# Infinite Symmetric Groups


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# Chapter 0

# Indroduction

This document is concerned with studying various properties of infinite symmetric groups. These groups are particularly interesting as all groups are isomorphic to a permutation group, by considering the right (or left) action of the group on itself, and every permutation group is a subgroup of an infinite symmetric group. Throughout this document we will be working using the ZFC axioms, and in particular, we will make use of the axiom of choice.
In this document we aim to prove many results concerning infinite symmetric groups including the following. If $\Omega$ is an infinite set then:

1. All elements of $\mathrm{Sym}(\Omega)$ can be written as a commutator of elements of $\mathrm{Sym}(\Omega)$.
2. The group $\mathrm{Sym}(\Omega)$, when viewed as a semigroup, satisfies the semigroup Bergman property.
3. The stong cofinality of $\mathrm{Sym}(\Omega)$ is uncountable.
4. The group $\mathrm{Sym}(\Omega)$ can we written as a product of 10 of its abelian subgroups.
5. There exists a family of $2^{2^{|\Omega|}}$ pairwise non-conjugate maximal subgroups of $\mathrm{Sym}(\Omega)$.
6. We will classify which finite partitions of $\Omega$ have maximal subgroups of $\mathrm{Sym}(\Omega)$ as their setwise stabilizers.

This project is organised as follows: We will introduce some notation and definitions that the reader will be expected to be familiar with. We will then spend a chapter proving various results which are not directly related to infinite symmetric groups but will be needed at various points in the document. We will spend the remaining chapters proving various interesting results about infinite symmetric groups.

## 0.1 Basic Definitions and Notation

In this section we will define various terms which are used throughout the doctument. It is expected that the reader will already be reasonably familiar with most of these terms and therefore we will omit some details and proofs of the validity of these definitions. We will make use of the following notations:

1. If $S$ is a set we use the notation $P(S)$ to denote the power set of $S$.
2. We use $\mathbb{N}$ to denote the set $\{1, 2, \ldots\}$ and $\mathbb{N}_0$ will be used to denote the set $\{0, 1, 2, \ldots\}$.
3. We use $\subseteq$ to denote a subset and $\subset$ to denote a strict subset.
4. If $S$ is a subset of some set which is clear from context, we use $S^c$ to denote the complement of $S$.
5. We will compose functions from left to right.
6. If $A, B$ are sets then we use $A^B$ to denote the set of functions from $B$ to $A$.

**Definition 0.1.1** If $f : X \to Y$ is a function then the *domain*, *image*, *fix* and *support* of $f$ are defined by:

$$\mathrm{dom}(f) := X \qquad \mathrm{fix}(f) := \{x \in X : (x)f = x\}$$
$$\mathrm{img}(f) := \{y \in Y : (x)f = y \text{ for some } x \in X\} \qquad \mathrm{supp}(f) := \{x \in X : (x)f \neq x\}.$$

**Definition 0.1.2** Let $\Omega$ be an infinite set and let $M \subseteq \Omega$. We call $M$ a *moiety* of $\Omega$ if $|M| = |M^c| = |\Omega|$.

**Definition 0.1.3** An *partially ordered set* is a pair $(P, \leq)$ where $\leq$ is a subset of $P \times P$ satisfying the below conditions. We use the notation $a \leq b$ to denote $(a, b) \in\leq$ and the notation $a < b$ to denote $a \leq b$ and $a \neq b$.

1. Reflexive: For all $p \in P$ we have $p \leq p$.
2. Anti-Symmetric: If $a \leq b$ and $b \leq a$ then $a = b$.
3. Transitive: If $a \leq b$ and $b \leq c$ then $a \leq c$.

**Definition 0.1.4** A *totally ordered set* $(T, \leq)$ is a partially ordered set in which we have, for all $a, b \in T$, that either $a \leq b$ or $b \leq a$.

**Definition 0.1.5** A *well ordered set* $(W, \leq)$ is a totally ordered set, in which for all $S \subseteq W$ there exists $m \in S$, such that $m \leq s$, for all $s \in S$.

If $R$ is a function or partial order we will use the notation $R|_S$ to denote $R$ restricted to the elements of $S$.

**Definition 0.1.6** Let $(P, \leq)$ be a partially ordered set and let $C \subseteq P$. We call $C$ a *chain* if $(C, \leq|_C)$ is a totally ordered set.

**Definition 0.1.7** Let $(W, <)$ be a well ordered set. We call $S \subseteq W$ an *initial segment* of $W$ if $S = \{x \in W : x < M\}$ for some $M \in W$.

**Definition 0.1.8** A *semigroup* is a pair $(S, *)$ where $S$ is a set and $* : S \times S \to S$ satisfying the condition below. If $a, b \in S$ then we use the notation $a * b$ (or sometimes just $ab$) to denote $*(a, b)$.

$$\text{Associativity: For all } a, b, c \in S \text{ we have that } (a * b) * c = a * (b * c)$$

**Definition 0.1.9** A *group* is a semigroup $(G, *)$ which satisfies the following conditions:

1. Identity: There is an element $e \in G$ such that for all $g \in G$ we have $eg = ge = g$.
2. Inverses: For all $g \in G$ there exists $g^{-1} \in G$ such that $gg^{-1} = g^{-1}g = e$.

**Definition 0.1.10** A *symmetric group* $\mathrm{Sym}(\Omega)$ is defined to be the set of bijections $f : \Omega \to \Omega$, under composition of functions.

An infinite symmetric group is simply a symmetric group which acts on an infinite set.

**Definition 0.1.11** Let $\Omega$ be a set and let $S$ be a set on which $\mathrm{Sym}(\Omega)$ acts.
The *pointwise stabiliser* of $S$ and *setwise stabiliser* of $S$ are defined to be the following:

$$\mathrm{Pstab}(S) = \{f \in \mathrm{Sym}(\Omega) : (x)f = x \quad \text{for all } x \in S\} \qquad \mathrm{Sstab}(S) = \{f \in \mathrm{Sym}(\Omega) : (x)f \in S \iff x \in S\}$$

It's not hard to verify that these are groups, and that the pointwise stabilizer is a normal subgroup of the setwise stabilizer.

**Definition 0.1.12** Let $\Omega$ be a set and let $S \subseteq \Omega$.

$$\mathrm{Sym}_\Omega(S) := \mathrm{Pstab}(S^c)$$

Note that $\mathrm{Sym}_\Omega(S)$ is isomorphic to $\mathrm{Sym}(S)$.

**Definition 0.1.13** Let $\Omega$ be a set, let $G \leq \mathrm{Sym}(\Omega)$ and let $S$ be a subset of $\Omega$. We say that $S$ is *full* in $G$ or $G$ *acts fully* on $S$ if for all $f \in \mathrm{Sym}(S)$ there exists $f' \in G$ such that $f'|_S = f$.

**Definition 0.1.14** If $\Omega$ is a set, $p \in \Omega$ and $S \subseteq \mathrm{Sym}(\Omega)$ then the *orbit* of $p$ with respect to $S$ is defined by:

$$\mathrm{orb}_S(p) := \{x \in \Omega : x = (p)f \text{ for some } f \in \langle S \rangle_G\}$$

**Definition 0.1.15** Let $\Omega$ be a set, and let $f \in \mathrm{Sym}(\Omega)$. The term *disjoint cycle shape* of $f$ is used to describe the partition of $\Omega$ into equivalence classes under the equivalence relation given by

$$a \sim b \iff a \in \mathrm{orb}_{\{f\}}(b)$$

In particular how many elements of $\Omega/\sim$ there are of each cardinality. If $\kappa$ is a non-zero cardinal, then the term *$\kappa$-cycle* will be used to refer to $f|_P$ where $P \in \Omega/\sim$ and $|P| = \kappa$.
Note that all cycles must have countable domain. We will also use the notation $(\ldots a_{-1}, a_0, a_1, a_2 \ldots)$ to denote a cycle which maps $a_i$ to $a_{i+1}$ and the notation $(a_0, a_1, a_2 \ldots a_{n-1})$ to denote a cycle which maps $a_i$ to $a_{i+1 \bmod n}$.

**Definition 0.1.16** A *metric space* is a pair $(X, d)$ where $d : X \times X \to \mathbb{R}$ is a function satisfying the following conditions:

1. Non-negativity: $\mathrm{img}(f) \subseteq [0, \infty)$.

2. Identity of indiscernibles: If $x, y \in X$ we have that $f(x, y) = 0$ if and only if $x = y$.

3. Symmetry: If $x, y \in X$ we have that $d(x, y) = d(y, x)$.

4. Triangle inequality: If $x, y, z \in X$ we have that $d(x, y) \leq d(x, z) + d(y, z)$.

**Definition 0.1.17** We say that a metric space is *complete* if every Cauchy sequence is convergent.

**Definition 0.1.18** A *topological space* is a pair $(X, \mathcal{T})$. Where $\mathcal{T} \subseteq P(X)$ satisfying the following conditions:

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.

2. If $S \subseteq \mathcal{T}$ is finite then $\cap S \in \mathcal{T}$.

3. If $S \subseteq \mathcal{T}$ then $\cup S \in \mathcal{T}$.

**Definition 0.1.19** If $(X, \mathcal{T})$ is a topological space then we use the following notations and terminology:

1. If $U \in \mathcal{T}$, we say that $U$ is *open*.

2. If $F^c \in \mathcal{T}$, we say that $F$ is *closed*.

3. If $S \subseteq X$ then we use $S^\circ$ to denote the largest open set contained in $S$. We call this the *interior* of $S$.

4. If $S \subseteq X$ then we use $\bar{S}$ to denote the smallest closed set containing $S$. We call this the *closure* of $S$.

5. If $(\bar{N})^\circ = \emptyset$ then we say that $N$ is *nowhere-dense*.

6. If $M$ is a countable union of nowhere dense sets then we say that $M$ is *meagre*.

7. If $B \subseteq \mathcal{T}$ satisfies the condition that $\mathcal{T} = \{\cup S : S \subseteq B\}$ then we say that $B$ is a *basis* for $\mathcal{T}$.

8. If $(X, d)$ is a metric space, and $\{\{y \in X : d(x, y) < \varepsilon\} : x \in X, \varepsilon > 0\}$ is a basis for $\mathcal{T}$ then we say that $\mathcal{T}$ is *induced* by $d$.

**Definition 0.1.20** If $\{(X_i, \mathcal{T}_i) : i \in I\}$ is a family of topological spaces then we define their *product topology* as follows:

$$\Pi_{i \in I}(X_i, \mathcal{T}_i) := (\Pi_{i \in I} X_i, \mathcal{T}).$$

Where $\mathcal{T}$ is the topology with basis $\{\Pi_{i \in I} U_i : U_i \in \mathcal{T}_i$ for all $i$ and all but finitely many $U_i$ are $X_i\}$

**Definition 0.1.21** If $(X, \mathcal{T})$ is a topological space, $x \in X$ and $S \subseteq X$, then we say that $x$ is a *limit point* of $S$, if every open set containing $x$ has non-empty intersection with $S$.

**Definition 0.1.22** A subset $D$ of a topological space $(X, \mathcal{T})$ is called *dense* if for all $U \in \mathcal{T} \backslash \{\emptyset\}$ we have $D \cap U \neq \emptyset$. Note this is equivalent to saying all points of $X$ are limit points of $D$.

**Definition 0.1.23** A *topological group* $(X, *, T)$ is a triple where $(X, *)$ is a group and $(X, T)$ is a topological space such that the functions: $* : X \times X \to X$ and $^{-1} : X \to X$ are both continuous.

We will use the notations $\langle S \rangle_G$, $\langle S \rangle_S$ and $\langle S \rangle_T$ to denote respectively the group, semigroup and topology generated by the set $S$.

# Chapter 1

# Background Topics

## 1.1 Topology and Baire Category

In this section we prove some well known results related to baire category and topology which we will need in chapter 2.

**Theorem 1.1.1.** *If* $((X_1, d_1), (X_2, d_2) \ldots)$ *are complete metric spaces, then* $(X_\pi, d_\pi)$ *where*

$$X_\pi := \prod_{i=1}^{\infty} X_i \qquad d_\pi((x_{1,1}, x_{1,2} \ldots), (x_{2,1}, x_{2,2} \ldots)) := \sum_{i \in \mathbb{N}} \frac{d_i(x_{1,i}, x_{2,i})}{2^i \times (1 + d_i(x_{1,i}, x_{2,i}))}$$

*is a complete metric space.*

*Proof.* We first show that $d_\pi$ is a metric. The non-negativity and symmetry conditions are clearly true from the definition. Let

$$x_1 = (x_{1,1}, x_{1,2} \ldots) \qquad x_2 = (x_{2,1}, x_{2,2} \ldots) \qquad x_3 = (x_{3,1}, x_{3,2} \ldots)$$

be arbitrary elements of $X$. Identity of indiscernibles:

$$d(x_1, x_1) = \sum_{i \in \mathbb{N}} \frac{d_i(x_{1,i}, x_{1,i})}{2^i \times (1 + d_i(x_{1,i}, x_{1,i}))} = \sum_{i \in \mathbb{N}} \frac{0}{2^i \times (1 + 0)} = 0$$

$$d(x_1, x_2) = 0 \implies \sum_{i \in \mathbb{N}} \frac{d_i(x_{1,i}, x_{2,i})}{2^i \times (1 + d_i(x_{1,i}, x_{2,i}))} = 0 \implies \frac{d_i(x_{1,i}, x_{2,i})}{2^i \times (1 + d_i(x_{1,i}, x_{2,i}))} = 0 \text{ for all } i \in \mathbb{N}$$

$$\implies d_i(x_{1,i}, x_{2,i}) = 0 \text{ for all } i \in \mathbb{N} \implies x_{1,i} = x_{2,i} \text{ for all } i \in \mathbb{N} \implies x_1 = x_2$$

Triangle inequality: Note that $f(x) := \frac{x}{1+x}$ is an increasing function on $[0, \infty)$ as $f'(x) = \frac{1}{(1+x)^2} > 0$.

$$\begin{aligned}
d_\pi(x_1, x_3) &= \sum_{i \in \mathbb{N}} \frac{d_i(x_{1,i}, x_{3,i})}{2^i \times (1 + d_i(x_{1,i}, x_{3,i}))} = \sum_{i \in \mathbb{N}} \frac{f(d_i(x_{1,i}, x_{3,i}))}{2^i} \\
&\leq \sum_{i \in \mathbb{N}} \frac{f(d_i(x_{1,i}, x_{2,i}) + d_i(x_{2,i}, x_{3,i}))}{2^i} = \sum_{i \in \mathbb{N}} \frac{d_i(x_{1,i}, x_{2,i}) + d_i(x_{2,i}, x_{3,i})}{2^i \times (1 + d_i(x_{1,i}, x_{2,i}) + d_i(x_{2,i}, x_{3,i}))} \\
&= \sum_{i \in \mathbb{N}} \Big( \frac{d_i(x_{1,i}, x_{2,i})}{2^i \times (1 + d_i(x_{1,i}, x_{2,i}) + d_i(x_{2,i}, x_{3,i}))} + \frac{d_i(x_{2,i}, x_{3,i})}{2^i \times (1 + d_i(x_{1,i}, x_{2,i}) + d_i(x_{2,i}, x_{3,i}))} \Big) \\
&= \sum_{i \in \mathbb{N}} \frac{d_i(x_{1,i}, x_{2,i})}{2^i \times (1 + d_i(x_{1,i}, x_{2,i}) + d_i(x_{2,i}, x_{3,i}))} + \sum_{i \in \mathbb{N}} \frac{d_i(x_{2,i}, x_{3,i})}{2^i \times (1 + d_i(x_{1,i}, x_{2,i}) + d_i(x_{2,i}, x_{3,i}))} \\
&\leq \sum_{i \in \mathbb{N}} \frac{d_i(x_{1,i}, x_{2,i})}{2^i \times (1 + d_i(x_{1,i}, x_{2,i}))} + \sum_{i \in \mathbb{N}} \frac{d_i(x_{2,i}, x_{3,i})}{2^i \times (1 + d_i(x_{2,i}, x_{3,i}))} = d_\pi(x_1, x_2) + d_\pi(x_2, x_3)
\end{aligned}$$

We now show that $(X_\pi, d_\pi)$ is complete. Let $S = (x_1, x_2 \ldots)$ where $x_i = (x_{i,1}, x_{i,2} \ldots)$ for all $i \in \mathbb{N}$ be a Cauchy sequence.

$$
\begin{aligned}
S \text{ is Cauchy} &\implies \text{for all } \varepsilon > 0 \text{ there exists an } N \in \mathbb{N} \text{ such that for all } n, m \geq N \text{ we have } d_\pi(x_n, x_m) \leq \varepsilon \\
&\implies \text{for all } \varepsilon > 0 \text{ there exists an } N \in \mathbb{N} \text{ such that for all } n, m \geq N \text{ we have } \sum_{i \in \mathbb{N}} \frac{f(d_i(x_{n,i}, x_{m,i}))}{2^i} \leq \varepsilon \\
&\implies \text{for all } i \in \mathbb{N} \text{ and } \varepsilon > 0 \text{ there exists an } N \in \mathbb{N} \text{ such that for all } n, m \geq N \text{ we have } \frac{f(d_i(x_{n,i}, x_{m,i}))}{2^i} \leq \varepsilon \\
&\implies \text{for all } i \in \mathbb{N} \text{ and } \varepsilon > 0 \text{ there exists an } N \in \mathbb{N} \text{ such that for all } n, m \geq N \text{ we have } f(d_i(x_{n,i}, x_{m,i})) \leq \varepsilon \\
&\implies \text{for all } i \in \mathbb{N} \text{ and } \varepsilon > 0 \text{ there exists an } N \in \mathbb{N} \text{ such that for all } n, m \geq N \text{ we have } d_i(x_{n,i}, x_{m,i}) \leq \varepsilon \\
&\implies (x_{1,i}, x_{2,i} \ldots) \text{ is Cauchy w.r.t } d_i \text{ for all } i \in \mathbb{N} \implies (x_{1,i}, x_{2,i} \ldots) \text{ is Convergent w.r.t } d_i \text{ for all } i \in \mathbb{N}
\end{aligned}
$$

Let $x_l = (x_{l,1}, x_{l,2} \ldots)$ be the sequence of these limits. It suffices to show that $S$ converges to $x_l$. Let $\varepsilon > 0$

$$
\sum_{i \in \mathbb{N}} \frac{d_i(x_{n,i}, x_{l,i})}{2^i \times (1 + d_i(x_{n,i}, x_{l,i}))} \leq \sum_{i \in \mathbb{N}} \frac{1}{2^i} \text{ for all } n \in \mathbb{N} \implies \text{there exists } k \in \mathbb{N} \text{ such that for all } n \in \mathbb{N}: \sum_{i=k+1}^{\infty} \frac{d_i(x_{n,i}, x_{l,i})}{2^i \times (1 + d_i(x_{n,i}, x_{l,i}))} \leq \frac{\varepsilon}{2}
$$

As $(x_{1,i}, x_{2,i} \ldots)$ converges to $x_{l,i}$ for all $i \in \mathbb{N}$ we have that there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and $i \in \{1, 2 \ldots k\}$ we have $\frac{d_i(x_{n,i}, x_{l,i})}{2^i \times (1 + d_i(x_{n,i}, x_{l,i}))} \leq d_i(x_{n,i}, x_{l,i}) \leq \frac{\varepsilon}{2k}$. For all $n \geq N$ it follows that:

$$
\begin{aligned}
d_\pi(x_n, x_l) = \sum_{i \in \mathbb{N}} \frac{d_i(x_{n,i}, x_{l,i})}{2^i \times (1 + d_i(x_{n,i}, x_{l,i}))} &= \sum_{i=1}^{k} \frac{d_i(x_{n,i}, x_{l,i})}{2^i \times (1 + d_i(x_{n,i}, x_{l,i}))} + \sum_{i=k+1}^{\infty} \frac{d_i(x_{n,i}, x_{l,i})}{2^i \times (1 + d_i(x_{n,i}, x_{l,i}))} \\
&\leq \Big( \sum_{i=1}^{k} \frac{d_i(x_{n,i}, x_{l,i})}{2^i \times (1 + d_i(x_{n,i}, x_{l,i}))} \Big) + \frac{\varepsilon}{2} \leq \Big( \sum_{i=1}^{k} \frac{\varepsilon}{2k} \Big) + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}
$$

So $S$ converges to $x_l$ as required. □

**Definition 1.1.2** A set $S$ in a topological space is said to be $G_\delta$ if $S$ is a countable intersection of open sets.

**Theorem 1.1.3.** *A $G_\delta$ subset of a completely metrizable topological space with the subspace topology is completely metrizable.*

*Proof.* The following proof is based on the proof of Theorem 1.2 in [3].
Let $S = \bigcap_{i \in \mathbb{N}} U_i$ for open sets $U_i$, be a $G_\delta$ subset of a completely metrizable topological space $(X, T)$ induced by complete metric space $(X, d)$.
Define the function $\phi' : S \to (X \times \mathbb{R} \times \mathbb{R} \ldots)$ by

$$
(x)\phi' = (x, \frac{1}{d(x, U_1^c)}, \frac{1}{d(x, U_2^c)}, \ldots)
$$

Note that here we do not divide by zero as the $U_i^c$ are closed so if $d(x, U_i^c) = 0$ then there is a sequence of points in $U_i^c$ converging to $x$ and thus $x \in U_i^c$ a contradiction. By Theorem 1.1.1 we have that $(X \times \mathbb{R} \times \mathbb{R} \ldots, d_\pi)$ is a complete metric space.
<u>Claim:</u> The function $\phi : S \to \mathrm{img}(\phi')$, defined by $(x)\phi = (x)\phi'$, is a homeomorphism.

Proof of claim: By construction $\phi$ is surjective, and clearly $\phi$ is also injective. We need to show $\phi$ is continuous. Let $\varepsilon > 0$ and $x \in S$. Let $k \in \mathbb{N}$ be such that $\sum_{i=k+1}^{\infty} \frac{1}{2^{i+1}} \leq \frac{\varepsilon}{3}$ and let

$$
\delta = \min\{\frac{\varepsilon}{3}, \frac{\min\{d(x, U_i^c) : i \in \{1, 2 \ldots k\}\}}{2}, \frac{\varepsilon \min\{d(x, U_i^c) : i \in \{1, 2 \ldots k\}\}^2}{6k}\}
$$

Let $y \in X$ be such that $d(x, y) < \delta$. For $\phi$ to be continuous it suffices to show that $d_\pi((x)\phi, (y)\phi) \leq \varepsilon$.

$$
\begin{aligned}
d_\pi((x)\phi, (y)\phi) = \frac{d(x, y)}{1 + d(x, y)} + \sum_{i=1}^{\infty} \frac{1}{2^{i+1}} \Big( \frac{|\frac{1}{d(x, U_i^c)} - \frac{1}{d(y, U_i^c)}|}{1 + |\frac{1}{d(x, U_i^c)} - \frac{1}{d(y, U_i^c)}|} \Big) &\leq d(x, y) + \frac{\varepsilon}{3} + \sum_{i=1}^{k} \frac{1}{2^{i+1}} \Big( \frac{|\frac{d(y, U_i^c) - d(x, U_i^c)}{d(x, U_i^c) d(y, U_i^c)}|}{1 + |\frac{1}{d(x, U_i^c)} - \frac{1}{d(y, U_i^c)}|} \Big) \\
\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \sum_{i=1}^{k} |\frac{d(y, U_i^c) - d(x, U_i^c)}{d(x, U_i^c) d(y, U_i^c)}| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \sum_{i=1}^{k} \frac{d(x, y)}{d(x, U_i^c) d(y, U_i^c)} &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \sum_{i=1}^{k} \frac{d(x, y)}{d(x, U_i^c)(d(x, U_i^c) - \delta)} \\
\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \sum_{i=1}^{k} \frac{d(x, y)}{d(x, U_i^c) \frac{d(x, U_i^c)}{2}} \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \sum_{i=1}^{k} \frac{2\delta}{d(x, U_i^c) d(x, U_i^c)} &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \sum_{i=1}^{k} \frac{2 \frac{\varepsilon \min\{d(x, U_i^c) : i \in \{1, 2 \ldots k\}\}^2}{6k}}{d(x, U_i^c)^2} \\
\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \sum_{i=1}^{k} \frac{\varepsilon}{3k} \leq \varepsilon &
\end{aligned}
$$

Finally we need to show $\phi^{-1}$ is continuous. Let $\varepsilon > 0$ and $x \in \text{img}(\phi)$, let $\delta = \min\{\frac{\varepsilon}{2}, \frac{1}{2}\}$ and let $y \in \text{img}(\phi)$.

$$d_\pi(x,y) < \delta \implies \frac{d((x)\phi^{-1},(y)\phi^{-1})}{1+d((x)\phi^{-1},(y)\phi^{-1})} < \delta \implies d((x)\phi^{-1},(y)\phi^{-1}) < \delta(1+d((x)\phi^{-1},(y)\phi^{-1}))$$
$$\implies d((x)\phi^{-1},(y)\phi^{-1})(1-\delta) < \delta \implies d((x)\phi^{-1},(y)\phi^{-1}) < \frac{\delta}{1-\delta} \leq \frac{\delta}{\frac{1}{2}} \leq \frac{\frac{\varepsilon}{2}}{\frac{1}{2}} = \varepsilon \quad \square$$

We have that S is homeomorphic to img($\phi$) which is contained in a complete metric space. So to show S is completely metrizable it suffices to show that img($\phi$) is closed and therefore complete.
Let $S = ((x_1)\phi, (x_2)\phi, \ldots)$ be a sequence in img($\phi$) which converges in $(X \times \mathbb{R} \times \mathbb{R} \ldots)$ to $(x, r_1, r_2 \ldots)$. For all $i \in \mathbb{N}$ we now have:

$$r_i = \lim_{n\to\infty} \frac{1}{d(x_n, U_i^c)} = \frac{1}{d(\lim_{n\to\infty} x_n, U_i^c)} = \frac{1}{d(x, U_i^c)}$$
$$\implies d(x, U_i^c) \neq 0 \implies x \notin U_i^c \implies x \in U_i$$

It follows that $x \in \bigcap_{i\in\mathbb{N}} U_i = S$. As $x \in S$ and $\frac{1}{d(x,U_i^c)} = r_i$ for all $i \in \mathbb{N}$ we have that $(x, r_1, r_2, \ldots) = (x)\phi \in \text{img}(\phi)$ as required. $\square$

In fact the converse of this theorem is also true and is proven in [3]. However the converse is not required in this document so the proof is omitted.

**Theorem 1.1.4.** *If $N$ is nowhere-dense in a topological space $(X,T)$, then $N^c$ is dense.*

*Proof.* Suppose for a contradiction that $U$ is a non-empty open set satisfying $U \cap N^c = \emptyset$.

$$U \cap N^c = \emptyset \implies U \backslash N = \emptyset \implies U \subseteq N \implies U \subseteq \overline{N} \implies U \subseteq (\overline{N})^\circ \implies (\overline{N})^\circ \neq \emptyset$$

The penultimate implication follows because $U$ is open. We therefore have that $(\overline{N})^\circ \neq \emptyset$ contradicting the nowhere-denseness of $N$. $\square$

**Theorem 1.1.5.** *Let $(X,T)$ be a topological space induced by a metric $d$. A set $N$ is nowhere-dense if and only if for all $x_1 \in X$ and $r_1 > 0$, there exists an $x_2 \in X$ and an $r_2 > 0$ such that $B(x_2, r_2) \subseteq B(x_1, r_1)\backslash N$*

*Proof.* ($\implies$) Let $N$ be nowhere-dense. Suppose for a contradiction that there exists $x_1 \in X$ and $r_1 > 0$ such that there are no $x_2 \in X$ and $r_2 > 0$ satisfying $B(x_2, r_2) \subseteq B(x_1, r_1)\backslash N$
Let $x \in B(x_1, r_1)$. As $B(x_1, r_1)$ is open there exists an $r > 0$ such that $B(x, r) \subseteq B(x_1, r_1)$. By assertion $B(x, r) \nsubseteq B(x_1, r_1)\backslash N$, so there exists $y \in N \cap B(x, r)$. As $r$ can be made arbitrarily small, we therefore have that $x$ is a limit point of $N$. As $x$ was arbitrary we have that all elements of $B(x_1, r_1)$ are limit points of $N$. It therefore follows that $B(x_1, r_1) \subseteq \overline{N}$ and therefore $B(x_1, r_1) \subseteq (\overline{N})^\circ$. So $x_1 \in (\overline{N})^\circ$, this contradicts the nowhere-denseness of $N$.
($\impliedby$) Suppose for a contradiction that $(\overline{N})^\circ \neq \emptyset$. Let $x \in (\overline{N})^\circ$. As $(\overline{N})^\circ$ is open there exists $r > 0$ such that $B(x, r) \subseteq (\overline{N})^\circ$. By assertion there exists $x_2 \in X$ and $r_2 > 0$ such that $B(x_2, r_2) \subseteq B(x, r)\backslash N \subseteq (\overline{N})^\circ\backslash N \subseteq \overline{N}\backslash N$. We therefore have $x_2 \in B(x_2, r_2) \subseteq \overline{N}\backslash N$ is a limit point of $N$.
But $B(x_2, r_2)$ is open and contains $x_2$ and $B(x_2, r_2) \subseteq B(x, r)\backslash N$ so $B(x_2, r_2) \cap N = \emptyset$. This contradicts the fact that $x_2$ is a limit point of $N$. $\square$

**Definition 1.1.6** A *Baire Space* is a topological space in which any countable collection of dense open sets $(U_n)_{n\in\mathbb{N}}$ has dense intersection.

**Theorem 1.1.7** (Baire Category Theorem)**.** *Baire Category Theorem.*

1. *Every completely metrizable topological space is a Baire Space.*

2. *A non-empty completely metrizable space cannot be expressed as a countable union of nowhere-dense sets, and is therefore not meagre.*

*Proof.* Part 1: The following proof is based on the proof of the Baire Category Theorem found in [1].
Let $(X,T)$ be a completely metrizable topological space induced by the complete metric $d$. Let $(U_n)_{n\in\mathbb{N}}$ be a countable collection of open sets.
We want to show that $I = \bigcap_{i\in\mathbb{N}} U_i$ is dense in $X$. Let $x \in X$. As $x$ is arbitrary it suffices to show that $x$ is a limit point of $I$.
Let $U_x$ be an open set containing $x$. It suffices to show that $I \cap U_x \neq \emptyset$. As $T$ is induced by $d$ there exists an $r_0 > 0$ such that $B(x, r_0) \subseteq U_x$.
Let $B_0 = V_0 = B(x, r_0)$. Let $V_n = U_n \cap B_{n-1}$ for all $n \in \mathbb{N}$ and $B_n = B(x_n, r_n)$ for all $n \in \mathbb{N}$ be such that $B_n \subseteq B(x_n, 2r_n) \subseteq V_n$ and $0 < r_n < \frac{r_{n-1}}{2}$.

Note that this construction is possible as each $V_i$ is constructed by intersecting an open set with an open set, so each $V_i$ is open. In addition $V_i$ is non-empty as the intersection of a dense set and an open set.
We will now show that $(x_n)_n$ is a Cauchy sequence. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ large enough that $\frac{r_0}{2^N} < \frac{\varepsilon}{2}$. For $n \geq N$ we have $x_n \in B_n \subseteq V_n \subseteq B_{n-1} \ldots \ldots \subseteq B_N = B(x_N, r_N) \subseteq B(x_N, \frac{r_0}{2^N})$. This final inclusion follows as each B has radius less than half the size of the previous one. So we have that $d(x_n, x_N) < \frac{r_0}{2^N} < \frac{\varepsilon}{2}$.
So for $n, m \geq N$ we have:

$$d(x_n, x_m) \leq d(x_n, x_N) + d(x_m, x_N) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

We therefore have that $(x_n)_n$ is Cauchy.
As $(X, d)$ is complete we have that $(x_n)_n$ is convergent. Let $y = \lim_{n\to\infty} x_n$. It suffices to show that $y \in U_x$ and $y \in I$. For $k \in \mathbb{N}$, the sequence $(x_{k+1}, x_{k+2}, \ldots)$ converges to $y$. For all $i > k$ we have

$$x_i \in B_i \subseteq B_{i-1} \ldots \subseteq B_{k+1} \subseteq \overline{B_{k+1}} = \{z \in X : d(x_{k+1}, z) \leq r_{k+1}\} \subseteq \{z \in X : d(x, z) < 2r_{k+1}\} \subseteq V_{k+1} \subseteq B_k.$$

As for all $i > k$ we have $x_i \in \overline{B_{k+1}}$ a closed set, we have that $y \in \overline{B_{k+1}} \subseteq B_k \subseteq V_k \subseteq U_k$. We therefore have that $y \in \bigcap_{i\in\mathbb{N}} U_i = I$, and $y \in B_1 \subseteq B_0 \subseteq U_x$ as required. □
Part 2: Suppose for a contradiction that $(X, T)$ is a non-empty completely metrizable space such that $X = \bigcup_{i\in\mathbb{N}} N_i (= \bigcup_{i\in\mathbb{N}} \overline{N_i})$ where the $N_i$ are nowhere-dense for all $i \in \mathbb{N}$. As $(X, T)$ is completely metrizable we have that $(X, T)$ is a Baire space by part 1. Consider the sets $(\overline{N_i}^c)_{i\in\mathbb{N}}$. As the complements of closed nowhere-dense sets, these sets are open and dense. We therefore have the following:

$$\bigcap_{i\in\mathbb{N}} \overline{N_i}^c \text{ is dense} \implies \left(\bigcup_{i\in\mathbb{N}} \overline{N_i}\right)^c \text{ is dense} \implies (X)^c \text{ is dense} \implies \emptyset \text{ is dense}$$

As $(X, T)$ is a non-empty topology, $X$ is a non-empty open set. But $X \cap \emptyset = \emptyset$, this contradicts the denseness of $\emptyset$. □

**Theorem 1.1.8.** *If $(X, *, T)$ is a topological group, then we have that for all $x \in X$ the functions given by:*

$$(y)\phi_{x_r} = yx \qquad (y)\phi_{x_l} = xy$$

*are homeomorphisms.*

*Proof.* Let $x \in X$, as $\phi_{x_r}$ has the inverse $\phi_{x^{-1}_r}$ and $\phi_{x_l}$ has inverse $\phi_{x^{-1}_l}$ we have that these are bijections. In addition we have by symmetry that if $\phi_{x_r}$ is continuous then $\phi_{x_r}^{-1}, \phi_{x_l}$ and $\phi_{x_l}^{-1}$ are continuous. It therefore suffices to show that $\phi_{x_r}$ is continuous. Let $U$ be open in $T$. We will show that $(U)\phi_{x_r}^{-1}$ is open.
We have that $(U)*^{-1}$ is open. By definition of the product topology, this means that there exists a collection of open sets $B$ such that $\cup B = (U)*^{-1}$ and for all $b \in B$ we have $b = U_{b,1} \times U_{b,2}$ for open sets $U_{b,1}$ and $U_{b,2}$. As $U_{b,1}$ is open for all $b \in B$, it suffices to show that $(U)\phi_{x_r}^{-1} = \cup\{U_{b,1} : x \in U_{b,2}\}$.

$$\begin{aligned} y \in (U)\phi_{x_r}^{-1} &\iff yx \in U \iff (y, x) \in (U)*^{-1} \iff (y, x) \in \cup B \\ &\iff \text{there exists } b \in B \text{ such that } (y, x) \in b \\ &\iff \text{there exists } b \in B \text{ such that } y \in U_{b,1} \text{ and } x \in U_{b,2} \\ &\iff y \in \cup\{U_{b,1} : x \in U_{b,2}\} \end{aligned}$$

We therefore have that $(U)\phi_1^{-1} = \cup\{U_{b,1} : x \in U_{b,2}\}$ and is thus open as required. □

## 1.2 Ordinals

**Definition 1.2.1** An *ordinal* is a set $\alpha$ such that satisfying the following conditions:

1. Transitivity: If $x \in \alpha$ and $y \in x$ then $x \in \alpha$.
2. Well ordered: The pair $(\alpha, \{(a, b) \in \alpha \times \alpha : \text{we have precisely one of } a \in b \text{ or } a = b)\})$ is a well ordered set.

It's not hard to see that if $\alpha$ is an ordinal then $\alpha^+ := \alpha \cup \{\alpha\}$ is also an ordinal. This is in fact the smallest ordinal containing alpha.

**Theorem 1.2.2.** *Every element of an ordinal is an ordinal.*

*Proof.* TODO Cite. Let $\alpha$ be an ordinal and let $x \in \alpha$.
Transitivity: If $z \in y \in x$ then we have that $y \in \alpha$ and $z \in \alpha$ by the transitivity of $\alpha$. Also $z < y < x$ so by the transitivity of $\alpha$ as a well ordered set we have $z < x$ and thus $z \in x$.
Well ordered: By the transitivity of $\alpha$ we have that $x \subseteq \alpha$ and therefore is well ordered. □

**Theorem 1.2.3.** *If $S$ is a non-empty set of ordinals then it has a least element (an element contained in all other elements).*

*Proof.* TODO Cite.
We will show that $I := \cap S$ is the desired element.
Claim: If $\alpha \in S$ and $\alpha \neq I$ then $\min(\alpha \backslash I) = I$.
Proof of Claim: Let $\alpha \in S \backslash I$. Let $x \in \min(\alpha \backslash I)$. We have that $x < \min(\alpha \backslash I)$ and $x \in \alpha$ by transitivity so $x \in I$. Conversely let $x \in I$. We want to show that $x \in \min(\alpha \backslash I)$. As $x \in I$ we have that $x \neq \min(\alpha \backslash I)$. Suppose for a contradiction that $x > \min(\alpha \backslash I)$. Then $\min(\alpha \backslash I) \in x$ and $x \in I$, so by transitivity of ordinals $\min(\alpha \backslash I)$ is an element of every element of $S$, and is thus an element of $I$, a contradiction. It follows that $x < \min(\alpha \backslash I)$ and thus $x \in \min(\alpha \backslash I)$ as required. □
It follows from the claim that if $I \in S$ then $I$ is contained in all other elements as required. Suppose for a contradiction that $I \notin S$. By the claim we have that $I$ is an element of every element of $S$ and thus $I \in \cap S = I$. As $I$ is an element of the elements of $S$ which are ordinals, we have that $I$ is an ordinal. Therefore $I$ is well ordered with $I < I$, a contradiction. □

**Corollary 1.2.4.** *All ordinals are compatible, so if $\alpha$ and $\beta$ are ordinals then we have that either $\alpha \in \beta$, $\beta \in \alpha$ or $\alpha = \beta$.*

**Corollary 1.2.5.** *All transitive sets of ordinals are ordinals.*

**Theorem 1.2.6.** *If $\alpha$ and $\beta$ are ordinals and there is a bijection $\phi : \alpha \to \beta$ which preserves order. Then $\phi$ is the identity map and $\alpha = \beta$.*

*Proof.* Let $\phi : \alpha \to \beta$ be an order preserving bijection. Suppose for a contradiction that an element of $\alpha$ is not fixed by $\phi$. Let $m$ be the minimum such element. It follows that $(m)\phi \neq m$. If $(m)\phi \in m$ then it follows that $(m)\phi < m$ and thus $(m)\phi\phi = (m)\phi$, contradicting the injectivity of $\phi$. It follows that $m \in (m)\phi \in \beta$. Thus we have $m \in \beta = \text{img}(\phi)$. So we have that $(m)\phi^{-1} > m$ and $m < (m)\phi$, a contradiction. □

**Definition 1.2.7** A *cardinal* is defined to be an ordinal which in not in bijective correspondence with any lesser ordinal.

It will be shown in the next section that all sets are in bijective correspondence with an ordinal so this is a reasonable way to view cardinality of sets in general.

## 1.3 Infinite sets

In this section we will prove various theorems concerning infinite sets which will be used throughout this document.

**Theorem 1.3.1** (Zorn's Lemma). *If $(P, \leq)$ is a partially ordered set such that every chain has an upper bound, then $P$ has a maximal element $M$ (an element such that there is no $x \in P$ with $x > M$).*

*Proof.* The following proof is based of the proofs of Lemma 3.3 and Theorem 4.2 in [12].
Suppose for a contradiction there is no maximal element. By the axiom of choice there is a choice function $c$ for $\mathcal{P}(P)$. For $a \in P$ let $G_a := \{x \in P : x > a\}$. For $C \subseteq P$, a chain, let $G_C := \{x \in P : x \text{ is an upper bound for } C\}$. As $P$ has no maximal element and every chain has an upper bound we have that these sets are non-empty.
We call a chain $C$ a c-chain if it satisfies the following:

1. $C$ is well-ordered by $<$.
2. If $C_M \subset C$ is an initial segment of $C$ with maximal element $M$ then $\min(C \backslash C_M) = c(G_M)$.
3. If $C_u \subset C$ is an initial segment of $C$ with no maximal element then $\min(C \backslash C_u) = c(G_{C_u})$.

Claim: If $C_1 \neq C_2$ are c-chains then one is an initial segment of the other.
Proof of claim: Suppose that $C_2$ is not an initial segment of $C_1$. It suffices to show that if all initial segments of $C_1$ are initial segments of $C_2$, then $C_1$ is an initial segment of $C_2$. This is because the initial segments of $C_1$ are well ordered by containment, and it would follow that there can't be a least initial segment of $C_1$, which is not an initial segment of $C_2$. Suppose that all initial segments of $C_1$ are initial segments of $C_2$. If $C_1$ has a maximal element $m$ then it follows from the definition of a c-chain that $m = \min(C_1 \backslash \{x \in C_1 : x < m\}) = \min(C_2 \backslash \{x \in C_1 : x < m\})$. It follows that $C_1 = \{x \in C_2 : x < \min(C_2 \backslash C_1)\}$ is an initial segment of $C_2$. If $C_1$ has no maximal element then it follows that $C_1 \subset C_2$, as all elements of $C_1$ are contained in an initial segment bounded above by a greater element of $C_1$. It again follows that $C_1$ is an initial segment of $C_2$. □
Let $S$ be the union of all c-chains of $P$. Note that for all elements $x \in S$ there exists a c-chain $C_x$ such that $x = \max(C_x)$. This follows as $x$ is contained in a c-chain, and if we restrict that c-chain to the elements less than or equal to $x$, then we also have a c-chain. We will now show that $S$ is a c-chain.

1. Let $s, t \in S$ and let $C_s$ and $C_t$ be c-chains with $s, t$ maximal respectively. By the claim, we have that either $C_s \subseteq C_t$ or $C_t \subseteq C_s$, so either $s = t, s < t$ or $t < s$ and therefore $S$ is totally ordered. Let $A \subseteq S$ be non-empty and let $t \in A$. Consider the set $B := \{x \in A : x < t\}$. If $B$ is empty then $t$ is minimal in $A$, otherwise if $B$ has a least element then $A$ has a least element and so $S$ is well ordered. All $x \in B$ are contained in a c-chain $C_x$ with $x$ as maximal element. If $C_x \supset C_t$ it follows that $t \leq x$, a contradiction, so we have $C_x \subseteq C_t$ and thus $B \subseteq C_t$ so $B$ has a minimal element and $S$ is well ordered.

2. Let $S_M \subset S$ be an initial segment with maximal element $M$. Let $s \in S \backslash S_M$. It follows that there is a c-chain $C_s \supseteq S_M$ with maximal element $s$, and thus $c(G_M) = \min\{y \in C_s : y \notin \{x \in C_s : x \leq M\}\} \in S$. As $c(G_M) \in C_s$ it follows that $c(G_M) \leq s$ but $s$ was arbitrary so we have that $\min\{x \in S : x \notin S_M\} = c(G_M)$ as required.
3. Let $S_u \subset S$ be an initial segment with no maximal element. Let $s \in S \backslash S_u$. Let $C_s$ be a c-chain with $s$ as its maximal element. If $x \in S_u$ it follows there is a c-chain with $x$ maximal, which is an initial segment of $C_s$ and thus $x \in C_s$. So we have that $S_u \subset C_s \subseteq S$. As $C_s$ is a c-chain it follows that $c(G_{S_u}) = \min\{y \in C_s : y \notin S_u\} \in S$ and, as $s$ was arbitrary and $s \geq c(G_{S_u})$, we have that $\min\{x \in S : x \notin C_u\} = c(G_{S_u})$.

Now we have that $S$ is a c-chain and there is no greater c-chain than $S$ by definition. However if $S$ has a maximal element $M$ then $S \cup \{c(G_M)\}$ is a strictly greater c-chain and if $S$ has no maximal element we have that $S \cup \{c(G_S)\}$ is a strictly greater c-chain so we have reached a contradiction. □

**Theorem 1.3.2.** *Every set $A$ is well orderable.*

*Proof.* The following proof is based of the proof of the same theorem given in [11].
If $A = \emptyset$ then the result is trivial, otherwise let $A_o := \{(S, <) : S \subseteq A, < \text{ is a well ordering of S}\}$. Let $A_o$ be partially ordered by: $(S_1, <_1) \leq (S_2, <_2)$ if $S_1 \subseteq S_2$, $<_2$ restricted to $S_1$ is equal to $<_1$ and for all $x_1 \in S_1, x_2 \in S_2 \backslash S_1$ we have $x_1 < x_2$ with respect to $<_2$. Note that $A_o$ is non-empty as singletons are trivially well orderable.
Let $C \subseteq A_o$ be a chain. We claim that the tuple:

$$(S_c, <_c) := (\{a \in A : a \in S \text{ for some } (S, <) \in C\}, a_1 < a_2 \iff \text{there exists } (S, <) \in C \text{ such that } a_1 < a_2 \text{ with respect to } <)$$

is an upper bound for $C$. This is well-defined as the orderings agree whenever they are defined. Given any two elements $x, y \in S_c$ there is a well ordered set containing both $x$ and $y$ and so we have precisely one of $x < y, x = y$ and $x > y$. Finally, if $D \subseteq S_c$ is non-empty then there exists $(S, <) \in C$ such that $S \cap D \neq \emptyset$. It follows that $\min(S \cap D)$ is minimal in $D$ as all elements of $D \backslash S$ are greater by the definition of the orderings.
By Zorn's Lemma $A_o$ has a maximal element $(X, <_x)$. If $X \neq A$ then there is an element $a \in A \backslash X$ and therefore we have that $(X \cup \{a\}, <_x$ (except $a$ is greater than every other element)) $> (X, <_x)$, a contradiction. Therefore we have that $A = X$, and is well ordered by $<_x$. □

**Theorem 1.3.3.** *Every well ordered set $(S, <)$ is order isomorphic to an ordinal.*

*Proof.* For $x \in S$, let $I_x := \{y \in S : y < x\}$. Note that the initial segments of $S$ are well ordered by $I_x < I_y \iff x < y$. Suppose for a contradiction that there are initial segments of $S$ which are not order isomorphic to an ordinal. Let $I_m$ be the minimal such initial segment. As ordinals are only order isomorphic if they are equal, it follows that for all $I_x < I_m$ there is a unique ordinal $\alpha_x$ order isomorphic to $I_x$. By the axiom of replacement it follows that $\alpha_m := \{\alpha_x : x < m\}$ is a set. Let $\phi : I_m \to \alpha_m$ be defined by $\phi(x) = \alpha_x$. We have that $\phi$ is injective as different initial segments can't be order isomorphic and it is surjective by definition of $\alpha_m$. It must also preserve order as the initial segments are initial segments of each other and thus the order-isomorphisms are extensions of each other. Let $\alpha \in \alpha_x \in \alpha_m$, let $\psi : I_x \to \alpha_x$ be an order isomorphism. It follows that $\psi|_{I_{\alpha\psi^{-1}}}$ is an order isomorphism between $\alpha$ and an initial segment of $S$, so $\alpha \in \alpha_m$. We now have that $\alpha_m$ is a transitive set of ordinals and is thus an ordinal. In addition $\phi$ is an order isomorphism from $I_m$ to $\alpha_m$ so we have contradicted then definition of $I_m$.
We now have that all initial segments of $S$ are order isomorphic to an ordinal. Therefore we can apply the same reasoning we did to $I_m$ to conclude that $S$ is order isomorphic to an ordinal as required. □

By the previous two theorems we now have that for every set, there exists an ordinal in bijective correspandance with it. Thus we may now safely assign cardinals to any set. If $S$ is a set we will use the notation $|S|$ to denote it's *cardinality* (the unique cardinal in bijective correspondence with it).

**Theorem 1.3.4.** *If $\Omega$ is an infinite set, then $|\Omega| = 2|\Omega|$.*

*Proof.* Without loss of generality we may assume that $\Omega$ is a cardinal. Let $\phi : \Omega \to \Omega \sqcup \Omega$ be defined by:

$$(x)\phi = \left\{ \begin{array}{ll} (\alpha + (k/2), 0) & \text{if } x = \alpha + k \text{ for a limit ordinal } \alpha \text{ and } k \in \aleph_0 \text{ even} \\ (\alpha + ((k-1)/2), 1) & \text{if } x = \alpha + k \text{ for a limit ordinal } \alpha \text{ and } k \in \aleph_0 \text{ odd} \end{array} \right\}$$

We have that $\phi$ is a bijection and therefore $|\Omega| = |\Omega \sqcup \Omega| = 2|\Omega|$. □

**Corollary 1.3.5.** *If $\Omega$ is an infinite set, then it has moiety subsets.*

**Theorem 1.3.6.** *If $\Omega$ is an infinite set then $|\Omega| = |\Omega|^2$.*

*Proof.* Without loss of generality we can assume $\Omega$ is a cardinal. Let $\Omega = \{(S, \phi) : S \subseteq \Omega, \phi : S \to S \times S \text{ is a bijection}\}$. Let $\Omega_o$ be partially ordered by: $(S_1, \phi_1) < (S_2, \phi_2) \iff S_1 \subset S_2$ and $\phi_2|_{S_1} = \phi_1$. Note that $\Omega_o$ is non-empty as there is a bijection from $\aleph_0$ to $\aleph_0 \times \aleph_0$.
Let $((S_i, \phi_i))_{i<\kappa}$ be a chain. We claim that $(\cup_{i<\kappa} S_i, \phi_U)$ where $\phi_U$ is defined by:

$$(x)\phi_U = (x)\phi_i \text{ for } x \in S_i$$

This is a well-defined bijection as the bijections agree whenever they are defined, and therefore it is clearly an upper bound.
By Zorn's Lemma $\Omega_o$ has a maximal element $(X, \phi_x)$. If $|X| = |\Omega|$ then we are done as there are bijections from $\Omega \to X$ and $\Omega \times \Omega \to X \times X$. Suppose for a contradiction that $|X| < |\Omega|$, then $|X| < |X^c|$ and so there exists $X' \subseteq X^c$ such that:

$$|X| = |X'| = |X' \times X'| = |X' \times X'| + |X' \times X| + |X \times X'| = |(X' \times X') \cup (X' \times X) \cup (X \times X')|$$

Let $\phi'_x : X' \to (X' \times X') \cup (X' \times X) \cup (X \times X')$ be a bijection. By adjoining the functions $\phi_x$ and $\phi'_x$ we can construct a bijection $\phi''_x : X \cup X' \to (X \cup X') \times (X \cup X')$. We therefore have that $(X, \phi_x) < (X \cup X', \phi''_x)$ a contradiction. ☐

**Theorem 1.3.7.** *An infinite symmetric group* $\mathrm{Sym}(\Omega)$ *has cardinality* $2^{|\Omega|}$.

*Proof.* Let $S$ be a subset of $\Omega$. If $S$ is finite, then there exists a bijection $\phi : S \to \{0 \ldots (n-1)\}$ for some $n \in \mathbb{N}_0$. We can construct a bijection $f_S : \Omega \to \Omega$ as follows:

$$(x)f_S = \left\{ \begin{array}{ll} ((x)\phi + 1 \text{ mod n})\phi^{-1} & x \in S \\ x & \text{otherwise} \end{array} \right\}$$

If $S$ is infinite then we can construct a partition $\{M_1, M_2\}$ of $S$ such that $M_1, M_2$ are moieties of $S$. There exists a bijection $\phi : M_1 \to M_2$ so we can construct a bijection $f_S : \Omega \to \Omega$ as follows:

$$(x)f_S = \left\{ \begin{array}{ll} (x)\phi & x \in M_1 \\ (x)\phi^{-1} & x \in M_2 \\ x & \text{otherwise} \end{array} \right\}$$

For $|S| \geq 2$ we have $supp(f_S) = S$ and therefore the $f_S$ are all distinct.

$$2^{|\Omega|} = 2^{|\Omega|} - |\Omega| = |P(\Omega)\backslash\{\{\phi\} \cup \{\{x\} : x \in \Omega\}\}| \leq |\mathrm{Sym}(\Omega)| \leq |\Omega^\Omega| \leq |(2^{|\Omega|})^{|\Omega|}| = |2^{|\Omega \times \Omega|}| = 2^{|\Omega|}$$

Therefore we have that $|\mathrm{Sym}(\Omega)| = 2^{|\Omega|}$. ☐

## 1.4 Ultrafilters

In this section we will prove various facts about ultrafilters which will be needed in the first section of chapter 4.

**Definition 1.4.1** Given a set $\Omega$, a *filter* on $\Omega$ is defined to be a collection of subsets $F$ of $\Omega$ satisfying:

1. We have $\Omega \in F$.
2. For all $A, B \in F$ we have $A \cap B \in F$.
3. If $A \subseteq B \subseteq \Omega$ and $A \in F$ then $B \in F$

**Example 1.4.2** If $\Omega$ is a set then the following are filters on $\Omega$.

1. The set $\{\Omega\}$.
2. The set $P(\Omega)$.
3. The set $\{X \subseteq \Omega : |\Omega \backslash X| < \kappa\}$, where $\kappa$ is any infinite cardinal.
4. The set $\{X \subseteq \Omega : S \subseteq X\}$, where $S$ is any subset of $\Omega$.

**Definition 1.4.3** Given a set $\Omega$, an *ultrafilter* on $\Omega$ is defined to be a filter $\mathcal{U}$ on $\Omega$ satisfying:

1. The empty set is not an element of $\mathcal{U}$.
2. There is no filter $\mathcal{U}'$ such that $\mathcal{U} \subset \mathcal{U}' \subset P(\Omega)$

**Example 1.4.4** If $\Omega$ is a set and $x \in \Omega$ then $\{X \subseteq \Omega : x \in X\}$ is an ultrafilter on $\Omega$.

**Theorem 1.4.5.** *Let $\Omega$ be a set, let $S \subseteq \Omega$ and let $\mathcal{U}$ be an ultrafilter on $\Omega$. Then precisely one of $S$ and $S^c$ is in $\mathcal{U}$.*

*Proof.* If we had that both $S$ and $S^c$ were in $\mathcal{U}$ then $S \cap S^c = \emptyset$ would also be in $\mathcal{U}$, a contradiction. Suppose for a contradiction that neither $S$ nor $S^c$ are in $\mathcal{U}$. Let $\mathcal{V}$ be defined as follows:

$$\mathcal{V} := \{V : V \supseteq S \cap U \text{ for some } U \in \mathcal{U}\}$$

We will now show that $\mathcal{V}$ is a filter contradicting condition 2 for ultrafilters.

1. As $\mathcal{U}$ is a filer, we have $\Omega \in \mathcal{U}$ and therefore as $\Omega \supseteq \Omega \cap S$ it follows that $\Omega \in \mathcal{V}$.
2. If $A \supseteq A' \cap S$ for some $A' \in \mathcal{U}$ and $B \supseteq B' \cap S$ for some $B' \in \mathcal{U}$, then $A \cap B \supseteq A' \cap B' \cap S$ and as $A' \cap B' \in \mathcal{U}$ it follows that $A \cap B \in \mathcal{V}$.
3. If $B \supseteq A$ for some $A \in \mathcal{V}$ then $A \supseteq A' \cap S$ for some $A' \in U$. It is clear that $B \supseteq A' \cap S$ as well and therefore $B \in \mathcal{V}$.

We therefore have that $\mathcal{V}$ is a filter. We have that $S \supseteq \Omega \cap S$ so $S \in \mathcal{V}$. In addition for all $U \in \mathcal{U}$ we have $U \supseteq U \cap S$. So $\mathcal{U} \subset \mathcal{V}$. It now suffices to show that $\mathcal{V} \subset P(\Omega)$. Suppose for a contradiction that $\mathcal{V} = P(\Omega)$. It follows that $\emptyset \in \mathcal{V}$ and so $\emptyset \supseteq U \cap S$ for some $U \in \mathcal{U}$. We therefore have that $\emptyset = U \cap S$ and so $U \subseteq S^c$ and $S^c \in \mathcal{U}$ a contradiction. □

**Theorem 1.4.6.** *Ultrafilters on an infinite set $\Omega$ are uniquely determined by their moiety elements.*

*Proof.* Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be ultrafilters with the same moiety elements and let $U \in \mathcal{U}_1$. If $U$ is the superset of some moiety then $U$ is also in $\mathcal{U}_2$. If not then $|\Omega \backslash U| = |\Omega|$ and we can therefore construct disjoint moieties $M_1$ and $M_2$ such that $M_1 \cup M_2 = \Omega \backslash U$. It therefore follows that $M_1 \cup U$ and $M_2 \cup U$ are moieties and elements of $\mathcal{U}_1$. So we also have that $M_1 \cup U$ and $M_2 \cup U$ are elements of $\mathcal{U}_2$ and therefore their intersection $U$ is in $\mathcal{U}_2$. We therefore have that $\mathcal{U}_1 \subseteq \mathcal{U}_2$ and by symmetry $\mathcal{U}_2 \subseteq \mathcal{U}_1$, so $\mathcal{U}_1 = \mathcal{U}_2$ as required. □

**Definition 1.4.7** We say that a set $S$ has the *finite intersection property* if all finite non-empty subsets of $S$ have non-empty intersection.

**Theorem 1.4.8** (Ultrafilter Lemma)**.** *Let $\Omega$ be a set and let $S$ be a collection of subsets of $\Omega$ with the finite intersection property. Then there is an ultrafilter $\mathcal{U}$ on $\Omega$ such that $S \subseteq \mathcal{U}$.*

*Proof.* The following proof is based on the proof of Theorem 1.8 in [9].
Let $S \subseteq P(\Omega)$ be a non-empty set with the finite intersection property. Let $F(S)$ denote the set of all filters which contain $S$ and don't have the empty set as an element. The set $F(S)$ is partially ordered by $\subseteq$. If $F(S)$ has a maximal element then this element must be an ultrafilter, therefore by Zorn's Lemma it suffices to show that every chain of $F(S)$ has an upper bound.
Consider the set $V_0 := \{U : U \supseteq (s_1 \cap s_2 \cap s_3 \ldots \cap s_n) \text{ for some } s_1, s_2 \ldots s_n \in S\}$.

1. We have $\Omega \in V_0$ as $S$ is non-empty and all elements of $S$ are subsets of $\Omega$.
2. If $A, B \in V_0$ then $A \supseteq (s_1 \cap s_2 \ldots \cap s_n)$ and $B \subseteq (s_{n+1} \cap s_{n+2} \ldots \cap s_m)$ for some $s_1, s_2 \ldots s_m \in S$. It follows that $A \cap B \supseteq (s_1 \cap s_2 \ldots \cap s_m)$ and thus $A \cap B \in V_0$.
3. If $A \in V_0$ and $A \subseteq B \subseteq \Omega$ then $A \supseteq (s_1 \cap s_2 \ldots \cap s_n)$ for some $s_1, s_2 \ldots s_n \in S$ and thus $B \supseteq (s_1 \cap s_2 \ldots \cap s_n)$ so $B \in V_0$.

We therefore have that $V_0$ is a filter, it clearly contains $S$ and as $S$ has the finite intersection property it also doesn't contain the empty set. So we have $V_0 \in F(S)$ so $V_0$ is an upper bound for the empty chain.
Let $(F_i)_{i<\alpha}$ be a non-empty chain of $F(S)$. Consider the set $V_1 := \bigcup_{i<\alpha} F_i$.

1. As the non-empty union of filters $\Omega \in V_1$.
2. Let $U_1, U_2 \in V_1$, then $U_1 \in F_{i_1}$ for some $i_1$ and $U_2 \in F_{i_2}$ for some $i_2$ and therefore $U_1, U_2 \in \max\{F_{i_1}, F_{i_2}\}$. So $U_1 \cap U_2 \in \max\{F_{i_1}, F_{i_2}\} \subseteq V_1$.
3. Let $U_1 \in V_1$ and let $U_2 \supseteq U_1$, then $U_1 \in F_i$ for some $i$ and therefore $U_2 \in F_i$ so we have $U_2 \in V_1$.

It follows that $V_1$ is a filter. As a non-empty union of elements of $F(S)$ we have also that $S \subseteq V_1$ and $\emptyset \notin V_1$ so we have $V_1 \in F(S)$. By construction $V_1 \supseteq F_i$ for all $i < \alpha$ and thus is an upper bound as required. □

**Definition 1.4.9** Let $\Omega, S$ be sets and $L \subseteq S^{\Omega}$. We say that $L$ has *large oscillation* if the following condition is satisfied:
If $n \in \mathbb{N}$ and $\{f_i : i < n\} \subseteq L$ are distinct and $\{s_i : i < n\} \subseteq S$ then there exists $\omega \in \Omega$ such that $(\omega)f_i = b_i$ for all $i < n$.

**Theorem 1.4.10.** *Let $\Omega$ be an infinite set. Then there exists $L \subseteq \{0,1\}^{\Omega}$ such that $|L| = 2^{|\Omega|}$ and $L$ has large oscillation.*

*Proof.* The following proof is based on the proof of Theorem 2.2 in [9].
Let $\Omega'$ be defined by:

$$\Omega' := \{(s, S, \phi) : s \text{ is a finite subset of } |\Omega|, S \in P(P(s)), \phi \in \{0,1\}^S\}$$

As $s$ is finite we have that $P(P(s))$ is finite, $S$ is finite and $\{0,1\}^S$ is finite. It follows that $|\Omega'| = |\Omega|$ as we have $|\Omega|$ choices for $s$ and finitely many choices for $S$ and $\phi$. We may therefore assume without loss of generality that $\Omega = \Omega'$ as $|\Omega| = |\Omega'|$ and $\Omega''$ would give the exact same set as $\Omega'$.
Let $f : P(|\Omega|) \to \{0,1\}^\Omega$ be defined by $(\Sigma)f = f_\Sigma$ where $f_\Sigma : \Omega \to \{0,1\}$ is defined by:

$$(s, S, \phi)f_\Sigma = \left\{ \begin{array}{ll} (\Sigma \cap s)\phi & \Sigma \cap s \in S \\ 0 & \Sigma \cap s \notin S \end{array} \right\}$$

Let $L := \text{img}(f)$. To show that $|L| = 2^{|\Omega|}$ it suffices to show that $f$ is injective as $|P(\Omega)| = 2^{|\Omega|}$. Let $\Sigma_1, \Sigma_2 \in P(|\Omega|)$ be distinct. Without loss of generality we have $\Sigma_1 \backslash \Sigma_2 \neq \emptyset$. Let $x \in \Sigma_1 \backslash \Sigma_2$. Let $s = \{x\}$, $S = \{s\}$ and $\phi$ be such that $(s)\phi = 1$ then:

$$(s, S, \phi)f_{\Sigma_1} = (\Sigma_1 \cap s)\phi = (s)\phi = 1 \neq 0 = (\emptyset)\phi = (s \cap \Sigma_2)\phi = (s, S, \phi)f_{\Sigma_2}$$

It follows that $(\Sigma_1)f \neq (\Sigma_2)f$ and thus $f$ is injective.
It remains to show that $L$ has large oscillation. Let $n \in \mathbb{N}$, $\{f_{A_m} : m < n\} \subseteq L$ be distinct and $\{k_m : m < n\} \subseteq \{0,1\}$. For each $(m_1, m_2)$ such that $m_1 < m_2 < n$ let $a_{m_1,m_2} \in A_{m_1} \triangle A_{m_2}$.
Let $s := \{a_{m_1,m_2} : m_1 < m_2 < n\}$, $S := \{A_m \cap s : m < n\}$ and let $\phi : S \to \{0,1\}$ be defined by $(A_m \cap s)\phi = k_m$.
To see that $\phi$ is well-defined (that no elements of $S$ can be represented as above in two ways) observe that if $A_{m_1} \neq A_{m_2}$ then $a_{m_1,m_2} \in A_{m_1} \triangle A_{m_2}$ and so is in precisely one of $s \cap A_{m_1}$ and $s \cap A_{m_2}$.
It is clear that $(s, S, \phi) \in \Omega$ and for all $m < n$ we have that $(s, S, \phi)f_{A_m} = (s \cap A_m)\phi = k_m$ as required. □

**Theorem 1.4.11.** *For all infinite sets $\Omega$ there are $2^{2^{|\Omega|}}$ ultrafilters on $\Omega$.*

*Proof.* The following proof is based on the proof of Theorem 2.5 in [9].
Let $\Omega$ be an infinite set. It is clear that any ultrafilter on $\Omega$ is an element of $P(P(\Omega))$ and therefore there are at most $|P(P(\Omega))| = 2^{2^{|\Omega|}}$ of them. To show there are at least $2^{2^{|\Omega|}}$ we will construct such a family of ultrafilters.
By Theorem 1.4.10 let $L \subseteq \{0,1\}^\Omega$ be a family of large oscillation such that $|L| = 2^{|\Omega|}$. For $S \in P(L)$ we define $B(S)$ by:

$$B(S) := \{(\{0\})f^{-1} : f \in S\} \cup \{(\{1\})f^{-1} : f \in S^c\}$$

Let $B_1, B_2 \dots B_k \in B(S)$ then it follows that for $i \in \{1, 2, \dots k\}$ we have $B_i = (b_i)f_i^{-1}$ for some $f_1, f_2 \dots f_k \in L$ and $b_1, b_2 \dots b_k \in \{0,1\}$. As $L$ has large oscillation it follows that for some $\omega \in \Omega$, $(\omega)f_i = b_i$ for all $i \in \{1, 2 \dots k\}$, and therefore $\omega \in \bigcap_{i \in \{1,2,\dots k\}} B_i$. Therefore $B(S)$ has the finite intersection property. By the Ultrafilter Lemma we have that for every $S \in P(L)$ we can extend $B(S)$ to an ultrafilter $\mathcal{U}(S)$.
Suppose for a contradiction that there exists distinct $S_1, S_2, \in P(L)$ such that $\mathcal{U}(S_1) = \mathcal{U}(S_2)$. Then without loss of generality there exists $f \in S_1 \backslash S_2$. We have that $(\{0\})f^{-1} \in \mathcal{U}(S_2)$ is the complement of $(\{1\})f^{-1} \in \mathcal{U}(S_2)$. So because $\mathcal{U}(S_1) = \mathcal{U}(S_2)$ we have a set in an ultrafilter who's complement is also in that ultrafilter, this contradicts theroem 1.4.5. It follows that $\mathcal{U}(S)$ for $S \in P(L)$ are distinct ultrafilters and there is $|P(L)| = 2^{2^{|\Omega|}}$ of them as required. □

It is noteworthy that it immediately follows from this Theorem that for any infinite set $\Omega$ there are $2^{2^{|\Omega|}}$ topologies on $\Omega$ as filters can be extended to topologies by adding the empty set.

# Chapter 2

# Topological Groups and Commutators

In this chapter we will be primarily focusing on $\mathrm{Sym}(\mathbb{N})$. We will be viewing this group as a topological group and will use its topological properties to show that all its elements can be written as a commutator. We will then generalise this property to all infinite symmetric groups.

## 2.1 Infinite Permutations

We start this chapter by introducing some useful notions about infinite permutations.

**Theorem 2.1.1.** *If $\Omega$ is an infinite set and $f, g \in \mathrm{Sym}(\Omega)$, then $f$ and $g$ are conjugate if and only if they have the same disjoint cycle shape.*

*Proof.* ( $\Longleftarrow$ ) Let $C_{f,i}$ be the set of i-cycles of $f$ and similarly $C_{g,i}$ be the set of i-cycles of $g$. We have that $|C_{f,i}| = |C_{g,i}|$ for all $i \in \aleph_0 \cup \{\aleph_0\}$. Label their elements such that:

$$C_{f,i} = \{C_{f,i,j} : j < |C_{f,i}|\} \qquad C_{g,i} = \{C_{g,i,j} : j < |C_{g,i}|\}$$

Let $C_{f,i,j} = (C_{f,i,j,0}, C_{f,i,j,1}, \dots C_{f,i,j,i-1})$ if $i \neq \aleph_0$ and $C_{f,i,j} = (\dots C_{f,i,j,-1}, C_{f,i,j,0}, C_{f,i,j,1} \dots)$ otherwise. Similarly let $C_{g,i,j} = (C_{g,i,j,0}, C_{g,i,j,1} \dots C_{g,i,j,i-1})$ if $i \neq \aleph_0$ and $C_{g,i,j} = (\dots C_{g,i,j,-1}, C_{g,i,j,0}, C_{g,i,j,1} \dots)$ otherwise. Note that the domains of the disjoint cycles of $f$ or $g$ partition $\Omega$ we can therefore define a bijection $h : \Omega \to \Omega$ by $(C_{f,i,j,k})h = (C_{g,i,j,k})$. We now have that $f = hgh^{-1}$ as required.
( $\Longrightarrow$ ) Suppose that $h^{-1}gh = f$ for some $h \in \mathrm{Sym}(\Omega)$. Let $C_f = (\dots c_{-1}, c_0, c_1 \dots)$ be a cycle of $f$, we have $h^{-1}C_f h = (\dots (c_{-1})h, (c_0)h, (c_1)h \dots)$. Similarly if $C_f = (c_1 \dots c_k)$ is a cycle of $f$. we have $h^{-1}C_f h = ((c_1)h, (c_2)h \dots (c_k)h)$.
We therefore have that all disjoint cycles in $f$ have a unique corresponding cycle in $g$ and therefore $g$ has the same number of cycles of each length as $f$ and so $g$ has the same disjoint cycle shape as $f$. $\square$

The desired result for this chapter will take some time to prove, however there is similar weaker result which can be shown with relatively little effort.

**Theorem 2.1.2.** *If $f \in \mathrm{Sym}(\mathbb{N}_0)$ and there exists $k \in \mathbb{N}_0$ such that $(i)f = i$ for all $i \geq k$, then $f$ can be written in the form:*

$$f = [g, h] = g^{-1}h^{-1}gh$$

*where $g, h \in \mathrm{Sym}(\mathbb{N}_0)$.*

*Proof.* Let $g, h \in \mathrm{Sym}(\mathbb{N}_0)$ be defined as follows:
For $n \in \mathbb{N}_0$ we write $n$ as $2kq + r$ where $q \in \mathbb{N}_0$ and $0 \leq r < 2k$

$$(n)g = \left\{ \begin{array}{ll} 2qk + (r)f^{-1} & r < k \\ n & \mathit{otherwise} \end{array} \right\} \qquad (n)h = \left\{ \begin{array}{ll} n - k & k \leq n < 2k \\ n + 2k & r < k \\ n - 2k & r \geq k \text{ and } q > 0 \end{array} \right\}$$

Note that $g, h$ have the following inverses and are therefore bijections:

$$(n)g^{-1} = \left\{ \begin{array}{ll} 2qk + (r)f & r < k \\ n & \mathit{otherwise} \end{array} \right\} \qquad (n)h^{-1} = \left\{ \begin{array}{ll} n + k & n < k \\ n - 2k & r < k \text{ and } q > 0 \\ n + 2k & r \geq k \end{array} \right\}$$

We now show that $f = [g, h]$.

Case 1: $n < k$

$$\begin{aligned}(n)[g,h] &= ((((n)g^{-1})h^{-1})g)h \\ &= ((((r)g^{-1})h^{-1})g)h \\ &= (((r)f+k)g)h \\ &= ((r)f+k)h \\ &= (r)f = (n)f\end{aligned}$$

Case 2: $q > 0$ and $r < k$

$$\begin{aligned}(n)[g,h] &= ((((n)g^{-1})h^{-1})g)h \\ &= ((((2qk+r)g^{-1})h^{-1})g)h \\ &= (((2qk+(r)f)h^{-1})g)h \\ &= ((2(q-1)k+(r)f)g)h \\ &= 2qk+r = n = (n)f\end{aligned}$$

Case 3: $k \le r$

$$\begin{aligned}(n)[g,h] &= ((((n)g^{-1})h^{-1})g)h \\ &= (((n)h^{-1})g)h \\ &= ((n+2k)g)h \\ &= (n+2k)h \\ &= n = (n)f\end{aligned}$$

□

## 2.2 Constructing a Topology on an Infinite Symmetric Group

In this section we will be constructing a completely metrizable topology on Sym($\mathbb{N}$) by first constructing one on the full transformation monoid and then considering the subspace topology. In doing this we will show some more general results concerning topologies and metric spaces.

**Definition 2.2.1** The countably infinite product of the discrete topology on $\mathbb{N}$ will be denoted by $T$. This topology is defined on the set $\mathbb{N}^{\mathbb{N}}$. The following notations will also be used:

$$\mathbb{N}^{<\mathbb{N}} := \{\sigma : \{1, 2 \dots n\} \to \mathbb{N} : n \in \mathbb{N}\} \qquad [\sigma] := \{f \in \mathbb{N}^{\mathbb{N}} : f|_{dom(\sigma)} = \sigma\} \qquad B := \{[\sigma] : \sigma \in \mathbb{N}^{<\mathbb{N}}\}$$

**Theorem 2.2.2.** *The set $B$ above is a basis for $T$.*

*Proof.* We have that $B_o := \{U_1 \times U_2 \dots \times U_k \times \mathbb{N} \times \mathbb{N} \times \dots : k \in \mathbb{N},\ U_i \subseteq \mathbb{N} \ \text{ for all } i \in \{1, 2 \dots k\}\}$ is a basis for T by the definition of an infinite product topology.
As $B \subseteq B_o$ it suffices to show that for all $U \in B_o$ there exists $G_U \subseteq B$ such that $U = \cup G_U$. Let $U := U_1 \times U_2 \dots \times U_k \times \mathbb{N} \times \mathbb{N} \times \dots$
Let $G_U := \{[\sigma] \in B : dom(\sigma) = \{1, 2 \dots k\}, \text{for all } i \in dom(\sigma)\ (i)\sigma \in U_i\}$.
We now show that $U = \cup G_U$.

$$\begin{gathered}(\subseteq) \text{ let } f \in U \\ (i)f \in U_i \text{ for all } i \in \{1, 2 \dots k\} \\ \textit{let } \sigma_f := f|_{\{1,2\dots k\}} \in \mathbb{N}^{<\mathbb{N}} \\ f \in [\sigma_f] \in G_U \implies f \in \cup G_U\end{gathered}$$

$$\begin{gathered}(\supseteq) \textit{ let } f \in \cup G_U \\ \text{there exists } [\sigma_f] \in G_U \text{ such that } f \in [\sigma_f] \\ (i)f = (i)\sigma_f \in U_i \ \text{ for all } i \in \{1, 2 \dots k\} \\ \implies f \in U\end{gathered}$$

□

**Definition 2.2.3** A topological space is called *completely metrizable* if it is induced by a complete metric space.

**Theorem 2.2.4.** *If $d : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{R}$ is defined by:*

$$d(f,g) = \left\{ \begin{array}{ll} 0 & f = g \\ \frac{1}{\min\{i \in \mathbb{N} : (i)f \neq (i)g\}} & f \neq g \end{array} \right\}$$

*then the tuple $(\mathbb{N}^{\mathbb{N}}, d)$ is a complete metric space which induces the topology $T$, and thus $T$ is completely metrizable.*

*Proof.* We first show that $d$ is a metric. The non-negativity, identity of indiscernibles and symmetry conditions are clearly true from the definition.
Triangle inequality: Let $f, g, h \in \mathbb{N}^{\mathbb{N}}$ :
First notice that if $j = \min\{i \in \mathbb{N} : (i)f \neq (i)h\}$ then either $(j)f \neq (j)g$ or $(j)g \neq (j)h$. It therefore follows:

$$\begin{aligned} & \min\{i \in \mathbb{N} : (i)f \neq (i)h\} \geq \min\{i \in \mathbb{N} : (i)f \neq (i)g \textit{ or } (i)g \neq (i)h\} \\ \implies & \min\{i \in \mathbb{N} : (i)f \neq (i)h\} \geq \min\{\min\{i \in \mathbb{N} : (i)f \neq (i)g\}, \min\{i \in \mathbb{N} : (i)g \neq (i)h\}\} \\ \implies & \frac{1}{d(f,h)} \geq \min\{\frac{1}{d(f,g)}, \frac{1}{d(g,h)}\} \\ \implies & \frac{1}{d(f,h)} \geq \frac{1}{\max\{d(f,g), d(g,h)\}} \\ \implies & d(f,h) \leq \max\{d(f,g), d(g,h)\} \\ \implies & d(f,h) \leq d(f,g) + d(g,h)\end{aligned}$$

We next show that $(\mathbb{N}^{\mathbb{N}}, d)$ is complete.
Let $S = (f_1, f_2, f_3 \ldots)$ be a Cauchy sequence.

Claim: for all $i \in \mathbb{N}$ there is a minimal $M_i \in \mathbb{N}$ such that for all $j \in \{1, 2 \ldots i\}$ and all $n \geq M_i$

$$(j)f_{M_i} = (j)f_n$$

Proof of Claim: Let $i \in \mathbb{N}$. As $S$ is Cauchy we have that for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $d(f_n, f_m) < \varepsilon$. By choosing $\varepsilon = \frac{1}{i+1}$ for all $i \in \mathbb{N}$ there exists $N_i$ such that for all $n, m \geq N_i$ we have $d(f_n, f_m) < \frac{1}{i+1}$.

$$\implies d(f_{N_i}, f_n) < \frac{1}{i+1} \quad \text{for all } n \geq N_i \qquad \implies (j)f_{N_i} = (j)f_n \text{ for all } n \geq N_i \text{ and } j \in \{1, 2 \ldots i\}$$

As there exists a natural number $N_i$ with the desired property and the natural numbers are well ordered there must exist a minimal such number $M_i$ with this property. $\square$

Define $l : \mathbb{N} \to \mathbb{N}$ by $(i)l = (i)f_{M_i}$. It suffices to show that $S$ converges to $l$. It is clear from the definition of $M_i$ that for all $i \in \mathbb{N}$ we have $M_i \leq M_{i+1}$.

$$\text{for all } i \in \mathbb{N} \text{ we have } M_i \leq M_{i+1} \implies \text{for all } i \in \mathbb{N} \text{ we have } (i)f_{M_{i+1}} = (i)f_{M_i} \implies \text{for all } j > i \text{ we have } (i)f_{M_i} = (i)f_{M_j}$$

We have that for all $n \geq M_i$ and all $j \in \{1, 2 \ldots i\} : (j)l = (j)f_{M_j} = (j)f_{M_i} = (j)f_n$ and therefore $d(l, f_n) < \frac{1}{i}$.
We have just shown that for all $i \in \mathbb{N}$ there exists $M_i \in \mathbb{N}$ such that for all $n \geq M_i$ we have that $d(l, f_n) < \frac{1}{i}$.
In particular for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have that $d(l, f_n) < \varepsilon$ and so $S$ converges to $l$ as required. Finally we show that $T$ is induced by $d$.

$$\begin{aligned}
\text{The topology induced by d} &= \langle \{\{g \in \mathbb{N}^{\mathbb{N}} : d(g, f) < \varepsilon\} : f \in \mathbb{N}^{\mathbb{N}}, \varepsilon > 0\} \rangle_T \\
&= \langle \{\{g \in \mathbb{N}^{\mathbb{N}} : (i)g = (i)f \text{ for all } i \in \{1, 2 \ldots \lfloor \frac{1}{\varepsilon} \rfloor\}\} : f \in \mathbb{N}^{\mathbb{N}}, \varepsilon > 0\} \rangle_T \\
&= \langle \{\{g \in \mathbb{N}^{\mathbb{N}} : (j)g = (j)f \text{ for all } j \in \{1, 2 \ldots i\}\} : f \in \mathbb{N}^{\mathbb{N}}, i \in \mathbb{N}\} \rangle_T \\
&= \langle \{\{g \in \mathbb{N}^{\mathbb{N}} : (j)g = (j)\sigma \text{ for all } j \in dom(\sigma)\} : \sigma \in \mathbb{N}^{<\mathbb{N}}\} \rangle_T \\
&= \langle \{\{g \in \mathbb{N}^{\mathbb{N}} : g|_{dom(\sigma)} = \sigma\} : \sigma \in \mathbb{N}^{<\mathbb{N}}\} \rangle_T \\
&= \langle \{[\sigma] : \sigma \in \mathbb{N}^{<\mathbb{N}}\} \rangle_T = T
\end{aligned}$$

$\square$

**Theorem 2.2.5.** *The infinite symmetric group* Sym($\mathbb{N}$) *is a* $G_\delta$ *subset of* $(\mathbb{N}^{\mathbb{N}}, T)$.

*Proof.* Let $U : \mathbb{N} \times P(\mathbb{N}) \times P(\mathbb{N}) \to P(\mathbb{N})$ be defined by:

$$U(i, S_1, S_2) := \left\{ \begin{array}{ll} S_2 & i \in S_1 \\ \mathbb{N} & i \notin S_1 \end{array} \right\}$$

Note that if $S_1$ is finite then $\prod_{i=1}^{\infty} U(i, S_1, S_2)$ is open as all but finitely many components are $\mathbb{N}$.
Note also that for $j, k, l \in \mathbb{N}$ we have

$$\prod_{i=1}^{\infty} U(i, \{j, k\}, \{l\}) = \mathbb{N}^{\mathbb{N}} \backslash \Big( (\prod_{i=1}^{\infty} U(i, \{j\}, \{\mathbb{N} \backslash \{l\}\})) \cup (\prod_{i=1}^{\infty} U(i, \{k\}, \{\mathbb{N} \backslash \{l\}\})) \Big)$$

Therefore $\prod_{i=1}^{\infty} U(i, \{j, k\}, \{l\})$ is clopen for all $j, k, l \in \mathbb{N}$.
Let $I$ denote the set of all injective functions from $\mathbb{N}$ to $\mathbb{N}$.

$$I = (I^c)^c = \Big( \bigcup_{\substack{j,k,l \in \mathbb{N} \\ j \neq k}} \prod_{i=1}^{\infty} U(i, \{j, k\}, \{l\}) \Big)^c = \bigcap_{\substack{j,k,l \in \mathbb{N} \\ j \neq k}} \Big( \prod_{i=1}^{\infty} U(i, \{j, k\}, \{l\}) \Big)^c = \bigcap_{\substack{j,k,l \in \mathbb{N} \\ j \neq k}} \Big( (\prod_{i=1}^{\infty} U(i, \{j\}, \{\mathbb{N} \backslash \{l\}\})) \cup (\prod_{i=1}^{\infty} U(i, \{k\}, \{\mathbb{N} \backslash \{l\}\})) \Big)$$

We have $I$ as a countable intersection of open sets.
Let $S$ denote the set of all surjective functions from $\mathbb{N}$ to $\mathbb{N}$. We have $S$ as a countable intersection of open sets as follows:

$$S = \bigcap_{k \in \mathbb{N}} \Big( \bigcup_{j \in \mathbb{N}} \prod_{i=1}^{\infty} U(i, \{j\}, \{k\}) \Big)$$

Therefore we have Sym($\mathbb{N}$) $= S \cap I$ as a countable intersection of open sets as required. $\square$

**Theorem 2.2.6.** *Let $T_s$ be the subspace topology of* $\mathrm{Sym}(\mathbb{N})$ *in* $(\mathbb{N}^{\mathbb{N}}, T)$. *The topological space* $(\mathrm{Sym}(\mathbb{N}), T_s)$ *is completely metrizable.*

*Proof.* By Theorems 1.1.1, 2.2.5 and 1.1.3 we have that $(\mathbb{N}^{\mathbb{N}}, T)$ is completely metrizable, $\mathrm{Sym}(\mathbb{N})$ is $G_\delta$ in $(\mathbb{N}^{\mathbb{N}}, T)$ and $G_\delta$ sets of a completely metrizable topology equipped with the subspace topology are completely metrizable. It therefore follows that $(\mathrm{Sym}(\mathbb{N}), T_s)$ is completely metrizable. □

**Theorem 2.2.7.** *The triple* $(\mathrm{Sym}(\mathbb{N}), \circ, T_s)$ *is a topological group.*

*Proof.* For $\sigma \in \mathbb{N}^{<\mathbb{N}}$ let $[\sigma]_s := \{f \in \mathrm{Sym}(\mathbb{N}) : f|_{dom(\sigma)} = \sigma\}$.
As $B$ is a basis for $T$ we have that $B_s := \{b \cap \mathrm{Sym}(\mathbb{N}) : b \in B\} = \{[\sigma]_s : \sigma \in \mathbb{N}^{<\mathbb{N}}\}$ is a basis for $T_s$. Let $\phi_i : \mathrm{Sym}(\mathbb{N}) \to \mathrm{Sym}(\mathbb{N})$ be defined by $(f)\phi_i = f^{-1}$. If suffices to show that $(b)\circ^{-1}$ and $(b)\phi_i^{-1} = (b)\phi_i$ are open for all $b \in B_s$.
Let $b = [\sigma]_s \in B_s$ and let $dom(\sigma) = \{1, 2 \ldots k\}$

$$(f, g) \in (b)\circ^{-1} \implies fg \in b = [\sigma]_s \implies ((i)f)g = (i)\sigma \text{ for all } i \in \{1, 2 \ldots k\}$$

Let $U_{f,g} := [f|_{\{1,2\ldots k\}}]_s \times [g|_{\{1,2\ldots \max\{(i)f : i \in \{1,2\ldots k\}\}\}}]_s$.
Clearly $(f, g) \in U_{f,g}$ and $U_{f,g}$ is open.
Let $(f_2, g_2) \in U_{f,g}$

$$\begin{aligned}
&((i)f_2)g_2 = ((i)f)g_2 \text{ for all } i \in \{1, 2 \ldots k\} \text{ as } f_2|_{\{1,2\ldots k\}} = f|_{\{1,2\ldots k\}} \\
&\implies ((i)f_2)g_2 = ((i)f)g \text{ for all } i \in \{1, 2 \ldots k\} \text{ as } g_2|_{\{1,2\ldots \max\{(i)f:i\in\{1,2\ldots k\}\}\}} = g|_{\{1,2\ldots \max\{(i)f:i\in\{1,2\ldots k\}\}\}} \\
&\implies ((i)f_2)g_2 = (i)\sigma \text{ for all } i \in \{1, 2 \ldots k\} = dom(\sigma) \\
&\implies (f_2, g_2) \in ([\sigma]_s)\circ^{-1} = (b)\circ^{-1} \\
&\implies U_{f,g} \subseteq (b)\circ^{-1}
\end{aligned}$$

So we have that for all $(f, g) \in (b)\circ^{-1}$ there is an open set $U_{f,g}$ such that $(f, g) \in U_{f,g} \subseteq (b)\circ^{-1}$ and therefore $(b)\circ^{-1}$ is open so $\circ$ is continuous.

$$f \in (b)\phi_i \implies f^{-1} \in [\sigma]_s \implies (i)f^{-1} = (i)\sigma \text{ for all } i \in \{1, 2 \ldots k\} \implies i = ((i)\sigma)f \text{ for all } i \in \{1, 2 \ldots k\}$$

Let $U_f := [f|_{\{1,2\ldots \max(\mathrm{img}(\sigma))\}}]_s$
Clearly $f \in U_f$ and $U_f$ is open. It therefore suffices to show that $U_f \subseteq (b)\phi_i$. Let $f_2 \in U_f$

$$\begin{aligned}
&((i)\sigma)f_2 = ((i)\sigma)f \text{ for all } i \in \{1, 2 \ldots k\} \quad \text{as } f_2|_{\{1,2\ldots \max(\mathrm{img}(\sigma))\}} = f|_{\{1,2\ldots \max(\mathrm{img}(\sigma))\}} \\
&\implies ((i)\sigma)f_2 = i \text{ for all } i \in \{1, 2 \ldots k\} \\
&\implies (i)\sigma = (i)f_2^{-1} \text{ for all } i \in \{1, 2 \ldots k\} \\
&\implies f_2^{-1} \in [\sigma]_s \implies (f_2^{-1})\phi_i \in ([\sigma]_s)\phi_i = (b)\phi_i \implies f_2 \in (b)\phi_i
\end{aligned}$$

□

## 2.3 Comeagre Conjugacy Class

For the rest of this chapter we will construct a comeagre conjugacy class of $\mathrm{Sym}(\mathbb{N})$ and use it, with the baire catagory theorem, to show the desired result that all elements of $\mathrm{Sym}(\mathbb{N})$ are commutators.

**Definition 2.3.1** Let the set $C$ be defined by:

$$C := \{f \in \mathrm{Sym}(\mathbb{N}) : \text{f has infinitely many cycles of all finite lengths but none of infinite length}\}$$

Note that by Theorem 2.1.1 we have that $C$ is a conjugacy class.

**Theorem 2.3.2.** *Let $I = (i_1, i_2, \ldots i_k)$ and $N = (n_1, n_2, \ldots n_k)$ be finite sequences of natural numbers with no repeats. there exists $\sigma \in \mathrm{Sym}(\mathbb{N})$ and $r > 0$ such that $(n_j)f = i_j$ for all $f \in B(\sigma, r)$ and for all $j \in \{1, 2 \ldots k\}$.*

*Proof.* As both $I$ and $N$ are finite, we have that $|\mathbb{N} \backslash I| = |\mathbb{N}| = |\mathbb{N} \backslash N|$. Therefore there exists a bijection $\phi : \mathbb{N} \backslash N \to \mathbb{N} \backslash I$.
Let $\sigma : \mathbb{N} \to \mathbb{N}$ be defined by:

$$(n)\sigma = \left\{ \begin{array}{ll} i_j & \text{if } n = n_j \text{ for some } j \in \{1, 2, \ldots k\} \\ (n)\phi & \text{otherwise} \end{array} \right\}$$

By construction $\sigma \in \mathrm{Sym}(\mathbb{N})$. Let $r = \frac{1}{\max(N)}$ and $f \in B(\sigma, r)$

$$\begin{aligned} d(\sigma, f) < \frac{1}{\max(N)} &\implies \frac{1}{\min(\{n \in \mathbb{N} : (n)\sigma \neq (n)f\})} < \frac{1}{\max(N)} \\ &\implies \max(N) < \min(\{n \in \mathbb{N} : (n)\sigma \neq (n)f\}) \\ &\implies (n_j)f = (n_j)\sigma \text{ for all } j \in \{1, 2, \dots k\} \\ &\implies (n_j)f = i_j \text{ for all } j \in \{1, 2, \dots k\} \end{aligned}$$

□

**Theorem 2.3.3.** *The sets $C_{1,i}$ defined by:*

$$C_{1,i} = \{f \in \mathrm{Sym}(\mathbb{N}) : f \textit{ has a cycle of infinite length with } i \textit{ in its domain}\}$$

*are nowhere-dense for all $i \in \mathbb{N}$*

*Proof.* Let $i \in \mathbb{N}$, $x_1 \in \mathrm{Sym}(\mathbb{N})$, $r_1 > 0$ and $x_r := x_1|_{\{1,2\dots\lfloor \frac{1}{r_1} \rfloor\}}$. We have $B(x_1, r_1) = \{f \in \mathrm{Sym}(\mathbb{N}) : f|_{\{1,2\dots\lfloor \frac{1}{r_1} \rfloor\}} = x_r\}$.
By Theorem 1.1.5 it suffices to find $\sigma \in \mathrm{Sym}(\mathbb{N})$ and $r > 0$ such that $B(\sigma, r) \subseteq B(x_1, r_1) \backslash C_{1,i}$.

$$m_1 := \min\{j \in \mathbb{N}_0 : (i)x_1^j \notin dom(x_r)\} \qquad m_2 := \min\{j \in \mathbb{N}_0 : (i)x_1^{-j} \notin \mathrm{img}(x_r)\}$$

Note that if $m_1$ and $m_2$ don't exist then $B(x_1, r_1) \subseteq B(x_1, r_1) \backslash C_{1,i}$ and we are done.
By Theorem 2.3.2 let $\sigma$ be a bijection satisfying:

$$(j)\sigma = (j)x_1 \text{ for all } j \in dom(x_r) \qquad ((i)x_1^{m_1})\sigma = (i)x_1^{-m_2}$$

and let $r > 0$ be such that for all $f \in B(\sigma, r)$ we have:

$$(j)\sigma = (j)f \text{ for all } j \in dom(x_r) \qquad ((i)x_1^{m_1})\sigma = ((i)x_1^{m_1})f$$

We now have that $\mathrm{orb}_{\{f\}}(i)$ is finite for all $f \in B(\sigma, r)$ and therefore $f \notin C_{1,i}$. So $B(\sigma, r) \subseteq B(x_1, r_1) \backslash C_{1,i}$ as required. □

**Theorem 2.3.4.** *The sets $C_{2,i,j}$ defined by:*

$$C_{2,i,j} = \{f \in \mathrm{Sym}(\mathbb{N}) : f \textit{ has } i \textit{ } j\textit{-cycles}\}$$

*are nowhere-dense for all $i \in \mathbb{N}_0$ and $j \in \mathbb{N}$.*

*Proof.* Let $i \in \mathbb{N}_0$ and $j \in \mathbb{N}$. Let $x_1 \in \mathrm{Sym}(\mathbb{N})$, $r_1 > 0$ and $x_r := x_1|_{\{1,2\dots\lfloor \frac{1}{r_1} \rfloor\}}$. We have $B(x_1, r_1) = \{f \in \mathrm{Sym}(\mathbb{N}) : f|_{\{1,2\dots\lfloor \frac{1}{r_1} \rfloor\}} = x_r\}$. By theorem 1.1.5 it suffices to find $\sigma \in \mathrm{Sym}(\mathbb{N})$ and $r > 0$ such that $B(\sigma, r) \subseteq B(x_1, r_1) \backslash C_{2,i,j}$.
Let $m := \max(dom(x_r) \cup \mathrm{img}(x_r)) + 1$. By Theorem 2.3.2 let $\sigma \in \mathrm{Sym}(\mathbb{N})$ be such that:

$$\begin{gathered} (k)\sigma = (k)x_1 \quad \textit{for all } k \in dom(x_r) \\ (m+1)\sigma = (m+2), (m+2)\sigma = (m+3) \dots (m+j)\sigma = (m+1), \\ (m+j+1)\sigma = (m+j+2) \dots, (m+2j)\sigma = (m+j+1), \\ (m+2j+1)\sigma = (m+2j+2) \dots, (m+3j)\sigma = (m+2j+1), \\ \vdots \\ (m+ij+1)\sigma = (m+ij+2) \dots (m+(i+1)j)\sigma = (m+ij+1) \end{gathered}$$

and let $r > 0$ be such that for all $f \in B(\sigma, r)$:

$$(k)f = (k)\sigma \quad \textit{for all } k \in \{1, 2 \dots (m+(i+1)j)\}$$

All $f \in B(\sigma, r)$ have at least $(i+1)$ cycles of length $k$ and therefore $f \notin C_{2,i,k}$. However $B(\sigma, r) \subseteq B(x_1, r_1)$ so we have $B(\sigma, r) \subseteq B(x_1, r_1) \backslash C_{2,i,j}$ as required. □

**Theorem 2.3.5.** *The conjugacy class $C$ is comeagre.*

*Proof.* Let $C_1$ and $C_2$ be defined by:

$$C_1 = \{f \in \mathrm{Sym}(\mathbb{N}) : f \textit{ has a cycle of infinite length}\} \quad C_2 = \{f \in \mathrm{Sym}(\mathbb{N}) : \text{f has finitly many cycles of some finite length }\}$$

Let $C_{2,i}$ be defined by:

$$C_{2,i} = \{f \in \mathrm{Sym}(\mathbb{N}) : \text{f has i cycles of some finite length }\}$$

Note the following:

$$C^c = C_1 \cup C_2 \quad C_1 = \bigcup_{i\in\mathbb{N}} C_{1,i} \quad C_2 = \bigcup_{i\in\mathbb{N}_0} C_{2,i} \quad C_{2,i} = \bigcup_{j\in\mathbb{N}} C_{2,i,j}$$

We therefore have that:

$$C^c = C_1 \cup C_2 = (\bigcup_{i\in\mathbb{N}} C_{1,i}) \cup (\bigcup_{i\in\mathbb{N}_0} C_{2,i}) = (\bigcup_{i\in\mathbb{N}} C_{1,i}) \cup (\bigcup_{i\in\mathbb{N}_0} (\bigcup_{j\in\mathbb{N}} C_{2,i,j}))$$

By Theorems 2.3.3 and 2.3.4 we have that all the $C_{1,i}$ and $C_{2,i,j}$ are nowhere-dense. Thus $C^c$ is a countable union of nowhere-dense sets and is therefore meagre. □

**Theorem 2.3.6.** *We have $CC = \mathrm{Sym}(\mathbb{N})$ and in particular all functions $f \in \mathrm{Sym}(\mathbb{N})$ can be written in the form:*

$$f = [g,h] = g^{-1}h^{-1}gh$$

*where $g, h \in \mathrm{Sym}(\mathbb{N})$.*

*Proof.* Let $f \in \mathrm{Sym}(\mathbb{N})$. By Theorems 2.3.5 and 1.1.8, we have that $C$ is comeagre, and right multiplication by $f$ is a homeomorphism. As homomorphisms preserve closures, interiors and unions it follows that $(C)f$ is comeagre. Therefore there exist nowhere dense sets $(N_{1,i})_i \in \mathbb{N}$ and $(N_{2,i})_{i\in\mathbb{N}}$ such that $C^c = \cup_{i\in\mathbb{N}} N_{1,i}$ and $((C)f)^c = \cup_{i\in\mathbb{N}} N_{2,i}$. It follows that $((C)f \cap C)^c = \cup_{i\in\mathbb{N}}(N_{1,i} \cup N_{2,i})$ and so $(C)f \cap C$ is comeagre. So by the Baire Catagory Theorem part 2 we have $C \cap (C)f \neq \emptyset$. For $x \in C \cap (C)f$, there exists $f_1, f_2 \in C$ such that $f_1 = x = f_2 f$. It follows that $f = f_2^{-1} f_1$. We have that $C$ is closed under taking inverses as the disjoint cycle structure is the same but with cycles reversed.
It therefore follows that $f_2^{-1}$ is conjugate to $f_2$, so $f_2^{-1} \in C$. Therefore we have that $f = f_2^{-1} f_1 \in CC$.
As $f_1, f_2 \in C$ we have that there exists $h \in \mathrm{Sym}(\mathbb{N})$ such that $f_1 = h^{-1} f_2 h$. Let $g := f_2$, we now have:

$$[g,h] = g^{-1}h^{-1}gh = f_2^{-1}h^{-1}f_2 h = f_2^{-1} f_1 = f$$

□

We now have that every element of $\mathrm{Sym}(\mathbb{N})$ is a commutator. It remains to show for any infinite set $\Omega$ that any element of $\mathrm{Sym}(\Omega)$ can be written as a commutator.

**Theorem 2.3.7.** *Let $\Omega$ be an infinite set. Then all functions $f \in \mathrm{Sym}(\Omega)$ can be written in the form:*

$$f = [g,h] = g^{-1}h^{-1}gh$$

*where $g, h \in \mathrm{Sym}(\Omega)$. In addition the conjugacy class*

$$C_\Omega := \{f \in \mathrm{Sym}(\Omega) : f \textit{ has } |\Omega| \textit{ cycles of all finite lengths and none of infinite length}\}$$

*satisfies $C_\Omega C_\Omega = \mathrm{Sym}(\Omega)$.*

*Proof.* If $|\Omega| = \aleph_0$ then we are done by Theorem 2.3.6. Otherwise let $f \in \mathrm{Sym}(\Omega)$. For any point $p \in \Omega$ we have that $|\operatorname{orb}_{\{f\}}(p)|$ is countable. Let $P_1$ be the partition of $\Omega$ into the orbits of points under $f$. We must have that $|P_1| = |\Omega|$ as if $|P_1| > |\Omega|$ we would have that $|\cup P_1| > |\Omega| = |\cup P_1|$, and if $|P_1| < |\Omega|$ then $|\cup P_1| \leq |P_1| \times \aleph_0 \leq \max\{|P_1|^2, \aleph_0\} = \max\{|P_1|, \aleph_0\} < |\Omega| = |\cup P_1|$. We can therefore index $P_1$ as $P_1 = \{S_i : i < |\Omega|\}$. Define an equivalence relation by $S_i \sim S_j \iff i = \alpha + a$ and $j = \alpha + b$ for some $a, b \in \aleph_0$ and limit ordinal $\alpha$. Let $P_2$ be the partition of $P_1$ into equivalence classes by this relation. For the same reasons as $P_1$ we have that $|P_2| = |\Omega|$. Let $P := \{\cup x : x \in P_2\}$. We therefore have that $P = \{P_i : i < |\Omega|\}$ is a partition of $\Omega$ into countably infinite sets such that for all $p \in \Omega$ we have that $p \in P_i \iff (p)f \in P_i$. We can therefore consider $f$ as an element of $\prod_{i<|\Omega|} \mathrm{Sym}_\Omega(P_i)$. As each $P_i$ is countably infinite it follows from Theorem 2.3.6 that $f|_{P_i}$ can be written as ${f_1}_i {f_2}_i$ for some ${f_1}_i, {f_2}_i \in \mathrm{Sym}(P_i)$ for all $i < |\Omega|$. From these we can define $f_1, f_2$ by:

$$(p)f_1 := (p){f_1}_i \text{ were i is the index of the unique } P_i \text{ containing p}$$

$$(p)f_2 := (p){f_2}_i \text{ were i is the index of the unique } P_i \text{ containing p}$$

Then we have that $f|_{P_i} = f_1 f_2|_{P_i}$ for all $i < |\Omega|$ and thus $f_1 f_2 = f$.
By the construction of $f_1$ and $f_2$ is is clear that they have $|\Omega| \times \aleph_0 = |\Omega|$ cycles of all finite lengths and $|\Omega| \times 0 = 0$ cycles of infinite length so $f_1, f_2 \in C_\Omega$ and $f \in C_\Omega C_\Omega$. It remains to show that $f$ is a commutator. As $f_1, f_2 \in C_\Omega$ it follows that there exists $h \in \mathrm{Sym}(\Omega)$ such that $h^{-1} f_1^{-1} h = f_2$. Let $g := f_1^{-1}$, we now have:

$$[g,h] = g^{-1}h^{-1}gh = f_1 h^{-1} f_1^{-1} h = f_1 f_2 = f$$

□

# Chapter 3

# Generating the Infinite Symmetric Groups

## 3.1 The Bergman Property

**Definition 3.1.1** A semigroup $S$ is said to have the *Bergman Property* if for all $U$ such that $\langle U\rangle_S = S$ there exists a natural number $n$ such that $\bigcup_{i=1}^{n} U^i = S$.

An equivalent way to view this concept is that the semigroup's cayley graph will always have bounded diameter. Note that all finite semigroups trivially have the semigroup Bergman Property. In this section we will show that given an infinite set $\Omega$ the group $\mathrm{Sym}(\Omega)$ has the Bergman Property. This is an unusual property among infinite semigroups.

**Example 3.1.2** For $S \in \{(\mathbb{N},+),(\mathbb{Z},+),(\mathbb{Q},+),(\mathbb{R},+),(\mathbb{C},+),(\mathbb{Q}\backslash\{0\},\times),(\mathbb{R}\backslash\{0\},\times),(\mathbb{C}\backslash\{0\},\times)\}$ there exists $U \subseteq S$ such that $\langle U\rangle_S = S$ and $\bigcup_{i=1}^{n} U^i \neq G$ for all $n \in \mathbb{N}$. So these semigroups don't have the Bergman Property.

*Proof.* Let $U$ be $B(0,r)\cap S$ for any $r > 1$. □

**Theorem 3.1.3.** *Let $\Omega$ be an infinite set and let $\{A, B, C\}$ be a partition of $\Omega$ into moieties of $\Omega$. Then we have:*

$$\mathrm{Sym}(\Omega) = \mathrm{Pstab}(A)\,\mathrm{Pstab}(B)\,\mathrm{Pstab}(A) \cup \mathrm{Pstab}(B)\,\mathrm{Pstab}(A)\,\mathrm{Pstab}(B)$$

*Proof.* The following proof is based on the proof of Lemma 2.1 found in [4].
Let $f \in \mathrm{Sym}(\Omega)$. Suppose that $|\Omega| = |C\backslash Af^{-1}|$, it follows that $|(B\cup C)f \cap (B\cup C)| = |\Omega|$. Let $\{M_1, M_2\}$ be a partition of $C$ into moieties and let $M_2' \subseteq M_2$ be such that $|M_2'| = |(C\cup B)\cap Af^{-1}|$. Let $b_1 : (C\cup B)\cap Af^{-1} \to M_2'$ and $b_2 : (C\cup B)\cap(C\cup B)f^{-1} \to (C\cup B)\backslash M_2'$ be bijections. Let $\phi_1 : \Omega \to \Omega$ be defined by:

$$(x)\phi_1 = \left\{\begin{array}{lr} x & x \in A \\ (x)b_1 & x \in (C\cup B)\cap Af^{-1} \\ (x)b_2 & x \in (C\cup B)\cap(C\cup B)f^{-1} \end{array}\right\}$$

Let $b_3 : M_1 \cup (M_2\backslash M_2') \cup (A\backslash Af^{-1}) \to C$ be a bijection (note this must exist as $|C| = |M_1| = |\Omega|$). Let $\phi_2 : \Omega \to \Omega$ be defined by:

$$(x)\phi_2 = \left\{\begin{array}{lr} x & x \in B \\ (x)b_3 & x \in M_1 \cup (M_2\backslash M_2') \cup (A\backslash Af^{-1}) \\ (x)b_1^{-1}f & x \in M_2' \\ (x)f & x \in A\cap(Af^{-1}) \end{array}\right\}$$

We have that $\phi_1 \in \mathrm{Pstab}(A)$ and $\phi_2 \in \mathrm{Pstab}(B)$. In addition $f^{-1}\phi_1\phi_2$ is a bijection which fixes $A$ pointwise so $f^{-1}\phi_1\phi_2 \in \mathrm{Pstab}(A)$. So there exists $\phi_3 \in \mathrm{Pstab}(A)$ such that $f^{-1}\phi_1\phi_2 = \phi_3$. It therefore follows that $f = \phi_1\phi_2\phi_3^{-1} \in \mathrm{Pstab}(A)\,\mathrm{Pstab}(B)\,\mathrm{Pstab}(A)$ as required.
Suppose that $|\Omega| = |C\backslash Bf^{-1}|$ then similarly we conclude that $f \in \mathrm{Pstab}(B)\,\mathrm{Pstab}(A)\,\mathrm{Pstab}(B)$
As $C = (C\backslash Af^{-1}) \cup (C\backslash Bf^{-1})$ and $|C| = |\Omega|$ we must be in one of these cases and we are done. □

**Definition 3.1.4** A semigroup $S$ is said to *quasi-bounded* if it satisfies the following:
Every function $\psi : S \to \mathbb{N}$ such that there is some constant $C_\psi$ satisfying:

$$(st)\psi \leq (s)\psi + (t)\psi + C_\psi \quad \textit{for all } s,t \in S$$

is bounded above.

**Theorem 3.1.5.** *Let $S$ be a quasi-bounded semigroup. Then $S$ also satisfies the semigroup Bergman property.*

*Proof.* Let $U$ be such that $\langle U\rangle_S = S$. Define $\psi : S \to \mathbb{N}$ by:

$$(s)\psi = \min\{n \in \mathbb{N} : s \in \bigcup_{i=1}^{n} U^i\}$$

Let $C_\psi = 0$ and let $s, t \in S$. Observe that if $(s)\psi = l_1$ and $(t)\psi = l_2$, then we have that $s = u_{s_1}u_{s_2}\dots u_{s_{l_1}}$ and $t = u_{t_1}u_{t_2}\dots u_{t_{l_2}}$ for some $u_{s_1}, u_{s_2}\dots u_{s_{l_1}}, u_{t_1}, u_{t_2}\dots u_{t_{l_2}} \in U$.
Therefore $st = u_{s_1}u_{s_2}\dots u_{s_{l_1}}u_{t_1}u_{t_2}\dots u_{t_{l_2}}$
So we have that $st \in \bigcup_{i=1}^{l_1+l_2} U^i \implies (st)\psi \le l_1 + l_2 = (s)\psi + (t)\psi + C_\psi$. As $S$ is quasi-bounded we therefore have that $\psi$ is bounded by some $N \in \mathbb{N}$ and therefore $(s)\psi \le N$ for all $s \in S$. This implies that $s \in \bigcup_{i=1}^{N} U^i$ for all $s \in S$ so we have $S = \bigcup_{i=1}^{N} U^i$ and therefore we have that $S$ satisfies the semigroup Bergman property. □

**Theorem 3.1.6.** *Let $\Omega$ be an infinite set. There is a sequence $(a_n)_{n\in\mathbb{N}}$ and $N \in \mathbb{N}$ such that for all $S = (s_n)_{n\in\mathbb{N}} \subset \mathrm{Sym}(\Omega)$ there are $g_1, g_2 \dots g_N \in \mathrm{Sym}(\Omega)$ such that for all $n \in \mathbb{N}$ we have $s_n = g_{n_1}, g_{n_2}\dots g_{n_k}$ for some $k \le a_n$.*

*Proof.* The following proof is based on the proof of Theorem 3.1 found in [4].
Choose $(a_n)_{n\in\mathbb{N}} = (36n + 6)_{n\in\mathbb{N}},\quad N = 8$.
Let $(s_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{Sym}(\Omega)$.
Without loss of generality we assume that $\Omega = \mathbb{Z} \times \mathbb{Z} \times \Omega'$ where $|\Omega'| = |\Omega|$.
Let $M$ be a moiety of $\Omega'$.

$$\Omega_0 := \{0\} \times \{0\} \times \Omega' \qquad \Omega_0^+ := \{(0,0,x) : x \in M\} \qquad \Omega_0^- := \{(0,0,x) : x \in M^c\}$$

By Theorem 3.1.3 build a sequence $S' = (s'_n)_{n\in\mathbb{N}} \subseteq \mathrm{Pstab}(\Omega_0^c) \cup \mathrm{Pstab}(\Omega_0^+)$ such that $s_n = s'_{3n-2}s'_{3n-1}s'_{3n}$ for all $n \in \mathbb{N}$.
Let $b_1 : \Omega_0^c \to \Omega_0^+$ be a bijection and let $\phi_1 : \Omega \to \Omega$, $\phi_2 : \Omega \to \Omega$ and $\phi_3 : \Omega \to \Omega$ be defined by:

$$(x)\phi_1 = \left\{ \begin{array}{ll} (x)b_1 & x \in \Omega_0^c \\ (x)b_1^{-1} & x \in \Omega_0^+ \\ x & x \in \Omega_0^- \end{array} \right\} \qquad \begin{array}{l} ((a,b,c))\phi_2 = (a+1,b,c) \\ ((a,b,c))\phi_3 = \left\{ \begin{array}{ll} (a,b+1,c) & a = 0 \\ (a,b,c) & a \neq 0 \end{array} \right\} \end{array}$$

Let $S'' := \{s''_i : i \in \mathbb{Z}\}$ where the bijections $b_2 : \mathbb{Z} \to \mathbb{N}$ and $s''_i$ are defined by:

$$(i)b_2 = \left\{ \begin{array}{ll} 2i & i \ge 0 \\ 2|i| - 1 & i < 0 \end{array} \right\} \qquad s''_i = \left\{ \begin{array}{ll} s'_{(i)b_2} & s'_{(i)b_2} \in \mathrm{Pstab}(\Omega_0^c) \\ \phi_1 s'_{(i)b_2}\phi_1 & s'_{(i)b_2} \in S' \setminus \mathrm{Pstab}(\Omega_0^c) \end{array} \right\}$$

As $S''$ stabilizes $\Omega_0^c$ pointwise for each $s''_i$ we can construct $\hat{s''_i} : \Omega' \to \Omega'$ such that $(0,0,p)s''_i = (0,0,(p)\hat{s''_i})$ for all $p \in \Omega'$ so $\phi_4 : \Omega \to \Omega$ can be defined by:

$$((a,b,c))\phi_4 = \left\{ \begin{array}{ll} (a,b,(c)\hat{s''_a}) & b \ge 0 \\ (a,b,c) & b < 0 \end{array} \right\}$$

Let $(g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8) = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_1^{-1}, \phi_2^{-1}, \phi_3^{-1}, \phi_4^{-1})$ and let $G = \{g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8\}$. Let $L(s)$ denote the minimum length of $s$ as a product of elements of $G$. It suffices to show that $L(s_n) \le a_n$ for all $n \in \mathbb{N}$.
Let $a, b, i \in \mathbb{Z}$ and $c \in \Omega'$:

$$\begin{aligned} ((a,b,c))\phi_2^i\phi_4\phi_2^{-i} &= ((a+i,b,c))\phi_4\phi_2^{-i} \\ &= \left\{ \begin{array}{ll} (a+i,b,(c)\hat{s''_{a+i}})\phi_2^{-i} & b \ge 0 \\ (a+i,b,c)\phi_2^{-i} & b < 0 \end{array} \right\} \\ &= \left\{ \begin{array}{ll} (a,b,(c)\hat{s''_{a+i}}) & b \ge 0 \\ (a,b,c) & b < 0 \end{array} \right\} \end{aligned}$$

$$\begin{aligned} ((0,b,c))\phi_3^{-1}\phi_2^i\phi_4^{-1}\phi_2^{-i}\phi_3 &= ((i,b-1,c))\phi_4^{-1}\phi_2^{-i}\phi_3 \\ &= \left\{ \begin{array}{ll} (i,b-1,(c)\hat{s_i''^{-1}})\phi_2^{-i}\phi_3 & b \ge 1 \\ (i,b-1,c)\phi_2^{-i}\phi_3 & b < 1 \end{array} \right\} \\ &= \left\{ \begin{array}{ll} (0,b,(c)\hat{s_i''^{-1}}) & b \ge 1 \\ (0,b,c) & b < 1 \end{array} \right\} \end{aligned}$$

We now have:

$$\begin{aligned} (a,b,c)\phi_2^i\phi_4\phi_2^{-i}(\phi_3^{-1}\phi_2^i\phi_4^{-1}\phi_2^{-i}\phi_3) &= \left\{ \begin{array}{lr} (a,b,c)\phi_2^i\phi_4\phi_2^{-i}(\phi_2^i\phi_4^{-1}\phi_2^{-i}\phi_3) & a \neq 0 \\ (a,b,c)\phi_3^{-1}\phi_2^i\phi_4^{-1}\phi_2^{-i}\phi_3 & a = 0 \text{ and } b < 0 \\ (a,b,(c)\hat{s''_{a+i}})\phi_3^{-1}\phi_2^i\phi_4^{-1}\phi_2^{-i}\phi_3 & a = 0 \text{ and } b > 0 \\ (a,b,(c)\hat{s''_{a+i}})\phi_3^{-1}\phi_2^i\phi_4^{-1}\phi_2^{-i}\phi_3 & a = b = 0 \end{array} \right\} \\ &= \left\{ \begin{array}{lr} (a,b,c) & a \neq 0 \\ (a,b,c) & a = 0 \text{ and } b < 0 \\ (a,b,c) & a = 0 \text{ and } b > 0 \\ (a,b,(c)\hat{s''_{a+i}}) & a = b = 0 \end{array} \right\} = s''_i \end{aligned}$$

It follows that for $i \geq 0$ we have $s''_i = g_2^i g_4 g_6^i g_7 g_2^i g_8 g_6^i g_3$ so $L(s''_i) \leq 4|i|+4$. In addition for $i < 0$ we have $s''_i = g_6^{|i|} g_4 g_2^{|i|} g_7 g_6^{|i|} g_8 g_2^{|i|} g_3$ so $L(s''_i) \leq 4|i| + 4$.
For all $n \in \mathbb{N}$ we have that $s'_n = s''_{(n)b_2^{-1}}$ or $s'_n = \phi_1 s''_{(n)b_2^{-1}} \phi_1 = g_1 s''_{(n)b_2^{-1}} g_1$ so $L(s'_n) \leq L(s''_{(n)b_2^{-1}}) + 2$.
Therefore we have that:

$$\begin{aligned} L(s_n) = L(s'_{3n-2} s'_{3n-1} s'_{3n}) &\leq L(s'_{3n-2}) + L(s'_{3n-1}) + L(s'_{3n}) \\ &\leq L(s''_{(3n-2)b_2^{-1}}) + L(s''_{(3n-1)b_2^{-1}}) + L(s''_{(3n)b_2^{-1}}) + 6 \\ &\leq 4|(3n-2)b_2^{-1}| + 4|(3n-1)b_2^{-1}| + 4|(3n)b_2^{-1}| + 18 \\ &\leq 4(3n-2) + 4(3n-1) + 4(3n) + 18 \\ &\leq 36n + 6 = a_n \end{aligned}$$

□

**Theorem 3.1.7.** *If $\Omega$ is an infinite set then* $\mathrm{Sym}(\Omega)$ *is quasi-bounded and satisfies the semigroup Bergman Property.*

*Proof.* The following proof is based on the proof of Lemma 2.4 in [5].
By Theorem 3.1.5 it suffices to show that $\mathrm{Sym}(\Omega)$ is quasi-bounded.
Let $(a_n)_{n\in\mathbb{N}}$ be as in Theorem 3.1.6.
We may assume $a_n$ is strictly increasing as $a_n$ can be replaced by $\max\{a_m + 1 : m \leq n\}$ and the required property clearly holds.
Suppose for a contradiction that there exists $\psi : \mathrm{Sym}(\Omega) \to \mathbb{N}$ such that there exists $C_\psi$ such that

$$(st)\psi \leq (s)\psi + (t)\psi + C_\psi \quad \textit{for all } s, t \in \mathrm{Sym}(\Omega)$$

and $\psi$ is unbounded.
As $\psi$ is unbounded for all $n \in \mathbb{N}$ there exists $s \in \mathrm{Sym}(\Omega)$ such that $(s)\psi > n$.
Therefore we can construct a sequence $(s_n)_{n\in\mathbb{N}}$ such that $(s_n)\psi > a_n^2$ *for all* $n \in \mathbb{N}$.
We now construct a set of generators $G = (g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8)$ for $(s_n)_{n\in\mathbb{N}}$ as done in Theorem 3.1.6.
Let $M = \max\{(g)\psi : g \in G\}$
Each $s_n$ can be written as a product of length at most $a_n$ in elements of $G$ it therefore follows from induction that:

$$(s_n)\psi \leq a_n C_\psi + a_n M \quad \textit{for all } n \in \mathbb{N}$$

For all sufficiently large n we have that $a_n > C_\psi + M$ and therefore:

$$(s_n)\psi \leq a_n C_\psi + a_n M = a_n(C_\psi + M) < a_n^2 < (s_n)\psi$$

So $(s_n)\psi < (s_n)\psi$. This is a contradiction. □

## 3.2 Cofinality and Strong Cofinality

In this section we will be exploring the cofinality and strong cofinality of infinite symmetric groups.

**Definition 3.2.1** Let $S$ be a semigroup. A *cofinal chain* of $S$, is defined to be a chain of strict subsemigroups $(S_i)_{i<\kappa}$ of $S$ indexed by the ordinals less than some cardinal $\kappa$ such that:

$$S = \bigcup_{i<\kappa} S_i$$

and $S_i \subseteq S_j$ for $i \leq j$.

**Definition 3.2.2** Let $S$ be a semigroup. A *strong cofinal chain* of $S$, is defined to be a chain of strict subsets $(S_i)_{i<\kappa}$ of $S$ indexed by the ordinals less than some cardinal $\kappa$ such that:

$$S = \bigcup_{i<\kappa} S_i$$

and $S_i \subseteq S_j$ for $i \leq j$ and for all $i < \kappa$ there exists $j < \kappa$ such that $S_i S_i \subseteq S_j$.

**Theorem 3.2.3.** *If $S$ is a non-finitely generated semigroup, then there exist cofinal and strong cofinal chains of $S$.*

*Proof.* The following proof is based on Note 3 of [6].
Let $S = \{t_i : i < |S|\}$ be an enumeration of $S$. Let $S_i := \langle\{t_j : j < i\}\rangle_S$ for $i < |S|$. We will show that $(S_i)_{i<|S|}$ is a cofinal chain. It is clear that $S = \bigcup_{i<|S|} S_i$ and $S_i \subseteq S_j$ for $i \leq j$. To see that the $S_i$ are strict subsemigroups observe that each $S_i$ is generated by a set indexed by an ordinal $i < |S|$ and therefore is generated by a set of strictly smaller cardinality. If $S$ is countable it follows that $S \neq S_i$ as $S_i$ is finitely generated, and if $S$ is uncountable it follows that $S \neq S_i$ as $|S_i| < |S|$. We therefore have that $(S_i)_{i<|S|}$ is a cofinal chain. As each $S_i$ is a semigroup it follows that $S_iS_i \subseteq S_i$ and therefore $(S_i)_{i<|S|}$ is also a strong cofinal chain as required. □

Note that the validity of the following two definitions follows from the previous theorem together with the fact that the cardinals are a subclass of the ordinals and thus any class of cardinals has a least element.

**Definition 3.2.4** Let $S$ be a non-finitely generated semigroup. The *cofinality* of $S$, denoted $cf(S)$, is defined to be the smallest cardinal $\kappa$ such that there exists a cofinal chain of $S$ indexed by $\kappa$.

**Definition 3.2.5** Let $S$ be a non-finitely generated semigroup. The *strong cofinality* of $S$, denoted $scf(S)$, is defined to be the smallest cardinal $\kappa$ such that there exists a strong cofinal chain of $S$ indexed by $\kappa$.

Note that $\text{Sym}(\Omega)$ is not finitely generated for an infinite $\Omega$ as it is uncountable by Theorem 1.3.7 and therefore we can assign it a cofinality and a strong cofinality. In addition by the proof of Theorem 3.2.3 we have:

$$scf(\text{Sym}(\Omega)) \leq cf(\text{Sym}(\Omega)) \leq |\text{Sym}(\Omega)| = 2^{|\Omega|}$$

**Theorem 3.2.6.** *If $\Omega$ is an infinite set, then $cf(\text{Sym}(\Omega)) > \aleph_0$.*

*Proof.* Suppose for a contradiction that $cf(\text{Sym}(\Omega)) \leq \aleph_0$. Then there is a sequence of strict subsemigroups $(S_i)_{i\in\mathbb{N}}$ of $\text{Sym}(\Omega)$ who's union is $\text{Sym}(\Omega)$. Let $\psi : \text{Sym}(\Omega) \to \mathbb{N}$ be defined by:

$$(f)\psi = \min\{n \in \mathbb{N} : f \in S_n\}$$

Let $s, t \in \text{Sym}(\Omega)$. Without loss of generality we may assume that $S_{\psi(s)} \subseteq S_{\psi(t)}$, it follows that $st \in S_{\psi(t)}$. Letting $C_\psi = 0$ we have $(st)\psi \leq (t)\psi \leq (t)\psi + (s)\psi + C_\psi$.
By Theorem 3.1.7 it follows that $\text{Sym}(\Omega)$ is quasi-bounded and so this function is bounded above by some natural number $N$. It follows then that for all $f \in \text{Sym}(\Omega)$, we have $f \in S_N$ so $S_N \not< \text{Sym}(\Omega)$ this is a contradiction. □

**Theorem 3.2.7.** *If $\Omega$ is an infinite set, then $scf(\text{Sym}(\Omega)) > \aleph_0$.*

*Proof.* The following proof is based on the proof of proposition 2.2 in [5].
Suppose for a contradiction that $scf(\text{Sym}(\Omega)) \leq \aleph_0$. Then there is a chain of strict subsets $(S_i)_{i\in\mathbb{N}}$ of $\text{Sym}(\Omega)$ such that $\text{Sym}(\Omega) = \bigcup_{i\in\mathbb{N}} S_i$, $S_i \subseteq S_j$ for $i \leq j$ and for all $i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that $S_iS_i \subseteq S_j$.
It is clear that $\text{Sym}(\Omega) = \bigcup_{i\in\mathbb{N}}\langle S_i\rangle_S$ and $\langle S_i\rangle_S \subseteq \langle S_j\rangle_S$ for $i \leq j$. From Theorem 3.2.6 we have that $cf(\text{Sym}(\Omega)) > \aleph_0$ and so we must have that $\langle S_j\rangle_S = \text{Sym}(\Omega)$ for some $j \in \mathbb{N}$. By Theorem 3.1.7 $\text{Sym}(\Omega)$ has the Bergman property so it follows that $\text{Sym}(\Omega) = \bigcup_{k=1}^n S_j^k$ for some $n \in \mathbb{N}$.
To reach the desired contradiction it suffices to show that $\text{Sym}(\Omega) = \bigcup_{k=1}^n S_j^k \subseteq S_N$ for some $N \in \mathbb{N}$.
We have that $S_jS_j \subseteq S_{j_2}$ for some $j_2 \in \mathbb{N}$. It follows that $S_jS_jS_j \subseteq S_{j_2}S_j \subseteq (S_{\max j,j_2})^2 \subseteq S_{j_3}$ for some $j_3 \in \mathbb{N}$. By induction for all $i \leq n$ we can construct $j_i$ such that $S_j^i \subseteq S_{j_i}$. It therefore follows that $\cup_{k=1}^n S_j^k \subseteq S_{\max\{j_i : i \leq n\}}$ as required. □

## 3.3 Shuffling the Plane

Throughout the next two sections we will without loss of generality consider $\Omega$ as $A \times A$ where $A$ is an abelian group of infinite order (note that there are abelian groups of all cardinalities). Let $A$ be indexed by $\{a_i : i < |A|\}$ with $a_0 = id_A$. We will use the functions $\pi_1, \pi_2$ to be the projection of a tuple onto its first and second coordinates respectively and if we have functions $S_i, S_{i+1} \ldots S_k$ we will use the notation $S_{i\to k}$ to denote $S_iS_{i+1}\ldots S_k$. In this section we try to write the elements of $\text{Sym}(\Omega)$ as the product of 'slides'.

**Definition 3.3.1** Let $f : A \to A$ be a function. Then a vertical slide $v_f : A \times A \to A \times A$ is defined by:

$$(x, y)v_f = (x, y + (x)f)$$

Similarly a horizontal slide $h_f : A \times A \to A \times A$ is defined by:

$$(x, y)h_f = (x + (y)f, y)$$

We will use the word slide to refer to either of these.

**Definition 3.3.2** Let $V$ be used to denote the group of all vertical slides of $\Omega$, and similarly let $H$ denote the group of all horizontal slides of $\Omega$. Note that these groups are abelian as $A$ is abelian.

We will start by showing any moiety can be mapped into the diagonal line $\{(x,y) \in A \times A : x = y\}$ and then show from this that we can construct any element of $\mathrm{Sym}(\Omega)$.

**Definition 3.3.3** Let $x \in A$. Then the vertical and horizonal lines of $x$ are defined respectively by:

$$v_x = \{(x,y) : y \in A\} \qquad h_x = \{(y,x) : y \in A\}$$

The word line will be used to describe any set of either of these types.

**Definition 3.3.4** Let $L \subseteq \Omega$ be a line and let $S \subseteq \Omega$. We say that $L$ is S-contained if we have that $L \subseteq S$, we say $L$ is S-disjoint if $L \subseteq S^c$ and we say that $L$ is S-sporadic if we have that neither $L \subseteq S$ nor $L \subseteq S^c$.

**Theorem 3.3.5.** *Let $M$ be a moiety of $\Omega$ then either $|\{x \in A : h_x \text{ is M-sporadic }\}| = |A|$ or $|\{x \in A : v_x \text{ is M-sporadic }\}| = |A|$*

*Proof.* The following proof is based on the proof of Lemma 1 in [14].
Suppose not, then we have $|\{x \in A : v_x \text{ is M-sporadic }\}| < |A|$ and $|\{x \in A : h_x \text{ is M-sporadic }\}| < |A|$ and thus also $|\{x \in A : v_x \text{ is M-contained or M-disjoint}\}| = |A|$.

Case 1: If $|\{x \in A : v_x \text{ is M-contained }\}| = |A|$ and $|\{x \in A : v_x \text{ is M-disjoint }\}| = |A|$ then it follows that for all $y \in A$ we have $h_y$ is M-sporadic a contradiction.

Case 2: If $|\{x \in A : v_x \text{ is M-contained }\}| < |A|$ it follows that we have $|\{x \in A : v_x \text{ is M-contained or M-sporadic }\}| < |A|$ and $|\{x \in A : v_x \text{ is M-disjoint }\}| = |A|$. It follows that $\{x \in A : h_x \text{ is M-contained }\} = \emptyset$. As $|\{x \in A : h_x \text{ is M-sporadic }\}| < |A|$ we have that $|\{x \in A : h_x \text{ is M-sporadic or M-contained }\}| < |A|$. However:

$$M \subseteq \{x \in A : v_x \text{ is M-sporadic or M-contained }\} \times \{x \in A : h_x \text{ is M-sporadic or M-contained }\}$$

Thus it follows that $|M| < |A \times A| = |A|$ as $M$ is a moiety this is a contradiction.

Case 3: If $|\{x \in A : v_x \text{ is M-disjoint }\}| < |A|$ it follows that we have $|\{x \in A : v_x \text{ is M-disjoint or M-sporadic }\}| < |A|$ and $|\{x \in A : v_x \text{ is M-contained }\}| = |A|$. It follows that $\{x \in A : h_x \text{ is M-disjoint }\} = \emptyset$. As $|\{x \in A : h_x \text{ is M-sporadic }\}| < |A|$ we have that $|\{x \in A : h_x \text{ is M-sporadic or M-disjoint }\}| < |A|$. However:

$$M^c \subseteq \{x \in A : v_x \text{ is M-sporadic or M-disjoint }\} \times \{x \in A : h_x \text{ is M-sporadic or M-disjoint }\}$$

Thus it follows that $|M^c| < |A \times A| = |A|$ as $M$ is a moiety this is a contradiction.

□

**Theorem 3.3.6.** *Let $M$ be a moiety, then there exists a slide $S$ such that either $h_x$ is MS-sporadic for all $x \in A$ or $v_x$ is MS-sporadic for all $x \in A$.*

*Proof.* The following proof is based on the proof of Lemma 2 in [14].
By Theorem 3.3.5 we may assume without loss of generality that $|\{x \in A : h_x \text{ is M-sporadic }\}| = |A|$.
Let $\{M_1, M_2\}$ be a partition of $\{x \in A : h_x \text{ is M-sporadic }\}$ into moieties . Let $\phi_1 : M_1 \to A$ and $\phi_2 : M_2 \to A$ be bijections. Let $\phi_3 : \{x \in A : h_x \text{ is M-sporadic }\} \to A$ and $\phi_4 : \{x \in A : h_x \text{ is M-sporadic }\} \to A$ be such that for all $x \in A$ such that $h_x$ is M-sporadic we have $((x)\phi_3, x) \in M$ and $((x)\phi_4, x) \notin M$.
Let $f : A \to A$ be defined by:

$$(a)f = \left\{ \begin{array}{ll} -(a)\phi_3 + (a)\phi_1 & a \in M_1 \\ -(a)\phi_4 + (a)\phi_2 & a \in M_2 \\ a & \text{otherwise} \end{array} \right\}$$

Let $S = h_f$. For all $x \in A$ we have

$$(x,(x)\phi_1^{-1}) = ((x)\phi_1^{-1}\phi_3 - (x)\phi_1^{-1}\phi_3 + (x)\phi_1^{-1}\phi_1, (x)\phi_1^{-1}) = ((x)\phi_1^{-1}\phi_3 + (x)\phi_1^{-1}f, (x)\phi_1^{-1}) = ((x)\phi_1^{-1}\phi_3, (x)\phi_1^{-1})h_f \in (M)S$$

$$(x,(x)\phi_2^{-1}) = ((x)\phi_2^{-1}\phi_4 - (x)\phi_2^{-1}\phi_4 + (x)\phi_2^{-1}\phi_2, (x)\phi_2^{-1}) = ((x)\phi_2^{-1}\phi_4 + (x)\phi_2^{-1}f, (x)\phi_2^{-1}) = ((x)\phi_2^{-1}\phi_4, (x)\phi_2^{-1})h_f \notin (M)S$$

and thus $v_x$ is MS-sporadic as required. □

**Definition 3.3.7** Let $P := \{(x, X, Y_0, Y_1) : x \in A, X \text{ is an initial segment of } A, Y_0, Y_1 \subseteq A \backslash X \text{ such that } Y_0 \cap Y_1 = \emptyset, \text{ and } |Y_0|, |Y_1| \leq 2\}$

**Theorem 3.3.8.** *The set $P$ can be indexed as $\{p_i = (x_i, X_i, Y_{0,i}, Y_{1,i}) : i < |P|\}$ such that for all initial segments $\{a_j : j < a_m\}$ of $A$ we have $\{p_i \in P : \{x_i\} \cup X_i \cup Y_{0,i} \cup Y_{1,i} \subseteq \{a_j : j < M\}\}$ is bounded above. In addition there exist $t_i \in A$ such that $\{t_i + X_i \cup Y_{0,i} \cup Y_{1,i} : i < |P|\}$ are pairwise disjoint.*

*Proof.* This proof is based on the proof of Lemma 3 in [14].
By the axiom of choice let $c_p : \mathcal{P}(P) \to P$ and $c_a : \mathcal{P}(A) \to A$ be choice functions.
By transfinite recursion let $A_i = \cup_{j<i}(\{x_k\} \cup X_k \cup Y_{0,k} \cup Y_{1,k}\})$, $p_i = c_p(\{(x, X, Y_0, Y_1) \in P : \{x\} \cup X \cup Y_0 \cup Y_1 \subseteq A_i\} \backslash \{p_j : j < i\})$ unless this set is empty in which case $p_i = (\min(A \backslash \cup_{j<i} A_j), \emptyset, \emptyset, \emptyset)$ in addition let $t_i := c_a(A \backslash \cup_{j<i} (t_j + X_j \cup Y_{0,j} \cup Y_{1,j} - X_i \cup Y_{0,i} \cup Y_{1,i}))$. Note that $A \backslash \cup_{j<i} (t_j + X_j \cup Y_{0,j} \cup Y_{1,j} - X_i \cup Y_{0,i} \cup Y_{1,i})$ is non-empty as we are removing less than $|A|$ points from $A$. Note that by construction we have $\{p_i \in P : \{x_i\} \cup X_i \cup Y_{0,i} \cup Y_{1,i} \subseteq \{a_j : j < M\}\}$ is bounded above by $(a_{m^+}, \emptyset, \emptyset, \emptyset)$, and if $\{t_i + X_i \cup Y_{0,i} \cup Y_{1,i} : i < |P|\}$ were not pairwise disjoint then we would have $(t_i + X_i \cup Y_{0,i} \cup Y_{1,i}) \cap (t_j + X_j \cup Y_{0,j} \cup Y_{1,j}) \neq \emptyset$ for some $i > j$ and therefore $t_i \in (t_j + X_j \cup Y_{0,j} \cup Y_{1,j} - X_i \cup Y_{0,i} \cup Y_{1,i})$ a contradiction. □

**Theorem 3.3.9.** *Let $M$ be a moiety of $\Omega$. Then there exist slides $S_1, S_2$ such that for all $p_i \in P$ we have $\{(x_i, t_i + b) : b \in Y_{1,i}\} \subseteq MS_{1\to2}$ and $\{(x_i, t_i + c) : c \in X_i \cup Y_{0,i}\} \subseteq (MS_{1\to2})^c$ or we have $\{(t_i + b, x_i) : b \in Y_{1,i}\} \subseteq MS_{1\to2}$ and $\{(t_i + c, x_i) : c \in X_i \cup Y_{0,i}\} \subseteq (MS_{1\to2})^c$*

*Proof.* The following proof is based on the proof of Lemma 3 in [14].
Let $S_1$ be as in Theorem 3.3.6 and without loss of generality assume that for all $x \in A$ we have $h_x$ is $MS_1$-sporadic. By Theorem 3.3.8 we have $\{t_i + X_i \cup Y_{0,i} \cup Y_{1,i} : i < |P|\}$ are pairwise disjoint. Let $\phi_1 : A \to A$ and $\phi_2 : A \to A$ be such that $((x)\phi_1, x) \in MS_1$ and $((x)\phi_2, x) \notin MS_1$. Let $f : A \to A$ be defined by:

$$(a)f = \left\{ \begin{array}{lr} -(a)\phi_1 + x_i & a \in t_i + Y_{1,i} \\ -(a)\phi_2 + x_i & a \in t_i + X_i \cup Y_{0,i} \\ a & \text{otherwise} \end{array} \right\}$$

Let $S_2 = h_f$. For all $i < |P|$, $b \in Y_{1,i}$ and $c \in X_i \cup Y_{0,i}$ we have

$$(x_i, t_i + b) = ((t_i + b)\phi_1 - (t_i + b)\phi_1 + x_i, t_i + b) = ((t_i + b)\phi_1, t_i + b)h_f \in MS_{1\to2}$$

$$(x_i, t_i + c) = ((t_i + c)\phi_2 - (t_i + c)\phi_2 + x_i, t_i + c) = ((t_i + c)\phi_2, t_i + c)h_f \notin MS_{1\to2}$$

□

**Theorem 3.3.10.** *Let $M$ be a moiety of $\Omega$, then there exist slides $S_1, S_2, S_3, S_4, S_5$ such that $MS_{1\to5} \subseteq \{(x, y) \in A \times A : x = y\}$.*

*Proof.* The following proof is based on the proof of Lemma 4 in [14].
Let $S_1, S_2$ be as in Theorem 3.3.9. Without loss of generality for all $p_i \in P$ we have $\{(x_i, t_i + b) : b \in Y_{1,i}\} \subseteq MS_{1\to2}$ and $\{(x_i, t_i + c) : c \in X_i \cup Y_{0,i}\} \subseteq (MS_{1\to2})^c$.

Case 1: $A$ is countable.
Let $k_0 = 0, X_0 = Y_0 = Z_0 = \{a_0\}, b_0 = c_0 = a_0$.

1. Let $X_{n+1} := \{a_i \in A : i \leq k_n \text{ and } (a_i, a_{n+1} + c_i) \in MS_{1\to2}\}$.
2. Let $k_{n+1} > k_n$ be such that $(X_{n+1} + \min(A \backslash \cup_{i=0}^n Z_i) - a_{k_{n+1}}) \cap (\cup_{i=0}^n Z_i) = \emptyset$. This must exist as $|(X_{n+1} + \min(A \backslash \cup_{i=0}^n Z_i)) - (\cup_{i=0}^n Z_i)|$ is finite and therefore its complement must contain elements greater than $a_{k_n}$.
3. Let $b_{n+1} = \min(A \backslash \cup_{i=0}^n Z_i) - a_{k_{n+1}}$. Note this means that $(X_{n+1} + b_{n+1}) \cap (\cup_{i=0}^n Z_i) = \emptyset$ by the definition of $k_{n+1}$.
4. Let $c_i$ for $k_n < i \leq k_{n+1}$ be defined by: $(a_{k_{n+1}}, a_{n+1} + c_{k_{n+1}}) \in MS_{1\to2}$ and for all other $k_n < i \leq k_{n+1}$ and $j \leq n+1$ we have $(a_i, a_j + c_i) \notin MS_{1\to2}$. This can be done by the definition of $S_1, S_2$.
5. Let $Y_{n+1} = X_{n+1} \cup \{a_{k_{n+1}}\}$
6. Let $Z_{n+1} = Y_{n+1} + b_{n+1}$. Note that $Z_{n+1}$ is disjoint from $\cup_{i=0}^n Z_i$ (by 3).

As each $Z_n$ contains $\min(A \backslash \cup_{i=0}^{n-1} Z_i)$ (by 3,5,6) we have that $\bigcup_{n\in\mathbb{N}} Z_n = A$. In addition (by 6) we have that the $Z_n$ are disjoint. So we have the $Z_n$ partition $A$.
Let $S_3 = v_{f_3}$ where $f_3$ is defined by $(a_i)f_3 = -c_i$. Let $S_4 = h_{f_4}$ where $f_4$ is defined by $(a_i)f_4 = b_i$. Let $S_5 = v_{f_5}$ where $f_5$ is defined by $(a_i)f_5 = a_i - a_n$ where $n$ corresponds to the unique $Z_n$ containing $a_i$
We have that $Y_n = \{a_i \in A : (a_i, a_n) \in MS_{1\to3}\}$ as if $a_i \in Y_n$ we have that $(a_i, a_n + c_i) \in MS_{1\to2}$ (by 1,4,5) and if $a_i \notin Y_n$ we have that $(a_i, a_n + c_i) \notin MS_{1\to2}$ (by 1,5 if $i \leq k_n$ and by 4,5 if $i > k_n$).
We therefore have (by 6) that $Z_n = \{a_i \in A : (a_i - b_n, a_n) \in MS_{1\to3}\} = \{a_i \in A : (a_i, a_n) \in MS_{1\to4}\}$.
So we have that if $(a_i, a_n) \in MS_{1\to4}$ then $a_i \in Z_n$ and thus $(a_i, a_n)S_5 = (a_i, a_n - a_n + a_i) = (a_i, a_i) \in \{(x, y) \in A \times A : x = y\}$ and thus $MS_{1\to5} \subseteq \{(x, y) \in A \times A : x = y\}$.

Case 2: $A$ is uncountable.
We re-index $A \backslash \{id_A\} = \{a_i : i < |A|\}$ and we define $(A_i)_{i<|A|}$ to be a cofinal chain for $A$ such that $A_0 = \{id_A\}$ and for all $i < |A|$ we have $A_i$ is a group, $a_{2i}, a_{2i+1} \in A_{i+1}$ and $[A_{i+1} : A_i] \geq 4$. This can be done as follows: TODO define index of group

1. if $i = 0$ then let $A_i = \{id_A\}$
2. if $i$ is a successor ordinal let $A_i := \langle\{a_j : j \leq \min\{k \geq 2i+1 : [\langle\{a_\alpha : \alpha \leq k\}\rangle_G : A_i] \geq 4\}\}\rangle_G$
3. if $i$ is a limit ordinal then $A_i := \cup_{j<i} A_j$

For $i < |A|$ we define $c_i, d_i, m_i$ by transfinite recursion.

1. Let $c_i \in A_{i+1} \backslash A_i$
2. Let $d_i \in A_{i+1} \backslash ((A_i + c_i) \cup (A_i + c_i^{-1}) \cup A_i)$. Note that we can do this as $[A_{i+1} : A_i] \geq 4$. It follows from the definition of $d_i$ that $id_A, c_i, d_i, c_i + d_i$ are in different cosets of $A_i$ in $A_{i+1}$.
3. For $a_j \in A_{i+1} \backslash A_i$ by the definition of $S_1, S_2$ we can define $m_j$ to be such that:
   (a) $(a_j, a_k - m_j) \notin MS_{1\to 2}$ for $k < 2i$ and $(a_j, -m_j) \notin MS_{1\to 2}$
   (b) If $a_j \in (A_i - c_i)$ then $(a_j, a_{2i} - m_j) \notin MS_{1\to 2}$
   (c) If $a_j = a_k - d_i - c_i$ for some $a_k \in A_i$ and $(a_k, a_{2i+1} - m_k) \notin MS_{1\to 2}$ then $(a_j, a_{2i} - m_j) \in MS_{1\to 2}$
   (d) If $a_j = a_k - d_i - c_i$ for some $a_k \in A_i$ and $(a_k, a_{2i+1} - m_k) \in MS_{1\to 2}$ then $(a_j, a_{2i} - m_j) \notin MS_{1\to 2}$
   (e) If $a_j \in A_{i+1} \backslash (A_i \cup (A_i - c_i) \cup (A_i - (d_i + c_i)))$ then $(a_j, a_{2i} - m_j) \in MS_{1\to 2}$
   (f) If $a_j = a_k + c_i + d_i$ for some $a_k \in A_i$ and $(a_k, a_{2i} - m_k) \notin MS_{1\to 2}$ then $(a_j, a_{2i+1} - m_j) \in MS_{1\to 2}$
   (g) If $a_j = a_k + c_i + d_i$ for some $a_k \in A_i$ and $(a_k, a_{2i} - m_k) \in MS_{1\to 2}$ then $(a_j, a_{2i+1} - m_j) \notin MS_{1\to 2}$
   (h) If $a_j \in A_{i+1} \backslash (A_i \cup (A_i + (c_i + d_i)))$ then $(a_j, a_{2i+1} - m_j) \notin MS_{1\to 2}$

Let $S_3 = v_{f_3}$ where $f_3$ is defined by $(a_i)f_3 = m_i$ and $(id_A)f_3$ is such that $(id_A, -(id_A)f_3) \in MS_{1\to 2}$. Let $S_4 = h_{f_4}$ where $f_4$ is defined by $(a_{2i})f_4 = c_i$, $(a_{2i+i})f_4 = -d_i$ and $(id_A)f_4 = id_A$. Note that for all non-identity elements $x \in A$ there is a unique $i < |A|$ such that $x \in A_{i+1} \backslash A_i$. We will now show that for all $x \in A$ there is at most one point in $MS_{1\to 4} \cap v_x$. For $i < |A|$ we have (by 3b and 3h) that:

$$(-c_i, a_{2i} - (-c_i)f_3) \notin MS_{1\to 2} \implies (-c_i, a_{2i}) \notin MS_{1\to 3} \implies (id_A, a_{2i}) \notin MS_{1\to 4}$$

$$(d_i, a_{2i+1} - (d_i)f_3) \notin MS_{1\to 2} \implies (d_i, a_{2i+1}) \notin MS_{1\to 3} \implies (id_A, a_{2i+1}) \notin MS_{1\to 4}$$

Let $a_j \in A_{i+1} \backslash A_i$.
For $k < i$ we have $a_j - c_k \in A_{i+1} \backslash A_i$ and $a_j + d_k \in A_{i+1} \backslash A_i$ therefore (by 3a) we have $(a_j - c_k, a_{2k} - (a_j - c_k)f_3) \notin MS_{1\to 2}$ and $(a_j + d_k, a_{2k+1} - (a_j + d_k)f_3) \notin MS_{1\to 2}$. For $k > i$ we also have these two conditions (by 3b and 3h).
Therefore for $i \neq k$ we have:

$$(a_j - c_k, a_{2k} - (a_j - c_k)f_3) \notin MS_{1\to 2} \implies (a_j - c_k, a_{2k}) \notin MS_{1\to 3} \implies (a_j, a_{2k}) \notin MS_{1\to 4} \implies (a_j, a_j + a_{2k} - a_{2i})$$

$$(a_j + d_k, a_{2k+1} - (a_j + d_k)f_3) \notin MS_{1\to 2} \implies (a_j + d_k, a_{2k+1}) \notin MS_{1\to 3} \implies (a_j, a_{2k+1}) \notin MS_{1\to 4}$$

If $a_j = a_k - d_i$ for some $a_k \in A_i$ then (by 3d) we have one of:

$$(a_k, a_{2i+1} - m_k) \notin MS_{1\to 2} \implies (a_k, a_{2i+1}) \notin MS_{1\to 3} \implies (a_j, a_{2i+1}) \notin MS_{1\to S_4}$$

$$(a_j - c_i, a_{2i} - (a_j - c_i)f_3) \notin MS_{1\to 2} \implies (a_j - c_i, a_{2i}) \notin MS_{1\to 3} \implies (a_j, a_{2i}) \notin MS_{1\to 4}$$

If $a_j = a_k + c_i$ for some $a_k \in A_i$. then (by 3g) we have one of:

$$(a_k, a_{2i} - m_k) \notin MS_{1\to 2} \implies (a_k, a_{2i}) \notin MS_{1\to 3} \implies (a_j, a_{2i}) \notin MS_{1\to 4}$$

$$(a_j + d_i, a_{2i+1} - (a_j + d_i)f_3) \notin MS_{1\to 2} \implies (a_j + d_i, a_{2i+1}) \notin MS_{1\to 3} \implies (a_j, a_{2i+1}) \notin MS_{1\to 4}$$

Otherwise by (3h) we have:

$$(a_j + d_i, a_{2i+1} - (a_j + d_i)f_3) \notin MS_{1\to 2} \implies (a_j + d_i, a_{2i+1}) \notin MS_{1\to 3} \implies (a_j, a_{2i+1}) \notin MS_{1\to 4}$$

We therefore have that $v_{id_A} \cap MS_{1\to 5}$ contains at most $(id_A, id_A)$ and for all $a_j$ we have at most one of $(a_j, a_{2i}), (a_j, a_{2i+1})$ in $v_{a_j} \cap M_{S_{1\to 4}}$ and no other points. Thus we can construct a vertical slide $S_5$ such that $MS_{1\to 5} \subseteq \{(x, y) \in A \times A : x = y\}$. $\square$

**Theorem 3.3.11.** *Let $M$ be a moiety and let $p \in \mathrm{Sym}_\Omega(M)$ then there exist 11 slides $S_1, S_2 \ldots S_{11}$ such that $S_{1\to 11}|_M = p|_M$.*

*Proof.* The following proof is based on the proof of claim 11 in [13].
Let $S_1, S_2, S_3, S_4, S_5$ be as in Theorem 3.3.10 and for $6 < n < 12$ let $S_n := S_{12-n}^{-1}$ . Without loss of generality assume that $S_5$ is a vertical slide. Let $I : M \to A$ be defined by $(x)I = (x)S_{1\to 5}\pi_1 (= (x)S_{1\to 5}\pi_2)$.
Let $f_1 : A \to A$ and $f_2 : A \to A$ be defined by:

$$(a)f_1 = \left\{ \begin{array}{ll} (a)I^{-1}pI - a & a \in \operatorname{img}(I) \\ a_0 & \text{otherwise} \end{array} \right\} \qquad (a)f_2 = \left\{ \begin{array}{ll} a - (a)I^{-1}p^{-1}I & a \in \operatorname{img}(I) \\ a_0 & \text{otherwise} \end{array} \right\}$$

Let $S'_5 := v_{f_1}$ and $S_6 := h_{f_2}$. We now have for $x \in M$:

$$\begin{aligned}
(x)S_{1\to 5}S'_5S_{6\to 11} &= ((x)I, (x)I)S'_5S_{6\to 11} \\
&= ((x)I, (x)I + (x)II^{-1}pI - (x)I)S_{6\to 11} \\
&= ((x)I, (x)pI)S_{6\to 11} \\
&= ((x)I + (x)pI - (x)pII^{-1}p^{-1}I, (x)pI)S_{7\to 11} \\
&= ((x)I + (x)pI - (x)I, (x)pI)S_{7\to 11} \\
&= ((x)pI, (x)pI)S_{7\to 11} \\
&= (x)p
\end{aligned}$$

Thus $S_{1\to 5}S'_5S_{6\to 11}|_M = p|_M$ and as $S_5$ and $S'_5$ are both vertical slides it follows that $S_5S'_5$ is also a single vertical slide ans thus we have the required result. $\square$

**Theorem 3.3.12.** *Let $M$ be a moiety of $\Omega$ and let $p \in \operatorname{Sym}_\Omega(M)$. Then there are slides $S_1, S_2 \ldots S_{44}$ such that $p = S_{1\to 44}$.*

*Proof.* The following proof is based on the proof of claim 12 in [13].
Let $\{M_1, M_2\}$ be a partition of $M^c$ into moieties. By Theorem 2.3.7 let $p_1, p_2 \in \operatorname{Sym}_\Omega(M)$ be such that $p_1^{-1}p_2^{-1}p_1p_2 = p$. By Theorem 3.3.11 let $S_1, S_2 \ldots S_{22}$ be such that $S_{1\to 11}|_{M\cup M_1} = p_1^{-1}|_{M\cup M_1}$ and $S_{12\to 22}|_{M\cup M_2} = p_2^{-1}|_{M\cup M_2}$. Let $S_n = S_{34-n}^{-1}$ for $23 \le n \le 33$ and $S_n = S_{56-n}^{-1}$ for $34 \le n \le 44$.
Let $(x, y) \in \Omega$.

$$\begin{aligned}
(x,y)S_{1\to 44} &= \left\{ \begin{array}{ll} (x,y)p_1^{-1}S_{12\to 44} & (x,y) \in M \\ (x,y)S_{12\to 44} & (x,y) \in M_1 \\ ((x,y)S_{1\to 11})S_{12\to 44} & (x,y) \in M_2 \end{array} \right\} \\
&= \left\{ \begin{array}{ll} (x,y)p_1^{-1}p_2^{-1}S_{23\to 44} & (x,y) \in M \\ ((x,y)S_{12\to 22})S_{23\to 44} & (x,y) \in M_1 \\ ((x,y)S_{1\to 11})S_{23\to 44} & (x,y) \in M_2 \end{array} \right\} \\
&= \left\{ \begin{array}{ll} (x,y)p_1^{-1}p_2^{-1}p_1S_{34\to 44} & (x,y) \in M \\ ((x,y)S_{12\to 22})S_{34\to 44} & (x,y) \in M_1 \\ (x,y)S_{34\to 44} & (x,y) \in M_2 \end{array} \right\} \\
&= \left\{ \begin{array}{ll} (x,y)p_1^{-1}p_2^{-1}p_1p_2 & (x,y) \in M \\ (x,y) & (x,y) \in M_1 \\ (x,y) & (x,y) \in M_2 \end{array} \right\} \\
&= (x,y)p
\end{aligned}$$

$\square$

**Theorem 3.3.13.** *Let $p \in \operatorname{Sym}(\Omega)$, then there exist 55 slides $S_1, S_2 \ldots S_{55}$ such that $p = S_{1\to 55}$.*

*Proof.* Let $P$ be the partition of $\Omega$ into disjoint cycles of $p$.
Case 1: If we have that $|P| = |\Omega|$ then let $M_p$ be a moiety of $P$ and let $M = \cup M_p$. We have that $p = p|_M p|_{M^c}$. By Theorem 3.3.11 let $S_1, S_2 \ldots S_{11}$ be such that $p|_M = S_{1\to 11}|_M$. By Theorem 3.3.12 we can find $S_{12\to 55}$ such that $(S_{1\to 11})^{-1}(p|_M)(p|_{M^c}) = S_{12\to 55}$ as this product fixes the moiety $M$ pointwise. We therefore have that

$$S_{1\to 55} = (S_{1\to 11})S_{12\to 55} = (S_{1\to 11})(S_{1\to 11})^{-1}(p|_M)(p|_{M^c}) = (p|_M)(p|_{M^c}) = p$$

Case 2: If we have that $|P| < |\Omega|$ then as each cycle is countable it follows than $\Omega$ is countable. As $|\Omega| = \aleph_0$ we have that $p$ has finitely many cycles and thus has at least one infinite cycle $C = (\ldots c_{-1}, c_0, c_1 \ldots)$. Let $p_1 : \Omega \to \Omega$ be defined by:

$$(x)p_1 = \left\{ \begin{array}{ll} c_{i+1} & x = c_i \text{ for some } i = 0 \bmod 3 \\ c_{i-1} & x = c_i \text{ for some } i = 1 \bmod 3 \\ x & \textit{otherwise} \end{array} \right\}$$

Let $M_1 := \{c_i : i \neq 2 \bmod 3\}$ and $M_2 := \{c_i : i = 1 \bmod 3\}$. By Theorem 3.3.11 let $S_1, S_2 \dots S_{11}$ be such that $p_1|_{M_1} = S_{1\to 11}|_{M_1}$. By Theorem 3.3.12 we can find $S_{12\to 55}$ such that $(S_{1\to 11})^{-1}p = S_{12\to 55}$ as this product fixes the moiety $M_2$ pointwise. We therefore have that

$$S_{1\to 55} = (S_{1\to 11})S_{12\to 55} = (S_{1\to 11})(S_{1\to 11})^{-1}p = p$$

□

**Corollary 3.3.14.** *The group* $\mathrm{Sym}(\Omega)$ *is equal to* $(HV)^{28}$.

*Proof.* Let $p \in \mathrm{Sym}(\Omega)$, we can now write $p$ as $S_{1\to 55}$ this is an alternating product of elements of $H$ and $V$ so if $S_1 \in H$ then $S_{1\to 55} \in (HV)^{27}H \subseteq (HV)^{28}$ and if $S_1 \in V$ then $S_{1\to 55} \in V(HV)^{27} \subseteq (HV)^{28}$ □

## 3.4 Products of Abelian Groups

In the previous section on shuffling the plane the works of Miklos Abert, Tamas Keleti and Peter Komjath gave us a means for expressing any element of an infinite symmetric group as a product of 'slides'. This gives us a way of writing any infinite symmetric group as a product of 56 abelian groups. In this section we utilise the techniques of Akos Seress found in [15] to decrease this bound further.

**Definition 3.4.1** Let $b \in A$, we call a set $D \subseteq \{(a, t + a) : a \in A\}$ a t-diagonal segment.

Observe that if $a \neq b$ then all a-diagonal segments are disjoint from all b-diagonal segments.

**Theorem 3.4.2.** *Let $t \in A$ and $D$ be a t-diagonal segment, then the group $HV$ acts fully on $D$.*

*Proof.* The following proof is based on the proof of Lemma 3 in [15]:
Let $g \in \mathrm{Sym}(D)$ and let $f_1 : A \to A$ and $f_2 : A \to A$ be defined by:

$$(a)f_1 = (a - t, a)g\pi_1 - a + t \qquad (a)f_2 = a + t - (a, a + t)g^{-1}\pi_2$$

Let $(p, p + t) \in D$, we have that:

$$\begin{aligned}
(p, p+t)h_{f_1}v_{f_2} &= (p + (p + t - t, p + t)g\pi_1 - (p + t) + t, p + t)v_{f_2} \\
&= ((p, p + t)g\pi_1, p + t)v_{f_2} \\
&= ((p, p + t)g\pi_1, p + t + (p, p + t)g\pi_1 + t - ((p, p + t)g\pi_1, (p, p + t)g\pi_1 + t)g^{-1}\pi_2) \\
&= ((p, p + t)g\pi_1, p + t + (p, p + t)g\pi_1 + t - ((p, p + t)g\pi_1, (p, p + t)g\pi_2)g^{-1}\pi_2) \\
&= ((p, p + t)g\pi_1, p + t + (p, p + t)g\pi_1 + t - (p, p + t)gg^{-1}\pi_2) \\
&= ((p, p + t)g\pi_1, p + t + (p, p + t)g\pi_1 + t - (p + t)) \\
&= ((p, p + t)g\pi_1, (p, p + t)g\pi_1 + t) \\
&= ((p, p + t)g\pi_1, (p, p + t)g\pi_2) \\
&= (p, p + t)g
\end{aligned}$$

So we have that $h_{f_1}v_{f_2} \in HV$ satisfies that $(h_{f_1}v_{f_2})|_D = g$ as required. □

**Theorem 3.4.3.** *Let $M$ be a moiety of $\Omega$ and $t \in A$, then there exists $h_1v_1h_2 \in HVH$ such that $Mh_1v_1h_2 \cap D = \emptyset$ for all t-diagonal segments $D$.*

*Proof.* By Theorem 3.3.6 we have that there is an element $S \in H \cup V$ such that $h_x$ is MS-sporadic for all $x \in A$ or $v_x$ is MS-sporadic for all $x \in A$.
Case 1: If $S \in H$ and we have $v_x$ is MS-sporadic for all $x \in A$ then let $p_x \in v_x \backslash MS$ for all $x \in A$. Let $f : A \to A$ be defined by $(a)f = a + t - p_a$. It follows that for all $(x, y) \in M$ we have $(x, y)Sv_f \notin D$ and thus we can choose $h_1 = S, v_1 = v_f$ and $h_2 = id_A$.
Case 2: If $S \in H$ and we have $h_x$ is MS-sporadic for all $x \in A$ then we can make a similar argument by using only an element of $H$ (viewed as a product of two elements of $H$)
Case 3: If $S \in V$ and we have $h_x$ is MS-sporadic for all $x \in A$ then we can make a similar argument by letting $h_1 = id_A$ and $v_1 = S$.
Case 4: If $S \in V$ and we have $v_x$ is MS-sporadic for all $x \in A$ then we can make a similar argument by using only an element of $V$ (viewed as a product of two elements of $V$) □

**Theorem 3.4.4.** *Let $M$ be a moiety of $\Omega$. Then there exist abelian groups $H_M, V_M$ such that $H_MV_M$ acts fully on $M$.*

*Proof.* Let $D := \{(x, y) \in A \times A : x = y\}$, we have by Theorem 3.4.2 that $HV$ acts fully on $D$. Let $\phi_1 : M \to D$ and $\phi_2 : M^c \to D^c$ be bijections. Let $I_M : \Omega \to \Omega$ be the bijection defined by:

$$(x)I_M = \left\{ \begin{array}{ll} (x)\phi_1 & x \in M \\ (x)\phi_2 & x \in M^c \end{array} \right\}$$

As $HV$ acts fully on $D$ it follows that $I_M(HV)I_M^{-1}$ acts fully on $M$. So we also have that $I_M H I_M^{-1} I_M V I_M^{-1} = (I_M H I_M^{-1})(I_M V I_M^{-1})$ acts fully on $M$. Let $H_M := I_M H I_M^{-1}$ and $V_M := I_M V I_M^{-1}$. As the conjugates of abelian groups these are abelian groups and we have the required result. □

Now that we have the required theorems and definitions, we will prove the main result of this section. In Lemma 5 of [15] Akos Seress makes use of a group $D$ which contains elements of all possible disjoint cycle shapes, this idea is a critical part of the following proof in which we use a very similar group $C$.

**Theorem 3.4.5.** *The group* $\mathrm{Sym}(\Omega)$ *can be expressed as the product of 10 abelian groups.*

*Proof.* Let $L$ be a moiety of $A$, $t \in A \backslash \{id_A\}$, $D := \{(x, y) \in A \times A : x = y\}$, $D_1 := \{(a, a) : a \in L\}$, $D_2 := \{(a, a + t) : a \in L\}$. Let $P = \{p_i : i < |\Omega|\}$ be a partition of $D_1^c$ into countable sets such that there are $|\Omega|$ sets of all cardinalities less that or equal to $\aleph_0$. For all $i < |\Omega|$ let $c_i$ be the group generated by a $|p_i|$-cycle on $p_i$. Let $C = \prod_{i<|\Omega|} c_i$ (fixing all points of $D_1$). Finally let $H_{D_1}, V_{D_1^c}, H_{D_2^c}, V_{D_2^c}$ be as in Theorem 3.4.4.
Note that all the above groups are abelian as $C$ is a product of cyclic groups acting on disjoint sets and the others are abelian by construction. We will show that $\mathrm{Sym}(\Omega) = HVHH_{D_1^c}V_{D_2^c}H_{D_1^c}V_{D_1^c}CV_{D_1^c}H_{D_1^c}$.
Let $g \in \mathrm{Sym}(\Omega)$ and $M = Dg^{-1}$. As the image of a moiety under a permutation we have that $M$ is a moiety. By Theorem 3.4.3 we have that there is an element $g' \in HVH$ such that $Mg' \cap \{(a, a + t) : a \in A\} = \emptyset$.
Observe that $\{(a, a + t) : a \in L^c\} \subseteq (Mg')^c \backslash D_2$ and $|\{(a, a + t) : a \in L^c\}| = |\Omega| = |(\{(x, y) \in A \times A : x = y\} \cup D_2)^c|$. Let $\phi : (Mg')^c \backslash D_2 \to (D \cup D_2)^c$ be a bijection. Let $f \in \mathrm{Sym}(D_2^c)$ be defined by:

$$(x)f = \left\{ \begin{array}{ll} (x)g'^{-1}g & x \in Mg' \\ (x)\phi & x \in (Mg')^c \end{array} \right\}$$

As $H_{D_2^c}V_{D_2^c}$ acts fully on $D_2^c$ there exists $g'' \in H_{D_2^c}V_{D_2^c}$ such that $g''|_{D_2^c} = f$. We now have that $(g'g'')|_M = g|_M$ and $g'g'' \in HVHH_{D_2^c}V_{D_2^c}$. It therefore follows that $s := (g'g'')^{-1}g \in \mathrm{Pstab}(D) = \mathrm{Sym}_\Omega(D^c) \leq \mathrm{Sym}_\Omega(D_1^c)$. As $s|_{D_1^c}$ fixes $D \backslash D_1$ it has $|\Omega|$ 1-cycles, therefore by definition of $C$ by choosing either the identity or generating element of appropriately many $c_i$, there is an element $s' \in C$ such that $s'|_{D_1^c}$ has the same disjoint cycle shape as $s|_{D_1^c}$. As we have $s, s' \in \mathrm{Sym}_\Omega(D_L^c)$ have the same disjoint cycle shape when restricted to $D_1^c$ there exists $w \in \mathrm{Sym}_\Omega(D_1^c)$ such that $ws'w^{-1} = s$. As $H_{D_1^c}V_{D_1^c}$ acts fully on $D_1^c$, we have that there exists $w' \in H_{D_1^c}V_{D_1^c}$ such that $w'|_{D_1^c} = w|_{D_1^c}$. For all $p \in \Omega$ we now have:

$$(p)w's'w'^{-1} = \left\{ \begin{array}{ll} (p)w'w'^{-1} & p \in D_1 \\ (p)s & otherwise \end{array} \right\} = (p)s$$

It follows that $s = w's'w'^{-1} \in H_{D_1^c}V_{D_1^c}CV_{D_1^c}H_{D_1^c}$. Therefore $g = g'g''(g'g'')^{-1}g = g'g''s \in HVHH_{D_2^c}V_{D_2^c}H_{D_1^c}V_{D_1^c}CV_{D_1^c}H_{D_L^c}$ as required. □

# Chapter 4

# Maximal Subgroups of Infinite Symmetric Groups

In this chapter we aim to construct large families of maximal subgroups of $\mathrm{Sym}(\Omega)$. In particular we will show that for any infinite set $\Omega$ there exists a family of $2^{2^{|\Omega|}}$ maximal subgroups of $\Omega$ which are pairwise non-conjugate. To do this we will first be building groups from ultrafilters using what was established in chapter 1.

## 4.1 Building Groups from Ultrafilters

The following Theorem will be useful when showing the maximality certain subgroups.

**Theorem 4.1.1.** *Let $\Omega$ be an infinite set and let $G \leq \mathrm{Sym}(\Omega)$, then if all moieties of $\Omega$ are full in $G$ then $G = \mathrm{Sym}(\Omega)$.*

*Proof.* The following proof is based on Note 3 of section 4 in [7].
Let $G$ be a group such that all moieties of $\Omega$ are full in $G$.
Let $M$ be a moiety of $\Omega$. We have that $M^c$ is also a moiety and therefore $G$ acts on fully on $M$ and $M^c$. Let $G_s := G \cap \mathrm{Sstab}(M)$. As $G$ acts fully on $M$ we have that $G_s$ acts fully on $M$ (as the required elements stabilise $M$ setwise).
Let $\{M_1, M_2\}$ be a partition of $M$ into moieties. Let $g \in \mathrm{Sym}_\Omega(M_1)$ be such that $g$ has $|\Omega|$ cycles of all finite lengths. As $M_1 \cup M^c$ is a moiety there exists $g' \in G$ such that $g'|_{M_1 \cup M^c} = g|_{M_1 \cup M^c}$. We have that $g' \in G_s$ (as it fixes $M^c$ pointwise).
We now construct an new element $g^* \in ker(\phi)$ by reversing all cycles of odd or infinite length of $g'$ and preserving the others. This permutation is an element of $G_s$ as it can be constructed by conjugating $g'$ by an element of $G_s$ with the required action on $M$ (noting $g'$ fixes $M^c$ so it's action on $M^c$ is unaffected by conjugation by elements of $G_s$).
In the product $g^* g'$ all cycles of odd or infinite length cancel and all cycles of even length are squared. The square of a cycle of length $2n$ gives two cycles of length $n$. It follows that $g^* g' \in G_s$ has $2 * |\Omega| = |\Omega|$ cycles of all finite lengths and none of infinite length.
Therefore $G_s$ contains an element of $\mathrm{Sym}_\Omega(M)$ which has $|\Omega|$ cycles of all finite lengths and none of infinite length. By conjugating this element by elements with the appropriate action on $M$ we have that $G_s$ contains all elements of $\mathrm{Sym}_\Omega(M)$ which have $|\Omega|$ cycles of all finite lengths and none of infinite length. By Theorem 2.3.7 $\mathrm{Sym}_\Omega(M) \subseteq G_s$. We now have that $\mathrm{Sym}_\Omega(M) \leq G$ for all moieties $M$ and therefore by Theorem 3.1.3 we have $G = \mathrm{Sym}(\Omega)$. ☐

**Definition 4.1.2** Given an ultrafilter $\mathcal{U}$ on an infinite set $\Omega$.

$$F_\mathcal{U} := \{f \in \mathrm{Sym}(\Omega) : fix(f) \in \mathcal{U}\}$$

**Theorem 4.1.3.** *Given an ultrafilter $\mathcal{U}$ on an infinite set $\Omega$, we have $F_\mathcal{U} \leq \mathrm{Sym}(\Omega)$.*

*Proof.* Let $f \in F_\mathcal{U}$ then $fix(f) = fix(f^{-1})$ so $f^{-1} \in F_\mathcal{U}$. We have that $F_\mathcal{U}$ is closed under inverses.
Let $f, g \in G_S$ we have that $fix(f)$ and $fix(g)$ are in $\mathcal{U}$ therefore $fix(f) \cap fix(g) \in \mathcal{U}$. As $fix(fg) \supseteq fix(f) \cap fix(g)$ we have $fix(fg) \in \mathcal{U}$ and $fg \in F_\mathcal{U}$ so $F_\mathcal{U}$ is also closed under multiplication. ☐

**Definition 4.1.4** Let $S$ be a set and let $G$ be a group which acts on $S$. Then we say $G$ is *transitive* on $S$ if for all $x, y \in S$ there is an $f \in G$ such that $(x)f = y$.

**Theorem 4.1.5.** *Let $\mathcal{U}$ be an ultrafilter on an infinite set $\Omega$, then $F_\mathcal{U}$ is transitive on the moieties of $\Omega$ in $\mathcal{U}$.*

*Proof.* The following proof is based on the proof of Theorem 6.4 in [6].
Let $M_1, M_2 \in \mathcal{U}$ be moieties of $\Omega$. We will show that there is an element of $F_\mathcal{U}$ mapping $M_1$ to $M_2$.

Case 1: If $M_1\backslash M_2$ and $M_2\backslash M_1$ are both moieties then it follows that $|M_1\backslash M_2| = |M_2\backslash M_1|$. Let $\phi : M_1\backslash M_2 \to M_2\backslash M_1$ be a bijection and let $f \in \mathrm{Sym}(\Omega)$ be defined by:

$$(x)f = \left\{ \begin{array}{lr} x & x \in M_1 \cap M_2 \\ x & x \in X\backslash(M_1 \cup M_2) \\ (x)\phi^{-1} & x \in M_2\backslash M_1 \\ (x)\phi & x \in M_1\backslash M_2 \end{array} \right\}$$

As $M_1 \cap M_2 \subseteq fix(f)$ it follows that $fix(f) \in \mathcal{U}$ and therefore $f \in F_{\mathcal{U}}$. We have that $(M_1)f = M_2$ as required.

Case 2: Suppose that $M_1\backslash M_2$ or $M_2\backslash M_1$ is not a moiety. Without loss of generality we assume that $M_1\backslash M_2$ is not a moiety. Then it follows that $|\Omega| = |(M_1\backslash M_2)^c|$ and $|\Omega| > |M_1\backslash M_2|$. We now have the following:

$$\begin{aligned} |M_1 \cap M_2| &= |M_1\backslash(M_1\backslash M_2)| = |M_1| - |M_1\backslash M_2| = |\Omega| \\ |(M_1 \cap M_2)^c| &= |M_1^c \cup M_2^c| = |\Omega| \\ |M_1 \cup M_2| &= |\Omega| \\ |(M_1 \cup M_2)^c| &= |(M_2 \cup (M_1\backslash M_2))^c| = |M_2^c \cap (M_1\backslash M_2)^c| = |M_2^c\backslash(M_1\backslash M_2)| = |\Omega| \end{aligned}$$

So both $M_1 \cap M_2$ and $M_1 \cup M_2$ are moieties. Let $\{M_3, M_4\}$ be a partition of $M_1 \cap M_2$ into moieties. By Theorem 1.4.5 either $M_3$ or $M_3^c$ is in $\mathcal{U}$. If $M_3^c \in \mathcal{U}$ then $M_4 = M_3^c \cap M_1 \cap M_2 \in \mathcal{U}$. So we have that either $M_3$ or $M_4$ is in $\mathcal{U}$.
Consider the set $M_5 := M_4 \cup (M_1 \cup M_2)^c$.

$M_5\backslash M_1 = (M_1 \cup M_2)^c$ a moiety.

$M_1\backslash M_5 = M_1\backslash M_4$ a moiety (as it contains $M_3$ and its complement contains the complement of $M_1$).

$M_5\backslash M_2 = (M_1 \cup M_2)^c$ a moiety.

$M_2\backslash M_5 = M_2\backslash M_4$ a moiety (as it contains $M_3$ and its complement contains the complement of $M_2$).

So by Case 1 there exist $f_1, f_2 \in F_{\mathcal{U}}$ such that $(M_1)f_1 = M_5$ and $(M_5)f_2 = M_2$, it follows that $(M_1)f_1f_2 = M_2$ and we have the required result. □

**Theorem 4.1.6.** *Let $\mathcal{U}$ be an ultrafilter on an infinite set $\Omega$, then $F_{\mathcal{U}}$ is transitive on the moieties of $\Omega$ not in $\mathcal{U}$.*

*Proof.* Let $M_1$ and $M_2$ be moieties not in $\mathcal{U}$ then both $M_1^c$ and $M_2^c$ are in $\mathcal{U}$ by Theorem 1.4.5. By Theorem 4.1.5 there is an element of $F_{\mathcal{U}}$ such that $(M_1^c)f = M_2^c$. It therefore follows that $(M_1)f = M_2$ as required. □

**Theorem 4.1.7.** *Let $\mathcal{U}$ be an ultrafilter on an infinite set $\Omega$, then $F_{\mathcal{U}} \leq \mathrm{Sstab}(\mathcal{U})$.*

*Proof.* Let $f \in F_{\mathcal{U}}$ and let $U \in \mathcal{U}$. We have that $U \cap fix(f) \in \mathcal{U}$ and $(U)f = (U \cap fix(f)) \cup (U\backslash fix(f))f \supseteq U \cap fix(f)$ so we also have $(U)f \in \mathcal{U}$.
Similarly let $(U)f \in \mathcal{U}$. We have that $(U)f \cap fix(f) \in \mathcal{U}$ and $U = ((U)f)f^{-1} = ((U)f \cap fix(f)) \cup ((U)f\backslash fix(f))f^{-1} \supseteq (U)f \cap fix(f)$ so we also have $U \in \mathcal{U}$. □

**Theorem 4.1.8.** *Let $\mathcal{U}$ be an ultrafilter on an infinite set $\Omega$, then* $\mathrm{Sstab}(\mathcal{U}) = F_{\mathcal{U}}$

*Proof.* The following proof is based on the proof of Theorem 6.4 in [6].
As by Theorem 4.1.7 we have $F_{\mathcal{U}} \leq \mathrm{Sstab}(\mathcal{U})$, it suffices to show that $\mathrm{Sstab}(\mathcal{U}) \leq F_{\mathcal{U}}$. Let $f \in \mathrm{Sstab}(\mathcal{U})$. We will construct two moieties $M_1$ and $M_2$ which partition $\Omega$. Let $(c_i)_{i<|\Omega|}$ be a the disjoint cycles of $f$. Let each $c_i$ of finite order $k > 1$ be given by $(c_{i,0}, c_{i,1} \ldots c_{i,k-1})$ and each $c_i$ of infinite order be given by $(\ldots c_{i,-1}, c_{i,0}, c_{i,1} \ldots)$. Then we define $M_1$ and $M_2$ as follows:

Case 1: If $|fix(f)| < |\Omega|$ then $M_1 = \{c_{i,j} : i < |\Omega|, j \text{ even }\} \cup fix(f)$ and $M_2 = \{c_{i,j} : i < |\Omega|, j \text{ odd }\}$. If there are $|\Omega|$ non-trivial cycles then each contributes at least one element to each of $M_1$ and $M_2$ thus $\{M_1, M_2\}$ is a partition of $\Omega$ into moieties. If not then as $|fix(f)| < |\Omega|$ we have $|supp(f)| = |\Omega|$ so we must have $|\Omega| = \aleph_0$ and one of the cycles is infinite so $\{M_1, M_2\}$ is still a partition of $\Omega$ into moieties

Case 2: If $|fix(f)| = |\Omega|$ then let $\{F_1, F_2\}$ be a partition of $fix(f)$ into moieties. Let $M_1 = \{c_{i,j} : i < |\Omega|, j \text{ even }\} \cup F_1$ and $M_2 = \{c_{i,j} : i < |\Omega|, j \text{ odd }\} \cup F_2$. It is clear that $\{M_1, M_2\}$ is a partition of $\Omega$ into moieties.

Suppose for a contradiction that $f \notin F_{\mathcal{U}}$. It follows that $fix(f) \notin \mathcal{U}$ and thus by Theorem 1.4.5 $supp(f) \in \mathcal{U}$. It follows from the definitiond of $M_1$ and $M_2$ that $supp(f) \subseteq M_1 \cup (M_1)f$ and similarly $supp(f) \subseteq M_2 \cup (M_2)f$ so we have that both of these sets are in $\mathcal{U}$.
By Theorem 1.4.5 we have that $(M_1 \cup (M_1)f)^c = M_2 \cap (M_2)f \notin \mathcal{U}$ and $(M_2 \cup (M_2)f)^c = M_1 \cap (M_1)f \notin \mathcal{U}$ so by the definition of a filter we have either $M_1$ or $(M_1)f$ is not in $\mathcal{U}$ and either $M_2$ or $(M_2)f$ not in $\mathcal{U}$. As $f \in \mathrm{Sstab}(\mathcal{U})$ it follows that none of $M_1, (M_1)f, M_2, (M_2)f$ are in $\mathcal{U}$ but as $M_1^c = M_2$ this contradicts Theorem 1.4.5. □

**Theorem 4.1.9.** *Let $\mathcal{U}$ be an ultrafilter on a set $\Omega$ and let $g$ be a permutation of $\Omega$. Then $\mathcal{V} := (\mathcal{U})g = \{(U)g : U \in \mathcal{U}\}$ is an ultrafilter on $\Omega$.*

*Proof.* We first show that $\mathcal{V}$ is a filter.

1. As $\Omega \in \mathcal{U}$ we have that $\Omega = (\Omega)g \in \mathcal{V}$.

2. For $A, B \in \mathcal{V}$ we have $A = (A')g$ and $B = (B')g$ for some $A', B' \in \mathcal{U}$. As $\mathcal{U}$ is a filter it follows that $A' \cap B' \in \mathcal{U}$ and therefore $A \cap B = (A')g \cap (B')g = (A' \cap B')g \in \mathcal{V}$.

3. Let $A \in \mathcal{V}$ and let $B \supseteq A$. Then $A = (A')g$ for some $A' \in \mathcal{U}$ and $A' \subseteq Bg^{-1}$ . So $Bg^{-1} \in \mathcal{U}$ and therefore $B \in \mathcal{V}$.

We now show that $\mathcal{V}$ is an ultrafilter. As $\emptyset \notin \mathcal{U}$ it follows that $\emptyset \notin \mathcal{U}$.
It therefore suffices to show that there is no filter $\mathcal{V}'$ such that $\mathcal{V} \subset \mathcal{V}' \subset P(\Omega)$. Suppose for a contradiction that there is such a $\mathcal{V}'$. Let $\mathcal{U}' := \{(V)g^{-1} : V \in \mathcal{V}'\}$. By using $g^{-1}$ with the previous part of the proof we have that $\mathcal{U}'$ is a filter and we have that $\mathcal{U} \subset \mathcal{U}' \subset P(X)$. This contradicts the fact that $\mathcal{U}$ is an ultrafilter. □

**Theorem 4.1.10.** *Let $\mathcal{U}$ be an ultrafilter on an infinite set $\Omega$. The group* $\mathrm{Sstab}(\mathcal{U})$ *is a maximal subgroup of* $\mathrm{Sym}(\Omega)$.

*Proof.* The following proof is based on the proof of Theorem 6.4 in [6].
First we show $\mathrm{Sstab}(\mathcal{U}) \neq \mathrm{Sym}(\Omega)$. Let $f \in \mathrm{Sym}(\Omega)$ be such that $fix(f) = \emptyset$, by Theorem 4.1.8 it follows that $f \notin F_{\mathcal{U}} = \mathrm{Sstab}(\mathcal{U})$. Let $g \in \mathrm{Sym}(\Omega) \setminus \mathrm{Sstab}(\mathcal{U})$. Suppose for a contradiction that for all $M \in \mathcal{U}$ which are moieties of $\Omega$ we have $(M)g \in \mathcal{U}$. We have by Theorem 4.1.9 that $(\mathcal{U})g$ is an ultrafilter who's moieties are all contained in $\mathcal{U}$. In addition by Theorem 1.4.5 one of $M$ and $M^c$ is in $(\mathcal{U})g$ and $M^c \notin \mathcal{U}$ so $M \in (\mathcal{U})g$. It follows that $\mathcal{U}$ and $(\mathcal{U})g$ have that same moieties and therefore by Theorem 1.4.6 we have that $(\mathcal{U})g = \mathcal{U}$ a contradiction as $g \notin \mathrm{Sstab}(\mathcal{U})$. So we have that there is a moiety $M_1 \in \mathcal{U}$ such that $(M_1)g \notin \mathcal{U}$. Let $M_2$ be a moiety of $\Omega$, by Theorem 1.4.5 precisely one of $M_2$ and $M_2^c$ is in $\mathcal{U}$. Without loss of generality suppose that $M_2^c \in \mathcal{U}$. We have that $F_{\mathcal{U}}$ acts fully on $M_2$ as for all $h \in \mathrm{Sym}_{\Omega}(M_2)$ we have $fix(h) \supseteq M_2^c \in \mathcal{U}$ and thus $fix(h) \in \mathcal{U}$. By Theorem 4.1.8 $F_{\mathcal{U}} = \mathrm{Sstab}(\mathcal{U})$ so $\mathrm{Sstab}(\mathcal{U})$ acts fully on $M_2$.
By Theorem 4.1.6 for all $M \notin \mathcal{U}$ which are moieties of $\Omega$ there exists $h \in F_{\mathcal{U}} = \mathrm{Sstab}(\mathcal{U})$ such that $(M)h = M_2$ it follows that $\langle \mathrm{Sstab}(\mathcal{U}), g\rangle_G$ acts fully on all moieties of $\Omega$ not in $\mathcal{U}$.
By Theorems 4.1.5 and 4.1.6 for all $M \in \mathcal{U}$ which are moieties of $\Omega$ there exists $h_1, h_2 \in F_{\mathcal{U}} = \mathrm{Sstab}(\mathcal{U})$ such that $(M)h_1 = M_1$ and $((M_1)g)h_2 = M_2$. We now have that $(M)h_1gh_2 = M_2$ and therefore $\langle \mathrm{Sstab}(\mathcal{U}), g\rangle_G$ acts fully on all moieties of $\Omega$ in $\mathcal{U}$.
We now have that $\langle \mathrm{Sstab}(\mathcal{U}), g\rangle_G$ acts fully on all moieties of $\Omega$ and therefore by Theorem 4.1.1 $\langle \mathrm{Sstab}(\mathcal{U}), g\rangle_G = \mathrm{Sym}(\Omega)$. □

**Theorem 4.1.11.** *Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be distinct ultrafilters on an infinite set $\Omega$. Then* $\mathrm{Sstab}(\mathcal{U}_1) \neq \mathrm{Sstab}(\mathcal{U}_2)$.

*Proof.* Let $U \in \mathcal{U}_1 \setminus \mathcal{U}_2$ be a moiety (this must exist by Theorem 1.4.6). By Theorem 4.1.8 it suffices to prove $F_{\mathcal{U}_1} \neq F_{\mathcal{U}_2}$. Choose $f \in \mathrm{Sym}(\Omega)$ such that $fix(f) = U$ then $f \in F_{\mathcal{U}_1} \setminus F_{\mathcal{U}_2}$ and therefore $F_{\mathcal{U}_1} \neq F_{\mathcal{U}_2}$ as required. □

**Theorem 4.1.12.** *For all infinite sets $\Omega$, there exists a family of $2^{2^{|\Omega|}}$ pairwise non-conjugate maximal subgroups of* $\mathrm{Sym}(\Omega)$.

*Proof.* The following proof is based on the proof of corollary 6.5 in [6].
By Theorem 1.4.11 there are $2^{2^{|\Omega|}}$ ultrafilters on $\Omega$. It follows from Theorems 4.1.11 and 4.1.10 that there exists a family of $2^{2^{|\Omega|}}$ maximal subgroups of $\mathrm{Sym}(\Omega)$. As there are only $2^{|\Omega|}$ elements of $\mathrm{Sym}(\Omega)$ it follows that this family can be partitioned into $2^{2^{|\Omega|}}$ conjugacy classes. By choosing one element of each we have the required result. □

## 4.2 Finite Partition Stabilisers

In this section we will explore more examples of maximal subgroups of infinite symmetric groups in the form of partition stabilisers.

**Theorem 4.2.1.** *Let $\Omega$ be an infinite set, and let $F \subseteq \Omega$ be finite. We then have that* $\mathrm{Sstab}(\{F, F^c\})$ *is a maximal subgroup of* $\mathrm{Sym}(\Omega)$.

*Proof.* Let $f \in \mathrm{Sym}(\Omega) \setminus \mathrm{Sstab}(F)$ and $g \in \mathrm{Sym}(\Omega)$. As $\mathrm{Sstab}(\{F, F^c\}) = \mathrm{Sstab}(F)$ it suffices to show that $g \in \langle \mathrm{Sstab}(F), f\rangle_G$. As $f \notin \mathrm{Sstab}(F)$ there must exist a point $p \in F$ such that $(p)f \in F^c$. Let $p_2 \in (F^c)f \cap F^c$. We have that $((p)f, p_2) \in \mathrm{Sstab}(F)$. It therefore follows that $(p, (p_2)f^{-1}) = f((p)f, p_2)f^{-1} \in \langle \mathrm{Sstab}(F), f\rangle_G$. For all $a \in F$ and $b \in F^c$ we have that $(a, b) = (a, p)(p, (p_2)f^{-1})((p_2)f^{-1}, b) \in \langle \mathrm{Sstab}(F), f\rangle_G$. Let $F$ be indexed by $F = \{p_i : i < k\}$. We now have that $g(p_1, (p_1)g)(p_2, (p_2)g) \ldots (p_k, (p_k)g) \in \mathrm{Sstab}(F)$. It therefore follows that $g \in \langle \mathrm{Sstab}(F), f\rangle_G$ as required. □

**Theorem 4.2.2.** *Let $\Omega$ be an infinite set and let $\Sigma_1$ and $\Sigma_2$ be infinite subsets of $\Omega$ such that $|\Sigma_1 \cap \Sigma_2| = |\Sigma_1 \cup \Sigma_2|$. We have that* $\mathrm{Sym}_{\Omega}(\Sigma_1 \cup \Sigma_2) = \langle \mathrm{Sym}_{\Omega}(\Sigma_1), \mathrm{Sym}_{\Omega}(\Sigma_2)\rangle_G$.

*Proof.* The following proof is based on the proof of the first lemma of [10].
Let $f \in \mathrm{Sym}_{\Omega}(\Sigma_1 \cup \Sigma_2)$. We have that either $|(\Sigma_1 \cap \Sigma_2)f \cap \Sigma_1| = |\Sigma_1 \cap \Sigma_2|$ or $|(\Sigma_1 \cap \Sigma_2)f \cap \Sigma_2| = |\Sigma_1 \cap \Sigma_2|$. Without loss of generality we assume that $|(\Sigma_1 \cap \Sigma_2)f \cap \Sigma_1| = |\Sigma_1 \cap \Sigma_2|$. Let $M$ be a moiety of $(\Sigma_1 \cap \Sigma_2)f \cap \Sigma_1$. It follows that $M$ is a moiety of $\Sigma_1$ and $(M)f^{-1}$ is a moiety of $\Sigma_1 \cap \Sigma_2$, $\Sigma_1$ and $\Sigma_2$. Let $f' \in \mathrm{Sym}_{\Omega}(\Sigma_1)$ be such that $(p)ff' = p$ for all $p \in (M)f^{-1}$. As

$|\Sigma_1 \cap \Sigma_2| = |\Sigma_1|$ we have that $\Sigma_1 \backslash \Sigma_2$ is contained in a moiety of $\Sigma_1$ and thus there exists an element $g \in \mathrm{Sym}_\Omega(\Sigma_1)$ such that $\Sigma_1 \backslash \Sigma_2 \subseteq M f^{-1} g$. As $ff'$ fixes $Mf^{-1}$ it follows that $g^{-1} ff' g \in \mathrm{Sym}_\Omega(\Sigma_1 \cup \Sigma_2)$ fixes $\Sigma_1 \backslash \Sigma_2$ and thus $g^{-1} ff' g \in \mathrm{Sym}_\Omega(\Sigma_2)$. So we have that $f \in g \,\mathrm{Sym}_\Omega(\Sigma_2) g^{-1} f'^{-1} \subseteq \langle \mathrm{Sym}_\Omega(\Sigma_1), \mathrm{Sym}_\Omega(\Sigma_2)\rangle_G$ as required. □

**Definition 4.2.3** Let $P$ be a partition of an infinite set $\Omega$ into finitely many sets and let $\kappa$ be an infinite cardinal. The $\kappa$-almost stabiliser of $P$ is defined by:

$$\mathrm{Sstab}(P)_{<\kappa} := \{f \in \mathrm{Sym}(\Omega) : \text{there exists } f' \in \mathrm{Sym}(P) \text{ such that for all } p \in P \text{ we have } |(p)f \triangle (p)f'| < \kappa \text{ and } |p| = |(p)f'|\}$$

**Theorem 4.2.4.** *Let $P$ be a partition of an infinite set $\Omega$ into finitely many sets and let $\kappa$ be an infinite cardinal. Then the $\kappa$-almost stabiliser of $P$ is a group.*

*Proof.* Let $f, g \in \mathrm{Sstab}(P)_{<\Omega}$ and let $p \in P$. As $(p)fg \triangle (p)f'g' \subseteq (((p)f \triangle (p)f')g \cup ((p)f'g \triangle pf'g'))$ we have that $|(p)fg \triangle (p)f'g'| \leq |((p)f \triangle (p)f')g| + |((p)f')g \triangle (p)f'g')| < \kappa + \kappa = \kappa$ and thus $fg \in \mathrm{Sstab}(P)_{<\kappa}$.
It now suffices to show that $f^{-1} \in \mathrm{Sstab}(P)_{<\kappa}$. Let $p \in P$. We have that $|(p)f'^{-1}| = |p|$ and $|(p)f^{-1} \triangle (p)f'^{-1}| = |(p)f^{-1} f \triangle (p)f'^{-1} f| = |(p) \triangle (p)f'^{-1} f| = |((p)f'^{-1})f' \triangle ((p)f'^{-1})f| < \kappa$. □

**Theorem 4.2.5.** *Let $\Omega$ be an infinite set, and let $P = \{M_0, M_1 \ldots M_k\}$ for $k > 1$ be a partition of $\Omega$ into finitely many moieties. We then have that $\mathrm{Sstab}(P)$ is not a maximal subgroup of $\mathrm{Sym}(\Omega)$. In particular $\mathrm{Sstab}(P) < \mathrm{Sstab}(P)_{<|\Omega|}$ which is maximal.*

*Proof.* The following proof is based on the proof of observation 6.2 in [6].
Let $x_0 \in M_0$ and $x_1 \in M_1$ we have that $(x_0, x_1) \in \mathrm{Sstab}(P)_{<|\Omega|} \backslash \mathrm{Sstab}(P)$ and thus $\mathrm{Sstab}(P) < \mathrm{Sstab}(P)_{<|\Omega|}$.
We now show that $\mathrm{Sstab}(P)_{<|\Omega|}$ is maximal. We first show that $\mathrm{Sstab}(P)_{<|\Omega|} \neq \mathrm{Sym}(\Omega)$. Let $\{M_{0,1}, M_{0,2}\}$ be a partition of $M_0$ into moieties of $M_0$ and $\{M_{1,1}, M_{1,2}\}$ be a partition of $M_1$ into moieties of $M_1$. Then we have that $|M_{0,1}| = |M_{1,1}|$ and thus there is a bijection $\phi : M_{0,1} \to M_{1,1}$. Let $g \in \mathrm{Sym}(\Omega)$ be defined by:

$$(x)g = \left\{ \begin{array}{ll} (x)\phi & x \in M_{0,1} \\ (x)\phi^{-1} & x \in M_{0,2} \\ x & \text{otherwise} \end{array} \right\}$$

We have that $g \notin \mathrm{Sstab}(P)_{<|\Omega|}$ as $|(M_0)g \triangle M_i| = |\Omega|$ for all $i \leq k$.
Let $f \in \mathrm{Sym}(\Omega) \backslash \mathrm{Sstab}(P)_{<|\Omega|}$. It suffices to show that $\langle f, \mathrm{Sstab}(P)_{<|\Omega|}\rangle_G$.

<u>Claim:</u> There exist $i, j_1, j_2$ such that $j_1 \neq j_2$, $|(M_i)f \cap M_{j_1}| = |\Omega|$ and $|(M_i)f \cap M_{j_2}| = |\Omega|$.
Proof of claim: Suppose for a contradiction that our claim is false. We partition each $M_i$ as $\{M_{i,1}, M_{i,2} \ldots M_{i,k}\}$ where $M_{i,j} := \{x \in M_i : (x)f \in M_j\}$. As each $M_i$ is a moiety we have that for every $i$ at least one of $M_{i,1}, M_{i,2} \ldots M_{i,k}$ is a moiety, but we also have at most one of these is a moiety as if not that would contradict our assumption that the claim is false. In addition for each $j$ there must be an $i$ such that $M_{i,j}$ is a moiety and $|M_j \backslash (M_{i,j})f| < |\Omega|$ as if this were not the case then $f$ would not be onto $M_j$. Let $f' \in \mathrm{Sym}(P)$ be defined by $(M_i)f' = M_j$ for $i, j$ such that $M_{i,j}$ is a moiety it follows that $f \in \mathrm{Sstab}(P)_{<|\Omega|}$ a contradiction. □
We now have

$$\mathrm{Sym}_\Omega((M_i)f \cap (M_{j_1} \cup M_{j_2})) \leq f^{-1} \mathrm{Sym}_\Omega(M_i) f \leq \langle f, \mathrm{Sstab}(P)_{<|\Omega|}\rangle_G$$

$$\mathrm{Sym}_\Omega(M_{j_1}) \leq \langle f, \mathrm{Sstab}(P)_{<|\Omega|}\rangle_G \qquad \mathrm{Sym}_\Omega(M_{j_1}) \leq \langle f, \mathrm{Sstab}(P)_{<|\Omega|}\rangle_G$$

It follows by Theorem 4.2.2 that $\mathrm{Sym}_\Omega(M_{j_1} \cup ((M_i)f \cap (M_{j_1} \cup M_{j_2}))) \leq \langle f, \mathrm{Sstab}(P)_{<|\Omega|}\rangle_G$. By using Theorem 4.2.2 again we then have that $\mathrm{Sym}_\Omega((M_{j_2} \cup M_{j_1} \cup ((M_i)f \cap (M_{j_1} \cup M_{j_2})))) = \mathrm{Sym}_\Omega(M_{j_1} \cup M_{j_2}) \leq \langle f, \mathrm{Sstab}(P)_{<|\Omega|}\rangle_G$.
As the elements of $P$ all have the same cardinality we can permute them using elements of $\mathrm{Sstab}(P)$. If follows that for all $i, j \leq k$ we have $\mathrm{Sym}_\Omega(M_i \cup M_j) \leq \langle f, \mathrm{Sstab}(P)_{<|\Omega|}\rangle_G$. By repeatedly applying Theorem 4.2.2 we will get the required result as follows:

$$\begin{aligned}
&\mathrm{Sym}_\Omega(M_0 \cup M_1) \leq \langle f, \mathrm{Sstab}(P)_{<|\Omega|}\rangle_G \text{ and } \mathrm{Sym}_\Omega(M_1 \cup M_2) \leq \langle f, \mathrm{Sstab}(P)_{<|\Omega|}\rangle_G \\
&\implies \mathrm{Sym}_\Omega(M_0 \cup M_1 \cup M_2) \leq \langle f, \mathrm{Sstab}(P)_{<|\Omega|}\rangle_G \text{ and } \mathrm{Sym}_\Omega(M_2 \cup M_3) \leq \langle f, \mathrm{Sstab}(P)_{<|\Omega|}\rangle_G \\
&\implies \mathrm{Sym}_\Omega(M_0 \cup M_1 \cup M_2 \cup M_3) \leq \langle f, \mathrm{Sstab}(P)_{<|\Omega|}\rangle_G \text{ and } \mathrm{Sym}_\Omega(M_3 \cup M_4) \leq \langle f, \mathrm{Sstab}(P)_{<|\Omega|}\rangle_G \\
&\ \ \vdots \\
&\implies \mathrm{Sym}_\Omega(M_0 \cup M_1 \ldots \cup M_k) \leq \langle f, \mathrm{Sstab}(P)_{<|\Omega|}\rangle_G \\
&\implies \mathrm{Sym}_\Omega(\cup P) \leq \langle f, \mathrm{Sstab}(P)_{<|\Omega|}\rangle_G \implies \mathrm{Sym}(\Omega) \leq \langle f, \mathrm{Sstab}(P)_{<|\Omega|}\rangle_G
\end{aligned}$$

□

**Theorem 4.2.6.** *Let $\Omega$ be an uncountable set, and let $P = \{N, N^c\}$ be a partition of $\Omega$ where $\aleph_0 \leq |N| < |\Omega|$. We then have that $\mathrm{Sstab}(P)$ is not a maximal subgroup of $\mathrm{Sym}(\Omega)$. In particular $\mathrm{Sstab}(P) < \mathrm{Sstab}(P)_{<|N|}$ which is maximal.*

*Proof.* Let $x_1 \in N$ and $x_2 \in N^c$ we have that $(x_1, x_2) \in \text{Sstab}(P)_{<|N|} \backslash Sstab(P)$ and thus $\text{Sstab}(P) < \text{Sstab}(P)_{<|N|}$.
We now show that $\text{Sstab}(P)_{<|N|}$ is maximal. We first show that $\text{Sstab}(P)_{<|N|} \neq \text{Sym}(\Omega)$. Let $N' \subseteq N^c$ be such that $|N| = |N'|$. Then we have that there is a bijection $\phi : N \to N'$. Let $h$ be defined by:

$$(x)h = \left\{ \begin{array}{ll} (x)\phi & x \in N \\ (x)\phi^{-1} & x \in N' \\ x & \text{otherwise} \end{array} \right\}$$

We have that $h \notin \text{Sstab}(P)_{<|N|}$ as $|(N)h \triangle N| = |N|$ and $|(N)h \triangle N^c| = |\Omega|$. Let $f \in \text{Sym}(\Omega) \backslash \text{Sstab}(P)_{<|N|}$, it suffices to show that $\langle f, \text{Sstab}(P)_{<|N|} = \text{Sym}(\Omega)$.
<u>Claim:</u> Either $|(N)f \cap N^c| = |N|$ or $|(N)f^{-1} \cap N^c| = |N|$.
Proof of claim: As $f \notin \text{Sstab}(P)_{<|N|}$ we have that either $|N \triangle (N)f| = |N|$ or $|N^c \triangle (N^c)f| = |N|$. We therefore have one of the following:

1. If $|N \backslash (N)f| = |N|$ then $|N| = |(N)f^{-1} \backslash N| = |(N)f^{-1} \cap N^c|$.
2. If $|(N)f \backslash N| = |N|$ then $|(N)f \cap N^c| = |N|$.
3. If $|N^c \backslash (N^c)f| = |N|$ then $|N| = |(N^c)f^{-1} \backslash N^c| = |(N^c)f^{-1} \cap N| = |N^c \cap (N)f|$.
4. If $|(N^c)f \backslash N^c| = |N|$ then $|N| = |(N^c)f \cap N| = |N^c \cap (N)f^{-1}|$. $\square$

As $\langle f, \text{Sstab}(P)_{<|N|} \rangle_G = \langle f^{-1}, \text{Sstab}(P)_{<|N|} \rangle_G$ we may assume without loss of generality that $|Nf \cap N^c| = |N|$. Let $\{N_0, N_1\}$ be a partition of $Nf \cap N^c$ into moieties. Let $N_2 \subset N^c$ be such that $N_2 \cap Nf \cap N^c = \emptyset$ and $|N_2| = |N|$. Let $\phi : N_1 \to N_2$ be a bijection and let $h \in \text{Sstab}(P)$ be defined by:

$$(x)h = \left\{ \begin{array}{ll} (x)\phi & x \in N_1 \\ (x)\phi^{-1} & x \in N_2 \\ x & \text{otherwise} \end{array} \right\}$$

It follows that $(N_1)h = N_2$ and thus $((N_1)f^{-1})fhf^{-1} = (N_2)f^{-1} \subset N^c$. As $(N_1)f^{-1}$ is a moiety of $N$ we have shown that there is an involution in $\langle f, \text{Sstab}(P)_{<|N|} \rangle_G$ which swaps a moiety of $N$ with a subset of $N^c$ and fixes all other points. It follows that, by conjugating this involution by elements of $\text{Sstab}(P)$, we can construct an involution which swaps any moiety of $N$ with any subset of $N^c$ with cardinality $|N|$ and fixes all other points. By partitioning $N$ and a subset of $N^c$ into moieties we can therefore construct an involution in $\langle f, \text{Sstab}(P)_{<|N|} \rangle_G$ which swaps $N$ with any subset of $N^c$ with cardinality $|N|$ and fixes all other points. Let $M$ be a moiety of $\Omega$ and let $N_M \subseteq M^c \backslash N$ be such that $|N_M| = |N|$. Let $g \in \langle f, \text{Sstab}(P)_{<|N|} \rangle_G$ be an involution swapping $N_M$ with $N$ and fixing all other points. We have that $g \, \text{Sstab}(P) g^{-1}$ acts fully on $M$ and thus as $M$ was arbitrary we have $\langle f, \text{Sstab}(P)_{<|N|} \rangle_G$ acts fully on all moieties of $\Omega$. We therefore have $\langle f, \text{Sstab}(P)_{<|N|} \rangle_G = \text{Sym}(\Omega)$ by Theorem 4.1.1. $\square$

**Theorem 4.2.7.** *Let $\Omega$ be an infinite set and let $P = \{\Omega_0, \Omega_1 \dots \Omega_k\}$ be a partition of $\Omega$ into finitely many sets. Then* $\text{Sstab}(P)$ *is a maximal subgroup of* $\text{Sym}(\Omega)$ *if and only if $P = \{F, F^c\}$ where $F$ is finite or $P = \{S_1, S_2 \dots S_{k-1}, \left(\bigcup_{i<k} S_i\right)^c\}$ where $S_1, S_2 \dots S_{k-1}$ are singletons.*

*Proof.* We will consider all the ways of partitioning $\Omega$ into finitely many sets.
Case 1: $P = \{S_0, S_1 \dots S_{k-1}, (\cup_{i<k} S_i)^c\}$ where $S_0, S_1 \dots S_{k-1}$ are singletons.
It follows that $\text{Sstab}(P) = \text{Sstab}(\{\cup_{i<k} S_i, (\cup_{i<k} S_i)^c\})$ and therefore by Theorem 4.2.1 that $\text{Sstab}(P)$ is maximal.
Case 2: $P = \{S_0, S_1 \dots S_{k_1}, \Omega_0, \Omega_1 \dots \Omega_{k_2}\}$ where $S_0, S_1 \dots S_{k_1}$ are singletons, $\Omega_0, \Omega_1 \dots \Omega_{k_2}$ are not singletons, $k_1, k_2 \in \mathbb{N}$ and $k_2 \geq 2$.
Let $x_1 \in \Omega_1$ and $x_2 \in \Omega_2$. We have that $(x_1, x_2) \in \text{Sstab}(\{\cup_{i \leq k_1} S_i, \cup_{i \leq k_2} \Omega_i\}) \backslash \text{Sstab}(P)$ and therefore we have $\text{Sstab}(P) < \text{Sstab}(\{\cup_{i \leq k_1} S_i, \cup_{i \leq k_2} \Omega_i\})$ which by Theorem 4.2.1 is maximal and therefore $\text{Sstab}(P)$ is not maximal.
Case 3: $P = \{F_0, F_1 \dots F_{k_1}, \Omega_0, \Omega_1 \dots \Omega_{k_2}\}$ where $F_1, F_2 \dots F_{k_1}$ are finite sets at least one of which has at least 2 elements, $\Omega_0, \Omega_1 \dots \Omega_{k_2}$ are infinite $k_1, k_2 \in \mathbb{N}$ and at least one of $k_1, k_2$ is not 1.
Similarly to case 2 if $k_2 \neq 1$ then $\text{Sstab}(P) < \text{Sstab}(\{\cup_{i \leq k_1} F_i, \cup_{i \leq k_2} \Omega_i\}$ which by Theorem 4.2.1 is maximal and therefore $\text{Sstab}(P)$ is not maximal. If $k_2 = 1$ then we have without loss of generality that $F_0$ has at least 2 elements and $k_1 > 1$. Therefore if $x_1 \in F_0$ and $x_2 \in F_1$ then we have that $(x_1, x_2) \in \text{Sstab}(\{\cup_{i \leq k_1} F_i, \cup_{i \leq k_2} \Omega_i\}) \backslash \text{Sstab}(P)$ and so $\text{Sstab}(P) < \text{Sstab}(\{\cup_{i \leq k_1} F_i, \cup_{i \leq k_2} \Omega_i\})$ which by Theorem 4.2.1 is maximal and therefore $\text{Sstab}(P)$ is not maximal.
Case 4: $P = \{F, F^c\}$ where $F$ is finite.
It follows immediately by Theorem 4.2.1 that $\text{Sstab}(P)$ is maximal.
Case 5: $P = \{N, \Omega_0, \Omega_1 \dots \Omega_k\}$ where $k \in \mathbb{N}$ and $\aleph_0 \leq |N| < |\Omega|$.
We have that $\text{Sstab}(P) \leq \text{Sstab}(\{\cup\{s \in P : |s| = |N|\}, \{\cup\{s \in P : |s| \neq |N|\}\}\})$ which is not maximal by Theorem 4.2.6 and therefore $\text{Sstab}(P)$ is not maximal.
Case 6: $P = \{M_1, M_2 \dots M_k\}$ a partition into moieties of $\Omega$.
It follows immediately from Theorem 4.2.5 that $\text{Sstab}(P)$ is not maximal. $\square$

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