(7) Suppose the elliptic fixed points of a Fuchsian group Γ are not a discrete set.

Let $\varepsilon > 0$ and find an elliptic fixed point z such that $B_{\mathbb{H}^2}(z, \varepsilon)$ contains infinitely many distinct elliptic fixed points z_n and let $g_n \in \Gamma$ be ~~the~~ an elliptic element fixing z_n, ie $g_n(z_n) = z_n$ $\forall n$.

Let $B = B_{\mathbb{H}^2}(z, 3\varepsilon)$ be the closed ball centred at z with radius 3ε.

We proved in lectures that

$$\{ g \in \Gamma : g(z) \in B \}$$

$$= \Gamma \cap \underbrace{\{ g \in PSL(2, \mathbb{R}) : g(z) \in B \}}_{(*)}$$

is finite since Γ is discrete, closed, and $(*)$ is compact. Hence, for some (in fact infinitely many) n we have

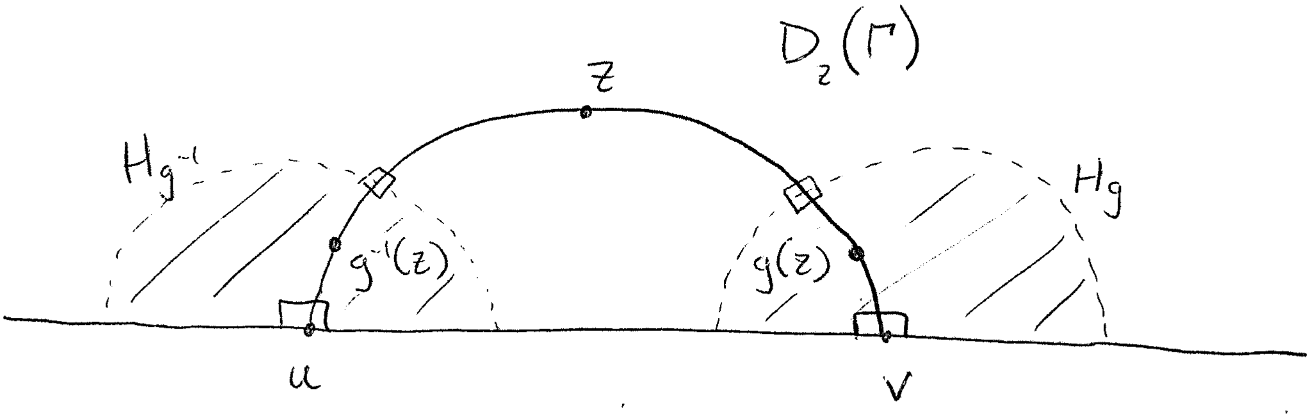
$$3\varepsilon < d_{\mathbb{H}^2}(z, g_n(z)) \leq d_{\mathbb{H}^2}(z, g_n(z_n)) + d_{\mathbb{H}^2}(g_n(z_n), g_n(z))$$
$$= 2\, d_{\mathbb{H}^2}(z, z_n) \leq 2\varepsilon, \text{ a contradiction.}$$

2

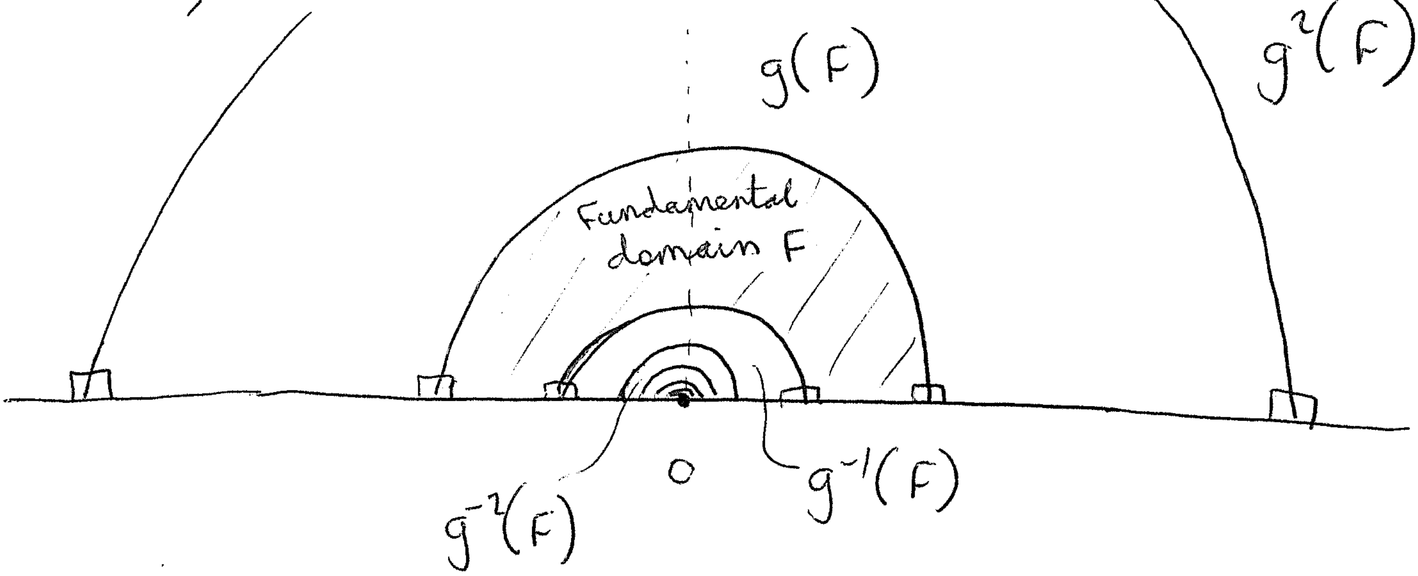
(8) (i) Let $g \in PSL(2, \mathbb{R})$ be hyperbolic with fixed points $u, v \in \mathbb{R} \cup \{\infty\}$ and let $\langle g \rangle = \Gamma$. Let $z \in \mathbb{H}^2$ be arbitrary and let C be the unique circle (or vertical line) containing $\{u, v, z\}$. Then $\Gamma(z)$ is a discrete subset of C (although it does accumulate in $|\cdot|$ at u, v).

Hence Γ is Fuchsian by part 6.

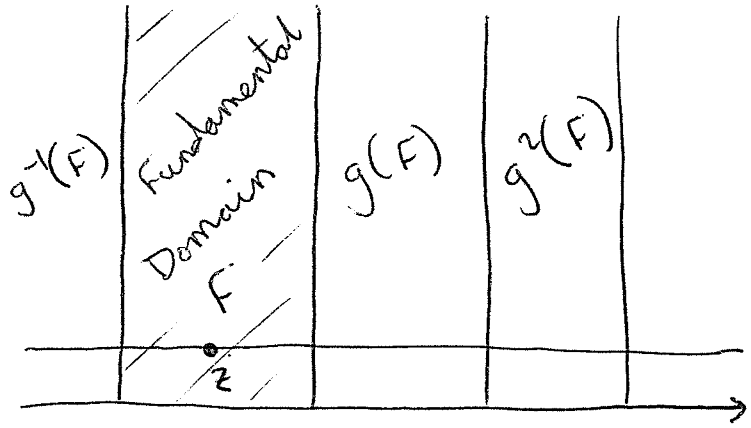
We construct a fundamental domain as a Dirichlet region using base point z contained in the geodesic ray from u to v.



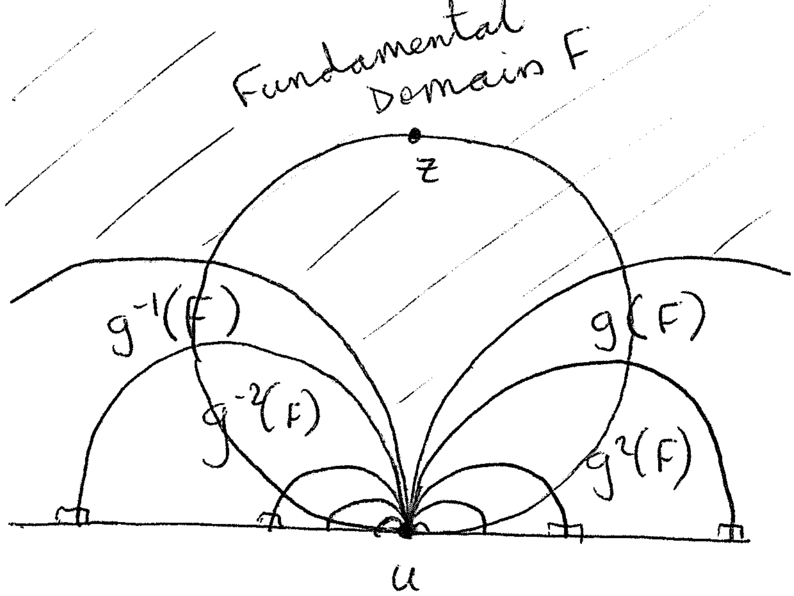
The tiling is easier to draw if we assume $u = 0$, $v = \infty$.



(ii) This time, let $g \in PSL(2, \mathbb{R})$ be parabolic with fixed point $u \in \mathbb{R} \cup \{\infty\}$. Let $\Gamma = \langle g \rangle$ and $z \in \mathbb{H}^2$ arbitrary. Then $\Gamma(z)$ is a discrete subset of the horocycle containing $\{z, u\}$, and so Γ Fuchsian by 6.



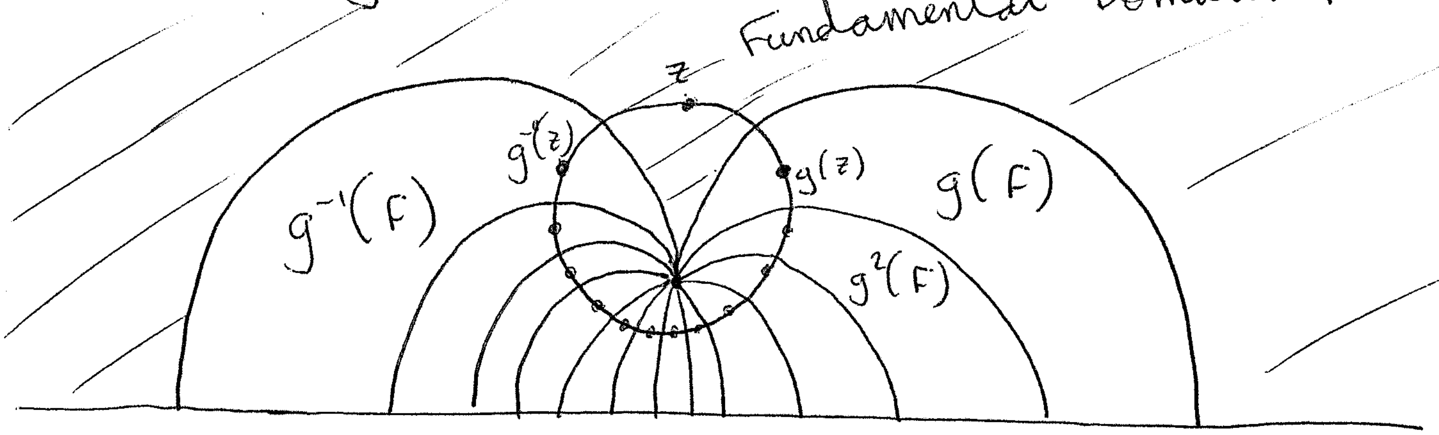
when $u = \infty$, base point z



when $u \neq \infty$, base point z.

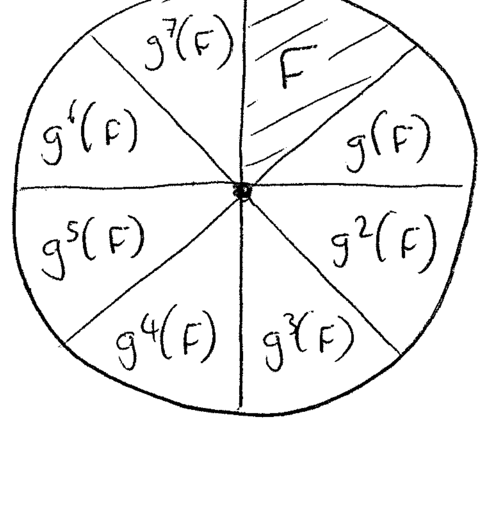
(iii) Let $g \in PSL(2,\mathbb{R})$ be an elliptic element with order $n < \infty$.

Then $\langle g \rangle = \{ Id, g, g^2, g^3, \ldots, g^{n-1} \}$ is the (finite) cyclic group of order n. Finite groups $\Gamma \leq PSL(2,\mathbb{R})$ are clearly Fuchsian.



Example fundamental domain when $n = 13$ and z is the base point.

Easier to visualise in $\mathbb{D}^2$, when fixed point $= 0$!



Here $n = 8$.

(8) continued...

The only case not covered is when g is an elliptic element with infinite order. Let g be such an element, with fixed point $u \in \mathbb{H}^2$ and $\Gamma = \langle g \rangle \leq PSL(2,\mathbb{R})$.

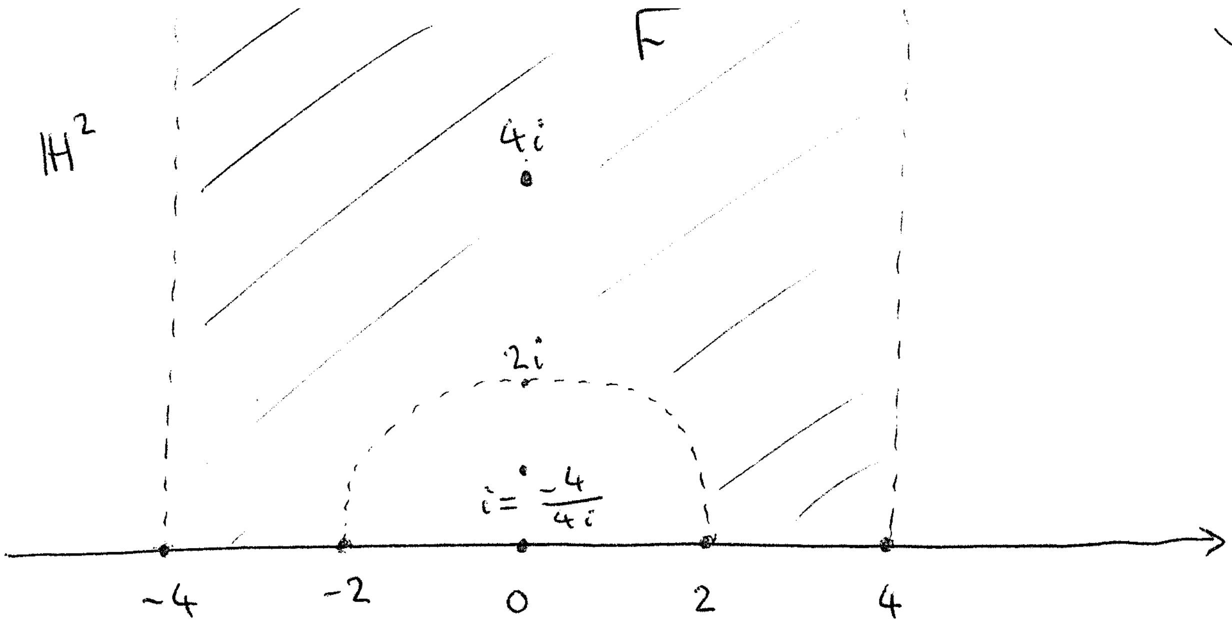
<u>claim:</u> For any $z \in \mathbb{H}^2 \setminus \{u\}$, $\Gamma(z)$ is infinite.

<u>Proof:</u> Suppose $m, n \in \mathbb{Z}$ with $m \neq n$ and $g^n(z) = g^m(z)$. Then $g^{n-m}(z) = g^n \circ g^{-m}(z) = z$

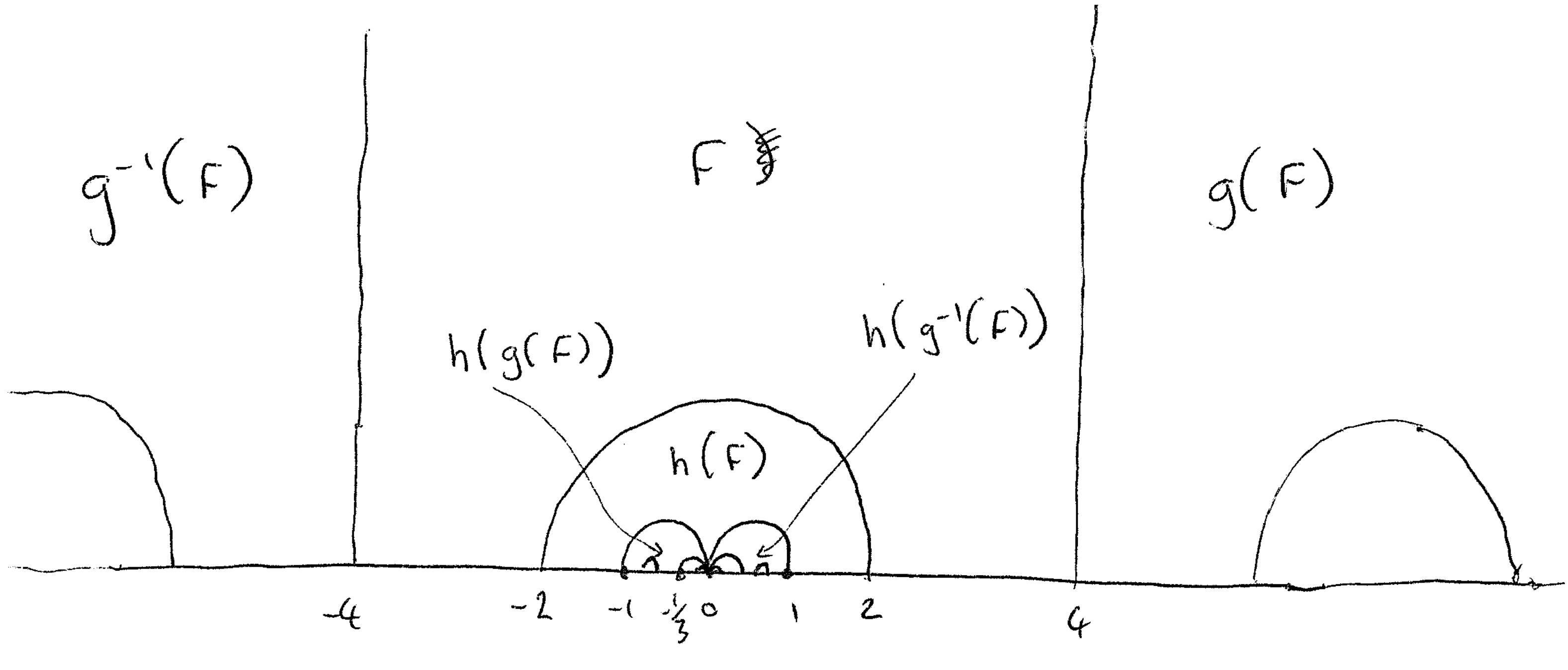
g^{n-m} therefore fixes z and u and is therefore the identity, implying g has finite order; a contradiction! The claim follows since $g^n(z)$ $(n \in \mathbb{Z})$ are distinct elements of $\Gamma(z)$.

Since $d_{\mathbb{H}^2}(u, g^n(z)) = d_{\mathbb{H}^2}(u, z)$ $\forall n$, we have $\Gamma(z) \cap B_{\mathbb{H}^2}(u, 2d_{\mathbb{H}^2}(u,v))$ infinite which ~~contradicts~~ implies Γ does not act properly discontinuously, and is therefore not Fuchsian.

9



Using the generators, the above area F is a good candidate for $D_{4i}(\Gamma)$. Clearly $D_{4i}(\Gamma) \subseteq F$ and, moreover, some simple calculations show $\Gamma(F)$ is a tiling. Therefore, if $D_{4i}(\Gamma) \subsetneq F$, then $\Gamma(D_{4i}(\Gamma))$ cannot be a tiling, which is a contradiction.



(10) Let $p, h \in PSL(2,\mathbb{R})$ be a parabolic & hyperbolic element which share a common fixed point. We may assume that the common fixed point is ∞ by conjugating if necessary. Therefore, by conjugating further, we may

assume

$$p(z) = z + 1 \qquad (\text{if } p(z) = z-1, \text{ use } p^{-1})$$

and

$$h(z) = \alpha z \qquad \text{for some } 1 > \alpha > 0. \quad (\text{if } \alpha > 1, \text{ use } h^{-1})$$

Consider $\Gamma = \langle p, h \rangle$ and $\Gamma(i)$.

For all m, n integers $h^n p^m h^{-n} \in \Gamma$

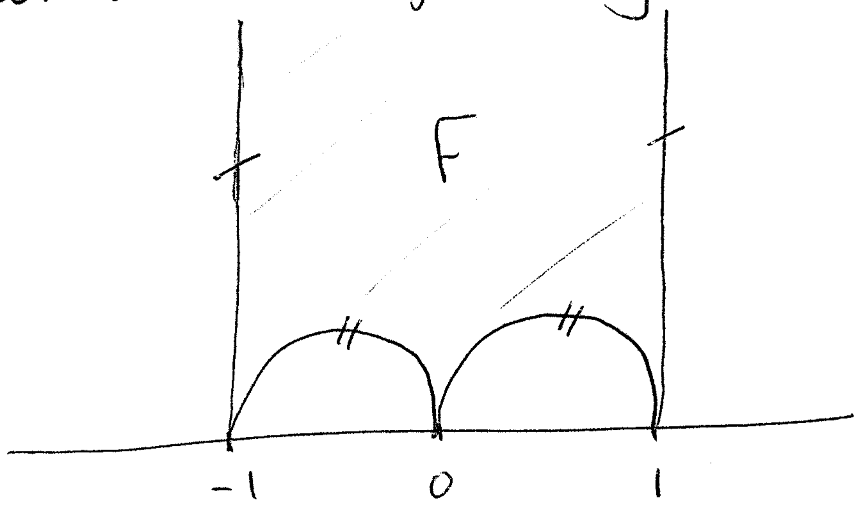
and

$$\begin{aligned} h^n p^m h^{-n}(i) &= h^n(p^m(\alpha^{-n} i)) \\ &= h^n(\alpha^{-n} i + m) \\ &= \alpha^n(\alpha^{-n} i + m) \\ &= i + \alpha^n m \end{aligned}$$

Hence $i \in \Gamma(i)$ is an accumulation point, $\Gamma(i)$ is not discrete, and so Γ is not Fuchsian

8

(11) Consider the following 4 sided polygon



sides / are paired by g and // are paired by h. Since all vertices are on the boundary, in order to apply Poincaré's Theorem we only need to check that cycle transformations are parabolic.

Consider (i) $-1 = v_1$

$$h(v_1) = 1 = v_2$$

$$g^{-1}(v_2) = -1 = v_3 = v_1 \text{ terminate}$$

therefore $-1 = g^{-1}h(-1)$

$$g^{-1}h(z) = g^{-1}\left(\frac{z}{2z+1}\right) = \frac{z}{2z+1} - 2 = \frac{3z+2}{-2z-1}$$

$\mathrm{tr}(g^{-1}h)^2 = 4$ and so $g^{-1}h$ is parabolic!

(11) continued ...

(ii) $0 = v_1$

$h(0) = 0 = v_2 = v_1$ terminate

$h(z) = \dfrac{z}{2z+1}$, so $tr(h)^2 = 4$

and h is parabolic.

(iii) $1 = v_1$

$h^{-1}(v_1) = -1 = v_2$

$g(v_2) = 1 = v_3 = v_1$ terminate

Therefore $1 = gh^{-1}(1)$

However $gh^{-1} = (hg^{-1})^{-1}$ and so is parabolic.

(iv) $\infty = v_1$

$g(v_1) = \infty = v_2 = v_1$ terminate

g is parabolic and so we are done!

(11) Continued...

Poincaré's Theorem tells us that

$\langle g, h \rangle$ is a Fuchsian group with F a fundamental domain.

Moreover, as there are no vertices in the interior, there are no cycle transformations which are elliptic.

Hence, if
$$\Gamma = \langle g, h \rangle \leq PSL(2, \mathbb{R})$$
we have the abstract presentation:

$$\Gamma \cong \langle g, h : \emptyset \rangle$$
$$\cong F_2$$

So Γ is the free group on 2 generators.