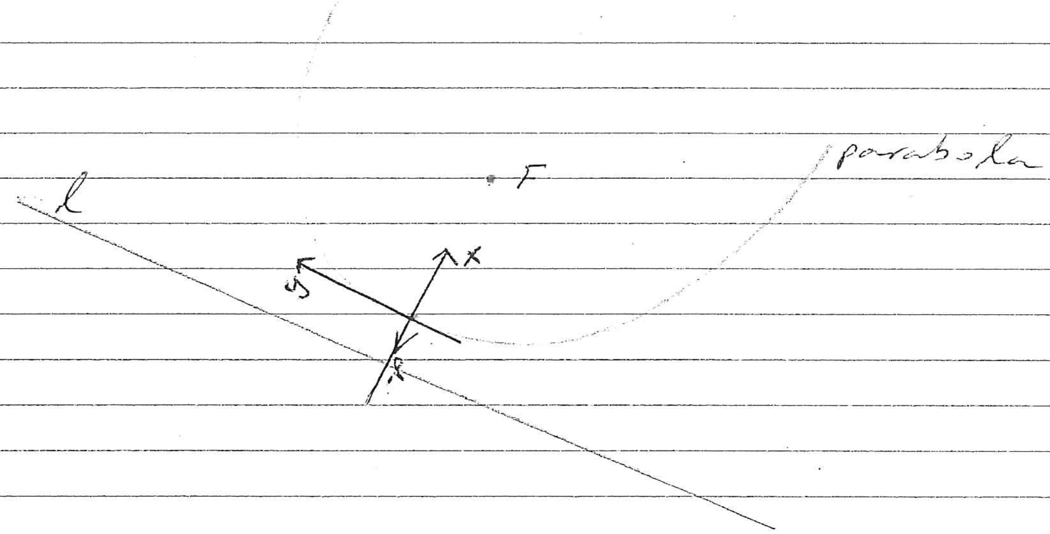
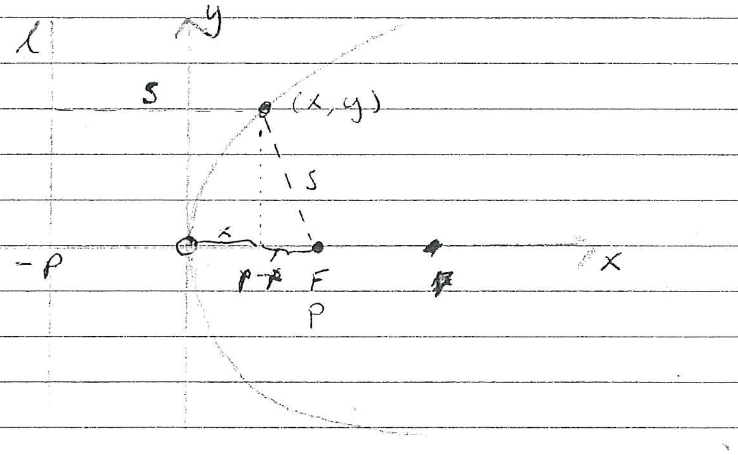
2) GEOMETRIC DEFINITIONS OF CONIC SECTIONS

2.1) THE PARABOLA

DEF$^{\text{N}}$: A parabola in the plane consists of all points equidistant from a fixed line l (the directrix) and a fixed point F (the focus) not on the line. The point V on the parabola closest to the directrix is the vertex of the parabola.



Let's choose axes through the vertex V so that the focus F is at $(p, 0)$ and the line l is the line $x = -p$.



If (x, y) is on the parabola then

$$S = \sqrt{(x-p)^2 + y^2} = x + p$$

$$\Rightarrow \quad (x-p)^2 + y^2 = (x+p)^2$$

$$\Rightarrow \quad x^2 - 2xp + p^2 + y^2 = x^2 + 2xp + p^2 \quad \Rightarrow \quad y^2 = 4px \quad ①$$

- The standard form for the eq$^{\text{n}}$ of a parabola.

EXAMPLE 1

Find the focus + directrix for $y^2 = 10x$

Sol^n

$4p = 10 \Rightarrow p = \frac{5}{2} \Rightarrow$ The focus is $(5/2, 0)$ and the directrix is the line $x = -5/2$

EXAMPLE 2

Find the vertex, focus + directrix for

$$x^2 + 4x + 3y = 2$$

Sol^n

Completing the square (check) we get

$(x+2)^2 = -3(y-2) \Rightarrow$ vertex at $(-2, 2)$

Let $\bar{x} = x+2$, $\bar{y} = y-2$ then

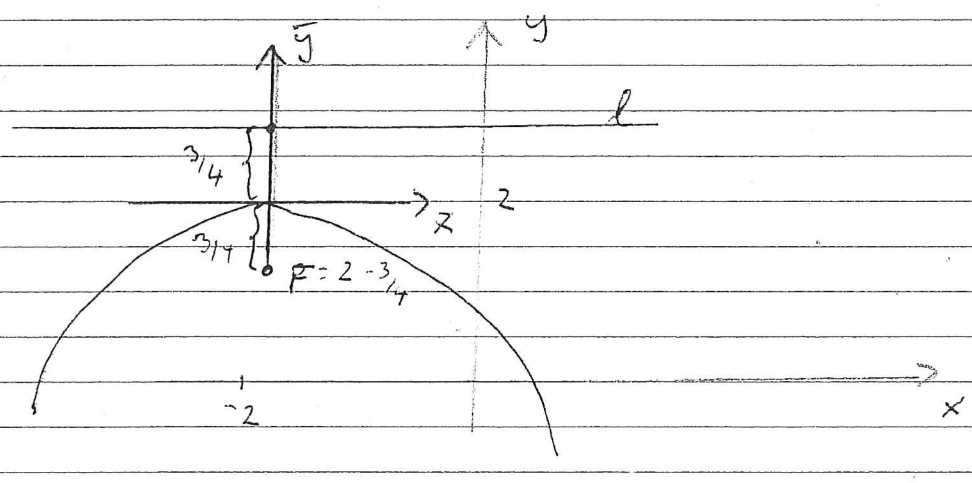
$\bar{x}^2 = -3\bar{y}$ [Translated axes through $(-2, 2)$]

Introducing the factor 4 to find p :-

$\bar{x}^2 = -4\left(\frac{3}{4}\right)\bar{y} \Rightarrow p = +\frac{3}{4}$

So we have $\bar{x}^2 = -4p\bar{y}$ with $\bar{x}, \bar{y}$ axes at $(-2, 2)$

$\therefore$ The focus is at $(0-2, -3/4+2) = (-2, 5/4)$ and the directrix is $y = \frac{3}{4} + 2 = \frac{11}{4}$



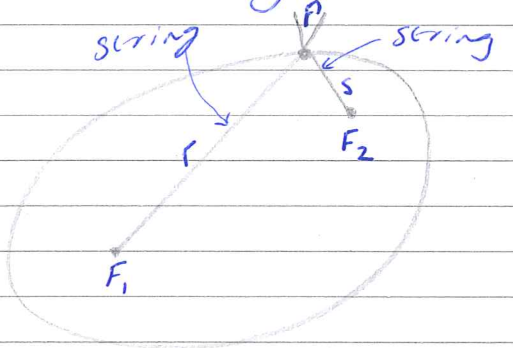
2.2) THE ELLIPSE

"Joe-sigh"

DEF$^{\underline{N}}$

An ellipse in the plane consists of all points the sum of whose distances from two fixed points F_1 and F_2 (the foci) remains a constant value $2a$ (greater than the distance between the foci).

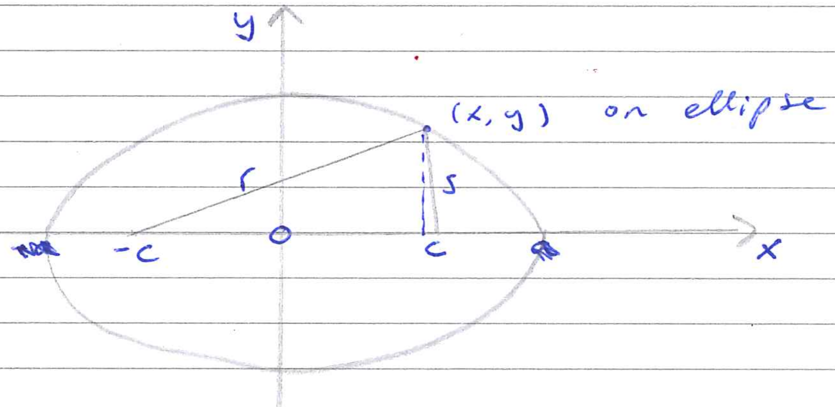
To draw an ellipse, place pencil at P + pull "string" taut then trace! (string fixed at F_1, F_2)



By def$^{\underline{n}}$

$r + s = 2a$

Let's choose axes so that the foci are at $(-c, 0)$ and $(c, 0)$ then



$$r + s = 2a$$

$$\Rightarrow \sqrt{(x+c)^2 + y^2} + \sqrt{(c-x)^2 + y^2} = 2a \qquad (2)$$

$$\Rightarrow \sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}$$

$$\Rightarrow \cancel{x^2} - 2cx + \cancel{c^2} + \cancel{y^2} = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + \cancel{x^2} + 2cx + \cancel{c^2} + \cancel{y^2}$$

$$\Rightarrow \cancel{4}a\sqrt{(x+c)^2 + y^2} = \cancel{4}a^2 + \cancel{4}cx$$

$$\Rightarrow a^2[x^2 + \cancel{2cx} + c^2 + y^2] = a^4 + \cancel{2a^2cx} + c^2x^2$$

$$\Rightarrow (a^2 - c^2)x^2 + a^2y^2 = a^4 - a^2c^2$$
$$= a^2(a^2 - c^2) \qquad (3)$$

The point where the ellipse intersects the +ve y axis is equidistant from the foci and must be equal to a:

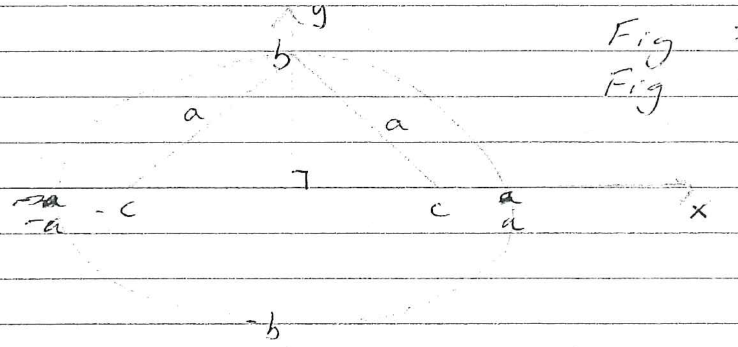


Fig $\Rightarrow$ $a > c$
Fig $\Rightarrow$ $a^2 = b^2 + c^2$
$\Rightarrow$ $b^2 = a^2 - c^2$ (4)
(>0 since $a > c$)

Sub $b^2 = a^2 - c^2$ into (3) then

$$b^2x^2 + a^2y^2 = a^2b^2$$

$$\Rightarrow \boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1} \qquad (5)$$ which is the standard form of the eq^n for an ellipse.

- The line segment from $(-a, 0)$ to $(a, 0)$ is the major axis
- " " " " $(0, -b)$ to $(0, b)$ " " minor axis

<u>NOTE</u>
We take the larger of the two numbers in the denominator ~~(a^2 or b^2)~~ to define the major axis. Usually denote this as a^2. (may swap a^2, b^2)

<u>EXAMPLE</u>
Find the foci and the lengths of major + minor axes for the ellipse

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

Sol^n

$a^2 = 25, \quad b^2 = 9$

(4) $\Rightarrow c^2 = a^2 - b^2$ (6) $= 25 - 9 = 16 \Rightarrow c = 4$

The foci are $(-4, 0)$ and $(0, 4)$

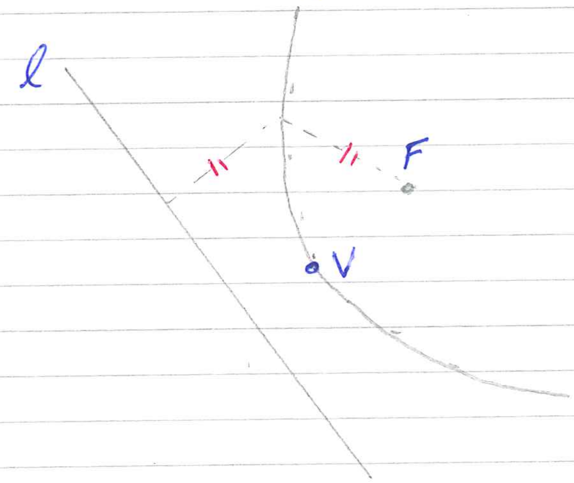
The length of the major axis is ~~2b = 6~~ $2a = 10$

" " " " minor axis is $2b = 6$

2) GEOMETRIC DEF$^{\text{NS}}$ OF CONIC SECTIONS

2.1) THE PARABOLA

DEF$^{\underline{n}}$: A *parabola* in the plane consists of all points equidistant from a fixed line l (the *directrix*) and a fixed point F (the *focus*) not on the line. The point V on the parabola closest to the directrix is the *vertex* of the parabola.



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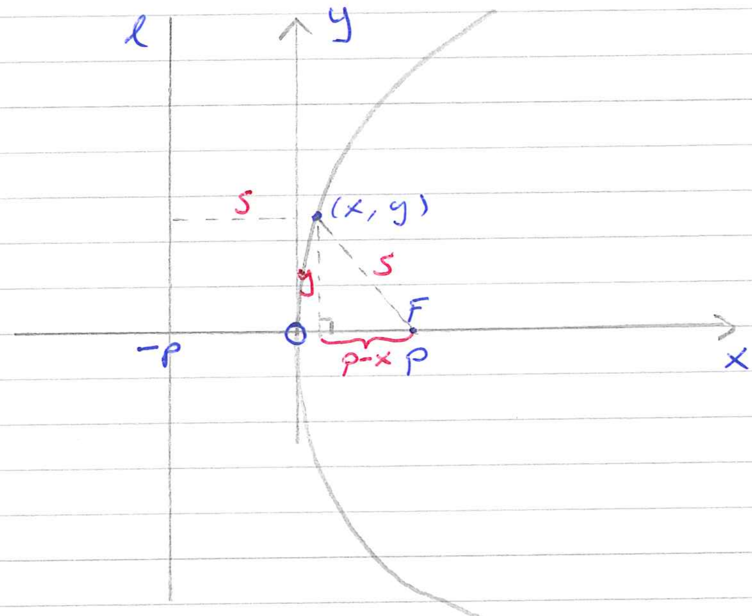


If (x, y) is on the parabola then

$$s = \sqrt{(p-x)^2 + y^2} = x + p$$

$$(p-x)^2 + y^2 = (x+p)^2$$

$$\cancel{p^2} - 2xp + \cancel{x^2} + y^2 = \cancel{x^2} + 2px + \cancel{p^2} \Rightarrow y^2 = 4px \quad (1)$$

The standard form for the eq$^{\underline{n}}$ of a parabola.

EXAMPLE 1

Find the focus + directrix for $y^2 = 10x$

Sol^{n}

$4p = 10 \Rightarrow p = \frac{5}{2} \Rightarrow$ The focus is $(5/2, 0)$ and the directrix is the line $x = -5/2$

EXAMPLE 2

Find the vertex, focus + directrix for $x^2 + 4x + 3y = 2$

Sol^{n}

$(x+2)^2 = -3(y-2) \Rightarrow$ vertex at $(-2, 2)$

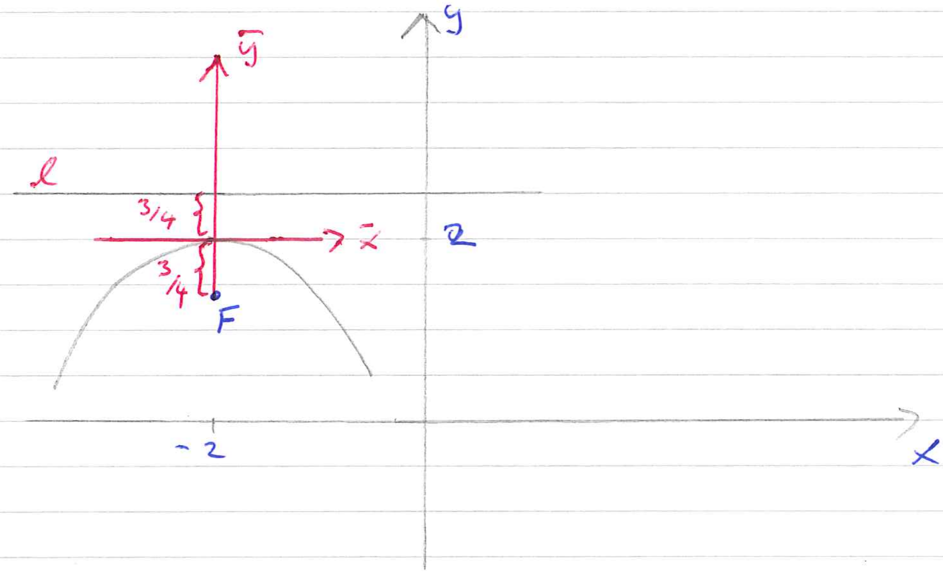
~~$(x-h)^2 = c(y-k)$~~ Let $\bar{x} = x + 2$, $\bar{y} = y - 2$

$\bar{x}^2 = -3\bar{y}$

Introducing a factor 4 to find p:-

$\bar{x}^2 = -4\left(\frac{3}{4}\right)\bar{y} \Rightarrow p = \frac{3}{4}$

$\bar{x}^2 = -4p\bar{y}$



Focus is at $(0-2, -3/4 + 2) = (-2, 5/4)$

Directrix is $y = 2 + \frac{3}{4} = \frac{11}{4}$

2.2) THE ELLIPSE

DEF$^{\underline{n}}$

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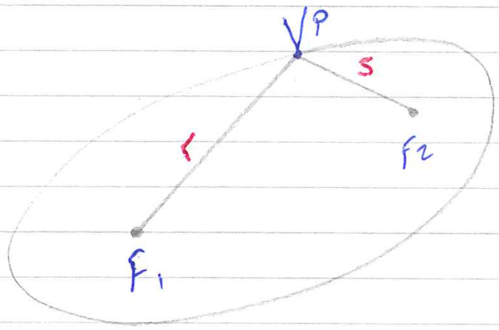

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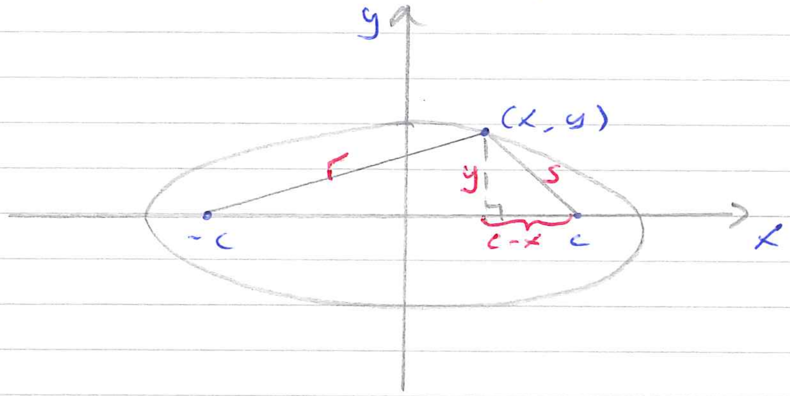

$$r + s = 2a$$

$$\Rightarrow \sqrt{(c+x)^2 + y^2} + \sqrt{(c-x)^2 + y^2} = 2a \qquad ②$$

$$\Rightarrow \sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}$$

$$\Rightarrow \cancel{x^2} - 2cx + \cancel{c^2} + \cancel{y^2} = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + \cancel{x^2} + 2cx + \cancel{c^2} + \cancel{y^2}$$

$$\Rightarrow \cancel{4}a\sqrt{(x+c)^2 + y^2} = \cancel{4}a^2 + \cancel{4}cx$$

$$\Rightarrow a^2\left[x^2 + 2cx + c^2 + y^2\right] = a^4 + 2a^2cx + c^2x^2$$

$$\Rightarrow (a^2 - c^2)x^2 + a^2y^2 = a^4 - a^2c^2$$
$$= a^2(a^2 - c^2) \qquad ③$$

The point where the ellipse intersects the +ve y axis is equidistant from the foci and must be equal to a:

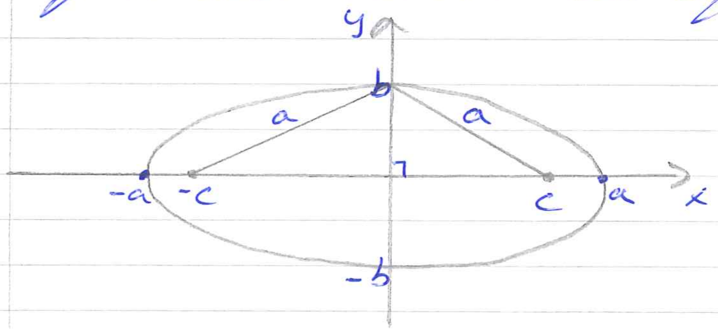

Fig $\Rightarrow$ $a > c$

Fig $\Rightarrow$ $a^2 = b^2 + c^2$

$b^2 = a^2 - c^2$ (4)

Take (4) sub into (3)

$\Rightarrow b^2x^2 + a^2y^2 = a^2b^2$

$\Rightarrow \boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$ (5)

the standard form of the eq^n for an ellipse

- The line segment from $(-a, 0)$ to $(a, 0)$ is the major axis
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NOTE: we take the larger of the two numbers in the denominator to define the major axis. Usually label it as a^2 (may swap $a^2 \rightarrow b^2$)

EXAMPLE

Find the foci and the lengths of the major + minor axes for the ellipse

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

Sol^n: $a^2 = 25 \Rightarrow a = 5$

$b^2 = 9 \Rightarrow b = 3$

(4) $\Rightarrow$ $c^2 = a^2 - b^2$ (6)

$= 25 - 9 = 16 \Rightarrow c = 4$

Foci: $(-4, 0)$, $(4, 0)$

Major axis length $= 2a = 10$

Minor $= 2b = 6$