MT5830 - 3 The upper half plane model

① a) This could be proved by letting $z \in \mathbb{H}^2$ and then proving $|\phi(z)| < 1$ (which would show $\phi(\mathbb{H}^2) \subseteq \mathbb{D}^2$) and then showing similarly that $\phi(\mathbb{H}^2) \supseteq \mathbb{D}^2$ / $\mathbb{H}^2 \supseteq \phi^{-1}(\mathbb{D}^2)$.

However, there is an easier way which avoids the algebra! Since $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a Möbius map we know $\phi(\mathbb{R} \cup \{\infty\})$ is either a circle or a line. Since:

$$\phi(\infty) = 1$$

$$\phi(0) = \frac{-i}{i} = -1$$

$$\phi(1) = \frac{1-i}{1+i} = -i$$

we conclude $\phi(\mathbb{R} \cup \{\infty\})$ is the unique circle passing through $1, -1, -i$, which is $S^1$. Since $\mathbb{R} \cup \{\infty\}$ cuts $\hat{\mathbb{C}}$ into two path connected disjoint pieces, the same must be true for $\phi(\mathbb{R} \cup \{\infty\}) = S^1$ and the pieces must map to each other. So either $\phi(\mathbb{H}^2) = \mathbb{D}^2$ or $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}^2}$.

Since $\phi(-i) = \infty$, we conclude $\phi(\mathbb{H}^2)$ is the bounded option, ie $\phi(\mathbb{H}^2) = \mathbb{D}^2$.

b) Solve for $w = \phi^{-1}(z)$ such that

$$\phi(w) = z, \text{ ie } \frac{w-i}{w+i} = z$$

so $$\phi^{-1}(z) = w = \frac{zi+i}{1-z} = \frac{iz+i}{-z+1}$$

which is a Möbius map since

$$i \times 1 - (-1) \times i = 2i \neq 0.$$

Note that if we identify $\phi$ with

$$\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \in GL(2, \mathbb{C})$$

then $$\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}^{-1} = \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix} \sim \phi^{-1}$$

c) Let $u, w \in \mathbb{D}^2$. Then by definition of $d_{\mathbb{H}^2}$ we have

$$d_{\mathbb{H}^2}(\phi^{-1}(u), \phi^{-1}(w)) = d_{\mathbb{D}^2}(\phi(\phi^{-1}(u)), \phi(\phi^{-1}(w)))$$
$$= d_{\mathbb{D}^2}(u, w)$$

② $$T_3(T_2(T_1(z))) = -i\left(i + \left(\frac{\sqrt{2}}{|\bar{z}-i|}\right)^2 (\bar{z}-i)\right)$$

$$= -i\left(i + \frac{2(\bar{z}-i)}{(\bar{z}-i)(z+i)}\right)$$

$$= -i\left(\frac{iz+i^2}{z+i} + \frac{2}{z+i}\right)$$

$$= -i\left(\frac{iz+1}{z+i}\right) = \frac{z-i}{z+i} = \phi(z).$$

③ a) Let $g \in PSL(2,\mathbb{R})$ be given by

$$g(z) = \frac{az+b}{cz+d} \quad \text{with } a,b,c,d \in \mathbb{R},\ ad-bc=1.$$

Then $$\phi g \phi^{-1}(z) = \phi\left(\frac{-ia\frac{z+1}{z-1}+b}{-ic\frac{z+1}{z-1}+d}\right)$$

$$= \phi\left(\frac{-iaz-ia+bz-b}{-icz-ic+dz-d}\right)$$

(3) a) cont...

$$= \frac{\dfrac{(b-ia)z-(b+ia)}{(d-ic)z-(d+ic)} - i}{\dfrac{(b-ia)z-(b+ia)}{(d-ic)z-(d+ic)} + i}$$

$$= \frac{\big(b-c-i(a+d)\big)z - \big(b+c+i(a-d)\big)}{\big(b+c-i(a-d)\big)z - \big(b-c+i(a+d)\big)}$$

$$\times\frac{\frac{i}{2}}{\frac{i}{2}} = \frac{\frac{1}{2}\big(a+d+i(b-c)\big)z + \overline{\frac{1}{2}\big(a-d+i(b+c)\big)}}{\frac{1}{2}\big(a-d+i(b+c)\big)z + \overline{\frac{1}{2}\big(a+d+i(b-c)\big)}}$$

and one can check

$$\left|\tfrac{1}{2}\big(a+d+i(b-c)\big)\right|^2 - \left|\tfrac{1}{2}\big(a-d+i(b+c)\big)\right|^2$$

$$= 1$$

and so $\phi g \phi^{-1} \in Con^{+}(1)$.

③ b) Let $\mathrm{Isom}(\mathbb{H}^2)$ denote the isometry group of $(\mathbb{H}^2, d_{\mathbb{H}^2})$. We know that:

$$\mathrm{Isom}(\mathbb{H}^2) \supseteq \phi^{-1}\, \mathrm{Isom}(\mathbb{D}^2)\, \phi$$

but it follows from ① c), that if $g \in \mathrm{Isom}(\mathbb{H}^2)$, then for all $u, w \in \mathbb{D}^2$

$$\begin{aligned} d_{\mathbb{D}^2}(u, w) &= d_{\mathbb{H}^2}(\phi^{-1}(u), \phi^{-1}(w)) \\ &= d_{\mathbb{H}^2}(g\phi^{-1}(u), g\phi^{-1}(w)) \\ &= d_{\mathbb{D}^2}(\phi g \phi^{-1}(u), \phi g \phi^{-1}(w)) \end{aligned}$$

and so $\phi g \phi^{-1} \in \mathrm{Isom}(\mathbb{D}^2)$ and we conclude that

$$\mathrm{Isom}(\mathbb{H}^2) \subseteq \phi^{-1}\, \mathrm{Isom}(\mathbb{D}^2)\, \phi.$$

Therefore

$$\mathrm{Isom}(\mathbb{H}^2) = \phi^{-1}\, \mathrm{Isom}(\mathbb{D}^2)\, \phi = \phi^{-1}\, \mathrm{Con}(1)\, \phi$$

$$= \phi^{-1} \langle \mathrm{Con}^{+}(1), z \mapsto \bar{z} \rangle \phi$$

③ b) cont...

$$= \langle \phi^{-1} \mathrm{con}^+(1) \phi, \ \phi^{-1}(z \mapsto \bar{z}) \phi \rangle$$

$$= \langle PSL(2, \mathbb{R}), \ z \mapsto -\bar{z} \rangle$$

Note that $\phi^{-1} \langle A \rangle \phi = \langle \phi^{-1} A \phi \rangle$

since any element in $\phi^{-1} \langle A \rangle \phi$ can be written as $\phi^{-1} a_1 \cdots a_k \phi$ for some $a_1, \ldots, a_k \in A \cup A^{-1}$ and

$$\phi^{-1} a_1 \cdots a_k \phi = \phi^{-1} a_1 \phi \ \phi^{-1} a_2 \phi \cdots \phi^{-1} a_k \phi$$

$$\in \langle \phi^{-1} A \phi \rangle$$

and vice versa.

(4) Let $u, w \in \mathbb{H}^2$ be given by

$$u = x + iy_1, \quad w = x + iy_2$$

where $x \in \mathbb{R}$ and we assume without loss of generality that

$$0 < y_1 < y_2.$$

Let $g \in PSL(2, \mathbb{R})$ be given by

$$g(z) = z - x = \frac{z - x}{0 \times z + 1}.$$

Then
$$\begin{aligned} d_{\mathbb{H}^2}(u, w) &= d_{\mathbb{H}^2}(g(u), g(w)) \\ &= d_{\mathbb{H}^2}(iy_1, iy_2) \\ &= d_{\mathbb{D}^2}(\phi(iy_1), \phi(iy_2)) \\ &= d_{\mathbb{D}^2}\left(\frac{y_1 - 1}{y_1 + 1}, \frac{y_2 - 1}{y_2 + 1}\right) \end{aligned}$$

$$= \log\left(\frac{1 + \frac{y_2 - 1}{y_2 + 1}}{1 - \frac{y_2 - 1}{y_2 + 1}}\right) - \log\left(\frac{1 + \frac{y_1 - 1}{y_1 + 1}}{1 - \frac{y_1 - 1}{y_1 + 1}}\right)$$

④ cont...

$$= \log \left( \frac{\left(1 + \frac{y_2 - 1}{y_2 + 1}\right)\left(1 - \frac{y_1 - 1}{y_1 + 1}\right)}{\left(1 - \frac{y_2 - 1}{y_2 + 1}\right)\left(1 + \frac{y_1 - 1}{y_1 + 1}\right)} \right)$$

$$= \log \left( \frac{\left[(y_2+1) + (y_2-1)\right]\left[(y_1+1) - (y_1-1)\right]}{\left[(y_2+1) - (y_2-1)\right]\left[(y_1+1) + (y_1-1)\right]} \right)$$

$$= \log \left( \frac{2y_2 \times 2}{2 \times 2y_1} \right)$$

$$= \log \frac{y_2}{y_1}.$$

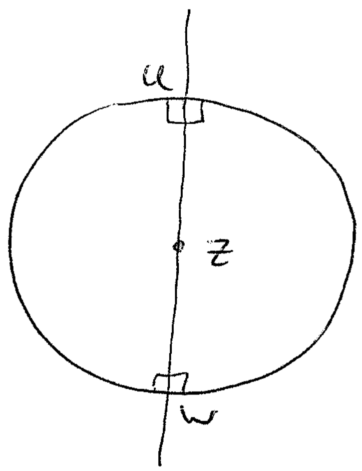
So generally ~~$d$~~

$$d_{\mathbb{H}^2}(u, w) = \left| \log \frac{\operatorname{Im}(u)}{\operatorname{Im}(w)} \right|$$

(5) Let $z \in \mathbb{H}^2$ and suppose we can find $r, R > 0$ such that

$$C_E(z, r) = C_{\mathbb{H}^2}(z, R). \quad (*)$$

Let $u, w \in \mathbb{H}^2$ be the two points of intersection of $C_E(z,r)$ with the vertical line through $z$.




It follows that

$$u = \mathrm{Re}(z) + i(\mathrm{Im}(z) + r)$$

$$w = \mathrm{Re}(z) + i(\mathrm{Im}(z) - r)$$

$(*)$ implies that $d_{\mathbb{H}^2}(w, z) = d_{\mathbb{H}^2}(z, u)$

writing $x = \mathrm{Im}(z)$, (4) implies

$$\log \frac{x + r}{x} = \log \frac{x}{x - r}$$

(5) cont...

Hence $\dfrac{\alpha+r}{\alpha} = \dfrac{\alpha}{\alpha - r}$

$\iff (\alpha+r)(\alpha-r) = \alpha^2$

$\iff \alpha^2 - r^2 = \alpha^2$

$\iff r = 0$

which is a contradiction.