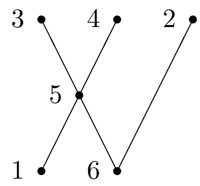
MT5823 Semigroup theory: Problem sheet 3 (James D. Mitchell)
Binary relations, equivalences, homomorphisms, and isomorphisms

Binary relations and equivalences

3-1. Let $X = \{1, 2, 3, 4, 5, 6\}$, let ρ be the equivalence relation on X with equivalence classes $\{1, 4\}$, $\{5\}$ and $\{2, 3, 6\}$, and let σ be the order relation on X given by the following Hasse diagram:



Write both ρ and σ as sets of ordered pairs. Find $\rho \cap \sigma$, $\rho \cup \sigma$, σ^{-1}, $\rho \circ \sigma$ and $\sigma \circ \rho$.

3-2. Prove the following statements about a binary relation ρ on a set X.

(a) ρ is reflexive if and only if $\Delta_X \subseteq \rho$ where $\Delta_X = \{ (x, x) : x \in X \}$;

(b) ρ is symmetric if and only if $\rho^{-1} \subseteq \rho$;

(c) ρ is transitive if and only if $\rho \circ \rho \subseteq \rho$.

3-3. Prove that the intersection $\rho \cap \sigma$ of two equivalence relations on a set X is again an equivalence relation. Describe the equivalence classes of this relation.

3-4. Find examples that show that neither the union nor composition of two equivalence relations needs to be an equivalence relation.

3-5. Let α, β be equivalence relations on a set X. Prove that $\alpha \circ \beta$ is an equivalence relation if and only if $\alpha \circ \beta = \beta \circ \alpha$.

3-6. Let X be a finite set with n elements and let B_X denote the semigroup of all binary relations on X. Prove that $|B_X| = 2^{n^2}$.

3-7. Let $S(n, r)$ $(1 \leq n \leq r)$ be the number of equivalence relations on X with precisely r equivalence classes. (The numbers $S(n, r)$ are called ***Stirling numbers of the second kind***.) Prove that

$$S(n, 1) = S(n, n) = 1$$
$$S(n, r) = S(n - 1, r - 1) + rS(n - 1, r) \ (2 \leq r \leq n - 1).$$

Use this to calculate $S(n, r)$ for $1 \leq r \leq n \leq 6$.

Homomorphisms and isomorphisms

3-8. Let $f : S \longrightarrow T$ be a homomorphism, and let $x \in S$. Prove that if x is an idempotent, then so is xf. Is it true that if x is the identity of S, then xf is the identity of T? Prove that if x is the identity and f is onto, then xf is the identity of T. If $P \leq S$, then prove that $Pf = \{pf : p \in P\}$ is a subsemigroup of T.

3-9. Let S be a semigroup such that $x^2 = x$ and $xyz = xz$ for all $x, y, z \in S$. Fix an arbitrary element $a \in S$. Let $I = Sa = \{sa : s \in S\}$ and $\Lambda = aS = \{as : s \in S\}$. Define a mapping f from S into the rectangular band $I \times \Lambda$ by $xf = (xa, ax)$. Prove that f is an isomorphism.

Further problems

3-10. Prove that a semigroup S is a rectangular band if and only if

$$(\forall a, b \in S)(ab = ba \Longrightarrow a = b).$$

3-11.* Prove that every finite cancellative semigroup is a monoid. Can you find an example of an infinite cancellative semigroup without identity that is not free?