2 The Poincaré disk model of hyperbolic space

1. Prove that $z \mapsto \overline{z}$ is an isometry of $\mathbb{D}^2$.

2. We saw in lectures that the inversion $z \mapsto 1/z$ maps circles and lines to circles and lines. Hence, or otherwise, deduce that general 'circle inversions' also have this property. A circle inversion can be viewed as reflecting $\hat{\mathbb{C}}$ in a given circle. The general formula for inversion in the circle centered at $w \in \mathbb{C}$ with radius $r > 0$ is given by
$$z \mapsto w + \left(\frac{r}{|z-w|}\right)^2 (z-w).$$
You should spend a moment thinking about the action of this map and why it corresponds to reflection in the given circle.

3. Prove that $\mathbb{D}^2$ is invariant under con(1), i.e. prove that for each $g \in \text{con}(1)$ we have $g(\mathbb{D}^2) = \mathbb{D}^2$.

4. Prove that con(1) is a group.

5. Prove that con(1) is generated by $\text{con}^+(1)$ together with conjugation $z \mapsto \overline{z}$.

6. Prove that for $z \in \mathbb{D}^2$ we have
$$|z| = \tanh \frac{d_{\mathbb{D}^2}(0,z)}{2}$$

7. Given $z \in \mathbb{D}^2$ find the hyperbolic midpoint of the geodesic joining 0 and z, i.e. find w lying on the geodesic such that
$$d_{\mathbb{D}^2}(0,w) = d_{\mathbb{D}^2}(w,z).$$

8. Prove that for $F \subset \mathbb{D}^2$ and $g \in \text{con}(1)$ we have
$$A_{\mathbb{D}^2}(F) = A_{\mathbb{D}^2}(g(F)).$$

9. Let C be a hyperbolic circle of radius r. Show that the hyperbolic circumference of C is given by
$$L_{\mathbb{D}^2}(C) = 2\pi \sinh r.$$

10. Let $B_{\mathbb{D}^2}(0,R)$ be a hyperbolic ball. Find r such that
$$A_{\mathbb{D}^2}(B_{\mathbb{D}^2}(0,r)) = A_{\mathbb{D}^2}(B_{\mathbb{D}^2}(0,R) \setminus (B_{\mathbb{D}^2}(0,r)).$$
Consider a similar problem for Euclidean balls and compare the two situations.