

## Recap

We had introduced Euler's axioms where we found that we needed to find the net external forces and net ext. moments acting on a system. Towards this goal, we had introduced the idea of equivalent force systems, using which we reduced the effect of several forces and couples acting on a system to an equivalent single resultant force and a single resultant couple, or an equivalent wrench.

Before we find the equivalent force system, it is needed that we identify all the forces and couples that are exerted on the system — the external forces and couples. In the last lecture, we started discussing how to draw FBDs where we identify all contact and body forces and contact moments acting externally to the system under consideration.

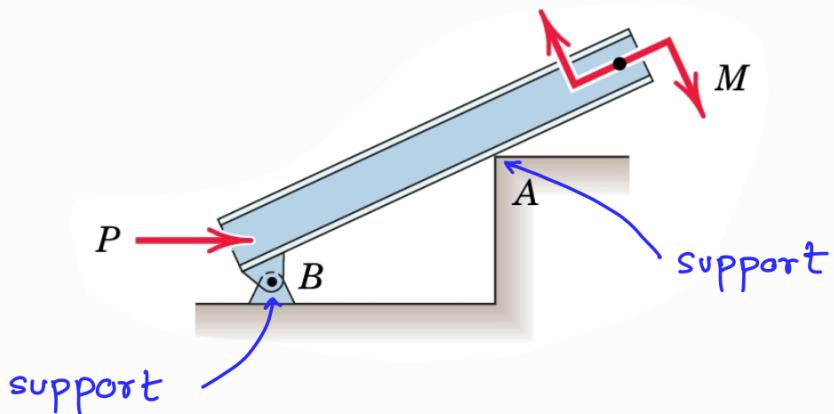
In this lecture, we will look at a class of ext. contact forces & moments that are generated (than directly applied) when

a system's movement is constrained by some supports.  
→ called reactions

## Reactions at Supports

In many situations, the motion of an RB or a system is constrained by some supports

Ex:



When a force is applied on the system, the support generates a force system (force and moment/couple)

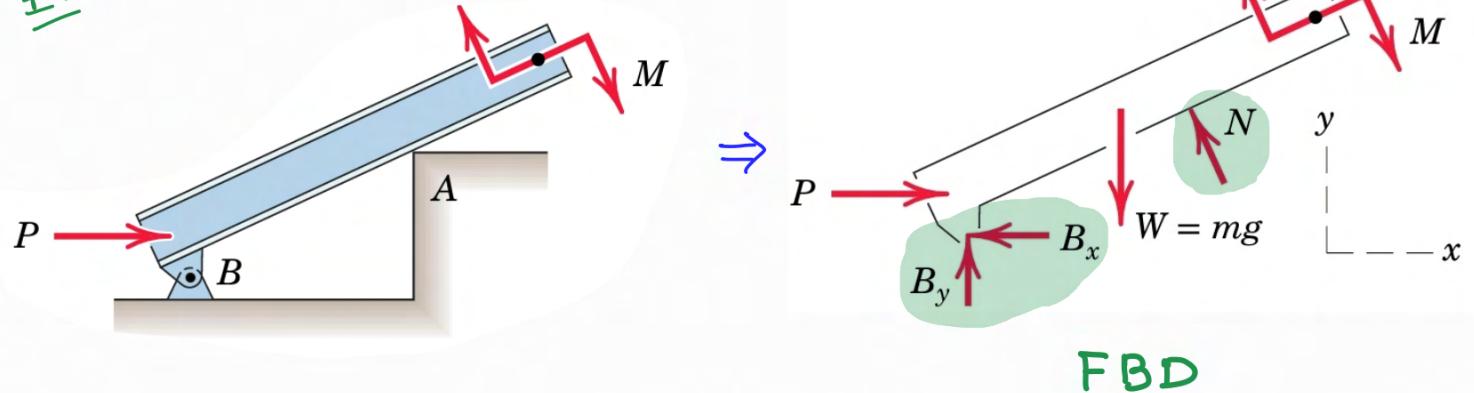
contact force & couple

The surface force system that develops at the supports

or points of contact on the body of interest is called the reaction force system (in short REACTIONS)

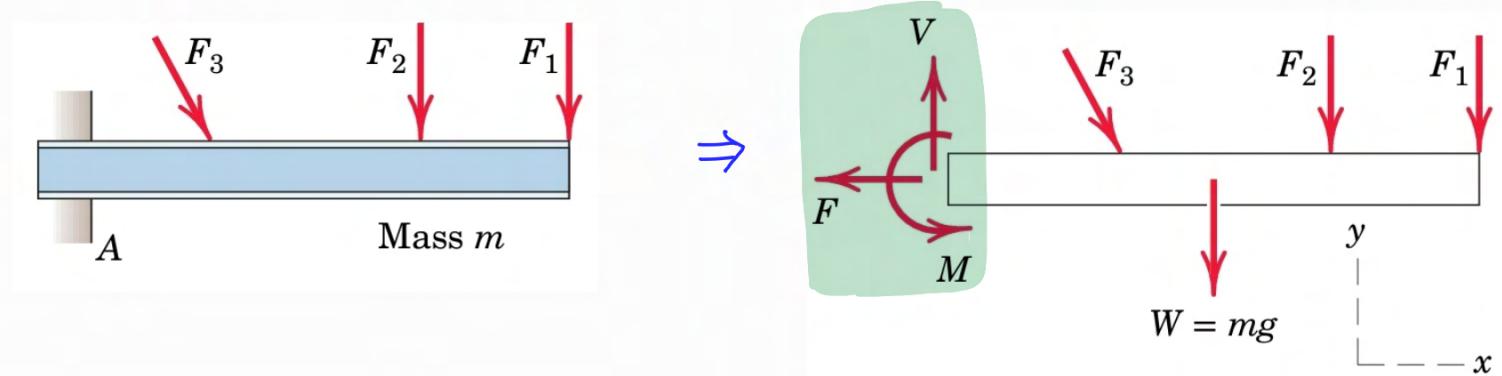
While drawing an FBD of a system, we should remove the physical details of the support and replace those details by a reaction force system!

Ex:



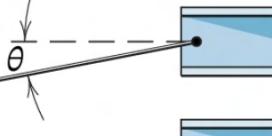
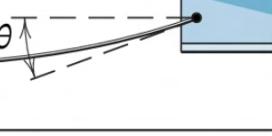
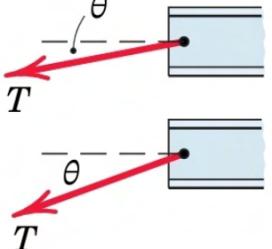
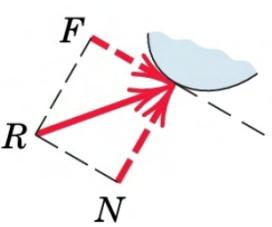
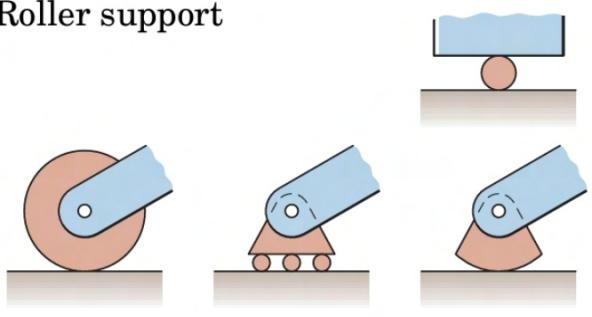
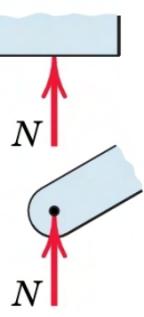
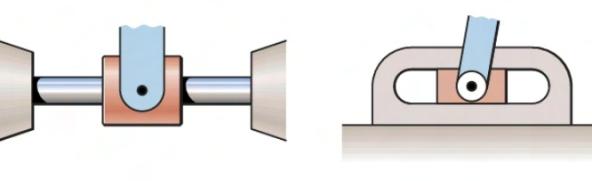
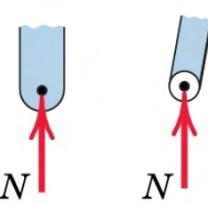
FBD

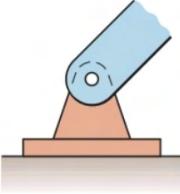
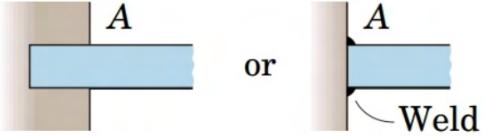
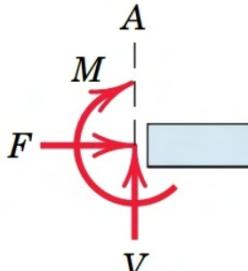
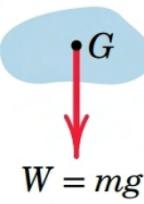
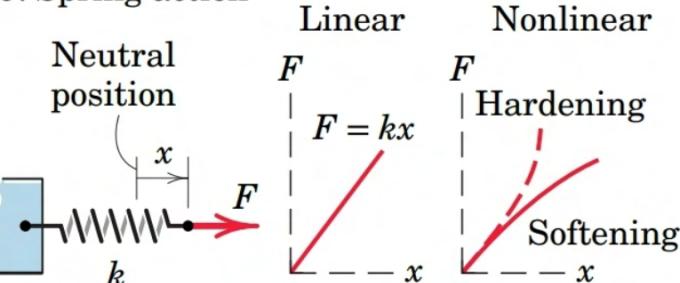
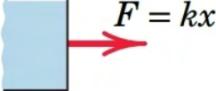
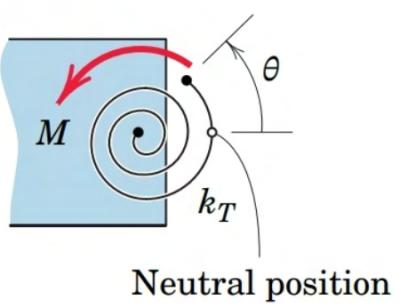
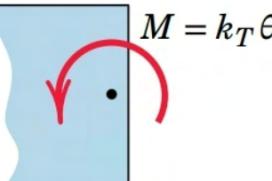
**General Rule:** If the support prevents translation in a given direction, then a force must be developed on the system in that direction. Likewise, if rotation is prevented, a couple moment must be exerted on the system.



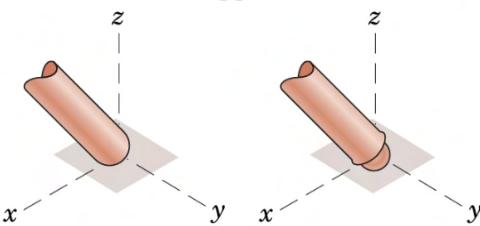
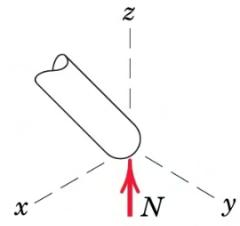
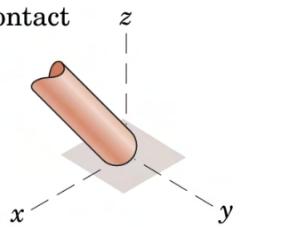
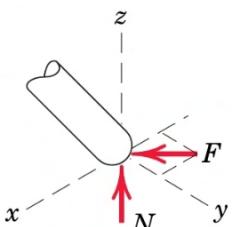
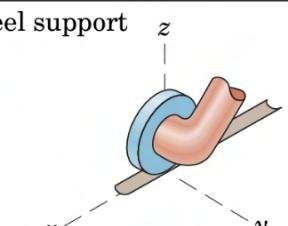
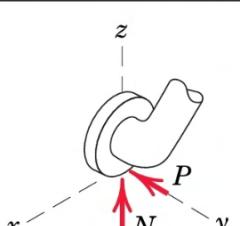
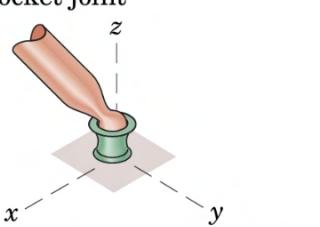
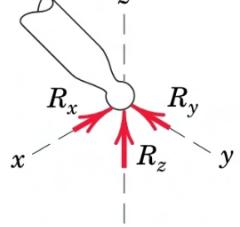
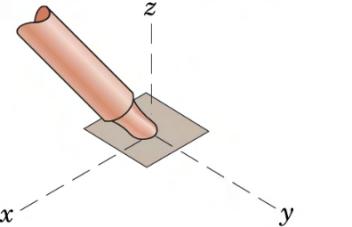
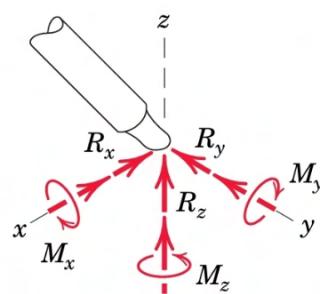
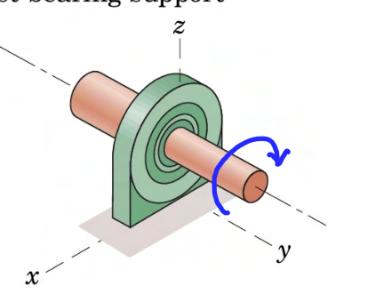
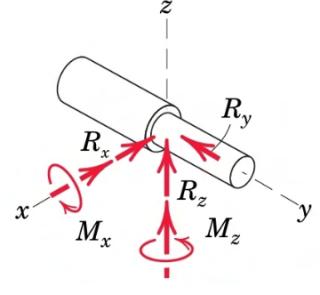
Let us consider few cases of different support systems in 2D first and examine the reactions generated.

# Support reactions for 2D analysis

Type of Contact and Force Origin	Action on Body to Be Isolated
<p>1. Flexible cable, belt, chain, or rope</p>  <p>Weight of cable negligible</p>  <p>Weight of cable not negligible</p> 	 <p>Force exerted by a flexible cable is always a tension away from the body in the direction of the cable.</p>
2. Smooth surfaces	 <p>Contact force is compressive and is normal to the surface.</p>
3. Rough surfaces	 <p>Rough surfaces are capable of supporting a tangential component <math>F</math> (frictional force) as well as a normal component <math>N</math> of the resultant contact force <math>R</math>.</p>
4. Roller support	  <p>Roller, rocker, or ball support transmits a compressive force normal to the supporting surface.</p>
5. Freely sliding guide	  <p>Collar or slider free to move along smooth guides; can support force normal to guide only.</p>

Type of Contact and Force Origin	Action on Body to Be Isolated
6. Pin connection	 <p>A freely hinged pin connection is capable of supporting a force in any direction in the plane normal to the pin axis. We may either show two components <math>R_x</math> and <math>R_y</math> or a magnitude <math>R</math> and direction <math>\theta</math>.</p>
7. Built-in or fixed support	  <p>A built-in or fixed support is capable of supporting an axial force <math>F</math>, a transverse force <math>V</math> (shear force), and a couple <math>M</math> (bending moment) to prevent rotation.</p>
8. Gravitational attraction	  <p>The resultant of gravitational attraction on all elements of a body of mass <math>m</math> is the weight <math>W = mg</math> and acts toward the center of the earth through the center of gravity <math>G</math>.</p>
9. Spring action	  <p>Spring force is tensile if the spring is stretched and compressive if compressed. For a linearly elastic spring the stiffness <math>k</math> is the force required to deform the spring a unit distance.</p>
10. Torsional spring action	  <p>For a linear torsional spring, the applied moment <math>M</math> is proportional to the angular deflection <math>\theta</math> from the neutral position. The stiffness <math>k_T</math> is the moment required to deform the spring one radian.</p>

# Reaction force systems in 3D analysis

Type of Contact and Force Origin	Action on Body to Be Isolated ( <b>Unknowns</b> )
1. Member in contact with smooth surface, or ball-supported member	  <p>Force must be normal to the surface and directed toward the member.</p>
2. Member in contact with rough surface	  <p>The possibility exists for a force <math>F</math> tangent to the surface (friction force) to act on the member, as well as a normal force <math>N</math>.</p>
3. Roller or wheel support with lateral constraint	  <p>A lateral force <math>P</math> exerted by the guide on the wheel can exist, in addition to the normal force <math>N</math>.</p>
4. Ball-and-socket joint	  <p>A ball-and-socket joint free to pivot about the center of the ball can support a force <math>\mathbf{R}</math> with all three components.</p>
5. Fixed connection (embedded or welded)	  <p>In addition to three components of force, a fixed connection can support a couple <math>\mathbf{M}</math> represented by its three components.</p>
6. Thrust-bearing support	  <p>Thrust bearing is capable of supporting axial force <math>R_y</math> as well as radial forces <math>R_x</math> and <math>R_z</math>. Couples <math>M_x</math> and <math>M_z</math>.</p>

In a 3D case, you may have a maximum of six unknown reaction force system (3 forces + 3 moment couples)

We have now learned

- a) how to draw an FBD of a system by tracking all  
contact forces and body forces,  
↓  
applied      reactions

- b) how to replace a force system by an equivalent  
resultant force system at a given point

These will be useful when we deal with Euler's axioms

Recall the two axioms:

1st axiom:

$$\dot{\underline{P}}|_I = \underline{F}_R \quad (\text{Rate of change of linear momentum} = \text{net external force})$$
$$\Rightarrow m \underline{\alpha}_{c/I} = \underline{F}_R$$

center of mass (COM)

2nd axiom:

$$\dot{\underline{H}}_{O/I} = \underline{M}_O \quad (\text{Rate of change of angular momentum} = \text{net external moment})$$

point O

must be a point  
fixed in the inertial  
reference frame I !!

[Note: Forces & moments  
are frame-invariant]

These two sets of relations leads to the equations of motion

which are effectively six scalar equations (equivalent to two vector equations of motion). Together with appropriate initial conditions, these are used to determine the general motion of an RB.

Determination of this motion, however, does not always split neatly into a translational part and a rotational part. In fact in many cases, these two parts are coupled together that give rise to strongly coupled system of nonlinear ODEs

Nonlinear ODE :  $\frac{d\dot{x}(t)}{dt} = \underline{g}(\underline{x}(t), \underline{F}_R(t), \underline{M}_o(t))$

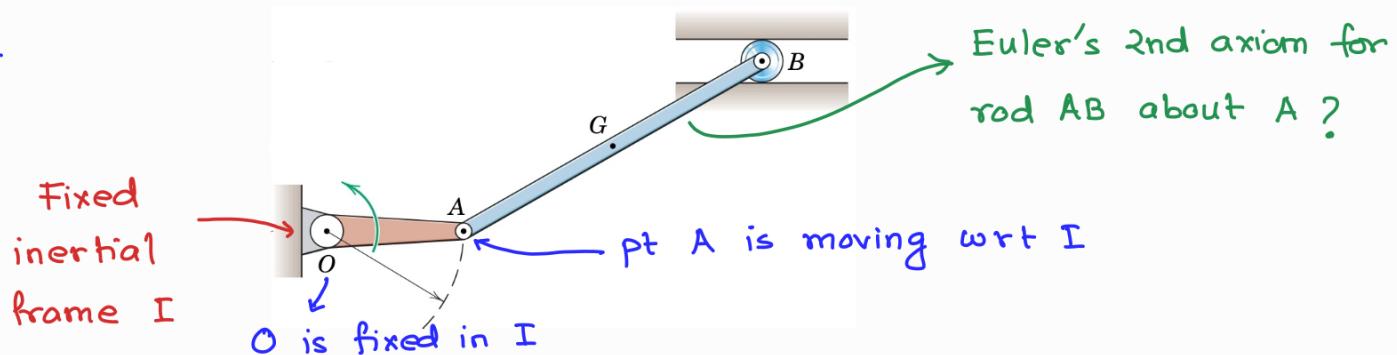
Initial condition :  $\underline{x}(t=0) = \underline{x}_0$   $\rightarrow \underline{x} = \begin{bmatrix} \underline{x}_{\text{tran}} \\ \underline{x}_{\text{rot}} \end{bmatrix}$

### Euler's 2nd axiom for a Moving Reference point

We turn to Euler's 2nd axiom:  $\dot{\underline{H}}_{O|I} = \underline{M}_o$

and recall that this relation holds only for a point O fixed in the inertial frame I.

Ex:



Given any general point 'A' moving w.r.t. I (see above figure)

Is  $\dot{H}_{A|I} = \underline{M}_A$  VALID? If so, under what circumstances?

To derive an alternate form of Euler's 2nd axiom valid for a moving point and to determine any restrictions on its use, let's first derive a relationship for transfer of angular momentum from one pt to another and then we will derive a relationship between the rate of change of angular momentum about a point A and the net external moment abt A.

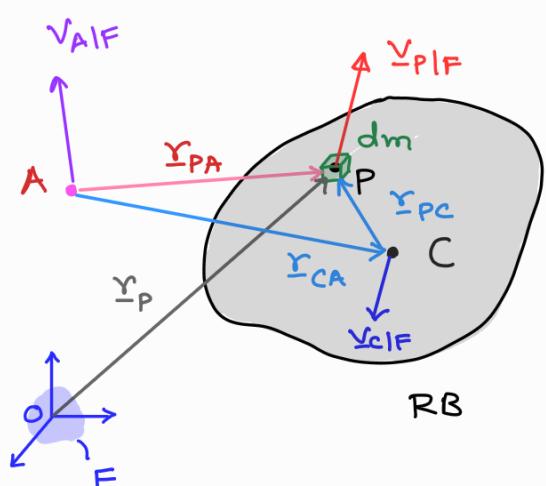
### Transformation rule for Angular Momentum

Angular momentum of an RB abt a moving pt A was defined earlier:

$$H_{A|F} = \int \underline{\tau}_{PA} \times \underline{v}_{PA|F} dm$$

C : Center of mass of RB

It can be shown that



$$H_{A|F} = H_{C|F} + m(r_{CA} \times v_{CA|F})$$

(Proved next)

angular mom.  
abt C      total mass of RB

Proof of  $H_{A|F} = H_{C|F} + m(\underline{r}_{CA} \times \underline{v}_{C|F})$

Start with the defn of angular momentum of RB about moving pt A

$$H_{A|F} = \int (\underline{r}_{PA} \times \underline{v}_{PA|F}) dm$$

$$= \int (\underline{r}_{PC} + \underline{r}_{CA}) \times \underline{v}_{PA|F} dm$$

$$= \int (\underline{r}_{PC} + \underline{r}_{CA}) \times (\underline{v}_{PC|F} + \underline{v}_{CA|F}) dm$$

Expand

$$= \int (\underline{r}_{PC} \times \underline{v}_{PC|F}) dm + \int (\underline{r}_{CA} \times \underline{v}_{PC|F}) dm$$

$$+ \int (\underline{r}_{PC} \times \underline{v}_{CA|F}) dm + \int (\underline{r}_{CA} \times \underline{v}_{CA|F}) dm$$

$$\underline{v}_{PA|F} = \underline{v}_{P|F} - \underline{v}_{A|F}$$

$$= \underline{v}_{P|F} - \underline{v}_{C|F}$$

$$+ \underline{v}_{C|F} - \underline{v}_{A|F}$$

$$= \underline{v}_{PC|F} + \underline{v}_{CA|F}$$

Simplify

Term :  $\int (\underline{r}_{PC} \times \underline{v}_{PC|F}) dm = H_{C|F}$  (angular momentum of RB abt C) by defn

Term :  $\int (\underline{r}_{CA} \times \underline{v}_{PC|F}) dm = \underline{r}_{CA} \times \int \underline{v}_{PC|F} dm$

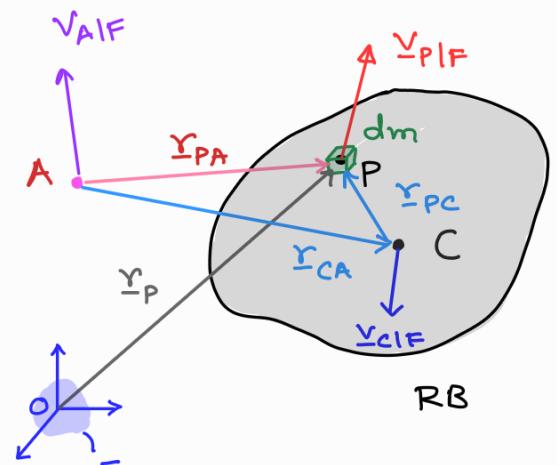
(II)

Constant =  $\underline{r}_{CA} \times \left\{ \underbrace{\int \underline{v}_{P|F} dm}_{\text{w.r.t. integ.}} - \underbrace{\int \underline{v}_{C|F} dm}_{\text{constant}}$

=  $\underline{r}_{CA} \times \left\{ \underbrace{\underline{P}_{|F}}_{= m \underline{v}_{C|F}} - \underline{v}_{C|F} \int dm \right\}$

=  $\underline{r}_{CA} \times \left\{ m \underline{v}_{C|F} - m \underline{v}_{C|F} \right\}$

=  $\underline{0}$



Term **III**

$$\int (\underline{r}_{pc} \times \underline{v}_{CA|F}) dm = \left( \int \underline{r}_{pc} dm \right) \times \underline{v}_{CA|F}$$

constant w.r.t  
the integ.

$$= \int (\underline{r}_{po} - \underline{r}_{co}) dm \times \underline{v}_{CA|F}$$

$$= \int (\underline{r}_p - \underline{r}_c) dm \times \underline{v}_{CA|F}$$

$$= \left( \underbrace{\int \underline{r}_p dm}_m - \underbrace{\underline{r}_c \int dm}_m \right) \times \underline{v}_{CA|F}$$

defn of COM  $\rightarrow m \underline{r}_c$

$$= (m \cancel{\underline{r}_c} - m \cancel{\underline{r}_c}) \times \underline{v}_{CA|F}$$

$$= \underline{0}$$

Term **IV**

$$\int (\underline{r}_{ca} \times \underline{v}_{CA|F}) dm = (\cancel{\underline{r}_{ca}} \times \cancel{\underline{v}_{CA|F}}) \int dm$$

constant term  
w.r.t integ.

↓  
total mass  $m$

$$= m \underline{r}_{ca} \times \underline{v}_{CA|F}$$

Therefore,

$$\underline{H}_{A|F} = \underline{H}_{C|F} + m \underline{r}_{ca} \times \underline{v}_{CA|F}$$

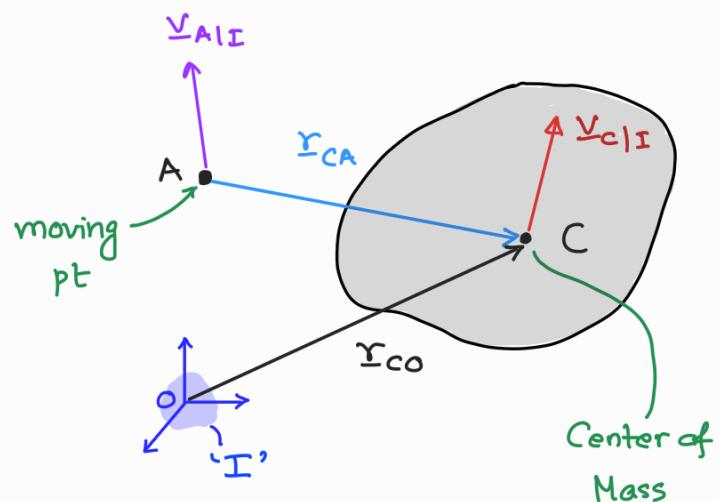
The above angular momentum transfer rule is valid for any reference frame F, and therefore is also valid for inertial frames of reference.

Next, we intend to find a relationship between

$$\dot{H}_{A|I} = \frac{d}{dt} \left\{ H_{A|I} \right\} \Big|_I \quad \xleftarrow{??} \quad M_A$$

$\dot{H}_{A|I} - M_A$  relationship

- Consider now 'O' as the origin of the inertial frame 'I'
- Pt A is moving reference pt
- Pt C is the COM of RB



Using angular momentum transfer rule, we can write :

$$① \quad H_{A|I} = H_{C|I} + m(r_{CA} \times v_{C|I})$$

$$② \quad H_{O|I} = H_{C|I} + m(r_{CO} \times v_{C|I})$$

Substitute Eq. ② in Eq. ① :

$$H_{A|I} = [H_{O|I} - m(r_{CO} \times v_{C|I})] + m(r_{CA} \times v_{C|I})$$

Differentiate both sides wrt time in the 'I' frame:

$$\dot{H}_{A|I} = \frac{d}{dt} \left\{ [H_{O|I} - m(r_{CO} \times v_{C|I})] + m(r_{CA} \times v_{C|I}) \right\} \Big|_I$$

$$\frac{d}{dt} \left\{ \underline{H}_{O|I} \right\} \Big|_I = \dot{\underline{H}}_{O|I} = \underline{M}_O \leftarrow \text{Net moment due to all ext. forces and couples about } O, O \in I$$

$$\frac{d}{dt} \left( \underline{r}_{CO} \times \underline{v}_{CO|I} \right) \Big|_I = \frac{d}{dt} \left\{ \underline{r}_{CO} \right\} \Big|_I \times \underline{v}_{CO|I} + \underline{r}_{CO} \times \frac{d}{dt} \left\{ \underline{v}_{CO|I} \right\} \Big|_I$$

$$= \underbrace{\underline{v}_{CO|I} \times \underline{v}_{CO|I}}_{\underline{\Omega}} + \underline{r}_{CO} \times \underline{\alpha}_{CO|I}$$

$$\frac{d}{dt} \left( \underline{r}_{CA} \times \underline{v}_{CA|I} \right) \Big|_I = \frac{d}{dt} \left\{ \underline{r}_{CA} \right\} \Big|_I \times \underline{v}_{CA|I} + \underline{r}_{CA} \times \frac{d}{dt} \left\{ \underline{v}_{CA|I} \right\} \Big|_I$$

$$= \underbrace{\underline{v}_{CA|I} \times \underline{v}_{CA|I}}_{\underline{\Omega}} + \underline{r}_{CA} \times \underline{\alpha}_{CA|I}$$

$$\dot{\underline{H}}_{A|I} = \dot{\underline{H}}_{O|I} - m (\underline{r}_{CO} \times \underline{\alpha}_{CO|I}) + m (\underline{r}_{CA} \times \underline{\alpha}_{CA|I})$$

$$\underline{r}_{CO} = \underline{r}_c - \cancel{\underline{r}_O}^{\underline{\Omega}} = \underline{r}_c, \quad \underline{r}_{CA} = \underline{r}_c - \underline{r}_A$$

$$\underline{\alpha}_{CO|I} = \underline{\alpha}_{CI|I} - \cancel{\underline{\alpha}_{OI|I}}^{\underline{\Omega}} = \underline{\alpha}_{CI|I}, \quad \underline{\alpha}_{CA|I} = \underline{\alpha}_{CI|I} - \underline{\alpha}_{AI|I}$$

$$\dot{\underline{H}}_{A|I} = \dot{\underline{H}}_{O|I} - m (\underline{r}_c \times \underline{\alpha}_{CI|I}) + m [\underline{r}_{CA} \times (\underline{\alpha}_{CI|I} - \underline{\alpha}_{AI|I})]$$

$$= \dot{\underline{H}}_{O|I} - m (\underline{r}_c - \underline{r}_{CA}) \times \underline{\alpha}_{CI|I} - m \underline{r}_{CA} \times \underline{\alpha}_{AI|I}$$

$$= \underbrace{\dot{\underline{H}}_{O|I}}_{= \underline{M}_O} - \underline{r}_A \times \underbrace{(m \underline{\alpha}_{CI|I})}_{= F_R \text{ (net force)}} - \underline{r}_{CA} \times (m \underline{\alpha}_{AI|I})$$

(Euler's  
2nd axiom)

(Euler's  
1st axiom)

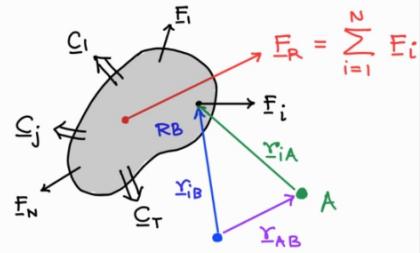
$$\Rightarrow \dot{H}_{A|I} = \underline{M}_A - (\underline{r}_A \times \underline{F}_R) - (\underline{r}_{CA} \times m \underline{a}_{CA|I})$$

Recall  
from Lec 8 :

$$\underline{M}_o = \underline{M}_A + \underline{r}_{AO} \times \underline{F}_R = \underline{M}_A + \underline{r}_A \times \underline{F}_R$$

Moment abt pt B due to force system 1

$$\begin{aligned}\underline{M}_B &= \sum_{i=1}^N \underline{r}_{iB} \times \underline{F}_i + \sum_{j=1}^T \underline{C}_j \\ &= \sum_{i=1}^N (\underline{r}_{AB} + \underline{r}_{iA}) \times \underline{F}_i + \sum_{j=1}^T \underline{C}_j \\ &= \underbrace{\sum_{i=1}^N \underline{r}_{iA} \times \underline{F}_i}_{\underline{M}_A} + \sum_{j=1}^T \underline{C}_j + \sum_{i=1}^N \underline{r}_{AB} \times \underline{F}_i \\ &= \underline{M}_A + \underline{r}_{AB} \times \left( \sum_{i=1}^N \underline{F}_i \right)\end{aligned}$$



$$\Rightarrow \underline{M}_B = \underline{M}_A + \underline{r}_{AB} \times \underline{F}_R \quad \text{--- } \otimes$$

$$\dot{H}_{A|I} = \underline{M}_A - \underline{r}_{CA} \times m \underline{a}_{CA|I}$$

← Alternative form  
of Euler's 2nd  
axiom for moving  
ref. point A

Revisiting the question:

Given any general point 'A' moving w.r.t. I

Is  $\dot{H}_{A|I} = \underline{M}_A$  VALID? If so, under what circumstances?

$\dot{H}_{AII} = \underline{M}_A$  is valid when

### Circumstances

$$\underline{r}_{CA} \times m \underline{\alpha}_{AII} = \underline{0}$$

①  $\underline{r}_{CA} = \underline{0} \Rightarrow A = C \equiv COM$

$\therefore \dot{H}_{cII} = \underline{M}_c$  is ALWAYS valid about COM

OR

②  $\underline{\alpha}_{AII} = \underline{0}$

(point 'A' is NOT accelerating in the inertial frame)

OR

③  $\underline{r}_{CA} \parallel \underline{\alpha}_{AII}$

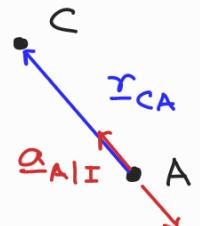
(point 'A' is accelerating towards or away from COM)

At least one of the three

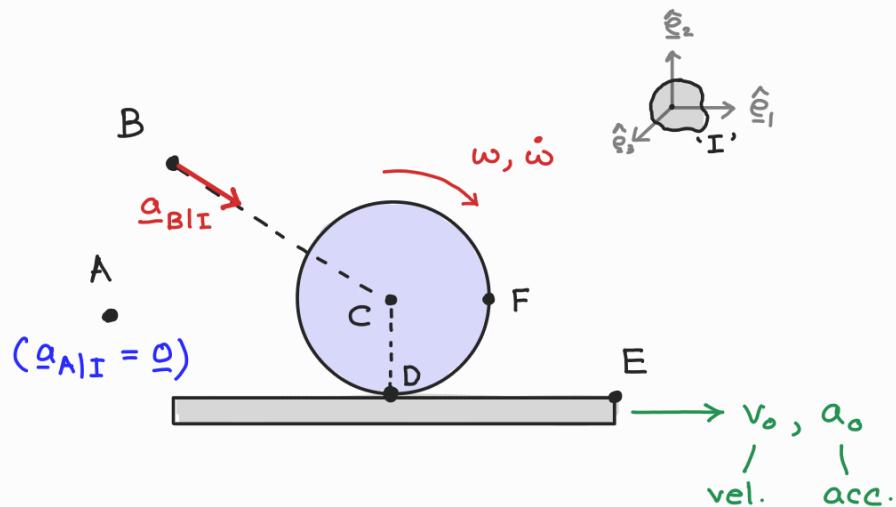
conditions must be met for

point A for equation  $\dot{H}_{AII} = \underline{M}_A$

to be valid for point A



Let's take an example of a uniform disc rolling without slip



$C \equiv$  Center of mass  
(assuming uniform mass distribution)

$D \equiv$  Contact point between plate and disc

In this example, let's list out the points ' $P$ ' for which

$$\dot{H}_{P/I} = M_P \text{ is valid!}$$

Points where valid

Case I:  $v_0 = 0$  and  $a_0 = 0$

A, B, C, D, E

Reason

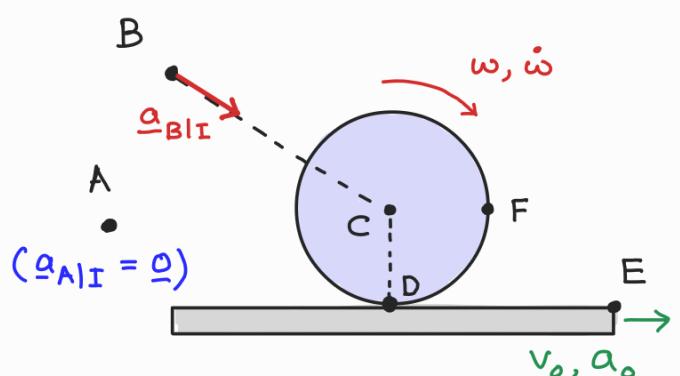
$$A \rightarrow \underline{a}_{A/I} = 0$$

$$B \rightarrow \underline{r}_{CB} \parallel \underline{a}_{B/I}$$

$$C \rightarrow \underline{r}_{CC} = 0 \text{ (COM)}$$

$$D \rightarrow \underline{r}_{CD} \parallel \underline{a}_{D/I} = \omega^2 r \hat{e}_2$$

$$E \rightarrow \underline{a}_{E/I} = 0$$



Points where valid

Case II:  $v_0 \neq 0$  and  $a_0 = 0$

A, B, C, D, E

Same reasons as case I

Points where valid

Case III:  $v_o \neq 0$  and  $a_o \neq 0$

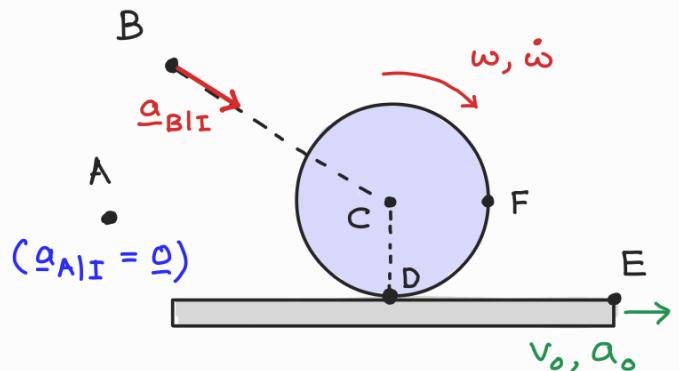
A, B, C

Reason

A  $\rightarrow \underline{a}_{A|I} = \underline{0}$

B  $\rightarrow \underline{r}_{CB} \parallel \underline{a}_{B|I}$

C  $\rightarrow \underline{r}_{cC} = \underline{0}$  (COM)



D X  $\underline{a}_{D|I} = a_o \hat{\underline{e}}_1 + \omega^2 r \hat{\underline{e}}_2$

E X  $\underline{a}_{E|I} = a_o \hat{\underline{e}}_1 \neq 0$

Point F:  $\underline{a}_{F|I} = (-\omega^2 r + a_o) \hat{\underline{e}}_1 - r \dot{\omega} \hat{\underline{e}}_2$  (Verify)

If  $\dot{\omega} = 0$ , then  $\underline{a}_{F|I}$  becomes parallel to  $\underline{r}_{CF}$  and pt.

F becomes a valid point!

accelerating towards  
or away from COM

$\therefore$  pt F is valid when  $\dot{\omega} = 0$