3 Integration

3.1 Line (contour) Integrals

We are familiar with the notion of line integrals in the calculus of real variables. This notion can be extended to the present case in a straightforward manner. Consider a curve (contour) C joining z_1 and z_2 in the complex plane. Let $t \in [t_1, t_2]$ be a real parameter. We can parametrically describe C by a one-to-one correspondence between $t \in [t_1, t_2]$ and points z on C as follows.

$$[t_1, t_2] \mapsto C : z = z(t) = x(t) + iy(t), \quad \text{where} \quad z(t_1) = z_1 \text{ and } z(t_2) = z_2.$$

In order to avoid unnecessary complications we assume that C is smooth (the derivatives of the above exist). The (contour) integral of a complex function $f(z)$ along C is defined by

$$\int_C f(z)\,dz = \int_{t_1}^{t_2} f(z(t))z'(t)\,dt.$$

The integral on the right-hand side is the usual integral over the real variable t from t_1 to t_2, albeit the integrand is complex. In practice, it is convenient to consider the above definition as a substitution formula in real calculus, where $dz = z'(t)\,dt$.

Suppose $|f(z)| \leq M$ for all z on C, whose length is L. Then it is clear that

$$\begin{aligned} \left|\int_C f(z)\,dz\right| &= \left|\int_{t_1}^{t_2} f(z(t))z'(t)\,dt\right| \leq \int_{t_1}^{t_2} |f(z(t))||z'(t)|\,dt \\ &\leq M\int_{t_1}^{t_2} \left(x'^2(t) + y'^2(t)\right)^{1/2}\,dt = ML. \end{aligned}$$

This estimate will be used repeatedly later.

Example 3.1.1 Evaluate

$$\int_C z^2\,dz,$$

where C is a semicircle (radius 1) defined by

$$x = \cos t,\ y = \sin t,\ 0 \leq t \leq \pi.$$

Solution On C we have

$$z = x + iy = \cos t + i\sin t = e^{it}, \quad 0 \leq t \leq \pi, \quad \text{with} \quad dz = ie^{it}\,dt.$$

Thus

$$\int_C z^2\,dz = \int_0^\pi e^{2it}ie^{it}\,dt = \int_0^\pi ie^{3it}\,dt = \frac{1}{3}\,e^{3it}\Big]_0^\pi = \frac{1}{3}(e^{3i\pi} - 1) = \frac{1}{3}(-1-1) = -\frac{2}{3}.$$

Observe that z^2 and z^3 are analytic functions (for all z) and

$$\frac{d}{dz}\left(\frac{z^3}{3}\right) = z^2.$$

It is no surprise that

$$\int_C z^2\,dz = \left[\frac{1}{3}z^3\right]_1^{-1} = -\frac{2}{3}.$$

We will return to this later. •

Also observe that in the above integral the parameter t increases from 0 to π, implying that we have integrated along the semi-circle in an anti-clockwise direction from $z = 1$ (when $t = 0$) to $z = -1$ (when $t = \pi$). Integrating in a clockwise direction along this semicircle from $z = -1$ to $z = 1$ (t decreases from π to 0) simply changes the sign of the result (the limits of integration are interchanged). Thus changing contour direction merely changes the sign of the contour integral:

$$\int_C f(z)\,dz = -\int_{C'} f(z)\,dz,$$

where C' is the path C traversed in the opposite direction.

Example 3.1.2 Evaluate

$$\int_C z^2\,dz,$$

where C is the straight line between $(0, 0)$ and $(1, 1)$.

Solution On C we have $z(t) = (1 + i)t$, where $0 \le t \le 1$. So $dz = (1 + i)\,dt$ and

$$\int_C z^2\,dz = \int_0^1 (1+i)^3 t^2\,dt = (1+i)^3 \int_0^1 t^2\,dt = \frac{(1+i)^3}{3}.$$

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It is straightforward to extend the above definition of contour integral to more complicated paths of integration. Suppose C consists of a finite number of smooth curves joined end-to-end:

$$C = C_1 \cup C_2 \cup \cdots C_n,$$

where C_j is between z_j and z_{j+1} joined by C_{j+1} between z_{j+1} and z_{j+2}. We can integrate $f(z)$ along C by considering each segment of the path in turn. Thus we can define

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_1} f(z)\,dz + \cdots \int_{C_n} f(z)\,dz.$$

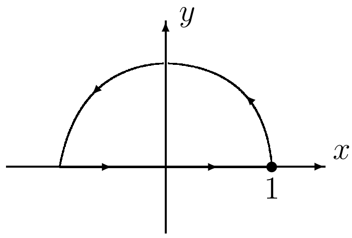

Figure 12: A semi-circular, closed contour C

Further, we can consider a contour that returns to its starting point - a **closed contour**. When C is closed we use the notation

$$\oint_C f(z)\,dz.$$

Example 3.1.3 Evaluate

$$\oint_C z^2\,dz,$$

where C is the closed, semi-circular contour starting at (and returning to) $(1,0)$ as illustrated in figure 12. Note that the contour is traversed in an anti-clockwise direction (a convenient convention to adopt).

Solution From a previous example we know that the integral along the semi-circle is -2/3. On the other hand, the integral over the interval $[-1,1]$ is just $2/3$. Hence

$$\oint_C z^2\,dz = -\frac{2}{3} + \frac{2}{3} = 0.$$

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Example 3.1.4 Evaluate

$$\oint_C e^z\,dz,$$

where C is the unit square, i.e. the square with vertices $z = 0, 1, 1+i, i$ (see figure 13).

Solution Let $C = C_1 \cup C_2 \cup C_3 \cup C_4$ as depicted by figure 13. Then

$$\oint_C e^z\,dz = \oint_{C_1} e^z\,dz + \oint_{C_2} e^z\,dz + \oint_{C_3} e^z\,dz + \oint_{C_4} e^z\,dz.$$

On C_1: $z = x$, $x \in [0,1]$, $dz = dx$.
On C_2: $z = 1 + iy$, $y \in [0,1]$, $dz = i\,dy$.
On C_3: $z = x + i$, $x \in [0,1]$, $dz = dx$.
On C_4: $z = iy$, $y \in [0,1]$, $dz = i\,dy$.

So

$$\begin{aligned}\oint_C e^z\,dz &= \int_0^1 e^x\,dx + \int_0^1 e^{1+iy} i\,dy + \int_1^0 e^{x+i}\,dx + \int_1^0 e^{iy} i\,dy \\ &= \int_0^1 e^x\,dx + ie\int_0^1 e^{iy}\,dy + e^i\int_1^0 e^x\,dx + i\int_1^0 e^{iy}\,dy \\ &= e-1+e(e^i-1)+e^i(1-e)+1-e^i = 0.\end{aligned}$$

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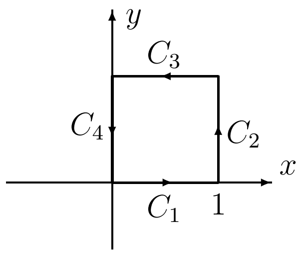

Figure 13: The unit square

The above examples are an illustration of a very important result known as Cauchy's Theorem.

3.2 Cauchy's Theorem

Let $f(z)$ be analytic on a domain D and let C be a closed contour within D then

$$\boxed{\oint_C f(z)\,dz = 0.}$$

That is the integral of an analytic function along a closed contour is zero.

By way of justification let

$$f(z) = u(x,y) + iv(x,y) \quad \text{and} \quad z = x + iy$$

so that $dz = dx + i\,dy$.

$$\oint_C f(z)\,dz = \oint_C (u+iv)(dx+i\,dy) = \oint_C (u\,dx - v\,dy) + i(v\,dx + u\,dy).$$

Assume that C is traversed in an anti-clockwise direction (starting at the extreme right hand side) and that C encloses a two-dimensional region A. Consider the first term of the real component:

$$\oint_C u\,dx = \int_b^a u(x,\phi_2(x))\,dx + \int_a^b u(x,\phi_1(x))\,dx$$

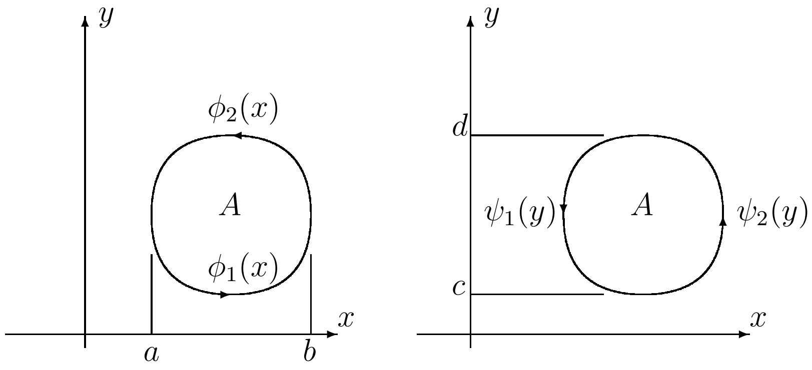

Figure 14: Integration around a simple closed contour

where $y = \phi_2(x)$ specifies the "top" part of curve C and $y = \phi_1(x)$ the "bottom" half (see figure 14). Thus

$$\begin{aligned}
\oint_C u\,dx &= -\int_a^b u(x,\phi_2(x))\,dx + \int_a^b u(x,\phi_1(x))\,dx \\
&= -\int_a^b \{u(x,\phi_2(x)) - u(x,\phi_1(x))\}\,dx \\
&= -\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \frac{\partial u}{\partial y}\,dy\,dx = -\iint_A \frac{\partial u}{\partial y}\,dA.
\end{aligned}$$

Now by using the Cauchy-Riemann equations we get

$$\begin{aligned}
\oint_C u\,dx &= \iint_A -\frac{\partial u}{\partial y}\,dA = \iint_A \frac{\partial v}{\partial x}\,dA \\
&= \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} \frac{\partial v}{\partial x}\,dx\,dy = \int_c^d v(\psi_2(y),y) - v(\psi_1(y),y)\,dy \\
&= \int_c^d v(\psi_2(y),y)\,dy + \int_d^c v(\psi_1(y),y)\,dy = \oint_C v\,dy.
\end{aligned}$$

Hence

$$\oint_C (u\,dx - v\,dy) = 0.$$

So the real part of the contour integral is zero. We can show the same thing happens with the imaginary part. Therefore, the integral of an analytic function around a closed curve (or **contour**) is automatically zero, by virtue of the Cauchy-Riemann equations. The above is basically a derivation of Green's Theorem (some will have seen in MT2003) with cancellation taking place due to the Cauchy–Riemann equations.

3.3 Consequences of Cauchy's Theorem

In the above we have assumed that C is a simple, closed curve and the domain D is **simply connected**, that is there are no holes in D (every simple closed contour encloses only points in D). We have not allowed any complications!

Now consider integrating (in an anti-clockwise direction) around a closed contour C starting at z_1 and passing through z_2 before returning to z_1, as illustrated in figure 15. If

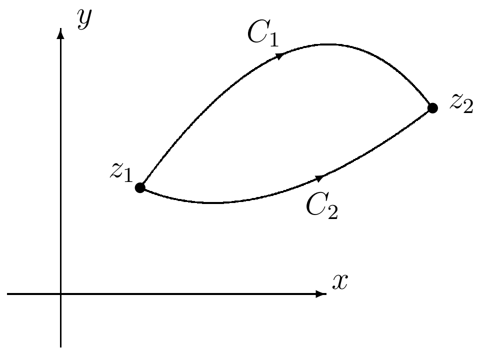

Figure 15: Paths with common ends.

$f(z)$ is analytic in a domain D which contains the contour C then by Cauchy's theorem

$$\oint_C f(z)\,dz = 0 = \int_{C_2} f(z)\,dz + \int_{C_1'} f(z)\,dz = 0,$$

where C_1' is the curve C_1 traversed in the opposite direction (i.e. right to left). Since

$$\int_{C_1'} f(z)\,dz = -\int_{C_1} f(z)\,dz,$$

we have

$$\int_{C_2} f(z)\,dz = \int_{C_1} f(z)\,dz.$$

This result implies that the integral of an analytic function $f(z)$ along a contour connecting z_1 and z_2 is independent of the shape of the contour.

For a function analytic in a simply connected domain D it now makes sense to write

$$\int_{z_1}^{z_2} f(z)\,dz,$$

since the result is independent of path between the two points within D. Thus, when dealing with analytic functions integration looks much like the integration of real functions. If $f(z)$ and $F(z)$ are analytic in D and such that

$$F'(z) = f(z),$$

then the anti-derivative (also known as primitive) $F(z)$ can be called the **indefinite integral** of $f(z)$ and for any contour $C \in D$ connecting z_1 and z_2 we have

$$\int_C f(z)\,dz = \int_{z_1}^{z_2} f(z)\,dz = F(z_2) - F(z_1).$$

We observed this result in the earlier example 3.1.1.

Remember, all the usual results apply, provided $f(z)$ is an analytic function.

Example 3.3.1 Evaluate

$$\int_C \frac{1}{z}\,dz,$$

where C is the unit quarter circle in the first quadrant defined by $z = e^{i\theta}$, θ varying between 0 and $\pi/2$.

Solution E •

valuating directly the line integral:

$$\int_C \frac{1}{z}\,dz = \int_0^{\pi/2} \frac{ie^{i\theta}\,d\theta}{e^{i\theta}} = \int_0^{\pi/2} i\,d\theta = \frac{i\pi}{2}$$

Now, the function z^{-1} is analytic except at $z = 0$. That means C can be contained in some neighbourhood where z^{-1} is analytic. So

$$\int_C \frac{1}{z}\,dz = \int_1^i \frac{1}{z}\,dz = [\log(z)]_1^i = \log(i) - \log(1).$$

Note that in the above equation, we can use any branch of log. Now $\log(i) = \log(e^{i(\pi/2+2n\pi)} = \ln(1) + i(\pi/2 + 2n\pi) = i(\pi/2 + 2n\pi)$ and $\log(1) = \ln(1) + i2n\pi = i2n\pi$. So

$$\int_C \frac{1}{z}\,dz = \frac{i\pi}{2}.$$

Example 3.3.2 Consider the complex function

$$f(z) = |z|^2 = x^2 + y^2$$

and the two points $z = 0$ and $z = 1 + i$.

First observe that

$$\frac{\partial u}{\partial x} = 2x \neq \frac{\partial v}{\partial y} = 0, \quad (v \equiv 0),$$

and

$$\frac{\partial u}{\partial y} = 2y \neq -\frac{\partial v}{\partial x} = 0.$$

Neither one of the Cauchy–Riemann equations is satisfied. Hence $f(z)$ is **not analytic**. Note that one needs just one equation not to hold (or the fact $\Delta u = 4 \neq 0$) to make this conclusion. As $f(z)$ is not analytic, its integral along a path connecting the given points $z = 0$ and $z = 1 + i$ depends on the shape of the path as will be seen presently.

Let C be the straight line segment connecting $z = 0$ and $z = 1 + i$. On C we have $y = x$. So $z = (1 + i)x$, $|z|^2 = 2x^2$, $dz = (1 + i)\, dx$ and

$$\int_C |z|^2\, dz = \int_0^1 2x^2(1 + i)\, dx = \frac{2}{3}\,(1 + i).$$

Now let C be the curve $y = x^2$ connecting $z = 0$ and $z = 1+i$. On C we have $y = x^2$. So $z = x + ix^2$, $|z|^2 = x^2 + x^4$, $dz = (1 + 2ix)\, dx$ and

$$\begin{aligned}\int_C |z|^2\, dz &= \int_0^1 (x^2 + x^4)(1 + 2ix)\, dx = \int_0^1 \left(x^2 + x^4 + 2i(x^3 + x^5)\right)\, dx \\ &= \left(\frac{1}{3} + \frac{1}{5}\right) + 2i\left(\frac{1}{4} + \frac{1}{6}\right) = \frac{8}{15} + \frac{5i}{6}.\end{aligned}$$

Thus the end points of the paths are the same but the values of the two line integrals are not the same.

3.4 Contour Deformation

Having considered the nature of integration of analytic functions in simply connected domains let us contemplate what happens when D is not simply connected. Suppose D contains a "hole": a small region contained within the boundary of D in which $f(z)$ is not analytic. An illustration is given in figure 16 where the shaded areas indicate where $f(z)$ is **not** analytic. The domain D has an annular shape and since it is not simply connected we say that it is multiply-connected. Suppose C_1 and C_2 are two simple, closed contours that lie within D and encircle the "hole", as shown. Then

$$\oint_{C_1} f(z)\, dz = \oint_{C_2} f(z)\, dz.$$

This is also a direct consequence of Cauchy's theorem. Imagine joining the two contours by a line C_3 and integrating (from the point A) along a new closed contour C made up of $C_1 \to C_3' \to C_2' \to C_3$ (back to A), as illustrated in figure 17. For ease of presentation the two passes over the joining line C_3 have been separated. Since the new closed contour C encloses only points within D (ie $f(z)$ is analytic at all points inside C), we know that

$$\oint_C f(z)\, dz = 0$$

by Cauchy's theorem. Therefore

$$\int_{C_1} f(z)\, dz + \int_{C_3'} f(z)\, dz + \int_{C_2'} f(z)\, dz + \int_{C_3} f(z)\, dz = 0,$$

where C_3' and C_2' are the contours C_3 and C_2 traversed in the oppositive direction. Clearly

$$\int_{C_3} f(z)\, dz + \int_{C_3'} f(z)\, dz = 0 \quad \text{and} \quad \int_{C_2'} f(z)\, dz = -\int_{C_2} f(z)\, dz,$$

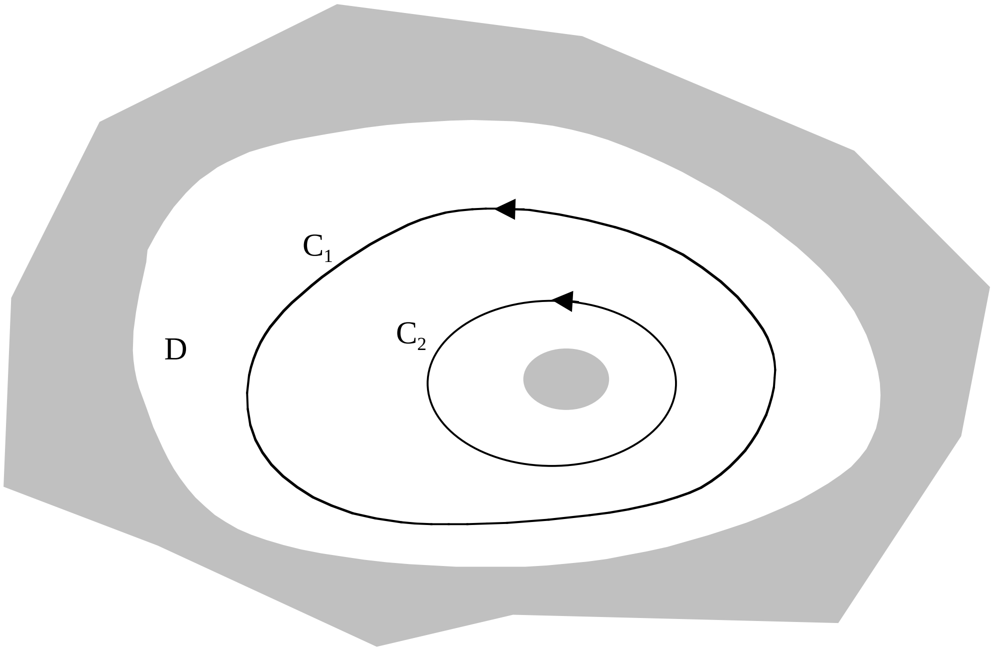

Figure 16: A multi-connected domain

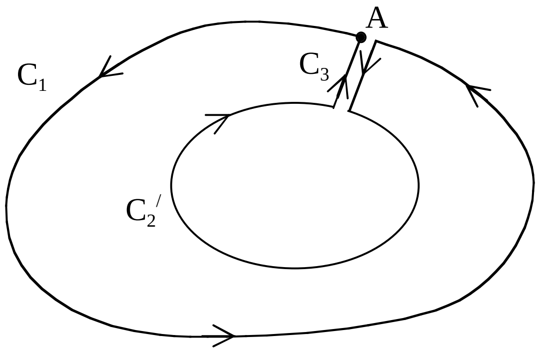

Figure 17: Deforming a contour

which gives the required result:

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz.$$

The value of a contour integral that encircles a non-analytic region is the same for all such contours, provided the contours themselves remain within D and have the same orientation (anti-clockwise). This allows us to choose a contour that simplifies the evaluation of the contour integral. This process is known as **contour deformation** since we can **deform** C_1 into C_2 (see figure 16).

Example 3.4.1 Evaluate

$$\oint_C \frac{dz}{z},$$

where C is any simple closed contour encircling the origin.

Solution Observe that C encloses $z=0$, which is a singular point (simple pole) of $f(z)=1/z$. So the integral may be nonzero. Deform C into a circle with centre at $z=0$ and radius R so that

$$z = Re^{i\theta}\ , \quad -\pi < \theta \leq \pi,$$

and $dz = Rie^{i\theta}\,d\theta$. Thus

$$\oint_C \frac{dz}{z}\,dz = \int_{-\pi}^{\pi} \frac{Rie^{i\theta}\,d\theta}{Re^{i\theta}} = \int_{-\pi}^{\pi} i\,d\theta = 2\pi i.$$

[Note that in terms of the log function we have jumped from one branch to the next and so the result is not zero.] •

Example 3.4.2 Evaluate

$$\oint_C \frac{dz}{z-a},$$

where C is any simple closed contour encircling the point $z=a$.

Solution Similar to the previous example, the function

$$f(z) = \frac{1}{z-a}$$

is analytic everywhere except at the point $z=a$. Deform C into a circle with centre at $z=a$ and radius R so that

$$z = a + Re^{i\theta}\ , \quad -\pi < \theta \leq \pi,$$

and $dz = Rie^{i\theta}\,d\theta$. Thus

$$\oint_C \frac{dz}{z-a} = \int_{-\pi}^{\pi} \frac{Rie^{i\theta}\,d\theta}{Re^{i\theta}} = \int_{-\pi}^{\pi} i\,d\theta = 2\pi i.$$

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Example 3.4.3 Evaluate

$$\oint_C \frac{dz}{(z-a)^2},$$

where C is any simple closed contour encircling the point $z=a$.

Solution Again $f(z)$ has a singularity at $z=a$ (a pole of order 2). So by the above approach we have

$$z = a + Re^{i\theta}\ , \quad -\pi < \theta \leq \pi,$$

so that $dz = Rie^{i\theta}\,d\theta$. Hence

$$\oint_C \frac{dz}{(z-a)^2} = \int_{-\pi}^{\pi} \frac{Rie^{i\theta}\,d\theta}{R^2 e^{2i\theta}} = \int_{-\pi}^{\pi} \frac{i}{R} e^{-i\theta}\,d\theta.$$

Therefore

$$\oint_C \frac{dz}{(z-a)^2} = \frac{i}{-iR} e^{-i\theta}\Big]_{-\pi}^{\pi} = -\frac{(e^{-i\pi} - e^{i\pi})}{R} = -\frac{(1-1)}{R} = 0.$$

We can easily generalise this result to $f(z) = (z-a)^{-n}$ with integer $n \geq 2$. •

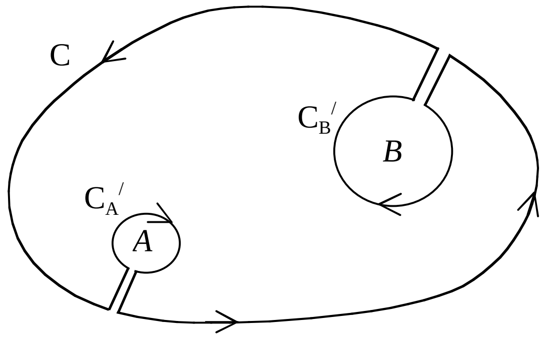

Figure 18: Deforming a contour about two "holes".

Example 3.4.4 Evaluate

$$\oint_C \frac{2z}{z^2-1}\, dz,$$

where C is the circle $|z| = 2$.

Solution Here

$$f(z) = \frac{2z}{z^2-1}$$

is analytic at all points inside C except for the two points $z = 1$ and $z = -1$. In order to apply the above result (only one singular point inside C) we write (partial fractions)

$$f(z) = \frac{2z}{z^2-1} = \frac{1}{z-1} + \frac{1}{z+1}.$$

It follows that

$$\oint_C \frac{2z}{z^2-1}\, dz = \oint_C \frac{dz}{z-1} + \oint_C \frac{dz}{z+1}.$$

From example 3.4.2, the value of each integral on the right-hand side is $2\pi i$. Hence

$$\oint_C \frac{2z}{z^2-1}\, dz = 4\pi i.$$

•

The idea of contour deformation can be extended to more complicated multi-connected domains. Consider a closed contour C encircling two separate "holes" at A and B as depicted in figure 18. We find that

$$\oint_C f(z)\, dz = \oint_{C_A} f(z)\, dz + \oint_{C_B} f(z)\, dz,$$

where all three closed contours are traversed in an anti-clockwise direction. This will shortly be seen to be convenient when the integrals on the right-hand side can be readily evaluated by Cauchy's integral formula while their counterpart on the left-hand side cannot.

Note that the above result can be generalized in a straightforward manner to cases with more than two holes.

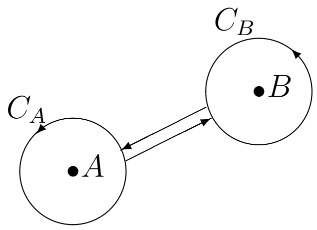

Figure 19: An alternative view of contour deformation

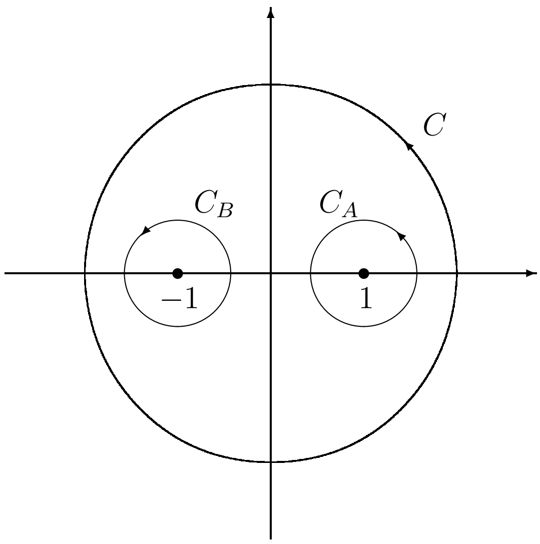

Figure 20: analytic everywhere except ± 1

Before moving on it is worth noting that contour deformation can also be applied to infinite contours. For an illustration, consider the Airy function $Ai(z)$ defined by

$$Ai(z) = \frac{1}{2\pi i} \int_C e^{zt - t^3/3}\, dt,$$

where C is an infinite contour in the complex t-plane coming in from $\infty e^{i4\pi/3}$ and going off to $\infty e^{i2\pi/3}$ (see figure 21). We show that the "directions" of the two infinite ends of C can be relaxed to some extent. In other words, $Ai(z)$ can be defined using contours coming in and going off along directions other than those described above.

To see how C can be deformed (onto another infinite contour) without affecting the value of the integral, we express t in standard polar form $t = re^{i\theta}$ and obtain

$$\begin{aligned}
\left|\exp\left\{zt - t^3/3\right\}\right| &= \left|\exp\left\{zre^{i\theta} - r^3 e^{3i\theta}/3\right\}\right| \\
&= \left|\exp\left\{zr(\cos\theta + i\sin\theta) - r^3(\cos 3\theta + i\sin 3\theta)/3\right\}\right| \\
&= \left|\exp\left\{zr(\cos\theta + i\sin\theta) - r^3\cos 3\theta/3\right\}\right|,
\end{aligned}$$

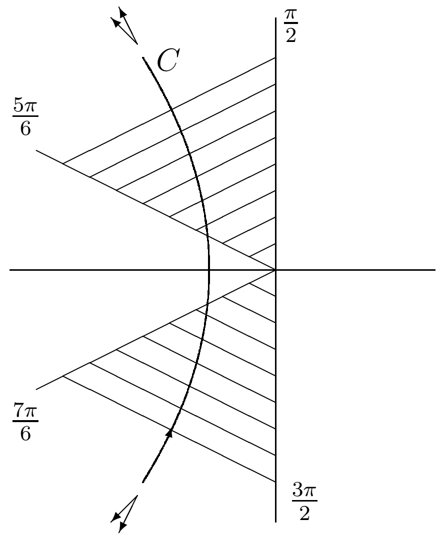

Figure 21: A typical contour C in the definition of the Airy function $Ai(z)$. The shaded sectors indicate permissible regions for the infinite ends of C.

which is guaranteed to vanish (exponentially) in the limit $r \to \infty$, provided that

$$\cos 3\theta > 0 \quad \Longrightarrow \quad \frac{2n\pi}{3} - \frac{\pi}{6} < \theta < \frac{\pi}{6} + \frac{2n\pi}{3}, \quad n = 0, 1, 2, \cdots . \tag{1}$$

For $n = 1, 2$, this corresponds to (see figure 21)

$$\frac{\pi}{2} < \theta < \frac{5\pi}{6}, \quad \frac{7\pi}{6} < \theta < \frac{3\pi}{2}. \tag{2}$$

Now consider an infinite contour C' coming in from $\infty e^{i\theta_1}$, where $7\pi/6 < \theta_1 < 3\pi/2$, and going off to $\infty e^{i\theta_2}$, where $\pi/2 < \theta_2 < 5\pi/6$. Imagine constructing a "closed" contour $\Gamma = C \cup \gamma_1 \cup C'' \cup \gamma_2$, where C'' is the contour C' but in the opposite direction and γ_1 and γ_2 connect the infinite ends of C and C'' (see figure 22). Since $\exp\{zt - t^3/3\}$ is an entire function we have

$$\oint_\Gamma e^{zt-t^3/3}\,dt = \int_C e^{zt-t^3/3}\,dt + \int_{\gamma_1} e^{zt-t^3/3}\,dt + \int_{C''} e^{zt-t^3/3}\,dt + \int_{\gamma_2} e^{zt-t^3/3}\,dt = 0.$$

Now the integrand $\exp\{zt - t^3/3\}$ vanishes exponentially on γ_1 and γ_2. This means that each of the two integrals along γ_1 and γ_2 is zero. It follows that

$$\int_C e^{zt-t^3/3}\,dt = -\int_{C''} e^{zt-t^3/3}\,dt = \int_{C'} e^{zt-t^3/3}\,dt.$$

Hence, $Ai(z)$ can be defined by using any infinite contour C' satisfying the given conditions.

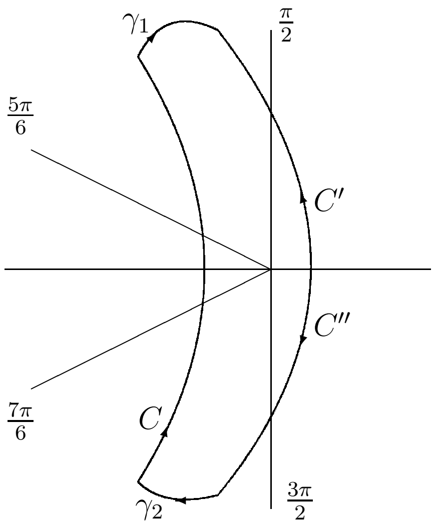

Figure 22: A finite version of $\Gamma = C \cup \gamma_1 \cup C'' \cup \gamma_2$ constructed to illustrate the deformation of C in the definition of $Ai(z)$.

3.5 Cauchy's Integral Formula

Suppose $f(z)$ is analytic in a simply connected domain D which contains the point z_0. Consider

$$\oint_C \frac{f(z)}{z - z_0}\, dz,$$

where C is a simple closed contour, within D, which encircles the point z_0. We have

$$\begin{aligned} \oint_C \frac{f(z)}{z - z_0}\, dz &= \oint_C \frac{f(z_0)}{z - z_0}\, dz + \oint_C \frac{f(z) - f(z_0)}{z - z_0}\, dz \\ &= 2\pi i f(z_0) + \oint_C \frac{f(z) - f(z_0)}{z - z_0}\, dz. \end{aligned}$$

Since $f(z)$ is analytic, it is continuous throughout D and hence given $\epsilon > 0$ there is a $\delta > 0$ such that

$$|f(z) - f(z_0)| < \epsilon \quad \text{for all} \quad |z - z_0| < \delta.$$

Therefore if we deform C onto a circle of radius $\rho < \delta$ (i.e. let C_ρ be the circle $|z - z_0| = \rho$) then

$$\begin{aligned} \left| \oint_C \frac{f(z) - f(z_0)}{z - z_0}\, dz \right| &= \left| \oint_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0}\, dz \right| = \int_0^{2\pi} \left| \frac{f(z_0 + \rho e^{i\theta}) - f(z_0)}{\rho e^{i\theta}} \rho i e^{i\theta} \right| d\theta \\ &\leq \int_0^{2\pi} |f(z_0 + \rho e^{i\theta}) - f(z_0)|\, d\theta < 2\pi\epsilon. \end{aligned}$$

Since we can choose the size of ϵ arbitrarily, the conclusion must be that

$$\oint_C \frac{f(z) - f(z_0)}{z - z_0}\, dz = 0.$$

Thus we have the following result, which is known as Cauchy's Integral formula:

$$\boxed{\oint_C \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0).}$$

This result enables us to easily evaluate any contour integral in which the integrand has a simple pole within the contour. Further, we can interpret the result as saying that if we know that a function is analytic within a domain D and that the function is known on a closed contour (within D) then we know the function everywhere within that contour, as the value of the function at any point z_0 is defined by this integral. If we take the contour as the boundary of the domain then an analytic function is determined by its values on the boundary. This is a very important result as we shall see.

Now let us look at some simple examples of how to use this result.

Example 3.5.1 Evaluate

$$\oint_C \frac{z}{z-3}\, dz$$

where C is (i) the circle $|z-2| = 2$; and (ii) the circle $|z| = 2$.

Solution (i) A circle centre $z = 2$ and radius 2 contains the point $z = 3$ therefore

$$\oint_C \frac{z}{z-3}\, dz = 2\pi i f(3) = 6\pi i\ , \text{ where } f(z) = z.$$

Whereas in (ii) the contour is a circle with centre at the origin and radius 2 so the point $z = 3$ **does not** lie inside C. Hence (by Cauchy's theorem)

$$\oint_C \frac{z}{z-3}\, dz = 0.$$

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Example 3.5.2 Evaluate

$$\oint_C \frac{e^{-z}}{z^3 - 9z}\, dz,$$

where C is a unit square centred on the origin.

Solution $z^3 - 9z = z(z-3)(z+3)$ therefore only the singularity at $z = 0$ lies inside C. Set

$$f(z) = \frac{e^{-z}}{z^2 - 9},$$

which is analytic within the unit square C, so that

$$\oint_C \frac{e^{-z}}{z^3 - 9z}\, dz = \oint_C \frac{f(z)}{z}\, dz = 2\pi i f(0) = \frac{2\pi i e^0}{0-9} = -\frac{2\pi i}{9}.$$

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Example 3.5.3 Let $f(z)$ be analytic on D and $z_0 \in D$ with $f'(z_0) \neq 0$. Show that

$$\oint_C \frac{dz}{f(z) - f(z_0)} = \frac{2\pi i}{f'(z_0)},$$

where C is a small circle (in D) centred at z_0.

Solution Consider the function

$$g(z) = \frac{z - z_0}{f(z) - f(z_0)}.$$

We have

$$\lim_{z \to z_0} g(z) = 1/f'(z_0).$$

So z_0 is a removable singularity of $g(z)$. If we define $g(z_0) = 1/f'(z_0)$, then $g(z)$ becomes analytic in a neighbourhood of z_0, particularly at z_0. Given C within this neighbourhood, Cauchy's integral formula is applicable to $g(z)$. Hence we have

$$\oint_C \frac{g(z)}{z - z_0}\, dz = 2\pi i g(z_0).$$

This immediately implies

$$\oint_C \frac{dz}{f(z) - f(z_0)} = \frac{2\pi i}{f'(z_0)}.$$

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It is possible to generalise Cauchy's formula. For this purpose let's rewrite the formula in the following form

$$\oint_C \frac{f(t)}{t - z}\, dt = 2\pi i f(z).$$

If we assume that differentiation with respect to z under the integral is permissible - we will not attempt to justify this process in this course - then we have

$$f'(z) = \frac{1}{2\pi i} \oint_C f(t) \frac{d}{dz} \left(\frac{1}{t - z} \right) dt = \frac{1}{2\pi i} \oint_C \frac{f(t)}{(t - z)^2}\, dt.$$

Unfortunately, although this 'trick' justifies the result, it does not provides a proof for it because we did not justify that we can swap the order. For a formal proof, see additional notes.

Repeating the process (proof by induction) we obtain

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(t)}{(t - z)^{n+1}}\, dt.$$

This is a remarkable result because it implies that an holomorphic function in a simple, closed contour C has derivatives that are also holomorphic. Also, all derivatives at points within C are determined by the values of the analytic function on the contour C (We could choose C to be the boundary of the domain D). This is what guaranties that the function is in fact infinitely differentiable as stated previously. We will see (very soon!) that we can also guaranty the convergence of the Taylor expansion, hence the fact that holomorphic functions are indeed analytic.

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0),$$

where $n = 0, 1, 2, \ldots$.

3.6 Some Illustrative Examples

First, consider a straightforward application of the above result.

Example 3.6.1 Evaluate

$$\oint_C \frac{e^{2z}}{(z+1)^4}\,dz,$$

where C is the circle $|z| = 3$.

Solution Here $z_0 = -1$ (within C) and $n = 3$. From the above we have

$$\oint_C \frac{e^{2z}}{(z+1)^4}\,dz = \frac{2\pi i}{6} f^{(3)}(-1),$$

where $f(z) = e^{2z}$. Differentiating yields

$$f^{(3)}(z) = 8e^{2z}.$$

So

$$\oint_C \frac{e^{2z}}{(z+1)^4}\,dz = \frac{2\pi i}{6}\, 8e^{-2} = \frac{8\pi i}{3e^2}.$$

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Let us now consider what at first appears to be an unrelated exercise.

Example 3.6.2 Evaluate

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}.$$

Solution Set $z = e^{i\theta}$, so $dz = ie^{i\theta}\,d\theta$.
Now recall

$$\cos\theta = (e^{i\theta} + e^{-i\theta})/2.$$

Hence

$$2 + \cos\theta = 2 + \frac{z + \frac{1}{z}}{2} = \frac{z^2 + 4z + 1}{2z}.$$

Therefore the given trigonometric integral is equivalent to the contour integral

$$\frac{2}{i} \oint_C \frac{dz}{z^2 + 4z + 1},$$

where C is the unit circle.
The function $z^2 + 4z + 1$ has 2 simple poles:

$$z_{\pm} = -2 \pm \sqrt{3}.$$

The root $z_- = -2 - \sqrt{3}$ is outside the unit circle while $z_+ = -2 + \sqrt{3}$ is inside. Hence the value of the integral (by Cauchy's formula) is

$$\frac{2}{i}\, 2\pi i \, \frac{1}{z_+ + 2 + \sqrt{3}} = \frac{2\pi}{\sqrt{3}}.$$

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Example 3.6.3 Show that

$$\oint_C \frac{f(z)}{(z-z_0)^2}\, dz = \oint_C \frac{f'(z)}{(z-z_0)}\, dz,$$

where z_0 is inside C and $f(z)$ is analytic inside and on C.

Solution Applying Cauchy's theorem with $n = 1$ to the left-hand integral yields

$$\oint_C \frac{f(z)}{(z-z_0)^2}\, dz = 2\pi i f'(z_0).$$

As $f(z)$ is analytic inside and on C, $f'(z)$ is analytic inside and on C. So by applying Cauchy's theorem with $n = 0$ to the right-hand integral we obtain

$$\oint_C \frac{f'(z)}{(z-z_0)}\, dz = 2\pi i f'(z_0),$$

hence the desired result. •

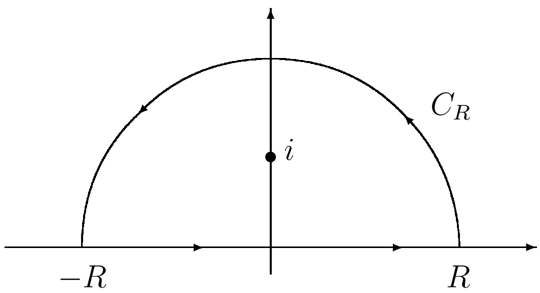

Figure 23: Semi-circular contour $C = [-R, R] \cup C_R$ in example 3.6.4

Example 3.6.4 Evaluate

$$\oint_C \frac{e^{iz}}{1+z^2}\,dz,$$

where C is the simple closed curve consisting of the semicircle $|z| = R,\ R > 1$ in the upper half plane and the real line $[-R, R]$ (see figure 23).

Solution We have

$$\oint_C \frac{e^{iz}}{1+z^2}\,dz = \oint_C \frac{e^{iz}}{(z+i)(z-i)}\,dz.$$

Applying Cauchy's integral formula with

$$f(z) = \frac{e^{iz}}{z+i}$$

$z_0 = i$ and $n = 0$ yields

$$\oint_C \frac{e^{iz}}{1+z^2}\,dz = \oint_C \frac{f(z)}{z-i}\,dz = 2\pi i\,\frac{e^{ii}}{2i} = \pi e^{-1}.$$

Note that we can write

$$\oint_C \frac{e^{iz}}{1+z^2}\,dz = \int_{-R}^{R} \frac{e^{ix}}{1+x^2}\,dx + \int_{C_R} \frac{e^{iz}}{1+z^2}\,dz.$$

Now since

$$\left|\frac{e^{iz}}{1+z^2}\right| = \frac{|e^{iz}|}{|1+z^2|} = \frac{|e^{ix-y}|}{|1+z^2|} \le \frac{1}{|z|^2-1},$$

the second integral on the right-hand side satisfies

$$\left|\int_{C_R} \frac{e^{iz}}{1+z^2}\,dz\right| \le \frac{\pi R}{R^2-1},$$

which tends to zero as $R \to \infty$. Thus in this limit we have

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}dx = \pi e^{-1},$$

or equivalently (the imaginary part vanishes)

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,dx = \pi e^{-1}.$$

This approach is the basis of a powerful method for evaluating certain kinds of real integrals. •

3.7 Further Results

Cauchy's Integral formula has far-reaching implications. Here are some of the most significant ones.

Cauchy's Inequality. If $f(z)$ is analytic inside and on a circular contour C of radius r and centre $z = z_0$, then

$$\left|f^{(n)}(z_0)\right| \leq \frac{Mn!}{r^n},$$

where $|f(z)| \leq M$ on C.

On C we have $z = z_0 + re^{i\theta}$, where $0 \leq \theta \leq 2\pi$. By just taking the modulus of Cauchy's integral formula we get

$$\begin{aligned}\left|f^{(n)}(z_0)\right| &= \left|\frac{n!}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}\right| = \left|\frac{n!}{2\pi i}\int_0^{2\pi} \frac{f(z_0+re^{i\theta})rie^{i\theta}\,d\theta}{r^{n+1}e^{i(n+1)\theta}}\right| \\ &\leq \frac{n!}{2\pi}\int_0^{2\pi}\frac{M\,d\theta}{r^n} = \frac{Mn!}{2\pi}\oint_C \frac{d\theta}{r^n} = \frac{Mn!}{r^n},\end{aligned}$$

which gives the required inequality. We will see shortly that this inequality guaranties the convergence of the Taylor series!

Louville's Theorem. If $f(z)$ is analytic everywhere (i.e. an entire function) and bounded, then $f(z)$ is a constant.

Let M be an upper bound for $|f(z)|$, i.e.

$$|f(z)| \leq M.$$

Cauchy's inequality for $n = 1$ is

$$|f'(z_0)| \leq \frac{M}{r}.$$

Since $f(z)$ is an entire function, we can take r to be as large as we like, so that

$$|f'(z_0)| \leq \frac{M}{r} \to 0$$

as $r \to \infty$, for any z_0. Hence $f'(z_0) = 0$. Thus, $f(z)$ must be a constant.

Fundamental theorem of Algebra. A polynomial in z of degree n has n zeros.

Consider the polynomial

$$p(z) = a_0 + a_1 z + \ldots + a_n z^n, \quad \text{where} \quad a_n \neq 0.$$

Suppose $p(z)$ has no zeros. Then the function $1/p(z)$ is bounded for all z and analytic everywhere and hence must be a constant! Clearly this is false. Hence $p(z) = 0$ at some point, say $z = z_1$. This allows us to write

$$p(z) = (z - z_1)q(z),$$

where $q(z)$ is a polynomial of degree $(n-1)$. Now repeat the argument for $q(z)$ we obtain another zero, say $z = z_2$. Hence by induction we arrive at

$$p(z) = a_n(z - z_1)(z - z_2)\ldots(z - z_n).$$

Maximum modulus. If $f(z)$ is analytic in D and on its boundary, then the maximum of the modulus $|f(z)|$ occurs on the boundary.

Suppose that $|f(z)|$ has a maximum at some $z = z_0$ within the interior of D. Let C be a circle in D centred at z_0 with radius r. Cauchy's integral formula gives

$$\oint_C \frac{f(z)}{(z - z_0)}\, dz = 2\pi i f(z_0).$$

It follows that

$$|f(z_0)| = \left| \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta}) r i e^{i\theta}}{re^{i\theta}}\, d\theta \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{i\theta})|\, d\theta.$$

This means that $|f(z_0)|$ is not greater than the mean of $|f(z)|$ around the circle. This contradicts the assumption that $|f(z_0)|$ is a maximum. Hence $|f(z)|$ cannot achieve its maximum at a point within the interior of D. Rather the maximum occurs on the boundary of D.