MT2507
Chapter 6: Fourier Series

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These notes are based on previous versions developed by Profs I. De Moortel, D. Dritschel, A. Hood and Dr R. Maclean.

Course Content

- Revision of geometric series (only for self-study and revision)
- Periodic functions
- Representation of functions by Fourier series
- Application of Fourier series

Chapter 1

Introduction

In this part of the lecture course we will encounter *Fourier series*, named after Jean-Baptiste Joseph Fourier (1768 – 1830) who used series of trigonometric functions for solving a particular partial differential equation called the heat equation. Fourier series are part of a wider field called Fourier analysis. Fourier series make use of discrete modes, i.e. the functions in a Fourier series can be labelled by integers. The continuous version is called *Fourier transform*, for which the infinite sum over integers is replaced by an integral.

The importance of Fourier analysis cannot be overstated. Applications include

- theory, e.g. solution methods for (partial) differential equations,
- data analysis, e.g. patterns in earthquake data, solar oscillations and many more,
- modern technology, e.g. image compression.

A very readable account of the importance of Fourier analysis and a bit of its history can be found in Chapter 9 of the book *17 Equations that changed the World* by Ian Stewart.

Chapter 2

Essential Prerequisites

This Chapter is for self-study and revision - it will not be explicitly covered in the lectures!

2.1 Trigonometric Identities

This part of the course deals with Fourier series, and the building blocks for Fourier series are the trigonometric functions, $\sin(kx)$ and $\cos(kx)$. The following are important and you really must ensure that you know these results.

1. $\cos(0) = 1, \qquad \sin(0) = 0.$
2. $\cos x$ is an *even* function and $\sin x$ is an *odd* function about $x = 0$. That is
$$\cos(-x) = \cos x, \qquad \sin(-x) = -\sin x.$$
3. $\sin x = 0$ at $x = n\pi$, for integer n. $\sin(kx) = 0$ at $x = n\pi/k$ for integer n and arbitrary non-zero k.
4. $\cos x = 0$ at $x = (n + \frac{1}{2})\pi$, for integer n. $\cos(kx) = 0$ at $x = (n + \frac{1}{2})\pi/k$ for integer n and arbitrary non-zero k.

The addition formulae for sine and cosines should be well-known to you. They are very useful in this course.

1. $\sin(A + B) = \sin A \cos B + \cos A \sin B.$
2. $\sin(A - B) = \sin A \cos B - \cos A \sin B.$
3. $\cos(A + B) = \cos A \cos B - \sin A \sin B.$
4. $\cos(A - B) = \cos A \cos B + \sin A \sin B.$

Using the above four trigonometric identities, we can form the identities for products of trigonometric functions.

1. $\cos A \cos B = \frac{1}{2}\cos(A+B) + \frac{1}{2}\cos(A-B)$.
2. $\sin A \sin B = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$.
3. $\sin A \cos B = \frac{1}{2}\sin(A+B) + \frac{1}{2}\sin(A-B)$.

Using these formulae, we can easily integrate products of sines and cosines. For example,

$$\begin{aligned}\int \sin x \sin 3x dx &= \frac{1}{2}\int (\cos(1-3)x - \cos(1+3)x)\, dx = \frac{1}{2}\int (\cos(2x) - \cos(4x))\, dx \\ &= \frac{1}{4}\sin(2x) - \frac{1}{8}\sin(4x) + C.\end{aligned}$$

Note that we used the fact that the cosine function is symmetric to express $\cos(-2x) = \cos(2x)$.

2.2 Integration by Parts

Throughout this part of the course, we will need to evaluate integrals of the forms $\int f(x)\cos(kx)dx$ and $\int f(x)\sin(kx)dx$. The choices of the function $f(x)$ we will use will allow these integrals to be evaluated exactly. Typical choices will be

$$f(x) = \begin{cases} x^m & m \text{ an integer,} \\ e^{ax} & a \text{ a real constant,} \\ \cos(lx) & l \text{ a real constant,} \\ \sin(lx) & l \text{ a real constant.} \end{cases}$$

These integrals can be evaluated using the technique of integration by parts, namely

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx.$$

Example 2.1

Use Integration by Parts to evaluate $\int x\cos(kx)dx$.

$$\begin{array}{ll} \int x\cos(kx)dx & u = x \quad \frac{dv}{dx} = \cos(kx), \\ = \frac{x}{k}\sin(kx) - \int \frac{1}{k}\sin(kx)dx & \frac{du}{dx} = 1 \quad v = \frac{1}{k}\sin(kx) \\ = \frac{x}{k}\sin(kx) + \frac{1}{k^2}\cos(kx) + C & \end{array}$$

Example End

Chapter 3

Revision of geometric series

This Chapter is for self-study and revision - it will not be explicitly covered in the lectures!

3.1 Geometric series

Consider the geometric series, S, where

$$S = 1 + x + x^2 + x^3 + \cdots + x^n.$$

Here there are a finite number of terms and each of the coefficients is equal to one. Multiplying both sides by x gives

$$xS = x + x^2 + x^3 + \cdots + x^n + x^{n+1}.$$

Now if we subtract the two equations from each other we obtain

$$\begin{aligned} S - xS &= 1 - x^{n+1} \\ (1-x)S &= 1 - x^{n+1} \\ S &= \frac{1 - x^{n+1}}{1 - x}. \end{aligned}$$

Note that all the terms cancelled apart from the first and last terms. Now if the absolute value of x is less than unity, namely $|x| < 1$, then the powers of x get smaller and smaller so that as $n \to \infty$ we have $x^{n+1} \to 0$. Thus,

$$\lim_{n\to\infty} S = \frac{1}{1-x} = (1-x)^{-1}.$$

Therefore, the power series for $(1-x)^{-1}$, about $x = 0$ is

$$1 + x + x^2 + x^3 + \cdots$$

and this is valid for x lying in the range $-1 < x < 1$. This is simply another way of writing $|x| < 1$. Note that $x_0 = 0$ in this example. Now we can never actually use the complete infinite series and so we tend to truncate the series after a finite number of terms. This means that we use the truncated series to *approximate* the value of $(1-x)^{-1}$.

This idea of approximating functions by a truncated series is extremely important and can be used to obtain approximate expressions for integrals and derivatives of complicated functions. It is this idea of approximating functions by finite number of terms in a Fourier Series that is central to this part of the course. Obviously, if we can actually use the infinite series, then we have an exact representation of the function. However, if we need to calculate the value of the function for a particular value of x, we need to truncate the series after a certain number of terms. Then, we only have an approximation and the question is how accurate is our approximate answer? Leading on from this question, if we only need an approximate answer, do we need to use a power series representation of the function?

Example 3.1

We can obtain a power series expansion for $(1+x)^{-1}$ by simply replacing x by $-x$ in our previous series. Thus,

$$f(x) = \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \cdots + (-1)^n x^n + \cdots \tag{3.1}$$

We can compare the truncated series with the exact value to see how good our approximation actually is. This is illustrated in Figure 3.1. The dotted curve corresponds to using the series only up to terms in x^2 and no further, i.e. $f(x) \approx 1 - x + x^2$. The dashed curve contains all the terms up to x^{10}, the dot-dashed curve terms up to x^{20} and the triple-dot-dashed curve terms up to x^{100}.

Figure 3.1 illustrates how the truncated series provides a better and better approximation as we take more and more terms, but only for $x < 1$. For $x > 1$, the series becomes worse as more terms are used. This shows graphically how a power series converges when x is less than the Radius of Convergence, R. However, a graphical demonstration of convergence is not the same as a rigorous proof.

Example End

Why do we use powers of x to approximate the function $1/(1+x)$? No reason really except that it gives a good approximation near $x = 0$. As you will see in Taylor series, when using a power series approximation, all the information about the function, $f(x)$, (in this case $f(x) = 1/(1+x)$) comes from the origin. Hence, the approximation is generally good near the origin but becomes less good further away. In this course, instead of using powers of x to approximate a function $f(x)$, we are going to use sines and cosines. This is particularly useful when the function we are approximating is periodic.

Example 3.2

As a final example of approximating a function by the geometric series, consider

$$f(x) = \frac{1}{1+x^2},$$

so that we can approximate this function by considering a finite number of terms, say n,

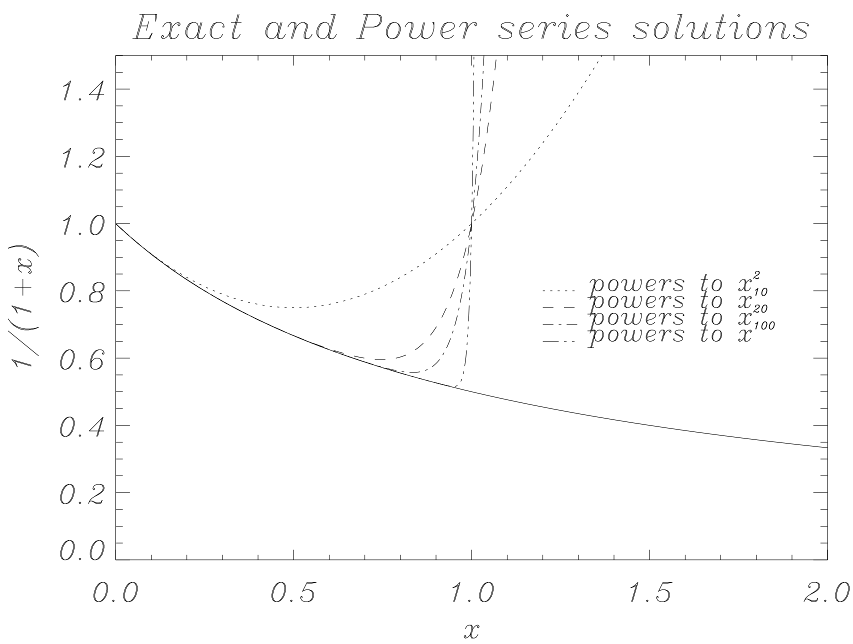

Figure 3.1: The power series approximation for various numbers of terms in the series, compared with the exact solution (the solid curve).

in the geometric series so that

$$f(x) = \frac{1}{1+x^2} \approx 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n}.$$

Note, the approximation sign is there to indicate that this is only an approximation to the function as we are only taking a finite number of terms. Also the approximation is only reasonable for $x^2 < 1$. If we wished to find a reasonable approximation over the domain $-2 < x < 2$, say, we would need to use another method. As we will see, Fourier series will provide an accurate approximation over the whole domain and not just over $-1 < x < 1$. For this example, the first three terms of the Fourier series turn out to be (we will see why later)

$$F(x) = 0.5536 + 0.3706\cos(\frac{\pi x}{2}) + 0.0536\cos(\pi x),$$

and this gives a reasonable fit to the function $f(x)$ over $-2 \leq x < 2$ (see Figure 3.2).

Example End

The general Fourier series expresses a function $f(x)$, defined over the domain $c \leq x < c + L$ as

$$f(x) = \frac{a_0}{2} + a_1\cos(\frac{2\pi}{L}x) + b_1\sin(\frac{2\pi}{L}x) + a_2\cos(\frac{4\pi}{L}x) + b_2\sin(\frac{4\pi}{L}x) + a_3\cos(\frac{6\pi}{L}x) + b_3\sin(\frac{6\pi}{L}x) + \cdots \tag{3.2}$$

The main aim is to determine the constants a_n and b_n for general integer n.

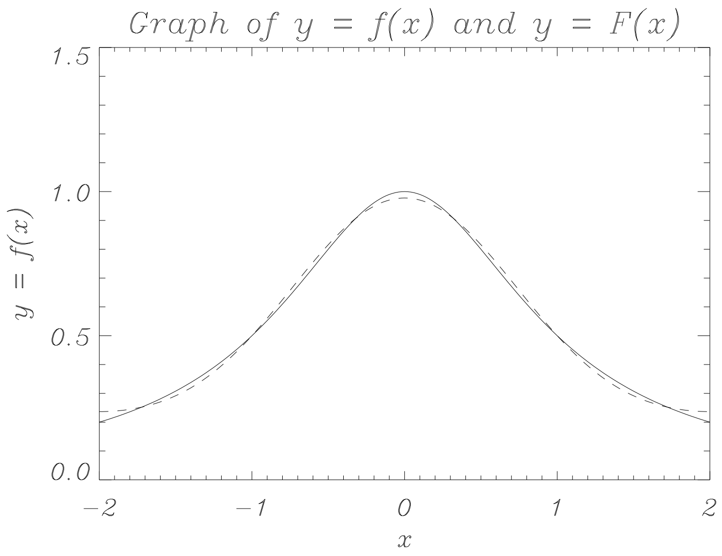

Figure 3.2: The function $f(x)$ (solid curve) and the three term Fourier approximation $F(x)$ (dashed curve).

3.2 Summation Notation

Rather than write out all the terms that we are using explicitly, as in the above expression for $f(x)$, it is much better to use the *summation notation* so that

$$\frac{1}{1+x^2} \approx 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} = \sum_{j=0}^{n} (-1)^j x^{2j}.$$

Please get used to this notation as it will save a large amount of writing later on. Thus, we use

$$\sum_{j=0}^{n} a_j x^j = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n. \tag{3.3}$$

Consider the example

$$\sin x = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} x^{2j+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Remember $3! = 3 \times 2 \times 1$ and so on.

Chapter 4

Periodic Functions

4.1 Definition

Any function $f(x)$ which satisfies

$$f(x+L) = f(x), \qquad \text{for all } x,$$

is said to be *periodic*. L is called the *period*. We assume that $L > 0$ and that L is finite. This is the basic definition of a periodic function. A simple example is $f(x) = \cos x$, since $\cos(x + 2\pi) = \cos(x)$ for all x. For this function the period is 2π. If instead we considered the periodic function $f(x) = \sin(4x)$, then the period would be $\pi/2$ since $\sin(4(x + \pi/2)) = \sin(4x + 2\pi) = \sin(4x)$.

1. Note: The function f need *not* be continuous. Continuity is defined in the other part of this course. An example of a discontinuous, periodic function is shown in Figure 4.1. To simplify matters, we assume that there is only a finite number of discontinuities in a single period L.

2. Note: A vast number of processes in nature and in technology are repetitive or nearly so to make this concept meaningful. For example, the Earth's rotation, the orbits of the planets, the number of sunspots on the Sun, tones of musical instruments, colour in light, some cloud patterns and water wave patterns and so on.

3. Note: Since $f(x+L) = f(x)$ for *any* x, we can replace x by $x+a$ in the above to get
$$f(x+a+L) = f(x+a),$$
for any a. For particular choices of a, we recover some useful results. For $a = -L$,
$$f(x) = f(x-L), \qquad \text{hence} \qquad f(x \pm L) = f(x).$$
For $a = L$,
$$f(x+2L) = f(x+L), \quad \text{but} \quad f(x+L) = f(x), \quad \text{and so} \quad f(x+2L) = f(x).$$

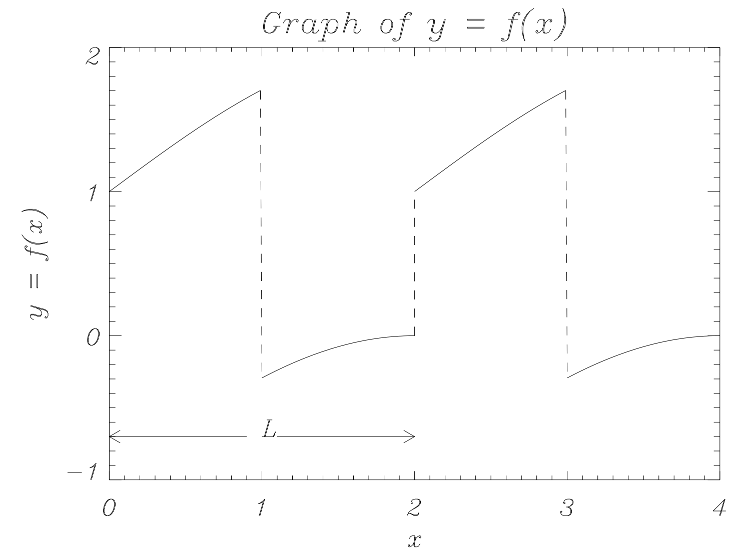

Figure 4.1: A function with one discontinuity per period L and a discontinuity at the end of the periodic interval.

Finally, if $a = jL$, for any integer j (either positive or negative),

$$f(x + jL) = f(x).$$

This just states that $f(x)$ repeats endlessly, which is really what is meant by a periodic function.

4. Note: L is called the *fundamental period* if the function, f, is not periodic on any shorter interval. $2L$, $3L$ and, in general, jL are called the *harmonics*.

In what follows the independent variable x will be used for both spatial position and time but it will be clear which is which from the context.

4.2 Construction from a function defined over a finite interval

Suppose we only know the values of f over a *finite* interval $[a, b)$, namely $a \le x < b$ as illustrated in Figure 4.2. For such a function, we can *always* construct a periodic function with period $L = b - a$, by just copying $f(x)$ to $f(x + L)$, then to $f(x + 2L)$ and so on before using $f(x - L)$, then $f(x - 2L)$ etc.

Let us illustrate, in Figure 4.3, this procedure of generating a periodic function for $f(x) = x$ which is initially defined over $0 \le x < 1$. The solid line shows the original function and the domain over which it is defined. The dashed lines are the copies that have been generated by shifting the original line by multiples of the period, L, which is unity in this case.

Thus, the point to take from this section is that *any function defined over a finite domain can be made periodic* over an infinite domain.

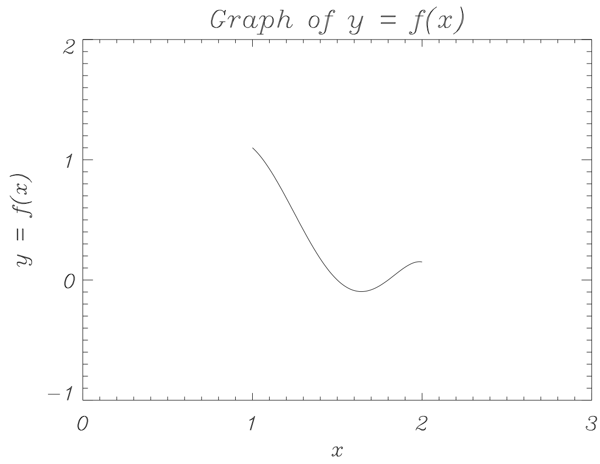

Figure 4.2: A function defined over a finite interval. Here $a = 1$ and $b = 2$.

4.3 The standard interval

The *simplest* form of the Fourier Series (to be defined later) applies to functions $f(x)$ defined over the *standard* interval $[-l, l)$. Note that the period of the function is $L = 2l$. In practice, this is **not** a restriction as we can always shift a function defined over $[a, b)$ onto the interval $[-l, l)$ by redefining the independent variable x. Consider the simple case where $f(x) = \sin(\pi x)$ and $0 \leq x < 1$. Define a new independent variable $X = x - \frac{1}{2}$ so that when $x = 0$ we have $X = -\frac{1}{2}$ and when $x = 1$ we have $X = \frac{1}{2}$. Thus, $f(x) = F(X) = \sin[\pi(X + \frac{1}{2})]$ is defined for $-\frac{1}{2} \leq X < \frac{1}{2}$.

How do we do this in general? If the original function is defined over the domain $a \leq x < b$, then the new independent variable, $X = x - (a + b)/2$ is defined over the domain $-l \leq X < l$ where $l = (b - a)/2$.

Example 4.1

Express $f(x) = 1/x$ defined over $1 \leq x < 3$ as a function defined over $-l \leq x < l$.

Solution 4.1

First we note that the period is the length of the domain, so that $L = (3 - 1) = 2$ and so $l = L/2 = 1$. Next we note that the centre of the original interval is at $x = 2$. Hence we need to shift x by 2 to the left. Thus, we define the new independent variable as

$$X = (x - 2),$$

and so

$$F(X) = \frac{1}{X + 2}, \qquad -1 \leq X < 1.$$

Note that the limits have been decreased in this example by the shift.

Example End

Example 4.2

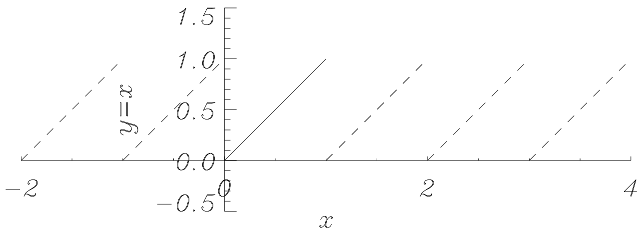

Figure 4.3: How to make $f(x) = x$ on $[0, 1)$ periodic over an infinite domain. Note that the resulting periodic function is discontinuous but still periodic.

Express $f(x) = x(1-x)$ over $0 \leq x < 1$ as a function defined over $-1/2 \leq x < 1/2$.

Solution 4.2

Note that the period is 1 this time and so we must shift the function to the left by half this amount, namely 1/2. Thus, we define the new independent variable as

$$X = x - 1/2$$

and put $x = X + 1/2$ in the original function. Thus,

$$F(X) = (X + \frac{1}{2})(1 - X - \frac{1}{2}) = (\frac{1}{2} + X)(\frac{1}{2} - X) = \frac{1}{4} - X^2, \qquad -\frac{1}{2} \leq X < \frac{1}{2}.$$

Example End

4.4 Orthogonality of two functions

Remember that two lines are defined to be orthogonal (or perpendicular) to each other if the product of the gradients equals minus unity. Two vectors, $\mathbf{a}$ and $\mathbf{b}$, are defined to be orthogonal if the scalar product is zero, namely $\mathbf{a} \cdot \mathbf{b} = 0$. How do we define the orthogonality of functions? Well if our vectors had n dimensions the orthogonality condition is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots + a_n b_n = \sum_{i=1}^{n} a_i b_i = 0.$$

If it helps to visualise things, consider a continuous function over a domain $a \le x < b$ as being a vector of infinite dimension. Then the summation would convert into an integral. Thus, two functions $f(x)$ and $g(x)$ are *orthogonal* over $a \le x < b$ if

$$\langle f, g\rangle \equiv \int_a^b f(x)g(x)dx = 0. \tag{4.1}$$

Note that any non-zero function, $f(x)$, for which the integral $\int_a^b f^2 dx$ exists, can never be orthogonal to itself as the integral of a positive quantity can never be zero. The integral

$$\langle f, f\rangle = \int_a^b f^2(x)dx \equiv \|f^2\|,$$

is called the *norm* of f (or more strictly the L^2 norm).

Example 4.3
Are $f = x$ and $g = 1$ orthogonal over the interval $-1 \le x < 1$?

Solution 4.3

$$\langle f, g\rangle = \int_{-1}^1 x \cdot 1dx = \left[\frac{1}{2}x^2\right]_{-1}^1 = (\frac{1}{2} - \frac{1}{2}) = 0.$$

Hence, f and g are orthogonal.
Example End

Example 4.4
Are $f = x$ and $g = x - c$ orthogonal over the interval $-1 \le x < 1$ for some constant c?

Solution 4.4

$$\langle f, g\rangle = \int_{-1}^1 x(x-c)dx = \int_{-1}^1 \left(x^2 - cx\right) dx = \left[\frac{1}{3}x^3 - \frac{1}{2}cx^2\right]_{-1}^1 = (\frac{1}{3} - \frac{1}{2}c) - (\frac{-1}{3} - \frac{1}{2}c) = \frac{2}{3}.$$

Hence, f and g are *not* orthogonal for any choice of c.
Example End

4.5 The fundamental periodic basis functions: $\sin kx$ and $\cos kx$

Sine and cosine turn out to be the *smoothest* periodic functions of all and this is why they are important. They turn up as solutions to a wide class of problems (motion of a pendulum, motion of a mass on a spring, tides, etc) governed by a second-order Ordinary Differential Equation (ODE) of the form

$$\frac{d^2y}{dx^2} + k^2y = 0.$$

This is called the *Simple Harmonic Oscillator*. The two linearly independent solutions are $y = \sin kx$ and $y = \cos kx$, where k is called the *frequency* or the *wave number*. The most general solution is an arbitrary combination of these functions, namely,

$$y = a \cos kx + b \sin kx,$$

where a and b are arbitrary constants.

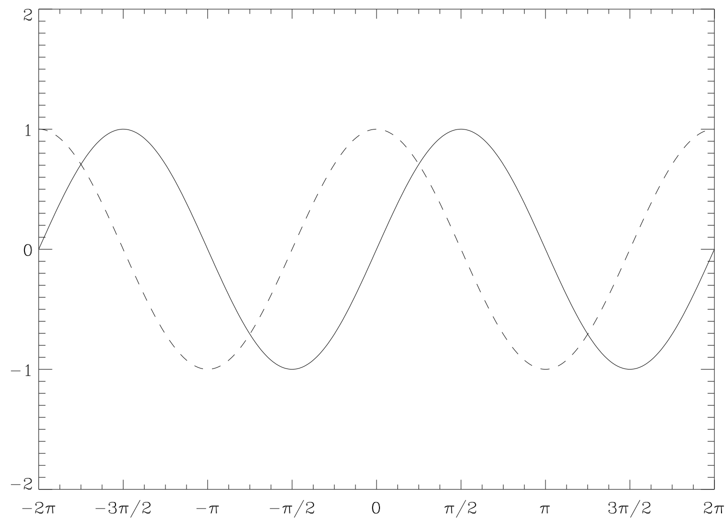

Figure 4.4: The fundamental periodic basis functions, $\cos kx$ (dashed curve) and $\sin kx$ (solid curve).

The fundamental (or smallest) period (or wavelength) is

$$L = \frac{2\pi}{k}.$$

Note that $\cos kx$ is *even* (or *symmetric*) and that $\sin kx$ is *odd* (or *anti-symmetric*) about $x = 0$. You must be able to draw these functions quickly, and being able to identify whether functions are even or odd will also be important.

Remember that any function satisfying $f(-x) = f(x)$ is called an *even* function. Examples are $f = 1, x^2, x^4, \cos kx, \cdots$. Any function satisfying $f(-x) = -f(x)$ is *odd*. Examples are $f = x, x^3, \sin kx, \cdots$. Finally, if you multiply two even functions together the answer is even. Multiply two odd functions together and the answer is even, and multiplying even and odd functions together gives an odd function. Thus,

$$\text{EVEN} \times \text{EVEN} = \text{EVEN}, \qquad \text{ODD} \times \text{ODD} = \text{EVEN}, \qquad \text{EVEN} \times \text{ODD} = \text{ODD}.$$

This can save you a tremendous number of unnecessary calculations.

We now go on to derive some important properties that will be used in determining the *Fourier Series* that represents a given function $f(x)$ as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{L}\right) + b_n \sin\left(\frac{2n\pi x}{L}\right),$$

where the constants a_n and b_n have to be determined.

4.5.1 Important Properties for Fourier Series: 1

The functions $\sin kx$ and $\cos kx$ are *orthogonal* over the interval $-\pi/k \le x < \pi/k$. This means

$$\langle \sin kx, \cos kx \rangle = \int_{-\pi/k}^{\pi/k} \sin kx \cos kx dx = 0. \tag{4.2}$$

Note that for this case the period is $L = 2\pi/k$ and so $l = L/2 = \pi/k$. Really there is no work to do. $\sin kx$ is ODD and $\cos kx$ is EVEN over this interval and so the product is ODD. The integral of an odd function over this standard interval is automatically zero.

However, to properly verify this we should undertake the integral. To do this we must express the product as a sine of a multiple angle. Here we need

$$\sin 2A = 2 \sin A \cos A.$$

Now we use $A = kx$ so that

$$\langle \sin kx, \cos kx \rangle = \int_{-\pi/k}^{\pi/k} \sin kx \cos kx dx = \int_{-\pi/k}^{\pi/k} \frac{1}{2} \sin 2kx dx = \left[-\frac{1}{4k} \cos 2kx \right]_{-\pi/k}^{\pi/k}.$$

Since $\cos 2kx$ is an EVEN function, we have

$$\int_{-\pi/k}^{\pi/k} \sin kx \cos kx dx = -\frac{1}{4k} \left(-\cos(2\pi) + \cos(-2\pi) \right) = 0.$$

Thus, the functions $\sin kx$ and $\cos kx$ are orthogonal to each other over the interval $-\pi/k \le x < \pi/k$. This is an important result since it will allow us to simplify some complicated expressions later on in this part of the course.

4.5.2 Important Properties for Fourier Series: 2

The next important property is

$$\langle \sin kx, \sin kx \rangle = \langle \cos kx, \cos kx \rangle = \frac{\pi}{k} = \frac{L}{2} = l. \tag{4.3}$$

To prove this property we need to use the trigonometric identity for $\cos 2A$, and also $\sin^2 A + \cos^2 A = 1$. Firstly, we remember that

$$\cos 2A = \cos^2 A - \sin^2 A = \cos^2 A - (1 - \cos^2 A) = 2\cos^2 A - 1,$$

and

$$\cos 2A = \cos^2 A - \sin^2 A = (1 - \sin^2 A - \sin^2 A) = 1 - 2\sin^2 A.$$

From these two expressions, we have the two results

$$\begin{aligned} \cos^2 kx &= \frac{1}{2} + \frac{1}{2} \cos 2kx, \\ \sin^2 kx &= \frac{1}{2} - \frac{1}{2} \cos 2kx. \end{aligned}$$

Thus, we can prove the important result above that

$$\begin{aligned}
\left\langle \cos^2 kx \right\rangle &= \int_{-\pi/k}^{\pi/k} \cos^2 kx dx = \int_{-\pi/k}^{\pi/k} \left(\frac{1}{2} + \frac{1}{2}\cos 2kx\right) dx = \left[\frac{x}{2} + \frac{1}{4k}\sin(2kx)\right]_{-\pi/k}^{\pi/k} = \frac{\pi}{k}, \\
\left\langle \sin^2 kx \right\rangle &= \int_{-\pi/k}^{\pi/k} \sin^2 kx dx = \int_{-\pi/k}^{\pi/k} \left(\frac{1}{2} - \frac{1}{2}\cos 2kx\right) dx = \left[\frac{x}{2} - \frac{1}{4k}\sin(2kx)\right]_{-\pi/k}^{\pi/k} = \frac{\pi}{k}.
\end{aligned}$$

4.5.3 Important Properties for Fourier Series: 3

Distinct 'harmonics' of the fundamental periodic functions $\sin kx$ and $\cos kx$ are *orthogonal* to each other. Thus, we have

$$\begin{aligned}
\langle \sin(mkx)\sin(nkx)\rangle &= 0, \qquad m \neq n, &(4.4)\\
\langle \cos(mkx)\cos(nkx)\rangle &= 0, \qquad m \neq n, &(4.5)
\end{aligned}$$

m and n are any integers (including zero) but they must *not* equal each other. These properties are extremely important and the proofs are tutorial exercises. The key steps make use of the trigonometric identities,

$$\begin{aligned}
\cos(A+B) &= \cos A\cos B - \sin A\sin B, \\
\cos(A-B) &= \cos A\cos B + \sin A\sin B.
\end{aligned}$$

Adding these two equations together gives

$$\begin{aligned}
2\cos A\cos B &= \cos(A+B) + \cos(A-B), \\
\Rightarrow \quad \cos A\cos B &= \frac{1}{2}\cos(A+B) + \frac{1}{2}\cos(A-B). \qquad (4.6)
\end{aligned}$$

Subtracting instead gives

$$\begin{aligned}
2\sin A\sin B &= \cos(A-B) - \cos(A+B), \\
\Rightarrow \quad \sin A\sin B &= \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B). \qquad (4.7)
\end{aligned}$$

To use these equations, we simply set $A = mkx$ and $B = nkx$.

4.5.4 Important Properties for Fourier Series: 4

Now property 4 is

$$\langle \sin(mkx), \cos(nkx)\rangle = 0, \qquad \text{for any integers } m \text{ and } n. \qquad (4.8)$$

This requires NO work to prove. Remember the definition

$$\langle \sin(mkx), \cos(nkx)\rangle = \int_{-L/2}^{L/2} \sin(mkx)\cos(nkx) dx,$$

and we immediately recognise that $\sin(mkx)\cos(nkx)$ is an odd function about the origin (mid-point of the range of integration) and so the integral is ALWAYS zero. Even though we know this must be true, this can be proved using the trigonometric identities for

$$\begin{aligned}\sin(A+B) &= \sin A\cos B + \cos A\sin B,\\ \sin(A-B) &= \sin A\cos B - \cos A\sin B.\end{aligned}$$

Adding and subtracting gives

$$\begin{aligned}\sin A\cos B &= \frac{1}{2}\sin(A+B) + \frac{1}{2}\sin(A-B),\\ \cos A\sin B &= \frac{1}{2}\sin(A+B) - \frac{1}{2}\sin(A-B).\end{aligned}$$

Thus, using $A = mkx$ and $B = nkx$, we have

$$\begin{aligned}\int_{-\pi/k}^{\pi/k}\sin(mkx)\cos(nkx)dx &= \int_{-\pi/k}^{\pi/k}\left\{\frac{1}{2}\sin(n+m)kx + \frac{1}{2}\sin(m-n)kx\right\}dx\\ &= \left[\frac{-1}{2(m+n)k}\cos(n+m)kx + \frac{-1}{2(m-n)k}\cos(m-n)kx\right]_{-\pi/k}^{\pi/k}\\ &= 0.\end{aligned}$$

Remember that

$$\cos(n+m)\pi = (-1)^{n+m}, \quad \cos(m-n)\pi = (-1)^{m+n-2n} = (-1)^{m+n}(-1)^{-2n} = (-1)^{m+n},$$

as $(-1)^{-2n} = 1$.

4.5.5 Important Properties for Fourier Series: 5

Our final properties are

$$\langle\sin(mkx),\sin(mkx)\rangle = \frac{\pi}{k} = \frac{L}{2} = l, \tag{4.9}$$

$$\langle\cos(mkx),\cos(mkx)\rangle = \frac{\pi}{k} = \frac{L}{2} = l, \tag{4.10}$$

for *any non-zero integer* m.

The proof is virtually identical to property 2 with k replaced by mk. It is interesting to note that the number of oscillations, m, does not matter as it does not appear in the answer.

4.6 Simple examples

Example 4.5

Consider the function $f(x) = \cos^2 x$. What is the period of this function?

Solution 4.5

Note that we can express $\cos^2 x$ in terms of $\cos(2x)$ using our trigonometric identity

$$\cos^2 x = \frac{1}{2}\left[1 + \cos(2x)\right].$$

Thus, the frequency k is $k = 2$ and so the function repeats twice as often as the fundamental function $\cos x$. This is shown in Figure 4.5. The solid curve for $\cos(2x)$ has twice as many repeats as the dashed curve for $\cos x$. Hence, when the frequency is twice the fundamental frequency, the period is half the fundamental period. The period of $\cos^2(x)$ is $L = 2\pi/k = 2\pi/2 = \pi$, while the period of $\cos x$ is $L = 2\pi/k = 2\pi/1 = 2\pi$.

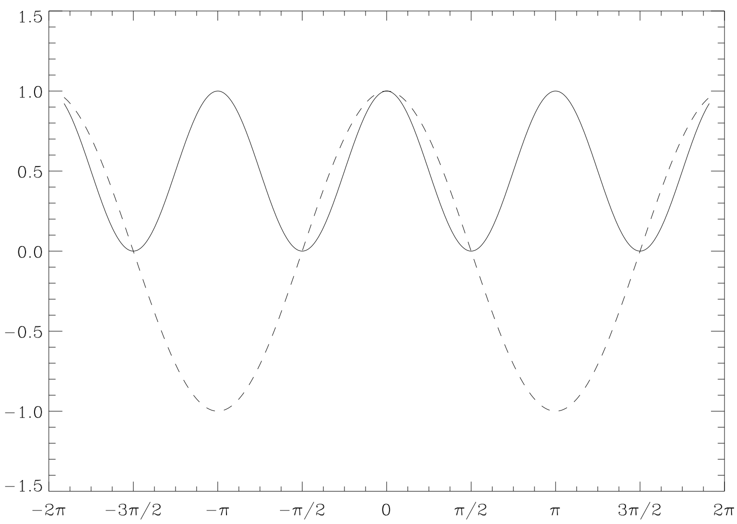

Figure 4.5: The function $\cos^2(x)$ is shown as a solid curve and the fundamental basis function $\cos(x)$ as a dashed curve. Note that $\cos^2(x)$ repeats twice as often as $\cos(x)$.

Example End

It is not too surprising to discover that the period of $\cos^n x$ is $L = 2\pi/n$.

Can we use what we already know to calculate $\langle \cos^2 x, \cos^2 x\rangle$? Choosing for this example $L = \pi$ so that $l = L/2 = \pi/2$ and from the definition of the norm, we have

$$\begin{aligned}
\left\langle \cos^2 x, \cos^2 x \right\rangle &= \int_{-\pi/2}^{\pi/2} \cos^2 x \cos^2 x dx, \\
&= \frac{1}{4}\int_{-\pi/2}^{\pi/2} (1+\cos(2x))^2\, dx, \\
&= \frac{1}{4}\int_{-\pi/2}^{\pi/2} 1 + 2\cos(2x) + \cos^2(2x) dx, \\
&= \frac{1}{4}\int_{-\pi/2}^{\pi/2} 1 + 2\cos(2x) + \frac{1}{2}\left(1+\cos(4x)\right) dx, \\
&= \frac{1}{4}\left[\frac{3}{2}x + \sin(2x) + \frac{1}{8}\sin(4x)\right]_{-\pi/2}^{\pi/2}, \\
&= \frac{1}{4}\left(\frac{3}{2}\pi/2 + \sin\pi + \frac{1}{8}\sin(2\pi)\right) - \frac{1}{4}\left(\frac{3}{2}(-\pi/2) + \sin(-\pi) + \frac{1}{8}\sin(-2\pi)\right), \\
&= \frac{3\pi}{8}.
\end{aligned}$$

Let's try this again and use the results that we already know.

$$\begin{aligned}
\left\langle \cos^2 x, \cos^2 x \right\rangle &= \frac{1}{4}\left\langle (1+\cos(2x)), (1+\cos(2x)) \right\rangle, \\
&= \frac{1}{4}\left\{ \langle 1, 1\rangle + \langle \cos(2x), \cos(2x)\rangle \right\}, \\
&= \frac{1}{4}\left\{ \pi + \frac{\pi}{2} \right\} = \frac{3\pi}{8}.
\end{aligned}$$

Note that we have used property 3, namely that 1 and $\cos(2x)$ are orthogonal for $m = 0$, $n = 2$ and $k = 2$, and property 5, that $\langle \cos^2(2x)\rangle = L/2 = \pi/2$ for $m = 2$ and $k = 2$.

The orthogonality properties of the sine and cosine functions are extremely useful in evaluating trigonometric integrals and form the cornerstone in calculating Fourier series.

Example 4.6

Consider the function $f(x) = 3\sin(2x) - 2\sin(3x)$. The function is periodic but what is the fundamental period?

Solution 4.6

The problem is that the fundamental periods are different, namely $2\pi/2 = \pi$ for $\sin(2x)$ and $2\pi/3$ for $\sin(3x)$. Hence, for $f(x)$ the fundamental period, L, must be longer still in order to accommodate both of these periods, while remaining continuous at $x = \pm L/2$. When uncertain what to do, we can always plot the functions on the same graph and see when they both repeat at the same place. This is done in Figure 4.6.

Both $\sin(2x)$ and $\sin(3x)$ are periodic on the interval $-\pi < x < \pi$ and their combination, i.e. $f(x)$ is not periodic on any shorter interval. Hence, $L = 2\pi$ is the fundamental period.

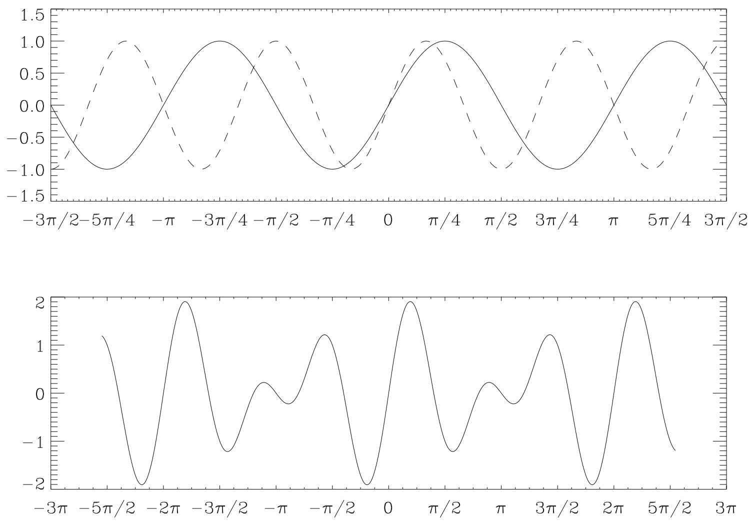

Figure 4.6: The functions $\sin(2x)$ and $\sin(3x)$ are shown as solid and dashed curves respectively in the top graph and $3\sin 2x - 2\sin 3x$ is shown in the bottom graph.

Example End

Note

As an aside to the above example, if we have the function $f(x) = a\sin(k_1x) + b\sin(k_2x)$ where

$$\frac{k_1}{k_2} = \frac{m}{n},$$

where integers m and n are the *smallest* integers satisfying this ratio, then the fundamental period is

$$L = \frac{2\pi m}{k_1} = \frac{2\pi n}{k_2}.$$

Example 4.7

What is the fundamental period of the function $f(x) = \cos(3x)\cos(5x)$?

Solution 4.7

Probably the easiest way to see this is to use our trigonometric identity for a product of cosines, namely

$$\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right].$$

Thus, with $A = 3x$ and $B = 5x$, we have

$$f(x) = \frac{1}{2}\cos(8x) + \frac{1}{2}\cos(2x).$$

Using the above note, we have

$$\frac{8}{2} = \frac{4}{1}, \quad \Rightarrow \quad L = \frac{2\pi 4}{8} = \pi.$$

Here we have $m = 4$ and $n = 1$.

Example End

Chapter 5

Representation of functions by Fourier Series

From now on, we consider a bounded function $f(x)$ defined on $[-l, l)$ with at most a finite number of discontinuities in this interval.

Let us now make use of the special properties of sine and cosine over this interval. First of all, the fundamental “Fourier” basis functions are

$$\sin\frac{\pi x}{l}, \qquad \cos\frac{\pi x}{l}$$

i.e. $k = \pi/l (= 2\pi/L)$ as in Section 4.5.

The functions $\sin\frac{\pi x}{l}$ and $\sin\frac{j\pi x}{l}$, for $j > 1$, when combined as in

$$a\sin\frac{\pi x}{l} + b\sin\frac{j\pi x}{l},$$

are periodic only on the fundamental period $L = 2l$. This is the *shortest* period for which the combination of these two sines is periodic. The same is true for any combination of cosines of the form

$$a\cos\frac{\pi x}{l} + b\cos\frac{j\pi x}{l}:$$

the fundamental period is $L = 2l$. As j is an arbitrary positive integer, the same is true for the sums

$$\sum_{j=1}^{\infty} a_j\cos\frac{j\pi x}{l} \text{ and } \sum_{j=1}^{\infty} b_j\sin\frac{j\pi x}{l}$$

for *arbitrary coefficients* $a_j, b_j,\ j = 1, 2, 3, \ldots$. At least one of a_1 or b_1 must be nonzero.

Note:

1. The first sum is an EVEN function about $x = 0$ while the second is an ODD one.

2. These are sums of *mutually orthogonal* functions so that for integers, m and n such that $m \neq n$

$$\langle\cos\frac{m\pi x}{l}, \cos\frac{n\pi x}{l}\rangle = \langle\sin\frac{m\pi x}{l}, \sin\frac{n\pi x}{l}\rangle = 0,$$

as well as

$$\langle \cos\frac{m\pi x}{l}, \sin\frac{n\pi x}{l}\rangle = \langle \sin\frac{n\pi x}{l}, \cos\frac{n\pi x}{l}\rangle = 0,$$

These orthogonality results are **extremely** *important* and are used in determining the values of the coefficients a_j and b_j when representing a specific function $f(x)$ by a Fourier Series (see Section 5.2.

3. Each function in the sum has a *norm* of $\frac{L}{2} = l$, i.e.

$$\langle \cos\frac{j\pi x}{l}, \cos\frac{j\pi x}{l}\rangle = \langle \sin\frac{j\pi x}{l}, \sin\frac{j\pi x}{l}\rangle = l$$

5.1 Definition of a Fourier Series

A *Fourier Series* is the function

$$S(x) \equiv \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos\frac{j\pi x}{l} + \sum_{j=1}^{\infty} b_j \sin\frac{j\pi x}{l}$$

NOTE:

1. I am using $S(x)$ to stand for the infinite (and assumed convergent) series. The $\frac{a_0}{2}$ and the cosine series represent an *even* function and the sine series an *odd* function.

2. It is convention to use a factor of 2 here, for reasons given below. $\frac{a_0}{2}$ multiplies $\cos\frac{0\pi x}{l} = 1$

   The a_j's and b_j's are the Fourier cosine and sine coefficients respectively.

3. **Crucial Point**: each component function, i.e. 1, $\cos\frac{\pi x}{l}, \cos\frac{2\pi x}{l}, .. \sin\frac{\pi x}{l}, \sin\frac{2\pi x}{l}, ..$ is orthogonal to every other one.

Moreover, the sum incorporates *every possible* sine and cosine function which is periodic on the fundamental period $L = 2\pi$. This means that the component functions form a *complete basis* on $[-l, l)$: and function $f(x)$ can be by represented by $S(x)$ *almost everywhere*:

$$f(x) = S(x),$$

except at isolated discontinuities in f. [More of this below]

5.2 Form of the Fourier Coefficients

Simply stating $f(x) = S(x)$ leaves us with the task of finding the a_j's and b_j's. But this is not so hard, if we use the basic properties of sines and cosines outlined in Section 4.5, and, in particular, the property that the individual sines and cosines in the series are orthogonal to each other.

Recipe for generating the a_j's and b_j's

This is the key part of Fourier Series, so make sure you understand this and can implement the recipe.

1. Replace $S(x)$ by $f(x)$ in Section 5.1 for the moment.

2. Multiply both sides by 1, yes 1!, and integrate over $[-l, l)$: all the component functions, i.e. $\cos(j\pi x/l)$ and $\sin(j\pi x/l)$ for $j \geq 1$, are orthogonal to 1 except 1 itself, and $< 1, 1 >= L = 2l$. So

$$< f(x), 1 >= \frac{a_0}{2} < 1 >= la_0,$$

and, hence,

$$a_0 = \frac{1}{l} < f(x), 1 >= \frac{1}{l}\int_{-l}^{l} f(x)dx \tag{5.1}$$

a_0 **is TWICE the average value of** $f(x)$ **over** $[-l, l)$. This explains $a_0/2$ in $S(x)$. $a_0/2$ is the average value of $f(x)$ over $[-1, 1)$.

3. Multiply both sides of the $S(x)$ formula by $\cos\frac{n\pi x}{l}, n \geq 1$ (integer). This will generate all the coefficients a_n for $n \geq 1$. Remember that the original summation covers all values of j from 0 to ∞. Thus, there is one term in the infinite series where $j = n$. That is the coefficient we are trying to calculate. Again, all component functions in $S(x)$ are orthogonal to $\cos\frac{n\pi x}{l}$ except for $\cos\frac{n\pi x}{l}$ itself! Hence,

$$\langle f(x), \cos\frac{n\pi x}{l}\rangle = a_n\langle\cos\frac{n\pi x}{l}, \cos\frac{n\pi x}{l}\rangle = a_n l$$

the only surviving coefficient.

$$\Rightarrow a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx \tag{5.2}$$

[Note that this also works also for $n = 0$.]

4. Repeat previous section for $\sin\frac{n\pi x}{l}$, $n \geq 1$.

$$\Rightarrow b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx. \tag{5.3}$$

5. Replace n by j and use in the expression for $S(x)$, or replace j by n there:

$$S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}.$$

This is equivalent to

$$S(x) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j\cos\frac{j\pi x}{l} + b_j\sin\frac{j\pi x}{l}.$$

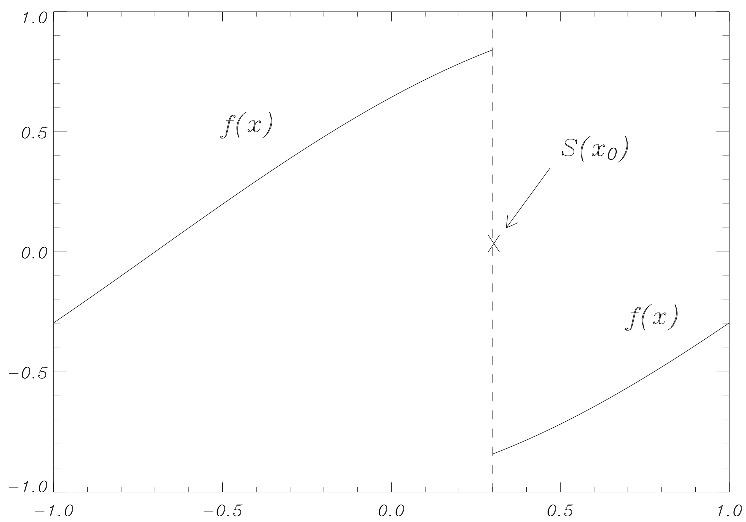

Figure 5.1: Example of a discontinuous function. The Fourier Series takes the average value of $f(x)$ at x_0.

6. Check the discontinuities in $f(x)$. Since $S(x) = f(x)$ everywhere except at the discontinuities (if there are any), *due to the completeness of the Fourier basis functions*, then $S(x) = f(x)$ arbitrarily close to either side of any discontinuity. By convention, we use the average value of f on either side of a discontinuity, say at $x = x_0$, as the value of S *at* the discontinuity

$$S(x_0) = \frac{1}{2}\left(\lim_{x \to x_0^-} f(x) + \lim_{x \to x_0^+} f(x) \right).$$

[This, by the way, can be more useful than you might imagine - see below.]

That is all there is to it.

5.3 Linear Superposition

This simply means we can add two Fourier series $S_1(x)$ and $S_2(x)$ and obtain another Fourier series, with the corresponding a_n's and b_n's added together.

This is useful, for instance, if say you know the Fourier series for one part, $f_1(x)$, of

$$f(x) = f_1(x) + f_2(x),$$

and want to focus on the other part, or if you want to tackle the two parts separately.

The Fourier Series, $S(x)$, for $f(x)$ is just $S(x) = S_1(x) + S_2(x)$. We'll demonstrate this in the next examples.

In addition, linear superposition of two Fourier Series can be used to prove (not given here but fairly straightforward) that the Fourier Series, $S(x)$, of a given function $f(x)$ is *unique*. Hence, there is only one way of generating the Fourier Series for $f(x)$.

5.4 Evaluating the coefficients: Some examples

Example 5.8

$f(x) = \cos^2 x$

Solution 5.8

We've seen this in section 1.6

$$f(x) = \frac{1}{2}(1 + \cos 2x)$$

so

$$L = \frac{2\pi}{2} = \pi \Rightarrow l = \frac{L}{2} = \frac{\pi}{2}.$$

Key: try to recognise that this is dead simple: f is already in the form of a Fourier Series! Just write down the coefficients by inspection!!

$$a_0 = 1, \qquad a_2 = \frac{1}{2},$$

and all other coefficients are *zero*.

Example End

Example 5.9

$f(x) = 3\sin 2x - 2\sin 3x$ - again this is an example from earlier. Recall that the fundamental period is $L = 2\pi$.

Solution 5.9

Anyway, again there is no work for this example: just write down the coefficients by inspection!

$$b_2 = 3, \qquad b_3 = -2,$$

all other coefficients are *zero*!

Example End

Example 5.10

$f(x) = x$ over $[-1, 1)$. The periodic extension of this function is shown in Figure 5.2.

Solution 5.10

Not all Fourier Series can be derived easily! This one involves an infinite series but the integration only uses standard techniques. We derive the coefficients in turn using our recipe.

Step 2

Let's start with a_0 but remember that the periodic length is 2 and so $l = 1$:

$$a_0 = \int_{-1}^{1} f(x)dx = \int_{-1}^{1} xdx = \left[\frac{1}{2}x^2\right]_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0,$$

but we could have guessed this since x is an odd function about $x = 0$.

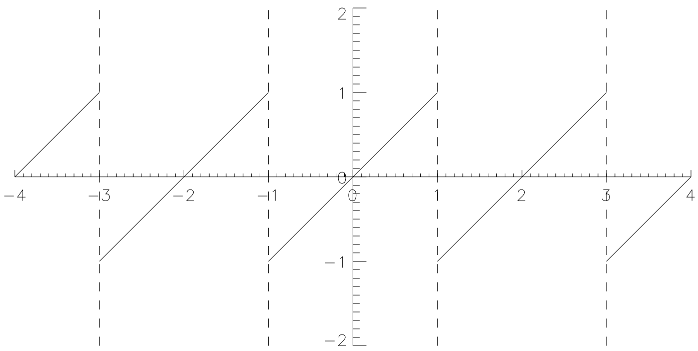

Figure 5.2: Periodic extension of $f(x) = x$. Note the discontinuities at $x = \pm 1$.

Step 3

Moving on, we calculate the general coefficient, a_n by multiplying by $\cos n\pi x$ and integrating from -1 to 1. Thus,

$$a_n = \int_{-1}^{1} f(x) \cos n\pi x dx = \int_{-1}^{1} x \cos n\pi x dx = 0.$$

Don't do unnecessary calculations: *ODD functions ALWAYS integrate to zero* over $[-l, l)$. You can check this by using integration by parts.

Step 4

So, we are finally left with

$$b_n = \int_{-1}^{1} f(x) \sin(n\pi x) dx = \int_{-1}^{1} x \sin(n\pi x) dx$$

This integration cannot be immediately done. However, it is easily done, using integration by parts.

Detour: integration by parts

The way to tackle these integrals, which arise *all too frequently* in the evaluation of the a_n's and b_n's, is to exploit the general formula for integration by parts,

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx. \tag{5.4}$$

Back to

$$b_n = \int_{-1}^{1} x \sin(n\pi x)$$

To use (5.4), we set $u(x) = x$ and $\frac{dv}{dx} = \sin(n\pi x)$. Thus, $\frac{du}{dx} = 1$ and $v = -\frac{1}{n\pi}\cos(n\pi x)$. So

$$\begin{aligned} b_n &= \left[x\left(-\frac{1}{n\pi}\cos(n\pi x)\right)\right]_{-1}^{1} - \int_{-1}^{1}\left(-\frac{1}{n\pi}\cos(n\pi x)\right).1.dx \\ &= -\frac{\cos(n\pi)}{n\pi} - \frac{\cos(-n\pi)}{n\pi} + \frac{1}{n\pi}\int_{-1}^{1}\cos n\pi x dx \\ &= -\frac{2\cos(n\pi)}{n\pi} \\ &= -\frac{2(-1)^n}{n\pi} \end{aligned}$$

Remember, $\cos n\pi = (-1)^n$.

Step 5

We are nearly finished now. All we do is insert these b_n into the summation, $S(x)$ [recall all $a_n = 0$] to get

$$\begin{aligned} S(x) &= -2\sum_{n=1}^{\infty}(-1)^n\frac{\sin(n\pi x)}{n\pi} \\ &= \frac{2}{\pi}[\frac{\sin \pi x}{1} - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} - \cdots] \end{aligned}$$

Step 6

$S(x) = x$ everywhere *except* at $x = 1$, where x is discontinuous. Thus, at the discontinuity at $x = 1$,

$$S(1) = \frac{2}{\pi}\left[\sin\pi - \frac{\sin 2\pi}{2} + \frac{\sin 3\pi}{3} - \cdots\right] = \frac{2}{\pi}[0 - 0 + 0 - 0 + \cdots] = 0.$$

This is correct as $S(1^+) + S(1^-) = (-1) + 1 = 0$. Hence, the Fourier Series correctly obtains the average value at the discontinuity. The function $f(x) = x$ and the Fourier Series, $S(x)$ (truncated to $5, 10$ and 15 terms) is shown in Figure 5.3.

Example End

Notice that the coefficients decay towards zero only as $1/n$. This behaviour of the Fourier Series coefficients is typical of functions which have a discontinuity. It does mean that the Fourier Series only converges slowly towards $f(x)$.

Now consider a function, $f(x)$ that is continuous everywhere. In this case it is piecewise continuous meaning that the derivative is not continuous.

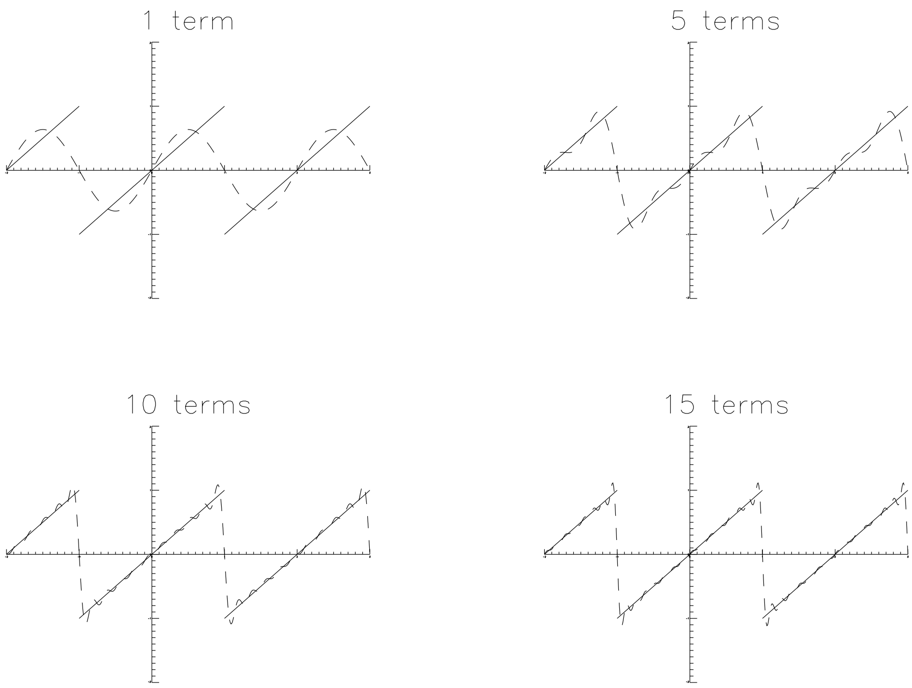

Figure 5.3: Fourier Series shown as dashed curves for 1, 5, 10 and 15 terms.

Example 5.11

$$f(x) = |x| \quad \text{over} \quad [-1, 1) \quad : \quad l = 1$$

Solution 5.11

Now compute the Fourier coefficients. Remembering that $l = 1$ and

$$|x| = \begin{cases} x & x \geq 0, \\ -x & x < 0, \end{cases}$$

we start with:

Step 2

$$\begin{aligned} a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx &= \int_{-1}^{1} |x|\, dx \\ &= 2\int_0^1 x dx, \qquad \text{by symmetry or by definition of } |x| \\ &= 2\left[\frac{1}{2}x^2\right]_0^1 \\ &= 1 \end{aligned}$$

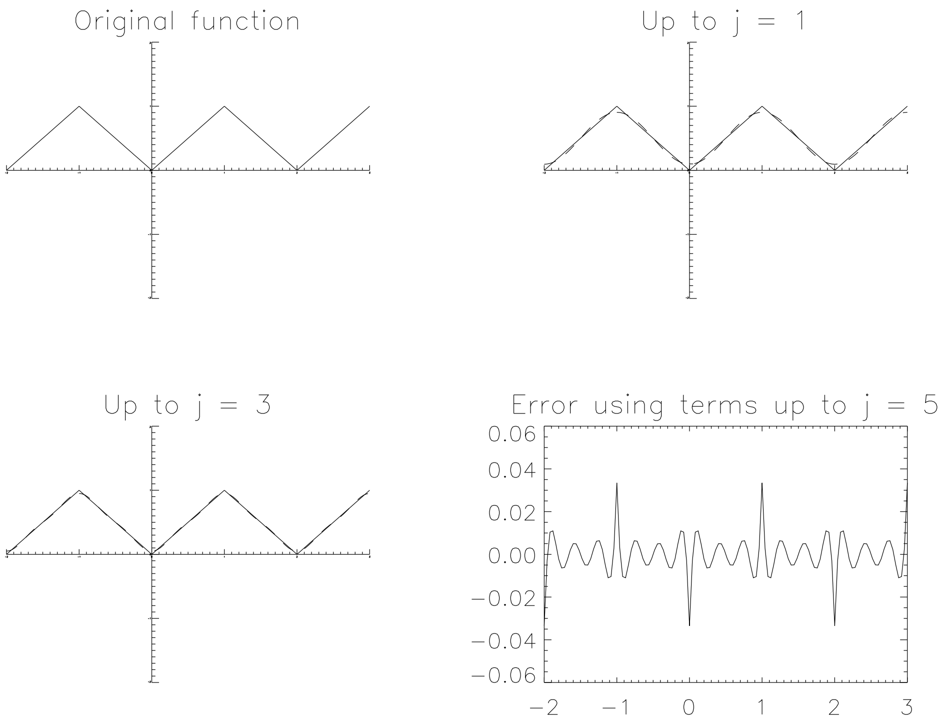

Figure 5.4: The periodic extension of $f(x) = |x|$ and the Fourier Series approximations up to $j = 1$ and $j = 3$. The error $f(x) - S(x)$ is shown for terms up to $j = 5$.

This makes sense since the average value of $f(x) = \frac{a_0}{2} = \frac{1}{2}$, which is clear geometrically.

Step 3

Now let's tackle a_n. Multiply by $\cos(n\pi x)$ and integrate from $x = -1$ to $x = 1$. Thus,

$$
\begin{aligned}
a_n &= \int_{-1}^{1} |x| \cos(n\pi x) dx \\
&= \int_{-1}^{0} (-x) \cos(-n\pi x) dx + \int_{0}^{1} x \cos(n\pi x) dx \\
&= -\int_{1}^{0} s \cos(n\pi s) ds + \int_{0}^{1} x \cos(n\pi x) dx, \quad \text{using the substitution } s = -x \\
&= \int_{0}^{1} s \cos(n\pi s) ds + \int_{0}^{1} x \cos(n\pi x) dx \\
&= 2\int_{0}^{1} x \cos(n\pi x) dx \qquad \text{since } s \text{ is a dummy variable and can be replaced by } x.
\end{aligned}
$$

Note that $|x|\cos(n\pi x)$ is an EVEN function about $x = 0$ and we could have got to the last line almost immediately. Now we use integration-by-parts:

$$\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}dx$$

Take $u = x$ and $\frac{dv}{dx} = \cos(n\pi x)$ so that $\frac{du}{dx} = 1$ and $v = \frac{1}{n\pi}\sin(n\pi x)$. Therefore,

$$\begin{aligned}
a_n &= 2\left\{x\left[\frac{1}{n\pi}\sin(n\pi x)\right]_0^1 - \int_0^1\left(\frac{1}{n\pi}\sin(n\pi x)\right).1.dx\right\} \\
&= 2\left(\frac{1}{n\pi}\sin(n\pi) - \frac{1}{n\pi}\int_0^1 \sin(n\pi x)dx\right) \\
&= \frac{2}{(n\pi)^2}\left[\cos(n\pi x)\right]_0^1 \\
&= \frac{2}{(n\pi)^2}[\cos(n\pi) - \cos 0] \\
&= \frac{2}{(n\pi)^2}[(-1)^n - 1] \\
&= 0 \quad \text{for } n \text{ even} \\
&= -\frac{4}{(n\pi)^2} \quad \text{for } n \text{ odd}
\end{aligned}$$

So, $a_n = 0$ for all positive even integers n and $-\frac{4}{(n\pi)^2}$ for all the odd integers. Don't forget $a_0 = 1$

Step 4

Now we have to compute the b_n, or do we?

$$b_n = \int_{-1}^{1} |x|\sin(n\pi x)dx = 0\ !$$

There is NO WORK TO DO! Just remember that the integral of an ODD function is *ALWAYS* zero over $[-l, l)$.

Step 5

So, finally, we construct our Fourier Series:

$$\begin{aligned}
S(x) &= \frac{1}{2} - 4\sum_{n \text{ odd}}\frac{\cos(n\pi x)}{(n\pi)^2} && (5.5) \\
&= \frac{1}{2} - \frac{4}{\pi^2}\left[\cos\pi x + \frac{\cos 3\pi x}{9} + \frac{\cos 5\pi x}{25} + \cdots\right] && (5.6)
\end{aligned}$$

Now here we have something rather nice! We know $S(x) = |x|$ everywhere over $[-1, 1)$. Let's then consider $S(1)$ which should be 1 of course from the original function:

$$1 = \frac{1}{2} - \frac{4}{\pi^2}[\cos\pi + \frac{\cos 3\pi}{9} + \frac{\cos 5\pi}{25} + ..]$$

But $\cos\pi = \cos 3\pi = \cos 5\pi = \cos n\pi = -1$, for odd integers n. So

$$\frac{1}{2} = \frac{4}{\pi^2}[1 + \frac{1}{9} + \frac{1}{25} + \cdots]$$

or, re-arranging,

$$1 + \frac{1}{9} + \frac{1}{25} + .. = \frac{\pi^2}{8}, \tag{5.7}$$

This is something you surely would not have guessed! *Moral*: Fourier Series can sometimes provide answers to unexpected problems!

Figure 5.3 shows how the Fourier Series quickly converges to the original function. Note that for continuous functions the coefficients decay like $1/n^2$ or faster. Again this is typical and indicates that continuity is *good*.

Example End

Example 5.12

Consider the function

$$f(x) = \begin{cases} x & x \geq 0, \\ 0 & x < 0, \end{cases}$$

defined over the domain $[-1, 1)$. This is shown in Figure 5.5.

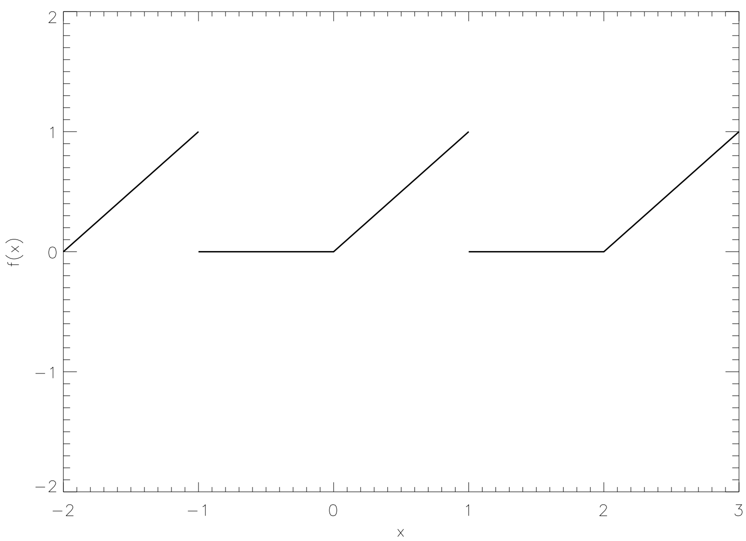

Figure 5.5: $f(x) = x$ for $x > 0$ and 0 for $x < 0$.

Solution 5.12

OK, there are two options, namely work out all the integrals as before or use previous results.

How do we use the previous results? Firstly, we must recognise that $f(x) = \frac{1}{2}(x+|x|)$,

$$\frac{1}{2}(x+|x|) = \begin{cases} \frac{1}{2}(x+x) = x & x \geq 0, \\ \frac{1}{2}(x-x) = 0 & x < 0. \end{cases}$$

So, by linear superposition, $S(x) = \frac{1}{2}(S_{(1)}(x)+S_{(2)}(x))$ where $S_{(1)}$ and $S_{(2)}$ are the Fourier Series for x and $|x|$ that were found earlier.

Therefore,

$$S(x) = \frac{1}{4} - 2\sum_{n \text{ odd}} \frac{\cos(n\pi x)}{(n\pi)^2} - \sum_{n=1}^{\infty}(-1)^n \frac{\sin(n\pi x)}{n\pi}$$

Using previous results and linear superposition, there was almost no work at all.

Example End

Example 5.13

$f(x) = e^x$ over $[-l, l)$, (l is arbitrary but finite). Note:

1. $f(x)$ is discontinuous at $x = l$ (and obviously at $x = -l$).

2. f is neither even nor odd: it is a *mixed* function. This means that we need to compute both the a's and the b's!

Solution 5.13

Step 2

Start with $a_0 = \frac{1}{l}\int_{-l}^{l} e^x dx$. But $\frac{d}{dx}(e^x) = e^x$, so

$$\begin{aligned} a_0 &= \frac{1}{l}e^x \,|_{-l}^{l} \\ &= \frac{1}{l}(e^l - e^{-l}) \end{aligned}$$

This looks like a *hyperbolic function*

$$\begin{aligned} \cosh l &= \frac{1}{2}(e^l + e^{-l}) \\ \sinh l &= \frac{1}{2}(e^l - e^{-l}) \\ \Rightarrow \qquad a_0 &= \left(\frac{2}{l}\right)\sinh l. \end{aligned}$$

Step 3

That was reasonable. How about the a_n's?

$$a_n = \frac{1}{l}\int_{-l}^{l} e^x \cos\frac{n\pi x}{l} dx?$$

This is simple by expressing $\cos(n\pi x/l)$ as a complex number but we will use integration-by-parts (repeatedly):

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

with $u = e^x$ and $\frac{dv}{dx} = \cos\frac{n\pi x}{l}$ Thus, $\frac{du}{dx} = e^x$ and $v = \frac{l}{n\pi}\sin\frac{n\pi x}{l}$. Hence,

$$\begin{aligned} a_n &= \frac{1}{l}\left\{\left[e^x\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right)\right]_{-l}^{l} - \frac{l}{n\pi}\int_{-l}^{l} e^x \sin\frac{n\pi x}{l}dx\right\} \\ &= \frac{l}{n\pi}[e^l\sin(n\pi) - e^{-l}\sin(-n\pi)] - \frac{1}{n\pi}\int_{-l}^{l} e^x\sin\frac{n\pi x}{l}dx \\ &= -\frac{1}{n\pi}\int_{-l}^{l} e^x \sin\frac{n\pi x}{l}dx \end{aligned}$$

Note that we have 'converted' the cosine term into a sine term. We need to use integration-by-parts again to generate a multiple of the original integral! Use $u = e^x$ and $\frac{dv}{dx} = -\sin\frac{n\pi x}{l}$ Thus, $\frac{du}{dx} = e^x$ and $v = \frac{l}{n\pi}\cos\frac{n\pi x}{l}$ Hence,

$$a_n = \frac{1}{n\pi}\left\{\left[e^x\left(\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right)\right]_{-l}^{l} - \frac{l}{n\pi}\int_{-l}^{l} e^x\cos\frac{n\pi x}{l}dx\right\}$$

But $\int_{-l}^{l} e^x \cos\frac{n\pi x}{l}dx = la_n$ is our original integral. We have almost gone in circles but not quite! What we have is a multiple of the original integral.

$$\begin{aligned} a_n &= \frac{1}{n\pi}\left[\frac{l}{n\pi}\left(e^l\cos(n\pi) - e^{-l}\cos(-n\pi)\right)\right] - \left(\frac{l}{n\pi}\right)^2 a_n \\ &= \frac{l}{(n\pi)^2}(-1)^2\left(e^l - e^{-l}\right) - \left(\frac{l}{n\pi}\right)^2 a_n \end{aligned}$$

So, collecting all the terms in a_n onto the left hand side of the equation, gives

$$\left[1 + \left(\frac{l}{n\pi}\right)^2\right] a_n = \frac{l}{(n\pi)^2}(-1)^n\left(e^l - e^{-l}\right)$$

Multiply both sides by $(n\pi)^2$:

$$[(n\pi)^2 + l^2]a_n = 2l(-1)^n\sinh l \tag{5.8}$$

$$a_n = \frac{2l(-1)^n\sinh l}{(n\pi)^2 + l^2} \tag{5.9}$$

Note, in this example $n = 0$ gives the same value calculated from $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$ - but this is *NOT* always true! Be careful.

Step 4

Next, we turn to the b_n's. These are done in exactly the same manner.

$$b_n = \frac{1}{l}\int_{-l}^{l} e^x \sin\frac{n\pi x}{l} dx.$$

Well, we've just done this in part. Let's do it again anyway: Use $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$ with $u = e^x$ and $\frac{dv}{dx} = \sin\frac{n\pi x}{l}$ So $\frac{du}{dx} = e^x$ and $v = -\frac{l}{n\pi}\cos\frac{n\pi x}{l}$ Thus,

$$b_n = \frac{1}{l}\left\{\left[e^x\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right)\right]_{-l}^{l} + \frac{l}{n\pi}\int_{-l}^{l} e^x \cos\frac{n\pi x}{l}dx\right\}$$

Note, in the integral on the right hand side we have $-(-) = +$. So

$$\begin{aligned}
b_n &= -\frac{1}{n\pi}\left[e^x\cos\frac{n\pi x}{l}\right]_{-l}^{l} + \frac{l}{n\pi}a_n \\
&= -\frac{1}{n\pi}\left(e^l\cos n\pi - e^{-l}\cos(-n\pi)\right) + \frac{l}{n\pi}a_n \\
&= -\frac{(-1)^n}{n\pi}(e^l - e^{-l}) + \frac{l}{n\pi}a_n \\
&= -\frac{2(-1)^n\sinh l}{n\pi} + \frac{l}{n\pi}a_n \\
&= \frac{1}{n\pi}[la_n - 2(-1)^n\sinh l]
\end{aligned}$$

or, using our result for a_n

$$\begin{aligned}
b_n &= \frac{1}{n\pi}\left[\frac{2l(-1)^n\sinh l}{(n\pi)^2 + l^2} - 2(-1)^n\sinh l\right] \\
&= \frac{2(-1)^n\sinh l}{n\pi}\left[\frac{l^2}{(n\pi)^2 + l^2} - 1\right] \\
&= \frac{2(-1)^n\sinh l}{n\pi}\left[\frac{l^2 - ((n\pi)^2 + l^2)}{(n\pi)^2 + l^2}\right] \\
&= \frac{2(-1)^n\sinh l}{n\pi}\left[-\frac{(n\pi)^2}{(n\pi)^2 + l^2}\right]
\end{aligned}$$

So, finally!

$$b_n = -\frac{2n\pi(-1)^n\sinh l}{(n\pi)^2 + l^2}.$$

Step 5

Now put them back into $S(x)$:

$$\begin{aligned}
S(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l} \\
&= \frac{\sinh l}{l} + 2\sinh l\sum_{n=1}^{\infty}\frac{(-1)^n}{(n\pi)^2 + l^2}\left(l\cos\frac{n\pi x}{l} - n\pi\sin\frac{n\pi x}{l}\right)
\end{aligned}$$

$S(x) = e^x$ everywhere except at the discontinuity, $x = l$. There $S(1) = \frac{1}{2}(e^l + e^{-l}) = \cosh l$.

This example is illustrated in Figure 5.6 for $l = 1$.

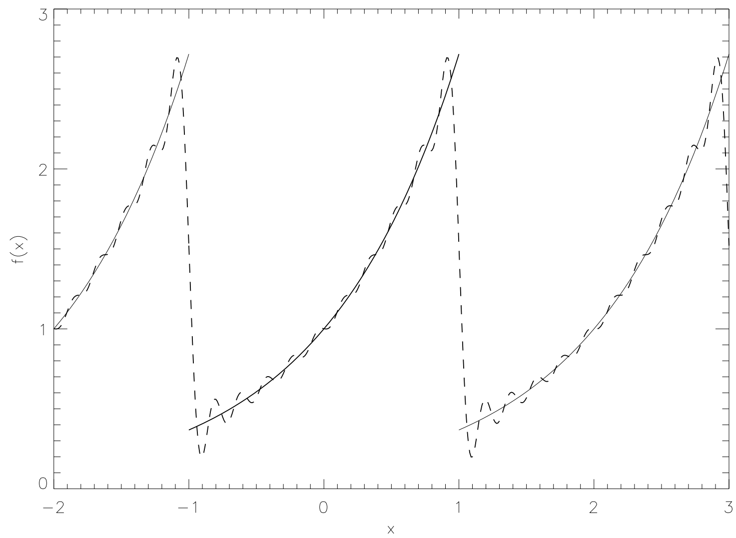

Figure 5.6: The solid curve shows $f(x) = e^x$ for $-1 < x \leq 1$ and the dashed curves shows the Fourier Series including all terms up to $j = 10$.

Example End

5.5 Simplifications for odd and even functions

In the previous section, we have already seen how Fourier Series can simplify if the function $f(x)$ is even or odd. Here's a reminder:

	EVEN functions	ODD functions
Definition	$f(-x) = f(x)$	$f(-x) = -f(x)$
Examples	x^2, $\lvert x\rvert$, $\cos x$	x, x^3, $\sin x$
Zero coefficients	b_n	a_n
Fourier Series	$S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$	$S(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$

Table 5.1: Which coefficients are zero for which type of function

In both cases, i.e. for a *pure* even or a *pure* odd function, you can save yourself work in evaluating the coefficients a_n or b_n by integrating *ONLY* from 0 to l and *doubling* the result:

1. *EVEN* functions

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx = \frac{2}{l}\int_{0}^{l} f(x)\cos\frac{n\pi x}{l}dx$$

2. *ODD* functions

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx = \frac{2}{l}\int_{0}^{l} f(x)\sin\frac{n\pi x}{l}dx$$

5.6 Half-range Fourier Series

Sometimes a function may be given *only* over the *half range*, say $[0, l)$ when we really need the function to be defined over $[-l, l)$. We could either shift it to lie in $[-l/2, l/2)$ as in section 1.3, or we can extend it as either a pure even function or a pure odd function:

- EVEN extension: $f(-x) = f(x)$
- ODD extension: $f(-x) = -f(x)$

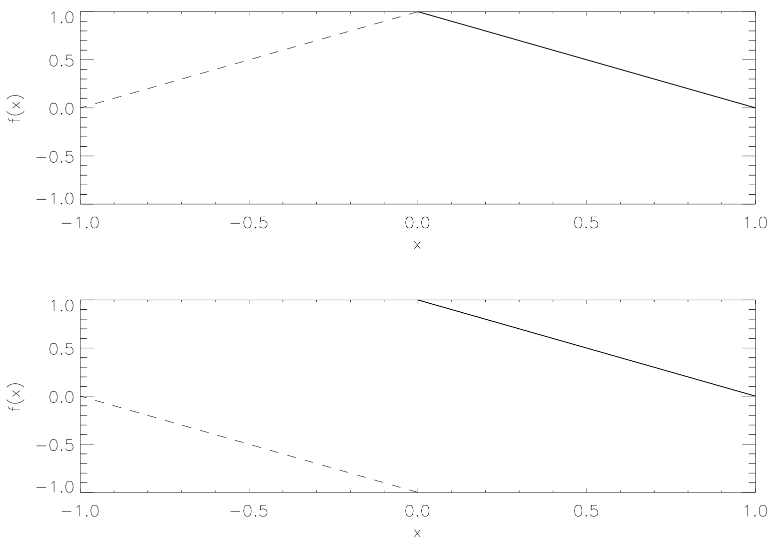

Figure 5.7: Even and odd extensions of $f(x) = 1 - x$ for $0 \leq x < 1$ into the domain $-1 \leq x < 1$.

Figure 5.7 is an example of even and odd extensions of a function. The even extension is sensible because the function is continuous at $x = 0$. This will converge quite rapidly. The odd extension is not sensible because it introduces a discontinuity and that, as we have already seen, will mean that the series will converge more slowly.

In each case, the Fourier Series (cosine series for even extension and sine series for odd extension) will give $S(x) = 1 - x$ over $(0, 1)$, despite having different forms (sums of cosines versus sums of sines)

In practice, you don't have to explicitly extend the function in order to find the Fourier Series. Simply deciding on an even or an odd extension before evaluating the coefficients is sufficient, since we can use the ideas outlined in section 5.5 to do the work.

Let us then go through the steps for $f(x) = 1 - x$, defined over $[0, 1)$.

There are two cases:

1. EVEN extension

   All $b_n = 0$ and the a_n are found from

$$\begin{aligned} a_n &= \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx \\ &= 2\int_0^1 (1-x)\cos(n\pi x)dx, \end{aligned}$$

   where $l = 1$, except for

$$\begin{aligned} a_0 &= 2\int_0^1 (1-x)dx \\ &= 2\left[x - \frac{1}{2}x^2\right]_0^1 = 1. \end{aligned}$$

   Returning to a_n, for $n > 0$,

$$a_n = 2\int_0^1 \{\cos(n\pi x) - x\cos(n\pi x)\}\, dx.$$

   So

$$\begin{aligned} a_n &= 2\left[\frac{1}{n\pi}\sin(n\pi x)\right]_0^1 - 2\int_0^1 x\cos(n\pi x)dx \\ &= 0 - 2\int_0^1 x\cos(n\pi x)dx \end{aligned}$$

   We tackle this integral by again using integration-by-parts,

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx. \tag{5.10}$$

   Choose $u = x$ and $\frac{dv}{dx} = \cos(n\pi x)$, so that $\frac{du}{dx} = 1$ and $v = \frac{1}{n\pi}\sin(n\pi x)$. Then

$$\begin{aligned} a_n &= -2\left\{\left[\frac{x}{n\pi}\sin(n\pi x)\right]_0^1 - \frac{1}{n\pi}\int_0^1 \sin(n\pi x)dx\right\} \\ &= \frac{-2}{(n\pi)^2}\cos(n\pi x)\,|_0^1 = \frac{-2}{(n\pi)^2}[\cos(n\pi) - \cos(0)] \\ &= \frac{-2}{(n\pi)^2}\left((-1)^n - 1\right) \end{aligned}$$

Hence, putting this into the Fourier *Cosine* Series, we have the *half-range* expansion

$$\begin{aligned} S(x) &= \frac{1}{2} + 4\sum_{n \text{ odd}}^{\infty} \frac{\cos(n\pi x)}{(n\pi)^2} \\ &= \frac{1}{2} + \frac{4}{\pi^2}\left(\cos(\pi x) + \frac{\cos(3\pi x)}{9} + \frac{\cos(5\pi x)}{25} + \cdots\right) \end{aligned}$$

which might seem familiar from one of our earlier examples. The extended even function here can be written $1 - |x|$, and the Fourier Series was derived for $|x|$.

Recall in this case, $S(x) = f(x) = 1 - x$ over $[0, 1]$ and the even extension has no discontinuities in the extension of f, making the extended function $f(x) = 1 - |x|$ over $[-1, 1)$.

2. Now consider the silly case: ODD extension. This is not a sensible extension to the function as it introduces a discontinuity at $x = 0$. Anyway, from the simplifications in section 5.5, we know all $a_n = 0$ while the b_n may be computed from

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx = 2\int_0^1 (1-x)\sin(n\pi x)dx \tag{5.11}$$

OK, let's do it!

$$\begin{aligned} b_n &= 2\int_0^1 [\sin(n\pi x) - x\sin(n\pi x)]dx \\ &= -\frac{2}{n\pi}\left[\cos(n\pi x)\right]_0^1 - 2\int_0^1 x\sin(n\pi x)dx \\ &= -\frac{2}{n\pi}[\cos(n\pi) - \cos(0)] - 2\int_0^1 x\sin(n\pi x)dx \\ &= +\frac{2}{n\pi}(1 - (-1)^n) - 2\int_0^1 x\sin(n\pi x)dx \end{aligned}$$

The last integral must be done by integration by parts. Take $u = x$ and $\frac{dv}{dx} = -\sin(n\pi x)$ so that $\frac{du}{dx} = 1$ and $v = \frac{1}{n\pi}\cos(n\pi x)$

Then

$$\begin{aligned} b_n &= \frac{2}{n\pi}(1 - (-1)^n) + 2\left\{\left[\frac{x}{n\pi}\cos(n\pi x)\right]_0^1 - \frac{1}{n\pi}\int_0^1 \cos(n\pi x)dx\right\} \\ &= \frac{2}{n\pi}(1 - (-1)^n) + 2\left\{\frac{1}{n\pi}\cos(n\pi) - 0 - \left[\frac{1}{(n\pi)^2}\sin(n\pi x)\right]_0^1\right\} \\ &= \frac{2}{n\pi} \end{aligned}$$

The final result is nice and simple due to cancellations!

It remains to slot these coefficients into our Fourier *Sine* Series for the *half-range* expansion for the *odd* extension of $f(x)$. Thus,

$$\begin{aligned} S(x) &= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \\ &= \frac{2}{\pi}\sum \frac{\sin(n\pi x)}{n}. \end{aligned}$$

At $x = 0$, $S(x) = 0$ which is indeed the average of the limits of $f(x)$ (i.e. its odd extensions) from either side of $x = 0$.

While this Fourier Series looks considerably simpler than that found in the even extension of $f(x)$, in practice it converges considerably more slowly (as $1/n$ as opposed to as $1/n^2$ for the even extension) with increasing number of coefficients retained in the sum. This is illustrated in Figure 5.8, where $f(x) = 1 - x$ is approximated by the half-range cosine series and the half-range sine series. Notice that the cosine series gives a better representation because the even extension is continuous.

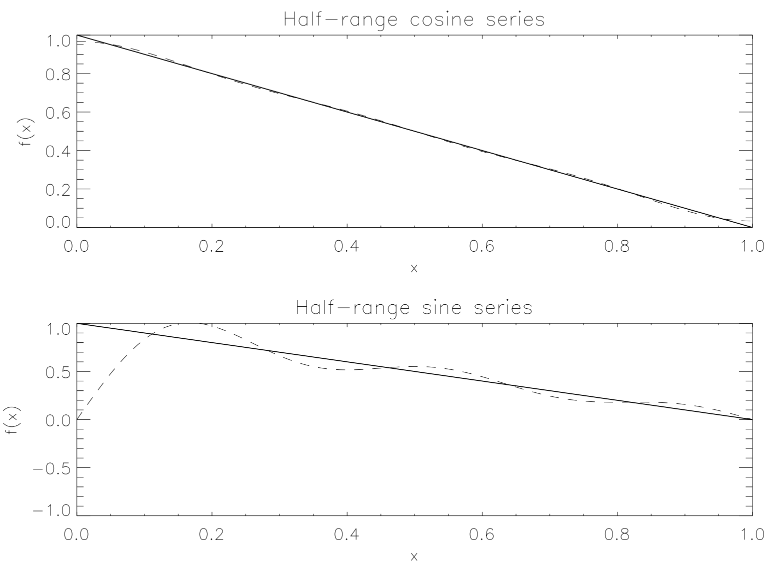

Figure 5.8: Fourier Series for even and odd extensions of $f(x) = 1-x$ for $0 \leq x < 1$ into the domain $-1 \leq x < 1$. Both series are taken up to terms $j = 5$.

5.7 Differentiation of a Fourier Series

For *smooth* functions $f(x)$, which have a Fourier Series, $S(x)$, it is generally safe to differentiate $S(x)$ to obtain the Fourier Series for $f' = \frac{df}{dx}$.

Hence if $S(x) = \frac{a_0}{2} + \sum a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)$ is the Fourier Series for $f(x)$, then

$$S'(x) = \sum a_n^* \cos\left(\frac{n\pi x}{l}\right) + b_n^* \sin\left(\frac{n\pi x}{l}\right) \tag{5.12}$$

is the Fourier Series for $f'(x) = df/dx$, if we identify $a_n^* = \frac{n\pi}{l} b_n$ and $b_n^* = -\frac{n\pi}{l} a_n$.

Note: $a_0^* = 0$.

Alas, many functions are not smooth, and you will generate a non-converging series if you differentiate any function that has a discontinuity. So, there is often a limit to how many times you can differentiate f and hence S.

Differentiation is useful for solving linear differential equations of the form

$$y'' + ay' + by = F(x) \tag{5.13}$$

over a periodic domain. The whole left hand side can be expressed as a Fourier Series, with coefficients to be determined, using the above derivative formulae. The r.h.s. can also be expressed as a Fourier series, with coefficients A_n, B_n. The solution then simply requires matching the coefficients of $\cos\left(\frac{n\pi x}{l}\right)$ and $\sin\left(\frac{n\pi x}{l}\right)$ on both sides.

Chapter 6

Fourier Series using Complex Notation

Using Euler's formula an alternative expression for the Fourier series of a function $f(x)$ can be found. With

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix}), \quad \sin x = \frac{1}{2i}(e^{ix} - e^{-ix}) = -\frac{i}{2}(e^{ix} - e^{-ix}),$$

we can rewrite a Fourier series in the following way:

$$\begin{aligned}
S(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \\
&= \frac{a_0}{2} + \frac{1}{2}\sum_{n=1}^{\infty} a_n \left[e^{i\frac{n\pi x}{l}} + e^{-i\frac{n\pi x}{l}}\right] - ib_n \left[e^{i\frac{n\pi x}{l}} - e^{-i\frac{n\pi x}{l}}\right] \\
&= \frac{a_0}{2} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n - ib_n)e^{i\frac{n\pi x}{l}} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n + ib_n)e^{-i\frac{n\pi x}{l}}.
\end{aligned}$$

Not much seems to have been gained so far, but we now simplify things in the following. First we change the summation variable n in the final sum to $-n$. Now the sum runs from $n = -1$ (previously $n = 1$) to $-\infty$ (previously ∞), but we can change the order of summation (which we can always do in a sum) to go from $-\infty$ to -1. The expression for $S(x)$ then is

$$S(x) = \frac{a_0}{2} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n - ib_n)e^{i\frac{n\pi x}{l}} + \frac{1}{2}\sum_{n=-\infty}^{-1}(a_{-n} + ib_{-n})e^{i\frac{n\pi x}{l}}.$$

Now the exponential functions in both sums look the same, but in the first sum the summation is carried out over the positive integers, whereas in the second sum the summation is over the negative integers!

Next we redefine the coefficients as

$$A_n = \frac{a_n - ib_n}{2}$$

for $n \geq 1$ and

$$A_n = A^*_{-n}$$

for $n \leq -1$, where A^* is the complex conjugate of A. We also define

$$A_0 = \frac{a_0}{2}$$

We then have

$$\begin{aligned} S(x) &= A_0 + \sum_{n=1}^{\infty} A_n e^{i\frac{n\pi x}{l}} + \sum_{n=-\infty}^{-1} A_n e^{i\frac{n\pi x}{l}} \\ &= \sum_{n=-\infty}^{\infty} A_n e^{i\frac{n\pi x}{l}}, \end{aligned}$$

with $A_n = A^*_{-n}$ for all n. This condition ensures that $S(x)$ is real, but can be relaxed if one would want to represent a complex valued function by a Fourier series.

The last infinite sum looks a lot more compact then our original expression with cosine functions, sine functions and a constant term. To get $S(x)$ into this form we had to use a summation over *all* integers (positive and negative) and complex coefficients A_n.

The next question to answer would be how we can determine the coefficients A_n from $f(x)$. As before we will use orthogonality to get an expression for A_n. Because our basis functions $\exp(in\pi x/l)$ are now complex valued functions, the scalar (inner) product has to be slightly modified to

$$< f(x), g(x) >= \int_{-l}^{l} f(x)g^*(x)dx.$$

So, here we use the *complex conjugate* of the second function in the inner product. Note that for real valued functions f and g this definition of the scalar product is the same as the one used before. For complex valued functions, however, we note that, for example, the order of the functions in the scalar product matters:

$$< f(x), g(x) >= \int_{-l}^{l} f(x)g^*(x)dx \neq < g(x), f(x) >= \int_{-l}^{l} f^*(x)g(x)dx,$$

but we have

$$< f(x), g(x) >=< g(x), f(x) >^* .$$

Now let us use this scalar product to establish the orthogonality of the complex exponential functions used in the Fourier series. We start with the case $n \neq m$, where n and

m can be any integers (negative, positive or zero). Then

$$\begin{aligned}
< e^{i\frac{n\pi x}{l}}, e^{i\frac{m\pi x}{l}} > &= \int_{-l}^{l} e^{i\frac{n\pi x}{l}} e^{-i\frac{m\pi x}{l}} dx \\
&= \int_{-l}^{l} e^{i\frac{(n-m)\pi x}{l}} dx \\
&= \frac{l}{i(n-m)\pi} \left[e^{i\frac{(n-m)\pi x}{l}} \right] \\
&= \frac{l}{i(n-m)\pi} \left[e^{i(n-m)\pi} - e^{-i(n-m)\pi} \right] \\
&= \frac{2l}{(n-m)\pi} \sin[(n-m)\pi] = 0
\end{aligned}$$

In the case $n = m$ (again it does not matter which integer number we pick here), we have

$$\begin{aligned}
< e^{i\frac{n\pi x}{l}}, e^{i\frac{n\pi x}{l}} > &= \int_{-l}^{l} e^{i\frac{n\pi x}{l}} e^{-i\frac{n\pi x}{l}} dx \\
&= \int_{-l}^{l} e^{i\frac{(n-n)\pi x}{l}} dx \\
&= \int_{-l}^{l} dx = 2l
\end{aligned}$$

If we apply this to

$$f(x) = S(x) = \sum_{n=-\infty}^{\infty} A_n e^{i\frac{n\pi x}{l}}$$

and take the scalar product with $\exp(im\pi x/l)$ we find that

$$A_n = \frac{1}{2l} \int_{-l}^{l} f(x) e^{-i\frac{n\pi x}{l}} dx$$

for all n!

We can actually recover our expressions for a_n and b_n by using the definition of A_n:

$$a_n = 2\mathrm{Re}(A_n), \quad b_n = -2\mathrm{Im}(A_n)$$

We will now illustrate how to use the complex version of the Fourier with example that we have already used before. This allows us to compare the results.

Example 6.14

Consider $f(x) = x$ over $[-1, 1)$, so $l = 1$ in this case.

Do not worry about the complex argument of the exponential function, just treat it like any other exponential function. We will again have to use integration by parts to evaluate the integral for A_n.

We have

$$\begin{aligned} A_n &= \frac{1}{2}\int_{-1}^{1} x e^{-in\pi x} dx \\ &= \frac{1}{2}\left\{ \left[-\frac{x}{in\pi} e^{-in\pi x}\right]_{-1}^{1} + \frac{1}{in\pi}\int_{-1}^{1} e^{-in\pi x} dx \right\} \\ &= \frac{1}{2}\left\{ \frac{i}{n\pi}\left[e^{-in\pi} + e^{in\pi}\right] + \frac{1}{n^2\pi^2}\left[e^{-in\pi x}\right]_{-1}^{1} \right\} \\ &= \frac{1}{2}\left\{ \frac{2i}{n\pi}(-1)^n + \frac{1}{n^2\pi^2}\left[e^{-in\pi} - e^{in\pi}\right] \right\} \\ &= i\frac{(-1)^n}{n\pi}. \end{aligned}$$

We have used here that

$$e^{-in\pi} + e^{in\pi} = 2\cos(n\pi) = 2(-1)^n$$

and

$$e^{-in\pi} - e^{in\pi} = -2i\sin(n\pi) = 0.$$

From

$$A_n = i\frac{(-1)^n}{n\pi}$$

we find that

$$a_n = 2\mathrm{Re}(A_n) = 0, \quad b_n = -2\mathrm{Im}(A_n) = -\frac{2(-1)^n}{n\pi}$$

which are exactly the coefficients we have found before.

Finally, we can write down the Fourier series for $f(x) = x$, $x \in [-1, 1)$ in complex notation as

$$S(x) = \frac{i}{\pi}\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n} e^{in\pi x}.$$

Example End

Chapter 7

Application of Fourier Series: Solving PDEs

Partial differential equations (PDEs) are equations which involve a function of two or more variables and its partial derivatives (ODEs: function of one variable and its derivatives). Here we will only discuss PDEs to illustrate how Fourier Series can be used to solve certain PDEs. PDEs will be treated in much more detail in MT3504 Differential Equations.

Some of the definitions we introduced for ODEs are also used for PDEs. For example, the *order* of a PDE is equal to the order of the highest partial derivative in the PDE. Similarly, we call a PDE linear if the function and its derivatives only appear in a linear combination in the PDE, just in the same way as for ODEs. Also homogeneous and inhomogeneous PDEs have the same definition as homogeneous and inhomogeneous ODEs.

In this section we will only discuss one particular PDE, the *heat equation.* This is the equation that Fourier tried to solve and while doing so found what we call Fourier series (or Fourier transform for the continuous version).

The heat equation is the PDE

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}, \tag{7.1}$$

where $T(x,t)$ is the temperature of some matter (gas, fluid or solid) at position x and time t. The constant κ is the heat conduction coefficient, which we will from now on set to one without loss of generality, because one can easily re-scale t and x in such a way that $\kappa = 1$. The heat equation describes how the spatial temperature profile changes due to the spatial variations in T. This is actually consistent with our experience: if we have a heat source, e.g. a radiator at one end of a cold room, which is suddenly switched on, we know that the heat will spread from there to the other parts of the room, but that this will take some time. The heat equation models mathematically how this happens!

The heat equation is a linear, second order PDE for $T(x,t)$, because it consists only of a linear combination of partial derivatives of T, and because the highest partial derivative in the PDE is a second order derivative. The heat equation is an example for a much wider class of equations which are called *diffusion equations.* For example, if we use the density $n(x,t)$ instead of the temperature $T(x,t)$ in Eq. (7.1) we get the diffusion equation for the density, which describes, for example, how a drop of milk in tea or coffee spreads over time (Brownian motion). Diffusion equations play an important role in many areas of science (mathematics, physics, chemistry, biology, economics, to name just a few).

7.1 Method of Solution: Separation of Variables

We now try to solve the PDE (7.1) using a method called *separation of variables.* This method can be applied to many linear PDEs, not just to the heat equation (7.1). We use the heat equation (7.1) as an example because in this case separation of variables leads us straight to Fourier series!

We start by trying to find a solution to Eq. (7.1) which has the general form

$$T(x,t) = X(x)Y(t), \tag{7.2}$$

i.e. a solution which is the product of a function of x only and a function of t only. If such a solution exists, it must of course satisfy the PDE and hence we substitute it into Eq. (7.1):

$$\frac{\partial T}{\partial t} = X\frac{dY}{dt} = \frac{\partial^2 T}{\partial x^2} = Y\frac{d^2X}{dx^2} \tag{7.3}$$

$$\implies \quad X\frac{dY}{dt} = Y\frac{d^2X}{dx^2}. \tag{7.4}$$

To make progress we divide the last equation by $T = XY$, so that

$$\frac{X}{XY}\frac{dY}{dt} = \frac{Y}{XY}\frac{d^2X}{dx^2} \quad \implies \quad \frac{1}{Y}\frac{dY}{dt} = \frac{1}{X}\frac{d^2X}{dx^2}. \tag{7.5}$$

In the resulting equation, the left hand side is a function of t only and the right hand side is a function of x only. Because t and x are independent variables the left hand side and the right hand side can only be identical *for every possible combination* of t and x if they are both equal to the same constant![1]

Hence we can write

$$\frac{1}{Y}\frac{dY}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = c, \tag{7.6}$$

where $c \in \mathbb{R}$ is a constant, but the value of c is at the moment arbitrary. We have replaced the PDE with two ODEs and separated the variables t and x (hence the name of the method). We first solve the ODE for Y:

$$\frac{1}{Y}\frac{dY}{dt} = c \quad \implies \quad \frac{dY}{dt} = cY.$$

[1] If you have difficulty understanding this argument, it may help to consider the following analogy. Imagine that you have two pieces of A4 paper and you want to put them on top of each other so that they touch each other at every single point. However, one of them is creased in the short direction, while the other one is creased in the long direction. To make them touch each other at every single point, you have to make both of them absolutely flat!

This ODE has the general solution

$$Y(t) = Y_0 \exp(ct). \tag{7.7}$$

We will see a bit later that we can set $Y_0 = 1$ without loss of generality. As we all know the temperature T will decrease over time if there is no heat source, so we have to generally choose $c < 0$ (we will also allow the possibility $c = 0$; this is needed later to get all terms of the Fourier series). If c is negative (or at most zero), we can rewrite it as

$$c = -k^2 \leq 0.$$

Looking at the ODE for X now, we have

$$\frac{1}{X}\frac{d^2X}{dx^2} = -k^2 \quad \Longrightarrow \quad \frac{d^2X}{dx^2} = -k^2 X$$

for which the general solution is

$$X(x) = a\cos(kx) + b\sin(kx). \tag{7.8}$$

So, we have found a set of solutions to the heat equation (7.1) that have the form

$$T(x,t) = X(x)Y(t) = \exp(-k^2 t)[a\cos(kx) + b\sin(kx)].$$

Here we have set $c = -k^2$ and $Y_0 = 1$, because it is only a multiplicative factor for the (free) constants a and b.

But how to we determine possible values for k, a and b? This can only be done if we have further information. For example, if we solve our PDE on a finite domain, $-l \leq x < l$, say, and the values $T(l,t)$ and $T(-l,t)$ were given as functions of t (boundary conditions), then we could find conditions for k and the coefficients a and b. We will in the following assume that $T(x,t)$ is a periodic function of x with a fundamental period $L = 2l$ and that we solve the PDE over the standard interval $[-l, l)$. This assumption immediately leads to a condition on the possible values for k, namely

$$kl = n\pi \quad \Longrightarrow \quad k = \frac{n\pi}{l}$$

with $n \geq 0$ integer. The solutions we have found now take the form

$$\begin{aligned} T_n(x,t) &= \exp(-n^2\pi^2 t/l^2)[a_n\cos(n\pi x/l) + b_n\sin(n\pi x/l)], \quad n \geq 1, \\ T_0(x,t) &= \frac{a_0}{2}, \quad n = 0, \end{aligned}$$

where we have added a subscript n to indicate that we can pick different values for a and b for every n. We have also chosen to write the solution for $n = 0$, which is a constant solution, as $a_0/2$.

However, we still do not know how to choose a value for n. Actually, all positive integer values for n (and $n = 0$) give valid solutions! So, which one is the "right" solution? The answer is: they all are (at this point)! Because we have a linear PDE, we can take arbitrary linear combinations of known solutions to get more solutions. Therefore, we can write down the *general* periodic solution of the heat equation over the domain $x \in [-l, l)$ as

$$T(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \exp(-n^2\pi^2 t/l^2)[a_n \cos(n\pi x/l) + b_n \sin(n\pi x/l)].$$

This already looks very similar to a Fourier series, but there is a time-dependent factor inside the sum. So how do we determine the coefficients a_n and b_n? To be able to do that we need more information again! The natural condition to use is to assume that we know the temperature profile at $t = 0$, $T(x, 0) = f(x)$ (initial condition). Then we have that

$$\begin{aligned} T(x,0) = f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \exp(0)[a_n \cos(n\pi x/l) + b_n \sin(n\pi x/l)] \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x/l) + b_n \sin(n\pi x/l). \end{aligned}$$

This now has *exactly* the form of the Fourier series that we have encountered before, and we know how calculate the coefficients a_n and b_n from $f(x)$!

The method of solution for PDEs that we have just illustrated using the heat equation as an example works for many linear PDEs. One can also use it for PDEs with more independent variables. For example, if one wanted to study how heat spreads in a three-dimensional space with Cartesian coordinates x, y, z, one would replace Eq. (7.1) by

$$\frac{\partial T}{\partial t} = \kappa \left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right] = \kappa \nabla^2 T.$$

If applied to different types of equations and/or different coordinate systems (for example, cylindrical or spherical polar coordinates) one does not always encounter series of trigonometric functions, but of often other special functions (Bessel functions, Legendre functions/polynomials, etc).

End of Fourier Series Lecture Notes
