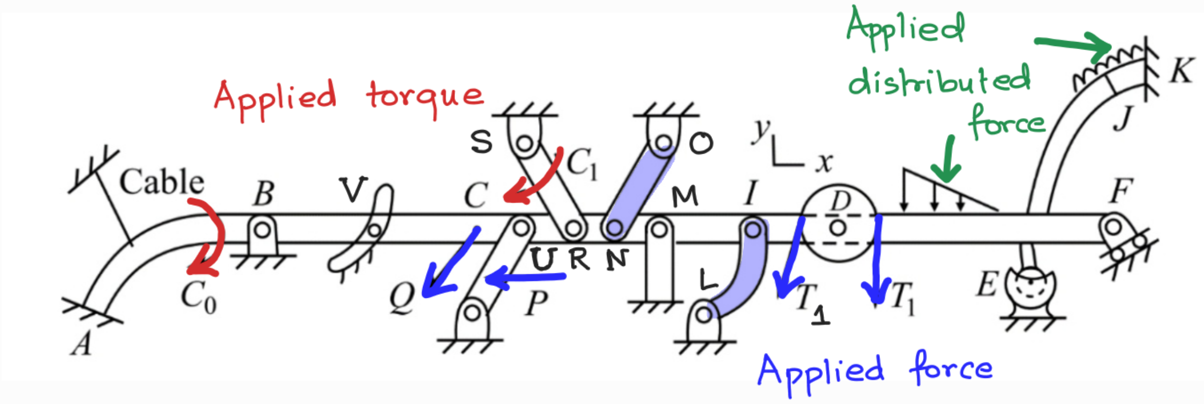
# Tutorial 5

## Part A

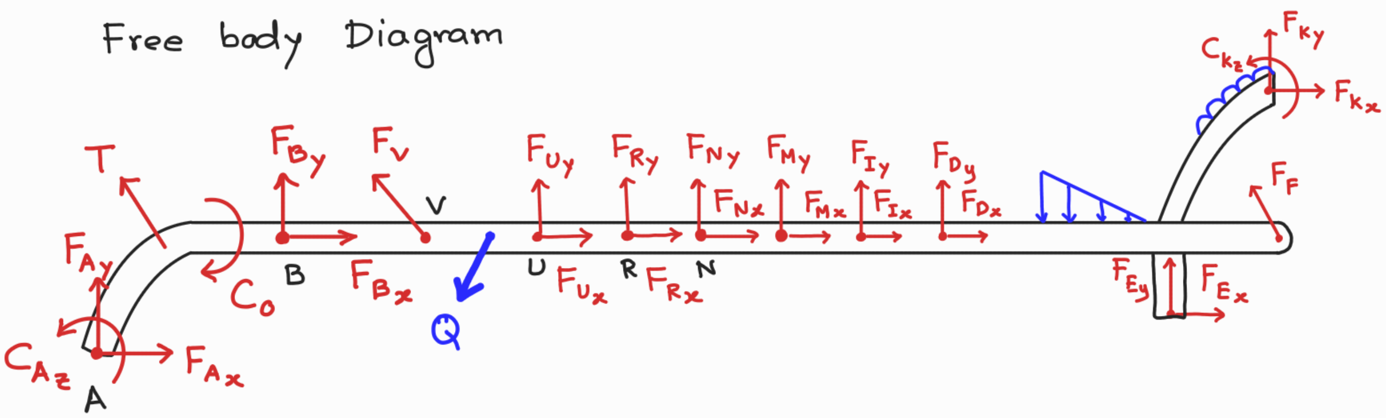
1> Draw the FBD and show the reaction force and torque components exerted by supports on member ABCDEF




Given: i> Coplanar loading ii> All members are light (massless)

iii> All contacts are smooth (i.e. frictionless)

Soln : Free body Diagram




A is a fixed joint $\Rightarrow F_{Ax}, F_{Ay}, C_{Az}$

T (tension) is along the cable

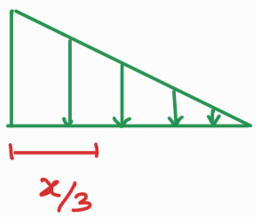
$C_0$ is the moment due to an applied couple

B is a pin joint

$F_v$ at point V is a slot connection

U, R, N, M, I, D are pin joints

The distributed force can be replaced by a single concentrated force at $x/3$ from left side



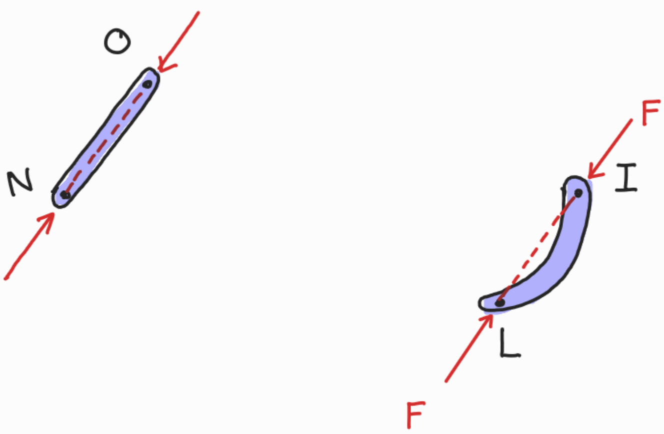

E is a 2D ball and socket joint, similar in functionality as a pin joint

F is a roller support. Reaction force is perpendicular to its direction of free translational movement

K is a fixed support: two forces and one moment

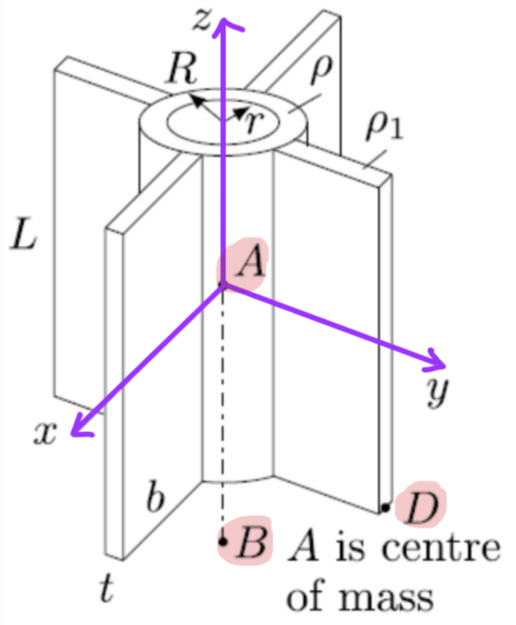
Note: The rod ABCDEFK is constrained and can't move it: it is stationary

Later, after studying statics, you will know that members 'NO' and 'IL' are "two-force" members



2) Find the inertia matrix $[\underline{\underline{I}}^A]$ of the composite RB at A relative to csys shown.

Density of annulus is $\rho$ and that of plates is $\rho_1$.




Find principal axes of the composite RB at point A, B, and D.

Solu:

Inertia matrix at COM A

Visual inspection → Is this a body of revolution? NO

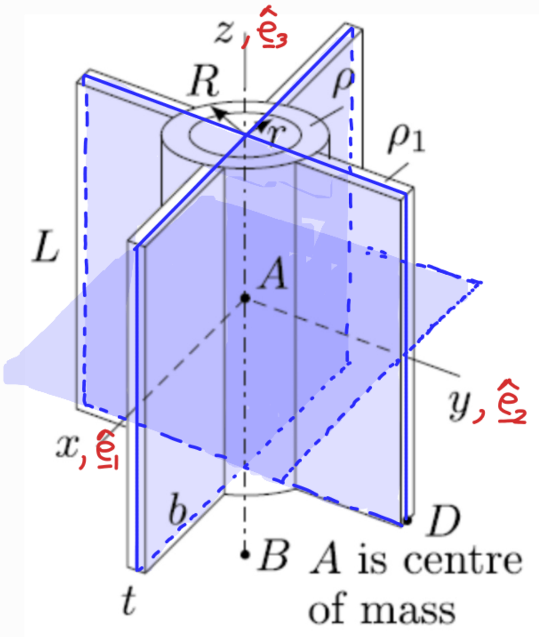
→ Does the body have planes of symmetry? YES

Three planes of symmetry through A

① xz-plane ⇒ y-axis as p-axis

② yz-plane ⇒ x-axis as p-axis

③ xy-plane ⇒ z-axis as p-axis



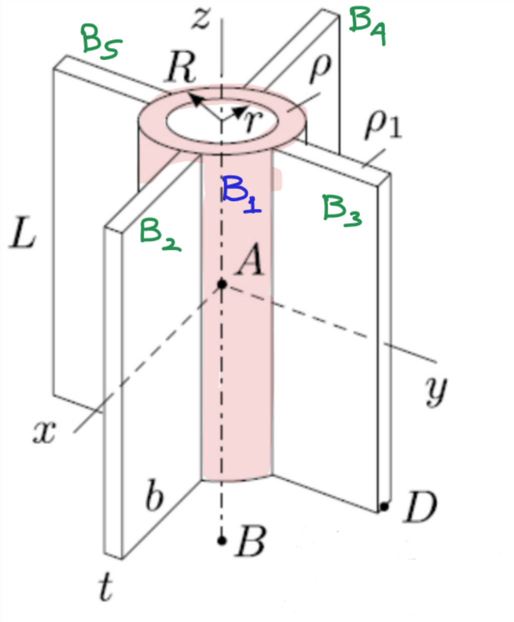

$$\therefore\ [\underline{\underline{I}}^A]_{\begin{pmatrix}\hat{e}_1\\ \hat{e}_2\\ \hat{e}_3\end{pmatrix}} = \begin{bmatrix} I^A_{xx} & 0 & 0 \\ 0 & I^A_{yy} & 0 \\ 0 & 0 & I^A_{zz} \end{bmatrix}$$

Further by symmetry, $I_{xx}^{A} = I_{yy}^{A}$

So we need to find only two scalars $I_{xx}^{A}$ and $I_{zz}^{A}$ to completely define $[\underline{\underline{I}}^{A}]$

$B_1 \equiv$ Hollow Cylinder

$B_{2-5} \equiv$ Rectangular plates (not thin)




## Calculation of $I_{xx}^{A}$ $(= I_{yy}^{A})$

$$I_{xx}^{A} = I_{xx}^{A}(B_1) + \sum_{i=2}^{5} I_{xx}^{A}(B_i)$$

Hollow circular cylinder (Lec 12)

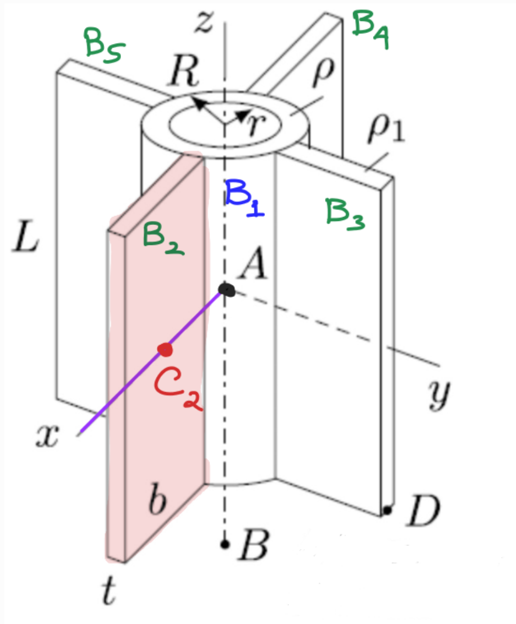
Mass, $m = \rho L \pi (R^2 - r^2)$

$$I_{xx}^{A}(B_1) = \frac{m}{4}(r^2 + R^2) + \frac{mL^2}{12}$$

Rectangular plates (Lec 12)

Mass, $m_1 = \rho_1 L t b$

// axes thm



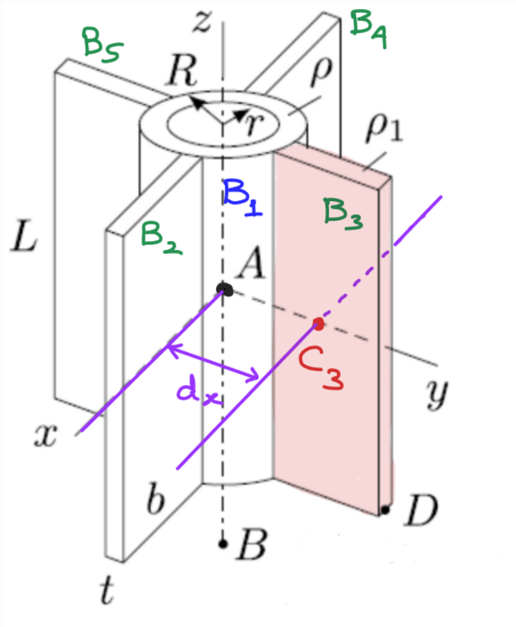

$$I_{xx}^{A}(B_2) = I_{xx}^{C_2}(B_2) + \cancelto{0}{m_1(\text{posi. of } C_2 \text{ from } A \perp \text{ to } x\text{-axis})^2}$$

$$= \frac{m_1(L^2 + t^2)}{12}$$

Note the $I^A_{xx}(B_4) = I^A_{xx}(B_2) = \dfrac{m_1(L^2+t^2)}{12}$ ✓

// axes thm

$$I^A_{xx}(B_3) = I^{C_3}_{xx}(B_3) + m_1(\text{posi. of } C_3 \text{ from A } \perp \text{ to } x\text{-axis})^2$$




$$= \frac{m_1(L^2+b^2)}{12} + m_1\left(R+\frac{b}{2}\right)^2$$

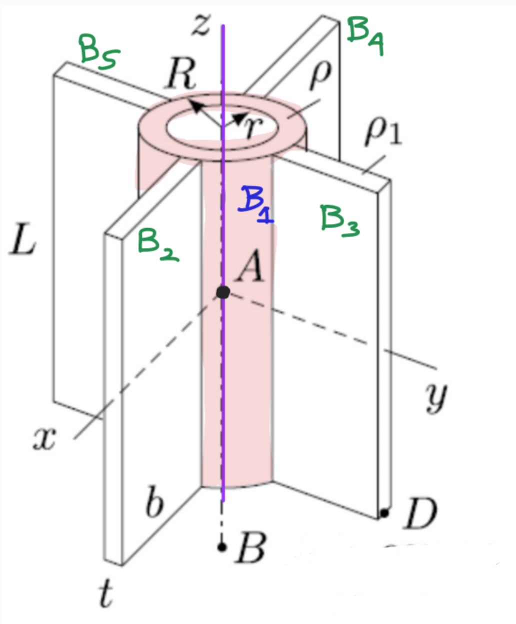
Similarly, the $I^A_{xx}(B_5) = I^A_{xx}(B_3) = \dfrac{m_1(L^2+t^2)}{12} + m_1\left(R+\dfrac{b}{2}\right)^2$

$$I^A_{xx} = I^A_{xx}(B_1) + \sum_{i=2}^{5} I^A_{xx}(B_i)$$

$$= \underbrace{\frac{m}{4}(r^2+R^2) + \frac{mL^2}{12}} + \underbrace{2\left(\frac{m_1(L^2+t^2)}{12}\right)} + \underbrace{2\left(\frac{m_1(L^2+t^2)}{12}\right)} + \underbrace{2\left(m_1\left(R+\frac{b}{2}\right)^2\right)}$$

## Calculation of $I^A_{zz}$

$$I^A_{zz} = I^A_{zz}(B_1) + \sum_{i=2}^{5} I^A_{zz}(B_i)$$

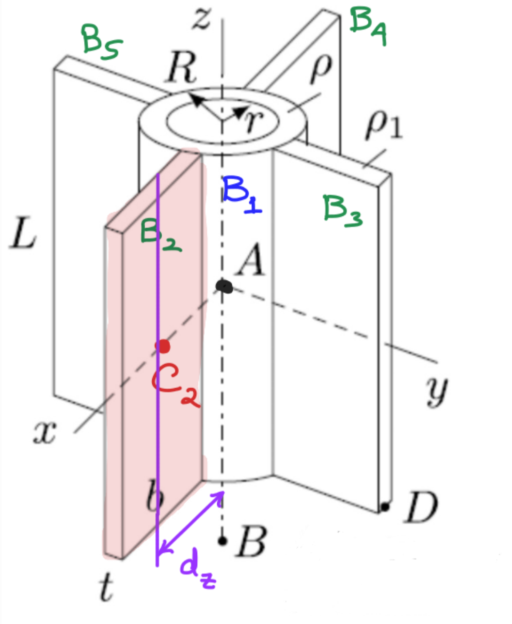



Hollow circular cylinder (Lec 12)

Mass, $m = \rho L \pi(R^2 - r^2)$, $A \equiv$ COM

$$I^A_{zz}(B_1) = \frac{m}{2}(r^2+R^2)$$

Rectangular plates (Lec 12)




Mass, $m_1 = \rho_1 L t b$

// axes thm

$$I_{zz}^{A}(B_2) = I_{zz}^{C_2}(B_2) + m_1(\text{posi. of } C_2 \text{ from } A \perp \text{ to } z\text{-axis})^2$$

$$= \frac{m_1(b^2+t^2)}{12} + m_1\left(R+\frac{b}{2}\right)^2$$

Recognize that $I_{zz}^{A}$ of all four rectangular plates are SAME

$$\therefore \; I_{zz}^{A}(B_2) = I_{zz}^{A}(B_3) = I_{zz}^{A}(B_4) = I_{zz}^{A}(B_5)$$

$$I_{zz}^{A} = I_{zz}^{A}(B_1) + \sum_{i=2}^{5} I_{zz}^{A}(B_i)$$

$$= \frac{m}{2}(r^2+R^2) + 4\left[\frac{m_1(b^2+t^2)}{12} + m_1\left(R+\frac{b}{2}\right)^2\right]$$

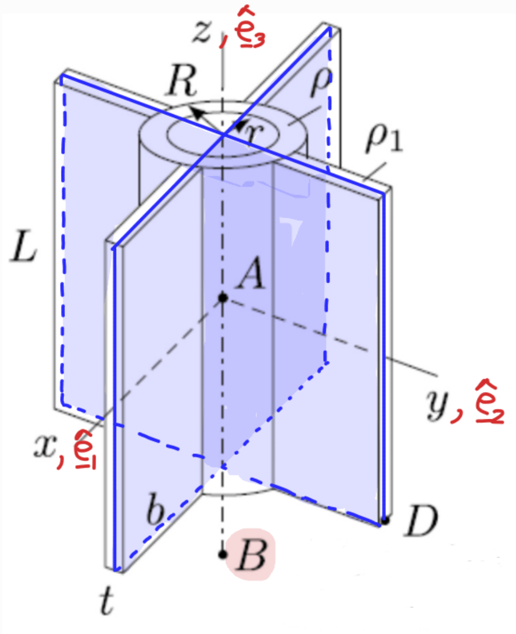
$$[\underline{\underline{I}}^{A}]_{(\hat{e}_1,\hat{e}_2,\hat{e}_3)} = \begin{bmatrix} I_{xx}^{A} & & 0 \\ & I_{yy}^{A}\ (= I_{xx}^{A}) & \\ 0 & & I_{zz}^{A} \end{bmatrix} \leftarrow \text{diagonal}$$

Since $[\underline{I}^{A}]$ matrix is diagonal wrt $\hat{e}_1$-$\hat{e}_2$-$\hat{e}_3$ csys, they are also the principal axes of inertia

# Identifying p-axes at pt B (by visual inspection)

The $e_3$-axis ($z$-axis) passes through point B.

Is the z-axis an axis of revolution for generating the composite body? NO




Are there identifiable planes of symmetry passing through B? YES

→ xz-plane → y-axis as p-axis of RB at B

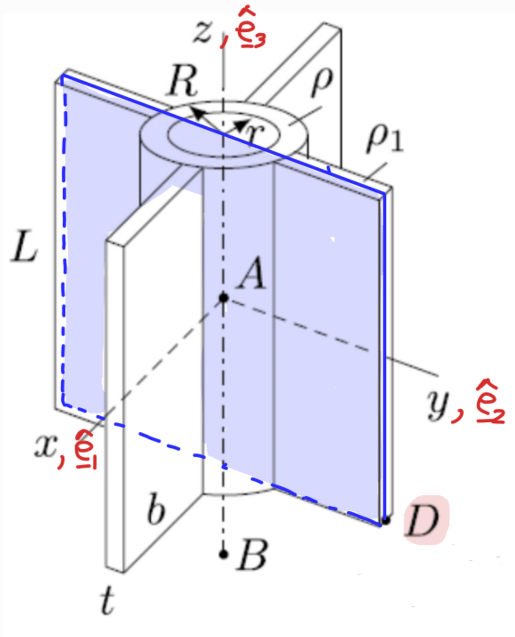
→ yz-plane → x-axis as p-axis of RB at B

Since there are two orthogonal planes of symmetry, therefore, the intersecting axis of the planes of symmetry is also a p-axis ⇒ z-axis is also a p-axis (Lec 12)

$$[\underline{\underline{I}}^B]_{\begin{pmatrix}\hat{e}_1\\ \hat{e}_2\\ \hat{e}_3\end{pmatrix}} = \begin{bmatrix} I_{xx}^B & & 0 \\ & I_{yy}^B \ (= I_{xx}^B) & \\ 0 & & I_{zz}^B \end{bmatrix}$$

# Identifying p-axes at pt D (by visual inspection)

Is any axis passing through D an axis of revolution for generating the composite body ? NO




Are there identifiable planes of symmetry passing through D ? YES

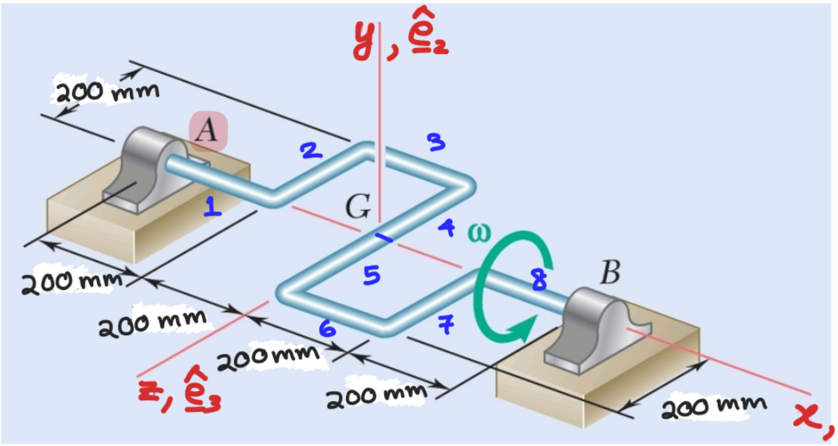
↳ yz-plane → x-axis as p-axis for RB at pt D

$$\left[\underline{\underline{I}}^D\right]_{\begin{pmatrix}\hat{e}_1\\ \hat{e}_2\\ \hat{e}_3\end{pmatrix}} = \begin{bmatrix} I^D_{xx} & 0 & 0 \\ 0 & I^D_{yy} & I^D_{yz} \\ 0 & I^D_{zy} & I^D_{zz} \end{bmatrix}$$

3) Find the angular momentum of the shaft about point A and w.r.t. ground with $\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3$ as csys.

Mass of the shaft = 8 kg

Angular velocity = 12 rad/s




Mass of each shaft segment $= \dfrac{8 \text{ kg}}{8} = 1$ kg

Solution :

$\underline{H}_{A|F}$ : Angular momentum of the composite shaft having 8 simple rods

$\underline{\underline{I}}^A$ : Inertia tensor of composite shaft

$\underline{\omega}_{m|F}$ : Angular velocity of composite shaft

Given : $\underline{\omega}_{m|F} = \omega_1 \hat{\underline{e}}_1$ (other components are zero)

$$\Rightarrow [\underline{\omega}_{m|F}]_{(\hat{\underline{e}}_1, \hat{\underline{e}}_2, \hat{\underline{e}}_3)} = \begin{bmatrix} \omega_1 \\ 0 \\ 0 \end{bmatrix}$$

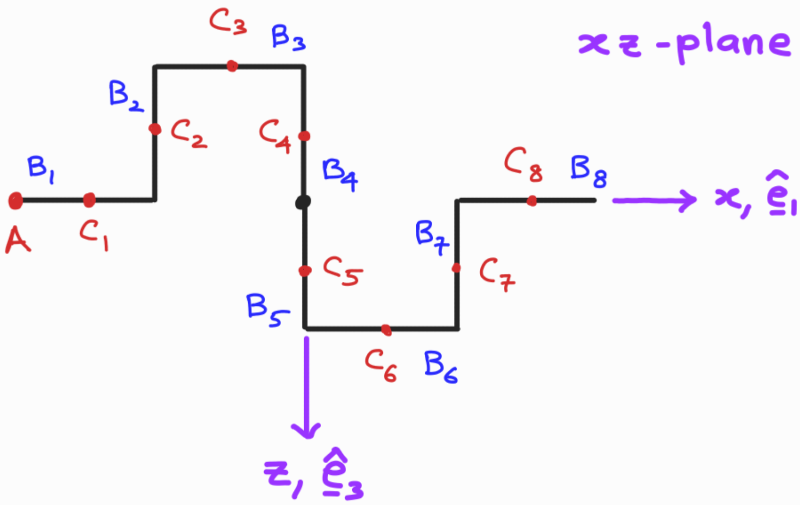
$$[\underline{H}_{A|F}] = \begin{bmatrix} I_{11}^A & I_{12}^A & I_{13}^A \\ I_{21}^A & I_{22}^A & I_{23}^A \\ I_{31}^A & I_{32}^A & I_{33}^A \end{bmatrix} \begin{bmatrix} \omega_1 \\ 0 \\ 0 \end{bmatrix}$$

only this col. matters since only $\omega_1 \neq 0$

We need to find $I_{11}^A$, $I_{21}^A$, and $I_{31}^A$

## Calculation of $I_{21}^A$

Since the shaft lies in the $xz$-plane, $I_{21}^A = 0$ (why?)

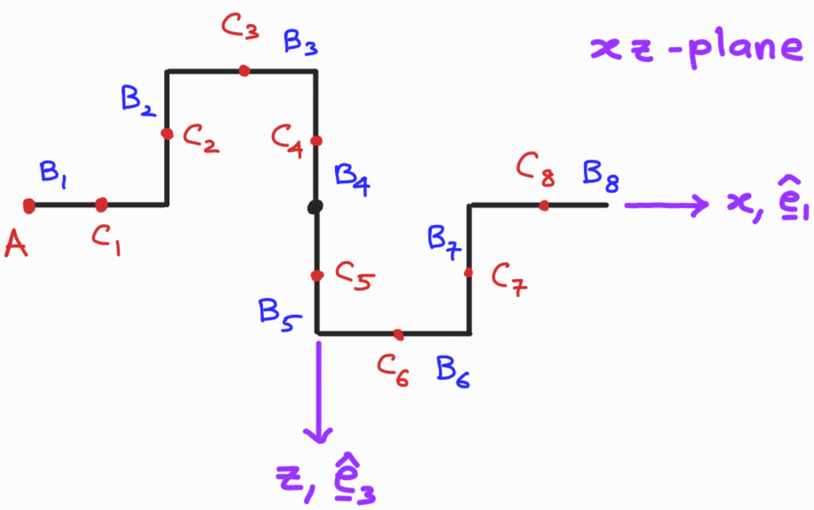



$y \approx 0$

$$I_{21}^A = \int x \cancelto{0}{y}\, dm = 0$$

## Calculation of $I_{31}^A$

We will first find $I_{31}^C$ ($I_{31}$ at COM of each rod) and then use // axes thm to transfer to pt A.




Mass of each rod, $m_i = 1$ kg

Length of each rod, $l_i = 0.2$ m

Composite shaft $B = \bigcup_{i=1}^{8} B_i$

$$\Rightarrow \quad I_{31}^{A} = \sum_{i=1}^{8} I_{31}^{A}(B_i)$$

We will use the following relation:

$$I_{31}^{A}(B_i) = I_{31}^{C}(B_i) - m_i \,(x_{c_1})_3\,(x_{c_1})_1$$

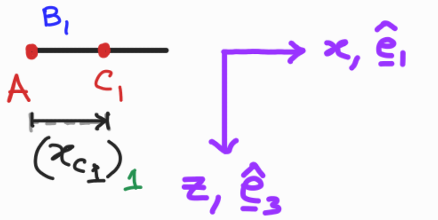
pos. from pt A to pt $C_i$ $\perp$ to $\hat{e}_1$

pos. from A to $C_i$ $\perp$ to $\hat{e}_3$

Note that $I_{31}^{C}(B_i)$ is zero for all rods

$$I_{31}^{A}(B_1) = -\, m_1 \,\overset{0}{\cancel{(x_{c_1})_3}}\,(x_{c_1})_1$$

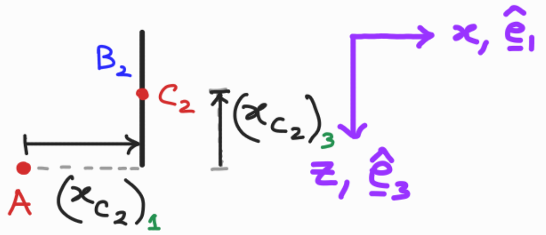
$$= 0$$




$$I_{31}^{A}(B_2) = -\, m_2 \,(x_{c_2})_3\,(x_{c_2})_1$$

$$= -\, m \left(-\frac{a}{2}\right)(a)$$

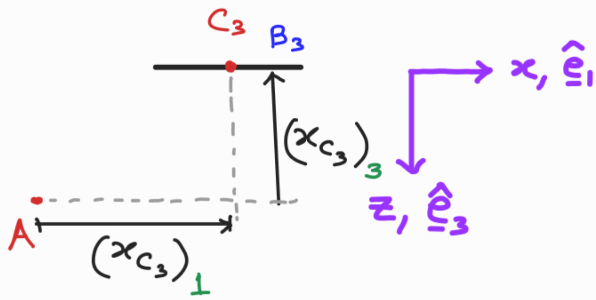
$$= m \,\frac{a^2}{2}$$



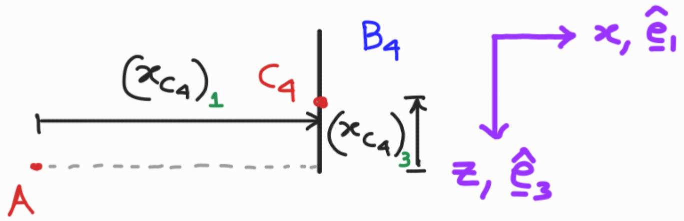

$$I_{31}^{A}(B_3) = -\, m_3 \,(x_{c_3})_3\,(x_{c_3})_1$$

$$= -\, m \,(-a)\left(a + \frac{a}{2}\right)$$

$$= m \,\frac{3a^2}{2}$$



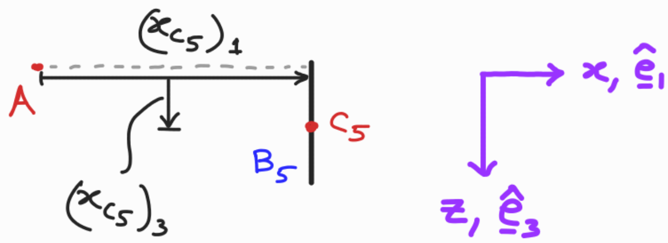
$$I_{31}^{A}(B_4) = -m_4 \,(x_{C_4})_3\,(x_{C_4})_1$$




$$= -m\left(\frac{-a}{2}\right)(2a)$$

$$= m a^2$$

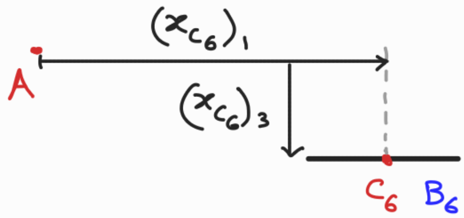
$$I_{31}(B_5) = -m_5\,(x_{C_5})_3\,(x_{C_5})_1$$




$$= -m\left(\frac{a}{2}\right)(2a)$$

$$= -m a^2$$

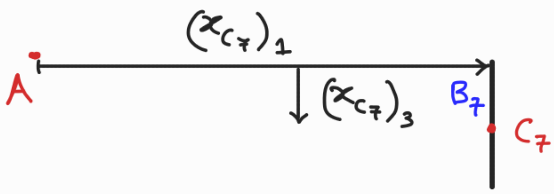
$$I_{31}(B_6) = -m_6\,(x_{C_6})_3\,(x_{C_6})_1$$




$$= -m\,(a)\left(2a + \frac{a}{2}\right)$$

$$= -m\left(\frac{5a^2}{2}\right)$$

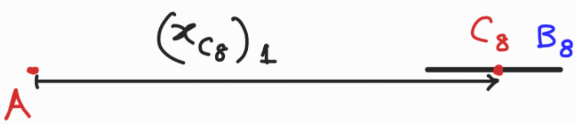
$$I_{31}^{A}(B_7) = -m_7\,(x_{C_7})_3\,(x_{C_7})_1$$




$$= -m\left(\frac{a}{2}\right)(3a)$$

$$= -m\left(\frac{3}{2}a^2\right)$$

$$I_{31}^{A}(B_8) = -m_8\,\cancel{(x_{C_8})_3}^{\,0}\,(x_{C_8})_1$$




$$= 0$$

$$I_{31}^{A}(B) = m\left[0 + \frac{a^2}{2} + \cancel{\frac{3a^2}{2}} + \cancel{a^2} - \cancel{a^2} - \frac{5a^2}{2} - \cancel{\frac{3a^2}{2}}\right]$$

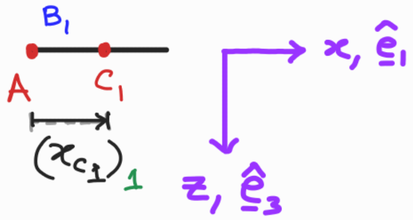
$$= -2ma^2$$

## Calculation of $I_{11}^{A}$

$$I_{11}^{A} = \sum_{i=1}^{8} I_{11}^{A}(B_i)$$

$$I_{11}^{A}(B_1) = \cancelto{0}{I_{11}^{C_1}(B_1)} + m\left[\cancelto{0}{(x_{c_1})_2^2} + (x_{c_1})_3^2\right]$$

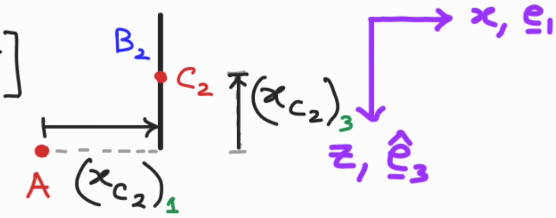
$$= 0$$




$$I_{11}^{A}(B_2) = I_{11}^{C_1}(B_2) + m\left[(x_{c_2})_2^2 + (x_{c_2})_3^2\right]$$

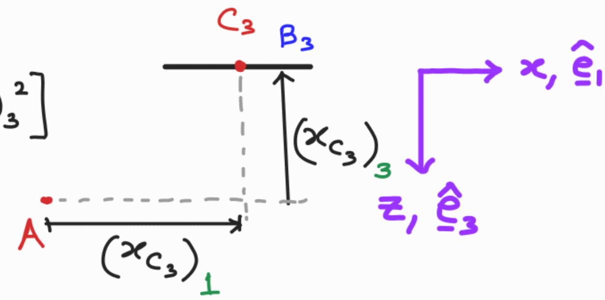
$$= \frac{ma^2}{12} + m\left[0 + \left(\frac{a}{2}\right)^2\right]$$

$$= \frac{ma^2}{3}$$




$$I_{11}^{A}(B_3) = \cancelto{0}{I_{11}^{C_3}(B_3)} + m\left[\cancelto{0}{(x_{c_3})_2^2} + (x_{c_3})_3^2\right]$$

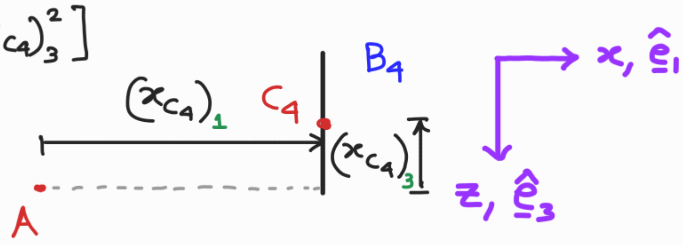
$$= ma^2$$




$$I_{11}^{A}(B_4) = I_{11}^{C_4}(B_4) + m\left[(x_{c_4})_2^2 + (x_{c_4})_3^2\right]$$

$$= \frac{ma^2}{12} + m\left(\frac{a}{2}\right)^2$$

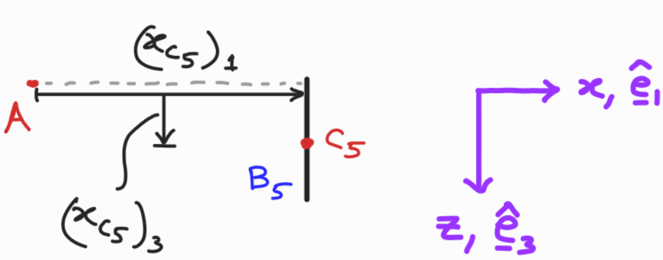
$$= \frac{ma^2}{3}$$




$$I_{11}^{A}(B_5) = I_{11}^{C_5}(B_5) + m\left[\cancelto{0}{(x_{c_5})_2^2} + (x_{c_5})_3^2\right]$$

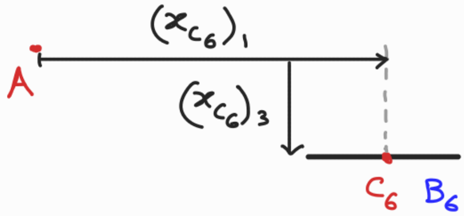
$$= \frac{ma^2}{12} + m\left(\frac{a}{2}\right)^2$$

$$= ma^2/3$$



$$I_{11}^{A}(B_6) = \cancelto{0}{I_{11}^{C_6}(B_6)} + m\left[\cancelto{0}{(x_{C_6})_2^2} + (x_{C_6})_3^2\right]$$

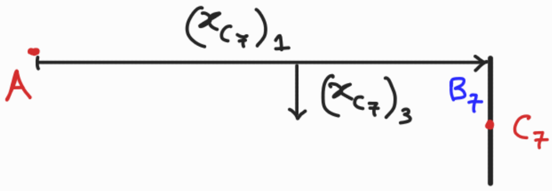
$$= m a^2$$




$$I_{11}^{A}(B_7) = I_{11}^{C_7}(B_7) + m\left[\cancelto{0}{(x_{C_7})_2^2} + (x_{C_7})_3^2\right]$$

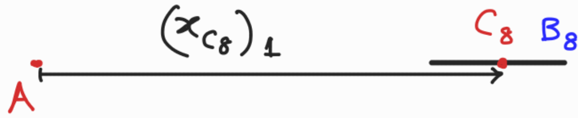
$$= \frac{ma^2}{12} + m\left(\frac{a}{2}\right)^2$$

$$= \frac{ma^2}{3}$$




$$I_{11}^{A}(B_8) = \cancelto{0}{I_{11}^{C_8}(B_8)} + m\left[\cancelto{0}{(x_{C_8})_2^2} + \cancelto{0}{(x_{C_8})_3^2}\right]$$

$$= 0$$




$$I_{11}^{A}(B) = \sum_{i=1}^{8} I_{11}^{A}(B_i)$$

$$= 0 + \frac{ma^2}{3} + ma^2 + \frac{ma^2}{3} + \frac{ma^2}{3} + ma^2 + \frac{ma^2}{3} + 0$$

$$= 2ma^2 + \frac{4}{3}ma^2$$

$$= \frac{10}{3}ma^2$$

Therefore, $$\left[\underline{H}_{A/F}\right]_{\begin{pmatrix}\underline{\hat{e}_1}\\ \underline{\hat{e}_2}\\ \underline{\hat{e}_3}\end{pmatrix}} = ma^2\omega \begin{bmatrix} 10/3 \\ 0 \\ -2 \end{bmatrix}$$