

Tutorial 5: Mohr's circle

APL 104 - 2022 (Solid Mechanics)

1. The stress tensor at a point is given by the following matrix in Cartesian coordinate system:

$$[\underline{\underline{\sigma}}] = \begin{bmatrix} -4 & 4 & 0 \\ 4 & -4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- (a) Draw Mohr's circle corresponding to this state for traction on planes whose normals lie in $(x - y)$ plane. What are the principal stress components and the corresponding principal normals? What is the maximum shear traction and on what plane does it act?
 - (b) Using Mohr's circle idea, find out the normal and shear tractions on a plane whose normal lies in $(x - y)$ plane and makes an angle of 7.5° from x-axis in clockwise direction.
2. The stress tensor at a point is denoted by the following matrix in Cartesian coordinate system:

$$[\underline{\underline{\sigma}}] = \begin{bmatrix} -7 & 6\sqrt{3} & 0 \\ 6\sqrt{3} & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- (a) Draw Mohr's circle corresponding to this state for tractions in $(x - y)$ plane. What are the principal stress components and the direction of principal planes? What is the maximum shear traction and on what plane does it act?
 - (b) Using Mohr's circle idea, find out the normal and shear tractions on a plane whose normal lies in $(x - y)$ plane and makes an angle of 15° from x -axis in clockwise direction.
 - (c) Find out the octahedral normal and shear stress components corresponding to this state of stress.
 - (d) Decompose the given stress matrix into hydrostatic and deviatoric part.
3. Suppose the state of stress at a point is as follows in $(x - y - z)$ coordinate system.

$$[\underline{\underline{\sigma}}] = \begin{bmatrix} -2 & 4\sqrt{3} & 0 \\ 4\sqrt{3} & 6 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

- (a) Find out the center and radius of corresponding Mohr's circle.

- (b) Find out (σ, τ) on a plane whose normal makes an angle 15° anti-clockwise from x -axis.
- (c) What are the values of the principal stress components?
- (d) Obtain the orientation of principal stress planes.

APL 104 Tutorial 5 solutions

Q1. The stress tensor at a point is given by the following matrix in Cartesian coordinate system:

$$[\underline{\underline{\sigma}}] = \begin{bmatrix} -4 & 4 & 0 \\ 4 & -4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- (a) Draw Mohr's circle corresponding to this state for traction on planes whose normals lie in $(x - y)$ plane. What are the principal stress components and the corresponding principal normals? What is the maximum shear traction and on what plane does it act?
- (b) Using Mohr's circle idea, find out the normal and shear tractions on a plane whose normal lies in $(x - y)$ plane and makes an angle of 7.5° from x -axis in clockwise direction.

Solution: Do it yourself!

Q2. The stress tensor at a point is denoted by the following matrix in Cartesian coordinate system:

$$[\underline{\underline{\sigma}}] = \begin{bmatrix} -7 & 6\sqrt{3} & 0 \\ 6\sqrt{3} & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- (a) Draw Mohr's circle corresponding to this state for tractions in $(x - y)$ plane. What are the principal stress components and the direction of principal planes? What is the maximum shear traction and on what plane does it act?
- (b) Using Mohr's circle idea, find out the normal and shear tractions on a plane whose normal lies in $(x - y)$ plane and makes an angle of 15° from x -axis in clockwise direction.
- (c) Find out the octahedral normal and shear stress components corresponding to this state of stress.
- (d) Decompose the given stress matrix into hydrostatic and deviatoric part.

Solution: The e_3 -axis (or the z -axis) corresponds to the principal axis and hence the stress tensor $\underline{\underline{\sigma}}$ can be readily represented by a Mohr's circle in $x - y$ plane.

- (a) The Mohr's circle is drawn in Fig. 1.

- Center: $\left(\frac{\sigma_{11} + \sigma_{22}}{2}, 0\right) = (-1, 0)$
- Draw point on circle corresponding to \underline{e}_1 -plane with coordinates $(-7, 6\sqrt{3})$

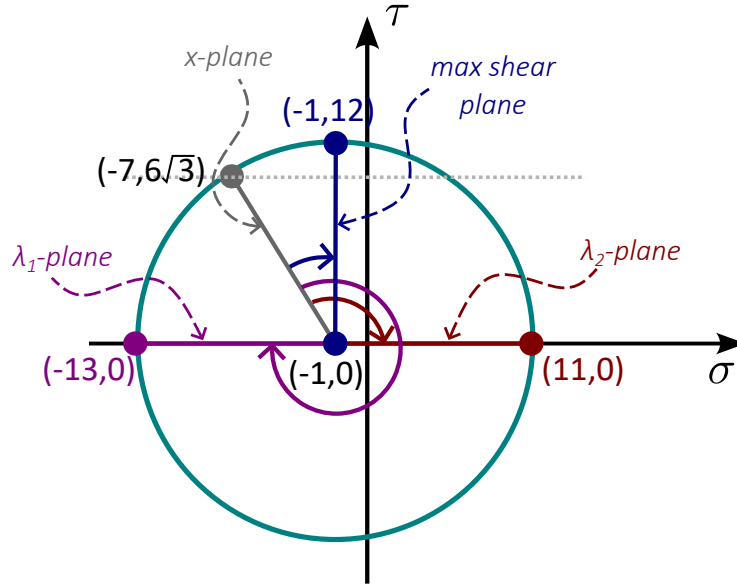


Figure 1: Mohr's circle for Q2(a)

- Radius of Mohr's circle: $R = \sqrt{(-7 + 1)^2 + (6\sqrt{3})^2} = 12$
- Principal Stresses:

$$\lambda_1 = -1 - 12 = -13 \text{ (extreme left point)}$$

$$\lambda_2 = -1 + 12 = 11 \text{ (extreme right point)}$$

$$\lambda_3 = 3 \text{ (already known)}$$

- Principal plane normals:

- λ_1 -plane occurs at $(360^\circ - 60^\circ) = 300^\circ$ clockwise from the e_1 -plane in Mohr's circle $\Leftrightarrow \lambda_1$ -plane occurs at 150° anti-clockwise from the e_1 -plane in physical coordinate system $\Rightarrow \alpha = 150^\circ$.

$$[\underline{n}_1] = \begin{bmatrix} \cos \alpha \\ \sin \alpha \\ 0 \end{bmatrix} = \begin{bmatrix} \cos 150^\circ \\ \sin 150^\circ \\ 0 \end{bmatrix} = \begin{bmatrix} -\cos 30^\circ \\ \sin 30^\circ \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

- λ_2 -plane occurs at $(180^\circ - 60^\circ) = 120^\circ$ clockwise from the e_1 -plane in Mohr's circle $\Leftrightarrow \lambda_2$ -plane occurs at 60° anti-clockwise from the e_1 -plane in physical coordinate system $\Rightarrow \alpha = 60^\circ$.

$$[\underline{n}_2] = \begin{bmatrix} \cos \alpha \\ \sin \alpha \\ 0 \end{bmatrix} = \begin{bmatrix} \cos 60^\circ \\ \sin 60^\circ \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix}$$

- λ_3 -plane occurs along e_3 -plane, as already mentioned.

- Maximum shear stresses and plane normals

- $\tau_{max}^{(1)} = \left| \frac{\lambda_1 - \lambda_2}{2} \right| = \left| \frac{-13 - 11}{2} \right| = 12$ which lies at 30° clockwise in Mohr's circle and hence will be at 15° anti-clockwise from e_1 -plane in physical coordinate system.

$$[\underline{n}^{(1)}] = \pm \begin{bmatrix} \cos 15^\circ \\ \sin 15^\circ \\ 0 \end{bmatrix} = \pm \begin{bmatrix} 0.9659 \\ 0.2588 \\ 0 \end{bmatrix}.$$

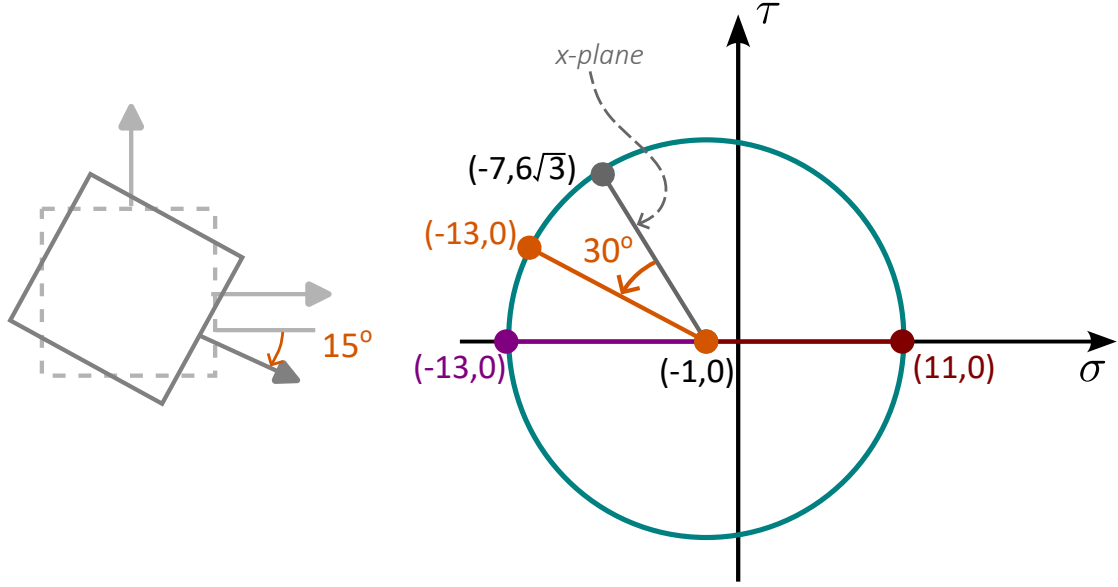


Figure 2: Mohr's circle for Q2(b)

(b) $\tau_{max}^{(2)} = \left| \frac{\lambda_1 - \lambda_3}{2} \right| = \left| \frac{-13 - 3}{2} \right| = 8.$

To derive $\tau_{max}^{(2)}$ using Mohr's circle, we would need to draw the Mohr's circle corresponding to $\lambda_1 - \lambda_3$ plane. Alternatively, we know that max-shear stress plane occurs at 45° from the two principal planes. The normal of λ_1 -plane was obtained earlier. Therefore, this max-shear plane will have the following normal:

$$\left[\underline{n}^{(2)} \right] = \frac{1}{\sqrt{2}} \left[\begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right] = \frac{1}{\sqrt{2}} \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}. \text{ There will be three more planes}$$

on which this shear stress will be realized.

(c) $\tau_{max}^{(3)} = \left| \frac{\lambda_2 - \lambda_3}{2} \right| = \left| \frac{11 - 3}{2} \right| = 4.$

The corresponding plane normal can be derived as we did earlier.

(b) 15° clockwise from \underline{e}_1 -plane physically $\Leftrightarrow 30^\circ$ anticlockwise from \underline{e}_1 -plane in Mohr's circle. Hence

$$\sigma = -1 - R \cos 30^\circ = -1 - 12 \frac{\sqrt{3}}{2} = -11.39$$

$$\tau = R \sin 30^\circ = 12 \cdot \frac{1}{2} = 6$$

(c) Octahedral normal stress:

$$\sigma_{oct} = \frac{1}{3} I_1 = \frac{1}{3} \frac{(\sigma_{11} + \sigma_{22} + \sigma_{33})}{3} = \frac{1}{3} (-7 + 5 + 3) = \frac{1}{3}.$$

Octahedral shear stress:

$$\begin{aligned}
\tau_{oct} &= \frac{1}{3} \sqrt{2I_1^2 - 6I_2} \\
I_2 &= \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \tau_{12}^2 - \tau_{23}^2 - \tau_{31}^2 \\
&= -149 \\
\tau_{oct} &= \frac{1}{3} \sqrt{2 \left(\frac{1}{3}\right)^2 - 6(-149)} \\
&= 9.977.
\end{aligned}$$

(d) Decomposition of stress tensor:

$$\begin{aligned}
[\underline{\sigma}_h] &= \frac{1}{3} [\underline{I}] = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \\
[\underline{\sigma}_d] &= [\underline{\sigma}] - [\underline{\sigma}_h] \\
&= \begin{bmatrix} -7 - \frac{1}{3} & 6\sqrt{3} & 0 \\ 6\sqrt{3} & 5 - \frac{1}{3} & 0 \\ 0 & 0 & 3 - \frac{1}{3} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{22}{3} & 6\sqrt{3} & 0 \\ 6\sqrt{3} & \frac{14}{3} & 0 \\ 0 & 0 & \frac{8}{3} \end{bmatrix}.
\end{aligned}$$

Q3. Suppose the state of stress at a point is as follows in $(x-y-z)$ coordinate system.

$$[\underline{\sigma}] = \begin{bmatrix} -2 & 4\sqrt{3} & 0 \\ 4\sqrt{3} & 6 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

- Find out the center and radius of corresponding Mohr's circle.
- Find out (σ, τ) on a plane whose normal makes an angle 15° anti-clockwise from x -axis.
- What are the values of the principal stress components?
- Obtain the orientation of principal stress planes.

Solution:

- Center: $\left(\frac{\sigma_{11} + \sigma_{22}}{2}, 0\right) = \left(\frac{-2+6}{2}, 0\right) = (2, 0)$
 - Draw point on circle corresponding to the e_1 -plane, with coordinates $(-2, 4\sqrt{3})$
 - Radius of Mohr's circle, $R = \sqrt{(-2-2)^2 + (4\sqrt{3})^2} = 8$

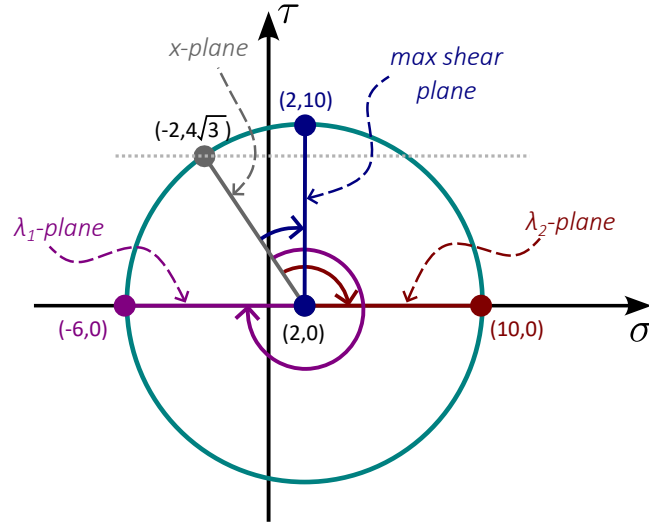


Figure 3: Mohr's circle for Q3(a)

- (b) 15° anti-clockwise from \underline{e}_1 physically $\Leftrightarrow 30^\circ$ clockwise from \underline{e}_1 -plane in Mohr's circle, which coincides with the plane of maximum shear, as can be seen from Fig. 4. The stresses on this plane is $(\sigma, \tau) = (2, 8)$.

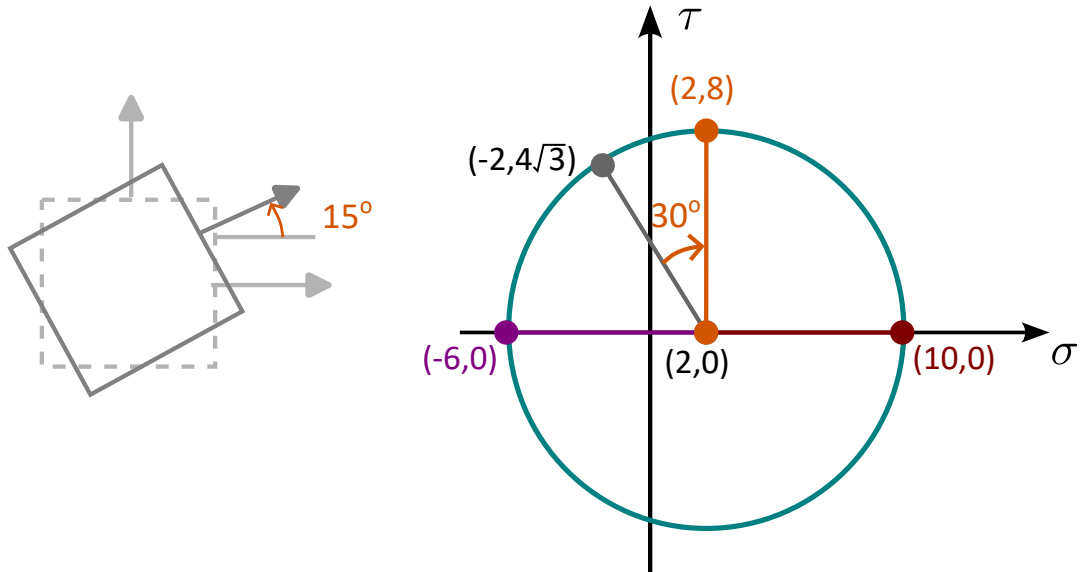


Figure 4: Mohr's circle for Q3(b)

- (c) Principal stresses:

$$\begin{aligned}\lambda_1 &= 2 + 8 = 10 \text{ (Extreme right point)} \\ \lambda_2 &= 2 - 8 = -6 \text{ (Extreme left point)} \\ \lambda_3 &= 4 \text{ (as given)}\end{aligned}$$

- (d) Orientation of principal planes: can be obtained as in previous problem.