Section 3

Direct sums

Definition and basic properties

The following construction is extremely useful.

Definition 3.1 Let V be a vector space over a field F. We say that V is the *direct sum* of two subspaces U_1 and U_2, written $V = U_1 \oplus U_2$ if every vector in V can be expressed *uniquely* in the form $u_1 + u_2$ where $u_1 \in U_1$ and $u_2 \in U_2$.

Proposition 3.2 *Let V be a vector space and U_1 and U_2 be subspaces of V. Then $V = U_1 \oplus U_2$ if and only if the following conditions hold:*

(i) $V = U_1 + U_2$,

(ii) $U_1 \cap U_2 = \{\mathbf{0}\}$.

Comment: Many authors use these two conditions to *define* what is meant by a direct sum and then show it is equivalent to our "unique expression" definition.

PROOF: By definition, $U_1 + U_2 = \{\, u_1 + u_2 \mid u_1 \in U_1,\ u_2 \in U_2 \,\}$, so certainly every vector in V can be expressed in the form $u_1 + u_2$ (where $u_i \in U_i$) if and only if $V = U_1 + U_2$. We must show that condition (ii) corresponds to the uniqueness part.

So suppose $V = U_1 \oplus U_2$. Let $u \in U_1 \cap U_2$. Then we have $u = u + \mathbf{0} = \mathbf{0} + u$ as two ways of expressing u as the sum of a vector in U_1 and a vector in U_2. The uniqueness condition forces $u = \mathbf{0}$, so $U_1 \cap U_2 = \{\mathbf{0}\}$.

Conversely, suppose $U_1 \cap U_2 = \{\mathbf{0}\}$. Suppose $v = u_1 + u_2 = u_1' + u_2'$ are expressions for a vector v where $u_1, u_1' \in U_1$ and $u_2, u_2' \in U_2$. Then

$$u_1 - u_1' = u_2' - u_2 \in U_1 \cap U_2,$$

so $u_1 - u_1' = u_2' - u_2 = \mathbf{0}$ and we deduce $u_1 = u_1'$ and $u_2 = u_2'$. Hence our expressions are unique, so (i) and (ii) together imply $V = U_1 \oplus U_2$. $\square$

Example 3.3 *Let $V = \mathbb{R}^3$ and let*

$$U_1 = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right) \qquad \textit{and} \qquad U_2 = \operatorname{Span}\left(\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right).$$

Show that $V = U_1 \oplus U_2$.

Solution: Let us solve

$$\alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We find

$$\alpha + 2\beta = \alpha + \beta + 3\gamma = \alpha + \gamma = 0.$$

Thus $\gamma = -\alpha$, so the second equation gives $\beta - 2\alpha = 0$; i.e., $\beta = 2\alpha$. Hence $5\alpha = 0$, so $\alpha = 0$ which implies $\beta = \gamma = 0$. Thus the three vectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$$

are linearly independent and hence form a basis for $\mathbb{R}^3$. Therefore every vector in $\mathbb{R}^3$ can be expressed (uniquely) as

$$\left[\alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right] + \left[\gamma \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right] = u_1 + u_2 \in U_1 + U_2.$$

So $\mathbb{R}^3 = U_1 + U_2$. If $\boldsymbol{v} \in U_1 \cap U_2$, then

$$\boldsymbol{v} = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \gamma \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$$

for some $\alpha, \beta, \gamma \in \mathbb{R}$ and we would have

$$\alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \gamma \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Linear independence forces $\alpha = \beta = \gamma = 0$. Hence $\boldsymbol{v} = \mathbf{0}$, so $U_1 \cap U_2 = \{\mathbf{0}\}$. Thus $\mathbb{R}^3 = U_1 \oplus U_2$. $\square$

The link between a basis for V and a direct sum decomposition $V = U_1 \oplus U_2$ has now arisen. We formalise this in the following observation.

Proposition 3.4 *Let $V = U_1 \oplus U_2$ be a finite-dimensional vector space expressed as a direct sum of two subspaces. If $\mathscr{B}_1$ and $\mathscr{B}_2$ are bases for U_1 and U_2, respectively, then $\mathscr{B}_1 \cup \mathscr{B}_2$ is a basis for V.*

PROOF: Let $\mathscr{B}_1 = \{u_1, u_2, \dots, u_m\}$ and $\mathscr{B}_2 = \{v_1, v_2, \dots, v_n\}$. If $v \in V$, then $v = x + y$ where $x \in U_1$ and $y \in U_2$. Since $\mathscr{B}_1$ and $\mathscr{B}_2$ span U_1 and U_2, respectively, there exist scalars α_i and β_j such that

$$x = \alpha_1 u_1 + \dots + \alpha_m u_m \qquad \text{and} \qquad y = \beta_1 v_1 + \dots + \beta_n v_n.$$

Then

$$v = x + y = \alpha_1 u_1 + \dots + \alpha_m u_m + \beta_1 v_1 + \dots + \beta_n v_n$$

and it follows that $\mathscr{B} = \{u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n\}$ spans V.

Now suppose

$$\alpha_1 u_1 + \dots + \alpha_m u_m + \beta_1 v_1 + \dots + \beta_n v_n = \mathbf{0}$$

for some scalars α_i, β_i. Put

$$x = \alpha_1 u_1 + \dots + \alpha_m u_m \in U_1 \qquad \text{and} \qquad y = \beta_1 v_1 + \dots + \beta_n v_n \in U_2.$$

Then $x + y = \mathbf{0}$ must be the unique decomposition of $\mathbf{0}$ produced by the direct sum $V = U_1 \oplus U_2$; that is, it must be $\mathbf{0} + \mathbf{0} = \mathbf{0}$. Hence

$$\alpha_1 u_1 + \dots + \alpha_m u_m = x = \mathbf{0} \qquad \text{and} \qquad \beta_1 v_1 + \dots + \beta_n v_n = y = \mathbf{0}.$$

Linear independence of $\mathscr{B}_1$ and $\mathscr{B}_2$ now give

$$\alpha_1 = \dots = \alpha_m = 0 \qquad \text{and} \qquad \beta_1 = \dots = \beta_n = 0.$$

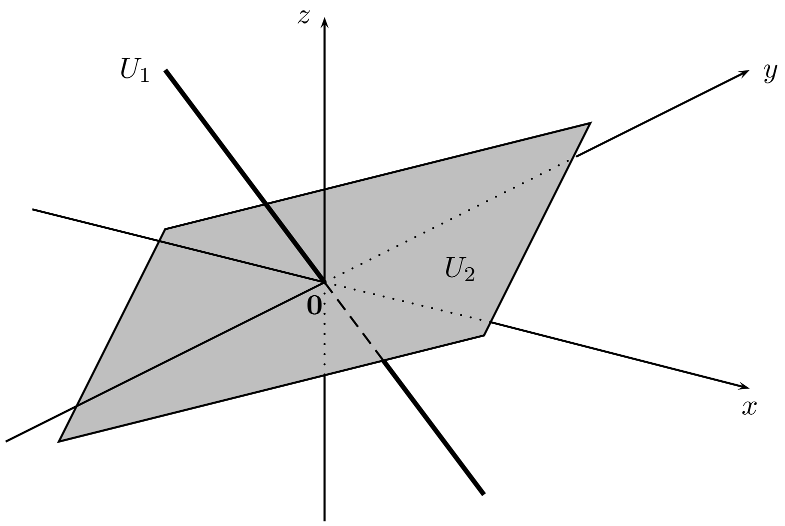
Hence $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$ is linearly independent and therefore a basis for V. $\square$

Corollary 3.5 *If $V = U_1 \oplus U_2$ is a finite-dimensional vector space expressed as a direct sum of two subspaces, then*

$$\dim V = \dim U_1 + \dim U_2.$$

$\square$

Example 3.3 is in some sense typical of direct sums. To gain a visual understanding, the following picture illustrates the 3-dimensional space $\mathbb{R}^3$ as the direct sum of a 1-dimensional subspace U_1 and a 2-dimensional subspace U_2 (these being a line and a plane passing through the origin, respectively).



Projection maps

Definition 3.6 Let $V = U_1 \oplus U_2$ be a vector space expressed as a direct sum of two subspaces. The two *projection maps* $P_1 \colon V \to V$ and $P_2 \colon V \to V$ onto U_1 and U_2, respectively, corresponding to this decomposition are defined as follows:

> if $v \in V$, express v uniquely as $v = u_1 + u_2$ where $u_1 \in U_1$ and $u_2 \in U_2$, then
>
> $$P_1(v) = u_1 \qquad \text{and} \qquad P_2(v) = u_2.$$

Note that the uniqueness of expression guarantees that precisely one value is specified for $P_1(v)$ and one for $P_2(v)$. (If we only had $V = U_1 + U_2$, then we would have choice as to which expression $v = u_1 + u_2$ to use and we would not have well-defined maps.)

Lemma 3.7 *Let $V = U_1 \oplus U_2$ be a direct sum of subspaces with projection maps $P_1 \colon V \to V$ and $P_2 \colon V \to V$. Then*

(i) *P_1 and P_2 are linear transformations;*

(ii) *$P_1(u) = u$ for all $u \in U_1$ and $P_1(w) = \mathbf{0}$ for all $w \in U_2$;*

(iii) *$P_2(u) = \mathbf{0}$ for all $u \in U_1$ and $P_2(w) = w$ for all $w \in U_2$;*

(iv) *$\ker P_1 = U_2$ and $\operatorname{im} P_1 = U_1$;*

(v) *$\ker P_2 = U_1$ and $\operatorname{im} P_2 = U_2$.*

Proof: We just deal with the parts relating to P_1. Those for P_2 are established by identical arguments. To simplify notation we shall discard the subscript and simply write P for the projection map onto U_1 associated to the direct sum decomposition $V = U_1 \oplus U_2$. This is defined by $P(v) = u_1$ when $v = u_1 + u_2$ with $u_1 \in U_1$ and $u_2 \in U_2$.

(i) Let $v, v' \in V$ and write $v = u_1 + u_2$, $v' = u_1' + u_2'$ where $u_1, u_1' \in U_1$ and $u_2, u_2' \in U_2$. Then

$$v + v' = (u_1 + u_1') + (u_2 + u_2')$$

and $u_1 + u_1' \in U_1$, $u_2 + u_2' \in U_2$. This must be the unique decomposition for $v + v'$, so

$$P(v + v') = u_1 + u_1' = P(v) + P(v').$$

Equally if $\alpha \in F$, then $\alpha v = \alpha u_1 + \alpha u_2$ where $\alpha u_1 \in U_1$, $\alpha u_2 \in U_2$. Thus

$$P(\alpha v) = \alpha u_1 = \alpha P(v).$$

Hence P is a linear transformation.

(ii) If $u \in U_1$, then $u = u + \mathbf{0}$ is the decomposition we use to calculate P, so $P(u) = u$.

If $w \in U_2$, then $w = \mathbf{0} + w$ is the required decomposition, so $P(w) = \mathbf{0}$.

(iv) For any vector v, $P(v)$ is always the U_1-part in the decomposition of v, so certainly $\operatorname{im} P \subseteq U_1$. On the other hand, if $u \in U_1$, then part (ii) says $u = P(u) \in \operatorname{im} P$. Hence $\operatorname{im} P = U_1$.

Part (ii) also says $P(w) = \mathbf{0}$ for all $w \in U_2$, so $U_2 \subseteq \ker P$. On the other hand, if $v = u_1 + u_2$ lies in $\ker P$, then $P(v) = u_1 = \mathbf{0}$, so $v = u_2 \in U_2$. Hence $\ker P = U_2$. □

The major facts about projections are the following:

Proposition 3.8 *Let $P \colon V \to V$ be a projection corresponding to some direct sum decomposition of the vector space V. Then*

(i) $P^2 = P$*;*

(ii) $V = \ker P \oplus \operatorname{im} P$*;*

(iii) $I - P$ *is also a projection;*

(iv) $V = \ker P \oplus \ker(I - P)$*.*

Here $I \colon V \to V$ denotes the identity transformation $I \colon v \mapsto v$ for $v \in V$.

Proof: As a projection map, P must be associated to a direct sum decomposition of V, so let us assume that $V = U_1 \oplus U_2$ and that $P = P_1$ is the corresponding projection onto the subspace U_1 (i.e., that P denotes the same projection as in the previous proof).

(i) If $v \in V$, then $P(v) \in U_1$, so by Lemma 3.7(ii),

$$P^2(v) = P(P(v)) = P(v).$$

Hence $P^2 = P$.

(ii) $\ker P = U_2$ and $\operatorname{im} P = U_1$, so

$$V = U_1 \oplus U_2 = \operatorname{im} P \oplus \ker P,$$

as required.

(iii) Let $Q\colon V \to V$ denote the projection onto U_2. If $v \in V$, say $v = u_1 + u_2$ where $u_1 \in U_1$ and $u_2 \in U_2$, then

$$Q(v) = u_2 = v - u_1 = v - P(v) = (I - P)(v).$$

Hence $I - P$ is the projection Q.

(iv) $\ker P = U_2$, while $\ker(I - P) = \ker Q = U_1$. Hence

$$V = U_1 \oplus U_2 = \ker(I - P) \oplus \ker P.$$

□

We give an example to illustrate how projection maps depend on both summands in the direct sum decomposition.

Example 3.9 *Let*

$$U_1 = \operatorname{Span}\left(\begin{pmatrix}1\\0\end{pmatrix}\right), \quad U_2 = \operatorname{Span}\left(\begin{pmatrix}0\\1\end{pmatrix}\right) \quad \textit{and} \quad U_3 = \operatorname{Span}\left(\begin{pmatrix}1\\1\end{pmatrix}\right).$$

Show that

$$\mathbb{R}^2 = U_1 \oplus U_2 \qquad \textit{and} \qquad \mathbb{R}^2 = U_1 \oplus U_3.$$

and, if $P\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the projection onto U_1 corresponding to the first decomposition and $Q\colon \mathbb{R}^2 \to \mathbb{R}^2$ is the projection onto U_1 corresponding to the second decomposition, that $P \neq Q$.

Solution: If $\boldsymbol{v} = \begin{pmatrix}x\\y\end{pmatrix} \in \mathbb{R}^2$, then

$$\boldsymbol{v} = \begin{pmatrix}x\\y\end{pmatrix} = x\begin{pmatrix}1\\0\end{pmatrix} + y\begin{pmatrix}0\\1\end{pmatrix}$$

and

$$\boldsymbol{v} = \begin{pmatrix}x\\y\end{pmatrix} = (x-y)\begin{pmatrix}1\\0\end{pmatrix} + y\begin{pmatrix}1\\1\end{pmatrix}.$$

Hence $\mathbb{R}^2 = U_1 + U_2 = U_1 + U_3$. Moreover,

$$U_1 = \left\{\begin{pmatrix}x\\0\end{pmatrix} \,\middle|\, x \in \mathbb{R}\right\} \qquad \text{and} \qquad U_2 = \left\{\begin{pmatrix}0\\y\end{pmatrix} \,\middle|\, y \in \mathbb{R}\right\},$$

so $U_1 \cap U_2 = \{\mathbf{0}\}$. Therefore we do have a direct sum $\mathbb{R}^2 = U_1 \oplus U_2$. Similarly, one can see $U_1 \cap U_3 = \{\mathbf{0}\}$, so the second sum is also direct.

We know by Lemma 3.7(ii) that

$$P(\boldsymbol{u}) = Q(\boldsymbol{u}) = \boldsymbol{u} \qquad \text{for all } \boldsymbol{u} \in U_1,$$

but if we take $\boldsymbol{v} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \in \mathbb{R}^2$, we obtain different values for $P(\boldsymbol{v})$ and $Q(\boldsymbol{v})$. Indeed

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

is the decomposition corresponding to $\mathbb{R}^2 = U_1 \oplus U_2$ which yields

$$P \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \in U_1$$

while

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

is that corresponding to $\mathbb{R}^2 = U_1 \oplus U_3$ which yields

$$Q \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in U_1.$$

Also note $\ker P = U_2 \neq \ker Q = U_3$, which is more information indicating the difference between these two transformations. $\square$

Example 3A *Let $V = \mathbb{R}^3$ and $U = \mathrm{Span}(\boldsymbol{v}_1)$, where*

$$\boldsymbol{v}_1 = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}.$$

(i) *Find a subspace W such that $V = U \oplus W$.*

(ii) *Let $P \colon V \to V$ be the associated projection onto W. Calculate $P(\boldsymbol{u})$ where*

$$\boldsymbol{u} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}.$$

Solution: (i) We first extend $\{\boldsymbol{v}_1\}$ to a basis for $\mathbb{R}^3$. We claim that

$$\mathscr{B} = \left\{ \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

is a basis for $\mathbb{R}^3$. We solve

$$\alpha \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix};$$

that is,

$$3\alpha + \beta = -\alpha + \gamma = 2\alpha = 0.$$

Hence $\alpha = 0$, so $\beta = -3\alpha = 0$ and $\gamma = \alpha = 0$. Thus $\mathscr{B}$ is linearly independent. Since $\dim V = 3$ and $|\mathscr{B}| = 3$, we conclude that $\mathscr{B}$ is a basis for $\mathbb{R}^3$.

Let $W = \mathrm{Span}(\boldsymbol{v}_2, \boldsymbol{v}_3)$ where

$$\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Since $\mathscr{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is a basis for V, if $\boldsymbol{v} \in V$, then there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$\boldsymbol{v} = (\alpha_1 \boldsymbol{v}_1) + (\alpha_2 \boldsymbol{v}_2 + \alpha_3 \boldsymbol{v}_3) \in U + W.$$

Hence $V = U + W$.

If $\boldsymbol{v} \in U \cap W$, then there exist $\alpha, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\boldsymbol{v} = \alpha \boldsymbol{v}_1 = \beta_1 \boldsymbol{v}_2 + \beta_2 \boldsymbol{v}_3.$$

Therefore

$$\alpha \boldsymbol{v}_1 + (-\beta_1)\boldsymbol{v}_2 + (-\beta_2)\boldsymbol{v}_3 = \mathbf{0}.$$

Since $\mathscr{B}$ is linearly independent, we conclude $\alpha = -\beta_1 = -\beta_2 = 0$, so $\boldsymbol{v} = \alpha \boldsymbol{v}_1 = \mathbf{0}$. Thus $U \cap W = \{\mathbf{0}\}$ and so

$$V = U \oplus W.$$

(ii) We write $\boldsymbol{u}$ as a linear combination of the basis $\mathscr{B}$. Inspection shows

$$\begin{aligned} \boldsymbol{u} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} &= 2 \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 6 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 6 \\ -2 \\ 4 \end{pmatrix} + \begin{pmatrix} -2 \\ 6 \\ 0 \end{pmatrix}, \end{aligned}$$

where the first term in the last line belongs to U and the second to W. Hence

$$P(\boldsymbol{u}) = \begin{pmatrix} -2 \\ 6 \\ 0 \end{pmatrix}$$

(since this is the W-component of $\boldsymbol{u}$). □

Direct sums of more summands

We briefly address the situation when V is expressed as a direct sum of more than two subspaces.

Definition 3.10 Let V be a vector space. We say that V is the *direct sum* of subspaces $U_1, U_2, \ldots, U_k$, written $V = U_1 \oplus U_2 \oplus \cdots \oplus U_k$, if every vector in V can be *uniquely* expressed in the form $u_1 + u_2 + \cdots + u_k$ where $u_i \in U_i$ for each i.

Again this can be translated into a condition involving sums and intersections, though the intersection condition is more complicated. We omit the proof.

Proposition 3.11 *Let V be a vector space with subspaces $U_1, U_2, \ldots, U_k$. Then $V = U_1 \oplus U_2 \oplus \cdots \oplus U_k$ if and only if the following conditions hold:*

(i) $V = U_1 + U_2 + \cdots + U_k$;

(ii) $U_i \cap (U_1 + \cdots + U_{i-1} + U_{i+1} + \cdots + U_k) = \{\mathbf{0}\}$ *for each i.*

We shall exploit the potential of direct sums to produce useful bases for our vector spaces. The following adapts quite easily from Proposition 3.4:

Proposition 3.12 *Let $V = U_1 \oplus U_2 \oplus \cdots \oplus U_k$ be a direct sum of subspaces. If $\mathscr{B}_i$ is a basis for U_i for $i = 1, 2, \ldots, k$, then $\mathscr{B}_1 \cup \mathscr{B}_2 \cup \cdots \cup \mathscr{B}_k$ is a basis for V.*