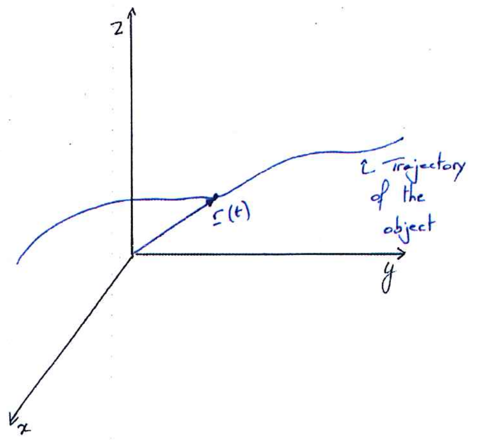
Chapter 1

Introduction to Vector Calculus

1.1 Introduction to Vector Calculus

Vector Calculus is the branch of mathematics that deals with differentiation and integration of *vector valued functions* in two or three dimensions.

It has applications whenever a subject involves quantities which are best represented by a vector, i.e. that have a direction and a magnitude that might depend on position and/or time. For example, we can think of the position of an object moving in space, winds on a weather map, magnetic fields,forces, ...



Areas of Applied Mathematics that make use of vector calculus are, for example

- Classical Mechanics
- Fluid Dynamics
- Electromagnetism
- Wave motion

- Oceanography
- Meteorology
- Elasticity
- Plasma Physics

Vector Calculus can be extended into more generally defined mathematical spaces that may be four or more dimensional or which might not be *flat* in the sense that $\mathbb{R}^2$ and $\mathbb{R}^3$ are. Such mathematics may be applied to the theories of Special and General Relativity and to modern Quantum Field Theories.

1.2 Definitions and Revision

In this course, we consider the real vector space $\mathbb{R}^3$, so a vector $\underset{\sim}{x} = \mathbf{x} \in \mathbb{R}^3$ is $\mathbf{x} = (x, y, z) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

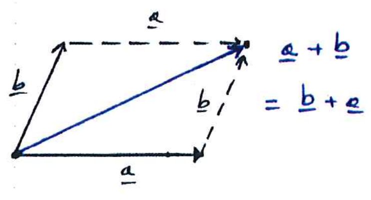
• Note the notation of a vector as either $\underset{\sim}{x}$ or $\mathbf{x}$ (typed notes).

• A scalar is a real number (magnitude).

• We can add vectors $\mathbf{x}_1,\ \mathbf{x}_2 \in \mathbb{R}^3$:

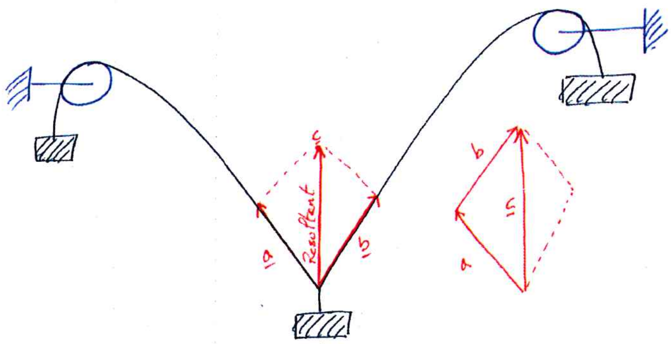
$$\mathbf{x}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \ \mathbf{x}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \Rightarrow \mathbf{x}_1 + \mathbf{x}_2 = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix}$$

" Parallelogram law "



It is clear from this definition that $\mathbf{x}_1 + \mathbf{x}_2 = \mathbf{x}_2 + \mathbf{x}_1$.

Example: Resultant of two forces (using the parallelogram law)



Resultant of two forces

Basic properties of vector addition:

(a) $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$
(b) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$
(c) $\mathbf{a}+\mathbf{0}=\mathbf{0}+\mathbf{a}=\mathbf{a}$
(d) $\mathbf{a}+(-\mathbf{a})=\mathbf{0}$

where $-\mathbf{a}$ denotes the vector having length $|\mathbf{a}|$ and direction opposite to that of $\mathbf{a}$.

• We can multiply a vector by a scalar

$$\lambda \in \mathbb{R},\ \mathbf{x}_1 \in \mathbb{R}^3,\ \mathbf{x}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \Rightarrow \lambda \mathbf{x}_1 = \begin{pmatrix} \lambda x_1 \\ \lambda y_1 \\ \lambda z_1 \end{pmatrix}.$$

Say $\mathbf{v}=\lambda\mathbf{x}$. Then $\mathbf{v}$ has the same direction as $\mathbf{x}$ and has length ('magnitude') $\lambda|\mathbf{x}|$ (where $|\mathbf{x}|$ is the norm of $\mathbf{x}$ - see below for definition).

Basic properties of scalar multiplication:

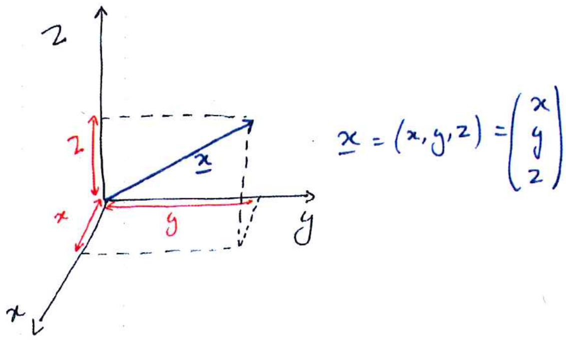
(a) $c(\mathbf{a}+\mathbf{b})=c\mathbf{a}+c\mathbf{b}$
(b) $(c+k)\mathbf{a}=c\mathbf{a}+k\mathbf{a}$
(c) $c(k\mathbf{a})=(ck)\mathbf{a}$
(d) $1\mathbf{a}=\mathbf{a}$

• Norm (length) of a vector $\mathbf{x}=(x,y,z)$ is given by

$$|\mathbf{x}|=\sqrt{x^2+y^2+z^2}\,.$$

1.3 Representation of a vector

Vectors are quantities that have a direction and a length (norm)



$\mathbf{x}$ can be written as

$$\mathbf{x} = |\mathbf{x}|\mathbf{e},$$

where $|\mathbf{x}|$ is the length (norm) and $\mathbf{e}$ is the unit vector indicating the direction.

By definition:

$$\mathbf{e} = \frac{\mathbf{x}}{|\mathbf{x}|}.$$

1.4 Coordinate system

Consider the vector $\mathbf{x} = (x, y, z)$.

$\mathbf{x}$ can be written as

$$\begin{aligned} \mathbf{x} &= (x,0,0) + (0,y,0) + (0,0,z) \\ &= x(1,0,0) + y(0,1,0) + z(0,0,1) \\ &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \end{aligned}$$

where $\mathbf{i} = (1,0,0)$, $\quad \mathbf{j} = (0,1,0)$, $\quad \mathbf{k} = (0,0,1)$.

Any vector $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$ can be written as a linear combination of the 3 unit vectors (vectors of length 1) $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$

$$\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ form the basis of the Cartesian coordinate system [from René Descartes (1596-1650)].

The basis is orthonormal, i.e.

$$|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1 \quad \text{and} \quad \mathbf{i}\cdot\mathbf{j} = \mathbf{i}\cdot\mathbf{k} = \mathbf{j}\cdot\mathbf{k} = 0\,.$$

Note to get one component out off a vector $\mathbf{x}$ (i.e. say the 1st one) we just need to take the scalar product of $\mathbf{x}$ with $\mathbf{i}$:

$$\mathbf{x}\cdot\mathbf{i} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = x$$

We can rewrite the same operation as

$$\mathbf{x}\cdot\mathbf{i} = (x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot\mathbf{i} \quad = x\underbrace{(\mathbf{i}\cdot\mathbf{i})}_{=|\mathbf{i}|^2=1} + y\underbrace{(\mathbf{i}\cdot\mathbf{j})}_{=0} + z\underbrace{(\mathbf{k}\cdot\mathbf{i})}_{=0}$$

1.5 The dot product (or scalar product)

Definition

The dot or scalar product of two vectors $\mathbf{x}_1 = (x_1, y_1, z_1)$ and $\mathbf{x}_2 = (x_2, y_2, z_2)$ is defined as:

$$\mathbf{x}_1 \cdot \mathbf{x}_2 = x_1x_2 + y_1y_2 + z_1z_2$$

The result is a scalar.

Note that 'dot product' refers to the symbol, 'scalar product' refers to the result.

Orthogonality

Two vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are orthogonal (or perpendicular) if and only if

$$\mathbf{x}_1 \cdot \mathbf{x}_2 = 0\,.$$

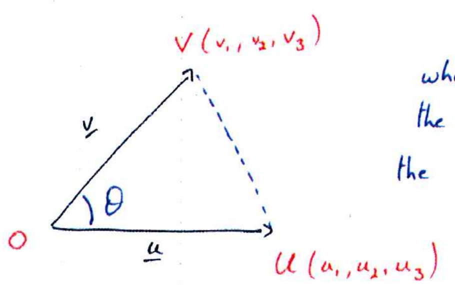
Norm

The norm of a vector $\mathbf{x}$ denoted $|\mathbf{x}|$ is defined by

$$|\mathbf{x}| = \sqrt{\mathbf{x}\cdot\mathbf{x}}$$

Geometrical Representation

Consider 2 vectors $\mathbf{u}$ and $\mathbf{v}$.



where θ is the angle between the vectors $\underline{u}$ and $\underline{v}$.

If θ is the angle between two non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ then

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= u_1v_1 + u_2v_2 + u_3v_3\\ &= |\mathbf{u}|\cdot|\mathbf{v}|\cos\theta\end{aligned}$$

Proof

(i) If $\mathbf{v} \neq c\mathbf{u}$ and applying the law of triangles to the triangle UOV (see figure above) gives:

$$|UV|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos\theta\,.$$

Hence, we get

$$(v_1-u_1)^2 + (v_2-u_2)^2 + (v_3-u_3)^2 = (u_1^2+u_2^2+u_3^2) + (v_1^2+v_2^2+v_3^2) - 2|\mathbf{u}||\mathbf{v}|\cos\theta\,,$$

which simplifies to

$$-2u_1v_1 - 2u_2v_2 - 2u_3v_3 = -2|\mathbf{u}||\mathbf{v}|\cos\theta\,,$$

or

$$-2(\mathbf{u}\cdot\mathbf{v}) = -2|\mathbf{u}||\mathbf{v}|\cos\theta\,.$$

(ii) If $\mathbf{v} = c\mathbf{u}$ then we have

$$\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot(c\mathbf{u}) = c(\mathbf{u}\cdot\mathbf{u}) = c|\mathbf{u}|^2$$

and

$$|\mathbf{u}||\mathbf{v}|\cos\theta = |\mathbf{u}||c\mathbf{u}|\cos\theta = |c||\mathbf{u}|^2\cos\theta$$

If $c > 0$, then $|c| = c$ and $\theta = 0$ such that $|c||\mathbf{u}|^2\cos\theta = c|\mathbf{u}|^2$ and hence

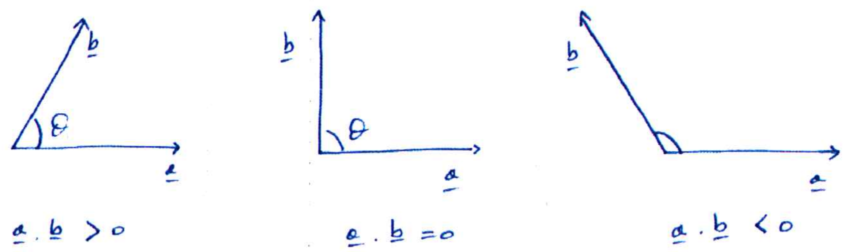
$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta\,.$$

If $c < 0$, then $|c| = -c$ and $\theta = \pi$ such that again $|c||\mathbf{u}|^2\cos\theta = c|\mathbf{u}|^2$. $\square$

We recover that orthogonal vectors corresponds geometrically to vectors making a right angle $\left(\cos\theta = 0\,,\ \theta = \pm\frac{\pi}{2}\right)$.

Hence, the dot product can be used to find the angle between two vectors as:

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\cdot|\mathbf{v}|}\,.$$



Examples

(i) Given

$$\mathbf{a} = (1, -2, 3)$$
$$\mathbf{b} = (2, 1, 1)$$
$$|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{1+4+9} = \sqrt{14}$$
$$|\mathbf{b}| = \sqrt{4+1+1} = \sqrt{6}$$
$$\mathbf{a}\cdot\mathbf{b} = 2-2+3 = 3$$
$$\cos\theta = \frac{3}{\sqrt{6}\sqrt{14}} = \sqrt{\frac{3}{28}} \qquad (\theta \simeq 71^\circ)$$

(ii) Given

$$\mathbf{a} = (1, -2, 1)$$
$$\mathbf{b} = (2, 3, 4)$$
$$\mathbf{a}\cdot\mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = 2-6+4 = 0$$

Hence, these vectors are perpendicular (or orthogonal).

1.6 The vector product (or cross product)

Definition

The vector product between 2 vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ is defined by:

$$\mathbf{x}_1 \times \mathbf{x}_2 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \times \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} y_1z_2 - z_1y_2 \\ z_1x_2 - x_1z_2 \\ x_1y_2 - y_1x_2 \end{pmatrix}$$

a compact notation for the vector product is

$$\mathbf{x}_1 \times \mathbf{x}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

is the "determinant" of the "matrix" whose 1st line is the basis $(\mathbf{i}, \mathbf{j}, \mathbf{k})$, 2nd line is the components of the 1st vector, 3rd line is the components of the second vector.

Properties of the vector product

- $(\mathbf{a}+\mathbf{b})\times\mathbf{c} = \mathbf{a}\times\mathbf{c} + \mathbf{b}\times\mathbf{c}$
- $(\alpha\mathbf{a})\times\mathbf{b} = \alpha(\mathbf{a}\times\mathbf{b})$
- $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$

- $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$

Proof

Start from

$$\mathbf{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \begin{pmatrix} a_y b_z - a_z b_y \\ -a_x b_z + a_z b_x \\ a_x b_y - a_y b_x \end{pmatrix}$$

$$\begin{aligned} \mathbf{a} \times \mathbf{b} \cdot \mathbf{a} = \; & a_y b_z a_x - a_z b_y a_x - a_x a_y b_z + a_y a_z b_x \\ & + a_x a_z b_y - a_y a_z b_x = 0 \end{aligned}$$

The same for $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b}$ $\square$

- $\mathbf{a} \times \mathbf{a} = 0$

Proof

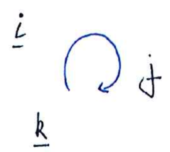
$$\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \times \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ a_x & a_y & a_z \end{vmatrix}$$

$$= \begin{pmatrix} a_y a_z - a_y a_z \\ -a_x a_z + a_x a_z \\ a_x a_y - a_x a_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0.$$

$\square$

$\Rightarrow$ Important consequence: the vector product between 2 parallel vectors is 0.

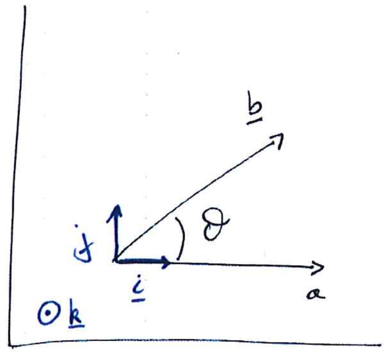
- $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$ (right-handed cyclic)



- $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$, $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$, $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$
- $\mathbf{i} \times \mathbf{i} = 0$, $\mathbf{j} \times \mathbf{j} = 0$, $\mathbf{k} \times \mathbf{k} = 0$

Geometrical interpretation

Two non-parallel vectors define a plane:



$\hat{k}$ unit vector perpendicular to the plane, pointing toward you

Let us take $\mathbf{i}$ the first unit vector parallel to a:

$$\mathbf{a} = |\mathbf{a}|\mathbf{i} = (a, 0, 0)$$

Geometrically $\mathbf{b} = |\mathbf{b}|\cos\theta\mathbf{i} + |\mathbf{b}|\sin\theta\mathbf{j}$

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ |\mathbf{a}| & 0 & 0 \\ |\mathbf{b}|\cos\theta & |\mathbf{b}|\sin\theta & 0 \end{vmatrix} = \begin{pmatrix} 0-0 \\ 0-0 \\ |\mathbf{a}||\mathbf{b}|\sin\theta \end{pmatrix}$$

$$\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\mathbf{k}$$

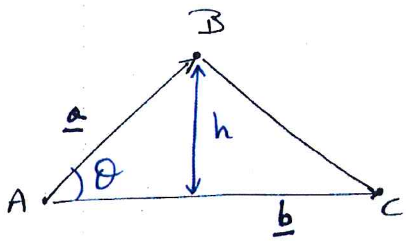
Hence, using the vector product between two vectors $\mathbf{u}$ and $\mathbf{v}$, we have

$$\sin\theta = \frac{|\mathbf{u}\times\mathbf{v}|}{|\mathbf{u}||\mathbf{v}|}$$

where θ is the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$.

Examples

(i) Area of a triangle:



$$\text{Area} = \frac{1}{2}\text{ base} \times \text{height} = \frac{1}{2}|\mathbf{b}|h$$

and $h = |\mathbf{a}|\sin\theta$ so we get

$$\text{Area} = \frac{1}{2}|\mathbf{a}||\mathbf{b}|\sin\theta = \frac{1}{2}|\mathbf{a}\times\mathbf{b}|\,.$$

(ii) Find $\mathbf{a}\times\mathbf{b}$ with $\mathbf{a} = (1, -2, 3)$, $\mathbf{b} = (2, 1, 1)$

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 3 \\ 2 & 1 & 1 \end{vmatrix} \\ &= \begin{pmatrix} -2-3 \\ -1+6 \\ 1+4 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \\ 5 \end{pmatrix}\end{aligned}$$

(iii) Find $\mathbf{a}\times\mathbf{b}$ with $\mathbf{a} = (1, 1, 1)$, $\mathbf{b} = (2, 2, 2)$

Obviously $\mathbf{a}\times\mathbf{b}$ will be 0 as $\mathbf{b} = 2\mathbf{a}$ ($\mathbf{a}$ and $\mathbf{b}$ are parallel).

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{vmatrix} = \begin{pmatrix} 2-2 \\ -2+2 \\ 2-2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0$$

Further properties

In many applications, we must consider the product of 3 (or more) vectors.

• *Triple scalar product*: $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$

Remember that the result of a cross product is a vector but the result of a dot product is a scalar!

Note that as long as the cycle $\mathbf{a}\to\mathbf{b}\to\mathbf{c}$ is maintained, cross product and scalar product can be exchanged.

Proof

Let us take

$$\mathbf{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} \ \text{and} \ \mathbf{c} = \begin{pmatrix} c_x \\ c_y \\ c_z \end{pmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \begin{pmatrix} a_y b_z - a_z b_y \\ -a_x b_z + a_z b_x \\ a_x b_y - a_y b_x \end{pmatrix}$$

$$\Rightarrow (\mathbf{a} \times \mathbf{b}).\mathbf{c} = \begin{pmatrix} a_y b_z - a_z b_y \\ -a_x b_z + a_z b_x \\ a_x b_y - a_y b_x \end{pmatrix} \cdot \begin{pmatrix} c_x \\ c_y \\ c_z \end{pmatrix} =$$

$$\underbrace{a_y b_z c_x - a_z b_y c_x}_{\text{from 1st components}} - \underbrace{a_x b_z c_y + a_z b_x c_y}_{\text{from 2nd components}} + \underbrace{a_x b_y c_z - a_y b_x c_z}_{\text{from 3rd components}}$$

Now take

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \ : \ \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \cdot \begin{pmatrix} b_y c_z - b_z c_y \\ -b_x c_z + b_z c_x \\ b_x c_y - b_y c_x \end{pmatrix}$$

$$= a_x b_y c_z - a_x b_z c_y - a_y b_x c_z + a_y b_z c_x + a_z b_x c_y - a_z b_z c_y$$

where all terms from the 2 expressions match.

The same will work for $\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$. (Try to verify this yourself!) □

• *Triple vector product:*

This becomes a bit more tricky as now the order (or the position of the brackets) really matters:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$$

For the triple vector product, we have that:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}\,.$$

This expression implies that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is a linear combination of vectors along $\mathbf{b}$ and $\mathbf{c}$.

Proof

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = \begin{pmatrix} b_y c_z - b_z c_y \\ -b_x c_z + b_z c_x \\ b_x c_y - b_y c_x \end{pmatrix}$$

So

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ (b_y c_z - b_z c_y) & (-b_x c_z + b_z c_x) & (b_x c_y - b_y c_x) \end{vmatrix}$$

and hence

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \begin{pmatrix} a_y[b_x c_y - b_y c_x] - a_z[-b_x c_z + b_z c_x] \\ -a_x[b_x c_y - b_y c_x] + a_z[b_y c_z - b_z c_y] \\ a_x[-b_x c_z + b_z c_x] - a_y[b_y c_z - b_z c_y] \end{pmatrix}$$

or, by collecting terms in $\mathbf{b}$ and $\mathbf{c}$:

$$\begin{pmatrix} (a_y c_y + a_z c_z) b_x - (a_y b_y + a_z b_z) c_x \\ (a_x c_x + a_z c_z) b_y - (a_x b_x + a_Z b_z) c_y \\ (a_x c_x + a_y b_y) b_z - (a_x b_x + a_y b_y) c_z \end{pmatrix} .$$

Now, by adding and subtracting the vector

$$\begin{pmatrix} a_x b_x c_x \\ a_y b_y c_y \\ a_z b_z c_z \end{pmatrix}$$

we find, for example for the 1st component:

$$(a_y c_y + a_z c_z) b_x - (a_y b_y + a_z b_z) c_x - (a_x b_x) c_x + (a_x c_x) b_x = \underbrace{(a_x c_x + a_y c_y + a_z c_z)}_{\mathbf{a}\cdot\mathbf{c}} b_x - \underbrace{(a_x b_x + a_y b_y + a_z b_z)}_{\mathbf{a}\cdot\mathbf{b}} c_x .$$

The same works for the other components, hence we have shown that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} .$$

Now take

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$$

and remember that

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = -\mathbf{c} \times (\mathbf{a} \times \mathbf{b}).$$

Consider

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

and exchange the variables such that

$$\begin{aligned} \mathbf{c} &\to \mathbf{a} \\ \mathbf{a} &\to \mathbf{b} \\ \mathbf{b} &\to \mathbf{c} \end{aligned}$$

then we get

$$-\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a} = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$$

which is a different vector!

We can confirm this further by combining the two expressions:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) - (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{b} \cdot \mathbf{c})\mathbf{a} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \neq 0!$$

$\square$

Examples

Given $\mathbf{a} = (1, 1, 2),\ \mathbf{b} = (2, -1, 1),\ \mathbf{c} = (1, -1, 3)$.

Find $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ and $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$

(i) Start with $(\mathbf{b} \times \mathbf{c})$

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 1 & -1 & 3 \end{vmatrix} = \begin{pmatrix} -3+1 \\ -6+1 \\ -2+1 \end{pmatrix} = \begin{pmatrix} -2 \\ -5 \\ -1 \end{pmatrix}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ -2 & -5 & -1 \end{vmatrix} = \begin{pmatrix} -1+10 \\ 1-4 \\ -5+2 \end{pmatrix} = \begin{pmatrix} 9 \\ -3 \\ -3 \end{pmatrix}$$

(ii) Start with $(\mathbf{a} \times \mathbf{b})$

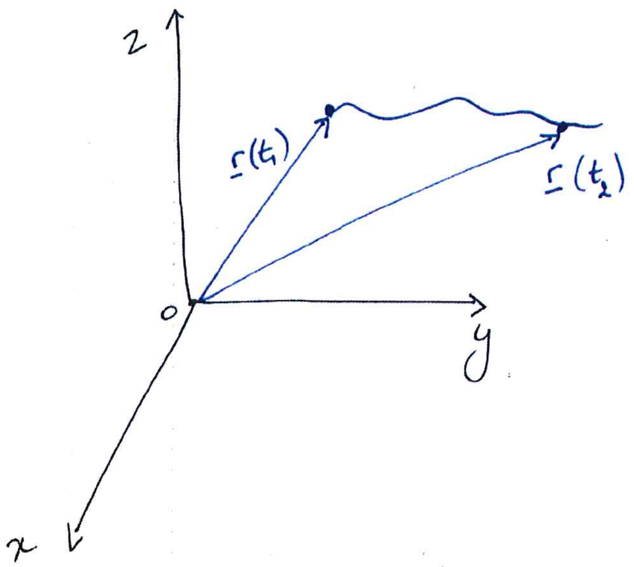
$$(\mathbf{a} \times \mathbf{b}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ 2 & -1 & 1 \end{vmatrix} = \begin{pmatrix} 1+2 \\ -1+4 \\ -1-2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix}$$

So $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 3 & -3 \\ 1 & -1 & 3 \end{vmatrix} = \begin{pmatrix} 9-3 \\ -9-3 \\ -3-3 \end{pmatrix} = \begin{pmatrix} 6 \\ -12 \\ -6 \end{pmatrix}$

which is obviously different from (i)!

1.7 Differentiation of a vector function

The position of a point moving in space can be conveniently traced by vectors.



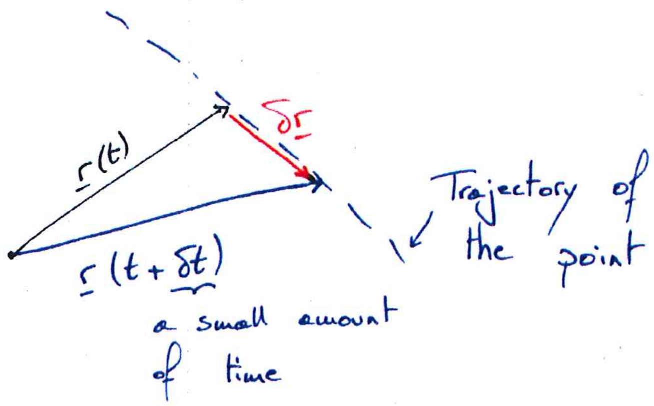
We can therefore define a position vector $\mathbf{r}$ which indicates the location of the moving object at any time. As this position is time dependent, we may write (for this object)

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

.

Remember that any vector can be expressed in the Cartesian coordinate system (using the fixed basis $\mathbf{i},\ \mathbf{j},\ \mathbf{k}$).

Consider a moving point:



By construction (parallelogram law), at $t + \delta t$, the position vector $\mathbf{r}$ has changed by an amount of $\delta \mathbf{r}$.

$$\mathbf{r}(t + \delta t) = \mathbf{r}(t) + \delta \mathbf{r}.$$

The velocity of the point is the rate of change of $\mathbf{r}$ in time

$$\frac{d\mathbf{r}}{dt} = \lim_{\delta t \to 0} \left(\frac{\delta \mathbf{r}}{\delta t} \right) = \lim_{\delta t \to 0} \left(\frac{\mathbf{r}(t + \delta t) - \mathbf{r}(t)}{\delta t} \right).$$

From the sketch it is obvious that the velocity is a vector, tangential to the trajectory.

Given the trajectory (time dependent position vector), $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, the velocity can be obtained easily as

$$\frac{d\mathbf{r}}{dt} = \frac{d}{dt}\left(x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}\right) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k},$$

as the basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are fixed vectors (independent of time).

It is occasionally convenient or necessary to use basis vector which do vary with position (i.e. are time dependent) - these are more troublesome to differentiate!

Example Find the velocity of a moving point with the trajectory

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix} = \cos t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}$$

The velocity is given by

$$\frac{d\mathbf{r}}{dt} = \begin{pmatrix} -\sin t \\ \cos t \\ 1 \end{pmatrix} = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}.$$

The acceleration of the moving point is the rate of change in time of its velocity:

$$\mathbf{a} = \frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right) = \frac{d^2\mathbf{r}}{dt^2}.$$

From the previous example

$$\mathbf{r} = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$$
$$\mathbf{u} = \frac{d\mathbf{r}}{dt} = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}$$
$$\mathbf{a} = \frac{d}{dt}(\mathbf{u}) = -\cos t\mathbf{i} - \sin t\mathbf{j}$$

Differentiating such vectors can be generalised, and if

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

then

$$\frac{d^n\mathbf{r}(t)}{dt^n} = \frac{d^n x(t)}{dt^n}\mathbf{i} + \frac{d^n y(t)}{dt^n}\mathbf{j} + \frac{d^n z(t)}{dt^n}\mathbf{k}.$$

For a moving point $\mathbf{r}(t)$,

$\dfrac{d\mathbf{r}}{dt}$ is its velocity (vector),

$\left|\dfrac{dr}{dt}\right|$ is its speed (norm of velocity),

$\dfrac{d^2\mathbf{r}}{dt^2}$ is its acceleration (vector),

$\left|\dfrac{d^2\mathbf{r}}{dt^2}\right|$ is the magnitude of the acceleration (scalar).

Examples

(i) Consider $\mathbf{r}(t) = t^2\mathbf{i} + (2-t^3)\mathbf{j} + t^4\mathbf{k}$ (position vector)

$$\frac{d\mathbf{r}}{dt} = \begin{pmatrix} 2t \\ -3t^2 \\ 4t^3 \end{pmatrix} \quad \text{velocity}$$

$$\frac{d^2\mathbf{r}}{dt^2} = \begin{pmatrix} 2 \\ -6t \\ 12t^2 \end{pmatrix} \quad \text{acceleration}$$

$$\left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{4t^2 + 9t^4 + 16t^6} = r\sqrt{4 + 9t^2 + 16t^4} \quad \text{speed}$$

$$\left|\frac{d^2\mathbf{r}}{dt^2}\right| = \sqrt{4 + 36t^2 + 144t^4} = 2\sqrt{1 + 9t^2 + 36t^4} \quad \text{magnitude of acceleration}$$

(ii) Consider $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \mathbf{k}$:

$$\frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \qquad \frac{d^2\mathbf{r}}{dt^2} = 2\mathbf{j}$$

$$\left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{1 + 4t^2} \qquad \left|\frac{d^2r}{dt^2}\right| = 2$$

For which times are $\mathbf{r}$ and $\frac{d\mathbf{r}}{dt}$ perpendicular?

The vectors $\mathbf{r}$ and $\frac{d\mathbf{r}}{dt}$ are perpendicular if and only if

$$\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = 0\,.$$

$$\Rightarrow \mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = \begin{pmatrix} t \\ t^2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2t \\ 0 \end{pmatrix} = t + 2t^3$$

$$\Rightarrow t(1 + 2t^2) = 0$$

Hence $\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = 0$ for $t = 0$ or $t^2 = -\frac{1}{2}$. However, as t represents time, we only consider the solution $t = 0$ (only consider real numbers).

At $t = 0$ we have $\begin{cases} \mathbf{r} = \mathbf{k} \\ \dfrac{d\mathbf{r}}{dt} = \mathbf{i} \end{cases}$

iii) Consider $\mathbf{r}(t) = \sin t\mathbf{i} + \cos t\mathbf{k}$:

$$\frac{d\mathbf{r}(t)}{dt} = \cos t\mathbf{i} - \sin t\mathbf{k}$$

and

$$\frac{d^2\mathbf{r}}{dt^2} = -\sin t\mathbf{i} - \cos t\mathbf{k} = -\mathbf{r}$$