Tutorial 9

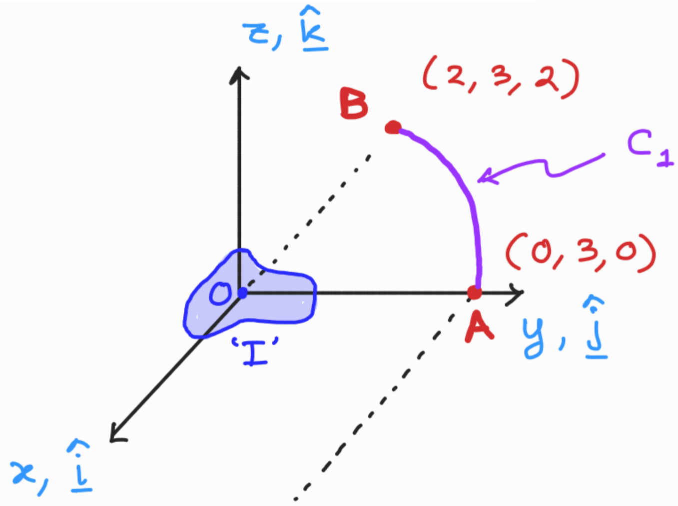
(Part A)

1) Is the following force conservative?

$$\boldsymbol{F} = (-2xy + yz)\hat{\boldsymbol{i}} + (-x^2 + xz - z)\hat{\boldsymbol{j}} + (xy - y)\hat{\boldsymbol{k}}$$

(a) If it is conservative, find its potential function V

(b) Find the work done by this force in moving a particle (say P) along an open quarter circular path C_1 (start at A and end at B)



Soln :

(a) To check if the force is conservative or not, we use the check the curl of the force:

$$\underbrace{\underline{\nabla}}_{\left(\frac{\partial}{\partial x}\hat{\underline{i}} + \frac{\partial}{\partial y}\hat{\underline{j}} + \frac{\partial}{\partial z}\hat{\underline{k}}\right)} \times \underline{F} = \begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -2xy + yz & -x^2 + xz - z & xy - y \end{vmatrix}$$

$$\underline{\nabla} \times \underline{F} = (\cancel{x}-\cancel{1}-\cancel{x}+\cancel{1})\,\hat{\underline{i}} - (\cancel{y}-\cancel{y})\,\hat{\underline{j}} + (-\cancel{2x}+\cancel{z}+\cancel{2x}-\cancel{z})\,\hat{\underline{k}}$$

$$= \underline{0} \quad \forall \; x, y, z$$

for all

$\Rightarrow$ $\underline{F}$ is CONSERVATIVE !

$\Rightarrow$ Work done by this force $\underline{F}$ in moving the particle P from pt A to pt B is independent of the path C_1.

(b) The potential function $V(x,y,z)$ is related to the conservative force $\underline{F}$ as:

$$\underbrace{F_x = -\frac{\partial V}{\partial x}}_{(i)}, \quad \underbrace{F_y = -\frac{\partial V}{\partial y}}_{(ii)}, \quad \underbrace{F_z = -\frac{\partial V}{\partial z}}_{(iii)}$$

Integrate (i) :

$$-2xy + yz = -\frac{\partial V}{\partial x}$$

$$\Rightarrow \int (-2xy + yz)\,dx = -V(x,y,z) + \underbrace{f(y,z)}_{\text{const w.r.t } x}$$

$$\Rightarrow \quad V(x,y,z) = x^2 y - xyz + f(y,z) \quad \text{—— (i)}$$

Let's now use this expression of $V(x,y,z)$ in (ii) and (iii)

$$-x^2 + xz - z = -\frac{\partial V}{\partial y} = -\frac{\partial}{\partial y}\left(x^2 y - xyz + f(y,z)\right)$$

$$\Rightarrow \quad \cancel{-x^2} + \cancel{xz} - z = - \cancel{x^2} + \cancel{xz} - \frac{\partial f}{\partial y}$$

$$\Rightarrow \quad \frac{\partial f}{\partial y} = z \quad \Rightarrow \quad f(y,z) = yz + \underset{\text{const. w.r.t. } y}{c(z)}$$

To determine the value of the constant c, use (iii)

$$F_z = -\frac{\partial V}{\partial z}$$

$$\Rightarrow \quad xy - y = -\frac{\partial}{\partial z}\left(x^2y - xyz + yz + c(z)\right)$$

$$\Rightarrow \quad \cancel{xy} - \cancel{y} = \cancel{xy} - \cancel{y} + \frac{\partial c}{\partial z}$$

$$\Rightarrow \quad \frac{\partial c}{\partial z} = 0 \quad \Rightarrow \quad c = \text{constant}$$

$$\therefore \quad V(x,y,z) = x^2y - xyz + yz + c$$

(c) Work done

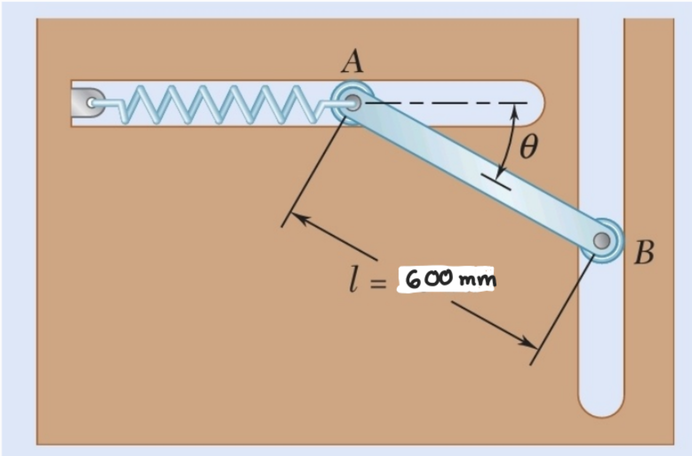
$$W_{A\to B} = -\left[V(\underline{r}_B) - V(\underline{r}_A)\right]$$

$$= -\left[V(2,3,2) - V(0,3,0)\right]$$

$$= -\left[\left(2^2\cdot 3 - 2\cdot 3\cdot 2 + 3\cdot 2 + c\right) - \left(0^2\cdot 3 - 0\cdot 3\cdot 0 + 3\cdot 0 + c\right)\right] = \underline{-6 \text{ Nm}}$$

2) **17.39** The ends of a 4.5 kg rod AB are constrained to move along slots cut in a vertical plate as shown. A spring of constant k = 600 N/m is attached to end A in such a way that its tension is zero when $\theta = 0$. If the rod is released from rest when θ = 0°, determine the angular velocity of the rod and the velocity of end B when θ = 30°

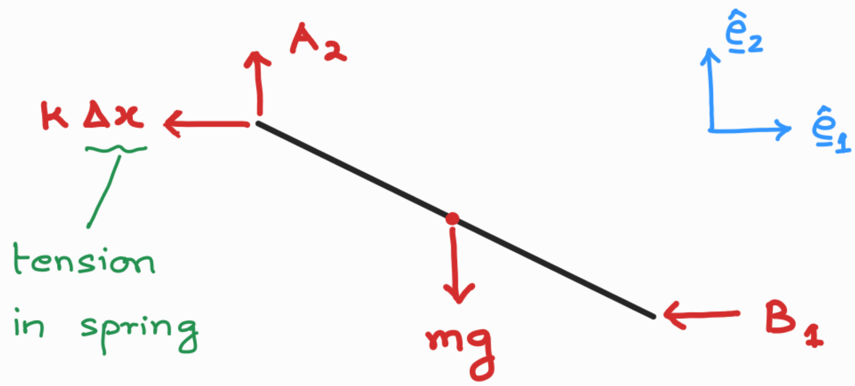
Neglect friction !



Soln : Let mass of rod be 'm'

spring constant be 'k'

Let's draw the FBD of the rod AB

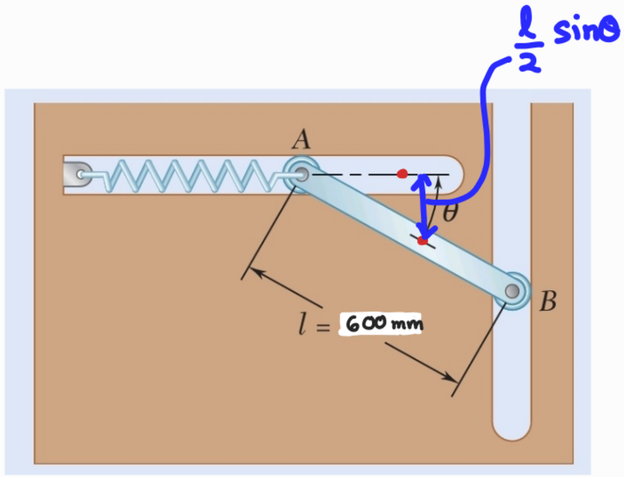


- A_2 and B_1 are workless forces
- Only the spring force and the gravitational weight mg do work. Both these forces are conservative forces.

$\Rightarrow$ Can make use of conservation of total energy

$$\Delta (T + V) = 0$$

$$\Rightarrow \quad \Delta V + \Delta T = 0$$

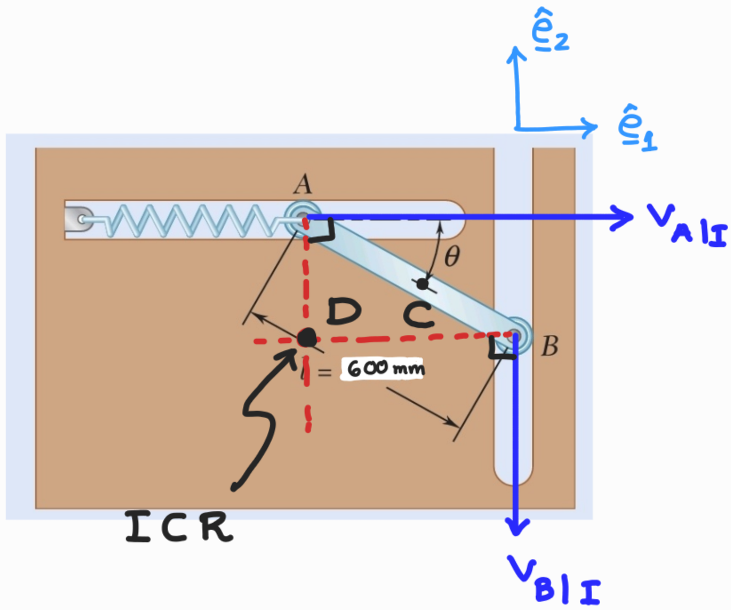


Change in potential energy, ΔV

$$= V^{\text{spring force}} + V^{\text{gravitational force}}$$

$$= \left(\frac{1}{2} k \left[l(1-\cos\theta)\right]^2\right) + \left(-mg\,\frac{l}{2}\sin\theta\right)$$

Change in kinetic energy



The motion starts from rest

$$\therefore \quad T_1 = 0$$

Point D is the instantaneous center of rotation of the rod

The velocity of the COM C of the rod is:

$$\underline{v}_{C|I} = \overset{0}{\underline{v}_{D|I}} + \underline{\omega}_{AB} \times \underline{r}_{CD}$$

$$= \omega\,\hat{e}_3 \times \left(\frac{l}{2}\cos\theta\,\hat{e}_1 + \frac{l}{2}\sin\theta\,\hat{e}_2\right)$$

$$= \frac{\omega l}{2}\left(-\sin\theta\ \hat{e}_1 + \cos\theta\ \hat{e}_2\right)$$

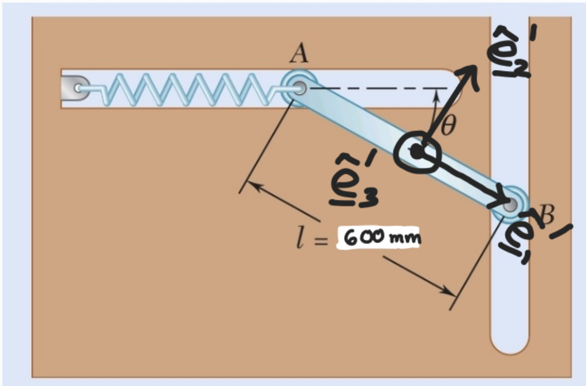
$$\therefore \quad \underline{v}_{c|I} \cdot \underline{v}_{c|I} = |\underline{v}_{c|I}|^2 = \left(\frac{\omega l}{2}\right)^2 = \frac{\omega^2 l^2}{4}$$

$$T_2 = \frac{1}{2} m\, \underline{v}_{c|I} \cdot \underline{v}_{c|I} + \underbrace{\frac{1}{2} \underline{\omega}_{AB} \cdot \underline{\underline{I}}^c \underline{\omega}_{AB}}_{\text{have to use a csys}}$$

$$= \frac{1}{2} m \frac{\omega^2 l^2}{4} +$$

$$\frac{1}{2} \omega^2 \frac{m l^2}{12}$$

$$= \frac{m \omega^2 l^2}{6}$$



$$[\underline{\underline{I}}^c]_{[\hat{e}_1', \hat{e}_2', \hat{e}_3']} = \begin{bmatrix} \frac{ml^2}{12} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{ml^2}{12} \end{bmatrix}$$

$$[\underline{\omega}_{AB}]_{[\hat{e}_1', \hat{e}_2', \hat{e}_3']} = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}$$

Now use $\Delta V + \Delta T = 0$

$$\Rightarrow \quad \frac{m l^2 \omega^2}{6} = m g \frac{l}{2} \sin\overset{30^\circ}{\cancel{\theta}} - \frac{k l^2}{2}(1-\cos\overset{30^\circ}{\cancel{\theta}})^2$$

$$\Rightarrow \quad \omega = \sqrt{\frac{6}{m l^2}\left[\frac{m g l \sin\theta}{2} - \frac{k l^2}{2}(1-\cos\theta)^2\right]}$$