6 Fuchsian Limit sets

1. Prove that if $p \in S^1$ is fixed by a parabolic map $g \in \mathrm{con}^+(1)$, then $g^n(0) \to p$ and if $h_1, h_2 \in S^1$ are the repelling and attracting fixed points of a hyperbolic map $g \in \mathrm{con}^+(1)$, then $g^n(0) \to h_2$ and $g^{-n}(0) \to h_1$.

2. Let Γ be a Fuchsian group. Prove that for any isometry g, we have $L\big(g\,\Gamma\,g^{-1}\big) = g(L(\Gamma)$.

3. Suppose $E \subseteq S^1$ is closed, contains at least 2 points, and is invariant under the action of a Fuchsian group $\Gamma \leqslant \mathrm{con}^+(1)$, i.e. $\Gamma(E) = E$. Prove that $L(\Gamma) \subseteq E$. Find an example of a Fuchsian group Γ and a closed, invariant, non-empty set E which does *not* contain $L(\Gamma)$. Deduce that the limit set of a non-elementary Fuchsian group is the smallest closed, invariant, non-empty subset of the boundary.

4. Show that if a Fuchsian group Γ has a bounded fundamental domain, then the limit set is the entire boundary.

5. (Harder bonus question) Show that if a Fuchsian group Γ has a fundamental domain which has *finite area*, then the limit set is the entire boundary. Hence, or otherwise, find the limit set of the modular group.

6. Prove that for a Fuchsian group $\Gamma \leqslant \mathrm{con}^+(1)$ the Poincaré exponent is equal to

$$\delta(\Gamma) = \inf\left\{ s \geqslant 0 \ : \ \sum_{z \in \Gamma(0)} (1 - |z|)^s < \infty \right\}.$$

7. Prove that if Γ is a Fuchsian group and $\Gamma' \leqslant \Gamma$, then Γ' is a Fuchsian group, $L(\Gamma') \subseteq L(\Gamma)$ and $\delta(\Gamma') \leqslant \delta(\Gamma)$.

8. Let $g, h : [0,1]^2 \to [0,1]^2$ be defined by $g(x) = x/3$ and $h(x) = x/3 + (2/3, 0)$ and let $\Gamma = \mathrm{Semi}\langle g, h, \mathrm{Id}\rangle$, i.e. the semigroup generated by g, h and the identity. Thus Γ consists of all maps which can be obtained as compositions of g and h only (not including their inverses). This semigroup acts properly discontinuously on the interior of Euclidean unit square, but not by isometry.

   (i) Draw a picture of a fundamental domain for the semigroup action and describe the limit set.
   (ii) Let $u = (1/2, 1/2)$, for example, and compute the 'exponent of convergence':

$$\delta(\Gamma) = \inf\left\{ s \geqslant 0 \ : \ \sum_{z \in \Gamma(u)} \beta(z)^s < \infty \right\}$$

   where for a point $z = (x, y) \in (0,1)^2$ the function $\beta(z) = \min\{x, y, 1-x, 1-y\}$ is a measure of how close the point z is to the boundary of the unit cube, and should be compared with the more familiar $(1 - |z|)$ which measures how close a point $z \in \mathbb{D}^2$ is to the boundary S^1.
   (iii) Comment on the geometrical significance of the number $\delta(\Gamma)$.