

Solid Mechanics
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Lecture - 6
Balance of Angular Momentum (Contd.)

Abstract

In the previous lecture, we had started with stress equilibrium equations. We derived the Linear Momentum Balance. We had also derived the traction and body force contributions for the Angular Momentum Balance. In this lecture, we will complete our derivation of Angular Momentum Balance and then discuss some properties and examples of the stress tensor.

1 Angular Momentum Balance (start time: 00:21)

1.1 Traction contribution (start time: 00:27)

In the last lecture, we had derived the torque due to traction forces to be

$$\underline{T}_{/\underline{x}}^{\text{trac}} = \sum_{i=1}^3 (\underline{e}_i \times \underline{\sigma} \underline{e}_i) \Delta V + o(\Delta V). \quad (1)$$

1.2 Body force contribution (start time: 00:35)

Similarly, we had derived the torque due to body force as

$$\underline{T}_{/\underline{x}}^{\text{body force}} = o(\Delta V). \quad (2)$$

1.3 Dynamics term (start time: 00:41)

We need to now derive $\frac{d}{dt}(\underline{H}_{/\underline{x}})$. In Figure 1, \underline{x} is the center of the cuboid as considered previously and \underline{y} is any arbitrary point in the cuboid. If we consider a particle of mass m moving with velocity \underline{v} , its angular momentum is given by:

$$\vec{r}_{/O} \times m \underline{v} \quad (3)$$

Here, $\vec{r}_{/O}$ is the position vector of the particle relative to the point 'O' about which we are measuring the angular momentum. To find the angular momentum of the mass in the cuboid about its center, we need to integrate over all the particles that it contains. As we are writing the equation in the center of mass frame, the momentum of the particle is simply mass dm times the velocity of the particle $\underline{v}(\underline{y})$ relative to the center of mass, i.e., $\underline{v}(\underline{y})_{/\underline{x}}$. This leads to

$$\vec{H}_{/\underline{x}} = \iiint (\underline{y} - \underline{x}) \times dm \underline{v}(\underline{y})_{/\underline{x}} = \iiint_{V(t)} (\underline{y} - \underline{x}) \times \rho \underline{v}_{/\underline{x}} d\gamma d\xi d\eta \quad (4)$$

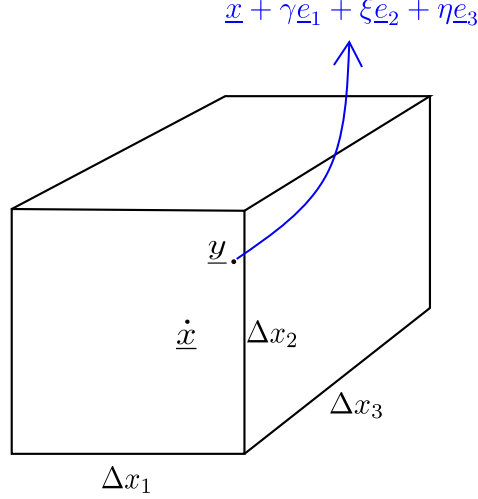


Figure 1: A cuboidal volume centered at the point of interest \underline{x} . \underline{y} is an arbitrary point in the volume of the cuboid

Note again that the first integral is over fixed/identifiable mass whereas the second integral is over changing volume domain since as the identifiable mass moves in space, the volume occupied by it changes with time. This volume happens to be our cuboid in the current time. Just like linear momentum balance, angular momentum balance is also applied always to a fixed/identifiable mass.¹ So, taking the time derivative in control volume setting isn't easy whereas in control mass setting, we can easily move the time derivative within the integral. Thus, taking the time derivative in control mass setting, we get

$$\frac{d}{dt}(\underline{H}_{/\underline{x}}) = \iiint_{mass} \underbrace{\frac{d}{dt}(\underline{y} - \underline{x})}_{\underline{v}(\underline{y})_{/\underline{x}}} \times dm + \iiint_{mass} (\underline{y} - \underline{x}) \times dm \underline{a}(\underline{y})_{/\underline{x}} \quad (5)$$

Here, the time derivative of velocity becomes acceleration. In the first term, the time derivative of $(\underline{y} - \underline{x})$ will give velocity of \underline{y} minus velocity of \underline{x} , i.e., $\underline{v}(\underline{y})_{/\underline{x}}$. So, this term cancels as it involves the cross product of a quantity with itself. Also note that dm , mass of the particle, is a constant and does not change with time: density and volume of a particle can change but mass cannot. For the second term, we can now switch back to the volume integral as the time derivative is already inside, i.e.,

$$\frac{d}{dt}(\underline{H}_{/\underline{x}}) = \iiint_V (\underline{y} - \underline{x}) \times \rho \underline{a}(\underline{y})_{/\underline{x}} d\gamma d\xi d\eta \quad (6)$$

Comparing this with the derivation of the torque due to body force term, we see that this is exactly similar to that term with just \underline{b} replaced with $\rho \underline{a}(\underline{y})_{/\underline{x}}$. So, upon integrating this dynamic term using Taylor's expansion for acceleration, we would get a similar result as that of the body force contribution to torque, i.e.,

$$\frac{d}{dt}(\underline{H}_{/\underline{x}}) = o(\Delta V) \quad (7)$$

¹If we just concentrate on the fixed cuboid in space, the mass contained in the cuboid will change with time since mass is transporting through its surfaces and hence we cannot apply linear or angular momentum balance to it.

1.4 Final balance (start time: 10:54)

So, now we can substitute equations (1),(2) and (7) in

$$\sum T_{/\underline{x}}^{ext} = \frac{d}{dt}(\underline{H}_{/\underline{x}}) \quad (8)$$

$$\Rightarrow \sum_{i=1}^3 (\underline{e}_i \times \underline{\underline{\sigma}} \underline{e}_i) \Delta V + o(\Delta V) = o(\Delta V) \quad (9)$$

As this equation is valid for a cuboid of any size, we can shrink it to its centroid (\underline{x}). So, first we divide by ΔV on both sides and then take the $\lim_{\Delta V \rightarrow 0}$:

$$\lim_{\Delta V \rightarrow 0} \left[\frac{\sum_{i=1}^3 (\underline{e}_i \times \underline{\underline{\sigma}} \underline{e}_i) \Delta V + o(\Delta V)}{\Delta V} = \frac{o(\Delta V)}{\Delta V} \right] \quad (10)$$

We finally get

$$\boxed{\sum_{i=1}^3 \underline{e}_i \times \underline{\underline{\sigma}} \underline{e}_i = \underline{0}} \rightarrow \text{Angular Momentum Balance(AMB)} \quad (11)$$

Note that this equation holds even if the body force is acting or if the body is accelerating.

1.5 Representation in a coordinate system (start time: 14:25)

Let us try to write the tensor equation (11) in $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ coordinate system. Consider the first term in the summation. The representation of \underline{e}_1 in $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ coordinate system is trivial and is given as $[1 \ 0 \ 0]^T$ while the representation of $\underline{\underline{\sigma}} \underline{e}_1$ will just be the first column of the stress matrix corresponding to $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ coordinate system. We thus have

$$\begin{aligned} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} \sigma_{11} \\ \tau_{21} \\ \tau_{31} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} \tau_{12} \\ \sigma_{22} \\ \tau_{32} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} \tau_{13} \\ \tau_{23} \\ \sigma_{33} \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \Rightarrow \begin{bmatrix} 0 \\ -\tau_{31} \\ \tau_{21} \end{bmatrix} + \begin{bmatrix} \tau_{32} \\ 0 \\ -\tau_{12} \end{bmatrix} + \begin{bmatrix} -\tau_{23} \\ \tau_{13} \\ 0 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned} \quad (12)$$

Thus, we get the following three scalar equations:

$$\begin{aligned} \tau_{32} &= \tau_{23} \\ \tau_{31} &= \tau_{13} \\ \tau_{21} &= \tau_{12} \end{aligned} \quad (13)$$

So, the final outcome of the angular momentum balance is that the stress matrix is symmetric which holds true even if the body is accelerating or body force is present.

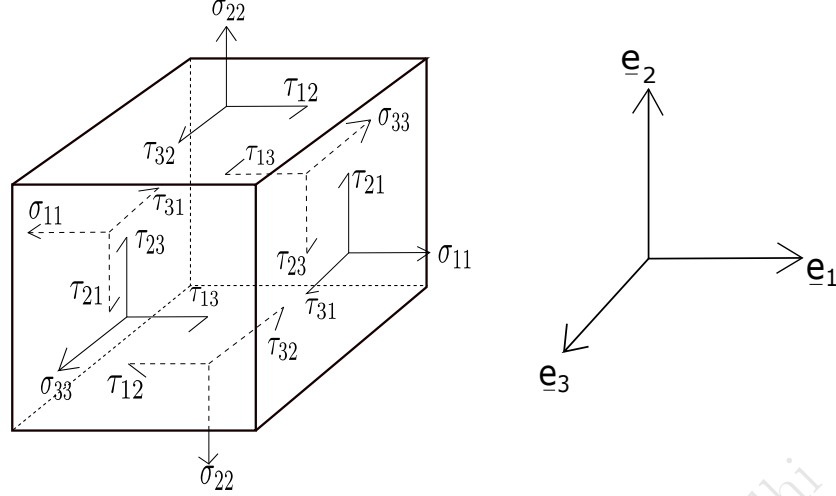


Figure 2: Traction components on various faces of the cuboid along with the coordinate system

1.6 An alternate method to derive AMB (start time: 19:02)

There is also a simpler but approximate way to come to this final outcome which is given in many textbooks. Consider the same cuboid again. The coordinate system $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ is shown on the right side of Figure 2. We have drawn stress matrix components on all its faces. The stress components act along positive direction on positive faces whereas they act along negative directions on negative faces. Let us now find the moment due to these tractions about \underline{e}_3 axis passing through the centroid of the cuboid. The normal components σ_{11} , σ_{22} and σ_{33} pass through the centroid and thus do not contribute to this moment. The components τ_{31} and τ_{32} also do not contribute to this moment as they are parallel or anti-parallel to \underline{e}_3 direction. Only contributions would come from τ_{12} and τ_{21} . Moment due to τ_{21} will be traction times the area of the face $(\Delta x_2 \Delta x_3)$ on which it acts times the distance from the center $(\frac{\Delta x_1}{2})$. The component τ_{21} has two contributions from \underline{e}_1 and $-\underline{e}_1$ planes both in the same direction $(+\underline{e}_3)$. The moment due to two τ_{12} components acts in $-\underline{e}_3$ direction. Finally, moment due to body force will be a smaller order term. The rate of change of angular momentum about \underline{e}_3 axis is also a smaller order term. When we shrink the cuboid, their contribution will vanish quickly than the contribution to moment due to traction components. The total moment about \underline{e}_3 axis (passing through the center of cuboid) will then be

$$\begin{aligned}
 (\tau_{21} \frac{\Delta x_2 \Delta x_3 \Delta x_1}{2}) \times 2 - (\tau_{12} \frac{\Delta x_1 \Delta x_3 \Delta x_2}{2}) \times 2 &= 0 \\
 \Rightarrow (\tau_{21} - \tau_{12}) \Delta V &= 0 \\
 \Rightarrow \tau_{21} &= \tau_{12}
 \end{aligned} \tag{14}$$

This is a simpler derivation but it involves a strong assumption of traction components not varying on cuboid's faces. Likewise, when we find the moment equation about \underline{e}_1 and \underline{e}_2 axis, we will get $\tau_{23} = \tau_{32}$ and $\tau_{13} = \tau_{31}$ respectively.

2 Relating externally applied distributed load on body's surface to stress tensor (start time: 27:22)

Suppose we have an arbitrary body which is clamped at some part of the boundary and a load is applied on some part of the boundary by an external agent. As the load is usually distributed over an area of the boundary of the body, it has the unit of traction which we write as \underline{t}_0 as shown in Figure 3. Our aim is to relate the state of stress (at the point where external load is acting) to the

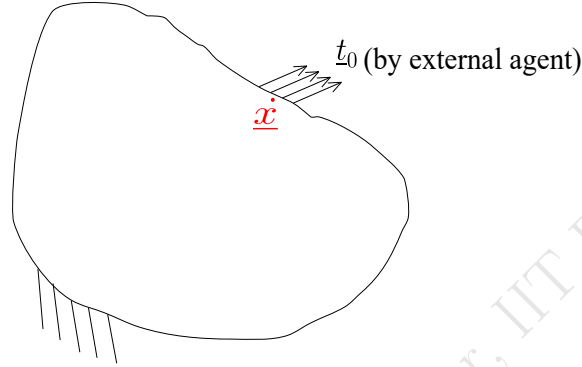


Figure 3: An arbitrary body subjected to an externally applied distributed load \underline{t}_0 over its surface

applied load itself. If we solve the stress equilibrium equations, we can find stress everywhere in the body. But at points on the boundary where the external load acts, we can find the state of stress partially without solving the stress equilibrium equations. In fact, the relation that we will derive here will be required as boundary condition in order to fully solve the stress equilibrium equations.

Let us set up our coordinate system such that we have \underline{e}_3 axis along the surface normal as shown in Figure 4. We have drawn the part of the body near the surface. On the top part of this surface,

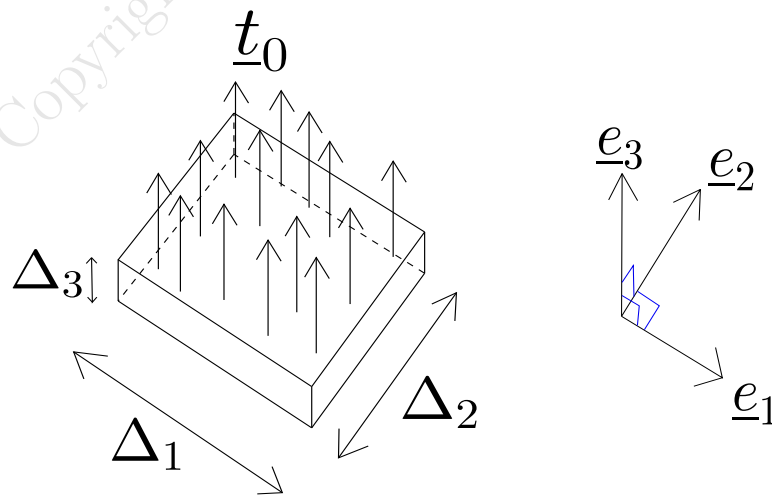


Figure 4: A zoomed view of the part of the body close to the surface point where the distributed external load acts

external load \underline{t}_0 acts. We are looking at an infinitesimal part of the body, so we can safely assume that \underline{t}_0 is constant on the top surface. The directions \underline{e}_1 and \underline{e}_2 are in the plane of the surface at that point and aligned along the edges of the small part of the body. Alternatively, we also say that the two directions are in the tangent plane at that point.² So, \underline{e}_3 becomes the normal to this tangent plane. The edges along $\underline{e}_1, \underline{e}_2$ and \underline{e}_3 directions are of length Δ_1, Δ_2 and Δ_3 respectively. The total force on this small part of the body will be the vector sum of the traction forces applied from remaining part of the body, externally applied load and the body force, i.e.,

$$\underline{t}^{-3}\Delta_1\Delta_2 + (\underline{t}^2 + \underline{t}^{-2})\Delta_1\Delta_3 + (\underline{t}^1 + \underline{t}^{-1})\Delta_2\Delta_3 + \underline{b}\Delta_1\Delta_2\Delta_3 + \underline{t}_0\Delta_1\Delta_2$$

We now apply Newton's 2^{nd} law to this small part of the body:

$$\underline{t}^{-3}\Delta_1\Delta_2 + (\underline{t}^2 + \underline{t}^{-2})\Delta_1\Delta_3 + (\underline{t}^1 + \underline{t}^{-1})\Delta_2\Delta_3 + \underline{b}\Delta_1\Delta_2\Delta_3 + \underline{t}_0\Delta_1\Delta_2 = \rho\Delta_1\Delta_2\Delta_3 \underline{a} \quad (15)$$

This small box is also called a 'pill box' in several books. We now divide both sides of the above equation by $\Delta_1\Delta_2$ and then take $\lim_{\Delta_3 \rightarrow 0}$. By taking this limit, we are essentially shrinking the pill box by pushing the bottom surface towards the top surface keeping the area of the surface ($\Delta_1\Delta_2$) constant.

$$\begin{aligned} \lim_{\Delta_3 \rightarrow 0} \frac{\underline{t}^{-3}\Delta_1\Delta_2 + (\underline{t}^2 + \underline{t}^{-2})\Delta_1\Delta_3 + (\underline{t}^1 + \underline{t}^{-1})\Delta_2\Delta_3 + \underline{b}\Delta_1\Delta_2\Delta_3 + \underline{t}_0\Delta_1\Delta_2}{\Delta_1\Delta_2} &= \frac{\rho\Delta_1\Delta_2\Delta_3 \underline{a}}{\Delta_1\Delta_2} \\ \Rightarrow \lim_{\Delta_3 \rightarrow 0} \left[\underline{t}^{-3} + \underline{t}_0 + (\underline{t}^2 + \underline{t}^{-2})\frac{\Delta_3}{\Delta_2} + (\underline{t}^1 + \underline{t}^{-1})\frac{\Delta_3}{\Delta_1} + (\underline{b} - \rho\underline{a})\Delta_3 \right] &= \underline{0} \end{aligned} \quad (16)$$

The terms containing Δ_3 will vanish in the limit and we get

$$\underline{t}^{-3} = -\underline{t}_0 \quad (17)$$

As \underline{t}^3 and \underline{t}^{-3} form an action reaction pair, we finally obtain

$$\underline{t}^3 = \underline{\sigma} \underline{e}_3 = \underline{t}_0 \quad (18)$$

As mentioned earlier, this equation is also used as boundary condition for solving the stress equilibrium equation. This result might look very intuitive and trivial. But, it is not as straightforward as it looks. Traction is the force per unit area applied by one part of the body on the other part of the body. It is applied by the body itself and is an internal force per unit area whereas \underline{t}_0 is applied by an external agent. We cannot conclude that $\underline{t}_0 = \underline{t}^3$ by intuition because they are two different things. But, this derivation tells us that the internal response of the body close to the surface is the same as what the external agent is applying. We won't be able to conclude anything about \underline{t}^1 or \underline{t}^2 from this derivation though.

2.1 Relating stress matrix at surface point with externally applied load (start time: 40:40)

Suppose, we want to find the stress matrix at the surface point with respect to the chosen coordinate system. We know that the columns of the stress matrix will be formed by the tractions on $\underline{e}_1, \underline{e}_2$ and

²The tangent plane of a surface is a plane that just touches the surface.

\underline{e}_3 planes respectively. Thus, using equation (18), we know that the third column is the externally applied load per unit area. The first and second columns of the stress matrix are still unknown. As a stress matrix has to be symmetric, the third component of the first and second column also become known to us. So, we get five entries of the stress matrix at surface point right away as shown below:

$$[\underline{\sigma}] = \begin{bmatrix} \times & \times & \underline{t}_0^1 \\ \times & \times & \underline{t}_0^2 \\ \underline{t}_0^1 & \underline{t}_0^2 & \underline{t}_0^3 \end{bmatrix}. \quad (19)$$

The rest four entries are unknown (out of which only three are independent as the stress matrix has to be symmetric) and can be found by solving the stress equilibrium equations. As a special case, if there are parts of the boundary where no external load is being applied ($\underline{t}_0=0$), then

$$[\underline{\sigma}] = \begin{bmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (20)$$

3 Traction and Stress inside a fluid body at rest (start time: 43:56)

Think of a static fluid, say bucket filled with water as shown in Figure 5. We know that the pressure

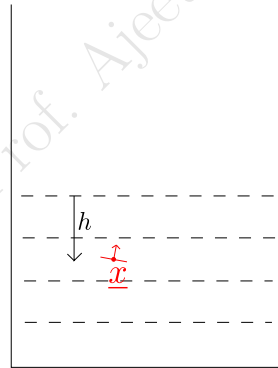


Figure 5: A bucket filled with water with a point \underline{x} at depth h inside water.

(p) inside water is given by ρgh where h is the depth from the top surface, g is the acceleration due to gravity and ρ is the density of water.

3.1 Traction (start time: 44:49)

We want to know the state of stress at any point in the water body. So, we first think of a small infinitesimal plane at that point. As fluids cannot sustain shear when they are in static equilibrium, there will be no shear component of traction on any plane. The traction would be the same as pressure (p) and would act along the plane normal but pointing into the plane due to compressive

nature of pressure. Thus, at an arbitrary point \underline{x} in the fluid and on an arbitrary plane at that point (with plane normal given by \underline{n}), traction \underline{t} will be given by

$$\underline{t}(\underline{x}; \underline{n}) = -p(\underline{x}) \underline{n}. \quad (21)$$

3.2 Stress matrix (start time: 48:37)

Stress matrix in $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ coordinate system can be found by writing the tractions as columns. Traction on \underline{e}_i plane will be equal to $-p\underline{e}_i$. So

$$[\underline{\underline{\sigma}}] = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix} = -p [\underline{\underline{I}}]. \quad (22)$$

We observe that all the shear components in the stress matrix are zero. We can verify as shown below that $\underline{\underline{\sigma}} \underline{n}$ will give us the traction given by equation (21):

$$\underline{t} = \underline{\underline{\sigma}} \underline{n} = -p \underline{\underline{I}} \underline{n} = -p \underline{n}. \quad (23)$$

Thus, the stress tensor for a fluid body in statics is $-p\underline{\underline{I}}$.

4 Objective questions to practise

1. We obtain stress tensor to be symmetric as a consequence of
 - (a) Balance of linear momentum
 - (b) Balance of angular momentum
 - (c) Balance of energy
 - (d) None of these
2. Stress tensor is not symmetric
 - (a) If the body is accelerating
 - (b) If body force acts on the body
 - (c) It is always symmetric
 - (d) None of these
3. Suppose a circular-cross-section straight beam is clamped at the one end and subjected to transverse load at other end. Its lateral surface is not subjected to any external load. What can we say about traction in its cross-section?
 - (a) Traction will vary in the cross-section
 - (b) Traction at peripheral points of the cross-section will not have any shear component
 - (c) The shear component of traction at peripheral points of the cross-section will have no radial component
 - (d) None of these
4. Suppose a beam having rectangular cross-section is subjected to normal distributed load (vertical along “-Y”) on its top surface.
 - (a) The shear component of traction in y-direction at mid point of cross-sectional plane is zero
 - (b) The shear component of traction in y-direction at the top points of cross-sectional plane is zero
 - (c) The shear component of traction in y-direction at the bottom points of cross-sectional plane is zero
 - (d) None of these
5. Suppose we want to write the stress matrix at a point in the \underline{e}_1 - \underline{e}_2 - \underline{e}_3 coordinate system and we know the traction components along the \underline{e}_2 plane. How many stress components of the stress matrix are still unknown?
 - (a) 6

- (b) 2
 - (c) 4
 - (d) 3
6. Which of the following is true for a stress matrix?
- (a) It is an orthogonal matrix
 - (b) It is a symmetric matrix
 - (c) It is a skew symmetric matrix
 - (d) None of the above
7. For a static problem, how many scalar unknowns are we left with in the linear momentum balance equation if we use results obtained by angular momentum balance in the linear momentum balance equation (given that body forces are known)?
- (a) 6
 - (b) 9
 - (c) 4
 - (d) 3
8. Which of the following is true with respect to traction at the outer surfaces of the boundary?
- (a) It is equal in magnitude but opposite in direction to the externally applied force per unit area
 - (b) It forms an action-reaction pair with the externally applied force per unit area
 - (c) It is equal to the externally applied force per unit area
 - (d) None of the above
9. When we derived the Angular Momentum Balance using the simpler method, what strong assumption/assumptions did we use?
- (a) Stress matrix is symmetric
 - (b) Stress matrix is skew symmetric
 - (c) Traction does not vary on the surfaces of the cuboid
 - (d) None of the above
10. Which of the following is true regarding stress equilibrium equations?
- (a) They help in finding the stress distribution in a body
 - (b) They can be solved without boundary conditions if angular momentum balance results are used with the stress equilibrium equations

- (c) They are derived using Newton's second law
 (d) All of the above
11. Stress tensor is not symmetric
- (a) if the body is accelerating.
 (b) if body force is present in the system.
 (c) it is always symmetric.
 (d) None of these.
12. The stress matrix for a body inside a fluid at rest has
- (a) diagonal components only
 (b) both diagonal and off diagonal components
 (c) only off diagonal components
 (d) None of these
13. Suppose we know the all the traction components on e_3 plane and we want to write stress matrix at a point in $(e_1 - e_2 - e_3)$ coordinate system. How many independent stress components of the stress matrix are still unknown?
- (a) 6
 (b) 2
 (c) 5
 (d) 3
14. In the absence of body forces, does the following stress distribution satisfy the equations of equilibrium?
- $$[\sigma]_{(e_1, e_2, e_3)} = \begin{bmatrix} x_2^2 + \vartheta(x_1^2 - x_2^2) & -2\vartheta x_1 x_2 & 0 \\ -2\vartheta x_1 x_2 & x_1^2 + \vartheta(x_2^2 - x_1^2) & 0 \\ 0 & 0 & \vartheta(x_1^2 + x_2^2) \end{bmatrix}$$
- (a) True
 (b) False
15. In the classical elasticity theory, the stress matrix is symmetric only when
- (a) the body is in static equilibrium
 (b) the body may be accelerating but not rotating
 (c) always
 (d) never.