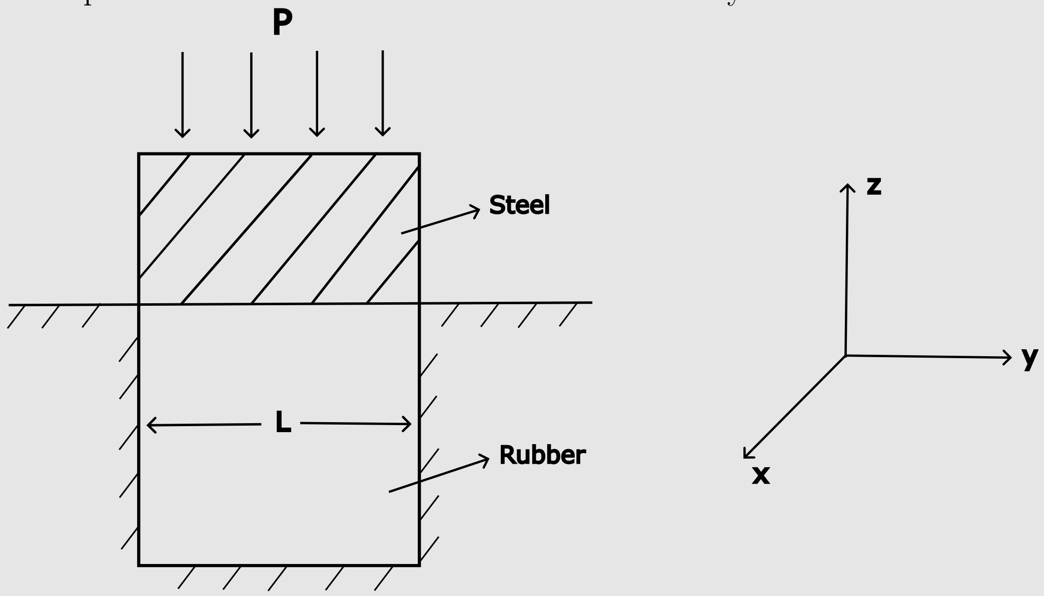
APL 104 Tutorial 7 solutions

Q1. Think of a rubber cube that is inserted within a cavity in a steel block - the cavity has the same form and size as that of a rubber cube (as shown below). The top surface of the rubber cube is pressed by another steel block with a pressure of p.
Assume the steel to be rigid and that there is no friction between steel and rubber. The steel block is rigid and the weight of the rubber block is negligible compared to the steel block. Treat rubber as an isotropic linear elastic material. The mass density of the steel block is ρ.



(a) Find the relation between normal stresses in the x and y directions in this problem.

(b) Find the volumetric strain.

Solution: In this problem, we will neglect the weight of the rubber block in comparison to the steel block.

GEOMETRIC COMPATIBILITY

Observe that the rubber block is constrained against expanding in x and y-directions and can only displace vertically along $z-$direction. This implies that

$$u_x = 0,\ u_y = 0,\ u_z(x, y, z)$$

Furthermore, since the applied pressure is uniform, the vertical displacement u_z at any C/S (in the $x-y$ plane) remains the same, and hence u_z can be treated as being independent of x and y, i.e.,

$$\therefore\ u_z(x, y, z)\ = u_z(z)$$

From the displacement field, we can write the strain tensor $\underset{=}{\epsilon}$ to be

$$\left[\underset{=}{\epsilon}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix}$$

since u_z is the only non-zero displacement component and it is only a function of z. Next, we want to obtain the stress components.

STRESS-STRAIN RELATION

Treating rubber as an isotropic linear elastic material, one obtains:

$$\tau_{xy} = \frac{\gamma_{xy}}{G} = 0, \quad \tau_{xz} = \frac{\gamma_{xz}}{G} = 0, \quad \tau_{yz} = \frac{\gamma_{yz}}{G} = 0$$

As of now, the remaining unknowns are σ_{xx}, σ_{yy}, σ_{zz}, and ϵ_{zz}. From the symmetry of the problem given, we can see that $\sigma_{xx} = \sigma_{yy}$, as the same reaction is generated from steel walls in both directions. Note that even though $\epsilon_{xx} = 0$ and $\epsilon_{yy} = 0$, it does not translate to σ_{xx} and σ_{yy} to be zero. Physically, as the steel cavity constrains the rubber cube from expanding, in the process, the wall of the steel cavity applies constraining traction on the $x-$ and $y-$ faces of the rubber cube leading to non-zero σ_{xx} and σ_{yy}. Using stress-strain relations, one can further write:

$$\cancelto{0}{\epsilon_{xx}} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\cancelto{\sigma_{xx}}{\sigma_{yy}} + \sigma_{zz}\right)\right]$$

$$\text{or } \sigma_{xx}\,(= \sigma_{yy}) = \frac{\nu\sigma_{zz}}{(1 - \nu)}.$$

EQUILIBRIUM CONDITION

For this case, σ_{zz} is the same throughout the body (if we ignore the weight of the rubber block). This can be proved from the equilibrium equations, which are valid for any point in the body:

$$\frac{\partial\sigma_{zz}}{\partial z} + \frac{\partial \cancelto{0}{\tau_{zx}}}{\partial x} + \frac{\partial \cancelto{0}{\sigma_{zy}}}{\partial y} + \cancelto{0}{\gamma_z} = 0$$

$$\Rightarrow \frac{\partial\sigma_{zz}}{\partial z} = 0$$

$$\text{or } \sigma_{zz} = \text{constant}.$$

Hence $\sigma_{zz} = -\left(p + \rho g h\right)$ is valid throughout the rubber cube.

Traction BC

Let's look at the traction boundary condition for finding σ_{zz}. Assuming the mass density of the steel block as ρ, and the height of the steel block as h, the total traction applied on the top face of the rubber block is

$$\text{Traction at the } \underline{\text{top face}} = \text{External pressure at } \underline{\text{top face}}$$

$$\Rightarrow \underline{\underline{\sigma}} \cdot \underline{e}_3 = (\rho g h + p)\cdot(-\underline{e}_3)$$

$$\text{or } \begin{bmatrix} \tau_{xz} \\ \tau_{yz} \\ \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\left(\rho g h + p\right) \end{bmatrix}$$

$$\Rightarrow \sigma_{zz} = -\left(p + \rho g h\right).$$

Therefore

$$\sigma_{xx} = \sigma_{yy} = \frac{\nu}{1 - \nu}\sigma_{zz} = -\frac{\nu\left(p + \rho g h\right)}{(1 - \nu)}.$$

Moreover

$$
\begin{aligned}
\epsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right] \\
&= \frac{1}{E}\left[\sigma_{zz} - 2\nu\sigma_{xx}\right] \\
&= \frac{1}{E}\left[\sigma_{zz} - \frac{2\nu^2}{(1 - \nu)}\sigma_{zz}\right] \\
&= \frac{1}{E}\left(\frac{1-\nu-2\nu^2}{1 - \nu}\right)\sigma_{zz}
\end{aligned}
$$

(b) Volumetric strain:

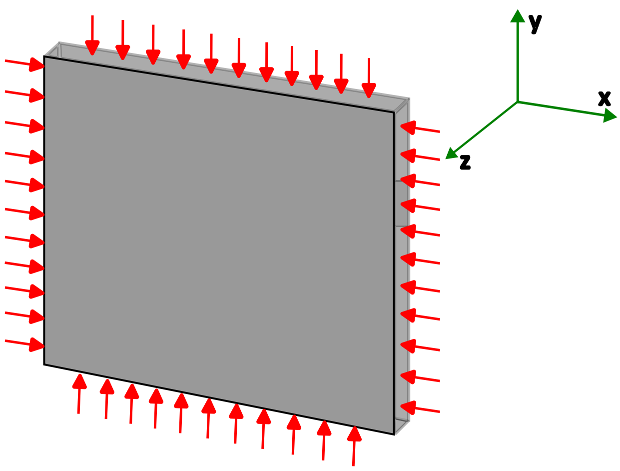
$$
\begin{aligned}
\epsilon_v &= \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} \\
&= 0 + 0 + \epsilon_{zz} \\
&= \frac{1}{E}\left(\frac{1-\nu-2\nu^2}{1 - \nu}\right)(p + \rho g h)
\end{aligned}
$$

Q2. Think of a thin rectangular plate being compressed along its four edges but not allowed to expand or contract in its thickness direction. Assume the thickness direction is along the z-axis. The following components of stress and strain matrices are given:

$$\sigma_{xx} = \sigma_{yy} = -p, \quad \tau_{xy} = \tau_{yz} = \tau_{zx} = 0, \quad \epsilon_{zz} = 0.$$

Assuming the material to be isotropic linear elastic, find out the remaining components of the strain matrix. Also, obtain the change in the area divided by the original area of the face of the plate (z-plane).

Solution:
This is a problem where the rectangular plate is under biaxial loading and furthermore



$$\sigma_{xx} = \sigma_{yy} = -p, \quad \tau_{xy} = \tau_{yz} = \tau_{zx} = 0.$$

The plate is not allowed to expand or contract in the thickness direction. Hence $u_z = 0$ and therefore $\epsilon_{zz} = 0$. To determine ϵ_{xx}, ϵ_{yy}, γ_{xy}, γ_{yz} and γ_{zx}, we use stress-strain relation for linear isotropic materials:

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = 0$$

and similarly

$$\gamma_{yz} = \gamma_{zx} = 0.$$

Using the following, we get a relation between σ_{xx} and σ_{zz}:

$$\begin{aligned}
\cancelto{0}{\epsilon_{zz}} &= \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right] \\
\Rightarrow \quad \sigma_{zz} &= \nu\left(\sigma_{xx} + \sigma_{yy}\right) \\
\Rightarrow \quad \sigma_{zz} &= -2\nu p
\end{aligned}$$

Note that σ_{zz} turns out to be negative, meaning it has a tendency to compress. As the rectangular block is being compressed from $x-$ and $y-$directions, the block tries to expand along the z-direction, however, the expansion is prevented by some kind of restraint, and this restraint imposes compressive traction on the plate to prevent expansion along z direction.

To obtain ϵ_{xx} and ϵ_{yy}, we use rest of the stress-strain relation:

$$\begin{aligned}
\epsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{zz} + \sigma_{yy}\right)\right] \\
\epsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right]
\end{aligned}$$

which leads to

$$\begin{aligned}
\epsilon_{xx} &= \frac{1}{E}\left[-p - \nu\left(-2\nu p - p\right)\right] \\
&= \frac{-p}{E}\left[1 - \nu\left(2\nu + 1\right)\right] \\
&= \frac{-\left(1 - \nu\left(2\nu + 1\right)\right)}{E}p
\end{aligned}$$

Also, we get $\epsilon_{yy} = \epsilon_{xx}$.

Change in the area over original area of the face of the plate ($xy-$plane) is given by:

$$\begin{aligned}
\frac{\Delta A_{xy}}{A_{xy}} &= \epsilon_{xx} + \epsilon_{yy} \\
&= \frac{-2\left(1 - \nu\left(2\nu + 1\right)\right)}{E}p
\end{aligned}$$

Q3. Show that in the case of isotropic bodies, the stress tensor and the strain tensor will both have the same set of principal directions. Further, show that the set of planes whose normals are parallel to one of the principal directions do not slide relative to each other.

Solution: Here, we need to show that both the stress tensor and strain tensor have the same set of principal directions for isotropic materials. The eigenvalue problem for strain gives us the relation:

$$\underline{\underline{\epsilon}}\,\underline{n} = \alpha \underline{n}$$

where $\underline{n}$ is the principal strain direction and α is the principal strain component. Now using the stress-strain relation, one can write the stress tensor as

$$\underline{\underline{\sigma}} = \lambda \operatorname{tr}(\epsilon)\,\underline{\underline{I}} + 2\mu\underline{\underline{\epsilon}}$$

Post-multiplying with principal strain normal $\underline{n}$, one obtains

$$\begin{aligned}
\underline{\underline{\sigma}}\,\underline{n} &= \lambda \operatorname{tr}\left(\underline{\underline{\epsilon}}\right)\underline{\underline{I}}\,\underline{n} + 2\mu\underline{\underline{\epsilon}}\,\underline{n} \\
&= \lambda \operatorname{tr}\left(\underline{\underline{\epsilon}}\right)\underline{n} + 2\mu\alpha\underline{n} \\
&= \underbrace{\left[\lambda \operatorname{tr}\left(\underline{\underline{\epsilon}}\right) + 2\mu\alpha\right]}_{\text{principal stress}}\underline{n}
\end{aligned}$$

From the above relation, we can see that $\underline{n}$ also turns out to be the principal direction for the stress tensor $\underline{\underline{\sigma}}$.

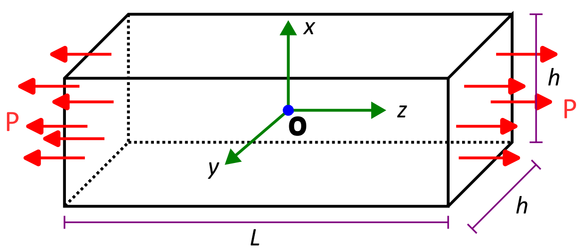
Q4. Think of a solid beam having a square cross-section of side length h and axial length L. The beam's axis lies along the z axis while its cross-section's sides lie along x and y axes. Suppose the beam is stretched by applying axial force P to it such that its cross-section remains square and planar even after deformation. Also, assume the deformation to be axially homogeneous. Let us think of using a Cartesian coordinate system.

(a) What coordinates (x, y, z) will the displacement functions $(u_x,\, u_y,\, u_z)$ depend on? Give reasons for your answer.

(b) Find out the strain matrix and the stress matrix in terms of displacement functions and the material parameters $(\lambda,\, \mu)$ in the Cartesian coordinate system.

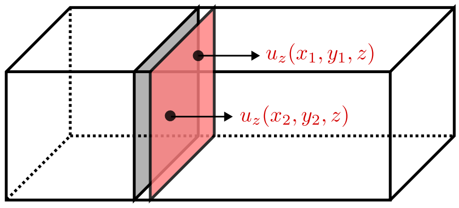
(c) Substitute the expressions for stress components in the equilibrium equation (assume no body force) and obtain the equations. Write down the boundary conditions too. Solve them to obtain the three displacement functions.

Solution: The problem given can be visualized as shown below:

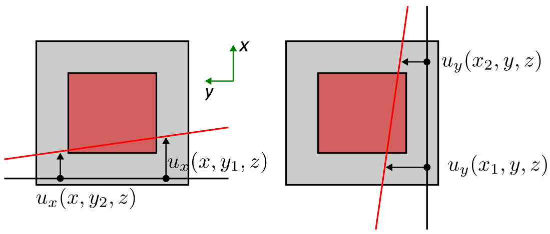


(a) • *Plane C/S remains plane before and after deformation*: It implies that the all points in a given cross-section (which lies in the $x-y$ plane) of the beam must have the same u_z, i.e., $u_z(x_1, y_1, z) = u_z(x_2, y_2, z)$ (having different u_z at different (x, y) points would lead to warping/bulging of the cross-section). Therefore, u_z must be independent of coordinates x and y, i.e.,

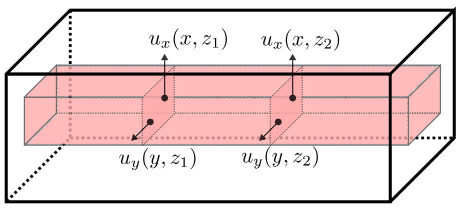
$$u_z(x, y, z) = u_z(z)$$



- *C/S remains square before and after deformation*: It implies that, at a given C/S, any hortizontal (or vertical) line elements must remain horizontal (or vertical) after deformation. From the following figure, it is clear that u_x for two different points having the same $x-$coordinate but different $y-$coordinates must have same u_x values after deformation, i.e. $u_x(x, y_1, z) = u_x(x, y_2, z)$, to prevent tilting/curving of the line element connecting the two points. Thus $u_x(x, y, z) = u_x(x, z)$. By a similar argument, $u_y(x, y, z) = u_y(y, z)$.



- *Deformation is axially homogeneous*: It implies that the beam cross-sections taken at different values of z along the axis of the beam will look the same. Physically, this implies the beam will deform uniformly along the axis of the beam, as shown in the figure below:



Alternatively, for any two cross-sections at z_1 and z_2, the displacements $u_x(x, z_1) = u_x(x, z_2)$ and $u_y(y, z_1) = u_y(y, z_2)$. Hence, we can say that

u_x and u_y are only functions of x and y, respectively!

(b) To this end, we have got the displacement components as follows:

$$u_x(x), \;\; u_y(y), \;\; u_z(z).$$

For the strain components, it is clear that the shear strain components $\gamma_{xy} = \gamma_{yz} = \gamma_{xz} = 0$: this is a rigorous proof of vanishing of shear strains during uniaxial loading of rectangular bars. Only the longitudinal strain components need to be determined now which are

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \epsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad \epsilon_{zz} = \frac{\partial u_z}{\partial z}.$$

Stress components can be obtained from stress-strain relations. The shear stresses, being directly proportional to the shear strains, are zero, i.e. $\tau_{xy} = \tau_{xz} = \tau_{yz} = 0$. The normal stress components can be related to longitudinal strains using Lame's constants (or with Elastic modulus and Poisson's ratio):

$$\begin{aligned}
\sigma_{xx} &= (\lambda + 2\mu)\,\epsilon_{xx} + \lambda\,(\epsilon_{yy} + \epsilon_{zz}) \\
\sigma_{yy} &= (\lambda + 2\mu)\,\epsilon_{yy} + \lambda\,(\epsilon_{xx} + \epsilon_{zz}) \\
\sigma_{zz} &= (\lambda + 2\mu)\,\epsilon_{zz} + \lambda\,(\epsilon_{xx} + \epsilon_{yy})
\end{aligned}$$

(c) Let us now use stress-equilibrium equations to obtain the displacement functions:

$$\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} &= 0 \quad \text{(rest all components are zero)} \\
\Rightarrow (\lambda + 2\mu)\,\frac{\partial \epsilon_{xx}}{\partial x} + \lambda \cancelto{0}{\frac{\partial}{\partial x}(\epsilon_{yy} + \epsilon_{zz})} &= 0 \quad \text{(since } \epsilon_{yy} \text{ and } \epsilon_{zz} \text{ are not functions of } x\text{)} \\
\Rightarrow \frac{\partial \epsilon_{xx}}{\partial x} &= 0 \\
\Rightarrow \frac{\partial^2 u_x}{\partial x^2} &= 0 \\
\Rightarrow \quad u_x &= ax + b
\end{aligned}$$

Following similarly for $y-$ and $z-$directions, we get

$$\begin{aligned}
\frac{\partial^2 u_y}{\partial y^2} &= 0, \quad \Rightarrow u_y = cy + d \\
\frac{\partial^2 u_z}{\partial z^2} &= 0, \quad \Rightarrow u_z = ey + f
\end{aligned}$$

It is to be noted that since the displacements $u_x,\ u_y,\ u_z$ are linear functions, the strain tensor $\underline{\underline{\epsilon}}$ will be constant, which also implies that stress tensor $\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}}\ \underline{\underline{\epsilon}}$ will also be constant throughout the beam (at all points) as $\underline{\underline{\underline{\underline{C}}}}$ is constant for isotropic linear elastic material.

<u>Boundary condtions</u>

The values of the constants a, b, c, d, e, f can be determined from displacement and traction boundary conditions.

- *Displacement BC*: Considering the origin of the coordinate system at the center of the beam, we can observe that the beam is pulled from both ends and therefore the displacements at the beam center would be zero. That is,

$$u_x|_{(0,0,0)} = u_y|_{(0,0,0)} = u_z|_{(0,0,0)} = 0$$

  and we obtain

$$b = d = f = 0.$$

- *Traction BC*: We observe that

$$\begin{aligned}
\sigma_{xx}|_{x=\pm h/2,y,z} &= 0 \\
\sigma_{yy}|_{x,y=\pm h/2,z} &= 0 \\
\sigma_{zz}|_{x,y,z=\pm L/2} &= \frac{P}{h^2}
\end{aligned}$$

  As we also know that the $\underline{\underline{\sigma}}$ is constant throughout the beam, it implies that $\sigma_{xx} = \sigma_{yy} = 0$ and $\sigma_{zz} = \frac{P}{h^2}$ at every point in the beam.

To determine a, c and e, we subsitute the values of $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$ in stress-strain relations:

$$\begin{aligned}
\cancelto{0}{\sigma_{xx}} &= (\lambda + 2\mu)\,\epsilon_{xx} + \lambda\,(\epsilon_{yy} + \epsilon_{zz}) \\
&\Rightarrow (\lambda + 2\mu)\,a + \lambda\,(c + e) = 0 \qquad (1)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\cancelto{0}{\sigma_{yy}} &= (\lambda + 2\mu)\,\epsilon_{yy} + \lambda\,(\epsilon_{xx} + \epsilon_{zz}) \\
&\Rightarrow (\lambda + 2\mu)\,c + \lambda\,(a + e) = 0 \qquad (2) \\
\cancelto{\frac{P}{h^2}}{\sigma_{zz}} &= (\lambda + 2\mu)\,\epsilon_{zz} + \lambda\,(\epsilon_{xx} + \epsilon_{yy}) \\
&(\lambda + 2\mu)\,e + \lambda\,(a + c) = \frac{P}{h^2} \qquad (3)
\end{aligned}$$

One can now use Eqns. (1), (2), and (3) to obtain the constants a, c, e and finally obtain the displacements u_x, u_y, u_z. For example, upon subtracting (1) and (2), we get $a = c$ which when substituted into (1) leads to

$$a = c = -\frac{\lambda}{2(\lambda+\mu)}e = -\nu\ e.$$

This simply implies that the ratio of lateral strain and longitudinal strain during uniaxial loading equals negative of the Poisson's ratio.