

## Recap

In the last 2-3 lectures, we have been talking a lot about Euler's 2nd axiom

- How it was defined for a point O fixed in the inertial frame 'I'

$$\dot{H}_{O|I} = \underline{M}_O \quad (O \in I) \quad \text{--- } \textcircled{1}$$

- Then we wanted to derive a modified Euler's 2nd axiom about a point A moving (not fixed) in 'I' and we arrived at the relation

$$\frac{d}{dt} \{ H_{A|I} \} \Big|_I = \dot{H}_{A|I} = \underline{M}_A - \underline{\tau}_{CA} \times m \underline{a}_{A|I} \quad \text{--- } \textcircled{2}$$

where C is the COM of the RB

The objective then was to identify ALL such points A for which Euler's 2nd axiom retains the same basic form as  $\textcircled{1}$

$$\dot{H}_{A|I} = \underline{M}_A \quad \text{if and only if}$$

and/or 1)  $A \equiv C \rightarrow \text{COM}$  (  $r_{CA} = 0$  ) → Most general case!

and/or 2)  $\underline{a}_{A|I} = 0 \rightarrow \text{point A is fixed in 'I'}$

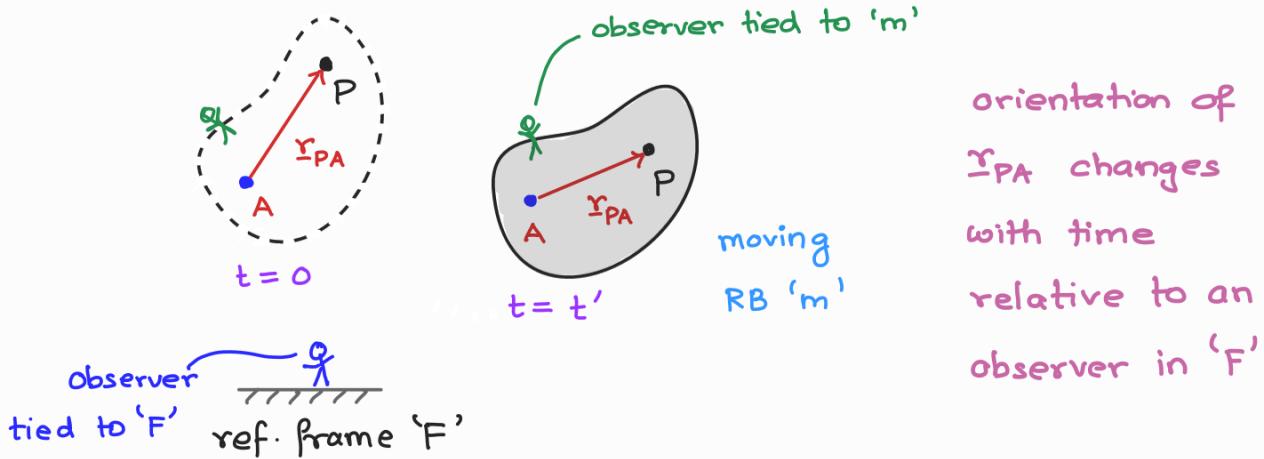
and/or 3)  $\underline{a}_{A|I} \parallel \underline{\tau}_{CA} \rightarrow \text{acc. of A is directed through}$   
 $\text{COM of RB}$

## Variation of Inertia Tensor with time

$$\underline{\underline{I}}^A = \int_m \left\{ (\underline{\underline{r}}_{PA} \cdot \underline{\underline{r}}_{PA}) \underline{\underline{I}} - \underline{\underline{r}}_{PA} \otimes \underline{\underline{r}}_{PA} \right\} dm$$

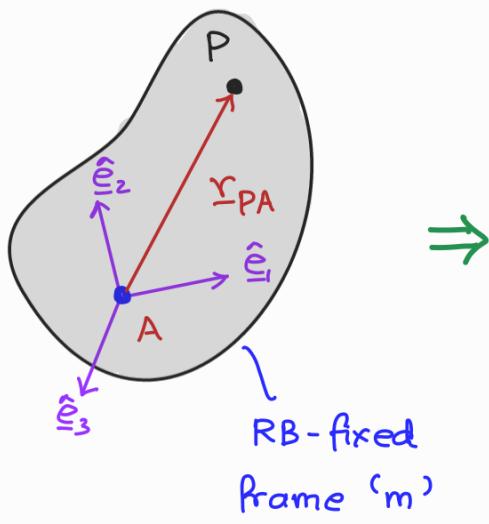
Observe that only the reference point A needs to be fixed on the RB (or its massless extension) for the calculation of  $\underline{\underline{I}}^A$ , but the reference frame 'F' need not be fixed to the RB

- a) Fixing pt A on RB  $\Rightarrow |\underline{r}_{PA}|$  (the magnitude) is constant
- b) But the orientation of  $\underline{\underline{r}}_{PA}$  can still change as the RB rotates relative to frame 'F'



$\Rightarrow$  Inertia Tensor will vary with time relative to frame 'F'

But for a moving ref. frame 'm', which is a frame attached to the RB itself, the inertia tensor  $\underline{\underline{I}}^A$  is a **CONSTANT** tensor. The matrix components of  $\underline{\underline{I}}^A$  will depend only on the choice of orientation of csys associated with the RB-fixed frame 'm'



$\underline{\underline{\underline{\underline{I}}}}^A \rightarrow$  Constant tensor relative  
to RB-fixed frame ' $m$ '

$\therefore \frac{d}{dt} \{\underline{\underline{\underline{\underline{I}}}}^A\}|_m = \underline{\underline{\underline{0}}} \text{ (zero tensor)}$

$\left[ \underline{\underline{\underline{\underline{I}}}}^A \right] \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}$  components will depend  
only on orientation of  
 $\hat{e}_1 - \hat{e}_2 - \hat{e}_3$  in frame ' $m$ '

# Euler's 2nd axiom in terms of Inertia Tensor

Having derived the following:

- 1) Angular momentum abt pt A as a product of inertia tensor at A and angular velocity of RB

$$\underline{H}_{A|F} = \underline{\underline{I}}^A \underline{\omega}_{m|F}$$

replace F with inertial frame 'I'

✓  $\underline{H}_{A|I} = \underline{\underline{I}}^A \underline{\omega}_{m|I} \quad \text{--- } ①$

- 2) Modified Euler's 2nd axiom

$$\frac{d}{dt} \left\{ \underline{H}_{A|I} \right\} \Big|_I = \dot{\underline{H}}_{A|I} = \underline{M}_A - \underline{\underline{\tau}}_{CA} \times m \underline{\alpha}_{A|I}$$

$C \equiv \text{COM of RB}$

our objective is to obtain modified Euler's 2nd axiom (also known as Angular Momentum Balance equation) for an RB using the inertia tensor  $\underline{\underline{I}}^A$ .

Let's consider a point 'A' on the RB where

$$② \quad \dot{\underline{H}}_{A|I} = \underline{M}_A \quad \text{if} \quad \begin{cases} A \equiv C \rightarrow \text{COM} \\ \underline{\alpha}_{A|F} = \underline{0} \\ \underline{\alpha}_{A|I} \text{ is directed through COM} \end{cases}$$

$$\Rightarrow \frac{d}{dt} \left\{ \underline{H}_{A|I} \right\} \Big|_I = \underline{M}_A$$

Recall that the time derivative of a vector in two different frames — fixed frame 'F' and moving frame 'm' — was related:

$$\frac{d \underline{A}}{dt} \Big|_F = \frac{d \underline{A}}{dt} \Big|_m + \underline{\omega}_{m|F} \times \underline{A}$$

'F' and 'm' can also be flipped around:

$$\frac{d \underline{A}}{dt} \Big|_m = \frac{d \underline{A}}{dt} \Big|_F + \underline{\omega}_{F|m} \times \underline{A}$$

Let's write the time derivative of  $\underline{H}_{A|I}$  relative to 'I' using equation ①

Treat  $\underline{A} \equiv \underline{H}_{A|I}$

$$\begin{aligned} \frac{d}{dt} \{ \underline{H}_{A|I} \} \Big|_I &= \underbrace{\frac{d}{dt} \{ \underline{H}_{A|I} \} \Big|_m}_{\text{pink wavy line}} + \underbrace{\underline{\omega}_{m|I} \times \underline{H}_{A|I}}_{\underline{\omega}_{m|I} \times (\underline{I}^A \underline{\omega}_{m|I})} \\ &\quad \curvearrowleft \text{pink curved arrow} \\ \frac{d}{dt} \{ \underline{I}^A \underline{\omega}_{m|I} \} \Big|_m &= \frac{d}{dt} \{ \underline{I}^A \} \Big|_m \underline{\omega}_{m|I} + \underline{I}^A \frac{d}{dt} \{ \underline{\omega}_{m|I} \} \Big|_m \\ &= \underline{\Omega} + \underline{I}^A \left\{ \frac{d}{dt} \{ \underline{\omega}_{m|I} \} \Big|_I + \underline{\omega}_{I|m} \times \underline{\omega}_{m|I} \right\} \\ &\quad - \underline{\omega}_{m|I} \\ &= \underline{\Omega} + \underline{I}^A \dot{\underline{\omega}}_{m|I} - \underline{\omega}_{m|I} \times \underline{\omega}_{m|I} \\ &= \underline{I}^A \dot{\underline{\omega}}_{m|I} \end{aligned}$$

Thus,

$$\dot{\underline{H}}_{A|I} = \underline{\underline{I}}^A \dot{\underline{\omega}}_{m|I} + \underline{\omega}_{m|I} \times (\underline{\underline{I}}^A \underline{\omega}_{m|I})$$

Note: The matrix components  $I_{ij}^A$  stay the same in RB-fixed frame 'm'

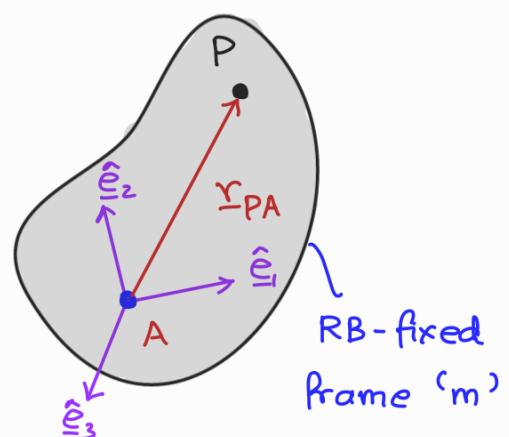
Using equation ②, we can write:

$$\underline{\underline{I}}^A \dot{\underline{\omega}}_{m|I} + \underline{\omega}_{m|I} \times (\underline{\underline{I}}^A \underline{\omega}_{m|I}) = \underline{M}_A$$

Expressing  $\underline{\omega}_{m|I}$  and  $\dot{\underline{\omega}}_{m|I}$  in terms of RB-fixed coordinate system  $\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3$

$$[\underline{\omega}_{m|I}] \begin{pmatrix} \hat{\underline{e}}_1 \\ \hat{\underline{e}}_2 \\ \hat{\underline{e}}_3 \end{pmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

$$\text{or, } \underline{\omega}_{m|I} = \omega_1 \hat{\underline{e}}_1 + \omega_2 \hat{\underline{e}}_2 + \omega_3 \hat{\underline{e}}_3$$



$$[\dot{\underline{\omega}}_{m|I}] \begin{pmatrix} \hat{\underline{e}}_1 \\ \hat{\underline{e}}_2 \\ \hat{\underline{e}}_3 \end{pmatrix} = \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} \quad \text{or, } \dot{\underline{\omega}}_{m|I} = \dot{\omega}_1 \hat{\underline{e}}_1 + \dot{\omega}_2 \hat{\underline{e}}_2 + \dot{\omega}_3 \hat{\underline{e}}_3$$

we get a GENERAL set of three coupled nonlinear (in  $\underline{\omega}$ ) ODEs:

$$\begin{bmatrix} I_{11}^A & I_{12}^A & I_{13}^A \\ I_{21}^A & I_{22}^A & I_{23}^A \\ I_{31}^A & I_{32}^A & I_{33}^A \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} + \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \times \left( \begin{bmatrix} I_{11}^A & I_{12}^A & I_{13}^A \\ I_{21}^A & I_{22}^A & I_{23}^A \\ I_{31}^A & I_{32}^A & I_{33}^A \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \right) = \begin{bmatrix} M_{A,1} \\ M_{A,2} \\ M_{A,3} \end{bmatrix}$$

The above system of ODEs are Euler's equations for rotational motion of a rigid body.

## Simplified cases of Euler's 2nd axiom

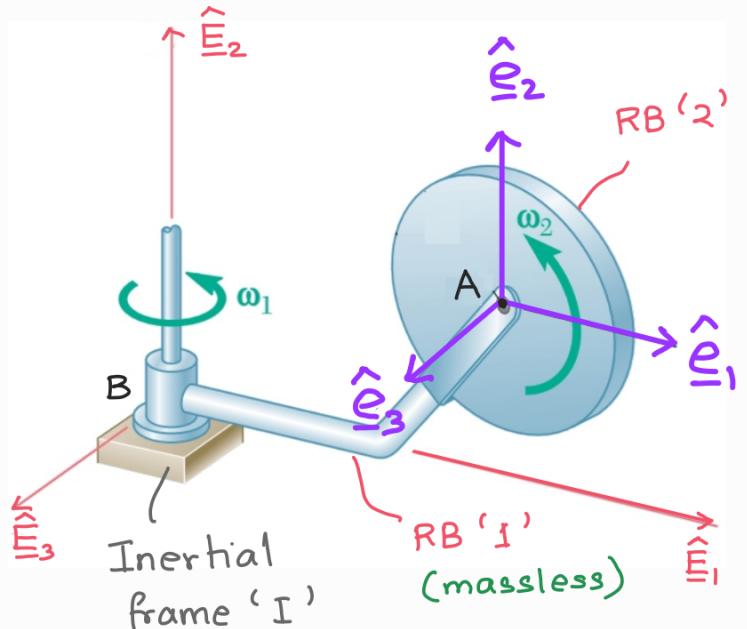
① When  $\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3$  are principal axes at A:

$$\Rightarrow [\underline{\underline{I}}^A]_{(\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3)} = \begin{bmatrix} I_{11}^A & 0 & 0 \\ 0 & I_{22}^A & 0 \\ 0 & 0 & I_{33}^A \end{bmatrix}$$

Note:

- $\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3$  is fixed to the disc (RB '2') in example and rotates with it

Example:



- Point A of the disc coincides with the COM of the disc
- The disc can be thought as a body of revolution with  $\hat{\underline{e}}_3$ -axis being the axis of revolution

Therefore,  $\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3$  axes are also the principal axes and in this case  $I_{11}^A = I_{22}^A$ !

Euler's 2nd axiom (or angular momentum balance) reduces to:

$$\begin{bmatrix} I_{11}^A & 0 & 0 \\ 0 & I_{22}^A & 0 \\ 0 & 0 & I_{33}^A \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} + \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \times \left( \begin{bmatrix} I_{11}^A & 0 & 0 \\ 0 & I_{22}^A & 0 \\ 0 & 0 & I_{33}^A \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \right) = \begin{bmatrix} M_{A,1} \\ M_{A,2} \\ M_{A,3} \end{bmatrix}$$

$$\begin{bmatrix} I_{11}^A & \dot{\omega}_1 \\ I_{22}^A & \dot{\omega}_2 \\ I_{33}^A & \dot{\omega}_3 \end{bmatrix} + \begin{bmatrix} - (I_{22}^A - I_{33}^A) \omega_2 \omega_3 \\ - (I_{33}^A - I_{11}^A) \omega_3 \omega_1 \\ - (I_{11}^A - I_{22}^A) \omega_1 \omega_2 \end{bmatrix} = \begin{bmatrix} M_{A,1} \\ M_{A,2} \\ M_{A,3} \end{bmatrix}$$

Therefore, the simplified Euler's 2nd axiom (when the RB-fixed says coincides with the principal axes triad at pt A) is:

$$M_A = [ I_{11}^A \dot{\omega}_1 - (I_{22}^A - I_{33}^A) \omega_2 \omega_3 ] \hat{e}_1$$

$$+ [ I_{22}^A \dot{\omega}_2 - (I_{33}^A - I_{11}^A) \omega_3 \omega_1 ] \hat{e}_2$$

$$+ [ I_{33}^A \dot{\omega}_3 - (I_{11}^A - I_{22}^A) \omega_1 \omega_2 ] \hat{e}_3$$

$\hat{e}_1 - \hat{e}_2 - \hat{e}_3$

are along  
principal axes

of  $\underline{I}^A$  at A

② Rotation of RB about an RB-fixed axis ( $\underline{\omega}_{m|I} = \omega \hat{\underline{e}}_3$ )  
(say  $\hat{\underline{e}}_3$ )

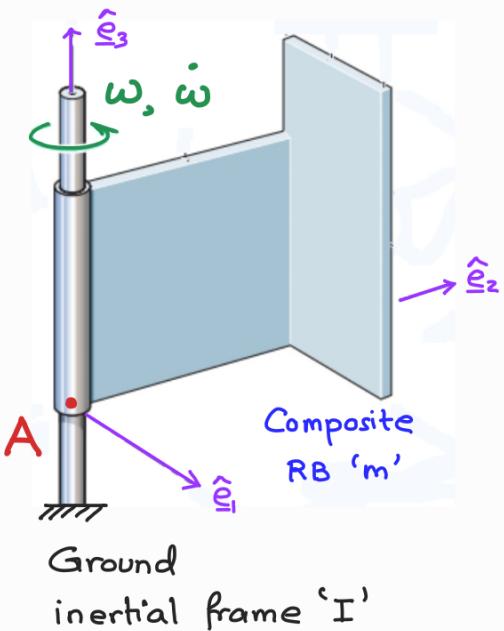
Here, the RB is constrained to rotate about an RB-fixed axis, say  $\hat{\underline{e}}_3$ , so that  $\underline{\omega}_{m|I} = \omega \hat{\underline{e}}_3$  and  $\dot{\underline{\omega}}_{m|I} = \dot{\omega} \hat{\underline{e}}_3$

Choose a fixed origin A at any e.g.

point on the axis of rotation,

which need not pass through the COM of RB

The axes  $\hat{\underline{e}}_1, \hat{\underline{e}}_2, \hat{\underline{e}}_3$  may not necessarily be principal axes



$$[\underline{\omega}_{m|I}] \begin{pmatrix} \hat{\underline{e}}_1 \\ \hat{\underline{e}}_2 \\ \hat{\underline{e}}_3 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \quad \text{or} \quad \underline{\omega}_{m|I} = \omega \hat{\underline{e}}_3$$

and

$$[\dot{\underline{\omega}}_{m|I}] \begin{pmatrix} \hat{\underline{e}}_1 \\ \hat{\underline{e}}_2 \\ \hat{\underline{e}}_3 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\omega} \end{bmatrix} \quad \text{or} \quad \dot{\underline{\omega}}_{m|I} = \dot{\omega} \hat{\underline{e}}_3$$

Euler's 2nd axiom (or angular momentum balance) reduces to:

$$\begin{bmatrix} I_{11}^A & I_{12}^A & I_{13}^A \\ I_{21}^A & I_{22}^A & I_{23}^A \\ I_{31}^A & I_{32}^A & I_{33}^A \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\omega}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \omega_3 \end{bmatrix} \times \left( \begin{bmatrix} I_{11}^A & I_{12}^A & I_{13}^A \\ I_{21}^A & I_{22}^A & I_{23}^A \\ I_{31}^A & I_{32}^A & I_{33}^A \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \omega_3 \end{bmatrix} \right) = \begin{bmatrix} M_{A,1} \\ M_{A,2} \\ M_{A,3} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} I_{13}^A \\ I_{23}^A \\ I_{33}^A \end{bmatrix} \dot{\omega} + \begin{bmatrix} -I_{23}^A \\ I_{13}^A \\ 0 \end{bmatrix} \omega^2 = \begin{bmatrix} M_{A,1} \\ M_{A,2} \\ M_{A,3} \end{bmatrix}$$

$$\Rightarrow M_A = \underbrace{(I_{13}^A \dot{\omega} - I_{23}^A \omega^2)}_{M_{A,1}} \hat{e}_1 + \underbrace{(I_{23}^A \dot{\omega} + I_{13}^A \omega^2)}_{M_{A,2}} \hat{e}_2 + \underbrace{I_{33}^A \dot{\omega}}_{M_{A,3}} \hat{e}_3$$

The above equation holds true for  $A \equiv \text{CoM}$ , or for any A lying on the fixed axis of rotation (in this case  $\hat{e}_3$ )

The axial component,

$$M_{A,3} = I_{33} \dot{\omega}(t)$$

relates the rotational motion to the externally applied torque  $M_{A,3}$  about the fixed axis.

The remaining components

$$M_{A,1} = I_{13} \dot{\omega} - I_{23} \omega^2, \quad M_{A,2} = I_{23} \dot{\omega} + I_{13} \omega^2$$

determine the reaction couples about the RB axes  $\hat{e}_1$  &  $\hat{e}_2$  required to control the rotation about  $\hat{e}_3$ .

Now, also if  $\hat{e}_3$  axis coincides with a principal axis at pt A,

then and only then:

$$I_{13}^A = 0 \quad \text{and} \quad I_{23}^A = 0$$

$$\Rightarrow M_{A,1} = 0 \quad \text{and} \quad M_{A,2} = 0$$

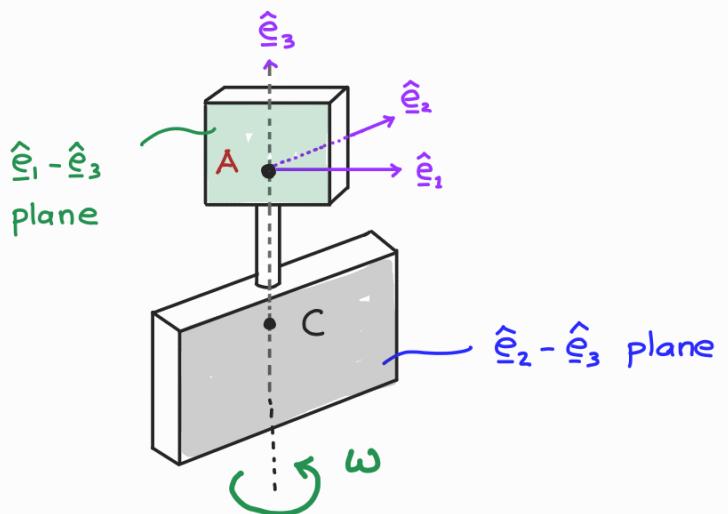
$$\text{and } M_{A,3} = I_{33}^A \dot{\omega}$$

$$\Rightarrow M_A = I_{33}^A \dot{\omega} \hat{e}_3$$

Recall:

$$\begin{bmatrix} I^A \\ \end{bmatrix} = \begin{bmatrix} \checkmark & \checkmark & 0 \\ \checkmark & \checkmark & 0 \\ 0 & 0 & I_{33}^A \end{bmatrix}$$

if  $\hat{e}_3$  is principal axis dir.



body rotates about  
an RB-fixed axis  $\hat{e}_3$ ,  
which also happens to be a  
principal axis in this case