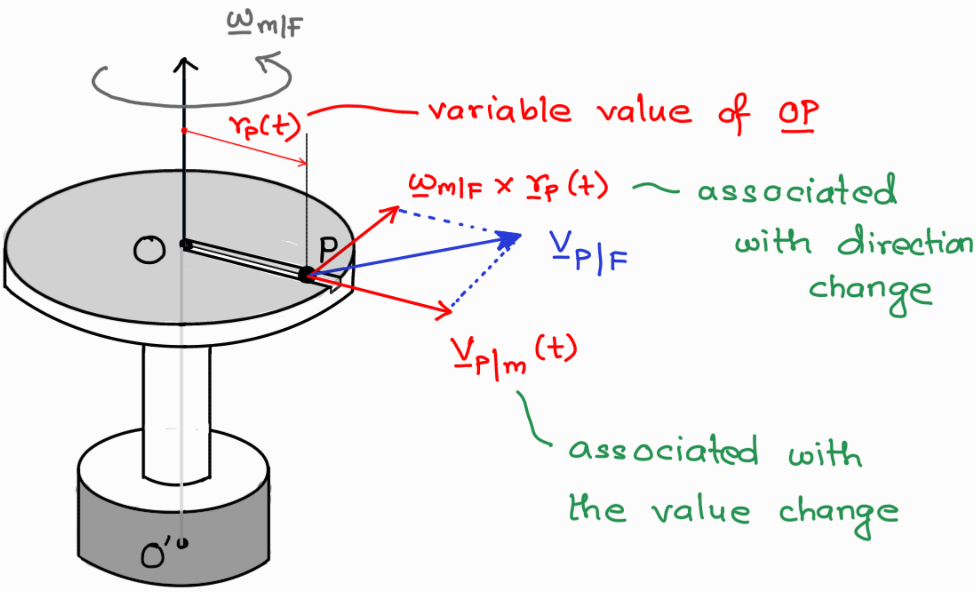
Substituting $\underline{A} = \underline{r}_P$, we get

$$\boxed{\underline{v}_{P|F} = \frac{d\underline{r}_P}{dt}\bigg|_F = \underline{v}_{P|m} + \underline{\omega}_{m|F} \times \underline{r}_P}$$

To consider a physical example, lets now consider that pt P is no longer fixed to 'm' but free to move. Thus, $\underline{r}_P(t)$ now not only changes direction but may also change its magnitude.







Angular acceleration vector of rotating frame 'm' wrt 'F'

Angular acceleration is defined as the time derivative of the angular velocity, usually denoted by $\underline{\alpha} = \dot{\underline{\omega}}$

The angular velocity of a rotating frame 'm' is a vector which can be expressed in terms of the csys $\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3$ of the frame 'm'

$$\underline{\omega}_{m|F} = \omega_1 \hat{\underline{e}}_1 + \omega_2 \hat{\underline{e}}_2 + \omega_3 \hat{\underline{e}}_3$$

You will notice that, even if the rotation rates ω_1, ω_2, ω_3 remain constant, there will be acceleration since the coordinate axes $\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3$ change with time for frame 'm'

To find the angular acceleration, we make use of the generic result $\dot{\underline{A}}|_F = \dot{\underline{A}}|_m + \underline{\omega}_{m|F} \times \underline{A}$ and substitute $\underline{A} = \underline{\omega}_{m|F}$

Diff w.r.t time

Angular acceleration

$$\underline{\alpha}|_F = \dot{\overline{(\underline{\omega}_{m|F})}}\Big|_F = \dot{\overline{(\underline{\omega}_{m|F})}}\Big|_m + \underbrace{\cancel{(\underline{\omega}_{m|F} \times \underline{\omega}_{m|F})}}_{\text{// to each other}} \rightarrow 0$$

$$= \dot{\overline{\underline{\omega}_{m|F}}}\Big|_m$$

$\therefore$ $\underline{\alpha}|_F$ (or $\dot{\underline{\omega}}_{m|F}$) is the same in 'F' or 'm'

We can also use analytical derivative of vectors to derive it.

The angular acceleration of the moving frame 'm' in terms of the csys of moving frame can be expressed as:

Using ESC

$$\dot{\underline{\omega}}_{m|F} = \frac{d}{dt}(\underbrace{\omega_i}_{\text{scalar}} \overbrace{\hat{\underline{e}}_i}^{\text{vector}})\Big|_F = \dot{\omega}_i \hat{\underline{e}}_i\Big|_F + \omega_i \dot{\hat{\underline{e}}}_i\Big|_F = \dot{\omega}_i \hat{\underline{e}}_i + \omega_i (\underline{\omega}_{m|F} \times \hat{\underline{e}}_i)$$

$$\dot{\underline{\omega}}_{m|F} = \underbrace{\dot{\omega}_1 \hat{\underline{e}}_1 + \dot{\omega}_2 \hat{\underline{e}}_2 + \dot{\omega}_3 \hat{\underline{e}}_3}_{\text{due to change of rotation rates w.r.t 'm'}} + \underbrace{\omega_1 (\underline{\omega}_{m|F} \times \hat{\underline{e}}_1) + \omega_2 (\underline{\omega}_{m|F} \times \hat{\underline{e}}_2) + \omega_3 (\underline{\omega}_{m|F} \times \hat{\underline{e}}_3)}_{\text{due to change of orientation of csys of 'm' w.r.t 'F'}}$$

$$= \dot{\underline{\omega}}_{m|F}\Big|_m + \underline{\omega}_{m|F} \times (\omega_1 \hat{\underline{e}}_1 + \omega_2 \hat{\underline{e}}_2 + \omega_3 \hat{\underline{e}}_3)$$

$$= \dot{\underline{\omega}}_{m|F}\Big|_m + \cancel{\underline{\omega}_{m|F} \times \underline{\omega}_{m|F}} \rightarrow 0$$

$$= \dot{\underline{\omega}}_{m|F}\Big|_m$$

Index Notation

Index notation, also known as Einstein summation convention [ESC] is widely used in engineering mechanics and other fields of applied mechanics because it simplifies representations of equations, system of equations, sums, etc. involving $\underbrace{\text{scalars and vectors}}_{\text{(Tensors, in general)}}$

will be introduced later

We will learn them so that we can use them to write more compactly

Examples: 1 Sum,

$$s = a_1 x_1 + a_2 x_2 + \dots + a_n x_n = \sum_{i=1}^{n} a_i x_i \xrightarrow{\text{ESC}} a_i x_i$$

2 Systems of linear eqns:

$$\left.\begin{aligned} \underline{\hat{E}}_1 &= a_{11}\,\underline{\hat{e}}_1 + a_{12}\,\underline{\hat{e}}_2 + a_{13}\,\underline{\hat{e}}_3 \\ \underline{\hat{E}}_2 &= a_{21}\,\underline{\hat{e}}_1 + a_{22}\,\underline{\hat{e}}_2 + a_{23}\,\underline{\hat{e}}_3 \\ \underline{\hat{E}}_3 &= a_{31}\,\underline{\hat{e}}_1 + a_{32}\,\underline{\hat{e}}_2 + a_{33}\,\underline{\hat{e}}_3 \end{aligned}\right\} \xrightarrow{\text{ESC}} \underline{\hat{E}}_i = a_{ij}\,\underline{\hat{e}}_j$$

compact!

Index Notation – Definitions

1 Non-repeating Index / free index:

occurs once and only once in a term

e.g. $\underline{\hat{E}}_i = \boxed{a_{ij}\,\underline{\hat{e}}_j}$ one term

$i \rightarrow$ free index

② Repeating index / dummy index / summing index:

(a) occurs twice in a term

(b) gets summed over the entire range of the index

Same

$$\hat{E}_i = a_{ij}\,\hat{e}_j$$

$j \to$ repeating index (occurs twice) (summed over the range of the index)

$\Downarrow$

typically '3' in this course

$$\hat{E}_i = a_{ik}\,\hat{e}_k$$

Some rules in ESC :

① No index must appear more than <u>twice</u>

e.g. (a) $a_{ij}\,b_{ij}$ ✓ vs $a_{ij}\,b_{jj}$ ✗ ($\because$ 'j' appear thrice)

(b) $a_i\,b_i\,c_i$ ✗ ($\because$ 'i' appears thrice)

② Number of free indices must match on both sides of an equation

e.g. (a) $x_i = a_{ij}\,b_j$ ✓

$i \to$ free

$i \to$ free index

$j \to$ summing/dummy index

$\Downarrow$

$$x_1 = a_{11}\,b_1 + a_{12}\,b_2 + a_{13}\,b_3$$

$$x_2 = a_{21}\,b_1 + a_{22}\,b_2 + a_{23}\,b_3$$

(b) $x_i = a_{ij}$ ✗ (Not allowed)

(ambiguous)

'i' or 'j' both look like free indices on RHS

(c) $\quad x_i = a_{jk}\, b_k + c_i \quad \times$

$i \rightarrow$ free

1 free index (LHS)

$k \rightarrow$ summing dummy index

$i \rightarrow$ free index

$j \rightarrow$ free index

2 free indices (RHS)

③ Each term must have the same free indices in any valid equation

e.g. $\quad F_i = m\, a_i \quad \checkmark$

$i \rightarrow$ free $\quad i \rightarrow$ free

$F_i = m\, a_j \quad \times$

$i \rightarrow$ free $\quad j \rightarrow$ free

$A_{ij} = B_{ik}\, C_{kj}$ [Matrix multiplication] $\checkmark$

$A_{12} = B_{11}\, C_{12} + B_{12}\, C_{22} + B_{13}\, C_{32}$

Kronecker Delta

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

two-variable function

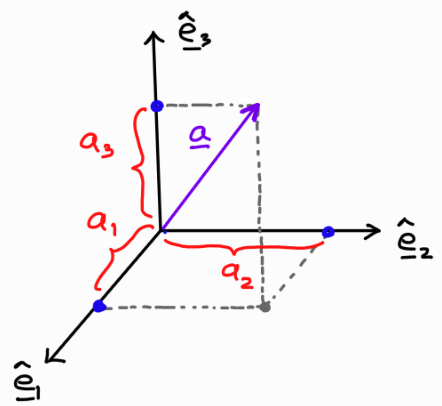
$$[\delta_{ij}] = \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_{3\times 3}$$

Suppose $\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3$ form a mutually perpendicular triad of unit vectors, then:

$$\left.\begin{array}{ll} \hat{\underline{e}}_1 \cdot \hat{\underline{e}}_1 = 1, & \hat{\underline{e}}_1 \cdot \hat{\underline{e}}_2 = 0 \\ \hat{\underline{e}}_2 \cdot \hat{\underline{e}}_2 = 1, & \hat{\underline{e}}_1 \cdot \hat{\underline{e}}_3 = 0 \\ \hat{\underline{e}}_3 \cdot \hat{\underline{e}}_3 = 1, & \hat{\underline{e}}_2 \cdot \hat{\underline{e}}_3 = 0 \end{array}\right\} \Rightarrow \underbrace{\hat{\underline{e}}_i \cdot \hat{\underline{e}}_j = \delta_{ij}}_{\text{Compact}}$$

and any vector can be represented by its components using the ESC notation as:

$$\underline{a} = a_i \, \hat{\underline{e}}_i \quad \Rightarrow \quad \underline{a} = a_1 \hat{\underline{e}}_1 + a_2 \hat{\underline{e}}_2 + a_3 \hat{\underline{e}}_3$$



Similarly, $\underline{b} = b_j \, \hat{\underline{e}}_j \quad \Rightarrow \quad \underline{b} = b_1 \hat{\underline{e}}_1 + b_2 \hat{\underline{e}}_2 + b_3 \hat{\underline{e}}_3$

and, an inner product between vectors $\underline{a}$ and $\underline{b}$ in ESC will be:

vectors

$$\underline{a} \cdot \underline{b} = (a_i \, \hat{\underline{e}}_i) \cdot (b_j \, \hat{\underline{e}}_j)$$

scalars

$$= a_i \, b_j \, (\hat{\underline{e}}_i \cdot \hat{\underline{e}}_j)$$

$$= a_i \, b_j \, \delta_{ij}$$

(Here, both i and j are dummy summing indices)