COMPLEX NUMBERS

by

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Course: MT1002 Mathematics

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Section 0 – Introduction

The history of complex numbers as it should have been.

One important aim by introducing new number systems is to increase the kinds of equations which can be solved. So if we start with the natural numbers

$$\mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \dots\}\,,$$

in order to be able to solve equations such as $4 + x = 2$ we have to create the integers

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}\,.$$

To deal with with equations like $2x = -3$ we need to entend the number system even further and introduce the rational numbers

$$\mathbb{Q} = \{\tfrac{p}{q} \,|\, p \in \mathbb{Z}\,,\; q \in \mathbb{N}\}\,.$$

There are still equations with no solution, for example $x^2 = 2$, and it is partly in response to this that we extend the number system once again to the real numbers

$$\mathbb{R}\,.$$

However, there are still equations that cannot be solved, e.g. $x^2 = -1$. A formal solution to the equation $x^2 = -1$ is $x = \sqrt{-1}$, but we know that the square of any non-zero real number is positive and so cannot equal -1. We are therefore required to extend the system one more time; this extension leads to the complex numbers

$$\mathbb{C} = \{a + b\sqrt{-1} \,|\, a, b \in \mathbb{R}\}\,.$$

The history of complex numbers as it really was.

It is interesting historically that the introduction of $\sqrt{-1}$ was not in connection with the natural extension of the real number system as described above. In stead, the symbol $\sqrt{-1}$ was introduced as an intermediate tool for solving cubics.

The solution of quadratics had been know for a very long time, and it would have been said that the quadratic $ax^2 + bx + c = 0$ had no roots when $b^2 - 4ac < 0$. With cubics the situation was different. In the 16'th century, Italian mathematicians discovered how to solve equations of degree 3. The chief figures involved were Bombelli, Scipione del ferro, Cardano, Tartaglia and Vieta. In *Ars Magna*, Cardano first published the solution to "cube plus cosa equals number", i.e. to a cubic of the form

$$x^3 + mx = n\,,$$

where m and n are given real numbers. He attacked the problem geometrically by decomposing three-dimensional cubes into various pieces, but in modern algebraic notation his solution is

$$x = \sqrt[3]{\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}} \;-\; \sqrt[3]{-\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}$$

This formula works nicely in solving, for example, the cubic

$$x^3 + 24x = 56\,.$$

Here $m = 24$ and $n = 56$, and we compute

$$\sqrt{\frac{n^2}{4} + \frac{m^3}{27}} = \sqrt{\frac{56^2}{4} + \frac{24^3}{27}} = \sqrt{1296} = 36\,.$$

Then, according to Cardano's formula above, the cubic's solution is

$$x = \sqrt[3]{\frac{56}{2} + 36} \; - \; \sqrt[3]{-\frac{56}{2} + 36} = \sqrt[3]{64} \; - \; \sqrt[3]{8} = 4 - 2 = 2\,.$$

But what happens if we consider the cubic

$$x^3 - 78x = 220\,?$$

Here we have $m = -78$ and $n = 220$, so that

$$\sqrt{\frac{n^2}{4} + \frac{m^3}{27}} = \sqrt{\frac{200^2}{4} + \frac{(-78)^3}{27}} = \sqrt{-5476}\,.$$

We have encountered the square root of a negative number, and it appears that Cardano's formula does not work in this case. However, what really puzzled mathematicians was that the cubic $x^3 - 78x = 220$ *does* have a (real) solution, namely 10. Moreover, we can find two other (real) solutions, namely $-5 + \sqrt{3}$ and $-5 - \sqrt{3}$. (It is easy the check that $10^3 - 78(10) = 220$, $(-5 + \sqrt{3})^3 - 78(-5 + \sqrt{3}) = 220$, and $(-5 - \sqrt{3})^3 - 78(-5 - \sqrt{3}) = 220$.) This situation is most unsatisfactory, for here is a cubic with three (real) solutions, yet Cardano's formula seemed incapable of finding any of them. Scholars were more puzzled than ever.

So matters stood for a generation until another Italian mathematician, Rafael Bombelli (1526–1572), had a strinking insight in his *Algebra* of 1572. He suggested that square roots of negative numbers could be introduced at least temporarily, in going from the cubic to its (real) solutions. In this way these strange numbers would serve as an intermediate tool for solving cubics. To see what Bombelli had in mind, we return to the cubic from before

$$x^3 - 78x = 220\,.$$

As noted before, here we have $m = -78$ and $n = 220$, so that

$$\sqrt{\frac{n^2}{4} + \frac{m^3}{27}} = \sqrt{\frac{200^2}{4} + \frac{(-78)^3}{27}} = \sqrt{-5476}\,.$$

Temporarily, following Bombelli's suggestion and suspending any prejudice against square roots of negative numbers, we write

$$\sqrt{-5476} = \sqrt{5476} \times \sqrt{-1} = 74\sqrt{-1}\,.$$

Then applying Cardano's formula to the cubic $x^3 - 78x = 220$ yields

$$\begin{aligned} x &= \sqrt[3]{\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}} \; - \; \sqrt[3]{-\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}} \\ &= \sqrt[3]{\frac{220}{2} + \sqrt{-5476}} \; - \; \sqrt[3]{-\frac{220}{2} + \sqrt{-5476}} \\ &= \sqrt[3]{110 + 74\sqrt{-1}} \; - \; \sqrt[3]{-110 + 74\sqrt{-1}}\,. \end{aligned}$$

This seems to have made matters worse, for we do not only retain the square root of -1 but now find it embedded within a cube root. However, Bombelli recognized that this expression could yet do the job. Indeed, Bombelli noted that

$$\begin{aligned}(5+\sqrt{-1})^3 &= (5+\sqrt{-1})(5+\sqrt{-1})(5+\sqrt{-1})\\ &= (5+\sqrt{-1})(5^2+5\sqrt{-1}+5\sqrt{-1}+(\sqrt{-1})^2)\\ &= (5+\sqrt{-1})(25+10\sqrt{-1}+(-1))\\ &= (5+\sqrt{-1})(24+10\sqrt{-1})\\ &= 5\times 24\ +\ 50\sqrt{-1}\ +\ 24\sqrt{-1}\ +\ 10(\sqrt{-1})^2\\ &= 120+74\sqrt{-1}+10(-1)\\ &= 110+74\sqrt{-1}\end{aligned}$$

using the fact that $(\sqrt{-1})^2 = -1$ and assuming that the mysterious $\sqrt{-1}$ satisfies all the usual rules of algebra. Having thus found that $(5+\sqrt{-1})^3 = 110+74\sqrt{-1}$, we take cube roots of each side to conclude that

$$5+\sqrt{-1} = \sqrt[3]{110+74\sqrt{-1}}\,.$$

A similar computation shows that $(-5+\sqrt{-1})^3 = -110+74\sqrt{-1}$, and so

$$-5+\sqrt{-1} = \sqrt[3]{-110+74\sqrt{-1}}\,.$$

Finally, substituting the cube roots just determined into Cardano's formula we find

$$\begin{aligned}x &= \sqrt[3]{110+74\sqrt{-1}}\ -\ \sqrt[3]{-110+74\sqrt{-1}}\\ &= (5+\sqrt{-1})-(-5+\sqrt{-1})\\ &= 5+\sqrt{-1}+5-\sqrt{-1}\\ &= 10\,.\end{aligned}$$

As we previously checked, $x = 10$ is a solution to the original cubic $x^3 - 78x = 220$. Hence:

> the mysterious symbol $\sqrt{-1}$ served as a crucial intermediate tool for solving cubics and was used in a purely formal sense without any meaning being attached to it.

Bombelli's approach – "a wild thought," he called it – appeared to work as much by magic as by logic. "The whole matter," he wrote, "seems to rest on sophistry rather than on truth". Cardano called them *numeri ficti*, and the fact that we still speak of imaginary numbers today is testimony to the persistence of such a philosophical attitude.

The philosophical question of what the square root of a negative number could possible mean was answered to some extent by showing that it could be represented geometrically. Like the algebraic development, this took some time, with a number of mathematicians generating the essential ideas, including Descartes (1637), Wallis (1673), Wessel (1797), Argand (1806) and Gauss who published an account of the representation of complex numbers in the plane in 1831. He had been aware of this some 20 years earlier, but it seems that his natural caution let him to be wary of publicly acknowledgning ideas which were still felt to be philosophical suspect.

It was the Irish mathematician, linguist and astronomer William Rowan Hamilton (1805–1865) who finally showed in 1833 how one could define the complex numbers solely in terms of real numbers without the need for a fictitious quantity like the non-existent square root of a negative number. The story of this and his subsequent investigations is a fascinating one, culminating in his discovery of a hypercomplex number system, the quaternions, and the famous (if not apocryphal) episode of his scrathing the crucial equations on a stone bridge. We shall explore Hamilton's construction of the complex numbers in some detail in the next section.

However, the story does no end there. A great deal has been done during the past two centuries in calculus and analysis involving complex numbers and functions, giving rise to one of the most powerful and satisfying branches of mathematics, which is a cornerstone of many undergraduate courses.

Section 1 – Definition of the complex numbers

The naive way to define the complex numbers is simply to "invent" a symbol $i = \sqrt{-1}$ and stipulate that

$$i^2 = (\sqrt{-1})^2 = -1\,.$$

To obtain an entire number system including the real numbers $\mathbb{R}$ and the new symbol $i = \sqrt{-1}$ we could naively define the complex numbers as expressions of the form

$$a + bi = a + b\sqrt{-1}$$

where a and b are real numbers, and agree that these expressions are added and multiplied as if they obeyed all the usual rules of algebra, i.e.

$$(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_1) + (b_1 + b_2)i \tag{1.1}$$

and

$$\begin{aligned}(a_1 + b_1 i)(a_2 + b_2 i) &= a_1a_2 + a_2b_1i + a_1b_2i + b_1b_2i^2 \\ &= a_1a_2 + a_2b_1i + a_1b_2i - b_1b_2 \\ &= (a_1a_2 - b_1b_2) + (a_1b_2 + a_2b_1)i\,, \end{aligned}\tag{1.2}$$

for all real numbers $a_1, a_2, b_1 b_2$.

The preceeding "definition" of the complex numbers doubtless seems somewhat eluse and unsatisfying. After all:

What is $i = \sqrt{-1}$?

Furthermore:

What is an expression of the form $a + b\sqrt{-1}$?

The solution to these problems arises from the work of Hamilton in 1833 referred to in the Introduction. He gave a formal definition of complex numbers in terms of pairs of real numbers. The fact that the geometrical representations involved coordinates fitted very well with the ordered pair idea. We will give his definition and look at some of the consequences, in particular showing how it leads to the traditional notation for complex numbers used in at the beginning of this section.

1.1. Definition of complex numbers.
The set $\mathbb{C}$ of complex numbers is defined by

$$\mathbb{C} = \{(a, b) \,|\, a, b \in \mathbb{R}\}\,.$$

We define addition and multiplication of complex numbers by

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_1, b_1 + b_2) \tag{1.3}$$

and

$$(a_1, b_1)(a_2, b_2) = (a_1a_2 - b_1b_2, a_1b_2 + a_2b_1)\,, \tag{1.4}$$

for all real numbers $a_1, a_2, b_1 b_2$.

Remark.
Naturally, Hamilton formulated his definitions of the arithmetic operations to reflect the way the tradional notation in (1.1) and (1.2) worked. He certainly did not invent arbitrarily what at first sight appears to be a bizarre multiplication. Indeed, if we identify the traditional notation $a + b\sqrt{-1}$ with the pair (a, b), then the reader will observe that (1.1) corresponds to (1.3) and that (1.2) corresponds to (1.4). (if this is not clear, then the reader is invited to look hard at the equations (1.1) and (1.3) (and the equations (1.2) and (1.4)) until this becomes crystal clear.)

Consider the subset of $\mathbb{C}$ for which the second coordinate is zero, i.e. the set

$$\{(a, 0) \,|\, a \in \mathbb{R}\}\,.$$

We see that for $(x, 0), (y, 0) \in \{(a, 0) \,|\, a \in \mathbb{R}\}$ we have using definitions (1.3) and (1.4),

$$(x, 0) + (y, 0) = (x + y, 0)$$

and

$$(x, 0) + (y, 0) = (xy - 0 \times 0, x \times 0 + y \times 0) = (xy, 0)\,.$$

Hence, the subset $\{(a, 0) \,|\, a \in \mathbb{R}\}$ behaves exactly like the real numbers in respect of algebra and arithmetic. We can therefore identify $\{(a, 0) \,|\, a \in \mathbb{R}\}$ with the set of real numbers, i.e.:

we indentify a with $(a, 0)$.

1.2. Definition of the $a + bi$ notation.
Define $i \in \mathbb{C}$ by

$$i = (0, 1)\,.$$

Then using definition (1.4)

$$i^2 = (0, 1)(0, 1) = (0 \times 0 - 1 \times 1, 0 \times 1 + 0 \times 1) = (-1, 0) = -1\,.$$

And for $a, b \in \mathbb{R}$ we have (recall that we have identified a with $(a, 0)$, and that we have identified b with $(b, 0)$)

$$\begin{aligned} a + bi &= (a, 0) + (b, 0)(0, 1) \\ &= (a, 0) + (b \times 0 - 1 \times 0, 0 \times 0 + 1 \times b) \\ &= (a, 0) + (0, b) \\ &= (a, b)\,. \end{aligned}$$

Section 2 – The algebra of the complex numbers

We will now see that the algebraic rules for the complex numbers are the same as those for the real numbers.

2.1. Addition is commutative.
For all $z_1, z_2 \in \mathbb{C}$ we have

$$z_1 + z_2 = z_2 + z_1 \,.$$

This is easily seen. Indeed, let $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ with $a_j, b_j \in \mathbb{R}$. Then

$$\begin{aligned} z_1 + z_2 &= (a_1 + b_1 i) + (a_2 + b_2 i) \\ &= (a_1, b_1) + (a_2, b_2) \\ &= (a_1 + a_2, b_1 + b_2) \,, \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} z_2 + z_1 &= (a_2 + b_2 i) + (a_1 + b_1 i) \\ &= (a_2, b_2) + (a_1, b_1) \\ &= (a_2 + a_1, b_2 + b_1) \,. \end{aligned}$$

Hence, $z_1 + z_2 = z_2 + z_1$.
Looking at (1.5), we also note the following useful formula which shows that we can add complex numbers in the $a + bi$ form as if the followed the usual rules of algebra,

$$(a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2) i \,.$$

2.2. Multiplication is commutative.
For all $z_1, z_2 \in \mathbb{C}$ we have

$$z_1 z_2 = z_2 z_1 \,.$$

Indeed, let $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ with $a_j, b_j \in \mathbb{R}$. Then

$$\begin{aligned} z_1 z_2 &= (a_1 + b_1 i)(a_2 + b_2 i) \\ &= (a_1, b_1)(a_2, b_2) \\ &= (a_1 a_2 - b_1 b_2, a_1 b_2 + a_2 b_1) \,, \end{aligned} \tag{1.6}$$

and

$$\begin{aligned} z_2 z_1 &= (a_2 + b_2 i)(a_1 + b_1 i) \\ &= (a_2, b_2)(a_1, b_1) \\ &= (a_2 a_1 - b_2 b_1, a_2 b_1 + a_1 b_2) \,. \end{aligned}$$

Hence, $z_1 z_2 = z_2 z_1$.
Looking at (1.6), we also note the following useful formula which shows that we can multiply complex numbers in the $a + bi$ form as if they followed the usual rules of algebra together with the fact that $i^2 = -1$,

$$(a_1 + b_1 i)(a_2 + b_2 i) = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1) i$$

2.3. Addition is associative.
For all $z_1, z_2, z_3 \in \mathbb{C}$ we have

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 \,.$$

This follows by a calculation similar to the calculation in 2.1 and is therefore omitted.

2.4. Multiplication is associative.
For all $z_1, z_2, z_3 \in \mathbb{C}$ we have

$$z_1(z_2 z_3) = (z_1 z_2) z_3 \,.$$

This follows by a lengthy calculation similar to the calculation in 2.2 and is therefore omitted.

2.5 0 is the additive identity.
For all $z \in \mathbb{C}$ we have

$$z + 0 = z \,.$$

Indeed, for $z = a + bi$ with $a, b \in \mathbb{R}$, we have $z + 0 = (a + ib) + 0 = (a, b) + (0, 0) = (a, b) = z$.

2.6. 1 is the multiplicative identity.
For all $z \in \mathbb{C}$ we have

$$z \times 1 = z \,.$$

Indeed, for $z = a + bi$ with $a, b \in \mathbb{R}$, we have $z \times 1 = (a + bi)(1, 0) = \ldots = z$. The reader is encouraged to fill in the missing calculation.

2.7. Multiplicative inverses.
For all $z \in \mathbb{C} \setminus \{0\}$ there exists $\frac{1}{z}$ such that

$$z \times \frac{1}{z} = 1 \,.$$

Indeed, if $z = a + bi$ with $a, b \in \mathbb{R}$, then we must find $\frac{1}{z} = x + yi$ with $x, y \in \mathbb{R}$ such that

$$(a + bi)(x + yi) = 1 \,,$$

i.e.

$$(ax - by) + (bx + ay)i = 1 \,.$$

This implies that

$$\begin{aligned} ax - by &= 1 \,, \\ bx + ay &= 0 \,. \end{aligned}$$

The solution x, y to this system of equations is

$$x = \frac{a}{a^2 + b^2} \,, \quad y = -\frac{b}{a^2 + b^2} \,.$$

Hence

$$\frac{1}{a + bi} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i \,.$$

Example.
Let $z = 2 + i$ and $w = 5 + 3i$. Then

$$z + w = (2 + i) + (5 + 3i) = 7 + 4i\,,$$

$$zw = (2 + i)(5 + 3i) = 10 + 5i + 6i + 3i^2 = 10 + 11i - 3 = 13 + 11i\,,$$

$$\begin{aligned}
\frac{z}{w} &= \frac{2+i}{5+3i} = \frac{(2+i)(5-3i)}{(5+3i)(5-3i)} \\
&= \frac{10+5i-6i-3i^2}{25+15i-15i-9i^2} = \frac{10-i+3}{25+9} \\
&= \frac{13-i}{34} = \frac{13}{34} - \frac{1}{34}i\,,
\end{aligned}$$

$$\begin{aligned}
\frac{w}{z} &= \frac{5+3i}{2+i} = \frac{(5+3i)(2-i)}{(2+i)(2-i)} \\
&= \frac{10+6i-5i-3i^2}{4+2i-2i-i^2} = \frac{10+i+3}{4+1} \\
&= \frac{13+i}{5} = \frac{13}{5} + \frac{1}{5}i\,.
\end{aligned}$$

Section 3 – The geometry of the complex numbers

We remarked in the introduction that geometrical respresentations for complex numbers were investigated during the 17'th–19'th centuries. The Hamilton notation itself is of the same form as that used in two-dimensional coordinate geometry, and so it seems natural to represent a complex number $x + yi$ for $x, y \in \mathbb{R}$ as the point (x, y) in the cartesian coordinate plane. This representation is referred to variously as the Argand diagram, the Gaussian plane et.c. Not wishing to single out any one particular mathematician above the others involved, we shall follow the practice of referring simply to the complex plane. In this context, the x-axis and the y-axis are called the real axis and the imaginary axis respectively.

Representing complex numbers in the the plane would not be especially significant unless the various operations and quanteties associated with complex numbers had a natural geometrical interpretation, and we shall explore this here and in the subsequent sections.Adding a visual component to the algebraic aspect of complex numbers will enable a much richer concept image for complex numbers to be established.

3.1. Definition. Real and imaginary part.
Let $z = x + yi$ be a complex number with $x, y \in \mathbb{R}$. The real part of z is defined by

$$\mathrm{Re}(z) = x\,.$$

The imaginary part of z is defined by

$$\mathrm{Im}(z) = y\,.$$

3.2. Definition. Complex conjugate.
Let $z = x + yi$ be a complex number with $x, y \in \mathbb{R}$. The complex conjugate of z is defined by

$$\overline{z} = x - iy\,.$$

3.3. Definition. Modulus.
Let $z = x + yi$ be a complex number with $x, y \in \mathbb{R}$. The modulus of z is defined by

$$|z| = \sqrt{x^2 + y^2}\,.$$

Example.
Let $z = 7 - 2i$ and $w = 5i$. Then

$$\mathrm{Re}(z) = 7\,, \quad \mathrm{Im}(z) = -2\,, \quad |z| = \sqrt{7^2 + (-2)^2} = \sqrt{49 + 4} = \sqrt{53}\,,$$

and

$$\mathrm{Re}(w) = 0\,, \quad \mathrm{Im}(w) = 5\,, \quad |w| = \sqrt{0^2 + 5^2} = \sqrt{25} = 5\,.$$

3.4. Theorem. Basic arithmetic of complex conjugates.
Let $z, w \in \mathbb{C}$.

(i) $\overline{z + w} = \overline{z} + \overline{w}$.
(ii) $\overline{zw} = \overline{z}\,\overline{w}$.
(iii) $\overline{\overline{z}} = z$.
(iv) If $w \neq 0$, then
$$\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}\,.$$
(v) $z + \overline{z} = 2\,\mathrm{Re}(z)$ and $z - \overline{z} = 2\,\mathrm{Im}(z)\,i$.
(vi) $|zw| = |z|\,|w|$.
(vii) $z\,\overline{z} = |z|^2$.

Proof
Let $z = x + yi$ and $w = u + vi$ for $x, y, u, v \in \mathbb{R}$.
(i) We have

$$\begin{aligned}\overline{z + w} &= \overline{x + yi + u + vi} = \overline{(x + u) + (y + v)i} \\ &= (x + u) - (y + v)i = x - yi + u - vi = \overline{z} + \overline{w}\,.\end{aligned}$$

(ii) We have

$$\begin{aligned}\overline{zw} &= \overline{(x + yi)(u + vi)} = \overline{(xu - yv) + (xv + yu)i} \\ &= (xu - yv) - (xv + yu)i\,,\end{aligned}$$

and

$$\begin{aligned}\overline{z}\,\overline{w} &= \overline{x+yi}\,\overline{u+iv} = (x-yi)(u-vi) \\ &= (xu-yv)-(xv+yu)i\,.\end{aligned}$$

Hence $\overline{zw} = \overline{z}\,\overline{w}$.
(iii)–(vi) The verification of (iii)–(v) is left as a small exercise for the student.
(vi) We have

$$\begin{aligned}|zw|^2 &= |(x+yi)(u+iv)|^2 = |(xu-yv)+(xv+yu)i|^2 \\ &= (xu-yv)^2+(xv+yu)^2 = x^2u^2+y^2v^2-2xuyv+x^2v^2+y^2u^2+2xvyu \\ &= x^2u^2+y^2v^2+x^2v^2+y^2u^2 = x^2(u^2+v^2)+y^2(u^2+v^2) \\ &= (x^2+y^2)(u^2+v^2) = |z|^2\,|w|^2\,.\end{aligned}$$

Section 4 – Polar form of complex numbers

If we look at the figure below, particular the interpretation of the modulus, we can perhaps see that, in fact, polar coordinates will be a useful device in the complex plane, and we shall consider this next.

4.1. Definition. Argument.
Let $z = x + yi \in \mathbb{C}$ and assume that $z \neq 0$. Let φ denote the angle from the real axis to the line joining 0 and z, i.e.

$$\cos\varphi = \frac{x}{|z|}\,, \quad \sin\varphi = \frac{y}{|z|}\,.$$

The argument of z, denoted by $\arg(z)$, is the set of angles from the real axis to the line joining 0 and z, i.e.

$$\arg(z) = \{\dots, \varphi - 4\pi, \varphi - 2\pi, \varphi, \varphi + 2\pi, \varphi + 4\pi, \dots\}\,.$$

If $\theta \in \arg(z)$, then θ is called an argument of z, and z can be written in the following polar form

$$z = x + yi = |z|\cos\theta + i|z|\sin\theta = |z|\big(\cos\theta + i\sin\theta\big)\,.$$

The unique argument $\theta \in \arg(z)$ with

$$-\pi < \theta \leq \pi$$

is called the principal argument of z and is denoted by

$$\theta = \mathrm{Arg}(z)\,.$$

Example.
Let $z = 1 - i$. The polar form of z is given by

$$\mathrm{Arg}(z) = -\frac{\pi}{4}\,,$$

and

$$\begin{aligned} z = 1 - i &= |z|\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) \\ &= \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) \end{aligned}$$

Example.
Let $z = 1 - \sqrt{3}i$. The polar form of z is given by

$$\mathrm{Arg}(z) = -\frac{\pi}{3}\,,$$

and

$$\begin{aligned} z = 1 - \sqrt{3}i &= |z|\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right) \\ &= 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right) \end{aligned}$$

Example.
Let $z = \frac{5-i}{2-3i}$. We will now find the polar form of z. We have

$$z = \frac{5-i}{2-3i} = \frac{(5-i)(2+3i)}{(2-3i)(2+3i)} = \frac{10-2i+15i+3}{4+9} = 1+i\,.$$

The polar form of z is given by

$$\mathrm{Arg}(z) = \frac{\pi}{4}\,,$$

and

$$\begin{aligned} z = 1 + i &= |z|\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) \\ &= \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) \end{aligned}$$

Section 5 – de Moivre's formula

5.1. Multiplication in polar form and.
Let z and w be two complex numbers written in polar form

$$\begin{aligned} z &= |z|\,(\cos\theta + i\sin\theta)\,, \\ w &= |w|(\cos\varphi + i\sin\varphi)\,. \end{aligned}$$

We are interesed in computing the polar form of the product zw. If we let

$$zw = |zw|\,(\cos\sigma + i\sin\sigma)\,, \tag{5.1}$$

then we are interested in finding the modulus $|zw|$ and the argument σ of zw.

When multiplying z and w together we obtain

$$\begin{aligned} zw &= |z|(\cos\theta + i\sin\theta)\,|w|(\cos\varphi + i\sin\varphi) \\ &= |z|\,|w|\Big(\big(\cos\theta\,\cos\varphi - \sin\theta\,\sin\varphi\big) - i\big(\sin\theta\,\cos\varphi + \cos\theta\,\sin\varphi\big)\Big) \\ &= |z|\,|w|\big(\cos(\theta+\varphi) + i\sin(\theta+\varphi)\big)\,. \end{aligned} \tag{5.2}$$

Comparing (5.1) and (5.2) we conclude that

$$\begin{aligned} |zw| &= |z|\,|w|\,, \\ \sigma &= \theta + \varphi\,. \end{aligned}$$

The first equation $|zw| = |z|\,|w|$ has already been verified in Section 3. The second equation says that the argument σ of the product equals $\theta + \varphi$, i.e. the argument of the product zw equals the sum of the arguments of z and w; this can be written as

$$\arg(zw) = \arg(z) + \arg(w)\,.$$

Example.
Let

$$z = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\,, \quad w = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\,.$$

We will now compute zw in the $a + bi$ form. Using (5.2) we see that

$$\begin{aligned} zw &= 2\sqrt{2}\left(\cos\left(\frac{2\pi}{3} + \frac{3\pi}{4}\right) + i\sin\left(\frac{2\pi}{3} + \frac{3\pi}{4}\right)\right) \\ &= 2\sqrt{2}\left(\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right) \\ &= 2\sqrt{2}\left(\frac{1-\sqrt{3}}{2\sqrt{2}} + i\frac{-1-\sqrt{3}}{2\sqrt{2}}\right) \\ &= (1-\sqrt{3}) - i(1+\sqrt{3})\,. \end{aligned}$$

5.2. de Moivre's formula.
Let z be a complex in the polar form

$$z = \cos\theta + i\sin\theta\,.$$

Using formula (5.1) we obtain

$$\begin{aligned} z^2 &= z\,z \\ &= \cos(\theta+\theta) + i\sin(\theta+\theta) \\ &= \cos(2\theta) + i\sin(2\theta)\,. \end{aligned}$$

Using (5.1) once again with $w = z^2$, we obtain

$$\begin{aligned} z^2 &= z\,z^2 \\ &= z\,w \\ &= \cos(\theta+2\theta) + i\sin(\theta+2\theta) \\ &= \cos(3\theta) + i\sin(3\theta)\,. \end{aligned}$$

More generally for a positive integer n we obtain after n applications of (5.1),

$$(\cos\theta + i\sin\theta)^n = z^n = \cos(n\theta) + i\sin(n\theta)\,. \tag{5.3}$$

This result is known as de Moivre's formula.

Example.
Let

$$z = 1 + i\,.$$

Find z^{19} in the $a + bi$ form.
Solution. We have

$$z = 1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\,.$$

de Moivre's formula now implies that

$$\begin{aligned} z^{19} &= (\sqrt{2})^{19}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{19} \\ &= 2^{9.5}\left(\cos\frac{19\pi}{4} + i\sin\frac{19\pi}{4}\right) \\ &= 2^9\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) \\ &= 2^9\sqrt{2}(-1+i)\,. \end{aligned}$$

Example.
Show that

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta\,, \quad \sin(2\theta) = 2\sin\theta\,\cos\theta\,,$$

for all real numbers θ.

Solution. Using de Moivre's formula we have

$$\begin{aligned}\cos(2\theta) + i\sin(2\theta) &= (\cos\theta + i\sin\theta)^2 \\ &= \cos^2\theta - \sin^2\theta + i2\sin\theta\,\cos\theta\,.\end{aligned}$$

Equating real and imaginary part now gives

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta\,, \quad \sin(2\theta) = 2\sin\theta\,\cos\theta\,.$$

Example.
Find $\cos(5\theta)$ in terms of $\cos\theta$ and $\sin\theta$.
Solution. Using de Moivre's formula we have

$$\begin{aligned}\cos(5\theta) + i\sin(5\theta) &= (\cos\theta + i\sin\theta)^5 \\ &= \big((\cos\theta + i\sin\theta)^2\big)^2\,(\cos\theta + i\sin\theta) \\ &= \big((\cos^2\theta - \sin^2\theta) + i2\sin\theta\,\cos\theta\big)^2\,(\cos\theta + i\sin\theta) \\ &= \big(\big((\cos^2\theta - \sin^2\theta)^2 - 4\cos^2\theta\,\sin^2\theta\big) + i4\cos\theta\,\sin\theta(\cos^2\theta - \sin^2\theta)\big)\,(\cos\theta + i\sin\theta) \\ &= \big(\cos\theta\,\big((\cos^2\theta - \sin^2\theta)^2 - 4\cos^2\theta\,\sin^2\theta\big) - 4\cos\theta\,\sin^2\theta(\cos^2\theta - \sin^2\theta)\big) \\ &\quad + i\,\big(\sin\theta\,\big((\cos^2\theta - \sin^2\theta)^2 - 4\cos^2\theta\,\sin^2\theta\big) + 4\cos^2\theta\,\sin\theta(\cos^2\theta - \sin^2\theta)\big)\,.\end{aligned}$$

Equating real and imaginary part now gives

$$\cos(5\theta) = \big(\cos\theta\,\big((\cos^2\theta - \sin^2\theta)^2 - 4\cos^2\theta\,\sin^2\theta\big) - 4\cos\theta\,\sin^2\theta(\cos^2\theta - \sin^2\theta)\big)\,.$$

Using the fact that

$$\cos^2\theta + \sin^2\theta = 1\,,$$

the above expression can be simplified to

$$\begin{aligned}\cos(5\theta) &= \big(\cos\theta\,\big((\cos^2\theta + \cos^2\theta - 1)^2 - 4\cos^2\theta\,(1 - \cos^2\theta)\big) - 4\cos\theta\,(1 - \cos^2\theta)(\cos^2\theta + \cos^2\theta - 1)\big) \\ &= 8\cos^5\theta + 8\cos^4\theta - 12\cos^3\theta + 5\cos\theta\,.\end{aligned}$$

Section 6 – Roots of complex numbers.

Much of the discussion about number systems has involved solving equations, and we continue that theme in this section. Here we shall explore the roots z of the equation

$$z^n = a \tag{6.1}$$

for some integer n and some complex number a. We will use polar form and use de Moivre's formula. Therefore assume that a has the polar form

$$a = r(\cos\theta + i\sin\theta)$$

for some given r and given θ. We are now trying to solve the equation

$$z^n = r(\cos\theta + i\sin\theta)\,.$$

We first write z in polar form

$$z = \rho(\cos\varphi + i\sin\varphi)\,.$$

Equation (6.1) now takes the form

$$\big(\,\rho(\cos\varphi + i\sin\varphi)\,\big)^n = r(\cos\theta + i\sin\theta)\,. \tag{6.2}$$

We are now trying to solve (6.2) for ρ and φ. Using de Moivre's formula we obtain

$$\begin{aligned}
& z^n = r(\cos\theta + i\sin\theta) \\
\Updownarrow\ & \\
& \big(\,\rho(\cos\varphi + i\sin\varphi)\,\big)^n = r(\cos\theta + i\sin\theta) \\
\Updownarrow\ & \\
& \rho^n(\cos\varphi + i\sin\varphi)^n = r(\cos\theta + i\sin\theta) \\
\Updownarrow\ & \\
& \rho^n(\cos(n\varphi) + i\sin(n\varphi)) = r(\cos\theta + i\sin\theta) \\
\Updownarrow\ & \\
& \begin{cases} \rho^n = r\,, \\ \cos(n\varphi) = \cos\theta \\ \sin(n\varphi) = \sin\theta \end{cases} \\
\Updownarrow\ & \\
& \begin{cases} \rho = \sqrt[n]{r}\,, \\ n\varphi = \theta + 2\pi k \qquad \text{for some } k \in \mathbb{Z} \end{cases} \\
\Updownarrow\ & \\
& \begin{cases} \rho = \sqrt[n]{r}\,, \\ \varphi = \frac{\theta + 2\pi k}{n} \qquad \text{for some } k \in \mathbb{Z} \end{cases} \\
\Updownarrow\ & \\
& z = \sqrt[n]{r}\left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right) \qquad \text{for some } k \in \mathbb{Z}.
\end{aligned}$$

This shows that:

$$
\begin{aligned}
&\text{the equation}\\
z^n = a \quad &\text{where} \quad a = r(\cos\theta + i\sin\theta)\\
&\text{has the following } n \text{ roots:}\\
z &= \sqrt[n]{r}\left(\cos\left(\frac{\theta}{n}\right) + i\sin\left(\frac{\theta}{n}\right)\right)\\
z &= \sqrt[n]{r}\left(\cos\left(\frac{\theta+2\pi}{n}\right) + i\sin\left(\frac{\theta+2\pi}{n}\right)\right)\\
z &= \sqrt[n]{r}\left(\cos\left(\frac{\theta+4\pi}{n}\right) + i\sin\left(\frac{\theta+4\pi}{n}\right)\right)\\
z &= \sqrt[n]{r}\left(\cos\left(\frac{\theta+6\pi}{n}\right) + i\sin\left(\frac{\theta+6\pi}{n}\right)\right)\\
&\vdots\\
z &= \sqrt[n]{r}\left(\cos\left(\frac{\theta+2(n-1)\pi}{n}\right) + i\sin\left(\frac{\theta+2(n-1)\pi}{n}\right)\right)
\end{aligned}
$$

We see that all the distinct solutions to the equations $z^n = a = r(\cos\theta + i\sin\theta)$, i.e. the following n complex numbers

$$\begin{aligned}
z_0 &= \sqrt[n]{r}\left(\cos\left(\frac{\theta}{n}\right) + i\sin\left(\frac{\theta}{n}\right)\right)\\
z_1 &= \sqrt[n]{r}\left(\cos\left(\frac{\theta+2\pi}{n}\right) + i\sin\left(\frac{\theta+2\pi}{n}\right)\right)\\
z_2 &= \sqrt[n]{r}\left(\cos\left(\frac{\theta+4\pi}{n}\right) + i\sin\left(\frac{\theta+4\pi}{n}\right)\right)\\
z_3 &= \sqrt[n]{r}\left(\cos\left(\frac{\theta+6\pi}{n}\right) + i\sin\left(\frac{\theta+6\pi}{n}\right)\right)\\
&\vdots\\
z_{n-1} &= \sqrt[n]{r}\left(\cos\left(\frac{\theta+2(n-1)\pi}{n}\right) + i\sin\left(\frac{\theta+2(n-1)\pi}{n}\right)\right),
\end{aligned}$$

have the same modulus, viz.

$$|z_0| = |z_1| = \ldots = |z_{n-1}| = \sqrt[n]{r} = \sqrt[n]{|a|}$$

(since $r = |a|$). We also see that the arguments increase from one to the other by the addition of $\frac{2\pi}{n}$. This implies that:

> The n distinct solutions $z_0, z_1, \ldots, z_{n-1}$ to the equation $z^n = a$ will be equally spaced around the circle with centre at 0 and radius $\sqrt[n]{|a|}$, at the vertices of a regular n sided polygon.

Example
Solve the equation

$$z^3 = 1\,. \tag{6.4}$$

Solution. Since

$$1 = 1(\cos 0 + i \sin 0)\,,$$

it follows from ??????, with $r = 1$ and $\theta = 0$, that the 3 solutions z_0, z_1, z_2 to (6.4) are given by

$$\begin{aligned}
z_0 &= \cos\left(\frac{0}{3}\right) + i \sin\left(\frac{0}{3}\right) = 1\,,\\
z_1 &= \cos\left(\frac{0+2\pi}{3}\right) + i \sin\left(\frac{0+2\pi}{3}\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}\,,\\
z_2 &= \cos\left(\frac{0+4\pi}{3}\right) + i \sin\left(\frac{0+4\pi}{3}\right) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}\,.
\end{aligned}$$

Example
Solve the equation

$$z^3 = -8i\,. \tag{6.5}$$

Solution. Since

$$-8i = 8\left(\cos\frac{-\pi}{2} + i\sin\frac{-\pi}{2}\right)\,,$$

it follows from ??????, with $r = 8$ and $\theta = \frac{-\pi}{2}$, that the 3 solutions z_0, z_1, z_2 to (6.5) are given by

$$\begin{aligned}
z_0 &= \sqrt[3]{8}\left(\cos\left(\frac{\frac{-\pi}{2}}{3}\right) + i\sin\left(\frac{\frac{-\pi}{2}}{3}\right)\right) = 2\left(\cos\left(\frac{-\pi}{6}\right) + i\sin\left(\frac{-\pi}{6}\right)\right) \\
&= 2\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = \sqrt{3} - i\,, \\
z_1 &= \sqrt[3]{8}\left(\cos\left(\frac{\frac{-\pi}{2} + 2\pi}{3}\right) + i\sin\left(\frac{\frac{-\pi}{2} + 2\pi}{3}\right)\right) = 2\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right) \\
&= 2i\,, \\
z_2 &= \sqrt[3]{8}\left(\cos\left(\frac{\frac{-\pi}{2} + 4\pi}{3}\right) + i\sin\left(\frac{\frac{-\pi}{2} + 4\pi}{3}\right)\right) = 2\left(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right) \\
&= 2\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = -\sqrt{3} - i\,.
\end{aligned}$$

Example
Solve the equation

$$z^2 = -3 + 4i\,. \tag{6.6}$$

Solution. We have

$$-3 + 4i = 5\left(\cos\theta + i\sin\theta\right)\,,$$

where θ is determined by

$$\cos\theta = -\frac{3}{5}\,, \quad \sin\theta = \frac{4}{5}\,. \tag{6.7}$$

It therefore follows from ??????, with $r = 5$ and θ determined by (6.7), that the 2 solutions z_0, z_1 to (6.6) are given by

$$\begin{aligned}
z_0 &= \sqrt{5}\left(\cos\left(\frac{0+\theta}{2}\right) + i\sin\left(\frac{0+\theta}{2}\right)\right)\\
&= \sqrt{5}\left(\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right)\,,\\
z_1 &= \sqrt{5}\left(\cos\left(\frac{\theta+2\pi}{2}\right) + i\sin\left(\frac{\theta+2\pi}{2}\right)\right)\\
&= \sqrt{5}\left(\cos\left(\frac{\theta}{2}+\pi\right) + i\sin\left(\frac{\theta}{2}+\pi\right)\right)\\
&= -\sqrt{5}\left(\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right)\,.
\end{aligned}$$

However, since

$$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}} = \sqrt{\frac{1-\frac{3}{5}}{2}} = \frac{1}{\sqrt{5}}\,, \quad \sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-\frac{4}{5}}{2}} = \frac{2}{\sqrt{5}}\,,$$

we conclude that

$$\begin{aligned}
z_0 &= \sqrt{5}\left(\frac{1}{\sqrt{5}} = i\frac{2}{\sqrt{5}}\right) = 1 + 2i\,,\\
z_1 &= -\sqrt{5}\left(\frac{1}{\sqrt{5}} = i\frac{2}{\sqrt{5}}\right) = -(1+2i)\,.
\end{aligned}$$

Example
Solve the equation

$$z^6 = 1\,. \tag{6.8}$$

Solution. Since

$$1 = 1(\cos 0 + i \sin 0)\,,$$

it follows from ??????, with $r = 1$ and $\theta = 0$, that the 6 solutions $z_0, z_1, \ldots, z_5$ to (6.8) are given by

$$\begin{aligned}
z_0 &= \cos\left(\frac{0}{6}\right) + i\sin\left(\frac{0}{6}\right) = 1\,,\\
z_1 &= \cos\left(\frac{0+2\pi}{6}\right) + i\sin\left(\frac{0+2\pi}{6}\right) = \frac{1}{2} + i\frac{\sqrt{3}}{2}\,,\\
z_2 &= \cos\left(\frac{0+4\pi}{6}\right) + i\sin\left(\frac{0+4\pi}{6}\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}\,,\\
z_3 &= \cos\left(\frac{0+6\pi}{6}\right) + i\sin\left(\frac{0+6\pi}{6}\right) = -1,\,,\\
z_4 &= \cos\left(\frac{0+8\pi}{6}\right) + i\sin\left(\frac{0+8\pi}{6}\right) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}\,,\\
z_5 &= \cos\left(\frac{0+10\pi}{6}\right) + i\sin\left(\frac{0+10\pi}{6}\right) = \frac{1}{2} - i\frac{\sqrt{3}}{2}\,.
\end{aligned}$$

Example
Solve the equation

$$z^3 = -1 + i\,. \tag{6.9}$$

Solution. Since

$$-1 + i = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\,,$$

it follows from ??????, with $r = \sqrt{2}$ and $\theta = \frac{3\pi}{4}$, that the 3 solutions z_0, z_1, z_2 to (6.9) are given by

$$\begin{aligned}
z_0 &= \sqrt[6]{2}\left(\cos\left(\frac{\frac{3\pi}{4}}{3}\right) + i\sin\left(\frac{\frac{3\pi}{4}}{3}\right)\right) = \sqrt[6]{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) \\
&= \sqrt[6]{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt[3]{2}\,(1+i)\,, \\
z_1 &= \sqrt[6]{2}\left(\cos\left(\frac{\frac{3\pi}{4} + 2\pi}{3}\right) + i\sin\left(\frac{\frac{3\pi}{4} + 2\pi}{3}\right)\right) = \sqrt[6]{2}\left(\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right) \\
&= \sqrt[6]{2}\left(\frac{-1-\sqrt{3}}{2\sqrt{2}} + i\frac{-1+\sqrt{3}}{2\sqrt{2}}\right) = \frac{\sqrt[3]{2}}{2}\left((-1-\sqrt{3}) + i(-1+\sqrt{3})\right)\,, \\
z_2 &= \sqrt[6]{2}\left(\cos\left(\frac{\frac{3\pi}{4} + 4\pi}{3}\right) + i\sin\left(\frac{\frac{3\pi}{4} + 4\pi}{3}\right)\right) = \sqrt[6]{2}\left(\cos\left(\frac{19\pi}{12}\right) + i\sin\left(\frac{19\pi}{12}\right)\right) \\
&= \sqrt[6]{2}\left(\frac{-1+\sqrt{3}}{2\sqrt{2}} + i\frac{-1-\sqrt{3}}{2\sqrt{2}}\right) = \frac{\sqrt[3]{2}}{2}\left((-1+\sqrt{3}) + i(-1-\sqrt{3})\right)\,.
\end{aligned}$$

Example
Solve the equation
$$z^4 = -2\sqrt{3} - 2i\,. \tag{6.10}$$

Solution. Since
$$-2\sqrt{3} - 2i = \sqrt[4]{4}\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)\,,$$

it follows from ??????, with $r = \sqrt[4]{4}$ and $\theta = \frac{7\pi}{6}$, that the 4 solutions z_0, z_1, z_2, z_3 to (6.10) are given by

$$\begin{aligned}
z_0 &= \sqrt{2}\left(\cos\left(\frac{\frac{7\pi}{6}}{4}\right) + i\sin\left(\frac{\frac{7\pi}{6}}{4}\right)\right) = \sqrt{2}\left(\cos\left(\frac{7\pi}{24}\right) + i\sin\left(\frac{7\pi}{24}\right)\right)\,,\\
z_1 &= \sqrt{2}\left(\cos\left(\frac{\frac{7\pi}{6}+2\pi}{4}\right) + i\sin\left(\frac{\frac{7\pi}{6}+2\pi}{4}\right)\right) = \sqrt{2}\left(\cos\left(\frac{19\pi}{24}\right) + i\sin\left(\frac{19\pi}{24}\right)\right)\,,\\
z_2 &= \sqrt{2}\left(\cos\left(\frac{\frac{7\pi}{6}+4\pi}{4}\right) + i\sin\left(\frac{\frac{7\pi}{6}+4\pi}{4}\right)\right) = \sqrt{2}\left(\cos\left(\frac{31\pi}{24}\right) + i\sin\left(\frac{31\pi}{24}\right)\right)\,,\\
z_3 &= \sqrt{2}\left(\cos\left(\frac{\frac{7\pi}{6}+6\pi}{4}\right) + i\sin\left(\frac{\frac{7\pi}{6}+6\pi}{4}\right)\right) = \sqrt{2}\left(\cos\left(\frac{43\pi}{24}\right) + i\sin\left(\frac{43\pi}{24}\right)\right)\,.
\end{aligned}$$

Section 7 – Complex exponential and trigonometric functions.

In this section we shall look at one of the most surprising and fundamental relationship between the exponential function and the trigonometric functions, namely Euler's formula. To derive Euler's formula we shall need to assume some results from calculus, namely the series expansion of the exponential, sine and cosine functions.

7.1. The series expansion of the exponential, sine and cosine functions.
For all $x \in \mathbb{R}$ we have

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \frac{x^{14}}{14!} + \cdots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \frac{x^{15}}{15!} + \cdots,$$

$$e^x = \exp x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots.$$

Motivated by Theorem 7.1 we now define $\cos(z)$, $\sin(z)$ and $\exp(z)$ for a complex number z as follows.

7.2. Definition of the exponential, sine and cosine functions of a complex variable.
For all $z \in \mathbb{C}$ we define

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \frac{z^{10}}{10!} + \frac{z^{12}}{12!} - \frac{z^{14}}{14!} + \cdots,$$

$$\sin z = x - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \frac{z^{11}}{11!} + \frac{z^{13}}{13!} - \frac{z^{15}}{15!} + \cdots,$$

$$e^z = \exp z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \frac{z^6}{6!} + \frac{z^7}{7!} + \cdots.$$

We will now study some of the properties of $\cos(z)$, $\sin(z)$ and $\exp(z)$.

7.3. Euler's formula.
For all $\theta \in \mathbb{R}$ we have

$$
\begin{aligned}
e^{i\theta} &= 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \cdots \\
&= 1 + i\frac{\theta}{1!} + i^2\frac{\theta^2}{2!} + i^3\frac{\theta^3}{3!} + i^4\frac{\theta^4}{4!} + i^5\frac{\theta^5}{5!} + i^6\frac{\theta^6}{6!} + i^7\frac{\theta^7}{7!} + \cdots \\
&= 1 + i\frac{\theta}{1!} - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + \cdots \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) \\
&\qquad + i\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right) \\
&= \cos\theta + i\sin\theta\,.
\end{aligned}
$$

We have now proved Euler's formula:

$$
e^{i\theta} = \cos\theta + i\sin\theta\,. \tag{7.1}
$$

It follows from Euler's formula that

$$
\begin{aligned}
e^{i\theta} &= \cos\theta + i\sin\theta\,, \\
e^{-i\theta} &= \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta\,.
\end{aligned}
$$

Adding and subtracting these equations gives

$$
\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}\,, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}\,. \tag{7.2}
$$

The formulas in (7.2) are often also referred to as Euler's formulas. Euler's formulas (7.2) should be compared with the defining equations for the hyperbolic functions $\cosh\theta$ and $\sinh\theta$,

$$
\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}\,, \quad \sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}\,.
$$

One particular case of Euler's formula (7.1) which is often quoted is obtained by putting $\theta = \pi$ in (7.1). Putting $\theta = \pi$ is (7.1) gives $e^{i\pi} = \cos\pi + i\sin\pi = -1 + i0 = -1$, i.e.

$$
e^{i\pi} + 1 = 0\,.
$$

This result is known as Euler's equation, and is thought to be quite remarkable since it connects in one simple formula the five most fundamental mathematical constants: 0, 1, π, e and i.

Example
We have

$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}e^{i\frac{\pi}{4}} \; .$$

$$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\frac{\pi}{2}} \; ,$$

$$1 + i\sqrt{3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2e^{i\frac{\pi}{3}} \; .$$

7.4. Multiplication of complex numbers in exponential form. de Moivre's theorem i exponential form

Consider the two complex numbers $e^{i\theta_1}$ and $e^{i\theta_2}$ for $\theta_1, \theta_2 \in \mathbb{R}$. It follows from (5.2) that

$$\begin{aligned} e^{i\theta_1}e^{i\theta_2} &= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) \\ &= e^{i(\theta_1+\theta_2)}, \end{aligned}$$

i.e. multiplication of complex numbers in exponential form is given by the rule:

$$e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}.$$

We will now look at de Moivre's theorem for complex numbers expressed in exponential form. It follows from de Moivre's theorem that

$$\begin{aligned} \left(e^{i\theta}\right)^n &= (\cos\theta + i\sin\theta)^n \\ &= \cos(n\theta) + i\sin(n\theta) \\ &= e^{in\theta}. \end{aligned}$$

Hence, de Moivre's theorem for complex numbers expressed in exponential form takes the following deceptively simple form:

$$\left(e^{i\theta}\right)^n = e^{in\theta}.$$

We will now illustrate how Eulers formulas (7.1) and (7.2) together with the above rules for multiplying complex numbers in exponential form can be used to obtaining various triginometric identities. The general rules for obtain trigonometric identities are as follows:

- For finding $\cos(n\theta)$ or $\sin(n\theta)$ in terms of $\cos\theta$ and $\sin\theta$ use (7.1):

$$e^{i\theta} = \cos\theta + i\sin\theta\,.$$

- For finding $\cos^n(\theta)$ or $\sin^n(\theta)$ in terms of $\cos\theta, \cos(2\theta), \cos(3\theta), \ldots$ and $\sin\theta, \sin(2\theta), \sin(3\theta), \ldots$ use (7.2):

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}\,, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}\,.$$

Example
Find $\cos^3(\theta)$ in terms of $\cos\theta, \cos(2\theta), \cos(3\theta), \ldots$ and $\sin\theta, \sin(2\theta), \sin(3\theta), \ldots$.
Solution. It follows from (7.2) that

$$\begin{aligned}
\cos^3(\theta) &= \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right) \\
&= \frac{1}{8}(e^{i\theta} + e^{-i\theta})(e^{i\theta} + e^{-i\theta})(e^{i\theta} + e^{-i\theta}) \\
&= \frac{1}{8}(e^{i2\theta} + 2 + e^{-i2\theta})(e^{i\theta} + e^{-i\theta}) \\
&= \frac{1}{8}(e^{i3\theta} + 2e^{i\theta} + e^{-i\theta} + e^{i\theta} + 2e^{-i\theta} + e^{-i3\theta}) \\
&= \frac{1}{8}\left(\left(e^{i3\theta} + e^{-i3\theta}\right) + 3\left(e^{i\theta} + e^{-i\theta}\right)\right) \\
&= \frac{1}{8}(2\cos(3\theta) + 6\cos\theta) \\
&= \frac{1}{4}(\cos(3\theta) + 3\cos\theta)\,.
\end{aligned}$$

Example
Find $\sin(4\theta)$ in terms of $\cos\theta$ and $\sin\theta$.
Solution. It follows from (7.1) that

$$\begin{aligned}
\cos(4\theta) + i\sin(4\theta) &= e^{i4\theta} \\
&= \left(e^{i\theta}\right)^4 \\
&= (\cos\theta + i\sin\theta)^4 \\
&= ((\cos^2\theta - \sin^2\theta) + i2\cos\theta\,\sin\theta)^2 \\
&= (\cos^2\theta - \sin^2\theta) - 4\cos\theta\,\sin\theta + i4\cos\theta\,\sin\theta(\cos^2\theta - \sin^2\theta)\,.
\end{aligned}$$

Equating real and imaginary parts now gives

$$\sin(4\theta) = 4\cos\theta\,\sin\theta(\cos^2\theta - \sin^2\theta)\,.$$

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EXERCISES

COMPLEX NUMBERS

1. Perform the required calculation and express the answer in the form $a + ib$ where a and b are real numbers.

(1) $(3 - 2i) - i(4 + 5i)$.

(2) $(7 - 2i)(3i + 5)$.

(3) $(1 + i)(2 + i)(3 + i)$.

(4) $(3 + i)(2 + i)$.

(5) $(i - 1)^3$.

(6) i^5.

(7) $\dfrac{1 + 2i}{3 - 4i} - \dfrac{4 - 3i}{2 - i}$.

(8) $(1 + i)^{-2}$.

(9) $\dfrac{(4 - i)(1 - 3i)}{-1 + 2i}$.

(10) $(1 + i\sqrt{3})(i + \sqrt{3})$.

2. Find the following quantities.

(1) $\mathrm{Re}((1 + i)(2 + i))$.

(2) $\mathrm{Im}((2 + i)(3 + i))$.

(3) $\mathrm{Re}\,\dfrac{4 - 3i}{2 - i}$.

(4) $\mathrm{Im}\,\dfrac{1 + 2i}{3 - 4i}$.

(5) $\mathrm{Re}((i - 1)^3)$.

(6) $\mathrm{Im}((1 + i)^{-2})$.

(7) For $x, y \in \mathbb{R}$, find $\mathrm{Re}((x - iy)^2)$.

(8) For $x, y \in \mathbb{R}$ with $(x, y) \neq (0, 0)$, find $\mathrm{Im}\,\dfrac{1}{x - iy}$.

(9) For $x, y \in \mathbb{R}$, find $\mathrm{Re}((x + iy)(x - iy))$.

(10) For $x, y \in \mathbb{R}$, find $\mathrm{Re}((x + iy)^3)$.

3. Show that

$$\operatorname{Re}(iz) = -\operatorname{Im}(z)$$

for all complex numbers z.

4. Show that addition of complex numbers is associative, i.e. show that

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$$

for all complex numbers z_1, z_2, z_3.

5. Show that multiplication of complex numbers is commutative and associative, i.e. show that

$$z_1 z_2 = z_2 z_1$$

and

$$z_1(z_2 z_3) = (z_1 z_2) z_3$$

for all complex numbers z_1, z_2, z_3.

6. Show that multiplication of complex numbers is distributive over addition, i.e. show that

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$$

and

$$(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$$

for all complex numbers z_1, z_2, z_3.

7. Find the following quatities.

(1) $|(1+i)(2+i)|$.

(2) $\overline{(2+i)(3+i)}$.

(3) $\left|\dfrac{4-3i}{2-i}\right|$.

(4) $\overline{\left(\dfrac{1+2i}{3-4i}\right)}$.

(5) $|(i+1)^{63}|$.

(6) $\overline{(1+i)^3}$

(7) For $z \in \mathbb{C}$, find $|z\overline{z}|$.

(8) For $z \in \mathbb{C}$, find $|z-1|^2$.

8. Sketch the set of points $z \in \mathbb{C}$ determined by the following relations.

(1) $|z+1-2i| = 2$.
(2) $\operatorname{Re}(z+1) = 0$.
(3) $|z+2i| \leq 1$.
(4) $\operatorname{Im}(z-2i) > 6$.

9. Show that,

(1) $\overline{u+v} = \overline{u} + \overline{v}$ for all $u, v \in \mathbb{C}$.

(2) $\overline{uv} = \overline{u}\,\overline{v}$ for all $u, v \in \mathbb{C}$.

(3) $\overline{\left(\dfrac{u}{v}\right)} = \dfrac{\overline{u}}{\overline{v}}$ for all $u, v \in \mathbb{C}$ with $v \neq 0$.

10. Let u, v be complex numbers. Show the that $u\,\overline{v} + \overline{u}\,v$ is a real number.

11. Show that $\overline{\overline{z}} = z$ for all complex numbers z.

12. Show that $\sqrt{2}|z| \geq |\operatorname{Re} z| + |\operatorname{Im} z|$ for all complex numbers z.

13. Show that $|uv| = |u|\,|v|$ for all complex numbers u, v.

14. Find $\operatorname{Arg} z$ for the following values of z.

(1) $i-1$.

(2) $-\sqrt{3}+i$.

(3) $(-1-i\sqrt{3})^2$.

(4) $(1-i)^3$.

(5) $\dfrac{2}{1+i\sqrt{3}}$.

(6) $\dfrac{2}{i-1}$.

(7) $\dfrac{1+i\sqrt{3}}{(1+i)^2}$.

(8) $(1+i\sqrt{3})(1+i)$.

15. Express the following complex numbers in the form $a + ib$ where a and b are real numbers.

(1) $e^{i\pi/2}$.

(2) $4e^{-i\pi/2}$.

(3) $8e^{i7\pi/3}$.

(4) $-2e^{i5\pi/6}$.

(5) $2ie^{-3\pi/4}$.

(6) $6e^{i2\pi/3}e^{i\pi}$

(7) $e^2e^{i\pi}$.

(8) $e^{i\pi/4}e^{-i\pi}$.

16. Let $u = -1 + i\sqrt{3}$ and $v = -\sqrt{3} + i$. Show that

$$\operatorname{Arg}(uv) \neq \operatorname{Arg} u + \operatorname{Arg} v\,.$$

17. Let u, v be complex numbers with $u, v \neq 0$. Show that

$$\arg u = \arg v$$

if and only there exists a number $c > 0$ such that

$$u = cv\,.$$

18. Let u, v be complex numbers with $v \neq 0$. Show that $\arg \dfrac{u}{v} = \arg u - \arg v$ and $\arg \dfrac{1}{v} = -\arg v$

19. Let u, v be complex numbers. Show that $\arg(u\,\overline{v}) = \arg u - \arg v$.

20. Show that $\operatorname{Arg}(z\,\overline{z}) = 0$ and $\operatorname{Arg}(z + \overline{z}) = 0$ for all complex numbers z.

21. Use De Moivre's Formula to prove the following identities for all real numbers θ.

(1) $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$.

(2) $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

22. Find all the following roots.
(1) $(5-12i)^{1/2}$.
(2) $(-3+4i)^{1/2}$.
(3) $(-2+2i)^{1/3}$.
(4) $(-64)^{1/4}$.
(5) $(-1)^{1/5}$.
(6) $(16i)^{1/4}$.
(7) $8^{1/6}$.

23. Find the solutions $z \in \mathbb{C}$ to the equation $z^2 + (1+i)z + 5i = 0$.

24. Find the solutions $z \in \mathbb{C}$ to the equation $(z+1)^3 = z^3$.

25. Find the solutions $z \in \mathbb{C}$ to the equation $z^3 + z^2 + 3z - 5 = 0$.

26. Show that $1+i$ is a solution to the equation $z^{17} + 2z^{15} - 512 = 0$.

27. Let
$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$
be a polynomial with real coefficients $a_n, a_{n-1}, \ldots, a_1, a_0$.
(1) Show that $\overline{P(\overline{z})} = P(z)$ for all $z \in \mathbb{C}$.
(2) Show that if $z \in \mathbb{C}$ is a root of P, then $\overline{z}$ is also a root of P.
(3) Consider the polynomial
$$Q(z) = z^4 - 4z^3 + 6z^2 - 4z + 5\,.$$
Show that i is a root of Q, and find all the roots of Q.

28. Let n be a positive integer.
(1) Let z be a complex number with $z \neq 1$. Show that
$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}\,.$$
(2) Use part (1) and De Moivre's Formula to derive Lagrange's identity,
$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin((n+\frac{1}{2})\theta)}{2\sin(\theta/2)}\,,$$
for all $0 < \theta < 2\pi$.

29. Let n be a positive integer with $n \geq 2$.

(1) Let z be a complex number. Show that

$$(1 + z + z^2 + \cdots + z^{n-1})(1 - z) = 1 - z^n \,.$$

(2) Let

$$\omega_k = e^{2\pi i k/n} \,, \quad k = 0, 1, 2, \ldots, n-1$$

denote the n different n'th unit roots. Deduce from part (1) that

$$\omega_0 + \omega_1 + \omega_2 + \cdots + \omega_{n-1} = 0 \,.$$

(This result is obviously not true for $n = 1$.)