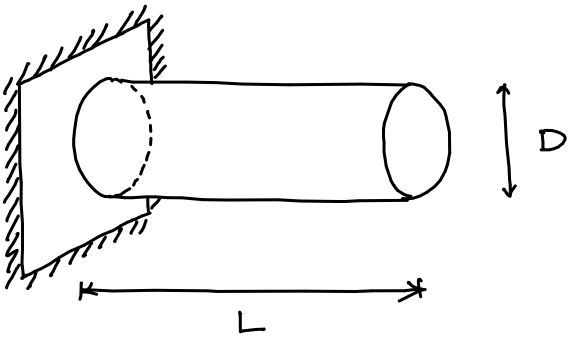
Deflections of beams by Euler-Bernoulli Beam Theory

Beams are usually members whose cross-sectional dimensions are usually much smaller than the length. Typically a beam is characterized by its aspect ratio or slenderness ratio

$$\text{Aspect ratio} = \frac{\text{length of beam}}{\text{representative size of its C/s}}$$

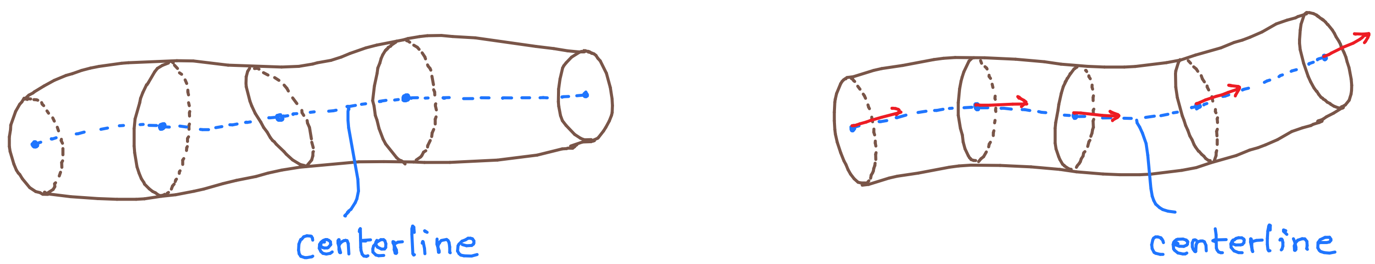


$$\text{Aspect ratio} = \frac{L}{D}$$

Usually, the aspect ratio for which beam theory works fine is in the range $\frac{L}{D} \geqslant 10$. Earlier, we had looked at bending strain and bending stress of a beam. However, we have not talked about the vertical deformation of a beam under the action of transverse loads.

One can solve all 15 PDE equations of elasticity along with boundary conditions to solve the deformation of beams. However, for beams significant deformations happen only in the transverse direction (i.e. directions perpendicular to the length of the beam).

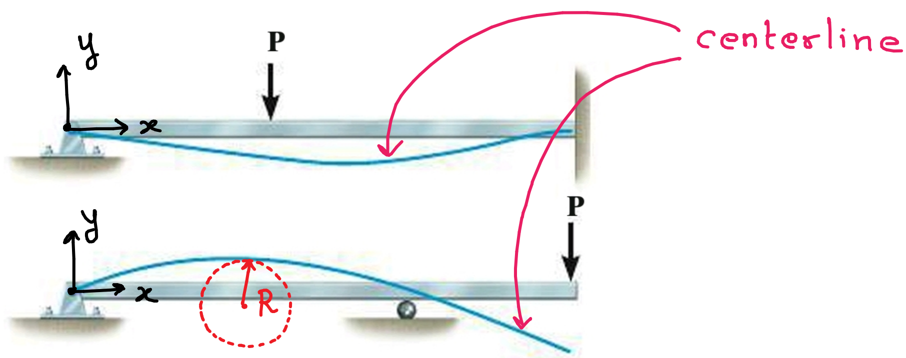
Beam theory places certain assumptions on deformation and provides an easy solution for calculating transverse deformation of beams. Instead of computing displacement at every point of the beam, beam theory finds only the displacement of its centerline — which is the line joining the geometric centroid of all its cross-sections — and ignores the cross-sectional deformation. The cross-sections are assumed to aligned along the centerline tangent



Exact deformed configuration

Approximated deformed configuration from Euler-Bernoulli beam theory

So in Euler-Bernoulli beam theory, we only consider the displacement of the centerline of the beam as the only unknown.



To get an equation for it, we use the bending formula derived earlier:

$$\frac{1}{R} = \frac{M}{EI}$$

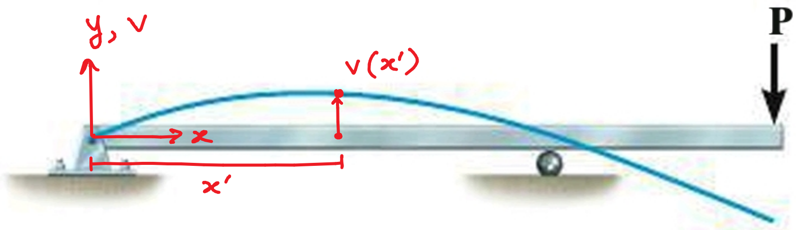
R — radius of curvature at a given c/s

M — Bending moment at a c/s

EI — flexural rigidity

If 'v' be the transverse deflection of the centerline of the beam directed along 'y', then the equation of the deflected centerline of the beam can be mathematically expressed as :

$$v = f(x)$$



To obtain this equation, we must get the radius of local curvature R in terms of x and displacement v. From differential geometry, the radius of curvature R of the centerline of the deflected beam can be written as :

$$\frac{1}{R(x)} = \frac{\frac{d^2 v}{dx^2}}{\left(1 + \left(\frac{dv}{dx}\right)^2\right)^{3/2}} = \frac{M(x)}{EI}$$

Slope of centerline

The above equation is a non-linear second-order differential equation. To linearize the expression, it is further assumed that the slope $\left|\frac{dv}{dx}\right| << 1$ (is very small). Therefore, we get :

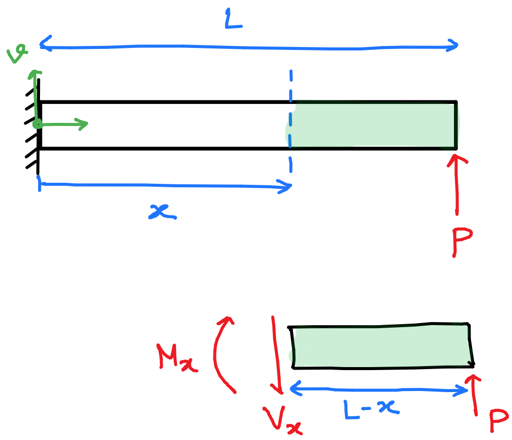
$$\frac{M(x)}{EI} \approx \frac{d^2 v}{dx^2}$$

$$\Rightarrow \boxed{M(x) = EI \frac{d^2 v}{dx^2}}$$

This is the governing equation of deflection of beam according to Euler-Bernoulli beam theory

Linear 2nd-order ODE
(not PDE)

Example: A cantilever clamped at one end and a transverse load is applied. Find the deflection of the beam using EB theory



Force balance of shaded region

$$+\uparrow \sum F_y = 0$$

$$\Rightarrow \quad P - V_x = 0 \quad \Rightarrow \quad V_x = P$$

Moment balance about left end of shaded region

$$\circlearrowleft + \sum M_{left} = 0$$

$$\Rightarrow -M_x + P(L-x) = 0$$

$$\Rightarrow \quad M_x = P(L-x)$$

$$EI \frac{d^2v}{dx^2} = M_x$$

$$\Rightarrow \quad \frac{d^2v}{dx^2} = \frac{P}{EI}(L-x)$$

Upon integrating twice, we will get two integration constants. So, we need two boundary conditions to find them.

Boundary conditions:

At fixed end $(x=0)$, the centerline can neither deflect nor rotate

deflection $\rightarrow$ $v(0) = 0$ —— 1

slope $\rightarrow$ $\frac{dv(0)}{dx} = 0$ —— 2

$$\frac{dy}{dx} = \int \frac{P}{EI}(L-x)\,dx + c_1 = \frac{P}{EI}\left(Lx - \frac{x^2}{2}\right) + c_1$$

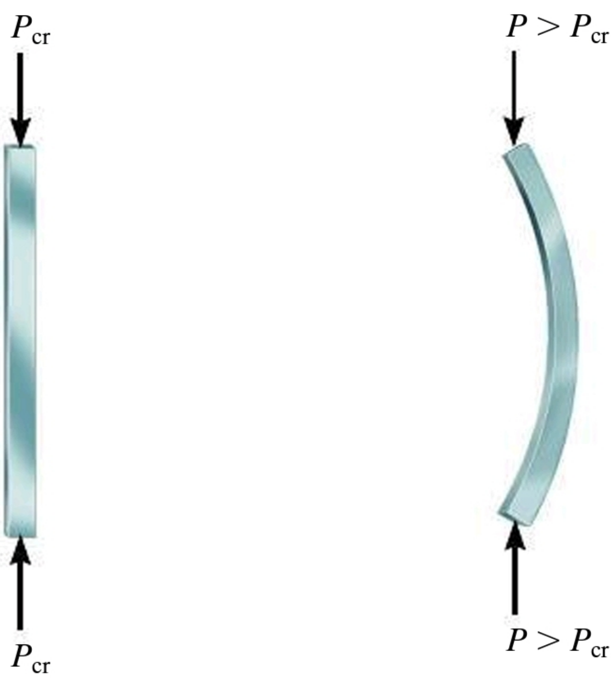
Using 2 $\Rightarrow$ $c_1 = 0$

$$y = \int \frac{P}{EI}\left(Lx - \frac{x^2}{2}\right)dx = \frac{P}{EI}\left(L\frac{x^2}{2} - \frac{x^3}{6}\right) + c_2$$

Using 1 $\Rightarrow$ $c_2 = 0$

Buckling of beams

When we try to compress a stick of a broom (or any long and thin rod), it initially remains straight, but as we keep increasing the compressive force above a critical value, the stick / beam bends instantaneously. This phenomenon is called BUCKLING and the critical value of compressive load at which it happens is called CRITICAL LOAD

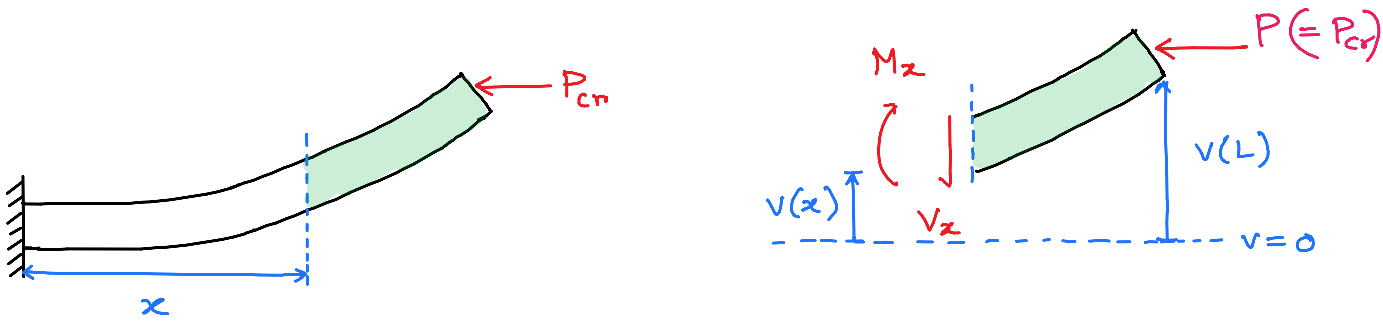


Whenever we design any structure having a beam-like element (such as pillars of a building) where the member has to resist compressive load, we have to make sure that the operational compressive load is less than the critical load. Otherwise, the beam-like member will fail by buckling.

How to find the critical load that causes buckling?

We will use the Euler-Bernoulli beam theory.

Consider a cantilever beam (fixed-free) subjected to an axial compressive load P. When the compressive load P reaches a critical value P_{cr}, the beam/column will bend as shown:



Balancing the moment for the shaded portion about the left end:

$$-M_x + P\left[v(L) - v(x)\right] = 0$$

$$\Rightarrow \quad EI\frac{d^2v}{dx^2} = P(v_L - v)$$

$$\Rightarrow \quad EI\frac{d^2v}{dx^2} + Pv = Pv_L$$

$$\Rightarrow \quad \frac{d^2v}{dx^2} + \frac{P}{EI}v = \frac{Pv_L}{EI}$$

A linear 2nd-order non-homogeneous ODE

To find a general solution, we need to first solve the correspond-ing homogeneous part and then add particular integral to it.

Homogeneous soln: Substitute $y = e^{\lambda x}$ in the equation

$$\underbrace{\lambda^2 + \frac{P}{EI} = 0}_{\text{Characteristic eqn}} \quad \Rightarrow \quad \lambda = \pm i\sqrt{\frac{P}{EI}}$$

$\sqrt{-1}$

As λ is imaginary, the solution has cosine and sine terms.

The complementary function can be written as

$$v(x) = c_1 \cos \lambda x + c_2 \sin \lambda x$$

$$= c_1 \cos\left(\sqrt{\frac{P}{EI}}\, x\right) + c_2 \sin\left(\sqrt{\frac{P}{EI}}\, x\right)$$

Particular Integral: In this case, we can get the particular integral just by observation. When we substitute $v(x) = v_L$ (constant)

$$EI \frac{d^2 v(x)}{dx^2} + P v(x) = P v_L$$

v_L satisfies the equation

Any function that satisfies the differential equation can be considered as a particular integral. Therefore, $v(x) = v_L$ is chosen as the particular integral