1) $$\underline{\underline{\sigma}}\ \underline{n}_1 = \lambda_1 \underline{n}_1 \quad — \ (1)$$

$$\underline{\underline{\sigma}}\ \underline{n}_2 = \lambda_2 \underline{n}_2 \quad — \ (2)$$

Take dot products of (1) with $\underline{n}_2$
(2) with $\underline{n}_1$

$$\underline{\underline{\sigma}}\ \underline{n}_1 \cdot \underline{n}_2 = \lambda_1 \underline{n}_1 \cdot \underline{n}_2 \quad — \ (3)$$

$$\underline{\underline{\sigma}}\ \underline{n}_2 \cdot \underline{n}_1 = \lambda_2 \underline{n}_2 \cdot \underline{n}_1 \quad — \ (4)$$

Subtract (4) from (3)

$$(\lambda_1 - \lambda_2)\ \underline{n}_1 \cdot \underline{n}_2 = 0$$

$\Rightarrow$ Either $\lambda_1 = \lambda_2$, or $\underline{n}_1 \cdot \underline{n}_2 = 0$

Since all eigenvalues are distinct, $\lambda_1 \neq \lambda_2$

Thus, the eigenvectors must be perpendicular to each other

2) $$\underline{\underline{\sigma}}\ \underline{n}_1 = \lambda \underline{n}_1 \quad — \ (1) \quad \times \alpha$$

$$\underline{\underline{\sigma}}\ \underline{n}_2 = \lambda \underline{n}_2 \quad — \ (2) \quad \times \beta$$

Add (1) and (2)

$$\underline{\underline{\sigma}}\ (\alpha \underline{n}_1 + \beta \underline{n}_2) = \lambda \underbrace{(\alpha \underline{n}_1 + \beta \underline{n}_2)}_{\text{this is also an eigenvector}}$$

$$\underline{\underline{\sigma}}\ \underline{n}_3 = \lambda' \underline{n}_3 \quad — \ (3)$$

From Problem 1, we know $\underline{n}_3 \perp \underline{n}_1, \underline{n}_2$

$\alpha \underline{n}_1 + \beta \underline{n}_2$ spans a plane $\perp$ to $\underline{n}_3$

3) $[\underline{\underline{\sigma}}] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ $\qquad [\underline{\underline{\sigma}} - \lambda \underline{\underline{I}}][\underline{n}] = \underline{0}$

$$\det([\underline{\underline{\sigma}} - \lambda \underline{\underline{I}}]) = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda(\lambda^2 - 1) - 1(-\lambda - 1) + 1(1 + \lambda) = 0$$

$$\Rightarrow -\lambda^3 + \lambda + \lambda + 1 + 1 + \lambda = 0$$

$$\Rightarrow -\lambda^3 + 3\lambda + 2 = 0$$

$$\Rightarrow -\lambda^3 - 1 + 3\lambda + 3 = 0$$

$$\Rightarrow -(\lambda^3 + 1) + 3(\lambda + 1) = 0$$

$$\Rightarrow -(\lambda + 1)(\lambda^2 - \lambda + 1) + 3(\lambda + 1) = 0$$

$$\Rightarrow (\lambda + 1)[-\lambda^2 + \lambda - 1 + 3] = 0$$

$$\Rightarrow (\lambda + 1)[-\lambda^2 + \lambda + 2] = 0$$

$$\Rightarrow (\lambda + 1)[-\lambda^2 + 2\lambda - \lambda + 2] = 0$$

$$\Rightarrow -(\lambda + 1)(\lambda + 1)(\lambda - 2) = 0$$

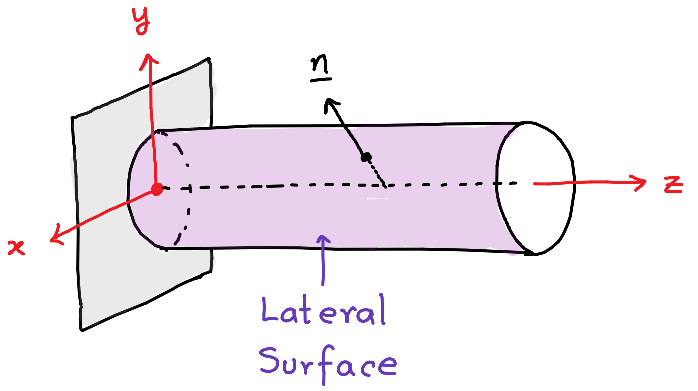
$$\Rightarrow \lambda_1 = \lambda_2 = -1, \quad \lambda_3 = 2$$

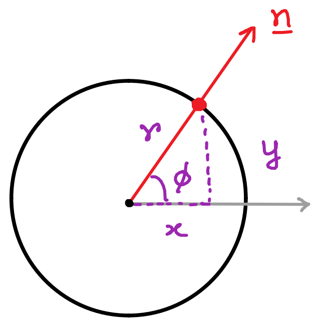
Two repeated eigenvalues, only $\underline{n}_3$ will be unique

$$\left.\begin{aligned} -2n_{11} + n_{12} + n_{13} &= 0 \\ n_{11} - 2n_{12} + n_{13} &= 0 \\ n_{11} + n_{12} - 2n_{13} &= 0 \\ n_{11}^2 + n_{12}^2 + n_{13}^2 &= 1 \end{aligned}\right] \rightarrow \begin{pmatrix} n_{11} \\ n_{12} \\ n_{13} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

4)
$$[\underline{\underline{\sigma}}]_{\begin{pmatrix}x\\y\\z\end{pmatrix}} = \begin{bmatrix} 0 & 0 & -G\theta y \\ 0 & 0 & G\theta x \\ -G\theta y & G\theta x & 0 \end{bmatrix}$$

Check yourself that the above stress tensor satisfies the equations of equilibrium. The lateral surface has outward normal $\underline{n}$





$\cos\phi = \frac{y}{r}$

$\sin\phi = \frac{x}{r}$

$$[\underline{n}]_{\begin{pmatrix}x\\y\\z\end{pmatrix}} = \begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{bmatrix} = \begin{bmatrix} y/r \\ x/r \\ 0 \end{bmatrix}$$

$$[\underline{T}^n] = [\underline{\underline{\sigma}}][\underline{n}]$$

$$= \begin{bmatrix} 0 & 0 & G\theta y \\ 0 & 0 & -G\theta x \\ G\theta y & -G\theta x & 0 \end{bmatrix} \begin{bmatrix} y/r \\ x/r \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ There is no external surface force on the lateral surface

For principal stresses, one needs to solve an eigenvalue problem

$$\det\left([\underline{\underline{\sigma}} - \lambda \underline{\underline{I}}]\right) = 0$$

$$\Rightarrow \det\left(\begin{bmatrix} -\lambda & 0 & G\theta y \\ 0 & -\lambda & -G\theta x \\ G\theta y & -G\theta x & -\lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow -\lambda^3 + \lambda G^2\theta^2(x^2+y^2) = 0$$

$$\Rightarrow -\lambda\left(\lambda^2 - G^2\theta^2(x^2+y^2)\right) = 0$$

$$\Rightarrow \lambda_1 = G\theta(x^2+y^2)^{1/2}, \quad \lambda_2 = 0, \quad \lambda_3 = -G\theta(x^2+y^2)^{1/2}$$

The first principal direction can be found as:

$$\begin{bmatrix} -\lambda_1 & 0 & G\theta y \\ 0 & -\lambda_1 & -G\theta x \\ G\theta y & -G\theta x & -\lambda_1 \end{bmatrix} \begin{bmatrix} n_{11} \\ n_{12} \\ n_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Below we write the three equations (of which only two are independent) plus one equation for normalization of the vector

$$-\lambda_1 n_{11} + G\theta y \, n_{13} = 0$$

$$-\lambda_1 n_{12} - G\theta x \, n_{13} = 0$$

$$G\theta y \, n_{11} - G\theta x \, n_{12} - \lambda_1 n_{13} = 0$$

$$n_{11}^2 + n_{12}^2 + n_{13}^2 = 1$$

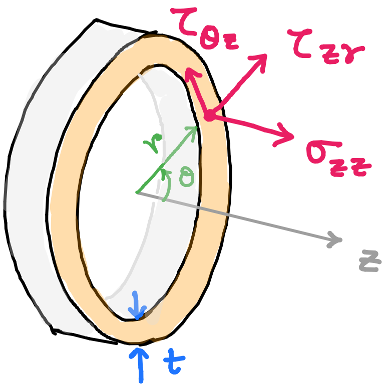
$$n_{11} = \mp \frac{G\theta y}{\sqrt{x^2+y^2}}$$

$$n_{12} = \mp \frac{G\theta x}{\sqrt{x^2+y^2}}$$

$$n_{13} = \pm 1/\sqrt{2}$$

You can derive the two other principal directions similarly.

5) Consider a circular strip from the middle of the cylinder



In general, stresses are functions of the coordinates, so

$$\left.\begin{matrix}\sigma_{zz}\\ \tau_{\theta z}\\ \tau_{zr}\end{matrix}\right\} \text{ functions of } r, \theta, z$$

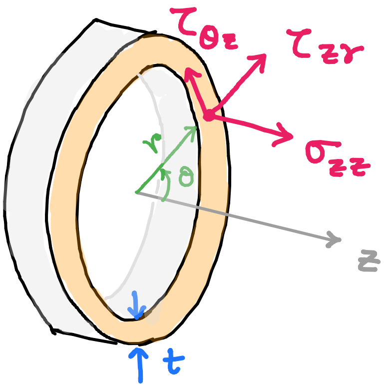
Due to axisymmetry of the geometry and that of the applied forces about the center of the cylinder, it can be fairly assumed that σ_{zz}, τ_{zr} and $\tau_{\theta z}$ are not going vary with θ

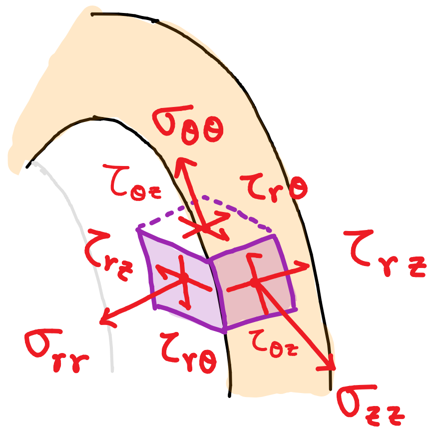
$$\left.\begin{matrix}\sigma_{zz}\\ \tau_{zr}\\ \tau_{\theta z}\end{matrix}\right\} \text{ functions of } r \text{ and } z \text{ only}$$

Due to coaxiality of the force applied in the z-direction and no moment, σ_{zz}, τ_{zr}, $\tau_{\theta z}$ must not change with z.

$$\left.\begin{matrix}\sigma_{zz}\\ \tau_{zr}\\ \tau_{\theta z}\end{matrix}\right\} \text{ functions of } r \text{ only}$$

In this case, the thickness is very small so variations across r is negligible, so we will consider constant values of σ_{zz}, τ_{zr}, and $\tau_{\theta z}$





$\left.\begin{matrix}\sigma_{zz}\\ \tau_{\theta z}\\ \tau_{zr}\end{matrix}\right\}$ constant

Since there is no external shear traction on the inner or outer boundary, $\boxed{\tau_{zr} = \tau_{rz} = 0}$

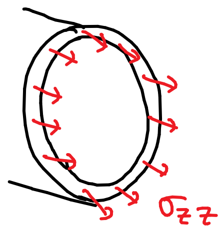
For $\tau_{r\theta}$, the resultant force generated at any C/s must be

Moment abt z $= \tau_{\theta z}\,(2\pi r t)$

$\Rightarrow \quad 0 = \tau_{\theta z}\,(2\pi r t)$

$\Rightarrow \quad \boxed{\tau_{\theta z} = 0}$

Using force balance in z-dir



$F = \sigma_{zz}\,(2\pi r t)$

$\Rightarrow \quad \boxed{\sigma_{zz} = \dfrac{F}{2\pi r t}}$

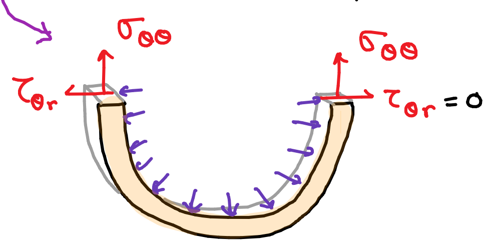
On the -ve r face, we have σ_{rr}, $\tau_{rz} = 0$, $\boxed{\tau_{\theta r} = 0}$ (to maintain symm.)

On the internal surface, we have $\sigma_{rr} = -p$ whereas on the +ve r face, we have no external force, so $\sigma_{rr} = 0$, i.e

$$\boxed{\sigma_{rr}\big|_{r} \text{ (internal face)} = -p, \quad \sigma_{rr}\big|_{r+t} \text{ (external face)} = 0}$$

On the +ve θ face, $\tau_{r\theta} = 0$, $\tau_{\theta z} = 0$, $\sigma_{\theta\theta}$ (?)

We cut the C/s



$2\sigma_{\theta\theta}\,(t) = p\,(2r)$

$\Rightarrow \quad \boxed{\sigma_{\theta\theta} = \dfrac{pr}{t}}$