Solid Mechanics
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Lecture - 7
Principal planes and Principal stress components

Abstract

In the last lecture, we had finished our discussion on stress equilibrium equations. Now, we will learn about principal stress planes, principal stress components and related properties.

1 Definition (start time: 00:30)

By now we have learnt that at any point in the body, we have different traction on different planes. Accordingly, each of the planes also has its own normal component of traction. Among these planes, the planes on which the normal component of traction becomes maximum or minimum are called principal planes and the values of the normal traction on these planes are called principal stress components. The knowledge of such planes and traction on them is important because one of the failure theories says that a body will fail at a point if the principal stress component reaches a threshold limit. Whenever we design a machine, the knowledge of principal stress components can help us to know whether our machine will be within the limits of failure or not.

2 Finding Principal Planes (start time: 1:56)

Let us suppose we are interested in finding principal planes at a point $\underline{x}$ in the body as shown in Figure 1. At this point, the normal component of traction on an arbitrary plane with normal $\underline{n}$ is

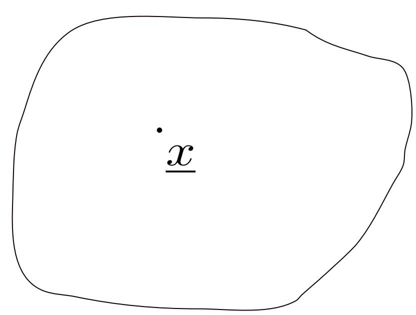

Figure 1: A body with an arbitrary point $\underline{x}$

given by

$$\sigma_{nn} = \underline{t}^n \cdot \underline{n} = (\underline{\underline{\sigma}}\,\underline{n}) \cdot \underline{n} \tag{1}$$

Our objective is to maximize/minimize it. We know from the first year calculus that once we have a mathematical formula for a quantity (in terms of variables) to be maximized/minimized, we set the derivative of the quantity with respect to all variables to zero and solve the resulting equations to obtain the variables. Let us choose a coordinate system $(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ and write formula for σ_{nn} in this coordinate system, i.e.,

$$
\begin{aligned}
\sigma_{nn} &= \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \cdot \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \\
&= \sum_i \sum_j \sigma_{ij} n_j n_i \qquad (2)
\end{aligned}
$$

The normal direction (or the three components (n_1, n_2, n_3)) is an unknown here while the stress matrix is known. However, the normal vector being of unit magnitude, its three components must satisfy

$$n_1^2 + n_2^2 + n_3^2 - 1 = 0 \qquad (3)$$

which implies that the three components are not all independent, e.g., n_3 can be calculated from the other two by setting

$$n_3 = \sqrt{1 - n_1^2 - n_2^2}. \qquad (4)$$

We can substitute the above formula for n_3 in equation (2) and then differentiate the resulting expression for σ_{nn} just with respect to n_1 and n_2. However, the modified expression for σ_{nn} becomes a bit complex differentiating which and further solving the resulting equations is not easy. Another way to maximize/minimize our function (2) is using the Lagrange multipliers which we now discuss.

2.1 Method of Lagrange Multipliers (start time: 06:33)

Whenever a function is to be maximized/minimized in presence of constraints, one uses the method of Lagrange multiplier. Basically, the objective function (the function to be minimized/maximized which is equation (2) in our case) is augmented by adding/subtracting to it the constraint equation (equation (3) here) multiplied with an unknown Lagrange multiplier. So, the augmented function to be maximized/minimized (denoted by f) now becomes a function of the 3 components (n_1, n_2, n_3) and the Lagrange multiplier λ as follows:

$$f(n_1, n_2, n_3, \lambda) = \sum_i \sum_j \sigma_{ij} n_j n_i - \lambda \Big(\sum_i n_i n_i - 1 \Big). \qquad (5)$$

The term $\lambda\Big(\sum_i n_i n_i - 1\Big)$ represents our constraint (equation (3)) and the negative sign in front of it could very well have been a positive sign. This sign does not make any difference to the overall formulation. We then take the derivative of f with respect to each of the unknowns (n_1,n_2,n_3,λ). Let us begin by taking the derivative with respect to n_1, i.e.,

$$\frac{\partial f}{\partial n_1} = \sum_i \sum_j \sigma_{ij} \frac{\partial n_j}{\partial n_1} n_i + \sum_i \sum_j \sigma_{ij} n_j \frac{\partial n_i}{\partial n_1} - \lambda \Big(\sum_i 2 \frac{\partial n_i}{\partial n_1} n_i \Big). \qquad (6)$$

As σ_{ij} is a constant here, it does not get differentiated. Furthermore, as n_1, n_2 and n_3 can be taken to be independent now[1], we can write

$$\frac{\partial n_i}{\partial n_j} = \delta_{ij}. \tag{7}$$

Now, taking the derivative of f with respect to a general component n_k and using equation (7), we get

$$\frac{\partial f}{\partial n_k} = \sum_i \sum_j \sigma_{ij}\delta_{jk}n_i + \sum_i \sum_j \sigma_{ij}n_j\delta_{ik} - 2\lambda \sum_i \delta_{ik}n_i = 0 \tag{8}$$

Using Kronecker delta property, we can remove one of the summations from each term and replace the index of summation by the other index in the Kronecker delta function, i.e.,

$$\sum_i \sigma_{ik}n_i + \sum_j \sigma_{kj}n_j - 2\lambda\ n_k = 0. \tag{9}$$

The first and second terms can be clubbed together because i and j are just dummy variables (variable of summation) which can be replaced with any other variable. Thus

$$\sum_i (\sigma_{ik} + \sigma_{ki})n_i - 2\lambda n_k = 0 \tag{10}$$

Finally, as the stress matrix is symmetric, we get

$$\boxed{\sum_i \sigma_{ki}n_i - \lambda n_k = 0 \text{ for } k = 1, 2, 3} \tag{11}$$

We now take the derivative of f with respect to λ, i.e.,

$$\frac{\partial f}{\partial \lambda} = \sum_i n_i n_i - 1 = 0. \tag{12}$$

So, we have 4 equations (3 from equation (11) and 1 from equation (12)) in 4 unknowns (n_1, n_2, n_3, λ). Equation (11) can be written in a matrix form since for each k, the first term on LHS of (11) can be obtained by multiplying the k^{th} row of $[\underline{\underline{\sigma}}]$ with the column of $\underline{n}$. This leads to

$$[\underline{\underline{\sigma}}]\ [\underline{n}] = \lambda\ [\underline{n}]\ . \tag{13}$$

We immediately see that this is an 'eigenvalue-eigenvector problem' with

$$\begin{aligned} \underline{n} &: \text{ eigenvector of } \underline{\underline{\sigma}} \\ \lambda &: \text{ eigenvalue of } \underline{\underline{\sigma}} \end{aligned}$$

We also know from first year mathematics that if $\underline{x}$ is an eigenvector, then a scalar multiple of $\underline{x}$ is also an eigenvector which can be proved as follows:

$$\underline{\underline{A}}\,\underline{x} = \lambda\underline{x} \Rightarrow \underline{\underline{A}}(b\underline{x}) = b(\underline{\underline{A}}\,\underline{x}) = b(\lambda\underline{x}) = \lambda(b\underline{x}) \tag{14}$$

[1]In this formulation, the three components of the normal direction are assumed to be independent of each other and the constraint equation automatically takes care of the relevant relation between them.

Thus both $\underline{x}$ and $b\underline{x}$ are the eigenvectors with the same eigenvalue λ for arbitrary scalar b. Thus, the magniude of our direction vector $\underline{n}$ could be anything as far as equation (13) is concerned but equation (12) restricts its magnitude to be unity. Thus, equations (12) and (13) together give us a unique solution for the direction vector $\underline{n}$. Also when we consider equation (13), the left hand side is nothing but the column representation of traction on plane with normal $\underline{n}$. Thus, we have:

$$\underline{t}^n = \underline{\underline{\sigma}}\,\underline{n} = \lambda\underline{n} \Rightarrow \sigma_{nn} = \underline{t}^n \cdot \underline{n} = \lambda\underline{n} \cdot \underline{n} = \lambda.$$

From this, we immediately infer that the traction on a plane with normal $\underline{n}$ (given by (13)) acts along the direction $\underline{n}$ itself and hence has no shear component. Summarizing, the principal planes of stress at a point have their normals equal to eigenvectors of the stress tensor whereas the principal stress components are given by the eigenvalues of the stress tensor.

3 Properties of Principal Planes at a point (start time: 28:12)

By definition, principal planes are the planes on which the normal component of traction is maximized/minimized. We want to know how many such planes exist at any given point in the body. As stress matrix is a 3×3 matrix, it will usually have three eigenvalues and eigenvectors but they need not all be real. However, being symmetric[2] ensures that these eigenvalues and eigenvectors are all real. In fact, for symmetric matrices, the eigenvectors corresponding to different eigenvalues are perpendicular to each other too. To prove this, consider two eigenvectors $\underline{n}_1$ and $\underline{n}_2$ of a symmetric tensor $\underline{\underline{\sigma}}$, with corresponding eigenvalues λ_1 and λ_2 (but distinct). Thus, we have

$$\begin{aligned} \underline{\underline{\sigma}}\,\underline{n}_1 &= \lambda_1\,\underline{n}_1, & (15)\\ \underline{\underline{\sigma}}\,\underline{n}_2 &= \lambda_2\,\underline{n}_2. & (16)\end{aligned}$$

We now dot the first equation with $\underline{n}_2$ and the second one with $\underline{n}_1$. So, we get

$$\begin{aligned} (\underline{\underline{\sigma}}\,\underline{n}_1)\cdot\underline{n}_2 &= \lambda_1(\underline{n}_1\cdot\underline{n}_2) & (17)\\ (\underline{\underline{\sigma}}\,\underline{n}_2)\cdot\underline{n}_1 &= \lambda_2(\underline{n}_2\cdot\underline{n}_1) & (18)\end{aligned}$$

Let us consider equation (18) now. From the matrix vector operations discussed in the first lecture, we can take $\underline{\underline{\sigma}}$ to the other side of the dot product by taking its transpose, i.e.,

$$\underline{n}_2\cdot(\underline{\underline{\sigma}}^T\underline{n}_1) = \lambda_2(\underline{n}_2\cdot\underline{n}_1). \qquad (19)$$

However, the stress tensor being symmetric ($\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$), we get

$$(\underline{\underline{\sigma}}\,\underline{n}_1)\cdot\underline{n}_2 = \lambda_2(\underline{n}_2\cdot\underline{n}_1) \qquad (20)$$

Now, we subtract equation (20) from equation (17) to get

$$\begin{aligned} (\underline{\underline{\sigma}}\,\underline{n}_1)\cdot\underline{n}_2 - (\underline{\underline{\sigma}}\,\underline{n}_1)\cdot\underline{n}_2 &= (\lambda_2-\lambda_1)(\underline{n}_1\cdot\underline{n}_2)\\ \Rightarrow (\lambda_1-\lambda_2)(\underline{n}_1\cdot\underline{n}_2) &= 0. & (21)\end{aligned}$$

[2] An $n \times n$ symmetric matrix has n real eigenvalues (not necessarily distinct).

As λ_1 and λ_2 are distinct, $\underline{n_1} \cdot \underline{n_2}$ has to be zero implying that the two normals are perpendicular. We have thus proved that principal planes at a point are three in number and are perpendicular to each other.

It is also easy to show that if two of the eigenvalues turn out to be the same, then any linear combination of the two eigenvectors is also an eigenvector. For example:

$$\underline{\underline{\sigma}}\,\underline{n}_1 = \lambda\ \underline{n}_1 \quad \Rightarrow \quad \underline{\underline{\sigma}}\,(\alpha\underline{n}_1) = \lambda\ (\alpha\underline{n}_1) \tag{22}$$

$$\underline{\underline{\sigma}}\,\underline{n}_2 = \lambda\ \underline{n}_2 \quad \Rightarrow \quad \underline{\underline{\sigma}}\,(\beta\underline{n}_2) = \lambda\ (\beta\underline{n}_2). \tag{23}$$

Summing the above two equations:

$$\underline{\underline{\sigma}}\,(\alpha\underline{n}_1 + \beta\underline{n}_2) = \lambda\ (\alpha\underline{n}_1 + \beta\underline{n}_2) \tag{24}$$

This implies that if two of the eigenvalues repeat, there exists infinite number of eigenvectors all in the plane formed by $\underline{n}_1$ and $\underline{n}_2$ all of which by definition are also the principal planes but all having the same eigenvalue or principal stress component.

4 Representation of stress tensor in the coordinate system of its eigenvectors (start time: 34:26)

If there are three distinct eigenvalues for a stress matrix, the corresponding three eigenvectors will all be perpendicular to each other (we proved it in the previous section). Thus, we can also choose them as the basis for our coordinate system. Let us represent our stress tensor in this coordinate system. We first need to find traction on the planes with normals along the basis vectors of this coordinate system. As the basis vectors are themselves eigenvectors of the stress tensor, traction on those planes will simply be $\lambda\underline{n}$ (no shear components present). Thus, the corresponding stress matrix will be a diagonal matrix[3] as shown below:

$$[\underline{\underline{\sigma}}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}. \tag{25}$$

Alternatively, given an arbitrary stress matrix in some coordinate system, we can always transform it to become diagonal in the coordinate system whose basis vectors are aligned along the eigenvectors of the stress matrix. In the general case as shown in Figure 2, both normal and shear components of traction are present on the faces of a cuboid element at a point in the body. But, if we choose the cuboid element in such a way that its faces are along the eigenvectors of the stress matrix as shown in Figure 3, its faces will have no shear component because the faces are also the principal planes. On these planes, we only have normal components (λ_1, λ_2, λ_3) present. So, the faces of this cuboid have no tendency to shear. They can either get pulled apart or pushed inside depending on the sign of λ.

[3] A diagonal matrix is one in which all the off diagonal components are zero.

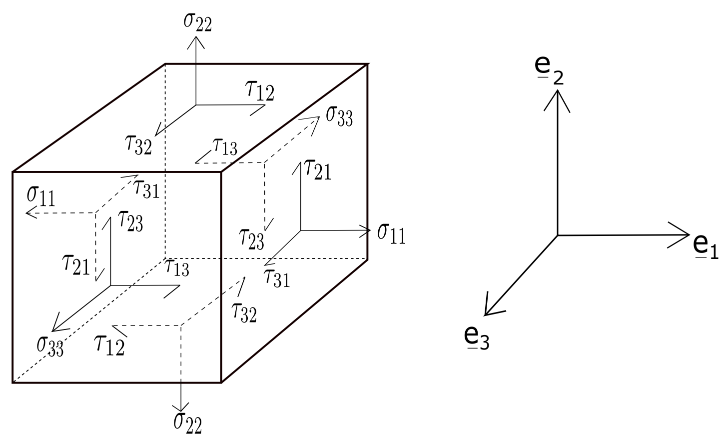

Figure 2: A cuboid element at a point in the body with all the stress components shown on it

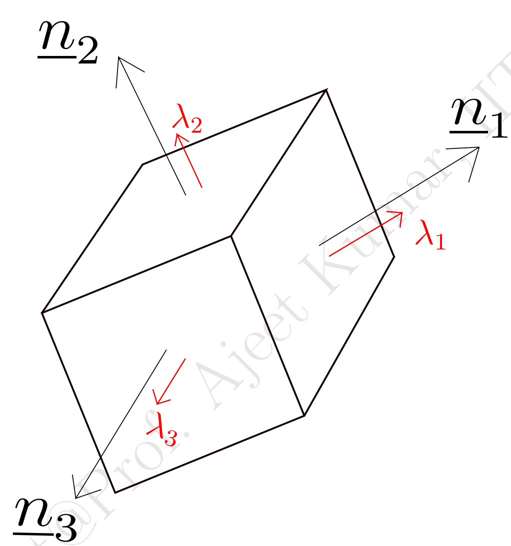

Figure 3: A cuboid element at a point in the body with its face normals along the eigenvectors of the stress matrix at that point: all the faces become shear free

5 Objective questions to practise

1. Given a stress tensor at a point in the body, what can you say about principal planes at that point?

   (a) There will be at least one principal plane
   (b) There will be at least three principal planes
   (c) The set of principal planes will be different in different coordinate system
   (d) None of these

2. The principal planes are perpendicular to each other because

(a) Their normals are eigenvectors of the stress tensor which happen to be perpendicular to each other for symmetric matrices

(b) They need not be perpendicular

(c) It is just a matter of choice

(d) None of the above

3. What can you say about shear component of traction on principal planes?

   (a) Shear components attain maximum value

   (b) Shear components always vanish on principal planes

   (c) Nothing can be said with the limited information in question

   (d) None of these

4. What can you say about the matrix form of stress tensor in the coordinate system of principal axes?

   (a) The matrix becomes diagonal

   (b) The matrix becomes proportional to identity

   (c) All components of the matrix are usually non-zero

   (d) All components of matrix become zero

5. State whether true or false:
   The principal stress components of a stress tensor depends on what coordinate system is used to represent the stress tensor?

   (a) True

   (b) False

6. Which of the following is true with respect to principal planes of stress tensor?

   (a) They are planes on which normal component of traction is maximized only.

   (b) They are planes on which normal component of traction is either maximized or minimized.

   (c) They are planes on which normal component of traction is minimized only.

   (d) All planes are principal planes.

7. The principal stress components are given by

   (a) Eigenvalues of the stress tensor

   (b) Eigenvectors of the stress tensor

   (c) The diagonal components of a general stress matrix

(d) None of the above

8. On principal planes:

    (a) Traction vector acts along the plane normal
    (b) Both shear stress components vanish
    (c) Only one of the shear stress components traction vanishes
    (d) None of the above

9. State whether True or False:
   All principal stress components are always greater than σ_{xx} (σ_{xx} represents the normal component of traction on the plane with normal along x-axis).

    (a) True
    (b) False

10. Which of the following is true about principal stress components?

    (a) The value of normal traction on principal planes are principal stress components.
    (b) The planes on which normal traction is maximized or minimized are principal planes
    (c) Principal planes also have shear component of traction
    (d) The value of shear traction on principal planes are principal stress components.

11. A stress matrix has

    (a) Real eigenvalues.
    (b) can have real or imaginary Eigenvalues.
    (c) Eigenvectors perpendicular to each other.
    (d) Eigenvectors parallel to each other.

12. Think of the matrix form of a stress tensor in a coordinate system whose basis vectors are same as eigenvectors of the stress. Which of the following is/are true?

    (a) The matrix form is a diagonal matrix.
    (b) The matrix form can be diagonal or non-diagonal matrix.
    (c) The diagonal elements of the matrix form are eigenvalues.
    (d) diagonal elements of the matrix need not be eigenvalues.

13. Suppose that two of the Eigen values of stress tensor at a point becomes the same. How many principal planes will exist in such a situation?

    (a) 3

(b) 2
(c) infinite
(d) none of these

14. The plane on which maximum traction acts
    (a) has no normal traction
    (b) does not exist
    (c) has zero normal traction if two principal stress components are equal and opposite
    (d) none of these

15. Suppose that two of the eigenvalues of the stress matrix at a point become the same. How many principal planes will exist in such a situation?
    (a) 3
    (b) 2
    (c) infinite
    (d) None of these

16. The principal planes of stress at a point
    (a) does not exist always
    (b) are at least three in number
    (c) cannot be defined
    (d) none of these

17. Given a stress tensor at a point in the body, what can you say about principal planes at that point?
    (a) There will be at least one principal plane
    (b) There will be at least three principal planes
    (c) The set of principal planes will be different in different coordinate system
    (d) None of these

18. The principal planes are perpendicular to each other because
    (a) Their normals are eigenvectors of the stress tensor which happen to be perpendicular to each other for symmetric matrices
    (b) They need not be perpendicular
    (c) It is just a matter of choice
    (d) None of the above

19. What can you say about the matrix form of stress tensor in the coordinate system of principal axes?

    (a) The matrix becomes diagonal
    (b) The matrix becomes proportional to identity
    (c) All components of the matrix are usually non-zero
    (d) All components of matrix become zero

20. State whether true or false:
    The principal stress components of a stress tensor depends on what coordinate system is used to represent the stress tensor?

    (a) True
    (b) False