APC523 Proposal: Numerical Integration of the Navier Stokes Equation

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1 Introduction

The Navier-Stokes equations are fundamental to fluid dynamics,[MV08] governing fluid motion across both laminar and turbulent flows. They describe the behavior of fluids based on the conservation of mass and momentum. The general form of the Navier-Stokes equations is as follows:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{1}$$

$$\frac{\partial}{\partial t}(\rho\, \mathbf{u}) + \nabla \cdot (\rho\, \mathbf{u} \otimes \mathbf{u}) = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \rho\, \mathbf{f}, \tag{2}$$

where ρ is the fluid density, $\mathbf{u}$ is the velocity vector, p is pressure, $\boldsymbol{\tau}$ represents the viscous stress tensor, and $\mathbf{f}$ denotes external body forces.
This project focuses on solving a **one-dimensional** form of the Navier-Stokes equation under the assumption of **non constant density**. The resulting equation is:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{\partial P}{\partial x \rho} + \nu \frac{\partial^2 u}{\partial x^2}, \tag{3}$$

where, $u(x,t)$ is the velocity in the x-direction,$P(x,t)$ is the pressure,ρ is the constant density, and , and $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity.
Overall, this equation captures the dynamics of viscous fluid flow in 1D, where the velocity depends only on x - direction and time.

2 Motivation

The Navier-Stokes equations serve as a fundamental framework for modeling fluid dynamics in both civil and chemical engineering, playing a critical role in our research. This equation governs the behavior of fluids in a wide range of applications, from hydrology to atmospheric dynamics, enabling a deeper understanding of fluid-driven natural .The Navier-Stokes equation is central to the study of complex systems, where fluid interactions shows nonlinear behaviors, turbulence, and chaotic dynamics. This is particularly relevant in atmospheric sciences, where they govern the motion of air masses, the formation of weather patterns, and large-scale circulation processes. Also, In hurricane modeling, solving the Navier-Stokes equations allows for accurate simulation of cyclonic systems, tracking their development, intensity, and potential impact on coastal regions.
In hydrology, the equations describe surface and subsurface water movement, capturing critical hydrodynamic processes such as infiltration, runoff, and sediment transport. This is essential for predicting flood events, designing sustainable water management strategies, and optimizing the operation of reservoirs and irrigation networks. The study of fluid dynamics more broadly extends to diverse applications in both civil and chemical engineering, including turbulence modeling, multiphase flow systems, and industrial processes.
The Navier-Stokes equations are critical for modeling transport phenomena, including fluid flow, heat transfer, and mass transport in various industrial applications. They are extensively used in chemical reactor engineering, where precise flow control ensures optimal mixing, reaction rates, and temperature regulation.

3 Scope

The primary objectives of this project include evaluating the stability of different numerical schemes, assessing their accuracy in approximating the solution, and determining the computational efficiency of each approach.

- **Stability Analysis:** Evaluate the stability of numerical methods used to solve the one-dimensional Navier-Stokes equation, ensuring that solutions remain bounded and physically meaningful over time.
- **Accuracy Assessment:** Analyze how well different numerical schemes approximate the true solution of the equation, determining their reliability in capturing fluid behavior.
- **Computational Cost Evaluation:** Assess the computational expense associated with each method, and implement an optimal numerical method by balancing accuracy, stability, and computational cost, particularly when addressing stiff problems where numerical precision and efficiency are crucial.

4 Governing Equations in Navier Stoke

The following equations describe the fundamental principles of mass, momentum, and energy conservation, which constitute the governing equations of the Navier-Stokes[Wik24]

Continuity Equation (Mass Conservation)

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0, \tag{4}$$

where $\rho(x,t)$ is the density and $u(x,t)$ is the velocity.

Momentum Equation

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2) = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\left(\mu \frac{\partial u}{\partial x}\right), \tag{5}$$

where $P(x,t)$ is the pressure and $\mu(x,t)$ is the dynamic viscosity.

Energy Conservation Equation

$$\frac{\partial}{\partial t}(\rho E) + \frac{\partial}{\partial x}\left((\rho E + P)u\right) = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \Phi, \tag{6}$$

where $E = e + \frac{1}{2}u^2$ is the total energy per unit mass, k is the thermal conductivity, T is the temperature, and Φ represents viscous dissipation. **The equation of state (EOS)** for an ideal gas is used:

$$P = \rho R T. \tag{7}$$

5 Methodology

The numerical solution of Navier Stoke requires the use of various discretization and time integration techniques to ensure accuracy, stability, and computational efficiency. A **finite volume method (FVM)** is employed due to its conservation properties. A **second-order central difference scheme** will may be utilized to improve numerical accuracy while minimizing dissipation. Moreover, A **second-order finite difference method** may also be incorporated to maintain precision in spatial approximations. For time integration, **Runge-Kutta methods** are applied to enhance stability and accuracy in time evolution. Additionally, the **Courant-Friedrichs-Lewy (CFL) condition** will be used to regulate the time step and ensure numerical stability. This combination of numerical techniques provides a structured and efficient computational framework.

6 Implementation

Dr. Edward Smith's code will serve as a starting point for implementing and evaluating numerical methods for solving the one-dimensional Navier-Stokes equation[Bar15]. Based on the results obtained from the stability and accuracy analysis, the numerical implementation will be modified, and an improved.

References

[Bar15] Lorena A. Barba. Cfd python: 12 steps to navier–stokes, 2015. [Online; accessed DATE].

[MV08] A. Mellet and A. Vasseur. Existence and uniqueness of global strong solutions for one-dimensional compressible navier–stokes equations. *SIAM Journal on Mathematical Analysis*, 39(4):1344–1365, 2008.

[Wik24] Wikipedia contributors. Navier–stokes equations — Wikipedia, The Free Encyclopedia, 2024. [Online; accessed DATE].