Solid Mechanics
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Lecture - 3
Stress Tensor and its Matrix Representation

Abstract

In the last lecture, we had learnt about the traction vector and how we could find the traction vector on any arbitrary plane at a point. In this lecture, we will learn about the stress tensor, its matrix representation and its physical meaning.

1 Traction vector (start time: 00:35)

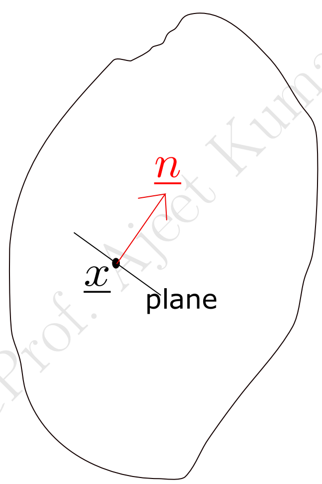

Figure 1: A plane with normal $\underline{n}$ at point $\underline{x}$

In the last lecture, we found the following formula for traction on an arbitrary plane with normal $\underline{n}$ at point $\underline{x}$ (see Figure 1):

$$\underline{t}^n = \sum_{i=1}^{3} \underline{t}^i (\underline{n} \cdot \underline{e}_i) \tag{1}$$

Physically, $\underline{t}^n$, the traction on the plane with normal $\underline{n}$, has to be the same irrespective of what three planes are used to find it, i.e.,

$$\underline{t}^n = \sum_{i=1}^{3} \underline{t}^i (\underline{n} \cdot \underline{e}_i) = \sum_{i=1}^{3} \underline{t}^{\hat{i}} (\underline{n} \cdot \underline{\hat{e}}_i) \tag{2}$$

In the first case, the planes used have normals along $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$ whereas in the second case, the planes used have normals along $\underline{\hat{e}}_1$, $\underline{\hat{e}}_2$, $\underline{\hat{e}}_3$.

2 Stress Tensor (start time: 05:24)

Let us now write equation (2) in a slightly different way using the commutative property of dot product:

$$\underline{t}^n = \sum_{i=1}^{3} \underline{t}^i (\underline{n} \cdot \underline{e}_i) = \sum_{i=1}^{3} \underline{t}^i (\underline{e}_i \cdot \underline{n}) \tag{3}$$

As a vector is represented as a column and the dot product $(\underline{a} \cdot \underline{b})$ in matrix form is given by $[\underline{a}]^T [\underline{b}]$ (derived in lecture 1), the general term of the above summation can be written as follows in the matrix form:

$$\underset{\underset{\underline{t}_i}{\downarrow}}{\begin{bmatrix} \\ \\ \\ \end{bmatrix}_{3\times1}} \left(\underset{\underset{\underline{e}_i^T}{\downarrow}}{[\quad\;]_{1\times3}} \underset{\underset{\underline{n}}{\downarrow}}{\begin{bmatrix} \\ \\ \\ \end{bmatrix}_{3\times1}} \right) \tag{4}$$

As matrix multiplication is associative, we can also write it alternatively as

$$\left(\underbrace{\underset{\underset{\underline{t}_i}{\downarrow}}{\begin{bmatrix} \\ \\ \\ \end{bmatrix}_{3\times1}} \underset{\underset{\underline{e}_i^T}{\downarrow}}{[\quad\;]_{1\times3}}}_{\text{tensor product}} \right) \underset{\underset{\underline{n}}{\downarrow}}{\begin{bmatrix} \\ \\ \\ \end{bmatrix}_{3\times1}} \tag{5}$$

However, note that $[\underline{a}][\underline{b}]^T$ is the matrix representation for tensor product $(\underline{a} \otimes \underline{b})$. Going back to the tensor notation but by using the alternate representation given by (5) in equation (3), we obtain

$$\underline{t}^n = \underbrace{\sum_{i=1}^{3} (\underline{t}^i \otimes \underline{e}_i)}_{\text{stress tensor}} \underline{n}. \tag{6}$$

This is just a different viewpoint but the interesting point in this notation is that the orientation $\underline{n}$ has been separated. The tensor that is multiplied with $\underline{n}$ is called the STRESS TENSOR. It is denoted by $\underline{\underline{\sigma}}$. So finally, we get:

$$\begin{aligned} \underline{t}^n &= \underline{\underline{\sigma}}\, \underline{n} \\ \text{Or, } \; \underline{t}^n(\underline{x}; \underline{n}) &= \underline{\underline{\sigma}}(\underline{x})\, \underline{n} \end{aligned} \tag{7}$$

We have thus found the expression for the stress tensor to be dependent on $\underline{x}$ alone, i.e.,

$$\underline{\underline{\sigma}}(\underline{x}) = \sum_{i=1}^{3} (\underline{t}^i \otimes \underline{e}_i) \tag{8}$$

Hence, to obtain the stress tensor at a point, choose three independent planes at the same point, find tractions on those planes, do their tensor product and sum! The resulting stress tensor will be independent of what three planes we choose!

3 Representation of stress tensor in a coordinate system (start time: 14:40)

Let us try to represent equation (8) in ($\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$) coordinate system. As we now know that the stress tensor is a second order tensor, it's representation is going to be a matrix, i.e.,

$$\left[\underline{\underline{\sigma}}\right]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} = \sum_{i=1}^{3} \left[\underline{t}^i\right] \left[\underline{e}_i\right]^T$$

Here, we have to represent $\underline{t}^i$ and $\underline{e}_i$ also in ($\underline{e}_1, \underline{e}_2, \underline{e}_3$) coordinate system:

$$\begin{aligned}
\Rightarrow \left[\underline{\underline{\sigma}}\right]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} &= \left[\underline{t}^1\right]\begin{bmatrix}1 & 0 & 0\end{bmatrix} + \left[\underline{t}^2\right]\begin{bmatrix}0 & 1 & 0\end{bmatrix} + \left[\underline{t}^3\right]\begin{bmatrix}0 & 0 & 1\end{bmatrix} \\
&= \begin{bmatrix} t_1^1 & 0 & 0 \\ t_2^1 & 0 & 0 \\ t_3^1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & t_1^2 & 0 \\ 0 & t_2^2 & 0 \\ 0 & t_3^2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & t_1^3 \\ 0 & 0 & t_2^3 \\ 0 & 0 & t_3^3 \end{bmatrix} \\
&= \begin{bmatrix} t_1^1 & t_1^2 & t_1^3 \\ t_2^1 & t_2^2 & t_2^3 \\ t_3^1 & t_3^2 & t_3^3 \end{bmatrix} \qquad (9)
\end{aligned}$$

where a general traction component signifies the following:

$$t_j^i = \underline{t}^i \cdot \underline{e}_j \qquad (10)$$

So, t_j^i represents the j^{th} component of traction on i^{th} plane. Thus, if we want to write the stress matrix corrresponding to ($\underline{e}_1$,$\underline{e}_2$,$\underline{e}_3$) coordinate system, its first column would the representation of traction on plane whose normal is along the first coordinate axis (which is $\underline{e}_1$ here). Similarly, the second column will be the representation of traction on plane with normal along second coordinate axis and the third column will be the representation of traction on plane with normal along third coordinate axis. Usually, the following notation is used to write the stress tensor:

$$\left[\underline{\underline{\sigma}}\right]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} = \begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33} \end{bmatrix} \qquad (11)$$

Basically, the off-diagonal elements are denoted by τ and diagonal components are denoted by σ. On comparing the above equation with (9), τ_{ij} denotes the i^{th} component of traction on j^{th} plane.[1]

3.1 Representation of stress tensor in Cartesian coordinate system (start time: 30:38)

A Cartesian coordinate system means that the coordinate axes are fixed in space and perpendicular to each other (such as $\underline{e}_1$, $\underline{e}_2$ and $\underline{e}_3$). Let us say we want to represent the stress tensor at a given point $\underline{x}$ in our body. Think of a cuboid around the point $\underline{x}$ as shown in Figure 2. It is centered at $\underline{x}$ and its six faces are chosen along $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$, -$\underline{e}_1$, -$\underline{e}_2$, -$\underline{e}_3$ respectively. The traction that acts on its $\underline{e}_1$

[1] In most books, τ_{ij} represents the j^{th} component of traction on i^{th} plane but we will follow our convention since it follows naturally. Our convention also aligns with the convention for components of other stress measures such as Piola-Kirchhoff stress tensor in nonlinear elasticity.

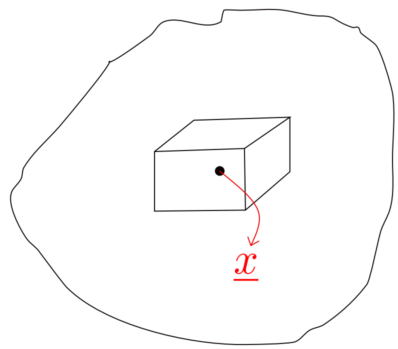

Figure 2: Visualizing a cuboid centered at the point of interest $\underline{x}$ in our body

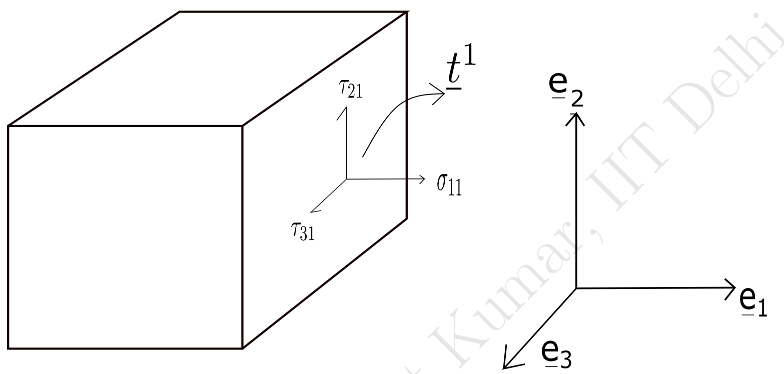

Figure 3: Components of traction that acts on $\underline{e}_1$ face of the cuboid

face is $\underline{t}^1$ whose three components are shown in Figure 3. From the figure, it can be concluded that σ_{11} acts normal to the plane whereas τ_{21} and τ_{31} are in the plane. Thus, σ_{11} is called the normal component of traction and τ_{21} and τ_{31} are called the shear components of traction since they try to shear the body. To visualize shearing, think of a section in the body having normal along $\underline{e}_1$. If the traction on this section has a non-zero component in the plane of the section, then the parts of the body on the two sides of the section will try to shear relative to each other, i.e., relatively displace along the plane of the section itself. Similarly, σ_{11} will lead to pushing or pulling between the two parts. If σ_{11} is positive (negative), we call it tensile (compressive) since the two parts will try to pull (push) themselves apart (into each other).

Going back to our cuboid, the components of traction that acts on all its faces are shown in Figure 4. Remember that second index denotes the plane normal and the first index denotes the direction. For faces with negative normals such as the bottom face, the plane normal is along -$\underline{e}_2$ and hence $\underline{t}^{-2}$ acts on it. We have already seen in the last lecture that

$$\underline{t}^{-2} = -\underline{t}^2. \tag{12}$$

So, on the bottom face, the same traction components act which also act on the top face but in the opposite direction.

Finally, if the various components drawn on the cuboid have to represent the components of stress matrix at point $\underline{x}$ in the body, the cuboid must be shrunk to this point and made infinitesimally

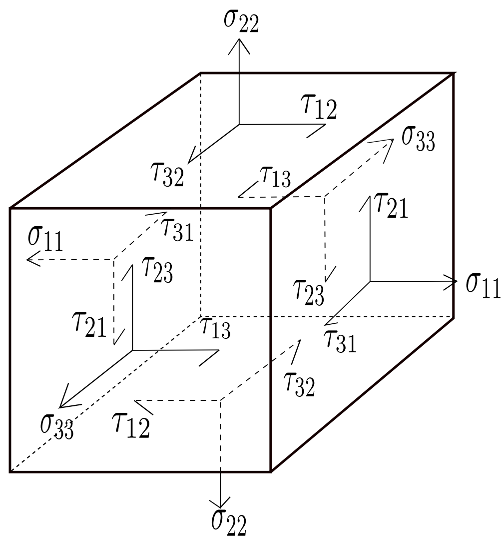

Figure 4: The traction components are shown on top, right, front and bottom faces of the cuboid. Notice that the traction components on the bottom face are in opposite direction compared to those on the top face

small so that all its faces pass through $\underline{x}$. In the figures however, we have drawn the cuboid bigger and the point $\underline{x}$ at the center of the cuboid just for visualization.