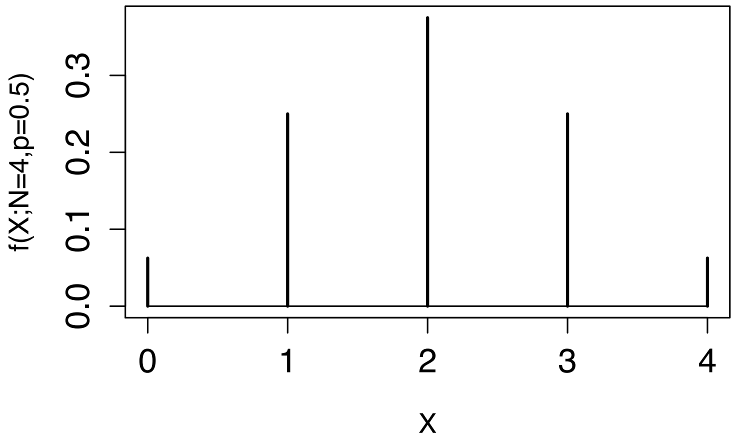
The function that returns the probability of the event "$X = n$" is called the binomial "probability mass function". (If you think of probabilities as masses, it gives the probability mass at the point $X = n$ for any $n \in \{0, 1, 2, \ldots, N\}$). It is as follows:

The binomial probability mass function:

$$\begin{aligned} f(n; N, p) \;=\; \mathbb{P}(X = n) \;&=\; \binom{N}{n} p^n (1-p)^{N-n} \qquad \text{for } n \in \{0, 1, 2, \ldots, N\} \\ &=\; 0 \qquad\qquad\qquad\qquad\quad \text{otherwise.} \end{aligned}$$

We say that X "has a binomial distribution" or "is binomially distributed". Figure 2 shows the binomial probability mass function (pmf) for $N = 4$ and $p = 0.5$.

Figure 2: The binomial probability mass function for $N = 4$, $p = 0.5$.



3.2 The binomial cumulative distribution function

The function that returns the probability of getting *no more than* n successes in N trials when the probability of succeeding on any one trial is p and trials are independent is called the binomial "cumulative distribution function". It is easily written down, but not so easily evaluated. It is (fairly self-evidently) the following:

The binomial cumulative distribution function:

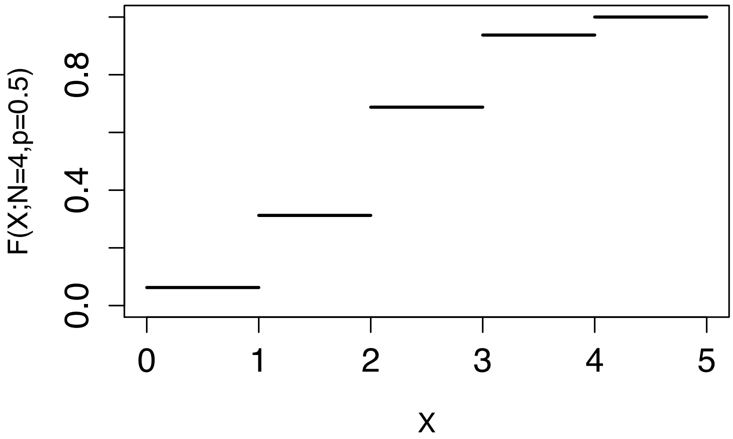
$$F(n; N, p) \;=\; \mathbb{P}(X \leq n) \;=\; \sum_{j=1}^{\lfloor n \rfloor} f(j; N, p) \;=\; \sum_{j=0}^{\lfloor n \rfloor} \binom{N}{j} p^j (1-p)^{N-j} \qquad \text{for } n \geq 0$$

where $\lfloor n \rfloor$ means "the greatest integer less than or equal to n", or equivalently "the integer part of n".

Figure 3 shows the binomial cumulative distribution function (cdf) for $N = 4$ and $p = 0.5$.

The probability of getting *at least* n successes in N trials is $1 - \mathbb{P}(X \leq (n-1)) = 1 - F(n-1; N, p)$. So the probability of at least 1 of N animals making 4 correct predictions when the probability of 4 correct predictions is $p = 1/16 = 0.0625$ is

Figure 3: The binomial cumulative distribution function for $N = 4$, $p = 0.5$.



$$
\begin{aligned}
\mathbb{P}(X \geq n) &= 1 - F(0; N = 20, p = 0.0625) \\
&= 1 - \sum_{j=0}^{0} f(0; N = 20, p = 0.0625) \\
&= 1 - \sum_{j=0}^{0} \binom{N}{j} 0.0625^j (1 - 0.0625)^{N-j} \\
&= 1 - f(0; N = 20, p = 0.0625) \\
&\approx 0.725.
\end{aligned}
$$

(Note that $F(0; n, p) = f(0; n, p)$ in the case of the binomial, since 0 is the lowest possible value that X can take.)

4 Why unlikely things are likely to happen

4.1 The birthday problem

A common example of an apparently unlikely event that is actually quite likely is at least two people in the same class having the same birthday. Assuming that birthdays occur randomly in the year (and forgetting about leap years), the probability of a randomly chosen person's birthday being on any particular day is $1/365 \approx 0.27\%$. And yet in any group of 23 randomly chosen people there is a 50% chance that at least two share the same birthday.

To see how we calculate this probability, let's take this year's MT1010 class as an example. It contains 18 students. What is the probability that at least one shares a birthday with someone else in the class? We can answer this question by calculating the probability that none share the same birthday and subtracting this from 1. (This is the easy way to to do it because there are many ways that people in the class can share birthdays, but only one way that none share a birthday.) We can calculate this by considering each person in turn and calculating the conditional probability that they do