MT4526 TOPOLOGY

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Contents

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We review some elementary notions from set theory that are required throughout the course. A set is just a collection of objects; the precise definition lies beyond the scope of this course. Not every collection of objects is a set (such as the collection of all sets not containing themselves as elements).

Henceforth, we use the words ***set***, ***collection***, and ***family*** interchangeably.

If X is a set, we write $x \in X$ to indicate that x is an element of X (or x belongs to X). We write $x \notin X$ to indicate that x is not an element of X. The symbol $\varnothing$ denotes the ***empty set***, that is, the set with no elements. The symbol $\varnothing$ used for the empty set is a letter in the Danish alphabet, this was introduced as notation for the empty set by the Bourbaki group (specifically André Weil) in 1939.

Let A and B be sets. Then we write $A \subseteq B$ to indicate that every element of A is an element of B (or A is a ***subset*** of B). We write $A \nsubseteq B$ to indicate that A is not a subset of B, and $A \subsetneq B$ to indicate that A is a subset of B but $A \neq B$. If $A \subsetneq B$, then we say that A is a ***proper subset*** of B. The ***complement*** of A in B is

$$B \setminus A = \{\, x \,:\, x \in B \text{ and } x \notin A \,\};$$

the sets

$$A \cap B = \{\, x \,:\, x \in A \text{ and } x \in B \,\}, \quad A \cup B = \{\, x \,:\, x \in A \text{ or } x \in B \,\}$$

are the ***intersection*** and ***union*** of A and B, respectively. If $A \cap B = \varnothing$, then we say that A and B are ***disjoint***. The ***cartesian product*** of A and B is:

$$A \times B = \{\, (a, b) \,:\, a \in A \text{ and } b \in B \,\}.$$

Let I be a set and let U_i be a set for all $i \in I$. Then the collection $\{\, U_i \,:\, i \in I \,\}$ is called the ***family of sets*** indexed by I. For example, if $U_1, U_2, \ldots$ are sets, then $I = \mathbb{N} = \{1, 2, \ldots\}$. Index sets can be finite or infinite. Another example is the family of intervals $U_a = [a, \infty) = \{\, x \in \mathbb{R} \,:\, a \leq x \,\}$, which is indexed by the real numbers $\mathbb{R}$.

Let X be any set and let $\{\, U_i \,:\, i \in I \,\}$ be a family of subsets of X indexed by some set I. Then ***de Morgan's laws*** are:

$$X \setminus \bigcap_{i \in I} U_i = \bigcup_{i \in I} X \setminus U_i \tag{0.1}$$

$$X \setminus \bigcup_{i \in I} U_i = \bigcap_{i \in I} X \setminus U_i. \tag{0.2}$$

A set X is ***countable*** if it is finite or there is a bijection from X to $\mathbb{N}$. The real numbers $\mathbb{R}$ are not countable. A set which is not countable is called ***uncountable***. The countable union of countable sets is countable.

Part 1. Fundamental concepts

1. The definition of a topological space

Definition 1.1. A ***topological space*** is a pair $(X, \mathcal{T})$ consisting of a set X and a collection $\mathcal{T}$ of subsets of X satisfying the following properties:

T1. $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$;
T2. $\mathcal{T}$ is closed under finite intersections;
T3. $\mathcal{T}$ is closed under arbitrary unions.

The subsets of X in $\mathcal{T}$ are called ***open sets*** and $\mathcal{T}$ is a ***topology*** on X.

We omit reference to $\mathcal{T}$ if there is no chance of confusion, and simply refer to X as a topological space.
Let's give some examples of topological spaces.

Example 1.2. [**Discrete and trivial topologies**.] Let X be any set. Then the family of all subsets of X is a topology on X; this topology is called the ***discrete topology***. The collection of $\{\varnothing, X\}$ is also a topology called the ***trivial topology***.

To verify that a set X and a collection of subsets $\mathcal{T}$ of X satisfies condition **T2**, it suffices to check that the intersection of any two sets in $\mathcal{T}$ belongs to $\mathcal{T}$.

Lemma 1.3. *Let X be any set and let $\mathcal{T}$ be any collection of subsets of X. If $U \cap V \in \mathcal{T}$ for all $U, V \in \mathcal{T}$, then $(X, \mathcal{T})$ satisfies* ***T2****.*

Proof. Let $U_1, \ldots, U_n \in \mathcal{T}$. If $U_1 \cap \cdots \cap U_k \in \mathcal{T}$ for some $k \geq 1$, then by assumption $(U_1 \cap \cdots U_k) \cap U_{k+1} \in \mathcal{T}$. The lemma follows by recursively applying this observation. $\square$

Example 1.4. [**Cofinite topology**.] Let X be any set and let $\mathcal{T}$ denote the family of subsets U of X such that $X \setminus U$ is finite or $U = \varnothing$. We show that $\mathcal{T}$ is a topology on X by verifying **T1**, **T2**, and **T3**.

T1: Certainly, $\varnothing \in \mathcal{T}$, and the set $X \setminus X = \varnothing$ is finite, and so $X \in \mathcal{T}$.

T2: If $U, V \in \mathcal{T}$, then, by de Morgan's law (0.1),

$$X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V)$$

So, if $X \setminus U = X$ or $X \setminus V = X$, then $(X \setminus U) \cup (X \setminus V) = X \in \mathcal{T}$. On the other hand, if $X \setminus U$ and $X \setminus V$ are finite, then so is $(X \setminus U) \cup (X \setminus V)$. In either case, $U \cap V \in \mathcal{T}$.

T3: Let $\{\, U_i \;:\; i \in I \,\}$ be a family of sets in $\mathcal{T}$ for some index set I. Then, by de Morgan's law (0.2),

$$X \setminus \bigcup_{i \in I} U_i = \bigcap_{i \in I} X \setminus U_i.$$

So, if $X \setminus U_i$ is finite for some $i \in I$, then certainly $\bigcap_{i \in I} X \setminus U_i$ is finite (it's a subset of a finite set). If $X \setminus U_i = X$ for all $i \in I$, then $\bigcap_{i \in I} X \setminus U_i = X$. In either case, $\bigcup_{i \in I} U_i \in \mathcal{T}$.

The topology $\mathcal{T}$ is sometimes referred to as the ***cofinite topology*** or the ***Zariski topology***.

Example 1.5. [**Finite topologies**.] Let $X = \{1, 2, 3\}$. If $\mathcal{T} = \{\varnothing, \{1\}, \{1, 2\}, \{3\}\}$, then $\mathcal{T}$ is not a topology on X since $\{1, 2, 3\} \notin \mathcal{T}$ and so (**T1**) is not satisfied. Also $\{1\} \cup \{3\} = \{1, 3\} \notin \mathcal{T}$ and so (**T3**) is not satisfied.

If $\mathcal{T} = \{\varnothing, \{1\}, \{1, 2\}, \{1, 2, 3\}, \{2\}, \{2, 3\}\}$, then:

T1: $\varnothing, \{1, 2, 3\} \in \mathcal{T}$;

T2: It suffices to show that the intersection of any two sets in $\mathcal{T}$ belongs to $\mathcal{T}$:

$\cap$	$\varnothing$	$\{1\}$	$\{1,2\}$	$\{1,2,3\}$	$\{2\}$	$\{2,3\}$
$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
$\{1\}$	–	$\{1\}$	$\{1\}$	$\{1\}$	$\varnothing$	$\varnothing$
$\{1,2\}$	–	–	$\{1,2\}$	$\{1,2\}$	$\{2\}$	$\{2\}$
$\{1,2,3\}$	–	–	–	$\{1,2,3\}$	$\{2\}$	$\{2,3\}$
$\{2\}$	–	–	–	–	$\{2\}$	$\{2\}$
$\{2,3\}$	–	–	–	–	–	$\{2,3\}$

T3: It is straightforward to verify that $\mathcal{T}$ is closed under taking unions.

It follows that $\mathcal{T}$ is a topology on X and it is not the discrete topology or the trivial topology.

Example 1.6. [**Particular point topology**.] Let X be any set, let $x \in X$ be arbitrary, and the collection $\mathcal{T}$ of subsets of X consisting of $\varnothing$ and those U such that $x \in U$. We will show that this is a topological space by verifying **T1, T2** and **T3**.

T1: By assumption $\varnothing \in X$, and since $x \in X$, it follows that $X \in \mathcal{T}$.

T2: Suppose that $U, V \in \mathcal{T}$. If $U = \varnothing$ or $V = \varnothing$, then $U \cap V = \varnothing \in \mathcal{T}$. If $U \neq \varnothing$ and $V \neq \varnothing$, then $x \in U$ and $x \in V$. Thus $x \in U \cap V$ and $U \cap V \in \mathcal{T}$.

T3: If $\{ U_i : i \in I \}$ is a collection of sets in $\mathcal{T}$ for some index set I, then $x \in U_i$ or $U_i = \varnothing$ for all $i \in I$. If $U_i = \varnothing$ for all $i \in I$, then

$$\bigcup_{i \in I} U_i = \varnothing \in \mathcal{T}.$$

If $U_i \neq \varnothing$ for some $i \in I$, then $x \in \bigcup_{i \in I} U_i$ and so $\bigcup_{i \in I} U_i \in \mathcal{T}$.

Example 1.7. [**Cocountable topology**.] Let X be a set and let $\mathcal{T}$ denote the collection of subsets U of X such that $U = \emptyset$ or $X \setminus U$ is countable. Then $\mathcal{T}$ is a topology (we will show this in tutorials) on X; called the ***cocountable topology***.

Example 1.8. [**Standard topology on** $\mathbb{R}$.] Let $\mathcal{T}$ denote the collection of all unions of intervals (a, b), $a, b \in \mathbb{R}$, such that $a \leq b$. Then $\mathcal{T}$ is a topology called the ***standard topology*** on $\mathbb{R}$. We will return to this example slightly later.

Example 1.9. [**Excluded point topology**.] Let X be any set and let $x \in X$ be arbitrary. Then the collection consisting of X and all subsets U of X such that $x \notin U$ is a topology on X (we will show this in tutorials).

For convenience, if X is a topological space and $x \in X$, then we refer to any open set containing x as an ***open nbhd*** of x.

2. Bases and Subbases for topological spaces

Definition 2.1. A ***basis for a topology*** is a collection $\mathcal{B}$ of open sets such that every open set is a union of sets belonging to $\mathcal{B}$. The members of $\mathcal{B}$ are referred to as ***basic open sets***.

If $\mathcal{B}$ is a basis for a topological space X, then, since arbitrary unions of open sets are open (**T3**) and $\mathcal{B}$ is a collection of open sets, it follows that the open sets in X are precisely the unions of elements of $\mathcal{B}$.

Example 2.2. Let X be any set. Then the family

$$\{ \{x\} : x \in X \}$$

is a basis for the discrete topology on X.

Example 2.3. Let $\mathcal{T}$ be the topology $\mathcal{T} = \{\varnothing, \{1\}, \{1, 2\}, \{1, 2, 3\}, \{2\}, \{2, 3\}\}$ from Example 1.5. Then

$$\mathcal{B} = \{\varnothing, \{1\}, \{2\}, \{2, 3\}\}$$

is a basis for $\mathcal{T}$.

Example 2.4. The set $\mathcal{B} = \{ (a, b) : a, b \in \mathbb{R},\ a \leq b \}$ is a basis for the standard topology on $\mathbb{R}$ by definition (although we still haven't shown that this is a topology!).

It is not true that if you take an arbitrary collection $\mathcal{B}$ of subsets of a set X, then $\mathcal{B}$ is a basis for a topology. In particular, the collection of unions of sets in $\mathcal{B}$ is not necessarily a topology. For example, let X be any set. Then

$$\mathcal{B} = \{ B \subseteq X : |B| = 2 \}$$

is not a basis for a topology. If $x, y, z \in X$ are distinct, then $\{x, y\}, \{y, z\} \in \mathcal{B}$ but $\{x, y\} \cap \{y, z\} = \{y\}$ is not a union of sets in $\mathcal{B}$.

An open set in a topology can be given in lots of different ways as a union of basis sets and so the notion of a basis of a topology has more in common with the notion of a generating set for a group and less in common with a basis for a vector space.

Proposition 2.5. *Let X be a topological space, let $\mathcal{B}$ be a basis for X, and let U be any subset of X. Then the following are equivalent:*

(i) *U is an open set in X;*
(ii) *for every $x \in U$ there is an open set V such that $x \in V \subseteq U$;*
(iii) *for every $x \in U$ there is $B \in \mathcal{B}$ such that $x \in B \subseteq U$.*

Proof. (**i**) $\Rightarrow$ (**ii**). This follows immediately by setting $V := U$.

(**ii**) $\Rightarrow$ (**iii**). Let $x \in U$ be arbitrary. Then by assumption there is an open set V such that $x \subseteq V \subseteq U$. But V is a union of sets from $\mathcal{B}$ and so there is a $B \in \mathcal{B}$ such that $x \in B \subseteq V \subseteq U$, as required.

(**iii**) ⇒ (**i**). Let $x \in U$ be arbitrary. Then by assumption there is $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U$. Hence

$$U = \bigcup_{x \in U} B_x$$

is a union of open sets, and is hence open. □

Corollary 2.6. *Let X be a topological space and let $\mathcal{B}$ be a collection of open sets in X. Then $\mathcal{B}$ is a basis for X if and only if for every $x \in X$ and for every open set U containing x there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.*

Proof. (⇒) Let $x \in X$ be arbitrary and let U be any open set containing x. Then U is a union of sets in $\mathcal{B}$ and so there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$, as required.

(⇐) Let U be any non-empty open set. For every $x \in U$, there exists, by assumption, $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U$. It follows that $U = \bigcup_{x \in U} B_x$ and so U is a union of basic open sets. Thus $\mathcal{B}$ is a basis for $\mathcal{T}$. □

Theorem 2.7. *Let X be a set and let $\mathcal{B}$ be a family of subsets of X. Then $\mathcal{B}$ is a basis for a topology on X if and only if*

B1. *for all $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$;*

B2. *if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.*

The topology having $\mathcal{B}$ as a basis is called the ***topology generated by the basis*** *$\mathcal{B}$; it consists of the unions of sets in $\mathcal{B}$.*

Proof. (⇐) Let $\mathcal{T}$ denote the set of all unions of sets from $\mathcal{B}$. If $\mathcal{T}$ is a topology, then it follows from the definition that $\mathcal{B}$ is a basis. So, we must show that $\mathcal{T}$ is a topology assuming that **B1** and **B2** hold.

$\varnothing \in \mathcal{T}$ **and** $X \in \mathcal{T}$**:** The empty set is the empty union of sets in $\mathcal{B}$, and is hence in $\mathcal{T}$. From **B1**, for every $x \in X$, there is $B_x \in \mathcal{B}$ such that $x \in B_x$. Hence $X = \bigcup_{x \in X} B_x \in \mathcal{T}$.

Arbitrary unions: This is trivially satisfied.

Finite intersections: Let $U, V \in \mathcal{T}$ and let $x \in U \cap V$ be arbitrary. Then, since U and V are unions of sets in $\mathcal{B}$, there exist $B_U, B_V \in \mathcal{B}$ such that $x \in B_U \subseteq U$ and $x \in B_V \subseteq V$. In particular, $x \in B_U \cap B_V$ and so by **B2** there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq B_U \cap B_V \subseteq U \cap V$. So, $U \cap V = \bigcup_{x \in U \cap V} B_x$ and hence $U \cap V \in \mathcal{T}$.

(⇒) Let $\mathcal{T}$ denote a topology having $\mathcal{B}$ as its basis. We verify conditions **B1** and **B2**.

B1. Let $x \in X$. Then, since X is open, it follows from Proposition 2.5(iii), that there is $B \in \mathcal{B}$ such that $x \in B \subseteq X$.

B2. Let $B_1, B_2 \in \mathcal{B}$ and let $x \in B_1 \cap B_2$. Then, since $\mathcal{T}$ is closed under finite intersections and $B_1, B_2 \in \mathcal{T}$, it follows that $B_1 \cap B_2 \in \mathcal{T}$, i.e. $B_1 \cap B_2$ is open. Hence by Proposition 2.5(iii) there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. □

Example 2.8. [**The lower limit topology**.] Let $\mathcal{B}$ denote the family of half-open intervals $[a, b) = \{\, x \in \mathbb{R} : a \leq x < b \,\}$ for all $a, b \in \mathbb{R}$ with $a < b$. We verify that $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$:

B1: Let $x \in \mathbb{R}$. Then $x \in [x-1, x+1) \in \mathcal{B}$.

B2: If $[a, b), [c, d) \in \mathcal{B}$ such that there is $x \in [a, b) \cap [c, d)$, then $x \in [\max\{a, c\}, \min\{b, d\}) = [a, b) \cap [c, d) \in \mathcal{B}$.

Hence, by Theorem 2.7, $\mathcal{B}$ is a basis for a topology on X and the topology generated by $\mathcal{B}$ is called the ***lower limit topology*** on $\mathbb{R}$.

Example 2.9. [**The K-topology**.] Let $K = \{\, 1/n : n \in \mathbb{N} \,\}$ and let $\mathcal{B}$ denote the collection of open intervals (a, b) for all $a, b \in \mathbb{R}$ with $a < b$ and all sets of the form $(a, b) \setminus K$. Again we verify that $\mathcal{B}$ is a basis for a topology on $\mathbb{R}$:

B1: Let $x \in \mathbb{R} \setminus K$. Then $x \in (x-1, x+1) \in \mathcal{B}$.

B2: Let $B_1, B_2 \in \mathcal{B}$ be such that $x \in B_1 \cap B_2$. If $x \in K$, then $B_1 = (a, b)$ and $B_2 = (c, d)$ for some $a, b, c, d \in \mathbb{R}$. Hence $x \in (\max\{a, c\}, \min\{b, d\}) = (a, b) \cap (c, d)$. If $x \notin K$, then there exist $a, b, c, d \in \mathbb{R}$ such that $(a, b) \setminus K \subseteq B_1 \subseteq (a, b)$ and $(c, d) \setminus K \subseteq B_2 \subseteq (c, d)$. Hence $x \in (\max\{a, c\}, \min\{b, d\}) \setminus K \subseteq B_1 \cap B_2$.

Hence, by Theorem 2.7, $\mathcal{B}$ is a basis for a topology on X; the topology generated by $\mathcal{B}$ is called the ***K-topology*** on $\mathbb{R}$.

It should be clear at this point that on a given set X we can define many topologies. Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on a set X. If $\mathcal{T} \subseteq \mathcal{T}'$, then we say that $\mathcal{T}$ ***is coarser*** than $\mathcal{T}'$. We also say that $\mathcal{T}'$ is ***finer*** than $\mathcal{T}$. A topology $\mathcal{T}$ is finer than another topology $\mathcal{T}'$ if and only if every open set in $\mathcal{T}'$ is also open in $\mathcal{T}$.

Example 2.10. Let X be any set. Then, since every finite set is countable, the cofinite topology of X is contained in the cocountable topology on X. Hence the cofinite topology is coarser than the cocountable topology and the cocountable topology is finer than the cofinite topology. If X is finite, then the two topologies coincide. If X is infinite, then the cocountable topology strictly contains the cofinite topology.

The K-topology contains more open sets than the standard topology, and hence is finer.

Example 2.11. Let's compare $\mathbb{R}$ with the lower limit topology and the standard topology. Let $x \in \mathbb{R}$ and let $x \in (a,b)$ for some $a, b \in \mathbb{R}$. Then there exists $c \in \mathbb{R}$ such that $a < c < x$ and so $x \in [c,b) \subseteq (a,b)$. Hence, by Propostion 2.5, the set (a,b) is open in the lower limit topology, and so every set which is open in the standard topology is open in the lower limit topology. In other words, the lower limit topology contains the standard topology on $\mathbb{R}$.

On the other hand, if $x \in \mathbb{R}$, then $x \in [x, x+1)$ but $x \notin (a,b)$ for all $a, b \in \mathbb{R}$ such that $(a,b) \subseteq [x, x+1)$. Thus the lower limit topology is strictly finer than the standard topology on $\mathbb{R}$.

Definition 2.12. A ***subbasis*** for a topology $\mathcal{T}$ is a collection $\mathcal{S} \subseteq \mathcal{T}$ such that the set of finite intersections of sets in $\mathcal{S}$ is a basis for $\mathcal{T}$.

Let $\mathcal{S}$ be an arbitrary collection of subsets of X whose union is X and let $\mathcal{B}$ denote the set of finite intersections of sets in $\mathcal{S}$. Then for all $x \in X = \bigcup\{ S \,:\, S \in \mathcal{S} \}$ there exists $S \in \mathcal{S} \subseteq \mathcal{B}$ such that $x \in S$. Hence $\mathcal{B}$ satisfies **B1**. If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then, since B_1 and B_2 are finite intersections of sets in $\mathcal{S}$, $B_1 \cap B_2$ is also a finite intersection of sets in $\mathcal{S}$ and so $x \in B_1 \cap B_2 \in \mathcal{B}$. Thus $\mathcal{B}$ satisfies **B2** and so $\mathcal{B}$ is the basis of topology on X. We refer to this topology as the ***topology generated by*** $\mathcal{S}$. By definition, $\mathcal{S}$ is a subbasis for this topology, which consists of the unions of finite intersections of sets belonging to $\mathcal{S}$.

Example 2.13. The collection $\mathcal{S}$ of intervals (a, ∞), $(-\infty, b)$ $(a, b \in \mathbb{R})$ is a subbasis for the standard topology on $\mathbb{R}$. This follows since the set of finite intersections of sets belonging to $\mathcal{S}$ includes

$$(a,b) = (-\infty, b) \cap (a, \infty),$$

and so the usual topology is contained in the topology generated by $\mathcal{S}$. On the other hand, the sets belonging to $\mathcal{S}$ are open in the standard topology on $\mathbb{R}$ and so $\mathcal{S}$ generates the standard topology.

However, note that $\mathcal{S}$ is not a basis for the usual topology on $\mathbb{R}$ since **B2** fails to hold.

Lemma 2.14. *Let X be any set and let $\mathcal{S}$ be a collection of subsets of X. Then the topology generated by $\mathcal{S}$ is the coarsest (smallest) topology on X containing $\mathcal{S}$.*

Proof. Let $\mathcal{T}$ denote the topology generated by $\mathcal{S}$ and let $\mathcal{T}'$ be any topology containing $\mathcal{S}$. Since $\mathcal{T}'$ satisfies **T2** and **T3**, it contains all unions of finite intersections of sets in $\mathcal{S}$. But this is just the definition of $\mathcal{T}$ and so $\mathcal{T} \subseteq \mathcal{T}'$. □

3. The metric topology

In this section we explore the connections between metric and topological spaces.

Definition 3.1. Let X be a set. Then a ***metric*** on X is a function $d : X \times X \longrightarrow \mathbb{R}^+$ satisfying the following:

- **M1:** $d(x,y) = 0$ if and only if $x = y$ for all $x, y \in X$;
- **M2:** $d(x,y) = d(y,x)$ for all $x, y \in X$.
- **M3:** (***the triangle inequality***) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

A ***metric space*** is a set X together with a metric d. As with topological spaces, where there is no opportunity for confusion we simply write 'Let X be a metric space...' and we reserve the letter d for the metric of this space.

Example 3.2. [**Euclidean metric on** $\mathbb{R}^n$.] Let $d_2 : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^+$ be defined by

$$d_2(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}|| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n), \mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. It is straightforward to verify that d_2 satisfies **M1** and **M2**. To verify **M3** is somewhat harder; see Appendix A for a full proof.

Example 3.3. [**Taxicab metric on** $\mathbb{R}^n$.] Let $\mathbf{x} = (x_1, x_2, \ldots, x_n), \mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. Then we define $d_1 : \mathbb{R}^n \longrightarrow \mathbb{R}^+$ by

$$d_1(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} |x_i - y_i|.$$

- **M1:** $\mathbf{x} = \mathbf{y}$ if and only if $x_i = y_i$ for all i if and only if $d_1(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} |x_i - y_i| = 0$.
- **M2:** Clearly, $d_1(\mathbf{x}, \mathbf{y}) = d_1(\mathbf{y}, \mathbf{x})$ by definition.
- **M3:** Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$. By the triangle inequality,

  $$|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$$

  for all i. Thus

  $$d_1(\mathbf{x}, \mathbf{z}) = \sum_{i=1}^{n} |x_i - z_i| \leq \sum_{i=1}^{n} |x_i - y_i| + \sum_{i=1}^{n} |y_i - z_i| = d_1(\mathbf{x}, \mathbf{y}) + d_1(\mathbf{y}, \mathbf{z}).$$

Hence d_1 is a metric on $\mathbb{R}^n$; d_1 is called the ***taxicab metric***. On $\mathbb{R}$ the taxicab metric and the usual euclidean metric coincide.

Example 3.4. [**Chebyshev metric**.] In this example we give yet another metric on $\mathbb{R}^n$. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n), \mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. Then we define $d_\infty : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^+$ by

$$d_\infty(\mathbf{x}, \mathbf{y}) = \max_{1 \le i \le n} \{|x_i - y_i|\}.$$

M1: Clearly, $\mathbf{x} = \mathbf{y}$ if and only if $|x_i - y_i| = 0$ for all i if and only if $d_\infty(\mathbf{x}, \mathbf{y}) = \max_{1 \le i \le n}\{|x_i - y_i|\} = 0$.

M2: Since $|x_i - y_i| = |y_i - x_i|$ for all i, it follows that $d_\infty(\mathbf{x}, \mathbf{y}) = d_\infty(\mathbf{y}, \mathbf{x})$.

M3: Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$. Then $d_\infty(\mathbf{x}, \mathbf{z}) = \max_{1 \le i \le n}\{|x_i - z_i|\} = |x_j - z_j|$ for some $j \in \{1, \ldots, n\}$. Hence

$$d_\infty(\mathbf{x}, \mathbf{z}) = |x_j - z_j| \le |x_j - y_j| + |y_j - z_j| \le \max_{1 \le i \le n} \{|x_i - y_i|\} + \max_{1 \le i \le n} \{|y_i - z_i|\} = d_\infty(\mathbf{x}, \mathbf{y}) + d_\infty(\mathbf{y}, \mathbf{z}).$$

Hence d_∞ is a metric; we may refer to d_∞ as the ***Chebyshev metric*** or the ***maximum metric***.

Example 3.5. [**Uniform metric**.] Let A be a subset of $\mathbb{R}$ and let $f, g : A \longrightarrow \mathbb{R}$. Then we say that f (or g) is ***bounded*** if there exists $M \in \mathbb{N}$ such that $|f(x)| \le M$ for all $x \in A$. Let X denote the set of all bounded functions on A and let $d : X \times X \longrightarrow \mathbb{R}^+$ be defined by

$$d(f, g) = \sup_{x \in A} |f(x) - g(x)|.$$

We start by noting that d is well-defined, since f and g are bounded (i.e. $f(x), g(x) \in [-M, M]$ and so $|f(x) - g(x)| \le 2M$ for all $x \in A$. Thus $\{\, |f(x) - g(x)| \,:\, x \in A \,\} \subseteq [0, 2M]$ and so $\{\, |f(x) - g(x)| \,:\, x \in A \,\}$ is bounded above. Hence, by the definition of the real numbers, the set $\{\, |f(x) - g(x)| \,:\, x \in A \,\}$ has a supremum).

M1: Clearly, $f = g$ if and only if $|f(x) - g(x)| = 0$ for all $x \in A$ if and only if $d(f, g) = \sup_{x \in A} |f(x) - g(x)| = 0$.

M2: Since, $|f(x) - g(x)| = |g(x) - f(x)|$ it follows that $d(f, g) = d(g, f)$.

M3: To prove the triangle inequality, let's first prove that if $f, g : A \longrightarrow \mathbb{R}$ are bounded, then

$$\sup_{x \in A} [f(x) + g(x)] \le \sup_{x \in A} f(x) + \sup_{x \in A} g(x).$$

Certainly, $f(x) \le \sup_{x \in A} f(x)$ and $g(x) \le \sup_{x \in A} g(x)$ for all $x \in A$. Hence $f(x) + g(x) \le \sup_{x \in A} f(x) + \sup_{x \in A} g(x)$ for all $x \in X$ and so $\sup_{x \in A} f(x) + \sup_{x \in X} g(x)$ is an upper bound for $\{\, f(x) + g(x) \,:\, x \in A \,\}$. But $\sup_{x \in A} [f(x) + g(x)]$ is the least upper bound for $\{\, f(x) + g(x) \,:\, x \in A \,\}$, and so $\sup_{x \in A} [f(x) + g(x)] \le \sup_{x \in A} f(x) + \sup_{x \in A} g(x)$, as required.

Let $f, g, h \in X$. Then

$$\begin{aligned} d(f, h) = \sup_{x \in A} |f(x) - h(x)| = \sup_{x \in A} |f(x) - g(x) + g(x) - h(x)| &\le \sup_{x \in A} \{|f(x) - g(x)| + |g(x) - h(x)|\} \\ &\le \sup_{x \in A} |f(x) - g(x)| + \sup_{x \in A} |g(x) - h(x)| = d(f, g) + d(g, h). \end{aligned}$$

Hence d is a metric on the space of bounded functions on $\mathbb{R}$; we may refer to this as the ***uniform metric***.

Let (X, d) be a metric space and let $x \in X$. If $\varepsilon \in \mathbb{R}^+$ and $\varepsilon > 0$, then we define the ***ball of radius*** ε ***centred on*** x to be:

$$B_d(x, \varepsilon) = \{\, y \in X \,:\, d(x, y) < \varepsilon \,\}. \tag{3.1}$$

We will omit the subscript d in the notation for a ball, so that $B_d(x, \varepsilon)$ becomes $B(x, \varepsilon)$, when possible.

Proposition 3.6. *If X is a metric space, then*

$$\{\, B(x, \varepsilon) \,:\, x \in X, \varepsilon > 0 \,\}$$

is a basis for a topology, called the ***metric topology*** *induced by d.*

Proof. We must verify condition **B1** and **B2** from Theorem 2.7.

B1: Certainly, $x \in B(x, 1)$ for all $x \in X$ and so this condition is satisfied.

B2: Let $x_1, x_2 \in X$ and let $\varepsilon_1, \varepsilon_2 > 0$. Suppose that $y \in B(x_1, \varepsilon_1) \cap B(x_2, \varepsilon_2)$. Let $\delta = \min\{\varepsilon_1 - d(x_1, y), \varepsilon_2 - d(x_2, y)\}$. Then for all $z \in B(y, \delta)$, it follows that

$$\begin{aligned} d(x_1, z) &\le d(x_1, y) + d(y, z) < \delta + d(x_1, y) \le \varepsilon_1 - d(x_1, y) + d(x_1, y) = \varepsilon_1 \\ d(x_2, z) &\le d(x_2, y) + d(y, z) < \delta + d(x_2, y) \le \varepsilon_2. \end{aligned}$$

In other words, $y \in B(y, \delta) \subseteq B(x_1, \varepsilon_1) \cap B(x_2, \varepsilon_1)$, as required. □

Corollary 3.7. *Let X be a metric space and let U be a subset of X. Then U is open in the metric topology if and only if for all $x \in U$ there exist $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq U$.*

Proof. This follows immediately from Proposition 2.5. □

So, every metric space is a topological space, but what about the converse?

Definition 3.8. Let X be a topological space. Then X is said to be ***metrizable*** if there exists a metric d on X such that the metric topology induced by d is the topology of X.

Not every topological space is metrizable, as we will see later on. Whether or not a topological space is metrizable depends only on the topology. However, some properties of metric spaces depend on the specific metric for the space and not only on the topology.

Proposition 3.9. *Let X be a set and let d and d' be metrics on X. Then the metric topology induced by d' is finer than the metric topology induced by d if and only if for every $x \in X$ and every $\varepsilon > 0$ there is $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \varepsilon)$.*

Proof. Let $\mathcal{T}$ denote the topology induced by d and let $\mathcal{T}'$ denote the topology induced by d'.

($\Rightarrow$) Let $x \in X$ and $\varepsilon > 0$ be arbitrary. Then $B_d(x, \varepsilon)$ is an open set in $\mathcal{T}$ and hence in $\mathcal{T}'$. Thus, by Proposition 2.5(iii), there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \varepsilon)$, as required.

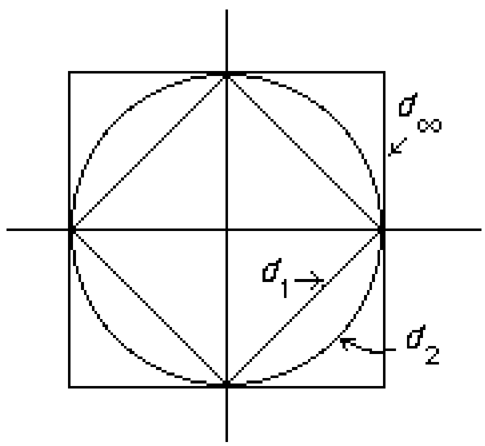
($\Leftarrow$) We must show that every open set in $\mathcal{T}$ is also open in $\mathcal{T}'$. Let U be any open set in $\mathcal{T}$. If $x \in U$ is arbitrary, then, by Corollary 3.7 applied to $\mathcal{T}$, there exists $\varepsilon > 0$ such that $B_d(x, \varepsilon) \subseteq U$. By assumption, there is $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq U$. Thus, by Corollary 3.7 applied to $\mathcal{T}'$, U is open in $\mathcal{T}'$ as required. □

Example 3.10. Let d_1 denote the taxicab metric, d_2 denote the euclidean metric, and d_∞ denote the Chebyshev metric on $\mathbb{R}^n$, $n \geq 1$. We will show that these metrics induce the same topology on $\mathbb{R}^n$.

To avoid using double subscripts we will write $B_i(\mathbf{x}, \varepsilon)$ instead of $B_{d_i}(\mathbf{x}, \varepsilon)$ for $i \in \{1, 2, \infty\}$ throughout the rest of this example. By Proposition 3.9, it suffices to prove that for any $\mathbf{x} \in \mathbb{R}^n$ and any $\varepsilon > 0$:

$$B_1(\mathbf{x}, \varepsilon) \subseteq B_\infty(\mathbf{x}, \varepsilon) \subseteq B_2(\mathbf{x}, n^{1/2}\varepsilon) \subseteq B_1(\mathbf{x}, n^{3/2}\varepsilon). \tag{3.2}$$

The picture to bear in mind is that of the unit circle in $\mathbb{R}^n$ under d_1, d_2, and d_∞:



Let $\mathbf{y} \in B_1(\mathbf{x}, \varepsilon)$. Then

$$d_1(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} |x_i - y_i| \leq \varepsilon \quad \text{and so} \quad d_\infty(\mathbf{x}, \mathbf{y}) = \max_{1 \leq i \leq n} \{|x_i - y_i|\} \leq \varepsilon.$$

Hence $B_1(\mathbf{x}, \varepsilon) \subseteq B_\infty(\mathbf{x}, \varepsilon)$, as required.

If $\mathbf{y} \in B_\infty(\mathbf{x}, \varepsilon)$, then

$$d_\infty(\mathbf{x}, \mathbf{y}) = \max_{1 \leq i \leq n} \{|x_i - y_i|\} \leq \varepsilon \quad \text{and so} \quad d_2(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \leq \sqrt{n \cdot d_\infty(\mathbf{x}, \mathbf{y})^2} \leq \sqrt{n}\varepsilon$$

and so $B_\infty(\mathbf{x}, \varepsilon) \subseteq B_2(\mathbf{x}, \sqrt{n}\varepsilon)$, as required.

Finally, if $\mathbf{y} \in B_2(\mathbf{x}, \sqrt{n}\varepsilon)$, then

$$d_2(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} < \sqrt{n}\varepsilon$$

and so

$$d_1(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n} |x_i - y_i| = \sum_{i=1}^{n} \sqrt{|x_i - y_i|^2} \leq \sum_{i=1}^{n} \sqrt{\sum_{j=1}^{n} |x_j - y_j|^2} = \sum_{i=1}^{n} d_2(\mathbf{x},\mathbf{y}) = n \cdot d_2(\mathbf{x},\mathbf{y}) < n^{\frac{3}{2}}\varepsilon.$$

Hence $B_2(\mathbf{x}, n^{1/2}\varepsilon) \subseteq B_1(\mathbf{x}, n^{3/2}\varepsilon)$, which concludes the proof.

Example 3.11. [**The Cantor set**.] Let $X = 2^{\mathbb{N}} = \{\, (x_1, x_2, \ldots) : (\forall i \in \mathbb{N})\,(x_i \in \{0,1\})\,\}$ (i.e. X is the space of sequences consisting of the values 0 and 1) and let $d : X \times X \longrightarrow \mathbb{R}^+$ be defined by

$$d(\mathbf{x},\mathbf{y}) = \begin{cases} 0 & \text{if } x = y \\ \dfrac{1}{\min\{\, i \in \mathbb{N} \,:\, x_i \neq y_i \,\}} & \text{if } x \neq y. \end{cases}$$

where $\mathbf{x} = (x_1, x_2, \ldots)$ and $\mathbf{y} = (y_1, y_2, \ldots)$. We will show that d is a metric on $2^{\mathbb{N}}$.

M1: By definition, $\mathbf{x} = \mathbf{y}$ if and only if $d(\mathbf{x},\mathbf{y}) = 0$.

M2: Since $\min\{\, i \in \mathbb{N} \,:\, x_i \neq y_i \,\} = \min\{\, i \in \mathbb{N} \,:\, y_i \neq x_i \,\}$, it follows that $d(\mathbf{x},\mathbf{y}) = d(\mathbf{y},\mathbf{x})$.

M3: Let $\mathbf{x},\mathbf{y},\mathbf{z} \in 2^{\mathbb{N}}$. We will prove that $d(\mathbf{x},\mathbf{z}) \leq \max\{d(\mathbf{x},\mathbf{y}), d(\mathbf{y},\mathbf{z})\} \leq d(\mathbf{x},\mathbf{y}) + d(\mathbf{y},\mathbf{z})$. Assume without loss of generality that $\max\{d(\mathbf{x},\mathbf{y}), d(\mathbf{y},\mathbf{z})\} = d(\mathbf{x},\mathbf{y})$ and $d(\mathbf{x},\mathbf{y}) = 1/m$ for some $m \in \mathbb{N}$. So, if $d(\mathbf{y},\mathbf{z}) = 1/n$, then

$$\frac{1}{m} \geq \frac{1}{n}$$

and so $m \leq n$. From the definition of d, $x_i = y_i$ for all $i < m$ and $y_j = z_j$ for all $j < n$. In particular, $x_i = z_i$ for all $i < m \leq n$. Hence

$$d(\mathbf{x},\mathbf{z}) \leq 1/m = d(\mathbf{x},\mathbf{y}) = \max\{d(\mathbf{x},\mathbf{y}), d(\mathbf{y},\mathbf{z})\} \leq d(\mathbf{x},\mathbf{y}) + d(\mathbf{y},\mathbf{z}),$$

as required. The proof is similar if $\max\{d(\mathbf{x},\mathbf{y}), d(\mathbf{y},\mathbf{z})\} = d(\mathbf{y},\mathbf{z})$

Hence d is a metric on $2^{\mathbb{N}}$.

[Aside: metrics d satisfying the stronger version $d(x,z) \leq \max\{d(x,y), d(y,z)\}$ of the triangle inequality are called ***ultrametrics***.]

We define a second metric on the Cantor set, let $d' : X \times X \longrightarrow \mathbb{R}^+$ be defined by

$$d'(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i}$$

where $\mathbf{x} = (x_1, x_2, \ldots), \mathbf{y} = (y_1, y_2, \ldots) \in 2^{\mathbb{N}}$. We will show that d' is a metric on $2^{\mathbb{N}}$. It follows from the Comparison Test that d' is well-defined, since the series $\sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i}$ is bounded above by $\sum_{i=1}^{\infty} \frac{1}{2^i} = 1$.

M1: $\mathbf{x} = \mathbf{y}$ if and only if $x_i = y_i$ for all $i \in \mathbb{N}$ if and only if $|x_i - y_i| = 0$ for all $i \in \mathbb{N}$ if and only if $|x_i - y_i|/2^i = 0$ for all $i \in \mathbb{N}$ if and only if $d'(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{\infty} |x_i - y_i|/2^i = 0$.

M2: Clearly, $d'(\mathbf{x},\mathbf{y}) = d'(\mathbf{y},\mathbf{x})$.

M3: Let $\mathbf{x},\mathbf{y},\mathbf{z} \in 2^{\mathbb{N}}$. Then

$$\begin{aligned} d'(\mathbf{x},\mathbf{z}) &= \sum_{i=1}^{\infty} \frac{|x_i - z_i|}{2^i} = \sum_{i=1}^{\infty} \frac{|x_i - y_i + y_i - z_i|}{2^i} \leq \sum_{i=1}^{\infty} \frac{|x_i - y_i| + |y_i - z_i|}{2^i} \\ &= \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i} + \sum_{i=1}^{\infty} \frac{|y_i - z_i|}{2^i} = d'(\mathbf{x},\mathbf{y}) + d'(\mathbf{y},\mathbf{z}). \end{aligned}$$

Thus d' is a metric on $2^{\mathbb{N}}$.

We will prove that the metrics d and d' induce the same topology on $2^{\mathbb{N}}$.

We start by showing that the topology induced by d' is finer than that induced by d. By Proposition 3.9, it suffices to show that for every $\mathbf{x} \in 2^{\mathbb{N}}$ and every $\varepsilon > 0$ that there exists $\delta > 0$ such that $B_{d'}(\mathbf{x},\delta) \subseteq B_d(\mathbf{x},\varepsilon)$.

Let $\mathbf{x} \in 2^{\mathbb{N}}$ and $\varepsilon > 0$ be arbitrary. Then there exists $n \in \mathbb{N}$ such that $1/n < \varepsilon$ and we set $\delta = 1/2^n$. If $\mathbf{y} \in B_{d'}(\mathbf{x}, 1/2^n)$, then

$$d'(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i} < \frac{1}{2^n}.$$

Suppose there exists $i \leq n$ such that $x_i \neq y_i$. Then

$$\sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i} \geq \frac{1}{2^i} \geq \frac{1}{2^n},$$

a contradiction. Hence $\min\{ i \in \mathbb{N} : x_i \neq y_i \} > n$ and so $d(\mathbf{x}, \mathbf{y}) < 1/n$. Therefore

$$B_{d'}(\mathbf{x}, \delta) = B_{d'}(\mathbf{x}, 1/2^n) \subseteq B_d(\mathbf{x}, 1/n) \subseteq B_d(\mathbf{x}, \varepsilon)$$

as required.

To show that the topology induced by d is finer than that induced by d, again by Proposition 3.9, it suffices to show that for every $\mathbf{x} \in 2^{\mathbb{N}}$ and every $\varepsilon > 0$ that there exists $\delta > 0$ such that $B_d(\mathbf{x}, \delta) \subseteq B_{d'}(\mathbf{x}, \varepsilon)$.

Let $\mathbf{x} \in 2^{\mathbb{N}}$ and $\varepsilon > 0$ be arbitrary. Then there exists $n \in \mathbb{N}$ such that $1/2^{n-1} < \varepsilon$ and we set $\delta = 1/n$. If $\mathbf{y} \in B_d(\mathbf{x}, 1/n) = \{ \mathbf{z} \in 2^{\mathbb{N}} : x_1 = z_1,\ x_2 = z_2, \ldots,\ x_n = z_n \}$, then

$$d'(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i} = \sum_{i=n+1}^{\infty} \frac{|x_i - y_i|}{2^i} \leq \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^n} < \frac{1}{2^{n-1}} < \varepsilon$$

and so $\mathbf{y} \in B_{d'}(\mathbf{x}, \varepsilon)$. Thus $B_d(\mathbf{x}, \delta) \subseteq B_{d'}(\mathbf{x}, \varepsilon)$, as required.

It follows that the topologies induced by d and d' on $2^{\mathbb{N}}$ coincide.

4. The subspace topology

Let X be any set, let Y be any subset of X and let $\mathcal{T}$ be a topology on X. Then

$$\mathcal{T}_Y = \{ U \cap Y : U \in \mathcal{T} \}$$

is a topology on Y called the ***subspace topology***. Let's double-check it really is a topology:

T1: Since $\varnothing, X \in \mathcal{T}$, it follows that $\varnothing = Y \cap \varnothing \in \mathcal{T}_Y$ and $Y = Y \cap X \in \mathcal{T}_Y$.

T2: Let $U \cap Y, V \cap Y \in \mathcal{T}_Y$ for some $U, V \in \mathcal{T}$. Then

$$(U \cap Y) \cap (V \cap Y) = (U \cap V) \cap Y \in \mathcal{T}_Y$$

since $U \cap V \in \mathcal{T}$.

T3: Let $\{ U_\alpha \cap Y : \alpha \in I \} \subseteq \mathcal{T}_Y$ where $U_\alpha \in \mathcal{T}$ for some index set I. Then

$$\bigcup_{i \in I} U_\alpha \cap Y = \left(\bigcup_{i \in I} U_\alpha \right) \cap Y \in \mathcal{T}_Y$$

since $\bigcup_{i \in I} U_\alpha \in \mathcal{T}$.

Example 4.1. The subspace topology on $Y := \mathbb{N}$ induced by the standard topology on $X := \mathbb{R}$ is the discrete topology. Let $X \subseteq \mathbb{N}$. Then $X = \mathbb{N} \cap \bigcup_{x \in X}(x - 1/2, x + 1/2)$ and since $\bigcup_{x \in X}(x - 1/2, x + 1/2)$ is open in $\mathbb{R}$, it follows that X is open in the subspace topology. Hence the subspace topology on $\mathbb{N}$ is discrete.

Let $Y := [0, 1) \subseteq \mathbb{R} := X$. Then note that $[0, 1/2) = (-1, 1/2) \cap [0, 1)$ is open in the subspace topology, but it is not open in $\mathbb{R}$ with the standard topology.

Proposition 4.2. *Let X be a topological space, let Y be a subset of X, and let $\mathcal{B}$ be a basis for X. Then*

$$\{ B \cap Y : B \in \mathcal{B} \}$$

is a basis for the subspace topology on Y.

Proof. It suffices to show that every open set in the subspace topology is a union of sets of the form $B \cap Y$, $B \in \mathcal{B}$. Let V be any open set in the subspace topology. Then there is an open set U in X such that $V = U \cap Y$. Hence, by Proposition 2.5(iii), for all $x \in V \subseteq U$ there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U$. But $x \in Y$ and so $x \in B_x \cap Y \subseteq U \cap Y = V$ and so $V = \bigcup_{x \in V} \left(B_x \cap Y\right)$, as required. □

Example 4.3. Let $Y := (0, 1)$ and let $X := \mathbb{R}$ with the usual topology. Then the collection of all intervals (a, b) for all $a, b \in \mathbb{R}$ such that $0 < a < b < 1$ is a basis for the subspace topology on $(0, 1)$ by Proposition 4.2.

Example 4.4. Let Y be the closed unit interval $[0, 1]$ in $\mathbb{R}$ with the subspace topology. Then, by Proposition 4.2, the collection of sets $(a, b) \cap Y$ where (a, b) is an open interval in $\mathbb{R}$ is a basis for Y. There are four possibilities:

$$\begin{array}{rclll rclll}
(a,b) \cap Y & = & (a,b) & \text{if } a, b \in [0,1], & & (a,b) \cap Y & = & [0,b) & \text{if } b \in [0,1], a \notin [0,1], \\
(a,b) \cap Y & = & (a,1] & \text{if } a \in [0,1], b \notin [0,1], & & [a,b] \cap Y & = & Y \text{ or } \varnothing & \text{if } a, b \notin [0,1].
\end{array}$$

Each of these sets is open in Y by definition, but sets of the form $[0, b)$ and $(a, 1]$ are not open in $\mathbb{R}$ with the standard topology.

Proposition 4.5. *Let Y be an open subspace of a topological space X. Then a subset U of Y is open in Y if and only if it is open in X.*

Proof. (⇒) By the definition of the subspace topology, U being open in Y implies that $U = Y \cap V$ for some open V in X. But Y is itself open, and so U, being the intersection of two open sets in X, is open in X.

(⇐) If U is open in X, then $U \cap Y = U$ is open in Y by definition of the subspace topology. □

Throughout the course if P is a property of topological spaces, and Y is a subset of a topological space X, then we will say that the ***subset*** Y ***has property*** P if Y has property P with respect to the subspace topology. For example, we say that a subset Y of a topological space X is discrete if Y the subspace topology on Y inherited from X is the discrete topology.

5. Closed sets

Definition 5.1. A subset F of a topological space X is called ***closed*** if its complement is open.

Example 5.2. Let $a, b \in \mathbb{R}$ be such that $a < b$. Then $[a, b]$ is closed in $\mathbb{R}$ with the usual topology since its complement is $(-\infty, a) \cup (b, \infty)$ which is open. The set $[a, \infty)$ is closed for all $a \in \mathbb{R}$ since its complement $(-\infty, a)$ is open.

The complement of the set $(0, 1)$ is $(-\infty, 0] \cup [1, \infty)$ and so the latter set is closed (but not open) in $\mathbb{R}$ with the standard topology.

The set $[0, 1)$ is neither open nor closed in $\mathbb{R}$ with the standard topology, but it is both open and closed in the lower limit topology (since its complement is $(-\infty, 0) \cup [1, \infty)$ is open).

The set $U = \{\, (x_1, x_2, \ldots) \in 2^{\mathbb{N}} \;:\; x_1 = 0 \,\}$ is open in the Cantor set, since if $\mathbf{y} \in U$, then $B(\mathbf{y}, 1) \subseteq U$. But it is also closed since $2^{\mathbb{N}} \setminus U = \{\, (x_1, x_2, \ldots) \in 2^{\mathbb{N}} \;:\; x_1 \neq 0 \,\} = \{\, (x_1, x_2, \ldots) \in 2^{\mathbb{N}} \;:\; x_1 = 1 \,\}$ is open.

In the cofinite topology on an arbitrary set X, the closed sets are the finite subsets of X and X itself.

So sets, unlike doors, can be both open and closed, open or closed, or neither open nor closed. A set that is open and closed is sometimes called ***clopen***.

Example 5.3. If X is any set with the discrete topology, then every subset of X is clopen.

Example 5.4. The set

$$\{\, (x, y) \in \mathbb{R}^2 \;:\; |x| \geq 1 \text{ or } |y| \geq 1 \,\}$$

in the metric topology on $\mathbb{R}$ induced by the Chebyshev metric d_∞ is closed, since its complement

$$\{\, (x, y) \in \mathbb{R}^2 \;:\; |x| < 1 \text{ and } |y| < 1 \,\} = B(\mathbf{0}, 1),$$

is the ball of radius 1 around the origin $\mathbf{0} = (0, 0) \in \mathbb{R}^2$, which is open by the definition of the metric topology.

Theorem 5.5. *Let X be a topological space. Then the following hold:*

(i) *$\varnothing$ and X are closed;*
(ii) *finite unions of closed sets are closed;*
(iii) *arbitrary intersections of closed sets are closed.*

Proof. **(i)** Since $\varnothing$ and X are open, it follows that their complements X and $\varnothing$, respectively, are closed.

(ii) If $F_1, F_2, \ldots, F_n$ are closed, then there are open sets $U_1, U_2, \ldots, U_n$ such that $F_i = X \setminus U_i$ for all i. Since topologies are closed under finite intersections, it follows that $\bigcap_{i=1}^{n} U_i$ is open and so

$$X \setminus \Big(\bigcap_{i=1}^{n} U_i\Big) = \bigcup_{i=1}^{n} X \setminus U_i = \bigcup_{i=1}^{n} F_i$$

is closed.

(iii) Let $\{\, F_\alpha \;:\; \alpha \in I \,\}$ be an arbitrary collection of closed sets where I is some index set. Then as in the previous part of the proof $F_\alpha = X \setminus U_\alpha$ for some open set U_α and for all $\alpha \in I$. Then

$$\bigcap_{\alpha \in I} F_\alpha = \bigcap_{\alpha \in I} X \setminus U_\alpha = X \setminus \Big(\bigcup_{\alpha \in I} U_\alpha\Big)$$

and since $\bigcup_{\alpha \in I} U_\alpha$ is open, it follows that $\bigcap_{\alpha \in I} F_\alpha$ is closed. □

Theorem 5.6. *Let Y be a subspace of X. Then F is closed in Y if and only if there exists a closed set F' of X such that $F = Y \cap F'$.*

Proof. (⇒) Since F is closed in Y, it follows that F is the complement of an open set U of Y. By definition, there is an open set U' of X such that $U = U' \cap Y$. But then

$$F = Y \setminus U = Y \setminus U' = (X \setminus U') \cap Y$$

and so $F' := X \setminus U'$ is the closed set we require.

(⇐) Since F' is closed in X, there exists an open set U of X such that $F' = X \setminus U$. It follows that $Y \setminus F = Y \cap U$ and so $Y \setminus F$ is open in Y. Thus F is closed in Y. □

Corollary 5.7. *A closed subset F of a closed subspace Y in a topological space X is closed in X.*

Proof. Since F is closed in Y, it follows from Theorem 5.6 that there exists closed set F' in X such that $F = F' \cap Y$. But then F is an intersection of the closed sets F' and Y in X and is hence closed in X. $\square$

Definition 5.8. The ***closure*** of a subset A of a topological space X is the least (with respect to containment) closed set containing A; this is denoted $\overline{A}$.

The ***interior*** of A is the greatest open set contained in A; this is denoted A°. Of course,

$$A^\circ \subseteq A \subseteq \overline{A}.$$

Example 5.9. The closure of $[0,1)$ in $\mathbb{R}$ is $[0,1]$ since the latter set is closed, $[0,1)$ is not closed, and the sets differ in only one point, 1.

Let A be an arbitrary subset of a topological space X and let $\mathcal{F}$ denote the family of closed subsets of X containing A. Since X is closed, it follows that $X \in \mathcal{F}$, and hence $\mathcal{F}$ is not empty. Hence $\bigcap_{F\in\mathcal{F}} F$ is a non-empty closed set (by Proposition 5.5(iii)) in X that contains A. On the other hand, $\overline{A}$ is closed and it contains A, and so $\overline{A} \in \mathcal{F}$. Thus $\bigcap_{F\in\mathcal{F}} F \subseteq \overline{A}$, but $\overline{A}$ is the least closed set containing A and so

$$\bigcap_{F\in\mathcal{F}} F \subseteq \overline{A}.$$

We have shown that the closure A of a subset of a topological space X always exists, and that it equals the intersection of all the closed sets in X containing A.

It is possible to prove that the interior of a set A always exists and that it is the union of the open sets contained in A.

Theorem 5.10. *Let X be a topological space, let $\mathcal{B}$ be a basis for X, let $A \subseteq X$, and let $x \in X$. Then the following are equivalent:*

(i) $x \in \overline{A}$;
(ii) *every open nbhd U of x satisfies $U \cap A \neq \varnothing$;*
(iii) *every basic open nbhd $B \in \mathcal{B}$ of x satisfies $B \cap A \neq \varnothing$.*

Proof. (i) $\Rightarrow$ (ii) We prove the contrapositive. Suppose there exists an open nbhd U of x such that $U \cap A = \varnothing$. Then $A \subseteq X \setminus U$ and since $X \setminus U$ is closed, it follows that $\overline{A} \subseteq X \setminus U$. But then $x \notin \overline{A}$.

(ii) $\Rightarrow$ (iii) Trivial since every basic open nbhd is an open nbhd.

(iii) $\Rightarrow$ (i) Again we prove the contrapositive. If $x \notin \overline{A}$, then $x \in X \setminus \overline{A}$ which is an open set. Hence, by Proposition 2.5(iii), it follows that there exists $B \in \mathcal{B}$ such that $x \in B \subseteq X \setminus \overline{A}$. In particular, $B \cap \overline{A} = \varnothing$ and so $B \cap A = \varnothing$. $\square$

Example 5.11. The closure of $(0,1)$ in $\mathbb{R}$ with the standard topology is $[0,1]$ since every open nbhd of 0 and 1 has non-empty intersection with $(0,1)$ and any $x \in \mathbb{R} \setminus [0,1]$ is contained in an interval disjoint from $(0,1)$.

The closure of $(0,1)$ is $[0,1)$ in the lower limit topology.

The closure of $\mathbb{Q}$ in $\mathbb{R}$ with the standard topology is $\mathbb{R}$: if $x \in \mathbb{R}$ and (a,b) is any basic open nbhd of x, then $(a,b) \cap \mathbb{Q} \neq \varnothing$, and so $x \in \overline{\mathbb{Q}}$ by Theorem 5.10. Thus $\mathbb{R} \subseteq \overline{\mathbb{Q}} \subseteq \mathbb{R}$.

The closure of the set $F := \{\, \sum_{i=1}^{n} 1/2^i \ : \ n \in \mathbb{N} \,\}$ in $\mathbb{R}$ is $F \cup \{1\}$.

If X is any topological space and F is closed in X, then $\overline{F} = F$.

When working with a subspace topology, you have to be careful about the use of notation such as $\overline{A}$ and A°, since there are two topologies (the parent topology and the subspace topology). We have to be clear about the topology where the closure or interior is defined.

Theorem 5.12. *Let Y be a subspace of X and let A be a subset of Y. Then the closure of A in Y equals the closure of A in X intersected Y.*

Proof. Let $\mathrm{Cl}_Y(A)$ and $\mathrm{Cl}_X(A)$ denote the closures of A in X and Y, respectively. We want to prove that $\mathrm{Cl}_Y(A) = \mathrm{Cl}_X(A) \cap Y$. Certainly, $\mathrm{Cl}_Y(A)$ is a closed subset of Y and so, by Theorem 5.6, there exists a closed subset F in X such that $\mathrm{Cl}_Y(A) = Y \cap F$. In particular, since $A \subseteq F$ and $\mathrm{Cl}_X(A)$ is the least closed set (in X) containing A, it follows that $\mathrm{Cl}_X(A) \subseteq F$ and so $\mathrm{Cl}_X(A) \cap Y \subseteq F \cap Y = \mathrm{Cl}_Y(A)$. But $\mathrm{Cl}_Y(A)$ is the least closed set of Y containing A and $\mathrm{Cl}_X(A) \cap Y$ is a closed subset of Y containing A and so $\mathrm{Cl}_Y(A) = \mathrm{Cl}_X(A) \cap Y$. $\square$

6. Limit points

Definition 6.1. Let X be a topological space, let A be a subset of X, and let $x \in X$. Then x is a ***limit point of*** A if for all open nbhds U of x there exists $y \in U \cap A$ such that $x \neq y$.

Example 6.2. The points 0 and 1 are limit points of $(0,1)$ in $\mathbb{R}$ with the standard topology since every open nbhd of 0 and 1 has non-empty intersection with $(0,1)$. Any $x \in \mathbb{R} \setminus [0,1]$ is not a limit point of $(0,1)$ since x is contained in an interval disjoint from $(0,1)$. Every point of $(0,1)$ is also a limit point of $(0,1)$.

The point 0 is a limit point of $(0,1)$ in the lower limit topology but 1 is not since $1 \in [1,2)$ and $[1,2) \cap (0,1) = \varnothing$.

Every point x in $\mathbb{R}$ is a limit point of $\mathbb{Q}$ since every open nbhd of x contains at least 2 rational numbers.

The point 1 is a limit point of the set $F := \{ \sum_{i=1}^{n} 1/2^i \ : \ n \in \mathbb{N} \}$ in $\mathbb{R}$.

Lemma 6.3. *Let X be a topological space, let A be a subset of X, and let $x \in X$. Then x is a limit point of A if and only if $x \in \overline{A \setminus \{x\}}$.*

Proof. ($\Rightarrow$) Let U be an open nbhd of x. Then there exists $y \in U \cap A$ such that $x \neq y$. In other words, $U \cap \big(A \setminus \{x\}\big) \neq \varnothing$. It follows, by Theorem 5.10(ii), that $x \in \overline{A \setminus \{x\}}$.

($\Leftarrow$) If U is any open nbhd of x, then again by Theorem 5.10(ii), $x \in \overline{A \setminus \{x\}}$ implies that $U \cap \big(A \setminus \{x\}\big) \neq \varnothing$. In particular, there exists $y \in U \cap \big(A \setminus \{x\}\big)$ and so certainly $y \in U \cap A$ and $y \neq x$. $\square$

Theorem 6.4. *Let A be a subset of a topological space X. Then $\overline{A}$ is the union of A and the set of all limit points of A.*

Proof. Let A' denote the set of all limit points of A. We must prove that $A \cup A' = \overline{A}$.

Clearly $A \subseteq \overline{A}$. If $x \in A'$, then $x \in \overline{A \setminus \{x\}} \subseteq \overline{A}$, by Lemma 6.3. Hence $A' \subseteq \overline{A}$ and so $A \cup A' \subseteq \overline{A}$.

Conversely, suppose that $x \in \overline{A}$. If $x \in A$, then certainly $x \in A \cup A'$. If $x \in \overline{A} \setminus A$, then $x \notin A$ and so $A \setminus \{x\} = A$. Hence $x \in \overline{A} = \overline{A \setminus \{x\}}$ and so x is a limit point of A by Lemma 6.3. Thus $\overline{A} = A \cup A'$, as required. $\square$

Corollary 6.5. *Let A be a subset of a topological space X. Then A is closed in X if and only if A contains all of its limit points.*

Proof. As in the proof of the last theorem, let A' denote the set of limit points of A.

($\Rightarrow$) Since A is closed, it follows that $A = \overline{A} = A \cup A'$ and so $A' \subseteq A$.

($\Leftarrow$) If $A' \subseteq A$, then $\overline{A} = A \cup A' = A$ and so A is closed. $\square$

7. Convergence in topological spaces

Definition 7.1. Let X be a metric space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points from X. Then $(x_n)_{n\in\mathbb{N}}$ ***converges*** to $x \in X$ if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n \geq N$.

If a sequence $(x_n)_{n\in\mathbb{N}}$ converges to x, then we write $x_n \longrightarrow x$ as $n \longrightarrow \infty$ or $\lim_{n\longrightarrow\infty} x_n = x$. The point x is called the ***limit*** of the sequence.

Example 7.2. [**Convergence in the Cantor set**.] Let $2^{\mathbb{N}}$ and d be as in Example 3.11, and let

$$\mathbf{x}_n = (\underbrace{1, 1, \ldots, 1}_{n \text{ times}}, 0, 0, 0, \ldots) \in 2^{\mathbb{N}}.$$

We will prove that $\mathbf{x}_n \longrightarrow \mathbf{1} = (1, 1, \ldots) \in 2^{\mathbb{N}}$ as $n \longrightarrow \infty$. Let $\varepsilon > 0$ and let $N \in \mathbb{N}$ be such that $N \geq 1/\varepsilon$. Then

$$d(\mathbf{x}_n, \mathbf{1}) = \frac{1}{n+1} \leq \frac{1}{N} \leq \varepsilon$$

for all $n \geq N$. Hence $\mathbf{x}_n \longrightarrow \mathbf{1}$ as $n \longrightarrow \infty$ in $(2^{\mathbb{N}}, d)$.

Let d' be the other metric on $2^{\mathbb{N}}$ defined in Example 3.11. Also let $\varepsilon > 0$ and let $N \in \mathbb{N}$ be such that $N \geq -\log_2(\varepsilon)$. Then

$$d'(\mathbf{x}_n, \mathbf{1}) = \sum_{i=1}^{\infty} \frac{|x_i - 1|}{2^i} = \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^n} \leq \frac{1}{2^N} \leq \varepsilon$$

since $N \geq -\log_2(\varepsilon) = \log_2(1/\varepsilon)$, it follows that $2^N \geq 2^{\log_2(1/\varepsilon)} = 1/\varepsilon$. Hence $\mathbf{x}_n \longrightarrow \mathbf{1}$ as $n \longrightarrow \infty$ in $(2^{\mathbb{N}}, d')$ also.

Definition 7.3. Let X be a topological space. A sequence $(x_n)_{n\in\mathbb{N}}$ in X ***converges*** to $x \in X$ if for every open nbhd U of x there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$.

Example 7.4. [**Limits are not always unique.**] Let $X = \{x_1, x_2, \ldots\}$ be a countably infinite set with the cofinite topology. We will show that the sequence $(x_n)_{n\in\mathbb{N}}$ converges to every point $x \in X$.

Let $x \in X$ be arbitrary and let U be an open nbhd of x. Then $X \setminus U$ is finite, and so, in particular, $X \setminus U = \{x_{i(1)}, x_{i(2)}, \ldots, x_{i(m)}\}$ for some $i(1), i(2), \ldots, i(m) \in \mathbb{N}$. Thus setting $N = \max\{i(1), i(2), \ldots, i(m)\}$, it follows that $x_n \in U$ for all $n \geq N$.

Example 7.5. [**Convergence in the discrete topology**.] Let X denote any set with the discrete topology and let $x \in X$ be arbitrary. Then $x_n \longrightarrow x$ as $n \longrightarrow \infty$ if and only if there exists $N \in \mathbb{N}$ such that $x_n = x$ for all $n \geq N$ (i.e. a sequence converges to x if and only if it is eventually constant).

($\Leftarrow$) Every open nbhd U of x contains the open nbhd $\{x\}$, and $x_n \in \{x\}$ for all $n \geq N$. Hence $x_n \in U$ for all $n \geq N$ and so $x_n \longrightarrow x$ as $n \longrightarrow \infty$.

($\Rightarrow$) We prove the contrapositive. Assume that for all $N \in \mathbb{N}$ there exists $n \geq N$ such that $x_n \neq x$. But $\{x\}$ is an open nbhd of x which does not satisfy the condition of Definition 7.3. Hence $x_n \not\longrightarrow x$ as $n \longrightarrow \infty$.

Proposition 7.6. *Let X be a metric space with metric d and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Then $(x_n)_{n\in\mathbb{N}}$ converges to $x \in X$ if and only if $(x_n)_{n\in\mathbb{N}}$ converges to x in the metric topology induced by d.*

Proof. ($\Rightarrow$) Let U be any open nbhd of x in X with the metric topology. Then since U is a union of basic open sets of the metric topology and these basic open sets are balls, it follows that there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq U$. Since $(x_n)_{n\in\mathbb{N}}$ converges in X, there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n \geq N$. In other words, $x_n \in B(x, \varepsilon) \subseteq U$ for all $n \geq N$, as required.

($\Leftarrow$) Let $\varepsilon > 0$. Then $B(x, \varepsilon)$ is an open nhbd of x and so there exists an $N \in \mathbb{N}$ such that $x_n \in B(x, \varepsilon)$ for all $n \geq N$. In other words, $d(x_n, x) < \varepsilon$ for all $n \geq N$ and so x_n converges to x in the metric space X. $\square$

Corollary 7.7. *Let X be a set with metrics d and d' that induce the same topology on X. Then a sequence converges in (X, d) if and only if it converges in (X, d').*

So, in particular, a sequence converges in $\mathbb{R}^n$ with respect to the metric d_1 if and only if it converges with respect to d_2 if and only if it converges with respect to d_∞ (as defined in Examples 3.3, 3.2, and 3.4, respectively).

The notion of convergence is not very useful in the context of general topological spaces. However, we will see that making a fairly mild assumption on the topological space can make the notion meaningful once again.

Definition 7.8. A topological space X is called ***Hausdorff*** if for all $x, y \in X$ there exist disjoint open nbhds U and V of x and y, respectively.

Example 7.9. [**Metric topologies are Hausdorff.**] Let X be a metric space and let $x, y \in X$. If $\varepsilon = d(x, y)/2$, then $B(x, \varepsilon)$ and $B(y, \varepsilon)$ are open nbhds of x and y in the metric topology. If $z \in B(x, \varepsilon) \cap B(y, \varepsilon)$, then

$$d(x, z) < d(x, y)/2 \text{ and } d(y, z) < d(x, y)/2$$

and so

$$d(x, y) \leq d(x, z) + d(y, z) < d(x, y),$$

a contradiction. Hence $B(x, \varepsilon)$ and $B(y, \varepsilon)$ are disjoint.

It follows that $\mathbb{R}$ with the standard topology is Hausdorff, the Cantor set $2^{\mathbb{N}}$ is Hausdorff, and so on.

Example 7.10. [**The cofinite topology is not Hausdorff (or metrizable).**] Let X be an infinite set with the cofinite topology, let $x, y \in X$, and let U and V be any open nbhds of x and y. Since U and V are cofinite, it follows that $U \cap V \neq \varnothing$ and so X is not Hausdorff and hence it is not metrizable.

Theorem 7.11. *Let X be a Hausdorff space. Then a sequence of points in X converges to at most one point of X.*

Proof. Suppose there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X that converges to $x, y \in X$ where $x \neq y$. Since X is Hausdorff, there exist disjoint open nbhds U and V of x and y, respectively. From the definition of convergence in a topological space, it follows that there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$. Hence $x_n \notin V$ for all $n \geq N$, and so V contains only finitely many points in $\{x_1, x_2, \ldots\}$, a contradiction. $\square$

Lemma 7.12. *Let X be a topological space and let A be any subset of X. If a sequence of points in A converges to $x \in X$, then $x \in \overline{A}$. If X is metrizable, then the converse also holds.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points in A that converge to $x \in X$. Then for all open nbhds U of x there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$. In particular, $A \cap U \neq \varnothing$ and so by Theorem 5.10, $x \in \overline{A}$.

Suppose that d is a metric that induces the topology on X and let $x \in \overline{A}$ be arbitrary. By Theorem 5.10, every open nbhd of x has non-empty intersection with A. Hence for every $n \in \mathbb{N}$, we may choose $x_n \in B_d(x, 1/n) \cap A$. We will show that the sequence $(x_n)_{n\in\mathbb{N}}$ converges to x. Let U be any open nbhd of x. Then there exists $N \in \mathbb{N}$ such that $B_d(x, 1/N) \subseteq U$. Thus for all $n \geq N$ it follows that $x_n \in B_d(x, 1/n) \subseteq B_d(x, 1/N) \subseteq U$, and so $(x_n)_{n\in\mathbb{N}}$ converges to x. $\square$

An example of a Hausdorff space that is not metrizable. It is not true that every Hausdorff space is metrizable. We will give an example of a Hausdorff space that is not metrizable.

Definition 7.13. A subset D of a topological space X is ***dense*** in X if $U \cap D \neq \varnothing$ for all non-empty open sets U in X.

Clearly, every topological space is a dense subset of itself.

Example 7.14. [**A dense subset can have fewer elements than the entire space.**] Let U be an non-empty open set in $\mathbb{R}$ with the standard topology. Then there exist $a, b \in \mathbb{R}$ such that $a < b$ and $(a, b) \subseteq U$. Since there exists $c \in \mathbb{Q}$ such that $a < c < b$, or $c \in (a, b)$ it follows that $U \cap \mathbb{Q} \neq \varnothing$. But U was arbitrary and so $\mathbb{Q}$ is dense in $\mathbb{R}$.

A similar argument shows that $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$ with the standard topology.

If U is any non-empty open set in $\mathbb{R}$ with the lower limit topology, Then there exist $a, b \in \mathbb{R}$ such that $a < b$ and $[a, b) \subseteq U$. Since there exists $c \in \mathbb{Q}$ such that $a < c < b$, or $c \in [a, b)$ it follows that $U \cap \mathbb{Q} \neq \varnothing$. But U was arbitrary and so $\mathbb{Q}$ is dense in $\mathbb{R}$ with the lower limit topology too.

Example 7.15. Let X be any infinite set and let $x \in X$ be fixed. Then the set $\{x\}$ is dense in X with the particular point topology.

If we consider the excluded point topology on X, then $X \setminus \{x\}$ is a dense subset of X. In fact, $X \setminus \{x\}$ is the least dense subset of X. If $x \in X$ is the excluded point and Y is a proper subset of $X \setminus \{x\}$, then $X \setminus (Y \cup \{x\})$ is open in X and has empty intersection with Y. Hence Y is not dense in X.

Definition 7.16. A topological space X is ***separable*** if it has a countable dense subset.

So, in Example 7.14, we saw that $\mathbb{R}$ with the standard topology is separable. Example 7.15 shows that the particular point topology on any set is separable, and that the excluded point topology on an uncountable set is not separable.

Theorem 7.17. *Let X be a separable metric space with metric d. Then the metric topology induced by d has a countable basis.*

Proof. Let D be a countable dense subset of X. We will show that the collection of balls $B_d(y, 1/n)$ where $y \in D$ and $n \in \mathbb{N}$ is a basis for X. It suffices by Proposition 2.5(iii) to show that for all $x \in X$, for all $\varepsilon > 0$, and for all $z \in B_d(x, \varepsilon)$, that there exists $y \in D$ and $n \in \mathbb{N}$ such that $z \in B_d(y, 1/n) \subseteq B_d(x, \varepsilon)$.

Suppose that $x \in X$ and $\varepsilon > 0$ are arbitrary and that $z \in B_d(x, \varepsilon)$. If $m \in \mathbb{N}$ is such that $0 < 2/m < \varepsilon - d(x, y)$, then $B_d(z, 2/m) \subseteq B_d(x, \varepsilon)$. Since D is dense, $D \cap B_d(z, 1/m) \neq \varnothing$ and so there is $y \in D \cap B_d(z, 1/m)$. So, $z \in B_d(y, 1/m) \subseteq B_d(z, 2/m) \subseteq B_d(x, \varepsilon)$, as required. □

Corollary 7.18. *The lower limit topology on $\mathbb{R}$ is Hausdorff but not metrizable.*

Proof. Let $\mathbb{R}_l$ denote the lower limit topology on $\mathbb{R}$. Since $\mathbb{R}_l$ is finer than $\mathbb{R}$ and $\mathbb{R}$ is Hausdorff, it follows that $\mathbb{R}_l$ is Hausdorff. We showed in Example 7.14 that $\mathbb{R}_l$ is separable. So, if $\mathbb{R}_l$ was metrizable, then $\mathbb{R}_l$ would have a countable basis. But in Problem **2-10** we showed that $\mathbb{R}_l$ has no countable basis, and so $\mathbb{R}_l$ is not metrizable. □

8. Continuous functions

Definition 8.1. Let X and Y be topological spaces. Then a function $f : X \longrightarrow Y$ is ***continuous*** if $f^{-1}(V) = \{\, x \in X : f(x) \in V \,\}$ is open in X for all open sets V in Y.

Note that $f^{-1}(V)$ in the above definition is called the ***preimage*** of V under f and it can be empty.

Example 8.2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = 2x + 1$ for all $x \in \mathbb{R}$ and let U be an open set in $\mathbb{R}$ with the standard topology. Then U is a union of intervals (a_i, b_i), $a_i, b_i \in \mathbb{R}$ and $i \in I$ for some index set I, i.e.

$$U = \bigcup_{i \in I} (a_i, b_i).$$

Hence

$$f^{-1}(U) = \bigcup_{i \in I} \left(\frac{a_i - 1}{2}, \frac{b_i - 1}{2} \right)$$

is open (being a union of open intervals) and so f is continuous.

Example 8.3. [**Constant functions are continuous**.] Let X be any topological space, let $x_0 \in X$ be fixed, and let $f : X \longrightarrow X$ be defined by $f(x) = x_0$. If U is any open set in X, then $f^{-1}(U) = \varnothing$ if $x_0 \notin U$ and $f^{-1}(U) = X$ if $x_0 \in X$. Hence, since $\varnothing$ and X are open, f is continuous.

Example 8.4. [**Continuous functions in the cofinite topology**.] Let X be any infinite set with the cofinite topology and let $f : X \longrightarrow X$ be any function such that $|f^{-1}(x)|$ is finite for all $x \in X$. If U is non-empty and open in X, then $X \setminus U = \{x_1, x_2, \dots, x_n\}$ for some $x_1, x_2, \dots, x_n \in X$. Thus

$$X = f^{-1}(X) = f^{-1}(U) \cup f^{-1}(X \setminus U) = f^{-1}(U) \cup f^{-1}(\{x_1, x_2, \dots, x_n\})$$

Hence $X \setminus f^{-1}(U) = f^{-1}(\{x_1, x_2, \dots, x_n\}) = f^{-1}(x_1) \cup f^{-1}(x_2) \cup \cdots f^{-1}(x_n)$, is finite (being a finite union of finite sets). Hence $f^{-1}(U)$ is cofinite and hence open. Thus the preimage of every open set is open and so f is continuous.

Proposition 8.5. *Let X and Y be topological spaces and let $f : X \longrightarrow Y$. Then the following are equivalent:*

(i) *f is continuous;*
(ii) *if $\mathcal{S}$ is a subbasis for Y, then $f^{-1}(S)$ is open in X for all $S \in \mathcal{S}$;*
(iii) *if $\mathcal{B}$ is a basis for Y, then $f^{-1}(B)$ is open in X for all $B \in \mathcal{B}$;*
(iv) *$f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$;*
(v) *$f^{-1}(F)$ is closed in X for all closed sets F in Y.*

Proof. (i)⇒(ii). Since every set in $\mathcal{S}$ is open in Y, it follows immediately from the definition of continuity that $f^{-1}(S)$ is open in X for all $S \in \mathcal{S}$.

(ii)⇒(iii). Since every basis for Y is also a subbasis for Y, this implication also follows immediately.

(iii)⇒(iv). Let $y \in f(\overline{A})$ be arbitrary. Then there is $x \in \overline{A}$ such that $f(x) = y$. Suppose that $B \in \mathcal{B}$ is any basic open nbhd of y. Then $f^{-1}(B)$ is open, and $x \in f^{-1}(B)$. Since $x \in \overline{A}$, it follows by Theorem 5.10(ii) that $f^{-1}(B) \cap A \neq \emptyset$ and so $B \cap f(A) \neq \emptyset$. Hence $y \in \overline{f(A)}$ (Theorem 5.10(iii)) and so $f(\overline{A}) \subseteq \overline{f(A)}$.

(iv)⇒(v). Let F be any closed subset of Y. If $x \in \overline{f^{-1}(F)}$ is arbitrary, then

$$f(x) \in f(\overline{f^{-1}(F)}) \subseteq \overline{f(f^{-1}(F))} = \overline{F} = F.$$

Hence $\overline{f^{-1}(F)} \subseteq f^{-1}(F)$ and so $f^{-1}(F) = \overline{f^{-1}(F)}$ and $f^{-1}(F)$ is closed.

(v)⇒(i). Let V be open in Y. Then $Y \setminus V$ is closed in Y and so $f^{-1}(Y \setminus V)$ is closed. But $X = f^{-1}(Y) = f^{-1}(Y \setminus V) \cup f^{-1}(V)$ and so, since $f^{-1}(Y \setminus V)$ and $f^{-1}(V)$ are disjoint, $f^{-1}(V) = X \setminus f^{-1}(Y \setminus V)$ and this is an open set (being the complement of closed set). Hence f is continuous. □

Example 8.6. Let $\mathbb{R}$ denote the real numbers with the standard topology and let $\mathbb{R}_l$ denote the real numbers with the lower limit topology. Then $f : \mathbb{R} \longrightarrow \mathbb{R}_l$ defined by $f(x) = x$ for all $x \in \mathbb{R}$ is not continuous since $f^{-1}[0, 1) = [0, 1)$, which is not open in $\mathbb{R}$ (no open interval containing 0 is contained in $[0, 1)$).

On the other hand, $g : \mathbb{R}_l \longrightarrow \mathbb{R}$ defined by $g(x) = x$ for all $x \in \mathbb{R}$ is continuous since

$$g^{-1}(a, b) = (a, b) = \bigcup_{a<c<b} [c, b)$$

is open in the lower limit topology for all $a, b \in \mathbb{R}$.

The sequence definition of continuity.

Theorem 8.7. *Let X and Y be topological spaces and let $f : X \longrightarrow Y$. If f is continuous, then for every sequence $(x_n)_{n\in\mathbb{N}}$ that converges to some $x \in X$, the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x) \in Y$. If X is metrizable, then the converse implication also holds.*

Proof. Let V be any open nbhd of $f(x)$ in Y. Then $f^{-1}(V)$ is open in X and $x \in f^{-1}(V)$. Thus there exists $N \in \mathbb{N}$ such that $x_n \in f^{-1}(V)$ for all $n \geq N$ and so $f(x_n) \in V$ for all $n \geq N$. In particular, $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x)$.

Suppose that X is metrizable and that for every sequence $(x_n)_{n\in\mathbb{N}}$ that converges to $x \in X$, the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x) \in Y$. We prove that $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq X$ so that, by Theorem 8.5, f is continuous. Let $A \subseteq X$ be arbitrary and let $x \in \overline{A}$. Since X is metrizable, it follows by Lemma 7.12 that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of points in A that converges to x. Hence by assumption $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x)$ and so by Lemma 7.12 again, it follows that $f(x) \in \overline{f(A)}$. Thus $f(\overline{A}) \subseteq \overline{f(A)}$, as required. □

Continuity at a point.

Definition 8.8. Let X and Y be topological spaces and let $f : X \longrightarrow Y$. Then f is ***continuous at*** $x \in X$ if for all open nbhds V of $f(x)$, there exists an open nbhd U of x such that $f(U) \subseteq V$.

Theorem 8.9. *Let X and Y be topological spaces and let $f : X \longrightarrow Y$. Then f is continuous if and only if f is continuous at every $x \in X$.*

Proof. ($\Rightarrow$) Let $x \in X$ be arbitrary and let V be any open nbhd of $f(x)$. Then $x \in f^{-1}(V) = \{ y \in X \,:\, f(y) \in V \}$ is an open nbhd of x.

($\Leftarrow$) Let V be any open set in Y. If $x \in f^{-1}(V)$, then $f(x) \in V$ and so, by assumption, there exists an open set U_x in X such that $x \in f(U_x) \subseteq V$. Thus $U_x \subseteq f^{-1}(V)$ and so $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$ is a union of open sets and is hence open. $\square$

Example 8.10. Let $\mathbb{N}^{\mathbb{N}} = \{ (x_n)_{n \in \mathbb{N}} \,:\, x_i \in \mathbb{N} \}$. We showed in Problem 2-7 that

$$d(\mathbf{x}, \mathbf{y}) = \begin{cases} 0 & \text{if } x = y \\ \dfrac{1}{\min\{ i \in \mathbb{N} \,:\, x_i \neq y_i \}} & \text{if } x \neq y \end{cases}$$

where $\mathbf{x} = (x_1, x_2, \ldots)$ and $\mathbf{y} = (y_1, y_2, \ldots)$ is a metric on $\mathbb{N}^{\mathbb{N}}$. Let $n \in \mathbb{N}$ be arbitrary and let $f : \mathbb{N}^{\mathbb{N}} \longrightarrow \mathbb{N}^{\mathbb{N}}$ be defined by

$$f(x_1, x_2, \ldots) = (x_n, x_{n+1}, \ldots).$$

Let $\mathbf{x} = (x_1, x_2, \ldots) \in \mathbb{N}^{\mathbb{N}}$ be arbitrary and let V be any open nbhd of $f(\mathbf{x})$. Then there exists $m \in \mathbb{N}$ such that $B_d(f(\mathbf{x}), 1/m) \subseteq V$. But $\mathbf{x} \in B_d(\mathbf{x}, 1/(m+n)) \subseteq f^{-1}(B_d(f(\mathbf{x}), 1/m))$ and so f is continuous at $\mathbf{x}$. But $\mathbf{x}$ was arbitrary and so f is continuous.

Continuity in metric spaces.

Definition 8.11. Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \longrightarrow Y$. Then f is ***continuous at*** $x \in X$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $d_X(x, y) \leq \delta$ implies that $d_Y(f(x), f(y)) \leq \varepsilon$.

If Y is any subset of X, then f is ***continuous on*** Y if f is continuous at y for all $y \in Y$.

The definition of continuity in a metric space is a direct analogue of the definition of continuity of a function from $\mathbb{R}$ to $\mathbb{R}$.

Example 8.12. [**Continuous functions on discrete spaces**.] We will prove that every function is continuous on a discrete metric space. Let $f : X \longrightarrow X$ be any function, let $x \in X$ be arbitrary, and let $\varepsilon > 0$. Then setting $\delta = 1/2$, it follows that if $y \in X$ such that $d(x, y) < 1/2$, then $x = y$ and so $d(x, y) = 0 < \varepsilon$. Thus f is continuous on X.

Example 8.13. [**A continuous function on the Cantor set**.] Let $\sigma : \{0, 1\} \longrightarrow \{0, 1\}$ be defined by

$$\sigma(x) = \begin{cases} 0 & \text{if } x = 1 \\ 1 & \text{if } x = 0, \end{cases}$$

and let $f : 2^{\mathbb{N}} \longrightarrow 2^{\mathbb{N}}$ be defined by

$$f(x_1, x_2, \ldots) = (\sigma(x_1), \sigma(x_2), \ldots)$$

for all $(x_1, x_2, \ldots) \in 2^{\mathbb{N}}$. We will prove f is continuous on $2^{\mathbb{N}}$ under the metrics d and d' defined in Example 3.11.

Let $\mathbf{x} = (x_1, x_2, \ldots) \in 2^{\mathbb{N}}$, let $\varepsilon > 0$, and let $\delta = \varepsilon$. Then if $\mathbf{y} = (y_1, y_2, \ldots) \in 2^{\mathbb{N}}$ such that $\mathbf{x} \neq \mathbf{y}$ and $d(\mathbf{x}, \mathbf{y}) < \delta$, then

$$d(f(\mathbf{x}), f(\mathbf{y})) = \frac{1}{\min\{ i \in \mathbb{N} \,:\, \sigma(x_i) \neq \sigma(y_i) \}} = \frac{1}{\min\{ i \in \mathbb{N} \,:\, x_i \neq y_i \}} = d(\mathbf{x}, \mathbf{y}) < \delta = \varepsilon$$

and so f is continuous on $2^{\mathbb{N}}$ under d. Also,

$$d'(f(\mathbf{x}), f(\mathbf{y})) = \sum_{i=1}^{\infty} \frac{|\sigma(x_i) - \sigma(y_i)|}{2^i} = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i} = d'(\mathbf{x}, \mathbf{y}) < \delta = \varepsilon$$

and so f is continuous on $2^{\mathbb{N}}$ under d'.

[**Aside:** a function $f : X \longrightarrow X$ on a metric space X that satisfies $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$ is called an ***isometry***.]

Proposition 8.14. *Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \longrightarrow Y$. Then f is continuous in the sense of Definition 8.11 if and only if f is continuous from the metric topology induced by d_X to the metric topology induced by d_Y.*

Proof. [As an exercise.] ($\Rightarrow$) It suffices by Lemma 8.9 to show that f is continuous at every point $x \in X$. Let V be any open nbhd of $f(x)$. Then there exists $\varepsilon > 0$ such that $f(x) \in B_{d_Y}(f(x), \varepsilon) \subseteq V$. Hence by Definition 8.11 there exists $\delta > 0$ such that if $d_X(x, y) < \delta$, then $d_Y(f(x), f(y)) < \varepsilon$. Rephrasing: if $y \in B_{d_X}(x, \delta)$, then $f(y) \in B_{d_Y}(f(x), \varepsilon)$. Thus $B_{d_X}(x, \delta)$ is an open nbhd of x such that $f(B_{d_X}(x, \delta)) \subseteq B_{d_Y}(f(x), \varepsilon) \subseteq V$ and f is continuous at x.

($\Leftarrow$) Let $x \in X$ be arbitrary and let $\varepsilon > 0$. Then, since f is continuous and $B_{d_Y}(f(x), \varepsilon)$ is open in the topology induced by d_Y, it follows that $f^{-1}(B_{d_Y}(f(x), \varepsilon))$ is open in the topology induced by d_X. Since $x \in f^{-1}(B_{d_Y}(f(x), \varepsilon))$, there exists $\delta > 0$ such that $B_{d_X}(x, \delta) \subseteq f^{-1}(B_{d_Y}(f(x), \varepsilon))$. Hence if $y \in X$ such that $d_X(x, y) < \delta$, then $y \in B_{d_X}(x, \delta)$ and so $f(y) \in B_{d_Y}(f(x), \varepsilon)$ implying that $d(f(x), f(y)) < \varepsilon$. Thus in the sense of Definition 8.11 f is continuous at x, and since x was arbitrary, f is continuous. $\square$

New continuous functions from old.

Proposition 8.15. *Let X, Y, and Z be topological spaces. Then:*

(i) *If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are continuous, then $g \circ f : X \longrightarrow Z$ is continuous.*
(ii) *If $f : X \longrightarrow Y$ is continuous and A is a subspace of X, then $f|_A : A \longrightarrow Y$, defined by $f|_A(a) = f(a)$ for all $a \in A$, is continuous.*
(iii) *$f : X \longrightarrow Y$ is continuous if $X = \bigcup_{i \in I} U_i$ where U_i is open and $f|_{U_i}$ is continuous for all $i \in I$.*

Proof. (i) Let V be open in Z. Then $g^{-1}(V)$ is open in Y and so $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is open in X. Hence $g \circ f$ is continuous.

(ii) Let V be open in Y. Then $f^{-1}(V)$ is open in X and so $f^{-1}(V) \cap A = f|_A^{-1}(V)$ is open in A. Hence $f|_A$ is continuous.

(iii) Let V be an open set in Y. Then
$$f^{-1}(V) = X \cap f^{-1}(V) = \bigcup_{i \in I} U_i \cap f^{-1}(V) = \bigcup_{i \in I} f|_{U_i}^{-1}(V),$$
which is a union of open sets and is hence open. $\square$

Homeomorphisms.

Definition 8.16. Let X and Y be topological spaces. If $f : X \longrightarrow Y$ is a bijection, and f and f^{-1} are continuous, then f is called a ***homeomorphism*** and X is said to be ***homeomorphic*** to Y.

It is straightforward to verify that "being homeomorphic" is an equivalence relation.

Example 8.17. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = 4x - 7$. Then f is a homeomorphism since if $g : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by
$$g(x) = \frac{1}{4}(x + 7),$$
then $f(g(x)) = x$ and $g(f(x)) = x$. Hence f is a bijection and $g = f^{-1}$. Proving that f and g are continuous is a routine exercise.

Proposition 8.18. *Let X and Y be topological spaces and let $f : X \longrightarrow Y$ be a bijection. Then the following are equivalent:*

(i) *f is a homeomorphism;*
(ii) *F is closed in X if and only if $f(F)$ is closed in Y;*
(iii) *U is open in X if and only if $f(U)$ is open in Y.*

Proof. [As an exercise.] (i)$\Rightarrow$(ii) Let F be a subset of X. If $f(F)$ is closed in Y, then $Y \setminus f(F)$ is open and so $f^{-1}(Y \setminus f(F)) = \{\, x \in X \,:\, f(x) \in Y \setminus f(F) \,\} = X \setminus F$ is open in X since f is continuous. Thus F is closed in X.
If F is closed in X, then $X \setminus F$ is open in X and so $(f^{-1})^{-1}(X \setminus F)$ is open in Y since f^{-1} is continuous. But $(f^{-1})^{-1} = f$ and so $f(X \setminus F) = f(X) \setminus f(F)$ is open in Y and $f(F)$ is closed.

(ii)$\Rightarrow$(iii) Let U be an arbitrary subset of X. Then U is open in X if and only if $X \setminus U$ is closed in X if and only if $f(X \setminus U) = f(X) \setminus f(U) = Y \setminus f(U)$ is closed in Y if and only if $f(U)$ is open in U.

(iii)$\Rightarrow$(i) Let U be an open set in X and let V be an open set in Y. Then $f(U) = (f^{-1})^{-1}(U)$ is open in Y and so f^{-1} is continuous. Likewise if $W = f^{-1}(V)$, then $f(W) = V$ is open in Y and so $W = f^{-1}(V)$ is open in X. Thus f is a homeomorphism. $\square$

As a consequence of Proposition 8.18, if $f : X \longrightarrow Y$ is a homeomorphism from the space X to Y, then f is a $1-1$ correspondence not only between X and Y but also between the open subsets of X and the open subsets of Y. It follows that any property of X that can be expressed entirely in terms of the opens sets of X is also satisfied by Y. In other words, from the point of view of topology the spaces can be considered to be the same. Much as we study of groups, semigroups, rings, fields, and so on, 'up to isomorphism', we study topological spaces 'up to homeomorphism'.

Example 8.19. [**All open intervals in $\mathbb{R}$, including those that are unbounded, are homeomorphic to $\mathbb{R}$.**] The function $f : (a,b) \longrightarrow (0,1)$ defined by

$$f(x) = \frac{x-a}{b-a}$$

is a homeomorphism. Moreover, if $a, b \in \mathbb{R}$, then the intervals (a,∞) and (b,∞) are homeomorphic via the mapping $x \mapsto x + (b-a)$. The intervals $(1,\infty)$ and $(0,1)$ are homeomorphic under the function $f(x) = 1/x$, and $(-\infty,-a)$ is homeomorphic to (a,∞) under the mapping $f(x) = -x$. Finally $\mathbb{R}$ is homeomorphic to $(-\pi/2,\pi/2)$ under the map $f(x) = \arctan(x)$.

Example 8.20. [**All closed bounded intervals in $\mathbb{R}$ are homeomorphic**.] The function $f : [a,b] \longrightarrow [0,1]$ defined by

$$f(x) = \frac{x-a}{b-a}$$

is a homeomorphism. We will see later on that no unbounded closed interval is homeomorphic to a bounded closed interval.

Example 8.21. [**A continuous bijection that is not a homeomorphism**.] Let $\mathbb{R}_l$ denote the real numbers $\mathbb{R}$ with the lower limit topology. Then the function $f : \mathbb{R}_l \longrightarrow \mathbb{R}$ in Example 8.6 is a continuous bijection but is not a homeomorphism.

Denote by S^1 the ***unit circle***

$$\{ (x,y) \in \mathbb{R}^2 \ : \ x^2 + y^2 = 1 \}$$

with the subspace topology inherited from the standard topology on $\mathbb{R}^n$ (the one induced by any of the metrics in Examples 3.3, 3.4, and 3.2). The function $f : [0,1) \longrightarrow S^1$ defined by $f(x) = (\cos 2\pi x, \sin 2\pi x)$ is a bijection and continuous but $f[0,1/4)$ is not open in S^1 because there is no open nbhd U of $f(0)$ such that $U \cap S^1 \subseteq f[0,1/4)$.

9. The finite product topology

Suppose X and Y are topological spaces. Then there is a natural way to define a topology on the Cartesian product $X \times Y = \{ (x,y) \ : \ x \in X, \ y \in Y \}$. Let $\mathcal{B} = \{ U \times V \ : \ U \text{ is open in } X, \ V \text{ is open in } Y \}$. We show that $\mathcal{B}$ is a basis for a topology on $X \times Y$.

B1.: Certainly, X is open in X and Y is open in Y and so $(x,y) \in X \times Y \in \mathcal{B}$ for all $(x,y) \in X \times Y$.

B2.: Suppose $U_1 \times V_1, U_2 \times V_2 \in \mathcal{B}$ and

$$(x,y) \in (U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2).$$

Then $U_1 \cap U_2$ is open in X and $V_1 \cap V_2$ is open in Y and so $(U_1 \cap U_2) \times (V_1 \cap V_2) \in \mathcal{B}$.

Definition 9.1. Let X and Y be topological spaces. Then the ***product topology*** on $X \times Y$ is the topology generated by the basis consisting of sets $U \times V$ where U is open in X and V is open in Y.

Proposition 9.2. *Let X and Y be topological spaces with bases $\mathcal{B}$ and $\mathcal{C}$, respectively. Then $\{ B \times C \ : \ B \in \mathcal{B}, \ C \in \mathcal{C} \}$ is a basis for the product topology on $X \times Y$.*

Proof. An arbitrary basic open set in $X \times Y$ is of the form $U \times V$ where U is open in X and V is open in Y. If $(x,y) \in U \times V$, then $x \in U$ and $y \in V$. Hence there exists $B \in \mathcal{B}$ and $C \in \mathcal{C}$ such that $x \in B \subseteq U$ and $y \in C \subseteq V$. Thus $(x,y) \in B \times C \subseteq U \times V$ and so every open set in the product topology is a union of sets in $\{ B \times C \ : \ B \in \mathcal{B}, \ C \in \mathcal{C} \}$ and this is a basis. □

Example 9.3. [**The standard topology on $\mathbb{R}^n$.**] Let $\mathbb{R}$ denote the standard topology on $\mathbb{R}$. Then using Proposition 9.2 it follows that the product topology on $\mathbb{R}^2$ has a basis consisting of sets of the form:

$$\{ (x,y) \in \mathbb{R}^2 \ : \ a < x < b \text{ and } c < y < d \}$$

for some $a, b, c, d \in \mathbb{R}$. It is straightforward to prove that the product topology coincides with the metric topology induced by any of the taxicab metric, the Chebyshev metric, or the Euclidean metric (these three metric induce the same topology; Example 3.10). We refer to the product topology on $\mathbb{R}^2$ derived from the standard topology on $\mathbb{R}$ as the ***standard topology*** on $\mathbb{R}^2$.

Definition 10.1. Let I be an arbitrary set indexing a family of topological spaces $(X_i)_{i\in I}$. Then the ***product topology*** on the Cartesian product $\prod_{i\in I} X_i$ is the topology generated by the basis $\mathcal{B}$ consisting of sets $\prod_{i\in I} U_i$ where U_i is open in X_i for all $i \in I$ and $U_i = X_i$ for all but finitely many $i \in I$.

It follows by a similar argument to that given above that the family of sets $\mathcal{B}$ in the definition actually are a basis for a topology:

B1.: Certainly, X_i is open in X_i for all $i \in I$ and so $\prod_{i\in I} X_i$ is open in $\prod_{i\in I} X_i$ and every $(x_i)_{i\in I}$ belongs to a set in the basis.

B2.: Suppose $\prod_{i\in I} U_i, \prod_{i\in I} V_i \in \mathcal{B}$ and

$$(x_i)_{i\in I} \in \prod_{i\in I} U_i \cap \prod_{i\in I} V_i = \prod_{i\in I}(U_i \cap V_i).$$

Then $U_i \cap V_i$ is open in X_i for all i, and there are only finitely many $i \in I$ such that $U_i \cap V_i \neq X_i$.

We will sometimes write X^I instead of $\prod_{i\in I} X_i$.

Example 10.2. [**The Cantor set**.] The set $\{0, 1\}$ is a topological space with the discrete topology. Hence we can consider the product topology on $\{0, 1\}^{\mathbb{N}} = \{\, (x_1, x_2, \ldots) : (\forall i \in \mathbb{N})\, (x_i \in \{0, 1\}) \,\}$ (i.e. the space of sequences consisting of the values 0 and 1). From the definition, a basis $\mathcal{B}$ for $\{0, 1\}^{\mathbb{N}}$ consists of sets of the form $\prod_{i\in\mathbb{N}} U_i$ where U_i is any subset of $\{0, 1\}$ for all $i \in \mathbb{N}$ and $U_i = \{0, 1\}$ for all but finitely many $i \in \mathbb{N}$. For example,

$$\{0, 1\} \times \{0\} \times \{1\} \times \{0, 1\} \times \{0, 1\} \times \cdots, \quad \{1\} \times \{1\} \times \{1\} \times \{0, 1\} \times \{0, 1\} \times \cdots$$

are open in $\{0, 1\}^{\mathbb{N}}$. Not every open set is of this form, for example,

$$(\{1\} \times \{0\} \times \{0, 1\} \times \{0, 1\} \times \cdots) \cup (\{0\} \times \{0, 1\} \times \{0, 1\} \times \cdots)$$

is open but is not an element of the basis.

We have seen the space $\{0, 1\}^{\mathbb{N}}$ before, it is the Cantor set $2^{\mathbb{N}}$. Let d be the metric introduced in Example 3.11:

$$d(\mathbf{x}, \mathbf{y}) = \begin{cases} 0 & \text{if } x = y \\ \dfrac{1}{\min\{\, i \in \mathbb{N} : x_i \neq y_i \,\}} & \text{if } x \neq y. \end{cases}$$

where $\mathbf{x} = (x_1, x_2, \ldots)$ and $\mathbf{y} = (y_1, y_2, \ldots)$. The metric topology on $2^{\mathbb{N}}$ induced by d is generated by the basis consisting of the balls

$$B_d(\mathbf{x}, 1/n) = \{x_1\} \times \{x_2\} \times \cdots \times \{x_n\} \times \{0, 1\} \times \cdots$$

for all $\mathbf{x} \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$.

We will show that the topology induced by d and the product topology are equal by showing that the balls $B_d(\mathbf{x}, 1/n)$ are a basis for $\{0, 1\}^{\mathbb{N}}$. By Proposition 2.5(iii), it suffices to prove that if $U := \prod_{i\in\mathbb{N}} U_i$ is in the basis for the product topology on $\{0, 1\}^{\mathbb{N}}$ and $\mathbf{x} = (x_1, x_2, \ldots) \in U$, then there exists $n \in \mathbb{N}$ such that $\mathbf{x} \in B_d(\mathbf{x}, 1/n) \subseteq U$. By definition, $U_i = \{0, 1\}$ for all but finitely many $i \in N$ and so

$$n = \max\{\, i \in \mathbb{N} : U_i \neq \{0, 1\} \,\}$$

exists. In particular, $U_i = \{0, 1\}$ for all $i > n$ and so

$$(x_1, x_2, \ldots) \in B_d(\mathbf{x}, 1/n) = \{x_1\} \times \{x_2\} \times \cdots \times \{x_n\} \times \{0, 1\} \times \{0, 1\} \times \cdots \subseteq U.$$

Hence the balls $B_d(\mathbf{x}, 1/n)$ are a basis for the product topology, and the metric topology, and so these two topologies are equal.

Henceforth whenever we write $\prod_{i\in I} X_i$ or X^I we will mean the Cartesian product of the collection of topological spaces $(X_i)_{i\in I}$ with the product topology, unless explicitly stated otherwise.

You might ask why the condition '$U_i = X_i$ for all but finitely many $i \in I$' appears in the definition? Good question. The answer is that certain results (such as Theorem 10.6) do not hold if this condition is omitted.

Proposition 10.3. *Let $(X_i)_{i\in I}$ be a collection of topological spaces such that X_i has basis $\mathcal{B}_i$ for all $i \in I$. Then the family of sets $\prod_{i\in I} B_i$ where $B_i \in \mathcal{B}_i$ for all i and $B_i = X_i$ for all but finitely many $i \in I$ is a basis for $\prod_{i\in I} X_i$.*

Proof. The proof of this result is similar to the proof of Proposition 9.2, and the proof is omitted. □

Example 10.4. [**The Baire space**.] The natural numbers $\mathbb{N} = \{1, 2, \ldots\}$ is a topological space with respect to the discrete topology. Hence we can consider the product topology on $\mathbb{N}^{\mathbb{N}} = \{ (x_1, x_2, \ldots) : (\forall i \in \mathbb{N})\,(x_i \in \mathbb{N}) \}$ (i.e. the space of sequences consisting of natural numbers). As in Example 10.2, the basis for $\mathbb{N}^{\mathbb{N}}$ given by the definition of the product topology consists of sets of the form $\prod_{i \in \mathbb{N}} U_i$ where U_i is any subset of $\mathbb{N}$ for all $i \in \mathbb{N}$ and $U_i = \mathbb{N}$ for all but finitely many $i \in \mathbb{N}$. Since the discrete topology on $\mathbb{N}$ has $\{ \{i\} : i \in \mathbb{N} \}$ as a basis, it follows from Proposition 10.3 that a basis for $\mathbb{N}^{\mathbb{N}}$ consists of sets of the form $\prod_{i \in \mathbb{N}} U_i$ where $|U_i| = 1$ for finitely many $i \in \mathbb{N}$ and $U_i = \mathbb{N}$ for the remaining $i \in \mathbb{N}$.

It can be shown (using a similar argument to that given in Example 10.2) that the sets of the form

$$\{x_1\} \times \{x_2\} \times \cdots \times \{x_n\} \times \mathbb{N} \times \cdots$$

are a basis for $\mathbb{N}^{\mathbb{N}}$ and that the metric d defined in Problem **2-7** by

$$d(\mathbf{x}, \mathbf{y}) = \begin{cases} 0 & \text{if } \mathbf{x} = \mathbf{y} \\ \dfrac{1}{\min\{ i \in \mathbb{N} \,:\, x_i \neq y_i \}} & \text{if } \mathbf{x} \neq \mathbf{y}. \end{cases}$$

(where $\mathbf{x} = (x_1, x_2, \ldots), \mathbf{y} = (y_1, y_2, \ldots) \in \mathbb{N}^{\mathbb{N}}$) induces the product topology on $\mathbb{N}^{\mathbb{N}}$.

Example 10.5. [$\mathbb{R}^{\mathbb{N}}$ **and the Hilbert cube**.] Let $\mathbb{R}$ denote the real numbers with the standard topology. Then we can consider the product topology on $\mathbb{R}^{\mathbb{N}}$ (the set of all sequences of real numbers). A basis for this topology consists of sets of the form:

$$(a_1, b_1) \times \cdots \times (a_n, b_n) \times \mathbb{R} \times \mathbb{R} \times \cdots.$$

If $n \in \mathbb{N}$, then the closed interval $[0, 1/n]$ is a topological space when considered as a subspace of $\mathbb{R}$. Hence we can consider the product $\prod_{n=1}^{\infty} [0, 1/n]$ with the product topology. The product topology on $\prod_{n=1}^{\infty} [0, 1/n]$ equals the subspace topology inherited from $\mathbb{R}^{\mathbb{N}}$ and the metric topology induced by the metric:

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=0}^{\infty} |x_i - y_i|^2}$$

from Problem **2-8**.

Let $(X_i)_{i \in I}$ be a collection of topological spaces and consider $\prod_{i \in I} X_i$ with the product topology. Several topological properties are passed from the X_i to $\prod_{i \in I} X_i$. For example, if X_i is Hausdorff for all $i \in I$, then $\prod_{i \in I} X_i$ is Hausdorff (see Problem **6-7**). Later in the course we will see some further examples of such properties.

So far, we have not seen any compelling reason to use the finiteness condition in the definition of the product topology. This is provided, in part, by the following theorem and the subsequent example.

Theorem 10.6. *Let Y be a topological space, let $(X_i)_{i \in I}$ be a collection of topological spaces, and let $f_i : Y \longrightarrow X_i$ for every $i \in I$. Then the function $f : Y \longrightarrow \prod_{i \in I} X_i$ defined by*

$$f(y) = (f_i(y))_{i \in I}$$

is continuous if and only if f_i is continuous for every $i \in I$.

Proof. ($\Rightarrow$) Let $i_0 \in I$ be arbitrary and let U_{i_0} be any open set in X_{i_0}. If $V := \prod_{i \in I} V_i$ where $V_i = X_i$ for all $i \neq i_0$ and $V_{i_0} = U_{i_0}$, then V is open in $\prod_{i \in I} X_i$. Hence

$$f^{-1}(V) = \{ y \in Y \,:\, f(y) \in V \} = \{ y \in Y \,:\, f_i(y) \in U_{i_0} \} = f_i^{-1}(U_{i_0})$$

is open in Y and so f_i is continuous.

($\Leftarrow$) Let $U := \prod_{i \in I} U_i$ be open in $\prod_{i \in I} X_i$. Then U_i is open in X_i for all $i \in I$ and so $f_i^{-1}(U_i)$ is open in Y for all $i \in I$. But

$$f^{-1}(U) = \{ y \in Y \,:\, f(y) \in U \} = \{ y \in Y \,:\, \forall i \in I,\ f_i(y) \in U_i \} = \bigcap_{i \in I} f_i^{-1}(U_i)$$

and so $f^{-1}(U)$ is open since it is a finite intersection of open sets. $\square$

What goes wrong in Theorem 10.6 if we drop the finiteness condition from the definition of the product topology?

Example 10.7. Let $\mathbb{R}$ denote the real numbers with the standard topology and let $\mathbb{R}^{\mathbb{N}}$ denote the space $\prod_{i \in \mathbb{N}} X_i$ where $X_i = \mathbb{R}$ for all $i \in \mathbb{N}$. Suppose that

$$\mathcal{U} = \left\{ \prod_{i \in \mathbb{N}} U_i \,:\, U_i \text{ open in } X_i \right\}$$

(i.e. $\mathcal{U}$ is the basis in the definition of the product topology but without the condition that $U_i = X_i$ for all but finitely many $i \in \mathbb{N}$). It is straightforward to verify that $\mathcal{U}$ is the basis for a topology on $\mathbb{R}^{\mathbb{N}}$; this topology is called the ***box topology*** on $\mathbb{R}^{\mathbb{N}}$.

Let $f_i : \mathbb{R} \longrightarrow \mathbb{R}$ be the identity function for all $i \in \mathbb{N}$. Then the function $f : \mathbb{R} \longrightarrow \mathbb{R}^{\mathbb{N}}$ defined in Theorem 10.6 is:

$$f(x) = (f_1(x), f_2(x), \dots) = (x, x, \dots).$$

If $\mathbb{R}^{\mathbb{N}}$ is given the product topology, then, since f_i is continuous for all $i \in \mathbb{N}$, it follows by Theorem 10.6 that f is continuous. However, if $\mathbb{R}^{\mathbb{N}}$ is given the box topology, then the function f is not continuous.

Suppose that

$$U = (-1, 1) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{3}, \frac{1}{3}\right) \times \cdots = \prod_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right) \in \mathcal{U}.$$

We will show that $f^{-1}(U)$ is not open in $\mathbb{R}$, and hence f is not continuous. If $x \in f^{-1}(U)$, then

$$f(x) = (x, x, x, \dots) \in U$$

and so

$$x \in \left(-\frac{1}{n}, \frac{1}{n}\right)$$

for all $n \in \mathbb{N}$. It follows that $x = 0$ and so $f^{-1}(U) = \{0\}$, which is not an open set in $\mathbb{R}$, as required.

Part 2. Properties of topological spaces

11. Preamble

You might recall the following theorems from real analysis.

Theorem 11.1 (Intermediate Value Theorem). *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function and let $x \in [f(a), f(b)]$. Then there exists $c \in [a, b]$ such that $f(c) = x$.*

Theorem 11.2 (Extremal Value Theorem). *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function. Then there exists $c \in [a, b]$ such that $f(c) \geq f(x)$ for all $x \in [a, b]$.*

Theorem 11.3 (Uniform Continuity Theorem). *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function. Then for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all $x, y \in [a, b]$ such that $|x - y| < \delta$.*

These theorems in some sense underlie the subject of calculus and they depend not only on the continuity of f but also on properties of the (closed, bounded) topological space $[a, b]$. The property of the space $[a, b]$ on which the Intermediate Value Theorem depends is called ***connectedness*** and the other two theorems depend on a property called ***compactness***. Connectedness can also often be used to prove that two topological spaces are not homeomorphic.

12. Connected spaces

Definition 12.1. A topological space X is ***connected*** if it cannot be given as a union of two disjoint non-empty open sets.

A non-empty subset Y of a topological space is ***connected*** if it is connected with the subspace topology.

Example 12.2. Let X be any set with the discrete topology. If $|X| > 1$, then X is not connected.

Example 12.3. Let X be any set with the trivial topology, then X is connected.

Theorem 12.4. *Let X be a topological space. Then X is connected if and only if the only clopen sets in X are $\varnothing$ and X.*

Proof. We prove the contrapositive. There exists a non-empty clopen set U such that $U \neq X$ if and only if $U \neq X, \varnothing$ is open and $X \setminus U$ is open if and only if X is not connected. $\square$

Example 12.5. Let $\mathbb{R}_l$ denote the real numbers with the lower limit topology. Then $\mathbb{R}_l$ is not connected since $[0, 1)$ is open (by definition) and closed since

$$\mathbb{R} \setminus [0, 1) = (-\infty, 0) \cup [1, \infty) = \bigcup_{x<0} [x, 0) \cup \bigcup_{y>1} [1, y)$$

is open. Hence, by Theorem 12.4, $\mathbb{R}_l$ is not connected.

Example 12.6. Let X be any infinite set with the cofinite topology. Then $U \subseteq X$ is open in X if and only if $X \setminus U$ is finite or $U = \varnothing$. Hence $F \subseteq X$ is closed in X if and only if F is finite or $F = X$. Hence the only clopen subsets of X are $\varnothing$ and X. Thus by Theorem 12.4, X is connected.

Example 12.7. The rational numbers $\mathbb{Q}$ as a subspace of $\mathbb{R}$ with the standard topology is not connected since

$$\mathbb{Q} \cap [-\sqrt{2}, \sqrt{2}] = \mathbb{Q} \cap (-\sqrt{2}, \sqrt{2})$$

is clopen in $\mathbb{Q}$.

Example 12.8. Let $[0, 1]$ denote the closed unit interval in $\mathbb{R}$ with the standard topology. We will show that $[0, 1]$ is connected. Suppose that U and V are disjoint non-empty open subsets of $[0, 1]$ such that $[0, 1] = U \cup V$ and $1 \in U$, say. Since U is open there exists $a \in \mathbb{R}$ such that $a < 1$ and $(a, 1] \subseteq U$. Hence $b := \sup(V) \leq a < 1$ and so $b \neq 1$. But $b \in U$ or $b \in V$ and, since these sets are open, there exist $c, d \in \mathbb{R}$ such that $b \in (c, d) \subseteq U$ or $b \in (c, d) \subseteq V$. But $(c, d) \cap U \neq \varnothing$ since no element of V is greater than $b \in (c, d)$ and $(c, d) \cap V \neq \varnothing$ since $b = \sup(V)$, a contradiction.

The set $X := [-1, 0) \cup (0, 1]$ is not connected as a subspace of $\mathbb{R}$ with the standard topology, since $[-1, 0)$ and $(0, 1]$ are disjoint open subsets whose union is X.

Subspaces of connected spaces can be connected or not.

Lemma 12.9. *Let X be a topological space such that $X = U \cup V$ where U and V are disjoint non-empty open sets. If Y is a connected subset of X, then Y is contained in U or V.*

Proof. We prove the contrapositive. If Y is not contained in either U or V, then $U \cap Y \neq \varnothing$ and $V \cap Y \neq \varnothing$ are non-empty open subsets of Y. Moreover, $Y = (U \cap Y) \cup (V \cap Y)$ and so Y is not connected. $\square$

Theorem 12.10. *Let X be a topological space and let $(Y_i)_{i \in I}$ be a family of connected subspaces of X such that $\bigcap_{i \in I} Y_i$ is non-empty. Then $\bigcup_{i \in I} Y_i$ is connected.*

Proof. Let Y denote $\bigcup_{i\in I} Y_i$ and let $y \in \bigcap_{i\in I} Y_i$ be arbitrary. Suppose that $Y = U \cup V$ where U and V are disjoint non-empty open sets in Y. Then $y \in U$ or $y \in V$ (but not both!), say $y \in U$. Since Y_i is connected and $y \in Y_i$ for all i, it follows, by Lemma 12.9, that $Y_i \subseteq U$ for all i. Hence $Y = \bigcup_{i\in I} Y_i \subseteq U$ and so $Y \cap V = \varnothing$, a contradiction. Thus $\bigcup_{i\in I} Y_i$ is connected. □

Theorem 12.11. *Let Y be a connected supspace of a topological space X. If $Y \subseteq Z \subseteq \overline{Y}$, then Z is connected.*

Proof. Let Y' denote the set of limit points of Y. By Theorem 6.4, it follows that the closure $\overline{Y}$ of Y equals $Y \cup Y'$ and so Z is the union of Y and a set of limit points of Y. Suppose that there are non-empty disjoint open subsets U and V of Z such that $Z = U \cup V$. Then by Lemma 12.9 it follows that Y (being a connected subspace of Z) is a subset of U or V, say U. So, if $z \in Z \setminus U \subseteq V$, then z is a limit point of Y. Hence every open nbhd of z contains an element of Y different from z. In particular, there exists $y \in V \cap Y$ such that $y \neq z$, which contradicts the fact that $Y \subseteq U$. Therefore Z is connected. □

Example 12.12. [**Connected subspaces of $\mathbb{R}$.**] In Example 12.8, we showed that $[0, 1]$ is connected. But $[0, 1]$ is homeomorphic to $[a, b]$ for all $a, b \in \mathbb{R}$ such that $a < b$ and so $[a, b]$ is connected also. If $a, b \in \mathbb{R}$ is arbitrary, then

$$[a, \infty) = \bigcup_{c>a} [a, c], \quad (-\infty, b] = \bigcup_{d<b} [d, b], \quad \mathbb{R} = \bigcup_{n\geq 1} [-n, n]$$

and so $[a, \infty)$, $(-\infty, b]$ and $\mathbb{R}$ are connected by Theorem 12.10. We also know that $\mathbb{R}$ is homeomorphic to every open interval (a, b), (a, ∞), $(-\infty, b)$, $a, b \in \mathbb{R}$ with $a < b$ and so each of these intervals is connected. Hence it follows from Theorem 12.11 that $(a, b]$ and $[a, b)$ are connected for all $a, b \in \mathbb{R}$ such that $a < b$.

We will show that the intervals are the only connected subsets of $\mathbb{R}$. Suppose that A is any subset of $\mathbb{R}$. If A is not an interval, then there exist $a, b \in A$ and $x \notin A$ such that $x \in (a, b)$. Hence $A = (A \cap (-\infty, x)) \cup (A \cap (x, \infty))$ and $A \cap (-\infty, x)$ and $A \cap (x, \infty)$ are disjoint open subsets of A (with the subspace topology).

Example 12.13. [**$\mathbb{R}^n$ is connected.**] Let $n > 1$. Then every line through the origin in $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}$ and so $\mathbb{R}^n$ is a union of connected subsets with non-empty intersection (the origin) and so Theorem 12.10 implies that $\mathbb{R}^n$ is connected.

Theorem 12.14. *Let X and Y be topological spaces and let $f : X \longrightarrow Y$ be continuous. If X is connected, then $f(X)$ is connected.*

Proof. We prove the contrapositive. Suppose that $f(X)$ is not connected. Then there exist disjoint non-empty open sets U and V in $f(X)$ such that $f(X) = U \cup V$. By the definition of the subspace topology, there exist open sets U' and V' in Y such that $U = U' \cap f(X)$ and $V = V' \cap f(X)$. Since f is continuous, it follows that $f^{-1}(U')$ and $f^{-1}(V')$ are non-empty disjoint open sets in X. Note that $x \in f^{-1}(U')$ if and only if $f(x) \in U' \cap f(X) = U$ if and only if $x \in f^{-1}(U)$. Hence $f^{-1}(U') = f^{-1}(U)$ and $f^{-1}(V') = f^{-1}(V)$ are non-empty disjoint open sets in X and so

$$X = f^{-1}(f(X)) = f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$$

is the union of the non-empty disjoint open sets $f^{-1}(U)$ and $f^{-1}(V)$. Therefore X is not connected. □

Theorem 12.15. *Let $(X_i)_{i\in I}$ be a family of topological spaces. Then $X^I = \prod_{i\in I} X_i$ with the product topology is connected if and only if X_i is connected for all $i \in I$.*

Proof. ($\Leftarrow$) We sketch the proof of this implication, the details are left as an exercise.

Let $\mathbf{a} = (a_i)_{i\in I} \in X^I$ be some fixed element.

(i) If J is any finite subset of I, then denote by X_J the set of those elements $\mathbf{x} = (x_i)_{i\in I} \in X^I$ such that $x_i = a_i$ for all $i \notin J$. Prove that X_J is connected.

(ii) Show that the union Y of the spaces X_J is connected.

(iii) Show that X^I is the closure of Y and deduce that X^I is connected.

($\Rightarrow$) Suppose that X_j is not connected for some $j \in I$. Then there exist disjoint non-empty open subsets U_j and V_j of X_j such that $X_j = U_j \cup V_j$. If $V_j = X_j \setminus U_j$ and let $U_i = V_i = X_i$ for all $i \in I$ such that $i \neq j$. Then $U := \prod_{i\in I} U_i$ and $V := \prod_{i\in I} V_i$ are disjoint non-empty open sets in X^I such that $X^I = U \cup V$. □

Example 12.16. [**$\mathbb{R}^n$ is connected.**] It is an immediate consequence of Theorem 12.15 and Example 12.12 that $\mathbb{R}^n$ and $\mathbb{R}^{\mathbb{N}}$ are connected.

We have reached a point where we can now prove the Intermediate Value Theorem stated at the start of Section 11.

Theorem 11.1 (Intermediate Value Theorem)**.** *Let X be a connected space, let $f : X \longrightarrow \mathbb{R}$ be a continuous function, let $a, b \in X$ be arbitrary, and let $x \in [f(a), f(b)]$. Then there exists $c \in X$ such that $f(c) = x$.*

Proof. The sets

$$A = f(X) \cap (-\infty, x) \quad \text{and} \quad B = f(X) \cap (x, \infty).$$

are disjoint, and since $f(a) \in A$ and $f(b) \in B$, it follows that A and B are non-empty. Also A and B are open in $f(X)$ with the subspace topology by definition. If $x \notin f(X)$, then $f(X) = A \cup B$ and so $f(X)$ is not connected. But $f(X)$ is connected, being the continuous image of a connected space (Theorem 12.14), a contradiction. Hence there exists $c \in X$ such that $f(c) = x$. □

13. Compact spaces

Definition 13.1. A collection $\mathcal{U}$ of open subsets of a topological space X is said to be an ***open cover*** of $Y \subseteq X$ if the union of the sets in $\mathcal{U}$ contains Y.

The collection $\mathcal{U}$ is also sometimes referred to as a ***open covering*** of Y or said to ***cover*** Y.

A subcollection of $\mathcal{U}$ that also covers Y is called a ***subcover***.

Definition 13.2. A topological space X is ***compact*** if every open covering of X contains a finite subcover for X.

Example 13.3. [**$\mathbb{R}$ is not compact**.] The real number $\mathbb{R}$ with the standard topology is not compact since

$$\mathcal{U} = \{ (-n, n) \, : \, n \in \mathbb{N} \}$$

is an open cover for $\mathbb{R}$ which has no finite subcover.

Example 13.4. We will show that the subspace

$$X = \{0\} \cup \{ 1/n \, : \, n \in \mathbb{N} \}$$

of $\mathbb{R}$ with the standard topology is compact. Let $\mathcal{U}$ be any open covering for X. Then there exists $U_0 \in \mathcal{U}$ such that $0 \in U_0$. Since $U_0 = (a, b) \cap X$ for some $a, b \in \mathbb{R}$, it follows that there exists $m \in \mathbb{N}$ such that $1/n \in U_0$ for all $n > m$. Hence if $U_1, \ldots, U_m$ are open subsets in $\mathcal{U}$ containing $1, 1/2, \ldots, 1/m$, respectively, then $U_0, U_1, \ldots, U_m$ covers X. Therefore every open covering for X contains a finite subcover and so X is compact.

Example 13.5. [**Every finite topological space is compact**.] Let X be a finite topological space. Then there are only finitely many open sets in X and so X is compact.

Example 13.6. Let X be the interval $(0, 1]$ in $\mathbb{R}$ with the standard topology. Then

$$\mathcal{U} = \{ (1/n, 1] \, : \, n \in \mathbb{N} \}$$

is an open covering of X that contains no finite subcollection covering X. Hence $(0, 1]$ is not compact. A similar argument shows that $(0, 1)$ is not compact.

Example 13.7. [**The closed unit interval is compact**.] Let $\mathcal{U}$ be any open cover for $[0, 1]$ as a subspace of $\mathbb{R}$ with the standard topology. Let K be the set of all points $x \in [0, 1]$ such that some finite subcollection of $\mathcal{U}$ covers $[0, x]$. We will show that $K = [0, 1]$. Since there exists $U \in \mathcal{U}$ such that $0 \in U$, it follows that $0 \in K$ (and so K is non-empty). If $x \in K$, then there exist $U_0, \ldots, U_n \in \mathcal{U}$ such that $[0, x]$ is contained in the union of $U_0, \ldots, U_n$. Hence if $y \leq x$, then

$$[0, y] \subseteq [0, x] \subseteq U_0 \cup \cdots \cup U_n$$

and so $y \in K$. Thus K is a subinterval of $[0, 1]$ containing 0.

If $x \in K$ such that $x < 1$, then since $x \in U_i$ for some $1 \leq i \leq n$ and U_i is open, there exists $\varepsilon > 0$ such that $x + \varepsilon \in U_i$ and so $[0, x + \varepsilon] \subseteq U_0 \cup \cdots \cup U_n$. In other words, for all $x \in K$, there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \cap [0, 1] \subseteq K$ and so K is open in $[0, 1]$. If $k = \sup(K)$, then there exists $V \in \mathcal{U}$ such that $k \in V$ and since V is open there exists $\varepsilon > 0$ such that $(k - \varepsilon, k] \subseteq V$. But $k - \varepsilon \in K$ and so there exist $U_0, \ldots, U_n \in \mathcal{U}$ such that

$$[0, k - \varepsilon] \subseteq U_0 \cup U_1 \cup \cdots \cup U_n$$

and so

$$[0, k] \subseteq V \cup U_0 \cup U_1 \cup \cdots \cup U_n.$$

This implies that $k \in K$ and so $K = [0, k]$, and in particular K is closed in $[0, 1]$.

We have shown that K is a clopen subinterval of $[0, 1]$ containing 0 and hence $K = [0, 1]$.

Theorem 13.8 (Heine-Borel Theorem)**.** *Every closed and bounded interval in $\mathbb{R}$ is compact.*

Proof. Let $a, b \in \mathbb{R}$ be such that $a < b$. Then $[a, b]$ is homeomorphic to $[0, 1]$ and so $[a, b]$ is compact. □

Lemma 13.9. *A subspace Y of a topological space X is compact if and only if every open cover of Y consisting of open sets in X contains a finite subcollection that also covers Y.*

Proof. (⇒) Let $\mathcal{U}_X = (U_i)_{i\in I}$ be an open cover for Y consisting of open sets in X. Then

$$\mathcal{U}_Y = \{\, U_i \cap Y \;:\; i \in I \,\}$$

is an open cover for Y consisting of open sets in Y. Hence $\mathcal{U}_Y$ contains a finite subcover

$$\{U_{i(1)} \cap Y, U_{i(2)} \cap Y, \ldots, U_{i(n)} \cap Y\}$$

for some $i(1), i(2), \ldots, i(n) \in I$. But then $\{U_{i(1)}, U_{i(2)}, \ldots, U_{i(n)}\}$ is a finite subcollection of $\mathcal{U}_X$ covering Y, as required.

(⇐) Let $\mathcal{U}_Y = (U_i)_{i\in I}$ be any open cover of Y consisting of open sets in Y. We will show that $\mathcal{U}_Y$ contains a finite subcover for Y, and hence Y is compact. For every $i \in I$, let U_i' be an open set in X such that $U_i = U_i' \cap Y$. Then $\mathcal{U}_X = (U_i')_{i\in I}$ is an open cover of Y consisting of open sets in X. Thus, by the hypothesis of this implication, there exists a finite subcover $\{U'_{i(1)}, U'_{i(2)}, \ldots, U'_{i(n)}\}$ for Y in $\mathcal{U}_X$. But then $\{U_{i(1)}, U_{i(2)}, \ldots, U_{i(n)}\}$ is a finite subcover of Y in $\mathcal{U}_Y$, as required. □

Theorem 13.10. *Every closed subspace of a compact space is compact.*

Proof. Let F be a closed subspace of the compact space X and let $\mathcal{U}$ be an open cover for F consisting of open sets in X. Then $\mathcal{U} \cup \{X \setminus F\}$ is an open cover for X and hence there is a finite subcover $\mathcal{U}'$ of X contained in $\mathcal{U} \cup \{X \setminus F\}$. Clearly F is contained in the union of the sets in $\mathcal{U}' \setminus \{X \setminus F\}$, which is hence a finite subcover for F contained in $\mathcal{U}$. Therefore F is compact, as required. □

Theorem 13.11. *Every compact subspace of a Hausdorff space is closed.*

Proof. Let Y be a compact subspace of the Hausdorff space X. We will prove that $X \setminus Y$ is open so that Y is closed. It suffices by Proposition 2.5(ii) to show that for all $x \in X \setminus Y$ there exists an open set U in X such that $x \in U \subseteq X \setminus Y$.

Let $x \in X \setminus Y$ be arbitrary. Since X is Hausdorff, for every $y \in Y$ there exist disjoint open sets U_y and V_y such that $x \in U_y$ and $y \in V_y$. Hence $(V_y)_{y\in Y}$ is an open cover for Y and hence it contains a finite subcover $\{V_{y_0}, V_{y_1}, \ldots, V_{y_n}\}$ for some $y_0, y_1, \ldots, y_n \in Y$. Let $V = V_{y_0} \cup V_{y_1} \cup \cdots \cup V_{y_n}$ and let $U = U_{y_0} \cap U_{y_1} \cap \cdots \cap U_{y_n}$. Then U and V are open sets and $x \in U$. We will show that $U \subseteq X \setminus Y$. If $y \in Y$, then $y \in V_{y_i}$ for some i and so $y \notin U_{y_i}$. In particular, $y \notin U$ for all $y \in Y$ and so $Y \cap U = \varnothing$, or, in other words, $U \subseteq X \setminus Y$, as required. □

Theorem 13.12. *The continuous image of a compact space is compact.*

Proof. Let X be a compact space, let Y be a topological space, and let $f : X \longrightarrow Y$ be a continuous function. Let $(U_i)_{i\in I}$ be an open cover for $f(X)$. Then $f^{-1}(U_i)$ is open in X for all $i \in I$. Thus $(f^{-1}(U_i))_{i\in I}$ is an open cover for X and hence it has a finite subcover

$$\{f^{-1}(U_{i(1)}), f^{-1}(U_{i(2)}), \ldots, f^{-1}(U_{i(n)})\}$$

for some $i(1), i(2), \ldots, i(n) \in I$. In particular,

$$X = f^{-1}(U_{i(1)}) \cup f^{-1}(U_{i(2)}) \cup \cdots \cup f^{-1}(U_{i(n)})$$

and so

$$f(X) = U_{i(1)} \cup U_{i(2)} \cup \cdots \cup U_{i(n)}.$$

In other words, $\{U_{i(1)}, U_{i(2)}, \ldots, U_{i(n)}\}$ is a finite subcover of $(U_i)_{i\in I}$ for $f(X)$, and so $f(X)$ is compact. □

Theorem 13.13 (Tychonoff's Theorem)**.** *Let $(X_i)_{i\in I}$ be a family of compact topological spaces indexed by some set I. Then the space $\prod_{i\in I} X_i$ with the product topology is compact.*

Example 13.14. In Example 13.5 we showed that every finite topological space is compact. In particular, $\{0, 1\}$ with the discrete topology is compact, and so by Tychonoff's Theorem 13.13, the Cantor set $2^{\mathbb{N}}$ is compact.

A subset A of a metric space X is ***bounded*** if there exists $M \in \mathbb{R}$ such that $d(x, y) \le M$ for all $x, y \in A$.

Theorem 13.15. *A subspace of $\mathbb{R}^n$ is compact if and only if it is closed and bounded in the euclidean metric (or the Chebyshev metric).*

Proof. Let d_∞ denote the maximum metric and let d_2 denote the euclidean metric. Then since

$$d_\infty(\mathbf{x}, \mathbf{y}) \le d_2(\mathbf{x}, \mathbf{y}) \le \sqrt{n}\, d_\infty(\mathbf{x}, \mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, it suffices to prove the theorem for the Chebyshev metric. For the sake of brevity we write d instead of d_∞ in the remainder of this proof.

(⇒) Let K be a compact subset of $\mathbb{R}^n$. Then Theorem 13.11 implies that K is closed in $\mathbb{R}^n$. It remains to prove that K is bounded under d_∞. But

$$\mathcal{U} = \{\, B_d(\mathbf{0}, m) \;:\; m \in \mathbb{R} \,\}$$

is an open cover for $\mathbb{R}^n$ and hence for K. Thus there exists a finite subcover of $\mathcal{U}$ for K. In particular, there exists $M \in \mathbb{R}$ such that $K \subseteq B_d(\mathbf{0}, M)$ and so K is bounded.

($\Leftarrow$) Let A be a closed and bounded subset of $\mathbb{R}^n$. Then there exists $M \in \mathbb{R}$ such that $d(\mathbf{x}, \mathbf{y}) \leq M$ for all $\mathbf{x}, \mathbf{y} \in A$. If $\mathbf{x}_0 \in A$ is fixed, then

$$d(\mathbf{x}, \mathbf{0}) \leq d(\mathbf{x}, \mathbf{x}_0) + d(\mathbf{x}_0, \mathbf{0}) \leq M + d(\mathbf{x}_0, \mathbf{0})$$

for all $\mathbf{x} \in A$. So, setting $N = M + d(\mathbf{x}_0, \mathbf{0})$ it follows that A is a subspace of $[-N, N]^n$. But $[-N, N]^n$ is compact (by Tychonoff's Theorem 13.13 and the Heine-Borel Theorem 13.8) and A is closed. Therefore A is compact by Theorem 13.10. □

We can now return to the theorems that motivated the definition of compactness.

Theorem 13.16 (Extreme Value Theorem)**.** *Let X be a compact space and let $f : X \longrightarrow \mathbb{R}$ be continuous. Then there exist $a, b \in X$ such that $f(a) \leq f(x) \leq f(b)$ for all $x \in X$.*

Proof. Since X is compact and f is continuous, it follows from Theorem 13.12 that $f(X)$ is compact. We prove that $\max f(X)$ exists, the proof that $\min f(X)$ exists follows by an analogous argument.

Seeking a contradiction assume that $\max f(X)$ does not exist. Then for all $f(x) \in f(X)$ there exists $y \in X$ such that $f(y) > f(x)$, and so $f(x) \in (-\infty, f(y))$. Thus

$$\{\, (-\infty, f(y)) \;:\; y \in X \,\}$$

is an open cover for $f(X)$. Since $f(X)$ is compact, there exist $y_1, y_2, \ldots, y_n \in X$ such that

$$\mathcal{U} = \{(-\infty, f(y_1)), \ldots, (-\infty, f(y_n))\}$$

is a cover for $f(X)$. Let $f(y_m) = \max\{f(y_1), f(y_2), \ldots, f(y_n)\}$. Then $f(y_m) \notin (-\infty, f(y_i))$ for any i but $f(y_m) \in f(X)$ and so $\mathcal{U}$ is not a cover for $f(X)$, a contradiction. □

In order to prove Theorem 11.3 we require the following lemma.

Lemma 13.17 (Lesbegue Number Lemma)**.** *Let (X, d) be a compact metric space and let $\mathcal{U}$ be any open covering for X. Then there exists $\delta > 0$ (called the **Lebesgue number of** $\mathcal{U}$) such that for every subset A of X with*

$$\sup\{\, d(x, y) \in \mathbb{R}^+ \;:\; x, y \in A \,\} < \delta$$

there exists $U \in \mathcal{U}$ containing A.

Proof. If $X \in \mathcal{U}$, then there is nothing to prove and so we may assume that $X \notin \mathcal{U}$. Since X is compact, there exists a finite subcover $\mathcal{U}' = \{U_1, U_2, \ldots, U_n\}$ of $\mathcal{U}$ for some n. Let $f : X \longrightarrow \mathbb{R}$ be defined by

$$f(x) = \frac{1}{n}\sum_{i=1}^{n} \inf\{\, d(x, y) \;:\; y \in X \setminus U_i \,\} \qquad \text{the average distance from } x \text{ to a set in } \mathcal{U}'.$$

We will prove that f is continuous so that we may apply the Extreme Value Theorem 13.16. It suffices to prove that the functions $g_i : X \longrightarrow \mathbb{R}$ defined by

$$g_i(x) = \inf\{\, d(x, y) \;:\; y \in X \setminus U_i \,\}$$

are continuous, since f is then a constant multiple of a sum of continuous functions, and is hence continuous.

Let $\varepsilon > 0$ and let $z \in X$ be such that $d(x, z) < \varepsilon$. Then

$$g_i(x) = \inf\{\, d(x, y) \;:\; y \in X \setminus U_i \,\} \leq d(x, y) \leq d(x, z) + d(z, y) < \varepsilon + d(z, y)$$

for all $y \in X \setminus U_i$. Hence $g_i(x) < \varepsilon + g_i(z)$ and so

$$|g_i(x) - g_i(z)| < \varepsilon,$$

and so g_i is continuous.

Since f is continuous and X is compact, it follows by the Extreme Value Theorem 13.16 that f has a minimum value δ. We show that $\delta > 0$. Let $x \in X$ be arbitrary. There exists i such that $x \in U_i$ and since U_i is open there is an $\varepsilon > 0$ such that $B_d(x, \varepsilon) \subseteq U_i$. Hence $\inf\{\, d(x, y) \;:\; y \in X \setminus U_i \,\} \geq \varepsilon$ and so $f(x) \geq \varepsilon/n$. In particular, $f(x) > 0$ for all $x \in X$ and so $\delta = \min f(X) > 0$.

Let A be a subset of X such that

$$\sup\{\, d(x, y) \in \mathbb{R}^+ \;:\; x, y \in A \,\} < \delta$$

and let $x_0 \in A$. If $y \in A$, then

$$d(x_0, y) \leq \sup\{\, d(x, y) \in \mathbb{R}^+ \;:\; x, y \in A \,\} < \delta$$

and so $y \in B_d(x_0, \delta)$. Thus $A \subseteq B_d(x_0, \delta)$. If $M \in \{1, 2, \ldots, n\}$ is such that

$$\inf\{\, d(x_0, y) \;:\; y \in X \setminus U_M \,\} = \max_{1 \leq i \leq n} \inf\{\, d(x_0, y) \;:\; y \in X \setminus U_i \,\},$$

then

$$\inf\{\, d(x_0, y) \;:\; y \in X \setminus U_M \,\} \geq f(x_0) \geq \delta$$

(the maximum is greater than the average). If $y \in B_d(x_0, \delta)$, then

$$d(x_0, y) < \delta < \inf\{\, d(x, z) \;:\; z \in X \setminus U_M \,\}$$

and so $y \in U_M$. It follows that $A \subseteq B_d(x_0, \delta) \subseteq U_M$, as required. $\square$

Theorem 13.18 (Uniform Continuity Theorem)**.** *Let X be a compact metric space and let $f : X \longrightarrow \mathbb{R}$ be a continuous function. Then for all $\varepsilon > 0$ there exists $\delta > 0$ such that $d(f(x), f(y)) < \varepsilon$ for all $x, y \in X$ such that $d(x, y) < \delta$ (i.e. f is **uniformly continuous**).*

Proof. Let $\varepsilon > 0$. Then $\{\, B(b, \varepsilon/2) \;:\; b \in \mathbb{R} \,\}$ is an open cover for $\mathbb{R}$ and hence

$$\mathcal{A} = \{\, f^{-1}(B(b, \varepsilon/2)) \;:\; b \in \mathbb{R} \,\}$$

is an open cover for X. Since X is compact, it follows by Lemma 13.17 that $\mathcal{A}$ has a Lebesgue number $\delta > 0$. If $x, y \in X$ such that $d(x, y) < \delta$, then certainly

$$\sup\{\, d(a, b) \;:\; a, b \in \{x, y\} \,\} < \delta.$$

Hence, by the Lebesgue Number Lemma 13.17 applied to the set $\{x, y\}$, it follows that there exists $b \in \mathbb{R}$ such that

$$\{f(x), f(y)\} \subseteq B(b, \varepsilon/2)$$

and so $|f(x) - f(y)| < \varepsilon$, and so f is uniformly continuous. $\square$

Part 3. **Appendices**

Appendix A. The triangle inequality in $\mathbb{R}^n$

In this appendix we show that the standard euclidean distance on $\mathbb{R}^n$, $n \geq 1$ is a metric on $\mathbb{R}^n$.

We start with the special case of $n = 1$. Let $d : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^+$ be defined by

$$d(x, y) = |x - y|$$

Let's verify that d is a metric:

M1: Let $x, y \in \mathbb{R}$ such that $d(x, y) = 0$. Then $|x - y| = 0$ and so $x = y$. Clearly, if $x = y$, then $d(x, y) = 0$.

M2: Clearly $d(x, y) = |x - y| = |y - x| = d(y, x)$ for all $x, y \in \mathbb{R}$.

M3: We start by proving that $|x + y| \leq |x| + |y|$. By definition, $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$ and so by adding these inequalities:

$$-(|x| + |y|) \leq x + y \leq |x| + |y|.$$

In particular, $x + y \leq |x| + |y|$ and $-(x + y) \leq |x| + |y|$. Hence

$$|x + y| = \begin{cases} x + y & \text{if } x + y \geq 0 \\ -(x + y) & \text{if } x + y < 0 \end{cases} \leq |x| + |y|.$$

So, if $x, y, z \in \mathbb{R}$ are arbitrary, it follows that

$$d(x, z) = |x - z| = |(x - y) + (y - z)| \leq |x - y| + |y - z| = d(x, y) + d(y, z).$$

When $n > 1$ it is more difficult to establish that **M3** holds. In Example 3.2, we defined $d_2 : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be defined by

$$d_2(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}|| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n), \mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n), \mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ and let $\alpha \in \mathbb{R}$. Then we define

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \in \mathbb{R}^n, \quad \mathbf{x} - \mathbf{y} = (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n) \in \mathbb{R}^n,$$

$$\alpha\mathbf{x} = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n) \in \mathbb{R}^n, \quad \langle\, \mathbf{x}, \mathbf{y} \,\rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \in \mathbb{R}, \quad ||\mathbf{x}|| = \sqrt{x_1^2 + x_2^2 + \cdots x_n^2} \in \mathbb{R}.$$

[Aside: $\langle\, \mathbf{x}, \mathbf{y} \,\rangle$ is called the ***inner product***, ***dot product***, or ***scalar product*** of $\mathbf{x}$ and $\mathbf{y}$; $||\mathbf{x}||$ is called the ***norm*** of $\mathbf{x}$.]

You might remember from Linear Mathematics MT3501 that an ***inner product*** satisfies:

(i) $\langle\, \mathbf{x} + \mathbf{y}, \mathbf{z} \,\rangle = \langle\, \mathbf{x}, \mathbf{y} \,\rangle + \langle\, \mathbf{y}, \mathbf{z} \,\rangle$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$;
(ii) $\langle\, \alpha\mathbf{x}, \mathbf{y} \,\rangle = \alpha\langle\, \mathbf{x}, \mathbf{y} \,\rangle$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and for all $\alpha \in \mathbb{R}$;
(iii) $\langle\, \mathbf{x}, \mathbf{y} \,\rangle = \langle\, \mathbf{y}, \mathbf{x} \,\rangle$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$;
(iv) $\langle\, \mathbf{x}, \mathbf{x} \,\rangle \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$;
(v) $\langle\, \mathbf{x}, \mathbf{x} \,\rangle = 0$ if and only if $\mathbf{x} = 0$.

Note that

$$\langle\, \mathbf{x}, \alpha\mathbf{y} \,\rangle = \langle\, \alpha\mathbf{y}, \mathbf{x} \,\rangle = \alpha\langle\, \mathbf{y}, \mathbf{x} \,\rangle = \alpha\langle\, \mathbf{x}, \mathbf{y} \,\rangle,$$

$||\mathbf{x}|| = \sqrt{\langle\, \mathbf{x}, \mathbf{x} \,\rangle}$, and so

$$||\alpha\mathbf{x}||^2 = \langle\, \alpha\mathbf{x}, \alpha\mathbf{x} \,\rangle = \alpha\langle\, \mathbf{x}, \alpha\mathbf{x} \,\rangle = \alpha^2\langle\, \mathbf{x}, \mathbf{x} \,\rangle$$

and so $||\alpha\mathbf{x}|| = |\alpha| \cdot ||\mathbf{x}||$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Finally note that $||\mathbf{x}|| \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$ and $||\mathbf{x}|| = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

Proposition A.1. *[Cauchy-Schwarz inequality] Let* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. *Then* $|\langle\, \mathbf{x}, \mathbf{y} \,\rangle| \leq ||\mathbf{x}|| \cdot ||\mathbf{y}||$.

Proof. If $\mathbf{y} = \mathbf{0} = (0, 0, \ldots, 0)$, then $||\mathbf{y}|| = 0$ and

$$\langle\, \mathbf{x}, \mathbf{y} \,\rangle = \langle\, \mathbf{x}, \mathbf{0} \,\rangle = \langle\, \mathbf{x}, 0 \cdot \mathbf{0} \,\rangle = 0 \cdot \langle\, \mathbf{x}, \mathbf{0} \,\rangle = 0.$$

Hence

$$|\langle\, \mathbf{x}, \mathbf{y} \,\rangle| = 0 = ||\mathbf{x}|| \cdot ||\mathbf{y}||,$$

as required. Hence we may assume that $\mathbf{y} \neq 0$.

Let $\alpha \in \mathbb{R}$. Then expanding $\langle\, \mathbf{x} + \alpha\mathbf{y}, \mathbf{x} + \alpha\mathbf{y} \,\rangle$ we obtain:

$$0 \leq \langle\, \mathbf{x} + \alpha\mathbf{y}, \mathbf{x} + \alpha\mathbf{y} \,\rangle = \langle\, \mathbf{x}, \mathbf{x} \,\rangle + 2\alpha\langle\, \mathbf{x}, \mathbf{y} \,\rangle + \alpha^2\langle\, \mathbf{y}, \mathbf{y} \,\rangle$$

Setting $\alpha = -\langle\, \mathbf{x}, \mathbf{y}\,\rangle/||\mathbf{y}||^2$, we deduce that

$$0 \leq \langle\, \mathbf{x}, \mathbf{x}\,\rangle + 2\alpha\langle\, \mathbf{x}, \mathbf{y}\,\rangle + \alpha^2\langle\, \mathbf{y}, \mathbf{y}\,\rangle = \langle\, \mathbf{x}, \mathbf{x}\,\rangle - \frac{2\langle\, \mathbf{x}, \mathbf{y}\,\rangle^2}{||\mathbf{y}||^2} + \frac{\langle\, \mathbf{x}, \mathbf{y}\,\rangle^2\langle\, \mathbf{y}, \mathbf{y}\,\rangle}{||\mathbf{y}||^4} = \langle\, \mathbf{x}, \mathbf{x}\,\rangle - \frac{2\langle\, \mathbf{x}, \mathbf{y}\,\rangle^2}{||\mathbf{y}||^2} + \frac{\langle\, \mathbf{x}, \mathbf{y}\,\rangle^2||\mathbf{y}||^2}{||\mathbf{y}||^4}$$

$$= ||\mathbf{x}||^2 - \frac{2\langle\, \mathbf{x}, \mathbf{y}\,\rangle^2}{||\mathbf{y}||^2} + \frac{\langle\, \mathbf{x}, \mathbf{y}\,\rangle^2}{||\mathbf{y}||^2} = ||\mathbf{x}||^2 - \frac{\langle\, \mathbf{x}, \mathbf{y}\,\rangle^2}{||\mathbf{y}||^2}$$

and so

$$0 \leq ||\mathbf{x}||^2 - \frac{\langle\, \mathbf{x}, \mathbf{y}\,\rangle^2}{||\mathbf{y}||^2}.$$

Rearranging

$$||\mathbf{x}||^2 \cdot ||\mathbf{y}||^2 \geq \langle\, \mathbf{x}, \mathbf{y}\,\rangle^2$$

and the result follows by taking square roots. □

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ be arbitrary. Then

$$||\mathbf{x} + \mathbf{y}||^2 = \langle\, \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y}\,\rangle = \langle\, \mathbf{x}, \mathbf{x}\,\rangle + \langle\, \mathbf{y}, \mathbf{y}\,\rangle + 2\langle\, \mathbf{x}, \mathbf{y}\,\rangle = ||\mathbf{x}||^2 + ||\mathbf{y}||^2 + 2\langle\, \mathbf{x}, \mathbf{y}\,\rangle \leq ||\mathbf{x}||^2 + ||\mathbf{y}||^2 + 2||\mathbf{x}|| \cdot ||\mathbf{y}||$$
$$= (||\mathbf{x}|| + ||\mathbf{y}||)^2$$

and so $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$. Therefore

$$d_2(\mathbf{x}, \mathbf{z}) = ||\mathbf{x} - \mathbf{z}|| = ||(\mathbf{x} - \mathbf{y}) + (\mathbf{y} - \mathbf{z})|| \leq ||\mathbf{x} - \mathbf{y}|| + ||\mathbf{y} - \mathbf{z}|| = d_2(\mathbf{x}, \mathbf{y}) + d_2(\mathbf{y}, \mathbf{z}).$$

It follows that d_2 satisfies **M3** and is hence a metric.