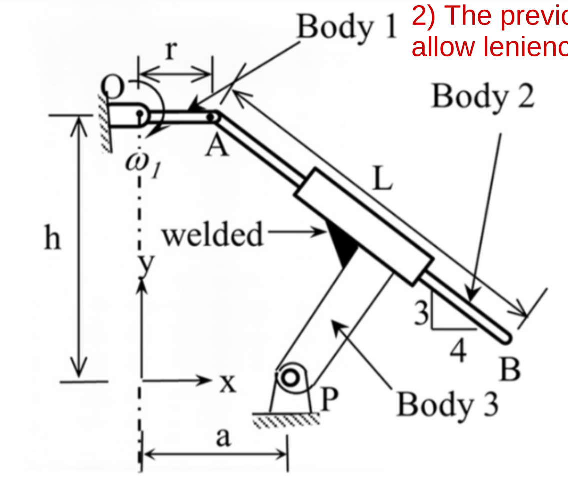
1) The marking of the questions follows the marking scheme provided in the solution.
2) The previously mentioned quantization of marks (0, 2, 4, 6,8) was not implemented, to allow leniency in the evaluation.



Q1) **[8 points]** Rigid rod OA (Body 1) rotates with constant angular velocity of = 2 rad/s (clockwise) with respect to ground when at the shown instant. O, A, and P are pin joints. Rigid rod AB (Body 2) can freely slide through the L-shaped rigid body (Body 3). Find the angular acceleration of body 3 (in $rads^{-1}$) with respect to ground. h = 14cm, a = 6 cm, r = 4 cm, L = 20 cm.

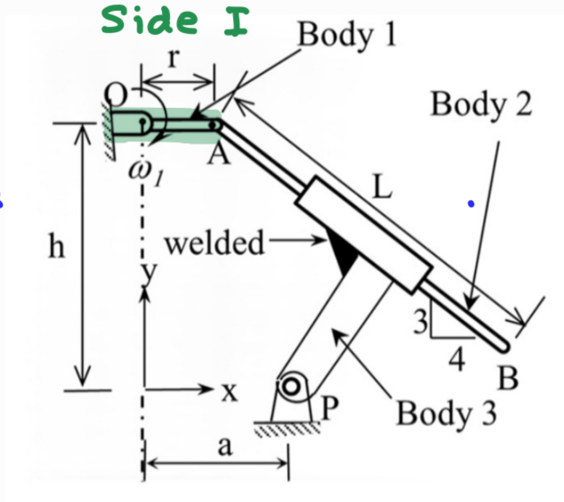
Find $\dot{\omega}_{3|I}$

$$\underline{v}_{A|I} = \underline{v}_{A|3} + \underline{v}_{P|I} + \omega_{3|I} \times \underline{r}_{AP}$$

Solution :

Acceleration of pt A from SIDE (I) :

Side I



$$\underline{a}^{(I)}_{A|I} = -\omega_1^2 \, r \, \hat{\underline{e}}_1$$

$$= -(2)^2(0.04)\,\hat{\underline{e}}_1$$

1

$\omega_1 = 2$ rad/s

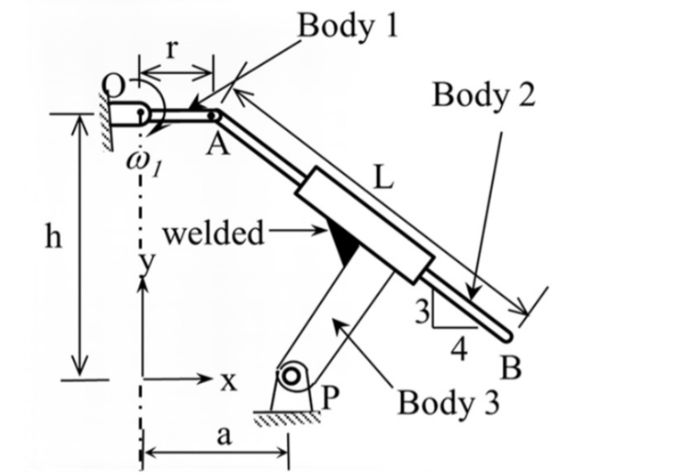
$r = 0.04$ m

Acceleration of pt A from SIDE (II) :

Note that RB 2 is only sliding (and not rotating) w.r.t RB 3 $\Rightarrow$ $\underline{\omega}_{2|3} = \underline{0}$ and $\dot{\underline{\omega}}_{2|3} = \underline{0}$

RB 3 is rotating with an unknown angular velocity & angular acceleration, $\underline{\omega}_{3|I}$ and $\dot{\underline{\omega}}_{3|I}$, respectively

unknown

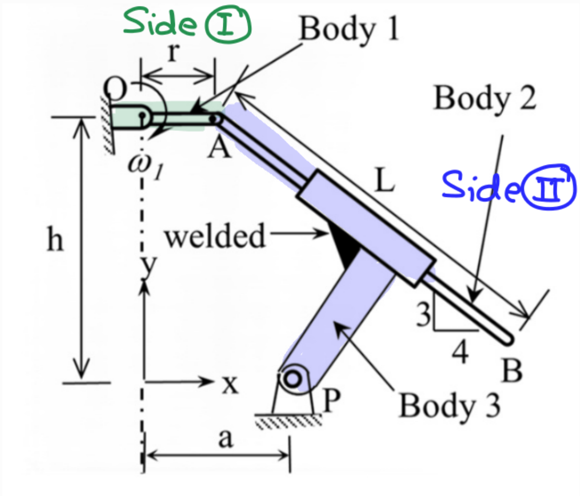


Q1) [**8 points**] Rigid rod OA (Body 1) rotates with constant angular velocity of = 2 rad/s (clockwise) with respect to ground when at the shown instant. O, A, and P are pin joints. Rigid rod AB (Body 2) can freely slide through the L-shaped rigid body (Body 3). Find the angular acceleration of body 3 (in $rads^{-1}$) with respect to ground. h = 14cm, a = 6 cm, r = 4 cm, L = 20 cm.

Find $\dot{\omega}_{3|I}$

Solution: We will need to know the velocity of rod AB (RB 2) w.r.t. RB 3 as well as the angular velocity of RB 3

To get these two unknowns $\underline{v}_{2|3}$ and $\underline{\omega}_{3|I}$, we will employ velocity matching at point A from two sides



Velocity of A from side (I) :

$$\underline{\omega}_1 = -2\,\hat{\underline{e}}_3$$

$$\underline{r}_{AO} = 0.04\,\hat{\underline{e}}_1$$

$$\Rightarrow \underline{v}^{(I)}_{A|I} = \underline{\omega}_1 \times \underline{r}_{AO}$$

$$= -0.08\,\hat{\underline{e}}_2$$

Velocity of point A from side (II) :

RB 2 is sliding (only translating) w.r.t RB 3, so pt A has

$\Rightarrow \underline{v}_{A|3} = v_{AB}\,\hat{\underline{e}}_{AB}$ where $\hat{\underline{e}}_{AB}$ ≡ unit normal vector directed from A to B

$\hat{e}_{AB}$ is obtained from the slope information

$$\hat{e}_{AB} = \frac{4}{5}\hat{e}_1 - \frac{3}{5}\hat{e}_2 = 0.8\,\hat{e}_1 - 0.6\,\hat{e}_2$$

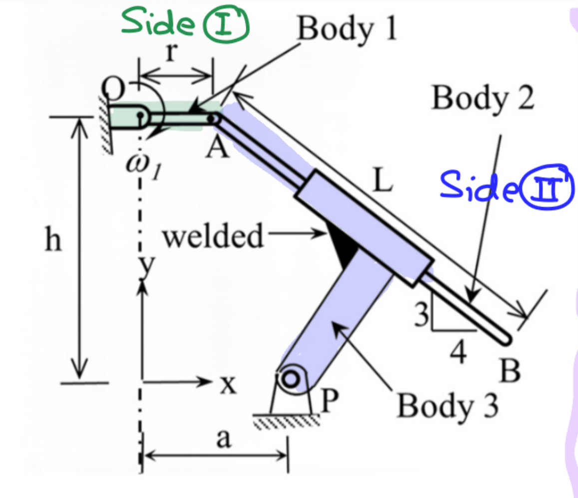
①

RB 3 is only rotating in a planar motion

$$\Rightarrow \quad \underline{\omega}_{3|I} = \omega_3\,\hat{e}_3$$

Now, using '$m \equiv$ RB 3' and I.P. $\equiv$ pt 'P'

$$\underline{v}^{(II)}_{A|I} = \underbrace{\cancel{\underline{v}_{P|I}}}_{0} + \underbrace{\cancel{\underline{v}_{A|3}}}_{v_{AB}\,\hat{e}_{AB}} + \underline{\omega}_{3|I} \times \underline{r}_{AP}$$



$\underline{r}_{AP}$

$$= 0.04\,\hat{e}_1 + 0.14\,\hat{e}_2 - 0.06\,\hat{e}_1$$

$$= -0.02\,\hat{e}_1 + 0.14\,\hat{e}_2$$

$$\underline{v}^{(II)}_{A|I} = 0.8\,v_{AB}\,\hat{e}_1 - 0.6\,v_{AB}\,\hat{e}_2 + \omega_3\,\hat{e}_3 \times (-0.02\,\hat{e}_1 + 0.14\,\hat{e}_2)$$

$$= (0.8\,v_{AB} - 0.14\,\omega_3)\,\hat{e}_1 + (-0.6\,v_{AB} - 0.02\,\omega_3)\,\hat{e}_2$$

Matching the velocities of pt A from two sides:

$$v^{(I)}_{A|I} = v^{(II)}_{A|I}$$

②

$$\Rightarrow \quad -0.08\,\hat{e}_2 = (0.8\,v_{AB} - 0.14\,\omega_3)\,\hat{e}_1 + (-0.6\,v_{AB} - 0.02\,\omega_3)\,\hat{e}_2$$

$$\Rightarrow \quad \left.\begin{aligned} 0.8\, v_{AB} - 0.14\, \omega_3 &= 0 \\ -0.6\, v_{AB} - 0.02\, \omega_3 &= -0.08 \end{aligned}\right\} \underset{\Longrightarrow}{\text{Solve}} \quad \begin{aligned} \omega_3 &= 0.64 \text{ rad/s} \\ V_{AB} &= 0.112 \text{ m/s} \end{aligned}$$

Analysis of acceleration by matching at point A

Acceleration of pt A from SIDE (I) :

$$\underline{a}^{(I)}_{A|I} = -\omega_1^2\, r\, \hat{\underline{e}}_1$$

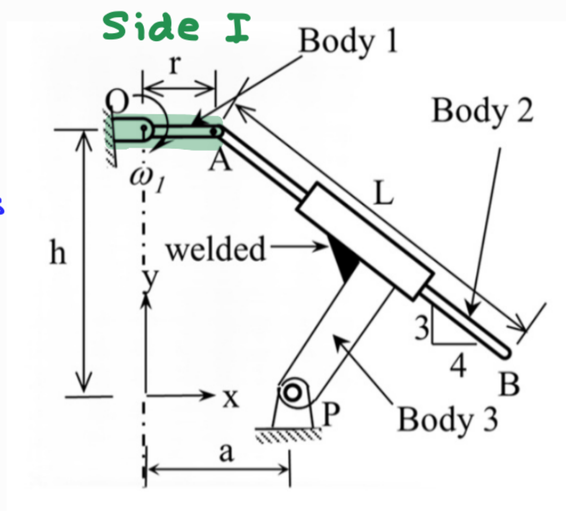
$$= -(2)^2(0.04)\, \hat{\underline{e}}_1$$

$$= -0.16\, \hat{\underline{e}}_1$$

$\omega_1 = 2$ rad/s

$r = 0.04$ m

1



Acceleration of pt A from SIDE (II) :

Note that RB 2 is only sliding (and not rotating) w.r.t RB 3 $\Rightarrow$ $\underline{\omega}_{2|3} = \underline{0}$ and $\underline{\dot{\omega}}_{2|3} = \underline{0}$

RB 3 is rotating with an unknown angular velocity & angular acceleration, $\underline{\omega}_{3|I}$ ✓ and $\underline{\dot{\omega}}_{3|I}$, respectively

Unknown

$$\underline{\omega}_{3|I} = 0.64\, \hat{\underline{e}}_3$$

Choosing 'm' ≡ 'RB3' and I.P ≡ 'P'

Unknown $= a_{AB}\,\hat{\underline{e}}_{AB}$

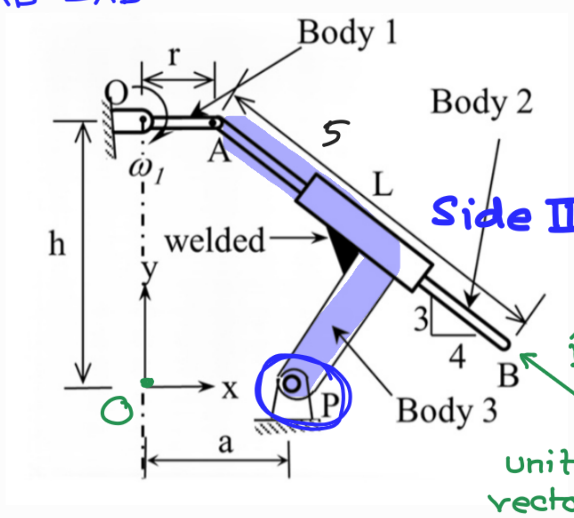
$$\underline{a}_{A|I}^{\text{II}} = \cancel{\underline{a}_{P|I}}^{\,0} + \boxed{\underline{a}_{A|3}} + \cancel{\underline{a}_{P|3}}^{\,0}$$

Unknown $\dot{\omega}_3\,\hat{\underline{e}}_3$

$$+\ \dot{\underline{\omega}}_{3|I} \times \underline{r}_{AP}$$

$$+\ \underbrace{\underline{\omega}_{3|I} \times \left(\underline{\omega}_{3|I} \times \underline{r}_{AP}\right)}_{\omega_3^2\,\underline{r}_{AP}}$$

$$+\ 2\,\underline{\omega}_{3|I} \times \underline{v}_{A|3}$$



unit vector along AB

$$= a_{AB}\left(0.8\,\hat{\underline{e}}_1 - 0.6\,\hat{\underline{e}}_2\right) + \dot{\omega}_3\,\hat{\underline{e}}_3 \times \left(-0.02\,\hat{\underline{e}}_1 + 0.14\,\hat{\underline{e}}_2\right)$$

$$+\ 0.64\,\hat{\underline{e}}_3 \times \left[0.64\,\hat{\underline{e}}_3 \times \left(-0.02\,\hat{\underline{e}}_1 + 0.14\,\hat{\underline{e}}_2\right)\right]$$

$$+\ 2\left(0.64\,\hat{\underline{e}}_3\right) \times (0.112)\left(0.8\,\hat{\underline{e}}_1 - 0.6\,\hat{\underline{e}}_2\right)$$

$$= 0.8\,a_{AB}\,\hat{\underline{e}}_1 - 0.6\,a_{AB}\,\hat{\underline{e}}_2 - 0.14\,\dot{\omega}_3\,\hat{\underline{e}}_1 - 0.02\,\dot{\omega}_3\,\hat{\underline{e}}_2$$

$$+\ 0.0082\,\hat{\underline{e}}_1 - 0.0573\,\hat{\underline{e}}_2$$

$$+\ 0.086\,\hat{\underline{e}}_1 + 0.1147\,\hat{\underline{e}}_2 \qquad (2)$$

$$= \left(0.8\,a_{AB} - 0.14\,\dot{\omega}_3 + 0.0082 + 0.086\right)\hat{\underline{e}}_1$$

$$+\ \left(-0.6\,a_{AB} - 0.02\,\dot{\omega}_3 - 0.0573 + 0.1147\right)\hat{\underline{e}}_2$$

Match the acceleration of pt A from both sides:

$$\underline{a}^{\mathrm{I}}_{A|I} = \underline{a}^{\mathrm{II}}_{A|I}$$

1

$$\Rightarrow -0.16\,\hat{\underline{e}}_1 = (0.8\,a_{AB} - 0.14\,\dot{\omega}_3 + 0.0942)\,\hat{\underline{e}}_1 + (-0.6\,a_{AB} - 0.02\,\dot{\omega}_3 + 0.0574)\,\hat{\underline{e}}_2$$

$$\Rightarrow \left.\begin{array}{l} 0.8\,a_{AB} - 0.14\,\dot{\omega}_3 = -0.254 \\ 0.6\,a_{AB} + 0.02\,\dot{\omega}_3 = 0.0574 \end{array}\right\} \underset{\Rightarrow}{\text{Solve}} \quad \begin{array}{l} a_{AB} = 0.0296\ m/s^2 \\ \dot{\omega}_3 = 1.98\ rad/s^2 \end{array}$$

1

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1) Velocity analysis & matching — 2

2) Velocity of A w.r.t RB 3 and unit normal along AB — 1

3) Acceleration of A from side I — 1

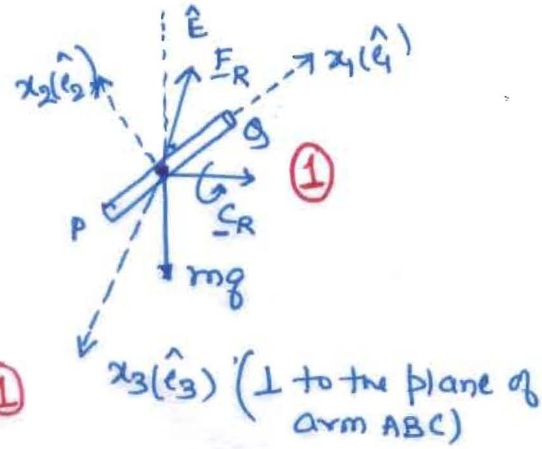
4) Acceleration of A from side II — 2

5) Acceleration matching I ≡ II — 1

6) Correct final answer $\dot{\omega}_3 = 0.98\ rad/s^2$ — 1

Q2

FBD of rod & working CSYS



~~Modified E~~

(2) rod PQ
(1) arm ABC
(I) ground.

$$\underline{\omega}_{2|I} = \underline{\omega}_{2|1} + \underline{\omega}_{1|I}$$

$$= \dot{\theta}\hat{e}_3 + \omega\cos\theta\,\hat{e}_2 + \omega\sin\theta\,\hat{e}_1$$

$$\Rightarrow \boxed{(\omega_1, \omega_2, \omega_3) = (\omega\sin\theta, \omega\cos\theta, \dot{\theta})} \text{ (i)} \quad (1)$$

$$\dot{\underline{\omega}}_{2|I} = \dot{\underline{\omega}}_{2|1} + \dot{\underline{\omega}}_{1|I} + \underline{\omega}_{1|I} \times \underline{\omega}_{2|1}$$

$$= \ddot{\theta}\hat{e}_3 + (\dot{\omega}\cos\theta\,\hat{e}_2 + \dot{\omega}\sin\theta\,\hat{e}_1) + (\omega\cos\theta\,\hat{e}_2 + \omega\sin\theta)\hat{e}_1 \times \dot{\theta}\hat{e}_3$$

$$\dot{\underline{\omega}}_{2|I} = \hat{e}_1(\dot{\omega}\sin\theta + \omega\dot{\theta}\cos\theta) + \hat{e}_2(\dot{\omega}\cos\theta - \omega\dot{\theta}\sin\theta)\hat{e}_2 + \ddot{\theta}\hat{e}_3.$$

$$\Rightarrow \boxed{(\dot{\omega}_1, \dot{\omega}_2, \dot{\omega}_3) = (\dot{\omega}\sin\theta + \omega\dot{\theta}\cos\theta,\ \dot{\omega}\cos\theta - \omega\dot{\theta}\sin\theta,\ \ddot{\theta})} \text{ (ii)} \quad (1)$$

Modified Euler's axiom for body-fixed CSYS with axes = p-axes of body.
Use formula sheet (about C: valid point)

$$M_{C3} = \underline{C}_R \cdot \hat{e}_3 = I^C_{33}\dot{\omega}_3 - (I^C_{11} - I^C_{22})\omega_1\omega_2 \quad \text{(iii)}$$

$mg, \underline{F}_R$ don't contribute.

$I^C_{11} = 0,\ I^C_{22} = \frac{mL^2}{12},\ I^C_{33} = mL^2/12$

Use (i) & (ii) and $\underline{C}_R \cdot \hat{e}_3 = 0$ (given)

$$\Rightarrow 0 = \frac{mL^2}{12}\ddot{\theta} - \left(0 - \frac{mL^2}{12}\right)\omega^2\sin\theta\cos\theta \Rightarrow \boxed{\ddot{\theta} = -\omega^2\sin\theta\cos\theta} \quad (2)$$

$$M_{C1} = (\underline{C}_R \cdot \hat{e}_1) = I^C_{11}\dot{\omega}_1 - (I^C_{22} - I^C_{33})\omega_2\omega_3$$

$$\Rightarrow (\underline{C}_R)_1 = (0)\dot{\omega}_1 - \left(\frac{mL^2}{12} - \frac{mL^2}{12}\right)\omega_2\omega_3 = 0 \Rightarrow C_{R1} = 0$$

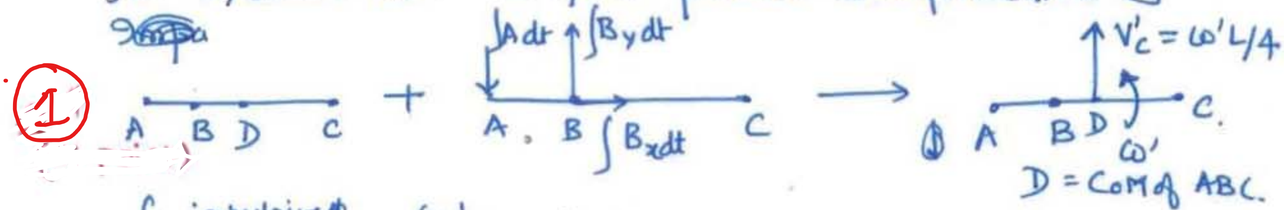
$$M_{C2} = (\underline{C}_R \cdot \hat{e}_2) = I^C_{22}\dot{\omega}_2 - (I^C_{33} - I^C_{11})\omega_3\omega_1$$

$$\text{or } (\underline{C}_R)_2 = \frac{mL^2}{12}(\dot{\omega}\cos\theta - \omega\dot{\theta}\sin\theta) - \left(\frac{mL^2}{12} - 0\right)\dot{\theta}\,\omega\sin\theta.$$

$$\text{or } (\underline{C}_R)_2 = \frac{mL^2}{12}\left\{\dot{\omega}\cos\theta - 2\omega\dot{\theta}\sin\theta\right\}$$

$$\therefore \boxed{|\underline{C}_R| = |(\underline{C}_R)_2| = \frac{mL^2}{12}\left|\dot{\omega}\cos\theta - 2\omega\dot{\theta}\sin\theta\right|} \quad (1)$$

Q3: System = rod: Impact process is represented as



$$\int \underline{M}_B^{impulsive}\,dt = (\underline{H}'_B)_3 - (\underline{H}_B)_3$$ B: "valid point"

1 $\Rightarrow \frac{L}{4}\int A\,dt = I_{33}^B\,\omega'$ (i), $I_{33}^B = I_{33}^D + M\left(\frac{L}{4}\right)^2 = \frac{7ML^2}{48}$ (ii) 1 M: mass of ABC

System = sphere: Impact process is represented as.

$\Rightarrow \int A\,dt = m\left(-v_1' - (-v_1)\right) = m(v_1 - v_1') \rightarrow$ (iii) 1

m: mass sphere

Eliminate $\int A\,dt$ from (i), (iii) $\Rightarrow$ $I_{33}^B\,\omega' = (v_1 - v_1')\frac{mL}{4}$ (iv)

Co-eff. of restitution:

1 $v_A' - v_1' = e(v_1 - \cancel{v_A})$ $\Rightarrow$ $\frac{\omega' L}{4} - v_1' = e v_1 \rightarrow$ (v)

(iv) & (v) have 2 unknowns: ω', v_1': Solve.

$$v_1' = \frac{-v_1\left(\frac{4e}{L} - \frac{12m}{7ML}\right)}{\left[\frac{4}{L} + \frac{12m}{7ML}\right]} \qquad \omega' = (e v_1 + v_1')\frac{4}{L}.$$

Set $v_1 = 4\,ms^{-1}$, $e = 0.5$, $m = 0.8$ kg, $M = 3.2$ kg, $L = .75$ m.

1 $\boxed{v_1' = -1.42\,ms^{-1} \;\&\; \omega' = 3.1\,rad\,s^{-1}}$