Solid Mechanics
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Lecture - 24
Bending of beams due to transverse load

Abstract

During the last lecture, we learnt about pure bending of beams in which bending moment was applied to deform a beam. In this lecture, we will analyze bending of beams when subjected to a transverse load.

1 Non-uniform Bending

In case of pure bending, we had the same moment acting on every cross-section. For this reason, pure bending is also called uniform bending. We will now move on to non-uniform bending of a beam wherein the bending moment is not uniform along the length of the beam. Let us consider a case of loading where we have distributed load $b(x)$ acting on the beam as shown in Figure 1a. The load $b(x)$ is assumed to act in $+y$ direction. Let us cut a small part of the beam of length Δx

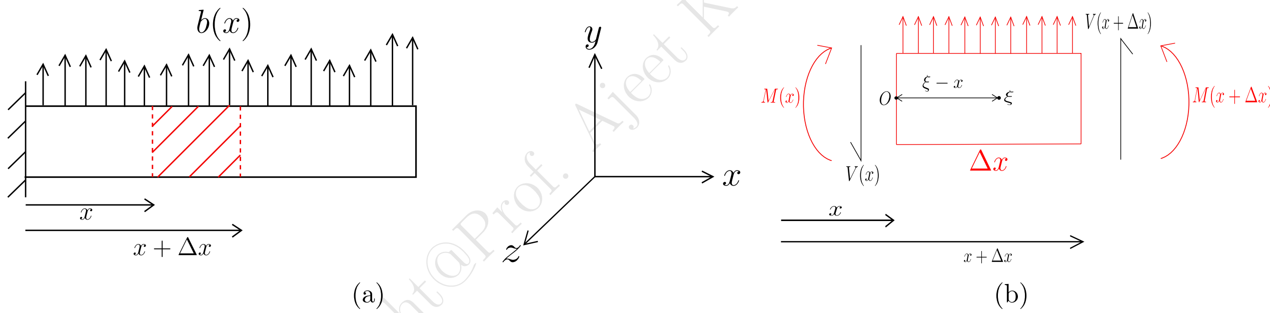


Figure 1: (a) A beam acted upon by a distributed load $b(x)$ with its small part shown in red (b) Free body diagram of the small part of the beam

at a distance x from the clamped end (see the red region in Figure 1a) and further draw its free body diagram as shown in Figure 1b. On this small part, apart from distributed load, there will be bending moment as well as shear force acting at the two ends. By convention, for the cross section having normal in $+x$ direction, shear force acting in $+y$ direction is considered as positive while for the cross section having normal in $-x$ direction, shear force acting in $-y$ direction is considered as positive. We denote this shear force by V. The center of the left cross-section is marked as O and the position of a general point in this part of the beam is denoted by ξ which varies from x to $x + \Delta x$. Due to static equilibrium, the net moment on this part of the beam must be zero about any point. In particular, let us do moment balance about O:

$$\sum \overrightarrow{M}/_O = \underline{0}$$

$$\Rightarrow \quad M(x+\Delta x)\ \underline{e}_3 - M(x)\ \underline{e}_3 + \Delta x\ V(x+\Delta x)\ \underline{e}_3 + \int_x^{x+\Delta x} (\xi - x) b(\xi) d\xi\ \underline{e}_3 = \underline{0}. \tag{1}$$

As everything is pointing in $\underset{\sim}{e}_3$ direction, we obtain the following scalar equation:

$$M(x+\Delta x) - M(x) + \Delta x\ V(x+\Delta x) + \int_x^{x+\Delta x} (\xi - x) b(\xi) d\xi = 0. \tag{2}$$

Since this equation holds for an arbitrary part of the beam having arbitrary length Δx, we can divide both the sides by Δx and take the limit $\Delta x \to 0$ in order to make the beam's small part reduce to a line, i.e.,

$$\lim_{\Delta x \to 0} \frac{M(x+\Delta x) - M(x) + \Delta x\ V(x+\Delta x) + \int_x^{x+\Delta x} (\xi - x) b(\xi) d\xi}{\Delta x} = 0$$

$$\Rightarrow \quad \frac{dM}{dx} + V(x) + \lim_{\xi \to x} (\xi - x) b(\xi) = 0 \qquad (\because \text{ as } \Delta x \to 0,\ \xi \to x)$$

$$\text{or} \quad \frac{dM}{dx} + V(x) = 0. \tag{3}$$

We have derived an important relation between the variation of bending moment and shear force. It says that whenever moment varies along the beam, there has to be a non-zero shear force acting on the beam's cross-section. The formula for σ_{xx} can still be taken to be the same as earlier. We just have to use the local value of bending moment in the formula, i.e.,

$$\sigma_{xx}(x, y, z) = \frac{-M(x) y}{I_{zz}} \tag{4}$$

where y represents distance from the neutral axis as earlier.

2 Variation of τ_{yx} in the cross-sectional plane

The shear components of traction in the cross section are τ_{yx} and τ_{zx}. As there is an overall shear force $V(x)$ acting on the cross section in y direction, τ_{yx} must be non-zero. Let us now try to obtain its distribution in the cross-section. We first make a simplifying assumption that τ_{yx} is only a function of y and not of z. This means that τ_{yx} would be the same at all points on lines parallel to z axis as shown in Figure 2 - different horizontal lines will have different τ_{yx} though. To obtain its distribution, we cut a small cuboid element from the beam as shown in green in Figure 3a. The bottom surface ($-y$ plane) of this element is at a distance of y from the neutral plane whereas its left face ($-x$ plane) is at a distance x from the beam's clamped end. A zoomed view of this green cuboid with all external loads acting on it is shown in Figure 3b. On the top face ($+y$ plane), distributed load $b(x)$ acts. The bottom face is the $-y$ plane and traction components τ_{xy}, σ_{yy} and τ_{zy} act on it in $-x$, $-y$ and $-z$ directions, respectively. The $+z$ and $-z$ faces (side faces) are part of the lateral surfaces of the original beam. As external forces are assumed to be applied only on $+y$-plane of the beam, the $+z$ and $-z$ faces of the element are traction free. On the $+x$ face, we have bending stress σ_{xx} that we have derived already. We also have τ_{yx} and τ_{zx} acting there. Similarly, σ_{xx}, τ_{yx} and τ_{zx} act on $-x$ plane but in negative directions. In order to find τ_{yx} (or τ_{xy}), we just need to balance the forces on this small cuboidal element in x direction. Let's first consider the force due to σ_{xx} on $+x$ and $-x$ faces:

$$\int_{\frac{-b}{2}}^{\frac{b}{2}} \int_y^{\frac{h}{2}} \sigma_{xx}(x+\Delta x, \eta, \gamma)\ d\eta d\gamma \quad - \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_y^{\frac{h}{2}} \sigma_{xx}(x, \eta, \gamma)\ d\eta d\gamma. \tag{5}$$

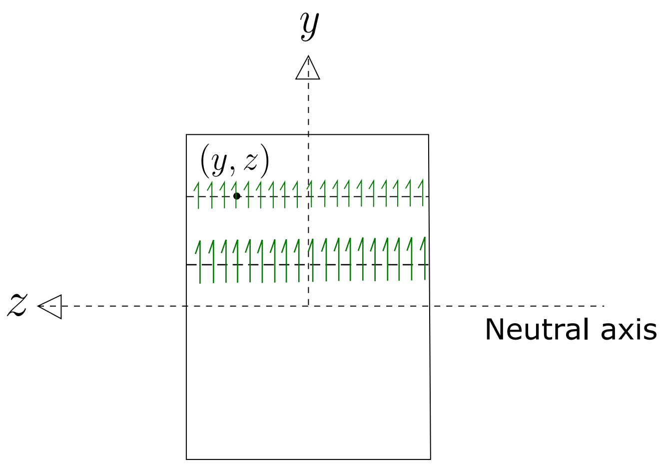


Figure 2: A typical cross section of the beam with variation of τ_{yx} shown such that it is a function of y alone

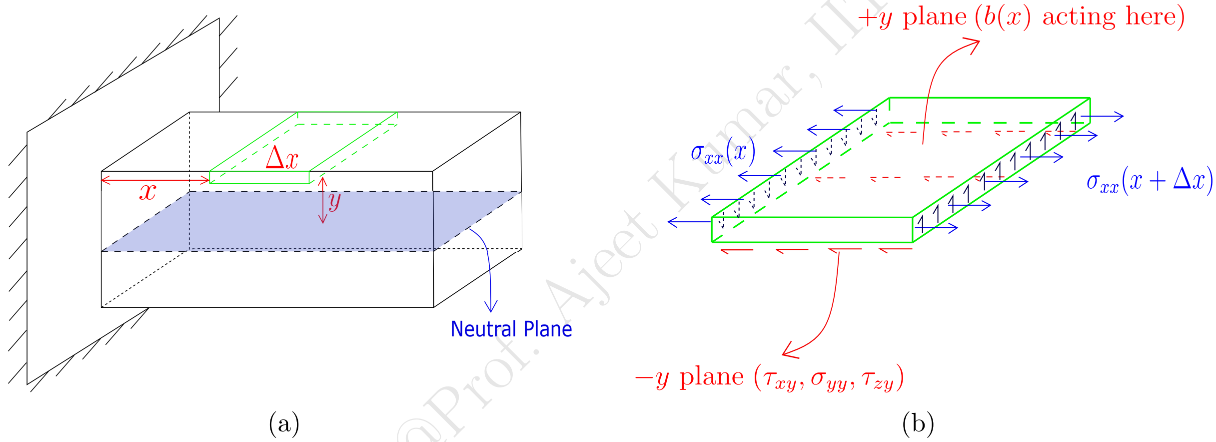


Figure 3: (a) A green cuboid element cut from the beam (b) Free body diagram of the cuboidal element cut from the beam

The force on $+y$ face has no component in x direction since the external distributed load acting there is assumed to act in y direction. However, $-y$ face has τ_{xy} acting on it which contributes the following force in x direction:

$$\int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{x}^{x+\Delta x} -\tau_{xy}(\xi, y, \gamma) d\xi d\gamma. \tag{6}$$

Summing all the forces in x direction to zero, we get

$$\int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{y}^{\frac{h}{2}} \Big[\sigma_{xx}(x+\Delta x, \eta, \gamma) - \sigma_{xx}(x, \eta, \gamma)\Big] d\eta d\gamma \quad - \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{x}^{x+\Delta x} \tau_{xy}(\xi, y, \gamma) d\xi d\gamma = 0. \tag{7}$$

Upon further substituting σ_{xx} from (4) in it, we get

$$\iint_{x \text{ plane}} \left[\frac{-M_z(x+\Delta x)\,\eta}{I_{zz}} - \frac{-M_z(x)\,\eta}{I_{zz}} \right] dA \quad - \int_{\frac{-b}{2}}^{\frac{b}{2}} \int_{x}^{x+\Delta x} \tau_{xy}(\xi, y, \gamma) d\xi d\gamma = 0. \tag{8}$$

As the first integral is over x plane, M_z and I_{zz} act as constants. We had also assumed initially that τ_{xy} does not vary with z or γ coordinate. All these simplifications lead to

$$-\left[\frac{M_z(x+\Delta x) - M_z(x)}{I_{zz}} \right] \iint_{x \text{ plane}} \eta dA \quad - \int_{x}^{x+\Delta x} \tau_{xy}(\xi, y) d\xi \int_{\frac{-b}{2}}^{\frac{b}{2}} d\gamma = 0. \tag{9}$$

Since the above equation holds for cuboid elements of any length Δx, we first divide the above equation by Δx and then take the limit $\Delta x \to 0$, i.e.,

$$\begin{aligned}
&-\left[\lim_{\Delta x \to 0} \frac{M_z(x+\Delta x) - M_z(x)}{\Delta x} \right] \frac{1}{I_{zz}} \iint_{x \text{ plane}} \eta dA \quad - b \lim_{\Delta x \to 0} \frac{\int_x^{x+\Delta x} \tau_{xy}(\xi, y) d\xi}{\Delta x} = 0 \\
&\Rightarrow -\frac{dM_z}{dx} \frac{1}{I_{zz}} \iint_{x \text{ plane}} \eta dA \quad - b\, \tau_{xy}(x, y) = 0 \\
&\Rightarrow \frac{V(x)}{I_{zz}} \iint_{x \text{ plane}} \eta dA \quad - b\, \tau_{xy}(x, y) = 0 \qquad \text{(using (3))}
\end{aligned} \tag{10}$$

The integral in the first term above is y-moment of x face of the cuboid element which is also shown as the shaded area in Figure 4a. We denote this moment by $Q(y)$: a function of y alone. Thus, the

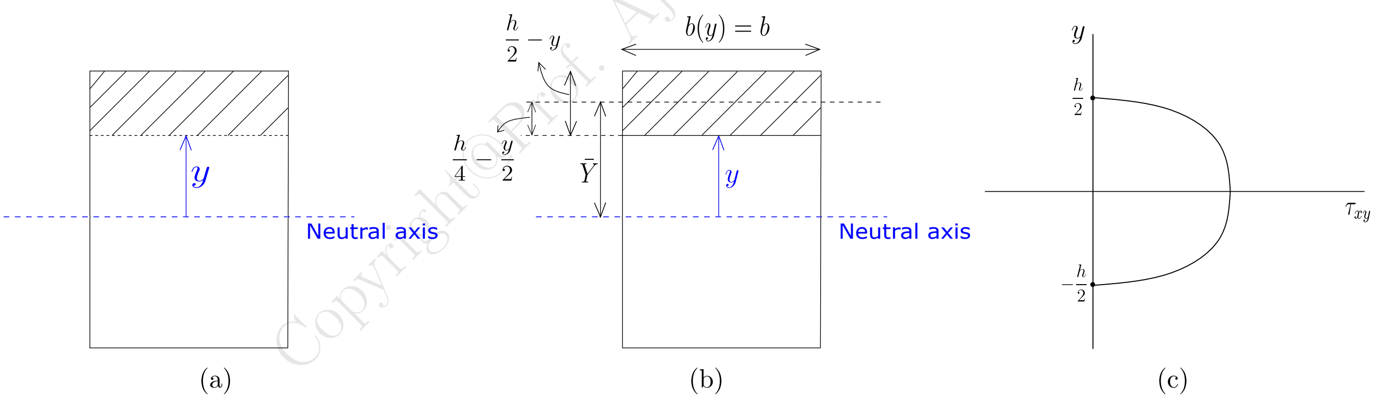


Figure 4: (a) The cross section of the beam with the shaded region representing the area of the x face of the small cuboidal element (b) Calculating second moment of shaded area (c) Plot of τ_{yx} vs y for a rectangular cross section

final expression for τ_{xy} becomes

$$\boxed{\tau_{xy}(x, y) = \frac{V(x) Q(y)}{I_{zz}\, b(y)}} \tag{11}$$

which is same as τ_{yx}, the shear component in the cross-sectional plane. Here, we have allowed b, the width of the cross section, to vary with y. If we compare equations (4) and (11), we see that while bending stress σ_{xx} is proportional to moment, shear stress τ_{yx} is proportional to shear force.

3 Variation in τ_{yx} for some representative cross-sections

3.1 Rectangular cross-section

A typical rectangular cross section is shown in Figure 4b and we want to find the value of τ_{yx} at a distance of y from the neutral axis. For applying equation (11), we need to find $Q(y)$, $b(y)$ and I_{zz}. As this is a rectangular cross section, width $b(y)$ is a constant and equal to b. We have already derived I_{zz} for a rectangular cross section in the preceding lecture which is

$$I_{zz} = \frac{1}{12} bh^3. \tag{12}$$

We only need to obtain expression for the first moment $Q(y)$ of the area above y line where τ_{yx} is to be calculated (shown as the shaded region in Figure 4b). The first moment will simply be y coordinate of the centroid of the shaded area multiplied by the shaded area. As the height of the shaded area is $\frac{h}{2} - y$, its centroid will be at half of this distance from the y line and hence at

$$\bar{Y} = y + \frac{1}{2}(\frac{h}{2} - y) = \frac{1}{2}\left(y + \frac{h}{2}\right) \tag{13}$$

from the neutral axis. Thus, $Q(y)$ becomes

$$Q(y) = \bar{Y}A = \frac{1}{2}\left(y + \frac{h}{2}\right) b \left(\frac{h}{2} - y\right) = \frac{1}{2} b \left(\frac{h^2}{4} - y^2\right) \tag{14}$$

which leads to

$$\tau_{yx} = \frac{V(x)\ \frac{1}{2} b \left(\frac{h^2}{4} - y^2\right)}{\frac{1}{12} bh^3\ b} = \frac{V}{bh} \times 6 \left(\frac{1}{4} - \left(\frac{y}{h}\right)^2\right). \tag{15}$$

As V is the total shear force on the cross section and bh is the area of the cross section, $\frac{V}{bh}$ equals average shear stress τ_{avg} while τ_{yx} at the neutral axis is

$$\tau_{yx}(y = 0) = \tau_{\text{avg}} \times 6 \left(\frac{1}{4} - 0\right) = \frac{3}{2} \tau_{\text{avg}}.$$

Likewise, at the periphery of the cross section $\left(y = \pm\frac{h}{2}\right)$, we have

$$\tau_{yx}(y = \pm\frac{h}{2}) = \tau_{\text{avg}} \times 6 \left(\frac{1}{4} - \frac{1}{4}\right) = 0. \tag{16}$$

This variation of τ_{yx} vs y is shown in Figure 4c. We observe that shear stress is maximum at centroid and vanishes at the two ends. There is another way to realise the vanishing of shear stress at the ends. The points $y = \pm\frac{h}{2}$ also lie on top and bottom surfaces of the beam, respectively. There is no external traction on the bottom surface whereas on the top surface, the distributed load $b(x)$ acts in y direction. Thus, τ_{xy} is zero at both top and bottom surfaces. However, due to τ_{xy} and τ_{yx} being equal, shear stress vanishes at $y = \pm\frac{h}{2}$ on the cross-sectional plane.

3.2 I-beam cross-section

Figure 5a shows the cross-section of an I-beam. The centroid of this section would be at the center because of symmetry. So, the neutral axis passes through the center. There are two different values

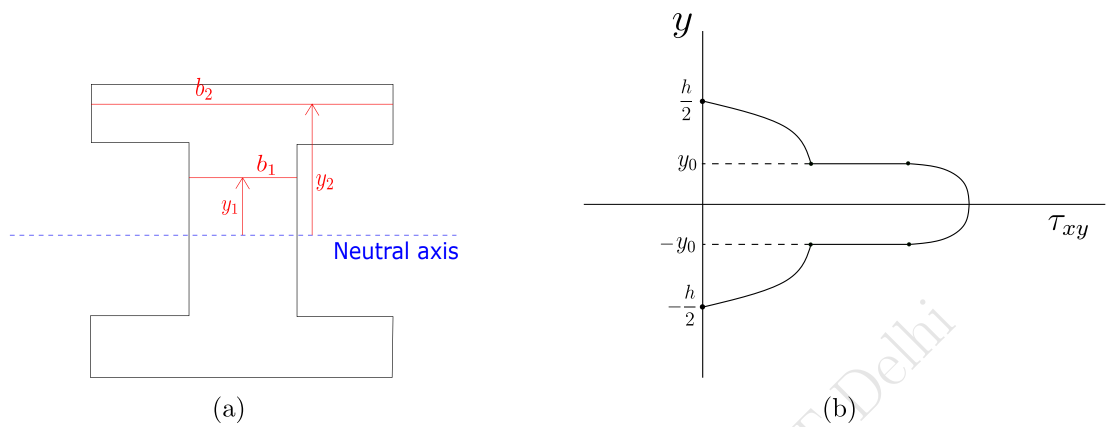


Figure 5: (a) The cross section of an I-beam (b) Variation in shear stress in an I-beam

of width possible in the cross section as shown in Figure 5a. To find τ_{yx} at a distance y from the neutral axis, we can again use equation (11). As the width changes abruptly in this case, the distribution of shear stress will also exhibit a jump corresponding to this abrupt change in width b (see Figure 5b).

3.3 Circular cross-section

In the derivation for rectangular beams, we had assumed that τ_{yx} is independent of z coordinate. For a circular beam however, this assumption does not hold which we now show. If τ_{yx} is constant along lines parallel to z axis, the shear stress will be as shown in Figure 6. Basically, it is non-zero even at the ends. Let us assume a vertical distributed load is acting on the beam's lateral surface which is also a radial plane for circular beams. In this case, the shear component of traction on the lateral surface along the cylinder axis is zero. So, due to symmetry of stress matrix, the radial component of shear traction on peripheral points of cross-sectional plane must also vanish. However, if we look at Figure 6, the assumption of τ_{yx} being independent of the z coordinate leads to a non-zero radial component at the periphery which is a contradiction. Thus, we conclude that for circular cross-sections, considering τ_{yx} independent of z is not a good assumption. Nevertheless, this assumption is often used even for circular beams since it gives an approximate distribution of shear stress.

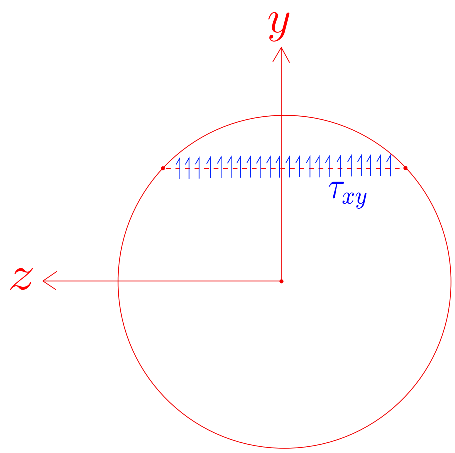


Figure 6: Shear stress τ_{yx} on a cross-section of a circular beam with the assumption that it is independent of z

4 Objective questions to recall concepts

1. Consider a beam which is clamped at one end and free at the other end. It is further subjected to some distributed load. The bending moment (in N-mm) along the length turns out to be $M(x) = 2x^3 - 5x^2$ where x is the distance (in mm) measured from the free end of the beam. The magnitude of shear force (in N) in the cross-section at $x = 5$mm is

   (a) 100
   (b) 50
   (c) 25
   (d) 10

2. When a beam, pinned at both the ends, is subjected to transverse load at its center

   (a) the maximum bending moment generates at the two ends
   (b) the maximum bending moment generates at the center
   (c) no bending moment generates in the beam
   (d) none

3. During non-uniform bending of rectangular beams (with the bending moment aligned along the cross-section's principal axis), which of the following hold true?

   (a) no shear stress acts in the cross-section
   (b) shear stress varies in the cross-section with its maximum value on the neutral axis
   (c) the ratio of shear stress at the neutral axis to the value of average shear stress equals 1.5
   (d) shear stress vanishes along all four edges of the cross-section

4. Consider an I-beam (with its axis along x-axis) clamped at one end. A uniformly distributed load is applied on the top face of the beam along y-axis. If we plot the shear stress τ_{xy} vs y for the cross section of the beam,

   (a) it would be a semicircle.
   (b) the variation would be quadratic throughout.
   (c) there would be a jumps in the value of τ_{xy} at the points where the width of the cross-section (length in z direction) changes.
   (d) None of the above.

5. Select True or False
   Consider a cylindrical beam (with its axis along the x axis) clamped at one end. A uniformly distributed load is applied on the top face of the beam along the y axis. For such a beam, the shear stress τ_{xy} is uniform at every point on y=constant lines in the cross-section.

   (a) True
   (b) False

6. The ratio of maximum shear stress to the average shear stress in a beam with square cross-section and subjected to transverse load is

   (a) 1
   (b) $\frac{2}{3}$
   (c) $\frac{3}{2}$
   (d) 2

7. When a beam, clamped at one end, is subjected to transverse load at the free end

   (a) the maximum bending moment generates at the free end
   (b) the maximum bending moment generates at the clamped end
   (c) the maximum bending moment generates in the middle of the beam
   (d) none

8. Why is the shear stress distribution in the cross-section of transversely loaded beams not uniform?

9. Think of a beam having rectangular cross-section. Suppose shear force is applied at one end of the beam.

   (a) Which of the following is true?
      i. The beam will always bend in the plane in which the beam and the shear force lies
      ii. The beam can bend in a different plane if the shear force does not act parallel to one of the edges of the cross-section

iii. shear traction is uniformly distributed on the beam's cross-section
iv. none of these

(b) Suppose the shear force acts parallel to one of the edges of the cross-section. Where will the shear traction be maximum?
   i. On the line passing through centroid and perpendicular to shear force direction
   ii. On the edge which is perpendicular to shear force direction
   iii. More information is required
   iv. None of these

(c) The ratio of average and maximum transverse shear stress (τ_{xy}) in a solid circular cross-section beam of diameter d subjected to shear force Q_y will be:
   i. 1.5
   ii. 4/3
   iii. 1
   iv. 2

10. Consider the following statement: Two beams of identical cross-section but of different materials carry the same bending moment and shear force at a particular section, then
   (a) The maximum bending stress at that section in the two beams will be same.
   (b) The maximum shear stress at that section in the two beams will be same.
   (c) Maximum bending stress at that section will depend upon the elastic modulus of the beam material.
   (d) Curvature of the beam having greater value of Young's modulus will be larger.

11. Two beams, one having square cross-section and another circular cross-section, are subjected to same amount of bending moment. If the cross-sectional area as well as material of both the beams are the same, then
   (a) maximum bending stress developed in both the beams is the same.
   (b) the circular beam experiences higher maximum bending stress than the square one.
   (c) the square beam experiences higher maximum bending stress than the circular one.
   (d) as the material is same, both beams experience same bending stress.

12. Think of a beam having rectangular cross-section of dimension ($b \times h$) and subjected to a transverse shear force $V(x)$.
   (a) Shear stress (τ_{yx}) at the center of cross-section will be
      i. $3V/2bh$
      ii. V/bh
      iii. $2V/3bh$

iv. zero

(b) Which of the following is true?

i. Shear stress will be continuous and vary linearly along the height of cross-section.

ii. Shear stress varies parabolically with minimum at the top and bottom, maximum at the middle of cross-section.

iii. Shear stress at the periphery of cross-section will be zero.

iv. Average shear stress is 3/2 times the maximum shear stress.

(c) While calculating shear stress distribution in the cross-section, we need to calculate the first moment of area 'Q'. Which of the following is/are true regarding 'Q'?

i. It is the first moment of area above $y = y_0$ line (here y_0 is the vertical distance from neutral axis) where τ_{yx} is to be calculated.

ii. It is y-coordinate of the centroid of the cross-section area multiplied by cross-section area.

iii. It is y-coordinate of the centroid of the shaded part of the crosssection above along $y = y_0$ line multiplied by the shaded area.

iv. None of these.

5 Subjective questions to practise

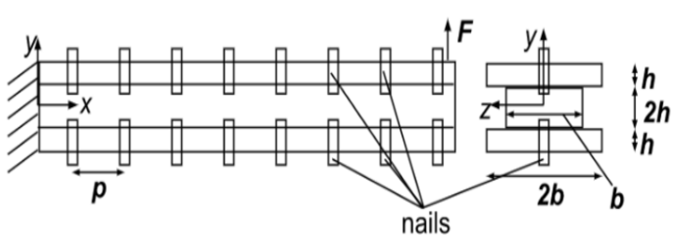
1. For an I-beam, assume the beam is subjected to transverse load

   (a) Obtain an expression for variation in shear stress τ_{xy} within its cross-section. You can use the formula $\tau_{xy} = \frac{VQ(y)}{I_{zz}T(y)}$

   (b) Using the expression above, draw a graph depicting qualitative variation in shear stress within the cross-section.

   (c) Where is the shear stress maximum? Find the ratio of the maximum shear stress to the average shear stress in the cross-section?

2. A cantilever beam (figure below) is made up of three wooden planks nailed together as shown. Find the expression for shear force in nails in terms of F, p, b and h. For the cross-section: $I_{zz} = 10bh^3$.



3. What will be the ratio of average and maximum transverse shear stress (τ_{xy}) in a solid circular cross-section beam of diameter d subjected to shear force S_y.