Introduction to Thompson's Groups F, T and V

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Abstract

We provide an introduction to Richard Thompson's groups F, T and V, heavily based on survey by Canon, Floyd and Parry. We provide additional detail in certain arguments, and replace others with our own proofs, or proofs from current literature.

1 Introduction

1.1 Declaration

I certify that this project report has been written by me, is a record of work carried out by me, and is essentially different from work undertaken for any other purpose or assessment. [UoS]

1.2 Introduction

Thompson's groups have generated interest since Richard Thompson discovered them in 1965. The groups T and V were the first examples of groups that were infinite, finitely presented, and simple. Thompson's group F has also generated significant interest, and currently a major open problem is whether or not F is amenable. In [Deh05] it is proved that F is the geometry group of associativity, and V is the geometry group of associativity and commutativity, which highlights the fundamental nature of these groups.

A widely used introduction to F, T and V is [CFP96]. We have rewritten many of the results in [CFP96], in order to use solely right actions, rather than a mixture of left and right actions, and to provide more detail in certain proofs. We have also replaced certain proofs with proofs from current literature, and some of our own. For example, our section on Thompson's group V is based on the use of small swaps, defined in [BQ17]. We include the proof from [BQ17] that V has an infinite presentation defined in that paper, but we replace the injectivity argument with our own, more direct approach. Our proof of the simplicity of V is that in [Ble17], which relies on the action of V, and the powerful generating set of small swaps. The use of small swaps allows us to exploit the highly transitive action of V on the Cantor space, and gives rise to a very natural proof of simplicity.

In addition, in [CFP96], a loose argument was provided for the tree representation of F being isomorphic to its tree representation, and the analogous proofs for T and V were left to the reader. We have included our own full proofs of these results.

Finally, we prove some basic metric properties of F, such as that F is a topological group, using the topology induced by d_1, and that it is dense in the group of homeomorphisms of $[0,\ 1]$.

1.3 Main Results

Our aim has been to provide an introduction to Thompson's groups F, T and V at an advanced undergraduate level. We do, however, require a basic understanding of group theory, real analysis, metric and topological spaces, graph theory and complex analysis.

In Section 2 we define Thompson's group F, and prove that it is a group. We include the definition of a standard dyadic partition, and prove some basic lemmas to be used in later sections. Finally, a proof that F is torsion-free is included. This section is largely based on [CFP96], although the majority of the proofs were not included in [CFP96].

In Section 3 we define an ordered rooted binary tree, a $\mathcal{T}$-tree, and set up the bijection between standard dyadic partitions and $\mathcal{T}$-trees. Again, this section is largely based on [CFP96].

In Section 4 we define $\mathcal{T}$-tree diagrams and $\mathcal{T}$-tree diagram classes. We set up a multiplication on $\mathcal{T}$-tree diagram classes. The proofs in the sections are mostly about well-definedness, and were not included in [CFP96].

In Section 5 we prove that a subset of the set of $\mathcal{T}$-tree diagram classes forms a group under the defined multiplication, and this group is isomorphic to F. This section includes a number of well-definedness proofs that were omitted from [CFP96]. However, the proof that there is a unique reduced $\mathcal{T}$-tree diagram (Lemma 5.2) comes from [CFP96].

Section 6 includes the proof that F can be represented by a finite and an infinite presentation. We also construct a normal form for elements of F. The proofs in this section are based on [CFP96].

In Section 7 we classify all elements of the commutator subgroup of F. We prove that the abelianisation of F is $\mathbb{Z} \times \mathbb{Z}$, that every proper quotient of F is abelian, and that the commutator subgroup of F is simple. The proofs in this section are based on [CFP96].

Section 8 proves that F is a topological group, under the topology induced by d_1. We also prove that F is dense in the group of increasing homeomorphisms of $[0,\ 1]$. This section is entirely original, although the definitions of topological groups come from [AT08].

Section 9 defines Thompson's group T and proves that it is a group, along with some basic lemmas. Although the definition is based on [CFP96], the majority of the proofs in this section are original, as they were omitted from [CFP96].

In Section 10 we prove that another subset of the set of $\mathcal{T}$-tree diagram classes is a group, and this subset is isomorphic to T. Although this result was stated in [CFP96], it was not proved, so this section is largely original.

Section 11 includes the proof that T is simple, and finitely presented. This section is based on [CFP96].

In Section 12 we define the Cantor space, and define Thompson's group V to be the set of prefix exchange maps. We prove that V and the set of $\mathcal{T}$-tree diagram classes are isomorphic groups. The definitions in this section are based on [BQ17] and [Ble17]. The proofs, however, are original.

Section 13 contains the definition of small swaps, the proof that they generate V, which comes from [Ble17], and the proof that V has an infinite presentation utilising them. The presentation is taken from [BQ17]. The proof, however, that the presentation is isomorphic to V is original, using a more direct injectivity argument than that which was included in [BQ17]. We also use the presentation for the finite symmetric groups, from [HB13].

Section 14 includes the proof that V is simple, using the small swaps. This is taken from [Ble17].

1.4 Notation

Throughout this paper the natural numbers will be defined to be the set of strictly positive integers. The symbol $\mathbb{N}_0$ will denote the non-negative integers. Group elements will act from the right. For such functions, we will often write xf, rather than $(x)f$. The empty word will be denoted by ε. If X is a set of formal symbols, we will define $W(X) = (X \cup \{x^{-1} \mid x \in X\})^*$, that is the set of words over $X \cup \{x^{-1} \mid x \in X\}$, where x^{-1} is defined to be a unique symbol, not in X, for each $x \in X$. A *partition* of an interval $[a,\ b] \subseteq \mathbb{R}$, where $a,\ b \in \mathbb{R},\ a < b$, will refer to a subset of $[a,\ b]$, which has a and b as elements. We will usually express partitions in order, and all partitions appearing will be finite.

2 Thompson's Group F

Definition 2.1 A *dyadic rational number* is a rational number that can be written as $\frac{a}{b}$, for $a \in \mathbb{Z},\ b \in \mathbb{N}$, with a and b coprime, and $b = 2^n$ for some $n \in \mathbb{N}_0$.

Let X be the set of piecewise linear homeomorphisms from $[0,\ 1]$ to itself, which are differentiable everywhere except at finitely many dyadic rational numbers, and such that where the functions are differentiable, their derivatives are a power of 2. Define *Thompson's group F*, usually denoted F, to be the tuple $(X,\ \circ)$, where $\circ$ represents composition of functions.

Lemma 2.2 *Let $f\colon [0,\ 1] \to [0,\ 1]$. Then $f \in F$ if and only if there are sets*

$$\{0 = x_0 < x_1 < \cdots < x_n = 1\},\ \{a_1,\ a_2, \cdots a_n\},\ \{b_1,\ b_2, \cdots b_n\},$$

for some $n \in \mathbb{N}$, such that for any $i \in \mathbb{N},\ i \leq n$, a_i is a power of 2, b_i and x_i are dyadic rational numbers, and

$$(0)f = 0, \quad (1)f = 1, \quad xf = a_i x + b_i$$

for all $x \in [x_{i-1},\ x_i]$.

Proof ($\Rightarrow$): Let $f \in F$. Let $0 = x_0 < x_1 < \cdots < x_n = 1$, for $n \in \mathbb{N}$ and $x_i \in \mathbb{R}$ for valid indices i, be the points where f is not differentiable. Note that these are dyadic rational numbers.

We will proceed by induction on $i \in \mathbb{N}$, to prove that $xf = a_i x + b_i$, where $x_{i-1} \leq x \leq x_i$, a_i is a power of 2 and b_i is a dyadic rational number. We will call this statement $\mathcal{P}(i)$. We have that $(0)f = 0$, and hence $xf = a_1 x + b_1$, where $x_0 \leq x \leq x_1$, a_1 is a power of 2 and b_1 is a dyadic rational number.

Let $i \in \mathbb{N}$, with $i \leq n$. Inductively suppose $\mathcal{P}(i)$ is true. Let $x \in [x_i,\ x_{i+1}]$. Since x_i is a dyadic rational number, and f is linear on $[x_i,\ x_{i+1}]$, whose derivative is a power of 2, we have that $xf = a_{i+1}x + b_{i+1}$, where a_{i+1} is a power of 2, and $b_{i+1} \in \mathbb{R}$. By induction, we have that $x_i f = a_i x_i + b_i$, where a_i is a power of 2, and b_i is a dyadic rational number.

Note that $x_{i+1}f = a_i x_{i+1} + b_i$. As a product of a power of 2 and a dyadic rational, $a_i x_{i+1}$ is a dyadic rational, and hence $x_{i+1}f$ is a dyadic rational, since it is a sum of dyadic rationals. We also have that $x_{i+1}f = a_{i+1}x_{i+1} + b_{i+1}$. This implies that $b_{i+1} = x_{i+1}f - a_{i+1}x_{i+1}$. We have already calculated that the $x_{i+1}f$ is a dyadic rational, and since $a_{i+1}x_{i+1}$ is also a dyadic rational, as a product of a dyadic rational and a power of 2, we can conclude that b_{i+1} is a dyadic rational, as required.

($\Leftarrow$): Let $f\colon [0,\ 1] \to [0,\ 1]$ and $n \in \mathbb{N}$. Let $X = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$. For any $i \in \mathbb{N}$ such that $i \leq n$, suppose there exists a dyadic rational number b_i, and a power of 2, a_i, such that

$$(0)f = 0, \quad (1)f = 1, \quad xf = a_i x + b_i, \quad a_{i-1}x_{i-1} + b_{i-1} = a_i x_{i-1} + b_i,$$

for all $x \in [x_{i-1},\ x_i]$.

Note that $a_{i-1}x_{i-1} + b_{i-1} = a_i x_{i-1} + b_i$ for all valid i, since f is well-defined on the overlapping points of the closed intervals $[x_{i-1},\ x_i]$. We have that f is piecewise linear and strictly increasing, since $a_i > 0$ for all valid i. Since linear pieces always meet at the same points, and f maps onto 0 and 1, we have that f is continuous, and hence a bijection. Let $i \in \mathbb{N}$ such that $i \leq n$. We have, for any $x \in [x_{i-1}f,\ x_i f]$ that

$$xf^{-1} = \frac{1}{a_i}x - \frac{b_i}{a_i}, \quad a_i x_i + b_i = a_{i+1}x_i + b_{i+1}$$

and hence

$$\begin{aligned}
\frac{1}{a_{i-1}}(x_{i-1}f) - \frac{b_{i-1}}{a_{i-1}} &= \frac{a_{i-1}x_{i-1} + b_{i-1}}{a_{i-1}} - \frac{b_{i-1}}{a_{i-1}} \\
&= \frac{a_{i-1}x_{i-1}}{a_{i-1}} \\
&= x_{i-1} \\
&= \frac{a_i x_{i-1}}{a_i} \\
&= \frac{a_i x_{i-1} + b_i}{a_i} - \frac{b_i}{a_i} \\
&= \frac{1}{a_i}(x_{i-1}f) - \frac{b_i}{a_i},
\end{aligned}$$

and it follows that the pieces of f^{-1} meet at the same points, and f^{-1} is continuous. We can conclude that f is a homeomorphism of $[0,\ 1]$.

Let $i \in \mathbb{N}$ such that $i \leq n$. We have $xf' = a_i$, for any $x \in [x_{i-1},\ x_i]$. So, except for finitely many dyadic rationals which comprise the set X, f is differentiable, and the derivative of f is a power of 2 at these points. We have that the set of points where f is not differentiable is a subset of X, and hence is a finite set of dyadic rational numbers. Hence f is differentiable everywhere, except finitely many dyadic rational numbers, and when differentiable, the derivative is a power of 2. Together with the fact that f is a homeomorphism of $[0,\ 1]$, it follows that $f \in F$. $\square$

Example 2.3 Let

$$A\colon [0,\ 1] \to [0,\ 1] \qquad x \mapsto \begin{cases} \frac{x}{2}, & 0 \le x \le \frac{1}{2} \\ x - \frac{1}{4}, & \frac{1}{2} \le x \le \frac{3}{4} \\ 2x - 1, & \frac{3}{4} \le x \le 1 \end{cases}$$

$$B\colon [0,\ 1] \to [0,\ 1] \qquad x \mapsto \begin{cases} x, & 0 \le x \le \frac{1}{2} \\ \frac{x}{2} + \frac{1}{4}, & \frac{1}{2} \le x \le \frac{3}{4} \\ x - \frac{1}{8}, & \frac{3}{4} \le x \le \frac{7}{8} \\ 2x - 1, & \frac{7}{8} \le x \le 1. \end{cases}$$

By Lemma 2.2, these are elements of F.

Theorem 2.4 *The tuple F forms a group.*

Proof We have that F is a subset of the group of homeomorphisms of $[0,\ 1]$, which is a group, so it suffices to show that F is closed under multiplication and inversion. Let $f,\ g \in F$. Let $\{0 = x_0 < x_1 < \cdots < x_n = 1\}$ and $\{0 = y_0 < y_1 < \cdots < y_m = 1\}$ be the points of indifferentiability of f and g respectively, where $m,\ n \in \mathbb{N}$. Let $a_i,\ b_i$ be defined for all $i \in \mathbb{N}$, where $i \le n$, by

$$xf = a_i x + b_i,$$

for all $x \in [x_i,\ x_{i+1}]$, where a_i is a power of 2 and b_i is a dyadic rational. Define $c_i,\ d_i$ for $i \in \mathbb{N},\ i \le m$ analogously for g. We can do this by Lemma 2.2. It follows that, given $i \in \mathbb{N}$, where $i \le n$,

$$xf = a_i x + b_i \implies x = \frac{xf}{a_i} - \frac{b_i}{a_i},$$

for any $x \in [x_i,\ x_{i+1}]$. Therefore,

$$xf^{-1} = \frac{1}{a_i}x - \frac{b_i}{a_i},$$

for all $x \in [x_i f,\ x_{i+1} f]$. Note that $\frac{1}{a_i}$ is a power of 2, and $\frac{b_i}{a_i}$ is a dyadic rational number, and hence $f^{-1} \in F$, by Lemma 2.2.

Let

$$X = \{x_0,\ x_1, \ldots,\ x_n\} \cup \{y_0 f^{-1},\ y_1 f^{-1},\ \ldots,\ y_m f^{-1}\}.$$

Inductively define z_i for $i \in \{1,\ \ldots,\ |X|\}$, by setting $z_1 = \min X$, and

$$z_{i+1} = \min(X \backslash \{z_1,\ z_2,\ \ldots,\ z_i\}).$$

Let $i \in \mathbb{N}$ and $x \in [z_{i-1},\ z_i]$. Then

$$xfg = (a_i x + b_i)g = (a_i x + b_i)c_i + d_i = a_i c_i x + (b_i c_i + d_i).$$

Since a_i and c_i are powers of 2, we have that $a_i c_i$ is a power of 2. Since b_i is a dyadic rational, and c_i is a power of 2, we have that $b_i c_i$ is a dyadic rational, and hence $b_i c_i + d_i$ is a dyadic rational. We conclude $fg \in F$, by Lemma 2.2. □

Lemma 2.5 *Elements of F map the set of dyadic rational numbers in $[0,\ 1]$ bijectively to itself.*

Proof Let $f \in F$ be differentiable everywhere except at $0 = x_0 < x_1 < \cdots < x_n = 1$, where $n \in \mathbb{N}$. Given $i \in \mathbb{N}$, define a_i and b_i as in Lemma 2.2. Let $q \in [0,\ 1]$ be a dyadic rational number. Then $q = \frac{a}{b}$, for some $a,\ b,\ k \in \mathbb{N}$, such that $b = 2^k$. Let $i \in \mathbb{N}$ such that $q \in [x_{i-1},\ x_i]$. Then $qf = a_i q + b_i$, which is also a dyadic rational number. Hence f maps dyadic rational numbers to dyadic rational numbers. As a homeomorphism, f is injective. Let $q \in [0,\ 1]$ be a dyadic rational number. Choose $i \in \mathbb{N}$, with $i \leq n$, such that $q \in [x_i f,\ x_{i+1} f]$. Then $\frac{q-b_i}{a_i} \in [0,\ 1]$, since

$$\left(\frac{q-b_i}{a_i}\right) f = \left(\frac{q-b_i}{a_i}\right) a_i + b_i = q \implies qf^{-1} = \frac{q-b_i}{a_i}.$$

We can therefore conclude that f maps the dyadic rationals onto the dyadic rationals. Since f is injective on $[0,\ 1]$, it is injective on a subset, such as the dyadic rationals. □

Definition 2.6 An interval I of $[0,\ 1]$ is a *standard dyadic interval* if there exist $a,\ n \in \mathbb{N}_0$ such that $a \leq 2^n - 1$ and

$$I = \left[\frac{a}{2^n},\ \frac{a+1}{2^n}\right].$$

Note that a and 2^n do not need to be coprime, and therefore this representation is not unique. If $J = [a,\ b]$ for some $a,\ b \in [0,\ 1],\ a < b$, then the *left half* and *right half* of J are defined as

$$\left[a,\ \frac{a+b}{2}\right], \quad \left[\frac{a+b}{2},\ b\right],$$

respectively. A *standard dyadic partition* of $[0,\ 1]$ is a partition $\{0 = x_0 < x_1 < \cdots < x_n = 1\}$, for some $n \in \mathbb{N}$, such that $[x_{i-1},\ x_i]$ is a standard dyadic interval for all $i \in \mathbb{N},\ i \leq n$.

Lemma 2.7 *If P is a partition of $[0,\ 1]$, such that all points in P are dyadic rationals, then there is a standard dyadic partition of $[0,\ 1]$ containing all points in P.*

Proof Set

$$P = \{0 = x_0 < x_1 < \cdots < x_n = 1\},$$

where $n \in \mathbb{N}$. For each valid index i, define $u_i \in \mathbb{N}_0$ and $v_i \in \mathbb{N}$ to be the numerator and denominator of x_i in its lowest terms respectively. Let $w_i = \log_2 v_i$. Note $w_i \in \mathbb{N}_0$, as v_i is a power of 2. Given $i \in \mathbb{N},\ i \leq n$, let

$$Q_i = \left\{x_{i-1} = \frac{u_{i-1}}{2^{w_{i-1}}} < \frac{u_{i-1}+1}{2^{w_{i-1}}} < \frac{u_{i-1}+2}{2^{w_{i-1}}} < \cdots < \frac{u_i}{2^{w_i}} = x_i\right\}.$$

Note that Q_i is a standard dyadic partition of $[x_{i-1}, x_i]$. Denote the jth element of Q_i by $y_{i,j}$, for any valid index j, starting with x_{i-1} as the zeroth index. Let $m_i = |Q_i| - 1$. Then the partition

$$\{0 = y_{0,1} < y_{0,2} < \cdots y_{0,m_0} < y_{1,0} < y_{1,1} \cdots < y_{n,m_n-1} < y_{n,m_n} = 1\},$$

is a standard dyadic partition of $[0,\ 1]$. □

Lemma 2.8 *For each $f \in F$ there exists a standard dyadic partition of $[0,\ 1]$, say*

$$\{0 = x_0 < x_1 < \cdots < x_n = 1\},$$

where $n \in \mathbb{N}$, such that for each $i \in \mathbb{N},\ i \leq n,\ f$ is linear on $[x_{i-1},\ x_i]$. In addition, for any such partition,

$$\{0 = x_0 f < x_1 f < \cdots < x_n f = 1\},$$

is a standard dyadic partition.

Proof Let $f \in F$. By Lemma 2.2, there is a partition P of $[0,\ 1]$, such that f is linear on every interval of P, and the elements of P are dyadic rational numbers. By Lemma 2.7, there is a standard dyadic partition

$$Q = \{0 = x_0 < x_1 < \cdots < x_n = 1\},$$

of $[0,\ 1]$, where $n \in \mathbb{N}$, such that Q contains all points in P. Define $a_i,\ b_i$ as stated with respect to f and x_i, as in Lemma 2.2, for all $i \in \mathbb{N},\ i \leq n$.

Let $i \in \mathbb{N},\ i \leq n$. We have

$$xf = a_i x + b_i,$$

for all $x \in [x_{i-1}, x_i]$, by Lemma 2.2. Note $P \subseteq Q$ and f is linear on every interval of P, and hence f is linear on every interval of Q. Since the derivative of f, where differentiable, is always a power of 2, f is differentiable everywhere in $[0,\ 1]$ except finitely many points, we have that f is increasing. Since f is an increasing continuous bijection, we have that $(0)f = 0$ and $(1)f = 1$. Therefore,

$$Qf = \{0 = x_0 f < x_1 f < \cdots < x_n f = 1\},$$

Let I be the ith interval of Q. Then there exist $a,\ m \in \mathbb{N}_0$ such that $a \leq 2^m - 1$ and

$$I = \left[\frac{a}{2^m},\ \frac{a+1}{2^m}\right].$$

There exists a $k \in \mathbb{Z}$, such that $a_i = 2^k$, since a_i is a power of 2. Then

$$\begin{aligned} If &= \left[\frac{aa_i + 2^m b_i}{2^m},\ \frac{a(a_i+1) + 2^m b_i}{2^m}\right] \\ &= \left[\frac{2^k a + 2^m b_i}{2^m},\ \frac{2^k a + 2^k + 2^m b_i}{2^m}\right] \\ &= \left[\frac{a + 2^{m-k} b_i}{2^{m-k}},\ \frac{a + 2^{m-k} b_i + 1}{2^{m-k}}\right] \end{aligned}$$

Suppose $k > m$. Then

$$\frac{a + 2^{m-k} b_i + 1}{2^{m-k}} > \frac{1}{2^{m-k}} > 1,$$

a contradiction. It follows that $k \leq m$, and If is a standard dyadic interval. Therefore, Qf is a standard dyadic partition. $\square$

Lemma 2.9 *Given two standard dyadic partitions P and Q of $[0,\ 1]$ such that $|P| = |Q|$, there exists a unique element f of F such that f is linear on the intervals of P, and $Pf = Q$.*

Proof Existence: Let $i \in \mathbb{N}_0$ such that $i \leq |P| - 1$. Let

$$I_i = \left[\frac{c_i}{2^{n_i}},\ \frac{c_i + 1}{2^{n_i}}\right], \quad J_i = \left[\frac{d_i}{2^{k_i}},\ \frac{d_i+1}{2^{k_i}}\right],$$

where $c_i,\ d_i \in \mathbb{N}_0$, and $n_i,\ k_i \in \mathbb{Z}$, such that I_i is the ith interval of P and J_i is the ith interval of Q. Let

$$\begin{aligned} f_i \colon I_i &\to J_i \\ x &\mapsto a_i x + b_i, \end{aligned}$$

where $a_i = 2^{n_i - k_i},\ b_i = 2^{-k}(d_i - c_i)$. Note a_i is a power of 2, and b_i is a dyadic rational number. In addition

$$
\begin{aligned}
I_i f_i &= \left[2^{n_i - k_i} \cdot \frac{c_i}{2^{n_i}} + 2^{-k}(d_i - c_i),\ 2^{n_i - k_i} \cdot \frac{c_i + 1}{2^{n_i}} + 2^{-k_i}(d_i - c_i)\right] \\
&= [2^{-k_i} c_i + 2^{-k_i}(d_i - c_i),\ 2^{-k_i}(c_i + 1) + 2^{-k_i}(d_i - c_i)] \\
&= \left[\frac{d_i}{2^{k_i}},\ \frac{d_i + 1}{2^{k_i}}\right] \\
&= J_i.
\end{aligned}
$$

Since the I_is are disjoint except at the endpoints, and on the endpoints of I_i and I_{i+1}, we have that

$$
\frac{c_i + 1}{2^{n_i}} f_i = \frac{d_i + 1}{2^{k_i}} = \frac{d_{i+1}}{2^{k_{i+1}}} = \frac{c_{i+1}}{2^{n_{i+1}}} f_{i+1},
$$

and hence we can define $f\colon [0,\ 1] \to [0,\ 1]$ as follows. Given $x \in [0,\ 1]$, find an $i \in \mathbb{N}_0$, such that $x \in I_i$, then map x to $x f_i$. This function satisfies the requirements of Lemma 2.2, and hence $f \in F$.

Uniqueness: Suppose $g \in F$ such that $Pg = Q$. Let $i \in \mathbb{N}_0$. By Lemma 2.2, there exists a power of 2 α_i, and a dyadic rational number β_i, such that for $x \in I_i$, we have $x_i g = \alpha_i x + \beta_i$. Since $I_i g = J_i$, we have

$$
\begin{aligned}
\alpha_i \cdot \frac{c_i}{2^{n_i}} + \beta_i &= \frac{d_i}{2^{k_i}} \\
\alpha_i \cdot \frac{c_i + 1}{2^{n_i}} + \beta_i &= \frac{d_i + 1}{2^{k_i}}
\end{aligned}
$$

Hence

$$
2^{k_i} \alpha_i (c_i + 1) + 2^{n_i} d_i - 2^{k_i} \alpha_i c_i = 2^{n_i}(d_i + 1) \implies 2^{k_i} \alpha_i = 2^{n_i} \implies \alpha_i = 2^{n_i - k_i}.
$$

In addition

$$
2^{n_i - k_i} \cdot \frac{c_i}{2^{n_i}} + \beta_i = \frac{d_i}{2^{k_i}} \implies 2^{-k_i} c_i + \beta_i = 2^{-k_i} d_i \implies \beta_i = 2^{-k_i}(d_i - c_i).
$$

Note $\alpha_i = a_i$ and $\beta_i = b_i$, and hence f and g coincide on I_i. Since I_i was an arbitrary interval, we can conclude that $f = g$. $\square$

Remark 2.10 A similar lemma is included in [CFP96], which uses partitions comprising dyadic rationals, rather than standard dyadic partitions. We have included it in Section 7 (Lemma 7.5).

Theorem 2.11 *Thompson's group F is torsion-free.*

Proof Let $f \in F$ be non-trivial. Then there exists $x \in [0,\ 1]$, such that $x \neq xf$. Note that as a piecewise linear function with positive derivatives, f is increasing. If $x < xf$, then $x < xf \leq xf^2 \leq \cdots$, so in particular, $x \neq xf^n$, for all $n \in \mathbb{N}$. Otherwise, if $x > xf$, then $x > xf \geq xf^2 \geq \cdots$, so again, x is not fixed by any positive power of f. We can conclude that the order of f is infinite. $\square$

3 Ordered Rooted Binary Trees and $\mathcal{T}$-trees

Definition 3.1 A non-empty tree G is called an *ordered rooted binary tree* if G has a vertex v_0 with degree two, called the *root*, and, if $v \in V(G)$ has degree strictly greater than one, then there are precisely two edges, $e_{v,L}$ and $e_{v,R}$, incident to v, that do not lie on the path from v_0 to v. We will call $e_{v,L}$ a *left edge* of G, and $e_{v,R}$ a *right edge*. Vertices of G with degree one will be called the *leaves* of G.

Given a vertex $v \in V(G)$, we will define a word associated with v. Let $(v_0 = w_0,\ w_1,\ \ldots,\ w_n = v)$, be the sequence of vertices in the path from v_0 to v, where $n \in \mathbb{N}_0$. Define a word $z_1 z_2 \cdots z_n$ over the alphabet $\{0,\ 1\}$ as follows. If $i \in \mathbb{N}$, with $i \leq n$, then $z_i = 0$ if the edge from w_{i-1} to w_i is a left edge, and $z_i = 1$ if it is a right edge.

We will define a total order on the leaves of G, by the lexicographic order on the set of words over $\{0,\ 1\}$, induced by the relation $0 \leq 1$. The *right-side* of G is the longest path in G, starting at v_0, such that every edge between vertices in this path is a right edge. The *left-side* is defined analogously.

We will also consider a graph with a single vertex and no edges to be an ordered rooted binary tree.

An *ordered rooted binary subtree* of G, is a subtree of G, whose left edges are left edges of G, and whose right edges are right edges of G. Note that the root of a subtree can be any vertex of G, and that such a subtree will be an induced subgraph of G. A *caret* is an ordered binary subtree of G with precisely two edges.

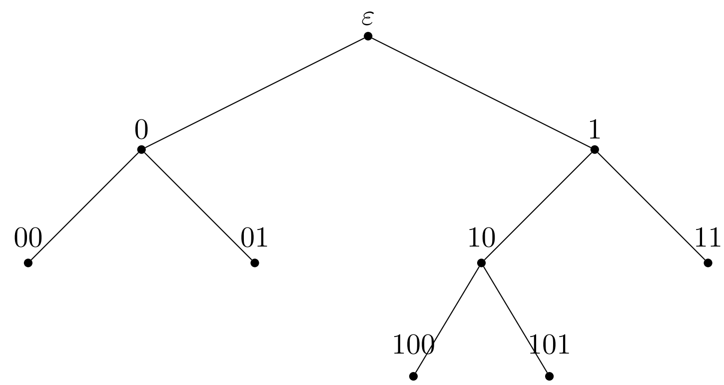
An *isomorphism* of ordered binary rooted trees is a graph isomorphism that takes left edges to left edges, and right edges to right edges.

If G is an ordered rooted binary tree, we will use the notation $\mathcal{L}(G)$, to denote the set of leaves of G, and $\mathbf{r}(G)$ will denote the root. We will also use $\mathcal{W}(G)$ to denote the set of associated words of the leaves.

Note that these trees can be represented graphically, with left and right edges pointing towards the left and right respectively, and the root at the top.

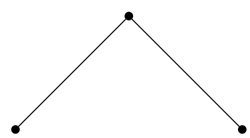
Example 3.2 We will give graphical representations of objects defined in Definition 3.1. To distinguish right and left edges, the edges will be drawn down and to their side.

1. An ordered rooted binary tree, which we will call G:



The leaves of G are 00, 01, 100, 101 and 11.

2. A caret:



Note that there are carets in G, such as the induced subgraphs of G with vertex sets $\{0,\ 00,\ 01\}$, $\{\varepsilon,\ 0,\ 1\}$ and $\{10,\ 101,\ 100\}$.

Example 3.3 Let $\mathcal{T}$ be a simple graph, whose vertices are the standard dyadic intervals of $[0,\ 1]$. Let $I,\ J \in V(\mathcal{T})$. Without loss of generality, we will suppose $\ell(I) \geq \ell(J)$. Let $a,\ b,\ m,\ n \in \mathbb{N}_0$, such that

$$I = \left[\frac{a}{2^n},\ \frac{a+1}{2^n}\right], \quad J = \left[\frac{b}{2^m},\ \frac{b+1}{2^m}\right].$$

If $\frac{a}{2^n} = \frac{b}{2^m}$ and $\frac{2a+1}{2^{n+1}} = \frac{b+1}{2^m}$, then $\{I,\ J\}$ is a left edge, and if $\frac{2a+1}{2^{n+1}} = \frac{b}{2^m}$ and $\frac{a+1}{2^n} = \frac{b+1}{2^m}$, then $\{I,\ J\}$ is a right edge. That is, if J is the left half of I then $\{I,\ J\}$ is a left edge and if J is the right half of I, then $\{I,\ J\}$ is a right edge.

The root of this graph will be $[0,\ 1]$, which has degree two, since its only neighbours are $\left[0,\ \frac{1}{2}\right]$ and $\left[\frac{1}{2},\ 1\right]$. Let $I \in V(\mathcal{T})$. Let $([0,\ 1] = I_0,\ I_1,\ \ldots,\ I_n = I)$ be a path from $[0,\ 1]$ to I, where $n \in \mathbb{N}_0$. By construction, I is either the right half or left half of I_{n-1}, so a single edge can occur to a superset of I. Similarly, there can exist no path from a subset of I to $[0,\ 1]$ that does not go through I, as any subset of I in $\mathcal{T}$, can only be reached by taking left or right halves. Since I_1 can only be a left half or a right half of $[0,\ 1]$, there is only one choice for I_1. Let $i \in \mathbb{N}_0$, $i \leq n$.

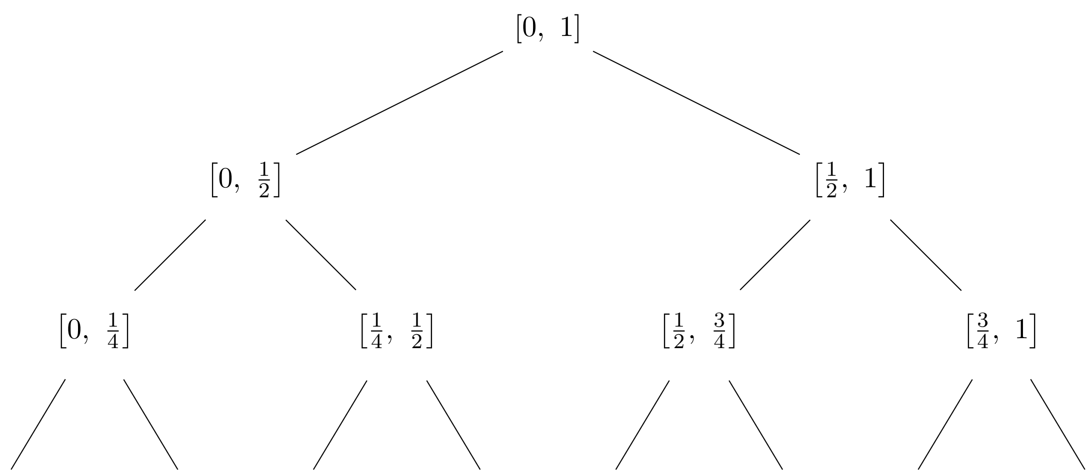
Inductively suppose that the only path from $[0,\ 1]$ to I_i is $(I_0,\ I_1,\ \ldots,\ I_i)$. We have that the only edge path going from I_i to any superset, is unique, since I_i can only be a left half or right half of an interval. It follows, by induction, that the path from $[0,\ 1]$ to I_n is unique. Since any path from an interval that is a subset of $\left[0,\ \frac{1}{2}\right]$ to a subset of $\left[\frac{1}{2},\ 1\right]$ must go through $[0,\ 1]$, we have that if there is a path from a vertex I to a vertex J that goes through $[0,\ 1]$, then it is unique. Since every path between any two vertices is an induced subgraph of such a path, we can conclude that a path between any two vertices is unique.

Let $I \in V(\mathcal{T})$. Then

$$I = \left[\frac{a}{2^n},\ \frac{a+1}{2^n}\right].$$

Any path from I to a superset J, which must be of the same form as I, except that the exponent of 2 in the denominator will be equal to $n-1$. Since n is finite, repeating this will eventually yield 0, and hence an interval of the form $[b,\ b+1] \subseteq [0,\ 1]$, for some $b \in \mathbb{N}_0$. This implies $b = 0$, and we have a path from our interval to $[0,\ 1]$, which implies $\mathcal{T}$ is connected. We can therefore conclude that $\mathcal{T}$ is a tree.

Note that the degree of $[0,\ 1]$ is 2, since the only adjacent vertices are $\left[0,\ \frac{1}{2}\right]$ and $\left[\frac{1}{2},\ 1\right]$. Since every interval has precisely one right and one left half, every vertex I will have two edges incident to it, not on the path from $[0,\ 1]$ to I. We can conclude that $\mathcal{T}$ is an ordered rooted binary tree. We will call $\mathcal{T}$ the *tree of standard dyadic intervals*. Part of it has been represented below:



Let a $\mathcal{T}$*-tree* be a finite ordered rooted binary subtree of $\mathcal{T}$, with root $[0,\ 1]$.

Remark 3.4 Given any $\mathcal{T}$-tree, it can be represented uniquely as a finite ordered rooted binary tree, with any 'names' for the vertices, by picking any ordered rooted binary tree isomorphic to it.

Conversely, given any finite ordered rooted binary tree, one can inductively construct the $\mathcal{T}$-tree isomorphic to it as follows. Rename the root as $[0,\ 1]$. Let v be a vertex that has not been labelled, but is adjacent to a vertex u that has. Since we have started at $[0,\ 1]$, and there is a unique path from $[0,\ 1]$ to v, we have that u must be on the path to v. If uv is a right edge, rename v as the right half of u, if it is a left edge, rename it as the left half.

From this point forward, it will suffice to show that a given graph is a finite ordered rooted binary tree, and then implicitly 'convert' it to the induced $\mathcal{T}$-tree, rather than construct it.

Lemma 3.5 *There is a bijection between the set of* $\mathcal{T}$*-trees, and the set of standard dyadic partitions of* $[0,\ 1]$.

Proof Let G be a $\mathcal{T}$-tree, with $n \in \mathbb{N}$ leaves, say $[x_{i-1},\ x_i]$, for $i \in \mathbb{N},\ i \leq n$. Note we have that $x_0 = 0$ and $x_n = 1$. Let ϕ be a function from the set of $\mathcal{T}$-trees to the set of standard dyadic partitions of $[0,\ 1]$, defined by

$$G\phi = \{0 = x_0 < x_1 < \cdots < x_n = 1\}.$$

which is a partition of $[0,\ 1]$, as the construction of $\mathcal{T}$-trees forces these inequalities to be strict. Since $[x_{i-1},\ x_i]$, is a standard dyadic interval, for all $i \in \mathbb{N},\ i \leq n$, by the definition of a $\mathcal{T}$-tree, we have that this is a standard dyadic partition. Let H be a $\mathcal{T}$-tree, with $m \in \mathbb{N}$ leaves $[y_{i-1},\ y_i]$, for $i \in \mathbb{N},\ i \leq m$. If $G\phi = H\phi$, then $m = n$, and hence

$$\{0 = x_0 < x_1 < \cdots < x_n = 1\} = \{0 = y_0 < y_1 < \cdots < y_n = 1\}.$$

Using the ordering on these sets, we can conclude that $x_i = y_i$ for all valid indices i. Hence the leaves of G and H are the same. Since G and H are rooted trees with root $[0,\ 1]$, they are connected and hence they have all the vertices on the unique path between $[0,\ 1]$ and each leaf. Let $I \in V(G)$. If I is a leaf, then $I \in V(H)$. If I is not a leaf, then it lies on a unique path from $[0,\ 1]$ to a leaf, and hence $I \in V(H)$. We can conclude $V(G) \subseteq V(H)$. By symmetry, we have that $V(H) \subseteq V(G)$. Since G and H are induced subgraphs of $\mathcal{T}$, we can conclude that $E(G) = E(H)$ and $G = H$. Hence ϕ is injective.

Let $P = \{0 = z_0 < z_1 < \cdots < z_n = 1\}$ be a standard dyadic partition of $[0,\ 1]$. Let

$$\mathscr{I} = \{[z_{i-1},\ z_i] \mid i \in \mathbb{N}\}.$$

Note every $I \in \mathscr{I}$ is a standard dyadic interval. We have that $\mathscr{I} \subseteq V(\mathcal{T})$, since every standard dyadic interval is a vertex of $\mathcal{T}$. Therefore, there is a unique path from $[0,\ 1]$ to I in $\mathcal{T}$, for every $I \in \mathscr{I}$, say P_I. Let K be the induced subgraph of $\mathcal{T}$, with vertex set

$$\bigcup_{I \in \mathscr{I}} P_I.$$

Note that K is an ordered binary subtree of $\mathcal{T}$, since we have not altered the nature of edges between intervals. We therefore have that $P = K\phi$, and ϕ is surjective. $\square$

4 Tree Diagrams and Classes

Definition 4.1 A $\mathcal{T}$*-tree diagram* is a tuple $(R,\ S,\ \sigma)$, where R and S are $\mathcal{T}$-trees, each with $n \in \mathbb{N}$ leaves, and $\sigma \in S_n$. We call R the *domain tree*, and S the *range tree* of the $\mathcal{T}$-tree diagram. We will say the ith leaf of R is *associated* with the $i\sigma$th leaf of S, for any $i \in \mathbb{N},\ i \leq n$. Note $n \geq 2$. They are often denoted

$$R \mapsto S,$$

and, if σ is not the identity permutation, the leaves of R are numbered 1 up to n, and the leaves of S are numbered $1\sigma^{-1}$ up to $n\sigma^{-1}$.

Let $(R,\ S,\ \sigma)$ be a $\mathcal{T}$-tree diagram. If there exists $i \in \mathbb{N}$ with $i \leq n$ such that the ith and $(i+1)$th leaves of R are part of a caret, and the $i\sigma$th and the $(i+1)\sigma$th leaves of S are part of a caret, define R' and S' to be the ordered rooted binary subtrees of R and S respectively, with the

ith and $(i+1)$th leaves, and $i\sigma$th and $(i+1)\sigma$th leaves, respectively and adjacent edges removed. Define

$$\begin{aligned}
\sigma' \colon \{1,\ 2,\ \ldots, n-1\} &\to \{1,\ 2,\ \ldots, n-1\} \\
i &\mapsto i\sigma \\
j &\mapsto \begin{cases} j\sigma & j < i,\ j\sigma < i\sigma \\ j\sigma - 1 & j < i,\ j\sigma > i\sigma \\ (j+1)\sigma & j > i,\ (j+1)\sigma < i\sigma \\ (j+1)\sigma - 1 & j > i,\ (j+1)\sigma > i\sigma. \end{cases}
\end{aligned}$$

We will call $(R',\ S',\ \sigma')$ an *elementary contraction* of $(R,\ S,\ \sigma)$. If no such $\mathcal{T}$-tree diagram exists, then $(R,\ S,\ \sigma)$ is called *reduced*. Otherwise $(R,\ S,\ \sigma)$ is called *reducible*.

Let $(R,\ S,\ \sigma)$ be a $\mathcal{T}$-tree diagram. Let $i \in \mathbb{N}$ such that $i \leq n$. Define R' and S' to be the ordered rooted binary trees, defined by taking R and S respectively, and adding two new vertices two new vertices to each; the left and right halves of the ith and $i\sigma$th leaves respectively, both adjacent only to the leaf, one by a left edge, the other by a right edge, in the ith and $i\sigma$th position respectively.

$$\begin{aligned}
\sigma' \colon \{1,\ 2,\ \ldots n+1\} &\to \{1,\ 2,\ \ldots n+1\} \\
i &\mapsto i\sigma \\
i+1 &\mapsto i\sigma + 1 \\
j &\mapsto \begin{cases} j\sigma & j < i,\ j\sigma < i\sigma \\ j\sigma + 1 & j < i,\ j\sigma > i\sigma \\ (j-1)\sigma & j > i+1,\ (j-1)\sigma < i\sigma \\ (j-1)\sigma + 1 & j > i+1,\ (j-1)\sigma > i\sigma. \end{cases}
\end{aligned}$$

We will call $(R',\ S',\ \sigma')$ an *elementary expansion* of $(R,\ S,\ \sigma)$. We will use *elementary operation* to refer to an elementary expansion or contraction.

Lemma 4.2 *If $(R,\ S,\ \sigma)$ is a $\mathcal{T}$-tree diagram, and $(R',\ S',\ \sigma')$ is an elementary expansion or contraction of $(R,\ S,\ \sigma)$, then $(R',\ S',\ \sigma')$ is a $\mathcal{T}$-tree diagram.*

Proof Elementary contraction: Let $(R,\ S,\ \sigma)$ be a $\mathcal{T}$-tree diagram, and $(R',\ S',\ \sigma')$ be an elementary contraction of $(R,\ S,\ \sigma)$. Let X be an ordered rooted binary tree, and $w \in V(X)$ such that the left edge of w is incident to a leaf u, and the right edge to a leaf v. Suppose $u,\ v$ and their incident edges are removed to yield a new graph X'. Note that X' is connected, as we have only removed leaves and their incident edges. All connected subgraphs of a tree are trees, so X' is a tree. The root of X' is the same as the root of X. Let $x \in V(X')\backslash\{w\}$. We have that the neighbours of x in X' are the same as the neighbours of x in X, and hence if the degree of x is greater than one, then x has precisely one left and one right edge emanating from it, not on the path from x to the root. The highest the degree of a vertex in X is three, since a vertex can only have two edges not on the path to the root, and only one on the path to the root, since if there were two, the path to the root would not be unique. We can conclude that the degree of w in X' is one, and hence X' is an ordered rooted binary tree. We can conclude that R' and S' are ordered rooted binary trees, and hence $\mathcal{T}$-trees.

Let $j,\ l \in \{1,\ 2,\ \ldots,\ n-1\}$, such that $j\sigma' = l\sigma'$. We will prove $j = l$ using three cases.

Case 1: $j,\ l < i$. If $j\sigma,\ l\sigma < i\sigma$, then

$$j\sigma = j\sigma' = l\sigma' = l\sigma,$$

which implies $j = l$, by injectivity of σ. If $j\sigma,\ l\sigma \geq i\sigma$, then

$$j\sigma - 1 = j\sigma' = l\sigma' = l\sigma - 1,$$

and hence $j = l$. Suppose one of $j\sigma$ and $l\sigma$ is less than $i\sigma$, and the other is greater than or equal to $i\sigma$. Without loss of generality, we will assume $l\sigma < i\sigma \leq j\sigma$. Then

$$j\sigma - 1 = j\sigma' = l\sigma' = l\sigma,$$

which implies $j\sigma = i\sigma$, since $l\sigma < i\sigma \leq j\sigma$. Therefore $j = i$, a contradiction to our case assumption.

Case 2: $j,\ l \geq i$. If $(j+1)\sigma,\ (l+1)\sigma < i\sigma$, then

$$(j+1)\sigma = j\sigma' = l\sigma' = (l+1)\sigma,$$

and therefore $j = l$. If $(j+1)\sigma,\ (l+1)\sigma \geq i\sigma$, then

$$(j+1)\sigma - 1 = j\sigma' = l\sigma' = (l+1)\sigma - 1,$$

which gives $j = l$. Suppose one of $(j+1)\sigma$ and $(l+1)\sigma$ is less than $i\sigma$, and the other is greater than or equal to $i\sigma$. Without loss of generality, let $(l+1)\sigma < i\sigma \leq (j+1)\sigma$. We have

$$(j+1)\sigma - 1 = j\sigma' = l\sigma' = (l+1)\sigma,$$

so $(j+1)\sigma = i\sigma$, since $(l+1)\sigma < i\sigma \leq (j+1)\sigma$. Hence $j+1 = i$, and $j < i$, a contradiction to our case assumption.

Case 3: One of j and l is greater than or equal to i, and the other is less. Without loss of generality, we will set $l < i \leq j$. If $(j+1)\sigma,\ l\sigma < i\sigma$ or $(j+1)\sigma,\ l\sigma \geq i\sigma$, then

$$(j+1)\sigma = j\sigma' = l\sigma' = l\sigma \text{ or } (j+1)\sigma - 1 = j\sigma' = l\sigma' = l\sigma - 1,$$

respectively, both of which imply $j+1 = l$, and hence $j < l$, a contradiction. If $(j+1)\sigma < i\sigma \leq l\sigma$, then

$$(j+1)\sigma = j\sigma' = l\sigma' = l\sigma - 1,$$

and hence $l\sigma = i\sigma$, and it follows that $l = i$, a contradiction. If $l\sigma < i \leq (j+1)\sigma$, then

$$(j+1)\sigma - 1 = j\sigma' = l\sigma' = l\sigma,$$

which implies $(j+1)\sigma = i\sigma$, and hence $j+1 = i$. Therefore $j < i$, a contradiction. We can conclude that σ' is injective. As it maps a finite set to itself, it is a permutation, and $(R',\ S',\ \sigma')$ is a $\mathcal{T}$-tree diagram.

Elementary expansion: Let $(R,\ S,\ \sigma)$ be a $\mathcal{T}$-tree diagram, and $(R',\ S',\ \sigma')$ be an elementary expansion of $(R,\ S,\ \sigma)$. Let X be a $\mathcal{T}$-tree. Suppose two new vertices are added to X, one vertex u is attached to a leaf w by a left edge, the other vertex v is attached to w by a right edge. We will call the resultant graph X'. Since X is a tree we have $|E(X)| = |V(X)| - 1$. We have attached two new vertices and two new edges. Therefore

$$|E(X')| = |E(X)| + 2 = |V(X)| - 1 + 2 = |V(X')| - 1,$$

and hence X' is a tree. We have that X' is rooted with the same root as X. If x is a vertex of X' that is not w, but has degree greater than 1, then its neighbours are the same as its neighbours in X, so it has precisely one right and one left edge emanating from it, that do not lie on the path to the root. The vertex w now has precisely one right edge and one left edge emanating from it, not on the path to the root. The vertices u and v have degree 1. We can conclude that X' is an ordered rooted binary tree and therefore that R' and S' are ordered rooted binary trees, and hence $\mathcal{T}$-trees.

Note that $|\mathcal{L}(R')| = |\mathcal{L}(S')| = |\mathcal{L}(R)|+1$. Let $j \in \{1,\ 2,\ \ldots,\ |\mathcal{L}(R)|+1\}$. If $j \in \{i\sigma,\ i\sigma+1\}$, then j is mapped onto by i or $i+1$, using σ'. Otherwise, if $j < i\sigma$, then there exists $l \in \{1,\ 2,\ \ldots |\mathcal{L}(R)|+1\}$ such that $l\sigma = j$, by surjectivity of σ. Note $l \neq i$, since if $l\sigma = i\sigma$, then $j = i\sigma$, a contradiction. If $l < i$, then $j = l\sigma = l\sigma'$. If $l > i$, then $j = l\sigma = (l+1)\sigma'$. Note $l+1 > i$, so this is valid. Suppose now $j > i\sigma$. By surjectivity of σ, there exists $l \in \{1,\ 2,\ \ldots |\mathcal{L}(R)|+1\}$, such that $l\sigma + 1 = j$. Note $l \neq i$, since if $l\sigma = i\sigma$, then $j = i\sigma + 1$, a contradiction, and σ' is surjective. Since σ' maps from a finite set to itself, we can conclude that σ' is a permutation, and hence $(R',\ S',\ \sigma')$ is a $\mathcal{T}$-tree diagram. $\square$

Lemma 4.3 *Let $(R,\ S,\ \sigma)$ be a $\mathcal{T}$-tree diagram. Let $i \in \{1,\ 2,\ \ldots,\ |\mathcal{L}(R)|\}$. Let $(R',\ S',\ \tau)$ be an elementary expansion, and $(R'',\ S'',\ \rho)$ be an elementary contraction of $(R,\ S,\ \sigma)$. Then applying an elementary expansion in the ith position, to $(R,\ S,\ \sigma)$, followed by an elementary contraction in the ith position yields $(R,\ S,\ \sigma)$, as does applying an elementary contraction in the ith position, followed by an elementary expansion in the ith position.*

In particular, $(R,\ S,\ \sigma)$ is an elementary contraction of $(R',\ S',\ \tau)$ and an elementary expansion of $(R'',\ S'',\ \rho)$.

Proof Since $(R',\ S',\ \tau)$ is an elementary expansion of $(R,\ S,\ \sigma)$, there exists $i \in \mathbb{N}$, $i \leq |\mathcal{L}(R')| = |\mathcal{L}(S')|$, such that the ith and $i\tau$th leaves of R' and S' respectively lie in a caret. It therefore suffices to show that $\sigma = \tau'$, where τ' is the permutation in an elementary contraction of a $\mathcal{T}$-tree diagram with permutation τ, as defined in Definition 4.1. We will show $\tau' = \sigma$.

Let $j \in \{1,\ 2,\ \ldots,\ |\mathcal{L}(R)|\}$. Note $i\sigma = i\tau = i\tau'$. If $j < i$ and $j\sigma < i\sigma$, then $j\tau = j\sigma$. This implies $j\tau < i\sigma = i\tau$. So $j\tau' = j\tau = j\sigma$. If $j < i$ and $j\sigma > i\sigma$, then $j\tau = j\sigma + 1$. We can conclude $j\tau > j\sigma > i\sigma$, and hence $j\tau' = j\tau - 1 = j\sigma$.

If $j > i + 1$ and $j\sigma < i\sigma$, then $(j+1)\tau = j\sigma$. It follows that $(j+1)\tau = j\sigma < i\sigma = i\tau$. Hence $j\tau' = (j+1)\tau = j\sigma$. If $j > i+1$ and $j\sigma > i\sigma$, then $(j+1)\tau = j\sigma + 1$. Therefore, $(j+1)\tau > j\sigma = i\sigma = i\tau$, and $j\tau' = (j+1)\tau - 1 = j\sigma$. We have proved that σ and τ' coincide on all points except $i+1$. Since they are both permutations of the same set, they must therefore be equal. We can conclude that $(R,\ S,\ \sigma)$ is an elementary contraction of $(R',\ S',\ \tau)$.

We also have that $(R'',\ S'',\ \rho)$ is an elementary contraction of $(R,\ S,\ \sigma)$. Then there exists $i \in \mathbb{N},\ i \leq |\mathcal{L}(R)|$, such that a caret of $(R,\ S,\ \sigma)$ was removed in the ith and $i\sigma$th positions, during the elementary contraction. Adding the carets back in the form of an elementary expansion of $(R'',\ S'',\ \rho)$ will give a $\mathcal{T}$-tree diagram whose $\mathcal{T}$-trees are R and S. It therefore suffices to show that the associated permutation, which we will call ρ', equals σ.

Note that $i\rho = i\sigma$, and hence $i\rho' = i\sigma$. Let $j \in \{1,\ 2,\ \dots,\ |\mathcal{L}(R)|\}$. If $j < i$ and $j\sigma < i\sigma$, then $j\rho = j\sigma < i\sigma = i\rho$. It follows that $j\rho' = j\rho = j\sigma$. If $j < i$ and $j\sigma > i\sigma$, then $j\rho = j\sigma - 1$. If $j\sigma - 1 \leq i\sigma$, we have that $j\sigma = i\sigma$, and hence $i = j$, a contradiction to the assumption that $j < i$. Hence $j\rho = j\sigma - 1 > i\sigma$ and $j\rho' = j\rho + 1 = j\sigma$.

If $j > i$, and $j\sigma < i\sigma$, then $(j+1)\rho = j\sigma$. Hence $(j+1)\rho = j\sigma > i\sigma = i\rho$ and $j\rho' = (j+1)\rho = j\sigma$. Finally, if $j > i$ and $j\sigma > i\sigma$, we have that $(j-1)\rho = j\sigma - 1$. By the argument for when $j < i$ and $j\sigma > i\sigma$, we have that $(j-1)\rho > i\rho$. Hence $j\rho' = (j-1)\rho + 1 = j\sigma$. We can conclude that $\sigma = \rho'$, and $(R,\ S,\ \sigma)$ is an elementary expansion of $(R'',\ S'',\ \rho)$. $\square$

Definition 4.4 We will define a relation $\sim$ on the set of $\mathcal{T}$-tree diagrams. Let $(R,\ S,\ \sigma)$ and $(U,\ V,\ \tau)$ be $\mathcal{T}$-tree diagrams. If there is a finite sequence of elementary expansions or contractions that takes $(R,\ S,\ \sigma)$ to $(U,\ V,\ \tau)$, then $(R,\ S,\ \sigma) \sim (U,\ V,\ \tau)$.

Lemma 4.5 *The relation $\sim$, as defined in Definition 4.4, is an equivalence relation on the set of $\mathcal{T}$-tree diagrams.*

Proof Let $X,\ Y$ and Z be $\mathcal{T}$-tree diagrams.

Reflexive: The empty sequence of elementary operations takes X to itself. Hence $X \sim X$.

Symmetric: Suppose $X \sim Y$. Then there is a finite sequence $(x_n)_n$ of elementary operations that takes X to Y. Using Lemma 4.3, we will construct a new sequence, starting at the end of $(x_n)_n$, and working towards the beginning as follows. Let $i \in \mathbb{N}$ be the index of the term considered. Let U be the $\mathcal{T}$-tree diagram obtained when applying $x_1,\ x_2,\ \dots, x_{i-1}$ to X, and V be the $\mathcal{T}$-tree diagram obtained when applying x_i to U. We will construct an elementary contraction or expansion, whichever x_i is not, to take V to U, using Lemma 4.3. This sequence will be of elementary operations, and will take Y to X. Hence $Y \sim X$.

Transitive: Suppose $X \sim Y$ and $Y \sim Z$. Concatenate the finite sequences of elementary operations that take X to Y, and Y to Z. This will take X to Z, via Y, and still be finite. Therefore $X \sim Z$. $\square$

Definition 4.6 We will call the equivalence classes of $\mathcal{T}$-tree diagrams under the equivalence relation $\sim$ from Definition 4.4, *$\mathcal{T}$-tree diagram classes.*

Lemma 4.7 *If $(R,\ S,\ \sigma)$ is an element of a $\mathcal{T}$-tree class X, such that σ is the identity permutation, then the permutation of every element of X is the identity permutation.*

Proof Let $(R,\ S,\ \sigma)$ be a $\mathcal{T}$-tree diagram, such that σ is the identity permutation. We will first show that σ', calculated by applying an elementary contraction, is the identity. Let $i \in$

$\{1,\ 2,\ \ldots,\ |\mathcal{L}(R)|\}$ be the point where the contraction is being applied. Let $j \in \{1,\ 2,\ \ldots,\ |\mathcal{L}(R)|-1\}$. If $j \leq i$, then $j\sigma = j \leq i = i\sigma$, and hence $j\sigma' = j\sigma = j$. If $j > i$, then $(j+1)\sigma = j+1 > i = i\sigma$, and it follows that $j\sigma' = (j+1)\sigma - 1 = j$.

We will now prove that σ', calculated by applying an elementary expansion, is the identity. Let $i \in \{1,\ 2,\ \ldots,\ |\mathcal{L}(R)|\}$ be the point where the elementary is being applied. Let $j \in \{1,\ 2,\ \ldots,\ |\mathcal{L}(R)|+1\}$. If $j \leq i$, then $j\sigma = j \leq i = i\sigma$, and $j\sigma' = j\sigma = j$. If $j > i+1$, then $(j-1)\sigma = j-1 \geq i$, and we have $j\sigma' = (j-1)\sigma + 1 = j$. Finally, $(i+1)\sigma' = i\sigma + 1 = i+1$.

Since a permutation remains the identity when an elementary operation is applied, the same can be said for when a finite sequence of elementary operations is applied. □

Remark 4.8 The proof of the following lemma includes lots of case work, involving similar cases. The complete proof has been included for the interested reader, however the method of the proof can be understood from Cases 7 and 8.

Lemma 4.9 *Let $(R,\ S,\ \rho)$ be a $\mathcal{T}$-tree diagram. Suppose ρ and τ are permutations in $S_{|\mathcal{L}(R)|}$. If f represents a finite sequence of elementary operations, and $f_1,\ f_2$ and f_3 denote the effect of f upon the three ordered parts of a $\mathcal{T}$-tree diagram, then*

$$(R,\ S,\ \rho\tau)f = (Rf_1,\ Sf_2,\ (\rho f_3)(\tau f_3)).$$

Note that, by τf_3, we mean to apply f_3 to τ, starting in the $i\rho$th position, where i was the position in which the first elementary operation was applied.

Proof We will first consider the case when the entire sequence comprises an elementary contraction. Let $i \in \{1,\ 2,\ \ldots,\ |\mathcal{L}(R)|\}$ be the point where the contraction is being applied. Let $j \in \{1,\ 2,\ \ldots |\mathcal{L}(R)|\}$. Case 1: $j = i$ or $j < i,\ j\rho < i\rho,\ j\rho\tau < i\rho\tau$.
Then $j\rho' = j\rho,\ j(\rho\tau)' = j\rho\tau$.

$$\implies j\rho'\tau' = j\rho\tau' = j\rho\tau = j(\rho\tau)'.$$

Case 2: $j < i,\ j\rho < i\rho,\ j\rho\tau > i\rho\tau$.
It follows that $j(\rho\tau)' = j\rho\tau - 1,\ j\rho' = j\rho$, and hence

$$j\rho'\tau' = j\rho\tau' = j\rho\tau - 1 = j(\rho\tau)'.$$

Case 3: $j < i,\ j\rho > i\rho,\ j\rho\tau < i\rho\tau$.
We have that $j(\rho\tau)' = j\rho\tau,\ j\rho' = j\rho - 1$, and $((j-1)\rho+1)\tau = j\rho\tau < i\rho\tau$. In addition, $j\rho' = j\rho - 1 \geq i\rho = i\rho'$. Since ρ' is a permutation, and $j \neq i$, this inequality must be strict, and $j\rho - 1 > i\rho$. Therefore,

$$j\rho'\tau' = (j\rho - 1)\tau' = (j\rho - 1 + 1)\tau = j\rho\tau = j(\rho\tau)'.$$

Case 4: $j < i,\ j\rho > i\rho,\ j\rho\tau > i\rho\tau$.
We can conclude that $j(\rho\tau)' = j\rho\tau - 1\ \ j\rho' = j\rho - 1$ and $((j\rho - 1) + 1)\tau = j\rho\tau > i\rho\tau$. We also have that $j\rho' = j\rho - 1 \geq i\rho = i\rho'$. Again, since $j \neq i$ and ρ' is a permutation, we have that $j\rho - 1 > i\rho$. Hence,

$$j\rho'\tau' = (j\rho - 1)\tau' = j\rho\tau - 1 = j(\rho\tau)'.$$

Case 5: $j > i,\ (j+1)\rho < i\rho,\ (j+1)\rho\tau < i\rho\tau$.
This implies $j(\rho\tau)' = (j+1)\rho\tau$, and $j\rho' = (j+1)\rho$. Therefore,

$$j\rho'\tau' = (j+1)\rho\tau' = (j+1)\rho\tau = j(\rho\tau)'.$$

Case 6: $j > i,\ (j+1)\rho < i\rho,\ (j+1)\rho\tau > i\rho\tau$.
We have $j\rho' = (j+1)\rho$, and $j(\rho\tau)' = (j+1)\rho\tau - 1$. It follows that

$$j\rho'\tau' = (j+1)\rho\tau' = (j+1)\rho\tau - 1 = j(\rho\tau)'.$$

Case 7: $j > i,\ (j+1)\rho > i\rho,\ (j+1)\rho\tau < i\rho\tau$.
Then $j(\rho\tau)' = (j+1)\rho\tau,\ j\rho' = (j+1)\rho - 1$, and $(((j+1)\rho-1)+1)\tau = (j+1)\rho\tau < i\rho\tau$. In addition $j\rho' = (j+1)\rho - 1 \geq i\rho = i\rho'$. This inequality must be strict as ρ' is a permutation and $i \neq j$, and it follows that $(j+1)\rho - 1 > i\rho$. Therefore

$$j\rho'\tau' = ((j+1)\rho - 1)\tau' = (j+1)\rho\tau = j(\rho\tau)'.$$

Case 8: $j > i,\ (j+1)\rho > i\rho,\ (j+1)\rho\tau > i\rho\tau$.
It follows that $j(\rho\tau)' = (j+1)\rho\tau - 1,\ j\rho' = (j+1)\rho - 1$, and $(((j+1)\rho-1)+1)\tau = (j+1)\rho\tau > i\rho\tau$. We also have that $j\rho' = (j+1)\rho - 1 \geq i\rho = i\rho'$. By the same argument as the previous case, $(j+1)\rho - 1 > i\rho$, and hence

$$j\rho'\tau' = ((j+1)\rho - 1)\tau' = (j+1)\rho\tau - 1 = j(\rho\tau)'.$$

We can conclude that the lemma is true if the entire sequence is an elementary contraction.

We will now consider the case when the entire sequence consists of an elementary expansion. Let $i \in \{1,\ 2,\ \ldots,\ |\mathcal{L}(R)|\}$ be the point where the contraction is being applied. Let $j \in \{1,\ 2,\ \ldots |\mathcal{L}(R)|\}$.
Case 1: $j = i$ or $j < i,\ j\rho < i\rho,\ j\rho\tau < i\rho\tau$.
We have $j\rho' = j\rho,\ j(\rho\tau)' = j\rho\tau$, and hence

$$j\rho'\tau' = j\rho\tau' = j\rho\tau = j(\rho\tau)'.$$

Case 2: Either $j = i+1$ or all three of the following inequalities hold: $j < i,\ j\rho < i\rho,\ j\rho\tau > i\rho\tau$.
Then $j\rho' = j\rho$ and $j(\rho\tau)' = j\rho\tau + 1$. Therefore,

$$j\rho'\tau' = j\rho\tau' = j\rho\tau + 1 = j(\rho\tau)'.$$

Case 3: $j < i,\ j\rho > i\rho,\ j\rho\tau < i\rho\tau$.
It follows that $j(\rho\tau)' = j\rho\tau$, $j\rho' = j\rho + 1$, $((j\rho+1)-1)\tau = j\rho\tau < i\rho\tau$, and $j\rho + 1 > i\rho + 1$. Hence,

$$j\rho'\tau' = (j\rho+1)\tau' = (j\rho+1-1)\tau = j\rho\tau = j(\rho\tau)'.$$

Case 4: $j < i,\ j\rho > i\rho,\ j\rho\tau > i\rho\tau$.
We have that $j(\rho\tau)' = j\rho\tau + 1,\ j\rho' = j\rho + 1,\ ((j\rho+1)-1)\tau = j\rho\tau > i\rho\tau$, and $j\rho + 1 > i\rho + 1$. It follows that

$$j\rho'\tau' = (j\rho+1)\tau' = j\rho\tau + 1 = j(\rho\tau)'.$$

Case 5: $j > i+1,\ (j-1)\rho < i\rho,\ (j-1)\rho\tau < i\rho\tau$.
This implies $j(\rho\tau)' = (j-1)\rho\tau,\ j\rho' = (j-1)\rho$, and hence,

$$j\rho'\tau' = (j-1)\rho\tau' = (j-1)\rho\tau = j(\rho\tau)'.$$

Case 6: $j > i+1,\ (j-1)\rho < i\rho,\ (j-1)\rho\tau > i\rho\tau$.
We have $j(\rho\tau)' = (j-1)\rho\tau + 1$ and $j\rho' = (j-1)\rho$. Therefore,

$$j\rho'\tau' = (j-1)\rho\tau' = (j-1)\rho\tau + 1 = j(\rho\tau)'.$$

Case 7: $j > i+1,\ (j-1)\rho > i\rho,\ (j-1)\rho\tau < i\rho\tau$.
Then $j(\rho\tau)' = (j-1)\rho\tau,\ j\rho' = (j-1)\rho+1,\ ((j\rho+1)-1)\tau = j\rho\tau < i\rho\tau$, and $(j-1)\rho+1 > i\rho+1$. We can conclude that

$$j\rho'\tau' = ((j-1)\rho+1)\tau' = (((j-1)\rho+1)-1)\tau = (j-1)\rho\tau = j(\rho\tau)'.$$

Case 8: $j > i+1,\ (j-1)\rho > i\rho,\ (j-1)\rho\tau > i\rho\tau$.
It follows that $j(\rho\tau)' = (j-1)\rho\tau+1,\ j\rho' = (j-1)\rho+1,\ ((j\rho+1)-1)\tau = j\rho\tau > i\rho\tau$, and $(j-1)\rho+1 > i\rho+1$. Then

$$j\rho'\tau' = ((j-1)\rho+1)\tau' = (j-1)\rho\tau+1 = j(\rho\tau)'.$$

We can conclude that the Lemma is true if the entire sequence is an elementary expansion.

In any other case, we have that the sequence comprises a finite number of elementary expansions and contractions, both of which preserve the two permutations ρ and τ, as stated in the Lemma. Therefore, repeatedly applying these expansions and contractions will preserve the two permutations, and the Lemma is true. $\square$

Remark 4.10 The proof of the following lemma repeatedly uses a similar argument for multiple cases, in Subcase 1.A. We have included the entire proof, however the spirit of the proof can be understood from the case when $j+1 < i+1 > k$ and $j\sigma < i\sigma < (k-2)\sigma$.

Lemma 4.11 *Let $(R,\ S,\ \sigma)$ and $(U,\ V,\ \tau)$ be $\mathcal{T}$-tree diagrams, such that f_1 and f_2 are both elementary expansions, with $(R,\ S,\ \sigma)f_1f_2 = (U,\ V,\ \tau)$. Let $i \in \{1,\ \dots,\ |\mathcal{L}(R)|\}$ and $j \in \{1,\ \dots,\ |\mathcal{L}(R)|+1\}$, be the positions in $(R,\ S,\ \sigma)$ and $(R,\ S,\ \sigma)f_1$, respectively, where f_1 and f_2 are applied. If $j \neq i,\ i+1$, then $(R,\ S,\ \sigma)f_3f_4 = (U,\ V,\ \tau)$, where f_3 and f_4 denote elementary expansions applied in the jth and $(i+1)$th positions, respectively, if $j < i$, and in the $(j-1)$th and ith positions if $j > i$.*

Proof Case 1: $j < i$.
If $j < i$, then, by applying f_1f_2, we expand ith leaf of R and $i\sigma$th leaf of S, followed by the jth leaf of R and $j\sigma'$th leaf of S. This leaf is the $(j\sigma-1)$th of Sf_1, if $j\sigma > i\sigma$, and the $j\sigma$th, if $j\sigma < i\sigma$. In both cases, this leaf is the $j\sigma$th of S.

By applying f_3f_4, we expand the jth leaf of R and the $j\sigma$th leaf of S, followed by the $(i-1)$th leaf of Rf_3, which is the ith leaf of R, since we have increased the index of the ith leaf by 1, by expanding below it. We also expand the $(i+1)\sigma'$th leaf of Sf_3, which is the same as the $i\sigma$th leaf of S, because of the change in indexing, and since $i\sigma' = i\sigma$.

We now need to show σ maps to the same permutation under f_1f_2 and f_3f_4. Let $k \in \{1,\ \dots,\ |\mathcal{L}(R)|+2\}$. Let

$$\rho = \sigma f_1,\ \tau = \sigma f_1 f_2,\ \theta = \sigma f_3,\ \mu = \sigma f_3 f_4.$$

We will consider two subcases, noting $j\sigma \neq i\sigma$.

Subcase 1.A: $j\sigma < i\sigma$.
Note that $j\sigma < i\sigma = ((i+1)-1)\sigma$, so

$$(i+1)\theta = ((i+1)-1)\sigma + 1 = i\sigma + 1.$$

If $k < j < i$ and $k\sigma < j\sigma < i\sigma$, then $k\rho = k\sigma < j\sigma = j\rho$, and $k\theta = k\sigma < i\sigma + 1 = (i+1)\theta$. Then

$$k\tau = k\rho = k\sigma = k\theta = k\mu,$$

which also holds if $j = k$. If $k < j < i$ and $j\sigma < k\sigma < i\sigma$, then $k\rho > j\rho$, and $k\theta = k\sigma < i\sigma + 1 = i\theta$, so

$$k\tau = k\rho + 1 = k\sigma + 1 = k\theta = k\mu.$$

If $k < j < i$ and $j\sigma < i\sigma < k\sigma$, then $k\rho = k\sigma + 1 > j\rho$, and $k\theta > (i+1)\theta$. Therefore,

$$k\tau = k\rho + 1 = k\sigma + 2 = k\theta + 1 = k\mu.$$

If $j + 1 < k < i$ and $(k-1)\sigma < j\sigma < i\sigma$, then $(k-1)\rho = (k-1)\sigma < j\sigma = j\rho$, and $k\theta = (k-1)\sigma < i\sigma + 1 = (i+1)\theta$. It follows that

$$k\tau = (k-1)\rho = (k-1)\sigma = k\theta = k\mu.$$

If $j + 1 < k < i$ and $j\sigma < (k-1)\sigma < i\sigma$, then $(k-1)\rho = (k-1)\sigma > j\rho$, and $k\theta = (k-1)\sigma + 1 < i\sigma + 1 = (i+1)\theta$. Hence

$$k\tau = (k-1)\rho + 1 = (k-1)\sigma + 1 = k\theta = k\mu.$$

If $j + 1 < k < i$ and $j\sigma < i\sigma < (k-1)\sigma$, then $(k-1)\rho = (k-1)\sigma > j\rho$, and $k\theta = (k-1)\sigma + 1 > i\sigma + 1 = (i+1)\theta$, and we have

$$k\tau = (k-1)\rho + 1 = (k-1)\sigma + 2 = k\theta + 1 = k\mu.$$

If $j + 1 < i + 1 < k$, and $(k-2)\sigma < j\sigma < i\sigma$, then $(k-1)\rho = (k-2)\sigma < j\sigma = j\rho$. In addition $k - 2 > i - 1 \geq j + 1$, and $(k-1)\theta = (k-2)\sigma < i\sigma + 1 = (i+1)\theta$. Therefore,

$$k\tau = (k-1)\rho = (k-2)\sigma = (k-1)\theta = k\mu.$$

If $j + 1 < i + 1 < k$, and $j\sigma < (k-2)\sigma < i\sigma$, then $(k-1)\rho = (k-2)\sigma < j\sigma = j\rho$. We also have that $k - 2 > i - 1 \geq j + 1$, and $(k-1)\theta = (k-2)\sigma + 1 > i\sigma + 1 = (i+1)\theta$. It follows that

$$k\tau = (k-1)\rho + 1 = (k-2)\sigma + 1 = (k-1)\theta = k\mu.$$

If $j + 1 < i + 1 < k$, and $j\sigma < i\sigma < (k-2)\sigma$, then $(k-1)\rho = (k-2)\sigma + 1 < j\sigma = j\rho$. We also have that $k - 2 > i - 1 \geq j + 1$, and $(k-1)\theta = (k-2)\sigma + 1 > i\sigma + 1 = (i+1)\theta$. Hence

$$k\tau = (k-1)\rho + 1 = (k-2)\sigma + 2 = (k-1)\theta + 1 = k\mu.$$

If $(i-1)\sigma > i\sigma > j\sigma$, then $(i-1)\rho > j\rho$. Therefore,

$$i\tau = (i-1)\rho + 1 = (i-1)\sigma + 2 = i\theta + 1 = i\mu.$$

If $i\sigma > (i-1)\sigma > j\sigma$, then $(i-1)\rho > j\rho$, and $(i+1)\rho = i\sigma + 1 > j\sigma = j\rho$. So

$$i\tau = (i-1)\rho + 1 = (i-1)\sigma + 1 = i\theta = i\mu.$$

If $i\sigma > j\sigma > (i-1)\sigma$, then

$$i\tau = (i-1)\rho = (i-1)\sigma = i\theta = i\mu.$$

In addition, since $i\rho = i\sigma > j\sigma = j\rho$,

$$(i+1)\tau = i\rho + 1 = i\sigma + 1 = (i+1)\theta = (i+1)\mu.$$

Subcase 1.B: $j\sigma > i\sigma$.
Let ϕ be any permutation of the domain of σ, such that $j\sigma\phi < i\sigma\phi$, and $j\phi < i\phi$. Applying Case A to $\sigma\phi$, and ϕ, gives $(\sigma\phi)f_1f_2 = (\sigma\phi)f_3f_4$, and $\phi f_1f_2 = \phi f_3f_4$. By Lemma 4.9, we have $(\sigma f_1f_2)(\phi f_1f_2) = (\sigma f_3f_4)(\phi f_3f_4)$, and hence $\sigma f_1f_2 = \sigma f_3f_4$.

Case 2: $j > i$. Let $m = j-1$, and $n = i$. We have that f_3 is an elementary expansion in the mth position, and f_4 is an elementary expansion in the nth position. In addition, f_1 is an elementary expansion in the nth position, and f_4 is an elementary expansion in the $(m+1)$th position. We have that $j \geq i+1$, and $j \neq i+1$, so $j > i+1$, and hence $m = j-1 > i = n$. The result follows now by Case 1. □

Corollary 4.12 *Let $(R,\ S,\ \sigma)$ and $(U,\ V,\ \tau)$ be $\mathcal{T}$-tree diagrams, such that f_1 and f_2 are both elementary contractions, with $(R,\ S,\ \sigma)f_1f_2 = (U,\ V,\ \tau)$. Let $i \in \{1,\ \ldots,\ |\mathcal{L}(R)|\}$ and $j \in \{1,\ \ldots,\ |\mathcal{L}(R)|-1\}$, be the positions in $(R,\ S,\ \sigma)$ and $(R,\ S,\ \sigma)f_1$, respectively, where f_1 and f_2 are applied. If $j \neq i$, then $(R,\ S,\ \sigma)f_3f_4 = (U,\ V,\ \tau)$, where f_3 and f_4 denote elementary contractions applied in the jth and $(i-1)$th positions, respectively, if $j < i$, and in the $(j+1)$th and ith positions if $j > i$.*

Proof Apply the sequence in reverse using Lemma 4.3, then we satisfy the conditions of Lemma 4.11. □

Remark 4.13 The proof of the following lemma is very similar to that of Lemma 4.11, and again includes the repetition in Subcase 1.A. We have, however, included the full proof for the interested reader.

Lemma 4.14 *Let $(R,\ S,\ \sigma)$ and $(U,\ V,\ \tau)$ be $\mathcal{T}$-tree diagrams, such that f_1 and f_2 represent an elementary expansion and contraction, respectively, with $(R,\ S,\ \sigma)f_1f_2 = (U,\ V,\ \tau)$. Let $i \in \{1,\ \ldots,\ |\mathcal{L}(R)|\}$ and $j \in \{1,\ \ldots,\ |\mathcal{L}(R)|+1\}$, be the positions in $(R,\ S,\ \sigma)$ and $(R,\ S,\ \sigma)f_1$, respectively, where f_1 and f_2 are applied. If $j \neq i,\ i+1$, then $(R,\ S,\ \sigma)f_3f_4 = (U,\ V,\ \tau)$, where f_3 and f_4 denote an elementary contraction and an expansion applied in the jth and $(i-1)$th positions, respectively, if $j < i$, and in the $(j-1)$th and ith positions if $j > i$.*

Proof We will consider two cases: when $j < i$ and when $j > i$:

Case 1: $j < i$.
By applying f_1f_2, we expand the ith leaf of R and the $i\sigma$th leaf of S, followed by contracting the jth leaf of Rf_1 and the $j\sigma f_1$th leaf of Sf_1, which are the jth and $j\sigma$th leaves of R and S.

By applying f_3f_4, we contract the jth and $j\sigma$th leaves of R and S, respectively, followed by expanding the $(i-1)$th and $(i-1)\sigma f_3$th leaves of Rf_3 and Sf_3. These are the ith and $i\sigma$th leaves of R and S.

Therefore, it suffices to show that resultant permutations are the same. Let

$$\rho = \sigma f_1,\ \tau = \sigma f_1f_2,\ \theta = \sigma f_3,\ \mu = \sigma f_3f_4.$$

Let $k \in \{1,\ \ldots,\ |\mathcal{L}(R)|\}$. We will consider two subcases:

Subcase 1.A: $j\sigma < i\sigma$.
Note that $i-1 \geq j$, and $i \neq j+1$ imply $i-1 > j$. In addition $((i-1)+1)\sigma - 1 = (i-1)\theta$. If $k < j < i$ and $k\sigma < j\sigma < i\sigma$, then $k\rho = k\sigma < j\sigma = j\rho$ and $k\theta = k\sigma \leq j\sigma - 1 < i\sigma - 1 = (i-1)\theta$. We also have that $k \leq j-1 < i+1$. Therefore

$$k\tau = k\rho = k\sigma = k\theta = k\mu.$$

If $k < j < i$ and $j\sigma < k\sigma < i\sigma$, then $k\rho = k\sigma > j\sigma = j\rho$, $k\theta = k\sigma < i\sigma - 1 = (i-1)\theta$, since $k \neq i$, and θ is a permutation. Also $k \leq j-1 < i+1$. So

$$k\tau = k\rho - 1 = k\sigma - 1 = k\theta = k\mu.$$

If $k < j < i$ and $j\sigma < i\sigma < k\sigma$, then $k\rho = k\sigma > j\sigma = j\rho$ and $k\theta = k\sigma > i\sigma - 1 = (i-1)\theta$. In addition, $k \leq j-1 < i+1$. This implies that

$$k\tau = k\rho - 1 = k\sigma = k\theta + 1 = k\mu.$$

If $j < k < i$ and $(k+1)\sigma < j\sigma < i\sigma$, then $(k+1)\rho = (k+1)\sigma < j\sigma = j\rho$, since $k+1 = i$ implies $(k+1)\rho = (k+1)\sigma$. In addition $k\theta = (k+1)\sigma \leq j$
$sigma - 1 < i\sigma - 1 = (i-1)\theta$, and if $k = i-1$, then $k\theta = k\mu$. It follows that

$$k\tau = (k+1)\rho = (k+1)\sigma = k\theta = k\mu.$$

If $j < k < i$ and $j\sigma < (k+1)\sigma < i\sigma$, then $(k+1)\rho = (k+1)\sigma > j\sigma = j\rho$, since $k+1 = i$ implies $(k+1)\rho = (k+1)\sigma$. Also, $k\theta = (k+1)\sigma < i\sigma - 1 = (i-1)\theta$, since $k \neq i$, and θ is a permutation. Also, if $k = i-1$, then $k\theta = k\mu$. Hence

$$k\tau = (k+1)\rho - 1 = (k+1)\sigma - 1 = k\theta = k\mu.$$

If $j < k < i-1$ and $j\sigma < i\sigma < (k+1)\sigma$, then $(k+1)\rho = (k+1)\sigma > j\sigma = j\rho$. We also have that $k\theta = (k+1)\sigma > i\sigma - 1 = (i-1)\theta$. Therefore

$$k\tau = (k+1)\rho - 1 = (k+1)\sigma = k\theta + 1 = k\theta.$$

If $j < k = i-1$ and $j\sigma < i\sigma < (k+1)\sigma$, then $(k+1)\rho = (k+1)\sigma > j\sigma = j\rho$. So

$$k\tau = (k+1)\rho - 1 = i\rho - 1 = i\sigma - 1 = (i-1)\theta = (i-1)\mu = k\mu.$$

If $j < i < k$ and $k\sigma < j\sigma < i\sigma$, then $(k+1)\rho = k\sigma < j\sigma$ and $(k-1)\theta = k\sigma \leq j\sigma - 1 < i\sigma - 1 = (i-1)\theta$. As a result,

$$k\tau = (k+1)\rho = k\sigma = (k-1)\theta = k\mu.$$

If $j < i < k$ and $j\sigma < k\sigma < i\sigma$, then $(k+1)\rho = k\sigma > j\sigma$ and $(k-1)\theta = k\sigma \leq i\sigma - 1 = (i-1)\theta$. If this inequality were strict, it would follow that $i < k = i-1$, a contradiction. Therefore,

$$k\tau = (k+1)\rho - 1 = k\sigma - 1 = (k-1)\theta = k\mu.$$

If $j < i < k$ and $j\sigma < k\sigma < i\sigma$, then $(k+1)\rho = k\sigma > j\sigma$ and $(k-1)\theta = k\sigma > i\sigma - 1 = (i-1)\theta$. Hence

$$k\tau = (k+1)\rho - 1 = k\sigma = (k-1)\theta + 1 = k\mu.$$

We have

$$j\tau = j\rho = j\sigma = j\theta = j\mu.$$

Since τ and μ are permutations of a finite set, agreeing on at least all but one point in their domain, they are equal.

Subcase 1.B: $j\sigma > i\sigma$.
Let ϕ be any permutation of the domain of σ, such that $j\sigma\phi < i\sigma\phi$, and $j\phi < i\phi$. Applying Case 1.A to $\sigma\phi$, and ϕ, gives $(\sigma\phi)f_1f_2 = (\sigma\phi)f_3f_4$, and $\phi f_1f_2 = \phi f_3f_4$. By Lemma 4.9, we have $(\sigma f_1f_2)(\phi f_1f_2) = (\sigma f_3f_4)(\phi f_3f_4)$, and hence $\sigma f_1f_2 = \sigma f_3f_4$.

Case 2: $j > i$. Let $m = j - 1$, and $n = i$. We have that f_3 is an elementary contraction in the mth position, and f_4 is an elementary expansion in the nth position. In addition, f_1 is an elementary expansion in the nth position, and f_4 is an elementary expansion in the $(m + 1)$th position. We have that $j \geq i + 1$, and $j \neq i + 1$, so $j > i + 1$, and hence $m = j - 1 > i = n$. The result follows now by Case 1. □

Corollary 4.15 *Let $(R,\ S,\ \sigma)$ and $(U,\ V,\ \tau)$ be $\mathcal{T}$-tree diagrams, such that one of f_1 and f_2 represent an elementary contraction and expansion, respectively, with $(R,\ S,\ \sigma)f_1f_2 = (U,\ V,\ \tau)$. Let $i \in \{1,\ \ldots,\ |\mathcal{L}(R)|\}$ and $j \in \{1,\ \ldots,\ |\mathcal{L}(R)|-1\}$, be the positions in $(R,\ S,\ \sigma)$ and $(R,\ S,\ \sigma)f_1$, respectively, where f_1 and f_2 are applied. If $j \neq i,\ i+1$, then $(R,\ S,\ \sigma)f_3f_4 = (U,\ V,\ \tau)$, where f_3 and f_4 denote an elementary expansion and a contraction applied in the jth and $(i-1)$th positions, respectively, if $j < i$, and in the $(j-1)$th and ith positions if $j > i$.*

Proof If we apply the sequence in reverse, using Lemma 4.3, then we satisfy the conditions of Lemma 4.14. □

Lemma 4.16 *Let $(R,\ S,\ \sigma)$ and $(U,\ V,\ \tau)$ be $\mathcal{T}$-tree diagrams such that $(R,\ S,\ \sigma) \sim (U,\ V,\ \tau)$. Then the following are equivalent:*

1. *$(R,\ S,\ \sigma) = (U,\ V,\ \tau)$,*
2. *$R = U$,*
3. *$S = V$.*

Proof (1) $\Rightarrow$ (2): If $(R,\ S,\ \sigma) = (U,\ V,\ \tau)$, then all three parts are equal, so in particular $R = U$.

(2) $\Leftrightarrow$ (3): We have that there is a finite sequence of elementary contractions and expansions that takes $(R,\ S,\ \sigma)$ to $(U,\ V,\ \tau)$. Then $U = R$ if and only if this sequence preserves R, which happens exactly when every caret it adds to R, it removes from R. Since, for every caret it adds to R, and later removes from R, it will do the same to S, this sequence will preserve R precisely when it preserves S, which happens if and only if $V = S$.

(3) $\Rightarrow$ (1): Suppose $S = V$. The argument above proves $R = U$. If the sequence has length 0, then $(R,\ S,\ \sigma) = (U,\ V,\ \tau)$, so, in particular, $\sigma = \tau$. Let $n \in \mathbb{N}$. Inductively suppose $\sigma = \tau$, if the sequence taking $(R,\ S,\ \sigma)$ to $(U,\ V,\ \tau)$ has length less than n. If the length is n, we will consider two cases.

Case 1: There are at least two leaves in $\mathcal{L}(R)$, to which an elementary contraction or expansion is applied. Note that here we are talking about the original set of leaves, so some of these may not be leaves after elementary expansions or contractions are applied.

By Lemma 4.11, Corollary 4.12, Lemma 4.14 and Corollary 4.15, we can reorder this sequence, such that we apply all contractions to one leaf, and subsequent 'new' leaves appearing from it, before we do any to the other. Let f_1 and f_2 be the two sequences of elementary contractions and expansions on the two leaves. Note that f_1 and f_2 must both take R to itself, since neither contracts or expands any leaves in $\mathcal{L}(R)$, except the two leaves, which are distinct, and $f_1 f_2$ preserves R. In addition, $|f_1|,\ |f_2| < n$, so by induction, they both preserve σ. Hence $\sigma = \tau$.

Case 2: Only one leaf in $\mathcal{L}(R)$ is ever expanded or contracted. We will call this leaf I. Since we begin and end at R, this leaf must be the point of expansion or contraction, of the first elementary operation, say f, and the last operation, say l. We will now consider two subcases:

Subcase 2.A: Precisely one of f and l is an elementary contraction.

Let $(R',\ S',\ \sigma') = (R,\ S,\ \sigma)f$, and $(U',\ V',\ \tau')$ be a $\mathcal{T}$-tree diagram, such that $(U',\ V',\ \tau')l = (U,\ V,\ \tau)$. We have that $U' = R'$, and $V' = S'$. In addition, we have a sequence of elementary contractions and expansions of length $n-2$, and hence $\sigma' = \tau'$, by induction. We now have an elementary contraction or expansion, followed by the other, applied on or to the same point. By Lemma 4.3, we have that $(R,\ S,\ \sigma)fl = (R,\ S,\ \sigma)$, and hence $\sigma = \tau$.

Subcase 2.B: Both of f and l are elementary contractions, or both are elementary expansions.

The only way this case can happen, is if the sequence takes R back to itself, then adds or removes a caret, and undoes the penultimate action. Therefore, we have a sequence of length $n-2$, taking $(R,\ S,\ \sigma)$ to $(U,\ V,\ \tau)$, and by induction $\tau = \sigma$. $\square$

Definition 4.17 We will define a partial multiplication on the set of $\mathcal{T}$-tree diagram classes as follows. Let $[(R,\ S,\ \sigma)]$ and $[(U,\ V,\ \tau)]$ be $\mathcal{T}$-tree diagram classes. If there exist elements $(R',\ S',\ \sigma') \in [(R,\ S,\ \sigma)]$ and $(U',\ V',\ \tau') \in [(U,\ V,\ \tau)]$, such that $S' = U'$, then define

$$[(R,\ S,\ \sigma)] \cdot [(U,\ V,\ \tau)] = [(R',\ V',\ \sigma'\tau')].$$

Lemma 4.18 *The multiplication in Definition 4.17 is well-defined. Moreover, if X and Y are $\mathcal{T}$-tree diagram classes, then XY exists.*

Proof Let X and Y be $\mathcal{T}$-tree diagram classes, $(R,\ S,\ \sigma) \in X$, and $(U,\ V,\ \tau) \in Y$. We will show XY exists. Let W be $\mathcal{T}$-tree with vertex set $V(S) \cup V(U)$. Note that as it is an ordered rooted binary subtree of $\mathcal{T}$, and contains all vertices of S and U, these trees will be ordered rooted binary subtrees of W. Therefore, W can be obtained from S and U by adjoining sequences of carets. Let $(\tilde{R},\ W,\ \tilde{\sigma})$ and $(W,\ \tilde{V},\ \tilde{\tau})$ be the $\mathcal{T}$-tree diagrams obtained by adjoining these carets to S and U in the form of elementary expansions. Therefore XY exists.

Now let $(R,\ S,\ \sigma),\ (U,\ V,\ \tau) \in X$ and $(S,\ W,\ \rho),\ (V,\ Z,\ \theta) \in Y$. We will show

$$[(R,\ S,\ \sigma)][(S,\ W,\ \rho)] = [(U,\ V,\ \tau)][(V,\ Z,\ \theta)].$$

We have

$$[(R,\ S,\ \sigma)][(S,\ W,\ \rho)] = [(R,\ W,\ \sigma\rho)]$$
$$[(U,\ V,\ \tau)][(V,\ Z,\ \theta)] = [(U,\ Z,\ \tau\theta)]$$

Since $(R,\ S,\ \sigma) \sim (U,\ V,\ \tau)$, there is a finite sequence of elementary operations that takes $(R,\ S,\ \sigma)$ to $(U,\ V,\ \tau)$. This sequence will take $(S,\ W,\ \rho)$ to an element of $[(S,\ W,\ \rho)]$ with domain tree V. Note that $(V,\ Z,\ \theta)$ is such an element. By Lemma 4.16, this element must be $(V,\ Z,\ \theta)$.

Therefore, applying this same sequence will take $(R,\ W,\ \sigma\rho)$ to $(U,\ Z,\ \tau\theta)$, using Lemma 4.9, and hence $[(R,\ W,\ \sigma\rho)] = [(U,\ Z,\ \tau\theta)]$. □

5 The Tree Representation of F

Theorem 5.1 *The set of $\mathcal{T}$-tree diagram classes, whose permutations are the identity, forms a group, and is isomorphic to F.*

Proof Let F_1 denote the magma $(X, \circ)$, where X is the set of $\mathcal{T}$-tree diagram classes, and $\circ$ represents composition of functions. Since bijections that preserve multiplications preserve the group axioms, it suffices to find a bijection that preserves multiplication, from F to F_1.

Let $f \in F$. Let P be a standard dyadic partition of $[0,\ 1]$, such that f is linear on every interval as stated in Lemma 2.8. We have that Pf is also a standard dyadic partition of $[0,\ 1]$. Let $R_{P,f}$ and $S_{P,f}$ be the $\mathcal{T}$-trees whose leaves are the intervals of P and Pf respectively. Let $\sigma_{P,f}$ be the identity of S_n where n is the number of intervals of P (and Pf). Note that, since P is not necessarily unique, these symbols are not necessarily unique either.

$$\begin{aligned}\phi\colon F &\to F_1\\ f &\mapsto [(R_{P,f},\ S_{P,f},\ \sigma_{P,f})]\end{aligned}$$

We will first show that ϕ is well-defined. Let $f \in F$, and P and Q be standard dyadic partitions of $[0,\ 1]$, such that F is linear on P and Q. Since $\sigma_{P,f}$ and $\sigma_{Q,f}$ are the identity permutations on their respective sets, we do not need to consider the effect of elementary expansions and contractions on the permutation in the tuples. Let D be any standard dyadic partition of $[0,\ 1]$ such that $P \cup Q \subseteq D$. We have that f is linear on D.

Let $E \in \{Q,\ P\}$. Since $E \subseteq D$, we can construct D from E, by replacing intervals in E with their right and left halves. This equates to adding a sequence of carets to $R_{E,f}$ to obtain $R_{D,f}$.

Let K be a standard dyadic partition such that $E \subseteq K$, and K can be obtained from E by replacing one interval I in E with its left and right halves I_L and I_R, respectively. Every interval of K that is an interval of E will be mapped to an element of Ef. The two remaining ones I_L and I_R, will be mapped into If, and their images will be standard dyadic intervals. Since If is a standard dyadic interval, and the only way to partition a standard dyadic interval into two standard dyadic intervals is to use the left and right halves, we have that I_Lf and I_Rf are the left and right halves of I, respectively. We can therefore conclude that Df can be obtained from Ef by taking left and right halves of the intervals in the same positions that were taken when obtaining D from E.

We now have a sequence of elementary expansions that take the $\mathcal{T}$-tree whose set of leaves is E to the $\mathcal{T}$-tree whose set of leaves is D, by adding carets to the leaves, that were the intervals which were replaced by their left and right halves. We can conclude that ϕ is well-defined.

Injectivity: Let $f,\ g \in F$ such that $f\phi = g\phi$. Then there exists standard dyadic partitions P and Q

of $[0,\ 1]$, such that f is linear on P and g is linear on Q, and $(R_{P,f},\ S_{P,f},\ \sigma_{P,f}) = (R_{Q,g},\ S_{Q,g},\ \sigma_{Q,g})$. Then $P = Q$, and f and g are both linear on P, and both map it to Pf. By Lemma 2.9, $f = g$.

Surjectivity. Let $(R,\ S,\ \sigma)$ be a $\mathcal{T}$-tree diagram, where σ is the identity permutation. Let P and Q be the standard dyadic partitions, whose intervals are the leaves of R and S, respectively. By Lemma 2.9, there is an element f in F, such that $Pf = Q$. Hence $f\phi = [(R_{P,f},\ S_{P,f},\ \sigma_{P,f})]$.

Homomorphism: Let $f,\ g \in F$. Let P be a standard dyadic partition such that f and g are linear on the intervals of P and Pf, respectively. The leaves of $S_{P,f}$ are the intervals of Pf, as are the leaves of $R_{Pf,g}$, and it follows that $R_{Pf,g} = S_{P,f}$. In addition, fg is linear on P, and takes it to Pfg, so the domain tree of fg with respect to P is equal to $R_{P,f}$, and the range tree is equal to $S_{Pf,g}$. Therefore

$$\begin{aligned}(f\phi)(g\phi) &= [(R_{P,f},\ S_{P,f},\ \sigma_{P,f})][(R_{Pf,g},\ S_{Pf,g},\ \sigma_{Pf,g})]\\ &= [(R_{P,f},\ S_{Pf,g},\ \sigma_{P,fg})]\\ &= (fg)\phi.\end{aligned}$$

□

Lemma 5.2 *Let $(R,\ S,\ \sigma)$ be a $\mathcal{T}$-tree diagram. There exists a unique element X of the equivalence class that $(R,\ S,\ \sigma)$ lies in, with respect to $\sim$ from Definition 4.4, that is reduced.*

Proof Existence: Let $n \in \mathbb{N}$ be the number of leaves of R. Applying an elementary contraction results in a $\mathcal{T}$-tree diagram with one fewer leaf. Therefore, a finite number of elementary contractions will either result in a $\mathcal{T}$-tree diagram which is reduced, or a $\mathcal{T}$-tree diagram with one leaf. Since a $\mathcal{T}$-tree diagram cannot have no leaves as it is an ordered rooted binary tree, we have that this $\mathcal{T}$-tree diagram is reduced.

Uniqueness: Let $(R,\ S,\ \sigma)$ be a reduced $\mathcal{T}$-tree diagram. Let $f \in F$ be the element associated with $(R,\ S,\ \rho)$, where ρ is the identity permutation on the domain of σ. Let I be a standard dyadic interval such that $I \in \mathcal{L}(R)$ or $I \notin V(R)$. If $I \in \mathcal{L}(R)$, then f is linear on I. If $I \notin V(R)$, then there is a standard dyadic interval $J \in \mathcal{L}(R)$, such that $I \subseteq J$. This is true since the leaves of R form the intervals of a partition of $[0,\ 1]$, and there is a unique path from any vertex in $\mathcal{T}$ to the root. We can conclude that f is linear on I.

Conversely, suppose that I is a standard dyadic interval such that f is linear on I. Since $(R,\ S,\ \rho)$ is reduced, if $J \in V(R)\backslash\mathcal{L}(R)$, and f is linear on J, then all vertices whose path to J does not go through the root, can be removed by elementary contractions. This contradicts $(R,\ S,\ \sigma)$ being reduced, since the permutation does not affect whether or not a $\mathcal{T}$-tree diagram can admit an elementary contraction. We therefore conclude that $I \in \mathcal{L}(R)$ or $I \notin V(R)$.

We now have $I \in \mathcal{L}(R)$ or $I \notin V(R)$ if and only if f is linear on I. Using this, we can construct R, and will have no choices. Therefore the domain of any reduced $\mathcal{T}$-tree diagram is unique. By Lemma 4.16, we have that $(R,\ S,\ \sigma)$ is the unique reduced $\mathcal{T}$-tree diagram in $[(R,\ S,\ \sigma)]$. □

Lemma 5.3 *Let $(R,\ S,\ \sigma)$ be a reduced $\mathcal{T}$-tree diagram. Then all elements of $[(R,\ S,\ \sigma)]$ can be obtained from $(R,\ S,\ \sigma)$ by applying a finite sequence of elementary expansions.*

Proof By Lemma 4.3, it suffices to show that there is a sequence of elementary contractions taking any element of $[(R,\ S,\ \sigma)]$ to $(R,\ S,\ \sigma)$. The proof in Lemma 5.2 of the existence of the reduced tree uses only elementary contractions, as required. □

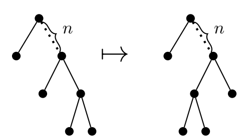
Example 5.4 We will give the reduced $\mathcal{T}$-tree diagrams in the $\mathcal{T}$-tree diagram classes representing A and B, from Example 2.3, using the isomorphism from Theorem 5.1. We have that A is linear on the intervals of the standard dyadic partition $\{0 < \frac{1}{2} < \frac{3}{4} < 1\}$, and maps it to $\{0 < \frac{1}{4} < \frac{1}{2} < 1\}$. In addition, B is linear on the standard dyadic partition $\{0 < \frac{1}{2} < \frac{3}{4} < \frac{7}{8} < 1\}$, and maps it to $\{0 < \frac{1}{2} < \frac{5}{8} < \frac{3}{4} < 1\}$. Therefore the following $\mathcal{T}$-tree diagrams are in the $\mathcal{T}$-tree class representing A and B, respectively:

These are reduced, as the domain and range trees never have a caret in the same position.

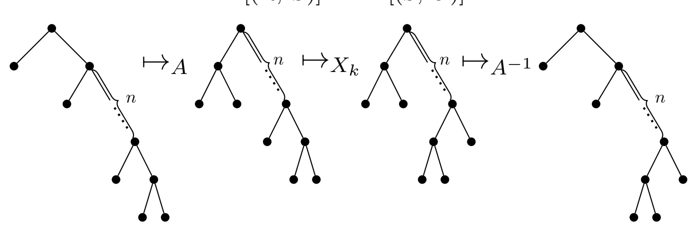
6 Presentations of F

Remark 6.1 Since the permutation part of a $\mathcal{T}$-tree diagram, in the class of an element of F is always the identity, we shall often omit the permutation part, taking it implicitly to be the identity, in the context of F.

Example 6.2 We will define new elements X_n of F for each $n \in \mathbb{N}_0$. Let $X_0 = A$, and let $X_n = B^{A^{-(n-1)}}$, for all $n \in \mathbb{N}$. Given $n \in \mathbb{N}_0$, we will show that X_n has $\mathcal{T}$-tree diagram:



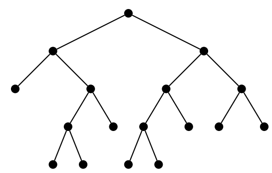
where the dots represent n edges, including the right edge from the root. If $n = 0$, then $X_n = A$, so it is true, by Example 2.3. We will proceed by induction on n. Our base case will be $n = 1$, when $X_n = B$. This is true by Example 2.3. Let $k \in \mathbb{N}$. Inductively suppose it is true for $n = k$. We will now show it is true if $n = k + 1$. We have that $X_{n+1} = AX_nA^{-1}$. Using the $\mathcal{T}$-tree class multiplication, where we will omit the class brackets, that is, where the dots will represent k edges, we will compute AX_kA^{-1}. In the following diagram, the multiplications of $\mathcal{T}$-tree diagrams $(R,\ S)$ and $(S,\ U)$, will simply be given as $R \mapsto_{[(R,\ S)]} S \mapsto_{[(S,\ U)]} U$, in order to avoid repeating trees.



Note that this is the $\mathcal{T}$-tree diagram that we hypothesised representing X_{n+1}, which we can see by observing the first $\mathcal{T}$-tree and the last $\mathcal{T}$-tree are the domain and range trees of the hypothesised representation of X_{n+1}. The result then follows by induction. Note that these $\mathcal{T}$-tree diagrams are reduced, as the domain and range trees have no carets in the same position.

Definition 6.3 Let R be a $\mathcal{T}$-tree. Let $I_0,\ \ldots,\ I_n$ be the leaves of R, in order, where $n \in \mathbb{N}$. For each $k \in \mathbb{N}_0,\ k \leq n$, define the *kth exponent* of R to be the length of the longest path in R, including I_k, which comprises entirely left edges, and never reaches a vertex in the right-side of R.

Example 6.4 Consider the following $\mathcal{T}$-tree.



The exponents of this $\mathcal{T}$-tree, in order, are

$$1,\ 2,\ 0,\ 0,\ 2,\ 0,\ 0,\ 0,\ 0.$$

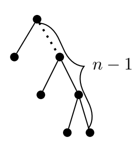
Theorem 6.5 *Let $(R,\ S)$ be a $\mathcal{T}$-tree diagram, such that σ is the identity permutation. Let $n = |\mathcal{L}(R)|$, and $a_0,\ \ldots,\ a_n$ and $b_0,\ \ldots,\ b_n$ be the exponents of R and S respectively, in order. Then the element of F, whose $\mathcal{T}$-tree class $(R,\ S)$ lies in, is*

$$X_0^{-a_0} X_1^{-a_1} \cdots X_n^{-a_n} X_n^{b_n} \cdots X_1^{b_1} X_0^{b_0}.$$

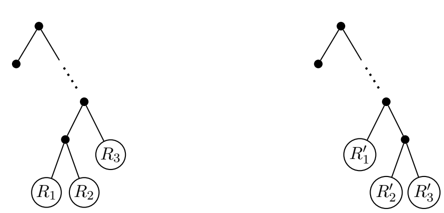
Furthermore, $(R,\ S)$ is reduced if and only if:

1. *If the last two leaves of R lie in the same caret, then the last two leaves of S do not,*
2. *If $a_k,\ b_k > 0$, for some $k \in \{0,\ \ldots,\ n\}$, then either $a_{k+1} > 0$ or $b_{k+1} > 0$ (but not both).*

Proof We will define a new $\mathcal{T}$-tree, where $n \in \mathbb{N}$, denoted $\mathcal{T}_n$, with n leaves, as the $\mathcal{T}$-tree whose exponents are all equal to 0:



Define $\mathcal{T}_0$ to be the $\mathcal{T}$-tree with one vertex. We will first show, by induction, that the element $f \in F$, that $(R,\ \mathcal{T}_n)$ lies in the $\mathcal{T}$-tree class of, is $X_0^{-a_0} X_1^{-a_1} \cdots X_n^{-a_n}$. We will induct on $a = a_0 + a_1 + \cdots + a_n$. If $a = 0$, then all exponents are equal to 0, so $R = \mathcal{T}_n$, and f is the identity function. Therefore $f = X_0^0 X_1^0 \cdots X_n^0$, as required. Let $b \in \mathbb{N}$, and inductively suppose that the result is true if $a < b$. Let i be the smallest index, such that $a_i > 0$. We can therefore conclude that the first m leaves, using the left to right ordering, are connected to the right-side of R by precisely one edge, and the $(m + 1)$th leaf is not. Therefore, R has ordered rooted binary subtrees $R_1,\ R_2$ and R_3, such that R is of the form of the left tree:



Here, the dots represent m edges, including the edge from the root. Let R' be the $\mathcal{T}$-tree, as represented by the diagram on the right, where $R'_1,\ R'_2$ and R'_3 are isomorphic, as ordered rooted binary trees, to $R_1,\ R_2$ and R_3, respectively. By Example 6.2, the element of F, whose $\mathcal{T}$-tree class contains $(R,\ R')$, is X_m^{-1}. Let $a'_1,\ \ldots,\ a'_n$ be the exponents of R'. We have that $a'_m = a_m - 1$, and $a'_k = 0 = a_k$, if $k < m$. If $k > m$ and the kth leaf is not in R_1, then $a_k = a'_k$. Otherwise, if the kth leaf is in R_1, then its maximal path of left edges cannot include the left edge emanating from R_1, on the path to the root, since then it would be on the path of left edges going to mth leaf. Therefore, its exponent will be unaffected by the mapping to R', and $a'_k = a_k$. The inductive hypothesis applies to R', as $a'_0 + \cdots a'_n = a - 1$, and so the function representing the $\mathcal{T}$-tree class of $(R',\ \mathcal{T}_n)$, is $X_0^{-a'_0} X_1^{-a'_1} \cdots X_n^{-a'_n}$. Note that

$$[(R,\ \mathcal{T}_n)] = [(R,\ R')][(R',\ \mathcal{T}_n)],$$

and hence the element of F, whose $\mathcal{T}$-tree class is $[(R,\ \mathcal{T}_n,\ \sigma)]$ is

$$\begin{aligned} X_m^{-1} X_0^{-a'_0} X_1^{-a'_1} \cdots X_n^{-a'_n} &= X_m^{-1} X_m^{-a_m+1} X_{m+1}^{-a_{m+1}} \cdots X_n^{-a_n} \\ &= X_m^{-a_m} X_{m+1}^{-a_{m+1}} \cdots X_n^{-a_n} \\ &= X_0^{-a_0} X_1^{-a_1} \cdots X_n^{-a_n}, \end{aligned}$$

as required.

We now have that the element of F whose $\mathcal{T}$-tree class is $[(R,\ \mathcal{T}_n)]$ is $X_0^{-a_0} X_1^{-a_1} \cdots X_n^{-a_n}$. Note that the element of F for $[(S,\ \mathcal{T}_n)]$ is $X_0^{-b_0} X_1^{-b_1} \cdots X_n^{-b_n}$, by our earlier inductive argument. Therefore, the element of F for $[(\mathcal{T}_n,\ S)]$ is the inverse of $X_0^{-b_0} X_1^{-b_1} \cdots X_n^{-b_n}$, which is $X_n^{b_n} \cdots X_1^{b_1} X_0^{b_0}$. Since

$$[(R,\ S)] = [(R,\ \mathcal{T}_n)][(\mathcal{T}_n,\ S)],$$

we have that the element of F, whose $\mathcal{T}$-tree class is $[(R,\ S,\ \sigma)]$ is

$$X_0^{-a_0} X_1^{-a_1} \cdots X_n^{-a_n} X_n^{b_n} \cdots X_1^{b_1} X_0^{b_0}.$$

($\Rightarrow$): Let $(R,\ S)$ be a reduced $\mathcal{T}$-tree diagram. If the last two leaves of both R and S lie in carets, then we can apply an elementary contraction, to remove these carets. This is a contradiction to the irreducibility of $(R,\ S)$, so we satisfy (1).

Let $k \in \{1,\ \ldots,\ n\}$. Suppose $a_k,\ b_k > 0$. Then the kth leaves of R and S both have a left edge emanating from them. If $a_{k+1} = b_{k+1} = 0$, then the $(k+1)$th leaves of R and S both have a right edge emanating from them. We can conclude that they lie in carets with the kth leaves, which allows an elementary contraction to be applied, a contradiction, so we satisfy (2).

($\Leftarrow$): Suppose $(R,\ S)$ is a $\mathcal{T}$-tree diagram, such that conditions (1) and (2) are satisfied. Suppose, for contradiction, that $(R,\ S)$ is reducible. Then there exists an index k, such that the kth and $(k+1)$th leaves of R and S lie in carets. Then kth leaves each have a left edge emanating from them.
It follows that $a_k,\ b_k > 0$. By (2), we have that one of a_{k+1} and b_{k+1} is non-zero. Therefore one of the $(k+1)$th leaves does not have a right edge emanating from it, so does not lie in a caret with one of the kth leaves, a contradiction. Hence $(R,\ S)$ is reduced. $\square$

Corollary 6.6 *Thompson's group F is generated by A and B.*

Proof By Theorem 6.5, we have that F is generated by $\{X_0,\ X_1,\ \ldots\}$. Since all these elements are products of As and Bs, and their inverses, it follows that $F = \langle A,\ B\rangle$. $\square$

Corollary 6.7 *Every non-trivial element of F can be expressed uniquely in the normal form*

$$X_0^{-a_0}X_1^{-a_1}\cdots X_n^{-a_n}X_n^{b_n}\cdots X_1^{b_1}X_0^{b_0},$$

where $n,\ a_0,\ \ldots,\ a_n,\ b_0,\ \ldots,\ b_n \in \mathbb{N}_0$, and

1. *Precisely one of a_n and b_n is non-zero,*
2. *If $a_k,\ b_k > 0$, for some $k \in \{0,\ \ldots,\ n\}$, then either $a_{k+1} > 0$ or $b_{k+1} > 0$.*

In addition, if an element of F can be expressed in the above normal form, then it is non-trivial.

Proof Let $f \in F$ be non-trivial. Let $(R,\ S)$ be the unique reduced $\mathcal{T}$-tree diagram in the $\mathcal{T}$-tree class of f, the existence and uniqueness of which follow from Lemma 5.2. By Theorem 6.5,

$$f = X_0^{-a_0}X_1^{-a_1}\cdots X_m^{-a_m}X_m^{b_m}\cdots X_1^{b_1}X_0^{b_0},$$

for some $m,\ a_0,\ \ldots,\ a_m,\ b_0,\ \ldots,\ b_m \in \mathbb{N}_0$, satisfying the conditions of the corollary.

Suppose now $f \in F$ can be expressed in the above normal form. If f were the identity, then its tree pair would be a single vertex ($[0,\ 1]$) taken to a single vertex. Therefore, all exponents would be zero, so the normal form would be $X_0^0X_0^0$, a contradiction to (1). $\square$

Definition 6.8 Let $f \in F$ have normal form

$$X_0^{-a_0}X_1^{-a_1}\cdots X_n^{-a_n}X_n^{b_n}\cdots X_1^{b_1}X_0^{b_0},$$

where $n,\ a_0,\ \ldots,\ a_n,\ b_0,\ \ldots,\ b_n \in \mathbb{N}_0$, as described in Corollary 6.7. If $a_0 = a_1 = \cdots = 0$, then f is called *positive.* If f is an inverse of a positive element, then f is called *negative.*

Lemma 6.9 *Positive elements of F, are precisely those whose $\mathcal{T}$-tree class contains a $\mathcal{T}$-tree class with domain tree $\mathcal{T}_n$, for some $n \in \mathbb{N}_0$. This follows from the proof of Theorem 6.5.*

Proof Let $f \in F$. We have that f is positive if and only if all exponents in the domain tree are zero. This is the case precisely when the domain tree is $\mathcal{T}_n$, for some $n \in \mathbb{N}_0$, as having any caret not on the right side, would give an exponent of 1. $\square$

Lemma 6.10 *If $f,\ g \in F$ are positive, then fg is positive.*

Proof Let $(\mathcal{T}_m,\ R)$ and $(\mathcal{T}_n,\ S)$ be $\mathcal{T}$-tree diagrams in the classes representing f and g, respectively, where $m,\ n \in \mathbb{N}_0$. Let k be the number of edges in the right side of R. By applying elementary expansions to $(\mathcal{T}_m,\ R)$, we can assume $k \geq n$, note that as these are applied to the mth leaf, they will increase m, but the domain tree will remain $\mathcal{T}_m$. In addition, by applying elementary expansions to $(\mathcal{T}_n,\ S)$, we can assume $k = n$, and similarly, the domain tree will remain $\mathcal{T}_n$.

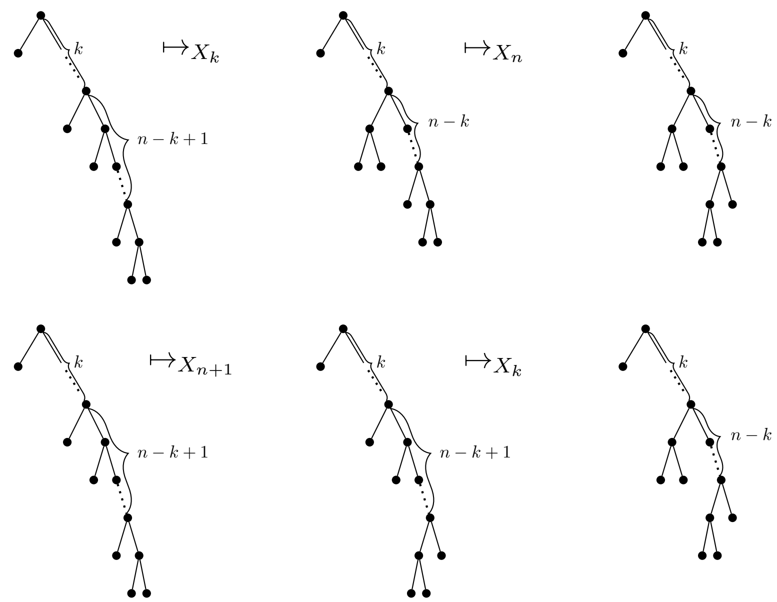
We now have that $\mathcal{T}_n$ is an ordered rooted binary subtree of R, with the same root. By applying elementary expansions to $(\mathcal{T}_n,\ S)$, we will have a $\mathcal{T}$-tree diagram in the class representing g, of the form $(R,\ Q)$, for some $\mathcal{T}$-tree Q. Hence the $\mathcal{T}$-tree diagram $(\mathcal{T}_m,\ Q)$ is in the class representing fg. By Lemma 6.9, we have that fg is positive. $\square$

Lemma 6.11 *The elements $X_0,\ X_1,\ X_2,\ \ldots \in F$ satisfy*

$$X_k X_n = X_{n+1} X_k,$$

for all $n,\ k \in \mathbb{N}_0$, such that $k < n$.

Proof We shall proceed by examining $\mathcal{T}$-tree diagrams in these functions' classes.



It follows that $X_k X_n = X_{n+1} X_k$. □

Theorem 6.12 (von Dyck) *Let X be a set and $R \subseteq W(X)$. Let $H = \langle X \mid R\rangle$ and G be a group. Then a function $\theta\colon X \to G$ extends uniquely to a group homomorphism from H to G if and only if $(x_1\theta)^{\varepsilon_1}(x_2\theta)^{\varepsilon_2}\cdots(x_n\theta)^{\varepsilon_n} = 1_G$, for all relations $r \in R$, being expressed over X, that is $r = x_1^{\varepsilon_1}\cdots x_n^{\varepsilon_n}$, $x_1,\ \ldots,\ x_n \in X$, $\varepsilon_1,\ \ldots,\ \varepsilon_n \in \{-1,\ 1\}$, and $n \in \mathbb{N}$.*

Proof For the proof of this theorem, we refer to [Bog08], Theorem 5.7. □

Theorem 6.13 *There is an isomorphism from F to*

$$\langle X_0,\ X_1,\ X_2,\ \ldots \mid \{X_{n+1}^{-1} X_k X_n X_k^{-1} \mid k,\ n \in \mathbb{N}_0,\ k < n\}\rangle,$$

that maps the functions $X_0,\ X_1,\ \ldots$ to the formal symbols of the same name.

Proof Let F_1 be the presented group. Let $X = \{X_0,\ X_1,\ \ldots\} \subseteq F_1$. Define $\phi\colon X \to F$ by mapping the formal symbols $X_0,\ X_1,\ \ldots$ to the functions in F with the same name. Lemma 6.11 shows that ϕ satisfies the conditions of von Dyck's Theorem (Theorem 6.12), and hence extends to a homomorphism $\tilde{\phi} : F_1 \to F$. By Theorem 6.5, X generates F, and it follows that $\tilde{\phi}$ is surjective.

We have from the relations in F_1, that for any $k,\ n \in \mathbb{N}_0$, with $k < n$,

$$X_n X_k^{-1} =_{F_1} X_k^{-1} X_{n+1}, \quad X_n^{-1} X_k^{-1} =_{F_1} X_k^{-1} X_{n+1}^{-1}.$$

We will proceed by induction on the length of a word, to show that every non-trivial element of F_1, given as an element of $W(X)$, can be expressed in the normal form mentioned in Corollary 6.7. If the word has length 1, then the word is already in its normal form. Let $k \in \mathbb{N}$, and suppose all words of length less than k can be put into the normal form. Let $w \in W(X)$ have length k. We have that $w = uv$, where $u \in X \cup X^{-1}$ and $v \in W(X)$, where $X^{-1} = \{X_0^{-1},\ X_1^{-1},\ \ldots\}$. Since $|v| < k$, by the inductive hypothesis, we have that v can be written as

$$v = X_0^{-a_0} X_1^{-a_1} \cdots X_n^{-a_n} X_n^{b_n} \cdots X_1^{b_1} X_0^{b_0},$$

where $n,\ a_0,\ \ldots,\ a_n,\ b_0,\ \ldots,\ b_n \in \mathbb{N}_0$ satisfy the conditions of Corollary 6.7.

Case 1: $u \in X$.
Let $l \in \mathbb{N}_0$ be such that $u = X_l$. We have

$$\begin{aligned} w &= X_l X_0^{-a_0} X_1^{-a_1} \cdots X_n^{-a_n} X_n^{b_n} X_{n-1}^{b_{n-1}} \cdots X_1^{b_1} X_0^{b_0} \\ &= X_0^{-a_0} X_{l+1} X_1^{-a_1} \cdots X_n^{-a_n} X_n^{b_n} X_{n-1}^{b_{n-1}} \cdots X_1^{b_1} X_0^{b_0} \\ &= \cdots = X_0^{-a_0} X_1^{-a_1} \cdots X_n^{-a_n} X_{l+n} X_n^{b_n} X_{n-1}^{b_{n-1}} \cdots X_1^{b_1} X_0^{b_0} \end{aligned}$$

We have that the indices of this expression, smaller than or equal to n, satisfy the second condition of Corollary 6.7. The indices strictly greater than $n+1$ and less than $l+n$ satisfy the condition as they are all equal to 0. The index of $n+1$ satisfies the conditions, as precisely one of a_n and b_n is non-zero, by the first condition of Corollary 6.7, so the second condition does not apply. The exponent of X_{l+n}^{-1} is 0 and of X_{l+n} is 1, therefore both conditions of Corollary 6.7 are satisfied, and this is a normal form.

Case 2: $u \in X^{-1}$.
Let $l \in \mathbb{N}_0$ be such that $u = X_l^{-1}$. It follows that

$$\begin{aligned} w &= X_l^{-1} X_0^{-a_0} X_1^{-a_1} \cdots X_n^{-a_n} X_n^{b_n} X_{n-1}^{b_{n-1}} \cdots X_1^{b_1} X_0^{b_0} \\ &= X_0^{-1} X_{l+1}^{-1} X_1^{-a_1} \cdots X_n^{-a_n} X_n^{b_n} X_{n-1}^{b_{n-1}} \cdots X_1^{b_1} X_0^{b_0} \\ &= \cdots \\ &= X_0^{-a_0} X_1^{-a_1} \cdots X_n^{-a_n} X_{l+n}^{-1} X_n^{b_n} X_{n-1}^{b_{n-1}} \cdots X_1^{b_1} X_0^{b_0} \end{aligned}$$

The second condition of Corollary 6.7 holds, by the same argument used in Case 1. In addition, the exponent of X_{n+l} is 0, and X_{n+l}^{-1} is 1, so we satisfy the first condition of Corollary 6.7.

We now have that every non-trivial element of F_1 can be expressed in the normal form, which maps to an element of F, represented in the normal form, under $\tilde{\phi}$, so the image is non-trivial. Therefore $\ker \tilde{\phi} = \{\varepsilon\}$, and $\tilde{\phi}$ is injective. $\square$

Theorem 6.14 *There is an isomorphism from F to*

$$\langle A,\ B \mid [B^{-1}A,\ ABA^{-1}],\ [B^{-1}A,\ A^2BA^{-2}] \rangle,$$

which maps the functions A and B to the formal symbols A and B.

Proof By Theorem 6.13, there is an isomorphism from F to the presentation

$$\langle X_0,\ X_1,\ X_2,\ \ldots \mid \{X_{n+1}^{-1}X_kX_nX_k^{-1} \mid k,\ n \in \mathbb{N}_0,\ k < n\}\rangle,$$

which we will call F_1. We will call the presentation mentioned in this theorem F_2. It suffices to show that there is an isomorphism from F_2 to F_1, that maps A to X_0, and B to X_1, as we can compose the inverse of this with the isomorphism from Theorem 6.13 to yield the isomorphism we want.

Define

$$\begin{aligned} \theta\colon \{A,\ B\} &\to F_1 \\ A &\mapsto X_0 \\ B &\mapsto X_1 \end{aligned}$$

Let $\tilde{\theta}$ be the linear extension of θ. From the presentation for F_1, we have

$$X_n^{-1}X_k^{-1} =_{F_1} X_k^{-1}X_{n+1}^{-1}, \quad X_kX_n = X_{n+1}X_k,$$

for all $k,\ n \in \mathbb{N}_0$, such that $k < n$. Then

$$\begin{aligned}
[B^{-1}A,\ ABA^{-1}]\tilde{\theta} &=_{F_1} (A^{-1}BAB^{-1}A^{-1}B^{-1}AABA^{-1})\tilde{\theta} \\
&=_{F_1} (A\theta)^{-1}(B\theta)(A\theta)(B\theta)^{-1}(A\theta)^{-1}(B\theta)^{-1}(A\theta)(A\theta)(B\theta)(A\theta)^{-1} \\
&=_{F_1} X_0^{-1}X_1X_0X_1^{-1}X_0^{-1}X_1^{-1}X_0X_0X_1X_0^{-1} \\
&=_{F_1} X_0^{-1}X_1X_0X_0^{-1}X_2^{-1}X_1^{-1}X_0X_2X_0X_0^{-1} \\
&=_{F_1} X_0^{-1}X_1X_2^{-1}X_1^{-1}X_0X_2 \\
&=_{F_1} X_0^{-1}X_1X_1^{-1}X_3^{-1}X_3X_0 \\
&=_{F_1} 1_{F_1} \\
[B^{-1}A,\ A^2BA^{-2}]\tilde{\theta} &=_{F_1} (A^{-1}BA^2B^{-1}A^2B^{-1}AA^2BA^{-2})\tilde{\theta} \\
&=_{F_1} (A\theta)^{-1}(B\theta)(A\theta)^2(B\theta)^{-1}(A\theta)^{-2}(B\theta)^{-1}(A\theta)(A\theta)^2(B\theta)(A\theta)^{-2} \\
&=_{F_1} X_0^{-1}X_1X_0X_0X_1^{-1}X_0^{-1}X_0^{-1}X_1^{-1}X_0X_0X_0X_1X_0^{-1}X_0^{-1} \\
&=_{F_1} X_0^{-1}X_1X_0X_0X_0^{-1}X_2^{-1}X_0^{-1}X_1^{-1}X_0X_0X_2X_0X_0^{-1}X_0^{-1} \\
&=_{F_1} X_0^{-1}X_1X_0X_2^{-1}X_0^{-1}X_1^{-1}X_0X_0X_2X_0^{-1} \\
&=_{F_1} X_0^{-1}X_1X_0X_0^{-1}X_3^{-1}X_1^{-1}X_0X_3X_0X_0^{-1} \\
&=_{F_1} X_0^{-1}X_1X_3^{-1}X_1^{-1}X_0X_3 \\
&=_{F_1} X_0^{-1}X_1X_1^{-1}X_4^{-1}X_4X_0 \\
&=_{F_1} 1_{F_1}.
\end{aligned}$$

It follows by von Dyck's Theorem (Theorem 6.12), that $\tilde{\theta}$ is a homomorphism. We have that $\{X_0,\ X_1\}$ generates F, by Corollary 6.6, and hence, using Theorem 6.13, the symbols X_0 and X_1, generate F_1. It follows that $\tilde{\theta}$ is surjective, as it maps onto a generating set.

It now suffices to find a surjective homomorphism from F_1 to F_2, that maps X_0 to A, and X_1 to B, since this will be the inverse of $\tilde{\theta}$, which will imply $\tilde{\theta}$ is a bijection. Let $Y_0 = A$ and $Y_n = A^{n-1}BA^{-(n-1)}$, for any $n \in \mathbb{N}$. We will consider, for $m \in \mathbb{N}$, and $m \geq 3$, the statements

$$[BA^{-1},\ Y_m] =_{F_2} 1_{F_2} \tag{1}$$
$$Y_k Y_n =_{F_2} Y_{n+1} Y_k, \tag{2}$$

where $k,\ n \in \mathbb{N}_0$, and $k < n$. Suppose $m \in \mathbb{N},\ m \geq 3$ such that (1) is true at m. Let $k,\ n \in \mathbb{N}_0$, with $k < n$ and $m = n - k + 2$. Then

$$\begin{aligned}
Y_k Y_n &= A^{k-1}BA^{-k+1}A^{n-1}BA^{-n+1} \\
&=_{F_2} A^{k-1}BA^{-1}A^{n-k+1}BA^{-(n-k+1)}A^{-k+2} \\
&=_{F_2} A^{k-1}BA^{-1}Y_m A^{-k+2} \\
&=_{F_2} A^{k-1}Y_m BA^{-k+1} \\
&=_{F_2} A^{k-1}A^{n-k+1}BA^{-(n-k+1)}BA^{-k+1} \\
&=_{F_2} A^n BA^{-n}A^{k-1}BA^{-(k-1)} \\
&=_{F_2} Y_{n+1}Y_k \quad (*)
\end{aligned}$$

We will now proceed by induction on m to show (1) and (2), where $n - k + 2 = m$. Our base cases will be $m = 3$ and $m = 4$. The first statement is true in these cases, since

$$\begin{aligned}
[B^{-1}A,\ ABA^{-1}] =_{F_2} 1_{F_2} &\implies A[B^{-1}A,\ ABA^{-1}]A^{-1} =_{F_2} 1_{F_2} \\
&\implies [AB^{-1},\ A^2BA^{-2}] =_{F_2} 1_{F_2} \\
&\implies [BA^{-1},\ A^2BA^{-2}] =_{F_2} 1_{F_2} \\
&\implies [BA^{-1},\ Y_3] =_{F_2} 1_{F_2} \\
[B^{-1}A,\ A^2BA^{-2}] =_{F_2} 1_{F_2} &\implies A[B^{-1}A,\ A^2BA^{-2}]A^{-1} =_{F_2} 1_{F_2} \\
&\implies [AB^{-1},\ A^3BA^{-3}] =_{F_2} 1_{F_2} \\
&\implies [BA^{-1},\ A^3BA^{-3}] =_{F_2} 1_{F_2} \\
&\implies [BA^{-1},\ Y_4] =_{F_2} 1_{F_2}.
\end{aligned}$$

The second statement is true in these cases as well, by $(*)$. Let $l \in \mathbb{N}$, with $l > 3$, and inductively suppose (1) and (2) are true if $m = l - 1$ and $m = l - 2$. Then if $m = l$, $n = l - 1$ and $k = 3$ imply that $k < n$, and $n - k + 2 = l - 2$, and hence $Y_k Y_n Y_k^{-1} = Y_{n+1} = Y_l$, by induction. Our inductive hypothesis also gives us that $[BA^{-1},\ Y_k] =_{F_2} 1_{F_2} = [BA^{-1},\ Y_n]$. Therefore

$$\begin{aligned}
[BA^{-1},\ Y_l] &=_{F_2} [BA^{-1},\ Y_k Y_n Y_k^{-1}] \\
&=_{F_2} Y_k[Y_k^{-1}BA^{-1}Y_k,\ Y_n]Y_k^{-1}
\end{aligned}$$

$$
\begin{aligned}
&=_{F_2} Y_k[Y_k^{-1}Y_kBA^{-1},\ Y_n]Y_k^{-1} \\
&=_{F_2} Y_k[BA^{-1},\ Y_n]Y_k^{-1} \\
&=_{F_2} Y_kY_k^{-1} \\
&=_{F_2} 1_{F_2}.
\end{aligned}
$$

Hence the function

$$
\begin{aligned}
\phi\colon \{X_n \mid n \in \mathbb{N}_0\} &\to F_2 \\
Y_n &\mapsto X_n,
\end{aligned}
$$

extends to a homomorphism $\tilde{\phi}$, by von Dyck's Theorem (Theorem 6.12). Since ϕ is onto the generators of F_1, we have that $\tilde{\phi}$ is surjective. In addition, note that $A\tilde{\theta}\tilde{\phi} = A$, and $B\tilde{\theta}\tilde{\phi} = B$, so any product over $\{A,\ B\}$ will be fixed by $\tilde{\theta}\tilde{\phi}$. Hence $\tilde{\theta}\tilde{\phi}$ is the identity function. In addition, if $n \in \mathbb{N}_0$, we have that $X_n\tilde{\phi}\tilde{\theta} = Y_n\tilde{\theta} = X_n$, so $\tilde{\phi}\tilde{\theta}$ is also the identity function. Hence $\tilde{\theta}$ and $\tilde{\phi}$ are each other's inverses, and hence bijections. □

7 The Commutator Subgroup, and Quotients of F

Definition 7.1 Let G be a group, and X be a generating set for G. We call X *irredundant* if $\langle X\backslash\{x\}\rangle \neq G$, for all $x \in X$.

Lemma 7.2 *Let G be a group with an irredundant generating set X. If an element $g \in G$, when represented as a word $w_1^{\varepsilon_1}w_2^{\varepsilon_2}\cdots w_n^{\varepsilon_n}$, over $X \cup \{x^{-1} \mid x \in X\}$, where $n \in \mathbb{N}$, $\varepsilon_i \in \mathbb{Z}$, and $w_i \in X$, for all valid i, satisfies*

$$
\sum_{\substack{i\in\{1,\ \ldots,\ n\} \\ w_i=x}} \varepsilon_i = 0,
$$

for all $x \in X$, then $g \in [G,\ G]$.

Proof Let $k = |X|$, and $X = \{x_1,\ x_2,\ \ldots,\ x_k\}$. Note that $[G,\ G] \trianglelefteq G$. Therefore, the quotient homomorphism $\phi\colon G \mapsto {G}/{[G,\ G]}$ has kernel $[G,\ G]$. Note also that $\operatorname{im}\phi$ is abelian. Let $g \in G$ be represented by a word $w_1^{\varepsilon_1}w_2^{\varepsilon_2}\cdots w_n^{\varepsilon_n}$, as in the statement of the lemma. Then

$$
\begin{aligned}
(w_1^{\varepsilon_1}w_2^{\varepsilon_2}\cdots w_n^{\varepsilon_n})\phi &= (w_1\phi)^{\varepsilon_1}(w_2\phi)^{\varepsilon_2}\cdots(w_n\phi)^{\varepsilon_n} \\
&= (x_1\phi)^{\epsilon_1}(x_2\phi)^{\epsilon_2}\cdots(x_m\phi)^{\epsilon_m},
\end{aligned}
$$

where

$$
\epsilon_i = \sum_{\substack{j\in\{1,\ \ldots,\ n\} \\ w_j=x_i}} \varepsilon_j = 0.
$$

We can conclude that $g \in \ker\phi = [G,\ G]$. □

Lemma 7.3 *Let G be a group, generated by a finite, irredundant generating set X, such that there is an epimorphism ϕ from G to $\mathbb{Z}^{|X|}$. Then*

$$
\ker\phi = [G,\ G].
$$

Proof We will use e to denote the identity of $\mathbb{Z}^n$. Note that $\mathbb{Z}^n$ is abelian, and $\ker\phi \trianglelefteq G$, so $[G,\ G] \leq \ker\phi$.

Let $g \in \ker\phi$, and represent g as a word $w_1^{\varepsilon_1} w_2^{\varepsilon_2} \cdots w_k^{\varepsilon_k}$ over X, where $k \in \mathbb{N}$. Let $n = |X|$, and $X = \{x_1,\ x_2,\ \ldots,\ x_n\}$. Then, since $\mathbb{Z}^n$ is abelian, we have

$$e = (w_1^{\varepsilon_1} w_2^{\varepsilon_2} \cdots w_k^{\varepsilon_k})\phi = (w_1\phi)^{\varepsilon_1}(w_2\phi)^{\varepsilon_2} \cdots (w_k\phi)^{\varepsilon_k} = (x_1\phi)^{\delta_1}(x_2\phi)^{\delta_2} \cdots (x_n\phi)^{\delta_n},$$

where for all valid i,

$$\delta_i = \sum_{\substack{j \in \{1,\ \ldots,\ n\} \\ w_j = x_i}} \varepsilon_j.$$

For any valid i, let $\gamma_i \in \mathbb{N}_0$, such that $x_i\phi = a_i^{\gamma_i}$, for some generator a_i of $\mathbb{Z}^n$, using a standard generating set Y (of size n). Suppose for contradiction that $a_i = a_j$, for some valid i and j. Then $|X\phi| < n$, so $\langle X\phi \rangle = \operatorname{im}\phi$ cannot be $\mathbb{Z}^n$, as every generating set must contain at least n elements. This contradicts ϕ being surjective, and we can conclude that the a_is are distinct. We have

$$e = a_1^{\gamma_1\delta_1} \cdots a_n^{\gamma_n\delta_n}.$$

Note that this corresponds to

$$(\gamma_1\delta_1,\ \gamma_2\delta_2,\ \ldots,\ \gamma_n\delta_n) = (0,\ 0,\ \ldots,\ 0),$$

using the additive notation for $\mathbb{Z}$, together with the direct product. Hence $\gamma_i\delta_i = 0$ for all i. Let $i \in \{1,\ \ldots,\ n\}$. Assume for contradiction, that $\gamma_i = 0$. Then $(x_i\phi) = e$, and so ϕ does not map onto one of the generators in $Y \cup \{y^{-1} \mid y \in Y\}$. This contradicts the surjectivity of ϕ, so we can conclude that $\gamma_i \neq 0$. Since $\gamma_i\delta_i = 0$, it follows that $\delta_i = 0$.

Since i was arbitrary, we have now satisfied the condition of Lemma 7.2, so $g \in [G,\ G]$. We can conclude that $\ker\phi \leq [G,\ G]$. $\square$

Theorem 7.4 *Let $f \in F$, and*

$$\{0 = x_0 < x_1 < \cdots < x_n = 1\},\ \{a_1,\ a_2, \cdots a_n\},\ \{b_1,\ b_2, \cdots b_n\},$$

be defined as in Lemma 2.2. Then $f \in [F,\ F]$ if and only if $a_1 = a_n = 1$. In such a case, $b_1 = b_n = 0$. In addition

$$F \big/ [F,\ F] \cong \mathbb{Z} \times \mathbb{Z}.$$

Proof Define

$$\begin{aligned} \phi\colon F &\to \mathbb{Z} \times \mathbb{Z} \\ f &\mapsto (\log_2 a_1,\ \log_2 a_n), \end{aligned}$$

where a_1 and a_n are defined as in Lemma 2.2, that is the right and left derivatives of f at 0 and 1, respectively. Note that ϕ maps into $\mathbb{Z} \times \mathbb{Z}$, as a_n and a_1 are powers of 2. Let $g \in F$, and c_1 and c_m be defined analogously to a_1 and a_n for g. Then the right and left derivatives of fg at 0 and 1, respectively, are a_1c_1 and a_nc_m. Hence

$$(fg)\phi = (\log_2(a_1c_1),\ \log_2(a_nc_m)) = (\log_2 a_1 + \log_2 c_1,\ \log_2 a_n + \log_2 c_m) = (f\phi)(g\phi),$$

and it follows that ϕ is a homomorphism. In addition,

$$A\phi = (-1,\ 1), \quad B\phi = (0,\ 1),$$

and $\mathbb{Z} \times \mathbb{Z} = \langle (-1,\ 1),\ (0,\ 1) \rangle$, we have that ϕ is surjective.

By Lemma 7.3, we have that the kernel of this epimorphism is $[F,\ F]$. By the First Isomorphism Theorem, we have

$$F \big/ [F,\ F] \cong \mathbb{Z} \times \mathbb{Z}.$$

Let $f \in F$. We have

$$f \in [F,\ F] \iff f \in \ker \phi \iff f\phi = (0,\ 0) \iff (\log_2 a_1,\ \log_2 a_n) = (0,\ 0) \iff a_1 = a_2 = 1.$$

We have $0 = (0)f = 1 \cdot 0 + b_1 = b_1$, and $1 = (1)f = 1 \cdot 1 + b_n$. Hence $b_1 = b_n = 0$. $\square$

Lemma 7.5 *Let $n \in \mathbb{N}$, and*

$$P = \{0 = x_0 < x_1 < \cdots < x_n = 1\}, \quad Q = \{0 = y_0 < y_1 < \cdots < y_n = 1\},$$

be partitions of $[0,\ 1]$ comprising dyadic rational numbers. Then there exists $f \in F$ that is linear on the intervals of P, such that

$$x_i f = y_i,$$

for every $i \in \{0,\ \ldots,\ n\}$. In addition, if $i \in \{1,\ \ldots,\ n\}$ such that $x_{i-1} = y_{i-1}$ and $x_i = y_i$, then f is trivial on $[x_{i-1},\ x_i]$.

Proof Let $i \in \{1,\ \ldots,\ n\}$, such that $[x_{i-1},\ x_i]$ is not a standard dyadic partition. We have that $x_{i-1} = \frac{a}{2^m}$ and $x_i = \frac{b}{2^n}$, for some $a,\ b \in \mathbb{N}_0,\ m,\ n \in \mathbb{Z}$. The partition

$$\left\{ x_{i-1} = \frac{2^n a}{2^{m+n}} < \frac{2^n a + 1}{2^{m+n}} < \cdots < \frac{2^m b}{2^{m+n}} \right\},$$

is a standard dyadic partition of $[x_{i-1},\ x_i]$. Therefore, there is a standard dyadic partition containing P, and another containing Q. We will call these P' and Q'. By adding additional points inside intervals, we can assume that $|P'| = |Q'|$, and that there are as many points between x_{i-1} and x_i as there are between y_{i-1} and y_i, for all valid i. Let R and S be the $\mathcal{T}$-trees, with leaves forming the intervals of P' and Q', respectively. There is a unique element f of F, whose $\mathcal{T}$-tree class contains $(R,\ S)$. We have that f maps the points in P' to the same positioned points in Q', with respect to the standard order on $\mathbb{R}$. Hence $x_i f = y_i$, for all $i \in \{0,\ \ldots,\ n\}$.

Let $i \in \mathbb{N},\ i \leq n$, such that $x_{i-1} = y_{i-1}$ and $x_i = y_i$. Let $a_i,\ b_i \in \mathbb{Q}$, be defined as in Lemma 2.2, for i. Then,

$$x_{i-1} = y_{i-1} = x_{i-1} a_i + b_i, \qquad x_i = x_i a_i + b_i.$$

So

$$b_i = x_{i-1}(1 - a_i) = x_i(1 - a_i).$$

Since $x_{i-1} < x_i$, it follows that $1 - a_i = 0$, and hence $a_i = 1$. Therefore, $x_i = x_i + b_i$, and $b_i = 0$. So f is trivial on $[x_{i-1},\ x_i]$. $\square$

Theorem 7.6 *The centre of F is trivial.*

Proof Let $f \in Z(F)$. Let $P = \{x_0 < x_1 < \cdots < x_n\} \subseteq [0,\ 1]$, consist of dyadic rationals. We have that P forms part of a partition of $[0,\ 1]$, comprising only dyadic rationals, by tacking on 0 and 1 to P. In addition, there exists $g \in F$, such that g fixes P, and no other points. Let $x \in P$. Then

$$xf = xgf = xfg,$$

so g fixes xf, and hence $xf \in P$. Since x was arbitrary, we have that f permutes P. But f is increasing, so f must stabilise every point in P.

Since P was arbitrary, we have that f fixes every dyadic rational in $[0,\ 1]$. There is a standard dyadic partition Q of $[0,\ 1]$ such that f is linear on its intervals. We have that f fixes all points in Q, and hence by Lemma 7.5, f is trivial on all of these intervals. Hence f is the identity function. $\square$

Theorem 7.7 *Proper quotients of F are abelian.*

Proof Let $N \trianglelefteq F$ be non-trivial, and $f \in N$ be non-trivial. We have for any $g \in F$, $g^{-1}f^{-1}g \in N$. Since N is non-trivial and, by Theorem 7.6, $Z(F)$ is trivial, we have that there is a $g \in F$ that does not commute with f^{-1}. Hence $g^{-1}f^{-1}gf \in N$ is a non-trivial commutator. Let the normal form of f, as in Corollary 6.7, be

$$f = X_0^{-a_0}X_1^{-a_1}\cdots X_n^{-a_n}X_n^{b_n}\cdots X_1^{b_1}X_0^{b_0},$$

where $n \in \mathbb{N}_0$. By Theorem 7.4, f is trivial in the neighbourhood of 0, so the first two leaves of the domain and range trees of any $\mathcal{T}$-tree diagram representing f, will be in carets, both of which will be on the same level of $\mathcal{T}$. Therefore $a_0 = b_0$. Let k be the smallest index, such that $a_k \neq b_k$. If $a_k > b_k$, we can redefine f as f^{-1}, in order to assume $b_k > a_k$. By redefining f as the conjugate of itself by

$$X_0^{-a_0}X_1^{-a_1}\cdots X_k^{-a_k},$$

we may also assume $a_0 = a_1 = \cdots = a_k = b_0 = b_1 = \cdots = b_{k-1} = 0$, and $b_k > 0$. By redefining f as the conjugate of f by X^{k-1}, we have, if $k > 1$, that

$$\begin{aligned}
f &= X_0^{1-k}X_{k+1}^{-a_{k+1}}X_{k+2}^{-a_{k+2}}\cdots X_n^{-a_n}X_n^{b_n}\cdots X_{k+1}^{b_{k+1}}X_k^{b_k}X_0^{k-1}\\
&= X_0^{1-k-1}X_k^{-a_{k+1}}X_0^{-1}X_{k+2}^{-a_{k+2}}\cdots X_n^{-a_n}X_n^{b_n}\cdots X_{k+1}^{b_{k+1}}X_0X_{k-1}^{b_k}X_0^{k-2}\\
&= \cdots\\
&= X_2^{-a_{k+1}}X_0^{-k+1}X_{k+2}^{-a_{k+2}}\cdots X_n^{-a_n}X_n^{b_n}\cdots X_{k+1}^{b_{k+1}}X_0^{k-1}X_1^{b_1}.
\end{aligned}$$

Hence $a_0 = a_1 = b_0 = 0$ and $b_1 > 0$. If $k = 1$, we also have this. From the presentation of F in Theorem 6.13, we have, if $k,\ n \in \mathbb{N}_0$ and $k < n$, that

$$X_kX_n^{-1} =_F X_{n+1}^{-1}X_k, \quad X_kX_n =_F X_{n+1}X_k, \quad X_n^{-1}X_k^{-1} =_F X_k^{-1}X_{n+1}^{-1}.$$

In the following two series of equalities, we will underline the expression that is being manipulated. We have

$$\begin{aligned}
N &\ni (X_1fX_1^{-1})^{-1}(X_0fX_0^{-1})\\
&=_F X_1\underline{f^{-1}}X_1^{-1}X_0fX_0^{-1}
\end{aligned}$$

$$
\begin{aligned}
&=_F \underline{X_1 X_1^{-b_1}} X_2^{-b_2} X_3^{-b_3} \cdots X_{n-1}^{-b_{n-1}} X_n^{-b_n} X_n^{a_n} X_{n-1}^{a_{n-1}} \cdots X_3^{a_3} X_2^{a_2} X_1^{-1} X_0 f X_0^{-1} \\
&=_F X_1^{-b_1} \underline{X_1 X_2^{-b_2}} X_3^{-b_3} \cdots X_{n-1}^{-b_{n-1}} X_n^{-b_n} X_n^{a_n} X_{n-1}^{a_{n-1}} \cdots X_3^{a_3} X_2^{a_2} X_1^{-1} X_0 f X_0^{-1} \\
&=_F X_1^{-b_1} X_3^{-b_2} \underline{X_1 X_3^{-b_3} \cdots X_{n-1}^{-b_{n-1}} X_n^{-b_n}} X_n^{a_n} X_{n-1}^{a_{n-1}} \cdots X_3^{a_3} X_2^{a_2} X_1^{-1} X_0 f X_0^{-1} \\
&=_F \cdots \\
&=_F X_1^{-b_1} X_3^{-b_2} X_4^{-b_3} \cdots X_n^{-b_{n-1}} X_{n+1}^{-b_n} \underline{X_1 X_n^{a_n}} X_{n-1}^{a_{n-1}} \cdots X_3^{a_3} X_2^{a_2} X_1^{-1} X_0 f X_0^{-1} \\
&=_F X_1^{-b_1} X_3^{-b_2} X_4^{-b_3} \cdots X_n^{-b_{n-1}} X_{n+1}^{-b_n} X_{n+1}^{a_n} \underline{X_1 X_{n-1}^{a_{n-1}} \cdots X_3^{a_3} X_2^{a_2}} X_1^{-1} X_0 f X_0^{-1} \\
&=_F \cdots \\
&=_F X_1^{-b_1} X_3^{-b_2} X_4^{-b_3} \cdots X_n^{-b_{n-1}} X_{n+1}^{-b_n} X_{n+1}^{a_n} X_n^{a_{n-1}} \cdots X_4^{a_3} X_3^{a_2} \underline{X_1 X_1^{-1}} X_0 f X_0^{-1} \\
&=_F X_1^{-b_1} X_3^{-b_2} X_4^{-b_3} \cdots X_n^{-b_{n-1}} X_{n+1}^{-b_n} X_{n+1}^{a_n} X_n^{a_{n-1}} \cdots X_4^{a_3} \underline{X_3^{a_2} X_0} f X_0^{-1} \\
&=_F X_1^{-b_1} X_3^{-b_2} X_4^{-b_3} \cdots X_n^{-b_{n-1}} X_{n+1}^{-b_n} \underline{X_{n+1}^{a_n} X_n^{a_{n-1}} \cdots X_4^{a_3} X_0} X_2^{a_2} f X_0^{-1} \\
&=_F \cdots \\
&=_F X_1^{-b_1} X_3^{-b_2} X_4^{-b_3} \cdots X_n^{-b_{n-1}} \underline{X_{n+1}^{-b_n} X_0} X_n^{a_n} X_{n-1}^{a_{n-1}} \cdots X_3^{a_3} X_2^{a_2} f X_0^{-1} \\
&=_F X_1^{-b_1} \underline{X_3^{-b_2} X_4^{-b_3} \cdots X_n^{-b_{n-1}} X_0} X_n^{-b_n} X_n^{a_n} X_{n-1}^{a_{n-1}} \cdots X_3^{a_3} X_2^{a_2} f X_0^{-1} \\
&=_F \cdots \\
&=_F X_1^{-b_1} X_0 \underline{1_{F_1}} X_2^{-b_2} X_3^{-b_3} \cdots X_{n-1}^{-b_{n-1}} X_n^{-b_n} X_n^{a_n} X_{n-1}^{a_{n-1}} \cdots X_3^{a_3} X_2^{a_2} f X_0^{-1} \\
&=_F X_1^{-b_1} X_0 X_1^{b_1} \underline{X_1^{-b_1} X_2^{-b_2} X_3^{-b_3} \cdots X_{n-1}^{-b_{n-1}} X_n^{-b_n} X_n^{a_n} X_{n-1}^{a_{n-1}} \cdots X_3^{a_3} X_2^{a_2}} f X_0^{-1} \\
&=_F X_1^{-b_1} \underline{X_0 X_1^{b_1}} \; \underline{f^{-1} f} X_0^{-1} \\
&=_F X_1^{-b_1} X_2^{b_1} \underline{X_0 X_0^{-1}} \\
&=_F X_1^{-b_1} X_2^{b_1}.
\end{aligned}
$$

Therefore, $X_2^{b_1} X_1^{-b_1} = X_1^{b_1} (X_1^{-b_1} X_2^{b_1}) X_1^{-b_1} \in N$. It follows that

$$
\begin{aligned}
N \ni\; & X_0^{-1} X_2^{1-b_1} ((\underline{X_1^{b_1} X_2^{-b_1}}) X_2^{-1} (X_2^{b_1} X_1^{-b_1}) X_2) X_2^{b_1-1} X_0 \\
=_F\; & X_0^{-1} X_2^{1-b_1} (X_{b_1+2}^{-b_1} \underline{X_1^{b_1} X_2^{-1}} X_2^{b_1} X_1^{-b_1} X_2) X_2^{b_1-1} X_0 \\
=_F\; & X_0^{-1} X_2^{1-b_1} (X_{b_1+2}^{-b_1} X_{b_1+2}^{-1} \underline{X_1^{b_1} X_2^{b_1}} X_1^{-b_1} X_2) X_2^{b_1-1} X_0 \\
=_F\; & X_0^{-1} X_2^{1-b_1} \underline{(X_{b_1+2}^{-b_1} X_{b_1+2}^{-1} X_{b_1+2}^{b_1}} \; \underline{X_1^{b_1} X_1^{-b_1}} X_2) X_2^{b_1-1} X_0 \\
=_F\; & X_0^{-1} \underline{X_2^{1-b_1} X_{b_1+2}^{-1}} X_2 X_2^{b_1-1} X_0 \\
=_F\; & X_0^{-1} X_3^{-1} \underline{X_2^{1-b_1} X_2 X_2^{b_1-1}} X_0
\end{aligned}
$$

$$
\begin{aligned}
&=_F \underline{X_0^{-1}X_3^{-1}}X_2X_0\\
&=_F X_2^{-1}\underline{X_0^{-1}X_2}X_0\\
&=_F X_2^{-1}X_1\underline{X_0^{-1}X_0}\\
&=_F X_2^{-1}X_1\\
&=_F AB^{-1}A^{-1}B.
\end{aligned}
$$

Hence $N \ni A^{-1}(AB^{-1}A^{-1}B)A =_F B^{-1}A^{-1}BA = [B,\ A]$. Therefore if $g,\ h \in F$, with words over $\{A,\ B\}$ being

$$g = A^{\varepsilon_1}B^{\delta_1}\cdots A^{\varepsilon_m}B^{\delta_m}, \quad h = A^{\alpha_1}B^{\beta_1}\cdots A^{\alpha_n}B^{\beta_n},$$

for some $m,\ n \in \mathbb{N}_0$, where $\varepsilon_i,\ \delta_i,\ \alpha_i,\ \beta_i \in \mathbb{Z}$, for all valid i. Then, since $[B,\ A] \in N$, we have

$$
\begin{aligned}
ghN &= A^{\varepsilon_1}B^{\delta_1}\cdots A^{\varepsilon_m}B^{\delta_m}A^{\alpha_1}B^{\beta_1}\cdots A^{\alpha_n}B^{\beta_n}N\\
&= A^{\alpha_1}B^{\beta_1}\cdots A^{\alpha_n}B^{\beta_n}A^{\varepsilon_1}B^{\delta_1}\cdots A^{\varepsilon_m}B^{\delta_m}N\\
&= hgN.
\end{aligned}
$$

□

Definition 7.8 Let G be a group acting on a set X. The *support* of an element $g \in G$, denoted $\operatorname{supt}(g)$, is defined by

$$\operatorname{supt}(g) = \{x \in X \mid xg \neq x\}.$$

Notation 7.9 Let $a,\ b \in [0,\ 1]$, with $a < b$. If $g : [a,\ b] \to [a,\ b]$, define $\dot{g}$ (ignoring the physicists) to be

$$
\begin{aligned}
\dot{g}\colon [0,\ 1] &\to [0,\ 1]\\
x &\mapsto \begin{cases} xg & x \in [a,\ b]\\ x & x \notin [a,\ b]. \end{cases}
\end{aligned}
$$

Lemma 7.10 *Let $a,\ b \in [0,\ 1],\ a < b$ be dyadic rationals, such that $b - a$ is a power of 2. Let*

$$G = \{f \in F \mid \operatorname{supt}(f) \subseteq [a,\ b]\}.$$

Then $G \cong F$.

Proof Define

$$
\begin{aligned}
\varphi\colon [a,\ b] &\to [0,\ 1]\\
x &\mapsto \frac{x-a}{b-a}.
\end{aligned}
$$

Note φ is linear and $a\varphi = 0,\ b\varphi = 1$, so φ is a bijection. As it is linear, it is also continuous, as is its inverse, so it is a homeomorphism. Let $f_\varphi = \varphi f \varphi^{-1}$, for any $f \in F$. Define

$$
\begin{aligned}
\phi\colon F &\to G\\
f &\mapsto \dot{f}_\varphi.
\end{aligned}
$$

Let $f_1,\ f_2 \in F$. Then, if $x \in [a,\ b]$,

$$x(f_1 f_2)\phi = x\varphi f_1 f_2 \varphi^{-1} = x\varphi f_1 \varphi^{-1} \varphi f_2 \varphi^{-1} = x(f_1\phi)(f_2\phi).$$

If $x \notin [a,\ b]$, then

$$x(f_1 f_2)\phi = x = x(f_1\phi) = x(f_1\phi)(f_2\phi),$$

and hence ϕ is a homomorphism. Let $f_1,\ f_2 \in F$, such that $f_1\phi = f_2\phi$. Then $\varphi f_1 \varphi^{-1} = \varphi f_2 \varphi^{-1}$, and hence $f_1 = f_2$. Let $g \in G$. Let $f = \varphi^{-1}(g \restriction_{[a,\ b]})\varphi$. Since $\varphi,\ g$ and φ^{-1} are homeomorphisms, we have that f is a homeomorphism. In addition, $f\colon [0,\ 1] \to [0,\ 1]$. Since g is piecewise linear, and φ and φ^{-1} are linear, we have that f is piecewise linear. Let $x \in [a,\ b]$. Again, using linearity, we can conclude that x is a point of indifferentiability of g if and only if $x\varphi$ is a point of indifferentiability of f. There are only finitely many such x, and since φ maps dyadic rationals to dyadic rationals, we have that the points of indifferentiability of f are dyadic rationals, and there are finitely many of them. Note, since $b - a$ is a power of 2, we have that conjugating g by φ preserves the gradients being powers of 2. Hence $f \in F$, $g = f\phi$, and ϕ is surjective. $\square$

Theorem 7.11 *The commutator subgroup of F is simple.*

Proof Let $N \trianglelefteq [F,\ F]$ be non-trivial. Then there is a non-identity element $f \in N$. By Theorem 7.4, f is trivial in the neighbourhoods of 0 and 1. Using Lemma 7.5, together with Theorem 7.4, we can construct $g \in [F,\ F]$, such that g maps $\left[0,\ \frac{1}{4}\right]$ and $\left[\frac{3}{4},\ 1\right]$ to these neighbourhoods. We can therefore conclude that gfg^{-1} is non-trivial and $\operatorname{supt}(gfg^{-1}) \subseteq \left[\frac{1}{4},\ \frac{3}{4}\right]$. By Lemma 7.10, F is isomorphic to the subgroup G of F, comprising all functions whose support lies in $\left[\frac{1}{4},\ \frac{3}{4}\right]$. Since the commutator subgroup of a group is the smallest normal subgroup with abelian quotient, and all proper quotients of G are abelian, by Theorem 7.7, we have that $[G,\ G] \subseteq N$. Hence N contains all elements of F that are trivial on $\left[0,\ \frac{1}{4}\right] \cup \left[\frac{3}{4},\ 1\right]$. Since $f \in [F,\ F]$ was conjugated by an element of $[F,\ F]$ into this subset of N, so can any element of $[F,\ F]$. Hence $[F,\ F] = N$. $\square$

8 Metrics on F

Definition 8.1 Let $(G,\ *)$ be a group. Let $\mathscr{T}$ be a topology on G. The tuple $(G,\ \mathscr{T},\ *)$ is a *topological group*, if:

1. The mapping

$$\begin{aligned} \operatorname{In}\colon G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

   is continuous, with respect to $\mathscr{T}$,

2. Given $h \in G$, the mappings

$$\begin{aligned} \rho_h\colon G &\to G \\ g &\mapsto gh \\ \lambda_h\colon G &\to G \\ g &\mapsto hg \end{aligned}$$

   are continuous, with respect to $\mathscr{T}$.

Theorem 8.2 *The group* $\mathrm{Homeo}^+([0,\ 1])$ *of increasing homeomorphisms of* $[0,\ 1]$ *induces a topological group, under the topology induced by* d_∞. *In addition,* d_∞ *is invariant under* λ_h*, for any* $h \in \mathrm{Homeo}^+([0,\ 1])$.

Proof Let $h \in \mathrm{Homeo}^+([0,\ 1])$. Let $\varepsilon > 0$ and $f \in \mathrm{Homeo}^+([0,\ 1])$. Theorem 3.17 of [How01] states that every continuous function on a closed interval of $\mathbb{R}$ is uniformly continuous on that closed interval. Thus h is uniformly continuous. Then there exists $\delta > 0$ such that if $x,\ y \in [0,\ 1]$, with $|x-y| < \delta$, then $|xh-yh| < \frac{\varepsilon}{2}$. Let $g \in B(f,\ \delta)$. Then for all $x \in [0,\ 1]$, we have $|xf-xg| < \delta$, and hence $|xfh - xgh| < \frac{\varepsilon}{2}$. Hence

$$d_\infty(f\rho_h,\ g\rho_h) = d_\infty(fh,\ gh) = \sup_{x\in[0,\ 1]} |xfh - xgh| \le \frac{\varepsilon}{2} < \varepsilon.$$

Hence ρ_h is continuous.

In addition, since h is a bijection, we have

$$d_\infty(f\lambda_h,\ g\lambda_h) = d_\infty(hf,\ hg) = \sup_{x\in[0,\ 1]} |xhf-xhg| = \sup_{x\in[0,\ 1]h} |xf-xg| = \sup_{x\in[0,\ 1]} |xf-xg| = d_\infty(f,\ g).$$

Hence d_∞ is invariant under λ_h, and it follows that λ_h is continuous.

Let $f \in \mathrm{Homeo}^+([0,\ 1])$ and $\varepsilon > 0$. By continuity of $\rho_{g^{-1}}$, there exists $\delta > 0$, such that $d_\infty(fg^{-1},\ \mathrm{id}_{[0,\ 1]}) < \varepsilon$, for all $g \in B(f,\ \delta)$. Hence,

$$d_\infty(f^{-1},\ g^{-1}) = d_\infty(ff^{-1},\ fg^{-1}) = d_\infty(\mathrm{id}_{[0,\ 1]},\ fg^{-1}) < \varepsilon.$$

Hence In is continuous. □

Corollary 8.3 *The group* $\mathrm{Homeo}^+([0,\ 1])$ *induces a topological group, under the topology induced by* d_1. *In addition,* d_1 *is invariant under* λ_h*, for any* $h \in \mathrm{Homeo}^+([0,\ 1])$.

Proof Since $d_1 \le d_\infty$, we have that $\lambda_h,\ \rho_h$ and In are continuous for any $h \in \mathrm{Homeo}^+([0,\ 1])$, using Theorem 8.2. In addition, if $f,\ g,\ h \in \mathrm{Homeo}^+([0,\ 1])$, then

$$d_1(f\lambda_h,\ g\lambda_h) = d_1(hf,\ hg) = \int_{[0,\ 1]} |hf - hg| = \int_{[0,\ 1]h} |f-g| = \int_{[0,\ 1]} |f-g| = d_1(f,\ g).$$

□

Lemma 8.4 *The dyadic rationals are dense in the reals.*

Proof Let $x \in \mathbb{R}$. We will write x as a base 2 expansion. That is, let $x_0 \in \mathbb{Z}$, and $x_i \in \{0,\ 1\}$, for all $i \in \mathbb{N}$, such that

$$x = \sum_{i=0}^{\infty} \frac{x_i}{2^i}.$$

Now consider the sequence $(y_n)_n$ of dyadic rationals, defined by

$$y_n = \frac{\lfloor 2^n x \rfloor}{2^n},$$

for any $n \in \mathbb{N}_0$. Let $\varepsilon > 0$. Let $N \in \mathbb{N}$ such that $N > \log_2 \frac{1}{\varepsilon}$. Then if $n \geq N$, we have

$$\begin{aligned}
|y_n - x| &= \left| \frac{\lfloor 2^n x \rfloor}{2^n} - x \right| \\
&= \left| \frac{\lfloor 2^n \sum_{i=0}^{\infty} \frac{x_i}{2^i} \rfloor}{2^n} - x \right| \\
&= \left| \frac{\lfloor \sum_{i=0}^{\infty} \frac{x_i}{2^{i-n}} \rfloor}{2^n} - x \right| \\
&= \left| \frac{\sum_{i=0}^{n} \frac{x_i}{2^{i-n}}}{2^n} - x \right| \\
&= \left| \sum_{i=0}^{n} \frac{x_i}{2^i} - \sum_{i=0}^{\infty} \frac{x_i}{2^i} \right| \\
&= \sum_{i=n+1}^{\infty} \frac{x_i}{2^i} \\
&\leq \frac{1}{2^n} \\
&\leq \frac{1}{2^N} \\
&< \varepsilon
\end{aligned}$$

□

Theorem 8.5 *Thompson's group F is dense in* $\mathrm{Homeo}^+([0,\ 1])$*, with respect to d_1 and d_∞.*

Proof Let $f \in \mathrm{Homeo}^+([0,\ 1])$. To prove the theorem, it suffices to show that there is a sequence $(f_n)_n \subseteq F$ such that $f_n \to f$ with respect to d_∞, as this will also converge with respect to d_1. Let $n \in \mathbb{N}$. Consider the partition

$$Q_n = \left\{ 0 < \frac{1}{2^{n+2}} < \frac{2}{2^{n+2}} < \cdots < \frac{2^{n+2}}{2^{n+2}} = 1 \right\}.$$

Let $x \in Q_n$. Consider the open set that is the pre-image of the open ball of radius $\frac{1}{2^{n+3}}$ and centre x, by f. By the denseness of the dyadic rationals in $\mathbb{R}$ (Lemma 8.4), there is a dyadic rational number in this set, which we will call $\tilde{x}$. Define the partition P_n of $[0,\ 1]$, by

$$P_n = \{\tilde{x} \mid x \in Q_n\}.$$

Note that since $\tilde{x}$ is not necessarily unique, there are multiple choices for P_n. We have that all elements of P_n are dyadic rationals, and that Q_n is a standard dyadic partition. Let P'_n be any standard dyadic partition of $[0,\ 1]$ containing all points in P_n. Let Q'_n be any standard dyadic partition of $[0,\ 1]$ containing all points in Q_n, and with the same number of points as P'_n, and such that the ith point of Q'_n is in Q_n if and only if the ith element of P'_n is in P_n, for all valid i. By Lemma 2.9, there exists $f_n \in F$, that sends the points of P'_n to those Q'_n, and hence the points of P_n to those of Q_n. Let $\mathscr{I}_n$ denote the intervals of P_n, and for each $I \in \mathscr{I}_n$, let l_I and r_I denote the left and right endpoints of I, respectively.

Let $I \in \mathscr{I}_n$. Since $|r_I f_n - r_I f|,\ |l_I f_n - l_I f| \leq \frac{1}{2^{n+3}}$, and $|r_I f_n - l_I f_n| \leq \frac{1}{2^{n+2}}$, we have that

$$\ell(If) = |l_I f - r_I f|$$

$$\begin{aligned}
&\leq |l_I f - l_I f_n| + |l_I f_n - r_I f_n| + |r_I f_n - r_I f| \\
&= \frac{1}{2^{n+3}} + \frac{1}{2^{n+2}} + \frac{1}{2^{n+3}} \\
&= \frac{1}{2^{n+1}}.
\end{aligned}$$

Note also that $\ell(If_n) = \frac{1}{2^{n+2}}$. Hence if $x \in I$, then since $xf \in If$ and $xf_n \in If_n$, and because $|l_I f_n - l_I f| \leq \frac{1}{2^{n+3}}$, we have that If and If_n overlap, and so $|xf_n - xf| \leq \ell(If_n) + \ell(If) \leq \frac{1}{2^n}$.

Let $\varepsilon > 0$. Let $N \in \mathbb{N}$ such that $N > \log_2 \frac{1}{\varepsilon}$. For all $n \geq N$, we have

$$\begin{aligned}
d_\infty(f_n,\ f) &= \sup_{x \in [0,\ 1]} |xf_n - xf| \\
&= \max_{I \in \mathscr{I}_n} \sup_{x \in I} |xf_n - xf| \\
&\leq \max_{I \in \mathscr{I}_n} \sup_{x \in I} \frac{1}{2^n} \\
&= \frac{1}{2^n} \\
&\leq \frac{1}{2^N} \\
&< \varepsilon.
\end{aligned}$$

□

9 Thompson's Group T

Definition 9.1 Let X be the set of piecewise linear homeomorphisms of S^1 that map images of dyadic rationals to images of dyadic rationals, under the map $t \mapsto e^{2\pi i t}$, from $[0,\ 1]$ to S^1, that are differentiable everywhere except at finitely many images of dyadic rational numbers, and where differentiable, the derivatives are powers of 2. Let *Thompson's group* T be the tuple $(X,\ \circ)$, where $\circ$ represents composition of functions.

Lemma 9.2 *The function*

$$\begin{aligned}
\phi\colon [0,\ 1] \mod 1 &\to S^1 \\
t &\mapsto e^{2\pi i t},
\end{aligned}$$

is a bijection, such that for all $u,\ v \in [0,\ 1] \mod 1$

$$d(u\phi,\ v\phi) = 2\pi|u - v|.$$

Proof Let $u,\ v \in [0,\ 1] \mod 1$. Then

$$d(u\phi,\ v\phi) = \int_v^u |(e^{2\pi i t})'| \mathrm{d}t = \int_v^u |2\pi i e^{2\pi i t}| \mathrm{d}t = 2\pi \int_{\max\{v,\ u\}}^{\min\{u,\ v\}} \mathrm{d}t = 2\pi|u - v|.$$

Hence ϕ is continuous.

If $u,\ v \in [0,\ 1] \mod 1$ such that $u\phi = v\phi$, then $e^{2\pi i u} = e^{2\pi i v}$, and so $2\pi u = 2\pi v \mod 2\pi$.

Hence $u = v \mod 1$, and ϕ is injective.

Let $z \in S^1$. Since S^1 is parametrised by $e^{2\pi it}$ for $t \in [0,\ 1)$ there exists $t \in [0,\ 1] \mod 1$, such that $t\phi = z$. So ϕ is surjective. $\square$

Remark 9.3 Using Lemma 9.2, we will often consider T as acting on $[0,\ 1] \mod 1$. Hence all elements of F induce elements of T as homeomorphisms of $[0,\ 1] \mod 1$ that fix 0.

Lemma 9.4 *Let $f\colon [0,\ 1] \mod 1 \to [0,\ 1] \mod 1$. Then $f \in T$ if and only if there are sets*

$$\{0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1 = x_0\},\ \{a_1,\ a_2, \cdots a_n\},\ \{b_1,\ b_2, \cdots b_n\},$$

for some $n \in \mathbb{N}$, such that for any $i \in \mathbb{N}$, $i \leq n$, a_i is a power of 2, b_i and x_i are dyadic rational numbers, and

$$xf = a_i x + b_i$$

for all $x \in [x_{i-1},\ x_i]$.

Proof ($\Rightarrow$): Let $f \in T$. Let $0 = x_0 < x_1 < \cdots < x_n = 1$, for $n \in \mathbb{N}$ and $x_i \in \mathbb{R}$ for valid indices i, be the points where f is not differentiable. Note that these are dyadic rational numbers.

We will proceed by induction on $i \in \mathbb{N}$, to prove that $xf = a_i x + b_i$, where $x_{i-1} \leq x \leq x_i$, a_i is a power of 2 and b_i is a dyadic rational number. We will call this statement $\mathcal{P}(i)$. Consider $\mathcal{P}(1)$. Let $b_1 = (0)f$. Since 0 is a dyadic rational, so is b_1. Since f is piecewise linear and differentiable on $[0,\ 1]$, we have that $xf = a_1 x + b_1$, where $x_0 \leq x \leq x_1$, a_1 is a power of 2 and b_1 is a dyadic rational number.

Let $i \in \mathbb{N}$, with $i \leq n$. Inductively suppose $\mathcal{P}(i)$ is true. Let $x \in [x_i,\ x_{i+1}]$. Since x_i is a dyadic rational number, and f is linear on $[x_i,\ x_{i+1}]$, with derivative 2, we have that $xf = a_{i+1}x + b_{i+1}$, where a_{i+1} is a power of 2, and $b_{i+1} \in [0,\ 1] \mod 1$. By induction, we have that $x_i f = a_i x_i + b_i$, where a_i is a power of 2, and b_i is a dyadic rational number. Since x_i is a dyadic rational number, we have that $a_i x_i$ is a dyadic rational number. The sum of two dyadic rationals is also a dyadic rational, and therefore $x_i f = a_i x_i + b_i$ is a dyadic rational number. Note that $a_{i+1}x_i$ is a dyadic rational number by the same argument. Hence $x_i f - a_{i+1}x_i = b_{i+1}$ is a dyadic rational.

($\Leftarrow$): Let $f\colon [0,\ 1] \mod 1 \to [0,\ 1] \mod 1$ and $n \in \mathbb{N}$. Let $X = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ comprise only dyadic rationals. For any $i \in \mathbb{N}$ such that $i \leq n$, suppose there exists a dyadic rational number b_i, and a power of 2, a_i, such that

$$xf = a_i x + b_i, \quad a_{i-1}x_{i-1} + b_{i-1} = a_i x_{i-1} + b_i,$$

for all $x \in [x_{i-1},\ x_i]$.

Note that $a_{i-1}x_{i-1} + b_{i-1} = a_i x_{i-1} + b_i$ for all valid i, since f is well-defined on the overlapping points of the closed intervals $[x_{i-1},\ x_i]$. We have that f is piecewise linear and strictly increasing, since $a_i > 0$ for all valid i. Since linear pieces always meet at the same points, we have that f is continuous, and hence a bijection. Let $i \in \mathbb{N}$ such that $i \leq n$. We have, for any $x \in [x_{i-1}f,\ x_i f]$, that

$$xf^{-1} = \frac{1}{a_i}x - \frac{b_i}{a_i}, \quad a_i x_i + b_i = a_{i+1}x_i + b_{i+1}$$

and hence

$$\begin{aligned}
\frac{1}{a_{i-1}}(x_{i-1}f) - \frac{b_{i-1}}{a_{i-1}} &= \frac{a_{i-1}x_{i-1} + b_{i-1}}{a_{i-1}} - \frac{b_{i-1}}{a_{i-1}} \\
&= \frac{a_{i-1}x_{i-1}}{a_{i-1}} \\
&= x_{i-1} \\
&= \frac{a_i x_{i-1}}{a_i} \\
&= \frac{a_i x_{i-1} + b_i}{a_i} - \frac{b_i}{a_i} \\
&= \frac{1}{a_i}(x_{i-1}f) - \frac{b_i}{a_i},
\end{aligned}$$

and it follows that the pieces of f^{-1} meet at the same points, and f^{-1} is continuous. We can conclude that f is a homeomorphism of $[0,\ 1] \mod 1$.

Let $i \in \mathbb{N}$ such that $i \leq n$. We have $xf' = a_i$, for any $x \in a_i x + b_i$. So, except for finitely many dyadic rationals which comprise the set X, f is differentiable, and the derivative of f is a power of 2 at these points. We have that the set of points where f is not differentiable is a subset of X, and hence is a finite set of dyadic rational numbers. Hence f is differentiable everywhere, except finitely many dyadic rational numbers, and when differentiable, the derivative is a power of 2. Together with the fact that f is a homeomorphism of $[0,\ 1] \mod 1$, it follows that $f \in T$. $\square$

Example 9.5 The function

$$\begin{aligned}
C\colon [0,\ 1] \mod 1 &\to [0,\ 1] \mod 1 \\
x &\mapsto \begin{cases} \frac{x}{2} + \frac{3}{4} & 0 \leq x \leq \frac{1}{2} \\ 2x - 1 & \frac{1}{2} \leq x \leq \frac{3}{4} \\ x - \frac{1}{4} & \frac{3}{4} \leq x \leq 1, \end{cases}
\end{aligned}$$

is an element of T, by Lemma 9.4.

Lemma 9.6 *The tuple T forms a group.*

Proof We have that T is a subset of the group of homeomorphisms of $[0,\ 1] \mod 1$. Let $f,\ g \in T$. Let $\{0 = x_0 < x_1 < \cdots < x_n = 1\}$ and $\{0 = y_0 < y_1 < \cdots < y_m = 1\}$ be the points of indifferentiability of f and g respectively, where $m,\ n \in \mathbb{N}$. Let $a_i,\ b_i$ be defined for all $i \in \mathbb{N}$, where $i \leq n$, by

$$xf = a_i x + b_i,$$

for all $x \in [x_i,\ x_{i+1}]$, where a_i is a power of 2 and b_i is a dyadic rational. Define $c_i,\ d_i$ for $i \in \mathbb{N},\ i \leq m$ analogously for g. We can do this by Lemma 9.4. It follows that, given $i \in \mathbb{N}$, where $i \leq n$,

$$xf = a_i x + b_i \implies x = \frac{xf}{a_i} - \frac{b_i}{a_i},$$

for any $x \in [x_i,\ x_{i+1}]$. Therefore,

$$xf^{-1} = \frac{1}{a_i}x - \frac{b_i}{a_i},$$

for all $x \in [x_i f,\ x_{i+1} f]$. Note that $\frac{1}{a_i}$ is a power of 2, $\frac{b_i}{a_i}$ is a dyadic rational number, and $x_i f$ and $x_{i+1} f$ are dyadic rationals, and hence $f^{-1} \in T$, by Lemma 9.4.

Let
$$X = \{x_0,\ x_1, \ldots,\ x_n\} \cup \{y_0 f^{-1},\ y_1 f^{-1},\ \ldots,\ y_m f^{-1}\}.$$
Inductively define z_i for $i \in \{1,\ \ldots,\ |X|\}$, by setting $z_1 = \min X$, and
$$z_{i+1} = \min(X \backslash \{z_1,\ z_2,\ \ldots,\ z_i\}).$$
Let $i \in \mathbb{N}$ and $x \in [z_{i-1},\ z_i]$. Then
$$xfg = (a_i x + b_i)g = (a_i x + b_i)c_i + d_i = a_i c_i x + (b_i c_i + d_i).$$
Since a_i and c_i are powers of 2, we have that $a_i c_i$ is a power of 2. Since b_i is a dyadic rational, and c_i is a power of 2, we have that $b_i c_i$ is a dyadic rational, and hence $b_i c_i + d_i$ is a dyadic rational. We conclude $fg \in T$, by Lemma 9.4. □

Lemma 9.7 *Elements of T map the set of dyadic rational numbers in* $[0,\ 1] \mod 1$ *bijectively to itself.*

Proof Let $f \in T$ be differentiable everywhere except at $0 = x_0 < x_1 < \cdots < x_n = 1$, where $n \in \mathbb{N}$. Given $i \in \mathbb{N}$, define a_i and b_i as in Lemma 9.4. By definition, f maps dyadic rational numbers to dyadic rational numbers.

As a homeomorphism, f is injective. Let $q \in [0,\ 1] \mod 1$ be a dyadic rational number. Choose $i \in \mathbb{N}$, with $i \leq n$, such that $q \in [x_i f,\ x_{i+1} f]$. Then $\frac{q - b_i}{a_i} \in [0,\ 1]$, since
$$\left(\frac{q - b_i}{a_i}\right) f = \left(\frac{q - b_i}{a_i}\right) a_i + b_i = q,$$
and thus
$$q f^{-1} = \frac{q - b_i}{a_i}.$$
Note that this is a dyadic rational, since the dyadic rationals are closed under subtraction, and division by powers of 2. We can therefore conclude that f maps the dyadic rationals onto the dyadic rationals. Since f is injective on $[0,\ 1] \mod 1$, it is injective on a subset, such as the dyadic rationals. □

Definition 9.8 Let $n \in \mathbb{N}$. We will use the notation $\mathrm{mod}_1\ n$ to map into $\{1,\ \ldots,\ n\}$, instead of $\{0,\ \ldots,\ n-1\}$. For example, $0 \mathrm{mod}_1\ n = n \mathrm{mod}_1\ n = n$, and $n + 1 \mathrm{mod}_1\ n = 1$, for any $n \in \mathbb{N}$. We call a permutation $\sigma \in S_n$ a *T-cycle* (of S_n), if there exists $k \in \{0,\ 1,\ \ldots,\ n-1\}$ such that $i\sigma = i + k \mathrm{mod}_1\ n$, for all valid i.

Lemma 9.9 *For each $f \in T$ there exists a T-cycle σ, and a standard dyadic partition of* $[0,\ 1]$*, say*
$$P = \{0 = x_0 < x_1 < \cdots < x_n = 1\},$$
such that for each $i \in \mathbb{N}$, $i \leq n$, f is linear on $[x_{i-1},\ x_i]$, and $xf = 0$, for some $x \in P$. In addition, for any such partition,
$$\{0 = x_0 f\sigma < x_1 f\sigma < \cdots < x_n f\sigma = 1\},$$
is a standard dyadic partition.

Proof Let $f \in T$. By Lemma 9.4, there is a partition P of $[0,\ 1]$ comprising dyadic rationals. By Lemma 2.7, there is a standard dyadic partition

$$Q = \{0 = x_0 < x_1 < \cdots < x_n = 1\},$$

where $n \in \mathbb{N}$, such that Q contains all points in P. Define $a_i,\ b_i$ as stated with respect to f and x_i, as in Lemma 9.4, for all $i \in \mathbb{N},\ i \leq n$.

Note $P \subseteq Q$ and f is linear on every interval of P, and hence f is linear on every interval of Q. Note also that the derivative of f, where differentiable, is always a power of 2, f is differentiable everywhere in $[0,\ 1] \mod 1$, except finitely many points. Thus we have that f is increasing, on every interval of $[0,\ 1]$, not containing $x = (0)f^{-1}$.

Therefore, for any valid i, the ith interval of Q is mapped to a standard dyadic interval, and the $(i+1)$th interval is mapped to an interval after the image of the ith interval, when considered as subsets of $[0,\ 1]$ modulo 1, and there will be precisely one j where the all points in the image of the jth interval under f are greater than all points in the image of the $(j+1)$th, except perhaps $1 = 0$. This will be when the image of the jth interval is $[y,\ 1]$ for some $y \in [0,\ 1]$, and hence the image of the $(j+1)$th interval will be $[0,\ z]$, for some $z \in [0,\ 1]$. We have

$$R = \{0 = x_{j+1}f < x_{j+2}f < \cdots < x_{n-1}f < x_0 f < x_1 f < \cdots < x_j f = x_{j+1}f = 1\},$$

will be a partition of $[0,\ 1]$, such that the ith element of Q is mapped by f to the $(i - j \bmod_1 n)$th interval of R. Define a function σ such that $x_i f\sigma = x_{i+j+1 \bmod_1 n} f$. Note that σ is a T-cycle of order n. Hence the partition

$$R = \{0 = x_0 f\sigma < x_1 f\sigma < \cdots < x_{n-1} f\sigma\}.$$

Let I be the ith interval of Q. Then there exist $a,\ k \in \mathbb{N}_0$ such that $a \leq 2^k - 1$ and

$$I = \left[\frac{a}{2^k},\ \frac{a+1}{2^k}\right].$$

There exists $q \in \mathbb{Z}$, such that $a_i = 2^q$, since a_i is a power of 2. Then

$$\begin{aligned} If &= \left[\frac{aa_i + 2^k b_i}{2^k},\ \frac{a_i(a+1) + 2^k b_i}{2^k}\right] \\ &= \left[\frac{2^q a + 2^k b_i}{2^k},\ \frac{2^q a + 2^q + 2^k b_i}{2^k}\right] \\ &= \left[\frac{a + 2^{k-q} b_i}{2^{k-q}},\ \frac{a + 2^{k-q} b_i + 1}{2^{k-q}}\right] \end{aligned}$$

Suppose $q > k$. Then

$$\frac{a + 2^{k-q} b_i + 1}{2^{k-q}} > \frac{1}{2^{k-q}} > 1,$$

a contradiction. It follows that $q \leq k$, and If is a standard dyadic interval. Therefore, R is a standard dyadic partition. □

Lemma 9.10 *Given two standard dyadic partitions P and Q of $[0,\ 1]$, and a T-cycle σ, such that $\sigma S_{|P|-1}$, there exists a unique element f of T such that f is linear on the intervals of P, and f maps the ith interval of P to the $i\sigma$th interval of Q.*

Proof Existence: Let $i \in \mathbb{N}$ such that $i \leq |P| - 1$.

$$I_i = \left[\frac{c_i}{2^{n_i}},\ \frac{c_i+1}{2^{n_i}}\right], \quad J_i = \left[\frac{d_i}{2^{k_i}},\ \frac{d_i+1}{2^{k_i}}\right],$$

where $c_i,\ d_i \in \mathbb{N}_0$, and $n_i,\ k_i \in \mathbb{Z}$, such that I_i is the ith interval of P and J_i is the $i\sigma$th interval of Q. Let

$$\begin{aligned} f_i \colon I_i &\to J_i \\ x &\mapsto a_i x + b_i, \end{aligned}$$

where $a_i = 2^{n_i - k_i}, \quad b_i = 2^{-k}(c_i - d_i)$. Note a_i is a power of 2, and b_i is a dyadic rational number. In addition

$$\begin{aligned} I_i f_i &= \left[2^{n_i-k_i} \cdot \frac{c_i}{2^{n_i}} + 2^{-k}(d_i - c_i),\ 2^{n_i-k_i} \cdot \frac{c_i+1}{2^{n_i}} + 2^{-k_i}(d_i - c_i)\right] \\ &= [2^{-k_i}c_i + 2^{-k_i}(d_i - c_i),\ 2^{-k_i}(c_i+1) + 2^{-k_i}(d_i - c_i)] \\ &= \left[\frac{d_i}{2^{k_i}},\ \frac{d_i+1}{2^{k_i}}\right] \\ &= J_i. \end{aligned}$$

Since the I_is are disjoint except at endpoints, and on the endpoints of I_i and I_{i+1}, we have that

$$\frac{c_i+1}{2^{n_i}} f_i = \frac{d_i+1}{2^{k_i}} = \frac{d_{i+1}}{2^{k_{i+1}}} = \frac{c_{i+1}}{2^{n_{i+1}}} f_{i+1},$$

and hence we can define $f \colon [0,\ 1] \mod 1 \to [0,\ 1] \mod 1$, such that if $x \in [0,\ 1] \mod 1$, find an $i \in \mathbb{N}_0$, such that $x \in I_i$, then map x to $x f_i$. This function satisfies the requirements of Lemma 9.4, and hence $f \in T$.

Uniqueness: Suppose $g \in T$ such that g maps the ith interval of P to the $i\sigma$th of Q, for all valid i. Let $i \in \mathbb{N}$. By Lemma 9.4, there exists a power of 2 α_i, and a dyadic rational number β_i, such that for $x \in I_i$, we have $x_i g = \alpha_i x + \beta_i$. Since $I_i g = J_i$, we have

$$\begin{aligned} \alpha_i \cdot \frac{c_i}{2^{n_i}} + \beta_i &= \frac{d_i}{2^{k_i}} \\ \alpha_i \cdot \frac{c_i+1}{2^{n_i}} + \beta_i &= \frac{d_i+1}{2^{k_i}}. \end{aligned}$$

Hence

$$2^{k_i}\alpha_i(c_i+1) + 2^{n_i}d_i - 2^{k_i}\alpha_i c_i = 2^{n_i}(d_i+1) \implies 2^{k_i}\alpha_i = 2^{n_i} \implies \alpha_i = 2^{n_i-k_i}.$$

In addition

$$2^{n_i-k_i} \cdot \frac{c_i}{2^{n_i}} + \beta_i = \frac{d_i}{2^{k_i}} \implies 2^{-k_i}c_i + \beta_i = 2^{-k_i}d_i \implies \beta_i = 2^{-k_i}(d_i - c_i).$$

Note $\alpha_i = a_i$ and $\beta_i = b_i$, and hence f and g coincide on I_i. Since I_i was an arbitrary interval, we can conclude that $f = g$. $\square$

10 The Tree Representation of T

Lemma 10.1 *Let $(R,\ S,\ \sigma)$ be a $\mathcal{T}$-tree diagram, such that σ is a T-cycle. If $(U,\ V,\ \tau) \in [(R,\ S,\ \sigma)]$, then τ is a T-cycle.*

Proof Note that throughout this proof we will use the injectivity of permutations implicitly; that is we will not mention the case of $j\rho = i\rho$, if we know $j \neq i$.

Case 1: $(U,\ V,\ \tau)$ is an elementary expansion of $(R,\ S,\ \sigma)$. Let $n = |\mathcal{L}(R)|$. Then there exists $k \in \mathbb{N},\ k \leq n$, such that for all $j \in \mathbb{N},\ j \leq n,\ j\sigma = j + k \bmod_1 n$. Let $i \in \{1,\ \ldots,\ n\}$ be the point of expansion. We have

$$\begin{aligned} \tau \colon \{1,\ 2,\ \ldots n+1\} &\to \{1,\ 2,\ \ldots n+1\} \\ i &\mapsto i\sigma \\ i+1 &\mapsto i\sigma + 1 \\ j &\mapsto \begin{cases} j\sigma & j < i,\ j\sigma < i\sigma \\ j\sigma + 1 & j < i,\ j\sigma > i\sigma \\ (j-1)\sigma & j > i+1,\ (j-1)\sigma < i\sigma \\ (j-1)\sigma + 1 & j > i+1,\ (j-1)\sigma > i\sigma. \end{cases} \end{aligned}$$

Hence

$$\begin{aligned} \tau \colon i &\mapsto i + k \bmod_1 n \\ i+1 &\mapsto (i + k \bmod_1 n) + 1 \\ j &\mapsto \begin{cases} j + k \bmod_1 n & j < i,\ j + k \bmod_1 n < i + k \bmod_1 n \\ (j + k \bmod_1 n) + 1 & j < i,\ j + k \bmod_1 n > i + k \bmod_1 n \\ j - 1 + k \bmod_1 n & j > i+1,\ j - 1 + k \bmod_1 n < i + k \bmod_1 n \\ (j - 1 + k \bmod_1 n) + 1 & j > i+1,\ j - 1 + k \bmod_1 n > i + k \bmod_1 n. \end{cases} \end{aligned}$$

We will consider two subcases:

Subcase 1.A: $i + k \leq n$.
First note that

$$\begin{aligned} i\tau &= i + k \bmod_1 n = i + k = i + k \bmod_1 (n+1) \\ (i+1)\tau &= (i + k \bmod_1 n) + 1 = (i+1) + k = (i+1) + k \bmod_1 (n+1). \end{aligned}$$

If $j < i$, then $j + k < i + k \leq n$, so $j + k \bmod_1 n < i + k \bmod_1 n$, and

$$j\tau = j + k \bmod_1 n = j + k = j + k \bmod_1 (n+1).$$

If $j > i + 1$, but $j + k \leq n + 1$, then $j - 1 + k \bmod_1 n = j - 1 + k > i + k = i + k \bmod_1 n$. So

$$j\tau = (j - 1 + k \bmod_1 n) + 1 = j + k = j + k \bmod_1 (n+1).$$

Finally, if $j > i+1$ and $j+k > n+1$, we have that $j-1+k \bmod_1 n < i+k \bmod_1 n$, otherwise $j-1+k-n > i+k$, which implies $j > i+n+1 > n+1$, a contradiction to j being in $\{1,\ 2,\ \ldots,\ n+1\}$. Therefore

$$j\tau = j-1+k \bmod_1 n = j+k-(n+1) = j+k \bmod_1 (n+1).$$

We can conclude that $j\tau = j+k \bmod_1 (n+1)$ for all $j \in \{1,\ 2,\ \ldots,\ n+1\}$, and hence τ is a T-cycle.

Subcase 1.B: $i+k > n$.
We have that

$$\begin{aligned} i\tau &= i+k \bmod_1 n = i+k-n = i+k+1-(n+1) = i+k+1 \bmod_1 (n+1) \\ (i+1)\tau &= (i+k \bmod_1 n)+1 = (i+1)+k = (i+1)+k+1 \bmod_1 (n+1). \end{aligned}$$

If $j < i$ and $j+k \leq n$, then it follows that $j+k \bmod_1 n > i+k \bmod_1 n$, since if not, we have that $j+k \leq i+k-n$, which implies $i \geq j+n$, a contradiction. Hence

$$j\tau = (j+k \bmod_1 n)+1 = j+k+1 = j+k+1 \bmod_1 (n+1).$$

If $j < i$, but $j+k > n$, then $j+k \bmod_1 n = j+k-n < i+k-n = i+k \bmod_1 n$, and it follows that

$$j\tau = j+k \bmod_1 n = j+k-n = j+k+1 \bmod_1 (n+1).$$

If $j > i+1$, then $j+k > i+k > n$, and so $j-1+k \bmod_1 n = j-1+k-n > i+k-n = i+k \bmod_1 n$. Therefore

$$j\tau = (j-1+k \bmod_1 n)+1 = j+k-n = j+k+1 \bmod_1 (n+1).$$

We can conclude that τ is a T-cycle.

Case 2: $(U,\ V,\ \tau)$ is an elementary contraction of $(R,\ S,\ \sigma)$. Using the same representation for σ as in Case 1, with i being the point of contraction, and the definition of an elementary contraction, we obtain:

$$\begin{aligned} \tau\colon \{1,\ 2,\ \ldots, n-1\} &\to \{1,\ 2,\ \ldots, n-1\} \\ i &\mapsto i+k \bmod_1 n \\ j &\mapsto \begin{cases} j+k \bmod_1 n & j < i,\ j+k \bmod_1 n < i+k \bmod_1 n \\ (j+k \bmod_1 n)-1 & j < i,\ j+k \bmod_1 n > i+k \bmod_1 n \\ j+1+k \bmod_1 n & j > i,\ (j+1)+k \bmod_1 n < i+k \bmod_1 n \\ (j+1+k \bmod_1 n)-1 & j > i,\ (j+1)+k \bmod_1 n > i+k \bmod_1 n. \end{cases} \end{aligned}$$

We will now consider three subcases:

Subcase 2.A: $i+k \leq n-1$.
First note that

$$i\tau = i+k \bmod_1 n = i+k = i+k \bmod_1 (n-1).$$

If $j < i$, we have that $j+k < i+k \leq n-1$, so $j+k \bmod_1 n = j+k < i+k = i+k \bmod_1 n$, and it follows that

$$j\tau = j+k \bmod_1 n = j+k = j+k \bmod_1 (n-1).$$

If $j > i$, but $j + k \leq n$, then $j + k \bmod_1 n = j + k > i + k = i + k \bmod_1 n$, so

$$j\tau = (j + 1 + k \bmod_1 n) - 1 = j + k = j + k \bmod_1 (n - 1).$$

Finally, if $j > i$ and $j + k > n$, we have that $j + 1 + k \bmod_1 n < i + k \bmod_1 n$; otherwise $j + 1 + k - n \geq i + k$, which would imply $j + 1 \geq i + n$, a contradiction, since $j \in \{1, \ldots, n - 1\}$. Hence

$$j\tau = j + 1 + k \bmod_1 n = j + 1 + k - n = j + k - (n - 1) = j + k \bmod_1 (n - 1).$$

As a result, τ is a T-cycle.

Subcase 2.B: $i + k > n$.
We have that

$$i\tau = i + k \bmod_1 n = i + k - n = i + k - 1 - (n - 1) = i + k - 1 \bmod_1 (n - 1).$$

If $j < i$ and $j + k \leq n$, then it follows that $j + k \bmod_1 n > i + k \bmod_1 n$, since if not, we have that $j + k \leq i + k - n$, which implies $i \geq j + n$, a contradiction. Hence

$$j\tau = (j + k \bmod_1 n) - 1 = j + k - 1 = j + k - 1 \bmod_1 (n - 1).$$

If $j < i$ and $j + k > n$, then $j + k \bmod_1 n = j + k - n < i + k - n = i + k \bmod_1 n$, and hence

$$j\tau = j + k \bmod_1 n = j + k - n = j + k - 1 \bmod_1 (n - 1).$$

Finally, if $j > i$, then $j+k > i+k > n$, and so $j+k+1 \bmod_1 n = j+k+1-n > i+k-n = i+k \bmod_1 n$. We can conclude that

$$j\tau = (j + 1 + k \bmod_1 n) - 1 = j + k - n = j + k - 1 \bmod_1 (n - 1).$$

Hence τ is a T-cycle.

Subcase 2.C: $i + k = n$. This case cannot occur, as it would mean that i was sent to a right leaf (n) in the range tree.

Case 3: There is a finite sequence of elementary contractions and expansions taking (R, S, σ) to (U, V, τ).
Note that if the length of the sequence is zero, then $\tau = \sigma$, so τ is a T-cycle. Let $n \in \mathbb{N}$. Inductively suppose the lemma holds, if the length of the sequence is n. Then if the sequence is length $n + 1$, the permutation of the $\mathcal{T}$-tree diagram obtained by applying all but the final term in the sequence is a T-cycle. Applying Case 1 or Case 2 to this $\mathcal{T}$-tree diagram, shows that τ is a T-cycle. $\square$

Theorem 10.2 *The set of $\mathcal{T}$-tree diagram classes whose permutations are T-cycles forms a group, isomorphic to Thompson's group T.*

Proof Let T_1 denote the set of $\mathcal{T}$-tree diagram classes whose permutations are T-cycles, together with the multiplication defined in Definition 4.17. Let $[(R, S, \sigma)]$ be such a $\mathcal{T}$-tree class. Let P_R and P_S be the standard dyadic partitions of $[0, 1]$ whose leaves are the intervals of R and S, respectively.

Let X be a $\mathcal{T}$-tree class, such that its permutation is a T-cycle, and $(R, S, \sigma) \in X$. Let P

and Q be the standard dyadic partitions, whose intervals are the leaves of R and S respectively. We have that there is a unique element $f \in T$ that sends the ith interval of P to the $i\sigma$th of Q, for all valid i, using Lemma 9.10. Let $(U,\ V,\ \tau)$ be an elementary expansion of $(R,\ S,\ \sigma)$, in the ith position. We will represent σ as $j \mapsto j+k \bmod_1 n$, where $n = |\mathcal{L}(R)|$. If $i+k \leq n$, then $\tau\colon j \mapsto j+k \bmod_1 (n+1)$, by Lemma 10.1.

Let I be the ith leaf of R. We have that the intervals that are the leaves of U are those of R, except I has been subdivided into its left and right halves, so f is linear on the intervals of U. In addition, we have that if $j \in \{1,\ 2,\ \ldots,\ i-1\}$, then the $j\sigma$th interval of S is the $j\tau$th interval of V, and if $j \in \{i+1,\ i+2,\ \ldots n\}$, then the $j\sigma$th interval of S is the $(j+1)\tau$th interval of V. Finally the $i\tau$th and $(i+1)\tau$th intervals of V are the left and right halves of the ith interval of S.

We have that for all valid j, the jth interval of R is mapped by f to the $j\sigma$th interval of S. Hence f maps the jth interval of U to the $j\tau$th interval of V, for all $j \in \{1,\ 2,\ \ldots,\ n+1\}\backslash\{i,\ i+1\}$. In addition, the left and right halves of the ith interval of S, which are the ith and $(i+1)$th intervals of U, are mapped to the left and right halves of the $i\sigma$th interval of S, which are the ith and $(i+1)$th intervals of V. We can conclude that f maps the jth interval of U to the $j\sigma$th of V, for all valid j, and hence the element of T determined by a $\mathcal{T}$-tree diagram is the same for all elements of X, as all elements of X can be made by repeatedly applying elementary expansions to the unique reduced element.

Let $X \in T_1$. Let f_X denote the unique element of T determined by X, which exists by Lemma 9.10, and is unique due to the arguments above. Define

$$\begin{aligned} \phi\colon T_1 &\to T \\ X &\mapsto f_X. \end{aligned}$$

Since f_X is unique given X, we have that ϕ is well defined.

Injective: Suppose $X,\ Y \in T_1$, such that $X\phi = Y\phi$. Hence $f_X = f_Y$. Let $(R,\ S,\ \sigma) \in X$ and $(U,\ V,\ \tau) \in Y$ be reduced. Let I be a standard dyadic interval. Let $(P,\ Q,\ \rho) \in \{(R,\ S,\ \sigma),\ (U,\ V,\ \tau)\}$. If $I \in \mathcal{L}(P)$, then f_X is linear on I. If $I \notin V(P)$, then there is a standard dyadic interval $J \in \mathcal{L}(P)$, such that $I \subseteq J$. This is true since the leaves of P form the intervals of a partition of $[0,\ 1]$, and there is a unique path from any vertex in $\mathcal{T}$ to the root. We can conclude that f_X is linear on I.

Conversely, suppose that I is a standard dyadic interval such that f_X is linear on I. Since $(P,\ Q,\ \rho)$ is reduced, if $J \in V(P)\backslash\mathcal{L}(P)$, and f_X is linear on J, then all vertices whose path to J does not intersect the path from J to the root (unique since P is a tree), can be removed by elementary contractions. This contradicts $(P,\ Q,\ \rho)$ being reduced, and it follows that $I \in \mathcal{L}(P)$ or $I \notin V(P)$.

We now have $I \in \mathcal{L}(P)$ or $I \notin V(P)$ if and only if f_X is linear on I. Using this, we can construct P, and will have no choices. Hence $U = R$. It follows that the images of the intervals of U and R under f_X are the same. Hence $V = S$ and $\tau = \sigma$. It follows that $X = Y$.

Surjective: Let $f \in T$. By Lemma 9.9, there exist standard dyadic partitions P and Q, and a permutation σ, such that f is linear on the intervals of P and maps the ith interval of P to the $i\sigma$th of Q. Let R and S be the $\mathcal{T}$-trees whose leaves are the intervals of P and Q, respectively.

Then $[(R,\ S,\ \sigma)]\phi = f$.

Respects multiplication: Let $X,\ Y \in T_1$. By picking a large enough common tree, in terms of number of tree levels, let $(R,\ S,\ \sigma) \in X$ and $(U,\ V,\ \tau) \in Y$, such that $S = U$. Let $f = ([(R,\ S,\ \sigma)]\phi)([(U,\ V,\ \tau)])\phi$. Then f will be linear on the intervals of U, and map the ith interval of R to the $i\sigma$th of $S = U$, to the $i\sigma\tau$th of V, for all valid i, and hence $f = [(R,\ V,\ \sigma\tau)]\phi$. So

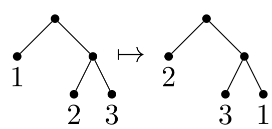
$$\begin{aligned}(XY)\phi &= ([(R,\ S,\ \sigma)][(U,\ V,\ \tau)])\phi \\ &= [(R,\ V,\ \sigma\tau)]\phi \\ &= ([(R,\ S,\ \sigma)]\phi)([(U,\ V,\ \tau)])\phi \\ &= (X\phi)(Y\phi).\end{aligned}$$

$\square$

Since the associativity, identity and inverses are preserved under bijections that respect multiplication, we have that T_1 is a group. Hence ϕ is a group isomorphism.

11 A Finite Presentation for T

Example 11.1 Recall the function $C \in T$ from Example 11.1. Note that $\{0 < \frac{1}{2} < \frac{3}{4} < 1\}$ is a standard dyadic partition of $[0,\ 1]$, and C maps the ith interval of this partition to the $i(1\ 3\ 2)$th interval, so a tree representation of C is



Note that this is reduced, so it is the unique reduced representation for C. For the remainder of this section, we will use A and B to denote the elements of T, induced by the action of the same named elements of F on $[0,\ 1] \mod 1$.

Lemma 11.2 *Thompson's group T is generated by A, B and C.*

Proof Let F' be the subgroup of T, isomorphic to F, generated by A and B. Note that F' comprises precisely the set of elements of T that fix 0. We have that $F' \subseteq \langle A,\ B,\ C\rangle$. Let $f \in T$, and $x = (0)f$. If $x = 0 \mod 1$, then $f \in F' \subseteq \langle A,\ B,\ C\rangle$. Otherwise, $x \neq 0 \mod 1$. Note that x is a dyadic rational number. Hence, by Lemma 7.5, there exists $h \in F' \subseteq \langle A,\ B,\ C\rangle$ such that $xh = \frac{3}{4}$. Then

$$(0)fhC^{-1} = xhC^{-1} = \left(\frac{3}{4}\right)C^{-1} = 0.$$

It follows that $fhC^{-1} \in F' \subseteq \langle A,\ B,\ C\rangle$. Therefore, since $h,\ C^{-1} \in \langle A,\ B,\ C\rangle$, we have that $f \in \langle A,\ B,\ C\rangle$. $\square$

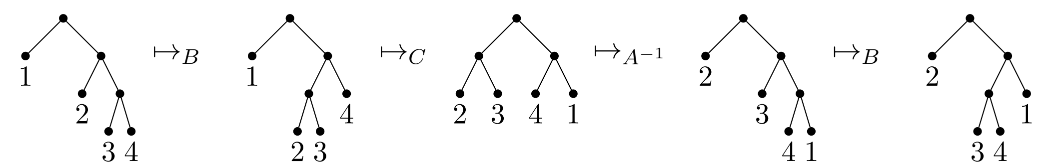
Lemma 11.3 *The generators A, B and C satisfy the relations*

1. $[B^{-1}A,\ ABA^{-1}] = 1_T$,
2. $[B^{-1}A,\ A^2BA^{-2}] = 1_T$,

3. $C = BCA^{-1}B$,
4. $B^2CA^{-2}B = ABA^{-1}BCA^{-1}$,
5. $AC = (BCA^{-1})^2$,
6. $C^3 = 1_T$.

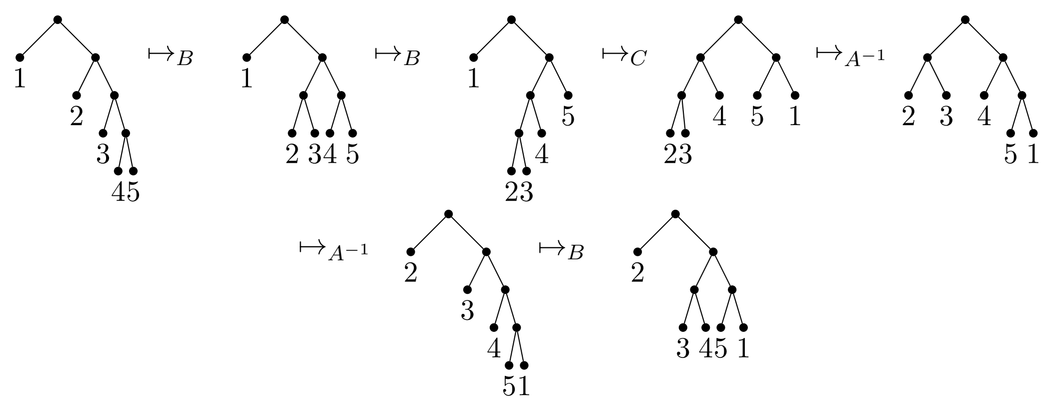
Proof

1. This follows from the finite presentation for F (Theorem 6.14),
2. This is also a relation in the finite presentation for F,
3. Consider the following sequence of $\mathcal{T}$-trees. Here, the $\mathcal{T}$-tree diagram representing C, is the one in Example 11.1 expanded in position 2, and the one for A^{-1} is the one for A in Example 5.4, expanded in position 3, and reversed to give the inverse.

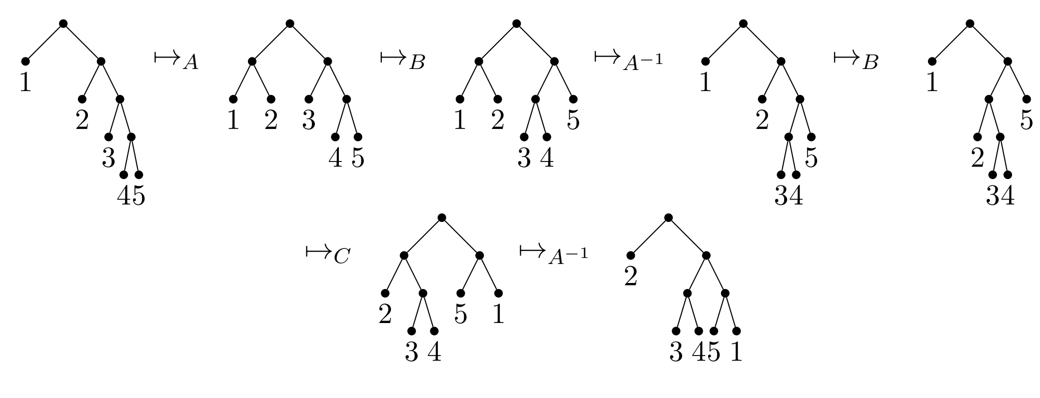


   Note that C can be represented by the first tree mapping to the last, by expanding the $\mathcal{T}$-tree diagram for C in Example 11.1 in position 3.

4. Consider the following sequence of $\mathcal{T}$-trees.



   Now consider



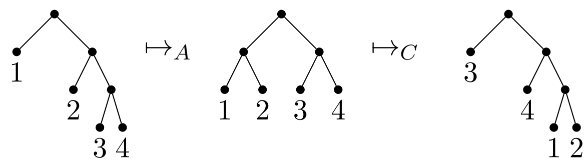
We can conclude that $B^2CA^{-2}B = ABA^{-1}BCA^{-1}$, as required.

6. From Example 11.1, we have that C has the tree representation $(R,\ R,\ (3\ 2\ 1))$, where R is the $\mathcal{T}$-tree given in Example 11.1. Since $(3\ 2\ 1)$ has order 3, we can conclude that $C^3 = 1_T$.

5. We have, using 3 and 6, that

$$(BCA^{-1})^2 = BCA^{-1}BCA^{-1} = BCA^{-1}B(C^{-1})^2A^{-1} = BCA^{-1}BC^{-1}C^{-1}A^{-1} = C^{-1}A^{-1},$$

so it suffices to show that $AC = C^{-1}A^{-1}$, that is $(AC)^2 = 1$. Consider the following sequence of $\mathcal{T}$-trees.



We can conclude that AC has the tree representation $(R,\ R,\ (1\ 3)(2\ 4))$, where R is the above (unlabelled) $\mathcal{T}$-tree. Since $(1\ 3)(2\ 4)$ is of order 2, we can conclude that $(AC)^2 = 1_T$.

□

Notation 11.4 For the remainder of this section, we will use T_1 to denote the group defined by the presentation $\langle \{A,\ B,\ C\} \mid R\rangle$, where A, B and C are formal symbols, and R is defined to be the set of relations stated in Lemma 11.3.

Lemma 11.5 *The subgroup $\langle A,\ B\rangle$ of T_1 is isomorphic to F.*

Proof Let $\phi\colon \{A,\ B\} \to T_1$ from the generating set of F to T_1, be defined by $A\phi = A$ and $B\phi = B$. By von Dyck's Theorem, the definition of T_1, and the finite presentation for F (Theorem 6.14), we have that ϕ extends to a homomorphism $\psi\colon F \to T_1$. By von Dyck's Theorem, and Lemma 11.3, there is an epimorphism θ from T_1 to T. We also have that $\operatorname{im}\psi\theta$ is precisely the subgroup of T (not T_1) generated by A and B, that is it is isomorphic to F, using the tree representation. Since F is not abelian, but Theorem 7.7 states that all proper quotients of F are abelian, the First Isomorphism Theorem implies that $\ker\psi\theta$ is trivial, and hence ψ is injective. It follows that $F \cong \operatorname{im}\psi = \langle A,\ B\rangle \leq T_1$. □

Definition 11.6 We will define elements X_n, for $n \in \mathbb{N}_0$, of T_1, as we did for F. Let $X_0 = A$, and $X_n = A^{n-1}BA^{-(n-1)}$, for $n \in \mathbb{N}$.

Let $C_0 = 1_{T_1}$ and $C_n = B^{n-1}CA^{-(n-1)}$, for $n \in \mathbb{N}$.

Let $f \in T_1$ be in the subgroup $\langle A,\ B\rangle$. Let f be expressible in the form

$$X_0^{-a_0}X_1^{-a_1}\cdots X_n^{-a_n}X_n^{b_n}\cdots X_1^{b_1}X_0^{b_0},$$

where $n,\ a_0,\ \ldots,\ a_n,\ b_0,\ \ldots,\ b_n \in \mathbb{N}_0$, satisfying the conditions of Corollary 6.7. If $a_0 = a_1 = \cdots = 0$, then f is called *positive*. If f is an inverse of a positive element, then f is called *negative*.

Lemma 11.7 *If $k,\ n \in \mathbb{N}$, such that $k \leq n$, then*

1. $X_k X_n =_{T_1} X_{n+1} X_k$, *if* $k < n$,

2. $C_n =_{T_1} C_{n+1} X_n$,

3. $X_k C_n =_{T_1} C_{n+1} X_{k-1}$,

4. $AC_n =_{T_1} C_{n+1}^2$.

Proof

1. From Lemma 11.5, we have that $\langle A,\ B\rangle$ is isomorphic to F, with A and B mapped to the functions in F of the same name. Thus the definition of X_n, for $n \in \mathbb{N}_0$, and the infinite presentation for F (Theorem 6.13) implies $X_k X_n =_{T_1} X_{n+1} X_k$, if $k < n$.

2. We have, using relation 3, that

   $C_n =_{T_1} B^{n-1} C A^{-(n-1)} =_{T_1} B^{n-1}(BCA^{-1}B)A^{-(n-1)} =_{T_1} B^n C A^{-n} A^{n-1} B A^{-(n-1)} =_{T_1} C_{n+1} X_n.$

3. We will first proceed by induction on n, when $k = 2$. If $k = 1$, then

   $$X_1 C_n =_{T_1} BC_n =_{T_1} BC_n A^{-1} A =_{T_1} C_{n+1} X_0.$$

   If $n = k = 2$, then, using relation 4, we have

   $$X_2 C_2 =_{T_1} ABA^{-1}BCA^{-1} =_{T_1} B^2 C A^{-2} B =_{T_1} C_3 X_1.$$

   Let $N \in \mathbb{N}$, and inductively suppose the result is true if $k = 2$ and $2 \leq n < N$. Then, if $n = N$, using Part 2, and the fact that $X_l X_m^{-1} = X_{m+1}^{-1} X_l$, for $l,\ m \in \mathbb{N}_0$, such that $l < m$, it follows that

   $$X_2 C_n =_{T_1} X_2 C_{n-1} X_{n-1}^{-1} =_{T_1} C_n X_1 X_{n-1}^{-1} =_{T_1} C_n X_n^{-1} X_1 =_{T_1} C_{n+1} X_1.$$

   We can conclude that if $k = 2$, then $X_k C_n = C_{n+1} X_{k-1}$, for all $n \in \mathbb{N},\ n \geq 2$. Recall that we have already shown this for $k = 1$. Let $K \in \mathbb{N}$, and inductively assume the result is true if $2 \leq k < K$. Then, if $k = K$, we have

   $$\begin{aligned} X_k C_n &=_{T_1} X_k B C_{n-1} A^{-1} \\ &=_{T_1} X_k X_1 C_{n-1} A^{-1} \\ &=_{T_1} X_1 X_{k-1} C_{n-1} A^{-1} \\ &=_{T_1} X_1 C_n X_{k-2} A^{-1} \\ &=_{T_1} BC_n A^{-1} A X_{k-2} A^{-1} \\ &=_{T_1} C_{n+1} X_{k-1}. \end{aligned}$$

4. We will proceed by induction on n. When $n = 1$, using relation 5, we have

   $$AC_1 =_{T_1} AC =_{T_1} (BCA^{-1})^2 =_{T_1} C_2^2.$$

   Let $N \in \mathbb{N}$. Assume inductively that the result is true if $n < N$. Then, if $n = N$ and $n \geq 2$, we have

   $$AC_n =_{T_1} ABC_{n-1}A^{-1}$$

$$
\begin{aligned}
&=_{T_1} ABA^{-1}AC_{n-1}A^{-1} \\
&=_{T_1} X_2C_n^2X_0^{-1} \\
&=_{T_1} C_{n+1}X_1C_nX_0^{-1} \\
&=_{T_1} C_{n+1}^2X_0X_0^{-1} \\
&=_{T_1} C_{n+1}^2.
\end{aligned}
$$

□

Lemma 11.8 *Let $n,\ m \in \mathbb{N}$ and $r \in \mathbb{N}_0$, such that $m \leq n+1$ and $r \leq n$. Then*

1.
$$
X_rC_n^m =_{T_1} \begin{cases} C_{n+1}^m X_{r-m} & r \geq m \\ C_{n+1}^{m+1} & r = m-1 \\ C_{n+1}^{m+1} X_{r+n+2-m} & r < m-1, \end{cases}
$$

2.
$$
C_n^m X_r^{-1} =_{T_1} \begin{cases} X_{r+m-(n+2)}^{-1} C_{n+1}^{m+1} & r \geq n+2-m \\ C_{n+1}^m & r = n+1-m \\ X_{r+m}^{-1} C_{n+1}^m & r \leq n-m, \end{cases}
$$

3. $C_n^m =_{T_1} C_{n+1}^m X_{n+1-m}$,
4. $C_n^m =_{T_1} X_{m-1}^{-1} C_{n+1}^{m+1}$,
5. $C_n^{n+2} =_{T_1} 1_{T_1}$.

Proof

1. We will first prove the case $r = m-1$. If $r = 0 = m-1$, then $r < n$, so by Lemma 11.7, Part 4, we have that
$$X_rC_n^m =_{T_1} X_0C_n =_{T_1} AC_n =_{T_1} C_{n+1}^2 =_{T_1} C_{n+1}^{m+1}.$$
If $r = m-1$ and $r > 0$, then $r \leq n$. So by Lemma 11.7, Part 3 and Part 4, we have
$$
\begin{aligned}
X_rC_n^m &=_{T_1} X_rC_n^{r+1} \\
&=_{T_1} C_{n+1}X_{r-1}C_n^r \\
&=_{T_1} \cdots \\
&=_{T_1} C_{n+1}^rX_0C_n \\
&=_{T_1} C_{n+1}^rAC_n \\
&=_{T_1} C_{n+1}^rC_{n+1}^2 \\
&=_{T_1} C_{n+1}^{m+1}.
\end{aligned}
$$

We will now prove the case $r \geq m$. Since $m > 0$, we have $r > 0$. Again, using Lemma 11.7, Part 3, we have

$$X_r C_n^m =_{T_1} C_{n+1} X_{r-1} C_n^{m-1} =_{T_1} \cdots =_{T_1} C_{n+1}^m X_{r-m}.$$

3. Using Lemma 11.7, Part 2 and Part 3, we can deduce that

$$C_n^m =_{T_1} C_n C_n^{m-1} =_{T_1} C_{n+1} X_n C_n^{m-1} =_{T_1} C_{n+1}^2 X_{n-1} C_n^{m-2} =_{T_1} \cdots =_{T_1} C_{n+1}^m X_{n-m+1}.$$

1. We finally show the case of when $r < m-1$. First note that $(r+1)-1 = r$. Therefore, by the case $r = m-1$, we have that $X_r C_n^{r+1} =_{T_1} C_{n+1}^{r+2}$. In addition, note that $r \leq n$. Using Part 3, and Lemma 11.7, Part 2, we have

$$\begin{aligned} X_r C_n^m &=_{T_1} X_r C_n^{r+1} C_n^{(m-1)-r} \\ &=_{T_1} C_{n+1}^{r+2} C_n^{(m-1)-r} \\ &=_{T_1} C_{n+1}^{r+2} C_{n+1}^{(m-1)-r} X_{n+1-((m-1)-r)} \\ &=_{T_1} C_{n+1}^{m+1} X_{r+n+2-m}. \end{aligned}$$

5. We will proceed by induction on n. First note $C_1^3 =_{T_1} 1_{T_1}$, by relation 6. Let $N \in \mathbb{N}$, and inductively suppose that $C_n^{n+2} =_{T_1} 1_{T_1}$ if $n = N$. If $n = N+1$, then, using Part 3, and Lemma 11.7, Part 4, we have

$$C_n^{n+2} =_{T_1} C_n^n C_n^2 =_{T_1} C_n^n X_0 C_{n-1} =_{T_1} C_{(n-1)+1}^n X_{(n-1)+1-n} C_{n-1} =_{T_1} C_{n-1}^{(n-1)+1} C_{n-1} =_{T_1} C_{n-1}^{n+1},$$

which is the identity, by induction.

2. We have, using Part 1, noting that $1 \leq n+2-m \leq n+1$, and Part 5, that

$$\begin{aligned} C_n^m X_r^{-1} &=_{T_1} C_n^{m-(n+2)} X_r^{-1} \\ &=_{T_1} (X_r C_{n+2-m})^{-1} \end{aligned}$$

$$=_{T_1} \begin{cases} (C_{n+1}^{n+2-m} X_{r-(n+2-m)})^{-1} & r \geq n+2-m \\ (C_{n+1}^{n+3-m})^{-1} & r = n+1-m \\ (C_{n+1}^{n+3-m} X_{r+n+2-(n+2-m)}^{-1})^{-1} & r < n-m \end{cases}$$

$$=_{T_1} \begin{cases} X_{r+m-(n+2)}^{-1} C_{n+1}^{m+1} & r \geq n+2-m \\ C_{n+1}^m & r = n+1-m \\ X_{r+m}^{-1} C_{n+1}^m & r < n-m. \end{cases}$$

4. By Part 1, noting that $0 \leq m-1 \leq n$, we have that $X_{m-1} C_n^m =_{T_1} C_{n+1}^{m+1}$. Left multiplying by X_{m-1}^{-1} gives $C_n^m =_{T_1} X_{m-1}^{-1} C_{n+1}^{m+1}$.

□

Lemma 11.9 *Every element of T_1 can be expressed in the form $p^{-1}C_n^m q$, where p and q are positive elements of T_1, and $n,\ m \in \mathbb{N}$, such that $m \leq n+1$.*

Proof Let H denote the set of all such elements of T_1. Let $i,\ j,\ k,\ l \in \mathbb{N}$. We will first show that $C_j^i C_l^k \in H$. By Lemma 11.8, we have that $C_j^{j+2} =_{T_1} C_l^{l+2} =_{T_1} 1_{T_1}$, so we if $i > j+2$ or $k > l+2$, we can redefine i and k to be $i \mod j+2$ and $k \mod l+2$, respectively, without changing $C_j^i C_k^l$. If $i = j+2$, or $k = l+2$, then at least one of C_j^i and C_l^k is the identity, so the expression is trivially in H, by taking $p = q = 1_{T_1}$. Therefore, we can assume $i < j+2$ and $k < l+2$.

By Lemma 11.8, Part 4, we have $C_j^i =_{T_1} X_{i-1}^{-1} C_{j+1}^{i+1}$, and by Part 3, $C_l^k =_{T_1} C_{l+1}^k X_{l+1-k}$. Thus

$$C_j^i C_l^k =_{T_1} X_{i-1}^{-1} C_{j+1}^{i+1} C_{l+1}^k X_{l+1-k}. \tag{3}$$

We will consider 3 cases:

Case 1: $j = l$.
It follows, using (3), that

$$C_j^i C_l^k =_{T_1} X_{i-1}^{-1} C_{j+1}^{i+1+k} X_{l+1-k},$$

which is in the form stated in the Lemma.

Case 2: $j < l$.
Using Lemma 11.8, Part 1, and (3) we can deduce the following series of equalities, noting that $i + r \leq j + r + 1$ and $(i + r + 1) - 1 = i + r$, for all $r \in \mathbb{N}_0$.

$$\begin{aligned} C_j^i C_l^k &=_{T_1} X_{i-1}^{-1} C_{j+1}^{i+1} C_{l+1}^k X_{l+1-k} \\ &=_{T_1} X_{i-1}^{-1} X_i^{-1} X_i C_{j+1}^{i+1} C_{l+1}^k X_{l+1-k} \\ &=_{T_1} X_{i-1}^{-1} X_i^{-1} C_{j+2}^{i+2} C_{l+1}^k X_{l+1-k} \\ &=_{T_1} \cdots \\ &=_{T_1} X_{i-1}^{-1} X_i^{-1} \cdots X_{(i-1)+(l-j)} C_{l+1}^{i+1+l-j} C_{l+1}^k X_{l+1-k} \\ &=_{T_1} X_{i-1}^{-1} X_i^{-1} \cdots X_{(i-1)+(l-j)} C_{l+1}^{i+1+k+l-j} X_{l+1-k}. \end{aligned}$$

Since the set of positive elements is closed under multiplication (Lemma 6.10), if this is not already in the desired form, the above positive elements can be rewritten to give the correct form.

Case 3: $j > l$.
Using Lemma 11.8, Part 2, and (3), and noting that $k + r = (k + r + 1) - k$ and $k + r \leq l + r + 1$, we have

$$\begin{aligned} C_j^i C_l^k &=_{T_1} X_{i-1}^{-1} C_{j+1}^{i+1} C_{l+1}^k X_{l+1-k} \\ &=_{T_1} X_{i-1}^{-1} C_{j+1}^{i+1} C_{l+1}^k X_{l-k}^{-1} X_{l-k} X_{l+1-k} \\ &=_{T_1} X_{i-1}^{-1} C_{j+1}^{i+1} C_{l+2}^k X_{l-k} X_{l+1-k} \end{aligned}$$

$$=_{T_1} \cdots$$

$$=_{T_1} X_{i-1}^{-1} C_{j+1}^{i+1} C_j^k X_{l+1-k+(j-l)} \cdots X_{l-k} X_{l+1-k}.$$

Again, since the set of positive elements is closed under multiplication, by Lemma 6.10, the above positive elements can be rewritten to give the correct form.

We can conclude that $C_j^i C_l^j \in H$. We will now prove that $H \leq T_1$. Lemma 11.8, Part 5 implies that H is closed under inversion. Let $g,\ h \in H$. We have that $g =_{T_1} p_1^{-1} C_j^i q_1$ and $h =_{T_1} p_2^{-1} C_l^k q_2$, where $p_1,\ p_2,\ q_1,\ q_2$ are positive, and $i,\ j,\ k,\ l \in \mathbb{N}_0$, such that $i < j+2$ and $k < l+2$. By Corollary 2.7, there exist positive elements p_3 and q_3, such that $q_1 p_2^{-1} =_{T_1} q_3^{-1} p_3$. We have

$$gh =_{T_1} p_1^{-1} C_j^i q_1 p_2^{-1} C_l^k q_2 =_{T_1} p_1^{-1} C_j^i q_3^{-1} p_3 C_l^k q_2.$$

If any of $i,\ j,\ k$ and l are equal to zero, then $C_j^i =_{T_1} 1_{T_1}$ or $C_l^k =_{T_1} 1_{T_1}$. Therefore, without loss of generality, if we assume i or j is zero, we have

$$gh =_{T_1} p_1^{-1} q_3^{-1} p_3 C_l^k q_2.$$

By Corollary 6.7, we have that $p_1^{-1} q_3^{-1} p_3 =_{T_1} p^{-1} q$, for positive elements p and q. Using Lemma 11.8, Part 1, we have that $q C_l^k =_{T_1} C_{l'}^{k'} q'$, where q' is a positive element, and $k',\ l' \in \mathbb{N}$. By Lemma 6.10, there is a positive element q_4, such that $q_4 = q' q_3$. Hence

$$gh =_{T_1} p^{-1} C_{l'}^{k'} q' q_3 =_{T_1} p^{-1} C_{l'}^{k'} q_4,$$

which is the desired form.

Now suppose $i,\ j,\ k$ and l are all strictly positive. Lemma 6.10 tells us that positive elements are closed under multiplication, so, using Lemma 11.8, Part 3 we may replace C_l^k with C_{l+1}^k, by redefining q_2. Hence we may assume that if $r \in \mathbb{N}_0$ is such that X_r occurs in p_3, then $l \geq r$. Similarly, using Lemma 11.8, Part 4, and redefining p_1, using Lemma 6.10, we may replace C_j^i with C_{j+1}^i, and therefore can assume that if $r \in \mathbb{N}_0$, such that X_r occurs in q_3, then $j \geq r$.

Thus, using Lemma 11.8, Part 1 and Part 2, and Lemma 6.10, we can rewrite gh as $p^{-1} C_{n_1}^{m_1} C_{n_2}^{m_2} q$, where p and q are positive elements, and $n,\ m \in \mathbb{N}$, such that $m \leq n+1$. We have already proven that $C_{n_1}^{m_1} C_{n_2}^{m_2} \in H$. It follows by Lemma 6.10, that $gh \in H$.

Note $C = C_1$, $A = C_0 X_0$, and $B = C_0 X_1$, so $T_1 = \langle A,\ B,\ C \rangle \leq H$. We can conclude that $H = T_1$. □

Theorem 11.10 *The group T_1 is simple.*

Proof Let N be a non-trivial normal subgroup of T_1, and let $\theta\colon T_1 \to {}^{T_1}\!/_{N}$ be the quotient homomorphism. Since N is non-trivial, there is a non-trivial element $g \in N$. Note $g\theta = 1$. By Lemma 11.9, we have that $g = p^{-1} C_n^m q$, for positive elements p and q, and $m,\ n \in \mathbb{N}_0$. Thus

$$(C_n^m)\theta = (pq^{-1})\theta.$$

In addition, by Lemma 11.8, Part 5, we have that $(C_n^m)^{n+2} =_{T_1} 1_{T_1}$. It follows that $((pq^{-1})^{n+2})\theta = 1$. Let $\phi\colon F \to T_1$ be the isomorphism from Lemma 11.5. Then $\phi\theta$ is an epimorphism from F

to ${}^{T_1}\!/_{N}$, so by the First Isomorphism Theorem, ${}^{T_1}\!/_{N}$ is a quotient of F. Since F is torsion-free (Theorem 2.11), if we consider the positive elements $p',\ q' \in F$, such that $p'\phi = p$ and $q'\phi = q$, then $p'q'^{-1} \neq 1$ implies $(p'q'^{-1})^{n+2} \neq 1$, so this is a proper quotient of F. Theorem 7.7 gives that ${}^{T_1}\!/_{N}$ is abelian.

Relation 4, gives that

$$(B^2CA^{-1})\theta(BA^{-1})\theta = (B^3CA^{-2})\theta = (B^2CA^{-2}B)\theta = (ABA^{-1}BCA^{-1})\theta = (B^2CA^{-1})\theta.$$

Hence $(BA^{-1})\theta = 1$, so $B\theta = A\theta$. Together with relation 3, we now have

$$(A\theta)(C\theta) = (BCA^{-1}B)\theta = (C\theta),$$

so $(A\theta) = (B\theta) = 1$, and hence $A,\ B \in N$. This together with relation 5, gives

$$(C\theta) = (AC)\theta = ((BCA^{-1})^2)\theta = (C\theta)(C\theta),$$

and so $C\theta = 1$, and we have $C \in N$. We can conclude that $T_1 = \langle A,\ B,\ C\rangle \leq N$, so $N = T_1$. $\square$

Corollary 11.11 *Thompson's group T has the presentation T_1.*

Proof Lemma 11.3 and von Dyck's Theorem (Theorem 6.12) imply that there is a homomorphism θ from T_1 to T, which is onto the generators of T, and hence is surjective. Since T is non-trivial, we have that $\ker\theta \neq T_1$. Since T_1 is simple, by Theorem 11.10, we have that $\ker\theta$ is trivial, and hence θ is injective. $\square$

Corollary 11.12 *Thompson's group T is simple.*

Proof Corollary 11.11 gives that $T \cong T_1$, and so Theorem 11.10 gives that T_1 is simple. $\square$

12 Thompson's Group V

Definition 12.1 Let $\{0,\ 1\}^*$ be the set of all finite sequences of elements of $\{0,\ 1\}$, and $\{0,\ 1\}^\omega$ be the set of all infinite sequences of elements of $\{0,\ 1\}$. Define the *Cantor set* to be $\{0,\ 1\}^\omega$. If $n \in \mathbb{N}$, we will use $\{0,\ 1\}^n$ to denote the set of sequences of length n.

Giving $\{0,\ 1\}$ the discrete topology, we can induce a topology on the Cantor set, under product topology. The Cantor set together with this topology, is called the *Cantor space*, denoted $\mathfrak{C}$.

Let $\alpha \in \{0,\ 1\}^*,\ \beta \in \{0,\ 1\}^* \cup \{0,\ 1\}^\omega$. We will use $\alpha\beta$ to denote concatenation, and $\alpha\mathfrak{C}$ to denote the set of all elements of $\mathfrak{C}$ with α as a prefix. We will use $\alpha \preceq \beta$ to state that α is a prefix of β. If $\alpha \not\preceq \beta$ and $\beta \not\preceq \alpha$, we will say α and β are *incomparable*, and denote this $\alpha \perp \beta$.

Remark 12.2 Recall the tree $\mathcal{T}$, from Example 3.3. Elements of $\{0,\ 1\}^*$ are the associated words of vertices of $\mathcal{T}$, and elements of $\mathfrak{C}$ are the associated words of 'boundary points'.

Lemma 12.3 *The relation $\preceq$ is a partial order on $\{0,\ 1\}^*$.*

Proof Reflexive: Let $\alpha \in \{0, 1\}^*$. Then $\alpha = \alpha \cdot \varepsilon$, so $\alpha \preceq \alpha$.

Anti-symmetric: Let $\alpha, \beta \in \{0, 1\}^*$, such that $\alpha \preceq \beta$ and $\beta \preceq \alpha$. Then $\alpha = \beta\gamma$ and $\beta = \alpha\delta$, for some $\gamma, \delta \in \{0, 1\}^*$. Then $\alpha = \alpha\delta\gamma$, so $\delta\gamma = \varepsilon$. It follows that $\delta = \gamma = \varepsilon$, so $\alpha = \beta$.

Transitive: Suppose $\alpha, \beta, \gamma \in \{0, 1\}^*$, such that $\alpha \preceq \beta \preceq \gamma$. Then $\beta = \alpha\delta_1$ and $\gamma = \beta\delta_2$, for some $\delta_1, \delta_2 \in \{0, 1\}^*$. Note $\delta_1\delta_2 \in \{0, 1\}^*$. Since $\alpha\delta_1\delta_2 = \gamma$, we have that $\alpha \preceq \gamma$. $\square$

Lemma 12.4 *Let $\alpha \in \{0, 1\}^*$. Then $\alpha\mathfrak{C}$ is homeomorphic to $\mathfrak{C}$.*

Proof Define

$$\begin{aligned} \phi\colon \mathfrak{C} &\to \alpha\mathfrak{C} \\ \beta &\mapsto \alpha\beta. \end{aligned}$$

We have that ϕ is surjective, by the definition of $\alpha\mathfrak{C}$. Let $\beta, \gamma \in \mathfrak{C}$, such that $\beta\phi = \gamma\phi$. Then $\alpha\beta = \alpha\gamma$, and $\beta = \gamma$. We now have that ϕ is a bijection.

Let $U \subseteq \mathfrak{C}$ be open. Then U is the Cartesian product of $\aleph_0$ subsets of $\{0, 1\}$, such that all but finitely many of them are $\{0, 1\}$. Then $U\phi$ is the Cartesian product of the singletons, the elements of which, when taken in order as a string, form α, and so themselves are finitely many subsets of $\{0, 1\}$. Hence $U\phi$ is the Cartesian product of $\aleph_0$ subsets of $\{0, 1\}$, all but finitely many of which are $\{0, 1\}$, and hence $U\phi$ is open. Hence ϕ^{-1} is a homeomorphism. $\square$

Definition 12.5 A subset $X \subseteq \{0, 1\}^*$ is called an *antichain*, if $\alpha \perp \beta$ for all $\alpha, \beta \in X$. If, in addition, for all $\gamma \in \mathfrak{C}$, there exists $\delta \in X$, such that $\delta \preceq \gamma$, then X is called a *complete antichain*.

Definition 12.6 Let f be a homeomorphism of $\mathfrak{C}$. We will define a partial action of f on elements of $\{0, 1\}^*$. Let $\alpha \in \{0, 1\}^*$. If there exists $\beta \in \{0, 1\}^*$, such that $(\alpha\gamma)f = \beta\gamma$, for all $\gamma \in \mathfrak{C}$, then we write $\alpha f = \beta$.

Let f be a homeomorphism of $\mathfrak{C}$. We will call f a *prefix exchange map* of the Cantor space, if there exist complete antichains $X, Y \subseteq \{0, 1\}^*$, under $\preceq$, such that f maps X bijectively to Y under the action of f on $\{0, 1\}^*$. The antichains X and Y are called the *domain antichain* and *range antichain* of f, respectively. Note that they are not unique.

Lemma 12.7 *The partial action of homeomorphisms of $\mathfrak{C}$ on $\{0, 1\}^*$, as defined in Definition 12.6, is well-defined.*

Proof Let f, g be homeomorphisms of $\mathfrak{C}$ and $\alpha \in \{0, 1\}^*$. Suppose the action of f on α is defined. Let $\beta = \alpha f$. Assume also that the action of g on β is defined, and let $\gamma = \beta g$. We must show $((\alpha)f)g = (\alpha)(fg)$. We have that $(\alpha f)g = \beta g = \gamma$. Let $\delta \in \mathfrak{C}$. We have that $(\alpha\delta)(fg) = (\beta\delta)g = \gamma\delta$, so $(\alpha)(fg) = \gamma$. $\square$

Lemma 12.8 *Two prefix exchange maps f and g are equal if and only if there exists a complete antichain $X \subseteq \{0, 1\}^*$ such that $\alpha f = \alpha g$, for all $\alpha \in X$, and Xf is a complete antichain.*

Proof ($\Rightarrow$): Suppose $f = g$. Since f is a prefix exchange map, there exist complete antichains $X, Y \subseteq \{0, 1\}^*$, such that $Xf = Y$. Since $f = g$, we have $\alpha f = \alpha g$, for all $\alpha \in X$.

($\Leftarrow$): Suppose there is a complete antichain $X \subseteq \{0,\ 1\}^*$ such that $Xf = Xg$ is a complete antichain. Let $\alpha \in \mathfrak{C}$. Since X is a complete antichain, there exists $\beta \in X$ such that $\beta \preceq \alpha$, and $\gamma \in \mathfrak{C}$ such that $\alpha = \beta\gamma$. Let $\delta = \beta f$, and note $\delta = \beta g$. Then

$$\alpha f = (\beta\gamma)f = \delta\gamma = (\beta\gamma)g = \alpha g,$$

so $f = g$. $\square$

Definition 12.9 We will define *Thompson's group* V to be the set of prefix exchange maps of the Cantor space, under composition of functions.

Theorem 12.10 *There is a bijection from Thompson's group V to the set of $\mathcal{T}$-tree diagram classes, under the multiplication defined in Definition 4.17, that preserves multiplication.*

Proof Throughout this proof, we will consider $\mathcal{W}(R)$, for any $\mathcal{T}$-tree R, as totally ordered under the lexicographic order induced by $0 \leq 1$. Since this set is finite, we can refer to the ith element of it, for any valid i.

Let V_1 denote the set of $\mathcal{T}$-tree diagram classes, under the stated multiplication. Let $X \in V_1$. Let $(R,\ S,\ \sigma) \in X$, and note that $\mathcal{W}(R)$ and $\mathcal{W}(S)$ are complete antichains of $\mathfrak{C}$. Let $f_X \in V$ be the element that maps the ith element of $\mathcal{W}(R)$ to the $i\sigma$th element of $\mathcal{W}(S)$. Define

$$\begin{aligned} \phi\colon V_1 &\to V \\ X &\mapsto f_X. \end{aligned}$$

We will first show ϕ is well-defined. Let $X \in V_1$, and $(R,\ S,\ \sigma),\ (U,\ W,\ \tau) \in X$.

Case 1: $(R,\ S,\ \sigma)$ is an elementary expansion of $(U,\ W,\ \tau)$.
Let u and v be the associated words of the leaves of U and W that are expanded. Then

$$\mathcal{W}(R) = (\mathcal{W}(U)\backslash\{u\}) \cup \{u0,\ u1\}, \quad \mathcal{W}(S) = (\mathcal{W}(W)\backslash\{v\}) \cup \{v0,\ v1\}.$$

Note that these are still complete antichains as we have replaced u and v with complete antichains of $u\mathfrak{C}$ and $v\mathfrak{C}$. Let f_X be the prefix exchange map that maps the ith element of $\mathcal{W}(R)$ to the $i\sigma$th of $\mathcal{W}(S)$, and g_X be the prefix exchange map that sends the jth element of U to the $j\tau$th of W. Let f_X and g_X be the images under ϕ of X, using the representatives $(R,\ S,\ \sigma)$ and $(U,\ W,\ \tau)$, respectively. We have that f_X agrees with g_X on all of $\mathcal{W}(U)$, except perhaps u. Since $(u0)f_X = v0$ and $(u1)f_X = v1$, we have that $(u0\alpha)f_X = v0\alpha$ and $(u1\alpha)f_X = v1\alpha$, for all $\alpha \in \mathfrak{C}$. Note that $\{1\alpha \mid \alpha \in \mathfrak{C}\} \cup \{0\alpha \mid \alpha \in \mathfrak{C}\} = \mathfrak{C}$. Hence $(u\alpha)f_X = v\alpha$, for all $\alpha \in \mathfrak{C}$, and we can conclude that $uf_X = v = g_X$. Since f_X and g_X agree on all points in $\mathcal{W}(U)$, which is a complete antichain, we have that $f_X = g_X$, by Lemma 12.8.

Case 2: There is any sequence of elementary contractions and expansions taking $(R,\ S,\ \sigma)$ to $(U,\ W,\ \tau)$. By Lemma 5.2, there is a reduced $\mathcal{T}$-tree diagram $(P,\ Q,\ \rho) \in X$. By Case 1, the images of ϕ calculated using $(R,\ S,\ \sigma)$ and $(P,\ Q,\ \rho)$ are the same. Similarly, the images calculated using $(U,\ W,\ \tau)$ and $(P,\ Q,\ \rho)$ are equal. We can conclude that the images calculated using $(R,\ S,\ \sigma)$ and $(U,\ W,\ \tau)$ coincide.

Injective: Let X and Y be $\mathcal{T}$-tree diagram classes such that $f_X = f_Y$. Let $(R,\ S,\ \sigma) \in X$ and $(U,\ W,\ \tau) \in Y$, such that $R = U$; we can do this by picking large enough tree pairs. Since $f_X = f_Y$, the ith element of $\mathcal{W}(R)$ is mapped by f to the $i\sigma$th element of $\mathcal{W}(S)$, and the $i\tau$th element of $\mathcal{W}(W)$. Hence $\mathcal{W}(S) = \mathcal{W}(W)$ and $\sigma = \tau$. We can conclude that $X = Y$.

Surjective: Let $f \in V$. Then there exist complete antichains $A,\ B \subseteq \{0,\ 1\}^*$, such that f maps A bijectively to B. Since this mapping is bijective, and A and B are finite, define $\sigma \in S_{|A|}$, such that f maps the ith element of A to the $i\sigma$th of B, for all valid i. In addition, since A and B are complete antichains, we can define $\mathcal{T}$-trees R and S, such that $\mathcal{W}(R) = A$ and $\mathcal{W}(S) = B$. Let $X = [(R,\ S,\ \sigma)]$. Then $X\phi$ will be a prefix exchange map that sends the ith element of A to the $i\sigma$th element of B. By Lemma 12.8, $f = X\phi$.

Respects multiplication: Let X and Y be $\mathcal{T}$-tree diagram classes. Using Lemma 4.18, we can choose $\mathcal{T}$-tree diagrams $(R,\ S,\ \sigma) \in X$ and $(U,\ W,\ \tau) \in Y$, such that $S = U$. Then $XY = [(R,\ W,\ \sigma\tau)]$. Hence $(XY)\phi$ will map the ith element of $\mathcal{W}(R)$ to the $i\sigma\tau$th element of W, for all valid i. In addition, $X\phi$ will map the ith element of $\mathcal{W}(R)$ to the $i\sigma$th of $\mathcal{W}(U)$, and $Y\phi$ will map the $i\sigma$th element of $\mathcal{W}(U)$ to the $i\sigma\tau$th element on $\mathcal{W}(W)$. Hence we have a complete antichain that $(XY)\phi$ and $(X\phi)(Y\phi)$ coincide on, and both map to a complete antichain. By Lemma 12.8, $(XY)\phi = (X\phi)(Y\phi)$. $\square$

Corollary 12.11 *Thompson's group V is a group.*

Proof Note that V is a subset of the homeomorphism group of the Cantor space, so it suffices to show that V is closed under multiplication and inversion. We will show the closure of multiplication using the definition of V, and the closure of inversion in the set of $\mathcal{T}$-tree classes, using Theorem 12.10, without proving directly that either is a group, but using the bijection that respects multiplication to conclude they are both groups.

Since the set of $\mathcal{T}$-tree diagram classes is closed under its multiplication, Theorem 12.10 implies that V is closed under multiplication. Let $f \in V$. Let X be the $\mathcal{T}$-tree class that is the image of f under the bijection in Theorem 12.10. Let $(R,\ S,\ \sigma) \in X$. Let $Y = [(S,\ R,\ \sigma^{-1})]$. Then

$$XY = [(R,\ R,\ \sigma\sigma^{-1})] = [(R,\ R,\ \mathrm{id})] = [(R,\ R,\ \sigma^{-1}\sigma)] = YX.$$

Let $g \in V$, such that the image of g under the bijection in Theorem 12.10 is $XY = YX$. Let $\alpha \in \mathfrak{C}$. Then there exists $\beta \in \mathcal{W}(R)$, such that $\beta \preceq \alpha$, since $\mathcal{W}(R)$ is a complete antichain. Let $\gamma \in \mathfrak{C}$, such that $\alpha = \beta\gamma$. Then $\alpha g = (\beta\gamma)g = \beta\gamma = \alpha$, so g is the identity function, which is the identity of the homeomorphism group of the Cantor space. Hence the pre-image of Y under the bijection in Theorem 12.10 is the inverse of f, and V is closed under taking inverses. $\square$

13 An Infinite Presentation for V

Definition 13.1 Let $\alpha,\ \beta \in \mathfrak{C}$, such that $\alpha \perp \beta$. The *swap* $t_{\alpha,\beta}$, is defined to be

$$t_{\alpha,\beta} \colon \mathfrak{C} \to \mathfrak{C}$$
$$x \mapsto \begin{cases} x & x \notin \alpha\mathfrak{C} \cup \beta\mathfrak{C} \\ \alpha y & x = \beta y \text{ for some } y \in \mathfrak{C} \\ \beta y & x = \alpha y \text{ for some } y \in \mathfrak{C} \end{cases}$$

If, in addition, $\{\alpha,\ \beta\} \neq \{0,\ 1\}$, then $t_{\alpha,\beta}$ is called a *small swap*.

Lemma 13.2 *Swaps are elements of V, and small swaps generate V.*

Proof Let $\alpha,\ \beta \in \{0,\ 1\}^*$, such that $\alpha \perp \beta$. Let $n = \max(|\alpha|,\ |\beta|)$, and

$$X = \{\alpha,\ \beta\} \cup \{\gamma \in \{0,\ 1\}^* \mid |\gamma| = n,\ \alpha \not\preceq \gamma,\ \beta \not\preceq \gamma\}.$$

Without loss of generality, assume $|\alpha| \geq |\beta|$. We have that $\alpha \perp \gamma$ for all $\gamma \in X \backslash \{\alpha\}$, since if two words have the same finite length, then neither can be a prefix of the other. Note that all elements of $X \backslash \{\beta\}$ cannot have β as a prefix. In addition, $\gamma \not\preceq \beta$, for any $\gamma \in X \backslash \{\beta\}$, since $|\gamma| \geq |\beta|$, and $\beta \not\preceq \gamma$. We can conclude that β is not comparable to any other element of X, and it follows that X is an antichain.

Let $\delta \in \mathfrak{C}$. Since δ is an infinite length word, it has a finite prefix γ of length n. Either $\gamma \in X \backslash \{\beta\}$, or it has β as a prefix. Therefore, δ has a prefix in X, and we can conclude that X is a complete antichain.

Let $\delta \in \mathfrak{C}$. Choose a prefix $\gamma \in X$ of δ, and write $\delta = \gamma\zeta$, where $\zeta \in \mathfrak{C}$. If $\gamma \notin \{\alpha,\ \beta\}$, then $(\gamma\zeta)t_{\alpha,\beta} = \gamma\zeta$, so $\gamma t_{\alpha,\beta} = \gamma$. Otherwise, we have

$$(\alpha\zeta)t_{\alpha,\beta} = \beta\zeta, \quad (\beta\zeta)t_{\alpha,\beta} = \alpha\zeta,$$

so $\alpha t_{\alpha,\beta} = \beta$ and $\beta t_{\alpha,\beta} = \alpha$. We can conclude that $X t_{\alpha,\beta} = X$, so $t_{\alpha,\beta}$ is a prefix exchange map, and hence an element of V.

We will proceed by induction on the length of complete antichains of elements, to prove that small swaps generate V. If $f \in V$ maps the complete antichain X to the complete antichain Y, where $|X| = 1$, then $X = \{\varepsilon\}$, and $f = \mathrm{id}_{\mathfrak{C}}$. We have that $f = t_{0,10}t_{0,10}$.

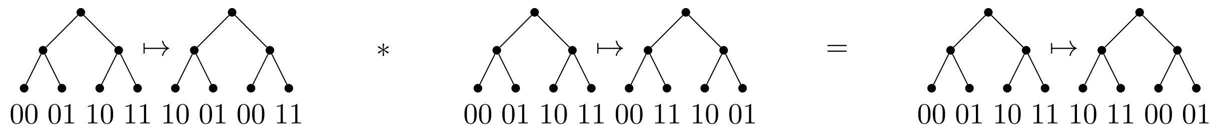
Let $n \in \mathbb{N}$, and inductively suppose that $f \in V$ is a product of small swaps, if there is a complete antichain X such that $|X| < n$, and f maps X bijectively to a complete antichain.

Let $f \in V$. Let D and R be complete antichains, such that $Df = R$, and $|D| = |R| = n$. We will also assume they are minimal, using the tree representation for V, and Lemma 4.7. If they are not in this form, then an elementary contraction can be applied to give a pair of antichains with cardinality smaller than n. Let

$$D = \{\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n\}, \quad R = \{\beta_1,\ \beta_2,\ \ldots,\ \beta_n\},$$

where $n \in \mathbb{N}$, such that the above elements are listed in lexicographic order. Let $\sigma \in S_n$, such that the ith element of D is mapped by f to the $i\sigma$th element of R. Using the tree representation for V, we can choose indices $i,\ j \in \{1,\ \ldots,\ n\}$, such that the leaves representing $\alpha_i,\ \alpha_{i+1}$ and $\beta_j,\ \beta_{j+1}$ lie in carets. Therefore, there exist $\alpha,\ \beta \in \{0,\ 1\}^*$, such that $\alpha_i = \alpha 0,\ \alpha_{i+1} = \alpha 1,\ \beta_j = \beta 0$ and $\beta_{j+1} = \beta 1$.

Case 1: $\alpha = \beta = \varepsilon$.
Then $\alpha_i = \beta_j 0,\ \alpha_{i+1} = \beta_{j+1} = 1$. Since this is a complete antichain, we have just listed all elements of D and R. Using the tree representation, we have



which implies $f = t_{00,10}t_{01,11}$.

Case 2: One of $\alpha,\ \beta$ is not ε, and $\beta_{(i+1)\sigma} \neq \beta_j$.
Since the complete antichains have the same length, if one is ε, then so is the other, so we have that both α and β are not ε. Note that since $\beta_{(i+1)\sigma} \neq \beta_j$, and these are part of an antichain, we have that $\beta_{(i+1)\sigma} \perp \beta_j = \beta 0$. Note also that $\beta_{i\sigma} \perp \beta_{(i+1)\sigma}$. Hence

$$(\alpha 0) f t_{\beta 0,\beta_{i\sigma}} t_{\beta 1,\beta_{(i+1)\sigma}} = \beta 0$$
$$(\alpha 1) f t_{\beta 0,\beta_{i\sigma}} t_{\beta 1,\beta_{(i+1)\sigma}} = \beta 1.$$

Let $\gamma \in D\backslash\{\alpha 0,\ \alpha 1\}$. Then $\gamma f \notin \{\beta_{i\sigma},\ \beta_{(i+1)\sigma}\}$, so

$$(\gamma) f t_{\beta 0,\beta_{i\sigma}} t_{\beta 1,\beta_{(i+1)\sigma}} \in R\backslash\{\beta 1,\ \beta 0\}.$$

We can therefore apply an elementary contraction to the $\mathcal{T}$-tree diagram representing $f t_{\beta 0,\beta_{i\sigma}} t_{\beta 1,\beta_{(i+1)\sigma}}$, with domain antichain D and range antichain R, to obtain a diagram with domain antichain $(D\backslash\{\alpha 0,\ \alpha 1\}) \cup \{\alpha\}$, and range antichain $(R\backslash\{\beta 0,\ \beta 1\}) \cup \{\beta\}$. Since these have cardinalities smaller than n, our inductive hypothesis applies, and $f t_{\beta 0,\beta_{i\sigma}} t_{\beta 1,\beta_{(i+1)\sigma}}$ is a product of small swaps. Multiplying on the right by $t_{\beta 1,\beta_{(i+1)\sigma}} t_{\beta 0,\beta_{i\sigma}}$, which are their own inverses, gives f as a product of small swaps.

Case 3: One of $\alpha,\ \beta$ is not ε, and $\beta_{(i+1)\sigma} = \beta_j$.
We can assume the above statement is true for all valid choices of α and β, otherwise by redefining α or β we are in Case 2. It therefore follows that the reduced tree representation for f has a unique caret; if there were two, we would be able to move one of α or β to be in Case 2. If this caret is the entire tree, then we are in Case 1. Otherwise, there is an element $\gamma \in R\backslash\{\beta 0,\ \beta 1\}$. Redefining f by post-multiplying by $t_{\gamma,\beta 0}$ would preserve the size of D and R, but result in

$$\beta_{(i+1)\sigma} = (\alpha 0) f = \gamma \neq \beta 0 = \beta_j,$$

so we are in Case 2. $\square$

Notation 13.3 We define a convenient notation for the partial action of swaps on $\{0,\ 1\}^*$. Define $(\alpha\ \beta)$ to be the swap $t_{\alpha,\beta}$, when acting on $\{0,\ 1\}^*$, for any $\alpha,\ \beta \in \{0,\ 1\}^*$, $\alpha \perp \beta$. Note that

$$\gamma(\alpha\ \beta) = \begin{cases} \beta\delta & \gamma = \alpha\delta,\ \text{for some } \delta \in \{0,\ 1\}^* \\ \alpha\delta & \gamma = \beta\delta,\ \text{for some } \delta \in \{0,\ 1\}^* \\ \gamma & \gamma \perp \alpha,\ \gamma \perp \delta \\ \text{undefined} & \text{otherwise,} \end{cases}$$

for any $\gamma \in \{0,\ 1\}^*$.

Lemma 13.4 *Let $\alpha,\ \beta,\ \gamma,\ \delta \in \{0,\ 1\}^*$, such that $\alpha \perp \beta$ and $\gamma \perp \delta$. Then*

1. $t^2_{\alpha,\beta} = 1_V$,

2. $t^{t_{\gamma,\delta}}_{\alpha,\beta} = t^{-1}_{\gamma,\delta} t_{\alpha,\beta} t_{\gamma,\delta} = t_{\alpha(\gamma\ \delta),\beta(\gamma\ \delta)}$, *whenever* $\alpha(\gamma\ \delta)$ *and* $\beta(\gamma\ \delta)$ *are defined,*

3. $t_{\alpha,\beta} = t_{\alpha 0,\beta 0} t_{\alpha 1,\beta 1}$.

Proof

1. Let $\gamma \in \mathfrak{C}$. If $\gamma = \alpha\delta$, for some $\delta \in \mathfrak{C}$, then

$$\gamma t^2_{\alpha,\beta} = (\alpha\delta) t^2_{\alpha,\beta} = (\beta\delta) t_{\alpha,\beta} = \alpha\delta = \gamma.$$

Similarly, if $\gamma = \beta\delta$, for some $\delta \in \mathfrak{C}$, then

$$\gamma t^2_{\alpha,\beta} = (\beta\delta) t^2_{\alpha,\beta} = (\alpha\delta) t_{\alpha,\beta} = \beta\delta = \gamma.$$

Otherwise, $\gamma \notin \alpha\mathfrak{C} \cup \beta\mathfrak{C}$, so

$$\gamma t^2_{\alpha,\beta} = \gamma.$$

2. Suppose $\alpha(\gamma\ \delta)$ and $\beta(\gamma\ \delta)$ are defined.

Case 1: $\alpha \perp \gamma,\ \alpha \perp \delta,\ \beta \perp \gamma,\ \beta \perp \delta$.
Then $\alpha(\gamma\ \delta) = \alpha$ and $\beta(\gamma\ \delta) = \beta$. Let $x \in \mathfrak{C}$. If $x = \alpha y$, for some $y \in \mathfrak{C}$, then

$$x t^{-1}_{\gamma,\delta} t_{\alpha,\beta} t_{\gamma,\delta} = (\alpha y) t_{\alpha,\beta} t_{\gamma,\delta} = (\beta y) t_{\gamma,\delta} = \beta y = (\alpha y) t_{\alpha,\beta} = x t_{\alpha(\gamma\ \delta),\ \beta(\gamma\ \delta)}.$$

Similarly, if $x = \beta y$, for some $y \in \mathfrak{C}$, then

$$x t^{-1}_{\gamma,\delta} t_{\alpha,\beta} t_{\gamma,\delta} = (\beta y) t_{\alpha,\beta} t_{\gamma,\delta} = (\alpha y) t_{\gamma,\delta} = \alpha y = (\alpha y) t_{\alpha,\beta} = x t_{\alpha(\gamma\ \delta),\ \beta(\gamma\ \delta)}.$$

If $x = \gamma y$ or $x = \delta y$, for some $y \in \mathfrak{C}$, then x is fixed by $t_{\alpha,\beta}$, so is fixed by $t^{-1}_{\gamma,\delta} t_{\alpha,\beta} t_{\gamma,\delta}$ and $t_{\alpha(\gamma\ \delta),\ \beta(\gamma\ \delta)}$. Finally, if $x \notin \alpha\mathfrak{C} \cup \beta\mathfrak{C} \cup \gamma\mathfrak{C} \cup \delta\mathfrak{C}$, then x is fixed by all of the mentioned swaps.

Case 2: One of α and β has either γ or δ as a prefix.
Without loss of generality, we will assume $\alpha \perp \gamma,\ \alpha \perp \delta$ and $\gamma\zeta = \beta$. Note that $\beta \perp \delta$. We have that $\alpha(\gamma\ \delta) = \alpha$, and $\beta(\gamma\ \delta) = \delta\zeta$. Let $x \in \mathfrak{C}$. If $x = \alpha y$, for some $y \in \mathfrak{C}$, then

$$x t^{-1}_{\gamma,\delta} t_{\alpha,\beta} t_{\gamma,\delta} = (\alpha y) t_{\alpha,\beta} t_{\gamma,\delta} = (\beta y) t_{\gamma,\delta} = \delta\zeta y = (\alpha y) t_{\alpha,\delta\zeta} = x t_{\alpha(\gamma\ \delta),\ \beta(\gamma\ \delta)}.$$

If $x = \beta y$, for some $y \in \mathfrak{C}$, then

$$x t^{-1}_{\gamma,\delta} t_{\alpha,\beta} t_{\gamma,\delta} = (\delta\zeta y) t_{\alpha,\beta} t_{\gamma,\delta} = (\delta\zeta y) t_{\gamma,\delta} = \beta y = (\beta y) t_{\alpha,\delta\zeta} = x t_{\alpha(\gamma\ \delta),\ \beta(\gamma\ \delta)}.$$

If $x \in \gamma\mathfrak{C} \backslash \beta\mathfrak{C}$, then x is fixed by $t_{\alpha,\beta}$, and hence by $t^{-1}_{\gamma,\delta} t_{\alpha,\beta} t_{\gamma,\delta}$. It is also fixed by $t_{\alpha(\gamma\ \delta),\ \beta(\gamma\ \delta)} = t_{\alpha,\delta\zeta}$.

If $x = \delta y$, for some $y \in \mathfrak{C}$, then

$$x t^{-1}_{\gamma,\delta} t_{\alpha,\beta} t_{\gamma,\delta} = (\gamma y) t_{\alpha,\beta} t_{\gamma,\delta} = (\gamma y) t_{\gamma,\delta} = \delta y = (\delta y) t_{\alpha,\delta\zeta} = x t_{\alpha(\gamma\ \delta),\ \beta(\gamma\ \delta)}.$$

Otherwise, if $x \notin \alpha\mathfrak{C} \cup \beta\mathfrak{C} \cup \gamma\mathfrak{C} \cup \delta\mathfrak{C}$, then it is fixed by all the swaps in question.

Case 3: One of γ and δ is a prefix of α, and the other is a prefix of δ.
Without loss of generality, we will assume $\alpha = \gamma\eta$ and $\beta = \delta\zeta$, where $\eta,\ \zeta \in \{0,\ 1\}^*$. Then $\alpha(\gamma\ \delta) = \delta\eta$, and $\beta(\gamma\ \delta) = \gamma\zeta$. Let $x \in \mathfrak{C}$. Note $\delta\eta \perp \gamma\zeta$, since $\gamma \perp \delta$. If $x = \alpha y$, for some $y \in \mathfrak{C}$, then

$$xt_{\gamma,\delta}^{-1}t_{\alpha,\beta}t_{\gamma,\delta} = (\delta\eta y)t_{\alpha,\beta}t_{\gamma,\delta} = (\delta\eta y)t_{\gamma,\delta} = \gamma\eta y = \alpha y = (\alpha y)t_{\delta\eta,\gamma\zeta} = xt_{\alpha(\gamma\ \delta),\ \beta(\gamma\ \delta)}.$$

If $x = \beta y$, for some $y \in \mathfrak{C}$, then

$$xt_{\gamma,\delta}^{-1}t_{\alpha,\beta}t_{\gamma,\delta} = (\gamma\zeta y)t_{\alpha,\beta}t_{\gamma,\delta} = (\gamma\zeta y)t_{\gamma,\delta} = \delta\zeta y = \beta y = (\beta y)t_{\delta\eta,\gamma\zeta} = xt_{\alpha(\gamma\ \delta),\ \beta(\gamma\ \delta)}.$$

If $x \in (\gamma\mathfrak{C} \cup \delta\mathfrak{C})\backslash(\alpha\mathfrak{C} \cup \beta\mathfrak{C})$, then x is fixed by $t_{\alpha,\beta}$, and hence by $t_{\gamma,\delta}^{-1}t_{\alpha,\beta}t_{\gamma,\delta}$. In addition, x is fixed by $t_{\delta\eta,\gamma\zeta} = t_{\alpha(\gamma\ \delta),\ \beta(\gamma\ \delta)}$.

Otherwise, if $x \notin \alpha\mathfrak{C} \cup \beta\mathfrak{C} \cup \gamma\mathfrak{C} \cup \delta\mathfrak{C}$, then it is fixed by all the swaps in question.

3. Let $x \in \mathfrak{C}$. If $x = \alpha 0 y$,

$$xt_{\alpha 0,\beta 0}t_{\alpha 1,\beta 1} = (\beta 0 y)t_{\alpha 1,\beta 1} = \beta 0 y = (\alpha 0 y)t_{\alpha,\beta} = xt_{\alpha,\beta}.$$

If $x = \alpha 1 y$, then

$$xt_{\alpha 0,\beta 0}t_{\alpha 1,\beta 1} = (\alpha 1 y)t_{\alpha 1,\beta 1} = \beta 1 y = (\alpha 1 y)t_{\alpha,\beta} = xt_{\alpha,\beta}.$$

By symmetry, if $x \in \beta\mathfrak{C}$, then $xt_{\alpha 0,\beta 0}t_{\alpha 1,\beta 1} = xt_{\alpha,\beta}$. Finally, if $x \notin \alpha\mathfrak{C} \cup \beta\mathfrak{C}$, then all the swaps in question fix x.

$\square$

Definition 13.5 The infinite families of relations defined in Lemma 13.4, are called

1. *Order relations*,
2. *Conjugacy relations*,
3. *Split relations.*

Let $\mathcal{A} = \{s_{\alpha,\beta} \mid \alpha,\ \beta \in \{0,\ 1\}^*,\ \alpha \perp \beta\}$. We will use P_∞ to denote the group whose presentation has the generating set $\mathcal{A}$, and the set of relations defined in Lemma 13.4, execpt reading $s_{\alpha,\beta}$, where Lemma 13.4 reads $t_{\alpha,\beta}$.

Remark 13.6 We note that although the symbols $s_{\alpha,\beta}$ and $s_{\beta,\alpha}$ are not equal as letters, they are equal in P_∞, since, using conjugacy relations, we have

$$s_{\alpha,\beta}s_{\beta,\alpha} =_{P_\infty} s_{\alpha,\beta}s_{\beta,\alpha}s_{\alpha,\beta}s_{\beta,\alpha},$$

and so $s_{\alpha,\beta}s_{\beta,\alpha} =_{P_\infty} 1_{P_\infty}$. The order relations give $s_{\alpha,\beta} =_{P_\infty} s_{\alpha,\beta}^{-1}$, and so we can conclude that $s_{\alpha,\beta} =_{P_\infty} s_{\beta,\alpha}$. This proof is thanks to Martyn Quick, and we will use this result implicitly throughout the remainder of this section.

Lemma 13.7 *Let $n \in \mathbb{N}$, and $X_n = \{1, \ldots, n\}$. Let*

$$\begin{aligned} A_n &= \{\sigma_{i,j} \mid i,\ j \in X_n,\ i \neq j\}, \\ R_{1,n} &= \{\sigma_{i,j}^2 = 1 \mid i,\ j \in X_n,\ i \neq j\}, \\ R_{2,n} &= \{\sigma_{i,j}^{\sigma_{k,l}} = \sigma_{i(k\ l),j(k\ l)} \mid i,\ j,\ k,\ l \in X_n,\ i \neq j,\ k \neq l\}. \end{aligned}$$

Here the symbols $\sigma_{i,j}$ and $\sigma_{j,i}$ are defined to be equal, for any $i,\ j \in \mathbb{N}$. The symmetric group on n points S_n has the presentation

$$\langle A_n \mid R_{1,n} \cup R_{2,n} \rangle.$$

Proof Let P_n be the presentation stated. Define

$$\begin{aligned} \xi_n \colon A_n &\to S_n \\ \sigma_{i,j} &\mapsto (i\ j). \end{aligned}$$

Let $i,\ j \in X_n$ such that $i \neq j$. Then $(\sigma_{i,j}\xi_n)^2 = (i\ j)^2 = ()$. Let $i,\ j,\ k \in X_n$ all be distinct. Let $i,\ j,\ k,\ l \in X_n$, such that $i \neq j$ and $k \neq l$. Then

$$(\sigma_{i,j}\xi_n)^{(\sigma_{k,l}\xi_n)} = (i\ j)^{(k\ l)} = (i(k\ l)\ j(k\ l)) = \sigma_{i(k\ l),j(k\ l)}\xi_n.$$

Therefore, by von Dyck's Theorem (Theorem 6.12), ξ_n extends to a homomorphism $\tilde{\xi}_n \colon P_n \to S_n$. Since $\tilde{\xi}$ is onto the generators of P_n, $\tilde{\xi}_n$ is surjective. We therefore have that $|P_n| \geq |S_n| = n!$.

From [HB13], we have that S_n has the presentation $\langle A \mid R \rangle$, where

$$\begin{aligned} A &= \{\tau_1,\ \ldots,\ \tau_{n-1}\}, \\ R &= \{\tau_i^2 \mid i \in \{1,\ \ldots,\ n-1\}\} \\ &\quad \cup \{(\tau_i\tau_{i+1})^3 \mid i \in \{1,\ \ldots,\ n-2\}\} \\ &\quad \cup \{(\tau_i\tau_j)^2 \mid i,\ j \in \{1,\ \ldots,\ n-1\},\ i < j,\ |j-i| > i\}. \end{aligned}$$

Define

$$\begin{aligned} \eta \colon A &\to P_n \\ \tau_i &\mapsto \sigma_{i,i+1}. \end{aligned}$$

Let $i \in \{1, \ldots, n-1\}$. Then

$$(\tau_i\eta)^2 =_{P_n} \sigma_i^2 =_{P_n} 1_{P_n}.$$

If $i \leq n-2$, then

$$\begin{aligned} ((\tau_i\eta)(\tau_{i+1}\eta))^3 &=_{P_n} (\sigma_{i,i+1}\sigma_{i+1,i+2})^3 \\ &=_{P_n} \sigma_{i,i+1}\sigma_{i+1,i+2}\sigma_{i,i+1}\sigma_{i+1,i+2}\sigma_{i,i+1}\sigma_{i+1,i+2} \\ &=_{P_n} \sigma_{i+1(i\ i+1),i+2(i\ i+1)}\sigma_{i(i+1\ i+2),i+1(i+1\ i+2)} \\ &=_{P_n} \sigma_{i,i+2}\sigma_{i,i+2} \\ &=_{P_n} 1_{P_n}. \end{aligned}$$

Let $i,\ j \in \{1, \ldots, n-1\}$, such that $i < j$ and $|j-i| > i$. Note $|j-i| > i \geq 1$, so $i+1 < j$. We have

$$((\tau_i\eta)(\tau_j\eta))^2 =_{P_n} \sigma_{i,i+1}\sigma_{j,j+1}\sigma_{i,i+1}\sigma_{j,j+1}$$

$$
\begin{aligned}
&=_{P_n} \sigma_{j(i\ i+1),\ j+1(i\ i+1}\sigma_{j,j+1}\\
&=_{P_n} \sigma_{j,j+1}\sigma_{j,j+1}\\
&=_{P_n} 1_{P_n}.
\end{aligned}
$$

By von Dyck's Theorem (Theorem 6.12), η extends to a homomorphism from S_n to P_n. Let $\sigma_{i,j} \in A_n$, and assume, without loss of generality, that $i < j$. Then

$$\sigma_{i,j} =_{P_n} \sigma_{j-1,j} \cdots \sigma_{i+1,i+2}\sigma_{i,i+1}\sigma_{i+1,i+2} \cdots \sigma_{j-1,j}.$$

So η is onto the generators of P_n, and hence the homomorphism it extends to is surjective. It follows that $|P_n| \leq |S_n|$. Since we already have that $|P_n| \geq |S_n|$, and P_n and S_n are finite, we can conclude that this homomorphism is injective. □

Lemma 13.8 *For each complete antichain X of $\{0,\ 1\}^*$, there exists a monomorphism ϕ_X from the Symmetric group S_X to P_∞, that maps $(\alpha_1\ \beta_1)(\alpha_2\ \beta_2)\cdots(\alpha_n\ \beta_n) \in S_X$ to $s_{\alpha_1,\beta_1}s_{\alpha_2,\beta_2}\cdots s_{\alpha_n,\beta_n}$, where $n \in \mathbb{N}$, and $\alpha_i,\ \beta_i \in X$, for all valid i.*

In addition, if $\theta\colon P_\infty \to V$ is the epimorphism defined using von Dyck's Theorem (Theorem 6.12) and Lemma 13.4, then $\theta \restriction_{\operatorname{im}\phi_X}$ is injective, for any complete antichain X.

Proof We have that $S_X \cong S_n$, where $n = |X|$, by conjugating by the bijection from X to $\{1,\ \ldots,\ n\}$, ie 'renaming' elements of X. Therefore, S_X is generated by $A_X = \{\sigma_{\alpha,\beta} \mid \alpha,\ \beta \in X,\ \alpha \neq \beta\}$ and has the relations stated in Lemma 13.7, except $i,\ j \in X_n$ can be replaced with $\alpha, \beta \in X$. Define

$$
\begin{aligned}
\phi'_X\colon S_X &\to P_\infty\\
\sigma_{\alpha,\beta} &\mapsto s_{\alpha,\beta}.
\end{aligned}
$$

Note that $R_{1,n}$ in Lemma 13.7 is satisfied after applying ϕ'_X by the order relations, and $R_{2,n}$ is satisfied after applying ϕ'_X by the conjugacy relations. So by von Dyck's Theorem (Theorem 6.12), ϕ'_X extends to a homomorphism $\phi_X\colon S_X \to P_\infty$. In addition, if $|X| = 2$, then $X = \{0,\ 1\}$, and $S_X\phi_X = \{1_{P_\infty},\ s_{0,\ 1}\}$. Since $S_2 = C_2$ is simple, ϕ_X is injective. Otherwise, if $|X| > 2$, then all potential proper quotients of S_X are the trivial group, the Klein four group, and the cyclic group of order two, all of which have no elements of order 3. Note, however, that if $\alpha,\ \beta,\ \gamma \in X$ are distinct, then

$$(s_{\alpha,\beta}s_{\beta,\gamma})\theta = t_{\alpha,\beta}t_{\beta,\gamma}.$$

We have that $t_{\alpha,\beta}$ and $t_{\beta,\gamma}$ both have the same domain and range trees, and apply non-equal transpositions to the leaves. Therefore $t_{\alpha,\beta}t_{\beta,\gamma} \neq_V 1_V$. In addition, using the conjugacy relations, and the fact that $\alpha \perp \gamma$, we have

$$(t_{\alpha,\beta}t_{\beta,\gamma})^2 = t_{\alpha,\beta}t_{\beta,\gamma}t_{\alpha,\beta}t_{\beta,\gamma} = t_{\alpha,\gamma}t_{\alpha,\beta},$$

which is not equal to 1_V by the same argument. Hence $s_{\alpha,\beta}s_{\beta,\gamma}$ has order at least 3. We also have

$$
\begin{aligned}
(s_{\alpha,\beta}s_{\beta,\gamma})^3 &=_{P_\infty} s_{\alpha,\beta}s_{\beta,\gamma}s_{\alpha,\beta}s_{\beta,\gamma}s_{\alpha,\beta}s_{\beta,\gamma}\\
&=_{P_\infty} s_{\beta(\alpha\ \beta),\gamma(\alpha\ \beta)}s_{\alpha(\beta\ \gamma),\beta(\beta\ \gamma)}\\
&=_{P_\infty} s_{\alpha,\gamma}s_{\alpha,\gamma}\\
&=_{P_\infty} 1_{P_\infty}.
\end{aligned}
$$

Therefore $\operatorname{im}\phi_X$ has an element of order 3, and hence $\ker\phi_X = \{1_{S_X}\}$.

Let X be any complete antichain of $\{0,\ 1\}^*$. Let $(\alpha_1\ \beta_1)\cdots(\alpha_n\ \beta_n) \in S_X$, where $n \in \mathbb{N}$, such that $(\alpha_1\ \beta_1)\cdots(\alpha_n\ \beta_n)\phi_X\theta =_V 1_V$. Then $t_{\alpha_1,\beta_1}\cdots t_{\alpha_n,\beta_n} =_V 1_V$. We have that t_{α_i,β_i} can be represented by the $\mathcal{T}$-tree diagram $(R,\ R,\ \rho_i)$, where ρ_i is the transposition corresponding to swapping α and β, and R is the tree whose leaves' associated words are X. Then the multiplication of the swaps in $t_{\alpha_1,\beta_1}\cdots t_{\alpha_n,\beta_n}$ is determined by the permutation part only. Since it is the identity, the product of the transpositions $(\alpha_1\ \beta_1)\cdots(\alpha_n\ \beta_n)$ must be the identity in S_X. So $((\alpha_1\ \beta_1)\cdots(\alpha_n\ \beta_n))\phi_X = s_{\alpha_1,\beta_1}\cdots s_{\alpha_n,\beta_n} =_{P_\infty} 1_{P_\infty}$, and we can conclude that $\theta\restriction_{\operatorname{im}\phi_X}$ is injective. □

Lemma 13.9 *For any $n \in \mathbb{N}_0$, let*

$$\lambda_n = \underbrace{11\cdots1}_{n\ times}.$$

There is a monomorphism ψ from Thompson's group F into P_∞, such that

$$X_n\psi = s_{\lambda_n0,\lambda_n1}s_{\lambda_n01,\lambda_n1}s_{\lambda_n00,\lambda_n01}.$$

In addition, if $\theta\colon P_\infty \to V$ is the epimorphism defined using von Dyck's Theorem (Theorem 6.12) and Lemma 13.4, then $\theta\restriction_{\operatorname{im}\psi}$ is injective.

Proof Define

$$\begin{aligned}\psi'\colon \{X_0,\ X_1,\ \ldots\} &\to P_\infty\\ X_n &\mapsto s_{\lambda_n0,\lambda_n1}s_{\lambda_n01,\lambda_n1}s_{\lambda_n00,\lambda_n01}.\end{aligned}$$

We will show that P_∞ satisfies the relations of F, in order to apply von Dyck's Theorem. Let $k,\ n \in \mathbb{N}$, such that $k < n$. Therefore,

1. $\lambda_k1 \preceq \lambda_n1$,
2. $\lambda_k1 \preceq \lambda_n00$,
3. $\lambda_k1 \preceq \lambda_n01$,
4. $\lambda_k00 \perp \lambda_n1$,
5. $\lambda_k00 \perp \lambda_n00$,
6. $\lambda_k00 \perp \lambda_n01$,
7. $\lambda_k01 \perp \lambda_n1$,
8. $\lambda_k01 \perp \lambda_n00$,
9. $\lambda_k01 \perp \lambda_n01$.

Using these, together with the conjugacy relations of P_∞, we can calculate the following. Note that, since $k < n$, we have that $n-k,\ n-k-1 \in \mathbb{N}_0$, so λ_{n-k} and λ_{n-k-1} are well defined. For clarity, we will underline the expression being manipulated.

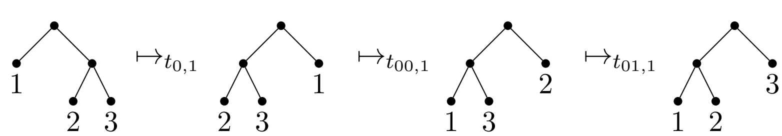
$(X_k\psi')(X_n\psi') =_{P_\infty} s_{\lambda_k0,\lambda_k1}s_{\lambda_k01,\lambda_k1}\underline{s_{\lambda_k00,\lambda_k01}s_{\lambda_n0,\lambda_n1}}s_{\lambda_n01,\lambda_n1}s_{\lambda_n00,\lambda_n01}$

$$
\begin{aligned}
&=_{P_\infty} s_{\lambda_k 0,\lambda_k 1} s_{\lambda_k 01,\lambda_k 1} \underline{s_{\lambda_n 0(\lambda_k 00\ \lambda_k 01),\lambda_n 1(\lambda_k 00\ \lambda_k 01)} s_{\lambda_k 00,\lambda_k 01}} s_{\lambda_n 01,\lambda_n 1} s_{\lambda_n 00,\lambda_n 01}\\
&=_{P_\infty} s_{\lambda_k 0,\lambda_k 1} \underline{s_{\lambda_k 01,\lambda_k 1} s_{\lambda_n 0,\lambda_n 1}} s_{\lambda_k 00,\lambda_k 01} s_{\lambda_n 01,\lambda_n 1} s_{\lambda_n 00,\lambda_n 01}\\
&=_{P_\infty} s_{\lambda_k 0,\lambda_k 1} \underline{s_{\lambda_n 0(\lambda_k 01\ \lambda_k 1),\lambda_n 1(\lambda_k 01\ \lambda_k 1)} s_{\lambda_k 01,\lambda_k 1}} s_{\lambda_k 00,\lambda_k 01} s_{\lambda_n 01,\lambda_n 1} s_{\lambda_n 00,\lambda_n 01}\\
&=_{P_\infty} s_{\lambda_k 0,\lambda_k 1} \underline{s_{\lambda_k 1\lambda_{n-k-1}0(\lambda_k 01\ \lambda_k 1),\lambda_k 1\lambda_{n-k}(\lambda_k 01\ \lambda_k 1)} s_{\lambda_k 01,\lambda_k 1}} s_{\lambda_k 00,\lambda_k 01} s_{\lambda_n 01,\lambda_n 1} s_{\lambda_n 00,\lambda_n 01}\\
&=_{P_\infty} \underline{s_{\lambda_k 0,\lambda_k 1} s_{\lambda_k 01\lambda_{n-k-1}0,\lambda_k 01\lambda_{n-k}}} s_{\lambda_k 01,\lambda_k 1} s_{\lambda_k 00,\lambda_k 01} s_{\lambda_n 01,\lambda_n 1} s_{\lambda_n 00,\lambda_n 01}\\
&=_{P_\infty} \underline{s_{\lambda_k 01\lambda_{n-k-1}0(\lambda_k 0\ \lambda_k 1),\lambda_k 01\lambda_{n-k}(\lambda_k 0\ \lambda_k 1)} s_{\lambda_k 0,\lambda_k 1}} s_{\lambda_k 01,\lambda_k 1} s_{\lambda_k 00,\lambda_k 01} s_{\lambda_n 01,\lambda_n 1} s_{\lambda_n 00,\lambda_n 01}\\
&=_{P_\infty} \underline{s_{\lambda_k 11\lambda_{n-k-1}0,\lambda_k 11\lambda_{n-k}} s_{\lambda_k 0,\lambda_k 1}} s_{\lambda_k 01,\lambda_k 1} s_{\lambda_k 00,\lambda_k 01} s_{\lambda_n 01,\lambda_n 1} s_{\lambda_n 00,\lambda_n 01}\\
&=_{P_\infty} s_{\lambda_{n+1}0,\lambda_{n+1}1} s_{\lambda_k 0,\lambda_k 1} s_{\lambda_k 01,\lambda_k 1} \underline{s_{\lambda_k 00,\lambda_k 01} s_{\lambda_n 01,\lambda_n 1}} s_{\lambda_n 00,\lambda_n 01}\\
&=_{P_\infty} s_{\lambda_{n+1}0,\lambda_{n+1}1} s_{\lambda_k 0,\lambda_k 1} s_{\lambda_k 01,\lambda_k 1} \underline{s_{\lambda_n 01(\lambda_k 00\ \lambda_k 01),\lambda_n 1(\lambda_k 00\ \lambda_k 01)} s_{\lambda_k 00,\lambda_k 01}} s_{\lambda_n 00,\lambda_n 01}\\
&=_{P_\infty} s_{\lambda_{n+1}0,\lambda_{n+1}1} s_{\lambda_k 0,\lambda_k 1} \underline{s_{\lambda_k 01,\lambda_k 1} s_{\lambda_n 01,\lambda_n 1}} s_{\lambda_k 00,\lambda_k 01} s_{\lambda_n 00,\lambda_n 01}\\
&=_{P_\infty} s_{\lambda_{n+1}0,\lambda_{n+1}1} s_{\lambda_k 0,\lambda_k 1} \underline{s_{\lambda_n 01(\lambda_k 01\ \lambda_k 1),\lambda_n 1(\lambda_k 01\ \lambda_k 1)} s_{\lambda_k 01,\lambda_k 1}} s_{\lambda_k 00,\lambda_k 01} s_{\lambda_n 00,\lambda_n 01}\\
&=_{P_\infty} s_{\lambda_{n+1}0,\lambda_{n+1}1} s_{\lambda_k 0,\lambda_k 1} \underline{s_{\lambda_k 1\lambda_{n-k-1}01(\lambda_k 01\ \lambda_k 1),\lambda_k 1\lambda_{n-k}(\lambda_k 01\ \lambda_k 1)} s_{\lambda_k 01,\lambda_k 1}} s_{\lambda_k 00,\lambda_k 01} s_{\lambda_n 00,\lambda_n 01}\\
&=_{P_\infty} s_{\lambda_{n+1}0,\lambda_{n+1}1} \underline{s_{\lambda_k 0,\lambda_k 1} s_{\lambda_k 01\lambda_{n-k-1}01,\lambda_k 01\lambda_{n-k}}} s_{\lambda_k 01,\lambda_k 1} s_{\lambda_k 00,\lambda_k 01} s_{\lambda_n 00,\lambda_n 01}\\
&=_{P_\infty} s_{\lambda_{n+1}0,\lambda_{n+1}1} \underline{s_{\lambda_k 01\lambda_{n-k-1}01(\lambda_k 0\ \lambda_k 1),\lambda_k 01\lambda_{n-k}(\lambda_k 0\ \lambda_k 1)} s_{\lambda_k 0,\lambda_k 1}} s_{\lambda_k 01,\lambda_k 1} s_{\lambda_k 00,\lambda_k 01} s_{\lambda_n 00,\lambda_n 01}\\
&=_{P_\infty} s_{\lambda_{n+1}0,\lambda_{n+1}1} \underline{s_{\lambda_k 11\lambda_{n-k-1}01,\lambda_k 11\lambda_{n-k}} s_{\lambda_k 0,\lambda_k 1}} s_{\lambda_k 01,\lambda_k 1} s_{\lambda_k 00,\lambda_k 01} s_{\lambda_n 00,\lambda_n 01}\\
&=_{P_\infty} s_{\lambda_{n+1}0,\lambda_{n+1}1} \underline{s_{\lambda_{n+1}01,\lambda_{n+1}1} s_{\lambda_k 0,\lambda_k 1}} s_{\lambda_k 01,\lambda_k 1} s_{\lambda_k 00,\lambda_k 01} s_{\lambda_n 00,\lambda_n 01}\\
&=_{P_\infty} s_{\lambda_{n+1}0,\lambda_{n+1}1} \underline{s_{\lambda_k 11\lambda_{n-k-1}01,\lambda_k 11\lambda_{n-k}} s_{\lambda_k 0,\lambda_k 1}} s_{\lambda_k 01,\lambda_k 1} s_{\lambda_k 00,\lambda_k 01} s_{\lambda_n 00,\lambda_n 01}\\
&=_{P_\infty} s_{\lambda_{n+1}0,\lambda_{n+1}1} s_{\lambda_{n+1}01,\lambda_{n+1}1} s_{\lambda_k 0,\lambda_k 1} s_{\lambda_k 01,\lambda_k 1} \underline{s_{\lambda_k 00,\lambda_k 01} s_{\lambda_n 00,\lambda_n 01}}\\
&=_{P_\infty} s_{\lambda_{n+1}0,\lambda_{n+1}1} s_{\lambda_{n+1}01,\lambda_{n+1}1} s_{\lambda_k 0,\lambda_k 1} s_{\lambda_k 01,\lambda_k 1} \underline{s_{\lambda_n 00(\lambda_k 00\ \lambda_k 01),\lambda_n 01(\lambda_k 00\ \lambda_k 01)} s_{\lambda_k 00,\lambda_k 01}}\\
&=_{P_\infty} s_{\lambda_{n+1}0,\lambda_{n+1}1} s_{\lambda_{n+1}01,\lambda_{n+1}1} s_{\lambda_k 0,\lambda_k 1} \underline{s_{\lambda_k 01,\lambda_k 1} s_{\lambda_n 00,\lambda_n 01}} s_{\lambda_k 00,\lambda_k 01}\\
&=_{P_\infty} s_{\lambda_{n+1}0,\lambda_{n+1}1} s_{\lambda_{n+1}01,\lambda_{n+1}1} s_{\lambda_k 0,\lambda_k 1} \underline{s_{\lambda_k 1\lambda_{n-k-1}00(\lambda_k 01\ \lambda_k 1),\lambda_k 1\lambda_{n-k-1}01(\lambda_k 01\ \lambda_k 1)} s_{\lambda_k 01,\lambda_k 1}} s_{\lambda_k 00,\lambda_k 01}\\
&=_{P_\infty} s_{\lambda_{n+1}0,\lambda_{n+1}1} s_{\lambda_{n+1}01,\lambda_{n+1}1} \underline{s_{\lambda_k 0,\lambda_k 1} s_{\lambda_k 01\lambda_{n-k-1}00,\lambda_k 01\lambda_{n-k-1}01}} s_{\lambda_k 01,\lambda_k 1} s_{\lambda_k 00,\lambda_k 01}\\
&=_{P_\infty} s_{\lambda_{n+1}0,\lambda_{n+1}1} s_{\lambda_{n+1}01,\lambda_{n+1}1} \underline{s_{\lambda_k 01\lambda_{n-k-1}00(\lambda_k 0\ \lambda_k 1),\lambda_k 01\lambda_{n-k-1}01(\lambda_k 0\ \lambda_k 1)} s_{\lambda_k 0,\lambda_k 1}} s_{\lambda_k 01,\lambda_k 1} s_{\lambda_k 00,\lambda_k 01}\\
&=_{P_\infty} s_{\lambda_{n+1}0,\lambda_{n+1}1} s_{\lambda_{n+1}01,\lambda_{n+1}1} \underline{s_{\lambda_k 11\lambda_{n-k-1}00,\lambda_k 11\lambda_{n-k-1}01} s_{\lambda_k 0,\lambda_k 1}} s_{\lambda_k 01,\lambda_k 1} s_{\lambda_k 00,\lambda_k 01}\\
&=_{P_\infty} s_{\lambda_{n+1}0,\lambda_{n+1}1} s_{\lambda_{n+1}01,\lambda_{n+1}1} \underline{s_{\lambda_{n+1}00,\lambda_{n+1}01} s_{\lambda_k 0,\lambda_k 1}} s_{\lambda_k 01,\lambda_k 1} s_{\lambda_k 00,\lambda_k 01}\\
&=_{P_\infty} (X_{n+1}\psi')(X_k\psi').
\end{aligned}
$$

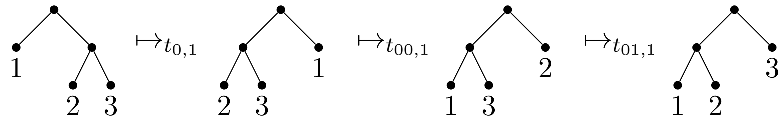
By von Dyck's Theorem (Theorem 6.12), we have that ψ' extends to a homomorphism ψ. Now let

$$a = s_{0,1}s_{01,1}s_{00,01},\ \ b = s_{10,11}s_{101,11}, s_{100,101}.$$

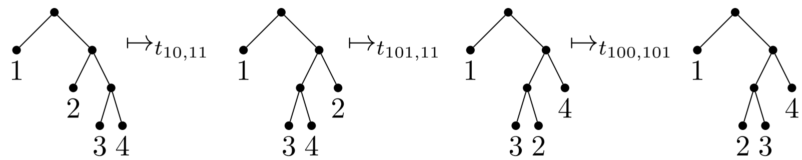
Note that $a,\ b \in \operatorname{im}\psi$, since $a = X_0\psi$ and $b = X_1\psi$. We have that the tree representation of $a\theta$ is



which is the same as that of the generator A of F. In addition, $b\theta$ has the tree representation



which is the same as that of the generator A of F. In addition, $t_{01,11}t_{101,11}t_{100,101}$ has the tree representation



which is the same as that of the generator B of F. The elements A and B do not commute; if they did then X_n would have the same tree representation as B, for all $n \in \mathbb{N}$, which we know to be false from Example 6.2. Hence $(ab)\theta = (a\theta)(b\theta) \neq (b\theta)(a\theta) = (ba)\theta$, so $ab \neq ba$, and we can conclude that $\operatorname{im}\psi$ is non-abelian. However, Theorem 7.7 tells us that all proper quotients of F are abelian, which implies that $\ker\psi = \{1_F\}$; that is ψ is injective.

Note that as $A\psi\theta$ and $B\psi\theta$ have the tree representation of A and B, we have that there is an isomorphism η from $\langle A\psi\theta,\ B\psi\theta\rangle$ to F, which maps $a\theta$ to A, and $b\theta$ to B. We have that $a\theta\restriction_{\operatorname{im}\psi}\eta\psi = A\psi = a$, and $b\theta\restriction_{\operatorname{im}\psi}\eta\psi = B\psi = b$. In addition, $(a\theta)\eta\psi\theta\restriction_{\operatorname{im}\psi} = A\psi\theta\restriction_{\operatorname{im}\psi} = a\theta\restriction_{\operatorname{im}\psi} = a\theta$ and $(b\theta)\eta\psi\theta\restriction_{\operatorname{im}\psi} = B\psi\theta\restriction_{\operatorname{im}\psi} = b\theta\restriction_{\operatorname{im}\psi} = b\theta$. As a and b generate $\operatorname{im}\psi$, and $a\theta$ and $b\theta$ generate $\operatorname{im}\psi\theta$, and the functions in question are homomorphisms, we have that $\eta\psi$ and $\theta\restriction_{\operatorname{im}\psi}$ are inverses. Hence $\theta\restriction_{\operatorname{im}\psi}$ is injective. $\square$

Lemma 13.10 *Let ψ denote the monomorphism from F to P_∞, as defined in Lemma 13.9, and ϕ_X denote the monomorphism from S_X into P_∞, for any complete antichain X of $\{0,\ 1\}^*$, as defined in Lemma 13.8. For every $g \in P_\infty$, there exists $f \in \operatorname{im}\psi$ and $\sigma \in \operatorname{im}\phi_X$, for some complete antichain X, such that*

$$g = f\sigma.$$

Proof Let $\theta\colon P_\infty \to V$ be the epimorphism defined using von Dyck's Theorem (Theorem 6.12) and Lemma 13.4. Using Lemma 13.9, $\psi\theta\restriction_{\operatorname{im}\psi}\colon F \to V$ is injective, mapping $t_{0,1}t_{00,1}t_{01,1}$ and $t_{10,11}t_{101,11}t_{100,101}$ to A and B, respectively. We can also induce an embedding of S_X into V, for any complete antichain X of $\{0,\ 1\}^*$, by mapping a permutation σ to the $\mathcal{T}$-tree diagram with domain and range trees both being the tree whose leaves' associated words are precisely X. The permutation is constructed by bijectively mapping $\{1,\ \ldots,\ |X|\}$ to X, in order (using $\preceq$), then defining the permutation as σ conjugated by this bijection.

It is clear, using the tree representation of V, that every element of V can be expressed as the product of a $\mathcal{T}$-tree diagram with the identity permutation, ie an element of the subgroup of V isomorphic to F, and a $\mathcal{T}$-tree diagram, whose domain and range trees are the same, and whose permutation is the permutation of the element.

Since $g\theta \in V$, there exist $f',\ \sigma' \in V$, such that f' is in the subgroup isomorphic to F, and σ' is in the subgroup isomorphic to S_X, where X is the complete antichain of $\{0,\ 1\}^*$ induced by the range tree of g. Since $f' \in \operatorname{im}(\psi\theta)$ we have that there exists $f \in \operatorname{im}\psi$, such that $f\theta = f'$. Similarly, since $\sigma' \in \operatorname{im}(\phi_X\theta)$, there exists $\sigma \in \operatorname{im}\phi_X$, such that $\sigma' = \sigma\theta$. We can conclude that $g\theta = (f\theta)(\sigma\theta)$.

We have that $(g\sigma^{-1})\theta = f\theta \in \mathrm{im}(\psi\theta)$. Since $\theta\restriction_{\mathrm{im}\,\psi}$ is injective, we can conclude that $g\sigma^{-1} = f$, and hence $g = f\sigma$. $\square$

Theorem 13.11 *Thompson's group V has the presentation P_∞.*

Proof Let ψ denote the monomorphism from F to P_∞, as defined in Lemma 13.9, and ϕ_X denote the monomorphism from S_X into P_∞, for any complete antichain X of $\{0,\ 1\}^*$, as defined in Lemma 13.8. By Lemma 13.4 and von Dyck's Theorem, there exists an epimorphism $\theta\colon P_\infty \to V$. Let $g \in \ker\theta$. By Lemma 13.10, there exists $f \in \mathrm{im}\,\psi$, and $\sigma \in \mathrm{im}\,\phi_X$, for some complete antichain X of $\{0,\ 1\}^*$, such that $g = f\sigma$. By Theorem 2.11, Thompson's group F is torsion-free, so we have that the order of f is infinite, or f is the identity. In addition, since S_X is finite, σ has finite order, say n.

We have that $(f\theta)(\sigma\theta) = g\theta = 1_V$, and hence $f^{-1}\theta = \sigma\theta$. It follows that $f^{-n}\theta = \sigma^n\theta = 1_V$. In addition, as $\mathrm{im}\,\psi \leq P_\infty$, we have that $f^{-n} \in \mathrm{im}\,\psi$, so by Lemma 13.9 it follows that $f^{-n} =_{P_\infty} 1_{P_\infty}$. Since f either has infinite order, or is the identity, we can conclude that $f =_{P_\infty} 1_{P_\infty}$. Hence $g = \sigma$. Since $\sigma\theta = 1_V$, Lemma 13.8 implies that $\sigma =_{P_\infty} 1_{P_\infty}$. Hence $g =_{P_\infty} 1_{P_\infty}$, and θ is injective. $\square$

14 Simplicity of V

Lemma 14.1 *Let $f \in V$, and $\alpha \in \{0,\ 1\}^*$, such that αf is defined. If $\gamma \in \mathfrak{C}$, then*

$$(\alpha\gamma)f = (\alpha f)\gamma.$$

Proof Since αf is defined, there is an element β in a domain antichain for f such that $\beta \preceq \alpha$. Hence there exists $\delta \in \{0,\ 1\}^*$, such that $\beta\delta = \alpha$. Let $\mu = \beta f$. We also have that $\alpha f = (\beta\delta)f = \mu\delta$. From the definition of a prefix exchange map $(\beta\lambda)f = \mu\lambda$, for all $\lambda \in \mathfrak{C}$. In particular,

$$(\alpha\gamma)f = (\beta\delta\gamma)f = \mu\delta\gamma = (\alpha f)\gamma.$$

$\square$

Lemma 14.2 *Let $f \in V$ have a domain antichain D. Let $\alpha,\ \beta \in \{0,\ 1\}^*$, such that $\alpha \perp \beta$, and αf and βf are both defined. Then*

$${t_{\alpha,\beta}}^f = t_{\alpha f,\beta f}.$$

Proof Let $\gamma \in \mathfrak{C}$. We will consider two cases: when $\gamma \in \alpha f\mathfrak{C} \cup \beta f\mathfrak{C}$, and when $\gamma \notin \alpha f\mathfrak{C} \cup \beta f\mathfrak{C}$.

Case 1: $\gamma \in \alpha f\mathfrak{C} \cup \beta f\mathfrak{C}$.
Without loss of generality, assume $\gamma \in \alpha f\mathfrak{C}$. Then $\gamma = (\alpha f)\delta$, for some $\delta \in \mathfrak{C}$. We can therefore compute the following, using Lemma 14.1, and noting that αf and βf are defined.

$$\gamma f^{-1}t_{\alpha,\beta}f = ((\alpha f)\delta)f^{-1}t_{\alpha,\beta}f = (\alpha\delta)t_{\alpha,\beta}f = (\beta\delta)f,$$
$$\gamma t_{\alpha f,\beta f} = ((\alpha f)\delta)t_{\alpha f,\beta f} = (\beta f)\delta = (\beta\delta)f.$$

Case 2: $\gamma \notin \alpha f\mathfrak{C} \cup \beta f\mathfrak{C}$.
It follows that $\gamma t_{\alpha f,\beta f} = \gamma$. We also have that $\gamma f^{-1} \notin \alpha\mathfrak{C} \cup \beta\mathfrak{C}$, using Lemma 14.1, so

$$\gamma f^{-1}t_{\alpha,\beta}f = \gamma f^{-1}f = \gamma.$$

$\square$

Lemma 14.3 *If $f \in V \backslash \{1_V\}$, then $\langle\langle f \rangle\rangle$ contains a small swap.*

Proof Since $f \neq 1_V$, there exists $\alpha \in \{0,\ 1\}^+$ in one of the domain antichains for f, such that $\alpha f \neq \alpha$, and $|\alpha| > 1$ or $|\alpha f| > 1$, since if $|\alpha| = |\alpha f| = 1$, we can pass to a larger pair of antichains for f, such as by applying an elementary expansion to α. If $\alpha \preceq \alpha f$, then replacing α by $\alpha 0$ and $\alpha 1$ in the domain antichain, and αf by $(\alpha f)0$ and $(\alpha f)1$ in the range antichain, yields a new pair of antichains for f. Noting that f is a prefix exchange map, either $(\alpha 0)f = (\alpha f)0 \perp \alpha 0$, or $(\alpha 1)f = (\alpha f)1 \perp \alpha 1$, so by redefining α to be one of $\alpha 0$ and $\alpha 1$, we may assume that α is an element of a domain antichain, such that $\alpha \perp \alpha f$. If $\alpha f \perp \alpha$, then we can redefine f to be f^{-1}, and apply the above case, noting that f^{-1} is non-trivial, as f is non-trivial.

We will now calculate the commutator of f with the small swap $t_{\alpha 00, \alpha 01}$, and use Lemma 14.2 to do so.

$$\begin{aligned} \langle\langle f \rangle\rangle &\ni [f,\ t_{\alpha 00,\alpha 01}] \\ &= f^{-1} t_{\alpha 00,\alpha 01} f t_{\alpha 00,\alpha 01} \\ &= {t_{\alpha 00,\alpha 01}}^f t_{\alpha 00,\alpha 01} \\ &= t_{\alpha 00 f,\alpha 01 f} t_{\alpha 00,\alpha 01} \\ &= t_{(\alpha f)00,(\alpha f)01} t_{\alpha 00,\alpha 01} \end{aligned}$$

Note that since $\alpha \perp \alpha f$, we have that

$$(\alpha f)00 \perp \alpha 00, \quad (\alpha f)00 \perp \alpha 01, \quad (\alpha f)01 \perp \alpha 00, \quad (\alpha f)01 \perp \alpha 01.$$

Note that these words form an antichain that is not complete. Hence there is a complete antichain D_g containing all of them. Let R_g be a complete antichain, with $|D_g|$ elements, and such that $\{000,\ 001,\ 100,\ 101\} \subseteq R_g$. Let $g \in V$ be the prefix exchange map, with domain antichain D_g and range antichain R_g, such that

$$((\alpha f)00)g = 000,\ ((\alpha f)01)g = 100,\ (\alpha 00)g = 001,\ (\alpha 01)g = 101.$$

Then, using Lemma 14.2 and the split relations, we have

$$\begin{aligned} \langle\langle f \rangle\rangle &\ni (t_{(\alpha f)00,(\alpha f)01} t_{\alpha 00,\alpha 01})^g \\ &= {t_{(\alpha f)00,(\alpha f)01}}^g {t_{\alpha 00,\alpha 01}}^g \\ &= t_{((\alpha f)00)g,((\alpha f)01)g} t_{(\alpha 00)g,(\alpha 01)g} \\ &= t_{000,100} t_{001,101} \\ &= t_{00,10}, \end{aligned}$$

which is a small swap. $\square$

Theorem 14.4 *Thompson's group V is simple.*

Proof Let N be a non-trivial normal subgroup in V. Then N contains a non-trivial element of V, and hence by Lemma 14.3 a small swap $t_{\alpha,\beta}$. Let $t_{\gamma,\delta}$ be a small swap. We have that there is a prefix exchange map $g \in V$ that maps α to γ and β to δ. This can be constructed by taking two complete antichains of the same size; the first containing $\{\alpha,\ \beta\}$ and the second containing $\{\gamma,\ \delta\}$. Since these swaps are small swaps, these containments will be proper, so there will be a pair of complete antichains of the same size, each containing one of these sets, and the prefix exchange map is defined using these. We can conclude that $t_{\gamma,\delta}$ is conjugate to $t_{\alpha,\beta}$, by Lemma 14.2. Hence N contains all small swaps, so Lemma 13.2 implies $N = V$. $\square$

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