Recap of traction decomposition

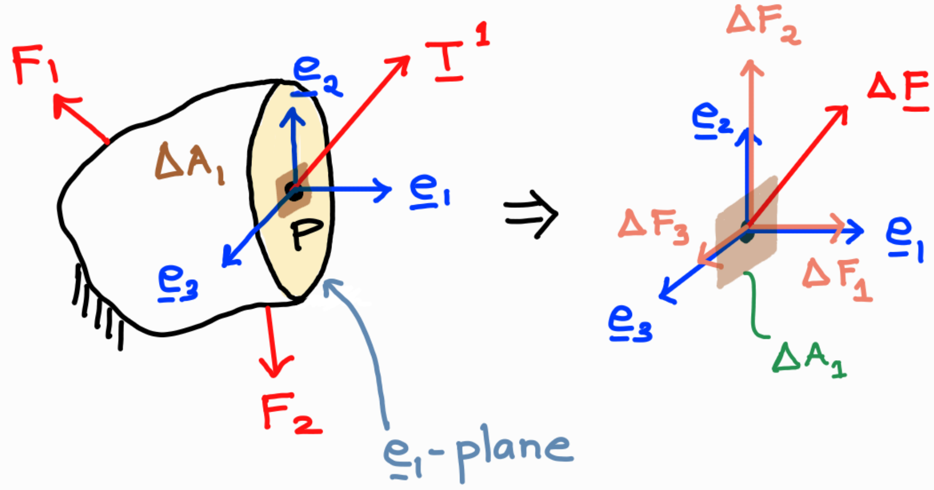
Traction vector $\underline{T}^n$ at a point $\underline{x}$ on a plane with outward normal $\underline{n}$ can be determined if we know the traction vectors on three mutually perpendicular planes

$$\underline{T}^n(\underline{x}) = \sum_{i=1}^{3} \underline{T}^i(\underline{x}) \, (\underline{n} \cdot \underline{e}_i)$$

What do the tractions $\underline{T}^1(\underline{x})$, $\underline{T}^2(\underline{x})$, $\underline{T}^3(\underline{x})$ represent?

Let's cut an $\underline{e}_1$-plane through point $\underline{x}$

(plane with outward normal $\underline{e}_1$)



⇒

$$T_1^1 = \lim_{\Delta A_1 \to 0} \frac{\Delta F_1}{\Delta A_1}$$

$$T_2^1 = \lim_{\Delta A_1 \to 0} \frac{\Delta F_2}{\Delta A_1}$$

$$T_3^1 = \lim_{\Delta A_1 \to 0} \frac{\Delta F_3}{\Delta A_1}$$

$$\underline{T}^1 = T_1^1 \, \underline{e}_1 + T_2^1 \, \underline{e}_2 + T_3^1 \, \underline{e}_3$$

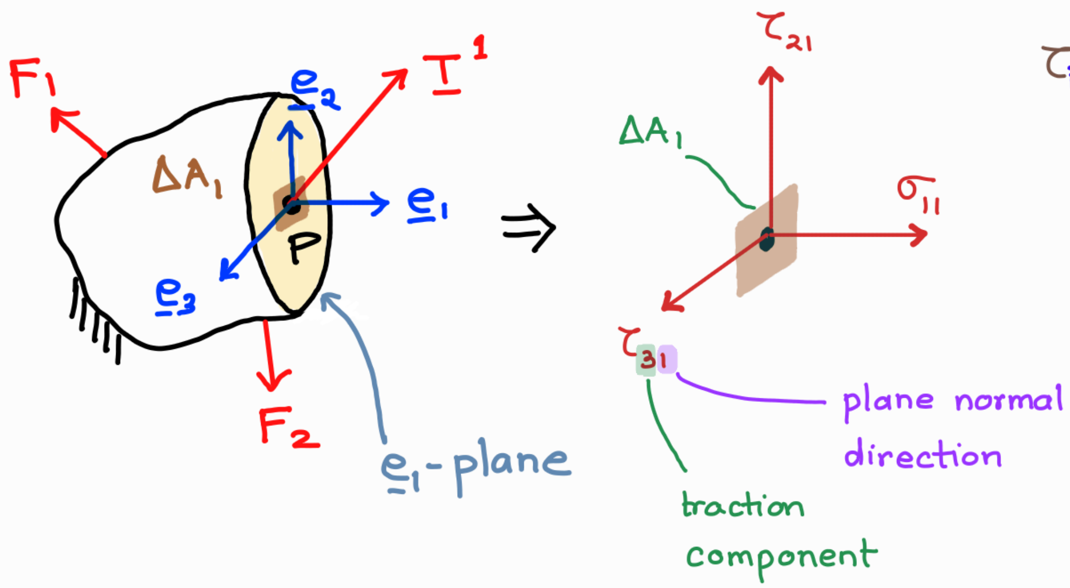
or

$$[\underline{T}^1]_{(\underline{e}_1, \underline{e}_2, \underline{e}_3)} = \begin{bmatrix} T_1^1 \\ T_2^1 \\ T_3^1 \end{bmatrix}$$

If we relate the components of $\underline{T}^1$ to normal and shear components of $\underline{T}^1$, we will get:

Normal stress component, $\sigma_{11} = T_1^1$ } tendency to pull or push

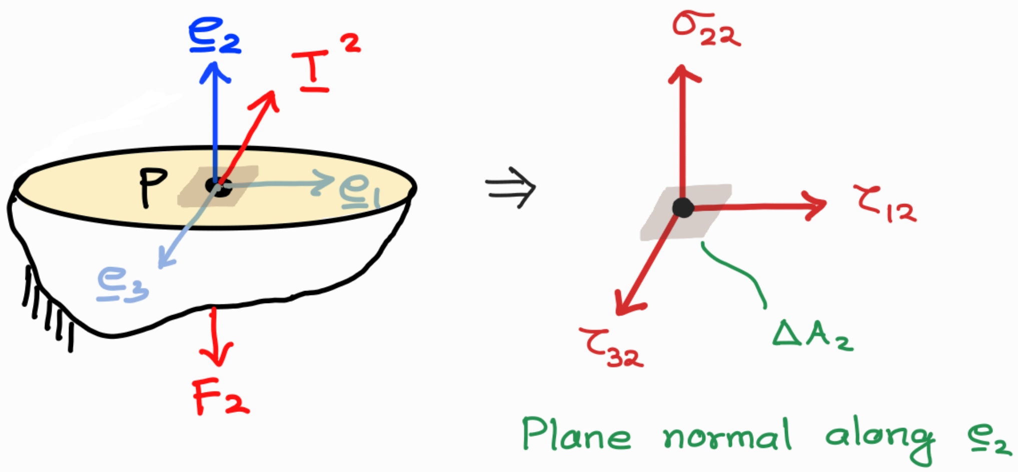
Shear stress component, $\tau_{21} = T_2^1$, $\tau_{31} = T_3^1$ } tendency to slide between two surfaces



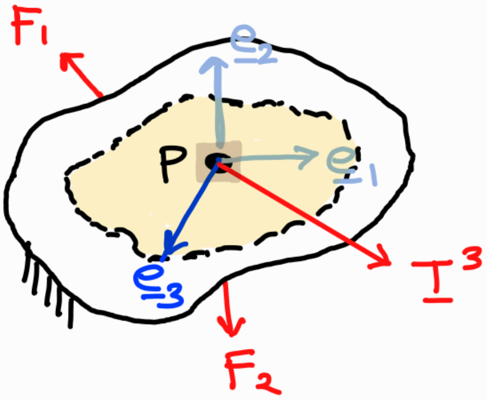
$\tau_{ij} \rightarrow$ represents the ith component of traction on the jth plane

$$\boxed{\underline{T}^1 = \sigma_{11}\,\underline{e}_1 + \tau_{21}\,\underline{e}_2 + \tau_{31}\,\underline{e}_3}$$

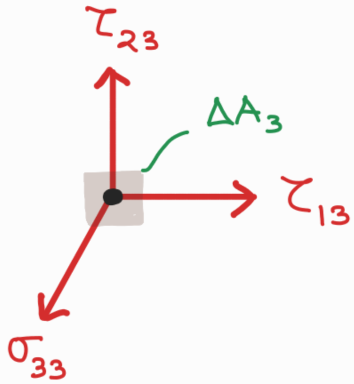
or, $[\underline{T}^1]_{(\underline{e}_1\,\underline{e}_2\,\underline{e}_3)} = \begin{bmatrix} \sigma_{11} \\ \tau_{21} \\ \tau_{31} \end{bmatrix}$



$[\underline{T}^2]_{(\underline{e}_1\,\underline{e}_2\,\underline{e}_3)} = \begin{bmatrix} \tau_{12} \\ \sigma_{22} \\ \tau_{32} \end{bmatrix}$



$\Rightarrow$



$$[\underline{T}^3]_{(\underline{e}_1\ \underline{e}_2\ \underline{e}_3)} = \begin{bmatrix} \tau_{13} \\ \tau_{23} \\ \sigma_{33} \end{bmatrix}$$

Plane normal along $\underline{e}_3$

$$\underline{T}^n = \underline{T}^1 n_1 + \underline{T}^2 n_2 + \underline{T}^3 n_3$$

$$\begin{bmatrix} T_1^n \\ T_2^n \\ T_3^n \end{bmatrix} = \begin{bmatrix} \sigma_{11} \\ \tau_{21} \\ \tau_{31} \end{bmatrix} n_1 + \begin{bmatrix} \tau_{12} \\ \sigma_{22} \\ \tau_{32} \end{bmatrix} n_2 + \begin{bmatrix} \tau_{13} \\ \tau_{23} \\ \sigma_{33} \end{bmatrix} n_3$$

Representing in the form of a matrix-vector product

$$\begin{bmatrix} T_1^n \\ T_2^n \\ T_3^n \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33} \end{bmatrix}}_{[\underline{\sigma}]} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

$$\underline{T}^n(\underline{x}) = \boxed{\underline{\underline{\sigma}}(\underline{x})}\ \underline{n}$$

Stress Tensor

$\downarrow$

independent of plane normal $\underline{n}$

State of Stress at a Point

An infinite number of traction vectors act at a given point. The totality of all traction vectors acting on every possible plane through a point is defined to be the STATE of STRESS at the point.

Since traction along any plane can be obtained from info of tractions on mutually perpendicular planes, therefore it is enough to know tractions on three mutually perpendicular planes to define the totality of all tractions acting at a pt.

The tractions on three mutually perpendicular planes can be further resolved into normal and tangential directions, which lead to the normal and shear stresses on each plane.

Thus the state of stress at a point is completely defined by the nine stress components acting on three mutually perpendicular planes (say $\underline{e}_1, \underline{e}_2, \underline{e}_3$ planes)

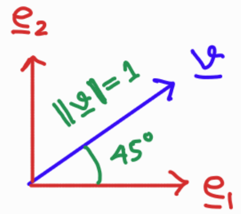
$$[\underline{\sigma}]_{\begin{pmatrix}\underline{e}_1\\ \underline{e}_2\\ \underline{e}_3\end{pmatrix}} = \begin{array}{c} \begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33} \end{bmatrix} \\ \begin{array}{ccc} \underline{T}^1 & \underline{T}^2 & \underline{T}^3 \end{array} \end{array}$$

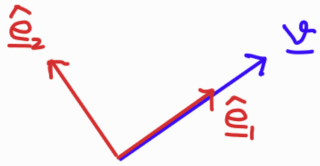
Matrix representation of the stress tensor in a choosen orthogonal coor. system ($\underline{e}_1 - \underline{e}_2 - \underline{e}_3$)

Tensor	Representation in a CSYS
Traction $\underline{T}^n$ (1st order tensor)	$[\underline{T}^n]_{\begin{pmatrix}\underline{e}_1\\ \underline{e}_2\\ \underline{e}_3\end{pmatrix}} = \begin{bmatrix} T_1^n \\ T_2^n \\ T_3^n \end{bmatrix}$
Stress $\underline{\underline{\sigma}}$ (2nd-order tensor)	$[\underline{\underline{\sigma}}]_{\begin{pmatrix}\underline{e}_1\\ \underline{e}_2\\ \underline{e}_3\end{pmatrix}} = \begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33} \end{bmatrix}$

Takeaway: A tensor does NOT depend upon a CSYS but its its component represent as a vector, matrix, etc. depends upon the choice of CSYS

Example:

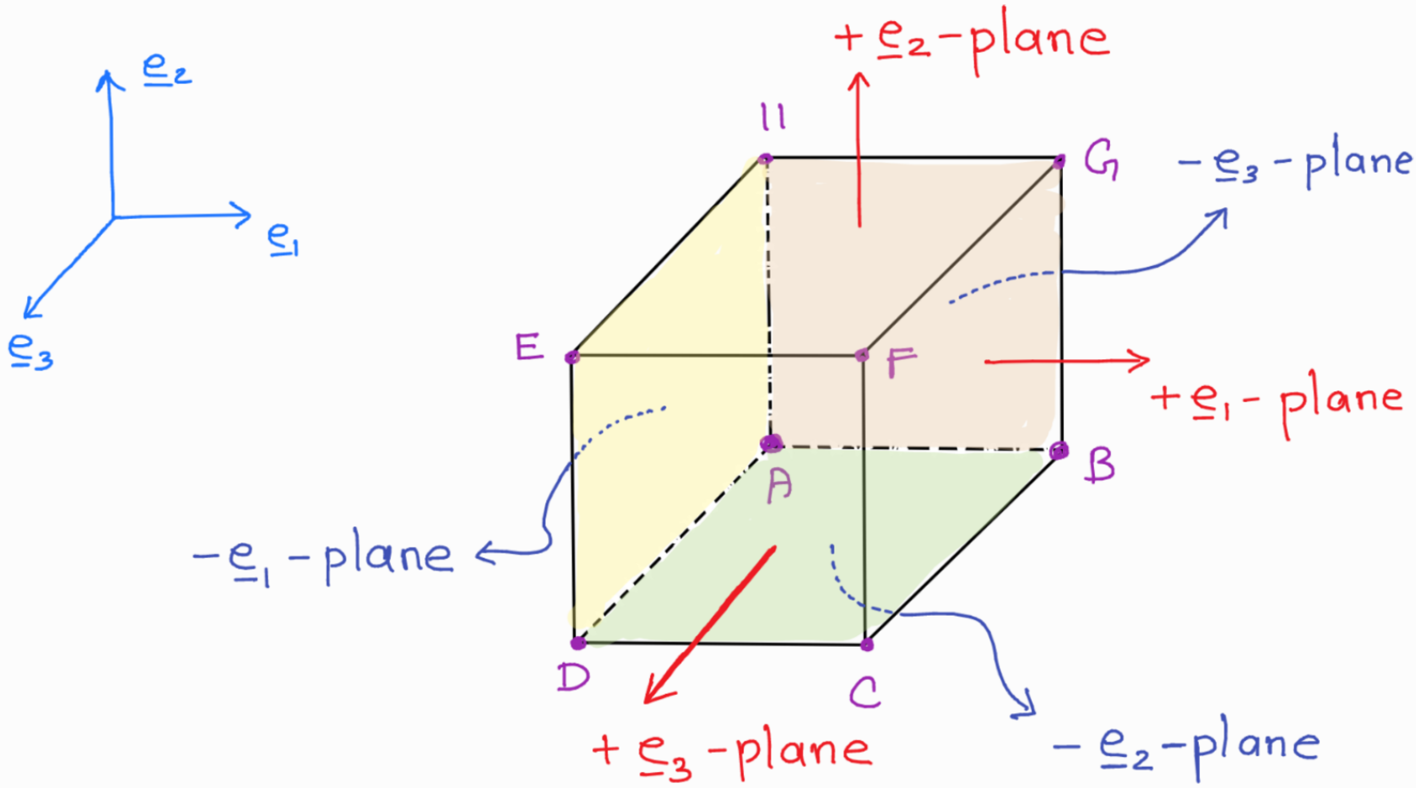




$$[\underline{v}]_{\begin{pmatrix}\underline{e}_1\\ \underline{e}_2\end{pmatrix}} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$[\underline{v}]_{\begin{pmatrix}\hat{\underline{e}}_1\\ \hat{\underline{e}}_2\end{pmatrix}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Sign conventions of stress components



- A face is +ve if the outward normal vector points in the direction of the +ve coordinate axis

- A face is −ve if the outward normal vector points in the direction of the −ve coordinate axis

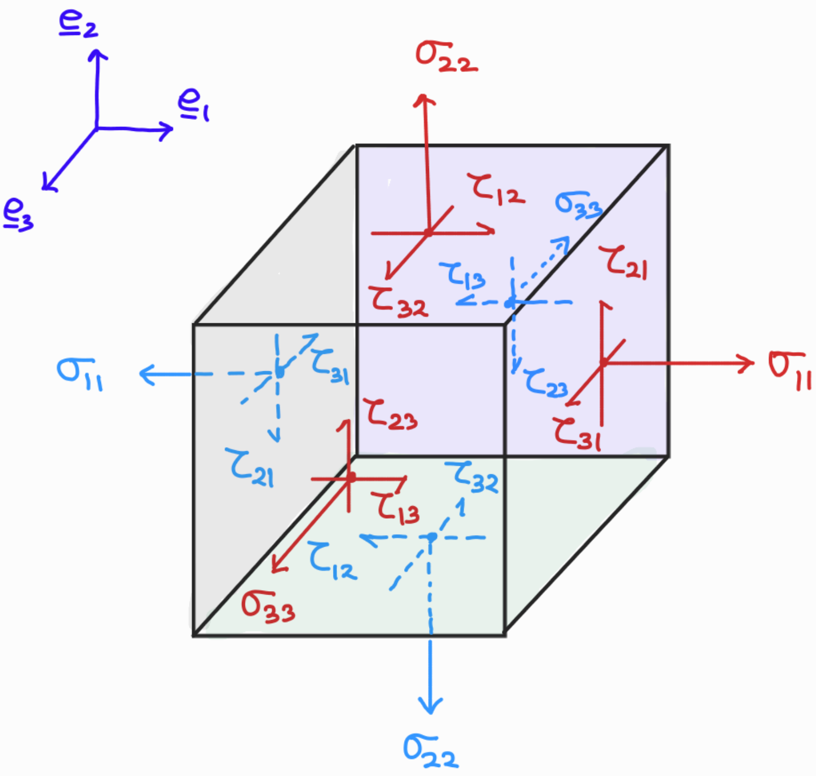
- The stress component is positive when a positively directed force component acts on a positive face

- The stress component is positive when a negatively directed force component acts on a negative face

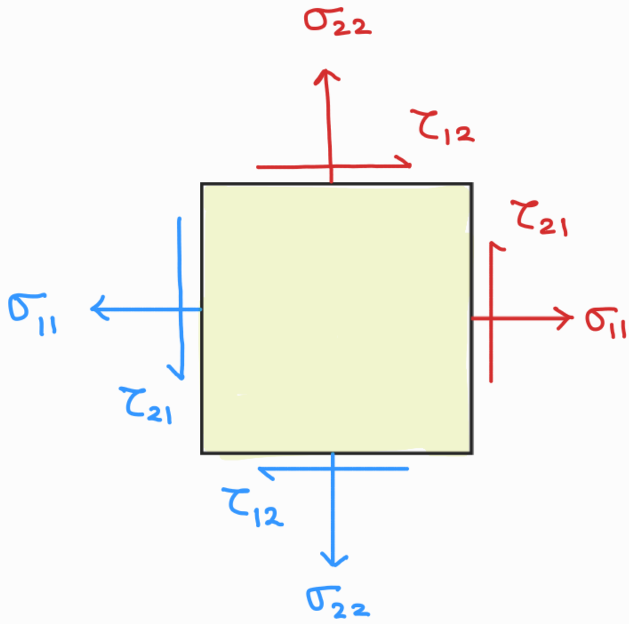
- When a positively directed force component acts on a negative face or a negatively directed force component acts on a positive face, the stress component is negative

Cuboidal representation of stress components

If we take out a small cuboid from the body such that it has its origin at point P inside the body, the state of stress is depicted as:



3D state of stress



2D state of stress

- The cuboid represents a small volume element from inside the body
- Note that all stress components shown are positive
  - Positively directed force component act on +ve plane
  - Negatively directed force component act on −ve plane
- The stress components are assumed to be uniform over the faces