Similarity between stress and strain tensors

Stress

$$\sigma_{nn} = (\underline{\underline{\sigma}}\,\underline{n})\cdot\underline{n}$$

$$\tau_{nm} = (\underline{\underline{\sigma}}\,\underline{n})\cdot\underline{m}$$

$\underline{n}$: plane normal

$\underline{m}$: direction along which we measure shear

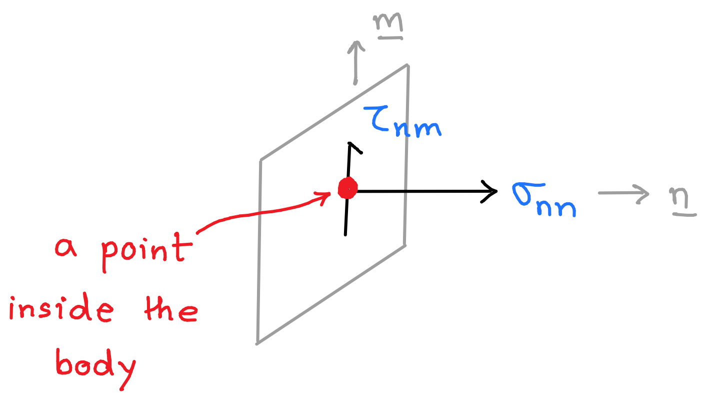
Strain

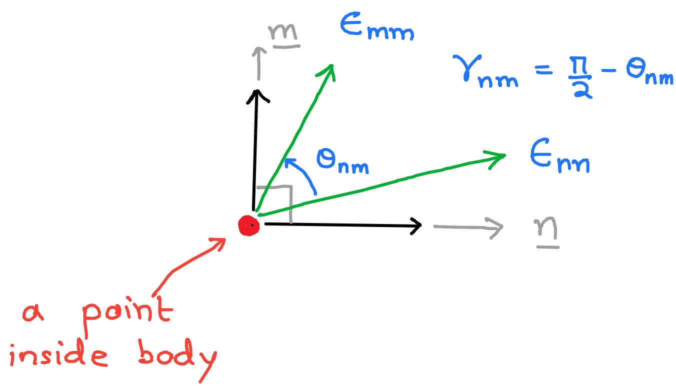
$$\epsilon_{nn} = (\underline{\underline{\epsilon}}\,\underline{n})\cdot\underline{n}$$

$$\epsilon_{nm} = (\underline{\underline{\epsilon}}\,\underline{n})\cdot\underline{m}$$

$\underline{n}$: direction of line element

$\underline{m}$: direction of line element perpendicular to $\underline{n}$





The previous results obtained for stress transformations (i.e. stress components on an arbitrarily inclined plane), principal stresses, Mohr's circle, etc. remain similar for strain as well.

Strain components associated with arbitrary sets of axes

As was the case with plane stress, where we found stress components on an arbitrary face inclined at an angle θ from the $\underline{e}_1$-plane and obtained

$$\sigma_{nn} = (\underline{\underline{\sigma}}\,\underline{n})\cdot\underline{n} = \sigma_{11}\, n_1^2 + \sigma_{22}\, n_2^2 + 2\,\tau_{12}\, n_1 n_2$$

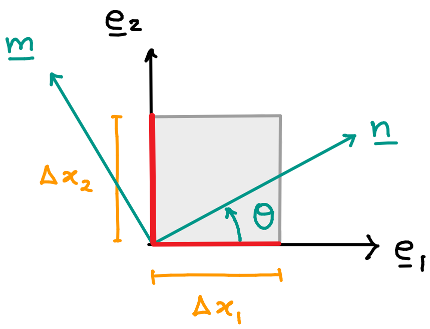
$$= \sigma_{11}\cos^2\theta + \sigma_{22}\sin^2\theta + 2\,\tau_{12}\cos\theta\sin\theta$$

$$\sigma_{nn} = \frac{\sigma_{11} + \sigma_{22}}{2} + \frac{\sigma_{11} - \sigma_{22}}{2} \cos 2\theta + \tau_{12} \sin 2\theta$$

$$\tau_{nn} = -\frac{\sigma_{11} - \sigma_{22}}{2} \sin 2\theta + \tau_{12} \cos 2\theta$$

imilarly, for plane strain case $[\underline{\underline{\epsilon}}] = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{12} & \epsilon_{22} \end{bmatrix}$, we get

(see Sec 4.11 from Crandall)



How are ϵ_{nn}, ϵ_{mm}, ϵ_{nm} related to ϵ_{11}, ϵ_{22}, ϵ_{12} ?

- $$\epsilon_{nn} = \epsilon_{11} \cos^2\theta + \epsilon_{22} \sin^2\theta + 2\epsilon_{12} \cos\theta \sin\theta$$

$$= \frac{\epsilon_{11} + \epsilon_{22}}{2} + \frac{\epsilon_{11} - \epsilon_{22}}{2} \cos 2\theta + \epsilon_{12} \sin 2\theta$$

- $$\epsilon_{mm} = \epsilon_{11} \cos\left(\frac{\pi}{2} + \theta\right) + \epsilon_{22} \sin^2\left(\frac{\pi}{2} + \theta\right) + 2\epsilon_{12} \cos\left(\frac{\pi}{2} + \theta\right) \sin\left(\frac{\pi}{2} + \theta\right)$$

$$= \epsilon_{11} \sin^2\theta + \epsilon_{22} \cos^2\theta - 2\epsilon_{12} \sin\theta \cos\theta$$

$$= \frac{\epsilon_{11} + \epsilon_{22}}{2} - \frac{\epsilon_{11} - \epsilon_{22}}{2} \cos 2\theta - \epsilon_{12} \sin 2\theta$$

- $$\epsilon_{nm} = -\frac{\epsilon_{11} - \epsilon_{22}}{2} \sin 2\theta + \epsilon_{12} \cos 2\theta$$

Principal strains and principal directions

Like principal stresses, we can define principal strains. However, unlike principal stress planes, here we donot have principal strain planes. Instead, we have principal directions.

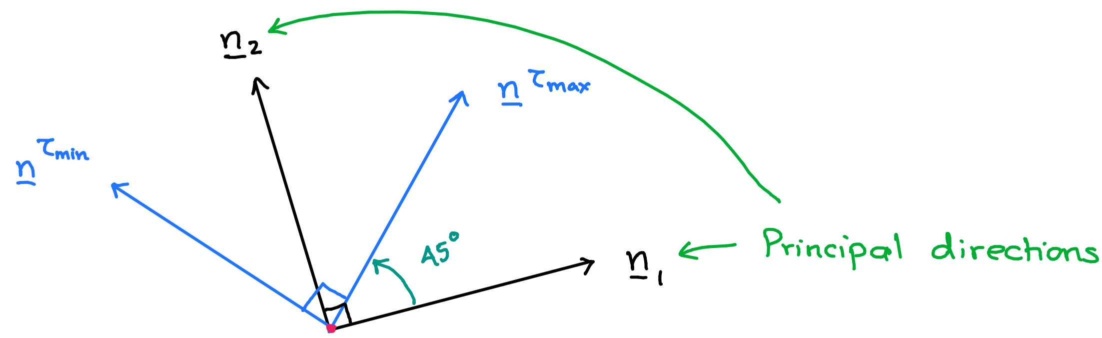
We know that at a point, principal stress planes are planes on which the normal component of traction is maximized or minimized. The value of the normal component of traction on these planes are principal stress components. Similarly, out of the numerous line elements at a point in the body, the directions of those line elements that experience maximum/minimum normal strain are called principal strain directions. The values of the normal strains in these directions are called principal strain components. To find them, we do the same as earlier, i.e. we obtain eigenvectors and eigenvalues of the strain tensor

$$\underline{\underline{\epsilon}}\,\underline{n} = \lambda\,\underline{n}$$

The strain matrix in the coordinate system of principal strain directions becomes diagonal. As the off-diagonal elements will be zero, this means if we take two line elements directed along the principal strain directions, there will not be any change in angle between them.

Maximum Shear strain

We can also maximize shear strain at a point just like we maximized the shear component of traction. We had found that the planes on which shear stress becomes max/min lie at an angle of 45° from the principal planes. Similarly, the pair of perpendicular line elements that undergo maximum change in angle (or max shear strain) will be directed at 45° from principal strain directions.

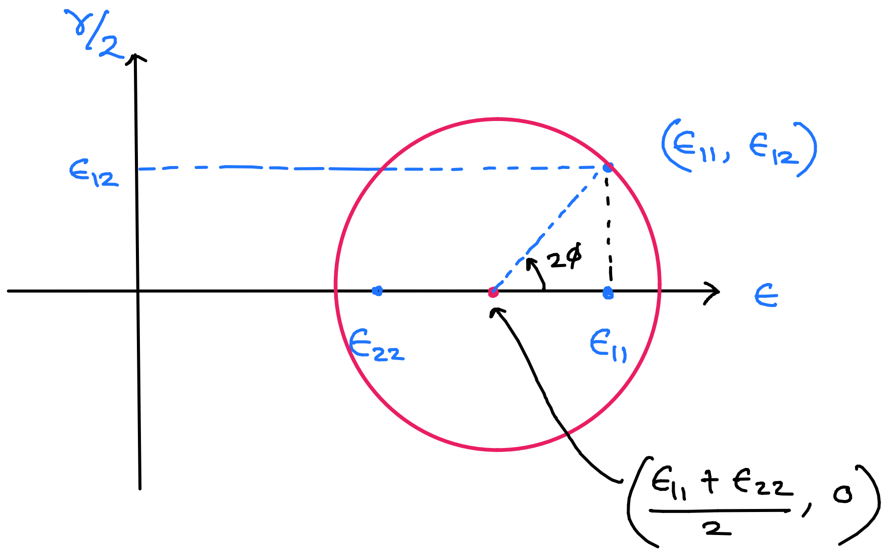


Mohr's Circle

Mohr's circle for stress gave the value of normal stress (σ) and shear stress (τ) on any arbitrary plane. Similarly, if we know the normal strain along two perpendicular directions say $\underline{e}_1$ and $\underline{e}_2$ and also know the shear strain between $\underline{e}_1$ and $\underline{e}_2$, then we can use Mohr's circle for strain to obtain normal and shear strain for two perpendicular line elements which are at an angle θ relative to $\underline{e}_1$ and $\underline{e}_2$ pair.

For 2D Mohr's circle, we need a state of strain s.t. at least one coordinate axis is along a principal direction

$$[\underline{\underline{\epsilon}}] = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ \epsilon_{12} & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix}$$



Strain Invariants

Just like we have invariants of stress tensor denoted by $I_1(\underline{\underline{\sigma}})$, $I_2(\underline{\underline{\sigma}})$, $I_3(\underline{\underline{\sigma}})$, we have invariants of strain tensor denoted by $J_1(\underline{\underline{\epsilon}})$, $J_2(\underline{\underline{\epsilon}})$, $J_3(\underline{\underline{\epsilon}})$

$$J_1(\underline{\underline{\epsilon}}) = \epsilon_{11} + \epsilon_{22} + \epsilon_{33}$$

$$J_2(\underline{\underline{\epsilon}}) = \epsilon_{11}\epsilon_{22} + \epsilon_{22}\epsilon_{33} + \epsilon_{33}\epsilon_{11} - \epsilon_{12}^2 - \epsilon_{13}^2 - \epsilon_{23}^2$$

$$J_3(\underline{\underline{\epsilon}}) = \det([\underline{\underline{\epsilon}}])$$

Decomposition of strain tensor

We had seen the decomposition of stress tensor into hydrostatic and deviatoric parts. We can also decompose the strain tensor into two parts in a similar way

$$\underline{\underline{\epsilon}} = \underbrace{\frac{1}{3} J_1(\underline{\underline{\epsilon}}) \, \underline{\underline{I}}}_{\substack{\text{volumetric/} \\ \text{spherical} \\ \text{strain tensor}}} + \underbrace{\left(\underline{\underline{\epsilon}} - \frac{1}{3} J_1(\underline{\underline{\epsilon}}) \, \underline{\underline{I}} \right)}_{\substack{\text{deviatoric strain} \\ \text{tensor}}}$$

The first part is proportional to identity matrix. It is similar to the hydrostatic part of stress. This part called volumetric (or spherical) strain tensor is responsible for volume change and does not affect the shape of the body. The deviatoric part is responsible for distorting the body and changing its shape. The trace of the deviatoric strain tensor is zero.