1) Think of $(\underline{e}_1-\underline{e}_2-\underline{e}_3)$ and $(\hat{\underline{e}}_1-\hat{\underline{e}}_2-\hat{\underline{e}}_3)$ are two coordinate sys.

$$\underline{e}_j = \sum_i (\underline{e}_j \cdot \hat{\underline{e}}_i)\, \hat{\underline{e}}_i$$

$$\underline{T}^n = \underline{T}^{\hat{1}}\, \hat{n}_1 + \underline{T}^{\hat{2}}\, \hat{n}_2 + \underline{T}^{\hat{3}}\, \hat{n}_3$$

$$= \underline{T}^{\hat{1}} (\underline{n}\cdot\hat{\underline{e}}_1) + \underline{T}^{\hat{2}} (\underline{n}\cdot\hat{\underline{e}}_2) + \underline{T}^{\hat{3}} (\underline{n}\cdot\hat{\underline{e}}_3)$$

$$= \sum_{i=1}^{3} \underline{T}^{\hat{i}} (\underline{n}\cdot\hat{\underline{e}}_i)$$

Lets express $\underline{T}^{\hat{i}}$ in terms of traction on planes $(e_1-e_2-e_3)$

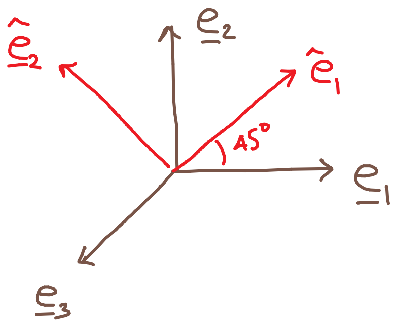
$$\underline{T}^{\hat{i}} = \sum_{j=1}^{3} \underline{T}^j (\hat{\underline{e}}_i \cdot \underline{e}_j)$$

$$\underline{T}^n = \sum_{i=1}^{3} \left(\sum_{j=1}^{3} \underline{T}^j (\hat{\underline{e}}_i \cdot \underline{e}_j) \right) (\underline{n}\cdot\hat{\underline{e}}_i)$$

$$= \sum_{j=1}^{3} \underline{T}^j\ \underline{n} \cdot \left(\sum_{i=1}^{3} (\underline{e}_j \cdot \hat{\underline{e}}_i)\, \hat{\underline{e}}_i \right)$$

$$= \sum_{j=1}^{3} \underline{T}^j (\underline{n}\cdot\underline{e}_j)$$

2) $[\underline{T}^1]_{e_i} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $[\underline{T}^2]_{e_i} = \begin{bmatrix}1\\5\\7\end{bmatrix}$, $[\underline{T}^3]_{e_i} = \begin{bmatrix}0\\7\\9\end{bmatrix}$



$$[\hat{e}_1]_{e_i} = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix}$$

$$\underline{T}^{\hat{1}} = \sum_{i=1}^{3} \underline{T}^i (\hat{e}_1 \cdot \underline{e}_i)$$

$$[\underline{T}^{\hat{1}}]_{e_i} = \sum_{i=1}^{3} [\underline{T}^i]_{e_i} \left([\hat{\underline{e}}_1]_{e_i} \cdot [\underline{e}_i]_{e_i}\right)$$

$$= \begin{bmatrix}0\\1\\0\end{bmatrix} 1/\sqrt{2} + \begin{bmatrix}1\\5\\7\end{bmatrix} 1/\sqrt{2} + \begin{bmatrix}0\\7\\9\end{bmatrix} 0$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix}1\\6\\7\end{bmatrix}$$

Normal component of traction $= \underline{T}^{\hat{1}} \cdot \hat{\underline{e}}_i$

Shear component of traction $= \underline{T}^{\hat{i}} - (\underline{T}^{\hat{1}} \cdot \hat{\underline{e}}_i) \cdot \hat{\underline{e}}_i$

3) $\underline{T}^n \cdot \underline{m} = [\underline{m}]^T [\underline{T}^n]$

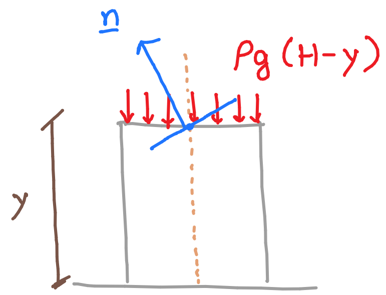
$$= [m]^T [\underline{\underline{\sigma}}] [\underline{n}]$$

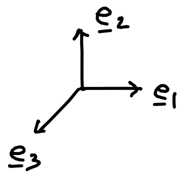
$$= [\underline{n}]^T [\underline{\underline{\sigma}}] [\underline{m}]$$

$$= [\underline{n}]^T [\underline{T}^m]$$

$$= \underline{T}^m \cdot \underline{n}$$

4)





$$\underline{T}^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \underline{T}^2 = \begin{bmatrix} 0 \\ -\rho g (H-y) \\ 0 \end{bmatrix}, \qquad \underline{T}^3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[\underline{T}^n] = \sum_{i=1}^{3} [\underline{T}^i] \left([\underline{n}] \cdot [\underline{e}_i] \right)$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} (-\sin\theta) + \begin{bmatrix} 0 \\ -\rho g (H-y) \\ 0 \end{bmatrix} (\cos\theta) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} (1)$$

$$= \begin{bmatrix} 0 \\ -\rho g (H-y) \cos\theta \\ 0 \end{bmatrix}$$

Normal component of traction

$$\sigma_n = \underline{T}^n \cdot \underline{n} = -\rho g (H-y) \cos\theta \; \underline{e}_2 \cdot \underline{n}$$

Shear component of traction

unit vector perpendicular to $\underline{n}$ lying in plane section

$$\tau_n = \underline{T}^n \cdot \underline{n}^{\perp} = -\rho g (H-y) \cos\theta \; \underline{e}_2 \cdot \underline{n}^{\perp}$$

$$= -\rho g (H-y) \cos\theta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^T \begin{bmatrix} \cos\theta \\ \sin\theta \\ 1 \end{bmatrix}$$

$$= -\rho g (H-y) \cos\theta \sin\theta$$

5) We want $\underline{T}^{n} = \underline{\underline{\sigma}}\,\underline{n} = \underline{0}$ for some $\underline{n}$

$$[\underline{\underline{\sigma}}]\,[\underline{n}] = \underline{0}$$

It implies $[\underline{\underline{\sigma}}]$ is rank-deficient $\Leftrightarrow$ $\det([\underline{\underline{\sigma}}]) = 0$

$$\Rightarrow \det([\underline{\underline{\sigma}}]) = \big(\sigma_{11}\,(-4)\big) - \big(2\,(-2)\big) + \big(1\,(4)\big) = 0$$

$$\Rightarrow -4\sigma_{11} + 4 + 4 = 0 \Rightarrow \sigma_{11} = 2$$

Now, use the relation:

$$\underline{T}^{n} = \underline{\underline{\sigma}}\,\underline{n}$$

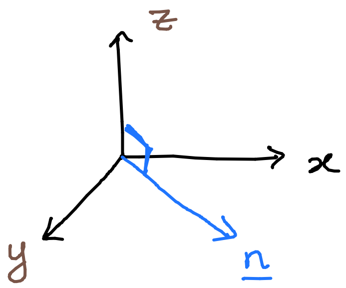
$$\Rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cancel{\sigma_{11}}\,2 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} 2n_1 + 2n_2 + n_3 &= 0 \\ 2n_1 + 2n_3 &= 0 \\ n_1 + 2n_2 &= 0 \end{aligned}\right\}$$ Solve these to get $\underline{n}$

In addition, use $n_1^2 + n_2^2 + n_3^2 = 1$ to get

$$\underline{n} = \begin{bmatrix} \pm 2/3 \\ \pm 1/3 \\ \pm 2/3 \end{bmatrix}$$

6) We have to find $[\underline{n}] = \begin{bmatrix} n_x \\ n_y \\ 0 \end{bmatrix}$ $\quad n_z = 0$ since $\underline{n} \perp z$



$\underline{T}^n \cdot \underline{n} = 0$ (normal component is zero)

$$\Rightarrow (\underline{\underline{\sigma}}\,\underline{n}) \cdot \underline{n} = 0$$

$$\Rightarrow [\underline{n}]^T [\underline{\underline{\sigma}}] [\underline{n}] = 0$$

$$\Rightarrow [n_x \;\; n_y \;\; 0] \begin{bmatrix} a & 0 & d \\ 0 & b & e \\ d & e & c \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ 0 \end{bmatrix} = 0$$

$$\Rightarrow a n_x^2 + b n_y^2 = 0 \quad \text{---} \; (1)$$

Also, $\underline{n}$ being an unit normal,

$$n_x^2 + n_y^2 + n_z^2 = 1$$

$$\Rightarrow n_x^2 + n_y^2 = 1$$

$$\Rightarrow n_y^2 = 1 - n_x^2 \quad \text{---} \; (2)$$

$$a n_x^2 + b(1 - n_x^2) = 0$$

$$\Rightarrow n_x = \pm \left(\frac{b}{b-a}\right)^{1/2}, \quad n_y = \pm \left(\frac{a}{a-b}\right)^{1/2}$$

$$n_z = 0$$

7)

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \cancel{\tau_{yx}}^{\,\tau_{xy}}}{\partial y} + \frac{\partial \cancel{\tau_{zx}}^{\,\tau_{xz}}}{\partial z} + \gamma_x = 0 \quad \text{—} \quad (1)$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \cancel{\tau_{zy}}^{\,\tau_{yz}}}{\partial z} + \gamma_y = 0 \quad \text{—} \quad (2)$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \gamma_z = 0 \quad \text{—} \quad (3)$$

$$\underbrace{\gamma_x = 0, \quad \gamma_z = 0}_{\substack{\text{no self-weight} \\ \text{in } x \text{ or } z \text{ directions}}}, \quad \gamma_y = \rho$$

specific weight (not density)

Stress equilibrium equations are automatically satisfied for (1) & (3). For eqn (2)

$$\left(\frac{\gamma}{\tan^2\beta} - \rho\right) - \frac{\gamma}{\tan^2\beta} + \rho = 0$$

(Satisfied)

Let's verify the traction boundary condition on face OB

Traction on face OB

$$T^{-1}\Big|_{(x=0,\,y,\,z)} = \underline{\underline{\sigma}} \cdot (-\underline{e}_1)$$

$$= -\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= -\begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \\ \tau_{xz} \end{bmatrix}_{\substack{x=0 \\ y,z}} = -\begin{bmatrix} -\gamma_y \\ -\frac{\gamma}{\tan^2\beta}x \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma_y \\ 0 \\ 0 \end{bmatrix}$$

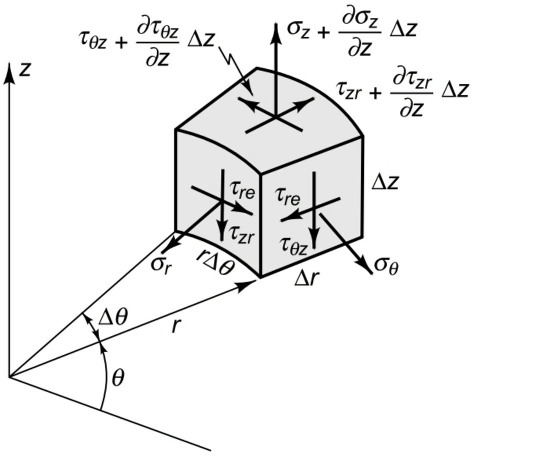
External load on OB

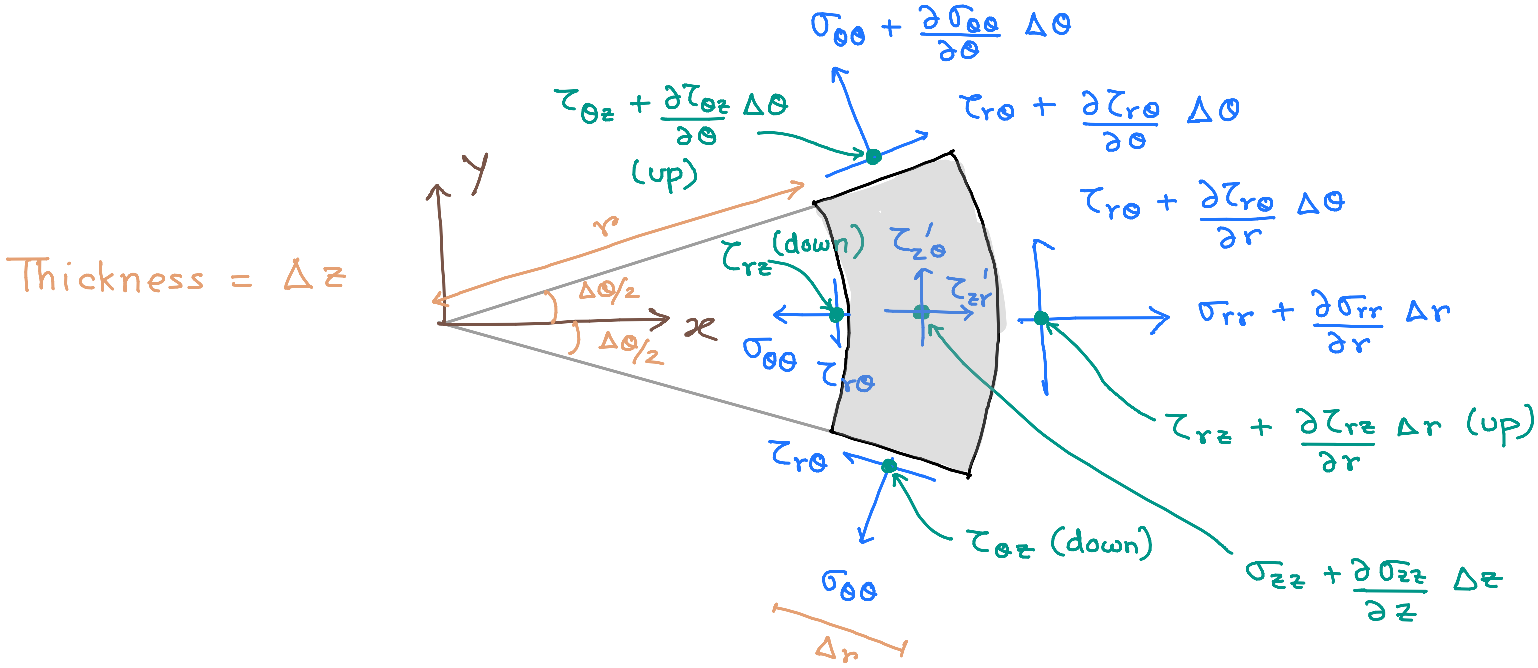
$$f^{ext}\Big|_{\text{face OB}} = \gamma_y\, \underline{e}_1$$

$$= \begin{bmatrix} \gamma_y \\ 0 \\ 0 \end{bmatrix}$$

matching

8) The elementary volume has three pairs of faces in the r, θ and z directions





$\xrightarrow{+} \sum F_x = 0$

$\cos\left(\frac{\Delta\theta}{2}\right) \approx 1$

$\sin\left(\frac{\Delta\theta}{2}\right) \approx \frac{\Delta\theta}{2}$

$$\Rightarrow \Big[\left(\cancel{\sigma_{rr}} + \frac{\partial \sigma_{rr}}{\partial r}\Delta r\right)\underbrace{\left((\cancel{r}+\Delta r)\,\Delta\theta\,\Delta z\right)}_{\text{area}} - \cancel{(\sigma_{rr})(r\,\Delta\theta\,\Delta z)}$$

$$+ \left(\cancel{\tau_{r\theta}} + \frac{\partial \tau_{r\theta}}{\partial r}\Delta\theta\right)\Delta r\,\Delta z\,\cancelto{1}{\cos\left(\frac{\Delta\theta}{2}\right)} - \cancel{\tau_{r\theta}\,\Delta r\,\Delta z\cos\left(\frac{\Delta\theta}{2}\right)}$$

$$+ \left(\cancel{\sigma_{\theta\theta}} + \frac{\partial \sigma_{\theta\theta}}{\partial \theta}\Delta\theta\right)\Delta r\,\Delta z\,\cancelto{\frac{\Delta\theta}{2}}{\sin\left(\frac{\Delta\theta}{2}\right)} - \cancel{\sigma_{\theta\theta}\,\Delta r\,\Delta z\,\sin\left(\frac{\Delta\theta}{2}\right)}$$

$$+ \left(\tau_{zr} + \frac{\partial \tau_{zr}}{\partial z}\Delta z\right) r\,\Delta\theta\,\Delta r - \tau_{zr}\, r\,\Delta\theta\,\Delta r + \gamma_x\, r\,\Delta\theta\,\Delta r\,\Delta z\Big]$$

$$= 0$$

Area of the shaded region $= \frac{1}{2}\left(r\Delta\theta + (r+\Delta r)\Delta\theta\right)\Delta r = \frac{1}{2}(2r+\Delta r)\,\Delta\theta\,\Delta r$

$\approx r\,\Delta\theta\,\Delta r$ (ignoring $(\Delta r)^2$)

$$\xrightarrow{+} \sum F_x = 0$$

$$\cos\left(\frac{\Delta\theta}{2}\right) \approx 1$$

$$\sin\left(\frac{\Delta\theta}{2}\right) \approx \frac{\Delta\theta}{2}$$

$$\Rightarrow \Bigg[\left(\cancel{\sigma_{rr}} + \frac{\partial \sigma_{rr}}{\partial r}\Delta r\right)\Big(\underbrace{(r+\Delta r)\,\Delta\theta\,\Delta z}_{\text{area}}\Big) - \cancel{(\sigma_{rr})(r\,\Delta\theta\,\Delta z)}$$

$$+ \left(\cancel{\tau_{r\theta}} + \frac{\partial \tau_{r\theta}}{\partial r}\Delta\theta\right)\Delta r\,\Delta z\,\cancelto{1}{\cos\left(\frac{\Delta\theta}{2}\right)} - \cancel{\tau_{r\theta}\,\Delta r\,\Delta z\,\cos\left(\frac{\Delta\theta}{2}\right)}$$

$$+ \left(\cancel{\sigma_{\theta\theta}} + \frac{\partial \sigma_{\theta\theta}}{\partial \theta}\Delta\theta\right)\Delta r\,\Delta z\,\cancelto{\frac{\Delta\theta}{2}}{\sin\left(\frac{\Delta\theta}{2}\right)} - \cancel{\sigma_{\theta\theta}\,\Delta r\,\Delta z\,\sin\left(\frac{\Delta\theta}{2}\right)}$$

$$+ \left(\cancel{\tau_{zr}} + \frac{\partial \tau_{zr}}{\partial z}\Delta z\right) r\,\Delta\theta\,\Delta r - \cancel{\tau_{zr}\, r\,\Delta\theta\,\Delta r} + \gamma_x\, r\,\Delta\theta\,\Delta r\,\Delta z\Bigg] = 0$$

$$\Rightarrow r\frac{\partial \sigma_{rr}}{\partial r}\Delta r\,\Delta\theta\,\Delta z + \frac{\partial \sigma_{rr}}{\partial r}(\Delta r)^2\Delta\theta$$

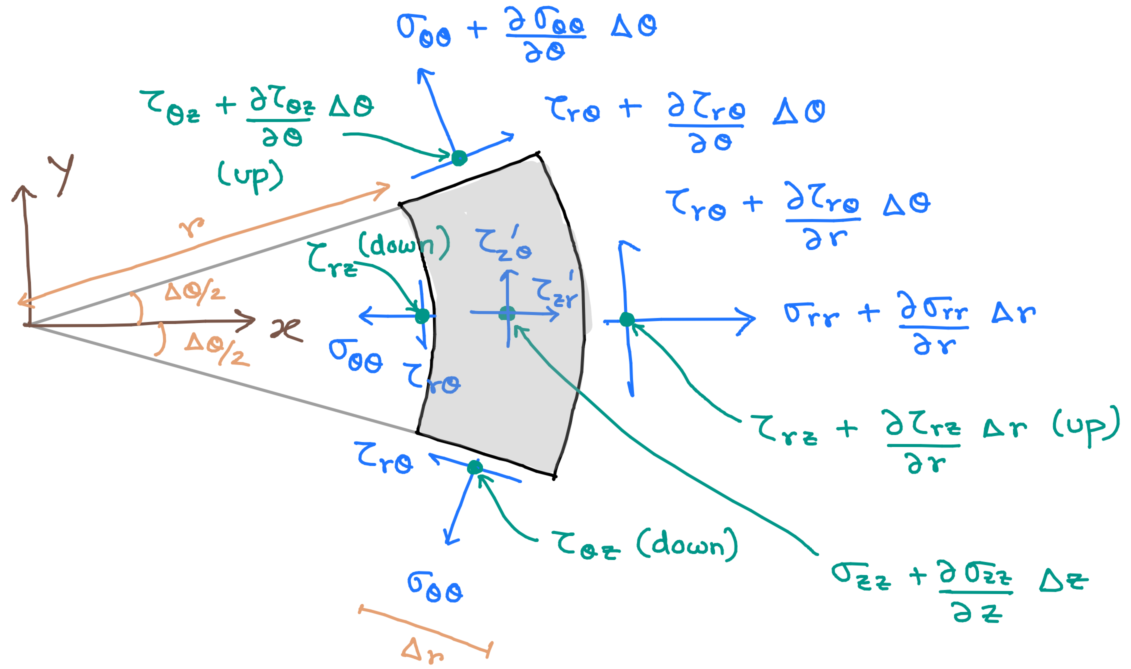
$$+ \frac{\partial \tau_{r\theta}}{\partial r}\Delta r\,\Delta\theta\,\Delta z + \frac{\partial \sigma_{\theta\theta}}{\partial \theta}\Delta r\frac{(\Delta\theta)^2}{2}\Delta z + r\frac{\partial \tau_{zr}}{\partial z}\Delta r\,\Delta\theta\,\Delta z$$

$$+ \gamma_r\, r\,\Delta r\,\Delta\theta\,\Delta z = 0$$

Dropping the higher order terms and dividing by $r\Delta r\Delta\theta\Delta z$

$$\Rightarrow \boxed{\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \gamma_r = 0}$$

Thickness $= \Delta z$



$+\uparrow \sum F_y = 0$

$$\Rightarrow \Bigg[\left(\cancel{\sigma_{\theta\theta}} + \frac{\partial \sigma_{\theta\theta}}{\partial \theta}\Delta\theta\right)\cancelto{1}{\cos\left(\frac{\Delta\theta}{2}\right)}\Delta r\,\Delta z - \cancel{\sigma_{\theta\theta}\cos\left(\frac{\Delta\theta}{2}\right)\Delta r\,\Delta z}$$

$$+ \left(\cancel{\tau_{r\theta}} + \frac{\partial \tau_{r\theta}}{\partial r}\Delta r\right)(r+\Delta r)\,\Delta\theta\,\Delta z - \cancel{\tau_{r\theta}\, r\,\Delta\theta\,\Delta z}$$

$$+ \left(\cancel{\tau_{r\theta}} + \frac{\partial \tau_{r\theta}}{\partial \theta}\Delta\theta\right)\cancelto{\frac{\Delta\theta}{2}}{\sin\left(\frac{\Delta\theta}{2}\right)}\Delta r\,\Delta z - \cancel{\tau_{r\theta}\sin\left(\frac{\Delta\theta}{2}\right)\Delta r\,\Delta z}$$

$$+ \left(\cancel{\tau_{\theta z}} + \frac{\partial \tau_{\theta z}}{\partial z}\Delta z\right) r\,\Delta r\,\Delta\theta - \cancel{\tau_{\theta z}\, r\,\Delta r\,\Delta\theta}$$

$$+ \gamma_\theta\, r\,\Delta r\,\Delta\theta\,\Delta z \Bigg] = 0$$

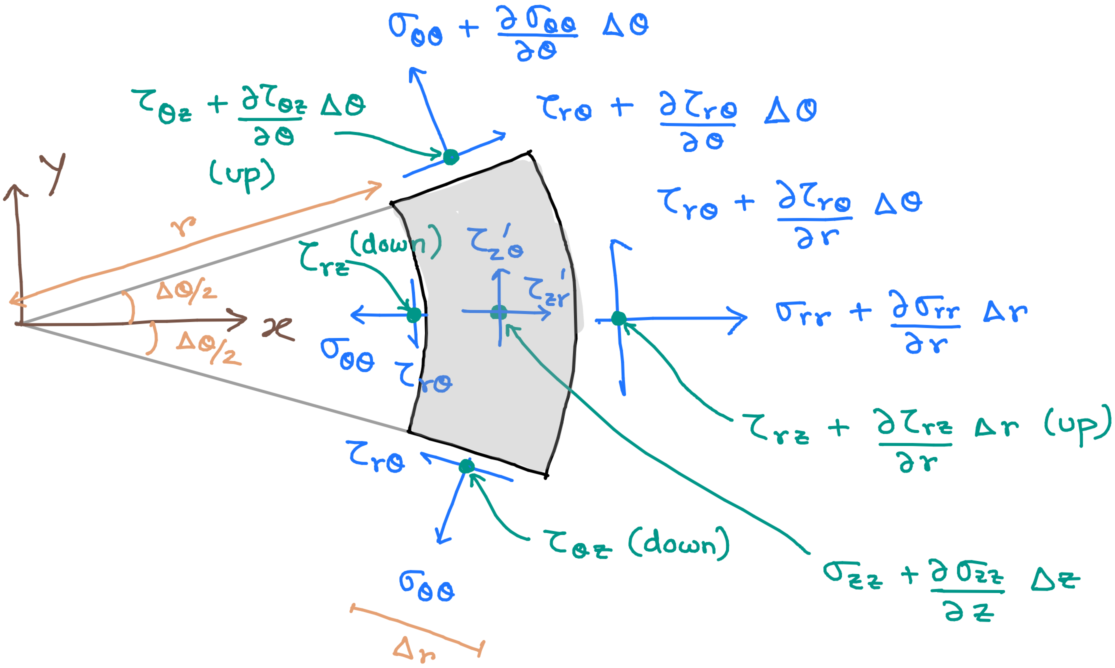
$$\Rightarrow \frac{\partial \sigma_{\theta\theta}}{\partial \theta}\Delta r\,\Delta\theta\,\Delta z + r\frac{\partial \tau_{r\theta}}{\partial r}\Delta r\,\Delta\theta\,\Delta z + \cancelto{0}{\frac{\partial \tau_{r\theta}}{\partial r}(\Delta r)^2\Delta\theta\,\Delta z}$$

$$+ \cancelto{0}{\frac{\partial \tau_{r\theta}}{\partial \theta}\Delta r\frac{(\Delta\theta)^2}{2}}\Delta z + r\frac{\partial \tau_{\theta z}}{\partial z}\Delta r\,\Delta\theta\,\Delta z + \gamma_\theta\, r\,\Delta r\,\Delta\theta\,\Delta z = 0$$

Dividing by $r\,\Delta r\,\Delta\theta\,\Delta z$

$$\Rightarrow \boxed{\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \gamma_\theta = 0}$$

Thickness = Δz



$\overset{+}{\swarrow} \sum F_z = 0$

$$\Rightarrow \Bigg[\left(\cancel{\sigma_{zz}} + \frac{\partial \sigma_{zz}}{\partial z} \Delta z\right) r \Delta r \Delta\theta - \cancel{\sigma_{zz}\, r \Delta r \Delta\theta}$$

$$+ \left(\cancel{\tau_{\theta z}} + \frac{\partial \tau_{\theta z}}{\partial \theta} \Delta\theta\right) \Delta r \Delta z - \cancel{\tau_{\theta z} \Delta r \Delta z}$$

$$+ \left(\cancel{\tau_{rz}} + \frac{\partial \tau_{rz}}{\partial r} \Delta r\right) (r+\Delta r) \Delta\theta \Delta z - \cancel{\tau_{rz}\, r \Delta\theta \Delta z}$$

$$+ \gamma_z\, r \Delta\theta \Delta r \Delta z \Bigg] = 0$$

$$\Rightarrow r \frac{\partial \sigma_{zz}}{\partial z} \Delta r \Delta\theta \Delta z + \frac{\partial \tau_{\theta z}}{\partial \theta} \Delta r \Delta\theta \Delta z + r \frac{\partial \tau_{rz}}{\partial r} \Delta r \Delta\theta \Delta z$$

$$+ \frac{\partial \tau_{rz}}{\partial r} \cancel{(\Delta r)^2}^{\,0} \Delta\theta \Delta z + \gamma_z\, r \Delta r \Delta\theta \Delta z = 0$$

Dividing by $r \Delta r \Delta\theta \Delta z$

$$\Rightarrow \boxed{\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \gamma_z = 0}$$