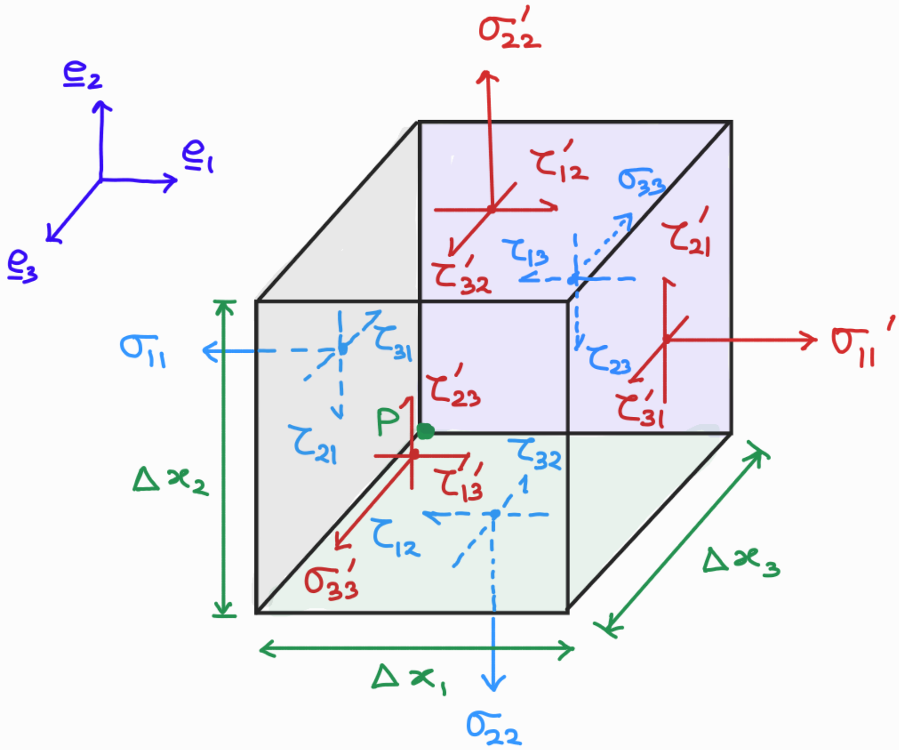
Stress Equilibrium (in Cartesian csys)

3D state of stress

at a point

in material body

$P(x_1, x_2, x_3)$



- Consider a small cuboid with point P at one corner. Assume uniform stresses σ_{ii} and τ_{ij} acting on the faces, $i, j = 1, 2, 3$.

- The stress components on faces at distances Δx_1, Δx_2, and Δx_3 are different from the components at the faces passing through point P. This is because these planes do not pass through point P.

- How are σ_{11}', τ_{21}', τ_{31}' related to σ_{11}, τ_{21}, τ_{31} ?

  To answer this, make use of Taylor series expansion about the local point $P(\underline{x})$

Using Taylor series expansion,

$$\sigma_{11}' = \sigma_{11}(\underline{x} + \Delta\underline{x})$$

$$= \sigma_{11}(x_1 + \Delta x_1, x_2, x_3)$$

$$= \sigma_{11}(\underline{x}) + \frac{\partial \sigma_{11}}{\partial x_1}\Delta x_1 + \boxed{\mathcal{O}(\Delta x_1^2)}$$

terms of order Δx_1^2 (or smaller)

Neglect higher order terms

$$\sigma_{11}' \approx \sigma_{11} + \frac{\partial \sigma_{11}}{\partial x_1}\Delta x_1$$

Similarly,

$$\tau_{21}' = \tau_{21}(\underline{x} + \Delta\underline{x}) = \tau_{21}(x_1 + \Delta x_1, x_2, x_3)$$

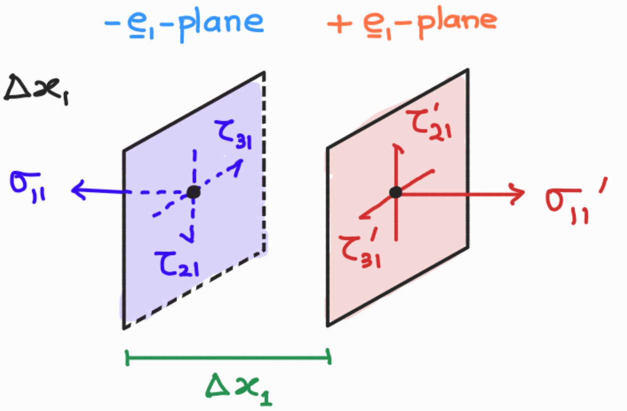
$$\approx \tau_{21}(\underline{x}) + \frac{\partial \tau_{21}}{\partial x_1}\Delta x_1$$

and

$$\tau_{31}' \approx \tau_{31}(\underline{x}) + \frac{\partial \tau_{31}}{\partial x_1}\Delta x_1$$

Stress components on RIGHT face

$$\sigma_{11} + \frac{\partial \sigma_{11}}{\partial x_1}\Delta x_1, \quad \tau_{21} + \frac{\partial \tau_{21}}{\partial x_1}\Delta x_1, \quad \tau_{31} + \frac{\partial \tau_{31}}{\partial x_1}\Delta x_1$$



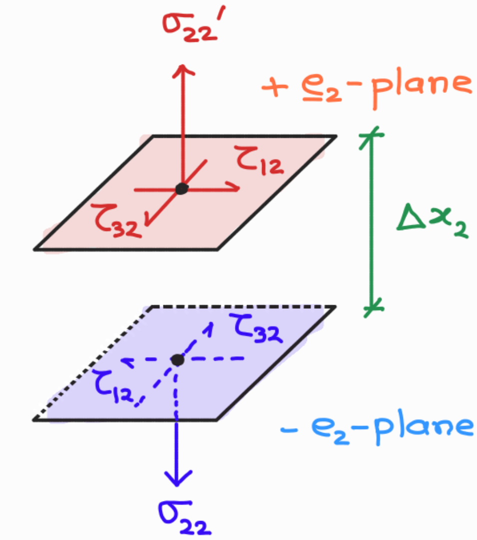
Stress components on LEFT face

σ_{11}, τ_{21}, τ_{31}

Area of both faces: $\Delta x_2 \, \Delta x_3$

Stress components on Top face

$$\tau_{12} + \frac{\partial \tau_{12}}{\partial x_2} \Delta x_2 , \quad \sigma_{22} + \frac{\partial \sigma_{22}}{\partial x_2} \Delta x_2 , \quad \tau_{32} + \frac{\partial \tau_{32}}{\partial x_2} \Delta x_2$$



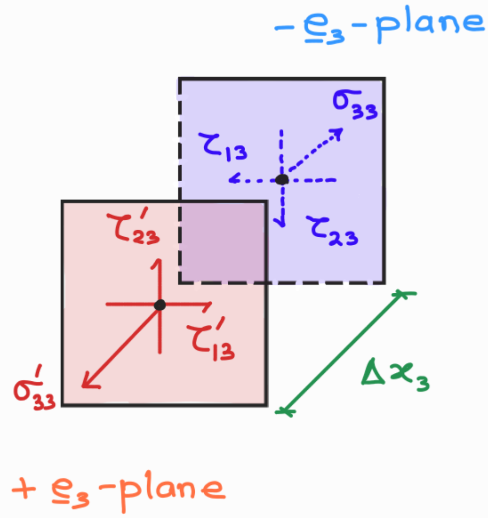
Stress components on BOTTOM face

τ_{12}, σ_{22}, τ_{32}

Area of both faces : $\Delta x_1 \, \Delta x_3$

Stress components on FRONT face

$$\tau_{13} + \frac{\partial \tau_{13}}{\partial x_3} \Delta x_3 , \quad \tau_{23} + \frac{\partial \tau_{23}}{\partial x_3} \Delta x_3 , \quad \sigma_{33} + \frac{\partial \sigma_{33}}{\partial x_3} \Delta x_3$$



Stress components on BACK face

τ_{13}, τ_{23}, σ_{33}

Area of both faces : $\Delta x_1 \, \Delta x_2$

If a deformable body is in equilibrium, then any isolate subsystem of the body must also be in equilibrium

Recall Euler's two axioms for equilibrium

1st axiom: Rate of change of linear momentum = Net ext. force

Inertial frame

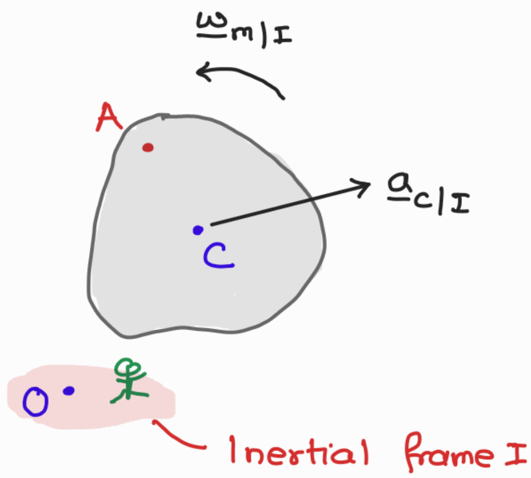
$$\Rightarrow \quad m\,\underline{a}_{c|I} = \underline{F}_R$$

2nd axiom: Rate of change of angular momentum about a point O fixed in 'I' = Net ext. moment abt O

$$\Rightarrow \quad \underline{\dot{H}}_{O|I} = \underline{M}_O$$

$$\Rightarrow \quad \underline{\underline{I}}^A\, \underline{\dot{\omega}}_{m|I} + \underline{\omega}_{m|I} \times \left(\underline{\underline{I}}^A\, \underline{\omega}_{m|I}\right) = \underline{M}_A$$

Modified Euler's 2nd axiom



For static equilibrium of the cuboid, it is at rest

$\Rightarrow$ No linear or angular motion

$$\underline{v}_{c|I} = \underline{0}, \quad \underline{a}_{c|I} = \underline{0}$$

$$\underline{\omega}_{m|I} = \underline{0}, \quad \underline{\dot{\omega}}_{m|I} = \underline{0}$$

Therefore, for the case of static equilibrium,

Balance of linear momentum $\Rightarrow$ $\underline{F}_R = \underline{0}$

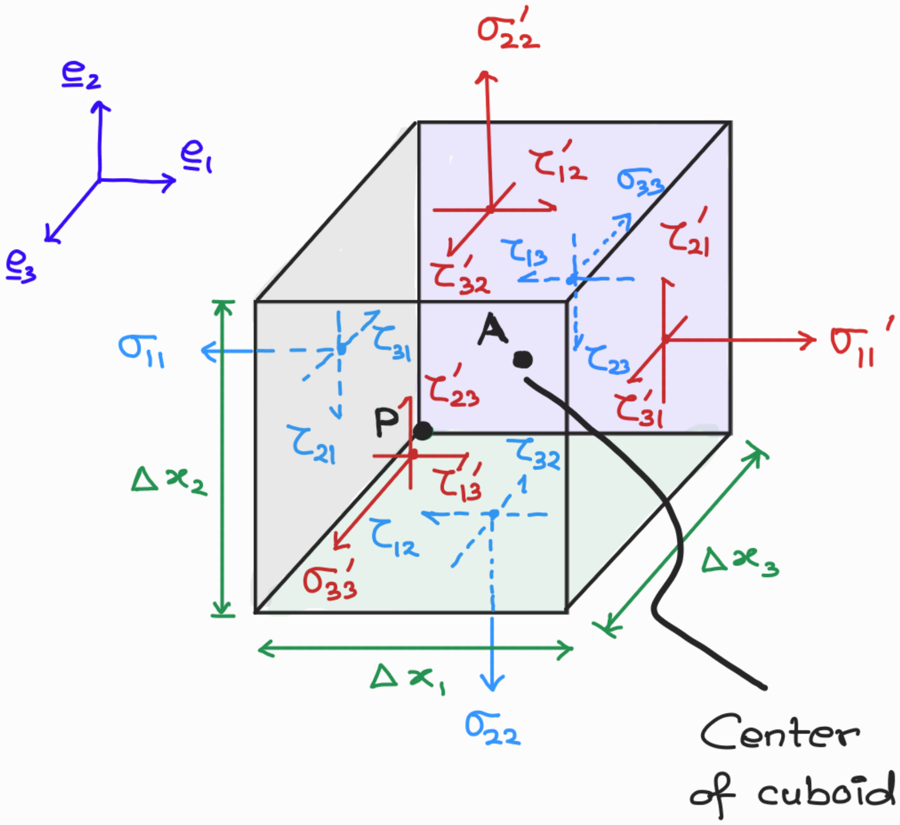
Balance of angular momentum $\Rightarrow$ $\underline{M}_A = \underline{0}$

$$\boxed{\underline{F}_R = 0}$$

$$\overset{+}{\rightarrow} \sum F_1 = 0$$

$$+\uparrow \sum F_2 = 0$$

$$\overset{+}{\swarrow} \sum F_3 = 0$$



$$\boxed{\underline{M}_A = \underline{0}}$$

$$\sum M_{A,1} = 0$$

$$\sum M_{A,2} = 0$$

$$\sum M_{A,3} = 0$$

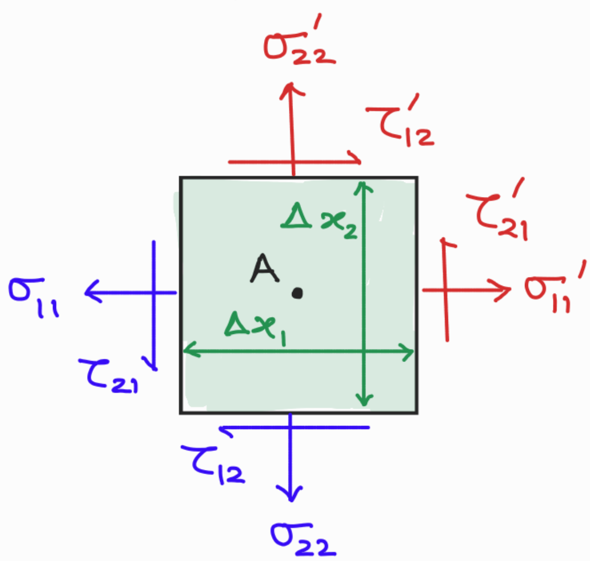
$$\sum M_{A,3} = 0 \quad \text{(net moment abt } \underline{e}_3\text{-axis)}$$

$$\Rightarrow \left(\tau'_{21}\, \Delta x_2\, \Delta x_3\right) \frac{\Delta x_1}{2}$$

$$+ \left(\tau_{21}\, \Delta x_2\, \Delta x_3\right) \frac{\Delta x_1}{2}$$

$$- \left(\tau'_{12}\, \Delta x_1\, \Delta x_2\right) \frac{\Delta x_2}{2}$$

$$- \left(\tau_{12}\, \Delta x_1\, \Delta x_2\right) \frac{\Delta x_2}{2} = 0$$



$$\Rightarrow \Bigg[\underbrace{\tau_{21} + \frac{\partial \tau_{21}}{\partial x_1}\Delta x_1 + \tau_{21} - \tau_{12} - \frac{\partial \tau_{12}}{\partial x_2}\Delta x_2 - \tau_{12}}_{\text{must be zero}} \Bigg] \times \frac{\Delta x_1 \Delta x_2 \Delta x_3}{2} = 0$$

$$\Rightarrow \quad 2\tau_{21} - 2\tau_{12} + \frac{\partial \tau_{21}}{\partial x_1}\Delta x_1 - \frac{\partial \tau_{12}}{\partial x_2}\Delta x_2 = 0$$

As Δx_1, Δx_2, and Δx_3 tend to zero

$$\boxed{\tau_{21} = \tau_{12}}$$

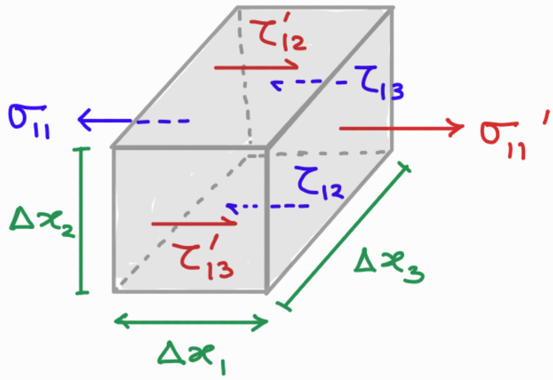
Similarly, taking moments abt the center of cuboid along $\underline{e}_1$- and $\underline{e}_2$- directions, we get:

$$\sum M_{A,1} = 0 \quad \Rightarrow \quad \boxed{\tau_{32} = \tau_{23}}$$

$$\sum M_{A,2} = 0 \quad \Rightarrow \quad \boxed{\tau_{31} = \tau_{13}}$$

Takeaway: Shear stress components on perpendicular faces are equal in magnitude

Next, we apply force equilibrium



$$\overset{+}{\rightarrow} \sum F_1 = 0$$

$$\Rightarrow \sigma_{11}' \Delta x_2 \Delta x_3 - \sigma_{11} \Delta x_2 \Delta x_3$$

$$+ \tau_{12}' \Delta x_1 \Delta x_3 - \tau_{12} \Delta x_1 \Delta x_3$$

$$+ \tau_{13}' \Delta x_1 \Delta x_2 - \tau_{13} \Delta x_1 \Delta x_2$$

$$+ b_1 (\Delta x_1 \Delta x_2 \Delta x_3) = 0$$

$$\Rightarrow \left(\cancel{\sigma_{11}} + \frac{\partial \sigma_{11}}{\partial x_1} \Delta x_1\right) \Delta x_2 \Delta x_3 - \cancel{\sigma_{11} \Delta x_2 \Delta x_3}$$

$$+ \left(\cancel{\tau_{12}} + \frac{\partial \tau_{21}}{\partial x_2} \Delta x_2\right) \Delta x_1 \Delta x_3 - \cancel{\tau_{12} \Delta x_1 \Delta x_3}$$

$$+ \left(\cancel{\tau_{13}} + \frac{\partial \tau_{31}}{\partial x_3} \Delta x_3\right) \Delta x_1 \Delta x_2 - \cancel{\tau_{13} \Delta x_1 \Delta x_2}$$

$$+ b_1 \Delta x_1 \Delta x_2 \Delta x_3 = 0$$

$$\Rightarrow \Bigg(\underbrace{\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} + \frac{\partial \tau_{13}}{\partial x_3} + b_1}_{\text{must vanish}}\Bigg) \Delta x_1 \Delta x_2 \Delta x_3 = 0$$

$$\boxed{\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} + \frac{\partial \tau_{13}}{\partial x_3} + b_1 = 0}$$

Similarly,

$$+\uparrow \sum F_2 = 0 \Rightarrow \frac{\partial \tau_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \tau_{23}}{\partial x_3} + b_2 = 0$$

$$\overset{+}{\swarrow} \sum F_3 = 0 \Rightarrow \frac{\partial \tau_{31}}{\partial x_1} + \frac{\partial \tau_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + b_3 = 0$$

Stress equilibrium equations (SIX RELATIONS)

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} + \frac{\partial \tau_{13}}{\partial x_3} + b_1 = 0$$

$$\frac{\partial \tau_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \tau_{23}}{\partial x_3} + b_2 = 0$$

$$\frac{\partial \tau_{31}}{\partial x_1} + \frac{\partial \tau_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + b_3 = 0$$

RHS is zero only under STATIC EQBM

$\begin{pmatrix} \rho a_1 \\ \rho a_2 \\ \rho a_3 \end{pmatrix}$ for DYN EQBM

$$\left.\begin{aligned} \tau_{21} &= \tau_{12} \\ \tau_{32} &= \tau_{23} \\ \tau_{31} &= \tau_{13} \end{aligned}\right\} \text{Complementary property of shear stresses}$$

Compact notation: $\mathrm{div}(\underline{\sigma}) + \underline{b} = \underline{0}$

$$\Rightarrow \underline{\nabla} \cdot \underline{\sigma} + \underline{b} = \underline{0}$$

where $\underline{\nabla} \equiv \frac{\partial}{\partial x_1} \underline{e}_1 + \frac{\partial}{\partial x_2} \underline{e}_2 + \frac{\partial}{\partial x_3} \underline{e}_3$ (gradient)

Index notation:

$$\boxed{\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = 0}$$

$i = 1, 2, 3$

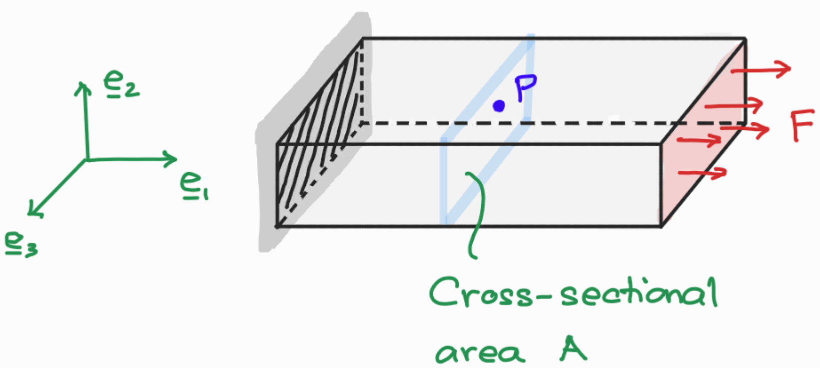
$j = 1, 2, 3$

Remarks

1) We have THREE equilibrium equations but SIX unknown stresses ($\sigma_{11}, \sigma_{22}, \sigma_{33}, \tau_{23}, \tau_{13}, \tau_{12}$), therefore, in general (but not always), we need more equations to solve for the unknowns.

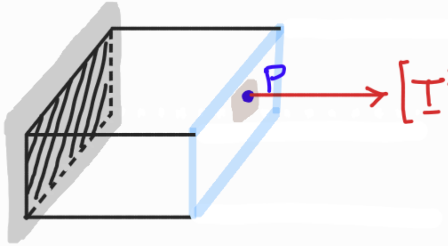
The general cases are statically indeterminate cases where the internal member stresses cannot be determined from solving equilibrium equations alone. However, for a statically determinate case, the member stresses can be determined even from the equilibrium relations because of additional stresses being known (often as zero)

Ex:

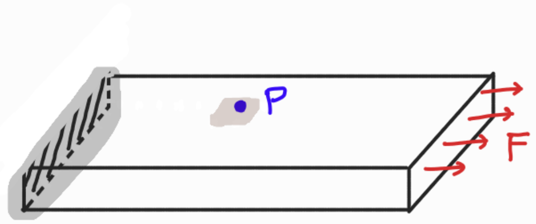


A rectangular bar with negligible weight being pulled by a uniformly distributed load F

Determine the stress matrix at point P.

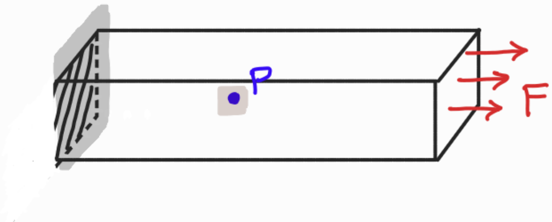


$$[T^1]_{(e_1, e_2, e_3)} = \begin{bmatrix} F/A \\ 0 \\ 0 \end{bmatrix} \qquad [\underline{\underline{\sigma}}]_{(e_1, e_2, e_3)} = \begin{bmatrix} F/A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



$$[T^2]_{(e_1, e_2, e_3)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We assume that the uniformly applied load in $\underline{e}_1$-direction will not cause any shear traction on the $\underline{e}_2$-plane or $\underline{e}_3$-plane.



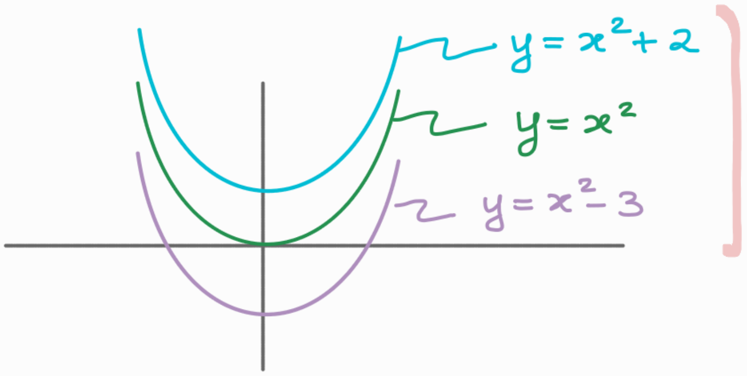
$$[T^3]_{(e_1, e_2, e_3)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

2) The extra sets of equations needed will be obtained as STRESS-STRAIN relations and STRAIN-DISPLACEMENT relations (which we earlier introduced as force-deformation and geometric compatibility relations in addition to equilibrium)

3) Note the equilibrium equations are PDEs. When we solve a PDE (or even an ODE), the general soln is usually a family of functions, not a single function

e.g $\quad \dfrac{dy}{dx} = 2x$

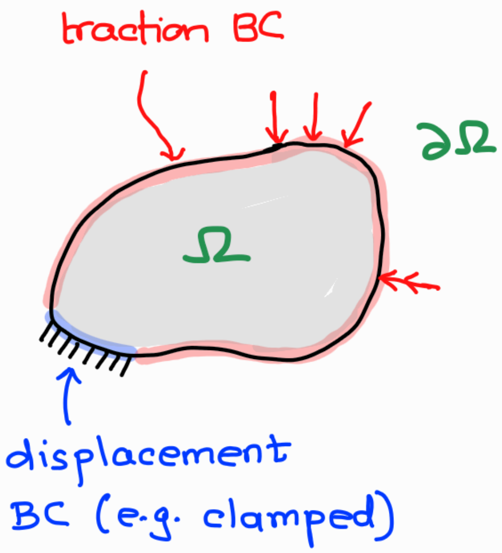
$\Rightarrow \quad y = x^2 + C \leftarrow$ which is a family of curves



To get a unique solution to a PDE, we would need BOUNDARY CONDITIONS (and also Initial Conditions if the PDE has a time-varying nature)

Boundary Conditions (BCs)

$\partial\Omega$: Set of all material point lying on the surface (or boundary) of the body



$\partial\Omega_u$: Set of material pts of Ω where the displacements are prescribed (or known)

→ Displacement (or Dirichlet) BCs

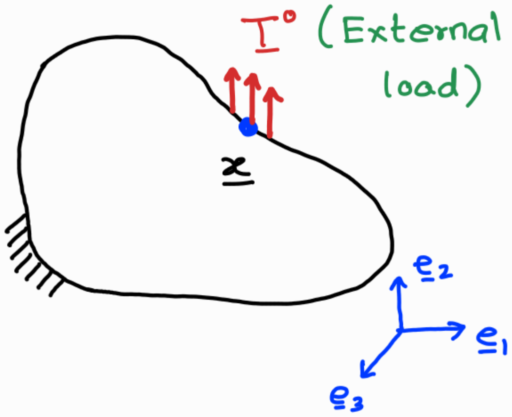
$\partial\Omega_t$: Set of material pts of Ω where the external loads are prescribed

→ Traction / force (or Neumann) BCs

$$\partial\Omega = \partial\Omega_u \cup \partial\Omega_t$$

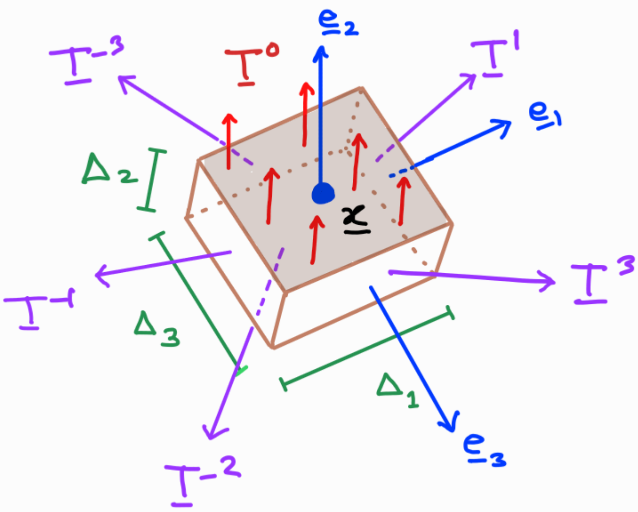
Derivation of Traction BCs

How to relate the external force applied on traction boundary $\partial\Omega_t$ to the internal stress at the interface of the boundary?



This applied load is usually distributed over an area; therefore has unit of traction. We denote this distributed $\underline{T}^\circ$

Let's take a small piece from the surface. The actual boundary is curved but if we take a tiny piece, it will be almost flat



On the top surface of this small body part acts the external uniform distributed traction $\underline{T}^\circ$

The $\underline{e}_1 - \underline{e}_3$ plane is the plane of the boundary surface and $\underline{e}_2$ is the normal to the plane.

What are the forces acting on this small body part?

- External force $\underline{T}^\circ$ (acting on exposed surface)
- Tractions (acting on the internal faces)
- Body force (acting at the COM of the volume)

The total force on this small body part must be also be in (static) equilibrium

$$\underbrace{\underline{T}^{0}\,\Delta_1\Delta_3}_{\text{Ext force}} + \left(\underline{T}^{1}+\underline{T}^{-1}\right)\Delta_2\Delta_3 + \left(\underline{T}^{3}+\underline{T}^{-3}\right)\Delta_1\Delta_2 + \underline{T}^{-2}\,\Delta_1\Delta_3 + \underbrace{\underline{b}\,\Delta_1\Delta_2\Delta_3}_{\text{Body force}} = \underline{0}$$

If you divide the above equation by $\Delta_1\Delta_3$, we get:

$$\Rightarrow \underline{T}^{0} + \left(\underline{T}^{1}+\underline{T}^{-1}\right)\cancelto{0}{\frac{\Delta_2}{\Delta_1}} + \left(\underline{T}^{3}+\underline{T}^{-3}\right)\cancelto{0}{\frac{\Delta_2}{\Delta_3}} + \underline{T}^{-2} + \underline{b}\,\cancelto{0}{\Delta_2} = \underline{0}$$

Now, let $\Delta_2 \to 0$, that is we shrink the height by pushing the bottom surface towards the top surface while keeping the surface area $\Delta_1\Delta_3$ constant.

$$\boxed{\underline{T}^{0} = -\underline{T}^{-2}} \quad\Rightarrow\quad \underline{T}^{0} = \underline{T}^{2} \Rightarrow \underline{T}^{0} = \underline{\underline{\sigma}}\,\underline{e}_2$$

$$\left[\underline{T}^{0}\right]_{(e_1,e_2,e_3)} = \left[\underline{\underline{\sigma}}\right]_{(e_1,e_2,e_3)}\left[\underline{e}_2\right]_{(e_1,e_2,e_3)}$$

$$\Rightarrow \begin{bmatrix} T_1^{0} \\ T_2^{0} \\ T_3^{0} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_{22} & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_{33} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} \tau_{12} &= T_1^{0} \\ \sigma_{22} &= T_2^{0} \\ \tau_{23} &= T_3^{0} \end{aligned}$$

Prescription of Boundary Value Problem (BVP)

Equilibrium condn : $\underline{\nabla} \cdot \underline{\underline{\sigma}}(\underline{x}) + \underline{b} = \underline{0}$ in Ω

Displacement BC : $\underline{u}(\underline{x}) = \tilde{\underline{u}}$ in $\partial\Omega_u$

Traction BC : $\underline{\underline{\sigma}}(\underline{x}) \cdot \underline{n} = \underline{T}^n(\underline{x})$ in $\partial\Omega_t$