Isoperimetric stability in lattices

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Abstract

We obtain isoperimetric stability theorems for general Cayley digraphs on $\mathbb{Z}^d$. For any fixed B that generates $\mathbb{Z}^d$ as a group, we characterise the approximate structure of large sets A that are approximately isoperimetric in the Cayley digraph of B: we show that A must be close to a set of the form $kZ \cap \mathbb{Z}^d$, where for the vertex boundary Z is the conical hull of B, and for the edge boundary Z is the zonotope generated by B.

1 Introduction

An important theme at the interface of Geometry, Analysis and Combinatorics is understanding the structure of approximate minimisers to isoperimetric problems. These problems take the general form of minimising surface area of sets with a fixed volume, for various meanings of 'area' and 'volume'. The usual meanings give the Euclidean Isoperimetric Problem considered since the ancient Greek mathematicians, where balls are the measurable subsets of $\mathbb{R}^d$ with a given volume which minimize the surface area. There is a large literature on its stability, i.e. understanding the structure of approximate minimisers, culminating in the sharp quantitative isoperimetric inequality of Fusco, Maggi and Pratelli [8].

In the discrete setting, isoperimetric problems form a broad area that is widely studied within Combinatorics (see the surveys [2, 14]) and as part of the Concentration of Measure phenomenon (see [15, 25]). Furthermore, certain particular settings have been intensively studied due to their applications; for example, there has been considerable recent progress (see [11, 12, 13, 22]) on isoperimetric stability in the discrete cube $\{0, 1\}^n$, which is intimately connected to the Analysis of Boolean Functions (see [20]) and the Kahn-Kalai Conjecture (see [10]) on thresholds for monotone properties (see [7] for the recent solution of Talagrand's fractional version). This paper concerns the setting of integer lattices, which is widely studied in Additive Combinatorics, where the Polynomial Freiman-Ruzsa Conjecture (see [9]) predicts the structure of sets with small doubling.

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For an isoperimetric problem on a digraph (directed graph) G, we measure the 'volume' of $A \subseteq V(G)$ by its size $|A|$, and its 'surface area' either by the *edge boundary* $\partial_{e,G}(A)$, which is the number of edges $\overrightarrow{xy} \in E(G)$ with $x \in A$ and $y \in V(G) \setminus A$, or by the *vertex boundary* $\partial_{v,G}(A)$, which is the number of vertices $y \in V(G) \setminus A$ such that $\overrightarrow{xy} \in E(G)$ for some $x \in A$. Here we consider Cayley digraphs: given a generating set B of $\mathbb{Z}^d$, we write G_B for the digraph on $\mathbb{Z}^d$ with edges $E(G_B) = \{(u, v) : v - u \in B\}$.

It is an open problem to determine the minimum possible value of $\partial_{v,G_B}(A)$ or $\partial_{e,G_B}(A)$ for $A \subseteq \mathbb{Z}^d$ of given size, let alone any structural properties of (approximate) minimisers; exact results are only known for a few instances of B (see [3, 4, 26, 23]). It is therefore natural to seek asymptotics. For the vertex boundary, this was achieved by Ruzsa [24], who showed that for $A \subseteq \mathbb{Z}^d$ of size $n \to \infty$, the minimum value $\partial_{v,G_B}(A)$ is asymptotic to that achieved by a set of the form $kC(B) \cap \mathbb{Z}^d$, where $C(B)$ is the conical hull of B, i.e. the convex hull of $B \cup \{0\}$. A corresponding result for the edge boundary was obtained by Barber and Erde [1]: the minimum value $\partial_{e,G_B}(A)$ is asymptotic to that achieved by a set of the form $kZ(B) \cap \mathbb{Z}^d$, where $Z(B) = \{\sum_{b \in B} bx_b : x \in [0, 1]^B\}$ is the zonotope generated by B.

We will prove stability versions of both these results, describing the approximate structure of asymptotic minimisers for both the vertex and edge isoperimetric problems in G_B. The statements require the following notation. Given $B \subseteq \mathbb{Z}^d$ and $n \in \mathbb{N}$ we write $B_n = (\kappa_B(n)C(B)) \cap \mathbb{Z}^d$, where $\kappa_B(n) = \min\{k \in \mathbb{N} : |kC(B) \cap \mathbb{Z}^d| \geq n\}$. We also write $Z(B) = C([B])$, where $[B] = \{\sum B' : B' \subseteq B\}$. We use μ to denote Lebesgue measure.

Theorem 1.1. *Let B be a generating set of $\mathbb{Z}^d$ with $d \geq 2$. Then there is $K \in \mathbb{N}$ so that for any $A \subseteq \mathbb{Z}^d$ with $|A| = n \geq K$ and $\partial_{v,G_B}(A) \leq d\mu(C(B))^{1/d} n^{1-1/d}(1 + \varepsilon)$, where $Kn^{-1/2d} < \varepsilon < K^{-1}$, there exists $v \in \mathbb{Z}^d$ with $|A \triangle (v + B_n)| < Kn\sqrt{\varepsilon}$.*

Theorem 1.2. *Let B be a generating set of $\mathbb{Z}^d$ with $d \geq 2$. For any $\delta > 0$ there are $\varepsilon > 0$ and $N \in \mathbb{N}$ so that for any $A \subseteq \mathbb{Z}^d$ with $|A| = n \geq N$ and $\partial_{e,G_B}(A) \leq d\mu(C([B]))^{1/d} n^{1-1/d}(1+\varepsilon)$, there exists $v \in \mathbb{Z}^d$ with $|A \triangle (v + [B]_n)| < \delta n$.*

We remark that the 'square root dependence' in Theorem 1.1 constitutes a tight quantitative stability result for the vertex isoperimetric inequality in G_B, as may be seen from an example where B consists of the corners of a cube and A is an appropriate cuboid. The stability result in Theorem 1.2 for the edge isoperimetric inequality in G_B, is purely qualitative, although we do obtain a tight quantitative stability result for certain B (see Theorem 3.1 below).

In proving our results, besides drawing on the methods of [24] (particularly Plünnecke's inequality for sumsets) and [1] (a probabilistic reduction to [24]), the most significant new contribution of our paper is a technique for transforming discrete problems to a continuous setting where one can apply results from Geometric Measure Theory. We will employ the sharp estimate on asymmetric index in terms of anisotropic perimeter with respect to any convex set K due to Figalli, Maggi and Pratelli [6] (building on the case when K is a ball, established in [8]). We consider vertex isoperimetry in the next section and then edge isoperimetry in the following section. We conclude the paper by discussing some potential directions for further research.

2 Vertex isoperimetry

This section contains the proof of our sharp tight quantitative stability result for the vertex isoperimetric inequality in general Cayley digraphs. We start in the first subsection with a summary of Ruzsa's approach in [24], during which we record some key lemmas on sumsets and fundamental domains of lattices that we will also use in our proof. In the second subsection we state the Geometric Measure Theory result of [6] (in a simplified setting that suffices for our purposes). The third subsection contains a technical lemma in elementary Real Analysis. We conclude in the final subsection by proving Theorem 1.1.

2.1 Ruzsa's approach

The sumset of $A, B \in \mathbb{Z}^d$ is defined by $A + B := \{a + b : a \in A, b \in B\}$. The vertex isoperimetric problem in the Cayley digraph G_B is equivalent to finding the minimum of $|A + B|$ over all sets A of given size. The following result of Ruzsa [24, Theorem 2] implies an asymptotic for this minimum.

Theorem 2.1. *Let B be a generating set of $\mathbb{Z}^d$ with $d \geq 2$. Then for any $A \subseteq \mathbb{Z}^d$ with $|A| = n \geq N$ we have $|A + B| \geq d\mu(C(B))^{1/d} n^{1-1/d}(1 - O(n^{-1/2d}))$.*

A key ingredient of Ruzsa's proof is the following well-known inequality of Plünnecke [21] (see [24, Statement 6.2]). We use the notation $\Sigma_k(A)$ for the k-fold sumset of A rather than the commonly used kA, which in this paper denotes the dilate of A by factor k.

Theorem 2.2. *Let $k \in \mathbb{N}$ and $A, B \subseteq \mathbb{Z}^d$ with $|A| = n$ and $|A + B| = \alpha n$. Then there is $\emptyset \neq A' \subseteq A$ with $|A' + \Sigma_k(B)| \leq \alpha^k |A'|$.*

Given A of size n with $|A + B|$ small and any $k' \in \mathbb{N}$, Theorem 2.2 provides $A' \subseteq A$ such that $|A' + \Sigma_{k'}(B)|$ is small. The next step is to convert this into smallness of a Minkowski sum $U + V = \{u + v : u \in U, v \in V\}$ of two bodies U, V in $\mathbb{R}^d$. The final bound in Theorem 2.1 will then follow from the Brunn-Minkowski inequality $\mu(U + V)^{1/d} \geq \mu(U)^{1/d} + \mu(V)^{1/d}$ (in the form due to Lusternik [16]).

The required U and V are obtained as $U = A' + Q$ and $V = kC(B)$, taking $k' = k + p$ with p and Q as in the next lemma, where k is a free parameter that can be optimised to obtain the $O(n^{-1/2d})$ error term in Theorem 2.1. This lemma, which is [24, Lemma 11.2], provides a certain set $Q \subseteq \mathbb{R}^d$ with four key properties. Firstly, it is *fundamental*, i.e. any $x \in \mathbb{R}^d$ has a unique representation as $x = y + z$ with $y \in Q$ and $z \in \mathbb{Z}^d$, and so $\mu(X + Q) = |X|$ for any $X \subseteq \mathbb{Z}^d$. Secondly, the conclusion of the lemma shows that $U + V$ as above is small when $A' + \Sigma_{k'}(B)$ is small. The remaining two properties are not explicitly stated in [24], but are clear from the proof; we have $Q \subseteq Z(B)$, and a certain technical condition, which for brevity we will call *nice*: Q is a finite union of bounded convex polytopes.

Lemma 2.3. *Let B be a generating set of $\mathbb{Z}^d$ with $d \geq 2$. Then there are $p \in \mathbb{N}$, $z \in \mathbb{Z}^d$ and a nice fundamental set $Q \subseteq Z(B)$ such that $kC(B) + Q + z \subseteq \Sigma_{k+p}(B) + Q$ for any $k \in \mathbb{N}$.*

2.2 Some Geometric Measure Theory

Here we give a brief account of the quantitative isoperimetric stability result of Figalli, Maggi and Pratelli [6]. We adopt simplified definitions that suffice for sets that are nice, as defined in the previous subsection (see [17, 18] for the general setting of sets of finite perimeter).

For a closed convex polytope $K \subseteq \mathbb{R}^d$ and a union E of disjoint (possibly non-convex) closed polytopes, the perimeter of E with respect to K is given by

$$\mathrm{Per}_K(E) = \lim_{\varepsilon \to 0^+} \frac{\mu(E + \varepsilon K) - \mu(E)}{\varepsilon}.$$

In our setting given a nice set A, for all $r \geq 0$, the measure of $A + rK$ and its closure $\overline{A + rK}$ are the same, that is $\mu(A + rK) = \mu(\overline{A + rK})$. Thus for all $r \geq 0$, (2.1) gives

$$\mathrm{Per}_K(\overline{A + rK}) = \lim_{\varepsilon \to 0^+} \frac{\mu(A + (r + \varepsilon)K) - \mu(A + rK)}{\varepsilon}. \tag{2.1}$$

The anisotropic isoperimetric problem was posed in 1901 by Wulff [27], who conjectured that minimisers of Per_K up to null sets are homothetic copies of K, giving $\mathrm{Per}_K(E) \geq d\mu(K)^{1/d}\mu(E)^{1-1/d}$. This was established for sets E with continuous boundary by Dinghas [5] and for general sets E of finite perimeter by Gromov [19]. It is equivalent to non-negativity of the *isoperimetric deficit* $\delta_K(E)$ of E with respect to K, defined by

$$\delta_K(E) := \frac{\mathrm{Per}_K(E)}{d\mu(K)^{1/d}\mu(E)^{1-1/d}} - 1.$$

We quantify the structural similarity between K and E via the *asymmetric index* (also known as Fraenkel asymmetry) of E with respect to K, which is given by

$$\mathcal{A}_K(E) = \inf\left\{\frac{\mu(E \triangle (x_0 + rK))}{\mu(E)} : x_0 \in \mathbb{R}^d \text{ and } r^d\mu(K) = \mu(E)\right\}.$$

The following is [6, Theorem 1.1].

Theorem 2.4. *For any $d \in \mathbb{N}$ there exists $D = D(d)$ such that for any bounded convex open set $K \subseteq \mathbb{R}^d$ and $E \subseteq \mathbb{R}^d$ of finite perimeter we have*

$$\mathcal{A}_K(E) \leq D\sqrt{\delta_K(E)}.$$

2.3 Some Real Analysis

In this subsection we establish the following technical lemma in elementary Real Analysis, which will allow us to pass to the setting of perimeters in Ruzsa's approach as described in the first subsection, so that we can apply the result of the second subsection. We presume the result is well-known, but we include a proof for completeness.

Lemma 2.5. *Let $f : [a,b] \to \mathbb{R}$ be continuous and right differentiable. Then for any $\varepsilon > 0$ there is $x \in [a,b)$ with $f'_+(x) \leq \frac{f(b)-f(a)}{b-a} + \varepsilon$.*

Proof. Without loss of generality we may assume $a = 0$, $b = 1$ and $f(0) = f(1) = 0$. Suppose for contradiction we have $f'_+(x) \geq \varepsilon > 0$ for all $x \in [0,1]$. Let $B = \{x : f(x) \geq \varepsilon x/2\}$. As $f(1) = 0$ we have $1 \notin B$. As f is continuous, B is closed. The required contradiction will thus follow if we show that B is open to the right, i.e. for any $x \in B$ there is $\delta > 0$ such that $(x, x+\delta) \subseteq B$. To see this, note that for small enough δ, by definition of $f'_+(x)$ we have $f(y) \geq f(x) + (y-x)\varepsilon/2 \geq \varepsilon y/2$ for any $y \in (x, x+\delta)$, so $y \in B$. □

2.4 Stability

In this final subsection we prove our theorem on stability for vertex isoperimetry in G_B.

For convenience we work with sumsets, which is an equivalent setting via the identity $\partial_{v,G_B}(n) = |A + (B \cup \{0\})| - |A|$ for $A, B \subseteq \mathbb{Z}^d$.

Proof of Theorem 1.1. Let B be a generating set of $\mathbb{Z}^d$ with $d \geq 2$, where we may assume $0 \in B$. Suppose $K = K(B,d)$ is sufficiently large and $A \subseteq \mathbb{Z}^d$ is such that $|A| = n \geq K$ and $|A + B| \leq \alpha |A|$, where

$$\alpha = 1 + (1+\varepsilon)\beta n^{-1/d}, \quad \text{with} \quad \beta = d\mu(C(B))^{1/d} \quad \text{and} \quad Kn^{-1/2d} < \varepsilon < K^{-1}.$$

We need to find $v \in \mathbb{Z}^d$ with $|A \triangle (v + B_n)| < Kn\sqrt{\varepsilon}$.

By Lemma 2.3, there are $p \in \mathbb{N}$, $z \in \mathbb{Z}^d$ and a nice fundamental set $Q \subseteq \mathbb{R}^d$ such that $kC(B) + Q + z \subseteq \Sigma_{k+p}(B) + Q$, where we choose $k = \lceil n^{1/2d} \rceil$. By Lemma 2.2 there is $\emptyset \neq A' \subseteq A$ with $|A' + \Sigma_{k+p}(B)| \leq \alpha^{k+p}|A'|$. It now suffices to prove the following claim.

Claim 2.6. We have $|A'| \geq (1+2\varepsilon)^{-d} n$ and $|A' \triangle (v + B_n)| \leq \frac{1}{2}Kn\sqrt{\varepsilon}$ for some $v \in \mathbb{Z}^d$.

To see the bound on $|A'|$, we use the choice of Q and Brunn-Minkowski to get

$$\begin{aligned} \alpha^{k+p}|A'| &\geq |A' + \Sigma_{k+p}(B)| = \mu(A' + \Sigma_{k+p}(B) + Q) \geq \mu(A' + kC(B) + Q) \\ &\geq \left(\mu(A'+Q)^{1/d} + \mu(kC(B))^{1/d}\right)^d = (|A'|^{1/d} + k\mu(C(B))^{1/d})^d. \end{aligned}$$

Expanding the last expression and dividing throughout by $|A'|$ then gives

$$1 + k\beta|A'|^{-1/d} \leq \alpha^{k+p} = 1 + (1+\varepsilon)k\beta n^{-1/d} + O(n^{-1/d}) < 1 + (1+2\varepsilon)k\beta n^{-1/d},$$

from which the required bound follows. For the remaining part of the claim, we start by bounding for some $r \in [0,k)$ the perimeter with respect to $C(B)$ of

$$A_r := \overline{A' + Q + rC(B)}.$$

Letting $f(r) = \mu(A_r)$, by (2.1) and niceness of Q, and so A_r, we have $\mathrm{Per}_{C(B)}(A_r) = f'_+(r)$ for all $r \geq 0$. By Lemma 2.5 with $\varepsilon = 1$, using $f(k) - f(0) < (\alpha^{k+p} - 1)|A'| < (1+2\varepsilon)k\beta n^{-1/d}|A'|$, there is an $r \in [0,k)$ such that

$$\mathrm{Per}_{C(B)}(A_r) \leq \frac{f(k) - f(0)}{k} + 1 < (1+3\varepsilon)\beta n^{-1/d}|A'|.$$

Theorem 2.4 now bounds the asymmetric index with respect to $C(B)$ of A_r: we have $\mathcal{A}_{C(B)}(A_r) \leq D\sqrt{3\varepsilon}$. Thus there is $t \in \mathbb{R}^d$ such that

$$\mu\big(A_r \triangle (t + r'C(B))\big) \leq D\sqrt{3\varepsilon}\mu(A_r),$$

or equivalently

$$\mu\big(A_r \triangle (t + r'C(B))\big) \leq D\sqrt{3\varepsilon}\mu(A_r),$$

where

$$r' = (\mu(A_r)/\mu(C(B))^{1/d} < q := (\alpha^{k+p}|A'|/\mu(C(B))^{1/d}.$$

Since $\alpha^{k+p} = 1 + o(1)$, we have $\mu\big((A' + Q) \setminus (t + qC(B))\big) \leq 2Dn\sqrt{\varepsilon}$. Now we fix $v \in \mathbb{Z}^d$ such that $t + (q + D)C(B) \subseteq v + (q + 2D)C(B)$, which is possible by taking D large enough that $DC(B)$ contains $[-1, 1]^d$. We may also assume that $DC(B)$ contains Q, as $Q \subseteq Z(B)$. We will show that v satisfies the claim.

To see this, we first note that

$$|A' \triangle (v + B_n)| \leq 2|A' \setminus (v + B_n)| + |B_n| - |A'| \leq 2|A' \setminus (v + B_n)| + 3d\varepsilon n,$$

using the bound on $|A'|$ and $|B_n| = n + O(n^{1-1/d})$, so it suffices to bound $|A' \setminus (v + B_n)|$.

As $(A' \setminus (t + (q + D)C(B))) - Q \subseteq (A' + Q) \setminus (t + qC(B))$, we have

$$\begin{aligned}|A' \setminus (v + (q + 2D)C(B))| &\leq |A' \setminus (t + (q + D)C(B))| \\ &= \mu(-(A' \setminus (t + (q + D)C(B))) + Q) \leq 2Dn\sqrt{\varepsilon}.\end{aligned}$$

Writing $X = \mathbb{Z} \cap (v + (q + 2D)C(B))$ we have

$$|A' \setminus (v + B_n)| \leq |A' \setminus (v + (q + 2D)C(B))| + |X| - n,$$

so it remains to bound $|X|$. As $X + [-1/2, 1/2]^d \subseteq v + (q + 3D)C(B)$, we have

$$|X| = \mu(X + [-1/2, 1/2]^d) \leq (q + 3D)^d\mu(C(B)) = (1 + 3D/q)^d\alpha^{k+p}|A'| < n + O(n^{1-1/2d}).$$

Thus v satisfies the claim, so the theorem follows. □

3 Edge isoperimetry

In this short section we deduce our stability result for edge isoperimetry from our stability result for vertex isoperimetry proved in the previous section (we use the reduction found by Barber and Erde [1]).

Proof of Theorem 1.2. Let B be a generating set of $\mathbb{Z}^d$ with $d \geq 2$. We adopt a parameter hierarchy $n^{-1} \ll \varepsilon \ll s^{-1} \ll \gamma \ll \delta \ll |B|^{-1}$, i.e. let δ be small given $|B|$, let γ be small given δ, let s be large given γ, let ε be small given s, and let n be large given ε. Suppose $A \subseteq \mathbb{Z}^d$ is such that $|A| = n$ and $\partial_{e,G_B}(A) \leq \beta n^{1-1/d}(1+\varepsilon)$, where $\beta = d\mu(Z)^{1/d}$ with $Z = Z(B) = C([B])$. We need to find $v \in \mathbb{Z}^d$ with $|A \triangle (v + [B]_n)| < \delta n$.

From the proof of [1, Theorem 1], identifying our parameters with theirs via $\eta = \gamma/2$ and $s = (1-\eta)t/k$, we have $|A + \Sigma_s([B])| - |A| < (1+\gamma)s\beta n^{1-1/d}$. As

$$|A + \Sigma_s([B])| - |A| = \sum_{j=0}^{s-1} |A + \Sigma_{j+1}([B])| - |A + \Sigma_j([B])|,$$

we can therefore fix $A_+ = A + \Sigma_j([B])$ for some $j < s$ such that

$$|A_+ + [B]| - |A_+| < (1+\gamma)\beta n^{1-1/d} \leq (1+\gamma)\beta n_+^{1-1/d},$$

where $n_+ = |A_+| \leq n + O(n^{1-1/d})$. By Theorem 1.1 we have $|A_+ \triangle (v + B_{n_+})| < \frac{1}{2}\delta n_+$ for some $v \in \mathbb{Z}^d$. Now $|A \triangle (v+[B]_n)| < |A_+ \triangle (v+[B]_{n_+})| + |A_+ \setminus A| + |\mathbb{Z} \cap [B]_{n_+}| - |\mathbb{Z} \cap [B]_n| < \delta n$. $\square$

Theorem 1.2 is purely qualitative, but there is a simple trick that gives the following tight quantitative result when the generating set B takes the form $\{\pm v : v \in \mathcal{B}\}$ for some integer basis $\mathcal{B}$ of $\mathbb{Z}^d$ (which may as well be the standard basis $\{e_1, \ldots, e_d\}$).

Theorem 3.1. *Let $B = \{\pm e_i : i \in [d]\} \subseteq \mathbb{Z}^d$ with $d \geq 2$. Then there is $K \in \mathbb{N}$ so that for any $A \subseteq \mathbb{Z}^d$ such that $|A| = n \geq K$ and $\partial_{e,G_B}(A) \leq 2dn^{1-1/d}(1+\varepsilon)$, where $Kn^{-1/2d} < \varepsilon < K^{-1}$, there exists $v \in \mathbb{Z}^d$ with $|A \triangle (v + [B]_n)| < Kn\sqrt{\varepsilon}$.*

Proof. Let $A' = A + [-1/2, 1/2]^d$. Then the edges of G_B counted by $\partial_{e,G_B}(A)$ are in bijection with those $(d-1)$-cubes that occur exactly once as $x + C$ with $x \in C$ and C a facet of $[-1/2, 1/2]^d$. Thus $\partial_{e,G_B}(A) = \mathrm{Per}_Z(A')$, where $Z = Z(B) = [-1,1]^d$. By Theorem 2.4 we have $\mathcal{A}_Z(A') \leq D\sqrt{\varepsilon}$, i.e. there is $x \in \mathbb{R}^d$ with $\mu(A' \triangle (x + rZ)) \leq nD\sqrt{\varepsilon}$, where $\mu(rZ) = (2r)^d = n$. We fix $v \in \mathbb{Z}^d$ with $x + (r+1)Z \subseteq v + (r+2)Z$. As

$$(A \setminus (x + (r+1)Z)) + [-1/2, 1/2]^d \subseteq A' \setminus (x + rZ)$$

we have $|A \setminus (v + (r+2)Z)| \leq nD\sqrt{\varepsilon}$. The theorem now follows from

$$A \triangle (v + [B]_n)| = 2|A \setminus (v + (r+2)Z)| + O(n^{1-1/d}) < Kn\sqrt{\varepsilon}. \qquad \square$$

We remark that by considering $A' = A + [-1/2, 1/2]^d$ as in the previous proof we can also improve the result of [1] on the edge isoperimetric problem in G_B. Indeed, by the anisotropic isoperimetric inequality, $\partial_{e,G_B}(A) = \mathrm{Per}_Z(A') \geq d\mu(Z)^{1/d}\mu(A')^{1-1/d} = 2dn^{1-1/d}$, which is tight whenever $n = k^d$ for some $k \in \mathbb{N}$.

4 Concluding Remarks

As mentioned in the introduction, there are several challenging and important open problems in isoperimetric stability, such as the Kahn-Kalai Conjecture and the Polynomial Freiman-Ruzsa Conjecture. We therefore find it rather striking that in this short paper we have been able to characterise isoperimetric stability for general Cayley graphs in lattices. Of course, the brevity of our paper masks the fact that we have greatly relied on previous work, particularly an analogous stability result of [6] in Geometric Measure Theory. This naturally suggests that further investigation of transformations between the discrete and continuous settings may be fruitful in future research.

For the isoperimetric problems considered in this paper, it is natural to ask if one can obtain tighter estimates than those in the asymptotic results of [1, 24], particularly for the edge boundary, where the probabilistic reduction in [1] introduces error terms that are presumably far from optimal. It would be interesting to determine whether our improved estimate for the edge isoperimetric inequality in the remark following Theorem 3.1 holds for general B: do we always have $\partial_{e,G_B}(A) \geq d\mu(Z(B))^{1/d}|A|^{1-1/d}$? Also, do we always have $\partial_{v,G_B}(A) \geq d\mu(C(B))^{1/d}|A|^{1-1/d}$?

References

[1] B. Barber and J. Erde. Isoperimetry in integer lattices. *Discrete Analysis*, 7, 2018.

[2] S. Bezrukov. Isoperimetric problems in discrete spaces. *Extremal Problems for Finite Sets, Bolyai Soc. Math. Stud. 3*, pages 59–91, 1994.

[3] B. Bollobás and I. Leader. Compressions and Isoperimetric Inequalities. *Journal of Combinatorial Theory Series A*, 56:47–62, 1991.

[4] B. Bollobás and I. Leader. Edge-isoperimetric inequalities in the grid. *Combinatorica*, 11:299–314, 1991.

[5] A. Dinghas. Über einen geometrischen Satz von Wulff für die Gleichgewichtsform von Kristallen. *Z. Kristallogr., Mineral. Petrogr.*, 105, 1944.

[6] A. Figalli, F. Maggi, and A. Pratelli. A mass transportation approach to quantitative isoperimetric inequalities. *Invent. Math.*, 182(1):167–211, 2010.

[7] K. Frankston, J. Kahn, B. Narayanan, and J. Park. Thresholds versus fractional expectation-thresholds. *arXiv:1910.13433*, 2019.

[8] N. Fusco, F. Maggi, and A. Pratelli. The sharp quantitative isoperimetric inequality. *Annals of Mathematics*, 168:941–980, 2008.

[9] B. J. Green. The Polynomial Freĭman-Ruzsa Conjecture. *Terence Tao's blog*, 2007.

[10] J. Kahn and G. Kalai. Thresholds and expectation thresholds. *Combinatorics, Probability and Computing*, 16:495–502, 2007.

[11] P. Keevash and E. Long. Stability for vertex isoperimetry in the cube. *arXiv:1807.09618*, 2018.

[12] P. Keevash and E. Long. A stability result for the cube edge isoperimetric inequality. *J. Combin. Theory Ser. A*, 155:360–375, 2018.

[13] N. Keller and N. Lifshitz. Approximation of biased Boolean functions of small total influence by DNF's. *Bulletin of the London Mathematical Society*, 50(4):667–679, 2018.

[14] I. Leader. Discrete isoperimetric inequalities. *Probabilistic Combinatorics and its Applications*, 1991.

[15] M. Ledoux. The concentration of measure phenomenon. *Mathematical Surveys and Monographs*, 89, 2005.

[16] L. Lusternik. Die Brunn-Minkowskische Ungleichung für beliebige messbare Mengen. *C. R. Acad. Sci. URSS*, 8:55–58, 1935.

[17] F. Maggi. Some methods for studying stability in isoperimetric type problems. *Bull. Amer. Math. Soc.*, 45:367–408, 2008.

[18] F. Maggi. *Sets of finite perimeter and geometric variational problems: an introduction to Geometric Measure Theory.* Number 135 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2012.

[19] V. Milman and G. Schechtman. *Asymptotic Theory of Finite-dimensional Normed Spaces. With an appendix by M. Gromov*, volume 1200 of *Lecture Notes in Mathematics.* Springer, Berlin, 1986.

[20] R. O'Donnell. *Analysis of Boolean functions.* Cambridge University Press, 2014.

[21] H. Plünnecke. Eine zahlentheoretische Anwendung der Graphentheorie. *J. Reine Angew. Math*, 243:171–183, 1970.

[22] M. Przykucki and A. Roberts. Vertex-isoperimetric stability in the hypercube. *Journal of Combinatorial Theory Series A*, 172, 2020.

[23] J. Radcliffe and E. Veomett. Vertex isoperimetric inequalities for a family of graphs on $\mathbb{Z}^k$. *Electron. J. Combin.*, 19(2):P45, 2012.

[24] I. Ruzsa. Sets of sums and commutative graphs. *Studia Sci. Math. Hungar*, 30:127–148, 1995.

[25] M. Talagrand. Concentration of measure and isoperimetric inequalities in product spaces. *Publ. Math. I.H.E.S.*, 81:73–203, 1995.

[26] D. Wang and P. Wang. Discrete isoperimetric problems. *SIAM J. Appl. Math.*, 32:860–870, 1977.

[27] G. Wulff. Zur Frage der Geschwindigkeit des Wachsrurms und der Auflösung der Kristallflächen. *Z. Kristallogr.*, 34:449–530, 1901.