⑥ Recall

$$\delta(\Gamma) = \inf\left\{ s \geq 0 : \underbrace{\sum_{z \in \Gamma(0)} \left(\frac{1-|z|}{1+|z|}\right)^s}_{P_\Gamma(s)} < \infty \right\}$$

Note that

$$P_\Gamma(s) \leq \sum_{z \in \Gamma(0)} (1-|z|)^s = \sum_{z \in \Gamma(0)} (1+|z|)^s \left(\frac{1-|z|}{1+|z|}\right)^s$$

$$\leq 2^s P_\Gamma(s)$$

and so

$$\sum_{z \in \Gamma(0)} (1+|z|)^s < \infty$$

if and only if $P_\Gamma(s) < \infty$

and therefore

$$\delta(\Gamma) = \inf\left\{ s \geq 0 : \sum_{z \in \Gamma(0)} (1-|z|)^s < \infty \right\}.$$

(7) Let Γ be Fuchsian and $\Gamma' \leq \Gamma$.

claim: Γ' is a Fuchsian group.

Let $g \in \Gamma' \subseteq PSL(2,\mathbb{R}) \equiv \mathbb{R}^4$. Then $g \in \Gamma$ and so for some $\varepsilon > 0$, we have $B(g,\varepsilon) \cap \Gamma = \{g\}$. Since $\Gamma' \subseteq \Gamma$, it follows that $B(g,\varepsilon) \cap \Gamma' = \{g\}$ and so Γ' is discrete. It is also a group by definition and so we are done.

claim: $L(\Gamma') \subseteq L(\Gamma)$.

Let $z \in L(\Gamma')$ and $g_n \in \Gamma'$ be such that $g_n(0) \to z$ (in $|\cdot|$). Since $g_n \in \Gamma' \subseteq \Gamma$ it follows that $z \in \overline{\Gamma(0)}$ and so $z \in L(\Gamma)$.

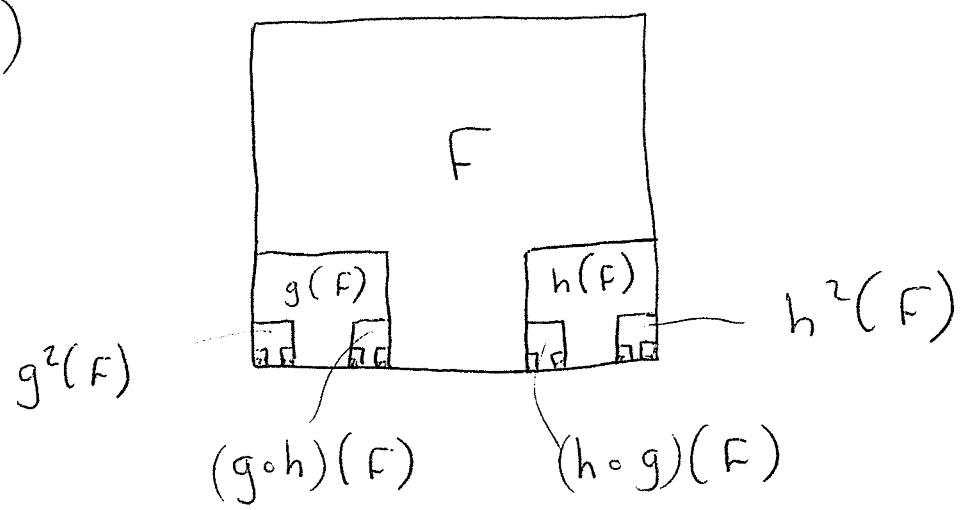
claim: $\delta(\Gamma') \leq \delta(\Gamma)$.

Let $s > \delta(\Gamma)$, which by definition means

$$P_{\Gamma'}(s) = \sum_{z \in \Gamma'(0)} \left(\frac{1-|z|}{1+|z|}\right)^s \leq \sum_{z \in \Gamma(0)} \left(\frac{1-|z|}{1+|z|}\right)^s = P_\Gamma(s) < \infty$$

and so $\delta(\Gamma') \leq s$. Since $s > \delta(\Gamma)$ arbitrary, result follows.

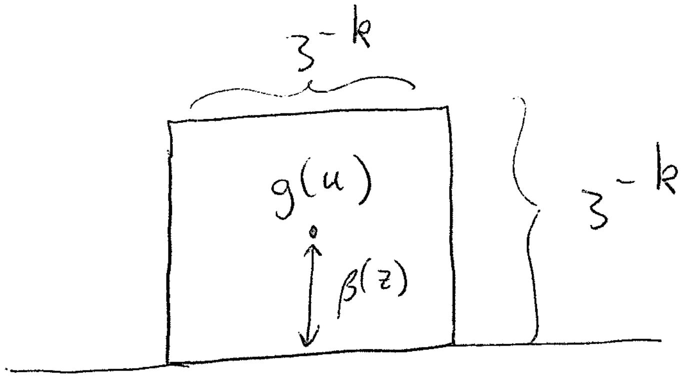
8 (i)



The limit set is the middle 3rd Cantor set!

(ii) Let $z = g(u) \in \Gamma(u)$ and note that if the 'length' of the word describing g is k, then $\beta(z) = \frac{3^{-k}}{2}$.

For example, the length of Id is 0, the words of length 1 are g, h, the words of length 2 are g^2, h^2, gh and hg.



Let Γ_k denote the words of length k and observe that $\Gamma = \bigcup_{k=0}^{\infty} \Gamma_k$ is a disjoint union and, for all k, we have $|\Gamma_k| = 2^k$.

(8) continued...

Hence $\displaystyle\sum_{z \in \Gamma(u)} \beta(z)^s = \sum_{k=0}^{\infty} \sum_{z \in \Gamma_k(u)} \beta(z)^s$

$$= \sum_{k=0}^{\infty} \sum_{z \in \Gamma_k(u)} \left(\frac{3^{-k}}{2}\right)^s$$

$$= \sum_{k=0}^{\infty} |\Gamma_k(u)| \left(\frac{3^{-k}}{2}\right)^s$$

$$= 2^{-s} \sum_{k=0}^{\infty} \left(2/3^s\right)^k$$

This is a geometric series, which converges if and only if $2/3^s < 1$ which holds if and only if $s > \dfrac{\log 2}{\log 3}$.

Therefore $\delta(\Gamma) = \dfrac{\log 2}{\log 3}$.

(iii) $\dfrac{\log 2}{\log 3}$ is the Hausdorff dimension of the limit set!