University of St Andrews

MAY 2016 EXAMINATION DIET

SCHOOL OF MATHEMATICS & STATISTICS

MODULE CODE: MT5823

MODULE TITLE: Semigroups

EXAM DURATION: 2 hours

EXAM INSTRUCTIONS: Attempt ALL questions.

The number in square brackets shows the maximum marks obtainable for that question or part-question.

Your answers should contain the full working required to justify your solutions.

PERMITTED MATERIALS: No calculators

YOU MUST HAND IN THIS EXAM PAPER AT THE END OF THE EXAM.

PLEASE DO NOT TURN OVER THIS EXAM PAPER UNTIL YOU ARE INSTRUCTED TO DO SO.

1. Let S be the semigroup generated by the boolean matrices

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Boolean matrices are multiplied in the usual way but where

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 1 \end{array} \qquad \begin{array}{c|cc} \times & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1. \end{array}$$

For example,

$$\begin{aligned} ab &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} (0 \times 1) + (1 \times 1) & (0 \times 0) + (1 \times 1) \\ (0 \times 1) + (0 \times 1) & (0 \times 0) + (0 \times 1) \end{pmatrix} \\ &= \begin{pmatrix} 0+1 & 0+1 \\ 0+0 & 0+0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

(a) Show that S has 6 elements. **[4]**

(b) Draw the left and the right Cayley graphs of S. **[4]**

(c) Find the $\mathscr{R}$-, $\mathscr{L}$-, and $\mathscr{D}$-classes of S. Draw the partial order of the $\mathscr{D}$-classes of S. **[4]**

(d) Show that S has 3 two-sided ideals and describe the Rees quotient of S by each of these ideals. **[5]**

(e) State the definitions of:

(i) a regular semigroup; **[1]**

(ii) the inverse of an element in a semigroup; **[1]**

(iii) an inverse semigroup. **[1]**

(f) Is S a regular semigroup? Is S an inverse semigroup? **[3]**

(g) Prove that an element of a semigroup is regular if and only if it has an inverse. **[2]**

2. A semigroup is called a *rectangular group* if it is isomorphic to a direct product of a group and a rectangular band. In other words, the semigroup S is a rectangular group if $S \cong G \times R = \{(g, r) \ : \ g \in G, \ r \in R\}$ where G is a group and R is a rectangular band, with multiplication defined by

$$(g, r)(h, s) = (gh, rs).$$

A semigroup is *orthodox* if its idempotents form a subsemigroup.

Let G denote a finite group and R denote a finite rectangular band.

(a) State the definition of a rectangular band. **[1]**

(b) Let e denote the identity of G. Prove that $\{e\} \times R$ is the set of idempotents in $G \times R$. Deduce that $G \times R$ is an orthodox semigroup. **[3]**

(c) State the definition of a simple semigroup. **[1]**

(d) Show that the rectangular group $G \times R$ is a simple semigroup. **[4]**

(e) State the Rees Theorem without giving a proof. **[2]**

Let $S = \mathscr{M}[G; I, \Lambda; P]$, where I and Λ are finite index sets, G is a finite group, and $P = (p_{\lambda,i})_{\lambda \in \Lambda, i \in I}$ is a $|\Lambda| \times I$ matrix with entries in G and suppose that S is orthodox.

(f) Prove that $\left\{(i, p_{\lambda,i}^{-1}, \lambda) : i \in I, \ \lambda \in \Lambda\right\}$ is the set of idempotents in S. **[3]**

(g) Show that for every $i \in I$ and $\lambda \in \Lambda$ there exist $q_i, r_\lambda \in G$ such that $p_{\lambda,i} = r_\lambda q_i$. **[5]**

(h) Let $I \times \Lambda$ be a rectangular band, and denote the direct product of G and the rectangular band $I \times \Lambda$ by $G \times (I \times \Lambda)$. Prove that $\phi : S \to G \times (I \times \Lambda)$ defined by

$$(i, g, \lambda)\phi = (q_i g r_\lambda, (i, \lambda))$$

is an isomorphism. Deduce that S is a rectangular group. **[4]**

(i) Show that a finite semigroup is simple and orthodox if and only if it is a rectangular group. **[2]**

END OF PAPER