Chapter 6

Introduction

6.1 Preliminaries

We assume that the independent variables are either x and y or x and t, depending on whether we are dealing with general functions of two variables or with functions that depend on space and time. The dependent variable is u and so either $u(x, y)$ or $u(x, t)$. We will mainly consider functions of two variables but everything discussed here can be extended to functions of more variables. For example, we could have $u(x, y, z, t)$, a function of all three space coordinates and time.

Given a known function, say $u(x, y)$, we can calculate the partial derivatives with respect to x and y. For example, the partial derivative of u with respect to x is written as

$$\frac{\partial u}{\partial x}.$$

We will frequently use the more compact alternative notation for first, second and subsequent derivatives, namely

$$\frac{\partial u}{\partial x} \equiv u_x, \quad \frac{\partial u}{\partial y} \equiv u_y \quad \frac{\partial^2 u}{\partial x^2} \equiv u_{xx}, \quad \frac{\partial^2 u}{\partial x \partial y} \equiv u_{xy} = u_{yx}, \text{ etc.}$$

In its most general form, a partial differential equation (PDE) is a relation between the partial derivatives of u. So, for example, if $u = u(x, y)$, then

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0, \tag{6.1}$$

is a PDE, involving the first and second derivatives of u.

The *order* of a PDE is the order of the highest derivative in F. In the above example, F depends on the second derivatives and so that equation is *second order*.

A PDE is *linear* if F is linear in u, u_x and u_y etc. There is no requirement that F be linear in the independent variables x, y and/or t.

6.2 Important examples

The following are standard PDEs that frequently occur in Applied Mathematics.

- The one dimensional wave equation (or advection equation) applies to problems where a disturbance is propagating in time at constant speed c is

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \qquad \text{or} \qquad u_t + cu_x = 0. \tag{6.2}$$

  This is a first order equation and linear.

- The two dimensional Laplace equation governs steady state problems and is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \text{or equivalently } \nabla^2 u = 0. \tag{6.3}$$

  This is second order and linear.

- The two dimensional Poisson equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = g(x) \text{ or equivalently } \nabla^2 u = g(x). \tag{6.4}$$

  This is a linear, second order, inhomogeneous equation. It can be solved by finding a partial integral and then adding on the solution to the homogeneous equation (which is identical to Laplace's equation). It is frequently solved using a Green's function approach analagous to that developed in chapter 3.

- The one dimensional diffusion equation (or heat conduction equation) describes how a non uniform initial distribution spreads out under the action of diffusion. It is expressed as

$$\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2}. \tag{6.5}$$

  It is a second order equation (because of the second derivatives in x) and linear.

- The second order, linear wave equation in one dimension is

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}. \tag{6.6}$$

- A simple example of a first order, non-linear equation is the advection equation when the speed of the wave is equal to the size of the function u. Thus,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0. \tag{6.7}$$

Chapter 7

First Order PDEs

7.1 Linear, homogeneous equations

We begin by restricting attention to two dimensional, first order, linear, homogeneous PDEs. The general equation can be written in the form

$$a(x,y)\frac{\partial u}{\partial x}+b(x,y)\frac{\partial u}{\partial y}=0 \tag{7.1}$$

This equation needs to be supplemented by an initial condition of the form $u = f(x, y)$ on some line $g(x, y) = 0$, for example $u = f(x)$ on the line $y = 0$. We require that a and b are continuous and, so that the PDE is non-trivial, that a and b are not identically zero. If the solution to this equation is $u(x, y)$ and if we think of the value of u at a particular point (x, y) as representing the height above the xy-plane, then the solution $u(x, y)$ describes a surface.

The idea of a *directional derivative* was introduced in MT2503, where the derivative of u in the direction of the vector $\boldsymbol{l} = (a, b)$ (here only considering a 2D vector) was defined by

$$\boldsymbol{l}\cdot\nabla u=a\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial y}. \tag{7.2}$$

Remember that the gradient vector of the scalar function u is defined as

$$\nabla u=\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right).$$

A quick comparison between (7.1) and (7.2) shows that the PDE is in fact equivalent to saying that the directional derivative of u in the direction of $\boldsymbol{l} = (a, b)$ is zero. In other words u is constant along the direction of $\boldsymbol{l}$. So along the curve defined by the direction vector $\boldsymbol{l}$, we have a simple solution to the PDE.

How do we interpret the curve generated by the direction vector $\boldsymbol{l}$? Consider the simple case where a and b are constants. Going along $\boldsymbol{l}$ is equivalent to moving distance a along in the x direction and distance b up in the y direction. This is, of course, a straight line $y = mx + c$ with gradient $m = b/a$.

We can extend this to the case were a and b, and hence $\boldsymbol{l}$, are functions of x and y. If we consider the curve $y = y(x)$ whose tangent at each point is given by the vector $\boldsymbol{l} = (a, b)$, then at the point (x, y), the slope of the curve is again given by $dy/dx = m = b(x, y)/a(x, y)$. The slope of the tangent vector is thus given by the ODE

$$\frac{dy}{dx}=\frac{b(x,y)}{a(x,y)}. \tag{7.3}$$

Solving this ODE therefore gives the curve $y = y(x)$ along which u is a constant. Assuming that a and b are such that (7.3) may be solved, we obtain a family of curves in the xy-plane, and along each curve u takes a constant value. The constant value will in general vary from one curve to the next. These curves are called the *characteristic curves* of (7.1).

An alternative description may be obtained by considering curves in the xy-plane defined by the parametric representation

$$x = x(s), \qquad y = y(s). \tag{7.4}$$

We construct the curve by selecting values for s and then determining the values of x and y for each s. If the functions $x(s)$ and $y(s)$ are determined as solutions to the differential equations,

$$\begin{aligned} \frac{dx}{ds} &= a(x, y) \\ \frac{dy}{ds} &= b(x, y), \end{aligned}$$

then the left-hand side of our original PDE (7.1) can be written as

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = \frac{dx}{ds}\frac{\partial u}{\partial x} + \frac{dy}{ds}\frac{\partial u}{\partial y}.$$

Writing $u = u(x, y) = u(x(s), y(s))$ and using the chain rule for partial derivatives,

$$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds}, \tag{7.5}$$

and we find that our original PDE reduces to

$$\frac{du}{ds} = 0.$$

Thus, we have reduced the original PDE to a system of three ODEs, namely

$$\begin{aligned} \frac{dx}{ds} &= a(x, y) \\ \frac{dy}{ds} &= b(x, y) \\ \frac{du}{ds} &= 0. \end{aligned}$$

The first pair define the characteristic curves in the xy-plane, while the third states that u is constant along each curve. Again, the particular constant value will depend on the particular characteristic curve.

If we think again of $u(x, y)$ as representing the height above the xy-plane, then contours of constant height (as on a topographic map) correspond to the characteristic curves along which u is constant.

The Cauchy initial value problem

A Cauchy problem defines initial data on any curve intersected by characteristic curves. The initial data fixes the value of the constant u on each characteristic.

Example 7.1 $u_x + c^{-1}u_t = 0, \;\; u(x, 0) = e^{-x^2}$.

Example 7.2 $u_x + c^{-1}u_t = 0, \;\; u(x, 0) = f(x), x \geq 0, \;\; u(x, 0) = g(x), t \geq 0.$

7.2 Semilinear PDEs

A first order PDE of the form

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = c(x,y,u), \tag{7.6}$$

is called a *semilinear* equation. Note that a and b only depend on x and y and do not depend on u. However, c can be any function of x, y and u. If this function has a non-linear dependence of u, then the equation is not strictly speaking linear (hence, the use of the word *semilinear*). However, the same approach as above can be applied to obtain the solution.

Again, thinking of a characteristic curve parametrized by s, $x = x(s)$ and $y = y(s)$, and defined by

$$\frac{dx}{ds} = a(x,y)$$
$$\frac{dy}{ds} = b(x,y),$$

we obtain

$$\frac{du}{ds} = \frac{dx}{ds}\frac{\partial u}{\partial x} + \frac{dy}{ds}\frac{\partial u}{\partial y} = a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = c(x,y,u).$$

We thus obtain the system of three ODEs

$$\frac{dx}{ds} = a(x,y)$$
$$\frac{dy}{ds} = b(x,y)$$
$$\frac{du}{ds} = c(x,y,u).$$

Now, u is no longer constant along each characteristic curve but will vary with x and y according to the solution of the third equation. However, as in the linear case, the characteristic curves are independent of the value of u and determined completely by the form of $a(x,y)$ and $b(x,y)$.

This system of equations can be more conveniently expressed in the compact form

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c} = ds. \tag{7.7}$$

Again, initial conditions should be specified on a curve that is intersected by characteristic curves (and not parallel to any characteristic curve).

7.3 Quasilinear PDEs

The above ideas can be extended further to the *quasilinear case*,

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u), \tag{7.8}$$

where now a, b and c may all be arbitrary functions of x, y, and u and where only the partial derivatives remain as linear terms. Again, the PDE can be solved by solving the associated ODEs for x, y and u

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c} = ds, \tag{7.9}$$

in the compact form, or as an explicit system

$$\begin{aligned} \frac{dx}{ds} &= a(x, y, u) \\ \frac{dy}{ds} &= b(x, y, u) \\ \frac{du}{ds} &= c(x, y, u). \end{aligned}$$

The complication here is that now u enters the expressions for a and b. This implies that the characteristic curves now depend not only on a and b but will also depend on the values of the solution u itself. In particular, this means that the characteristic curves will also depend on the initial conditions that are specified.

Geometrical Interpretation

If we consider a surface given by $u = u(x, y)$ (again thinking of u as the height of the surface above the point (x, y)) then the *normal* to the surface is

$$\boldsymbol{n} = (u_x, u_y, -1) \, .$$

Equation (7.8) can be expressed as

$$\boldsymbol{n} \cdot (a, b, c) = 0, \qquad \Longrightarrow \qquad au_x + bu_y - c = 0$$

so that the vector (a, b, c) is tangent to the surface. Equation (7.9) implies that the characteristic curves are parallel to (a, b, c). Hence, this ensures that the characteristic curves all lie in the surface $u(x, y)$.

7.4 Examples

Example 7.3 $xu_x - yu_y = 0$.

Example 7.4 $yu_x - xu_y = 0, \;\; u(x, 0) = f(x), \; x > 0$.

Example 7.5 $2xyu_x + u_y = u, \;\; u(x, 0) = f(x)$.

Example 7.6 $xu_x + yu_y = u, \;\; u = f(x)$ on $y = 1$.

Example 7.7 $x^2u_x + uu_y = 1, \;\; u = 0$ on $x + y = 1$.

Example 7.8 $tu_x + (x - u)u_t = -t, \;\; u(1, t) = t$.

Example 7.9 $u^2u_x + u_y = 0, \;\; u(x, 0) = \sqrt{x}, \; x > 0$.

Example 7.10 $u_t + [F(u)]_x = 0, \;\; u(x, 0) = f(x)$.

Example 7.12 $2uu_x + (x - u^2)u_y = 1, \;\; u(x, 0) = 0, \; x \geq 1$.

Example 7.13 $u_y + uu_x = 1, \;\; u(x, 0) = g(x)$.