SOLUTIONS TO THE 2010 EXAM FOR MT5823

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(1) (a) [Easy - Routine and similar to tutorial questions] Using the algorithm from lectures, using 14 multiplications we compute that the elements of S are:

$$t_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 2 & 4 & 5 & 6 & 7 & 1 \end{pmatrix}, t_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 1 & 4 & 5 & 6 & 7 & 2 \end{pmatrix},$$

$$t_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 1 & 4 & 5 & 6 & 7 & 3 \end{pmatrix}, t_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 6 & 6 & 4 & 4 & 6 & 7 & 6 \end{pmatrix},$$

$$t_5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 6 & 6 & 4 & 4 & 6 & 7 & 7 \end{pmatrix}, t_6 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 6 & 7 & 4 & 4 & 6 & 7 & 4 \end{pmatrix}$$

$$t_7 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 6 & 7 & 4 & 4 & 6 & 7 & 5 \end{pmatrix}$$

To show that S is not a monoid, note that t_3 is the unique element in S with rank 7 and so $t_3s \neq t_3$ and $st_3 \neq t_3$ for all $s \in S \setminus \{t_3\}$. Also t_3 does not fix its image, and so it is not an idempotent. Thus S has no identity and is not a monoid.
Idempotents in S are precisely those elements that fix their images. Thus, by inspection, t_1, t_4, t_5, and t_6 are the idempotents of S.

(b) [Moderate - not seen before] The images of t_4, t_5, and t_6 are equal to $\{4, 6, 7\}$ and since they all fix this image, the set $\{t_4, t_5, t_6\}$ is a subsemigroup of S. The image of t_1 is $\{1, 2, 4, 5, 6, 7\}$ and so $st_1 = t_1$ for all $s \in \{t_4, t_5, t_6\}$. On the other hand, $t_1s = t_4$ for all $s \in \{t_4, t_5, t_6\}$. It follows that $\{t_1, t_4, t_5, t_6\}$ is a subsemigroup of S.

(c) [Easy - book work] The Vagner's Representation Theorem states that every inverse semigroup S is isomorphic to an (inverse) subsemigroup of some symmetric inverse semigroup I_X.
It is not true that every subsemigroup of I_X is inverse. Let

$$f = \begin{pmatrix} 1 & 2 \\ - & 1 \end{pmatrix}.$$

Then $\langle\, f\, \rangle = \{f, \emptyset\}$ where $\emptyset$ denote the partial bijection

$$\begin{pmatrix} 1 & 2 \\ - & - \end{pmatrix}.$$

It follows that for all $g \in \langle\, f \rangle$, $fgf = \emptyset \neq f$ and so f has no inverse in $\langle\, f\, \rangle$.

(d) [Moderate - Similar to tutorial questions] The semigroup S is not inverse. There are several ways to answer this question. We saw in part (a) that $t_3st_3 \neq t_3$ for all $s \in S$. It follows that t_3 does not have an inverse in S and so S is not an inverse semigroup. From the algorithm used to find the elements of S in part (a) we can draw the left and right Cayley graphs of S. Since the strongly connected components of the left and right Cayley graphs correspond to the $\mathcal{L}$- and $\mathcal{R}$-classes of S, it follows that the $\mathcal{L}$-classes of S are:

$$\{t_1, t_2\}, \{t_3\}, \{t_4, t_5, t_6\}, \{t_7\}$$

and the $\mathcal{R}$-classes of S are:

$$\{t_1, t_2\}, \{t_3\}, \{t_4\}, \{t_5, t_6\}, \{t_7\}.$$

Hence the number of $\mathcal{L}$-classes of S is 4, and the number of $\mathcal{R}$-classes is 5. [Note that in an inverse semigroup the numbers of $\mathcal{L}$-classes and $\mathcal{R}$-classes are equal and so this provides an alternative way of showing that S is not an inverse semigroup.]

Date: 2010.

(e) [Moderate - Similar to tutorial questions] A semigroup is *simple* if it has one $\mathcal{J}$-class and a semigroup is *Clifford* if it is inverse and every $\mathcal{H}$-class of S is a group.
Green's $\mathcal{J}$- and $\mathcal{D}$-relation coincide on finite semigroups, and so to show that S is not simple it suffices to prove that S has more than one $\mathcal{D}$-class. Since $\mathcal{D}$-classes are a union of $\mathcal{L}$-classes, it follows that the $\mathcal{D}$-classes of S are:

$$\{t_1, t_2\}, \{t_3\}, \{t_4, t_5, t_6\}, \{t_7\}.$$

In particular, S is not simple.
The semigroup S is not inverse and so it is not Clifford.

(2) (a) [Easy - Similar to tutorial questions] By a theorem from lectures, a semigroup U is *simple* if and only if for all $a, b \in U$ there exist $s, t \in U$ such that $sat = b$.
Since T is simple and $p_{\lambda i}sp_{\lambda i}, t \in T$, there exist $u, v \in T$ such that $u(p_{\lambda i}sp_{\lambda i})v = t$. Hence

$$(j, u, \lambda)(i, s, \lambda)(i, v, \mu) = (j, up_{\lambda i}sp_{\lambda i}v, \mu) = (j, t, \mu),$$

and so S is simple.
Let T be a group. Then T is simple since $g^{-1}gh = h$ for all $g, h \in T$. It follows from the first part of the question that S is simple.

(b) [Moderate - Similar to tutorial questions] A semigroup U is *regular* if for all $x \in U$ there exists $y \in U$ such that $xyx = x$.
Let $t \in T$ be arbitrary. Since S is simple, for all $i \in I$ and $\lambda \in \Lambda$ there exists $(j, s, \mu) \in S$ such that

$$(i, t, \lambda)(j, s, \mu)(i, t, \lambda) = (i, t, \lambda).$$

Hence $(i, tp_{\lambda j}sp_{\mu,i}t, \lambda) = (i, t, \lambda)$ and, in particular, $tp_{\lambda,j}sp_{\mu,i}t = t$, as required.

(c) [Moderate - Similar to tutorial questions] The semigroup T is regular since every element is an idempotent and so $xxx = x$ for all $x \in T$.
Let P denotes the 1×1 sandwich matrix:

$$\begin{pmatrix} a \end{pmatrix}.$$

Then for any element $(1, x, 1) \in S$ we have that

$$\begin{aligned} (1, b, 1)(1, x, 1)(1, b, 1) &= (1, bp_{11}xp_{11}b, 1) = (1, b(axab), 1) \\ &= (1, ba, 1) = (1, a, 1) \end{aligned}$$

since $axab = a$. Hence $(1, b, 1)$ is not regular, and so S is not regular.

(d) [Hard - Not seen before] Recall that if $xyx = x$, then yxy is an inverse for x.
($\Rightarrow$) $(i, x, \lambda) \in S$ is regular implies there exists $(j, y, \mu) \in S$ such that

$$(i, x, \lambda)(j, y, \mu)(i, x, \lambda) = (i, x, \lambda).$$

Hence $xp_{\lambda j}yp_{\mu i}x = x$ and so $p_{\lambda j}yp_{\mu i}xp_{\lambda j}yp_{\mu i} \in p_{\lambda j}Tp_{\mu i}$ is an inverse for x.
($\Leftarrow$) Let $y \in T$ such that $p_{\lambda j}yp_{\mu i}$ is an inverse for x. Then

$$(i, x, \lambda)(j, y, \mu)(i, x, \lambda) = (i, xp_{\lambda j}yp_{\mu i}x, \lambda) = (i, x, \lambda)$$

and so (i, x, λ) is regular.

(3) (a) [Moderate - similar to tutorial questions] Let H be an $\mathcal{H}$-class of S and let $x \in H$. Then, by a theorem from lectures, either $H^2 \cap H = \emptyset$ or H is a group. Since $x^2 = x \in H$, it follows that H is a group. Hence H contains at most one idempotent, its identity. Since every element of H is an idempotent, it follows that H contains only one element of S.

(b) [Easy - book work] Since S is a band, S is periodic and so a theorem from lectures tells us that $\mathcal{D} = \mathcal{J}$.

(c) [Moderate - not seen before] Since $x = x^2 \in Sx$, we have that

$$\begin{aligned}(x,y) \in \mathcal{L} \quad &\iff \quad S^1x = S^1y \iff Sx \cup \{x\} = Sy \cup \{y\}\\ &\iff \quad Sx = Sy\end{aligned}$$

and

$$\begin{aligned}(x,y) \in \mathcal{R} \quad &\iff \quad xS^1 = yS^1 \iff xS \cup \{x\} = yS \cup \{y\}\\ &\iff \quad xS = yS.\end{aligned}$$

Since $Sx = Sx^2 \subseteq SxS$ and $xS = x^2S \subseteq SxS$, we have that

$$\begin{aligned}(x,y) \in \mathcal{D} \quad &\iff \quad (x,y) \in \mathcal{J} \iff S^1xS^1 = S^1yS^1\\ &\iff \quad SxS \cup Sx \cup xS \cup \{x\} = SyS \cup Sy \cup yS \cup \{y\}\\ &\iff \quad SxS = SyS,\end{aligned}$$

as required.

(d) [Hard - not seen before] (i)$\Rightarrow$(ii) Let $x \in S$ be fixed. If $xy \in xS$ is arbitrary, then $xy = xyx \in xSx$ and so $xS \subseteq xSx$. Also, clearly, $xSx \subseteq xS$ and so $xS = xSx$, as required.

(ii)$\Rightarrow$(iii) Certainly, $Sx = Sxx \subseteq SxS$. If $yxz \in SxS$ is arbitrary, then $xz \in xS = xSx$ and so there exists $t \in S$ such that $xz = xtx$. Hence $y(xz) = yxtx \in Sx$.

(iii)$\Rightarrow$(iv) Since $\mathcal{L} \subseteq \mathcal{D}$, it suffices to show that $\mathcal{D} \subseteq \mathcal{L}$. So, if $(x,y) \in \mathcal{D}$, then since S is periodic, $(x,y) \in \mathcal{J}$. Hence $SxS = SyS$ and so $Sx = Sy$ by (iii). That is, $(x,y) \in \mathcal{L}$.

(iv)$\Rightarrow$(v) Let $(x,y) \in \mathcal{R}$. Then $(x,y) \in \mathcal{D}$ and so $(x,y) \in \mathcal{L}$. It follows that $(x,y) \in \mathcal{H}$ and so, by part (a), $x = y$.

(v)$\Rightarrow$(vi) Since S is a band, $x^2 = x$ and so $(x,x) \in \rho$ for all $x \in S$. Hence ρ is reflexive.
If $(x,y),(y,x) \in \rho$, then $xy = y$ and $yx = x$. Hence $(xy)x = x(yx) = x^2 = x$ and $xy = xy$. Thus $(xy,x) \in \mathcal{R}$ and so $x = xy = y$ since $\mathcal{R}$ is trivial. Hence ρ is antisymmetric.
If $(x,y),(y,z) \in \rho$, then $xy = y$ and $yz = z$. Thus $xz = xyz = yz = z$ and so $(x,z) \in \rho$.

(vi)$\Rightarrow$(i) Since ρ is a partial order, it follows that it is antisymmetric and so if $(s,t),(t,s) \in \rho$, then $s = t$. In other words, if $st = t$ and $ts = s$, then $s = t$. So, if $s,t \in S$ are arbitrary, then $sts \cdot st = st \cdot st = st$ and $st \cdot sts = sts$. Thus $st = sts$, as required.