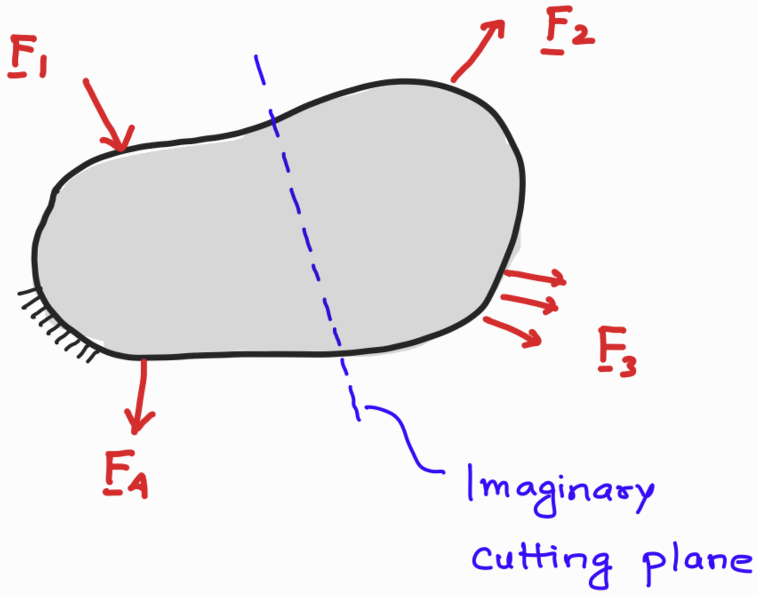
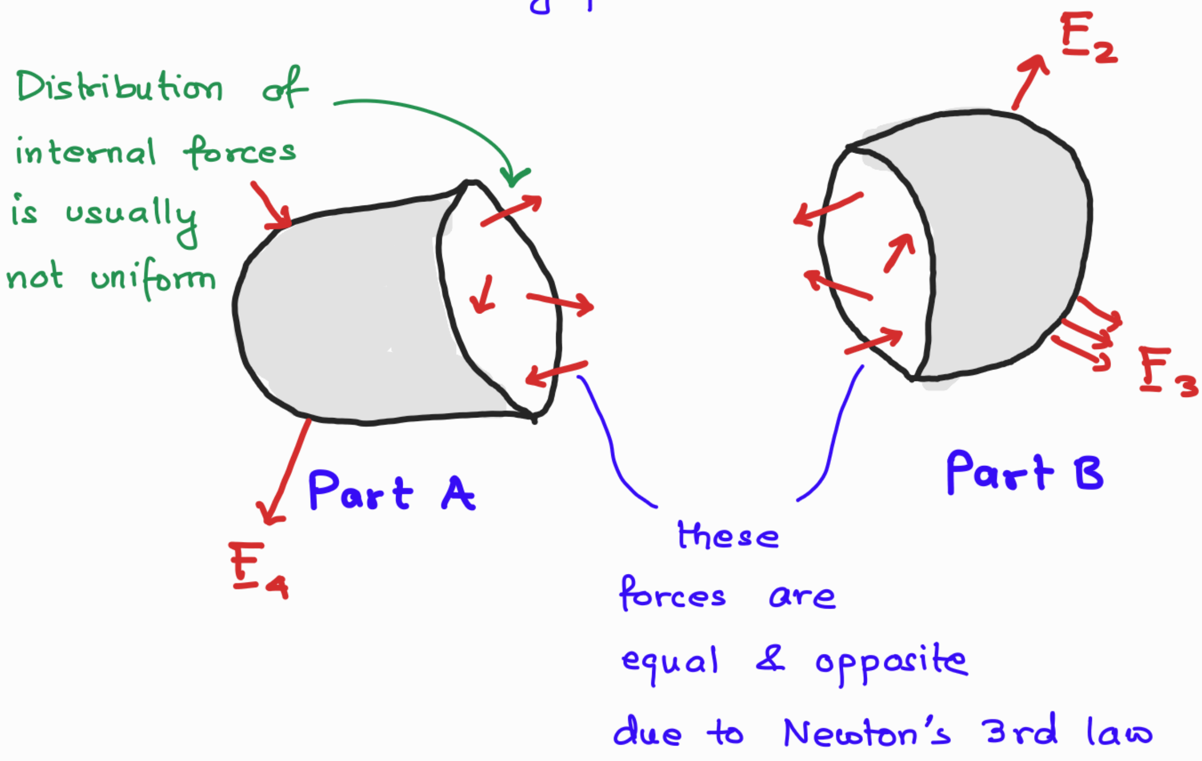
# Concept of Traction (or Stress) vector

Consider that we want to investigate the distribution of internal forces in a 3D material body




We cut a plane to look at the internal forces

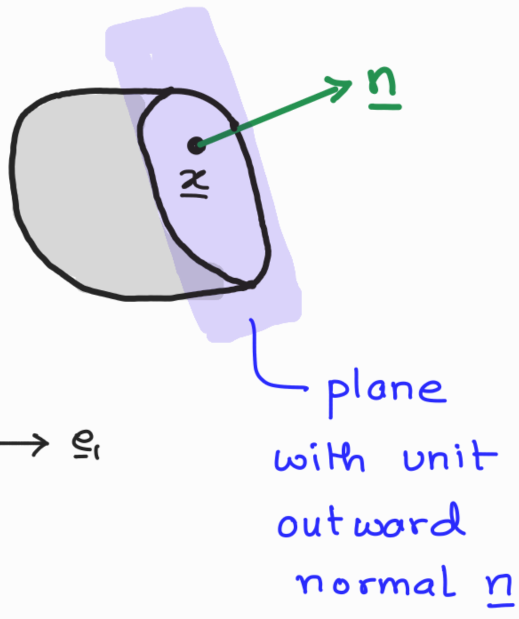


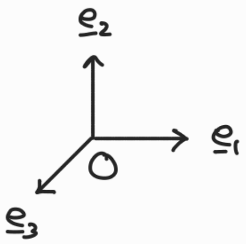

The distribution of internal forces on the exposed cut surface is usually not uniform and the individual small internal force vectors (shown by small red arrows) vary from point to point on the cut surface.

Obtaining the distribution of internal resistance in a deformable body is very important in solid mechanics.

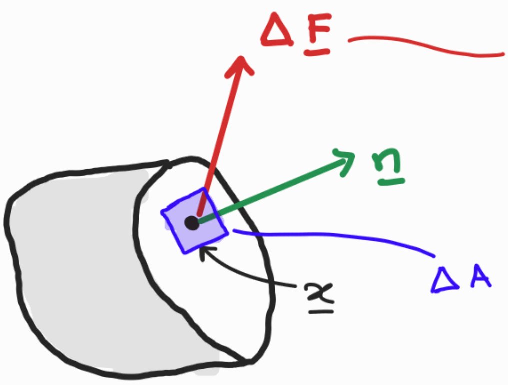
To get this distribution, we establish the concept of TRACTION (or STRESS) vector and STRESS TENSOR.

To define traction, we first consider a point with coordinates $[\underline{x}]_{(\underline{e}_1-\underline{e}_2-\underline{e}_3)} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in the material body and cut the body through $\underline{x}$ along a plane with unit outward normal $\underline{n}$.




We now focus on the internal force distribution over a small area $\Delta A$ encompassing the point $\underline{x}$




Traction vector $\quad \underline{T}^n(\underline{x}) = \lim_{\Delta A \to 0} \dfrac{\Delta \underline{F}}{\Delta A}$

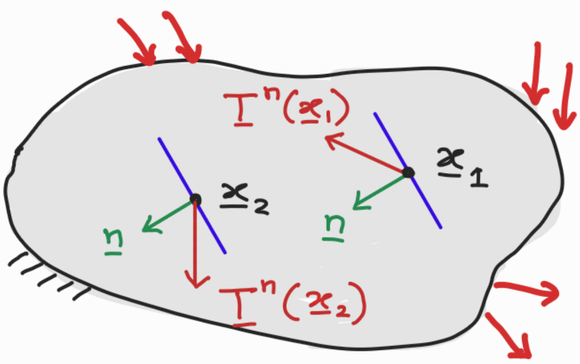
If we shrink the area $\Delta A$ such that area $\Delta A$ always contains the point $\underline{x}$, then the force acting at the pt. $\underline{x}$ in the limiting case of $\Delta A \to 0$ is called the **TRACTION vector** (also called **STRESS vector**)

$$\underline{T}^n(\underline{x}) = \lim_{\Delta A \to 0} \frac{\underline{\Delta F}}{\Delta A}$$

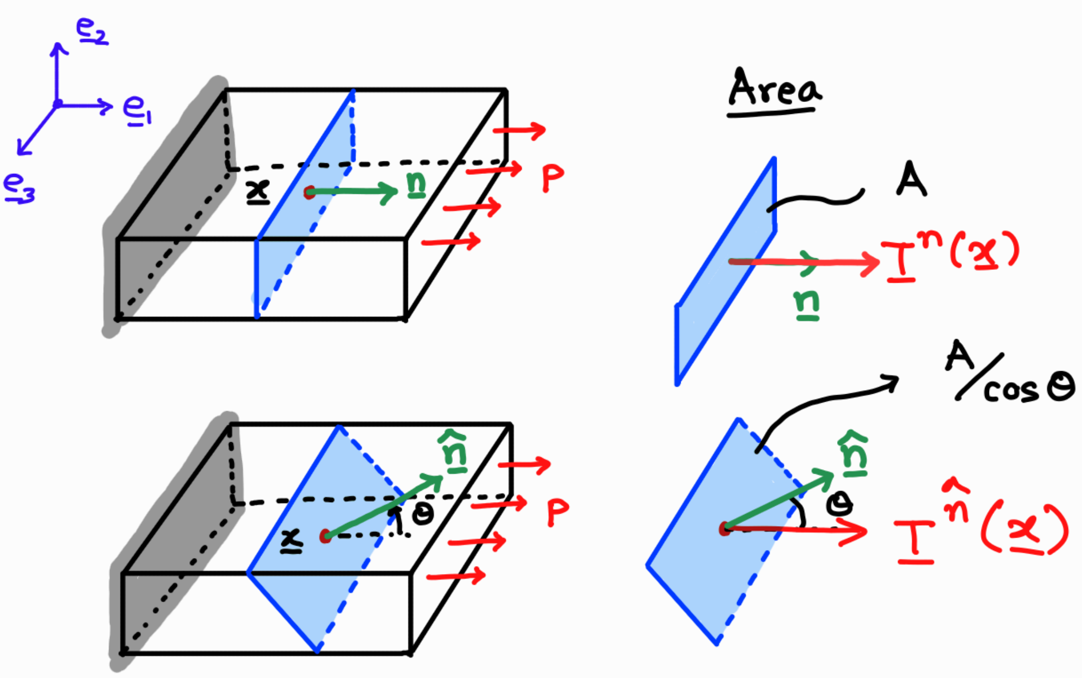
Traction vector has units of pressure, but it is more general than pressure: pressure always acts in the direction opposite to the outward plane normal whereas traction can act in an arbitrary direction


## Remarks on Traction

1) In general, traction changes from point to point



2) Traction vector depends upon the plane orientation

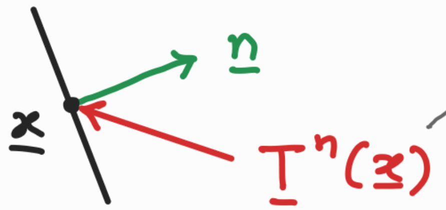



**Traction**

$$\underline{T}^n(\underline{x}) = \frac{P\underline{e}_1}{A}$$

$$\underline{T}^{\hat{n}}(\underline{x}) = \frac{P\,\underline{e}_1}{\left(\frac{A}{\cos\theta}\right)} = \frac{P\cos\theta}{A}\,\underline{e}_1$$

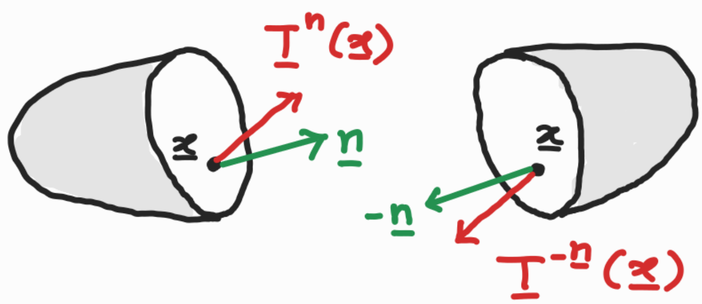
3) Traction at a point on a given plane can have any arbitrary direction (i.e. it need not be pointing in the direction of the outward plane normal)




need not be aligned with $\underline{n}$

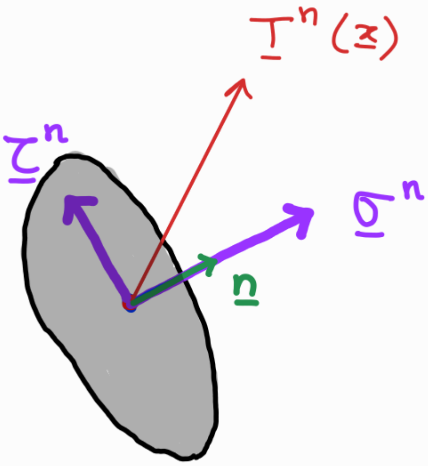
4) Tractions are equal and opposite at a point on planes having opposite outward normals

$$\underline{T}^n(\underline{x}) = -\underline{T}^{-\underline{n}}(\underline{x})$$



## Normal and Shear components of Traction vector

Since traction is a vector, one can obtain components of the vector using a user choice of coordinate sys.




Traction vector

$\underline{T}^n(\underline{x})$

resolved along $\underline{n}$ → $\underline{\sigma}^n$

resolved ⊥ to $\underline{n}$ → $\underline{\tau}^n$

$$\underline{\sigma}^n = (\underline{T}^n \cdot \underline{n})\, \underline{n}$$

$$\underline{\tau}^n = \underline{T}^n - \underline{\sigma}^n$$

$$|\underline{T}^n|^2 = |\underline{\sigma}^n|^2 + |\underline{\tau}^n|^2$$

## Importance of Traction

By defn, it gives us the intensity of force with which one part of the body (say part A) pulls or pushes the other part (part B).

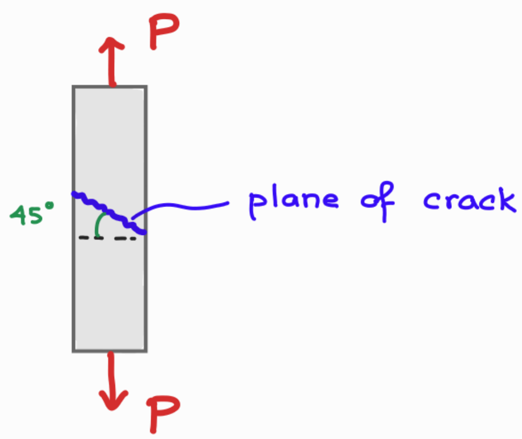
* If the value of this traction is lower than the material's threshold strength, the body will not fracture/fail.

* At a given point, as the traction varies from one plane to the other, the failure chance is higher on the plane on which the traction has got a larger value

Thus, traction tells us at what point in the body and on what plane at that point would the body fail!

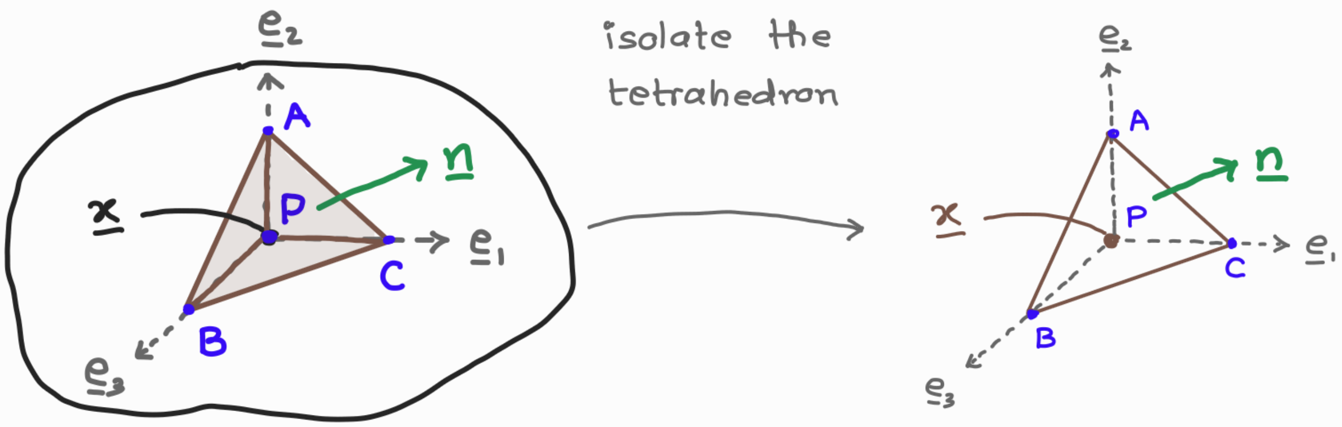
Ex: It is experimentally observed that during tensile testing of a steel bar, the steel bar develops a crack that is inclined at around 45° to the direction of loading.

In this case, the shear component of traction causes a shear failure of the material



# Relating traction on different planes at a pt.

We will now prove that if we know traction vector on three mutually perpendicular planes at a pt, then we can find traction vector on ANY PLANE at that point



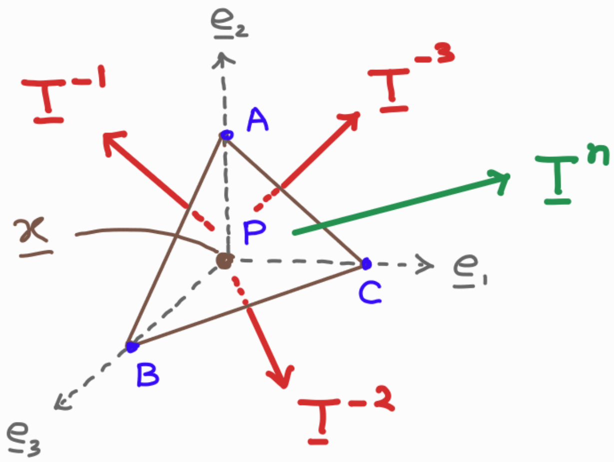

| Faces of Tetrahedron | | Outward normals |
|---|---|---|
| ABC | : | $\underline{n}$ |
| PAB | : | $-\underline{e}_1$ |
| PBC | : | $-\underline{e}_2$ |
| PAC | : | $-\underline{e}_3$ |

What are the various forces acting on this tetrahedron?

- Weight (body force) due to gravity acting on every point of the tetrahedron

- Internal (surface forces) acting on the surfaces of the tetrahedron (applied by the rest of the body)

Let's assume that the traction vector is uniformly acting on each face of the tetrahedron, so we have



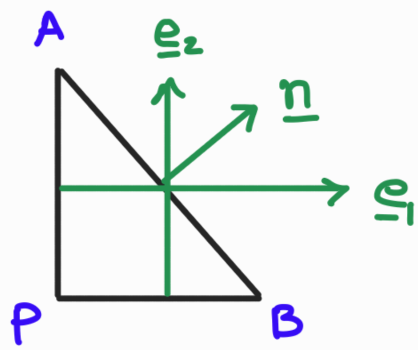

| Plane | Outward normal | Traction |
|---|---|---|
| ABC | $\underline{n}$ | $\underline{T}^{n}$ |
| PAB | $-\underline{e}_1$ | $\underline{T}^{-1}$ |
| PBC | $-\underline{e}_2$ | $\underline{T}^{-2}$ |
| PAC | $-\underline{e}_3$ | $\underline{T}^{-3}$ |

For static equilibrium of the tetrahedron,

$$\sum \underline{F} = \underline{0} \quad \text{(Resultant of all forces)}$$

$$\Rightarrow \underline{T}^{-1} A_{PAB} + \underline{T}^{-2} A_{PBC} + \underline{T}^{-3} A_{PAC} + \underline{T}^{n} A_{ABC} + \underbrace{\rho}_{\text{density}} \underbrace{V}_{\text{vol}} \underline{g} = \underline{0}$$

We can write the area of the inclined face ABC in terms of other areas $A_{PAB}$, $A_{PBC}$, and $A_{PAC}$




$$A_{AP} = A_{AB} \left(\underline{n} \cdot \underline{e}_1\right)$$

$$A_{PB} = A_{AB} \left(\underline{n} \cdot \underline{e}_2\right)$$

$$\Rightarrow \underline{T}^{-1} \cancel{A_{ABC}} (\underline{n} \cdot \underline{e}_1) + \underline{T}^{-2} \cancel{A_{ABC}} (\underline{n} \cdot \underline{e}_2) + \underline{T}^{-3} \cancel{A_{ABC}} (\underline{n} \cdot \underline{e}_3)$$

$$+ \underline{T}^{n} \cancel{A_{ABC}} + \rho \left(\frac{1}{3} \cancel{A_{ABC}}\, h\right) \underline{g} = \underline{0}$$

$$\Rightarrow \underline{T}^{-1} (\underline{n} \cdot \underline{e}_1) + \underline{T}^{-2} (\underline{n} \cdot \underline{e}_2) + \underline{T}^{-3} (\underline{n} \cdot \underline{e}_3) + \underline{T}^{n} + \frac{1}{3} \rho h \underline{g} = \underline{0}$$

Now we shrink the volume of the tetrahedron to bring it to the point P by sending $h \to 0$ (thus body force term does not affect the relation)

$$\Rightarrow \sum_{i=1}^{3} \underline{T}^{-i} (\underline{n} \cdot \underline{e}_i) + \underline{T}^{n} = 0$$

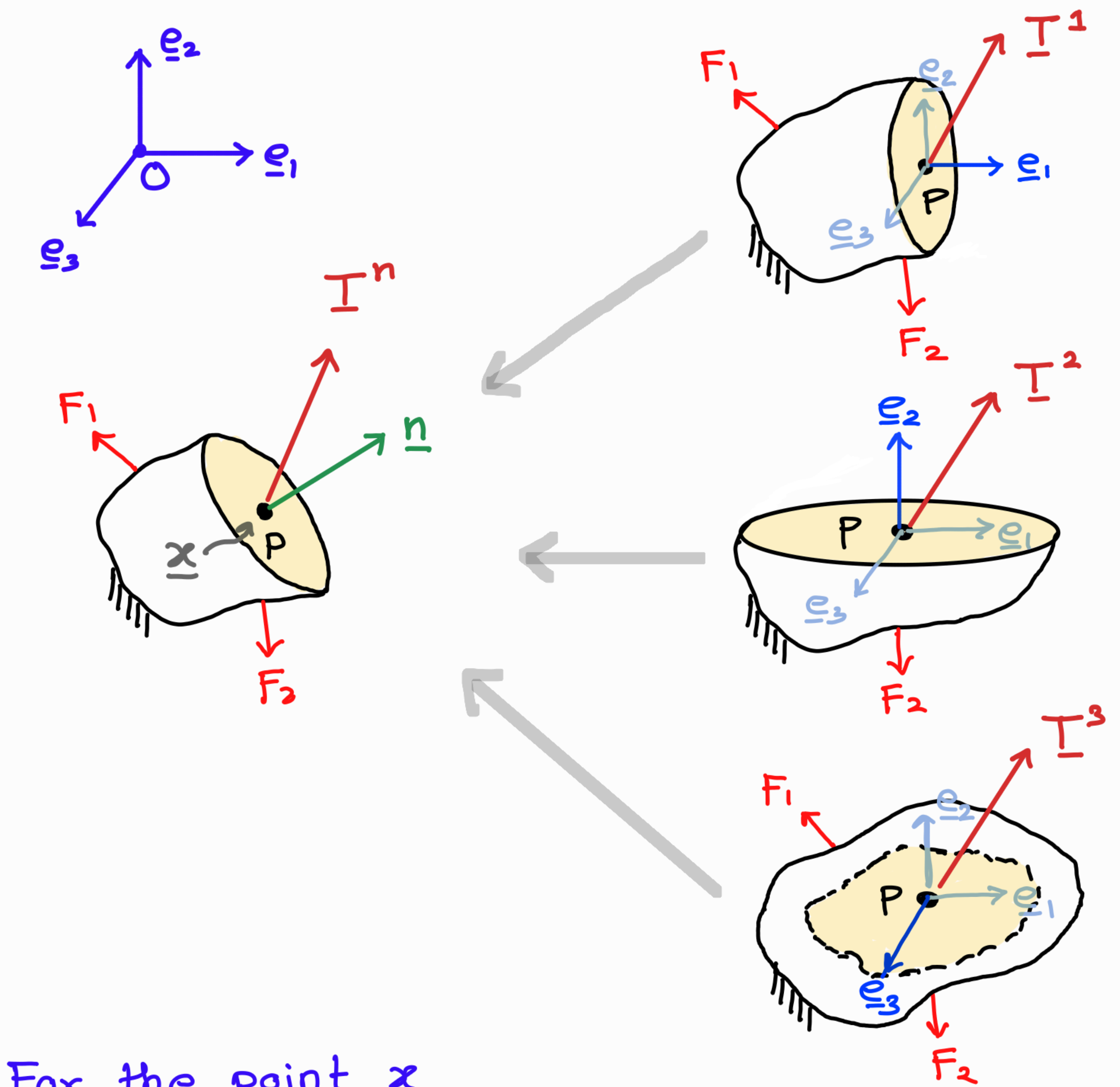
$$\Rightarrow \underline{T}^{n}(\underline{x}) = \sum_{i=1}^{3} - \underline{T}^{-i} (\underline{n} \cdot \underline{e}_i)$$

We have earlier stated that $\underline{T}^{n}(\underline{x}) = -\underline{T}^{-n}(\underline{x})$

Using this idea, we get

$$\boxed{\underline{T}^{n}(\underline{x}) = \sum_{i=1}^{3} \underline{T}^{i} (\underline{n} \cdot \underline{e}_i)}$$

If we know the traction vectors $\underline{T}^{i}$ on three mutually perpendicular planes at a pt., then the traction vector $\underline{T}^{n}$ on any plane passing through the point can be obtained.




For the point $\underline{x}$

$$\underline{T}^n = \underline{T}^1(\underline{n}\cdot\underline{e}_1) + \underline{T}^2(\underline{n}\cdot\underline{e}_2) + \underline{T}^3(\underline{n}\cdot\underline{e}_3)$$

$$= \underline{T}^1 n_1 + \underline{T}^2 n_2 + \underline{T}^3 n_3$$

direction cosines
(not vectors)