MT5830 - 6 Limit sets

① Let $g \in \mathrm{Con}^+(1)$ be a parabolic element fixing $p \in S^1$. Let H be the (unique) horocycle containing p and 0. We know from lectures that $g^n(0) \in H$ for all n. Moreover, the points $g^n(0)$ are all distinct. (suppose $g^n(0) = g^m(0)$, then $0 = g^{m-n}(0)$ and since g^{m-n} is either parabolic or the identity, it must be the identity and so $m=n$).

Since $\langle g \rangle$ is Fuchsian, the orbit $g^n(0)$ cannot accumulate in $\mathbb{D}^2$ and so it must accumulate at $H \cap S^1 = \{p\}$.

The hyperbolic case is similar. Let $g \in \mathrm{Con}^+(1)$ be hyperbolic fixing $h_1, h_2 \in S^1$. Let C be the unique circle (or straight line) passing through $0, h_1$ & h_2. We know $g^n(0) \in C \cap \mathbb{D}^2$ for all n.

Similarly $g^n(0)$ are all distinct, and since $\langle g \rangle$ is Fuchsian the orbit $g^n(0)$ must accumulate on $C \cap S^1 = \{h_1, h_2\}$. Moreover, if $g^n(0)$ accumulates on h_2, then $g^{-n}(0)$ accumulates on h_1.

② Let $\Gamma \leq \mathrm{Con}^+(1)$ be Fuchsian and $g \in \mathrm{Con}^+(1)$ be arbitrary.

Let $z \in L(\Gamma)$, which means we can find $g_n \in \Gamma$ such that $g_n(0) \to z$.

Then
$$g\, g_n\, g^{-1}\, g(0) = g(g_n(0))$$
$$\to g(z) \in S^1$$

since g is continuous (in 1.1).

This means that $g(z) \in (g \Gamma g^{-1})(g(0))$ since $\cancel{g(z) \in}$ $(g g_n g^{-1}) \in g \Gamma g^{-1}$.

Therefore $g(z) \in L(g\Gamma g^{-1})$, where we have used the fact that limit sets are independent of base and so we can use $g(0)$ instead of 0.

② cont...

So far we have proved $g(L(\Gamma)) \subseteq L(g\Gamma g^{-1})$.

In the other direction, let $z \in L(g\Gamma g^{-1})$ and hence we can find $g_n \in \Gamma$ such that $g g_n g^{-1}(0) \to z$ (in $|\cdot|$).

Therefore, since g^{-1} is continuous

$$g_n(g^{-1}(0)) \to g^{-1}(z)$$

and so

$$g^{-1}(z) \in \overline{\Gamma(g^{-1}(0))} \setminus \Gamma(g^{-1}(0))$$
$$= L(\Gamma).$$

Finally, this gives

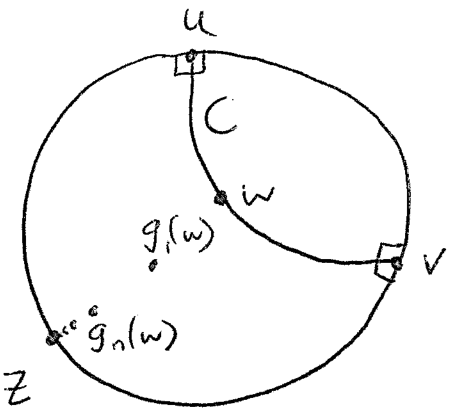
$$z \in g(L(\Gamma))$$

as required.

(Note that this question is only interesting if $g \in \Gamma$).

③ Since $|E| \geq 2$, we may choose $u, v \in E$ with $u \neq v$. Let C be the geodesic ray joining u and v and let $w \in C \cap \mathbb{D}^2$. Fix $z \in L(\Gamma)$ with the aim of showing $z \in E$.

Let $g_n \in \Gamma$ such that $g_n(w) \to z$ in $|\cdot|$ and using compactness, extract a subsequence



such that $g_n(u) \to \tilde{u} \in S^1$ and $g_n(v) \to \tilde{v} \in S^1$ (both in $|\cdot|$). If both $\tilde{u}, \tilde{v} \neq z$, then $g_n(w) \not\to z$ and so without loss of generality assume $g_n(u) \to \tilde{u} = z$. By Γ-invariance, $g_n(u) \in E$ for all n and by closedness of E, $z = \lim_{n \to \infty} g_n(u) \in E$ as required.

③ continued.

Let $\Gamma = \langle g \rangle$ where g is a hyperbolic element fixing $z_1, z_2 \in S^1$. We have already seen Γ is Fuchsian and $E = \{z_1\}$ is clearly non-empty, closed and Γ-invariant. However

$$L(\Gamma) = \{z_1, z_2\} \not\subseteq E.$$

Now suppose Γ is non-elementary, ie $|L(\Gamma)| = \infty$, and let $E \subseteq S^1$ be non-empty, closed, and Γ-invariant.

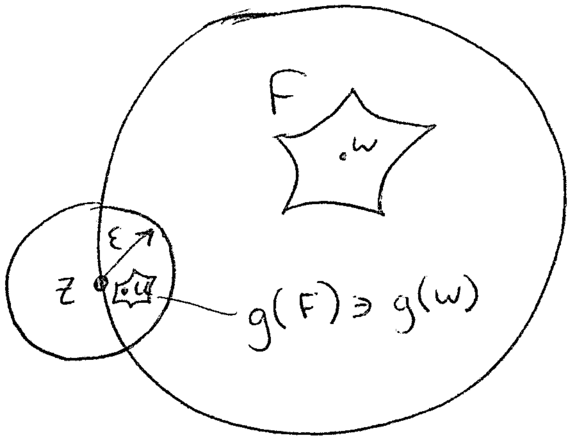
case 1: $|E| \geq 2$. Then it follows from the above that $L(\Gamma) \subseteq E$ (and so E must be infinite).

case 2: $|E| = 1$. Suppose $E = \{z\}$ and so $g(z) = z$ for all $g \in \Gamma$. Hence all non-identity elements of Γ are parabolic or hyperbolic, fixing z. Since Γ is non-elementary it cannot be monogenic and so we can find $h \in \Gamma$ hyperbolic and $g \in \Gamma$ either parabolic or hyperbolic (with a different fixed point from h). In either case Γ cannot be Fuchsian, which is a contradiction

(4) Let $\Gamma \leq \mathrm{Con}^+(1)$ be a Fuchsian group and suppose $F \subseteq \mathbb{D}^2$ is a bounded fundamental domain, ie $F \subseteq B_{\mathbb{D}^2}(0, r)$ for some $r > 0$.

Suppose $z \in S^1$ and let $\varepsilon > 0$ and consider $B_E(z, \varepsilon) \cap \mathbb{D}^2$ (Euclidean ball).

Let $w \in F$ and $u \in B_E(z, \varepsilon) \cap \mathbb{D}^2$ be such that $d_{\mathbb{D}^2}(u, y) > 2r$ for any $y \notin B_E(z, \varepsilon)$.

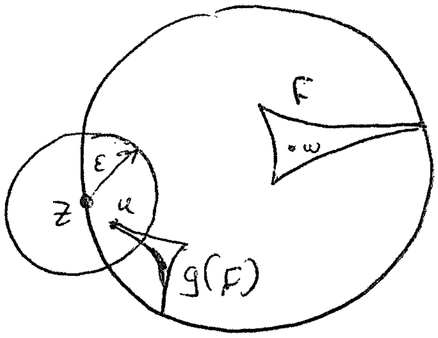


We can find such a u since $B_E(z, \varepsilon)$ contains some of the boundary S^1. Since $\Gamma(\bar{F})$ is a tiling of $\mathbb{D}^2$ we can find $g \in \Gamma$ such that $u \in g(\bar{F})$. It follows that $g(\bar{F}) \subseteq B_E(z, \varepsilon)$ and hence $g(w) \in B_E(z, \varepsilon)$. It follows that $z \in \overline{\Gamma(w)} \setminus \Gamma(w) = L(\Gamma)$.

(5) (Sketch proof!) A similar strategy will work in this more general case. The aim is to show that if $z \in S^1$ and $\varepsilon > 0$ arbitrary then we can find $g \in \Gamma$ such that $g(\bar{F})$ tiles an area close to z such that

$$g(\bar{F}) \subseteq B_E(z, \varepsilon).$$

This was easy to achieve in (4) because F was assumed to be bounded! The worry this time is a situation like this:



ie $g(F) \ni u$
but $g(w) \notin B_E(z, \varepsilon)$.

The key to adapting the proof is to prove that if $g_n \in \Gamma$ is a sequence of elements of (any) Fuchsian group, then

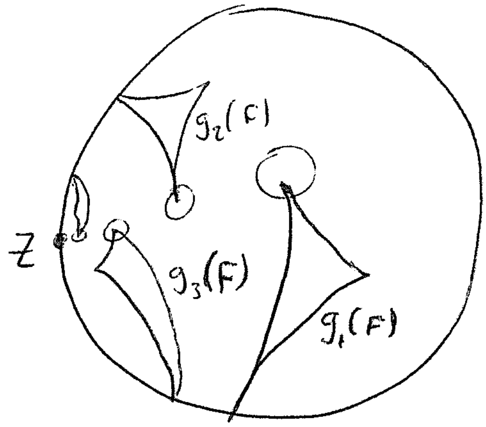
$$|g_n(F)| \to 0 \quad \text{as } n \to \infty$$

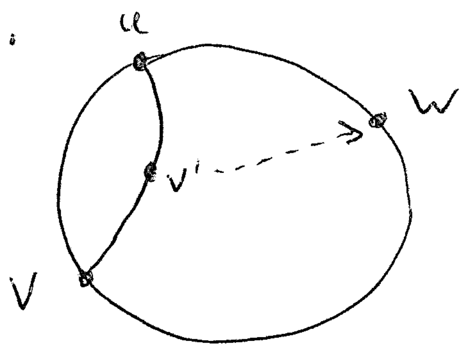
where F is (any) Fundamental domain and $|\cdot|$ denotes Euclidean diameter (of course the hyperbolic diameter is a constant!!)

5 continued...

Let us build a sequence $g_n \in \Gamma$ directly. Since $g(\bar{F})$ has finite volume for all g, by taking a sequence of balls converging to $z \in S^1$ which each require a different tile, we define a sequence g_n such that $g_n(F)$ at least somewhat approaches z:

Using compactness, we can assume $g_n(0) \to w \in S^1$ and this implies that $g_n(v) \to w$ for all $v \in \mathbb{D}^2$ and even for all but at most one $v \in S^1$! To see this latter point we use the standard "3-points trick". Suppose $v \in S^1$ is such that $g_n(v) \not\to w$ and let $u \in S^1$ be arbitrary. Since $g_n(v') \to w$ for any $v' \in \mathbb{D}^2$ on the geodesic ray joining v and u, we must have $g_n(u) \to w$.





⑤ continued...

Provided we can show that the (potential) bad point $v \in S^1$ (such that $g_n(v) \not\to w$) is not in $\overline{F}$, it follows that $|g_n(\overline{F})| \to 0$ as $n \to \infty$. Doing this precisely could invoke the closedness of $\overline{F}$ & the Arzela-Ascoli theorem (MT4515). (Here $\overline{F}$ denotes the Euclidean closure of F). Suppose $v \in \overline{F}$ and assume (using compactness) that $g_n(v) \to \tilde{w} \neq w$. Let C be the geodesic joining $\tilde{w}$ and w. Since all points in the space are pulled towards w, once n is very large $g_n(F)$ is pulled tighter and tighter towards C. This contradicts the action being properly discontinuous since we can find a compact ball which intersects infinitely many $g_n(F)$.

Consider the modular group $PSL(2,\mathbb{Z})$ and the fundamental domain from lectures. This is a hyperbolic triangle and Gauss-Bonnet tells us that its area is $\pi - 0 - \frac{\pi}{3} - \frac{\pi}{3} = \frac{\pi}{3} < \infty$ and so the ~~boundary is~~ limit set is the whole boundary! (It is also fun to prove this directly!)