Chapter 5

Qualitative study of ODEs

5.1 Introduction

Recall, for the 1st order ODE, $y' = f(x, y)$, solutions are *integral curves* in xy-plane. The *direction field* is the field of line segments at each point with slope $\frac{dy}{dx} = f(x, y)$. These are tangent to the integral curves at each point. We will adopt a similar graphical representation for the solutions of 2nd order *nonlinear* ODEs.

In this chapter we use a different notation: the independent variable is t and dependent variables are x and y, thus $x = x(t)$ and $y = y(t)$. We use the shorthand notation $\dot{x} = \frac{dx}{dt}$, $\ddot{x} = \frac{d^2x}{dt^2}$, etc. The general 2nd order ODE is then

$$\ddot{x} = f(t, x, \dot{x}) \tag{5.1}$$

where now f may be nonlinear in all arguments.

First, note that this ODE can always be reduced to a *system* of first order ODEs, by setting $\dot{x} = y$. This gives

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= f(t, x, y).\end{aligned}$$

If f is does not depend *explicitly* on t, that is if $f = f(x, y)$, then we say that the system is *autonomous*. A more general autonomous system is

$$\begin{aligned}\dot{x} &= F(x, y)\\ \dot{y} &= G(x, y),\end{aligned} \tag{5.2}$$

where neither F nor G depend explicitly on t, but both can be nonlinear functions of x and y.

Solutions of the system (5.2) (and hence the second order ODE (5.1), may again be represented in the xy-plane. The solution $x(t), y(t)$ defines a solution curve in the plane, which is now parameterized by the independent variable t. If we think of t as representing time, we move along the solution curve in time and the curve represents the evolution of the system. Each curve $[x(t), y(t)]$ is referred to as a *trajectory* and arrows can be drawn to indicate the evolution as t increases. Here we refer to the xy-plane as the *phase plane* and the sketch of all integral curves or trajectories as the *phase portrait*. Note that the condition that F and G do not depend explicitly on t means that the phase portrait does not change in time.

By writing $\dot{y}/\dot{x} = \frac{dy}{dx}$, the system (5.2) reduces further to the first order ODE,

$$\frac{dy}{dx} = \frac{G}{F};$$

from this an approximate direction field and phase portrait can be constructed (sheet 5, questions 1 and 2).

A *critical point* is a point where $\dot{x} = \dot{y} = 0$. Such a point is sometimes also referred to as a *singular point* (since the slope of a trajectory through the point is undefined) or a *stationary point* (since the evolution along the trajectory slows to zero). A critical point (x_0, y_0) is *stable* if all nearby trajectories approach (x_0, y_0) as $t \to \infty$; it is *unstable* if all nearby trajectories approach (x_0, y_0) as $t \to -\infty$.

5.2 Classification of critical points

The general *linear* autonomous system is

$$\begin{aligned} \dot{x} &= ax + by \\ \dot{y} &= cx + dy, \end{aligned} \tag{5.3}$$

where a, b, c, d are constants. Note that this is equivalent to a second order linear ODE with constant coefficients: take d/dt of $\dot{x}$ and eliminate y and $\dot{y}$. So the nature of the (two) solutions depends on the roots of the characteristic equation. Here we take a different approach and write (5.3) in vector form:

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} \quad \text{where} \quad \boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{5.4}$$

Provided $\det(A) \neq 0$, the only critical point of the system (5.3) is at $(x, y) = (0, 0)$ and its nature depends on the eigenvalues of A.

To analyse the nature of the critical points of (5.4) we make a transformation to a new set of dependent variables

$$\boldsymbol{\xi} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

defined by $\boldsymbol{x} = S\boldsymbol{\xi}$, or $\boldsymbol{\xi} = S^{-1}\boldsymbol{x}$, where S is a transformation matrix. In terms of these variables, the system (5.4) becomes

$$\begin{aligned} & S\dot{\boldsymbol{\xi}} = AS\boldsymbol{\xi} \\ \implies \quad & \dot{\boldsymbol{\xi}} = S^{-1}AS\boldsymbol{\xi} = \Lambda\boldsymbol{\xi} \end{aligned} \tag{5.5}$$

We choose the matrix S in such a way that the system in the new variables is in *Jordan normal form*, that is, where the matrix $\Lambda = S^{-1}AS$ is in one of the following forms, depending on the eigenvalues of A. In the following, spec(A), the spectrum of A, is the set of all eigenvalues of A.

1. spec$(A) = \{\lambda_1, \lambda_2\}$, $\lambda_i \in \mathbb{R}$, $\lambda_1 \neq \lambda_2 \quad \implies \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

   In this case, the system (5.5) reduces to

$$\dot{\xi} = \lambda_1 \xi \qquad \dot{\eta} = \lambda_2 \eta$$

giving

$$\xi = \xi_0 e^{\lambda_1 t} \qquad \eta = \eta_0 e^{\lambda_2 t}$$

or

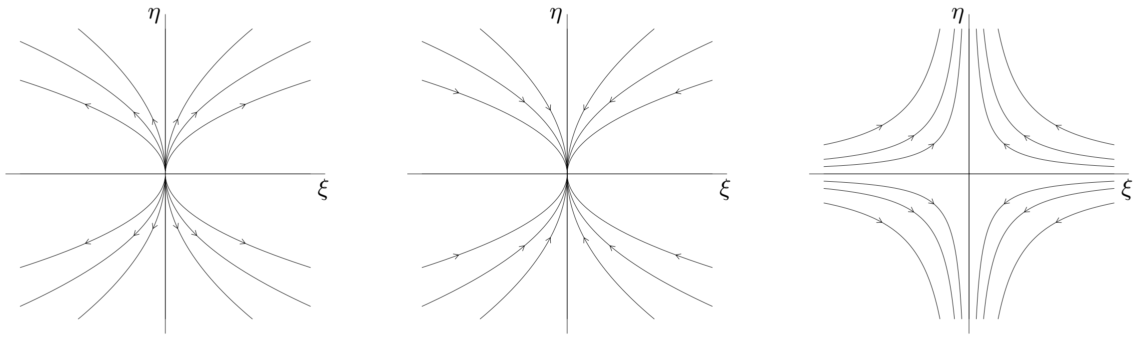
$$\eta = C\xi^{\lambda_2/\lambda_1}.$$

The nature of the critical point depends now on the signs of the eigenvalues, as follows:

(i) $\lambda_1 > \lambda_2 > 0$, *unstable node*

(ii) $\lambda_1 < \lambda_2 < 0$, *stable node*

(iii) $\lambda_1 < 0 < \lambda_2$, *saddle point*



(i) unstable node (ii) stable node (iii) saddle

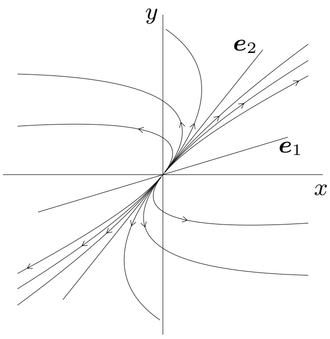
To understand how these pictures transform back to the xy-plane, it is helpful to note that the matrix S is here made up of the two eigenvectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ corresponding to λ_1 and λ_2: in particular the two columns of S are $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$: $S = (\boldsymbol{e}_1\ \boldsymbol{e}_2)$. This can be verified from the relation $AS = S\Lambda$, that is

$$A(\boldsymbol{e}_1\ \boldsymbol{e}_2) = (\boldsymbol{e}_1\ \boldsymbol{e}_2)\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$$

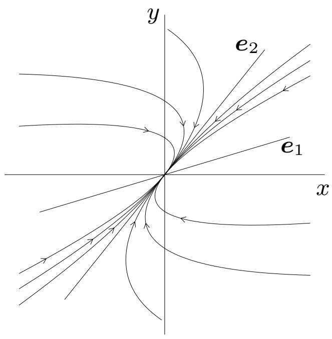
which reduces to the eigenvalue relation $A\boldsymbol{e}_i = \lambda_i \boldsymbol{e}_i$, $i = 1, 2$, as required.

Now, in cases (i) and (ii), the trajectories are tangent to the η axis, that is, to the vector $\begin{pmatrix}0\\1\end{pmatrix}$ in the $\xi\eta$-plane. This vector transforms to $S\begin{pmatrix}0\\1\end{pmatrix} = (\boldsymbol{e}_1\ \boldsymbol{e}_2)\begin{pmatrix}0\\1\end{pmatrix} = \boldsymbol{e}_2$ in the xy-plane, that is, to the eigenvector of A corresponding to λ_2. Thus, the trajectories in the xy-plane are tangent to the eigenvector corresponding to λ_2, the smallest eigenvalue in modulus.

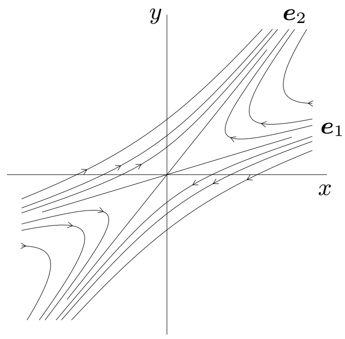
In (iii) the trajectories are tangent to $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ in $\xi\eta$-plane. In a similar way, these vectors transform to the eigenvectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ in xy-plane, respectively.



(i) unstable node



(ii) stable node



(iii) saddle

2. $\mathrm{spec}(A) = \{\lambda\}$, $\lambda \in \mathbb{R}$ (repeated eigenvalue). There are two possibilities.

   (i) If the original matrix A is already diagonal then the new matrix Λ is also be diagonal and takes the form $\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$. In this case, the system (5.5) becomes

   $$\dot{\xi} = \lambda\xi$$
   $$\dot{\eta} = \lambda\eta,$$

   which integrate separately to $\xi = \xi_0 e^{\lambda t}$ and $\eta = \eta_0 e^{\lambda t}$. These can be combined to give $\eta = C\xi$ and so trajectories are straight lines passing throught the critical point. The critical point is called a *star* and is stable for $\lambda < 0$ and unstable for $\lambda > 0$.

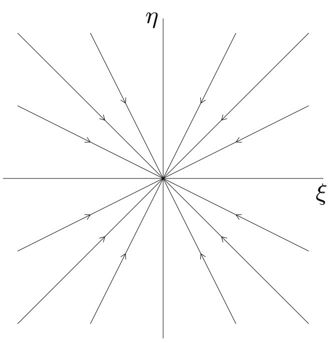
   (ii) If the original matrix A is *not* diagonal then the new matrix Λ takes the form $\Lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. The critical point is called and *improper node*. The system (5.5) becomes

   $$\dot{\xi} = \lambda\xi + \eta$$
   $$\dot{\eta} = \lambda\eta.$$

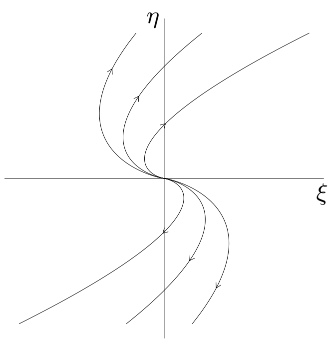
   The second of these can be integrated to give $\eta = \eta_0 e^{\lambda t}$; then substituting into the first gives an equation for ξ that can be solved (using an integrating factor) to give

   $$\xi = \xi_0 e^{\lambda t} + \eta_0 t e^{\lambda t}.$$

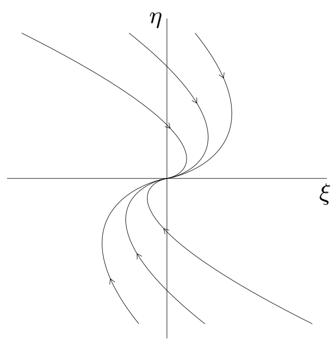
   This case can be compared to the case of a linear second order ODE when the roots of the characteristic equation are equal. Note that the trajectories in the $\xi\eta$-plane are vertical along the straight line given by $\eta = -\lambda\xi$. Thus the stable ($\lambda < 0$) and unstable ($\lambda > 0$) improper notes are reflections in the η axis.



(i) star



(ii) unstable improper node



(iii) stable improper node

3. $\operatorname{spec}(A) = \{\rho \pm i\omega\}$. Now $\Lambda = \begin{pmatrix} \rho & -\omega \\ \omega & \rho \end{pmatrix}$.

   The system (5.5) becomes

$$\begin{aligned} \dot{\xi} &= \rho\xi - \omega\eta \\ \dot{\eta} &= \omega\xi + \rho\eta \end{aligned}$$

   but the behaviour near the critical point is easiest to describe in polar coordinates. Let

$$\begin{aligned} \xi &= r\cos\theta \qquad \implies \dot{\xi} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ \eta &= r\sin\theta \qquad \implies \dot{\eta} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \end{aligned}$$

   Eliminating the $r\dot{\theta}$ term gives

$$\dot{\xi}\cos\theta + \dot{\eta}\sin\theta = \dot{r}\cos^2\theta + \dot{r}\sin^2\theta = \dot{r}$$

   and so

$$\begin{aligned} \dot{r} &= (\rho\xi - \omega\eta)\cos\theta + (\omega\xi + \rho\eta)\sin\theta \\ &= \rho r\cos^2\theta - \omega r\sin\theta\cos\theta + \omega r\cos\theta\sin\theta + \rho r\sin^2\theta \\ &= \rho r. \end{aligned}$$

   Similarly, eliminating the $\dot{r}$ term gives

$$\dot{\xi}\sin\theta - \dot{\eta}\cos\theta = -r\dot{\theta}\sin^2\theta - r\dot{\theta}\cos^2\theta = -r\dot{\theta}$$

   and so

$$\begin{aligned} r\dot{\theta} &= -(\rho\xi - \omega\eta)\sin\theta + (\omega\xi + \rho\eta)\cos\theta \\ &= -\rho r\cos\theta\sin\theta + \omega r\sin^2\theta + \omega r\cos^2\theta + \rho r\sin\theta\cos\theta \\ &= \omega r. \end{aligned}$$

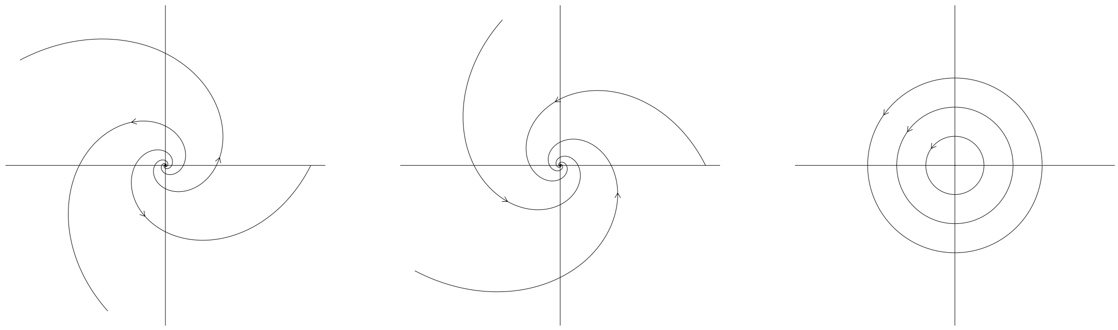
   Thus the system becomes

$$\begin{aligned} \dot{r} &= \rho r \\ \dot{\theta} &= \omega, \end{aligned}$$

   which can be integrated directly to give

$$\begin{aligned} r &= r_0 e^{\rho t} \\ \theta &= \theta_0 + \omega t. \end{aligned}$$

   The critical point is therefore stable for $\rho < 0$ and unstable for $\rho > 0$; the critical point is called a stable or unstable *focus*. For $\rho = 0$, trajectories neither approach nor recede from the critical point but lie on concentric circles; the crictical point is called a *centre*. The examples shown are all for the case $\omega > 0$, so θ is increasing with t. In practice, when determining whether the trajectories circle in a clockwise or anticlockwise direction the simplest approach is to consider the sign of $\dot{x}$ on the line $x = 0$ (or the sign of $\dot{y}$ on the line $y = 0$). Note that in these cases there are no special axes and so no need to consider the eigenvectors of A. In other words, the local phase portrait looks the same in both the $\xi\eta$-plane and the xy-plane.



(i) unstable focus (ii) stable focus (iii) centre

5.3 Linearization about a critical point

We will use without proof the following "theorem" for a general nonlinear autonomous system:

Theorem 5.1 The local phase portrait in a neighbourhood of a critical point (x_0, y_0) is the same as the phase portrait of the linear system obtained by linearizing about (x_0, y_0).

By linearizing we mean tranforming to a local coordinate system centred on (x_0, y_0) and considering a sufficiently small neighbourhood of (x_0, y_0) in which nonlinear terms may be neglected.

To see how this works in practice, we write the general autonomous system

$$\begin{aligned}\dot{x} &= F(x, y)\\ \dot{y} &= G(x, y),\end{aligned}$$

in vector form

$$\dot{\boldsymbol{x}} = \boldsymbol{F}$$

where

$$\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad \boldsymbol{F} = \begin{pmatrix} F(x, y) \\ G(x, y) \end{pmatrix}.$$

Suppose this system has a critical point (x_0, y_0), i.e. a point where $F = G = 0$. We consider a Taylor expansion of $\boldsymbol{F}$ about the point (x_0, y_0), which takes the form

$$\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{F}(\boldsymbol{x}_0) + \left.\frac{d\boldsymbol{F}}{d\boldsymbol{x}}\right|_{\boldsymbol{x}_0} (\boldsymbol{x} - \boldsymbol{x}_0) + O(|\boldsymbol{x} - \boldsymbol{x}_0|^2),$$

where $\frac{d\boldsymbol{F}}{d\boldsymbol{x}}$ is the *derivative matrix* (or Jacobian matrix) of $\boldsymbol{F}$, defined by

$$\frac{d\boldsymbol{F}}{d\boldsymbol{x}} = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}.$$

Since $\boldsymbol{F}(\boldsymbol{x}_0) = \boldsymbol{0}$, near the critical point the system may be approximated by

$$\dot{\boldsymbol{x}} = \boldsymbol{F} \approx \left.\frac{d\boldsymbol{F}}{d\boldsymbol{x}}\right|_{\boldsymbol{x}_0} (\boldsymbol{x} - \boldsymbol{x}_0),$$

for small $|\boldsymbol{x} - \boldsymbol{x}_0|$ or, writing $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{x}_0$,

$$\dot{\boldsymbol{u}} \approx \left.\frac{d\boldsymbol{F}}{d\boldsymbol{x}}\right|_{\boldsymbol{x}_0} \boldsymbol{u}.$$

This is a linear system with a critical point at $\boldsymbol{u} = (0, 0)$ and with the matrix A of the previous section given by the derivative matrix $\frac{d\boldsymbol{F}}{d\boldsymbol{x}}\big|_{\boldsymbol{x}_0}$. To determine the local phase portrait near the point $\boldsymbol{x}_0$, therefore, we determine the eigenvalues and eigenvectors (if appropriate) of the derivative matrix $\frac{d\boldsymbol{F}}{d\boldsymbol{x}}$ evaluated at $\boldsymbol{x}_0$ and use the classification of section 5.2 to determine the nature of the point.

The global phase portrait of the full nonlinear system may then be constructed by sketching the local phase portraits around all critical points of the system and joining these in a smooth way. The procedure is summarized as follows:

1. Identify all critical points.
2. Construct the derivative matrix $\frac{d\boldsymbol{F}}{d\boldsymbol{x}}$.
3. Evaluate $\frac{d\boldsymbol{F}}{d\boldsymbol{x}}$ at each critical point. This gives the matrix A of the linearized system at each point.
4. Determine the nature of each critical point by finding the eigenvalues of A; for nodes and saddles determine the orientation from the eigenvectors of A.
5. Sketch local phase portraits about each critical point and join together in a smooth way.
6. Use any global information from the original system to constrain the slopes of trajectories between critical points. Such global information typically involves lines where $\dot{x} = 0$ or $\dot{y} = 0$ (sometimes called nullclines), where the trajectories are vertical or horizontal, respectively.

Example 5.1

$$\begin{aligned}\dot{x} &= (1 + x - 2y)x\\ \dot{y} &= (x - 1)y\end{aligned}$$

Example 5.2
Nonlinear pendulum: $\ddot{x} + \sin x = 0.$
Damped pendulum: $\ddot{x} + \epsilon\dot{x} + \sin x = 0, \qquad \epsilon > 0.$