Chapter 2

Functions of one variable: Revision

{chap:2}

2.1 Functions

The idea of a function was first introduced by Leibnitz in 1673 to denote the dependence of one quantity on another. The modern definition is

Definition 1

A *function* is a rule that assigns to each element in a set A one and only one element in a set B.
Note that

1. A and B need not be sets of real numbers, but in this course we will take them as real.
2. The set A is called the *domain* of the function. In one variable, if x is an element of A, then we denote the corresponding element in B by $f(x)$. The set of all possible values of $f(x)$ is called the *range* of f.

Definition 2

The *graph* of a function f is defined to be the Cartesian graph of

$$y = f(x).$$

Example 2.3

Consider the function defined by

$$f(x) = \sqrt{x-1}$$

The graph of $f(x)$ is shown in Figure 2.1. From Figure 2.1 the domain is given by

$$x \in [1, \infty) \qquad \text{or} \qquad x \geq 1,$$

and the range is

$$y \in [0, \infty) \qquad \text{or} \qquad y \geq 0.$$

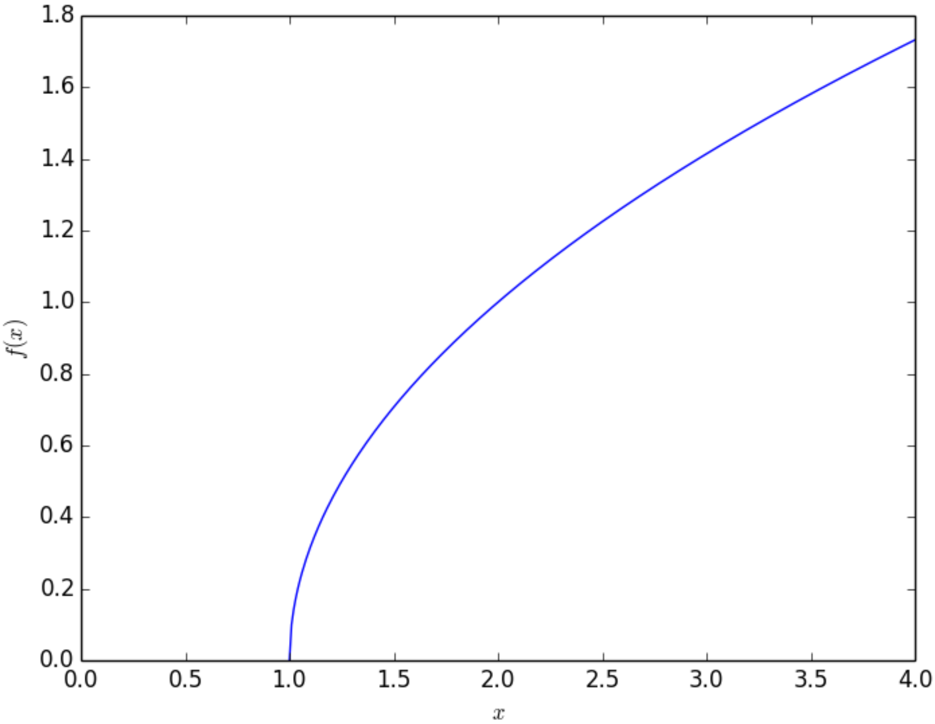


Figure 2.1: The graph of the function $f(x) = \sqrt{x-1}$. {fig:1.2}

Example End

Remember the notation

$$x \in [a, b] \qquad \Rightarrow \qquad a \leq x \leq b,$$

and

$$x \in (a, b) \qquad \Rightarrow \qquad a < x < b.$$

2.2 Limit of a function and continuity

A limit of a function is the value that the function $f(x)$ approaches as x approaches a given value. For real variables this value may be approached from the left or the right (i.e. from below or above). Suppose that $f(x)$ is defined over an open interval (a, b), namely $a < x < b$, and $f(x_0) = L$. x_0 is the value x approaches and L is the value of the function of the function at $x = x_0$. We want to see what the limit of $f(x)$ is as x approaches x_0 from below and above. The limit from the *left* is

$$L_- = \lim_{x \to x_0^-} f(x),$$

where x_0^- indicates that we are approaching x_0 from the left (or below). The limit from the *right* is

$$L_+ = \lim_{x \to x_0^+} f(x),$$

where x_0^+ indicates that this time we approach x_0 from the right (or above). The function $f(x)$ is *continuous* at $x = x_0$ if $L_+ = L_- = L$. This means that the limit is independent of how we approach x_0 and is equal to $f(x_0) = L$. We then say

$$\lim_{x \to x_0} f(x) = L.$$

Sometimes we write

$$\lim_{h \to 0} f(x + h) = L.$$

This form of the limit is slightly reminiscent of the limit we use in evaluating derivatives from first principles.

If $L_+ \neq L_-$, then the function is not continuous and is called *discontinuous*. An example of a discontinuous function is shown in Figure 2.2. Notice how the function, f, jumps as x crosses x_0. In

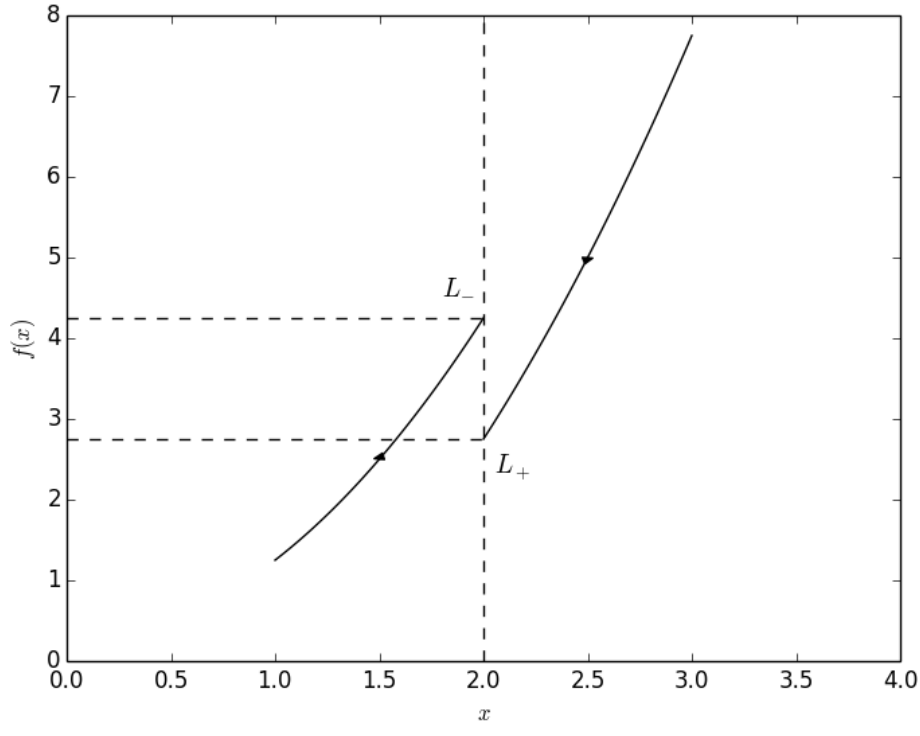


Figure 2.2: An example of a discontinuous function. {fig:1.3a}

most cases when $L_- = L_+$, (having the value L, say) we find that

$$f(x_0) = L.$$

However, this is not always true. Consider

$$f(x) = \frac{\sin x}{x}.$$

This is *not* defined at $x = 0$ (putting $x = 0$ gives $0/0$) but the limit as $x \to 0$ *does* exist. For this example

$$L_- = L_+ = 1.$$

This is an example of an *indeterminate form*. The limit can be found using *L'Hôpital's Rule* or by using the power series for $\sin(x)$.

Thus, we can make the function continuous at $x = 0$ if we modify the definition so that

$$f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

Defining the function in this manner means that it is continuous at $x = x_0 = 0$ since

$$L_+ = L_- = f(0).$$

Of course, if $L_+ \neq L_-$ then the function is necessarily discontinuous at $x = x_0$.

Example 2.4

Which of the following functions are continuous at $x = 0$? You may often find that a sketch of the function is very helpful. If you can draw the graph of the function without lifting your pen then it is continuous.

Solution 2.4

(a) $f(x) = x^2 + 2$. Hence, we can clearly see that

$$L_+ = L_- = 2 = f(0).$$

(b) $f(x) = \frac{|x|}{x}$. Remember that the modulus function is defined as

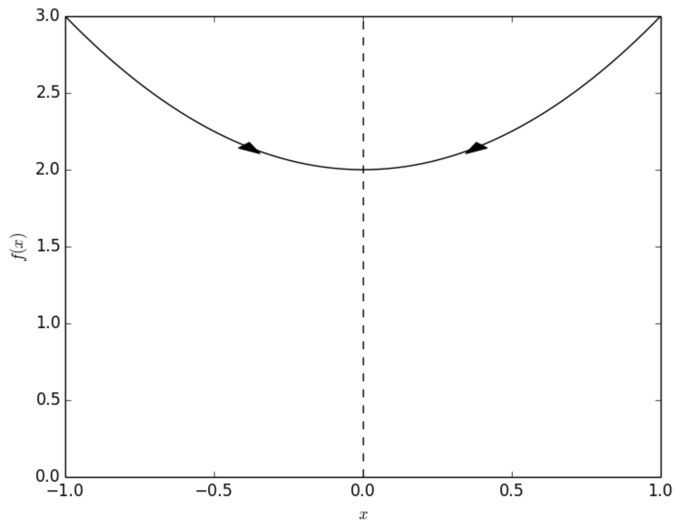
$$|x| = \begin{cases} -x & x < 0 \\ x & x > 0 \end{cases}$$

In this case

$$L_+ = 1, \qquad L_- = -1, \qquad \Rightarrow \qquad \text{function is discontinuous.}$$

(c) $f(x) = \frac{e^x - 1}{x}$. This function is indeterminate at $x = 0$, since we have $0/0$. We will determine the left and right limit by using L'Hôpital's rule. Thus, we can calculate the left and right limits as

$$\begin{aligned} L_- &= \lim_{x \to 0_-} \left\{ \frac{e^x - 1}{x} \right\} = \lim_{x \to 0_-} \left\{ \frac{e^x}{1} \right\} = 1, \\ L_+ &= \lim_{x \to 0_+} \left\{ \frac{e^x - 1}{x} \right\} = \lim_{x \to 0_+} \left\{ \frac{e^x}{1} \right\} = 1. \end{aligned}$$



{fig:1.3} Figure 2.3: The graph of $x^2 + 2$.

Hence, since $L_- = L_+ = 1$, we can define $f(0) = 1$. So when the function is indeterminate at a point, if the limit as $x \to x_0$ from the left equals the limit from the right, we can define the function to have that value at $x = x_0$. Thus,

$$f(x) = \begin{cases} \frac{e^x - 1}{x} & x \neq 0, \\ 1 & x = 0, \end{cases}$$

which is continuous at $x = 0$.

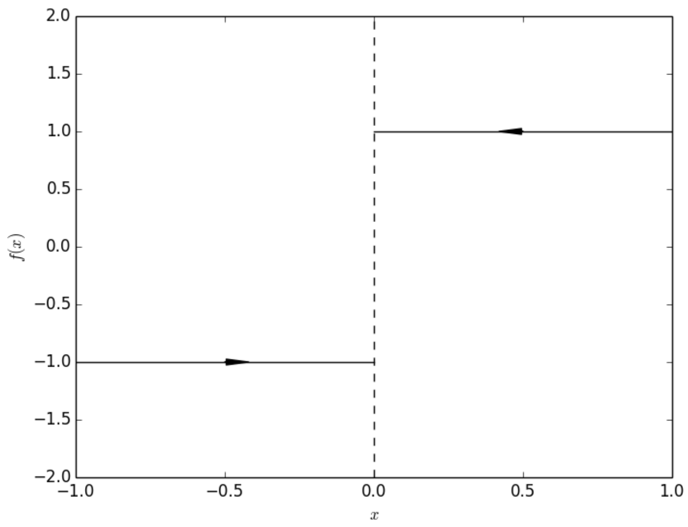
Example End

Notes

1. Continuity at a *point* can be extended to *regions*. If $f(x)$ is continuous at all points where $x \in (a, b)$, i.e. $a < x < b$, then $f(x)$ is said to be continuous on the interval (a, b).

2. Continuity in a region can be established pictorially by drawing a graph of the function. For example,
$$f(x) = \frac{1}{x}$$
is shown in Figure 2.5. This is obviously continuous everywhere except at $x = 0$, where it jumps from $-\infty$ to $+\infty$.

3. If you sketch the graph of $y = f(x)$ over a given domain and your pencil has to leave the paper, then $f(x)$ is discontinuous.



1.4b} Figure 2.4: The graph of $|x|/x$.

2.3 Differentiability

The derivative of $f(x)$, denoted by

$$\frac{df}{dx}, \qquad \text{or} \qquad f'(x),$$

at a point x_0 is defined as the limit

$$\frac{df}{dx}(x_0) = f'(x_0) = \lim_{x \to x_0} \left\{ \frac{f(x) - f(x_0)}{x - x_0} \right\},$$

provided the limit exists in the sense of the previous section as $x \to x_0^+$ and $x \to x_0^-$. A function for which such a limit exists is said to be *differentiable* at $x = x_0$.

The derivative can also be defined as *a function of* x,

$$\frac{df}{dx} = f'(x) = \lim_{h \to 0} \left\{ \frac{f(x+h) - f(x)}{h} \right\},$$

and can have continuity properties itself. If $f'(x)$ is continuous in a given interval, we say that $f(x)$ is a *continuously differentiable* function in that interval.

The derivative at a point, say $x = x_1$, can be interpreted as the slope of the function $f(x)$ at that point and it gives the *gradient of the tangent* to the curve $y = f(x)$ at the point $x = x_1$ and $y = f(x_1)$. This is illustrated in Figure 2.6.

The slope/ gradient/ derivative of the straight line passing through the points $(x_1, f(x_1))$ and $(x_1 + h, f(x_1 + h))$ is

$$\text{slope} = \text{gradient} = \frac{f(x_1 + h) - f(x_1)}{h}.$$

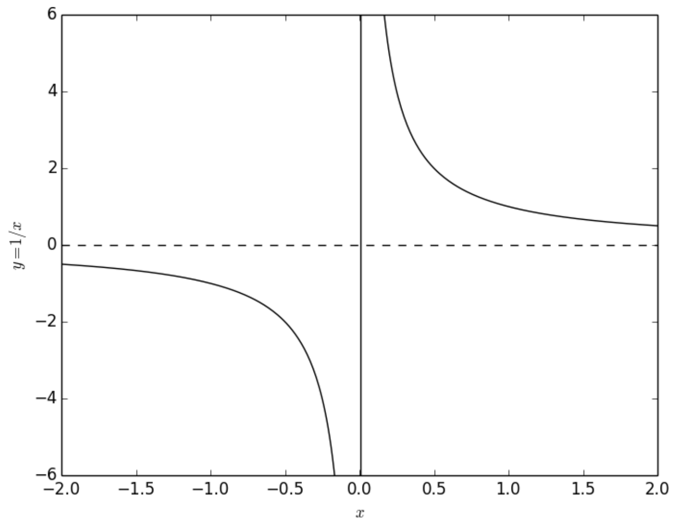


Figure 2.5: Graph of $1/x$ {fig:1.5}

As $h \to 0$ the straight line becomes tangential to $f(x)$ at $x = x_1$, i.e. the slope of f at this point, and that of the straight line are equal.

The derivative can also be interpreted as giving the rate of change of f with x. For example, if f is height and x is the horizontal coordinate, then the *slope* or *gradient* is

$$\frac{df}{dx}.$$

However, if f is the position and t is the time, then the rate of change of position with time is the speed. Thus,

$$\frac{df}{dt} = \text{speed}.$$

Since the *derivative* can be interpreted as the *slope* of the graph of $f(x)$, the existence of a continuous derivative in an interval implies the graph is smooth, that is it is continuous and does not have any sharp corners. Three possible cases are shown in Figure 2.7. In Figure 2.7, the limit of $f'(x)$ (that is the derivative of $f(x)$) does not exist in (b) at $x = x_0$ and in shown in (c) to be discontinuous. In this case we have $x_0 = 1$. Clearly, we have

- f differentiable implies f continuous (smoothness implies continuous).
- f continuous does *not* imply f differentiable (continuous does not imply smoothness).

Example 2.5

(1) Is

$$g(x) = \begin{cases} 1 + x^2, & x \geq 0, \\ 1 - x^2 & x < 0, \end{cases}$$

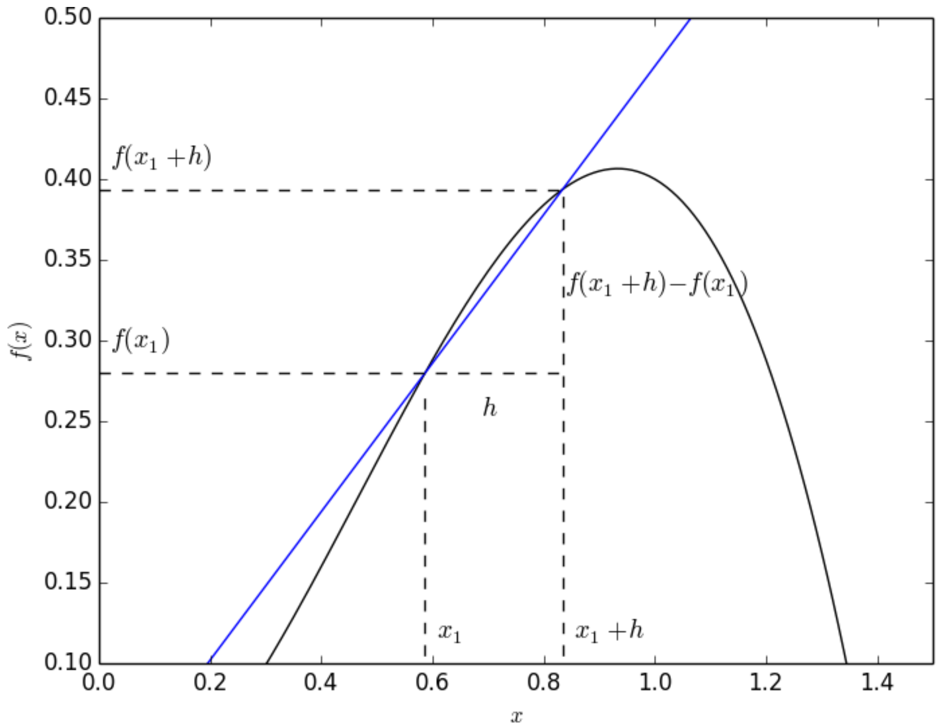


Figure 2.6: The derivative of the function $f(x)$ at $x = x_1$ is illustrated as the slope of the straight line
;:1.6} joining the points $(x_1, f(x_1))$ and $(x_1 + h, f(x_1 + h))$.

continuous and differentiable at $x = 0$?

For continuity we need

$$\lim_{x\to 0^+}\left\{1 + x^2\right\} = 1, \qquad \lim_{x\to 0^-}\left\{1 - x^2\right\} = 1.$$

Thus, $L_+ = L_- = 1 = g(0)$, (since $g(x)$ is defined at $x = 0$).

For differentiability we need to check that

$$g'(0) = \lim_{h\to 0}\left\{\frac{g(0 + h) - g(0)}{h}\right\}$$

gives the same limit for $h \to 0^+$ and 0^-.

$$\begin{aligned} L_- &= \lim_{h\to 0^-}\left\{\frac{(1 - h^2) - 1}{h}\right\} = \lim_{h\to 0^-}\{-h\} = 0, \\ L_+ &= \lim_{h\to 0^+}\left\{\frac{(1 + h^2) - 1}{h}\right\} = \lim_{h\to 0^+}\{h\} = 0. \end{aligned}$$

Since $L_- = L_+ = 0$, g is differentiable at $x = 0$.

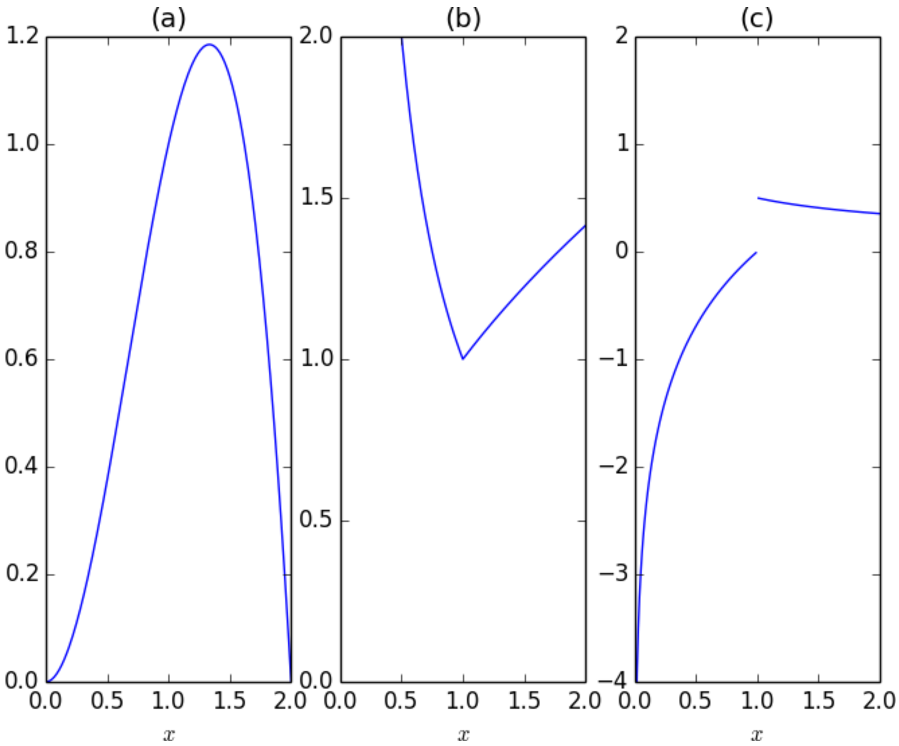


Figure 2.7: (a) A continuous and differentiable function. (b) A continuous but not differentiable function at the sharp corner. (c) The derivative of (b) is discontinuous. {fig:1.7}

(2) Is

$$f(x) = |x| = \begin{cases} x & x \geq 0, \\ -x & x < 0. \end{cases}$$

differentiable at $x = 0$? To check we need to evaluate the limits for L_- and L_+. Thus,

$$\begin{aligned} L_- &= \lim_{x \to 0^-} \{-x\} = 0, \\ L_+ &= \lim_{x \to 0^+} \{x\} = 0, \end{aligned}$$

shows that $f(x)$ is continuous at $x = 0$ and $L_- = L_+ = f(0) = 0$. For the derivative we have

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \left\{ \frac{f(0+h) - f(0)}{h} \right\}, \\ L_- &= \lim_{h \to 0^-} \left\{ \frac{-h - 0}{h} \right\} = -1, \\ L_+ &= \lim_{h \to 0^+} \left\{ \frac{h - 0}{h} \right\} = +1. \end{aligned}$$

Thus, since $L_- \neq L_+$, $f(x)$ is *not differentiable* at $x = 0$.

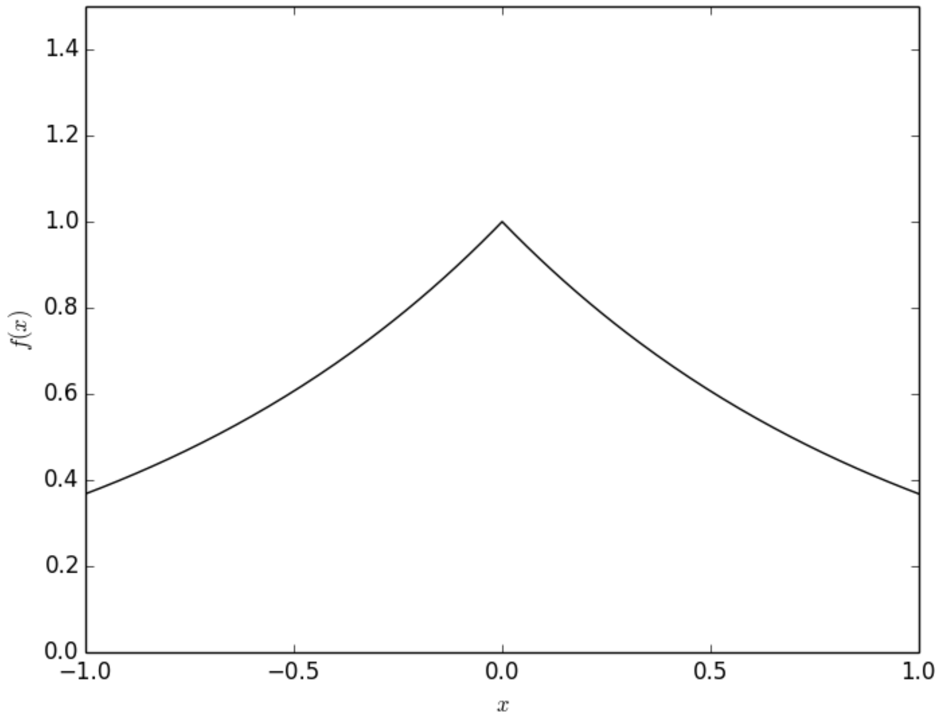
(3) Consider

$$f(x) = e^{-|x|}.$$

Thus, we can express $f(x)$ as

$$f(x) = \begin{cases} e^{-x} & x \geq 0, \\ e^{x} & x < 0. \end{cases}$$

This function is shown in Figure 2.8. Clearly, there is no jump at $x = 0$, so the function is continuous,



:1.8} Figure 2.8: The function $e^{-|x|}$.

but it is not smooth, so it is not differentiable. We prove these statements as follows.

$$\begin{aligned} L_- &= \lim_{x\to 0^-} \left\{ e^{+x} \right\} = 1, \\ L_+ &= \lim_{x\to 0^+} \left\{ e^{-x} \right\} = 1, \end{aligned}$$

and since $f(0) = 1$, this implies that f is continuous. For differentiability at $x = 0$ we take

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \left\{ \frac{f(0+h) - f(0)}{h} \right\}, \\ L_+ &= \lim_{h\to 0^+} \left\{ \frac{e^{-h} - 1}{h} \right\} = \lim_{h\to 0^+} \left\{ \frac{-e^{-h}}{1} \right\} = -1, \qquad \text{L'Hôpital's rule} \\ L_- &= \lim_{h\to 0^-} \left\{ \frac{e^{+h} - 1}{h} \right\} = \lim_{h\to 0^-} \left\{ \frac{+e^{+h}}{1} \right\} = +1. \end{aligned}$$

Since $L_- \neq L_+$ so $f(x)$ is not differentiable at $x = 0$.
Example End

2.4 Rolle's Theorem

If $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), with

$$f(a) = f(b),$$

then there exists a point c, $a < c < b$ such that

$$f'(c) = 0.$$

This is an important theorem and is illustrated in Figure 2.9.

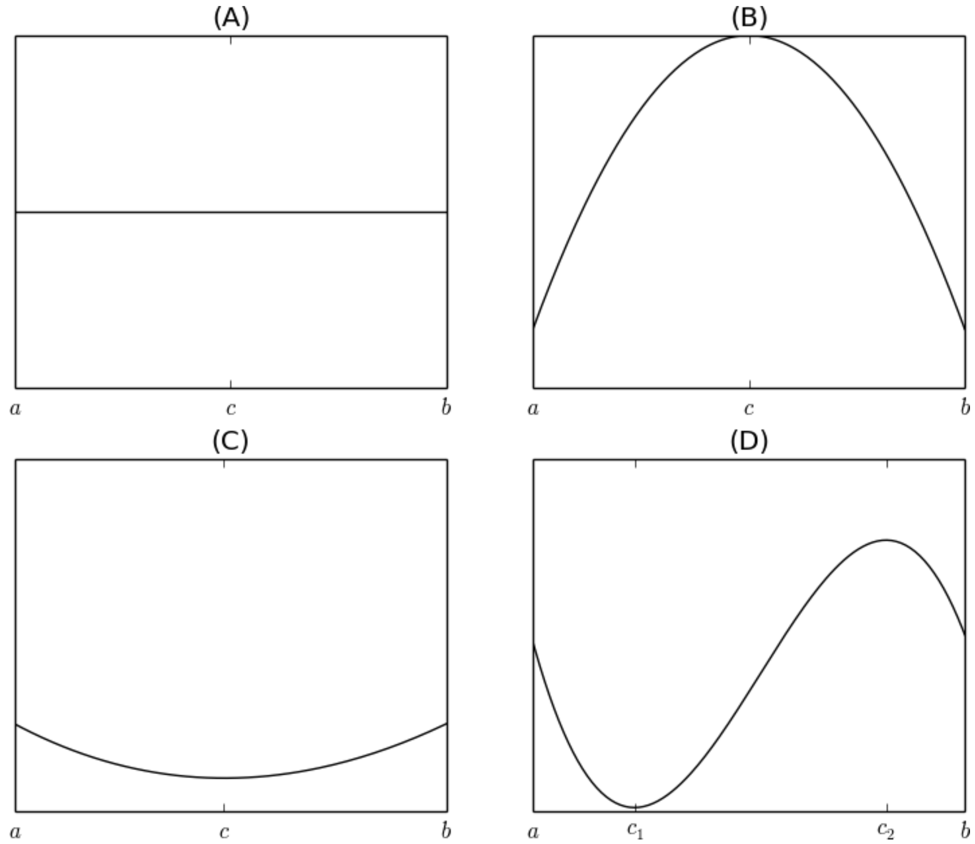


Figure 2.9: Four possible situations are illustrated. {fig:1.9}

A If $f(x) = f(a) = f(b)$ everywhere in $[a, b]$, then the function is constant and $f'(x) = 0$ (see previous definition) for all $x \in (a, b)$.

B & C If $f(x)$ is different from $f(a)$ and $f(b)$, then it must have either a minimum (B) or a maximum (C) in (a, b) at which point

$$f'(c) = 0$$

D There may, of course, be more than one value of c for which $f'(c) = 0$. If there are two points, then one point is a maximum and one point is a minimum.

2.5 Mean Value Theorem

If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b), then there exists a number c, $a < c < b$, such that

:1.7}

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{2.1}$$

For a smooth curve, this is obvious from the graph shown in Figure 2.10. Geometrically, c is the point

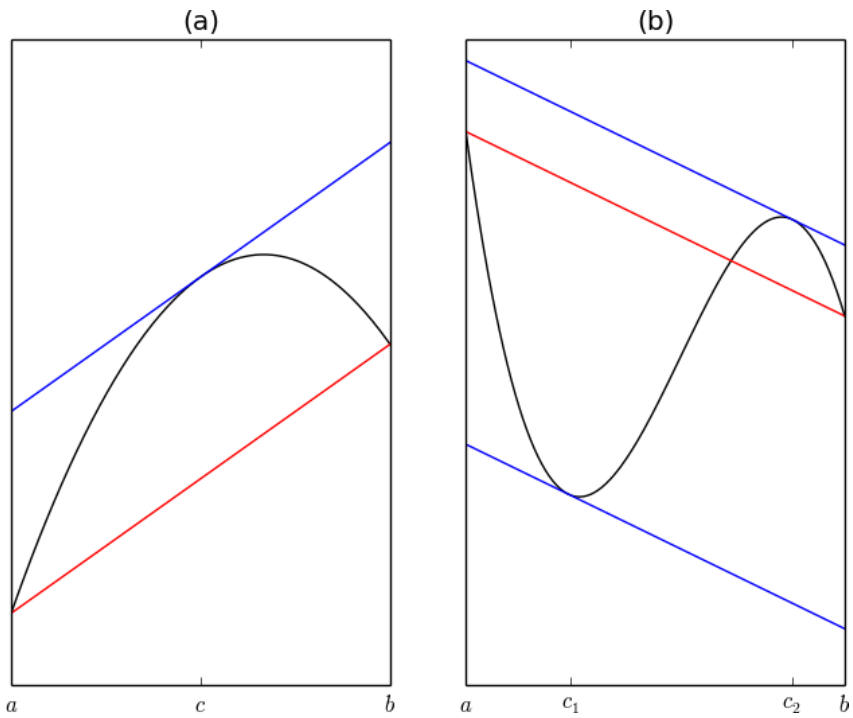


Figure 2.10: (a) The tangent at $x = c$ is parallel to the straight line joining the end points. (b) There
1.10} are two points with parallel tangents that have the same gradient as the straight line between a and b.

where the tangent of the curve is parallel to the straight line joining a and b. There will always be at least one value of c for which this is true.

Proof of the Mean Value Theorem

The proof uses Rolle's theorem. To see this we consider a new function made up of the original function minus the straight line joining the end points. Thus, let

{eq:1.8}
$$g(x) = f(x) - (x-a)\frac{(f(b) - f(a))}{b-a} - f(a). \tag{2.2}$$

Note that $g(a) = 0$ and

$$g(b) = f(b) - (b-a)\frac{(f(b) - f(a))}{b-a} - f(a) = f(b) - (f(b) - f(a)) - f(a) = 0.$$

Hence, $g(a) = g(b)$ and, since g is continuous because f and $x - a$ are continuous, Rolle's theorem applies. This means that there exists a point c such that

$$g'(c) = 0.$$

Using the definition of g this implies that

$$g'(c) = f'(c) - \frac{(f(b) - f(a))}{b-a} = 0,$$

and so

$$f'(c) = \frac{(f(b) - f(a))}{b-a}.$$

Graphically, the function $g(x)$ that satisfies Rolle's theorem is the difference between f and the mean slope. Note that we can write the *average* or mean gradient of f over the interval $[a, b]$ as

$$f'(c) = \frac{1}{b-a}\int_a^b f'(x)dx = \frac{(f(b) - f(a))}{b-a}.$$

Example 2.6

By considering the Mean Value Theorem, show that

$$(1+x)^{1/2} < 1 + \frac{1}{2}x, \qquad \text{for all } x > 0.$$

Solution 2.6

Take the interval $[0, x]$ and let $f(x) = (1+x)^{1/2}$. Thus,

$$f'(x) = \frac{1}{2}(1+x)^{-1/2}.$$

Use the Mean Value Theorem with $a = 0$ and $b = x$. Thus, there exists a c such that

$$f'(c) = \frac{(f(b) - f(a))}{b-a} = \frac{f(x) - f(0)}{x-0} = \frac{f(x) - 1}{x}.$$

Note that $0 < c < x$. Since we have an expression for the derivative we must have

$$f'(c) = \frac{1}{2}(1+c)^{-1/2}.$$

The aim is to find a bound on this derivative. Hence, for $c > 0$ we have

$$\frac{1}{1+c} < 1, \qquad \Rightarrow \qquad \frac{1}{(1+c)^{1/2}} < 1.$$

Thus, we have

$$f'(c) = \frac{1}{2}\frac{1}{(1+c)^{1/2}} < \frac{1}{2}.$$

However, we know that

$$f'(c) = \frac{f(x)-1}{x} < \frac{1}{2},$$

and so

$$f(x) - 1 < \frac{1}{2}x.$$

Thus,

$$\begin{aligned} f(x) &< 1 + \frac{1}{2}x, \\ (1+x)^{1/2} &< 1 + \frac{1}{2}x, \end{aligned}$$

as required.

Example End