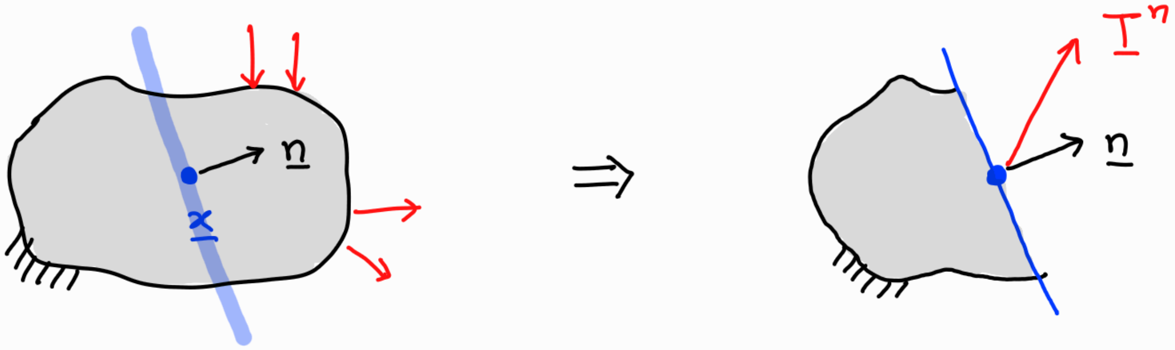
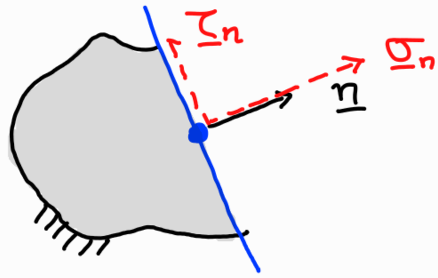
Principal Stresses and Principal Planes



decompose $\underline{T}^n$ into
normal & shear components



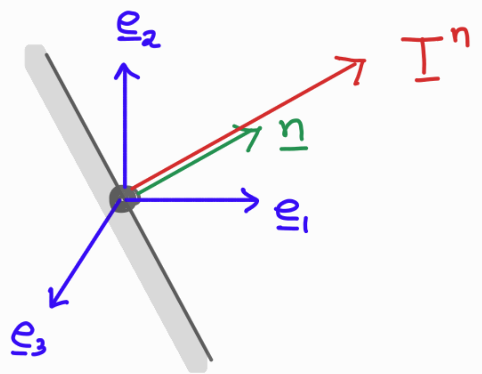
From failure considerations of materials, it is of interest to know the following:

a) If there are any planes passing through a given point on which traction vector is wholly normal ?
(i.e. traction vector has only normal component and zero shear component)

b) On which plane does the normal stress become maximum? What will be its magnitude?

c) On which plane does the shear stress become maximum? What will be its magnitude?

Consider a plane with normal $\underline{n}$ s.t. the traction vector is oriented along the normal vector



$$T^n = \lambda \underline{n} \quad \text{— (1)}$$

Also, we know

$$T^n = \underline{\underline{\sigma}}\, \underline{n} \quad \text{— (2)}$$

Equating (1) and (2), we can write:

$$\boxed{[\underline{\underline{\sigma}} - \lambda \underline{\underline{I}}]\, \underline{n} = \underline{0}} \longrightarrow \underline{n} = \underline{0} \text{ is a TRIVIAL SOLN}$$

called as **Stress eigenvalue problem**

For non-trivial solution $\longrightarrow$ $\det([\underline{\underline{\sigma}} - \lambda \underline{\underline{I}}]) = 0$

Writing in the matrix form using $\underline{e}_1 - \underline{e}_2 - \underline{e}_3$ csys:

$$\det\left(\begin{bmatrix} \sigma_{11} - \lambda & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_{22} - \lambda & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_{33} - \lambda \end{bmatrix}\right) = 0$$

$$\lambda^3 - (\sigma_{11} + \sigma_{22} + \sigma_{33})\lambda^2$$

$$+ (\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{33} - \tau_{21}^2 - \tau_{31}^2 - \tau_{32}^2)\lambda$$

$$- (\sigma_{11}\sigma_{22}\sigma_{33} + 2\tau_{21}\tau_{32}\tau_{31} - \sigma_{11}\tau_{32}^2 - \sigma_{22}\tau_{31}^2 - \sigma_{33}\tau_{21}^2) = 0$$

Cubic polynomial → has THREE roots

↳ $\underbrace{\lambda_1, \lambda_2, \lambda_3}_{\text{3 eigenvalues}}$

Substituting each eigenvalue one-by-one in the stress eigenvalue problem would give us the corresponding normal vector $\underline{n}_i$

Principal plane normal $\underline{n}_1$ ← eigenvector associated with eigenvalue λ_1

Substitute $\lambda = \lambda_1$ and solve for $\underline{n}_1$

$$\underbrace{\begin{bmatrix} \sigma_{11} - \lambda_1 & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_{22} - \lambda_1 & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_{33} - \lambda_1 \end{bmatrix}}_{\substack{\text{Rank-deficient} \\ \text{(need more equations)}}} \underbrace{\begin{bmatrix} n_{11} \\ n_{21} \\ n_{31} \end{bmatrix}}_{[\underline{n}_1]_{(e_1, e_2, e_3)}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Assuming UNIT normal gives one extra eqn : $\|\underline{n}_1\|_2^2 = 1$

$$\Rightarrow n_{11}^2 + n_{21}^2 + n_{31}^2 = 1$$

Terminology:

λ_1 – 1st principal stress

$\underline{n}_1$ – 1st principal plane normal

Principal plane normal $\underline{n}_2$

eigenvector associated with eigenvalue λ_2

Substitute $\lambda = \lambda_2$ and solve for $\underline{n}_2$

$$\begin{bmatrix} \sigma_{11} - \lambda_2 & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_{22} - \lambda_2 & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_{33} - \lambda_2 \end{bmatrix} \underbrace{\begin{bmatrix} n_{12} \\ n_{22} \\ n_{32} \end{bmatrix}}_{[\underline{n}_2]_{(e_1, e_2, e_3)}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$n_{12}^2 + n_{22}^2 + n_{32}^2 = 1$$

λ_2 – 2nd principal stress

$\underline{n}_2$ – 2nd principal plane normal

Principal plane normal $\underline{n}_3$

eigenvector associated with eigenvalue λ_3

Substitute $\lambda = \lambda_3$ and solve for $\underline{n}_3$

$$\begin{bmatrix} \sigma_{11} - \lambda_3 & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_{22} - \lambda_3 & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_{33} - \lambda_3 \end{bmatrix} \underbrace{\begin{bmatrix} n_{13} \\ n_{23} \\ n_{33} \end{bmatrix}}_{[\underline{n}_3]_{(e_1, e_2, e_3)}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$n_{13}^2 + n_{23}^2 + n_{33}^2 = 1$$

λ_3 – 3rd principal stress

$\underline{n}_3$ – 3rd principal plane normal

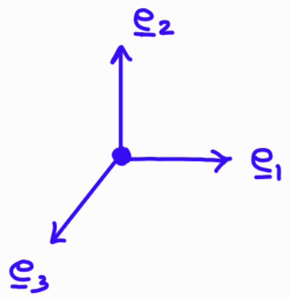
Representation of stress tensor in the coordinate system of its eigenvectors

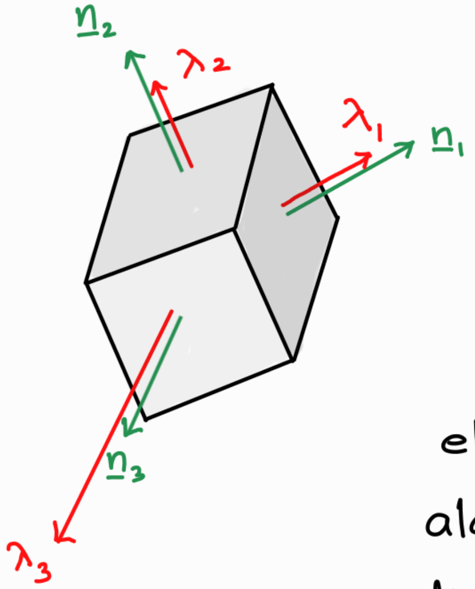
If you choose three mutually perpendicular eigenvectors to be the basis vectors of a coordinate system and then represent the stress tensor in the coordinate system

By definition, the traction on the principal planes will be $\lambda \underline{n}$ (no shear components would be present)

The stress matrix will be diagonal when expressed in the coordinate system spanned by principal directions

$$[\underline{\underline{\sigma}}]_{\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$





With a cuboid element's faces along the principal directions, there will be no shear components

Properties of principal stress/planes at a point

- Principal planes are also planes on which the normal component of traction reaches extremum (maximum/minimum)

  (will be proved in next lecture)

- The principal plane normals are the eigenvectors of the stress tensor

Stress matrix is 3 x 3 matrix → 3 eigenvalues, 3 eigenvectors

→ SYMMETRIC

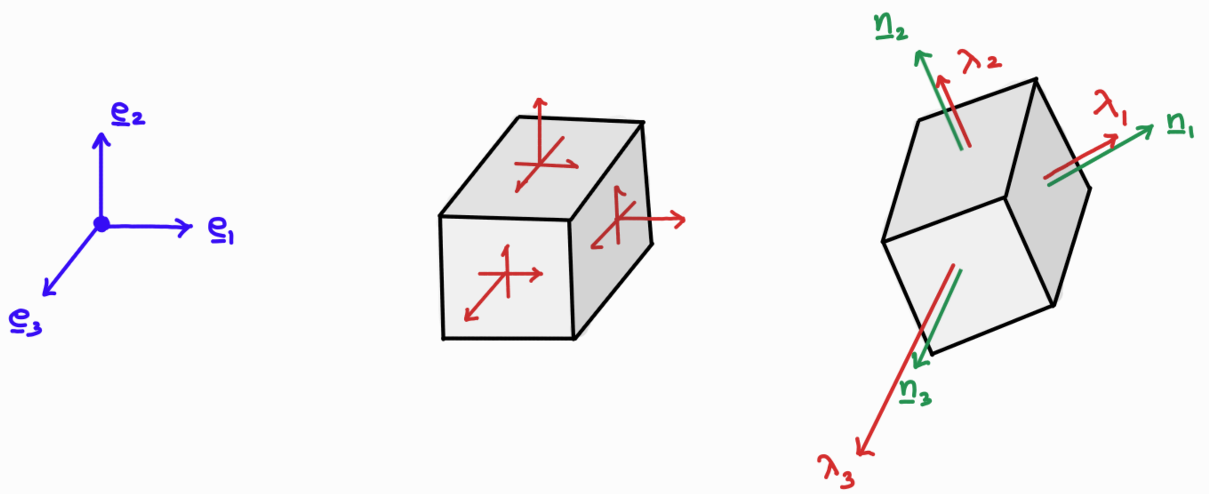
⇓

eigenvalues/eigenvectors are always REAL-valued

1) If $\lambda_1 \neq \lambda_2 \neq \lambda_3$ (distinct eigenvalues)

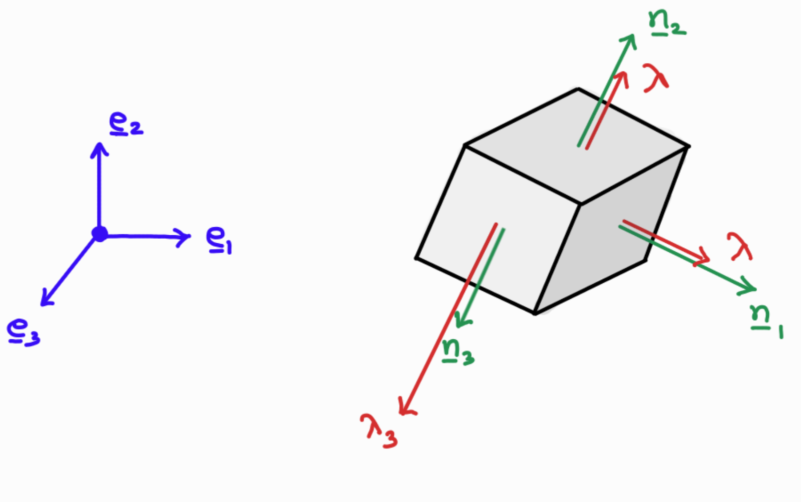
   then the associated eigenvectors are UNIQUE and they are perpendicular to each other

$$\underline{n}_1 \perp \underline{n}_2 \perp \underline{n}_3$$ (prove in Tut 4)



2) If $\lambda_1 = \lambda_2 \neq \lambda_3$ (two repeated eigenvalues)

Only the plane normal corresponding to non-repeated eigenvalue is unique (e.g. $\underline{n}_3$ is unique) and every direction perpendicular to $\underline{n}_3$ is a principal direction



3) If $\lambda_1 = \lambda_2 = \lambda_3$ (all eigenvalues are repeated)

then every direction is a principal direction

From failure considerations of materials, it is of interest to know the following:

(a) If there are any planes passing through a given point on which traction vector is wholly normal?

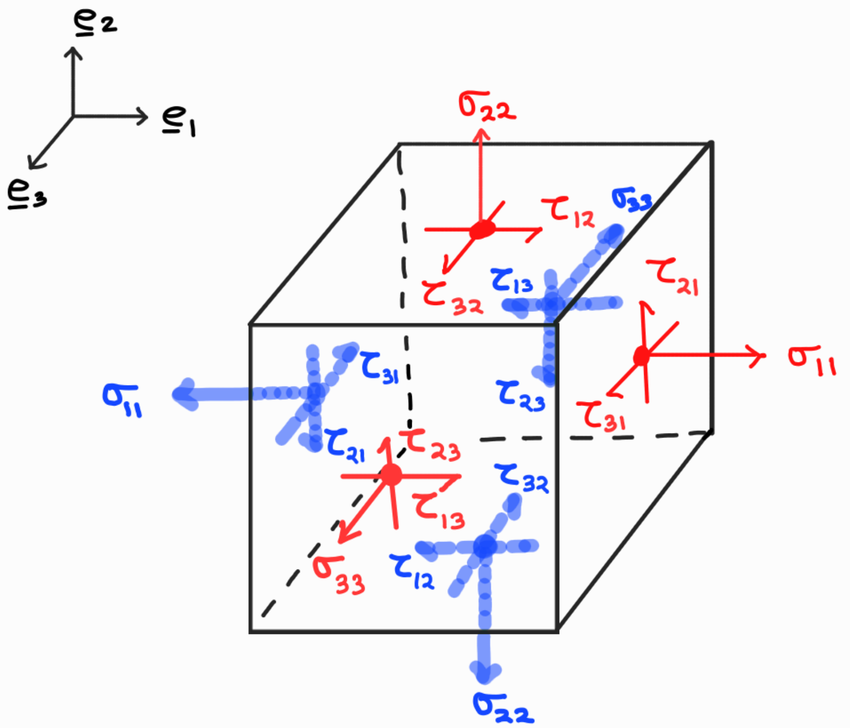
(i.e. traction vector has zero shear component & non-zero normal component) Principal planes ✓

(b) On which plane does the normal stress become maximum? What will be its magnitude?

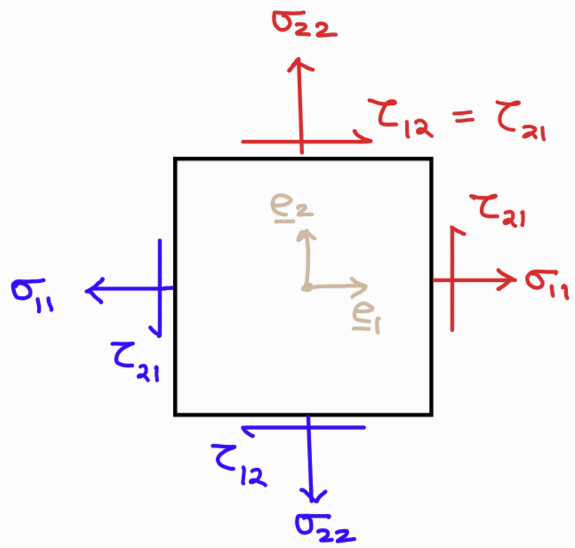
(c) On which plane does the shear stress become maximum? What will be its magnitude?

We will now look at 2D plane stress case (instead of 3D) as this condition is most commonly assumed in practice.

At the end, we will discuss a method for finding the absolute maximum normal & shear stress at a point when the material is subject to both plane & 3D states of stress



3D state of stress

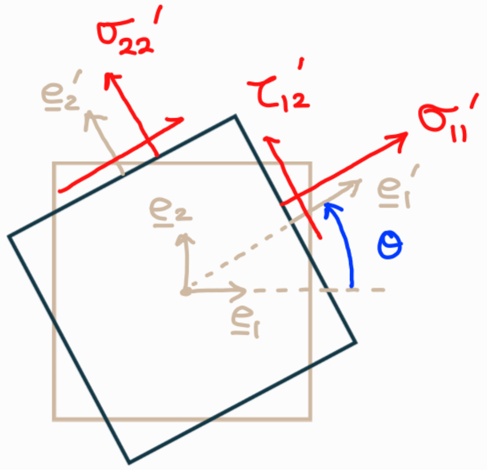


2D/plane state of stress

The general state of PLANE STRESS at a point is represented by two normal stress components, σ_{11} and σ_{22}, and one shear stress component, τ_{12}.

The values of these components will be different for different orientation of the plane stress element.

That is to say, if we rotate the plane stress element by an angle θ (say counterclockwise), then the stress component values will change to σ_{11}', σ_{22}', and τ_{12}'

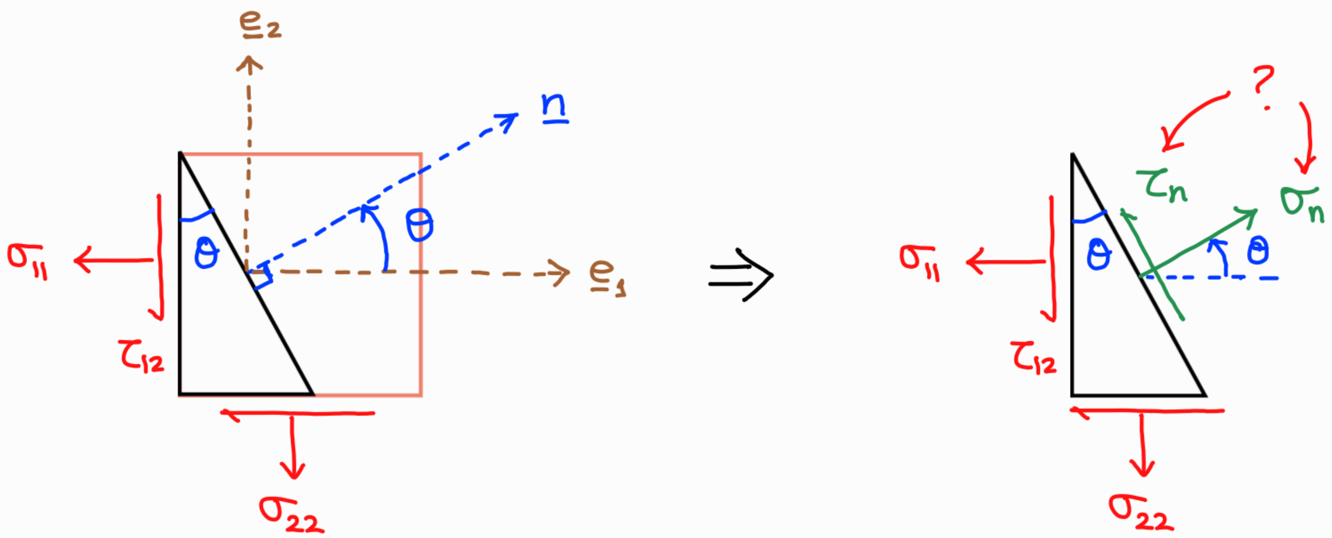


How are these plane stress components σ_{11}', σ_{22}', τ_{12}' related to σ_{11}, σ_{22}, τ_{12} via orientation θ?

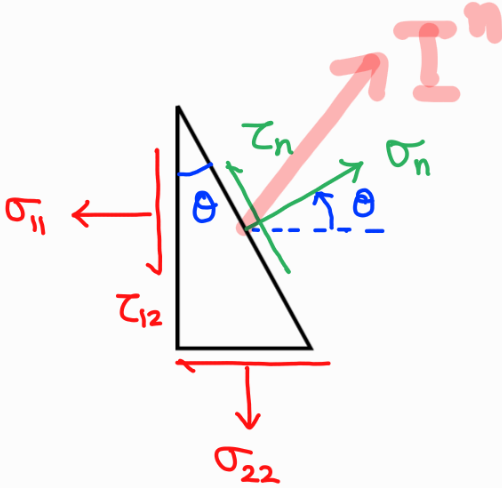
Can we find a θ for which normal stresses σ_{11}', σ_{22}' become max/min?

Can we find a θ for which shear stress τ_{12}' becomes absolute maximum?

Plane Stress Transformation

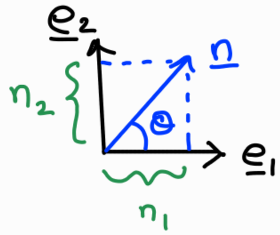


Traction on $\underline{n}$-plane at that point



$$\underline{T}^n = \underline{\underline{\sigma}}\, \underline{n}$$

Normal component of traction



$$\sigma_n = \underline{T}^n \cdot \underline{n}$$

$$= (\underline{\underline{\sigma}} \cdot \underline{n}) \cdot \underline{n}$$

$$= [\underline{n}]^T [\underline{\underline{\sigma}}] [\underline{n}]$$

$$= \begin{bmatrix} n_1 & n_2 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \tau_{12} \\ \tau_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$

$$= \sigma_{11} n_1^2 + \sigma_{22} n_2^2 + 2\tau_{12} n_1 n_2$$

$$= \sigma_{11} \cos^2\theta + \sigma_{22} \sin^2\theta + 2\tau_{12} \cos\theta \sin\theta$$

$$\boxed{\sigma_n(\theta) = \sigma_{11}\left(\frac{1+\cos 2\theta}{2}\right) + \sigma_{22}\left(\frac{1-\cos 2\theta}{2}\right) + \tau_{12} \sin 2\theta}$$

To find max/min values of $\sigma_n(\theta)$, set:

$$\frac{d\sigma_n(\theta)}{d\theta} = 0$$

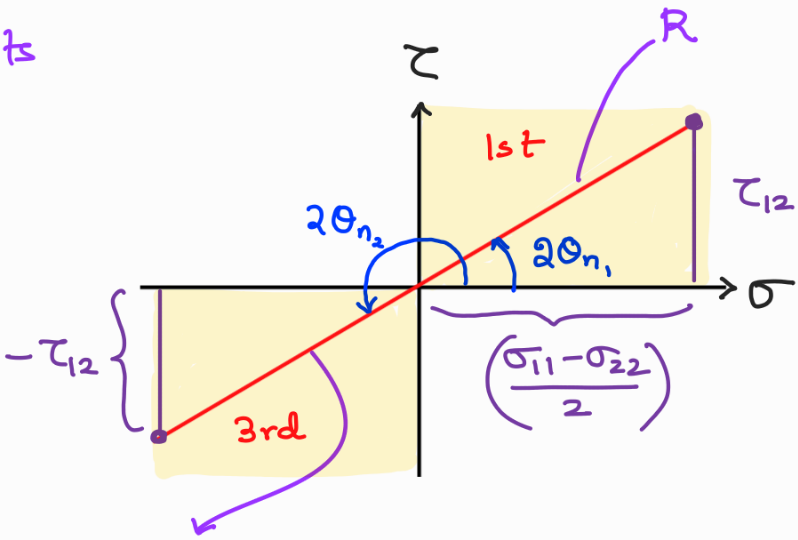
$$\frac{d\sigma_n}{d\theta} = -\frac{\sigma_{11} - \sigma_{22}}{2}(2\sin 2\theta) + 2\tau_{12} \cos 2\theta = 0$$

$$\tan 2\theta = \frac{\tau_{12}}{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)}$$

+ve in 1st & 3rd quadrants

1st $$2\theta_{n_1} = \frac{\tau_{12}}{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)}$$

3rd $$2\theta_{n_2} = \frac{-\tau_{12}}{-\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)}$$



$$R = \sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \tau_{12}^2}$$

The two solutions $2\theta_{n_1}$ and $2\theta_{n_2}$ are 180° apart, so in physical space, θ_{n_1} and θ_{n_2} are 90° apart.

To obtain maximum and minimum normal stress, we must substitute the angles θ_{n_1} and θ_{n_2} respectively.

$$\sigma_n(\theta) = \sigma_{11}\left(\frac{1+\cos 2\theta}{2}\right) + \sigma_{22}\left(\frac{1-\cos 2\theta}{2}\right) + \tau_{12}\sin 2\theta$$

$$= \frac{\sigma_{11}+\sigma_{22}}{2} + \frac{\sigma_{11}-\sigma_{22}}{2}\cos 2\theta + \tau_{12}\sin 2\theta$$

Sub. $\theta = \theta_{n_1}, \theta_{n_2}$

$$\cos 2\theta_{n_1,n_2} = \pm \frac{\sigma_{11}-\sigma_{22}}{2R}, \qquad \sin 2\theta_{n_1,n_2} = \pm \frac{\tau_{12}}{R}$$

$\sigma_{n_1} \leftarrow$ max normal stress, $\sigma_{n_2} \leftarrow$ min normal stress

$\theta_{n_1} \leftarrow$ corresponding angle w.r.t. $\underline{e}_1$ $\theta_{n_2} \leftarrow$ corresponding angle w.r.t. $\underline{e}_1$

$$\sigma_{n_1, n_2} = \frac{\sigma_{11} + \sigma_{22}}{2} \pm \frac{\sigma_{11} - \sigma_{22}}{2}\left(\frac{\sigma_{11} - \sigma_{22}}{2R}\right) \pm \tau_{12}\left(\frac{\tau_{12}}{R}\right)$$

$$= \frac{\sigma_{11} + \sigma_{22}}{2} \pm \left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 \frac{1}{R} \pm \frac{\tau_{12}^2}{R}$$

$$= \frac{\sigma_{11} + \sigma_{22}}{2} \pm \frac{1}{R}\left[\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \tau_{12}^2\right]$$

$$R = \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \tau_{12}^2}$$

$$\sigma_{n_1} = \frac{\sigma_{11} + \sigma_{22}}{2} + \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \tau_{12}^2}$$ (maximum normal stress)

$$\sigma_{n_2} = \frac{\sigma_{11} + \sigma_{22}}{2} - \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \tau_{12}^2}$$ (minimum normal stress)

What if we set $\sigma_{11} = \lambda_1$ (1st prin. stress) and $\sigma_{22} = \lambda_2$?

$$\sigma_{n_1, n_2} = \frac{\overset{\lambda_1}{\sigma_{11}} + \overset{\lambda_2}{\sigma_{22}}}{2} \pm \sqrt{\left(\frac{\overset{\lambda_1}{\sigma_{11}} - \overset{\lambda_2}{\sigma_{22}}}{2}\right)^2 + \overset{0}{\tau_{12}^2}}$$

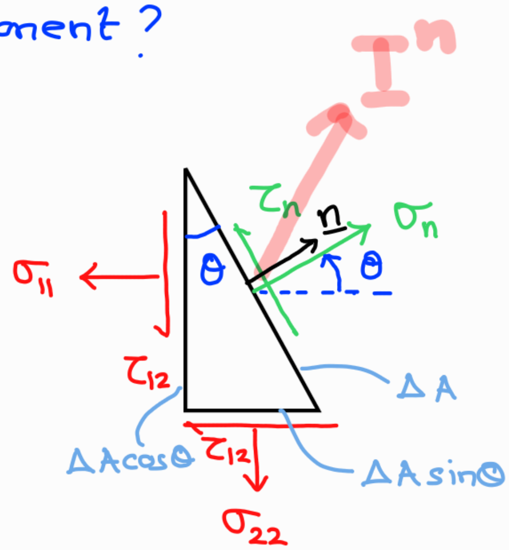
$$= \frac{\lambda_1 + \lambda_2}{2} \pm \frac{\lambda_1 - \lambda_2}{2} = \lambda_1, \lambda_2$$

Let us also check what are the shear stresses on planes where normal stress components are maximum or minimum

For this, we can put the value of $\sin 2\theta$ and $\cos 2\theta$ in the relation for shear stress component τ_n :

How to derive the shear stress component?

Method 1: Equilibrium of forces along the inclined plane



$$+\nwarrow \sum F_{\underline{n}\perp} = 0$$

$$\Rightarrow \tau_n \Delta A + (\tau_{12}\,\Delta A\sin\theta)\sin\theta - (\sigma_{22}\,\Delta A\sin\theta)\cos\theta + (\sigma_{11}\,\Delta A\cos\theta)\sin\theta - (\tau_{12}\,\Delta A\cos\theta)\cos\theta = 0$$

$$\Rightarrow \tau_n = (\sigma_{22} - \sigma_{11})\sin\theta\cos\theta + \tau_{12}(\cos^2\theta - \sin^2\theta)$$

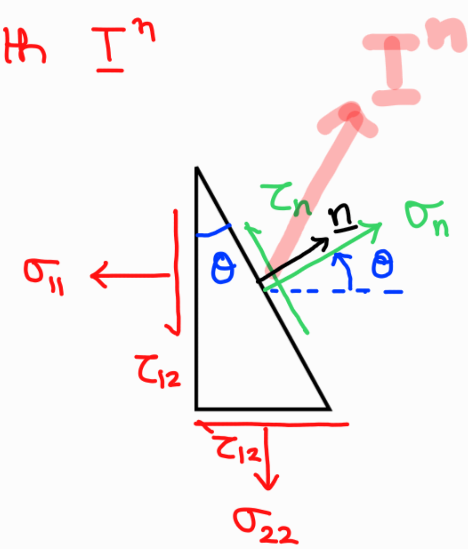
$$= -\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)\sin 2\theta + \tau_{12}\cos 2\theta$$

Method 2 : Dot product of $\underline{n}^\perp$ with $\underline{T}^n$

$$\tau_n = \underline{T}^n \cdot \underline{n}^\perp$$

$$= \left([\underline{\underline{\sigma}}][\underline{n}]\right)\cdot[\underline{n}^\perp]$$

$$= \left(\begin{bmatrix}\sigma_{11} & \tau_{12}\\ \tau_{12} & \sigma_{22}\end{bmatrix}\begin{bmatrix}n_1\\ n_2\end{bmatrix}\right)^T \begin{bmatrix}n_1^\perp\\ n_2^\perp\end{bmatrix}$$

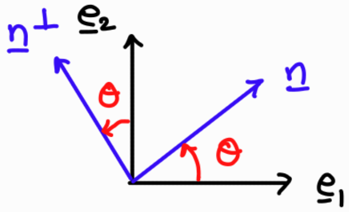


$$= \begin{bmatrix} n_1 & n_2 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \tau_{12} \\ \tau_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} n_1^{\perp} \\ n_2^{\perp} \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11} n_1 + \tau_{12} n_2 & \tau_{12} n_1 + \sigma_{22} n_2 \end{bmatrix} \begin{bmatrix} n_1^{\perp} \\ n_2^{\perp} \end{bmatrix}$$

$$= \sigma_{11} n_1 n_1^{\perp} + \tau_{12} n_2 n_1^{\perp} + \tau_{12} n_1 n_2^{\perp} + \sigma_{22} n_2 n_2^{\perp}$$

$$= \sigma_{11} n_1 n_1^{\perp} + \sigma_{22} n_2 n_2^{\perp} + \tau_{12} \left(n_1 n_2^{\perp} + n_2 n_1^{\perp} \right)$$



$$\underline{n} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \qquad \underline{n}^{\perp} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$= -\sigma_{11} \cos\theta \sin\theta + \sigma_{22} \sin\theta \cos\theta + \tau_{12} \left(\cos^2\theta - \sin^2\theta \right)$$

$$= -\left(\frac{\sigma_{11} - \sigma_{22}}{2} \right) \sin 2\theta + \tau_{12} \cos 2\theta$$

derived previously

$$\sigma_n = \frac{\sigma_{11} + \sigma_{22}}{2} + \frac{\sigma_{11} - \sigma_{22}}{2} \cos 2\theta + \tau_{12} \sin 2\theta$$

$$\tau_n = -\frac{\sigma_{11} - \sigma_{22}}{2} \sin 2\theta + \tau_{12} \cos 2\theta$$

Let us also check what are the shear stresses on planes where normal stress components are maximum or minimum

$$\left(\pm \frac{\tau_{12}}{R} \right) \qquad\qquad \pm \left(\frac{\sigma_{11} - \sigma_{22}}{2R} \right)$$

$$\tau_n = - \left(\frac{\sigma_{11} - \sigma_{22}}{2} \right) \sin 2\theta_{n_1, n_2} + \tau_{12} \cos 2\theta_{n_1, n_2}$$

$$= \mp \left(\frac{\sigma_{11} - \sigma_{22}}{2} \right) \left(\frac{\tau_{12}}{R} \right) \pm \tau_{12} \left(\frac{\sigma_{11} - \sigma_{22}}{2R} \right) = 0$$

- Therefore, we find that the shear stress components are zero on planes where normal stress components are maximized or minimized.

- Coincidentally principal planes are also planes where there are only normal stress components and zero shear stresses.

Thus, principal planes are also planes where the normal stresses are max/min and shear stresses are zero!

Back to our questions:

(a) If there are any planes passing through a given point on which traction vector is wholly normal?
(i.e. traction vector has zero shear component & non-zero normal component) Principal planes ✓

(b) On which plane does the normal stress become maximum? What will be its magnitude? Principal planes ✓
Principal stresses ✓

(c) On which plane does the shear stress become maximum? What will be its magnitude?

Maximum shear stress and corresponding planes

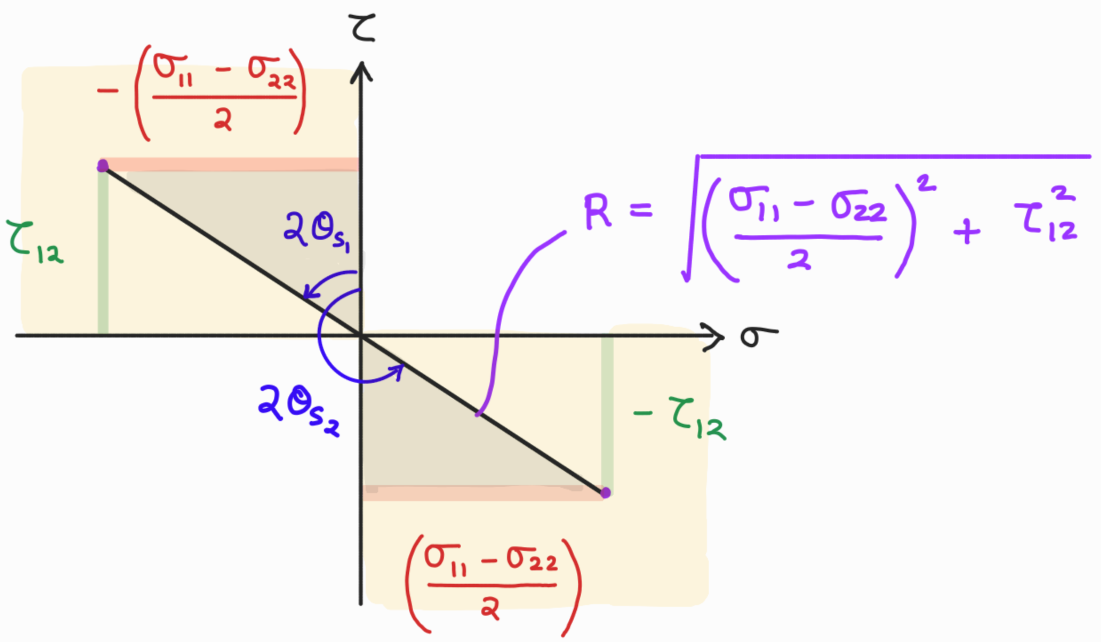
To find the orientation of the plane that is subject to maximum/minimum shear stress, we will take the derivative of the following w.r.t. θ and set it equal to zero

$$\tau_n(\theta) = -\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)\sin 2\theta + \tau_{12}\cos 2\theta$$

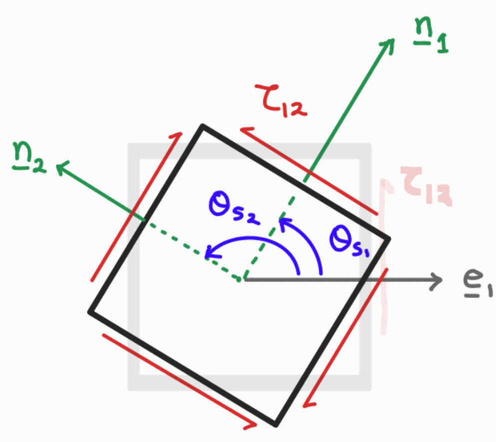
$$\frac{d\tau_n}{d\theta} = 0 \Rightarrow \boxed{\tan 2\theta_s = -\left(\frac{\sigma_{11}-\sigma_{22}}{2\tau_{12}}\right)}$$

Assume $\sigma_{11} > \sigma_{22}$

$-ve$ in → 2nd quad, → 4th quad } two roots to this equation $2\theta_{s_1}$ & $2\theta_{s_2}$



There are two planes $2\theta_{s_1}$ and $2\theta_{s_2}$, where $\tan 2\theta_s$ is $-ve$. These planes are 180° apart, but, in reality, they are 90° apart



Max/Min normal stress plane orientation

$$\tan 2\theta_n = \frac{\tau_{12}}{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)}$$

Max/Min shear stress plane orientation

$$\tan 2\theta_s = -\frac{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)}{\tau_{12}}$$

By comparison, $\tan 2\theta_s$ is the negative reciprocal of $\tan 2\theta_n$, so each plane $2\theta_s$ must be $90°$ from $2\theta_n$ and therefore, in reality, the planes of max/min shear stress and the principal planes (with max/min normal stresses) occur at angles of $45°$ to each other

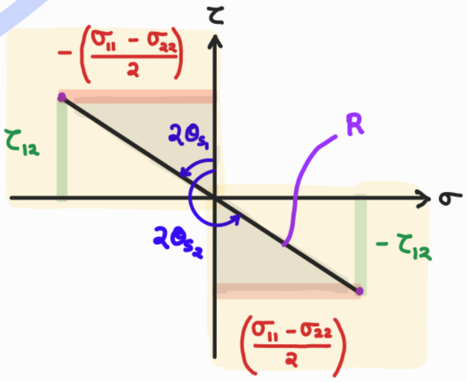
Maximum value of shear stress

They are obtained by putting the values of θ_s in the relation of $\tau_n(\theta)$

$$\tau_n(\theta) = -\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right) \sin 2\theta_s + \tau_{12} \cos 2\theta_s$$

$$\sin 2\theta_s = -\frac{\sigma_{11} - \sigma_{22}}{2R}$$

$$\cos 2\theta_s = \frac{\tau_{12}}{R}$$



$$\tau_n^{max/min} = \frac{1}{R}\left[\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \tau_{12}^2\right] \rightarrow (\pm R)^2$$

$$= \pm R$$

$$= \pm\sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \tau_{12}^2}$$

$$\boxed{\tau^{max/min} = \pm\sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \tau_{12}^2}}$$

If one uses the coor. sys. with coordinate lines oriented along principal directions, we will get

$$\sigma_{11} = \lambda_1 \quad , \quad \sigma_{22} = \lambda_2 \quad , \quad \tau_{12} = 0$$

and

$$\tau^{max/min} = \pm\left|\frac{\lambda_1 - \lambda_2}{2}\right|$$

What are the normal stresses on the planes of maximum/minimum shear stresses?

Set the values of θ_{s_1} and θ_{s_2} in the relation of $\sigma_n(\theta)$

$$\sigma_n(\theta) = \frac{\sigma_{11}+\sigma_{22}}{2} + \frac{\sigma_{11}-\sigma_{22}}{2}\cos 2\theta + \tau_{12}\sin 2\theta$$

$$\cos 2\theta_s = \frac{\tau_{12}}{R} \qquad \sin 2\theta_s = -\frac{\sigma_{11}-\sigma_{22}}{2R}$$

$$= \frac{\sigma_{11}+\sigma_{22}}{2} + \cancel{\frac{\sigma_{11}-\sigma_{22}}{2}\left(\frac{\tau_{12}}{R}\right)} - \cancel{\tau_{12}\left(\frac{\sigma_{11}-\sigma_{22}}{2R}\right)}$$

$$= \frac{\sigma_{11}+\sigma_{22}}{2} = \sigma_{avg}$$ (average normal stress)