Minor Exam Solutions

APL 104 - 2022 (Solid Mechanics)

Q1. We have learnt in the class relation between stress and strain components based on the following two approaches: (5)

$$\sigma_{ij} = \lambda \ trace(\underline{\epsilon})\delta_{ij} + 2\mu \ \epsilon_{ij}$$

or

$$\epsilon_{11} = \frac{1}{E}(\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})), \ \epsilon_{22} = ..., \ \epsilon_{33} = ...$$
$$\gamma_{12} = \tau_{12}/G, \ \gamma_{23} = ..., \ \gamma_{13} = ...$$

Using the above two equivalent approaches, express (E, G, ν) in terms of (λ, μ).

Solution:
The two formulations can be written as follows in a matrix form

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\end{bmatrix} = \begin{bmatrix}\gamma \ + \ 2\mu & \gamma & \gamma \\ \gamma & \gamma \ + \ 2\mu & \gamma \\ \gamma & \gamma & \gamma \ + \ 2\mu\end{bmatrix}\begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{33}\end{bmatrix} \tag{1}$$

or

$$\begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ \epsilon_{33}\end{bmatrix} = \frac{1}{E}\begin{bmatrix}1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1\end{bmatrix}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\end{bmatrix} \tag{2}$$

Taking the inverse of Eq. (1) and equating to Eq. (2) will give relations $E\,(\gamma, \ \mu)\,, \ \nu\,(\gamma, \ \mu)$. Also note that

$$\sigma_{12} \ = \ 2\mu\epsilon_{12} \ = \ \mu\gamma_{12} \quad \text{(using first approach)} \tag{3}$$
$$\gamma_{12} \ = \ \frac{\tau_{12}}{G} \quad \text{(using first approach)} \tag{4}$$

On comparing Eq. (3) and Eq. (4), we get $\mu \ = \ G$.

Alternative approach

We know $E \ = \ \dfrac{\sigma_{11}}{\epsilon_{11}}$ and $\nu \ = \ -\dfrac{\epsilon_{22}}{\epsilon_{11}} \ = \ -\dfrac{\epsilon_{33}}{\epsilon_{11}}$ from unixial loading. So, we can write

$$\begin{aligned}\sigma_{11} \ &= \ E\epsilon_{11}, \ \sigma_{22} \ = \ \sigma_{33} \ = \ 0\\ \epsilon_{22} \ &= \ \epsilon_{33} \ = \ -\nu\epsilon_{11}.\end{aligned} \tag{5}$$

Let us substitute this special case in the below general equation:

$$\sigma_{ij} \; = \; \lambda \, tr \left(\underline{\underline{E}} \right) \delta_{ij} \; + \; 2\mu\epsilon_{ij}. \tag{6}$$

For $i \; = \; j \; = \; 1$, we get

$$\begin{aligned} \sigma_{11} \; &= \; (\lambda \; + \; 2\mu) \, \epsilon_{11} \; + \; \lambda \, (\epsilon_{22} \; + \; \epsilon_{33}) \\ \Rightarrow \; E\epsilon_{11} \; &= \; (\lambda \; + \; 2\mu) \, \epsilon_{11} \; - \; 2\lambda\nu\epsilon_{11} \\ \Rightarrow \; E \; &= \; (\lambda \; + \; 2\mu) \; - \; 2\lambda\nu. \end{aligned} \tag{7}$$

For $i \; = \; j \; = \; 2$

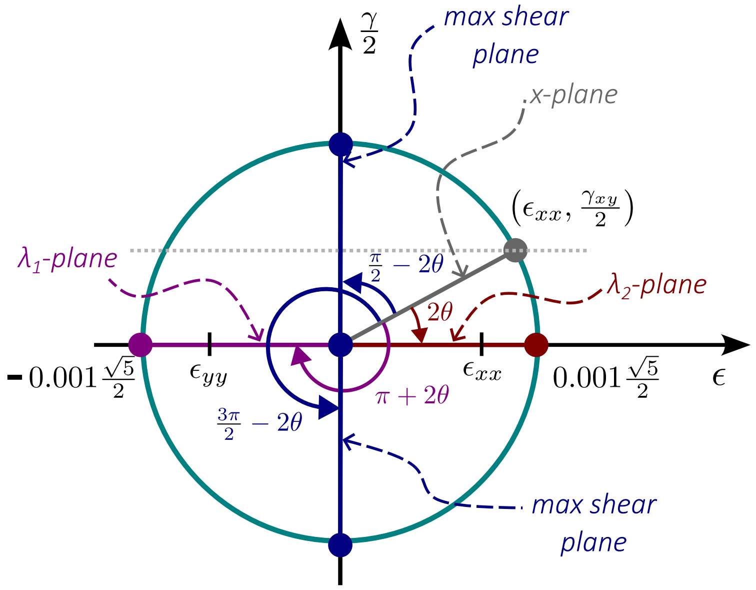
$$\begin{aligned} \sigma_{22} \; &= \; (\lambda \; + \; 2\mu) \, \epsilon_{22} \; + \; \lambda \, (\epsilon_{11} \; + \; \epsilon_{33}) \\ \Rightarrow \; 0 \; &= \; - (\lambda \; + \; 2\mu) \, \nu\epsilon_{11} \; + \; \lambda \, (1 \; - \; \nu) \, \epsilon_{11} \\ \Rightarrow \; \nu \; &= \; \frac{\lambda}{2 \, (\lambda \; + \; \mu)} \end{aligned} \tag{8}$$

From Eq. (7) and Eq. (8),

$$\begin{aligned} E \; &= \; \lambda \; + \; 2\mu \; - \; 2\lambda \, \frac{\lambda}{2 \, (\lambda \; + \; \mu)} \\ &= \; \frac{\lambda^2 \; + \; 3\lambda\mu \; + \; 2\mu^2 \; - \; \lambda^2}{(\lambda \; + \mu)} \\ &= \; \frac{\mu \, (3\lambda \; + \; 2\mu)}{\lambda \; + \; \mu}. \end{aligned}$$

Q2. A thin rectangular sheet is strained with uniform strain components: $\epsilon_{xx} = 0.001$, $\epsilon_{yy} = -0.001$, $\gamma_{xy} = 0.001$. All other strain components are zero implying it is a plane strain condition.
(i) Draw Mohr's circle for this case. (3)
(ii) How much will be the percentage change in the area of thin sheet? (2)
(iii) Which line elements undergo maximum and minimum longitudinal/normal strain? (3)
(iv) What will be the orientation of the two perpendicular lines in the plane of sheet which undergo maximum angle change between them? (2)

Solution:
$\epsilon_{xx} = 0.001$, $\epsilon_{yy} = 0.001$, $\gamma_{xy} = 0.001$, others are all zero!



(i) The Mohr's circle is drawn above.

$$
\begin{aligned}
\tan 2\theta &= \frac{\gamma_{xy}/2}{(\epsilon_{xx} - \epsilon_{yy})/2} \\
&= \frac{0.0005}{0.001} = \frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
\text{Center of the circle} &= \frac{\epsilon_{xx} + \epsilon_{yy}}{2} = 0 \\
\text{Radius} &= \sqrt{\left(\frac{\epsilon_{xx} - \epsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \\
&= \sqrt{0.001^2 + \frac{0.001^2}{4}} \\
&= 0.001\frac{\sqrt{5}}{2}
\end{aligned}
$$

(ii) Percentage change in area of the sheet $= (\epsilon_{xx} + \epsilon_{yy}) \times 100 = 0\%$

(iii) To find line elements undergoing max. and min. longitudinal strain (principal strains), we use the Mohr's circle. The principal strains are:

- $\dfrac{\epsilon_{xx} + \epsilon_{yy}}{2} + R = R$: the line element that undergoes maximum longitudinal strain lie along the normal of maximum principal strain plane. Its normal is oriented at

$$2\theta = \tan^{-1}\left(\frac{1}{2}\right) \text{ clockwise from X-axis in Mohr's cirle}$$
$$\Rightarrow \theta = \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right) \text{ anti-clockwise from X-axis physically.}$$

- $\dfrac{\epsilon_{xx} + \epsilon_{yy}}{2} - R = -R$; the line element that undergoes minimum longitudinal strain lie along the normal of minimum principal strain plane. Its normal is oriented at

$$\pi + 2\theta = \pi + \tan^{-1}\left(\frac{1}{2}\right) \text{ clockwise from X-axis in Mohr's cirle}$$
$$\Rightarrow \frac{\pi}{2} + \theta = \frac{\pi}{2} + \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right) \text{ anti-clockwise from X-axis physically.}$$

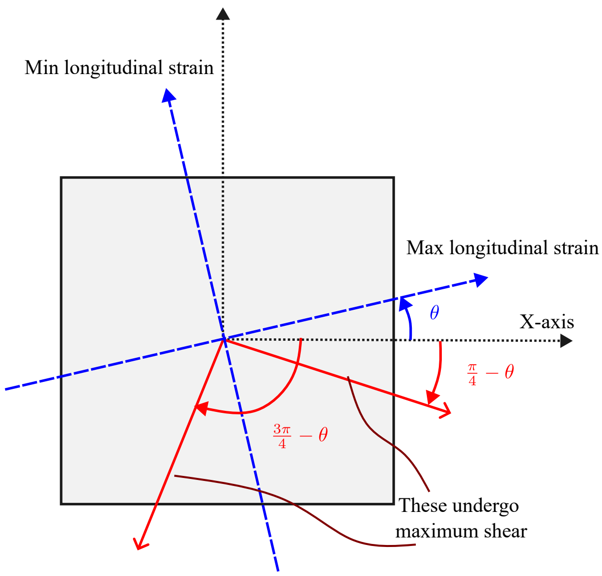
(iv) Maximum shear strain will occur between those two line elements which correspond to top and bottom points in the circle. The corresponding directions are:

- First line element oriented along

$$\left(\frac{\pi}{2} - 2\theta\right) \text{ anti-clockwise from X-axis in Mohr's circle}$$
$$\Rightarrow \left(\frac{\pi}{4} - \theta\right) \text{ clockwise from X-axis physically}$$

- Second line element oriented along

$$\left(\frac{3\pi}{2} - 2\theta\right) \text{ anti-clockwise from X-axis in Mohr's circle}$$
$$\Rightarrow \left(\frac{3\pi}{4} - \theta\right) \text{ clockwise from X-axis physically}$$

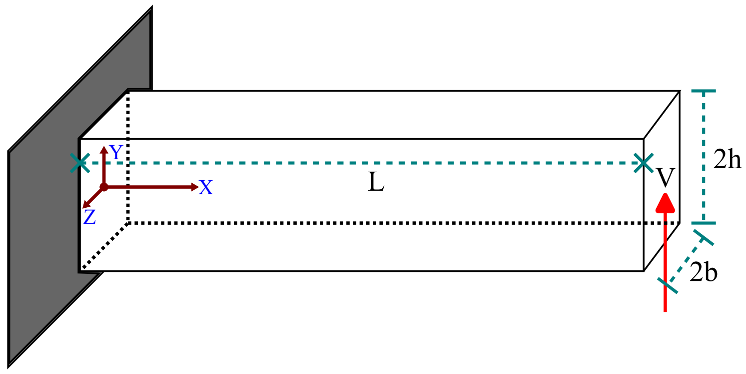


Q3. Think of a rectangular beam clamped at one end and subjected to a transverse load V at the other end. The beam length is L, height $2h$ and breadth $2b$. Think of a coordinate system wherein the origin lies at the center of the clamped end cross-section and the beam runs along x-axis while the load is applied in y-direction. One can show that the following state of stress generates in the beam:

$$\sigma_{xx} = c_1 V(L-x)y, \quad \tau_{xy} = \tau_{yx} = c_2 V(h^2 - y^2).$$

Here c_1 and c_2 are constants. The other stress components are zero.
(i) Draw Mohr's circle for this state of stress. (3)
(ii) What are the principal stress components and maximum value of shear stress in the beam? (3)
(iii) On what plane is the maximum shear stress realized? (2)

Solution: The problem is shown below:

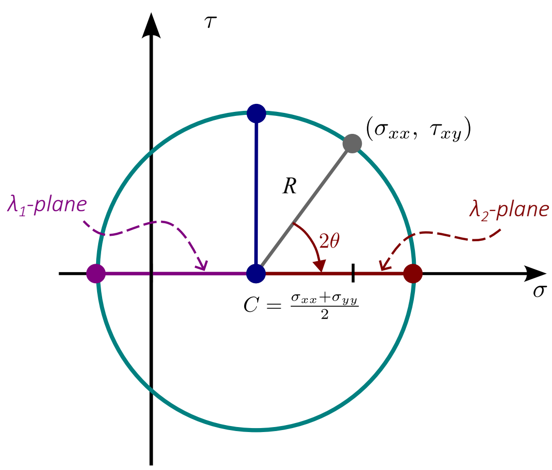


Given

$$\sigma_{xx} = C_1 V (L - X) y, \quad \tau_{xy} = C_2 V \left(h^2 - y^2\right)$$

Notice that $\tau_{xy} > 0$ always but $\sigma_{xx} > 0$ for $y > 0$ and < 0 for $y < 0$!

(i) Using the Mohr's circle



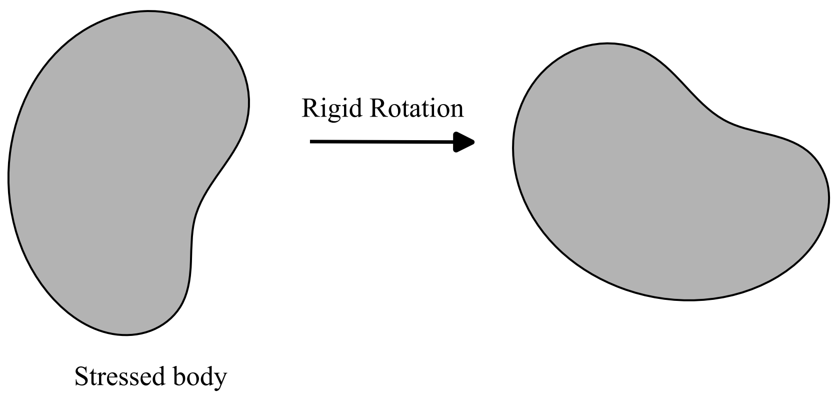
$$
\begin{aligned}
C &= \frac{\sigma_{xx} + \sigma_{yy}}{2} = \frac{C_1}{2} V (L - X) y, \\
R &= \sqrt{\left(\frac{C_1}{2} V (L - X) y\right)^2 + \left(C_2 V \left(h^2 - y^2\right)\right)^2}
\end{aligned}
$$

(ii) Principal stress component: $\frac{C_1}{2} V (L - X) y \pm R,\ 0$
The two non-zero principal stress components will always be of opposite sign irrespective of the sign of y! Hence, this pair will lead to generation of maximum shear stress and equal to R.

(iii) Max shear is realized at $\left(\frac{\pi}{4} - \theta\right)$ clockwise from x-axis where $\tan 2\theta = \frac{\tau_{xy}}{(\sigma_{xx} - \sigma_{yy})/2}$.

Q4. Suppose a body is under stress and the stress tensor at an arbitrary point $\underline{x}$ in the body is denoted by $\underline{\underline{\sigma}}$. Further, suppose the stressed body is then rigidly rotated where the rotation is denoted by the tensor $\underline{\underline{R}}$. Let us think of a coordinate system attached with the stressed body which also rotates when the stressed body rotates.
(i) What can you say about the stress matrix in the body before and after rotation in the coordinate system attached with the body? (3)
(ii) What happens to the stress matrix in a fixed coordinate system not attached to the body? Does it change after rotation? If so, what does it change to? (2)
(iii) What happens to the stress tensor due to rotation of the stressed body? Does it change? If so, what does it change to? (2)

Solution:



(i) Keep in mind that rigid rotation does not induce extra stain in the body! Further, stress being proportional to strain, stress components in a body-fixed coordinate system should not change! Hence, no change in stress matrix in the coordinate system fixed to body!

(ii) Suppose the rotation tensor for rigid rotation is $\underline{\underline{R}}$. Furthermore, let $\underline{\underline{\sigma}} = \sum_i \underline{t}^i \otimes \underline{e}_i$ before rotation! Concentrate on these three planes with their normals along coordinate axis.

After rotation

$$\underline{e}_i \;\rightarrow\; \underline{\underline{R}}\,\underline{e}_i, \quad \underline{t}^i \;\rightarrow\; \underline{\underline{R}}\,\underline{t}^i$$

So,

$$\underline{\underline{\sigma}}^R(\text{after rotation}) \;=\; \sum_i \underline{t}^{i,R} \otimes \underline{e}_i^R = \;\sum_i\; \underline{\underline{R}}\,\underline{t}^i \otimes \underline{\underline{R}}\,\underline{e}_i = \;\underline{\underline{R}}\left(\sum_i \underline{t}^i \otimes \underline{e}_i\right)\underline{\underline{R}}^T = \underline{\underline{R}}\,\underline{\underline{\sigma}}\,\underline{\underline{R}}^T.$$

(iii) Stress tensor does change. In fact, it transforms to $\underline{\underline{R}}\,\underline{\underline{\sigma}}\,\underline{\underline{R}}^T$.
In a fixed coordinate system (say $\underline{E}_1 - \underline{E}_2 - \underline{E}_3$):

$$\begin{aligned}\left[\underline{\underline{\sigma}}^R\right]_{[\underline{E}_1-\underline{E}_2-\underline{E}_3]} &= \left[\underline{\underline{R}}\,\underline{\underline{\sigma}}\,\underline{\underline{R}}^T\right]_{[\underline{E}_1-\underline{E}_2-\underline{E}_3]} \\ &= \left[\underline{\underline{R}}\right]_{[\underline{E}_1-\underline{E}_2-\underline{E}_3]}\left[\underline{\underline{\sigma}}\right]_{[\underline{E}_1-\underline{E}_2-\underline{E}_3]}\left[\underline{\underline{R}}^T\right]_{[\underline{E}_1-\underline{E}_2-\underline{E}_3]}.\end{aligned}$$

Hence, Stress matrix in $(\underline{E}_1 - \underline{E}_2 - \underline{E}_3)$ coordinate system changes!