Solid Mechanics
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Lecture - 5
Stress equilibrium equations

Abstract

In this lecture, we will derive the stress equilibrium equations which are used to find how stress tensor or stress components vary in a body.

1 Introduction (start time: 00:35)

Let us suppose we have a body which could be clamped at some part of the boundary as shown in Figure 1. We have talked about traction and stress at a point but not about how stress is varying

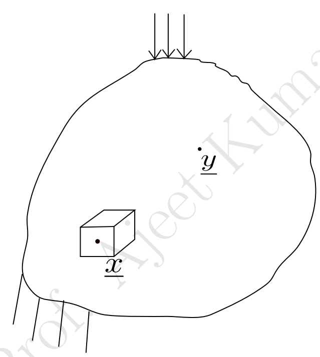

Figure 1: A body under the action of some boundary load: a cuboid is considered centered at $\underline{x}$

in the body. As we apply a load at some part of the boundary, the stress would be different at different points within the body. To decide where the body is going to fail, we need to know the value of stress at every point in the body. Higher the stress more the chance of failure. The stress equilibrium equations allow us to obtain the distribution of stress in a body.

2 Linear Momentum Balance (start time: 01:40)

Think of a small cuboid with its centroid at an arbitrary point $\underline{x}$ as shown in Figure 1 and apply Newton's second law to this cuboid:

$$\sum \underline{F}^{ext} = \frac{d}{dt}(\overrightarrow{P}). \tag{1}$$

There will be traction on the six faces of this cuboid that will be applied by the other part of the body. Suppose the edges of the cuboid are aligned along $\underline{e}_1, \underline{e}_2$ and $\underline{e}_3$ directions as shown in Figure 2. The edges along $\underline{e}_1, \underline{e}_2$ and $\underline{e}_3$ are of length Δx_1, Δx_2 and Δx_3 respectively. To find the net force on this cuboid, we need to find the forces due to tractions and body force.

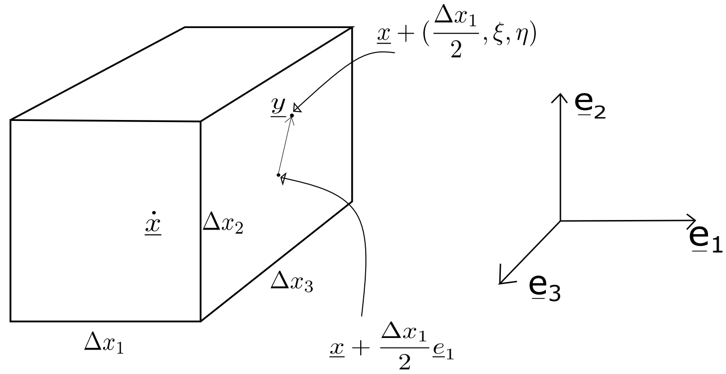

Figure 2: The coordinates of an arbitrary point $\underline{y}$ on the cuboid's +ve $\underline{e}_1$ face

2.1 Traction contribution (start time: 04:32)

Starting with the $\underline{e}_1$ plane, an arbitrary point on this face will have coordinate $\underline{x}+\left(\frac{\Delta x_1}{2}, \xi, \eta\right)$ in the $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ coordinate system. This is because we need to move by $\frac{\Delta x_1}{2}\underline{e}_1$ from the centroid to first reach the center of the $\underline{e}_1$ face. Then, any arbitrary point on this face will have some coordinate along $\underline{e}_2$ and $\underline{e}_3$ given by ξ and η respectively. In general, traction will vary from point to point on this face. For the particular point under consideration, traction will be given by:

$$\underline{t}^n = \underline{\underline{\sigma}}\, \underline{n} \tag{2}$$

$$\Rightarrow \underline{t}^1(\underline{x} + \frac{\Delta x_1}{2}\underline{e}_1 + \xi\underline{e}_2 + \eta\underline{e}_3) = \underline{\underline{\sigma}}(\underline{x} + \frac{\Delta x_1}{2}\underline{e}_1 + \xi\underline{e}_2 + \eta\underline{e}_3)\ \underline{e}_1 \tag{3}$$

Let us write this in terms of the stress tensor at the centroid of the cuboid using Taylor's expansion, i.e.,

$$\underline{\underline{\sigma}}(\underline{x} + \frac{\Delta x_1}{2}\underline{e}_1 + \xi\underline{e}_2 + \eta\underline{e}_3)\ \underline{e}_1 = \left[\underline{\underline{\sigma}} + \frac{\partial\underline{\underline{\sigma}}}{\partial x_1}\frac{\Delta x_1}{2} + \frac{\partial\underline{\underline{\sigma}}}{\partial x_2}\xi + \frac{\partial\underline{\underline{\sigma}}}{\partial x_3}\eta + ...\right]\underline{e}_1 \tag{4}$$

We neglect higher order terms to simplify the analysis. Also keep in mind that $\underline{\underline{\sigma}}$ and its derivatives are evaluated at the cuboid's centroid. The total force due to traction on $\underline{e}_1$ plane will be obtained by integration of traction over the area of this plane. As $\xi = 0$ and $\eta = 0$ corresponds to the center of this surface, we need to integrate from $\frac{-\Delta x_2}{2}$ to $\frac{\Delta x_2}{2}$ for ξ and from $\frac{-\Delta x_3}{2}$ to $\frac{\Delta x_3}{2}$ for η. Thus

$$\underline{F}^1 = \int_{\frac{-\Delta x_3}{2}}^{\frac{\Delta x_3}{2}}\int_{\frac{-\Delta x_2}{2}}^{\frac{\Delta x_2}{2}} \underline{t}^1 d\xi d\eta = \int_{\frac{-\Delta x_3}{2}}^{\frac{\Delta x_3}{2}}\int_{\frac{-\Delta x_2}{2}}^{\frac{\Delta x_2}{2}} \left[\underline{\underline{\sigma}} + \frac{\partial\underline{\underline{\sigma}}}{\partial x_1}\frac{\Delta x_1}{2} + \frac{\partial\underline{\underline{\sigma}}}{\partial x_2}\xi + \frac{\partial\underline{\underline{\sigma}}}{\partial x_3}\eta\right]\underline{e}_1 d\xi d\eta \tag{5}$$

The first two terms within square brackets in the above equation are independent of ξ and η. So, they can come out of the integration and integration of remaining integrand simply gives the total area of $\underline{e}_1$ face. The third term is independent of η and the fourth is independent of ξ. So, we get

$$\underline{F}^1 = \left[\underline{\underline{\sigma}} + \frac{\partial \underline{\underline{\sigma}}}{\partial x_1}\frac{\Delta x_1}{2}\right]\underline{e}_1 \Delta x_2 \Delta x_3 + \frac{\partial \underline{\underline{\sigma}}}{\partial x_2}\underline{e}_1 \Delta x_3 \int\limits_{\frac{-\Delta x_2}{2}}^{\frac{\Delta x_2}{2}} \xi d\xi + \frac{\partial \underline{\underline{\sigma}}}{\partial x_3}\underline{e}_1 \Delta x_2 \int\limits_{\frac{-\Delta x_3}{2}}^{\frac{\Delta x_3}{2}} \eta d\eta \tag{6}$$

The two single integrals give $\dfrac{\xi^2}{2}$ and $\dfrac{\eta^2}{2}$ respectively. Thus, when we put their corresponding limits, we see that both these integrals vanish. On further noting that $\Delta x_1 \Delta x_2 \Delta x_3$ equals volume of the cuboid (denoted by ΔV), we obtain:

$$\underline{F}^1 = \underline{\underline{\sigma}}\ \underline{e}_1 \Delta x_2 \Delta x_3 + \frac{\partial \underline{\underline{\sigma}}}{\partial x_1}\underline{e}_1 \frac{\Delta V}{2} \tag{7}$$

This is the force on positive $\underline{e}_1$ plane. The force on $-\underline{e}_1$ plane will accordingly be

$$\begin{aligned}
\underline{F}^{-1} = \int\limits_{\frac{-\Delta x_3}{2}}^{\frac{\Delta x_3}{2}} \int\limits_{\frac{-\Delta x_2}{2}}^{\frac{\Delta x_2}{2}} \underline{t}^{-1} d\xi d\eta \quad &= \int\limits_{\frac{-\Delta x_3}{2}}^{\frac{\Delta x_3}{2}} \int\limits_{\frac{-\Delta x_2}{2}}^{\frac{\Delta x_2}{2}} \left[\underline{\underline{\sigma}}(\underline{x} - \frac{\Delta x_1}{2}\underline{e}_1 + \xi \underline{e}_2 + \eta \underline{e}_3)\right](-\underline{e}_1) d\xi d\eta \\
&= \int\limits_{\frac{-\Delta x_3}{2}}^{\frac{\Delta x_3}{2}} \int\limits_{\frac{-\Delta x_2}{2}}^{\frac{\Delta x_2}{2}} \left[\underline{\underline{\sigma}} - \frac{\partial \underline{\underline{\sigma}}}{\partial x_1}\frac{\Delta x_1}{2} + \frac{\partial \underline{\underline{\sigma}}}{\partial x_2}\xi + \frac{\partial \underline{\underline{\sigma}}}{\partial x_3}\eta\right](-\underline{e}_1) d\xi d\eta \\
&= -\underline{\underline{\sigma}}\ \underline{e}_1 \Delta x_2 \Delta x_3 + \frac{\partial \underline{\underline{\sigma}}}{\partial x_1}\underline{e}_1 \frac{\Delta V}{2}.
\end{aligned} \tag{8}$$

The total force on $\underline{e}_1$ and -$\underline{e}_1$ faces are thus

$$\underline{F}^1 + \underline{F}^{-1} = \frac{\partial \underline{\underline{\sigma}}}{\partial x_1}\underline{e}_1\ \Delta V. \tag{9}$$

We can then find out the total force on $+\underline{e}_2$ and $-\underline{e}_2$ planes using similar analysis and we will find that only the index '1' changes to '2' in equation (9). Similarly, for total force on $+\underline{e}_3$ and $-\underline{e}_3$ planes, the index '1' changes to '3'. So, the total traction force (denoted by $\underline{F}^{tra}$) on all the six faces will be

$$\underline{F}^{tra} = \sum_{i=1}^{3} \frac{\partial \underline{\underline{\sigma}}}{\partial x_i}\underline{e}_i\ \Delta V \tag{10}$$

2.2 Body force contribution (start time: 22:53)

We also need the body force contribution. As the body force is defined as force per unit volume, we need to integrate over the volume of the cuboid to get the total force which we denote by $\underline{F}^b$. The body force, denoted by $\underline{b}$, can also vary within the volume of the cuboid in general. So, it will be a function of the position vector of the point of interest in the cuboid. A general point in the cuboid will have position vector $(\underline{x} + \gamma\underline{e}_1 + \xi\underline{e}_2 + \eta\underline{e}_3)$. Thus, the total force would be

$$\begin{aligned}
\underline{F}^b &= \int_{\frac{-\Delta x_1}{2}}^{\frac{\Delta x_1}{2}} \int_{\frac{-\Delta x_2}{2}}^{\frac{\Delta x_2}{2}} \int_{\frac{-\Delta x_3}{2}}^{\frac{\Delta x_3}{2}} \underline{b}(\underline{x} + \gamma\underline{e}_1 + \xi\underline{e}_2 + \eta\underline{e}_3) d\gamma d\xi d\eta \\
&= \int_{\frac{-\Delta x_1}{2}}^{\frac{\Delta x_1}{2}} \int_{\frac{-\Delta x_2}{2}}^{\frac{\Delta x_2}{2}} \int_{\frac{-\Delta x_3}{2}}^{\frac{\Delta x_3}{2}} \left[\underline{b} + \frac{\partial \underline{b}}{\partial x_1}\gamma + \frac{\partial \underline{b}}{\partial x_2}\xi + \frac{\partial \underline{b}}{\partial x_3}\eta + ...\right] d\gamma d\xi d\eta \qquad \text{(using Taylor's expansion)} \\
&= \underline{b}\,\Delta V + \frac{\partial \underline{b}}{\partial x_1}\Delta x_2 \Delta x_3 \int_{\frac{-\Delta x_1}{2}}^{\frac{\Delta x_1}{2}} \gamma d\gamma + \frac{\partial \underline{b}}{\partial x_2}\Delta x_1 \Delta x_3 \int_{\frac{-\Delta x_2}{2}}^{\frac{\Delta x_2}{2}} \gamma d\gamma + \frac{\partial \underline{b}}{\partial x_3}\Delta x_1 \Delta x_2 \int_{\frac{-\Delta x_3}{2}}^{\frac{\Delta x_3}{2}} \gamma d\gamma + o(\Delta V) \\
&= \underline{b}\,\Delta V + 0 + 0 + 0 + o(\Delta V) \\
&= \underline{b}\,\Delta V + o(\Delta V) \qquad (11)
\end{aligned}$$

The higher order terms in the Taylor's expansion when integrated will give terms of order less than the volume of the cuboid. In mathematics, 'o' is used to denote a smaller order term. So, if we have terms like $o(\Delta V)$ and if ΔV approaches zero, $o(\Delta V)$ goes to zero faster than ΔV, i.e.,

$$\lim_{\Delta V \to 0} \frac{o(\Delta V)}{\Delta V} = 0. \qquad (12)$$

2.3 Dynamic term (start time: 28:50)

The total linear momentum ($\overrightarrow{P}$) will be obtained by the volume integration of the linear momentum of a small volumetric point of the cuboid:

$$\overrightarrow{P} = \int\int\int dm\,\underline{v} = \int\int\int (\rho dV)\underline{v} = \int_{\frac{-\Delta x_3}{2}}^{\frac{\Delta x_3}{2}} \int_{\frac{-\Delta x_2}{2}}^{\frac{\Delta x_2}{2}} \int_{\frac{-\Delta x_1}{2}}^{\frac{\Delta x_1}{2}} \rho\underline{v}\, d\gamma d\xi d\eta \qquad (13)$$

As this is for a deforming body, the velocity and density could also be changing in space. So, we again need to use Taylor's expansion of $\rho\underline{v}$ about the centroid. When we use the expansion and solve the integration similar to what was done for the body force, we get:

$$\overrightarrow{P} = \rho\underline{v}(\underline{x})\Delta V + o(\Delta V) \qquad (14)$$

For the rate of change of linear momentum, we need to expand $\rho\underline{a}$ using Taylor's expansion about the centroid. Here $\underline{a}$ represents acceleration which may also be changing within the cuboid. Thus

$$\begin{aligned}\frac{d}{dt}\overrightarrow{P} &= \frac{d}{dt}\int\int\int dm\,\underline{v} = \int\int\int dm\,\frac{d}{dt}\underline{v} = \int\int\int dm\,\underline{a} = \int\int\int(\rho dV)\underline{a} \\ &= \int_{\frac{-\Delta x_3}{2}}^{\frac{\Delta x_3}{2}}\int_{\frac{-\Delta x_2}{2}}^{\frac{\Delta x_2}{2}}\int_{\frac{-\Delta x_1}{2}}^{\frac{\Delta x_1}{2}}\rho\underline{a}\,d\gamma d\xi d\eta = \rho\underline{a}(\underline{x})\Delta V + o(\Delta V)\end{aligned} \tag{15}$$

Note that the total time derivative $\frac{d}{dt}$ could be moved inside the integral easily because we have written the first integral as an integration over mass domain which does not change due to the choice of our system: Newton's 2nd law is applied for an identifiable/fixed set of mass particles. In case, the first integral were over the volume, then we could not have moved the total time derivative easily inside the integral since the corresponding volume domain changes with time for a fixed/identifiable mass.

2.4 Final balance (start time: 33:15)

Substituting equations (10),(11) and (15) in Newton's second law given by equation (1), we get

$$\sum_{i=1}^{3}\frac{\partial\underline{\underline{\sigma}}}{\partial x_i}\underline{e}_i\Delta V + \underline{b}(\underline{x})\Delta V + o(\Delta V) = \rho\underline{a}(\underline{x})\Delta V + o(\Delta V). \tag{16}$$

The $o(\Delta V)$ terms can be combined together, i.e.,

$$\sum_{i=1}^{3}\frac{\partial\underline{\underline{\sigma}}}{\partial x_i}\underline{e}_i\Delta V + \underline{b}(\underline{x})\Delta V + o(\Delta V) = \rho\underline{a}(\underline{x})\Delta V. \tag{17}$$

This is Newton's second law applied to a cuboid volume. Let us now divide both sides by ΔV and shrink the volume of the cuboid to its center ($\Delta V \to 0$).

$$\lim_{\Delta V\to 0}\left[\sum_{i=1}^{3}\frac{\partial\underline{\underline{\sigma}}}{\partial x_i}\underline{e}_i + \underline{b}(\underline{x}) + \frac{o(\Delta V)}{\Delta V} - \rho\underline{a}(\underline{x})\right] = \underline{0} \tag{18}$$

We can do this shrinking because the above equation is valid for any cuboid irrespective of its volume. Now, using equation (12), the term containing $o(\Delta V)$ vanishes and we finally get:

$$\boxed{\sum_{i=1}^{3}\frac{\partial\underline{\underline{\sigma}}}{\partial x_i}\underline{e}_i + \underline{b} - \rho\underline{a} = \underline{0}} \rightarrow \textbf{Linear Momentum Balance(LMB)} \tag{19}$$

This equation is called Linear Momentum Balance equation because we get it by applying Newton's second law which relates the rate of change of linear momentum of a body to the net external force acting on the body.

2.5 Representation in a coordinate system (start time: 38:33)

Let us write the LMB equation in $(\underline{e}_1 - \underline{e}_2 - \underline{e}_3)$ coordinate system. For the first three terms, $\underline{\underline{\sigma}}$ will be represented by a matrix. When we take the derivative of a matrix by a scalar, we take the derivative of each term in the matrix. Then we need to multiply by column of $\underline{e}_i$ in $(\underline{e}_1 - \underline{e}_2 - \underline{e}_3)$ coordinate system (e.g., in this coordinate system, $\underline{e}_1$ will simply be $[1\ \ 0\ \ 0]^T$). Following this way, we finally obtain

$$\begin{bmatrix} \frac{\partial \sigma_{11}}{\partial x_1} \\ \frac{\partial \tau_{21}}{\partial x_1} \\ \frac{\partial \tau_{31}}{\partial x_1} \end{bmatrix} + \begin{bmatrix} \frac{\partial \tau_{12}}{\partial x_2} \\ \frac{\partial \sigma_{22}}{\partial x_2} \\ \frac{\partial \tau_{32}}{\partial x_2} \end{bmatrix} + \begin{bmatrix} \frac{\partial \tau_{13}}{\partial x_3} \\ \frac{\partial \tau_{23}}{\partial x_3} \\ \frac{\partial \sigma_{33}}{\partial x_3} \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} - \begin{bmatrix} \rho a_1 \\ \rho a_2 \\ \rho a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{20}$$

We get in total three scalar equations from the three rows of the above matrix equation, i.e.,

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} + \frac{\partial \tau_{13}}{\partial x_3} + b_1 = \rho a_1 \tag{21}$$

$$\frac{\partial \tau_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \tau_{23}}{\partial x_3} + b_2 = \rho a_2 \tag{22}$$

$$\frac{\partial \tau_{31}}{\partial x_1} + \frac{\partial \tau_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + b_3 = \rho a_3 \tag{23}$$

Equation (19) is the tensor form of the Linear Momentum Balance and equations (21), (22) and (23) represent it in $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ coordinate system. The first of these (equation (21)) is actually just the balance of forces along $\underline{e}_1$. This is because all the terms present in this equation act in $\underline{e}_1$ direction. Similar interpretation can be made for the remaining two equations (22) and (23). There is an easy way to remember these scalar equations: for the k^{th} scalar equation, the first subscript of the stress components will always be 'k' as it denotes the direction. The second subscript is for the plane on which the traction is being considered. So, it will take all the three values (1,2,3) in each equation and the derivative of the stress component has to be taken with respect to x_i corresponding to the second subscript.
These equations contain derivatives in space. Thus, we would also need boundary conditions to solve them. The solution so obtained would give us the distribution of stress components in the body.

3 Angular Momentum Balance (start time: 47:50)

We know from first year mechanics course that a rigid body has to also obey the balance of angular momentum apart from the Newton's second law. For a rigid body, net external torque on it equals the rate of change of its angular momentum. We'll use $\underline{T}$ to denote Torque and $\overrightarrow{H}$ to represent angular momentum. Let 'O' denote a fixed point and 'cm' the center of mass. We can do the balance of angle momentum either about a fixed point or about the centre of mass according to which[1]:

$$\sum \underline{T}^{ext}_{/O,cm} = \frac{d}{dt}(\overrightarrow{H}_{/O,cm}) \tag{24}$$

[1] If we do the balance of angular momentum about a moving point, we get an extra term in the equation

We will carry out our derivation about the centre of mass of the cuboid (i.e. $\underline{x}$). So, we have to first find the net external torque about this point which again comes from traction on the six faces and body force.

3.1 Traction contribution (start time: 50:54)

Let us begin with torque due to traction on the plane with normal $\underline{e}_1$ (denoted by $\underline{T}^1_{/\underline{x}}$). For an arbitrary point $\underline{y}$ on this face, the arm of the force $\underline{r}$ will be $\underline{y}-\underline{x}$ and force on a small area around $\underline{y}$ would be $t^1 dA$. Hence

$$\begin{aligned}
\underline{T}^1_{/\underline{x}} &= \int_{\frac{-\Delta x_2}{2}}^{\frac{\Delta x_2}{2}} \int_{\frac{-\Delta x_3}{2}}^{\frac{\Delta x_3}{2}} (\underline{y}-\underline{x}) \times \underline{t}^1 d\xi d\eta \\
&= \int_{\frac{-\Delta x_2}{2}}^{\frac{\Delta x_2}{2}} \int_{\frac{-\Delta x_3}{2}}^{\frac{\Delta x_3}{2}} (\frac{\Delta x_1}{2}\underline{e}_1 + \xi \underline{e}_2 + \eta \underline{e}_3) \times \left[\underline{\underline{\sigma}} + \frac{\partial \underline{\underline{\sigma}}}{\partial x_1}\frac{\Delta x_1}{2} + \frac{\partial \underline{\underline{\sigma}}}{\partial x_2}\xi + \frac{\partial \underline{\underline{\sigma}}}{\partial x_3}\eta + ...\right]\underline{e}_1 d\xi d\eta \\
&= \underline{e}_1 \times \underline{\underline{\sigma}}\,\underline{e}_1 \frac{\Delta V}{2} + \underline{e}_1 \times \frac{\partial \underline{\underline{\sigma}}}{\partial x_1}\underline{e}_1 \Delta x_1 \frac{\Delta V}{2} + ... \qquad (25)
\end{aligned}$$

We have again used Taylor's expansion here about the center of the cuboid. In the integration above, the first term within the small bracket crossed with first two terms within the square brackets will be constants and can come out of the integration. So, their integration just gives surface area of $\underline{e}_1$ face for these terms. Other terms will either cancel or will be of lower order than ΔV. In fact, the second term in the final expression is also of lower order than ΔV as it is multiplied by Δx_1 along with ΔV. So, this term together with all terms after it can be clubbed together as $o(\Delta V)$, i.e.,

$$\underline{T}^1_{/\underline{x}} = \underline{e}_1 \times \underline{\underline{\sigma}}\,\underline{e}_1 \frac{\Delta V}{2} + o(\Delta V). \qquad (26)$$

Doing a similar analysis for $-\underline{e}_1$ plane:

$$\begin{aligned}
\underline{T}^{-1}_{/\underline{x}} &= \int_{\frac{-\Delta x_2}{2}}^{\frac{\Delta x_2}{2}} \int_{\frac{-\Delta x_3}{2}}^{\frac{\Delta x_3}{2}} (\frac{-\Delta x_1}{2}\underline{e}_1 + \xi \underline{e}_2 + \eta \underline{e}_3) \times \left[\underline{\underline{\sigma}} - \frac{\partial \underline{\underline{\sigma}}}{\partial x_1}\frac{\Delta x_1}{2} + \frac{\partial \underline{\underline{\sigma}}}{\partial x_2}\xi + \frac{\partial \underline{\underline{\sigma}}}{\partial x_3}\eta + ...\right](-\underline{e}_1) d\xi d\eta \\
&= \underline{e}_1 \times \underline{\underline{\sigma}}\,\underline{e}_1 \frac{\Delta V}{2} + o(\Delta V). \qquad (27)
\end{aligned}$$

Adding torques on $\underline{e}_1$ and $-\underline{e}_1$ faces, we get

$$\underline{T}^1_{/\underline{x}} + \underline{T}^{-1}_{/\underline{x}} = \underline{e}_1 \times \underline{\underline{\sigma}}\,\underline{e}_1 \Delta V + o(\Delta V). \qquad (28)$$

Doing similar analysis for the remaining four faces, we obtain

$$\underline{T}^{\text{tra}}_{/\underline{x}} = \sum_{i=1}^{3} (\underline{e}_i \times \underline{\underline{\sigma}}\,\underline{e}_i)\Delta V + o(\Delta V) \qquad (29)$$

3.2 Body force contribution (start time: 59:17)

Now, consider torque due to body force about the centroid. The point $\underline{y}$ will now represent a general point in the volume of the cuboid. The volumetric integration together with Taylor's expansion leads to

$$\begin{aligned}
\underline{T}^{\text{body force}}_{/\underline{x}} &= \iiint_V (\underline{y}-\underline{x}) \times \underline{b}(\underline{y})\, d\gamma d\xi d\eta \\
&= \iiint_V (\gamma \underline{e}_1 + \xi \underline{e}_2 + \eta \underline{e}_3) \times \left[\underline{b}(\underline{x}) + \frac{\partial \underline{b}}{\partial x_1}\gamma + ... \right] d\gamma d\xi d\eta \\
&= \underline{e}_1 \times \underline{b} \iiint_V \gamma dV + \underline{e}_2 \times \underline{b} \iiint_V \xi dV + \underline{e}_3 \times \underline{b} \iiint_V \eta dV + ... \qquad (30)
\end{aligned}$$

Noting the formula for centroid $(\bar{x}, \bar{y}, \bar{z})$ for an arbitrary volume given by

$$\bar{x} = \frac{\int_V x\, dV}{V},\ \bar{y} = \frac{\int_V y\, dV}{V},\ \bar{z} = \frac{\int_V z\, dV}{V}, \qquad (31)$$

equation (30) becomes

$$\underline{T}^{\text{body force}}_{/\underline{x}} = \underline{e}_1 \times \underline{b}\ x_{1c}\ \Delta V + \underline{e}_2 \times \underline{b}\ x_{2c}\ \Delta V + \underline{e}_3 \times \underline{b}\ x_{3c}\ \Delta V.... \qquad (32)$$

Here x_{1c} represents x_1 coordinate of centroid with respect to the cuboid's centroid itself. Thus, it will be zero. Similarly, all other terms are either going to vanish or will be of the order smaller than the cuboid's volume. So, we can club all the terms together to obtain

$$\underline{T}^{\text{body force}}_{/\underline{x}} = o(\Delta V). \qquad (33)$$

Thus, the body force contribution to the torque has all terms of the order less than the volume of the cuboid. We will continue from this point in the next lecture.

4 Objective questions to practise

1. Suppose no body force acts on a body. Further, the body is under static equilibrium

   (a) The stress tensor in that body remains constant throughout
   (b) The stress matrix in the body (for a given coordinate system) remains constant throughout
   (c) More information is needed
   (d) None of these

2. While deriving the stress equilibrium equations, we shrink the cuboid volume to a point because

   (a) Our goal is to obtain stress equilibrium equations which hold at every point in the body
   (b) Newton's law does not apply to a finite cuboid, it only applies to particles.
   (c) The problem is vague
   (d) None of these

3. The stress equilibrium equations tell us the

   (a) Stress at a point in a body.
   (b) Traction at a point in a body.
   (c) Both stress and traction at a point in a body.
   (d) Distribution of stresses at every point in a body.