4 Laurent Series and the Residue Theorem

4.1 Complex Series

Consider the complex power series

$$S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

We can express convergence of a complex power series in much the same way as we do with real power series. Thus if the limit

$$\lim_{N\to\infty} S_N(z) = \lim_{N\to\infty} \sum_{n=0}^{N} a_n (z - z_0)^n$$

exists for $|z - z_0| < R$, then the series is said to be convergent. Here R is the **radius of convergence**. In the context of complex series, R is the radius of a circle (with centre z_0) in the complex plane.

We can use the **ratio test** in the usual way. Let L be defined by

$$L = \lim_{n\to\infty} \frac{|a_{n+1}(z - z_0)^{n+1}|}{|a_n (z - z_0)^n|} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} |z - z_0|.$$

Then the series coverges and diverges when $L < 1$ and $L > 1$, respectively. If $L = 1$ or L fails to exist, then the ratio test is inconclusive. Clearly, convergence is guaranteed for all z such that

$$|z - z_0| < \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|},$$

where the limit (which is the radius of convergence) is assumed to exist.

Example 4.1.1 $f(z) = (1 - z)^{-1} = 1 + z + z^2 + \dots$ provided $|z| < 1$. Note that $f(z)$ is not analytic at $z = 1$.

Example 4.1.2 $f(z) = e^z$ is analytic everywhere. The series

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

has an infinite radius of convergence.

Example 4.1.3 Show that the series

$$1 - z^2 + z^4 - z^6 + \dots$$

converges to $(1 + z^2)^{-1}$ for $|z| < 1$.

Solution Let

$$S_{2N} = 1 - z^2 + z^4 - z^6 + \dots + (-1)^N z^{2N}$$

so that

$$z^2 S_{2N} = z^2 - z^4 + z^6 - \cdots + (-1)^N z^{2N+2}$$

and adding the two together gives

$$S_{2N} + z^2 S_{2N} = 1 + (-1)^N z^{2N+2}$$

$$S_{2N} = \frac{1}{1+z^2} + \frac{(-1)^N z^{2N+2}}{1+z^2},$$

provided $z^2 \neq -1$. Therefore

$$\left| S_{2N} - \frac{1}{1+z^2} \right| = \frac{|z|^{2N+2}}{|1+z^2|}.$$

If $|z| < 1$ then $|z|^{2N+2} \to 0$ as $N \to \infty$ so that

$$\lim_{N\to\infty} \left| S_{2N} - \frac{1}{1+z^2} \right| = 0.$$

Thus $S_{2N} \to (1+z^2)^{-1}$. Again note that the function $f(z) = (1+z^2)^{-1}$ is analytic everywhere except at two isolated points ($z = \pm i$) and the radius of convergence for the Taylor series about the origin is the distance to these points (i.e. $|\pm i| = 1$). •

We can justify the Taylor series of an holomorphic function from Cauchy's Integral formula in the following way. Let C be a circle centred at z_0 and $f(z)$ be holomorphic in and on C. Recall Cauchy's Integral formula (for $n = 0$)

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(s)}{s-z}\, ds,$$

where z lies within C. Put $z = z_0 + w$, we have

$$f(z_0 + w) = \frac{1}{2\pi i} \oint_C \frac{f(s)}{s - (z_0 + w)}\, ds = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s - z_0)\left(1 - \frac{w}{s - z_0}\right)}\, ds.$$

Now since

$$\left| \frac{w}{s - z_0} \right| = \left| \frac{z - z_0}{s - z_0} \right| < 1,$$

we can write

$$\frac{1}{1 - \frac{w}{s-z_0}} = \sum_{n=0}^{\infty} \left(\frac{w}{s - z_0} \right)^n.$$

Substituting this series into the above integral yields

$$\begin{aligned} f(z_0 + w) &= \frac{1}{2\pi i} \oint_C \frac{f(s)}{s - z_0} \sum_{n=0}^{\infty} \left(\frac{w}{s - z_0} \right)^n ds \\ &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} w^n \oint_C \frac{f(s)}{(s - z_0)^{n+1}}\, ds = \sum_{n=0}^{\infty} \frac{w^n}{n!} f^{(n)}(z_0), \end{aligned}$$

where we have used Cauchy's integral formula in its general form. Hence

$$f(z) = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!} f^{(n)}(z_0).$$

Note that the coefficient of $(z-z_0)^n$, say a_n, is just

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz.$$

Now you can prove that the series converges and that therefore f is analytic.

by Cauchy's inequality we have

$$|f(z)| \leq \sum_{n=0}^{\infty} \frac{|z-z_0|^n}{n!} |f^{(n)}(z_0)| \leq \sum_{n=0}^{\infty} \frac{|z-z_0|^n}{n!} \frac{Mn!}{r^n} = \sum_{n=0}^{\infty} M \frac{|z-z_0|^n}{r^n}.$$

By the ratio test, the right-hand power series has a radius of convergence

$$|z-z_0| < r,$$

which can be up to the radius of the largest circle centred at z_0 and contained within D. Thus, the Taylor series about z_0 converges within a circle of radius equal to the distance to the nearest non-analytic (singular) point.

We have finally proved that the idea of a function f being differentiable once in an open domain D (i.e. holomorphic in D) and being analytic in D (i.e. having a Taylor expansion) are equivalent in complex analysis. Recall that this is not true in real analysis!

4.2 Laurent Series

Let us return to example 4.1.1, where the series

$$f(z) = \frac{1}{1-z} = 1 + z + z^2 + z^3 + \dots, \tag{1}$$

is valid for $|z| < 1$ only. However, since $f(z)$ is analytic for $|z| > 1$, we wish to express $f(z)$ in terms of series valid for $|z| > 1$. Toward this end, we rewrite $f(z)$ as follows.

$$f(z) = \frac{1}{1-z} = -\frac{1}{z} \frac{1}{1-\frac{1}{z}} = -\frac{1}{z} \left(1 + \frac{1}{z} + \frac{1}{z^2} + \dots\right),$$

provided $|1/z| < 1$. Hence

$$\frac{1}{1-z} = -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \dots, \tag{2}$$

which converges provided $|z| > 1$. Therefore we have created two series: one valid within the unit circle and the other valid outside the unit circle. Both can be thought of as series expansions about the point $z = 0$. In principle we can expand a function about any point in terms of power serise (see example 4.2.1 below). The series containing inverse powers of z derived above is an example of a **Laurent series**. The other containing powers of z can be considered as a special case of a Laurent series: the coefficients of the inverse powers are all zero.

Example 4.2.1 Express $(1-z)^{-1}$ as a Laurent series about the point $z=-1$.

Solution First let us determine the Taylor series about $z=-1$. Set $w=z+1$, We have

$$\begin{aligned}\frac{1}{1-z} &= \frac{1}{1-(w-1)} = \frac{1}{2-w} = \frac{1}{2\left(1-\frac{w}{2}\right)} \\ &= \frac{1}{2}\left(1+\left(\frac{w}{2}\right)+\left(\frac{w}{2}\right)^2+\ldots\right) \\ &= \frac{1}{2}+\frac{1+z}{4}+\frac{(1+z)^2}{8}+\ldots,\end{aligned}$$

which is valid for all z satisfying

$$\left|\frac{w}{2}\right| < 1 \implies |1+z| < 2,$$

i.e. for all z within the disk centred at $z=-1$ with radius 2. Outside that disk we can obtain another valid series by writing

$$\frac{1}{1-z} = \frac{1}{2-w} = \frac{1}{-w\left(1-\frac{2}{w}\right)} = \frac{-1}{w}\left(1+\frac{2}{w}+\ldots\right),$$

for $|2/w| < 1$. Hence

$$\frac{1}{1-z} = -\frac{1}{1+z} - \frac{2}{(1+z)^2} - \frac{4}{(1+z)^3}\ldots,$$

for $|1+z| > 2$. These are the Laurent series for the function about the point $z=-1$, although we usually reserve the description "Laurent" for the second series and merely refer to the first as the Taylor series. •

A Taylor series is valid within a circle of convergence whereas a Laurent series is, in general, valid within an annulus. In example 4.2.1 the annulus is infinite [valid for $|z+1|$ in $(2,\infty)$]. The following example illustrates convergence within and outside an annular region.

Example 4.2.2 Determine the Laurent series about the origin for the complex function

$$f(z) = \frac{1}{(1-z)(3-z)}.$$

Solution The easiest approach is to express $f(z)$ in partial fractions

$$\begin{aligned}\frac{1}{(1-z)(3-z)} &= \frac{1/2}{1-z} - \frac{1/2}{3-z} = \frac{1}{2}\frac{1}{1-z} - \frac{1}{6}\frac{1}{1-z/3} \\ &= \frac{1}{2}\left(1+z+z^2+\ldots\right) - \frac{1}{6}\left(1+\frac{z}{3}+\frac{z^2}{9}+\ldots\right) \\ &= \frac{1}{3}+\frac{4z}{9}+\frac{13z^2}{27}+\ldots, \qquad (3)\end{aligned}$$

which is valid for the smaller of $|z| < 1$ and $|z/3| < 1$. Hence we have the Laurent series about $z = 0$ (more commonly known as the Maclaurin series) valid for $|z| < 1$.

Now express $f(z)$ as

$$f(z) = \frac{-1/2}{z(1-1/z)} + \frac{-1/6}{1-z/3}$$

and expand

$$f(z) = -\frac{1}{2z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \dots\right) - \frac{1}{6}\left(1 + \frac{z}{3} + \frac{z^2}{9} + \dots\right),$$

which is valid for $|1/z| < 1$ and $|z/3| < 1$, i.e. for z in the annulus $1 < |z| < 3$. So we have a second Laurent series in an annulus:

$$f(z) = \dots - \frac{1}{2z^2} - \frac{1}{2z} - \frac{1}{6} - \frac{z}{18} - \dots,$$

which contains an infinite number of both positive and negative powers of z.

Finally, for $|z| > 3$ we can write

$$\begin{aligned} f(z) &= \frac{-1/2}{z(1-1/z)} + \frac{1/2}{z(1-3/z)} \\ &= \frac{-1}{2z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \dots\right) + \frac{1}{2z}\left(1 + \frac{3}{z} + \frac{9}{z^2} + \dots\right) \\ &= \frac{1}{z^2} + \frac{4}{z^3} + \dots, \end{aligned}$$

which contains only negative powers. •

4.3 Laurent's Theorem

The results illustrated by the above example are encapsulated in the following theorem.

If $f(z)$ is analytic in an annulus

$$r_1 < |z - z_0| < r_2,$$

then

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_0)^k,$$

where

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{k+1}}\, dz,$$

for $k = 0, \pm 1, \pm 2, \dots$ and C is the circle $|z - z_0| = \rho,\ r_1 < \rho < r_2$.

This series is called the Laurent series for $f(z)$ in the annulus $r_1 < |z - z_0| < r_2$. The sum of terms for which $k \geq 0$ is called the **analytic part** and the sum of terms with negative powers is called the **principal part**. A Taylor series has no principal part. The result is largely of theoretical interest as we do not use it to construct Laurent series — instead we adopt the approach in the preceding subsection. For a formal proof of Laurent's Theorem, see Jeffrey *Complex Analysis & Applics* page 500.

4.4 Laurent expansions about isolated singularities

Laurent's series when expanded about an isolated singularity plays a key role in the theory of integration. Consider the Laurent expansion of $f(z)$ about a point $z = z_0$

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_0)^k,$$

for z satisfying $0 < |z - z_0| < r$, i.e. for z in the disk centred at z_0 with radius r (excluding the centre and the disk boundary). If z_0 is a pole of order n then by definition, the limit

$$\lim_{z \to z_0} (z - z_0)^n f(z)$$

is finite and non-zero. Hence, the principal part of the Laurent series must contain only n terms, i.e.

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-(n-1)}}{(z - z_0)^{n-1}} + \ldots \frac{a_{-1}}{(z - z_0)} + \sum_{k=0}^{+\infty} a_k (z - z_0)^k$$

where $a_{-n} \neq 0$.

Example 4.4.1 Consider the function

$$f(z) = \frac{1}{1 + z^2}.$$

Determine the Laurent series for $f(z)$ about the point $z_0 = -i$, which is valid for $0 < |z + i| < 2$.

Solution It is convenient to shift the origin by setting $w = z - z_0 = z + i$ so that

$$\begin{aligned} \frac{1}{1+z^2} &= \frac{1}{(z+i)(z-i)} = \frac{1}{w(w-2i)} = \frac{1}{-2iw(1 - w/2i)} \\ &= \frac{1}{-2iw}\left(1 + \frac{w}{2i} + \frac{w^2}{-4} + \ldots\right) = \frac{i}{2w} + \frac{1}{4} - \frac{iw}{8} + \ldots, \end{aligned}$$

which is valid for $w \neq 0$ and $|w/2i| < 1$, that is $0 < |w| < 2$. Hence

$$\frac{1}{1+z^2} = \frac{i}{2(z+i)} + \frac{1}{4} - \frac{i(z+i)}{8} + \ldots,$$

which valid for $0 < |z + i| < 2$. The principal part contains only one term. •

A function with a simple pole will have a Laurent series (about that pole) with just one term in the principal part. The preceding example 4.4.1: $(1+z^2)^{-1}$ is a testament to this fact. Indeed, the Laurent expansion about the simple pole $z=-i$ contains just one term with coefficient $a_{-1}=i/2$. Note that $z=+i$ is also a simple pole.

If the principal part of the Laurent expansion of $f(z)$ about a singularity $z=z_0$ has an infinite number of terms then $z=z_0$ is an essential singularity (the limit of $(z=z_0)^n f(z)$ as $z\to z_0$ does not exist for any integer n).

Finally, it is important to note that the Laurent expansion for a function about a removable singularity will have no principal part.

4.5 The Residue Theorem

Recall that in section 3.4 we have shown that

$$\oint_C \frac{dz}{(z-z_0)^n} = \begin{cases} 2\pi i & n=1 \\ 0 & n\geq 2, \end{cases}$$

where C is a closed contour encircling the point $z=z_0$. Now suppose we expand $f(z)$ in a Laurent series about an isolated singularity at $z=z_0$ and consider the integral of $f(z)$ on a simple, closed contour C around the point z_0 so that z_0 is the only singularity within C. From above we see that

$$\oint_C f(z)\,dz = 2\pi i a_{-1}.$$

Thus the coefficient a_{-1} plays an important role in complex integration theory and provides a powerful method for evaluating integrals (both complex and real). This coefficient is called the **residue** of $f(z)$ at the pole $z=z_0$.

The Residue Theorem. Let $f(z)$ be analytic in a domain D except for isolated singularites at $z_1, z_2, \ldots, z_m$. Let C be a simple, closed contour in D enclosing all m singular points. Then

$$\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{m} a_{-1}^{(j)},$$

where $a_{-1}^{(j)}$ is the residue of $f(z)$ at $z=z_j$, $j=1,2,\ldots,m$.

To see why this is true refer to section 3.4 concerning contour deformation. We can deform the contour C into m small circles with one circle around each singular point. This is basically the method of example 3.4.4 but expressed in a different way.

Thus the **value of the integral of a complex function** around a simple, closed contour is simply the **sum of the residues within that contour** times $(2\pi i)$. All we need to do is determine the coefficient of the first negative power in the Laurent series about each singular point.

It is worth mentioning that Cauchy's integral formula can be derived from the residue theorem. Indeed, let $f(z)$ be analytic in a domain containing a simple closed contour C. Consider a point z_0 inside C. The function $g(z) = f(z)/(z - z_0)^n$ has a pole of order n at z_0 (assuming $f(z_0) \neq 0$). By applying the residue theorem to $g(z) = f(z)/(z - z_0)^n$ we obtain

$$\oint_C \frac{f(z)}{(z - z_0)^n}\, dz = \oint_C g(z)\, dz = 2\pi i Res|_{z_0}.$$

Now the residue $Res|_{z_0}$ is given by (see later)

$$Res|_{z_0} = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \{(z - z_0)^n g(z)\} = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} f(z) = \frac{f^{(n-1)}(z_0)}{(n-1)!}.$$

Substituting this into the preceding equation yields

$$\oint_C \frac{f(z)}{(z - z_0)^n}\, dz = \frac{2\pi i}{(n-1)!}\, f^{(n-1)}(z_0),$$

which is just Cauchy's integral formula.

When $f(z_0) = 0$, the above argument does not apply because z_0 becomes a pole of order lower than n. Nevertheless, one can still derive Cauchy's integral formula from the residue formula. This problem is a good exercise.

4.6 Calculating Residues

As the determination of residues via Laurent expansion is usually a tedious excercise, we seek alternative methods. In what follows we derive three formulae for residue calculation.

For a simple pole at $z = z_0$, we know that

$$(z - z_0) f(z) = a_{-1} + a_0(z - z_0) + a_1(z - z_0)^2 + \cdots .$$

So

$$\boxed{a_{-1} = \lim_{z \to z_0} (z - z_0) f(z).}$$

Note that without a prior knowledge of the order of the pole $z = z_0$, one can proceed to calculate the limit $\lim_{z \to z_0}(z - z_0)f(z)$. If this limit exists, the pole is simple and its residue is obtained. If the limit does not exist, the pole is not simple and we need to determine its order before using the formula below.

For a pole of order n at $z = z_0$, we have

$$(z - z_0)^n f(z) = a_{-n} + a_{-n+1}(z - z_0) + \ldots + a_{-1}(z - z_0)^{n-1} + a_0(z - z_0)^n + \cdots .$$

So

$$\frac{d^{n-1}}{dz^{n-1}}\left[(z - z_0)^n f(z)\right] = (n-1)! a_{-1} + n! a_0 (z - z_0) + \cdots$$

and hence

$$\boxed{a_{-1} = \frac{1}{(n-1)!} \lim_{z \to z_0} \left\{ \frac{d^{n-1}}{dz^{n-1}}\left[(z - z_0)^n f(z)\right] \right\}.}$$

Note that this may be as difficult to evaluate as the actual Laurent expansion.

Example 4.6.1 Determine the residue at the pole $z = i$ for the function

$$f(z) = \frac{1}{(1 + z^2)^3}.$$

Solution We will do this in two ways, first by the above formula and then by expanding $f(z)$ in terms of Laurent series and read off the coefficient a_{-1} from the series.

Rewrite

$$f(z) = \frac{1}{(1 + z^2)^3} = \frac{1}{(z - i)^3 (z + i)^3}.$$

So $z = i$ is a triple pole (pole of order 3) and the formula gives

$$a_{-1} = \frac{1}{2} \lim_{z \to i} \frac{d^2}{dz^2} \frac{1}{(z + i)^3} = \frac{1}{2} \lim_{z \to i} \frac{12}{(z + i)^5} = \frac{3}{16i} = -\frac{3i}{16}.$$

Alternatively, we have

$$\begin{aligned}
f(z) &= \frac{1}{(1 + z^2)^3} = \frac{1}{(z - i)^3 (z + i)^3} \\
&= \frac{1}{(z - i)^3 [(z - i) + 2i]^3} = \frac{1}{[2i(z - i)]^3 [1 + (z - i)/2i]^3} \\
&= \frac{-1}{8i(z - i)^3} \left(1 - 3\,\frac{z - i}{2i} + \frac{(-3)(-4)}{2} \frac{(z - i)^2}{-4} + \cdots \right),
\end{aligned}$$

where we have expressed $1/[1 + (z - i)/2i]^3$ in terms of power series. It is easy to read off a_{-1} from the above expression. Thus we have

$$a_{-1} = \frac{1}{-8i} \frac{12}{-8} = -\frac{3i}{16}.$$

•

There is no simple formula for evaluating the residue at an essential singularity (since there are an infinite number of terms in the principal part). For such a singularity, we may have to determine the residue from the Laurent expansion of the function. For a removable singularity the residue is always zero since there is no principal part in the Laurent expansion.

Finally, a variation of the earlier result is that if

$$f(z) = \frac{g(z)}{h(z)},$$

where

$$h(z_0) = 0\ ,\ h'(z_0) \neq 0 \text{ and } g(z_0) \neq 0,$$

then $z = z_0$ is a **simple pole with residue**

$$\boxed{a_{-1} = \frac{g(z_0)}{h'(z_0)}.}$$

For a proof of this formula, observe that the limit

$$\lim_{z\to z_0} (z - z_0) f(z) = \lim_{z\to z_0} (z - z_0)\frac{g(z)}{h(z)} = g(z_0) \lim_{z\to z_0} \frac{(z - z_0)}{h(z) - h(z_0)} = \frac{g(z_0)}{h'(z_0)}$$

exists and is nonzero. Hence $z = z_0$ is a simple pole with residue $g(z_0)/h'(z_0)$.

4.7 Examples of the residue theorem

First let us re-evaluate example (3.4.4) using the residue theorem.

Example 4.7.1 Determine the value of

$$\oint_C \frac{2z}{z^2 - 1}\, dz,$$

where C is the circle $|z| = 2$.

Solution Here $f(z)$ has two simple poles at $z = \pm 1$ and both lie inside the contour C. Either the first or third formula can be used for this example. Here we use the third with $g(z) = 2z$ and $h(z) = z^2 - 1$.

Residue at $z = 1$ is given by

$$\frac{g(1)}{h'(1)} = \frac{2}{2} = 1.$$

Likewise

Residue at $z = -1$ is given by

$$\frac{g(-1)}{h'(-1)} = \frac{-2}{-2} = 1.$$

Therefore the value of the integral is

$$2\pi i(1 + 1) = 4\pi i$$

as before. •

Example 4.7.2 Evaluate

$$\oint_C \frac{\sin z\, dz}{z^2(z^2 + 4)},$$

where C is the circle $|z| = 3$.

Solution The integrand

$$f(z) = \frac{\sin z}{z^2(z^2 + 4)}$$

has 3 simple poles at $z = \pm 2i$ and $z = 0$. To see that $z = 0$ is a simple pole note that

$$\sin z = z - \frac{z^3}{3!} + \dots,$$

so that $(\sin z)/z \to 1$ as $z \to 0$. Therefore $z = 0$ is indeed a simple pole. All three simple poles are inside C. We calculate their residues using the first formula.

Residue at $2i$

$$\lim_{z \to 2i} \frac{(z - 2i)\sin z}{z^2(z - 2i)(z + 2i)} = \frac{\sin(2i)}{(2i)^2(2i + 2i)} = \frac{i \sinh 2}{-16i} = -\frac{\sinh 2}{16}.$$

Residue at $-2i$

$$\lim_{z \to -2i} \frac{(z + 2i)\sin z}{z^2(z - 2i)(z + 2i)} = \frac{\sin(-2i)}{(-2i)^2(-2i - 2i)} = \frac{-i \sinh 2}{16i} = -\frac{\sinh 2}{16}.$$

Residue at 0

$$\lim_{z \to 0} \frac{z \sin z}{z^2(z - 2i)(z + 2i)} = \left(\lim_{z \to 0} \frac{\sin z}{z}\right) \frac{1}{(-2i)(2i)} = \frac{1}{4}.$$

Therefore

$$\oint_C \frac{\sin z dz}{z^2(z^2 + 4)} = 2\pi i \left(\frac{1}{4} - 2\,\frac{\sinh 2}{16}\right) = \pi i \left(\frac{1}{2} - \frac{\sinh 2}{4}\right).$$

•

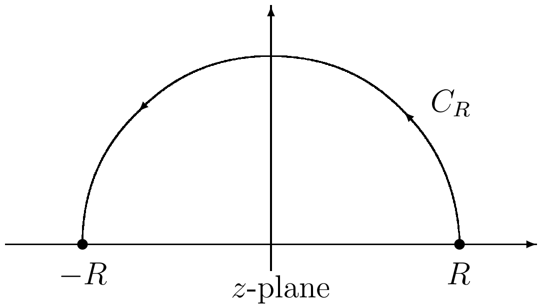

Figure 24: Semi-circular arc C_R

4.8 Evaluation of real integrals using the residue theorem

Earlier in example 3.6.3, you were given a glimpse of how contour integration can be used to evaluate certain real integrals. We will complete this section with a series of examples of how the study of certain complex integrals reveals the value of real definite integrals, which are otherwise difficult to evaluate. The residue theorem plays a key role here. As a general application of this theorem we have the following formula.

Let $f(z)$ be an analytic function on the upper half plane, except at a finite number of poles, none of which lie on the real line. Suppose that

$$\lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0,$$

where C_R is a semi-circular arc of radius R in the upper half plane (see figure 24). Then by the residue theorem we have

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \text{(Residues of } f(z) \text{ in the upper half plane)}\,.$$

The above formula applies to a variety of families of $f(z)$. Here we consider a couple of popular cases.

(I) First, suppose that there exists a number $M > 0$ such that for all $|z|$ sufficiently large, the inequality

$$|f(z)| \le \frac{M}{|z|^{1+\epsilon}},$$

where $\epsilon > 0$, holds. Then

$$\lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0.$$

For a proof of this result, observe that

$$\begin{aligned}\left|\int_{C_R} f(z)\,dz\right| &= \left|i\int_0^{\pi} f(Re^{i\theta})Re^{i\theta}\,d\theta\right| \le \int_0^{\pi} |f(Re^{i\theta})|R\,d\theta \\ &\le \frac{MR}{R^{1+\epsilon}}\int_0^{\pi} d\theta = \frac{\pi M}{R^{\epsilon}}. \qquad (4)\end{aligned}$$

This tends to zero as $R \to \infty$, hence the assertion is proven.

Example 4.8.1 Consider $f(z) = (3z-1)/(z^4+1)$. We have

$$|f(z)| = \left|\frac{3z-1}{z^4+1}\right| \leq \frac{3|z|+1}{|z|^4-1},$$

since $|a-b| \leq |a|+|b|$ and $|a+b| \geq |a|-|b|$. Thus

$$|f(z)| \leq \frac{3|z|}{|z|^4}\left(\frac{1+1/3|z|}{1-1/|z|^4}\right).$$

It follows that

$$\left|\int_{C_R} f(z)\,dz\right| \leq \int_0^\pi \frac{3R}{R^4}\left(\frac{1+1/3R}{1-1/R^4}\right) R\left|ie^{i\theta}\right|\,d\theta = \frac{3\pi}{R^2}\left(\frac{1+1/3R}{1-1/R^4}\right)$$

and as $R \to \infty$ the right hand side tends to zero.

(II) Second, suppose that $f(z)$ has the form

$$f(z) = e^{iz}g(z),$$

where

$$|g(z)| \leq \frac{M}{|z|^k}\,, \quad k > 0,$$

for all $|z|$ sufficiently large, then

$$\lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0.$$

This is known as **Jordan's lemma**, which requires a less restrictive condition on $f(z)$ for large $|z|$ than the preceding case.

Proof of Jordan's lemma. On C_R we have $z = R(\cos\theta + i\sin\theta)$. So

$$|f(z)| = |e^{iz}||g(z)| \leq \left|e^{iR(\cos\theta+i\sin\theta)}\right|\frac{M}{R^k} = e^{-R\sin\theta}\,\frac{M}{R^k}.$$

It follows that

$$\left|\int_{C_R} f(z)\,dz\right| \leq \frac{M}{R^k}\int_0^\pi e^{-R\sin\theta}R\,d\theta = \frac{M}{R^{k-1}}\int_0^{\pi/2} e^{-R\sin\theta}\,d\theta + \frac{M}{R^{k-1}}\int_{\pi/2}^{\pi} e^{-R\sin\theta}\,d\theta.$$

Set $\phi = \pi - \theta$ in the second integral to see that

$$\int_{\pi/2}^{\pi} e^{-R\sin\theta}\,d\theta = -\int_{\pi/2}^{0} e^{-R\sin\phi}\,d\phi = \int_0^{\pi/2} e^{-R\sin\theta}\,d\theta.$$

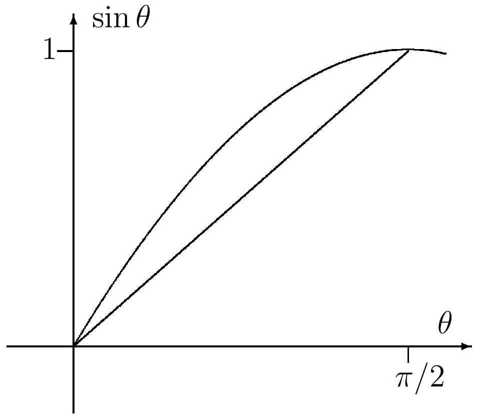

Figure 25: An illustration of the inequality $\sin\theta \geq 2\theta/\pi$.

Hence
$$\left|\int_{C_R} f(z)\,dz\right| \leq \frac{2M}{R^{k-1}}\int_0^{\pi/2} e^{-R\sin\theta}\,d\theta.$$
Now for $0 \leq \theta \leq \pi/2$, we have (as illustrated by figure 25)
$$\sin\theta \geq \frac{2\theta}{\pi}.$$

Substituting this result into the above yields
$$\left|\int_{C_R} f(z)\,dz\right| \leq \frac{2M}{R^{k-1}}\int_0^{\pi/2} e^{-2R\theta/\pi}\,d\theta = \frac{2M}{R^{k-1}}\left[\frac{\pi e^{-2R\theta/\pi}}{-2R}\right]_0^{\pi/2} < \frac{M\pi}{R^k}.$$
This clearly tends to zero as $R \to \infty$, and the lemma is proven. Note that Jordan's lemma still holds if the condition $|g(z)| \leq \frac{M}{R^k},\ k > 0$ is replaced by $|g(z)| \to 0$ as $|z| \to \infty$.

We now apply the above results to some examples, where $f(z)$ satisfies either (I) or (II).

Example 4.8.2 Evaluate the real integral
$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1}.$$

Solution Consider
$$\oint_C \frac{dz}{z^4+1},$$
where C is the closed contour in figure 24. The integrand has simple poles at
$$z_j = e^{i(\pi+2\pi j)/4} \quad j = 0, 1, 2, 3,$$

of which z_0 and z_1 lie inside C when R is large enough. Hence

$$\oint_C \frac{dz}{z^4+1} = \int_{-R}^{R} \frac{dx}{x^4+1} + \int_{C_R} \frac{dz}{z^4+1} = 2\pi i \left(Res|_{z_0} + Res|_{z_1} \right).$$

The integrand $1/(z^4+1)$ clearly satisfies (I). Hence, in the limit $R \to \infty$ the integral along C_R vanishes, thereby rendering

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = -2\pi i \left(Res|_{z_0} + Res|_{z_1} \right).$$

All that remains is to evaluate the two residues. For $z_0 = e^{i\pi/4}$ we have

$$Res|_{z_0} = \frac{1}{4\left(e^{i\pi/4}\right)^3} = \frac{1}{4e^{3i\pi/4}} = \frac{e^{-3i\pi/4}}{4} = \frac{-1-i}{4\sqrt{2}}.$$

On the other hand, for $z_1 = e^{3i\pi/4}$ we have

$$Res|_{z_1} = \frac{1}{4\left(e^{3i\pi/4}\right)^3} = \frac{1}{4e^{9i\pi/4}} = \frac{e^{-9i\pi/4}}{4} = \frac{1-i}{4\sqrt{2}}.$$

The sum of the residues is

$$Res|_{z_0} + Res|_{z_1} = -\frac{i}{2\sqrt{2}}$$

and hence

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = 2\pi i \, \frac{-i}{2\sqrt{2}} = \frac{\pi}{\sqrt{2}}.$$

•

Example 4.8.3 Evaluate the real integral

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{x^4+1}.$$

Solution Consider

$$\oint_C \frac{z^2\,dz}{z^4+1},$$

where C is the closed contour in figure 24. The integrand has simple poles at

$$z_j = e^{i(\pi+2\pi j)/4} \quad j = 0, 1, 2, 3,$$

of which z_0 and z_1 lie inside C when R is large enough. Hence

$$\oint_C \frac{z^2\,dz}{z^4+1} = \int_{-R}^{R} \frac{x^2\,dx}{x^4+1} + \int_{C_R} \frac{z^2\,dz}{z^4+1} = 2\pi i \left(Res|_{z_0} + Res|_{z_1} \right).$$

The integrand $z^2/(z^4+1)$ clearly satisfies (I). Hence, in the limit $R \to \infty$ the integral along C_R vanishes, thereby rendering

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{x^4+1} = -2\pi i\left(Res|_{z_0} + Res|_{z_1}\right).$$

All that remains is to evaluate the two residues. For $z_0 = e^{i\pi/4}$ we have

$$Res|_{z_0} = \frac{e^{i\pi/2}}{4\left(e^{i\pi/4}\right)^3} = \frac{e^{-i\pi/4}}{4} = \frac{1-i}{4\sqrt{2}}.$$

On the other hand, for $z_1 = e^{3i\pi/4}$ we have

$$Res|_{z_1} = \frac{e^{3i\pi/2}}{4\left(e^{3i\pi/4}\right)^3} = \frac{e^{-3i\pi/4}}{4} = \frac{-1-i}{4\sqrt{2}}.$$

The sum of the residues is

$$Res|_{z_0} + Res|_{z_1} = -\frac{i}{2\sqrt{2}}$$

and hence

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{x^4+1} = 2\pi i\,\frac{-i}{2\sqrt{2}} = \frac{\pi}{\sqrt{2}}.$$

•

Example 4.8.4 Evaluate

$$\int_{-\infty}^{\infty} \frac{dz}{z^2+i}$$

and use the result to recover the values of the integrals

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1} \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{x^2\,dx}{x^4+1}.$$

Solution Since the integrand $1/(z^2+i)$ satisfies (I), we have

$$\int_{-\infty}^{\infty} \frac{dz}{z^2+i} = \oint_C \frac{dz}{z^2+i},$$

where C is the closed contour in the preceding examples (in the limit $R \to \infty$). The integrand $1/(z^2+i)$ has 2 simple poles $z_1 = e^{-i\pi/4}$ and $z_2 = e^{i3\pi/4}$, the latter of which lies inside C. So by the residue theorem we have

$$\begin{aligned}
\oint_C \frac{dz}{z^2+i} &= 2\pi i Res|_{z_2} = \frac{2\pi i}{2z_2} = \frac{\pi i}{e^{i3\pi/4}} \\
&= \frac{\pi i}{\cos 3\pi/4 + i\sin 3\pi/4} = \frac{\pi i}{-1/\sqrt{2} + i/\sqrt{2}} \\
&= \frac{\pi(1-i)}{\sqrt{2}}.
\end{aligned}$$

Now observe that
$$\int_{-\infty}^{\infty} \frac{dz}{z^2+i} = \int_{-\infty}^{\infty} \frac{z^2-i}{z^4+1}\,dz.$$
So
$$\int_{-\infty}^{\infty} \frac{z^2-i}{z^4+1}\,dz = \frac{\pi(1-i)}{\sqrt{2}}.$$
Equating the real and imaginary parts yields
$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{x^4+1} = \frac{\pi}{\sqrt{2}}$$
and
$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}.$$

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Example 4.8.5 Evaluate
$$\int_{0}^{\infty} \frac{\cos x}{(1+x^2)(4+x^2)}\,dx.$$

Solution Observe that the function is an even function so that
$$\int_{0}^{\infty} \frac{\cos x}{(1+x^2)(4+x^2)}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos x}{(1+x^2)(4+x^2)}\,dx.$$
Consider
$$\oint_C \frac{e^{iz}}{(1+z^2)(4+z^2)}\,dz,$$
where C is as before (see figure 24). The function
$$f(z) = \frac{e^{iz}}{(1+z^2)(4+z^2)}$$
satisfies Jordan's lemma and has simple poles at $z = \pm i$ and $z = \pm 2i$. The only poles inside C are $z = i$ and $z = 2i$.

The residue at $z = i$ is
$$Res|_{z=i} = \frac{e^{ii}}{2i(4+i^2)} = \frac{e^{-1}}{6i} = \frac{-i}{6e}.$$
The residue at $z = 2i$ is
$$Res|_{z=2i} = \frac{e^{i2i}}{(1+(2i)^2)2(2i)} = \frac{e^{-2}}{-12i} = \frac{i}{12e^2}.$$
Therefore
$$\oint_C \frac{e^{iz}}{(1+z^2)(4+z^2)}\,dz = 2\pi i\left(\frac{-i}{6e} + \frac{i}{12e^2}\right) = \frac{\pi(2e-1)}{6e^2}.$$

By Jordan's lemma, the integral around $C_R \to 0$ as $R \to \infty$. So in this limit, we obtain

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(1+x^2)(4+x^2)}\,dx = \frac{\pi(2e-1)}{6e^2}.$$

Equating the real parts yields

$$\int_{-\infty}^{\infty} \frac{\cos x}{(1+x^2)(4+x^2)}\,dx = \frac{\pi(2e-1)}{6e^2}.$$

Note that the imaginary part is zero as the integrand is an odd function. The required result is therefore

$$\int_{0}^{\infty} \frac{\cos x}{(1+x^2)(4+x^2)}\,dx = \frac{\pi(2e-1)}{12e^2}.$$

Note that a direct consideration of the contour integral

$$\oint_C \frac{\cos z}{(1+z^2)(4+z^2)}\,dz$$

would fail to render the above result. The reason is that the value of the integral

$$\int_{C_R} \frac{\cos z}{(1+z^2)(4+z^2)}\,dz$$

is not known even if it coverges the limit $R \to \infty$. •

4.9 Further illustrative examples

Things can become complicated as the following two examples illustrate. However, with a little ingenuity problems can often be overcome.

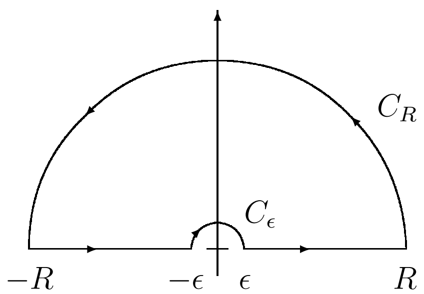

Figure 26: Indented contour

Example 4.9.1 Evaluate

$$\int_0^{\infty} \frac{\sin x}{x}\,dx.$$

Solution Consider the function $f(z) = e^{iz}/z$ and the usual semi-circular contour of figure 24. Here $f(z)$ satisfies Jordan's lemma and has a simple pole at $z = 0$. But this point is neither inside nor outside C: it is **on** C for all values of R. We must therefore choose a new contour, which either includes or excludes the origin, yet allowing the earlier method to be adopted. Consider the indented contour C illustrated in figure 26. Then

$$\oint_C f\,dz = \int_{C_R} f dz + \int_{-R}^{-\epsilon} f\,dz + \int_{C_\epsilon} f\,dz + \int_{\epsilon}^{R} f\,dz = 0.$$

So

$$\int_{-R}^{-\epsilon} f\,dz + \int_{\epsilon}^{R} f\,dz = -\int_{C_R} f dz - \int_{C_\epsilon} f\,dz.$$

We consider the limits $R \to \infty$ and $\epsilon \to 0$. By Jordan's lemma the integral around C_R tends to zero as $R \to \infty$. What happens to the integral around C_ϵ? On C_ϵ we have $z = \epsilon e^{i\theta}$, so

$$\int_{C_\epsilon} \frac{e^{iz}}{z}\,dz = \int_{\pi}^{0} \frac{e^{i\epsilon(\cos\theta + i\sin\theta)}\epsilon i e^{i\theta}\,d\theta}{\epsilon e^{i\theta}} = \int_{\pi}^{0} e^{i\epsilon(\cos\theta + i\sin\theta)} i\,d\theta.$$

In the limit $\epsilon \to 0$, we have

$$\lim_{\epsilon\to 0} \int_{C_\epsilon} \frac{e^{iz}}{z}\,dz = \int_{\pi}^{0} i\,d\theta = -i\pi.$$

Hence

$$\begin{aligned}
\lim_{\epsilon\to 0}\left(\int_{-\infty}^{-\epsilon} \frac{e^{ix}}{x}\,dx + \int_{\epsilon}^{\infty} \frac{e^{ix}}{x}\,dx\right) &= \lim_{\epsilon\to 0}\left(\int_{-\infty}^{-\epsilon} \frac{\cos x + i\sin x}{x}\,dx + \int_{\epsilon}^{\infty} \frac{\cos x + i\sin x}{x}\,dx\right) \\
&= \lim_{\epsilon\to 0}\left(\int_{\epsilon}^{\infty} \frac{-\cos x + i\sin x}{x}\,dx + \int_{\epsilon}^{\infty} \frac{\cos x + i\sin x}{x}\,dx\right) \\
&= i\pi.
\end{aligned}$$

Obviously, the real parts cancel, while the imaginary parts give

$$\int_0^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

•

Example 4.9.2 Evaluate

$$\int_0^{\infty} \frac{x^{a-1}}{1+x}\,dx, \quad 0 < a < 1.$$

Solution There are different ways of tackling this problem. The present approach illustrates the significance of branch points (and cuts). Consider

$$\oint_\Gamma \frac{z^{a-1}}{1-z}\,dz,$$

where Γ is the closed contour in figure 27. The integrand $f(z) = z^{a-1}/(1-z)$ has a simple pole at $z = 1$. Since $0 < a < 1$, there is also a branch point at $z = 0$. The choice of the "key hole" shaped contour Γ is to avoid the pole $z = 1$ and the negative real axis (a branch cut) altogether. We now proceed to evaluate the contour integral and consider the limits $R \to \infty$ and $\epsilon \to 0$. The contour Γ consists of four pieces:

$$\Gamma \equiv C_R \cup AB \cup C_\epsilon \cup CD.$$

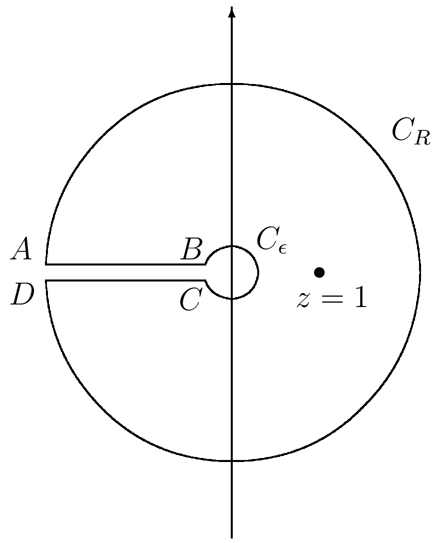

Figure 27: a keyhole contour

Inside Γ, $z = 1$ is the only pole with residue

$$Res|_{z=1} = \lim_{z\to 1}(z-1)\,\frac{z^{a-1}}{1-z} = -1.$$

For the integral along AB, where $z = re^{i\pi}$, we have

$$\int_{AB} \frac{z^{a-1}}{1-z}\,dz = \int_R^\epsilon \frac{(re^{i\pi})^{a-1}}{1-re^{i\pi}} e^{i\pi}\,dr = \int_R^\epsilon \frac{r^{a-1}}{1+r}\,e^{ia\pi}\,dr = -e^{ia\pi}\int_\epsilon^R \frac{r^{a-1}}{1+r}\,dr.$$

Similarly, for the integral along CD, where $z = re^{-i\pi}$, we have

$$\int_{CD} \frac{z^{a-1}}{1-z}\,dz = \int_\epsilon^R \frac{(re^{-i\pi})^{a-1}}{1-re^{-i\pi}}\,e^{-i\pi}\,dr = \int_\epsilon^R \frac{r^{a-1}}{1+r}\,e^{-ia\pi}\,dr = e^{-ia\pi}\int_\epsilon^R \frac{r^{a-1}}{1+r}\,dr.$$

It can be shown that the integrals along C_R and along C_ϵ tend to zero as $R \to \infty$ and $\epsilon \to 0$, respectively. Indeed, for the former we have

$$\left|\int_{C_R} \frac{z^{a-1}}{1-z}\,dz\right| = \left|\int_{-\pi}^{\pi} \frac{R^{a-1}e^{(a-1)i\theta}}{1-Re^{i\theta}}\,Rie^{i\theta}\,d\theta\right| \le \int_{-\pi}^{\pi} \frac{R^a}{R-1}\,d\theta = \frac{2\pi R^a}{R-1},$$

which clearly tends to zero in the limit $R \to \infty$. For the latter we have

$$\left| \int_{C_\epsilon} \frac{z^{a-1}}{1-z}\, dz \right| = \left| \int_{\pi}^{-\pi} \frac{\epsilon^{a-1} e^{(a-1)i\theta}}{1-\epsilon e^{i\theta}}\, \epsilon i e^{i\theta}\, d\theta \right| \le \int_{-\pi}^{\pi} \frac{\epsilon^a}{1-\epsilon}\, d\theta = \frac{2\pi\epsilon^a}{1-\epsilon},$$

which clearly tends to zero in the limit $\epsilon \to 0$. So in the limits $R \to \infty$ and $\epsilon \to 0$ we have

$$-e^{ia\pi} \int_0^\infty \frac{r^{a-1}}{1+r}\, dr + e^{-ia\pi} \int_0^\infty \frac{r^{a-1}}{1+r}\, dr = -2\pi i,$$

which immediately implies

$$\int_0^\infty \frac{r^{a-1}}{1+r}\, dr = \frac{\pi}{\sin a\pi}.$$

•

Example 4.9.3 Given $a > 1$, evaluate

$$\int_0^{2\pi} \frac{d\theta}{a + \sin\theta}.$$

Solution From the identity

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i},$$

we can write

$$\int_0^{2\pi} \frac{d\theta}{a+\sin\theta} = 2i \int_0^{2\pi} \frac{d\theta}{2ai + e^{i\theta} - e^{-i\theta}}. \tag{5}$$

Put $z = e^{i\theta}$, then $dz = ie^{i\theta}\, d\theta$ or $d\theta = -iz^{-1}\, dz$ and

$$\int_0^{2\pi} \frac{d\theta}{a+\sin\theta} = 2 \oint_C \frac{z^{-1}\, dz}{2ai + z - z^{-1}} = 2 \oint_C \frac{dz}{z^2 + 2aiz - 1},$$

where C is the unit circle. Now $z^2 + 2aiz - 1 = 0$ has roots

$$z_\pm = -ai \pm i(a^2-1)^{1/2},$$

with $z_+ = -ai + i(a^2-1)^{1/2}$ lying in the unit circle. So z_+ is the only pole of the integrand in the above contour integral. Its residue is given by

$$Res|_{z_+} = \lim_{z \to z_+} \frac{z - z_+}{z^2 + 2aiz - 1} = \frac{1}{z_+ - z_-} = \frac{1}{2i(a^2-1)^{1/2}}.$$

So

$$\oint_C \frac{dz}{z^2 + 2aiz - 1} = 2\pi i Res|_{z_+} = \frac{\pi}{(a^2-1)^{1/2}}$$

and

$$\int_0^{2\pi} \frac{d\theta}{a+\sin\theta} = \frac{2\pi}{(a^2-1)^{1/2}}.$$

•

The above method can be used to evaluate integrals of the form

$$I = \int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta,$$

where $R(x, y)$ is a rational function, which is continuous on the unit circle $x^2 + y^2 = 1$. As above, we put $z = e^{i\theta}$. Then

$$\cos\theta = \frac{z + z^{-1}}{2}, \quad \sin\theta = \frac{z - z^{-1}}{2i} \quad \text{and} \quad d\theta = -iz^{-1}\, dz.$$

It follows that

$$I = -i \oint_C R\left(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}\right) z^{-1}\, dz,$$

where C is the unit circle. By the residue theorem we have

$$I = 2\pi S,$$

where S is the sum of the residues of

$$R\left(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}\right) z^{-1}$$

at all poles within the unit circle.

4.10 Fourier transform pair

An integral of the form

$$F(k) = \int_{-\infty}^{\infty} f(s)e^{-iks}\, ds$$

for real k, is called the Fourier transform of $f(s)$. In general, $F(k)$ exists if $f(s)$ is absolutely integrable. Furthermore, $F(k)$ is square integrable if $f(s)$ is square integrable. Given the transform $F(k)$, $f(s)$ can be retrieved from the inversion formula

$$f(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k)e^{iks}\, dk.$$

The above equations are called Fourier transform pair. An important application of the residue theorem is to the handling of the inversion formula.

Let $F(z)$ be an analytic function, except at a finite number of poles not lying on the real axis. Suppose that there exist a constant $K > 0$ such that

$$|F(z)| \le \frac{K}{|z|},$$

for all sufficiently large z. Let $s > 0$, then

$$\int_{-\infty}^{\infty} F(k)e^{iks}\, dk = 2\pi i \sum \left(\text{Residues of } F(k)e^{iks} \text{ in the upper half plane}\right).$$

Remark. For $s < 0$, the lower half plan is used instead. More precisely

$$\int_{-\infty}^{\infty} F(k)e^{iks}\,dk = -2\pi i \sum \big(\textit{Residues of } F(k)e^{iks} \textit{ in the lower half plane}\big)\,.$$

For a proof of this formula, consider a rectangular contour

$$C = C_1 \cup C_2 \cup C_3 \cup C_4,$$

where $C_1 = \{x : -R \le x \le R\}$, $C_2 = \{R + iy : 0 \le y \le R\}$, $C_3 = \{x + iR : -R \le x \le R\}$ and $C_4 = \{-R + iy : 0 \le y \le R\}$ (see figure 28).

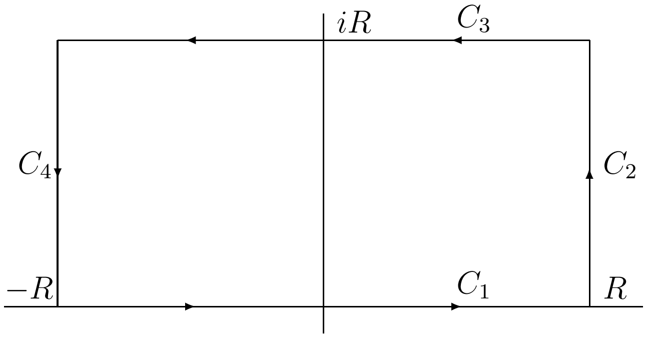

Figure 28: A rectangular contour used in the inversion of the Fourier transform

By the residue theorem we have

$$\begin{aligned}
\oint_C F(z)e^{izs}\,dz &= \int_{C_1} F(z)e^{izs}\,dz + \int_{C_2} F(z)e^{izs}\,dz + \int_{C_3} F(z)e^{izs}\,dz + \int_{C_4} F(z)e^{izs}\,dz \\
&= 2\pi i \sum \left(\text{Residues of } F(z)e^{izs} \text{ in the upper half plane}\right),
\end{aligned}$$

provided that R is large enough for C to contain all the poles of $F(z)e^{izs}$. The requirement is to prove that the 3 integrals along the 3 sides other than the bottom side of the rectangular contour C tend to zero as $R \to \infty$. Indeed, we have

$$\begin{aligned}
\left|\int_{C_2} F(z)e^{izs}\,dz\right| &= \left|\int_0^R F(R+iy)e^{is(R+iy)}\,idy\right| = \left|\int_0^R F(R+iy)e^{isR-sy}\,idy\right| \\
&\le \int_0^R |f(R+iy)|e^{-sy}\,dy \le \frac{K}{R}\int_0^R e^{-sy}\,dy = \frac{K}{sR}(1-e^{-sR}),
\end{aligned}$$

which clearly vanishes in the limit $R \to \infty$. Similarly,

$$\begin{aligned}
\left|\int_{C_4} F(z)e^{izs}\,dz\right| &= \left|\int_R^0 F(-R+iy)e^{ik(-R+iy)}\,idy\right| = \left|\int_R^0 F(-R+iy)e^{-ikR-sy}\,idy\right| \\
&\le \int_0^R |f(-R+iy)|e^{-sy}\,dy \le \frac{K}{R}\int_0^R e^{-sy}\,dy = \frac{K}{sR}(1-e^{-sR}),
\end{aligned}$$

which also vanishes in the limit $R \to \infty$. Finally,

$$\begin{aligned}
\left|\int_{C_3} F(z)e^{izs}\,dz\right| &= \left|\int_R^{-R} F(x+iR)e^{is(x+iR)}\,dx\right| = \left|\int_R^{-R} F(x+iR)e^{isx-sR}\,dx\right| \\
&\le \int_{-R}^{R} |F(x+iR)|e^{-sR}\,dx \le e^{-sR}\int_{-R}^{R}\frac{K}{R}\,dx \\
&= e^{-sR}\frac{K}{R}2R = 2Ke^{-sR},
\end{aligned}$$

which vanishes in the limit $R \to \infty$. This completes the proof.

Example 4.10.1 Calculate the Fourier transform of

$$f(s) = e^{-|s|}$$

and use the inversion formula to retrive $f(s)$.

Solution For the transform we have

$$\begin{aligned}
F(k) &= \int_{-\infty}^{\infty} e^{-|s|-iks}\,ds = \int_{-\infty}^{0} e^{s-iks}\,ds + \int_0^{\infty} e^{-s-iks}\,ds \\
&= \frac{1}{1-ik}[e^{s-iks}]_{-\infty}^{0} - \frac{1}{1+ik}[e^{-s-iks}]_0^{\infty} = \frac{1}{1-ik} + \frac{1}{1+ik} = \frac{2}{1+k^2}.
\end{aligned}$$

Given the above $F(k)$, the inversion formula gives (for $s > 0$)

$$\begin{aligned}
f(s) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2e^{iks}}{1+k^2}\,dk = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{iks}}{1+k^2}\,dk \\
&= \frac{1}{\pi}2\pi i Res|_{k=i} = 2i\,\frac{e^{-s}}{2i} = e^{-s}.
\end{aligned}$$

Similarly, for $s < 0$, we used the lower half plane and obtain

$$\begin{aligned}
f(s) &= -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2e^{iks}}{1+k^2}\,dk = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{iks}}{1+k^2}\,dk \\
&= -\frac{1}{\pi}2\pi i Res|_{k=-i} = -2i\,\frac{e^{s}}{-2i} = e^{s}.
\end{aligned}$$

Finally, for $s = 0$, we have

$$f(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2}{1+k^2}\,dk = \frac{2}{\pi}\int_0^{\infty}\frac{1}{1+k^2}\,dk = \frac{2}{\pi}[\tan^{-1}]_0^{\infty} = 1.$$

Putting all the results together we obtain $f(s) = e^{-|s|}$. •