MT5823 Semigroup theory: Problem sheet 2 (James D. Mitchell)
Rectangular bands, cancellative semigroups, subsemigroups, monogenic semigroups, and idempotents

Rectangular bands

Let I and Λ be two sets. Define multiplication on the set $S = I \times \Lambda = \{ (i, \lambda) : i \in I,\ \lambda \in \Lambda \}$ by $(i, \lambda)(j, \mu) = (i, \mu)$. Such a semigroup is called a ***rectangular band***.

2-1. Prove that a rectangular band S is a semigroup in which every element is idempotent. Also prove that $xyz = xz$ for any $x, y, z \in S$.

2-2. Prove that a rectangular band has a left zero if and only if $|\Lambda| = 1$, in which case every element of S is a left zero.

Cancellative semigroups

A semigroup is called ***cancellative*** if $ax = ay \Rightarrow x = y$ and $xa = ya \Rightarrow x = y$ for all $a, x, y \in S$.

2-3. Let e, a be elements of a cancellative semigroup S such that $ea = a$. Prove that e is an idempotent. Prove that e is the identity of S.

2-4. Does there exist a cancellative semigroup without an identity element?

Subsemigroups

2-5. Let S be the subsemigroup of the semigroup of partial mappings P_5 generated by the mappings

$$x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & - & - \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 4 & 5 & 4 \end{pmatrix}.$$

List the elements of S, determine the order of S and draw the (right) Cayley graph of S.

2-6. Prove that every subset of a semigroup of right (or left) zeros is a subsemigroup. Can you find another semigroup with the same property? Is it true that every non-empty subset of a zero semigroup is a subsemigroup? How many subsemigroups does the zero semigroup of order n have?

2-7. Let G be a finite group. Prove that any subsemigroup of G is also a subgroup of G. Find an example of a subsemigroup of an infinite group that is not a subgroup.

Monogenic semigroups and idempotents

A semigroup is ***monogenic*** if it is generated by a single element.

2-8. If S is a finite monogenic semigroup, then prove that there exist $m, r > 0$ such that $a^{m+r} = a^m$, and that there is an idempotent power of a.

2-9. Prove that every finite semigroup contains an idempotent.

2-10. Does there exist a finite semigroup with exactly one idempotent? Does there exist an infinite semigroup without idempotents?

Further problems

2-11.* Let $\mathbb{N} \times \mathbb{N}$ denote the set $\{ (x, y) : x, y \in \mathbb{N} \}$. Prove that $\mathbb{N} \times \mathbb{N}$ under the operation $(x, y) + (z, t) = (x + z, y + t)$ forms a semigroup. Is $\mathbb{N} \times \mathbb{N}$ finitely generated?

2-12.* Prove that the rectangular band $I \times \Lambda$ can be generated using $\max(|I|, |\Lambda|)$ elements. Is there a smaller generating set?

2-13.* Let S be an infinite semigroup with the property that every countable subset is contained in a monogenic subsemigroup. Prove that S is isomorphic to the natural numbers under addition.