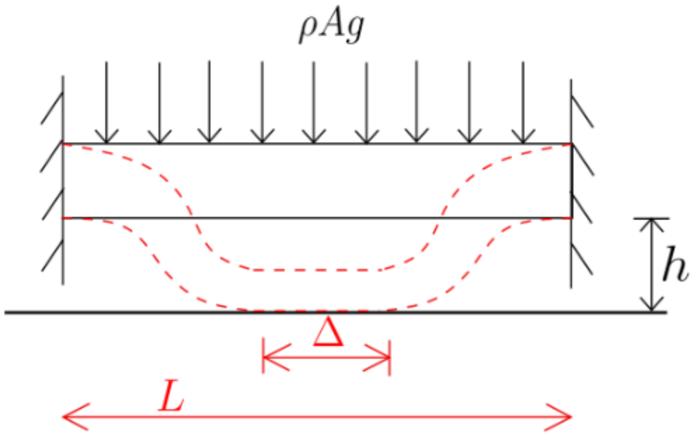
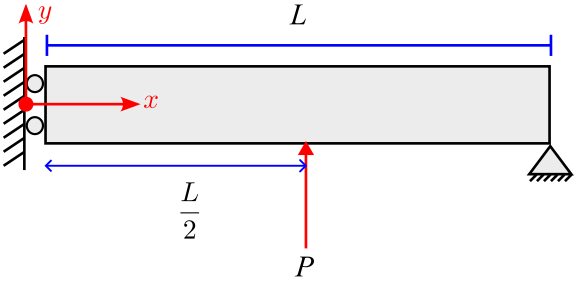
Tutorial 11

APL 104 - 2022 (Solid Mechanics)

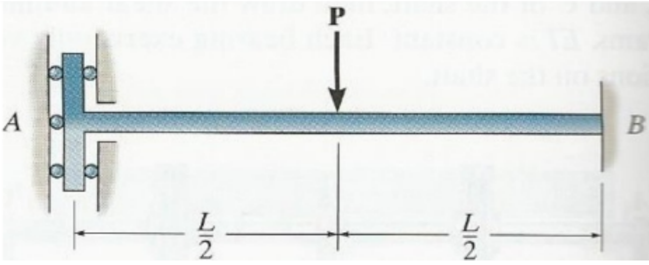
1. Consider a beam clamped at both its ends. The beam sags down due its own weight as shown in Figure 10, the distributed weight being $\rho A g$. However, the ground position (h below the beam) is such that the some part of the beam rests on the ground upon deformation while the remaining part just hangs. Find the length of the beam Δ which will rest on the ground.



2. Suppose a beam is kept with roller support at one end ($X = 0$) constrained to only move in y-direction and pinned at the other end ($X = L$) as shown. Beam is subjected to transverse load (P) at the middle of the beam. Find the deflection of the beam using Euler-Bernoulli beam theory.



3. For the beam shown, find the reaction moment and deflection at A (use Timoshenko beam theory).



4. Think of a beam which is clamped against both transverse deflection as well as rotation at both the ends (see figure).

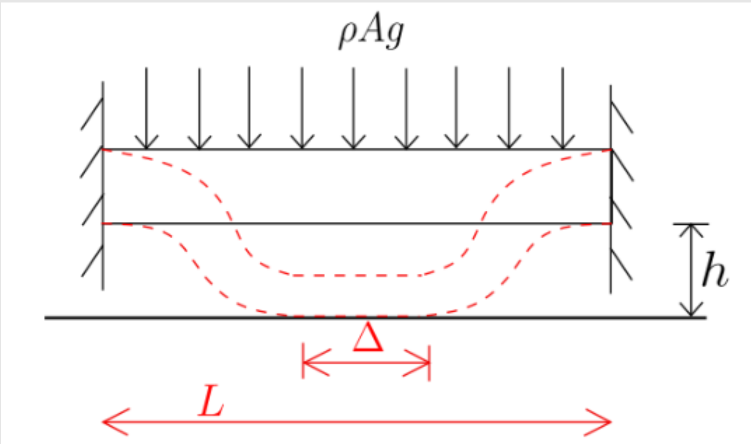
   (a) Write down the two governing equations to obtain transverse deflection of this beam if it behaves as a Timoshenko beam. Also obtain the boundary conditions.

   (b) Can you think of reducing the two equations in part (a) into a single ordinary differential equation in terms of just the transverse deflection? Similarly, also write down all the boundary condition such that the cross-sectional rotation variables doesn't show.

   (c) Can you deduce the equation which gives us the critical buckling load of the beam? (You don't have to solve it)

APL 104 Tutorial 11 solutions

Q1. Consider a beam clamped at both its ends. The beam sags down due its own weight as shown in Figure 10, the distributed weight being ρAg. However, the ground position (h below the beam) is such that the some part of the beam rests on the ground upon deformation while the remaining part just hangs. Find the length of the beam Δ which will rest on the ground.



Solution: Notice that the problem is symmetrical with respect to the center of the beam ($X = \frac{L}{2}$). Let us consider the left hanging part of the beam, i.e., from $X = 0$ to $X = \frac{L-\Delta}{2}$. We draw its free body diagram as shown in Figure 1. Bending moment $M(0)$ and shear

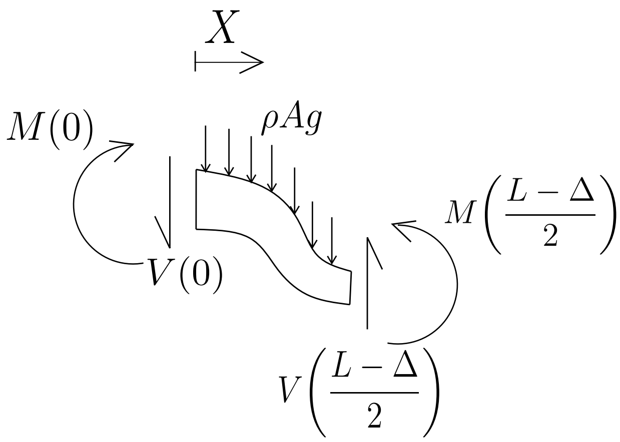


Figure 1: Free body diagram of the left hanging part of the beam.

force $V(0)$ act on its left-end cross-section (applied by the clamped end). A shear force and a bending moment also acts on the right-end of this part (applied by the remaining part of the beam). All of these four quantities are unknowns. To proceed with the Euler-Bernoulli beam equation

$$EI\frac{d^2y}{dX^2} = M(X)$$

we need to first find the bending moment profile $M(X)$. We thus cut a section at a distance X from the left end and draw the free body diagram of the right part as shown in Figure 2.

Shear force and bending moment act on left and right-ends whereas distributed load acts

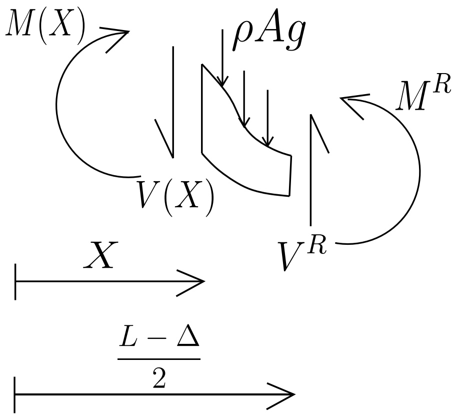


Figure 2: Free body diagram of a section of the hanging part of the beam at a distance X from the left end.

throughout its length. Its force balance gives us

$$V(X) = V^R - \rho A g\left(\frac{L-\Delta}{2} - X\right). \tag{1}$$

We then do moment balance about the centroid of its left-end cross-section which yields

$$\begin{aligned} -M(X) + M^R + V^R\left(\frac{L-\Delta}{2} - X\right) - \frac{\rho A g}{2}\left(\frac{L-\Delta}{2} - X\right)^2 &= 0 \\ \implies M(X) = M^R + V^R\left(\frac{L-\Delta}{2} - X\right) - \frac{\rho A g}{2}\left(\frac{L-\Delta}{2} - X\right)^2. \end{aligned} \tag{2}$$

Plugging this in equation (**??**), we get

$$\frac{d^2 y}{dX^2} = \frac{1}{EI}\left(M^R + V^R\left(\frac{L-\Delta}{2} - X\right) - \frac{\rho A g}{2}\left(\frac{L-\Delta}{2} - X\right)^2\right). \tag{3}$$

We now think of boundary conditions. As there are three additional unknown parameters (V^R, M^R, Δ), a total of five boundary conditions will be required. Out of these, two can be obtained as earlier from the clamped end at $X = 0$, i.e.,

$$y(0) = 0, \quad \frac{dy}{dX}(0) = 0. \tag{4}$$

We cannot use the clamped boundary condition of the original beam at $X = L$ because we are only analyzing the left hanging portion of the full beam. If we carefully observe, we can see that at $X = \frac{L-\Delta}{2}$, the beam starts to get into contact with the ground surface. Thus, the deflection of this point is $-h$, i.e.,

$$y\left(\frac{L-\Delta}{2}\right) = -h. \tag{5}$$

Also, as the ground is flat, the beam's deflection is uniform throughout the resting portion of the beam. Thus, the first and second derivatives of deflection in the resting portion will be zero which by continuity will also be zero at $X = \frac{L-\Delta}{2}$. Thus, we get our fourth boundary condition as

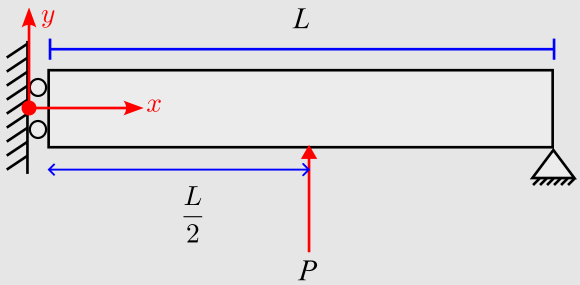
$$\frac{dy}{dX}\left(\frac{L-\Delta}{2}\right) = 0. \tag{6}$$

As the second derivative $\frac{d^2y}{dX^2}$ vanishes throughout in the resting region, the bending curvatue and hence the bending moment will be zero in the entire resting region of the beam. Thus, M^R which is the bending moment at the edge of the flat region must also be zero.[1] This gives us the final boundary condition, i.e.,

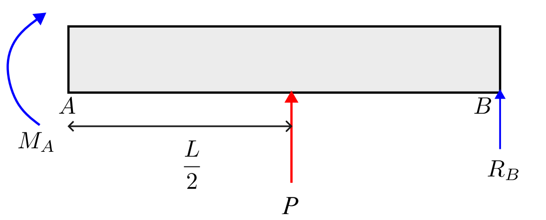
$$M^R = 0. \tag{7}$$

Using all the five boundary conditions in the EBT equation, we can solve for the deflection of the beam and also the length of the resting portion of the beam, i.e., Δ.

Q2. Suppose a beam is kept with roller support at one end ($X = 0$) constrained to only move in y-direction and pinned at the other end ($X = L$) as shown. Beam is subjected to transverse load (P) at the middle of the beam. Find the deflection of the beam using Euler-Bernoulli beam theory.



Solution: The constraint at the left end has frictionless rollers which allows free translation in the vertical direction but no rotation. So there is a reaction moment at the left end but no vertical reaction force. The right end is sitting on a pin support which allows free rotation but does not allow translation. Thus, at the right end, the reaction moment is zero but the vertical reaction force is non-zero. The free body diagram of the full beam thus looks as follows:



Since there are two unknown reactions in this case, we can determine them using the two static equilibirum equations, i.e., by doing force balance in y-direction and moment balance about (say) the left end.

$$+\uparrow \sum F_y = 0 \;\Rightarrow R_B = -P$$

[1]There can also be no jump discontinuity in bending moment at $X = \frac{L-\Delta}{2}$ because of line contact of beam with ground due to which the ground does not exert any reactive bending moment at this point.

$$\circlearrowleft\!\!+ \sum M_{\text{left end}} = 0 \;\Rightarrow -M_A + P\,\frac{L}{2} + \overset{-P}{\cancel{R_B}}\,L = 0 \Rightarrow M_A = -\frac{PL}{2}$$

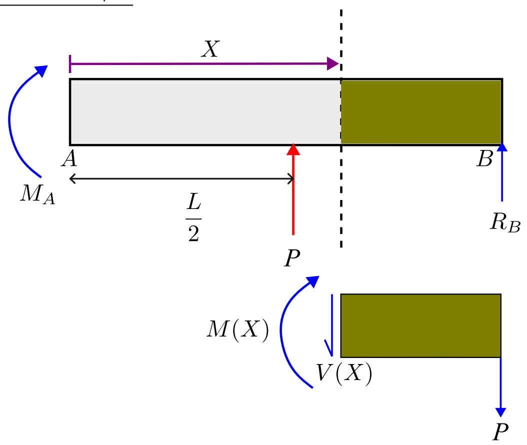
The reaction force R_B is $-$ve meaning it should act in the downward direction, and the moment M_A is $-$ve meaning that it should act in anti-clockwise sense (opposite to what is shown in the free-body diagram).

Consider the origin of the coordinate system to be at the left end A. The delfection equation for Euler-Bernoulli beam theory is given by

$$EI\frac{d^2y}{dX^2} = M\,(X)$$

To use this formula, we need to find the expression for $M\,(X)$. Note that $M(X)$ in the cross section will have different expressions for $X < \frac{L}{2}$ and $X \geq \frac{L}{2}$. So, we will consider the two cases separately:

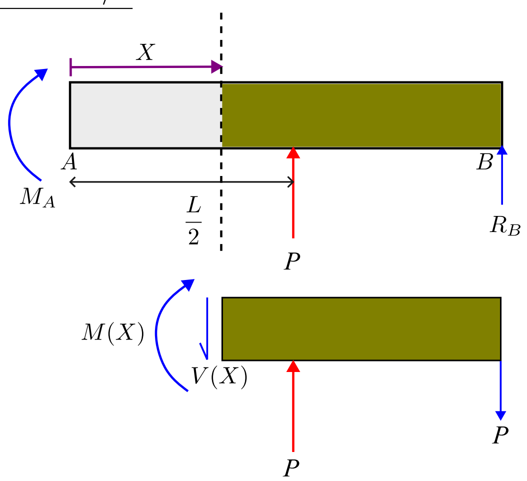
For $X \geq L/2$:



$$+\uparrow \sum F_y = 0 \;\Rightarrow\; V(X) = -P$$

$$\circlearrowleft\!\!+ \sum M_{\text{left end}} = 0$$
$$\Rightarrow\; -M\,(X) - P\,(L - X) = 0$$
$$\Rightarrow\; M\,(X) = -P\,(L - X)$$

For $X < L/2$:



$$+\uparrow \sum F_y = 0 \;\Rightarrow\; V\,(X) = -P + P = 0$$

$$\circlearrowleft\!\!+ \sum M_{\text{left end}} = 0$$
$$\Rightarrow\; -M\,(X) + P\left(\frac{L}{2} - X\right) - P\,(L - X) = 0$$
$$\Rightarrow\; M(X) = -\frac{PL}{2}$$

The boundary conditions for solving the problem are:

1. **BC 1**: For $X < \frac{L}{2}$: Slope at $X = 0$ is zero, $\frac{dy}{dX}(0) = 0$
2. **BC 2**: For $X \geq \frac{L}{2}$: Displacement at $X = L$ is zero, $y(L) = 0$

Let us find the deformation for each case separately:

- For $X < L/2$:

$$EI\frac{d^2y}{dX^2} = \frac{-PL}{2}$$

$$\Rightarrow\ EI\frac{dy}{dX} = \frac{-PL}{2}X + C_1$$

$$\text{Use } \mathbf{BC\ 1}\text{:}\ \frac{dy}{dX}(0) = 0,\ \text{we get } C_1 = 0$$

$$\Rightarrow\ EI\frac{dy}{dX} = \frac{-PL}{2}X \tag{8}$$

$$\Rightarrow\ y(X) = \frac{-PLX^2}{4EI} + C_2 \tag{9}$$

- For $X \geq L/2$:

$$M(X) = -P(L - X)$$

$$EI\frac{d^2y}{dX^2} = -P(L - X)$$

$$\Rightarrow\ EI\frac{dy}{dX} = -PLX + \frac{PX^2}{2} + D_1 \tag{10}$$

$$\Rightarrow\ y(X) = \frac{1}{EI}\left(-\frac{PLX^2}{2} + \frac{PX^3}{6} + D_1X + D_2\right) \tag{11}$$

Here, we can find D_1 using continuity of slope at $X = \frac{L}{2}$

$$\frac{dy}{dX}\left(X = \frac{L^-}{2}\right) = \frac{dy}{dX}\left(X = \frac{L^+}{2}\right)$$

$$\Rightarrow\ \frac{-PL^2}{4} = \frac{-PL^2}{2} + \frac{PL^2}{8} + D_1$$

$$\Rightarrow\ D_1 = \frac{PL^2}{8}$$

Substituting the value of D_1 in Eq.(11), we get

$$\Rightarrow\ y(X) = \frac{1}{EI}\left(-\frac{PLX^2}{2} + \frac{PX^3}{6} + \frac{PL^2X}{8} + D_2\right)$$

$$\text{Use } \mathbf{BC\ 2}\ : y(L) = 0,\ \rightarrow D_2 = \frac{1}{EI}\left[\frac{PL^3}{2} - \frac{PL^3}{6} - \frac{PL^3}{8}\right] = \frac{5PL^3}{24}$$

$$\Rightarrow\ y(X) = \frac{1}{EI}\left(-\frac{PLX^2}{2} + \frac{PX^3}{6} + \frac{PL^2X}{8} + \frac{5PL^3}{24}\right)$$

To determine C_2 of Eq.9, we use the continuity of deflection at $X = \frac{L}{2}$

$$y\left(X = \frac{L^-}{2}\right) = y\left(X = \frac{L^+}{2}\right)$$

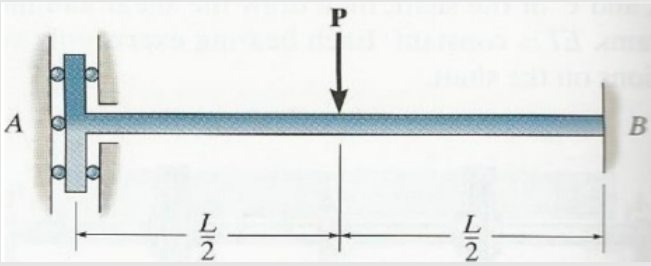
$$\Rightarrow -\frac{PL^3}{16EI} + C_2 = \frac{1}{EI}\left(-\frac{PL^3}{8} + \frac{PL^3}{48} + \frac{PL^3}{16} + \frac{5PL^3}{24}\right)$$

$$\Rightarrow C_2 = \frac{11PL^3}{48EI}$$

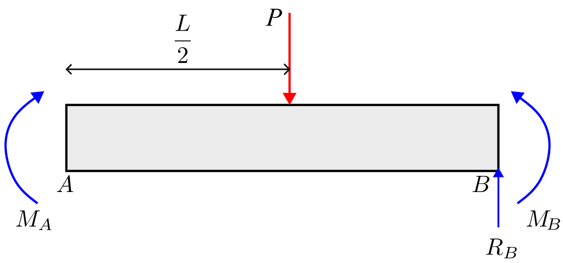
The deflection at any point is therefore

$$y(X) = \begin{cases} \dfrac{-PLX^2}{4EI} + \dfrac{11PL^3}{48EI} & \text{for} X < \dfrac{L}{2} \\ \\ \dfrac{1}{EI}\left(-\dfrac{PLX^2}{2} + \dfrac{PX^3}{6} + \dfrac{PL^2X}{8} + \dfrac{5PL^3}{24}\right) & \text{for} X \geq \dfrac{L}{2} \end{cases}$$

Q3. For the beam shown, find the reaction moment and deflection at A (use Timoshenko beam theory).



Solution: The constraint at the left end has frictionless rollers which allows free translation in the vertical direction but no rotation. So there is an unknown reaction moment at the left end. The right end has clamped/fixed support, which does not allow rotation or translation. Thus, at the right end, there is an unknown reaction moment and an unknown vertical reaction force. The free body diagram of the entire beam thus looks as follows:



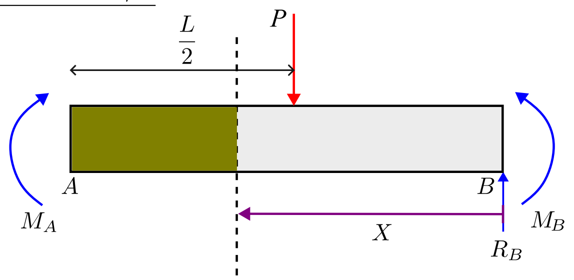
Since there are three unknown reactions and two equilibrium equations, ($\sum F_y = 0$ and $\sum M = 0$) so we can't determine all the three reactions in this case.

Consider the origin of the coordinate system at the right end B for this problem (you could also choose the origin at the left end). The delfection equation for Timonshenko beam theory is given by

$$EI\frac{d\theta}{dX} = M\left(X\right), \qquad \frac{dy}{dX} - \theta(X) = V(X)$$

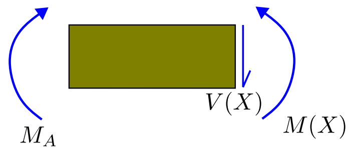
To use this formula, we need to find the expression for $M\left(X\right)$ and $V(X)$. Note that $M(X)$ and $V(X)$ in the cross section will have different expressions for $X < \dfrac{L}{2}$ and $X \geq \dfrac{L}{2}$. So we will consider two cases:

For $X \geq L/2$:

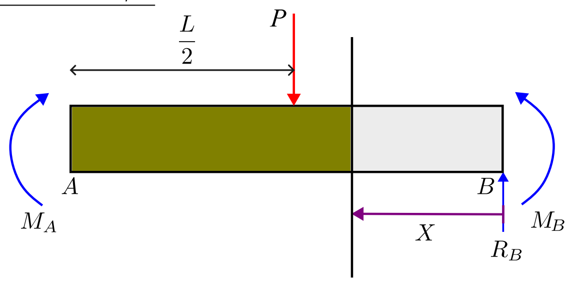


$$+\uparrow \sum F_y = 0 \;\Rightarrow\; V(X) = 0$$

$$\circlearrowleft\!\!+ \sum M_{\text{left end}} = 0$$
$$\Rightarrow\; M\left(X\right) - M_A = 0$$
$$\Rightarrow\; M\left(X\right) = M_A$$

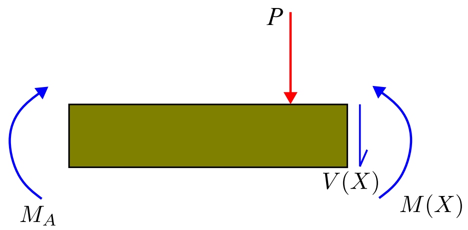


For $X < L/2$:



$$+\uparrow \sum F_y = 0 \;\Rightarrow\; V\left(X\right) = -P$$

$$\circlearrowleft\!\!+ \sum M_{\text{left end}} = 0$$
$$\Rightarrow\; M\left(X\right) - \frac{PL}{2} - \cancelto{-P}{V(X)}\left(L - X\right) - M_A = 0$$
$$\Rightarrow\; M(X) = M_A - P\left(\frac{L}{2} - X\right)$$



We observe that in the expression for $M(X)$, M_A is unknown. So, in total, we need three (two + one extra) BCs to solve for the displacement.

- For $X < L/2$: $y(0) = 0$, and $\theta(0) = 0$
- For $X \geq L/2$: $\theta(L) = 0$

Let us first solve it for the region $X = 0 \to L/2$

$$EI\frac{d\theta}{dX} = M_A - P\left(\frac{L}{2} - X\right)$$
$$EI\theta\left(X\right) = M_A X - P\left(\frac{LX}{2} - \frac{X^2}{2}\right) + C_1$$
$$\text{Use } \theta(0) = 0 \to C_1 = 0$$
$$\theta\left(X\right) = \frac{M_A}{EI}X - \frac{PX}{2EI}\left(L - X\right) \tag{12}$$

Let's substitute $\theta(X)$ in the equation below:

$$
\begin{aligned}
\frac{dy}{dX} - \theta(X) &= \frac{-P}{kGA} \\
\frac{dy}{dX} &= \frac{M_A}{EI}X - \frac{PX}{2EI}(L-X) - \frac{P}{kGA} \\
y(X) &= \frac{M_A}{EI}\frac{X^2}{2} - \frac{P}{2EI}\left[L\frac{X^2}{2} - \frac{X^3}{3}\right] - \frac{PX}{kGA} + C_2
\end{aligned}
\tag{13}
$$

Upon further using $y(0) = 0$, we get $C_2 = 0$. The above $y(X)$ and $\theta(X)$ now have M_A as unknowns!
Let us know solve in the range $X = L/2 < X < L$

$$
\begin{aligned}
EI\frac{d\theta}{dX} &= M_A \\
\theta(X) &= \frac{M_A}{EI}X + D_1 \\
\text{Use } \theta(L) = 0 \;\Rightarrow\; D_1 &= \frac{-M_AL}{EI} \\
\theta(X) &= \frac{M_A}{EI}(X-L)
\end{aligned}
\tag{14}
$$

Expression 12 and 14 must match at $X = L/2$

$$
\begin{aligned}
\frac{-M_AL}{2EI} &= \frac{M_AL}{2EI} - \frac{PL^2}{8EI} \\
\Rightarrow M_A &= \frac{PL^2}{8}
\end{aligned}
$$

Next, using the second part of equation for $X \geq L/2$

$$
\begin{aligned}
\frac{dy}{dX} - \theta &= 0 \\
\frac{dy}{dX} &= \frac{M_A}{EI}(X-L) \;\; \text{(from Eq.14)} \\
y(X) &= \frac{PL}{8EI}\left(\frac{X^2}{2} - LX\right) + D_2
\end{aligned}
\tag{15}
$$

D_2 can be obtained by matching 13 and 15 at $X = L/2$

$$
\begin{aligned}
\frac{PL}{8EI}\left(\frac{L^2}{8} - \frac{L^2}{2}\right) + D_2 &= \frac{PL^3}{64EI} - \frac{PL^3}{24EI} - \frac{PL}{2kGA} \\
D_2 &= \frac{PL^3}{EI}\left(\frac{1}{64} + \frac{3}{64} - \frac{1}{24}\right) - \frac{PL}{2kGA} \\
&= \frac{PL^3}{48EI} - \frac{PL}{2kGA}
\end{aligned}
$$

Therefore, the deflection and reaction moment at point A are:

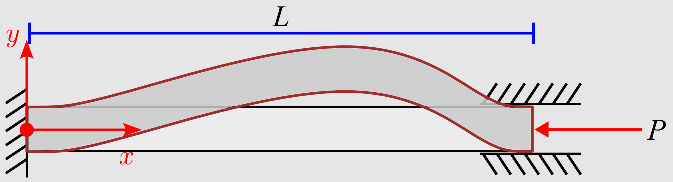
$$
\begin{aligned}
y(L) &= \frac{PL}{8EI}\left(\frac{X^2}{2} - Lx\right) + D_2 \text{ at } X = L \\
&= \frac{-PL^3}{24EI} - \frac{PL}{2kGA}
\end{aligned}
$$

We thus have

$$M_A = \frac{PL}{8}$$
$$y\left(L\right) = \frac{-PL^3}{24EI} - \frac{PL}{2kGA}$$

Q4. Think of a beam which is clamped against both transverse deflection as well as rotation at both the ends (see figure).

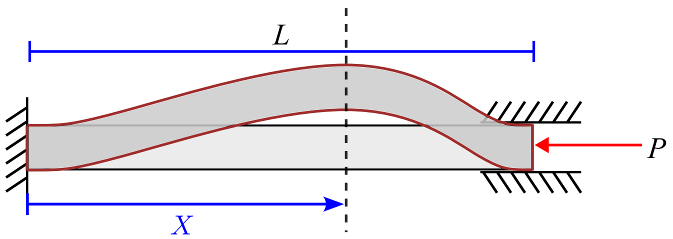
(a) Write down the two governing equations to obtain transverse deflection of this beam if it behaves as a Timoshenko beam. Also obtain the boundary conditions.

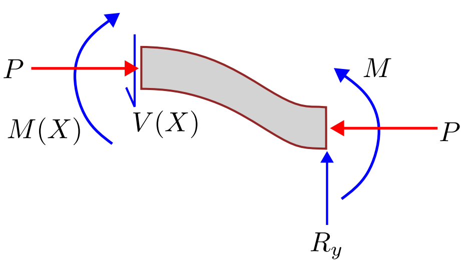


(b) Can you think of reducing the two equations in part (a) into a single ordinary differential equation in terms of just the transverse deflection? Similarly, also write down all the boundary condition such that the cross-sectional rotation variables doesn't show.

(c) Can you deduce the equation which gives us the critical buckling load of the beam? (You don't have to solve it)

Solution:





a)

Moment balance about X

$$
\begin{aligned}
&-M(X)+M-Py(X)+R(L-X)=0\\
&\Rightarrow M(X)=M-Py+R(L-X) \qquad (16)\\
&V(X)=R \qquad (17)
\end{aligned}
$$

The two governing equations of TBT are:

$$
\begin{aligned}
EI\frac{d\theta}{dX}&=M(X)\\
\frac{dy}{dX}&=\theta+\frac{V(X)}{kGA}\\
\text{from eq 16 and 17 :}&\\
EI\frac{d\theta}{dX}&=M-Py+R(L-X)\\
\frac{dy}{dX}&=\theta+\frac{R}{kGA}
\end{aligned}
$$

We have two unknowns in M and R at the end and two more unknowns of integrating constants! So, four BCs are required which are

$$
\begin{aligned}
y(0)&=y(L)=0\\
\theta(0)&=\theta(L)=0
\end{aligned}
$$

b)

$$
\begin{aligned}
&\theta=\frac{dy}{dX}-\frac{R}{kGA}\\
&\Rightarrow\frac{d\theta}{dX}=\frac{d^2y}{dX^2}-\frac{d}{dX}\left(\frac{R}{kGA}\right)=\frac{d^2y}{dX^2}\\
&\text{substitute this in bending eq, we get}\\
&EI\frac{d^2y}{dX^2}=M-Py+R(L-X)
\end{aligned}
$$

Now, we write boundary condition completely in terms of y:

$$
\begin{aligned}
y(0)&=y(L)=0\\
\theta&=\frac{dy}{dX}-\frac{R}{kGA}\\
\Rightarrow\frac{dy}{dX}(0)-\frac{R}{kGA}&=0\\
\frac{dy}{dX}(L)-\frac{R}{kGA}&=0
\end{aligned}
$$

c) Let's first obtain general solution.

Particular integral: $y(X)=\dfrac{M+R(L-X)}{P}$

Complimentary function: $y(X)=C_1\cos wx+c_2\sin wx$

General solution: $y(X)=C_1\cos wx+C_2\sin wx+\dfrac{M+R(L-X)}{P}$

i) $y(0) = 0 \;\Rightarrow\; C_1 + \dfrac{M+RL}{P} = 0 \;\Rightarrow\; C_1 = -\dfrac{M+RL}{P}$

ii) $y(L) = 0 \;\Rightarrow\; C_1 \cos wL + C_2 \sin \omega L + \dfrac{M}{P} = 0$

iii) $\dfrac{dy}{dX}(0) = \dfrac{R}{kGA} \;\Rightarrow\; C_2\omega - \dfrac{R}{P} = \dfrac{R}{kGA} \;\Rightarrow\; C_2 = \dfrac{R}{\omega}\left(\dfrac{1}{P} + \dfrac{1}{kGA}\right)$

iv) $\dfrac{dy}{dX}(L) = \dfrac{R}{kGA} \;\Rightarrow\; -C_1\omega \sin \omega L + C_2\omega \cos \omega L - \dfrac{R}{P} = \dfrac{R}{kGA}$

Equation (ii) thus becomes

$$\frac{M}{P}(1-\cos\omega L) + R\left[\left(\frac{1}{P\omega} + \frac{1}{kGA\omega}\right)\sin\omega L - \frac{L}{P}\cos\omega L\right] = 0$$

Similarly eq (iv) becomes

$$\frac{M}{P}\omega\sin\omega L + R\left[\left(\frac{1}{P} + \frac{1}{kGA}\right)(\cos\omega L - 1) + \frac{L}{P}\omega\sin\omega L\right] = 0$$

Writing them together in a matrix form, we get

$$\begin{bmatrix} \dfrac{1-\cos\omega L}{P} & \left(\dfrac{1}{P\omega} + \dfrac{1}{kGA\omega}\sin\omega L\right) - \dfrac{L}{P}\cos\omega L \\ \dfrac{\omega\sin\omega L}{P} & \left(\dfrac{1}{P} + \dfrac{1}{kGA}(\cos\omega L - 1)\right) + \dfrac{L}{P}\omega\sin\omega L \end{bmatrix} \begin{bmatrix} M \\ R \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For this to have non-zero solution, its determinant should be zero. This gives us the equation to obtain buckling load!