

Inertia matrix for RBs with symmetry planes in mass distribution

Recall the moment of inertia integrals about an axis n passing through point A

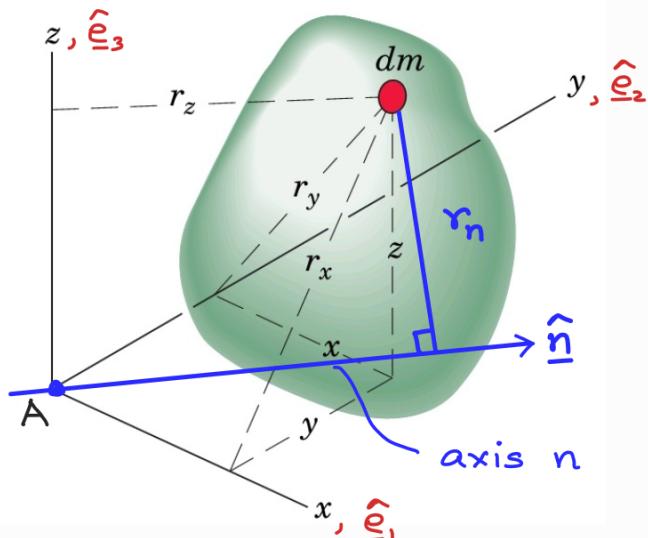
$$I_{nn}^A = \int r_n^2 dm$$

Set

$$\hat{n} = \hat{e}_1 \rightarrow I_{11}^A = \int r_x^2 dm$$

$$\hat{n} = \hat{e}_2 \rightarrow I_{22}^A = \int r_y^2 dm$$

$$\hat{n} = \hat{e}_3 \rightarrow I_{33}^A = \int r_z^2 dm$$



These integrals are always positive, whereas the products of inertia $I_{12}^A = \int xy dm$, $I_{13}^A = \int xz dm$, and $I_{23}^A = \int yz dm$ may be positive, negative, or zero.

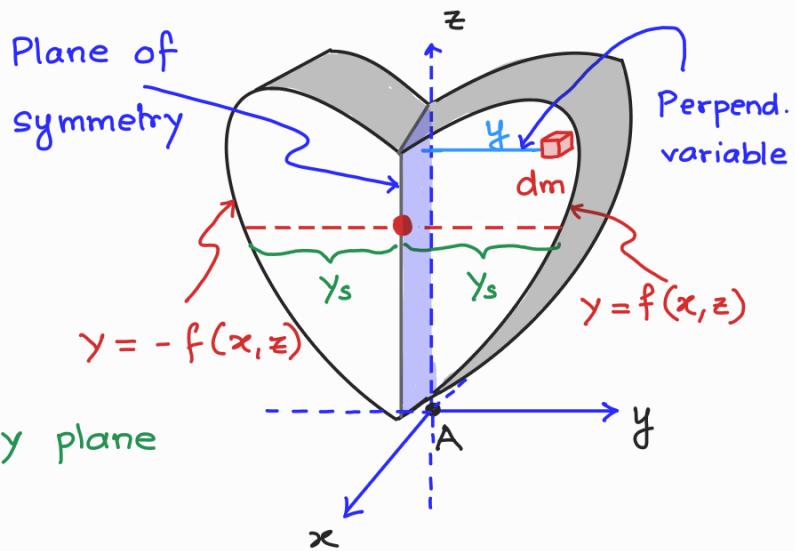
For a homogeneous body having a plane of symmetry, if one of the coordinate planes contains the body plane symmetry, the products of inertia involving the coordinate variable perpendicular to this plane will vanish.

Ex: A homogeneous body

for which the xz -plane

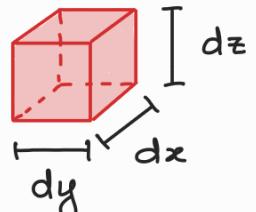
is a body symmetry plane.

point 'A' lies on the symmetry plane



$$I_{12}^A = - \int xy \, dm$$

$dm = \rho \, dx \, dz \, dy$



$$I_{12}^A = - \int xy \, \rho \, dx \, dz \, dy$$

$$= - \rho \iint_{\substack{z \\ x}} \left[\begin{array}{c} \int_{-f(x,z)}^{f(x,z)} y \, dy \\ 0 \end{array} \right] x \, dx \, dz = 0$$

Similarly, $I_{23}^A = \int yz \, dm$

$$= - \int yz \, \rho \, dx \, dz \, dy$$

$$= - \rho \iint_{\substack{x \\ z}} \left[\begin{array}{c} \int_{-f(x,z)}^{f(x,z)} y \, dy \\ 0 \end{array} \right] z \, dz \, dx = 0$$

With a xz -plane of symmetry, y is the coordinate

variable \perp to plane of symmetry $\Rightarrow I_{12}^A = 0$ and $I_{23}^A = 0$

Thus, the inertia matrix at point A in the chosen coordinate system :

$$[\underline{\underline{I}}^A] = \begin{bmatrix} I_{11}^A & 0 & I_{13}^A \\ 0 & I_{22}^A & 0 \\ I_{31}^A & 0 & I_{33}^A \end{bmatrix}$$

Inertia tensor for some special homogeneous RBs

We focus on three specific homogeneous (meaning uniform mass distribution) bodies →

- 1) Rectangular body
- 2) Circular body
- 3) Spherical body

1) Inertia tensor for a rectangular (cuboidal) RB

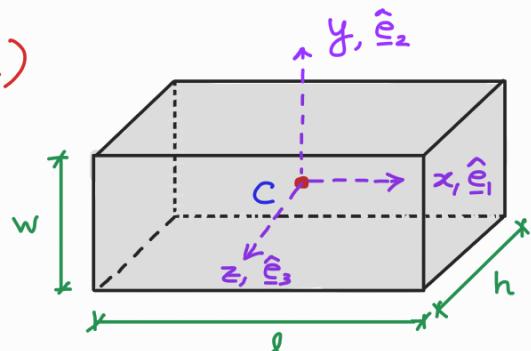
$C \rightarrow$ Center of mass (also centroid)

$\rho \rightarrow$ density (uniform)

Find inertia matrix of RB

at C w.r.t. csys $\hat{\underline{\underline{e}}}_1 - \hat{\underline{\underline{e}}}_2 - \hat{\underline{\underline{e}}}_3$

(parallel to the edges of cuboid)



Notice the planes of symmetry at pt C !!

$$[\underline{\underline{I}}^C] \begin{pmatrix} \hat{\underline{\underline{e}}}_1 \\ \hat{\underline{\underline{e}}}_2 \\ \hat{\underline{\underline{e}}}_3 \end{pmatrix} = ?$$

$$\begin{aligned}
 I_{11}^c &= \int (y^2 + z^2) \frac{dm}{PdV} = P \int_{-h/2}^{h/2} \int_{-w/2}^{w/2} \int_{-l/2}^{l/2} (y^2 + z^2) dx dy dz \\
 &= P l \int_{-h/2}^{h/2} \int_{-w/2}^{w/2} (y^2 + z^2) dy dz \\
 \text{mass of block } 'm' &\rightarrow = P \frac{lwh}{12} (w^2 + h^2) \\
 &= \frac{m(w^2 + h^2)}{12}
 \end{aligned}$$

Note the correspondence $(x, y, z) \sim (l, w, h)$

We can permute the symbols and get I_{22}^c , I_{33}^c

$$I_{22}^c = \frac{m(l^2 + h^2)}{12}, \quad I_{33}^c = \frac{m(l^2 + w^2)}{12}$$

Note: All products of inertia terms are ZERO

$I_{12}^c = I_{13}^c = I_{23}^c = 0$ (due to plane symmetry of cuboid about $x-y$, $y-z$, $x-z$ planes through c)

Thus,

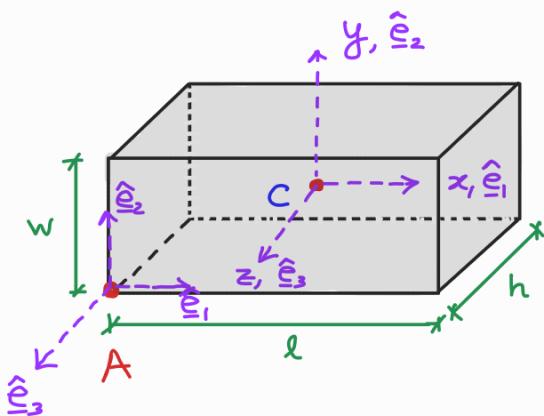
$$\begin{bmatrix} I^c \\ \end{bmatrix} = \begin{bmatrix} \frac{m(w^2 + h^2)}{12} & 0 & 0 \\ 0 & \frac{m(l^2 + h^2)}{12} & 0 \\ 0 & 0 & \frac{m(l^2 + w^2)}{12} \end{bmatrix}$$

How do you find $[I^A]$ about point A (at corner)?

The csys bases at COM C

and at corner A are //

\therefore Use parallel axes theorem



$$I_{11}^A = I_{11}^C + m \underbrace{(x_{c_2}^2 + x_{c_3}^2)}_{d_1^2}$$

$$\Rightarrow I_{12}^A = I_{12}^C - m x_{c_1} x_{c_2}$$

Similarly,

$$\text{and, } I_{23}^A = I_{23}^C - m x_{c_2} x_{c_3}$$

$$I_{22}^A = I_{22}^C + m \underbrace{(x_{c_1}^2 + x_{c_3}^2)}_{d_2^2}$$

$$\text{and, } I_{13}^A = I_{13}^C - m x_{c_1} x_{c_3}$$

$$I_{33}^A = I_{33}^C + m \underbrace{(x_{c_1}^2 + x_{c_2}^2)}_{d_3^2}$$

from Lec 11

$$I_{11}^A = \frac{m(\omega^2 + h^2)}{12} + m \left[\left(\frac{w}{2} \right)^2 + \left(\frac{h}{2} \right)^2 \right]$$

$$I_{22}^A = \frac{m(l^2 + h^2)}{12} + m \left[\left(\frac{l}{2} \right)^2 + \left(\frac{h}{2} \right)^2 \right]$$

$$I_{33}^A = \frac{m(l^2 + w^2)}{12} + m \left[\left(\frac{l}{2} \right)^2 + \left(\frac{w}{2} \right)^2 \right]$$

$$I_{12}^A = 0 - m \left(\frac{l}{2} \right) \left(\frac{w}{2} \right)$$

$$\hat{\underline{e}}_1 \rightarrow \textcircled{1} \rightarrow l$$

$$\hat{\underline{e}}_2 \rightarrow \textcircled{2} \rightarrow w$$

$$\hat{\underline{e}}_3 \rightarrow \textcircled{3} \rightarrow h$$

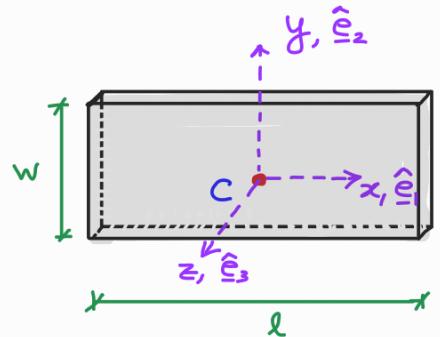
$$I_{23}^A = 0 - m \left(\frac{w}{2} \right) \left(\frac{h}{2} \right)$$

$$I_{13}^A = 0 - m \left(\frac{l}{2} \right) \left(\frac{h}{2} \right)$$

Inertia tensor for thin rectangular plate

$$h \approx 0$$

$$[\underline{\underline{I}}^c] = \begin{bmatrix} \frac{m(\omega^2 + h^2)}{12} & 0 & 0 \\ 0 & \frac{m(l^2 + h^2)}{12} & 0 \\ 0 & 0 & \frac{m(l^2 + w^2)}{12} \end{bmatrix}$$

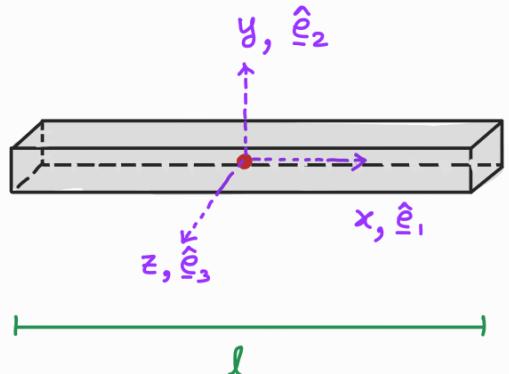


$$[\underline{\underline{I}}^c] = \begin{bmatrix} \frac{m\omega^2}{12} & 0 & 0 \\ 0 & \frac{m\ell^2}{12} & 0 \\ 0 & 0 & \frac{m(\ell^2 + w^2)}{12} \end{bmatrix}$$

Inertia matrix of a thin rod

$$h \approx 0, w \approx 0$$

$$[\underline{\underline{I}}^c] = \begin{bmatrix} \frac{m\omega^2}{12} & 0 & 0 \\ 0 & \frac{m\ell^2}{12} & 0 \\ 0 & 0 & \frac{m(\ell^2 + w^2)}{12} \end{bmatrix}$$



$$[\underline{\underline{I}}^c] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{m\ell^2}{12} & 0 \\ 0 & 0 & \frac{m\ell^2}{12} \end{bmatrix}$$

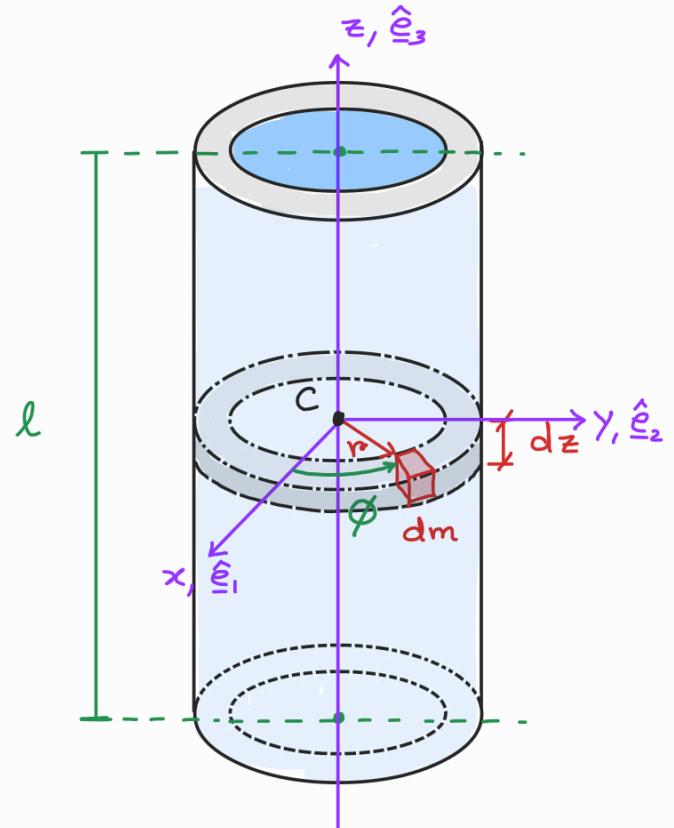
2) Inertia tensor for a circular cylindrical RB

Inner radius : r_i

Outer radius : r_o

Length : l

Here, the z -axis is the central axis about which this body can be generated by revolution.



Symmetry planes :

- $\rightarrow x-y$ plane
- $\rightarrow y-z$ plane
- $\rightarrow x-z$ plane

$$\Rightarrow \left. \begin{matrix} I_{12} \\ I_{23} \\ I_{13} \end{matrix} \right\} = 0$$

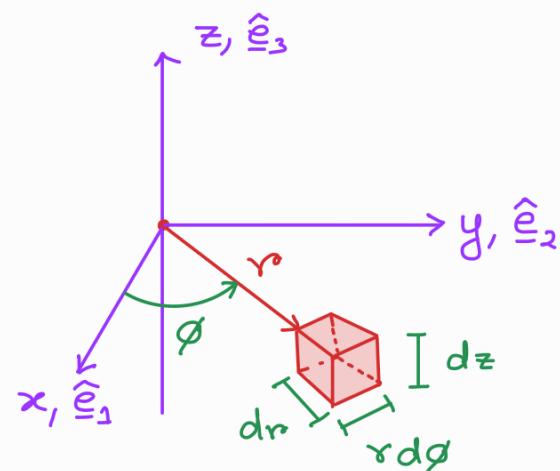
Any infinitesimally small mass 'dm' can be written as :

$$dm = \rho dr d\phi dz$$

$$I_{33}^C = \int r^2 dm$$

$$= \int_{-l/2}^{l/2} \int_0^{2\pi} \int_{r_i}^{r_o} r^2 \rho dr d\phi dz$$

$$= \rho 2\pi l \int_{r_i}^{r_o} r^3 dr = \rho 2\pi l \left(\frac{r_o^4 - r_i^4}{4} \right) = \frac{\pi}{2} \rho l (r_o^4 - r_i^4)$$



With cross-sectional area $A = \pi(r_o^2 - r_i^2)$ and $m = PA$

$$I_{33}^c = \frac{m}{2} (r_o^2 + r_i^2)$$

Note that the general form of I_{11}^c , I_{22}^c , I_{33}^c are:

$$I_{11}^c = \int (y^2 + z^2) dm$$

$$I_{22}^c = \int (x^2 + z^2) dm$$

$$I_{33}^c = \int (x^2 + y^2) dm$$

$$\begin{aligned} \therefore I_{11}^c + I_{22}^c &= \int (x^2 + y^2) dm + 2 \int z^2 dm \\ &= I_{33}^c + 2 \int z^2 dm \quad \text{--- (EQ)} \end{aligned}$$

Clearly, for the cylinder, since z -axis is the central axis about which the RB is symmetric (also called AXISYMMETRY)

$$I_{11}^c = I_{22}^c$$

$$\int z^2 dm = \int_{-\frac{l}{2}}^{\frac{l}{2}} \int_0^{2\pi} \int_{r_i}^{r_o} z^2 P dr rd\phi dz = P / 2\pi \left(\frac{r_o^2 - r_i^2}{2} \right) \int_{-\frac{l}{2}}^{\frac{l}{2}} z^2 dz$$

$$= P \underbrace{\pi(r_o^2 - r_i^2)}_{C/S \text{ area } A} \left[\frac{z^3}{3} \right]_{-\frac{l}{2}}^{\frac{l}{2}}$$

$$= \rho A \frac{l^3}{12} = \underbrace{(\rho A l)}_m \frac{l^2}{12} = \frac{ml^2}{12}$$

Using (EQ), we get:

$$\underbrace{I_{11}^c + I_{22}^c}_{2I_{11}^c} = I_{33}^c + 2\left(\frac{ml^2}{12}\right)$$

$$\Rightarrow I_{11}^c = I_{22}^c = \frac{m}{4} (r_o^2 + r_i^2) + \frac{ml^2}{12}$$

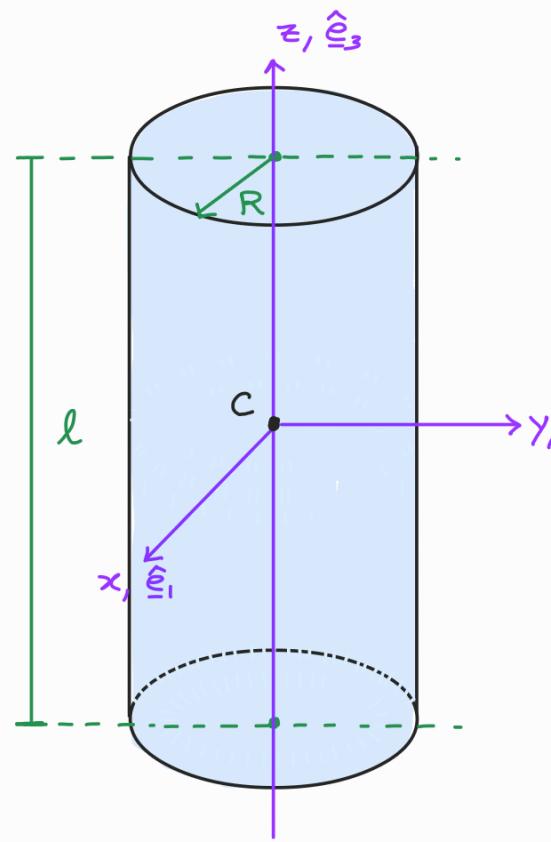
$$[\underline{\underline{I}}^c] = \begin{bmatrix} \frac{m}{4} (r_o^2 + r_i^2) + \frac{ml^2}{12} & 0 & 0 \\ 0 & \frac{m}{4} (r_o^2 + r_i^2) + \frac{ml^2}{12} & 0 \\ 0 & 0 & \frac{m}{4} (r_o^2 + r_i^2) \end{bmatrix}$$

Inertia tensor for a solid circular cylinder

$$\text{Put } r_o = R, m = \rho \pi r^2 l$$

$$r_i = 0$$

$$[\underline{\underline{I}}^c] = \begin{bmatrix} \frac{mR^2}{4} + \frac{ml^2}{12} & 0 & 0 \\ 0 & \frac{mR^2}{4} + \frac{ml^2}{12} & 0 \\ 0 & 0 & \frac{mR^2}{2} \end{bmatrix}$$



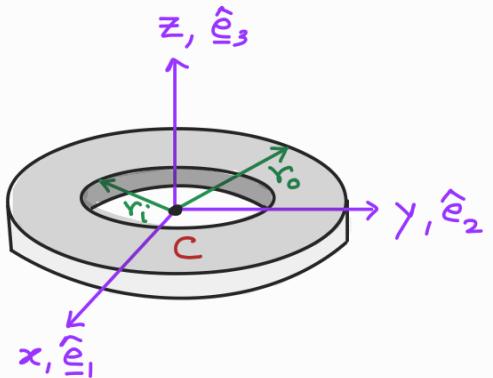
Inertia Tensor for Annular Plate

$\varepsilon \approx 0$ (negligible thickness)

$$\text{Area, } A = \pi(r_o^2 - r_i^2)$$

$$\text{Mass density, } m = n A$$

$\underbrace{}_{\text{mass per unit area}}$



$$[\underline{\underline{I}}^c] = \begin{bmatrix} \frac{m(r_o^2 + r_i^2)}{4} & 0 & 0 \\ 0 & \frac{m(r_o^2 + r_i^2)}{4} & 0 \\ 0 & 0 & \frac{m(r_o^2 + r_i^2)}{2} \end{bmatrix}$$

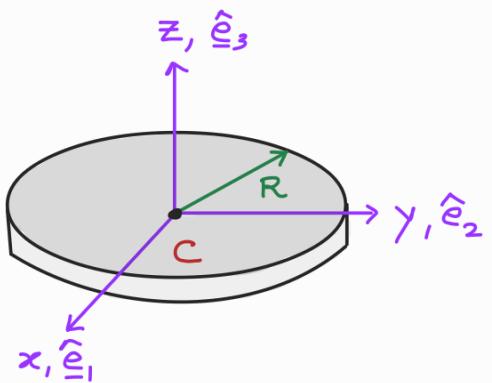
Inertia Tensor for a Thin Circular Disk

$\varepsilon \approx 0$ (negligible thickness)

$$\text{Area, } A = \pi R^2$$

$$\text{Mass density, } m = n A$$

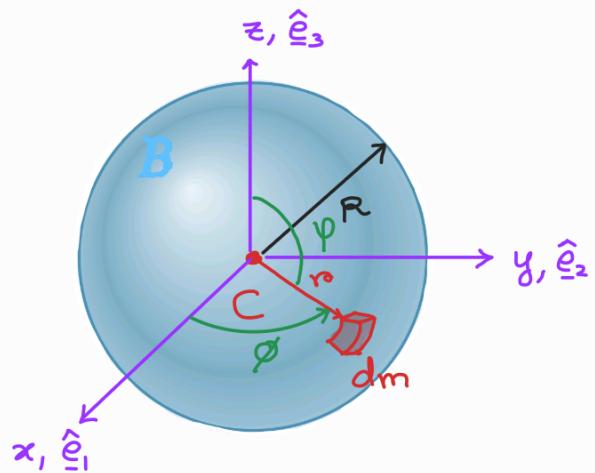
$\underbrace{}_{\text{mass per unit area}}$



$$[\underline{\underline{I}}^c] = \begin{bmatrix} \frac{mR^2}{4} & 0 & 0 \\ 0 & \frac{mR^2}{4} & 0 \\ 0 & 0 & \frac{mR^2}{2} \end{bmatrix}$$

3) Inertia Tensor for a Sphere

Observe that every plane through COM C is a plane of symmetry, so all products of inertia vanish



and $I_{11}^c = I_{22}^c = I_{33}^c$ (since the sphere is symmetric about any axis passing through c)

Recall:

$$I_{11}^c = \int (y^2 + z^2) dm \quad dm = \rho dr (r d\phi) (r d\psi)$$

$$I_{22}^c = \int (x^2 + z^2) dm$$

$$I_{33}^c = \int (x^2 + y^2) dm$$

Add

$$I_{11}^c + I_{22}^c + I_{33}^c = 2 \int (x^2 + y^2 + z^2) dm$$

$$\Rightarrow 3 I_{11}^c = 2 \int r^2 dm$$

$$\Rightarrow I_{11}^c = \frac{2}{3} \int_0^{2\pi} \int_0^{2\pi} \int_0^R r^2 \rho dr r d\phi r d\psi$$

$$= \frac{2}{3} \rho 4\pi^2 \int_0^R r^4 dr$$

$$\therefore [I^c] = \frac{2}{5} m R^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{8\pi^2 \rho}{3} \left(\frac{R^5}{5} \right) = \left(\rho \underbrace{\frac{4}{3}\pi R^3}_m \right) \frac{2}{5} R^2 = \frac{2}{5} m R^2$$

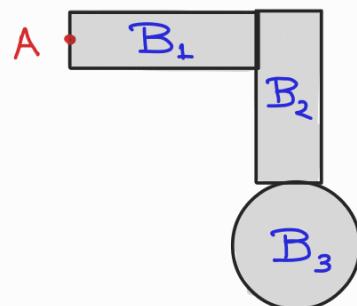
Inertia Tensor of a Composite RB

A composite RB is obtained by composition of several geometrically simple bodies whose inertia tensors are known or maybe easily determined.

If there are 'n' RBs B_k , then the inertial tensor for the composite RB $B = \bigcup_{k=1}^n B_k$, maybe be written at a point A as:

$$\underline{\underline{I}}^A(B) = \sum_{i=1}^n \underline{\underline{I}}^A(B_k)$$

↑ composite body
↓ inertia tensor
of ith RB at pt A

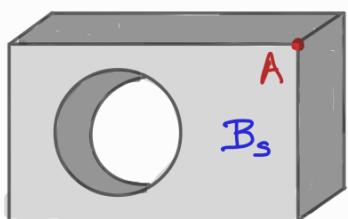


If B has 'p' cavities/holes ξ_k , we may imagine the each cavity as an RB with negative mass, and the inertia tensor of the composite RB with holes/cavity is:

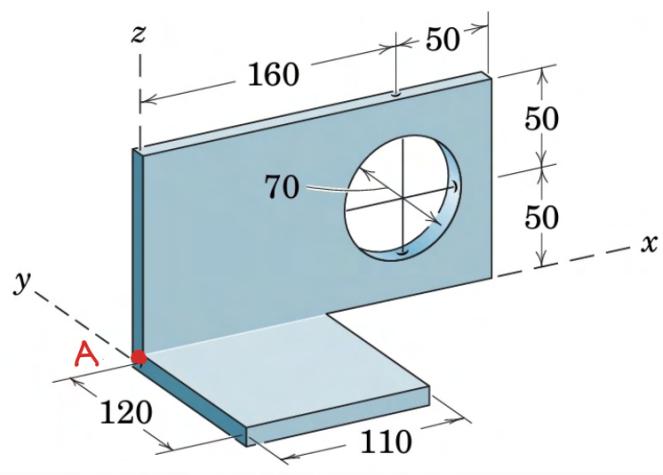
$$\underline{\underline{I}}^A(B_s) = \underline{\underline{I}}^A(B) - \sum_{k=1}^p \underline{\underline{I}}^A(\xi_k)$$

↓ RB with holes/cavity ↓ RB with no cavity/hole ↓ inertia tensor
of ith RB at pt A

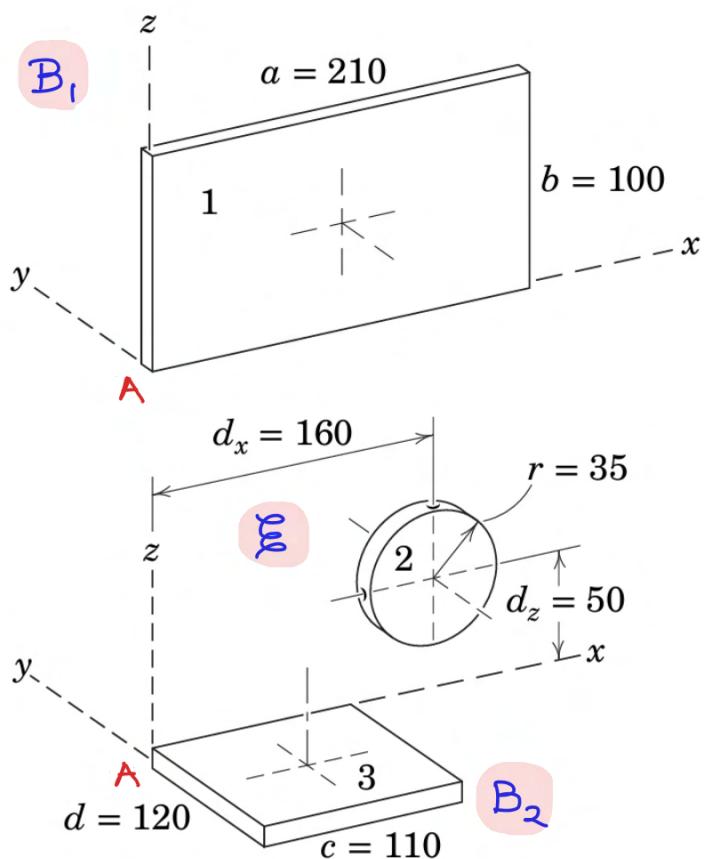
e.g.



Example:



$$\begin{aligned}
 [\underline{\underline{I}}^A](B_s) &= [\underline{\underline{I}}^A](B_1) \\
 &+ [\underline{\underline{I}}^A](B_2) \\
 &- [\underline{\underline{I}}^A](\xi)
 \end{aligned}$$



$$[\underline{\underline{I}}^A](B_1) = \begin{bmatrix} \frac{1}{3}mb^2 & 0 & -\frac{mab}{4} \\ 0 & \frac{1}{3}m(a^2+b^2) & 0 \\ -\frac{mab}{4} & 0 & \frac{1}{3}ma^2 \end{bmatrix}$$

$$[\underline{\underline{I}}^A](B_2) = \begin{bmatrix} \frac{1}{3}md^2 & -m\frac{c}{2}\left(-\frac{d}{2}\right) & 0 \\ -m\frac{c}{2}\left(\frac{d}{2}\right) & \frac{1}{3}mc^2 & 0 \\ 0 & 0 & \frac{1}{3}m(c^2+d^2) \end{bmatrix}$$

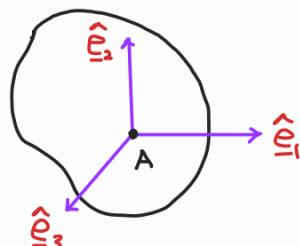
$$[\underline{\underline{I}}^A](\xi) = \begin{bmatrix} \frac{1}{4}mr^2 + md_z^2 & 0 & -md_x dy \\ 0 & \frac{1}{2}mr^2 + m(d_x^2 + d_z^2) & 0 \\ -md_x dy & 0 & \frac{1}{4}mr^2 + m d_x^2 \end{bmatrix}$$

Transformation Rule of Inertia Matrix of RB

We know that the inertia tensor $\underline{\underline{I}}^A$ does not depend on the orientation of the coordinate system. Infact, the same is true for all tensor quantities. However, the second-order inertia tensor, when expressed in a matrix form using an orthonormal csys, the matrix components depend on the orientation of the csys lines.

for csys with axes $\hat{\underline{\underline{e}}}_1 - \hat{\underline{\underline{e}}}_2 - \hat{\underline{\underline{e}}}_3$

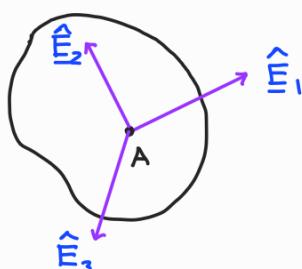
\Rightarrow



$$\left[\underline{\underline{I}}^A \right] = \begin{bmatrix} I_{11}^A & I_{12}^A & I_{13}^A \\ I_{21}^A & I_{22}^A & I_{23}^A \\ I_{31}^A & I_{32}^A & I_{33}^A \end{bmatrix}$$

$$\begin{bmatrix} \hat{\underline{\underline{e}}}_1 \\ \hat{\underline{\underline{e}}}_2 \\ \hat{\underline{\underline{e}}}_3 \end{bmatrix}$$

for csys with axes $\hat{\underline{\underline{E}}}_1 - \hat{\underline{\underline{E}}}_2 - \hat{\underline{\underline{E}}}_3$



$$\left[\underline{\underline{I}}^A \right] = \begin{bmatrix} I_{11}^{A'} & I_{12}^{A'} & I_{13}^{A'} \\ I_{21}^{A'} & I_{22}^{A'} & I_{23}^{A'} \\ I_{31}^{A'} & I_{32}^{A'} & I_{33}^{A'} \end{bmatrix}$$

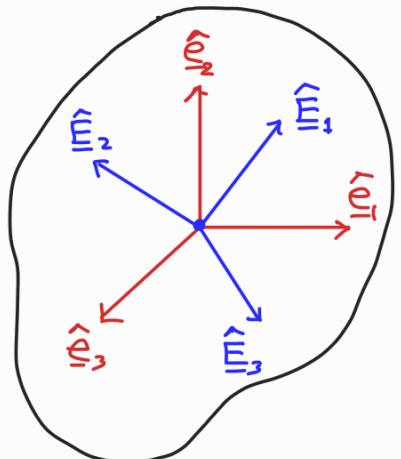
$$\begin{bmatrix} \hat{\underline{\underline{E}}}_1 \\ \hat{\underline{\underline{E}}}_2 \\ \hat{\underline{\underline{E}}}_3 \end{bmatrix}$$

We ask the question: How are the inertia matrix components related?

$$I_{ij}^A \leftrightarrow ?? \quad I_{ij}^{A'}$$

Relating inertia matrix in two different coordinate systems

Two sets of orthonormal triads can be related through a unique rotation tensor



$$\hat{E}_i = \underline{\underline{R}} \hat{e}_i \quad \text{for all } i = 1, 2, 3$$

Orthonormal tensor

in a csys

Orthonormal matrix $[\underline{\underline{R}}]$

Property of
orthonormal matrix

$$[\underline{\underline{R}}] [\underline{\underline{B}}]^T = [\underline{\underline{I}}] \quad \text{and} \quad [\underline{\underline{B}}]^T [\underline{\underline{R}}] = [\underline{\underline{I}}]$$

$$[\hat{E}_i]_{(\hat{e}_1, \hat{e}_2, \hat{e}_3)} = [\underline{\underline{R}}]_{(\hat{e}_1, \hat{e}_2, \hat{e}_3)} [\hat{e}_i]_{(\hat{e}_1, \hat{e}_2, \hat{e}_3)}$$

Let's consider the position vector \underline{r}_{PA} in two csys

$$[\underline{r}_{PA}] \begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{pmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$[\underline{r}_{PA}] \begin{pmatrix} \hat{E}_1 \\ \hat{E}_2 \\ \hat{E}_3 \end{pmatrix} = \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix}$$

$$\underline{r}_{PA} = x_1 \hat{e}_1 + x_2 \hat{e}_2 + x_3 \hat{e}_3 = x'_1 \hat{E}_1 + x'_2 \hat{E}_2 + x'_3 \hat{E}_3$$

$$= x'_1 \underline{\underline{R}} \hat{e}_1 + x'_2 \underline{\underline{R}} \hat{e}_2 + x'_3 \underline{\underline{R}} \hat{e}_3$$

$$= \underline{\underline{R}} (x'_1 \hat{e}_1 + x'_2 \hat{e}_2 + x'_3 \hat{e}_3)$$

In other words, writing $[\underline{r}_{PA}]$ w.r.t $\underline{\underline{e}}_1 - \underline{\underline{e}}_2 - \underline{\underline{e}}_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [\underline{\underline{R}}] \begin{pmatrix} \underline{\underline{e}}_1 \\ \underline{\underline{e}}_2 \\ \underline{\underline{e}}_3 \end{pmatrix} \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix}$$

Let's now use this transformation in the definition of inertia matrix:

$$[\underline{\underline{I}}^A] \begin{pmatrix} \underline{\underline{e}}_1 \\ \underline{\underline{e}}_2 \\ \underline{\underline{e}}_3 \end{pmatrix} = \left\{ \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) [\underline{\underline{I}}] - \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T \right\} dm$$

$$= \left\{ \left(\begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix}^T [\underline{\underline{R}}]^T [\underline{\underline{R}}] \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} \right) [\underline{\underline{I}}] - \left([\underline{\underline{R}}] \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} \right) \left([\underline{\underline{R}}] \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} \right)^T \right\} dm$$

$$= \left\{ \left(\begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix}^T \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} \right) [\underline{\underline{R}}] [\underline{\underline{I}}] [\underline{\underline{R}}]^T - [\underline{\underline{R}}] \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix}^T [\underline{\underline{R}}]^T \right\} dm$$

$$= [\underline{\underline{R}}] \left[\left\{ \left([\underline{r}_{PA}]^T [\underline{r}_{PA}] \right) [\underline{\underline{I}}] - [\underline{r}_{PA}] [\underline{r}_{PA}]^T \right\} dm \right] [\underline{\underline{R}}]^T \begin{pmatrix} \underline{\underline{e}}_1 \\ \underline{\underline{e}}_2 \\ \underline{\underline{e}}_3 \end{pmatrix}$$

$$[\underline{\underline{I}}^A]_{(\underline{\underline{e}}_1 - \underline{\underline{e}}_2 - \underline{\underline{e}}_3)} = [\underline{\underline{R}}]_{(\underline{\underline{e}}_1 - \underline{\underline{e}}_2 - \underline{\underline{e}}_3)} [\underline{\underline{I}}^A]_{(\underline{\underline{e}}_1 - \underline{\underline{e}}_2 - \underline{\underline{e}}_3)} [\underline{\underline{R}}]_{(\underline{\underline{e}}_1 - \underline{\underline{e}}_2 - \underline{\underline{e}}_3)}^T$$

$$\Rightarrow \begin{bmatrix} \underline{\underline{I}}^A \end{bmatrix}_{(\underline{\underline{\underline{E}}}_1 - \underline{\underline{\underline{E}}}_2 - \underline{\underline{\underline{E}}}_3)} = [\underline{\underline{R}}]_{(\underline{\underline{\underline{e}}}_1 - \underline{\underline{\underline{e}}}_2 - \underline{\underline{\underline{e}}}_3)}^T \begin{bmatrix} \underline{\underline{I}}^A \end{bmatrix}_{(\underline{\underline{\underline{E}}}_1 - \underline{\underline{\underline{E}}}_2 - \underline{\underline{\underline{E}}}_3)} [\underline{\underline{R}}]_{(\underline{\underline{\underline{e}}}_1 - \underline{\underline{\underline{e}}}_2 - \underline{\underline{\underline{e}}}_3)}$$

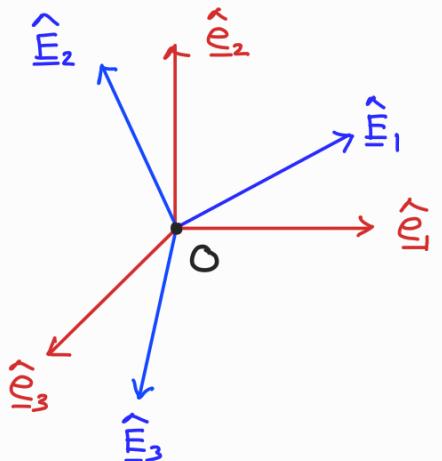
Using index notation, we can write the components of orthonormal matrix as:

$$\underline{\underline{\underline{E}}}_i \cdot \underline{\underline{\underline{e}}}_j = \cos \left(\underbrace{\angle(\underline{\underline{\underline{E}}}_i, \underline{\underline{\underline{e}}}_j)}_{\text{angle between } \underline{\underline{\underline{E}}}_i \text{ and } \underline{\underline{\underline{e}}}_j} \right)$$

$$= (\underline{\underline{R}} \underline{\underline{\underline{e}}}_i) \cdot \underline{\underline{\underline{e}}}_j$$

$$= [\underline{\underline{\underline{e}}}_j]^T [\underline{\underline{R}}] [\underline{\underline{\underline{e}}}_i]$$

$$= R_{ji} \equiv A_{ij}$$



(Note $\underline{\underline{\underline{E}}}_j \cdot \underline{\underline{\underline{e}}}_i \neq \underline{\underline{\underline{E}}}_i \cdot \underline{\underline{\underline{e}}}_j$)

Therefore,

$$I_{ij}' = A_{ip} I_{pq}^A A_{jq}$$

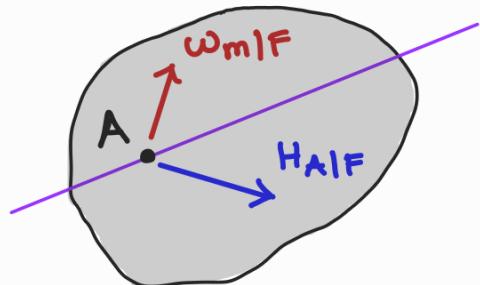
Principal Axes of Inertia of RB at point A

We now ask the question that is there a set of bases using which if we express the inertia matrix, it will be DIAGONAL? (i.e. the products of inertia terms will vanish)

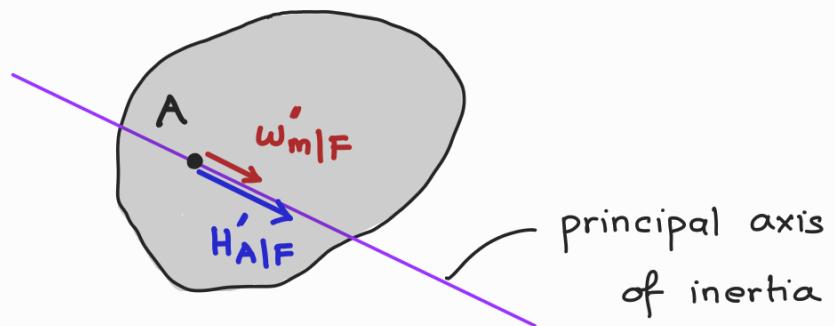
In a rigid body, the two vectors $\underline{\omega}_{mif}$ (angular velocity) and \underline{H}_{Aif} (angular momentum), related through the inertia tensor $\underline{\underline{I}}^A$, are not parallel in general.

$$\underline{H}_{Aif} = \underline{\underline{I}}^A \underline{\omega}_{mif}$$

not //
in general



However, for each point A, there are some directions/axes for which the two vectors $\underline{\omega}_{mif}$ and \underline{H}_A are parallel. These directions are called principal directions and the axes along these directions are called principal axes of inertia at that point.



Along principal axis of inertia

$$\underline{H}_{Aif} = \underline{\underline{I}}^A \underline{\omega}'_{mif} = \lambda \underline{\omega}'_{mif}$$

↑
scalar

λ values are called the eigenvalues of the inertia matrix and they are the roots of the characteristic equation:

$$\det \left(\begin{bmatrix} I^A \\ \hline 3 \times 3 \end{bmatrix} - \lambda \begin{bmatrix} I \\ \hline 3 \times 3 \end{bmatrix} \right) = 0$$

identity
matrix

As $\begin{bmatrix} I^A \\ \hline 3 \times 3 \end{bmatrix}$ is a symmetric and positive definite matrix,
 There are always three real positive eigenvalues.

There is an eigenvector \underline{n} associated with each eigenvalue λ .

They can be determined through the equation:

$$\left(\begin{bmatrix} I^A \\ \hline 3 \times 3 \end{bmatrix} - \lambda \begin{bmatrix} I \\ \hline 3 \times 3 \end{bmatrix} \right) \underline{n} = 0$$

These eigenvectors $\underline{n}^{(1)}, \underline{n}^{(2)}, \underline{n}^{(3)}$ give the principal axes of the RB at point A.

When we express the inertia matrix using a csys whose bases are $\underline{n}^{(1)}, \underline{n}^{(2)}, \underline{n}^{(3)}$, the inertia matrix becomes diagonal :

$$\begin{bmatrix} I^A \\ \hline 3 \times 3 \end{bmatrix} \begin{pmatrix} \underline{n}^{(1)} \\ \underline{n}^{(2)} \\ \underline{n}^{(3)} \end{pmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} I'_{11} & 0 & 0 \\ 0 & I'_{22} & 0 \\ 0 & 0 & I'_{33} \end{bmatrix}$$

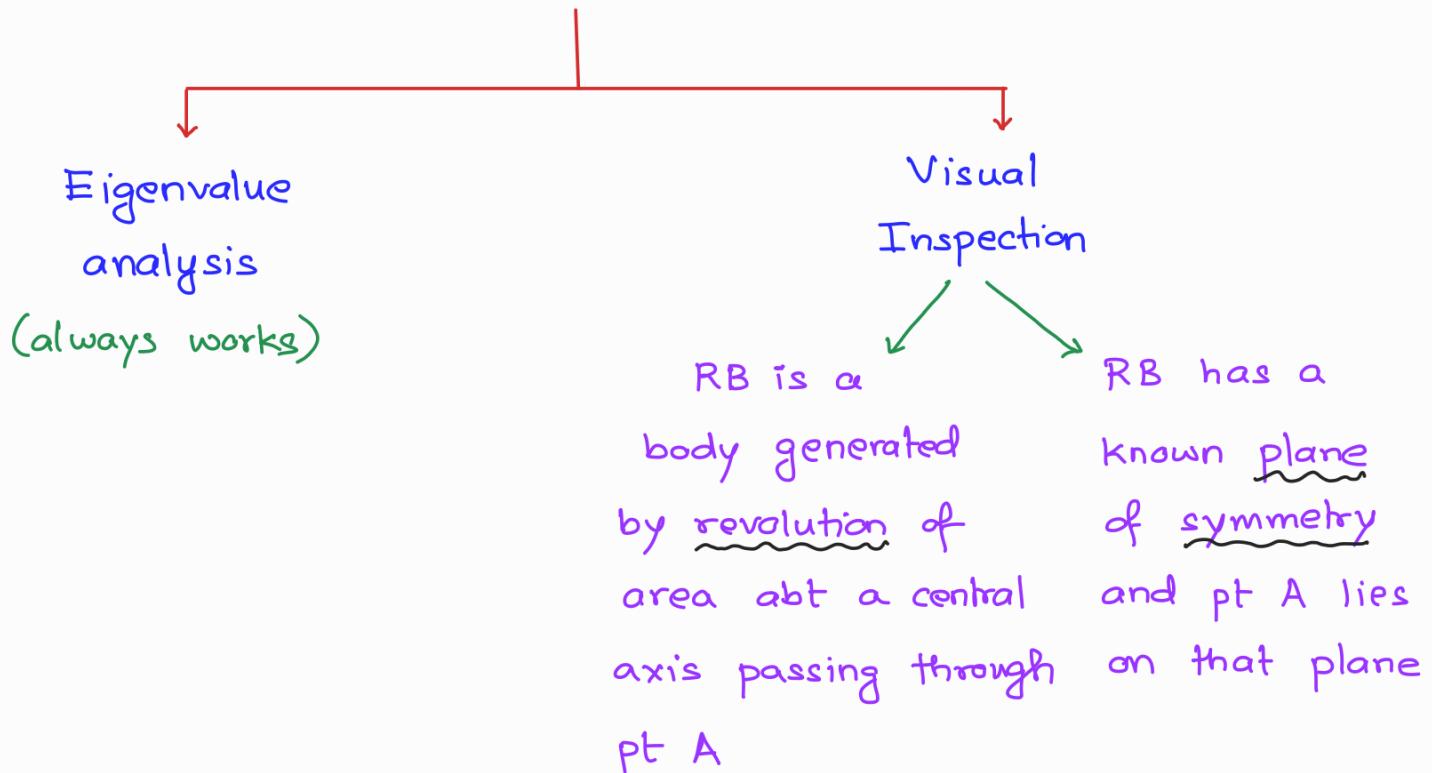
$$\text{Thus, } H'_{A_1} = I'_{11} \omega'_{1/F} \quad H'_{A_2} = I'_{22} \omega'_{1/F} \quad H'_{A_3} = I'_{33} \omega'_{1/F}$$

Cross-coupling is removed, and algebra is simpler!

Note that at least one mutually perpendicular set of three principal axes **ALWAYS EXISTS** at every point A of the RB

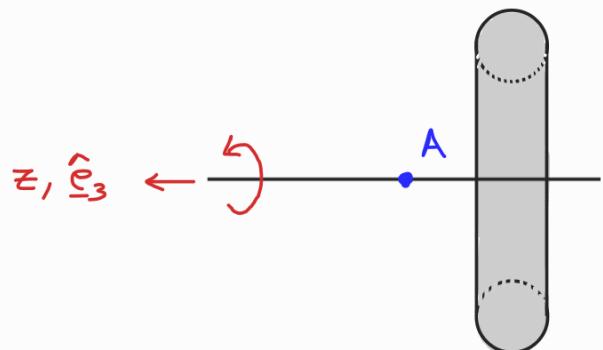
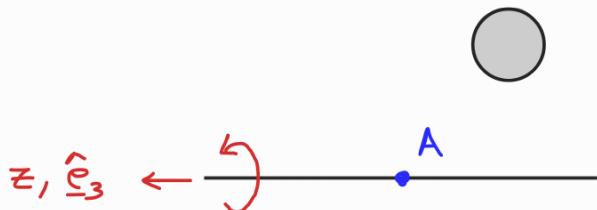
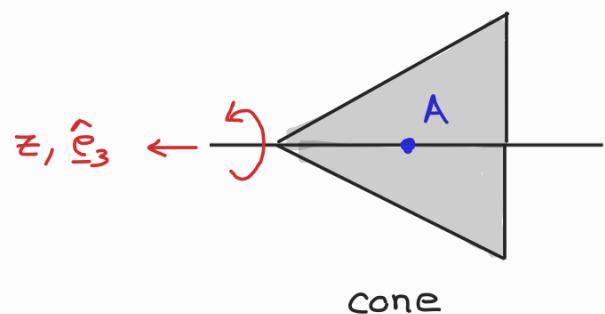
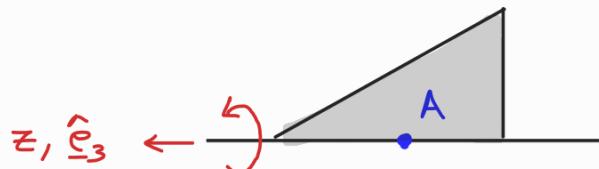
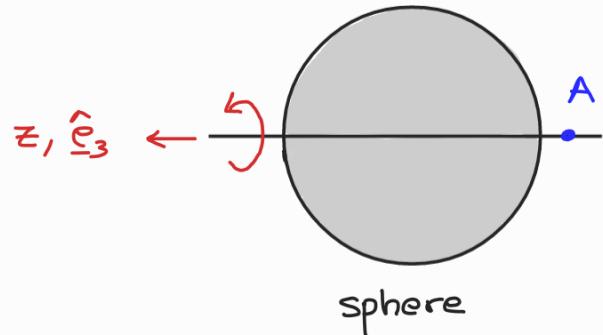
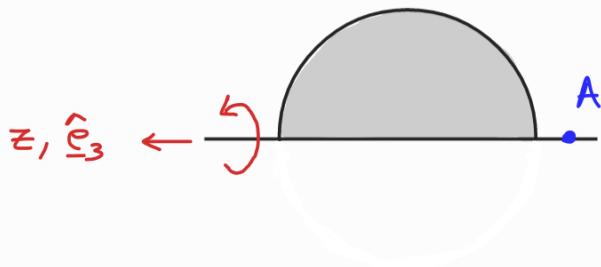
Two ways of finding
principal axes of inertia of

RB at pt A



Bodies of revolution

A body generated by rotating a planar area through 360° about an axis (say z, \hat{e}_3)



torus

- The central axis of revolution is one principal axis
- Any other two mutually perpendicular axes passing through pt A and lying in the plane perpendicular to central axis of revolution will result in two other principal axes

Therefore, for bodies of revolution, the inertia matrix at pt A lying on the central axis of revolution (say $\hat{z}, \hat{\epsilon}_3$) will have the form

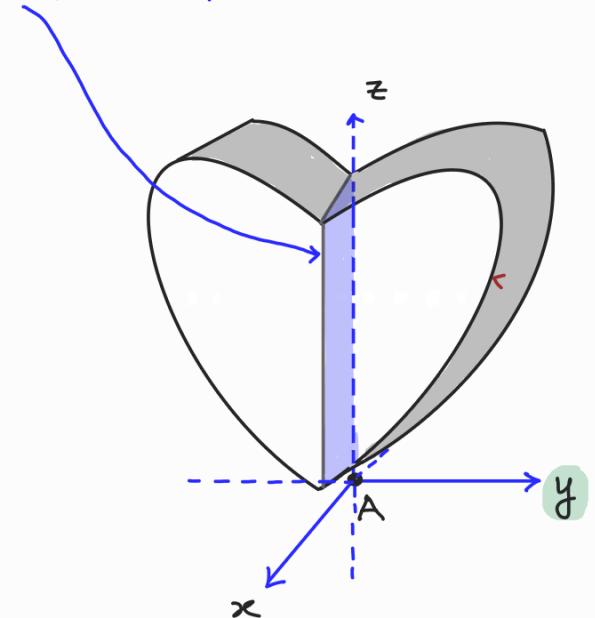
$$[\underline{\underline{I}}^A] = \begin{bmatrix} I_{11}^A & 0 & 0 \\ 0 & I_{22}^A = I_{11}^A & 0 \\ 0 & 0 & I_{33}^A \end{bmatrix}$$

Bodies with plane of symmetry

For an RB with point A lying on a plane of symmetry, any axis perpendicular to this plane of symmetry is a principal axis of inertia

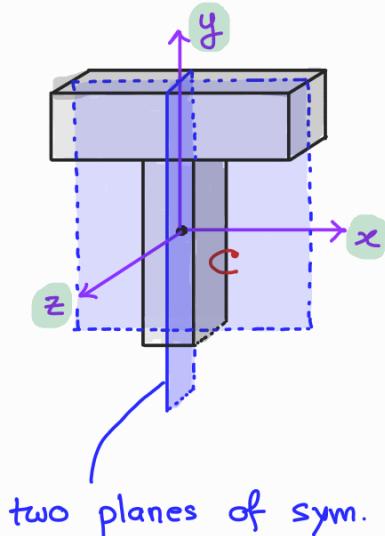
Case 1 : RB with one plane of symmetry

- Suppose an RB has a symmetry plane (say $x-z$ plane) passing point A
- Then, the y -axis (perpendicular to the plane) is a principal axis



Case 2: RB with two planes of symmetry

- If the RB has two symmetry planes (say xy -plane and yz -plane), then their intersection (y -axis) is also a principal axis
- The axes perpendicular to each sym. plane (i.e. z -axis and x -axis) also become principal axes.
- In this case, all the three coordinate axes themselves are the principal axes.



Case 3: RB with three mutually \perp planes of symmetry

Bodies like sphere have three mutually \perp symmetry planes. Hence, the coordinate axes aligned with these planes are the principal axes.

