5 Locating zeros of complex functions

5.1 Multiplicity of zeros

A function $f(z)$ has a zero of order m at $z = z_0$ if

$$\lim_{z \to z_0} \left\{ \frac{f(z)}{(z - z_0)^m} \right\} = A \neq 0.$$

We also say that the zero $z = z_0$ has multiplicity m. Obviously, in this case, we may write

$$f(z) = (z - z_0)^m g(z) \quad \text{where} \quad g(z_0) = A \neq 0.$$

Example 5.1.1 $f(z) = \sin^2 z$ has a zero of multiplicity 2 at the origin since

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots ,$$

so that

$$\sin^2 z = z^2 \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots \right)^2 = z^2 g(z),$$

where $g(0) = 1$.

Note that the term "multiplicity" is sometimes used in place of "order" in relation to the poles of a function $f(z)$. For the remainder of this section, we use "multiplicity" in relation to both zeros and poles.

Example 5.1.2 The function $f(z) = \dfrac{(z-1)^3}{(z+2)^2}$ has a zero of multiplicity 3 at $z = 1$ and a pole of multiplicity 2 at $z = -2$.

Note that a function can be analytic at its zeros but is not analytic at its poles.

5.2 Logarithmic derivative

If $f(z)$ is analytic in a domain D, then the function $F(z)$ defined by

$$F(z) = \frac{d}{dz}[\log f(z)] = \frac{f'(z)}{f(z)}$$

is called the logarithmic derivative of $f(z)$. The derivative $F(z)$ is analytic except at the zeros of $f(z)$.

Example 5.2.1 The logarithmic derivative of the function in example 5.1.2 is

$$F(z) = \frac{3}{z-1} - \frac{2}{z+2}.$$

It is interesting to note that the zero $z = 1$ of multiplicity 3 and the pole $z = -2$ of multiplicity 2 become simple poles of $F(z)$ with residues 3 and -2, respectively. This is true in general, not just for this particular case, as will be seen shortly.

Consider $f(z)$ with a zero of multiplicity m at $z = z_0$. We write

$$f(z) = (z - z_0)^m g(z),$$

where $g(z_0) \neq 0$. Differentiating yields

$$f'(z) = (z - z_0)^m g'(z) + m(z - z_0)^{m-1} g(z),$$

so that

$$F(z) = \frac{f'(z)}{f(z)} = \frac{(z - z_0)^m g'(z) + m(z - z_0)^{m-1} g(z)}{(z - z_0)^m g(z)} = \frac{m}{(z - z_0)} + \frac{g'(z)}{g(z)}.$$

Hence the logarithmic derivative $F(z)$ has a simple pole at $z = z_0$ with residue m.

Now consider the situation where $f(z)$ has a pole of multiplicity p at $z = z_*$. Its Laurent series about z_* has the form

$$\begin{aligned} f(z) &= \frac{a_{-p}}{(z - z_*)^p} + \frac{a_{-(p-1)}}{(z - z_*)^{p-1}} + \ldots \\ &= \frac{1}{(z - z_*)^p} \left(a_{-p} + a_{-(p-1)}(z - z_*) + a_{-(p-2)}(z - z_*)^2 + \ldots \right) = \frac{g(z)}{(z - z_*)^p}, \end{aligned}$$

where $g(z_*) = a_{-p} \neq 0$. Differentiating gives

$$f'(z) = (z - z_*)^{-p} g'(z) - p(z - z_*)^{-(p+1)} g(z).$$

So

$$F(z) = \frac{f'(z)}{f(z)} = \frac{(z - z_*)^{-p} g'(z) - p(z - z_*)^{-(p+1)} g(z)}{(z - z_*)^{-p} g(z)} = \frac{-p}{z - z_*} + \frac{g'(z)}{g(z)}.$$

This means that $F(z)$ has a simple pole at $z = z_*$ with residue $-p$.

5.3 The Argument principle

Consider the above situation where $f(z)$ has a pole of multiplicity p at $z = z_*$ and a zero of multiplicity m at $z = z_0$. The logarithmic derivative $F(z) = f'(z)/f(z)$ has simple poles at z_* and z_0 with residues $-p$ and m, respectively. If a simple closed contour C encircles both points z_* and z_0 but no other poles or zeros of $f(z)$, then by the residue theorem we have

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)}\, dz = m - p,$$

provided that $f(z) \neq 0$ on C. This extends in a straightforward manner to more complicated situations where f possesses several zeros and poles within C, and the precise statement is as follows.

If $f(z)$ is analytic in a domain D except for a finite number of poles and if $f(z) \neq 0$ on a simple closed contour C inside D that encircles these poles, then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = N_0 - N_p,$$

where

$$N_0 = \text{total number of zeros of } f(z) \text{ inside } C$$

and

$$N_p = \text{total number of poles of } f(z) \text{ inside } C.$$

Note that the count includes multiplicities.

Now the logarithmic derivative $f'(z)/f(z)$ is analytic in a neighbourhood of C. So

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\left[\log f(z)\right]_C = \frac{1}{2\pi i}\left[\ln|f(z)| + i\arg f(z)\right]_C.$$

Since C is a simple closed contour, $|f(z)|$ returns to the same value and $[\ln|f(z)|]_C = 0$. Hence the value of the integral is just the change in the value of the argument of $f(z)$ as one circuit of C is completed. That is

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi}\Delta_C\left\{\arg(f(z))\right\} = N_0 - N_p,$$

where $\Delta_C\left\{\arg(f(z))\right\}$ is the change in the argument of $f(z)$ as we traverse C once in an anti-clockwise direction. For this reason the above result is known as the argument principle.

Example 5.3.1 The function $f(z) = z^2$ has a double zero at the origin. Its logarithmic derivative is $2z/z^2$. The argument principle implies

$$\frac{1}{2\pi i}\oint_C \frac{2z}{z^2}\,dz = \frac{1}{2\pi i}\oint_C \frac{2}{z}\,dz = 2,$$

provided C encloses the origin. Thus we recover the well-known result

$$\oint_C \frac{dz}{z} = 2\pi i.$$

Also, note that since C encloses the origin, during one circuit of C, $\arg(z)$ moves through 2π and $\arg(z^2) = 2\arg(z)$ changes by 4π and hence

$$\frac{1}{2\pi}\Delta_C\left\{\arg(z^2)\right\} = 2.$$

5.4 Locating roots of equations

Let us consider applying the argument principle to the problem of determining the location of roots of an equation. First let us choose a simple (obvious) problem.

Example 5.4.1 How many zeros of the equation

$$f(z) = z^3 + 1 = 0$$

lie in the first quadrant?

Solution Consider the contour illustrated in figure 29 where AB is a circular arc of radius R and we will let R tend to infinity.

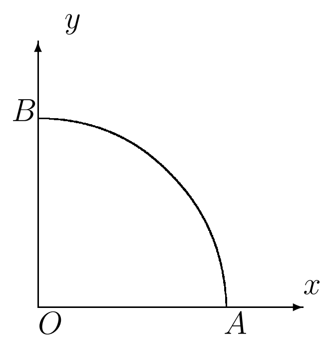

Figure 29: Contour OA→AB→BO

It is easy to see that for sufficiently large R, there are no roots on C. On OA we have $z = x$, so

$$f(z) = x^3 + 1 > 0 \quad \text{for all } x \geq 0,$$

Since $f(z)$ is real, $\arg(f(z))$ remains zero from O to A. Thus, let $\phi = \arg(f(z))$, then

$$\Delta_{OA}\phi = 0.$$

On AB $z = Re^{i\theta}$, so that

$$f(z) = R^3 e^{3i\theta} + 1 = (R^3 \cos 3\theta + 1) + iR^3 \sin 3\theta.$$

It follows that

$$\tan \phi = \frac{R^3 \sin 3\theta}{R^3 \cos 3\theta + 1} \approx \tan 3\theta\ , \quad \phi \approx 3\theta,$$

when R is large. Since θ varies from 0 to $\pi/2$ on AB, we have

$$\Delta_{AB}\phi = 3 \times \frac{\pi}{2} = \frac{3\pi}{2}.$$

Now, along BO $z = iy$, so that

$$f(z) = -iy^3 + 1$$

and

$$\tan\phi = -y^3.$$

As y varies from ∞ to 0, $\tan\phi$ varies from $-\infty$ to 0, implying that $\Delta_{BO}\phi = \pi/2$. Hence

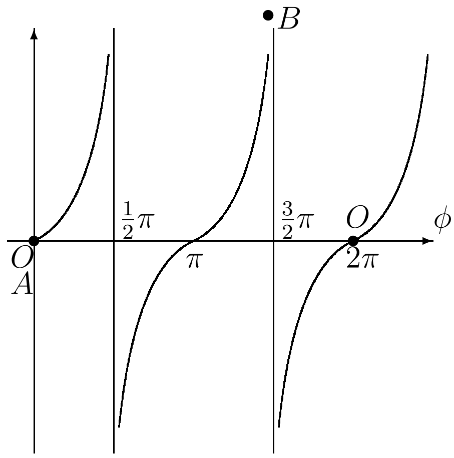

Figure 30: $\tan\phi$ on C

$$\frac{1}{2\pi}\Delta_C\left\{\arg f(z)\right\} = \frac{1}{2\pi}\left(0 + \frac{3\pi}{2} + \frac{\pi}{2}\right) = 1.$$

There is just one root in the first quadrant as expected. •

Now let us attempt a problem where we do not a priori know the answer.

Example 5.4.2 Consider the equation

$$f(z) = z^3 - iz - i = 0.$$

Determine the number of roots in the first quadrant.

Solution Again use the contour illustrated in figure 29. First we check that there are no zeros on the contour. Indeed, when $z = x$ (real), the equation $x^3 - i(x+1) = 0$ has no solution. Likewise, when $z = iy$ (y real), the equation $y - i(y^3 + 1) = 0$ has no solution. Thus there are no real and no purely imaginary solutions. For roots within the first quadrant we can always make R large enough to avoid them and in practice we consider $R \to \infty$. So there are no roots on the contour in figure 29 in this limit. Now consider the change of $\phi = \arg(f(z))$ along the contour.

<u>On OA</u>

$$f(z) = x^3 - i(x+1) \quad \text{and } \tan\phi = \frac{-(x+1)}{x^3}.$$

As x goes from 0 to ∞, $\tan\phi$ (which remains negative throughout) goes from $-\infty$ to 0^-. So ϕ goes from $-\pi/2$ to 0, i.e. a change of $\pi/2$. Thus $\Delta_{OA}\phi = \pi/2$.

<u>On AB</u>

$$f(z) = R^3 e^{3i\theta} - iRe^{i\theta} - i = R^3\cos 3\theta + iR^3 \sin 3\theta - iR\cos\theta + R\sin\theta - i.$$

So

$$\tan\phi = \frac{R^3\sin 3\theta - R\cos\theta - 1}{R^3\cos 3\theta + R\sin\theta} \approx \tan 3\theta \implies \phi \approx 3\theta,$$

for R large. The change of θ from 0 to $\pi/2$ on AB induces a change of $3\pi/2$ for ϕ. Thus $\Delta_{AB}\phi = 3\pi/2$.

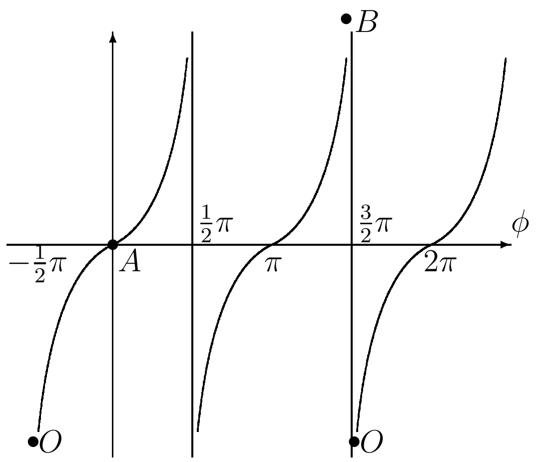

Figure 31: $\tan(\phi)$ on C

<u>On BO</u>

$$f(z) = y - i(y^3+1) \quad \text{so that } \tan\phi = \frac{-(y^3+1)}{y}.$$

As y varies from ∞ to 0, $\tan\phi$ (which remains negative throughout) increases from $-\infty$ to a maximum (< 0) and then decreases back to $-\infty$, giving an overall change along BO of 0. Thus $\Delta_{BO}\phi = 0$. Putting these results together yields

$$\frac{1}{2\pi}\Delta_C\{\arg f(z)\} = \frac{1}{2\pi}\left(\frac{\pi}{2} + \frac{3\pi}{2} + 0\right) = 1.$$

Hence there is one root in the first quadrant. •

Example 5.4.3 Show that the equation

$$f(z) = z^4 + 3z^3 + 5z^2 + 4z + 2 = 0$$

has no roots in the first and fourth quadrants.

Solution First note that

$$\overline{f(z)} = \overline{z}^4 + 3\overline{z}^3 + 5\overline{z}^2 + 4\overline{z} + 2 = 0.$$

Hence, if z_0 is a root then $\overline{z}_0$ is also a root. This is the case for polynomials with real coefficients. So it is sufficient to show that there exists no roots in the first quadrant.

Consider the contour in figure 29. Obviously, there are no roots on OA because $x^4 + 3x^3 + 5x^2 + 4x + 2 \geq 2$, for $x \geq 0$. On AB, $|f(z)| \approx R^4$ for large R, so there are no roots on AB for sufficiently large R. Suppose there are roots on BO, i.e. there exist real y such that

$$y^4 - 3iy^3 - 5y^2 + 4iy + 2 = 0.$$

This requires that both the real and imaginary parts vanish, i.e.

$$y^4 - 5y^2 + 2 = 0 \quad \text{and} \quad -3y^3 + 4y = 0.$$

The first equation requires $y = \pm\sqrt{5 \pm \sqrt{17}}/\sqrt{2}$, while the second equation requires $y = 0$ or $y = \pm 2/\sqrt{3}$. So there are no solutions to the system, i.e. no roots on BO. Hence $f(z) = 0$ has no roots on the contour in figure 29, and the argument principle applies.

The change of $\phi = \arg(f(z))$ on OA is zero. On AB we have

$$f(z) = R^4 e^{4i\theta} + 3R^3 e^{3i\theta} + 5R^2 e^{2i\theta} + 4Re^{i\theta} + 2 = 0.$$

So

$$\tan\phi = \frac{R^4 \sin 4\theta + 3R^3 \sin 3\theta + 5R^2 \sin 2\theta + 4R \sin\theta}{R^4 \cos 4\theta + 3R^3 \cos 3\theta + 5R^2 \cos 2\theta + 4R\cos\theta + 2} \approx \tan 4\theta\ , \quad \phi \approx 4\theta,$$

for R large. So in the limit $R \to \infty$, the change of ϕ on AB is 2π. On BO, ϕ changes in a highly nontrivial manner. From the real and imaginary parts given above, we have

$$\tan\phi = \frac{-3y^3 + 4y}{y^4 - 5y^2 + 2}.$$

The chart below describes the behaviour of $\tan\phi$, and hence the change of ϕ, on BO.

As y goes from ∞ to $\sqrt{5+\sqrt{17}}/\sqrt{2}$, $\tan\phi$ goes from 0^- to $-\infty$, corresponding a change of $-\pi/2$ in ϕ. As y goes from $\sqrt{5+\sqrt{17}}/\sqrt{2}$ to $2/\sqrt{3}$, $\tan\phi$ goes from $+\infty$ to 0^+, corresponding a change of $-\pi/2$ in ϕ. As y goes from $2/\sqrt{3}$ to $\sqrt{5-\sqrt{17}}/\sqrt{2}$, $\tan\phi$ goes from to 0^- to $-\infty$, corresponding a change of $-\pi/2$ in ϕ. Finally, As y goes from $\sqrt{5-\sqrt{17}}/\sqrt{2}$ to 0, $\tan\phi$ goes from to $+\infty$ to 0^+, corresponding a change of $-\pi/2$ in ϕ. The total change of ϕ on BO is -2π. The net change in ϕ along the closed contour C is zero. So there are no roots of $f(z) = 0$ in the first quadrant. •

Remark. It can be seen that the 4 roots of

$$z^4 + 3z^3 + 5z^2 + 4z + 2 = 0$$

are $-1 \pm i$ and $(-1 \pm \sqrt{3}\,i)/2$, all lying in the second and third quadrants.

y	0		$\frac{\sqrt{5-\sqrt{17}}}{\sqrt{2}}$		$\frac{2}{\sqrt{3}}$		$\frac{\sqrt{5+\sqrt{17}}}{\sqrt{2}}$		∞
$-3y^3+4y$	0		$+$		0		$-$		$-\infty$
y^4-5y^2+2		$+$	0		$-$		0	$+$	$+\infty$
$\tan\phi$	0	$\swarrow$	$+\infty-$	$\swarrow$	0	$\swarrow$	$+\infty-$	$\swarrow$	0

5.5 Rouche's theorem

Rouche's theorem is an immediate consequence of the argument principle and can be stated as follows.

Let $f(z)$ and $g(z)$ be analytic inside and on a simple closed contour C. If $|g(z)| < |f(z)|$ at each point on C, then $f(z)+g(z)$ and $f(z)$ have the same number of zeros, counting multiplicities, inside C.

For a proof of this theorem, consider the function $h(z)$ defined by

$$h(z) = \frac{f(z)+g(z)}{f(z)}.$$

It is easy to see that $h(z)$ has no poles or zeros on C because the condition $|g(z)| < |f(z)|$ implies

$$|f(z)| > 0 \quad \text{and} \quad |f(z)+g(z)| \geq |f(z)| - |g(z)| > 0.$$

Applying the argument principle to $h(z)$ and C yields

$$\frac{1}{2\pi}\Delta_C\{\arg h(z)\} = N_0(h) - N_p(h).$$

Now

$$|h(z)-1| = \left|\frac{g(z)}{f(z)}\right| < 1.$$

This means that the image of C under the map $h(z)$ lies within the circle centred at 1 having radius less than 1. Clearly this image does not enclose the origin. Hence

$$\Delta_C\{\arg h(z)\} = 0.$$

So

$$N_0(h) = N_p(h).$$

But

$$N_0(h) = N_0(f+g)$$

and

$$N_p(h) = N_0(f).$$

Therefore

$$N_0(f+g) = N_0(f).$$

Example 5.5.1 Show that the polynomial

$$P_5(z) = z^5 + 3z^2 + 6z + 1$$

has one zero inside the unit circle.

Solution Let $f(z) = 6z + 1$ and $g(z) = z^5 + 3z^2$. On the unit circle we have

$$|f(z)| = |6z + 1| \geq 6|z| - 1 = 5 \quad \text{and} \quad |g(z)| = |z^5 + 3z^2| \leq |z|^5 + 3|z|^2 = 4.$$

Hence $|g(z)| < |f(z)|$, and all conditions for Rouche's theorem are satisfied. Now $z = -1/6$ is the only zero of $f(z)$ in the unit circle. Rouche's theorem implies that $P_5(z) = g(z)+f(z)$ has only one zero in the unit circle. •

Example 5.5.2 Show that all roots of

$$P_7(z) = z^7 - 2z^2 + 8 = 0$$

lie inside the annulus $1 < |z| < 2$.

Solution Let $f(z) = 8$ and $g(z) = z^7 - 2z^2$. On the unit circle we have

$$|g(z)| = |z^7 - 2z^2| \leq |z|^7 + 2|z|^2 = 3 < |f(z)| = 8.$$

Hence all conditions for Rouche's theorem are satisfied. Now $f(z) = 8$ clearly has no zero inside the unit circle (in fact, no zeros anywhere). So Rouche's theorem implies that $P_7(z) = f(z) + g(z)$ has no zero inside (and on) the unit circle.

Now let $f(z) = z^7$ and $g(z) = -2z^2 + 8$. On the circle $|z| = 2$ we have

$$|g(z)| = | - 2z^2 + 8| \leq 2|z|^2 + 8 = 16 < |f(z)| = 128.$$

Hence all conditions for Rouche's theorem are satisfied. Since $f(z) = z^7 = 0$ has 7 roots (counting multiplicity of course) within the circle $|z| = 2$, Rouche's theorem implies that $P_7(z) = f(z) + g(z)$ has 7 zeros inside the circle $|z| = 2$. This, together with the earlier result, implies that $P_7(z) = 0$ has all 7 roots inside the annulus $1 < |z| < 2$. •

Example 5.5.3 Show that no roots of

$$P_4(z) = z^4 + 5z^3 + z^2 + 2 = 0$$

lie in the annulus $1 \leq |z| \leq 4$.

Solution Let $f(z) = 5z^3$ and $g(z) = z^4 + z^2 + 2$. On the unit circle $|z| = 1$ we have

$$|f(z)| = |5z^3| = 5 \quad \text{and} \quad |g(z)| = |z^4 + z^2 + 2| \leq |z|^4 + |z|^2 + 2 = 4.$$

So all conditions for Rouche's theorem are satisfied. Now since $f(z) = 5z^3$ has 3 zeros inside the unit circle, $P_4(z) = f(z) + g(z)$ also has 3 zeros inside the unit circle.

On the other hand, on the circle $|z| = 4$ we have

$$|f(z)| = |5z^3| = 320 \quad \text{and} \quad |g(z)| = |z^4 + z^2 + 2| \leq |z|^4 + |z|^2 + 2 = 256 + 16 + 2 = 274.$$

So by Rouche's theorem, $P_4(z)$ has 3 zeros inside the circle $|z| = 4$. But we see from above that there are already three zeros inside the unit circle. So no zeros lie in the annulus $1 \leq |z| \leq 4$.

The above analysis also reveals that $P_4(z) = 0$ has at least 2 real roots. The reason is that the fourth root, which alone lies in the region $|z| > 4$, must be real. Furthermore, at least one of the roots inside the unit circle must be real. Thus at least 2 roots are real. •