Chapter 3

Second order linear ODEs

3.1 General Theory

We start with general second order linear differential operator in the form

$$L = a_2(x)\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_0(x). \tag{3.1}$$

This is an operator on the space of twice differentiable functions on an interval $[a, b]$, which may be the entire real line, i.e.

$$L[y] = a_2(x)y'' + a_1(x)y' + a_0(x)y$$

for $y(x) \in C^2[a, b]$. If $a_2(x)$ is non-zero on the interval $[a, b]$ then the operator is regular and we can divide through by a_2. If $a_2(x)$ is zero at a point in $[a, b]$ then the operator is singular.

The general second order linear differential equation is then

$$L[y] = a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x). \tag{3.2}$$

If $g(x) \equiv 0$ on $[a, b]$ then the ODE (3.2) is homogeneous; otherwise it is inhomogeneous.

The *general solution* to (3.2) is

$$y = y_{CF} + y_{PI}$$

where:

(i) $y_{CF} = C_1 y_1 + C_2 y_2$ is the *complementary function*, a linear combination of any two linearly independent solutions $y_1(x)$, $y_2(x)$ to the homogeneous equation $L[y] = 0$, and

(ii) y_{PI} is a *particular integral*, any solution to the inhomogeneous equation $L[y] = g$.

The constants C_1 and C_2 are determined either by: (i) initial values, y and y' specified at a single point; or (ii) by boundary values, some combination of y and y' specified at the two points $x = a$ and $x = b$. In the first case, the initial value problem, there are existence and uniqueness theorems similar to the one examined in chapter 1. However, in this chapter we will consider the second case only, the two-point boundary value problem.

The two functions y_1, y_2 are *linearly dependent* if one is proportional to the other, in other words if the equation $\alpha y_1 + \beta y_2 = 0$ can be satisfied for α and β not both zero. We define the *Wronskian* of y_1 and y_2 by

$$W[y_1, y_2] = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$$

Then if y_1, y_2 are linearly dependent we can right $y_1 = Cy_2$ for some constant C, so $y_1' = Cy_2'$ and so $W[y_1, y_2] = 0$. Thus,

$$\begin{aligned} y_1, y_2 \text{ linearly dependent } &\implies W[y_1, y_2] = 0 \\ W[y_1, y_2] \neq 0 &\implies \text{ linearly independent} \end{aligned}$$

Be careful, however, because

$$W[y_1, y_2] = 0 \;\not\Longrightarrow\; \text{linearly dependent}$$

Example 3.1 Establish whether the following are linearly independent and compute $W[y_1, y_2]$:

$$\begin{aligned} &y_1 = e^x, \quad y_2 = e^{x+1} \\ &y_1 = e^x, \quad y_2 = e^{-x} \\ &y_1 = \max(0, e^x - 1), \quad y_2 = \max(0, e^{-x-1}) \end{aligned}$$

There are always exactly two linearly independent solutions to $L[y] = 0$. So if $L : A \to B$ is considered as a linear map between vector spaces A and B, and $K = \ker(L) = \{v \in A : L[v] = 0\}$ is the kernel of L, then $\dim(K) = 2$.

The homogeneous equation $L[y] = 0$ can be easily solved when L is one of the following two standard forms.

1. Constant coefficients (revision of MT1002)

When $a_i(x)$ are all constant, the homogeneous equation can be solved by letting $y(x) = e^{\lambda x}$, substituting into $L[y] = 0$ and solving the resulting quadratic equation (characteristic equation) for λ. The form of the solution depends on the roots of the characteristic equation.

If the ODE is

$$ay'' + by' + cy = 0, \qquad a, b, c = \text{ constant}$$

then $y = e^{\lambda x}$ gives $a\lambda^2 + b\lambda + c = 0$. The cases are:

(i) $\lambda_1, \lambda_2 \in \mathbb{R}$, distinct $\longrightarrow$ $y_1 = e^{\lambda_1 x}, y_2 = e^{\lambda_2 x}$

(ii) $\lambda_1 = \lambda_2 \in \mathbb{R}$ $\longrightarrow$ $y_1 = e^{\lambda_1 x}, y_2 = xe^{\lambda_1 x}$

(iii) $\lambda_\pm = \alpha \pm i\beta$ complex $\longrightarrow$ $y_1 = e^{\alpha x}\cos(\beta x), y_2 = e^{\alpha x}\sin(\beta x)$

For the inhomogeneous equation

$$ay'' + by' + cy = g(x),$$

the fastest way to find y_{PI} is to guess (sometimes called the method of undertermined coefficients; see MT1002 and revision questions on sheet 3).

Note that if $L[y] = g_1(x) + g_2(x)$, then we can find separately the particular integrals y_{PI1} to $L[y] = g_1(x)$ and y_{PI2} to $L[y] = g_2(x)$. Then linearity of L means that we can take $y_{PI} = y_{PI1} + y_{PI2}$.

Example 3.2 $y'' + 5y' + 6y = 3e^{-2x}$

2. Equidimensional equations

Another standard case is when the ODE has the form

$$x^2y'' + axy' + by = 0.$$

Such an equation is called equidimensional and has solutions of the form $y = x^l$. Substituting in gives:

$$\begin{aligned} & l(l-1) + al + b = 0 \\ \Longrightarrow\ & l^2 + l(a-1) + b = 0 \\ \Longrightarrow\ & l = -\tfrac{1}{2}(a-1) \pm \tfrac{1}{2}\sqrt{(a-1)^2 - 4b} \end{aligned}$$

(i) $l_1, l_2 \in \mathbb{R}$, distinct $\longrightarrow$ $y_1 = x^{l_1}, y_2 = x^{l_2}$

(ii) $l_1 = l_2 \in \mathbb{R}$ $\longrightarrow$ $y_1 = x^{l_1}, y_2 = x^{l_1}\log x$ (see example 3.5)

(iii) $l_\pm = \alpha \pm i\beta$ complex $\longrightarrow$ $y_1 = x^\alpha \cos(\beta \log x), y_2 = x^\alpha \sin(\beta \log x)$ (see sheet 3, 5(iii))

Example 3.3 $x^2y'' + 2xy' - 2y = 0$

In the remainder of this chapter we will consider the general linear operator (3.1) and assume that we can simply divide through by $a_2(x)$. Thus we consider L in the form

$$L = \frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x). \tag{3.3}$$

If L is regular so that $a_2(x) \neq 0$ on $[a, b]$ this is fine. If L is singular, so that $a_2(x) = 0$ at some point in $[a, b]$, then we absorb the singularity into $p(x)$ and $q(x)$ and proceed with caution.

Reduction of order (*non-examinable)

For the general second order linear homogeneous ODE,

$$L[y] = y'' + p(x)y' + q(x)y = 0,$$

if we know one solution, $y_1(x)$, a second, linearly independent solution can be found by setting $y = v(x)y_1(x)$ for some function $v(x)$ to be determined. Differentiating gives

$$\begin{aligned} y' &= v'y_1 + vy_1' \\ y'' &= v''y_1 + 2v'y_1' + vy_1'' \end{aligned}$$

and substituting back into $L[y] = 0$ gives

$$\begin{aligned} & v''y_1 + 2v'y_1' + vy_1'' + p(x)(v'y_1 + vy_1') + q(x)vy_1 = 0 \\ \Longrightarrow\ & v''y_1 + v'(2y_1' + py_1) + v\,\underbrace{(y_1'' + py_1' + qy_1)}_{=0 \text{ due to } L[y]=0} = 0 \\ \Longrightarrow\ & v''y_1 + v'(2y_1' + py_1) = 0 \\ \Longrightarrow\ & y_1V' + (2y_1' + py_1)V = 0 \qquad \text{where } V = v'. \end{aligned}$$

This is a first order linear and separable ODE for V. Solve for V, then integrate once to get v. For y_2 take the part of vy_1 that is not proportional to y_1.

Example 3.4 $y'' - 2y' + y = 0$, given $y_1 = e^x$

Example 3.5 $x^2y'' - xy' + y = 0$, given $y_1 = x$

Variation of Parameters

This is a general method to find y_{PI}. If we already know two linearly independent solutions to $L[y] = 0$, y_1 and y_2, we look for a solution to the inhomogeneous problem $L[y] = g$ of the form

$$y = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where u_1, u_2 are arbitrary functions that satisfy the additional conditions

$$u_1'y_1 + u_2'y_2 = 0.$$

Substitute this expression for y into $L[y] = g(x)$, using

$$\begin{aligned} y &= u_1y_1 + u_2y_2 \\ y' &= u_1y_1' + u_1'y_1 + u_2y_2' + u_2'y_2 \\ &= u_1y_1' + u_2y_2' && \text{(since } u_1'y_1 + u_2'y_2 = 0\text{)} \\ y'' &= u_1y_1'' + u_2y_2'' + u_1'y_1' + u_2'y_2' \end{aligned}$$

so that $L[y] = g(x)$ becomes

$$\begin{aligned} & u_1y_1'' + u_2y_2'' + u_1'y_1' + u_2'y_2' + p(u_1y_1' + u_2y_2') + q(u_1y_1 + u_2y_2) = g \\ \implies & u_1 \underbrace{(y_1'' + py_1' + qy_1)}_{=L[y_1]=0} + u_2 \underbrace{(y_2'' + py_2' + qy_2)}_{=L[y_2]=0} + u_1'y_1' + u_2'y_2' = g \end{aligned}$$

Hence,

$$u_1'y_1' + u_2'y_2' = g(x) \tag{3.4}$$
$$u_1'y_1 + u_2'y_2 = 0. \qquad \text{(the condition given above).} \tag{3.5}$$

These are simultaneous equations for u_1' and u_2', so eliminate u_2' to get an equation for $u_1' = \dots$ and integrate to find u_1. Then repeat similarly to find u_2.

Example 3.6 $y'' - y = g(x)$.

Note that the $g(x)$ used in the above equation must be that from the ODE $y''+p(x)y'+q(x)y = g(x)$; that is, the coefficient of y'' should be 1.

Example 3.7 $x^2y'' + xy' - y = x\log x$.

3.2 Two-point boundary value problems: the Green's function

We will consider the two-point boundary value problem (BVP)

$$L[y] = g(x)$$

with boundary conditions given at $x = a$ or $x = b$. In general the boundary conditions may involve both y and y'. Here we will restrict attention to the simplest case of *unmixed*, *homogeneous* boundary conditions, of the form

$$y(a) = y(b) = 0.$$

Not all boundary value problems of this form have a unique or non-trivial solutions. For example, the BVP

$$y'' + y = 0 \qquad y(0) = y(\tfrac{\pi}{2}) = 0$$

has only the trivial solution, $y(x) \equiv 0$, while the BVP

$$y'' + y = 0 \qquad y(0) = y(\pi) = 0$$

has infinitely many solutions, $y(x) = C \sin x$, with arbitrary C. We will touch upon some aspects of solvability in the next chapter.

In the following, we assume that y_1 and y_2 are two linearly independent solutions to $L[y] = 0$, that have already been found. *These do not necessarily satisfy any boundary conditions.* We can use the variation of parameters to find the general solution to $L[y] = g(x)$. Solving the simultaneous equations (3.4) and (3.5) gives:

$$\begin{aligned} u_1'(y_1 y_2' - y_1' y_2) &= -g y_2 \\ u_2'(y_1 y_2' - y_1' y_2) &= g y_1 \end{aligned}$$

Provided the Wonskian $W[y_1, y_2] = y_1 y_2' - y_1' y_2 \neq 0$, these can be integrated from $x = a$ to x to give:

$$\begin{aligned} u_1(x) - u_1(a) &= \int_a^x \frac{-g(s) y_2(s)}{W(s)} ds \\ u_2(x) - u_2(a) &= \int_a^x \frac{g(s) y_1(s)}{W(s)} ds \end{aligned}$$

where $W(s)$ is a shorthand for $W[y_1(s), y_2(s)]$. The solution $y = u_1 y_1 + u_2 y_2$ is then

$$y = u_1(a) y_1(x) + u_2(a) y_2(x) + \int_a^x \frac{g(s)(-y_1(x) y_2(s) + y_1(s) y_2(x)}{W(s)} ds.$$

Since $u_1(a)$ and $u_2(a)$ are just constants of integration, this is

$$y = C_1 y_1(x) + C_2 y_2(x) + \int_a^x \frac{g(s)(-y_1(x) y_2(s) + y_1(s) y_2(x))}{W(s)} ds,$$

which can be recognized as $y_{CF} + y_{PI}$. The constants C_1 and C_2 are then determined from the boundary conditions $y(a) = y(b) = 0$ giving

$$\begin{aligned} C_1 y_1(a) + C_2 y_2(a) &= 0 \\ C_1 y_1(b) + C_2 y_2(b) &= \int_a^b \frac{g(s)(y_1(b) y_2(s) - y_1(s) y_2(b))}{W(s)} ds \end{aligned}$$

These can be solved for C_1 and C_2 provided

$$\det \begin{pmatrix} y_1(a) & y_2(a) \\ y_1(b) & y_2(b) \end{pmatrix} \neq 0,$$

that is provided $y_1(a)y_2(b)-y_1(b)y_2(a)\neq 0$. All BVPs that we will encounter in this chapter will satisfy this requirement.

The above formulation of the solution is cumbersome. To simplify, we note that the y_1 and y_2 used in the variation of parameters process can be *any* two linearly independent solutions. In particular, we are free to choose y_1 and y_2 to be such that they separately satisfy the conditions $y_1(a)=0$ and $y_2(b)=0$. The equations for C_1 and C_2 then simplify to

$$\begin{aligned} C_2y_2(a) &= 0 \\ C_1y_1(b) &= \int_a^b \frac{g(s)y_1(b)y_2(s)}{W(s)}ds \end{aligned}$$

giving

$$\begin{aligned} C_2 &= 0 \\ C_1 &= \int_a^b \frac{g(s)y_2(s)}{W(s)}ds. \end{aligned}$$

The solution is then

$$\begin{aligned} y &= \int_a^b \frac{g(s)y_1(x)y_2(s)}{W(s)}ds + \int_a^x \frac{g(s)(-y_1(x)y_2(s)+y_1(s)y_2(x))}{W(s)}ds \\ &= \int_x^b \frac{g(s)y_1(x)y_2(s)}{W(s)}ds + \int_a^x \frac{g(s)y_1(s)y_2(x)}{W(s)}ds \\ &= \int_a^b g(s)G(x,s)ds \end{aligned} \tag{3.6}$$

where

$$G(x,s) = \frac{1}{W(s)}\begin{cases} y_1(x)y_2(s) & a\leq x<s\leq b \\ y_1(s)y_2(x) & a\leq s<x\leq b. \end{cases}$$

The function $G(x,s)$ is called the Green's function for the BVP $L[y]=g$ and $y(a)=y(b)=0$. The Green's function provides a means to solve the BVP for any inhomogeneous term $g(x)$ once the solutions to the homogeneous problem have been established.

Note that our condition $y_1(a)y_2(b)-y_1(b)y_2(a)\neq 0$ for the solution of the constants C_1 and C_2 means that solutions to the homogeneous problem $L[y]=0$ must not satisfy both boundary conditions at $x=a$ and $x=b$. If they do then the inhomogeneous problem $L[y]=g$ has either no solutions or infinitely many. We'll return to this point in chapter 4.

Suppose, to begin with we have two linearly independent solutions to $L[y]=0$, say v_1 and v_2, but these do not satisfy the requirement $v_1(a)=0$ and $v_2(b)=0$. We can construct two new linearly independent solutions, y_1 and y_2 by

$$\begin{aligned} y_1(x) &= v_1(x)v_2(a)-v_1(a)v_2(x) \\ y_2(x) &= v_1(x)v_2(b)-v_1(b)v_2(x). \end{aligned}$$

Clearly these are solutions to $L[y]=0$, since they are linear combinations of v_1 and v_2. Also $y_1(a)=0$ and $y_2(b)=0$, as required. Finally, it can be shown that the Wronskian is nonzero, and hence y_1 and y_2 are linearly independent, again provided that $v_1(a)v_2(b)-v_1(b)v_2(a)\neq 0$.

Example 3.8 Find two linearly independent solutions to $y'' - y = 0$ that satisfy $y_1(0) = 0$ and $y_2(1) = 0$. Sketch the solutions.

Example 3.9 Find the Green's function for the BVP $y'' - y = 0$ with $y(0) = y(1) = 0$.

Example 3.10 Using the Green's function, solve the BVP $y'' - y = f(x)$ with $y(0) = y(1) = 0$ for the case

$$f(x) = \begin{cases} 0 & 0 \leq x \leq \frac{1}{2} \\ 1 & \frac{1}{2} < x \leq 1 \end{cases}$$

3.3 Properties of the Green's function

The Green's function has several important properties that are independent of the particular boundary value problem under consideration:

1. $L_x[G(x,s)] = 0$ provided $x \neq s$
2. $G(a,s) = G(b,s) = 0$
3. $G(x,s)$ is continuous in x
4. $\frac{dG}{dx}(x,s)$ is continuous except at $x = s$ where it has a jump and
$$\left.\frac{dG}{dx}\right|_{s^+} - \left.\frac{dG}{dx}\right|_{s^-} = 1$$

1. follows because in each range $x < s$ and $s < x$, G is proportional to $y_1(x)$ or $y_2(x)$, which are solution of $L[y] = 0$. Note that we use the notation L_x to mean that L operates on the x dependence in $G(x,s)$.

2. follows from $y_1(a) = 0$ and $y_2(b) = 0$

3. follows from continuity of $y_1(x)$ and $y_2(x)$ for $x < s$ or $s < x$ and by direct evaluation of $G(x,s)$ when $x = s$.

4. follows from

$$\frac{dG}{dx}(x,s) = \frac{1}{W(s)} \begin{cases} y_1'(x)y_2(s) & a \leq x < s \leq b \\ y_1(s)y_2'(x) & a \leq s < x \leq b, \end{cases}$$

which gives

$$\left.\frac{dG}{dx}\right|_{s^+} - \left.\frac{dG}{dx}\right|_{s^-} = \frac{1}{W(s)}(y_1(s)y_2'(s) - y_1'(s)y_2(s)) = 1.$$

The theory of Green's functions is often presented in the reverse order from the way we have presented it here. First, construct a function $G(x,s)$ that satisfies the properties (1)–(4), then show that the solution to the BVP is given by (3.6):

Theorem 3.1: There is a unique function $G(x,s)$ that satisfies the properties (1)–(4).

Theorem 3.2: Let $G(x,s)$ satisfy the properties (1)–(4); then the solution to the boundary value problem

$$L[y] = g(x)$$
$$y(a) = y(b) = 0$$

is given by

$$y(x) = \int_a^b g(s)G(x,s)ds$$

Proof of Theorem 3.1. The proof is by construction. Let

$$G(x,s) = \begin{cases} c_1(s)y_1(x) & a \le x < s \le b \\ c_2(s)y_2(x) & a \le s < x \le b, \end{cases}$$

for some unknowns $c_1(s)$ and $c_2(s)$, and where $y_1(x)$ and $y_2(x)$ are solutions to $L[y] = 0$ with $y_1(a) = y_2(b) = 0$. The properties (1) and (2) are clearly satisfied. We use properties (3) and (4) to determine c_1 and c_2. Property (3) implies that

$$c_1(s)y_1(s) = c_2(s)y_2(s)$$

Property (4) implies that

$$c_2(s)y_2'(s) - c_1(s)y_1'(s) = 1$$

Solving for $c_1(s)$ and $c_2(s)$ then gives

$$c_1(s) = \frac{y_2(s)}{W(s)} \qquad \text{and} \qquad c_2(s) = \frac{y_1(s)}{W(s)}.$$

□

Proof of Theorem 3.2. By theorem 3.1 the function $G(x,s)$ is given by

$$G(x,s) = \frac{1}{W(s)} \begin{cases} y_1(x)y_2(s) & a \le x < s \le b \\ y_1(s)y_2(x) & a \le s < x \le b. \end{cases}$$

We show that $y(x)$ defined by

$$y(x) = \int_a^b g(s)G(x,s)ds$$

is a solution to the BVP. We have

$$\begin{aligned} y(x) &= \int_a^x \frac{g(s)}{W(s)} y_1(s)y_2(x)ds + \int_x^b \frac{g(s)}{W(s)} y_1(x)y_2(s)ds \\ &= y_2(x)\int_a^x \frac{g(s)}{W(s)} y_1(s)ds + y_1(x)\int_x^b \frac{g(s)}{W(s)} y_2(s)ds \end{aligned}$$

Differentiating with respect to x, using the product rule and fundamental theorem of calculus, gives

$$\begin{aligned} y'(x) &= y_2'(x)\int_a^x \frac{g(s)}{W(s)} y_1(s)ds + y_2(x)\frac{g(x)}{W(x)} y_1(x) + y_1'(x)\int_x^b \frac{g(s)}{W(s)} y_2(s)ds - y_1(x)\frac{g(x)}{W(x)} y_2(x) \\ &= y_2'(x)\int_a^x \frac{g(s)}{W(s)} y_1(s)ds + y_1'(x)\int_x^b \frac{g(s)}{W(s)} y_2(s)ds, \end{aligned}$$

and differentiating again gives:

$$\begin{aligned} y''(x) &= y_2''(x)\int_a^x \frac{g(s)}{W(s)} y_1(s)ds + y_2'(x)\frac{g(x)}{W(x)} y_1(x) + y_1''(x)\int_x^b \frac{g(s)}{W(s)} y_2(s)ds - y_1'(x)\frac{g(x)}{W(x)} y_2(x) \\ &= y_2''(x)\int_a^x \frac{g(s)}{W(s)} y_1(s)ds + y_1''(x)\int_x^b \frac{g(s)}{W(s)} y_2(s)ds + g(x). \end{aligned}$$

Recalling that $L[y] = y'' + p(x)y' + q(x)y$, we have

$$\begin{aligned} L[y] &= L[y_2]\int_a^x \frac{g(s)}{W(s)} y_1(s)ds + L[y_1]\int_x^b \frac{g(s)}{W(s)} y_2(s)ds + g(x). \\ &= g(x) \end{aligned}$$

as required. It is straightforward to verify that this $y(x)$ also satisfies the boundary conditions $y(a) = y(b) = 0$. □

3.4 Delta function formulation

This is an informal introduction to the *Dirac delta function*. The delta function is not a function. It is usually written $\delta(x - x_0)$, for some fixed x_0 and may be defined variously as follows:

(i) Explicitly, it can be thought of as defined by

$$\delta(x - x_0) = \begin{cases} 0 & x \neq x_0 \\ \infty & x = x_0 \end{cases}$$

with the normalization

$$\int_{-\infty}^{\infty} \delta(x - x_0)dx = 1.$$

(ii) More rigorously, it may be defined by its action on a suitable test function $f(x)$, namely, $\delta(x - x_0)$ is such that

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)dx = f(x_0)$$

for all $f(x)$.

(iii) The explicit definition (i) may be considered as the limit of various actual functions. For example, define

$$\delta_\epsilon(x - x_0) = \begin{cases} 0 & |x - x_0| > \epsilon/2 \\ \frac{1}{\epsilon} & |x - x_0| \leq \epsilon/2 \end{cases}$$

Or, if you prefer continous functions, define

$$\delta_\epsilon(x - x_0) = \frac{1}{\sqrt{\pi\epsilon}} e^{-(x-x_0)^2/\epsilon}.$$

Both of these satisfy

$$\int_{-\infty}^{\infty} \delta_\epsilon(x - x_0)dx = 1,$$

for any $\epsilon > 0$ and in both cases we can define $\delta(x - x_0) = \lim_{\epsilon\to 0} \delta_\epsilon(x - x_0)$.

(iv) It may also be considered as the derivative of the *Heaviside step function*, defined by

$$h(x - x_0) = \begin{cases} 0 & x < x_0 \\ 1 & x > x_0. \end{cases}$$

Then

$$\delta(x - x_0) = \frac{dh}{dx}(x - x_0)$$

Example 3.11 Show that both functions $\delta_\epsilon(x - x_0)$ defined in (iii) above satisfy $\int_{-\infty}^{\infty} \delta_\epsilon(x - x_0)dx = 1$ and $\lim_{\epsilon \to 0} \delta_\epsilon(x - x_0) = 0$ for $x \neq x_0$.

The Green's function $G(x, s)$ to the inhomogeneous equation $L[x] = g(x)$ can be defined as the solution to the equation

$$L_x[G(x, s)] = \delta(x - s). \tag{3.7}$$

Defining $G(x, s)$ in this way, the solution to $L[y] = g(x)$ is again given by

$$y(x) = \int_a^b g(s)G(x, s)ds.$$

To see this we let L operate on both sides:

$$\begin{aligned} L[y] &= L\left[\int_a^b g(s)G(x, s)ds\right] \\ &= \int_a^b g(s)L_x[G(x, s)]ds \\ &= \int_a^b g(s)\delta(x - s)]ds \\ &= g(x) \qquad\qquad \text{(using definition (ii) above)} \end{aligned}$$

as required.

Requiring $G(x, s)$ to be a solution of (3.7) is equivalent to specifying the properties (1,3,4) of the previous section. Property (1) follows because $\delta(x-s) = 0$ for $x \neq s$. Properties (3) and (4) follow from the interpetation of the delta function as a derivative of the Heaviside step function. The most singular term on the left-hand-side of (3.7) is the second derivative of $G(x, s)$. Therefore, the first derivative of $G(x, s)$ can be equated with the Heaviside function, which has a jump of 1 at $x = s$. This is the same as integrating (3.7) with respect to x from $s - \epsilon$ to $s + \epsilon$, which gives

$$\lim_{\epsilon \to 0} \left(\left.\frac{dG}{dx}\right|_{x=s+\epsilon} - \left.\frac{dG}{dx}\right|_{x=s-\epsilon} \right) = 1.$$