Chapter 5

Eigenvalues, Eigenvectors and Diagonalisation

We have seen that, given a linear transformation $T : V \to W$, different choices of basis for our vector space V will lead to different forms of matrix for $\mathrm{Mat}(T)$. It would be particularly helpful if we could find a choice of basis which gives rise to a diagonal matrix for $\mathrm{Mat}(T)$.

In this section, we shall be mainly concerned with the case that we have a vector space V and a linear transformation $T\colon V \to V$ (that is, from V to itself). A particular case will be when we have a square $n \times n$ matrix A viewed as a linear transformation $A\colon F^n \to F^n$.

Eigenvalues and eigenvectors

Definition 5.1 A linear transformation $T\colon V \to V$ of a finite dimensional vector space V is said to be *diagonalisable* if there is a basis for V with respect to which the matrix of T is represented by a diagonal matrix.

By considering how $\mathrm{Mat}(T)$ is formed, we see that a linear transformation T will be diagonalisable precisely if there is a basis $\mathscr{B} = \{v_1, v_2, \dots, v_n\}$ such that the matrix of T with respect to $\mathscr{B}$ has the form

$$\mathrm{Mat}_{\mathscr{B},\mathscr{B}}(T) = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{pmatrix}.$$

This means that we require $T(v_i) = \lambda_i v_i$ for each i — i.e. each element of the basis is mapped to a scalar multiple of itself by T. This motivates the following definition.

Definition 5.2 Let V be a vector space and $T\colon V \to V$ be a linear transformation. We say that a scalar λ is an *eigenvalue* of T if there is some *non-zero* vector v in V such that

$$T(v) = \lambda v.$$

Any non-zero vector satisfying this equation will be called an *eigenvector* for T associated to the eigenvalue λ.

Eigenvalues and eigenvectors of a square $n \times n$ matrix A with entries from a field F are defined as the eigenvalues and eigenvectors of the linear transformation m_A. Thus a scalar $\lambda \in F$ is an eigenvalue of A with corresponding eigenvector $\boldsymbol{v} \in F^n$ if

$$A\boldsymbol{v} = \lambda\boldsymbol{v} \qquad \text{and} \qquad \boldsymbol{v} \neq \mathbf{0}.$$

Definition 5.3 For each eigenvalue λ of a given matrix A, consider the set $\{\boldsymbol{v} : A\boldsymbol{v} = \lambda\boldsymbol{v}\}$. This comprises all eigenvectors of A together with the zero vector. It is a subspace of F^n called the eigenspace of A corresponding to λ.

Example 5.4 *Let $V = \mathbb{R}^2$ and consider the matrix*

$$A = \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}$$

Show that the vectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are eigenvectors of A.

Solution: We calculate that

$$\begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \end{pmatrix} = -4\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Hence

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ is an eigenvector for } A \text{ with eigenvalue } 2$$

and

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ is an eigenvector for } A \text{ with eigenvalue } -4.$$

In fact, it is clear that all non-zero scalar multiples $\alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ($\alpha \in \mathbb{R}$, $\alpha \neq 0$) are eigenvectors of A with eigenvalue 2, and all non-zero scalar multiples $\beta\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ($\beta \in \mathbb{R}$, $\beta \neq 0$) are eigenvectors of A with eigenvalue -4.

Note that the two eigenvectors above are two linearly independent vectors in $\mathbb{R}^2$ and hence form a basis for $\mathbb{R}^2$.

Example 5.5 *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by*

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x + 3y \\ 3x - y \end{pmatrix}.$$

Determine $\mathrm{Mat}(T)$ *with respect to*

(i) the standard basis of $\mathbb{R}^2$*;*

(ii) the basis $\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$ *of* $\mathbb{R}^2$.

Solution: For (i), since

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-1\\3\end{pmatrix} = (-1)\mathbf{e}_1 + 3\mathbf{e}_2$$

and

$$T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix} = 3\mathbf{e}_1 + (-1)\mathbf{e}_2,$$

the matrix is

$$\operatorname{Mat}(T) = \begin{pmatrix}-1 & 3\\3 & -1\end{pmatrix}$$

For (ii), since

$$T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix} = 2\begin{pmatrix}1\\1\end{pmatrix} + 0\begin{pmatrix}1\\-1\end{pmatrix}$$

and

$$T\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}-4\\4\end{pmatrix} = 0\begin{pmatrix}1\\1\end{pmatrix} + (-4)\begin{pmatrix}1\\-1\end{pmatrix},$$

we have

$$\operatorname{Mat}(T) = \begin{pmatrix}2 & 0\\0 & -4\end{pmatrix}$$

This is a diagonal matrix with the eigenvalues as diagonal entries.

Geometrically, T is a linear transformation of $\mathbb{R}^2$ with the following properties: it takes each point (α, α) on the line $x - y = 0$ and "stretches it" to a point on the same line but twice as far away from the origin. Similarly it takes each point $(\beta, -\beta)$ on the line $x + y = 0$ and stretches it to a point four times as far away from, and on the other side of, the origin.

Finding eigenvalues and eigenvectors

We now proceed to investigate how such eigenvalues and eigenvectors may be obtained.

Let $T\colon V \to V$ and let $A = \operatorname{Mat}(T)$ be the matrix of T with respect to some basis $\mathscr{B}$ for V. We seek to find a scalar λ and a non-zero vector $v \in V$ such that $T(v) = \lambda v$.

Theorem 5.6 *Let $T\colon V \to V$ and let $A = \operatorname{Mat}(T)$ be the matrix of T with respect to some basis $\mathscr{B}$ for V. A scalar λ is an eigenvalue of T if and only if* $\det(\lambda I - A) = 0$.

Proof: Observe that $T(\boldsymbol{v}) = \lambda\boldsymbol{v}$ can be rearranged into the form

$$(\lambda i_V - T)(\boldsymbol{v}) = \mathbf{0}$$

where $i_V : V \to V$ is the identity transformation ($i_V : \boldsymbol{v} \mapsto \boldsymbol{v}$). Correspondingly we find

$$(\lambda I - A)\boldsymbol{v} = \mathbf{0}$$

for some $\boldsymbol{v} \neq \mathbf{0}$ in F^n. Then λ is an eigenvalue if and only if the homogeneous system of equations

$$(\lambda I - A)\boldsymbol{v} = \mathbf{0}$$

has a non-zero solution. This is the case precisely if the matrix $(\lambda I - A)$ is non-invertible, i.e. precisely if $\det(\lambda I - A) = 0$. $\square$

We therefore make the following definition.

Definition 5.7 Let $T\colon V \to V$ be a linear transformation and let A be the matrix of T with respect to some basis $\mathscr{B}$ for V. The *characteristic polynomial* of T is

$$\det(xI - A);$$

that is, we expand the determinant of the matrix involving the variable x to obtain a polynomial in x.

Proposition 5.8 *The characteristic polynomial of $T : V \to V$ does not depend on the choice of basis $\mathscr{B}$ for V.*

PROOF: Omitted here; see MT3501. $\square$

We note that an eigenvalue must lie in the given field of scalars. If, for example, the field of scalars $F = \mathbb{R}$ but all roots of the characteristic polynomial lie in $\mathbb{C} \setminus \mathbb{R}$, then T has no eigenvalues over F.

Example 5.9 *Consider the matrix*

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Its characteristic polynomial is $x^2 + 1$. If we are working over $\mathbb{R}$, then A has no eigenvalues, whereas if we are working over $\mathbb{C}$, it has two eigenvalues, i and $-i$.

We have observed that the eigenvalues of a linear transformation are the roots of its characteristic polynomial. (Note that this is *not* the definition of the eigenvalues: it is the *method* to find the eigenvalues. The definition of eigenvalue — given in Definition 5.2 — makes sense even over infinite-dimensional vector spaces while we cannot calculate a determinant of a matrix in such a setting.) We then find the eigenvectors by solving the equation $T(v) = \lambda v$ for each eigenvalue λ that we have determined.

Remark 5.10 *In the examples that follow, we shall consider a matrix A as a linear transformation $A\colon F^n \to F^n$ (as in Example 4.5). Then the matrix of the linear transformation A with respect to the standard bases is A itself and so we find the eigenvalues by solving the equation* $\det(xI - A) = 0$.

Example 5.11 *Find the eigenvalues and corresponding eigenvectors of the matrix*

$$A = \begin{pmatrix} 1 & -4 & 0 \\ 2 & 7 & 3 \\ -2 & -4 & -2 \end{pmatrix}.$$

Solution: The characteristic polynomial of A is

$$\begin{aligned}
\det(xI - A) &= \det \begin{pmatrix} x-1 & 4 & 0 \\ -2 & x-7 & -3 \\ 2 & 4 & x+2 \end{pmatrix} \\
&= (x-1)\big((x-7)(x+2)+12\big) - 4\big(-2(x+2)+6\big) \\
&= (x-1)(x^2-5x-14+12) - 4(2-2x) \\
&= (x-1)(x^2-5x-2) + 8(x-1) \\
&= (x-1)(x^2-5x-2+8) \\
&= (x-1)(x^2-5x+6) \\
&= (x-1)(x-2)(x-3).
\end{aligned}$$

Hence the roots of the characteristic polynomial are 1, 2 and 3 and these are the eigenvalues of A.

We now solve the equation $A\boldsymbol{v} = \lambda\boldsymbol{v}$ for each eigenvalue λ in turn seeking a non-zero solution $\boldsymbol{v}$.

Case $\boldsymbol{\lambda = 1}$: Our equation $A\boldsymbol{v} = \lambda\boldsymbol{v} = \boldsymbol{v}$ rearranges to $(A - I)\boldsymbol{v} = \mathbf{0}$; that is,

$$\begin{pmatrix} 0 & -4 & 0 \\ 2 & 6 & 3 \\ -2 & -4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus we arrive at three equations:

$$-4y = 0, \qquad 2x + 6y + 3z = 0, \qquad -2x - 4y - 3z = 0.$$

We can either apply Gaussian elimination/row-reduction to reduce the system, or proceed directly. In this case, since the first equation tells us that $y = 0$, we can proceed directly; the last two equations now both reduce to $2x + 3z = 0$. (This is a general phenomenon of this process. The fact that λ is an eigenvector will always ensure that there is a non-zero solution and hence some redundancy in the equations.) We can take x to be any non-zero value; an eigenvector is then given by $\begin{pmatrix} x \\ 0 \\ -2x/3 \end{pmatrix}$. e.g. taking $x = 3$, $\begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}$ is an eigenvector for A with eigenvalue 1.

Case $\boldsymbol{\lambda = 2}$: We solve $(A - 2I)\boldsymbol{v} = \mathbf{0}$; that is,

$$\begin{pmatrix} -1 & -4 & 0 \\ 2 & 5 & 3 \\ -2 & -4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

To solve this equation, let us apply row operations to the matrix (more precisely, the augmented matrix):

$$\left(\begin{array}{ccc|c} -1 & -4 & 0 & 0 \\ 2 & 5 & 3 & 0 \\ -2 & -4 & -4 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} -1 & -4 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 4 & -4 & 0 \end{array}\right) \qquad \begin{array}{l} r_2 \mapsto r_2 + 2r_1 \\ r_3 \mapsto r_3 - 2r_1 \end{array}$$

Hence we obtain essentially two equations:

$$-x - 4y = 0, \qquad y - z = 0.$$

Given any non-zero choice of y, we now obtain a solution: an eigenvector is $\begin{pmatrix} -4y \\ y \\ y \end{pmatrix}$.

Take $y = 1$; then $\begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector for A with eigenvalue 2.

Case $\boldsymbol{\lambda = 3}$: We solve $(A - 3I)\boldsymbol{v} = \boldsymbol{0}$; that is,

$$\begin{pmatrix} -2 & -4 & 0 \\ 2 & 4 & 3 \\ -2 & -4 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Row-reducing, we have

$$\left(\begin{array}{ccc|c} -2 & -4 & 0 & 0 \\ 2 & 4 & 3 & 0 \\ -2 & -4 & -5 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} -2 & -4 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & -5 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus we obtain two equations

$$x + 2y = 0, \qquad z = 0.$$

Any non-zero value of y yields an eigenvector $\begin{pmatrix} -2y \\ y \\ 0 \end{pmatrix}$.

Eg take $y = 1$; then $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector for A with eigenvalue 3.

Note that in each case, we always have some choice for our eigenvector. This reflects the (easily verified) fact that if v is an eigenvector with eigenvalue λ for a linear transformation T, then any *non-zero* scalar multiple of v is also an eigenvector with the same eigenvalue.

Geometrically, what is happening here? The linear transformation of $\mathbb{R}^3$ given by $\boldsymbol{v} \mapsto A\boldsymbol{v}$ leaves 3 lines in $\mathbb{R}^3$ invariant — those through the origin which contain the points $(3, 0, -2)$, $(-4, 1, 1)$ and $(-2, 1, 0)$.

Example 5.12 *Find the eigenvalues and eigenvectors of the matrix*

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution: First we find the characteristic polynomial:

$$\det(xI - A) = \det \begin{pmatrix} x-2 & -1 & -1 \\ 0 & x-1 & 0 \\ 0 & 0 & x-1 \end{pmatrix}$$

$$= (x-2)(x-1)^2.$$

Hence the eigenvalues (roots of the characteristic polynomial) are

$$\lambda = 2 \text{ and } 1 \text{ (twice)}.$$

We now find the eigenvectors.

Case $\boldsymbol{\lambda = 2}$: We solve $(A - 2I)\boldsymbol{v} = \mathbf{0}$; that is,

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

From this, $y = z = 0$, while x may be arbitrary. For any non-zero x, $\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}$ is an eigenvector for A with eigenvalue 2. For example, taking $x = 1$ yields $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Case $\boldsymbol{\lambda = 1}$: We solve $(A - I)\boldsymbol{v} = \mathbf{0}$; that is,

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence $x + y + z = 0$. So two of the variables, say x and y, can be arbitrary; the third is then determined. An eigenvector is given by any $\begin{pmatrix} x \\ y \\ -x-y \end{pmatrix}$ where x, y are not both 0.

Taking $x = 1$ and $y = 0$ gives the eigenvector

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

while taking $x = 0$ and $y = 1$ gives the eigenvector

$$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

It is easy to check that these vectors are linearly independent. Hence, although we have a repeated eigenvalue, we have managed to find two *linearly independent* eigenvectors with eigenvalue 1.

When the eigenvalue appears as a repeated root of the characteristic polynomial, it is not always the case that we can find as many linearly independent eigenvectors. This is a key topic in linear algebra and it will be considered in greater detail in MT3501.

Example 5.13 *Recall the matrix A from Example 5.11:*

$$A = \begin{pmatrix} 1 & -4 & 0 \\ 2 & 7 & 3 \\ -2 & -4 & -2 \end{pmatrix}.$$

Show that we can diagonalise A.

Solution: In Example 5.11, we found three eigenvectors

$$\boldsymbol{v}_1 = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

with eigenvalues 1, 2 and 3, respectively. Using the methods from earlier in the course, it is not difficult to verify that these three vectors are linearly independent. Hence $\mathscr{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is a basis for $\mathbb{R}^3$. We can therefore consider the matrix of the linear transformation $A\colon \boldsymbol{v} \mapsto A\boldsymbol{v}$ with respect to the basis $\mathscr{B}$. Since the three vectors in $\mathscr{B}$ are *eigenvectors*, when we express the image of each of them in terms of $\mathscr{B}$ we obtain

$$A\boldsymbol{v}_1 = \boldsymbol{v}_1, \quad A\boldsymbol{v}_2 = 2\boldsymbol{v}_2, \quad A\boldsymbol{v}_3 = 3\boldsymbol{v}_3.$$

We write the coefficients down the columns of the matrix:

$$\mathrm{Mat}_{\mathscr{B},\mathscr{B}}(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

yielding a diagonal matrix.

We have the following general result.

Theorem 5.14 *Let T be a linear transformation from V to V (alternatively a square matrix). Eigenvectors of T corresponding to distinct eigenvalues are linearly independent.*

PROOF: The proof is by induction on the number of eigenvalues.

If T has just one eigenvalue λ, then any corresponding eigenvector v is non-zero by definition, hence $\{v\}$ is a LI set.

For the induction step, suppose that every set of n eigenvectors that correspond to n distinct eigenvalues is linearly independent. Let $v_1, \dots, v_{n+1}$ be eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots, \lambda_{n+1}$. If we have the linear combination

$$a_1 v_1 + \cdots + a_n v_n + a_{n+1} v_{n+1} = \mathbf{0}$$

then applying T (if it is a transformation) or multiplying by T (if it is a matrix) yields:

$$a_1 T(v_1) + \cdots + a_n T(v_n) + a_{n+1} T(v_{n+1}) = \mathbf{0}$$

by linearity, i.e.

$$a_1 \lambda_1 v_1 + \cdots + a_n \lambda_n v_n + a_{n+1} \lambda_{n+1} v_{n+1} = \mathbf{0}$$

(using the fact that the v_i's are eigenvectors). From this, subtract λ_{n+1} times the first equation to get

$$a_1(\lambda_1 - \lambda_{n+1})v_1 + \cdots + a_n(\lambda_n - \lambda_{n+1})v_n = \mathbf{0}.$$

By the induction hypothesis, $\{v_1, \ldots, v_n\}$ is a linearly independent set, so all coefficients must be zero, i.e.

$$a_i(\lambda_i - \lambda_{n+1}) = 0, \text{ for } i = 1, \ldots, n.$$

Since $\lambda_1, \ldots, \lambda_{n+1}$ are all distinct, we deduce that

$$a_1 = \cdots = a_n = 0.$$

We must then have $a_{n+1}v_{n+1} = \mathbf{0}$, and since $v_{n+1} \neq \mathbf{0}$, this shows $a_{n+1} = 0$. So the vectors $v_1, \ldots, v_{n+1}$ are linearly independent. $\square$

Corollary 5.15 *Let T be an $n \times n$ matrix. If the characteristic polynomial of T has n distinct roots then T is diagonisable.*

NB: if the n roots are *not* distinct, the matrix may or may not be diagonisable.

Change of basis

Given a diagonalisable matrix, it is natural to ask: what is the relationship between the original matrix A and the new diagonal matrix which represents m_A? Is there a simple relationship?

We can explore this question through the more general one: if we have a linear transformation T, and two matrices which represent T with respect to two different bases, how are these related?

Theorem 5.16 *Let V be a vector space of dimension n over a field F and let $T \colon V \to V$ be a linear transformation. Let $\mathscr{A}$ and $\mathscr{B}$ be bases for V and let A and B be the matrices of T with respect to $\mathscr{A}$ and $\mathscr{B}$, respectively. Then there exists an invertible matrix P such that*

$$B = P^{-1}AP.$$

Let $\mathscr{A} = \{v_1, v_2, \ldots, v_n\}$ and $\mathscr{B} = \{w_1, w_2, \ldots, w_n\}$. The jth column of P contains the coefficients of w_j when it is written in terms of the basis $\mathscr{A}$ (i.e. if we express w_j as a linear combination of $\{v_1, \ldots, v_n\}$ then the coefficient of v_i is the (i, j)th entry of P).

PROOF: We shall determine how $A = \mathrm{Mat}_{\mathscr{A},\mathscr{A}}(T)$ and $B = \mathrm{Mat}_{\mathscr{B},\mathscr{B}}(T)$ are related. Suppose that

$$\mathrm{Mat}_{\mathscr{A},\mathscr{A}}(T) = [\alpha_{ij}] \qquad \text{and} \qquad \mathrm{Mat}_{\mathscr{B},\mathscr{B}}(T) = [\beta_{ij}].$$

This means that

$$T(v_j) = \sum_{i=1}^{n} \alpha_{ij} v_i \qquad \text{and} \qquad T(w_j) = \sum_{i=1}^{n} \beta_{ij} w_i$$

for $j = 1, 2, \dots, n$.

To determine how the α_{ij} and β_{ij} are related, the important thing to remember is that any vector in V can be uniquely expressed as a linear combination of the members of a basis. In particular, we can write

$$w_j = \sum_{k=1}^{n} \lambda_{kj} v_k \tag{5.1}$$

and

$$v_\ell = \sum_{i=1}^{n} \mu_{i\ell} w_i \tag{5.2}$$

(for some coefficients $\lambda_{kj}, \mu_{i\ell} \in F$), so expressing each basis vector w_j from $\mathscr{B}$ in terms of the basis $\mathscr{A}$ and *vice versa*. From this, we calculate

$$\begin{aligned}
T(w_j) &= T\Bigl(\sum_{k=1}^{n} \lambda_{kj} v_k\Bigr) \\
&= \sum_{k=1}^{n} \lambda_{kj} T(v_k) \\
&= \sum_{k=1}^{n} \lambda_{kj} \sum_{\ell=1}^{n} \alpha_{\ell k} v_\ell \\
&= \sum_{\ell=1}^{n} \sum_{k=1}^{n} \alpha_{\ell k} \lambda_{kj} \sum_{i=1}^{n} \mu_{i\ell} w_i \\
&= \sum_{i=1}^{n} \sum_{\ell=1}^{n} \sum_{k=1}^{n} \mu_{i\ell} \alpha_{\ell k} \lambda_{kj} w_i \\
&= \sum_{i=1}^{n} \Bigl(\sum_{\ell=1}^{n} \sum_{k=1}^{n} \mu_{i\ell} \alpha_{\ell k} \lambda_{kj}\Bigr) w_i.
\end{aligned}$$

This must be the *unique* expression for $T(w_j)$ as a linear combination of the vectors in the basis $\mathscr{B}$. Hence

$$\beta_{ij} = \sum_{\ell=1}^{n} \sum_{k=1}^{n} \mu_{i\ell} \alpha_{\ell k} \lambda_{kj}.$$

This formula is simply that expressing the multiplication of the matrices involved. Specifically, if we write

$$\begin{aligned}
A = \operatorname{Mat}_{\mathscr{A},\mathscr{A}}(T) &= [\alpha_{ij}], \qquad & B &= \operatorname{Mat}_{\mathscr{B},\mathscr{B}}(T) = [\beta_{ij}] \\
P &= [\lambda_{ij}], \qquad & Q &= [\mu_{ij}],
\end{aligned}$$

then the above formula says

$$B = QAP.$$

However, it turns out that Q and P are also linked. Substituting (5.1) into (5.2) gives

$$v_\ell = \sum_{i=1}^{n} \mu_{i\ell} \sum_{k=1}^{n} \lambda_{ki} v_k = \sum_{k=1}^{n} \Bigl(\sum_{i=1}^{n} \lambda_{ki} \mu_{i\ell}\Bigr) v_k.$$

This must be the unique expression for v_ℓ as a linear combination of the vectors in $\mathscr{A} = \{v_1, v_2, \ldots, v_n\}$. Thus

$$(PQ)_{kl} = \sum_{i=1}^{n} \lambda_{ki}\mu_{i\ell} = \delta_{k\ell} = \begin{cases} 1 & \text{if } k = \ell \\ 0 & \text{if } k \neq \ell. \end{cases}$$

(This $\delta_{k\ell}$ is called the *Kronecker delta.*) So

$$PQ = I,$$

the $n \times n$ identity matrix. Similarly, substituting (5.2) into (5.1) yields $QP = I$ by the same argument. Hence

$$Q = P^{-1}.$$

$\square$

Example 5.17 *Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by*

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ x \\ y \end{pmatrix}.$$

Let $\mathscr{A}$ be the standard basis of $\mathbb{R}^3$, and let $\mathscr{B}$ be the basis

$$\mathscr{B} = \{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \}.$$

Obtain the matrix A of T w.r.t. basis $\mathscr{A}$, the matrix B of T w.r.t. basis $\mathscr{B}$ and the change of basis matrix P (from $\mathscr{A}$ to $\mathscr{B}$). Verify that $B = P^{-1}AP$.

Solution: We have

$$T(e_1) = e_2, \quad T(e_2) = e_3, \quad T(e_3) = e_1.$$

So

$$A = \mathrm{Mat}_{\mathscr{A},\mathscr{A}}(T) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Denote the vectors in $\mathscr{B}$ by $v_1 = (1, 1, 1), v_2 = (0, 1, 1), v_3 = (0, 0, 1)$. Then

$$T(v_1) = v_1, \quad T(v_2) = v_1 - v_2 + v_3, \quad T(v_3) = v_1 - v_2.$$

So

$$B = \mathrm{Mat}_{\mathscr{B},\mathscr{B}}(T) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Following the recipe from Theorem 5.16 above, we write the basis vectors in $\mathscr{B}$ as linear combinations of the vectors in $\mathscr{A}$, and write the coefficients down the columns. Since $\mathscr{A}$ is the standard basis, we get

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

By using the adjugate, or otherwise, we find that the inverse of P is given by

$$P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

Then we can verify that

$$P^{-1}A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

and thence $P^{-1}AP = B$.

We mention briefly that the same sort of argument as for Theorem 5.16 (though slightly more complicated because two change of bases are involved) establishes what happens if we have a linear transformation $T\colon V \to W$ and we consider different bases for V and for W:

Theorem 5.18 *Let V and W be finite-dimensional vector spaces over a field F and let $T\colon V \to W$ be a linear transformation. Suppose that $\mathscr{B}$ and $\mathscr{B}'$ are bases for V and $\mathscr{C}$ and $\mathscr{C}'$ are bases for W. Then there exist invertible matrices P and Q such that*

$$\mathrm{Mat}_{\mathscr{B}',\mathscr{C}'}(T) = Q^{-1} \cdot \mathrm{Mat}_{\mathscr{B},\mathscr{C}}(T) \cdot P.$$

Moreover, the (i,j)th entry of P is the coefficient when the jth vector of $\mathscr{B}'$ is written in terms of the basis $\mathscr{B}$ and the (i,j)th entry of Q is the coefficient when the jth vector of $\mathscr{C}'$ is written in terms of the basis $\mathscr{C}$.

Theorem 5.16 motivates us to make the following definition:

Definition 5.19 For $A, B \in M_{n\times n}(F)$, we say B is *similar to* A if and only if $B = P^{-1}AP$ for some invertible matrix $P \in M_{n\times n}(F)$.

We may then state the following result:

Theorem 5.20 *If $T : V \to V$ and A is the matrix of T with respect to some basis of V, then the set of all matrices arising as the matrix of T w.r.t. the different possible bases of V is the set of matrices similar to A.*

Note that the proof that the characteristic polynomial is independent of choice of basis follows by showing that similar matrices have the same characteristic polynomial.

We can now summarize the following characterizations for a transformation and a matrix to be diagonalisable:

Theorem 5.21

- *A linear transformation $T\colon V \to V$ is diagonalisable if and only if there is a basis for V consisting of eigenvectors for T.*
- *A matrix A is diagonalisable if and only if there exists an invertible matrix P such that $P^{-1}AP$ is diagonal.*

Example 5.22 *Is the matrix*

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

diagonalisable? If so, find the change of basis matrix P such that $P^{-1}AP$ is diagonal.

Solution: In Example 5.12, we determined the eigenvalues and linearly independent eigenvectors for this matrix, namely

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

with eigenvalues 2, 1 and 1 respectively. The entries appearing in these vectors are the coefficients when we write them in terms of the standard basis for $\mathbb{R}^3$. Hence the change of basis matrix is

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}$$

and $P^{-1}AP$ is the diagonal matrix whose entries are the eigenvalues:

$$P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

[Exercise: Check via hand calculation this equation.]

Powers of matrices

One advantage of diagonal matrices is that it is much easier to calculate powers of a diagonal matrix. Suppose that A is a matrix which is diagonalisable, say $P^{-1}AP = D$, where D is diagonal. Rearranging we have

$$A = PDP^{-1}.$$

Calculating successive powers:

$$A^2 = PDP^{-1} \cdot PDP^{-1} = PD^2P^{-1}$$

and

$$A^3 = PDP^{-1} \cdot PDP^{-1} \cdot PDP^{-1} = PD^3P^{-1},$$

and in general

$$A^k = PD^kP^{-1}.$$

Thus, we can calculate powers of A by calculating powers of D and multiplying by P and P^{-1}. Calculating powers of D is very easy, for if

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{pmatrix},$$

then

$$D^k = \begin{pmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n^k \end{pmatrix}.$$

So provided we have found all the eigenvalues λ_i of A and the change of basis matrix P (via calculating the eigenvectors), we can calculate A^k very quickly — far more quickly than performing all the matrix multiplications!

Symmetric matrices

We know that diagonalisable transformations and matrices exist — we have seen examples — and we even have a sufficient condition based on the factorization of the characteristic polynomial. But a natural question is: which classes of matrices are diagonalisable, and are there any for which the process is simpler than usual? Here we will explore a specific class: the real symmetric matrices, which themselves are a special case of hermitian matrices. Not only is any real symmetric matrix A diagonalisable, but its "diagonalizing" matrix P has a particularly nice form.

Recall from Chapter 1 that an $n \times n$ matrix M

- is called *symmetric* if $M^{\mathrm{T}} = M$, and
- is called *orthogonal* if $M^{\mathrm{T}} = M^{-1}$.

The following result holds:

Theorem 5.23 *If A is a real symmetric matrix, then there exists an orthogonal matrix P such that $P^{\mathrm{T}}AP = D$ is diagonal.*

To prove this theorem, we need the following lemmas.

Lemma 5.24 *Let A be an $n \times n$ symmetric matrix with real entries. Then eigenvectors v_i and v_j of A corresponding to distinct eigenvalues λ_i and λ_j are orthogonal; that is,*

$$\boldsymbol{v}_i \cdot \boldsymbol{v}_j = 0,$$

where $\cdot$ denotes the usual scalar (or "dot") product for vectors in $\mathbb{R}^n$.

Proof: Recall that

$$\boldsymbol{v}_i \cdot \boldsymbol{v}_j = \boldsymbol{v}_i^{\mathrm{T}} \boldsymbol{v}_j$$

when we consider the v_i as column vectors (specifically, the right-hand side denotes the matrix multiplication of the vector $\boldsymbol{v}_i$ (viewed as a $1 \times n$ matrix) and the transpose of the vector $\boldsymbol{v}_j$ (viewed as an $n \times 1$ matrix). Taking the transpose of $A\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i$ gives

$$\boldsymbol{v}_i^{\mathrm{T}} A^T = \lambda_i \boldsymbol{v}_i^{\mathrm{T}},$$

and since A is symmetric this becomes

$$\boldsymbol{v}_i^{\mathrm{T}} A = \lambda_i \boldsymbol{v}_i^{\mathrm{T}}.$$

The trick is to post-multiply both sides by $\boldsymbol{v}_j$, and obtain two different expressions for the resulting left-hand side.

$$\boldsymbol{v}_i^{\mathrm{T}} A \boldsymbol{v}_j = \lambda_i \boldsymbol{v}_i^{\mathrm{T}} \boldsymbol{v}_j.$$

On the other hand, using the fact that $\boldsymbol{v}_j$ is also an eigenvector, we deduce

$$\boldsymbol{v}_i^{\mathrm{T}} A \boldsymbol{v}_j = \boldsymbol{v}_i^{\mathrm{T}} \lambda_j \boldsymbol{v}_j = \lambda_j \boldsymbol{v}_i^{\mathrm{T}} \boldsymbol{v}_j.$$

Subtracting these last two equations gives

$$(\lambda_i - \lambda_j) \boldsymbol{v}_i^{\mathrm{T}} \boldsymbol{v}_j = 0.$$

Since $\lambda_i \neq \lambda_j$, we can divide by $\lambda_i - \lambda_j$ and conclude

$$\boldsymbol{v}_i \cdot \boldsymbol{v}_j = \boldsymbol{v}_i^{\mathrm{T}} \boldsymbol{v}_j = 0,$$

as claimed. □

Lemma 5.25 *If A is a symmetric $n \times n$ matrix, then there is a basis for $\mathbb{R}^n$ consisting of orthogonal eigenvectors for A.*

Proof: (Sketch) If A has n distinct eigenvalues, then by Lemma 5.24, a set S of n eigenvectors corresponding to these eigenvalues is an orthogonal set. It can be proved that an orthogonal set is also a linearly independent set, and hence S forms a basis for $\mathbb{R}^n$. (Note that Corollary 5.15 guarantees diagonalisability of A but not the orthogonal part.) For the case when A has a repeated eigenvalue, the desired result can be established via the Gram-Schmidt orthogonalization process, but we omit the details here. □

Now we are ready to prove Theorem 5.23:

Proof: Let $\mathscr{B} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$ be a basis for $\mathbb{R}^n$ consisting of orthogonal eigenvectors for A. Let

$$k_i = |\boldsymbol{v}_i| = \sqrt{\boldsymbol{v}_i \cdot \boldsymbol{v}_i}$$

and replace each $\boldsymbol{v}_i$ by $\frac{1}{k_i}\boldsymbol{v}_i$. This has the consequence that each vector $\boldsymbol{v}_i$ now has unit length. Thus we may assume $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$ is an *orthonormal set*:

$$\boldsymbol{v}_i \cdot \boldsymbol{v}_j = \boldsymbol{v}_i^{\mathrm{T}} \boldsymbol{v}_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Let P be the change of basis matrix from the standard basis of $\mathbb{R}^n$ to $\mathscr{B}$; that is, we write the entries of each vector $\boldsymbol{v}_i$ down the columns of P. Thus

$$P = \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \dots & \boldsymbol{v}_n \end{pmatrix}$$

(the ith column of P is the vector $\boldsymbol{v}_i$). Consider the product $P^{\mathrm{T}}P$:

$$P^{\mathrm{T}}P = \begin{pmatrix} \boldsymbol{v}_1^{\mathrm{T}} \\ \boldsymbol{v}_2^{\mathrm{T}} \\ \vdots \\ \boldsymbol{v}_n^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \dots & \boldsymbol{v}_n \end{pmatrix}$$

$$= \begin{pmatrix} \boldsymbol{v}_1^{\mathrm{T}}\boldsymbol{v}_1 & \boldsymbol{v}_1^{\mathrm{T}}\boldsymbol{v}_2 & \dots & \boldsymbol{v}_1^{\mathrm{T}}\boldsymbol{v}_n \\ \boldsymbol{v}_2^{\mathrm{T}}\boldsymbol{v}_1 & \boldsymbol{v}_2^{\mathrm{T}}\boldsymbol{v}_2 & \dots & \boldsymbol{v}_2^{\mathrm{T}}\boldsymbol{v}_n \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{v}_n^{\mathrm{T}}\boldsymbol{v}_1 & \boldsymbol{v}_n^{\mathrm{T}}\boldsymbol{v}_2 & \dots & \boldsymbol{v}_n^{\mathrm{T}}\boldsymbol{v}_n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

Hence $P^{\mathrm{T}}P = I$, the identity matrix. Rearranging, we deduce that $P^{-1} = P^{\mathrm{T}}$.

Therefore, applying change of basis to the matrix A we conclude

$$P^{\mathrm{T}}AP = D$$

where D is the diagonal matrix containing the eigenvalues of A. $\square$

Example 5.26 *Find an orthogonal matrix which diagonalises*

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

Solution: The characteristic polynomial of A is

$$\begin{aligned} \det \begin{pmatrix} x-3 & -1 \\ -1 & x-3 \end{pmatrix} &= (x-3)^2 - 1 \\ &= x^2 - 6x + 8 \\ &= (x-2)(x-4). \end{aligned}$$

Hence the eigenvalues of A are 2 and 4. We must now find orthonormal eigenvectors.

Case $\boldsymbol{\lambda = 2}$: We solve $(A - 2I)\boldsymbol{v} = \boldsymbol{0}$:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$

that is, $x + y = 0$. Hence

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is an eigenvector for A with eigenvalue 2. We now normalise:

$$\left| \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right|^2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1^2 + (-1)^2 = 2.$$

Hence

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is an eigenvector for A with eigenvalue 2 and *unit length.*

Case $\boldsymbol{\lambda} = \mathbf{4}$: We solve $(A - 4I)\boldsymbol{v} = \mathbf{0}$:

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$

that is, $x - y = 0$. Hence

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is an eigenvector for A with eigenvalue 4. Its length is $\sqrt{1^2 + 1^2} = \sqrt{2}$, so

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is an eigenvector for A with eigenvalue 4 and unit length.

Hence

$$\left\{ \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \right\}$$

is an orthonormal basis for $\mathbb{R}^2$ consisting of eigenvectors for A. The change of basis matrix is

$$P = \begin{pmatrix} 1/\sqrt{2} & 1\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

[Let us verify that this matrix P does indeed solve the problem:

$$P^{\mathrm{T}} P = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = I.$$

Then

$$\begin{aligned} P^{\mathrm{T}} A P &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 2 & -2 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 4 & 0 \\ 0 & 8 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}, \end{aligned}$$

which is indeed the diagonal matrix containing the eigenvalues of A.]

Hermitian matrices

The previous section addresses a special case of a more general one in which the matrices have entries that are complex numbers.

Definition 5.27 Let A be a matrix whose entries are complex numbers. Write $A^\dagger$ for the matrix obtained by taking the complex conjugate of each entry of A and then the transpose of the resulting matrix. That is,

$$A^\dagger = (\bar{A})^{\mathrm{T}}.$$

Definition 5.28 A *Hermitian matrix* is a matrix A with complex numbers as entries such that $A^\dagger = A$.

Proposition 5.29 *If A is a Hermitian matrix, then all its eigenvalues are real numbers.*

PROOF: Omitted. □

We observe that

Proposition 5.30 *Every real symmetric matrix is a Hermitian matrix.*

We previously considered real symmetric matrices and observed that they could be diagonalised using orthogonal matrices. The corresponding type of matrix to be considered here is:

Definition 5.31 A *unitary matrix* is a matrix U whose inverse is $U^\dagger$.

Thus, a unitary matrix is a square matrix U satisfying

$$UU^\dagger = U^\dagger U = I.$$

In the same way that a real symmetric matrix can be diagonalised by an orthogonal matrix, here a Hermitian matrix can be diagonalised by a unitary matrix:

Theorem 5.32 *If A is a Hermitian matrix, then there exists a unitary matrix U such that $U^\dagger AU = D$ is diagonal.*

We do not go into the details, which are similar in spirit to those explored in the previous section. We do, however, give a worked example.

Example 5.33 *Find a unitary matrix that diagonalises the Hermitian matrix*

$$A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}.$$

Note that

$$\bar{A} = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix},$$

so $A^\dagger = (\bar{A})^{\mathrm{T}} = A$. Thus the previous theorem does indeed apply here. The method of solution is the same as for real symmetric matrices: we find eigenvectors that are of unit length.

Solution: The characteristic polynomial is

$$\begin{aligned} \det \begin{pmatrix} x-1 & -i \\ i & x-1 \end{pmatrix} &= (x-1)^2 + i^2 \\ &= x^2 - 2x + 1 - 1 \\ &= x^2 - 2x = x(x-2). \end{aligned}$$

Hence the eigenvalues of A are 0 and 2.

Case $\boldsymbol{\lambda} = 0$: We solve $A\boldsymbol{v} = \boldsymbol{0}$:

$$\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$

that is, $z + iw = 0$. Hence

$$\begin{pmatrix} -i \\ 1 \end{pmatrix}$$

is an eigenvector with eigenvalue 0. Its magnitude is

$$\sqrt{|-i|^2 + |1|^2} = \sqrt{2},$$

so the unit eigenvector is

$$\frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

Case $\boldsymbol{\lambda} = 2$: We solve $(A - 2I)\boldsymbol{v} = \boldsymbol{0}$:

$$\begin{pmatrix} -1 & i \\ -i & -1 \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix};$$

that is, $-z + iw = 0$. Hence

$$\begin{pmatrix} i \\ 1 \end{pmatrix}$$

is an eigenvector with eigenvalue 2 and the unit eigenvector is

$$\frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

The required unitary matrix is

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}.$$

[Let us now verify the claim. Note

$$U^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}^{\mathrm{T}} = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix},$$

so

$$U^\dagger U = \frac{1}{2} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = I$$

(using the fact that $i^2 = -1$). Then

$$\begin{aligned} U^\dagger A U &= \frac{1}{2} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 0 & 0 \\ -2i & 2 \end{pmatrix} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \end{aligned}$$

which (as required) is a diagonal matrix whose diagonal entries are the eigenvalues of A.]

Applications (non-examinable)

We end the chapter with two applications of diagonalisation. The first is to quadratic forms, which you may encounter in your further studies — they arise in physics and statistics, as well as being important in many areas of maths. The other is a perhaps surprising application to number theory — namely the Fibonacci sequence.

Quadratic forms

A quadratic form of order n over F is a mapping from F^n to F specified by a formula of the form

$$q(x_1, \ldots, q_n) = \sum_{i=1}^{n} \alpha_{ii} x_i^2 + \sum_{i,k=1,\, i<k}^{n} 2\alpha_{ik} x_i x_k.$$

Its defining expression is a polynomial in $x_1, \ldots, x_n$ in which *every* term is of degree two — either a "square term" $\alpha_{ii} x_i^2$ or a "mixed product term" $2\alpha_{ik} x_i x_k$. An example of order 2 in x, y is given by

$$q(x, y) = x^2 + 3y^2 + 4xy$$

The key thing about quadratic forms is that we can write them in the form

$$q(x_1, \ldots, x_n) = X^T A X$$

where A is a symmetric matrix in $M_{n \times n}(F)$ and $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. Conversely, given a symmetric matrix A, it defines a quadratic form $q_A(X) = X^T A X$. For example, the matrix corresponding to the quadratic form $q(x, y) = 3x^2 + 3y^2 + 2xy$ is $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$.

Observe that, if D is a diagonal matrix $D = \operatorname{diag}(\gamma_1, \ldots, \gamma_n)$, then its associated quadratic form has a nice simple form:

$$q_D(x_1, \ldots, x_n) = \gamma_1 x_1^2 + \cdots + \gamma_n x_n^2.$$

Just as with our previous section on matrices, we are interested in the process of diagonalizing: this time, taking an arbitrary quadratic form and converting it into the "shape" above (corresponding to a change of coordinates). Because of the matrix connection, we can apply our previous work on matrices in order to achieve this. For real quadratic forms, the associated matrices are real symmetric, so we can diagonalize them using orthogonal matrices.

We will not go into details, but instead illustrate the main idea via a worked example.

Example 5.34 *Consider the quadratic form $q(x, y) = 3x^2 + 3y^2 + 2xy$ whose associated real symmetric matrix is $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$.*
By the previous section, we can find a real orthogonal matrix M such that

$$M^{-1} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} M \text{ is a diagonal matrix } \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

Specifically, $M = \begin{pmatrix} 1/\sqrt{2} & 1\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ *and the diagonal matrix is* $\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$.
Writing $\begin{pmatrix} X \\ Y \end{pmatrix} = M^{-1} \begin{pmatrix} x \\ y \end{pmatrix} M$ *changes* $x^2 + 3y^2 + 4xy$ *into the form* $\alpha X^2 + \beta Y^2$*, i.e.* $2X^2 + 4Y^2$.

This process is particularly helpful if we are given an equation defining a conic section in the plane. It may be impossible to tell by inspection what type of conic we have (ellipse, hyperbola, circle, etc), but after diagonalization we can immediately read off what type of conic we have from the new equation.

The Fibonacci sequence

A nice application of the diagonalization process is to obtain a formula for the nth term of the Fibonacci sequence $1, 1, 3, 5, 8, 11, \ldots$.

Recall that the Fibonacci sequence is defined recursively by $a_0 = 1$, $a_1 = 1$, and

$$a_{n+2} = a_{n+1} + a_n.$$

This can be written as a system of difference equations:

$$a_{n+2} = a_{n+1} + b_{n+1}, \quad b_{n+2} = a_{n+1}.$$

We can represent this in matrix form: $\boldsymbol{x_{n+2}} = A\boldsymbol{x_{n+1}}$ where

$$\boldsymbol{x_{n+2}} = \begin{pmatrix} a_n \\ b_n \end{pmatrix} \text{ and } A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $\boldsymbol{x_2} = A\boldsymbol{x_1}$, $\boldsymbol{x_3} = A\boldsymbol{x_2} = A^2\boldsymbol{x_1}$, and in general $\boldsymbol{x_{n+1}} = A^n\boldsymbol{x_1}$. We will use our diagonalisation techniques to obtain an expression for A^n.

The characteristic polynomial of A is $x^2 - x - 1$, so the eigenvalues are the roots of this: $\lambda_1 = \frac{1}{2}(1+\sqrt{5})$ and $\lambda_2 = \frac{1}{2}(1-\sqrt{5})$. Since A is a 2×2 matrix with 2 distinct eigenvalues, it is diagonalisable, and we can obtain the eigenvectors

$$\begin{pmatrix} 1 \\ \lambda_1 - 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -\lambda_2 \end{pmatrix}, \begin{pmatrix} 1 \\ \lambda_2 - 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -\lambda_1 \end{pmatrix}.$$

Proceeding as usual, we can use these two (clearly LI) vectors to form the invertible matrix

$$P = \begin{pmatrix} 1 & 1 \\ -\lambda_2 & -\lambda_1 \end{pmatrix} \text{ such that } P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = D, \text{ say.}$$

We can calculate the inverse of P to be

$$P^{-1} = \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} -\lambda_1 & -1 \\ \lambda_2 & 1 \end{pmatrix}.$$

We now use our result on powers of matrices, which tells us that $A^n = PD^nP^{-1}$. Note that $\lambda_1\lambda_2 = -1$.

$$A^n = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} 1 & 1 \\ -\lambda_2 & -\lambda_1 \end{pmatrix} \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \begin{pmatrix} -\lambda_1 & -1 \\ \lambda_2 & 1 \end{pmatrix}$$

$$= \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} 1 & 1 \\ -\lambda_2 & -\lambda_1 \end{pmatrix} \begin{pmatrix} -\lambda_1^{n+1} & -\lambda_1^n \\ \lambda_2^{n+1} & \lambda_2^n \end{pmatrix}$$
$$= \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} \lambda_2^{n+1} - \lambda_1^{n+1} & \lambda_2^n - \lambda_1^n \\ \lambda_2^n - \lambda_1^n & \lambda_2^{n-1} - \lambda_1^{n-1} \end{pmatrix}.$$

Since by definition $b_1 = a_0 = 0$ and $a_1 = 1$, we have that

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = A^n \boldsymbol{x_1} = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} \lambda_2^{n+1} - \lambda_1^{n+1} \\ \lambda_2^n - \lambda_1^n \end{pmatrix}.$$

Thus, finally, we see that

$$a_n = \frac{1}{\lambda_2 - \lambda_1}(\lambda_2^n - \lambda_1^n);$$

in other words,

$$a_n = \frac{1}{\sqrt{5}}[\frac{1}{2}(1+\sqrt{5})]^n - \frac{1}{\sqrt{5}}[\frac{1}{2}(1-\sqrt{5})]^n.$$

So we have used diagonalization of matrices to obtain a formula for the nth Fibonacci number in terms of n.