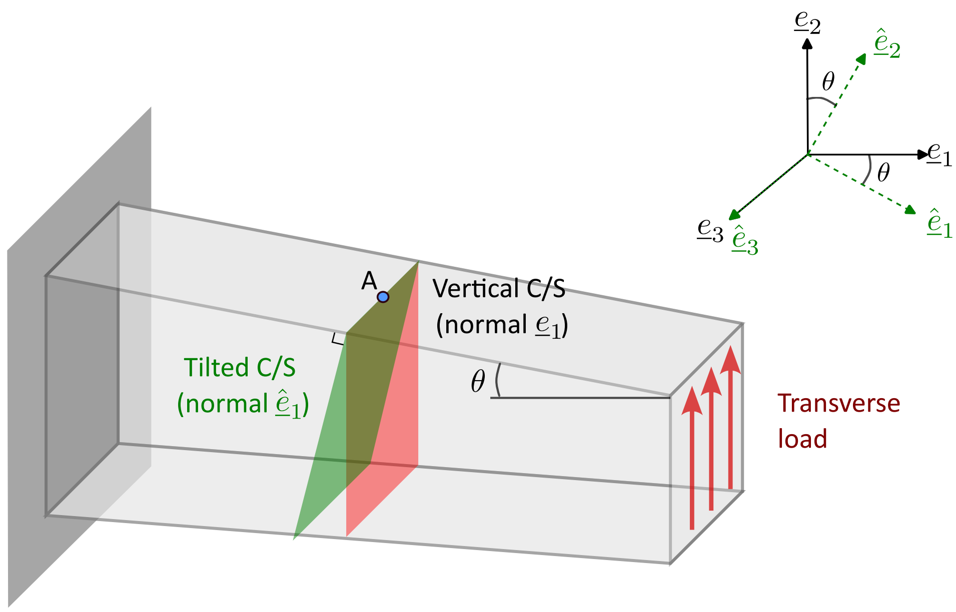
Tutorial 3: Stress tensor and its transformation

APL 104 - 2022 (Solid Mechanics)

Q1. A tapered beam is clamped at one end and subjected to transverse load (along $\underline{e}_2$) at the other end. Think of a point A on the top slanted surface of the beam. What can you say about the state of stress at point A?

Suppose that we know $\hat{\sigma}_{11}$ at pt A. Can we find the components $\tau_{21}, \sigma_{11}, \hat{\tau}_{21}$ at pt A?

Assume that traction has no components along $\underline{e}_3$ at any point in the body.



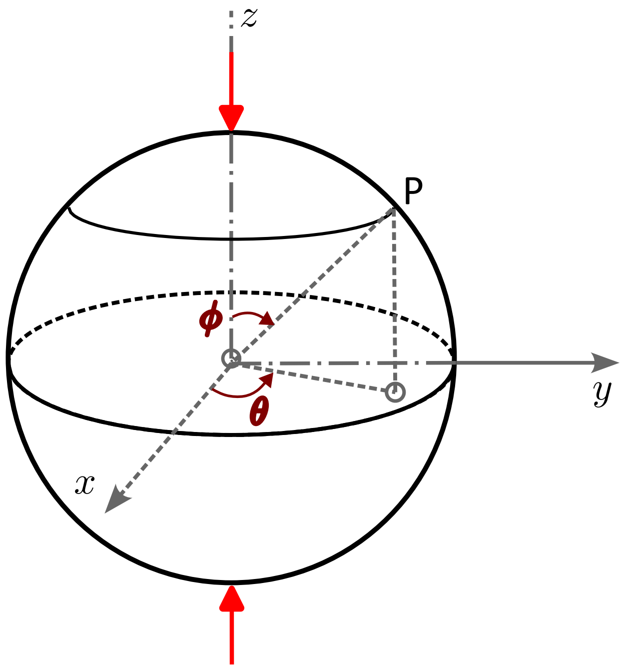
Q2. The state of stress at a point is given by $\left[\underline{\underline{\sigma}}\right] = \begin{bmatrix} \sigma_{11} & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$.

What should be σ_{11} such that there is at least one plane at that point on which the traction vanishes? Also, find the corresponding plane normal.

Q3. Suppose the stress matrix at a point equals $\left[\underline{\underline{\sigma}}\right] = \begin{bmatrix} a & 0 & d \\ 0 & b & e \\ d & e & c \end{bmatrix}$.

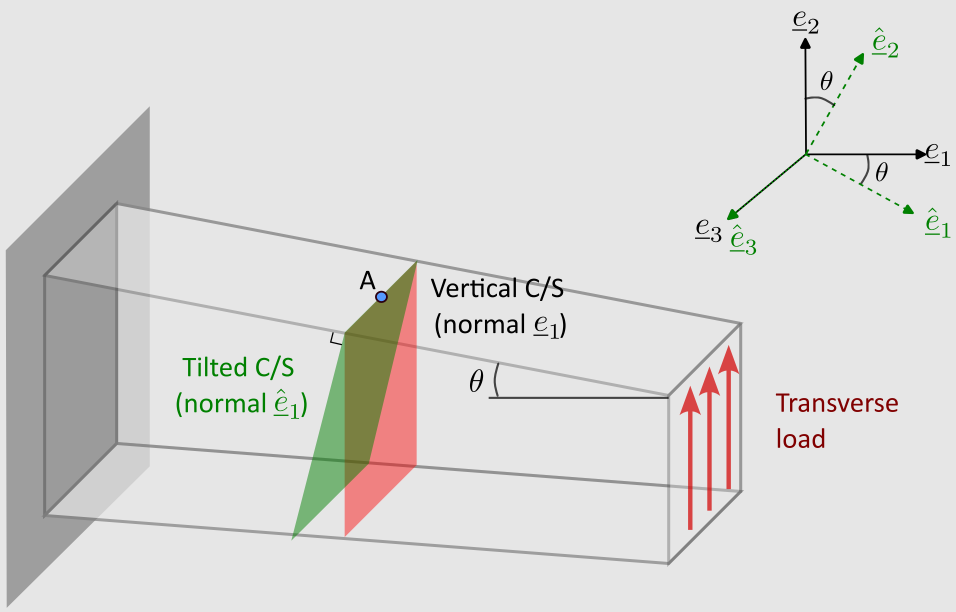
Determine the plane having its normal perpendicular to z-axis such that the traction on that plane is tangential to the plane.

Q4. Consider a sphere of radius R subjected to diametrical compression as shown in the figure. Let σ_{rr}, $\sigma_{\theta\theta}$, and $\sigma_{\phi\phi}$ be the normal stresses and $\tau_{r\theta}$, $\tau_{\theta\phi}$ and $\tau_{\phi r}$ the shear stresses at any point in the sphere. At point $P(x, y, z)$ on the sphere's surface and lying in the $y - z$ plane, determine the rectangular normal stress components σ_{xx}, σ_{yy} and σ_{zz} in terms of the spherical stress components.



APL 104 Tutorial 3 solutions

Q1. A tapered beam is clamped at one end and subjected to transverse load (along $\underline{e}_2$) at the other end. Think of a point A on the top slanted surface of the beam. What can you say about the state of stress at point A?
Suppose we know $\hat{\sigma}_{11}$ at point A. Can we find the components $\tau_{21}, \sigma_{11}, \hat{\tau}_{21}$ at point A?
Assume that traction has no components along $\underline{e}_3$ at any point in the body.



Solution:
Notice that point A lies on the slanted surface where no external load is being applied! Furthermore, the slanted surface has normal $\hat{\underline{e}}_2$.

$$\text{This implies } \underline{t}^{\hat{2}} = \underline{0} \text{ at point A (this does not mean } \underline{t}^2 = \underline{0})$$

If we write the stress matrix in a coordinate system of $(\hat{\underline{e}}_1, \hat{\underline{e}}_2, \hat{\underline{e}}_3)$, then the 2^{nd} column will be zero ($\underline{t}^{\hat{2}} = \underline{0}$). Furthermore, the third row will also be zero since tractions have zero components along $\underline{e}_3$ (which equals $\hat{\underline{e}}_3$). Thus,

$$\left[\underline{\underline{\sigma}}\,(A)\right]_{(\hat{\underline{e}}_1,\hat{\underline{e}}_2,\hat{\underline{e}}_3)} = \begin{bmatrix} X & 0 & X \\ X & 0 & X \\ 0 & 0 & 0 \end{bmatrix}$$

But, we know that stress matrix is also symmetric

$$\left[\underline{\underline{\sigma}}(A)\right]_{(\hat{\underline{e}}_1,\hat{\underline{e}}_2,\hat{\underline{e}}_3)} = \begin{bmatrix} X & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $\hat{\sigma}_{11}$ is the only non-zero quantity!

$$\text{Note that } \hat{\tau}_{21}{=}0 \text{ (since } \hat{\tau}_{12} = 0)$$

From here, we can then transform the stress matrix to $(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ coordinate system

$$\text{If } \hat{\underset{\sim}{e}}_i = \underset{\approx}{R}\, \underset{\sim}{e}_i \;\Rightarrow \left[\underset{\approx}{R}\right]_{(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \widehat{\left[\underset{\approx}{\sigma}\right]} = \left[\underset{\approx}{R}\right]^T \left[\underset{\approx}{\sigma}\right] \left[\underset{\approx}{R}\right]$$

$$\begin{aligned}
\Rightarrow \left[\underset{\approx}{\sigma}\right] &= \left[\underset{\approx}{R}\right] \widehat{\left[\underset{\approx}{\sigma}\right]} \left[\underset{\approx}{R}\right]^T \\
&= \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\sigma}_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\sigma}_{11} c\theta & -\hat{\sigma}_{11} s\theta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \hat{\sigma}_{11} \cos^2\theta & -\hat{\sigma}_{11} s\theta c\theta & 0 \\ -\hat{\sigma}_{11} s\theta c\theta & \hat{\sigma}_{11} \sin^2\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \sigma_{11} &= \hat{\sigma}_{11} \cos^2\theta \\
\tau_{12} &= \tau_{21} = -\hat{\sigma}_{11} \sin\theta \cos\theta \\
\sigma_{22} &= \hat{\sigma}_{11} \sin^2\theta
\end{aligned}$$

There is another way to obtain these components!
Let us first obtain the traction on $\underset{\sim}{e}_1$-plane using $\underset{\sim}{t}^1 = \underset{\approx}{\sigma}\, \underset{\sim}{e}_1$. As we have $\left[\underset{\approx}{\sigma}\right]$ readily available in the 'hat' coordinate system, we can write the tensor formula to obtain $[\underset{\sim}{t}^1]$ in the 'hat'-coordinate system.

$$\Rightarrow [\underset{\sim}{t}^1]_{(\hat{\underset{\sim}{e}}_1, \hat{\underset{\sim}{e}}_2, \hat{\underset{\sim}{e}}_3)} = \left[\underset{\approx}{\sigma}\right]_{(\hat{\underset{\sim}{e}}_1, \hat{\underset{\sim}{e}}_2, \hat{\underset{\sim}{e}}_3)} [\underset{\sim}{e}_1]_{(\hat{\underset{\sim}{e}}_1, \hat{\underset{\sim}{e}}_2, \hat{\underset{\sim}{e}}_3)} = \begin{bmatrix} \hat{\sigma}_{11} & 0 & 0 \\ 0 & 0 & \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{\sigma}_{11} \cos\theta \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \sigma_{11} = \underset{\sim}{t}^1 \cdot \underset{\sim}{e}_1 = [\underset{\sim}{t}^1]_{(\hat{\underset{\sim}{e}}_1, \hat{\underset{\sim}{e}}_2, \hat{\underset{\sim}{e}}_3)} \cdot [\underset{\sim}{e}_1]_{(\hat{\underset{\sim}{e}}_1, \hat{\underset{\sim}{e}}_2, \hat{\underset{\sim}{e}}_3)} = \begin{bmatrix} \hat{\sigma}_{11} \cos\theta \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} = \hat{\sigma}_{11} \cos^2\theta!$$

Caution: Don't make a mistake by writing $[\underset{\sim}{e}_1]_{(\hat{\underset{\sim}{e}}_1, \hat{\underset{\sim}{e}}_2, \hat{\underset{\sim}{e}}_3)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Q2. The state of stress at a point is given by $\left[\underline{\underline{\sigma}}\right] = \begin{bmatrix} \sigma_{11} & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$.

What should be σ_{11} such that there is at least one plane at that point on which the traction vanishes? Also, find the corresponding plane normal.

Solution:

We basically want $\underline{\underline{\sigma}}\,\underline{n} = \underline{0}$ for at least one $\underline{n}$!

$$\text{or,} \quad \begin{bmatrix} \sigma_{11} & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From matrix property, we know that this happens only when the determinant of the matrix vanishes

$$\Rightarrow \det\left(\left[\underline{\underline{\sigma}}\right]\right) = \sigma_{11} \times [-4] - 2 \times [-2] + 1 \times 4 = 0$$
$$\Rightarrow -4\sigma_{11} + 4 + 4 = 0$$
$$\Rightarrow \sigma_{11} = 2$$

The corresponding $\underline{n}$ turns out to be $\underline{n} = \begin{bmatrix} \pm 2/3 \\ \pm 1/3 \\ \pm 2/3 \end{bmatrix}$

Q3. Suppose the stress matrix at a point equals $\left[\underline{\underline{\sigma}}\right] = \begin{bmatrix} a & 0 & d \\ 0 & b & e \\ d & e & c \end{bmatrix}$.

Determine the plane having its normal perpendicular to z-axis such that the traction on that plane is tangential to the plane.

Solution:

We have to find $[\underline{n}] = \begin{bmatrix} n_x \\ n_y \\ 0 \end{bmatrix}$ $\qquad$ $(n_z = 0 \text{ since } \underline{n} \perp \underline{e}_3)$

It is given that $\underline{t}^n$ has only tangential component

$$\Rightarrow \underline{t}^n \cdot \underline{n} = 0, \; or \left(\underline{\underline{\sigma}}\,\underline{n}\right) \cdot \underline{n} = \underline{0}$$
$$\Rightarrow \left(\begin{bmatrix} a & 0 & d \\ 0 & b & e \\ d & e & c \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ 0 \end{bmatrix} \right) \cdot \begin{bmatrix} n_x \\ n_y \\ 0 \end{bmatrix}$$
$$\Rightarrow a n_x^2 + b n_y^2 = 0$$

At the same time $n_x^2 + n_y^2 = 1 \Rightarrow n_y^2 = 1 - n_x^2$

$$\therefore a n_x^2 + b\left(1 - n_x^2\right) = 0$$
$$\Rightarrow n_x = \pm\left(b/\left(b-a\right)\right)^{1/2}, \; n_y = \pm\left(a/\left(a-b\right)\right)^{1/2}, n_z = 0$$