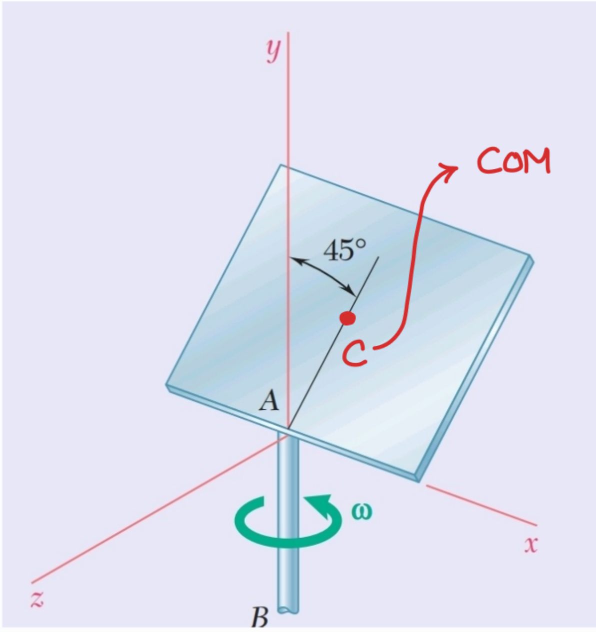
Part A solutions

1) Find kinetic energy of plate

Recall that computation of KE requires careful selection of a point on the body for which the calculation becomes easy.



Two choices for computing KE

$T|_F$

pt A has zero linear velocity, $\underline{v}_{A|F} = \underline{0}$

$\Rightarrow$ can use $T|_F = \frac{1}{2}\, \underline{\omega}_{m|F}\, \underline{\underline{I}}^A\, \underline{\omega}_{m|F}$

Calculation of $\underline{\underline{I}}^A$ needs the use of parallel axes thm to transfer from C to A

pt $C \equiv$ COM

$\Rightarrow$ use $T|_F = \frac{1}{2} m\, \underline{v}_{C|F} \cdot \underline{v}_{C|F}$

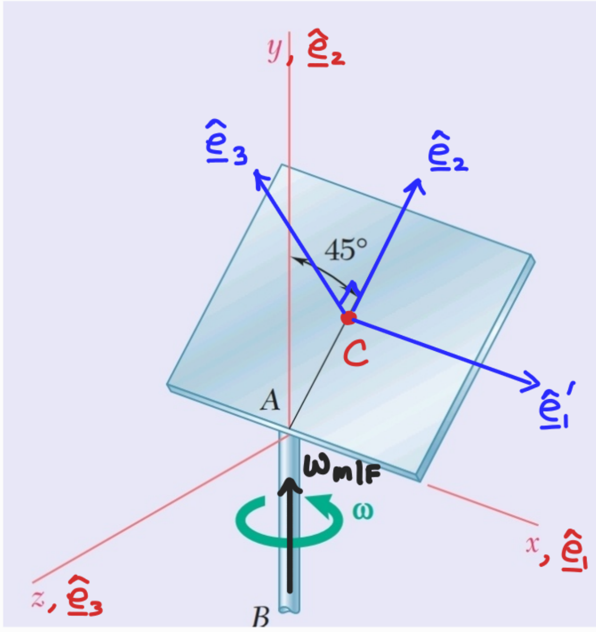
$+ \frac{1}{2}\, \underline{\omega}_{m|F}\, \underline{\underline{I}}^C\, \underline{\omega}_{m|F}$

$\underline{\underline{I}}_C$ can be easily compute but $\underline{v}_{C|F}$ needs to be calculated as well

For the csys given in the figure,

$[\underline{\underline{I}}^C]_{\begin{pmatrix}\hat{e}_1\\\hat{e}_2\\\hat{e}_3\end{pmatrix}}$ is not easily calculated.

as they donot coincide with the principal directions of the plate $\hat{e}_1' - \hat{e}_2' - \hat{e}_3'$



$T_{|F}$ computation using C as base point

in $\hat{e}_1 - \hat{e}_2 - \hat{e}_3$ csys

$$[\underline{\omega}_{m|F}] = \begin{bmatrix} 0 \\ \omega \\ 0 \end{bmatrix}$$

$$[\underline{\underline{I}}^C] = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

Get $[\underline{\underline{I}}^C]_{\begin{pmatrix}\hat{e}_1'\\\hat{e}_2'\\\hat{e}_3'\end{pmatrix}}$ in prin. csys $\hat{e}_1'-\hat{e}_2'-\hat{e}_3'$

and then use transformation rule

$$[\underline{\underline{I}}^C]_{\begin{pmatrix}\hat{e}_1\\\hat{e}_2\\\hat{e}_3\end{pmatrix}} = [\underline{\underline{A}}]\,[\underline{\underline{I}}^C]_{\begin{pmatrix}\hat{e}_1'\\\hat{e}_2'\\\hat{e}_3'\end{pmatrix}}\,[\underline{\underline{A}}]^T$$

need to calculate $[\underline{\underline{A}}]$

in principal csys $\hat{e}_1' - \hat{e}_2' - \hat{e}_3'$

$$[\underline{\underline{I}}^C] = \begin{bmatrix} \frac{ma^2}{12} & 0 & 0 \\ 0 & \frac{ma^2}{12} & 0 \\ 0 & 0 & \frac{ma^2}{6} \end{bmatrix}$$

$$[\underline{\omega}_{m|F}] = \begin{bmatrix} 0 \\ \omega\cos 45^\circ \\ \omega\sin 45^\circ \end{bmatrix}$$

Working in the principal csys at COM is easier!

Compute $v_{C|F}$ using velocity transfer relations

$$\underline{v}_{C|F} = \cancelto{0}{\underline{v}_{A|F}} + \underline{\omega}_{m|F} \times \underline{r}_{CA}$$

$$= \left(\omega/\sqrt{2}\ \hat{\underline{e}}_2' + \omega/\sqrt{2}\ \hat{\underline{e}}_3' \right) \times a/2\ \hat{\underline{e}}_2'$$

$$= -a\omega/2\sqrt{2}\ \hat{\underline{e}}_1'$$

Finally compute KE using C as base pt (using $\hat{\underline{e}}_1' - \hat{\underline{e}}_2' - \hat{\underline{e}}_3'$)

$$T|_F = \frac{1}{2} m\, \underline{v}_{C|F} \cdot \underline{v}_{C|F} + \frac{1}{2} \underline{\omega}_{m|F} \cdot \underline{\underline{I}}^C \underline{\omega}_{m|F}$$

$$= \frac{1}{2} m \left[\underline{v}_{C|F}\right]^T \left[\underline{v}_{C|F}\right] + \frac{1}{2} \left[\underline{\omega}_{m|F}\right] \left[\underline{\underline{I}}^C\right] \left[\underline{\omega}_{m|F}\right]$$

$$= \frac{m}{2} \cdot \frac{a^2\omega^2}{8} + \frac{1}{2} \begin{bmatrix} 0 \\ \omega/\sqrt{2} \\ \omega/\sqrt{2} \end{bmatrix}^T \begin{bmatrix} \frac{ma^2}{12} & 0 & 0 \\ 0 & \frac{ma^2}{12} & 0 \\ 0 & 0 & \frac{ma^2}{6} \end{bmatrix} \begin{bmatrix} 0 \\ \omega/\sqrt{2} \\ \omega/\sqrt{2} \end{bmatrix}$$

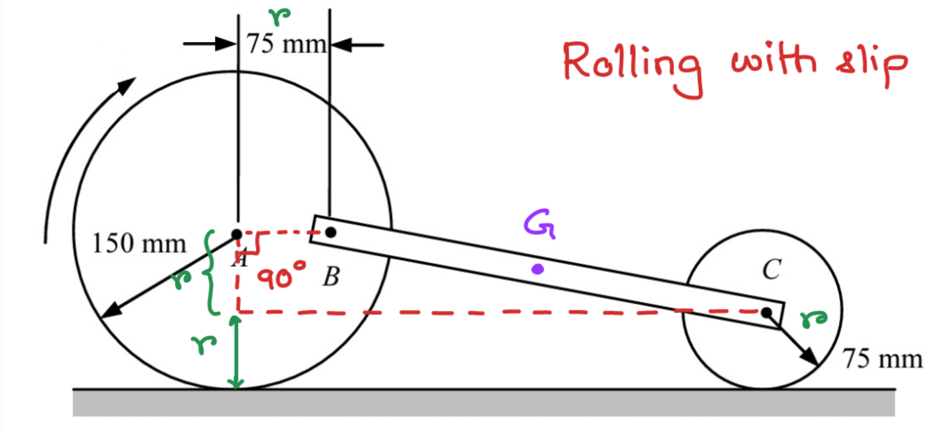
$$= \frac{ma^2\omega^2}{16} + \frac{1}{2} \left[\frac{\omega^2}{2} \cdot \frac{ma^2}{12} + \frac{\omega^2}{2} \cdot \frac{ma^2}{6} \right]$$

$$= \frac{ma^2\omega^2}{16} + \frac{3ma^2\omega^2}{48}$$

$$= \frac{ma^2\omega^2}{8}$$

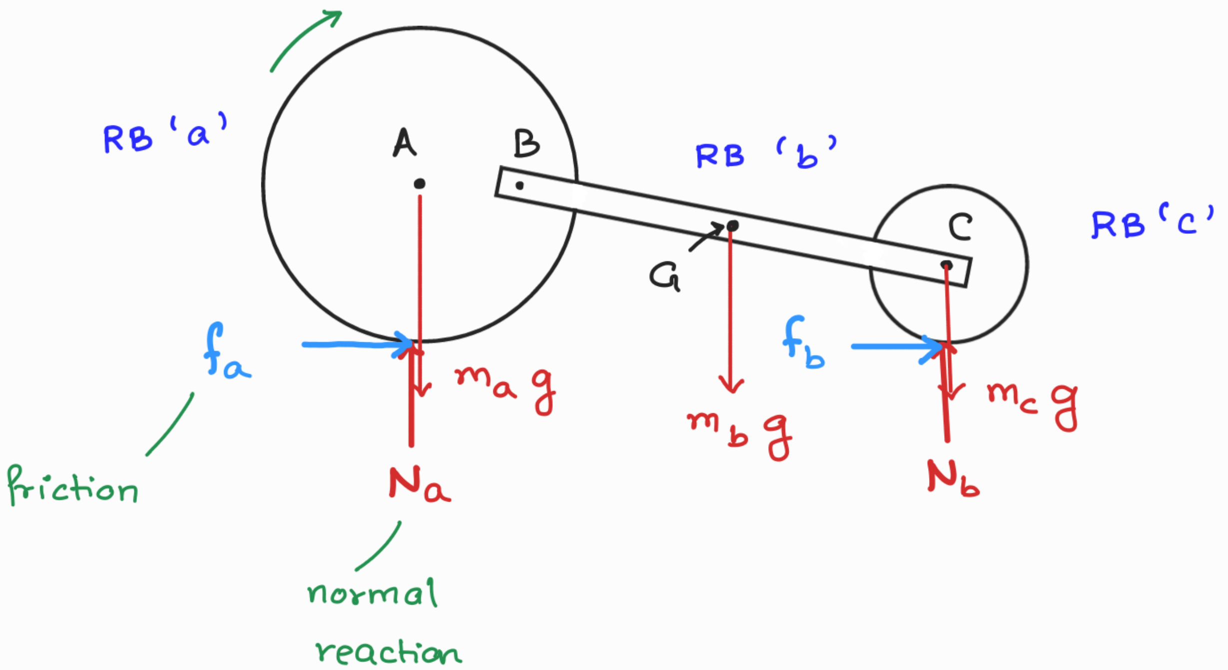
$$\boxed{T|_F = \frac{ma^2\omega^2}{8}}$$

2) Determine velocity of rod BC after disk has rotated by 90°



- Since there are Two positions of rod BC : initial – 1 and final – 2, and we are interested in knowing the velocity of rod, we use the work-energy principle.

- Consider system = rod + two disks, and draw FBD for the system The work done by internal forces are equal and opposite and hence cancel out. We therefore consider only forces/moments external to the "system"



Work energy principle $\Rightarrow$ $W_{1\to 2} = \Delta T = T_{2|F} - T_{1|F}$

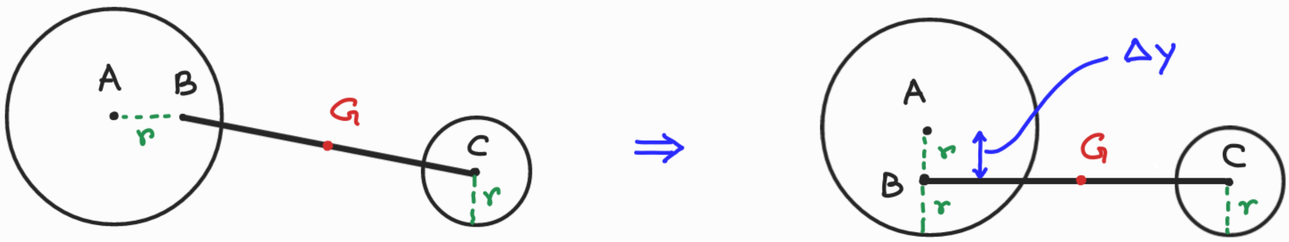
Work done by ext. forces in moving the body from state 1 → 2

1) Frictional forces f_a and f_b do zero work as velocities at the no-slip contact points are zero

2) The normal reactions N_a and N_b are a part of reaction force system, and reaction force systems do not any work as they arise due to constraints in motion.

3) The forces that do non-zero work in this problem are the weights due to gravity (which are constant forces, hence also conservative forces) : $m_a g$, $m_b g$, and $m_c g$

Work done by weights due to gravity



State 1 $\Rightarrow$ State 2

$$\text{Work done } W_{1\to 2} = m_b g \cdot \Delta y$$

vertical disp of G going from state 1 to state 2

$$= m_b g \cdot r$$

According to work-energy principle,

$$W_{1\to 2} = \Delta T = \overbrace{T_2}^{\text{KE at state 2}} - \underbrace{T_1}_{\text{KE at state 1}}$$

$$\Rightarrow \quad m_b\, g \cdot r = (T^a + T^b + T^c)_2 - \cancelto{0}{(T^a + T^b + T^c)_1}$$

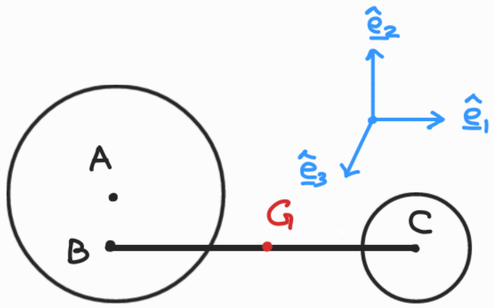
("system" was at rest initially)

$$\Rightarrow \quad (T^a + T^b + T^c)_2 = m_b\, g \cdot r \quad — \quad (\ast\ast)$$

Kinetic energy of the system at state 2

$$(T^a)_2 = ? \qquad (T^b)_2 = ? \qquad (T^c)_2 = ?$$

Using $\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3$ csys for calculation



State 2

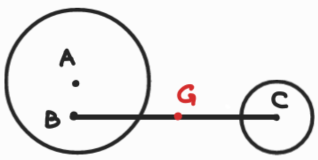
$$(T^a)_2 = \frac{1}{2}\, m_a\, [\underline{v}_{A|I}]^T [\underline{v}_{A|I}] + \frac{1}{2} [\underline{\omega}_{a|I}]^T [\underline{\underline{I}}^A] [\underline{\omega}_{a|I}]$$

$$(T^b)_2 = \frac{1}{2}\, m_b\, [\underline{v}_{G|I}]^T [\underline{v}_{G|I}] + \frac{1}{2} [\underline{\omega}_{b|I}]^T [\underline{\underline{I}}^G] [\underline{\omega}_{b|I}]$$

$$(T^c)_2 = \frac{1}{2}\, m\, [\underline{v}_{C|I}]^T [\underline{v}_{C|I}] + \frac{1}{2} [\underline{\omega}_{c|I}]^T [\underline{\underline{I}}^C] [\underline{\omega}_{c|I}]$$

Use kinematics to relate $\underline{v}_{G|I}$, $\underline{v}_{A|I}$, and $\underline{v}_{C|I}$

In state 2, $\boxed{\underline{v}_{G|I} = \underline{v}_{B|I} = \underline{v}_{C|I}}$ $(\ast)$

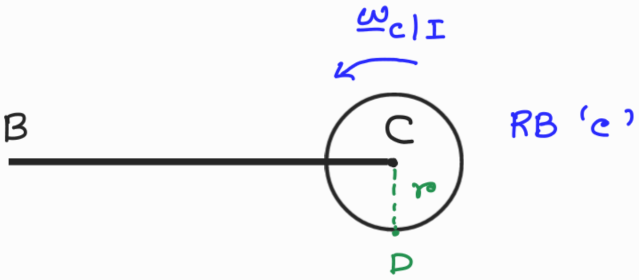


State 2

Therefore, at the instant of state 2, $\underline{\omega}_{b|I} = \underline{0}$

$\Rightarrow$ every point on rod BC has identical velocity

$$\overset{v_c \hat{e}_1}{\underline{v}_{c|I}} = \overset{0}{\cancel{\underline{v}_{D|I}}} + \overset{\omega_c \hat{e}_3}{\underline{\omega}_{c|I}} \times \overset{r \hat{e}_2}{\underline{r}_{CD}}$$

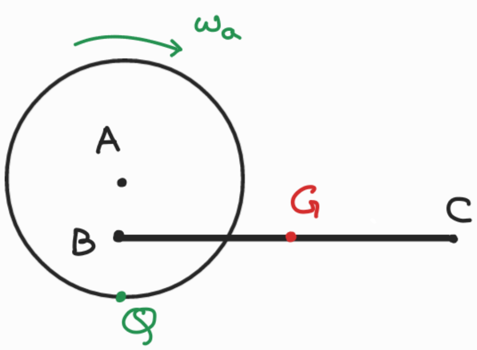


$$\Rightarrow \quad v_c \hat{e}_1 = \omega_c r \hat{e}_1$$

$$\Rightarrow \quad v_c = \omega_c r$$

$$\underline{v}_{A|I} = \overset{0}{\underline{v}_{Q|I}} + \overset{-\omega_a \hat{e}_3}{\underline{\omega}_{a|I}} \times \overset{2r \hat{e}_2}{\underline{r}_{AQ}}$$

$$= 2\omega_a r \, \hat{e}_1$$



$$\underline{v}_{B|I} = \overset{0}{\underline{v}_{Q|I}} + \overset{-\omega_a \hat{e}_3}{\underline{\omega}_{a|I}} \times \overset{r \hat{e}_2}{\underline{r}_{AQ}}$$

from ⊛

$$= \omega_a r \, \hat{e}_1 = \underline{v}_{G|I} = \underline{v}_{c|I} \Rightarrow \omega_a r \hat{e}_1 = \omega_c r \hat{e}_1$$

$$\Rightarrow \boxed{\omega_a = \omega_c}$$

$$\left(T^a_{|I}\right)_2 = \frac{1}{2} m_a \left[\underline{v}_{A|I}\right]^T \left[\underline{v}_{A|I}\right] + \frac{1}{2} \left[\underline{\omega}_{a|I}\right]^T \left[\underline{\underline{I}}^A\right] \left[\underline{\omega}_{a|I}\right]$$

$$= \frac{1}{2} m_a \begin{bmatrix} 2r\omega_a \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 2r\omega_a \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ -\omega_a \end{bmatrix}^T \begin{bmatrix} \checkmark & 0 & 0 \\ 0 & \checkmark & 0 \\ 0 & 0 & \frac{m_a R^2}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -\omega_a \end{bmatrix}$$

$$= \frac{1}{2} m_a \cdot (2 r \omega_a)^2 + \frac{1}{2} (\omega_a)^2 \cdot \left(\frac{m_a (2r)^2}{2} \right) \qquad [R = 2r]$$

$$= \frac{m_a \; 4 r^2 \omega_a^2}{2} + \frac{m_a \; 4 r^2 \omega_a^2}{4}$$

$$= 3 \, m_a \, r^2 \omega_a^2$$

$$(T^b_{|I})_2 = \frac{1}{2} m_b \left[\underline{v}_{G|I}\right]^T \left[\underline{v}_{G|I}\right] + \frac{1}{2} \left[\underline{\omega}_{b|I}\right]^T \left[\underline{\underline{I}}^G\right] \left[\underline{\omega}_{b|I}\right]$$

$$= \frac{1}{2} m_b \begin{bmatrix} r\omega_a \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} r\omega_a \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} \checkmark & 0 & 0 \\ 0 & \checkmark & 0 \\ 0 & 0 & \checkmark \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2} m_b \, r^2 \omega_a^2$$

$$(T^c_{|I})_2 = \frac{1}{2} m_c \left[\underline{v}_{c|I}\right]^T \left[\underline{v}_{c|I}\right] + \frac{1}{2} \left[\underline{\omega}_{c|I}\right]^T \left[\underline{\underline{I}}\right] \left[\underline{\omega}_{c|I}\right]$$

$$= \frac{1}{2} m_c \begin{bmatrix} r\omega_a \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} r\omega_a \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ \omega_c \end{bmatrix}^T \begin{bmatrix} \checkmark & 0 & 0 \\ 0 & \checkmark & 0 \\ 0 & 0 & \frac{m_c r^2}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \omega_c \end{bmatrix}$$

$$= \frac{1}{2} m_c \, r^2 \omega_a^2 + \frac{1}{2} \omega_a^2 \frac{m_c r^2}{2} \qquad [\omega_c = \omega_a]$$

$$= \frac{3}{4} m_c \, r^2 \omega_a^2$$

Adding $(T^a_{|I})_2$, $(T^b_{|I})_2$, and $(T^c_{|I})_2$

$$(T_{1I}^{a} + T_{1I}^{b} + T_{1I}^{c})_2 = 3\,\overset{6}{m_a}\, r^2 \omega_a^2 + \frac{1}{2}\,\overset{5}{m_b}\, r^2 \omega_a^2 + \frac{3}{4}\,\overset{1.5}{m_c}\, r^2 \omega_a^2$$

$[v_B = \omega_a r]$

$$= 18\, v_B^2 + \frac{5}{2}\, v_B^2 + \frac{9}{8}\, v_B^2$$

$$= 21.6\, v_B^2$$

Using work-energy relation (**)

$$21.6\, v_B^2 = \overset{5}{m_b}\, g\, \overset{0.075}{r} \qquad g = 9.81$$

$$\Rightarrow \quad v_B = 0.412 \text{ m/s}$$