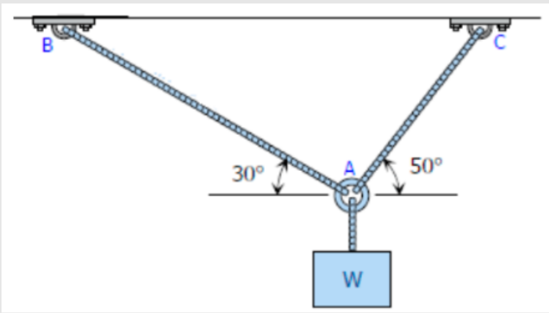
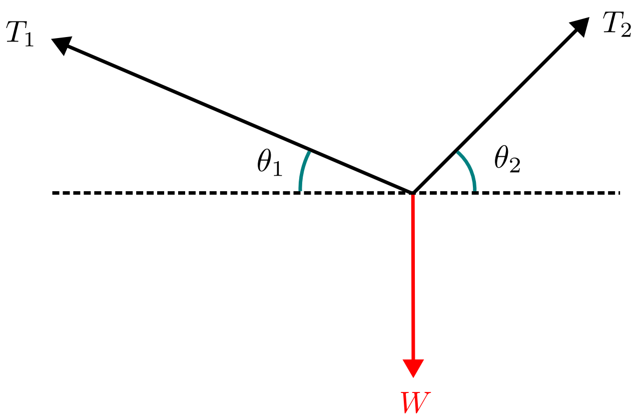
Tutorial 8 solution

APL 104 - 2022 (Solid Mechanics)

Q1. Determine the largest weight W that can be supported by wire shown in figure shown below. The stress in either wire is not to exceed $207 \mathrm{N/mm^2}$. The cross-sectional area s of wires AB and AC are $258 \mathrm{mm^2}$ and $323 \mathrm{mm^2}$, respectively.



Solution:
Using free-body diagram and doing force balance, find tension in wires.



$$\overset{+}{\rightarrow} \sum F_x = 0 \Rightarrow T_2 \cos\theta_2 - T_1 \cos\theta_1 = 0$$
$$\Rightarrow T_2 = T_1 \frac{\cos\theta_1}{\cos\theta_2}$$

$$
\begin{aligned}
+\uparrow \sum F_y \;=\; 0 \Rightarrow\; & T_1 \sin\theta_1 \;+\; T_2 \sin\theta_2 \;=\; W \\
\Rightarrow\; & T_1 \sin\theta_1 \;+\; T_1 \frac{\cos\theta_1}{\cos\theta_2} \sin\theta_2 \;=\; W \\
\Rightarrow\; & T_1 \sin\theta_1 \;+\; T_1 \cos\theta_1 \tan\theta_2 \;=\; W \\
\Rightarrow\; & T_1 \;=\; \frac{W}{\cos\theta_1 \left(\tan\theta_1 \;+\; \tan\theta_2\right)} \\
\text{and } T_2 \;=\; & \frac{W}{\cos\theta_2 \left(\tan\theta_1 \;+\; \tan\theta_2\right)}
\end{aligned}
$$

The stress in each wire should not exceed the prescribed limit, so check that

$$
\begin{aligned}
\sigma_1 \;=\; \frac{T_1}{A_1} \;=\; & \frac{W}{A_1 \cos\theta_1 \left(\tan\theta_1 \;+\; \tan\theta_2\right)} \;\leq\; \sigma_{tol} \\
\Rightarrow\; W \;\leq\; & \sigma_{tol} \left[A_1 \cos\theta_1 \left(\tan\theta_1 \;+\; \tan\theta_2\right)\right] \qquad (1)
\end{aligned}
$$

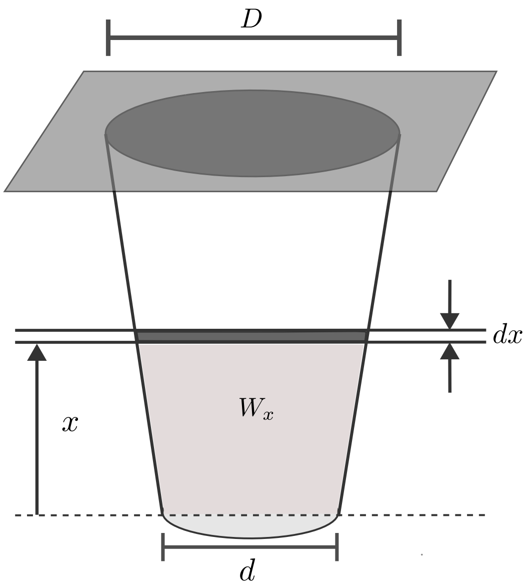
Similarly,

$$
\begin{aligned}
\sigma_2 \;=\; \frac{T_2}{A_2} \;=\; & \frac{W}{A_2 \cos\theta_2 \left(\tan\theta_1 \;+\; \tan\theta_2\right)} \;\leq\; \sigma_{tol} \\
\Rightarrow\; W \;\leq\; & \sigma_{tol} \left[A_2 \cos\theta_2 \left(\tan\theta_1 \;+\; \tan\theta_2\right)\right] \qquad (2)
\end{aligned}
$$

The maximum permissible load W would be the minimum of the loads obtained from relations 1 and 2. Note that as the wires undergo extension, the angles (θ_1, θ_2) would also change. However, we have assumed that such changes are negligible here.

> **Q2.** A round bar of length L, which tapers uniformly from a diameter D at one end to a smaller diameter d at the other, is suspended vertically from the large end. If w is the weight per unit volume, find the elongation of the rod caused by its own weight.

Solution: Note that this problem represents a uniaxial loading scenario where the suspended bar is subjected to tension (due to its self-weight) along the vertical direction.



Note the downward force at any point in a given section is same. Hence, the elongation $d\Delta$ of an incremental section of thickness dx at a distance x from the bottom, is given by the formula:

$$d\Delta \;=\; \frac{F\,l}{AE} \;=\; \frac{W_x\,dx}{A_x E}$$

where W_x is weight of the shaded portion below the incremental section of thickness dx and A_x is the cross-sectional area of the incremental section both of which varying with x. The diameter of the relevant section and its area A_x can be found as

$$d_x \;=\; d \;+\; \frac{D \;-\; d}{L}x, \quad A_x \;=\; \frac{\pi\,d_x^2}{4}.$$

The weight $W_x = V_x \times w$ where V_x is the volume of the shaded truncated cone. This volume can be easily obtained as the sum of following two volumes: (a) volume of an internal cylinder of diameter d, and (b) volume of an outer cone of base diameter $d_x - d$, i.e.,

$$V_x = \frac{\pi}{4}d^2x + \frac{1}{3}\frac{\pi}{4}(d_x - d)^2x = \frac{\pi}{4}d^2x + \frac{1}{3}\frac{\pi}{4}\frac{(D-d)^2}{L^2}x^3$$

Accordingly, the downward weight W_x becomes

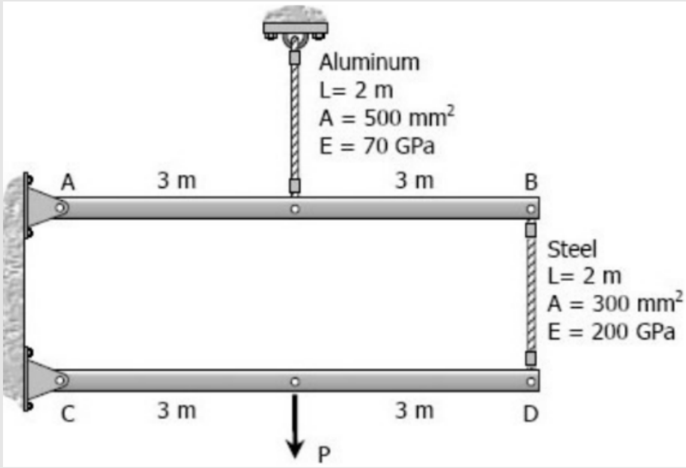
$$W_x = \left(\frac{\pi}{4}d^2x + \frac{1}{3}\frac{\pi}{4}\frac{(D-d)^2}{L^2}x^3\right) \times w$$

and the total elongation of the suspended bar is

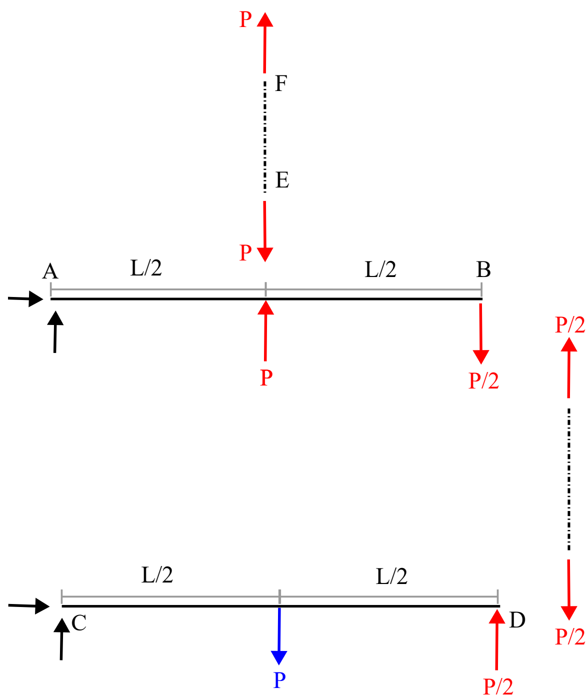
$$\delta L = \int_0^L \frac{W_x dx}{A_x E}$$

.

Q3. The rigid bars AB and CD shown in figure below are supported by pins at A and C and the two rods. Determine the maximum force P that can be applied as shown if its vertical movement is limited to 5mm. Neglect the wights of all members.



Solution: The bars AB and CD are rigid weightless bars. Since the aluminium and steel ropes are stretchable, they will deform and the assembly will undergo displacement.
First find the force acting in elements EF and BD to find how much they will stretch. To do that, we draw free diagrams and do force and moments balance for all members. Note that all joints are pinned joints an dhence no moment acts at these joints.



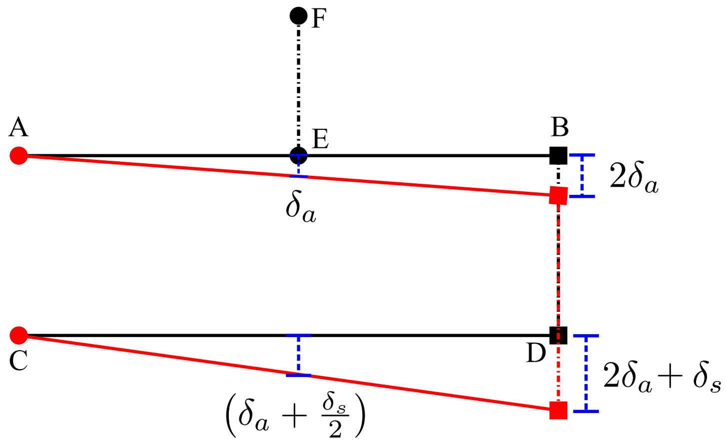
Next let's find the extensions of the ropes EF and BD.

Aluminium rope EF:

$$\delta_a = \frac{Fl}{EA} = \frac{PL_a}{E_a A_a}$$

This implies point E goes down by δ_a whereas point B will accordingly go down by $2\delta_a$.
Steel rope BD:

$$\delta_s = \frac{Fl}{EA} = \frac{(P/2)L_s}{E_s A_s}$$



This implies point D goes down by $2\delta_a + \delta_s$ and hence the the displacement of the point of application of load P turns out to be

$$\begin{aligned}
\delta &= \delta_a + \frac{\delta_s}{2} \leq \delta_c \\
&= \frac{PL_a}{E_a A_a} + \frac{PL_s}{4E_s A_s} \leq \delta_c \\
\therefore P &\leq \frac{\delta_c}{\dfrac{L_{Al}}{E_{Al} A_{Al}} + \dfrac{L_s}{4E_s A_s}}
\end{aligned}$$

Q4. A 50.8mm diameter steel tube with a wall thickness of 1.27mm just fits in a rigid hole. Find the hoop stress if an axial compressive load of 1424kg is applied.

Solution:

Note that it might seem like a uniaxial loading problem, however, this is a biaxial loading problem, since an unknown pressure is acting on the outer lateral surface. The axial load causes the steel tube to compress axially and expand radially but the wall being rigid does not allow any expansion. So, in turn, the wall applies a compressive pressure on the outer surface of the tube.

We can say that $u_\theta = 0$, and $u_z \neq 0$, $u_r \neq 0$. We need to find out $\sigma_{\theta\theta}$. Let us first find out σ_{zz}

$$\sigma_{zz} = \frac{F}{\frac{\pi(D^2-d^2)}{4}} = \frac{F}{\frac{\pi(D+d)(D-d)}{4}} = \frac{F}{\frac{\pi(d+2t+d)(2t)}{4}} = \frac{F}{\pi(d+t)t}$$

.

here we used the fact that σ_{zz} does not vary through the wall thickness as proved in the class. The solution of σ_{rr} and $\sigma_{\theta\theta}$ from radial stress equilibrium equation in cylindrical coordinates are as follows:

$$\sigma_{rr} = \frac{A}{2} + \frac{B}{r^2}, \quad \sigma_{\theta\theta} = \frac{A}{2} - \frac{B}{r^2}$$

The values of the integrating constants A and B are obtained using the boundary conditions!

<u>Traction BC</u>: No internal pressure:

$$\Rightarrow \sigma_{rr}(r_1) = 0$$

We can't say $\sigma_{rr}(r_2) = 0$ since the outer constraining pressure is an unknown.

<u>Displacement BC</u>: No radial expansion along the outer boundary:

$$u_r(r_2) = 0.$$

The radial displacement u_r is given by

$$u_r = \frac{Cr}{2} + \frac{D}{r}$$

and the integration constants A and B of the radial stress σ_{rr} are dependent on C and D in the following fashion:

$$\begin{aligned}
\sigma_{rr} &= \lambda\left(u_r^{'} + \frac{u_r}{r} + u_z^{'}\right) + 2\mu u_r^{'} \\
&= \lambda\left(\frac{C}{2} - \frac{D}{r^2} + \frac{C}{2} + \frac{D}{r^2} + u_z^{'}\right) + 2\mu\left(\frac{C}{2} - \frac{D}{r^2}\right) \\
&= \underbrace{(\lambda+\mu)C + \lambda u_z^{'}}_{A/2} - \frac{2\mu D}{r^2} \\
&= \frac{A}{2} - \frac{B}{r^2}
\end{aligned}$$

Thus, A and B are related to C and D as follows:

$$A = (\lambda + \mu)C + \lambda u_z^{'}, \quad B = 2\mu D.$$

We also have $u_z' = 1/E(\sigma_{zz} - \nu(\sigma_{rr} + \sigma_{\theta\theta})) = 1/E(\sigma_{zz} - \nu\ A)$ which when substituted in above equation leads to

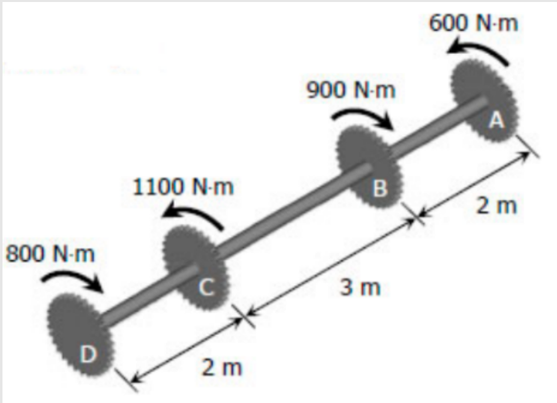
$$C = 1/(\lambda + \mu) \left[A(1 + \lambda\nu/E) - \lambda/E\ \sigma_{zz}\right],\ D = B/2\mu.$$

So, the final equations to obtain A and B are

$$\begin{aligned}
&\sigma_{rr}(r_1) = 0 \Rightarrow \frac{A}{2} + \frac{B}{r_1^2} = 0, \\
&u_r(r_2) = 0 \Rightarrow A\frac{1 + \lambda\nu/E}{2(\lambda + \mu)}r_2 + \frac{B}{2\mu r_2} = \nu/E\ \frac{F}{\pi(d+t)t}r_2
\end{aligned}$$

One can then also evaluate $\sigma_{rr}(r_2)$ to obtain the unknown pressure load applied by the rigid hole.

Q5. An aluminum shaft with a constant diameter of 50mm is loaded by torques applied to gears attached to its as shown in Fig. Using G = 28GPa, determine the relative angle of twist of gear relative to gear A.

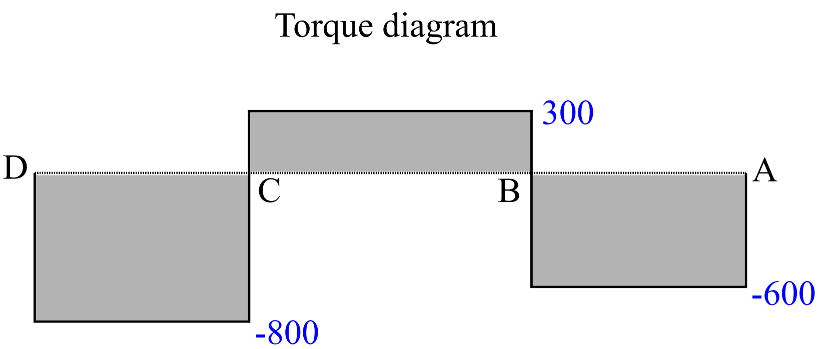


Solution: First note that the net torque on the shaft $T_A + T_B + T_C + T_D$ is zero. Hence, it is a static problem. The $+\underset{\sim}{e}_3$ is taken along the axis pointing from D towards A. Accordingly, positive/negative torques are determined by the right-hand-thumb rule, with the thumb pointing along $+\underset{\sim}{e}_3$ direction.

To determine the relative rotation of point A w.r.t. point D, we need to determine relative rotation of individual segments AB, BC, and CD using the formula

$$\Omega = \frac{TL}{GJ}.$$

The torques in each segment is different here though which can be determined from the below torque diagram - it depicts variation in torque along the shaft.

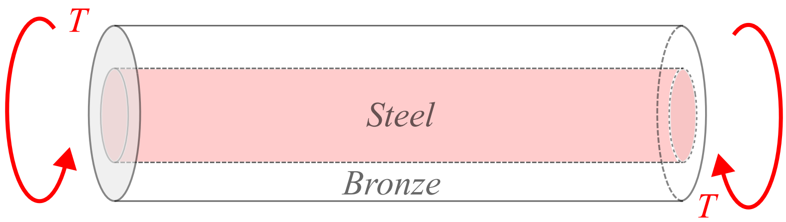


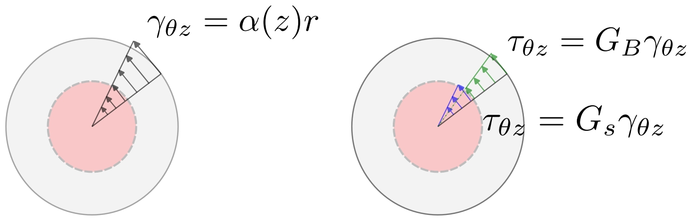
If Ω_{AD} is the relative rotation of point A w.r.t. point D, then

$$\Omega_{AD} = \Omega_{AB} + \Omega_{BC} + \Omega_{CD} = \frac{T_{AB}L_{AB} + T_{BC}L_{BC} + T_{CD}L_{CD}}{GJ}.$$

Q6. A hollow bronze shaft of 76.2mm outer diameter and 50.8mm inner diameter is slipped observer a solid steel shaft 50.8mm in diameter and of the same length as the hollow shaft. The two shafts are then fastened rigidly together at their ends. For bronze $G_B = 7\text{kN/mm}^2$, and for steel, $G_S = 12\text{kN/mm}^2$. What torque can be applied to the composite shaft without exceeding a shearing stress of 55N/mm^2 in the bronze or 82N/mm^2 in the steel?

Solution:





The shear strain $\gamma_{\theta z}$ varies linearly with r in both the materials since the two parts are fastened together rigidly. Hence

$$\gamma_{\theta z} = \frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta} = \kappa r$$

Not that the twist value κ would be the same in the two materials. The shear stress $\tau_{\theta z}$ is proportional to shear strain but shear modulus being different for two materials, it will exhibit a jump across the interface for the two materials - see figure above. Accordingly, total torque in the cross-section equals

$$\int\int \tau_{\theta z}\, dA = \int\int_{Steel} G_S\, \kappa\, r^2 dA + \int\int_{Bronze} G_B\, \kappa\, r^2 dA = G_S J_S\, \kappa + G_B J_B\, \kappa$$
$$\Rightarrow T = T_S + T_B.$$

As the twist κ is same across the cross-section, this allows us to write the following relating T_S and T_B:

$$\frac{T_S}{G_S J_S} = \frac{T_B}{G_B J_B} \Rightarrow T_S = T_B\left(\frac{G_S J_S}{G_S J_B}\right)$$
$$\text{or } T = T_S + T_B = T_B\left(\frac{G_S J_S}{G_B J_B}\right) + T_B = T_B\frac{(G_S J_S + G_B J_B)}{G_B J_B}$$
$$\text{or } T = T_S + T_B = T_S + T_S\left(\frac{G_B J_B}{G_S J_S}\right) = T_S\frac{(G_S J_S + G_B J_B)}{G_S J_S}$$

Here $J_S = \pi d^4/32$ and $J_B = \pi\left(D^4 - d^4\right)/32$. Next we need to check for the maximum shear stress in each of the parts:

Steel

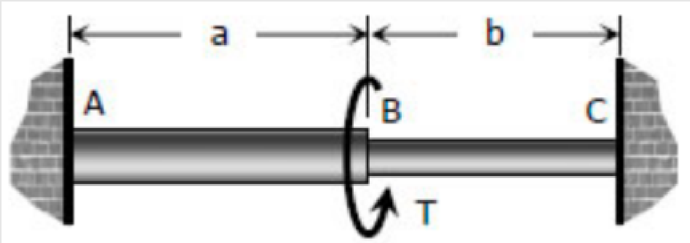
$$\tau_{S,max} \;=\; \frac{T_S r_S}{J_s} \;=\; \frac{T_S d}{2J_S} \;\leq\; \tau_{S,tol} \Rightarrow T_S \;\leq\; \tau_{S,tol}\frac{2J_S}{d}$$

Bronze

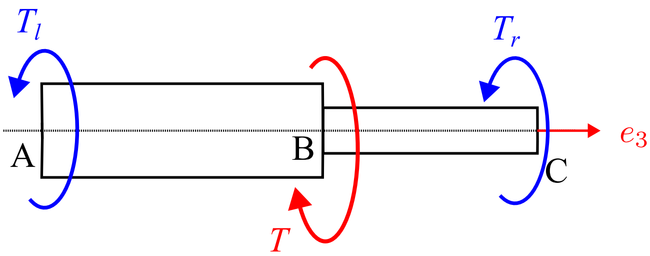
$$\tau_{B,max} \;=\; \frac{T_B r_B}{J_B} \;=\; \frac{T_B D}{2J_B} \;\leq\; \tau_{B,tol} \Rightarrow\; T_B \;\leq\; \tau_{B,tol}\frac{2J_B}{D}$$

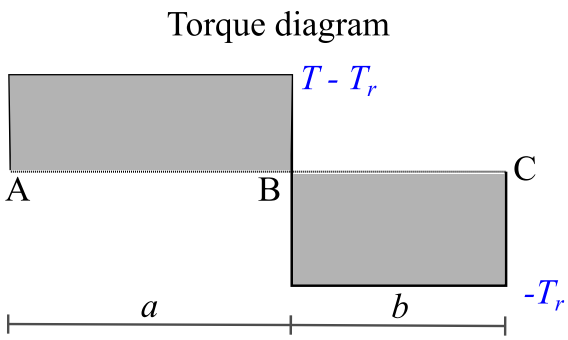
$$T \;=\; \min\left\{\tau_{B,tol}\frac{2J_B}{D}\frac{(G_S J_S \;+\; G_B J_B)}{G_B J_B},\; \tau_{S,tol}\frac{2J_S}{d}\frac{(G_S J_S \;+\; G_B J_B)}{G_S J_S}\right\}$$

Q7. The compound shaft shown in figure below is attached to rigid supports. For the bronze segment AB, the diameter is 75mm, $\tau \leq$ 60MPa, and G = 35GPa. For the steel segment BC, the diameter is 50mm, $\tau \leq$ 80MPa, and G = 83GPa IF a = 2m and b = 1.5m, compute the maximum torque T that can be applied.



Solution: Let the end reaction torques be T_L and T_R, respectively, at left and right fixed ends as shown in the figure below.





The two reaction torques are unknowns. We can use moment balance of the full beam as follows:

$$\sum \text{Moment about } \underset{\sim}{e}_3\text{-axis} = 0$$
$$\Rightarrow -T_R + T - T_L = 0 \quad \text{or} \quad T = T_R + T_L$$

which gives us one equation. It turns out we do not get any further equation from statics to obtain the two reaction torques. Hence, it is also called a statically indeterminate problem. We need to use torque-twist relation to obtain the two reaction torques which we now show. Let

$$\begin{aligned}
\Omega_{BC} &= \text{rotation of end C w.r.t end B,} \\
\Omega_{AB} &= \text{rotation of end B w.r.t end A,} \\
\Omega_{AC} &= \text{rotation of end C w.r.t end A.}
\end{aligned}$$

Since the two ends are fixed, no rotations are allowed at A and C. Therefore $\Omega_{AC} = 0$. However

$$\begin{aligned}
\Omega_{AC} &= \Omega_{AB} + \Omega_{BC} \\
\Rightarrow \Omega_{AB} &= -\Omega_{BC} \\
\Rightarrow \frac{T_{AB}L_{AB}}{G_{AB}J_{AB}} &= -\frac{T_{BC}L_{BC}}{G_{BC}J_{BC}} \\
\Rightarrow \frac{T_L a}{G_B J_B} &= \frac{T_R b}{G_S J_S}.
\end{aligned}$$

So,

$$T_L = T_R \left(\frac{b}{a}\right)\left(\frac{G_B J_B}{G_S J_S}\right) \text{ or } T_R = T_L \left(\frac{a}{b}\right)\left(\frac{G_S J_S}{G_B J_B}\right) \tag{3}$$

So, total torque

$$T = T_L + T_R = T_L \left(\frac{G_B J_B b + G_S J_S a}{G_B J_B b}\right) \text{ or } T = T_R \left(\frac{G_B J_B b + G_S J_S a}{G_S J_S a}\right) \tag{4}$$

<u>Check for bronze</u> ($\tau_{B,max} \leq \tau_{tol,B}$)

$$\tau_{b,max} = \frac{T_L\ r_B}{J_B} \Rightarrow T_L \leq \frac{\tau_{tol,b} J_B}{r_B}$$

So, using Eq. 4, Total torque

$$T \leq \left(\frac{\tau_{tol,B}\ J_B}{r_B}\right)\left(\frac{G_B J_B b + G_S J_S a}{G_B J_B b}\right)$$

<u>Check for steel</u> ($\tau_{s,max} \leq \tau_{tol,S}$)

$$\tau_{s,max} = \frac{T_R\ r_S}{J_S} \Rightarrow T_R \leq \frac{\tau_{tol,S} J_S}{r_S}$$

Again, using Eq. 4, total torque

$$T \leq \left(\frac{\tau_{tol,S}\ J_S}{r_S}\right)\left(\frac{G_B J_B b + G_S J_S a}{G_S J_S a}\right)$$

Maximum permissible torque T will be minimum of above two values.

Q8. A composite shaft is clamped at both it ends , It is composed of three sub-shafts as shown in Figure 2. The first part is made up of bronze (shear modulus G_B) and has length L_B and radius R_B. The second part is made up of aluminium (shear modulus G_A) and has length L_A and radius R_A. The third part is made up of steel (shear modulus G_S) and has length L_S and radius R_S. There is a torque T_1 acting at the interface of aluminium and steel parts. Both T_1 and T_2 act along $+\underline{e}_1$ direction. What is the maximum shear component of traction in each of the three shafts?

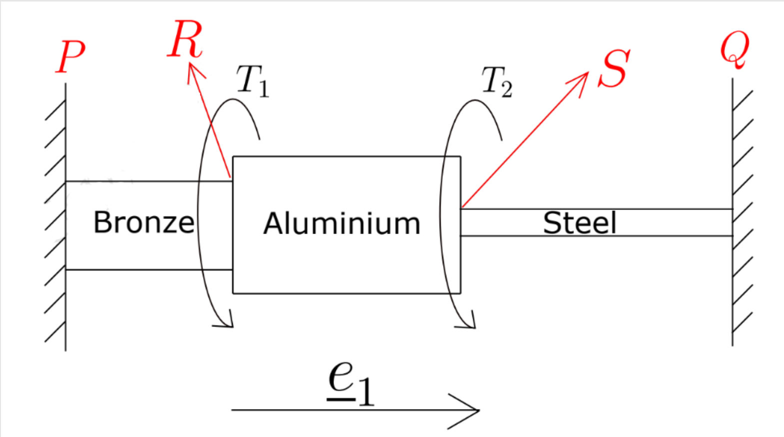


Figure 1: A composite shaft made up of bronze, aluminium and steel: torques T_1 and T_2 acts on the shaft as shown

Solution:
For finding shear component of traction, we need to find the shear strain for which we require the torque acting on each of the three shafts. In this problem, we will not be able to find the torques just by balance of forces and moments. This is a statically indeterminate problem as we will see when we try to solve it. Fig. 2 shows the free body diagram of the entire shaft. The torque T_1 and T_2 are externally applied torques and point in the $\underline{e}_1$ direction

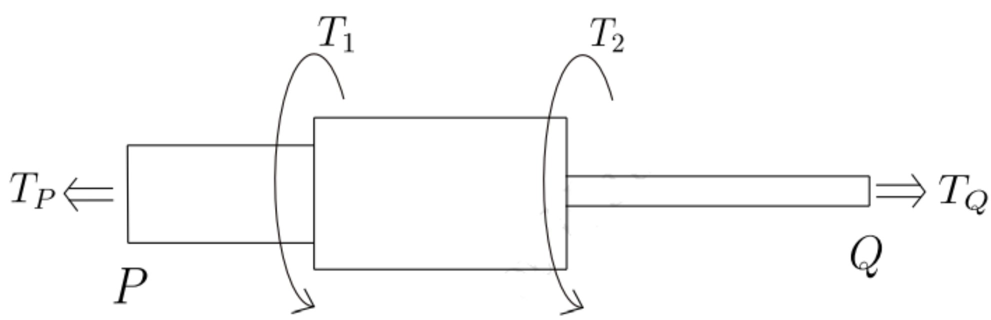


Figure 2: Free body diagram of the entire shaft

while T_P and T_Q are reaction torques applied by the support at the two ends. The torque T_P is shown pointing in the negative direction. The moment balance of the entire shaft gives us

$$
\begin{aligned}
T_Q\,\underset{=}{e}_1 + (T_1 + T_2)\,\underset{=}{e}_1 - T_P\,\underset{=}{e}_1 &= 0 \\
\Rightarrow T_Q - T_P + T_1 + T_2 &= 0
\end{aligned}
\tag{5}
$$

As there are no forces acting on the shaft, force balance will not give us anything. So, we have one equation and two unknowns T_P and T_Q. Thus, this is a statically indeterminate problem To solve such problems, we need to take deformation into account. We know that whenever equal and opposite torques are applied at different points in a beam, the beam undergoes torsion as shown in Fig. 3. A straight line on the outer surface of the cylinder (parallel to the axis) become a helix after deformation due to twisting of the cylinder. We had derived the displacement function for the extension-torsion-inflation scenario and the result is summarized below:

$$
\begin{aligned}
u_r &= -\nu\epsilon_z r + \left(\frac{P}{2(\lambda + \mu)}\frac{r_1^2}{r_2^2 - r_1^2}\right) r + \frac{P}{2\mu r}\frac{r_1^2 r_2^2}{r_2^2 - r_1^2} \\
u_\theta &= \frac{\Omega}{L} r z \\
u_z &= \epsilon_z z
\end{aligned}
$$

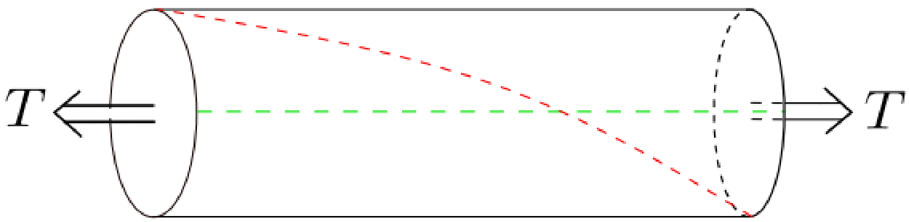


Figure 3: Equal and opposite torques on the end cross-sections of a cylinder causes torsion in the cylinder

As there is no internal pressure applied, $P = 0$. The beam is clamped at the two ends and no forces are applied anywhere. So, there is no axial strain ϵ_z either. Thus, for the pure torsion scenario, we get displacements as

$$
u_r = 0, \;\; u_\theta = \frac{\Omega}{L} r z, \;\; u_z = 0
$$

We can note that torsion alone does not generate any radial or axial displacement. The strain matrix accordingly simplifies to

$$
\left[\underset{=}{\epsilon}\right]_{(r,\theta,z)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \dfrac{\Omega r}{2L} \\ 0 & \dfrac{\Omega r}{2L} & 0 \end{bmatrix}
\tag{6}
$$

The only non zero term is $\epsilon_{\theta z}$ which in terms of twist can be written as

$$
\begin{aligned}
\epsilon_{\theta z} &= \frac{1}{2}\frac{\Omega}{L} r = \frac{1}{2}\kappa r \\
\Rightarrow \gamma_{\theta z} &= \kappa r
\end{aligned}
\tag{7}
$$

The stress matrix can then be written as

$$\left[\underline{\underline{\sigma}}\right]_{r,\theta,z} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & G\kappa r \\ 0 & G\kappa r & 0 \end{bmatrix} \tag{8}$$

This is the stress matrix for pure torsion and it only has $\tau_{\theta z} \neq 0$. It is the state of stress at any point in the composite shaft. At a point, we have infinite planes. We need the plane on which the shear component of traction is maximum. In the lecture on Mohr's circle, we had discussed that for finding the maximum shear component, we need to obtain the difference of the maximum and minimum principal stresses and divide that by 2. To find the principal stress components, we can use the concept of Mohr's circle. As the first column of the stress matrix is all zero,$\underline{e}_1$ is a principal axis. Thus, we can draw the Mohr's circle for those planes whose normals are of the form:

$$\underline{n} = \cos\theta \epsilon_\theta + \sin\theta \underline{e}_z \tag{9}$$

The center of the circle will be at midpoint of $\sigma_{\theta\theta}$ and σ_{zz} on the σ-axis. As both of them are zero, the center will be at origin. W then mark the value of $\tau_{\theta z}$ $(= G\kappa r)$on the τ-axis. This point also corresponding to θ plane. With center and θ plane known, we can get the radius by joining these two points and equal $G\kappa r$. The Mohr's circle so drawn is shown in Fig.4. We can now

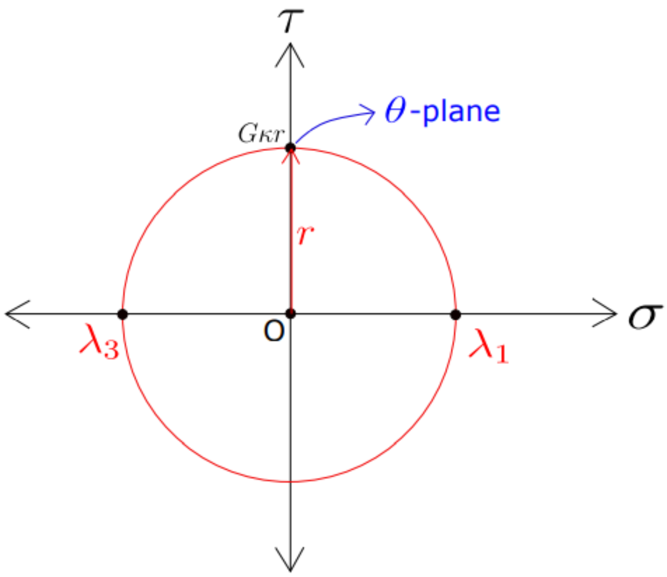


Figure 4: Mohr's circle for the state of stress given in (8)

extract the principal stress components from the circle, i.e.,

$$\lambda_1 = G\kappa r, \quad \lambda_3 = -G\kappa r \tag{10}$$

and λ_2 is zero corresponding to $\underline{e}_r$ principal direction. Thus: The maximum shear component of traction is thus

$$\tau_{\max} = \frac{\lambda_1 - \lambda_3}{2} = G\kappa r \tag{11}$$

We have to find the maximum shear component of traction for each of the three sub-shafts. From our above analysis, it turns out to be $G\kappa r$ and r is maximized for points on the lateral surface. We also need to find the variation of twist along the axis for each for each of the three shifts. The torque at an arbitrary cross section is given by

$$T(z) = G(z)\,\kappa(z)\,J(z) \tag{12}$$

This time z denotes the coordinate along the axis. The twist κ is then

$$\kappa(z) = \frac{T(z)}{G(z)\,J(z)} \tag{13}$$

As each sub-shaft is made of a single material and has a constant cross section radius, the denominator $G(z)\,J(z)$ is constant for the three sub-shafts separately. We just need to find the variation of torque in each of them. We first draw the free body diagram of a section of the bronze shaft. We make a cut in the bronze shaft and consider its left part as shown in Fig. 5. T_P acts as a reaction

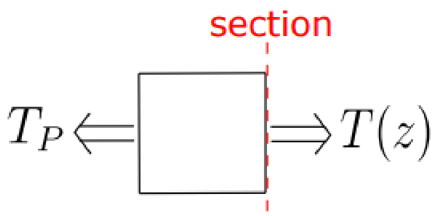


Figure 5: Free body diagram of the left part of the bronze shaft

torque from the left clamping and $T(z)$ is exerted by other part of bronze shaft. Balance of moment for this section gives

$$T_B = T_P \tag{14}$$

where T_B represents the torque within the bronze shaft and it does not vary within the bronze shaft. Now, we cut section in the aluminium shaft such that bronze and aluminium shafts are included in the left part and draw its free body diagram (see Fig. 6). This time we also have external

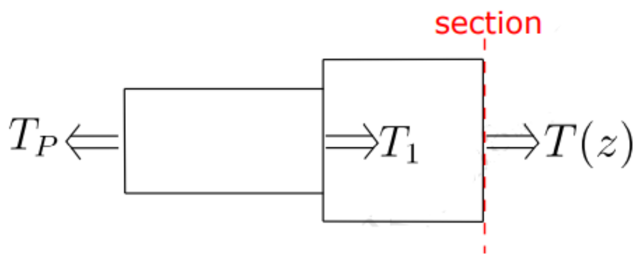


Figure 6: Free body diagram of the left body diagram of composite shaft with a section cut in the aluminium part of the composite shaft

torque T_1 acting in the free body diagram. Moment balance gives torque in the Aluminium section (T_A) as

$$T_A = T_P - T_1 \tag{15}$$

Finally, we cut a section in the steel shaft such that all thee materials are included in the left part. We draw the free body diagram of this left part as shown in Figure 9. Moment balance gives torque

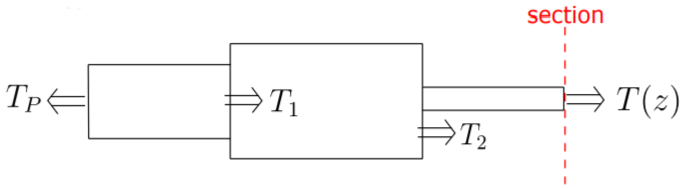


Figure 7: Free body diagram of the left part of the composite shaft when a section is cut in the steel part

in the steel section(T_S) as

$$T_S = T_P - T_1 - T_2 \tag{16}$$

We can see that torque within each of the three shafts does not vary. But, we can also observe that

$$T_B \neq T_A \neq T_S \tag{17}$$

If we now come back to equation 17, all the quantities on the RHS are constant within each sub-shaft separately. Thus, twist (κ) is constant within each of the three sub-shafts. At the same time, its value is different for the three sub-shafts as their radii, G and T values are distinct. To find the maximum shear components of traction($= G\kappa r$), we need the value twist in the three regions. We have expressed the torques in the three sub-shafts in terms of T_P which is still an unknown. To find this unknown T_P, we need to use the constraint that the two-ends of the shaft are clamped. So, both the ends do not undergo any rotation and the end-to-end rotation for the composite shaft must be zero. Thus, the sum of the end-to-end rotations for the three sub-shafts should be zero. This is the extra equation that required taking deformation into account. The end-to-end rotations for each of of the sub-shafts will be

$$\Omega_R - \Omega_P = \frac{T_B L_B}{G_B J_B},\ \Omega_S - \Omega_R = \frac{T_A L_A}{G_A J_A},\ \Omega_Q - \Omega_S = \frac{T_S L_S}{G_S J_S} \tag{18}$$

Adding the three equations, we finally get

$$\begin{aligned} \Omega_R - \Omega_P + \Omega_S - \Omega_R + \Omega_Q - \Omega_S &= \frac{T_B L_B}{G_B J_B} + \frac{T_A L_A}{G_A J_A} + \frac{T_S L_S}{G_S J_S} \\ \Rightarrow \Omega_Q - \Omega_P &= \frac{T_B L_B}{G_B J_B} + \frac{T_A L_A}{G_A J_A} + \frac{T_S L_S}{G_S J_S} \end{aligned} \tag{19}$$

However, $\Omega_Q - \Omega_P$ is zero since it equals end-to-end rotation of the doubly clamped composite shaft. We can now substitute T_B, T_A and T_S from equations 14,15 and 16 into 19 to obtain

$$\begin{aligned} \frac{T_P L_B}{G_B J_B} + \frac{T_P - T_1}{G_A J_A} + \frac{(T_P - T_1 - T_2)\, L_S}{G_S J_S} &= 0 \\ \Rightarrow T_P \left(\frac{L_B}{G_B J_B} + \frac{L_A}{G_A J_A} + \frac{L_S}{G_S J_S} \right) &= \frac{T_1 L_A}{G_A J_A} + \frac{(T_1 + T_2)\, L_S}{G_S J_S} \end{aligned} \tag{20}$$

We can solve this equation for T_P. Once we get T_P, we can find torque in the three sub-shafts from equations 14, 15 and 16 and hence twist in them using equation 17. Then, we can use equation 11 to calculate the maximum shear components of traction in the three sub-shafts. We need to use the outer radius of the three sub-shafts to get the maximum shear component.