Tutorial 8 solutions

Q1. Suppose a solid disk of radius 'R' and a hollow disk of inner radius R^1 and outer radius R^2 are shrunk fit together (assume R $>$ R^1). Further, assume that no external pressure is applied on the outer hollow disk and that no axial displacement is allowed in the two disks. Let the two disks be made of the same material. Only radial displacement u_r is generated in this case.

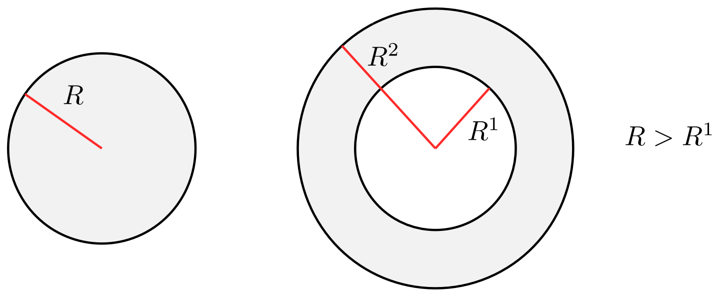
(a) Write down all the boundary conditions/ interface conditions required to obtain variation in radial stress σ_{rr} in the two disks.

(b) Solve the governing equations and obtain expression for σ_{rr} in the two disks.

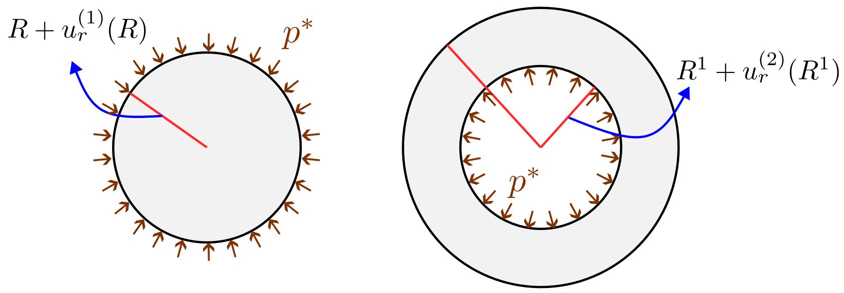
(c) Obtain an expression for circumferential/hoop stress too.

(d) Draw plots for variation of both radial and circumferential stress.

Solution:



(a) Before shrink fit



(b) On shrink fitting

Shrink-fit insertion is usually performed by heat treatment, i.e., by heating the hollow disc (thereby expanding its diameter), fitting it over the solid disc, and then cooling it. This results in the hollow cylinder to *prestress* itself and sit tightly over the solid disc. Alternatively, you could imagine that shrink-fit results in the application of pressure p^* over the external surface of the solid disk and over the inner surface of the hollow disk, as required to eliminate the overlap $R - R^1$.

We denote the solid disc and the hollow disc by superscripts (1) and (2) respectively. For example, $\sigma_{rr}^{(1)}$ and $\sigma_{rr}^{(2)}$ represent radial stress in solid and hollow discs, respectively. For the problem at hand, the discs will only undergo displacement in the radial direction, i.e. $u_\theta = u_z = 0$. As no body force acts, the radial equation when solved leads to the same solution for radial displacement or radial stress as in the class. In terms of radial stress, e.g.,

$$\sigma_{rr}^{(1)}(r) = A^{(1)}/2 + B^{(1)}/r^2, \quad \sigma_{rr}^{(2)}(r) = A^{(2)}/2 + B^{(2)}/r^2$$

or, in terms of radial displacement, we have

$$u_r^{(1)}(r) = C^{(1)}r/2 + D^{(1)}/r, \quad u_r^{(2)}(r) = C^{(2)}r/2 + D^{(2)}/r.$$

For the current problem where axial strain $\epsilon = 0$, we had derived in the class

$$A^{(i)} = 2(\lambda + \mu)C^{(i)}, \quad B^{(i)} = -2\mu\ D^{(i)}.$$

We have basically four independent integrating constants which will require four boundary conditions as mentioned below:

(a) Boundary conditions:

$$\begin{aligned}
\sigma_{rr}^{(1)}(R) &= \sigma_{rr}^{(2)}\left(R^1\right) \text{ (continuity of radial stress)} \\
R + u_r^{(1)}(R) &= R_1 + u_r^{(2)}\left(R^1\right) \text{ (Shrink-fit condition)} \\
\sigma_{rr}^{(2)}(R_2) &= 0 \text{ (Traction-free boundary condition)} \\
\sigma_{rr}^{(1)}(0) &= \text{finite!} \quad \text{or } u_r^{(1)}(0) = 0.
\end{aligned}$$

Note that radial stress is continuous at the interface. The same cannot be said about hoop stress. In fact, the three traction components on the radial plane will only get continuous across the interface because they form action-reaction pairs on the two radial faces of solid and hollow cylinders at the interface.

(b) In terms of the unknown constants appearing in the displacement formula above, we can write the following for radial stress:

$$\sigma_{rr}^{(1)}(r) = (\lambda + \mu)C^{(1)} - 2\mu\ D^{(1)}/r^2, \quad \sigma_{rr}^{(2)}(r) = (\lambda + \mu)C^{(2)} - 2\mu\ D^{(2)}/r^2.$$

Next, we apply the boundary conditions to determine the four integrating constants.

Using $\sigma_{rr}^{(1)}(0) = \text{finite} \Rightarrow D^{(1)} = 0$.

Applying first boundary condition: $\sigma_{rr}^{(1)}(R) = \sigma_{rr}^{(2)}\left(R^1\right)$

$$\Rightarrow (\lambda + \mu)\, C^{(1)} = (\lambda + \mu)\, C^{(2)} - 2\mu \frac{D^{(2)}}{\left(R^1\right)^2}, \tag{1}$$

Applying second boundary condition: $u^{(1)}(R) - u^{(2)}\left(R^1\right) = R^1 - R$

$$\Rightarrow C^{(1)}\frac{R}{2} - C^{(2)}\frac{R^1}{2} - \frac{D^{(2)}}{R^1} = R^1 - R, \tag{2}$$

Applying third boundary condition: $\sigma_{rr}^{(2)}(R_2) = 0$

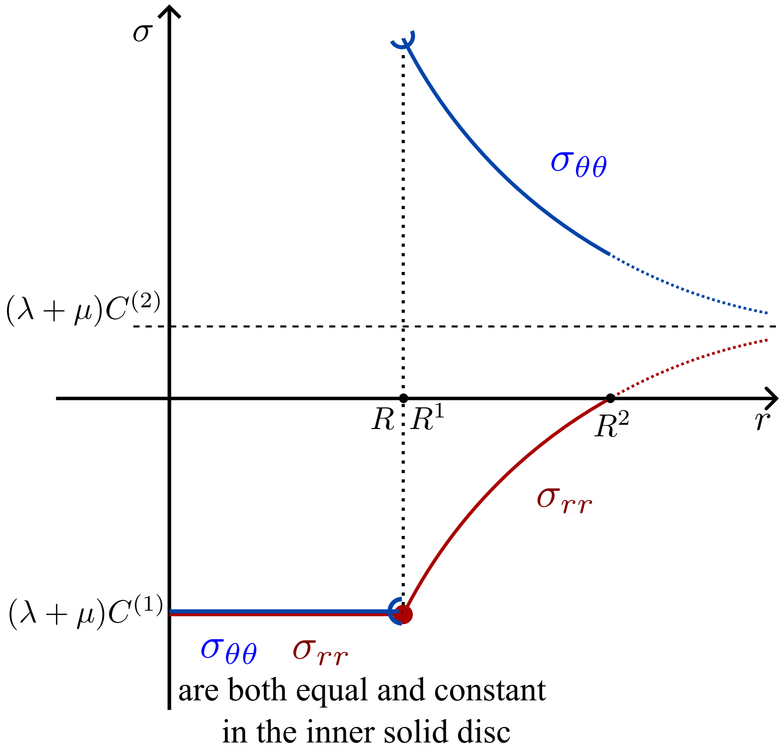
$$\Rightarrow (\lambda + \mu)\, C^{(2)} - 2\mu \frac{D^{(2)}}{\left(R^2\right)^2} = 0. \tag{3}$$

Solving equations 1, 2 and 3, we obtain get $C^{(1)},\ C^{(2)},\ D^{(2)}$.

(c) Once the constants are known, we can get circumferential stress $\sigma_{\theta\theta}$ as follows:

$$\sigma_{\theta\theta}^{(1)}(r) = (\lambda + \mu)C^{(1)}, \quad \sigma_{\theta\theta}^{(2)}(r) = (\lambda + \mu)C^{(2)} + 2\mu\ D^{(2)}/r^2.$$

(d) The plot of radial and circumferential stress variation would be as shown below. Note that σ_{rr} is continuous but $\sigma_{\theta\theta}$ is discontinuous!



Q2. Think of an isotropic solid cylinder that is glued to a rigid plate at both ends. The two rigid plates are then pulled apart along the axis of the cylinder such the normals of the rigid plates remain aligned with the axis of the cylinder. It turns out that u_θ is zero in this case but u_r and u_z do arise. Furthermore, (u_r, u_z) are not functions of θ coordinate. Assume that the deformation of the cylinder is such that every planar cross-section of the cylinder (z-plane or axial planes) remains planar even after deformation but it does change its radius.

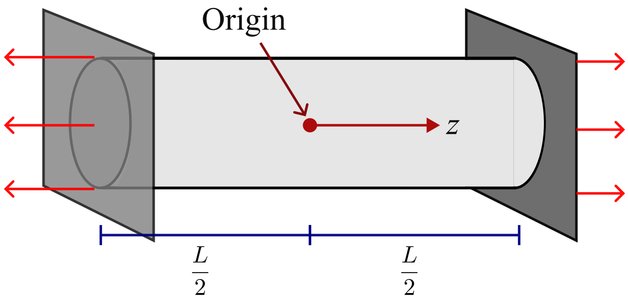
(a) What can you say about the dependence of u_r and u_z on radial and axial coordinates (r, z)?

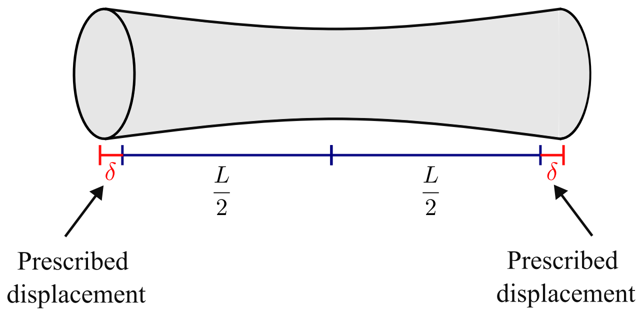
(b) Obtain the strain matrix and stress matrix for the above problem.

(c) Show that the θ-component of the stress equilibrium equation is automatically satisfied.

(d) What boundary condition will be used in order to solve the above deformation problem?

Solution:





(a) As the rigid plates are glued to the ends of the cylinder, the end cross-section can neither shrink nor expand. But, as the plates are pulled apart, the cylinder undergoes extension which will mean that the radius of the cylinder will change non-uniformly! Therefore, u_r is a function of z. Also, u_r will depend on r since the value of u_r at $r = 0$ is zero but it is non-zero for other values of r.
Finally, u_z does not depend on r otherwise a plane cross-section becomes non-planar. It will depend on z though otherwise there will be no axial strain! (Note $\partial u_z/\partial z$ is axial strain).

(b) Strain matrix

$$\left[\underline{\underline{\epsilon}}\right] = \begin{bmatrix} \dfrac{\partial u_r}{\partial r} & \dfrac{1}{2}\left(\dfrac{1}{r}\cancelto{0}{\dfrac{\partial u_r}{\partial \theta}} + \cancelto{0}{\dfrac{\partial u_\theta}{\partial r}} - \cancelto{0}{\dfrac{u_\theta}{r}}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_r}{\partial z} + \cancelto{0}{\dfrac{\partial u_z}{\partial r}}\right) \\ & \dfrac{1}{r}\cancelto{0}{\dfrac{\partial u_\theta}{\partial \theta}} + \dfrac{u_r}{r} & \dfrac{1}{2}\left(\cancelto{0}{\dfrac{\partial u_\theta}{\partial z}} + \dfrac{1}{r}\cancelto{0}{\dfrac{\partial u_z}{\partial \theta}}\right) \\ \text{sym} & & \dfrac{\partial u_z}{\partial z} \end{bmatrix}$$

$$\Rightarrow \left[\underline{\underline{\epsilon}}\right] = \begin{bmatrix} \dfrac{\partial u_r}{\partial r} & 0 & \dfrac{1}{2}\dfrac{\partial u_r}{\partial z} \\ 0 & \dfrac{u_r}{r} & 0 \\ \dfrac{1}{2}\dfrac{\partial u_r}{\partial z} & 0 & \dfrac{\partial u_z}{\partial z} \end{bmatrix}.$$

Notice that $\gamma_{rz} \neq 0$ which is a departure from the case of uniform extension-torsion-inflation case. To obtain stress matrix, use $\sigma_{ij} = \lambda\, tr\left(\underline{\underline{\epsilon}}\right)\delta_{ij} + 2\mu\epsilon_{ij}$ which leads to

$$
\begin{aligned}
\sigma_{rr} &= \lambda\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right) + 2\mu\frac{\partial u_r}{\partial r} \\
\sigma_{\theta\theta} &= \lambda\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right) + 2\mu\frac{u_r}{r} \\
\sigma_{zz} &= \lambda\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right) + 2\mu\frac{\partial u_z}{\partial z} \\
\tau_{r\theta} &= 2G\epsilon_{r\theta} = 0 \\
\tau_{rz} &= 2G\epsilon_{rz} = G\frac{\partial u_r}{\partial z}\ ,\ \tau_{\theta z} = 0
\end{aligned}
$$

(c) The θ-component of stress-equilibrium equation is

$$
\frac{\partial \tau_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + 2\frac{\tau_{\theta r}}{r} + b_\theta = \rho a_\theta .
$$

In this equation,

$$
\begin{aligned}
&\tau_{\theta r} = \tau_{r\theta} = 0,\ \text{so}\ \frac{\partial \tau_{\theta r}}{\partial r} = 0 \text{ and } \frac{\tau_{\theta r}}{r} = 0, \\
&\sigma_{\theta\theta} \text{ is not dependent on } \theta,\ \text{so}\ \frac{\partial \sigma_{\theta\theta}}{\partial \theta} = 0, \\
&\tau_{\theta z} = 0,\ \text{so}\ \frac{\partial \tau_{\theta z}}{\partial z} = 0 \\
&\text{no body force and statics condition, so } b_\theta = a_\theta = 0
\end{aligned}
$$

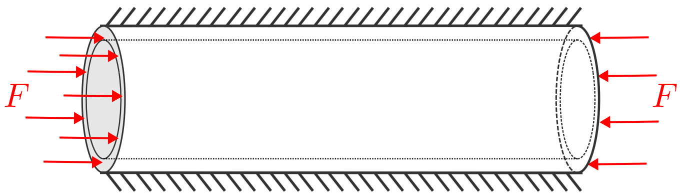
So, θ-equation is automatically satisfied! It happened due to the assumption of axisymmetry which allows the reduction of equilibrium equation from the whole three-dimensional cylindrical domain to just its $r-z$ plane.

(d) The boundary conditions along the four edges of $r-z$ plane are:

$$
\begin{aligned}
&\text{At } z = L/2 \\
&\quad u_r(r, z = L/2) = 0 \\
&\quad u_z(r, z = L/2) = \text{prescribed displacement!} \\
&\text{At } z = -L/2 \\
&\quad u_r(r, z = -L/2) = 0 \\
&\quad u_z(r, z = -L/2) = \text{prescribed displacement!} \\
&\text{At } r = 0 \\
&\quad u_r(r = 0, z) = 0 \text{ (this also implies } \frac{\partial u_r}{\partial z}(r = 0, z) = 0 \text{ or } \tau_{rz}(r = 0, z) = 0) \\
&\quad \sigma_{rr}(r = 0, z) \text{ is bounded!} \\
&\text{At } r = R \text{ (outer surface) traction free} \\
&\Rightarrow \begin{bmatrix} \sigma_{rr} \\ \tau_{\theta r} \\ \tau_{zr} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \sigma_{rr} = \tau_{zr} = 0 \ (\tau_{\theta r} = 0 \text{ is trivially satisfied})
\end{aligned}
$$

Q3. A 50.8mm diameter steel tube with a wall thickness of 1.27mm just fits in a rigid hole. Find the hoop stress if an axial compressive load of 1424kg is applied.

Solution:



Note that it might seem like a uniaxial loading problem, however, this is a biaxial loading problem, since an unknown pressure is acting on the outer lateral surface. The axial load causes the steel tube to compress axially and expand radially but the wall being rigid does not allow any expansion. So, in turn, the wall applies a compressive pressure on the outer surface of the tube.

We can say that $u_\theta = 0$, and $u_z \neq 0$, $u_r \neq 0$. We need to find out $\sigma_{\theta\theta}$. Let us first find out σ_{zz}

$$\sigma_{zz} = \frac{F}{\frac{\pi(D^2-d^2)}{4}} = \frac{F}{\frac{\pi(D+d)(D-d)}{4}} = \frac{F}{\frac{\pi(d+2t+d)(2t)}{4}} = \frac{F}{\pi(d+t)t}$$

.

here we used the fact that σ_{zz} does not vary through the wall thickness as proved in the class. The solution of σ_{rr} and $\sigma_{\theta\theta}$ from radial stress equilibrium equation in cylindrical coordinates are as follows:

$$\sigma_{rr} = \frac{A}{2} + \frac{B}{r^2}, \quad \sigma_{\theta\theta} = \frac{A}{2} - \frac{B}{r^2}$$

The values of the integrating constants A and B are obtained using the boundary conditions!

Traction BC: No internal pressure:

$$\Rightarrow \sigma_{rr}(r_1) = 0$$

We can't say $\sigma_{rr}(r_2) = 0$ since the outer constraining pressure is an unknown.

Displacement BC: No radial expansion along the outer boundary:

$$u_r(r_2) = 0.$$

The radial displacement u_r is given by

$$u_r = \frac{Cr}{2} + \frac{D}{r}$$

and the integration constants A and B of the radial stress σ_{rr} are dependent on C and D in the following fashion:

$$\begin{aligned}
\sigma_{rr} &= \lambda\left(u_r' + \frac{u_r}{r} + u_z'\right) + 2\mu u_r' \\
&= \lambda\left(\frac{C}{2} - \frac{D}{r^2} + \frac{C}{2} + \frac{D}{r^2} + u_z'\right) + 2\mu\left(\frac{C}{2} - \frac{D}{r^2}\right) \\
&= \underbrace{(\lambda+\mu)C + \lambda u_z'}_{A/2} - \frac{2\mu D}{r^2} \\
&= \frac{A}{2} - \frac{B}{r^2}
\end{aligned}$$

Thus, A and B are related to C and D as follows:

$$A = (\lambda + \mu)C + \lambda u_z', \quad B = 2\mu D.$$

We also have $u_z' = 1/E(\sigma_{zz} - \nu(\sigma_{rr} + \sigma_{\theta\theta})) = 1/E(\sigma_{zz} - \nu\ A)$ which when substituted in above equation leads to

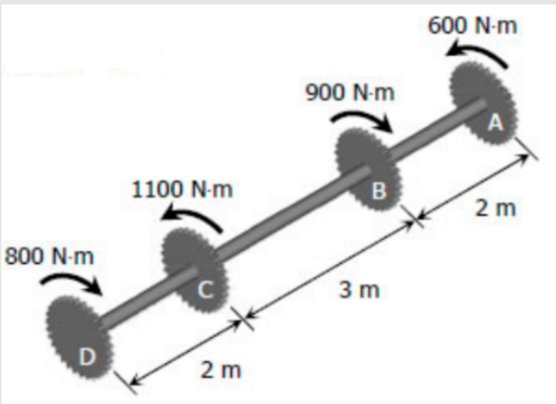
$$C = 1/(\lambda + \mu)\left[A(1 + \lambda\nu/E) - \lambda/E\ \sigma_{zz}\right],\ D = B/2\mu.$$

So, the final equations to obtain A and B are

$$\sigma_{rr}(r_1) = 0 \Rightarrow \frac{A}{2} + \frac{B}{r_1^2} = 0,$$
$$u_r(r_2) = 0 \Rightarrow A\frac{1 + \lambda\nu/E}{2(\lambda + \mu)}r_2 + \frac{B}{2\mu r_2} = \nu/E\ \frac{F}{\pi(d+t)t}r_2$$

One can then also evaluate $\sigma_{rr}(r_2)$ to obtain the unknown pressure load applied by the rigid hole.

Q4. An aluminum shaft with a constant diameter of 50mm is loaded by torques applied to gears attached to it as shown in Fig. Using G = 28GPa, determine the relative angle of twist of gear relative to gear A.



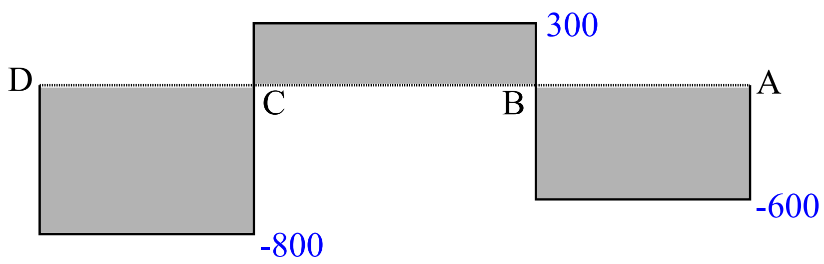
Solution: First note that the net torque on the shaft $T_A + T_B + T_C + T_D$ is zero. Hence, it is a static problem. The $+\underline{e}_3$ is taken along the axis pointing from D towards A. Accordingly, positive/negative torques are determined by the right-hand-thumb rule, with the thumb pointing along the $+\underline{e}_3$ direction.

To determine the relative rotation of point A w.r.t. point D, we need to determine the relative rotation of individual segments AB, BC, and CD using the formula

$$\Omega = \frac{TL}{GJ}.$$

The torque in each segment is different here though which can be determined from the below torque diagram - it depicts variation in torque along the shaft.

Torque diagram

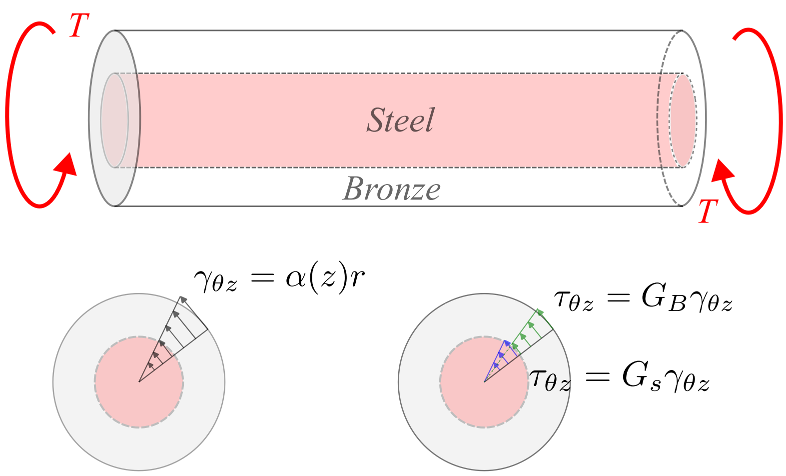


If Ω_{AD} is the relative rotation of point A w.r.t. point D, then

$$\Omega_{AD} = \Omega_{AB} + \Omega_{BC} + \Omega_{CD} = \frac{T_{AB}L_{AB} + T_{BC}L_{BC} + T_{CD}L_{CD}}{GJ}.$$

Q5. A hollow bronze shaft of 76.2mm outer diameter and 50.8mm inner diameter is slipped observer a solid steel shaft 50.8mm in diameter and of the same length as the hollow shaft. The two shafts are then fastened rigidly together at their ends. For bronze $G_B = 7\text{kN/mm}^2$, and for steel, $G_S = 12\text{kN/mm}^2$. What torque can be applied to the composite shaft without exceeding a shearing stress of 55N/mm^2 in the bronze or 82N/mm^2 in the steel?

Solution:



The shear strain $\gamma_{\theta z}$ varies linearly with r in both materials since the two parts are fastened together rigidly. Hence

$$\gamma_{\theta z} = \frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta} = \kappa r$$

Not that the twist value κ would be the same in the two materials. The shear stress $\tau_{\theta z}$ is proportional to shear strain but shear modulus being different for two materials, it will exhibit a jump across the interface for the two materials - see figure above. Accordingly, total torque in the cross-section equals

$$\int\int \tau_{\theta z}\, dA = \int\int_{Steel} G_S\, \kappa\, r^2 dA + \int\int_{Bronze} G_B\, \kappa\, r^2 dA = G_S J_S\, \kappa + G_B J_B\, \kappa$$
$$\Rightarrow T \;=\; T_S \;+\; T_B.$$

As the twist κ is same across the cross-section, this allows us to write the following relating T_S and T_B:

$$\begin{aligned}
\frac{T_S}{G_S J_S} \;&=\; \frac{T_B}{G_B J_B} \Rightarrow T_S \;=\; T_B\left(\frac{G_S J_S}{G_S J_B}\right) \\
\text{or } T \;&=\; T_S \;+\; T_B \;=\; T_B\left(\frac{G_S J_S}{G_B J_B}\right) \;+\; T_B \;=\; T_B\frac{(G_S J_S \;+\; G_B J_B)}{G_B J_B} \\
\text{or } T \;&=\; T_S \;+\; T_B \;=\; T_S \;+\; T_S\left(\frac{G_B J_B}{G_S J_S}\right) \;=\; T_S\frac{(G_S J_S \;+\; G_B J_B)}{G_S J_S}
\end{aligned}$$

Here $J_S \;=\; \pi d^4/32$ and $J_B \;=\; \pi\left(D^4 \;-\; d^4\right)/32$. Next we need to check for the maximum shear stress in each of the parts:
<u>Steel</u>

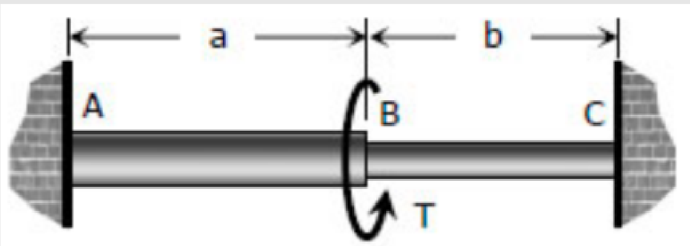
$$\tau_{S,max} \;=\; \frac{T_S r_S}{J_s} \;=\; \frac{T_S d}{2J_S} \;\leq\; \tau_{S,tol} \Rightarrow T_S \;\leq\; \tau_{S,tol}\frac{2J_S}{d}$$

<u>Bronze</u>

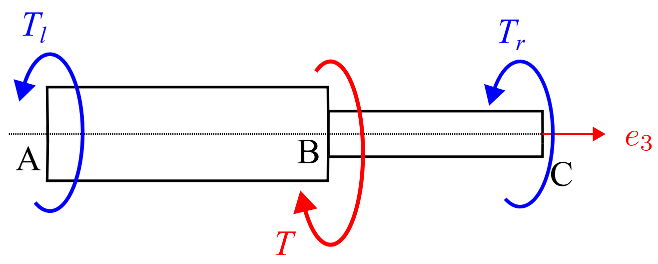
$$\tau_{B,max} \;=\; \frac{T_B r_B}{J_B} \;=\; \frac{T_B D}{2J_B} \;\leq\; \tau_{B,tol} \Rightarrow\; T_B \;\leq\; \tau_{B,tol}\frac{2J_B}{D}$$

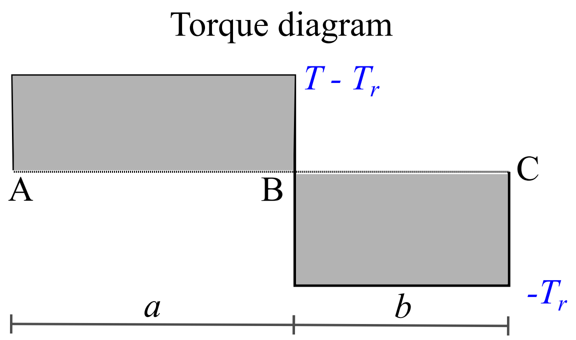
$$T \;=\; \min\left\{\tau_{B,tol}\frac{2J_B}{D}\frac{(G_S J_S \;+\; G_B J_B)}{G_B J_B},\; \tau_{S,tol}\frac{2J_S}{d}\frac{(G_S J_S \;+\; G_B J_B)}{G_S J_S}\right\}$$

Q6. The compound shaft shown in the figure below is attached to rigid supports. For the bronze segment AB, the diameter is 75mm, $\tau \leq$ 60MPa, and G = 35GPa. For the steel segment BC, the diameter is 50mm, $\tau \leq$ 80MPa, and G = 83GPa IF a = 2m and b = 1.5m, compute the maximum torque T that can be applied.



Solution: Let the end reaction torques be T_L and T_R, respectively, at left and right fixed ends as shown in the figure below.





The two reaction torques are unknown. We can use the moment balance of the full beam as follows:

$$\begin{aligned}
&\sum \text{Moment about } \underline{e}_3\text{-axis} \;=\; 0 \\
&\Rightarrow -T_R \,+\, T \,-\, T_L \;=\; 0 \quad \text{or } \; T \;=\; T_R \,+\, T_L
\end{aligned}$$

which gives us one equation. It turns out we do not get any further equation from statics to obtain the two reaction torques. Hence, it is also called a statically indeterminate problem. We need to use torque-twist relation to obtain the two reaction torques that we now show. Let

$$\begin{aligned}
\Omega_{BC} &= \text{rotation of end C w.r.t end B,} \\
\Omega_{AB} &= \text{rotation of end B w.r.t end A,} \\
\Omega_{AC} &= \text{rotation of end C w.r.t end A.}
\end{aligned}$$

Since the two ends are fixed, no rotations are allowed at A and C. Therefore $\Omega_{AC} \;=\; 0$. However

$$\begin{aligned}
\Omega_{AC} \;&=\; \Omega_{AB} \,+\, \Omega_{BC} \\
\Rightarrow \Omega_{AB} \;&=\; -\Omega_{BC} \\
\Rightarrow \frac{T_{AB}L_{AB}}{G_{AB}J_{AB}} \;&=\; -\frac{T_{BC}L_{BC}}{G_{BC}J_{BC}} \\
\Rightarrow \frac{T_L a}{G_B J_B} \;&=\; \frac{T_R b}{G_S J_S}.
\end{aligned}$$

So,

$$T_L \;=\; T_R\left(\frac{b}{a}\right)\left(\frac{G_B J_B}{G_S J_S}\right) \;\text{or}\; T_R \;=\; T_L\left(\frac{a}{b}\right)\left(\frac{G_S J_S}{G_B J_B}\right) \tag{4}$$

So, total torque

$$T \;=\; T_L \,+\, T_R = T_L\left(\frac{G_B J_B b \,+\, G_S J_S a}{G_B J_B b}\right) \;\text{or}\; T \;=\; T_R\left(\frac{G_B J_B b \,+\, G_S J_S a}{G_S J_S a}\right) \tag{5}$$

Check for bronze ($\tau_{B,max} \leq \tau_{tol,B}$)

$$\tau_{b,max} = \frac{T_L \; r_B}{J_B} \Rightarrow T_L \leq \frac{\tau_{tol,b} J_B}{r_B}$$

So, using Eq. 5, Total torque

$$T \leq \left(\frac{\tau_{tol,B} \; J_B}{r_B}\right)\left(\frac{G_B J_B b \; + \; G_S J_S a}{G_B J_B b}\right)$$

Check for steel ($\tau_{s,max} \leq \tau_{tol,S}$)

$$\tau_{s,max} = \frac{T_R \; r_S}{J_S} \Rightarrow T_R \leq \frac{\tau_{tol,S} J_S}{r_S}$$

Again, using Eq. 5, total torque

$$T \leq \left(\frac{\tau_{tol,S} \; J_S}{r_S}\right)\left(\frac{G_B J_B b \; + \; G_S J_S a}{G_S J_S a}\right)$$

Maximum permissible torque T will be minimum of above two values.