Chapter 3

Differential Vector Calculus

3.1 Scalar and Vector Function

In this chapter, we will consider both scalar and vector functions.

A scalar function is a function

$$\begin{aligned} f: \ & \mathbb{R}^3 \to \mathbb{R} \\ & \mathbf{x} \to f(\mathbf{x}) = f(x, y, z) \end{aligned}$$

For example, $f(\mathbf{x}) = x^2 + y^2 + z^2$ or $f(\mathbf{x}) = x \cosh y \ \tan z$, etc.

A vector function is a function

$$\begin{aligned} f: \ & \mathbb{R}^3 \to \mathbb{R}^3 \\ & \mathbf{x} \to \mathbf{F}(\mathbf{x}) \end{aligned}$$

For example, $\mathbf{F}(\mathbf{x}) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ or $\mathbf{F}(\mathbf{x}) = \cos x\mathbf{i} + \tan y\mathbf{j} + \ln \mathbf{x}\mathbf{k}$

Practical examples (in fluid dynamics)

- The pressure field at a given instant in time is a scalar function $p(\mathbf{x})$

- The velocity field at a given instant in time is a vector function $\mathbf{u}(\mathbf{x})$

3.2 Gradient of a scalar function

Consider the scalar function $f : \mathbb{R}^3 \to \mathbb{R}$.

The gradient of f is defined as the vector function:

$$\mathbf{grad} f = \nabla f = \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \\ \dfrac{\partial}{\partial y} \\ \\ \dfrac{\partial}{\partial z} \end{pmatrix} f = \begin{pmatrix} \dfrac{\partial f}{\partial x} \\ \\ \dfrac{\partial f}{\partial y} \\ \\ \dfrac{\partial f}{\partial z} \end{pmatrix} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

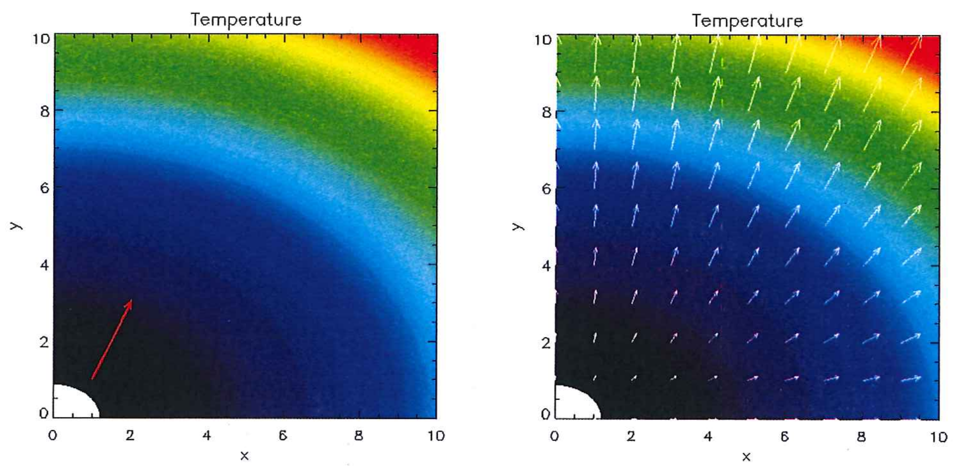
where the operator ∇ is called 'Del', 'grad', or 'Nabla' .

Each component gives the rate of change of f in the direction considered. Hence the gradient of the scalar function f (∇f) is a vector function which points in the direction of the greatest rate of increase of the function and its magnitude is the slope of the graph in that direction.

In practice, the gradient or 'grad' is a very important quantity. For example in fluid dynamics, pressure appears through its gradient - it is the change in pressure in space that matters for fluid motion.

Examples

(i) Consider the function $T(x, y) = 5 + 0.05x^2 + 0.1y^2$ where $0 \le x \le 10$ and $0 \le y \le 10$.



You can think of the function T as the temperature in the Maths building. You can work out that the minimum temperature occurs at $(x, y)_{min} = (0, 0)$ and is $T_{min} = 5$ degrees and $T_{max} = 20$ degrees is at $(x, y)_{max} = (10, 10)$.

Let us assume my office is at $(x, y)_{office} = (1, 1)$, where $T_{office} = 5.15$ degrees. As you can well imagine, I want to get away from there quite quickly.

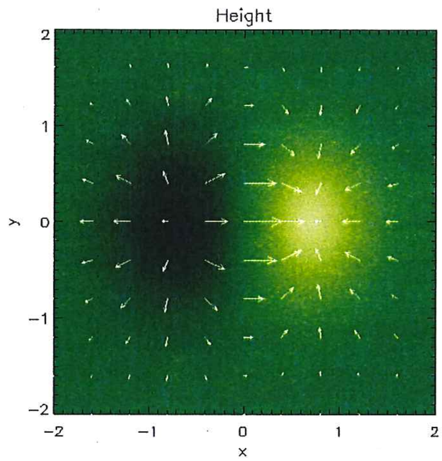
To work our the direction in which I need to go to have the fastest increase in temperature, we can work out ∇T:

$$\nabla T = \begin{pmatrix} 0.1x \\ 0.2y \end{pmatrix}$$

so $(\nabla T)_{office}$ = (0.1,0.2). This direction is indicated on the LHS figure by the red arrow.

The direction and magnitude of ∇T is represented by the white arrows on the RHS figure.

(ii) Consider the function $h(x, y) = x \exp(-x^2 - y^2)$ and let us say this function $h(x, y)$ represents the height above sea level at a point (x, y).



The gradient of h is given by:

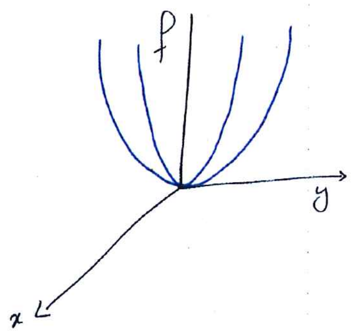
$$\nabla h = \begin{pmatrix} (1 - 2x^2)\, e^{-x^2-y^2} \\ -2xy\, e^{-x^2-y^2} \end{pmatrix}.$$

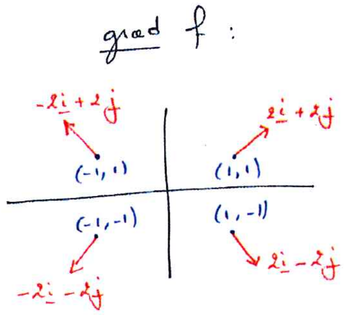
Here ∇h is a vector function pointing in the direction of the steepest slope. The steepness of the slope is given by the magnitude of ∇h.

(iii) Consider $f(x) = x^2 + y^2 + z^2$

$$\Rightarrow \nabla f = \begin{pmatrix} \dfrac{\partial}{\partial x}(x^2+y^2+z^2) \\ \\ \dfrac{\partial}{\partial y}(x^2+y^2+z^2) \\ \\ \dfrac{\partial}{\partial z}(x^2+y^2+z^2) \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} = 2(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

For visualisation purposes, let us take the z-component in f and ∇f to be zero. Then f defines a paraboloid with its minimum at the origin.





Note that ∇f points away from the origin. Also, as (x, y) is located further away from the origin, the magnitude of ∇f increases (in other words, the rate at which f changes increases).

(iv) Consider $f(x) = x \cosh y \tan z$

$$\Rightarrow \nabla f = \begin{pmatrix} \dfrac{\partial}{\partial x}(x\cosh y \tan z) \\ \\ \dfrac{\partial}{\partial y}(x\cosh y \tan z) \\ \\ \dfrac{\partial}{\partial z}(x\cosh y \tan z) \end{pmatrix} = \begin{pmatrix} \cosh y \tan z \\ x \sinh y \tan z \\ \dfrac{x\cosh y}{\cos^2 z} \end{pmatrix} = \cosh y \tan z\mathbf{i} + x \sinh y \tan z\mathbf{j} + \frac{x \cosh y}{\cos^2 z}\mathbf{k}$$

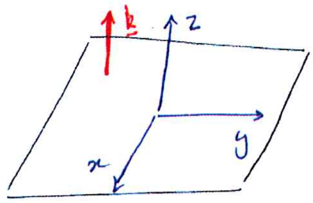
Normal to a surface

If a surface in space is defined by $f(x, y, z) =$ constant, then ∇f is perpendicular to this surface.

Simple examples

(i) Consider the horizontal plane $z = 0$.

- Plane $z=0$
  (the horizontal plane)



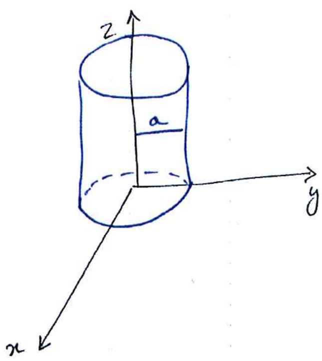
Obviously, the direction normal to the horizontal plane is along $\mathbf{k}$.

Here $f(x, y, z) = z$.

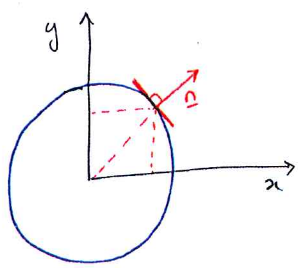
$$\Rightarrow \nabla f = \begin{pmatrix} \dfrac{\partial z}{\partial x} \\[2ex] \dfrac{\partial y}{\partial z} \\[2ex] \dfrac{\partial z}{\partial z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mathbf{k}$$

(ii) Consider a cylinder of radius a.

- Side of a cylinder of radius a:



Top view



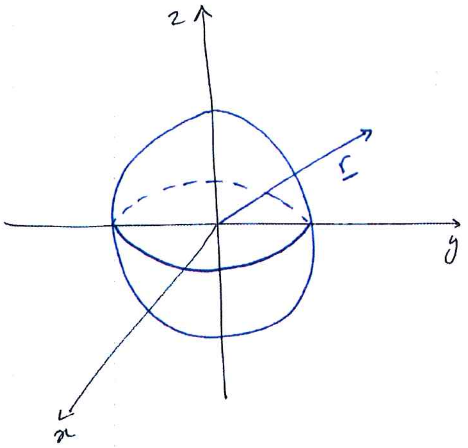
The equation of the surface is $f(x, y, z) = x^2 + y^2 = a^2$.

Taking a horizontal cross-section clearly shows that the normal to the surface is the radial direction. In other words, the normal is along $x\mathbf{i} + y\mathbf{j}$ where (x, y) is a point on the surface.

$$\Rightarrow \nabla f = \begin{pmatrix} \dfrac{\partial(x^2+y^2)}{\partial x} \\ \\ \dfrac{\partial(x^2+y^2)}{\partial y} \\ \\ \dfrac{\partial(x^2+y^2)}{\partial z} \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 0 \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = 2x\mathbf{i} + 2y\mathbf{j}\,,$$

i.e. ∇f is normal to the surface.

(iii) Consider a sphere of radius a.



Now, the normal is along $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ at the surface.

The equation of the sphere is

$$f(x,y,z) = x^2 + y^2 + z^2 = a^2$$

$$\Rightarrow \nabla f = \begin{pmatrix} \dfrac{\partial}{\partial x}(x^2+y^2+z^2) \\ \\ \dfrac{\partial}{\partial y}(x^2+y^2+z^2) \\ \\ \dfrac{\partial}{\partial z}(x^2+y^2+z^2) \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} = 2\mathbf{r}$$

Directional Derivatives

The directional derivative of a function f at a point $\mathbf{x} = (x, y, z)$ in the direction of a vector $\mathbf{v}$ is given by

$$\nabla_{\mathbf{v}} f(\mathbf{x}) = (\nabla f)_{\mathbf{x}} \cdot \frac{\mathbf{v}}{|\mathbf{v}|}\,.$$

Intuitively, the directional derivative of f at a point $\mathbf{x}$ represents the rate of change (or slope) of f in the direction of $\mathbf{v}$.

Examples

(i) Show that the rate of change of a function $f(x, y, z)$ in the direction of the unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ is simply given by $\partial f/\partial x$, $\partial f/\partial y$ and $\partial f/\partial z$.

Set $\mathbf{v}$ in the definition above equal to $\mathbf{i}$. Then

$$\nabla_{\mathbf{i}} f = \nabla f \cdot \mathbf{i} = \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) \cdot \mathbf{i} = \frac{\partial f}{\partial x}$$

Similarly, setting $\mathbf{v} = \mathbf{j}$ gives $\partial f/\partial y$ and $\mathbf{v} = \mathbf{k}$ gives $\partial f/\partial z$.

(ii) Consider $f(x, y, z) = xy\, e^{x^2+z^2-5}$.

Calculate ∇f and $\nabla_{\mathbf{u}} f$ at $\mathbf{x} = (1, 3, -2)$, for $\mathbf{u} = (3, -1, 4)$.

$$\nabla f = \begin{pmatrix} \dfrac{\partial z}{\partial x} \\[2ex] \dfrac{\partial y}{\partial z} \\[2ex] \dfrac{\partial z}{\partial z} \end{pmatrix} = \begin{pmatrix} y + 2x^2 y \\ x \\ 2xyz \end{pmatrix} e^{x^2+z^2-5}$$

Then $\nabla f(\mathbf{x}) = \nabla f(1, 3, -2) = (9, 1, -12)\,.$

Using $|\mathbf{u}| = \sqrt{9 + 1 + 16} = \sqrt{26}$, we have:

$$\nabla_{\mathbf{u}} f(\mathbf{x}) = (\nabla f)_{\mathbf{x}} \cdot \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{1}{\sqrt{26}}(9, 1, -12) \cdot (3, -1, 4) = \frac{27 - 1 - 48}{\sqrt{26}} = \frac{-22}{\sqrt{26}}\,.$$

3.3 Divergence of a vector function

Consider the vector function, $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$

$$\mathbf{F}(x, y, z) = \begin{pmatrix} F_x(x,y,z) \\ F_y(x,y,z) \\ F_z(x,y,z) \end{pmatrix}.$$

The x component of $\mathbf{F}$ is denoted F_x. F_x can be seen as a *scalar* function of x, y, z.

The y component of $\mathbf{F}$ is denoted F_y. F_y can be seen as a *scalar* function of x, y, z.

The z component of $\mathbf{F}$ is denoted F_z. F_z can be seen as a *scalar* function of x, y, z.

The *divergence* of the vector function $\mathbf{F}$ is defined as:

$$\text{div}\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_Z}{\partial z}.$$

Note that the divergence of $\mathbf{F}$ is a *scalar* function of (x, y, z).

Notation

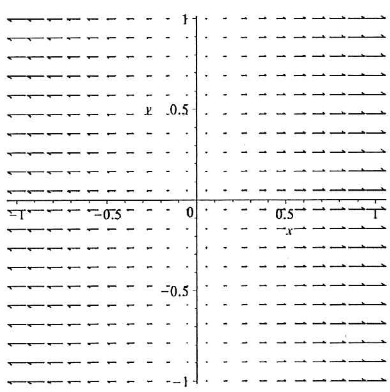
$$\nabla \cdot \mathbf{F} = \begin{pmatrix} \dfrac{\partial}{\partial x} \\[2ex] \dfrac{\partial}{\partial y} \\[2ex] \dfrac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

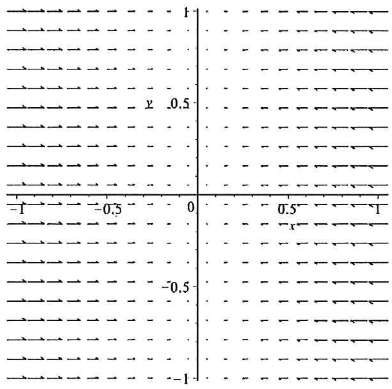
We will see that the divergence is useful to define fluxes through surfaces (i.e. the amount of fluid entering and leaving a given surface).

3.3.1 Examples

(i) $\mathbf{F}(\mathbf{x}) = x\mathbf{i}(+0\mathbf{j} + 0\mathbf{k})$

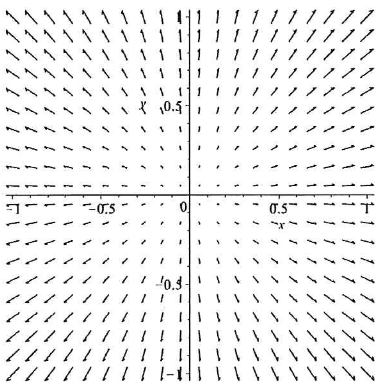
$$\nabla \cdot \mathbf{F} = \frac{\partial(x)}{\partial x} + \frac{\partial(0)}{\partial y} + \frac{\partial(0)}{\partial z} = 1 + 0 + 0 = 1$$





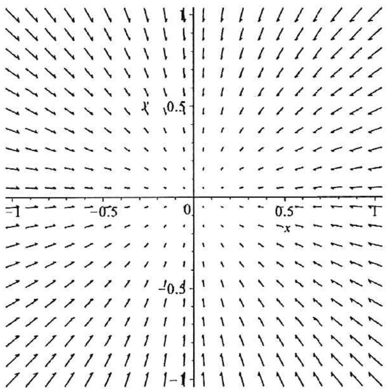
(ii) $\mathbf{F}(\mathbf{x}) = -x\mathbf{i}$

$$\nabla \cdot \mathbf{F} = \frac{\partial(-x)}{\partial x} + \frac{\partial(0)}{\partial y} + \frac{\partial(0)}{\partial z} = -1 + 0 + 0 = -1$$



(iii) $\mathbf{F}(\mathbf{x}) = 2x\mathbf{i} + 2y\mathbf{j}(+0\mathbf{k})$

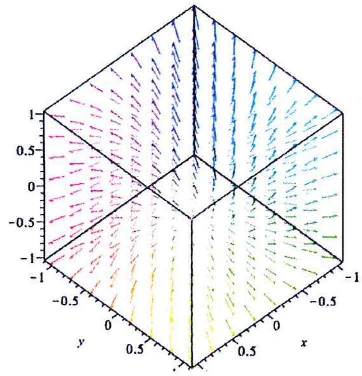
$$\nabla \cdot \mathbf{F} = \frac{\partial(2x)}{\partial x} + \frac{\partial(2y)}{\partial y} + \frac{\partial(0)}{\partial z} = 2 + 2 + 0 = 4$$



(iv) $\mathbf{F}(\mathbf{x}) = -2x\mathbf{i} - 2y\mathbf{j}(+0\mathbf{k})$

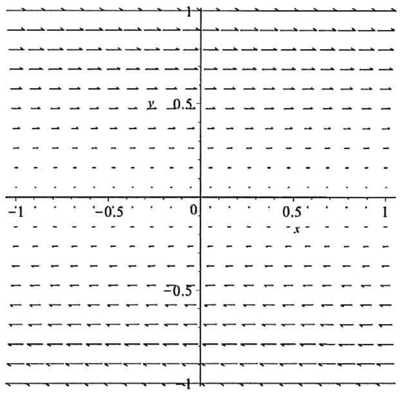
$$\nabla \cdot \mathbf{F} = \frac{\partial(-2x)}{\partial x} + \frac{\partial(-2y)}{\partial y} + \frac{\partial(0)}{\partial z} = -2 - 2 + 0 = -4$$

(v) $\mathbf{F}(\mathbf{x}) = 2x\mathbf{i} + (2y + x)\mathbf{j} + az\mathbf{k}$



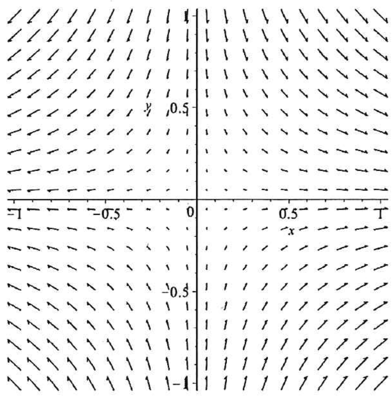
$$\nabla\cdot\mathbf{F} = \frac{\partial(2x)}{\partial x} + \frac{\partial(2y+x)}{\partial y} + \frac{\partial az}{\partial z} = 2+2+a = 4+a$$

Note that the term $x\mathbf{j}$ of $\mathbf{F}$ did not contribute to the divergence.



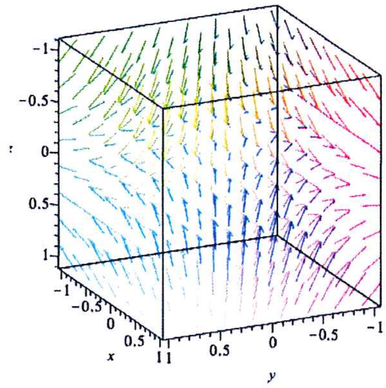
(vi) $\mathbf{F}(\mathbf{x}) = y\mathbf{i}$

$$\nabla\cdot\mathbf{F} = \frac{\partial y}{\partial x} = 0$$



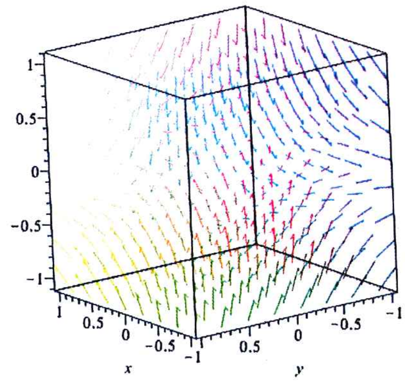
(vii) $\mathbf{F}(\mathbf{x}) = \alpha x\mathbf{i} - \alpha y\mathbf{j}$

$$\nabla\cdot\mathbf{F} = \frac{\partial(\alpha x)}{\partial x} + \frac{\partial(-\alpha y)}{\partial y} = \alpha - \alpha = 0$$



(viii) $\mathbf{F}(\mathbf{x}) = x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k}$

$$\nabla \cdot \mathbf{F} = \frac{\partial(x)}{\partial x} + \frac{\partial(y)}{\partial y} + \frac{\partial(-2z)}{\partial z} = 1 + 1 - 2 = 0$$



(ix) $\mathbf{F}(\mathbf{x}) = 2x\mathbf{i} + y\mathbf{j} - 3z\mathbf{k}$

$$\nabla \cdot \mathbf{F} = \frac{\partial(2x)}{\partial x} + \frac{\partial(y)}{\partial y} + \frac{\partial(-3z)}{\partial z} = 2 + 1 - 3 = 0$$

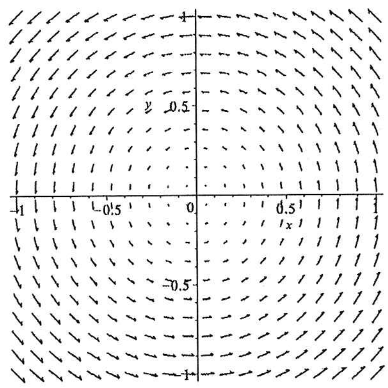
(x) $\mathbf{F}(\mathbf{x}) = \cos(z)\tan^{-1}(y)\mathbf{i} + \dfrac{\log(1+x)}{\sqrt{1-z^2}}\mathbf{j} + \dfrac{\tan(\exp(-x^2+\sin(y)))}{1+\cosh^{-1}(x-y)}\mathbf{k}$

$$\nabla \cdot \mathbf{F} = 0$$

because the $x-$component F_x of $\mathbf{F}$ does not depend on x, nor the y (resp. z)-component of $\mathbf{F}$ depend on y (resp. z).

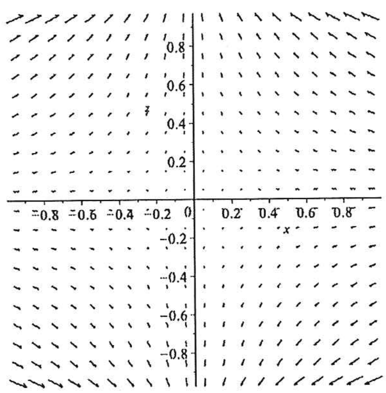
(xi) $\mathbf{F}(\mathbf{x}) = -y\mathbf{i} + x\mathbf{j}$

$$\nabla \cdot \mathbf{F} = \frac{\partial(-y)}{\partial x} + \frac{\partial(x)}{\partial y} = 0 + 0 = 0$$



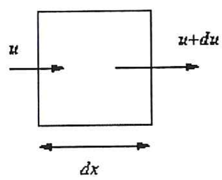
(xii) $\mathbf{F}(\mathbf{x}) = -\frac{x}{\sqrt{1-z^2}}\mathbf{i} + \sin^{-1} z\mathbf{k}$

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \frac{\partial}{\partial x}\left(-\frac{x}{\sqrt{1-z^2}}\right) + \frac{\partial \sin^{-1} z}{\partial z} \\ &= -\frac{1}{\sqrt{1-z^2}} + \frac{1}{\sqrt{1-z^2}} = 0\end{aligned}$$



3.3.2 Think local

looking at a fixed infinitesimal volume



looking at a fixed infinitesimal amount of fluid

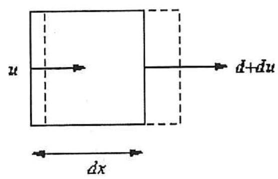
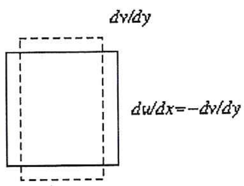


Illustration of du/dx+dv/dy=0



3.3.3 Linearity

Because of the linearity of partial differentiation, the divergence is a linear operator.

If $\mathbf{F} = (F_x, F_y, F_z)$ and $\mathbf{G} = (G_x, G_y, G_z)$ are two vector functions, and a is a real constant, then

$$\begin{aligned}
\nabla \cdot (a\mathbf{F} + \mathbf{G}) &= \frac{\partial(aF_x + G_x)}{\partial x} + \frac{\partial(aF_y + G_y)}{\partial y} + \frac{\partial(aF_z + G_z)}{\partial z} \\
&= a\frac{\partial F_x}{\partial x} + \frac{\partial G_x}{\partial x} + a\frac{\partial F_y}{\partial y} + \frac{\partial G_y}{\partial y} + a\frac{\partial F_z}{\partial z} + \frac{\partial G_z}{\partial z} \\
&= a\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) + \left(\frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z}\right) \\
&= a\nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}.
\end{aligned}$$

Example

Consider the vector functions

$$\mathbf{F} = (\tan x + 3y)\mathbf{i} + 2\sin y\mathbf{j} + (\ln(x^2+1) + \sinh^{-1} z)\mathbf{k}$$

$$\mathbf{G} = -\frac{1}{2}\tan x\mathbf{i} - (\sin y + \cos x)\mathbf{j} + \cos^{-1} z\mathbf{k}$$

Verify the linearity of the divergence on $\mathbf{H} = \mathbf{F} + 2\mathbf{G}$.

$$\begin{aligned}
\nabla \cdot \mathbf{F} &= \frac{\partial(\tan x + 3y)}{\partial x} + \frac{\partial(2\sin y)}{\partial y} + \frac{\partial(\sinh^{-1} z)}{\partial z} \\
&= \frac{1}{1+x^2} + 2\cos y + \frac{1}{\sqrt{1+z^2}}
\end{aligned}$$

$$\begin{aligned}
\nabla \cdot \mathbf{G} &= \frac{\partial(-\frac{1}{2}\tan x)}{\partial x} + \frac{\partial(-\sin y - \cos x)}{\partial y} + \frac{\partial(\cos^{-1} z)}{\partial z} \\
&= -\frac{1}{2}\frac{1}{1+x^2} - \cos y - \frac{1}{\sqrt{1-z^2}}.
\end{aligned}$$

So

$$\begin{aligned}
\nabla \cdot \mathbf{F} + 2\nabla \cdot \mathbf{G} &= \frac{1}{1+x^2} + 2\cos y + \frac{1}{\sqrt{1+z^2}} - \frac{1}{1+x^2} - 2\cos y - \frac{2}{\sqrt{1-z^2}} \\
&= \frac{1}{\sqrt{1+z^2}} - \frac{2}{\sqrt{1-z^2}} \\
&= \frac{\sqrt{1-z^2} - 2\sqrt{1+z^2}}{\sqrt{1-z^4}}.
\end{aligned}$$

While

$$\begin{aligned}
\mathbf{F} + 2\mathbf{G} &= (\tan x + 3y)\mathbf{i} + 2\sin y\mathbf{j} + (\ln(x^2+1) + \sinh^{-1} z)\mathbf{k} - \tan x\mathbf{i} - 2(\sin y + \cos x)\mathbf{j} + 2\cos^{-1} z\mathbf{k} \\
&= 3y\mathbf{i} - 2\cos x\mathbf{j} + \left(\ln(x^2+1) + \sinh^{-1} z + 2\cos^{-1} z\right)\mathbf{k}
\end{aligned}$$

and

$$
\begin{aligned}
\nabla \cdot (\mathbf{F}+2\mathbf{G}) &= \frac{\partial(3y)}{\partial x} + \frac{\partial(-2\cos x)}{\partial y} + \frac{\partial\left(\ln(x^2+1)+\sinh^{-1} z + 2\cos^{-1} z\right)}{\partial z} \\
&= \frac{\partial(\sinh^{-1} z + 2\cos^{-1} z)}{\partial z} \\
&= \frac{1}{\sqrt{1+z^2}} - 2\frac{1}{\sqrt{1-z^2}} \\
&= \frac{\sqrt{1-z^2} - 2\sqrt{1+z^2}}{\sqrt{1-z^4}}
\end{aligned}
$$

Additional examples:

(i) $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$

$$\Rightarrow \text{div}\mathbf{F} = 2x + 2y + 2z$$

(ii) $\mathbf{F} = \cos x\mathbf{i} + \tan y\mathbf{j} + \ln z\mathbf{k}$

$$\Rightarrow \text{div}\mathbf{F} = -\sin x + \frac{1}{\cos^2 y} + \frac{1}{z}$$

(iii) $\mathbf{F} = \tan z\mathbf{i} + \cos x\mathbf{j} + \ln y\mathbf{k}$

$$\Rightarrow \text{div}\mathbf{F} = 0$$

(iv) $\mathbf{F} = (3y\ln x + 3z)\mathbf{i} + (2\tan(1+y^2) + y\sin x)\mathbf{j} + (e^{xyz} + \frac{1}{y}\tan^{-1}(xyz))\mathbf{k}$

$$\Rightarrow \text{div}\mathbf{F} = \frac{3y}{x} + 4y\sec^2(1+y^2) + \sin x + xye^{xyz} + x\frac{1}{1+(xyz)^2}.$$

Note

From above (and from the definition of the divergence) we see that if the vector function $\mathbf{F}$ has the special form

$$\mathbf{F} = F_x(y,z)\mathbf{i} + F_y(x,z)\mathbf{j} + F_z(x,y)\mathbf{k}$$

however complicated the terms F_x, F_y and F_z may look, $\text{div}\mathbf{F} = \nabla \cdot \mathbf{F} = 0$.

This is extremely important for integration, as we will see at the end of the course.

3.4 The curl of a vector function

Let us again consider the vector function, $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$

$$\mathbf{F}(x,y,z) = \begin{pmatrix} F_x(x,y,z) \\ F_y(x,y,z) \\ F_z(x,y,z) \end{pmatrix}.$$

The *curl* of $\mathbf{F}$ is the vector function defined by

$$\mathrm{curl}\mathbf{F} = \boldsymbol{\nabla} \times \mathbf{F} = \begin{pmatrix} \dfrac{\partial F_z}{\partial y} - \dfrac{\partial F_y}{\partial z} \\ \\ \dfrac{\partial F_x}{\partial z} - \dfrac{\partial F_z}{\partial x} \\ \\ \dfrac{\partial F_y}{\partial x} - \dfrac{\partial F_x}{\partial y} \end{pmatrix}.$$

Notation

This can be rewritten as

$$\boldsymbol{\nabla} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \\ F_x & F_y & F_z \end{vmatrix}$$

"Proof"

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_y & F_z \end{vmatrix} \mathbf{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ F_x & F_z \end{vmatrix} \mathbf{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ F_x & F_y \end{vmatrix} \mathbf{k}$$

$$= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{i} - \left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k} = \boldsymbol{\nabla} \times \mathbf{F}$$

Examples

Example 1: Solid rotation

The curl of a vector function is related to the notion of rotation.

Consider solid rotation in the $xy-$plane with constant angular velocity ω. Points have circular trajectories at fixed radius r_0 and a time-dependent angle $\theta = \omega t + \theta_0$.

In Cartesian coordinates we have

$$\mathbf{x} = (r_0 \cos(\omega t + \theta_0), r_0 \sin(\omega t + \theta_0), z_0)$$

where z_0 is the fixed height of the point.

Hence, we can find the velocity $\mathbf{u}$ as

$$\mathbf{u} = \frac{d\mathbf{x}}{dt} = \begin{pmatrix} r_0(-\omega)\sin(\omega t + \theta_0) \\ r_0\omega\cos(\omega t + \theta_0) \\ 0 \end{pmatrix} = \begin{pmatrix} -\omega y \\ \omega x \\ 0 \end{pmatrix}$$

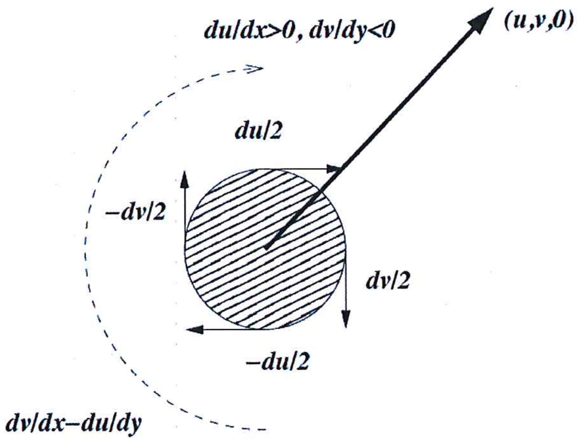
This velocity is valid for all points in space.

The curl of the velocity is

$$\nabla \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \omega x & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ -\omega y & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ -\omega y & \omega x \end{vmatrix} \mathbf{k} = 0\mathbf{i} + 0\mathbf{j} + 2\omega\mathbf{k} = 2\omega\mathbf{k}$$

Here, the magnitude of the curl gives twice the angular velocity and the direction of the curl gives the axis of rotation.



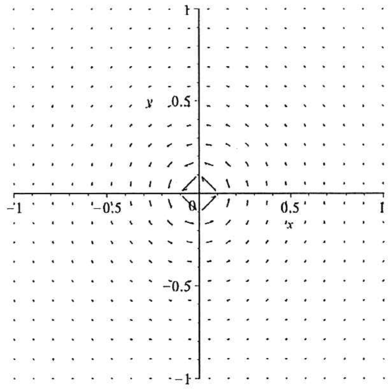
Think local

Example 2

$$\mathbf{F} = y\mathbf{i} \qquad \Rightarrow \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & 0 & 0 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} - \frac{\partial y}{\partial y}\mathbf{k} = -\mathbf{k}$$

Example 3

$$\mathbf{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$$



$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} & 0 \end{vmatrix} \\
&= \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{x}{x^2+y^2} & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ \dfrac{-y}{x^2+y^2} & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{vmatrix} \mathbf{k} \\
&= \left(\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\right)\mathbf{k} \\
&= \left(\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}\right)\mathbf{k} \\
&= \left(\frac{2}{x^2+y^2} - 2\frac{x^2+y^2}{(x^2+y^2)^2}\right)\mathbf{k} = \mathbf{0}
\end{aligned}$$

In this example, even if the trajectories are circles, locally there is no rotation (think about a Ferris wheel)!

Note: The curl "in 2D"

By construction the curl only applies to three-dimensional vectors (as it is related to the vector product which only works in 3D).

However, we can define the curl in 2D by making the two-dimensional vectors three-dimensional (just add a zero component!).

So if

$$\mathbf{F}(x, y) = F_x(x, y)\mathbf{i} + F_y(x, y)\mathbf{j}$$

write it as

$$\mathbf{F} = (F_x, F_y, 0)\,.$$

It follows

$$
\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x(x,y) & F_y(x,y) & 0 \end{vmatrix} \\
&= \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_y(x,y) & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ F_x(x,y) & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ F_x(x,y) & F_y(x,y) \end{vmatrix} \mathbf{k} \\
&= \left(0 - \frac{F_y(x,y)}{\partial z}\right) \mathbf{i} - \left(0 - \frac{F_x(x,y)}{\partial z}\right) \mathbf{j} + \left(\frac{F_y(x,y)}{\partial x} - \frac{F_x(x,y)}{\partial y}\right) \mathbf{k} \\
&= \left(\frac{F_y(x,y)}{\partial x} - \frac{F_x(x,y)}{\partial y}\right) \mathbf{k}
\end{aligned}
$$

Note that the curl here has only 1 non-zero component, along $\mathbf{k}$. Because of this (only one non-trivial component) it is sometimes replaces by a scalar by using the orthogonal 2D grad operator

$$
\nabla_{\perp} = \begin{pmatrix} -\dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial x} \end{pmatrix}
$$

and

$$
\nabla_{\perp} \cdot \mathbf{F} = \begin{pmatrix} -\dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial x} \end{pmatrix} \cdot \begin{pmatrix} F_x \\ F_y \end{pmatrix} = \frac{F_y(x,y)}{\partial x} - \frac{F_x(x,y)}{\partial y}
$$

Note

$$
\nabla \cdot \nabla_{\perp} = \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} -\dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial x} \end{pmatrix} \text{ “ = ” } -\frac{\partial}{\partial x}\frac{\partial}{\partial y} + \frac{\partial}{\partial y}\frac{\partial}{\partial x} \text{ “ = ”} 0
$$

which is why $\nabla_{\perp}$ is referred to as the orthogonal 2D grad operator.

Additional Examples

(i)

$$\mathbf{F} = (3y^2 - 3x\tan z)\mathbf{i} + (2z\sin z - x^3)\mathbf{j} + (z^4 - \cosh x)\mathbf{k}$$

$$\begin{aligned}
\boldsymbol{\nabla}\times\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3y^2 - 3x\tan z & 2z\sin z - x^3 & z^4 - \cosh x \end{vmatrix} \\
&= \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2z\sin z - x^3 & z^4 - \cosh x \end{vmatrix}\mathbf{i} - \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ 3y^2 - 3x\tan z & z^4 - \cosh x \end{vmatrix}\mathbf{j} \\
&\qquad + \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ 3y^2 - 3x\tan z & 2z\sin z - x^3 \end{vmatrix}\mathbf{k} \\
&= \left(\frac{\partial(z^4 - \cosh x)}{\partial y} - \frac{\partial(2z\sin z - x^3)}{\partial z}\right)\mathbf{i} - \left(\frac{\partial(z^4 - \cosh x)}{\partial x} - \frac{\partial(3y^2 - 3x\tan z)}{\partial z}\right)\mathbf{j} \\
&\qquad + \left(\frac{\partial(2z\sin z - x^3)}{\partial x} - \frac{\partial(3y^2 - 3x\tan z)}{\partial y}\right)\mathbf{k} \\
&= -(\sin z + z\cos z)\mathbf{i} + (\sinh x + 3x\sec^2 z)\mathbf{j} - (3x^2 + 6y)\mathbf{k}
\end{aligned}$$

Note that

$$\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\times\mathbf{F} = \frac{\partial}{\partial x}(-\sin z - z\cos z) + \frac{\partial}{\partial y}(\sinh x + 3x\sec^2 z) + \frac{\partial}{\partial z}\left(3x^2 + 6y\right) = 0$$

(ii)

$$\mathbf{F} = xyz(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

$$\boldsymbol{\nabla}\times\mathbf{F} = \begin{pmatrix} xz - xy \\ xy - yz \\ yz - xz \end{pmatrix}$$

Note that

$$\boldsymbol{\nabla}\cdot[(xz - xy)\mathbf{i} + (xy - yz)\mathbf{j} + (yz - xz)\mathbf{k}] = z - y + x - z + y - x = 0$$

Linearity

Consider two vector functions $\mathbf{F}$ and $\mathbf{G}$ and a constant scalar α, then

$$\boldsymbol{\nabla} \times (\alpha\mathbf{F} + \mathbf{G}) = \alpha\boldsymbol{\nabla} \times \mathbf{F} + \boldsymbol{\nabla} \times \mathbf{G}$$

Proof: See tutorial.

Note

If $\mathbf{F}$ has the special form

$$\mathbf{F}(x, y, z) = F_x(x)\mathbf{i} + F_y(y)\mathbf{j} + F_z(z)\mathbf{k}$$

then $\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}$.

Example

$$\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$$

$$\boldsymbol{\nabla} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

3.5 Laplacian of a scalar function

The Laplacian of the scalar function f is

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

The Laplacian is also written as ∇^2 ('del squared') as

$$\Delta = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} = \boldsymbol{\nabla}^2$$

Proof

$$\boldsymbol{\nabla} f = \begin{pmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \\ \dfrac{\partial f}{\partial z} \end{pmatrix}$$

$$\Rightarrow \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \Delta f$$

□

Examples

Find Δf for

(i) $f = x^3 - 6y^4 + 3z^2$

$$\Rightarrow \nabla f = \begin{pmatrix} 3x^2 \\ -24y^3 \\ 6z \end{pmatrix} \quad \Rightarrow \quad \Delta f = 6x - 72y^2 + 6$$

(ii) $f = x^2y^2e^{-4z}$

$$\Rightarrow \nabla f = \begin{pmatrix} 2xy^2e^{-4z} \\ 2x^2ye^{-4z} \\ -4x^2y^2e^{-4z} \end{pmatrix} \quad \Rightarrow \quad \Delta f = (2y^2 + 2x^2 + 16x^2y^2)e^{-4x}$$

(iii) $f = ax^2 + by^2 + cz^2$ $\quad (a, b, c$ constant)

$$\Rightarrow \nabla f = \begin{pmatrix} 2ax \\ 2by \\ 2cz \end{pmatrix} \quad \Rightarrow \quad \Delta f = 2(a + b + c)$$

3.6 Properties of differential operators

Consider two vector functions $\mathbf{F}$ and $\mathbf{G}$ from $\mathbb{R}^3$ to $\mathbb{R}^3$, and two scalar functions f and g from $\mathbb{R}^3$ to $\mathbb{R}$.

The following is a list of fundamental identities satisfied by the differential operators (see also Section 5.3 in the Handbook):

(a*) $\nabla(fg) = f\nabla g + g\nabla f$

(b) $\nabla \cdot (f\mathbf{F}) = f\nabla \cdot \mathbf{F} + \nabla f \cdot \mathbf{F}$

(c) $\nabla \times (f\mathbf{F}) = f\nabla \times \mathbf{F} + \nabla f \times \mathbf{F}$

(d) $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \nabla \times \mathbf{F} - \mathbf{F} \cdot \nabla \times \mathbf{G}$

(e*) $\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{G} \cdot \nabla)\mathbf{F} + (\mathbf{F} \cdot \nabla)\mathbf{G} + \mathbf{G} \times (\nabla \times \mathbf{F}) + \mathbf{F} \times (\nabla \times \mathbf{G})$

(f) $\nabla \times (\mathbf{F} \times \mathbf{G}) = (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G} + \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F})$

(g*) $\nabla \cdot (\nabla f) = \nabla^2 f = \Delta f$

(h*) $\nabla \times (\nabla f) = 0$

(i*) $\nabla \cdot (\nabla \times \mathbf{F}) = 0$

(j*) $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$

where the operator $(\mathbf{F}\cdot\boldsymbol{\nabla})$ is defined by

$$(\mathbf{F}\cdot\boldsymbol{\nabla}) = F_x\frac{\partial}{\partial x} + F_y\frac{\partial}{\partial y} + F_z\frac{\partial}{\partial z}\,.$$

The starred expressions are ones that are used frequently in subsequent courses and hence you need to memorise them.

• Gradient of a product

Consider 2 scalar functions f and g

$$\boldsymbol{\nabla}(fg) = f\boldsymbol{\nabla}g + g\boldsymbol{\nabla}f.$$

Proof

This is a direct, straightforward application of the product rule:

$$\boldsymbol{\nabla}(fg) = \begin{pmatrix} \dfrac{\partial(fg)}{\partial x} \\[2ex] \dfrac{\partial(fg)}{\partial y} \\[2ex] \dfrac{\partial(fg)}{\partial z} \end{pmatrix} = \begin{pmatrix} f\dfrac{\partial g}{\partial x} + g\dfrac{\partial f}{\partial x} \\[2ex] f\dfrac{\partial g}{\partial y} + g\dfrac{\partial f}{\partial y} \\[2ex] f\dfrac{\partial g}{\partial z} + g\dfrac{\partial f}{\partial z} \end{pmatrix} = f\begin{pmatrix} \dfrac{\partial g}{\partial x} \\[2ex] \dfrac{\partial g}{\partial y} \\[2ex] \dfrac{\partial g}{\partial z} \end{pmatrix} + g\begin{pmatrix} \dfrac{\partial f}{\partial x} \\[2ex] \dfrac{\partial f}{\partial y} \\[2ex] \dfrac{\partial f}{\partial z} \end{pmatrix} = f\boldsymbol{\nabla}g + g\boldsymbol{\nabla}f.$$

□

• Curl of a gradient

Consider the scalar function f, then

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}f = 0\,.$$

Proof: See Examples Class.

• Divergence and curl of a "product"

Consider a scalar function f and a vector function $\mathbf{F}$ then

$$\boldsymbol{\nabla}\cdot(f\mathbf{F}) = f(\boldsymbol{\nabla}\cdot F) + (\boldsymbol{\nabla}f)\cdot\mathbf{F}$$

and

$$\boldsymbol{\nabla}\times(f\mathbf{F}) = f\boldsymbol{\nabla}\times\mathbf{F} + \boldsymbol{\nabla}f\times\mathbf{F}\,.$$

Proof: See tutorial.

• **Divergence of a curl**

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0$$

Proof: See Examples Class.

• **Divergence of a cross-product**

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \nabla \times \mathbf{F} - \mathbf{F} \cdot \nabla \times \mathbf{G}$$

Proof:

Start with

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ F_x & F_y & F_z \\ G_x & G_y & G_z \end{vmatrix} = \begin{pmatrix} F_y G_z - G_y F_z \\ -F_x G_z + G_x F_z \\ F_x G_y - F_y G_x \end{pmatrix}$$

$$\begin{aligned}
\nabla \cdot (\mathbf{F} \times \mathbf{G}) &= \frac{\partial F_y}{\partial x} G_z - \frac{\partial F_z}{\partial x} G_y + F_y \frac{\partial G_z}{\partial x} - F_z \frac{\partial G_y}{\partial x} - \frac{\partial F_x}{\partial y} G_z + \frac{\partial F_z}{\partial y} G_x - F_x \frac{\partial G_z}{\partial y} + F_z \frac{\partial G_x}{\partial y} \\
&\quad + \frac{\partial F_x}{\partial z} G_y - \frac{\partial F_y}{\partial z} G_x + F_x \frac{\partial G_y}{\partial z} - F_y \frac{\partial G_x}{\partial z} \\
&= G_x \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial x} \right) + G_y \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + G_z \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \\
&\quad + F_x \left(\frac{\partial G_y}{\partial z} - \frac{\partial G_z}{\partial y} \right) + F_y \left(\frac{\partial G_z}{\partial x} - \frac{\partial G_x}{\partial z} \right) + F_z \left(\frac{\partial G_x}{\partial y} - \frac{\partial G_y}{\partial x} \right) \\
&= \mathbf{G} \cdot \nabla \times \mathbf{F} - \mathbf{F} \cdot \nabla \times \mathbf{G}
\end{aligned}$$

□

• $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$.

Here, the vector Laplacian is given by

$$\nabla^2 \mathbf{F} = \begin{pmatrix} \nabla^2 F_x \\ \nabla^2 F_y \\ \nabla^2 F_z \end{pmatrix}.$$

Note that the vector Laplacian is applied to a vector and also returns a vector!

Let us demonstrate that the vector Laplacian is indeed of this form.

Using the expression above, we can write

$$\nabla^2 \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F}).$$

Considering the $x-$component of the RHS terms:

-

$$[\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{F})]_x = \frac{\partial}{\partial x}\left(\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}\right)$$

-

$$\begin{aligned}[\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{F})]_x &= \frac{\partial}{\partial y}[\boldsymbol{\nabla}\times\mathbf{F}]_z - \frac{\partial}{\partial z}[\boldsymbol{\nabla}\times\mathbf{F}]_y \\ &= \frac{\partial}{\partial y}\left[\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right] - \frac{\partial}{\partial z}\left[\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right] \\ &= \frac{\partial^2 F_y}{\partial y\partial x}-\frac{\partial^2 F_x}{\partial y^2}-\frac{\partial^2 F_x}{\partial z^2}+\frac{\partial^2 F_z}{\partial z\partial x}\end{aligned}$$

$$\Rightarrow [\boldsymbol{\nabla}^2\mathbf{F}]_x = \frac{\partial^2 F_x}{\partial x^2}+\frac{\partial^2 F_x}{\partial y^2}+\frac{\partial^2 F_x}{\partial z^2} = \nabla^2 F_x\,.$$

Similar for the y and z components so

$$\boldsymbol{\nabla}^2\mathbf{F} = (\nabla^2 F_x, \nabla^2 F_y, \nabla^2 F_z)\,.$$

□

• Potential

+ Consider a vector function $\mathbf{F}$ which satisfies

$$\boldsymbol{\nabla}\cdot\mathbf{F} = 0$$

Then there exists a vector function $\mathbf{G}$ (a *'vector potential'*) such that $\mathbf{F} = \boldsymbol{\nabla}\times\mathbf{G}$.

[Recall property (i) above, namely that $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{G}) = 0$.]

+ If

$$\boldsymbol{\nabla}\times\mathbf{F} = 0\,,$$

then there exists a scalar function φ (a *'scalar potential'*) such that $\mathbf{F} = \boldsymbol{\nabla}\varphi$.

[Recall property (h) above, namely that $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}f) = 0$.]

SUMMARY:

- Gradient $\boldsymbol{\nabla}f$ is a vector function *(scalar → vector)*.
- Divergence $\boldsymbol{\nabla}\cdot\mathbf{F}$ is a scalar function *(vector → scalar)*.
- Curl $\boldsymbol{\nabla}\times\mathbf{F}$ is a vector function *(vector → vector)*.
- Scalar Laplacian Δf is a scalar function *(scalar → scalar)*.
- Vector Laplacian $\boldsymbol{\nabla}^2 F$ is a vector function *(vector → vector)*.

3.7 Other coordinate systems

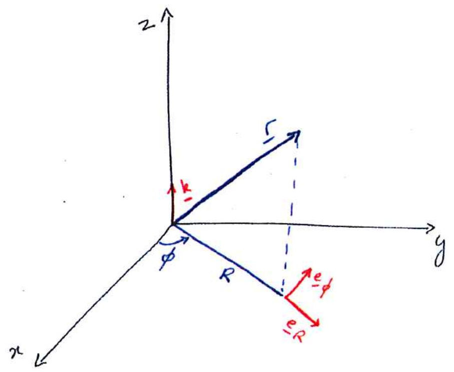
Sometimes it is easier to work in a non-Cartesian coordinate system, so in this section we will derive expressions for the differential operators in cylindrical and spherical coordinates. These expressions are summarised in the Handbook. You do not need to know these by heart but you should be able to re-derive the expressions. Note that the Handbook also contains the expressions for a general orthogonal coordinate system.

3.7.1 Cylindrical polar coordinates

Let us first consider cylindrical polar coordinates (R, ϕ, z).

Remember that

$$
\begin{aligned}
R &= \sqrt{x^2 + y^2} \\
\phi &= \tan^{-1}\left(\frac{y}{x}\right) \\
z &= z
\end{aligned}
$$



and

$$
\begin{aligned}
x &= R\cos\phi \\
y &= R\sin\phi \\
z &= z
\end{aligned}
$$

In cylindrical coordinates, the unit vectors are given by

$$
\begin{aligned}
\mathbf{e}_R &= \cos\phi\mathbf{i} + \sin\phi\mathbf{j} \\
\mathbf{e}_\phi &= -\sin\phi\mathbf{i} + \cos\phi\mathbf{j} \\
\mathbf{k} &= \mathbf{k}\,.
\end{aligned}
$$

Hence, we can write down the following expressions:

$$
\begin{aligned}
\frac{\partial \mathbf{e}_R}{\partial R} &= 0 & \frac{\partial \mathbf{e}_\phi}{\partial R} &= 0 & \frac{\partial \mathbf{k}}{\partial R} &= 0 \\
\frac{\partial \mathbf{e}_R}{\partial \phi} &= -\sin\phi\mathbf{i} + \cos\phi\mathbf{j} = \mathbf{e}_\phi & \frac{\partial \mathbf{e}_\phi}{\partial \phi} &= -\mathbf{e}_R & \frac{\partial \mathbf{k}}{\partial \phi} &= 0 \\
\frac{\partial \mathbf{e}_R}{\partial z} &= 0 & \frac{\partial \mathbf{e}_\phi}{\partial z} &= 0 & \frac{\partial \mathbf{k}}{\partial z} &= 0
\end{aligned}
$$

Recall from earlier in the course that in cylindrical coordinates a point $P(R, \phi, z)$ has a position vector given by

$$\mathbf{r} = R\mathbf{e}_R + z\mathbf{k}\,.$$

Remember that the unit vectors depend on ϕ!

Let us now consider the infinitesimal increment $\delta\mathbf{r}$:

$$\begin{aligned}\delta\mathbf{r} &= \delta(R\mathbf{e}_R + z\mathbf{k}) = \delta R\,\mathbf{e}_R + R\,\delta\mathbf{e}_R + \delta z\,\mathbf{k} + z\,\delta\mathbf{k}\\ &= \delta R\,\mathbf{e}_R + R\left(\frac{\partial\mathbf{e}_R}{\partial R}\delta R + \frac{\partial\mathbf{e}_R}{\partial\phi}\delta\phi + \frac{\partial\mathbf{e}_R}{\partial z}\delta z\right) + \delta z\,\mathbf{k}\\ &= \delta R\,\mathbf{e}_R + R\,\delta\phi\,\mathbf{e}_\phi + \delta z\,\mathbf{k}\,.\end{aligned}$$

Gradient

Let us consider the scalar function $f(R, \phi, z)$.

Then we can write

$$\delta f = \frac{\partial f}{\partial R}\delta R + \frac{\partial f}{\partial\phi}\delta\phi + \frac{\partial f}{\partial z}\delta z\,.$$

But we also have that the directional derivative would give us

$$\delta f = \delta\mathbf{r}\cdot\boldsymbol{\nabla} f$$

where $\delta\mathbf{r} = \delta R\,\mathbf{e}_R + R\,\delta\phi\,\mathbf{e}_\phi + \delta z\,\mathbf{k}$.

Hence, we have:

$$\delta f = \frac{\partial f}{\partial R}\delta R + \frac{\partial f}{\partial\phi}\delta\phi + \frac{\partial f}{\partial z}\delta z = (\boldsymbol{\nabla} f)_R\delta R + (\boldsymbol{\nabla} f)_\phi R\delta\phi + (\boldsymbol{\nabla} f)_z\delta z$$

and thus

$$\begin{aligned}(\boldsymbol{\nabla} f)_R &= \frac{\partial f}{\partial R}\\ (\boldsymbol{\nabla} f)_\phi &= \frac{1}{R}\frac{\partial f}{\partial\phi}\\ (\boldsymbol{\nabla} f)_z &= \frac{\partial f}{\partial z}\end{aligned}$$

or, in terms of the differential operator,

$$\boldsymbol{\nabla} = \left(\frac{\partial}{\partial R}, \frac{1}{R}\frac{\partial}{\partial\phi}, \frac{\partial}{\partial z}\right)\,.$$

Note on the directional derivative

Consider the unit vector (a, b, c), where $a^2 + b^2 + c^2 = 1$. Now assume that the small changes to R, ϕ and z are given by

$$\delta R = a\delta s, \quad R\delta\phi = b\delta s, \quad \text{and} \quad \delta z = c\delta s.$$

Note that as $\delta s \to 0$, we have $\delta R \to 0$, $\delta\phi \to 0$ and $\delta z \to 0$. The distance between the points given by $P = (R, \phi, z)$ and $Q = (R+\delta R, \phi+\delta\phi, z+\delta z)$ can be written as

$$\delta\mathbf{r} = \delta R\mathbf{e}_R + R\delta\phi\mathbf{e}_\phi + \delta z\mathbf{e}_z = (a\mathbf{e}_R + b\mathbf{e}_\phi + c\mathbf{e}_z)\,\delta s\ .$$

Hence, the unit vector defining the direction is

$$\frac{\delta\mathbf{r}}{|\delta\mathbf{r}|} = \frac{\delta\mathbf{r}}{\delta s}.$$

Thus, the directional derivative is given by

$$\frac{df}{ds} = \lim_{\delta s\to 0}\frac{\delta f}{\delta s} = \lim_{\delta s\to 0}\boldsymbol{\nabla} f\cdot\frac{\delta\mathbf{r}}{\delta s}$$

Hence,

$$\delta f = \boldsymbol{\nabla} f\cdot\delta\mathbf{r} = ((\boldsymbol{\nabla} f\cdot\mathbf{e}_R)\mathbf{e}_R + (\boldsymbol{\nabla} f\cdot\mathbf{e}_\phi)\mathbf{e}_\phi + (\boldsymbol{\nabla} f\cdot\mathbf{e}_z)\mathbf{e}_z)\cdot(a\mathbf{e}_R + Rb\mathbf{e}_\phi + c\mathbf{e}_z)\,\delta s$$

Expanding out the brackets and comparing with the previous expression for δf, we have

$$\begin{aligned}
\delta f &= \frac{\partial f}{\partial R}\delta R + \frac{\partial f}{\partial\phi}\delta\phi + \frac{\partial f}{\partial z}\delta z\\
&= (\boldsymbol{\nabla} f\cdot\mathbf{e}_R)a\delta s + (\boldsymbol{\nabla} f\cdot\mathbf{e}_\phi)bR\delta s + (\boldsymbol{\nabla} f\cdot\mathbf{e}_z)c\delta s\\
&= (\boldsymbol{\nabla} f\cdot\mathbf{e}_R)\delta R + (\boldsymbol{\nabla} f\cdot\mathbf{e}_\phi)R\delta\phi + (\boldsymbol{\nabla} f\cdot\mathbf{e}_z)\delta z\\
&= (\boldsymbol{\nabla} f)_R\delta R + (\boldsymbol{\nabla} f)_\phi R\delta\phi + (\boldsymbol{\nabla} f)_z\delta z
\end{aligned}$$

Comparing the coefficients of δR, $\delta\phi$ and δz, we have that the R, ϕ and z components of $\boldsymbol{\nabla} f$ are

$$\begin{aligned}
(\boldsymbol{\nabla} f)_R &= \frac{\partial f}{\partial R},\\
(\boldsymbol{\nabla} f)_\phi &= \frac{1}{R}\frac{\partial f}{\partial\phi},\\
(\boldsymbol{\nabla} f)_z &= \frac{\partial f}{\partial z}.
\end{aligned}$$

Alternative derivation of the differential operator

An alternative derivation of the differential operator can be obtained as follows.

In Cartesian coordinates, grad is defined as

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\,.$$

We can convert this expression into cylindrical coordinates by converting both the partial derivatives and the basis vectors **i**, **j** *and* **k** *into cylindrical coordinates.*

From

$$\begin{aligned}\mathbf{e}_R &= \cos\phi\mathbf{i} + \sin\phi\mathbf{j}\\ \mathbf{e}_\phi &= -\sin\phi\mathbf{i} + \cos\phi\mathbf{j}\\ \mathbf{k} &= \mathbf{k}\,.\end{aligned}$$

we can easily obtain

$$\begin{aligned}\mathbf{i} &= \cos\phi\mathbf{e}_R - \sin\phi\mathbf{e}_\phi\\ \mathbf{j} &= \sin\phi\mathbf{e}_R + \cos\phi\mathbf{e}_\phi\\ \mathbf{k} &= \mathbf{k}\,.\end{aligned}$$

Next, we work out the partial derivatives $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ *and* $\dfrac{\partial f}{\partial z}$ *in cylindrical coordinates:*

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial R}\frac{\partial R}{\partial x} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x}\\ &= \cos\phi\frac{\partial f}{\partial R} - \frac{\sin\phi}{R}\frac{\partial f}{\partial \phi},\end{aligned}$$

where we have used

$$\begin{aligned}\frac{\partial R}{\partial x} &= \frac{x}{R} = \frac{R\cos\phi}{R} = \cos\phi\\ \frac{\partial \phi}{\partial x} &= \frac{-y}{x^2+y^2} = -\frac{R\sin\phi}{R^2} = -\frac{\sin\phi}{R}\\ \frac{\partial z}{\partial x} &= 0\end{aligned}$$

from $R = \sqrt{x^2+y^2}$ *and* $\phi = \tan^{-1}(y/x)$.

Similarly, we can work out that

$$\frac{\partial \phi}{\partial y} = \sin\phi\frac{\partial f}{\partial R} + \frac{\cos\phi}{R}\frac{\partial f}{\partial \phi}.$$

Substituting back into the expression for ∇f, *we find:*

$$\begin{aligned}\nabla f &= \left(\cos\phi\frac{\partial f}{\partial R} - \frac{\sin\phi}{R}\frac{\partial f}{\partial \phi}\right)(\cos\phi\mathbf{e}_R - \sin\phi\mathbf{e}_\phi) + \left(\sin\phi\frac{\partial f}{\partial R} + \frac{\cos\phi}{R}\frac{\partial f}{\partial \phi}\right)(\sin\phi\mathbf{e}_R + \cos\phi\mathbf{e}_\phi) + \frac{\partial f}{\partial z}\mathbf{k}\\ &= (\cos^2\phi + \sin^2\phi)\frac{\partial f}{\partial R}\mathbf{e}_R + \frac{1}{R}(\sin^2\phi + \cos^2\phi)\frac{\partial f}{\partial \phi}\mathbf{e}_\phi + \frac{\partial f}{\partial z}\mathbf{k}\\ &= \frac{\partial f}{\partial R}\mathbf{e}_R + \frac{1}{R}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi + \frac{\partial f}{\partial z}\mathbf{k}\,.\end{aligned}$$

Writing this in terms of the differential operator we have, as before,

$$\nabla = \left(\frac{\partial}{\partial R}, \frac{1}{R}\frac{\partial}{\partial \phi}, \frac{\partial}{\partial z}\right).$$

Divergence

Now that we have an expression for ∇ in cylindrical coordinates, we can use this to derive the expression for 'div' in cylindrical coordinates.

Consider a vector function $\mathbf{F} = F_R\mathbf{e}_R + F_\phi\mathbf{e}_\phi + F_z\mathbf{k}$.

Then we can write

$$\begin{aligned}\nabla\cdot\mathbf{F} &= \left(\mathbf{e}_R\frac{\partial}{\partial R} + \mathbf{e}_\phi\frac{1}{R}\frac{\partial}{\partial\phi} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot\mathbf{F}\\ &= \underbrace{\mathbf{e}_R\cdot\frac{\partial\mathbf{F}}{\partial R}}_{(1)} + \underbrace{\frac{1}{R}\mathbf{e}_\phi\cdot\frac{\partial\mathbf{F}}{\partial\phi}}_{(2)} + \underbrace{\mathbf{k}\cdot\frac{\partial\mathbf{F}}{\partial z}}_{(3)}.\end{aligned}$$

We will now consider each of the terms on the RHS in turn:

$$\begin{aligned}(1) &= \mathbf{e}_R\cdot\frac{\partial}{\partial R}(F_R\mathbf{e}_R + F_\phi\mathbf{e}_\phi + F_z\mathbf{k})\\ &= \mathbf{e}_R\cdot\left[\frac{\partial F_R}{\partial R}\mathbf{e}_R + F_R\underbrace{\frac{\partial\mathbf{e}_R}{\partial R}}_{=0} + \frac{\partial F_\phi}{\partial R}\mathbf{e}_\phi + F_\phi\underbrace{\frac{\partial\mathbf{e}_\phi}{\partial R}}_{=0} + \frac{\partial F_z}{\partial R}\mathbf{k} + F_z\underbrace{\frac{\partial\mathbf{k}}{\partial R}}_{=0}\right]\\ &= \frac{\partial F_R}{\partial R}.\end{aligned}$$

Remember that $\mathbf{e}_R\cdot\mathbf{e}_R = 1$, $\mathbf{e}_R\cdot\mathbf{e}_\phi = 0$ and $\mathbf{e}_R\cdot\mathbf{k} = 0$.

Similarly, we have

$$\begin{aligned}(2) &= \frac{1}{R}\mathbf{e}_\phi\cdot\frac{\partial}{\partial\phi}(F_R\mathbf{e}_R + F_\phi\mathbf{e}_\phi + F_z\mathbf{k})\\ &= \frac{1}{R}\mathbf{e}_\phi\cdot\left[\frac{\partial F_R}{\partial\phi}\mathbf{e}_R + F_R\underbrace{\frac{\partial\mathbf{e}_R}{\partial\phi}}_{=\mathbf{e}_\phi} + \frac{\partial F_\phi}{\partial\phi}\mathbf{e}_\phi + F_\phi\underbrace{\frac{\partial\mathbf{e}_\phi}{\partial\phi}}_{=-\mathbf{e}_R} + \frac{\partial F_z}{\partial\phi}\mathbf{k} + F_z\underbrace{\frac{\partial\mathbf{k}}{\partial\phi}}_{=0}\right]\\ &= \frac{1}{R}\mathbf{e}_\phi\cdot\left[\frac{\partial F_R}{\partial\phi}\mathbf{e}_R + F_R\mathbf{e}_\phi + \frac{\partial F_\phi}{\partial\phi}\mathbf{e}_\phi - F_\phi\mathbf{e}_R + \mathbf{k}\frac{\partial F_z}{\partial\phi}\right]\\ &= \frac{1}{R}\left(F_R + \frac{\partial F_\phi}{\partial\phi}\right)\end{aligned}$$

And finally

$$\begin{aligned}(3) &= \mathbf{k}\cdot\frac{\partial}{\partial z}(F_R\mathbf{e}_R + F_\phi\mathbf{e}_\phi + F_z\mathbf{k})\\ &= \mathbf{k}\cdot\left[\frac{\partial F_R}{\partial z}\mathbf{e}_R + \frac{\partial F_\phi}{\partial z}\mathbf{e}_\phi + \frac{\partial F_z}{\partial z}\mathbf{k}\right]\\ &= \frac{\partial F_z}{\partial z}.\end{aligned}$$

Combining (1), (2) and (3), we find

$$\nabla\cdot\mathbf{F} = \frac{\partial F_R}{\partial R} + \frac{1}{R}\left(F_R + \frac{\partial F_\phi}{\partial\phi}\right) + \frac{\partial F_z}{\partial z}$$

or, we can rewrite this as

$$\boldsymbol{\nabla}\cdot\mathbf{F} = \frac{1}{R}\frac{\partial}{\partial R}(RF_R) + \frac{1}{R}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}$$

Curl

We follow the same general method to derive an expression for 'curl' in cylindrical coordinates.

$$\begin{aligned}\boldsymbol{\nabla}\times\mathbf{F} &= \left(\mathbf{e}_R\frac{\partial}{\partial R} + \mathbf{e}_\phi\frac{1}{R}\frac{\partial}{\partial \phi} + \mathbf{k}\frac{\partial}{\partial z}\right)\times\mathbf{F}\\ &= \underbrace{\mathbf{e}_R\times\frac{\partial\mathbf{F}}{\partial R}}_{(1)} + \frac{1}{R}\underbrace{\mathbf{e}_\phi\times\frac{\partial\mathbf{F}}{\partial \phi}}_{(2)} + \underbrace{\mathbf{k}\times\frac{\partial\mathbf{F}}{\partial z}}_{(3)}\end{aligned}$$

Let us again work out each of the RHS terms in turn:

(1):

$$\begin{aligned}\frac{\partial\mathbf{F}}{\partial R} &= \frac{\partial}{\partial R}(F_R\mathbf{e}_R + F_\phi\mathbf{e}_\phi + F_z\mathbf{k})\\ &= \mathbf{e}_R\frac{\partial F_R}{\partial R} + \mathbf{e}_\phi\frac{\partial F_\phi}{\partial R} + \mathbf{k}\frac{\partial F_z}{\partial R}\end{aligned}$$

$$\begin{aligned}\Rightarrow \mathbf{e}_R\times\frac{\partial\mathbf{F}}{\partial R} &= \underbrace{(\mathbf{e}_R\times\mathbf{e}_R)}_{=0}\frac{\partial F_R}{\partial R} + \underbrace{(\mathbf{e}_R\times\mathbf{e}_\phi)}_{=\mathbf{k}}\frac{\partial F_\phi}{\partial R} + \underbrace{(\mathbf{e}_R\times\mathbf{k})}_{=-\mathbf{e}_\phi}\frac{\partial F_z}{\partial R}\\ &= \frac{\partial F_\phi}{\partial R}\mathbf{k} - \frac{\partial F_z}{\partial R}\mathbf{e}_\phi\end{aligned}$$

where we have used $\mathbf{e}_R\times\mathbf{e}_\phi = \mathbf{k}$, $\mathbf{e}_\phi\times\mathbf{k} = \mathbf{e}_R$ and $\mathbf{k}\times\mathbf{e}_R = \mathbf{e}_\phi$.

(2):

$$\begin{aligned}\frac{\partial\mathbf{F}}{\partial \phi} &= \frac{\partial}{\partial \phi}(F_R\mathbf{e}_R + F_\phi\mathbf{e}_\phi + F_z\mathbf{k})\\ &= \mathbf{e}_\phi F_R + \mathbf{e}_R\frac{\partial F_R}{\partial \phi} + \mathbf{e}_\phi\frac{\partial F_\phi}{\partial \phi} - \mathbf{e}_R F_\phi + \mathbf{k}\frac{\partial F_z}{\partial \phi}\end{aligned}$$

where we have again used that $\dfrac{\partial\mathbf{e}_R}{\partial\phi} = \mathbf{e}_\phi$ and $\dfrac{\partial\mathbf{e}_\phi}{\partial\phi} = -\mathbf{e}_R$.

$$\begin{aligned}\Rightarrow \mathbf{e}_\phi\times\frac{\partial\mathbf{F}}{\partial R} &= \underbrace{(\mathbf{e}_\phi\times\mathbf{e}_R)}_{=-\mathbf{k}}\frac{\partial F_R}{\partial \phi} - \underbrace{(\mathbf{e}_\phi\times\mathbf{e}_R)}_{=-\mathbf{k}}F_\phi + \underbrace{(\mathbf{e}_\phi\times\mathbf{k})}_{=\mathbf{e}_R}\frac{\partial F_z}{\partial \phi}\\ &= -\frac{\partial F_R}{\partial \phi}\mathbf{k} + F_\phi\mathbf{k} + \frac{\partial F_z}{\partial \phi}\mathbf{e}_R\,.\end{aligned}$$

(3):

$$\begin{aligned}\frac{\partial\mathbf{F}}{\partial z} &= \frac{\partial}{\partial z}(F_R\mathbf{e}_R + F_\phi\mathbf{e}_\phi + F_z\mathbf{k})\\ &= \mathbf{e}_R\frac{\partial F_R}{\partial z} + \mathbf{e}_\phi\frac{\partial F_\phi}{\partial z} + \mathbf{k}\frac{\partial F_z}{\partial z}\end{aligned}$$

and

$$
\begin{aligned}
\Rightarrow \mathbf{k} \times \frac{\partial \mathbf{F}}{\partial z} &= \underbrace{(\mathbf{k} \times \mathbf{e}_R)}_{=\mathbf{e}_\phi} \frac{\partial F_R}{\partial z} + \underbrace{(\mathbf{k} \times \mathbf{e}_\phi)}_{=-\mathbf{e}_R} \frac{\partial F_\phi}{\partial z} \\
&= \frac{\partial F_R}{\partial z} \mathbf{e}_\phi - \frac{\partial F_\phi}{\partial z} \mathbf{e}_R .
\end{aligned}
$$

Finally, combining (1), (2) and (3), we find

$$
\begin{aligned}
\nabla \times \mathbf{F} &= \left[\frac{1}{R} \frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z} \right] \mathbf{e}_R + \left[-\frac{\partial F_z}{\partial R} + \frac{\partial F_R}{\partial z} \right] \mathbf{e}_\phi + \left[\frac{\partial F_\phi}{\partial R} - \frac{1}{R} \frac{\partial F_R}{\partial \phi} + \frac{1}{R} F_\phi \right] \mathbf{k} \\
&= \frac{1}{R} \left[\frac{\partial F_z}{\partial \phi} - R \frac{\partial F_\phi}{\partial z} \right] \mathbf{e}_R + \left[\frac{\partial F_R}{\partial z} - \frac{\partial F_z}{\partial R} \right] \mathbf{e}_\phi + \frac{1}{R} \left[\frac{\partial}{\partial R} (R F_\phi) - \frac{\partial F_R}{\partial \phi} \right] \mathbf{k} .
\end{aligned}
$$

By working out the determinant, you can show that this expression for curl in cylindrical coordinates can also be written as

$$
\nabla \times \mathbf{F} = \frac{1}{R} \begin{vmatrix} \mathbf{e}_R & R\mathbf{e}_\phi & k \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ F_R & RF_\phi & F_z \end{vmatrix}
$$

Laplacian

Finally, we can derive an expression for the (scalar) Laplacian ($\nabla^2 f$) of a scalar function $f(R, \phi, z)$ by considering $\nabla \cdot \mathbf{F}$, where

$$
\mathbf{F} = \nabla f = \left(\frac{\partial f}{\partial R}, \frac{1}{R} \frac{\partial f}{\partial \phi}, \frac{\partial f}{\partial z} \right) .
$$

Hence, we get

$$
\begin{aligned}
\nabla^2 f &= \nabla \cdot \nabla f \\
&= \frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial f}{\partial R} \right) + \frac{1}{R} \frac{\partial}{\partial \phi} \left(\frac{1}{R} \frac{\partial f}{\partial \phi} \right) + \frac{\partial}{\partial z} \left(\frac{\partial f}{\partial z} \right) \\
&= \frac{1}{R} \frac{\partial f}{\partial R} + \frac{\partial^2 f}{\partial R^2} + \frac{1}{R^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} .
\end{aligned}
$$

Summary

Gradient:

$$
\nabla f = \frac{\partial f}{\partial R} \mathbf{e}_R + \frac{1}{R} \frac{\partial f}{\partial \phi} \mathbf{e}_\phi + \frac{\partial f}{\partial z} \mathbf{k}
$$

Divergence:

$$
\nabla \cdot \mathbf{F} = \frac{1}{R} \frac{\partial}{\partial R} (R F_R) + \frac{1}{R} \frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}
$$

Curl:

$$\nabla \times \mathbf{F} = \frac{1}{R}\begin{vmatrix} \mathbf{e}_R & R\mathbf{e}_\phi & \mathbf{k} \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ F_R & RF_\phi & F_z \end{vmatrix}$$

Laplacian:

$$\nabla^2 f = \frac{1}{R}\frac{\partial f}{\partial R} + \frac{\partial^2 f}{\partial R^2} + \frac{1}{R^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

Examples:

(i)

$$\nabla \times (\phi \nabla R^3)$$

$$\nabla R^3 = \frac{\partial (R^3)}{\partial R}\mathbf{e}_R = 3R^2\mathbf{e}_R$$

$$\nabla \times (\phi \nabla R^3) = \nabla \times (3\phi R^2 \mathbf{e}_R) = \frac{1}{R}\begin{vmatrix} \mathbf{e}_R & R\mathbf{e}_\phi & \mathbf{k} \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 3\phi R^2 & 0 & 0 \end{vmatrix}$$

$$= \frac{1}{R}\left(0 \cdot \mathbf{e}_R + \frac{\partial}{\partial z}(3\phi R^2) R\mathbf{e}_\phi - \frac{\partial}{\partial \phi}(3\phi R^2)\mathbf{k}\right) = -3R\mathbf{k}$$

(ii)

$$\nabla \cdot (\phi \nabla R^3)$$

$$3\phi R^2 \mathbf{e}_R = F_R \mathbf{e}_R$$

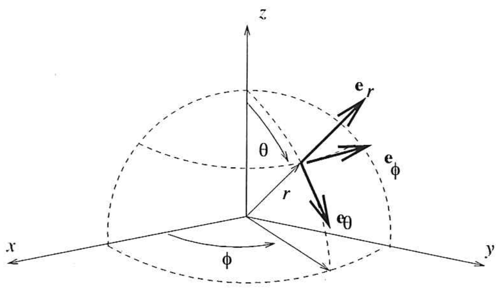
$$\nabla \cdot (3\phi R^2 \mathbf{e}_R) = \frac{1}{R}\frac{\partial}{\partial R}(3\phi R^3) = \frac{1}{R} 9\phi R^2 = 9\phi R$$

3.7.2 Spherical Coordinates

We now consider spherical coordinates (r, θ, ϕ).

Remember that

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$$
$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$



and

$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\theta$$

In spherical coordinates, the unit vectors are given by

$$\mathbf{e}_r = \sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k}$$
$$\mathbf{e}_\theta = \cos\theta\cos\phi\mathbf{i} + \cos\theta\sin\phi\mathbf{j} - \sin\theta\mathbf{k}$$
$$\mathbf{e}_\phi = -\sin\phi\mathbf{i} + \cos\phi\mathbf{j}\,.$$

Hence, we can write down the following expressions:

$$\frac{\partial \mathbf{e}_r}{\partial r} = 0 \qquad \frac{\partial \mathbf{e}_\theta}{\partial r} = 0 \qquad \frac{\partial \mathbf{e}_\phi}{\partial r} = 0$$
$$\frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta \qquad \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r \qquad \frac{\partial \mathbf{e}_\phi}{\partial \theta} = 0$$
$$\frac{\partial \mathbf{e}_r}{\partial \phi} = \sin\theta\mathbf{e}_\phi \qquad \frac{\partial \mathbf{e}_\theta}{\partial \phi} = \cos\theta\mathbf{e}_\phi \qquad \frac{\partial \mathbf{e}_\phi}{\partial \phi} = -\cos\phi\mathbf{i} - \sin\phi\mathbf{j} = -\sin\theta\mathbf{e}_r - \cos\theta\mathbf{e}_\theta$$

In spherical coordinates a point $P(r, \theta, \phi)$ has a position vector given by

$$\mathbf{r} = r\mathbf{e}_r\,.$$

Remember that now the unit vectors depend on θ and ϕ.

Let us now consider the infinitesimal increment $\delta\mathbf{r}$:

$$\begin{aligned}\delta\mathbf{r} = \delta(r\mathbf{e}_r) &= \delta r\,\mathbf{e}_r + r\,\delta\mathbf{e}_r \\ &= \delta r\,\mathbf{e}_r + r\left(\frac{\partial \mathbf{e}_r}{\partial r}\delta r + \frac{\partial \mathbf{e}_r}{\partial \theta}\delta\theta + \frac{\partial \mathbf{e}_r}{\partial \phi}\delta\phi\right) \\ &= \delta r\,\mathbf{e}_r + r\,\delta\theta\,\mathbf{e}_\theta + r\,\sin\theta\delta\phi\,\mathbf{e}_\phi\,.\end{aligned}$$

Gradient

Let us consider the scalar function $f(r, \theta, \phi)$.

Then we can write

$$\delta f = \frac{\partial f}{\partial r}\delta r + \frac{\partial f}{\partial \theta}\delta\theta + \frac{\partial f}{\partial \phi}\delta\phi .$$

But we also have that the directional derivative would give us

$$\delta f = \delta\mathbf{r}\cdot\boldsymbol{\nabla} f$$

where $\delta\mathbf{r} = \delta r\,\mathbf{e}_r + r\,\delta\theta\,\mathbf{e}_\theta + r\,\sin\theta\delta\phi\,\mathbf{e}_\phi$.

Hence, we have:

$$\delta f = \frac{\partial f}{\partial r}\delta r + \frac{\partial f}{\partial \theta}\delta\theta + \frac{\partial f}{\partial \phi}\delta\phi = (\boldsymbol{\nabla} f)_r\delta r + (\boldsymbol{\nabla} f)_\theta r\delta\theta + (\boldsymbol{\nabla} f)_\phi r\sin\theta\delta\phi$$

and thus

$$\begin{aligned}
(\boldsymbol{\nabla} f)_r &= \frac{\partial f}{\partial r}\\
(\boldsymbol{\nabla} f)_\phi &= \frac{1}{r}\frac{\partial f}{\partial \theta}\\
(\boldsymbol{\nabla} f)_\phi &= \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}
\end{aligned}$$

or, in terms of the differential operator,

$$\boldsymbol{\nabla} = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right) .$$

Divergence

Again, we can use the expression for the differential operator to derive 'div' in spherical coordinates.

Consider a vector function $\mathbf{F} = F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_\phi\mathbf{e}_\phi$.

Then we can write

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{F} &= \left(\mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\cdot\mathbf{F}\\
&= \underbrace{\mathbf{e}_r\cdot\frac{\partial\mathbf{F}}{\partial r}}_{(1)} + \frac{1}{r}\underbrace{\mathbf{e}_\theta\cdot\frac{\partial\mathbf{F}}{\partial\theta}}_{(2)} + \frac{1}{r\sin\theta}\underbrace{\mathbf{e}_\phi\cdot\frac{\partial\mathbf{F}}{\partial\phi}}_{(3)} .
\end{aligned}$$

We will now consider each of the terms on the RHS in turn:

$$\begin{aligned}
(1) &= \mathbf{e}_r\cdot\frac{\partial}{\partial r}(F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_\phi\mathbf{e}_\phi)\\
&= \mathbf{e}_r\cdot\left[\frac{\partial F_r}{\partial r}\mathbf{e}_r + F_r\underbrace{\frac{\partial\mathbf{e}_r}{\partial r}}_{=0} + \frac{\partial F_\theta}{\partial r}\mathbf{e}_\theta + F_\theta\underbrace{\frac{\partial\mathbf{e}_\theta}{\partial r}}_{=0} + \frac{\partial F_\phi}{\partial r}\mathbf{e}_\phi + F_\phi\underbrace{\frac{\partial\mathbf{e}_\phi}{\partial r}}_{=0}\right]\\
&= \frac{\partial F_r}{\partial r} .
\end{aligned}$$

Remember that $\mathbf{e}_r \cdot \mathbf{e}_r = 1$, $\mathbf{e}_r \cdot \mathbf{e}_\theta = 0$ and $\mathbf{e}_r \cdot \mathbf{e}_\phi = 0$.

Similarly, we have

$$\begin{aligned}
(2) &= \mathbf{e}_\theta \cdot \frac{\partial}{\partial \theta}(F_r \mathbf{e}_r + F_\theta \mathbf{e}_\theta + F_\phi \mathbf{e}_\phi) \\
&= \mathbf{e}_\theta \cdot \left[\frac{\partial F_r}{\partial \theta}\mathbf{e}_r + F_r \underbrace{\frac{\partial \mathbf{e}_r}{\partial \theta}}_{=\mathbf{e}_\theta} + \frac{\partial F_\theta}{\partial \theta}\mathbf{e}_\theta + F_\theta \underbrace{\frac{\partial \mathbf{e}_\theta}{\partial \theta}}_{=-\mathbf{e}_r} + \frac{\partial F_\phi}{\partial \theta}\mathbf{e}_\phi + F_\phi \underbrace{\frac{\partial \mathbf{e}_\phi}{\partial \theta}}_{=0} \right] \\
&= F_r + \frac{\partial F_\theta}{\partial \theta} .
\end{aligned}$$

And finally

$$\begin{aligned}
(3) &= \mathbf{e}_\phi \cdot \frac{\partial}{\partial \phi}(F_r \mathbf{e}_r + F_\theta \mathbf{e}_\theta + F_\phi \mathbf{e}_\phi) \\
&= \mathbf{e}_\phi \cdot \left[\frac{\partial F_r}{\partial \phi}\mathbf{e}_r + F_r \underbrace{\frac{\partial \mathbf{e}_r}{\partial \phi}}_{=\sin\theta \mathbf{e}_\phi} + \frac{\partial F_\theta}{\partial \phi}\mathbf{e}_\theta + F_\theta \underbrace{\frac{\partial \mathbf{e}_\theta}{\partial \phi}}_{=\cos\theta \mathbf{e}_\phi} + \frac{\partial F_\phi}{\partial \phi}\mathbf{e}_\phi + F_\phi \underbrace{\frac{\partial \mathbf{e}_\phi}{\partial \phi}}_{=-\sin\theta \mathbf{e}_r - \cos\theta \mathbf{e}_\theta} \right] \\
&= F_r \sin\theta + F_\theta \cos\theta + \frac{\partial F_\phi}{\partial \phi} .
\end{aligned}$$

Combining (1), (2) and (3), we find

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{F} &= \frac{\partial F_r}{\partial r} + \frac{1}{r}\left(F_r + \frac{\partial F_\theta}{\partial \theta}\right) + \frac{1}{r\sin\theta}\left(F_r \sin\theta + F_\theta \cos\theta + \frac{\partial F_\phi}{\partial \phi}\right) \\
&= \left(\frac{2}{r}F_r + \frac{\partial F_r}{\partial r}\right) + \left(\frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\cos\theta}{r\sin\theta}F_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi} \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(F_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi}
\end{aligned}$$

Curl

Finally, we look at 'curl' in spherical coordinates.

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{F} &= \left(\mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right) \times \mathbf{F} \\
&= \underbrace{\mathbf{e}_r \times \frac{\partial \mathbf{F}}{\partial r}}_{(1)} + \frac{1}{r}\underbrace{\mathbf{e}_\theta \times \frac{\partial \mathbf{F}}{\partial \theta}}_{(2)} + \frac{1}{r\sin\theta}\underbrace{\mathbf{e}_\phi \times \frac{\partial \mathbf{F}}{\partial \phi}}_{(3)} .
\end{aligned}$$

$$
\begin{aligned}
(1) &= \mathbf{e}_r \times \frac{\partial}{\partial r}(F_r \mathbf{e}_r + F_\theta \mathbf{e}_\theta + F_\phi \mathbf{e}_\phi) \\
&= \mathbf{e}_r \times \left[\frac{\partial F_r}{\partial r}\mathbf{e}_r + F_r \underbrace{\frac{\partial \mathbf{e}_r}{\partial r}}_{=0} + \frac{\partial F_\theta}{\partial r}\mathbf{e}_\theta + F_\theta \underbrace{\frac{\partial \mathbf{e}_\theta}{\partial r}}_{=0} + \frac{\partial F_\phi}{\partial r}\mathbf{e}_\phi + F_\phi \underbrace{\frac{\partial \mathbf{e}_\phi}{\partial r}}_{=0} \right] \\
&= \underbrace{(\mathbf{e}_r \times \mathbf{e}_\theta)}_{=\mathbf{e}_\phi} \frac{\partial F_\theta}{\partial r} + \underbrace{(\mathbf{e}_r \times \mathbf{e}_\phi)}_{=-\mathbf{e}_\theta} \frac{\partial F_\phi}{\partial r} \\
&= \frac{\partial F_\theta}{\partial r}\mathbf{e}_\phi - \frac{\partial F_\phi}{\partial r}\mathbf{e}_\theta
\end{aligned}
$$

where we have used $\mathbf{e}_r \times \mathbf{e}_\theta = \mathbf{e}_\phi$, $\mathbf{e}_\theta \times \mathbf{e}_\phi = \mathbf{e}_r$ and $\mathbf{e}_\phi \times \mathbf{e}_r = \mathbf{e}_\theta$.

$$
\begin{aligned}
(2) &= \mathbf{e}_\theta \times \frac{\partial}{\partial r}(F_r \mathbf{e}_r + F_\theta \mathbf{e}_\theta + F_\phi \mathbf{e}_\phi) \\
&= \mathbf{e}_\theta \times \left[\frac{\partial F_r}{\partial \theta}\mathbf{e}_r + F_r \underbrace{\frac{\partial \mathbf{e}_r}{\partial \theta}}_{=\mathbf{e}_\theta} + \frac{\partial F_\theta}{\partial \theta}\mathbf{e}_\theta + F_\theta \underbrace{\frac{\partial \mathbf{e}_\theta}{\partial \theta}}_{=-\mathbf{e}_r} + \frac{\partial F_\phi}{\partial \theta}\mathbf{e}_\phi + F_\phi \underbrace{\frac{\partial \mathbf{e}_\phi}{\partial \theta}}_{=0} \right] \\
&= \underbrace{(\mathbf{e}_\theta \times \mathbf{e}_r)}_{=-\mathbf{e}_\phi} \frac{\partial F_r}{\partial \theta} - \underbrace{(\mathbf{e}_\theta \times \mathbf{e}_r)}_{=-\mathbf{e}_\phi} F_\theta + \underbrace{(\mathbf{e}_\theta \times \mathbf{e}_\phi)}_{=\mathbf{e}_r} \frac{\partial F_\phi}{\partial \theta} \\
&= \left(F_\theta - \frac{\partial F_r}{\partial \theta} \right) \mathbf{e}_\phi + \frac{\partial F_\phi}{\partial \theta}\mathbf{e}_r
\end{aligned}
$$

$$
\begin{aligned}
(3) &= \mathbf{e}_\phi \times \frac{\partial}{\partial r}(F_r \mathbf{e}_r + F_\theta \mathbf{e}_\theta + F_\phi \mathbf{e}_\phi) \\
&= \mathbf{e}_\phi \times \left[\frac{\partial F_r}{\partial \phi}\mathbf{e}_r + F_r \underbrace{\frac{\partial \mathbf{e}_r}{\partial \phi}}_{=\sin\theta \mathbf{e}_\phi} + \frac{\partial F_\theta}{\partial \phi}\mathbf{e}_\theta + F_\theta \underbrace{\frac{\partial \mathbf{e}_\theta}{\partial \phi}}_{=\cos\theta \mathbf{e}_\phi} + \frac{\partial F_\phi}{\partial \phi}\mathbf{e}_\phi + F_\phi \underbrace{\frac{\partial \mathbf{e}_\phi}{\partial \phi}}_{=-\sin\theta \mathbf{e}_r - \cos\theta \mathbf{e}_\theta} \right] \\
&= \underbrace{(\mathbf{e}_\phi \times \mathbf{e}_r)}_{=\mathbf{e}_\theta} \frac{\partial F_r}{\partial \phi} + \underbrace{(\mathbf{e}_\phi \times \mathbf{e}_\theta)}_{=-\mathbf{e}_r} \frac{\partial F_\theta}{\partial \phi} - F_\phi \sin\theta \underbrace{(\mathbf{e}_\phi \times \mathbf{e}_r)}_{=\mathbf{e}_\theta} - F_\phi \cos\theta \underbrace{(\mathbf{e}_\phi \times \mathbf{e}_\theta)}_{=-\mathbf{e}_r} \\
&= \left(F_\phi \cos\theta - \frac{\partial F_\theta}{\partial \phi} \right) \mathbf{e}_r + \left(\frac{\partial F_r}{\partial \phi} - F_\phi \sin\theta \right) \mathbf{e}_\theta
\end{aligned}
$$

Finally, combining (1), (2) and (3), we get

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{F} &= \left[\frac{1}{r}\frac{\partial F_\phi}{\partial \theta} - \frac{1}{r\sin\theta}\frac{\partial F_\theta}{\partial \phi} + F_\phi \frac{\cos\theta}{r\sin\theta}\right]\mathbf{e}_r + \left[-\frac{\partial F_\phi}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial F_r}{\partial \phi} - \frac{F_\phi}{r}\right]\mathbf{e}_\theta \\
&\quad + \left[\frac{\partial F_\theta}{\partial r} - \frac{1}{r}\frac{\partial F_r}{\partial \theta} + \frac{F_\theta}{r}\right]\mathbf{e}_\phi \\
&= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(F_\phi \sin\theta) - \frac{\partial F_\theta}{\partial \phi}\right]\mathbf{e}_r + \frac{1}{r\sin\theta}\left[\frac{\partial F_r}{\partial \phi} - \sin\theta \frac{\partial}{\partial r}(rF_\phi)\right]\mathbf{e}_\theta \\
&\quad + \frac{1}{r}\left[\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial \theta}\right]\mathbf{e}_\phi .
\end{aligned}$$

Curl in spherical coordinates can also be written as

$$\boldsymbol{\nabla} \times \mathbf{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta\mathbf{e}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix}$$

Laplacian

The (scalar) Laplacian ($\nabla^2 f$) of a scalar function $f(r,\theta,\phi)$ can again be obtained from $\nabla^2 f = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} f$:

$$\begin{aligned}
\nabla^2 f &= \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} f \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left(\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\right) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}.
\end{aligned}$$

Summary

Gradient:

$$\boldsymbol{\nabla} f = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\right)$$

Divergence:

$$\boldsymbol{\nabla}\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(F_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi}$$

Curl:

$$\boldsymbol{\nabla} \times \mathbf{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta\mathbf{e}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix}$$

Laplacian:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}.$$

Examples:

(i) Show that $f(r,\theta,\phi) = \frac{1}{r}$ is a solution of Laplace's equation (i.e. $\nabla^2 f = 0$)

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial(1/r)}{\partial r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left(-\frac{1}{r^2}\right)\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(-1) = 0$$

(ii) Consider $\mathbf{v}(r,\theta,\phi) = r\sin\theta\mathbf{e}_\theta + r\cos\theta\mathbf{e}_\phi$. Determine $\boldsymbol{\nabla}\times\mathbf{v}$.

$$\begin{aligned}
\boldsymbol{\nabla}\times\mathbf{v} &= \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta\mathbf{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ 0 & r(r\sin\theta) & r\sin\theta(r\cos\theta) \end{vmatrix}, \\
&= \frac{1}{r^2\sin\theta}\left(r^2(\cos^2\theta - \sin^2\theta), r(-2r\sin\theta\cos\theta), r\sin\theta\,(2r\sin\theta)\right), \\
&= \left(\frac{\cos 2\theta}{\sin\theta}, -2\cos\theta, 2\sin\theta\right).
\end{aligned}$$

If $\mathbf{v}$ represents the velocity of a flow, then $\boldsymbol{\nabla}\times\mathbf{v}$ is called the vorticity.

(iii) Consider $\mathbf{v}(r,\theta,\phi) = r\sin\theta\mathbf{e}_\theta + r\cos\theta\mathbf{e}_\phi$. Determine $(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}$.

$$\begin{aligned}
(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} &= \underbrace{\left((r\sin\theta)\frac{1}{r}\frac{\partial}{\partial\theta} + (r\cos\theta)\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)}_{(\mathbf{v}\cdot\boldsymbol{\nabla})-\text{scalar operator}} \underbrace{(r\sin\theta\mathbf{e}_\theta + r\cos\theta\mathbf{e}_\phi)}_{\mathbf{v}-\text{vector to be operated on}} \\
&= \sin\theta\left(\frac{\partial}{\partial\theta}(r\sin\theta)\right)\mathbf{e}_\theta + r\sin^2\theta\left(\frac{\partial}{\partial\theta}\mathbf{e}_\theta\right) + \frac{\cos\theta}{\sin\theta}\left(\frac{\partial}{\partial\phi}(r\sin\theta)\right)\mathbf{e}_\theta + \\
&\quad + r\cos\theta\left(\frac{\partial}{\partial\phi}\mathbf{e}_\theta\right) + \sin\theta\left(\frac{\partial}{\partial\theta}(r\cos\theta)\right)\mathbf{e}_\phi + r\sin\theta\cos\theta\left(\frac{\partial}{\partial\theta}\mathbf{e}_\phi\right) + \\
&\quad + \frac{\cos\theta}{\sin\theta}\left(\frac{\partial}{\partial\phi}(r\cos\theta)\right)\mathbf{e}_\phi + \frac{r\cos^2\theta}{\sin\theta}\left(\frac{\partial}{\partial\phi}\mathbf{e}_\phi\right) \\
&= \sin\theta\,(r\cos\theta)\,\mathbf{e}_\theta + r\sin^2\theta(-\mathbf{e}_r) + 0 + r\cos\theta(\cos\theta\mathbf{e}_\phi) + \\
&\quad + \sin\theta(-r\sin\theta)\mathbf{e}_\phi + 0 + 0 + \frac{r\cos^2\theta}{\sin\theta}(-\sin\theta\mathbf{e}_r - \cos\theta\mathbf{e}_\theta) \\
&= -r(\sin^2\theta + \cos^2\theta)\mathbf{e}_r + r\frac{\cos\theta}{\sin\theta}(\sin^2\theta - \cos^2\theta)\,\mathbf{e}_\theta + r(\cos^2\theta - \sin^2\theta)\mathbf{e}_\phi \\
&= -r\mathbf{e}_r - r\cot\theta\cos 2\theta\mathbf{e}_\theta + r\cos 2\theta\mathbf{e}_\phi
\end{aligned}$$