Complex Numbers

Brief Notes

Definitions

A *complex number* z is an expression of the form:

$$z = a + bi$$

where a and b are real numbers and i is thought of as $\sqrt{-1}$. We call a the *real part* of z, written $\mathrm{Re}(z)$, and b the *imaginary part* of z, written $\mathrm{Im}(z)$.

The *complex number* $z = a + bi$ is called *real* if $b = 0$ and *purely imaginary* if $a = 0$. Two complex numbers are *equal* if they have both real and imaginary parts the same.

The *complex conjugate* of $z = a + bi$ is defined to be $\overline{z} = a - bi$.

Thus $z = 3 + 6i$ is a complex number with $\mathrm{Re}(z) = 3$ and $\mathrm{Im}(z) = 6$, and $\overline{z} = 3 - 6i$.

Complex arithmetic

The golden rule for arithmetic of complex numbers is : *the usual rules of algebra hold with the proviso that* $i^2 = ii = -1$, i.e whenever you encounter a product of two is you replace it by -1. Thus we define:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

Examples:

$$(2 - 3i) + (4 - 5i) = 6 - 8i, \quad (2 - 3i) - (4 - 5i) = -2 + 2i,$$

$$(2 - 3i)(4 - 5i) = 2.4 + (-3)(-5)i^2 + 2(-5)i - 3.4i = 8 - 15 - 10i - 12i = -7 - 22i.$$

Note that if $z = a + bi$ then $\overline{z} = a - bi$ so

$$z\overline{z} = (a + bi)(a - bi) = a^2 + b^2 \quad \text{which is real.}$$

This property give us the rule for dividing complex numbers: we multiply both numerator and denominator by the conjugate of the denominator, e.g.

$$\frac{2 - 3i}{1 - 2i} = \frac{(2 - 3i)(1 + 2i)}{(1 - 2i)(1 + 2i)} = \frac{2 + 4i - 3i - 6i^2}{(1 - 4i^2)} = \frac{8 + i}{5} = \frac{8}{5} + \frac{1}{5}i.$$

The usual rules of arithmetic hold, e.g. for z_1, z_2, z_3 complex numbers:

$$(z_1 + z_2)z_3 = z_1z_3 + z_2z_3, \quad z_1z_2 = z_2z_1, \quad (z_1z_2)z_3 = z_1(z_2z_3).$$

Thus we can solve equations, simultaneous equations, quadratic equations, etc., in the usual way, provided we always replace i^2 by -1.

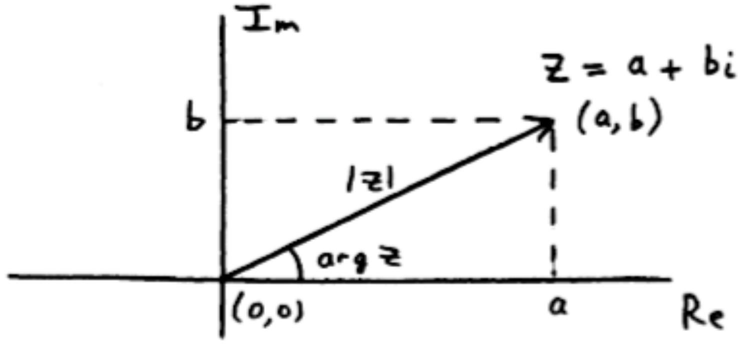
Example Solve the quadratic equation $z^2 + (1-2i)z - 1 - i = 0$.
By the formula for solution of a quadratic equation,

$$z = \frac{-(1-2i) \pm \sqrt{(1-2i)^2 - 4(-1-i)}}{2} = \frac{-1+2i \pm \sqrt{1-4i-4+4+4i}}{2}$$

$$= \frac{-1+2i \pm \sqrt{1}}{2} = i \text{ or } i-1.$$

The complex plane

We represent $z = a + bi$ by the point (a, b) in the coordinate plane.



This picture is called the *complex plane* or *Argand diagram.* The x-axis is called the *real* axis and the y-axis is called the *imaginary* axis.

We sometimes think of $z = a + bi$ as the vector from the origin $(0, 0)$ to the point with coordinates (a, b). The length of this vector (i.e. the distance of (a, b) from the origin) is called the *modulus* or *absolute value* of z, written $|z|$. The angle made by this vector with the real axis is the *argument* of z, written $\arg z$ (note that 0 has no argument). By Pythagoras' theorem

$$|z|^2 = a^2 + b^2 = z\bar{z} \quad \text{so } |z| = \sqrt{a^2 + b^2} = \sqrt{z\bar{z}}.$$

Complex numbers are often very convenient to describe geometric properties of the plane. Very useful is:

$$(\text{distance between } z = a + bi \text{ and } w = c + di) = |z - w|.$$

To see this note that

$$|z - w| = |(a + bi) - (c + di)| = |(a - c) + (b - d)i| = \sqrt{(a-c)^2 + (b-d)^2}$$

which, by Pythagoras' theorem, is the distance between the points (a, b) and (c, d) in the coordinate plane.

Certain lines and curves in the plane have very simple equations in terms of complex numbers. For example, $|z - (2 + i)| = 3$ is the equation of the circle with centre $(2 + i)$ and radius 3, since z will satisfy $|z - (2 + i)| = 3$ precisely when the distance of z from the point $(2 + i)$ equals 3.

Using the same idea:

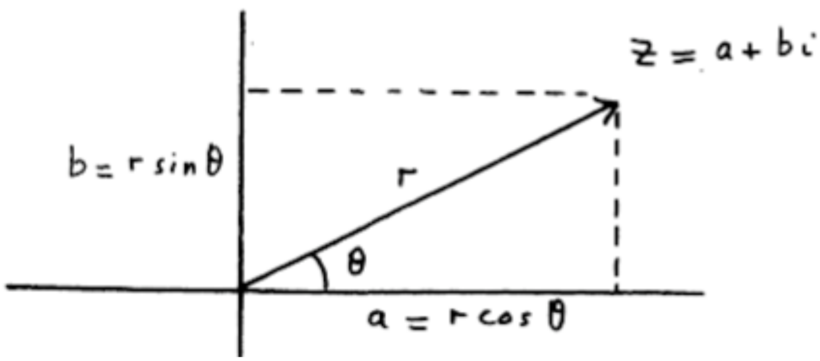
$$|z - w| = r \text{ is the circle with centre } w \text{ and radius } r$$

$$|z - w| = |z - u| \text{ is the perpendicular bisector between the points } w \text{ and } u$$

$$|z - w| + |z - u| = c \text{ is an ellipse with foci } w \text{ and } u.$$

Modulus-argument form and multiplication

Let $z = a + bi$ have modulus $r = |z|$ and argument $\theta = \arg z$.



From the diagram

$$a = r\cos\theta \text{ and } b = r\sin\theta$$

and also

$$r = |z| = \sqrt{a^2 + b^2} \text{ and } \tan\theta = b/a.$$

In particular

$$z = a + bi = r(\cos\theta + i\sin\theta).$$

We call $r(\cos\theta + i\sin\theta)$, where $r = |z|$ and $\theta = \arg z$ the *modulus-argument* form or *polar* form of z. Thus, we have

$$1 + \sqrt{3}i = 2(\cos(\pi/3) + i\sin(\pi/3)), \quad -2 + 2i = 2\sqrt{2}(\cos(3\pi/4) + i\sin(3\pi/4)).$$

[it is advisable to draw a diagram when putting a number into modulus-argument form.]

Note that a complex number has many arguments: if $\arg z = \theta$ then $\ldots, \theta - 4\pi, \theta - 2\pi, \theta, \theta + 2\pi, \theta + 4\pi, \ldots$ are also valid arguments. The *principal argument* is the value with $-\pi < \theta \leq \pi$.

Modulus-argument form is very useful when multiplying complex numbers, squaring, cubing, etc., since:

$$|z_1 z_2| = |z_1||z_2|$$

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2.$$

Thus: *to multiply complex numbers, multiply the moduli and add the arguments.*

Proof: Write z_1 and z_2 in modulus-argument form, so that

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \text{ and } z_2 = r_2(\cos\theta_2 + i\sin\theta_2).$$

Then

$$\begin{aligned}
z_1 z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) r_2(\cos\theta_2 + i\sin\theta_2) \\
&= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\
&= r_1 r_2\big((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\big) \\
&= r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)),
\end{aligned}$$

using the addition formulae from trigonometry. But this is just the complex number with modulus $r_1 r_2$ and argument $\theta_1+\theta_2$, that is $|z_1 z_2| = r_1 r_2$ and $\arg(z_1 z_2) = \theta_1+\theta_2$, as required.

Example: From above,

$$(1+\sqrt{3}i)(-2+2i) = 2(\cos(\pi/3) + i\sin(\pi/3))2\sqrt{2}(\cos(3\pi/4) + i\sin(3\pi/4))$$

$$= 2.2\sqrt{2}(\cos(\pi/3 + 3\pi/4) + i\sin(\pi/3 + 3\pi/4)) = 4\sqrt{2}(\cos(13\pi/12) + i\sin(13\pi/12)).$$

Powers and de Moivre's theorem

From above, for a positive integer power n:

$$|z^n| = |z|^n, \quad \arg(z^n) = n\,\arg z. \qquad (*)$$

For each positive integer m, we have that $1 = z^m z^{-m}$; thus

$$1 = |z^m z^{-m}| = |z|^m |z^{-m}| \quad \text{so} \quad |z^{-m}| = |z|^{-m}$$

and

$$0 = \arg 1 = \arg(z^m z^{-m}) = m\,\arg z + \arg(z^{-m}) \quad \text{so} \quad \arg(z^{-m}) = -m\,\arg z.$$

Thus $(*)$ is also true for negative integers $n = -m$. In particular, taking $z = \cos\theta + i\sin\theta$, this gives

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \quad \text{(de Moivre's theorem)},$$

which is valid for any positive or negative integer n.

de Moivre's theorem has many important applications.

(i) *Powers of numbers.* e.g. to find $(1+\sqrt{3}i)^8$. In modulus-argument form $(1+\sqrt{3}i) = 2(\cos(\pi/3) + i\sin(\pi/3))$ so

$$(1+\sqrt{3}i)^8 = 2^8(\cos(\pi/3) + i\sin(\pi/3))^8 = 2^8(\cos(8\pi/3) + i\sin(8\pi/3))$$

$$= 2^8(\cos(2\pi/3) + i\sin(2\pi/3)) = 256(-1/2 + i\sqrt{3}/2) = -128 + 128\sqrt{3}i.$$

(ii) *Trigonometric functions.* e.g. to expand $\cos 3\theta$, using de Moivre's theorem and multiplying out gives:

$$\begin{aligned}
\cos 3\theta + i\sin 3\theta &= (\cos\theta + i\sin\theta)^3 \\
&= \cos^3\theta + 3\cos^2\theta\; i\sin\theta + 3\cos\theta\; i^2\sin^2\theta + i^3\sin^3\theta \\
&= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta.
\end{aligned}$$

Equating real and imaginary parts gives

$$\begin{aligned}\cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta \\ &= \cos^3\theta - 3\cos\theta(1-\cos^2\theta) \\ &= 4\cos^3\theta - 3\cos\theta\end{aligned} \qquad \begin{aligned}\sin 3\theta &= 3\cos^2\theta\sin\theta - \sin^3\theta \\ &= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta \\ &= 3\sin\theta - 4\sin^3\theta.\end{aligned}$$

Similarly for $\cos 4\theta, \sin 4\theta$, etc.

(iii) *nth roots of complex numbers.* Care is required since complex numbers have 2 square roots, 3 cube roots, etc. Let n be a positive integer. Note that by de Moivre's theorem, for every integer k

$$\left(\cos\frac{\theta+2\pi k}{n} + i\sin\frac{\theta+2\pi k}{n}\right)^n = \cos(\theta+2\pi k) + i\sin(\theta+2\pi k) = \cos\theta + i\sin\theta.$$

Hence

$$\cos\left(\frac{\theta+2\pi k}{n}\right) + i\sin\left(\frac{\theta+2\pi k}{n}\right)$$

is an nth root for *every* integer k. Taking $k = 0, 1, 2, \dots, n-1$ gives the n different values.

Example: To find the cube roots of i: Note that

$$i = \cos(\pi/2) + i\sin(\pi/2) = \cos(\pi/2 + 2\pi k) + i\sin(\pi/2 + 2\pi k)$$

for all k. By de Moivre's theorem

$$\big(\cos((\pi/2+2\pi k)/3) + i\sin((\pi/2+2\pi k)/3)\big)^3 = \cos(\pi/2+2\pi k) + i\sin(\pi/2+2\pi k) = i,$$

so taking $k = 0, 1, 2$ gives the three cube roots as

$$\cos(\pi/6) + i\sin(\pi/6) = \sqrt{3}/2 + i/2,$$
$$\cos(5\pi/6) + i\sin(5\pi/6) = -\sqrt{3}/2 + i/2,$$
$$\cos(9\pi/6) + i\sin(9\pi/6) = -i.$$

Note that the nth roots of any complex number are symmetrically arranged at the vertices of a regular n-sided polygon centered at the origin.

Euler's formula

The exponential series is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots.$$

Assume this series is valid for *complex* x as well as real x. Substitute $x = i\theta$ to get

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \cdots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right) \\ &= \cos\theta + i\sin\theta \end{aligned}$$

recalling the series for $\cos\theta$ and $\sin\theta$.
Thus

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{1}$$

This is *Euler's formula* and is the reason why the expression '$\cos\theta + i\sin\theta$' is so important.

There are many consequences of this formula:

1. Putting $\theta = \pi$ in (1) gives $e^{i\pi} = \cos\pi + i\sin\pi = -1$. Thus we get Euler's identity

$$e^{i\pi} + 1 = 0.$$

2. We have

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$$

on replacing θ by $-\theta$ in (1). Adding and subtracting this to (1)

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta \quad \text{and} \quad e^{i\theta} - e^{-i\theta} = 2i\sin\theta,$$

so

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

This is reminiscent of the definitions of hyperbolic functions.

3. Using (1) and the rules of exponents

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i\sin n\theta$$

which is a quick derivation of de Moivre's Theorem.

4. Again using the rules of exponents

$$e^{i(\theta+\phi)} = e^{i\theta + i\phi} = e^{i\theta}e^{i\phi}$$

so applying (1) to both sides

$$\cos(\theta+\phi) + i\sin(\theta+\phi) = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$$

$$= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\phi)$$

so equating real and imaginary parts gives the addition formulae for trigonometry.