TUTORIAL 2 - The Poincaré disk model of hyperbolic space - Solutions

① Let $\phi : \mathbb{D}^2 \to \mathbb{D}^2$ be given by

$$\phi(z) = \bar{z}.$$

Let $u = u_1 + u_2 i$, $v = v_1 + v_2 i \in \mathbb{D}^2$ and $C = \gamma([0,1])$ be a curve joining u, v where $\gamma(t) = \alpha(t) + i\beta(t)$ with $\alpha'(t), \beta'(t)$ continuous. Then $\bar{C} = \bar{\gamma}([0,1])$ is a curve joining $\bar{u}$ to $\bar{v}$ where $\bar{\gamma}$ is defined by

$$\bar{\gamma}(t) = \alpha(t) - i\beta(t)$$
$$= \alpha(t) + i(-\beta(t))$$

clearly $\alpha'(t)$ and $\frac{d}{dt}(-\beta(t)) = -\beta'(t)$ exist and are continuous. Then

$$L(\bar{c}) = \int_{\bar{C}} \frac{2}{1-|z|^2}|dz| = \int_0^1 \frac{2\sqrt{\alpha'(t)^2 + (-\beta'(t))^2}\,dt}{1-(\alpha(t)^2 + (-\beta(t))^2}$$

$$= \int_0^1 \frac{2\sqrt{\alpha'(t)^2 + \beta'(t)}}{1-(\alpha(t)^2+\beta(t)^2)}\,dt = \int_C \frac{2}{1-|z|^2}|dz|$$

$= L(C)$. Hence $d_{\mathbb{D}^2}(u,v) = d_{\mathbb{D}^2}(\phi(u), \phi(v))$.

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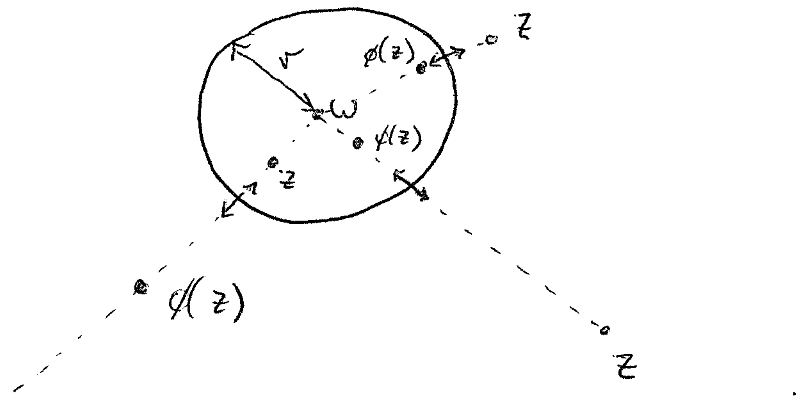
(2) Action of map $\phi : z \mapsto w + \left(\frac{r}{|z-w|}\right)^2 (z-w)$



we have

$$\phi(z) = w + \frac{r^2}{(z-w)(\overline{z-w})}(z-w)$$

$$= w + \frac{r^2}{\bar{z} - \bar{w}}$$

$$= \frac{\bar{z}w - \bar{w}w + r^2}{\bar{z} - \bar{w}}$$

$$= \frac{w\bar{z} + (r^2 - |w|^2)}{\bar{z} + (-\bar{w})} \in \underbrace{\langle \text{Möb}^+, z \mapsto \bar{z} \rangle}$$

group generated by Möb$^+$ and $z \mapsto \bar{z}$.

since $w \times (-\bar{w}) - (r^2 - |w|^2) = -r^2 \neq 0$

Result follows by result for Möb.

(3) $\mathbb{D}^2$ is clearly invariant under $z \mapsto \bar{z}$ and so it suffices to only consider $\mathrm{con}^+(1)$. Let $g \in \mathrm{con}^+(1)$ be given by

$$g(z) = \frac{az + \bar{c}}{cz + \bar{a}}$$ for some $a, c \in \mathbb{C}$ with $|a|^2 - |c|^2 = 1$.

Let $z \in \mathbb{D}^2$. We want to show that $g(z) \in \mathbb{D}^2$. It suffices to prove that $|g(z)|^2 < 1$.

$$|g(z)|^2 = \frac{(az + \bar{c})(\bar{a}\bar{z} + c)}{|cz + \bar{a}|^2}$$

$$= \frac{|a|^2|z|^2 + acz + \bar{a}\bar{c}\bar{z} + |c|^2}{|cz + \bar{a}|^2}$$

$$= \frac{|c|^2|z|^2 + |z|^2 + acz + \bar{a}\bar{c}\bar{z} + |a|^2 - 1}{|cz + \bar{a}|^2}$$

$$= \frac{|cz + \bar{a}|^2}{|cz + \bar{a}|^2} - \frac{1 - |z|^2}{|cz + \bar{a}|^2}$$

$$= 1 - \frac{1 - |z|^2}{|cz + \bar{a}|^2} < 1 \quad \text{since } |z|^2 < 1.$$

③ cont...

We have shown that $g(\mathbb{D}^2) \subseteq \mathbb{D}^2$. The other inclusion is similar... or follows directly using the fact that $\mathrm{con}^+(1)$ is a group!

④ First we show that $\mathrm{Con}(1)$ is closed under composition of functions. Let $g, f \in \mathrm{Con}^+(1)$ be given by:

$$g(z) = \frac{az + \bar{c}}{cz + \bar{a}} \quad \text{for } a, c \in \mathbb{C} \text{ with } |a|^2 - |c|^2 = 1$$

$$f(z) = \frac{bz + \bar{d}}{dz + \bar{b}} \quad \text{for } b, d \in \mathbb{C} \text{ with } |b|^2 - |d|^2 = 1.$$

Then

$$(f \circ g)(z) = f(g(z)) = \frac{b \dfrac{az + \bar{c}}{cz + \bar{a}} + \bar{d}}{d \dfrac{az + \bar{c}}{cz + \bar{a}} + \bar{b}}$$

$$= \frac{abz + b\bar{c} + c\bar{d}z + \bar{a}\bar{d}}{adz + \bar{c}d + \bar{b}cz + \bar{a}\bar{b}}$$

$$= \frac{(ab + c\bar{d})z + (\bar{a}\bar{d} + b\bar{c})}{(ad + \bar{b}c)z + (\bar{a}\bar{b} + \bar{c}d)}$$

$$= \frac{(ab + c\bar{d})z + \overline{(ad + \bar{b}c)}}{(ad + \bar{b}c)z + \overline{(ab + c\bar{d})}}$$

(4) cont...

Also
$$(ab+c\bar{d})(\bar{a}\bar{b}+\bar{c}d) - (ad+\bar{b}c)(\bar{a}\bar{d}+b\bar{c})$$

$$= a\bar{a}b\bar{b} + ab\bar{c}d + \bar{a}\bar{b}c\bar{d} + c\bar{c}d\bar{d} - a\bar{a}d\bar{d} - ab\bar{c}d - \bar{a}\bar{b}c\bar{d} - b\bar{b}c\bar{c}$$

$$= |a|^2|b|^2 + |c|^2|d|^2 - |a|^2|d|^2 - |b|^2|c|^2$$

$$= (|a|^2-|c|^2)(|b|^2-|d|^2) = 1$$

and so $f \circ g \in Con^+(1)$

To show $Con(1)$ is closed we also have to consider f, g of the other form. This is similar and the details are omitted. There are four cases:

(i) $f, g \in Con^+(1)$

(ii) $f \in Con^+(1)$, $g \in Con(1) \setminus Con^+(1)$

(iii) $f \in Con(1) \setminus Con^+(1)$, $g \in Con^+(1)$

(iv) $f, g \in Con(1) \setminus Con^+(1)$.

④ cont...

Now we prove the group axioms hold:

(i) <u>associativity</u>. composition is always associative (think about why?)

(ii) <u>identity element</u>: we already proved in the lectures that

$$z \mapsto z = \frac{1 \times z + \bar{0}}{0 \times z + \bar{1}} \in \mathrm{con}^+(1)$$

(iii) <u>inverses</u>: we want $w = g^{-1}(z)$ such that

$$g(w) = z, \quad \text{where } g(z) = \frac{az + \bar{c}}{cz + \bar{a}}$$
$$(|a|^2 - |c|^2 = 1)$$

ie
$$\frac{aw + \bar{c}}{cw + \bar{a}} = z$$

$$\Leftrightarrow \quad (a - cz)w = \bar{a}z - \bar{c}$$

$$\Leftrightarrow \quad w = \frac{\bar{a}z - \bar{c}}{-cz + a}.$$

Define g^{-1} by $g^{-1}(z) = \dfrac{\bar{a}z + \overline{(-c)}}{-cz + \bar{\bar{a}}}$

since $|\bar{a}|^2 - |-c|^2 = |a|^2 - |c|^2 = 1$ we indeed have $g^{-1} \in \mathrm{con}^+(1)$

(4) cont...

Finally, we have checked that

$$g(g^{-1}(z)) = z$$

but we should also check that

$$g^{-1}(g(z)) = z.$$

The details are omitted.

Also, inverses of elements $g \in \text{con}(1) \setminus \text{con}^+(1)$ are found similarly.

(5) Since $\text{con}^+(1) \leq \text{con}(1)$ and

$$z \mapsto \bar{z} = \frac{1 \times \bar{z} + \bar{0}}{0 \times \bar{z} + \bar{1}} \in \text{con}(1)$$

we immediately have

$$\langle \text{con}^+(1), z \mapsto \bar{z} \rangle \leq \text{con}(1).$$

However, an arbitrary element $g \in \text{con}(1) \setminus \text{con}^+(1)$ given by $g(z) = \dfrac{a\bar{z} + \bar{c}}{c\bar{z} + \bar{a}}$ is given by composition of $z \mapsto \bar{z}$ and $z \mapsto \dfrac{az + \bar{c}}{cz + \bar{a}} \in \text{Con}^+(1)$ and so $\langle \text{con}^+(1), z \mapsto \bar{z} \rangle \supseteq \text{con}(1)$ and we are done.