Vertex-isoperimetric stability in the hypercube

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Abstract

Harper's Theorem states that in a hypercube the Hamming balls have minimal vertex boundaries with respect to set size. In this paper we prove a stability-like result for Harper's Theorem: if the vertex boundary of a set is close to minimal in the hypercube, then the set must be very close to a Hamming ball around some vertex.

1 Introduction

For all natural numbers n, we define the n-dimensional hypercube $Q_n = (V, E)$ where $V = \{0,1\}^n$ and $uv \in E$ if the two vertices differ in exactly one co-ordinate. For a vertex $u \in V$ inductively we let $\Gamma^0(u) = \{u\}$, $\Gamma^1(u) = \Gamma(u)$, and for $k \geq 2$ we have $\Gamma^k(u) = \bigcup_{v \in \Gamma^{k-1}(u)} \Gamma(v) \setminus \Gamma^{k-2}(u)$ (so $\Gamma^k(v)$ is the set of vertices which have shortest path length to v equal to k). For a subset of the vertices $U \subseteq V$, we also write $\Gamma(U) = \bigcup_{v \in U} \Gamma(v)$, and we define the *vertex boundary* of U to be $U \cup \Gamma(U)$, the set of vertices in U together with the neighbourhood of U.

Let $A, B \subseteq [n]$ and let $<_L$ be the ordering of subsets of $[n]$ such that $A <_L B$ if $|A| < |B|$ or if $|A| = |B|$ and $\min((A \cup B) \setminus (A \cap B)) \in A$. (This is known as the lexicographic, or lex, ordering.) Since with every vertex $v = (v_1, \ldots, v_n) \in V(Q_n)$ we can naturally associate a set $Z_v = \{i \in [n] : v_i = 1\}$, the ordering $<_L$ induces an ordering on $V(Q_n)$: for $u, w \in V(Q_n)$ we have $u <_L w$ if $Z_u <_L Z_w$. The following well known result of Harper [12] (see also [2, §16]) shows that initial segments of $<_L$ have minimal vertex boundaries.

Theorem 1.1. *For each $\ell \in \mathbb{N}$, let S_ℓ be the first ℓ elements of $V(Q_n)$ according to $<_L$. If $D \subset V(Q_n)$ with $|D| = \ell$, then*

$$|\Gamma(D) \cup D| \geq |\Gamma(S_\ell) \cup S_\ell|.$$

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A direct corollary of Harper's Theorem is that a set of size $\binom{n}{k}$ with minimal vertex boundary must closely resemble a k-th neighbourhood (the set of vertices at distance k from a vertex). A natural question is what happens when a set of size $\binom{n}{k}$ has close to the minimal vertex boundary? We provide a stability theorem when k is not too large. Note that we consider neighbourhoods rather than vertex boundaries, but since these differ by at most $\binom{n}{k}$ vertices this does not change the nature of our result.

Theorem 1.2. *Let $k(n) : \mathbb{N} \to \mathbb{N}$ and $p(n) : \mathbb{N} \to \mathbb{R}_+$ be functions such that $k(n) \leq \frac{\log n}{3 \log \log n}$, $\frac{k(n)}{p(n)}$ is bounded, and $\frac{p(n)k(n)^2}{n} \to 0$ as $n \to \infty$. Then there exists a constant C (which may depend on k and p) such that the following holds: If $A \subseteq V(Q_n)$ with $|A| = \binom{n}{k(n)}$ and $|\Gamma(A)| \leq \binom{n}{k(n)+1} + \binom{n}{k(n)}p(n)$, then there exists some $w \in V(Q_n)$ for which we have*

$$|\Gamma^{k(n)}(w) \cap A| \geq \binom{n}{k(n)} - C\binom{n}{k(n)-1}p(n)k(n).$$

Remark 1.3. The fact that $k(n) : \mathbb{N} \to \mathbb{N}$ and $k(n) = O(p(n))$ together imply that $p(n)$ is bounded away from 0.

Throughout the paper we use the notation $f(n) = O(g(n))$ to mean that there exists some constant $C > 0$ such that $|\frac{f(n)}{g(n)}| \leq C$ for all n, and $f(n) = o(g(n))$ to say that $\frac{f(n)}{g(n)} \to 0$ as $n \to \infty$. For the ease of notation, we shall often denote $k = k(n)$ and $p = p(n)$.

The strongly related edge-boundary version of the isoperimetric problem (see, e.g., Harper [11], Bernstein [1], and Hart [13]) has been considered in the stability context by Ellis [6], Ellis, Keller and Lifshitz [7], Friedgut [10], and others.

There are many other fundamental stability-type results in graph theory: for example, the Erdős-Simonovits Stability Theorem [8] states that an H-free graph that is close to maximum in size must in fact be close to a Turán graph. The famous Erdős-Ko-Rado Theorem [9] concerning the maximum size of intersecting set systems has been extended using stability results by, among others, Dinur and Friedgut [5], Bollobás, Narayanan and Raigorodskii [3], and Devlin and Khan [4].

The stability versions of extremal results can often be applied even more widely that the statements they extend; indeed, the motivation for this work came from the authors' forthcoming paper with Alex Scott [18] on the shotgun assembly of the hypercube.

The paper is organised as follows. In Section 2 we prove some preparatory lemmas including a tightening of the Local LYM Lemma, and in Section 3 we prove Theorem 1.2.

We also remark that Peter Keevash and Eoin Long have independently been working on a similar problem [15]. They use very different techniques and their results give weaker bounds for the set-sizes we consider but work in somewhat greater generality (i.e., for $k \gg \frac{\log n}{3 \log \log n}$, although with $p = O(1/k)$).

2 Preliminaries

Another important ordering in finite set theory is the colexicographic, or colex, ordering $<_C$ of layers $[n]^{(r)}$. For $A, B \in [n]^{(r)}$ we have $A <_C B$ if $A \neq B$ and $\max((A \cup B) \setminus (A \cap B)) \in B$. An important fact connecting the orderings $<_L$ and $<_C$ on $[n]^{(r)}$ is that if $\mathcal{F}$ is the initial segment of $<_L$ on $[n]^{(r)}$ then $\mathcal{F}^c$ is isomorphic to the initial segment of colex on $[n]^{(n-r)}$ (more precisely, it is the initial segment of colex on $[n]^{(n-r)}$ using the "reversed alphabet" where $n < n-1 < \ldots < 1$). Indeed, if $|A| = |B| = r$ and $A <_L B$ then by definition we have $\min((A \cup B) \setminus (A \cap B)) \in A$, which implies that $\min((A^c \cup B^c) \setminus (A^c \cap B^c)) \in B^c$. Treating the alphabet as "reversed" we see that indeed $A^c <_C B^c$.

Let us now fix some more notation that will be used throughout this paper. For $\mathcal{F} \subseteq [n]^{(r)}$ we write

$$\partial(\mathcal{F}) = \{A \in [n]^{(r-1)} : \exists B \in \mathcal{F}, A \subseteq B\}$$

for the shadow of $\mathcal{F}$, and similarly

$$\partial^+(\mathcal{F}) = \{A \in [n]^{(r+1)} : \exists B \in \mathcal{F}, B \subseteq A\}$$

for the upper shadow of $\mathcal{F}$. For a set system $\mathcal{F} \subseteq \mathcal{P}(n)$ we write $\mathcal{F}^c = \{[n] \setminus A : A \in \mathcal{F}\}$.

It will be useful to be able to bound from below the size of the neighbourhood of a subset of $[n]$ by some function of the size of the subset itself. A good starting point for this is the local LYM-inequality [17, Ex. 13.31(b)].

Lemma 2.1. *Let $\mathcal{A} \subseteq [n]^{(r)}$, then*

$$\frac{|\partial(\mathcal{A})|}{\binom{n}{r-1}} \geq \frac{|\mathcal{A}|}{\binom{n}{r}}, \tag{2.1}$$

and

$$\frac{|\partial^+(\mathcal{A})|}{\binom{n}{r+1}} \geq \frac{|\mathcal{A}|}{\binom{n}{r}}. \tag{2.2}$$

Theorem 1.1 and Lemma 2.1 give us the following corollary.

Corollary 2.2. *Let $k \in \mathbb{N}$ and let $B \subseteq V(Q_n)$ with $|B| \leq \binom{n}{k}$. Then*

$$|\Gamma(B)| \geq |B| \frac{n}{k+1} - 2\binom{n}{k}.$$

Proof. We have

$$|\Gamma(B)| \geq |B \cup \Gamma(B)| - |B| \geq |B \cup \Gamma(B)| - \binom{n}{k}.$$

Let $\ell = |B|$. By Theorem 1.1 we can bound further to obtain

$$|B \cup \Gamma(B)| \geq |\Gamma(S_\ell) \cup S_\ell| \geq |\Gamma(S_\ell)| = \sum_{i=0}^{k} |\Gamma(S_\ell \cap [n]^{(i)})| \geq \sum_{i=0}^{k} |\partial^+(S_\ell \cap [n]^{(i)})|.$$

Applying (2.2) we then have

$$\sum_{i=0}^{k} |\partial^+(S_\ell \cap [n]^{(i)})| \geq \sum_{i=0}^{k} |S_\ell \cap [n]^{(i)}| \frac{n-i}{i+1} \geq |B| \frac{n-k}{k+1} \geq |B| \frac{n}{k+1} - \binom{n}{k},$$

completing the proof. $\square$

Unfortunately the well-known inequality (2.2) is not quite strong enough for our purpose, and so we will need the following result.

Lemma 2.3. *Let $m, r, i \in \mathbb{N}$. If $\mathcal{F} \subseteq [n]^{(r)}$ has order*

$$|\mathcal{F}| \in \left[\binom{n}{r} - \binom{n-i+1}{r} + 1, \binom{n}{r} - \binom{n-i}{r} \right], \tag{2.3}$$

then

$$|\partial^+(\mathcal{F})| \geq |\mathcal{F}| \frac{\binom{n}{r+1} - \binom{n-i}{r+1}}{\binom{n}{r} - \binom{n-i}{r}}. \tag{2.4}$$

We do not claim that Lemma 2.3 is unknown, but we have been unable to find a reference and so we provide a proof here. The proof uses the following celebrated result of Kruskal and Katona [14, 16].

Theorem 2.4. *Let $\mathcal{F} \subseteq [n]^{(r)}$ and let $\mathcal{A}$ be the first $|\mathcal{F}|$ elements of $[n]^{(r)}$ according to $<_C$. Then $|\partial(\mathcal{F})| \geq |\partial(\mathcal{A})|$.*

Proof of Lemma 2.3. Let $m, r, i \in \mathbb{N}$ and suppose $\mathcal{F} \subseteq [n]^{(r)}$ satisfies (2.3). It is easy to see that $\partial^+(\mathcal{F}) = (\partial(\mathcal{F}^c))^c$, and so it suffices to estimate $|\partial(\mathcal{F}^c)|$. By Theorem 2.4, the size of the shadow of $\mathcal{F}^c$ is at least the size of the shadow of the initial segment of size $|\mathcal{F}|$ in the $<_C$ order on $[n]^{(n-r)}$.

So suppose that $\mathcal{H} \subset [n]^{(n-r)}$ is an initial segment of $<_C$ order of size as in (2.3). We first want to claim that

$$|\mathcal{H}| = \sum_{j=0}^{i-2} \binom{n-j-1}{r-1} + s,$$

where $1 \leq s \leq \binom{n-i}{r-1}$. Indeed, observe that the first $\binom{n}{r} - \binom{n-i}{r}$ elements in the $<_L$ order on $[n]^{(r)}$ are the sets that are not fully contained in $[n] \setminus [i]$. These can be listed as the $\binom{n-1}{r-1}$ sets that contain 1, followed by the $\binom{n-2}{r-1}$ sets that contain 2 but do not contain 1, etc., followed finally by the $\binom{n-i}{r-1}$ sets A such that $A \cap [i] = i$. A similar argument holds for the lower bound in (2.3), which proves our claim.

For $j = 0, \ldots, i-2$, let

$$\mathcal{H}_j = \left\{ A \cup \{n+1-j, n+2-j, \ldots, n\} : A \in [n-j-1]^{n-r-j} \right\},$$

so that $|\mathcal{H}_j| = \binom{n-j-1}{n-r-j} = \binom{n-j-1}{r-1}$. Then $\mathcal{H}$, being the initial segment of the $<_C$ order on $[n]^{(n-r)}$, can be expressed as the disjoint union $\mathcal{H} = \bigcup_{j=0}^{i-2} \mathcal{H}_j \cup \mathcal{S}$, where

$$\mathcal{S} \subset \left\{ A \cup \{n+2-i, \ldots, n\} : A \in [n-i]^{(n-r-(i-1))} \right\}$$

has size s. We may then write the shadow of $\mathcal{H}$ as the disjoint union

$$\partial\mathcal{H} = \bigcup_{j=0}^{i-2} (\partial\mathcal{H}_j \setminus (\partial\mathcal{H}_0 \cup \ldots \cup \partial\mathcal{H}_{j-1})) \cup (\partial\mathcal{S} \setminus (\partial\mathcal{H}_0 \cup \ldots \cup \partial\mathcal{H}_{i-2}))\,.$$

For each j, $\partial\mathcal{H}_j \setminus (\partial\mathcal{H}_0 \cup \ldots \cup \partial\mathcal{H}_{j-1})$ contains exactly the sets of the form $A \cup \{n+1-j, n+2-j, \ldots, n\}$ where $A \in [n-j-1]^{(n-r-j-1)}$. Writing $\mathcal{S} = \{A \cup \{n+2-i, \ldots, n\} : A \in \mathcal{A}\}$ (so $\mathcal{A} \subseteq [n-i]^{(n-r-(i-1))}$ has $|\mathcal{A}| = s$) we similarly see that

$$\partial\mathcal{S} \setminus (\partial\mathcal{H}_0 \cup \ldots \cup \partial\mathcal{H}_{i-2}) = \{A \cup \{n+2-i, \ldots, n\} : A \in \partial\mathcal{A}\}.$$

Hence $\partial\mathcal{H}$ is the disjoint union

$$\begin{aligned}\partial\mathcal{H} &= \bigcup_{j=0}^{i-2} \{A \cup \{n+1-j, n+2-j, \ldots, n\} : A \in [n-j-1]^{(n-r-j-1)}\} \\ &\quad \cup \{A \cup \{n+2-i, \ldots, n\} : A \in \partial\mathcal{A}\} \\ &= \sum_{j=0}^{i-2} \binom{n-j-1}{n-r-j-1} + |\partial\mathcal{A}|.\end{aligned}$$

Observing that $(n-j-1) - (n-r-j-1) = r$ and applying (2.1), we see

$$\begin{aligned}|\partial\mathcal{H}| &\geq \sum_{j=0}^{i-2} \binom{n-j-1}{r} + \frac{n-r-(i-1)}{r}|\mathcal{A}| \\ &= \sum_{j=0}^{i-2} \frac{n-r-j}{r}\binom{n-j-1}{r-1} + \frac{n-r-(i-1)}{r}s.\end{aligned}$$

If we divide the above expression by $|\mathcal{H}|$, we can think of this lower bound as a "weighted average", with the weights of the elements of $\mathcal{H}_j$ equal to $\frac{n-r-j}{r}$, and the weights of the elements of $\mathcal{S}$ equal to $\frac{n-r-(i-1)}{r}$. This last weight is the smallest, hence increasing s only decreases this average. Therefore we get

$$\begin{aligned}\frac{|\partial\mathcal{H}|}{|\mathcal{H}|} &\geq \frac{\sum_{j=0}^{i-1} \frac{n-r-j}{r}\binom{n-j-1}{r-1}}{\sum_{j=0}^{i-1}\binom{n-j-1}{r-1}} \\ &= \frac{\sum_{j=0}^{i-1}\binom{n-j-1}{r}}{\sum_{j=0}^{i-1}\binom{n-j-1}{r-1}} \\ &= \frac{\binom{n}{r+1} - \binom{n-i}{r+1}}{\binom{n}{r} - \binom{n-i}{r}},\end{aligned} \tag{2.5}$$

completing the proof of the lemma. $\square$

Corollary 2.5. *The sequence* $\frac{\binom{n}{r+1}-\binom{n-i}{r+1}}{\binom{n}{r}-\binom{n-i}{r}}$ *in* (2.4) *is decreasing in* i.

Proof. By (2.5) we have

$$\frac{\binom{n}{r+1}-\binom{n-i}{r+1}}{\binom{n}{r}-\binom{n-i}{r}} = \frac{\sum_{j=0}^{i-1} \frac{n-r-j}{r}\binom{n-j-1}{r-1}}{\sum_{j=0}^{i-1} \binom{n-j-1}{r-1}}.$$

If we move from i to $i+1$ on the left-hand side, in the weighted average on the right-hand side we obtain another term $\binom{n-i-1}{r-1}$ with weight $\frac{n-r-i}{r}$; this weight is smaller than all the preceding weights and so the average decreases. □

The next lemma somewhat cleans up the multiplicative factor in Lemma 2.3.

Lemma 2.6. *Suppose* $\alpha, c \in (0,1)$ *are such that* $\binom{n}{r} - \binom{\alpha n}{r} = c\binom{n}{r}$. *Then*

$$\frac{\binom{n}{r+1}-\binom{\alpha n}{r+1}}{\binom{n}{r}-\binom{\alpha n}{r}} \geq \frac{n-r}{r+1}\left(1+\frac{1-c}{r}\right).$$

Proof. Suppose that $\binom{\alpha n}{r} = (1-c)\binom{n}{r}$. Then

$$\begin{aligned}(1-c) &= \prod_{i=0}^{r-1} \frac{\alpha n - i}{n-i} \\ &= \prod_{i=0}^{r-1}\left(\alpha - (1-\alpha)\frac{i}{n-i}\right) \\ &\geq \prod_{i=0}^{r-1}\left(\alpha - (1-\alpha)\frac{r}{n-r}\right) \\ &= \left(\frac{\alpha n - r}{n-r}\right)^r.\end{aligned}$$

Hence we have that $\frac{\alpha n - r}{n-r} \leq (1-c)^{1/r}$. Thus

$$\begin{aligned}\binom{\alpha n}{r+1} &= \frac{\alpha n - r}{r+1}(1-c)\binom{n}{r} \\ &= (1-c)\frac{\alpha n - r}{n-r}\frac{n-r}{r+1}\binom{n}{r} \\ &\leq (1-c)^{1+1/r}\binom{n}{r+1}.\end{aligned}$$

We therefore have

$$\begin{aligned}
\frac{\binom{n}{r+1}-\binom{\alpha n}{r+1}}{\binom{n}{r}-\binom{\alpha n}{r}} &= \frac{\left(1-(1-c)^{1+1/r}\right)\binom{n}{r+1}}{c\binom{n}{r}} \\
&= \frac{n-r}{r+1}\frac{c+(1-c)\left(1-(1-c)^{1/r}\right)}{c} \\
&= \frac{n-r}{r+1}\left(1+\frac{1-c}{c}\left(1-(1-c)^{1/r}\right)\right).
\end{aligned}$$

A generalisation of Bernoulli's inequality says that if $x \geq -1$ and $t \in [0,1]$, then we have $(1+x)^t \leq 1+tx$. Applying this to the above formula we obtain

$$\begin{aligned}
\frac{\binom{n}{r+1}-\binom{\alpha n}{r+1}}{\binom{n}{r}-\binom{\alpha n}{r}} &\geq \frac{n-r}{r+1}\left(1+\frac{1-c}{c}\cdot\frac{c}{r}\right) \\
&\geq \frac{n-r}{r+1}\left(1+\frac{1-c}{r}\right).
\end{aligned}$$

□

In the proof of Theorem 1.2 we first delete sets of vertices with too many unique neighbours. The next lemma will allow us to impose that after this deletion, we get larger and larger layers around vertices in our set.

Lemma 2.7. *Let $k = o(\log n)$. For sufficiently large n the following holds. Let J be a subset of the hypercube such that for all $S \subseteq J$,*

$$|\Gamma(S) \setminus \Gamma(J \setminus S)| \leq |S|\frac{n}{k+1}\left(1+\frac{1}{8k}\right). \tag{2.6}$$

Then for any vertex v and $j \leq 2k$, if $|J \cap \Gamma^j(v)| \in [1, \frac{1}{2}\binom{n}{k}]$, then

$$|J \cap \Gamma^{j+2}(v)| \geq \frac{n}{64k^3}|J \cap \Gamma^j(v)|.$$

Proof. Without loss of generality, throughout this proof we assume that $v = (0,\ldots,0)$, so $Z_v = \emptyset$ and for all j we have $\Gamma^j(v) = [n]^{(j)}$. Let $k = o(\log n)$ and let J be a subset of the vertex set of the hypercube such that (2.6) holds for all $S \subseteq J$.

Assume that we have $j \leq 2k$ with $|J \cap \Gamma^j(v)| \in [1, \frac{1}{2}\binom{n}{k}]$. If $j \leq k-1$, then we may appeal to (2.2) to see that for sufficiently large n,

$$\begin{aligned}
\frac{|\partial^+(J \cap \Gamma^j(v))|}{|J \cap \Gamma^j(v)|} &\geq \frac{n-j}{j+1} \\
&\geq \frac{n}{k}-1 \\
&= \frac{n}{k+1}\left(1+\frac{1}{k}-\frac{k+1}{n}\right) \\
&\geq \frac{n}{k+1}\left(1+\frac{1}{4k}\right).
\end{aligned}$$

Now suppose that $j \geq k$. By Theorem 2.4 and the relation between the orders $<_C$ and $<_L$, $|\partial^+(J \cap \Gamma^j(v))|$ is minimised when $J \cap \Gamma^j(v)$ is the initial segment of size $|J \cap \Gamma^j(v)|$ in the $<_L$ order on $[n]^{(j)}$.

First suppose that $|J \cap \Gamma^j(v)| \leq \binom{n-(j+i)}{k-i}$ for some $i \geq 1$. Then all elements of the initial segment of length $|J \cap \Gamma^j(v)|$ in the $<_L$ order on $[n]^{(j)}$ contain the set $[j-k+i]$. So remove $[j-k+i]$ from all sets in $J \cap \Gamma^j(v)$ and instead work in $[n] \setminus [j-k+i]$. We now have an initial segment of size $|J \cap \Gamma^j(v)|$ in the $<_L$ order in $([n] \setminus [j-k+i])^{(k-i)}$ and so (2.2), together with the fact that $j \leq 2k$ and $i \geq 1$, give

$$\begin{aligned}|\partial^+(J \cap \Gamma^j(v))| &\geq |J \cap \Gamma^j(v)|\frac{n-j}{k-i+1}\\ &\geq |J \cap \Gamma^j(v)|\frac{n}{k+1}\left(1+\frac{1}{4k}\right).\end{aligned}$$

Finally let us consider the case when $|J \cap \Gamma^j(v)| > \binom{n-(j+1)}{k-1}$. Since $k = o(\log n)$, we have $|J \cap \Gamma^j(v)| \leq \frac{1}{2}\binom{n}{k} \leq \frac{3}{5}\binom{n-j+k}{k}$ for sufficiently large n. Therefore we see that all elements of the initial segment of length $|J \cap \Gamma^j(v)|$ in the $<_L$ order on $[n]^{(j)}$ contain the set $[j-k]$. Hence remove $[j-k]$ from all sets and instead work in $[n] \setminus [j-k]$. For convenience, we relabel our ground set so that we work with the initial segment of $<_L$ order in $[m]^{(k)}$ where $m = n-j+k$ instead. For n (and so also m) large enough we have

$$\binom{m}{k} - \binom{m(\frac{1}{3})^{1/k}}{k} \geq \binom{m}{k} - \frac{m^k}{3k!} \geq \frac{3}{5}\binom{m}{k} = \frac{3}{5}\binom{n-j+k}{k} \geq |J \cap \Gamma^j(v)|.$$

By Corollary 2.5, we can apply Lemma 2.3 with $\mathcal{F} = J \cap \Gamma^j(v)$, $n = m$, $n-i = m(\frac{1}{3})^{1/k}$, and $r = k$, to get

$$|\partial^+(J \cap \Gamma^j(v))| \geq |J \cap \Gamma^j(v)|\frac{\binom{m}{k+1} - \binom{m(\frac{1}{3})^{1/k}}{k+1}}{\binom{m}{k} - \binom{m(\frac{1}{3})^{1/k}}{k}}.$$

(We note that $m(\frac{1}{3})^{1/k}$ should be an integer to apply Lemma 2.3. This can be fixed by considering the floor of $m(\frac{1}{3})^{1/k}$, but for ease of reading we refrain from doing this.) Now,

$$\begin{aligned}\binom{m(\frac{1}{3})^{1/k}}{k} &= \frac{m(\frac{1}{3})^{1/k}(m(\frac{1}{3})^{1/k}-1)\dots(m(\frac{1}{3})^{1/k}-k+1)}{k!}\\ &\leq \frac{1}{3}\frac{m(m-1)\dots(m-k+1)}{k!} = \frac{1}{3}\binom{m}{k},\end{aligned}$$

so for n large enough we have $\binom{m}{k} - \binom{m(\frac{1}{3})^{1/k}}{k} \geq \frac{2}{3}\binom{m}{k}$ and we can apply Lemma 2.6 to find

$$\begin{aligned}|\partial^+(J \cap \Gamma^j(v))| &\geq |J \cap \Gamma^j(v)|\frac{m-k}{k+1}\left(1+\frac{1-\frac{2}{3}}{k}\right)\\ &= |J \cap \Gamma^j(v)|\frac{n-j}{k+1}\left(1+\frac{1}{3k}\right)\\ &\geq |J \cap \Gamma^j(v)|\frac{n}{k+1}\left(1+\frac{1}{4k}\right).\end{aligned}$$

In all cases, we see that

$$|\partial^+(J \cap \Gamma^j(v))| \geq |J \cap \Gamma^j(v)| \frac{n}{k+1} \left(1 + \frac{1}{4k}\right). \tag{2.7}$$

Since $j \leq 2k$, each vertex in $\Gamma^{j+2}(v)$ is adjacent to at most $2(k+1)$ vertices in $\partial^+(J \cap \Gamma^j(v))$. Together with (2.7), this gives

$$\begin{aligned}|\Gamma(J \cap \Gamma^j(v)) \setminus \Gamma(J \setminus \Gamma^j(v))| &\geq |\partial^+(J \cap \Gamma^j(v))| - (2k+2)|J \cap \Gamma^{j+2}(v)| \\ &\geq |J \cap \Gamma^j(v)| \frac{n}{k+1} \left(1 + \frac{1}{4k}\right) - (2k+2)|J \cap \Gamma^{j+2}(v)|.\end{aligned}$$

On the other hand, by assumption,

$$|\Gamma(J \cap \Gamma^j(v)) \setminus \Gamma(J \setminus \Gamma^j(v))| \leq |J \cap \Gamma^j(v)| \frac{n}{k+1} \left(1 + \frac{1}{8k}\right).$$

Together these inequalities give

$$(2k+2)|J \cap \Gamma^{j+2}(v)| \geq |J \cap \Gamma^j(v)| \frac{n}{(k+1)8k},$$

and so $|J \cap \Gamma^{j+2}(v)| \geq \frac{n}{16k(k+1)^2}|J \cap \Gamma^j(v)| \geq \frac{n}{64k^3}|J \cap \Gamma^j(v)|$. $\square$

3 Proof of Theorem 1.2

In this section we prove Theorem 1.2. The nature of the proof is much like that of the Erdős-Simonovits stability arguments [8]. Starting with a set A with close to minimal neighbourhood size, we first delete sets of vertices which contribute too many unique neighbours (neighbours unseen by the rest of A). We then build up, layer by layer, a rough structure around a vertex of A. If A has many vertices in the j-th neighbourhood of a vertex v, then there must be many vertices of A in $\Gamma^{j+2}(v)$ (else $A \cap \Gamma^j(v)$ has too many unique neighbours). This will mean that for each vertex $v \in A$, there is some $j(v)$ such that almost all of A is contained in $\Gamma^{2j(v)}(v)$, and we then show that $j(v) = k$ for almost all $v \in A$. This means we find two vertices $u, v \in A$ at distance $2k$ from one another with $j(u) = j(v) = k$. A pigeonhole argument then reveals a vertex w between u and v for which A is almost entirely contained in $\Gamma^k(w)$.

Proof of Theorem 1.2. Let $k : \mathbb{N} \to \mathbb{N}$ and $p : \mathbb{N} \to \mathbb{R}_+$ be functions with $k \leq \frac{\log n}{\log \log n}$, $k = O(p)$, and $p = o(n/k^2)$ as $n \to \infty$. Suppose $A \subseteq V(Q_n)$ with $|A| = \binom{n}{k}$ and $|\Gamma(A)| \leq \binom{n}{k+1} + \binom{n}{k}p$. For ease of reading, we now state the following two claims here which we will prove later.

Claim 3.1. *There exists $B \subseteq A$ with $|B| \geq \binom{n}{k} - D\binom{n}{k-1}pk$, where $D > 0$ is a constant depending on p, such that for all $S \subseteq B$ we have*

$$|\Gamma(S) \setminus \Gamma(B \setminus S)| \leq |S| \frac{n}{(k+1)} \left(1 + \frac{1}{8k}\right).$$

Claim 3.2. *Let $B \subseteq A$ be a set which satisfies Claim 3.1. Suppose that there is a vertex $u \in V(Q_n)$ and an integer $\ell \in [k, 2k]$ such that $|B \cap \Gamma^\ell(u)| \geq \frac{65k^3}{n}\binom{n}{k}$. Then*

$$|B \cap \Gamma^\ell(u)| = \binom{n}{k} - O\left(\binom{n}{k-1}pk\right).$$

Fix a set $B \subseteq A$ which satisfies Claim 3.1. We additionally claim that for all $v \in B$, there exists a $j(v) \leq k$ such that $|\Gamma^{2j}(v) \cap B| \geq |B| - O(\binom{n}{k-1}pk)$. Fix a vertex $v \in B$ and let j be the least integer such that

$$|B \cap \Gamma^{2(j+1)}(v)| < \frac{n}{64k^3}|B \cap \Gamma^{2j}(v)|$$

(note that since $v \in B$, we have $|B \cap \Gamma^0(v)| = |B \cap \{v\}| = 1$). If $j \leq k$ then, by Lemma 2.7, $|B \cap \Gamma^{2j}(v)|$ must be at least $\frac{1}{2}\binom{n}{k}$, which means that we must have $2j \geq k$. Since for n large enough we have $\frac{1}{2}\binom{n}{k} \geq \frac{65k^3}{n}\binom{n}{k}$, by Claim 3.2 we obtain $|B \cap \Gamma^{2j}(v)| = \binom{n}{k} - O(\binom{n}{k-1}pk)$.

Suppose now that $j \geq k+1$. Then, by the choice of j, we obtain

$$|B \cap \Gamma^{2(k+1)}(v)| \geq \left(\frac{n}{64k^3}\right)^{k+1} = \exp\{(k+1)\log n - 3k\log k + O(k)\}.$$

On the other hand,

$$|B \cap \Gamma^{2(k+1)}(v)| \leq |B| \leq \binom{n}{k} \leq \frac{n^k}{k!} = \exp\{k \log n - k \log k + O(k)\}.$$

Putting these together, we get

$$\log n - 2k \log k + O(k) \leq 0.$$

Since $k \leq \frac{\log n}{3 \log\log n}$, we have a contradiction and so $j \leq k$.

For $j \leq k$, let $H(j) = \{v \in B : j(v) = j\}$. Fix $j < k$, and suppose that there are distinct vertices $u, w \in H(j)$ such that $d(u, w) = 2j$. Without loss of generality, we may assume that $Z_u = \emptyset$ and $Z_w = [2j]$. Observe that

$$\Gamma^{2j}(u) \cap \Gamma^{2j}(w) = \{U \cup W : U \in [2j]^{(j)}, W \in ([n] \setminus [2j])^{(j)}\}.$$

The size of this set is clearly $\binom{2j}{j}\binom{n-2j}{j}$. On the other hand

$$\begin{aligned}\Gamma^{2j}(u) \cap \Gamma^{2j}(w) &\supseteq \Gamma^{2j}(u) \cap \Gamma^{2j}(w) \cap B \\ &= B \setminus (B \setminus \Gamma^{2j}(w) \cup B \setminus \Gamma^{2j}(u)).\end{aligned}$$

Recall that by the definition of $j = j(u) = j(w)$ we have $|B \setminus \Gamma^{2j}(u)| = O(\binom{n}{k-1}pk)$ and $|B \setminus \Gamma^{2j}(w)| = O(\binom{n}{k-1}pk)$, therefore

$$|\Gamma^{2j}(u) \cap \Gamma^{2j}(w)| \geq \binom{n}{k} - O\left(\binom{n}{k-1}pk\right).$$

Putting these bounds together gives $\binom{2j}{j}\binom{n-2j}{j} \geq \binom{n}{k} - O(\binom{n}{k-1}pk)$. But $j < k$ and so

$$\begin{aligned}\binom{2j}{j}\binom{n-2j}{j} &\leq 4^j\binom{n}{j} \\ &\leq 4^k\binom{n}{k}\frac{k}{n-k} \\ &= \binom{n}{k}\exp\{O(k) - \log n\}.\end{aligned}$$

Since $k = o(\log n)$, we have $\binom{2j}{j}\binom{n}{j} = o(\binom{n}{k})$. We have a contradiction and so no two vertices from $H(j)$ can be at distance $2j$ from each other.

Since for any $v \in H(j)$ by definition we have $|B \setminus \Gamma^{2j}(v)| = O(\binom{n}{k-1}pk)$, and no two vertices from $H(j)$ can be at distance $2j$ from each other, we obtain $|H(j)| = O(\binom{n}{k-1}pk)$. Summing over $j < k$, we see

$$\begin{aligned}|H(k)| &= |B| - \sum_{j=0}^{k-1}|H(j)| \\ &\geq |B| - O\left(\binom{n}{k-1}pk\right)k \\ &\geq \binom{n}{k} - O\left(\binom{n}{k-1}pk^2\right).\end{aligned}$$

Since "most" of B lies in $H(k)$ and for a vertex $v \in H(k)$, "most" of B lies in $\Gamma^{2k}(v)$, there must exist two vertices in $H(k)$ at distance $2k$ from each other. Let $u, v \in V$ be such vertices and without loss of generality, suppose that $Z_u = \emptyset$ and $Z_v = [2k]$.

Any vertex in $\Gamma^{2k}(u) \cap \Gamma^{2k}(v) \cap B$ must be of the form $X \cup Y$, where $X \in [2k]^{(k)}$ and $Y \in ([n] \setminus [2k])^{(k)}$, and so any such vertex must be at distance k from some vertex in $[2k]^{(k)}$. For $w \in [2k]^{(k)}$, let $f(w) = |\{z \in \Gamma^{2k}(u) \cap \Gamma^{2k}(v) \cap B : d(w,z) = k\}|$. Then we have

$$\begin{aligned}\sum_{w\in[2k]^{(k)}} f(w) &= |\Gamma^{2k}(u) \cap \Gamma^{2k}(v) \cap B| \\ &\geq \binom{n}{k} - O\left(\binom{n}{k-1}pk\right).\end{aligned}$$

Hence by the pigeonhole principle, there exists a vertex $w \in [2k]^{(k)}$ for which we have

$$|\Gamma^k(w) \cap B| \geq \frac{\binom{n}{k}}{\binom{2k}{k}} - O\left(\frac{\binom{n}{k-1}}{\binom{2k}{k}}pk\right).$$

Recall that $k \leq \frac{\log n}{3\log\log n}$ and so $\binom{2k}{k} \leq 4^k = n^{o(1)} = o(\frac{n}{65k^3})$. Since we have $p = o(n/k^2)$, by Claim 3.2 we have $|\Gamma^k(w) \cap B| = \binom{n}{k} - O(\binom{n}{k-1}pk)$, proving Theorem 1.2. $\square$

We now complete our argument by proving Claims 3.1 and 3.2.

Proof of Claim 3.1. Let us run the following algorithm.

Initialization *Set* $i = 0$, $B_0 = A$;
while $\exists S \subseteq B_i$ *such that* $|\Gamma(S) \setminus \Gamma(B_i \setminus S)| > |S|\frac{n}{(k+1)}\left(1+\frac{1}{8k}\right)$ **do**
  pick such an S;
  set $i = i + 1$;
  set $L_i = S$;
  set $B_i = B_{i-1} \setminus S$;
end

Suppose that the algorithm terminates when $i = m$. An easy induction gives

$$|\Gamma(A)| = \sum_{i=1}^{m} |\Gamma(L_i) \setminus \Gamma(B_{i-1} \setminus L_i)| + |\Gamma(B_m)|.$$

Recall that for each i we have $|\Gamma(L_i) \setminus \Gamma(B_{i-1} \setminus L_i)| > |L_i|\frac{n}{k+1}(1+\frac{1}{8k})$, and so

$$|\Gamma(A)| \geq |A \setminus B_m|\frac{n}{k+1}\left(1+\frac{1}{8k}\right) + |\Gamma(B_m)|.$$

Corollary 2.2 gives $|\Gamma(B_m)| \geq |B_m|\frac{n}{k+1} - 2\binom{n}{k}$. Therefore

$$\begin{aligned}
|\Gamma(A)| &\geq |A \setminus B_m|\frac{n}{k+1} + |A \setminus B_m|\frac{n}{8k(k+1)} + |B_m|\frac{n}{k+1} - 2\binom{n}{k} \\
&= |A|\frac{n}{k+1} + |A \setminus B_m|\frac{n}{8k(k+1)} - 2\binom{n}{k} \\
&= \frac{n!}{k!(n-k)!}\frac{n}{k+1} + |A \setminus B_m|\frac{n}{8k(k+1)} - 2\binom{n}{k} \\
&\geq \binom{n}{k+1} + |A \setminus B_m|\frac{n}{8k(k+1)} - 2\binom{n}{k}.
\end{aligned}$$

Since by assumption $|\Gamma(A)| \leq \binom{n}{k+1} + \binom{n}{k}p$ and $p \geq 1$, we obtain

$$\begin{aligned}
|A \setminus B_m| &\leq \left(\binom{n}{k}p + 2\binom{n}{k}\right)\frac{8k(k+1)}{n} \\
&\leq \frac{Dk^2}{n}\binom{n}{k}p \\
&\leq D\binom{n}{k-1}pk,
\end{aligned}$$

where $D > 0$ is such that $Dp \geq 16p + 32$. Setting $B = B_m$ we obtain the desired result. $\square$

Proof of Claim 3.2. Let B be the set given by Claim 3.1 (so $|B| \geq \binom{n}{k} - D\binom{n}{k-1}pk$). Let $v \in V(Q_n)$ be such that for some $\ell \in [k, 2k]$ we have $|B \cap \Gamma^\ell(v)| \geq \frac{65k^3}{n}\binom{n}{k}$. (Without loss of generality we again assume that $v = (0, \dots, 0)$, so that $Z_v = \emptyset$.) If we also have $|B \cap \Gamma^\ell(v)| \leq \frac{1}{2}\binom{n}{k}$ then by Lemma 2.7 we have

$$|B \cap \Gamma^{\ell+2}(v)| \geq \frac{n}{64k^3}\frac{65k^3}{n}\binom{n}{k} > \binom{n}{k}$$

which contradicts the fact that $|B| \leq \binom{n}{k}$. Therefore we may assume that $|B \cap \Gamma^\ell(v)| \geq \frac{1}{2}\binom{n}{k}$ and so $|A \cap \Gamma^\ell(v)| \geq \frac{1}{2}\binom{n}{k}$ and $|A \setminus \Gamma^\ell(v)| \leq \frac{1}{2}\binom{n}{k}$. If $|A \cap \Gamma^\ell(v)| = \binom{n}{k} - O(\binom{n}{k-1}pk)$ then we are done. Hence, throughout the proof, we assume $|A \setminus \Gamma^\ell(v)| \geq \binom{n}{k-1}(pk+2)$ (recall that $k \geq 1$ and p is bounded away from 0).

We can undercount the neighbourhood of A as follows: We count the neighbours of $A \cap \Gamma^\ell(v)$ in $\Gamma^{\ell+1}(v)$ (ignoring the neighbours in $\Gamma^{\ell-1}(v)$). We then add the neighbours of $A \setminus \Gamma^\ell(v)$ not in $\Gamma^{\ell+1}(v)$. Since any vertex in $A \setminus \Gamma^\ell(v)$ has at most $\ell+2$ neighbours in $\Gamma^{\ell+1}(v)$ we have

$$|\Gamma(A)| \geq |\Gamma(A \cap \Gamma^\ell(v)) \cap \Gamma^{\ell+1}(v)| + |\Gamma(A \setminus \Gamma^\ell(v))| - |A \setminus \Gamma^\ell(v)|(\ell+2). \tag{3.1}$$

As we remarked at the beginning of the proof, we may assume $|A \setminus \Gamma^\ell(v)| \geq \binom{n}{k-1}(pk+2)$. By Theorem 1.1, $|\Gamma(A \setminus \Gamma^\ell(v))|$ is at least the upper shadow of the first $|\Gamma(A \setminus \Gamma^\ell(v))| - \sum_{i=0}^{k-1}\binom{n}{i}$ elements of $[n]^k$ according to the $<_L$ order. Write

$$c\binom{n}{k} = |A \setminus \Gamma^\ell(v)| - \sum_{i=0}^{k-1}\binom{n}{i}, \tag{3.2}$$

and observe that by the assumption that $|A \setminus \Gamma^\ell(v)| \leq \frac{1}{2}\binom{n}{k}$ we have $c \leq 1/2$.

Let $\alpha \in (0,1)$ be such that

$$c\binom{n}{k} = \binom{n}{k} - \binom{\alpha n}{k}.$$

Since $|A \setminus \Gamma^\ell(v)| > \binom{n}{k} - \binom{\alpha n}{k}$, by Lemma 2.3 and Corollary 2.5 we have

$$\begin{aligned} |\Gamma(A \setminus \Gamma^\ell(v))| &\geq |\partial^+(A \setminus \Gamma^\ell(v))| \\ &\geq |A \setminus \Gamma^\ell(v)| \frac{\binom{n}{k+1} - \binom{\alpha n}{k+1}}{\binom{n}{k} - \binom{\alpha n}{k}}. \end{aligned}$$

(As in Lemma 2.7 we refrain from ensuring things are integer valued for ease of reading.)

Recalling the relation between α and c, Lemma 2.6 gives

$$|\Gamma(A \setminus \Gamma^\ell(v))| \geq c\binom{n}{k}\frac{n-k}{k+1}\left(1 + \frac{1-c}{k}\right). \tag{3.3}$$

We clearly have
$$|\Gamma(A\cap\Gamma^{\ell}(v))\cap\Gamma^{\ell+1}(v)| = |\partial^{+}(A\cap\Gamma^{\ell}(v))|.$$
As we mentioned earlier, for a family $\mathcal{A}\subseteq[n]^{(\ell)}$ we have $\partial^{+}\mathcal{A}=(\partial\mathcal{A}^{c})^{c}$, thus by Theorem 2.4 the size of the upper shadow of $\mathcal{A}$ is minimised when $\mathcal{A}^{c}$ is isomorphic to the initial segment of colex $<_{C}$ on $[n]^{(n-\ell)}$, i.e., when $\mathcal{A}$ is isomorphic to the initial segment of lex $<_{L}$ on $[n]^{(\ell)}$.

If $|A\cap\Gamma^{\ell}(v)|\geq\binom{n}{k}-O(\binom{n}{k-1}pk)$ then the claim holds and there is nothing to prove. Hence, since p is bounded away from 0, we may assume that $\frac{1}{2}\binom{n}{k}\leq|A\cap\Gamma^{\ell}(v)|\leq\binom{n}{k}-\binom{n}{k-1}k$. Applying the Pascal's rule k times, for n large enough we have

$$\begin{aligned}|A\cap\Gamma^{\ell}(v)| &\leq \binom{n}{k}-\binom{n}{k-1}k\\ &= \binom{n-1}{k}+\binom{n-1}{k-1}-\binom{n}{k-1}k\\ &\leq \binom{n-1}{k}-\binom{n}{k-1}(k-1)\leq\ldots\leq\binom{n-k}{k}.\end{aligned}$$

Recall also that we have $k\leq\ell\leq 2k$. This implies that $\binom{n-k}{k}\leq\binom{n-(\ell-k)}{k}$. Hence every set in the initial segment of size $|A\cap\Gamma^{\ell}(v)|$ of $<_{L}$ on $[n]^{(\ell)}$ consists of the set $[\ell-k]$ union one of the $\binom{n-(\ell-k)}{k}$ subsets of $[n]\setminus[\ell-k]$ of size k. Hence we can again imagine removing $[\ell-k]$ from all sets in our segment and instead working in $[n]\setminus[\ell-k]$. We now have an initial segment of size $|A\cap\Gamma^{\ell}(v)|$ in the $<_{L}$ order in $([n]\setminus[\ell-k])^{(k)}$ which we denote by $\mathcal{H}$. Then (2.2), together with the fact that $\ell\leq 2k$, gives

$$\begin{aligned}|\partial^{+}(A\cap\Gamma^{\ell}(v))| &\geq |\partial^{+}(\mathcal{H})|\\ &\geq |A\cap\Gamma^{\ell}(v)|\frac{n-(\ell-k)-k}{k+1} \qquad (3.4)\\ &= |A\cap\Gamma^{\ell}(v)|\frac{n-k}{k+1}+O\left(\binom{n}{k}\right).\end{aligned}$$

We know that $|A\setminus\Gamma^{\ell}(v)|(\ell+2)=O(\binom{n}{k}k)$. Substituting (3.3) and (3.4) into (3.1) then gives

$$\begin{aligned}|\Gamma(A)| &\geq |A\cap\Gamma^{\ell}(v)|\frac{n-k}{k+1}+O\left(\binom{n}{k}\right)+c\binom{n}{k}\frac{n-k}{k+1}\left(1+\frac{1-c}{k}\right)+O\left(\binom{n}{k}k\right)\\ &= \left(|A\cap\Gamma^{\ell}(v)|+c\binom{n}{k}\right)\frac{n-k}{k+1}+\frac{c(1-c)}{k}\binom{n}{k}\frac{n-k}{k+1}+O\left(\binom{n}{k}k\right).\end{aligned}$$

Since we defined $c\binom{n}{k}=|A\setminus\Gamma^{\ell}(v)|-\sum_{i=0}^{k-1}\binom{n}{i}$, and also we have $c\leq 1/2$, we obtain

$$\begin{aligned}|\Gamma(A)| &\geq \left(|A|-\sum_{i=0}^{k-1}\binom{n}{i}\right)\frac{n-k}{k+1}+\frac{c}{2k}\binom{n}{k+1}+O\left(\binom{n}{k}k\right)\\ &\geq \binom{n}{k+1}+\frac{c}{2k}\binom{n}{k+1}+O\left(\binom{n}{k}k\right),\end{aligned}$$

Since we assume $|\Gamma(A)| \leq \binom{n}{k+1} + O(\binom{n}{k}p)$, and $k = O(p)$, we must have $c = O(\frac{pk^2}{n})$. By the definition of c in (3.2), we then have $|A \setminus \Gamma^\ell(v)| = O\left(\binom{n}{k-1}pk\right)$ and so $|B \setminus \Gamma^\ell(v)| = O\left(\binom{n}{k-1}pk\right)$. Since $|B| \geq \binom{n}{k} - D\binom{n}{k-1}pk$, we then have $|B \cap \Gamma^\ell(v)| = \binom{n}{k} - O\left(\binom{n}{k-1}pk\right)$. $\square$

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