School of Mathematics and Statistics

MT5836 Galois Theory

Problem Sheet II: Field extensionions: Algebraic elements, minimum polynomials, simple extensions

1. Let K be an extension of the field F such that the degree $|K : F|$ is a prime number. Show that there are no *intermediate* fields between F and K; that is, no fields L satisfying $F \subset L \subset K$.

2. For all values of $a, b \in \mathbb{Q}$, determine the minimum polynomial of $a + b\sqrt{2}$ over $\mathbb{Q}$.

3. (a) Show that $\mathbb{C}$ is a simple extension of $\mathbb{R}$.
   (b) What are the irreducible polynomials over $\mathbb{C}$?
   (c) Show that if α is algebraic over $\mathbb{C}$, then $\mathbb{C}(\alpha) = \mathbb{C}$.

4. Let α be algebraic over the base field F. Show that every element of the simple extension $F(\alpha)$ is algebraic over F.

5. Show that the polynomial $f(X) = X^4 - 16X^2 + 4$ is irreducible over $\mathbb{Q}$.

   Let α be a root of $f(X)$ in some field extension of $\mathbb{Q}$. Determine the minimum polynomials of α^2 and of $\alpha^3 - 14\alpha$ over $\mathbb{Q}$.

6. Determine the following degrees of field extensions:

   (a) $|\mathbb{Q}(\sqrt[5]{3}) : \mathbb{Q}|$
   (b) $|\mathbb{Q}(\mathrm{e}^{2\pi i/5}) : \mathbb{Q}|$
   (c) $|\mathbb{Q}(\sqrt{2}, i) : \mathbb{Q}|$
   (d) $|\mathbb{Q}(\sqrt{2}i) : \mathbb{Q}|$
   (e) $|\mathbb{Q}(\sqrt{2}, \sqrt{5}) : \mathbb{Q}|$
   (f) $|\mathbb{Q}(\sqrt{6}, i) : \mathbb{Q}(i)|$

7. Let $\alpha \in \mathbb{C}$ be a root of the polynomial $X^2 + 2X + 5$. Express the element

   $$\frac{\alpha^3 + \alpha - 2}{\alpha^2 - 3}$$

   of $\mathbb{Q}(\alpha)$ as a linear combination of the basis $\{1, \alpha\}$.

8. Show that $\mathbb{Q}(\sqrt{2}, \sqrt{5}) = \mathbb{Q}(\sqrt{2} + \sqrt{5})$.

   Determine the minimum polynomial of $\sqrt{2} + \sqrt{5}$ over the following subfields:

   (i) $\mathbb{Q}$; (ii) $\mathbb{Q}(\sqrt{2})$; (iii) $\mathbb{Q}(\sqrt{5})$.

9. Let α and β be algebraic elements over the base field F. Suppose that the minimum polynomial of α over F has degree m, the minimum polynomial of β over F has degree n, and that m and n are coprime. Show that $|F(\alpha, \beta) : F| = mn$.

10. Let α be transcendental over the field F. Show that there is an isomorphism ϕ from the field $F(X)$ of rational functions in the indeterminate X over F to the simple extension $F(\alpha)$ satisfying $X\phi = \alpha$ and $b\phi = b$ for all $b \in F$.

11. (a) Show that the field $\mathbb{A}$ of algebraic numbers over $\mathbb{Q}$ is countable.
    (b) Show that $\mathbb{C}$ is an infinite degree extension of $\mathbb{A}$.
    (c) Show that $\mathbb{C}$ contains elements that are transcendental over $\mathbb{Q}$.