Solid Mechanics
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Lecture - 9
Mohr's Circle

Abstract

In the last two lectures, we learnt about planes on which normal and shear components of traction are maximized/minimized. In this lecture, we will discuss about Mohr's circle, a graphical technique to find components of traction on arbitrary planes at a point.

1 Conditions for applying Mohr's Circle (start time: 00:26)

Mohr's circle is a graphical way to find normal and shear components of traction on arbitrary planes. The only restriction is that the plane normal has to be perpendicular to one of the principal normals. Accordingly, let us consider a coordinate system such that the third coordinate axis is along one of the principal normals (the rest two coordinate axes need not be along any principal direction, also see Figure 1) and we are looking at planes whose normals are perpendicular to the third principal direction. Then, all such plane normals will have $n_3 = 0$. On such planes, we want to find the normal and shear components of traction (see Figure 2). The stress matrix in this coordinate system will be such that its third column will be formed by traction on the plane along third coordinate axis. But that being a principal plane, the third column will not have any shear component. Hence, the third column will have its last entry nonzero but the other two zero. Symmetry of stress matrix will further force the first two entries of last row to be zero, i.e.,

$$[\underline{\sigma}] = \begin{bmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & \times \end{bmatrix}. \tag{1}$$

Also, the normal vectors of the plane on which we want to obtain normal and shear components will look like

$$[\underline{n}] = \begin{bmatrix} \times \\ \times \\ 0 \end{bmatrix}. \tag{2}$$

2 Deriving formulas for normal and shear components on such planes (start time: 04:19)

Let us draw a cuboid element at the point of interest with its face normals along the coordinate system (see Figure 1). As discussed earlier, the third coordinate axis ($\underline{e}_3$) is along the third principal direction. We will call $\underline{e}_1$, $\underline{e}_2$ and $\underline{e}_3$ axes as x, y and z axes, respectively. On the $\underline{e}_1$ plane, we

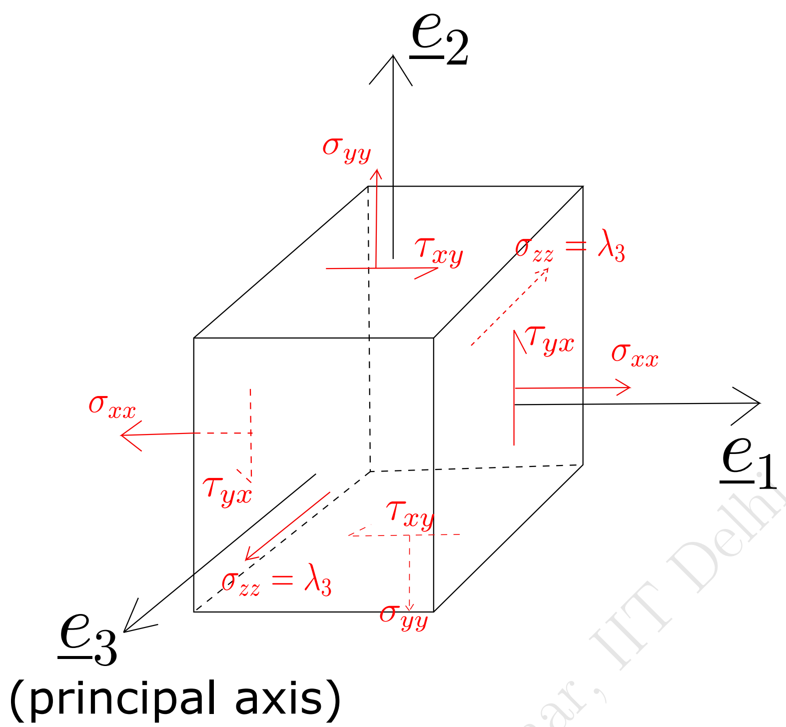

Figure 1: The cuboid with face normals along the three coordinate axes. The third coordinate axis is also a principal direction. The traction components are also shown.

have normal component of traction denoted as σ_{xx} and shear component of traction denoted as τ_{yx} pointing towards y axis. The third component (τ_{zx}) is absent as per the stress representation (1). Similarly, on the $\underline{e}_2$ plane, we have τ_{xy} (same as τ_{yx}) and on the $\underline{e}_3$ plane, we have σ_{zz} (equal to λ_3) only. On the $-\underline{e}_1$ plane, the normal component is σ_{xx} in $-x$ direction and shear component is τ_{yx} in $-y$ direction and likewise for other two negative planes.

2.1 Reducing the cuboidal representation of state of stress to a square (start time: 07:53)

There is another simpler way to draw such a state of stress for whom there are no shear components in the third direction: we can just draw a square instead of a cube where the sides of the square denote the faces of the cuboid as shown in Figure 2. The right edge of the square represents $+\underline{e}_1$ face. Similarly, the top edge represents $+\underline{e}_2$ face and the plane of the square itself represents $\underline{e}_3$ face. So, the plane containing the square has just one traction component σ_{zz}. We use a dot enclosed by a circle to denote the traction component coming out of the plane as shown in Figure 2. On the edges of the square, we have both the shear and the normal components of traction. On the right edge, representing $\underline{e}_1$ face, we have σ_{xx} and τ_{yx} (which is equal to τ_{xy} in magnitude). So, we will just write τ_{xy} to denote both τ_{yx} and τ_{xy}. On the top edge, representing $\underline{e}_2$ face, we have τ_{xy} and σ_{yy}. Keep in mind that such a reduce to square to represent stress matrix is possible only when the third coordinate axis lies along one of the principal directions.

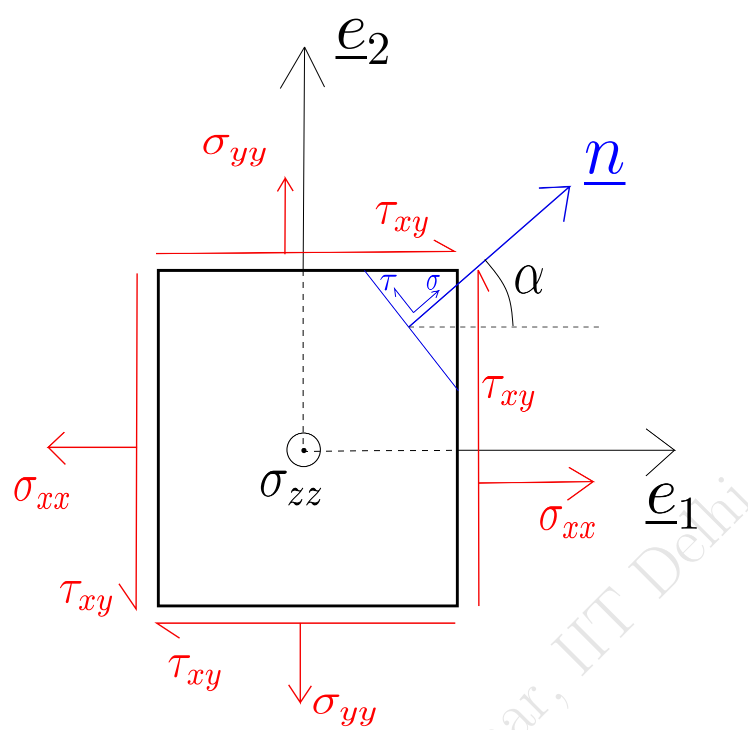

Figure 2: A square with its sides representing the faces of the cuboid. Traction components are also drawn.

2.2 Trigonometric formula for σ and τ (start time: 11:04)

Now, our goal is to calculate the normal and shear components of traction on planes whose normals are perpendicular to $\underline{e}_3$. The blue line in Figure 2 shows a general plane of such kind. The normal to this plane is represented by $\underline{n}$ and assume that it makes an angle α with $\underline{e}_1$ axis. To get to this arbitrary plane $\underline{n}$, we can rotate our $\underline{e}_1$ plane by α about $\underline{e}_3$ axis. The column representation of $\underline{n}$ in this coordinate system $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ will be

$$[\underline{n}] = \begin{bmatrix} cos(\alpha) \\ sin(\alpha) \\ 0 \end{bmatrix}. \tag{3}$$

An important point to note here is that we have assumed that the direction of τ on this plane makes $90°$ (anti clockwise) from the normal vector $\underline{n}$ (also see Figure 2). Now σ will be given by

$$\begin{aligned} \sigma = \left([\underline{\sigma}]\ [\underline{n}] \cdot [\underline{n}] \right) &= \left(\begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix} \begin{bmatrix} cos(\alpha) \\ sin(\alpha) \\ 0 \end{bmatrix} \right) \cdot \begin{bmatrix} cos(\alpha) \\ sin(\alpha) \\ 0 \end{bmatrix} \\ &= \sigma_{xx} cos^2(\alpha) + 2\tau_{xy} sin(\alpha) cos(\alpha) + \sigma_{yy} sin^2(\alpha). \end{aligned} \tag{4}$$

To get τ, we need to first represent the direction along which it is acting which we denote by $\underline{n}^{\perp}$. If we look at Figure 2, we can find the angle that $\underline{n}^{\perp}$ makes with all the three axes. The representation of $\underline{n}^\perp$ will thus be

$$\underline{n}^\perp = \begin{bmatrix} -sin(\alpha) \\ cos(\alpha) \\ 0 \end{bmatrix}. \tag{5}$$

Now, to get τ, we need to take the component of total traction ($[\underline{\sigma}]\ [\underline{n}]$) on this plane along the $\underline{n}^\perp$ direction, i.e.,

$$\begin{aligned} \tau = \Big([\underline{\sigma}]\ [\underline{n}] \Big) \cdot [\underline{n}^\perp] &= \left(\begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix} \begin{bmatrix} cos(\alpha) \\ sin(\alpha) \\ 0 \end{bmatrix} \right) \cdot \begin{bmatrix} -sin(\alpha) \\ cos(\alpha) \\ 0 \end{bmatrix} \\ &= -\sigma_{xx} cos(\alpha) sin(\alpha) - \tau_{xy} sin^2(\alpha) + \tau_{xy} cos^2(\alpha) + \sigma_{yy} cos(\alpha) sin(\alpha) \quad (6) \end{aligned}$$

Upon doing some algebraic manipulation in equation (4), we get

$$\sigma = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \sigma_{xx}\left(cos^2(\alpha) - \frac{1}{2}\right) + \sigma_{yy}\left(sin^2(\alpha) - \frac{1}{2}\right) + 2\tau_{xy} sin(\alpha) cos(\alpha). \tag{7}$$

Further, using the following trigonometric identities:

$$cos(2\alpha) = cos^2(\alpha) - sin^2(\alpha) = 2cos^2(\alpha) - 1 = 1 - 2sin^2(\alpha), \tag{8}$$
$$sin(2\alpha) = 2sin(\alpha)cos(\alpha), \tag{9}$$

we obtain

$$\begin{aligned} \sigma &= \frac{\sigma_{xx} + \sigma_{yy}}{2} + \sigma_{xx}\frac{cos(2\alpha)}{2} - \sigma_{yy}\frac{cos(2\alpha)}{2} + \tau_{xy} sin(2\alpha) \\ &= \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2} cos(2\alpha) + \tau_{xy} sin(2\alpha) \quad (10) \end{aligned}$$

Similar rearrangements and simplifications using trigonometric identities (8) and (9) in equation (6) gives us

$$\tau = -\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right) sin(2\alpha) + \tau_{xy} cos(2\alpha) \tag{11}$$

2.3 Introducing Graphical Parameters (start time: 23:42)

If we look at equations (10) and (11), we notice that σ and τ both have $cos(2\alpha)$ and $sin(2\alpha)$ terms. Let us define a scalar R as follows:

$$R = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} \tag{12}$$

We can now think of a right angled triangle with hypotenuse R and the two perpendicular arms as $\dfrac{\sigma_{xx} - \sigma_{yy}}{2}$ and τ_{xy} as shown in Figure 3. Let us denote the angle between the hypotenuse and the

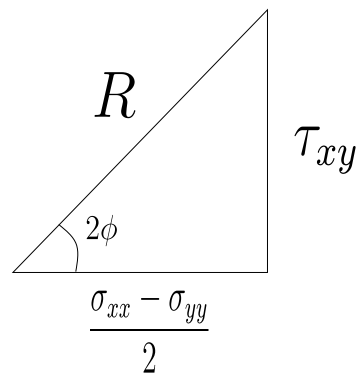

Figure 3: A right angled triangle with the two arms as $\dfrac{\sigma_{xx}-\sigma_{yy}}{2}$ and τ_{xy}.

base as 2ϕ. Thus, from basic trigonometry, we see that

$$cos(2\phi) = \frac{\sigma_{xx}-\sigma_{yy}}{2R}, \quad sin(2\phi) = \frac{\tau_{xy}}{R}, \quad tan(2\phi) = \frac{2\tau_{xy}}{\sigma_{xx}-\sigma_{yy}} \tag{13}$$

Using equation (13) in equations (10) and (11), we get

$$\begin{aligned} \sigma &= \frac{\sigma_{xx}+\sigma_{yy}}{2} + R\Big(cos(2\phi)cos(2\alpha) + sin(2\phi)sin(2\alpha)\Big) \\ \tau &= R\Big[-cos(2\phi)sin(2\alpha) + sin(2\phi)sin(2\alpha)\Big] \end{aligned} \tag{14}$$

Upon further using following trigonometric identities:

$$cos(a-b) = cos(a)cos(b) + sin(a)sin(b), \tag{15}$$
$$sin(a-b) = sin(a)cos(b) - cos(a)sin(b), \tag{16}$$

we get

$$\boxed{\begin{aligned} \sigma &= \frac{\sigma_{xx}+\sigma_{yy}}{2} + Rcos(2\phi-2\alpha) \\ \tau &= Rsin(2\phi-2\alpha) \end{aligned}} \tag{17}$$

These are the formulae obtained for getting σ and τ on a plane making an angle α with $\underline{e}_1$ axis. For a given stress matrix, we can find out R and ϕ using equations (12) and (13) respectively. Then, using equation (17), we can find σ and τ on the plane which is obtained by rotating $\underline{e}_1$ by angle α about $\underline{e}_3$ axis. Let us try to represent the two formulas in the box graphically.

3 Graphical representation of the derived formulas (start time: 31:07)

We need to see what equation (17) means. Let us think of a $\sigma-\tau$ plane and plot σ and τ for each α in this plane. The plane with σ on the x-axis and τ on the y-axis is shown in Figure 4.

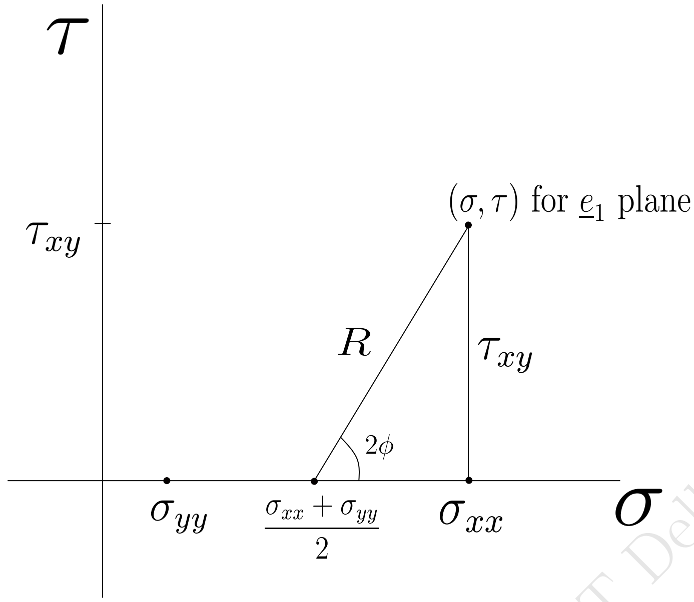

Figure 4: $\sigma - \tau$ plane with various parameters plotted on it.

From equation (17), we can see that by plotting all points, we will get a circle centered on the σ axis at $\left(\frac{\sigma_{xx}+\sigma_{yy}}{2}, 0\right)$. Let us start by placing σ_{xx} and σ_{yy} on the σ axis. The center of the circle is thus at the mid point of the two points. We then place τ_{xy} on the τ axis. Now, we can plot the point (σ_{xx}, τ_{xy}) which corresponds to $\underset{\sim}{e}_1$ plane. If we join this point with the center, the line obtained will give us the radius of the circle. This is because the circle has to pass through the point corresponding to $\underset{\sim}{e}_1$ plane: the circle is the locus of all (σ, τ) when α is varied and the point corresponding to $\underset{\sim}{e}_1$ plane is obtained for $\alpha = 0$. We can also verify that the radius R that we obtained graphically also matches with the formula for R in equation (12).[1].

3.1 Mohr's Circle (start time: 34:09)

Once we have obtained the radius and the center of the circle, we can draw the complete circle as shown in Figure 5. This circle is called the Mohr's circle. Upon comparing the right angled triangle in Figure 5 with Figure 3, we see that the angle that the line from center to the point corresponding to $\underset{\sim}{e}_1$ plane makes with the σ axis is 2ϕ. To find σ and τ on any arbitrary plane (for general α), let us look at equation (17): the angle in the cosine and sine terms there is $(2\phi - 2\alpha)$. So, for a general plane, the argument in the trigonometric function there reduces by 2α. So, the radial line from the center to the point corresponding to the α-plane on Mohr's circle should be at an angle of $(2\phi - 2\alpha)$ from the x-axis. Thus, we can obtain the point corresponding to α-plane on Mohr's circle by going in the clockwise direction by angle 2α from the $\underset{\sim}{e}_1$-plane point as shown in Figure 5.

We summarize below the steps involved in drawing the Mohr's circle and finding the point coresponding to α-plane:

[1]The right angled traingles in Figures 3 and 4 turn out to be the same!

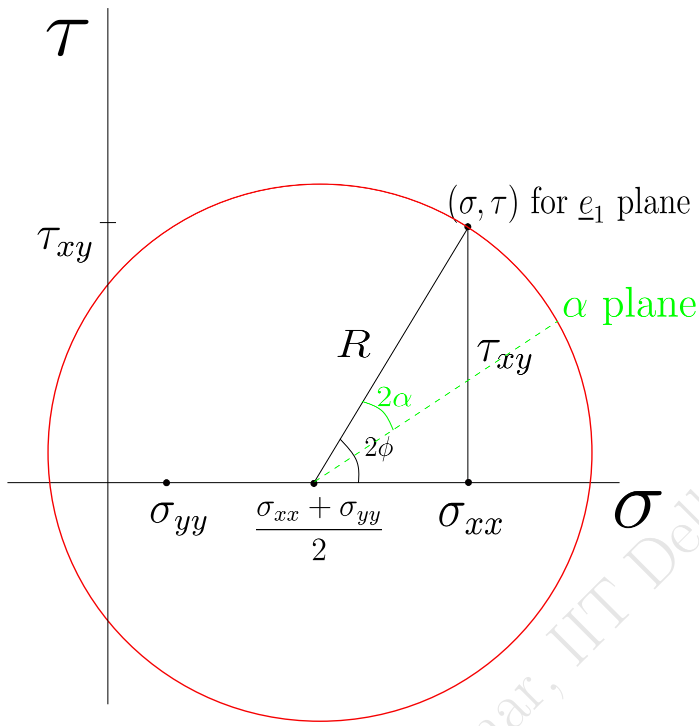

Figure 5: Mohr's circle plot

1. Draw the center of the circle at $\left(\dfrac{\sigma_{xx}+\sigma_{yy}}{2}, 0\right)$.
2. Draw (σ, τ) for $\underset{\sim}{e}_1$ plane, i.e., the point (σ_{xx}, τ_{xy}).
3. the line joining the center and the point for $\underset{\sim}{e}_1$ plane forms the radius of the circle.
4. With center and radius known, draw your circle!
5. To find (σ, τ) for α-plane, rotate the radial line of $\underset{\sim}{e}_1$ plane by 2α clockwise.

Notice that the normal to the required plane made an angle α with $\underset{\sim}{e}_1$ in the counter clockwise direction (also see Figure 2). But on the Mohr's circle, we draw that point by rotating by 2α in the clockwise direction from the point corresponding to the $\underset{\sim}{e}_1$ plane. This is because in equation (17), we have 2α with a minus sign in the trigonometric functions. So, clockwise rotation of the plane corresponds to counter-clockwise rotation in the Mohr's circle and vice versa.

4 Sign convention while using Mohr's cirle (start time: 42:40)

Let us draw the Mohr's circle again as shown in Figure 6. We know the point corresponding to $\underset{\sim}{e}_1$ plane. To get to $\underset{\sim}{e}_2$ plane, α should be 90° in the counter-clockwise direction. So, in the Mohr's circle, we need to go 180° in the clockwise direction from $\underset{\sim}{e}_1$ plane. Thus, we get $\underset{\sim}{e}_2$ plane at the

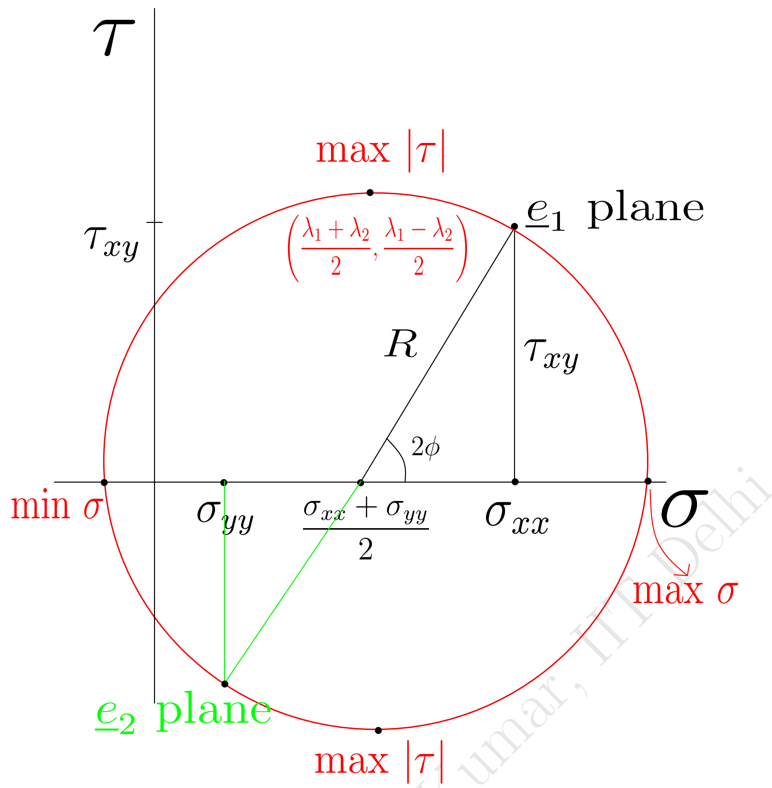

Figure 6: Mohr's circle with important quantities marked on it.

diametrically opposite point with respect to $\underline{e}_1$-plane point. The two right angled triangles in Figure 6 are similar and thus, we get σ at this point as σ_{yy} as required but τ as $-\tau_{xy}$. However, in Figure 2, we see that on $\underline{e}_2$ plane, shear component is τ_{xy}. So, why are we getting $-\tau_{xy}$ from the Mohr's circle? This is because of our convention for the shear component of traction. In Figure 2, we had defined positive τ when we go 90° in the counter clockwise direction from $\underline{n}$. So, to get shear component on $\underline{e}_2$ plane, we go 90° in the anti-clockwise direction from $\underline{e}_2$ direction and thus, get to the $-\underline{e}_1$ direction. So, Mohr's circle is giving us τ on $\underline{e}_2$ plane in the $-\underline{e}_1$ direction whereas τ_{xy}, by definition, is the shear component in the $+\underline{e}_1$ direction. Therefore, Mohr's circle gives us $-\tau_{xy}$ as the shear traction on $\underline{e}_2$ plane. Essentially, we get the negative sign because $\underline{n}^{\perp}$ for $\underline{e}_2$ plane is along $-\underline{e}_1$ direction by our convention.

5 Other conclusions that can be drawn using Mohr's circle (start time: 48:57)

We can also get the maximum and minimum values of σ and τ using Mohr's circle. The maximum and minimum values of σ are attained on the σ-axis itself. These are plotted in Figure 6. The maximum and minimum values of τ are at the top and bottom of the circle respectively as highlighted in Figure 6. We can verify that these values match with the ones derived in the last lecure. Suppose that the principal stress components λ_1, λ_2 and λ_3 are defined such that $\lambda_1 > \lambda_2 > \lambda_3$. Thus, λ_1 and λ_2 will correspond to the points of maximum σ and minimum σ respectively. From the circle,

λ_1 and λ_2 will be obtained by adding and subtracting R to the σ for center respectively, i.e.,

$$\lambda_1 = \frac{\sigma_{xx} + \sigma_{yy}}{2} + R \tag{18}$$

$$\lambda_2 = \frac{\sigma_{xx} + \sigma_{yy}}{2} - R \tag{19}$$

This allows us to get the value of principal stress components directly from the Mohr's circle: otherwise one has to solve an eigenvalue problem. Writing the center of the circle in terms of principal stress components, we obtain

$$\text{Center}=\left(\frac{\lambda_1 + \lambda_2}{2}, 0\right) \tag{20}$$

We can also get the radius of the circle in terms of principal stress components by subtracting equation (19) from equation (18), i.e.,

$$R = \frac{\lambda_1 - \lambda_2}{2} \tag{21}$$

Notice that in the previous lecture, we had derived the maximum value of shear component of traction to be $\frac{\lambda_1 - \lambda_2}{2}$. And from Mohr's circle too, we get $\tau_{\max} = R = \frac{\lambda_1 - \lambda_2}{2}$. From the Mohr's cicrle, we can also see that σ corresponding to the point where we have maximum shear will be the same as σ for the center of the circle, i.e. $\frac{\lambda_1 + \lambda_2}{2}$. This is the same value that we had derived in the previous lecture. We have thus verified our results of the Mohr's circle.

6 Objective questions to practise

1. Suppose the stress tensor is proportional to identity. What will be the radius of the Mohr's circle for such a stress tensor?

   (a) zero
   (b) equal to the value of diagonal entry in the stress matrix
   (c) more information is needed
   (d) None of these

2. Think of a stress matrix such that it has no diagonal component and its third column corresponds to traction on one of the principal planes. Which of the following holds true?

   (a) The center of the corresponding Mohr's circle will be at origin
   (b) Two of the principal stress components will be equal in magnitude but of opposite sign.
   (c) The normal component of traction on the plane where shear traction is maximized is zero.
   (d) None of these

3. A point on the Mohr's circle, which is at an angle of θ (clockwise) from the point corresponding to the $\underline{e}_1$ plane, will correspond to

   (a) The plane at an angle of 2θ (anticlockwise) from the $\underline{e}_1$ plane.
   (b) The plane at an angle of 2θ (clockwise) from the $\underline{e}_1$ plane.
   (c) The plane at an angle of $\frac{\theta}{2}$ (anticlockwise) from the $\underline{e}_1$ plane.
   (d) None of the above

4. Suppose we are given the radius of the Mohr's circle. Which of the following can we find using this information?

   (a) Principal stress components
   (b) Maximum shear component of traction
   (c) Magnitude of total traction at any plane
   (d) Difference of the normal components of traction on $\underline{e}_1$ and $\underline{e}_2$ plane.

5. At a point inside a fluid body:

   (a) Every plane is a principal plane.
   (b) All principal stress components are equal.
   (c) The Mohr's circle reduces to a point.
   (d) None of the above

6. If at a point, we choose a coordinate system such that none of the coordinate axes is aligned along the principal plane's normal, then:
   (a) At that point, principal planes do not exist.
   (b) We cannot apply the concept of Mohr's circle to the stress matrix in the above mentioned coordinate system.
   (c) We can apply the concept of Mohr's circle but cannot find the principal planes.
   (d) None of the above

7. Suppose we first choose a coordinate system such that one of the axes is normal to one of the principal planes and draw the Mohr's circle. Now, we choose another coordinate system such that one of the coordinate axis is aligned along a different principal plane's normal and draw the Mohr's circle. Which of the following is true?
   (a) The radius of the two Mohr circles must be same.
   (b) The radius of the two Mohr circles can be different.
   (c) The center of the two Mohr circles must be same.
   (d) None of the above

8. What are the conditions on planes where the concepts of Mohr's circle can be applied?
   (a) No condition is required.
   (b) The normal to these set of planes must be inclined at $45°$ to one of the principal stress directions.
   (c) The normal to these set of planes must be parallel to one of principal stress direction.
   (d) The normal to these set of planes must be all perpendicular to one of the principal stress direction.

9. The stress matrix for applying Mohr's Circle will look like (X represent any constant value)
   (a) $\begin{bmatrix} X & X & 0 \\ X & X & 0 \\ 0 & 0 & X \end{bmatrix}$
   (b) $\begin{bmatrix} X & X & X \\ X & X & X \\ 0 & 0 & X \end{bmatrix}$
   (c) $\begin{bmatrix} X & X & X \\ X & X & X \\ X & X & X \end{bmatrix}$
   (d) $\begin{bmatrix} X & 0 & 0 \\ 0 & X & X \\ 0 & X & X \end{bmatrix}$

10. If we have to go 90° counter-clockwise from e_1 plane to get to e_2 plane, then in Mohr's circle, we need to

    (a) go 180° clockwise from e_1 plane to get e_2 plane.
    (b) go 45° clockwise from e_1 plane to get e_2 plane.
    (c) go 180° counter-clockwise from e_1 plane to get e_2 plane.
    (d) go 45° counter-clockwise from e_1 plane to get e_2 plane.

11. The following state of stress exists at a point:

$$\sigma = \begin{bmatrix} -2 & 4 & 0 \\ 4 & 6 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

    The principal stresses are:

    (a) $(2+4\sqrt{2},\ 2-4\sqrt{2},\ 10)$
    (b) (-2, 6, 10)
    (c) more information is needed
    (d) none of these

12. For the state of stress in problem 11, what are the normal and shear components of traction on a plane whose normal lies in $(x-y)$ plane and makes an angle of 7.5° clockwise from x-axis?

    (a) $2(1+\sqrt{2}),\ 2\sqrt{6}$
    (b) 6, 4
    (c) $2(1+\sqrt{6}),\ 2\sqrt{2}$
    (d) none of these

13. Suppose the stress tensor is proportional to identity. What will be the radius of the Mohr's circle for such a stress tensor?

    (a) zero
    (b) equal to the value of diagonal entry in the stress matrix
    (c) more information is needed
    (d) None of these

14. The max value of shear component of traction equals

    (a) the radius of Mohr's circle
    (b) the center of radius circle

(c) diameter of Mohr's circle

(d) None of these

15. If we have to go 90° counter-clockwise from e_1 plane to get to e_2 plane, then in Mohr's circle, we need to

    (a) go 180° clockwise from e_1 plane to get to e_2 plane

    (b) go 45° clockwise from e_1 plane to get to e_2 plane

    (c) go 180° counter-clockwise from e_1 plane to get to e_2 plane

    (d) go 45° counter-clockwise from e_1 plane to get to e_2 plane