Deformation and strain

We have seen force balance at a point within a stressed body

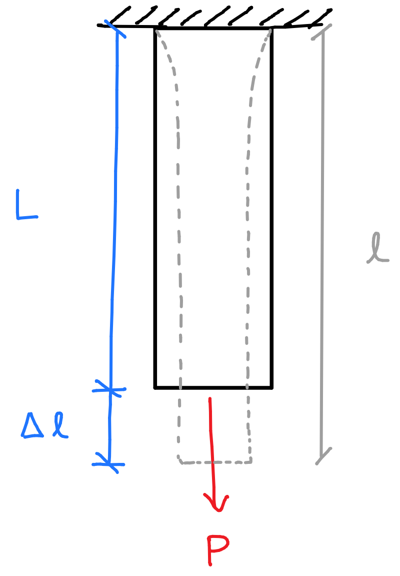
- the results we obtained were based on requirements of equilibrium only
- However, we found that the stress equilibrium equations that we derived were not enough to solve for the stresses; we need more equations
- In essence, we have only looked at force-equilibrium relations for a deformable body so far. Deformation-compatibility & force-deformation relations still remain to be explored.

In this lecture, we will starting to look at deformation and requirement of geometric compatibility for a continuously distributed body.

Geometric compatibility of deformation implies that deformation should occur is such a fashion that there is no overlap or void created in the deformed body.

Here, we assess the problem of deformation and geometric compatibility from purely a geometrical perspective, and independent of the equilibrium requirements.

Consider an example where you have a bar hanging under the action of gravitational force due to self-weight



We defined longitudinal strain

$$= \frac{\text{Change in length}}{\text{Total length}}$$

$$= \frac{\Delta \ell}{\ell} \quad \text{or} \quad \frac{\Delta \ell}{L}$$

Deformed length

original length

For most engineering structures, deformations are very small,

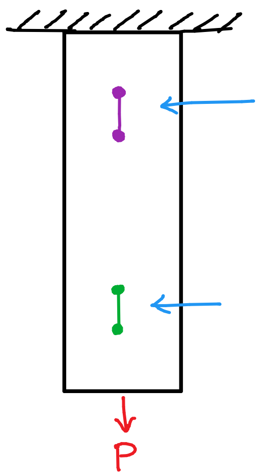
$$\Delta \ell << L$$

$$\therefore \quad \frac{\Delta \ell}{L} \approx \frac{\Delta \ell}{\ell}$$

This definition of longitudinal strain is the AVERAGE longitudinal strain of the entire bar.

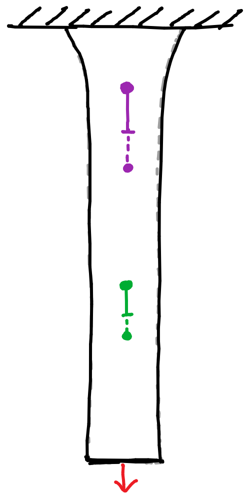
In general, "strains" in a body varies from point to point.

Take for example two line very small line elements at two different points in the body. We find that the tractions at the two levels are different. Hence, the line elements will undergo different amounts of elongation (and $\therefore$ different local strain)



traction is higher, hence this element will undergo a higher strain

traction is smaller, hence this element will undergo a smaller strain



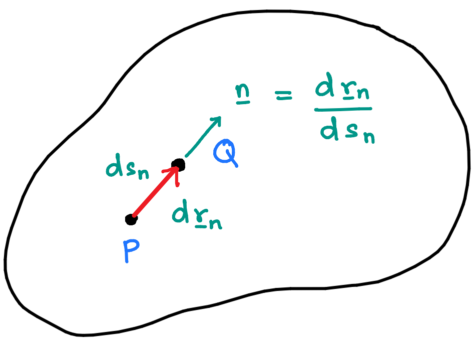
Definition of strain components

As we see from the previous example that we can describe deformations of a body in terms of changes in length of a small line element. We will now describe two types of strains that are commonly used to characterize these changes.

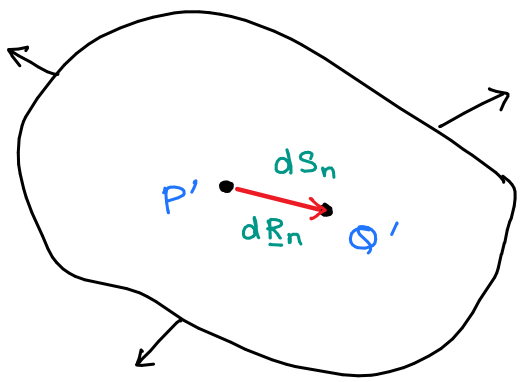
NORMAL STRAIN

When a body deforms, line elements inside the body may elongate or contract. The quantity that we use to measure the changes of length of a small line element at a point inside the body is called normal strain in simple terms.

Consider a small directed line segment, $d\underline{r}_n$ between two very close points P and Q in an undeformed body

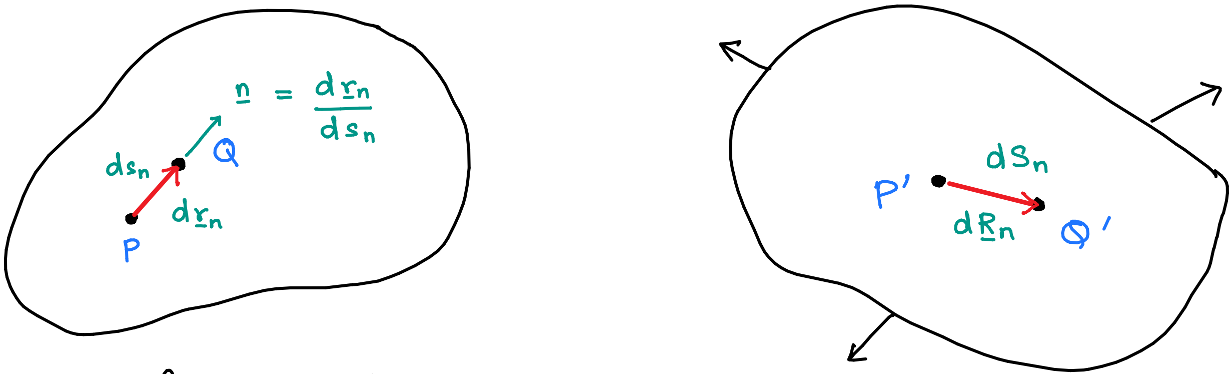


Undeformed body



Deformed body

When the body deforms, point P will move to a new point P' and Q will move to Q' and the original line segment will change to a new line segment, $d\underline{R}_n$, of length dS_n.



Undeformed body   Deformed body

The normal strain at the point P is defined as

$$E_{nn} = \frac{1}{2}\,\frac{dS_n^2 - ds_n^2}{ds_n^2}$$

$$= \frac{1}{2}\,\frac{d\underline{R}_n \cdot d\underline{R}_n - d\underline{r}_n \cdot d\underline{r}_n}{ds_n^2}$$

$$= \frac{1}{2}\left(\frac{d\underline{R}_n}{ds_n} \cdot \frac{d\underline{R}_n}{ds_n} - \overbrace{\cancel{\frac{d\underline{r}_n}{ds_n}}}^{\underline{n}} \cdot \overbrace{\cancel{\frac{d\underline{r}_n}{ds_n}}}^{\underline{n}}\right) \qquad \underline{n}\cdot\underline{n} = \|\underline{n}\|^2 = 1$$

$$= \frac{1}{2}\left(\left\|\frac{d\underline{R}_n}{ds_n}\right\|^2 - 1\right)$$

This definition of normal strain holds true regardless of the amount of strain — large or small.

Linearized Normal Strain

If the change in the length of the line segment is very small, the normal strain reduces to linearized normal strain

$$E_{nn} = \frac{1}{2}\,\frac{dS_n^2 - ds_n^2}{ds_n^2} = \frac{1}{2}\,\frac{(dS_n - ds_n)(\overset{ds_n}{\cancel{dS_n}} + ds_n)}{ds_n^2}$$

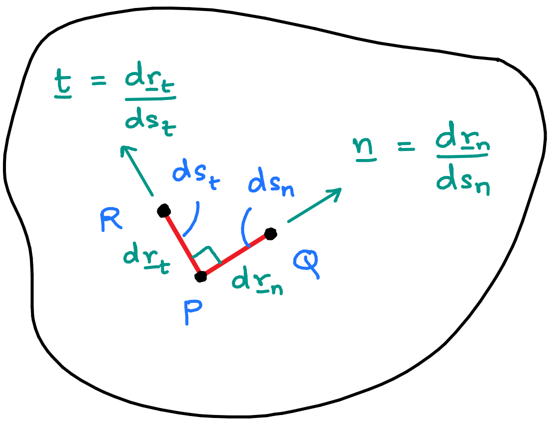
$$= \frac{1}{\cancel{2}}\,\frac{(dS_n - ds_n)}{ds_n}\,\frac{\cancel{2}\,d\cancel{s_n}}{d\cancel{s_n}}$$

$$\boxed{\epsilon_{nn} = \lim_{ds_n \to 0} \frac{dS_n - ds_n}{ds_n}}$$

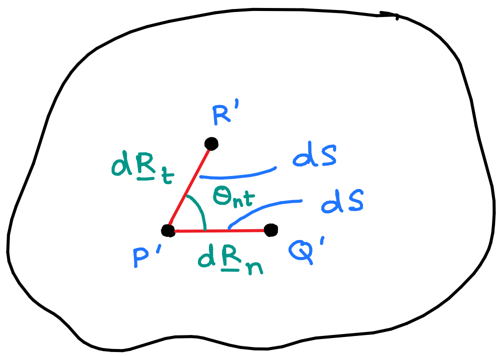
Shear strain

In addition to changes in length occuring in a deformable body, there are distortions which correspond to change in angles between two line segments. These changes in angles is defined using shear strain.

To define shear strain, we consider two small line segments $d\underline{r}_n$ and $d\underline{r}_t$ in the undeformed bod along $\underline{n}$ and $\underline{t}$ unit normal directions, respectively. The two units normals $\underline{n}$ & $\underline{t}$ are initially perpendicular to each other.



Undeformed body



deformed body

However in the deformed state, the line segments $(d\underline{R}_n, dR_t)$ will no longer be orthogonal. Let θ_{nt} be the new angle between the two rotated line segments. The shear strain is defined as

$$E_{nt} = \frac{1}{2} \frac{d\underline{R}_n}{ds_n} \cdot \frac{d\underline{R}_t}{ds_t}$$

This shear strain is related to the angular distortions occuring in a body since the cosine of the angle betn any two unit vectors is just the dot product of those unit vectors,

$$\cos \theta_{nt} = \sin\left(\frac{\pi}{2} - \theta_{nt}\right) = \frac{\frac{d\underline{R}_n}{ds_n} \cdot \frac{d\underline{R}_t}{ds_t}}{\left\| \frac{d\underline{R}_n}{ds_n} \right\| \left\| \frac{d\underline{R}_t}{ds_t} \right\|}$$

Using the definitions of normal and shear strains for $\underline{n}$ & $\underline{t}$ directions, we get

$$\sin\left(\frac{\pi}{2} - \theta_{nt}\right) = \frac{2\, E_{nt}}{\sqrt{1+2E_{nn}}\,\sqrt{1+2E_{tt}}}$$

which shows that if $\theta_{nt} = \pi/2$, then $E_{nt} = 0$ implying no angular distortion of these two lines.

When the changes of lengths and changes in angles are small then,

$$\sin\left(\frac{\pi}{2} - \theta_{nt}\right) \approx \left(\frac{\pi}{2} - \theta_{nt}\right)$$

$$E_{nn} << 1, \quad E_{tt} << 1$$

and so, we get linearized shear strain ϵ_{nt}

$$\epsilon_{nt} = \frac{1}{2}\left(\frac{\pi}{2} - \theta_{nt}\right)$$

There is also engineering strain

$$\gamma_{nt} = 2\,\epsilon_{nt} = \left(\frac{\pi}{2} - \theta_{nt}\right)$$