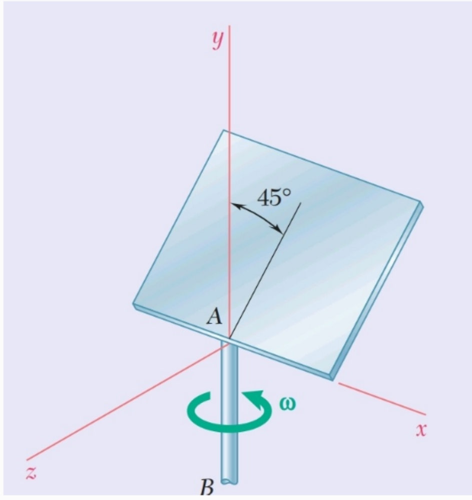
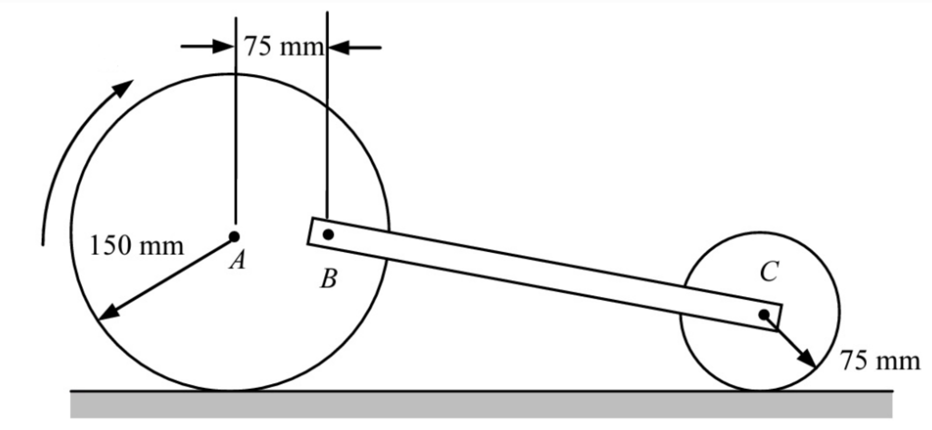
Tutorial Set 8

(Part A)

1) Determine the kinetic energy of the thin, homogeneous square plate of mass 'm' and side 'a' welded to a vertical shaft AB with which it forms a 45° angle.



2) The 5-kg rod BC is attached by pins to two uniform disks as shown. The mass of the 150 mm radius disk is 6 kg and that of the 75 mm radius is 1.5 kg. Knowing that the system is released from rest in the position shown, determine the velocity of the rod after disk A has rotated through 90°. Assume that disks roll without slip



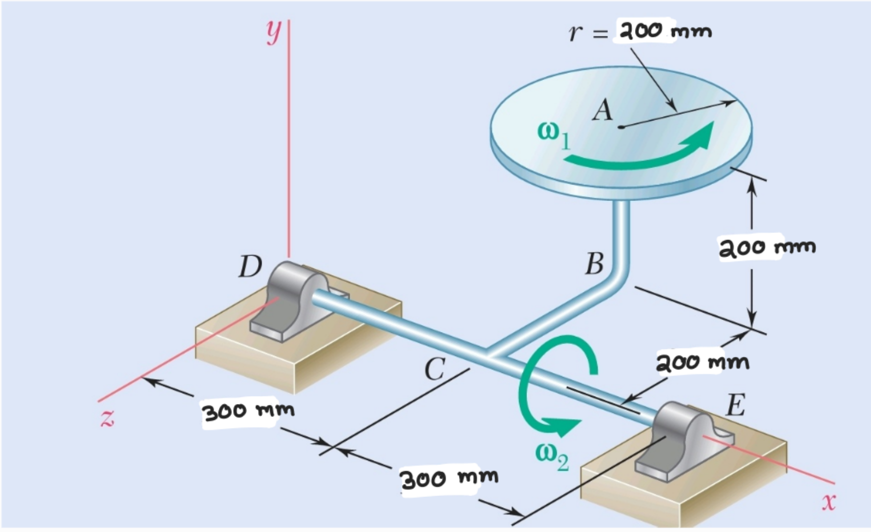
Part B

1) Determine kinetic energy of the disk

Mass = 3.92 kg

ω_1 = 16 rad/s

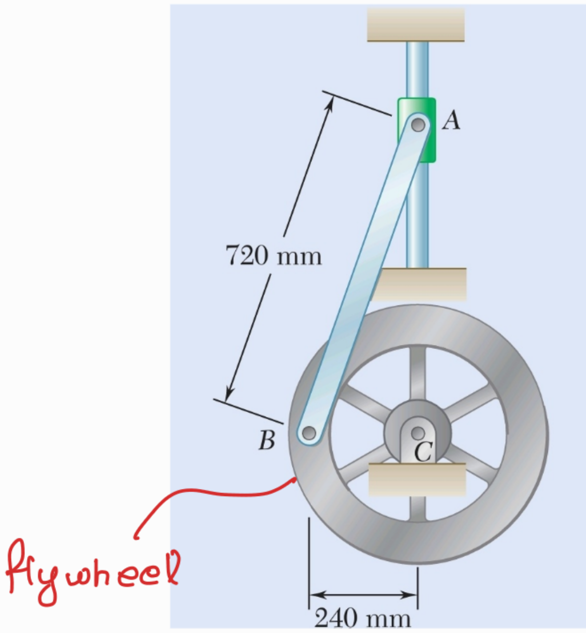
ω_2 = 8 rad/s



Ans: T = 16.32 Nm

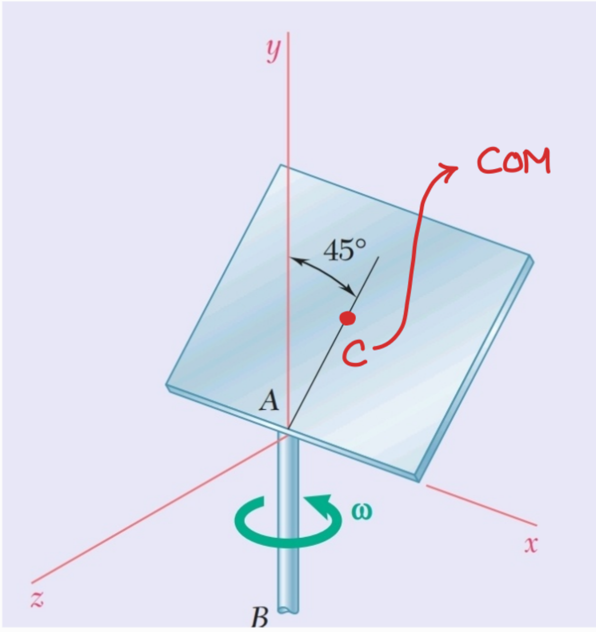
2) The 4-kg rod *AB* is attached to a collar of negligible mass at *A* and to a flywheel at *B*. The flywheel has a mass of 16 kg and a radius of gyration of 180 mm. Knowing that in the position shown the angular velocity of the flywheel is 60 rpm clockwise, determine the angular velocity of the flywheel when point *B* is directly below *C*.

Ans: ω = 84.7 rad/s



Part A solutions

1) Find kinetic energy of plate



Recall that computation of KE requires careful selection of a point on the body for which the calculation becomes easy.

Two choices for computing KE

$T|_F$

pt A has zero linear velocity, $\underline{v}_{A|F} = \underline{0}$

$\Rightarrow$ can use $T|_F = \frac{1}{2}\, \underline{\omega}_{m|F}\, \underline{\underline{I}}^A\, \underline{\omega}_{m|F}$

Calculation of $\underline{\underline{I}}^A$ needs the use of parallel axes thm to transfer from C to A

pt $C \equiv$ COM

$\Rightarrow$ use $T|_F = \frac{1}{2} m\, \underline{v}_{C|F} \cdot \underline{v}_{C|F}$

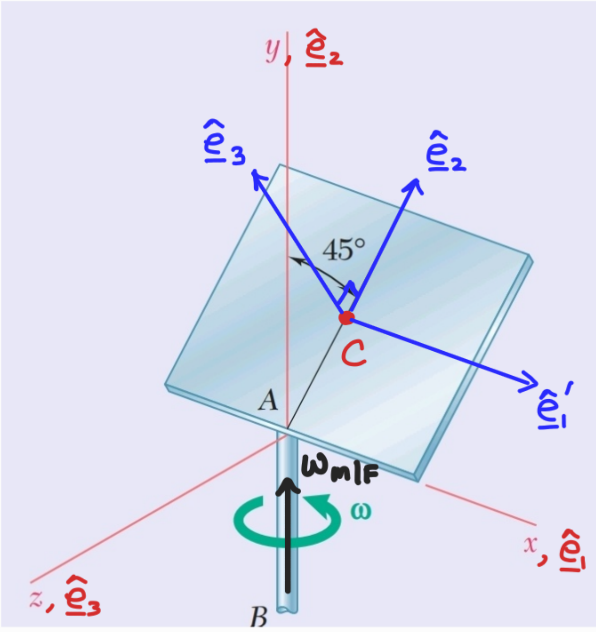
$+ \frac{1}{2}\, \underline{\omega}_{m|F}\, \underline{\underline{I}}^C\, \underline{\omega}_{m|F}$

$\underline{\underline{I}}_C$ can be easily compute but $\underline{v}_{C|F}$ needs to be calculated as well

For the csys given in the figure,

$[\underline{\underline{I}}^C]_{\begin{pmatrix}\hat{e}_1\\ \hat{e}_2\\ \hat{e}_3\end{pmatrix}}$ is not easily calculated.

as they donot coincide with the principal directions of the plate $\hat{e}_1' - \hat{e}_2' - \hat{e}_3'$



$T_{|F}$ computation using C as base point

in $\hat{e}_1 - \hat{e}_2 - \hat{e}_3$ csys

$$[\underline{\omega}_{m|F}] = \begin{bmatrix} 0 \\ \omega \\ 0 \end{bmatrix}$$

$$[\underline{\underline{I}}^C] = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

Get $[\underline{\underline{I}}^C]_{\begin{pmatrix}\hat{e}_1'\\ \hat{e}_2'\\ \hat{e}_3'\end{pmatrix}}$ in prin. csys $\hat{e}_1'-\hat{e}_2'-\hat{e}_3'$

and then use transformation rule

$$[\underline{\underline{I}}^C]_{\begin{pmatrix}\hat{e}_1\\ \hat{e}_2\\ \hat{e}_3\end{pmatrix}} = [\underline{A}]\,[\underline{\underline{I}}^C]_{\begin{pmatrix}\hat{e}_1'\\ \hat{e}_2'\\ \hat{e}_3'\end{pmatrix}}\,[\underline{A}]^T$$

need to calculate $[\underline{A}]$

in principal csys $\hat{e}_1' - \hat{e}_2' - \hat{e}_3'$

$$[\underline{\underline{I}}^C] = \begin{bmatrix} \frac{ma^2}{12} & 0 & 0 \\ 0 & \frac{ma^2}{12} & 0 \\ 0 & 0 & \frac{ma^2}{6} \end{bmatrix}$$

$$[\underline{\omega}_{m|F}] = \begin{bmatrix} 0 \\ \omega\cos 45^\circ \\ \omega\sin 45^\circ \end{bmatrix}$$

Working in the principal csys at COM is easier!

Compute $\underline{v}_{C|F}$ using velocity transfer relations

$$\underline{v}_{C|F} = \cancel{\underline{v}_{A|F}}^{\,0} + \underline{\omega}_{m|F} \times \underline{r}_{CA}$$

$$= \left(\omega/\sqrt{2}\ \hat{\underline{e}}_2' + \omega/\sqrt{2}\ \hat{\underline{e}}_3'\right) \times a/2\ \hat{\underline{e}}_2'$$

$$= -a\omega/2\sqrt{2}\ \hat{\underline{e}}_1'$$

Finally compute KE using C as base pt (using $\hat{\underline{e}}_1'-\hat{\underline{e}}_2'-\hat{\underline{e}}_3'$)

$$T|_F = \frac{1}{2} m\, \underline{v}_{C|F} \cdot \underline{v}_{C|F} + \frac{1}{2} \underline{\omega}_{m|F} \cdot \underline{\underline{I}}^C \underline{\omega}_{m|F}$$

$$= \frac{1}{2} m \left[\underline{v}_{C|F}\right]^T \left[\underline{v}_{C|F}\right] + \frac{1}{2} \left[\underline{\omega}_{m|F}\right] \left[\underline{\underline{I}}^C\right] \left[\underline{\omega}_{m|F}\right]$$

$$= \frac{m}{2} \cdot \frac{a^2\omega^2}{8} + \frac{1}{2} \begin{bmatrix} 0 \\ \omega/\sqrt{2} \\ \omega/\sqrt{2} \end{bmatrix}^T \begin{bmatrix} \frac{ma^2}{12} & 0 & 0 \\ 0 & \frac{ma^2}{12} & 0 \\ 0 & 0 & \frac{ma^2}{6} \end{bmatrix} \begin{bmatrix} 0 \\ \omega/\sqrt{2} \\ \omega/\sqrt{2} \end{bmatrix}$$

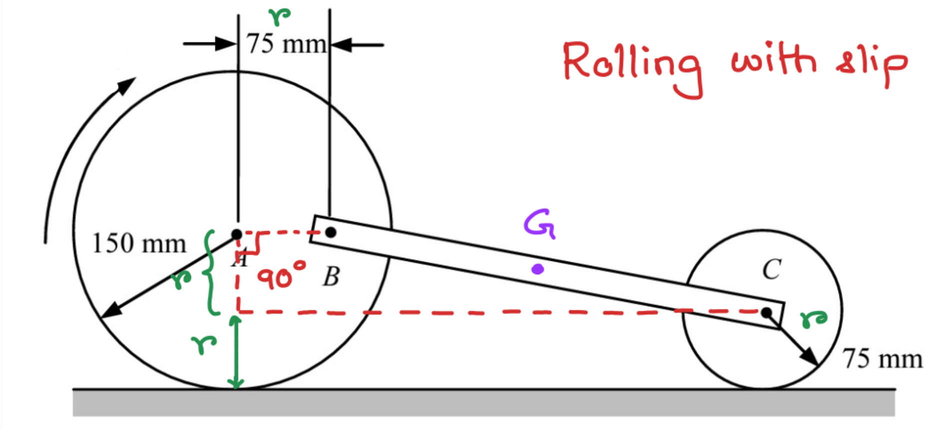
$$= \frac{ma^2\omega^2}{16} + \frac{1}{2}\left[\frac{\omega^2}{2} \cdot \frac{ma^2}{12} + \frac{\omega^2}{2} \cdot \frac{ma^2}{6}\right]$$

$$= \frac{ma^2\omega^2}{16} + \frac{3ma^2\omega^2}{48}$$

$$= \frac{ma^2\omega^2}{8}$$

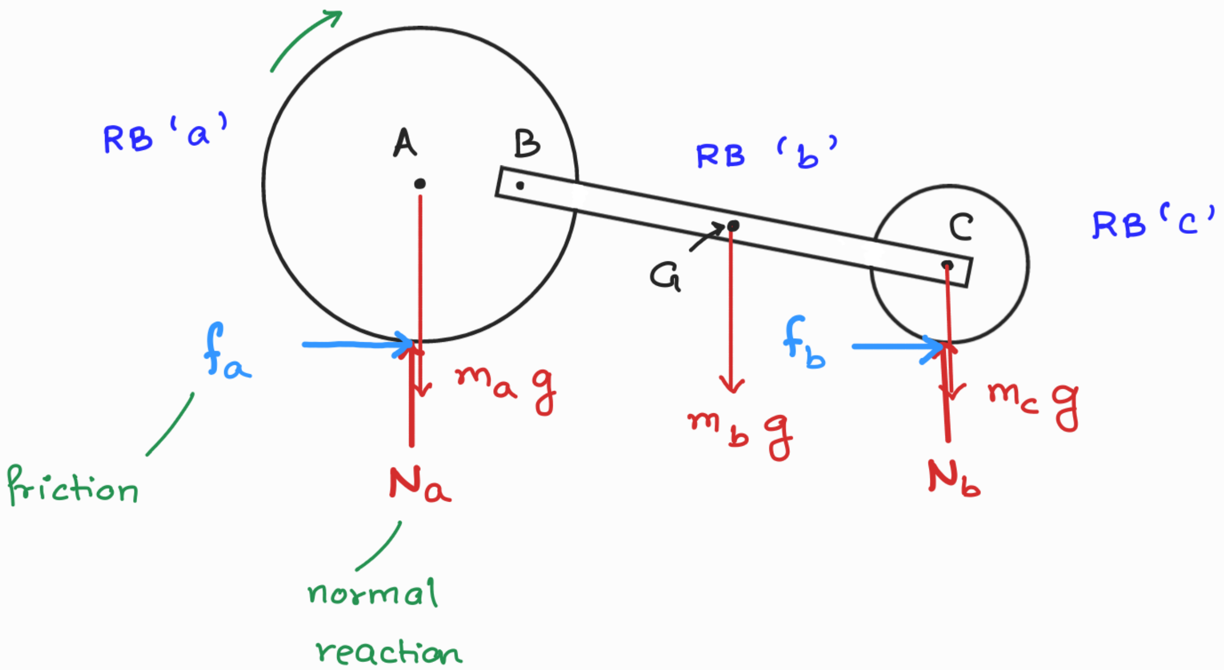
$$\boxed{T|_F = \frac{ma^2\omega^2}{8}}$$

2) Determine velocity of rod BC after disk has rotated by 90°



- Since there are Two positions of rod BC : initial – 1 and final – 2 , and we are interested in knowing the velocity of rod, we use the work-energy principle.

- Consider system = rod + two disks , and draw FBD for the system The work done by internal forces are equal and opposite and hence cancel out. We therefore consider only forces/moments external to the "system"



Work energy principle $\Rightarrow$ $\underset{\sim\sim\sim}{W_{1\to 2}} = \Delta T = T_{2|F} - T_{1|F}$

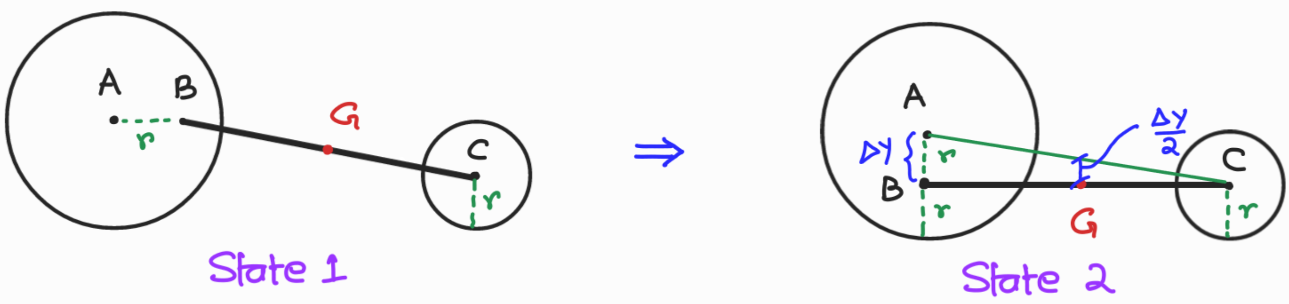
Work done by ext. forces in moving the body from state 1 → 2

1) Frictional forces f_a and f_b do zero work as velocities at the no-slip contact points are zero

2) The normal reactions N_a and N_b are a part of reaction force system, and reaction force systems do not any work as they arise due to constraints in motion.

3) The forces that do non-zero work in this problem are the weights due to gravity (which are constant forces, hence also conservative forces) : $m_a g$, $m_b g$, and $m_c g$

Work done by weights due to gravity



Work done $= m_b g \cdot \dfrac{\Delta y}{2}$ ← vertical disp of G going from state 1 to state 2

$W_{1\to 2}$

$= m_b g \cdot \dfrac{r}{2}$

According to work-energy principle,

KE at state 2

$$W_{1\to 2} = \Delta T = T_2 - T_1$$ — KE at state 1

$$\Rightarrow \; m_b\, g \cdot r = \left(T^a + T^b + T^c\right)_2 - \cancel{\left(T^a + T^b + T^c\right)_1}^{\,0}$$

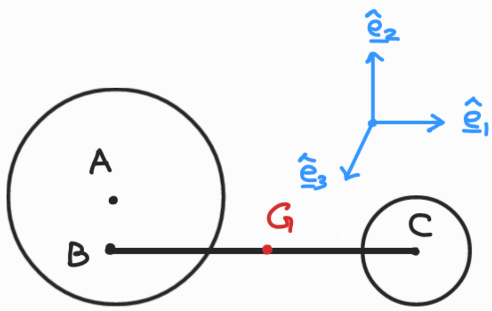
("system" was at rest initially)

$$\Rightarrow \; \left(T^a + T^b + T^c\right)_2 = m_b\, g \cdot r \quad \text{—} \; (**)$$

Kinetic energy of the system at state 2

$$(T^a)_2 = ? \qquad (T^b)_2 = ? \qquad (T^c)_2 = ?$$

Using $\hat{e}_1 - \hat{e}_2 - \hat{e}_3$ csys for calculation



State 2

$$(T^a)_2 = \frac{1}{2} m_a \left[\underline{v}_{A|I}\right]^T \left[\underline{v}_{A|I}\right] + \frac{1}{2} \left[\underline{\omega}_{a|I}\right]^T \left[\underline{\underline{I}}^A\right] \left[\underline{\omega}_{a|I}\right]$$

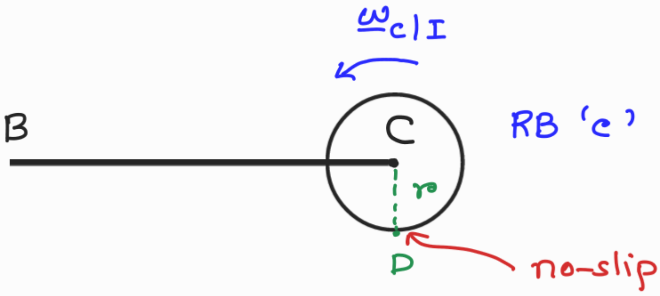
$$(T^b)_2 = \frac{1}{2} m_b \left[\underline{v}_{a|I}\right]^T \left[\underline{v}_{a|I}\right] + \frac{1}{2} \left[\underline{\omega}_{b|I}\right]^T \left[\underline{\underline{I}}^a\right] \left[\underline{\omega}_{b|I}\right]$$

$$(T^c)_2 = \frac{1}{2} m \left[\underline{v}_{c|I}\right]^T \left[\underline{v}_{c|I}\right] + \frac{1}{2} \left[\underline{\omega}_{c|I}\right]^T \left[\underline{\underline{I}}^c\right] \left[\underline{\omega}_{c|I}\right]$$

Use kinematics to relate $\underline{v}_{a|I}$, $\underline{v}_{A|I}$, and $\underline{v}_{c|I}$

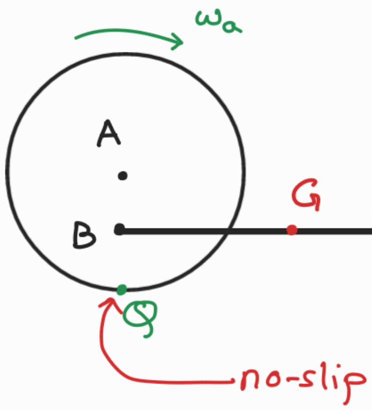
Compute $\underline{v}_{c|I}$

$$\overset{v_c\,\hat{e}_1}{\underline{v}_{C|I}} = \overset{0}{\cancel{\underline{v}_{D|I}}} + \overset{\omega_c\,\hat{e}_3}{\underline{\omega}_{C|I}} \times \overset{r\,\hat{e}_2}{\underline{r}_{CD}}$$



$$\Rightarrow \quad \underline{v}_{C|I} = \omega_c\, r\, \hat{e}_1$$

Compute $\underline{v}_{A|I}$



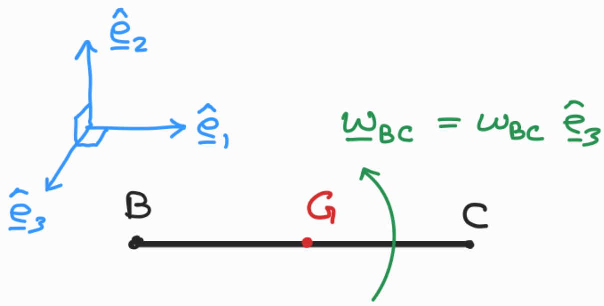
$$\underline{v}_{A|I} = \overset{0}{\underline{v}_{Q|I}} + \overset{-\omega_a\,\hat{e}_3}{\underline{\omega}_{a|I}} \times \overset{2r\,\hat{e}_2}{\underline{r}_{AQ}}$$

$$= 2\,\omega_a\, r\, \hat{e}_1$$

Compute $\underline{v}_{B|I}$

$$\underline{v}_{B|I} = \overset{0}{\underline{v}_{Q|I}} + \overset{-\omega_a\,\hat{e}_3}{\underline{\omega}_{a|I}} \times \overset{r\,\hat{e}_2}{\underline{r}_{AQ}}$$

$$= \omega_a\, r\, \hat{e}_1$$



Compute $\underline{\omega}_{BC}$

$$\underline{v}_{B|I} = \underline{v}_{C|I} + \underline{\omega}_{BC} \times \underline{r}_{BC}$$

$$\Rightarrow \quad \omega_a\, r\, \hat{e}_1 = \omega_c\, r\, \hat{e}_1 + \omega_{BC}\, \hat{e}_3 \times l\, \hat{e}_1$$

$$\Rightarrow \quad \boxed{\omega_a\, r}\, \hat{e}_1 = \boxed{\omega_c\, r}\, \hat{e}_1 + \boxed{\omega_{BC}\, l}\, \hat{e}_2 \quad \Rightarrow \quad \omega_{BC} = 0 \quad (\hat{e}_2\text{-comp})$$

$$\& \quad \omega_a = \omega_c \quad (\hat{e}_1\text{-comp})$$

Compute $\underline{v}_{G|I}$

$$\underline{v}_{G|I} = \underline{v}_{C|I} + \overset{0}{\cancel{\underline{\omega}_{BC}}} \times \underline{r}_{GC} = \omega_a\, r\, \hat{e}_1$$

Compute kinetic energies of the individual RBs 'a', 'b', 'c'

$$(T^{a}_{|I})_2 = \frac{1}{2} m_a [\underline{v}_{A|I}]^T [\underline{v}_{A|I}] + \frac{1}{2} [\underline{\omega}_{a|I}]^T [\underline{\underline{I}}^A] [\underline{\omega}_{a|I}]$$

$$= \frac{1}{2} m_a \begin{bmatrix} 2r\omega_a \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 2r\omega_a \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ -\omega_a \end{bmatrix}^T \begin{bmatrix} \checkmark & 0 & 0 \\ 0 & \checkmark & 0 \\ 0 & 0 & \frac{m_a R^2}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -\omega_a \end{bmatrix}$$

$$= \frac{1}{2} m_a \cdot (2r\omega_a)^2 + \frac{1}{2} (\omega_a)^2 \cdot \left(\frac{m_a (2r)^2}{2} \right) \qquad [R = 2r]$$

$$= \frac{m_a \; 4r^2\omega_a^2}{2} + \frac{m_a \; 4r^2\omega_a^2}{4}$$

$$= 3 \, m_a \, r^2 \omega_a^2$$

$$(T^{b}_{|I})_2 = \frac{1}{2} m_b [\underline{v}_{G|I}]^T [\underline{v}_{G|I}] + \frac{1}{2} [\underline{\omega}_{b|I}]^T [\underline{\underline{I}}^G] [\underline{\omega}_{b|I}]$$

$$= \frac{1}{2} m_b \begin{bmatrix} r\omega_a \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} r\omega_a \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} \checkmark & 0 & 0 \\ 0 & \checkmark & 0 \\ 0 & 0 & \checkmark \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2} m_b \, r^2 \omega_a^2$$

$$(T^{c}_{|I})_2 = \frac{1}{2} m_c [\underline{v}_{C|I}]^T [\underline{v}_{C|I}] + \frac{1}{2} [\underline{\omega}_{c|I}]^T [\underline{\underline{I}}] [\underline{\omega}_{c|I}]$$

$$= \frac{1}{2} m_c \begin{bmatrix} r\omega_a \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} r\omega_a \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ \omega_c \end{bmatrix}^T \begin{bmatrix} \checkmark & 0 & 0 \\ 0 & \checkmark & 0 \\ 0 & 0 & \frac{m_c r^2}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \omega_c \end{bmatrix}$$

$$= \frac{1}{2} m_c r^2 \omega_a^2 + \frac{1}{2} \omega_a^2 \frac{m_c r^2}{2} \qquad [\omega_c = \omega_a]$$

$$= \frac{3}{4} m_c r^2 \omega_a^2$$

Adding $(T_{1I}^a)_2$, $(T_{1I}^b)_2$, and $(T_{1I}^c)_2$ for KE of system

$$(T_{1I}^a + T_{1I}^b + T_{1I}^c)_2 = 3 \overset{6}{m_a} r^2 \omega_a^2 + \frac{1}{2} \overset{5}{m_b} r^2 \omega_a^2 + \frac{3}{4} \overset{1.5}{m_c} r^2 \omega_a^2$$

$$[v_B = \omega_a r]$$

$$= 18 v_B^2 + \frac{5}{2} v_B^2 + \frac{9}{8} v_B^2$$

$$= 21.6 \, v_B^2$$

Using work-energy relation (**)

$$21.6 \, v_B^2 = \overset{5}{m_b} g \frac{\overset{0.075}{r}}{2} \qquad g = 9.81$$

$$\Rightarrow \quad v_B = 0.29 \text{ m/s}$$