Tutorial 9 solution

APL 104 - 2022 (Solid Mechanics)

Q1. Suppose a solid disk of radius 'R' and a hollow disk of inner radius R^1 and outer radius R^2 are shrunk fit together (assume R $>$ R^1). Further assume that no external pressure is applied on the outer hollow disk and that no axial displacement is allowed in the two disks. Let the two disks be made of same material. Only radial displacement u_r generates in this case.

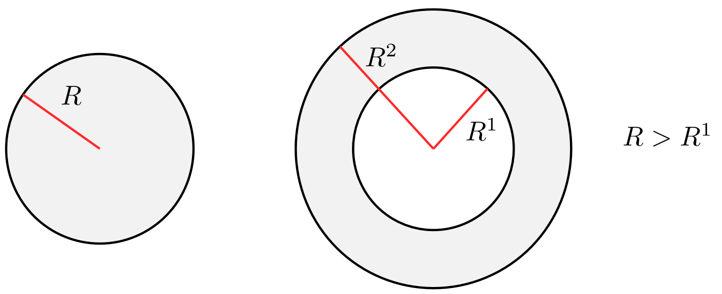
(a) Write down all the boundary conditions/ interface conditions required to obtain variation in radial stress σ_{rr} in the two disks.

(b) Solve the governing equations and obtain expression for σ_{rr} in the two disks.

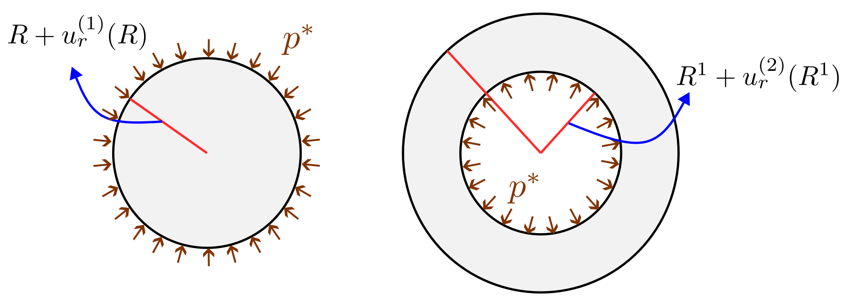
(c) Obtain expression for circumferential/hoop stress too.

(d) Draw plots for variation of both radial and circumferential stress.

Solution:



(a) Before shrink fit



(b) On shrink fitting

Shrink-fit insertion is usually performed by heat treatment, i.e., by heating the hollow disc (thereby expanding its diameter), fitting it over the solid disc and then cooling it. This results in the hollow cylinder to *prestress* itself and sit tightly over the solid disc. Alternatively, you could imagine that shrink-fit results in application of a pressure p^* over the external surface of the solid disk and over the inner surface of the hollow disk, as required to eliminate the overlap $R - R^1$.

We denote the solid disc and the hollow disc by superscripts (1) and (2) respectively. For example, $\sigma_{rr}^{(1)}$ and $\sigma_{rr}^{(2)}$ represent radial stress in solid and hollow discs, respectively. For the problem at hand, the discs will only undergo displacement in the radial direction, i.e. $u_\theta = u_z = 0$. As no body force acts, the radial equation when solved leads to the same solution for radial displacement or radial stress as in the class. In terms of radial stress, e.g.,

$$\sigma_{rr}^{(1)}(r) = A^{(1)}/2 + B^{(1)}/r^2, \quad \sigma_{rr}^{(2)}(r) = A^{(2)}/2 + B^{(2)}/r^2$$

or, in terms of radial displacement, we have

$$u_r^{(1)}(r) = C^{(1)}r/2 + D^{(1)}/r, \quad u_r^{(2)}(r) = C^{(2)}r/2 + D^{(2)}/r.$$

For the current problem where axial strain $\epsilon = 0$, we had derived in the class

$$A^{(i)} = 2(\lambda + \mu)C^{(i)}, \quad B^{(i)} = -2\mu\ D^{(i)}.$$

We have basically four independent integrating constants which will require four boundary conditions as mentioned below:

(a) Boundary conditions:

$$\begin{aligned}
\sigma_{rr}^{(1)}(R) &= \sigma_{rr}^{(2)}\left(R^1\right) \text{ (continuity of radial stress)}\\
R + u_r^{(1)}(R) &= R_1 + u_r^{(2)}\left(R^1\right) \text{ (Shrink-fit condition)}\\
\sigma_{rr}^{(2)}(R_2) &= 0 \text{ (Traction-free boundary condition)}\\
\sigma_{rr}^{(1)}(0) &= \text{finite!} \quad \text{or } u_r^{(1)}(0) = 0.
\end{aligned}$$

Note that radial stress is continuous at the interface. The same cannot be said about hoop stress. In fact, the three traction components on radial plane will only get continuous across the interface because they form action-reaction pairs on the two radial faces of solid and hollow cylinders at the interface.

(b) In terms of the unknown constants appearing in the displacement formula above, we can write the following for radial stress:

$$\sigma_{rr}^{(1)}(r) = (\lambda + \mu)C^{(1)} - 2\mu\ D^{(1)}/r^2, \quad \sigma_{rr}^{(2)}(r) = (\lambda + \mu)C^{(2)} - 2\mu\ D^{(2)}/r^2.$$

Next, we apply the boundary conditions to determine the four integrating constants.

Using $\sigma_{rr}^{(1)}(0) = \text{finite} \Rightarrow D^{(1)} = 0$.

Applying first boundary condition: $\sigma_{rr}^{(1)}(R) = \sigma_{rr}^{(2)}\left(R^1\right)$

$$\Rightarrow (\lambda + \mu)\,C^{(1)} = (\lambda + \mu)\,C^{(2)} - 2\mu\,\frac{D^{(2)}}{(R^1)^2}, \tag{1}$$

Applying second boundary condition: $u^{(1)}(R) - u^{(2)}\left(R^1\right) = R^1 - R$

$$\Rightarrow C^{(1)}\frac{R}{2} - C^{(2)}\frac{R^1}{2} - \frac{D^{(2)}}{R^1} = R^1 - R, \tag{2}$$

Applying third boundary condition: $\sigma_{rr}^{(2)}(R_2) = 0$

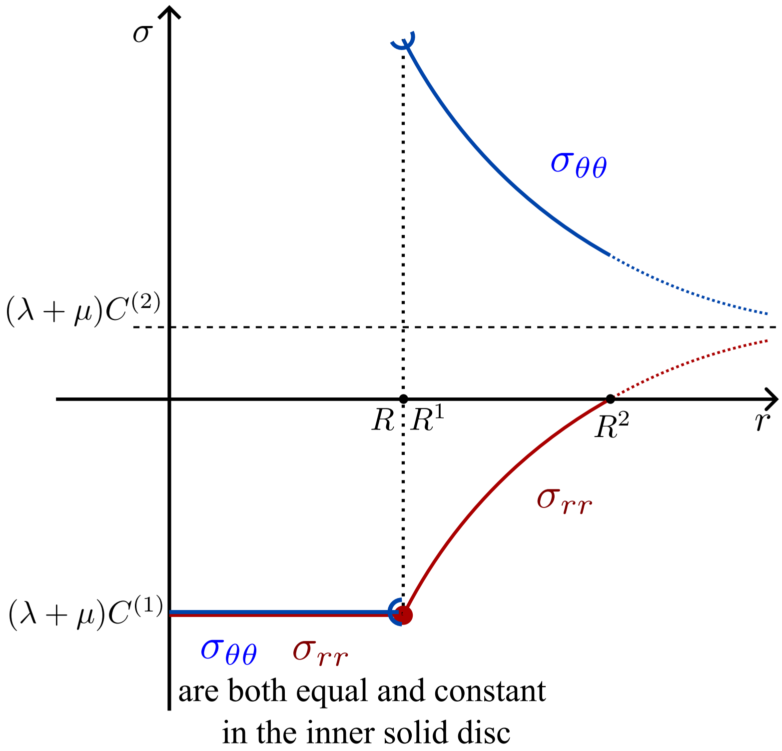
$$\Rightarrow (\lambda + \mu)\,C^{(2)} - 2\mu\,\frac{D^{(2)}}{(R^2)^2} = 0. \tag{3}$$

Solving equations 1, 2 and 3, we obtain get $C^{(1)},\ C^{(2)},\ D^{(2)}$.

(c) Once the constants are known, we can get circumferential stress $\sigma_{\theta\theta}$ as follows:

$$\sigma_{\theta\theta}^{(1)}(r) = (\lambda+\mu)C^{(1)}, \quad \sigma_{\theta\theta}^{(2)}(r) = (\lambda+\mu)C^{(2)} + 2\mu\ D^{(2)}/r^2.$$

(d) The plot of radial and circumferential stress variation would be as shown below. Note that σ_{rr} is continuous but $\sigma_{\theta\theta}$ is discontinuous!



Q2. Think of an isotropic solid cylinder which is glued to a rigid plate at both its ends. The two rigid plates are then pulled apart along the axis of the cylinder such the normals of the rigid plates remain aligned with the axis of the cylinder. It turns out that u_θ is zero in this case but u_r and u_z do arise. Furthermore, (u_r, u_z) are not functions of θ coordinate. Assume that the deformation of the cylinder is such that every planar cross-section of the cylinder (z-plane or axial planes) remains planar even after deformation but it does change its radius.

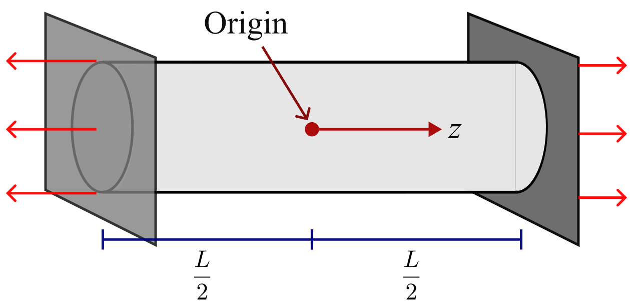
(a) What can you say about the dependence of u_r and u_z on radial and axial coordinates (r, z)?

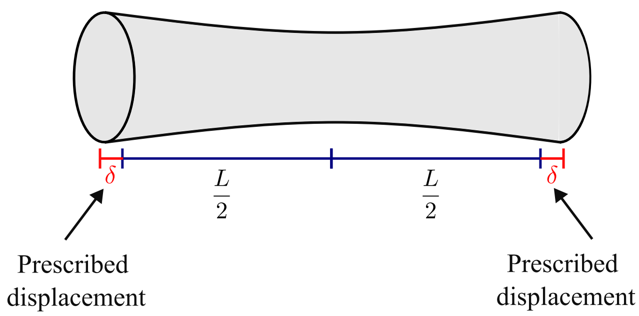
(b) Obtain the strain matrix and stress matrix for the above problem.

(c) Show that the θ-component of stress equilibrium equation is automatically satisfied.

(d) What boundary condition will be used in order to solve the above deformation problem?

Solution:





(a) As the rigid plates are glued to ends of the cylinder, the end cross-section can neither shrink or expand. But, as the plates are pulled apart, the cylinder undergoes extension which will mean that the radius of cylinder will change non-uniformly!

Therefore, u_r is a function of z. Also, u_r will depend on r since the value of u_r ar $r = 0$ is zero but it is non-zero for other values of r.

Finally, u_z does not depend or r otherwise a plane cross-section becomes non-planar (recall from Tutorial 7 Q5). It will depend on z though otherwise there will be no axial strain! (Note $\partial u_z/\partial z$ is axial strain).

(b) Strain matrix

$$
\underline{\underline{\epsilon}} = \begin{bmatrix} \dfrac{\partial u_r}{\partial r} & \dfrac{1}{2}\left(\dfrac{1}{r}\cancelto{0}{\dfrac{\partial u_r}{\partial \theta}} + \cancelto{0}{\dfrac{\partial u_\theta}{\partial r}} - \cancelto{0}{\dfrac{u_\theta}{r}}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_r}{\partial z} + \cancelto{0}{\dfrac{\partial u_z}{\partial r}}\right) \\ & \dfrac{1}{r}\cancelto{0}{\dfrac{\partial u_\theta}{\partial \theta}} + \dfrac{u_r}{r} & \dfrac{1}{2}\left(\cancelto{0}{\dfrac{\partial u_\theta}{\partial z}} + \dfrac{1}{r}\cancelto{0}{\dfrac{\partial u_z}{\partial \theta}}\right) \\ \text{sym} & & \dfrac{\partial u_z}{\partial z} \end{bmatrix}
$$

$$
\Rightarrow \underline{\underline{\epsilon}} = \begin{bmatrix} \dfrac{\partial u_r}{\partial r} & 0 & \dfrac{1}{2}\dfrac{\partial u_r}{\partial z} \\ 0 & \dfrac{u_r}{r} & 0 \\ \dfrac{1}{2}\dfrac{\partial u_r}{\partial z} & 0 & \dfrac{\partial u_z}{\partial z} \end{bmatrix}.
$$

Notice that $\gamma_{rz} \neq 0$ which is a departure from the case of uniform extension-torsion-inflation case. To obtain stress matrix, use $\sigma_{ij} = \lambda\, tr\left(\underline{\underline{\epsilon}}\right)\delta_{ij} + 2\mu\epsilon_{ij}$ which leads to

$$
\begin{aligned}
\sigma_{rr} &= \lambda\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right) + 2\mu\frac{\partial u_r}{\partial r} \\
\sigma_{\theta\theta} &= \lambda\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right) + 2\mu\frac{u_r}{r} \\
\sigma_{zz} &= \lambda\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right) + 2\mu\frac{\partial u_z}{\partial z} \\
\tau_{r\theta} &= 2G\epsilon_{r\theta} = 0 \\
\tau_{rz} &= 2G\epsilon_{rz} = G\frac{\partial u_r}{\partial z}\ ,\ \tau_{\theta z} = 0
\end{aligned}
$$

(c) The θ-component of stress-equilibrium equation is

$$
\frac{\partial \tau_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + 2\frac{\tau_{\theta r}}{r} + b_\theta = \rho a_\theta.
$$

In this equation,

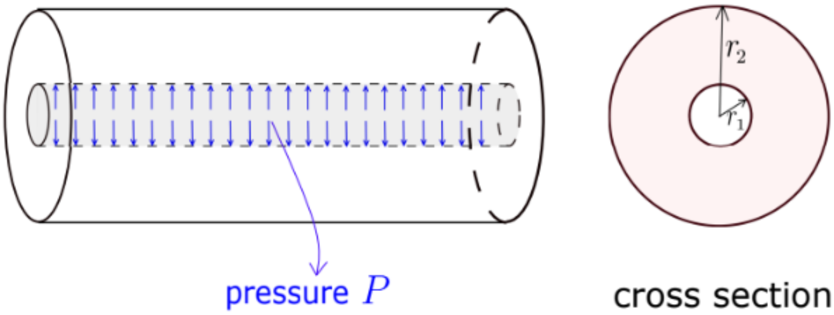
$$
\begin{aligned}
&\tau_{\theta r} = \tau_{r\theta} = 0,\ \text{so}\ \frac{\partial \tau_{\theta r}}{\partial r} = 0 \text{ and } \frac{\tau_{\theta r}}{r} = 0, \\
&\sigma_{\theta\theta} \text{ is not dependent on } \theta,\ \text{so}\ \frac{\partial \sigma_{\theta\theta}}{\partial \theta} = 0, \\
&\tau_{\theta z} = 0,\ \text{so}\ \frac{\partial \tau_{\theta z}}{\partial z} = 0 \\
&\text{no body force and statics condition, so } b_\theta = a_\theta = 0
\end{aligned}
$$

So, θ-equation is automatically satisfied! It happened due to the assumption of axisymmetry which allows the reduction of equilibrium equation from the whole three-dimensional cylindrical domain to just its $r - z$ plane.

(d) The boundary conditions along the four edges of $r - z$ plane are:

$$
\begin{aligned}
\text{At } z &= L/2 \\
u_r\,(r, z = L/2) &= 0 \\
u_z\,(r, z = L/2) &= \text{prescribed displacement!} \\
\text{At } z &= -L/2 \\
u_r\,(r, z = -L/2) &= 0 \\
u_z\,(r, z = -L/2) &= \text{prescribed displacement!} \\
\text{At } r &= 0 \\
u_r\,(r = 0, z) &= 0 \text{ (this also implies } \frac{\partial u_r}{\partial z}(r = 0, z) = 0 \text{ or } \tau_{rz}(r = 0, z) = 0) \\
\sigma_{rr}\,(r = 0, z) &\text{ is bounded!} \\
\text{At } r &= R \text{ (outer surface) traction free} \\
\Rightarrow \begin{bmatrix} \sigma_{rr} \\ \tau_{\theta r} \\ \tau_{zr} \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \sigma_{rr} = \tau_{zr} = 0 \ (\tau_{\theta r} = 0 \text{ is trivially satisfied})
\end{aligned}
$$

Q3. In class, we worked out distribution of the radial stress, the hoop stress as well as the axial stress for thick tubes. Work out their thin tube approximations which is useful for pressure vessels. Can you obtain them directly from free body diagram of various sections of pressure vessels and doing their force balance?



Solution: In class, it was derived that for an internally pressurized hollow cylinder

$$\sigma_{rr} = \frac{A}{2} + \frac{B}{r^2}, \quad \sigma_{\theta\theta} = \frac{A}{2} - \frac{B}{r^2}$$

where the integration constants were worked out using the boundary conditions for internally pressurized cylinder to be

$$A = 2P\frac{r_1^2}{r_2^2 - r_1^2}, \qquad B = -P\frac{r_1^2 r_2^2}{r_2^2 - r_1^2}.$$

For thin cylinders, one can approximate the outer radius r_2 to be almost equal to r_1, i.e., $r_2 \approx r_1$ and the thickness being $t = r_2 - r_1$. Using these, we can arrive at simpler values of A and B as follows:

$$\begin{aligned} A &= 2P\frac{r_1^2}{r_2^2 - r_1^2} = 2P\frac{r_1^2}{(r_2 + r_1)(r_2 - r_1)} \approx 2P\frac{r_1^2}{2r_1 t} = \frac{Pr_1}{t} \\ B &= -P\frac{r_1^2 r_2^2}{r_2^2 - r_1^2} = -P\frac{r_1^2 r_2^2}{(r_2 + r_1)(r_2 - r_1)} \approx -P\frac{r_1^2 r_1^2}{2r_1 t} = -\frac{Pr_1^3}{2t} \end{aligned}$$

Now, substituting the values of A and B back in the expression for σ_{rr} and $\sigma_{\theta\theta}$, and letting $r \approx r_1$ we get:

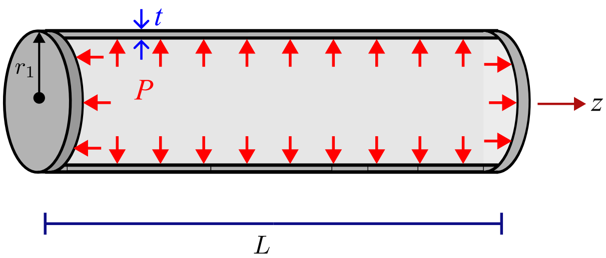
$$\begin{aligned} \sigma_{rr} &= \frac{A}{2} + \frac{B}{r_1^2} = \frac{Pr_1}{2t} - \frac{Pr_1^3}{2tr_1^2} = \frac{Pr_1}{2t} - \frac{Pr_1}{2t} = 0, \\ \sigma_{\theta\theta} &= \frac{A}{2} - \frac{B}{r_1^2} = \frac{Pr_1}{2t} + \frac{Pr_1^3}{2tr_1^2} = \frac{Pr_1}{2t} + \frac{Pr_1}{2t} = \frac{Pr_1}{t}. \end{aligned}$$

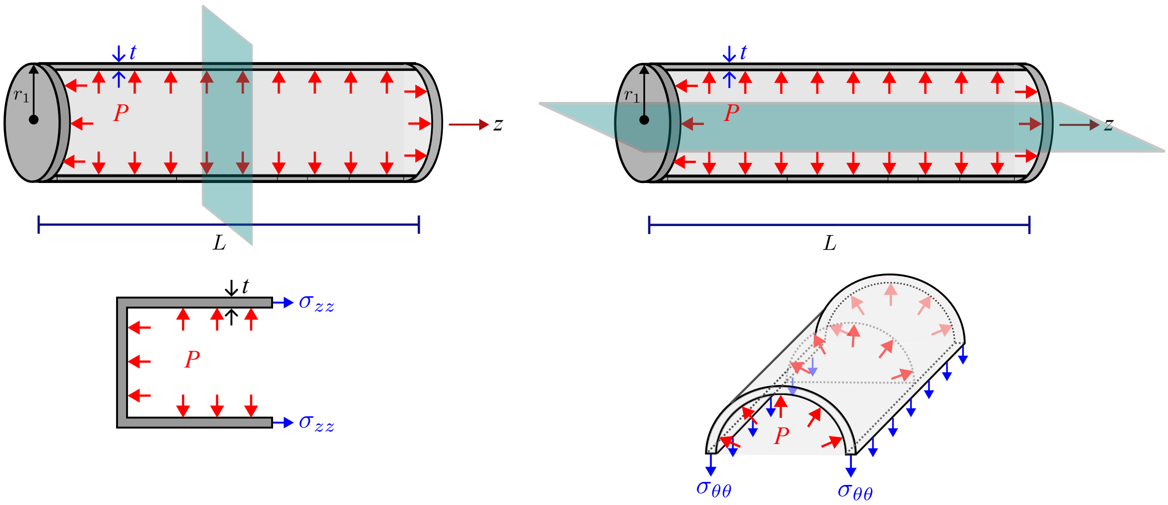
To obtain thin tube approximation of σ_{zz}, we had seen in class that σ_{zz} does not vary through the wall thickness. Accordingly when no axial force acts on the cylinder, we have

$$\sigma_{zz} = \frac{\pi\, r_1^2\, P}{\pi(r_2^2 - r_1^2)} \approx \frac{Pr_1}{2t}.$$

Instead of deriving the thin-tube approximations from the formula of thick cylinders, we will now consider deriving them directly from force balance of various sections of thin cylinder.

Let us consider an internally pressurized cylinder *with capped ends* and having radius r_1 and thickness t (see below).





To obtain σ_{zz}, we take a vertical section through the cylinder and do force balance in $z-$direction, i.e.,

$$\sigma_{zz}\left(2\pi r_1 t\right) = P\left(\pi r_1^2\right) \text{ or } \sigma_{zz} = \frac{P r_1}{2t}.$$

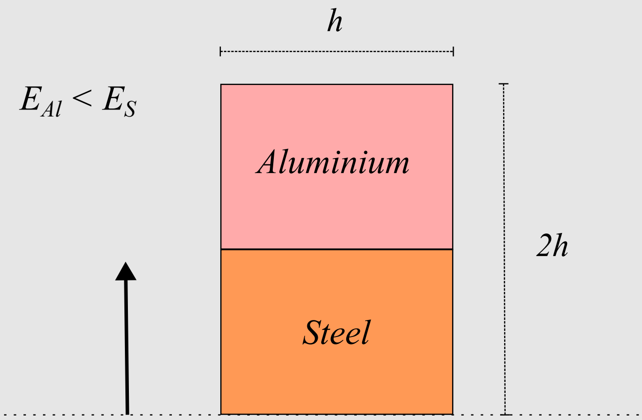
To obtain $\sigma_{\theta\theta}$, we take a horizontal section through the cylinder, and do force balance in vertical direction, i.e.,

$$\sigma_{\theta\theta}\left(2Lt\right) = P\left(2r_1 L\right) \text{ or } \sigma_{\theta\theta} = \frac{P r_1}{t}.$$

Note that pressure can be thought to be acting over the projected area of $2r_1L$.

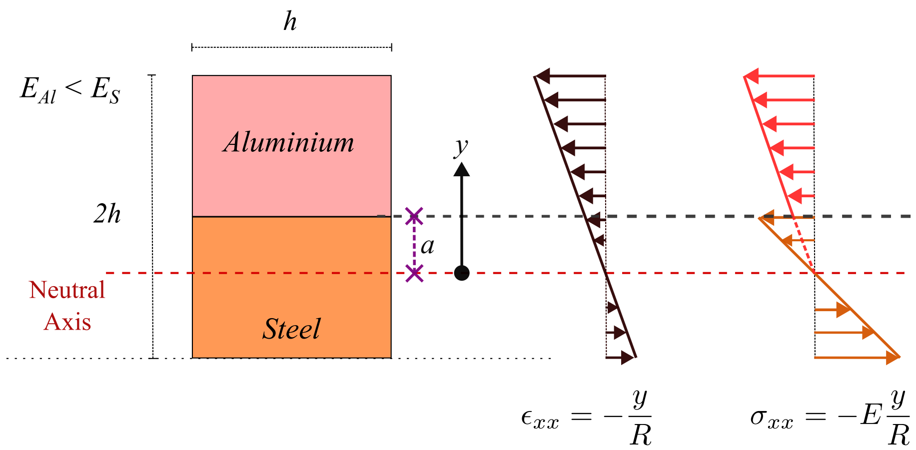
To obtain radial stress σ_{rr}, consider the outer traction-free surface $(\sigma_{rr} = 0)$ and the internally pressurized surface $(\sigma_{rr} = -P)$. Through the thickness of thin tube, the radial stress drops sharply from $-P$ at inner surface to zero at the outer surface. It must be emphasized that $\dfrac{r_1}{t} >> 1$ for thin tubes. Accordingly, the values of $\dfrac{Pr_1}{t}$ and $\dfrac{Pr_1}{2t}$ for hoop and axial stresses are very large compared to P. Accordingly, σ_{rr} is neglected and set to zero.

Q4. Think of a composite beam having rectangular cross-section such that one half of the cross-section (having square shape) is aluminium while the other half is steel. When such a beam is bent, where will the neutral axis lie in the cross-section (calculated from the bottom line of cross-section)?



Solution: Composite beams are constructed from more than one materials to increase their strength. In class, you have only seen beams made up of just one material. In general, the neutral axis of a composite beam will not lie at the geometric centroid of the cross-section.

Let us suppose that the neutral axis (NA) lies at a distance 'a' from the midline of the cross-section. The bending strain profile will be linear, continuous and passes through zero at the NA as shown below.



It obeys the following formula:

$$\epsilon_{xx} = -\frac{y}{R}$$

where y is the distance from neutral axis. However, the bending stress profile will be discontinuous at the location where the material changes. It follows the folowing formula:

$$\sigma_{xx} = E\epsilon_{xx} = \begin{cases} \sigma_{xx}^S = -E_S\frac{y}{R} & (-(h-a) < y < a) \\ \sigma_{xx}^A = -E_A\frac{y}{R}, & (a < y < h+a) \end{cases}$$

To obtain 'a', we use the fact that the total axial force must vanish in the cross-section (due to pure bending), i.e.,

$$
\begin{aligned}
&\int\int_{A_S} \sigma_{xx}^S dA + \int\int_{A_A} \sigma_{xx}^A dA = 0 \\
&\Rightarrow\ -\frac{E_S}{R}\int_{-(h-a)}^{a} y dy \ -\ \frac{E_A}{R}\int_{a}^{h+a} y dy \ =\ 0 \\
&\Rightarrow\ \left[(a)^2\ -\ (h-a)^2\right]\ +\ \frac{E_A}{E_S}\left[(h+a)^2\ -\ (a)^2\right]\ =\ 0 \\
&\Rightarrow\ -h^2\left(1\ -\ \frac{E_A}{E_S}\right)\ +\ 2ha\left(1\ +\ \frac{E_A}{E_S}\right)\ =\ 0 \\
&\Rightarrow\ a\ =\ \frac{h}{2}\left(\frac{E_S\ -\ E_A}{E_S\ +\ E_A}\right).
\end{aligned}
$$

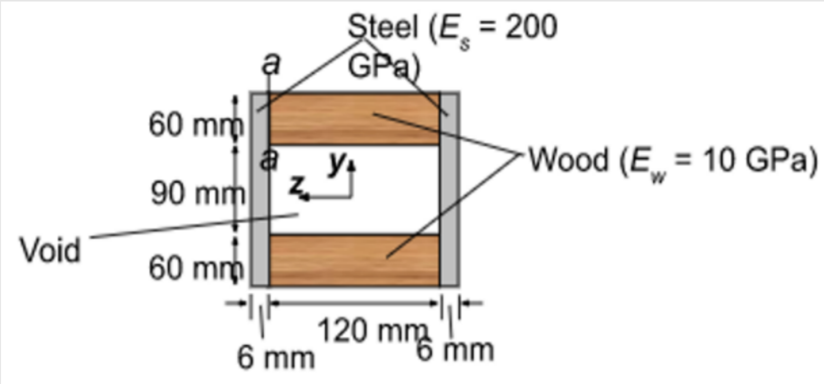
Note that when we set $E_s = E_A$ assuming both the materials to be the same, we indeed get $a = 0$ or the neutral axis then passes through the geometric center.

Q5. A beam of composite cross-section is subjected to bending moment $M_z = 30$kN. Find:

(a) The curvature induced in the beam

(b) Maximum bending stress in wood

(c) Maximum bending stress in steel



Solution:
We first obtain bending stiffness of the cross-section. The cross-section being symmetrical, the neutral axis will be the mid horizontal line in the cross-section. The total bending stiffness of the cross-section will simply be

$$\begin{aligned}(EI)_{tot} &= (EI)_{steel} + (EI)_{wood} \\ &= 2E_s\frac{1}{12}0.006 \times 0.21^3 + E_w\frac{1}{12}\left(.12 \times .21^3 - .12 \times .09^3\right).\end{aligned}$$

(i) The curvature induced in the beam would then simply be

$$\kappa = M_z/(EI)_{tot}$$

(ii) Maximum bending stress in steel would be in its topmost/bottom-most fiber, i.e.,

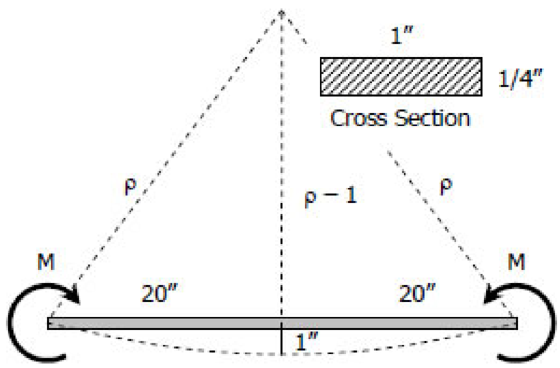
$$\sigma_s^{top} = E_s\kappa\ (0.045 + 0.060).$$

(iii) Likewise, the maximum bending stress in the wood fiber will be in its topmost/bottom-most fiber, i.e.,

$$\sigma_w^{top} = E_w\kappa\ (0.045 + 0.060).$$

Q6. A flat steel bar, 1 inch wide by 0.25 inch thick and 40 inch long, is bent by couples applied at the ends so that the midpoint deflection is 1 inch. Compute the stress in the bar and the magnitude of the applied couples. Use $E = 200$GPa.

Solution:



Note that the deflection of the beam is very small compared to its length. Therefore, in the above picture, we have drawn the ends of the deformed beam to also coincide with the ends of the undeformed beam. In reality, deformed beam and undeformed beam will be of the same length due to pure bending deformation. However, the deformed beam being curved, its two ends will be slightly inward compared to the undeformed beam. Neglecting this mismatch, we can write based on geometry that

$$
\begin{aligned}
&(\rho - \delta)^2 + L^2/4 = \rho^2 \\
&\Rightarrow \rho^2 - 2\rho\delta + \delta^2 + L^2/4 = \rho^2 \\
&\text{or } \rho = \frac{L^2 + 4\delta^2}{8\delta} = 200.5 \text{ in.}
\end{aligned}
$$

Another way to obtain radius of curvature ρ is as follows. Let the angle subtended by the deformed beam at the center is $\theta/2$. So, $\rho\theta = L$. One can then write using trigonometry that

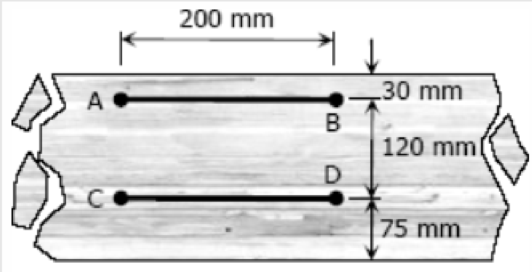
$$
\begin{aligned}
&\rho\cos(\theta/2) = \rho - \delta \\
&\Rightarrow \rho(1 - \theta^2/8) = \rho - \delta \quad \text{(assuming } \theta \text{ to be very small)} \\
&\Rightarrow \rho\left(1 - \frac{L^2}{8\rho^2}\right) = \rho - 1 \\
&\text{or } \rho = \frac{L^2}{8\delta} = 200 \; in.
\end{aligned}
$$

With the above value of ρ, θ turns out to be 0.2 radian or approximately 11 degrees. This validates our assumption. The couple required to generate this bending will be

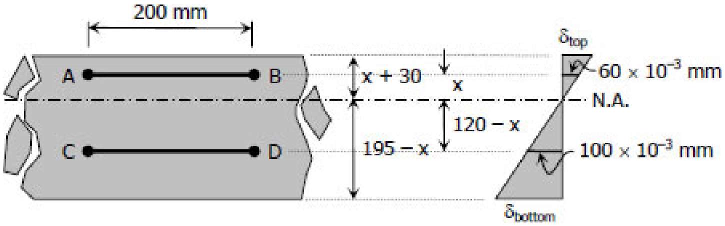
$$
M = \frac{EI}{\rho} = \frac{(29 \times 10^6)\,\frac{1\,(1/4)^3}{12}}{200.5} = 188.3 \text{ lb.in (answer)}.
$$

Here the young's modulus E has been converted into the units of lb/in^2.

Q7. In a laboratory test of a beam loaded by end couples, the longitudinal fibers at layer AB in the figure below are found to increase 60×10^{-3}mm whereas those at CD decrease $100 \times 10^{-3} mm$ in the 200mm-gauge length. Using $E = 70GPa$, determine the flexural stress in the top and bottom fibers.



Solution:
The picture above is that of the section of the beam along its length. The shape of the cross-section is not mentioned here. Accordingly, we can not assume that the neutral axis lies at the center. Let it lie at a distance x below the longitudinal fiber AB (see the figure below).



We know that $\epsilon_{xx} = -\frac{y}{R}$. Hence $\frac{y}{\epsilon_{xx}}$ must be a constant, i.e.,

$$\begin{aligned}
\frac{x}{\epsilon_{AB}} &= \frac{x - 120}{\epsilon_{CD}} \\
\Rightarrow \frac{x}{\frac{60 \times 10^{-3}}{200}} &= \frac{x - 120}{-\frac{100 \times 10^{-3}}{200}} \\
\Rightarrow x &= 0.6\,(120 - x) \\
\text{or } x &= 45\text{mm}.
\end{aligned}$$

We now obtain flexural/bending stress in the top fiber. We can write the following for the bending strain in the top fiber:

$$\frac{x}{\epsilon_{AB}} = \frac{x + 30}{\epsilon_{top}} \Rightarrow \epsilon_{top} = \frac{x + 30}{x}\epsilon_{AB} = \frac{75}{45}\frac{60 \times 10^{-3}}{200} = 5 \times 10^{-4}.$$

The bending stress in the top fiber will simply be $E\epsilon_{top}$ since the longitudinal fibers are under uniaxial loading during pure bending. Hence

$$\sigma_{top} = E\epsilon_{top} = 70 \times 10^9 \ \times \ 5 \times 10^{-4} = 35 \ MPa.$$

One can likewise obtain bending stress in the bottom-most fiber.