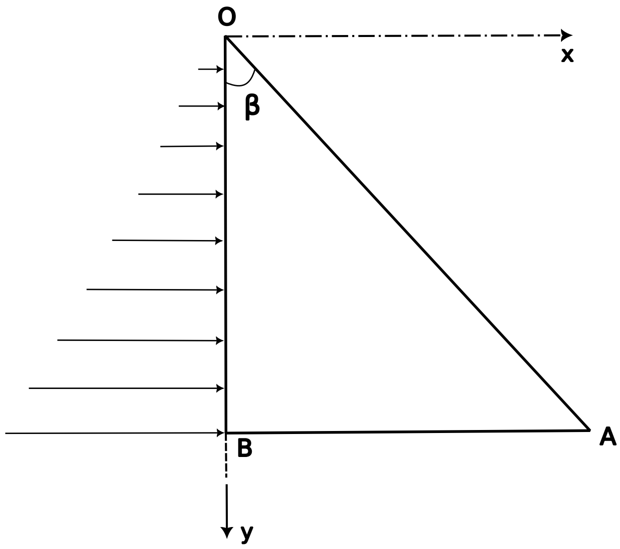
Tutorial 4: Stress equilibirum equations and Principal Stresses

APL 104 - 2022 (Solid Mechanics)

1. The sectional view of a dam is shown in Fig.1.



The pressure of water on vertical face (denoted by line OB) is also shown. With the axes Ox and Oy, as shown in Fig.1, the stress components at any point (x, y) are given by (γ = specific weight of water and ρ = specific weight of dam material)

$$\sigma_{xx} = -\gamma\, y$$

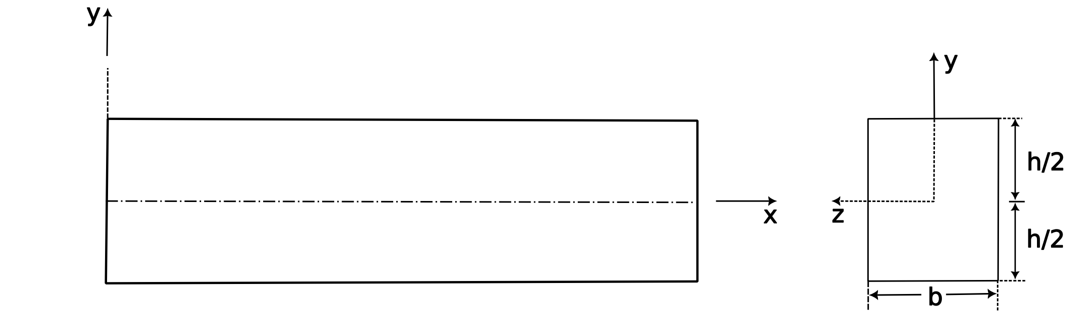
$$\sigma_{yy} = \left(\frac{\rho}{\tan\beta} - \frac{2\gamma}{\tan^3\beta}\right) x + \left(\frac{\gamma}{\tan^2\beta} - \rho\right) y$$

$$\tau_{xy} = \tau_{yx} = -\frac{\gamma}{\tan^2\beta} x$$

$$\tau_{yz} = 0,\ \tau_{zx} = 0,\ \sigma_{zz} = 0$$

Check if these stress components satisfy the differential equations of equilibrium. Also, verify if the boundary conditions are satisfied on the vertical face OB.

2. Consider the rectangular beam shown in Fig.2. According to the elementary theory of bending, the 'fiber stress' within the elastic range due to bending is given by

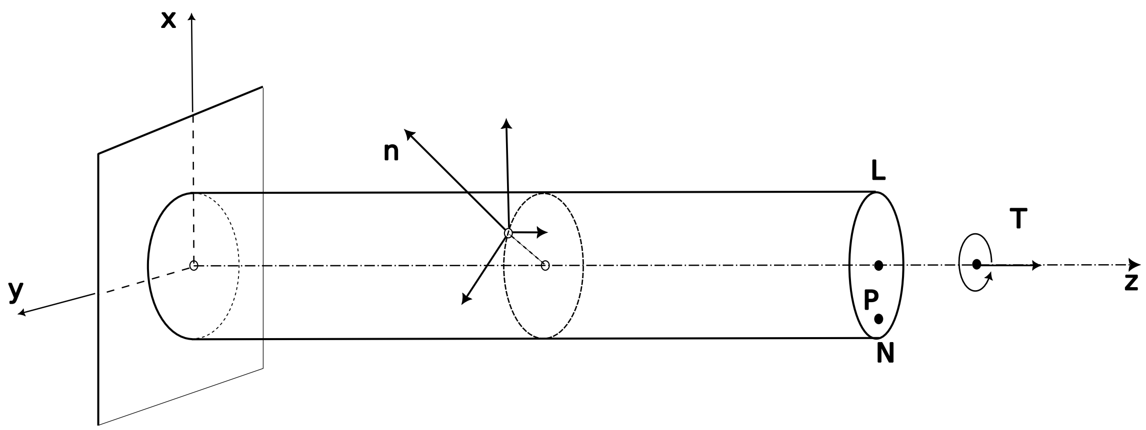


$$\sigma_{xx} = -\frac{My}{I} = -\frac{12My}{bh^3}$$

where M is the bending moment in the beam's cross-section and is a function of x. Assume that $\sigma_{zz} = \tau_{zx} = \tau_{zy} = 0$ everywhere. Furthermore $\tau_{xy} = 0$ on the top and bottom face and $\sigma_{yy} = 0$ on the bottom face. Using the differential equations of equilibrium, determine τ_{xy} and σ_{yy}. Compare these with the values given in the elementary strength of materials.

3. A cylindrical rod (Fig.3) is subjected to a torque T. At any point P of the cross-section LN, the following stress components exist:

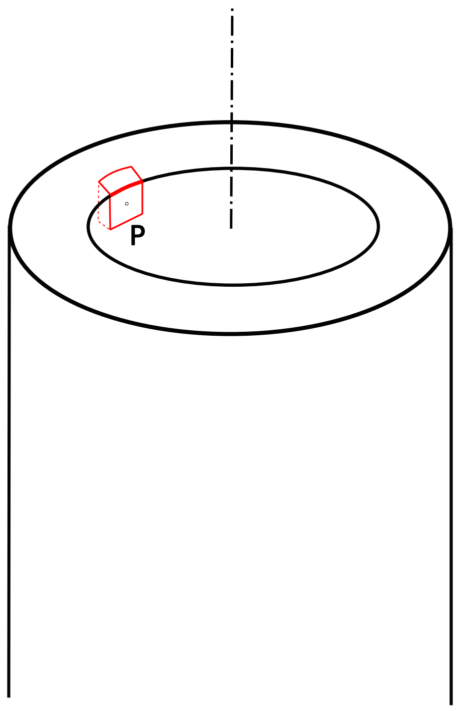
$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \tau_{xy} = \tau_{yx} = 0, \; \tau_{xz} = \tau_{zx} = -G\theta y, \; \tau_{yz} = \tau_{zy} = G\theta x$$

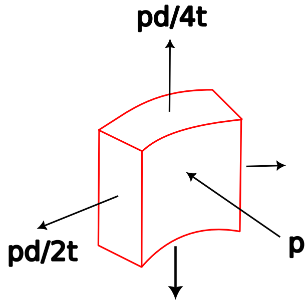


Check whether these satisfy the equations of equilibrium. Also show that the above distribution implies that the lateral surface should be free of external load.

4. For the state of stress given in Q3, determine the principal stress components, maximum shear stress values and the associated plane normals on which they are realized.

5. A cylindrical boiler, 180cm in diameter, is made of plates 1.8cm thick and is subjected to an internal pressure of 1400 kPa. Determine the maximum shearing stress in the plate at point P and the plane on which it acts.





6. **Divergence operator**
Divergence of a tensor is defined as follows:

$$\underline{\nabla} \cdot (\circ) = \sum_i \frac{\partial}{\partial x_i} (\circ) \cdot \underline{e}_i$$

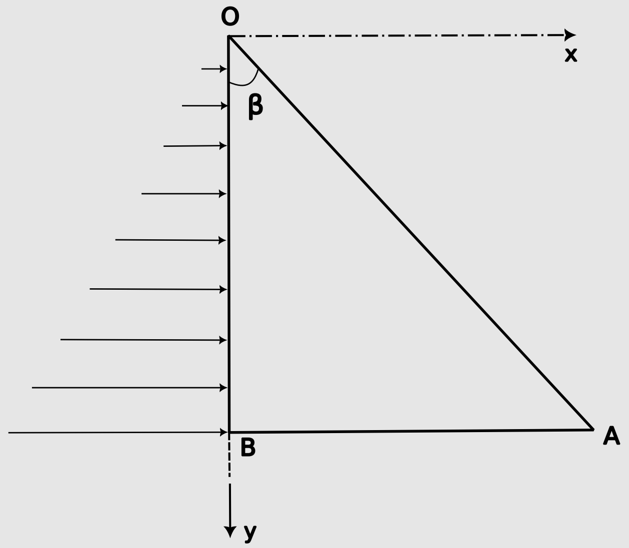
For example

$$\begin{aligned}\underline{\nabla} \cdot \underline{v} = \sum_i \frac{\partial}{\partial x_i} (\underline{v}) \cdot \underline{e}_i &= \sum_i \frac{\partial}{\partial x_i} \left(\sum_j v_j \underline{e}_j \right) \cdot \underline{e}_i \\ &= \sum_i \sum_j \frac{\partial v_j}{\partial x_i} \delta_{ij} = \sum_i \frac{\partial v_i}{\partial x_i}.\end{aligned}$$

Show that $\underline{\nabla} \cdot \underline{\underline{\sigma}} = \sum_i \sum_j \frac{\partial \sigma_{ji}}{\partial x_i} \underline{e}_j$

APL 104 Tutorial 4 solutions

Q1. The sectional view of a dam is shown in Fig.1.



The pressure of water on vertical face (denoted by line OB) is also shown. With the axes Ox and Oy, as shown in Fig.1, the stress components at any point (x, y) are given by (γ = specific weight of water and ρ = specific weight of dam material)

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$$\sigma_{yy} = \left(\frac{\rho}{\tan\beta} - \frac{2\gamma}{\tan^3\beta}\right)x + \left(\frac{\gamma}{\tan^2\beta} - \rho\right)y$$

$$\tau_{xy} = \tau_{yx} = -\frac{\gamma}{\tan^2\beta}x$$

$$\tau_{yz} = 0,\ \tau_{zx} = 0,\ \sigma_{zz} = 0$$

Check if these stress components satisfy the differential equations of equilibrium. Also, verify if the boundary conditions are satisfied on the vertical face OB.

Solution:
Writing the equations of equilibrium component-wise, we have

$$\underline{X - direction}: \quad \frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} + b_x = \left(\frac{\rho}{g}\right)a_x \tag{1}$$

$$\underline{Y - direction}: \quad \frac{\partial\tau_{yx}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\tau_{yz}}{\partial z} + b_y = \left(\frac{\rho}{g}\right)a_y \tag{2}$$

$$\underline{Z - direction}: \quad \frac{\partial\tau_{zx}}{\partial x} + \frac{\partial\tau_{zy}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z} + b_z = \left(\frac{\rho}{g}\right)a_z \tag{3}$$

As the dam is at rest, we have $a_x = a_y = a_z = 0$. Next note that body forces (force per unit volume) are $b_x = b_z = 0$ and $b_y = \rho$ since the self-weight per unit volume acts downwards along +ve y axis. Accordingly, Eq. (1) and Eq. (3) are satisfied automatically when we substitute corresponding formulas for stress components. For Eq. (2), we get

$$\left(\frac{\gamma}{\tan^2\beta} - \rho\right) - \frac{\gamma}{\tan^2\beta} + \rho = 0$$

We thus see that all equilibrium equations are satisfied by the given formulas of stress components.

Next we need to verify if the stress components satisfy the boundary condition at face OB. Recall that there are two types of BCs: <u>displacement BC</u> and <u>traction BC</u>. On face OB, we have traction BC, i.e., the traction at any point on the face OB (with outward normal $-\underline{e}_1$) should be equal to the externally applied load (hydrostatic pressure due to water).

<u>External load on face OB</u>

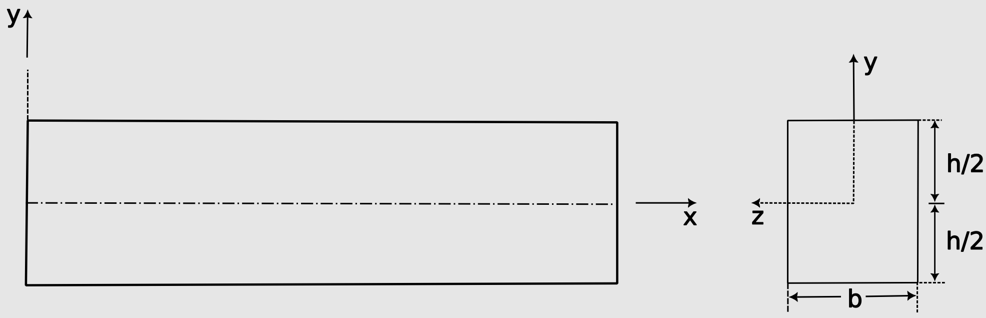
$$\underline{f}^{\text{ext}}\,|_{(\text{face OB})} = \gamma y\,\underline{e}_1 = \begin{bmatrix}\gamma y\\0\\0\end{bmatrix}$$

<u>Traction on face OB</u>

$$\begin{aligned}\underline{t}^{-1}\,|_{(x=0,y,z)} &= \underline{\underline{\sigma}}\cdot(-\underline{e}_1)\\ &= -\begin{bmatrix}\sigma_{xx} & \tau_{yx} & \tau_{zx}\\ \tau_{yx} & \sigma_{yy} & \tau_{zy}\\ \tau_{zx} & \tau_{zy} & \sigma_{zz}\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}\\ &= \begin{bmatrix}-\sigma_{xx}\\-\tau_{yx}\\-\tau_{zx}\end{bmatrix}_{(x=0,y,z)} = \begin{bmatrix}-(-\gamma y)\\ -\left(-\frac{\gamma}{\tan^2\beta}\,x\right)\\ 0\end{bmatrix}_{(x=0,y,z)} = \begin{bmatrix}\gamma y\\0\\0\end{bmatrix}.\end{aligned}$$

Hence, the boundary conditions are also satisfied.

Q2. Consider the rectangular beam shown in Fig.2. According to the elementary theory of bending, the 'fiber stress' within the elastic range due to bending is given by



$$\sigma_{xx} = -\frac{My}{I} = -\frac{12My}{bh^3}$$

where M is the bending moment in the beam's cross-section and is a function of x. Assume that $\sigma_{zz} = \tau_{zx} = \tau_{zy} = 0$ everywhere. Furthermore $\tau_{xy} = 0$ on the top and bottom face and $\sigma_{yy} = 0$ on the bottom face. Using the differential equations of equilibrium, determine τ_{xy} and σ_{yy}. Compare these with the values given in the elementary strength of materials.

Solution:
Let us assume the the weight of the beam to be negligible. Hence, there is no body force along any direction. Furthermore, the beam is at rest. Hence the acceleration components along X-, Y- and Z-directions are also zero.
Using the equilibrium equation along X-direction, we have

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0.$$

Since $\tau_{xz} = 0$ and M is a function of x, we get:

$$\begin{aligned}
-\frac{12y}{bh^3}\frac{\partial M}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0 \\
\Rightarrow \frac{\partial \tau_{xy}}{\partial y} &= \frac{12}{bh^3}\frac{\partial M}{\partial x} y \\
\Rightarrow \tau_{xy} &= \frac{6}{bh^3}\frac{\partial M}{\partial x} y^2 + f(x, z).
\end{aligned}$$

Here $f(x, z)$ is the integration constant which will be a function of x and z. To obtain its formula, we use the condition that $\tau_{xy} = 0$ at the top and bottom face of the cross-section, i.e., at $(x, y = \pm\frac{h}{2}, z)$. It leads to

$$\begin{aligned}
f(x, z) &= -\frac{6}{bh^3}\frac{h^2}{4}\frac{\partial M}{\partial x} \quad (f(x, z) = f(x) \text{ since the RHS is only a function of } x) \\
\Rightarrow f(x) &= -\frac{3}{2bh}\frac{\partial M}{\partial x} \\
\therefore \ \tau_{xy} &= \frac{3}{2bh}\frac{\partial M}{\partial x}\left(\frac{4y^2}{h^2} - 1\right).
\end{aligned}$$

To obtain σ_{yy}, we will use equilibrium equation in Y-direction:

$$\begin{aligned}
\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} &= 0 \\
\Rightarrow \frac{\partial \sigma_{yy}}{\partial y} &= -\frac{3}{2bh}\left(\frac{4y^2}{h^2} - 1\right)\frac{\partial^2 M}{\partial x^2} \\
\Rightarrow \sigma_{yy} &= -\frac{3}{2bh}\frac{\partial^2 M}{\partial x^2}\left(\frac{4y^3}{3h^2} - y\right) + g(x,z)
\end{aligned}$$

where $g\left(x,z\right)$ is the integrating constant and it has to be determined using the condition $\sigma_{yy} = 0$ on the bottom face, i.e., at $(x, y = -\frac{h}{2}, z)$. It leads to

$$\begin{aligned}
g(x,z) &= \frac{3}{2bh}\frac{\partial^2 M}{\partial x^2}\left(\frac{4\left(-\frac{h}{2}\right)^3}{3h^2} - \left(-\frac{h}{2}\right)\right) \quad (g(x,z) = g(x) \text{ again since RHS is a function of only } x) \\
\Rightarrow g\left(x\right) &= \frac{3}{2bh}\frac{\partial^2 M}{\partial x^2}\frac{h}{3} = \frac{1}{2b}\frac{\partial^2 M}{\partial x^2}
\end{aligned}$$

Substituting it back in the expression of σ_{xx}, we get

$$\sigma_{yy} = -\frac{3}{2bh}\frac{\partial^2 M}{\partial x^2}\left(\frac{4y^3}{3h^2} - y - \frac{h}{3}\right)$$

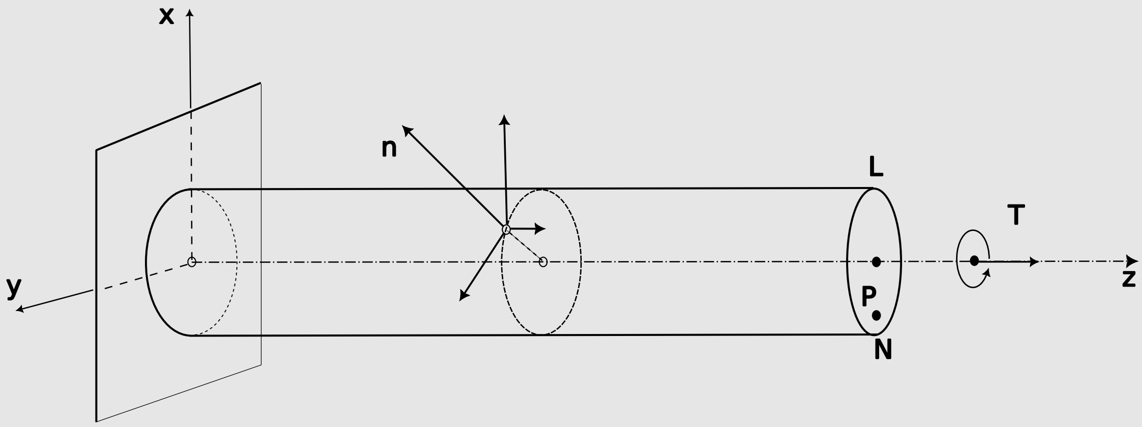
Using the above formula, at $y = +h/2$, the value of σ_{yy} turns out to be

$$\sigma_{yy} = \frac{1}{b}\frac{\partial^2 M}{\partial x^2} = \frac{w}{b}$$

where $w = \dfrac{\partial^2 M}{\partial x^2}$ denotes the distributed load acting on the top face of the beam. Since b is the width of the beam, the stress will be $^w/_b$ as obtained above.

Q3. A cylindrical rod (Fig.3) is subjected to a torque T. At any point P of the cross-section LN, the following stress components exist:

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \tau_{xy} = \tau_{yx} = 0, \ \tau_{xz} = \tau_{zx} = -G\theta y, \ \tau_{yz} = \tau_{zy} = G\theta x$$



Check whether these satisfy the equations of equilibrium. Also show that the above distribution implies that the lateral surface should be free of external load.

Solution: Assuming again that self-weight is zero, the body force components are all zero. Moreover, the rod is not in motion. Hence, the three acceleration components are also zero.

Substituting the given values of stress components in equilibrium equations, we get:

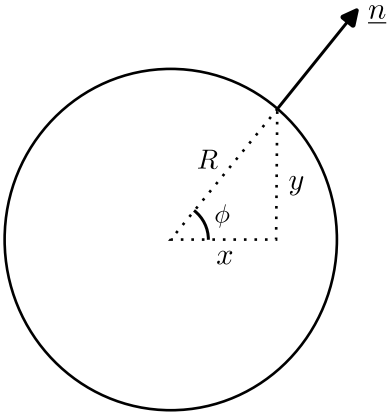
$$(LHS) \ \frac{\partial \cancelto{0}{\sigma_{xx}}}{\partial x} + \frac{\partial \cancelto{0}{\tau_{xy}}}{\partial y} + \cancelto{0}{\frac{\partial \tau_{xz}}{\partial z}} \ = \ 0 \ \ (RHS)$$

$$(LHS) \ \frac{\partial \cancelto{0}{\tau_{yx}}}{\partial x} + \frac{\partial \cancelto{0}{\sigma_{yy}}}{\partial y} + \cancelto{0}{\frac{\partial \tau_{yz}}{\partial z}} \ = \ 0 \ \ (RHS)$$

$$(LHS) \ \cancelto{0}{\frac{\partial \tau_{zx}}{\partial x}} + \cancelto{0}{\frac{\partial \tau_{zy}}{\partial y}} + \frac{\partial \cancelto{0}{\sigma_{zz}}}{\partial z} \ = \ 0 \ \ (RHS)$$

The lateral surface of the rod consists of the circumferencial outer surface of the rod spanning along the z-axis. To show that the lateral surface is free of external load, we invoke the relation of traction boundary condition with the external load. If the traction at any point on the lateral surface turns out to be a $\underline{0}$ vector, then the external load must be zero.

The unit outward normal on lateral surface will be perpendicular to Z-direction and will be given by



$$[\underline{n}] = \begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{bmatrix} = \begin{bmatrix} x/R \\ y/R \\ 0 \end{bmatrix}.$$

Hence, the traction there becomes

$$\begin{aligned}[\underline{t}^n] &= \left[\underline{\underline{\sigma}}\right][\underline{n}] \\ &= \begin{bmatrix} 0 & 0 & -G\theta y \\ 0 & 0 & G\theta x \\ -G\theta y & G\theta x & 0 \end{bmatrix} \begin{bmatrix} x/R \\ y/R \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.\end{aligned}$$

Q4. For the state of stress given in Q3, determine the principal stress components, maximum shear stress values and the associated plane normals on which they are realized.

Solution: To obtain the principal components, we can set the det $\left(\left[\underline{\underline{\sigma}}\right] - \lambda\left[\underline{\underline{\mathrm{I}}}\right]\right) = 0$ and obtain the characteristic equation:

$$\begin{aligned}\det\left(\begin{bmatrix} -\lambda & 0 & -G\theta y \\ 0 & -\lambda & G\theta x \\ -G\theta y & G\theta x & -\lambda \end{bmatrix}\right) &= 0 \\ \Rightarrow -\lambda^3 + \lambda G^2\theta^2\left(x^2+y^2\right) &= 0 \\ \Rightarrow \lambda\left(\lambda^2 - G^2\theta^2\left(x^2+y^2\right)\right) &= 0\end{aligned}$$

Therefore, we get the principal stress components to be (in decreasing order):

$$\lambda_1 = G\theta\sqrt{x^2+y^2}, \quad \lambda_2 = 0, \quad \lambda_3 = -G\theta\sqrt{x^2+y^2}.$$

The <u>first principal stress plane</u> can be found as follows:

$$\begin{bmatrix} -\lambda_1 & 0 & -G\theta y \\ 0 & -\lambda_1 & G\theta x \\ -G\theta y & G\theta x & -\lambda_1 \end{bmatrix} \begin{bmatrix} n_1^1 \\ n_2^1 \\ n_3^1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Below we write the three equations (of which only two are independent), plus one equation for normalization of the vector:

$$\begin{aligned} -\lambda_1 n_1^1 - G\theta y\, n_3^1 &= 0 \\ -\lambda_1 n_2^1 - G\theta x\, n_3^1 &= 0 \\ -G\theta y\, n_1^1 - G\theta x\, n_2^1 - \lambda_1 n_3^1 &= 0 \\ \left(n_1^1\right)^2 + \left(n_2^1\right)^2 + \left(n_3^1\right)^2 &= 1 \end{aligned}$$

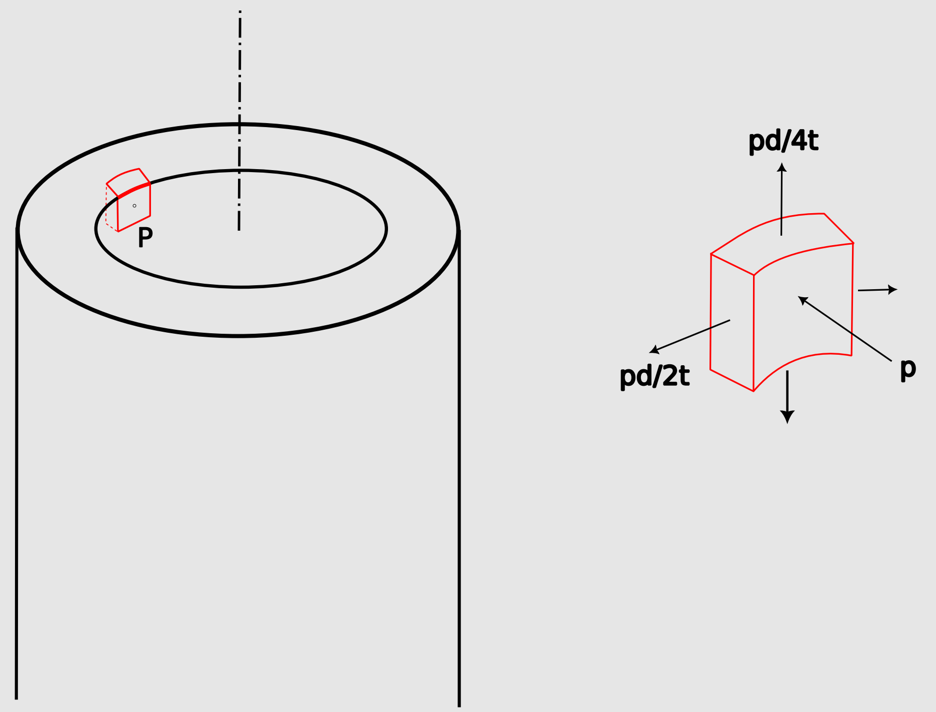
Solving the above, we get:

$$n_3^1 = \pm\frac{1}{\sqrt{2}}$$
$$n_1^1 = \mp\frac{G\theta y}{\sqrt{x^2+y^2}}$$
$$n_2^1 = \mp\frac{G\theta x}{\sqrt{x^2+y^2}}$$

Similarly, one can also solve for the normals corresponding to other two principal stress components (not worked out here). One of them will be radial direction corresponding to zero principal stress component (easy to figure out from the last problem - check the boundary condition on lateral surface!).
For the maximum shear stresses, we get

$$\tau_1 = \left|\frac{\sigma_{11}-\sigma_{33}}{2}\right| = G\theta\sqrt{x^2+y^2},$$
$$\tau_2 = \left|\frac{\sigma_{11}-\sigma_{22}}{2}\right| = \frac{1}{2}G\theta\sqrt{x^2+y^2},$$
$$\tau_3 = \left|\frac{\sigma_{22}-\sigma_{33}}{2}\right| = \frac{1}{2}G\theta\sqrt{x^2+y^2}.$$

Q5. A cylindrical boiler, 180cm in diameter, is made of plates 1.8cm thick and is subjected to an internal pressure of 1400 kPa. Determine the maximum shearing stress in the plate at point P and the plane on which it acts.



Solution:

Q6. **Divergence operator**
Divergence of a tensor is defined as follow

$$\underline{\nabla} \cdot (\circ) = \sum_i \frac{\partial}{\partial x_i} (\circ) \cdot \underline{e}_i$$

For example

$$\begin{aligned}
\underline{\nabla} \cdot \underline{v} = \sum_i \frac{\partial}{\partial x_i} (\underline{v}) \cdot \underline{e}_i &= \sum_i \frac{\partial}{\partial x_i} \left(\sum_j v_j \underline{e}_j \right) \cdot \underline{e}_i \\
&= \sum_i \sum_j \frac{\partial \underline{v}}{\partial \underline{x}_j} \delta_{ij} \\
&= \sum_i \frac{\partial v_i}{\partial x_i}
\end{aligned}$$

Show that $\underline{\nabla} \cdot \underline{\underline{\sigma}} = \sum_i \sum_j \frac{\partial \sigma_{ji}}{\partial x_i} \underline{e}_j$

Solution:

$$\begin{aligned}
\underline{\nabla} \cdot \left(\underline{\underline{\sigma}} \right) &= \sum_i \frac{\partial \underline{\underline{\sigma}}}{\partial x_i} \cdot \underline{e}_i \\
&= \sum_i \sum_j \sum_k \frac{\partial \sigma_{jk}}{\partial x_i} \left(\underline{e}_j \otimes \underline{e}_k \right) \cdot \underline{e}_i \\
&= \sum_i \sum_j \sum_k \frac{\partial \sigma_{jk}}{\partial x_i} \underline{e}_j \delta_{ik} \\
&= \sum_i \sum_j \frac{\partial \sigma_{ji}}{\partial x_i} \underline{e}_j
\end{aligned}$$