Chapter 8

Double Integrals

{chap:8}

Swokowski Chapter 13.1

A *double integral* is

$$I = \int\int_A f(x,y)dA,$$

where A is the *region* or *domain* or *area* of integration and dA is the infinitesimal 'area' element. In simple integrals of one variable we had dx as the infinitesimal length in the x direction. Now we have an area. In *Cartesian* coordinates (x, y), we have

$$dA = dxdy. \tag{8.1}$$ {eq:8.1}

This is illustrated in Figure 8.1

We will consider other coordinates later.

Notes:

1. I is a number when the area A is fixed. It is not a function of x and y. Like the definite integrals, we have 'integrated out' the dependence on x and y. However, as before, indefinite integrals are often used in intermediate steps when evaluating I.

2. f can be a function of other coordinates, such as *polar coordinates* (R, ϕ), defined over the region. The only difference from the cartesian case is the calculation of dA.

3. When $f(x, y) = 1$, I is simply the area of the region. Only in this case is

$$\text{Area} = \int\int_A dA.$$

   Otherwise it is *not* the area.

4. When $f(x, y)$ is the height of a surface above the x-y plane, I is the *volume* under the surface. However, do *not* always think in these terms. $f(x, y)$ can be many other things. For example

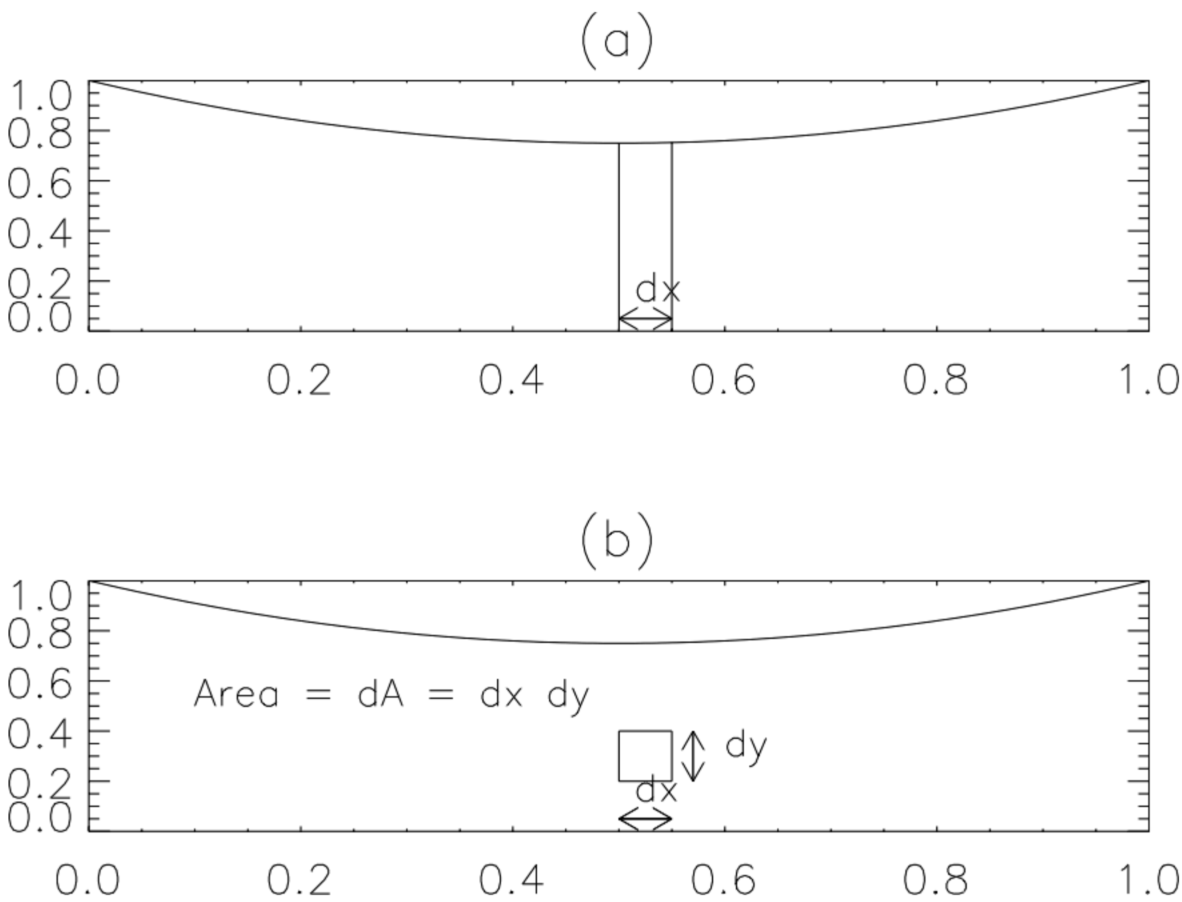


Figure 8.1: (a) The infinitesimal line element dx. (b) The infinitesimal area element $dA = dxdy$. {fig:8.2}

$f(x, y)$ could be the temperature at a location (x, y). Then if A is the whole of the UK, we could calculate the mean temperature by integrating the temperature over A and dividing by the area.

$$\text{Mean temperature } = \frac{\int\int_A f(x,y)dA}{\int\int_A dA}.$$

In addition, note that $f(x, y)$ can be locally positive or negative. (It would always have to be positive if we only thought in terms of volume under a surface!!)

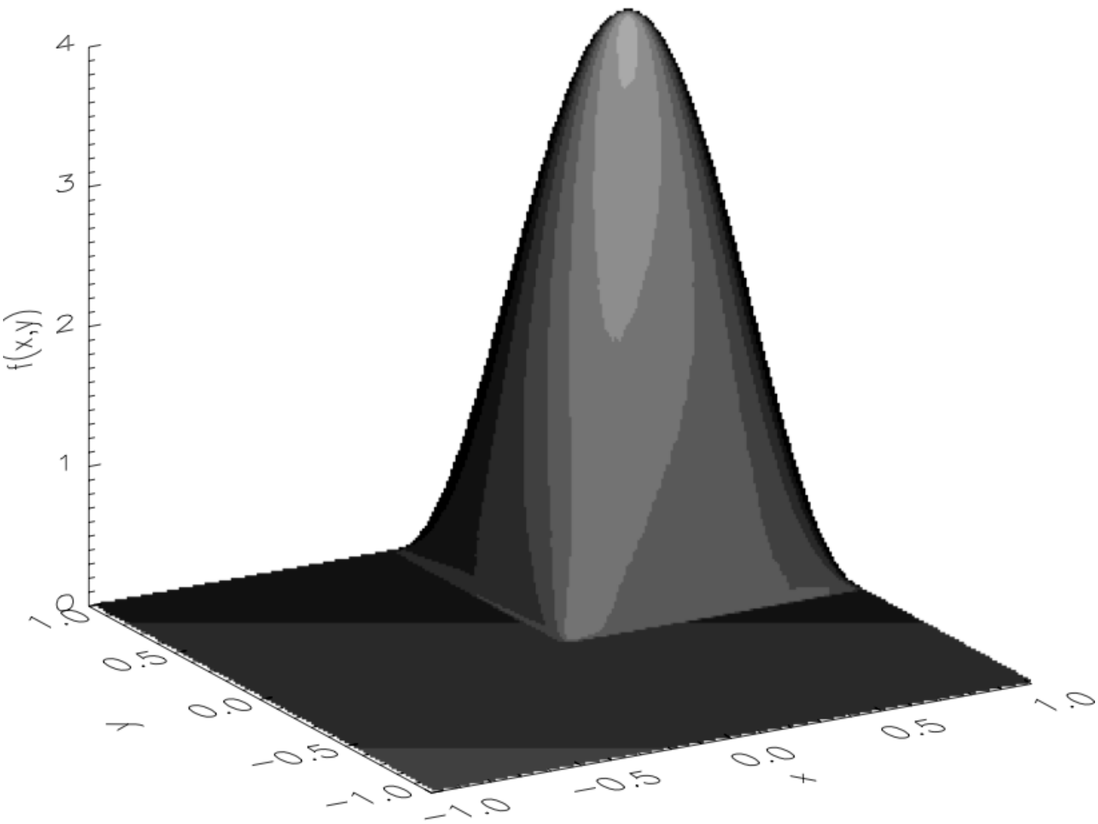


Figure 8.2: The height of the surface above the x-y plane is given by $f(x, y)$. {fig:8.3}

8.1 Rectangular Base Areas

{sec:8.1}

A rectangular base will have a form similar to Figure 8.3. These are the easiest, as they can always be done as a *composition* of two single integrals. In this case the limits of integration are simple, namely

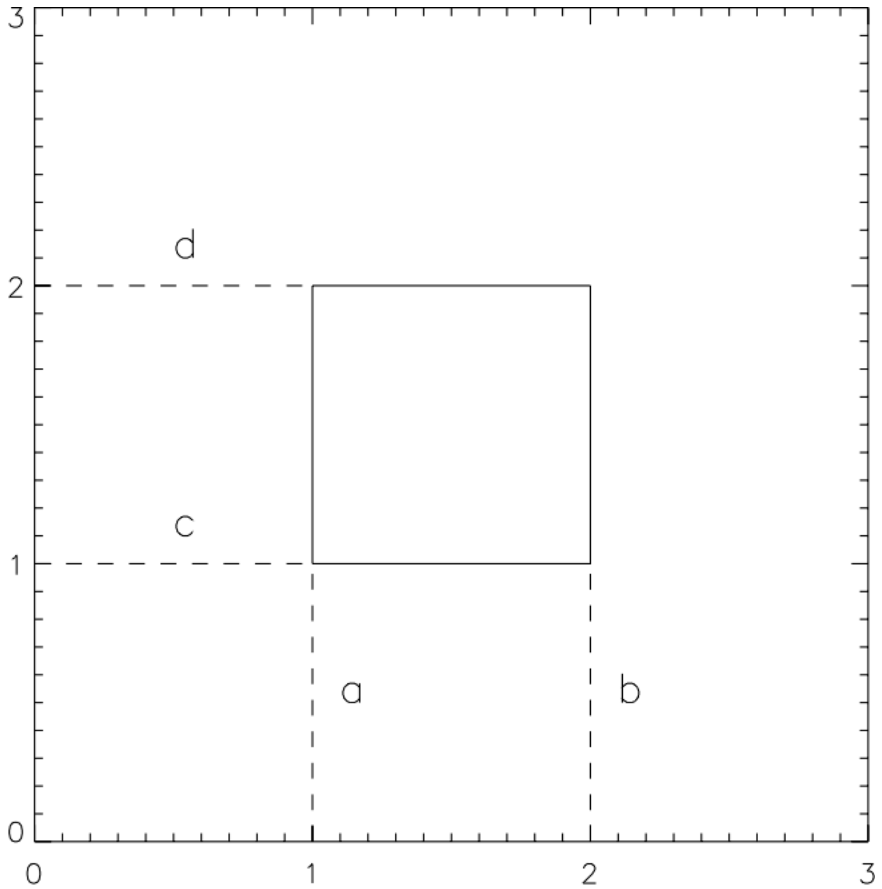
{eq:8.2}
$$I = \int_{y=c}^{d} \int_{x=a}^{b} f(x, y) dx dy = \int_{y=c}^{d} \left(\int_{x=a}^{b} f(x, y) dx \right) dy. \tag{8.2}$$

Notice that the x integration is done first, where the y in $f(x, y)$ is treated as a constant and the y integration is done second. There is no x appearing after the first integration, as it has already been 'integrated out'.

Consider the *inner integral*

$$\int_{x=a}^{b} f(x, y) dx \equiv g(y).$$

Remembering that y is held constant at present. Then, if the indefinite integral of $f(x, y)$ **with respect**



:8.4} Figure 8.3: A rectangular base, where x lies between a and b and where y lies between c and d.

to x is $F(x,y)$, then

$$g(y) = F(b,y) - F(a,y) = \int_{x=a}^{b} f(x,y)dx.$$

This is exactly the same as in single integrals. Since f depends on x and y, we use *partial* derivatives to define F through

$$\frac{\partial F}{\partial x} = f(x,y).$$

The point is that, if we can find *any* function satisfying $\partial F/\partial x = f(x,y)$, then $g(y) = F(b,y) - F(a,y)$ is unique. The proof is that F is known only to within an arbitrary function of y, say $h(y)$. Thus, if

$$\frac{\partial F}{\partial x} = f(x,y), \quad \Rightarrow \quad \frac{\partial (F + h(y))}{\partial x} = f(x,y)$$

as well. However, when we apply the limits, we have

$$g(y) = [F(b,y) + h(y)] - [F(a,y) + h(y)] = F(b,y) - F(a,y),$$

is *independent* of $h(y)$.

Having completed this inner integral, we now know $g(y)$. Then, our original integral has reduced to

$$\int_c^d g(y)dy,$$

and is now a definite integral over y. If $G(y)$ is the indefinite integral, namely

$$\frac{dG}{dy} = g(y),$$

then

$$I = G(d) - G(c) = [G(y)]_c^d .$$

Note $G(y)$ only depends on y and so it is *not* a partial derivative but the standard 'straight d ' derivative. This is the final answer.

Note that the order of the integration, namely x first and then y second or y first and x second, does not matter. You get the same answer both ways. However, sometimes the integrals are simpler one way than the other.

Example 8.55

Find the integral of $f(x,y) = \sqrt{xy}$ over the rectangle $1 \le x \le 4$ and $4 \le y \le 9$. Thus, we must evaluate

$$I = \int_{y=4}^{9} \int_{x=1}^{4} \sqrt{xy}dxdy.$$

Solution 8.55

Let us do the x integration first, as indicated above. Firstly, we integrate $\sqrt{xy} = x^{1/2}y^{1/2}$ with respect to x keeping y fixed. Thus,

$$\int_{x=1}^{4} x^{1/2}y^{1/2}dx = \left[\frac{2}{3}x^{3/2}y^{1/2}\right]_1^4 = \left(\frac{2}{3}4^{3/2}y^{1/2}\right) - \left(\frac{2}{3}1^{3/2}y^{1/2}\right).$$

Hence, the first integral gives

$$\int_{x=1}^{4} x^{1/2}y^{1/2}dx = \frac{2}{3}y^{1/2}\,(8-1) = \frac{14}{3}y^{1/2}.$$

Now we can do the second integral over y, so that

$$I = \int_{y=4}^{9} \int_{x=1}^{4} \sqrt{xy}dxdy = \frac{14}{3}\int_{y=4}^{9} y^{1/2}dy = \frac{14}{3}\left[\frac{2}{3}y^{3/2}\right]_4^9.$$

Hence,

$$I = \frac{28}{9}\left(9^{3/2} - 4^{3/2}\right) = \frac{28}{9}\,(27-8) = \frac{541}{9}.$$

In this example, reversing the order of integration is simple since $f(x,y)$ is separable in x and y. Hence, there is no advantage in changing the order of integration.

Example End

Example 8.56

In this example, the integrand is *not* separable.

$$I = \int\int_A \frac{y}{(x+y^2)^2} dxdy, \qquad \text{where } A \text{ is the rectangle } 0 \le x \le 1, \quad 1 \le y \le 2.$$

Solution 8.56

We will do this example twice. Firstly we do the x integration first and the y integration second. Secondly, we will reverse the order of integration.

(a)

$$I = \int_{y=1}^{2} \left(\int_{x=0}^{1} \frac{y}{(x+y^2)^2} dx \right) dy.$$

We find F satisfying $\partial F/\partial x = y/(x+y^2)^2$. This is equivalent to integrating $a(x^2+b)^{-2}$ with respect to x where $a = y$ and $b = y^2$ are assumed constant for the x integration. Thus,

$$F = -\frac{y}{(x+y^2)},$$

is a good choice as we can check by differentiating. Hence,

$$\int_{x=0}^{1} \frac{y}{(x+y^2)^2} dx = g(y) = F(1,y) - F(0,y) = -\frac{y}{1+y^2} + \frac{1}{y}.$$

Next we find G satisfying $dG/dy = g$. Thus,

$$G = -\frac{1}{2}\log(1+y^2) + \log y,$$

will do, as we can again check by differentiating. If you cannot spot that G is the integral of g, then we could use a substitution $u = 1 + y^2$. Finally, we have

$$I = \int_{y=1}^{2} \left(-\frac{y}{1+y^2} + \frac{1}{y} \right) dy = \left[-\frac{1}{2}\log(1+y^2) + \log y \right]_1^2 = \left(-\frac{1}{2}\log(5) + \log 2 \right) - \left(-\frac{1}{2}\log 2 + \log 1 \right).$$

Thus,

$$I = -\frac{1}{2}\log 5 + \frac{3}{2}\log 2 = -\frac{1}{2}\log 5 + \frac{1}{2}\log 2^3 = \frac{1}{2}\log\left(\frac{8}{5}\right).$$

(b) Now we reverse the order, doing the y integration first and the x integration second.

$$I = \int_{x=0}^{1} \left(\int_{y=1}^{2} \frac{y}{(x+y^2)^2} dy \right) dx.$$

Find $F(x,y)$ satisfying $\partial F/\partial y = f(x,y)$.

$$F(x,y) = -\frac{1}{2}\frac{1}{(x+y^2)},$$

will do, as simple differentiation will verify. However, the substitution $u = x + y^2$ could also have been used. Thus

$$\int_{y=1}^{2} \frac{y}{(x+y^2)^2} dy = g(x) = \left[-\frac{1}{2}\frac{1}{x+y^2}\right]_{y=1}^{2} = -\frac{1}{2(x+4)} + \frac{1}{2(x+1)}.$$

Next, we find $G(x)$ such that $dG/dx = g$.

$$G = -\frac{1}{2}\log(x+4) + \frac{1}{2}\log(x+1)$$

is a good choice. Again we can verify this by differentiation. The final answer is

$$\begin{aligned} I &= \int_{x=0}^{1} -\frac{1}{2(x+4)} + \frac{1}{2(x+1)} dx \\ &= \left[-\frac{1}{2}\log(x+4) + \frac{1}{2}\log(x+1)\right]_0^1 \\ &= -\frac{1}{2}\log 5 + \frac{1}{2}\log 2 + \frac{1}{2}\log 4 - \frac{1}{2}\log 1 \\ &= \frac{1}{2}\log\left(\frac{8}{5}\right), \end{aligned}$$

as before.

Example End

8.2 Base Areas Bounded by Lines and Curves

{sec:8.2}

Now we move onto more interesting situations, when the base area, A, in $\int\int_A f(x,y)dxdy$ is *not* rectangular but is formed from areas bounded by curves. Some careful thinking is required! In particular, the order of the integration can be crucial.

The key steps every time (until you become expert) are

- Sketch the base area, A. If you can't do this, you will struggle to get the correct limits of integration.
- Decide on the order of integration from the form of $f(x,y)$ **and** the shape of A.
- Work out the inner integral's limits of integration. **Warning**: these limits will depend on the outer variable. The outer integral's limits are always constants. In a sense, *this step is the crux of the whole problem.*
- Perform the integrations as usual in the correct order. This will only involve integrals and integration techniques you have seen before.

There is no automatic procedure: setting up the integrals and, in particular, the limits of integration are the hardest parts. These depend on *both* the form of $f(x,y)$ and the shape of the area, A.

We will learn by doing lots of examples.

Example 8.57

Find

$$I = \int\int_A dA = \int\int_A dxdy,$$

where A is the region in the first quadrant bounded by the line $x + y = 1$. So the area is contained by the lines $y = 1 - x$, $x = 0$ and $y = 0$. This shown in Figure 8.4

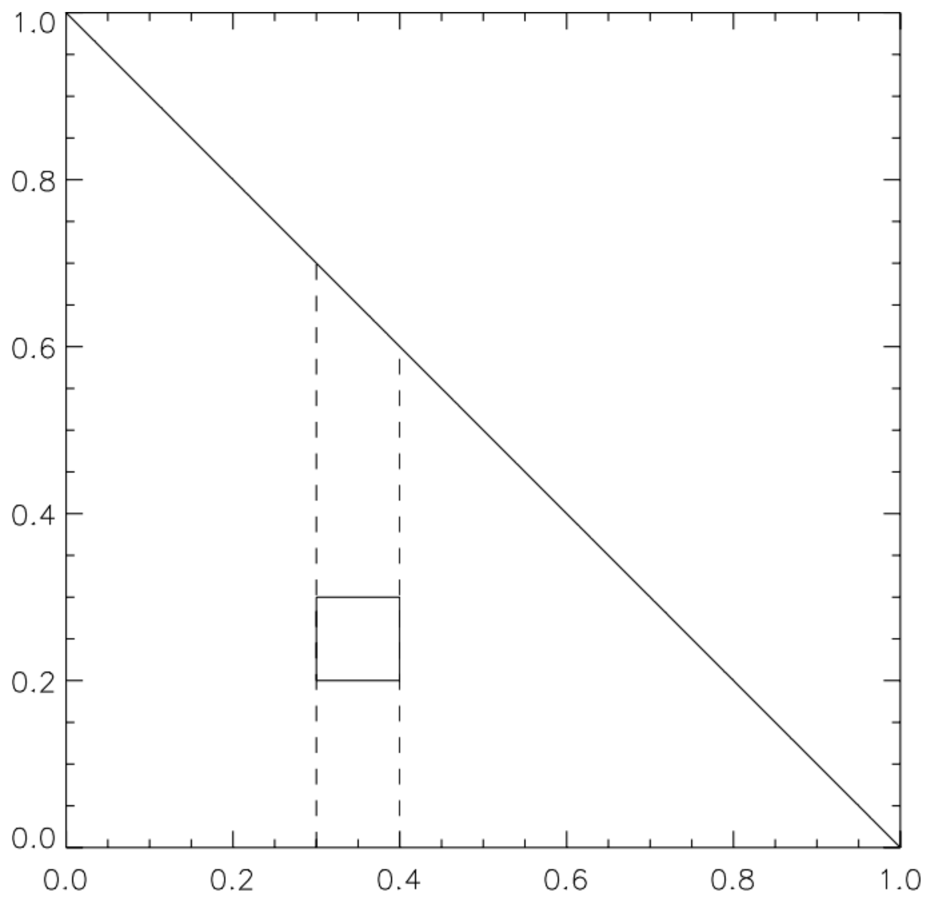


Figure 8.4: The area contained by the x and y axes and the straight line $y = 1 - x$. The area of the small square is $dxdy$ and we add up the contributions from $y = 0$ up to $y = 1 - x$. {fig:8.5}

Solution 8.57

Here, we go through the steps outlined above.

- Sketch the area A. This is shown in Figure 8.4. It is clearly a triangle and so the area is simply (half base times height) 1/2. However, we continue to illustrate the method and confirm that we do indeed get the same answer.

- Decide on the order of integration. Here it is does not matter as $f(x, y) = 1$ in this case. I have chosen to do the y integration first and the x integration second.

- In Figure 8.4, I have drawn a typical rectangle of area $dxdy$. I now add up all the contributions from similar rectangles going from $y = 0$ up to the line where $y = 1 - x$. This gives me the limits on the x integration (i.e the inner integration). Having got the area of this single strip, I now need to add up all the area of each strip (as shown in Figure 8.5), going from $x = 0$ to $x = 1$. I should have completely covered the base area *only once*. Thus,

$$I = \int_{x=0}^{1} \left(\int_{y=0}^{1-x} dy \right) dx.$$

- Now we actually do the integrals. Thus,

$$I = \int_{x=0}^{1} [y]_{y=0}^{1-x}\, dx = \int_{x=0}^{1} \{(1-x) - 0\}\, dx.$$

  Thus, we have

$$I = \int_{x=0}^{1} (1-x)\, dx = \left[x - \frac{x^2}{2} \right]_0^1 = 1 - \frac{1}{2} = \frac{1}{2}.$$

Finally, this example can be easily done in the other order. This time the limits of the integration will give us,

$$I = \int_{y=0}^{1} \left(\int_{x=0}^{1-y} dx \right) dy.$$

If you cannot understand why the limits are as given, please see me as soon as possible.

Example End

Example 8.58

Evaluate,

$$I = \int \int_A \frac{1-y}{1-x} dA,$$

over the same area as above. This is a little more challenging. Note that $f(x, y) \neq 1$ but is $f(x, y) = (1 - y)/(1 - x)$. Hence, we are **not** finding the area this time.

Solution 8.58

Go through the same steps as before.

- Well the area is exactly the same as shown in Figure 8.4.

- Decide on the order of integration. If we choose x first we will get logarithms. If we choose y first we will get powers of y. The latter seems easier, so we will do the y integration first and the x integration second.

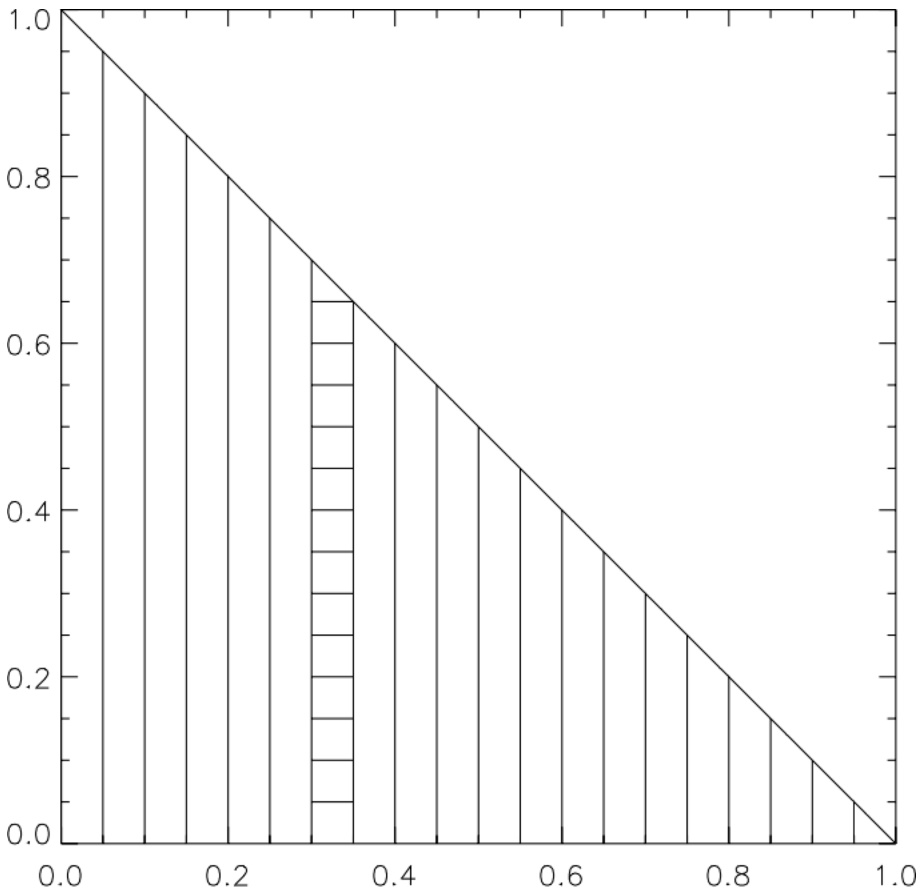


Figure 8.5: Having calculated the area of each strip, we add them up from $x = 0$ to $x = 1$. {fig:8.6}

- Decide on limits. As above we imagine drawing a small rectangle of area $dA = dxdy$ and add all these contribution up from $y = 0$ up to the line at $y = 1 - x$. Then we add up all the vertical strips going from $x = 0$ to $x = 1$. Thus,

$$\int_{x=0}^{1} \left(\int_{y=0}^{1-x} \frac{1-y}{1-x} dy \right) dx.$$

- Let's do the y integration first, remembering that x is kept fixed.

$$\int_{y=0}^{1-x} \frac{1-y}{1-x} dy = \left[\frac{y - y^2/2}{1-x} \right]_0^{1-x} = \left(\frac{1 - x - (1-x)^2/2}{1-x} \right) - 0 = 1 - \frac{1-x}{2}.$$

Now we do the x integration

$$I = \int_{x=0}^{1} \frac{1}{2} + \frac{x}{2} dx = \frac{1}{2} \left[x + \frac{x^2}{2} \right]_0^1 = \frac{1}{2} \left(1 + \frac{1}{2} \right) - 0 = \frac{3}{4}.$$

Challenge: Can anyone do the integral by reversing the order of integration?

Example End

Example 8.59

Evaluate

$$I = \int\int_A e^{-y^2} dA,$$

where A is the area bounded by the y axis, the line $y = x$ and the line $y = 1$.

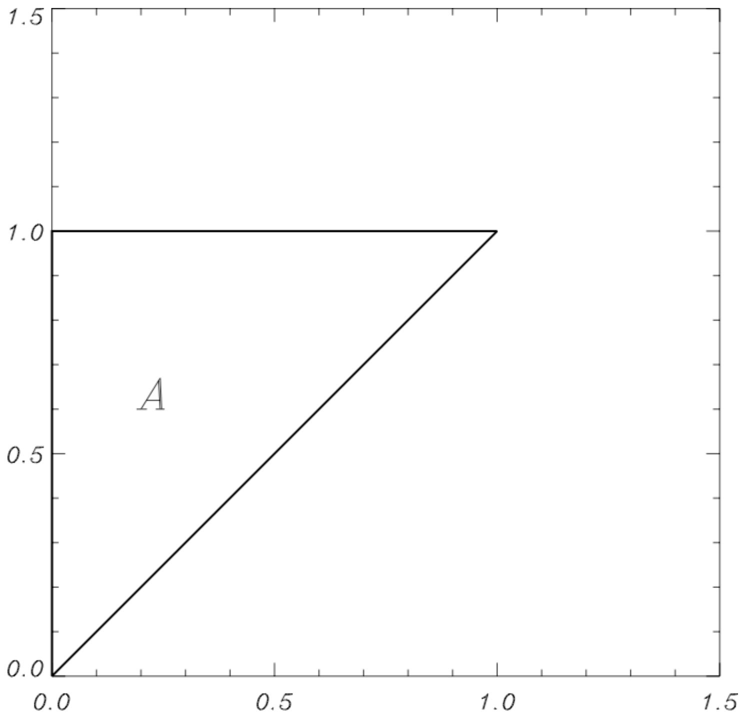


Figure 8.6: The area contained by the y axis, $y = 1$ and the straight line $y = x$. {fig:8.7}

Solution 8.59

We follow the usual steps.

- Firstly, we draw the area A. This is shown in Figure 8.6.
- Decide on the order of integration. From your knowledge of standard integrals, you cannot integrate e^{-y^2} analytically (over a finite domain) no matter how imaginative you are. So the best bet is to try the x integration first.

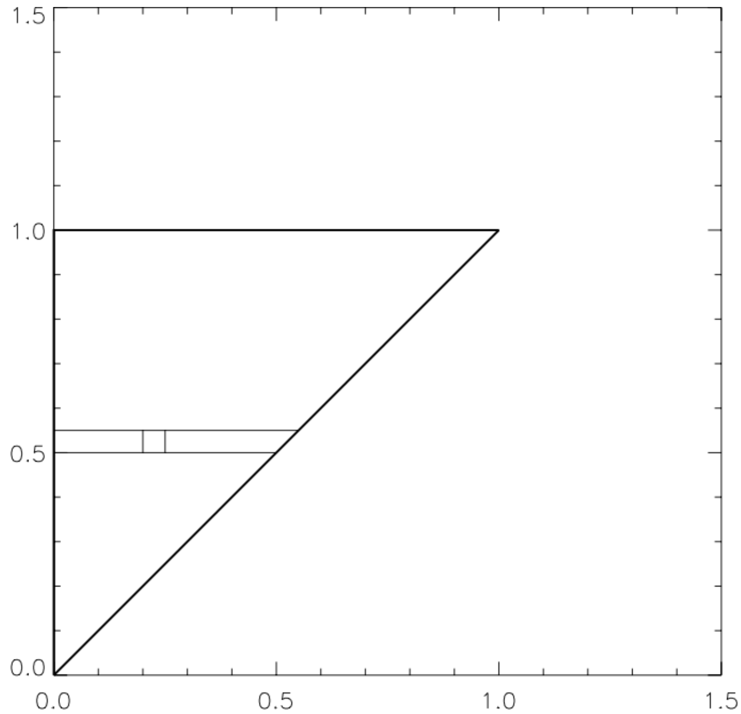


Figure 8.7: The area contained by the y axis, $y = 1$ and the straight line $y = x$. The area of the small square is $dxdy$ and we add up the contributions from $x = 0$ up to $x = y$. {fig:8.8}

- Decide on the limits. From Figure 8.7, we can see that we must add up the contributions from the infinitesimal areas from $x = 0$ to $x = y$. This gives us a strip. We then add up all the contributions from all the strips from $y = 0$ to $y = 1$. This will completely cover the area once.

$$I = \int_{y=0}^{1} \left(\int_{x=0}^{y} e^{-y^2} dx \right) dy.$$

- Do the integration. Well the first integral is straightforward, since we assume that y is fixed while we integrate in the x direction.

$$\int_{x=0}^{y} e^{-y^2} dx = \left[x e^{-y^2} \right]_0^y = y e^{-y^2}.$$

That is very useful. The original integral has now reduced to

$$I = \int_{y=0}^{1} y e^{-y^2} dy.$$

This can be done either directly, if you can spot the indefinite integral, or by a substitution $u = y^2$. So $du/dy = 2y$ and this means $du/2 = ydy$. Remembering to change the limits we have $u = 0$

when $y = 0$ and $u = 1$ when $y = 1$. Therefore,

$$\begin{aligned} I &= \frac{1}{2}\int_{u=0}^{1} e^{-u} du \\ &= -\frac{1}{2}\left[e^{-u}\right]_0^1 = -\frac{1}{2}\left(e^{-1} - e^0\right) \\ &= \frac{1}{2}\left(1 - \frac{1}{e}\right). \end{aligned}$$

Example End

Note that not all integrals are straightforward. Not all integrals can be done analytically. However, even if the integrals need to be done numerically, the derivation of the limits always follows the same pattern.

Example 8.60

Find the mean values of x and y over the area A bounded by the parabola $y = y_1(x) = 4x(1 - x)$ and the line $y = y_2(x) = x$.

Solution 8.60

First of all, we need to know what is meant by $\bar{x}$, the mean of x. This is defined as

$$\bar{x} = \frac{\int\int_A x dx dy}{\int\int_A dx dy}. \quad (8.3) \quad \{eq:8.3\}$$

Thus, we have two integrals to evaluate, namely the area of the base

$$\int\int_A dx dy,$$

and

$$I = \int\int_A x dx dy.$$

Similarly, $\bar{y}$, the mean of y is defined by

$$\bar{y} = \frac{\int\int_A y dx dy}{\int\int_A dx dy}. \quad (8.4) \quad \{eq:8.4a\}$$

Again we need to evaluate two integrals but at least the area is the same. Thus, we need

$$J = \int\int_A y dx dy.$$

$(\bar{x}, \bar{y})$ gives the *centre of mass* of a plate with uniform density.

Thus, there are three integrals to evaluate, I, J and the area, A. They are all straightforward once they are correctly set up.

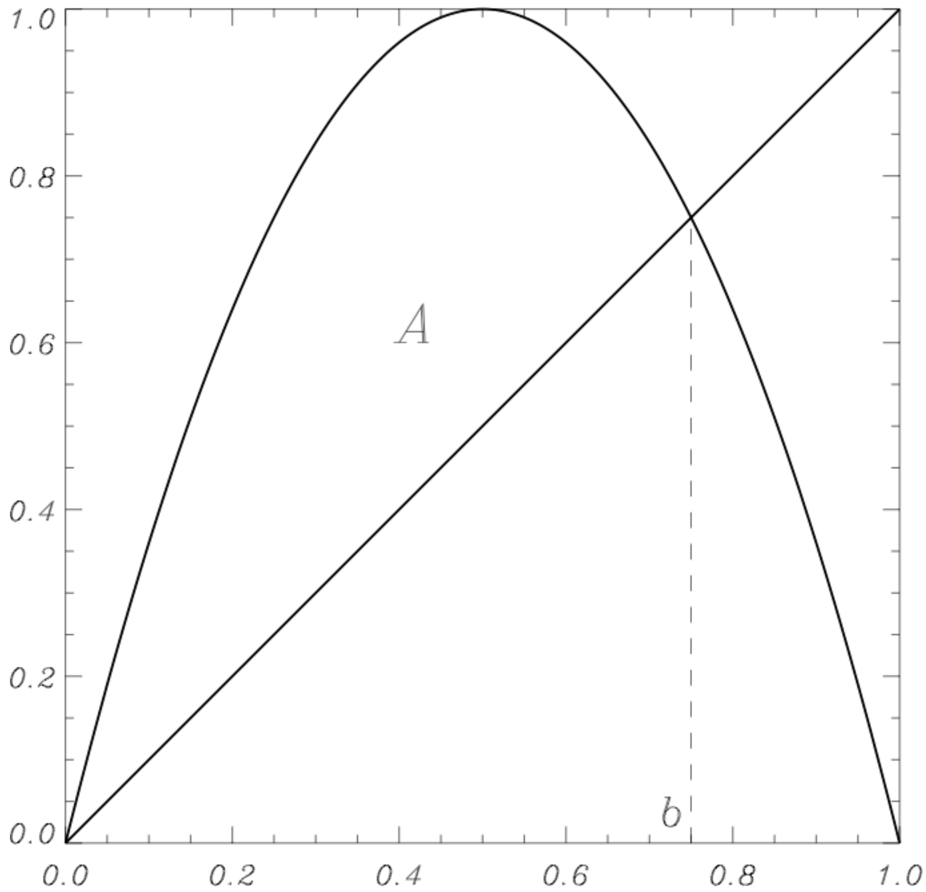


Figure 8.8: The area contained by the $y = x$ and the parabola $y = 4x(1 - x)$. {fig:8.9}

- Draw the area, A. This is shown in Figure 8.8. We see that the two curves intersect at two values of x, namely $x = 0$ and $x = b$. These occur at the same values of y. Thus, setting $y = x$ equal to $y = 4x(1 - x)$ we have

$$x = 4x(1-x) \quad \Rightarrow \quad x = 0 \text{ or } 1 = 4(1-x) \quad \Rightarrow \quad x = \frac{3}{4}.$$

  Hence, $b = \frac{3}{4}$.

- Now we decide on the order of integration. Really there is only one straightforward choice, y first.

- Decide on the limits. Integrating in y first means we imagine adding up contributions from infinitesimal boxes (of area $dxdy$) from $y = x$ up to $y = 4x(1 - x)$. This generates a strip. Then we add up all the strips from $x = 0$ to $x = 3/4$ so that we cover the area exactly once.

- Do the integration. We have three integrals to evaluate. Let's do them in turn.

1. Firstly, we evaluate the area. Thus,

$$A = \int_{x=0}^{3/4} \left(\int_{y=x}^{4x(1-x)} dy \right) dx.$$

Doing the inner integral first, we get

$$\int_{y=x}^{4x(1-x)} dy = [y]_{y=x}^{4x(1-x)} = (4x(1-x)) - x = 3x - 4x^2.$$

Now we can do the outer integral.

$$A = \int_{x=0}^{3/4} \left(3x - 4x^2\right) dx = \left[\frac{3}{2}x^2 - \frac{4}{3}x^3\right]_0^{3/4}$$

Substituting in the limits we get

$$A = \frac{3}{2}\frac{9}{16} - \frac{4}{3}\frac{27}{64} = \frac{27}{32} - \frac{9}{16} = \frac{9}{32}.$$

2. Next we evaluate, I.

$$I = \int_{x=0}^{3/4} \left(\int_{y=x}^{4x(1-x)} x dy \right) dx.$$

The inner integral gives, since x is constant while integrating with respect to y,

$$\int_{y=x}^{4x(1-x)} x dy = [xy]_{y=x}^{4x(1-x)} = \left(4x^2(1-x)\right) - \left(x^2\right) = 3x^2 - 4x^3.$$

Now we can do the outer integral.

$$I = \int_0^{3/4} \left(3x^2 - 4x^3\right) dx = \left[x^3 - x^4\right]_0^{3/4} = \frac{27}{64} - \frac{81}{256}.$$

This simplifies to

$$I = \frac{27}{256}.$$

3. The final integral is J. Here

$$J = \int_{x=0}^{3/4} \left(\int_{y=x}^{4x(1-x)} y dy \right) dx.$$

The inner integral gives

$$\int_{y=x}^{4x(1-x)} y dy = \left[\frac{1}{2}y^2\right]_{y=x}^{4x(1-x)} = 8x^2(1-x)^2 - \frac{1}{2}x^2 = \frac{15}{2}x^2 - 16x^3 + 8x^4.$$

Again, we are ready to evaluate the outer integral to get

$$J=\int_{x=0}^{3/4}\left(\frac{15}{2}x^2-16x^3+8x^4\right)dx=\left[\frac{5}{2}x^3-4x^4+\frac{8}{5}x^5\right]_0^{3/4}.$$

Thus,

$$J=\frac{135}{128}-\frac{81}{64}+\frac{243}{640}=\frac{27}{160}.$$

Now we can work out the mean of x and y as

$$\bar{x}=\frac{27}{256}\frac{32}{9}=\frac{3}{8},\qquad \bar{y}=\frac{27}{160}\frac{32}{9}=\frac{3}{5}.$$

Thus, the centre of mass of plate with this shape is at $(\frac{3}{8},\frac{3}{5})$.

Interestingly, if I was to throw a plate with this shape, it would travel in exactly the same manner as a point mass located at the centre of mass (ignoring air resistance)!

Example End

8.3 Base areas of circular or part circular form

:8.3}

When the base area, A, is a circle or a sector of a circle, it is usually more effective to change coordinates from Cartesian coordinates to polar coordinates when evaluating double integrals. This is particularly useful when the integrand is a function of x^2+y^2.

The first example will illustrate the basic ideas and importantly how to calculate the infinitesimal area dA.

Example 8.61

Calculate

$$I=\int\int_A\sqrt{x^2+y^2}dA,$$

where A is the sector of a circular disk of radius $R=a$ between the angles $\pi/6(=30^\circ)$ and $\pi/3(=60^\circ)$. Remember we must work in radians!

Solution 8.61

We follow the usual approach.

- Draw the area A. This is shown in Figure 8.9. This would be extremely complicated in Cartesian coordinates. So we switch from (x,y) to (R,ϕ), where

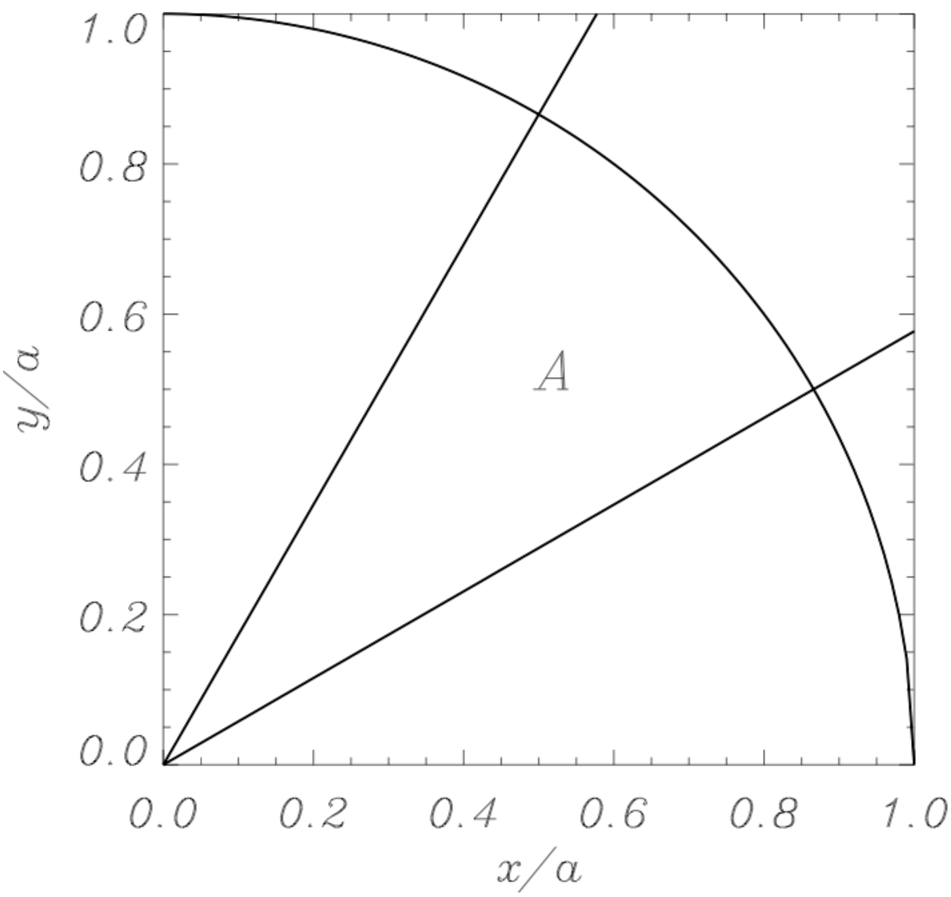
:8.4}
:8.5}

$$x = R\cos\phi, \tag{8.5}$$
$$y = R\sin\phi. \tag{8.6}$$

Equations (8.5) and (8.6) give x and y in terms of R and ϕ. The inverse relationships are

:8.6}
:8.7}

$$R = \sqrt{x^2+y^2}, \tag{8.7}$$
$$\phi = \tan^{-1}\left(\frac{y}{x}\right). \tag{8.8}$$



{fig:8.10} Figure 8.9: The area contained by the circle $R = a$ and the lines $y = x\tan(\pi/6)$ and $y = x\tan(\pi/3)$.

We are nearly ready to tackle the integral I but we need to get an expression for dA. There are several ways to obtaining dA and I will illustrate two.

Firstly, Figure 8.10 shows the area dA. The difference in radii is dR and the angle subtended is $d\phi$. The area is simply ('length times height'). Now the distance around the circle is the appropriate fraction of the circumference, which at a general radius, R is

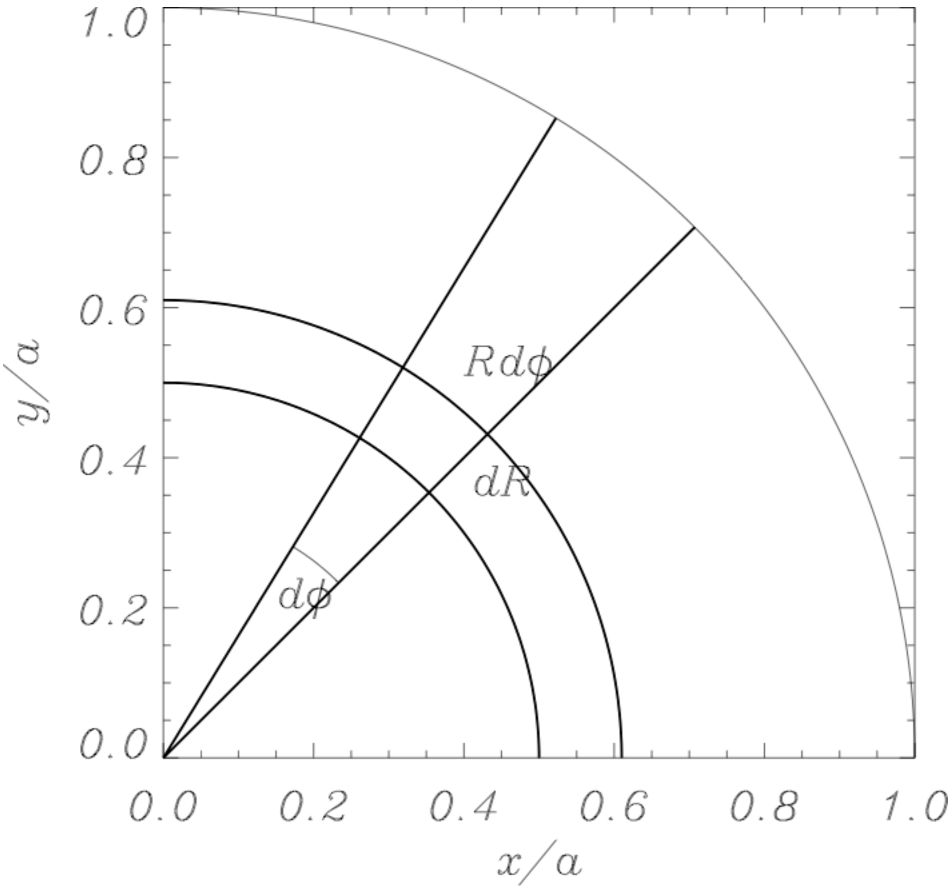
$$\frac{d\phi}{2\pi} \times 2\pi R = Rd\phi.$$

Thus, the infinitesimal area is

$$dA = RdRd\phi. \tag{8.9}$$ {eq:8.8}

Remember this!

The second method involves using the differences between the area of a sector of a circle of radius, $R + dR$ and one of radius R. The area of a sector is the appropriate fraction of the area of a circle.



8.11} Figure 8.10: The infinitesimal area, dA has sides of length dR and $Rd\phi$.

Thus, the area of the larger sector is

$$A_2 = \frac{d\phi}{2\pi}\pi(R+dR)^2 = \frac{1}{2}d\phi(R+dR)^2,$$

and the area of the smaller sector is

$$A_1 = \frac{d\phi}{2\pi}\pi R^2 = \frac{1}{2}d\phi R^2.$$

Thus, the required area is

$$dA = A_2 - A_1 = \frac{1}{2}d\phi\left\{R^2 + 2RdR + dR^2 - R^2\right\}.$$

Cancelling the terms in R^2 and neglecting dR^2 in comparison of the size of dR, we find that the infinitesimal area in polar coordinates is again

$$dA = RdRd\phi.$$

A third method, using the *Jacobian*, works for any change of 2D coordinates and is presented in Section 8.4.

Now return to our example.

$$I = \int\int R dA = \int\int R(RdRd\phi) = \int\int R^2 dRd\phi,$$

where we have used $R = \sqrt{x^2 + y^2}$.

- Decide order of integration. Here it does not matter.
- Decide on limits. Again it does matter how we determine the limits. Assume we have our infinitesimal area, dA, we add up all the contributions round the sector of the circle *at the same radius*. Thus, we add up from $\phi = \pi/6$ to $\phi = \pi/3$. This creates a small curved strip located at a certain radius. Now we add up all the strips from $R = 0$ to $R = a$. So,

$$I = \int_{R=0}^{a} \left(\int_{\phi=\pi/6}^{\pi/3} R^2 d\phi \right) dR.$$

- The inner integral is straightforward (since R is held fixed while integrating in the ϕ direction). Thus, the inner integral gives,

$$\int_{\phi=\pi/6}^{\pi/3} R^2 d\phi = R^2 \left[\phi\right]_{\phi=\pi/6}^{\pi/3} = R^2 \left(\frac{\pi}{3} - \frac{\pi}{6} \right) = \frac{\pi}{6} R^2.$$

Now we evaluate the outer integral to get

$$I = \int_{R=0}^{a} \frac{\pi}{6} R^2 dR = \frac{\pi}{6} \left[\frac{1}{3} R^3 \right]_{R=0}^{a} = \frac{\pi}{6} \frac{a^3}{3} = \frac{\pi a^3}{18}.$$

Example End

Example 8.62

Obtain

$$I = \int\int_A x^2 dA,$$

where A is the circular disk given by $R \leq a$.

Solution 8.62

Use polar coordinates! It is easier. Remember,

$$\begin{aligned} x &= R\cos\phi \\ y &= R\sin\phi \\ R^2 &= x^2 + y^2 \\ \phi &= \tan^{-1}\left(\frac{y}{x}\right) \\ dA &= RdRd\phi. \end{aligned}$$

So,

$$I = \int\int_A R^2 \cos^2 \phi R dR d\phi = \int\int_A R^3 \cos^2 \phi dR d\phi.$$

The order does not matter and the limits are fairly obvious. We add up our infinitesimal area round the circle so that ϕ goes from 0 to 2π. Then we add up all the strips in the radial direction from $R = 0$ to $R = a$. Thus,

$$I = \int_{R=0}^{a} \left(\int_{\phi=0}^{2\pi} R^3 \cos^2 \phi d\phi \right) dR.$$

Remember to express $\cos^2 \phi$ in terms of multiple angles as $\cos^2 \phi = (1 + \cos 2\phi)/2$. Thus, the inner integral gives,

$$\int_{\phi=0}^{2\pi} R^3 \frac{1}{2} (1 + \cos 2\phi) \, d\phi = \frac{R^3}{2} \left[\phi + \frac{1}{2} \sin 2\phi \right]_{\phi=0}^{2\pi} = \pi R^3.$$

Now the outer integral can be done,

$$I = \int_{R=0}^{a} \pi R^3 dR = \pi \left[\frac{R^4}{4} \right]_0^a = \frac{\pi a^4}{4}.$$

Example End

Example 8.63

Sometimes the order of integration does matter in polar coordinates. Consider finding the area

$$I = \int\int_A dA$$

where A is the area between the two *spirals*

$$R = R_1(\phi) = a\phi^{1/2}, \qquad R = R_2(\phi) = b\phi^{1/2},$$

with

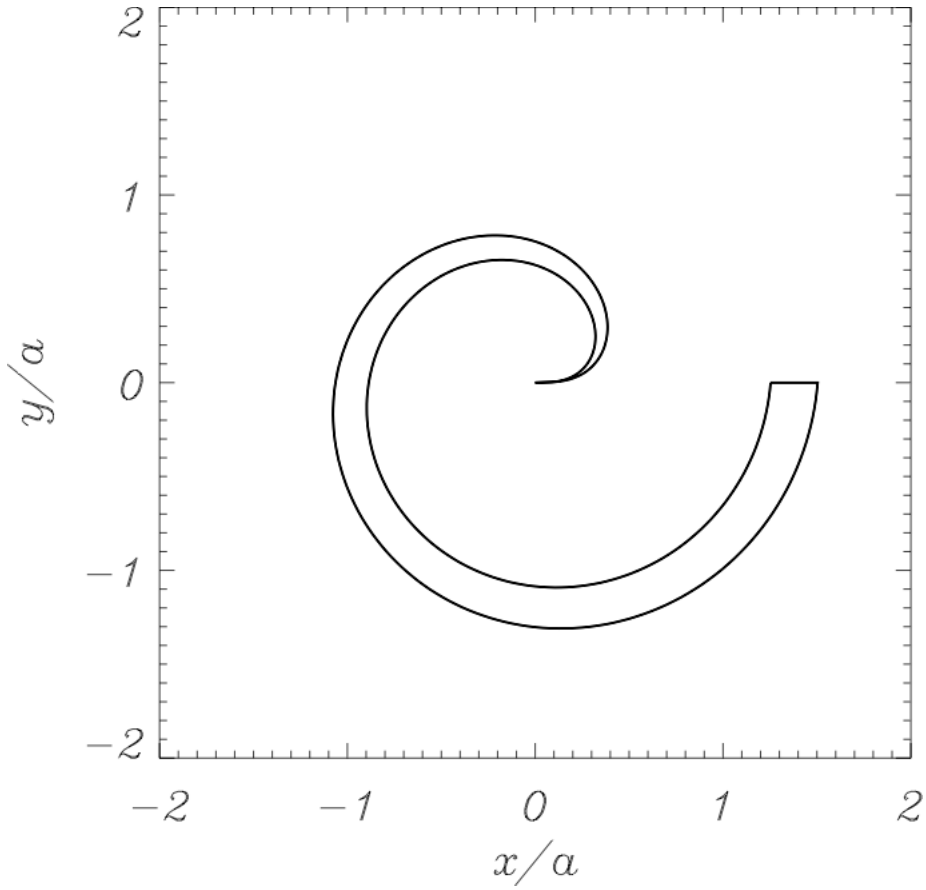
$$0 < a < b, \quad \text{and} \quad 0 \leq \phi \leq 2\pi.$$

Solution 8.63

We follow the usual steps.

- Draw the area A. This you may find difficult but you could always use MAPLE to help you. The area is shown in Figure 8.11.
- Decide on the order of integration. Because we are given $R = R_1(\phi)$ and $R = R_2(\phi)$ it makes sense to take R as the inner integral and ϕ as the outer integral.
- Determine the limits. We've just determined the order and the limits follow immediately. The radial integral goes from $R = R_1(\phi)$ to $R = R_2(\phi)$. In the ϕ direction we have $0 \leq \phi \leq 2\pi$. Thus,

$$I = \int_{\phi=0}^{2\pi} \left(\int_{R=a\sqrt{\phi}}^{b\sqrt{\phi}} R dR \right) d\phi.$$



{fig:8.12} Figure 8.11: The area enclosed by two spirals $r = a\sqrt{\phi}$ and $r = b\sqrt{\phi}$.

- Do the inner integral first. Thus,

$$\int_{R=a\sqrt{\phi}}^{b\sqrt{\phi}} RdR = \frac{1}{2}\left[R^2\right]_{R=a\sqrt{\phi}}^{b\sqrt{\phi}} = \frac{1}{2}\left(b^2\phi - a^2\phi\right)$$

Thus, the outer integral can now be done.

$$I = \int_{\phi=0}^{2\pi} \frac{1}{2}\left(b^2 - a^2\right)\phi d\phi = \frac{1}{4}\left(b^2 - a^2\right)\left[\phi^2\right]_{\phi=0}^{2\pi}.$$

This can be simplified to

$$I = \frac{1}{4}\left(b^2 - a^2\right)4\pi^2 = \pi^2\left(b^2 - a^2\right).$$

Example End

Example 8.64

Find the area of an ellipse,

$$I = \int\int_A dA,$$

where A is the area enclosed by

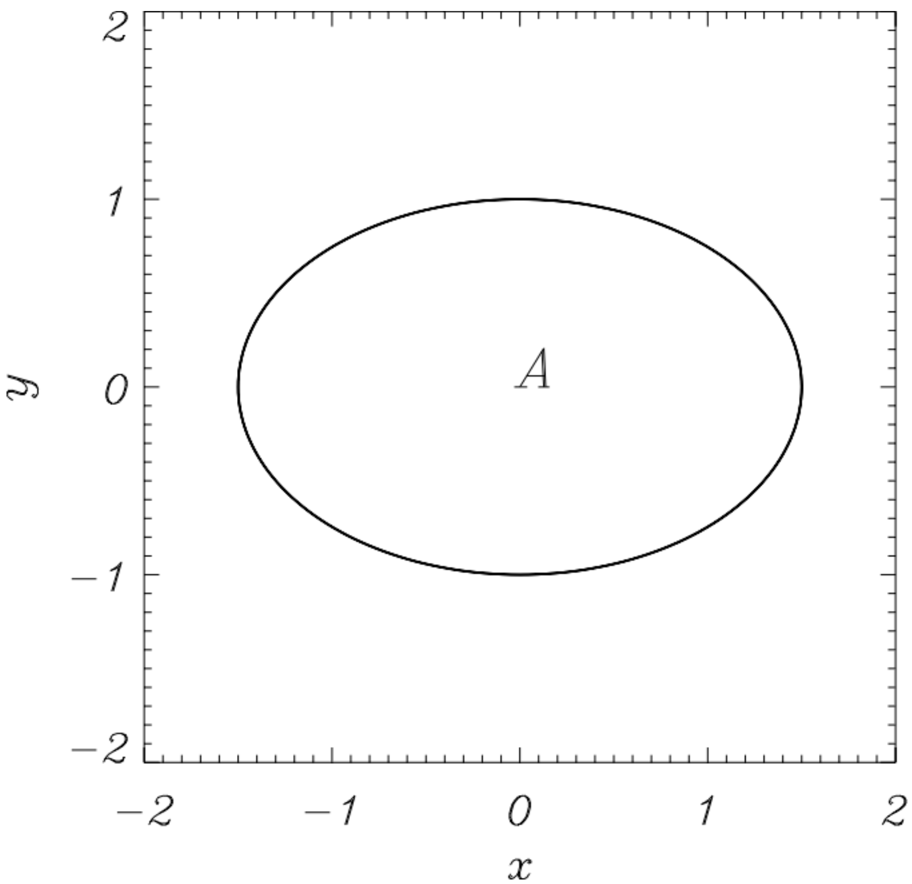
8.8a} $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{8.10}$$

a and b are constants. If $a = b$, then we have a circle of radius a.

Solution 8.64

We follow the usual approach.

- Draw the area. This is shown in Figure 8.12. It is not obvious but 'modified' polar coordinates



8.13} Figure 8.12: The area enclosed by an ellipse.

are the sensible choice of coordinates. Given that any point lying inside the ellipse satisfies

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = R^2 \leq 1,$$

we can try the following change of variables

$$\frac{x}{a} = R\cos\phi, \qquad \frac{y}{b} = R\sin\phi.$$

Then the area is covered by $0 \le \phi \le 2\pi$ and $0 \le R \le 1$.

- Decide on order of integration. The order of integration does not matter in this case.
- Choose the limits. As mentioned above, the changes of variables indicate that the limits are $0 \le \phi \le 2\pi$ and $0 \le R \le 1$
- Do the integration. Now we must stop and think. We do not yet have the correct expression for dA and this not quite as simple as the standard Cartesian and polar cases. A detailed derivation is given in the next section. For now we take

$$dA = abRdRd\phi.$$

Thus,

$$I = \int_{R=0}^{1} \left(\int_{\phi=0}^{2\pi} abRd\phi \right) dR.$$

The inner integral gives

$$\int_{\phi=0}^{2\pi} abRd\phi = abR\,[\phi]_{\phi=0}^{2\pi} = 2\pi abR.$$

The outer integral gives the area of the ellipse as

$$I = \int_{R=0}^{1} 2\pi abRdR = 2\pi ab\left[\frac{1}{2}R^2\right]_{R=0}^{1} = \pi ab.$$

Note that if $a = b$ we get the area of a circle of radius a as πa^2. That is a useful check!

Example End

Example 8.65

Find

$$I = \int\int_A \left(x^2 + y^2\right) dA,$$

where A is the elliptical region in the previous example.

Solution 8.65

Here you might be tempted to use ordinary polar coordinates because the integrand has the 'polar' form $x^2 + y^2$. However, getting the region properly treated rather than the integrand is more important in this example. Again we are going to use

$$x = aR\cos\phi \qquad y = bR\sin\phi$$

The order of the integration does not matter since the limits are fixed ($0 \leq \phi \leq 2\pi$ and $0 \leq R \leq 1$) and do not depend on the other variable. Thus, we must evaluate

$$I = \int_{\phi=0}^{2\pi} \left(\int_{R=0}^{1} \left[(aR\cos\phi)^2 + (bR\sin\phi)^2 \right] abRdR \right) d\phi.$$

The inner integral can be expressed as

$$\int_{R=0}^{1} ab \left[a^2 \cos^2\phi + b^2 \sin^2\phi \right] R^3 dR = ab \left\{ a^2 \cos^2\phi + b^2 \sin^2\phi \right\} \left[\frac{1}{4} R^4 \right]_{R=0}^{1} = \frac{ab}{4} \left\{ a^2 \cos^2\phi + b^2 \sin^2\phi \right\}$$

The outer integral is

$$I = \int_{\phi=0}^{2\pi} \frac{ab}{4} \left\{ a^2 \cos^2\phi + b^2 \sin^2\phi \right\} d\phi.$$

Remember the trigonometric identities,

$$\begin{aligned} \cos^2\phi &= \frac{1}{2}(1 + \cos 2\phi) \\ \sin^2\phi &= \frac{1}{2}(1 - \cos 2\phi) \end{aligned}$$

Thus,

$$\int_{\phi=0}^{2\pi} \cos^2\phi \; d\phi = \pi,$$

and

$$\int_{\phi=0}^{2\pi} \sin^2\phi \; d\phi = \pi.$$

Hence,

$$I = \frac{ab}{4} \left\{ a^2\pi + b^2\pi \right\} = \frac{\pi ab}{4} \left(a^2 + b^2 \right).$$

Example End

8.4 Generating the area element dA for any 2D coordinate system

:8.4}

If we have a change of coordinates so that the Cartesian coordinates are expressed in terms of the new coordinates, denoted here by u and v, then the infinitesimal area

$$dA = dxdy$$

can be expressed as

8.20}

$$dA = dxdy = \mid J \mid dudv, \tag{8.11}$$

where J is the *Jacobian.* If we have the Cartesian coordinates expressed explicitly in terms of the new coordinates as

$$\begin{aligned} x &= x(u,v), \\ y &= y(u,v), \end{aligned}$$

then the following partial derivatives can be calculated

$$\begin{array}{cc} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{array}$$

The Jacobian is given by evaluating the determinant,

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

This is straightforward to obtain and reduces to

$$|J| = \left| \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right| \tag{8.12}$$ {eq:8.21}

Let's try this out for the elliptic coordinates used in the last example of the last section. Here

$$x = aR\cos\phi, \qquad y = bR\sin\phi.$$

The partial derivatives (where $u = R$ and $v = \phi$ in the above expressions) are

$$\frac{\partial x}{\partial R} = a\cos\phi, \qquad \frac{\partial x}{\partial \phi} = -aR\sin\phi$$

and

$$\frac{\partial y}{\partial R} = b\sin\phi \qquad \frac{\partial y}{\partial \phi} = bR\cos\phi.$$

Therefore, the modulus of the Jacobian is

$$|J| = |(a\cos\phi bR\cos\phi) - (-aR\sin\phi b\sin\phi)| = abR\left(\cos^2\phi + \sin^2\phi\right) = abR.$$

Hence, the infinitesimal area is

$$dA = dxdy = abRdRd\phi$$

as used above.

You can check that the change of coordinates to polar coordinates does indeed give the correct form for the infinitesimal area. The same idea can be extended to volume integrals where we will need to evaluate the determinant of a 3×3 matrix based on all the partial derivatives.