# MT5823 SEMIGROUP THEORY - SOLUTIONS TO MAY 2008 EXAM

(1) (a) [Easy - not from lectures] From the Cayley table, $\{b, c, c^2, bc, cb\} = \{a, b, c, d, e\}$. Now, $cd = f$ and $be = g$.
The semigroup $S$ is not generated by a single element since $\langle a\rangle = \{a\}$, $\langle b\rangle = \{b\}$, $\langle c\rangle = \{a, c\}$, $\langle d\rangle = \{d, g\}$, $\langle e\rangle = \{e, g\}$, $\langle f\rangle = \{f\}$, and $\langle g\rangle = \{g\}$.
(Alternatively, since $bc \neq cb$, $S$ is not commutative and so not generated by a single element.)

(b) [Easy - from lectures] The right Cayley graph is given in Figure 1 and the left Cayley graph is given in Figure 2.

(c) [Medium - from lectures] It was shown in lectures that the $\mathcal{R}$-classes of a semigroup correspond to the strongly connected components of the right Cayley graph and the analogous statement for $\mathcal{L}$-classes. Hence the $\mathcal{R}$-classes of $S$ are $\{a, c\}, \{b, d\}, \{e, f\}, \{g\}$ and the $\mathcal{L}$-classes are $\{a, c\}, \{b, e\}, \{d, f\}, \{g\}$.
Since $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, the $\mathcal{H}$-classes of $S$ are $\{a, c\}, \{b\}, \{d\}, \{e\}, \{f\}, \{g\}$. Likewise, since $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$, the $\mathcal{D}$-classes of $S$ are $\{a, c\}, \{b, d, e, f\}, \{g\}$.
The eggbox diagram of the $\mathcal{D}$-classes can be seen in Figure 3.

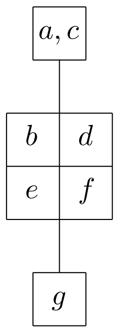

Figure 3.

(d) [Medium - from lectures] A semigroup is *inverse* if it is regular and its idempotents commute. We saw in lectures that a semigroup is inverse if and only if every $\mathcal{R}$-class and every $\mathcal{L}$-class contains exactly one idempotent.
It is easy to see from the eggbox diagrams given above that this is true.
Alternatively from the Cayley table, the idempotents in the $\mathcal{R}$-classes $\{a, c\}, \{b, d\}, \{e, f\}, \{g\}$ are $a$, $b$, $f$, and $g$, respectively, and in the $\mathcal{L}$-classes $\{a, c\}, \{b, e\}, \{d, f\}, \{g\}$. are $a$, $b$, $f$, and $g$.

(e) [Hard - similar to tutorial questions] We start by finding the elements of $T$. Using the algorithm given in lectures

$$t_1 = x, t_2 = y$$

$$t_1x = x^2 = x, t_1y = xy = t_3$$

$$t_2x = yx = t_4, t_2y = y^2 = t_5$$

$$t_3x = xyx = t_6, t_3y = xy^2 = x$$

$$t_4x = yx^2 = yx, t_4y = yxy = t_7$$

$$t_5x = y^2x = x, t_5y = y^3 = y$$

$$t_6x = xyx^2 = x^2 = x, t_6y = xyxy = xyx$$

$$t_7x = yxyx = xyx, t_7y = yxy^2 = yx.$$

Hence the elements of $T$ are $\{x, y, xy, yx, y^2, xyx, yxy\}$.
Let $\phi : T \longrightarrow S$ be the mapping defined by $x\phi = b$ and $y\phi = c$. Since $b$ and $c$ generate $S$, $\phi$ is a surjective mapping and since $T$ is finite $\phi$ is a bijection.

It suffices by a theorem from lectures to show that $S$ satisfies the relations given in the presentation for $T$. So, in $S$

$$b^2 = b, bc^2 = ba = b, c^2b = ab = b, c^3 = c^2.c = ac = c, (bc)^2 = d^2 = g = db = bcb$$
$$(cb)^2 = e^2 = g = bcb.$$

It follows that $\phi$ is a bijective homomorphism.

(f) [Very hard - not from lectures] Let $I_2$ denote the symmetric inverse semigroup on a 2 element set. The elements of $I_2$ are

$$m = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, p = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, n = \begin{pmatrix} 1 & 2 \\ 1 & - \end{pmatrix}, q = \begin{pmatrix} 1 & 2 \\ 2 & - \end{pmatrix}, r = \begin{pmatrix} 1 & 2 \\ - & 2 \end{pmatrix},$$
$$s = \begin{pmatrix} 1 & 2 \\ - & 1 \end{pmatrix}, t = \begin{pmatrix} 1 & 2 \\ - & - \end{pmatrix}$$

and from tutorials we know that $I_2$ is generated by $p$ and $n$. Define $\phi : S \to I_2$ by

$$\begin{pmatrix} a & b & c & d & e & f & g \\ m & n & p & q & r & s & t \end{pmatrix}.$$

It suffices, by part (e) above, to prove that $I_2$ satisfies the relations of the presentation defining $T$. Now,

$$n^2 = n, np^2 = n, p^2n = n, p^3 = p,$$
$$(np)^2 = \begin{pmatrix} 1 & 2 \\ 2 & - \end{pmatrix}^2 = \begin{pmatrix} 1 & 2 \\ - & - \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & - \end{pmatrix} n = npn, (pn)^2 = npn.$$

It follows that $\phi$ is an isomorphism.

(2) (a) [Easy - from lectures] A semigroup is *simple* if it has no proper two-sided ideals. The Rees theorem states that a finite semigroup is simple if and only if it is isomorphic to a Rees matrix semigroup $\mathcal{M}[G; I, \Lambda; P]$ where $G$ is a finite group and $I$ and $\Lambda$ are finite sets.

(b) [Easy - from lectures] ($\Rightarrow$) Let $(i, g, \lambda)\mathcal{R}(j, h, \mu)$. Then there exists $(i', g', \lambda') \in \mathcal{M}[G; I, \Lambda; P]$ such that

$$\begin{aligned} (i, g, \lambda)(i', g', \lambda') &= (j, h, \mu) \\ \Rightarrow (i, gp_{\lambda i'}g', \lambda') &= (j, h, \mu) \\ \Rightarrow i &= j. \end{aligned}$$

($\Leftarrow$) Let $(i, g, \lambda), (i, h, \mu) \in \mathcal{M}[G; I, \Lambda; P]$. Then

$(i, g, \lambda)(i, p_{\lambda i}^{-1}g^{-1}h, \mu) = (i, h, \mu)$ & $(i, h, \mu)(i, p_{\mu i}^{-1}h^{-1}g, \lambda) = (i, g, \lambda)$.

Hence $(i, g, \lambda)\mathcal{R}(i, h, \mu)$.

The analogous statement for Green's $\mathcal{L}$-relation can be proved using a similar argument.

Since $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, it follows that $(i, g, \lambda)\mathcal{H}(j, h, \mu)$ if and only if $i = j$ and $\lambda = \mu$.

(c) [Easy - not from lectures] We know from lectures that the intersection of two equivalence relations is again an equivalence relation. Hence it remains to show that $\rho \cap \sigma$ is a congruence. Let $s \in S$ and $(x, y) \in \rho \cap \sigma$ be arbitrary. Then $(xs, ys), (sx, sy) \in \rho$ and $(xs, ys), (sx, sy) \in \sigma$ since $\rho$ and $\sigma$ are congruences. It follows that $(xs, ys), (sx, sy) \in \rho \cap \sigma$ and so $\rho \cap \sigma$ is a congruence.

(d) [Moderate - similar to lectures] We know from lectures that $\mathcal{R}$ is a left congruence and so it suffices to prove that it is also a right congruence. Let $(i, g, \lambda)\mathcal{R}(j, h, \lambda)$ and $(k, t, \mu) \in \mathcal{M}[G; I, \Lambda; P]$ be arbitrary.

$$(i, g, \lambda)(k, t, \mu) = (i, gp_{\lambda k}t, \mu) \;\;\&\;\; (j, h, \lambda)(k, t, \mu) = (j, hp_{\lambda k}t, \mu).$$

Since $i = j$, it follows from part (b) above that $(i, g, \lambda)(k, t, \mu)\mathcal{R}(j, h, \lambda)(k, t, \mu)$.

That $\mathcal{L}$ is a 2-sided congruence follows by an analogous argument.

Since $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ by definition, it follows that from part (c) that $\mathcal{H}$ is a congruence on $\mathcal{M}[G; I, \Lambda; P]$.

(e) [Hard - not from lectures] From part (b), $(i, g, \lambda)/\mathcal{R} = (j, h, \mu)/\mathcal{R}$ if and only if $i = j$. It follows that

$$(i, g, \lambda)/\mathcal{R}\,(j, h, \mu)/\mathcal{R} = (i, gp_{\lambda j}h, \mu)/\mathcal{R} = (i, g, \mu)/\mathcal{R}$$

for any $(i, g, \lambda), (j, h, \mu) \in \mathcal{M}[G; I, \Lambda; P]$. It follows that $\mathcal{M}[G; I, \Lambda; P]$ is a semigroup of left zeros.

It suffices from lectures to prove that $x^2 = x$ and $xyz = xz$ for all $x, y, z \in \mathcal{M}[G; I, \Lambda; P]/\mathcal{H}$. Let $(i, g, \lambda)/\mathcal{H}, (j, h, \mu)/\mathcal{H}, (k, t, \mu)/\mathcal{H} \in \mathcal{M}[G; I, \Lambda; P]/\mathcal{H}$. Then

$$(i, g, \lambda)/\mathcal{H}\,(i, g, \lambda)/\mathcal{H} = (i, gp_{\lambda i}g, \lambda)/\mathcal{H}.$$

It follows by part (b) that $(i, gp_{\lambda i}g, \lambda)/\mathcal{H} = (i, g, \lambda)/\mathcal{H}$. Now,

$$(i, g, \lambda)/\mathcal{H}\,(j, h, \mu)/\mathcal{H}\,(k, t, \mu)/\mathcal{H} = (i, gp_{\lambda j}hp_{\mu k}t, \mu)/\mathcal{H} = (i, g, \lambda)/\mathcal{H}\,(k, t, \mu)/\mathcal{H}$$

again by part (b). We have shown that $\mathcal{M}[G; I, \Lambda; P]/\mathcal{H}$ is a rectangular band.

(f) [Easy] A semigroup is *Clifford* if it is regular and its idempotents commute with all elements.

(g) [Moderate - not from lectures] It is obvious that (iii) implies (ii) and that (ii) implies (i). To see that (i) implies (iii), assume using the Rees theorem that $S = \mathcal{M}[G; I, \Lambda; P]$ where $G$ is a group and $I$ and $\Lambda$ are finite. Now, $S$ is an inverse semigroup and so every $\mathcal{R}$-class and every $\mathcal{L}$-class contains exactly one idempotent. By part (b), the number of $\mathcal{R}$-classes in $S$ is $|I|$ and by definition every $\mathcal{R}$-class contains $|\Lambda|$ idempotents. It follows that $|\Lambda| = 1$ and by symmetry $|I| = 1$. Thus $S$ is isomorphic to the group $G$.

(3) (a) [Easy - similar to lectures] A mapping $h$ is idempotent if and only if $xh = x$ for all $x \in (h)$. It follows that from inspecting the elements that:

$$f^2, gf^2, gf, g^2f^2, f^2g^2, fg, f^2g, fg^2f, g^3$$

are the idempotents of $S$.

(b) [Easy - from to lectures] A semigroup is regular if and only if all of its elements are regular (that is, for all $x$ there exists $y$ such that $xyx = x$).

(c) [Moderate - not from lectures] A $\mathcal{D}$-class $D$ is regular if and only if there exists an element in $D$ that is regular. Every idempotent $e$ is regular as $e^3 = e$. Now, $fg \in D_f$, $f^2 \in D_{f^2}$ and $g^3 \in D_{g^3}$. By part (a), $fg$, $f^2$ and $g^3$ are idempotents and so every $\mathcal{D}$-class of $S$ is regular. It follows that $S$ is regular.

(d) [Moderate - from lectures] ($\Rightarrow$) If $x \in S$ has an inverse, then there exists $y$ such that $xyx = x$ and $yxy = y$. Hence $x$ is regular.

($\Leftarrow$) If $x \in S$ is regular, then there exists $y \in S$ such that $xyx = x$. Let $z = yxy$. Then

$$xzx = xyxyx = xyx = x \ \ \& \ \ zxz = y(xyx)yxy = y(xyx)y = yxy = z.$$

Thus $z$ is an inverse for $x$.

(e) [Easy - similar to lectures] Inverses of $f$, $f^2$, and $g^3$ are $g$, $f^2$ and $g^3$, respectively.

(f) [Hard - similar to lectures] Since $S$ is regular, its Green's $\mathcal{L}$- and $\mathcal{R}$-classes are restrictions of those in $T_5$. From lectures we know that $x\mathcal{L}y$ if and only if $(x) = (y)$, and $x\mathcal{R}y$ if and only if $\ker(x) = \ker(y)$.

It follows that the number of $\mathcal{L}$-classes in $D_f$ is 2 corresponding to the images $\{1, 4, 5\}$ and $\{3, 4, 5\}$, and the number of $\mathcal{R}$-classes in $D_f$ is 2 corresponding to the kernels $\{\{1, 2, 4\}, \{3\}, \{5\}\}$ and $\{\{1\}, \{2, 4\}, \{3, 5\}\}$.

Likewise, the number of $\mathcal{L}$-classes in $D_{f^2}$ is 3 corresponding to the images

$$\{1, 5\}, \{4, 5\}, \{3, 5\},$$

and the number of $\mathcal{R}$-classes in $D_{f^2}$ is 3 corresponding to the kernels

$$\{\{\{1\}, \{2, 3, 4, 5\}\}, \{\{1, 2, 3, 4\}, \{5\}\}, \{\{1, 2, 4\}, \{3, 5\}\}\}.$$

Since $D_{g^3}$ contains only 1 element, the number of $\mathcal{L}$-classes is 1 and the number of $\mathcal{R}$-classes is 1.

(g) [Hard - not similar to lectures] A semigroup is inverse if and only if every $\mathcal{R}$-class and every $\mathcal{L}$-class contains precisely 1 idempotent. The $\mathcal{R}$-class $R_{f^2}$ of $f^2$ is $\{f^2, f^2g, f^2g^2\}$ using the same argument as that used in part (e). Hence by part (a), $R_{f^2}$ contains 3 idempotents and $S$ is not an inverse semigroup.