Tutorial 2: Traction vector

APL 104 - 2022 (Solid Mechanics)

Q1. Show that $\underline{t}^n = \sum_i \underline{t}^i \, (\underline{n} \cdot \underline{e}_i) = \sum_i \underline{t}^{\hat{i}} \, (\underline{n} \cdot \underline{\hat{e}}_i)$, i.e, the formula is independent of what three planes are chosen to determine $\underline{t}^n$!

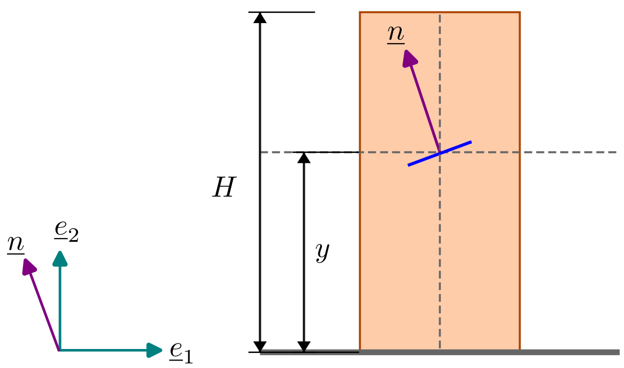
Q2. Suppose $[\underline{t}^1] = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $[\underline{t}^2] = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}$, $[\underline{t}^3] = \begin{bmatrix} 0 \\ 7 \\ 9 \end{bmatrix}$ in $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ coordinate system.
What will be the traction on a plane with normal $\underline{n} = \underline{\hat{e}}_1$ where $(\underline{\hat{e}}_1, \underline{\hat{e}}_2, \underline{\hat{e}}_3)$ is obtained from rotation of $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ about $\underline{e}_3$ by $45°$? What are the normal and shear components of traction on this plane?

Q3. Show that the component of a traction vector on $\underline{n}$-plane in the direction $\underline{m}$ equals the component of the traction on $\underline{m}$-plane in the direction $\underline{n}$, i.e, $\underline{t}^n \cdot \underline{m} = \underline{t}^m \cdot \underline{n}$.

Q4. Consider a vertical bar having mass density ρ. Assume its length be to H and is subjected to uniform body force due to gravity. Find the traction vector on an infinitesimal internal section of the bar located at the center of its cross-section with outward normal

$$\underline{n} = -\sin\theta \; \underline{e}_1 + \cos\theta \; \underline{e}_2$$

and at a height of y from the base (see figure). Also find the normal and tangential components of the traction vector on this plane.



Q5. Think of a bar lying along $\underline{e}_1$ axis and loaded axially. We will learn later in the class that during the tensile loading of a bar, the traction on a section with normal along $\underline{e}_1$ has no shear components of traction. Also, as the bar is allowed to contract freely in the transverse direction, the traction on sections having normals perpendicular to $\underline{e}_1$ completely vanish. What will be the traction on the plane whose normal makes an angle θ from axial direction. What are the normal and shear components of traction on this plane?

APL 104 Tutorial 2 solutions

Q1. Show that $\underline{t}^n = \sum_i \underline{t}^i\,(\underline{n}\cdot\underline{e}_i) = \sum_i \underline{t}^{\hat{i}}\,(\underline{n}\cdot\underline{\hat{e}}_i)$, i.e, the formula is independent of what three planes are chosen to determine $\underline{t}^n$!

Solution:

Think of $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ and $(\underline{\hat{e}}_1, \underline{\hat{e}}_2, \underline{\hat{e}}_3)$ system which form the plane normals in the two cases. Let us expand the vector $\underline{e}_1$ in $(\underline{\hat{e}}_1, \underline{\hat{e}}_2, \underline{\hat{e}}_3)$ basis, i.e., $\underline{e}_1 = \sum_{i=1}^{3} (\underline{e}_1\cdot\underline{\hat{e}}_i)\,\underline{\hat{e}}_i$. Here $(\underline{e}_1\cdot\underline{\hat{e}}_i)$ is the component of $\underline{e}_1$ along $\underline{\hat{e}}_i$. By the same logic, we can write

$$\underline{e}_j = \sum_i \left(\underline{e}_j\cdot\underline{\hat{e}}_i\right)\underline{\hat{e}}_i$$

. Now

$$\begin{aligned}
\underline{t}^n &= \sum_i \underline{t}^{\hat{i}}\,(\underline{n}\cdot\underline{\hat{e}}_i)\\
&= \sum_i \left(\sum_j \underline{t}^j \left(\underline{\hat{e}}_i\cdot\underline{e}_j\right)\right)(\underline{n}\cdot\underline{\hat{e}}_i) \qquad (\underline{t}^{\hat{i}} \text{ is expressed using tractions on } \underline{e}_1,\ \underline{e}_2, \text{ and } \underline{e}_3 \text{ planes})\\
&= \sum_j \underline{t}^j \sum_i (\underline{n}\cdot\underline{\hat{e}}_i)\left(\underline{\hat{e}}_i\cdot\underline{e}_j\right) \quad \text{(upon changing the order of summation)}\\
&= \sum_j \underline{t}^j \left(\underline{n}\cdot\sum_i \underline{\hat{e}}_i\left(\underline{\hat{e}}_i\cdot\underline{e}_j\right)\right)\\
&= \sum_j \underline{t}^j \left(\underline{n}\cdot\underline{e}_j\right) \quad \left(\because \underline{e}_j = \sum_i \underline{\hat{e}}_i\left(\underline{\hat{e}}_i\cdot\underline{e}_j\right) \text{ as derived earlier}\right).
\end{aligned}$$

NOTE: This also proves that stress tensor is independent of what three planes are chosen to form it, i.e., $\underline{\underline{\sigma}} = \sum_i \underline{t}^i \otimes \underline{e}_i = \sum_i \underline{t}^{\hat{i}} \otimes \underline{\hat{e}}_i$.

Q2. Suppose $[\underline{t}^1] = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $[\underline{t}^2] = \begin{bmatrix}1\\5\\7\end{bmatrix}$, $[\underline{t}^3] = \begin{bmatrix}0\\7\\9\end{bmatrix}$ in $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ coordinate system.
What will be the traction on a plane with normal $\underline{n} = \underline{\hat{e}}_1$ where $(\underline{\hat{e}}_1, \underline{\hat{e}}_2, \underline{\hat{e}}_3)$ is obtained from rotation of $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ about $\underline{e}_3$ by 45°? What are the normal and shear components of traction on this plane?

Solution:
We simply have to use the formula

$$\underline{t}^n = \sum_i \underline{t}^i\,(\underline{n}\cdot\underline{e}_i)$$

Keep in mind that the above is a tensor formula. To use it, we must write every quantity involved in the above formula in the same coordinate system! Let us use $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ coordinate system here:

$$\Rightarrow [\underline{n}]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} = [\underline{\hat{e}}_1]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

Therefore,

$$\begin{aligned} [\underline{t}^n] &= \sum_i \left[\underline{t}^i\right] ([\underline{n}] \cdot [\underline{e}_i]) \\ &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} 1/\sqrt{2} + \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} 1/\sqrt{2} + \begin{bmatrix} 0 \\ 7 \\ 9 \end{bmatrix} 0 \\ &= 1/\sqrt{2} \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix}. \end{aligned}$$

CAUTION: If you use $[\underline{n}] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, you will get wrong result!

Never mix the coordinate system! Either use $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ for all or use $(\underline{\hat{e}}_1, \underline{\hat{e}}_2, \underline{\hat{e}}_3)$ for all! The final result for $\underline{t}^n$ will be in the coordinate system that you choose.

In order to get the normal component of traction, we need to find $\underline{t}^n \cdot \underline{n}$ or $\underline{t}^{\hat{1}} \cdot \underline{\hat{e}}_1$ which when expressed in $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ coordinate system yields

$$1/\sqrt{2} \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \frac{7}{2}. \tag{1}$$

One can likewise obtain the shear components by taking dot product of traction vector with $\underline{\hat{e}}_2$ and $\underline{\hat{e}}_3$.

Q3. Show that the component of a traction vector on $\underline{n}$-plane in the direction $\underline{m}$ equals the component of the traction on $\underline{m}$-plane in the direction $\underline{n}$, i.e, $\underline{t}^n \cdot \underline{m} = \underline{t}^m \cdot \underline{n}$.

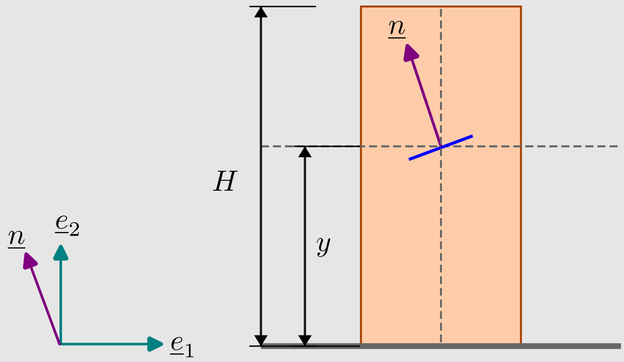
Solution:
To prove this, we will use the definition of stress tensor as follows:

$$\begin{aligned} \underline{t}^n \cdot \underline{m} &= \left(\underline{\underline{\sigma}}\ \underline{n}\right) \cdot \underline{m} \\ &= \underline{n} \cdot \left(\underline{\underline{\sigma}}^T\ \underline{m}\right) \text{ (see Tutorial 1)} \\ &= \underline{n} \cdot \left(\underline{\underline{\sigma}}\ \underline{m}\right) \text{ (due to symmetry of } \underline{\underline{\sigma}} \text{ which will be proved later)} \\ &= \underline{n} \cdot \underline{t}^m . \end{aligned}$$

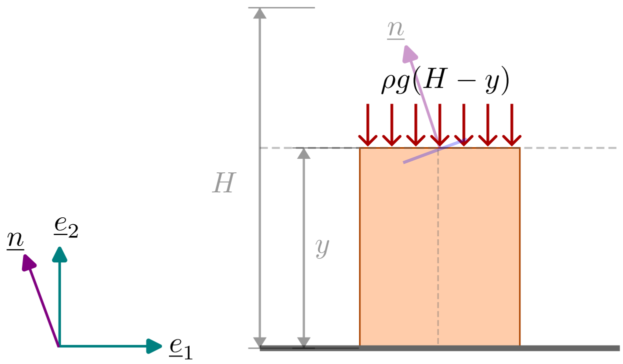
Q4. Consider a vertical bar having mass density ρ. Assume its length be to H and is subjected to uniform body force due to gravity. Find the traction vector on an infinitesimal internal section of the bar located at the center of its cross-section with outward normal

$$\underline{n} = -\sin\theta\,\underline{e}_1 + \cos\theta\,\underline{e}_2$$

and at a height of y from the base (see figure). Also find the normal and tangential components of the traction vector on this plane.



Solution:



Taking a horizontal section at height y and writing force balance of top part of bar yields

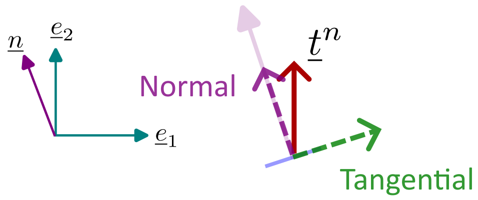
$$\left[\underline{t}^2\right] = \begin{bmatrix} 0 \\ -\rho g(H-y) \\ 0 \end{bmatrix}.$$

Similarly taking a vertical section all along the bar and doing force balance either of left or right portion of the bar yields

$$\left[\underline{t}^1\right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

One can likewise show that $\left[\underline{t}^3\right]$ also vanishes although it turns out this information is not required. In order to obtain traction on an infinitesimal inclined plane at the center of the horizontal section of the bar, we can use the tetrahedron formula, i.e.,

$$
\begin{aligned}
[\underline{t}^n] &= \sum_i \left[\underline{t}^i\right] ([\underline{n}] \cdot [\underline{e}_i]) \\
&= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} (-\sin\theta) + \begin{bmatrix} 0 \\ -\rho g(H-y) \\ 0 \end{bmatrix} (\cos\theta) + \left[\underline{t}^3\right] \ 0 \\
&= \begin{bmatrix} 0 \\ -\rho g(H-y)\cos\theta \\ 0 \end{bmatrix}.
\end{aligned}
$$



The normal component of traction vector is given by the projection of $\underline{t}^n$ along $\underline{n}$:

$$
\begin{aligned}
t_{\text{normal}} &= \underline{t}^n \cdot \underline{n} \\
&= -\rho g(H-y)\cos\theta \ \underline{e}_2 \cdot \underline{n} \\
&= -\rho g(H-y)\cos\theta \ (\underline{e}_2 \cdot \underline{n}) \\
&= -\rho g(H-y)\cos^2\theta
\end{aligned}
$$

Note that the normal component of traction acts along $\underline{n}$, and hence can be written as a vector as

$$
\underline{t}^n_{\text{normal}} = (\underline{t}^n \cdot \underline{n})\, \underline{n}
$$

The shear or tangential component of traction can then be obtained as follows:

$$
\begin{aligned}
t_{\text{tangential}} &= \underline{t}^n \cdot \underline{n}^{\perp} \\
&= -\rho g(H-y)\cos\theta \ \underline{e}_2 \cdot (\cos\theta \ \underline{e}_1 + \sin\theta \ \underline{e}_2) \\
&= -\rho g(H-y)\cos\theta \sin\theta
\end{aligned}
$$

where $\underline{n}^{\perp}$ is a unit vector perpendicular to $\underline{n}$ and lies in the plane of inclined section.