Sequences and Series

Brief Notes

Sequences

A *sequence* or *infinite sequence* is an infinite list of real numbers

$$(a_1, a_2, a_3, a_4, \ldots),$$

often written as (a_n), $\{a_n\}$, $(a_n)_{n=1}^{\infty}$ or even just a_n. The number a_n is the n-th *term* of the sequence.

Examples.

1. $\left(\frac{n}{2n+1}\right)_{n=1}^{\infty} = \left(\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \ldots\right)$.
2. $(2^{-n}) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\right)$.
3. A constant sequence: $(5) = (5, 5, 5, \ldots)$.
4. A sequence obtained from a function: $(f(n))_{n=1}^{\infty}$, e.g. $(\sin n) = (\sin 1, \sin 2, \ldots)$.
5. A sequence defined recursively, with a_n expressed in terms of a_{n-1}, e.g.
$$a_1 = 1,\ a_n = \tfrac{1}{2}\left(a_{n-1} + \tfrac{2}{a_{n-1}}\right),$$
so $(a_n) = (1, 1.5, 1.4166, 1.4121, \ldots)$, which approaches $\sqrt{2}$ as n gets large.

Convergence of sequences

The sequence (a_n) is said to *converge* to a or have *limit* a if (a_n) gets closer and closer to a as n gets larger and larger. This is denoted by

$$a_n \to a \quad \text{or} \quad \lim_{n\to\infty} a_n = a.$$

Equivalently, the 'error' $|a_n - a|$ becomes arbitraily small as n gets arbirtarily large.

A sequence that does not converge is said to *diverge*.

[Note that this is an informal definition of convergence. A formal definition will be met in later courses.]

Examples.

1. $\left(\frac{2^n-1}{2^n}\right)_{n=1}^{\infty} = \left(\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \ldots\right)$. converges to 1, that is $\dfrac{2^n-1}{2^n} = 1 - \dfrac{1}{2^n} \to 1$ (since $\frac{1}{2^n}$ becomes arbitraily small as n gets large).
2. $(4(1 + (-1)^n)) = (0, 8, 0, 8, 0, 8, \ldots)$ is divergent.

Some important limits.

3. $\dfrac{1}{n} \to 0$.
4. For each $\alpha > 0$, $\dfrac{1}{n^\alpha} \to 0$.
5. For a given c, if $|c| < 1$ then $c^n \to 0$, if $c = 1$ then $c^n \to 1$, and if $c \le -1$ or $c > 1$ then c^n diverges. e.g. $\left(\frac{1}{3}\right)^n \to 0$ but $(-2)^n$ diverges.

Rules for limits

Suppose $a_n \to a$ and $b_n \to b$, and λ is a real number. Then
(a) $a_n + b_n \to a + b$,
(b) $a_n - b_n \to a - b$,
(c) $a_n b_n \to ab$,
(d) if $b_n \neq 0$ and $b \neq 0$ then $a_n/b_n \to a/b$,
(e) $\lambda a_n \to \lambda a$.

To find the limit of a sequence, think about the 'dominant' contributions (e.g. which terms 'swamp' the numerator and denominator of a fraction when n is large), divide through by an appropriate term, and use the above rules.

Examples.

1. Consider the sequence $\left(\dfrac{2n+1}{3n+1}\right)$. Intuitively the terms are dominated by $\dfrac{2n}{3n} = \dfrac{2}{3}$ when n is large, so we expect the limit to be $\frac{2}{3}$. More precisely:

$$\frac{2n+1}{3n+1} = \frac{2+\frac{1}{n}}{3+\frac{1}{n}} \to \frac{2+0}{3+0} = \frac{2}{3}$$

since $\frac{1}{n} \to 0$ and using rules (a) and (d).

2. $$\frac{4^n+3^n}{4^n+2^n} = \frac{1+(\frac{3}{4})^n}{1+(\frac{2}{4})^n} \to \frac{1+0}{1+0} = 1$$

since $(\frac{3}{4})^n \to 0$ and $(\frac{2}{4})^n \to 0$.

Infinite series

A *series* or *infinite series* is an expression of the form

$$a_1 + a_2 + a_3 + \ldots \quad \text{often written as} \quad \sum_{n=1}^{\infty} a_n$$

where the (a_n) are real numbers.

Examples.

1. $\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \ldots = \displaystyle\sum_{n=1}^{\infty} 2^{-n}$.
2. $\displaystyle\sum_{n=1}^{\infty} \frac{n+2}{n^3+1}$.
3. $3 + \dfrac{1}{10} + \dfrac{4}{100} + \dfrac{1}{1000} + \dfrac{5}{10000} + \dfrac{9}{100000} \ldots$.

To give a meaning to such infinite sums we define the *N-th partial sum* of the series as

$$S_N = a_1 + a_2 + \ldots + a_N = \sum_{n=1}^{N} a_n.$$

If the sequence (S_N) of partial sums converges to S, that is if $S_N \to S$, we say that the series $\sum_{n=1}^{\infty} a_n$ is *convergent with sum* S and write $\sum_{n=1}^{\infty} a_n = S$.

Thus the partial sums $\sum_{n=1}^{N} a_n$ get close to the sum $\sum_{n=1}^{\infty} a_n$ when N is large. If a series is not convergent it is called *divergent*.

Examples.

4. Consider the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{n=1}^{\infty} 2^{-n}$. Here $S_1 = \frac{1}{2}, S_2 = \frac{3}{4}, S_3 = \frac{7}{8}, \ldots, S_N = 1-2^{-N}$. Since $S_N = 1 - 2^{-N} \to 1$ as $N \to \infty$, the series is convergent with $\sum_{n=1}^{\infty} 2^{-n} = 1$.

5. For the series $1 - 1 + 1 - 1 + 1 - \ldots = \sum_{n=1}^{\infty} (-1)^{n+1}$, $S_N = 1$ if N is odd and $S_N = 0$ if N even, so S_N is not a convergent sequence and the series is divergent.

Note that whilst it is sometimes possible to find an exact expression for the partial sums and take the limit, often all we can do is to show that a series converges and then compute an aproximation to the sum.

Geometric series

For $a \neq 0$, the series $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \ldots$ is the *geometric series* with *first term* a and *ratio* r. We may calculate the partial sums:

$$\begin{aligned} S_N &= a + ar + ar^2 + \ldots + ar^{N-1} \\ rS_N &= \phantom{a + {}} ar + ar^2 + \ldots + ar^{N-1} + ar^N \end{aligned}$$

so subtracting

$$S_N(1-r) = a - ar^N = a(1-r^N),$$

giving for $r \neq 1$

$$S_N = \frac{a(1-r^N)}{1-r} = \frac{a}{1-r} - \frac{ar^N}{1-r}.$$

By looking at the right hand terms, if $|r| < 1$ then $r^N \to 0$ so $S_N \to a/(1-r)$ and the series is convergent with sum $a/(1-r)$. Similarly, if $|r| > 1$ the series is divergent. (If $r = \pm 1$ the series is clearly divergent.)

If $|r| < 1$ the geometric series is convergent with sum

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \ldots = \frac{a}{1-r}.$$

If $|r| \geq 1$ the series is divergent.

Example.

The series $9 + 3 + 1 + \frac{1}{3} + \frac{1}{9} + \ldots$ is geometric with first term $a = 9$ and ratio $r = 1/3$, so is convergent with sum $9/(1 - 1/3) = 27/2$.

Partial fractions method

Sometimes it is possible to express the terms of a series in partial fractions and use cancellation to obtain the partial sums.

Example.

Consider $\dfrac{1}{1.2} + \dfrac{1}{2.3} + \dfrac{1}{3.4} + \ldots = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Decomposing into partial fractions, $\dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$. Thus

$$\sum_{n=1}^{N} \frac{1}{n(n+1)} = \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1 - \frac{1}{N+1} \to 1 \quad \text{as } N \to \infty.$$

Hence the series is convergent with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

The harmonic series

The harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent (but very slowly). More generally,

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^p}$$

is convergent if $p > 1$ and divergent if $0 < p < 1$.

[To see that the harmonic series diverges, note that $\displaystyle\sum_{n=2^k+1}^{2^{k+1}} \frac{1}{n} > \frac{1}{2}$ for each k.]

Tests for convergence or divergence of series

We often need to determine whether a series converges or diverges even if we cannot find the partial sums or sum explicitly. Once we can show a series converges, the sum can usually be estimated computationally.

n-th term test for divergence

If $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent then $a_n \to 0$. In particular if $a_n \not\to 0$ then the series is divergent.

[Proof: If $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent with sum S and $S_N = \displaystyle\sum_{n=1}^{N} a_n$, then

$$a_N = \sum_{n=1}^{N} a_n - \sum_{n=1}^{N-1} a_n = S_N - S_{N-1} \to S - S = 0.]$$

Note that the converse is false: we can *not* conclude that a series is convergent just because $a_n \to 0$. (Consider the harmonic series.)

Examples.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{3n+1}{5n+3}$ diverges since $\displaystyle\frac{3n+1}{5n+3} = \frac{3+\frac{1}{n}}{5+\frac{3}{n}} \to \frac{3}{5} \neq 0$.

2. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} = 1 - 1 + 1 - 1 + \dots$ diverges since the sequence $1, -1, 1, -1, 1, \dots$ is divergent.

3. $\displaystyle\frac{1}{n^2+n} \to 0$, so the n-th term test tells us nothing about the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n}$.

Comparison test

Let $0 \leq a_n \leq b_n$. $(*)$

(a) If $\displaystyle\sum_{n=1}^{\infty} b_n$ converges then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

(a) If $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges then $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges.

[Proof: Since $a_n \geq 0$, the partial sums $S_N = \sum_{n=1}^{N} a_n$ are increasing, i.e. $S_1 \leq S_2 \leq S_3 \leq \dots$. But $\displaystyle S_N = \sum_{n=1}^{N} a_n \leq \sum_{n=1}^{N} b_n \leq \sum_{n=1}^{\infty} b_n = B$, say. Thus S_N is an increasing sequence with $S_N \leq B$ for all N, so S_N converges to some number $S \leq B$.]

Note that the test is still valid if $(*)$ holds just for all sufficiently large n or if $(*)$ is replaced by $0 \leq a_n \leq cb_n$ for some constant $c > 0$.

Examples.

1. Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n n}$ converge?

   Since $\displaystyle 0 \leq \frac{1}{3^n n} \leq \frac{1}{3^n}$, and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ is convergent (geometric series, ratio $1/3$), the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n n}$ is convergent by the comparison test (a).

2. Does $\displaystyle\sum_{n=1}^{\infty} \frac{2+\sqrt{n}}{n}$ converge?

   Since $\displaystyle 0 \leq \frac{1}{n} \leq \frac{2+\sqrt{n}}{n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (harmonic series), the series $\displaystyle\sum_{n=1}^{\infty} \frac{2+\sqrt{n}}{n}$ is divergent by the comparison test (b).

3. Does $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+3^n}{3^n+4^n}$ converge?

Since $0 \leq \dfrac{2^n+3^n}{3^n+4^n} \leq \dfrac{3^n+3^n}{4^n} = 2\left(\dfrac{3}{4}\right)^n$, and $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n$ is convergent (geometric series, ratio 3/4), the series $\displaystyle\sum_{n=1}^{\infty}\frac{2^n+3^n}{3^n+4^n}$ is convergent by the comparison test (a).

Absolute convergence

The series $\displaystyle\sum_{n=1}^{\infty} a_n$ is called *absolutely convergent* if $\displaystyle\sum_{n=1}^{\infty} |a_n|$ is convergent. Every absolutely convergent series is convergent.

[Proof: Note that $0 \leq a_n + |a_n| \leq 2|a_n|$. If $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges then $\displaystyle\sum_{n=1}^{\infty}(a_n+|a_n|)$ converges by the comparison test, so the partial sums $\displaystyle\sum_{n=1}^{N} a_n = \sum_{n=1}^{N}(a_n+|a_n|) - \sum_{n=1}^{N}|a_n|$ converge as $N \to \infty$.]

Examples.

1. Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}$ is convergent, $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}$ is absolutely convergent and so is convergent.
2. $\displaystyle\sum_{n=1}^{\infty}\frac{\sin n}{3^n}$ is absolutely convergent and thus convergent since $\displaystyle\sum_{n=1}^{\infty}\frac{|\sin n|}{3^n}$ is convergent by comparison with the geometric series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{3^n}$.

Ratio test

Let $\displaystyle\sum_{n=1}^{\infty} a_n$ be a series with $a_n \neq 0$ for all n. Suppose $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$. Then
(a) if $L < 1$ the series is absolutely convergent and thus convergent,
(b) if $L > 1$ the series is divergent,
(c) if $L = 1$ no conclusion can be drawn.

The ratio test follows from comparison with a geometric series.

[Proof: (a) Take a number r with $L < r < 1$. Then there is some integer N such that $|a_{n+1}|/|a_n| \leq r$ if $n \geq N$. Applying this repeatedly,

$$|a_N| \geq \frac{|a_{N+1}|}{r} \geq \frac{|a_{N+2}|}{r^2} \geq \dots$$

i.e. $|a_{N+m}| \leq r^m|a_N|$ for $m = 1, 2, 3, \dots$. Since $\displaystyle\sum_{m=1}^{\infty} r^m|a_N|$ is convergent (geometric series ratio $r < 1$), $\displaystyle\sum_{m=1}^{\infty}|a_{N+m}|$ is convergent by the comparison test. Including N further terms at the start of the series does not affect convergence, so $\sum_{n=1}^{\infty} |a_n|$ is convergent and $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

For (b), note that $|a_{n+1}| \geq |a_n| > 0$ if n is large enough, so $a_n \not\to 0$ and divergence follows from the n-th term test.]

The ratio test is particularly useful when the terms involve powers c^n or factorials $n!$.

Examples.

1. Consider $\sum_{n=1}^{\infty} \frac{3^n}{n!}$. Then $\left|\frac{a_{n+1}}{a_n}\right| = \frac{3^{n+1}}{(n+1)!} \Big/ \frac{3^n}{n!} = \frac{3^{n+1} n!}{3^n (n+1)!} = \frac{3}{n+1} \to 0$, so the series is convergent by the ratio test.
2. Consider $\sum_{n=1}^{\infty} \frac{5^n}{n^3}$. Then $\left|\frac{a_{n+1}}{a_n}\right| = \frac{5^{n+1}}{(n+1)^3} \Big/ \frac{5^n}{n^3} = \frac{5^{n+1} n^3}{5^n (n+1)^3} = \frac{5n^3}{(n+1)^3} \to 5$, so the series is divergent by the ratio test.

There are many variants of the ratio test, for example rather than requiring the limit to exist, conclusion (a) holds if $|a_{n+1}/a_n| < L$ for all sufficiently large n.

Alternating series test

Let a_n be a decreasing sequence with $a_n \to 0$. Then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \dots$ is convergent.

[Proof: Consider the 'zig-zag' behaviour of the partial sums.]

Examples.

1. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$ is convergent by the alternating series test, since $\frac{1}{n} \searrow 0$.
2. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n + n^{1/2}}$ is convergent by the alternating series test, since $1/(n + n^{1/2}) \searrow 0$.

Power series

A *power series* in x is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots.$$

For a power series $\sum_{n=0}^{\infty} a_n x^n$ one of the following is true:

(a) the series is absolutely convergent for all x,

(b) the series converges only when $x = 0$,

(c) there is a number $R > 0$ called the *radius of convergence* such that the series converges if $|x| < R$ and diverges if $|x| > R$.

The number R can often be found using the ratio test which should always be tried first. Other tests are needed to determine whether the series converges at $x = \pm R$. Thus, in case (c), the series converges for x in an *interval of convergence* which may be $(-R, R), [-R, R], (-R, R]$ or $[-R, R)$.

Examples.

1. For which x is $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{3^n n}$ convergent? To apply the ratio test we examine

$$\left|\frac{(n+1)\text{-th term}}{n\text{-th term}}\right| = \left|\frac{x^{n+1}}{3^{n+1}(n+1)} \Big/ \frac{x^n}{3^n n}\right| = \left|\frac{x^{n+1}3^n n}{x^n 3^{n+1}(n+1)}\right| = \frac{|x|n}{3(n+1)} = \frac{|x|}{3+\frac{3}{n}} \to \frac{|x|}{3}.$$

By the ratio test, the series converges if $|x|/3 < 1$ and diverges if $|x|/3 > 1$ so the radius of convergence is 3.

When $x = 3$ the series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ which is the divergent harmonic series. When $x = -3$ the series becomes $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$ which is convergent by the alternating series test. Thus the interval of convergence is $[-3, 3)$.

2. The exponential series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x by the ratio test, since for all x

$$\left|\frac{(n+1)\text{-th term}}{n\text{-th term}}\right| = \left|\frac{x^{n+1}}{(n+1)!} \Big/ \frac{x^n}{n!}\right| = \frac{|x|}{n+1} \to 0$$

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Taylor or Maclaurin series

We often wish to express a given function $f(x)$ as a convergent power series in x, that is to find coefficients a_n such that $f(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$, at least for a range of x. Then the first few terms of the series may give a good approximation to $f(x)$ for small x.

If $f(x)$ has a power series representation

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

with positive radius of convergence, then $a_n = \dfrac{f^{(n)}(0)}{n!}$, i.e.

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \ldots;$$

this series is called the *Taylor* or *Maclaurin series* for $f(x)$.

[Sketch of proof. Assuming that the differentiations are valid we have

$$\begin{aligned}
f(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots \\
f'(x) &= a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots \\
f''(x) &= 2a_2 + 3.2a_3x + 4.3a_4x^2 + \dots \\
f^{(3)}(x) &= 3.2a_3 + 4.3.2a_4x + \dots \\
&\vdots \\
f^{(n)}(x) &= n!a_n + \text{terms in } x \text{ and higher powers.}
\end{aligned}$$

Putting $x = 0$ gives

$$f(0) = a_0,\ f'(0) = a_1,\ f''(0) = 2a_2,\ f^{(3)}(0) = 3.2a_3, \dots, f^{(n)}(0) = n!a_n,$$

so $a_n = \dfrac{f^{(n)}(0)}{n!}$.]

Examples.

1. Find the Taylor series for $f(x) = \sin x$.
   Here $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$, $f^{(3)}(x) = -\cos x$, $f^{(4)}(x) = \sin x, \ldots$ so $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f^{(3)}(0) = -1$, $f^{(4)}(0) = 0, \ldots$. Thus the Taylor series is

$$\boxed{\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \quad \text{valid and convergent for all } x}$$

Similarly

$$\boxed{\begin{aligned} \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \qquad \text{for all } x \\ e^x &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \qquad \text{for all } x. \end{aligned}}$$

2. Find the Taylor series for $f(x) = (1+x)^p$ where p is a real number, that is the *binomial series*. Here $f^{(n)}(x) = p(p-1)(p-2)\ldots(p-n+1)(1+x)^{p-n}$ so $f^{(n)}(0) = p(p-1)(p-2)\ldots(p-n+1)$. Hence:

Binomial series

$$(1+x)^p = \sum_{n=0}^{\infty} \frac{p(p-1)(p-2)\ldots(p-n+1)}{n!} x^n,$$

convergent for $|x| < 1$, unless p is a non-negative integer in which case the series terminates at $n = p$ and is trivially convergent for all x.

3. Find the Taylor series for $f(x) = \log(1+x)$.
   Here $f(x) = \log(1+x)$, $f'(x) = (1+x)^{-1}$, $f''(x) = -(1+x)^{-2}$, $f^{(3)}(x) = 2(1+x)^{-3}$, $f^{(4)}(x) = -2.3(1+x)^{-4}$, $f^{(5)}(x) = 2.3.4(1+x)^{-5}, \ldots$, so $f(0) = 0$, $f'(0) = 1$, $f''(0) = -1$, $f^{(3)}(0) = 2$, $f^{(4)}(0) = -2.3$, $f^{(5)}(0) = -2.3.4, \ldots$. Thus the Taylor series is

$$\log(1+x) = \frac{x}{1!} - \frac{x^2}{2!} + \frac{2x^3}{3!} - \frac{2.3x^4}{4!} + \ldots$$

i.e.

Logarithmic series

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots \qquad \text{for } -1 < x \leq 1.$$

We sometimes wish to expand a function about a number a other than 0. By writing $f(x) = g(x+a)$ and then $y = x + a$ in the Taylor series above, we get the Taylor series of g about a:

$$g(y) = g(a) + \frac{g'(a)}{1!}(y-a) + \frac{g''(a)}{2!}(y-a)^2 + \frac{g^{(3)}(a)}{3!}(y-a)^3 + \ldots$$

which will generally be convergent if $|y-a| < R$ and divergent if $|y-a| > R$ for some R.