# MT 1010

# LECTURE NOTES

## 2014

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# Contents

# Chapter 1

# Solutions to Algebraic Equations

## 1.1 Introduction

The problem of finding solutions of the algebraic equation $f(x) = 0$ is a common and important one. In some cases there exists a formula for computing solutions.

e.g., the quadratic $f(x) = ax^2 + bx + c = 0$ has an explicit formula for the roots:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} .$$

A *root* is an algebraic value of $x$ such that $f(x) = 0$.

However, in the vast majority of practical situations we do not have a formula available and so we must use some other strategy. An important tool in such situations is the use of *difference equations* in the form of *iterative schemes*.

A *difference equation* is a relationship giving $x_{n+1}$ in terms of $x_n,\ x_{n-1}, ..., x_0$, where $x_i$ are a sequence of numbers. The simplest case is where

$$x_{n+1} = F(x_n)$$

and $x_{n+1}$ just depends on the previous value $x_n$. This is called a first-order difference equation. We will restrict attention to iterative schemes of this order.

## 1.2 Iterative Schemes

The first type of iterative scheme we will consider depends on rewriting the original equations in the form

$$\underbrace{f(x)}_{given} \equiv \underbrace{x - F(x)}_{rewriting} = 0 \ . \tag{1.1}$$

Now $f(x) = 0$ implies

$$x = F(x) \ . \tag{1.2}$$

**Example 1.2.1** *Question: Rewrite* $f(x) = x^2 - 2x + 1 = 0$ , *in the form* $x = F(x)$.

*Answer:*

$$x = \frac{x^2 + 1}{2} \ .$$

Note: The form of $F(x)$ is not unique. We could also have written

$$x = \sqrt{2x - 1} \ .$$

The idea now is to construct the first order difference equation.

$$x_{n+1} = F(x_n) \ , \tag{1.3}$$

based on (1.2) given an initial value of $x_0$.

In this way we can construct the sequence $\{x_n\}$.

If $x_n \longrightarrow x$ as $n \longrightarrow \infty$ then from (1.3) $x$ will satisfy $x = F(x)$ (1.2) and so will be a root of $f(x) = 0$. This is a called an iterative method.

Notes:

1. By sketching $f(x)$ we can usually find intervals in which the roots lie. This informs our choice of the initial guess $x_0$.

2. We can construct the rearrangement (1.2) in a number of ways (see example 1.2.1):

   e.g. $x^2 - 2x + 1 = 0$

   $$x = \frac{x^2 + 1}{2} \quad \Longrightarrow \quad x_{n+1} = \frac{x_n^2 + 1}{2} \ .$$
   $$x = \sqrt{2x - 1} \quad \Longrightarrow \quad x_{n+1} = \sqrt{2x_n - 1} \ .$$

   We want the rearrangement which produces a limit as $n \longrightarrow \infty$ in (1.3).

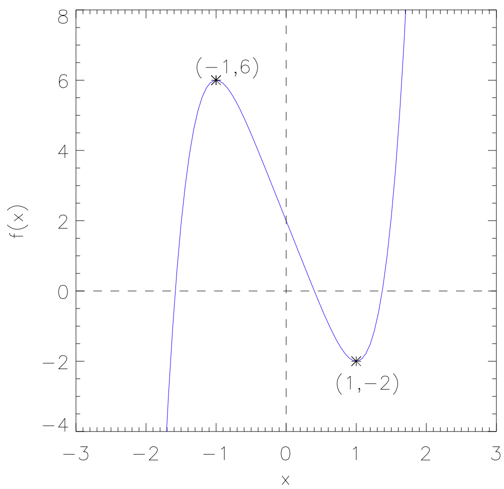



Figure 1.1: Graph of $f(x) = x^5 - 5x + 2$ showing that there is only one root in the interval $(0, 1)$.

**Example 1.2.2** *Question: Find the root of* $f(x) = x^5 - 5x + 2 = 0$ *that lies in the interval* $(0, 1)$*..*

*Answer: No direct formula for solving this, so need to undertake the following steps:*

***Step 1:*** *First draw* $f(x)$*. i.e., need to find the turning points of* $f(x)$ *and their nature. Also need to know what happens when* $x \longrightarrow \pm\infty$ *and the zeros.*

*Turning points:*

$$\begin{aligned} f'(x) &= 5x^4 - 5 = 0 \,, \qquad \textit{when} \quad x = \pm 1 \,, \\ f''(x) &= 20x^3 \,. \\ x = 1: & \qquad f(1) = -2, \quad f''(1) = 20 > 0 \,, \quad \textit{minimum} \,. \\ x = -1: & \qquad f(-1) = 6, \quad f''(-1) = -20 < 0 \,, \quad \textit{maximum} \,. \end{aligned}$$

*Behaviour at infinity and zeros:*

$$\begin{aligned} x \longrightarrow \infty, & \qquad f(x) \longrightarrow \infty \,, \\ x \longrightarrow -\infty, & \qquad f(x) \longrightarrow -\infty \,, \\ x = 0, & \qquad f(0) = 2 \,, \end{aligned}$$

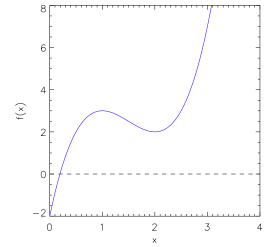
*Note: To show there is only a single root in the interval* $a < x < b$ *we need to show the following:*

*(i)* $f(a)$ *- on one side of* $x$*-axis (either positive or negative)*

$f(b)$ *- other side of* $x$*-axis (i.e., of opposite sign)* *(ii) First derivative sign changes?*

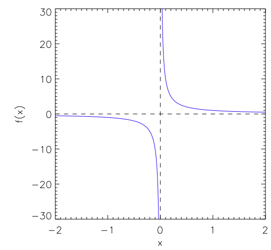
*If $f'(x) > 0 (< 0)$ $\forall x$ in $a < x < b$, then $f(x)$ is increasing (decreasing) across the interval $\Longrightarrow$ no turning point $\Longrightarrow$ no second root.*

*If the first derivative does change sign, this does not mean there IS a second root (e.g. see graph on right), but if it does NOT change sign there CANNOT be a second root.*




*(iii) Continuity? i.e., the graph should have no asymptotes in the interval.*

*Won't crop up in this course but consider $f(x) = 1/x$.*
*$f(-1) = -1$ , $f(1) = 1$. no root on $-1 < x < 1$.*




***Step 2:*** *Determine interval in which root lies, such as between turning points, or known points where function cuts the y-axis.*

*From Fig 1.1 its clear that there is only one root in the interval $(0, 1)$. We will try and find this root.*

***Step 3:*** *Rearrange equation.*

*Two possible rearrangements:*

*(a)* $x = (x^5 + 2)/5 = F_1(x)$

*(b)* $x = (5x - 2)^{1/5} = F_2(x)$

***Step 4:*** *Apply iterative scheme*

*(a) Scheme 1: Iterative scheme is* $x_{n+1} = F_1(x_n) = (x_n^5 + 2)/5$ .

*Take $x_0 = 1/2$ which is the middle of our interval $(0, 1)$, so is a reasonable estimate.*

| $n$ | $x_n$ |
|---|---|
| 0 | $x_0 = 0.5$ |
| 1 | $x_1 = F_1(x_0) = (0.5^5 + 2)/5 = 0.406$ |
| 2 | $x_2 = F_1(x_1) = (0.406^5 + 2)/5 = 0.402$ |
| 3 | $x_3 = F_1(x_2) = (0.402^5 + 2)/5 = 0.402$ |

*Scheme 1 quickly converges to root at $x = 0.402$.*

*(b) Scheme 2: Iterative scheme is* $x_{n+1} = F_2(x_n) = (5x_n - 2)^{1/5}$ .

| $n$ | $x_n$ |
|---|---|
| *0* | $x_0 = 0.5$ |
| *1* | $x_1 = F_2(x_0) = (5 * 0.5 - 2)^{1/5} = 0.871$ |
| *2* | $x_2 = F_2(x_1) = (5 * 0.871 - 2)^{1/5} = 1.187$ |
| *3* | $x_3 = F_2(x_2) = (5 * 1.187 - 2)^{1/5} = 1.315$ |
| *4* | $x_3 = F_2(x_2) = (5 * 1.315 - 2)^{1/5} = 1.355$ |
| *5* | $x_3 = F_2(x_2) = (5 * 1.355 - 2)^{1/5} = 1.367$ |
| *6* | $x_3 = F_2(x_2) = (5 * 1.367 - 2)^{1/5} = 1.371$ |
| *7* | $x_3 = F_2(x_2) = (5 * 1.371 - 2)^{1/5} = 1.372$ |

*Converges to* $x = 1.372$. *This is the root in* $x > 1$.

*This iterative scheme does not converge to the root in* $0 < x < 1$.

To decide whether a particular iterative scheme $x_{n+1} = F(x_n)$ will converge to a particular root $x = r$ without going through the actual calculation we can check that

$$|F'(x)|_{x=r} < 1.$$

We discard the iterative scheme for which this condition *does not* hold.

e.g., Check schemes from Example 1.2.2. We do not know the actual root, so instead we check values over the interval near where we expect to find the root.

(a) $F_1(x) = (x^5 + 2)/5$ so $F_1'(x) = x^4$

Note: $F_1'(0.402) = 0.026 < 1$.

We wished to determine the root in the interval $(0, 1)$:

$$|F_1'(x)| = |x^4| < 1 , \quad \text{for } -1 < x < 1 .$$

Which is why our first scheme worked.

(b) $F_2(x) = (5x - 2)^{1/5}$ so $F_2'(x) = (5x - 2)^{-4/5}$

$F_2'(0.402) = 39.81 > 1$. So iterative scheme will fail.

### 1.2.1 Steps for Computing Roots of Algebraic Equations

- **Step 1**: Sketch graph (or look at the behaviour) of $y = f(x)$ to find intervals in which roots lie.

- **Step 2**: Seek a rearrangement of $f(x)$ of the form $x = F(x)$ for which $|F'(r)| < 1$ at the required root $r$.

In general, we do not a priori know $r$ so this involves finding an interval containing $x = r$ for which $|F'(x)| < 1$.

- **Step 3**: Compute the root using the difference equation $x_{n+1} = F(x_n)$, from a suitable start point chosen with reference to the graph.

## 1.3 Newton's Method

Have looked at iterative schemes of the form $x_{n+1} = F(x_n)$. A disadvantage of this type of scheme is how slowly it converges. A much faster iterative method for solving algebraic equations is Newton's method.

We want to find a root $x = r$ of the equation $f(x) = 0$, i.e., the point $x = r$ where the graph of $f(x)$ crosses the $x$-axis. We take an initial estimate $x = x_0$.

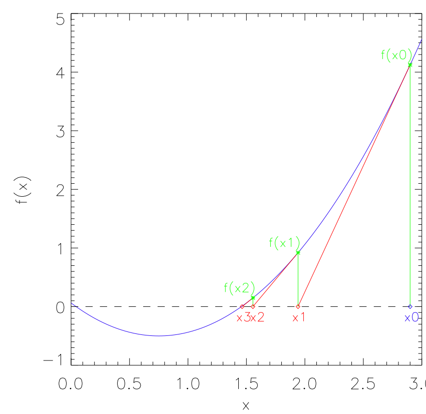



Figure 1.2: Graph illustrating how Newton's method to find the root of an algebraic equation works.

Initial estimate for root $x_0 \longrightarrow f(x_0) \longrightarrow$ tangent to $f(x)$ at $x_0$ cuts the $x$-axis at $x = x_1$: this gives us a new (better) approximation for the root.

To get a better approximation we take more iterations: Want an expression for $x_1$.

*Formulae: the equation of a straight line with slope $m$ and containing the point $(a, b)$ is:* $y - b = m(x - a)$.

From the graph (Fig 1.2) the tangent to $f(x)$ at $x = x_0$ has slope $f'(x_0)$ and contains the point $(x_0, f(x_0))$.

$\Longrightarrow$ equation of tangent is $y - f(x_0) = f'(x_0)[x - x_0]$.

This line intersects the $x$-axis at the point $(x_1, 0)$.

$$\begin{aligned} 0 - f(x_0) &= f'(x_0)[x_1 - x_0] \ , \\ \Longrightarrow \quad x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \ . \end{aligned}$$

Continuing like this to get successive approximations leads to the difference equation:

Newtons Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \ . \qquad (1.4)$$

This is Newton's method which we will compare with our earlier method.

**Example 1.3.1** *Question: Use Newton's method to find the roots of*

$$f(x) = x + \ln x \ , \qquad x_0 = \frac{1}{2}.$$

*Answer:*

$$f'(x) = 1 + \frac{1}{x} \ .$$

*Newton's method becomes*

$$\begin{aligned} x_{n+1} &= x_n - \frac{x_n + \ln(x_n)}{1 + 1/x_n} \ , \\ &= x_n - \frac{x_n(x_n + \ln(x_n))}{x_n + 1} \ , \\ &= \frac{x_n(1 - \ln(x_n))}{x_n + 1} \ . \end{aligned}$$

| $n$ | $x_n$ |
|---|---|
| *0* | *0.5* |
| *1* | *0.564* |
| *2* | *0.567* |

*Newton's method takes 2 iterations (previous method takes 10 see problem class 1).*

## 1.4 Solutions of Ordinary Differential Equations using Difference Equations

A common problem is that of solving a differential equation of the form $\frac{dy}{dx} = f(x,y)$ given the value $y(x_0) = y_0$ , $x_0$ , $y_0-$ constants.

**Example 1.4.1** *Question:*

$$\frac{dy}{dx} = xy \ , \quad y(0) = 1.$$

*Answer: This can be solved. Separating the variables and integrating:*

$$\begin{aligned} \int \frac{1}{y} dy &= \int x dx \ , \\ \implies \qquad \ln \|y\| &= \frac{x^2}{2} + C \ , \\ y &= e^{x^2/2+c} = Ae^{x^2/2} \ , \quad e^C = A \ . \end{aligned}$$

$y(0) = 1 \quad \implies A = 1$

*The explicit solution is*

$$y = e^{x^2/2} \ .$$

*However, in the vast majority of practical situations $f(x,y)$ is such that we cannot obtain an explicit solution. Again, we must resort to other methods.*

*Suppose we are given an interval $a < x < b$ in which we require a solution or an approximate to it. In geometric terms, we have to construct an approximation to the graph of $y(x)$ on this interval.*

*Divide the interval $a < x < b$ into equidistant points. Choose some constant $h \ll 1$. (h-stepsize).*

$$x_0 = a \ , \quad x_1 = x_0 + h \ , \quad x_2 = x_0 + 2h \ , \qquad\qquad , \quad x_n = x_0 + nh = b \ .$$

*Given the value of $y = y_0$ at $x = x_0$, if we can find some way of computing $y = y_i$ at $x = x_i$ then we could generate an approximate graph of the solution.*

*This is a huge area of mathematics. There are many ways of approximating the graph of differing complexity and accuracy. Such methods are called algorithms. We consider the two simplest algorithms, but the basic idea is common to all computer software for solving IVPs.*

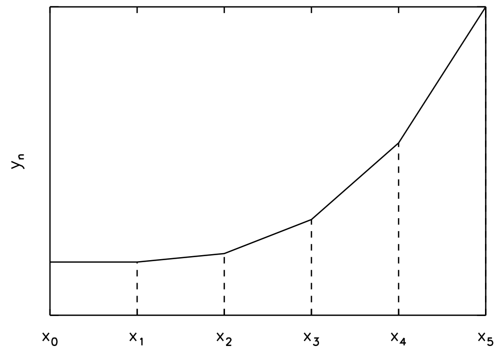



Figure 1.3: Graph illustrating the division of an interval into equidistant points.

## 1.5 The Euler Algorithm

The simplest method is to construct the graph in straight line segments with the slope computed from the left-hand end of each interval.

Recall: we are given.

$$\frac{dy}{dx} = f(x,y)\ , \qquad y(x_0) = y_0\ , \quad x_0\ , \ y_0 - \text{given constants}$$

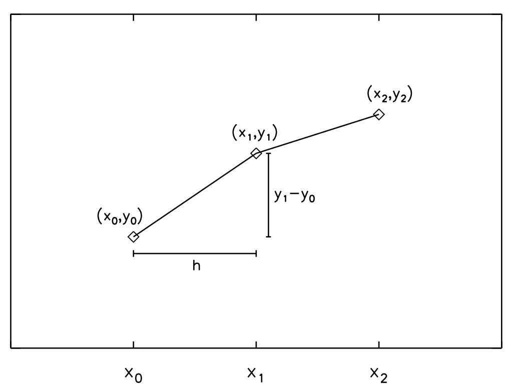



Figure 1.4: Graph illustrating how the Euler Algorithm approximates a graph in straight line segments.

1.

$$\begin{aligned} \left(\frac{dy}{dx}\right)_{x=x_0} &= f(x_0,y_0) = \frac{y_1-y_0}{h}\ , \quad \text{(slope of straight line)} \\ \Longrightarrow \qquad y_1 &= y_0 + hf(x_0,y_0) \end{aligned} \tag{1.5}$$

2.

$$\begin{aligned}\left(\frac{dy}{dx}\right)_{x=x_1} &= f(x_1,y_1) = f(x_0+h,y_1) = \frac{y_2-y_1}{h}\ , \\ \Longrightarrow \qquad y_2 &= y_1 + hf(x_0+h,y_1)\end{aligned} \tag{1.6}$$

3.

$$\left(\frac{dy}{dx}\right)_{x=x_n} = f(x_n,y_n) = f(x_0+nh,y_n) = \frac{y_{n+1}-y_n}{h}\ , \tag{1.7}$$

Euler Algorithm:

$y_{n+1} = y_n + hf(x_0+nh, y_n)\ , \qquad n = 0, 1, 2, \ldots$

$x_0\ ,\ \ y_0\ ,\ \ h$ all given.

In this way we can construct an approximate graph of $y(x)$ as an approximate solution to the differential equation.

The Euler algorithm is a non-linear, first-order difference equation for $y_n$.

**Example 1.5.1** *Question:*

*Use the Euler algorithm to compute the solution at intervals of 0.1 on* $[0,1]$ *of*

$$\frac{dy}{dx} = xy\ , \qquad y(0) = 1.$$

*Answer:*

*This is a simple problem - we know the exact solution* $y = e^{x^2/2}$. *We can test the accuracy of the Euler approximations against the true solution.*

*Do first 3 steps - computer loop for rest:*

$$y_{n+1} = y_n + hf(x_0+nh, y_n)\ .$$

$$h = 0.1\ , \qquad x_0 = 0\ , \quad y_0 = 1\ , \qquad f(x,y) = xy\ .$$

$$\begin{aligned} y_{n+1} &= y_n + h(x_0+nh)y_n\ , \\ y_{n+1} &= y_n + 0.1(0.1n)y_n\ , \\ n=0: \qquad y_1 &= 1 + (0.1)(0.1*0)(1) = 1\ , \\ n=1: \qquad y_2 &= 1 + (0.1)(0.1)(1) = 1.01\ , \\ n=2: \qquad y_3 &= 1.01 + (0.1)(0.2)(1.01) = 1.0302\ , \end{aligned}$$

| $n$ | $x_n$ | $y_n$ | Actual value | Error |
|---|---|---|---|---|
| 0 | 0.00000 | 1.00000 | 1.00000 | 0.00000 |
| 1 | 0.10000 | 1.00000 | 1.00501 | 0.00501 |
| 2 | 0.20000 | 1.01000 | 1.02020 | 0.01020 |
| 3 | 0.30000 | 1.03020 | 1.04603 | 0.01583 |
| 4 | 0.40000 | 1.06111 | 1.08329 | 0.02218 |
| 5 | 0.50000 | 1.10355 | 1.13315 | 0.02960 |
| 6 | 0.60000 | 1.15873 | 1.19722 | 0.03849 |
| 7 | 0.70000 | 1.22825 | 1.27762 | 0.04937 |
| 8 | 0.80000 | 1.31423 | 1.37713 | 0.06290 |
| 9 | 0.90000 | 1.41937 | 1.49930 | 0.07993 |
| 10 | 1.00000 | 1.54711 | 1.64872 | 0.10161 |

Table 1.1: Table of values for the Euler algorithm applied to $\frac{dy}{dx} = xy$, $y(0) = 1$ on $[0, 1]$, with step size $h = 0.1$.

*We can use a computer to compute the remaining $y_n$ in the Euler Algorithm. The values are given in Table 1.1, along with the actual value and the error at each step. As you can see, the error increases with every iteration. This can also be seen in the graph, Figure 1.5, which shows both the Euler Algorithm approximation and the exact solution.*

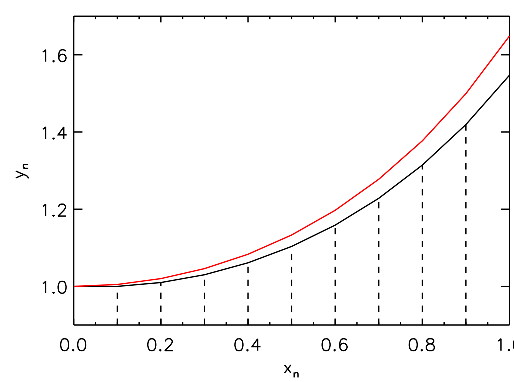



Figure 1.5: Graph for the Euler algorithm applied to $\frac{dy}{dx} = xy$, $y(0) = 1$ on $[0, 1]$, with step size $h = 0.1$ (black line). The exact solution, $y = e^{x^2/2}$, is shown in red.

*Why does the error increase as we move across the interval? We are taking the derivative only at the left-hand end point, but the derivative changes across the interval.*

*One way to decrease the error is the decrease the step size, $h$, Another is to improve the algorithm.*

| $n$ | $x_n$ | $y_n$ | Actual value | Error |
|---|---|---|---|---|
| 0 | 0.00000 | 1.00000 | 1.00000 | 0.00000 |
| 1 | 0.05000 | 1.00000 | 1.00125 | 0.00125 |
| 2 | 0.10000 | 1.00250 | 1.00501 | 0.00251 |
| 3 | 0.15000 | 1.00751 | 1.01131 | 0.00380 |
| 4 | 0.20000 | 1.01507 | 1.02020 | 0.00513 |
| 5 | 0.25000 | 1.02522 | 1.03174 | 0.00652 |
| 6 | 0.30000 | 1.03803 | 1.04603 | 0.00799 |
| 7 | 0.35000 | 1.05361 | 1.06316 | 0.00956 |
| 8 | 0.40000 | 1.07204 | 1.08329 | 0.01124 |
| 9 | 0.45000 | 1.09348 | 1.10655 | 0.01307 |
| 10 | 0.50000 | 1.11809 | 1.13315 | 0.01506 |
| 11 | 0.55000 | 1.14604 | 1.16329 | 0.01725 |
| 12 | 0.60000 | 1.17756 | 1.19722 | 0.01966 |
| 13 | 0.65000 | 1.21288 | 1.23522 | 0.02234 |
| 14 | 0.70000 | 1.25230 | 1.27762 | 0.02532 |
| 15 | 0.75000 | 1.29613 | 1.32478 | 0.02865 |
| 16 | 0.80000 | 1.34474 | 1.37713 | 0.03239 |
| 17 | 0.85000 | 1.39853 | 1.43512 | 0.03660 |
| 18 | 0.90000 | 1.45796 | 1.49930 | 0.04134 |
| 19 | 0.95000 | 1.52357 | 1.57027 | 0.04670 |
| 20 | 1.00000 | 1.59594 | 1.64872 | 0.05278 |

Table 1.2: Table of values for the Euler algorithm applied to $\frac{dy}{dx} = xy$, $y(0) = 1$ on $[0, 1]$, with step size $h = 0.05$.

## 1.6 The Modified Euler Algorithm

The main limitation of the Euler Algorithm is that we approximate the slope across the whole interval $[x_n, y_n]$ by the constant value at the left end of the interval, at $x = x_n$, $y = y_n$.

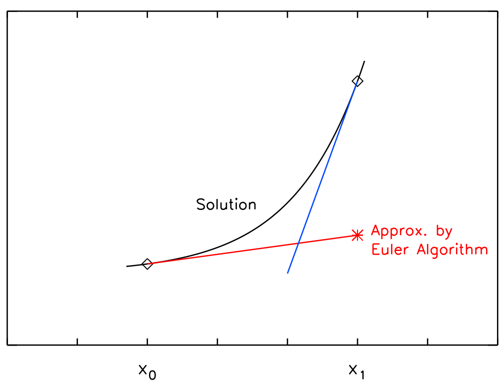
We are using a constant to approximate a variable.

In reality the slope is obviously changing. To try and accommodate this, the modified Euler method takes the average value of the slope at the two end points of $[x_n, x_{n+1}]$.

Have $\dfrac{dy}{dx} = f(x, y)$ :

| $n$ | $x_n$ | $y_n$ | Actual value | Error |
|---|---|---|---|---|
| 0 | 0.00000 | 1.00000 | 1.00000 | 0.00000 |
| 1 | 0.05000 | 1.00000 | 1.00125 | 0.00125 |
| 2 | 0.10000 | 1.00250 | 1.00501 | 0.00251 |
| 3 | 0.15000 | 1.00751 | 1.01131 | 0.00380 |
| 4 | 0.20000 | 1.01507 | 1.02020 | 0.00513 |
| ... | ... | ... | ... | ... |
| 16 | 0.80000 | 1.34474 | 1.37713 | 0.03239 |
| 17 | 0.85000 | 1.39853 | 1.43512 | 0.03660 |
| 18 | 0.90000 | 1.45796 | 1.49930 | 0.04134 |
| 19 | 0.95000 | 1.52357 | 1.57027 | 0.04670 |
| 20 | 1.00000 | 1.59594 | 1.64872 | 0.05278 |

Table 1.3: Table of values for the Euler algorithm applied to $\frac{dy}{dx} = xy$, $y(0) = 1$ on $[0, 1]$, with step size $h = 0.05$.




$$\begin{aligned}\left(\frac{dy}{dx}\right)_{\text{ave}} &= \frac{1}{2}\left\{\left(\frac{dy}{dx}\right)_{(x_n,y_n)} + \left(\frac{dy}{dx}\right)_{(x_{n+1},y_{n+1})}\right\} ,\\ &= \frac{1}{2}\left\{f(x_n, y_n) + f(x_n + h, y_{n+1})\right\} .\end{aligned}$$

Use the Euler Algorithm (unmodified version) itself to compute $y_{n+1}$ from $y_n$:

$$y_{n+1} = y_n + hf(x_n, y_n) .$$

[Two step process: Euler - initial estimate modified Euler - refined estimate.]

Thus we take

$$\left(\frac{dy}{dx}\right)_{\text{ave}} = \frac{1}{2}\left\{f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))\right\} ,$$

as the slope of $y$ in $[x_n, x_{n+1}]$.

Thus the algorithm becomes

$$\frac{y_{n+1} - y_n}{h} = \left(\frac{dy}{dx}\right)_{\text{ave}}$$

or

Modified Euler Algorithm:

$$y_{n+1} = y_n + \frac{h}{2}\left\{f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))\right\} \ ,$$

where $x_n = x_0 + nh \ , \quad x_{n+1} = x_0 + (n+1)h \ .$

First order difference equation

**Example 1.6.1** *We apply this to our test problem, where*

$$\frac{dy}{dx} = xy \; (= f(x,y)) \ , \quad x_0 = 0 \ , \quad y_0 = 1 \ , \quad h = 0.1.$$

$$\begin{aligned} f(x_n, y_n) &= x_n y_n \ , \\ y_n + hf(x_n, y_n) &= y_n + hx_n y_n \ , \\ f(x_n + h, y_n + hf(x_n, y_n)) &= (x_n + h)(y_n + hx_n y_n) \ , \end{aligned} \tag{1.8}$$

$$\begin{aligned} y_{n+1} &= y_n + \frac{h}{2}\left\{x_n y_n + (x_n + h)(y_n + hx_n y_n)\right\} \ , \\ x_n = nh \ , &\qquad x_n + h = (n+1)h \ . \end{aligned} \tag{1.9}$$

$$\begin{aligned} n = 0: \qquad y_1 &= 1 + \frac{0.1}{2}\left\{0 * 1 + 0.1(1))\right\} = 1.005 \ , \\ n = 1: \qquad y_2 &= 1.005 + \frac{0.1}{2}\left\{(0.1) * (1.005) + (0.2)(1.005 + 0.1(0.1(1.005)))\right\} = 1.0202 \ , \end{aligned}$$

*to four decimal places.*

*Computing the remaining* $y_n$ *on a computer, we obtain the values given in Table 1.4. Again, the error increases with every iteration, but is much smaller than the error resulting from the Euler Algorithm. Figure 1.6 shows a plot of the approximation from the Modified Euler Algorithm in black, with the actual solution over-plotted in red. Here you can see that the Modified Euler Algorithm gives an approximation that is very close to the actual solution - the curves almost completely overlap.*

Tables 1.5 and 1.6 give the solution to Example 1.5.1 at $y(1)$ for different step sizes, $h$, for the Euler Algorithm (1.5) and Modified Euler Algorithm (1.6). The errors for the Modified Euler Algorithm

| $n$ | $x_n$ | $y_n$ | Actual value | Error |
|---|---|---|---|---|
| 0 | 0.00000 | 1.00000 | 1.00000 | 0.0000000 |
| 1 | 0.10000 | 1.00500 | 1.00501 | 0.0000125 |
| 2 | 0.20000 | 1.02018 | 1.02020 | 0.0000257 |
| 3 | 0.30000 | 1.04599 | 1.04603 | 0.0000420 |
| 4 | 0.40000 | 1.08322 | 1.08329 | 0.0000641 |
| 5 | 0.50000 | 1.13305 | 1.13315 | 0.0000972 |
| 6 | 0.60000 | 1.19707 | 1.19722 | 0.0001487 |
| 7 | 0.70000 | 1.27739 | 1.27762 | 0.0002294 |
| 8 | 0.80000 | 1.37677 | 1.37713 | 0.0003548 |
| 9 | 0.90000 | 1.49876 | 1.49930 | 0.0005474 |
| 10 | 1.00000 | 1.64788 | 1.64872 | 0.0008401 |

Table 1.4: Table of values for the Modified Euler algorithm applied to $\frac{dy}{dx} = xy$, $y(0) = 1$ on $[0, 1]$, with step size $h = 0.1$.

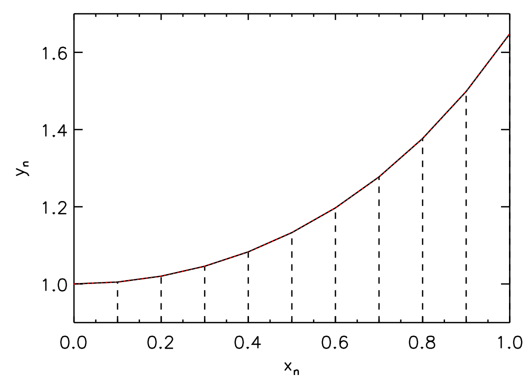



Figure 1.6: Graph for Euler algorithm applied to $\frac{dy}{dx} = xy$, $y(0) = 1$ on $[0, 1]$, with step size $h = 0.1$ (black line). The exact solution, $y = e^{x^2/2}$, is shown in red.

are much smaller than those of the unmodified version, but for both algorithms the error decreases with decreasing step size $h$.

Considering the size of the error for the case of the Euler Algorithm, the error appears to be proportional to the step size: $E(h) = |\text{Error}|_{x=1} \propto kh$, $k$ constant. This is demonstrated in the right-hand column of Table 1.5.

Considering the size of the error for the Modified Euler Algorithm, the error decreases more rapidly, and appears to be proportional to the step size squared: $E(h) = |\text{Error}|_{x=1} \propto kh^2$, $k$ constant. This

| $h$ | $y(1)$ | \|Error\| | | |
|---|---|---|---|---|
| 0.1 | 1.5471 | $a$ =0.1016 | | |
| 0.05 | 1.5959 | 0.0528 | $a/2 =$ | 0.0508 |
| 0.025 | 1.6218 | 0.0269 | $a/4 =$ | 0.0254 |
| 0.0125 | 1.6351 | 0.0136 | $a/8 =$ | 0.0127 |
| exact | 1.6487 | – | | |

Table 1.5: Table of values for the Euler algorithm applied to $\frac{dy}{dx} = xy$, $y(0) = 1$ at $x = 1$, with step sizes $h = 0.1$, 0.05, 0.025 and 0.0125.

| $h$ | $y(1)$ | \|Error\| | | |
|---|---|---|---|---|
| 0.1 | 1.647881 | $a$ =0.00084 | | |
| 0.05 | 1.648529 | 0.00019 | $a/4 =$ | 0.00021 |
| 0.025 | 1.648676 | 0.000045 | $a/16 =$ | 0.00005 |
| 0.0125 | 1.648710 | 0.000011 | $a/64 =$ | 0.00001 |
| exact | 1.648721 | – | | |

Table 1.6: Table of values for the Modified Euler algorithm applied to $\frac{dy}{dx} = xy$, $y(0) = 1$ at $x = 1$, with step sizes $h = 0.1$, 0.05, 0.025 and 0.0125.

is demonstrated in the right-hand column of Table 1.6.

## 1.7 Summary of Iterative Methods

- ***Computing the Roots of Algebraic Equations***

1a. Sketch a graph (or look at the behaviour) of $y = f(x)$ to find the intervals in which roots lie.

1b. *Showing there is a single root in the interval $a < x < b$.*

Given interval $(a, b)$, show that only a single root lies in that interval:

(i) Check that $f(a)$ has the opposite sign to $f(b)$ – this implies that $f(x)$ crosses the $x-$axis at least once in the interval $(a, b)$, therefore at least one root exists $a < x < b$.

(ii) Check the sign of the derivative $f'(x)$ on $(a, b)$. If $f'(x) > 0(< 0)$ $\forall x \in (a, b)$ then $f(x)$ is increasing (decreasing) across the interval $\Rightarrow$ no turning point $\Rightarrow$ no second root. (Remember that if $f'(x)$ does change sign on $(a, b)$, this does not necessarily mean that there is a second root in the interval $(a, b)$, but no sign change means that there cannot be a second root.)

(iii) Check that the graph is continuous on the interval, i.e. has no asymptotes.

2. Seek a rearrangement of $f(x)$ of the form $x = F(x)$ for which $\|F'(r)\| < 1$ at the required root $r$.

   (In general, we do not already know $r$ so this involves finding an interval containing $x = r$ for which $\|F'(x)\| < 1$.)

3. Compute the root using the difference equation $x_{n+1} = F(x_n)$, from a suitable start point chosen with reference to the graph.

- ***Newton's Method for Finding Roots***

Finding a root $x = r$ of the equation $f(x) = 0$ (a point where the graph of $f(x)$ crosses the $x-$axis):

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- ***Solutions of Ordinary Differential Equations Using Difference Equations***

Solving a differential equation of the form $\frac{dy}{dx} = f(x, y)$ given the condition $y(x_0) = y_0$ ($x_0$, $y_0$ constants) and step size $h$.

1. *Euler Algorithm:*

   $x_{n+1} = x_n + h$

   $y_{n+1} = y_n + hf(x_n, y_n)$

   $n = 0, 1, 2, ...$

2. *Modified Euler Algorithm:*

$$x_{n+1} = x_n + h$$
$$y_{n+1} = y_n + \frac{h}{2}\{f(x_n, y_n) + f(x_{n+1}, y_n + hf(x_n, y_n))\}$$
$$n = 0, 1, 2, ...$$