1 Introductory ideas

1. Show that if $C \subseteq X$ is a geodesic between $x, z \in X$, then for all $y \in C$ we have
$$d(x, y) + d(y, z) = d(x, z).$$

2. Let (X, d) be a metric space such that X is countable. Show that there are no geodesics in X.

3. A metric space X is called *geodesic* if there exists a geodesic between any two points and is called *path connected* if for any two points $x, y \in X$ there exists a continuous function $\gamma : [0, 1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

   Prove that every geodesic metric space is path connected, but that path connected metric spaces are not necessarily geodesic.

4. A metric space X is called *uniquely geodesic* if there exists a unique geodesic between any two points, i.e. given $w, z \in X$ there exists a geodesic between x and z and if C_1 and C_2 are both geodesics between x and z then $C_1 = C_2$.

   Give an example of (i) a uniquely geodesic metric space, (ii) a geodesic metric space where geodesics are sometimes unique and sometimes not, (iii) a geodesic metric space where geodesics are never unique, (iv) a geodesic metric space where for every $k \in \mathbb{N}$ there exists a pair of points x, y with precisely k geodesics joining them.

5. Prove that a subset of a geodesic is itself a geodesic if and only if it is path connected and not equal to a single point.

6. Prove that if $C \subseteq X$ is a geodesic in an arbitrary metric space and ϕ is an isometry of X, then $\phi(C)$ is a geodesic.

7. Describe the isometry group of (i) an equilateral triangle, (ii) an isosceles triangle, (iii) a scalene triangle.

8. Prove that rotations (about arbitrary points), reflections (in arbitrary straight lines), and translations, are isometries of $\mathbb{C}$ equipped with the usual Euclidean metric. You may wish to first show that all of these maps can be expressed as combinations of translations, rotations about the origin, and reflections in the real axis (i.e. conjugation).