Chapter 2

Vector Spaces

Often in mathematics, things which appear different on the surface actually have an underlying similarity. One role of algebra is to identify and clarify what this similarity is. This can be helpful in many ways: we get a clearer picture of what is going on, and save ourselves the work of repeatedly performing what is essentially the same process in different settings.

We frequently encounter sets of objects that have the property that when we "add" two of the objects together, or "multiply" one such object by a number, we are guaranteed that the resulting object is also a member of the original set. (The inverted commas are used, as we still need to define properly what these operations would mean.)

Some examples:

- Consider the (homogeneous) system of equations:

$$\begin{aligned} w + 3x + 2y - z &= 0 \\ 2w + x + 4y + 3z &= 0 \\ w + x + 2y + z &= 0. \end{aligned}$$

  Two solutions are $(2, -1, 0, -1)$ and $(2, 0, -1, 0)$. Notice that their (component-wise) sum $(4, -1, -1, -1)$ is also a solution, as is any constant multiple of the original solutions, e.g. $(8, -4, 0, -4)$ $(= 4(2, -1, 0, -1))$. In fact, we may verify that the sum of any two solutions of this system is again a solution, and so is any constant multiple of a solution.

- Recall that a magic square is a square array of numbers in which each row, each column and both diagonals add up to the same number. Consider the set of 3×3 magic squares; two members of the set are

$$\begin{pmatrix} 2 & 7 & 6 \\ 9 & 5 & 1 \\ 4 & 3 & 8 \end{pmatrix} \text{ and } \begin{pmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{pmatrix}$$

  We can verify that multiplying each entry in a 3×3 magic square by the same number yields another magic square (zero is allowed if we are prepared to accept the "trival" all-zero magic square), as does adding two 3×3 magic squares component-wise.

Definition 2.1 Let F be a field. Suppose we are given a set S on which it is possible to define

- a *sum* $s_1 + s_2$ of any two elements $s_1, s_2 \in S$, and
- a *scalar product* λs of any element $s \in S$ with any scalar $\lambda \in F$.

We say that S is *closed under addition* if $s_1 + s_2$ is always in S, and *closed under multiplication by scalars* if λs is always in S.

So the sets of solutions to our system of equations, and the set of 3×3 magic squares, are each closed under addition and scalar multiplication. In fact, we can find many other such examples.

Example 2.2 *Consider the set of all n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers (may be written as either a row or a column). This set is denoted $\mathbb{R}^n$, and may be identified with our usual n-dimensional space by associating each point in that space with its n-tuple of coordinates. We define addition component-wise:*

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

and multiplication by a scalar $c \in \mathbb{R}$ by

$$c(x_1, x_2, \ldots, x_n) = (cx_1, cx_2, \ldots, cx_n).$$

We can see that $\mathbb{R}^n$ is closed under addition and scalar multiplication.

Note that $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$ may be represented by the set of points in the $x - y$ plane, while $\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$ may be viewed as the set of points in three-dimensional space, and so on.

We give an example relating to systems of equations.

Example 2.3 *Let P be the set of all triples (x, y, z) in $\mathbb{R}^3$ such that $x+2y+3z = 0$. Show that P is closed under addition and scalar multiplication.*

Solution: Let $(a, b, c), (d, e, f) \in P$. So $a + 2b + 3c = 0$ and $d + 2e + 3f = 0$. Then $(a, b, c) + (d, e, f) = (a + d, b + e, c + f)$ and

$$(a + d) + 2(b + e) + 3(c + f) = (a + 2b + 3c) + (d + 2e + 3f) = 0 + 0 = 0.$$

So their sum lies in P. Similarly, let $\lambda \in R$. Then $\lambda(a, b, c) = (\lambda a, \lambda b, \lambda c)$ and

$$(\lambda a + 2(\lambda b) + 3(\lambda c)) = \lambda(a + 2b + 3c) = \lambda \cdot 0 = 0.$$

So the scalar product lies in P. So P is closed under both operations.

Matrices and polynomials are key examples.

Example 2.4 *Consider the set $M_{n\times n}(F)$ of $n \times n$ matrices with entries from a field F (as always, $F = \mathbb{R}$ or $\mathbb{C}$ in this course). Using the usual component-wise addition and scalar multiplication of matrices, if $A = [a_{ij}]$, $B = [b_{ij}]$ and $\lambda \in F$, then we have*

$$A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}] \text{ and } \lambda A = \lambda[a_{ij}] = [\lambda a_{ij}],$$

which are both matrices in $M_{n\times n}(F)$. Hence $M_{n\times n}(F)$ is closed under addition and scalar multiplication.

Example 2.5 *Consider the set $\mathcal{P}_n$ of polynomials*

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

of degree at most n with coefficients from $\mathbb{R}$. We take the usual addition (add corresponding coefficients) and scalar multiplication (multiply each coefficient by the scalar). Let $f(x), g(x) \in \mathcal{P}_n$, say

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \quad \text{and} \quad g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n,$$

where all $a_i, b_i \in \mathbb{R}$. Then

$$f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots + (a_n + b_n)x^n \in \mathcal{P}_n$$

and, if $\alpha \in \mathbb{R}$,

$$\alpha f(x) = (\alpha a_0) + (\alpha a_1)x + (\alpha a_2)x^2 + \cdots + (\alpha a_n)x^n \in \mathcal{P}_n.$$

So $\mathcal{P}_n$ is closed under addition and scalar multiplication.

We also give a non-example:

Example 2.6 *Let $\mathcal{P}_3$ be the set of polynomials of degree at most 3 with coefficients from $\mathbb{R}$, and let T be the subset of all polynomials of degree exactly 3. Show that T is not closed under either operation.*

Solution: The set T contains the polynomials x^3 and $1 + x^2 - x^3$, but

$$x^3 + (1 + x^2 - x^3) = 1 + x^2 \notin T.$$

Also, $x^3 \in T$ but $0 \cdot x^3 = 0 \notin T$.

Example 2.7 *Consider the set $\mathcal{F}_{\mathbb{R}}$ of all real-valued functions of a real variable $f\colon \mathbb{R} \to \mathbb{R}$, with addition and scalar multiplication given by*

$$(f + g)(x) = f(x) + g(x) \qquad (\alpha f)(x) = \alpha f(x).$$

This set is clearly closed under addition and scalar multiplication.

From the above examples, we can see that there are many occasions in mathematics when we find ourselves working with sets of objects which are closed under two operations describable as "addition" and "scalar multiplication". This provides motivation to treat all these sets as examples of some more general type of structure. Our next step is to identify a list of basic properties which all these examples possess. We shall give our "general structure" the name *vector space*, the individual elements in it the name *vectors*, and the list of properties will be called the *vector space axioms*.

Definition 2.8 Let F be a field of scalars (typically $\mathbb{R}$ and $\mathbb{C}$ for this course). A *vector space* over F is a non-empty set V together with two operations

$$\begin{aligned} V \times V &\to V \\ (u, v) &\mapsto u + v \end{aligned} \qquad\qquad \begin{aligned} F \times V &\to V \\ (\alpha, v) &\mapsto \alpha v, \end{aligned}$$

called *addition* and *scalar multiplication*, respectively, such that

V1 $u + v = v + u$ for all $u, v \in V$;

V2 $(u + v) + w = u + (v + w)$ for all $u, v, w \in V$;

V3 there exists a vector $\mathbf{0}$ in V such that $v + \mathbf{0} = v$ for all $v \in V$;

V4 for each $v \in V$, there exists a vector $-v$ in V such that $v + (-v) = \mathbf{0}$;

V5 $\alpha(u + v) = \alpha u + \alpha v$ for all $u, v \in V$ and all scalars α;

V6 $(\alpha + \beta)v = \alpha v + \beta v$ for all $v \in V$ and all scalars α, β;

V7 $(\alpha\beta)v = \alpha(\beta v)$ for all $v \in V$ and all scalars α, β;

V8 $1v = v$ for all $v \in V$.

It is important to distinguish between the zero vector $\mathbf{0}$ in the vector space V and the zero scalar in our field F. We shall use bold-face notation in these lecture notes (and an underlined zero on the white-board) to denote the zero vector.

The examples above satisfy the axioms. We give the proof for the set of $n \times n$ matrices, and leave the details of the others as exercises.

Example 2.9 *Consider the set $M_{m\times n}(F)$ of $m \times n$ matrices with entries from field F. We have seen that this set is closed under addition and scalar multiplication. We now show that it satisfies the vector space axioms.*

(i) *Let $A, B \in M_{m\times n}(F)$. If $A = [a_{ij}]$ and $B = [b_{ij}]$, then $A + B = [a_{ij} + b_{ij}] = [b_{ij} + a_{ij}] = B + A$.*

(ii) *Let $A, B, C \in M_{m\times n}(F)$. If $A = [a_{ij}], B = [b_{ij}]$ and $C = [c_{ij}]$, then $(A + B) + C = [(a_{ij} + b_{ij}) + c_{ij}] = [a_{ij} + (b_{ij} + c_{ij})] = A + (B + C)$.*

(iii) *Let $\mathbf{0}$ denote the matrix all of whose entries are zero. Then $A+\mathbf{0} = [a_{ij}+0] = [a_{ij}] = A$ for all $A \in M_{m\times n}(F)$.*

(iv) *Given $A = [a_{ij}]$, take $-A = [-a_{ij}]$, i.e. the matrix whose entries are the negatives of those in A. Then $A + (-A) = [a_{ij} + (-a_{ij})] = \mathbf{0}$, for all $A \in M_{m\times n}(F)$.*

(v) *Let $A, B \in M_{m\times n}(F)$. If $A = [a_{ij}]$ and $B = [b_{ij}]$, then*

$$\begin{aligned}\alpha(A + B) = \alpha([a_{ij}] + [b_{ij}]) = \alpha[a_{ij} + b_{ij}] &= [\alpha(a_{ij} + b_{ij})]\\ &= [\alpha a_{ij}] + [\alpha b_{ij}] = \alpha[a_{ij}] + \alpha[b_{ij}] = \alpha A + \alpha B.\end{aligned}$$

(vi) *Let $\alpha, \beta \in F$ and $A = [a_{ij}] \in M_{m\times n}(F)$. Then $(\alpha + \beta)A = (\alpha + \beta)[a_{ij}] = [(\alpha + \beta)a_{ij}] = [\alpha a_{ij} + \beta a_{ij}] = [\alpha a_{ij}] + [\beta a_{ij}] = \alpha A + \beta A$*

(vii) *Equally $(\alpha\beta)A = \alpha\beta[a_{ij}] = \alpha[\beta a_{ij}] = \alpha(\beta A)$ for all $\alpha, \beta \in F$ and $A \in M_{m\times n}(F)$.*

(viii) *Finally $1A = 1[a_{ij}] = [1a_{ij}] = [a_{ij}] = A$ for all $A \in M_{m\times n}(F)$.*

Hence the space of $m \times n$ matrices forms a vector space over F.

We give the zero vector and negatives for some of the other examples; it is left as an exercise to check that the vector space axioms hold in each case.

- The set $\mathbb{R}^n$ of Example 2.2 is a vector space over $\mathbb{R}$. The zero vector is

$$\mathbf{0} = (0, 0, \ldots, 0)$$

  and negatives are given by

$$-(x_1, x_2, \ldots, x_n) = (-x_1, -x_2, \ldots, -x_n).$$

- The set $\mathcal{P}_n$ of Example 2.5 is a vector space over $\mathbb{R}$. The zero vector is

$$\mathbf{0} = 0 + 0x + 0x^2 + \cdots + 0x^n$$

  and the negative of $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ is given by

$$-f(x) = (-a_0) + (-a_1)x + (-a_2)x^2 + \cdots + (-a_n)x^n.$$

- The set $\mathcal{F}_{\mathbb{R}}$ of Example 2.7 is a vector space over $\mathbb{R}$. The zero vector is the function f_0 given by

$$f_0(x) = 0 \text{ for all } x \in \mathbb{R}$$

  and the negative of f is $-f$ given by

$$(-f)(x) = -f(x) \text{ for all } x \in \mathbb{R}$$

We end with two other examples of vector spaces, which we will return to later.

Example 2.10 *Consider the set P of all triples $(x, y, z) \in \mathbb{R}^3$ such that $x+2y+3z = 0$. We have already checked that P is closed under addition and scalar multiplication, and we can verify that it satisfies the vector space axioms, so it is a vector space over $\mathbb{R}$. It also has a geometric meaning: it is a plane in three-dimensional space which passes through the origin.*

We will see later that there is a quicker way of verifying that P is a vector space, which does not require us to check all the axioms.

Example 2.11 *The field of complex numbers $\mathbb{C}$ is a vector space over the field $\mathbb{R}$.*

This example may seem slightly confusing at first sight, as we are more used to seeing $\mathbb{C}$ play the role of the field, not the set of vectors! However, if you think of every complex number $z = a + bi$ as corresponding to the ordered pair (a, b) of real numbers, it is easier to see how $\mathbb{C}$ is a vector space over $\mathbb{R}$.

Basic properties of vector spaces

We now give a set of consequences which follow from the vector space axioms. These may seem "obvious" intuitively, but this is mainly due to the fact that they hold in settings we are familiar with. In our general set-up, these require proof.

Proposition 2.12 *Let V be a vector space over a field F. Let $v \in V$ and $\alpha \in F$. Then*

(i) $\alpha\mathbf{0} = \mathbf{0}$;

(ii) $0v = \mathbf{0}$;

(iii) *if* $\alpha v = \mathbf{0}$, *then either* $\alpha = 0$ *or* $v = \mathbf{0}$;

(iv) $(-\alpha)v = -\alpha v = \alpha(-v)$.

PROOF: (i) Let $w = \alpha\mathbf{0}$. Use V3 and V5 to get:

$$w = \alpha\mathbf{0} = \alpha(\mathbf{0} + \mathbf{0}) = \alpha\mathbf{0} + \alpha\mathbf{0} = w + w.$$

Now add $-w$ to both sides to yield

$$w + (-w) = (w + w) + (-w)$$

i.e.

$$\mathbf{0} = w + (w + (-w)) = w + \mathbf{0} = w$$

(using V2 and V4).

(ii) Use V6 to give

$$0v = (0 + 0)v = 0v + 0v$$

and then add $-0v$ just as in part (i) to give $\mathbf{0} = 0v$.

(iii) Suppose that $\alpha v = \mathbf{0}$, but that $\alpha \neq 0$. Then we can multiply by $1/\alpha$ to give

$$\tfrac{1}{\alpha}(\alpha v) = \tfrac{1}{\alpha}\mathbf{0} = \mathbf{0} \qquad \text{(by part (i)).}$$

Now use conditions V7 and V8 to give

$$v = 1v = \left(\tfrac{1}{\alpha} \cdot \alpha\right) v = \tfrac{1}{\alpha}(\alpha v) = \mathbf{0}.$$

Hence if $\alpha v = \mathbf{0}$, either $\alpha = 0$ or $v = \mathbf{0}$.

(iv)

$$\alpha v + (-\alpha)v = (\alpha + (-\alpha))\, v = 0v = \mathbf{0},$$

so if we add $-\alpha v$ to both sides so as to cancel the first term on the left, we deduce

$$(-\alpha)v = -\alpha v.$$

Similarly,

$$\alpha v + \alpha(-v) = \alpha(v + (-v)) = \alpha\mathbf{0} = \mathbf{0}$$

and again adding $-\alpha v$, we deduce

$$\alpha(-v) = -\alpha v.$$

$\square$

Subspaces

Another natural concept in mathematics is the idea of a "substructure"; a (usually smaller) part of our big structure, which itself operates as a structure of the same type. Consider Example 2.10: the set of solutions in the real vector space $\mathbb{R}^3$, to the equation $2x - 3y + 11z = 0$, forms a vector space itself over $\mathbb{R}$. Here, the big vector space can be thought of as three-dimensional space, and the smaller vector space is a plane lying inside this space.

Definition 2.13 Let V be a vector space over a field F. A *subspace* W of V is a non-empty subset of V which itself forms a vector space under the same operations.

Hence, a subspace must obey all the axioms of its "parent" space.

To check whether a set forms a subspace, we do not actually need to check every axiom. Most of them are immediately inherited from the fact that they hold in the parent space. Actually it is sufficient to ensure the closure of the operations within W.

Theorem 2.14 *Let V be a vector space over a field F. Let W be a subset of V which is* non-empty. *Then W is a subspace of V if and only if*

- *if $v, w \in W$, then $v + w \in W$;*
- *if $v \in W$ and α is any scalar, then $\alpha v \in W$.*

PROOF: ($\Rightarrow$) If W is a subspace of V, then the closure conditions must hold since W is a vector space in its own right.
($\Leftarrow$) Suppose the above conditions hold. We claim that all the other vector space axioms also hold. Most of these hold simply because they are inherited from the parent space V. For example, [V1]: since [V1] holds in V, then $u + v = v + u$ for all $u, v \in V$, and since $W \subseteq V$, then [V1] clearly holds in W. Similar reasoning holds in all other cases except [V3] and [V4]. Take $w \in W$ (which we can do since W is non-empty). Since W is closed under scalar multiplication, $(-1)w = -w \in W$; since W is closed under addition, $w + (-w) = 0 \in W$ (where 0 is the zero element of V). So V3 and V4 hold in W, and hence W is a vector space, i.e. a subspace of V. $\square$

We give some examples, and non-examples, of subspaces. Note that checking the "non-empty" condition is conceptually important; it is often straightforward in our examples, but can be non-trivial if a subset is defined in a less direct way (e.g. as the intersection of two other subsets, or in terms of a property possessed by its elements).

Example 2.15 *In any vector space V, $\{\mathbf{0}\}$ is always a subspace, as is V itself.*

Example 2.3 is a subspace of $\mathbb{R}^3$. We have checked it is closed under the operations, and is clearly non-empty (it contains $(0, 0, 0)$).

Example 2.16 *Let F be a field and $V = M_{2\times 2}(F)$ be the vector space of all 2×2 matrices with entries from F. Then the subset W of all 2×2 diagonal matrices is a subspace of V.*

Solution: Here
$$W = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \;\middle|\; a, b \in F \right\} \subseteq M_{2\times 2}(F).$$

This set is non-empty since, for example, it contains the zero matrix (take $a = b = 0$). Now let $A, B \in W$, say

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}.$$

Then
$$A + B = \begin{pmatrix} a+c & 0 \\ 0 & b+d \end{pmatrix} \in W$$

and, if α is a scalar from F, then

$$\alpha A = \begin{pmatrix} \alpha a & 0 \\ 0 & \alpha b \end{pmatrix} \in W.$$

Hence W is closed under addition and scalar multiplication, so we conclude W is a subspace of $V = M_{2\times 2}(F)$.

Example 2.17 *Consider the subset W of $\mathbb{R}^3$ given by:*

$$W = \{(x, y, z) : 2x + 4y - 3z = 0\}.$$

Geometrically, W is a plane in $\mathbb{R}^3$. We claim that W is a subspace of $\mathbb{R}^3$.

Solution: We need to show that W is non-empty and satisfies the above two conditions (i.e., is closed under addition and scalar multiplication).

Take $x = y = z = 0$, then $2.0 + 4.0 - 3.0 = 0$ and so the zero vector $(0, 0, 0)$ is in W. Hence W is non-empty.

Now, let $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ be vectors in W. By definition of W, we must have that

$$2v_1 + 4v_2 - 3v_3 = 0 \text{ and } 2w_1 + 4w_2 - 3w_3 = 0. \tag{2.1}$$

To check closure under addition, consider $v + w = (v_1 + w_1, v_2 + w_2, v_3 + w_3)$. Adding the above equations from (2.1) shows that

$$2(v_1 + w_1) + 4(v_2 + w_2) - 3(v_3 + w_3) = 0 + 0 = 0,$$

and so $v + w \in W$. To check closure under scalar multiplication, let $\alpha \in \mathbb{R}$. Then $\alpha v = (\alpha v_1, \alpha v_2, \alpha v_3)$, and from the first equation in (2.1) we see

$$2\alpha v_1 + 4\alpha v_2 - 3\alpha v_3 = \alpha(2v_1 + 4v_2 - 3v_3) = 0,$$

so $\alpha v \in W$. Hence W is a subspace of $\mathbb{R}^3$.

You might wonder whether all non-empty subsets of a vector space are actually subspaces, but it is easy to find examples of subsets which are not closed under addition or are not closed under scalar multiplication.

Example 2.18 *Consider the subset of* $\mathbb{R}^3$ *given by*

$$S = \{(x, y, z) : 2x + 4y - 3z = 1\}$$

Geometrically, this is a plane in $\mathbb{R}^3$*. Show that* S *is not a subspace of* $\mathbb{R}^3$*.*

Solution: We show that S is not closed under addition. It suffices to exhibit a counterexample. The vectors $v = (0, 1, 1)$ and $w = (-1, 0, -1)$ belong to S, since $2.0 + 4.1 - 3.1 = 1 = 2(-1) + 4.0 - 3(-1)$. But their sum $v + w = (-1, 1, 0)$ does not belong to S, since $2(-1) + 4.1 - 3.0 = 2 \neq 1$. Hence S is not a subspace of $\mathbb{R}^3$.

In fact, it may be shown that the only subspaces of $\mathbb{R}^3$ are $\mathbb{R}^3$ itself, $\{\mathbf{0}\}$, lines through the origin and planes through the origin.

The split between the homogeneous and non-homogeneous examples is no coincidence — mapping to zero is important! The following subspace will play an important role later in the course.

Example 2.19 *Let* $A \in M_{m\times n}(\mathbb{R})$*. Let* S *be the subset of* $\mathbb{R}^n$ *given by*

$$S := \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} = \mathbf{0}\}.$$

We claim that S *is a subspace of* $\mathbb{R}^n$

Solution: S is non-empty since $A\mathbf{0} = \mathbf{0}$ and so $\mathbf{0} \in S$. For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, we have $A\boldsymbol{x} = \mathbf{0}$ and $A\boldsymbol{y} = \mathbf{0}$ by definition. Then $A(\boldsymbol{x} + \boldsymbol{y}) = A\boldsymbol{x} + A\boldsymbol{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}$, so $\boldsymbol{x} + \boldsymbol{y} \in S$. Let $\alpha \in \mathbb{R}$; then $A(\alpha\boldsymbol{x}) = \alpha(A\boldsymbol{x}) = \alpha\mathbf{0} = \mathbf{0}$. Hence S is a subspace.

Some combinations of subspaces create new subspaces, some do not.

Theorem 2.20 *Let* S *and* T *be subspaces of a vector space* V*. Then*

- *their intersection* $S \cap T := \{x : x \in S \text{ and } x \in T\}$ *is a subspace of* V*;*
- *their union* $S \cup T = \{x : x \in S \text{ or } x \in T\}$ *is not necessarily a subspace of* V*.*

PROOF: Exercise. □