MT2507

Mathematical Modelling

2014–2015

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Lecture Notes for MT2507 Mathematical Modelling

2014/15, Semester 2

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based on lecture notes provided by
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1 What is Mathematical Modelling?

1.1 What is Mathematical Modelling?

Definition: *Mathematical modelling* is the use of mathematics to describe a *system*.

This system could, for example, be from any of the following scientific areas:

- physics (e.g. radioactive decay, dynamics of particles, etc)
- chemistry (e.g. chemical reactions)
- biology (e.g. population dynamics)
- engineering (e.g. stability of buildings, artificial intelligence, networks, etc)
- social sciences (e.g. economics, sociology, etc)
- others ...

There are two major aspects of mathematical modelling:

- find a/the set of equations describing the system (a model)
- use/solve these equations to make predictions or to gain a better understanding of the system (extract information from the model)

This leads to an important distinction between *qualitative* and *quantitative* models:

- Quantitative models can make accurate predictions about the system and the equations are usually derived from "first principles" (e.g. the laws of physics).
- Qualitative models usually model some of the main properties of the system correctly and are often a first step for a better understanding, but do not necessarily make very accurate (quantitative) predictions; the underlying equations maybe heuristic and not derived from "first principles" (e.g. population dynamics).

The aim of this section of the course is to:

- provide an accurate description of the model using mathematics.
- predict (using the mathematical model) what will happen in the future.

Very often, although not always, differential equations are used in mathematical modelling and this course will deal only with mathematical models described by differential equations. We will mainly use ordinary differential equations (ODEs), e.g. to model population dynamics and particle mechanics, but will briefly touch upon partial differential equations (PDEs) in the second half of the course. Thus, differential equations are highly important for the course.

Example 1.1: For example, consider that if $N(t)$ is the number of birds as a function of time, t. Then the rate of change of the number of birds is given by

$$\frac{dN}{dt}$$

Thus, the equation involves derivatives and therefore requires differential equations.

1.2 Examples

1.2.1 Single-species population dynamics

Example 1.2: A population of birds exists in a place where the food supply is limited

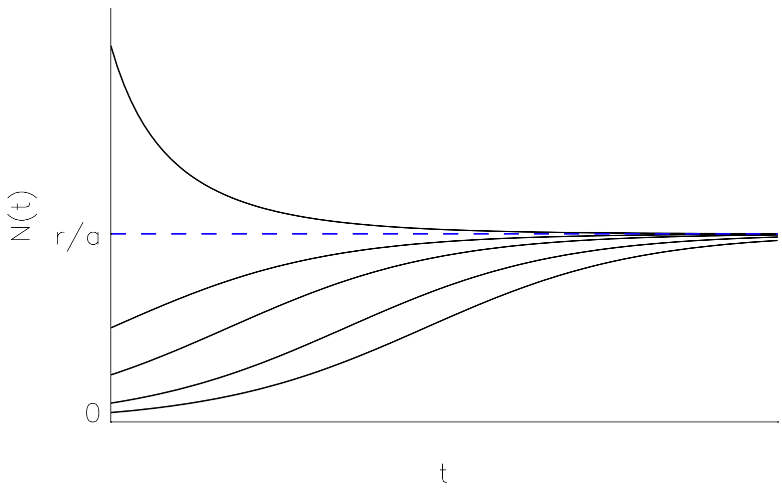
Hence, when the population is

- low, there is plenty of food, so the population increases.
- high, there is not enough food, so the population decreases.

This type of problem can be dealt with in terms of the single variable $N(t)$, which represents the bird population as a function of time. One option for the equation is the *Logistic Equation*,

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN - aN^2, \tag{1}$$

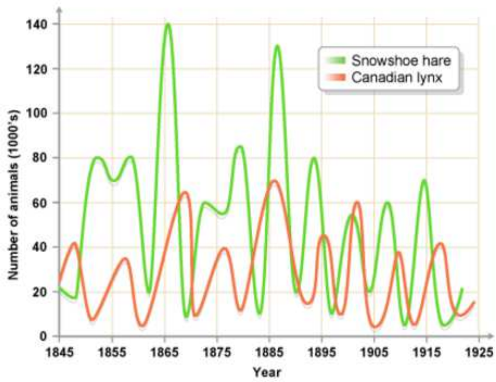
where $r > 0$ and $a > 0$ specify the rates of growth and decay..



This may determine how future population levels depend on the supply of food.

1.2.2 Predator-prey models

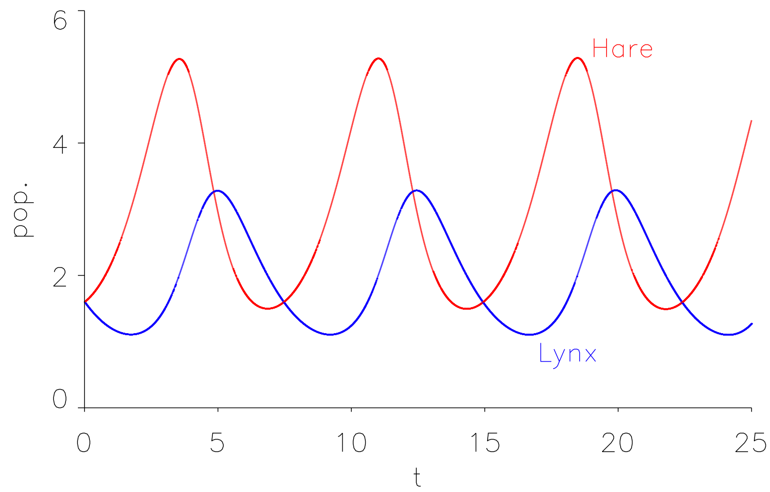
Example 1.3: The populations of lynx and hare.



This consider a pair of creatures, where the hares (x) are hunted by the lynxes (y). This can be described by the system of differential equations:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= x\left(1-\tfrac{1}{2}y\right),\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= y\left(-\tfrac{3}{4}+\tfrac{1}{4}x\right).\end{aligned}$$

A solution of these equations, gives the following graph:



1.2.3 Mechanics

Example 1.4: Launching a rocket to Mars.

This considers the motion of the (accelerated) rocket (of velocity v) under the force of gravity,

$$v\frac{\mathrm{d}v}{\mathrm{d}r} = F = -\frac{GM}{r^2},$$

where r is the distance to the centre of the Earth, M is the mass of the Earth and G is the gravitational constant. We will demonstrate that if the initial velocity of the rocket at the Earth's surface ($r = R_E$) exceeds

$$\sqrt{\frac{2GM}{R_E}},$$

the rocket will escape the Earth's gravity (and hence will continue onwards to Mars).

These three models are used to describe problems in nature, and we will discuss them in more detail later in the course.

2 Revision

2.1 Curve Sketching

Definition: For many problems, we need to interpret the solution through its long-term stability. For this, we need to *sketch the curve*, which may be termed "Draw a rough sketch".

A simple method to draw a curve sketch is to find the key features of the curve first:

1. The *zeros* of x and y. That is where the curve crosses the x and y axis. To find this we find all values of y when $x = 0$ and all values of x when $y = 0$.
2. The *infinities* of x and y.
3. The *maxima* and *minima*. These are locations where
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$ if it is a mimima, or $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$ if it is a maxima. If $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$, then we need to do further analysis.

In many cases, only 1 and 2 need to be considered.

Example 2.1: Sketch the curves of:

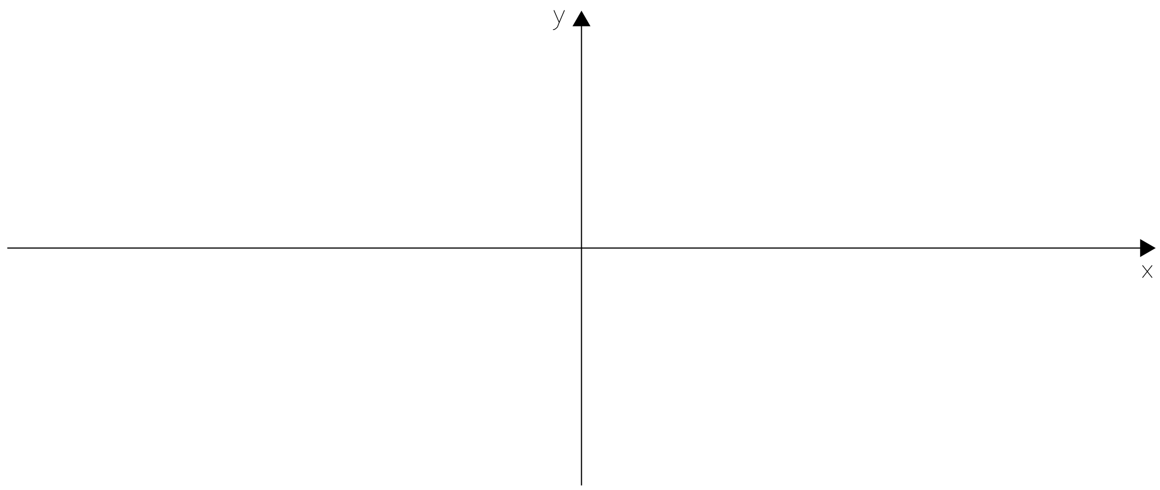
(a) $y = \dfrac{x}{1-x^2}$,

(b) $y = \dfrac{1-x^2}{x}$,

(c) $y = \dfrac{x}{1+x^2}$.

(a) $y = \dfrac{x}{1-x^2}$: It is important to have a sketch and then build up.

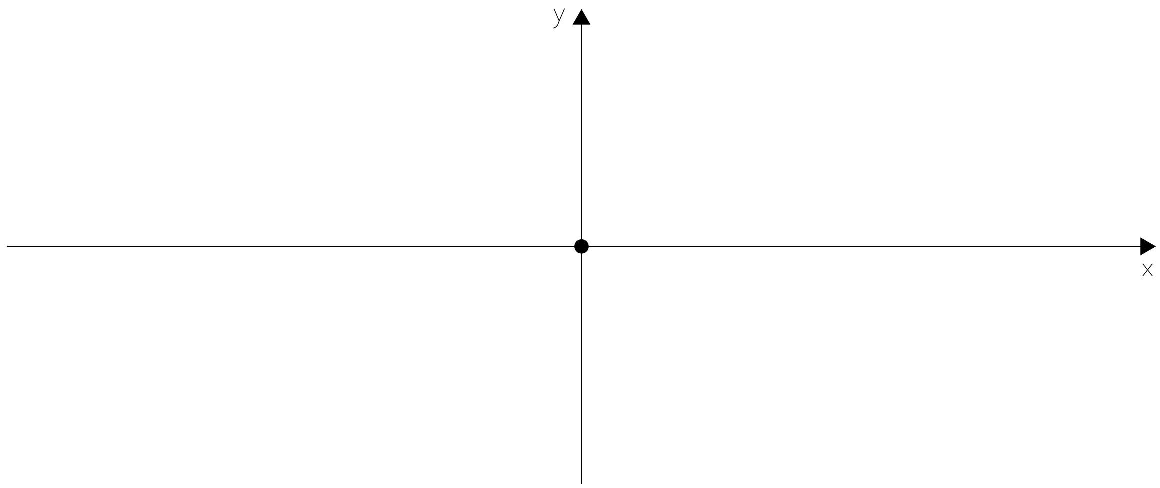
0. We start by drawing the **axes**.



1. Then we consider the **zeros**, that is the locations where $x =$ or $y = 0$.

   - When $x = 0$, we must have $y = 0$.
   - When $y = 0$, we must have $x = 0$.

   So the curve only crosses the axes at a single point, namely $(0, 0)$.



2. We then find the **Infinities**.
   We start by considering when $x \to \pm\infty$. For "large" x, we approximate so

$$\begin{aligned} y &\approx \frac{x}{-x^2} = -\frac{1}{x} \\ &\to 0 \qquad \text{(as } x \to \infty). \end{aligned}$$

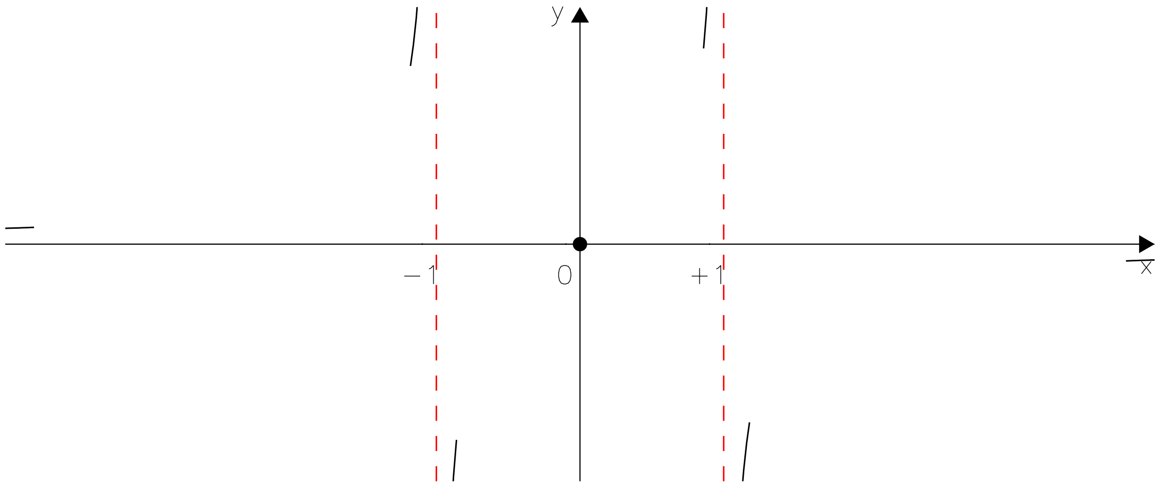
   But this does not explain how y tends to zero. Does it tend to zero from above or below? So we must consider $x \to \pm\infty$ independently.

   - As $x \to +\infty$, $y \to 0^-$. (i.e. y tends to zero from below.)
   - As $x \to -\infty$, $y \to 0^+$. (i.e. y tends to zero from above.)

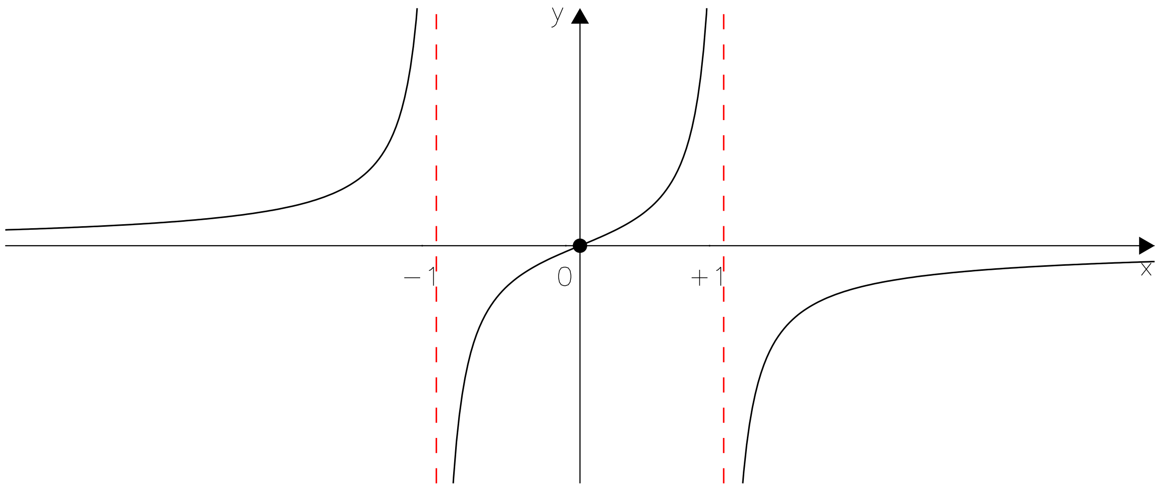
   We now consider when $y \to \pm\infty$. This happens $x = \pm 1$, i.e. the denominator is zero. (This needs to be considered carefully using each of the four approaches.)

- $x \to -1^-$: the fraction has signs $\frac{-}{-}$ $\implies$ $y \to +\infty$.
- $x \to -1^+$: the fraction has signs $\frac{-}{+}$ $\implies$ $y \to -\infty$.
- $x \to +1^-$: the fraction has signs $\frac{+}{+}$ $\implies$ $y \to +\infty$.
- $x \to +1^+$: the fraction has signs $\frac{+}{-}$ $\implies$ $y \to -\infty$.

Note: $+1^+ > +1$, i.e. its magnitude is larger.



4. **Graph**: Using this information we can now sketch the curve.



Limits

When sketching curves, we often consider the limit as

$$x \to \pm\infty,$$

and find

$$\frac{1}{0} \quad , \quad \frac{1}{\infty} \quad , \quad \frac{0}{0} \quad , \quad \frac{\infty}{\infty} \quad , \quad 0 \times \infty.$$

occur. When these occur we need to be very careful.

If we come across,

(i) $\dfrac{1}{0}$: This is infinity, but need to check if it is $+\infty$ or $-\infty$.

(ii) $\dfrac{1}{\infty}$: This is zero, but need to check if it is 0^+ or 0^-. (I.e. does it approach zero from above/below?)

(iii) $\dfrac{0}{0}$, $\dfrac{\infty}{\infty}$ or $0 \times \infty$: These are indeterminate and need to be considered carefully.

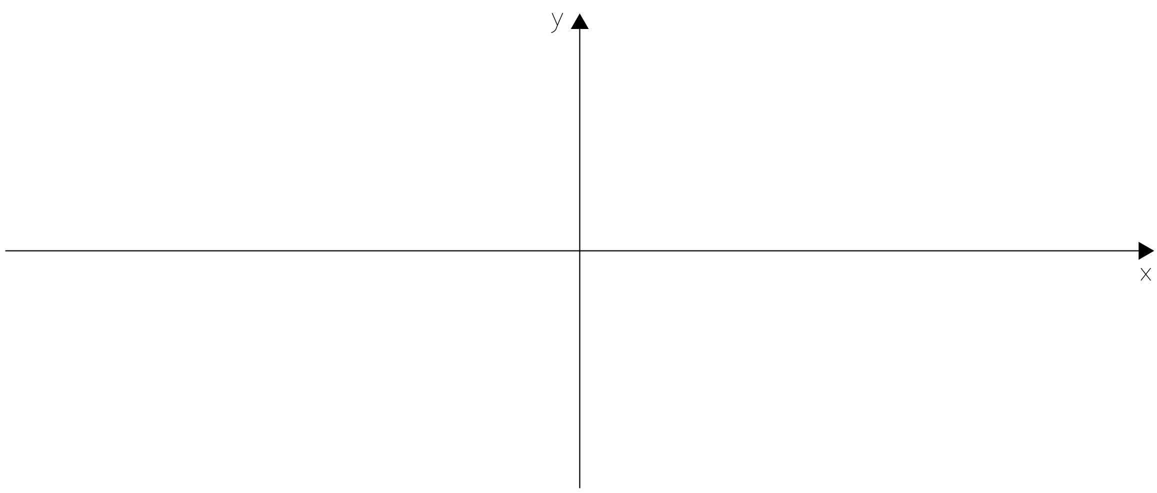
For example, as $x \to 0$,

- $\dfrac{93x}{x} \quad \to \quad 93.$
- $\dfrac{x^2}{x^4} \quad \approx \quad \dfrac{1}{x^2} \quad \to \quad +\infty.$
- $\dfrac{x^4}{x^2} \quad \approx \quad x^2 \quad \to \quad 0^+.$

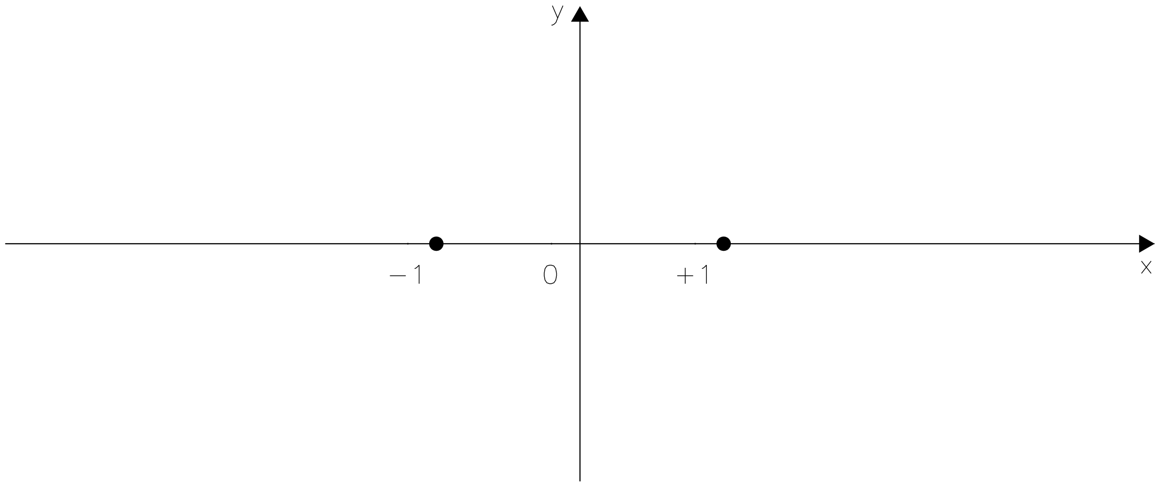
All of these are examples of the form $\dfrac{0}{0}$.

(b) $\quad y = \dfrac{1-x^2}{x}$:

0. **Axes**:

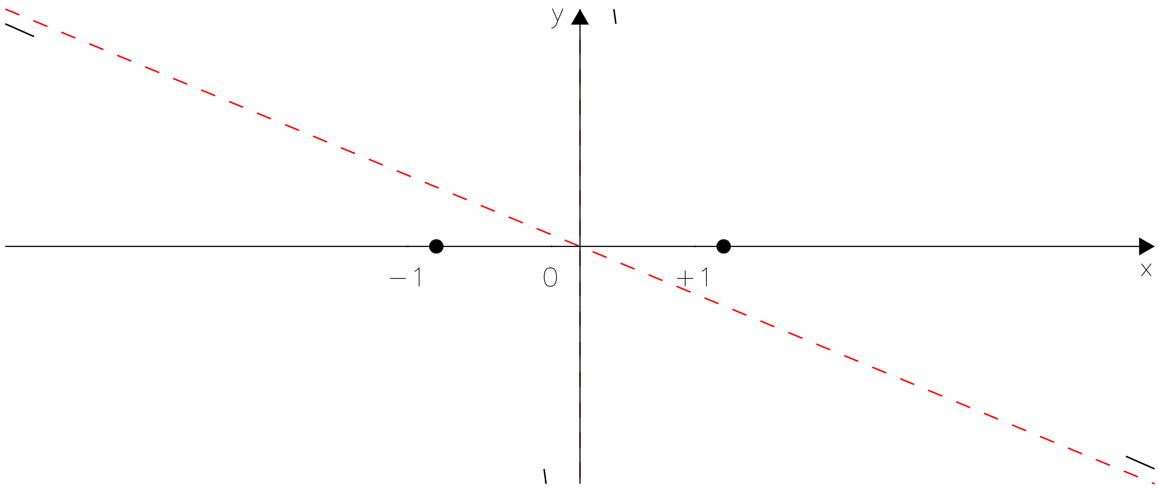


1. **Zeros**:
   - At $x = 0$, y is not defined, so we consider when $x \to 0$. So, as $x \to 0$,
   $$y \quad \to \quad \frac{1}{x}.$$
   What about the sign? Does $y \to \pm\infty$?
     - $x \to 0^+ \quad \implies \quad y \quad \to \quad +\infty.$
     - $x \to 0^- \quad \implies \quad y \quad \to \quad -\infty.$
   - When $y = 0$, $x = \pm 1$.

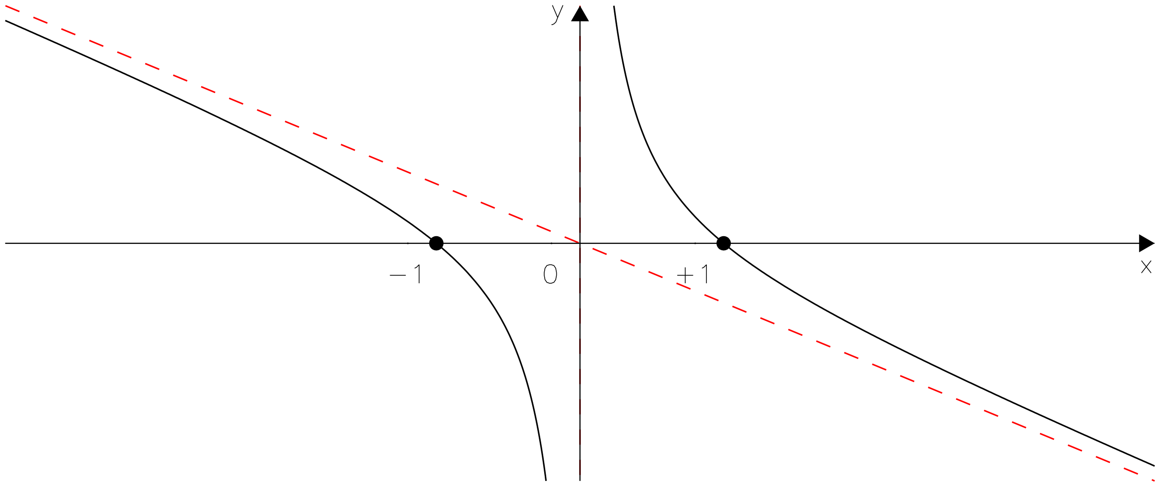


2. **Infinities**: (Remember that $\frac{\infty}{\infty}$ is meaningless.)

   - As $x \to \infty$, $y \;=\; \dfrac{1-x^2}{x} \;=\; \dfrac{1}{x} - x \;\to\; -x$. Hence y approaches the line $y = -x$ as $x \to \infty$. But do we approach from above/below?
     - As $x \to +\infty$, $y \;=\; \underbrace{\dfrac{1}{x}}_{+ve} -x \;\to\; -x$ from above.
     - As $x \to -\infty$, $y \;=\; \underbrace{\dfrac{1}{x}}_{-ve} -x \;\to\; -x$ from below.

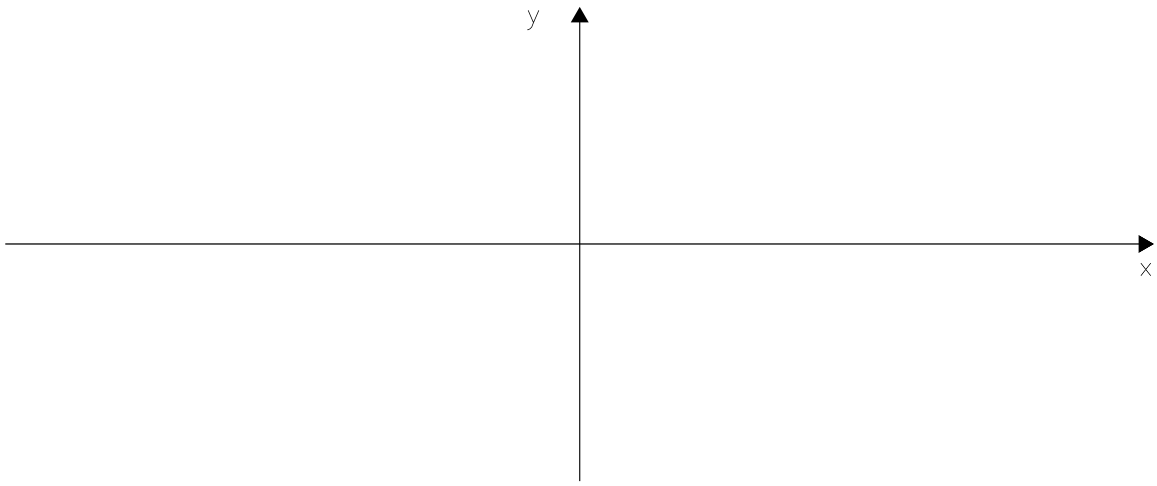


4. **Graph**:



(c) $y = \dfrac{x}{1+x^2}$:

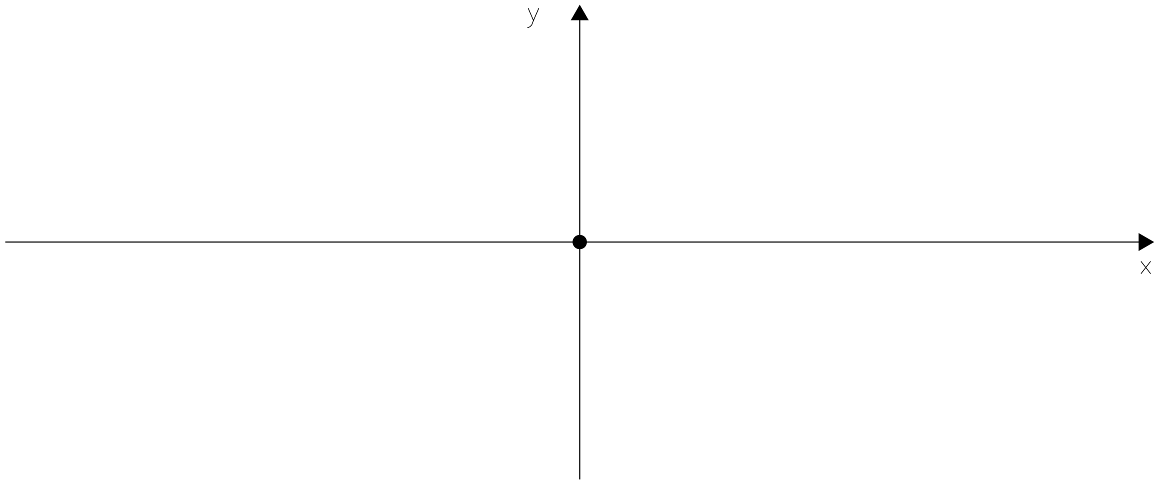
0. **Axes**:



1. **Zeros**:

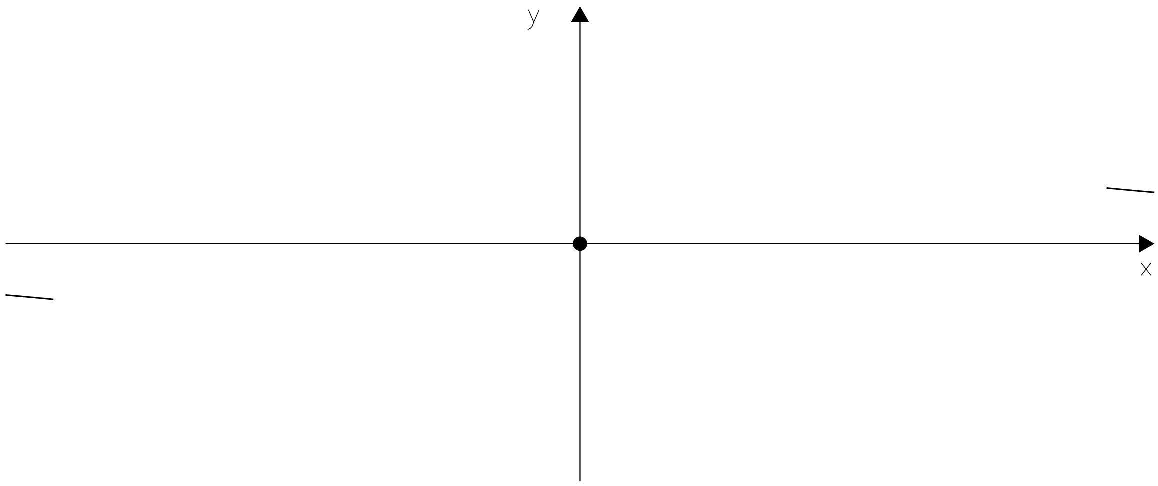
    - When $x = 0$, we must have $y = 0$.
    - When $y = 0$, we must have $x = 0$.

    So the curve only crosses the axes at $(0, 0)$.



2. **Infinities**:

   - $x \rightarrow +\infty \implies y = \dfrac{x}{1+x^2} \rightarrow \dfrac{x}{x^2} = \dfrac{1}{x} \rightarrow 0^+$.
   - $x \rightarrow -\infty \implies y = \dfrac{x}{1+x^2} \rightarrow \dfrac{x}{x^2} \rightarrow \dfrac{1}{x} \rightarrow 0^-$.



3. **Maxima/minima**: Out sketch suggests turning points, i.e. maxima and/or minima. We find these by differentiation.

   First we remember the quotient rule,

$$y = \frac{u}{v} \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}.$$

   In this question,

$$y = \frac{x}{1+x^2} = \frac{u}{v},$$

   where $u = x$ and $v = 1 + x^2$, so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1\left(1+x^2\right) - x\,(2x)}{\left(1+x^2\right)^2} = \frac{1-x^2}{\left(1+x^2\right)^2}.$$

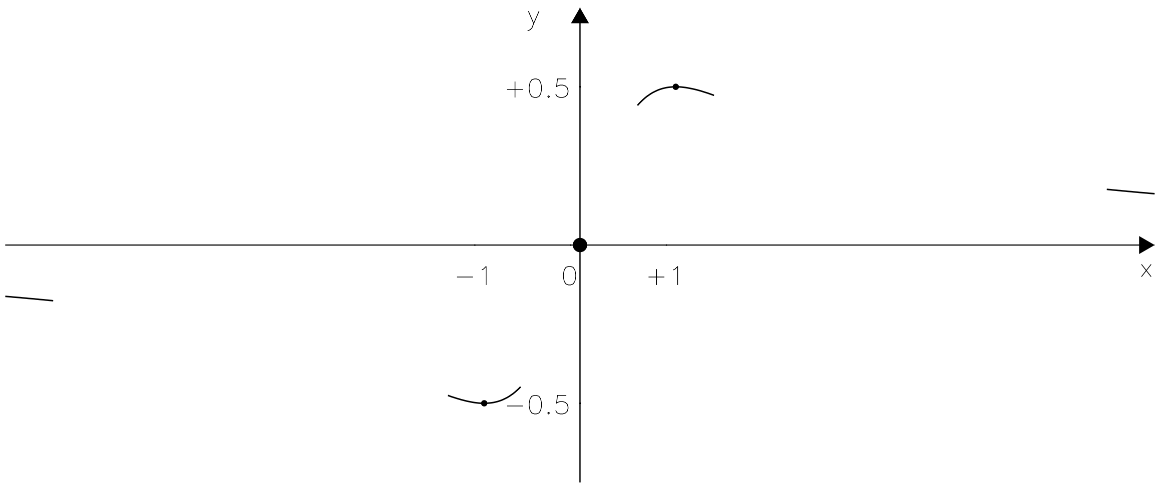
   By solving

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

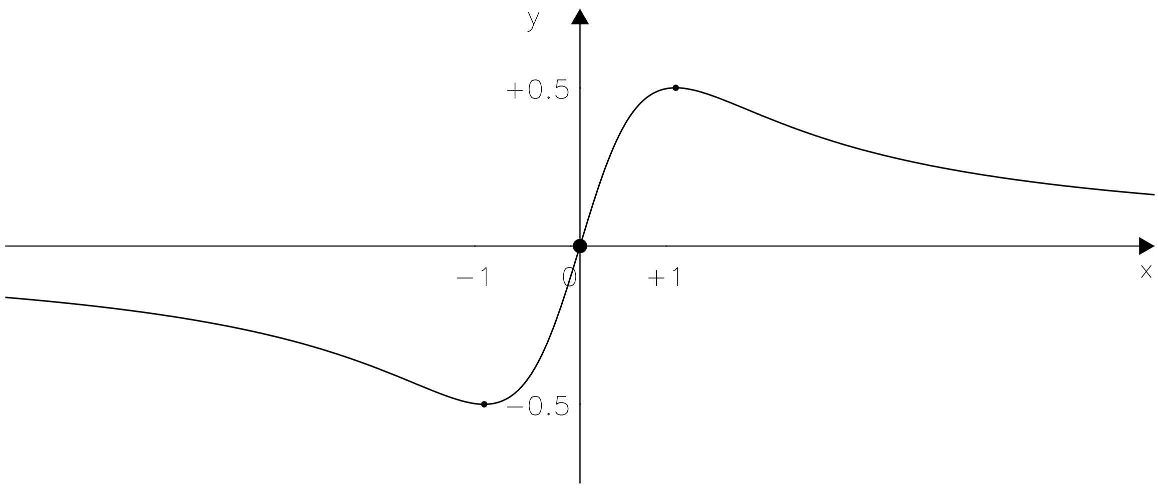
we find that the turning points are at $x = \pm 1$. By considering our graph, we can conclude that $(x, y) = (1, \frac{1}{2})$ is a maximum and $(-1, -\frac{1}{2})$ is a minimum.

Alternatively, one could use the second derivative,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{(-2x)\left(1+x^2\right)^2 - \left(1-x^2\right)\left(2\left(1+x^2\right)(2x)\right)}{\left(1+x^2\right)^4} = \frac{2x^3 - 6x}{\left(1+x^2\right)^3}.$$



4. **Graph**:



2.2 Standard Integrals

Remember the following indefinite standard integrals:

$$\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$$

$$\int \frac{1}{x} \, \mathrm{d}x = \log_e x + C$$

$$
\begin{aligned}
\int \sin x \, \mathrm{d}x &= -\cos x + C \\
\int \cos x \, \mathrm{d}x &= \sin x + C \\
\int \tan x \, \mathrm{d}x &= -\log_e |\cos x| + C \\
\int \sec^2 x \, \mathrm{d}x &= \tan x + C \\
\int e^x \, \mathrm{d}x &= e^x + C \\
\int \frac{\mathrm{d}x}{\sqrt{a^2 - x^2}} &= \sin^{-1} \frac{x}{a} + C \\
\int \frac{\mathrm{d}x}{\sqrt{a^2 + x^2}} &= \sinh^{-1} \frac{x}{a} + C \\
\int \frac{\mathrm{d}x}{a^2 + x^2} &= \frac{1}{a} \tan^{-1} \frac{x}{a} + C
\end{aligned}
$$

These standard integrals should be memorised, as they appear frequently. (N.B. No marks are given for deriving them in the exam.)

2.3 Integration by Substitution

In substitution, we try by trial and error to spot a substitution (i.e. a change of variable) that simplifies the integral and converts it to one of the standard forms (see Section 2.2).

We will consider a few examples.

Example 2.2:

$$I = \int \cos x \left(\sin x + \sin^2 x\right) \, \mathrm{d}x.$$

We try

$$u = \sin x \quad \Longrightarrow \quad \mathrm{d}u = \cos x \, \mathrm{d}x,$$

so that

$$
\begin{aligned}
I &= \int (u + u^2) \, \mathrm{d}u \\
&= \frac{u^2}{2} + \frac{u^3}{3} + C \\
&= \tfrac{1}{2} \sin^2 x + \tfrac{1}{3} \sin^3 x + C.
\end{aligned}
$$

Another form of the integral to look out for is

$$
\begin{aligned}
I &= \int \frac{f'(x)}{f(x)} \, \mathrm{d}x \\
&= \log_e |f(x)| + C.
\end{aligned}
$$

In this case we have used the substitution $u = f(x)$ so that $\mathrm{d}u = f'(x) \, \mathrm{d}x$ and remembering that

$$\int \frac{\mathrm{d}u}{u} = \log_e |u| + C.$$

Example 2.3:
$$I = \int \frac{\cos x}{\sin x} \, \mathrm{d}x.$$

If we let
$$u = \sin x,$$
then
$$\mathrm{d}u = \cos x \, \mathrm{d}x,$$
so that
$$\begin{aligned} I &= \int \frac{\mathrm{d}u}{u} = \log_e |u| + C \\ &= \log_e |\sin x| + C. \end{aligned}$$

Similarly,
$$\int \tan x \, \mathrm{d}x = \int \frac{\sin x}{\cos x} \, \mathrm{d}x = -\log_e |\cos x| + C.$$

Example 2.4:
$$I = \int x^2 e^{x^3} \, \mathrm{d}x.$$

If we let
$$u = x^3,$$
then
$$\mathrm{d}u = 3x^2 \, \mathrm{d}x,$$
so that
$$\begin{aligned} I &= \int \frac{e^u}{3} \, \mathrm{d}u = \tfrac{1}{3} e^u + C \\ &= \tfrac{1}{3} e^{x^3} + C. \end{aligned}$$

2.4 Integration by Parts

This technique uses a standard formula to reduce the integral into a form that can be:

1. Calculated,
2. Applied again.

The standard formula for integration by parts is
$$\int u \frac{\mathrm{d}v}{\mathrm{d}x} \, \mathrm{d}x = uv - \int \frac{\mathrm{d}u}{\mathrm{d}x} v \, \mathrm{d}x. \tag{2}$$

This starts with the product of two functions ($u(x)$ and $\frac{\mathrm{d}v}{\mathrm{d}x}$) and uses the derivative of the first and the integral of the second.

This may be used for integral of the form:

- $\int xf(x)\,\mathrm{d}x$, where $f(x)$ may be integrated. ($u = x$, $\dfrac{dv}{dx} = f(x)$ so $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 1$)
- $\int x^2 f(x)\,\mathrm{d}x$, where Equation 2 is applied twice. ($u = x^2$, $\dfrac{dv}{dx} = f(x)$ so $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 2x$)
- $\int f(x)\,\mathrm{d}x = \int 1 \cdot f(x)\,\mathrm{d}x$, where $f(x)$ may be differentiated.($u = f(x)$, $\dfrac{\mathrm{d}v}{\mathrm{d}x} = 1$ so $v = x$)

Example 2.5:

$$\int x\cos x\,\mathrm{d}x.$$

Letting

$$u = x \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \cos x,$$

gives

$$\frac{\mathrm{d}u}{\mathrm{d}x} = x \quad \text{and} \quad v = \sin x.$$

So

$$\begin{aligned}\int x\cos x\,\mathrm{d}x &= x\sin x - \int \sin x\,\mathrm{d}x\\ &= x\sin x + \cos x + C.\end{aligned}$$

Example 2.6:

$$\int \log_e|x|\,\mathrm{d}x.$$

Letting

$$u = \log_e|x| \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = 1,$$

gives

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x} \quad \text{and} \quad v = x.$$

So

$$\begin{aligned}\int \log_e|x|\,\mathrm{d}x &= \int 1\cdot\log_e|x|\,\mathrm{d}x\\ &= x\log_e|x| - \int \frac{x}{x}\,\mathrm{d}x\\ &= x\log_e|x| - x + C.\end{aligned}$$

2.5 Partial Fractions

Sometimes a complicated function can be split into a few simpler functions. One technique to do this is *partial fractions*.

Consider the integral,

$$\int \frac{p(x)}{(x-a_1)(x-a_2)\cdots(x-a_n)}\,\mathrm{d}x,$$

where $a_1,\ a_2,\ \ldots,\ a_n$ are distinct constants and $p(x)$ is a polynomial of degree at most $n-1$, then there exists coefficients $A_1,\ A_2,\ \ldots,\ A_n$ such that

$$\begin{aligned}\int \frac{p(x)}{(x-a_1)\,(x-a_2)\cdots(x-a_n)}\,\mathrm{d}x &= \int\left(\frac{A_1}{x-a_1}+\frac{A_2}{x-a_2}+\cdots+\frac{A_n}{x-a_n}\right)\,\mathrm{d}x \\ &= A_1\log_e|x-a_1|+A_2\log_e|x-a_2|+\cdots+A_n\log_e|x-a_n|+C.\end{aligned}$$

These coefficients ($A_1,\ A_2,\ \ldots,\ A_n$) may be determined by the "cover-up" method.

Example 2.7:

$$\int \frac{x}{x^2-2x-3}\,\mathrm{d}x$$

$$\begin{aligned}\int \frac{x}{x^2-2x-3}\,\mathrm{d}x &= \int \frac{x}{(x-3)\,(x+1)}\,\mathrm{d}x \\ &= \int\left(\frac{A}{x-3}+\frac{B}{x+1}\right)\,\mathrm{d}x \\ &= A\log_e|x-3|+B\log_e|x+1|+C.\end{aligned}$$

How do we determine A and B? Either:

I. **(The long way)**

$$\begin{aligned}\frac{x}{(x-3)(x+1)} &= \frac{A}{x-3}+\frac{B}{x+1} \\ &= \frac{A(x+1)+B(x-3)}{(x-3)(x+1)} \\ \Longrightarrow \qquad x &= A(x+1)+B(x-3) \\ \Longrightarrow \qquad x &= (A+B)x+(A-3B).\end{aligned}$$

Now, we equate the coefficients of the powers of x on each side, so:

$$\begin{array}{lcrcl} \text{coeff. } x^0 & : & 0 = A-3B & \Longrightarrow & A = 3B \\ \text{coeff. } x^1 & : & 1 = A+B & \Longrightarrow & 1 = 4B \quad \Longrightarrow \quad B = \frac{1}{4}\ ,\ A = \frac{3}{4}. \end{array}$$

II. **(The medium way)** As before,

$$\begin{aligned}\frac{x}{(x-3)(x+1)} &= \frac{A}{x-3}+\frac{B}{x+1} \\ &= \frac{A(x+1)+B(x-3)}{(x-3)(x+1)} \\ \Longrightarrow \qquad x &= A(x+1)+B(x-3)\end{aligned}$$

Now substitute in any two values of x to find A and B, for example:

- Using $x=-1$ reduces equation to $-1=-4B$, so $B=\frac{1}{4}$.
- Using $x=3$ reduces equation to $3=4A$, so $A=\frac{3}{4}$.

III. **("Cover-Up" method/The short way)** As before,

$$\frac{x}{(x-3)(x+1)} = \frac{\frac{3}{4}}{x-3}+\frac{\frac{1}{4}}{x+1}.$$

1. Determine the roots of the denominator.
2. Then substitute each value into the left-hand side, whilst "covering-up" the factor that is equal to zero.
3. This is the coefficient of the corresponding right-hand term.

Therefore,

$$\int \frac{x}{x^2-2x-3}\,\mathrm{d}x \;=\; \tfrac{3}{4}\log_e|x-3| + \tfrac{1}{4}\log_e|x+1| + C.$$

Example 2.8:

$$\int \frac{\mathrm{d}x}{a^2-x^2}$$

$$\begin{aligned}
\int \frac{\mathrm{d}x}{a^2-x^2} &= \int \frac{\mathrm{d}x}{(a-x)\,(a+x)} \\
&= \int \left(\frac{\frac{1}{2a}}{a-x} + \frac{\frac{1}{2a}}{a+x}\right)\,\mathrm{d}x \\
&= -\frac{1}{2a}\log_e|a-x| + \frac{1}{2a}\log_e|a+x| + C \\
&= \frac{1}{2a}\log_e\left|\frac{a+x}{a-x}\right| + C.
\end{aligned}$$

For the "cover-up" method to work, we need to factorise the denominator, which may be more difficult for cubics or quartics. But, we can remember that if $p(x)$ is a polynomial of degree n and we **can** find an a such that $p(a)=0$, then

$$p(x) \;=\; (x-a)q(x),$$

for some polynomial $q(x)$ of degree $n-1$.

Example 2.9:

$$\int \frac{\mathrm{d}x}{x^3-2x^2-5x+6}.$$

Our polynomial is $p(x)=x^3-2x^2-5x+6$, and we look for values of x such that $p(x)=0$. One such value is $x=1$, so $(x-1)$ is a factor.

We determine the coefficients of $q(x)$ so that $(x-1)\,q(x)$ reproduces the original expression, so

$$\begin{aligned}
p(x) &= (x-1)(x^2-x-6) \\
&= (x-1)(x+2)(x-3).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int \frac{\mathrm{d}x}{(x-1)(x+2)(x-3)} &= \int \left(\frac{-\frac{1}{6}}{x-1} + \frac{\frac{1}{15}}{x+2} + \frac{\frac{1}{10}}{x-3}\right)\,\mathrm{d}x \\
&= -\tfrac{1}{6}\log_e|x-1| + \tfrac{1}{15}\log_e|x+2| + \tfrac{1}{10}\log_e|x-3| + C
\end{aligned}$$

2.6 The Complex Exponential Function

The complex exponential function will be very useful for this module, but also for many other applications and modules. Let i be the imaginary unit, with $i^2 = -1$.

For $x \in \mathbb{R}$ Euler's formula states that

$$\exp(ix) = \cos(x) + i\sin(x).$$

("the most remarkable formula in mathematics", Richard Feynman)

We will come across cases where the argument of the exponential function is a complex number, $z = x+iy$, say. Then

$$\exp(z) = \exp(x+iy) = \exp(x)\exp(iy) = \exp(x)(\cos(y) + i\sin(y)),$$

where we have used the property $\exp(x+y) = \exp(x)\exp(y)$ in the first step.

There are many useful applications of Euler's formula. As one example we derive the double angle formulae for sin and cos using Euler's formula. Suppose $x \in \mathbb{R}$, then we have

$$\begin{aligned}
\exp(2ix) &= \cos(2x) + i\sin(2x) = \exp(ix)\exp(ix) \\
&= [\cos(x) + i\sin(x)][\cos(x) + i\sin(x)] \\
&= \cos^2(x) - \sin^2(x) + 2i\sin(x)\cos(x) \\
\Longrightarrow \quad & \cos(2x) = \cos^2(x) - \sin^2(x) \quad \text{and} \quad \sin(2x) = 2\sin(x)\cos(x).
\end{aligned}$$

In the last step we have identified the real and the imaginary parts of the equation with each other. The other forms of the double-angle formula for $\cos(2x)$ can be found using the identity $\sin^2(x)+\cos^2(x) = 1$.

We can also use Euler's formula to write down expressions for $\cos(x)$ and $\sin(x)$ in terms of the complex exponential function:

$$\cos(x) = \frac{1}{2}[\exp(ix) + \exp(-ix)], \quad \sin(x) = \frac{1}{2i}[\exp(ix) - \exp(-ix)].$$

(Compare this to the definition of the hyperbolic functions:

$$\cosh(x) = \frac{1}{2}[\exp(x) + \exp(-x)], \quad \sinh(x) = \frac{1}{2}[\exp(x) - \exp(-x)];$$

the functions are basically defined in the same way and if regarded as complex functions are (almost) identical.)

2.7 Separable First-Order ODEs

Definition: An *differential equation* is an equation involving derivatives of a function. It is an *ordinary* differential equation *(ODE)* if the function is a function of only one variable, e.g. $y(x)$. The *order* of a differential equation is the highest order of its derivatives in the equation. We consider a differential equation to be *linear* if it only has linear powers of the function (y) and its derivatives.

We consider the first-order non-linear ODE of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y),$$

where y is a function of x. Now, suppose that we can separate $f(x, y)$ into two functions $g(x)$ and $h(y)$ such that

$$f(x, y) = g(x)h(y).$$

Then

$$
\begin{aligned}
&& \frac{\mathrm{d}y}{\mathrm{d}x} &= f(x,y) = g(x)h(y) \\
\Longrightarrow && \frac{\mathrm{d}y}{h(y)} &= g(x)\,\mathrm{d}x \\
\Longrightarrow && \int \frac{\mathrm{d}y}{h(y)} &= \int g(x)\,\mathrm{d}x + C.
\end{aligned}
$$

This is the general solution of the ODE, and to determine C, we will need an initial condition, e.g. $y(0) = y_0$.

Example 2.10:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -xy^{1/3}$$

where $y(0) = 1$, $x \geq 0$ and $y \geq 0$.

Then (with $g(x) = -x$ and $h(y) = y^{1/3}$),

$$
\begin{aligned}
&& \int \frac{\mathrm{d}y}{y^{1/3}} &= \int (-x)\,\mathrm{d}x + C \\
\Longrightarrow && \frac{3}{2}y^{2/3} &= -\frac{x^2}{2} + C.
\end{aligned}
$$

Using the initial condition $y(0) = 1$, gives $C = \frac{3}{2}$, so

$$y = \left[1 - \frac{x^2}{3}\right]^{3/2}.$$

Example 2.11:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2\sqrt{1-y^2}$$

So (with $g(x) = x^2$ and $h(y) = \sqrt{1-y^2}$)

$$
\begin{aligned}
&& \int \frac{\mathrm{d}y}{\sqrt{1-y^2}} &= \int x^2\,\mathrm{d}x + C \\
\Longrightarrow && \sin^{-1} y &= \frac{x^3}{3} + C \\
\therefore && y &= \sin\left(\frac{x^3}{3} + C\right).
\end{aligned}
$$

2.8 Linear First-Order ODEs

Definition: All *linear first-order ODEs* must take the form,

$$\frac{\mathrm{d}y}{\mathrm{d}x} + f(x)y = g(x),$$

where the coefficients are functions of x.

We define the *integrating factor*,

$$I(x) = \exp\left\{\int f(x)\,\mathrm{d}x\right\},$$

so that

$$\frac{\mathrm{d}I}{\mathrm{d}x} = f(x)I(x)$$

and

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(I(x)y\right) &= I(x)\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}I}{\mathrm{d}x}y = I(x)\left(\frac{\mathrm{d}y}{\mathrm{d}x} + f(x)y\right) = I(x)g(x) \\
\implies \qquad I(x)y &= \int I(x)g(x)\,\mathrm{d}x + C \\
\implies \qquad y &= \frac{1}{I(x)}\int I(x)g(x)\,\mathrm{d}x + \frac{C}{I(x)}.
\end{aligned}$$

Example 2.12:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = x$$

Here, $f(x) = 4$, so

$$I(x) = \exp\left\{\int f(x)\,\mathrm{d}x\right\} = \exp\left\{\int 4\,\mathrm{d}x\right\} = e^{4x}.$$

Hence,

$$\begin{aligned}
y &= \frac{1}{I(x)}\int I(x)g(x)\,\mathrm{d}x + \frac{C}{I(x)} \\
&= \frac{1}{e^{4x}}\int xe^{4x}\,\mathrm{d}x + \frac{C}{e^{4x}} \\
&= \frac{1}{e^{4x}}\left(\tfrac{1}{4}xe^{4x} - \int \tfrac{1}{4}e^{4x}\,\mathrm{d}x\right) + \frac{C}{e^{4x}} \\
&= \frac{1}{e^{4x}}\left(\tfrac{1}{4}xe^{4x} - \tfrac{1}{16}e^{4x}\right) + \frac{C}{e^{4x}} \\
&= \tfrac{1}{4}x - \tfrac{1}{16} + Ce^{-4x}.
\end{aligned}$$

2.9 Second-Order ODEs

2.9.1 Homogeneous Equations

Definition: A *second-order homogeneous ODE* takes the form

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0,$$

where a, b and c are constants and $a \neq 0$. The equation is *homogeneous* as the right-hand side is zero.

To find the solution, we try

$$y = Ae^{\lambda x}$$

so that

$$a\lambda^2 Ae^{\lambda x} + b\lambda Ae^{\lambda x} + cAe^{\lambda x} = 0$$

and reduces to the *characteristic equation*,

$$a\lambda^2 + b\lambda + c \;=\; 0.$$

This is solved to give two solutions (λ_1 and λ_2) so that the *general solution* is given by one of:

(a) λ_1 and λ_2 are real and distinct,

$$y(x) \;=\; Ae^{\lambda_1 x} + Be^{\lambda_2 x}.$$

(b) $\lambda_1 = \lambda_2$ (i.e. real and repeated),

$$y(x) \;=\; (A + Bx)e^{\lambda_1 x}.$$

(c) λ_1 and λ_2 are complex conjugates (i.e. $\lambda_1 = u + vi$ and $\lambda_2 = u - vi$),

$$y(x) \;=\; e^{ux}\left(A\cos vx + B\sin vx\right).$$

Example 2.13: Find the solution of

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 3y \;=\; 0$$

when $y(0) = 0$ and $y'(0) = 1$.

First, we try $y(x) = e^{\lambda x}$, so

$$\lambda^2 + 4\lambda + 3 \;=\; 0$$

which gives $\lambda_1 = -1$, $\lambda_2 = -3$. (These are real and distinct roots.)

Therefore,

$$y(x) \;=\; Ae^{-x} + Be^{-3x}.$$

Using the boundary conditions,

$$\left.\begin{array}{rcccr} y(0) & = & 0 & = & A + \;B \\ y'(0) & = & 1 & = & -A - 3B \end{array}\right\} \implies A = \tfrac{1}{2} \quad \text{and} \quad B = -\tfrac{1}{2}.$$

So the solution is

$$y(x) \;=\; \tfrac{1}{2}e^{-x} - \tfrac{1}{2}e^{-3x}.$$

Example 2.14: Find the general solution of

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + 5y \;=\; 0$$

First, we try $y(x) = e^{\lambda x}$, so the characteristic equation is

$$\lambda^2 + 2\lambda + 5 \;=\; 0$$

which gives $\lambda_1 = -1 + 2i$, $\lambda_2 = -1 - 2i$. (These are complex conjugates.)

Therefore,

$$y(x) \;=\; e^{-x}\left(A\cos 2x + B\sin 2x\right).$$

is the general solution.

2.9.2 Inhomogeneous Equations

Definition: The *second-order inhomogeneous equation* takes the form

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = f(x).$$

The *general solution* can be written as

$$y(x) = y_c(x) + y_p(x)$$

where

- the *complementary function*, $y_c(x)$, is the general solution of the homogeneous equation

$$a\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0.$$

- and the *particular integral*, $y_p(x)$, is any solution of the inhomogeneous equation.

Example 2.15: Find the general solution of

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = e^{-2x}.$$

Complementary function: Using the homogeneous equation,

$$\begin{aligned} \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} - 4y &= 0 \\ \Longrightarrow \qquad \lambda^2 + 3\lambda - 4 &= 0, \end{aligned}$$

so $\lambda_1 = 1$ and $\lambda_2 = -4$ gives

$$y_c(x) = Ae^{x} + Be^{-4x}.$$

Particular Integral: We try a solution of the form $y_p = Ae^{-2x}$ where A is unknown:

$$\begin{aligned} 4Ae^{-2x} - 6Ae^{-2x} - 4Ae^{-2x} &= e^{-2x} \\ \Longrightarrow \qquad A &= -\tfrac{1}{6}. \end{aligned}$$

So

$$y_p(x) = -\tfrac{1}{6}e^{-2x}.$$

Adding together gives the general solution

$$y(x) = Ae^{x} + Be^{-4x} - \tfrac{1}{6}e^{-2x}.$$

Other possible forms we may need for the particular integral are:

$f(x)$	Try $y_p(x)$
e^{kx}	Ae^{kx}
$a_0 + a_1x + a_2x^2 + \ldots a_nx^n$, ($n$ integer)	$A_nx^n + \cdots + A_1x + A_0$
$a\sin(kx) + b\cos(kx)$	$A\cos(kx) + B\sin(kx)$
$a\sinh(kx) + b\cosh(kx)$	$A\cosh(kx) + B\sinh(kx)$
any linear combination of the functions above	a linear combination of the functions above

2.10 Taylor Series

Taylor series (Taylor expansion) are often used to approximate functions of one or two variables in the vicinity of a given point x_0 (or x_0, y_0 in the case of two variables).

2.10.1 Functions of One Variable

The Taylor series of a function $f(x)$ about $x = x_0$ is defined as

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \ldots = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

The Taylor series with remainder the is given by

$$f(x) = P_n(x) + R_n(x),$$

where the Taylor polynomial $P_n(x)$ is given the Taylor series up to power n,

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

and the remainder term $R_n(x)$ is defined as

$$R_(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

Here c is an unknown real number with a value between x and x_0.

The remainder term can be used for estimating the error made by replacing the exact function by a Taylor series approximation.

2.10.2 Functions of Two Variables

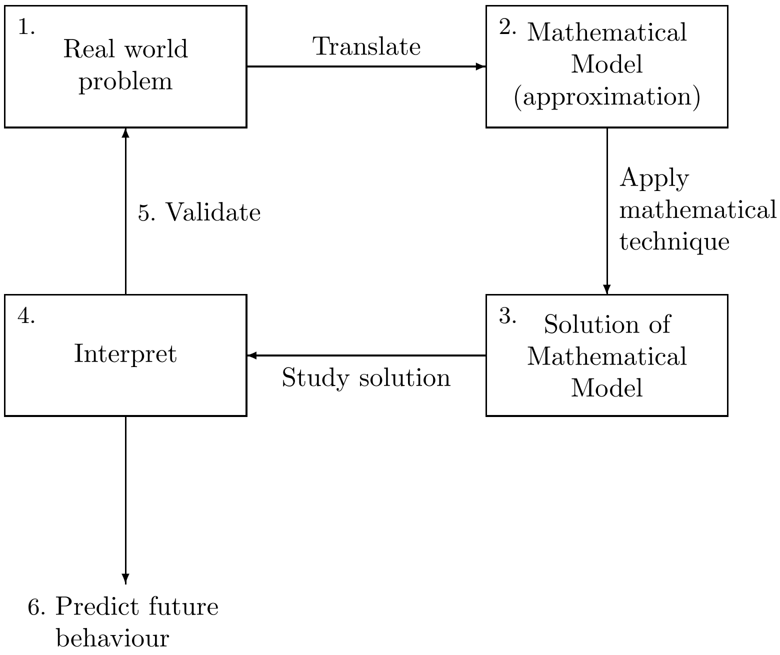
The first three terms of the Taylor series of a function $f(x, y)$ about $(x, y) = (x_0, y_0)$ are given by

$$\begin{aligned} f(x,y) &= f(x_0, y_0) + \left(\frac{\partial f}{\partial x}\right)_{(x_0,y_0)} (x - x_0) + \left(\frac{\partial f}{\partial y}\right)_{(x_0,y_0)} (y - y_0) \\ &\quad + \frac{1}{2!}\left[\left(\frac{\partial^2 f}{\partial x^2}\right)_{(x_0,y_0)} (x - x_0)^2 + 2\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(x_0,y_0)} (x - x_0)(y - y_0) + \left(\frac{\partial^2 f}{\partial y^2}\right)_{(x_0,y_0)} (y - y_0)^2\right] + \ldots \end{aligned}$$

3 Simple Mathematical Models

3.1 Introduction

Definition: When a real world problem (in non-mathematical terms) is **restated** in mathematical terms, the resulting description is a *mathematical model*.



To create the mathematical model (step 2), we need to make assumptions on the observed behaviour, e.g.

$$\frac{\mathrm{d}N}{\mathrm{d}t} \propto N,$$

that ensure the key features of the problem are modelled, as it is normally too complicated to identically reproduce the problem. All of the models we will study will involve differential equations, normally of function of space or time, e.g.

$$y(x) \quad \longrightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} \quad , \quad N(t) \quad \longrightarrow \quad \frac{\mathrm{d}N}{\mathrm{d}t}.$$

3.2 Decay Processes

Suppose the rate of decay of a substance is proportional to a power of the amount $y(t)$ of the substance present,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -ky^p, \tag{3}$$

where power $p > 0$, decay rate $k > 0$ and amount $y > 0$.

3.2.1 Radioactive Decay

Definition: For a *radioactive substance* (isotope), the rate of decay is directly proportional to the amount of the substance (N), i.e.

$$\frac{\mathrm{d}N}{\mathrm{d}t} \propto -N$$

or

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -kN \tag{4}$$

where $N(t) > 0$ is the mass of substance present after t days and $k > 0$ is the *decay rate*.

By separating the variables,

$$\begin{aligned} & \frac{\mathrm{d}N}{N} &=& -k\,\mathrm{d}t \\ \Longrightarrow \quad & \log_e |N| &=& -kt + C \\ \therefore \quad & N &=& Ae^{-kt}, \end{aligned}$$

where $A = e^C$ is a constant.

There are two cases, we need to consider:

- k given: One unknown, so we need one observation (or boundary condition) to determine $N(t)$.
- k **not** given: Two unknowns, so we need two observations to determine $N(t)$.

Example 3.1: Suppose there is initially 100 mg of Americium-241 in a smoke detector and that it decays to 85.2 mg in a century (100 years). Determine A and k.

There are two observations (in milligrammes):

$$\begin{aligned} N(0) &= 100, \\ N(100) &= 85.2. \end{aligned}$$

Using $N(0) = 100$ in Equation 4 gives

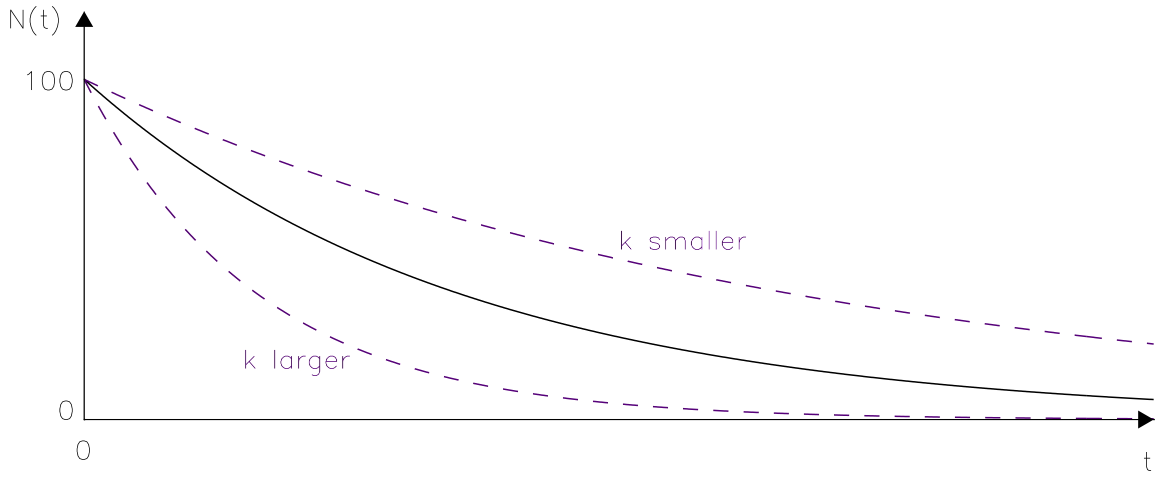
$$\begin{aligned} & 100 &=& Ae^0 \\ \therefore \quad & A &=& 100, \end{aligned}$$

and if we measure time in years,

$$\begin{aligned} & 85.2 &=& 100e^{-100k} \\ \Longrightarrow \quad & e^{-100k} &=& \tfrac{85.2}{100}, \\ \therefore \quad & k &=& -\tfrac{1}{100}\log_e 0.852 \quad \approx \quad 0.00160\,\mathrm{year}^{-1} \end{aligned}$$

So,

$$N(t) = 100e^{-0.00160t}.$$



Definition: An important feature of a decay process is its *half-life*, the time (τ) for the substance to decay to half its original size.

So if the original amount of substance is $N(0) = N_0$ then τ is given by

$$\begin{aligned} & & \tfrac{1}{2}N_0 &= N_0 e^{-k\tau} \\ \Longrightarrow & & e^{-k\tau} &= \frac{1}{2} \\ \therefore & & \tau &= \frac{1}{k}\log_e 2. \end{aligned}$$

So in our previous example,

$$\tau \approx 432.8\,\text{years.}$$

We should note that

- The half-life τ depends only on k and, thus, is independent of the initial amount of material.
- When the decay is proportional to N^1 it will take an infinite length of time before $N = 0$.

3.2.2 Decay of a Contaminant

Suppose a contaminant in the soil is decaying (or biodegrading) at a rate

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -ky^{1-q}, \tag{5}$$

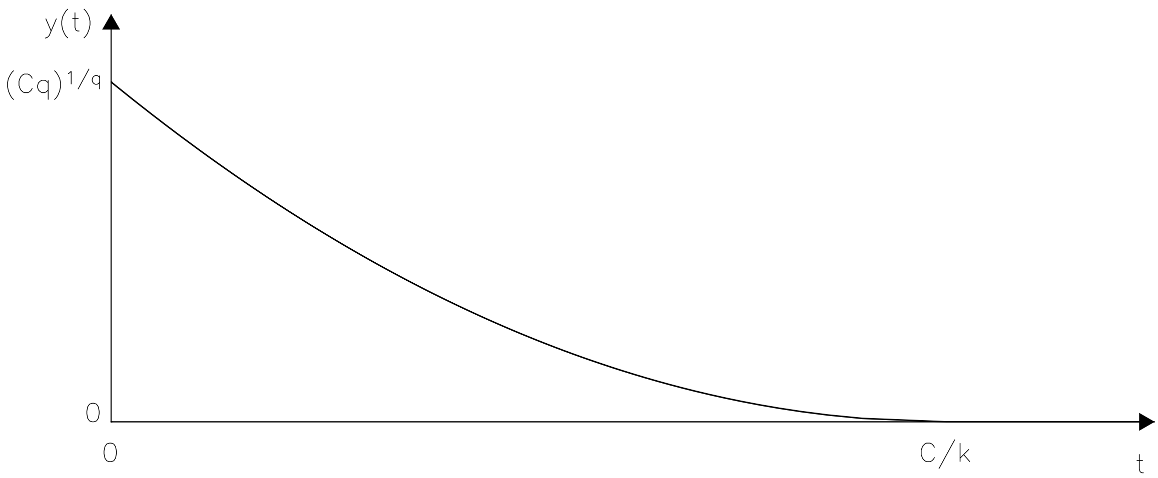
where $0 < q < 1$.

Using separation of variables,

$$\begin{aligned} & & \int y^{q-1}\,\mathrm{d}y &= -\int k\,\mathrm{d}t + C \\ \Longrightarrow & & \frac{y^q}{q} &= -kt + C \\ \therefore & & y &= q^{1/q}\left[C - kt\right]^{1/q} \end{aligned}$$

so $y = 0$ when

$$t = \frac{C}{k}.$$



Example 3.2: Determine the time for a contaminant to disappear, given that $q = \frac{1}{2}$, $y(0) = 100\,\text{kg}$ and that it decays to $81\,\text{kg}$ in $1000\,\text{days}$.

We know that $q = \frac{1}{2}$ and that

$$\begin{aligned} y(0) &= 100, \\ y(1000) &= 81, \end{aligned}$$

where $y(t)$ is in kilogrammes. Then using Equation 5,

$$y(t) = q^{1/q}\left[C - kt\right]^{1/q},$$

at $t = 0$:

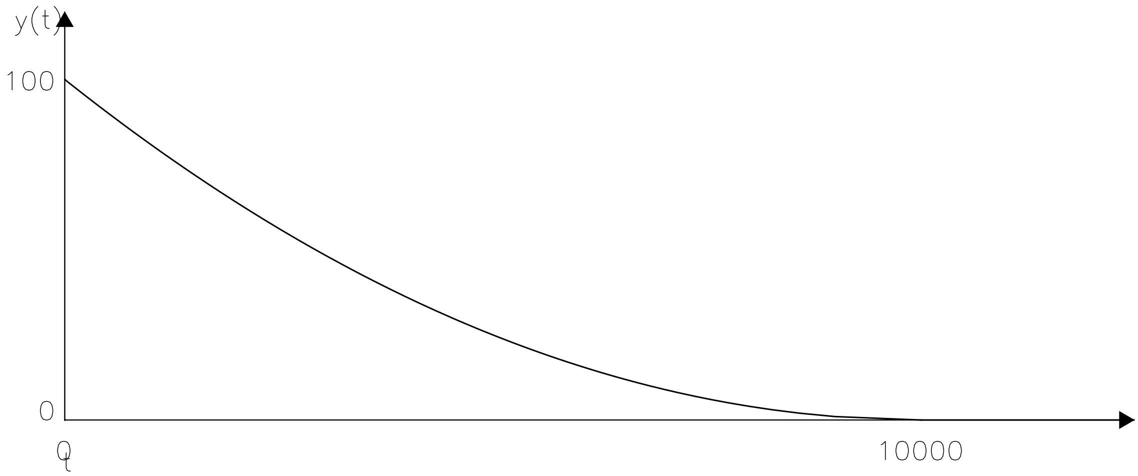
$$\begin{aligned} y(0) = 100 &= \left(\tfrac{1}{2}\right)^2 C^{\,2} \\ \Longrightarrow \qquad C &= 20. \end{aligned}$$

At $t = 1000$,

$$\begin{aligned} y(1000) = 81 &= \left(\tfrac{1}{2}\right)^2 (20 - 1000k)^2 \\ \Longrightarrow \qquad 20 - 1000k &= 18 \\ k &= \frac{2}{1000}\,\text{day}^{-1}. \end{aligned}$$

Now the contaminant will disappear $(y(t) = 0)$when

$$t = \frac{C}{k} = \frac{20}{\frac{2}{1000}} = 10,000\,\text{days} \approx 27.4\,\text{years}.$$



3.2.3 Newton's Law of Cooling

Definition: *Newton's law of cooling* is

'The instantaneous rate of change, of the temperature of an object, is directly proportional to the difference between its own temperature and that of the ambient surroundings."

$T(t)$ $T_a(t)$

In other words, the temperature, $T(t)$, of an object changes at a rate α to the difference $(T(t) - T_a(t))$ in the temperature of the object and its surroundings.

Thus,

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -k\,(T(t) - T_a(t))\,, \tag{6}$$

where $k > 0$ is a constant. If $T_a(t)$ is given, then we have a linear differential equation of $T(t)$.

We can see that

- $T(t) > T_a(t)$ then $\dfrac{\mathrm{d}T}{\mathrm{d}t} < 0$, so object is cooled.
- $T(t) < T_a(t)$ then $\dfrac{\mathrm{d}T}{\mathrm{d}t} > 0$, so object is heated.

Example 3.3: Find $T(t)$ for a cup of coffee initially with a temperature of $80\,^\circ$C at $t = 0$ in a room of temperature $20\,^\circ$C.

Using Equation 6,

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -k\,(T - T_a)$$

with $T(t)$ in Celsius and $T_a(t) = 20$ gives

$$\begin{aligned} \frac{\mathrm{d}T}{\mathrm{d}t} &= -k\,(T - 20) \\ \Longrightarrow \qquad \frac{\mathrm{d}T}{\mathrm{d}t} + kT &= -20k. \end{aligned}$$

Then, the integrating factor is

$$I(t) = \exp\left\{\int k \;\mathrm{d}t\right\} = e^{kt},$$

so

$$\begin{aligned} T(t) &= \frac{1}{I(t)} \int I(t)(20k)\;\mathrm{d}t + \frac{C}{I(t)} \\ &= e^{-kt} \int 20ke^{kt}\;\mathrm{d}t + \frac{C}{e^{kt}} \qquad (7) \\ &= e^{-kt}(20e^{kt}) + Ce^{-kt} \\ &= 20 + Ce^{-kt}. \end{aligned}$$

Inserting $T = 80$ when $t = 0$ gives the constant $C = 80 - 20 = 60$, so that

$$T(t) = 20 + (80 - 20)e^{-kt} = 20 + 60e^{-kt}.$$

We note that

- If we were to repeat with any arbitrary constant T_a and initial temperature $T(0) = T_0$ then the general solution to the problem is

$$T(t) = T_a + (T_0 - T_a)\,e^{-kt}.$$

- If k has not been given, then we'll need a further measurement, say $T(t_1) = T_1$ at $t = t_1 > 0$, during the cooling process. Then

$$\begin{aligned} T(t_1) = T_1 &= T_a + (T_0 - T_a)\,e^{-kt_1} \\ \Longrightarrow \qquad e^{-kt_1} &= \frac{T_1 - T_a}{T_0 - T_a} \\ \therefore \qquad k &= -\frac{1}{t_1}\log_e\left|\frac{T_1 - T_a}{T_0 - T_a}\right|. \end{aligned}$$

- If $T_a = T_a(t)$ is not constant, then we'll need to evaluate

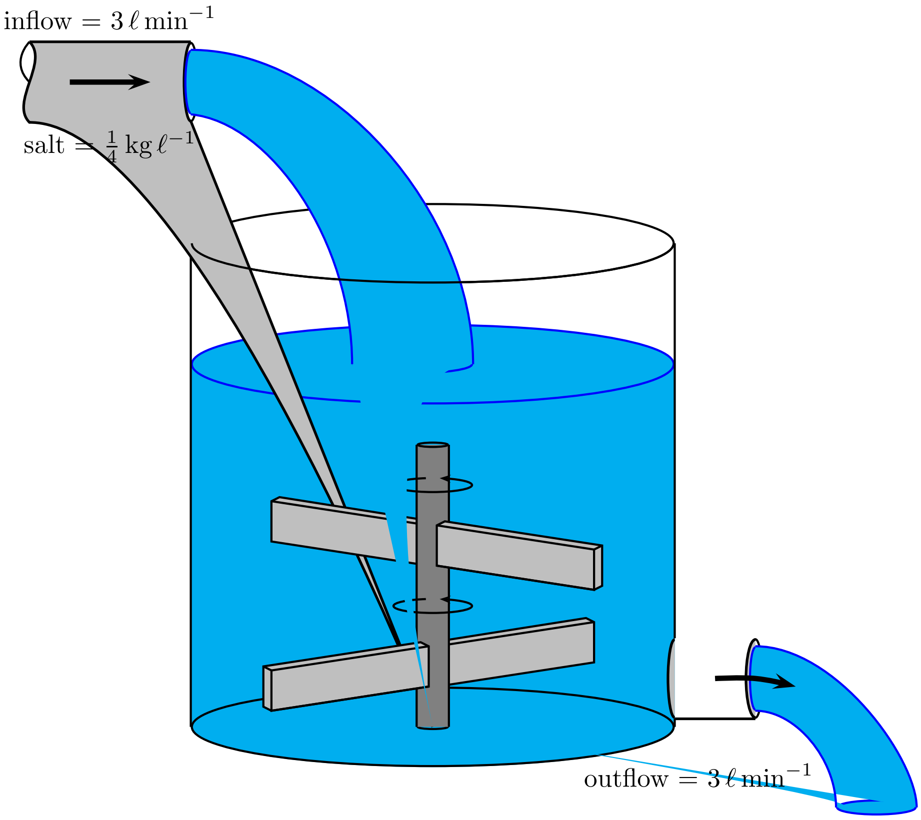
$$\int kT_a(t)e^{kt}\;\mathrm{d}t$$

at Equation 7 of the solution.

3.3 Mixing Problem

Example 3.4:

At $t = 0$, a tank contains Q_0 kg of salt dissolved in 100 litres of water. Water containing $\frac{1}{4}$ kg of salt per litre enters the tank at a rate of 3 litres min^{-1}. A well-stirred solution leaves the tank at the same rate, so the volume of water remains constant. Find an expression for the amount of salt, $Q(t)$ kg, in the tank at time t.



The important information is that at $t = 0$: Q_0 kg dissolved in 100 litres of water. Throughout:

- $\frac{1}{4}$ kg of salt per litre enters at 3 litres per minute.
- A well stirred solution also leaves at 3 litres per minute.
- Therefore, volume of water remains constant.

We then build the equation from

$$\underbrace{\left\{\begin{array}{c}\text{Rate of change}\\ \text{of salt in tank}\end{array}\right\}}_{\frac{\mathrm{d}Q}{\mathrm{d}t}\,\mathrm{kg\,min}^{-1}} = \underbrace{\left\{\begin{array}{c}\text{Rate that salt}\\ \text{enters}\end{array}\right\}}_{\frac{1}{4}\mathrm{kg}\,\ell^{-1}\times 3\ell\,\mathrm{min}^{-1}} - \underbrace{\left\{\begin{array}{c}\text{Rate that salt}\\ \text{leaves}\end{array}\right\}}_{\frac{Q(t)\,\mathrm{kg}}{100\,\ell}\times 3\,\ell\,\mathrm{min}^{-1}}$$

So,

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \tfrac{3}{4} - \tfrac{3}{100}Q(t) \qquad (8)$$

$$\Longrightarrow \qquad \frac{\mathrm{d}Q}{\mathrm{d}t} + \tfrac{3}{100}Q(t) \;=\; \tfrac{3}{4}.$$

Using the integrating factor,

$$I(t) \;=\; \exp\left\{\int \tfrac{3}{100}\,\mathrm{d}t\right\} \;=\; e^{0.03t}$$

gives

$$\begin{aligned} Q(t) &= \frac{1}{I(t)}\int g(t)I(t)\,\mathrm{d}t + \frac{C}{I(t)} \\ &= e^{-0.03t}\int \tfrac{3}{4}e^{0.03t}\,\mathrm{d}t + Ce^{-0.03t} \\ &= e^{-0.03t} \times \tfrac{3}{4}\cdot\tfrac{1}{0.03}e^{0.03t} + Ce^{-0.03t} \\ &= 25 + Ce^{-0.03t} \end{aligned}$$

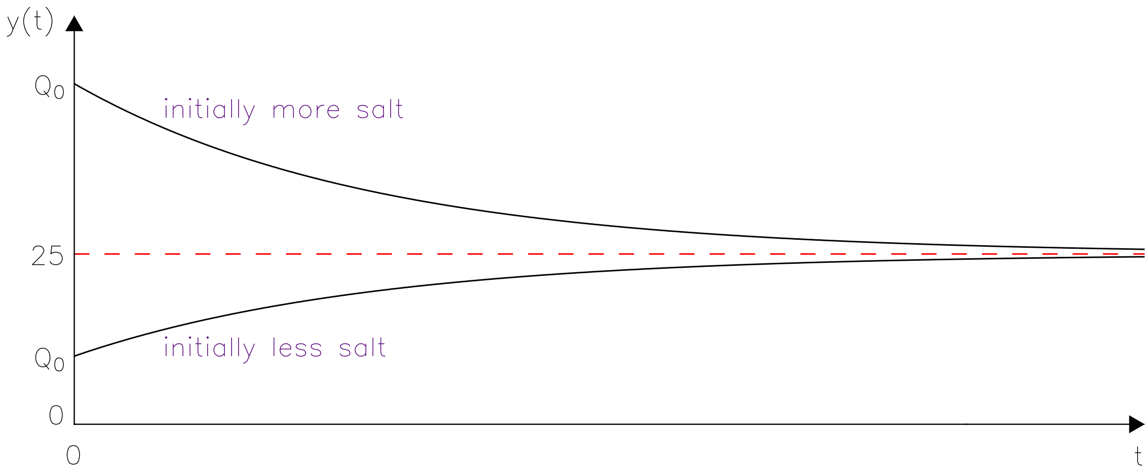
If we now apply the initial condition $Q(0) = Q_0$ then

$$C \;=\; Q_0 - 25.$$

Therefore,

$$\begin{aligned} Q(t) &= 25 + (Q_0 - 25)\,e^{-0.03t} \\ Q(t) &= \underbrace{25\left(1 - e^{-0.03t}\right)}_{\substack{\text{amount of salt}\\ \text{at time } t \text{ due to}\\ \text{flow process}}} \quad + \quad \underbrace{Q_0 e^{-0.03t}}_{\substack{\text{amount of}\\ \text{original salt at}\\ \text{time } t}} \quad . \end{aligned}$$

Finally, note that as $t \to \infty$, $Q(t) \to 25\,\mathrm{kg}$.



3.4 Population Dynamics

In biological mathematics (or medicine, ecology or economics) it is desirable to predict the future growth or decline of a population, $N(t)$ of a species, for example insects, bacteria, mammals or people. In this section, we will discuss models for population growth or decline.

3.4.1 Exponential Growth

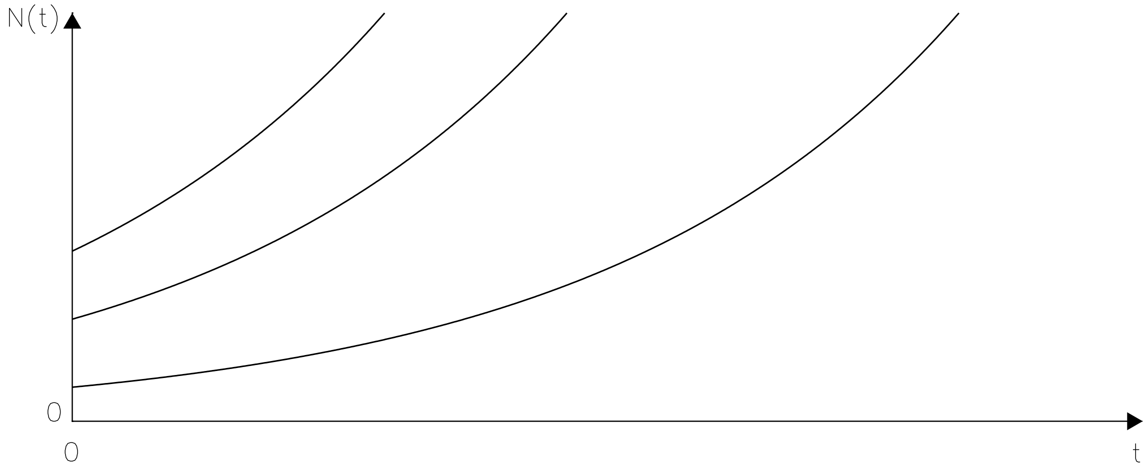
Definition: The simplest hypothesis regarding $N(t)$ is that the rate of change of $N(t)$ is proportional to its current value,

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN,$$

where $r > 0$ is the natural *growth rate* and $r < 0$ is the *decline rate*. This called *exponential growth/decay*.

If $N(0) = N_0$ then

$$N(t) = N_0 e^{rt}. \tag{9}$$



So when $r > 0$, growth continues for all time so $N(t) \to \infty$ as $t \to \infty$.

In general, growth cannot continue indefinitely due to limited resources which will eventually reduce the growth rate. So we will need a more realistic model. For example, the population of rabbits on North Haugh will now grow indefinitely due to limited food and space.

3.4.2 Logistic Growth

Consider models of the form

$$\frac{\mathrm{d}N}{\mathrm{d}t} = R(N)N$$

where the growth rate R is a function of N. We now chose $R(N)$ such that

$$\begin{array}{lllll} R(N) & \approx & r > 0 & : & \text{when } N \text{ is small..} \\ R(N) & > & 0 & : & \text{and decreases as } N \text{ increases.} \\ R(N) & < & 0 & : & \text{when } N \text{ is very large} \end{array}$$

The simplest from for $R(N)$ to take is

$$R(N) = r - aN,$$

where $r > 0$ is the natural growth rate and $a > 0$ is the inhibition of growth when population is large, so

$$\frac{\mathrm{d}N}{\mathrm{d}t} = (r - aN)\, N. \tag{10}$$

We will solve this both mathematically and graphically.

Mathematical Solution

Assuming $N > 0$, $r - aN > 0$ and then separating Equation 10 gives

$$\begin{aligned}
& \int \frac{\mathrm{d}N}{(r-aN)N} &=& \int \mathrm{d}t \\
\Longrightarrow \quad & \int \left\{ \frac{ar^{-1}}{r-aN} + \frac{r^{-1}}{N} \right\} \mathrm{d}N &=& t + C \\
\Longrightarrow \quad & -\frac{1}{r}\log_e |r - aN| + \frac{1}{r}\log_e |N| &=& t + C \\
\Longrightarrow \quad & \frac{1}{r}\log_e \left| \frac{N}{r-aN} \right| &=& t + C \\
\Longrightarrow \quad & \log_e \left| \frac{N}{r-aN} \right| &=& rt + rC \\
\Longrightarrow \quad & \frac{N}{r-aN} &=& Ae^{rt} \\
\Longrightarrow \quad & N &=& (r-aN)\, Ae^{rt} \\
\Longrightarrow \quad & N\left(1 + aAe^{rt}\right) &=& rAe^{rt} \\
\therefore \quad & N &=& \frac{rAe^{rt}}{1 + aAe^{rt}} \\
& &=& \frac{rA}{aA + e^{-rt}},
\end{aligned}$$

where $A = e^{rC} = \text{const.}$. This solution leads to a number of properties:

I. When the initial condition, $N(0) = N_0$ is used,

$$A = \frac{N_0}{r - aN_0}.$$

II. When $N \ll \frac{r}{a}$ (or $aN \ll r$),

$$N(t) \approx N_0 e^{rt},$$

and hence grows exponentially.

III. Any value for which $\frac{\mathrm{d}N}{\mathrm{d}t} = 0$ is called a *steady state*. Using Equation 10 this occurs when either

$$N = 0 \qquad \text{or} \qquad N = \frac{r}{a}.$$

IV. As $T \longrightarrow \infty$, we find that

$$N(t) \longrightarrow \frac{r}{a}.$$

This value is independent of the initial value N_0 and is called the *saturation level*.

V. When $rt \gg 1$ we can write,

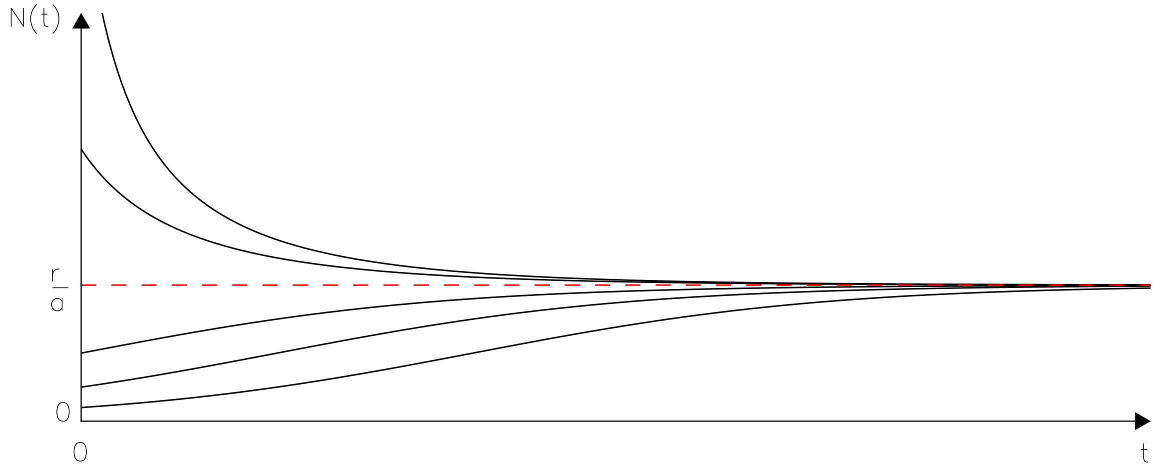
$$N = \frac{\frac{r}{a}}{1 + \frac{1}{aA}e^{-rt}} = \frac{r}{a} \cdot \left(1 + \frac{1}{aA}e^{-rt}\right)^{-1},$$

and if we use $(1+x)^{-1} = 1 - x + \cdots$ (provided $|x| \ll 1$) then

$$N = \frac{\frac{r}{a}}{1 + \frac{1}{aA}e^{-rt}} = \frac{r}{a} \cdot \left\{1 - \frac{1}{aA}e^{-rt} + \cdots \right\}.$$

So the steady state of $\frac{r}{a}$ is approached by a **negative exponential**, and is **stable**.

VI. Graph:



Graphical Solution

Remember that the Logistic Equation is

$$\frac{\mathrm{d}N}{\mathrm{d}t} = (r - aN)\,N. \tag{10}$$

We consider the graph of $\frac{\mathrm{d}N}{\mathrm{d}t}$ against N:

I. **Steady States:**

$$\frac{\mathrm{d}N}{\mathrm{d}t} = 0 \implies N \in \left\{0, \frac{r}{a}\right\}$$

II. Consider the behaviour between steady states:

- If $0 < N < \frac{r}{a}$, then $\frac{\mathrm{d}N}{\mathrm{d}t} > 0$, so N is **increasing**.
- If $N > \frac{r}{a}$, then $\frac{\mathrm{d}N}{\mathrm{d}t} < 0$, so N is **decreasing**.

III. Consider what happens at each steady state?

(i) Near $N = 0$ Let

$$N = N_{ss} + N_1(t),$$

where N_{ss} is the value of N at the steady state, and $N_1(t)$ is small. Then Equation 10 implies that

$$\begin{aligned}
\frac{\mathrm{d}N}{\mathrm{d}t} &= \frac{\mathrm{d}(N_{ss}+N_1)}{\mathrm{d}t} = \frac{\mathrm{d}N_1}{\mathrm{d}t} = (r - aN_1)\,N_1 = rN_1 - aN_1^2 \approx rN_1. \\
\implies \quad \frac{\mathrm{d}N_1}{\mathrm{d}t} &\approx rN_1. \\
\therefore \quad N_1 &\approx e^{rt},
\end{aligned}$$

so solution increases exponentially away from the steady point. For this question, we have linearised the expression on the right-hand (i.e. neglected squares and higher powers) side since $N_1 \gg N_1^2 \gg N_1^3 \gg \cdots$, because we assumed N_1 to be small (i.e. close to the steady state).

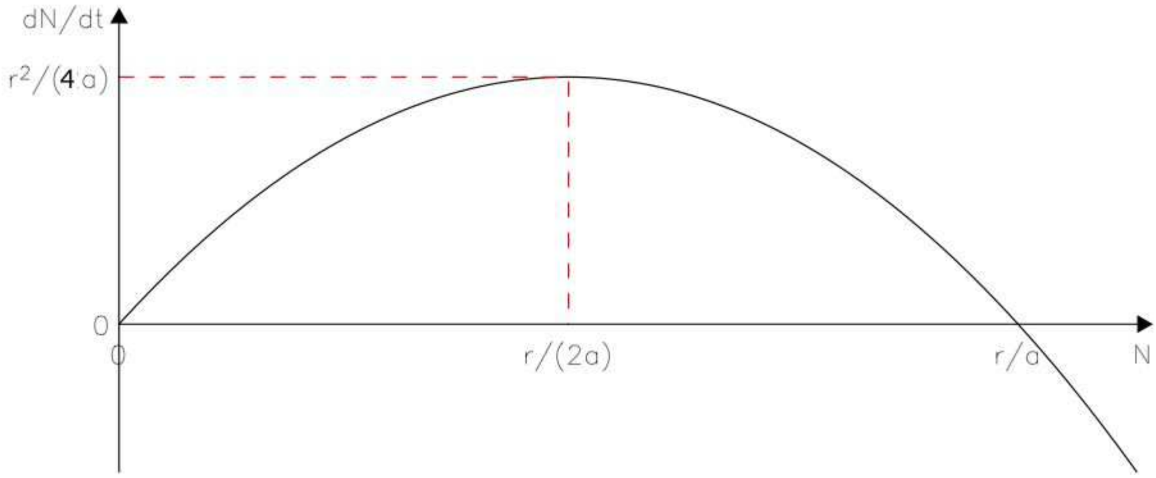
(ii) Near $N = \frac{r}{a}$: We write

$$N = N_{ss} + N_1(t) = \frac{r}{a} + N_1(t)$$

where $N_1 \ll \frac{r}{a}$. So, Equation 10 becomes

$$\begin{aligned}
\frac{\mathrm{d}N_1}{\mathrm{d}t} &= \left(r - a\left\{\frac{r}{a} + N_1\right\}\right)\left(\frac{r}{a} + N_1\right) \\
&= -aN_1\left(\frac{r}{a} + N_1\right) \\
&= -rN_1 - aN_1^2 \\
&\approx -rN_1 \\
\therefore \qquad N_1 &\approx e^{-rt}.
\end{aligned}$$

So as $t \longrightarrow \infty$, $N_1 \longrightarrow 0$ and $N \longrightarrow \frac{r}{a}$. Therefore, this is a **stable** steady state.



3.4.3 Linear Stability Analysis of Steady States of General 1st Order ODEs

The linear stability analysis we just carried out can be also be carried out in a general way for 1st order ODEs of the type

$$\frac{dN}{dt} = F(N), \tag{11}$$

with $F(N)$ just a function of N. Obviously the logistic equation is one example for this type of ODE.

Steady states N_{ss} of (11) are given by the solutions of

$$F(N_{ss}) = 0.$$

Note that this equation can have multiple solutions and you need to carry out a linear stability analysis for each of the solutions.

We want to see how small perturbations of a steady state N_{ss} develop in time. If the perturbation decays in time the steady is stable and if the perturbation grows in time the steady state is unstable. Let us start by setting

$$N(t) = N_{ss} + N_1(t),$$

as for the logistic equation. Then

$$\frac{dN}{dt} = \frac{dN_{ss}}{dt} + \frac{dN_1}{dt} = \frac{dN_1}{dt},$$

because a steady state does not depend on time, hence the time derivative vanishes. Substituting this and our expression for $N(t)$ into the ODE, we obtain

$$\frac{dN_1}{dt} = F(N_{ss} + N_1(t)).$$

So far, no approximations have been made, but now we assume that $N_1(t)$ is small and we Taylor expand the right hand side of the ODE about the steady state N_{ss}:

$$\frac{dN_1}{dt} = F(N_{ss} + N_1(t)) \approx F(N_{ss}) + \left(\frac{dF}{dN}\right)_{N=N_{ss}} N_1 + \ldots = \left(\frac{dF}{dN}\right)_{N=N_{ss}} N_1 + \ldots$$

In the last step we have used that steady states satisfy $F(N_{ss}) = 0$. The derivative of F is evaluated at the steady state N_{ss} and hence is a constant. So, the (small) perturbation $N_1(t)$ satisfies the linear ODE (hence the name linearisation)

$$\frac{dN_1}{dt} = \left(\frac{dF}{dN}\right)_{N=N_{ss}} N_1,$$

with solution

$$N_1(t) = N_{10} \exp\left[\left(\frac{dF}{dN}\right)_{N=N_{ss}} t\right],$$

where $N_{10} = N_1(0)$ is the initial value of N_1 at $t = 0$.

We now see that we have a solution $N_1(t)$ which is exponentially growing in time if $\left(\frac{dF}{dN}\right)_{N=N_{ss}} > 0$ and this means that such a steady state N_{ss} is unstable (solutions starting close to N_{ss} move away from N_{ss}). On the other hand, if $\left(\frac{dF}{dN}\right)_{N=N_{ss}} < 0$, the steady state N_{ss} is stable, because the perturbation $N_1(t)$ decreases exponentially with time and solutions starting close to N_{ss} approach the steady state.

We also see that this result matches exactly with what we saw in the graph representing the right hand side of the logistic equation: there, the steady state for which $F(N)$ crossed the N-axis with a positive slope ($N = 0$) was found be unstable and the steady state for which $F(N)$ crossed the N-axis with a negative slope ($N = r/a$) was found to be stable. Our calculation above shows that this is actually not a coincidence, but generally true for 1st order ODEs of this type.

3.5 Two Species Model

3.5.1 Competing Species (CS) Models

Consider two competing species (x and y) which compete for a limited food supply. Suppose each species, in the absence of the other is governed by the logistic equation,

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= xR_1(x) = x\,(a_1 - b_1 x)\,, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= yR_2(y) = y\,(a_2 - b_2 y)\,, \end{aligned}$$

where a_1 and a_2 are the natural growth rates and $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$ are the saturation levels of the two species.

When both species are present, they compete for the available food which inhibits $R_1(x)$ and $R_2(y)$. The simplest way to include this is to change R_1 and R_2 to

$$\begin{aligned} R_1(x,y) &= a_1 - b_1 x - c_1 y, \\ R_2(x,y) &= a_2 - b_2 y - c_2 x \end{aligned}$$

where a_1, a_2, b_1, b_2, c_1, c_2 are positive. The addition term $c_1 y$ is the effect on species y on the growth of species x, and similarly $c_2 x$ is the effect on species x on the growth of species y.

Thus,

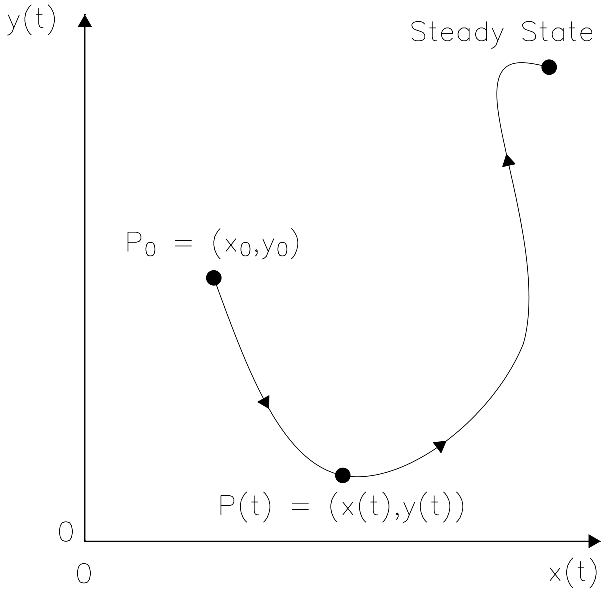
$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(\underbrace{a_1}_{\substack{\text{natural}\\ \text{growth of } x}} - \underbrace{b_1 x}_{\substack{\text{inhibition of } x\\ \text{by itself}}} - \underbrace{c_1 y}_{\substack{\text{inhibition of } x\\ \text{by } y}}) \tag{12}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = x(\underbrace{a_2}_{\substack{\text{natural}\\ \text{growth of } y}} - \underbrace{b_2 y}_{\substack{\text{inhibition of } y\\ \text{by itself}}} - \underbrace{c_2 x}_{\substack{\text{inhibition of } y\\ \text{by } x}}) \tag{13}$$

(14)

This system is very difficult to solve, so normally we only consider using graphical solutions.

We first introduce parametrised plots of $(x(t), y(t))$.



We start at a point $P_0 = (x_0, y_y)$ and consider how the it moves in the x–y plane. A line is draw along using the function of position

$$P(t) = (x(t), y(t))$$

so that the position in x and y are both functions of t (time). This line continues until it reaches a steady state: a point at which $P(t)$ remains the same for the rest of time. This occurs when

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0 \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = 0.$$

But there may be more than one steady state, so we have to determine which one any initial starting point (P_0) will head to. (This may not be the same for different starting points.)

3.5.2 Graphical Solving Method

For this method we shall use four distinct stages:

I. Identify **steady states**.

II. Deduce behaviour on **axes**.

III. **Analyse** each steady state and determine their **stability**.

IV. **Draw** better sketch.

V. Deduce long-term behaviour as $t \longrightarrow \infty$.

Identify steady states: There are four steady states of Equations 12 and 13. They exist when

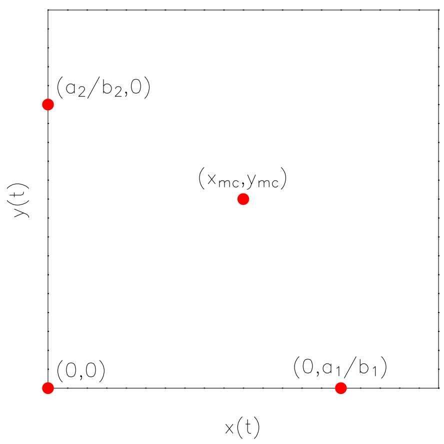
$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} = 0 \quad &\Longrightarrow \quad x\,(a_1 - b_1 x - c_1 y) = 0 \\ &\Longrightarrow \quad \text{(i)} \quad x = 0 \quad \text{or} \quad \text{(ii)} \quad a_1 - b_1 x - c_1 y = 0.\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} = 0 \quad &\Longrightarrow \quad y\,(a_2 - b_2 y - c_2 x) = 0 \\ &\Longrightarrow \quad \text{(iii)} \quad y = 0 \quad \text{or} \quad \text{(iv)} \quad a_2 - b_2 y - c_2 x = 0.\end{aligned}$$

We consider the four options:

- (i) and (iii):
  $$x = 0 \quad \text{and} \quad y = 0,$$
  so both die out.
- (i) and (iv):
  $$x = 0 \quad \text{and} \quad y = \frac{a_2}{b_2},$$
  So only x dies out.
- (ii) and (iii):
  $$y = 0 \quad \text{and} \quad x = \frac{a_1}{b_1},$$
  So only y die out.
- (ii) and (iv):
  $$y = \frac{1}{c_1}(a_1 - b_1 x) \quad \text{and} \quad y = \frac{1}{b_2}(a_2 - c_2 x)$$
  $$\Longrightarrow \quad x = x_{mc} = \frac{a_1 b_2 - a_2 c_1}{b_1 b_2 - c_1 c_2}$$
  $$\text{or} \quad y = y_{mc} = \frac{a_2 b_1 - a_1 c_2}{b_1 b_2 - c_1 c_2}.$$
  So both mutually coexist.



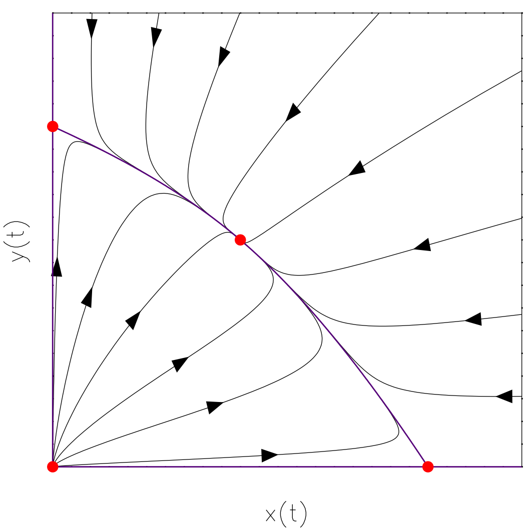
Behaviour on axes: We start with the x-axis on which $y = 0$ (and $\frac{dy}{dt} = 0$) and consider the sign of $\frac{dx}{dt}$. The sign can only change at a steady state point, of which only $(0, 0)$ and $(\frac{a_1}{b_1}, 0)$ are on the x-axis. By deduction (e.g. using test values), we conclude that

$$\begin{aligned} \frac{dx}{dt} &> 0 \quad \text{when} \quad 0 < x < \frac{a_1}{b_1}, \\ \frac{dx}{dt} &< 0 \quad \text{when} \quad x > \frac{a_1}{b_1}. \end{aligned}$$

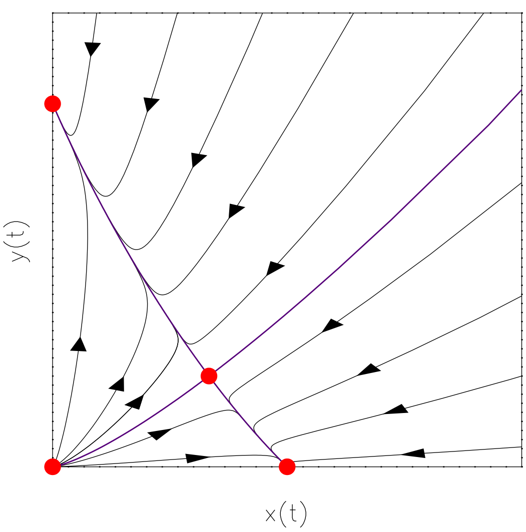
Similarly, on the y-axis (i.e. $x = 0$),

$$\begin{aligned} \frac{dy}{dt} &> 0 \quad \text{when} \quad 0 < y < \frac{a_2}{b_2}, \\ \frac{dy}{dt} &< 0 \quad \text{when} \quad y > \frac{a_2}{b_2}. \end{aligned}$$

Using the information gives the paths/trajectories as:



or



Analyse steady states: Close to each steady state $((x_0, y_0))$, we let

$$x(t) = x_0 + x_1(t) \qquad \text{and} \qquad y(t) = y_0 + y_1(t), \tag{15}$$

where $x_1(t)$ and $y_1(t)$ are small. Substituting into Equations 12 and 13 gives

$$\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= (a_1 - 2b_1x_0 - c_1y_0)\,x_1 - c_1x_0y_1 - b_1x_1^2 - c_1x_1y_1 \\
\frac{\mathrm{d}y_1}{\mathrm{d}t} &= -c_2y_0x_1 + (a_2 - 2b_2y_0 - c_2x_0)\,y_1 - c_2x_1y_1 - b_2y_1^2.
\end{aligned}$$

Linearising (i.e. neglecting x_1^2, x_1y_1, y_1^2, etc.) gives

$$\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= d_1x_1 + d_2y_1, \\
\frac{\mathrm{d}y_1}{\mathrm{d}t} &= d_3x_1 + d_4y_1.
\end{aligned}$$

where

$$\begin{aligned}
d_1 &= a_1 - 2b_1x_0 - c_1y_0, \\
d_2 &= -c_1x_0, \\
d_3 &= -c_2y_0, \\
d_4 &= a_2 - 2b_2y_0 - c_2x_0.
\end{aligned}$$

These equations may be rewritten as

$$\begin{pmatrix} \frac{\mathrm{d}x_1}{\mathrm{d}t} \\ \frac{\mathrm{d}y_1}{\mathrm{d}t} \end{pmatrix} = A \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \tag{16}$$

where

$$A = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix}.$$

Without proof, it is known that Equation 16 allows solutions of the form

$$\begin{aligned}
x_1 &= \alpha e^{\lambda_1 t} + \beta e^{\lambda_2 t}, \\
y_1 &= \gamma e^{\lambda_1 t} + \delta e^{\lambda_2 t},
\end{aligned}$$

where λ_1 and λ_2 are eigenvalues of A and α, β, γ and δ are (real or complex) constants.

To find the eigenvalues of A (i.e. λ_1 and λ_2), we solve

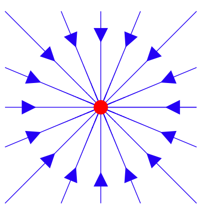
$$|A - \lambda I| = 0, \tag{17}$$

where I is the identity matrix, i.e.

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We consider the stability for various cases of λ_1 and λ_2:

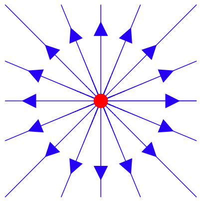
1. λ_1 and λ_2 are **real** and **negative**:
   As $t \to \infty$, $x_1 \to 0$ and $y_1 \to 0$, so $x \to x_0$ and $y \to y_0$ so all trajectories approach (x_0, y_0).
   Therefore the fixed point is **stable**.



2. λ_1 and λ_2 are **real** and **positive**:

   As $t \to \infty$, $|x_1| \to \infty$ and $|y_1| \to \infty$ so all trajectories diverge from (x_0, y_0).

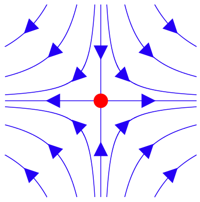
   Therefore the fixed point is **unstable**.



3. λ_1 and λ_2 are **real** and of **different sign**: (Assume $\lambda_1 < 0 < \lambda_2$)

   This has a diverging part and converging part to the trajectories. Only along a pair of specific paths does the trajectory approach the fixed point. In all other cases it diverges from the fixed point.

   Therefore the fixed point is **unstable** and called a *saddle point*.



4. λ_1 and λ_2 are **complex**: Since A is a real matrix, they are complex conjugates so

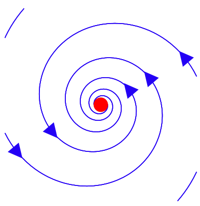
$$\begin{aligned} \lambda_1 &= u + vi, \\ \lambda_2 &= u - vi. \end{aligned}$$

   Then

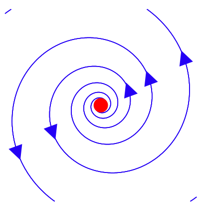
$$\begin{aligned} x_1 &= e^{ut} \left(C_1 \cos vt + C_2 \sin vt \right) \\ y_1 &= e^{ut} \left(C_3 \cos vt + C_4 \sin vt \right) \end{aligned}$$

   where C_1, C_2, C_3 and C_4 are constants.

   If $u < 0$ then the trajectories spiral inwards, so the fixed point is **stable**.



   If $u > 0$ then the trajectories spiral outwards, so the fixed point is **unstable**.



	Properties of λ_1 and λ_2			Type of fixed point		
	Complex	sign($Re(\lambda_1)$)	sign($Re(\lambda_2)$)	Stable	Spiral	Saddle Point
1	No	+	+	No	No	No
2	No	+	−	No	No	Yes
3	No	−	+	No	No	Yes
4	No	−	−	Yes	No	No
5	Yes	+	+	No	Yes	No
6	Yes	−	−	Yes	Yes	No

Draw better sketch: Using all this information, we draw a better sketch.

Long-term behaviour: From the sketch, we determine the long-term behaviour as $t \to \infty$.

3.5.3 Steady States and Linearisation for Two Coupled First Order ODEs: The General Case

The type of linear stability analysis we have just carried out for the two-species population model applies to a much wider class of coupled first order ODE systems. Let us assume that we have the system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(x,y), \tag{18}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = G(x,y). \tag{19}$$

Steady states (x_0, y_0) are solutions of

$$F(x_0,y_0) = 0, \quad G(x_0,y_0) = 0.$$

Now we assume that we perturb the steady state slightly, i.e.

$$x(t) = x_0 + x_1(t), \quad y(t) = y_0 + y_1(t),$$

with x_1 and y_1 assumed to be small (note that we can rewrite these two equations in the form $x_1 = x - x_0$, $y_1 = y - y_0$, so x_1 and y_1 really are the deviations of x and y from the steady state values x_0 and y_0).

We now substitute our expressions for $x(t)$ and $y(t)$ into the differential equations (18) and (19). Using that the time derivatives of the steady state value vanish (otherwise they would not be steady states), we obtain

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= F(x_0+x_1, y_0+y_1),\\ \frac{\mathrm{d}y_1}{\mathrm{d}t} &= G(x_0+x_1, y_0+y_1).\end{aligned}$$

So far, we have not made any approximations. We use the Taylor expansions for $F(x,y)$ and $G(x,y)$ about the steady state $(x_0, y_)$ up to linear order, i.e. we neglect quadratic terms (and higher order terms), and get

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= F(x_0,y_0) + \frac{\partial F}{\partial x}x_1 + \frac{\partial F}{\partial y}y_1,\\ \frac{\mathrm{d}y_1}{\mathrm{d}t} &= G(x_0,y_0) + \frac{\partial G}{\partial x}x_1 + \frac{\partial G}{\partial y}y_1,\end{aligned}$$

where the derivatives of F and G are evaluated at the steady state (x_0, y_0). Because $F(x_0,y_0) = G(x_0,y_0) = 0$, we get

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= \frac{\partial F}{\partial x}x_1 + \frac{\partial F}{\partial y}y_1,\\ \frac{\mathrm{d}y_1}{\mathrm{d}t} &= \frac{\partial G}{\partial x}x_1 + \frac{\partial G}{\partial y}y_1,\end{aligned}$$

which can be written as

$$\begin{pmatrix} \frac{\mathrm{d}x_1}{\mathrm{d}t} \\ \\ \frac{\mathrm{d}y_1}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{pmatrix} \begin{pmatrix} x_1 \\ \\ y_1 \end{pmatrix}.$$

The matrix

$$\begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} \end{pmatrix}_{(x_0,y_0)}$$

is called the Jacobian matrix at the steady state (x_0, y_0). We emphasize again that all derivatives are to be evaluated at the steady state.

One can now proceed in exactly the same way as for the two-species model above, namely calculate the eigenvalues to determine the stability of the steady state and draw a phase plane sketch using the information one can deduce from the calculation together with behaviour on the axes and the null clines etc, but this method is applicable to a much wider class of systems of first order ODEs.

3.5.4 Eigenvalues of Real 2×2 Matrices

The eigenvalues of a matrix A are the roots of the characteristic equation $\det(A - \lambda I) = 0$, where I is the identity (or unit) matrix. For a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the characteristic equation becomes

$$\det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = 0,$$

which results in the quadratic equation

$$\lambda^2 - (a + d)\lambda + ad - bc = 0.$$

The coefficient of the linear term in λ is known as *trace* of the matrix A – for any matrix the trace is defined as the sum of its diagonal entries. We will abbreviate the trace by τ. We also notice that the constant term is the *determinant* D of A itself. Hence the characteristic equation of A can be written as

$$\lambda^2 - \tau\lambda + D = 0.$$

The eigenvalues of A are then given by

$$\lambda_1 = \frac{1}{2}(\tau + \sqrt{\tau^2 - 4D}), \quad \lambda_2 = \frac{1}{2}(\tau - \sqrt{\tau^2 - 4D}).$$

Hence, instead of re-calculating the eigenvalues for different 2×2 matrices A over and over again, we can simply calculate the trace and the determinant of A and use the formulae above.

The calculation of eigenvalues becomes even easier of either $b = 0$, $c = 0$ or both vanish. Then the characteristic equation becomes

$$\lambda^2 - (a + d)\lambda + ad = 0 \quad \Longrightarrow \quad (\lambda - a)(\lambda - d) = 0 \quad \Longrightarrow \quad \lambda_1 = a, \ \lambda_2 = d,$$

and the eigenvalues of matrix A are just its diagonal entries!

3.5.5 Examples

Example 3.5: In Africa, lions and cheetahs both compete for the same prey. If the number of lions ($x(t)$) and cheetahs ($y(t)$) satisfy the equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x\,(1 - x - y)\,, \tag{20}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y\left(\tfrac{3}{4} - y - \tfrac{1}{2}x\right), \tag{21}$$

draw the phase plane of the two species and consider their long term behaviour.

Find Steady States: Consider the steady states:

The first equation implies that

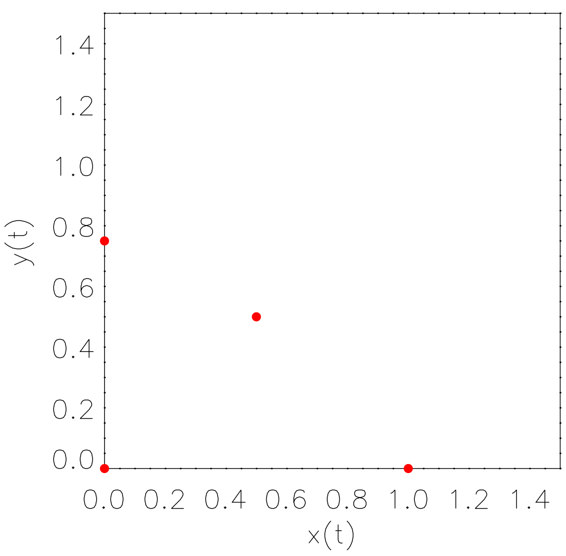
$$\text{(i)} \quad x = 0 \qquad \text{or} \qquad \text{(ii)} \quad 1 - x - y = 0.$$

The second implies that

$$\text{(iii)} \quad y = 0 \qquad \text{or} \qquad \text{(iv)} \quad \frac{3}{4} - y - \frac{x}{2} = 0.$$

The four steady states are at:

- (i) and (iii): $x = 0$, $y = 0$. (both die)
- (i) and (iv): $x = 0$, $y = \frac{3}{4}$. (lions die out)
- (ii) and (iii): $x = 1$, $y = 0$. (cheetahs die out)
- (ii) and (iv): $x + y = 1$ and $\frac{1}{2}x + y = \frac{3}{4}$ $\implies$ $x = \frac{1}{2}$, $y = \frac{1}{2}$. (mutual coexistence of lions and cheetahs)



Behaviour on axis On the x-axis (i.e. $y = 0$),

$$\frac{dx}{dt} = x(1-x),$$
$$\frac{dy}{dt} = 0.$$

Therefore,

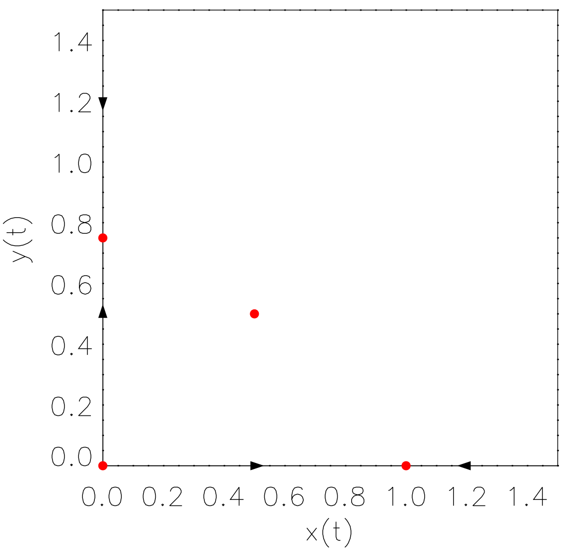
$$\frac{dx}{dt} > 0 \quad \text{when} \quad 0 < x < 1,$$
$$\frac{dx}{dt} < 0 \quad \text{when} \quad x > 1.$$

Also, on the y-axis (i.e. $x = 0$),

$$\frac{dx}{dt} = 0,$$
$$\frac{dy}{dt} = y\left(\tfrac{3}{4} - y\right).$$

Therefore,

$$\frac{dy}{dt} > 0 \quad \text{when} \quad 0 < y < \tfrac{3}{4},$$
$$\frac{dy}{dt} < 0 \quad \text{when} \quad y > \tfrac{3}{4}.$$



Analyse Steady States and determine their stability:

- $(0,0)$: Putting

$$x = 0 + x_1,$$
$$y = 0 + y_1$$

into the original equations so that

$$\frac{dx_1}{dt} = x_1(1 - x_1 - y_1),$$
$$\frac{dy_1}{dt} = y_1\left(\tfrac{3}{4} - y_1 - \tfrac{1}{2}x_1\right).$$

Linearising gives

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1,$$
$$\frac{\mathrm{d}y_1}{\mathrm{d}t} = \tfrac{3}{4}y_1.$$

So

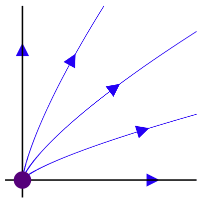
$$A = \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{4} \end{pmatrix}.$$

We find the eigenvalues:

$$\begin{aligned} |A - \lambda I| &= 0 \\ \Longrightarrow \quad \begin{vmatrix} 1-\lambda & 0 \\ 0 & \frac{3}{4}-\lambda \end{vmatrix} &= 0 \\ \Longrightarrow \quad (1-\lambda)\left(\tfrac{3}{4}-\lambda\right) &= 0 \end{aligned}$$

which gives $\lambda_1 = \frac{3}{4}$, $\lambda_2 = 1$.

Since $\lambda_1 > 0$ and $\lambda_2 > 0$, all trajectories diverge. Hence fixed point $(0,0)$ is **unstable**.



- $(0, \frac{3}{4})$: Putting

$$x = 0 + x_1,$$
$$y = \tfrac{3}{4} + y_1$$

into the original equations so that

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_1\left(1 - x_1 - \tfrac{3}{4} - y_1\right) &= x_1\left(\tfrac{1}{4} - x_1 - y_1\right), \\ \frac{\mathrm{d}y_1}{\mathrm{d}t} &= \left(\tfrac{3}{4} + y_1\right)\left(\tfrac{3}{4} - \tfrac{3}{4} - y_1 - \tfrac{1}{2}x_1\right) &= \left(\tfrac{3}{4} + y_1\right)\left(-\tfrac{1}{2}x_1 - y_1\right). \end{aligned}$$

and linearising gives

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \frac{x}{4},$$
$$\frac{\mathrm{d}y_1}{\mathrm{d}t} = -\tfrac{3}{8}x_1 - \tfrac{3}{4}y_1.$$

So

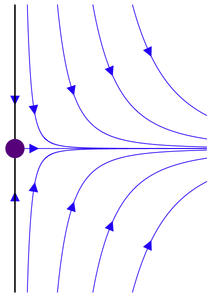
$$A = \begin{pmatrix} \frac{1}{4} & 0 \\ -\frac{3}{8} & -\frac{3}{4} \end{pmatrix}$$

We determine the eigenvalues:

$$\begin{aligned} \begin{vmatrix} \frac{1}{4}-\lambda & 0 \\ -\frac{3}{8} & -\frac{3}{4}-\lambda \end{vmatrix} &= 0 \\ \Longrightarrow \quad \left(\tfrac{1}{4}-\lambda\right)\left(-\tfrac{3}{4}-\lambda\right) - \left(-\tfrac{3}{8}\times 0\right) &= 0 \end{aligned}$$

which gives $\lambda_1 = -\frac{3}{4}$ $\lambda_2 = \frac{1}{4}$.

Since $\lambda_1 < 0$ and $\lambda_2 > 0$, some trajectories approach but most trajectories diverge. Hence fixed point $(0, \frac{3}{4})$ is an unstable **saddle point**.



- $(1, 0)$: Putting

$$\begin{aligned} x &= 1 + x_1, \\ y &= 0 + y_1, \end{aligned}$$

into Equations 20 and 21 gives

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= (1 + x_1)(1 - 1 - x_1 - y_1) &&= (1 + x_1)(-x_1 - y_1), \\ \frac{\mathrm{d}y_1}{\mathrm{d}t} &= y_1\left(\tfrac{3}{4} - y_1 - \tfrac{1}{2} - \tfrac{1}{2}x_1\right) &&= y_1\left(\tfrac{1}{4} - y_1 - \tfrac{1}{2}x_1\right). \end{aligned}$$

Linearising gives

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= -x_1 - y_1, \\ \frac{\mathrm{d}y_1}{\mathrm{d}t} &= \tfrac{1}{4}y_1, \end{aligned}$$

so

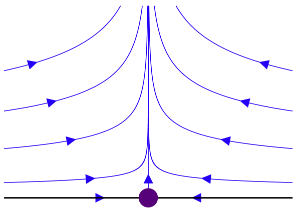
$$A = \begin{pmatrix} -1 & -1 \\ 0 & \frac{1}{4} \end{pmatrix}.$$

We now find the eigenvalues:

$$\begin{aligned} |A - \lambda I| &= 0 \\ \Longrightarrow \quad \begin{vmatrix} -1 - \lambda & -1 \\ 0 & \frac{1}{4} - \lambda \end{vmatrix} &= 0 \\ \Longrightarrow (-1 - \lambda)\left(\tfrac{1}{4} - \lambda\right) - (-1 \times 0) &= 0, \end{aligned}$$

which gives $\lambda_1 = -1$ and $\lambda_2 = \frac{1}{4}$.

Since $\lambda_1 < 0$ and $\lambda_2 > 0$, some trajectories approach but most trajectories diverge. Hence fixed point $(1, 0)$ is also an unstable **saddle point**.



- $(\frac{1}{2}, \frac{1}{2})$: Putting

$$\begin{aligned} x &= \tfrac{1}{2} + x_1, \\ y &= \tfrac{1}{2} + y_1, \end{aligned}$$

into Equations 20 and 21 gives

$$\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= \left(\tfrac{1}{2}+x_1\right)\left[1-\left(\tfrac{1}{2}+x_1\right)-\left(\tfrac{1}{2}+y_1\right)\right] &&= \left(\tfrac{1}{2}+x_1\right)\left(-x_1-y_1\right),\\
\frac{\mathrm{d}y_1}{\mathrm{d}t} &= \left(\tfrac{1}{2}+y_1\right)\left[\tfrac{3}{4}-\left(\tfrac{1}{2}+y_1\right)-\tfrac{1}{2}\left(\tfrac{1}{2}+x_1\right)\right] &&= \left(\tfrac{1}{2}+y_1\right)\left(-\tfrac{1}{2}x_1-y_1\right).
\end{aligned}$$

Linearising gives

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= -\tfrac{1}{2}x_1-\tfrac{1}{2}y_1,\\
\frac{\mathrm{d}y}{\mathrm{d}t} &= -\tfrac{1}{4}x_1-\tfrac{1}{2}y_1.
\end{aligned}$$

so

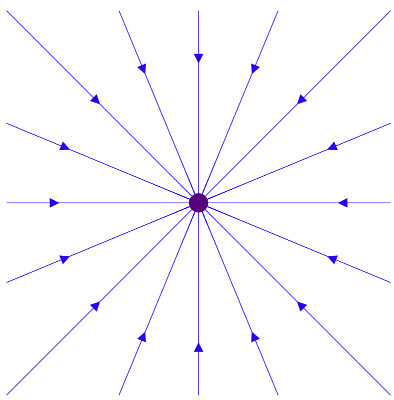
$$A = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix}.$$

We now find the eigenvalues:

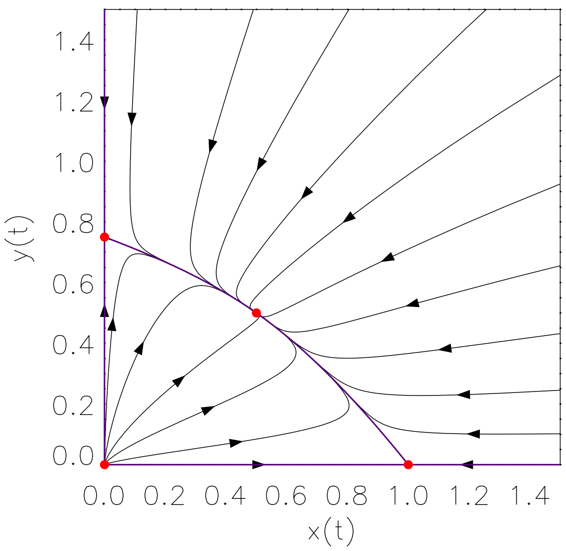
$$\begin{aligned}
& & |A-\lambda I| &= 0\\
&\Longrightarrow & \begin{vmatrix} -\frac{1}{2}-\lambda & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{2}-\lambda \end{vmatrix} &= 0\\
&\Longrightarrow & \left(-\tfrac{1}{2}-\lambda\right)^2-\left(-\tfrac{1}{2}\right)\left(-\tfrac{1}{4}\right) &= 0\\
&\Longrightarrow & \left(-\tfrac{1}{2}-\lambda\right)^2 &= \tfrac{1}{8}\\
&\Longrightarrow & -\tfrac{1}{2}-\lambda &= \pm\tfrac{1}{2\sqrt{2}}\\
&\therefore & \lambda &= -\tfrac{1}{2}\pm\tfrac{1}{2\sqrt{2}}
\end{aligned}$$

which gives $\lambda_1 \approx -0.853$ and $\lambda_2 \approx -0.147$.

Since $\lambda_1 < 0$ and $\lambda_2 < 0$, all trajectories approach the fixed point. Hence fixed point $(\frac{1}{2}, \frac{1}{2})$ is **stable**.



Draw better sketch and determine long term behaviour: We have determined that $(0,0)$ is an unstable steady state, $(0,\frac{3}{4})$ and $(1,0)$ are saddle points (and unstable). Finally, $(\frac{1}{2},\frac{1}{2})$ is the only stable steady state. So we sketch:



So for any initial population x_0 and y_0, the populations of lions and cheetahs as $t \to \infty$ tends to the steady state of $x = \frac{1}{2}$ and $y = \frac{1}{2}$. Therefore both lions and cheetahs will mutually coexist.

Example 3.6: In the woods and forests of Scotland, red and grey squirrels compete for food. Consider that two species of squirrel in a forest governed by the equations,

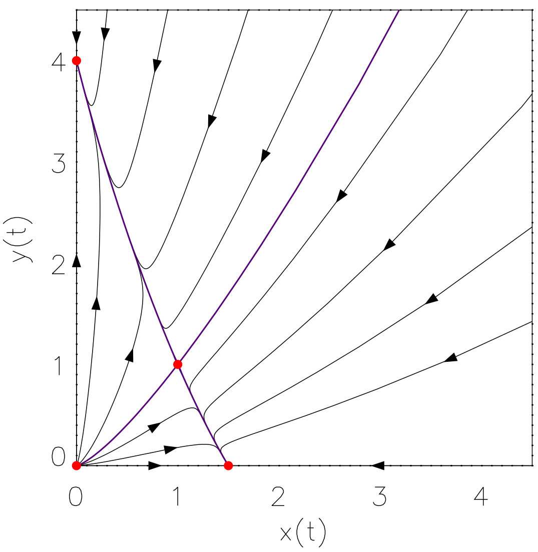
$$\frac{\mathrm{d}x}{\mathrm{d}t} = x\left(\tfrac{3}{2} - x - \tfrac{1}{2}y\right), \tag{22}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y\left(2 - \tfrac{1}{2}y - \tfrac{3}{2}x\right), \tag{23}$$

where $x(t)$ is the number of grey squirrels and $y(t)$ is the number of red squirrels.

Using a similar approach, we find the four fixed points:

- $(0,0)$ — unstable $\quad(\lambda_1 = \frac{3}{2},\ \lambda_2 = 2)$
- $(0,4)$ — stable $\quad(\lambda_1 = -2,\ \lambda_2 = -\frac{1}{2})$
- $(\frac{3}{2},0)$ — stable $\quad(\lambda_1 = -\frac{3}{2},\ \lambda_2 = -\frac{1}{8})$
- $(1,1)$ — unstable (saddle point) $\quad(\lambda_1 \approx -1.65,\ \lambda_2 \approx 0.151)$



3.6 Predator-Prey (PP) Models

3.6.1 Introduction

In the previous section we considered the interaction of two species where interaction was through the same food source that they ate (i.e. no direct interaction). We now consider a situation where one species eats another, for example

- foxes and rabbits.
- wolves and deer.

Definition: A *predator-prey* model is a model of prey and predators which satisfies the following rules:

1. In the absence of the predator, prey follows the simple logistic model.
2. In the absence of the prey (i.e. $x = 0$), the predator dies out.
3. The number of encounters of predators and prey is proportional to the product of the populations. Each encounter
   (a) promotes the growth of the predator, and
   (b) inhibits growth of the prey.

Now we consider the interaction of a predator with a prey mathematically. So if we let $x(t)$ be the number of prey and $y(t)$ be the number of predators, then Condition 1 implies that when $y = 0$,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x\,(a - bx)\,, \tag{24}$$

where $x(t) > 0$, $a > 0$, $b > 0$ and a and b are constants.

Condition 2 implies that when $x = 0$,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -cy, \tag{25}$$

where constant $c > 0$.

Adding an additional term to each of Equations 24 and 25 to satisfy Condition 3, gives the predator-prey equations as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax - bx^2 - \alpha xy, \qquad (26)$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -cy + \beta xy, \qquad (27)$$

where $\alpha > 0$ and $\beta > 0$ are constants.

We now rewrite into the form

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= x\left(a - bx - \alpha y\right), \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= y\left(-c + \beta x\right). \end{aligned}$$

Identify steady states: Setting the time derivatives to zero gives

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} = 0 \quad &\Longrightarrow \quad x\left(a - bx - \alpha y\right) = 0 \\ &\Longrightarrow \quad \text{(i)} \quad x = 0 \quad \text{or} \quad \text{(ii)} \quad a - bx - \alpha y = 0. \end{aligned}$$

and

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} = 0 \quad &\Longrightarrow \quad y\left(-c + \beta x\right) = 0 \\ &\Longrightarrow \quad \text{(iii)} \quad y = 0 \quad \text{or} \quad \text{(iv)} \quad -c + \beta x = 0. \end{aligned}$$

There are four cases:

- (i) and (iii):
$$x = 0 \quad \text{and} \quad y = 0,$$
so both die out.
- (i) and (iv):
$$x = 0 \quad \text{and} \quad x = \frac{c}{\beta},$$
So impossible as $c > 0$!
- (ii) and (iii):
$$x = \frac{a}{b} \quad \text{and} \quad y = 0,$$
so predators (y) die out.
- (ii) and (iv):
$$x = \frac{c}{\beta} \quad \text{and} \quad y = \frac{a\beta - bc}{\alpha\beta}.$$
Provided $a\beta - bc > 0$, both species mutually coexist.

In these models, there is either two or three states states (compared to four in the competing species models).

To analyse these models, we use the same approach as for the competing species models.

3.6.2 Examples

Example 3.7: In the Late Cretaceous period in the western part of what is now known as North America, Tyrannosaurus Rex preyed upon smaller dinosaurs like Triceratops. If the number of these two species is governed by the equations,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x\,(1 - x - y)\,, \tag{28}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y\left(-\tfrac{1}{4} + x\right). \tag{29}$$

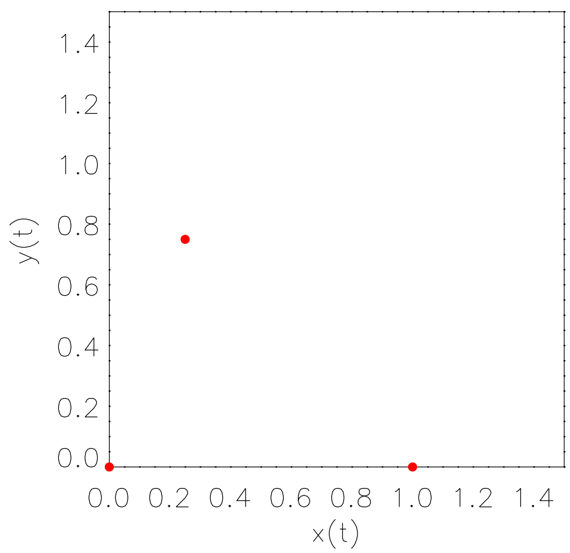
Using graphical means, determine their long-term behaviour.

Find steady states: By solving the equations subject to

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0 \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = 0,$$

we find the three fixed points:

$$(0,0) \quad , \quad (1,0) \quad \text{and} \quad (\frac{1}{4}, \frac{3}{4}).$$



Behaviour on axes: On the y-axis (i.e. $x = 0$), Equations 28 and 29 reduce to

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0,$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\tfrac{1}{4}y,$$

so for all $y > 0$ on the x-axis,

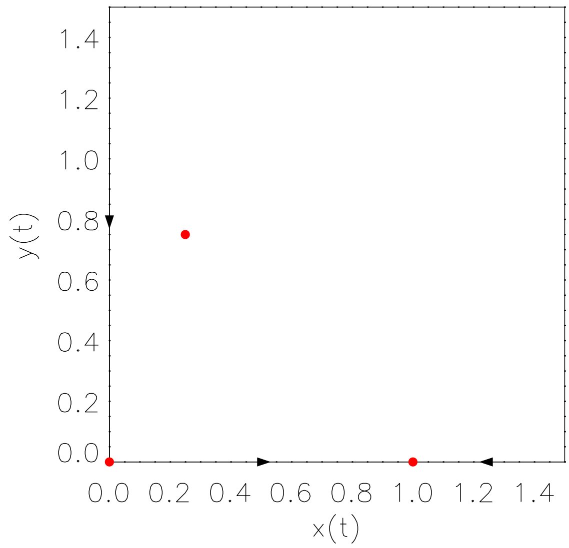
$$\frac{\mathrm{d}y}{\mathrm{d}t} < 0.$$

Also, on the x-axis (i.e. $y = 0$), Equations 28 and 29 reduce to

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= x\,(1-x)\,,\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= 0.\end{aligned}$$

so

$$\begin{array}{ccccccccc}\dfrac{\mathrm{d}x}{\mathrm{d}t} & > & 0 & \text{when} & 0 & < & x & < & 1,\\ \dfrac{\mathrm{d}x}{\mathrm{d}t} & < & 0 & \text{when} & & & x & > & 1.\end{array}$$



Analyse steady states and determine their stability: We consider each steady state in turn:

- $(0, 0)$: Putting

  $$\begin{aligned}x(t) &= 0 + x_1(t),\\ y(t) &= 0 + y_1(t)\end{aligned}$$

  into Equations 28 and 29 gives,

  $$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_1\,(1 - x_1 - y_1)\,,\\ \frac{\mathrm{d}y_1}{\mathrm{d}t} &= y_1\left(-\tfrac{1}{4} + x_1\right).\end{aligned}$$

  Linearising reduces these equations to

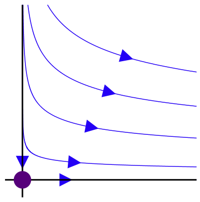
  $$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_1,\\ \frac{\mathrm{d}y_1}{\mathrm{d}t} &= -\tfrac{1}{4}y_1,\end{aligned}$$

  so

  $$A = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{4}\end{pmatrix},$$

which has the eigenvalues $\lambda_1 = -\frac{1}{4}$ and $\lambda_2 = 1$.

As the eigenvalues of A are real and of opposite sign, this fixed point $(0, 0)$ is an **unstable** saddle point.



- $(1, 0)$: Putting

$$\begin{aligned} x(t) &= 1 + x_1(t), \\ y(t) &= 0 + y_1(t) \end{aligned}$$

into Equations 28 and 29 gives,

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= (1 + x_1)\,[1 - (1 + x_1) - y_1] &&= (1 + x_1)\,(-x_1 - y_1) \\ \frac{\mathrm{d}y_1}{\mathrm{d}t} &= y_1\left[-\tfrac{1}{4} + (1 + x_1)\right] &&= y_1\left(\tfrac{3}{4} + x_1\right). \end{aligned}$$

Linearising reduces these equations to

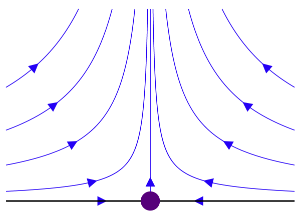
$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= -x_1 - y_1, \\ \frac{\mathrm{d}y_1}{\mathrm{d}t} &= \tfrac{3}{4} y_1 \end{aligned}$$

so

$$A = \begin{pmatrix} -1 & -1 \\ 0 & \frac{3}{4} \end{pmatrix},$$

which has the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = \frac{3}{4}$.

As the eigenvalues of A are real and of opposite sign, this fixed point $(1, 0)$ is again an **unstable** saddle point.



- $(\frac{1}{4}, \frac{3}{4})$: Putting

$$\begin{aligned} x(t) &= \tfrac{1}{4} + x_1(t), \\ y(t) &= \tfrac{3}{4} + y_1(t) \end{aligned}$$

into Equations 28 and 29 gives,

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= \left(\tfrac{1}{4} + x_1\right)\left[1 - \left(\tfrac{1}{4} + x_1\right) - \left(\tfrac{3}{4} + y_1\right)\right], \\ \frac{\mathrm{d}y_1}{\mathrm{d}t} &= \left(\tfrac{3}{4} + y_1\right)\left[-\tfrac{1}{4} + \left(\tfrac{1}{4} + x_1\right)\right]. \end{aligned}$$

Linearising reduces these equations to

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= -\tfrac{1}{4}x_1 - \tfrac{1}{4}y_1,\\ \frac{\mathrm{d}y_1}{\mathrm{d}t} &= \tfrac{3}{4}x_1.\end{aligned}$$

so

$$A = \begin{pmatrix} -\frac{1}{4} & -\frac{1}{4} \\ \frac{3}{4} & 0 \end{pmatrix}.$$

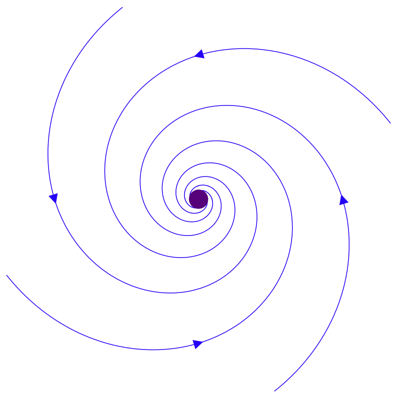
The eigenvalues of A are given by

$$\begin{aligned} |A - \lambda I| &= 0 \\ \Longrightarrow \quad \begin{vmatrix} -\frac{1}{4} - \lambda & -\frac{1}{4} \\ \frac{3}{4} & -\lambda \end{vmatrix} &= 0 \\ \Longrightarrow \left(-\tfrac{1}{4} - \lambda\right)(-\lambda) - \left(-\tfrac{1}{4}\right)\left(\tfrac{3}{4}\right) &= 0 \\ \Longrightarrow \quad \lambda^2 + \tfrac{1}{4}\lambda + \tfrac{3}{16} &= 0 \\ \Longrightarrow \quad \lambda &= \frac{-\frac{1}{4} \pm \sqrt{\left(\frac{1}{4}\right)^2 - 4\cdot 1\cdot\left(\frac{3}{16}\right)}}{2} \\ &= \frac{-1 \pm \sqrt{-11}}{8} \end{aligned}$$

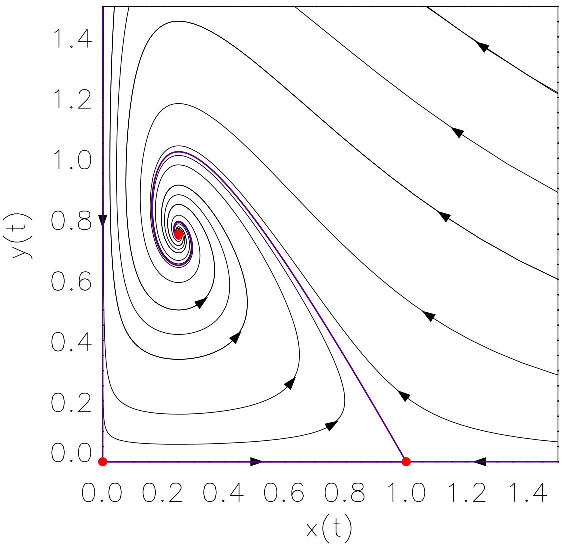
so

$$\lambda_1 = -\frac{1}{8} - \frac{\sqrt{11}}{8}i \qquad \text{and} \qquad \lambda_2 = -\frac{1}{8} + \frac{\sqrt{11}}{8}i$$

As the eigenvalues of A are complex and the real part is of negative sign, the fixed point $(\frac{1}{4}, \frac{3}{4})$ has a spiral structure and is **stable**.



Draw better sketch and determine long term behaviour:



Provided both the triceratopses and T-rexes initially exist, the system will evolve as $t \to \infty$ to a mutually coexisting state where both species survive ($x = \frac{1}{4}$, $y = \frac{3}{4}$).

Example 3.8: Consider the competing species of hare and lynx governed by the equations,

$$\frac{dx}{dt} = x\left(1 - \tfrac{1}{2}y\right), \tag{30}$$

$$\frac{dy}{dt} = y\left(-\tfrac{3}{4} + \tfrac{1}{4}x\right), \tag{31}$$

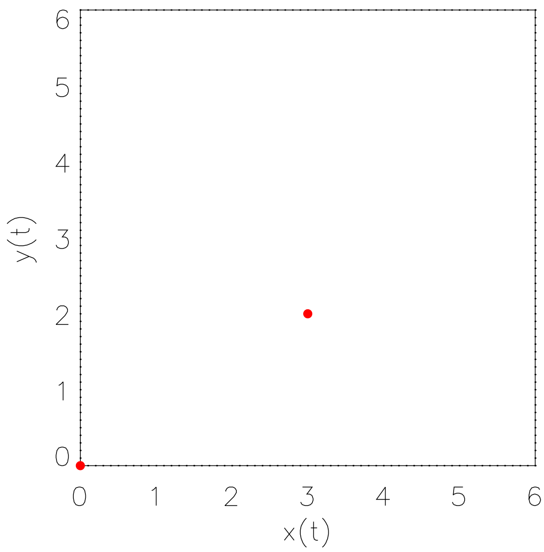
where $x(t)$ is the number of snowshoe hare and $y(t)$ is the number of Canadian lynx.

Find steady states: By solving the equations subject to

$$\frac{dx}{dt} = 0 \qquad \text{and} \qquad \frac{dy}{dt} = 0,$$

we find the two fixed points:

$$(0,0) \qquad \text{and} \qquad (3,2).$$

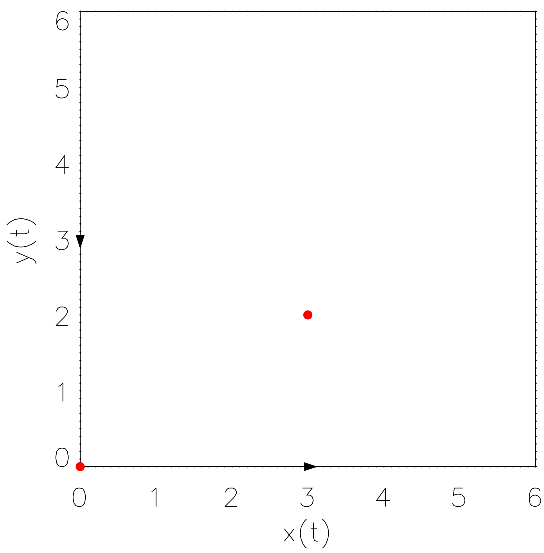


Behaviour on axes: On the y-axis (i.e. $x = 0$) for all $y > 0$,

$$\frac{\mathrm{d}y}{\mathrm{d}t} < 0.$$

Similarly, on the x-axis (i.e. $y = 0$) for all $x > 0$,

$$\frac{\mathrm{d}x}{\mathrm{d}t} > 0.$$



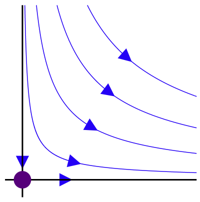
Analyse steady states and determine their stability: We consider each steady state in turn:

- $(0, 0)$: Linearising about $(0, 0)$ gives

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{3}{4} \end{pmatrix},$$

so the eigenvalues are $\lambda_1 = -\frac{3}{4}$ and $\lambda_2 = 1$.

As the eigenvalues of A are real and of opposite sign, the fixed point $(0,0)$ in a saddle point and hence **unstable**.



- $(3,2)$: Linearising about $(0,0)$ gives

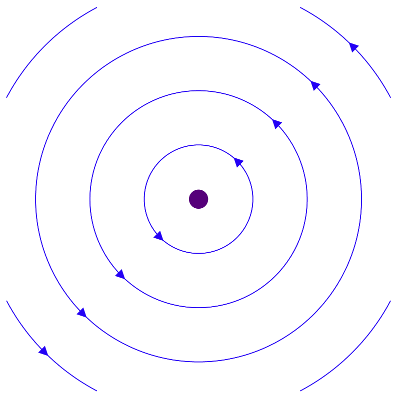
$$A = \begin{pmatrix} 0 & -\frac{3}{2} \\ \frac{1}{2} & 0 \end{pmatrix},$$

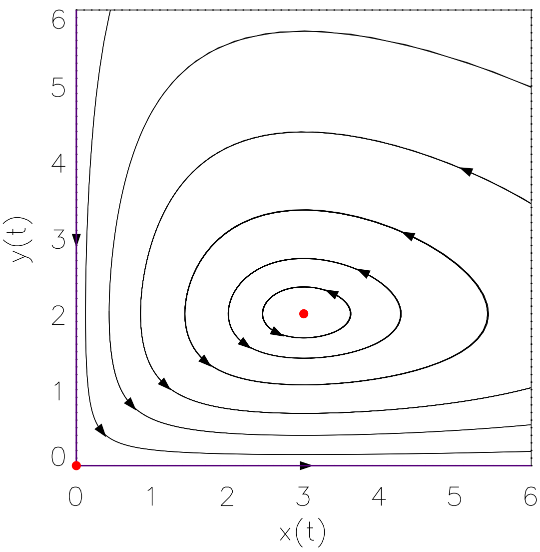
so the eigenvalues are $\lambda_1 = -\frac{\sqrt{3}}{2}i$ and $\lambda_2 = \frac{\sqrt{3}}{2}i$.

These eigenvalues are complex and have no real part, so the local perturbations about $(3,2)$ vary as:

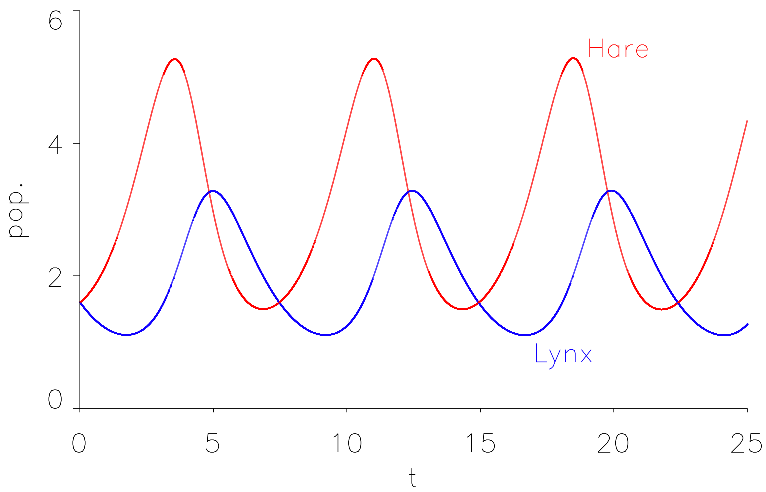
$$\begin{aligned} x_1 &\approx C_1 \cos\frac{\sqrt{3}t}{2} + C_2 \sin\frac{\sqrt{3}t}{2} \\ y_1 &\approx C_3 \cos\frac{\sqrt{3}t}{2} + C_4 \sin\frac{\sqrt{3}t}{2} \end{aligned}$$

So there is a cyclic variation about the fixed point with no decay or growth (as $u = 0$).





This gives an type of interaction where the numbers of both hare and lynx follow a periodic behaviour.



4 Dynamics

4.1 Introduction

4.1.1 Definitions

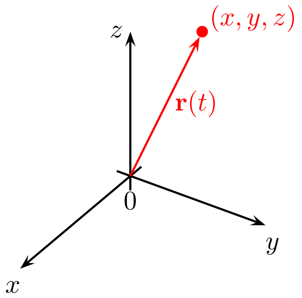
Definition:

- A *particle* is any object that may be represented by a point. It has a mass m. Therefore, its size is much smaller than the path taken.
- The *position*, $\mathbf{r}$, is the location (x, y, z) of the particle in space, i.e.

$$\mathbf{r}(t) \equiv x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$$

Position is

- Defined relative to the origin.
- A function of time.



This is a right-hand coordinate system (RHCS).

Definition:

- The *velocity*, $\mathbf{v}$, the rate of change of position, i.e.

$$\mathbf{v}(t) \equiv \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}.$$

- The *speed* is the magnitude of the velocity, i.e.

$$|\mathbf{v}|.$$

- The *acceleration* is the rate of change of velocity, i.e.

$$\mathbf{a}(t) \equiv \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}.$$

- The *momentum*, $\mathbf{p}$, is the product of mass and velocity, i.e.

$$\mathbf{p}(t) \equiv m\mathbf{v}.$$

Notes:

- Position, velocity, momentum and acceleration are **vectors**.
- Mass and speed are **scalars**.
- Velocity may be written

$$\begin{aligned} \mathbf{v}(t) &= v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k} \\ &= \frac{\mathrm{d}x}{\mathrm{d}t}\mathbf{i} + \frac{\mathrm{d}y}{\mathrm{d}t}\mathbf{j} + \frac{\mathrm{d}z}{\mathrm{d}t}\mathbf{k} \\ &= \dot{\mathbf{r}}(t) \quad = \quad \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}. \end{aligned}$$

  or

$$v_x = \frac{\mathrm{d}x}{\mathrm{d}t} \quad , \quad v_y = \frac{\mathrm{d}y}{\mathrm{d}t} \quad , \quad v_z = \frac{\mathrm{d}z}{\mathrm{d}t}.$$

- Similarly, acceleration may be written

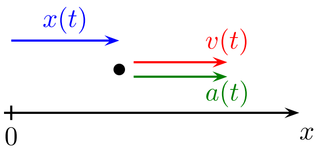
$$\begin{aligned} \mathbf{a}(t) &= a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} \\ &= \frac{\mathrm{d}v_x}{\mathrm{d}t}\mathbf{i} + \frac{\mathrm{d}v_y}{\mathrm{d}t}\mathbf{j} + \frac{\mathrm{d}v_z}{\mathrm{d}t}\mathbf{k} \\ &= \frac{\mathrm{d}^2x}{\mathrm{d}t^2}\mathbf{i} + \frac{\mathrm{d}^2y}{\mathrm{d}t^2}\mathbf{j} + \frac{\mathrm{d}^2z}{\mathrm{d}t^2}\mathbf{k} \\ &= \ddot{\mathbf{r}}(t) \quad = \quad \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k}. \end{aligned}$$

- v is maybe used for **both** speed and velocity in 1D. For speed,

$$v = |\mathbf{v}| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

  Note: Speed is always non-negative, velocity is signed.

4.1.2 1D Interpretation



In a single dimension (e.g. in x-direction), we can use scalar expressions $x(t)$, $v(t)$ and $a(t)$ based on the vector quantities of:

- Position:

$$\mathbf{r}(t) = x(t)\mathbf{i}$$

- Velocity:

$$\mathbf{v}(t) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = v(t)\mathbf{i} = \frac{\mathrm{d}x}{\mathrm{d}t}\mathbf{i}$$

  Note that velocity is signed, whereas speed

$$|v(t)| = |\mathbf{v}(t)|$$

  is its absolute value.

- Acceleration:
$$\mathbf{a} = a(t)\mathbf{i} = \frac{\mathrm{d}v}{\mathrm{d}t}\mathbf{i} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}\mathbf{i}$$

4.1.3 Technique

We solve dynamical system problems using the following process:

1. Draw diagram, which includes
   - label axes,
   - forces,
   - vectors of velocity and forces.
2. Write down equation and boundary conditions
3. Solve equations

4.1.4 Units

The primary units of measurement are length ($[L]$), mass ($[M]$) and time ($[T]$). From these we derive other units, e.g.

- Velocity: $\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$ has units $\frac{[L]}{[T]}$.
- Momentum: $\mathbf{p} = m\mathbf{v}$ has units $\frac{[M][L]}{[T]}$.
- Acceleration: $\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$ has units $\frac{[L]}{[T]^2}$.

Standard units are:

	MKS	CGS	Imperial
Length	metres (m)	centimetres (cm)	feet (ft)
Mass	kilogrammes (kg)	grammes (g)	pounds (lb)
Time	second (s)	second (s)	second (s)
Velocity	ms^{-1}	$\mathrm{cm\,s}^{-1}$	ft/s
Acceleration	ms^{-2}	$\mathrm{cm\,s}^{-2}$	$\mathrm{ft/s}^2$
Momentum	$\mathrm{kg\,ms}^{-1}$	$\mathrm{cm\,s}^{-1}$	lb ft/s

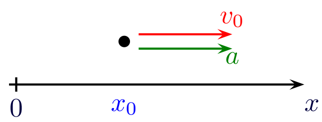
We normally work in MKS units.

4.1.5 Examples

Example 4.1

Consider the case of 1D motion at constant acceleration, such that at $t = 0$, $x = x_0$ and $v = v_0$. Determine $x(t)$ and $v(t)$ for all $t > 0$.

Diagram:



Equations:

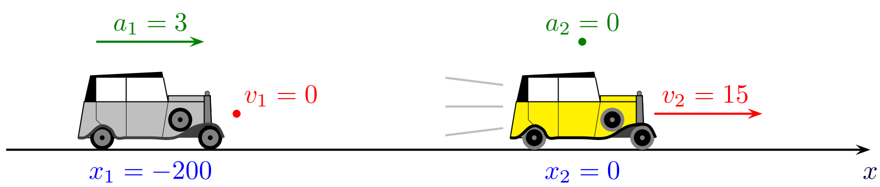
Since
$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{\mathrm{d}v}{\mathrm{d}t} = a,$$
we integrate to get
$$\frac{\mathrm{d}x}{\mathrm{d}t} = v = at + C_1.$$
Since $v(0) = v_0$ then $C_1 = v_0$ so that
$$v(t) = v_0 + at.$$
Differentiate again to get
$$x(t) = C_2 + v_0t + \tfrac{1}{2}at^2$$
and using $x(0) = x_0$ gives $C_2 = x_0$ and hence
$$x(t) = x_0 + v_0t + \tfrac{1}{2}at^2.$$

Example 4.2

Consider two cars travelling along a road (in the x-axis) such that at $t = 0$,

- James Bond's silver car is initially stationary at $x = -200\,\mathrm{m}$ and accelerates at $a = 3\,\mathrm{ms}^{-2}$ *(0 to 60 in 8.94 seconds).*
- Goldfinger's gold car is passing $x = 0m$ whilst travelling at $v = 15\,\mathrm{ms}^{-1}$ *(33.6 mph)* with no acceleration.

When does James Bond catch up with Goldfinger?
(Assume acceleration is constant throughout.)



Silver Car: Since
$$\frac{\mathrm{d}^2x_1}{\mathrm{d}t^2} = a_1 = 3,$$
then
$$v_1(t) = \frac{\mathrm{d}x_1}{\mathrm{d}t} = 3t + C_1$$
$$\therefore \quad v_1(t) = 3t,$$

since $v_1(0) = 0$. And

$$\begin{aligned} x_1(t) &= \tfrac{3}{2}t^2 + C_2 \\ &= \tfrac{3}{2}t^2 - 200, \end{aligned}$$

since $x_1(0) = -200$.

Gold Car Similarly, since

$$\frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} = a_2 = 0,$$

then

$$\begin{aligned} v_2(t) &= \frac{\mathrm{d}x_2}{\mathrm{d}t} = C_3 \\ \therefore \qquad v_2(t) &= 15, \end{aligned}$$

since $v_2(0) = 15$. And

$$\begin{aligned} x_2(t) &= 15t + C_4 \\ \therefore \qquad x_2(t) &= 15t, \end{aligned}$$

since $x_2(0) = 0$.

The silver car meets the gold car meet when $x_1(t) = x_2(t)$, so

$$\begin{aligned} x_1(t) &= x_2(t) \\ \Longrightarrow \qquad \tfrac{3}{2}t^2 - 200 &= 15t \\ \Longrightarrow \qquad 3t^2 - 30t - 400 &= 0 \\ \therefore \qquad t &= 5 \pm \tfrac{5}{3}\sqrt{57}. \end{aligned}$$

For this to be realistic, t must be positive so

$$t = 5\left(1 + \tfrac{5}{3}\sqrt{57}\right) \approx 17.6\,\mathrm{s}.$$

which is when they are both at

$$x = 75\left(1 + \tfrac{1}{3}\sqrt{57}\right) \approx 263.75\,\mathrm{m}.$$

4.2 Newton's Laws of Motion

Definition: *Newton's Laws of Motion* are:

I. The velocity of a objects remains constant unless the body is acted upon by an external net force.

II. The acceleration $\mathbf{a}$ of an object is in the direction of and directly proportional to the net force $\mathbf{F}$ acting against it and inversely proportional to the mass m of the object, that is

$$\mathbf{a} = \frac{1}{m}\mathbf{F} \quad \Longleftrightarrow \quad \mathbf{F} = m\mathbf{a}.$$

III. Every force has an equal and opposite reaction force.

Laws I and II can be written as the the *Equation of Motion*,

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}$$

where m is mass of the object.

Since

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$$

then we can solve for each component, i.e.

$$F_x = m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2},$$
$$F_y = m\frac{\mathrm{d}^2 y}{\mathrm{d}t^2},$$
$$F_z = m\frac{\mathrm{d}^2 z}{\mathrm{d}t^2}.$$

What about the units?

Since

$$\mathbf{F} = m\mathbf{a}$$

then

$$[F] = [M]\cdot\frac{[L]}{[T]^2} = \mathrm{kg\,ms}^2$$

or

$$1\,\mathrm{Newton} = 1\,\mathrm{N} = 1\,\mathrm{kg\,ms}^2.$$

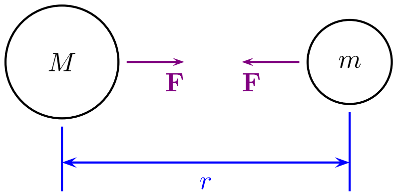
4.3 Newton's Model for Gravitational Attraction

Every object in the Universe with non-zero mass attracts every other object.

Definition: Two particles of masses M and m are a distance r apart. Then the force of *gravitational attraction*, $\mathbf{F}$, on each object has magnitude

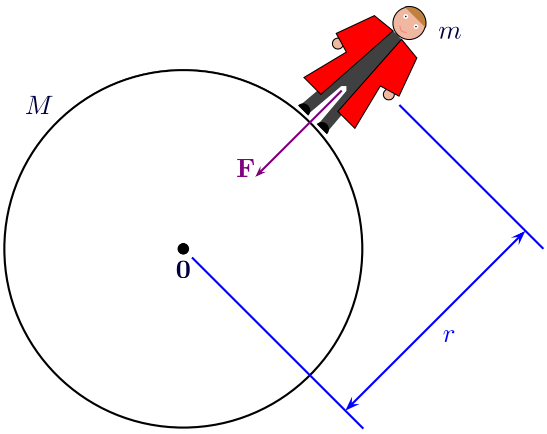
$$F = |\mathbf{F}| = \frac{GMm}{r^2}$$

in the direction of the other object. Here $G = 6.67 \times 10^{-11}\,\mathrm{N\,m^2kg^{-2}}$ is the *universal gravitational constant*.



Example 4.3

What is the force of attaction by the Earth on a student of mass 70 kg (11 stones) at the Earth's surface?



Consider Earth as a particle with all mass at the centre, so

$$\begin{aligned} M &= 5.89 \times 10^{24}\,\mathrm{kg}, && \text{(Mass of Earth)} \\ m &= 70\,\mathrm{kg}, && \text{(Mass of Student)} \\ r &= 6.378 \times 10^{6}\,\mathrm{m}. && \text{(Radius of Earth)} \end{aligned}$$

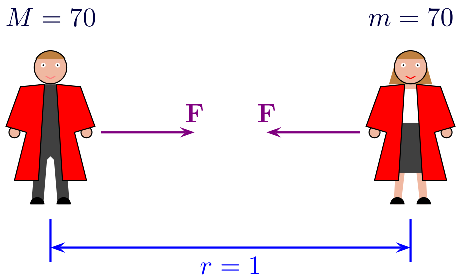
Hence the gravitational force is

$$\begin{aligned} |\mathbf{F}| &= \frac{GMm}{r^2} \\ &= \frac{6.67 \times 10^{-11} \cdot 5.89 \times 10^{24} \cdot 70}{(6.378 \times 10^6)^2} \\ &= 676.0\,\mathrm{N}. \end{aligned}$$

This is also the force that the student exerts on the Earth. But, since $M \gg m$, the Earth's acceleration is small and so the person does most of the moving.

Example 4.4

What is the force of attaction between a couple of students, a metre apart?



Here,

$$\begin{aligned} M &= 70\,\text{kg}, && \text{(Mass of Student 1)} \\ m &= 70\,\text{kg}, && \text{(Mass of Student 2)} \\ r &= 1\,\text{m}. && \text{(Distance)} \end{aligned}$$

Hence the gravitational force is

$$\begin{aligned} |\mathbf{F}| &= \frac{GMm}{r^2} \\ &= \frac{6.67 \times 10^{-11} \cdot 70 \cdot 70}{1^2} \\ &= 3.2683 \times 10^{-7}\,\text{N}. \end{aligned}$$

Clearly, the Earth has a much greater attraction than the other student.

We now consider the force due to gravity and and equation of motion, so

$$\begin{aligned} ma \;=\; F \;=\; |\mathbf{F}| &= \frac{GMm}{r^2} \\ \implies \qquad a &= \frac{GM}{r^2}. \end{aligned}$$

If r is close to r_0, so that

$$\frac{r - r_0}{r_0}$$

is small, then

$$\begin{aligned} a &= \frac{GM}{r_0^2} - \frac{2GM}{r_0^3}(r - r_0) + \frac{3GM}{r_0^4}(r - r_0)^2 + \cdots \\ &= \frac{GM}{r_0^2}\left(1 - \frac{2(r - r_0)}{r_0} + \frac{3(r - r_0)^2}{r_0^2} + \cdots\right) \\ &\approx \frac{GM}{r_0^2} \end{aligned}$$

using a Taylor expansion. Hence we can assume a is constant when $r \approx r_0$.

Definition: The *gravitational acceleration* of the Earth, g, at the surface is given by

$$g = \frac{GM_E}{R_E^2} = 9.81\,\text{ms}^{-2}$$

where M_E is the mass of the Earth and R_E is the radius of the Earth.

4.4 2D Motion in Uniform Gravity

4.4.1 Basic Equations

Consider a particle of constant mass (m) that moves in a 2D plane comprising of

- Horizontal motion : x-direction,
- Vertical motion : y or z-direction.

Example 4.5

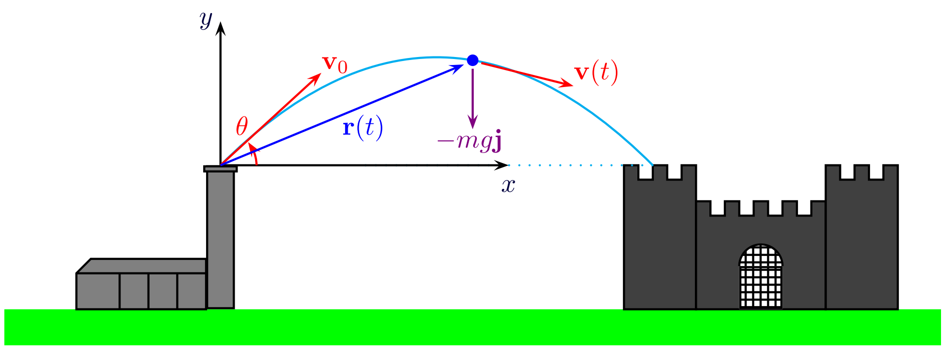
In 1547, during the siege of St Andrews Castle, a cannon has been placed on top of the tower of St Salvator's College, 300 m away. If a cannon ball is fired at $t = 0$ with an angle θ from the horizontal with a speed v_0, determine

i. The position, $\mathbf{r}(t)$ of the cannon ball.

ii. The angle required for the cannon ball to hit.

Assume the only force acting is uniform gravity and that the top of the towers of the college and castle are of the same height and the cannon is at the origin.

i. **Determine $\mathbf{r}(t)$.**



We have

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j},$$

with $x = 0$, $y = 0$, $|\mathbf{v}| = v_0$ at $t = 0$.

The force acting (i.e. gravity) is

$$\mathbf{F} = -mg\mathbf{j}.$$

The equation of motion is:

$$\begin{aligned}
\mathbf{F} &= m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}, \\
\implies \quad -mg\mathbf{j} &= m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\mathbf{i} + m\frac{\mathrm{d}^2y}{\mathrm{d}t^2}\mathbf{j}, \\
\therefore \quad \frac{\mathrm{d}^2x}{\mathrm{d}t^2}\mathbf{i} + \frac{\mathrm{d}^2y}{\mathrm{d}t^2}\mathbf{j} &= -g\mathbf{j}.
\end{aligned}$$

Resolving the individual components gives

$$\begin{aligned}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= 0,\\ \frac{\mathrm{d}^2y}{\mathrm{d}t^2} &= -g,\end{aligned}$$

where $x(0)=0$, $y(0)=0$, $\frac{\mathrm{d}x}{\mathrm{d}t}(0)=v_0\cos\theta$ and $\frac{\mathrm{d}y}{\mathrm{d}t}(0)=v_0\sin\theta$.

We now solve x and y separately.

- x-direction:

$$\begin{aligned} & \frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= 0,\\ \Longrightarrow\quad & \frac{\mathrm{d}x}{\mathrm{d}t} &= C_1 = v_0\cos\theta\\ \Longrightarrow\quad & x(t) &= v_0t\cos\theta + C_2\\ \therefore\quad & x(t) &= v_0t\cos\theta.\end{aligned}$$

- y-direction:

$$\begin{aligned} & \frac{\mathrm{d}^2y}{\mathrm{d}t^2} &= -g,\\ \Longrightarrow\quad & \frac{\mathrm{d}y}{\mathrm{d}t} &= -gt + C_3\\ \Longrightarrow\quad & \frac{\mathrm{d}y}{\mathrm{d}t} &= -gt + v_0\sin\theta\\ \Longrightarrow\quad & y(t) &= -\tfrac{1}{2}gt^2 + v_0t\sin\theta + C_4\\ \therefore\quad & y(t) &= v_0t\sin\theta - \tfrac{1}{2}gt^2.\end{aligned}$$

So

$$\mathbf{r}(t) = v_0t\cos\theta\mathbf{i} + \left(v_0t\sin\theta - \tfrac{1}{2}gt^2\right)\mathbf{j}.$$

4.4.2 Properties of Motion

- The *time of flight*, T, is the time taken for the cannon ball to fall back to $y=0$. So (with $T>0$)

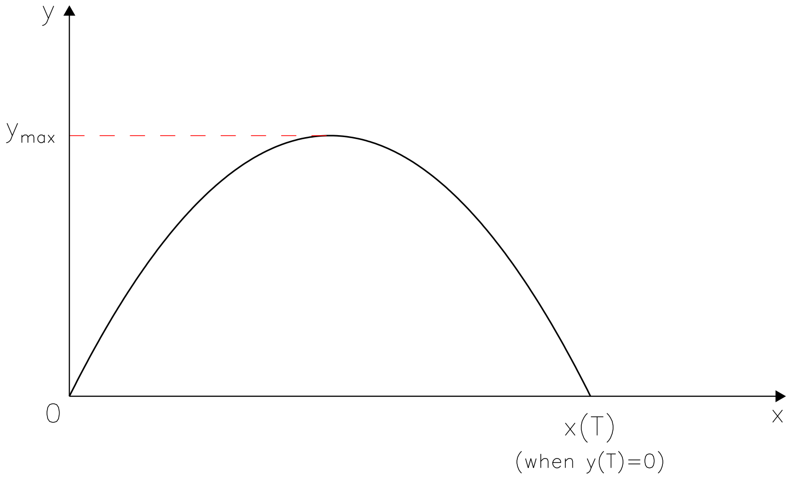
$$\begin{aligned} & y(T) &= 0\\ \Longrightarrow\quad & v_0T\sin\theta - \tfrac{1}{2}gT^2 &= 0\\ \Longrightarrow\quad & T\left(v_0\sin\theta - \tfrac{1}{2}gT\right) &= 0\\ \Longrightarrow\quad & \tfrac{1}{2}gT &= v_0\sin\theta \qquad (\because\quad T>0)\\ \therefore\quad & T &= \frac{2v_0}{g}\sin\theta.\end{aligned}$$

  For example, if we throw the cannon ball vertically upwards, i.e.

$$\theta = \frac{\pi}{2} = \tfrac{\pi}{2}$$

  then

$$T = \frac{2v_0}{g}.$$



- The *maximum height*, y_{max} occurs at time $t = t_{max}$ when

$$v_y = \frac{\mathrm{d}y}{\mathrm{d}t} = 0.$$

Hence this happens when

$$\begin{aligned} \implies && \frac{\mathrm{d}y}{\mathrm{d}t}(t_{max}) &= 0 \\ \implies && -gt_{max} + v_0 \sin\theta &= 0 \\ \therefore && t_{max} &= \frac{v_0 \sin\theta}{g}. \end{aligned}$$

And the maximum height reached is

$$\begin{aligned} y_{max} = y(t_{max}) &= y\left(\frac{v_0 \sin\theta}{g}\right) \\ &= v_0\left(\frac{v_0 \sin\theta}{g}\right)\sin\theta - \tfrac{1}{2}g\left(\frac{v_0 \sin\theta}{g}\right)^2 \\ &= \frac{v_0^2 \sin^2\theta}{g} - \frac{v_0^2 \sin^2\theta}{2g} \\ &= \frac{v_0^2 \sin^2\theta}{2g}. \end{aligned}$$

For example, in 1D with $\theta = \frac{\pi}{2}$ we have

$$\begin{aligned} t_{max} &= \frac{v_0}{g}, \\ y_{max} &= \frac{v_0^2}{2g}. \end{aligned}$$

- The *range*, R is the distance travelled, i.e.

$$R = x(T)$$

where

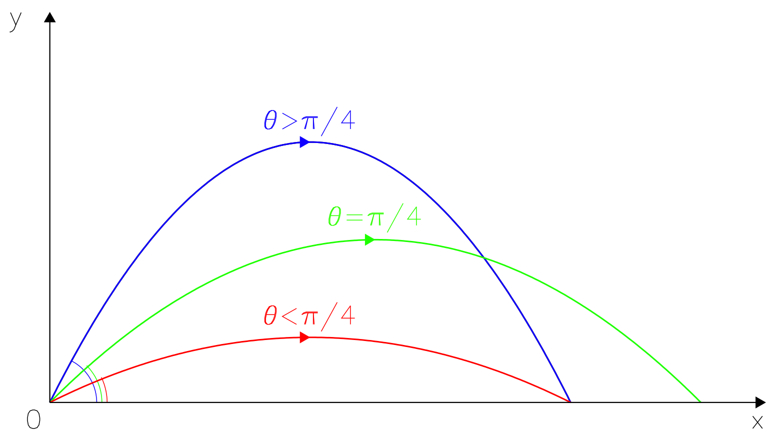
$$T = \frac{2v_0 \sin\theta}{g}.$$

Thus,

$$R = x(T)$$

$$
\begin{aligned}
&= x\left(\frac{2v_0\sin\theta}{g}\right)\\
&= v_0\left(\frac{2v_0\sin\theta}{g}\right)\cos\theta\\
&= \frac{2v_0^2\sin\theta\cos\theta}{g}\\
&= \frac{v_0^2\sin 2\theta}{g}.
\end{aligned}
$$

From this we can clearly see that that the maximum range is achieved when $\theta = \frac{\pi}{4}$ so that $\sin 2\theta = 1$.



- The *Equation of the Trajectory*, is the path of the cannonball, $y(x)$.

  To obtain $y(x)$, first invert $x(t)$ (i.e. solve for t) to get

$$
t = \frac{x}{v_0\cos\theta}
$$

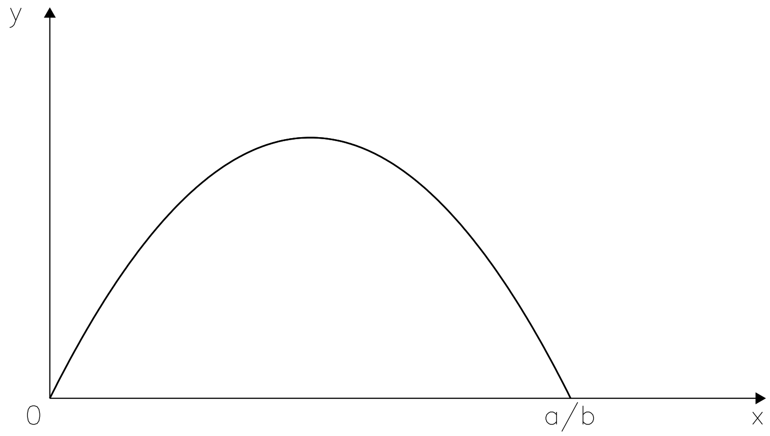
then

$$
\begin{aligned}
y(x) &= y(t(x))\\
&= y\left(\frac{x}{v_0\cos\theta}\right)\\
&= v_0\left(\frac{x}{v_0\cos\theta}\right)\sin\theta - \tfrac{1}{2}g\left(\frac{x}{v_0\cos\theta}\right)^2\\
&= \underbrace{(\tan\theta)}_{a}\,x - \underbrace{\left(\frac{g}{2v_0^2\cos^2\theta}\right)}_{b}x^2\\
&= ax - bx^2,
\end{aligned}
$$

where

$$
\begin{aligned}
a &= \tan\theta,\\
b &= \frac{g}{2v_0^2\cos^2\theta}.
\end{aligned}
$$

Hence the trajectory of the cannon ball is a parabola.



Since the range is given by the distance $R = x(T)$ such that $y(x) = 0$, then we can find the range by this alternative method:

$$\begin{array}{rrcl} & 0 & = & ax - bx^2 \\ \Longrightarrow & 0 & = & a - bx \qquad (\because x \neq 0) \\ \Longrightarrow & x & = & \dfrac{a}{b} \\ & & = & \dfrac{\sin\theta}{\cos\theta} \cdot \dfrac{2v_0^2\cos^2\theta}{g} \\ & & = & \dfrac{2v_0^2 \sin\theta\cos\theta}{g} \\ \therefore & R(T) & = & \dfrac{v_0^2 \sin 2\theta}{g}, \end{array}$$

which agrees with previous result.

- The *return velocity* is the velocity the cannon ball hits the castle ($y = 0$). This occurs when $y = 0$, so we have wither the initial velocity at $t = 0$ or the return velocity at $t = T$.

  Since
  $$T = \frac{2v_0}{g}\sin\theta.$$
  and
  $$\mathbf{v}(t) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = v_0\cos\theta\mathbf{i} + (v_0\sin\theta - gt)\,\mathbf{j}$$
  then the the return velocity is
  $$\mathbf{v}(T) = v_0\cos\theta\mathbf{i} - v_0\sin\theta\mathbf{j}.$$
  So the cannon ball returns at same speed it was fired at but with the vertical component of the velocity of opposite sign.

Returning to the second part of the question.

ii. **Angle to fire at:**

Since the range is given by
$$R(T) = \frac{v_0^2 \sin 2\theta}{g}$$

then since $R(T) = 300$ is the distance to the castle, we can rearrange to get $\sin(2\theta)$ as:

$$\begin{aligned} 300 &= \frac{v_0^2 \sin 2\theta}{g} \\ \Longrightarrow \quad \sin 2\theta &= \frac{300g}{v_0^2} \\ \Longrightarrow \quad \theta &= \tfrac{1}{2}\sin^{-1}\left(\frac{300g}{v_0^2}\right). \end{aligned}$$

We also see that if $v_0 < \sqrt{300g} \approx 54.25\, ms^{-1}$ it is not possible to hit the tower for any angle θ!

> Archbishop Hamilton was not pleased that the tower was used for this purpose, so he had the current stone spire built above the belfry.

4.4.3 1D Example

It may also be useful to consider $v(x)$ instead of $v(t)$, i.e.

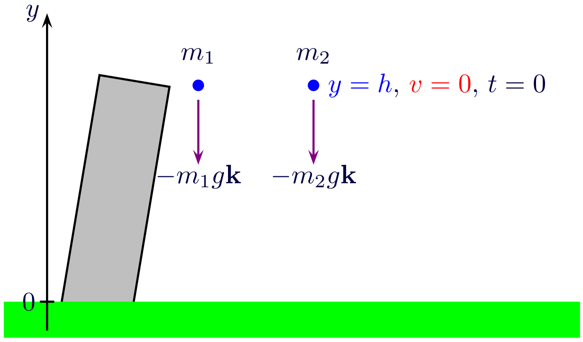
$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}x}$$

so that

$$a = v\frac{\mathrm{d}v}{\mathrm{d}x}.$$

> Note: Integrating both sides gives
>
> $$\tfrac{1}{2}v^2 = ax + C$$
>
> If $v = v_0$ when $x = x_0$ then
>
> $$C = \tfrac{1}{2}v_0^2 - ax_0$$
>
> and so
>
> $$v^2 - v_0^2 = 2ax - 2ax_0.$$

> **Example 4.6**
>
> Galileo Galilei (1564–1642) is rumoured to have dropped two cannon balls from the Leaning Tower of Pisa with masses m_1 and $m_2 = 10m_1$.
> At
>
> i. what time
>
> ii. what velocity
>
> do the two cannon balls hit the ground if the tower has height h?
> *Note: Assume gravity is constant.*



Let $\mathbf{r} = y\mathbf{j}$ and $\mathbf{v} = v\mathbf{j}$. Clearly,

$$\mathbf{a} = -g\mathbf{j},$$

so

$$\begin{aligned} m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} &= m\mathbf{a} \\ \Longrightarrow \quad m\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\mathbf{j} &= -mg\mathbf{j} \\ \Longrightarrow \quad \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} &= -g \\ \Longrightarrow \quad v(t) = \frac{\mathrm{d}y}{\mathrm{d}t} &= -gt + C \\ \Longrightarrow \quad y(t) &= -\tfrac{1}{2}gt^2 + Ct + D. \end{aligned}$$

Initially, $v(0) = 0$ and $y(0) = h$, so

$$\begin{aligned} v(0) &= 0 \\ \Longrightarrow \quad -g\cdot 0 + C &= 0 \\ \Longrightarrow \quad C &= 0, \end{aligned}$$

and

$$\begin{aligned} y(0) &= h \\ \Longrightarrow \quad -\tfrac{1}{2}g\cdot 0^2 + 0\cdot 0 + D &= h \\ \Longrightarrow \quad D &= h. \end{aligned}$$

So

$$\begin{aligned} y(t) &= h - \tfrac{1}{2}gt^2, \\ v(t) &= -gt. \end{aligned}$$

Hence the two cannon balls fall at the same rate.

i. When $y = 0$, we get

$$\begin{aligned} 0 &= h - \tfrac{1}{2}gt^2 \\ \Longrightarrow \quad gt^2 &= 2h \\ \Longrightarrow \quad t &= \sqrt{\frac{2h}{g}}. \qquad (\because t > 0) \end{aligned}$$

ii. Also

$$\begin{aligned} v &= -g\left(\sqrt{\frac{2h}{g}}\right) \\ &= -\sqrt{2gh}. \end{aligned}$$

Alternatively,

$$\begin{array}{rrcl}
 & \frac{\mathrm{d}v}{\mathrm{d}t} & = & -g \\
\Longrightarrow & \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}v}{\mathrm{d}y} & = & -g \\
\Longrightarrow & v\frac{\mathrm{d}v}{\mathrm{d}y} & = & -g \\
\Longrightarrow & v\,\mathrm{d}v & = & -g\mathrm{d}y \\
\therefore & \frac{1}{2}v^2 & = & -gy + C.
\end{array}$$

Since $v = 0$ when $y = h$, then

$$\begin{array}{rrcl}
\Longrightarrow & \frac{1}{2}\cdot 0^2 & = & -g\cdot h + C \\
\Longrightarrow & C & = & gh,
\end{array}$$

and hence

$$v^2 = 2g(h - y).$$

So when the object hits the ground (i.e. $y = 0$),

$$\begin{array}{rrcl}
 & v^2 & = & 2g(h - 0) \\
\Longrightarrow & v & = & \pm\sqrt{2gh} \\
\therefore & v & = & -\sqrt{2gh},
\end{array}$$

as the object is falling (i.e. $v < 0$).

Since the Leaning Tower of Pisa is $h = 56\,\mathrm{m}$ tall, it would have landed at a speed of $33.15\,\mathrm{ms}^{-1}$ after 3.38 seconds

4.5 Air Resistance

Definition: *Air resistance*, R, is a force which acts against an object as it travels through the air. Depeneding on the model used it can be written

$$\mathbf{R} = -mk\mathbf{v}$$

or

$$\mathbf{R} = -mk|\mathbf{v}|\mathbf{v}$$

where $k > 0$ is a constant.
We use both in this course – form required will always be given.

4.5.1 1D Motion with Air Resistance

The effect of air resistance $\mathbf{R}$ opposing 1D vertical motion is

$$\mathbf{R} = -mk\mathbf{v} = -mkv\mathbf{j}$$

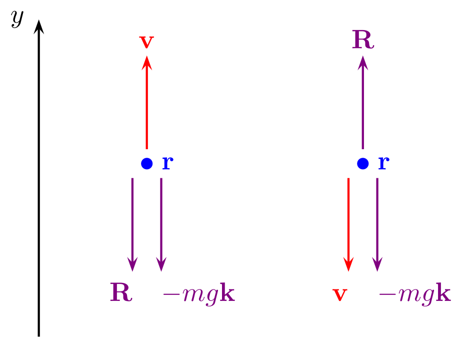
or

$$\mathbf{R} = -mkv^2\mathbf{j}$$

and $k > 0$.

So when

- Object is falling ($v < 0$), air resistance force is upwards ($R > 0$),
- Object is rising ($v > 0$), air resistance force is downwards ($R < 0$).



> **Example 4.7**
>
> An trampolinist of mass m moves vertically upwards with initial velocity $v(0) = v_0 > 0$ from $y = 0$ under uniform gravity and an air resistance of
>
> $$\mathbf{R} = -mk\mathbf{v}.$$
>
> i. What are the units of k?
>
> ii. Find their height as a function of time, $y(t)$,
>
> iii. Find their height as a function of the velocity, $y(v)$.
>
> iv. What happens to $y(t)$ as $k \to 0$?

i. **Units of k:**

Since

$$v \sim \frac{[L]}{[T]} \quad , \quad F \sim \frac{[M][L]}{[T]^2} \quad \text{and} \quad m \sim [M]$$

then

$$\begin{aligned} \frac{[M][L]}{[T]^2} &\sim [M] \cdot k \cdot \frac{[L]}{[T]} \\ \implies \qquad k &\sim \frac{1}{[T]}. \end{aligned}$$

ii. **Position as a function of time $y(t)$:**

Starting with the equation of motion,

$$\mathbf{F} = m\mathbf{a},$$

with

$$\begin{aligned} \mathbf{a} &= \frac{\mathrm{d}v}{\mathrm{d}t}\mathbf{j}, \\ \mathbf{F} &= -mg\mathbf{j} - mkv\mathbf{j} \end{aligned}$$

gives

$$m\frac{\mathrm{d}v}{\mathrm{d}t}\mathbf{j} = -mg\mathbf{j} - mkv\mathbf{j}$$

$$\begin{aligned}
\Longrightarrow \qquad \frac{\mathrm{d}v}{\mathrm{d}t} &= -g - kv \\
\Longrightarrow \qquad \frac{\mathrm{d}v}{\mathrm{d}t} + kv &= -g.
\end{aligned}$$

Using the integrating factor,

$$I(t) = \exp\left\{\int k\,\mathrm{d}t\right\} = e^{kt}$$

gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(e^{kt}v) &= -ge^{kt} \\
\Longrightarrow \qquad e^{kt}v &= -\int ge^{kt}\,\mathrm{d}t \\
\Longrightarrow \qquad e^{kt}v &= -\frac{g}{k}e^{kt} + C \\
\therefore \qquad v(t) &= -\frac{g}{k} + Ce^{-kt}.
\end{aligned}$$

Initially $v(0) = v_0$, so

$$\begin{aligned}
v_0 &= -\frac{g}{k} + Ce^{-k\cdot 0} \\
\Longrightarrow \qquad C &= v_0 + \frac{g}{k}
\end{aligned}$$

hence

$$v(t) = -\frac{g}{k} + \left(v_0 + \frac{g}{k}\right)e^{-kt}$$

Integrating gives

$$y(t) = -\frac{gt}{k} - \frac{1}{k}\left(v_0 + \frac{g}{k}\right)e^{-kt} + C$$

Since $y = 0$ at $t = 0$ then

$$C = 0 + \frac{g + kv_0}{k^2}.$$

Therefore

$$\begin{aligned}
y(t) &= -\frac{gt}{k} - \frac{g + kv_0}{k^2}e^{-kt} + \frac{g + kv_0}{k^2} \\
&= \frac{g + kv_0}{k^2}\left(1 - e^{-kt}\right) - \frac{gt}{k}.
\end{aligned}$$

iii. **Position as a function of velocity $y(v)$:**

Either integrate

$$\Longrightarrow \qquad \frac{\mathrm{d}v}{\mathrm{d}t} = -g - kv$$

with

$$\frac{\mathrm{d}v}{\mathrm{d}t} = v\,\frac{\mathrm{d}v}{\mathrm{d}y}$$

or we use the previous solutions to find $y(v)$ by elimating t. Since

$$\begin{aligned}
v &= -\frac{g}{k} + \left(v_0 + \frac{g}{k}\right)e^{-kt} \\
\Longrightarrow \qquad v + \frac{g}{k} &= \left(v_0 + \frac{g}{k}\right)e^{-kt} \\
\Longrightarrow \qquad e^{-kt} &= \frac{g + kv}{g + kv_0} \\
\Longrightarrow \qquad t &= -\frac{1}{k}\log_e\left|\frac{g + kv}{g + kv_0}\right|,
\end{aligned}$$

then

$$\begin{aligned} y &= \frac{g+kv_0}{k^2}\left(1-e^{-kt}\right) - \frac{gt}{k} \\ &= \frac{g+kv_0}{k^2}\left(1-\frac{g+kv}{g+kv_0}\right) + \frac{g}{k^2}\log_e\left|\frac{g+kv}{g+kv_0}\right| \\ &= \frac{g+kv_0}{k^2}\left(\frac{kv_0-kv}{g+kv_0}\right) + \frac{g}{k^2}\log_e\left|\frac{g+kv}{g+kv_0}\right| \\ &= \frac{v_0-v}{k} + \frac{g}{k^2}\log_e\left|\frac{g+kv}{g+kv_0}\right|. \end{aligned}$$

iv. **When air resistance tends to zero, i.e. $k \to 0$:**

By Taylor Expansion,

$$e^{-kt} = 1 - kt + \tfrac{1}{2}k^2t^2 - \cdots$$

so as $k \to 0$

$$\begin{aligned} y(t) &= \frac{g+kv_0}{k^2}\left(1-\left\{1-kt+\tfrac{1}{2}k^2t^2-\cdots\right\}\right) - \frac{gt}{k} \\ &= \frac{g+kv_0}{k^2}\left(kt-\tfrac{1}{2}k^2t^2+\cdots\right) - \frac{gt}{k} \\ &= \frac{gt}{k} + v_0t - \tfrac{1}{2}gt^2 - \underbrace{\tfrac{1}{2}kv_0t^2}_{=0} - \frac{gt}{k} - \cdots \\ &\approx v_0t - \tfrac{1}{2}gt^2, \end{aligned}$$

which is same as our previous expression.

4.5.2 Properties of motion (1D)

- The object is at *maximum height*, $y = y_{max}$ at $t = t_{max}$ when $v = 0$.

  Hence

$$\begin{aligned} & & v(t_{max}) &= 0 \\ \Longrightarrow & & -\frac{g}{k} + \left(v_0 + \frac{g}{k}\right)e^{-kt} &= 0 \\ \Longrightarrow & & \left(v_0 + \frac{g}{k}\right)e^{-kt} &= \frac{g}{k} \\ \Longrightarrow & & e^{-kt} &= \frac{g}{g+kv_0} \\ \therefore & & t &= -\frac{1}{k}\log_e\left|\frac{g}{g+kv_0}\right| \\ & & &= \frac{1}{k}\log_e\left|1+\frac{kv_0}{g}\right|. \end{aligned}$$

  And so

$$\begin{aligned} y_{max} &= \frac{v_0-0}{k} + \frac{g}{k^2}\log_e\left|\frac{g+k\cdot 0}{g+kv_0}\right| \\ &= \frac{v_0}{k} - \frac{g}{k^2}\log_e\left|1+\frac{kv_0}{g}\right| \end{aligned}$$

  Now consider what happens when $k \to 0$:

Since (provided $|x| < 1$)

$$\log_e |1 + x| = x - \frac{x^2}{2} + \cdots$$

then

$$\begin{aligned} t &= \frac{1}{k} \log_e \left| 1 + \frac{kv_0}{g} \right| \\ &= \frac{1}{k} \left\{ \frac{kv_0}{g} - \frac{k^2 v_0^2}{2g^2} + \cdots \right\} \\ &= \frac{v_0}{g} - \frac{kv_0^2}{2g^2} + \cdots \\ &\to \frac{v_0}{g} \end{aligned}$$

and

$$\begin{aligned} y_{max} &= \frac{v_0}{k} - \frac{g}{k^2} \log_e \left| 1 + \frac{kv_0}{g} \right| \\ &= \frac{v_0}{k} - \frac{g}{k^2} \left\{ \frac{kv_0}{g} - \frac{k^2 v_0^2}{2g^2} + \cdots \right\} \\ &= \frac{v_0}{k} - \frac{v_0}{k} + \frac{v_0^2}{2g} - \cdots \\ &= \frac{v_0^2}{2g} - \cdots \\ &\to \frac{v_0^2}{2g}. \end{aligned}$$

These are the same as the previous expressions (without air resistance).

- The *speed of return*, $|v_r|$, occurs when $y = 0$ (and $t > 0$), so we find that it must satisfy

$$\begin{aligned} v_r &= v_0 + \frac{g}{k} \log_e \left| \frac{g + kv_r}{g + kv_0} \right| \\ &= v_0 + \frac{g}{k} \log_e \left| \frac{1 + \frac{kv_r}{g}}{1 + \frac{kv_0}{g}} \right| \\ \implies \quad v_r - \frac{g}{k} \log_e \left| 1 + \frac{kv_r}{g} \right| &= v_0 - \frac{g}{k} \log_e \left| 1 + \frac{kv_0}{g} \right| \\ \therefore \quad \frac{kv_r}{g} - \log_e \left| 1 + \frac{kv_r}{g} \right| &= \frac{kv_0}{g} - \log_e \left| 1 + \frac{kv_0}{g} \right|, \end{aligned}$$

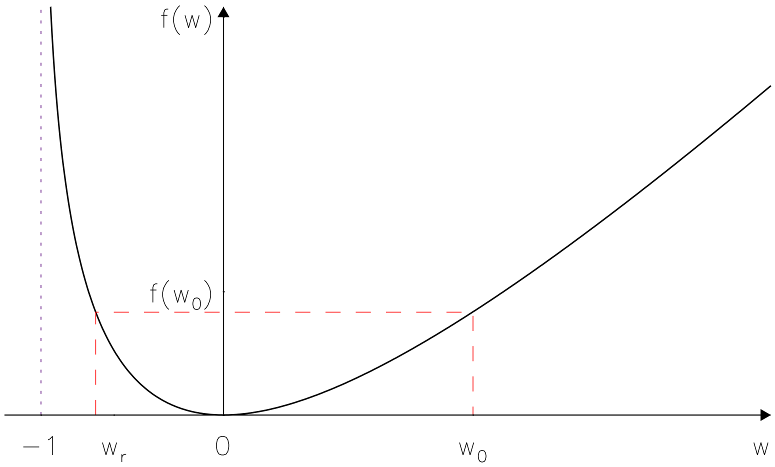
which cannot be solved for v_r! Using an implicit approach and substituting

$$w_r = \frac{kv_r}{g} \quad \text{and} \quad w_0 = \frac{kv_0}{g}$$

so that

$$\begin{aligned} f(w_r) &= f(w_0) \\ \implies \quad w_r - \log_e |1 + w_r| &= w_0 - \log_e |1 + w_0| . \end{aligned}$$

Clearly one solution is $w_r = w_0$ (which is the initial condition). Is there another solution?



Note on the graph that:

- $w - \log_e|1+w| \to +\infty$ as $w \to -1$.
- $f(0) = 0$.
- since $w \gg \log_e|1+w|)$ then $f(w) \to w$ as $w \to \infty$.

So as $f(w_0)$ cuts the curve twice, there are two solutions:

1. $v = v_0$
2. $v = v_r$ where $|v_r| < |v_0|$ due to linear dependence in $f(w)$.

Hence the projectile returns at a slower velocity.

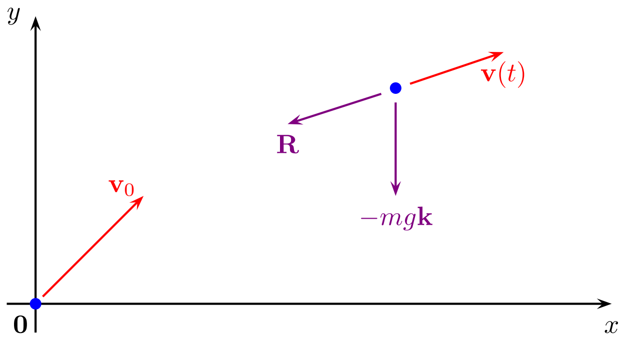
4.5.3 2D Motion with Air Resistance

Example 4.8

Consider the motion of a golf ball (of mass m) towards the 18th hole of the Old Course, which experiences:

- gravity $\mathbf{F}_g = -mg\mathbf{j}$ and
- air resistance, $\mathbf{R} = -mk\mathbf{v}$.

Assume that the golf ball is leaves the club at a speed v_0 and angle θ from the horizontal above flat ground.



The equation of motion is now

$$m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = -mg\mathbf{j} - mk\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}$$

Taking components gives

$$\begin{aligned}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= -k\frac{\mathrm{d}x}{\mathrm{d}t},\\ \frac{\mathrm{d}^2y}{\mathrm{d}t^2} &= -g - k\frac{\mathrm{d}y}{\mathrm{d}t},\end{aligned}$$

or in terms of velocity components

$$\begin{aligned}\frac{\mathrm{d}v_x}{\mathrm{d}t} &= -kv_x,\\ \frac{\mathrm{d}v_y}{\mathrm{d}t} &= -g - kv_y.\end{aligned}$$

First we solve the x-component (using $v_x(0) = v_0\cos\theta$ and $x(0) = 0$):

$$\begin{aligned}\frac{\mathrm{d}v_x}{\mathrm{d}t} &= -kv_x\\ \Longrightarrow \qquad v_x &= Ce^{-kt}\\ \Longrightarrow \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = v_x &= v_0e^{-kt}\cos\theta\\ \Longrightarrow \qquad x(t) &= -\frac{v_0\cos\theta}{k}\,e^{-kt} + D.\end{aligned}$$

Using the initial condition $x(0) = 0$ gives

$$\begin{aligned}-\frac{v_0\cos\theta}{k}\,e^{-k\cdot 0} + D &= 0\\ \Longrightarrow \qquad D &= \frac{v_0\cos\theta}{k}.\end{aligned}$$

So

$$x(t) = \frac{v_0\cos\theta}{k}\left(1 - e^{-kt}\right).$$

Now solve for the y-component (using $v_x(0) = v_0\cos\theta$ and $y(0) = 0$):

$$\begin{aligned}\frac{\mathrm{d}v_y}{\mathrm{d}t} &= -g - kv_y\\ \Longrightarrow \qquad \frac{\mathrm{d}v_y}{\mathrm{d}t} + kv_y &= -g\end{aligned}$$

Using the integrating factor

$$I(t) = \exp\left\{\int k\mathrm{d}t\right\} = e^{kt}$$

gives

$$\begin{aligned}v_y &= \frac{1}{I(t)}\int -gI(t)\,\mathrm{d}t + \frac{C}{I(t)}\\ &= e^{-kt}\int -ge^{kt}\,\mathrm{d}t + Ce^{-kt}\\ &= e^{-kt}\cdot\left(-\frac{g}{k}e^{kt}\right) + Ce^{-kt}\\ &= -\frac{g}{k} + Ce^{-kt}.\end{aligned}$$

Using $v_y(0) = v_0 \sin\theta$ gives

$$
\begin{aligned}
-\frac{g}{k} + Ce^{-k\cdot 0} &= v_0 \sin\theta \\
\Longrightarrow \qquad C &= v_0 \sin\theta + \frac{g}{k}
\end{aligned}
$$

So

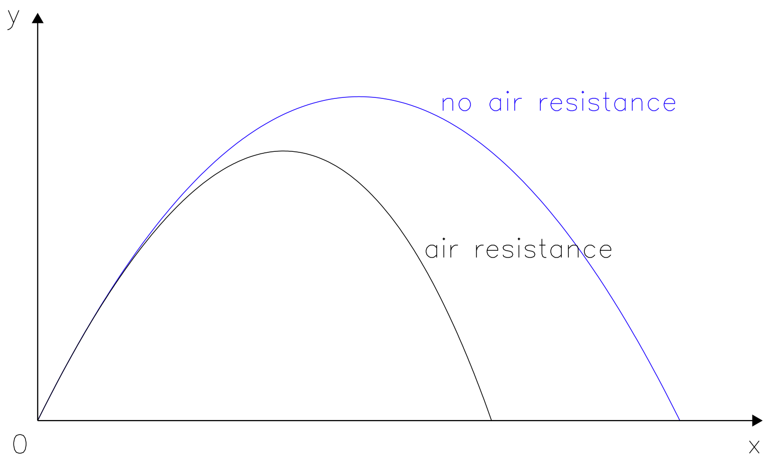
$$
\begin{aligned}
v_y = \frac{\mathrm{d}y}{\mathrm{d}t} &= \left(v_0 \sin\theta + \frac{g}{k}\right) e^{-kt} - \frac{g}{k} \\
\Longrightarrow \qquad y &= -\frac{1}{k}\left(v_0 \sin\theta + \frac{g}{k}\right) e^{-kt} - \frac{gt}{k} + D
\end{aligned}
$$

Since $y(0) = 0$ then

$$
\begin{aligned}
0 &= -\frac{1}{k}\left(v_0 \sin\theta + \frac{g}{k}\right) e^{-k\cdot 0} - \frac{g\cdot 0}{k} + D \\
\Longrightarrow \qquad D &= \frac{1}{k}\left(v_0 \sin\theta + \frac{g}{k}\right)
\end{aligned}
$$

And hence

$$
\begin{aligned}
y(t) &= -\frac{1}{k}\left(v_0 \sin\theta + \frac{g}{k}\right) e^{-kt} - \frac{gt}{k} + \frac{1}{k}\left(v_0 \sin\theta + \frac{g}{k}\right) \\
&= \left(\frac{v_0}{k} \sin\theta + \frac{g}{k^2}\right)\left(1 - e^{-kt}\right) - \frac{gt}{k}
\end{aligned}
$$



4.6 1D Motion with Variable Gravity

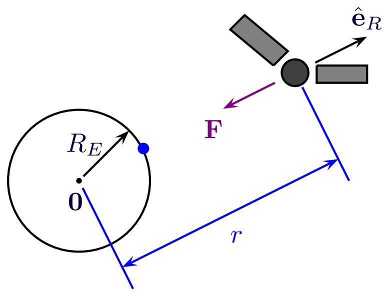
If a object's height varies by more than a small fraction of the Earth's radius then gravity is no longer constant, and hence we need to use

$$
\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}},
$$

where G is the gravitational constant, M and m are the masses of the particles and r is their separation.

Example 4.9

Consider the motion of the flight of the a space probe as it departs from Earth (under the influence of Earth's gravity only).



Since

$$g = \frac{GM}{R_E^2}$$

then

$$\mathbf{F} = -GM\frac{m}{r^2}\hat{\mathbf{r}} = -gR_E^2 \cdot \frac{m}{r^2}\hat{\mathbf{r}} = -mg\left(\frac{R_E}{r}\right)^2\hat{\mathbf{r}}.$$

So the equation of motion becomes

$$m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = \mathbf{F}(\mathbf{r})$$

where the origin is the centre of the Earth. So

$$\begin{aligned} m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} &= -mg\left(\frac{R_E}{r}\right)^2\hat{\mathbf{r}} \\ \Longrightarrow \quad \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} &= -g\left(\frac{R_E}{r}\right)^2\hat{\mathbf{r}} \\ \Longrightarrow \quad \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 r}{\mathrm{d}t^2} &= -g\left(\frac{R_E}{r}\right)^2. \end{aligned}$$

Since

$$\frac{\mathrm{d}v}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}r}$$

then

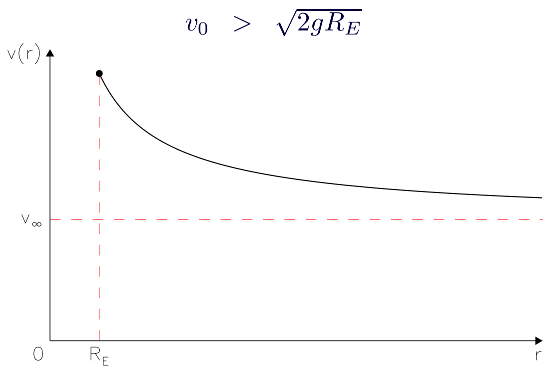
$$\begin{aligned} v\frac{\mathrm{d}v}{\mathrm{d}r} &= -g\left(\frac{R_E}{r}\right)^2 \\ \Longrightarrow \quad \int v\,\mathrm{d}v &= -gR_E^2\int\frac{\mathrm{d}r}{r^2} \\ \Longrightarrow \quad \tfrac{1}{2}v^2 &= \frac{gR_E^2}{r} + C \end{aligned}$$

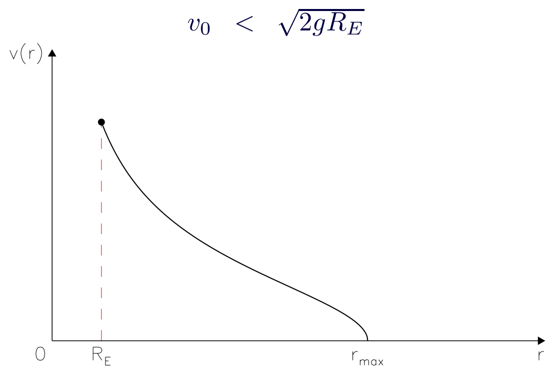
Assuming the rocket takes off at $v = v_0$ from $r = R_E$ then

$$\begin{aligned} \tfrac{1}{2}v_0^2 &= \frac{gR_E^2}{r_E} + C \\ \Longrightarrow \quad C &= \tfrac{1}{2}v_0^2 - gR_E \end{aligned}$$

Therefore

$$v^2 = v_0^2 + 2gR_E\left(\frac{R_E}{r} - 1\right).$$





For particle to escape to ∞ then $v > 0$ for all r. Since

$$\lim_{r\to\infty}\left\{\frac{R_E}{r}\right\} \;=\; 0$$

then we must have

$$\begin{aligned} v_0^2 - 2gR_E &> 0 \\ \Longrightarrow \qquad v_0 &> \sqrt{2gR_E}. \end{aligned}$$

Also when particle does not escape then when $v = 0$ and $r = r_{max}$:

$$\begin{aligned} 0^2 &= v_0^2 + 2gR_E\left(\frac{R_E}{r_{max}} - 1\right) \\ 2gR_E\left(\frac{R_E}{r_{max}} - 1\right) &= -v_0^2 \\ \Longrightarrow \qquad \frac{R_E}{r_{max}} - 1 &= -\frac{v_0^2}{2gR_E} \\ \Longrightarrow \qquad \frac{R_E}{r_{max}} &= 1 - \frac{v_0^2}{2gR_E} \\ \therefore \qquad r_{max} &= \frac{gR_E^2}{gR_E - \frac{1}{2}v_0^2}. \end{aligned}$$

Definition: The *escape velocity*,

$$v_e \;=\; \sqrt{2gR_E} \;=\; \sqrt{\frac{2GM}{R}},$$

is the minimum velocity required (at Earth's surface) to fully escape from Earth's gravity.

Example 4.10

What are the escape velocities of

i. The Earth $\quad (g_E = 9.81\,\mathrm{ms}^{-2},\ R_E = 6.378\times 10^6\,\mathrm{m})$

ii. The Moon $\quad (M_M = 7.35\times 10^{22}\,\mathrm{kg},\ R_M = 1.74\times 10^6\,\mathrm{m})$

Earth:

$$v_e \;=\; \sqrt{2\cdot 9.81\cdot 6.378\times 10^6} \;=\; 11,186\,\mathrm{ms}^{-1}.$$

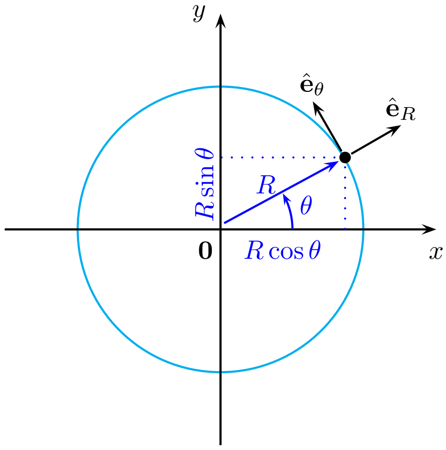
Moon:

$$v_e = \sqrt{\frac{2 \cdot 6.67 \times 10^{-11} \cdot 7.35 \times 10^{22}}{1.74 \times 10^6}} = 2,374\,\mathrm{ms}^{-1}.$$

So the Moon's escape velocity is much lower than the Earth's.

4.7 Circular Motion

Consider an object at a constant speed v, moving around a circular orbit with constant radius R.



Definition:

- The position on a *circular orbit* is given by

$$R(t) = R\hat{\mathbf{e}}_R(t) = x\mathbf{i} + y\mathbf{j}$$

  where

$$\hat{\mathbf{e}}_R(t) = \cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j}$$

  rotates at a constant rate, i.e. ω is constant.

- The *angular position*, θ (in radians), is given by

$$\theta = \omega t.$$

- The *angular velocity*, ω (in $\mathrm{rad\,s}^{-1}$), is the rate of change of angular position

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t}.$$

- Finally, the *period* for one rotation, T (in seconds), is given by

$$T = \frac{2\pi}{\omega}.$$

Using this, we can determine that the velocity is

$$\begin{aligned}\mathbf{v}(t) &= \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = R\frac{\mathrm{d}\hat{\mathbf{R}}}{\mathrm{d}t}\\ &= R\frac{\mathrm{d}}{\mathrm{d}t}(\cos\omega t\mathbf{i} + \sin\omega t\mathbf{j})\end{aligned}$$

$$\begin{aligned} &= \omega R\,(-\sin\omega t\mathbf{i} + \cos\omega t\mathbf{j}) \\ &= \omega R\,\hat{\mathbf{e}}_\theta(t) \end{aligned}$$

where

$$\hat{\mathbf{e}}_\theta(t) \;=\; -\sin\omega t\mathbf{i} + \cos\omega t\mathbf{j}.$$

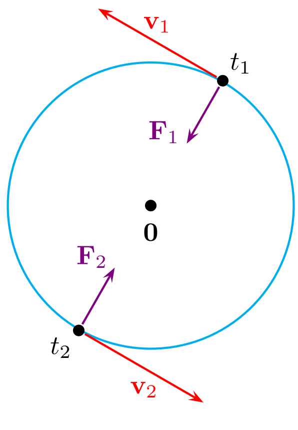
> Hence the speed is
> $$v \;=\; |\mathbf{v}| \;=\; \omega R.$$

Finally, the acceleration is

$$\begin{aligned} \mathbf{a}(t) &= \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \;=\; \frac{\mathrm{d}}{\mathrm{d}t}(\omega R\,\hat{\mathbf{e}}_\theta) \\ &= \omega R\,\frac{\mathrm{d}}{\mathrm{d}t}(-\sin\omega t\mathbf{i} + \cos\omega t\mathbf{j}) \\ &= -\omega^2 R\,(\cos\omega t\mathbf{i} + \sin\omega t\mathbf{j}) \\ &= -\omega^2 R\,\hat{\mathbf{e}}_R. \end{aligned}$$

> As said above, ω =constant has been assumed, otherwise there would be more terms as ω would also have to be differentiated with respect to time.
> The acceleration is always directed towards the centre and has magnitude:
> $$a \;=\; \omega^2 R \;=\; \frac{v^2}{R}.$$

This means that even though the particle has a constant speed v, there is an acceleration which is directed towards the centre. Therefore a force must be acting on the particle. The produce the angular motion the force always changes the direction of velocity, not its magnitude.



So $|\mathbf{v}_1| = |\mathbf{v}_2|$ and $|\mathbf{F}_1| = |\mathbf{F}_2|$.

Example 4.11

The *Death Star*, a giant space station, orbits around an Earth-like planet at a radius r.

i. What speed is it travelling at?

ii. At what orbit does it stay about the same point on the plane, i.e. when the period,

$$T = 86,400\,\mathrm{s} = 1\,\mathrm{day}.$$

i. From Newton's second law,

$$\begin{aligned} \mathbf{F} &= m\mathbf{a} \\ \implies \quad \frac{GMm}{r^2} &= \frac{mv^2}{r} \\ \therefore \quad v &= \sqrt{\frac{GM}{r}}. \end{aligned}$$

ii. Since

$$T = \frac{2\pi}{\omega} \quad \text{and} \quad v = \omega r$$

then

$$\begin{aligned} v^2 &= \frac{GM}{r} \\ \implies \quad \omega^2 r^2 &= \frac{GM}{r} \\ \implies \quad \left(\frac{2\pi}{T}\right)^2 r^2 &= \frac{GM}{r} \\ \implies \quad r^3 &= \frac{GMT^2}{4\pi^2} \\ \therefore \quad r &= \sqrt[3]{\frac{GMT^2}{4\pi^2}}. \end{aligned}$$

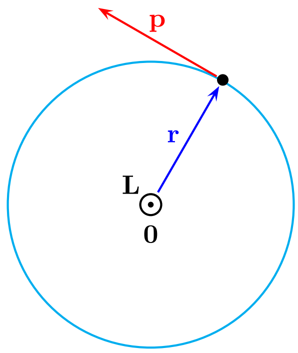
So using values gives

$$r \approx \sqrt[3]{\frac{6.67\times 10^{-11}\cdot 5.89\times 10^{24}\cdot 86400^2}{4\pi^2}} \approx 42,037,464\,\mathrm{m}.$$

Definition: As the particle rotates its *angular momentum* ($\mathbf{L}$) defined relative to the origin is

$$\mathbf{L} = \mathbf{r}\times\mathbf{p} = \mathbf{r}\times(m\mathbf{v})$$

and is normal to the plane of the orbit.



Now,

$$
\begin{aligned}
\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\{\mathbf{r}\times(m\mathbf{v})\} \\
&= \underbrace{\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}}_{\mathbf{v}}\times(m\mathbf{v}) + \mathbf{r}\times\frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v}) \\
&= \mathbf{r}\times\left(m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right) \\
&= \mathbf{r}\times\mathbf{F},
\end{aligned}
$$

where $\mathbf{F}$ is the force acting on the particle.

Definition: The quatity

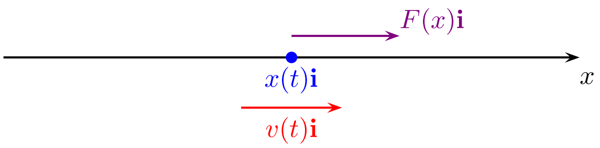
$$\boldsymbol{\tau} = \mathbf{r}\times\mathbf{F}$$

is called the *torque*.

4.8 Energy Equation

4.8.1 Derivation of Energy Equation

Definition: Consider the 1D motion of a particle acted upon by a (conservative) force $F(x)\mathbf{i}$ along the x-axis.



Then let the *potential function* $V(x)$ be defined as

$$F(x) = -\frac{\mathrm{d}V}{\mathrm{d}x},$$

so that

$$V(x) = -\int F(x)\,\mathrm{d}x + C.$$

This satisfies the *energy equation*,

$$\underbrace{\tfrac{1}{2}mv^2}_{\text{kinetic}} + \underbrace{V(x)}_{\text{potential}} = \underbrace{E}_{\text{total}}$$

where the total energy (E) is constant.

We now derive the energy equation. Since in 1D

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = F(x)$$

and

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}v}{\mathrm{d}x} = v\frac{\mathrm{d}v}{\mathrm{d}x}$$

then

$$\begin{aligned}
& & mv\frac{\mathrm{d}v}{\mathrm{d}x} &= F(x) \\
&\Longrightarrow & mv\,\mathrm{d}v &= F(x)\,\mathrm{d}x \\
&\Longrightarrow & \int mv\,\mathrm{d}v &= \int F(x)\,\mathrm{d}x \\
&\Longrightarrow & \tfrac{1}{2}mv^2 &= -V(x) + E \\
&\therefore & \tfrac{1}{2}mv^2 + V(x) &= E,
\end{aligned}$$

where E is the constant of integration.

To find the total energy, all we need is an initial condition, for example, at $x = x_0$ with $v = v_0$ at $t = 0$ has

$$E = \tfrac{1}{2}mv_0^2 + V_0$$

where $V_0 = V(x_0)$.

4.8.2 General Method

Example 4.12

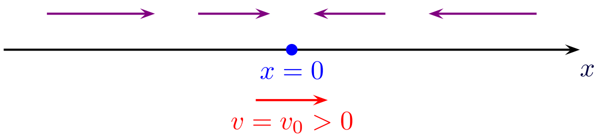
A ball of mass $m = 1$ attached to a spring moves along the x-axis under a force $\mathbf{F} = -x\mathbf{i}$ from $x = 0$ at $t = 0$ and $v = v_0 > 0$.
Find

(a) the potential if $V(0) = 0$.

(b) the energy equation

(c) the total energy (in terms of v_0).

(d) the behaviour of the particle.

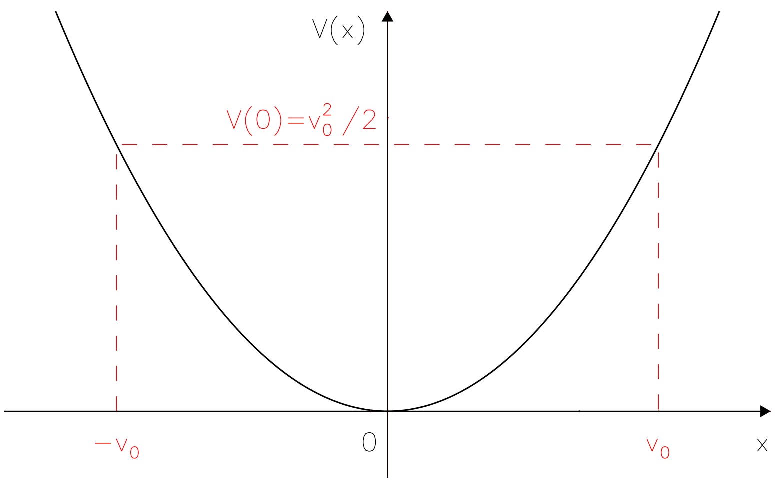


(a) The potential function is given by

$$\begin{aligned} V(x) &= -\int F(x)\,\mathrm{d}x \\ &= -\int (-x)\,\mathrm{d}x \\ &= \tfrac{1}{2}x^2 + C. \end{aligned}$$

Since $V(0) = 0$ then $C = 0$ so

$$V(x) = \tfrac{1}{2}x^2.$$



(b) The energy equation is

$$\begin{aligned} \tfrac{1}{2}mv^2 + V(x) &= E \\ \Longrightarrow \qquad \tfrac{1}{2}v^2 + \tfrac{1}{2}x^2 &= E. \end{aligned}$$

(c) Since $x = 0$ and $v = v_0$ at $t = 0$ then the total energy is

$$\begin{aligned} E &= \tfrac{1}{2}v^2 + \tfrac{1}{2}x^2 \\ &= \tfrac{1}{2}v_0^2 + \tfrac{1}{2}\cdot 0^2 \\ &= \tfrac{1}{2}v_0^2. \end{aligned}$$

(d) Rearranging the energy equation gives

$$\tfrac{1}{2}v^2 = E - V(x)$$

But since we must have

$$\tfrac{1}{2}v^2 \geq 0$$

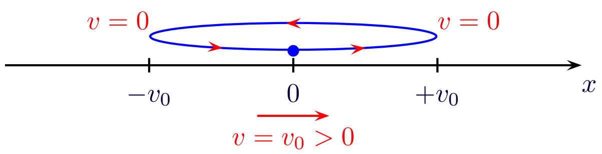
then

$$\begin{aligned} && V(x) &\leq E \\ \Longrightarrow && \tfrac{1}{2}x^2 &\leq \tfrac{1}{2}v_0^2. \end{aligned}$$

So the particle **oscillates** in the range

$$-v_0 \leq x \leq v_0.$$

and its motion follows:



In this case it is easy to integrate. Since

$$v = \sqrt{v_0^2 - x^2}$$

then

$$\begin{aligned} && v = \frac{\mathrm{d}x}{\mathrm{d}t} &= \sqrt{v_0^2 - x^2} \\ \Longrightarrow && \int \frac{\mathrm{d}x}{\sqrt{v_0^2 - x^2}} &= \int \mathrm{d}t \\ \Longrightarrow && \sin^{-1}\left(\frac{x}{v_0}\right) &= t + C \\ \Longrightarrow && x(t) &= v_0 \sin(t + C). \end{aligned}$$

As $x(0) = 0$ then $C = 0$ so

$$x(t) = v_0 \sin t.$$

This is an example of simple harmonic motion with an amplitude of v_0.

4.8.3 General Method

Given the energy equation

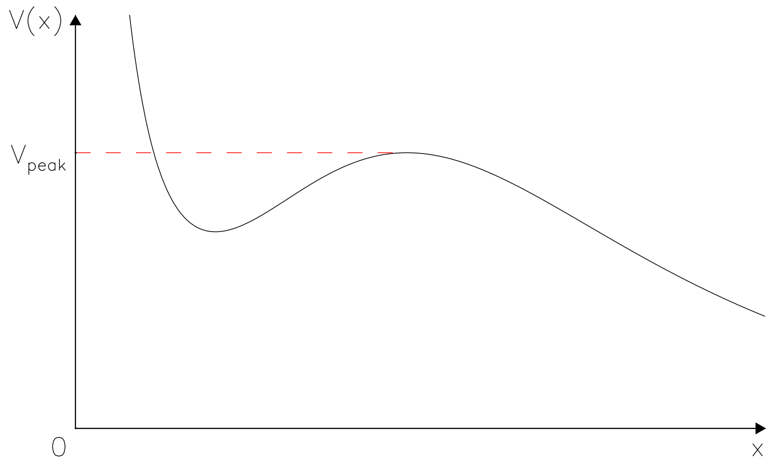
$$\tfrac{1}{2}mv^2 + V(x) = E$$

do the following:

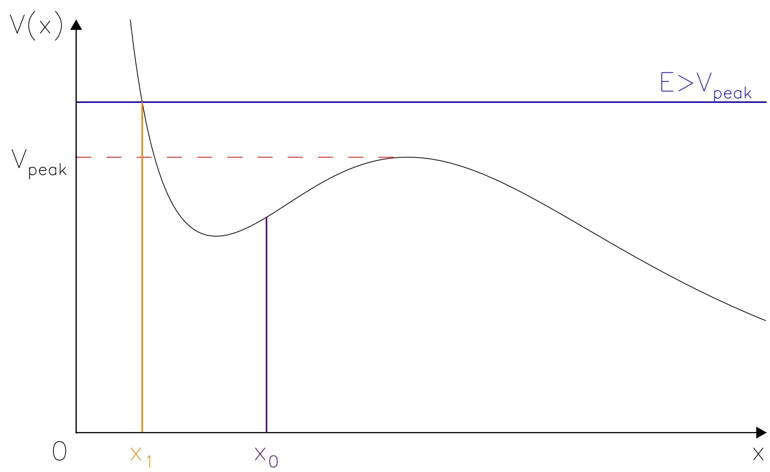
1. Given $F(x)$ determine
$$V(x) \;=\; -\int F(x)\,\mathrm{d}x + C$$
and apply any conditions.
2. Determine E from the initial conditions, e.g. $x = x_0$ and $v = v_0$ when $t = 0$, so that
$$E \;=\; \tfrac{1}{2}mv_0^2 + V(x_0).$$
3. Determine behaviour
   (a) The range of values of $V(x)$ such that $v^2 \geq 0$ (from $V(x) \leq E$).
   From this, determine if particle (i) oscillates, or (ii) heads to infinity.
   (b) The velocity $v(x)$ at any x and/or
$$\lim_{x\to\pm\infty} v(x).$$

Example 4.13

What behaviours for a particle are there for the potential function:



Case 1: $E > V_{peak}$

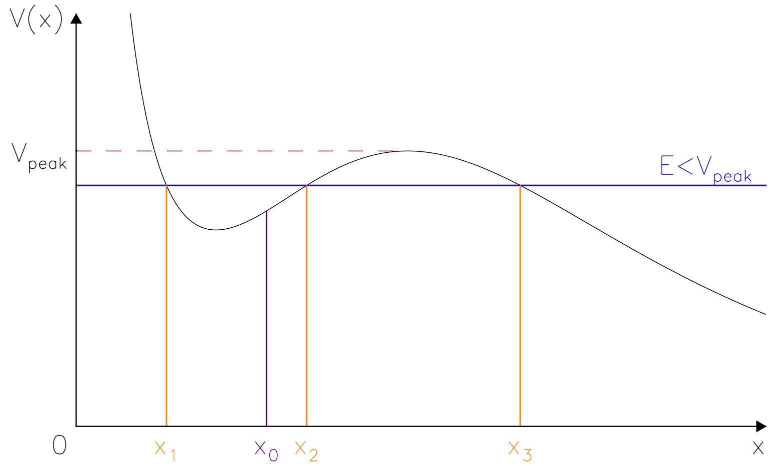


Assume particle starts at $x = x_0$ at $t = 0$ then

i. if $v_0 > 0$ particle moves to $+\infty$ and all energy will be kinetic as $V(x) \to 0$.

ii. if $v_0 < 0$ then particle moves left. It slows to stop at $x = x_1$ where $V(x_1) = E$, changes directions and heads to $+\infty$.

Case 2: $E < V_{peak}$



Let $x \in \{x_1, x_2, x_3\}$ be such that $V(x) = E$ then if particle starts at $x = x_0$ when $t = 0$ then

i. if $x_1 < x_0 < x_2$ then particle oscillates between x_1 and x_2 in the well.

ii. if $x_0 > x_3$ then particle tends to $+\infty$ (changing direction at $x = x_3$ if $v_0 < 0$).

Example 4.14

A particle of unit mass moves along the x-axis subject to a force

$$F(x) = -\frac{4x}{(1 + 2x^2)^2}$$

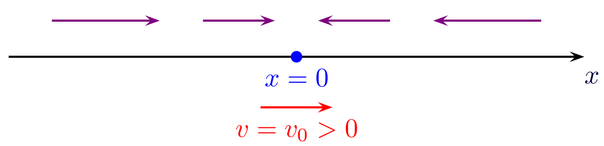
Let $x = x_0$ and $v = v_0 > 0$ when $t = 0$. Determine

(a) $V(x)$ subject to $V(0) = 0$,

(b) the energy equation,

(c) motion if $E > 1$ and $v_\infty = \lim_{x\to\infty} v(x)$, and

(d) motion if $E < 1$ and values of x allowed.



(a) $V(x)$: Simply

$$\begin{aligned} V(x) &= -\int F(x)\,\mathrm{d}x \\ &= \int \frac{4x\,\mathrm{d}x}{(1+2x^2)^2} \\ &= \int \frac{\mathrm{d}u}{u^2} \\ &= -\frac{1}{u} + C \\ &= -\frac{1}{1+2x^2} + C \end{aligned}$$

where

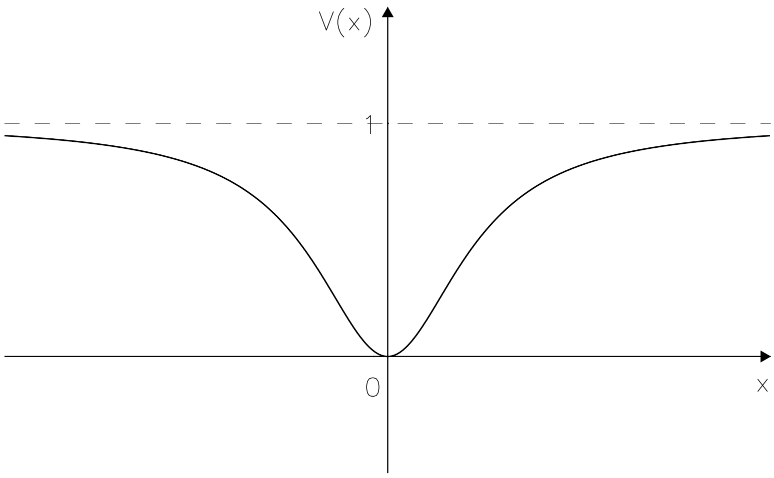
$$u = 1 + 2x^2 \qquad \Longrightarrow \qquad \mathrm{d}u = 4x\,\mathrm{d}x.$$

Since $V(0) = 0$ then

$$\begin{aligned} 0 &= -\frac{1}{1+2\cdot 0^2} + C \\ \Longrightarrow \qquad C &= 1, \end{aligned}$$

so

$$V(x) = 1 - \frac{1}{1+2x^2} = \frac{2x^2}{1+2x^2}.$$



Here, we can see that

$$\lim_{x\to\pm\infty} V(x) = 1$$

(b) Starting with

$$\tfrac{1}{2}mv^2 + V(x) = E$$

with $x = 0$, $v = v_0$ at $t = 0$ and $m = 1$ then

$$E = \tfrac{1}{2}\cdot 1 \cdot v_0^2 + V(0) = \tfrac{1}{2}v_0^2,$$

so **energy equation** is

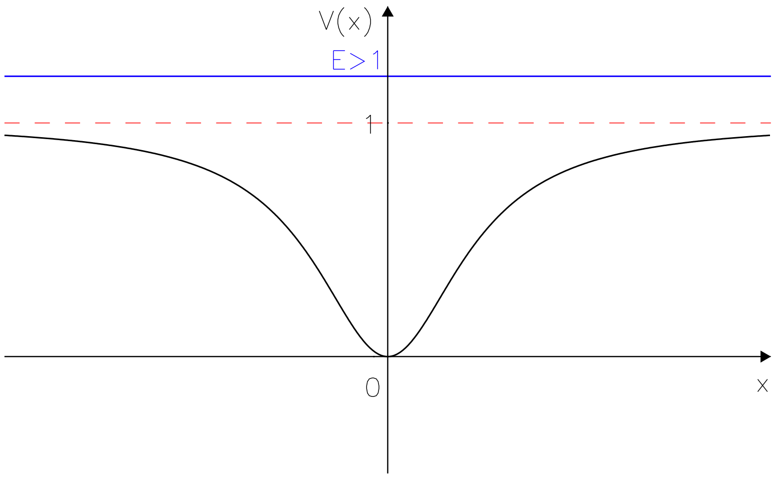
$$\tfrac{1}{2}v^2 + \frac{2x^2}{1+2x^2} = \tfrac{1}{2}v_0^2.$$

From this we see that motion is allowed for all x such that

$$V(x) \leq \tfrac{1}{2}v_0^2$$

and when $V(x) = \frac{1}{2}v_0^2$ then all energy is potential so $v(x) = 0$.

(c) $E > 1$:



Here we have

$$\tfrac{1}{2}v_0^2 > 1$$

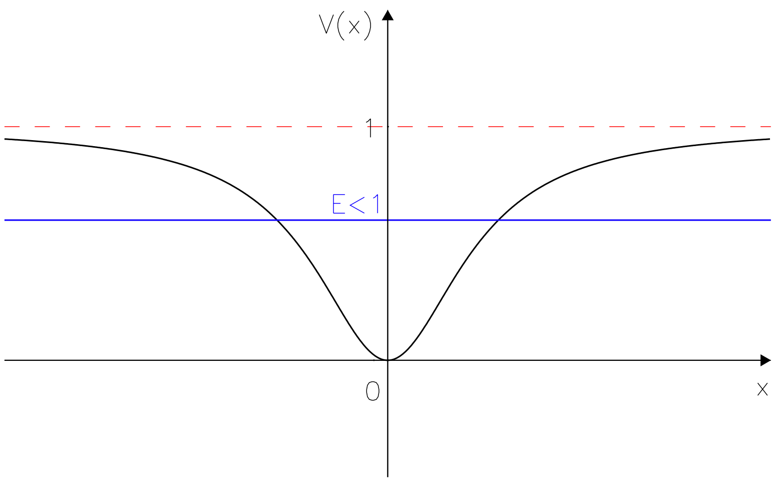
or $v_0 > \sqrt{2}$. Since $V(x) \leq 1$ for all x then if $E > 1$, $V(x) < E$ for all x. So particle moves right to infinity.

Finally, at infinity, $v = v_\infty$ and $V(\infty) = 1$ so

$$\begin{array}{rrcl} & \tfrac{1}{2}v_\infty^2 + V(\infty) & = & \tfrac{1}{2}v_0^2 \\ \Longrightarrow & v_\infty^2 & = & v_0^2 - 2V(\infty) \\ & & = & v_0^2 - 2 \\ \therefore & v_\infty & = & \sqrt{v_0^2 - 2}. \end{array}$$

In the case when $E = 1$, $v_0 = \sqrt{2}$ and $v_\infty = 0$.

(d) $0 < E < 1$:



Here

$$E = \tfrac{1}{2}v_0^2$$

where $0 < v_0 < \sqrt{2}$. So particle oscillates between $\pm x_1$ (due to symmetry), where $V(x_1) = E$.

What is x_1? Clearly,

$$\begin{aligned}
\frac{2x_1^2}{1+2x_1^2} &= E \\
\implies \quad 2x_1^2 &= E\left(1+2x_1^2\right) \\
\implies \quad 2\left(1-E\right)x_1^2 &= E \\
\implies \quad x_1^2 &= \frac{E}{2(1-E)} \\
&= \frac{\frac{1}{2}v_0^2}{2\left(1-\frac{1}{2}v_0^2\right)} \\
&= \frac{v_0^2}{4-2v_0^2} \\
\therefore \quad x_1 &= \frac{v_0}{\sqrt{4-2v_0^2}}
\end{aligned}$$

So if

- $E = 0$ then $v_0 = 0$ and $x_1 = 0$ (i.e. stationary).
- $E = 1$ then $v_0 = \sqrt{2}$ and $x_1 \to \infty$ – (i.e. escapes).

Example 4.15

A particle moves along the x-axis under a force

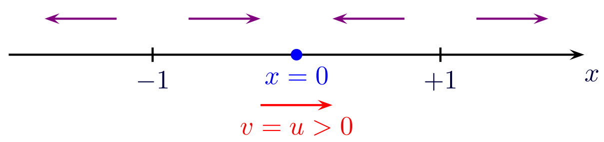
$$F(x) = 4\left(x^3 - x\right).$$

(a) Find $V(x)$ given that $V(0) = 0$.

(b) If $m = 1$ and $x = 0$ and $v = u > 0$ at $t = 0$, show that it never returns to the origin if $u > \sqrt{2}$.

(c) Find its speed at $x = \sqrt{2}$.

(d) Between what values of x does the particle oscillate if $u = \sqrt{\frac{7}{8}}$.



(a) Clearly,

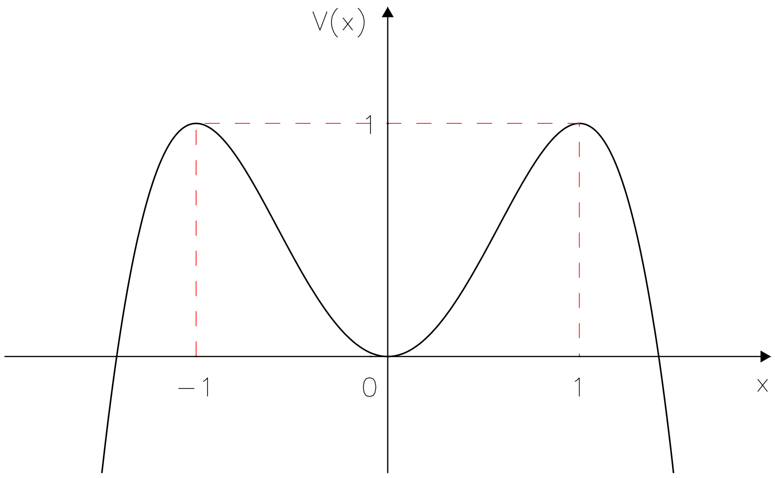
$$\begin{aligned}
V(x) &= -\int F(x)\,\mathrm{d}x \\
&= -\int 4\left(x^3 - x\right)\,\mathrm{d}x \\
&= -x^4 + 2x^2 + C.
\end{aligned}$$

Since $V(0) = 0$ then

$$\begin{aligned}
0 &= -0^4 + 2\cdot 0^2 + C \\
\implies \quad C &= 0.
\end{aligned}$$

So

$$V(x) \;=\; x^2\left(2-x^2\right).$$



Note:

- Minimum at $(0,0)$.
- Maxima at $(-1,1)$ and $(1,1)$.
- Zeros at $(-\sqrt{2},0)$, $(0,0)$ and $(\sqrt{2},0)$.
- $V(x)\to-\infty$ as $x\to\pm\infty$.

(b) Using $m=1$, $x=0$, $v=u$ when $t=0$ and the energy equation

$$\tfrac{1}{2}v^2+V(x) \;=\; E$$

implies

$$E \;=\; \tfrac{1}{2}u^2+V(0) \;=\; \tfrac{1}{2}u^2+0 \;=\; \tfrac{1}{2}u^2.$$

So

$$\tfrac{1}{2}v^2+V(x) \;=\; \tfrac{1}{2}u^2$$

The particle never returns (i.e. escapes to infinity) provided that

$$\begin{array}{lrcl} & E & > & V(1) \\ \Longrightarrow & E & > & 1 \\ \Longrightarrow & \tfrac{1}{2}u^2 & > & 1 \\ \Longrightarrow & u & > & \sqrt{2} \end{array}$$

since $u>0$.

(c) At $x=\sqrt{2}$, the speed $(v=v(\sqrt{2}))$ is given by

$$\begin{array}{lrcl} & \tfrac{1}{2}v^2+V(\sqrt{2}) & = & \tfrac{1}{2}u^2 \\ \Longrightarrow & v^2 & = & u^2-2V(\sqrt{2}) \\ & & = & u^2-2\cdot 0 \\ & & = & u^2 \\ \therefore & v & = & u \end{array}$$

as $u>0$.

(d) Since $u = \sqrt{\frac{7}{8}}$ then

$$\tfrac{1}{2}v^2 + V(x) = \tfrac{7}{16}.$$

The particle oscillates between the points x_1 and x_2 ($x_1 < x_0 < x_2$) such that

$$v(x_1) = v(x_2) = 0$$

and

$$V(x) < \tfrac{7}{16}$$

for all $x_1 < x < x_2$.
First solve equation for x when $v = 0$:

$$\begin{aligned}
& \tfrac{1}{2}v^2 + V(x) &=& \tfrac{7}{16} \\
\implies & 0 + \left(-x^4 + 2x^2\right) &=& \tfrac{7}{16} \\
\implies & x^4 - 2x^2 + \tfrac{7}{16} &=& 0 \\
\implies & \left(x^2 - \tfrac{7}{4}\right)\left(x^2 - \tfrac{1}{4}\right) &=& 0
\end{aligned}$$

so

$$x \in \left\{-\tfrac{1}{2}\sqrt{7}, -\tfrac{1}{2}, +\tfrac{1}{2}, +\tfrac{1}{2}\sqrt{7}\right\}.$$

Clearly (using above requirements) the particle must oscillate between $x_1 = -\frac{1}{2}$ and $x_2 = +\frac{1}{2}$.

Example 4.16

At $t = 0$ a rocket (of mass m) is lanched from the Earth at an initial speed u on its long journey to Mars. Assume the rocket is subject only to the force of Earth's gravity,

$$\mathbf{F}(r) = -\frac{GMm}{r^2}\hat{\mathbf{r}}$$

where M is the mass of the Earth.

(a) Sketch $V(r)$ for $r \in [R_R, \infty)$, labelling the value of $V(r)$ at $r = R_E$, the radius of the Earth.

(b) Show that the energy equation for the rocket is given by

$$\tfrac{1}{2}v^2 - \frac{GM}{r} = \tfrac{1}{2}u^2 - \frac{GM}{U_E}.$$

(c) By considering the behaviour of the rocket, use the energy equation to show that the rocket to escapes Earth's graviaty if

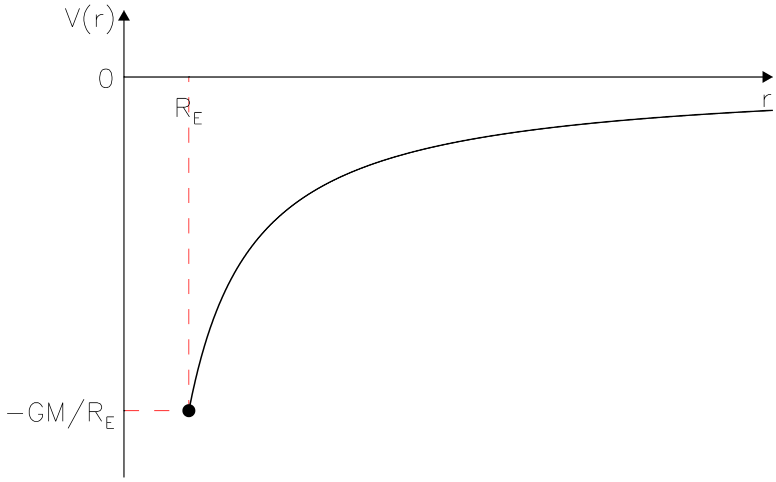
$$u \geq \sqrt{\frac{2GM}{R_E}}.$$

(a) Simply,

$$\begin{aligned}
V(r) &= -\int F(r)\,\mathrm{d}r \\
&= -\int \left(-\frac{GMm}{r^2}\right)\mathrm{d}r \\
&= -\frac{GMm}{r} + C.
\end{aligned}$$

Without loss of generality, let $C = 0$ so

$$V(r) = -\frac{GMm}{r}.$$



(b) Starting with

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{F}(r)$$

and since

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}v}{\mathrm{d}r} = v\frac{\mathrm{d}v}{\mathrm{d}r}$$

and

$$F(r) = -\frac{\mathrm{d}V}{\mathrm{d}r}$$

then

$$\begin{aligned} mv\frac{\mathrm{d}v}{\mathrm{d}r} &= -\frac{\mathrm{d}V}{\mathrm{d}r} \\ \implies \quad \frac{\mathrm{d}}{\mathrm{d}r}\left(\tfrac{1}{2}mv^2\right) &= -\frac{\mathrm{d}V}{\mathrm{d}r} \\ \implies \quad \frac{\mathrm{d}}{\mathrm{d}r}\left(\tfrac{1}{2}mv^2 + V(r)\right) &= 0 \\ \implies \quad \tfrac{1}{2}mv^2 + V(r) &= E \\ \therefore \quad \tfrac{1}{2}mv^2 - \frac{GMm}{r} &= E \end{aligned}$$

where E is constant.
Since at launch ($t = 0$), we have $v = u$ and $r = R_E$ then

$$E = \tfrac{1}{2}mu^2 + V(R_E) = \tfrac{1}{2}mu^2 - \frac{GMm}{R_E}$$

Therefore

$$\begin{aligned} \tfrac{1}{2}mv^2 - \frac{GMm}{r} &= \tfrac{1}{2}mu^2 - \frac{GMm}{R_E} \\ \implies \quad \tfrac{1}{2}v^2 - \frac{GM}{r} &= \tfrac{1}{2}u^2 - \frac{GM}{R_E}, \end{aligned}$$

as required.

(c) To escape Earth's gravity, first note that

$$\lim_{r\to\infty}\left\{\frac{GM}{r}\right\} = 0,$$

and let

$$v_\infty = \lim_{r\to\infty} v(x)$$

so that

$$\begin{aligned} & & \tfrac{1}{2}v_\infty^2 &> 0 \\ \Longrightarrow & & \tfrac{1}{2}u^2 - \frac{GM}{R_E} &\geq 0 \\ \Longrightarrow & & u^2 &\geq \frac{2GM}{R_E} \\ \therefore & & u &\geq v_e = \sqrt{\frac{2GM}{R_E}}, \end{aligned}$$

where v_e is the escape velocity. Provided that the rocket takes off with a speed greater than v_e then the rocket will escape Earth's gravity and can proceed onwards to Mars!

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5 Numerical Methods

Numerical methods are very important for solving many problems in the sciences. In this module we focus on two types of problems for which numerical methods are frequently used: root finding and the solution of (ordinary) differential equations.

5.1 Forms of Numbers

This section has been put in for information and should be read as background information, but will not be covered in the lectures themselves.

5.1.1 Significant Figures (sig. figs.)

The first non-zero digit is called the **first significant figure** (sig. fig.).

Example 5.1:: First significant figures for a range of numbers.

For the number 3.496 the first sig. fig. is 3.

For the number 0.0047 the first sig. fig. is 4.

To state number to n sig. figs. count from the first sig. fig. and apply rounding to determine the last sig. fig.

$315,814$ is 315800 to four sig. figs. or $316,000$ to three sig. figs.

0.004723 is 0.00472 to 3 sig. figs.

1360, 1.360 and 0.001360 all have four sig. figs.

5.1.2 Floating Point Numbers

Let $x = r \times N^s$, where s is the exponent, N the base (usually 2 or 10), r is a real number (usually first fig. is between 1 and 9, or first decimal place is between 1 and 9).

5.1.3 Errors

The numerical evaluation of a function has errors that could arise from:

- An approximate from of the function (e.g. a Taylor series) is used (truncation error).
- Computers only carry r to, typically, 8 or 16 sig, figs. (rounding error).

5.2 Root Finding

Often one can use a Taylor series expansion to find an estimate for the root of a function. It is important to choose an appropriate value for the point $x = x_0$ about which the function is to be expanded.

Example 5.2:: Find the first positive root of $\cos x$.

We can use the first two non-zero terms of the Taylor expansion of $\cos(x)$ about $x = 0$ to estimate the first positive ($x > 0$) root of $\cos(x)$ (we know that the exact root is $x = \pi/2 \approx 1.571$). The approximate root is then calculated by solving

$$0 = 1 - \frac{x^2}{2} \quad \Longrightarrow \quad x = \sqrt{2} \approx 1.414.$$

We can improve this estimate by taking more terms of the Taylor expansion into account. Let us try

$$0 = 1 - \frac{x^2}{2} + \frac{x^4}{24} \quad \Longrightarrow \quad x^4 - 12x^2 + 24 = 0$$

Solutions:

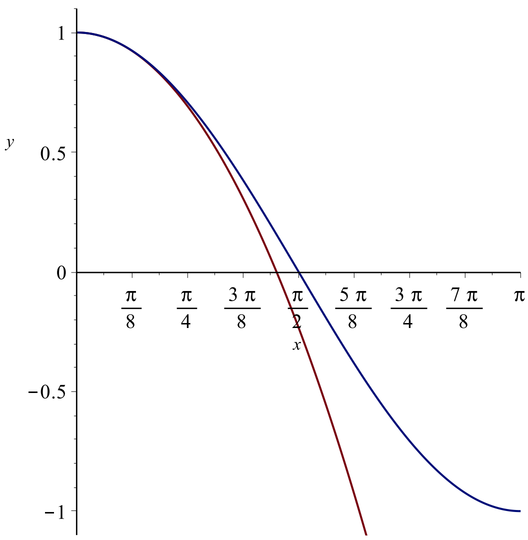
$$x^2 = \frac{12 \pm \sqrt{144 - 96}}{2} = 6 \pm \sqrt{12}.$$

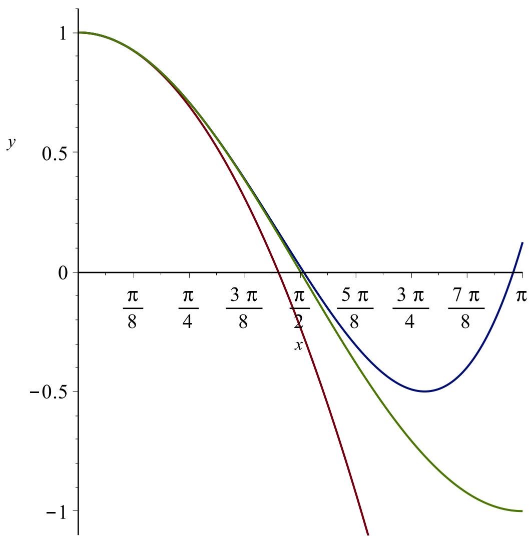
To obtain the first root, we take the solution with the negative root, so

$$x^2 = 6 - \sqrt{12}, \quad x = \sqrt{6 - \sqrt{12}} \approx 1.592,$$

which is obviously closer to the exact root 1.571 than the previous approximation.

This is also shown in the figure below. The left panel shows $y = \cos x$ in blue and the Taylor expansion up to quadratic order about $x = 0$ in red. The right panel shows $y = \cos x$ in green, the Taylor expansion up to quadratic order in red and the Taylor expansion up to fourth order in blue.

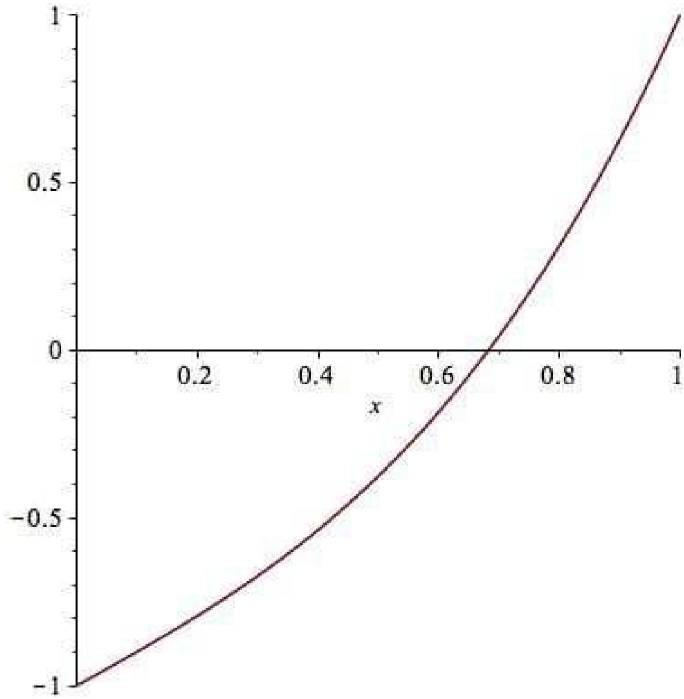




5.3 Root Finding: General Thoughts

Suppose we wish to find the roots of $f(x) = 0$ for a continuous function $f(x)$. Set $y = f(x)$ and sketch the function to help identify the location of the roots. Also one can often use simple considerations to identify restrict the interval in which one has to search for a root.

Example 5.3: Show that $f(x) = x^3 + x - 1$ has only one real root in $[0, 1]$.:



$f(x)$ is continuous and has no singularities.

$f(0) = -1$ and $f(1) = 1 \Longrightarrow$ because $f(x)$ is continuous, there is at least one real root in $[0, 1]$, but there could be three as well.

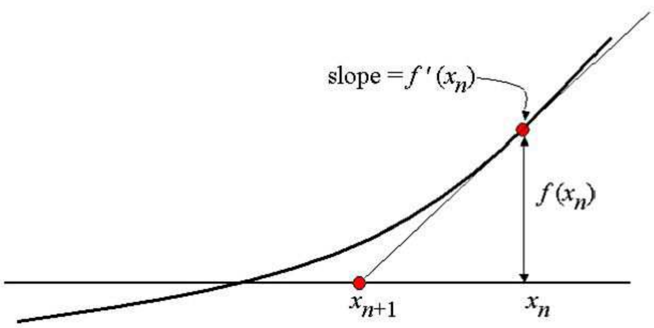
But, $f'(x) = 3x^2 + 1 \geq 0$ on $[0, 1]$ (actually for all x), hence $f(x)$ is a monotonic function (in this case increasing) with no turning points.

$\Longrightarrow$ There is only one real root in $[0, 1]$

5.4 Newton-Raphson Method

(named after Isaac Newton, 1642-1727, and Joseph Raphson, 1648-1715)

We wish to solve the equation $f(x) = 0$, assuming that $f(x)$ is continuous and differentiable. We also assume that $f(x)$ has a root at $x = r$, but that we do not know the exact value of r. However, we assume that we have an initial guess x_0 for the root r.



We now aim to improve the initial guess x_0. The Newton-Raphson (NR) method uses the tangent to the curve $y = f(x)$ at the point $x_0, f(x_0)$ to improve the estimate for the value of the root r. The NR method does that by setting the next estimate for the root r, x_1, equal to the value of x where the tangent crosses the x-axis (see figure). The slope of the tangent is given by

$$f'(x_0) \equiv \frac{\Delta y}{\Delta x} = \frac{f(x_0) - f(x_1)}{x_0 - x_1} = \frac{f(x_0)}{x_0 - x_1},$$

where we have used that $f(x_1) = 0$ by definition. Rearranging this equation for x_1 gives

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)},$$

assuming that $f'(x_0) \neq 0$.

One can now repeat this process using x_1 as initial guess to obtain a new (hopefully improved) estimate x_2:

$$x_2 = x_2 - \frac{f(x_1)}{f'(x_1)}.$$

Generally, we can continue this iterative process as many times as we want and the n^{th} step would be

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{32}$$

Equation (32) is what we call the Newton-Raphson Method.

Here $n = 0, 1, 2, 3, \ldots$ produces a sequence of root estimates $\{x_0, x_1, x_2, x_3, \ldots\}$ which may (or may not) converge.

If the sequence converges, it usually does so rapidly (see error analysis below). Furthermore, if the limit of the sequence is x_∞, then Eq. (32) reads

$$x_\infty = x_\infty - \frac{f(x_\infty)}{f'(x_\infty)} \quad \Longrightarrow \quad f(x_\infty) = 0,$$

i.e. the sequence converges to the root, $x_\infty = r$.

The NR method is widely used in many areas of science and engineering.

Example 5.4::
Find the root of $x^3 + x - 1 = 0$ in $[0, 1]$.

With $f(x) = x^3 + x - 1$ and $f'(x) = 3x^2 + 1$ the NR formula for this problem is

$$x_{n+1} = x_n - \frac{x_n^3 + x_n - 1}{3x_n^2 + 1}.$$

Suppose we want to determine the value of the root r accurate to 3 decimal places. To start the calculation we need to choose an initial guess for the root x_0. Any value within the interval $[0, 1]$ would do, but here we choose $x_0 = 1.0$.

Carrying out the NR calculation gives:

n	x_n
0	1.0000
1	0.7500
2	0.6861
3	0.6823
4	0.6823

Hence the root is $r = 0.682$ to 3 d.p.

Note: Evaluate x_n to a higher accuracy than 3 d.p. to check convergence of the method to the value of the root to the required accuracy.

Example 5.5: Evaluate $\sqrt{2}$ to 5 sig. figs.

Let $f(x) = x^2 - 2$, then the roots of $f(x)$ are $x = \pm\sqrt{2}$. So, in this case we actually know the root, but we use the NR method to get a numerical approximation to the irrational number $\sqrt{2}$.

Obviously, $f'(x) = 2x$, hence the NR equation becomes

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

Starting again with $x_0 = 1.0$ (because we are interested in the positive root of $f(x)$) and including 7 sig. figs., the NR method gives

n	x_n
0	1.000000
1	1.500000
2	1.416667
3	1.414216
4	1.414214

Hence, $\sqrt{2} = 1.4142$ to 5 sig. figs. (a standard pocket calculator gives $\sqrt{2} = 1.414213562\ldots$).

Note: Rounding errors in evaluating $f(x_n)$ and $f'(x_n)$ should be so small as to not change the first 5 sig. figs. of x_{n+1}.

Example 5.6: Determine the root of $\sin(x)$ for $x \in [-\pi/2, \pi/2]$ (which is, of course, $r = 0.0$).

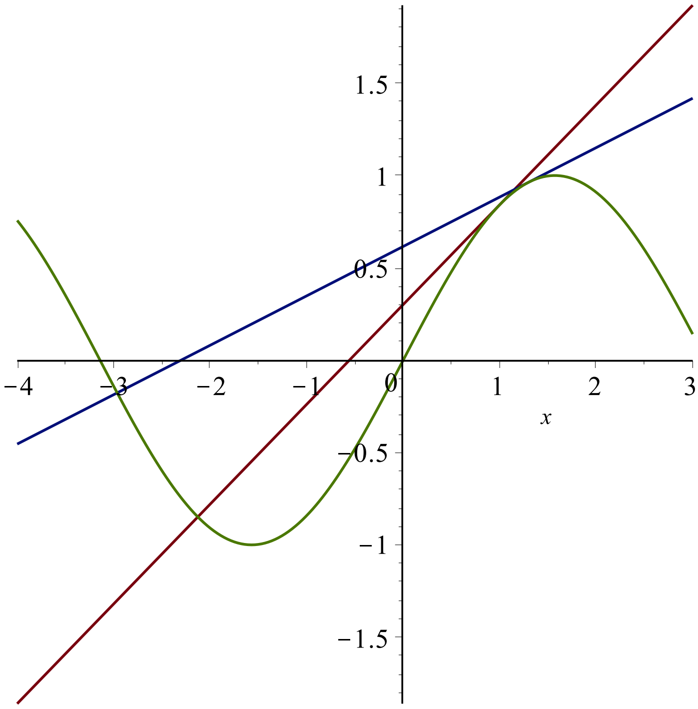
We choose two different values for x_0, namely $x_0 = 1.0$ and $x_0 = 1.3$. Applying the NR method we get

$x_0 = 1.0$

n	x_n
0	1.0000
1	-0.5574
2	0.0659
3	0.0001

$x_0 = 1.3$

n	x_n
0	1.3000
1	-2.3021
2	-3.4166
3	-3.1344



The green curve is the sine function, the red line is the tangent to $\sin(x)$ at $x_0 = 1.0$ and the blue line the tangent to $\sin(x)$ at $x_0 = 1.3$. Because $x = 1.3$ is too close to the maximum of $sin(x)$ the blue tangent crosses the x-axis too far to the left and the NR method converges to the "wrong" root.

Example 5.7:: This example demonstrates that it is important to calculate $f(x)$ and $f'(x)$ to sufficient accuracy.

Let $f(x) = \sinh(ax)$, then $f'(x) = a\cosh(ax)$. We want to use the NR method to find the root of $\sinh(ax)$ accurate to 3 decimal places when $a = 0.01$. Evidently the exact value of the root is $r = 0.0$.

Starting with $x_0 = 10.0$, we get in the first NR step

n	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}
0	10.00000	0.10017	0.01005	0.03320
1	0.03320	0.00033	0.01000	x_2

What value would we find for x_2? Well, according to the NR method we have

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)},$$

but if f and f' would be calculated to 3 d.p.s only we would have $f(x_1) = 0.000$ and hence $x_2 = x_1$!

The method would have "converged" to the root 0.033, which is obviously wrong! To obtain the "correct" root, or at least a better approximation to it, we need to evaluate $f'(x)$ and $f(x)$ to more d.p.s.

Other errors which could affect the correctness or accuracy of the results of the NR method are:

- $f(x)$ is approximated by e.g. a Taylor polynomial.
- Truncation/round-off errors in calculators or computers.

5.5 Error Analysis for the Newton-Raphson Method

We would like to know how the error (deviation from the exact solution) that we are making when using the NR method evolves with an increasing number of iterations (steps). To do this in general, we write

$$r = x_n + \epsilon_n, \tag{33}$$

where r is the exact root, x_n its n^{th} approximation and ϵ_n the error in the n^{th} approximation.

We can rewrite Eq. (33) as

$$x_n = r - \epsilon_n \quad \text{and} \quad x_{n+1} = r - \epsilon_{n+1}.$$

We now ask how ϵ_n and ϵ_{n+1} are related to each other. Using the expressions for x_n and x_{n+1} from above and substituting them into the NR equation (32) we obtain

$$r - \epsilon_{n+1} = r - \epsilon_n - \frac{f(r-\epsilon_n)}{f'(r-\epsilon_n)}$$

$$\implies \quad \epsilon_{n+1} = \epsilon_n + \frac{f(r-\epsilon_n)}{f'(r-\epsilon_n)} = \frac{\epsilon_n f'(r-\epsilon_n) + f(r-\epsilon_n)}{f'(r-\epsilon_n)} \tag{34}$$

So far, we have not made any approximations, so Eq. (34) is an exact equation for the error ϵ_n.

We now approximate the right hand side of Eq. (34) by Taylor expanding it about $x = r$ for small ϵ_n and retaining only the first non-vanishing term. Because we have a relatively complicated expression involving a fraction as well as the derivative of $f(x)$, we do the calculation term by term

We start expanding $f(x)$ and $f'(x)$ about r. The Taylor expansion of $f(x)$ about a point x_0 up to quadratic order is:

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \ldots$$

and by taking the first derivative d/dx we can find the Taylor expansion for $f'(x)$ about x_0:

$$f'(x) = f'(x_0) + f''(x_0)(x-x_0) + \ldots$$

We now rewrite this by using

$$x_0 = r, \quad x = r - \epsilon_n \implies x - x_0 = -\epsilon_n$$

and thus

$$f(r-\epsilon_n) = f(r) + f'(r)(-\epsilon_n) + \frac{f''(r)}{2!}\epsilon_n^2 + \ldots = f'(r)(-\epsilon_n) + \frac{f''(r)}{2!}\epsilon_n^2 + \ldots,$$

where we have used that $f(r) = 0$ because r is a root of f. Also

$$f'(r-\epsilon_n) = f'(r) + f''(r)(-\epsilon_n) + \ldots$$

Hence the numerator of Eq. (34) becomes

$$\epsilon_n\{f'(r) - f''(r)\epsilon_n \ldots\} - f'(r)\epsilon_n + \frac{f''(r)}{2!}\epsilon_n^2 + \ldots = -\frac{f''(r)}{2}\epsilon_n^2 + O(\epsilon_n^3) \approx -\frac{1}{2}f''(r)\epsilon_n^2.$$

The lowest non-vanishing term of the Taylor expansion of the numerator is quadratic in ϵ_n.

Can the denominator change this result? Using the quotient rule, we can expand the denominator as

$$\frac{1}{f'(r-\epsilon_n)} \approx \frac{1}{f'(r)} - \frac{f''(r)}{[f'(r)]^2}(-\epsilon_n) + \ldots$$

So, if we multiply our previous result for the numerator with this we get

$$\frac{\epsilon_n f'(r-\epsilon_n)+f(r-\epsilon_n)}{f'(r-\epsilon_n)} \approx \left\{-\frac{1}{2}f''(r)\epsilon_n^2\right\}\left\{\frac{1}{f'(r)}-\frac{f''(r)}{[f'(r)]^2}(-\epsilon_n)+\dots\right\} \approx -\frac{f''(r)}{f'(r)}\frac{\epsilon_n^2}{2}+O(\epsilon_n^3),$$

and so, to lowest order in ϵ_n, Eq. (34) reads

$$\epsilon_{n+1} \approx -\frac{f''(r)}{f'(r)}\frac{\epsilon_n^2}{2},$$

i.e., $\epsilon_{n+1} \propto \epsilon_n^2$. The power of 2 means the scheme is "second order" and this implies very rapid convergence, because the error decreases quadratically (if it is small).

General remarks about the NR Method

- The NR method (as many other root finding methods) is ideal for a computer, because the equation

  $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

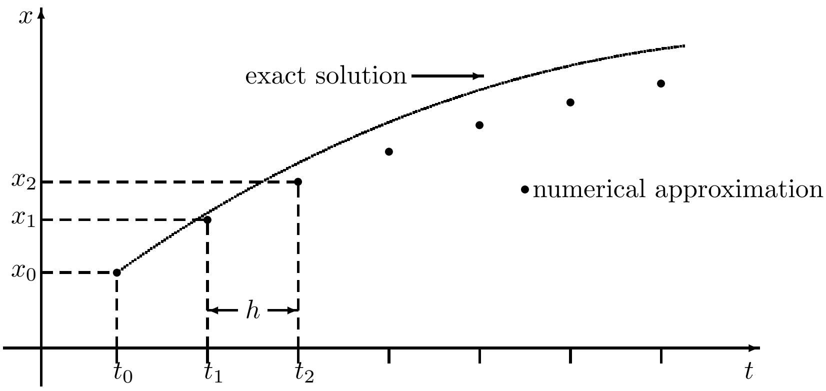
  can easily by coded up.
- The NR method is an iterative algorithm and like all iterative algorithms it requires a stopping criterion, e.g. either $|f(x_n)| < \epsilon$ or $|x_{n+1} - x_n| < \epsilon$, where ϵ determines the desired numerical accuracy. An alternative stopping criterion would be to simply stop after a certain fixed number of iterations, N, say.
- You should always work to more accuracy than the answer is required to have.
- The NR method usually converges very fast (if it converges).
- It is important to have a reasonable initial guess, x_0 (the NR method can be shown to converge if the initial guess is "close" to the exact root). In particular, it is important to stay away from values of x_0 for which $f'(x_0) \approx 0$.
- The NR method is insensitive to "small" errors in the numerical calculation.
- The NR method is easy to remember (e.g. use "dimensions" to note that the f' term has to be in the denominator, or use the graphical interpretation of NR).

5.6 Numerical Solution to Ordinary Differential Equations

In this section we consider ODEs of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,t) \tag{35}$$

which has the solution $x(t)$, subject to the initial condition $x = x_0$ at $t = t_0$. This is an initial value problem. Note that we will occasionally abbreviate $f(x(t),t)$ as $f(t)$.



An approximate (numerical) solution consists of a set of *discrete* grid points

$$(t_0, x_0), (t_1, x_1), (t_2, x_2), \ldots, (t_n, x_n), \ldots,$$

often with a fixed t increment:

$$\begin{aligned}
&t_1 = t_0 + h\\
&t_2 = t_1 + h = t_0 + 2h\\
&\vdots\\
&t_n + t_0 + nh
\end{aligned}$$

This *discretisation* of the problem makes it possible to use numerical methods, which can be implemented on a computer.

5.6.1 Euler's Method

The Taylor series for the solution $x(t)$ around $t = t_0$ is

$$x(t) = \underbrace{x(t_0)}_{x_0} + \underbrace{\left(\frac{dx}{dt}\right)_{t_0}}_{=f(x_0,t_0)} (t - t_0) + O[(t - t_0)^2] \tag{36}$$

and can be used to estimate the value of x when t increases from t_0 by h. Evidently the new point (t_1, x_1) will have

$$t_1 = t_0 + h.$$

x_1 may be estimated using the first two terms in (36) when we set $t = t_1 \equiv t_0 + h$:

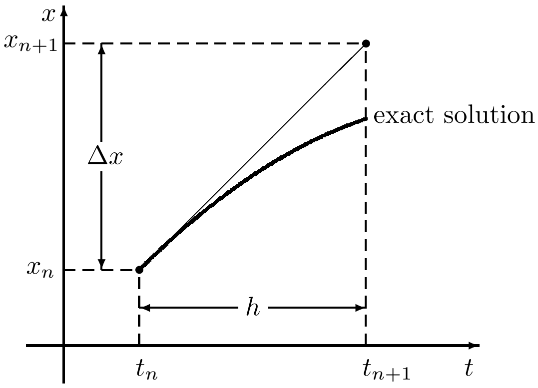
$$x_1 = x(t_1) \approx x_0 + hf(x_0, t_0).$$

We can then repeat the process starting with (t_1, x_1) to estimate (t_2, x_1). In general

$$\begin{aligned}
t_{n+1} &= t_n + h\\
x_{n+1} &= x_n + hf(x_n, t_n)
\end{aligned}$$

This is Euler's Method.

Graphical interpretation of Euler's Method



$$\frac{\Delta x}{h} = f(x_0, t_0) \quad \Longrightarrow \quad \Delta x = h f(x_0, t_0)$$

$$t_1 = t_0 + h$$

$$x_1 = x_0 + \Delta x = x_0 + h f(x_0, t_0),$$

as before.

Example 5.8: : Solve the ODE $\dot{x} - x = t$ numerically using Euler's Method.

The ODE

$$\frac{\mathrm{d}x}{\mathrm{d}t} - x = t$$

with initial conditions $x = 0$ for $t = 0$, may be solved (using an integrating factor or complimentary function and particular integral) to give the exact solution

$$x(t) = \mathrm{e}^t - t - 1.$$

We can check that this is the correct solution by direct substitution into the ODE:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{e}^t - 1,$$

so

$$\frac{\mathrm{d}x}{\mathrm{d}t} - x = \mathrm{e}^t - 1 - (\mathrm{e}^t - t - 1) = t,$$

as required.

We now write the ODE as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, t),$$

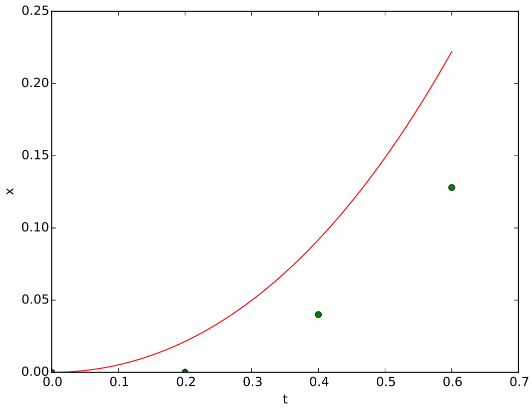
with $f(x, t) = x + t$.

Applying Euler's method with $h = 0.2$ to estimate x at $t = 0.6 = 3h$ starting from the initial condition $x_0 = 0$ at $t_0 = 0$ we get

$$\begin{aligned} t_{n+1} &= t_n + 0.2 \\ x_{n+1} &= x_n + f(x_n, t_n)\, h = x_n + 0.2\,(x_n + t_n) \end{aligned}$$

and the values in the following table:

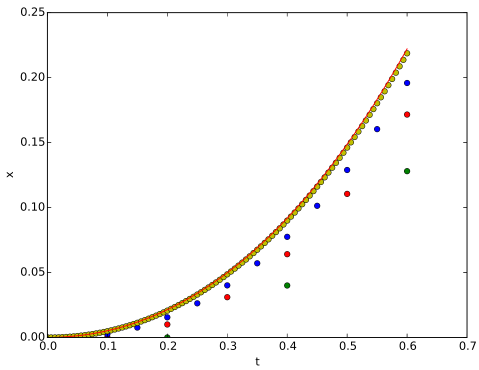
n	t_n	x_n	$f(x_n, t_n) = x_n + t_n$	$f(x_n, t_n)\,h$	x_{n+1}
0	0.0	0.0	0.0	0.0	$0.0 + 0.2 \cdot 0.0 = 0.0$
1	0.2	0.0	0.2	0.04	$0.0 + 0.04 = 0.04$
2	0.4	0.04	0.44	0.088	$0.04 + 0.088 = 0.128$
3	0.6	0.128	...	...	...

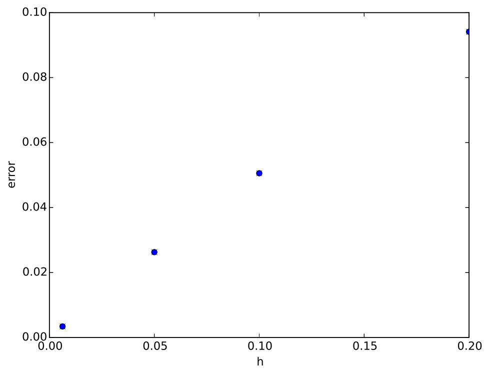


This suggests that $x(t = 0.6) = x_3 = 0.128$. The exact value is $x_{exact}(0.6) = 0.22212$ (compare figure).

The estimate from Euler's method can be improved by using a smaller step size h (which in turn makes it necessary to take a larger number of time steps to get to $t = 0.6$):

h	$x(0.6)$	no. of steps	error	colour
0.2	0.128	3	0.094	green
0.1	0.172	6	0.050	red
0.05	0.196	12	0.026	blue
$6.25 \cdot 10^{-3}$	0.219	96	0.003	yellow





Here the error is defined as the modulus of the difference between the exact value $x_{exact}(0.6)$ and the value $x(0.6)$ obtained using Euler's method, so

$$\text{error} = |x_{exact}(0.6) - x(0.6)|.$$

It seems that Euler's method converges slowly to the exact value as the step size h is reduced. Actually the error seems to be proportional to h.

5.6.2 Heun's Method

This method is sometimes also called the improved Euler method, so how does it "improve" Euler's method? In Euler's method we started at a point (t_0, x_0), where $x_0 = x(t_0)$ and used the slope of $x(t)$, given by the right-hand side of the ODE

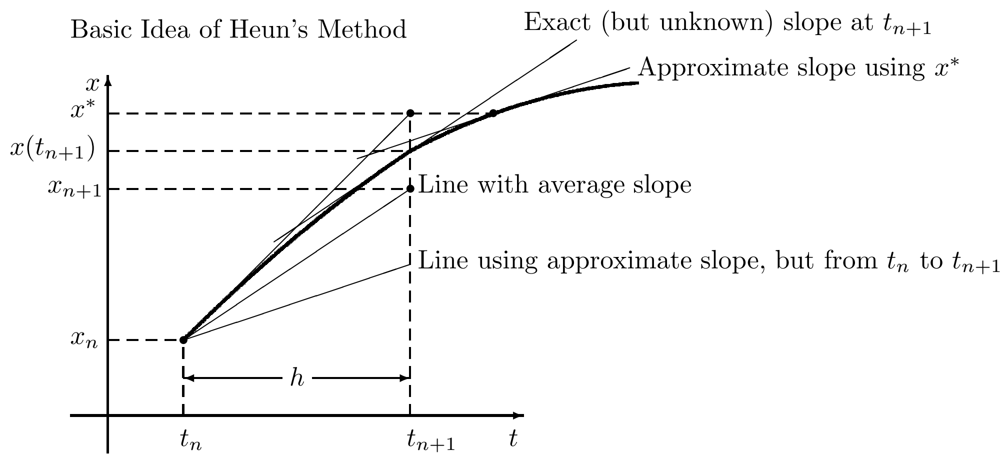
$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x(t), t)$$

to make one time step forward to $t_1 = t_0 + h$. In Euler's method we use the slope at (t_0, x_0), i.e. the slope on the left side of the interval $[t_0, t_1]$:

$$x_1 \approx x_0 + f(x_0, t_0)h.$$

However, one could argue that one could alternatively use the slope on the right side of the interval to increment x, i.e. at t_1. Then

$$x_1 \approx x_0 + f(x_1, t_1)h.$$



As shown in the figure one of the two methods usually overestimates the slope whereas the other one underestimates the slope and hence one method will give a value of x_1 which is too large and the other one a value for x_1 which is too small. Thus, one can improve the estimate for x_1 by taking the average value between the two methods. This in essence means that we would take the average of the two slopes to calculate the value of x_1:

$$x_1 \approx x_0 + \frac{1}{2}[f(x_0, t_0) + f(x_1, t_1)]h.$$

We now face the problem that the expression on the right-hand side that we want to use to calculate the new estimate x_1 contains x_1 itself, i.e. we need to know the value in order to calculate it! How do we overcome this problem?

In Heun's method, one replaces the unknown value x_1 by an approximate value x_1^* that is calculated using a single Euler step:

$$x_1^* = x_0 + f(x_0, t_0)h.$$

This can be interpreted in the following way: we use Euler's method to *predict* $x(t_1)$ and call our estimate x_1^*. This is then substituted into Eq. (5.6.2) instead of x_1 and using Eq. (5.6.2) with x_1^* on the right-hand side gives a *corrected* value for x_1. This method is therefore a member of a family of methods called *predictor-corrector methods*.

Starting at (t_0, x_0) we carry out the steps

1. $t_1 = t_0 + h$ (advance t)

2. $x_1^* = x_0 + f(x_0, t_0)h$ (estimate x_1)
3. $x_1 = x_0 + [f(x_0, t_0) + f(x_1^*, t_1)]h/2$ (correct x_1)

This allows us to advance from (t_0, x_0) to (t_1, x_1). We then repeat the process to step from (t_1, x_1) to (t_2, x_2) with $t_2 = t_1 + h$.

In general the time step for Heun's method is given by

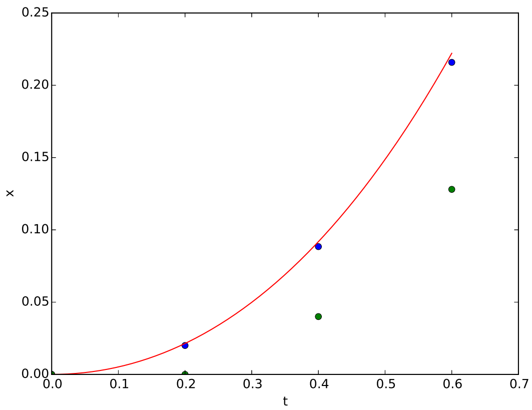
$$\begin{aligned} t_{n+1} &= t_n + h \\ x_{n+1}^* &= x_n + f(x_n, t_n)h \\ x_{n+1} &= x_n + \frac{1}{2}[f(x_n, t_n) + f(x_{n+1}^*, t_{n+1})]h \end{aligned}$$

Exampe 5.9: Use Heun's method with a time step $h = 0.2$ to estimate $x(0.6)$ when

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x + t$$

with initial conditions $x = 0$ at $t = 0$.

n	t_n	x_n	$f(x_n, t_n) = x_n + t_n$	x_{n+1}^*	t_{n+1}	$f(x_{n+1}^*, t_{n+1})$	x_{n+1}
0	0.0	0.0	0.0	0.0	0.2	0.2	0.02
1	0.2	0.02	0.22	0.064	0.4	0.464	0.0884
2	0.4	0.0884	0.4884	0.18608	0.6	0.78608	0.2159
3	0.6	0.2159	...	...	...	...	...



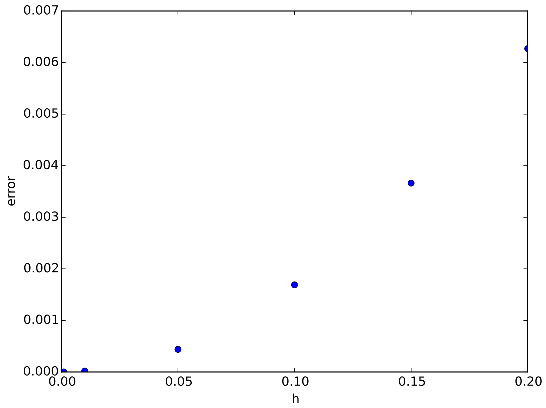
We have

- exact solution: $x(0.6) = 0.22212$,
- Heun's method: $x(0.6) = 0.2159$ (blue dots in figure)
- Euler's method: $x(0.6) = 0.128$ (green dots in figure).

We see that Heun's method gives a better approximation to the exact value than Euler's method if we use the same time step h.

As for Euler's method, decreasing the time step h also decreases the error:

h	$x(0.6)$	no. of steps	error
0.1	0.220429	6	$1.60 \cdot 10^{-3}$
0.01	0.2221007	60	$1.8 \cdot 10^{-5}$
0.001	0.2221186	600	$2 \cdot 10^{-7}$



The difference between the exact value of $x(0.6)$ and the approximations calculated with Heun's method (the error) is proportional to h^2, whereas it was proportional to h for Euler's method.

5.6.3 Error Analysis

To find out how a method scales with time step h we have to carry out an error analysis. To start, we use the Taylor expansion of the *exact* solution $x(t)$ about $t = t_0$ ($x(t_0) = x_0$):

$$x(t) = x_0 + \left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{t_0} (t - t_0) + \left.\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right|_{t_0} \frac{(t-t_0)^2}{2!} + + \left.\frac{\mathrm{d}^3x}{\mathrm{d}t^3}\right|_{t_0} \frac{(t-t_0)^3}{3!} + \dots$$

If we evaluate this series at $t = t_1$ and use $t_1 = t_0 + h$, i.e. $t_1 - t_0 = h$ and that $\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,t)$, we get

$$\begin{aligned} x(t_1) &= x_0 + \left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{t_0} h + \left.\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right|_{t_0} \frac{h^2}{2!} + + \left.\frac{\mathrm{d}^3x}{\mathrm{d}t^3}\right|_{t_0} \frac{h^3}{3!} + \dots \\ &= x_0 + f(x_0, t_0)h + \left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{t_0} \frac{h^2}{2!} + \left.\frac{\mathrm{d}^2f}{\mathrm{d}t^2}\right|_{t_0} \frac{h^3}{3!} + \dots \qquad (37) \end{aligned}$$

We now compare Euler's method and Heun's method with the Taylor expansion of $x(t_1)$: For Euler's method we have

$$x_1 = x_0 + f(x_0, t_0)h,$$

and this agrees with Eq. (37) for the first two terms, Hence the error made if we use Euler's method to approximate the solution is of order h^2 ($O(h^2)$).

For Heun's method $f(x_1^*, t_1)$ is expanded as

$$f(x_1^*, t_1) = f(x_0 + f(x_0, t_0)h, t_0 + h) = f(x_0, t_0) + \left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{t_0} h + \dots,$$

with the time derivative of f evaluated at t_0, x_0. Then we get for Heun's method

$$x_1 = x_0 + f(t_0)\frac{h}{2} + [f(x_0, t_0) + \left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{t_0} h + \dots]\frac{h}{2}$$

so

$$x_1 = x_0 + f(t_0)h + \left.\frac{\mathrm{d}f}{\mathrm{d}t}\right|_{t_0} \frac{h^2}{2} + \ldots$$

This agrees with Eq. (37) for the first three terms, and the error in x_1 occurs in the h^3 term.

5.6.4 Local and Global Errors

The error that is made in one single time step of a method is called *the local error*. We have just seen that the local error for Euler's method is $\propto h^2$ and for Heun's method it is $\propto h^3$. When carrying out a numerical calculation to solve an ODE we usually carry out many time steps to reach a (fixed) final time. Because in each individual step an error is made, these errors will accumulate and give rise to a global error. The global error is the error in the value of $x(t_f)$ where t_f is the final time.

The number of steps we need to reach this time is given by t_f/h and the global error is given by the product of the number of steps with the local error in each step. So, for Euler's method the global error is $\propto h$ and for Heun's method it is $\propto h^2$, as we have already seen in our examples discussed above.

5.6.5 Other Methods

There are numerous methods for solving ODEs and systems of ODEs. A particularly useful and popular method is the so-called Runge-Kutta Method of Fourth Order (often abbreviated as RK4). As the name suggests this method is accurate to fourth order in the time step h, but is numerically still sufficiently efficient to be not too slow. Without going into any details, we give the equations for one single time step. The quantities k_1 to k_4 are approximations to the slope of $x(t)$ at different points in the interval $[t_n, t_{n+1}]$: k_1 is the slope at t_n, k_2 and k_3 are two different approximations to the slope of $x(t)$ at $t_n + h/2$ and k_4 is an approximation to the slope at t_{n+1}:

$$\begin{aligned}
t_{n+1} &= t_n + h, \\
k_1 &= f(x_n, t_n), \\
k_2 &= f(x_n + k_1 h/2, t_n + h/2), \\
k_3 &= f(x_n + k_2 h/2, t_n + h/2), \\
k_4 &= f(x_n + k_3 h, t_n + h), \\
\\
x_{n+1} &= x_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4).
\end{aligned}$$