(6) $$A_{\mathbb{D}^2}(\phi(F)) = \int_{\phi(F)} \frac{4\,|dz|}{(1-|z|^2)^2}$$

use substitution

$z = \phi(w)$

$(\,|dz| = |\phi'(w)|^2\,|dw|\,)$

$$= \int_F \frac{4\,|\phi'(w)|^2\,|dw|}{(1-|\phi(w)|^2)^2}$$

$$= \int_F \left(\frac{2\,|\phi'(w)|}{1-|\phi(w)|^2}\right)^2 |dw|$$

from proof of Theorem 3.4 in notes

$$= \int_F \left(\frac{1}{\mathrm{Im}(w)}\right)^2 |dw|$$

$$= A_{\mathbb{H}^2}(F)$$

as required.

7 We have already proved that for $h \in \text{Con}^+(1)$ $h(\mathbb{D}^2) = \mathbb{D}^2$.

Let $g \in PSL(2, \mathbb{R}) = \phi^{-1}\,\text{Con}^+(1)\,\phi$;

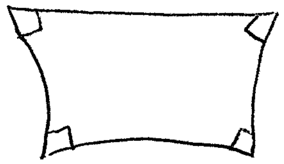
and so $g = \phi^{-1} h \phi$ for some $h \in \text{Con}^+(1)$.

Therefore

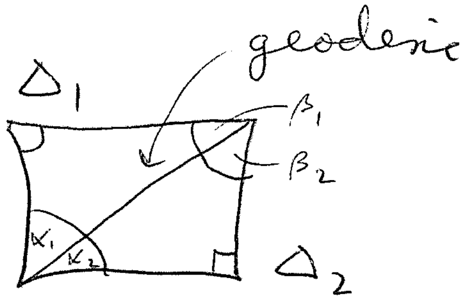
$$
\begin{aligned}
g(\mathbb{H}^2) &= \phi^{-1} h \phi(\mathbb{H}^2) \\
&= \phi^{-1} h(\mathbb{D}^2) \\
&= \phi^{-1}(\mathbb{D}^2) \\
&= \mathbb{H}^2
\end{aligned}
$$

which completes the proof.

(8) Let R be a hyperbolic rectangle:



and divide it into two triangles as follows:

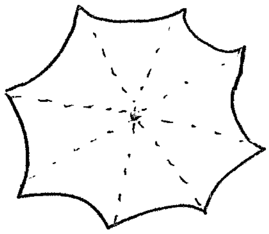


By the Gauss-Bonnet Theorem

$$
\begin{aligned}
A(R) &= A(\Delta_1) + A(\Delta_2) \\
&= \pi - \tfrac{\pi}{2} - \alpha_1 - \beta_1 \\
&\quad + \pi - \tfrac{\pi}{2} - \alpha_2 - \beta_2 \\
&= \pi - (\alpha_1 + \alpha_2) - (\beta_1 + \beta_2) \\
&= \pi - \tfrac{\pi}{2} - \tfrac{\pi}{2} = 0
\end{aligned}
$$

and so hyperbolic rectangles do not exist!

(9) P convex hyperbolic polygon:



Let z be any point in the interior of P. Since P is convex, we can join z to each of the n vertices of P via a geodesic which only intersects P at the vertex. Therefore, for each edge of P, we have associated a triangle which shares one side with P and the vertex opposite this side is z. The geodesic joining z to a vertex v, splits the angle at v into two (not necessarily equal) angles, ie if α_i is the angle at v, then the geodesic splits α_i as

$$\alpha_i = \beta_i^1 + \beta_i^2.$$

Applying the Gauss-Bonnet Theorem to each of the n triangles partitioning P we get:

$$A(P) = \sum_{i=1}^{n} A(\Delta_i)$$

$$= \sum_{i=1}^{n} \pi - \theta_i - \gamma_i^1 - \gamma_i^2$$

where θ_i is the angle at z and γ_i^1, γ_i^2 are the other interior angles of Δ_i.

Continuing, we get

$$A(P) = n\pi - \left(\sum_{i=1}^{n} \theta_i\right) - \sum_{i=1}^{n} (\gamma_i^1 + \gamma_i^2)$$

$$= n\pi - 2\pi - \sum_{i=1}^{n} \alpha_i$$

$$= (n-2)\pi - \sum_{i=1}^{n} \alpha_i$$

as required.

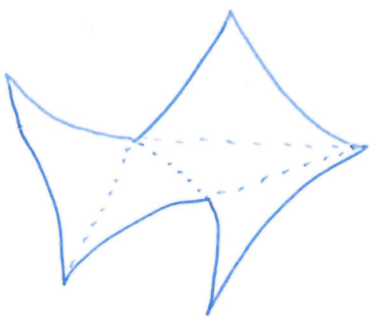
(9) continued...

Convexity was used in the previous proof, but is it necessary?

Here is a possible strategy:

Let P be a hyperbolic polygon with n sides and find a "triangulation". This means cut it up into N triangles whose vertices are all vertices of P.

e.g.



Applying the Gauss-Bonnet theorem to each of the triangles $\Delta_1, \dots, \Delta_N$ we get:

$$A(P) = \sum_{i=1}^{N} A(\Delta_i) = \sum_{i=1}^{N} \pi - \alpha_i - \beta_i - \gamma_i$$

9 cont...

$$= N\pi - \sum_{i=1}^{N} (\alpha_i' + \beta_i' + \gamma_i')$$

$$= N\pi - \sum_{i=1}^{n} \alpha_i$$

(since all vertices of triangles are vertices of P).

Therefore, we want $N = n-2$.

This follows from Euler's identity for planar graphs: $v - e + f = 2$.

Viewing the sides of all triangles as edges of a planar graph, this gives

$$n - e + N = 1$$

(we ignore the "unbounded" face). Also,

$$3N = \sum_{i=1}^{N} \#\text{edges of } \Delta_i = n + 2\overbrace{(e-n)}^{\text{internal edges counted twice!}}$$

$$= 2e - n$$

Putting these together we get $N = n-2$

All that remains is to show that such a triangulation always exists!