Tree Matchings

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Abstract

An (s,t)-matching in a bipartite graph $G = (U, V, E)$ is a subset of the edges F such that each component of $G[F]$ is a tree with at most t edges and each vertex in U has s neighbours in $G[H]$. We give sharp conditions for a bipartite graph to contain an (s,t)-matching. As a special case, we prove a conjecture of Bonacina, Galesi, Huynh and Wollan [1].

1   Introduction

Let $G = (U, V, E)$ be a bipartite graph. A *matching* from U to V is a subset F of pairwise disjoint edges from E such that each vertex from U is incident to an edge in F. For $\alpha > 0$ we will say that G satisfies the α-*neighbourhood condition* if $|\Gamma(S)| \geq \alpha|S|$ for each $S \subset U$. A fundamental result in matching theory is Hall's Theorem.

Theorem 1.1 (Hall's Theorem [2])**.** *Let $G = (U, V, E)$ be a bipartite graph, then G has a matching from U to V iff G satisfies the 1-neighbourhood condition.*

It follows easily from Hall's Theorem that if G satisfies the h-neighbourhood condition then G has an (h, h)-matching, or in other words a collection of vertex disjoint stars $K_{1,h}$ centred on the vertices of U. But what happens if G does not quite satisfy the h-neighbourhood condition? G no longer has an h-matching, but perhaps we can choose h edges incident with each vertex of U so that the resulting graph has only small components.

Definition 1.2. Let $t \geq s$ be positive integers and $G = (U, V, E)$ be a bipartite graph. An (s,t)-matching is a subset F of E such that in $H = (U, V, F)$, each component is a tree with at most t edges, and $d_H(u) = s$ for each $u \in U$.

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A special case of this question was raised in a paper of Bonacina, Galesi, Huynh and Wollan [1]. That paper considered a covering game on a bipartite graph. It turned out that which player wins is strongly linked to the existence of a $(2, 4)$-matching in G. Bonacina, Galesi, Huynh and Wollan showed that for $\epsilon < \frac{1}{23}$ the $(2 - \epsilon)$-neighbourhood condition is sufficient for the existence of a $(2, 4)$-matching in a bipartite graph G with maximal left degree at most 3. They conjectured that the result should hold for $\epsilon = \frac{1}{3}$. In this paper, we prove their conjecture as a special case of a much more general result.

We will give sufficient neighbourhood conditions for the existence of (h, hk)-matchings for general h, k.

Theorem 1.3. *Let $k \geq 1$ and $h \geq 2$ be positive integers and let $G = (U, V, E)$ be a bipartite graph. Suppose that for all $S \subset U$,*

$$|\Gamma(S)| \geq \left(h - 1 + \frac{1}{\lceil k/h \rceil} \right) |S|.$$

Then G has an (h, hk)-matching.

We will actually prove a stronger result which conditions on the maximum left degree of the bipartite graph.

Theorem 1.4. *Let $k \geq 1$ and $d > h \geq 2$ be positive integers and let $G = (U, V, E)$ be a bipartite graph. Suppose that $d(u) \leq d$ for all $u \in U$ and, for all $S \subset U$,*

$$|\Gamma(S)| \geq \left(h - 1 + \frac{d - h + 1}{k + 1 + (d - h - 1)\lceil k/h \rceil} \right) |S|.$$

Then G has an (h, hk)-matching.

By taking the limit as d tends to infinity, one can see that Theorem 1.3 follows directly from Theorem 1.4. Taking $h, k = 2$ and $d = 3$, we also see that Theorem 1.4 proves the conjecture of Bonacina, Galesi, Huynh and Wollan [1] mentioned above.

Showing these α-bounds is a little tricky; unlike the case of Hall's Theorem, the conditions in Theorem 1.4 are sufficient but not necessary (for example, $K_{2,3}$ contains a $(2, 4)$-matching but does not satisfy the $\frac{5}{3}$-neighbourhood condition); and it is necessary to provide an infinite family of examples for α increasing to the relevant threshold as the example increases in size. Bonacina, Galesi, Huynh and Wollan [1] provide a example to show that for any $\alpha < \frac{5}{3}$, there exists a bipartite graph G with maximal left degree 3 which satisfies the α-neighbourhood condition but does not contain a $(2, 4)$-matching. We will modify this particular family of examples to give examples for all values of d, k and h. These examples show that the sufficient neighbourhood conditions given in Theorem 1.4 are in fact optimal.

Proposition 1.5. *Let $k \geq 2$ and $d > h \geq 1$ be positive numbers and $\alpha < h - 1 + \frac{d - h + 1}{k + 1 + (d - h - 1)\lceil k/h \rceil}$. Then there exists a bipartite graph G with maximum left degree at most d which satisfies the α-neighbourhood condition but does not contain an (h, hk)-matching.*

The paper is organised as follows. In Section 2 we prove some preliminary results regarding bipartite graphs that satisfy a neighbourhood condition which is no longer satisfied upon the deletion of any edge. In Section 3 we prove Theorem 1.4. In Section 4 we expose the examples which prove Proposition 1.5 and so demonstrate the bounds given in Theorem 1.4 are tight. Finally, in Section 5, we consider a related covering problem.

2   Preliminary results

As stated before, we will prove Theorem 1.4 by induction on the number of edges in the graph. For a bipartite graph $G = (U, V, E)$, we will call an edge, $e \in E$, α-redundant if $G - e$ satisfies the α-neighbourhood condition. In other words, an edge is redundant if it is not necessary for the satisfaction of neighbourhood constraints. This section will show that if a connected bipartite graph satisfies the α-neighbourhood condition and has no redundant edge, then (subject to a couple of other restraints) it must be a tree. We will start the section by introducing some notation which will be used throughout the paper.

Let $G = (U, V, E)$ be a bipartite graph. For $A \subset U$ and $\alpha > 0$, let $h(A, \alpha) = |\Gamma(A)| - \alpha|A|$ (so $G = (U, V, E)$ satisfies the α-neighbourhood condition if and only if $h(A, \alpha) \geq 0$ for each $A \subset U$). Then for $uv \in E$, $u \in U, v \in V$, let $F_{uv} = \{A \subset U : u \in A, v \notin \Gamma(A \setminus u)\}$ and $G_{uv} = \{A \subset U \setminus u : v \in \Gamma(A)\}$. Then we define functions f and g:

$$
\begin{aligned}
f(uv, \alpha) &= \min_{A \in F_{uv}} h(A, \alpha) \\
g(uv, \alpha) &= \min_{A \in G_{uv}} h(A, \alpha)
\end{aligned}
$$

where we put $g(uv, \alpha) = 1$ if $d(v) = 1$ (and so $G_{uv} = \emptyset$). We will drop the α when obvious or when it's value is inconsequential.

$f(uv)$ can be thought of as a measure of how redundant an edge uv is and $g(uv)$ can be thought of as a measure of how redundant the vertex v is to the graph $G - u$; in other words, how little is it required by other vertices. For a graph satisfying the α-neighbourhood condition it is clear that $f(uv, \alpha), g(uv, \alpha) \geq 0$ for each $uv \in E$. The next proposition analyses some properties of f and g on a graph satisfying the α-neighbourhood condition.

Proposition 2.1. *Let $\alpha > 0$ and $G = (U, V, E)$ be a bipartite graph which satisfies the α-neighbourhood condition.*

   (i) An edge uv is α-redundant if and only if $f(uv, \alpha) \geq 1$.

   (ii) For $v \in V$, $u, w \in \Gamma(v)$, $F_{wv} \subset Guv$ and so $g(uv, \alpha) \leq f(wv, \alpha)$.

   (iii) Suppose further that G does not contain a redundant edge. For an edge $uv \in E$, $g(uv, \alpha) \leq 1$ with equality if and only if $d(v) = 1$.

Proof. Let $\alpha > 0$ and $G = (U, V, E)$ be a bipartite graph which satisfies the α-neighbourhood condition. Suppose $uv \in E$ and let $H = G - uv$.

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(i) First suppose that $f(uv, \alpha) < 1$ and let $A \in F_{uv}$ be such that $h_G(A) = f(uv, \alpha)$. Note that by definition of F_{uv}, $\Gamma_H(A) = \Gamma_G(A) \setminus v$ and so $h_H(A) = h_G(A) - 1 < 0$. We can then conclude that uv is not redundant since H does not satisfy the α-neighbourhood condition.

Now suppose that uv is not redundant. By definition, there must be some $A \subset U$ such that $h_H(A) < 0$. Note that since G satisfies the α-neighbourhood condition, such a subset A must contain u and that $v \notin \Gamma(A \setminus u)$. It is then clear that $A \in F_{uv}$ and so $f(uv, \alpha) \leq h_G(A) = h_H(A) + 1 < 1$.

(ii) Note that if $S \in F_{wv}$, then $S \subset U \setminus u$ and $v \in \Gamma(S)$ and so $S \in G_{uv}$. It follows that if $u, w \in \Gamma(v)$, then $F_{wv} \subset G_{uv}$ and so $g(uv, \alpha) \leq f(wv, \alpha)$.

(iii) Recall that if $d(v) = 1$, then $g(uv, \alpha) = 1$ by definition. So suppose that $d(v) \geq 2$ and pick some $w \in \Gamma(v) \setminus u$. Then since vw is not a redundant edge, $f(wv, \alpha) < 1$ and the results follows from (ii).

$\square$

The following lemma considers the effect of applying h to a union of two sets and will be used extensively in the remainder of the section.

Lemma 2.2. *Let* $G = (U, V, E)$ *be a bipartite graph and fix some* $\alpha > 0$. *Then for* $A, B \subset U$,

$$h(A \cup B) = h(A) + h(B) - h(A \cap B) - (|\Gamma(A) \cap \Gamma(B)| - |\Gamma(A \cap B)|).$$

Proof. Note that $|\Gamma(A \cup B)| = |\Gamma(A)| + |\Gamma(B)| - |\Gamma(A) \cap \Gamma(B)|$ and $|A \cup B| = |A| + |B| - |A \cap B|$, so

$$
\begin{aligned}
h(A \cup B) &= |\Gamma(A \cup B)| - \alpha|A \cup B| \\
&= |\Gamma(A)| + |\Gamma(B)| - |\Gamma(A) \cap \Gamma(B)| - \alpha(|A| + |B| - |A \cap B|) \\
&= h(A) + h(B) - (|\Gamma(A) \cap \Gamma(B)| - \alpha|A \cap B|) \\
&= h(A) + h(B) - (|\Gamma(A) \cap \Gamma(B)| - |\Gamma(A \cap B)|) - (|\Gamma(A \cap B)| - \alpha|A \cap B|) \\
&= h(A) + h(B) - h(A \cap B) - (|\Gamma(A) \cap \Gamma(B)| - |\Gamma(A \cap B)|).
\end{aligned}
$$

$\square$

We are now in a position to show that, under additional constraints regarding h, a bipartite graph satisfying the α-neighbourhood condition with no redundant edges, must be a tree.

Lemma 2.3. *Let* $\alpha > 0$ *and let* $G = (U, V, E)$ *be a bipartite graph with no isolated vertices. Suppose that* $h(U, \alpha) \in [0, 1)$, $h(S, \alpha) > h(U, \alpha)$ *for each* $S \subsetneq U$, *and that* G *contains no* α-redundant edges. *Then* G *is a tree.*

Proof. Since α is fixed, we will write $h(S)$, $f(uv)$ and $g(uv)$ in place of $h(S, \alpha)$, $f(uv, \alpha)$ and $g(uv, \alpha)$ respectively.

First suppose that G is not connected. Let $A \cup \Gamma(A)$ be the vertex set of a component of G with $A \subset U$ and let $B = U \setminus A$. As $\Gamma(A)$ and $\Gamma(B)$ are disjoint we have $h(U) = h(A) + h(B)$ by Lemma 2.2. Note that $h(B) > h(U) > 0$ by assumption and so $h(U) > h(A) + h(U)$. We have arrived at a contradiction since $h(A) > 0$. So G is connected.

Now suppose that G contains a cycle. Choose an edge uv that belongs to a cycle which has $g(uv)$ as small as possible. Choose $A \in F_{uv}$ and $B \in G_{uv}$ such that $h(A) = f(uv)$ and $h(B) = g(uv)$.

Suppose that $G[A \cup \Gamma(A)]$ is disconnected and $J \subset A$ is such that $G[J \cup \Gamma(J)]$ is a component of $G[A \cup \Gamma(A)]$. $\Gamma(J)$ and $\Gamma(A \setminus J)$ are disjoint, and so $h(A) = h(J) + h(J \setminus A)$. Note that since both J and $J \setminus A$ are non-trivial subsets of U, we have $\min\{h(J), h(J \setminus A)\} > 0$ and so $h(A) > \max\{h(J), h(J \setminus A)\}$. On the other hand, assuming without loss of generality that $u \in J$, we have that $J \in F_{uv}$ and so $h(J) \geq h(A)$, a contradiction. Similarly, suppose that $G[B \cup \Gamma(B)]$ is disconnected and that $K \subset B$ is such that $G[K \cup \Gamma(K)]$ is a component of $G[B \cup \Gamma(B)]$. $\Gamma(K)$ and $\Gamma(B \setminus K)$ are disjoint and so $h(B) = h(K) + h(B \setminus K)$. Since K and $B \setminus K$ are non-trivial subset of U, we have $\min\{h(K), h(B \setminus K)\} > 0$ and so $\max\{h(K), h(B \setminus K)\} < h(B)$. On the other hand, assuming without loss of generality that $v \in \Gamma(K)$, we see that $K \in G_{uv}$ and so $h(K) \geq h(B)$, a contradiction. Thus we may assume that both $G[A \cup \Gamma(A)]$ and $G[B \cup \Gamma(B)]$ are connected.

Letting $C = A \cup B$ and $D = A \cap B$, an application of Lemma 2.2 gives

$$\begin{aligned} h(C) &= h(A) + h(B) - h(D) - (|\Gamma(A) \cap \Gamma(B)| - |\Gamma(A \cap B)|) \\ &= f(uv) + g(uv) - h(D) - (|\Gamma(A) \cap \Gamma(B)| - |\Gamma(A \cap B)|). \end{aligned} \tag{2.1}$$

Note that G satisfies the α-neighbourhood condition, has no redundant edges, and $d(v) \geq 2$. The conditions for Proposition 2.1 are therefore satisfied and so $f(uv), g(uv) < 1$. If $|\Gamma(A) \cap \Gamma(B)| - |\Gamma(A \cap B)| \geq 2$, then $h(C) < 0$ and we arrive at a contradiction. Therefore $|\Gamma(A) \cap \Gamma(B)| - |\Gamma(A \cap B)| \leq 1$ and so

$$\Gamma(A) \cap \Gamma(B) = \Gamma(A \cap B) \cup \{v\}, \tag{2.2}$$

as $v \in \Gamma(A) \cap \Gamma(B)$ but $v \notin \Gamma(A \cap B)$. In particular, $|\Gamma(A) \cap \Gamma(B)| - |\Gamma(A \cap B)| = 1$. Putting this into (2.1) gives

$$h(C) = f(uv) + g(uv) - h(D) - 1. \tag{2.3}$$

Now suppose that $D \neq \emptyset$ and choose some vertex $w \in D$. Since $G[A \cup \Gamma(A)]$ and $G[B \cup \Gamma(B)]$ are both connected, there exists a $v - w$ path, $v = a_1 \cdots a_r = w$ in $G[A \cup \Gamma(A)]$ and a $v - w$ path $v = b_1 \cdots b_t = w$ in $G[B \cup \Gamma(B)]$. Note that $A \in F_{uv}$ and so $A \cap \Gamma(v) = \{u\}$, forcing $a_2 = u$. This means that $a_2 \neq b_2$ and so the two $v - w$ paths are distinct. Let $i > 1$ be minimal such that $a_i = b_j$ for some $j > 1$ and fix j minimal with $b_j = a_i$. Note then that $a_1, a_2, \ldots, a_i, b_2, b_3 \ldots b_{j-1}$ are distinct vertices and so $a_1 a_2 \cdots a_i b_{j-1} b_{j-2} \cdots b_2$ is a cycle in $G[C \cup \Gamma(C)]$.

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Note that either $a_i \in A \cap B = D$ or $a_i \in \Gamma(A) \cap \Gamma(B)$. If the latter is the case, then $a_i \in \Gamma(D)$ since $a_i \neq v$, and, by (2.2), $\Gamma(A) \cap \Gamma(B) = \Gamma(D) \cup \{v\}$. In either case, $a_i \in D \cup \Gamma(D)$. Then since $a_2 = u \notin D \cup \Gamma(D)$, there must be some $s < i$ such that $a_s \notin D \cup \Gamma(D)$, but $a_{s+1} \in D \cup \Gamma(D)$. If $a_{s+1} \in D$, then $a_s \in \Gamma(D)$, which gives a contradiction. So $a_s \in U \setminus D$ and $a_{s+1} \in \Gamma(D)$. This means that $D \in G_{a_s a_{s+1}}$ and so $h(D) \geq g(a_s a_{s+1})$. Finally, since $a_s a_{s+1}$ is an edge in a cycle and we have chosen uv to minimise $g(uv)$, it must be that $g(a_s a_{s+1}) \geq g(uv)$. If we put this inequality into (2.3) we get

$$
\begin{aligned}
h(C) &\leq f(uv) + g(uv) - g(uv) - 1 \\
&= f(uv) - 1.
\end{aligned}
\tag{2.4}
$$

Since $f(uv) < 1$, we have that $h(C) < 0$, a contradiction. It must therefore be the case that $D = \emptyset$.

Now suppose uv is in some cycle $F = u_1 v_1 \ldots u_m v_m$ with $(u_1, v_m) = (u, v)$ and let $Q = \{u_1, \ldots, u_m\}$. Note that $u_m \notin A$ and $u_1 \notin B$. So if $Q \subset C = A \cup B$, then there exists some $j \neq m$ with $u_j \in A$, $u_{j+1} \in B$. It is then the case $v_j \in \Gamma(A) \cap \Gamma(B) = \{v\}$, which gives a contradiction. So there exists some j such that $u_{j+1} \notin C$ but $u_j \in C$. This means that $C \in G_{u_{j+1} v_j}$ and so $h(C) \geq g(u_{j+1} v_j) \geq g(uv)$. Since $D = \emptyset$ and $\Gamma(A) \cap \Gamma(B) = \{v\}$, (2.3) gives

$$
\begin{aligned}
h(C) &= f(uv) + g(uv) - 0 - 1 \\
&< g(uv).
\end{aligned}
$$

But this again gives a contradiction since then $g(u_{j+1} v_j) < g(uv)$. It follows that G cannot contain a cycle and so must be a tree. $\square$

3   Proof of Theorem 1.4

We now come to proving Theorem 1.4. We will prove this by considering an edge-minimal counterexample and arriving at a contradiction. The first half of the proof will show that this counterexample must be a tree and thus acyclic. The second half will show that the counterexample, at the same time, must contain a cycle.

Proof of Theorem 1.4. Let $k \geq 1$ and $d > h \geq 2$ be positive integers, fix $r = d - h$ and let

$$
\alpha = h - 1 + \frac{r + 1}{k + 1 + (r - 1)\lceil k/h \rceil}.
$$

Since α is fixed, we will write $h(S)$, $f(uv)$ and $g(uv)$ in place of $h(S, \alpha)$, $f(uv, \alpha)$ and $g(uv, \alpha)$ respectively.

Suppose that there exists a bipartite graph with maximum left degree at most $h + r$ which satisfies the α-neighbourhood condition but doesn't have an (h, hk)-matching. Let $G = (U, V, E)$ be edge-minimal with these properties and assume without loss of generality that there are no isolated vertices in G. Note that G has minimum left degree at least h

since $h(\{u\}) \geq 0$ for each $u \in U$. Suppose that there exists some non-empty $A \subsetneq U$ such that $h(A) \leq h(U)$ and suppose that A is such a set with minimal $h(A)$. It is clear that the subgraph $H = G[A \cup \Gamma(A)]$ satisfies the α-neighbourhood condition and has fewer edges than G and so by assumption must have an (h, hk)-matching. If $B \subset U \setminus A$, then

$$
\begin{aligned}
|\Gamma_{G-(A \cup \Gamma(A))}(B)| - \alpha|B| &\geq (|\Gamma_G(B \cup A)| - \alpha|B \cup A|) - (|\Gamma_G(A)| - \alpha|A|) \\
&= h(A \cup B) - h(A) \\
&\geq 0.
\end{aligned}
$$

Thus $G - (A \cup \Gamma(A))$ satisfies the α-neighbourhood condition and so by assumption must contain an (h, hk)-matching. The two (h, hk)-matchings are vertex-disjoint and so their union is an (h, hk)-matching in G. This gives a contradiction and so we must have $h(A) > h(U)$ for each non-empty $A \subsetneq U$.

If there exists an α-redundant edge uv in G (equivalently $f(uv) \geq 1$), then we may simply delete it to find a counter-example with fewer edges which contradicts the minimality of G. So $f(uv) \in [0, 1)$ for each edge uv in E. If we pick some edge uv, we have that $f(uv) < 1$ and so there must be some $A \in F_{uv}$ with $h(A) < 1$. Then since $h(U) \leq h(A)$ for non-empty $A \subset U$, it must be the case that $h(U) < 1$. We have now shown that G satisfies all the conditions for Lemma 2.3 and so G must be a tree.

For each positive integer i, let $U_i = \{u \in U : d(u) = i\}$ and $V_i = \{v \in V : d(v) = i\}$. Then let $F = \{u \in U_h : |\Gamma(u) \cap V_1| = h - 1\}$ and $Z = U_h \setminus F$. Suppose that $C \cup \Gamma(C)$ is a component of $G[F \cup \Gamma(F)]$. For each $u \in C$, let $X_u = \Gamma(u) \cap V_1$ and $Y_u = \Gamma(u) \setminus V_1$. Note that by pruning the leaves of $G[C \cup \Gamma(C)]$ contained in V_1, we get $G[C \cup \bigcup_{u \in C} Y_u]$. We can then see that $G[C \cup \bigcup_{u \in C} Y_u]$ is a tree and so $e(G[C \cup \bigcup_{u \in C} Y_u]) = |C \cup \bigcup_{u \in C} Y_u| - 1 = |C| + |\bigcup_{u \in C} Y_u| - 1$. On the other hand $|Y_u| = 1$ for each $u \in C$ and so $e(G[C \cup \bigcup_{u \in C} Y_u]) = |C|$. Comparing these two expressions we see that $|\bigcup_{u \in C} Y_u| = 1$.

G is connected and so $\Gamma(C)$ and $\Gamma(U \setminus C)$ must have a non-empty intersection. Note however that $\Gamma(U \setminus C) \cap \Gamma(C) \subset \bigcup_{u \in C} Y_u$ since all vertices in $\bigcup_{u \in C} X_u$ are leaves. Therefore each component of $G[F \cup \Gamma(F)]$ has exactly one vertex in $\Gamma(U \setminus F)$. In this case, we will say that F satisfies the *critical link property*.

The following algorithm adds vertices from Z to F as long as it is possible to do so under the constraint that F must always satisfy the critical link property.

Initialization *Set* $\eta = \emptyset$;
while $Z \neq \emptyset$ **do**
    Pick $u \in Z$;
    if $|\Gamma(u) \cap \Gamma(U \setminus (F \cup \{u\}))| = 1$ **then**
        set $Z = \eta \cup (Z \setminus \{u\})$, $F = F \cup \{u\}$ and $\eta = \emptyset$;
    else
        Set $Z = Z \setminus \{u\}$ and $\eta = \eta \cup \{u\}$;
    end
end

We claim that after each iteration of the loop, F still satisfies the critical link property. This is true initially and can only be changed in the loop, if we add a vertex to F. Suppose u is added to F at a certain stage and let $C \cup \Gamma(C)$ be the component of $(F \cup \{u\}) \cup \Gamma(F \cup \{u\})$ containing u. Note that all other components of F will remain unchanged and so we only have to consider $C \cup \Gamma(C)$. Let $B = C \setminus \{u\}$ and note that $B \cup \Gamma(B)$ is the collection of components which are joined together by the addition of u to F. Let $R = \Gamma(B) \cap \Gamma(U \setminus B)$ be the set of vertices in V which connect the components of $B \cup \Gamma(B)$ to the rest of G and note that by assumption R must be a subset of the neighbourhood of u. Note that since we have added u to F, it must be the case that $|\Gamma(u) \cap \Gamma(U \setminus (F \cup \{u\}))| = 1$. Further note that $\Gamma(C) \cap \Gamma(U \setminus (F \cup \{u\}))$ is a subset of $\Gamma(u)$ by the above argument and so $|\Gamma(C) \cap \Gamma(U \setminus C)| = 1$ as required.

So let us suppose we have augmented F as far as we can by running the algorithm described above (so we have a subset $F \subset U_h$ such that $G[F \cup \Gamma(F)]$ is a forest which satisfies the critical link property and further that we cannot maintain this property if we add any vertex from U_h). Note that since each component of $G[F \cup \Gamma(F)]$ has exactly one vertex in the neighbourhood of $U \setminus F$, we know that $U \setminus F$ cannot be the empty set. So let $W = \{v \in V : |\Gamma(v) \setminus F| \geq 2\}$ be the subset of V with at least two neighbours in $U \setminus F$. We claim that each $u \in U \setminus F$ has at least two neighbours in W which will in turn mean that $G[(U \setminus F) \cup W]$ is a subgraph of G with minimum degree at least 2. We will then have arrived at a contradiction since this subgraph of the tree G must then contain a cycle.

So pick $u \in U \setminus F$ and first suppose that $d(u) = h$. Since we have not added u to F whilst running the algorithm, either $\Gamma(u) \cap \Gamma(U \setminus (F \setminus \{u\})) = \emptyset$ or $|\Gamma(u) \cap \Gamma(U \setminus (F \cup \{u\}))| \geq 2$. In the latter case, note that $\Gamma(u) \cap \Gamma(U \setminus (F \cup \{u\}) \subset W$ and so $|\Gamma(u) \cap W| \geq 2$. In the former case, let $F^+ = F \cup \{u\}$ and consider the component, $\mathcal{C} = Q \cup \Gamma(Q)$, of $G[F^+ \cup \Gamma(F^+)]$ with $u \in Q \subset F^+$. Note that $\Gamma(Q) \cap \Gamma(U \setminus F^+) = \Gamma(u) \cap \Gamma(F^+)$ since u must be a neighbour of each vertex in $\Gamma(Q \setminus \{u\}) \cap \Gamma(U \setminus (Q \setminus \{u\}))$ and so $Q \cup \Gamma(Q)$ must be disconnected from the rest of the graph. Since G is connected, it must then be the case that $Q = U$. Recall that G is a bipartite tree. Counting edges two ways, we see that $h|U| = |U| + |V| - 1$ and so $|V| = (h - 1 + \frac{1}{|U|})|U|$. On the other hand, recall that G satisfies the α-neighbourhood condition and so $|V| \geq \alpha|U|$. This in turn forces $(h - 1 + \frac{1}{|U|}) \geq \alpha$. We can then bound the size of U:

$$
\begin{aligned}
|U| &\leq \frac{k + 1 + (r - 1)\lceil k/h \rceil}{r + 1} \\
&< \frac{k + 1}{2} + \left\lceil \frac{k}{h} \right\rceil \\
&\leq \frac{k + 1}{2} + \frac{k + 1}{2}.
\end{aligned}
$$

It must therfore be the case that $|U| \leq k$. Now we have a contradiction since G is already an (h, hk)-matching. Therefore any vertex $u \in U \setminus F$ with $d(u) = h$ has at least two neighbours in W.

Now suppose we have picked some $u \in U \setminus F$ with $d(u) \geq h + 1$ but $|\Gamma(u) \cap W| < 2$. First consider what happens with $|\Gamma(u) \cap W| = 0$. Let, $\mathcal{C} = Q \cup \Gamma(Q)$, be the component

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of $G[F \cup \{u\} \cup \Gamma(F \cup \{u\})]$ with $u \in Q \subset F$. As argued before, it must be the case that $\Gamma(Q \setminus \{u\}) \cap \Gamma(U \setminus (Q \setminus \{u\}))$ is a subset of $\Gamma(u)$. But note that $\Gamma(u) \cap \Gamma(U \setminus Q) = \Gamma(u) \cap W = \emptyset$ and so $\mathcal{C}$ must be the vertex set of a component in G. Since G is connected, it must then be the case that $U = Q$. As in the case when $d(u) = h$, we can now count edges two ways to realise $h(|U| - 1) + d(u) = |U| + |V| - 1$ and so $|V| = (h - 1 + \frac{d(u)+1-h}{|U|})|U|$. Again, we recall that G satisfies the α-neighbourhood condition and so $h - 1 + \frac{d(u)+1-h}{|U|} \geq \alpha$. We can bound the size of U:

$$|U| \;\leq\; \frac{(d(u) + 1 - h)(k + 1 + (r - 1)\lceil k/h \rceil)}{r + 1}. \tag{3.1}$$

On the other hand, order the vertices of $\Gamma(u) = \{v_1, \ldots v_{d(u)}\}$ such that if $i < j$, then in $G - u$ the size of the component containing v_i is at most the size of the component containing v_j (alternatively, consider G as a tree with root u and order the branches by increasing size). If the h shortest branches collectively contain at most $k - 1$ vertices in U, then we can construct a matching, simply by cutting the edges $uv_{h+1}, uv_{h+2}, \ldots uv_{d(h)}$. Therefore the union of the smallest h branches contain at least k vertices from U. It must also be the case that all other branches contain at least $\lceil \frac{k}{h} \rceil$ left vertices. We now bound the size of U by counting u, the vertices in the h smallest branches, and the vertices in other branches:

$$|U| \geq 1 + k + (d(u) - h)\lceil k/h \rceil. \tag{3.2}$$

After some algebra, we can reformulate (3.2) to get

$$\begin{aligned} |U| \;\geq\; & \frac{(d(u) + 1 - h)(k + 1 + (r - 1)\lceil k/h \rceil)}{r + 1} \\ & + \frac{(r + h - d(u))(k + 1 - 2\lceil k/h \rceil) + (r + 1)\lceil k/h \rceil}{r + 1}. \end{aligned} \tag{3.3}$$

Since $k + 1 - 2\lceil \frac{k}{h} \rceil \geq 0$, we see that the second term in (3.3) is positive and so our lower bound for U here is strictly larger than the upper bound we have at (3.1). So we have a contradiction and so it cannot be the case that $|\Gamma(u) \cap W| = 0$.

All that remains is to consider the case that $u \in U \setminus F$ is such that $d(u) \geq h + 1$ and $|\Gamma(u) \cap W| = 1$. Suppose that $\Gamma(u) \cap W = \{w\}$ and let $Y = F \cup \{u\}$, $Z = \Gamma(Y) \setminus \{w\}$. Further let $A \cup B$ be the component of $G[Y \cup Z]$ such that $u \in A \subset Y$. Recall that $d(w) \geq 2$ by assumption and so there must exist some $u' \in U \setminus Y$. It is then the case that $A \neq U$, so that $h(A) > 0$ and $|B| > \alpha|A| - 1$. Note that $G \setminus (A \cup B)$ will still satisfy the α-neighbourhood condition and so by assumption, must contain an (h, hk)-matching. Let $H = G[A \cup B]$. Then if H contains an (h, hk)-matching, it is independent of any (h, hk)-matching in $G \setminus (A \cup B)$ and their union is an (h, hk)-matching in G. Therefore H does not contain an (h, hk)-matching. As in the previous case, we will now bound $|A|$ above and below to reach a contradiction. Firstly, since H is a tree and we know the degrees of all the vertices in A, we can count the number of edges two ways to get that $h(|A| - 1) + d(u) - 1 = |A| + |B| - 1$ and so

$|B| = (h - 1 + \frac{d(u)-h}{|A|})|A|$. Using the fact that $|B| > \alpha|A| - 1$, we get an upper bound for $|A|$:

$$|A| < \frac{(d(u) - h + 1)(k + 1 + (r - 1)\lceil k/h \rceil)}{r + 1}. \tag{3.4}$$

On the other hand, order the vertices of $\Gamma(u) \setminus \{w\} = \{v_1, \ldots v_{d(u)}\}$ such that if $i < j$, then in $H - u$ the size of the component containing v_i is at most the size of the component containing v_j. Since we cannot have a matching, the h smallest branches must collectively have at least k vertices of A in them and the $(d(u) - 1 - h)$ other branches must each contain at least $\lceil \frac{k}{h} \rceil$ vertices of A. Therefore we get a lower bound for $|A|$:

$$
\begin{aligned}
|A| &\geq 1 + k + (d(u) - 1 - h)\lceil k/h \rceil \\
&= \frac{(r + 1)(1 + k + (d(u) - 1 - h)\lceil k/h \rceil)}{r + 1} \\
&= \frac{(r + 1)(k + 1 + (r - 1)\lceil k/h \rceil) - (r + h - d(u))(r + 1)\lceil k/h \rceil}{r + 1} \\
&= \frac{(d(u) - h + 1)(k + 1 + (r - 1)\lceil k/h \rceil)}{r + 1} \\
&\quad + \frac{(r + h - d(u))(k + 1 - 2\lceil k/h \rceil)}{r + 1}. \tag{3.5}
\end{aligned}
$$

Again, since $k + 1 - 2\lceil \frac{k}{h} \rceil \geq 0$, we see that our lower bound for $|A|$ at (3.5) is at least the strict upper bound given at (3.4). This is a contradiction and so it must be the case that $|\Gamma(u) \cap W| \geq 2$.

We have now shown that $G[(U \setminus F) \cup W]$ is a graph with at least one vertex and minimum degree at least 2. Therefore G must contain a cycle, contradicting that G is a tree and so acyclic. So we can finally conclude that there can be no such counterexample and so the result holds. $\square$

4 Optimality

In this section we give examples to show that the bounds given in Theorem 1.4 are tight. Much of the material in this section builds on the work given in the paper of Bonacina, Galesi, Huynh and Wollan [1] (this is very clear for the case $k = h$). For ease of notation, for a bipartite graph $G = (U, V, E)$ and a set $S \subset U$, we let $R_G(S) = \frac{|\Gamma(S)|}{|S|}$.

Rather than drawing the bipartite graph $G = (U, V, E)$, we will give pictorial representations of the hypergraph $H = (V, F)$ where $F = \{\Gamma(u) : u \in U\}$. So H is the hypergraph on the right vertices V of G, where each edge is the neighbourhood of a left vertex of G. Throughout the section, an ellipse represents the neighbourhood of a vertex in U (i.e. a hyperedge of H), a small circle represents a single vertex in V, and a rectangle with a number x inside represents a collection of x vertices in V. For all the figures that follow, we will

assume that parameters a, b, h, q and r are given. We give a toy example below where Figure 2 is the hypergraph representation of Figure 1:
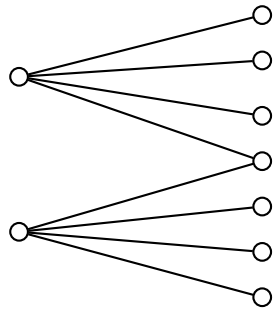

Figure 1

Figure 2

To make the graph representations more digestible, we will use a hexagon so that Figures 3 and 4 represent the same graphs, which we will call I_q. Thus I_q is a chain of q hyperedges $A_1, \ldots, A_q$ each containing h vertices such that A_i and A_{i+1} overlap in one vertex for each $i \leq q - 1$ and the A_i are disjoint otherwise. Another way of thinking of I_q is to start with a path consisting of q left vertices and $q + 1$ right vertices, adding another $h - 2$ distinct leaf-neighbours to each left vertex in the path and then taking the hyperedge representation.
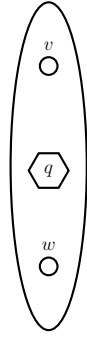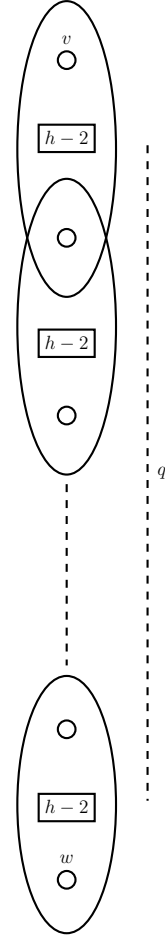
Figure 3

Figure 4

Given our representation of the graph I_q, we will use a star so that Figures 5 and 6 represent the same graph and a triangle so that Figures 7 and 8 represent the same graphs.
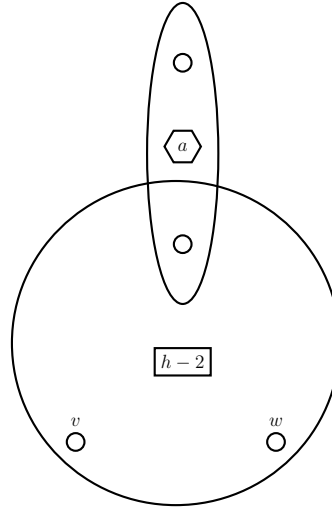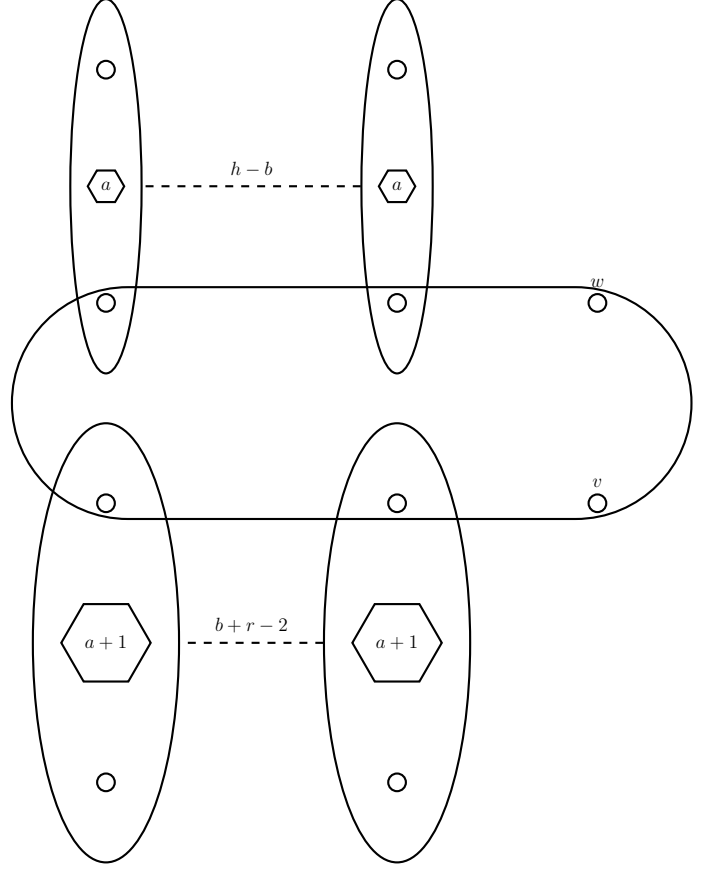

Figure 5

Figure 6

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Figure 7

Figure 8

Given these new pieces of notation, we are now in a position to prove Proposition 1.5

Proof of Proposition 1.5. Let $k \geq 2$ and $d > h \geq 2$, fix $r = d-h$ and suppose that $a \in \mathbb{N}$ and $b \in [h]$ are such that $k = ah+b$. We will have to construct a sequence of bipartite graphs G_n each satisfying the α_n-neighbourhood condition with no (h, hk)-matching where α_n tends to $\alpha = h-1+\frac{r+1}{k+1+(r-1)\lceil k/h \rceil} = h-1+\frac{r+1}{k+1+(r-1)(a+1)}$. We will do this starting with a small graph H which does not contain a (h, hk)-matching and then replacing a copy of I_{a+1} connected to the rest of H through v_1 with a large graph J_n in which in every (h, hk)-matching, v_1 is in a component with at least $h(a+1)$ edges. We give the base graph $H = (U, V, E)$ below.
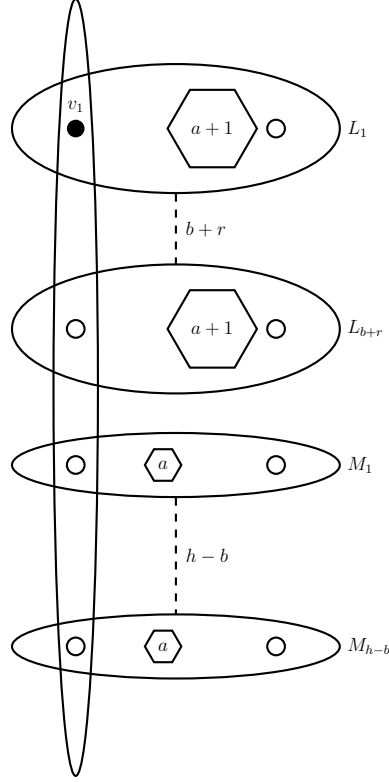
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Figure 9: Base Graph H

The augmenting gadget J_n can be thought of an odd cycle where the edges are replaced with copies of the graphs given in figures 5 and 7 alternately with two "star" edges next to each other.
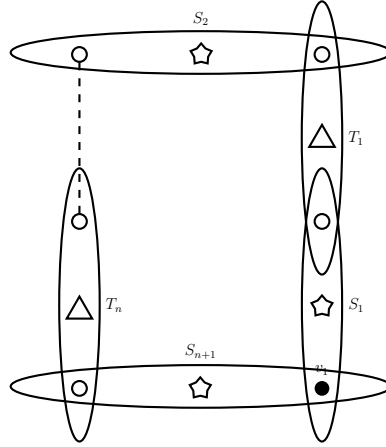

Figure 10: Augmenting Gadget J_n

To form G_n, we first remove $L_1 \setminus \{v_1\}$ from H to give H'. We will then identify the vertices labelled v_1 in H' and the augmenting gadget J_n to form $G_n = (U_n, V_n, E_n)$. To prove our proposition, it suffices to show that G_n does not contain a (h, hk)-matching and that it satisfies the α_n-neighbourhood condition where α_n increases to α as n tends to infinity.

So suppose that G_n contains a (h, hk)-matching. This induces a (h, hk)-matching on J_n.

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One can verify that in any (h, hk)-matching on J_n that v_1 is in a component with at least $(a+1)h$ edges. Then the matching on G_n must induce a (h, hk)-matching on H' where v is a component with at most $(k-(a+1))h$ edges. This is a contradiction though, since we could extend this to a (h, hk)-matching on H by adding back L_1. So G_n cannot contain a (h, hk)-matching.

It is also easy to verify that $R_{G_n}(S)$ is minimised over $S \subseteq U_n$ when U_n. G_n then satisfies the $R_{G_n}(U_n)$-neighbourhood condition where

$$
\begin{aligned}
R_{G_n}(U_n) &= \frac{|V_n|}{|U_n|} \\
&= \frac{|V| - (a+1)(h-1) + |B_n| - 1}{|U| - (a+1) + |A_n|}.
\end{aligned}
$$

After some simplification this becomes

$$
\begin{aligned}
R_{G_n}(U_n) &= \frac{(h-1)(a(h+r)+b+r+1)+r+1+n(r+1)+n(a(h+r-1)+b+r)}{1+a(h+r)+b+r+n(a(h+r-1)+b+r)} \\
&= h-1+\frac{n(r+1)+r+1}{1+a(h+r)+b+r+n(a(h+r-1)+b+r)} \\
&= h-1+\frac{(n+1)(r+1)}{(n+1)(ah+b+ar-a+r)+1} \\
&= h-1+\frac{(n+1)(r+1)}{(n+1)(k+1+(a+1)(r-1))+1}.
\end{aligned}
$$

We now see that $R_{G_n}(U_n)$ tends to α as n tends to infinity and so we are done.

$\square$

5 k-star covering

A naturally related problem is the following: Under what conditions can you cover a graph with trees of bounded size?. It is clear that only stars will be necessary since, for any tree with diameter at least three contains an edge between two non-leaf vertices which we may delete.

Definition 5.1. Let $k \geq 1$ be a positive integer and $G = (U, V, E)$ be a bipartite graph. A *k-star covering* is a subset F of E such that in $H = (U, V, F)$, each component is a star with at most k edges in it, and $d_H(x) \geq 1$, for each $x \in U \cup V$.

The following result gives a necessary and sufficient condition for a bipartite graph to have a k-star covering.

Theorem 5.2. *Let $G = (U, V, E)$ be a bipartite graph. Then G has a k-star covering iff $|\Gamma(S)| \geq \frac{1}{k}|S|$ for all $S \subset U$ and $S \subset V$.*

Note that we require all vertices to be covered in a star covering and so the above is not equivalent to the existence of a $(1, k)$-matching (which may only cover the left vertices).

Proof of Theorem 5.2. For a bipartite graph $G = (U, V, E)$, we shall say that G satisfies the *double-sided α-neighbourhood condition* if $|\Gamma(S)| \geq \alpha|S|$ for any $S \subset U \cup V$. Theorem 5.2 can be reformulated as $G = (U, V, E)$ contains a k-star covering if and only if it satisfies the double-sided $\frac{1}{k}$-neighbourhood condition.

The necessity of the double-sided neighbourhood condition can be seen by counting edges. Suppose that $G = (U, V, E)$ is a bipartite graph with a k-star covering $F \subset E$. If we let $H = (U, V, F)$, and $S \subset U$, then $|S| \leq e_H(S, \Gamma(S)) \leq e_H(U, \Gamma(S)) \leq k|\Gamma(S)|$, since each vertex v in V has degree $d_H(v) \leq k$ and similarly if $S \subset V$, $|S| \leq k|\Gamma(S)|$. Therefore, G satisfies the doubled-sided $\frac{1}{k}$-neighbourhood condition.

It remains to show sufficiency. Let $G = (U, V, E)$ be a bipartite graph such that $|\Gamma(S)| \geq \frac{1}{k}|S|$ for each $S \subset U$ and $S \subset V$ and suppose that G is minimal with respect to $|E|$ such that it does not have a k-star covering. G is minimal with respect to edges so it must be connected and for all $uv \in E$, where $u \in U$ and $v \in V$, it must be the case that there is either a set $S \subset U$ such that $u \in S$, $v \notin \Gamma(S \setminus u)$ and $h(U, \frac{1}{k}) < 1$, or a set $T \subset V$ such that $v \in T$, $u \notin \Gamma(T \setminus \{v\})$ and $h(U, \frac{1}{k}) < 1$. Now suppose that there exists an edge $uv \in E$ such that $d(u), d(v) \geq 2$ and assume without loss of generality that $S \subset U$ is such that $u \in S$, $v \notin \Gamma(S \setminus u)$ and $h(S, \frac{1}{k}) < 1$. Note that since $h(S, \frac{1}{k})$ must be a multiple of $\frac{1}{k}$, $h(S, \frac{1}{k}) \leq \frac{k-1}{k}$. Further note that $h(\{u\}, \frac{1}{k}) = d(u) - \frac{1}{k} \geq 1$ and so $S \setminus u \neq \emptyset$, and

$$
\begin{aligned}
h(S \setminus u, \frac{1}{k}) &= |\Gamma(S \setminus u)| - \frac{1}{k}|S \setminus u| \\
&= |\Gamma(S)| - |\Gamma(u) \setminus \Gamma(S)| - \frac{1}{k}(|S| - 1) \\
&= h(S, \frac{1}{k}) + \frac{1}{k} - |\Gamma(u) \setminus \Gamma(S)|. \qquad (5.1)
\end{aligned}
$$

Note that $h(S, \frac{1}{k}) \leq \frac{k-1}{k}$ and $v \in \Gamma(u) \setminus \Gamma(S)$, so $|\Gamma(u) \setminus \Gamma(S)| \geq 1$. Putting these into (5.1) gives us that $h(S \setminus u, \frac{1}{k}) \leq 0$ and so since h is positive, $h(S \setminus u, \frac{1}{k}) = 0$. Consider $H = G[(S \setminus u) \cup \Gamma(S \setminus u)]$. $|\Gamma_H(A)| \geq \frac{1}{k}|A|$ for all $A \subset (S \setminus u)$ and so there must be a $(1, k)$-matching on H (consider blowing up the right vertices and applying Hall's Theorem). However, since $|\Gamma(S \setminus u)| = \frac{1}{k}|S \setminus u|$, each vertex in $\Gamma(S \setminus u)$ must be used in this covering and so this $(1, k)$-matching must in fact be a k-star covering. It must then be the case that $J = G[(U \setminus (S \setminus u)) \cup (V \setminus \Gamma(S \setminus u))]$ cannot have a k-star covering, else we may take the union of the vertex disjoint k-star coverings of H and J to get a k-star covering of G. On the other hand, if $A \subset (U \setminus (S \setminus u))$, then

$$
\begin{aligned}
|\Gamma_J(A)| &\geq |\Gamma_G(A \cup (S \setminus u))| - |\Gamma_G(S \setminus u)| \\
&= |\Gamma_G(A \cup (S \setminus u))| - \frac{1}{k}|S \setminus u| \\
&\geq \frac{1}{k}|A \cup (S \setminus u)| - \frac{1}{k}|S \setminus u| \\
&\geq \frac{1}{k}|A|.
\end{aligned}
$$

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If $B \subset V \setminus (\Gamma(S \setminus u))$, then $\Gamma_J(B) = \Gamma_G(B)$ and so $|\Gamma_J(B)| \geq \frac{1}{k}|B|$. Therefore J satisfies the double-sided $\frac{1}{k}$-neighbourhood condition but does not have a k-star covering. This contradicts the edge-minimality of G and so there can be no such edge uv with $d(u), d(v) \geq 2$.

So it must be the case that each edge in G must be incident to a leaf. The only such connected graphs are stars and so G must be a star. G would then already be a k-star covering and so we arrive at a contradiction. $\square$

It would be interesting to find a "Tutte-style" result for the existence of a k-star covering in a general (non-bipartite) graph.

References

[1] I. Bonacina, N. Galesi, T. Huynh and P. Wollan. Space proof complexity for random 3-CNFs via a $(2 - \epsilon)$-Hall's Theorem. *arXiv:1411.1619*

[2] P. Hall. On Representatives of Subsets. *J. London Math. Soc. 10 (1): 26-30, 1935*