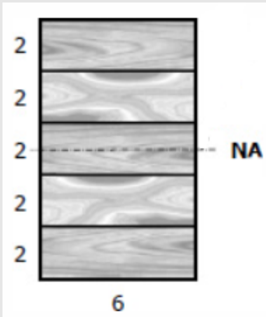
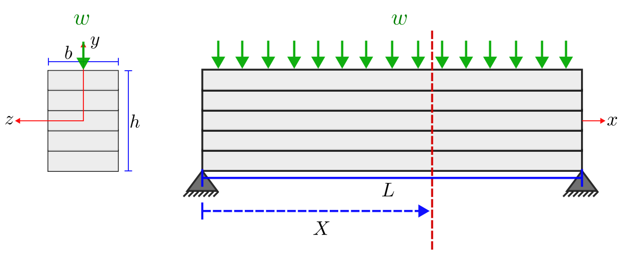
APL 104 Tutorial 10 solutions

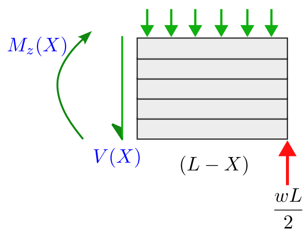
Q1. A laminated beam is composed of five planks, each 6 in. by 2 in. glued together to form a section 6 in. wide by 10 in. high.



The allowable shear stress in the glue is 90 psi, the allowable shear stress in the wood is 120 psi, and the allowable flexural stress in the wood is 1200 psi. Determine the maximum uniformly distributed load that can be carried by a simply supported beam on a 6ft simple span.

Solution: Assume that the uniformly-distributed load is applied along the y-axis. Note the rectangular cross-section is symmetrical about the y and z axes. Hence, they are also the principal axes.

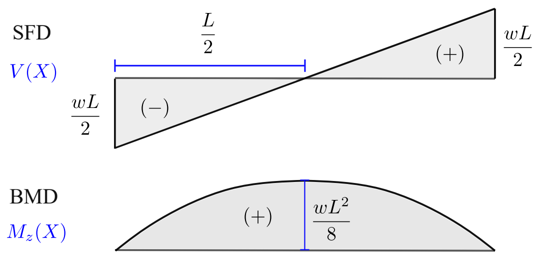




- $\uparrow + \sum F_y \ = \ 0$

$$-V_x - w\left(L \ - \ x\right) + \frac{wl}{2} = 0$$

$$\Rightarrow V_x = -w\left(\frac{L}{2} - x\right)$$



- $\sum M_{\text{left end}} = 0$ (CCW sense +ve)

$$-M_z\left(x\right) + \frac{wl}{2}\left(L - x\right) - \frac{w\left(L-x\right)^2}{2} = 0$$

$$\Rightarrow M_z\left(x\right) = \frac{wl}{2}\left(L - x\right) - \frac{w\left(L-x\right)^2}{2}$$

Since the load is applied along the principal axis, it will result in symmetrical bending. Also, the UDL will lead to both bending moment and shear force at any cross-section, hence, a non-uniform symmetrical bending would be observed.

To find the maximum flexural and shear stress, we need to find the bending and shear stress distribution. For symmetrical bending, the formula for bending and shear stresses is relatively simple:

$$\sigma_{xx} = \frac{M_z(x)\, y}{I_{zz}}, \quad \tau_{yz} = \frac{V_y(x)\, Q(y)}{I_{zz} b}$$

Note: If the load was applied along non-principal axes, it would have resulted in non-uniform unsymmetrical bending.

Distributions of $M_z(x)$ and $V_y(x)$ along the length of the beam are obtained from the bending moment diagram (BMD) and shear force diagram (SFD) as shown in the previous figure.

Max bending stress

$$\begin{aligned}
\sigma^{\text{wood}}_{xx,\max} &= \max_{x,y} \left\{ \frac{M_z(x)\, y}{I_{zz}} \right\} \\
&= \max_{x} \frac{M_z(x)}{I_{zz}} \max_{y} y \\
&= \frac{\left(\frac{wL^2}{8}\right)\left(\frac{h}{2}\right)}{\frac{1}{12} b h^3} \\
&= \frac{3wL^2}{4bh^2} \leq \sigma^{\text{wood}}_{xx,\text{tol}} \quad \ldots \quad (1)
\end{aligned}$$

Max shear stress

$$\tau_{yx}(x,y) = \frac{V(x) Q(y)}{I_{zz} b}$$

$$\tau_{yx,\max} = \max_{x,y} \frac{V(x)\; Q(y)}{I_{zz}\, b} = \frac{1}{I_{zz} b} \max_{x} V(x) \quad \max_{y} Q(y)$$

$\rightarrow \max_{x} V(x)$ occurs at $x = 0$ or $x = L$

$$\begin{aligned}
\rightarrow Q(y) &= \text{moment of area} \\
&= \text{area of the shaded region} \times \text{centroid of the shaded area from the NA} \\
&= b\left(\frac{h}{2} - y\right) \times \left\{ y + \frac{\frac{h}{2} - y}{2} \right\} \\
&= b\left(\frac{h}{2} - y\right) \times \frac{1}{2}\left(\frac{h}{2} + y\right) \\
&= \frac{b}{2}\left(\frac{h^2}{4} - y^2\right) \quad [Q(y) \text{ becomes maximum at } y = 0]
\end{aligned}$$

$$\rightarrow \max_{y} Q(y) = \frac{b}{2}\frac{h^2}{4}$$

$$\therefore\ \tau^{\text{wood}}_{yx,\max} = \frac{V(x = L/2)}{I_{zz}} \frac{Q(y=0)}{b} = \frac{\frac{wL}{2}\left(\frac{b}{2}\frac{h^2}{4}\right)}{\frac{1}{12} b h^3\, b} = \frac{3wL}{4bh} \leq \tau^{\text{wood}}_{yx,\text{tol}} \quad \ldots \quad (2)$$

The glue between the wooden planks resist the shear stress generated between the planks. The shear stress at the level of the glue also should not exceed the maximum tolerable shear stress of glue.

$$\therefore\ \tau_{yx,\max}^{\text{glue}} = \frac{V\left(x=\dfrac{L}{2}\right)}{I_{zz}}\frac{Q\left(y=\dfrac{h}{10}\right)}{b} = \frac{\dfrac{wL}{2}\left(\dfrac{3bh^2}{25}\right)}{\dfrac{1}{12}bh^3\, b} = \frac{18wL}{25bh} \leq \tau_{yx,\text{tol}}^{glue} \quad \ldots \quad (3)$$

From Eqs. (1), (2), and (3), we would obtain three values of w. The maximum value of w should be minimum of the values obtained from the three equations.

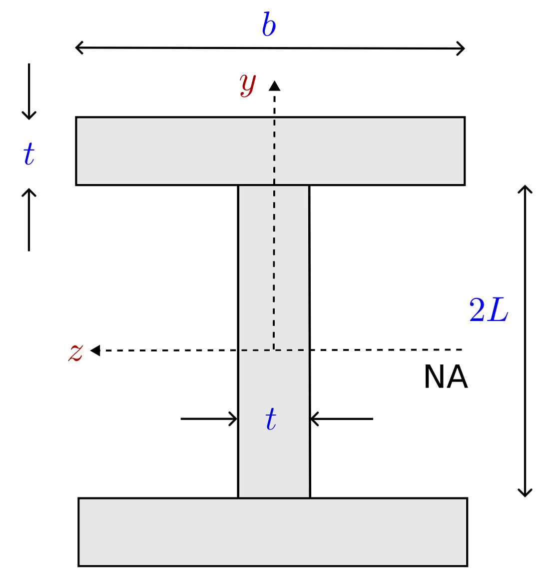
Q2. For an I-beam, assume the beam is subjected to tranverse load

(a) Obtain an expression for variation in shear stress τ_{xy} within its cross-section. You can use the formula $\tau_{xy} = \dfrac{VQ\,(y)}{I_{zz}b\,(y)}$.

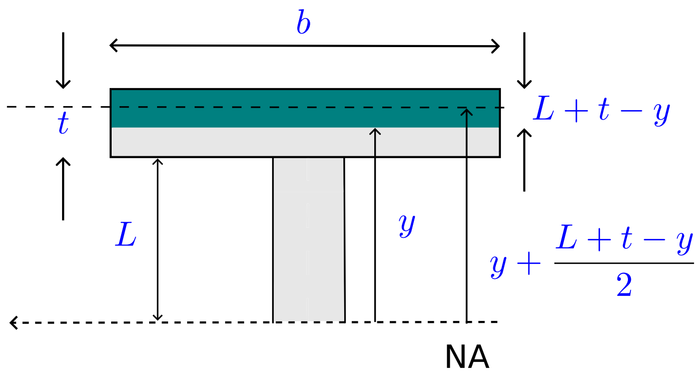
(b) Using the expression above, draw a graph depicting qualitative variation in shear stress within the cross-section.

(c) Where is the shear stress maximum? Find the ratio of maximum shear stress to average shear stress in the cross-section.

Solution: We will assume that the transverse load is acting along the principal axis of the beam resulting in symmetrical bending. Therefore, we are instructed to use the simple formula $\tau_{yx}\,(x,y) = \dfrac{VQ(y)}{I_{zz}\,b\,(y)}$. Note that, unlike the beam with rectangular cross-section, the width of the I-beam changes in the flange and the web. Hence, in this case, $b(y)$ in the denominator of the formula is taken to be a function of y.



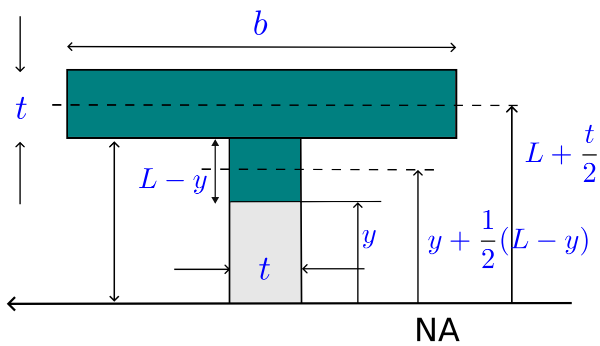
The shear force V is already given. Since we are looking at a section x, only $Q\,(y)$ and $b\,(y)$ will differ in the cross-section as a function of y. So lets find the distribution of $Q\,(y)$, which is the moment of the area. The Neutral Axis (NA) will be at the center aligned with the z-axis. It is clear from figure of I-beam that the geometry of the areas of the flange and the web are to be taken separately and will therefore lead to different distributions of shear stress.



Flange:

$$\begin{aligned} Q^f(y) &= b\,(t + L - y) \times \left[y + \frac{t + L - y}{2}\right] \\ &= \frac{b}{2}\,(t + L - y)\,(t + L + y) \\ &= \frac{b}{2}\left[(t + L)^2 - y^2\right] \end{aligned}$$

$$\begin{aligned} \therefore\ \tau_{yx} &= \frac{V_x}{I_{zz}}\frac{(b/2)}{b}\left[(t + L)^2 - y^2\right] \quad (L \le y \le L+t) \\ &= \frac{V_x}{2I_{zz}}\left[(t + L)^2 - y^2\right] \end{aligned}$$



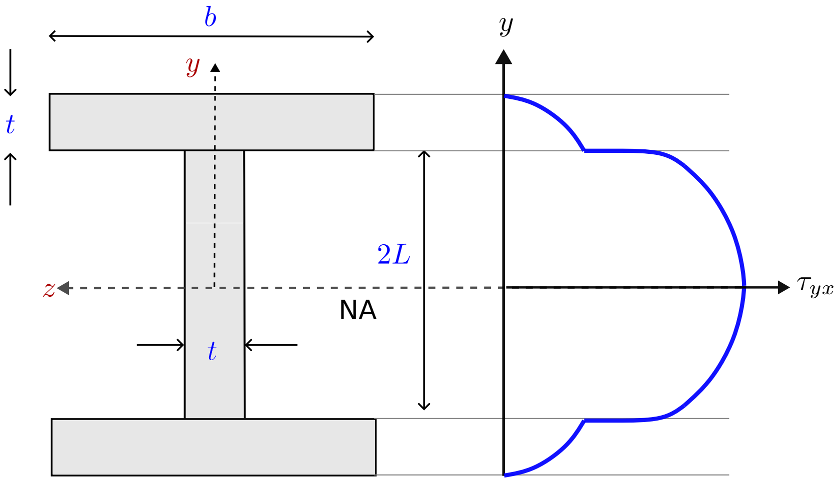
Web:

$$\begin{aligned} Q^w(y) &= b\,t \times \left(L + \frac{t}{2}\right) + t\,(L-y) \times \frac{1}{2}\,(L+y) \\ &= b\,t\left(L + \frac{t}{2}\right) + \frac{t}{2}\left(L^2 - y^2\right) \\ \therefore\ \tau_{yx} &= \frac{V_x}{I_{zz}\,t}\left[b\,t\left(L + \frac{t}{2}\right) + \frac{t}{2}\left(L^2 - y^2\right)\right] \end{aligned}$$

$\underline{I_{zz} \text{ for full I-beam}}$

$$
\begin{aligned}
I_{zz}^{f} \text{ (for flange)} &= \left[\frac{1}{12}bt^3 + bt\left(L + \frac{t}{2}\right)^2\right] \\
I_{zz}^{w} \text{ (for full web)} &= \frac{1}{12} + (2L)^3 \\
&= \frac{8tL^3}{12} \\
I_{zz} &= 2I_{zz}^{f} + I_{zz}^{w}
\end{aligned}
$$

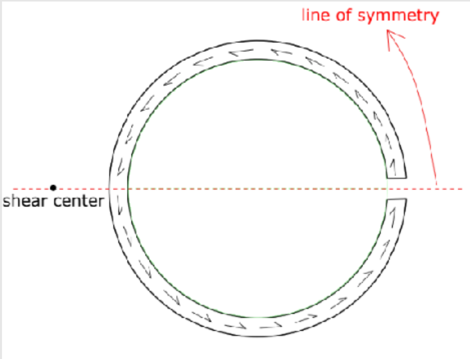
(b) The variation of shear stress with respect to y is shown below:



(c) Shear-stress is maximum at the NA

Avg shear stress, $\tau_{xy,\text{avg}} = \dfrac{V}{\text{Area of c/s of the I-beam}}$. Do the rest on your own.

Q4. Think of a beam whose cross-section has the shape of a thin annular disc but having a cut (along a radial line on z-axis). How will the shear stress distribution be in the cross-section? Will such a beam undergo just bending or it can also twist?



Solution: Here, s_c is the shear center of the cross-section, which can be worked out.

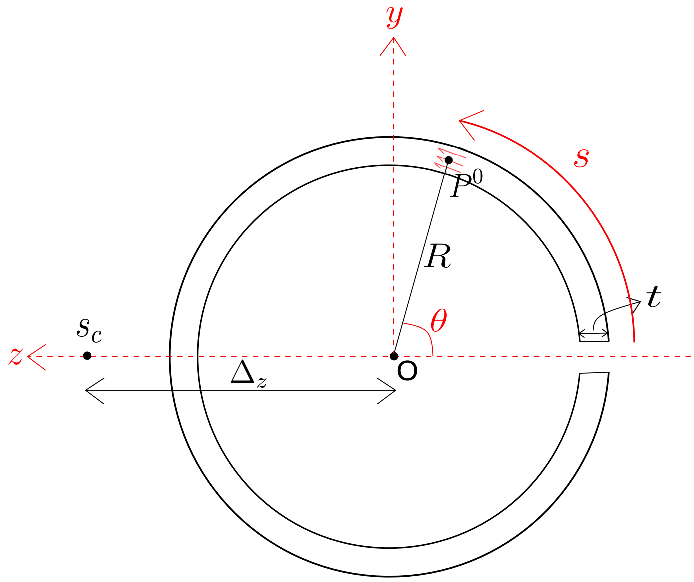


Figure 1: Variables for the analysis of the cross-section shown.

We need to know find Q_y^s for the region $[0, \theta]$ (see Figure 1) which has also been drawn separately in Figure 2. Let us identify a tiny strip shown in red in Figure 2 which subtends an angle $d\phi$ at the center. The y-coordinate of the centroid of this tiny strip is $R \sin \phi$ and its area is $tRd\phi$. Thus, for this tiny strip, we can write:

$$dQ_y = \bar{Y} dA = R^2 t \sin \phi \; d\phi \tag{1}$$

We can now find Q_y^s by integrating over such tiny strips, i.e.

$$Q_y^s = \int_0^\theta dQ_y = R^2 t \int_0^\theta \sin \phi \, d\phi = R^2 t(1 - \cos \theta) = R^2 t \left(1 - \cos\left(\frac{s}{R}\right)\right). \tag{2}$$

Let us now find I_{zz} for the cross-section about its centroid O. As the cut is really thin, the annulus is approximately complete. So, to calculate I_{xx}, I_{yy} and I_{zz}, we can forget about

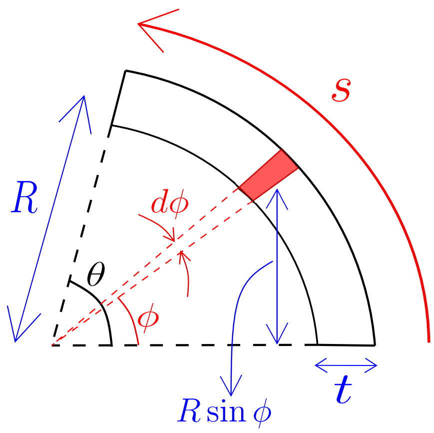


Figure 2: The part of the cross-section from 0 up to θ

the thin cut. We thus have

$$I_{zz} = I_{yy} = 1/2\ I_{xx} = 1/2R^2\ (2\pi R\ t) = \pi R^3\ t. \tag{3}$$

$$\tau_{sx} = -\frac{V_y Q_y^s}{I_{zz}\ t} = -\frac{V_y R^2 t \left(1 - \cos\left(\frac{s}{R}\right)\right)}{\pi R^3 t \times t} = -\frac{V_y \left(1 - \cos\left(\frac{s}{R}\right)\right)}{\pi R t}$$

If you work out the shear center of the cross-section, you will see that it does not lie on the cross-section. As such, if we want to apply shear force on beams having such cross-sections so that the beam does not twist, we must apply shear force so that its line of action passes through shear center. For the present cross-section, we need to create an extension of it (e.g., by a thin rod as shown in red in Figure 3) and apply shear force there so that the beam indeed does not twist.

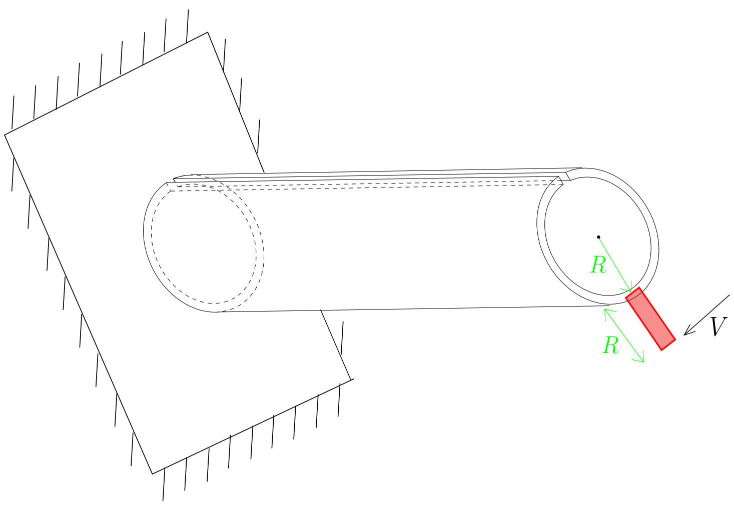
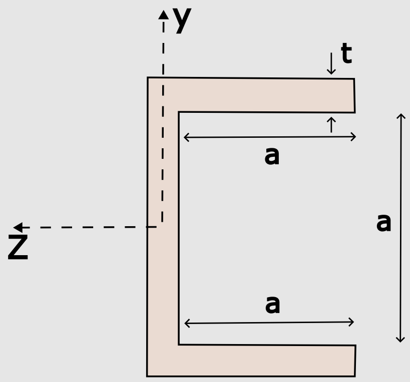


Figure 3: An extension made to the beam's cross-section to avoid twisting of the beam due to application of transverse load

Q5. Think of a beam having thin and open cross-section. The shape of the cross-section is shown in the figure. Suppose the beam is subjected to a transverse load at one of its end such that the force applied is along 'Y' axis.

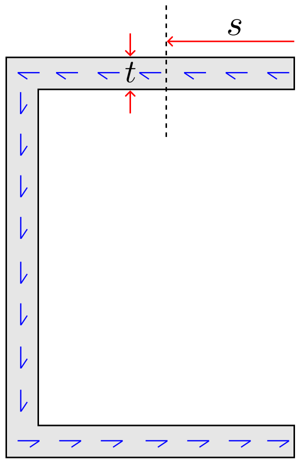


(a) Where should the load be applied in the cross-sectional plane at the tip so that the beam does not twist?

(b) Find out shear stress distribution in the beam's cross-section assuming the beam does not twist.

Solution: Since, it is a thin tube $t << a$, therefore

$$\tau_{sx} = \frac{VQ_{sy}}{I_{zz}\, t}$$

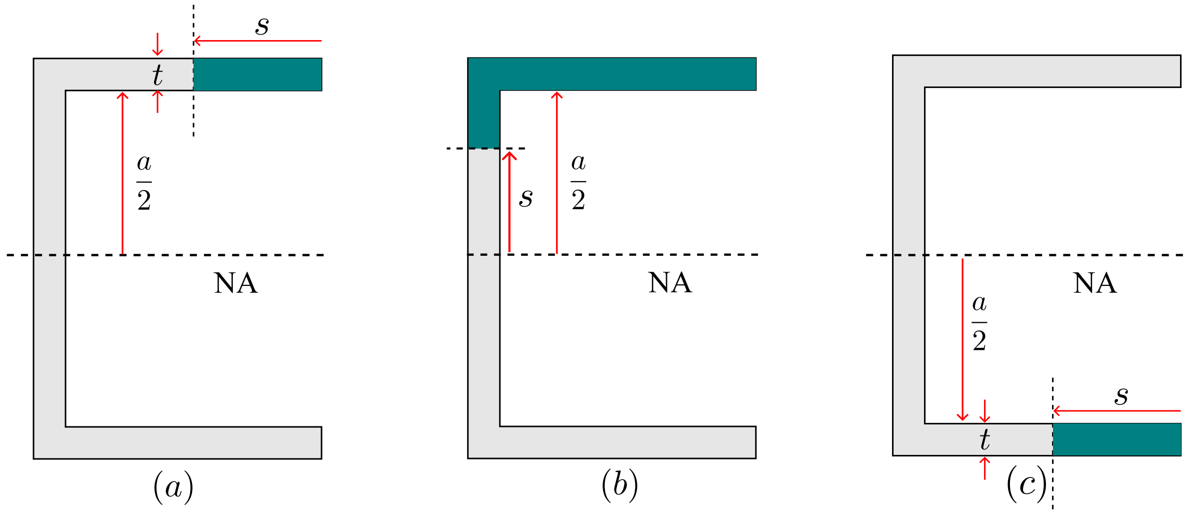


As the tube is thin, shear stress lies along the cross section's periphery! For the top horizontal flange portion (see subfigure (a)), the shear stress is:

$$\Rightarrow \tau_{sx} = \frac{V\left(\frac{a}{2}st\right)}{I_{zz}t} = \frac{Vast}{2I_{zz}t}$$

For the middle vertical web portion (see subfigure (b)), the shear stress is:

$$\begin{aligned}
\Rightarrow \tau_{sx} &= \frac{V\left(\frac{a}{2}at + \left[s + \left(\frac{a}{2} - s\right)/2\right]\left(\frac{a}{2} - s\right)t\right)}{I_{zz}t} \\
&= \frac{V\left(\frac{a^2t}{2} + \frac{1}{2}\left(\frac{a^2}{4} - s^2\right)t\right)}{I_{zz}t}
\end{aligned}$$



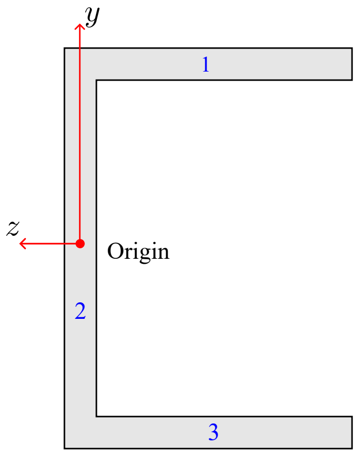
For the bottom horizontal flange portion (see subfigure (a)), the shear stress is:

$$\begin{aligned}
\Rightarrow \tau_{sx} &= \frac{VQ}{I_{zz}t} \\
&= \frac{V\left(-\frac{a}{2}st\right)}{I_{zz}t} \\
&= \frac{-Vas}{2I_{zz}}
\end{aligned}$$

The moment of inertia can be calculated separately for each of the three blocks (top, middle and lower) and then shifted to the NA respectively.

$$\begin{aligned}
I_{zz} &= I_{zz}(1) \times 2 + I_{zz}(2) \\
&= \left[\frac{1}{12}at^3 + at\left(\frac{a}{2}\right)^2\right] \times 2 + \frac{1}{12}a^3t \\
&= \frac{1}{6}at^3 + a^3t\frac{7}{12}
\end{aligned}$$

The shear center is the location of the point in/outside the cross-section at which a load applied parallel to the plane of the section will produce no twisting. To obtain shear center, first get the torque due to shear stress distribution about origin



$$
\begin{aligned}
T &= 1 + 2 + 3 \\
&= 1 \times 2 + 2 \quad (1 \text{ and } 3 \text{ contribute same torque}) \\
&= \left[\frac{Vat}{2I_{zz}} \int_0^a s ds \times \frac{a}{2}\right] \times 2 + 0 \quad (\text{torque due to } 2 \text{ is zero!}) \\
&= \frac{Va^4 t}{8I_{zz}} \times 2 = \frac{Va^3}{4\,(t^2/6 + 7/12a^2)}
\end{aligned}
$$

$$
\text{Location of shear center} = \frac{a^3}{4\,(t^2/6 + 7/12a^2)} \text{ (along } -z\text{-axis)}
$$