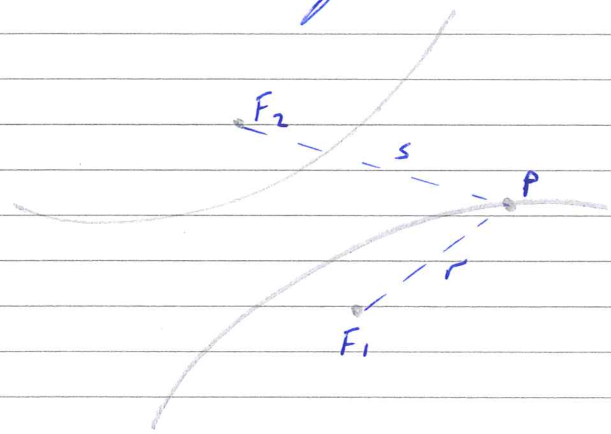
2.3) THE HYPERBOLA

DEF^{n}

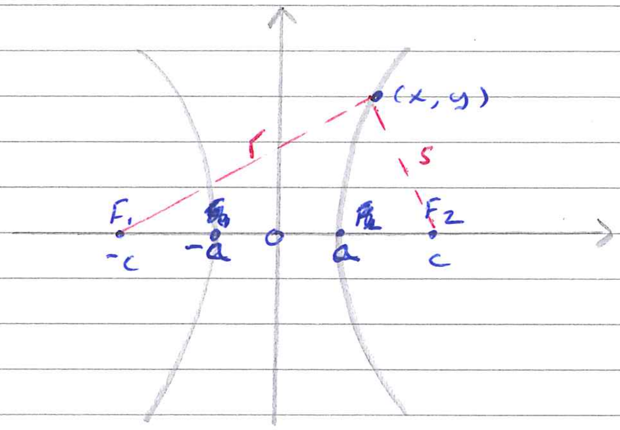
A hyperbola in the plane consists of all points the difference of whose distances from two fixed points F_1 and F_2 (the foci) remains at a constant value of $2a$ (which must be less than the distance between the foci).



By def^{n}

$|r-s| = 2a$

To determine the curve analytically we choose axes so that F_1 and F_2 are at $(-c,0)$ and $(c,0)$



$a < c$

(Proof to be done in tutorial ~~problem~~ class)

For a point (x,y) on the hyperbola and r and s as shown we have

$$|r-s| = 2a$$

$$\Rightarrow \left|\sqrt{(x+c)^2+y^2} - \sqrt{(c-x)^2+y^2}\right| = 2a \quad (7)$$

The simplification of eqn (7) is similar to that of eqn (2) and this is left as an exercise. again results in eq^{n} (5)

$$(a^2-c^2)x^2 + a^2y^2 = a^2(a^2-c^2)$$

Note this time $c > a$ so we set

$$b^2 = c^2 - a^2 \quad (8)$$ + sub into (3) to get

$$-b^2x^2 + a^2y = a^2(-b^2)$$

$$\Rightarrow \boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1} \quad (9)$$ which is the standard form of the eq^n of the hyperbola.

NOTE

For a hyperbola, we think of a^2 as the number under the quadratic variable with positive sign.

EXAMPLE 1

Sketch the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$

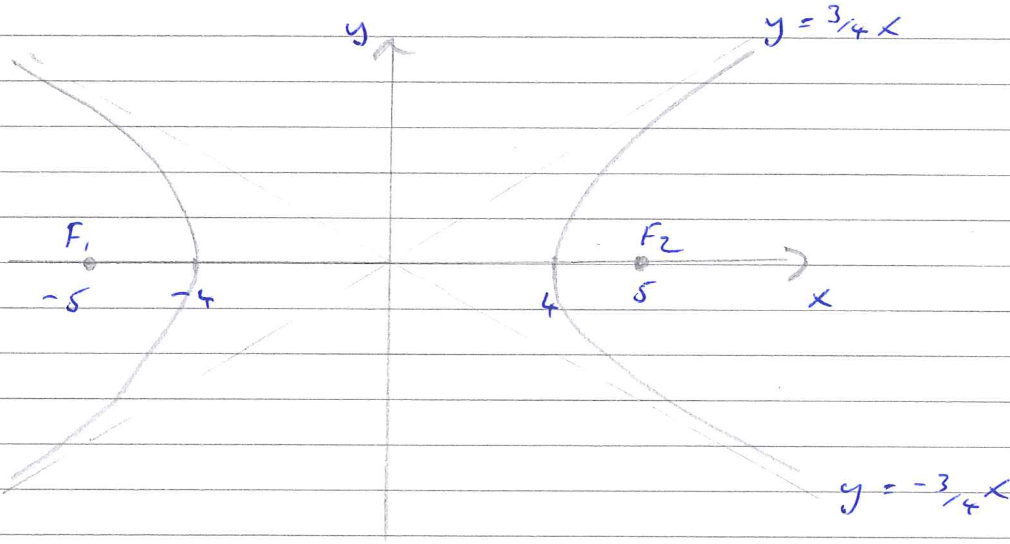
Showing the asymptotes + foci.

Sol^n

Asymptotes $y = \pm(3/4)x$. From (8)

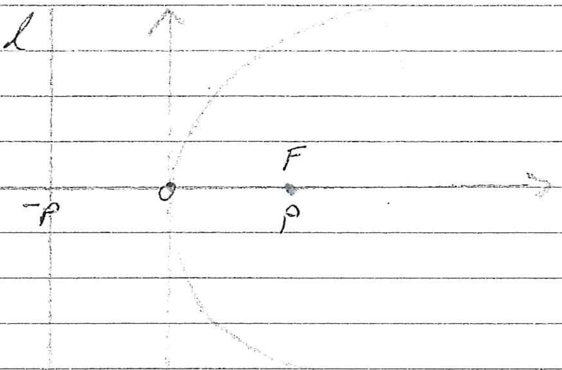
$c^2 = a^2 + b^2 = 16 + 9 = 25 \Rightarrow c = 5$

$\therefore$ The foci are $(-5, 0)$ and $(5, 0)$



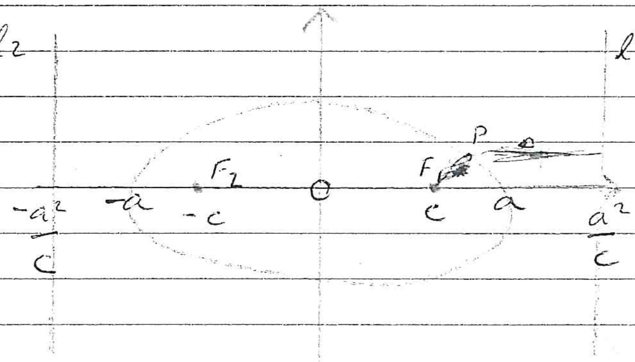
3) ECCENTRICITY AND DIRECTRICES

Like the parabola, for the ellipse + hyperbola, each focus has an associated line as directrix.



PARABOLA

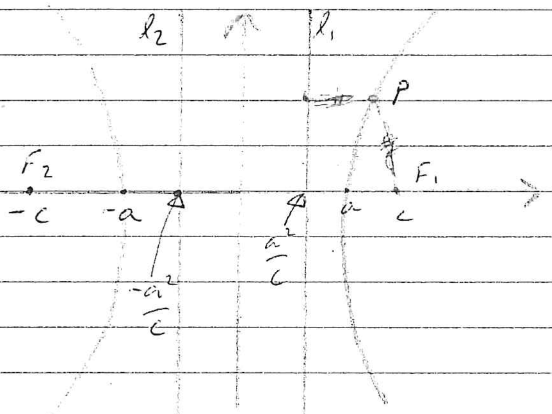
Directrix: $x = -p$



ELLIPSE

Directrixes:

$$x = \pm \frac{a^2}{c} = \pm \frac{a}{e}$$



HYPERBOLA

Directrices

$$x = \pm \frac{a^2}{c} = \pm \frac{a}{e}$$

do in tutorials (4) for ellipse
exercise for hyperbola

$\underline{\text{TH}^{\underline{M}}\ 3 \cdot 1}$ (Proof exercise)

If P is a point on an ellipse or a hyperbola then

$$\frac{\text{Distance from } P \text{ to a focus}}{\text{Distance from } P \text{ to the associated directrix}} = \frac{c}{a} \qquad (11)$$

(12)

DEF$^{\underline{N}}$: The number $e = c/a$ is called the eccentricity of the ellipse or hyperbola.

EXAMPLE

Find the eccentricity + directrices for the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$

Sol^n

$a = 4, \ b = 3 \Rightarrow c = 5$ (saw earlier)

$\therefore \ e = 5/4$ and directrices are the lines

$$x = \pm \frac{a^2}{c} = \pm \frac{16}{5}.$$

$$= \pm \frac{a}{e}$$

In view of eq^{ns} 11 and 12 and the def^n of a parabola given in Chp 2, all 3 conic sections can be defined concisely as follows :-

DEF^N : SYNTHETIC DESCRIPTIONS

Let F be a point in the plane and l a line not containing F. Let e be any positive number. The curve in the plane consisting of all points P such that

$$\frac{\text{Distance from } P \text{ to } F}{\text{Distance from } P \text{ to } l} = e$$

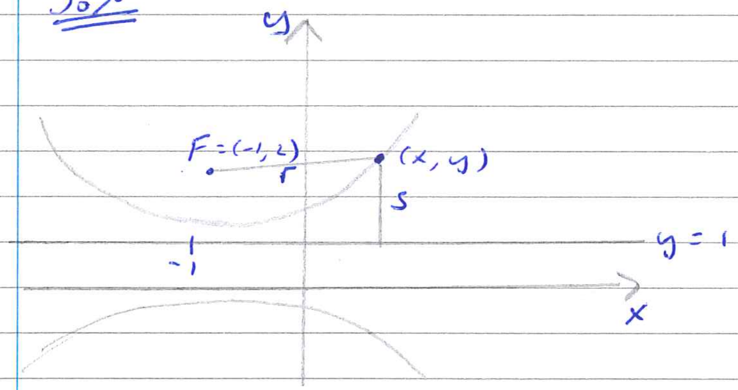
is

1) a parabola if $e = 1$
2) an ellipse if $e < 1$
3) a hyperbola if $e > 1$

EXAMPLE

Find the eq^n of the hyperbola with a focus at $(-1, 2)$ associated directix $y = 1$ and with eccentricity $3/2$.

Sol^n



$$e = \frac{r}{s} = \frac{\sqrt{(x+1)^2 + (y-2)^2}}{y-1} = \frac{3}{2}$$

$$\Rightarrow (x+1)^2 + (y-2)^2 = \frac{9}{4}(y-1)^2$$

$$\Rightarrow 4x^2 - 5y^2 + 8x + 2y + 11 = 0$$