Chapter 5

Integral Vector Calculus: Surface Integrals

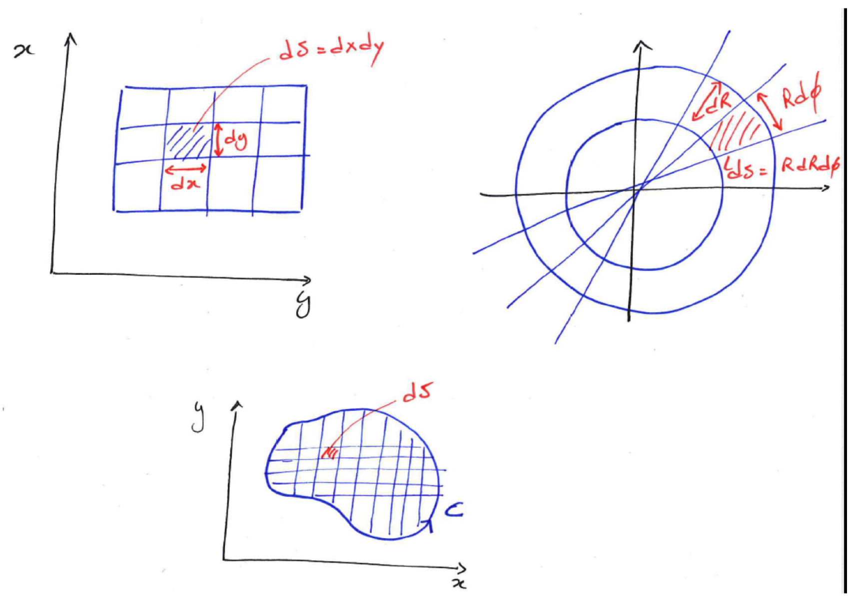
See appendice at the end of Chapter 6 for an schematic overview of the integration part

5.1 Surface Integrals

5.1.1 Introduction

Surface integrals are a generalisation of the idea integration over a 2D planar area.

For example, suppose we want to integrate a function $f(x, y)$ over a given area S of x-y plane.



Then the sum of f over each element of the mesh multiplied by the differential surface area of each element

$$\sum f\ ds \to \iint_S f(x,y)ds.$$

This is the simplest example of a surface integral, the surface being a portion of the x-y plane.

5.1.2 Rapid examples

- Calculate

$$\iint_A xydxdy,$$

  where A is the triangle of vertices $(0,0)$, $(1,0)$ and $(0,1)$.

  Integrating with respect to x first from 0 to the side where $y = 1 - 1x$ to $x = 1 - y$,

$$\iint_A xydxdy = \int_0^1 y \int_0^{1-y} xdxdy = \frac{1}{2}\int_-^1 y(1-y)^2 dy = \frac{1}{2}\int_0^1 (y-2y^2+y^3)dy = \frac{1}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{1}{24}$$

- Calculate

$$\iint_A 3x^2 dxdy$$

  where A is the disk centred at $(0,0)$ and radius 2.

$$\iint_A 3x^2 dxdy = 3\int_0^{2\pi}\int_0^2 r^2\cos^2\theta r dr d\theta$$

  using $x = r\cos\theta$ and $dS = rdrd\theta$

$$\iint_3 x^2 dxdy = 3\int_0^2 r^3 dr \int_0^{2\pi} \cos^2\theta d\theta = 12\pi$$

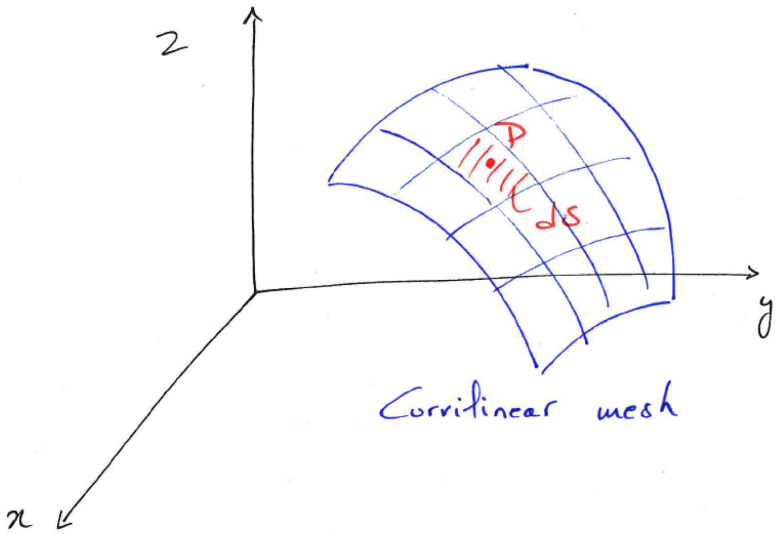
  Recall

$$\int_0^{2\pi} \cos^2\theta d\theta = \frac{1}{2}\int_0^{2\pi}(1+\cos 2\theta)d\theta = \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_{\theta=0}^{\theta=2\pi} = \pi$$

  or using symmetry arguments: We integrate over a period and $\sin$ and $\cos$ are the same functions with the argument is just shifted by $\pi/2$, therefore we must have $\int_0^{2\pi}\cos^2\theta d\theta = \int_0^{2\pi}\sin^2\theta d\theta$ and $\int_0^{2\pi}(\cos^2\theta + \sin^2\theta)d\theta = \int_0^{2\pi} d\theta = 2\pi = 2\int_0^{2\pi}\cos^2\theta d\theta$)

5.1.3 General case

We now want to generalise this notion to non-planar or curved surfaces in space.



The function f to be integrated is defined at points P on the surface S. Again, the surface integral is just the sum of f at the points P, multiplied by the local surface element dS at P.

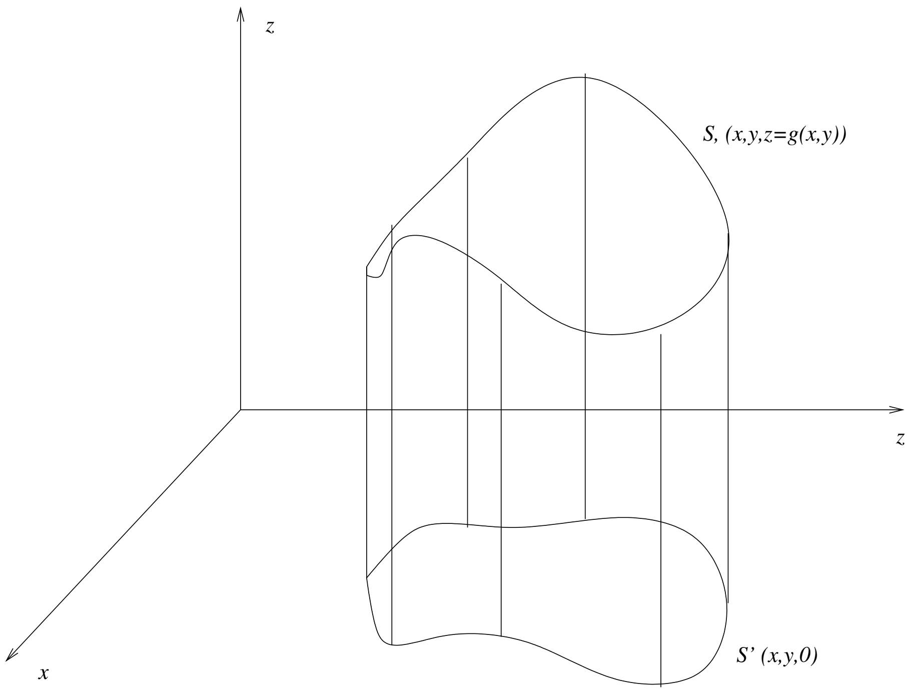
$$\sum_P f dx \to \iint_S f ds$$

Here again the double integral sign implies that S can be represented by 2 coordinates, generally curvilinear coordinates.

5.2 Method of Projections

5.2.1 Formula

Consider a surface S defined by $z = g(x, y)$, where g is a differentiable real function of 2 variables, x and y.

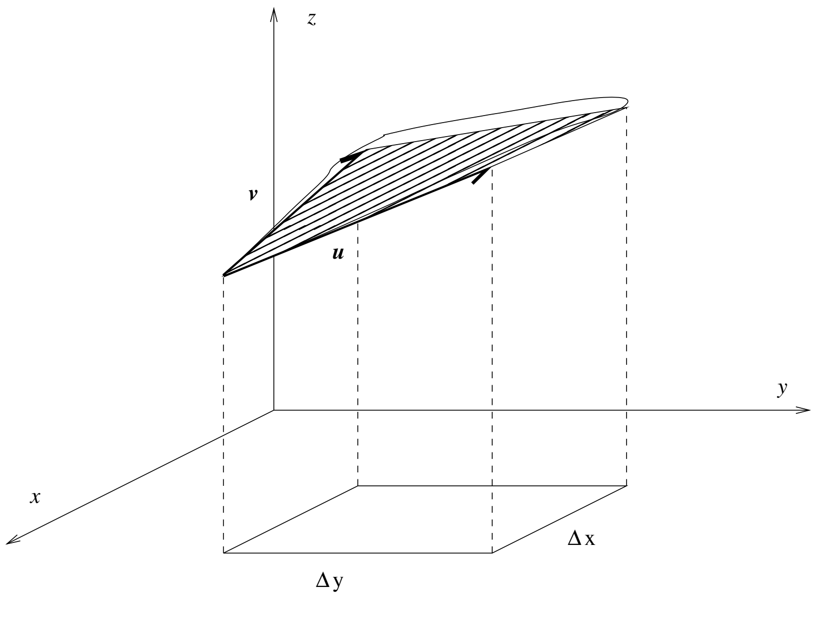


Consider the projection S' of the surface S onto the plane $z = 0$, we have

$$\iint_S f(x,y,z)\,dS = \iint_{S'} f(x,y,g(x,y))\sqrt{\left(\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1\right)}\,dxdy \qquad (1.1)$$

5.2.2 Elements of proof

We approximate dS by the area of a small parallelogram, such that the differences are of higher order than $dxdy$ (i.e. remaining term/$dxdy \to 0$ as $dxdy \to 0$.



$\Delta S = |\mathbf{u} \times \mathbf{v}|$ + higher order terms which are negligible when taking the limit

with $\mathbf{u} = (\Delta x, 0, \Delta x \frac{\partial g}{\partial x})$, and $\mathbf{v} = (0, \Delta y, \Delta y \frac{\partial g}{\partial y})$. And

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x & 0 & \Delta x \frac{\partial g}{\partial x} \\ 0 & \Delta y & \Delta y \frac{\partial g}{\partial y} \end{vmatrix}$$

$$= (-\Delta y \Delta x \frac{\partial g}{\partial x}, -\Delta x \Delta y \frac{\partial g}{\partial y}, \Delta x \Delta y)$$

as Δx, $\Delta y \longrightarrow dx$, dy

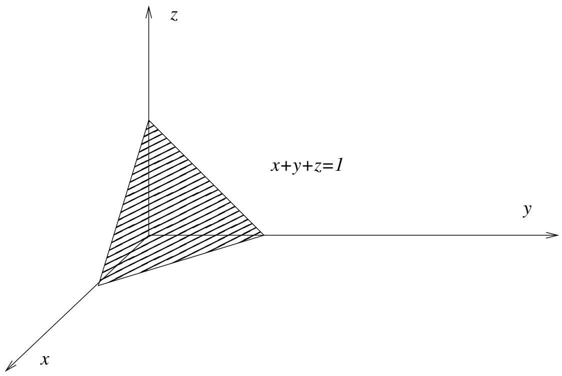
$$\mathbf{u} \times \mathbf{v} \longrightarrow dxdy \left(-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1 \right)$$

and

$$dS \longrightarrow \sqrt{\left(\left(\frac{\partial g}{\partial x} \right)^2 + \left(\frac{\partial g}{\partial y} \right)^2 + 1 \right)}\, dxdy$$

5.2.3 Examples

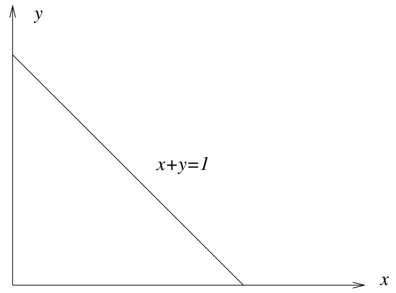
- **Ex. 1** Consider the scalar function $f(x, y, z) = 2xy + z$ and the surface S defined by the portion of the plane $x + y + z = 1$ for $x, y, z \geq 0$.



To calculate this area, we can use the projection S' of S on the plane $z = 0$, and use $z = g(x, y) = 1 - x - y$,

$$\iint_S f(x, y, z) dS = \iint_{S'} f(x, y, g(x, y)) \sqrt{\left(\frac{\partial g}{\partial x} \right)^2 + \left(\frac{\partial g}{\partial y} \right)^2 + 1}\, dxdy$$

$$= \iint_{S'} (2xy + (1 - x - y)) \sqrt{(-1)^2 + (-1)^2 + 1} dxdy. = \sqrt{3} \iint_{S'} (2xy + (1 - x - y)) dxdy$$



The surface S' is shown on the figure above, and is delimited by the axis $y = 0$, $x = 0$, and $x + y = 1$. Integrating first with respect to y (*note: here the order does not matter*) we have

$$\sqrt{3}\int_0^1\int_0^{1-x}(2xy+1-x-y)\,dy\,dx = \sqrt{3}\int_0^1\int_0^{1-x}(y(2x-1)+(1-x))\,dy\,dx$$

$$= \sqrt{3}\int_0^1\left[(2x-1)\frac{y^2}{2}+(1-x)y\right]_{y=0}^{y=(1-x)}dx = \sqrt{3}\int_0^1\left((2x-1)\frac{(1-x)^2}{2}+(1-x)^2\right)dx$$

$$= \sqrt{3}\int_0^1\left(x^3-\frac{3}{2}x^2+\frac{1}{2}\right)dx = \frac{\sqrt{3}}{4}-\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}=\frac{\sqrt{3}}{4}$$

Alternatively we could project onto the plane $y = 0$ and we would have $y = h(x, z) = 1 - x - z$, and $\sqrt{(\partial h/\partial x)^2 + (\partial h/\partial y)^2 + 1} = \sqrt{3}$,

$$\iint_S f dS = \sqrt{3}\int_0^1\int_0^{1-x}(2(x(1-x-z))+z)\,dzdx$$

$$= \sqrt{3}\int_0^1\left(2x(1-x)(1-x)+(1-2x)\frac{(1-x)^2}{2}\right)dx$$

$$= \int_0^1\left(\sqrt{3}x^3-\sqrt{3}\frac{3}{2}x^2+\frac{\sqrt{3}}{2}\right)dx = \frac{\sqrt{3}}{4}$$

Ex. 2 Calculate the surface of the paraboloid $z = 1 - (x^2 + y^2) = f(x, y)$

Here we want to calculate

$$\iint_S dS = \iint_{S'}\sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+1}dA$$

where S' is th surface of the disk of radius 1, $x^2 + y^2 = 1$, on the plane $z = 0$.

$$\iint_S dS = \iint_{S'}((2x)^2+(2y)^2+1)^{1/2}dxdy = \int_0^{2\pi}\int_0^1(\frac{1}{4}+r^2)^{1/2}2rdrd\theta = 2\pi\int_{1/4}^{5/4}u^{1/2}du$$

$$= 2\pi\frac{2}{3}\left[u^{3/2}\right]_{1/4}^{5/4} = \frac{4\pi}{3}\left(\left(\frac{5}{4}\right)^{3/2}-\left(\frac{1}{2}\right)^{3/2}\right) = \frac{\pi}{6}\left(5^{3/2}-1\right)$$

using the substitution $u = \dfrac{1}{4} + r^2$.

- **Ex. 3** Consider the scalar function $f(x,y,z) = z^2$ over the surface of the upper hemisphere $x^2+y^2+z^2=1$, $z \geq 0$.

  We project the surface onto the disk $x^2+y^2=1$ on the plane $z=0$. We use $z=g(x,y)=\sqrt{1-(x^2+y^2)}$ on the hemisphere. We have

$$\sqrt{\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial x}\right)^2+1}=\sqrt{\frac{x^2}{1-(x^2+y^2)}+\frac{y^2}{1-(x^2+y^2)}+1}=\frac{1}{\sqrt{1-(x^2+y^2)}}$$

  So we have

$$\iint_S z^2 dS = \iint_{S'}(1-(x^2+y^2))\frac{1}{\sqrt{1-(x^2+y^2)}}dxdy = \iint_{S'}\sqrt{1-(x^2+y^2)}dxdy$$

  Using polar co-ordinates : $x^2+y^2=r^2$, $dxdy \to rdrd\theta$

$$\iint_S z^2 dS = \int_0^{2\pi}\int_0^1 \sqrt{1-r^2}rdrd\theta = 2\pi\int_1^0 -\frac{1}{2}u^{1/2}du = 2\pi\left[-\frac{1}{3}u^{3/2}\right]_0^1 = \frac{2\pi}{3}$$

  using the substitution $u=1-r^2$.

  Note: Using spherical polar.... we would have

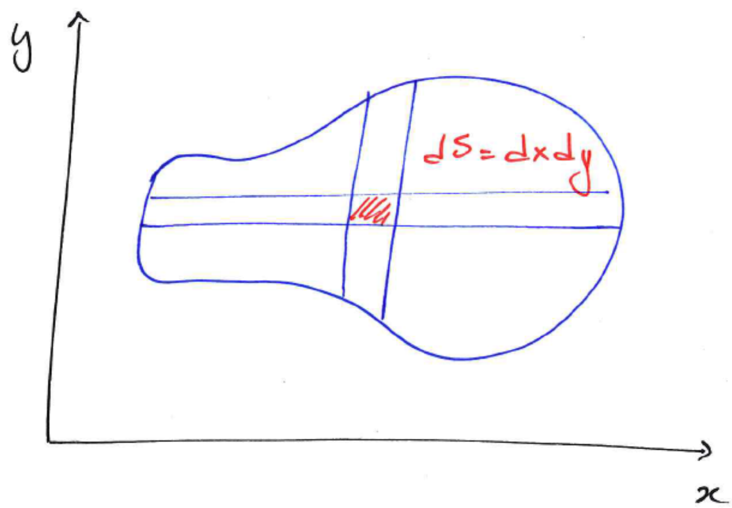
$$\begin{aligned}
x &= r\cos\phi\sin\theta\\
y &= r\sin\phi\sin\theta\\
z &= r\cos\theta\\
dS &= r^2\sin\theta d\theta d\phi \quad \text{where } r = const \text{ at the surface of a sphere}\\
dV &= r^2\sin\theta drd\theta d\phi
\end{aligned}$$

  For the upper hemisphere of radius $r=1$, we have $0<\phi<2\pi$ and $0<\theta<\pi/2$ for $z \geq= 0$ so the integral is

$$\int_0^{2\pi}\int_0^{\pi/2}\cos^2\theta\sin\theta d\theta d\phi = 2\pi\left[-\frac{\cos^3\theta}{3}\right]_0^{\pi/2} = \frac{2\pi}{3}$$

5.2.4 Surface Integrals: common surfaces in classical co-ordinate systems

(i) xy-plane (z=constant)



$$\iint_S f dS = \iint_S f dx dy$$

or

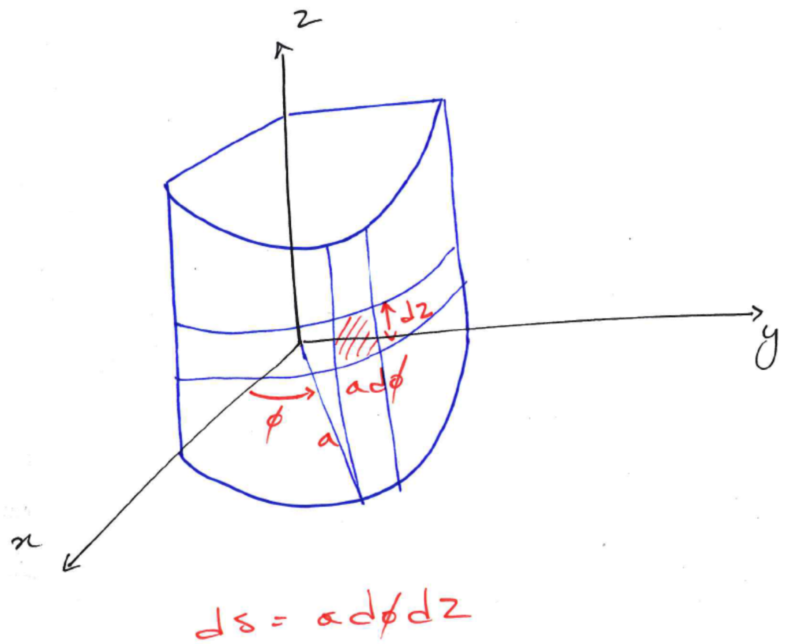
$$\iint_f (r,\theta) dS = \iint f(r,\theta) r dr d\theta$$

Indeed, recall that

$$dS = dxdy = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} dr d\theta$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} dr d\theta = r(\cos^2\theta + \sin^2\theta) dr d\theta.$$

(ii) Side of a cylinder of radius a



$$\iint_S f dS = \iint_S f(R,\phi,z) dS = \int_\phi \int_z f(a,\phi,z) dz a d\phi$$

This can be proved using the method of projection: Consider the side surface for $y > 0$ (generalise to $y < 0$ afterwards) for which we can express y as a simple function of x, y:

$$y = g(x,z) = \sqrt{a^2 - x^2}$$

According to the method of projection

$$dS = \sqrt{\left(\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial z}\right)^2 + 1\right)} dxdz$$

with in this case

$$\frac{\partial g}{\partial x} = \frac{-x}{\sqrt{a^2 - x^2}} \quad \text{and} \quad \frac{\partial g}{\partial z} = 0.$$

So

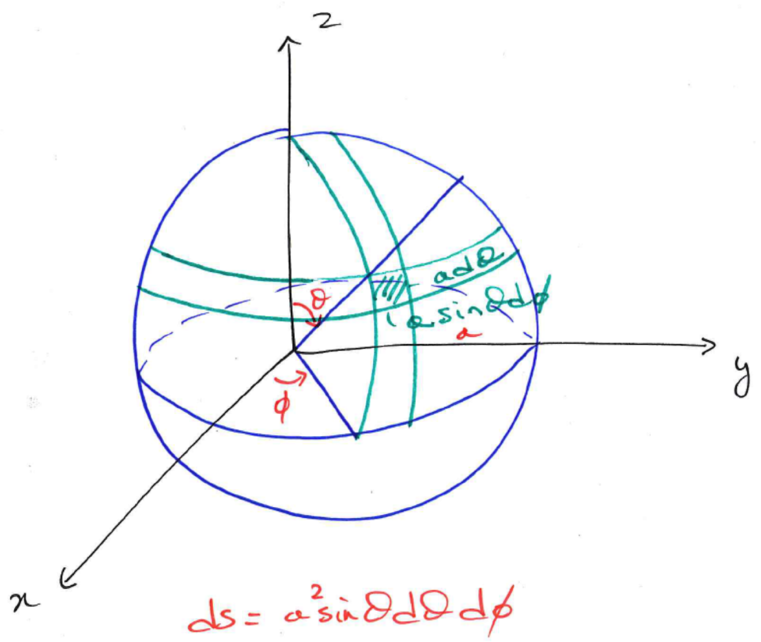
$$dS = \sqrt{\frac{x^2}{a^2 - x^2} + 1 dxdz} = \sqrt{\frac{a^2}{a^2 - x^2}} dxdz$$

where $-a \leq x \leq a$. Make the substitution $a = a\cos\phi$, ϕ is an angle, $|dx| = asin\phi d\phi$. Note: it is OK because the substitution affects only one variable! z is independent of ϕ. We will figure the sign later.

$$dS = \sqrt{\frac{1}{1 - \cos^2\phi}} a \sin\phi d\phi dz = \frac{1}{\sin\phi} a \sin\phi d\phi dz = ad\phi dz.$$

Note: The sign (-) of dx we ignored comes from when $d\phi > 0$, $dx < 0$ for $0 < \phi < \pi$, so we would have to change the limits to come back to the correct sign.

(iii) Surface of a sphere of radius a



$$\iint f ds = \iint f(r, \theta, \phi) dS = \int_\phi \int_\theta f(a, \theta, \phi) a^2 \sin\theta d\theta d\phi$$

5.2.5 Link between volume elements and surface elements

Cylindrical $\rightarrow$ polar

$$dV = RdRd\theta dz$$

$$dS = ad\theta dz \text{ on the side (no variation of } R)$$

$$dS = RdRd\theta \text{ on the top and bottom (no variation of 2)}$$

Spherical

$$dV = r^2 dr \sin\theta d\theta d\phi$$

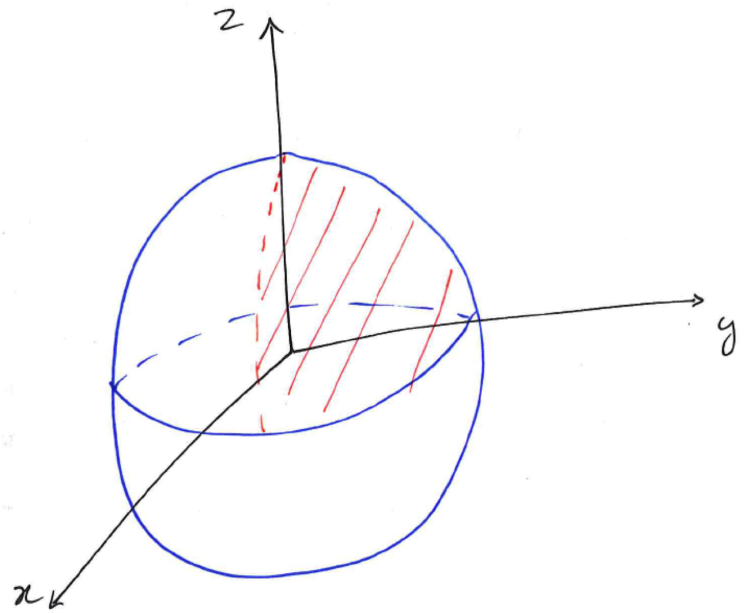
$$dS = a^2 \sin\theta d\theta d\phi$$

Examples

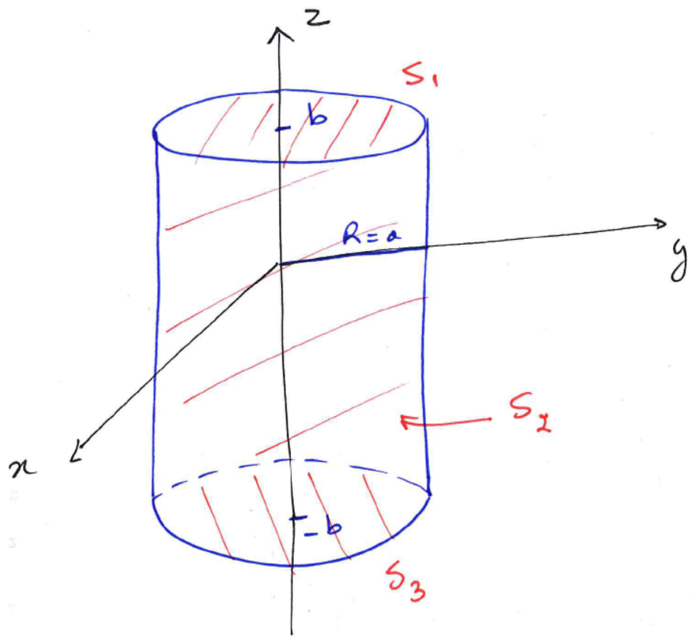
a) Evaluate $\iint_S dS$ where S is the surface area of the sphere for which $x, y, z > 0$ (first octant):

$$S = a^2 \int_0^{\pi/2} \sin\theta d\theta \quad \int_0^{\pi/2} d\phi =$$

$$a^2 \frac{\pi}{2} \left[-\cos\theta\right]_0^{\pi/2} = a^2 \frac{\pi}{2}$$

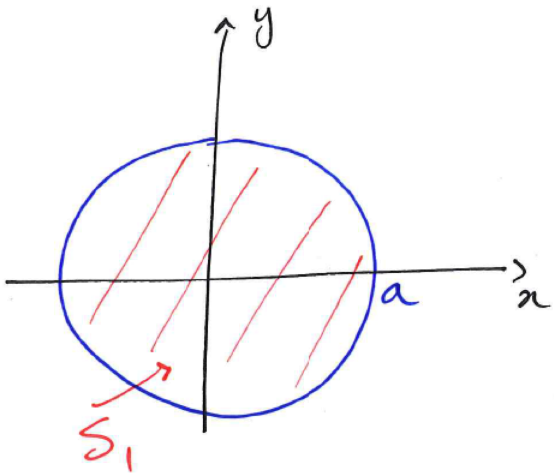


b) Evaluate $\iint_S f ds$ where f is $f = Rz^2$ expressed in cylindrical polar coordinates and S is the entire surface of a cylinder of radius a between $z = -b$ and b



$$\iint_S f dS = \iint_{S_1} f dS + \iint_{S_2} f dS + \iint_{S_3} f ds$$

Top View :



On S_1, $f = Rb^2$ on the surface of a disk (located at $z = +b$) of radius a and centred at $(0, 0, b)$.

Using cylindrical polars (equivalent here to 2D polars)

$$I_1 = \iint_{S_1} f dS = \int_0^a \int_0^{2\pi} Rb^2 R d\phi dR = b^2 \int_0^{2\pi} d\phi \int_0^a R^2 dR = 2\pi b^2 \frac{a^3}{3}$$

I_3 is the same as $z = +b \Rightarrow f = Rb^2$ is the same and the surface is "identical" (disk of radius a)

$$I_3 = \frac{2}{3}\pi b^2 a^3$$

Finally I_2: Here $R = a$ does not vary

$$dS = ad\phi dz$$

$$f = az^2 \qquad -b \leq z \leq b \text{ and } 0 \leq \phi \leq 2\pi$$

$$I_2 = \int_0^{2\pi} \int_{-b}^{b} az^2 adzd\phi = 2\pi a^2 \int_{-b}^{b} z^2 dz = 2\pi a^2 \left[\frac{z^3}{3}\right]_{-b}^{b} = \frac{4\pi a^2 b^3}{3}$$

$$I = I_1 + I_2 + I_3 = \frac{2\pi b^2 a^3}{3} + \frac{4\pi b^3 a^2}{3} + \frac{2\pi b^2 a^3}{3} = \frac{4\pi}{3} b^2 a^2 (b + a)$$

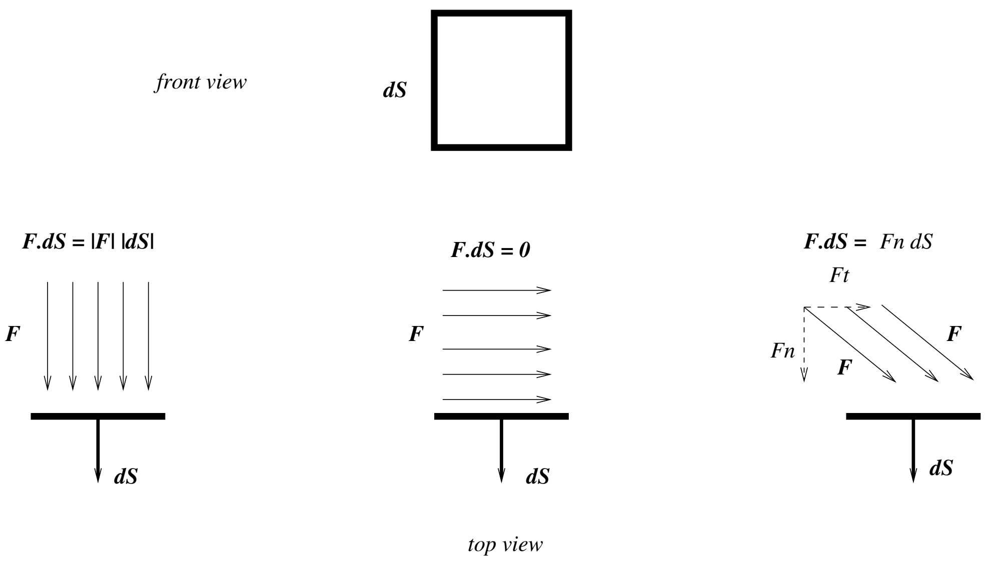
5.3 Vector form of a surface integral

5.3.1 Why? - Notion of Flux

The flux of the vector quantity $\mathbf{F}$ across the surface dS is the 'amount' of $\mathbf{F}$ *crossing* the elementary surface dS. Only the component F_n of $\mathbf{F}$ normal to dS will cross it, the tangential part F_t just slides/goes along it. In over words, if we consider the unit vector $\mathbf{n}$ normal to the surface dS, the local flux is

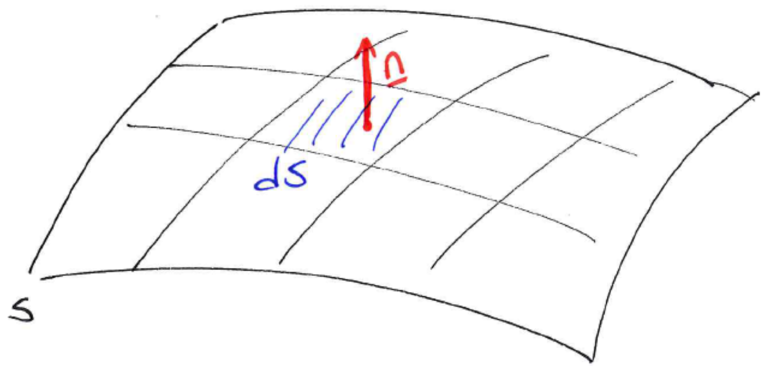
$$\mathbf{F} \cdot \mathbf{n} dS = \mathbf{F} \cdot \mathbf{dS}$$

by defining $\mathbf{dS} = \mathbf{n} dS$. See figure below for illustraion.



5.3.2 Definition

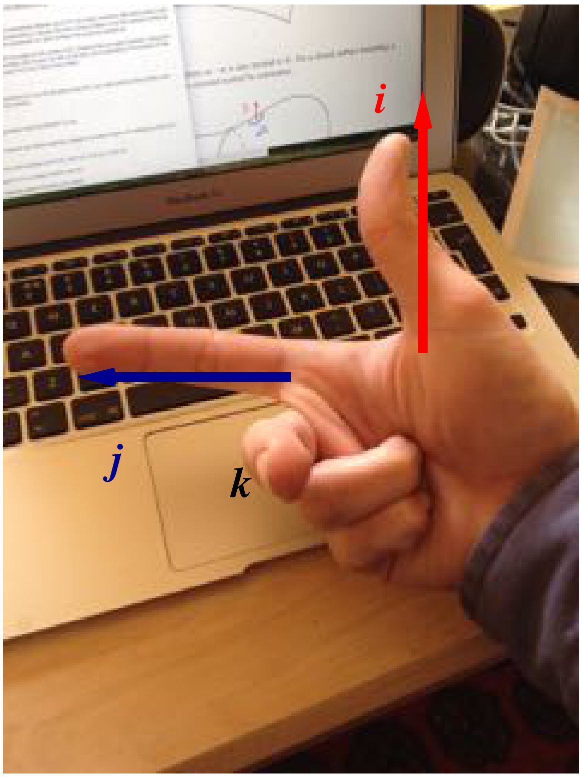
Consider a surface S with a unit normal vector $\mathbf{n}$



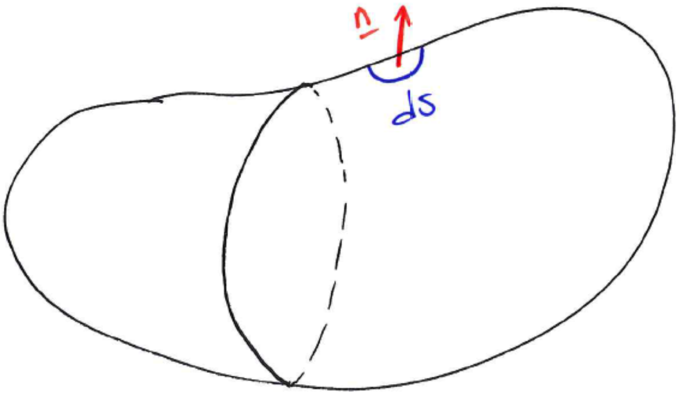
Here the direction $\mathbf{n}$ is somewhat arbitrary as $-\mathbf{n}$ is also normal to S and is also a unit vector. For a closed surface bounding a volume V the normal is taken to be the outward normal by convention. This follows the convention for open surface that could be worded as

Consider a surface S bounded by the contour C. Imagine that you walk (forward!) along C such that you keep the surface S to your **left** *at all time, you're standing in the direction of the normal, i.e. the normal goes from your feet to your head.*

... alternatively



The right-hand convention: If your thumb goes along C, direction $\mathbf{i}$ in the figure, you index points at the surface, direction $\mathbf{j}$ in the figure, your middle finger gives the normal., denoted $\mathbf{k}$ in the figure



We have seen that we can define a surface integral of a scalar function f (defined on S) by $\iint_S f dS$.

Now given the vector function $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$.

$\mathbf{F} \cdot \mathbf{n}$ is a scalar function, say f.

We can therefore define

$$\mathbf{ds} = \mathbf{n} dS$$

$$\iint_S \mathbf{F} \cdot \mathbf{dS} = \iint_S (\mathbf{F} \cdot \mathbf{n}) dS = \iint_S f ds$$

Note that $\mathbf{ds}$ only depends on the surface S.

Examples

(i) Evaluate

$$I = \iint_S \mathbf{F} \cdot \mathbf{dS}$$

for

$$\mathbf{F} = \mathbf{r}$$

and S is the surface of the upper hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$.

Here $dS = \sin\theta d\theta d\phi \mathbf{e}_r$ as $r = 1$ here. On the upper hemisphere of radius 1, $0 < \phi < 2\pi$ and $0 < \theta < \pi/2$. On the surface of the hemisphere $\mathbf{F} = \mathbf{e}_r$ so the integral is

$$I = \int_0^{2\pi} \int_0^{\pi/2} \mathbf{e}_r \cdot \mathbf{e}_r \sin\theta d\theta d\phi = 2\pi.$$

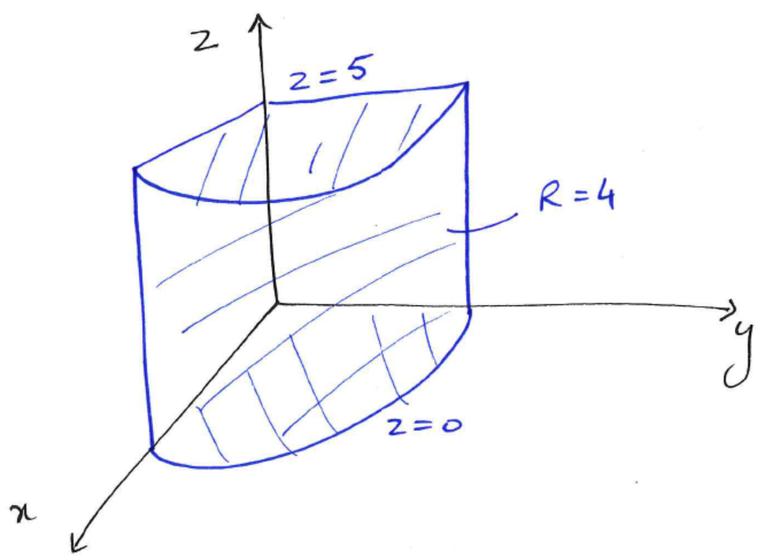
(ii) Evaluate

$$I = \iint_S \mathbf{F} \cdot \mathbf{dS}$$

where

$$\mathbf{F} = y^2 \mathbf{j}$$

And S is the portion of the surface of a cylinder $x^2 + y^2 = 16$ between $z = 0$ and $z = 5$ in the first octant $(x, y, z \geq 0)$.

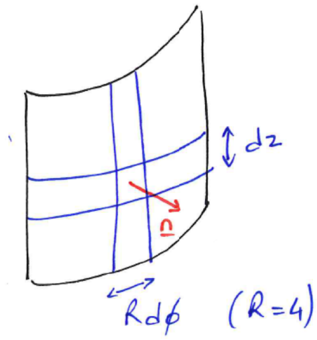


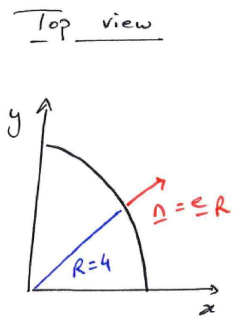
We will take the normal vectors pointing <u>outward</u>.

We have 5 sides in this surface:

Let's define them

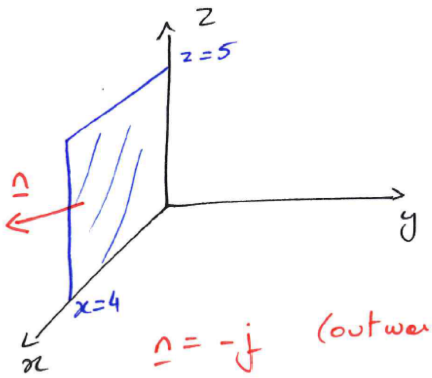
S_1:





$$d\underline{S} = 4\, dz\, d\phi\, \underline{e}_R$$

S_2:



$y = 0$

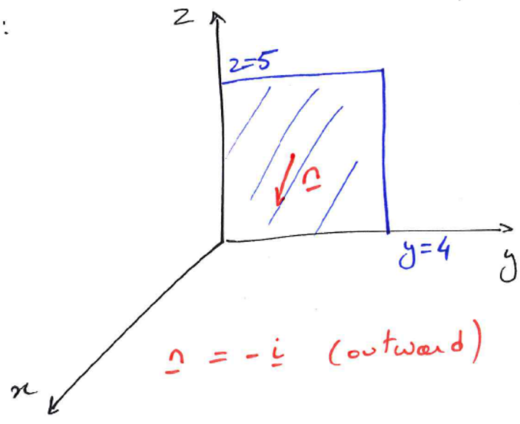
$0 \leq x \leq 4$

$0 \leq z \leq 5$

$\underline{n} = -\underline{j}$ (outward)

$$d\underline{S} = -\, dx\, dz\, \underline{j}$$

S_3:



$x = 0$

$0 \leq y \leq 4$

$0 \leq z \leq 5$

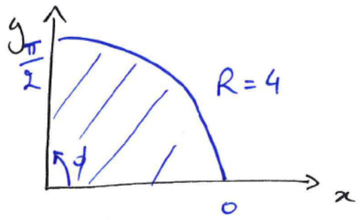
$\underline{n} = -\underline{i}$ (outward)

$$d\underline{S} = -dy\, dz\, \underline{i}$$

S_4 : The top surface

$0 \leq \phi \leq \frac{\pi}{2}$

$0 \leq R \leq 4$



$\underline{n} = \underline{k}$

$\underline{ds} = R\,dR\,d\phi\,\underline{k}$

S_5 : The bottom surface

$z = 0$

$0 \leq R \leq 4$

$0 \leq \phi \leq \frac{\pi}{2}$

$\underline{n} = -\underline{k}$ (downward)

$\underline{ds} = -R\,dR\,d\phi\,\underline{k}$

Let's integrate now

$$I_1 = \iint_{S_1} \mathbf{F} \cdot \mathbf{ds} = \int_0^{\pi/2} \int_0^5 /y^2\mathbf{j}) \cdot (4dzd\phi)\mathbf{e}_R$$

$$y^2 = (R\sin\theta)^2 \text{ with } R = 4$$

$$I_1 = \int_0^{\pi/2} \int_0^5 4^3 \sin^2\phi \mathbf{j} \cdot \mathbf{e}_R dzd\phi$$

$$\mathbf{e}_R = \cos\phi\mathbf{i} + \sin\phi\mathbf{j}$$

$$\Rightarrow \mathbf{j} \cdot \mathbf{e}_R = \sin\phi$$

$$I_1 = \int_0^{\pi/2} \int_0^5 64\sin^3\phi dzd\phi$$

$$= 320 \int_0^{\pi/2} \sin^3\phi\phi\phi$$

Use

$$\sin^3\phi = \sin\phi\sin^2\phi$$

$$= \sin\phi(1-\cos^2\phi)$$

and substitute $u = \cos\phi,\ du = -\sin\phi d\phi$

$$I_1 = 320\int_0^{\pi/2}\sin^3\phi d\phi = 320\int_1^0 (1-u^2)(-du)$$

$$I_1 = 320\int_0^1 (1-u^2)du = 320\left[u - \frac{u^3}{3}\right]_0^1 = \frac{640}{3}$$

$$I_2 = y = 0 \text{ on } S_2 \Rightarrow \mathbf{F} = 0$$

$$\Rightarrow \iint_{S_2}\mathbf{F}\cdot\mathbf{ds} = 0$$

$$I_3 = \iint_{S_3}\mathbf{S}\cdot\mathbf{ds} = \int_0^4\int_0^5 y^2\mathbf{j} - (-\mathbf{i}d2dy) = 0 \text{ as } \mathbf{i}\cdot\mathbf{j} = 0$$

$I_4 = I_5 = 0$ as well as $\mathbf{F}\cdot\mathbf{n} = 0 \quad (\mathbf{n} = \pm\mathbf{k})$

$$\Rightarrow I = I_1 = \frac{640}{3}$$

5.3.3 General case: parametric form

In the general case, a surface may be described by two independent (curvilinear) co-ordinates, say u, v

$$S = \{\mathbf{r} = (x(u,v), y(u,v), z(u,v)), (u,v) \in D\}$$

where D is the domain of (u,v) (e.g. $\mathbb{R}^2$).

$$dS = \left|\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right| d\,udv$$

You can recover again all most common dS's this way!

Note that for a planar surface $\mathbf{r}(u,v) = (x(u,v), y(u,v), 0)$, we recover the Jacobian!