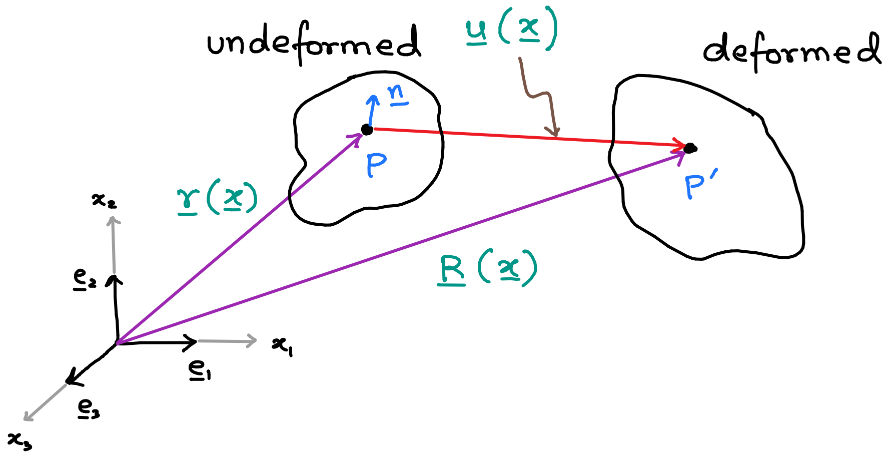
Strain displacement relations

Now we will look at the relation between displacement and strains

A convenient way of defining deformations in a body is to define the displacement vector $\underline{u}(\underline{x})$ for every point $P(\underline{x})$ in the undeformed body.



From geometry, we see that

$$\underline{R}(\underline{x}) = \underline{r}(\underline{x}) + \underline{u}(\underline{x})$$

Now if $\underline{r}$ changes by a small amount $d\underline{r}_n$ in the direction of unit normal $\underline{n}$, then we can write

$$d\underline{R}_n = d\underline{r}_n + d\underline{u}$$

Dividing by the length of $d\underline{r}_n$ which ds_n, we get

$$\frac{d\underline{R}_n}{ds_n} = \frac{d\underline{r}_n}{ds_n} + \frac{d\underline{u}}{ds_n}$$

$$= \underline{n} + \frac{d\underline{u}}{ds_n}$$

Using this result in the definition of normal strain

$$E_{nn} = \frac{1}{2}\left(\frac{d\underline{R}_n}{ds_n} \cdot \frac{d\underline{R}_n}{ds_n} - 1\right)$$

$$= \frac{1}{2}\left[\left(\underline{n} + \frac{d\underline{u}}{ds_n}\right) \cdot \left(\underline{n} + \frac{d\underline{u}}{ds_n}\right) - 1\right]$$

$$= \frac{1}{2}\left[\cancelto{1}{\underline{n} \cdot \underline{n}} + 2\underline{n} \cdot \frac{d\underline{u}}{ds_n} + \frac{d\underline{u}}{ds_n} \cdot \frac{d\underline{u}}{ds_n} - 1\right]$$

$$= \underline{n} \cdot \frac{d\underline{u}}{ds_n} + \frac{1}{2}\frac{d\underline{u}}{ds_n} \cdot \frac{d\underline{u}}{ds_n}$$

Similarly, if we let $\underline{r}$ change by a small amt $d\underline{r}_t$ in the unit normal direction $\underline{t}$, we get

$$\frac{d\underline{R}_t}{ds_t} = \frac{d\underline{r}_t}{ds_t} + \frac{d\underline{u}}{ds_t} = \underline{t} + \frac{d\underline{u}}{ds_t}$$

and using the results in the definition of shear strain

$$E_{nt} = \frac{1}{2}\frac{d\underline{R}_n}{ds_n} \cdot \frac{d\underline{R}_t}{ds_t}$$

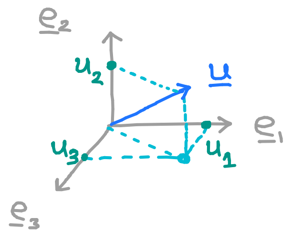
$$= \frac{1}{2}\left[\left(\underline{t} + \frac{d\underline{u}}{ds_t}\right) \cdot \left(\underline{n} + \frac{d\underline{u}}{ds_n}\right)\right]$$

$$= \frac{1}{2}\left[\underline{t} \cdot \frac{d\underline{u}}{ds_n} + \underline{n} \cdot \frac{d\underline{u}}{ds_t} + \frac{d\underline{u}}{ds_t} \cdot \frac{d\underline{u}}{ds_n}\right]$$

If the strains are small enough, we can neglect the products of the displacement gradients, and we see that for linearized normal and shear strains

$$\epsilon_{nn} = \underline{n} \cdot \frac{d\underline{u}}{ds_n}, \quad \epsilon_{nt} = \frac{1}{2}\left(\underline{n} \cdot \frac{d\underline{u}}{ds_t} + \underline{t} \cdot \frac{d\underline{u}}{ds_n}\right)$$

The displacement vector $\underline{u}$ can be written in terms of its scalar components along $(\underline{e}_1 - \underline{e}_2 - \underline{e}_3)$ axes



$$\underline{u} = u_1\, \underline{e}_1 + u_2\, \underline{e}_2 + u_3\, \underline{e}_3$$
$$= \sum_{i=1}^{3} u_i\, \underline{e}_i$$

$$\frac{\partial \underline{u}}{\partial x_j} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_j}\, \underline{e}_i \qquad (j = 1, 2, 3)$$

The gradient of displacement vector can be written in matrix form:

$$\nabla \underline{u} = \frac{\partial \underline{u}}{\partial \underline{x}} = \frac{\partial \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}}{\partial \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

Note the relations of ϵ_{nn} and ϵ_{nt} are valid for any directions $\underline{n}$ and $\underline{t}$, we can compute small strains for line segments oriented along $\underline{e}_1$, $\underline{e}_2$, and $\underline{e}_3$ directions.

$\underline{n} = \underline{e}_1$ $\quad \epsilon_{11} = \underline{e}_1 \cdot \frac{\partial \underline{u}}{\partial x_1} = \frac{\partial u_1}{\partial x_1}$

$\underline{n} = \underline{e}_1$, $\underline{t} = \underline{e}_2$

$$\epsilon_{12} = \epsilon_{21} = \frac{1}{2}\left(\underline{e}_1 \cdot \frac{\partial \underline{u}}{\partial x_2} + \underline{e}_2 \cdot \frac{\partial \underline{u}}{\partial x_1}\right) = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

$\underline{n} = \underline{e}_2$ $\quad \epsilon_{22} = \underline{e}_2 \cdot \frac{\partial \underline{u}}{\partial x_2} = \frac{\partial u_2}{\partial x_2}$

$\underline{n} = \underline{e}_2$, $\underline{t} = \underline{e}_3$

$$\epsilon_{23} = \epsilon_{32} = \frac{1}{2}\left(\underline{e}_3 \cdot \frac{\partial \underline{u}}{\partial x_2} + \underline{e}_2 \cdot \frac{\partial \underline{u}}{\partial x_3}\right) = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right)$$

$\underline{n} = \underline{e}_3$ $\quad \epsilon_{33} = \underline{e}_3 \cdot \frac{\partial \underline{u}}{\partial x_3} = \frac{\partial u_3}{\partial x_3}$

$\underline{n} = \underline{e}_1$, $\underline{t} = \underline{e}_3$

$$\epsilon_{13} = \epsilon_{31} = \frac{1}{2}\left(\underline{e}_3 \cdot \frac{\partial \underline{u}}{\partial x_1} + \underline{e}_1 \cdot \frac{\partial \underline{u}}{\partial x_3}\right) = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right)$$

We can write the strain tensor using displacement gradient,

$$\underline{\underline{\epsilon}} = \frac{1}{2}\left(\nabla \underline{u} + \nabla \underline{u}^T\right)$$

$$= \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

State of strain at a point

We have nine strain components : 3 normal strains & 6 shear strains, out of which only six components $\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \epsilon_{12}, \epsilon_{13}, \epsilon_{23}$ are independent since $\epsilon_{12} = \epsilon_{21}$, $\epsilon_{13} = \epsilon_{31}$, $\epsilon_{23} = \epsilon_{32}$

These strains define the STATE OF STRAIN at a point in a body (just like state of stress at a point).

The state of strain at a point is unique and is given by a strain tensor $\underset{\approx}{\epsilon}$, which can be represented by a matrix using a chosen coordinate system

$\underset{\approx}{\epsilon} \leftarrow$ 2nd-order tensor

unique at a point

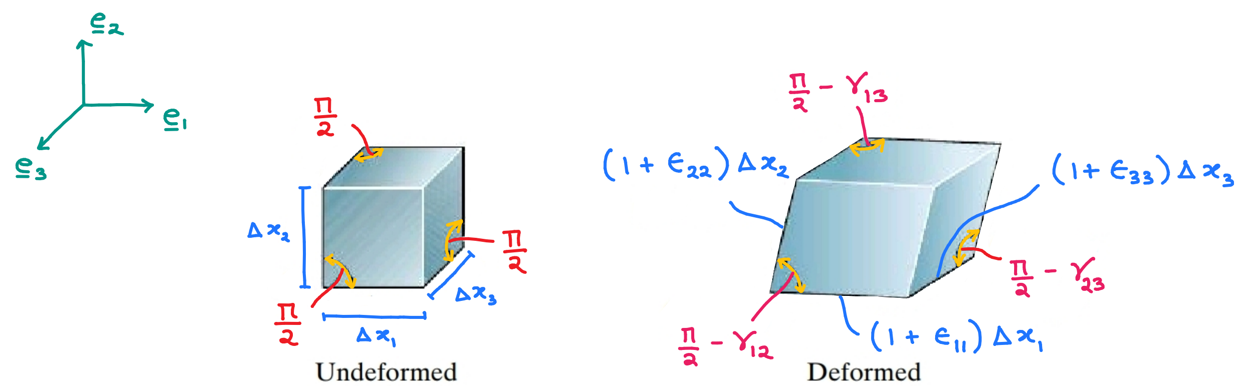
Matrix representation

$$[\underset{\approx}{\epsilon}]_{(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33} \end{bmatrix}$$

SYMMETRIC

Note : The matrix components depend upon the choice of coordinate sys.

If the state of strain at a point is known, one can describe the deformation of a small cuboidal element at that point — whose face normals are oriented along the coordinate axes — is completely defined by the state of strain $\underset{\approx}{\epsilon}$



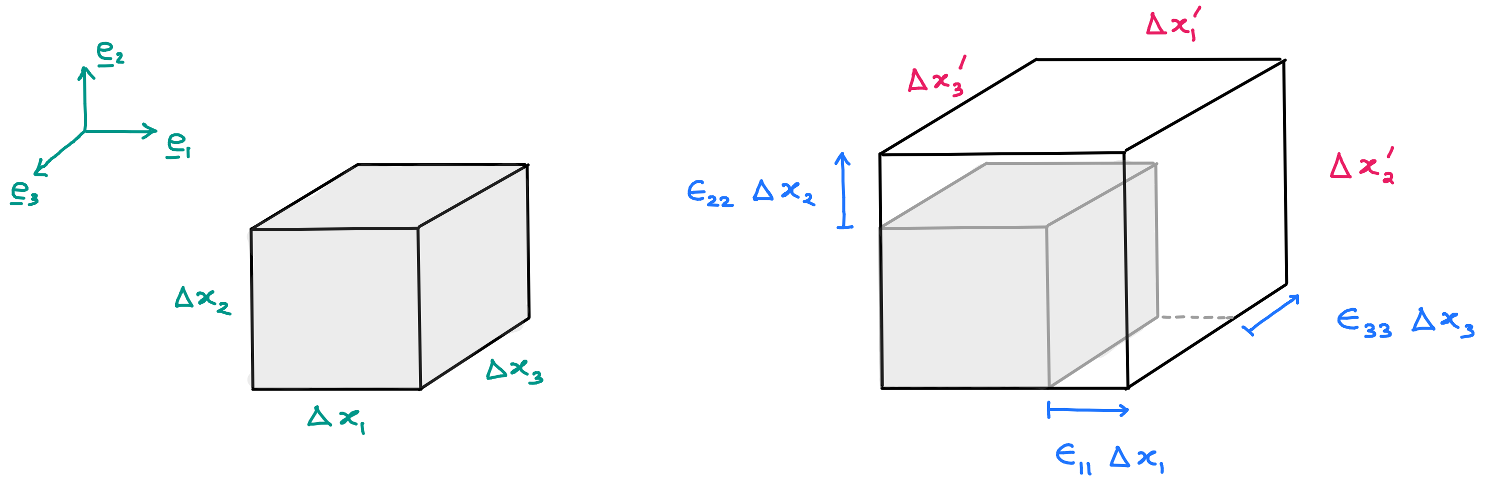
Undeformed element

Deformed element

Local Volumetric Strain (or Dilatational Strain)

As a body deforms, the volume of every small region (called local volume element) of the body also changes. We can then define a quantity called volumetric strain because the change in volume per unit volume will be different for different parts in a body.

Normal strain leads to change in volume, so we will consider a small local volume element (in the form of a cuboid) at a point in the undeformed body.



$$\left.\begin{aligned}
\Delta x_1' &= \Delta x_1 + \epsilon_{11}\,\Delta x_1 \\
\Delta x_2' &= \Delta x_2 + \epsilon_{22}\,\Delta x_2 \\
\Delta x_3' &= \Delta x_3 + \epsilon_{33}\,\Delta x_3
\end{aligned}\right\}\ \text{for small strains}$$

Volume of original local volume element, $V = \Delta x_1\,\Delta x_2\,\Delta x_3$

Volume of the element after deformation, $v = \Delta x_1'\,\Delta x_2'\,\Delta x_3'$

$$\text{Volumetric strain} = \frac{v - V}{V} = \frac{(1+\epsilon_{11})(1+\epsilon_{22})(1+\epsilon_{33})\,V - V}{V}$$

$$\approx \epsilon_{11} + \epsilon_{22} + \epsilon_{33} \quad \text{(neglecting products of small strains)}$$

Local average rotation tensor

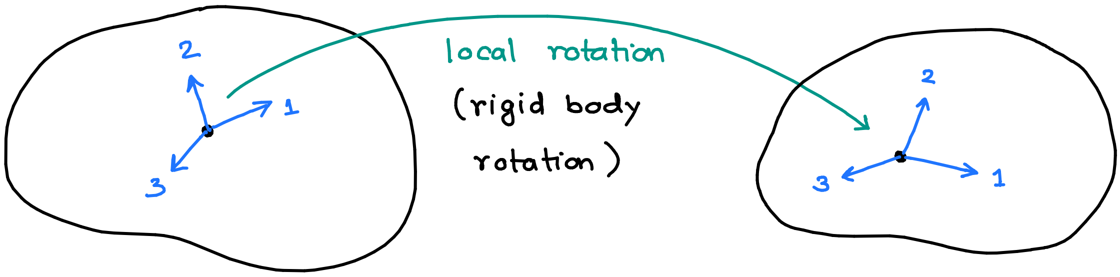
Note that the displacement gradient $\underline{\nabla}\,\underline{u}$ can be written as:

$$\underline{\nabla}\,\underline{u} = \frac{1}{2}\left(\underline{\nabla}\underline{u} + \underline{\nabla}\,\underline{u}^T\right) + \frac{1}{2}\left(\underline{\nabla}\,\underline{u} - \underline{\nabla}\,\underline{u}^T\right)$$

$$= \underline{\underline{\epsilon}} + \underline{\underline{W}}$$

small strain tensor (induces strain at a point)

local average rotation tensor (causes rigid-body rotation of line elements at a pt)



$$[\underline{\underline{W}}]_{\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}} = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \\ -\frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \\ -\frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) & -\frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) & 0 \end{bmatrix}$$

ANTI-SYMMETRIC MATRIX

The local average rotation tensor is responsible for rigid-body rotation of line elements. So if the displacement is such that the strain tensor $\underline{\underline{\epsilon}}$ is $\underline{\underline{0}}$ at a oint, then there will be no strain of any kind (normal / shear strain) at that point. However, due to $\underline{\underline{W}}$, the line elements may undergo rigid rotation. $\underline{\underline{W}}$ can vary from point to point, meaning rigid body rotation will be different for different points in the body.