Solid Mechanics
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Lecture - 4
Transformation of stress matrix

Abstract

Till now, we have learnt about the traction vector, the stress tensor and its representation as a stress matrix. In this lecture, we will discuss how to transform the stress matrix corresponding to a stress tensor from one coordinate system to another.

1 Stress matrix (start time: 00:31)

It is the representation (matrix form) of the stress tensor in a coordinate system. The stress matrix changes from one coordinate system to another but the stress tensor itself remains the same. If we want to get the stress matrix of a stress tensor, then we first choose the coordinate system (let's say $\underline{e}_1$,$\underline{e}_2$,$\underline{e}_3$). We then find traction on the three planes whose normals are $\underline{e}_1$, $\underline{e}_2$ and $\underline{e}_3$ and form columns out of those tractions. In case we have a different coordinate system, we simply have to find traction on planes whose normals are the basis vectors of the new coordinate system and then form columns from these tractions. Remember to also represent the traction vectors in column form in the same coordinate system in which stress matrix is desired. This is why stress matrix changes from one coordinate system to other but the stress tensor itself remains the same.

2 Formula for transformation of a stress matrix (start time: 03:00)

The transformation of a stress matrix refers to finding out a relationship between the stress matrices in two different coordinate systems. The stress matrix in ($\underline{e}_1$,$\underline{e}_2$,$\underline{e}_3$) coordinate system is:

$$[\underline{\underline{\sigma}}]_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} = \begin{array}{c} \left[\begin{pmatrix} \\ \\ \end{pmatrix} \begin{pmatrix} \\ \\ \end{pmatrix} \begin{pmatrix} \\ \\ \end{pmatrix} \right] \\ \downarrow \quad \downarrow \quad \downarrow \\ \underline{t}^1 \quad \underline{t}^2 \quad \underline{t}^3 \end{array}$$

$$\text{Furthermore, } \sigma_{ij} = \underline{t}^j \cdot \underline{e}_i \tag{1}$$

Here, $\underline{t}^j$ represents the traction on j^{th} plane. So, the first column is the representation of $\underline{t}^1$ in $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ coordinate system. Also, notice that i^{th} row corresponds to component of that particular traction along i^{th} basis vector.

Similarly, writing down the stress matrix in a new coordinate system $(\hat{\underline{e}}_1, \hat{\underline{e}}_2, \hat{\underline{e}}_3)$, we get:

$$\left[\underline{\underline{\sigma}}\right]_{(\hat{\underline{e}}_1,\hat{\underline{e}}_2,\hat{\underline{e}}_3)} = \left[\underset{\substack{\downarrow \\ \underline{t}^{\hat{1}}}}{\Big(\quad\Big)} \underset{\substack{\downarrow \\ \underline{t}^{\hat{2}}}}{\Big(\quad\Big)} \underset{\substack{\downarrow \\ \underline{t}^{\hat{3}}}}{\Big(\quad\Big)} \right]$$

$$\text{where } \hat{\sigma}_{ij} = \underline{t}^{\hat{j}} \cdot \hat{\underline{e}}_i \neq \underline{t}^{\hat{j}} \cdot \underline{e}_i$$

2.1 Relating σ_{ij} and $\hat{\sigma}_{ij}$ (start time: 09:01)

In the second lecture, we had found that traction on an arbitrary plane can be written as:

$$\underline{t}^n = \sum_{i=1}^{3} \underline{t}^i (\underline{n} \cdot \underline{e}_i) \tag{2}$$

$$\begin{aligned} \Rightarrow \hat{\sigma}_{ij} = \underline{t}^{\hat{j}} \cdot \hat{\underline{e}}_i &= \underbrace{\sum_{k=1}^{3} \underline{t}^k (\hat{\underline{e}}_j \cdot \underline{e}_k)}_{\underline{t}^{\hat{j}}} \cdot \hat{\underline{e}}_i \\ &= \sum_{k=1}^{3} (\underline{t}^k \cdot \hat{\underline{e}}_i)(\hat{\underline{e}}_j \cdot \underline{e}_k) \end{aligned} \tag{3}$$

Now, we need to define a relationship between $(\hat{\underline{e}}_1,\ \hat{\underline{e}}_2,\ \hat{\underline{e}}_3)$ and $(\underline{e}_1,\ \underline{e}_2,\ \underline{e}_3)$. We know that we can always find a rotation tensor $\underline{\underline{R}}$ that relates one set of basis vectors to another set of basis vectors :

$$\Rightarrow \hat{\underline{e}}_i = \underline{\underline{R}}\, \underline{e}_i \tag{4}$$

This is a tensor equation and thus coordinate free. It does not correspond to any one coordinate system and can be used with respect to any coordinate system. Let's write this equation in $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ coordinate system. This means that we have to write each of the vectors and tensors in this coordinate system[1], i.e.,

$$\left[\hat{\underline{e}}_i\right]_{(\underline{e}_1,\underline{e}_2.\underline{e}_3)} = \left[\underline{\underline{R}}\right]_{(\underline{e}_1,\underline{e}_2.\underline{e}_3)} \left[\underline{e}_i\right]_{(\underline{e}_1,\underline{e}_2.\underline{e}_3)} \tag{5}$$

Representation of $\underline{e}_i$ in its own coordinate system is trivial. $\underline{e}_1$ will be represented as $\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\underline{e}_2$ as $\begin{bmatrix}0\\1\\0\end{bmatrix}$ and $\underline{e}_3$ as $\begin{bmatrix}0\\0\\1\end{bmatrix}$.

[1] $\hat{\underline{e}}_i$ will be represented as a column vector having entries as components of $\hat{\underline{e}}_i$ along $\underline{e}_1, \underline{e}_2$ and $\underline{e}_3$. $\underline{\underline{R}}$ is a second order tensor, so it will be represented by a matrix

When $i = 1$, $\underline{e}_1$ gets rotated to $\hat{\underline{e}}_1$ as:

$$\Rightarrow \left[\hat{\underline{e}}_1\right]_{(\underline{e}_1,\underline{e}_2.\underline{e}_3)} = \left[\underline{\underline{R}}\right]_{(\underline{e}_1,\underline{e}_2.\underline{e}_3)} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R_{11} \\ R_{21} \\ R_{31} \end{bmatrix} \tag{6}$$

So, rotated $\underline{e}_1$(i.e., $\hat{\underline{e}}_1$) expressed in column form is just the first column of the rotation matrix. Similarly, $\hat{\underline{e}}_2$ is then the second column of rotation matrix and $\hat{\underline{e}}_3$ is the third column of rotation matrix. Also

$$R_{11} = \hat{\underline{e}}_1 \cdot \underline{e}_1, R_{21} = \hat{\underline{e}}_1 \cdot \underline{e}_2, R_{31} = \hat{\underline{e}}_1 \cdot \underline{e}_3, \tag{7}$$

$$\Rightarrow R_{ij} = \hat{\underline{e}}_j \cdot \underline{e}_i \tag{8}$$

Another way to obtain this equation is to realize that the rotation tensor can also be written as $\underline{\underline{R}} = \sum_{i=1}^{3} \hat{\underline{e}}_i \otimes \underline{e}_i$. Further, using the formula derived in lecture 1 to obtain the matrix component of a second order tensor, i.e., $C_{mn} = \left(\underline{\underline{C}}\ \underline{e}_n\right) \cdot \underline{e}_m$, we obtain the above equation. Now, our equation for stress matrix component from equation (3) becomes:

$$\begin{aligned} \hat{\sigma}_{ij} &= \sum_{k=1}^{3} (\underline{t}^k \cdot \hat{\underline{e}}_i) \underbrace{(\hat{\underline{e}}_j \cdot \underline{e}_k)}_{R_{kj}} \quad \text{(using (8))} \\ &= \sum_{k=1}^{3} \underbrace{\left(\sum_{m=1}^{3} t_m^k\ \underline{e}_m \right)}_{\underline{t}^k} \cdot \hat{\underline{e}}_i\ R_{kj} = \sum_k \sum_m t_m^k\ R_{mi} R_{kj}. \end{aligned} \tag{9}$$

However, by definition, $t_m^k = \sigma_{mk}$ because t_m^k represents the m^{th} component of traction $\underline{t}^k$ and will thus represent the m^{th} row in the k^{th} column of stress matrix. Hence

$$\hat{\sigma}_{ij} = \sum_k \sum_m \sigma_{mk} R_{mi} R_{kj} = \sum_k \sum_m R_{im}^T \sigma_{mk} R_{kj} \tag{10}$$

Finally, writing this equation in matrix form, we get

$$\left[\underline{\underline{\sigma}}\right]_{(\hat{\underline{e}}_1,\hat{\underline{e}}_2,\hat{\underline{e}}_3)} = \underbrace{\left[\underline{\underline{R}}\right]^T \left[\underline{\underline{\sigma}}\right] \left[\underline{\underline{R}}\right]}_{(\underline{e}_1,\underline{e}_2,\underline{e}_3)} \tag{11}$$

This is the relation that we use to transform stress matrix from one coordinate system to another. In fact, the above relation holds for transformation of matrix form of any second order tensor.

3 Transformation of vector components (start time: 24:56)

Let us now see how the column form of a vector transform when we change the coordinate system just like we found the transformation for a second order tensor's matrix representation. If we have a vector $\underline{v}$, we can represent it in column form with respect to ($\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$) coordinate system as well as with respect to $(\underline{\hat{e}}_1, \underline{\hat{e}}_2, \underline{\hat{e}}_3)$ coordinate system. To relate these two representations, one can easily deduce:

$$\begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \\ \hat{v}_3 \end{bmatrix} = \left[\underline{\underline{R}}\right]^T \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{12}$$

What is to be noticed here is that for transforming the vector components, we need to premultiply by $\left[\underline{\underline{R}}\right]^T$ while for relating the basis vectors, we need to premultiply by $\underline{\underline{R}}$ (see equation (4)). This is because the vector $\underline{v}$ has to remain the same, we are just trying to write it in two different coordinate systems, i.e.,:

$$\underline{v} = \sum v_i \underline{e}_i = \sum \hat{v}_i \underline{\hat{e}}_i \tag{13}$$

Therefore, the basis vector $\underline{\hat{e}}_i$ gets transformed with rotation $\underline{\underline{R}}$ whereas the components $\hat{v}_i$ get transformed with $\underline{\underline{R}}^T$. $\underline{\underline{R}}^T \underline{\underline{R}}$ gets cancelled and becomes $\underline{\underline{I}}$.

4 An example for stress matrix transformation (start time: 28:30)

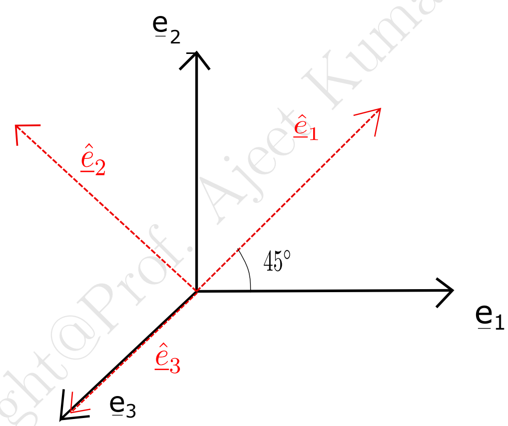

Figure 1: Two coordinate systems with the red one obtained by rotation of the black one by an angle 45° about $\underline{e}_3$

Suppose the stress matrix in $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ coordinate system is given by:

$$\left[\underline{\underline{\sigma}}\right]_{(\underline{e}_1, \underline{e}_2, \underline{e}_3)} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{14}$$

We define a new coordinate system by rotation of the old coordinate system by 45° about $\underline{e}_3$ as shown in Figure 1. Our objective is to find out the stress matrix representation in this new coordinate system. First, we need to find the Rotation matrix. As derived in previous lecture, this rotation matrix is:

$$\left[\underline{\underline{R}}\right] = \begin{bmatrix} cos(45°) & -sin(45°) & 0 \\ sin(45°) & cos(45°) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{15}$$

Now, we can apply the formula for stress matrix transformation, i.e., equation (11). We get:

$$\left[\underline{\underline{\sigma}}\right]_{(\hat{\underline{e}}_1,\hat{\underline{e}}_2,\hat{\underline{e}}_3)} = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{16}$$

From this, we observe that as the second column has all zeros, there is no traction on $\hat{\underline{e}}_2$ plane whereas the traction on $\hat{\underline{e}}_1$ plane only has the first component non zero, i.e., along $\hat{\underline{e}}_1$. So, it has only got a normal component.

4.1 Verification of stress matrix (start time: 34:03)

Let us verify the new stress matrix by directly finding the traction on one of the planes. Consider traction on the plane with normal as $\hat{\underline{e}}_2$:

$$\underline{t}^{\hat{2}} = \sum_{i=1}^{3} \underline{t}^i (\hat{\underline{e}}_2 \cdot \underline{e}_i) \tag{17}$$

This is again a vector equation. Let's write this down in the $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ coordinate system. To get the traction vectors, we get back to our stress matrix given in $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ coordinate system and take its columns. To obtain the column form of $\hat{\underline{e}}_2$, we look at the second column of our rotation matrix. To get $\hat{\underline{e}}_2 \cdot \underline{e}_i$, we take the i^{th} component in this column. Alternatively, we can also find this dot product by looking at the orientation of the two coordinate systems shown in figure 1 and using the basic dot product formula:

$$\underline{a} \cdot \underline{b} = ||\underline{a}||\ ||\underline{b}||\ cos(\text{angle between them}). \tag{18}$$

For example, $\hat{\underline{e}}_2 \cdot \underline{e}_1$ would be $cos(135°) = -\frac{1}{\sqrt{2}}$.

$$\Rightarrow \underline{t}^{\hat{2}} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} (-\frac{1}{\sqrt{2}}) + \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} (\frac{1}{\sqrt{2}}) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} (0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{19}$$

This verifies what we had gotten using transformation formula.
Remark : This zero column represents $\underline{t}^{\hat{2}}$ in $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$ coordinate system. But being a 'zero vector' means that its representation in any coordinate system will be a zero column. Therefore, the 2^{nd} column of $\left[\underline{\underline{\sigma}}\right]_{(\hat{\underline{e}}_1,\hat{\underline{e}}_2,\hat{\underline{e}}_3)}$ is a zero column!