MT5823 Semigroup theory: Solutions 5 (James D. Mitchell)
Bicyclic monoid, ideals, Green's relations

Bicyclic monoid

5-1. Let c^ib^j, $i \geq j$ be an idempotent in B. Then $(c^ib^j)^2 = c^ib^j$ and so $c^ib^jc^ib^j = c^ib^j$. Repeatedly applying $bc = 1$ to the left hand side of the last equality, we obtain $c^{2i-j}b^j = c^ic^{i-j}b^j = c^ib^j$. Hence $2i - j = i$ and so $i = j$. A similar argument proves that $i = j$ in the case that c^ib^j, $i \leq j$, is an idempotent.

Conversely, $(c^ib^i)^2 = c^ib^ic^ib^i = c^ib^i$ (by applying $bc = 1$ a total of i times). Hence every element c^ib^i is an idempotent.

Let $c^ib^i, c^jb^j \in E$. Then

$$c^ib^ic^jb^j = \begin{cases} c^ib^{i-j}b^j = c^ib^i & i \geq j \\ c^ic^{j-i}b^j = c^jb^j & j \geq i \end{cases}.$$

Thus E is closed and so a subsemigroup of B. If $X \subseteq E$ and $\langle X \rangle = E$, then from the above equation $X = E$. Hence E is not finitely generated (although B is finitely generated). □

5-2. Let $c^{4i+5}b^{4j+5}, c^{4k+5}b^{4l+5} \in S_1$. Then

$$c^{4i+5}b^{4j+5}c^{4k+5}b^{4l+5} = \begin{cases} c^{4i+5}b^{4j-4k}b^{4l+5} & j \geq k \\ c^{4i+5}c^{4k-4j}b^{4l+5} & k > j \end{cases} = \begin{cases} c^{4i+5}b^{4j-4k+4l+5} & j \geq k \\ c^{4i+4k-4j-5}b^{4l+5} & k > j. \end{cases}$$

Thus S_1 is closed and hence a subsemigroup of B.

A similar argument proves that S_2 is a subsemigroup of B also.

The mappings $\phi_1 : B \longrightarrow S_1$ and $\phi_2 : B \longrightarrow S_2$ defined by

$$(c^ib^j)\phi_1 = c^{4i+5}b^{4j+5} \quad i, j \geq 0$$

$$(c^ib^j)\phi_2 = c^{4i+7}b^{4j+7} \quad i, j \geq 0$$

are isomorphisms.

To prove that $S = S_1 \cup S_2$ is a subsemigroup it suffices to show that if $x \in S_1$ and $y \in S_2$, then $xy, yx \in S$. Let $x = c^{4i+5}b^{4j+5}$ and $y = c^{4k+7}b^{4l+7}$ where $k \geq j$. Then $xy = c^{4(i+k-j)+7}b^{4l+7} \in S_2 \subseteq S$. Analogous arguments prove that $xy \in S$ when $k < j$ and $yx \in S$.

Since B is finitely generated (by b and c), $B \cong S_1$, and $B \cong S_2$, it follows that S_1 and S_2 are finitely generated also, and so too is $S_1 \cup S_2 = S$. A finite generating set for S_1 is $\{c^5b^9, c^9b^5\} = \{b, c\}\phi_1$ and for S_2 is $\{c^7b^{11}, c^{11}b^7\} = \{b, c\}\phi_2$. □

Ideals

5-3. Let I be a left ideal and J be a right ideal. If $i \in I$ and $j \in J$, then $ji \in I$ and $ji \in J$. Hence $IJ \subseteq I \cap J$ and in particular, $I \cap J$ is nonempty.

Let S be a right zero semigroup. Then for $x, y \in S$, $x \neq y$, both $\{x\}$ and $\{y\}$ are left ideals, but $\{x\} \cap \{y\} = \varnothing$. □

5-4. Let S be a rectangular band and let $I \subseteq S$ be any 2-sided ideal. Then for any $y \in I$ and $x \in S$ we have $xyx \in I$. But $xyx = x^2 = x$ and so $I = S$.

Consider $S = I \times \Lambda = \{ (i, \lambda) : i \in I, \lambda \in \Lambda \}$. Each set $L_i = \{ (i, \lambda) : \lambda \in \Lambda \}$ is a right ideal and each set $R_\lambda = \{ (i, \lambda) : i \in I \}$ is a left ideal. □

Green's relations

5-5. We have that $\mathrm{im}(f) = \{3, 4\}$ and $\mathrm{im}(g) = \mathrm{im}(h) = \{2, 3\}$. Thus, by Theorem 9.4, $(g, h) \notin \mathscr{L}$ in T_4 but $(f, g) \notin \mathscr{L}$ and $(f, h) \notin \mathscr{L}$.

On the other hand,

$$\ker(f) = \{\{1, 4\}, \{2, 3\}\}, \ker(g) = \{\{1, 4\}, \{2, 3\}\}, \text{ and } \ker(h) = \{\{1, 2, 3\}, \{4\}\}.$$

Again it follows by Theorem 9.4 that $(f, g) \in \mathscr{R}$ but $(f, h) \notin \mathscr{R}$ and $(g, h) \notin \mathscr{R}$.

Now,

$$fh = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 2 & 3 \end{pmatrix} \text{ and } gh = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 3 \end{pmatrix}.$$

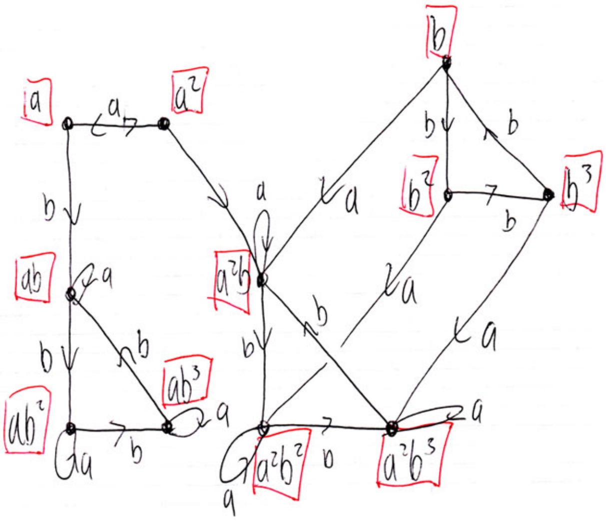

Figure 1: The right Cayley graph of S.

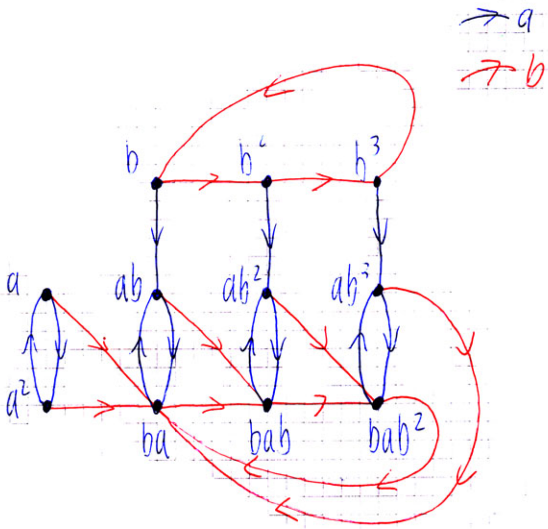

Figure 2: The left Cayley graph of S.

Thus $\ker(fh) = \{\{1,4\},\{2,3\}\}$ and $\ker(gh) = \{\{1,2,3\},\{4\}\}$. Hence $(fh, gh) \notin \mathscr{R}$ and so $\mathscr{R}$ is not right congruence. $\square$

5-6. The right Cayley graph of S is shown in Figure 1. From Theorem **??**, the strongly connected components of the right Cayley graph correspond to the $\mathscr{R}$-classes of S. From Figure 1 it is straightforward to deduce that the strongly connected components are:

$$\{b, b^2, b^3\}, \{a^2b, a^2b^2, a^2b^3\}, \{a, a^2\}, \{ab, ab^2, ab^3\}.$$

The left Cayley graph of S is shown in Figure 2. From Theorem **??**, the strongly connected components of the left Cayley graph correspond to the $\mathscr{L}$-classes of S. From Figure 2 it is straightforward to deduce that the strongly connected components are:

$$\{b, b^2, b^3\}, \{a, a^2\}, \{ab, ab^2, ab^3, a^2b, a^2b^2, a^2b^3\}.$$

$\square$

5-7. If $i = k$, then $c^i = c^k$ and so $c^i \mathscr{R} c^k$.

If $c^i \mathscr{R} c^k$, then there exist $c^x b^y, c^z b^t \in B$ $(x, y, z, t \geq 0)$ such that $c^i c^x b^y = c^k$ and $c^k b^z b^t = c^i$. Thus $i + x = k$ and $k + z = i$. It follows that $x = z = 0$ and so $i = k$.

We have that $c^i \cdot b^j = c^ib^j$ and $c^ib^j \cdot c^j = c^i$. Hence $c^ib^j \mathscr{R} c^i$.

Finally, $c^ib^j \mathscr{R} c^kb^l$ if and only if $c^i \mathscr{R} c^ib^j \mathscr{R} c^kb^l \mathscr{R} c^k$ if and only if $c^i \mathscr{R} c^k$ if and only if $i = k$.

The analogous criterion for two elements of B to be $\mathscr{L}$-related is $c^ib^j \mathscr{L} c^kb^l$ if and only if $j = l$. □

5-8. Let $e^2 = e \in S$ and $x \in R_e$ (the $\mathscr{R}$-class of e). Then there exist $u, v \in S^1$ such that $eu = x$ and $xv = e$. Thus $ex = eeu = e^2u = eu = x$. Hence e is a left identity of R_e.

Let $x \in L_e$ (the $\mathscr{L}$-class of e). Then there exist $u, v \in S^1$ such that $ux = e$ and $ve = x$. Thus $xe = vee = ve^2 = ve = x$. □

5-9. Using the algorithm from lectures we find that the elements of S are:

$$x, y, x^2, xy, yx, y^2, x^3, x^2y, xy^2, x^3y, x^2y^2, x^3y^2$$

(12 elements in total). By drawing the left and right Cayley graphs of S we find that the $\mathscr{R}$- classes of S are:

$$\{yx\}, \{x^3y, x^3y^2\}, \{x^2y, x^2y^2\}, \{xy, xy^2\}, \{x, x^2, x^3\}, \{y, y^2\}$$

the $\mathscr{L}$- classes of S are:

$$\{yx\}, \{x, x^2, x^3\}, \{xy, x^2y, x^3, y\}, \{xy^2, x^2y^2, x^3y^2\}, \{y, y^2\}$$

the only $\mathscr{H}$- classes of S with more than one element are:

$$\{x, x^2, x^3\}, \{y, y^2\}.$$

Taking the composition of the $\mathscr{L}$- and $\mathscr{R}$- relations, we obtain Green's $\mathscr{D}$- relation:

$$\{yx\}, \{x, x^2, x^3\}, \{y, y^2\}, \{xy, x^2y, x^3, y, xy^2, x^2y^2, x^3y^2\}.$$

Since S is finite $\mathscr{J} = \mathscr{D}$. □

5-10. In Problem **5-7** we proved that

$$c^ib^j \mathscr{R} c^kb^l \text{if and only if} i = k \tag{1}$$

$$c^ib^j \mathscr{L} c^kb^l \text{if and only if} j = l. \tag{2}$$

Hence $c^ib^j \mathscr{H} c^kb^l$ if and only if $i = k$ and $j = l$ if and only if $c^ib^j = c^kb^l$. It follows that $\mathscr{H} = \Delta_B$.

On the other hand, if $c^ib^j, c^kb^l \in B$ are arbitrary, then

$$c^ib^j \mathscr{R} c^ib^l \mathscr{L} c^kb^l$$

from (1) and (2). Thus $\mathscr{D} = B \times B$.

Finally, $(c^kb^i)c^ib^j(c^jb^l) = c^kb^l$ and $(c^ib^k)c^kb^l(c^lb^j) = c^ib^j$. Thus $c^ib^j \mathscr{J} c^kb^l$ and $\mathscr{J} = B \times B$. (Note that B is infinite.) □

5-11. Let S be a semigroup and suppose that S is defined by a presentation $\langle A|R \rangle$ where $|A| > |R|$. Let I and J be index sets such that $|I| = |A|$ and $|J| = |R|$, and write $A = \{\, a_i : i \in I \,\}$ and $R = \{\, (u_j, v_j) \in A^+ \times A^+ : j \in J \,\}$. We define a $|R| \times |A|$ matrix $Q = (q_{j,i})_{j \in J, i \in I}$ where $q_{j,i}$ is the number of times a_i occurs in u_j minus the number of times a_i occurs in v_j. For example, if S is the semigroup defined by the presentation

$$\langle\, a_1, a_2, a_3 \mid a_1a_2a_1 = a_2a_3, a_3a_1 = a_2 \,\rangle,$$

then the matrix is

$$\begin{pmatrix} 2 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

Since Q is a matrix with entries in $\mathbb{Q}$, it is the matrix of a linear transformation $\mathbf{q} : \mathbb{Q}^{|A|} \longrightarrow \mathbb{Q}^{|R|}$ with respect to any basis for $\mathbb{Q}^{|A|}$. Hence, by the Rank-Nullity Theorem,

$$\dim(\mathbb{Q}^{|A|}) = |A| = \dim \ker(\mathbf{q}) + \dim \operatorname{im}(\mathbf{q}).$$

Clearly, $\dim \operatorname{im}(\mathbf{q}) < \dim(\mathbb{Q}^{|R|}) = |R|$ and, since $|R| < |A|$, it follows that $\dim \ker(\mathbf{q}) = |A| - \dim \operatorname{im}(\mathbf{q}) \geq |A| - |R| > 0$.

Suppose that $\vec{x} \in \ker(\mathbf{q}) \setminus \{\vec{0}\}$ and that the entries of (the column vector) $\vec{x}$ are $x_1, x_2, \ldots, x_{|I|}$. We define $f : A \longrightarrow \mathbb{Q}$ by $(a_i)f = x_i$ for all $i \in I$. If $(b_1 \cdots b_k, c_1 \cdots c_l) \in R$, then since $\vec{x} \in \ker(\mathbf{q})$ it follows that

$$(b_1)f + \cdots + (b_k)f - ((c_1)f + \cdots + (c_l)f) = 0$$

and so

$$(b_1)f + \cdots + (b_k)f = (c_1)f + \cdots + (c_l)f.$$

In other words, the subsemigroup U of the additive semigroup $\mathbb{Q}$ generated by $x_1, \ldots, x_{|I|}$ satisfies the relations R defining S, and so U is a homomorphic image of S by Theorem 6.4. But at least one of $x_1, \ldots, x_{|I|}$ is non-zero, and so U is infinite. A finite semigroup cannot have an infinite homomorphic image and so S must be infinite too.