Chapter 9

Triple Integrals

{chap:9}

Swokowski Chapter 13.5

We now move onto triple integrals. While these are generally more work than double integrals, the procedure is exactly the same: draw the diagram, decide on the order, decide on the limits and then do the integrals in turn.

A triple integral is the expression

$$I = \int\int\int_V f(x,y,z)dV, \qquad (9.1) \quad \{eq:9.1\}$$

where V is the *region* or *domain* or *volume* of integration and dV is an infinitesimal volume element. In Cartesian coordinates,

$$dV = dxdydz. \qquad (9.2) \quad \{eq:9.2\}$$

Of course, V is only volume if x, y, z are spatial coordinates; they could be other quantities. For example, the *total mass*, M, crossing a surface in a certain time interval could be given by

$$M = \int\int\int \rho(x,y,t)v(x,y,t)dV,$$

where ρ is the mass density, v is the speed of the material, x and y are spatial coordinates and t is time. For this case, $dV = dxdydt$. The only point to take from this is that triple integrals involve *three integrals.*

Notes

1. I is just a number, not a function of x, y, z.
2. f can be expressed in other coordinates if convenient, for example *cylindrical polars* (R, ϕ, z).
3. When $f(x,y,z) = 1$, I is the **volume**,

$$\text{Volume } = I = \int\int\int_V dV. \qquad (9.3) \quad \{eq:9.3\}$$

Otherwise, it is **NOT**.

4. When f is the density (or sometimes called the mass density) of a material within V, then I is the *total mass*. We will consider such an example later.

9.1 Rectangular Volumes

:9.1}

This follows closely Section 8.1 for double integration – here you just have one more integral to evaluate.

The limits of integration are all fixed in this Section and independent of each other. In general, the order of integration is not important (you can do them in any order). A general integral of this type can be written

$$I = \int\int\int_V f(x,y,z)dxdydz = \int_{z=e}^{g} \left(\int_{y=c}^{d} \left(\int_{x=a}^{b} f(x,y,z)dx \right) dy \right) dz.$$

As the order of integration does not really matter (do the simplest first), this expression for I can be written in many other ways (but also equivalent in the sense of giving the same answer). As an example, I can also be written as

$$I = \int\int\int_V f(x,y,z)dxdydz = \int_{x=a}^{b} \left(\int_{y=c}^{d} \left(\int_{z=e}^{g} f(x,y,z)dz \right) dy \right) dx.$$

Note how important it is that you indicate clearly which variable has which limits. Just one example is given for rectangular volumes (as they really are so similar to the double integrals in rectangular areas).

Example 9.66

Suppose $f(x,y,z) = (x+y+z))^{-5/2}$ and the volume is

$$\begin{array}{ccccc} 0 & \leq & x & \leq & 1, \\ 0 & \leq & y & \leq & 2. \\ 0 & \leq & z & \leq & 3. \end{array}$$

Solution 9.66

Then, the integral is

$$I = \int_{z=0}^{3} \left(\int_{y=0}^{2} \left(\int_{x=0}^{1} \{x+y+z\}^{-5/2}\, dx \right) dy \right) dz$$

During the x integration, y and z are held fixed. During the y integration, z is held fixed (remember x does not appear as we have already integrated it out). Finally, the z integration is a simple integral only involving one variable. I am going to remove the brackets indicating the order of the integration. It is assumed that we will always start with the innermost integral and work our way out to finally do the outermost one.

$$\begin{aligned}
I &= \int_{z=0}^{3}\int_{y=0}^{2}\int_{x=0}^{1}\{x+y+z\}^{-5/2}\,dxdydz \\
&= \int_{z=0}^{3}\int_{y=0}^{2}\left[-\frac{2}{3}(x+y+z)^{-3/2}\right]_{x=0}^{1}dydz \\
&= -\frac{2}{3}\int_{z=0}^{3}\int_{y=0}^{2}\left\{(1+y+z)^{-3/2}-(y+z)^{-3/2}\right\}dydz \\
&= -\frac{2}{3}\int_{z=0}^{3}\left\{\left[-2(1+y+z)^{-1/2}\right]_{y=0}^{2}+\left[2(y+z)^{-1/2}\right]_{y=0}^{2}\right\}dz \\
&= -\frac{4}{3}\int_{z=0}^{3}\left\{\frac{-1}{(3+z)^{1/2}}+\frac{1}{(1+z)^{1/2}}+\frac{1}{(2+z)^{1/2}}-\frac{1}{z^{1/2}}\right\}dz \\
&= -\frac{8}{3}\left[-(3+z)^{1/2}+(1+z)^{1/2}+(2+z)^{1/2}-z^{1/2}\right]_{z=0}^{3} \\
&= -\frac{8}{3}\left\{\left(-\sqrt{6}+\sqrt{4}+\sqrt{5}-\sqrt{3}\right)-\left(-\sqrt{3}+1+\sqrt{2}-0\right)\right\} \\
&= -\frac{8}{3}\left\{1+\sqrt{5}-\sqrt{6}-\sqrt{2}\right\}
\end{aligned}$$

This cannot be simplified any further but it is approximately 1.67.

Example End

9.2 Volumes bounded by Oblique Planes and Simple Surfaces

{sec:9.2}

For all but rectangular volumes, we need to work out the *order of integration* and the *limits*. Doing the integrals themselves is no harder than any integrals you have seen before.

The key steps are the same as for double integrals:

1. Sketch the region. This is likely going to be the hardest part! It is hard to visualise regions in 3D and some practice is needed. If you really find this difficult, you should use MAPLE or Python initially to help you visualise the region.

2. Decide on the order of integration. This can sometimes be crucial and you should always reconsider the order of integration if your first approach runs into difficulties.

3. Decide on the *limits*. In a sense, this is the key to triple integrals and, once correctly chosen, everything else falls into place. Since we are dealing with functions of **three** variables, the innermost limits can depend on *BOTH* of the outer two variables (not the innermost as that is the one we are integrating over). The middle limits can depend **ONLY** on the outer variable (cannot depend on either the middle variable or the inner variable). The outer limits **MUST** be constants. Evaluating the triple integral results in a **CONSTANT**.

4. Do the integration, starting from the innermost integration, through the middle integration to end with the outer integration.

We learn how to do triple integrals by examples.

Example 9.67

Find the *centre*, $\bar{x}$ of the volume, V defined by the region in the first **OCTANT** bounded by the *plane* $x+y+z=1$.

The centre is defined by

$$\bar{x} = \frac{\int\int\int_V x dV}{\int\int\int_V dV}. \tag{9.4}$$

So we have to evaluate two integrals. However, both are easy once they are set up properly.

Solution 9.67

- Sketch the volume V. This is a little tricky to show clearly and an attempt is shown in Figure 9.1

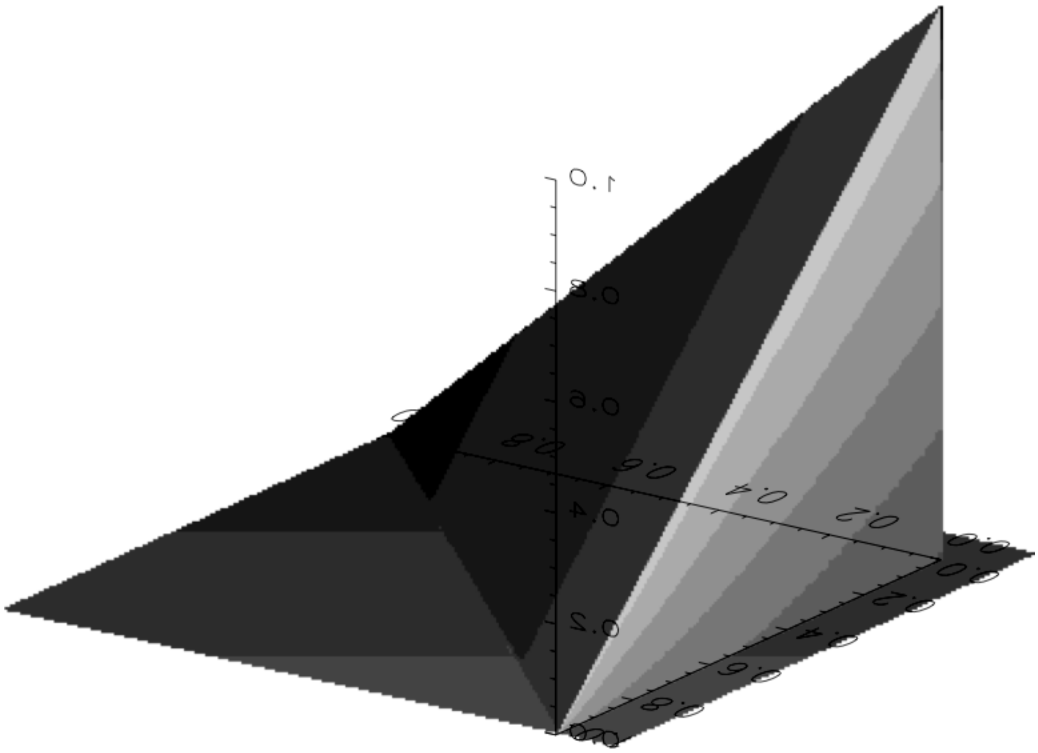
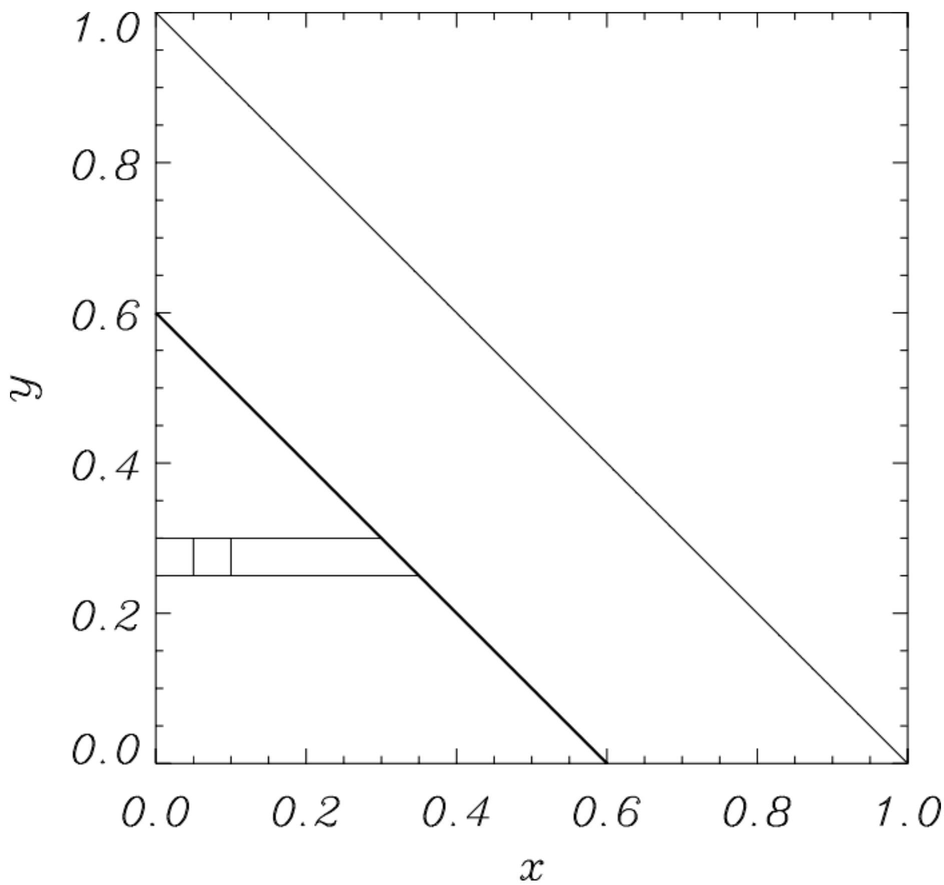


Figure 9.1: The volume enclosed by the plane $x+y+z=1$ and lying in the first octant.

- Decide on the order. Here there are six possible orders for doing the integrations but luckily all are equally simple. Let us choose z as the outer variable, y as the middle one and x as the inner variable.

- We imagine an infinitesimal volume given by $dV = dxdydz$ placed at an arbitrary location (x, y, z). Now, if we are integrating in x first, then we must add up all the contributions from all the infinitesimal boxes from $x = 0$ to $x = 1 - y - z$. This give the innermost limits. This is shown in Figure 9.2. Now we have generated a strip (in exactly the same way as we did with double



{fig:9.2} Figure 9.2: The volume viewed from above. The outer triangle represents the base at $z = 0$. The thick line is located at an arbitrary height z (in this case $z = 0.4$). The contribution to the integrals at this height is given by adding up the infinitesimal volumes from $x = 0$ to $x = 1 - z - y$ (the bold line), as shown in the Figure.

integrals). To integrate in y, we add up all the strips from $y = 0$ to $y = 1 - z$.

At this stage, we have a plane located at a height z and to complete the integration, we add up all the planes from $z = 0$ to $z = 1$. Thus, we have added up all the infinitesimal boxes so that we cover the volume exactly once. So the integrals we need to evaluate are

$$I = \int_{z=0}^{1} \left(\int_{y=0}^{1-z} \left(\int_{x=0}^{1-y-z} x dx \right) dy \right) dz$$

and

$$V = \int_{z=0}^{1} \left(\int_{y=0}^{1-z} \left(\int_{x=0}^{1-y-z} dx \right) dy \right) dz.$$

It is much neater if I leave out the brackets and write the integrals as,

$$I = \int_{z=0}^{1} \int_{y=0}^{1-z} \int_{x=0}^{1-y-z} x dx dy dz$$

and

$$V = \int_{z=0}^{1} \int_{y=0}^{1-z} \int_{x=0}^{1-y-z} dx dy dz.$$

Remember, we start with the innermost integrals and work our way out.

- Do the integration. This is now relatively easy and we do them in turn. To evaluate I, we start with the x integration

$$\int_{x=0}^{1-y-z} x dx = \frac{1}{2} \left[x^2\right]_{x=0}^{1-y-z} = \frac{1}{2} (1-y-z)^2 .$$

Now the y integration gives

$$\int_{y=0}^{1-z} \frac{1}{2} (1-y-z)^2 \, dy = -\frac{1}{2}\frac{1}{3} \left[(1-y-z)^3\right]_{y=0}^{1-z} = -\frac{1}{6} \left\{0 - (1-z)^3\right\}.$$

Finally, we can do the z integration to get I

$$I = \int_{z=0}^{1} \frac{1}{6} (1-z)^3 \, dz = -\frac{1}{24} \left[(1-z)^4\right]_{z=0}^{1} = -\frac{1}{24} \{0-1) = \frac{1}{24}.$$

Now we repeat the integration to determine the volume. Firstly the x integration gives

$$\int_{x=0}^{1-y-z} dx = (1-y-z) .$$

next the y integration gives

$$\int_{y=0}^{1-z} (1-y-z) \, dy = -\frac{1}{2} \left[(1-y-z)^2\right]_{y=0}^{1-z} = \frac{1}{2} (1-z)^2$$

Finally, the z integration gives the volume as

$$V = \int_{z=0}^{1} \frac{1}{2} (1-z)^2 \, dz = -\frac{1}{6} \left[(1-z)^3\right]_{z=0}^{1} = \frac{1}{6}.$$

Hence, the centre $\bar{x}$ is

$$\bar{x} = \frac{1}{24}\frac{6}{1} = \frac{1}{4}.$$

It is fairly obvious, by symmetry, that

$$\bar{x} = \bar{y} = \bar{z} = \frac{1}{4}.$$

Note that $(\bar{x}, \bar{y}, \bar{z})$ is called the *centre of mass* of the volume V if it is made up of a material of uniform density.

Example End

Example 9.68

Find

$$I = \int\int\int_V z dV,$$

where V is the volume within a conical 'mountain' of width $2a$ and height h, above the plane $z = 0$.

Solution 9.68

- Draw the volume V. A cone can be generated as a surface of revolution about the z axis. At any height between $z = 0$ and $z = h$, the edge of the cone appears circular. The radius of the circle is $R_e = a$ at $z = 0$ and $R_e = 0$ at $z = h$. These points can be joined by a straight line to give

$$R_e = a\left(1 - \frac{z}{h}\right).$$

The equation of a circle (and hence the equation of the surface of the cone) is

$$x^2 + y^2 = R_e^2 = a^2\left(1 - \frac{z}{h}\right)^2.$$

Thus, we have a description of the volume within the cone, namely that

$$x^2 + y^2 \leq R_e^2.$$

The volume is shown in Figure 9.3

- Decide on the order of integration. It seems plausible that z should be the outer variable, i.e. do the z integration last. What do we choose for the inner variable, x or y? Actually it does not matter. What are we calculating with the integrals over x and y? Remember that z is held fixed, during these integrations, and so we have

$$I = \int_{z=0}^{h} z\left(\int\int_A dxdy\right),$$

where A is the area of a circle of radius R_e. This is, of course, just πR_e^2 and so

$$I = \int_{z=0}^{h} z\pi a^2\left(1 - \frac{z}{h}\right)^2 dz.$$

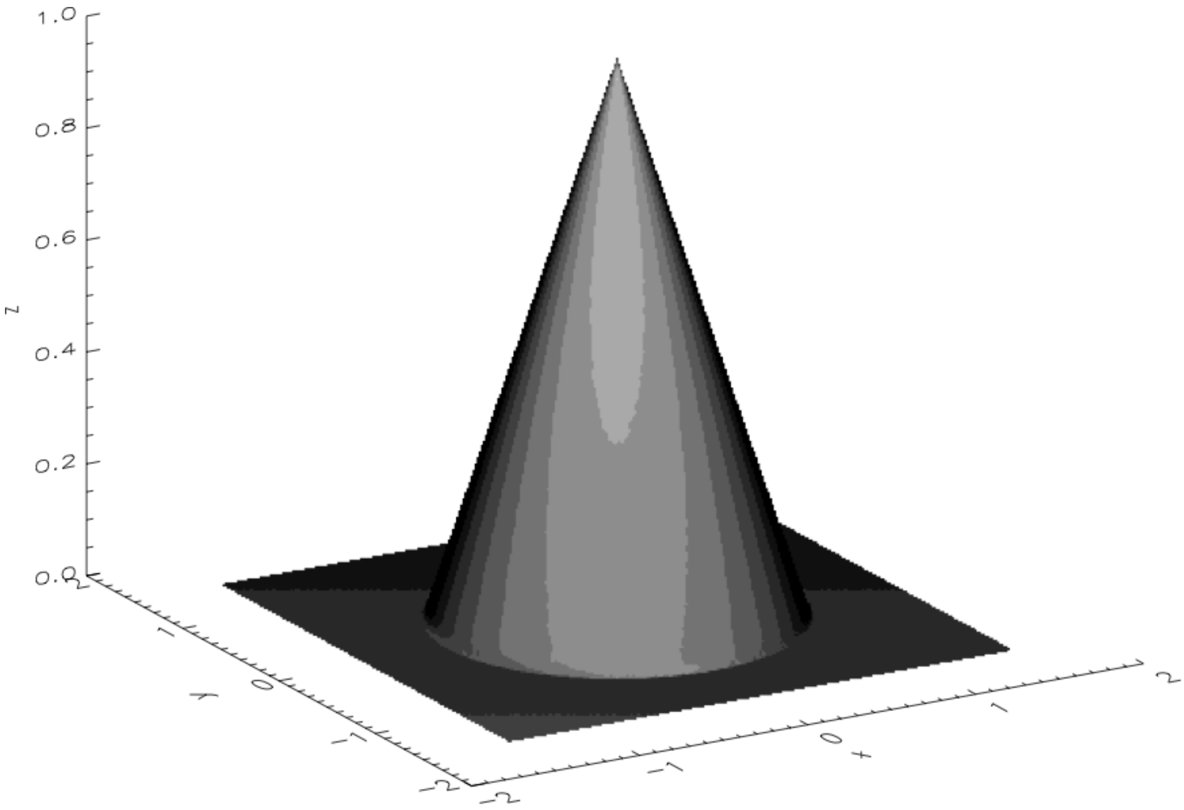


Figure 9.3: The volume defined by an inverted cone. {fig:9.3}

- Decide on the limits. While I have just said what the integral will turn out to be, it is worthwhile getting the limits for the x, y and z integrals as practice. Thus, we imagine that we have an infinitesimal box of volume $dV = dxdydz$ at a general point (x, y, z). Now we add up all the contributions in the x direction. Remember that the equation describing the edge of the circle (at height z) is $x^2 + y^2 = R_e^2$, the x integration will be from

$$x = -\sqrt{R_e^2 - y^2}, \text{ to } x = +\sqrt{R_e^2 - y^2}.$$

This gives a horizontal strip. Now we add up the contributions from all the strips in the y direction to generate the contribution from the circle at height z. Thus, the y integration goes from

$$y = -R_e \text{ to } y = +R_e.$$

Finally, we add up all the circle contributions from $z = 0$ to $z = h$. Thus,

$$I = \int_{z=0}^{h} \left(\int_{y=-R_e}^{R_e} \left(\int_{x=-sqrtR_e^2-y^2}^{\sqrt{R_e^2-y^2}} zdx \right) dy \right) dz.$$

- Do the integration in the correct order. We start with the x integration. Thus, we have

$$\int_{x=-\sqrt{R_e^2-y^2}}^{\sqrt{R_e^2-y^2}} z dx = 2z\sqrt{R_e^2 - y^2}.$$

The y integral becomes

$$\int_{-R_e}^{R_e} 2z\sqrt{R_e^2 - y^2} dy.$$

Now the y integration, requires the substitution $y = R_e \sin\phi$. Thus, $dy = R_e \cos\phi d\phi$. The y limits now give $y = -R_e$ implies $\phi = -\pi/2$ and $y = R_e$ implies $\phi = \pi/2$. Therefore,

$$\int_{y=-R_e}^{R_e} 2z\sqrt{R_e^2 - y^2} dy = \int_{\phi=-\pi/2}^{\pi/2} 2zR_e^2 \cos^2\phi d\phi = \pi R_e^2 z,$$

as predicted above. Finally, the z integration gives

$$I = \int_{z=0}^{h} \pi z R_e^2 dz = \int_{z=0}^{h} \pi z a^2 \left(1 - \frac{z}{h}\right)^2 dz.$$

Expanding the brackets gives

$$\begin{aligned} I &= \int_{z=0}^{h} \pi a^2 z \left(1 - 2\frac{z}{h} + \frac{z^2}{h^2}\right) dz \\ &= \pi a^2 \left[\frac{z^2}{2} - \frac{2}{3}\frac{z^3}{h} + \frac{1}{4}\frac{z^4}{h^2}\right]_{z=0}^{h} \\ &= \pi a^2 \left(\frac{h^2}{2} - \frac{2}{3}h^2 + \frac{1}{4}h^2\right) \\ &= \frac{1}{12}\pi a^2 h^2 \end{aligned}$$

Similarly, it is easy to show that the volume of this cone is

$$V = \frac{1}{3}\pi a^2 h.$$

So the *mean height* of the cone (or *centre*) is

$$\bar{z} = \frac{\int\int\int z dV}{\int\int\int dV} = \frac{1}{4}h.$$

Example End

The above example illustrates a tremendous simplification to triple integrals when

1. V is a volume of revolution – obtained by taking a curve in the y-z plane and rotating it about the z axis and

2. $f(x, y, z) = f(z)$ only.

The cone is just one of many examples. All we are doing is stacking circles or discs of radius $R_e(z)$, where R_e is the radius to the edge of the shape and whose value depends on the height z, and thickness dz. Thus,

$$\int\int\int_V f(z)dV = \pi \int_{z_1}^{z_2} f(z)R_e(z)^2 dz,$$

over the range of z, namely z_1 and z_2 that defines the volume V.

Let us do one example of a volume of revolution.

Example 9.69

Find,

$$I = \int\int\int_V e^{-z} dz,$$

where V is the volume enclosed by a *paraboloid*,

$$az = x^2 + y^2,$$

and the plane $z = b$. Here, a is a constant.

Solution 9.69

- Draw V. The paraboloid only exists for $z > 0$. Along $y = 0$, $az = x^2$ and this is just a parabola. Along $x = 0$, $az = y^2$ and this again a parabola. Since

  $$R^2 = x^2 + y^2,$$

  in polar coordinates, we can define

  $$R_e^2(z) = az.$$

  Note that the cross-sectional area, πR_e^2 just becomes

  $$\pi R_e^2 = \pi az,$$

  for a paraboloid at a fixed height z. The paraboloid is shown in Figure 9.4

- Limits. These are simply $z = 0$ and $z = b$. Hence, the integrals become

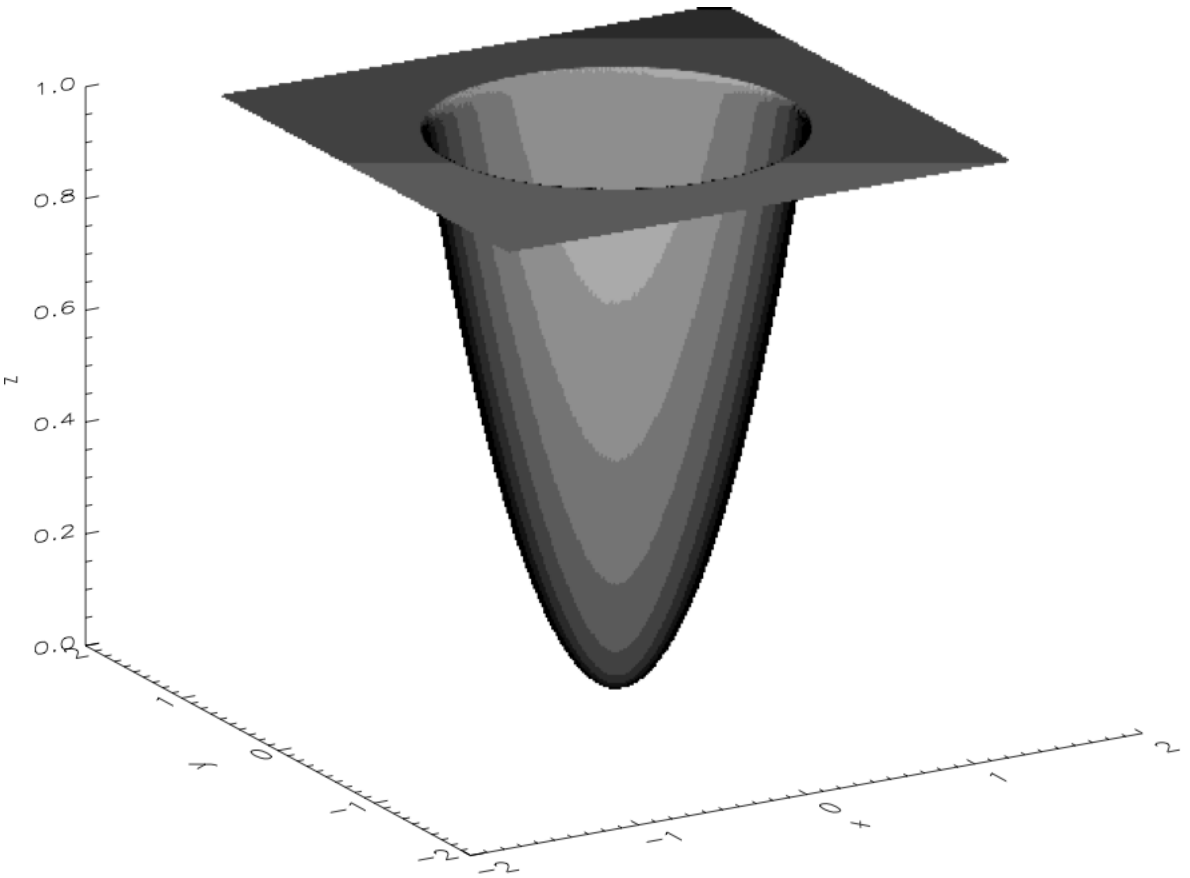
  $$I = \int\int\int_V e^{-z} dz = \pi \int_{z=0}^{b} e^{-z} R_e^2(z) dz = \pi a \int_{z=0}^{b} z e^{-z} dz.$$

- Do the integration. This integration will require integration by parts. We choose,

  $$\begin{aligned} u &= z, & \frac{dv}{dz} = e^{-z} \\ \frac{du}{dz} &= 1, & v = -e^{-z}. \end{aligned}$$

  Hence,

  $$I = \pi a \int_{z=0}^{b} z e^{-z} dz = \pi a \left\{ \left[-z e^{-z} \right]_{z=0}^{b} + \int_{z=0}^{b} e^{-z} dz \right\}.$$



{fig:9.4} Figure 9.4: The volume of a paraboloid is illustrated.

Thus,

$$\begin{aligned} I &= -\pi a\left[(z+1)e^{-z}\right]_{z=0}^{b} \\ &= -\pi a\left((b+1)e^{-b}-1\right) \\ &= \pi a\left(1-(b+1)e^{-b}\right) > 0. \end{aligned}$$

Example End

9.3 Volumes with Circular or Part Circular Cross Sections {sec:9.3}

Briefly, we consider a generalisation of the two examples just given above, namely integrals of the form

$$I = \int\int\int_V f(x,y,z)dV,$$

where V is circular (at least in part) in a cross-section. To keep things simple, we assume that the circular cross-section is always in the plane z =constant. Some typical examples are

1. $R =$ constant: the **cylinder**.
2. $R = az$: the **cone**.
3. $R = a\sqrt{z}$: the **paraboloid**.
4. $R = \sqrt{a^2 - z^2}$: the **sphere**. This is probably more easily recognised in the form $x^2+y^2+z^2 = a^2$.
5. $R = \sqrt{a^2 + z^2}$: the **hyperboloid**.

In all of these examples,

$$R^2 = x^2 + y^2.$$

Unlike the examples of Section 9.2, we now allow the integrand to be a function of all three spatial coordinates, namely $f(x, y, z)$. Now we cannot simplify the triple integral so easily as above. Care is needed in deciding the order of integration.

We illustrate the idea through two examples.

Example 9.70

Obtain

$$I = \int\int\int_V x^2 dV,$$

where V is the volume enclosed by the *inverted cone* considered previously, namely

$$R = R_e(z) = a\left(1 - \frac{z}{h}\right),$$

for $0 \le z \le h$ and the plane $z = 0$.

Solution 9.70

- Draw V. This has been done before in Figure 9.3.
- Decide on order. Since the integrand in independent of z, we can always do this one at any time. Hence, take z as the outer variable. Thus, we are looking at the integral in the form

  $$I = \int\left(\int\int_A x^2 dA\right) dz,$$

  where A is a circular area of radius $R_e(z)$. This is exactly the same as the double integrals with circular cross-section that we saw in Section 8.3. We need to use *polar coordinates*, with

  $$\begin{aligned} x &= R\cos\phi, \\ y &= R\sin\phi, \\ dA &= RdRd\phi. \end{aligned}$$

  We can do either R or ϕ first. It does not matter. Choose R as the inner variable and ϕ as the middle one. Thus,

  $$I = \int\left(\int\left(\int R^2\cos^2\phi RdR\right) d\phi\right) dz.$$

- Decide on limits. These are fairly straightforward. For R we have
$$0 \leq R \leq R_e(z).$$
For ϕ we have
$$0 \leq \phi \leq 2\pi.$$
Finally for z we have
$$0 \leq z \leq h.$$
Hence, the integral is
$$I = \int_{z=0}^{h} \left(\int_{\phi=0}^{2\pi} \left(\int_{R=0}^{R_e(z)} R^2 \cos^2 \phi R dR \right) d\phi \right) dz.$$

- Do the integrals. Starting with the R integral, remembering that ϕ is constant during this integration so that the $\cos^2 \phi$ term can be brought outside the integral sign, we have
$$\int_{R=0}^{R_e} R^3 dR = \frac{1}{4} \left[R^4 \right]_{R=0}^{R_e} = \frac{1}{4} a^4 \left(1 - \frac{z}{h} \right)^4.$$
The ϕ integration can now be done. Remember the double angle formula for terms like $\cos^2 \phi$, this gives,
$$\int_{\phi=0}^{2\pi} \cos^2 \phi d\phi = \frac{1}{2} \int_{\phi=0}^{2\pi} (1 + \cos 2\phi) \, d\phi = \frac{1}{2} \left[\phi + \frac{1}{2} \sin 2\phi \right]_{\phi=0}^{2\pi} = \pi.$$
Hence, the final integration gives
$$I = \int_{z=0}^{h} \frac{\pi}{4} a^4 \left(1 - \frac{z}{h} \right)^4 dz = -\frac{\pi}{20} a^4 h \left[\left(1 - \frac{z}{h} \right)^5 \right]_{z=0}^{h}.$$
This can be evaluated to give
$$I = \frac{1}{20} \pi a^4 h.$$

Example End

Example 9.71

Find
$$I = \int \int \int_V xyz dV,$$
where V is the portion of the sphere of radius a in the first octant, i.e. $x, y, z > 0$.

Solution 9.71

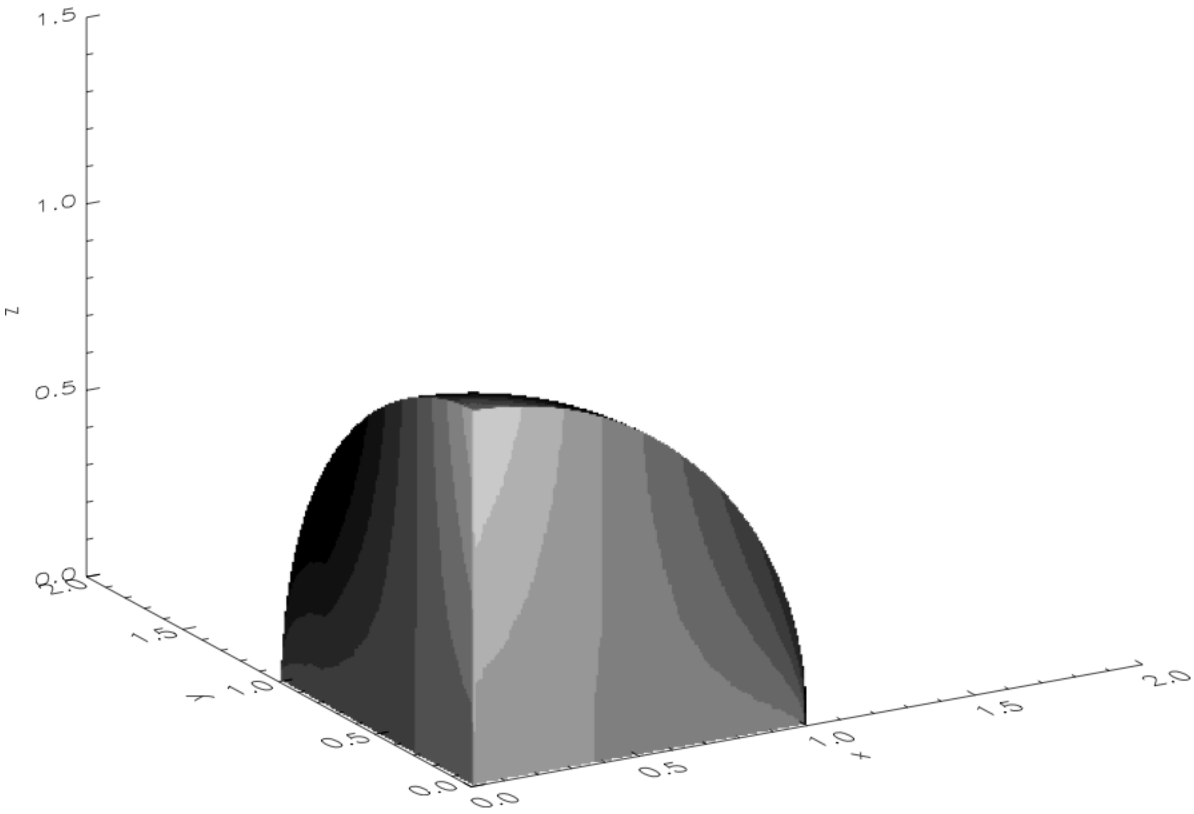


Figure 9.5: The sphere is the first octant. {fig:9.5}

- Draw V. This is shown in Figure 9.5.

  We will use polar coordinates for x and y or more correctly we switch from Cartesian coordinates (x, y, z) to *cylindrical coordinates* (R, ϕ, z) as shown in Figure 9.6. This is sensible since the cross-section at fixed height z is part of a circle. In any horizontal plane the cross-section of the volume is shown in Figure 9.6. In cylindrical coordinates, R varies from 0 to $R_e(z)$ where

  $$R^2 = x^2 + y^2 = a^2 - z^2 = R_e(z)^2.$$

  Thus, $R_e(z) = \sqrt{a^2 - z^2}$. ϕ varies from 0 to 2π and z varies from 0 to a.

  Using the same approach as in Section 9.3, we have

  $$dV = dxdydz = dAdz = RdRd\phi dz.$$

- Decide on order of integration. Clearly z should be the outer variable and done last. R and ϕ are the inner and middle variables, since the R limits in particular depend on z through $R_e(z)$.

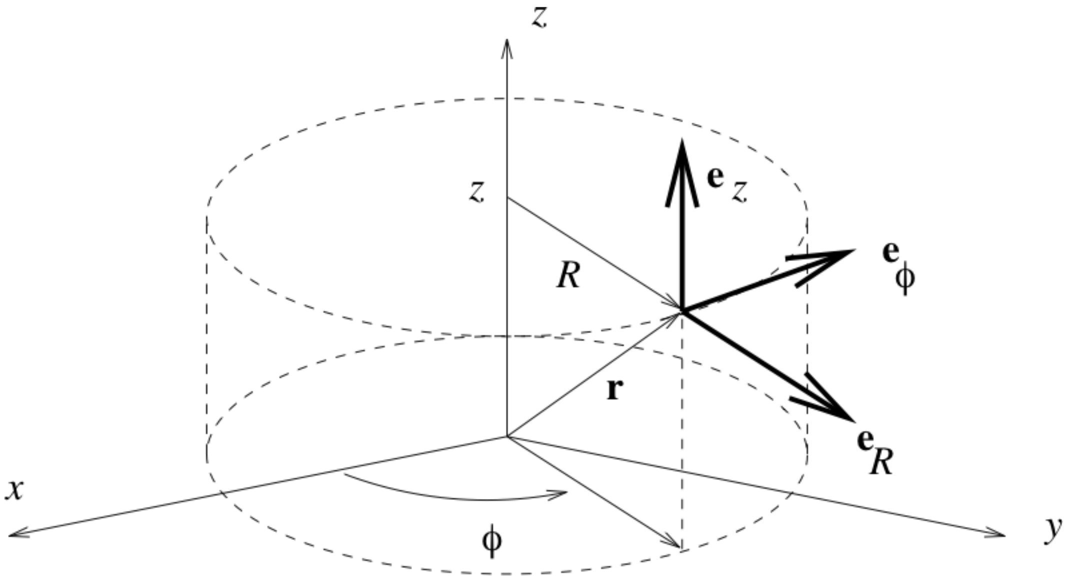


Figure 9.6: Cylindrical coordinates (R, ϕ, z). {fig:9.6}

- Decide on limits. These have more or less been worked out already.

$$\begin{aligned} 0 &\leq z \leq h \\ 0 &\leq \phi \leq \pi/2 \\ 0 &\leq R \leq R_e(z) = \sqrt{a^2 - z^2}. \end{aligned}$$

  Thus,

$$I = \int_{z=0}^{h} \int_{\phi=0}^{\pi/2} \int_{R=0}^{R_e(z)} R\cos\phi \; R\sin\phi \; zRdRd\phi dz,$$

  where we have replaced x and y in the integrand by $R\cos\phi$ and $R\sin\phi$ respectively and dV by $RdRd\phi dz$.

- Do the integrals. First we integrate with respect to R. Thus,

$$\int_{R=0}^{R_e(z)} R^3 dR = \frac{1}{4}\left[R^4\right]_{R=0}^{\sqrt{a^2-z^2}} = \frac{1}{4}a^4\left(a^2 - z^2\right)^2.$$

  Next, integration with respect to ϕ gives

$$\int_{\phi=0}^{\pi/2} \cos\phi \sin\phi \; d\phi = \frac{1}{2}\int_{\phi=0}^{\pi/2} \sin 2\phi \; d\phi = \frac{1}{4}\left[-\cos 2\phi\right]_{\phi=0}^{\pi/2} = \frac{1}{2}.$$

  Finally, bringing all the results together and doing the z integration gives

$$I = \int_{z=0}^{a} \frac{1}{8}a^4 z\left(a^2 - z^2\right)^2 dz.$$

We can do this integral by expanding the brackets (with care) and integrating term by term. However, we can also do this using a substitution, $u = a^2 - z^2$. Thus,

$$u = a^2 - z^2, \quad \Rightarrow \quad du = -2z dz.$$

Thus,

$$I = \int_{u=a^2}^{0} -\frac{1}{8}u^2 \frac{1}{2} du = \frac{1}{48}\left[u^3\right]_{u=0}^{a^2} = \frac{a^6}{48}.$$

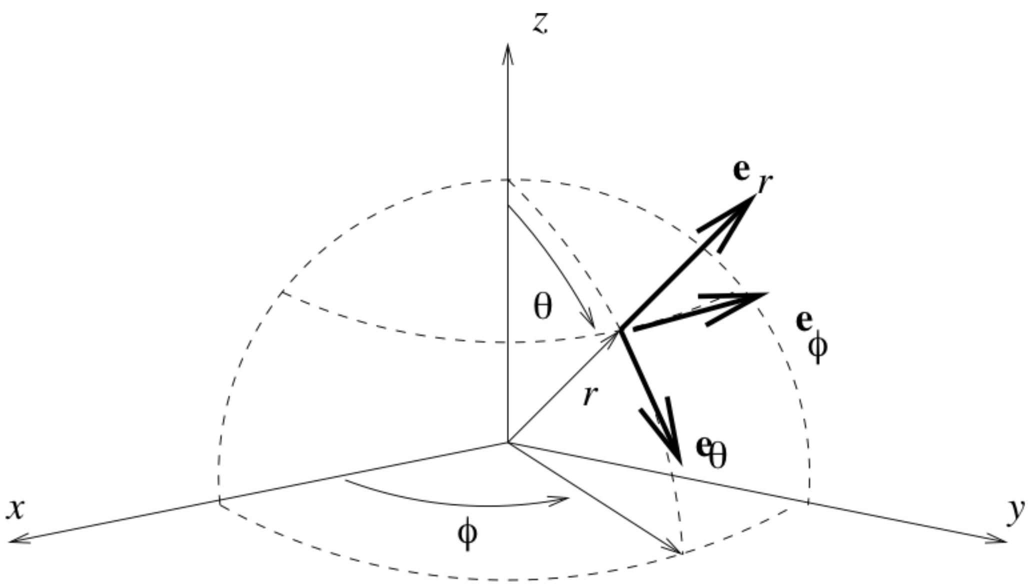
Example End

This example brings us to *spherical coordinates* which often prove useful for evaluating integrals over spherical volumes, as discussed in the next section.

9.4 Volumes which are Portions of Spheres

:9.4}

Sometimes, and it occurs more often than you might imagine, integrals arise over spherical volumes. For example, in meteorology finding the mean temperature of the Earth and in Solar Physics finding the mass of the Sun, requires the evaluation of integrals over a spherical volume.



;9.7} Figure 9.7: Spherical coordinates, (r, θ, ϕ).

Spherical coordinates consist of the three variables (r, θ, ϕ), where r is the radius, θ is the *co-latitude* and is measured from the North pole $\theta = 0$ down to the South pole $\theta = \pi$ and ϕ is the *longitude*. This is shown in Figure 9.7. We have,

{eq:9.5}
{eq:9.6}
{eq:9.7}

$$\begin{aligned} x &= r\sin\theta\cos\phi & (9.5)\\ y &= r\sin\theta\sin\phi & (9.6)\\ z &= r\cos\theta & (9.7)\end{aligned}$$

where $0 \le \phi < 2\pi$ and $0 \le \theta \le \pi$.

Important notes: We can readily calculate the infinitesimal volume element by a similar method to that introduced in Section 8.4.

9.4.1 Volume Element

{subsec:9.4.1}

We take the partial derivatives and form the matrix

$$\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix}$$

This gives

$$\begin{pmatrix} \cos\phi\sin\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\sin\phi\cos\theta & r\cos\phi\sin\theta \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}$$

Now we calculate the determinant of this 3×3 matrix. It is not too difficult if we expand it by the first row. This gives the Jacobian, J, as

$$\begin{aligned} J &= \cos\phi\sin\theta \begin{vmatrix} r\sin\phi\cos\theta & r\cos\phi\sin\theta \\ -r\sin\theta & 0 \end{vmatrix} - r\sin\phi\sin\theta \begin{vmatrix} \sin\phi\sin\theta & r\cos\phi\sin\theta \\ \cos\theta & 0 \end{vmatrix} \\ & -r\sin\phi\sin\theta \begin{vmatrix} \sin\phi\sin\theta & r\cos\phi\sin\theta \\ \cos\theta & -r\sin\theta \end{vmatrix} \end{aligned}$$

This can be simplified to

$$\begin{aligned} J &= r^2\cos^2\phi\sin^3\theta + r^2\cos^2\phi\cos^2\theta\sin\theta + r^2\sin^2\phi\sin\theta(\sin^2\theta+\cos^2\theta) \\ &= r^2\sin\theta(\cos^2\phi\sin^2\theta + \cos^2\phi\cos^2\theta + \sin^2\phi) \\ &= r^2\sin\theta(\cos^2\phi + \sin^2\phi) = r^2\sin\theta \end{aligned}$$

Therefore, the volume element is given by

$$dV = dxdydz = r^2\sin\theta dr d\theta d\phi \tag{9.8}$$ {eq:9.8}

9.4.2 Examples

{subsec:9.1.2}

Now consider integrals of the form

$$I = \int\int\int_V f(r,\phi,\theta)dV, \tag{9.9}$$ {eq:9.8a}

where V is a sphere (or part of a sphere). We are nearly ready to start the integration process but first we need to know the volume element dV in terms of spherical coordinates.

Let us consider two examples.

Example 9.72

Find the total mass of the Earth's atmosphere, taking for simplicity the density of the air to vary like

$$\rho = f(r) = \rho_0 e^{-(r-R_e)/h},$$

where ρ_0 is the density of the air at ground level, denoted by $R = R_e$. h is the scale height. Typical values for these constants are

$$\rho_0 = 1.3\text{kg/m}^3, \quad h = 7.4\text{km}$$

However, although we know these values, don't insert them until all the integration has been performed. Why? Well if you want to repeat the example for a different planet you can use the same integration and just change the values of the constants in the final answer.

Solution 9.72

The mass is given by integrating the density over the volume of the atmosphere,

$$\begin{aligned} I &= \int\int\int_V \rho(r)dV \\ &= \int\int\int_V f(r)r^2 \sin\theta dr d\theta d\phi \end{aligned}$$

This is not so hard as the density is only a function of radius. Hence, the order of integration is not really important. What about the limits? To cover the volume of atmosphere above ground level, we must integrate ϕ from 0 to 2π and θ from 0 to π (see Figure 9.7). The radial integration goes from ground level, $r = R_e$ to infinity. Thus,

$$I = \int_{r=R_e}^{\infty} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \rho_0 e^{-(r-R_e)/h} r^2 \sin\theta d\phi d\theta dr.$$

The ϕ integration is easy

$$\int_{\phi=0}^{2\pi} \rho_0 e^{-(r-R_e)/h} r^2 \cos\theta d\phi = 2\pi\rho_0 e^{-(r-R_e)/h} r^2 \sin\theta$$

Now the integration in θ gives,

$$\int_{\theta=0}^{\pi} 2\pi\rho_0 e^{-(r-R_e)/h} r^2 \sin\theta d\theta = 2\pi\rho_0 e^{-(r-R_e)/h} r^2 \times 2$$

Finally, the r integration, after expanding the brackets in the exponential, gives

$$I = 4\pi\rho_0 \int_{r=R_e} r^2 e^{-r/h} e^{R_e/h} dr.$$

This requires integration by parts (twice). We choose $u = r^2$ so that $du/dr = 2r$ and $dv/dr = e^{-r/h}$ so that $v = -he^{-r/h}$. Hence.

$$\begin{aligned}
I &= 4\pi\rho_0 e^{R_e/h} \int_{r=R_e}^{\infty} r^2 e^{-r/h} dr \\
&= 4\pi\rho_0 e^{R_e/h} \left\{ \left[-hr^2 e^{-r/h} \right]_{R_e}^{\infty} + h \int_{r=R_e}^{\infty} 2re^{-r/h} dr \right\} \\
&= 4\pi\rho_0 e^{R_e/h} h \left\{ R_e^2 e^{-R_e/h} + 2 \left[-hre^{-r/h} \right]_{R_e}^{\infty} + 2h \int_{r=R_e}^{\infty} e^{-r/h} dr \right\} \\
&= 4\pi\rho_0 e^{R_e/h} h \left\{ R_e^2 e^{-R_e/h} + 2hR_e e^{-R_e/h} + 2h^2 e^{-R_e/h} \right\}.
\end{aligned}$$

This simplifies to give

$$I = 4\pi\rho_0 h \left(R_e^2 + 2hR_e + 2h^2 \right).$$

Example End