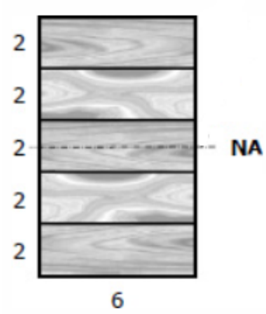
Tutorial 10

APL 104 - 2022 (Solid Mechanics)

1. A laminated beam is composed of five planks, each 6 in. by 2 in. glued together to form a section 6 in. wide by 10 in. high.



The allowable shear stress in the glue is 90 psi, the allowable shear stress in the wood is 120 psi, and the allowable flexural stress in the wood is 1200 psi. Determine the maximum uniformly distributed load that can be carried by a simply supported beam on a 6ft simple span.

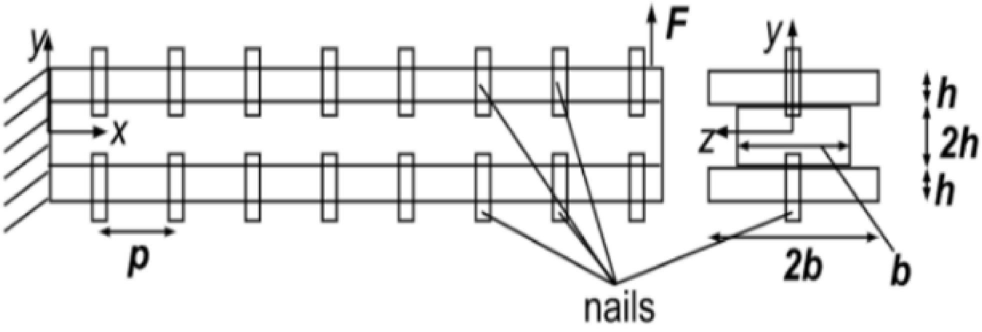
2. For an I-beam, assume the beam is subjected to tranverse load

    (a) Obtain an expression for variation in shear stress τ_{xy} within its cross-section. You can use the formula $\tau_{xy} = \dfrac{VQ\,(y)}{I_{zz}T\,(y)}$.

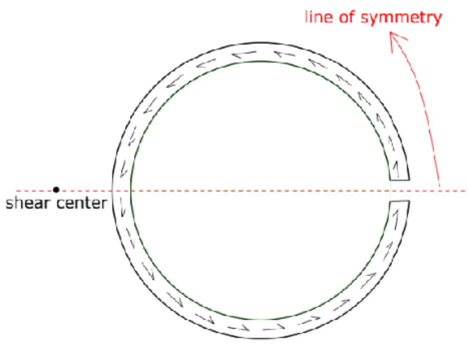
    (b) Using the expression above, draw a graph depicting qualitative variation in shear stress within the cross-section.

    (c) Where is the shear stress maximum? Find the ration of maximum shear stress to average shear stress in the cross-section.

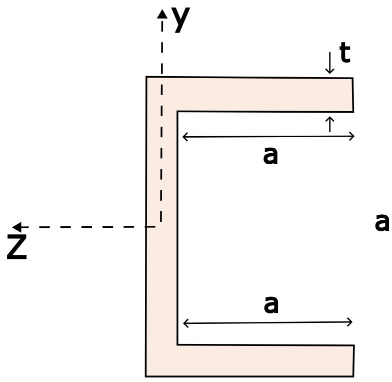
3. A cantilever beam (figure below) is made up of three wooden planks nailed together as shown. Find the expression for shear force in nails in terms of F, p, b and h. For the cross-section: $I_{zz} = 10bh^3$.



4. Think of a beam whose cross-section has the shape of a thin annular disc but having a cut (along a radial line on z-axis). How will the shear stress distribution be in the cross-section? Will such a beam undergo just bending or it can also twist?



5. Think of a beam having thin and open cross-section. The shape of the cross-section is shown in the figure. Suppose the beam is subjected to a transverse load at one of its end such that the force applied is along 'Y' axis.

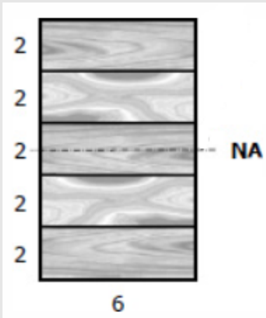


(a) Where should the load be applied in the cross-sectional plane at the tip so that the beam does not twist?

(b) Find out shear stress distribution in the beam's cross-section assuming the beam does not twist.

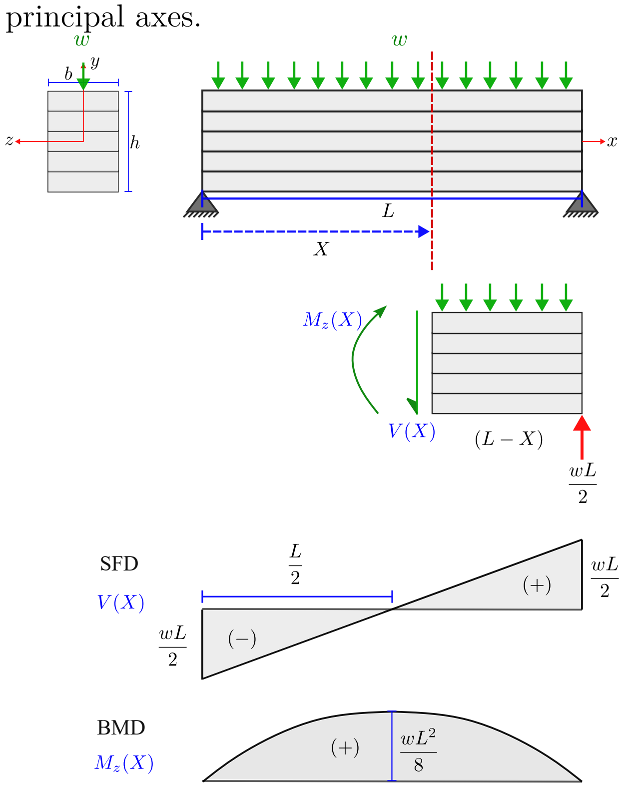
APL 104 Tutorial 10 solutions

Q1. A laminated beam is composed of five planks, each 6 in. by 2 in. glued together to form a section 6 in. wide by 10 in. high.



The allowable shear stress in the glue is 90 psi, the allowable shear stress in the wood is 120 psi, and the allowable flexural stress in the wood is 1200 psi. Determine the maximum uniformly distributed load that can be carried by a simply supported beam on a 6ft simple span.

Solution: Assume that the uniformly-distributed load is applied along y-axis. Note that the rectangular cross-section is symmetrical about y and z axes. Hence, they are also the principal axes.



- $$\uparrow + \sum F_y = 0$$
  $$\Rightarrow -V_x - w(L - x) + \frac{wl}{2} = 0$$
  $$\Rightarrow V_x = -w\left(\frac{L}{2} - x\right)$$

- $$\sum M_{\text{left end}} = 0 \text{ (CCW sense +ve)}$$
  $$\Rightarrow -M_z(x) + \frac{wl}{2}(L - x) - \frac{w(L-x)^2}{2} = 0$$
  $$\Rightarrow M_z(x) = \frac{wl}{2}(L - x) - \frac{w(L-x)^2}{2}$$

Since the load is applied along one of the principal axes, it will result in in-plane bending. Also, the UDL will lead to both bending moment and shear force at any cross-section leading to non-uniform bending. To find the maximum flexural and shear stress, we need to find the bending and shear stress distribution. Due to symmetrical bending, the formula for bending and shear stresses is relatively simple:

$$\sigma_{xx} = \frac{M_z(x)\, y}{I_{zz}}, \quad \tau_{yx} = \frac{V(x)\, Q(y)}{I_{zz} b}$$

If the load were not applied along any of the principal axes, it would have resulted in unsymmetrical bending.

The distribution of $M_z(x)$ and $V(x)$ along the length of the beam are obtained from the bending moment diagram (BMD) and shear force diagram (SFD) as shown in the previous figure. Note that bending moment is maximum at the center and vanishes at the two ends as is the case for simply-supported beams.

Max bending stress

$$\sigma_{xx,\max}^{\text{wood}} = \max_{x,y}\left\{\frac{M_z\,(x)\, y}{I_{zz}}\right\} = \max_{x}\frac{M_z\,(x)}{I_{zz}}\max_{y} y = \frac{\left(\frac{wL^2}{8}\right)\left(\frac{h}{2}\right)}{\frac{1}{12}bh^3} = \frac{3wL^2}{4bh^2} \le \sigma_{xx,\text{tol}}^{\text{wood}} \quad \ldots \ (1)$$

Max shear stress

$$\tau_{yx,\max} = \max_{x,y}\frac{V\,(x)\;\; Q\,(y)}{I_{zz}\; b} = \frac{1}{I_{zz} b}\max_{x} V(x) \quad \max_{y} Q(y)$$

$$\begin{aligned} Q\,(y) &= \text{first moment of shaded area} \\ &= \text{area of the shaded region} \times \text{centroid of the shaded area from the NA} \\ &= b\left(\frac{h}{2} - y\right) \times \left\{y + \frac{\frac{h}{2} - y}{2}\right\} = \frac{b}{2}\left(\frac{h^2}{4} - y^2\right) \end{aligned}$$

$$\Rightarrow \max_{y} Q\,(y) = \frac{b}{2}\frac{h^2}{4}.$$

$$\therefore\ \tau_{yx,\max}^{\text{wood}} = \frac{V\,(x = L)}{I_{zz}}\ \frac{Q\,(y = 0)}{b} = \frac{\frac{wL}{2}\left(\frac{b}{2}\frac{h^2}{4}\right)}{\frac{1}{12}bh^3\; b} = \frac{3wL}{4bh} \le \tau_{yx,\text{tol}}^{\text{wood}} \quad \ldots \ (2)$$

The glue between the wooden planks resist the shear stress generated between the planks. The shear stress at the level of the glue also should not exceed the maximum tolerable shear stress of glue.

$$\therefore\ \tau_{yx,\max}^{\text{glue}} = \frac{V\,(x = L)}{I_{zz}}\ \frac{Q\left(y = \frac{h}{10}\right)}{b} = \frac{\frac{wL}{2}\left(\frac{3bh^2}{25}\right)}{\frac{1}{12}bh^3\; b} = \frac{18wL}{25bh} \le \tau_{yx,\text{tol}}^{glue} \quad \ldots \ (3)$$

From Eqs. (1), (2), and (3), we would obtain three values of w. The maximum allowed value of w would be minimum of the values obtained from these three equations.

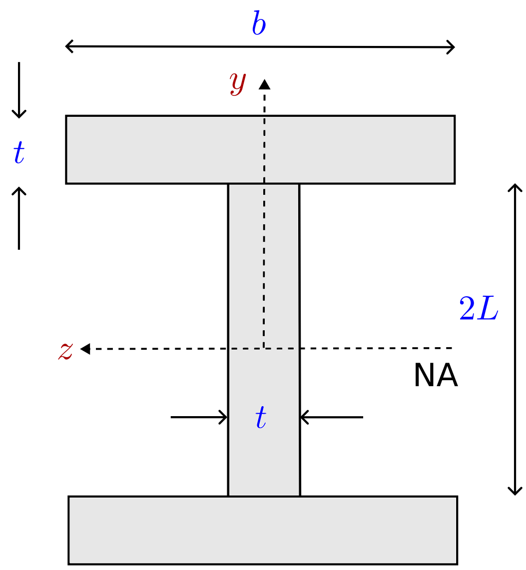
Q2. For an I-beam, assume the beam is subjected to tranverse load

(a) Obtain an expression for variation in shear stress τ_{yx} within its cross-section. You can use the formula $\tau_{yx} = \dfrac{VQ(y)}{I_{zz} b(y)}$.

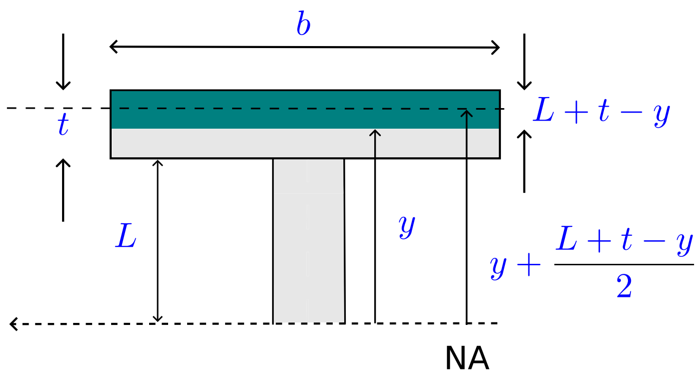
(b) Using the expression above, draw a graph depicting qualitative variation in shear stress within the cross-section.

(c) Where is the shear stress maximum? Find the ratio of maximum shear stress to average shear stress in the cross-section.

Solution: We will assume that the transverse load is acting along the principal axis of the beam resulting in symmetrical bending. Therefore, we can use the simpler formula $\tau_{yx}(x, y) = \dfrac{VQ(y)}{I_{zz}\, b(y)}$. Note that unlike the beam with rectangular cross-section, the width of the I-beam is different in the flange and the web. Hence, in this case, $b(y)$ in the denominator of the formula is taken to be a function of y.



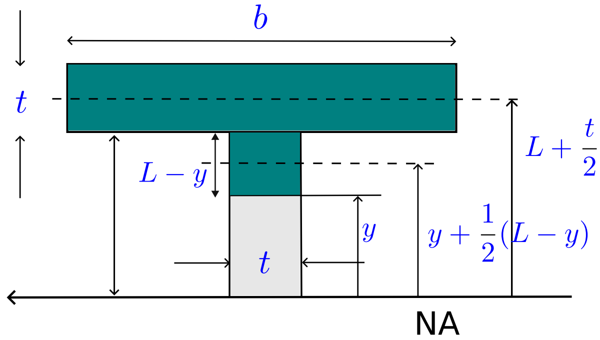
The shear force V is already given. Since we are looking at a section with fixed x, only $Q(y)$ and $b(y)$ will differ in the cross-section as y varies. Lets first find the distribution of $Q(y)$. The neutral axis (NA) will be at the center aligned with the z-axis.



<u>**Flange**</u>:

$$
\begin{aligned}
Q^f(y) &= b\,(t + L - y) \times \left[y + \frac{t + L - y}{2}\right] \\
&= \frac{b}{2}\,(t + L - y)\,(t + L + y) \\
&= \frac{b}{2}\left[(t + L)^2 - y^2\right]
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ \tau_{yx} &= \frac{V_x}{I_{zz}}\frac{(b/2)}{b}\left[(t + L)^2 - y^2\right] \quad (L \le y \le L + t) \\
&= \frac{V_x}{2I_{zz}}\left[(t + L)^2 - y^2\right]
\end{aligned}
$$



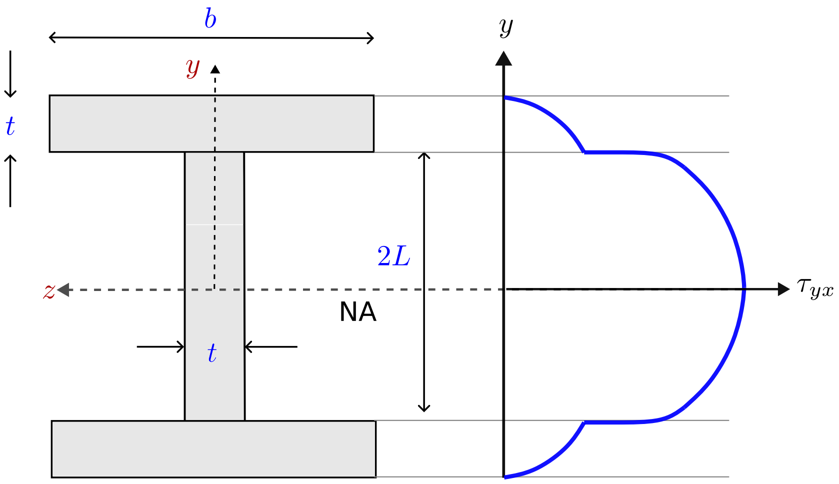
<u>**Web**</u>:

$$
\begin{aligned}
Q^w(y) &= b\,t \times \left(L + \frac{t}{2}\right) + t\,(L - y) \times \frac{1}{2}\,(L + y) \\
&= b\,t\left(L + \frac{t}{2}\right) + \frac{t}{2}\left(L^2 - y^2\right) \\
\therefore\ \tau_{yx} &= \frac{V_x}{I_{zz}\,t}\left[b\,t\left(L + \frac{t}{2}\right) + \frac{t}{2}\left(L^2 - y^2\right)\right]
\end{aligned}
$$

<u>I_{zz} for full I-beam</u>

$$
\begin{aligned}
I^f_{zz}\ \text{(for flange)} &= \left[\frac{1}{12}bt^3 + bt\left(L + \frac{t}{2}\right)^2\right] \\
I^w_{zz}\ \text{(for full web)} &= \frac{1}{12} + (2L)^3 \\
&= \frac{8tL^3}{12} \\
I_{zz} &= 2I^f_{zz} + I^w_{zz}
\end{aligned}
$$

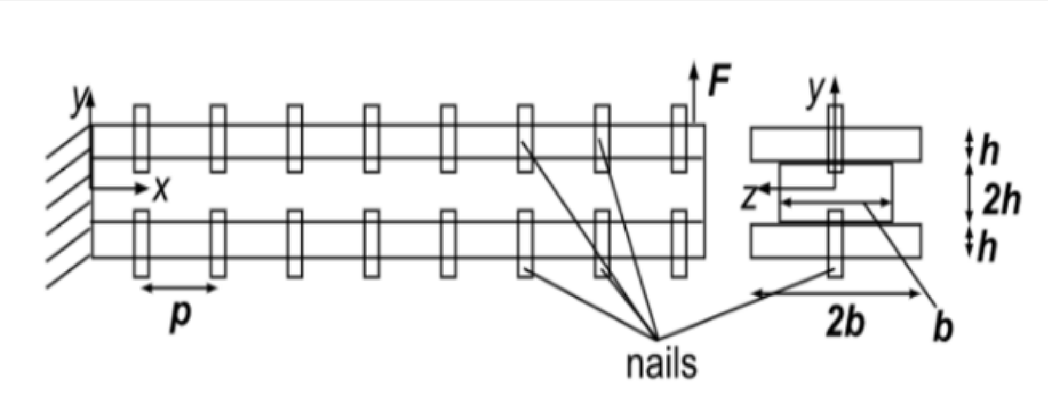
(b) The variation of shear stress with respect to y is shown below:



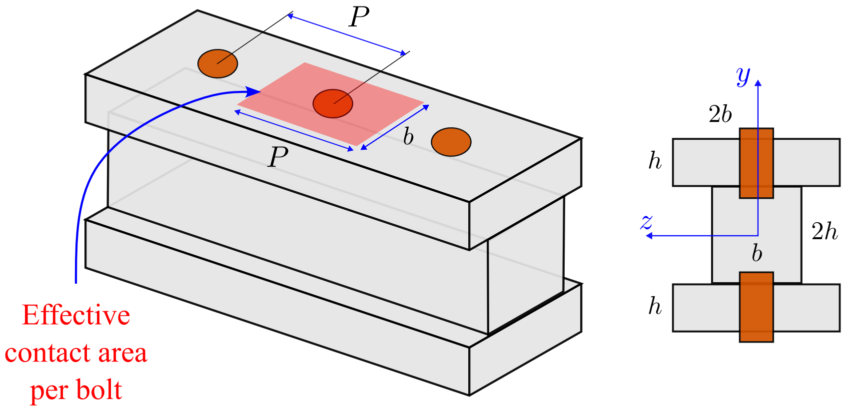
(c) Shear-stress is maximum at the NA

Avg shear stress, $\tau_{yx,\text{avg}} = \dfrac{V}{\text{Area of c/s of the I-beam}}$. Do the rest on your own.

Q3. A cantilever beam (figure below) is made up of three wooden planks nailed together as shown. Find the expression for shear force in nails in terms of F, p, b and h. For the cross-section: $I_{zz} = 10bh^3$.



Solution:

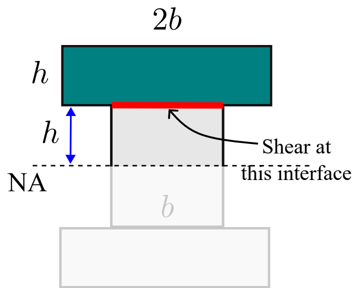


The effective area covered by each bolt has a length equal to the spacing between bolts. The total shearing force between the two planks must be resisted by the bolts.

$$\begin{aligned}
\text{Resistive shear force per bolt} &= \tau_{yx}(\text{at interface}) \times \text{Effective contact area per bolt} \\
&= \tau_{yx}(\text{near interface in flange}) \times (2Pb) \\
&= \tau_{yx}(\text{near interface in web}) \times (Pb)
\end{aligned}$$

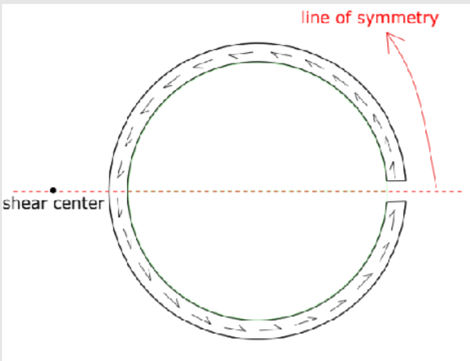
Notice that although shear stress exhibits a jump at the interface, the force experienced by each bolt will not exhibit such a jump. It can be easily apreciated from the above formula. We need to now find shear stress at the interface of the two planks. Lets first find $Q(y = h)$ which will only have a contribution from the full shaded region of flange as shown below.

$$
\begin{aligned}
Q(y=h) &= \text{Moment of shaded area from NA} \\
&= \text{Area of shaded region} \\
&\quad \times \text{centroid of that area from NA} \\
&= (2bh) \times \left(\frac{3h}{2}\right) = 3bh^2
\end{aligned}
$$



$$
\begin{aligned}
\tau_{yx}(\text{near interface in web}) &= \frac{V\,(3bh^2)}{10bh^3 \times b} = \frac{3V}{10bh} \\
\text{Resistive shear force/bolt} &= \frac{3V}{10bh} \times Pb = \frac{3VP}{10h}
\end{aligned}
$$

Q4. Think of a beam whose cross-section has the shape of a thin annular disc but having a cut (along a radial line on z-axis). How will the shear stress distribution be in the cross-section? Will such a beam undergo just bending or it can also twist?



Solution: Here s_c is the shear center of the cross-section which can be worked out.
We need to know find Q_y^s for the region $[0, \theta]$ (see Figure 1) which has also been drawn separately in Figure 2. Let us identify a tiny strip shown in red in Figure 2 which subtends an angle $d\phi$ at the center. The y-coordinate of the centroid of this tiny strip is $R \sin\phi$ and its area is $tRd\phi$. Thus, for this tiny strip, we can write

$$
dQ_y = \bar{Y}\,dA = R^2 t \sin\phi \; d\phi. \tag{1}
$$

We can now find Q_y^s by integrating over such tiny strips, i.e.,

$$
Q_y^s = \int_0^\theta dQ_y = R^2 t \int_0^\theta \sin\phi \, d\phi = R^2 t(1-\cos\theta) = R^2 t\left(1 - \cos\left(\frac{s}{R}\right)\right). \tag{2}
$$

Let us now find I_{zz} for the cross-section about its centroid O. As the cut is really thin, the annulus is approximately complete. So, to calculate I_{xx}, I_{yy} and I_{zz}, we can forget about the thin cut. We thus have

$$
I_{zz} = I_{yy} = 1/2\ I_{xx} = 1/2\ R^2\ (2\pi R\ t) = \pi R^3\ t. \tag{3}
$$

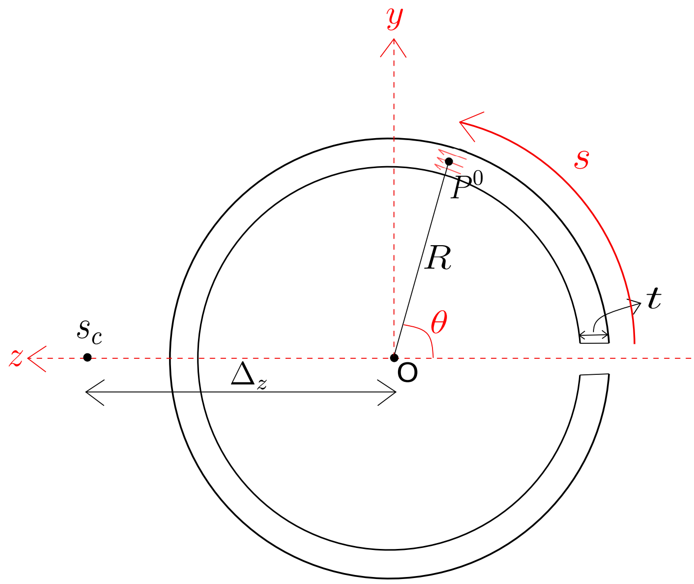


Figure 1: Variables for the analysis of the cross-section shown.

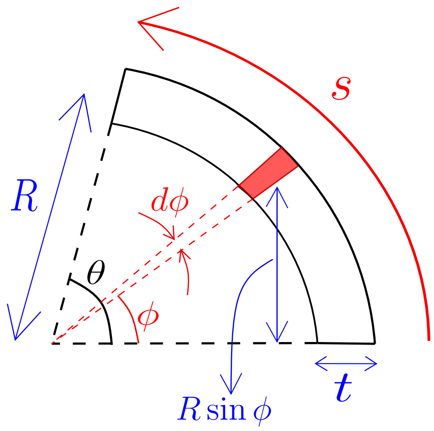


Figure 2: The part of the cross-section from 0 up to θ

$$\tau_{sx} = -\frac{V_y Q_y^s}{I_{zz}\ t} = -\frac{V_y R^2 t \left(1 - \cos\left(\frac{s}{R}\right)\right)}{\pi R^3 t \times t} = -\frac{V_y \left(1 - \cos\left(\frac{s}{R}\right)\right)}{\pi R t}$$

If you work out the shear center of the cross-section, you will see that it does not lie on the cross-section. As such, if we want to apply shear force on beams having such cross-sections so that the beam does not twist, we must apply shear force so that its line of action passes through shear center. For the present cross-section, we need to create an extension of it (e.g., by a thin rod as shown in red in Figure 3) and apply shear force there so that the beam indeed does not twist.

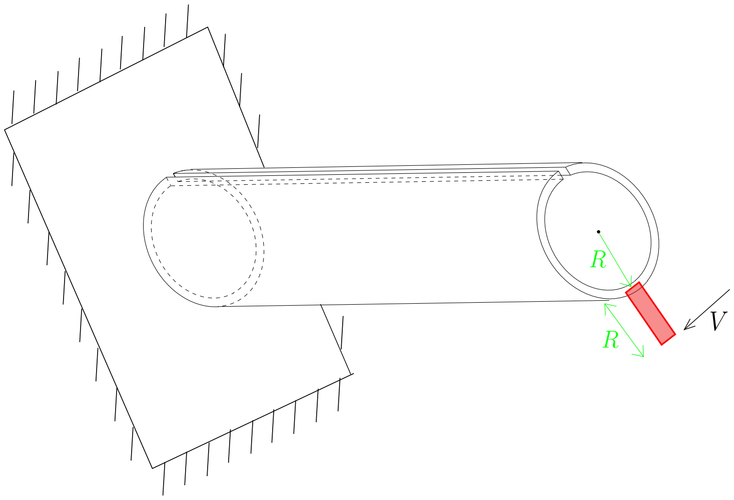
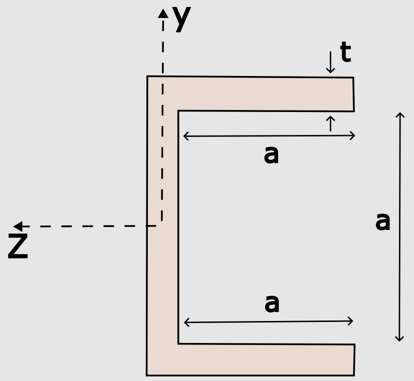


Figure 3: An extension made to the beam's cross-section to avoid twisting of the beam due to application of transverse load

Q5. Think of a beam having thin and open cross-section ($t << a$). The shape of the cross-section is shown in the figure. Suppose the beam is subjected to a transverse load at one of its end such that the force applied is along 'Y' axis.

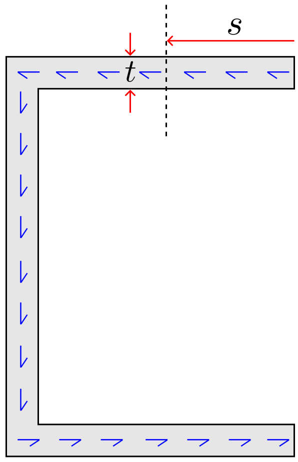


(a) Where should the load be applied in the cross-sectional plane at the tip so that the beam does not twist?

(b) Find out shear stress distribution in the beam's cross-section assuming the beam does not twist.

Solution: Since it is a thin tube ($t << a$) and that the load is acting along the princial axis, we can use the formula

$$\tau_{sx} = -\frac{VQ_{sy}}{I_{zz}\, t}$$

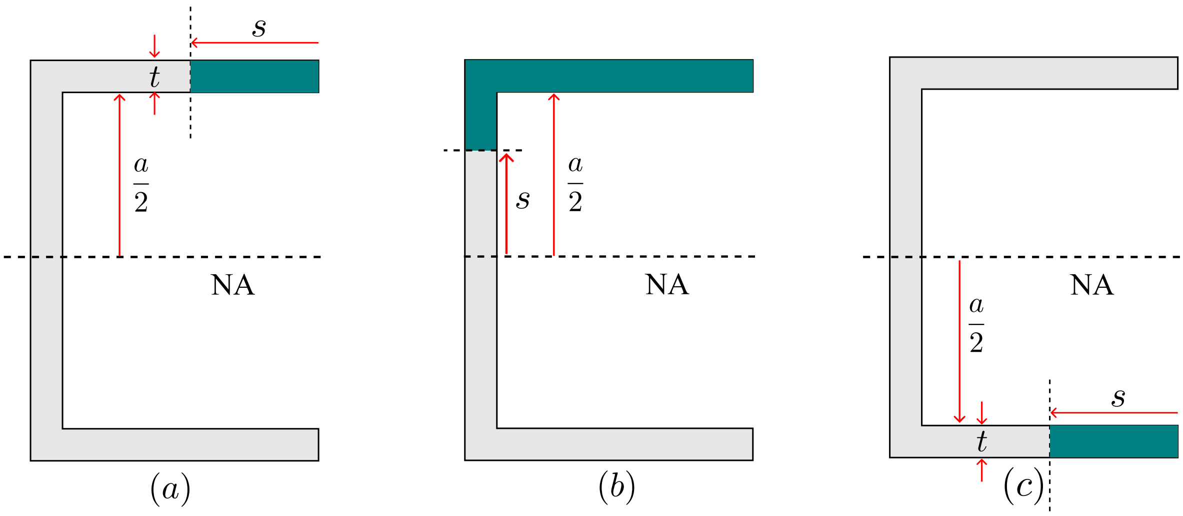


As the tube is thin, shear stress lies along the cross section's periphery! For the top horizontal flange portion (see subfigure (a) below), the shear stress is

$$\tau_{sx} = \frac{V\left(\frac{a+t}{2}st\right)}{I_{zz}t} \approx \frac{Vas}{2I_{zz}}.$$

For the middle vertical web portion (see subfigure (b) below), the shear stress is

$$\tau_{sx} = \frac{V\left(\frac{a}{2}at + \left[s + \left(\frac{a}{2} - s\right)/2\right]\left(\frac{a}{2} - s\right)t\right)}{I_{zz}t} = \frac{V\left(\frac{a^2}{2} + \frac{1}{2}\left(\frac{a^2}{4} - s^2\right)\right)}{I_{zz}}.$$



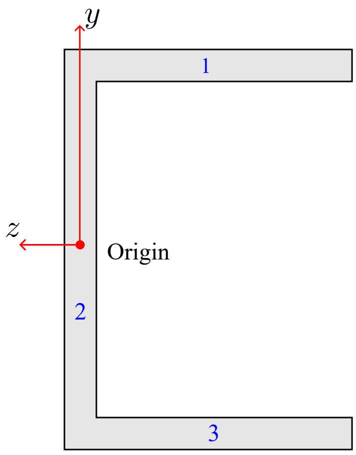
For the bottom horizontal flange portion (see subfigure (c) above), the shear stress is

$$\tau_{sx} = \frac{VQ}{I_{zz}t} = \frac{V\left(-\frac{a}{2}st\right)}{I_{zz}t} = -\frac{Vas}{2I_{zz}}.$$

The moment of inertia can be calculated separately for each of the three blocks (top, middle and lower) about their own centroidal axis and then shifted to the NA as shown below.

$$\begin{aligned} I_{zz} &= I_{zz}(1) \times 2 + I_{zz}(2) \\ &= \left[\frac{1}{12}at^3 + at\left(\frac{a}{2}\right)^2\right] \times 2 + \frac{1}{12}a^3t \\ &= \frac{1}{6}at^3 + a^3t\frac{7}{12} \end{aligned}$$

The shear center is the location of the point in/outside the cross-section at which a load applied parallel to the plane of the section will produce no twisting. To obtain shear center, first get the torque due to shear stress distribution about origin



$$\begin{aligned} T &= (1) + (2) + (3) \\ &= (1) \times 2 + (2) \quad ((1) \text{ and } (3) \text{ contribute same torque}) \\ &= \left[\frac{Vat}{2I_{zz}}\int_0^a s ds \times \frac{a}{2}\right] \times 2 + 0 \\ &= \frac{Va^4t}{8I_{zz}} \times 2 = \frac{Va^3}{4\left(t^2/6 + 7/12a^2\right)} \end{aligned}$$

Note that torque due to region (2) is zero because shear stress there passes through origin!

Location of shear center $= \dfrac{a^3}{4\left(t^2/6 + 7/12a^2\right)}$ (along $-z$-axis)