Section 5

Jordan normal form

In the previous section we discussed at great length the diagonalisation of linear transformations. This is useful since it is much easier to work with diagonal matrices than arbitrary matrices. However, as we saw, not every linear transformation can be diagonalised. In this section, we discuss an alternative which, at least in the case of vector spaces over $\mathbb{C}$, can be used for any linear transformation or matrix.

Definition 5.1 A *Jordan block* is an $n \times n$ matrix of the form

$$J_n(\lambda) = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & \text{\Large 0} & \\ & & \ddots & \ddots & \\ & \text{\Large 0} & & \ddots & 1 \\ & & & & \lambda \end{pmatrix}$$

for some positive integer n and some scalar λ.

A linear transformation $T \colon V \to V$ (of a vector space V) has *Jordan normal form* A if there exists a basis for V with respect to which the matrix of T is

$$\mathrm{Mat}(T) = A = \begin{pmatrix} J_{n_1}(\lambda_1) & 0 & \cdots & \cdots & 0 \\ 0 & J_{n_2}(\lambda_2) & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & J_{n_k}(\lambda_k) \end{pmatrix}$$

for some positive integers $n_1, n_2, \dots, n_k$ and scalars $\lambda_1, \lambda_2, \dots, \lambda_k$. (The occurrences of 0 here indicate zero matrices of appropriate sizes.)

Comment: Blyth & Robertson use the term "elementary Jordan matrix" for what we have called a Jordan block and use the term "Jordan block matrix" for something that is a hybrid between our two concepts above. I believe the terminology above is most common.

Theorem 5.2 *Let V be a finite-dimensional vector space and $T\colon V \to V$ be a linear transformation of V such that the characteristic polynomial $c_T(x)$ is a product of linear factors with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, then there exist a basis for V with respect to which $\mathrm{Mat}(T)$ is in Jordan normal form where each Jordan block has the form $J_m(\lambda_i)$ for some m and some i.*

In particular, this theorem applies when our field is $\mathbb{C}$, since every polynomial is a product of linear factors over $\mathbb{C}$. When $c_T(x)$ is not a product of linear factors, Jordan normal form cannot be used. Instead, one uses something called *rational normal form*, which I shall not address here.

Corollary 5.3 *Let A be a square matrix over $\mathbb{C}$. Then there exist an invertible matrix P (over $\mathbb{C}$) such that $P^{-1}AP$ is in Jordan normal form.*

This corollary follows from Theorem 5.2 and Theorem 2.13 (which tells us that change of basis corresponds to forming $P^{-1}AP$).

We shall not prove Theorem 5.2. It is reasonably hard to prove and is most easily addressed by developing more advanced concepts and theory. Instead, we shall use this section to explain how to calculate the Jordan normal form associated to a linear transformation or matrix.

First consider a Jordan block

$$J = J_n(\lambda) = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & \text{\huge 0} & \\ & & \ddots & \ddots & \\ & \text{\huge 0} & & \ddots & 1 \\ & & & & \lambda \end{pmatrix}.$$

We shall first determine its characteristic polynomial and minimum polynomial.

The characteristic polynomial of J is

$$c_J(x) = \det \begin{pmatrix} x-\lambda & -1 & & & \\ & x-\lambda & -1 & \text{\huge 0} & \\ & & \ddots & \ddots & \\ & \text{\huge 0} & & \ddots & -1 \\ & & & & x-\lambda \end{pmatrix}$$

$$= (x-\lambda)\det\begin{pmatrix} x-\lambda & -1 & & \text{\Large 0} \\ & \ddots & \ddots & \\ & \text{\Large 0} & \ddots & -1 \\ & & & x-\lambda \end{pmatrix}$$

$$\vdots$$

$$= (x-\lambda)^n.$$

When we turn to calculating the minimum polynomial, we note that $m_J(x)$ divides $c_J(x)$, so $m_J(x) = (x-\lambda)^k$ for some value of k with $1 \leqslant k \leqslant n$. Our problem is to determine what k must be.

We note

$$J - \lambda I = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & \text{\Large 0} & \\ & & \ddots & \ddots & \\ & \text{\Large 0} & & \ddots & 1 \\ & & & & 0 \end{pmatrix}.$$

Let us now calculate successive powers of $J - \lambda I$:

$$(J-\lambda I)^2 = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & \text{\Large 0} & \\ & & \ddots & \ddots & \\ & \text{\Large 0} & & \ddots & 1 \\ & & & & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & \text{\Large 0} & \\ & & \ddots & \ddots & \\ & \text{\Large 0} & & \ddots & 1 \\ & & & & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & \ddots & 1 & 0 \\ \vdots & \vdots & & & & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

Repeatedly multiplying by $J - \lambda I$ successively moves the diagonal of 1s one level higher in the matrix. Thus

$$(J-\lambda I)^3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & & \ddots & 1 & 0 \\ \vdots & \vdots & & & & & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

Finally, we find

$$(J-\lambda I)^{n-1} = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad (J-\lambda I)^n = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}.$$

So $(J-\lambda I)^n = 0$ but $(J-\lambda I)^{n-1} \neq 0$. Therefore

$$m_J(x) = (x-\lambda)^n.$$

In particular, we see the characteristic and minimum polynomials of a Jordan block coincide. We record these observations for future use:

Proposition 5.4 *Let $J = J_n(\lambda)$ be an $n \times n$ Jordan block. Then*

(i) $c_J(x) = (x-\lambda)^n$;

(ii) $m_J(x) = (x-\lambda)^n$;

(iii) *the eigenspace E_λ of J has dimension* 1.

PROOF: It remains to prove part (iii). To find the eigenspace E_λ, we solve $(J-\lambda I)(v) = \mathbf{0}$. We have calculated $J - \lambda I$ above, so we solve

$$\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & \text{\Large 0} & \\ & & \ddots & \ddots & \\ & \text{\Large 0} & & \ddots & 1 \\ & & & & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix};$$

that is,

$$\begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.$$

Hence $x_2 = x_3 = \cdots = x_n$, while x_1 may be arbitrary. Therefore

$$E_\lambda = \left\{ \begin{pmatrix} x \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\} = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right).$$

Hence $\dim E_\lambda = 1$, as claimed. □

We now use the information obtained in the previous proposition to tell us how to embark on solving the following general proposition.

Problem: Let V be a finite-dimensional vector space and let $T \colon V \to V$ be a linear transformation. If the characteristic polynomial $c_T(x)$ is a product of linear factors, find a basis $\mathscr{B}$ with respect to which T is in Jordan normal form and determine what this Jordan normal form is.

If $\mathscr{B}$ is the basis solving this problem, then

$$\operatorname{Mat}_{\mathscr{B},\mathscr{B}}(T) = A = \begin{pmatrix} J_{n_1}(\lambda_1) & & & \\ & J_{n_2}(\lambda_2) & & \mbox{\Large 0} \\ & \mbox{\Large 0} & \ddots & \\ & & & J_{n_k}(\lambda_k) \end{pmatrix},$$

where $J_{n_1}(\lambda_1)$, $J_{n_2}(\lambda_2)$, ..., $J_{n_k}(\lambda_k)$ are the Jordan blocks. When we calculate the characteristic polynomial $c_T(x)$ using this matrix, each block $J_{n_i}(\lambda_i)$ contributes a factor of $(x - \lambda_i)^{n_i}$ (see Proposition 5.4(i)). Collecting all the factors corresponding to the same eigenvalue, we conclude:

Observation 5.5 *The algebraic multiplicity of λ as an eigenvalue of T equals the sum of the sizes of the Jordan blocks $J_n(\lambda)$ (associated to λ) occurring in the Jordan normal form for T.*

This means, of course, that the number of times that λ occurs on the diagonal in the Jordan normal form matrix A is precisely the algebraic multiplicity r_λ of λ.

If particular, if $r_\lambda = 1$, a single 1×1 Jordan block occurs in A, namely $J_1(\lambda) = (\lambda)$. If $r_\lambda = 2$, then either two 1×1 Jordan blocks occur or a 2×2 Jordan block $J_2(\lambda)$ occurs in A. Thus A either contains

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

Similar observations may be made for other small values of r_λ, but the possibilities grow more complicated as r_λ increases.

To distinguish between these possibilities, we first make use of the minimum polynomial. To ensure the block $J_{n_i}(\lambda_i)$ becomes 0 when we substitute A into the polynomial, we must have at least a factor $(x - \lambda_i)^{n_i}$ (see Proposition 5.4(ii)). Consequently:

Observation 5.6 *If λ is an eigenvalue of T, then the power of $(x - \lambda)$ occurring in the minimum polynomial $m_T(x)$ is $(x - \lambda)^m$ where m is the largest size of a Jordan block associated to λ occurring in the Jordan normal form for T.*

Observations 5.5 and 5.6 are enough to determine the Jordan normal form in small cases.

Example 5.7 *Let $V = \mathbb{R}^4$ and let $T \colon V \to V$ be the linear transformation given by the matrix*

$$B = \begin{pmatrix} 2 & 1 & 0 & -3 \\ 0 & 2 & 0 & 4 \\ 4 & 5 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

Determine the Jordan normal form of T.

Solution: We first determine the characteristic polynomial of T:

$$\begin{aligned} c_T(x) &= \det \begin{pmatrix} x-2 & -1 & 0 & 3 \\ 0 & x-2 & 0 & -4 \\ -4 & -5 & x+2 & -1 \\ 0 & 0 & 0 & x+2 \end{pmatrix} \\ &= (x+2) \det \begin{pmatrix} x-2 & -1 & 0 \\ 0 & x-2 & 0 \\ -4 & -5 & x+2 \end{pmatrix} \\ &= (x+2)^2 \det \begin{pmatrix} x-2 & -1 \\ 0 & x-2 \end{pmatrix} \\ &= (x-2)^2(x+2)^2. \end{aligned}$$

So the Jordan normal form contains either a single Jordan block $J_2(2)$ corresponding to eigenvalue 2 or two blocks $J_1(2)$ of size 1. Similar observations apply to the Jordan block(s) corresponding to the eigenvalue -2. To determine which occurs, we consider the minimum polynomial.

We now know the minimum polynomial of T has the form $m_T(x) = (x-2)^i(x+2)^j$ where $1 \leqslant i, j \leqslant 2$ by Corollary 4.19 and Theorem 4.20. Now

$$B - 2I = \begin{pmatrix} 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 4 \\ 4 & 5 & -4 & 1 \\ 0 & 0 & 0 & -4 \end{pmatrix} \qquad \text{and} \qquad B + 2I = \begin{pmatrix} 4 & 1 & 0 & -3 \\ 0 & 4 & 0 & 4 \\ 4 & 5 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so

$$(B-2I)(B+2I) = \begin{pmatrix} 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0$$

and

$$\begin{aligned} (B-2I)^2(B+2I) &= \begin{pmatrix} 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 4 \\ 4 & 5 & -4 & 1 \\ 0 & 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

The first calculation shows $m_T(x)$ is not equal to the only possibility of degree 2, so we conclude from the second calculation that

$$m_T(x) = (x-2)^2(x+2).$$

Hence at least one Jordan block $J_2(2)$ of size 2 occurs in the Jordan normal form of T, while all Jordan blocks corresponding to the eigenvalue -2 have size 1.

We conclude the Jordan normal form of T is

$$\begin{pmatrix} J_2(2) & 0 & 0 \\ 0 & J_1(-2) & 0 \\ 0 & 0 & J_1(-2) \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

Example 5.8 *Let $V = \mathbb{R}^4$ and let $T \colon V \to V$ be the linear transformation given by the matrix*

$$C = \begin{pmatrix} 3 & 0 & 1 & -1 \\ 1 & 2 & 1 & -1 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Determine the Jordan normal form of T.

Solution:

$$\begin{aligned}
c_T(x) &= \det \begin{pmatrix} x-3 & 0 & -1 & 1 \\ -1 & x-2 & -1 & 1 \\ 1 & 0 & x-1 & -1 \\ 0 & 0 & 0 & x-2 \end{pmatrix} \\
&= (x-2) \det \begin{pmatrix} x-3 & 0 & -1 \\ -1 & x-2 & -1 \\ 1 & 0 & x-1 \end{pmatrix} \\
&= (x-2)^2 \det \begin{pmatrix} x-3 & -1 \\ 1 & x-1 \end{pmatrix} \\
&= (x-2)^2 \big((x-3)(x-1) + 1 \big) \\
&= (x-2)^2 (x^2 - 4x + 3 + 1) \\
&= (x-2)^2 (x^2 - 4x + 4) \\
&= (x-2)^4.
\end{aligned}$$

Now we calculate the minimum polynomial:

$$C - 2I = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so

$$\begin{aligned}
(C-2I)^2 &= \begin{pmatrix} 1 & 0 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

Hence $m_T(x) = (x-2)^2$. We now know the Jordan normal form for T includes at least one block $J_2(2)$ but we cannot tell whether the remaining blocks are a single block of size 2 or two blocks of size 1.

To actually determine which, we need to go beyond the characteristic and minimum polynomials, and consider the eigenspace E_2. We shall describe this in general and return to complete the solution of this example later.

Consider a linear transformation $T\colon V \to V$ with Jordan normal form A. Each block $J_n(\lambda)$ occurring in A contributes one linearly independent eigenvector to a basis for the eigenspace E_λ (see Proposition 5.4(iii)). Thus the number of blocks in A corresponding to a particular eigenvalue λ will equal

$$\dim E_\lambda = n_\lambda,$$

the geometric multiplicity of λ. In summary:

Observation 5.9 *The geometric multiplicity of λ as an eigenvalue of T equals the number of Jordan blocks $J_n(\lambda)$ occurring in the Jordan normal form for T.*

Example 5.8 (cont.) Let us determine the eigenspace E_2 for our transformation T with matrix C. We solve $(T - 2I)(v) = \mathbf{0}$; that is,

$$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This reduces to a single equation

$$x + z - t = 0,$$

so

$$E_2 = \left\{ \begin{pmatrix} x \\ y \\ z \\ x+z \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\} = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right).$$

Hence $n_2 = \dim E_2 = 3$. It follows that the Jordan normal for T contains three Jordan blocks corresponding to the eigenvalue 2. Therefore the Jordan normal form of T is

$$\begin{pmatrix} J_2(2) & 0 & 0 \\ 0 & J_1(2) & 0 \\ 0 & 0 & J_1(2) \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Our three observations are enough to determine the Jordan normal form for the linear transformations that will be encountered in this course. They are sufficient for small matrices, but will not solve the problem for all possibilities. For example, they do not distinguish between the 7×7 matrices

$$\begin{pmatrix} J_3(\lambda) & 0 & 0 \\ 0 & J_2(\lambda) & 0 \\ 0 & 0 & J_2(\lambda) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} J_3(\lambda) & 0 & 0 \\ 0 & J_3(\lambda) & 0 \\ 0 & 0 & J_1(\lambda) \end{pmatrix},$$

which both have characteristic polynomial $(x-\lambda)^7$, minimum polynomial $(x-\lambda)^3$ and geometric multiplicity $n_\lambda = \dim E_\lambda = 3$. To deal with such possible Jordan normal forms one needs to generalise Observation 5.9 to consider the dimension of generalisations of eigenspaces:

$$\dim\ker(T-\lambda I)^2, \quad \dim\ker(T-\lambda I)^3, \quad \dots .$$

We leave the details to the interested and enthused student.

We finish this section by returning to the final part of the general problem: finding a basis with respect to which a linear transformation is in Jordan normal form.

Example 5.10 *Let*

$$C = \begin{pmatrix} 3 & 0 & 1 & -1 \\ 1 & 2 & 1 & -1 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

(the matrix from Example 5.8). Find an invertible matrix P such that $P^{-1}CP$ is in Jordan normal form.

Solution: We have already established the Jordan normal form of the transformation $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ with matrix C is

$$A = \begin{pmatrix} J_2(2) & 0 & 0 \\ 0 & J_1(2) & 0 \\ 0 & 0 & J_1(2) \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Our problem is equivalent (by Theorem 2.13) to finding a basis $\mathscr{B}$ with respect to which the matrix of T equals A. Thus $\mathscr{B} = \{v_1, v_2, v_3, v_4\}$ such that

$$T(v_1) = 2v_1, \qquad T(v_2) = v_1 + 2v_2, \qquad T(v_3) = 2v_3, \qquad T(v_4) = 2v_4.$$

So we need to choose v_1, v_3 and v_4 to lie in the eigenspace E_2 (which we determined earlier).

On the face of it, the choice of v_2 appears to be less straightforward: we require $(T-2I)(v_2) = v_1$, some non-zero vector in E_2 and this indicates we also probably do not have total freedom in the choice of v_1. In Example 5.8, we calculated

$$E_2 = \left\{ \begin{pmatrix} x \\ y \\ z \\ x+z \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\}.$$

Let us solve

$$(T - 2I)(v) = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 1 & 0 & 1 & -1 \\ -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ x + z \end{pmatrix}.$$

We need to establish for which values of x, y and z this has a non-zero solution (and in the process we determine possibilities for both v_1 and v_2). The above matrix equation implies

$$\alpha + \gamma - \delta = x = y = -z$$

and

$$x + z = 0.$$

Any value of x will determine a possible solution, so let us choose $x = 1$. Then $y = 1$ and $z = -1$. Hence we shall take

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

and then the equation $(T - 2I)(v_2) = v_1$ has non-zero solutions, namely

$$v_2 = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \qquad \text{where} \quad \alpha + \gamma - \delta = 1.$$

There are many possible solutions, we shall take $\alpha = 1$, $\beta = \gamma = \delta = 0$ and hence

$$v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

will be good enough.

To find v_3 and v_4, we need two vectors from E_2 which together with v_1 form a basis for E_2. We shall choose

$$v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad v_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

Indeed, note that an arbitrary vector in E_2 can be expressed as

$$\begin{aligned}\begin{pmatrix} x \\ y \\ z \\ x+z \end{pmatrix} &= x \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} + (y-x) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + (x+z) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \\ &= xv_1 + (y-x)v_3 + (x+z)v_4,\end{aligned}$$

so $E_2 = \mathrm{Span}(v_1, v_3, v_4)$.

We now have our required basis

$$\mathscr{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

and the required change of basis matrix is

$$P = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

[Exercise: Calculate $P^{-1}CP$ and verify it has the correct form.]

Example 5.11 *Let $T \colon \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation given by the matrix*

$$D = \begin{pmatrix} -3 & 2 & \frac{1}{2} & -2 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Determine the Jordan normal form J of T and find an invertible matrix P such that $P^{-1}DP = J$.

Solution:

$$\begin{aligned} c_T(x) &= \det \begin{pmatrix} x+3 & -2 & -\frac{1}{2} & 2 \\ 0 & x & 0 & 0 \\ 0 & 3 & x+3 & 3 \\ 0 & 0 & 0 & x \end{pmatrix} \\ &= (x+3) \det \begin{pmatrix} x & 0 & 0 \\ 3 & x+3 & 3 \\ 0 & 0 & x \end{pmatrix} \\ &= x(x+3) \det \begin{pmatrix} x & 0 \\ 3 & x+3 \end{pmatrix} \end{aligned}$$

$$= x^2(x+3)^2.$$

So the eigenvalues of T are 0 and -3. Then $m_T(x) = x^i(x+3)^j$ where $1 \leqslant i, j \leqslant 2$. Since

$$D + 3I = \begin{pmatrix} 0 & 2 & \frac{1}{2} & -2 \\ 0 & 3 & 0 & 0 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & 0 & 3 \end{pmatrix},$$

we calculate

$$D(D+3I) = \begin{pmatrix} 0 & -\frac{3}{2} & -\frac{3}{2} & -\frac{3}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0$$

and

$$\begin{aligned} D(D+3I)^2 &= \begin{pmatrix} 0 & -\frac{3}{2} & -\frac{3}{2} & -\frac{3}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 & \frac{1}{2} & -2 \\ 0 & 3 & 0 & 0 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & 0 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Hence $m_T(x) = x(x+3)^2$. Therefore the Jordan normal form of T is

$$J = \begin{pmatrix} J_1(0) & 0 & 0 \\ 0 & J_1(0) & 0 \\ 0 & 0 & J_2(-3) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

We now seek a basis $\mathscr{B} = \{v_1, v_2, v_3, v_4\}$ with respect to which the matrix of T is J. Thus, we require $v_1, v_2 \in E_0,\ \ v_3 \in E_{-3}$ and

$$T(v_4) = v_3 - 3v_4.$$

We first solve $T(v) = \mathbf{0}$:

$$\begin{pmatrix} -3 & 2 & \frac{1}{2} & -2 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

so

$$-3x + 2y + \tfrac{1}{2}z - 2t = 0$$

$$-3y - 3z - 3t = 0.$$

Hence given arbitrary $z, t \in \mathbb{R}$, we have $y = -z - t$ and

$$\begin{aligned} x &= \tfrac{1}{3}(2y + \tfrac{1}{2}z - 2t) \\ &= \tfrac{1}{3}(-\tfrac{3}{2}z - 4t) \\ &= -\tfrac{1}{2}z - \tfrac{4}{3}t. \end{aligned}$$

So

$$E_0 = \left\{ \begin{pmatrix} -\frac{1}{2}z - \frac{4}{3}t \\ -z - t \\ z \\ t \end{pmatrix} \,\middle|\, z, t \in \mathbb{R} \right\} = \mathrm{Span}\left(\begin{pmatrix} -\frac{1}{2} \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{4}{3} \\ -1 \\ 0 \\ 1 \end{pmatrix} \right).$$

Take

$$v_1 = \begin{pmatrix} -\frac{1}{2} \\ -1 \\ 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad v_2 = \begin{pmatrix} -\frac{4}{3} \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

Now solve $(T + 3I)(v) = \mathbf{0}$:

$$\begin{pmatrix} 0 & 2 & \frac{1}{2} & -2 \\ 0 & 3 & 0 & 0 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} z \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

so

$$\begin{aligned} 2y + \tfrac{1}{2}z - 2t &= 0 \\ 3y \phantom{{}+ \tfrac{1}{2}z - 2t} &= 0 \\ -3y \phantom{{}+ \tfrac{1}{2}z} - 3t &= 0 \\ 3t &= 0. \end{aligned}$$

Hence $y = t = 0$ and we deduce $z = 0$, while x may be arbitrary. Thus

$$E_{-3} = \left\{ \begin{pmatrix} x \\ 0 \\ 0 \\ 0 \end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\} = \mathrm{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right).$$

Take

$$v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We finally solve $T(v_4) = v_3 - 3v_4$; that is, $(T + 3I)(v_4) = v_3$:

$$\begin{pmatrix} 0 & 2 & \frac{1}{2} & -2 \\ 0 & 3 & 0 & 0 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence

$$\begin{aligned} 2y + \tfrac{1}{2}z - 2t &= 1 \\ 3y \qquad\qquad &= 0 \\ -3y \qquad - 3t &= 0 \\ 3t &= 0, \end{aligned}$$

so $y = t = 0$ and then $\frac{1}{2}z = 1$, which forces $z = 2$, while x may be arbitrary. Thus

$$v_4 = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}$$

is one solution. Thus

$$\mathscr{B} = \left\{ \begin{pmatrix} -\frac{1}{2} \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{4}{3} \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix} \right\}$$

and our change of basis matrix is

$$P = \begin{pmatrix} -\frac{1}{2} & -\frac{4}{3} & 1 & 0 \\ -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

This last example illustrates some general principles. When seeking the invertible matrix P such that $P^{-1}AP$ is in Jordan normal form, we seek particular vectors to form a basis. These basis vectors can be found by solving appropriate systems of linear equations (though sometimes care is needed to find the correct system to solve as was illustrated in Example 5.10).

Example 5A *Let $V = \mathbb{R}^5$ and let $T\colon V \to V$ be the linear transformation given by the matrix*

$$E = \begin{pmatrix} 1 & 0 & -1 & 0 & -8 \\ 0 & 1 & 4 & 0 & 29 \\ -1 & 0 & 1 & 1 & 5 \\ 0 & 0 & -1 & 1 & -11 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

Determine a Jordan normal form J of T and find an invertible matrix P such that $P^{-1}EP = J$.

SOLUTION: We first determine the characteristic polynomial of T:

$$\begin{aligned}
c_T(x) &= \det \begin{pmatrix} x-1 & 0 & 1 & 0 & 8 \\ 0 & x-1 & -4 & 0 & -29 \\ 1 & 0 & x-1 & -1 & -5 \\ 0 & 0 & 1 & x-1 & 11 \\ 0 & 0 & 0 & 0 & x+2 \end{pmatrix} \\
&= (x+2) \det \begin{pmatrix} x-1 & 0 & 1 & 0 \\ 0 & x-1 & -4 & 0 \\ 1 & 0 & x-1 & -1 \\ 0 & 0 & 1 & x-1 \end{pmatrix} \\
&= (x-1)(x+2) \det \begin{pmatrix} x-1 & 1 & 0 \\ 1 & x-1 & -1 \\ 0 & 1 & x-1 \end{pmatrix} \\
&= (x-1)(x+2) \left((x-1) \det \begin{pmatrix} x-1 & -1 \\ 1 & x-1 \end{pmatrix} - \det \begin{pmatrix} 1 & -1 \\ 0 & x-1 \end{pmatrix} \right) \\
&= (x-1)(x+2) \Big((x-1)\big((x-1)^2+1\big) - (x-1) \Big) \\
&= (x-1)^2(x+2)\big((x-1)^2+1-1\big) \\
&= (x-1)^4(x+2).
\end{aligned}$$

We now know that the Jordan normal form for T contains a single Jordan block $J_1(-2)$ corresponding to eigenvalue -2 and some number of Jordan blocks $J_n(1)$ corresponding to eigenvalue 1. The sum of the sizes of these latter blocks equals 4.

Let us now determine the minimum polynomial of T. We know $m_T(x) = (x-1)^i(x+2)$ where $1 \leqslant i \leqslant 4$ by Corollary 4.19 and Theorem 4.20. Now

$$E - I = \begin{pmatrix} 0 & 0 & -1 & 0 & -8 \\ 0 & 0 & 4 & 0 & 29 \\ -1 & 0 & 0 & 1 & 5 \\ 0 & 0 & -1 & 0 & -11 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix} \quad \text{and} \quad E + 2I = \begin{pmatrix} 3 & 0 & -1 & 0 & -8 \\ 0 & 3 & 4 & 0 & 29 \\ -1 & 0 & 3 & 1 & 5 \\ 0 & 0 & -1 & 3 & -11 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

so

$$(E-I)(E+2I) = \begin{pmatrix} 1 & 0 & -3 & -1 & -5 \\ -4 & 0 & 12 & 4 & 20 \\ -3 & 0 & 0 & 3 & -3 \\ 1 & 0 & -3 & -1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \neq 0,$$

$$(E-I)^2(E+2I) = \begin{pmatrix} 0 & 0 & -1 & 0 & -8 \\ 0 & 0 & 4 & 0 & 29 \\ -1 & 0 & 0 & 1 & 5 \\ 0 & 0 & -1 & 0 & -11 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 & -3 & -1 & -5 \\ -4 & 0 & 12 & 4 & 20 \\ -3 & 0 & 0 & 3 & -3 \\ 1 & 0 & -3 & -1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & 0 & -3 & 3 \\ -12 & 0 & 0 & 12 & -12 \\ 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \neq 0$$

and

$$(E-I)^3(E+2I) = \begin{pmatrix} 0 & 0 & -1 & 0 & -8 \\ 0 & 0 & 4 & 0 & 29 \\ -1 & 0 & 0 & 1 & 5 \\ 0 & 0 & -1 & 0 & -11 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 & -3 & 3 \\ -12 & 0 & 0 & 12 & -12 \\ 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence $m_T(x) = (x-1)^3(x+2)$. As a consequence, the Jordan normal form of T must contain at least one Jordan block $J_3(1)$ of size 3. Since the sizes of the Jordan blocks associated to the eigenvalue 1 has sum equal to 4 (from earlier), there remains a single Jordan block $J_1(1)$ of size 1.

Our conclusion is a Jordan normal form of T is

$$J = \begin{pmatrix} J_3(1) & 0 & 0 \\ 0 & J_1(1) & 0 \\ 0 & 0 & J_1(-2) \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{0} & 0 & 0 \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{-2} \end{pmatrix}.$$

We now want to find a basis $\mathscr{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5\}$ for $\mathbb{R}^5$ such that $\mathrm{Mat}_{\mathscr{B},\mathscr{B}}(T) = J$. In particular, $\boldsymbol{v}_1$ and $\boldsymbol{v}_4$ are required to be eigenvectors with eigenvalue 1. Let us first find the eigenspace E_1 by solving $(T-I)(\boldsymbol{v}) = \mathbf{0}$; that is,

$$\begin{pmatrix} 0 & 0 & -1 & 0 & 8 \\ 0 & 0 & 4 & 0 & 29 \\ -1 & 0 & 0 & 1 & 5 \\ 0 & 0 & -1 & 0 & -11 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The fifth row yields $-3u = 0$; that is, $u = 0$. It follows from the first row that $-z+8u = 0$ and hence $z = 0$. The only row yielding further information is the third which says $-x + t + 5u = 0$ and so $x = t$. Hence

$$E_1 = \left\{ \begin{pmatrix} x \\ y \\ 0 \\ x \\ 0 \end{pmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}.$$

From this we can read off a basis for the eigenspace E_1, but this does not tell us which vector to take as $\boldsymbol{v}_1$. We need $\boldsymbol{v}_1$ to be a suitable choice of eigenvector so that $T(\boldsymbol{v}_2) = \boldsymbol{v}_1 + \boldsymbol{v}_2$, that is, $(T - I)(\boldsymbol{v}_2) = \boldsymbol{v}_1$, is possible. We solve for $(T - I)(\boldsymbol{v}) = \boldsymbol{w}$ where $\boldsymbol{w}$ is a typical vector in E_1. So consider

$$\begin{pmatrix} 0 & 0 & -1 & 0 & 8 \\ 0 & 0 & 4 & 0 & 29 \\ -1 & 0 & 0 & 1 & 5 \\ 0 & 0 & -1 & 0 & -11 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \\ u \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ 0 \\ \alpha \\ 0 \end{pmatrix}$$

for some non-zero scalars α and β. Thus $-3u = 0$ and so $u = 0$. We then obtain three equations

$$-z = \alpha, \qquad 4z = \beta, \qquad -x + t = 0. \tag{5.1}$$

Thus to have a solution we must have $-4\alpha = \beta$. This tells us what to take as $\boldsymbol{v}_1$: we want a vector of the form

$$\begin{pmatrix} \alpha \\ -4\alpha \\ 0 \\ \alpha \\ 0 \end{pmatrix} \qquad \text{where } \alpha \in \mathbb{R}, \text{ but } \alpha \neq 0.$$

So take $\alpha = 1$ and

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ -4 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Then Equations (5.1) tell us that vector $\boldsymbol{v}_2$ is given by $z = -1$, $x = t$ and y and x can be arbitrary. We shall take $x = y = 0$ (mainly for convenience):

$$\boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$$

The vector $\boldsymbol{v}_3$ is required to satisfy $T(\boldsymbol{v}_3) = \boldsymbol{v}_2 + \boldsymbol{v}_3$, so to find $\boldsymbol{v}_3$ we solve $(T - I)(\boldsymbol{v}) = \boldsymbol{v}_2$:

$$\begin{pmatrix} 0 & 0 & -1 & 0 & 8 \\ 0 & 0 & 4 & 0 & 29 \\ -1 & 0 & 0 & 1 & 5 \\ 0 & 0 & -1 & 0 & -11 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$$

Thus $u = 0, \ \ z = 0$ and $-x + t = -1$. Any other choices are arbitrary, so we shall take $x = 1, \ \ y = 0$ and then $t = 0$. So we take

$$\boldsymbol{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

For $\boldsymbol{v}_3$, we note that $\boldsymbol{v}_1$, $\boldsymbol{v}_4$ should be linearly independent vectors (as they form part of a basis) in the eigenspace

$$E_1 = \left\{ \begin{pmatrix} x \\ y \\ 0 \\ x \\ 0 \end{pmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}.$$

Given our choice of $\boldsymbol{v}_1$, there remains much choice for $\boldsymbol{v}_4$ (essentially it must not be a scalar multiple of $\boldsymbol{v}_1$). We shall take

$$\boldsymbol{v}_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

(i.e., take $x = 1, \ \ y = 0$).

Finally, we require $\boldsymbol{v}_5$ to be an eigenvector for T with eigenvalue -2, so we solve $T(\boldsymbol{v}) = -2\boldsymbol{v}$ or, equivalently, $(T + 2I)(\boldsymbol{v}) = \mathbf{0}$:

$$\begin{pmatrix} 3 & 0 & -1 & 0 & -8 \\ 0 & 3 & 4 & 0 & 29 \\ -1 & 0 & 3 & 1 & 5 \\ 0 & 0 & -1 & 3 & -11 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We apply row operations to solve this system of equations:

$$\left(\begin{array}{ccccc|c} 3 & 0 & -1 & 0 & -8 & 0 \\ 0 & 3 & 4 & 0 & 29 & 0 \\ -1 & 0 & 3 & 1 & 5 & 0 \\ 0 & 0 & -1 & 3 & -11 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccccc|c} 0 & 0 & 8 & 3 & 7 & 0 \\ 0 & 3 & 4 & 0 & 29 & 0 \\ -1 & 0 & 3 & 1 & 5 & 0 \\ 0 & 0 & -1 & 3 & -11 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad r_1 \mapsto r_1 + 3r_3$$

$$\longrightarrow \left(\begin{array}{ccccc|c} 0 & 0 & 0 & 27 & -81 & 0 \\ 0 & 3 & 0 & 12 & -15 & 0 \\ -1 & 0 & 0 & 10 & -28 & 0 \\ 0 & 0 & -1 & 3 & -11 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{array}{l} r_1 \mapsto r_1 + 8r_4 \\ r_2 \mapsto r_2 + 4r_4 \\ r_3 \mapsto r_3 + 3r_4 \end{array}$$

$$\longrightarrow \left(\begin{array}{ccccc|c} 0 & 0 & 0 & 1 & -3 & 0 \\ 0 & 3 & 0 & 4 & -5 & 0 \\ -1 & 0 & 0 & 10 & -28 & 0 \\ 0 & 0 & -1 & 3 & -11 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{array}{l} r_1 \mapsto \frac{1}{27} r_1 \\ r_2 \mapsto \frac{1}{3} r_2 \end{array}$$

Hence

$$\begin{aligned} t - 3u &= 0 \\ y + 4t - 5u &= 0 \\ -x + 10t - 28u &= 0 \\ -z + 3t - 11u &= 0. \end{aligned}$$

Take $u = 1$ (it can be non-zero, but otherwise arbitrary, when producing the eigenvector $\boldsymbol{v}_5$). Then

$$\begin{aligned} t &= 3u = 3 \\ y &= 5u - 4t = -7 \\ x &= 10t - 28u = 2 \\ z &= 3t - 11u = -2. \end{aligned}$$

So we take

$$\boldsymbol{v}_5 = \begin{pmatrix} 2 \\ -7 \\ -2 \\ 3 \\ 1 \end{pmatrix}.$$

With the above choices, the matrix of T with respect to the basis $\mathscr{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5\}$ is then our Jordan normal form J. The change of basis matrix P such that $P^{-1}EP = J$ is found by writing each $\boldsymbol{v}_j$ in terms of the standard basis and placing the coefficients in the jth column of P. Thus

$$P = \begin{pmatrix} 1 & 0 & 1 & 1 & 2 \\ -4 & 0 & 0 & 0 & -7 \\ 0 & -1 & 0 & 0 & -2 \\ 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

□