Solid Mechanics
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Lecture - 2
Traction Vector

Abstract

In the previous lecture, we had discussed about vectors, tensors and some operations involving them. In this lecture, we will learn about the concept of traction vector.

1 Introduction (start time: 01:07)

Consider an arbitrary body which is clamped at one part of the boundary and some force acts on another part of the boundary as shown in Figure 1. The dashed lines are used to denote clamping

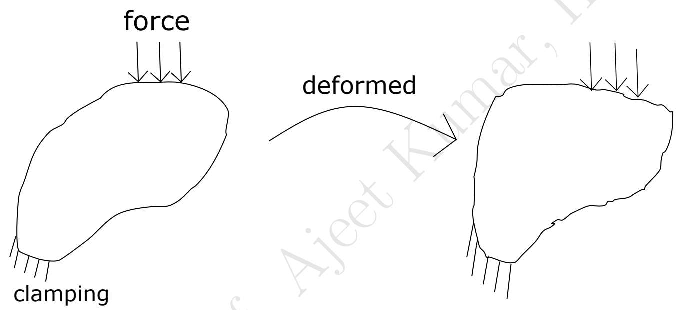

Figure 1: A body under the action of a force gets deformed

of a part of the boundary which does not move during deformation. In the deformed configuration, we say that the body is under some stress. Let us take a section that divides the deformed body into two parts: Part A and Part B as shown in Figure 2. This section is shown by dashed lines

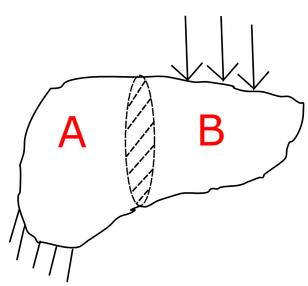

Figure 2: An internal section is cut that divides the deformed body into two Parts:A and B

as it is an internal section. Let's analyze Part A. Part B exerts some force on Part A that will be distributed over the cut section as shown in Figure 3. The distributed load on the section can be in any direction. In Figure 3 for example, the upper portion of the section is being pulled while the bottom section is being pushed.

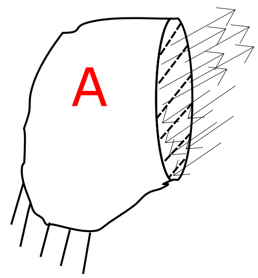

Figure 3: Part A of the body when considered alone is acted upon by forces applied by Part B

2 Definition of traction vector (start time: 05:21)

Traction is defined as the intensity of the force (force per unit area) with which Part B is pulling or pushing part A. At different points in the section, the intensity of the force would be different (see Figure 3). Let us try to find the intensity at some point $\underline{x}$ in this section (see Figure 4). We

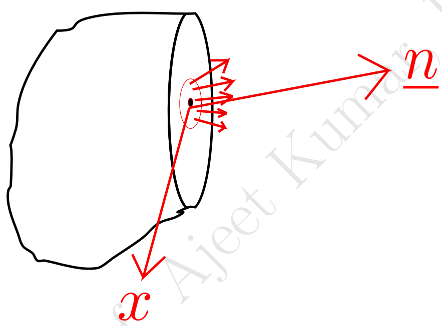

Figure 4: A small circular area around the point of interest $\underline{x}$ on the section with normal as $\underline{n}$

draw a circle around this point in the plane of the section. This plane has its normal vector shown as $\underline{n}$. Let the area enclosed by the circle be ΔA and total force acting on this circular area be $\Delta \overrightarrow{F}$. To find the intensity at point $\underline{x}$, we need to shrink the area of the circle such that the circle always contains $\underline{x}$. As we shrink the area, the total force acting on it also decreases but their ratio attains a limiting value which is called traction. We represent it by $\underline{t}(\underline{x};\underline{n})$ where $\underline{x}$ represents the point at which traction is being measured and $\underline{n}$ is the normal to the plane on which traction is being measured:

$$\underline{t}(\underline{x};\underline{n}) = \lim_{\Delta A \to 0} \frac{\Delta \overrightarrow{F}}{\Delta A}. \tag{1}$$

It has the unit of pressure but it is more general than pressure: pressure always acts along plane normal whereas traction can have arbitrary direction.

2.1 Parameters on which traction depends (start time: 11:58)

Consider an arbitrary body under the influence of external load and look at three points inside the body as shown in Figure 5. Traction will be different at these three points because by definition, it is a function of the location at which it is being measured. At a given point also, say $\underline{x}$, several

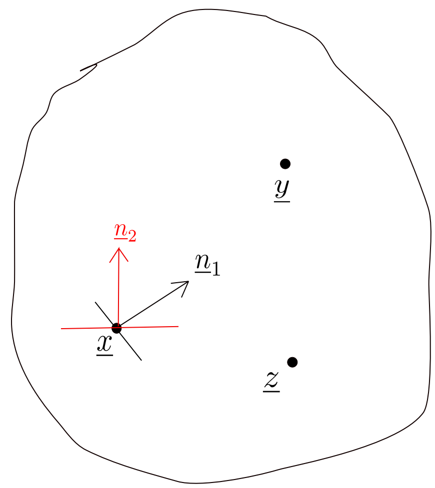

Figure 5: Three arbitrary points ($\underline{x}, \underline{y}, \underline{z}$) shown on a body and two different planes at point $\underline{x}$ with plane normals $\underline{n}_1$ and $\underline{n}_2$ respectively

planes exist and traction on each of these planes will be different. In Figure 5 for example, traction on the plane having normal $\underline{n}_1$ will be different from traction on plane with normal $\underline{n}_2$. To see this more clearly, let us consider a rectangular beam clamped at one end as shown in Figure 6. A force P is being applied on its other end. The cross-sectional area of the beam is A. We first consider a

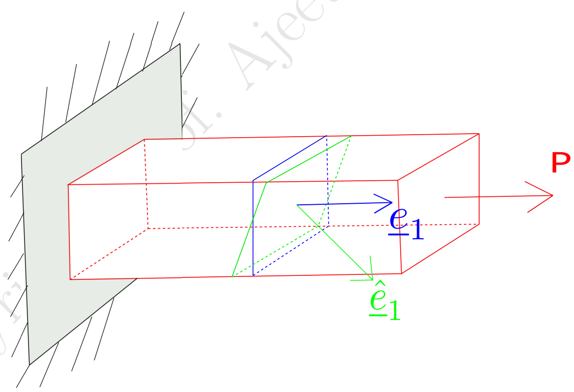

Figure 6: A cantilever beam which is acted upon by force P. Two sections are shown: one horizontal and the other tilted both having same centroidal location

section whose normal vector (denoted by $\underline{e}_1$) is along the axis of the beam (also shown as the blue plane in Figure 6). The traction on this plane (denoted by $\underline{t}^1$) is given by:

$$\underline{t}^1 = \frac{P\underline{e}_1}{A} = \frac{P}{A}\,\underline{e}_1. \tag{2}$$

For simplicity, we have assumed here that we have the same traction at every point on this section although that is not the usual case. Now, let us take a different section (shown in green in Figure 6) but at the same axial location as the blue plane. The unit normal to this new section ($\hat{\underline{e}}_1$) makes an angle θ with the beam's axis. So, the area of this section ($\hat{A}$) is given by:

$$\hat{A} = \frac{A}{cos\theta} \tag{3}$$

The total force is still P$\underline{e}_1$. So, the traction on this section (denoted by $\underline{t}^{\hat{1}}$) will be:

$$\underline{t}^{\hat{1}} = \frac{P\underline{e}_1}{A/cos\theta} = \frac{Pcos\theta}{A}\underline{e}_1 \tag{4}$$

Thus, we see that the traction on two different planes at the same point are different. There are infinite number of planes at a point. Hence, if we want to know traction even at one point in the body, we need to know traction on all the planes at that point. We will see later how we can conveniently obtain traction on an arbitrary plane at a point by the information of traction on just three planes at the same point.

2.2 Importance of traction (start time: 20:30)

By definition, it gives us the intensity of force with which one part of the body is pulling or pushing the other part of the body. If the value of this traction is lower than a threshold limit, the body will not fracture/fail. At a given point, as traction varies from one plane to the other, the probability of failure is higher on the plane on which traction has got larger value. Thus, traction can tell us at what point in the body and on what plane at that point would the body fail!

3 Relating traction on different planes at a point (start time: 21:58)

We will now prove that if we know traction on three independent planes at a point in the body, we can find traction on any plane at the same point. This means that we need to just store traction on three planes and then use a formula to get traction on any other plane. Let us work out this formula. Consider a point $\underline{x}$ in the body as shown in Figure 7 and think of a small volume there in the shape of a tetrahedron whose vertex is at $\underline{x}$ and the tetrahedron's three edges at this point are perpendicular and along the coordinate axes. The tetrahedron has four faces having the following outward normals (by outward we mean pointing in a direction out of the tetrahedron volume):

$$\begin{array}{rcl} \underline{Plane} & : & \underline{Outward\ normal} \\ OAB & : & -\underline{e}_3 \\ OBC & : & -\underline{e}_1 \\ OAC & : & -\underline{e}_2 \\ ABC & : & \underline{n}(say) \end{array}$$

Suppose, we know tractions on its the three planes with outward normals ($-\underline{e}_1, -\underline{e}_2, -\underline{e}_3$) and we want to find traction on the tilted plane ABC (having normal $\underline{n}$). Let us apply Newton's 2^{nd} law of motion to the mass contained in the tetrahedron:

$$\sum \underline{F}^{ext} = \frac{d}{dt}(\overrightarrow{P}) \tag{5}$$

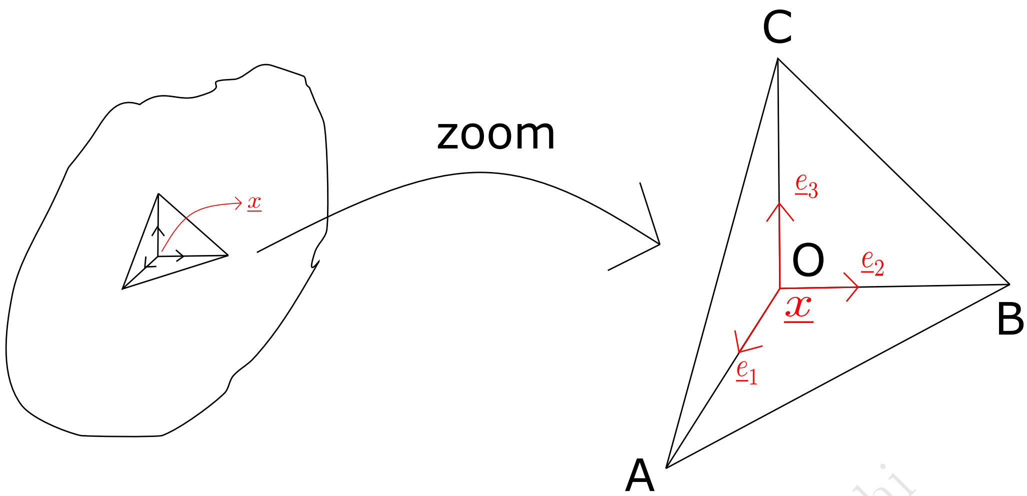

Figure 7: A tetrahedron is considered at the point of interest whose three edges are along the three coordinate axes

Here, $\overrightarrow{P}$ denotes the momentum of the tetrahedron's mass. To visualize the external forces acting on the tetrahedron, we can imagine taking out this tetrahedron from the original body as shown on the right of Figure 7. The remaining body exerts traction force on the four faces of the tetrahedron which are also categorized as contact forces. However, the total external force will consist of both contact and non-contact forces:

$$\begin{array}{cc} \multicolumn{2}{c}{\sum \underline{F}^{ext}} \\ \swarrow & \searrow \\ contact\ \ force & non-contact\ \ force \\ (traction\ \ force) & (body\ \ force) \\ \text{units: [force/area]} & \text{units: [force/volume]} \end{array}$$

The body force acts on every point of the body, e.g., the gravitational force is applied by the Earth on every point of the tetrahedron. Its unit is force/volume as it acts on the entire volume in a distributed manner. and is given by:

$$\text{Gravitational body force} = (\rho \underline{g} V)/V = \rho \underline{g} \tag{6}$$

Here ρ denotes density, $\underline{g}$ is the acceleration due to gravity and V is the volume of the tetrahedron. To get the total contact force, we need to integrate it over the area on which it is acting whereas to get the total body force, we need to integrate it over the volume on which it is acting. Let the area of face OBC be A_1 and traction on it be $\underline{t}^{-1}$, area of face OAC be A_2 and traction on it be $\underline{t}^{-2}$, area of face OAB be A_3 and traction on it be $\underline{t}^{-3}$ and finally the area of face ABC be A_n and traction on it be $\underline{t}^n$. Assuming that traction does not vary over different points in a plane, the total external force on the tetrahedron can be written as

$$\sum \underline{F}^{ext} = \underbrace{\underline{t}^{-1}A_1 + \underline{t}^{-2}A_2 + \underline{t}^{-3}A_3 + \underline{t}^{n}A_n}_{\text{traction force}} + \underbrace{\rho \underline{g} V}_{\text{body force}} = \underbrace{\rho V}_{\text{mass of the tetrahedron}} \times \underline{a}_{CM} \tag{7}$$

Here $\underline{a}_{CM}$ denotes the acceleration of the center of mass of the tetrahedron. Suppose, h is the perpendicular distance of the plane ABC from the tetrahedron's vertex. The volume V is then given by:

$$V = \frac{A_n h}{3} \tag{8}$$

We can also relate the areas A_1, A_2, A_3 in terms of A_n. If we project the area A_n along the direction $\underline{e}_1$, it will turn out to be the same as the area of OBC which is A_1. Using geometry, we can prove that

$$A_i = A_n(\underline{n} \cdot \underline{e}_i) \tag{9}$$

Upon plugging equations (8) and (9) in equation (7), we get

$$A_n\left(\underline{t}^{-1}(\underline{n} \cdot \underline{e}_1) + \underline{t}^{-2}(\underline{n} \cdot \underline{e}_2) + \underline{t}^{-3}(\underline{n} \cdot \underline{e}_3) + \underline{t}^n\right) + \frac{\rho \underline{g} A_n h}{3} = \frac{\rho A_n h\ \underline{a}_{CM}}{3}$$

$$\Rightarrow \boxed{\underline{t}^n + \sum_{i=1}^{3} \underline{t}^{-i}(\underline{n} \cdot \underline{e}_i) + \frac{\rho h(\underline{g} - \underline{a}_{CM})}{3} = 0} \tag{10}$$

Note that our original plan was to obtain traction on an arbitrary plane ($\underline{n}$) at a point in terms of traction on other planes at the same point. However, in our tetrahedron example, three planes pass through the point O but the plane with normal $\underline{n}$ (the tilted plane) does not pass through point O. So,we need to shrink the tetrahedron such that in the limiting case, all the four planes pass through O. We can achieve this by letting the perpendicular distance h go to zero. When we apply this limit ($\lim_{h \to 0}$) to equation (10), the terms proportional to h vanish. Thus, in the limit of tetrahedron shrinking to point $\underline{x}$ where all the four planes exist now, the terms corresponding to body force and acceleration vanish and we get the following desired result:

$$\underline{t}^n = -\sum_{i=1}^{3} \underline{t}^{-i}(\underline{n} \cdot \underline{e}_i). \tag{11}$$

3.1 Relation between tractions on planes with opposite normals (start time: 47:00)

By definition, $\underline{t}^{-i}$ is the traction on $-\underline{e}_i$ plane whereas $\underline{t}^i$ is the traction on $\underline{e}_i$ plane. The two tractions basically act on the same plane but with normals pointing in opposite direction. Let us understand it in the context of an internal section in the body from Figure 2. We redraw the two parts so created in Figure 8. The original internal section now forms external surface of the two parts of the body and have outward normals pointing opposite to each other.[1] The tractions on these two planes (which were the same in the original body) will be equal and opposite due to Newton's third law since they form an action-reaction pair. We can thus write

$$\underline{t}^{-i} = -\underline{t}^i. \tag{12}$$

[1]Note that we always consider outward normals.

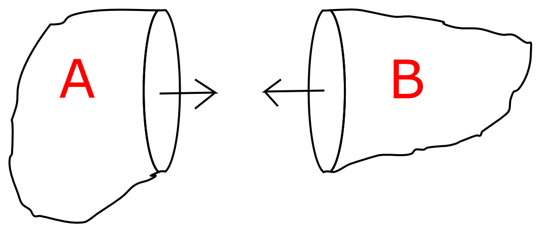

Figure 8: The planes of the two parts of the body at the internal section have outward normals pointing in opposite direction

Using above equation (12) in equation (11), we finally obtain:

$$\boxed{\underline{t}^n = \sum_{i=1}^{3} \underline{t}^i \; (\underline{n} \cdot \underline{e}_i)} \tag{13}$$

We emphasize that the body force and the acceleration terms dropped out from the above formula! Thus, the above formula holds even if the body force is present or the body is accelerating!