Recap

Relation of vectors in rotating ref. frames w.r.t. fixed frames

$$\dot{\underline{A}}\big|_F = \dot{\underline{A}}\big|_m + \underline{\omega}_{m|F} \times \underline{A}$$

Multiple frames

Angular velocity

$$\underline{\omega}_{3|1} = \underline{\omega}_{3|2} + \underline{\omega}_{2|1} \quad \text{(Addivity holds)}$$

Angular acceleration

$$\dot{\underline{\omega}}_{3|1} = \dot{\underline{\omega}}_{3|2} + \dot{\underline{\omega}}_{2|1} + \underline{\omega}_{2|1} \times \underline{\omega}_{3|2}$$

Velocity & Acceleration vectors of a pt in different ref-frames

Velocity

$$\underline{v}_{P|F} = \underline{v}_{A|F} + \underline{v}_{P|m} + \underline{\omega}_{m|F} \times \underline{r}_P$$

Acceleration

$$\underline{a}_{P|F} = \underline{a}_{A|F} + \underline{a}_{P|m} + \underline{\omega}_{m|F} \times (\underline{\omega}_{m|F} \times \underline{r}_P)$$
$$+ \dot{\underline{\omega}}_{m|F} \times \underline{r}_{P|m}$$
$$+ 2(\underline{\omega}_{m|F} \times \underline{v}_{P|m})$$

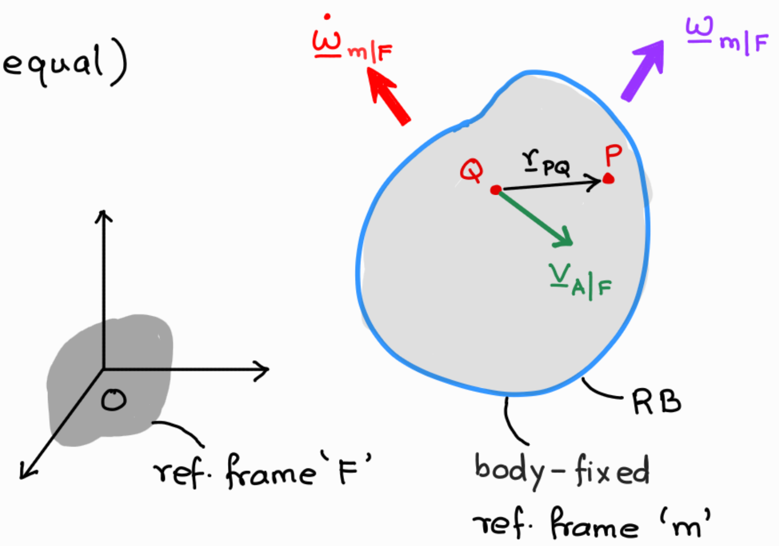
Kinematics of Rigid Bodies

The point/particle model is not suitable when orientation is relevant to the problem being studied. The simplest model incorporating the orientation is the rigid body, which is defined as a set of material points with constant mutual distances.

Recall that the defns of reference frames and rigid bodies are very close from kinematics pt. of view. Infact, the ref. frame of points fixed to the rigid body is called body-fixed ref. frame. The kinematics of the rigid body corresponds to the velocities and accelerations associated with this body-fixed ref. frame.

Velocity & Acceleration distributions in a Rigid body

The constant distances between points in an RB 'm' imply that their motion (velocity and acceleration) are related (but not necessarily equal)



Since the points of the RB are fixed wrt body-fixed ref frame 'm', their motion wrt frame 'F' can be obtained by composition formulae:

Consider two points P and Q fixed in the RB that is translating and rotating (with $\underline{\omega}_{m|F}$ and $\dot{\underline{\omega}}_{m|F}$), then:

Velocity of point P in RB wrt 'F'

$$\underline{v}_{P|F} = \underline{v}_{Q|F} + \underline{\omega}_{m|F} \times \underline{r}_{PQ} \quad \text{——} \; (1)$$

Acceleration of point P in RB wrt 'F'

$$\underline{a}_{P|F} = \underline{a}_{Q|F} + \dot{\underline{\omega}}_{m|F} \times \underline{r}_{PQ} + \underline{\omega}_{m|F} \times (\underline{\omega}_{m|F} \times \underline{r}_{PQ}) \quad \text{——} \; (2)$$

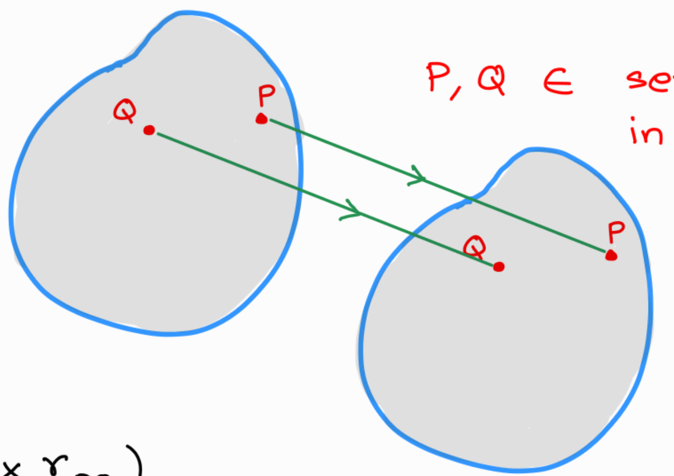
Note: The velocities and accelerations of the two points P & Q are usually different

Case of PURE TRANSLATION

When the rigid body is purely translating (no rotation), all points have the same velocity and acceleration, and they all describe equal paths.

$$\underline{v}_{P|F} = \underline{v}_{Q|F} + \cancelto{0}{\underline{\omega}_{m|F}} \times \underline{r}_{PQ}$$

$$\Rightarrow \quad \underline{v}_{P|F} = \underline{v}_{Q|F}$$



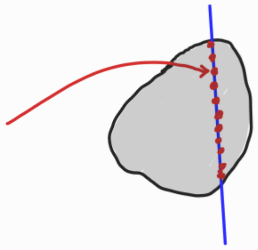
$$\underline{a}_{P|F} = \underline{a}_{Q|F} + \cancelto{0}{\dot{\underline{\omega}}_{m|F}} \times \underline{r}_{PQ}$$
$$+ \cancelto{0}{\underline{\omega}_{m|F}} \times (\cancelto{0}{\underline{\omega}_{m|F}} \times \underline{r}_{PQ})$$

$$\Rightarrow \quad \underline{a}_{P|F} = \underline{a}_{Q|F}$$

Instantaneous axis of rotation (IAR)

An RB is said to have an instantaneous axis of rotation at a given time 't' if all its $\underbrace{\text{points}}$ on such an axis have zero velocity at that instant.

material pts in the RB



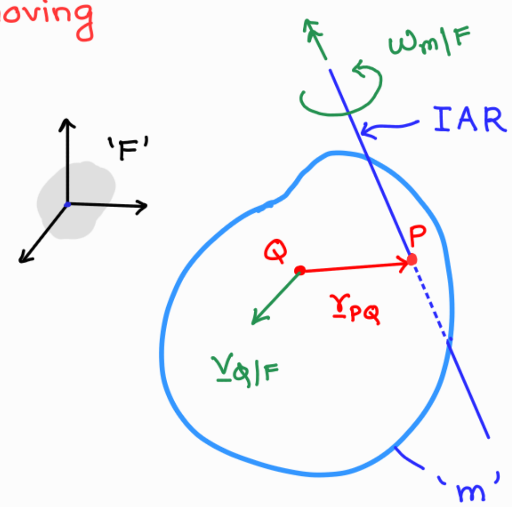
→ On IAR, velocity of its points is zero, but acceleration may or may not be zero

→ IAR may be on the RB itself, or on its rigid massless extension

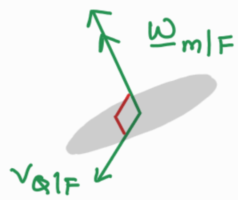
→ IAR does <u>NOT</u> always exist, in general

Mathematically, it means that for a moving (translating and rotating) RB, an IAR will exist if a point (say P) has zero velocity w.r.t 'F'

$$\underline{v}_{P|F} = \underline{0}$$



It can be shown that non-zero velocity vectors (e.g. $\underline{v}_{Q|F}$) of all other points of the RB must be $\perp$ to $\underline{\omega}_{m|F}$, if there exists an IAR through (say) point P of the RB.



$$\underline{v}_{P|F} = 0$$

Using velocity vector formula:

$$\Rightarrow \underline{v}_{Q|F} + \underline{\omega}_{m|F} \times \underline{r}_{PQ} = 0 \quad \Rightarrow \underline{v}_{Q|F} = -\left(\underline{\omega}_{m|F} \times \underline{r}_{PQ}\right)$$

Now, lets consider inner (dot) product of $\underline{v}_{Q|F}$ with $\underline{\omega}_{T|F}$

$$\underline{v}_{Q|F} \cdot \underline{\omega}_{m|F} = -\underbrace{\left(\underline{\omega}_{m|F} \times \underline{r}_{PQ}\right)}_{\perp \text{ to } \underline{\omega}_{m|F}} \cdot \underline{\omega}_{m|F} = \underline{0}$$

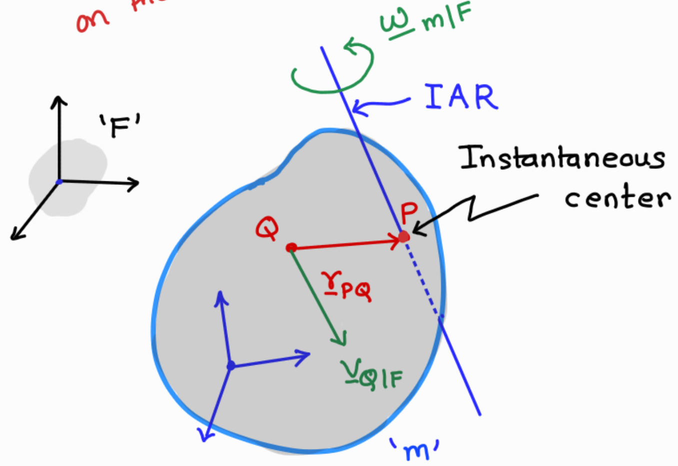
Infact, $\underline{v}_{Q|F} \cdot \underline{\omega}_{m|F} = \underline{0}$ is the required condition for IAR to exist.

For planar motion of RB, the IAR always exists. This axis is $\perp$ to the motion of RB and // to $\underline{\omega}_{m|F}$

Instantaneous Centre (I)

Point where the IAR intersects the motion plane of point P of the RB (or on its rigid massless extension)

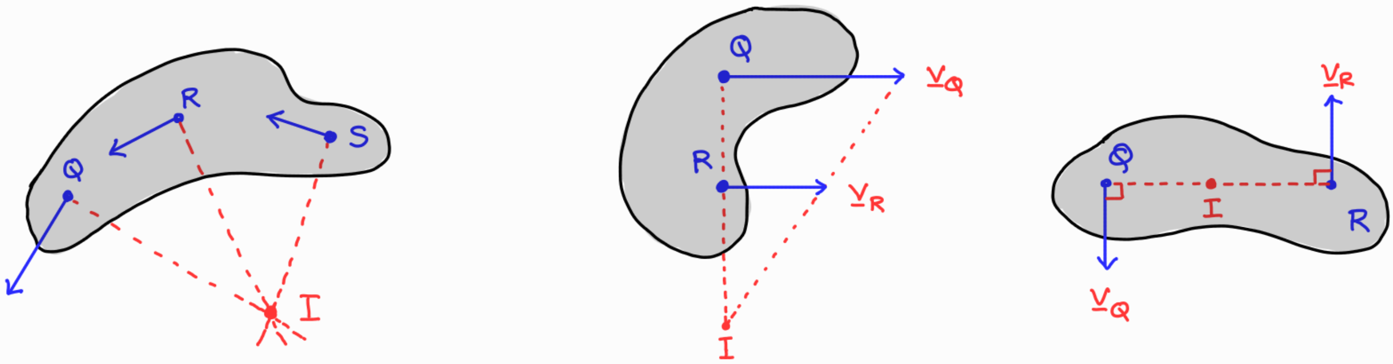
Does it have to be on the RB or its rigid massless extension?



$\underline{v}_I = \underline{0}$ at that instant

(but $\underline{a}_I \neq \underline{0}$ in general)

If $\underline{v}_Q$ and $\underline{v}_R$ are known, I can be located



Basic Constraint Conditions: Contact, No Slip, Impending slip

Any restriction on the movement of rigid bodies constitutes a constraint condition. The conditions of contact (no slip, impending slip) in a single-point contact between two rigid bodies are called basic constraint conditions.

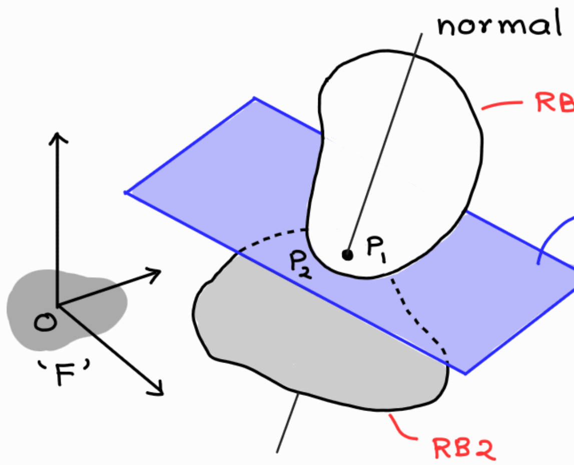
The kinematic equations that express these limitations are called constraint equations. When dealing with rigid bodies, these eqns are linear relationships among the variables used to describe the velocity of rigid bodies separately.

In a contact between the points P_1 and P_2 of the RBs: RB1 and RB2, respectively, the constraints are expressed as:

1> Contact

$$\underline{v}_{P_1|F}(t)\Big|_n = \underline{v}_{P_2|F}(t)\Big|_n$$

normal

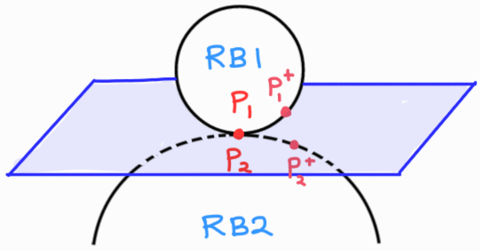


2> No slip (implies contact)

$$\underline{v}_{P_1|F}(t) = \underline{v}_{P_2|F}(t)$$

Points P_1 (of RB1) and P_2 (of RB2) are touching each other at an instant 't'

3) Rolling without slipping



$$\underline{v}_{P_1|F}(t) = \underline{v}_{P_2|F}(t)$$

and

$$\underline{v}_{P_1^+|F}(\underbrace{t+\Delta t}_{t^+}) = \underline{v}_{P_2^+|F}(\underbrace{t+\Delta t}_{t^+})$$

Along the tangential plane of contact, the acceleration components of P_1 and P_2 are the same at time instant 't'

i.e. $$\underline{a}_{P_1|F}(t)\Big|_{tg} = \underline{a}_{P_2|F}(t)\Big|_{tg}$$

tangential comp. of acc.

but, $$\underline{a}_{P_1|F}(t)\Big|_{n} \neq \underline{a}_{P_2|F}\Big|_{n} \quad \text{(in general)}$$

$$\therefore \quad \underline{a}_{P_1|F}(t) \neq \underline{a}_{P_2|F}(t) \quad \text{(in general)}$$

Note 'F' can be any ref. frame (fixed/moving)

↳ Proved later

4) Rolling with impending slip

$$\underline{v}_{P_1|F}(t) = \underline{v}_{P_2|F}(t)$$

but

$$\underline{v}_{P_1^+|F}(t^+) \neq \underline{v}_{P_2^+|F}(t^+)$$

$$\underline{a}_{P_1|F}(t)\Big|_{tg} = \underline{a}_{P_2|F}(t)\Big|_{tg}$$

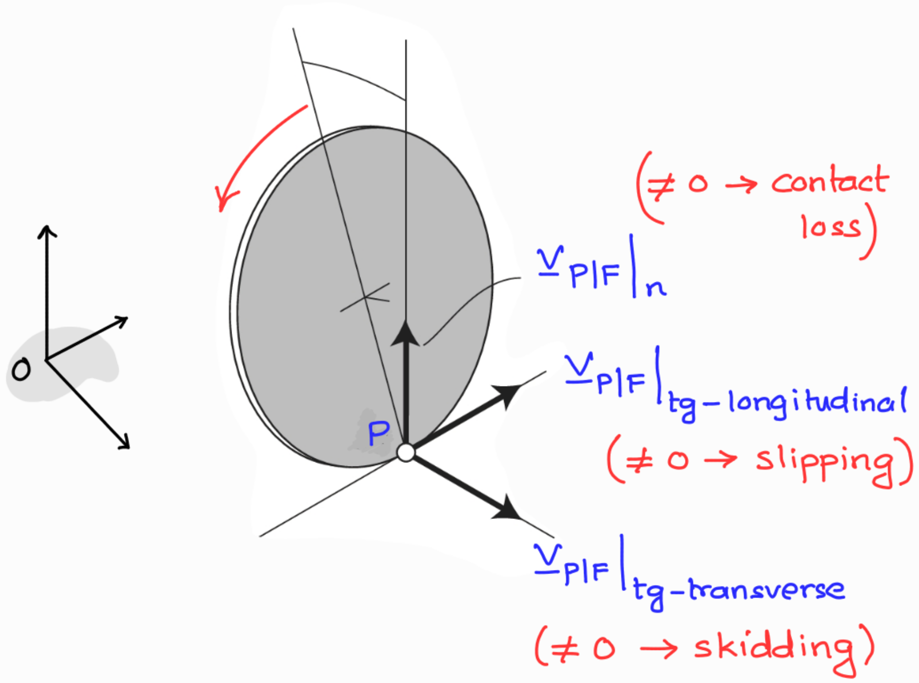
but

$$\underline{a}_{P_1|F}(t) \neq \underline{a}_{P_1|F}(t)$$

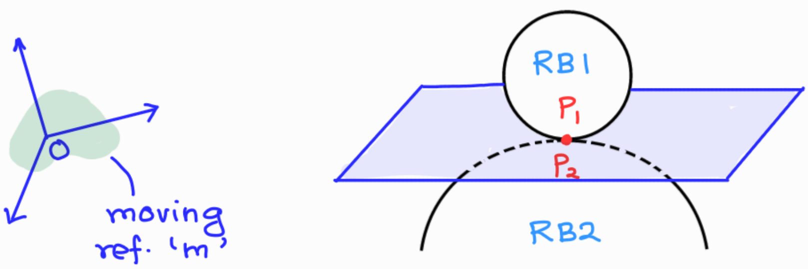
5) Slipping

$$\Rightarrow \quad \underline{v}_{P_1|F}(t) \neq \underline{v}_{P_2|F}(t) \quad \text{for any frame } F$$

For instance, when a wheel has rolling without slipping on the ground frame 'F', the wheel point P in contact with the surface has zero velocity relative to it, and the instantaneous motion of the wheel is just a rotation around the contact point P



Would the condition of rolling without slipping change if we switch to a moving reference frame 'm'?



Let's use the composition of velocity formula:

$$\underline{v}_{P_1|F} = \underline{v}_{P_1|m} + \underline{\omega}_{m|F} \times \underline{r}_{P_1O}$$

$$\underline{v}_{P_2|F} = \underline{v}_{P_2|m} + \underline{\omega}_{m|F} \times \underline{r}_{P_2O}$$

Rolling without slip

$$\Rightarrow \quad \underline{v}_{P_1|F} = \underline{v}_{P_2|F}$$

$$\Rightarrow \quad \underline{v}_{P_1|m} + \cancel{\underline{\omega}_{m|F} \times \underline{r}_{P_1O}} = \underline{v}_{P_2|m} + \cancel{\underline{\omega}_{m|F} \times \underline{r}_{P_2O}}$$

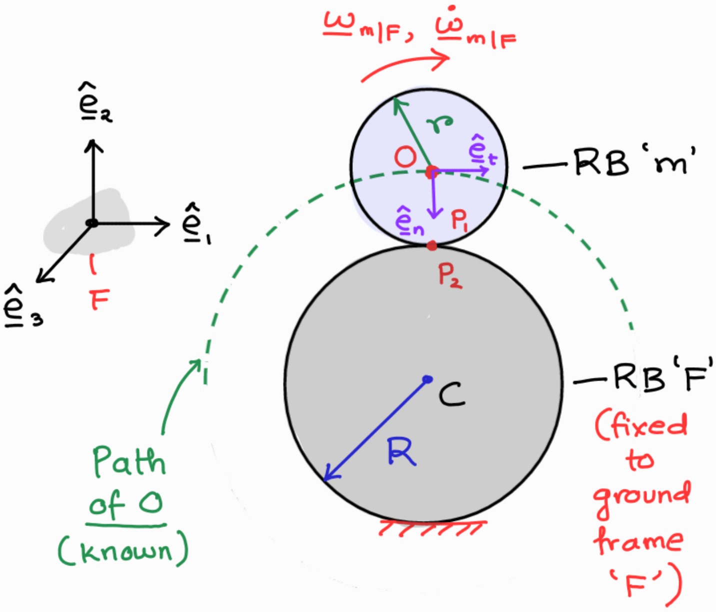
$[\because \ \underline{r}_{P_1O} = \underline{r}_{P_2O}]$

$$\Rightarrow \quad \underline{v}_{P_1|m} = \underline{v}_{P_2|m}$$

$\therefore$ No slip condition stays the SAME in all reference frames

Problem: A cylinder of radius 'r' rolls without slip on a fixed cylinder of radius 'R'. At this instant of time, the speed and rate of change of speed of the centre 'O' of the rolling cylinder are v and a respectively

Find $\underline{\omega}_{m|F}$, $\dot{\underline{\omega}}_{m|F}$, and $\underline{a}_{P_1}$



① Since the path of pt O is known and curved, we use path coordinates

Radius of curvature of path of 'O'

$$\rho_O = (R + r)$$

② Angular velocity of RB 'm' = $\underline{\omega}_{m|F} = -\omega\, \hat{e}_3$

Angular acceleration of RB 'm' = $\dot{\underline{\omega}}_{m|F} = -\dot{\omega}\, \hat{e}_3$

values to be found!

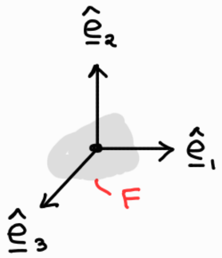
$$\underline{v}_{O|F} = \dot{s}\,\hat{\underline{e}}_t = v\,\hat{\underline{e}}_t = v\,\hat{\underline{e}}_1$$

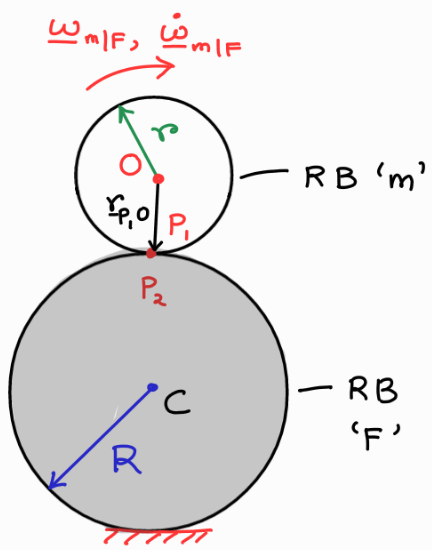
$$\underline{a}_{O|F} = \ddot{s}\,\hat{\underline{e}}_t + \frac{\dot{s}^2}{\rho_O}\,\hat{\underline{e}}_n$$

$$= \underset{\text{(given)}}{a}\,\hat{\underline{e}}_t + \frac{v^2}{(R+r)}\,\hat{\underline{e}}_n$$

$$= a\,\hat{\underline{e}}_1 - \frac{v^2}{(R+r)}\,\hat{\underline{e}}_2 \qquad \left[\because\ \hat{\underline{e}}_t = \hat{\underline{e}}_1 \text{ and } \hat{\underline{e}}_2 = -\hat{\underline{e}}_n\right]$$

(3) Fixed cylinder $\Rightarrow$ $\left.\begin{array}{l} \underline{v}_{P_2|F} = \underline{0} \\ \underline{a}_{P_2|F} = \underline{0} \end{array}\right\}$ $\because$ RB 'F' is stationary





Due to rolling with no slip

$$\underline{v}_{P_1|F} = \underline{v}_{P_2|F} = \underline{0}$$

$$\left.\underline{a}_{P_1|F}\right|_{tg} = \left.\underline{a}_{P_2|F}\right|_{tg} = \underline{0}$$

tg ~ tangential (along $\hat{\underline{e}}_1 - \hat{\underline{e}}_3$)

Use composition formula:

$$\underline{v}_{P_1|F} = \cancelto{0}{\underline{v}_{P_1|m}} + \underline{v}_{O|F} + \underline{\omega}_{m|F} \times \underline{r}_{P_2O} \qquad [\because\ P_2 \text{ is fixed to RB 'm'}]$$

$$\Rightarrow\ \underline{0} = v\,\hat{\underline{e}}_1 + \left[\left(-\omega\,\hat{\underline{e}}_3\right) \times \left(-r\,\hat{\underline{e}}_2\right)\right]$$

$$\Rightarrow\ \underline{0} = (v - \omega r)\,\hat{\underline{e}}_1$$

$$\Rightarrow\ \omega = \frac{v}{r}$$

$$\left.\underline{a}_{P_1|F}\right|_{tg} = \Big(\cancelto{0}{\underline{a}_{P_1|m}} + \underline{a}_{O|F} + \underline{\omega}_{m|F} \times (\underline{\omega}_{m|F} \times \underline{r}_{P_1O}) + \dot{\underline{\omega}}_{m|F} \times \underline{r}_{P_1O}$$

$$\left. + 2\,(\underline{\omega}_{m|F} \times \cancelto{0}{\underline{v}_{P_1|m}})\Big)\right|_{tg}$$

[$\because$ Point P_1 is fixed to RB 'm']

$$\Rightarrow \underline{0} = \underline{a}_{O|F} + \underline{\omega}_{m|F} \times (\underline{\omega}_{m|F} \times \underline{r}_{P_1O}) + \underline{\dot{\omega}}_{m|F} \times \underline{r}_{P_1O}$$

$$\Rightarrow \underline{0} = \left(a\,\hat{\underline{e}}_1 - \frac{v^2}{(R+r)}\hat{\underline{e}}_2\right) + (-\omega\,\hat{\underline{e}}_3) \times \left((-\omega\,\hat{\underline{e}}_3) \times (-r\,\hat{\underline{e}}_2)\right)$$

$$+ (-\dot{\omega}\,\hat{\underline{e}}_3) \times (-r\,\hat{\underline{e}}_2)$$

$$\Rightarrow \underline{0} = \underbrace{a\,\hat{\underline{e}}_1}_{\text{tang.}} - \frac{\omega^2 r^2}{(R+r)}\hat{\underline{e}}_2 - \omega^2 r\,\hat{\underline{e}}_2 - \underbrace{\dot{\omega} r\,\hat{\underline{e}}_1}_{\text{tang.}} \qquad \left[\because v = \omega r\right]$$

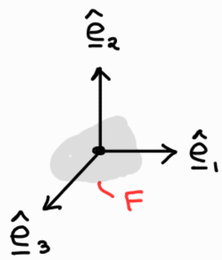
$$\Rightarrow \underbrace{(a - \dot{\omega} r)}_{\text{tangential component only}} = 0 \Rightarrow \dot{\omega} = \frac{a}{r}$$

$$\boxed{\omega = \frac{v}{r}} \qquad \boxed{\dot{\omega} = \frac{a}{r}}$$

Valid $\forall\, t$ as long as RB 'm' rolls without slip on RB 'F'

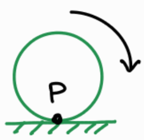
Finally the acceleration of point P_1 is

$$\underbrace{\underline{a}_{P_1|F}}_{\text{has components only in the normal direction } (\hat{\underline{e}}_2)} = -\left[\frac{\omega^2 r^2}{(R+r)} + \omega^2 r\right]\hat{\underline{e}}_2 = \frac{\omega^2}{\left(\frac{1}{R} + \frac{1}{r}\right)}\hat{\underline{e}}_2$$



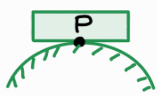
Special cases:

① $R \to \infty$



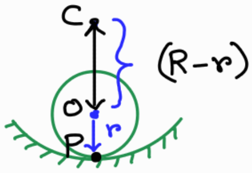
$$\underline{a}_{P|F} = -\omega^2 r\,\hat{\underline{e}}_2$$

② $r \to \infty$



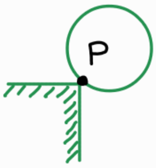
$$\underline{a}_{P|F} = \omega^2 R\,\hat{\underline{e}}_2$$

③ $R \rightarrow -R$



$$\underline{a}_{P|F} = \frac{\omega^2}{\left(-\frac{1}{R}+\frac{1}{r}\right)} \, \underline{\hat{e}}_2$$

④ $R \rightarrow 0$



Corner

Cusp

$$\underline{a}_{P|F} = \underline{0}$$