Theorem 0.0.1 *There exist group epimorphisms $\phi_p : GL_n(\mathbb{Z}) \to GL_n(\mathbb{Z}_p)$.*

Proof Let $\phi_p : GL_n(\mathbb{Z}) \to GL_n(\mathbb{Z}_p)$ be defined by:

$$\phi_p(\begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} & \dots & x_{1,n} \\ x_{2,1} & x_{2,2} & x_{2,3} & \dots & x_{2,n} \\ \dots & \dots & \dots & \dots & \dots \\ x_{n,1} & x_{n,2} & x_{n,3} & \dots & x_{n,n} \end{bmatrix}) = \begin{bmatrix} x_{1,1} \bmod p & x_{1,2} \bmod p & x_{1,3} \bmod p & \dots & x_{1,n} \bmod p \\ x_{2,1} \bmod p & x_{2,2} \bmod p & x_{2,3} \bmod p & \dots & x_{2,n} \bmod p \\ \dots & \dots & \dots & \dots & \dots \\ x_{n,1} \bmod p & x_{n,2} \bmod p & x_{n,3} \bmod p & \dots & x_{n,n} \bmod p \end{bmatrix}$$

Note that ϕ_p is surjective for all p as it fixes non-negative coordinates less than p.

$$\phi_p(\begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} & \dots & x_{1,n} \\ x_{2,1} & x_{2,2} & x_{2,3} & \dots & x_{2,n} \\ \dots & \dots & \dots & \dots & \dots \\ x_{n,1} & x_{n,2} & x_{n,3} & \dots & x_{n,n} \end{bmatrix})\phi_p(\begin{bmatrix} y_{1,1} & y_{1,2} & y_{1,3} & \dots & y_{1,n} \\ y_{2,1} & y_{2,2} & y_{2,3} & \dots & y_{2,n} \\ \dots & \dots & \dots & \dots & \dots \\ y_{n,1} & y_{n,2} & y_{n,3} & \dots & y_{n,n} \end{bmatrix})$$

$$= \begin{bmatrix} x_{1,1} \bmod p & x_{1,2} \bmod p & \dots & x_{1,n} \bmod p \\ x_{2,1} \bmod p & x_{2,2} \bmod p & \dots & x_{2,n} \bmod p \\ \dots & \dots & \dots & \dots \\ x_{n,1} \bmod p & x_{n,2} \bmod p & \dots & x_{n,n} \bmod p \end{bmatrix} \begin{bmatrix} y_{1,1} \bmod p & y_{1,2} \bmod p & \dots & y_{1,n} \bmod p \\ y_{2,1} \bmod p & y_{2,2} \bmod p & \dots & y_{2,n} \bmod p \\ \dots & \dots & \dots & \dots \\ y_{n,1} \bmod p & y_{n,2} \bmod p & \dots & y_{n,n} \bmod p \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^n x_{1,i}\ y_{i,1} \bmod p & \sum_{i=1}^n x_{1,i}\ y_{i,2} \bmod p & \dots & \sum_{i=1}^n x_{1,i}\ y_{i,n} \bmod p \\ \sum_{i=1}^n x_{2,i}\ y_{i,1} \bmod p & \sum_{i=1}^n x_{2,i}\ y_{i,2} \bmod p & \dots & \sum_{i=1}^n x_{2,i}\ y_{i,n} \bmod p \\ \dots & \dots & \dots & \dots \\ \sum_{i=1}^n x_{n,i}\ y_{i,1} \bmod p & \sum_{i=1}^n x_{n,i}\ y_{i,2} \bmod p & \dots & \sum_{i=1}^n x_{n,i}\ y_{i,n} \bmod p \end{bmatrix}$$

$$= \phi_p(\begin{bmatrix} \sum_{i=1}^n x_{1,i}\ y_{i,1} & \sum_{i=1}^n x_{1,i}\ y_{i,2} & \dots & \sum_{i=1}^n x_{1,i}\ y_{i,n} \\ \sum_{i=1}^n x_{2,i}\ y_{i,1} & \sum_{i=1}^n x_{2,i}\ y_{i,2} & \dots & \sum_{i=1}^n x_{2,i}\ y_{i,n} \\ \dots & \dots & \dots & \dots \\ \sum_{i=1}^n x_{n,i}\ y_{i,1} & \sum_{i=1}^n x_{n,i}\ y_{i,2} & \dots & \sum_{i=1}^n x_{n,i}\ y_{i,n} \end{bmatrix})$$

$$= \phi_p(\begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} & \dots & x_{1,n} \\ x_{2,1} & x_{2,2} & x_{2,3} & \dots & x_{2,n} \\ \dots & \dots & \dots & \dots & \dots \\ x_{n,1} & x_{n,2} & x_{n,3} & \dots & x_{n,n} \end{bmatrix} \begin{bmatrix} y_{1,1} & y_{1,2} & y_{1,3} & \dots & y_{1,n} \\ y_{2,1} & y_{2,2} & y_{2,3} & \dots & y_{2,n} \\ \dots & \dots & \dots & \dots & \dots \\ y_{n,1} & y_{n,2} & y_{n,3} & \dots & y_{n,n} \end{bmatrix})$$

So we have that ϕ_p are homomorphisms as required. ∎

Theorem 0.0.2 *$GL_n(\mathbb{Z})$ is residually finite.*

Proof It suffices to show that for all $x \in GL_n(\mathbb{Z})\backslash\{id_{GL_n(\mathbb{Z})}\}$ there exists p prime such that $\phi_p(x) \neq id_{GL_n(\mathbb{Z}_p)}$.
Let $x \in GL_n(\mathbb{Z})\backslash\{id_{GL_n(\mathbb{Z})}\}$

$$id_{GL_n(\mathbb{Z})} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \quad x = \begin{bmatrix} x_{1,1} & x_{1,2} & \dots & x_{1,n} \\ x_{2,1} & x_{2,2} & \dots & x_{2,n} \\ \dots & \dots & \dots & \dots \\ x_{n,1} & x_{n,2} & \dots & x_{n,n} \end{bmatrix} \quad id_{GL_n(\mathbb{Z}_p)} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

As $x \neq id_{GL_n(\mathbb{Z})}$ there exist $i, j \leq n$ such that either ($i \neq j$ and $x_{i,j} \neq 0$) or ($i = j$ and $x_{i,j} \neq 1$).
Let p be an prime such that $p > 2|x_{i,j}|$.
We have that $x_{i,j} \bmod p$ is either $x_{i,j}$ or $p - |x_{i,j}|$, as $x_{i,j}$ is not equal the i,j-coordinate of $id_{GL_n(\mathbb{Z})}$ by assumption and $p - |x_{i,j}| > \frac{p}{2} \geq 1$ we have that $\phi_p(x) \neq id_{GL_n(\mathbb{Z}_p)}$ as required.