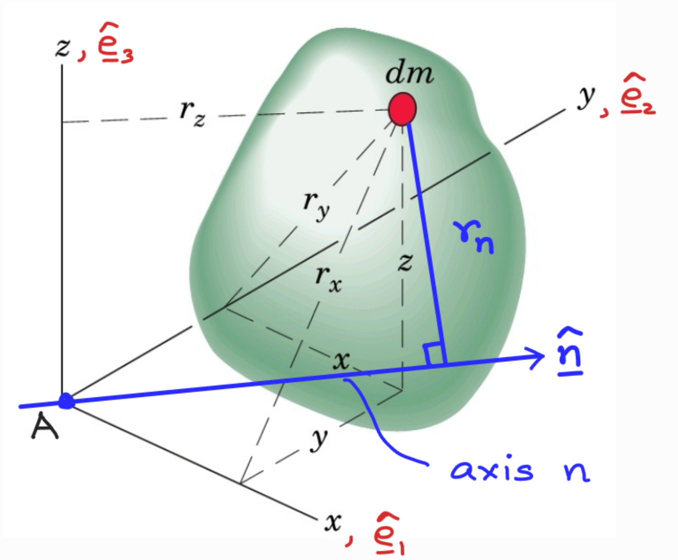
Inertia matrix for RBs with symmetry planes in mass distribution

Recall the moment of inertia integrals about an axis n passing through point A



$$I_{nn}^{A} = \int r_n^{2}\, dm$$

Set

$\hat{n} = \hat{e}_1 \longrightarrow I_{11}^{A} = \int r_x^{2}\, dm$

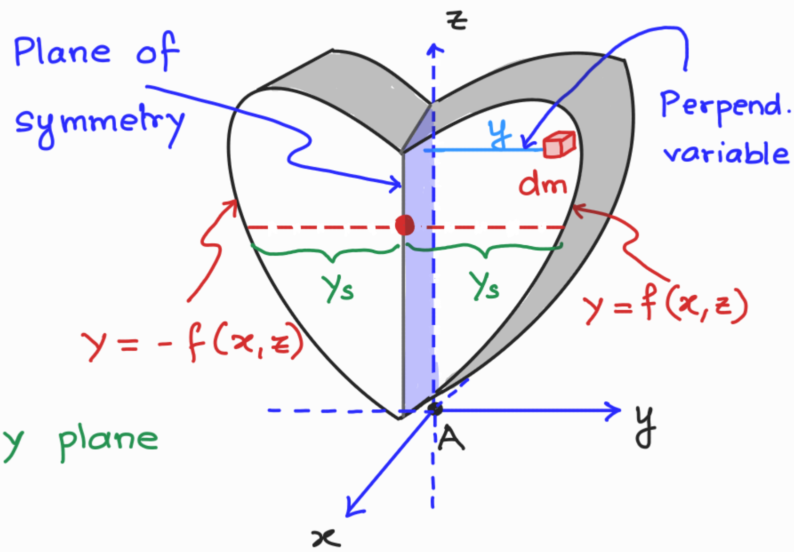
$\hat{n} = \hat{e}_2 \longrightarrow I_{22}^{A} = \int r_y^{2}\, dm$

$\hat{n} = \hat{e}_3 \longrightarrow I_{33}^{A} = \int r_z^{2}\, dm$

These integrals are always positive, whereas the products of inertia $I_{12}^{A} = \int xy\, dm$, $I_{13}^{A} = \int xz\, dm$, and $I_{23}^{A} = \int yz\, dm$ may be positive, negative, or zero.

For a homogeneous body having a plane of symmetry, if one of the coordinate planes contains the body plane symmetry, the products of inertia involving the coordinate variable perpendicular to this plane will vanish.

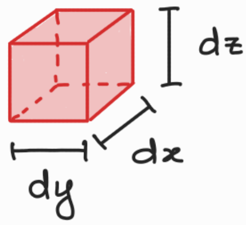
Ex: A homogeneous body for which the xz-plane is a body symmetry plane.



point 'A' lies on the symmetry plane

$$I_{12}^{A} = -\int xy\, dm$$

$dm = \rho\, dx\, dz\, dy$



$$I_{12}^{A} = -\int xy\, \rho\, dx\, dz\, dy$$

$$= -\rho \int_z \int_x \underbrace{\boxed{\int_{-f(x,z)}^{f(x,z)} y\, dy}}_{0}\; x\, dx\, dz = 0$$

Similarly, $I_{23}^{A} = \int yz\, dm$

$$= -\int yz\, \rho\, dx\, dz\, dy$$

$$= -\rho \int_x \int_z \underbrace{\boxed{\int_{-f(x,z)}^{f(x,z)} y\, dy}}_{0}\; z\, dz\, dx = 0$$

With a xz-plane of symmetry, y is the coordinate variable $\perp$ to plane of symmetry $\Rightarrow I_{12}^{A} = 0$ and $I_{23}^{A} = 0$

Thus, the inertia matrix at point A in the chosen coordinate system :

$$[\underline{\underline{I}}^A] = \begin{bmatrix} I_{11}^A & 0 & I_{13}^A \\ 0 & I_{22}^A & 0 \\ I_{31}^A & 0 & I_{33}^A \end{bmatrix}$$

Inertia tensor for some special homogeneous RBs

We focus on three specific homogeneous (meaning uniform mass distribution) bodies $\longrightarrow$ 1) Rectangular body

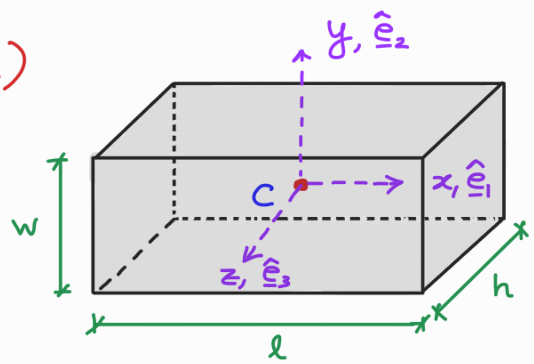
2) Circular body

3) Spherical body

1) Inertia tensor for a rectangular (cuboidal) RB

C $\rightarrow$ Center of mass (also centroid)

ρ $\rightarrow$ density (uniform)



Find inertia matrix of RB at C w.r.t. csys $\hat{e}_1$-$\hat{e}_2$-$\hat{e}_3$ (parallel to the edges of cuboid)

Notice the planes of symmetry at pt C !!

$$[\underline{\underline{I}}^C]_{\begin{pmatrix}\hat{e}_1\\ \hat{e}_2\\ \hat{e}_3\end{pmatrix}} = \ ?$$

$$I_{11}^c = \int (y^2+z^2)\, \underbrace{dm}_{\rho\, dV} = \rho \int_{-h/2}^{h/2} \int_{-w/2}^{w/2} \int_{-\ell/2}^{\ell/2} (y^2+z^2)\, dx\, dy\, dz$$

$$= \rho \ell \int_{-h/2}^{h/2} \int_{-w/2}^{w/2} (y^2+z^2)\, dy\, dz$$

mass of block 'm'

$$= \frac{\rho \ell w h}{12} (w^2+h^2)$$

$$= \frac{m(w^2+h^2)}{12}$$

Note the correspondence $(x, y, z) \sim (\ell, w, h)$

We can permute the symbols and get I_{22}^c, I_{33}^c

$$I_{22}^c = \frac{m(\ell^2+h^2)}{12}, \quad I_{33}^c = \frac{m(\ell^2+w^2)}{12}$$

Note: All products of inertia terms are ZERO

$$I_{12}^c = I_{13}^c = I_{23}^c = 0$$

(due to plane symmetry of cuboid about x-y, y-z, x-z planes through c)

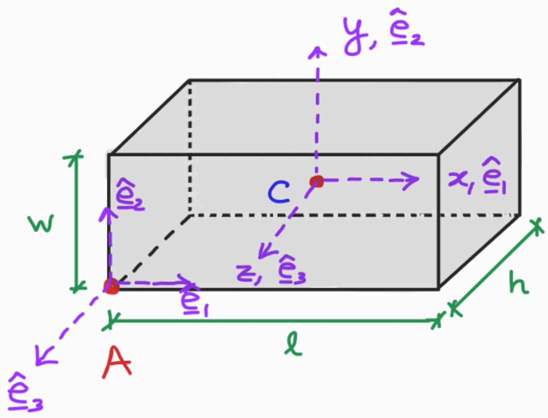
Thus,

$$[\underline{\underline{I}}^c] = \begin{bmatrix} \frac{m(w^2+h^2)}{12} & 0 & 0 \\ 0 & \frac{m(\ell^2+h^2)}{12} & 0 \\ 0 & 0 & \frac{m(\ell^2+w^2)}{12} \end{bmatrix}$$

How do you find $[\underline{\underline{I}}^A]$ about point A (at corner) ?

The csys bases at COM C

and at corner A are //



$\therefore$ Use parallel axes theorem

$$I_{11}^A = I_{11}^C + m\underbrace{(x_{C_2}^2 + x_{C_3}^2)}_{d_1^2}$$

$$\Rightarrow I_{12}^A = I_{12}^C - m\, x_{C_1} x_{C_2}$$

Similarly,

and, $I_{23}^A = I_{23}^C - m\, x_{C_2} x_{C_3}$

$$I_{22}^A = I_{22}^C + m\underbrace{(x_{C_1}^2 + x_{C_3}^2)}_{d_2^2}$$

and, $I_{13}^A = I_{13}^C - m\, x_{C_1} x_{C_3}$

$$I_{33}^A = I_{33}^C + m\underbrace{(x_{C_1}^2 + x_{C_2}^2)}_{d_3^2}$$

from Lec 11

$$I_{11}^A = \frac{m(w^2+h^2)}{12} + m\left[\left(\frac{w}{2}\right)^2 + \left(\frac{h}{2}\right)^2\right]$$

$$I_{22}^A = \frac{m(\ell^2+h^2)}{12} + m\left[\left(\frac{\ell}{2}\right)^2 + \left(\frac{h}{2}\right)^2\right]$$

$$I_{33}^A = \frac{m(\ell^2+w^2)}{12} + m\left[\left(\frac{\ell}{2}\right)^2 + \left(\frac{w}{2}\right)^2\right]$$

$$I_{12}^A = 0 - m\left(\frac{\ell}{2}\right)\left(\frac{w}{2}\right)$$

$\hat{e}_1 \rightarrow 1 \rightarrow \ell$

$\hat{e}_2 \rightarrow 2 \rightarrow w$

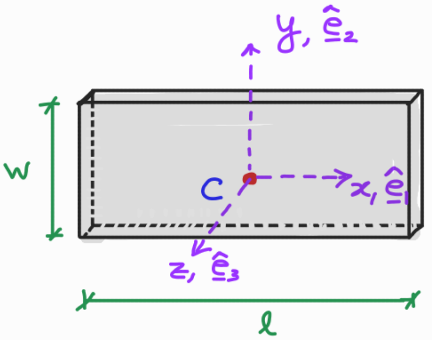
$\hat{e}_3 \rightarrow 3 \rightarrow h$

$$I_{23}^A = 0 - m\left(\frac{w}{2}\right)\left(\frac{h}{2}\right)$$

$$I_{13}^A = 0 - m\left(\frac{\ell}{2}\right)\left(\frac{h}{2}\right)$$

Inertia tensor for thin rectangular plate

$h \approx 0$

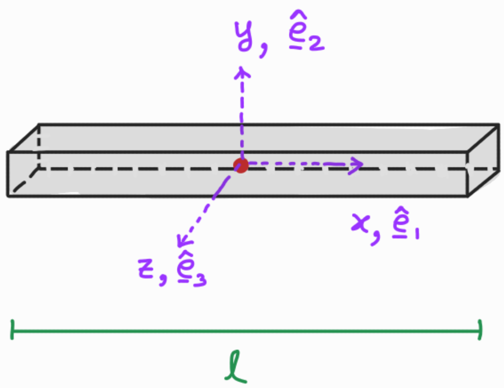


$$[\underline{\underline{I}}^c] = \begin{bmatrix} \frac{m(w^2+\cancelto{0}{h^2})}{12} & 0 & 0 \\ 0 & \frac{m(\ell^2+\cancelto{0}{h^2})}{12} & 0 \\ 0 & 0 & \frac{m(\ell^2+w^2)}{12} \end{bmatrix}$$

$$[\underline{\underline{I}}^c] = \begin{bmatrix} \frac{m w^2}{12} & 0 & 0 \\ 0 & \frac{m \ell^2}{12} & 0 \\ 0 & 0 & \frac{m(\ell^2+w^2)}{12} \end{bmatrix}$$

Inertia matrix of a thin rod

$h \approx 0, \quad w \approx 0$



$$[\underline{\underline{I}}^c] = \begin{bmatrix} \frac{m \cancelto{0}{w^2}}{12} & 0 & 0 \\ 0 & \frac{m \ell^2}{12} & 0 \\ 0 & 0 & \frac{m(\ell^2+\cancelto{0}{w^2})}{12} \end{bmatrix}$$

$$[\underline{\underline{I}}^c] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{m \ell^2}{12} & 0 \\ 0 & 0 & \frac{m \ell^2}{12} \end{bmatrix}$$

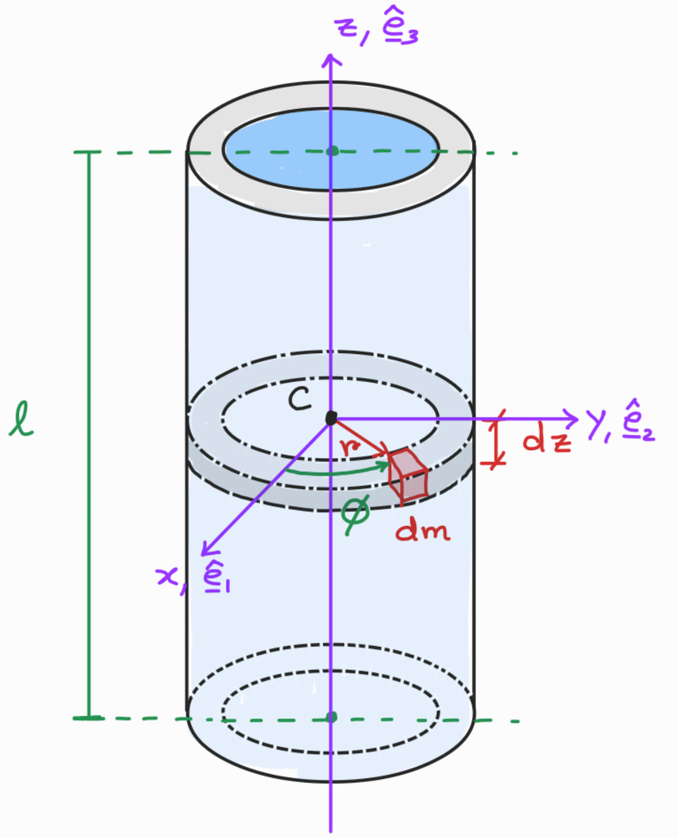
2) Inertia tensor for a circular cylindrical RB

Inner radius : r_i

Outer radius : r_o

Length : ℓ

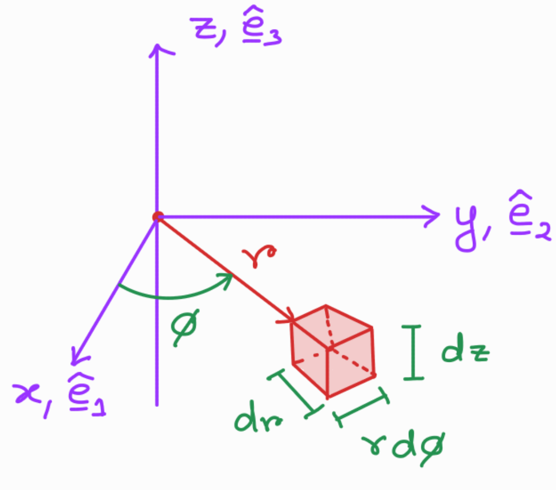
Here, the z-axis is the central axis about which this body can be generated by revolution.



$$\text{Symmetry planes: } \begin{cases} x\text{-}y \text{ plane} \\ y\text{-}z \text{ plane} \\ x\text{-}z \text{ plane} \end{cases} \Rightarrow \left.\begin{matrix} I_{12} \\ I_{23} \\ I_{13} \end{matrix}\right\} = 0$$

Any infinitesimally small mass 'dm' can be written as :

$$dm = \rho \, dr \, d\phi \, dz$$



$$I_{33}^{C} = \int r^2 \, dm$$

$$= \int_{-\ell/2}^{\ell/2} \int_{0}^{2\pi} \int_{r_i}^{r_o} r^2 \rho \, dr \, r d\phi \, dz$$

$$= \rho \, 2\pi \ell \int_{r_i}^{r_o} r^3 \, dr = \rho \, 2\pi \ell \left(\frac{r_o^4 - r_i^4}{4} \right) = \frac{\pi}{2} \rho \ell \left(r_o^4 - r_i^4 \right)$$

With cross-sectional area $A = \pi(r_o^2 - r_i^2)$ and $m = \rho A \ell$

$$I_{33}^c = \frac{m}{2}(r_o^2 + r_i^2)$$

Note that the general form of I_{11}^c, I_{22}^c, I_{33}^c are:

$$I_{11}^c = \int (y^2 + z^2)\, dm$$

$$I_{22}^c = \int (x^2 + z^2)\, dm$$

$$I_{33}^c = \int (x^2 + y^2)\, dm$$

$$\therefore \quad I_{11}^c + I_{22}^c = \int (x^2 + y^2)\, dm + 2\int z^2\, dm$$

$$= I_{33}^c + 2\int z^2\, dm \quad \text{——— (EQ)}$$

Clearly, for the cylinder, since z-axis is the central axis about which the RB is symmetric (also called AXISYMMETRY)

$$\boxed{I_{11}^c = I_{22}^c}$$

$$\int z^2\, dm = \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \int_0^{2\pi} \int_{r_i}^{r_o} z^2 \rho\, dr\, r d\phi\, dz = \rho\, \cancel{2}\pi \left(\frac{r_o^2 - r_i^2}{\cancel{2}}\right) \int_{-\ell/2}^{\ell/2} z^2\, dz$$

$$= \rho\, \underbrace{\pi(r_o^2 - r_i^2)}_{\text{C/s area } A} \left[\frac{z^3}{3}\right]_{-\ell/2}^{\ell/2}$$

$$= \frac{\rho A \ell^3}{12} = \underbrace{(\rho A \ell)}_{m} \frac{\ell^2}{12} = \frac{m\ell^2}{12}$$

Using (EQ), we get:

$$\underbrace{I^c_{11} + I^c_{22}}_{2 I^c_{11}} = I^c_{33} + 2\left(\frac{m\ell^2}{12}\right)$$

$$\Rightarrow \quad I^c_{11} = I^c_{22} = \frac{m}{4}\left(r_o^2 + r_i^2\right) + \frac{m\ell^2}{12}$$

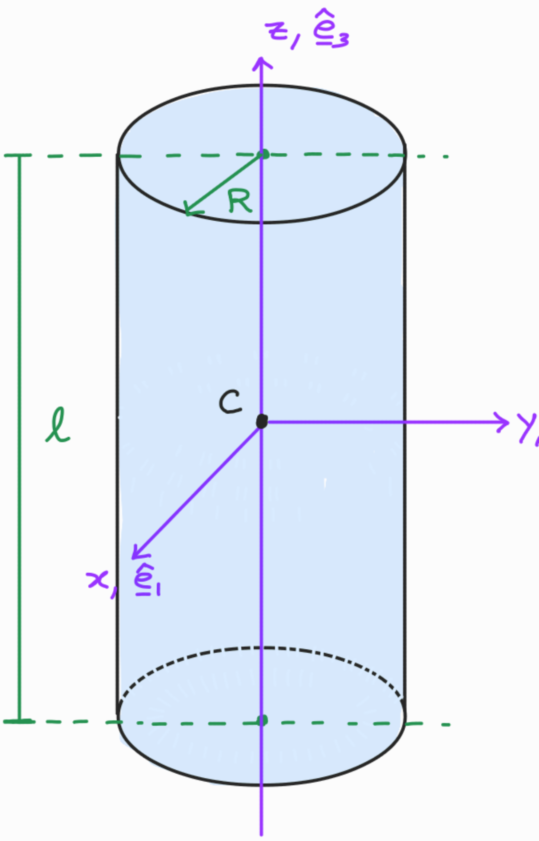
$$\left[\underline{\underline{I}}^c\right] = \begin{bmatrix} \frac{m}{4}\left(r_o^2+r_i^2\right) + \frac{m\ell^2}{12} & 0 & 0 \\ 0 & \frac{m}{4}\left(r_o^2+r_i^2\right) + \frac{m\ell^2}{12} & 0 \\ 0 & 0 & \frac{m}{2}\left(r_o^2+r_i^2\right) \end{bmatrix}$$

Inertia tensor for a solid circular cylinder

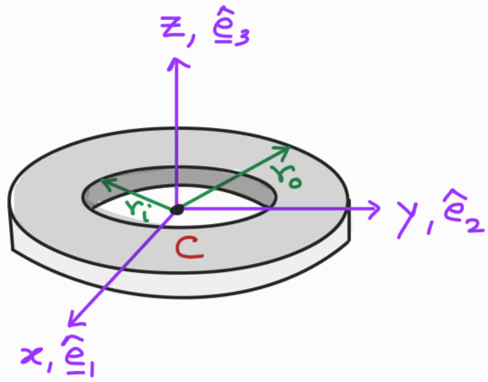
Put $r_o = R$, $m = \rho \pi r^2 \ell$

$r_i = 0$

$$\left[\underline{\underline{I}}^c\right] = \begin{bmatrix} \frac{mR^2}{4} + \frac{m\ell^2}{12} & 0 & 0 \\ 0 & \frac{mR^2}{4} + \frac{m\ell^2}{12} & 0 \\ 0 & 0 & \frac{mR^2}{2} \end{bmatrix}$$



Inertia Tensor for Annular Plate



$z \approx 0$ (negligible thickness)

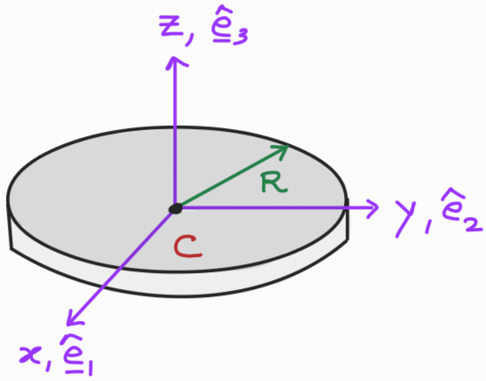
Area, $A = \pi(r_o^2 - r_i^2)$

Mass density, $m = \eta A$

mass per unit area

$$[\underline{\underline{I}}^c] = \begin{bmatrix} \frac{m(r_o^2 + r_i^2)}{4} & 0 & 0 \\ 0 & \frac{m(r_o^2 + r_i^2)}{4} & 0 \\ 0 & 0 & \frac{m(r_o^2 + r_i^2)}{2} \end{bmatrix}$$

Inertia Tensor for a Thin Circular Disk



$z \approx 0$ (negligible thickness)

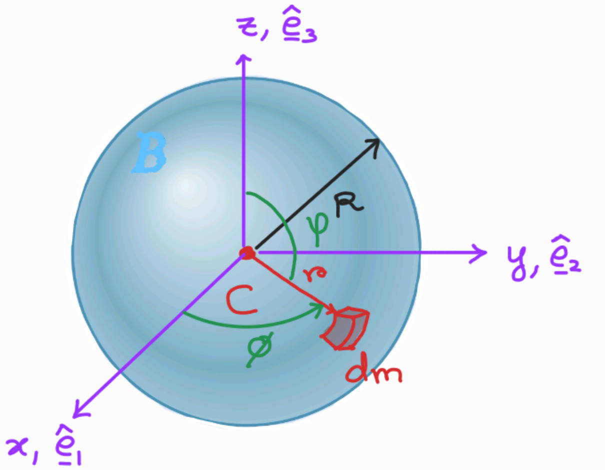
Area, $A = \pi R^2$

Mass density, $m = \eta A$

mass per unit area

$$[\underline{\underline{I}}^c] = \begin{bmatrix} \frac{mR^2}{4} & 0 & 0 \\ 0 & \frac{mR^2}{4} & 0 \\ 0 & 0 & \frac{mR^2}{2} \end{bmatrix}$$

3) Inertia Tensor for a Sphere



Observe that every plane through COM C is a plane of symmetry, so all products of inertia vanish and $\boxed{I^c_{11} = I^c_{22} = I^c_{33}}$ (since the sphere is symmetric about any axis passing through C)

Recall:

$$I^c_{11} = \int (y^2 + z^2)\, dm \qquad dm = \rho\, dr\, (r\, d\phi)\, (r\, d\varphi)$$

$$I^c_{22} = \int (x^2 + z^2)\, dm$$

$$I^c_{33} = \int (x^2 + y^2)\, dm$$

Add

$$I^c_{11} + I^c_{22} + I^c_{33} = 2 \int (x^2 + y^2 + z^2)\, dm$$

$$\Rightarrow \quad 3\, I^c_{11} = 2 \int r^2\, dm$$

$$\Rightarrow \quad I^c_{11} = \frac{2}{3} \int_0^{2\pi} \int_0^{2\pi} \int_0^{R} r^2 \rho\, dr\; r\, d\phi\; r\, d\varphi$$

$$= \frac{2}{3} \rho\, 4\pi^2 \int_0^R r^4\, dr$$

$$= \frac{8\pi^2 \rho}{3} \left(\frac{R^5}{5}\right) = \underbrace{\left(\rho\, \overbrace{\tfrac{4}{3}\pi R^3}^{\text{Volume}}\right)}_{m} \frac{2}{5} R^2$$

$$= \frac{2}{5} m R^2$$

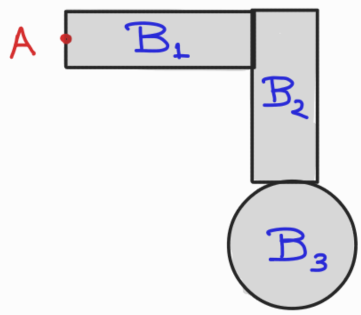
$$\therefore \quad [\underline{\underline{I}}^c] = \frac{2}{5} m R^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Inertia Tensor of a Composite RB

A composite RB is obtained by composition of several geometrically simple bodies whose inertia tensors are known or maybe easily determined.

If there are 'n' RBs $\mathcal{B}_k$, then the inertial tensor for the composite RB $\mathcal{B} = \bigcup_{k=1}^{n} \mathcal{B}_k$, maybe be written at a point A as:

$$\underset{=}{I}^{A}(\underbrace{\mathcal{B}}_{\text{composite body}}) = \sum_{i=1}^{n} \underbrace{\underset{=}{I}^{A}}_{\text{inertia tensor of ith RB at pt A}} (\mathcal{B}_k)$$

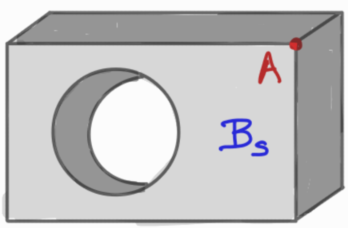


If $\mathcal{B}$ has 'p' cavities/holes $\mathcal{E}_k$, we may imagine the each cavity as an RB with negative mass, and the inertia tensor of the composite RB with holes/cavity is:

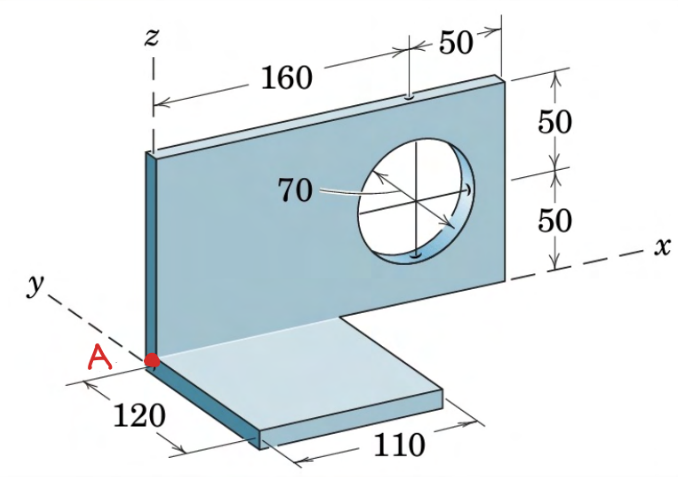
$$\underset{=}{I}^{A}(\underbrace{\mathcal{B}_s}_{\text{RB with holes/cavity}}) = \underset{=}{I}^{A}(\underbrace{\mathcal{B}}_{\text{RB with no cavity/hole}}) - \sum_{k=1}^{p} \underset{=}{I}^{A}(\mathcal{E}_k)$$

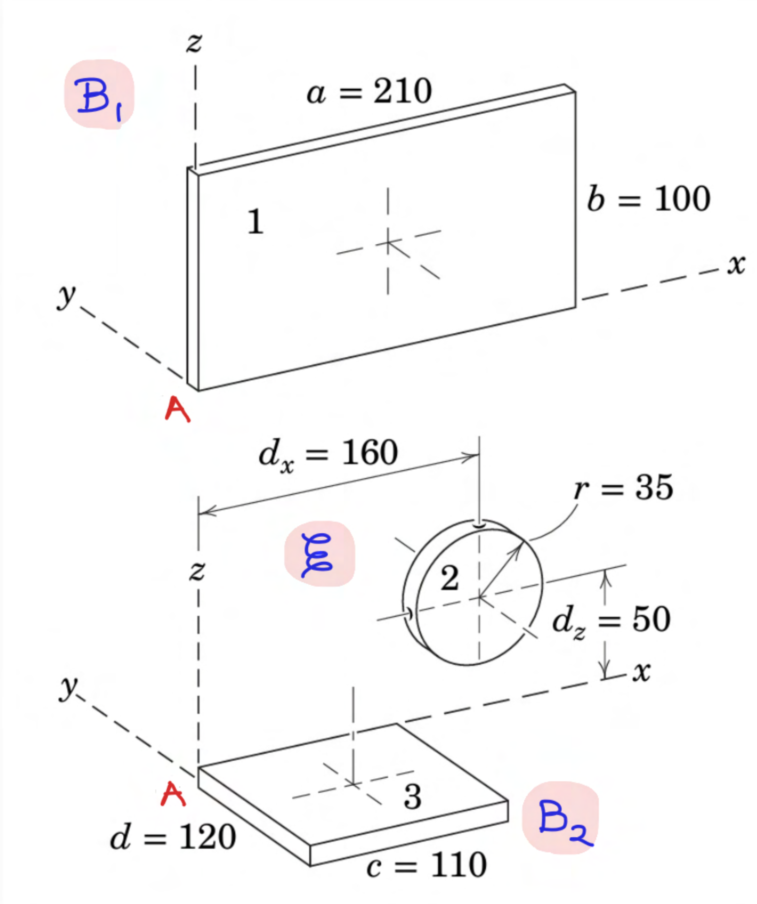
inertia tensor of ith RB at pt A

e.g.



Example:





$$[\underline{\underline{I}}^A](B_s) = [\underline{\underline{I}}^A](B_1) + [\underline{\underline{I}}^A](B_2) - [\underline{\underline{I}}^A](\xi)$$

why?

$$[\underline{\underline{I}}^A](B_1) = \begin{bmatrix} \frac{1}{3}mb^2 & 0 & -\frac{mab}{4} \\ 0 & \frac{1}{3}m(a^2+b^2) & 0 \\ -\frac{mab}{4} & 0 & \frac{1}{3}ma^2 \end{bmatrix}$$

directed opp. to y-axis

why?

$$[\underline{\underline{I}}^A](B_2) = \begin{bmatrix} \frac{1}{3}md^2 & -m\frac{c}{2}\left(-\frac{d}{2}\right) & 0 \\ -m\frac{c}{2}\left(-\frac{d}{2}\right) & \frac{1}{3}mc^2 & 0 \\ 0 & 0 & \frac{1}{3}m(c^2+d^2) \end{bmatrix}$$

why?

$$[\underline{\underline{I}}^A](\xi) = \begin{bmatrix} \frac{1}{4}mr^2 + md_z^2 & 0 & -md_x d_y \\ 0 & \frac{1}{2}mr^2 + m(d_x^2+d_z^2) & 0 \\ -md_x d_y & 0 & \frac{1}{4}mr^2 + md_x^2 \end{bmatrix}$$

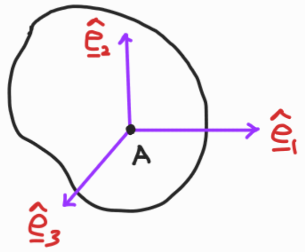
Transformation Rule of Inertia Matrix of RB

We know that the inertia tensor $\underline{\underline{I}}^A$ does not depend on the orientation of the coordinate system. Infact, the same is true for all tensor quantities. However, the second-order inertia tensor, when expressed in a matrix form using an orthonormal csys, the matrix components depend on the orientation of the csys lines.

for csys with axes $\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3$

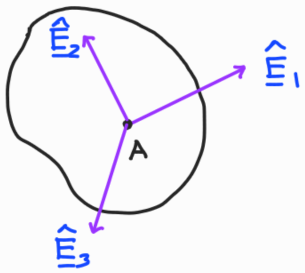
$$\left[\underline{\underline{I}}^A\right]_{\begin{bmatrix}\hat{\underline{e}}_1\\ \hat{\underline{e}}_2\\ \hat{\underline{e}}_3\end{bmatrix}} = \begin{bmatrix} I_{11}^A & I_{12}^A & I_{13}^A \\ I_{21}^A & I_{22}^A & I_{23}^A \\ I_{31}^A & I_{32}^A & I_{33}^A \end{bmatrix}$$

$\Rightarrow$



for csys with axes $\hat{\underline{E}}_1 - \hat{\underline{E}}_2 - \hat{\underline{E}}_3$

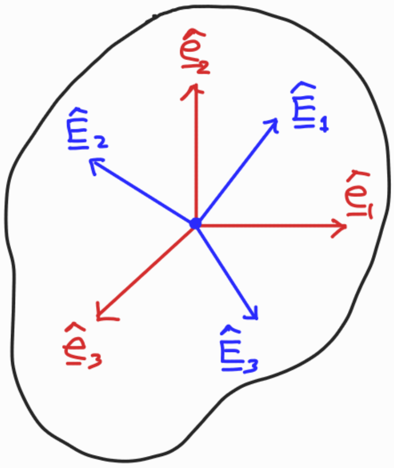
$$\left[\underline{\underline{I}}^A\right]_{\begin{bmatrix}\hat{\underline{E}}_1\\ \hat{\underline{E}}_2\\ \hat{\underline{E}}_3\end{bmatrix}} = \begin{bmatrix} I_{11}^{A'} & I_{12}^{A'} & I_{13}^{A'} \\ I_{21}^{A'} & I_{22}^{A'} & I_{23}^{A'} \\ I_{31}^{A'} & I_{32}^{A'} & I_{33}^{A'} \end{bmatrix}$$



We ask the question: How are the inertia matrix components related ?

$$I_{ij}^A \overset{??}{\longleftrightarrow} I_{ij}^{A'}$$

Relating inertia matrix in two different coordinate systems



Two sets of orthonormal triads can be related through a unique rotation tensor

for all

$$\hat{\underline{E}}_i = \underline{\underline{R}}\,\hat{\underline{e}}_i \qquad \forall \; i = 1, 2, 3$$

Orthonormal tensor

$\big\downarrow$ in a csys

Orthonormal matrix $[\underline{\underline{R}}]$

Property of orthonormal matrix

$$[\underline{\underline{R}}][\underline{\underline{R}}]^T = [\underline{\underline{I}}] \quad \text{and} \quad [\underline{\underline{R}}]^T[\underline{\underline{R}}] = [\underline{\underline{I}}]$$

$$[\hat{\underline{E}}_i]_{(\hat{e}_1-\hat{e}_2-\hat{e}_3)} = [\underline{\underline{R}}]_{(\hat{e}_1-\hat{e}_2-\hat{e}_3)} \; [\hat{\underline{e}}_i]_{(\hat{e}_1-\hat{e}_2-\hat{e}_3)}$$

Lets consider the position vector $\underline{r}_{PA}$ in two csys

$$[\underline{r}_{PA}]_{\begin{pmatrix}\hat{\underline{e}}_1\\ \hat{\underline{e}}_2\\ \hat{\underline{e}}_3\end{pmatrix}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad\qquad [\underline{r}_{PA}]_{\begin{pmatrix}\hat{\underline{E}}_1\\ \hat{\underline{E}}_2\\ \hat{\underline{E}}_3\end{pmatrix}} = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}$$

$$\underline{r}_{PA} = x_1\hat{\underline{e}}_1 + x_2\hat{\underline{e}}_2 + x_3\hat{\underline{e}}_3 = x_1'\hat{\underline{E}}_1 + x_2'\hat{\underline{E}}_2 + x_3'\hat{\underline{E}}_3$$

$$= x_1'\,\underline{\underline{R}}\,\hat{\underline{e}}_1 + x_2'\,\underline{\underline{R}}\,\hat{\underline{e}}_2 + x_3'\,\underline{\underline{R}}\,\hat{\underline{e}}_3$$

$$= \underline{\underline{R}}\left(x_1'\hat{\underline{e}}_1 + x_2'\hat{\underline{e}}_2 + x_3'\hat{\underline{e}}_3\right)$$

In other words, writing $[\underline{r}_{PA}]$ w.r.t $\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [\underline{\underline{R}}]_{\begin{pmatrix}\hat{\underline{e}}_1 \\ \hat{\underline{e}}_2 \\ \hat{\underline{e}}_3\end{pmatrix}} \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}$$

Let's now use this transformation in the definition of inertia matrix:

$$[\underline{\underline{I}}^A]_{\begin{pmatrix}\hat{\underline{e}}_1 \\ \hat{\underline{e}}_2 \\ \hat{\underline{e}}_3\end{pmatrix}} = \int \left\{ \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) [\underline{\underline{I}}] - \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^T \right\} dm$$

$$= \int \left\{ \left(\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}^T \underbrace{[\underline{\underline{R}}]^T [\underline{\underline{R}}]}_{[\underline{\underline{I}}]} \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} \right) [\underline{\underline{I}}] - \left([\underline{\underline{R}}] \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} \right) \left([\underline{\underline{R}}] \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} \right)^T \right\} dm$$

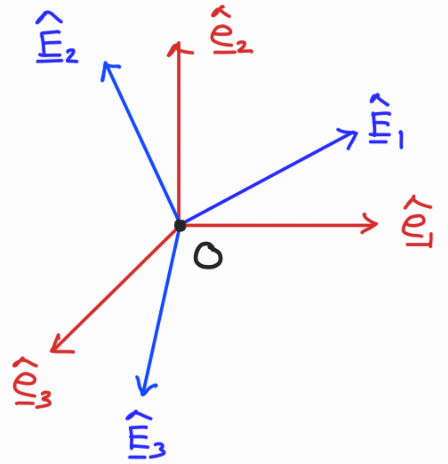
$$= \int \left\{ \left(\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}^T \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} \right) [\underline{\underline{R}}][\underline{\underline{I}}][\underline{\underline{R}}]^T - [\underline{\underline{R}}] \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}^T [\underline{\underline{R}}]^T \right\} dm$$

$$= [\underline{\underline{R}}] \left[\int \left\{ \left([\underline{r}_{PA}]^T [\underline{r}_{PA}] \right) [\underline{\underline{I}}] - [\underline{r}_{PA}] [\underline{r}_{PA}]^T \right\} dm \right]_{\begin{pmatrix}\hat{\underline{e}}_1 \\ \hat{\underline{e}}_2 \\ \hat{\underline{e}}_3\end{pmatrix}} [\underline{\underline{R}}]^T$$

$$[\underline{\underline{I}}^A]_{(\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3)} = [\underline{\underline{R}}]_{(\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3)} [\underline{\underline{I}}^A]_{(\hat{\underline{E}}_1 - \hat{\underline{E}}_2 - \hat{\underline{E}}_3)} [\underline{\underline{R}}]^T_{(\hat{\underline{e}}_1 - \hat{\underline{e}}_2 - \hat{\underline{e}}_3)}$$

$$\Rightarrow \left[\underline{\underline{I}}^A\right]_{(\hat{E}_1-\hat{E}_2-\hat{E}_3)} = \left[\underline{\underline{R}}\right]^T_{(\hat{e}_1-\hat{e}_2-\hat{e}_3)} \left[\underline{\underline{I}}^A\right]_{(\hat{e}_1-\hat{e}_2-\hat{e}_3)} \left[\underline{\underline{R}}\right]_{(\hat{e}_1-\hat{e}_2-\hat{e}_3)}$$

Using index notation, we can write the components of orthonormal matrix as:



$$\hat{\underline{E}}_i \cdot \hat{\underline{e}}_j = \cos\Big(\underbrace{\measuredangle(\hat{\underline{E}}_i, \hat{\underline{e}}_j)}_{\text{angle between } \hat{\underline{E}}_i \text{ and } \hat{\underline{e}}_j}\Big)$$

$$= (\underline{\underline{R}}\,\hat{\underline{e}}_i)\cdot\hat{\underline{e}}_j$$

$$= [\hat{\underline{e}}_j]^T[\underline{\underline{R}}][\hat{\underline{e}}_i]$$

$$= R_{ji} \equiv A_{ij}$$

(Note $\hat{\underline{E}}_j \cdot \hat{\underline{e}}_i \neq \hat{\underline{E}}_i \cdot \hat{\underline{e}}_j$)

Therefore,

$$\boxed{I^{A'}_{ij} = A_{ip}\, I^A_{pq}\, A_{jq}}$$

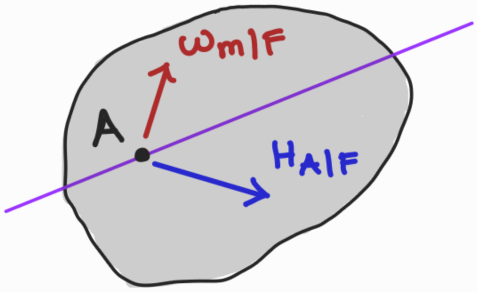
Principal Axes of Inertia of RB at point A

We now ask the question that is there a set of bases using which if we express the inertia matrix, it will be DIAGONAL?

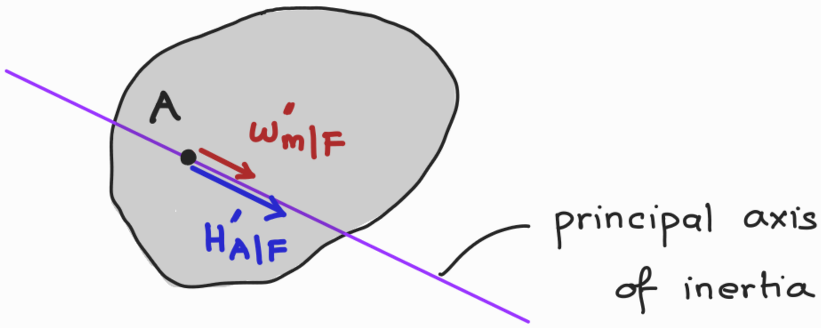
(i.e. the products of inertia terms will vanish)

In a rigid body, the two vectors $\underline{\omega}_{m|F}$ (angular velocity) and $\underline{H}_{A|F}$ (angular momentum), related through the inertia tensor $\underline{\underline{I}}^A$, are not parallel in general.

$$\underbrace{\underline{H}_{A|F} = \underline{\underline{I}}^A \, \underline{\omega}_{m|F}}_{\text{not // in general}}$$



However, for each point A, there are some directions/axes for which the two vectors $\underline{\omega}_{m|F}$ and $\underline{H}_A$ are parallel. These directions are called principal directions and the axes along these directions are called principal axes of inertia at that point.



Along principal axis of inertia

$$\underline{H}_{A|F} = \underline{\underline{I}}^A \, \underline{\omega}'_{m|F} = \underbrace{\lambda}_{\text{scalar}} \, \underline{\omega}'_{m|F}$$

λ values are called the eigenvalues of the inertia matrix and they are the roots of the characteristic equation:

$$\det\left(\underset{3\times 3}{[\underline{\underline{I}}^A]} - \lambda \underset{\substack{3\times 3\\ \text{identity}\\ \text{matrix}}}{[\underline{\underline{I}}]}\right) = 0$$

As $[\underline{\underline{I}}^A]$ is a symmetric and positive definite matrix, there are always three real positive eigenvalues.

There is an eigenvector $\underline{n}$ associated with each eigenvalue λ.

They can be determined through the equation:

$$\left([\underline{\underline{I}}^A] - \lambda [\underline{\underline{I}}]\right) \underline{n} = \underline{0}$$

These eigenvectors $\underline{n}^{(1)}$, $\underline{n}^{(2)}$, $\underline{n}^{(3)}$ give the principal axes of the RB at point A.

When we express the inertia matrix using a csys whose bases are $\underline{n}^{(1)}$, $\underline{n}^{(2)}$, $\underline{n}^{(3)}$, the inertia matrix becomes diagonal:

$$[\underline{\underline{I}}^A]_{\begin{pmatrix}\underline{n}^{(1)}\\ \underline{n}^{(2)}\\ \underline{n}^{(3)}\end{pmatrix}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} I'_{11} & 0 & 0 \\ 0 & I'_{22} & 0 \\ 0 & 0 & I'_{33} \end{bmatrix}$$

Thus, $H'_{A_1} = I'_{11}\,\omega'_{1|F}$ $\quad H'_{A_2} = I'_{11}\,\omega'_{1|F}$ $\quad H'_{A_3} = I'_{11}\,\omega'_{1|F}$

Cross-coupling is removed, and algebra is simpler!

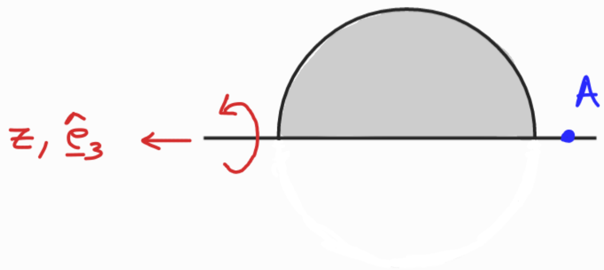
Note that at least one mutually perpendicular set of three principal axes ALWAYS EXISTS at every point A of the RB

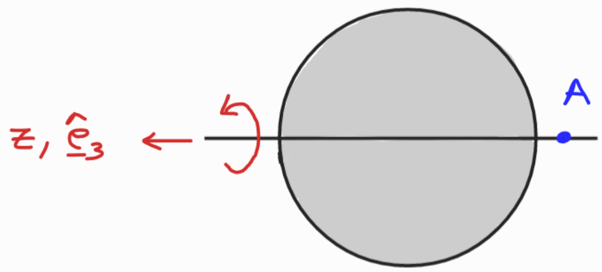
Two ways of finding principal axes of inertia of RB at pt A

- Eigenvalue analysis (always works)
- Visual Inspection
  - RB is a body generated by revolution of area abt a central axis passing through pt A
  - RB has a known plane of symmetry and pt A lies on that plane

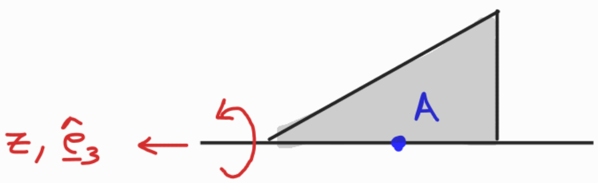
Bodies of revolution

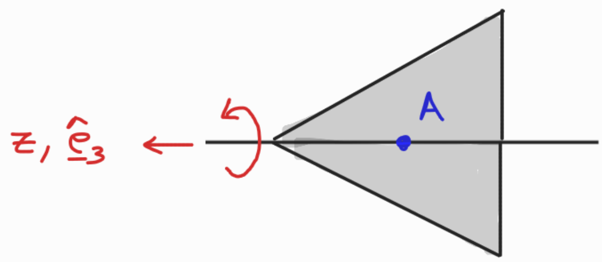
A body generated by rotating a planar area through $360°$ about an axis (say $z, \hat{e}_3$)



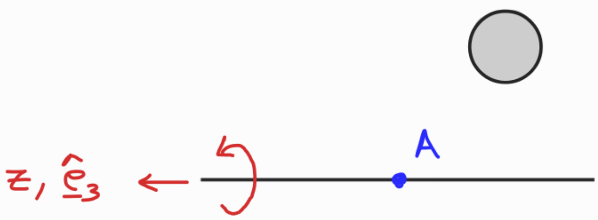


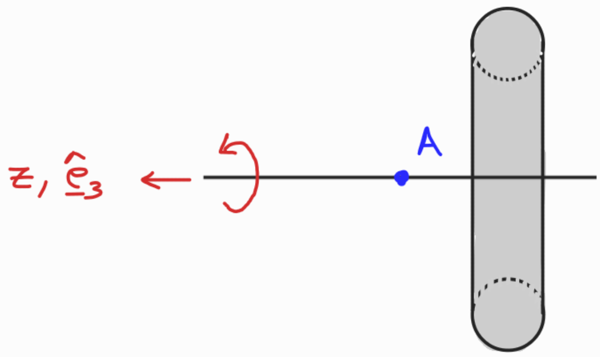
sphere





cone





torus

- The central axis of revolution is one principal axis
- Any other two mutually perpendicular axes passing through pt A and lying in the plane perpendicular to central axis of revolution will result in two other principal axes

Therefore, for bodies of revolution, the inertia matrix at pt A lying on the central axis of revolution (say z, $\hat{e}_3$) will have the form

$$[\underline{\underline{I}}^A] = \begin{bmatrix} I_{11}^A & 0 & 0 \\ 0 & I_{22}^A = I_{11}^A & 0 \\ 0 & 0 & I_{33}^A \end{bmatrix}$$

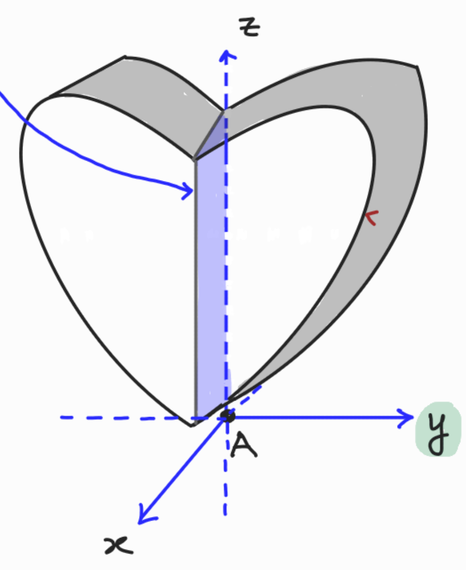
Bodies with plane of symmetry

For an RB with point A lying on a plane of symmetry, any axis perpendicular to this plane of symmetry is a principal axis of inertia

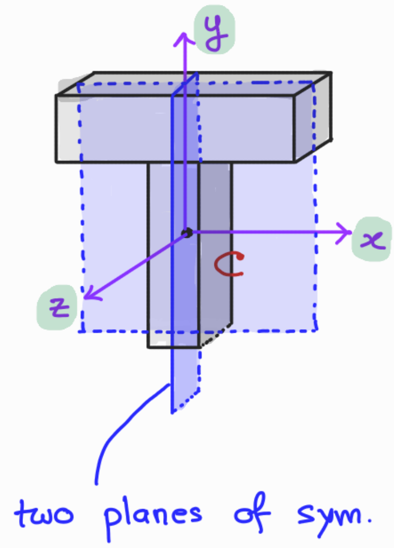
Case 1 : RB with one plane of symmetry

- Suppose an RB has a symmetry plane (say $x-z$ plane) passing point A

- Then, the y-axis (perpendicular to the plane) is a principal axis



Case 2 : RB with two planes of symmetry



- If the RB has two symmetry planes (say xy-plane and yz-plane), then their intersection (y-axis) is also a principal axis

- The axes perpendicular to each sym. plane (i.e. z-axis and x-axis) also become principal axes.

- In this case, all the three coordinate axes themselves are the principal axes.

Case 3 : RB with three mutually ⊥ planes of symmetry

Bodies like sphere have three mutually ⊥ symmetry planes Hence, the coordinate axes aligned with these planes are the principal axes.