# Chapter 7

# Integration of a function of one variable: Revision

{chap:7}

## 7.1 Indefinite Integrals: Swokowski Chapter 4

{sec:7.1}

The integral of a function $f(x)$ is denoted by $\int f(x)dx$. It is called the *indefinite integral* because the limits are not specified.

What does it mean? What does it tells us?

Some people like to think of it as the area under a curve $y = f(x)$ but this is not always helpful. For example, $f(x)$ could be negative! We will come back to this later. An indefinite integral is itself a function of $x$, denoted by $F(x)$, where $F$ satisfies

$$\frac{dF}{dx} = f(x).$$

So, the indefinite integral is the function $F(x)$ whose derivative is the integrand, $f(x)$. This is why $F(x)$ is sometimes called the *anti-derivative*. Note that if you can find a function $F(x)$ satisfying $\frac{dF}{dx} = f(x)$, then $F(x) + C$, where $C$ is any constant, *also* satisfies $d(F + C)/dx = f(x)$ because $dC/dx = 0$. $F$ is, therefore, only known to within an arbitrary constant.

Sometimes, in say some physical application, we are given additional information that allows us to choose the constant $C$ uniquely.

Why is integration an art? $F(x)$ is not known for every $f(x)$ because some integrals are just too tough. Not in this course, however! On the other hand, you can take the derivative of every (reasonable) function $F(x)$ to get $f(x)$. For these $f(x)$, $F(x)$ is obviously known.

There is no systematic procedure to integrate analytically (we can always do definite integrals numerically to whatever accuracy we desire), but there are a finite number of methods to try when faced with a new integral.

## 7.2 Techniques of Integration: Swokowski Chapter 7

{sec:7.2}

Here are a few standard techniques of integration that you have met either at school or in earlier modules. They are important to know and require practice to become proficient in integration.

### 7.2.1 Integration by parts

{subsec:7.2.1}

We express

$$f(x) = u(x)\frac{dv(x)}{dx},$$

and use

:7.1}

$$\int u\frac{dv}{dx}dx = uv - \int \frac{du}{dx}vdx. \tag{7.1}$$

By suitable choices of $u$ and $v$ (usually based on experience that comes from practice), the integral on the r.h.s. can be made into a simpler integral. This is particularly effective when $u$ is an integer power of $x$ and $dv/dx$ is either a trigonometric function or an exponential (including hyperbolic) function.

.2.2}

### 7.2.2 Partial Fractions

This is useful when $f(x)$ is the ratio of two polynomials, for example,

$$f(x) = \frac{a_1x + a_0}{b_2x^2 + b_1x + b_0},$$

and when the degree of the numerator is less than the degree of the denominator, namely 1 and 2 respectively in this example. The idea is to factorise the denominator in linear factors. This can always be done *provided we allow for complex factors*. Then, we rewrite $f(x)$ as a sum of fractions where the denominators only contain polynomials of degree 1.

Let's illustrate these ideas with a couple of examples.

## Example 7.46

Integrate

$$f(x) = \frac{1}{(x-1)(x+1)}.$$

## Solution 7.46

Here we split the expression into the two fractions

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1},$$

where we need to find the values for A and $B$. The cover-up rule can be used but I prefer to use the fail-safe method of recombining the fractions with a common denominator (basically the product of all the linear, and possible quadratic, factors). Thus, we express

$$\frac{A}{x-1} + \frac{B}{x+1} = \frac{A(x+1) + B(x-1)}{(x-1)(x+1)}.$$

Comparing with the original function, we must have

$$A(x+1) + B(x-1) = 1, \qquad \text{for all } x.$$

Since this is true for all $x$, it must be true for $x = 1$ and $x = -1$. Choosing $x = 1$ we have

$$2A = 1, \qquad \Rightarrow \qquad A = \frac{1}{2}.$$

Choosing $x = -1$, we have

$$-2B = 1, \qquad \Rightarrow \qquad B = -\frac{1}{2}.$$

Thus, we have

$$\int \frac{1}{(x-1)(x+1)}dx = \int \frac{1}{2}\frac{1}{x-1} - \frac{1}{2}\frac{1}{x+1}dx = \frac{1}{2}\int \frac{1}{x-1}dx - \frac{1}{2}\int \frac{1}{x+1}dx.$$

Finally, the integrals give logarithms and so

$$\int \frac{1}{(x-1)(x+1)}dx = \frac{1}{2}\log \mid x-1 \mid - \frac{1}{2}\log |x+1| + C = \frac{1}{2}\log \frac{|x-1|}{|x+1|} + C = \log \left(\frac{|x-1|}{|x+1|}\right)^{1/2} + C$$

## Example End

## Example 7.47

Integrate

$$f(x) = \frac{1}{x^2 - 3x + 2}.$$

## Solution 7.47

First of all, we need to factorise the denominator. Thus,

$$x^2 - 3x + 2 = (x-1)(x-2).$$

If this is not obvious, then you obtain the roots of the quadratic $x^2 - 3x + 2 = 0$. Hence, the roots are $(3 \pm \sqrt{9-8})/2 = 2$ or 1. Now we use partial fractions

$$\frac{1}{x^2-3x+2} = \frac{A}{x-1} + \frac{B}{x-2} = \frac{A(x-2)+B(x-1)}{(x-1)(x-2)} \qquad \Rightarrow \qquad A(x-2) + B(x-1) = 1.$$

As this must be true for all $x$, we set $x = 1$ to get $A = -1$ and $x = 2$ to get $B = 1$. Thus,

$$\int \frac{1}{x^2-3x+2}dx = \int -\frac{1}{x-1} + \frac{1}{x-2}dx = -\log \mid x-1 \mid + \log \mid x+2 \mid + C = \log\left(\frac{\mid x-2 \mid}{\mid x-1 \mid}\right) + C.$$

## Example End

Probably, it is worth reminding you that the integral of $1/x$ is $\log \mid x \mid$. Obviously we can drop the modulus signs when $x$ is positive.

### 7.2.3 Substitution

{subsec:7.2.3}

This is one of the most effective methods and will be used in many subsequent modules. The general idea is to make a change of variable or *substitution* of the form $s = g(x)$ (or maybe $x = g(s)$), for some cleverly chosen function $g$, which reduces

$$\int f(x)dx \text{ to } \int h(s)ds,$$

where $h(s)$ is simpler to integrate. How do you know what function to choose for the substitution? Well this really comes down to knowing the standard integrals. If the integral is not a standard integral, then you need to choose a substitution to put it into a standard form. We illustrate the idea through a few examples.

**Example 7.48**

Integrate $f(x) = x/(x^2 - 1)$ with respect to $x$. Note, I am going to emphasise which variable we are integrating with respect to now. This is important when using substitutions. Thus, we wish to evaluate

$$I = \int \frac{x}{x^2 - 1} dx.$$

**Solution 7.48**

We could do this by partial fractions as above but instead we try the substitution

$$s = x^2 - 1, \quad \Rightarrow \quad \frac{ds}{dx} = 2x, \quad \Rightarrow \quad ds = 2xdx.$$

Thus, we replace $x^2 - 1$ by $s$ and $dx$ by $ds/2$ to obtain

$$\int \frac{x}{x^2 - 1} dx = \int \frac{ds/2}{s} = \frac{1}{2} \int \frac{1}{s} ds = \frac{1}{2} \log \mid s \mid + C.$$

Finally, we replace $s$ by $x^2 - 1$ to get the answer in terms of $x$. Hence,

$$\int \frac{x}{x^2 - 1} dx = \frac{1}{2} \log \mid x^2 - 1 \mid + C.$$

Note that when we did the substitution, *all* $x$'s were replaced by $s$'s. Thus the integration was done with respect to $s$. This is important. You *cannot* substitute only some of the integrand in terms of $s$ and leave some in terms of $x$. That just *does not* work.

**Example End**

**Example 7.49**

Integrate

$$\int \frac{1}{x^2 + 1} dx.$$

This looks like the last example but the simple substitution $s = x^2 + 1$ *does not* help here. Instead we need some inspiration, or better recognise that this form can always be done using a trigonometric substitution.

## Solution 7.49

First of all, we remind you of a few trigonometric identities.

$$\begin{aligned}\sin^2\phi + \cos^2\phi &= 1,\\ \text{divide both sides by } \cos^2\phi & \\ \tan^2\phi + 1 &= \sec^2\phi.\end{aligned}$$

Remember that

$$\frac{\sin\phi}{\cos\phi} = \tan\phi \text{ and } \sec\phi = \frac{1}{\cos\phi}.$$

Finally, the last reminder before completing this example involves the derivatives of trigonometric functions. These should be memorised!

$$\begin{aligned}\frac{d}{d\phi}\sin\phi &= \cos\phi,\\ \frac{d}{d\phi}\cos\phi &= -\sin\phi,\\ \frac{d}{d\phi}\tan\phi &= \sec^2\phi,\\ \frac{d}{d\phi}\sec\phi &= \sec\phi\tan\phi.\end{aligned}$$

Now we are ready to progress! Set

$$x = \tan\phi, \quad \Rightarrow \quad \frac{dx}{d\phi} = \sec^2\phi \quad \Rightarrow \quad dx = \sec^2\phi d\phi.$$

Using $\tan^2\phi + 1 = \sec^2\phi$, we have $x^2 + 1 = \sec^2\phi$. Therefore,

$$\frac{1}{x^2+1} = \frac{1}{\sec^2\phi} = \cos^2\phi.$$

Thus,

$$\int \frac{1}{x^2+1}dx = \int \cos^2\phi(\sec^2\phi d\phi) = \int d\phi = \phi + C.$$

Hence, substituting back for $x$ (using the fact that if $x = \tan\phi$, then $\phi = \tan^{-1}x$ where the power *minus one* means *INVERSE FUNCTION* and not reciprocal) we have

$$\int \frac{1}{x^2+1}dx = \tan^{-1}x + C.$$

The natural extension of this gives the standard integral

$$\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C.$$

## Example End

## Example 7.50

This example uses the same idea of trigonometric substitutions. Consider

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx.$$

## Solution 7.50

For expressions involving purely $\sqrt{a^2 - x^2}$ (and powers of this square root) always try

$$x = a \sin \phi.$$

[As an aside, note that the integrand $x/\sqrt{a^2 - x^2}$ could use the simpler substitution $u = a^2 - x^2$.]

Since,

$$\begin{aligned}
\sin^2 \phi + \cos^2 \phi &= 1, \\
a^2 \sin^2 \phi + a^2 \cos^2 \phi &= a^2, \\
x^2 + a^2 \cos^2 \phi &= a^2, \\
a^2 \cos^2 \phi &= a^2 - x^2 \quad \Rightarrow \quad \sqrt{a^2 - x^2} = a \cos \phi.
\end{aligned}$$

So the denominator simplifies to $a \cos \phi$. What about the $dx$? Thus,

$$x = a \sin \phi \quad \Rightarrow \quad \frac{dx}{d\phi} = a \cos \phi \quad \Rightarrow \quad dx = a \cos \phi d\phi.$$

Substituting these into the original integral gives,

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \int \frac{1}{a \cos \phi} a \cos \phi d\phi = \int d\phi = \phi + C.$$

Finally, substituting back in terms of $x$ gives

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left( \frac{x}{a} \right) + C.$$

## Example End

## Example 7.51

One last common trigonometric substitution involves integrals of the form

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx.$$

Note that we cannot use $x = a \sin \phi$ because $\sin \phi$ is less than or equal to one and so $x$ is less than or equal to $a$. Therefore, $x^2 - a^2$ is less than or equal to zero. Hence, we cannot take the square root. The assumption must be that $x$ is greater or equal to $a$. We need another substitution.

## Solution 7.51

Instead we make use of

$$\tan^2\phi + 1 = \sec^2\phi.$$

Rearranging this gives

$$\tan^2\phi = \sec^2\phi - 1.$$

Multiplying by $a^2$ and comparing with the denominator suggests trying

$$x = a\sec\phi \quad \Rightarrow x^2 - a^2 = a\sec^2\phi - a^2 = a^2\tan^2\phi.$$

So far so good. Now we need to replace $dx$ by the correct expression involving only $\phi$ and $d\phi$. Thus,

$$\begin{aligned} \frac{dx}{d\phi} &= a\sec\phi\tan\phi \\ dx &= a\sec\phi\tan\phi d\phi. \end{aligned}$$

Thus we have converted the original integral into

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \int \frac{1}{a\tan\phi}a\sec\phi\tan\phi d\phi = \int \sec\phi d\phi = \int \frac{d\phi}{\cos\phi}.$$

This integral may appear simpler but still requires a bit of effort to get a final answer. This illustrates that you may need to do several substitutions to get to the actual result.

To proceed, and I suspect you would never guess this, consider the two derivatives discussed above,

$$\begin{aligned} \frac{d}{d\phi}\tan\phi &= \sec^2\phi, \\ \frac{d}{d\phi}\sec\phi &= \sec\phi\tan\phi. \end{aligned}$$

Add these equations together to get

$$\begin{aligned} \frac{d}{d\phi}\left(\tan\phi + \sec\phi\right) &= \sec^2\phi + \sec\phi\tan\phi, \\ &= \sec\phi\left(\tan\phi + \sec\phi\right). \end{aligned}$$

Hence, we have the result

$$\frac{1}{v}\frac{dv}{d\phi} = \sec\phi,$$

where

$$v = \tan\phi + \sec\phi.$$

Therefore,

$$\int \sec\phi d\phi = \int \frac{1}{\cos\phi}d\phi = \int \frac{1}{v}dv = \log \mid v \mid + C = \log \mid \tan\phi + \sec\phi \mid + C.$$

This is not a result I would expect you know.

**Example End**

**Example 7.52**
Consider

$$\int \frac{x}{\sqrt{x^2 - a^2}} dx.$$

**Solution 7.52**
It is the $xdx$ that appears in the numerator that should warn you that a simple trigonometric substitution will not work. The correct substitution to use is

$$u = x^2 - a^2 \quad \Rightarrow \quad \frac{du}{dx} = 2x \quad \Rightarrow \quad \frac{1}{2}du = xdx.$$

Hence, the original integral can be re-written as

$$\int \frac{x}{\sqrt{x^2 - a^2}} dx = \frac{1}{2} \int \frac{1}{\sqrt{u}} du = \frac{1}{2} \int u^{-1/2} du.$$

Therefore,

$$\int \frac{x}{\sqrt{x^2 - a^2}} dx = u^{1/2} + C = \sqrt{x^2 - a^2} + C.$$

Note that you can always check that you have done the integration correctly by differentiating and retrieving the original integrand. Remember

$$F(x) = \int f(x)dx \qquad \Longleftrightarrow \qquad \frac{dF}{dx} = f(x).$$

**Example End**

## 7.3 Definite Integrals

:7.3}
These refer to integrals with *limits*, i.e.

:7.2}
$$\int_a^b f(x)dx. \tag{7.2}$$

Remember that the answer is a *number* if $a$ and $b$ are numbers. It is *NOT* a function of $x$. However, we can use the indefinite integral, which is a function of $x$, as an intermediate step. Let

$$F(x) = \int f(x)dx, \quad \text{or equivalently} \quad \frac{dF}{dx} = f(x),$$

Then

:7.3}
$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a). \tag{7.3}$$

Note we could think of the integral as being a function of the end points. This will be useful when the integrals are double integrals and the limits in, say $x$, involve functions of $y$. More on this later.

### Example 7.53

Consider

$$I = \int_0^{\pi/4} \tan x dx.$$

What is the value of the *number* $I$?

### Solution 7.53

First step is to find the indefinite integral.

$$F(x) = \int \tan x dx = \int \frac{\sin x}{\cos x} dx.$$

Use the substitution $u = \cos x$, so that

$$u = \cos x \quad \Rightarrow \quad \frac{du}{dx} = -\sin x \quad \Rightarrow \quad du = -\sin x dx.$$

Hence,

$$F = \int \frac{\sin x}{\cos x} dx = -\int \frac{1}{u} du = -\log \mid u \mid + C.$$

Finally, we can replace $u$ by $\cos x$ in the answer. Remember that the constant $C$ is *NOT* important when evaluating definite integrals. Thus,

$$I = \int_0^{\pi/4} \tan x dx = [-\log \mid \cos x \mid]_0^{\pi/4} = -\log \mid \cos(\pi/4) \mid + \log \mid \cos(0) \mid .$$

Hence,

$$I = -\log\left(\frac{1}{\sqrt{2}}\right) + \log 1 = \log\left(\frac{1}{\sqrt{2}}\right)^{-1} + 0 = \log \sqrt{2}$$

There are, of course, other ways of writing this answer, such as $I = (1/2)\log 2$, but the final number is $I \approx 0.346$.

### Example End

## 7.4 Linearity

{sec:7.4}

This section contains some obvious (and perhaps not so obvious) properties of integrals.

1. $$\int (f(x) + g(x))\, dx = \int f(x)dx + \int g(x)dx.$$

2. $$\int cf(x)dx = c \int f(x)dx$$

   where $c$ is a constant. This means that Integration is a *vector space.*

3.
$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$
Thus, you can always split the integration range into smaller pieces and still get the same result. In a sense, this comes from the fact the *integration is the limit of a sum.* The formal definition of an integral is
$$\int f(x)dx = \lim_{\delta x \to 0} \sum_{i=1}^{n} f(x_i)\delta x,$$
where $x_0 = a$ and $x_n = b$, which means that $n\delta x = b - a$. Thus, as $\delta x \to 0$, $n \to \infty$.

4.
$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$
Hence, switching the limits switches the sign in front of the integral (but the value of $I$ *remains* the same). From above with $b = a$, we have
$$\int_a^a f(x)dx = 0 = \int_a^c f(x)dx + \int_c^a f(x)dx, \quad \Rightarrow \quad \int_a^c f(x)dx = -\int_c^a f(x)dx,$$
as stated above.

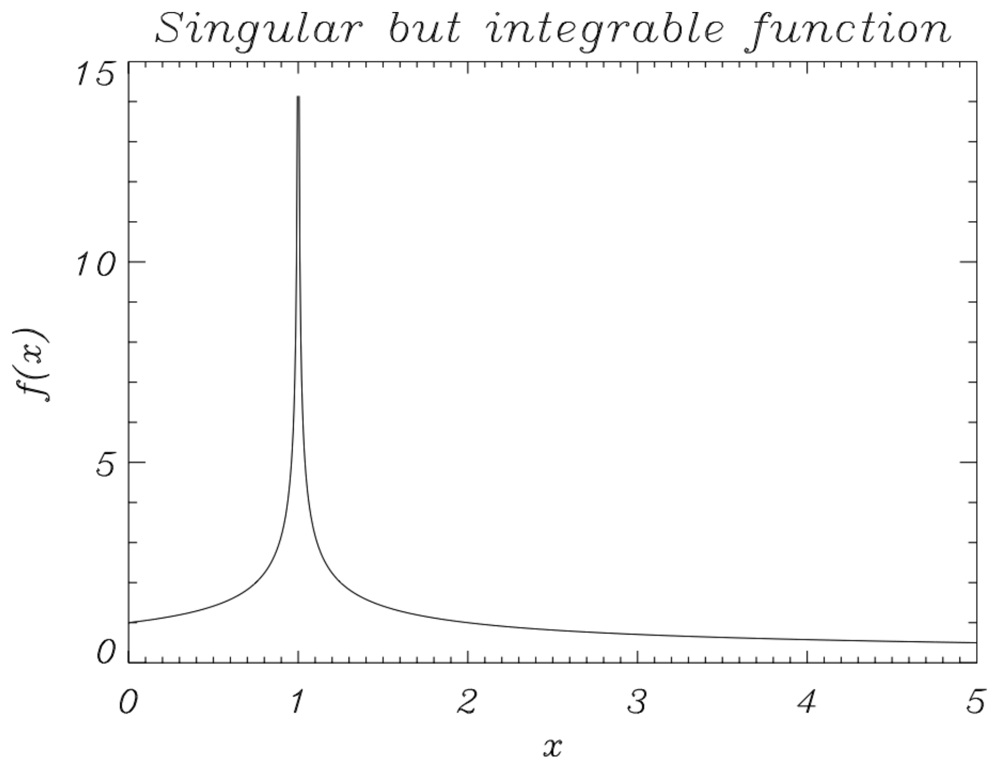
Chopping the range of integration up can be very useful. One particular instance is when $f(x) \to \infty$ at a point $c$, with $a < c < b$, but the integral still exists. It just depends on how 'singular'$f$ is at $c$. The integral exists if $\lim_{x\to c}\{(x-c)f(x)\} = 0$.

## Example 7.54

An example is
$$f(x) = \frac{1}{\sqrt{\mid x-1 \mid}}, \quad a = 0, b = 5.$$
Note that we are using the modulus of $x - 1$ instead the square root so that $\mid x - 1 \mid$ is always positive and we can always take the square root. The integrand is shown in Figure 7.1.

## Solution 7.54

Thus, we split the range of integration at $x = 1$.
$$\begin{aligned} I &= \int_0^5 \frac{1}{\sqrt{\mid x-1 \mid}} dx, \\ &= \int_0^1 \frac{1}{\sqrt{1-x}} dx + \int_1^5 \frac{1}{\sqrt{x-1}} dx \\ \text{since} \quad & \mid x-1 \mid = \begin{cases} 1-x & \text{for } x \le 1, \\ x-1 & \text{for } x \ge 1. \end{cases} \end{aligned}$$
The first integral has an indefinite integral, $F(x) = -2\sqrt{1-x}$ and so its contribution to $I$ is $F(1) - F(0) = 0 - (-2) = 2$. For the second integral, $F(x) = +2\sqrt{x-1}$ and so its contribution to $I$ is $F(5) - F(1) = 2\sqrt{4} - 0 = 4$. Hence,
$$I = 2 + 4 = 6.$$




{fig:7.1} Figure 7.1: A singular function, $f(x) = 1/\sqrt{\mid x - 1 \mid}$, but it is integrable.

**Example End**