MT5823 Semigroup theory: Solutions 4 (James D. Mitchell)
Congruences and presentations

Congruences

4-1. Let S be a left zero semigroup and let ρ be an equivalence relation on S. Then for $(x, y) \in \rho$ and $z \in S$ we have

$$(xz, yz) = (x, y) \in \rho$$

$$(zx, zy) = (z, z) \in \rho.$$

Hence ρ is a (2-sided) congruence.

Let R be the rectangular band $\{1,2\}\times\{1,2\}$. Then the equivalence relation ρ with classes $\{(1,1),(2,2)\}$, $\{(1,2)\}$, and $\{(2,1)\}$ is not a congruence as $((1,1),(2,2)) \in \rho$ but

$$((2,1)(1,1),(2,1)(2,2)) = ((2,1),(2,2)) \notin \rho. \quad \Box$$

4-2. We want to show that if $e^2 = e$, then $e/\rho \leq S$. That is, we want to show that if $e^2 = e$, then $xy \in e/\rho$ for all $x, y \in e/\rho$. Hence it suffices to show that if $e^2 = e$ and $x, y \in S$ are such that $(x, e), (y, e) \in \rho$, then $(xy, e) \in \rho$.

Let $x, y \in S$ be such that $(x, e), (y, e) \in \rho$. Then since ρ is a congruence, $(xy, ey), (ey, e^2) \in \rho$. Hence $(xy, e^2) = (xy, e) \in \rho$ (ρ is transitive!), as required.

Also,

$$(e/\rho)^2 = (e/\rho)(e/\rho) = e^2/\rho = e/\rho$$

and so e/ρ is an idempotent in S/ρ.

If S is finite and $(x/\rho)^2 = x/\rho$, then $(x/\rho)^i = x/\rho$ for all $i \in \mathbb{N}$. Hence $(x^i)/\rho = x/\rho$ and so $x^i \in x/\rho$ for all i. By Problem **2-8**, every element of finite semigroup has an idempotent power. Since S is finite and $x \in S$, it follows that x^m is an idempotent and $x^m \in x/\rho$ for some $m \in \mathbb{N}$. $\Box$

4-3. (a) Let $i \in \{1, 2\}$ be arbitrary. Then

$$(i, 1) = (i, i)(1, 1)\rho(i, i)(2, 2) = (i, 2)$$

and

$$(1, i) = (1, 1)(i, i)\rho(2, 2)(i, i) = (2, i).$$

Hence $(1,1)\rho(1,2)$, $(2,1)\rho(2,2)$, $(1,1)\rho(2,1)$, $(1,2)\rho(2,2)$, and so $\rho = S \times S$.

(b) The classes of the least congruence ρ on S such that $(1,1)\rho(1,2)$ are:

$$\{(1,1),(1,2)\}, \quad \{(2,2),(2,1)\}.$$

Let σ denote the equivalence relation with classes given above. Then, certainly, since

$$(2,2) = (2,2)(1,2)\rho(2,2)(1,1) = (2,1),$$

$\sigma \subseteq \rho$. On the other hand, it is routine to verify that σ is a congruence and so $\rho \subseteq \sigma$.

(c) The four congruences on S are: $S \times S$, Δ_S, ρ from part (b), and the congruence σ with classes:

$$\{(1,1),(2,1)\}, \quad \{(1,2),(2,2)\}.$$

Let τ be any congruence on S such that $\tau \neq \Delta_S$. Then there exist $((i,\lambda),(j,\mu)) \in \tau$ such that $(i,\lambda) \neq (j,\mu)$. If $(1,1)\tau(2,2)$, then $\tau = S \times S$; if $(1,1)\tau(2,1)$, then $\tau = \rho$ or $\tau = S \times S$; if $(1,1)\tau(1,2)$, then $\tau = \sigma$ or $\tau = S \times S$.

4-4. We start by verifying that σ/ρ is an equivalence relation.

Reflexive: if $x/\rho = y/\rho$, then $x/\sigma = y/\sigma$ since $\rho \subseteq \sigma$. Hence $(x/\rho, y/\rho) \in \sigma/\rho$.

Symmetric: if $(x/\rho, y/\rho) \in \sigma/\rho$, then $(x, y) \in \sigma$ and so $(y, x) \in \sigma$ and so $(y/\rho, x/\rho) \in \sigma/\rho$.

Transitive: if $(x/\rho, y/\rho), (y/\rho, z/\rho) \in \sigma/\rho$, then $(x, y), (y, z) \in \sigma$ and so $(x, z) \in \sigma$, which implies that $(x/\rho, z/\rho) \in \sigma/\rho$.

If $(x/\rho, y/\rho) \in \sigma/\rho$ and $s/\rho \in S/\rho$, then $(x, y) \in \sigma$ and so $(xs, ys), (sx, sy) \in \sigma$. Thus $(xs/\rho, ys/\rho), (sx/\rho, sy/\rho) \in \sigma/\rho$ and σ/ρ is a ***congruence***.

We define $\phi : (S/\rho)/(\sigma/\rho) \longrightarrow S/\sigma$ by

$$\big((x/\rho)/(\sigma/\rho)\big)\phi = x/\sigma.$$

Well-defined: If $(x/\rho)/(\sigma/\rho) = (y/\rho)/(\sigma/\rho)$, then $(x/\rho, y/\rho) \in \sigma/\rho$ and so $(x, y) \in \sigma$. In other words, $x/\sigma = y/\sigma$ and ϕ is well-defined.

Injective: Suppose that $\big((x/\rho)/(\sigma/\rho)\big)\phi = \big((y/\rho)/(\sigma/\rho)\big)\phi$. Then $x/\sigma = y/\sigma$ and so $(x/\rho, y/\rho) \in \sigma/\rho$. In other words, $(y/\rho)/(\sigma/\rho) = (y/\rho)/(\sigma/\rho)$, and ϕ is injective.

Surjective: Trivial.

Homomorphism: Let $(x/\rho)/(\sigma/\rho), (y/\rho)/(\sigma/\rho) \in (S/\rho)/(\sigma/\rho)$ be arbitrary. Then

$$\begin{aligned}
\big((x/\rho)/(\sigma/\rho)\big)\phi \cdot \big((y/\rho)/(\sigma/\rho)\big)\phi &= x/\sigma \cdot y/\sigma \\
&= xy/\sigma \\
&= ((xy/\rho)/(\sigma/\rho))\phi \\
&= (((x/\rho \cdot y/\rho)/(\sigma/\rho))\phi \\
&= \big((x/\rho)/(\sigma/\rho) \cdot (y/\rho)/(\sigma/\rho)\big)\phi
\end{aligned}$$

and so ϕ is a homomorphism.

Presentations

4-5. Using the algorithm from lectures:

$$\begin{array}{lllll}
t_1 & = a & \textbf{(new)} & t_2 = b & \textbf{(new)} \\
t_1a & = a^2 = t_3 & \textbf{(new)} & t_1b = ab = t_4 & \textbf{(new)} \\
t_2a & = ba = a^2b = t_5 & \textbf{(new)} & t_2b = b^2 = t_6 & \textbf{(new)} \\
t_3a & = a^3 = a = t_1 & \textbf{(old)} & t_3b = a^2b = t_5 & \textbf{(old)} \\
t_4a & = aba = a^3b = ab = t_4 & \textbf{(old)} & t_4b = ab^2 = t_7 & \textbf{(new)} \\
t_5a & = a^2(ba) = a^4b = a^2b = t_5 & \textbf{(old)} & t_5b = a^2b^2 = t_8 & \textbf{(new)} \\
t_6a & = b^2a = ba^2b = a^4b = a^2b = t_5 & \textbf{(old)} & t_6b = b^3 = t_9 & \textbf{(new)} \\
t_7a & = ab^2a = aba^2b = a^5b = ab^2 = t_7 & \textbf{(old)} & t_7b = ab^3 = t_{10} & \textbf{(new)} \\
t_8a & = a^2b^2a = a^2b^2 = t_8 & \textbf{(old)} & t_8b = a^2b^3 = t_{11} & \textbf{(new)} \\
t_9a & = b^3a = a^6b^3 = a^2b^3 = t_{11} & \textbf{(old)} & t_9b = b^4 = b = t_2 & \textbf{(old)} \\
t_{10}a & = ab^3a = a^7b^3 = ab^3 = t_{10} & \textbf{(old)} & t_{10}b = ab^4 = ab = t_4 & \textbf{(old)} \\
t_{11}a & = a^2b^3a = a^6b^3 = a^2b^3 = t_{11} & \textbf{(old)} & t_{11}b = a^2b^4 = a^2b = t_5 & \textbf{(old)}.
\end{array}$$

Hence:

$$S = \{a, b, a^2, ab, a^2b, b^2, ab^2, a^2b^2, b^3, ab^3, a^2b^3\}$$

and, by squaring every element, the idempotents are:

$$E(S) = \{a^2, b^3, ab^3, a^2b^3\}.$$

The right Cayley graph of S is shown in Figure 1. □

4-6. Let $w \in \{a, b, 0\}^+$ and let ρ be the least congruence on $\{a, b\}^+$ containing the relations of the presentation defining S. We will prove that w/ρ equals one of $0/\rho, a/\rho, b/\rho, ab/\rho, ba/\rho$ in S. If w contains a^2 or b^2 as a factor, then by the relations $a^2 = b^2 = 0$ and $a0 = 0a = b0 = 0b = 0$ we obtain that $w/\rho = 0/\rho$. Otherwise, $w/\rho = a(ba)^n/\rho$, $(ba)^n/\rho$, $(ba)^nb/\rho$, or $(ab)^n/\rho$ for some n. In any case, applying the relations $aba = a$ and $bab = b$, we deduced that $w/\rho \in \{a/\rho, b/\rho, ab/\rho, ba/\rho\}$. Thus $|S| \leq 5$.

[Note that you could omit the $/\rho$ in the above and simply say that w equals 0, a, b, ab, ba *in* S. But remember that every time you write a or b you have to say if you are considering them as elements of $\{a, b, 0\}^+$ or S.]

We will show that the elements $0/\rho, a/\rho, b/\rho, ab/\rho, ba/\rho$ are distinct. [If you prefer, another way of saying this is that we want to prove that the words 0, a, b, ab, ba represent different elements of S.] To prove that $a/\rho \neq b/\rho$, say, then we must show that $w \neq w'$ in $\{a, b, 0\}^+$ for all $w, w' \in \{a, b, 0\}^+$ such that $w/\rho = a/\rho$ and $w'/\rho = b/\rho$.

If $a/\rho = w/\rho$ for some $w \in \{a, b, 0\}^+$, then there exists an elementary sequence

$$a = w_0, w_1, \ldots, w_m = w$$

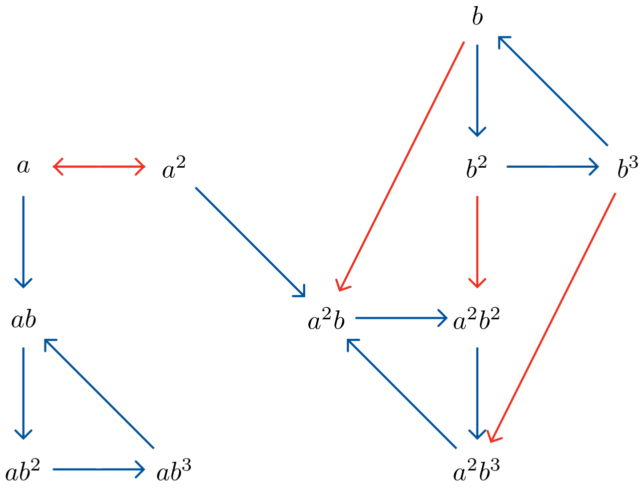

Figure 1: The right Cayley graph of the semigroup in Problem **4-5** (red is for a and blue is for b, loops are omitted).

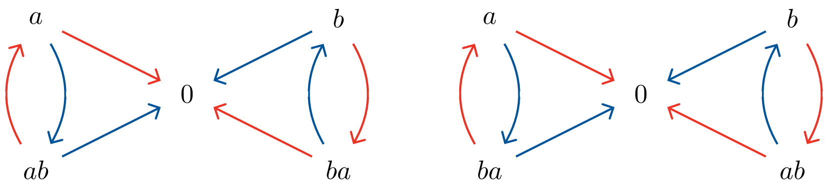

Figure 2: The left and right Cayley graphs of the semigroup in Problem **4-6** (red is for a and blue is for b, loops are omitted).

of words in $\{a, b, 0\}^+$. The only relation that can be applied to the word $a = w_0$ is $a = aba$ and so $w_1 = aba$. By the same reasoning, $w_i = au_ia$ for some $u_i \in \{a, b\}^*$ (u_i could be the empty word) for all i. Hence $w = aw_ma$ in $\{a, b, 0\}^+$. Using similar arguments, you can show that if $b/\rho = w/\rho$, then the first and last letters of $w \in \{a, b, 0\}^+$ are b; if $ab/\rho = w/\rho$, then the first letter of w is a and the last letter is b; if $ba/\rho = w/\rho$, then the first letter of w is b and the last letter is a; if $0/\rho = w/\rho$, then w contains a^2, b^2, or 0 as a factor. It follows that $0/\rho, a/\rho, b/\rho, ab/\rho, ba/\rho$ are distinct and so $|S| = 5$.

The multiplication table of S is:

$\cdot$	0	a	b	ab	ba
0	0	0	0	0	0
a	0	0	ab	0	a
b	0	ba	0	b	0
ab	0	a	0	ab	0
ba	0	0	b	0	ba

The left and right Cayley graphs of S are shown in Figure 2. $\square$

4-7. If $i \leq j$, then

$$\begin{aligned} a_ia_j &= a_ia_{i+1}a_j = a_ia_{i+1}a_{i+2}a_j = \cdots = a_ia_{i+1}a_{i+2}\cdots(a_{j-1}a_j) \\ &= a_ia_{i+1}a_{i+2}\cdots a_{j-1} = \cdots = a_ia_{i+1} = a_i. \end{aligned}$$

The proof follows by a similar argument if $i > j$.

We must prove that $a_i \neq a_j$ in S if $i \neq j$. Using a similar argument to that given in the solution to Problem **4-6**, if w is any word in $A^+ = \{a_1, \ldots, a_n\}^+$ representing a_i, then the first letter of w must be a_i. It follows that the elements $a_1, a_2, \ldots, a_n$ are distinct and so S has n elements. $\square$

4-8. The relation

$$xy = xy1 = xy(xyx) = (xyx)yx = 1yx = yx$$

holds in S.

To prove that every element in S has the form x^i, y^j, xy^j $(i \geq 0,\ j \geq 1)$ it suffices to prove that the relation $x^2y = 1$ holds in S. So,

$$x^2y = x(xy) = xyx = 1$$

by the first part of the question.

Let $p = -1$ and $q = 2$. Then $2p + q = 0$ and $\langle -1, 2 \rangle = \mathbb{Z}$. Hence $(\mathbb{Z}, +)$ satisfy the relations of the presentation defining S with respect to the mapping f such that $x \mapsto p$ and $y \mapsto q$. It follows by Theorem **??** that the mapping $\phi : S \longrightarrow \mathbb{Z}$ defined by $x^i \mapsto -i$, $y^j \mapsto 2j$, and $xy^j \mapsto 2j - 1$ is an isomorphism. $\square$