3 The upper half plane model of hyperbolic space

Let ϕ be the Cayley map.

1. a) Prove that $\phi(\mathbb{H}^2) = \mathbb{D}^2$ and that $\phi(\mathbb{R} \cup \{\infty\}) = S^1$.
   b) Prove that ϕ is invertible and give a simple expression for its inverse (for example in the standard form for a Möbius map).
   c) Prove that for $u, w \in \mathbb{D}^2$ that $d_{\mathbb{D}^2}(u, w) = d_{\mathbb{H}^2}(\phi^{-1}(u), \phi^{-1}(w))$.

2. Let
$$T_1(z) = \overline{z} \qquad \text{(reflection in } \mathbb{R}\text{)},$$
$$T_2(z) = i + \left(\frac{\sqrt{2}}{|z-i|}\right)^2 (z - i) \qquad \text{(reflection in the circle } C(i, \sqrt{2})\text{)}$$
   and
$$T_3(z) = -iz \qquad \text{(clockwise rotation by } \pi/2\text{)}.$$
   Prove that $\phi(z) = T_3(T_2(T_1(z)))$.

3. a) Let $g \in \mathrm{PSL}(2, \mathbb{R})$. Prove that $\phi g \phi^{-1} \in \mathrm{con}^+(1)$.
   b) Using the fact that $\mathrm{con}(1)$ is *precisely* the isometry group of $\mathbb{D}^2$ and that $\mathrm{PSL}(2, \mathbb{R}) = \phi^{-1}\mathrm{con}^+(1)\phi$, prove that the isometry group of $\mathbb{H}^2$ is *precisely* the group generated by $\mathrm{PSL}(2, \mathbb{R})$ together with the reflection $z \mapsto -\overline{z}$.

4. Prove that for $u, w \in \mathbb{H}^2$ with $\mathrm{Re}(u) = \mathrm{Re}(w)$ that
$$d_{\mathbb{H}^2}(u, w) = \left|\log \frac{\mathrm{Im}(u)}{\mathrm{Im}(w)}\right|.$$

5. Recall that the hyperbolic centre and Euclidean centre of a circle in $\mathbb{D}^2$ agree if and only if the common center is the origin. Prove that in $\mathbb{H}^2$ hyperbolic centres and Euclidean centers *never* agree.

6. Prove that
$$A_{\mathbb{H}^2}(F) = A_{\mathbb{D}^2}(\phi(F))$$
   for reasonable sets $F \subset \mathbb{H}^2$.

7. Prove that $\mathbb{H}^2$ is invariant under $\mathrm{PSL}(2, \mathbb{R})$, i.e. prove that for each $g \in \mathrm{PSL}(2, \mathbb{R})$ we have $g(\mathbb{H}^2) = \mathbb{H}^2$.

8. Define a *hyperbolic rectangle* to be a polygon formed by 4 distinct geodesics which all meet at right angles. Prove that hyperbolic rectangles do not exist.

9. Let P be a *convex hyperbolic polygon* with n sides all formed by distinct geodesics. Recall that a set $P \subset \mathbb{H}^2$ is *convex* if for any $w, z \in P$ the hyperbolic geodesic between w and z is contained in P. Suppose that the internal angles are given by $\alpha_1, \ldots, \alpha_n$. Prove that
$$A_{\mathbb{H}^2}(P) = (n-2)\pi - \sum_{i=1}^{n} \alpha_i.$$
   Hard bonus question: Is the convexity assumption necessary?