Part 1. History and motivation

A brief history of groups

What follows is an abridged version of The development of group theory from the MacTutor History of Mathematics archive.

Roughly speaking, modern algebra is the study of sets with operations defined on them. A *group* is a set together with a single operation that satisfies certain conditions given in Section 4. We will see that many familiar mathematical objects are groups, such as number systems, matrices, functions and sets. The study of groups or *group theory* considers groups in the abstract, removed from any particular example. Such examples pervade mathematics and other fields of science involving symmetry. Groups can be used in the physics of special relativity and the study of crystals in molecular chemistry.

The notion of a group arose as an abstraction of ideas in the separate, but more or less simultaneous, studies of geometry, number theory, and the theory of algebraic equations from the late 18th century to the late 19th century. Early protagonists of group theory in geometry were Gauss, Möbius and Steiner. In 1761 Euler studied modular arithmetic (see Section 7) and this was continued by Gauss in 1801. Euler and Gauss' work can be thought of as the origin of the study of abelian groups (see Definition 4.5). Permutations (see Section 8) were studied by Lagrange in a paper from 1770 on the theory of algebraic equations.

Lagrange's main object was to find out why cubic and quartic equations could be solved algebraically. In 1799 based on ideas of Lagrange, Ruffini published a work whose purpose was to demonstrate the insolubility of the general quintic equation. Ruffini's proof of the insolubility of the quintic had some gaps and, disappointed with the lack of reaction to his paper Ruffini published further proofs. Cauchy played a major role in developing the theory of permutations. His first paper on the subject was in 1815 but at this stage Cauchy was motivated by permutations of roots of equations. However, in 1844, Cauchy published a major work which set up the theory of permutations as a subject in its own right.

Abel, in 1824, gave the first accepted proof of the insolubility of the quintic, and he used the existing ideas on permutations of roots but little new in the development of group theory. Galois in 1831 was the first to really understand that the algebraic solution of an equation was related to the structure of a group related to the equation. By 1832 Galois had discovered that special subgroups (now called normal subgroups, see Section 18) are fundamental. Galois' work was not known until Liouville published Galois' papers in 1846. Liouville saw clearly the connection between Cauchy's theory of permutations and Galois' work. Betti began in 1851 publishing work relating permutation theory and the theory of equations.

Serret published an important work discussing Galois' work, still without seeing the significance of the group concept. Jordan, however, in papers of 1865, 1869 and 1870 realised the significance of groups of permutations. Klein proposed the Erlangen Program in 1872 which was the group theoretic classification of geometry.

As early as 1849 Cayley published a paper linking his ideas on permutations with Cauchy's. In 1854 Cayley wrote two papers which are remarkable for the insight they have of abstract groups. At that time the only known groups were groups of permutations and even this was a radically new area, yet Cayley defines an abstract group and gives a table to display the group multiplication. He gives the 'Cayley tables' (see Section 5) of some special permutation groups but, much more significantly for the introduction of the abstract group concept, he realised that matrices and quaternions were groups.

Cayley's papers of 1854 were so far ahead of their time that they had little impact. However when Cayley returned to the topic in 1878 with four papers on groups, one of them called The theory of groups, the time was right for the abstract group concept to move towards the centre of mathematical investigation. Cayley proved, among many other results, that every finite group can be represented as a group of permutations.

In the late 1800s the work of Frobenius, Netto (a student of Kronecker), von Dyck, Burnside, Weber (a student of Dedekind) influenced the next generation of mathematicians and brought group theory to the forefront of 20th Century and 21st century mathematics.

How to solve it

The following is some general advice about how to solve problems in mathematics from [**?**].

Understand the problem: *What is the unknown? What are the data? What is the condition?* Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory? Draw a figure; separate the various parts of the conditions; can you write them down?

Devise a plan: Have you seen the problem before? Or have you seen the same problem in a slightly different form? You may be obliged to consider intermediate or auxiliary problems if an immediate connection cannot be found. You should eventually obtain a *plan* of the solution.

Do you know a related problem? Do you know a theorem that could be useful?

Look at the unknown! And try to think of a familiar problem having the same or a similar unknown.

Here is a problem related to yours and solved before. Could you use it? Could you use a result? Could you use its method? Should you introduce some auxiliary element in order to make its use possible?

Can you restate the problem? Can you restate it again still differently? Go back to the definitions.

If you cannot solve the proposed problem try to solve a related problem first. Could you imagine a more accessible related problem? A more general problem? A more special problem? An analogous problem? Could you solve a part of the problem? Keep only a part of the condition, drop the other part; how far is the unknown then determined, how can it vary? Could you derive something useful from the data? Could you think of other data appropriate to determine the unknown? Could you change the unknown or the date, or both, so that the new unknown and the new data are nearer to each other?

Did you use all the data? Did you use the whole condition? Have you taken into account all essential notions involved in the problem?

Carry out your plan: Carrying out your plan of the solution, *check each step.* Can you see clearly that the step is correct? Can you prove that it is correct?

Examine the solution obtained: Can you *check the result?* Can you check the argument?

Can you derive the result differently? Can you see it at a glance?

Can you use the result, or the method, for some other problem?