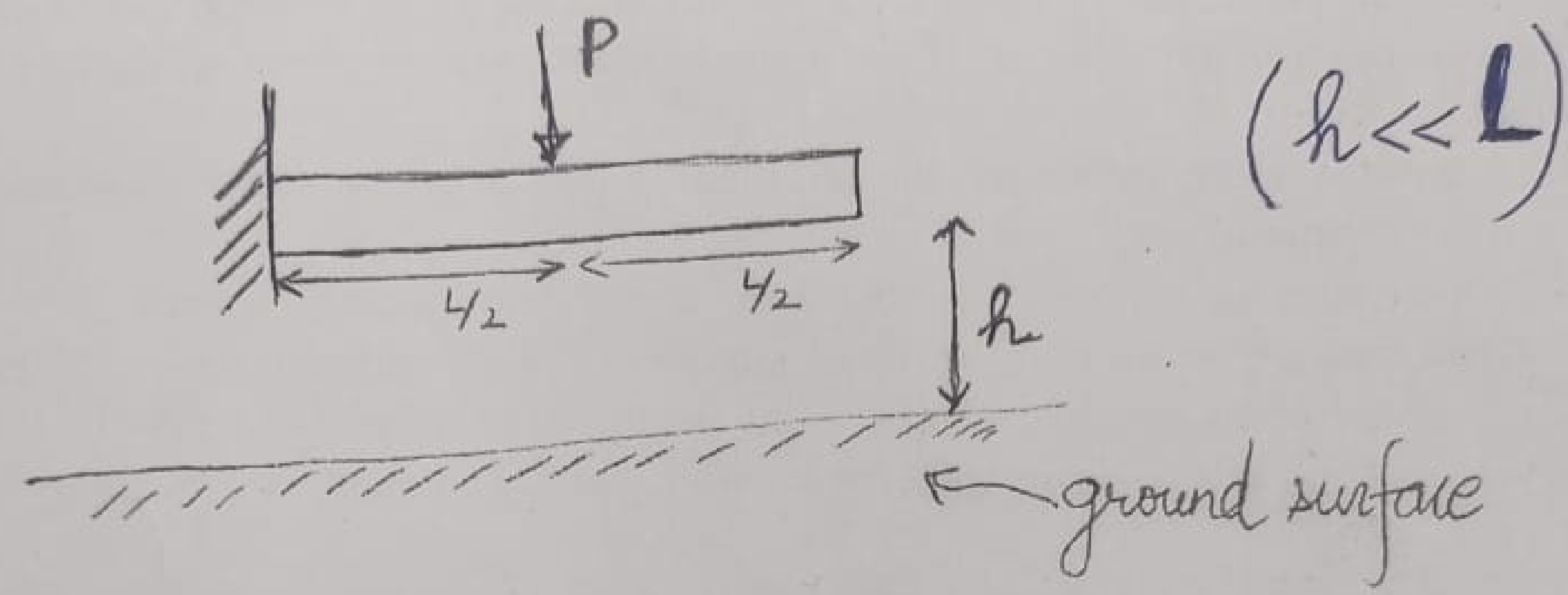
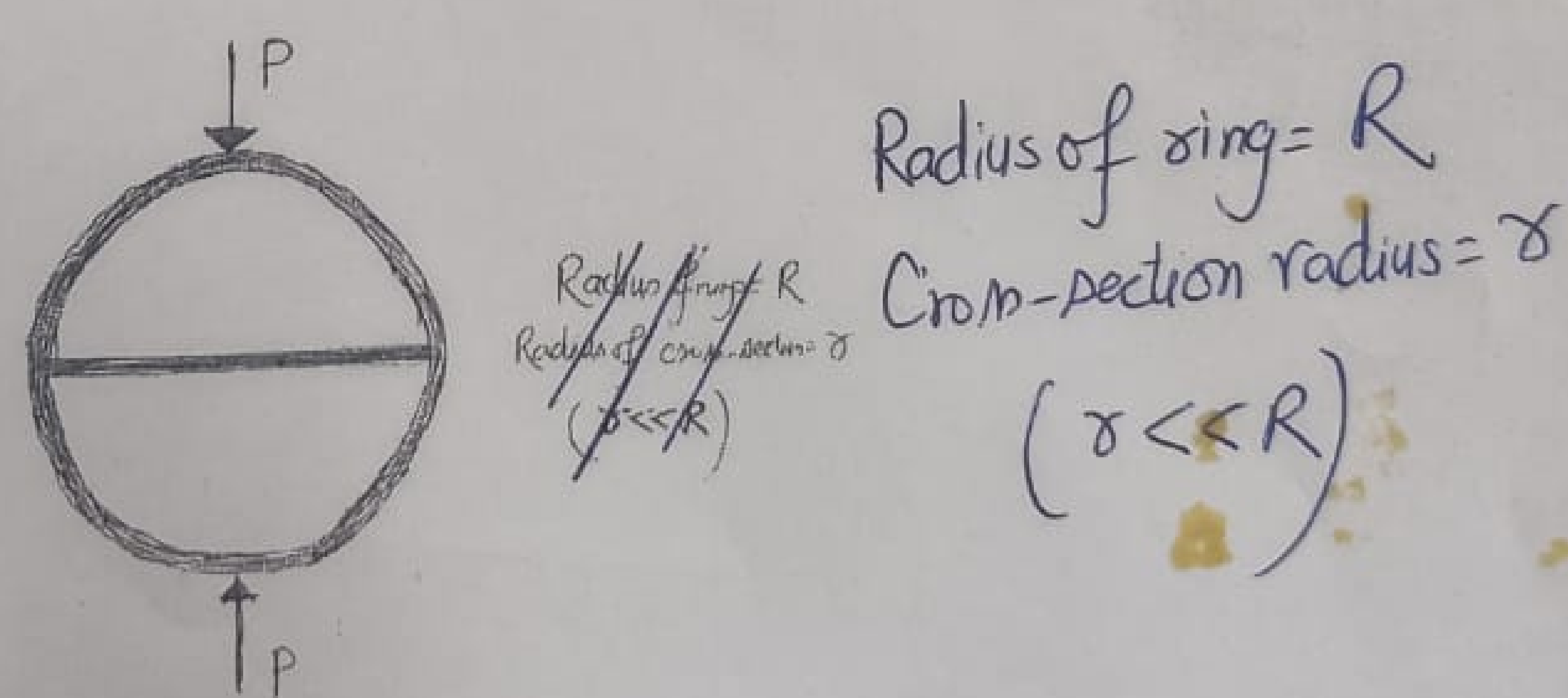
APL104: Major Exam (20/11/2022) Full Marks: 45

* Assume all the constants that you feel would be required!

Q1. Think of a cantilever beam which is loaded in the middle. How much of the beam would be in contact with the surface below? Neglect weight of the beam. Model the cantilever as a Timoshenko beam. Assume the load applied is such that the portion of the beam in contact with the surface is less than the beam's half length. (10)



Q2. In the class, we had learnt about deformation of a circular ring subjected to equal and opposite diametrical loads. Suppose the ring has another straight member along the other diameter as shown in the figure. By how much will the ring contract along the line of applied load? By how much will the straight member elongate? (10)

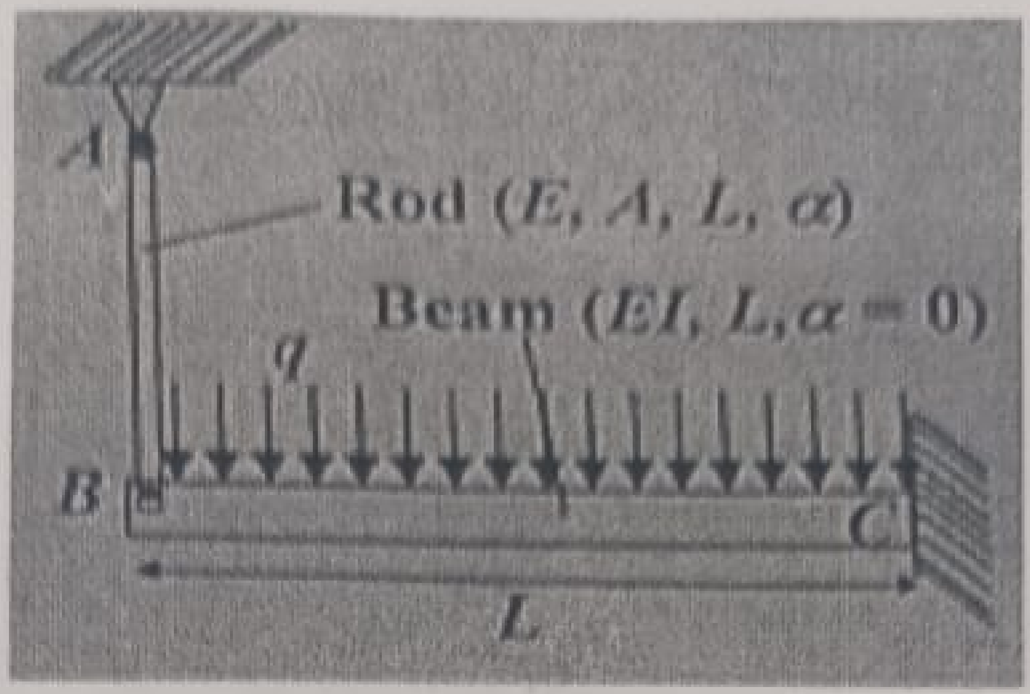


Q3. A steel bar of length **L** and cross-sectional area **A** is pinned at both its ends. It is connected to a cantilever which is subjected to a uniformly distributed loaded. Let the temperature of the entire system be increased by $\Delta\mathbf{T}$ and the cantilever beam be subjected to a uniformly distributed load of **q** N/m^2. It is also given that the thermal expansion coefficient of the material used in cantilever beam is zero but for the vertical bar is non-zero. (10)

(a) What will be the deflection of the tip of the cantilever?

(b) Suppose the junction between vertical bar and the cantilever were clamped instead of being pinned. How would this affect the solution to part (a)?

(PTO)



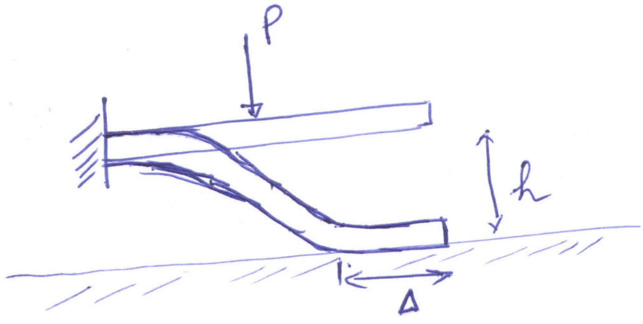
Q4. A circular shaft of radius 'R' spins with certain angular velocity. One can think of this problem being a statics problem wherein there is a distributed body force acting in the shaft (radially outward). (15)

(a) Find out the distribution of all stress components in the shaft. Assume the shaft to be axially free to expand.

(b) At what point in the shaft is the shear stress maximum (take all planes at a point into account!)? Obtain this value in terms of angular velocity and material constants.

(c) If the shaft had to fail due to maximum shear stress theory and say the critical shear stress is τ_y? What is the critical spin velocity beyond which the shaft will fail?

1)



Note that over the region Δ: $y(x) = -h \quad , \quad x > L-\Delta$

$\Rightarrow y' = 0, \; y'' = 0$

$\theta(x) = 0 \quad , \quad x > L-\Delta$

$\Rightarrow \theta' = 0 \quad ,$

2

Now, look at TBT equation

$EI\theta' = M(x) \Rightarrow M(x) = 0 \quad , \quad x > L-\Delta$

$$\frac{dy}{dx} - \theta = \frac{V(x)}{KGA} \Rightarrow V(x) = 0, \quad x > L-\Delta$$

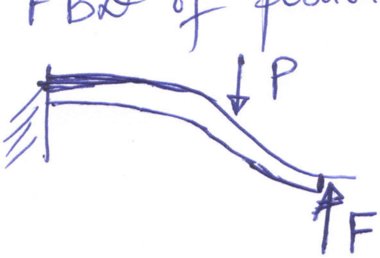
So, $V(x) = M(x) = 0$ in the region $x > L-\Delta$

1

$\Rightarrow$ It also implies no reaction from ground for $x > L-\Delta$. However, at $x = L-\Delta$, ground will exert a force on the beam which will be distributed along beam width! Due to this line force, ground will not exert any effective moment at $x = L-\Delta$.

* Also from geometry, y' will suffer discontinuity at $x = L-\Delta$, but y & θ will be continuous! 1

Let's draw FBD of portion not in touch with beam



We then obtain shear force & bending moment distribution in beam

$$V(x) = F \quad , L-\Delta > x > L/2$$
$$= F-P, \quad x < L/2$$

1

$$M(x) = F(L-\Delta-x) \quad , \quad L-\Delta > x > L/2$$
$$= F(L-\Delta-x) - P(L/2 - x) \quad , \quad x < L/2$$

2

Let's put them in TBT equation

$\boxed{\text{for} \quad 0 < x < L/2}$

$$EI\,\theta' = F(L-\Delta-x) - P(L/2 - x)$$

$$\Rightarrow EI\left[\theta(x) - \cancel{\theta(0)}\right] = F(L-\Delta)x - F\frac{x^2}{2} - \frac{PL}{2}x + \frac{Px^2}{2}$$

from B.C

0.5

$$\text{or,} \quad \theta(x) = \frac{1}{EI}\left[F(L-\Delta)x - \frac{Fx^2}{2} - \frac{PL}{2}x + \frac{Px^2}{2}\right], \quad x < L/2$$

$$\Downarrow$$

$$\theta(L/2) = \frac{1}{EI}\left[F(L-\Delta)\frac{L}{2} - \frac{FL^2}{8} - \frac{PL^2}{4} + \frac{PL^2}{8}\right]$$
$$= \frac{1}{EI}\left[\frac{3FL^2}{8} - \frac{F\Delta L}{2} - \frac{PL^2}{8}\right]$$

0.5

$$\frac{dy}{dx} = \theta + \frac{V}{kGA} = \theta(x) + \frac{F-P}{kGA}$$

$$\Rightarrow y(x) - \cancel{y(0)}^{\,0} = \int_0^x \theta(\xi)\,d\xi + \frac{(F-P)x}{kGA}$$

B.C.

$$\Rightarrow y(x) = \frac{1}{EI}\left[F(L-\Delta)\frac{x^2}{2} - \frac{Fx^3}{6} - \frac{PLx^2}{4} + \frac{Px^3}{6}\right]$$

$$\Rightarrow y(L/2) = \frac{1}{EI}\left[\frac{F(L-\Delta)L^2}{8} - \frac{FL^3}{48} - \frac{PL^3}{16} + \frac{PL^3}{48}\right] = \frac{1}{EI}\left[\frac{5FL^3}{48} - \frac{F\Delta L^2}{8} - \frac{PL^3}{24}\right]$$

Note that F (reaction from ground) and Δ are remaining two unknowns which can be obtained by applying bc.

$$y(L-\Delta) = -h$$

$$\theta(L-\Delta) = 0$$

2

for, $L-\Delta > x > L/2$

$$EI\,\theta' = F(L-\Delta-x)$$

$$\Rightarrow \theta(x) = \theta(L/2) + \frac{1}{EI}\left[\cancel{F(L-\Delta)(x-L/2) - \frac{Fx^2}{2}}\right]$$

$$+ \frac{1}{EI}\left[F(L-\Delta)(x-L/2) - \frac{F}{2}\left(x^2 - \frac{L^2}{4}\right)\right]$$

apply here

Similarly

$$\frac{dy}{dx} = \theta + \frac{F}{KGA}$$

$$\Rightarrow y(x) = y\left(\tfrac{L}{2}\right) + \int_{L/2}^{x} \theta(\xi)\,d\xi + \frac{F(x-L/2)}{KGA}$$

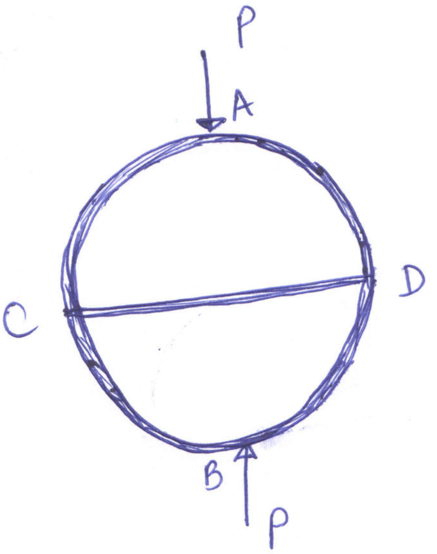
$\Downarrow$

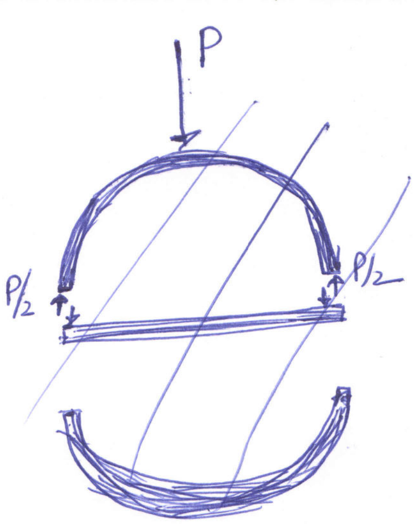
Evaluate ~~y(L)~~ $y(L-\Delta)$ and set to $-h$

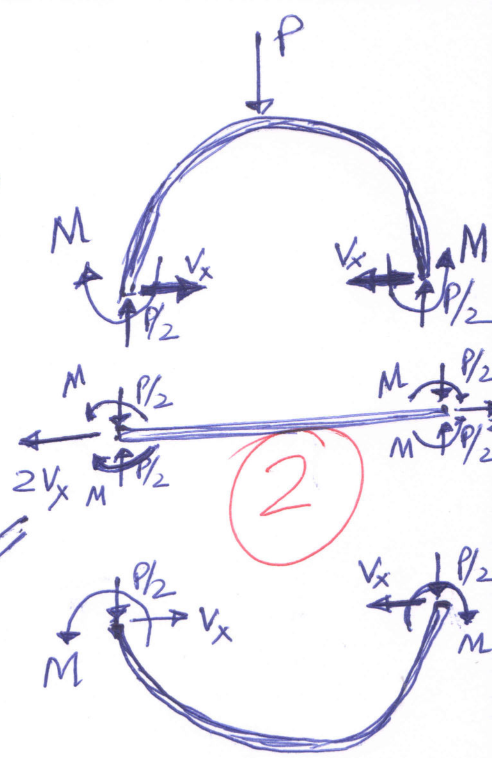
Solve to obtain Δ!

* No marks will be cut for not solving the differential equation!!

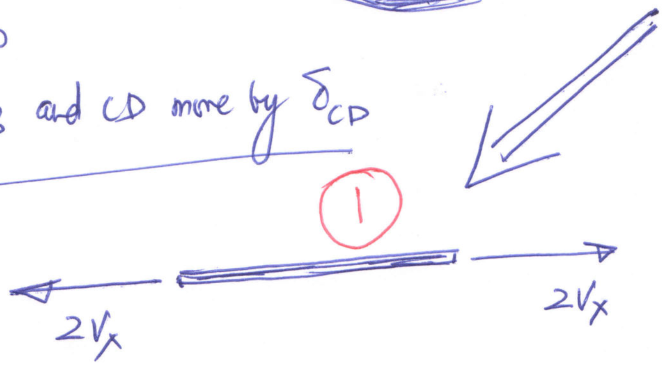
(2)

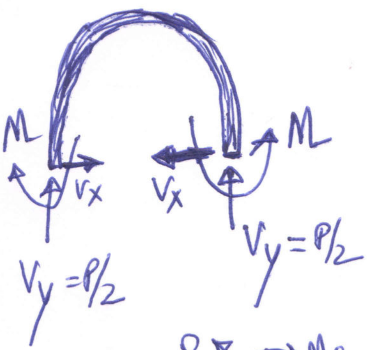




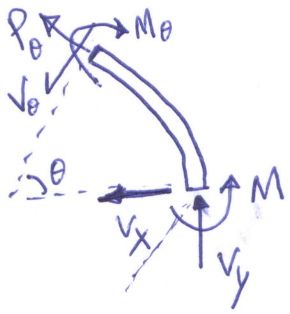


Let AB move by δ_{AB} and CD move by δ_{CD}





* Note that on straight member $2V_x$ is present whereas in curved member V_x acts
* V_x is not zero unlike in tutorial questions
* Straight member is under pure tensile load!



⇓ Balance of force & moment

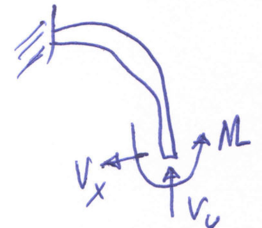
$$P_\theta = -(V_y \cos\theta + V_x \sin\theta)$$

2+1

$$V_\theta = V_y \sin\theta - V_x \cos\theta$$

$$M_\theta = M + V_y R(1-\cos\theta) - V_x R \sin\theta$$

* Let's look at one quarter in isolation.



* Let its energy be $E_{(V_x, V_y, M)}$

Eq. to obtain unknown M

$$\Rightarrow \frac{\partial E}{\partial M} = 0 \,!$$

2

$$\left\{ \begin{array}{l} \dfrac{\partial E}{\partial V_y} = \delta_{AB}/2 \\[2ex] \dfrac{\partial E}{\partial V_x} = -\delta_{CD}/2 \end{array} \right.$$

Looking at straight member

$$\delta_{CD} = \frac{2V_x}{EA} 2R = \frac{4V_xR}{EA}$$

1

$$\Rightarrow \boxed{\frac{\partial E}{\partial V_x} = -\frac{\delta_{CD}}{2} = -\frac{2V_xR}{EA}} \rightarrow$$ equation to obtain unknown V_x!

Let us now write energy of quarter ring

1

$$E = \int_0^{\pi/2} \left[\frac{(V_y \sin\theta - V_x \cos\theta)^2}{2kGA} + \frac{(V_y \cos\theta + V_x \sin\theta)^2}{2EA} + \frac{\{M + V_yR(1-\cos\theta) - V_xR\sin\theta\}^2}{2EI} \right] R\,d\theta$$

$$\Rightarrow \frac{\partial E}{\partial M} = \int_0^{\pi/2} \frac{M + V_yR(1-\cos\theta) - V_xR\sin\theta}{EI} R\,d\theta = 0$$

$$\Rightarrow M \cdot \frac{\pi}{2} + V_yR\left(\frac{\pi}{2} - 1\right) + V_xR\cos\theta \Big|_0^{\pi/2} = 0$$

$$\Rightarrow M = \frac{-2V_xR}{\pi} - V_yR\left(1 - 2/\pi\right) = -\frac{2V_xR}{\pi} - \frac{PR}{2}\left(1 - \frac{2}{\pi}\right) \quad \text{--- (i)}$$

Bonus marks for doing these steps!

$$\frac{\partial E}{\partial V_x} = \int_0^{\pi/2} \left[\frac{V_y \sin\theta - V_x\cos\theta}{kGA}(-\cos\theta) + \frac{V_y\cos\theta + V_x\sin\theta}{EA}\sin\theta + \frac{M + V_yR(1-\cos\theta) - V_xR\sin\theta}{EI}(-R\sin\theta) \right] R\,d\theta$$

$$= -\frac{2V_xR}{EA} \quad \text{-- (ii)}$$

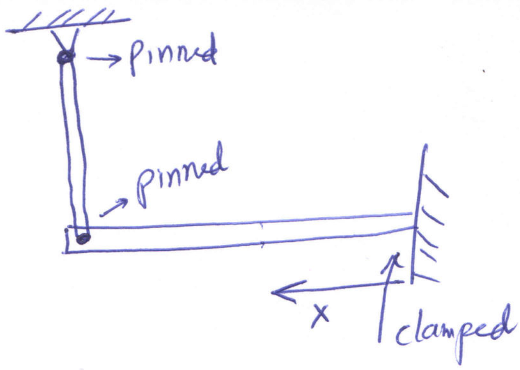
Substitute (i) & in (ii) which will finally yield equation for V_x!

Once we get V_x and further M from (i), we can then obtain

$$\delta_{CD} = \frac{4V_xR}{EA} \quad \& \quad \delta_{AB} = 2\frac{\partial E}{\partial V_y}!$$

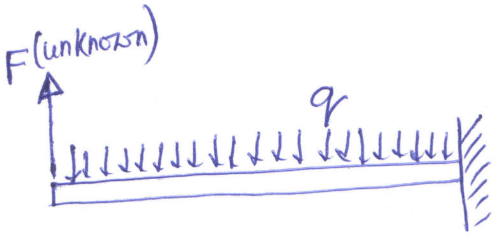
3

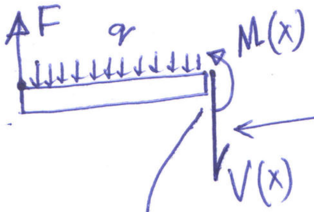
(a)



$\boxed{\text{No moment is exerted through pinned joint}}$ 1

F.B.D. of Cantilever





1

Moment balance about (this point) $\Rightarrow M(x) + \frac{q(L-x)^2}{2} - F(L-x) = 0$

$$\Rightarrow M(x) = F(L-x) - \frac{q(L-x)^2}{2}$$ 1

Using EBT,

$$EI\frac{d^2y}{dx^2} = F(L-x) - \frac{q(L-x)^2}{2}$$

$$\Rightarrow EI\left(\frac{dy}{dx} - \cancel{\frac{dy}{dx}(0)}^{\,0}\right) = FLx - \frac{Fx^2}{2} - \frac{qL^2x}{2} - \frac{qx^3}{6} + \frac{qLx^2}{2}$$

B.C. 0.5

Integrate again

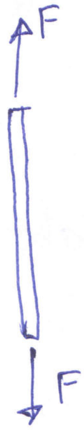
$$EI\left(y(x) - \cancel{y(0)}^{\,0}\right) = \frac{FLx^2}{2} - \frac{Fx^3}{6} - \frac{qL^2x^2}{4} - \frac{qx^4}{24} + \frac{qLx^3}{6}$$

B.C. 0.5

$$\therefore \Rightarrow y(x) = \frac{1}{EI}\left[\frac{FLx^2}{2} - \frac{Fx^3}{6} - \frac{qL^2x^2}{4} - \frac{qx^4}{24} + \frac{qLx^3}{6}\right]$$

$$\text{or, } y(L) = \frac{1}{EI}\left[\frac{FL^3}{3} - \frac{qL^4}{8}\right] \quad 1$$

Now look at vertical bar



$\Rightarrow$ Uniaxial loading

$$\Rightarrow \epsilon = \frac{F}{EA} + \alpha \Delta T \quad 2$$

$$\Rightarrow \delta = \frac{FL}{EA} + \alpha L \Delta T$$

Now δ should be equal to $y(L)$ 1

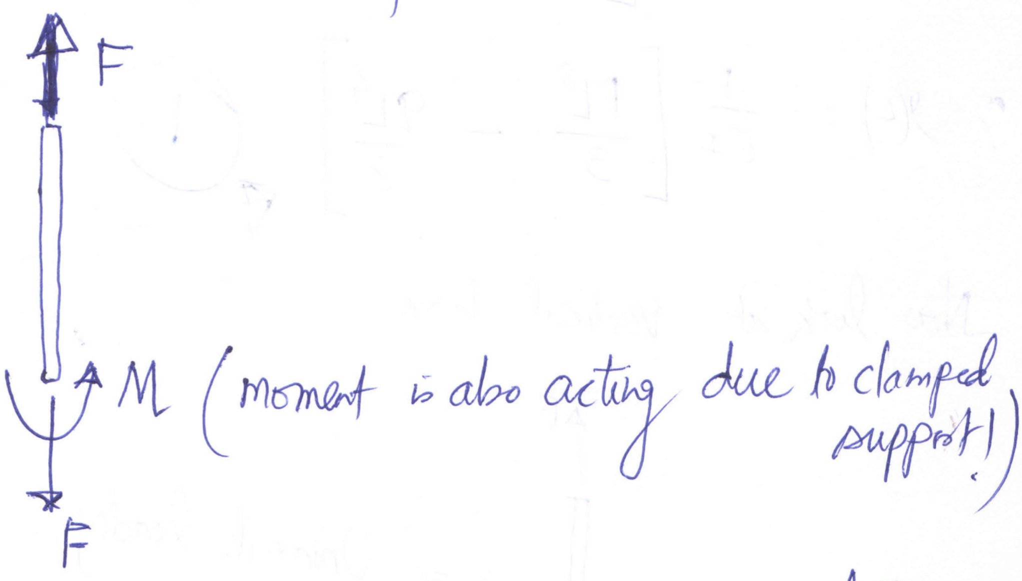
$$\Rightarrow \left(\frac{F}{EA} + \alpha\Delta T\right) L = \frac{L^3}{EI}\left(\frac{F}{3} - \frac{qL}{8}\right)$$

$$\Rightarrow F\left(\frac{1}{EA} - \frac{L^2}{3EI}\right) = -\alpha\Delta T - \frac{qL^3}{8EI}$$

$$\text{or, } F = \frac{\alpha\Delta T + qL^3/8EI}{\left(\frac{L^2}{3EI} - \frac{1}{EA}\right)}$$

Substitute it in above formula for $y(L)$ to finally obtain $y(L)$

(b) It says the junction between vertical bar and cantilever to be clamped! Let's draw FBD of vertical bar.



2

However, when we do moment balance about either top or bottom point, we get $M=0$!

So, the junction becoming clamped does not affect solution to last problem!

4(a) Assuming axisymmetry and axial homogeneity, we can write

$u_r(r), \quad u_\theta = 0, \quad u_z(z)$ 1

$$\Rightarrow [\underline{\underline{\epsilon}}] = \begin{bmatrix} u_r' & 0 & 0 \\ 0 & u_r/r & 0 \\ 0 & 0 & u_z' \end{bmatrix} \Rightarrow \begin{cases} \sigma_{rr} = \lambda\left(u_r' + \frac{u_r}{r} + u_z'\right) + 2\mu u_r' \\ \sigma_{\theta\theta} = \quad " \quad + 2\mu u_r/r \end{cases}$$ 1

1

$\epsilon_{zz} = u_z'$ will turn out to be a constant when using z-equation!

Let us look at radial equation now:-

$$\sigma_{rr}' + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \rho\omega^2 r = 0$$ 1

$$\Rightarrow \lambda\left(u_r' + \frac{u_r}{r}\right)' + 2\mu u_r'' + 2\mu \frac{u_r' - u_r/r}{r} + \rho\omega^2 r = 0$$

$$\Rightarrow (\lambda + 2\mu)\left(u_r' + \frac{u_r}{r}\right)' = -\rho\omega^2 r$$

$$\text{or,} \quad (\lambda+2\mu)\left(u_r' + \frac{u_r}{r}\right) = -\rho\omega^2\frac{r^2}{2} + C$$ 2

$$\Rightarrow \sigma_{rr} + \sigma_{\theta\theta} = (2\lambda + 2\mu)\left(u_r' + u_r/r\right) + 2\lambda\epsilon_{zz}$$

$$= -\frac{(\lambda+\mu)}{(\lambda+2\mu)}\rho\omega^2 r^2 + \tilde{C}$$ → another constant 1

Let us re-look at radial equation:-

$$\sigma_{rr}' + 2\frac{\sigma_{rr}}{r} - \frac{\sigma_{rr} + \sigma_{\theta\theta}}{r} + \rho\omega^2 r = 0$$

$$\Rightarrow \frac{1}{r^2}\left(\sigma_{rr} r^2\right)' = -\rho\omega^2 r\left(1 + \frac{\lambda+\mu}{\lambda+2\mu}\right) + \frac{\tilde{C}}{r}$$ 2

$$\Rightarrow \sigma_{rr} = -\frac{\rho\omega^2 r^2}{4}\left(\frac{2\lambda+3\mu}{\lambda+2\mu}\right) + \frac{\tilde{C}}{2} + \frac{\tilde{D}}{r^2}$$

Apply B.C.,

$$\sigma_{rr}(r=0) = \text{finite} \Rightarrow \tilde{D} = 0$$

1

$$\sigma_{rr}(r=R) = 0 \Rightarrow \tilde{C} = \frac{\rho\omega^2 R^2}{2}\left(\frac{2\lambda+3\mu}{\lambda+2\mu}\right)$$

$$\Rightarrow \sigma_{rr} = \frac{\rho\omega^2}{4}\left(\frac{2\lambda+3\mu}{\lambda+2\mu}\right)\left(R^2-r^2\right)$$

$$\sigma_{\theta\theta} = \sigma_{rr} + \sigma_{\theta\theta} - \sigma_{rr}$$

1

$$= -\frac{(\lambda+\mu)}{(\lambda+2\mu)}\rho\omega^2 r^2 + \frac{\rho\omega^2 R^2}{2}\left(\frac{2\lambda+3\mu}{\lambda+2\mu}\right) - \frac{\rho\omega^2}{4}\left(\frac{2\lambda+3\mu}{\lambda+2\mu}\right)\left(R^2-r^2\right)$$

$$= \frac{\rho\omega^2 R^2}{4}\left(\frac{2\lambda+3\mu}{\lambda+2\mu}\right) + \frac{\rho\omega^2 r^2}{4}\left[\frac{2\lambda+3\mu}{\lambda+2\mu} - \frac{4\lambda+4\mu}{\lambda+2\mu}\right]$$

$$= \frac{\rho\omega^2 R^2}{4}\left(\frac{2\lambda+3\mu}{\lambda+2\mu}\right) - \frac{\rho\omega^2 r^2}{4}\left(\frac{2\lambda+\mu}{\lambda+2\mu}\right)$$

$\sigma_{zz} = 0$ (shaft is free to expand in axial direction)

$$\Rightarrow [\underline{\underline{\sigma}}] = \begin{bmatrix} \sigma_{rr} & 0 & 0 \\ 0 & \sigma_{\theta\theta} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

1

$\Rightarrow \sigma_{rr}, \sigma_{\theta\theta}, 0$ become principal stress components at a point!

$$\tau_{max} = \max\left\{\frac{\sigma_{rr}-\sigma_{\theta\theta}}{2}, \frac{\sigma_{rr}}{2}, \frac{\sigma_{\theta\theta}}{2}\right\}$$

1

$$\tau^{critical} = \max_r \tau_{max} < \tau_y$$

Note that $\sigma_{rr}, \sigma_{\theta\theta} > 0$ for all r.

Furthermore, both are decreasing with r.

Also, $\sigma_{rr} - \sigma_{\theta\theta} = -\frac{\rho\omega^2 r^2}{2} \cdot \frac{\mu}{\lambda+2\mu}$

$$\Rightarrow \max_r \left|\frac{\sigma_{rr}-\sigma_{\theta\theta}}{2}\right| = \frac{\rho\omega^2 R^2}{4}\frac{\mu}{\lambda+2\mu}$$

1

$$\max_{r} \frac{\sigma_{rr}}{2} = \frac{\sigma_{rr}(r=0)}{2} = \frac{\rho\omega^2R^2}{4}\left(\frac{\lambda+1.5\mu}{\lambda+2\mu}\right) = \max_{r} \frac{\sigma_{\theta\theta}}{2}$$

Hence $$\tau^{critical} = \frac{\rho\omega^2R^2}{4}\left(\frac{\lambda+1.5\mu}{\lambda+2\mu}\right) < \tau_y$$ 1

$$\Rightarrow \omega < \sqrt{\frac{4\tau_y}{\rho R^2}\left(\frac{\lambda+2\mu}{\lambda+1.5\mu}\right)}$$