Tutorial 6 (Part A)

A thin homogeneous rectangular plate, as shown, rotates about a diagonal axis with angular velocity $\underline{\omega}$ and angular acceleration $\underline{\dot{\omega}}$.

(i) Determine the total moment $\underline{M}_C$ exerted on the plate about the COM C, using the coordinate system $\underline{\hat{e}}_1', \underline{\hat{e}}_2', \underline{\hat{e}}_3'$ in terms of the rotational motion $\underline{\omega}$ and $\underline{\dot{\omega}}$.

(ii) Find the relation between a drive torque $\underline{T} = T\underline{\hat{e}}_3'$ (applied about $\underline{\hat{e}}_3'$-axis) to the rotational motion. Also, determine the bearing support reaction forces (assuming bearing support reaction couples are zero).

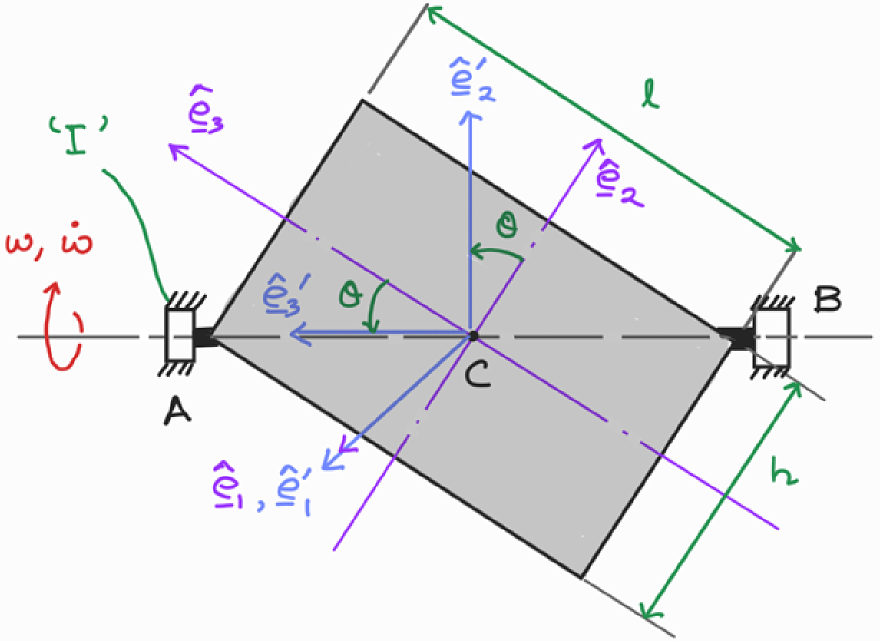
Given:

$$\underline{\omega}_{m|I} = \omega\, \underline{\hat{e}}_3'$$

$$\underline{\dot{\omega}}_{m|I} = \dot{\omega}\, \underline{\hat{e}}_3'$$

plate lies in the plane of $x_2(\hat{e}_2) - x_3(\hat{e}_3)$ / $x_2'(\hat{e}_2') - x_3'(\hat{e}_3')$

Both CSYS are body-fixed.



Solution: RB 'm' $\leftarrow$ rectangular plate rotates about an axis

We can make use of the simplified Euler's 2nd equation for an RB rotating about a fixed axis.

Recall the simplified Euler's 2nd equation at point A of the RB rotating about a fixed body axis $\hat{\underline{e}}_3$

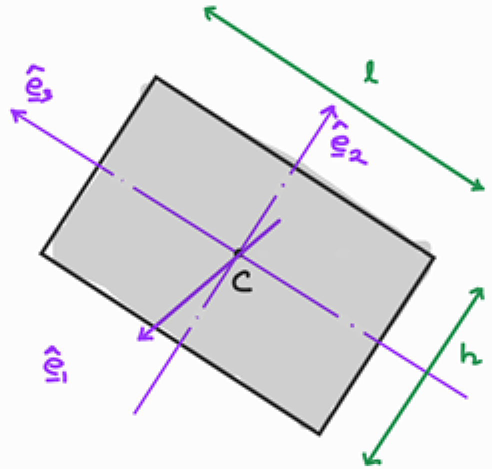
$$\underline{M}_A = \underbrace{\left(I_{13}^A \dot{\omega} - I_{23}^A \omega^2\right)}_{M_{A,1}} \hat{\underline{e}}_1 + \underbrace{\left(I_{23}^A \dot{\omega} + I_{13}^A \omega^2\right)}_{M_{A,2}} \hat{\underline{e}}_2 + \underbrace{I_{33}^A \dot{\omega}}_{M_{A,3}} \hat{\underline{e}}_3$$

With the body-fixed csys being $\hat{\underline{e}}_1{}' - \hat{\underline{e}}_2{}' - \hat{\underline{e}}_3{}'$, we rewrite the equation as: $A \equiv C$, $\hat{\underline{e}}_i \rightarrow \hat{\underline{e}}_i{}'$

$$\underline{M}_C = \underbrace{\left(I_{13}^C \, \dot{\omega} - I_{23}^C \, \omega^2\right)}_{M'_{C,1}} \hat{\underline{e}}_1{}' + \underbrace{\left(I_{23}^C \, \dot{\omega} + I_{13}^C \, \omega^2\right)}_{M'_{C,2}} \hat{\underline{e}}_2{}' + \underbrace{I_{33}^C \, \dot{\omega}}_{M'_{C,3}} \hat{\underline{e}}_3{}'$$

To determine $\underline{M}_C$, we need to calculate the inertia matrix components in $\hat{\underline{e}}_1{}' - \hat{\underline{e}}_2{}' - \hat{\underline{e}}_3$ csys.

Note: $\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3$ csys coincide with the principal axes of inertia of the rectangular plate (due to planes of symmetry)

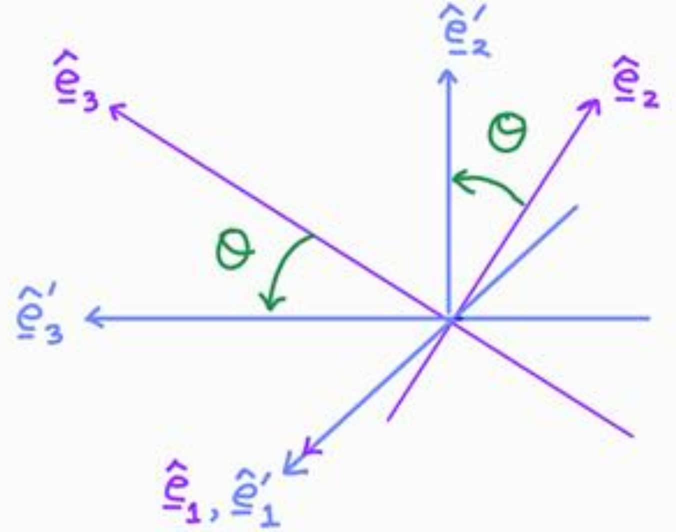


$$\Rightarrow \left[\underline{\underline{I}}^C\right]_{(\hat{\underline{e}}_1 \hat{\underline{e}}_2 \hat{\underline{e}}_3)} = \begin{bmatrix} I_{11}^C & 0 & 0 \\ 0 & I_{22}^C & 0 \\ 0 & 0 & I_{33}^C \end{bmatrix} = \begin{bmatrix} \frac{m(\ell^2+h^2)}{12} & 0 & 0 \\ 0 & \frac{m\ell^2}{12} & 0 \\ 0 & 0 & \frac{mh^2}{12} \end{bmatrix}$$

Next we will use inertia matrix transformation law to find the components of $\underline{\underline{I}}^C$ in $\hat{\underline{e}}_1' - \hat{\underline{e}}_2' - \hat{\underline{e}}_3'$ csys

$$[\underline{\underline{I}}^c]_{(\hat{\underline{e}}_1', \hat{\underline{e}}_2', \hat{\underline{e}}_3')} = [\underline{\underline{A}}]\,[\underline{\underline{I}}_c]_{(\hat{\underline{e}}_1, \hat{\underline{e}}_2, \hat{\underline{e}}_3)}\,[\underline{\underline{A}}]^T$$

Find transformation matrix $[\underline{\underline{A}}]$: Use $A_{ij} = \hat{\underline{e}}_i' \cdot \hat{\underline{e}}_j$



$$A_{22} = \hat{\underline{e}}_2' \cdot \hat{\underline{e}}_2 = \cos\theta = c$$

$$A_{23} = \hat{\underline{e}}_2' \cdot \hat{\underline{e}}_3 = \sin\theta = s$$

$$A_{32} = \hat{\underline{e}}_3' \cdot \hat{\underline{e}}_2 = -\sin\theta = -s$$

$$A_{33} = \hat{\underline{e}}_3' \cdot \hat{\underline{e}}_3 = \cos\theta = c$$

$$A_{11} = 1, \quad A_{21} = A_{31} = 0$$

$$[\underline{\underline{I}}^c]_{\begin{pmatrix}\hat{e}_1' \\ \hat{e}_2' \\ \hat{e}_3'\end{pmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{bmatrix} \begin{bmatrix} I_{11}^c & 0 & 0 \\ 0 & I_{22}^c & 0 \\ 0 & 0 & I_{33}^c \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{bmatrix}$$

$$= \begin{bmatrix} I_{11}^c & 0 & 0 \\ 0 & c^2 I_{22}^c + s^2 I_{33}^c & cs(-I_{22}^c + I_{33}^c) \\ 0 & \text{sym} & s^2 I_{22}^c + c^2 I_{33}^c \end{bmatrix}$$

From geometry of plate: $\sin\theta = \dfrac{h}{\sqrt{\ell^2+h^2}}$, $\cos\theta = \dfrac{\ell}{\sqrt{\ell^2+h^2}}$

Given: $[\underline{\omega}]_{\begin{pmatrix}\hat{e}_1' \\ \hat{e}_2' \\ \hat{e}_3'\end{pmatrix}} = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}$ & $[\underline{\dot{\omega}}]_{\begin{pmatrix}\hat{e}_1' \\ \hat{e}_2' \\ \hat{e}_3'\end{pmatrix}} = \begin{bmatrix} 0 \\ 0 \\ \dot{\omega} \end{bmatrix}$ 0

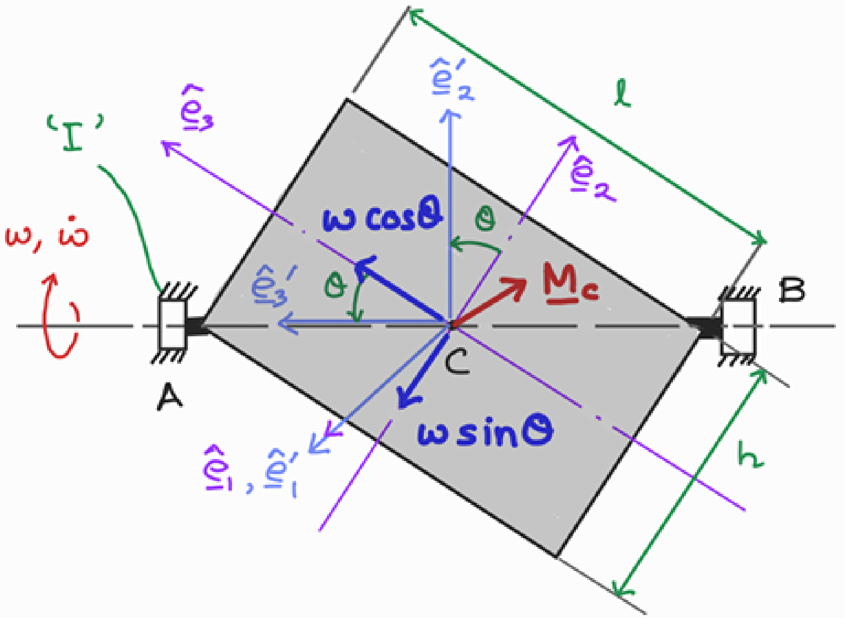
Let's now use simplified Euler's 2nd equation:

$$\underline{M}_c = \underbrace{\left(\cancelto{0}{I_{13}^c \dot{\omega}} - I_{23}^c \omega^2\right)}_{M'_{c,1}} \hat{\underline{e}}_1' + \underbrace{\left(I_{23}^c \dot{\omega} + \cancelto{0}{I_{13}^c \omega^2}\right)}_{M'_{c,2}} \hat{\underline{e}}_2' + \underbrace{I_{33}^c \dot{\omega}}_{M'_{c,3}} \hat{\underline{e}}_3'$$

$$\left.\begin{aligned}
M'_{c,1} &= -\frac{1}{12} m \left(-\ell^2 + h^2\right) \sin\theta \cos\theta \; \omega^2 \\
&= -\frac{m \ell h}{12} \frac{\left(-\ell^2 + h^2\right)}{\left(\ell^2 + h^2\right)} \omega^2 \\
M'_{c,2} &= \frac{m \ell h}{12} \frac{\left(-\ell^2 + h^2\right)}{\left(\ell^2 + h^2\right)} \dot{\omega} \\
M'_{c,3} &= \left(\sin^2\theta \; \frac{m \ell^2}{12} + \cos^2\theta \; \frac{m h^2}{12}\right) \dot{\omega} \\
&= \frac{m}{12} \left(\frac{h^2 \ell^2}{\ell^2 + h^2} + \frac{\ell^2 h^2}{\ell^2 + h^2} \right) \dot{\omega} \\
&= \frac{m \, h^2 \ell^2}{6\left(\ell^2 + h^2\right)} \dot{\omega}
\end{aligned}\right\} \; *$$

Alternative way : Using simplified Euler's 2nd equation in csys $\hat{e}_1-\hat{e}_2-\hat{e}_3$ (coinciding with p-axes of plate)

Let's solve the problem using p-axes csys!



$$[\underline{\omega}]_{(\hat{e}_1\,\hat{e}_2\,\hat{e}_3)} = \begin{bmatrix} 0 \\ -\omega\sin\theta \\ \omega\cos\theta \end{bmatrix}$$

$$[\underline{\dot{\omega}}]_{(\hat{e}_1\,\hat{e}_2\,\hat{e}_3)} = \begin{bmatrix} 0 \\ -\dot{\omega}\sin\theta \\ \dot{\omega}\cos\theta \end{bmatrix}$$

where $\sin\theta = \dfrac{h}{\sqrt{\ell^2+h^2}} \equiv s$

$\cos\theta = \dfrac{\ell}{\sqrt{\ell^2+h^2}} \equiv c$

Since $\underline{\hat{e}}_1 - \underline{\hat{e}}_2 - \underline{\hat{e}}_3$ coincides with p-axes of inertia of the rectangular plate

$$[\underline{\underline{I}}^C]_{\begin{pmatrix}\underline{\hat{e}}_1\\ \underline{\hat{e}}_2\\ \underline{\hat{e}}_3\end{pmatrix}} = \begin{bmatrix} \frac{m(l^2+h^2)}{12} & 0 & 0 \\ 0 & \frac{ml^2}{12} & 0 \\ 0 & 0 & \frac{mh^2}{12} \end{bmatrix}$$

Now let's use simplied Euler's 2nd equation at pt C in the $\underline{\hat{e}}_1 - \underline{\hat{e}}_2 - \underline{\hat{e}}_3$ csys (coinciding with p-axes of RB at C)

$$\underline{M}_c = M_{c,1}\,\hat{\underline{e}}_1 + M_{c,2}\,\hat{\underline{e}}_2 + M_{c,3}\,\hat{\underline{e}}_3$$

$$\begin{aligned}\underline{M}_c = {} & \left[I_{11}^c\, \cancelto{0}{\dot{\omega}_1} - \left(I_{22}^c - I_{33}^c\right)\omega_2\,\omega_3 \right]\hat{\underline{e}}_1 \\ & + \left[I_{22}^c\, \cancelto{-\dot{\omega}s}{\dot{\omega}_2} - \left(I_{33}^c - I_{11}^c\right)\omega_3\,\cancelto{0}{\omega_1} \right]\hat{\underline{e}}_2 \\ & + \left[I_{33}^c\, \cancelto{\dot{\omega}c}{\dot{\omega}_3} - \left(I_{11}^c - I_{22}^c\right)\cancelto{0}{\omega_1}\,\omega_2 \right]\hat{\underline{e}}_3 \end{aligned}$$

$$= -\underbrace{\frac{m\,c\,s}{12}\,(\ell^2 - h^2)\,\omega^2}_{M_{c,1}}\,\hat{\underline{e}}_1 + \underbrace{\frac{m}{12}\,(-s\,\ell^2}_{M_{c,2}}\,\hat{\underline{e}}_2 + \underbrace{c\,h^2}_{M_{c,3}}\,\hat{\underline{e}}_3)\,\dot{\omega}$$

Note that $M_{c,1}$, $M_{c,2}$, and $M_{c,3}$ are components of $\underline{M}_c$ in the $\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3$ csys.

To get the components of $\underline{M}_c$ in $\hat{\underline{e}}_1'$-$\hat{\underline{e}}_2'$-$\hat{\underline{e}}_3$ csys, one needs to use a transformation of coordinate sys :

$$[\underline{M}_c]_{(\hat{\underline{e}}_1', \hat{\underline{e}}_2', \hat{\underline{e}}_3')} = [\underline{\underline{A}}]\,[\underline{M}_c]_{(\hat{\underline{e}}_1, \hat{\underline{e}}_2, \hat{\underline{e}}_3)}$$

$$\Rightarrow \begin{bmatrix} M_{c,1}' \\ M_{c,2}' \\ M_{c,3}' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{bmatrix} \begin{bmatrix} M_{c,1} \\ M_{c,2} \\ M_{c,3} \end{bmatrix}$$

$$\therefore \quad M_{c,1}' = M_{c,1} = -\frac{m\,\ell h}{12}\,\frac{(-\ell^2 + h^2)}{(\ell^2 + h^2)}\,\omega^2$$

$$M'_{c,2} = c\, M_{c,2} + s\, M_{c,3}$$

$$= \frac{m}{12} c s \left(-\ell^2 + h^2\right) \dot{\omega}$$

$$= \frac{m}{12} \ell h \frac{\left(-\ell^2 + h^2\right)}{\left(\ell^2 + h^2\right)} \dot{\omega}$$

$$M'_{c,3} = -s\, M_{c,2} + c\, M_{c,3}$$

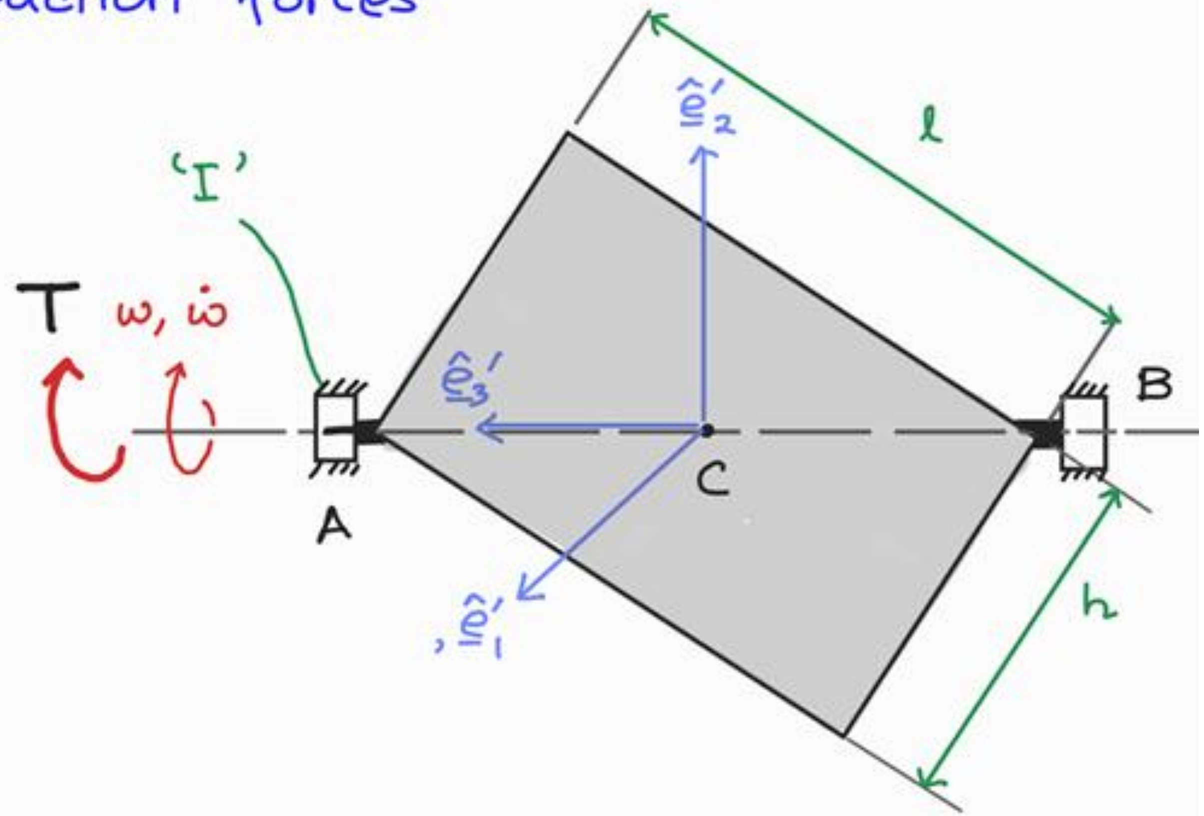
$$= \frac{m}{12} \left(s^2 \ell^2 + c^2 h^2 \right) \dot{\omega} = \frac{m\, h^2 \ell^2}{6\left(\ell^2 + h^2\right)} \dot{\omega}$$

These are the same components we obtained using the 1st method.

Note that even if $\dot{\omega} = 0$, still $\underline{M}_c \neq 0$ for a rotational motion with constant angular velocity ω.

ii) Equation relating drive torque $\underline{T}$ to the rotation and dynamic bearing reaction forces

$$\underline{T} = T\,\hat{\underline{e}}_3'$$



Draw the FBD

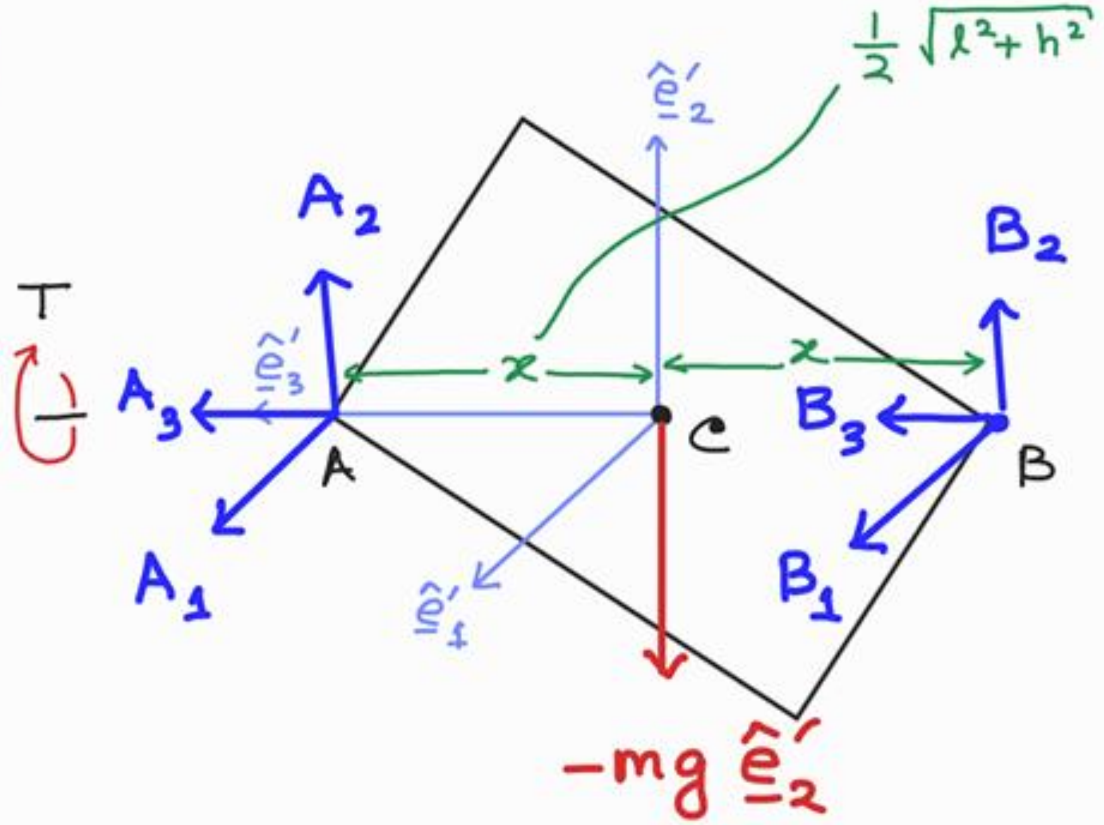
Assume zero reaction couples at supports (Given)

Reaction forces at supports A and B

$\underline{A} = A_1 \hat{\underline{e}}_1' + A_2 \hat{\underline{e}}_2' + A_3 \hat{\underline{e}}_3$

$\underline{B} = B_1 \hat{\underline{e}}_1' + B_2 \hat{\underline{e}}_2' + B_3 \hat{\underline{e}}_3$

FBD



From Euler's 1st eqn:

$$\underline{A} + \underline{B} - mg\,\hat{\underline{e}}_2' = \cancelto{0}{m\,\underline{a}_{c|I}} = 0$$

because C is on the $\hat{\underline{e}}_3'$-axis itself at all times!

$$\Rightarrow \quad \underline{A} + \underline{B} = mg\,\hat{\underline{e}}_2$$

Component-wise:

$$B_1 = -A_1$$

$$A_2 + B_2 = mg$$

$$B_3 = -A_3$$

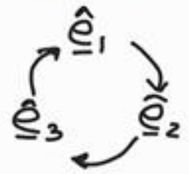
Express $\underline{M}_c$ in terms of the external force system

$$\underline{M}_c = \underline{T} + \underline{r}_{AC} \times \underline{A} + \underline{r}_{BC} \times \underline{B}$$

($-mg\,\hat{\underline{e}}_2'$ passes through C)

$$= T\,\hat{\underline{e}}_3' + \left(x\,\hat{\underline{e}}_3'\right) \times \left(A_1\,\hat{\underline{e}}_1' + A_2\,\hat{\underline{e}}_2' + A_3\,\hat{\underline{e}}_3'\right)$$

$$+ \left(-x\,\hat{\underline{e}}_3'\right) \times \left(\cancelto{-A_1}{B_1}\,\hat{\underline{e}}_1' + B_2\,\hat{\underline{e}}_2' + \cancelto{-A_3}{B_3}\,\hat{\underline{e}}_3'\right)$$



$$= T\,\hat{e}_3' + A_1 x\,\hat{e}_2 - A_2 x\,\hat{e}_1' + A_1 x\,\hat{e}_2 + B_2 x\,\hat{e}_1'$$

$$= \underbrace{(-A_2 + B_2)\,x}_{= M_{c,1}}\,\hat{e}_1' + \underbrace{A_1\,(2x)}_{= M_{c,2}}\,\hat{e}_2' + \underbrace{T}_{= M_{c,3}}\,\hat{e}_3'$$

Comparing these relations with the values obtained in (*)

$$\Rightarrow \quad T = \frac{m\,h^2 \ell^2}{6(\ell^2 + h^2)}\,\dot{\omega}\,, \qquad A_1 = -B_1 = -\frac{m\,\ell h}{12}\,\frac{(-\ell^2 + h^2)}{(\ell^2 + h^2)^{3/2}}\,\omega^2$$

and, finally

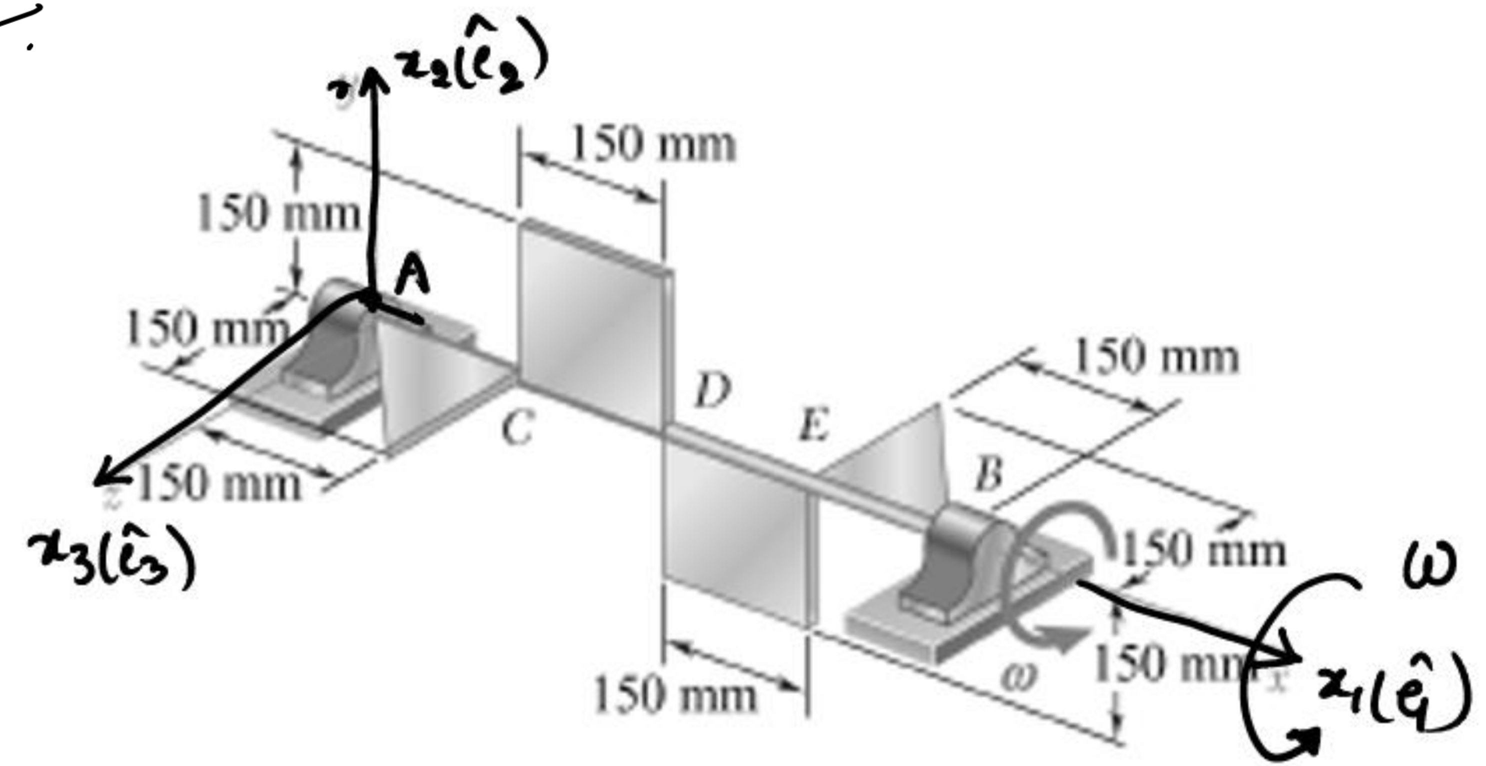
$$\left.\begin{aligned} A_2 + B_2 &= mg \\ A_2 - B_2 &= \frac{m\,\ell h}{24}\,\frac{(-\ell^2 + h^2)}{(\ell^2 + h^2)^{3/2}}\,\dot{\omega} \end{aligned}\right\}$$

Solve to get values of A_2 & B_2

The axial reaction force components along $\hat{e}_3'$, A_3 & B_3 cannot be determined from this analysis.

Q1. Relate the reaction forces and moments of couples at A and B to the the motion of the shaft. Angular velocity of the shaft is constant.

Set 6 B



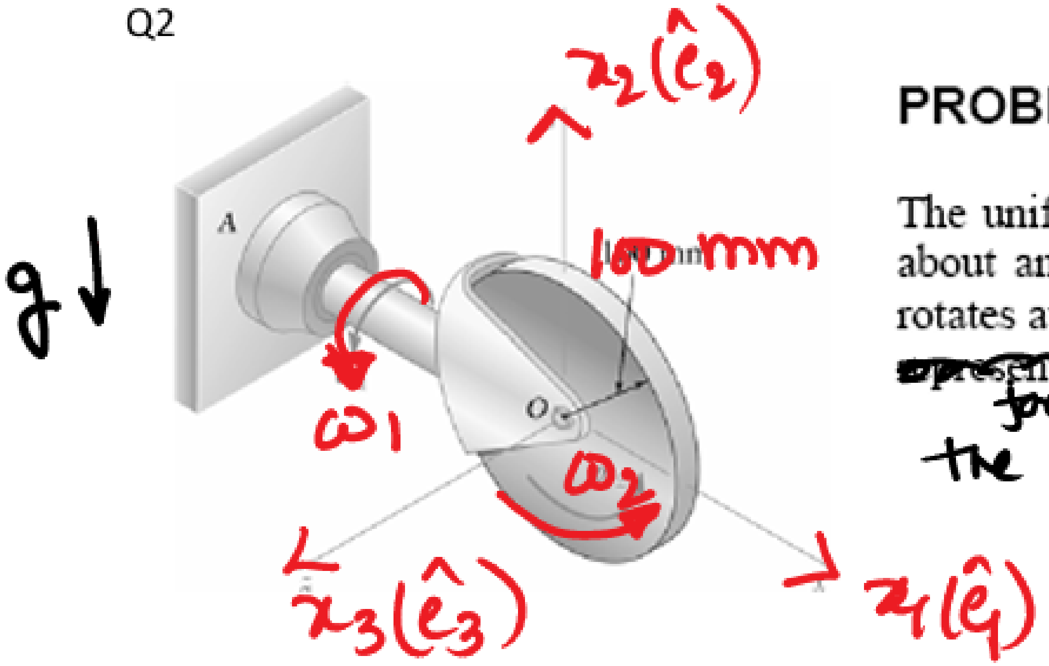
Answer: $A_1 + B_1 = 0$, $A_2 + B_2 = mg$, $A_3 + B_3 = 0$

$$0.3A_3 - 0.3B_3 + C_{A2} + C_{B2} = -I^D_{31}\omega^2$$

$$-0.3A_2 + 0.3B_2 + C_{A3} + C_{B3} = I^D_{21}\omega^2.$$

Set 6B : Q 2 (Type II: problem)

Q2



PROBLEM 18.83 (Beer Johnston)

The uniform thin 2.5-kg disk spins at a constant rate $\omega_2 = 6$ rad/s about an axis held by a housing attached to a horizontal rod that rotates at the constant rate $\omega_1 = 3$ rad/s. ~~Determine the couple which represents the dynamic reaction at the support A.~~ Find the force and moment of couple acting on the rod at A. (AO is massless) (at this instant).

AO = 1m

$\underline{F} = 24.5$ N vertically up

$\underline{C} = -0.225 \text{ Nm}\, \hat{e}_2 + 24.5 \text{ Nm}\, \hat{e}_3$

$O\, x_1(\hat{e}_1)\, x_2(\hat{e}_2)\, x_3(\hat{e}_3)$ is fixed to the disk.