Tutorial 1: Mathematical Preliminaries

APL 104 - 2022 (Solid Mechanics)

Q1. Show that $\underline{a} \cdot \left(\underline{\underline{A}}\, \underline{b}\right) = \left(\underline{\underline{A}}^T \underline{a}\right) \cdot \underline{b}$

Q2. There exists a tensor $\underline{\underline{A}}$ such that $\underline{\underline{A}} \cdot \underline{e}_1 = \underline{a}$, $\underline{\underline{A}} \cdot \underline{e}_2 = \underline{b}$, $\underline{\underline{A}} \cdot \underline{e}_3 = \underline{c}$. What will be the matrix form of $\underline{\underline{A}}$ in $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ coordinate system?

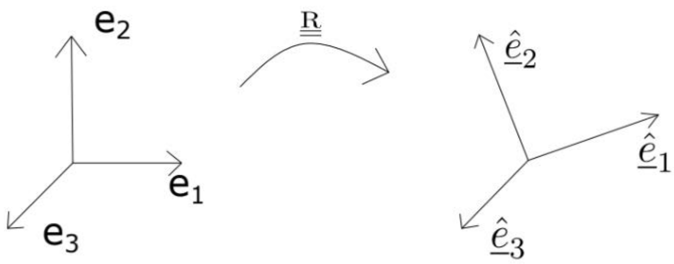
Q3. Show that

(a) $(\underline{a} \otimes \underline{b})\, \underline{\underline{C}} = \underline{a} \otimes \left(\underline{\underline{C}}^T \underline{b}\right)$

(b) $\underline{\underline{C}}\, (\underline{a} \otimes \underline{b}) = \left(\underline{\underline{C}}\, \underline{a}\right) \otimes \underline{b}$

Q4. Given an anti-symmetric tensor $\underline{\underline{A}}$, prove that $\left(\underline{\underline{A}}\, \underline{x}\right) \cdot \underline{x} = 0 \;\; \forall \, \underline{x}$

Q5. In class we learnt that a unique rotation tensor $\underline{\underline{R}}$ can be associated with transforming a set of orthonormal triad into anoother say $(\underline{e}_1, \underline{e}_2, \underline{e}_3) \to (\hat{\underline{e}}_1, \hat{\underline{e}}_2, \hat{\underline{e}}_3)$. In particular, we discussed a specific case where $(\hat{\underline{e}}_1, \hat{\underline{e}}_2, \hat{\underline{e}}_3)$ is obtained by rotation of $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ about $\underline{e}_3$ axis by angle θ. Find the matrix form of this rotation tensor $\underline{\underline{R}}$ in $(\hat{\underline{e}}_1, \hat{\underline{e}}_2, \hat{\underline{e}}_3)$ coordinate system.



APL 104 Tutorial 1 solutions

Definition of "transpose" of a 2nd-order tensor

For a second order tensor $\underline{a} \otimes \underline{b}$, its **transpose** is defined to be $\underline{b} \otimes \underline{a}$. For a general second order tensor of the form

$$\underline{\underline{A}} = \sum_i \sum_j A_{ij}\ \underline{e}_i \otimes \underline{e}_j, \tag{1}$$

we can thus write its transpose to be

$$\underline{\underline{A}}^T = \sum_i \sum_j A_{ij}\ \underline{e}_j \otimes \underline{e}_i. \tag{2}$$

Upon further renaming $i \to j$ and $j \to i$, we get

$$\underline{\underline{A}}^T = \sum_j \sum_i A_{ji}\ \underline{e}_i \otimes \underline{e}_j. \tag{3}$$

From definitions (1) and (3), one can then realize that the matrix form of transpose of $\underline{\underline{A}}$ is equal to the transpose of the matrix form of $\underline{\underline{A}}$, i.e.,

$$\left[\underline{\underline{A}}^T\right] = \left[\underline{\underline{A}}\right]^T.$$

Q1. Show that $\underline{a} \cdot \left(\underline{\underline{A}}\ \underline{b}\right) = \left(\underline{\underline{A}}^T \underline{a}\right) \cdot \underline{b}$

Method 1 (using indicial notation):

$$\begin{aligned}
\underline{a} \cdot \left(\underline{\underline{A}}\ \underline{b}\right) &= \sum_i a_i \underline{e}_i \cdot \left[\left(\sum_j \sum_k A_{jk}\ \underline{e}_j \otimes \underline{e}_k\right) \cdot \sum_l b_l \underline{e}_l\right] \\
&= \sum_i a_i \underline{e}_i \cdot \sum_j \sum_k \sum_l a_i A_{jk} b_l \delta_{kl}\ \underline{e}_j \\
&= \sum_i \sum_j \sum_k \sum_l a_i A_{jk} b_l \delta_{kl} \delta_{ij} \\
&= \sum_j \sum_l a_j A_{jl} b_l
\end{aligned}$$

$$\begin{aligned}
\left(\underline{\underline{A}}^T \underline{a}\right) \cdot \underline{b} &= \left[\left(\sum_j \sum_k A_{kj}\ \underline{e}_j \otimes \underline{e}_k\right) \cdot \sum_i a_i \underline{e}_i\right] \cdot \sum_l b_l \underline{e}_l \\
&= \sum_i \sum_j \sum_k a_i A_{kj} \delta_{ki}\ \underline{e}_j \cdot \sum_l b_l \underline{e}_l \\
&= \sum_i \sum_j \sum_k \sum_l a_i A_{kj} b_l \delta_{ki} \delta_{lj} \\
&= \sum_i \sum_j a_i A_{ij} b_j = \sum_j \sum_l a_j A_{jl} b_j \text{ (due to renaming of dummy summation indices)}
\end{aligned}$$

Method 2 (using their matrix forms):

$$\begin{aligned}
\underline{a} \cdot (\underline{\underline{A}}\, \underline{b}) &= [\underline{a}]^T \left[\underline{\underline{A}}\right] \; [\underline{b}] \\
&= \left([\underline{a}]^T \left[\underline{\underline{A}}\right] \; [\underline{b}]\right)^T \quad (\because \text{ transpose of a scalar remains the same}) \\
&= [\underline{b}]^T \left[\underline{\underline{A}}\right]^T [\underline{a}] \\
&= [\underline{b}]^T \left(\left[\underline{\underline{A}}\right]^T [\underline{a}]\right) \\
&= [\underline{b}] \cdot \left[\underline{\underline{A}}^T \underline{a}\right] = \underline{b} \cdot \left(\underline{\underline{A}}^T \underline{a}\right).
\end{aligned}$$

> **Q2**. There is a tensor $\underline{\underline{A}}$ such that $\underline{\underline{A}} \cdot \underline{e}_1 = \underline{a}$, $\underline{\underline{A}} \cdot \underline{e}_2 = \underline{b}$, $\underline{\underline{A}} \cdot \underline{e}_3 = \underline{c}$. What will be the matrix form of $\underline{\underline{A}}$ in $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ coordinate system?

Solution: As per information provided:

$$\underline{\underline{A}} \cdot \underline{e}_1 = \underline{a}$$

$$\text{or} \quad \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \text{(expressed in the sought coordinate system)}$$

$$\therefore \begin{bmatrix} A_{11} \\ A_{21} \\ A_{31} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \left(\text{1st column of } \left[\underline{\underline{A}}\right] = [\underline{a}]\right)$$

One can similarly show that

$$\begin{bmatrix} A_{12} \\ A_{22} \\ A_{32} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \; \left(\text{2nd column of } \left[\underline{\underline{A}}\right] = [\underline{b}]\right) \text{ and } \begin{bmatrix} A_{13} \\ A_{23} \\ A_{33} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \; \left(\text{3rd column of } \left[\underline{\underline{A}}\right] = [\underline{c}]\right).$$

Extra material: To see how the tensor form of $\underline{\underline{A}}$ looks like, one can derive as follows:

$$\begin{aligned}
\left[\underline{\underline{A}}\right] &= \begin{bmatrix} A_{11} & 0 & 0 \\ A_{21} & 0 & 0 \\ A_{31} & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & A_{12} & 0 \\ 0 & A_{22} & 0 \\ 0 & A_{32} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & A_{13} \\ 0 & 0 & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \\
\text{or } \left[\underline{\underline{A}}\right] &= [\underline{a}]\,[\underline{e}_1]^T + [\underline{b}]\,[\underline{e}_2]^T + [\underline{c}]\,[\underline{e}_3]^T \\
\text{or } \underline{\underline{A}} &= \underline{a} \otimes \underline{e}_1 + \underline{b} \otimes \underline{e}_2 + \underline{c} \otimes \underline{e}_3
\end{aligned}$$

> **Q3**. Show that
>
> (a) $(\underline{a} \otimes \underline{b})\, \underline{\underline{C}} = \underline{a} \otimes \left(\underline{\underline{C}}^T \underline{b}\right)$
>
> (b) $\underline{\underline{C}}\, (\underline{a} \otimes \underline{b}) = \left(\underline{\underline{C}}\, \underline{a}\right) \otimes \underline{b}$

Solution:

(a) Let's start with the matrix form of tensor expression $(\underline{a} \otimes \underline{b})\,\underline{\underline{C}}$

$$\begin{aligned}
\left([\underline{a}]\ \ [\underline{b}]^T\right)\left[\underline{\underline{C}}\right] &= [\underline{a}]\left([\underline{b}]^T\left[\underline{\underline{C}}\right]\right) \quad \text{(using associative rule of matrix multiplication)} \\
&= [\underline{a}]\left(\left[\underline{\underline{C}}\right]^T [\underline{b}]\right)^T \\
&= \underline{a} \otimes \left(\underline{\underline{C}}^T \underline{b}\right)
\end{aligned}$$

(b) Starting with the matrix form of tensor expression $\underline{\underline{C}}\,(\underline{a} \otimes \underline{b})$

$$\begin{aligned}
\left[\underline{\underline{C}}\,(\underline{a} \otimes \underline{b})\right] &= \left[\underline{\underline{C}}\right]\left([\underline{a}]\,[\underline{b}]^T\right) \\
&= \left(\left[\underline{\underline{C}}\right][\underline{a}]\right)[\underline{b}]^T \\
&= \left(\underline{\underline{C}}\,\underline{a}\right) \otimes \underline{b}.
\end{aligned}$$

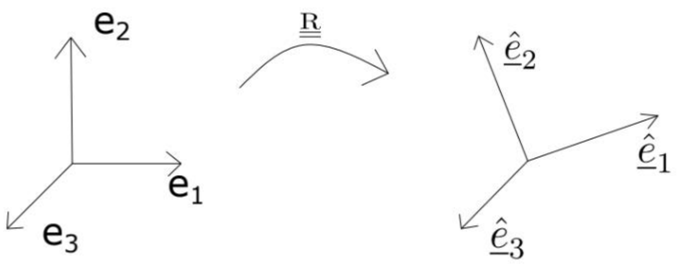
Q4. Given an anti-symmetric tensor $\underline{\underline{A}}$, prove that $\left(\underline{\underline{A}}\,\underline{x}\right) \cdot \underline{x} = 0 \ \ \forall\, \underline{x}$

Solution:

$$\begin{aligned}
\left(\underline{\underline{A}}\,\underline{x}\right) \cdot \underline{x} &= \underline{x} \cdot \left(\underline{\underline{A}}^T \underline{x}\right) \\
&= -\underline{x} \cdot \left(\underline{\underline{A}}\,\underline{x}\right) \\
&= -\left(\underline{\underline{A}}\,\underline{x}\right) \cdot \underline{x} \ \text{(from commutative rule of dot-product of two vectors)} \\
&= 0 \ \text{(since only a zero can be the negative of itself!)}
\end{aligned}$$

Disclaimer: The commutative rule ($\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$) applies only for 1st-order tensors and does not extend to higher order tensors. For example, $\underline{\underline{A}} \cdot \underline{b} \neq \underline{b} \cdot \underline{\underline{A}}$ or $\underline{\underline{A}} \cdot \underline{\underline{C}} \neq \underline{\underline{C}} \cdot \underline{\underline{A}}$.

Q5. In class we learnt that a unique rotation tensor $\underline{\underline{R}}$ can be associated with transforming a set of orthonormal triad into anoother say $(\underline{e}_1, \underline{e}_2, \underline{e}_3) \to (\underline{\hat{e}}_1, \underline{\hat{e}}_2, \underline{\hat{e}}_3)$. In particular, we discussed a specific case where $(\underline{\hat{e}}_1, \underline{\hat{e}}_2, \underline{\hat{e}}_3)$ is obtained by rotation of $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ about $\underline{e}_3$ axis by angle θ. Find the matrix form of this rotation tensor $\underline{\underline{R}}$ in $(\underline{\hat{e}}_1, \underline{\hat{e}}_2, \underline{\hat{e}}_3)$ coordinate system.



Solution: Recall that a tensor $\underline{\underline{R}}$ remains invariant in different coordinate systems. However, its representation or its matrix form is different in different coordinate systems. Also, rotation tensors are orthonormal and satisfy the following property: $\underline{\underline{R}}^{-1} = \underline{\underline{R}}^T$.

Since $\underline{\underline{R}}$ in the question maps $(\underline{e}_1, \underline{e}_2, \underline{e}_3) \to (\underline{\hat{e}}_1, \underline{\hat{e}}_2, \underline{\hat{e}}_3)$, we can write

$$\begin{aligned} &\underline{\hat{e}}_i = \underline{\underline{R}}\, \underline{e}_i \\ &\Rightarrow \underline{\underline{R}}^{-1} \underline{\hat{e}}_i = \underline{e}_i \text{ or } \underline{\underline{R}}^T \underline{\hat{e}}_i = \underline{e}_i. \end{aligned}$$

One can write

$$\underline{\underline{R}} = \sum_i \sum_j \hat{R}_{ij}\; \underline{\hat{e}}_i \otimes \underline{\hat{e}}_j$$

where the components $\hat{R}_{ij}$ can be obtained as follows:

$$\begin{aligned} \hat{R}_{ij} &= \left(\underline{\underline{R}}\, \underline{\hat{e}}_j\right) \cdot \underline{\hat{e}}_i \\ &= \underline{\hat{e}}_j \cdot \underline{\underline{R}}^T \underline{\hat{e}}_i \\ &= \underline{\hat{e}}_j \cdot \underline{e}_i. \end{aligned}$$

We had derived the same expression for R_{ij} in the class. This result is non-intuitive as we do not expect the representation of a tensor in two different coordinate systems to be the same in general. However, this case happens to be a special one since the given tensor also transforms the first coordinate system into other.