MT5823 Semigroup theory: Problem sheet 5 (James D. Mitchell)
Bicyclic monoid, ideals, Green's relations

Bicyclic monoid

The ***bicyclic semigroup*** B is defined by the presentation

$$\langle b, c \mid bc = 1 \rangle$$

and its elements are $\{c^i b^j \ : \ i, j \geq 0\}$.

5-1. Prove that an element $c^i b^j$ of B is an idempotent if and only if $i = j$. Prove that the set E of all idempotents is a subsemigroup which is not finitely generated.

5-2. Consider the following two subsets of the bicyclic monoid B:

$$\begin{aligned} S_1 &= \{c^{4i+5} b^{4j+5} \ : \ i, j \geq 0\} \\ S_2 &= \{c^{4i+7} b^{4j+7} \ : \ i, j \geq 0\}. \end{aligned}$$

Prove that both S_1 and S_2 are subsemigroups and that they are isomorphic to B. Prove that their union $S = S_1 \cup S_2$ is also a subsemigroup. Prove that S is finitely generated.

Ideals

5-3. Prove that the intersection of a left ideal and a right ideal of a semigroup is always non-empty. Show by way of an example that the intersection of two left ideals may be empty. (Hint: right zero semigroup.)

5-4. Prove that a rectangular band has no proper two-sided ideals. Does it have proper left or right ideals?

Green's relations

5-5. Consider the following three mappings

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 4 & 3 \end{pmatrix}, \quad g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 3 & 2 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 2 \end{pmatrix},$$

from T_4. Determine which of them are $\mathscr{R}$-equivalent, and which of them are $\mathscr{L}$-equivalent. Is $fh\mathscr{R}gh$? Conclude that $\mathscr{R}$ is not necessarily a right congruence.

5-6. Let S be the semigroup defined by the presentation

$$\langle\, a, b \mid a^3 = a,\ b^4 = b,\ ba = a^2 b \,\rangle.$$

The right Cayley graph of this semigroup was determined in Problem **4-5**. Use the right Cayley graph to determine the $\mathscr{R}$-classes of S. Draw the left Cayley graph and determine the $\mathscr{L}$-classes of S.

5-7. Consider the bicyclic monoid $B = \{c^i b^j \ : \ i, j \geq 0\}$ $(bc = 1)$. Prove that $c^i \mathscr{R} c^k$ if and only if $i = k$. Prove that $c^i b^j \mathscr{R} c^i$. Conclude that $c^i b^j \mathscr{R} c^k b^l$ if and only if $i = k$. State and prove an analogous criterion for two elements of B to be $\mathscr{L}$-equivalent.

5-8. Prove that an idempotent is a left identity in its $\mathscr{R}$-class. State and prove an analogous assertion about $\mathscr{L}$-classes.

5-9. Determine the $\mathscr{R}$-, $\mathscr{L}$-, $\mathscr{H}$, $\mathscr{D}$-, and $\mathscr{J}$-classes of the semigroup S generated by the two (partial) mappings

$$x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & - & - \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 4 & 5 & 4 \end{pmatrix}.$$

5-10. Prove that in the bicyclic monoid B the following equalities hold: $\mathscr{H} = \Delta_B$ and $\mathscr{D} = \mathscr{J} = B \times B$.

Further problems

5-11.* If a semigroup S is defined by a finite presentation $\langle A \mid R \rangle$ with $|A| > |R|$ then prove that S is infinite. (Hint: prove that there is a homomorphism from S onto a non-trivial subsemigroup of the additive semigroup $\mathbb{Q}$.)