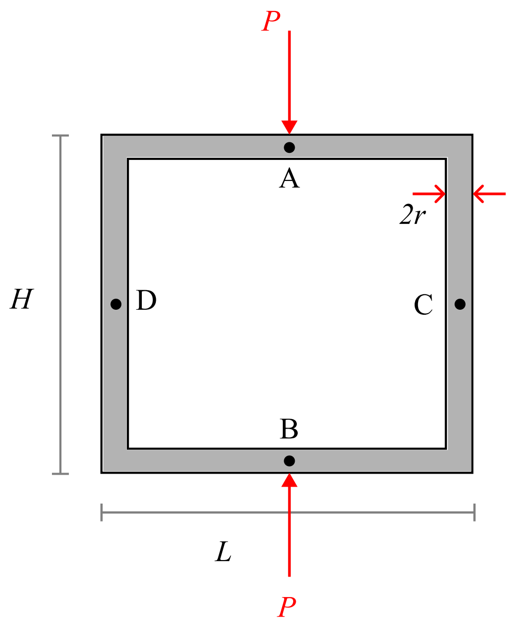
Tutorial 12

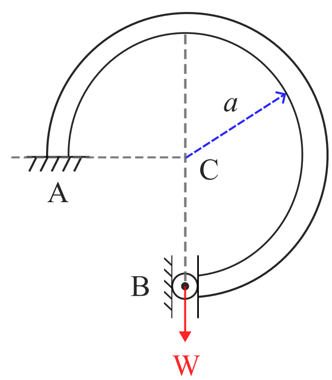
APL 104 - 2022 (Solid Mechanics)

1. Think of a rectangular beam as shown. Asusme its cross-section to be solid circular. Suppose the ring is subjected to equal and opposite forces at ‘A’ and ‘B’. Neglect energy in the beam’s cross-section due to shear force and axial force.

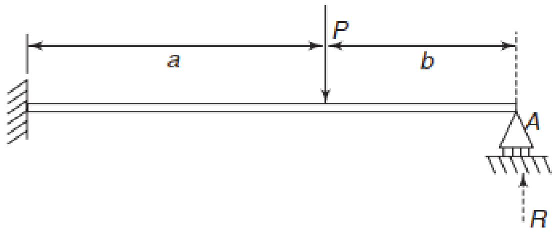


   (a) By how much will points A and B get closer to each other?
   (b) By how much will points C and D get farther apart?
   (c) What is internal moment in the cross-section at C?

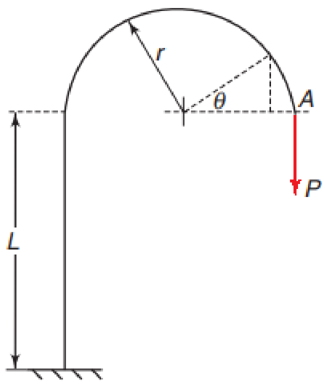
2. Using energy method, determine (i) the vertical deflection of point B under the action of load W and (ii) the horizontal reaction force at B. The end B is free to rotate but can move only in a vertical direction. Consider all forms of energy, i.e. bending, twisting, stretching as well as shearing energy.



3. Determine the support reaction for the propped cantilever.

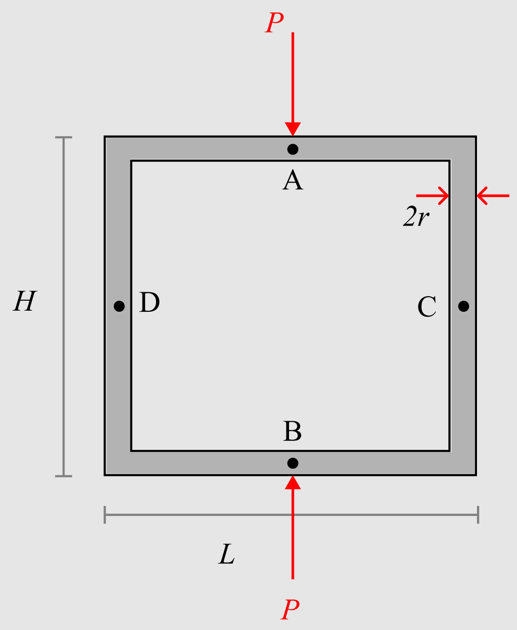


4. For the structure shown, what is the vertical deflection at end A? Also, determine the ratio of L to r if the horizontal and vertical deflections of the loaded end A are equal. P is the only force acting



APL 104 Tutorial 12 solutions

Q1. Think of a rectangular beam as shown. Assume its cross-section to be solid circular. Suppose the ring is subjected to equal and opposite forces at 'A' and 'B'. Neglect energy in the beam's cross-section due to shear force and axial force.

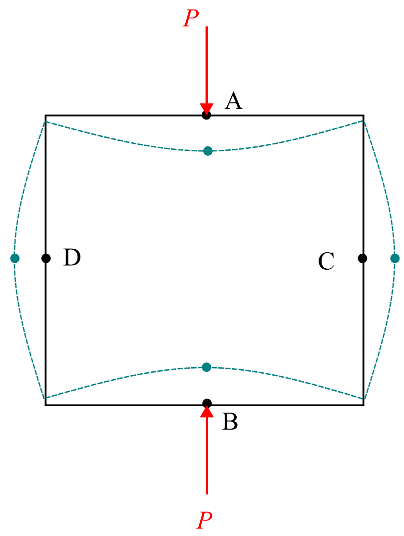


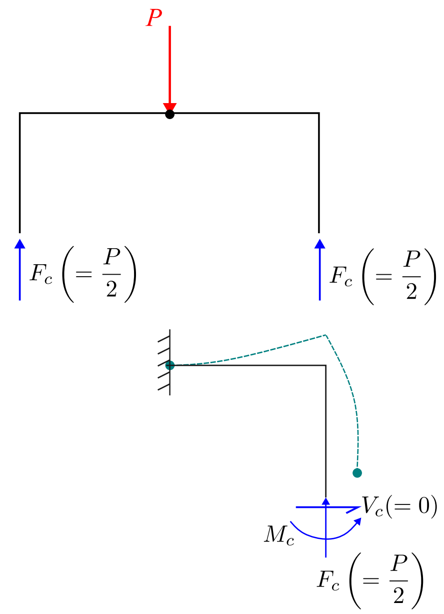
i. By how much will points A and B get closer to each other?

ii. By how much will points C and D get farther apart?

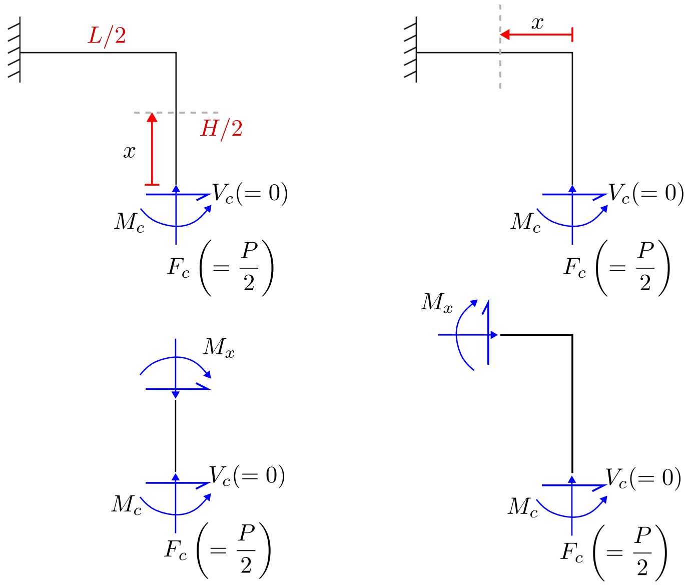
iii. What is internal moment in the cross-section at C?

Solution:





Just like in class, we can take one-quarter of the ring!
– Shear force V_c would be zero by symmetry
– $F_c = P/2$ (from force balance in the y-direction)
– There would be no rotation at C, implying $\dfrac{\partial \theta_c}{\partial M_c} = 0$, where θ_c is the rotation at C



Vertical part $\rightarrow M_x = M_c + V_c x$ Horizontal part $\rightarrow M_x = M_c + V_c H/2 + F_c x$

Total energy due to bending,

$$E(F_c, V_c, M_c) = \int_0^{H/2} \frac{(M_c + V_c x)^2}{2EI} dx + \int_0^{L/2} \frac{(M_c + V_c H/2 + F_c x)^2}{2EI} dx$$

In the above expression, the known variables are $F_c(= P/2)$ and $V_c(= 0)$, while M_c is unknown. Note that M_c is the corresponding moment for rotation at C, which we know is zero. Thus, we can obtain the value of M_c by setting $\dfrac{\partial E}{\partial M_c} = 0$ (and then set $V_c = 0,\ F_c = P/2$ after differentiation)

$$\Rightarrow \frac{\partial E}{\partial M_c} = 0$$

$$\Rightarrow \int_0^{H/2} \frac{\left(M_c + \cancelto{0}{V_c} x\right)}{EI} dx + \int_0^{L/2} \frac{\left(M_c + \cancelto{0}{V_c} H/2 + \cancelto{P/2}{F_c} x\right)}{EI} dx = 0$$

$$\Rightarrow \frac{M_c H}{2EI} + \frac{M_c L}{2EI} + \frac{PL^2}{16EI} = 0$$

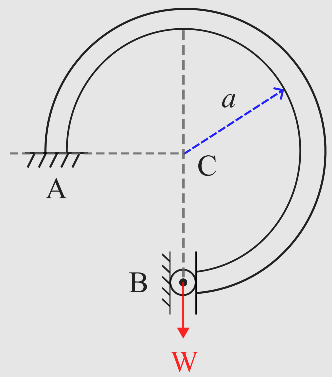
$$\Rightarrow M_c = \frac{PL^2}{8(L+H)}$$

$\delta_{CD} = 2\delta_{Cx}$, where δ_{Cx} is the horizontal displacement of point C.
Get $\delta_{Cx} = \dfrac{\partial E}{\partial V_c}$ (and set $V_c = 0,\ F_c = P/2,\ M_c = \dfrac{PL^2}{8(L+H)}$ after differentiation)

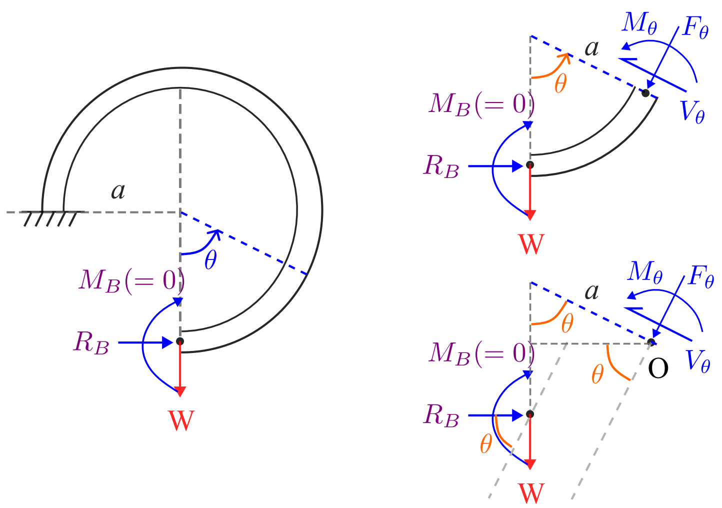
$\delta_{AB} = 2\delta_{Cy}$, where δ_{Cy} is the vertical displacement of point C.

Get $\delta_{Cy} = \dfrac{\partial E}{\partial F_c}$ (and set $V_c = 0$, $F_c = P/2$, $M_c = \dfrac{PL^2}{8(L+H)}$ after differentiation)

Q2. Using energy method, determine (i) the vertical deflection of point B under the action of load W and (ii) the horizontal reaction force at B. The end B is free to rotate but can move only in a vertical direction. Consider all forms of energy, i.e. bending, twisting, stretching as well as shearing energy.



Solution:



At location B of roller support, there is only horizontal reaction R_B; there is no vertical reaction or moment.

Since we are required to find out the corresponding rotation at point B, we will introduce a corresponding dummy moment $M_B(=0)$ at B. The knowns are the load W and dummy moment $M_B(=0)$, and the unknown is the reaction R_B.

Let's find the moment, shear force, and axial force ($M_\theta,\ V_\theta,\ F_\theta$) at a section at an angle θ; these sectional forces would be used in writing the total stored energy expression later.

Using force balance and moment balance:

$$
\begin{aligned}
F_\theta &= R_B \cos\theta - W \sin\theta \\
V_\theta &= R_B \sin\theta + W \cos\theta \\
M_\theta &= M_B - Wa\sin\theta - R_B a(1-\cos\theta)
\end{aligned}
$$

The total energy is,

$$
\begin{aligned}
E(F_B, R_B, M_B) &= \int_0^{3\pi/2} \left[\frac{F_\theta^2}{2EA} + \frac{V_\theta^2}{2kGA} + \frac{M_\theta^2}{2EI} \right] a d\theta \\
&= \int_0^{3\pi/2} \left[\frac{(R_B\cos\theta - W\sin\theta)^2}{2EA} + \frac{(R_B\sin\theta + W\cos\theta)^2}{2kGA} \right. \\
&\qquad \left. + \frac{(M_B - Wa\sin\theta - R_B a(1-\cos\theta))^2}{2EI} \right] a d\theta
\end{aligned}
$$

Since the beam has no horizontal deflection at B, we can determine the corresponding force R_B by setting $\dfrac{\partial E}{\partial R_B} = 0$ and evaluating at $M_B = 0$ and given W.

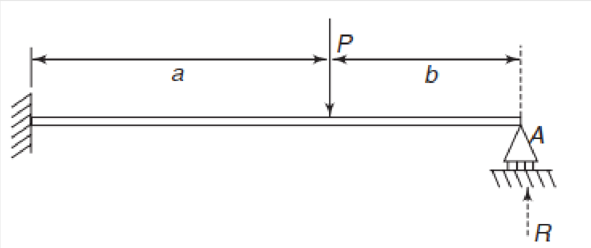
$$
\begin{aligned}
&\Rightarrow \frac{\partial E}{\partial R_B} = 0 \\
&\Rightarrow \int_0^{3\pi/2} \left[\frac{F_\theta}{EA}\frac{\partial F_\theta}{\partial R_B} + \frac{V_\theta}{kGA}\frac{\partial V_\theta}{\partial R_B} + \frac{M_\theta}{EI}\frac{\partial M_\theta}{\partial R_B} \right] a d\theta = 0 \\
&\Rightarrow \int_0^{3\pi/2} \frac{R_B\cos\theta - W\sin\theta}{EA}\cos\theta + \frac{R_B\sin\theta + W\cos\theta}{kGA}\sin\theta \\
&\qquad + \frac{\cancelto{0}{M_B} - Wa\sin\theta - R_B a(1-\cos\theta)}{EI}\left(-a(1-\cos\theta)\right) a d\theta = 0 \\
&\Rightarrow a\left[\frac{3\pi R_B - 2W}{4EA} + \frac{3\pi R_B + 2W}{4kGA} + \frac{Wa^2}{2EI} + R_B a^2\left(2 + \frac{9\pi}{4}\right) \right] = 0 \\
&\Rightarrow R_B \left[\frac{3\pi}{4EA} + \frac{3\pi}{4kGA} + a^2\left(2+\frac{9\pi}{4}\right) \right] = W\left[\frac{1}{2EA} - \frac{1}{2kGA} - \frac{a^2}{2EI} \right] \\
&\Rightarrow R_B = W \frac{\left[\dfrac{1}{2EA} - \dfrac{1}{2kGA} - \dfrac{a^2}{2EI} \right]}{\left[\dfrac{3\pi}{4EA} + \dfrac{3\pi}{4kGA} + a^2\left(2+\dfrac{9\pi}{4}\right) \right]}
\end{aligned}
$$

To obtain vertical deflection at B, differentiate the energy with respect to the corresponding force W

$$
\begin{aligned}
\Rightarrow \delta_B &= \frac{\partial E}{\partial W} \\
&= \int_0^{3\pi/2} \left[\frac{F_\theta}{EA}\frac{\partial F_\theta}{\partial W} + \frac{V_\theta}{kGA}\frac{\partial V_\theta}{\partial W} + \frac{M_\theta}{EI}\frac{\partial M_\theta}{\partial w} \right] a d\theta \\
&= \int_0^{3\pi/2} \frac{R_B\cos\theta - W\sin\theta}{EA}(-\sin\theta) + \frac{R_B\sin\theta + W\cos\theta}{kGA}\cos\theta \\
&\qquad + \frac{\cancelto{0}{M_B} - Wa\sin\theta - R_B a(1-\cos\theta)}{EI}\left(a\sin\theta)\right) a d\theta
\end{aligned}
$$

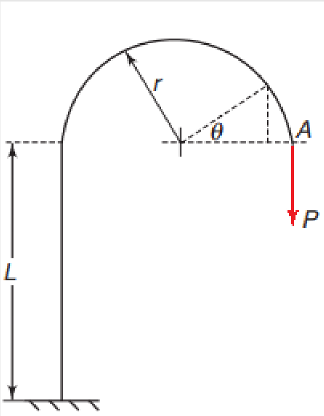
Put the value of R_B as obtained previously and solve this in a similar way to obtain δ_B!

Q3. Determine the support reaction for the propped cantilever.

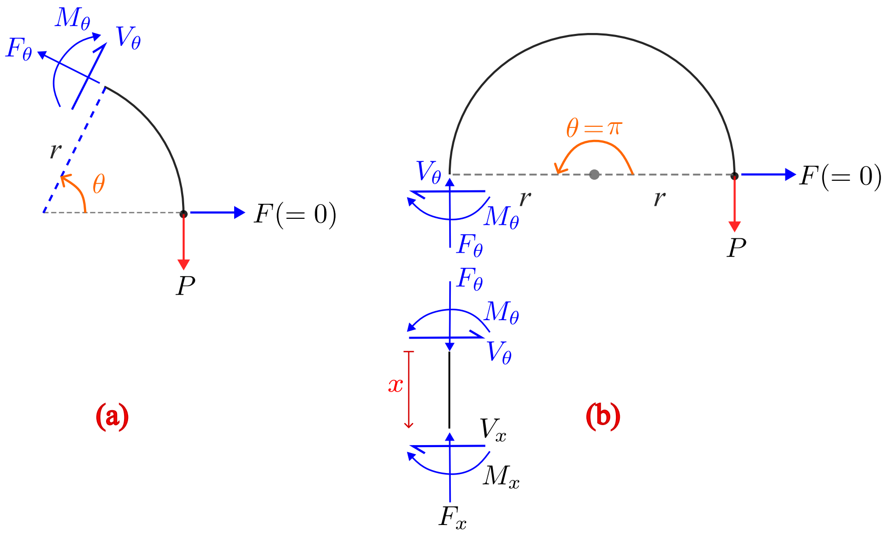


Solution: The solution has been worked out in pdf notes of Lecture 30 Section 5 Example 2.

Q4. For the structure shown, what is the vertical deflection at end A? Also, determine the ratio of L to r if the horizontal and vertical deflections of the loaded end A are equal. P is the only force acting



Solution: We are required to find both vertical and horizontal deflection for the beam at point A. We are given a corresponding force P at A to obtain the corresponding vertical deflection, however, there is no corresponding force for the corresponding horizontal displacement. Hence, we apply a dummy horizontal force $F(=0)$.



Next we find the sectional forces and moments at any arbitrary cross-sections. We take two cross-sections, one at an angle θ from A (see subfigure (a)) and another at a distance x (see subfigure (b)).

The procedure is as follows:
– First find out shear force, axial force, moment, twist.
– Write down energy
– Then do derivative with respect to the corresponding force to find the corresponding displacement.

In this problem, we neglect the energy due to shearing and stretching and only focus on the energy due to bending. So we will only use the sectional bending moment for calculating the energy.

Bending moment at any section on the curved semi-circle:

$$\begin{aligned} &\circlearrowleft\!\!+ \sum M_o = 0 \\ &\Rightarrow\ -M_\theta - Pr\left(1-\cos\theta\right) + Fr\sin\theta = 0 \\ &\Rightarrow\ M_\theta = -Pr\left(1-\cos\theta\right) + Fr\sin\theta \end{aligned}$$

Bending moment at any section on the straight portion:

$$\begin{aligned} &\circlearrowleft\!\!+ \sum M_o = 0 \\ &\Rightarrow\ M_x - M_\theta\,|_{\theta=\pi} + V_\theta\,|_{\theta=\pi}\,x = 0 \\ &\Rightarrow\ M_x = M_\theta\,|_{\theta=\pi} = -2Pr - Fx \end{aligned}$$

$$\begin{aligned} E\left(P,F\right) &= \int_0^\pi \frac{M_\theta^2}{2EI} r\,d\theta + \int_0^L \frac{M_x^2}{2EI} dx \\ &= \int_0^\pi \frac{\left(-Pr\left(1-\cos\theta\right)+Fr\sin\theta\right)^2}{2EI} r\,d\theta + \int_0^L \frac{\left(-2Pr-Fr\right)^2}{2EI} dx \end{aligned}$$

To obtain vertical deflection, δ_v, at point A, we take partial derivative of the energy w.r.t to the corresponding force P:

$$\begin{aligned}
\frac{\partial E}{\partial P}\Big|_{F=0} &= \int_0^\pi \frac{\left(-Pr\left(1-\cos\theta\right)+\cancelto{0}{F}x\sin\theta\right)}{EI}\left(-r(1-\cos\theta)\right)r d\theta + \int_0^L \frac{-2Pr-\cancelto{0}{F}r}{EI}(-2r)dx \\
&= \frac{Pr^3}{EI}\int_0^\pi \left(1-\cos\theta\right)^2 d\theta + \frac{4Pr^2L}{EI} \\
&= \frac{Pr^3}{EI}\left(\frac{3\theta}{2}-2\sin\theta+\frac{1}{4}\sin 2\theta\right)\Big|_0^\pi + \frac{4Pr^2L}{EI} \\
&= \frac{3\pi Pr^3}{2EI}+\frac{4Pr^2L}{EI} \\
&= \frac{Pr^2}{EI}\left[\frac{3\pi r}{2}+4L\right]
\end{aligned}$$

To obtain horizontal deflection, δ_h, at point A, we take the partial derivative of the energy w.r.t to the corresponding dummy force F

$$\begin{aligned}
\frac{\partial E}{\partial F}\Big|_{F=0} &= \int_0^\pi \frac{\left(-Pr\left(1-\cos\theta\right)+\cancelto{0}{F}r\sin\theta\right)}{EI}\left(r\sin\theta\right)r d\theta + \int_0^L \frac{-2Pr-\cancelto{0}{F}x}{EI}(-x)dx \\
&= -\frac{Pr^3}{EI}\int_0^\pi \left(1-\cos\theta\right)\sin\theta d\theta + \frac{2Pr}{EI}\int_0^L x dx \\
&= -\frac{Pr^3}{EI}\left(\frac{\cos^2\theta}{2}-\cos\theta\right)\Big|_0^\pi + \frac{2Pr}{EI}\left(\frac{x^2}{2}\right)\Big|_0^L \\
&= -\frac{2Pr^3}{EI}+\frac{2PrL^2}{EI} \\
&= \frac{Pr}{EI}\left[-2r^2+2L^2\right]
\end{aligned}$$

Equating δ_v to δ_h, we get

$$\begin{aligned}
\frac{Pr^2}{EI}\left[\frac{3\pi r}{2}+4L\right] &= \frac{Pr}{EI}\left[-2r^2+2L^2\right] \\
L^2-4Lr-r^2\left(\frac{3\pi}{2}+2\right) &= 0
\end{aligned}$$

Dividing by r^2 and putting $\frac{L}{r}=\rho$, we get

$$\rho^2-4\rho-\left(\frac{3\pi}{2}+2\right)=0$$

Upon solving the above, we get:

$$\rho=\frac{4\pm\sqrt{16+4(3\pi/2+2)}}{2}=2+\sqrt{6+\frac{3}{2}\pi}$$