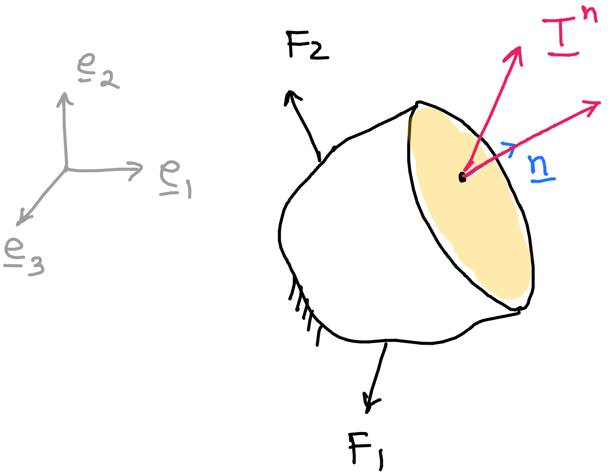
Decomposition of traction vector

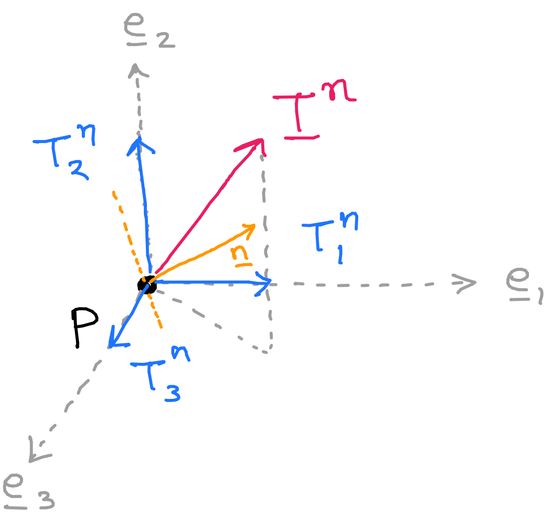
In the last lecture, we introduced the traction vector as the intensity of force at a point on a given plane.



Since traction is a vector, one can obtain components of the traction vector using a choice of coordinate system.

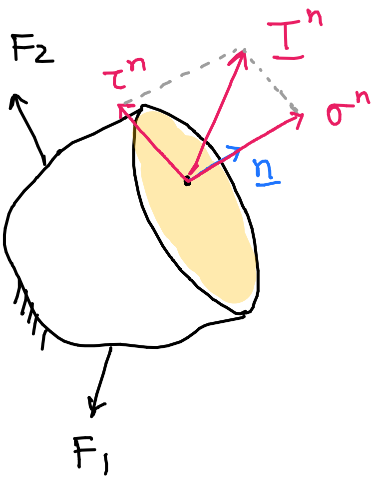
$$\underline{T}^n(\underline{x}) = T_1^n \underline{e}_1 + T_2^n \underline{e}_2 + T_3^n \underline{e}_3$$

Note the a vector is independent of the coordinate system chosen to represent its components



$$|T^n|^2 = |T_1^n|^2 + |T_2^n|^2 + |T_3^n|^2$$

Normal and shear components of traction



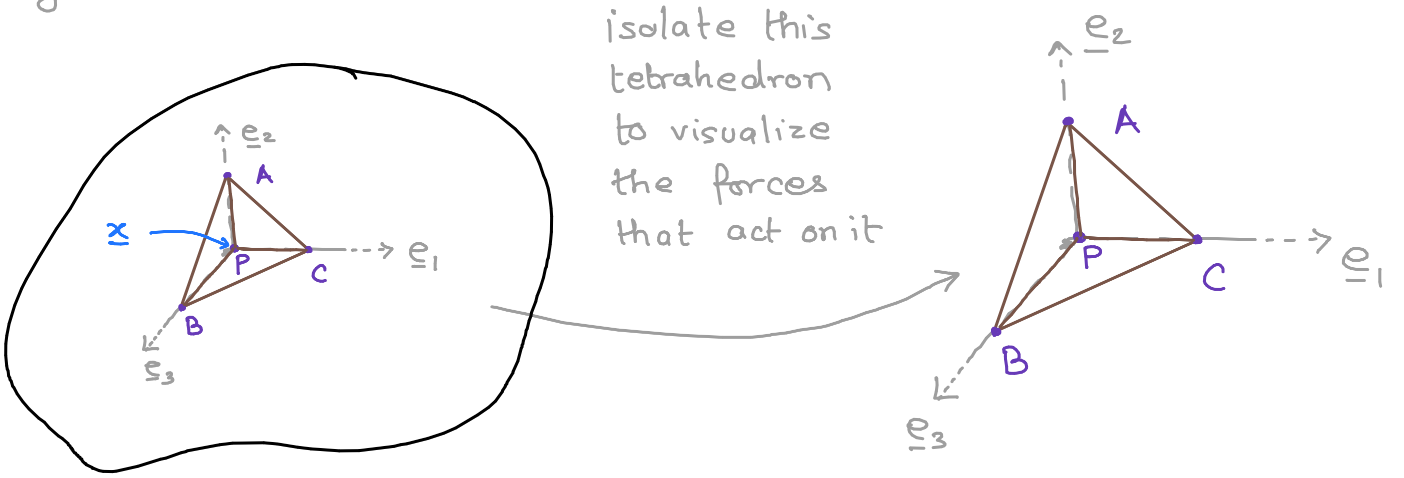
The traction vector could also be resolved along the direction of normal $\underline{n}$ and a direction tangential to the plane of cut such that

$$|\underline{T}^n|^2 = (\sigma^n)^2 + (\tau^n)^2$$

$$\sigma^n = T_1^n n_1 + T_2^n n_2 + T_3^n n_3$$

Now we will derive how a traction vector at a point for a given plane orientation may be found using tractions on three mutually perpendicular planes at the same point

Consider a small tetrahedron at a point P inside the body



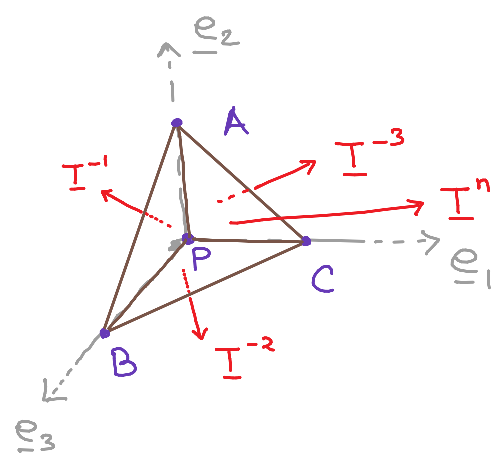
Faces of tetrahedron : Outward normals

ABC	—	$\underline{n}$
PAB	—	$-\underline{e}_1$
PBC	—	$-\underline{e}_2$
PAC	—	$-\underline{e}_3$

What are the various forces acting on this tetrahedron?

- Weight (body force) due to gravity acting on every point of the tetrahedron. The total weight of the tetrahedron can be assumed to act through its COM (center of mass)

- Internal (surface) forces acting on surfaces of the tetrahedron. These internal surface forces are applied by the rest of the body on the sides of the tetrahedron



Lets assume that the traction vector acting on each face is uniform, so we have

Plane	Outward Normal	Traction
ABC	$\underline{n}$	$\underline{T}^n$
PAB	$-\underline{e}_1$	$\underline{T}^{-1}$
PBC	$-\underline{e}_2$	$\underline{T}^{-2}$
PAC	$-\underline{e}_3$	$\underline{T}^{-3}$

For static equilibrium of the tetrahedron

$$\sum \underline{F} = 0$$ (Resultant of all forces must be zero)

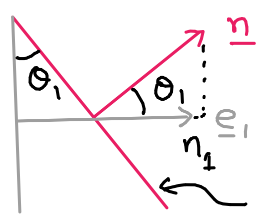
$$\Rightarrow \underline{T}^{-1} A_{PAB} + \underline{T}^{-2} A_{PBC} + \underline{T}^{-3} A_{PAC} + \underline{T}^{n} A_{ABC} + \rho V \underline{g} = \underline{0}$$

density volume

acceleration vector due to gravity

We can write the area of the inclined face ABC in terms of other areas A_{PAB}, A_{PBC}, and A_{PAC}.

For example, the area of ABC if projected along $\underline{e}_1$-axis is equal to A_{PAB}



$$A_{PAB} = A_{ABC} (\underline{n} \cdot \underline{e}_1)$$
$$= A_{ABC} \; n_1$$

$$\Rightarrow \underline{T}^{-1} A_{PAB} + \underline{T}^{-2} A_{PBC} + \underline{T}^{-3} A_{PAC} + \underline{T}^{n} A_{ABC} + \rho V \underline{g} = \underline{0}$$

So we can rewrite the above relation as follows:

$$\Rightarrow \underline{T}^{-1} \cancel{A_{ABC}} (\underline{n} \cdot \underline{e}_1) + \underline{T}^{-2} \cancel{A_{ABC}} (\underline{n} \cdot \underline{e}_2) + \underline{T}^{-3} \cancel{A_{ABC}} (\underline{n} \cdot \underline{e}_3) + \underline{T}^{n} \cancel{A_{ABC}} + \rho \underline{g} (\cancel{A_{ABC}} \cdot h) = \underline{0}$$

$$\Rightarrow \sum_{i=1}^{3} \underline{T}^{-i} (\underline{n} \cdot \underline{e}_i) + \underline{T}^{n} + \rho \underline{g} h = \underline{0}$$

Our goal is find the tractions at the point P, instead of the faces (which are not a point). So, we reduce 'h' to zero s.t. the tetrahedron shrinks to the point P

Also, note that as $h \to 0$, the term $\rho \underline{g} h \to \underline{0}$ and we have a simple result:

$$\sum_{i=1}^{3} \underline{T}^{-i} (\underline{n} \cdot \underline{e}_i) + \underline{T}^{n} = 0$$

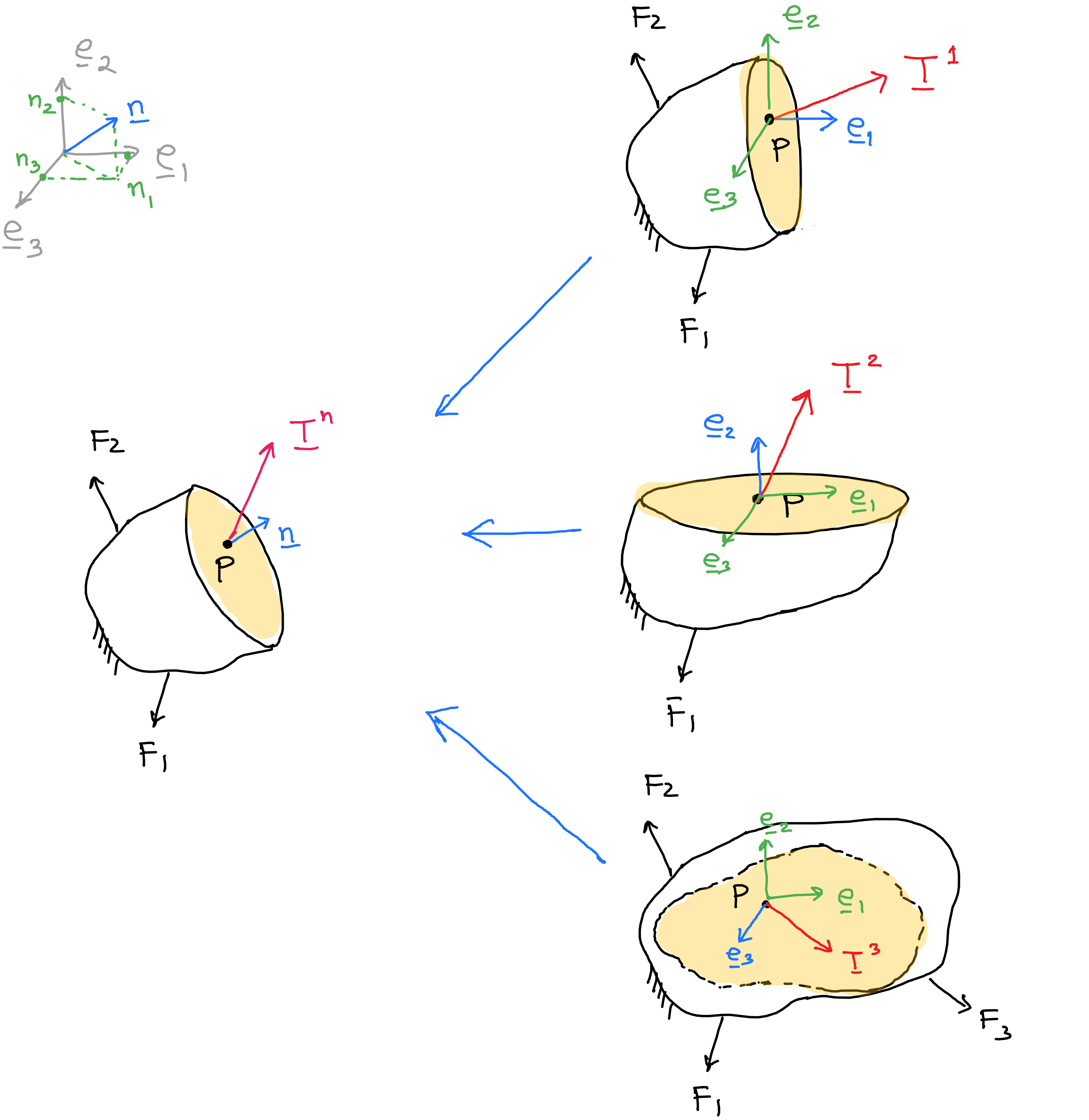
Furthermore, we know that $\underline{T}^{i} = -\underline{T}^{-i}$ (see prev. lecture)

So, we get

$$\boxed{\underline{T}^{n} = \sum_{i=1}^{3} \underline{T}^{i} (\underline{n} \cdot \underline{e}_i)}$$

(expressed in a coordinate system e_1, e_2, e_3)

If we know traction vectors on three mutually perpendicular planes, the the traction vector on any plane passing through the point P may be obtain by the above relation

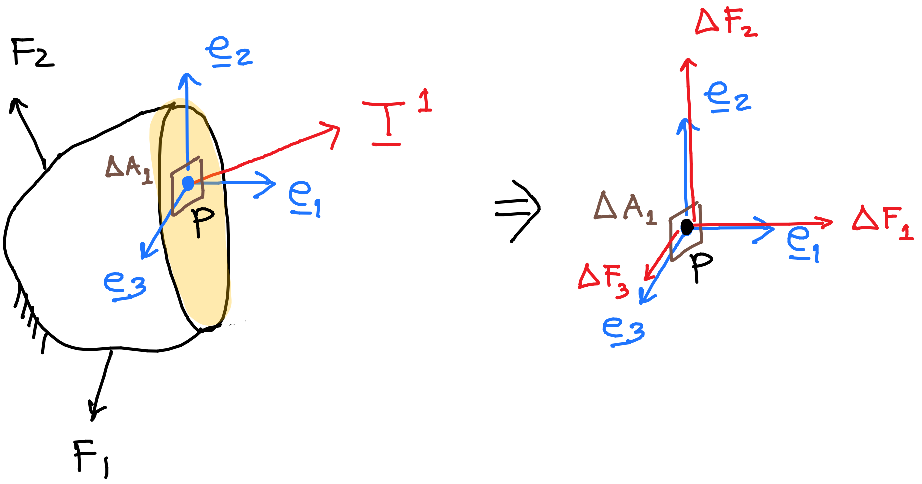


$$\underline{T}^n = \underline{T}^1 n_1 + \underline{T}^2 n_2 + \underline{T}^3 n_3$$

$$\begin{bmatrix} T_1^n \\ T_2^n \\ T_3^n \end{bmatrix} = \underbrace{\begin{bmatrix} T_1^1 \\ T_2^1 \\ T_3^1 \end{bmatrix}}_{\text{Traction on plane with normal } \underline{e}_1} n_1 + \begin{bmatrix} T_1^2 \\ T_2^2 \\ T_3^2 \end{bmatrix} n_2 + \underbrace{\begin{bmatrix} T_1^3 \\ T_2^3 \\ T_3^3 \end{bmatrix}}_{\text{Traction on plane with normal } \underline{e}_3} n_3$$

Traction $\underline{T}^1$ acts on a plane with outward normal $\underline{e}_1$

If we cut the body with a $\underline{e}_1$-plane through pt P then we get a traction $\underline{T}^1$ whose components in the three directions of the coordinate system are given by T_1^1, T_2^1, and T_3^1



$\Rightarrow$

$$T_1^1 = \lim_{\Delta A_1 \to 0} \frac{\Delta F_1}{\Delta A_1}$$

$$T_2^1 = \lim_{\Delta A_1 \to 0} \frac{\Delta F_2}{\Delta A_1}$$

$$T_3^1 = \lim_{\Delta A_1 \to 0} \frac{\Delta F_3}{\Delta A_1}$$

NORMAL and SHEAR stresses

For defining these quantities, lets consider a plane with outward normal in the +ve $\underline{e}_1$-direction. With traction $\underline{T}^1$ acting on the $\underline{e}_1$-plane, we can use its components to define:

Normal stress, $\sigma_{11} = T_1^1$ } tendency to pull or push

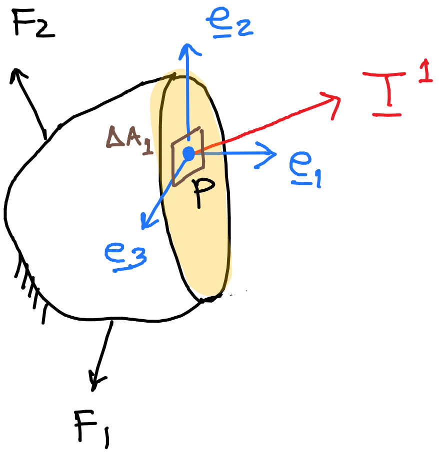
Shear stress, $\tau_{12} = T_2^1$

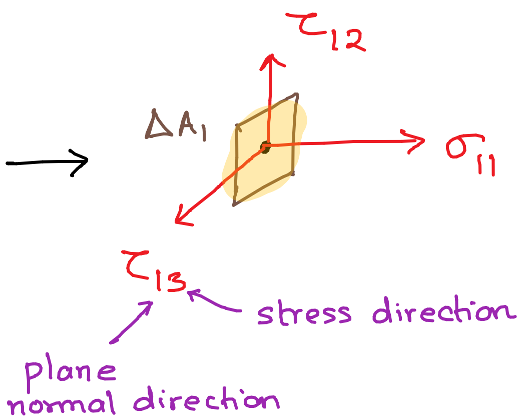
Shear stress, $\tau_{13} = T_3^1$

} tendency to slide between two surfaces

Similarly, we can define $\underline{T}^2$ and $\underline{T}^3$

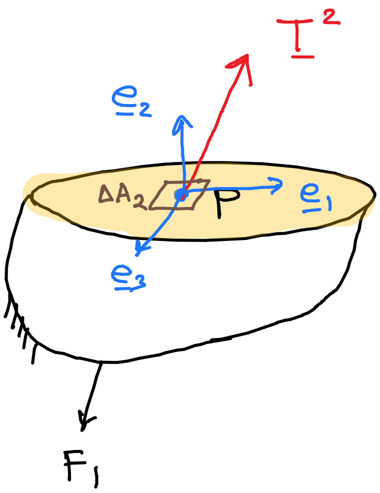
Plane normal along $\underline{e}_1$

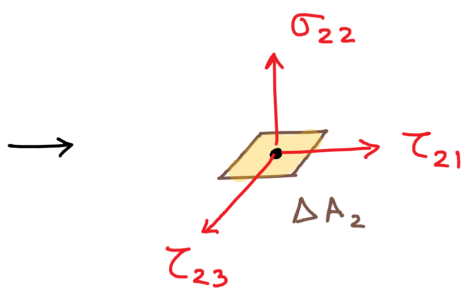




$$\underline{T}^1 = \begin{bmatrix} T_1^1 \\ T_2^1 \\ T_3^1 \end{bmatrix} = \begin{bmatrix} \sigma_{11} \\ \tau_{12} \\ \tau_{13} \end{bmatrix}$$

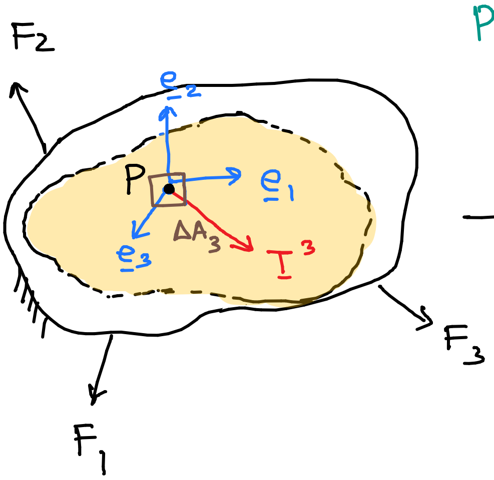
Plane normal along $\underline{e}_2$

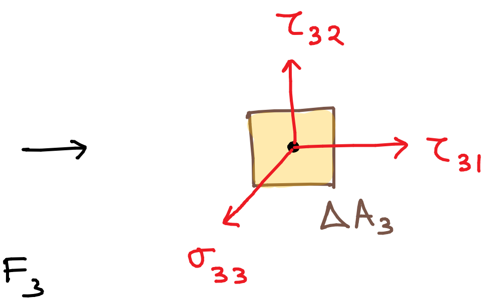




$$\underline{T} = \begin{bmatrix} T_1^2 \\ T_2^2 \\ T_3^2 \end{bmatrix} = \begin{bmatrix} \tau_{21} \\ \sigma_{22} \\ \tau_{23} \end{bmatrix}$$

Plane normal along $\underline{e}_3$

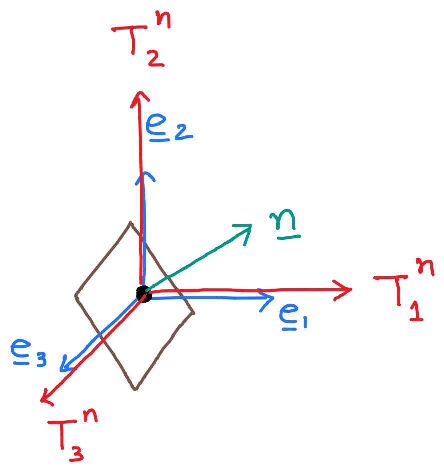




$$\underline{T}^3 = \begin{bmatrix} T_1^3 \\ T_2^3 \\ T_3^3 \end{bmatrix} = \begin{bmatrix} \tau_{31} \\ \tau_{32} \\ \sigma_{33} \end{bmatrix}$$

$$\underline{T}^n = \underline{T}^1 n_1 + \underline{T}^2 n_2 + \underline{T}^3 n_3$$

$$\begin{bmatrix} T_1^n \\ T_2^n \\ T_3^n \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma_{11} \\ \tau_{12} \\ \tau_{21} \end{bmatrix}}_{\text{Traction on plane with normal } \underline{e}_1} n_1 + \begin{bmatrix} \tau_{21} \\ \sigma_{22} \\ \tau_{23} \end{bmatrix} n_2 + \underbrace{\begin{bmatrix} \tau_{31} \\ \tau_{32} \\ \sigma_{33} \end{bmatrix}}_{\text{Traction on plane with normal } \underline{e}_3} n_3$$



Componentwise

$$T_1^n = \sigma_{11} n_1 + \tau_{21} n_2 + \tau_{31} n_3$$

$$T_2^n = \tau_{12} n_1 + \sigma_{22} n_2 + \tau_{32} n_3$$

$$T_3^n = \tau_{13} n_1 + \tau_{23} n_2 + \sigma_{33} n_3$$

Representing in the form of a matrix:

$$\begin{bmatrix} T_1^n \\ T_2^n \\ T_3^n \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma_{11} & \tau_{21} & \tau_{31} \\ \tau_{12} & \sigma_{22} & \tau_{32} \\ \tau_{13} & \tau_{23} & \sigma_{33} \end{bmatrix}}_{[\underline{\underline{\sigma}}]} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

Stress tensor

$$\boxed{\underline{T}^n = \underline{\underline{\sigma}}\ \underline{n}}$$

State of stress at a point

An infinite number of traction vectors act at a given point. The totality of all traction vectors acting on every possible plane through a point is defined to be the STATE of STRESS at the point.

Since traction along any plane can be obtained from info of tractions on mutually perpendicular planes, therefore it is enough to know tractions on three mutually perpendicular planes to define the totality of all tractions acting at a pt.

The tractions on three mutually perpendicular planes can be further resolved into normal and tangential directions, which lead to the normal and shear stresses on each plane.

Thus the state of stress at a point is completely defined by the nine stress components acting on three mutually perpendicular planes (say $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$ planes)

$$[\underline{\sigma}]_{(e_1-e_2-e_3)} = \begin{array}{c} \begin{bmatrix} \sigma_{11} & \tau_{21} & \tau_{31} \\ \tau_{12} & \sigma_{22} & \tau_{32} \\ \tau_{13} & \tau_{23} & \sigma_{33} \end{bmatrix}_{(e_1-e_2-e_3)} \\ \begin{matrix} \underline{T}^1 & \quad \underline{T}^2 & \quad \underline{T}^3 \end{matrix} \end{array}$$

$\underline{\sigma}$ → STRESS TENSOR

$[\underline{\sigma}]_{(e_1-e_2-e_3)}$ → Representation of stress tensor in $(e_1-e_2-e_3)$ Coor. Sys.