MT5823 Semigroups: Exam Solutions May 2016 (James D. Mitchell)

1. (a) **[Easy - but time consuming]** We apply the algorithm from lectures:

$$\begin{array}{rcll rcll}
a^2 & = & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & & ab & = & \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} & \\
ba & = & \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} & & b^2 & = & \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & = b \\
a^3 & = & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & = a^2 & a^2b & = & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & = a^2 \\
aba & = & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & = a & ab^2 & = & \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} & = ab \\
ba^2 & = & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & = a^2 & bab & = & \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & \\
baba & = & \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} & = ba & ab^2 & = & \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & = bab.
\end{array}$$

Hence the elements of $S$ are:

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad a^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad ab = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad ba = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad bab = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

and $S$ satisfies the following relations:

$$b^2 = b, \quad a^3 = a^2, \quad a^2b = a^2, \quad aba = a, \quad ba^2 = a^2.$$

(b) **[Easy - but time consuming]** The right Cayley graph is obtained directly from the algorithm above and the left Cayley graph is obtained from the relations given above:

$$\begin{array}{rcl rcl}
a \cdot a & = & a^2 & b \cdot a & = & ba \\
a \cdot b & = & ab & b \cdot b & = & b \\
a \cdot a^2 & = & a^3 = a^2 & b \cdot a^2 & = & a^2 \\
a \cdot ab & = & a^2b = a^2 & b \cdot ab & = & bab \\
a \cdot ba & = & a & b \cdot ba & = & ba \\
a \cdot bab & = & ab & b \cdot bab & = & bab
\end{array}$$

The Cayley graphs are shown in Figure 1.

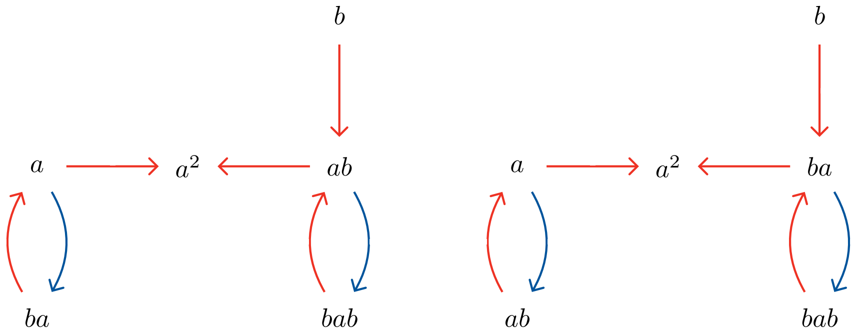


Figure 1: The left and right Cayley graphs (red is for $a$ and blue is for $b$, loops are omitted).

(c) **[Moderate - similar to tutorial problems]** The $\mathscr{L}$-classes are the strongly connected components of the left Cayley graph:

$$\{a, ba\}, \quad \{a^2\}, \quad \{ab, bab\}, \quad \{b\}$$

and, similarly, the $\mathscr{R}$-classes are the strongly connected components of the right Cayley graph:

$$\{a, ab\}, \quad \{a^2\}, \quad \{ba, bab\}, \quad \{b\}.$$

Since $\mathscr{D} = \mathscr{L} \circ \mathscr{R}$, it follows that the $\mathscr{D}$-classes are:

$$D_a = \{a, ab, ba, bab\}, \quad D_{a^2} = \{a^2\}, \quad D_b = \{b\}.$$

From the Cayley graphs it is clear that $D_b > D_a > D_{a^2}$. The eggbox picture is shown in Figure 2.

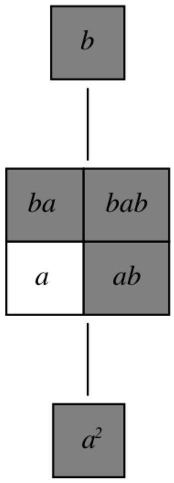


Figure 2: The eggbox picture.

(d) [**Hard - unseen**] Since $S$ is finite $\mathscr{D} = \mathscr{J}$, and so there is a 1-1 correspondence between the $\mathscr{D}$-classes of $S$ and the principal 2-sided ideals. Hence there are only three principal ideals. Since every ideal is a union of principal ideals, and the $\mathscr{D}$-order is linear, it follows that there are only three ideals. They are:

$$S, \quad I = \{a, ab, ba, bab, a^2\}, \quad J = \{a^2\}.$$

Since $|J| = 1$, it follows that $|S/J| = |S|$, and since $S/J$ is a homomorphic image of $S$, $S/J \cong S$. Clearly $S/S$ is trivial, and $S/I$ has two elements $\{x, y\}$ corresponding to $S \setminus I = \{b\}$ and $I$. Hence the Cayley table of $S/I$ is:

| | $x$ | $y$ |
|---|---|---|
| $x$ | $x$ | $y$ |
| $y$ | $y$ | $y$ |

(e) [**Easy - definitions**]

(i) A semigroup $S$ is ***regular*** if for every $x \in S$ there exists $y \in S$ such that $xyx = x$.

(ii) If $S$ is a semigroup and $x, y \in S$, then $y$ is an inverse for $x$ if and only if $xyx = x$ and $yxy = y$.

(iii) A semigroup $S$ is ***inverse*** if for every element of $S$ has a unique inverse.

(f) [**Moderate**] The $\mathscr{R}$-classes of $S$ are:

$$\{a, ab\}, \quad \{a^2\}, \quad \{ba, bab\}, \quad \{b\}$$

and the only non-idempotent element of $S$ is $a$. Hence every $\mathscr{R}$-class of $S$ contains an idempotent and so, by a theorem from lectures, $S$ is a regular semigroup.

By a theorem from lectures a semigroup is inverse if and only if every $\mathscr{R}$-class and every $\mathscr{L}$-class contains exactly 1 idempotent. The $\mathscr{R}$-class $\{ba, bab\}$ contains 2 idempotents and so $S$ is not an inverse semigroup.

(g) [**Moderate**] If $x \in S$ has an inverse, then it is clearly regular by definition. If $x, y \in S$ are such that $xyx = x$, then we will show that $yxy$ is an inverse for $x$:

$$x(yxy)x = (xyx)yx = xyx = x$$

and

$$(yxy)x(yxy) = y(xyx)yxy = yxyxy = yxy.$$

2. (a) **[Easy - definition]** A semigroup $S$ is a ***rectangular band*** if $xyz = xz$ and $x^2 = x$ for all $x, y, z \in S$. Equivalently, $S$ is a rectangular band if it is isomorphic to $I \times \Lambda$ with multiplication

$$(i, \lambda)(j, \mu) = (i, \mu)$$

for all $(i, \lambda), (j, \mu) \in I \times \Lambda$.

(b) **[Moderate - requires remembering several definitions.]** If $(g, r) \in S$ is an idempotent, then $(g, r)^2 = (g^2, r^2) = (g, r)$ and so, in particular, $g^2 = g$ and $r^2 = r$. Since $R$ is a rectangular band, it follows that $x^2 = x$ for all $x \in R$. On the other hand, since $G$ is a group, $g^2 = g$ implies that $g = e$, the identity of $G$. Thus $(g, r) \in S$ is an idempotent if and only if $g = e$, or, in other words, $\{e\} \times R$ is the set of idempotents in $S$.

If $(e, r), (e, s) \in \{e\} \times R$, then $(e, r)(e, s) = (e, rs) \in \{e\} \times R$, and so the set of idempotents $\{e\} \times R$ is a subsemigroup of $S$.

(c) **[Easy - definition]** A semigroup $S$ is ***simple*** if the only 2-sided ideal is the semigroup itself. Or equivalently, if $x \mathscr{J} y$ for all $x, y \in S$.

(d) **[Moderate - unseen]** It suffices to show that $(g, r) \mathscr{J} (h, s)$ for every $(g, r), (h, s) \in S$. Recall that in a rectangular band the identity $xyz = xz$ holds for all $x, y, z$.

Suppose that $(g, r), (h, s) \in S$ are arbitrary. Then

$$(hg^{-1}, s)(g, r)(e, s) = (hg^{-1}g, srs) = (h, s)$$

and

$$(gh^{-1}, r)(h, s)(e, r) = (gh^{-1}h, rsr) = (g, r).$$

Hence $(g, r) \mathscr{J} (h, s)$ and $S$ is simple as required.

(e) **[Easy - definition]** Let $S$ be a semigroup. Then $S$ is finite and simple if and only if it is isomorphic to a Rees matrix semigroup $\mathscr{M}[G; I, \Lambda; P]$, where $I$ and $\Lambda$ are finite index sets, $G$ is a finite group, and $P$ is a $|\Lambda| \times I$ matrix with entries in $G$.

(f) **[Moderate - unseen]** Since

$$(i, p_{\lambda,i}^{-1}, \lambda)^2 = (i, p_{\lambda,i}^{-1} p_{\lambda,i} p_{\lambda,i}^{-1}, \lambda) = (i, p_{\lambda,i}^{-1}, \lambda),$$

$(i, p_{\lambda,i}^{-1}, \lambda)$ is an idempotent for all $i \in I$ and $\lambda \in \Lambda$. Conversely, if $(i, g, \lambda) \in S$ is an idempotent, then $(i, g, \lambda)^2 = (i, g p_{\lambda,i} g, \lambda)$ which implies that $g p_{\lambda,i} g = g$, and hence $g = p_{\lambda,i}^{-1}$. Therefore the set of idempotents in $S$ is

$$\{\, (i, p_{\lambda,i}^{-1}, \lambda) \;:\; i \in I,\ \lambda \in \Lambda \,\}.$$

(g) **[Hard - unseen]** Assume without loss of generality that $I = \{1, \ldots, m\}$ and $\Lambda = \{1, \ldots, n\}$ for some $m, n \in \mathbb{N}$. We set

$$r_\lambda = p_{\lambda,1} \quad \text{and} \quad q_i = p_{1,1}^{-1} p_{1,i}.$$

Since $S$ is orthodox, if $(i, p_{\mu,i}^{-1}, \mu), (j, p_{\lambda,j}^{-1}, \lambda) \in S$ are arbitrary, then

$$(i, p_{\mu,i}^{-1}, \mu)(j, p_{\lambda,j}^{-1}, \lambda) = (i, p_{\mu,i}^{-1} p_{\mu,j} p_{\lambda,j}^{-1}, \lambda)$$

is an idempotent. Hence

$$p_{\lambda,i}^{-1} = p_{\mu,i}^{-1} p_{\mu,j} p_{\lambda,j}^{-1}$$

for all $i, j \in I$ and $\lambda, \mu \in \Lambda$. In particular,

$$p_{\lambda,i} = p_{\lambda,1} p_{1,1}^{-1} p_{1,i} = q_\lambda r_i,$$

as required.

(h) **[Hard - unseen]** Suppose that $(i, g, \lambda), (j, h, \mu) \in S$ are such that $(i, g, \lambda)\phi = (j, h, \mu)\phi$. Then $i = j$ and $\lambda = \mu$, and $q_i g r_\lambda = q_i h r_\lambda$. Hence since $G$ is a group $g = h$ and $\phi$ is ***injective***.

If $(g, (i, \lambda)) \in G \times (I \times \Lambda)$ is arbitrary, then $(i, q_i^{-1} g r_\lambda^{-1}, \lambda)\phi = (g, (i, \lambda))$, and so $\phi$ is ***surjective***.

If $(i, g, \lambda), (j, h, \mu) \in S$, then

$$\begin{aligned}
(i, g, \lambda)\phi (j, h, \mu)\phi &= (q_i g r_\lambda, (i, \lambda))(q_j g r_\mu, (j, \mu)) \\
&= (q_i g r_\lambda q_j g r_\mu, (i, \mu)) \\
&= (q_i g p_{\lambda,j} g r_\mu, (i, \mu)) \\
&= (i, g p_{\lambda,j} h, \mu)\phi \\
&= \big((i, g, \lambda)(j, h, \mu)\big)\phi
\end{aligned}$$

for all $(i, g, \lambda), (j, h, \mu) \in I \times G \times \Lambda$. Hence $\phi$ is a ***homomorphism***.

It follows that $\phi$ is an isomorphism from $S$ to a rectangular group, and hence $\phi$ is a rectangular group.

(i) **[Easy - just putting the pieces together]** Let $S$ be a finite simple orthodox semigroup. Then, by the Rees Theorem, $S$ is isomorphic to a finite orthodox Rees matrix semigroup. It follows from part (iii) that $S$ is a rectangular group.

Conversely we showed that a rectangular group is simple in part (c) and that it is orthodox in part (a).