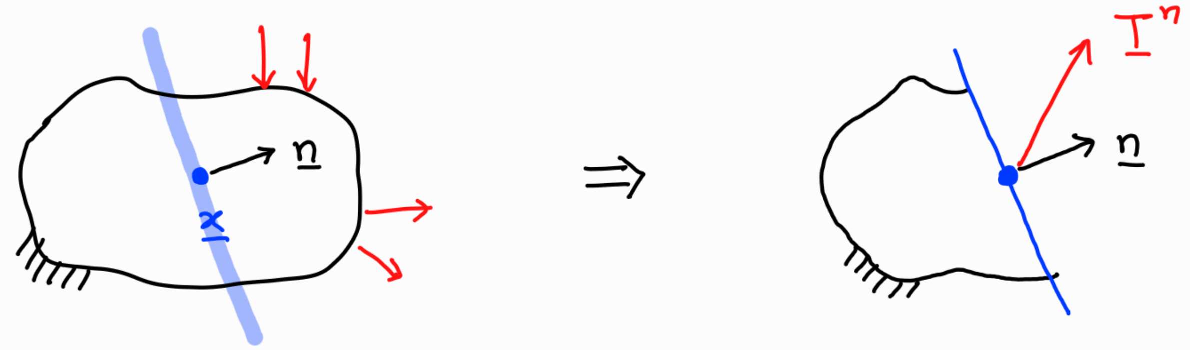
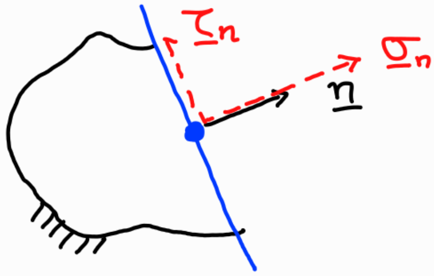


decompose $\underline{T}^n$ into

normal & shear components



From failure considerations of materials, it is of interest to know the following:

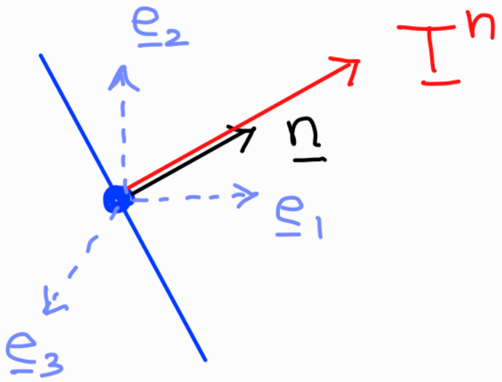
(a) If there are any planes passing through a given point on which traction vector is wholly normal?

(i.e. traction vector has zero shear component & non-zero normal component)

(b) On which plane does the normal stress become maximum? What will be its magnitude?

(c) On which plane does the shear stress become maximum? What will be its magnitude?

Consider a plane with normal $\underline{n}$ s.t. the traction vector is oriented along the normal vector



$$[\underline{T}^n] = \lambda [\underline{n}] \quad — ①$$

$$[\underline{T}^n] = [\underline{\underline{\sigma}}][\underline{n}] \quad — ②$$

$$[\underline{\underline{\sigma}}][\underline{n}] = \lambda [\underline{n}]$$

$$\Rightarrow \quad [\underline{\underline{\sigma}} - \lambda \underline{\underline{I}}][\underline{n}] = \underline{0}$$

$$\Rightarrow \quad \begin{bmatrix} \sigma_{11} - \lambda & \tau_{21} & \tau_{31} \\ \tau_{21} & \sigma_{22} - \lambda & \tau_{32} \\ \tau_{31} & \tau_{32} & \sigma_{33} - \lambda \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

A trivial soln is going to be $n_1 = n_2 = n_3 = 0$

For non-trivial soln, determinant must vanish

$$\lambda^3 - (\sigma_{11} + \sigma_{22} + \sigma_{33})\lambda^2 + (\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{33} - \tau_{21}^2 - \tau_{31}^2 - \tau_{32}^2)\lambda$$

$$- (\sigma_{11}\sigma_{22}\sigma_{33} + 2\tau_{21}\tau_{32}\tau_{31} - \sigma_{11}\tau_{32}^2 - \sigma_{22}\tau_{31}^2 - \sigma_{33}\tau_{21}^2) = 0$$

Cubic eqn. → three roots

$$\left.\lambda_1, \; \lambda_2, \; \lambda_3 \right\} \text{ 3 eigenvalues}$$

Substitute $\lambda = \lambda_1$ $\Rightarrow$ $[\underline{n}] = [\underline{n}_1] = \begin{bmatrix} n_{11} \\ n_{21} \\ n_{31} \end{bmatrix}$ ← eigenvector

1st Principal stress

$$\begin{bmatrix} \sigma_{11} - \lambda_1 & \tau_{21} & \tau_{31} \\ & \sigma_{22} - \lambda_1 & \tau_{32} \\ \text{sym} & & \sigma_{33} - \lambda_1 \end{bmatrix} \underbrace{\begin{bmatrix} n_{11} \\ n_{21} \\ n_{31} \end{bmatrix}}_{[\underline{n}_1]} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$n_{11}^2 + n_{21}^2 + n_{31}^2 = 1$ ← the magnitude of the eigenvector is taken as 1

$[\underline{n}_1]$ → is the outward normal of the 1st principal plane

Substitute $\lambda = \lambda_2$ $\Rightarrow$ $[\underline{n}] = [\underline{n}_2] = \begin{bmatrix} n_{12} \\ n_{22} \\ n_{32} \end{bmatrix}$ ← eigenvector 2

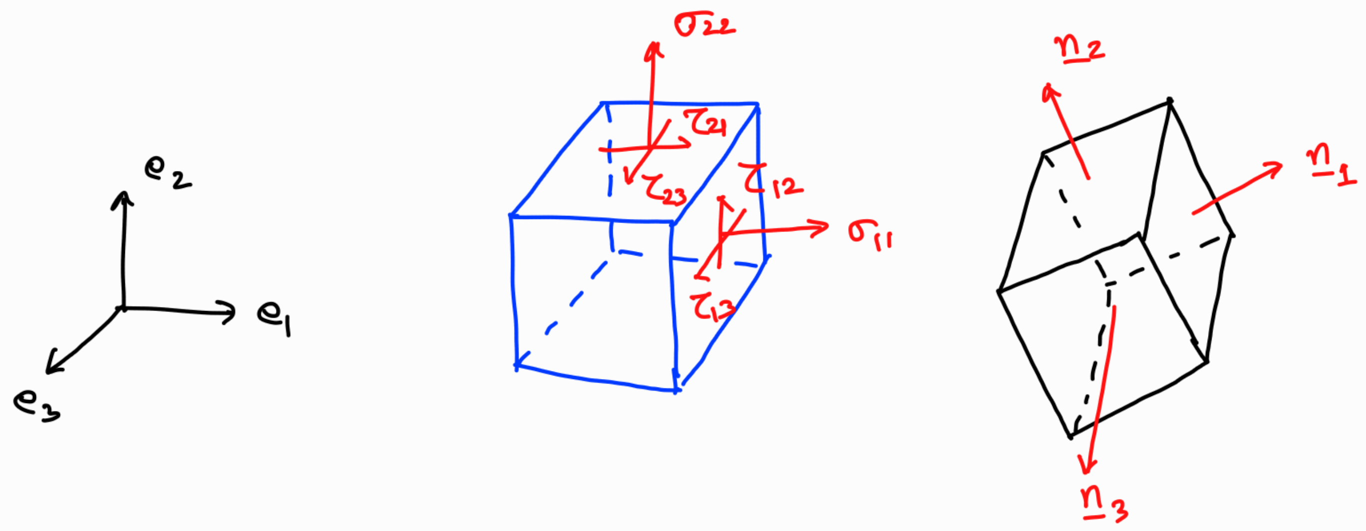
1st Principal stress

$$\begin{bmatrix} \sigma_{11} - \lambda_2 & \tau_{21} & \tau_{31} \\ & \sigma_{22} - \lambda_2 & \tau_{32} \\ \text{sym} & & \sigma_{33} - \lambda_2 \end{bmatrix} \underbrace{\begin{bmatrix} n_{12} \\ n_{22} \\ n_{32} \end{bmatrix}}_{[\underline{n}_2]} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$n_{12}^2 + n_{22}^2 + n_{32}^2 = 1$ ← the magnitude of the eigenvector is taken as 1

$[\underline{n}_2]$ → is the outward normal of the 2nd principal plane

Properties of principal planes at a point



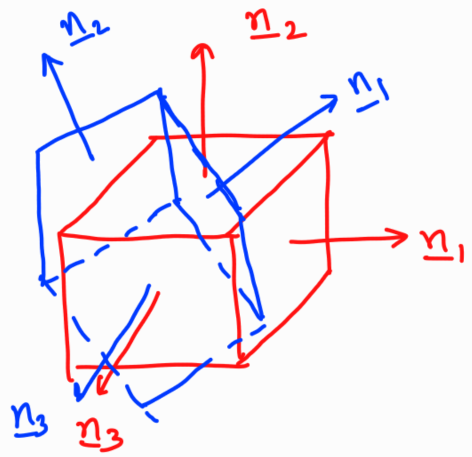
1) If eigenvalues $\lambda_1 \neq \lambda_2 \neq \lambda_3$ (distinct)

$$\underline{n}_1 \perp \underline{n}_2 \perp \underline{n}_3$$

2) $\lambda_1 = \lambda_2 \neq \lambda_3$ (two eigenvalues repeated)

$\downarrow$

$\underline{n}_3$ is unique



3) $\lambda_1 = \lambda_2 = \lambda_3$ (all eigenvalues repeated)

Every direction is a principal direction

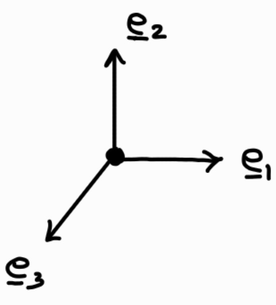
Representation of stress tensor in the coordinate system of its eigenvectors

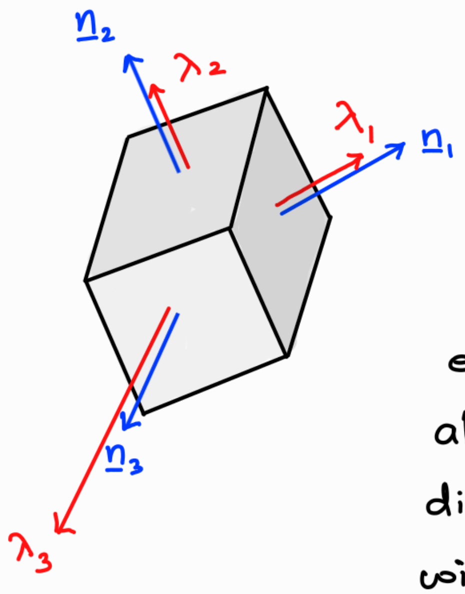
If you choose three mutually perpendicular eigenvectors to be the basis vectors of a coordinate system and then represent the stress tensor in the coordinate system

By definition, the traction on the principal planes will be $\lambda \underline{n}$ (no shear components would be present)

The stress matrix will be diagonal when expressed in the coordinate system spanned by principal directions

$$[\underline{\underline{\sigma}}]_{\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$





With a cuboid element's faces along the principal directions, there will be no shear components