# Chapter 2

# Particular methods of solution

There is no universal method to obtain closed form solutions to the general first order ODE (1.1). Here we illustrate some particular methods that can be used to solve the ODE for certain forms of $f(x, y)$.

## 2.1 Separable equations

A first order ODE (1.1) is *separable* if $f(x, y)$ can be written in the form $f(x, y) = F(x)G(y)$. The equation can be solved by direct integration:

$$\int \frac{dy}{G(y)} = \int F(x)dx.$$

**Example 2.1** $\frac{dy}{dx} = \frac{2x+1}{2y}$ with $y(2) = 2$

**Example 2.2** $\frac{dy}{dx} = \frac{y \cos x}{1+y^2}$ with $y(0) = 1$

## 2.2 Bernoulli equations

A first order ODE is called a *Bernoulli equation* if it can be written in the form

$$y' + p(x)y = q(x)y^n,$$

that is, if the only nonlinearity occurs as a power of $y(x)$. This equation can be solved by the substitution $v(x) = y^{1-n}$:

$$\begin{aligned} \frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx} &= (1-n)y^{-n}(qy^n - py) \\ &= (1-n)(q - pv) \end{aligned}$$

that is

$$v' + (1-n)pv = (1-n)q,$$

which is a linear first order ODE for $v$. Solve using an integrating factor to obtain $v(x)$, then obtain $y$ from $y = v^{1/(1-n)}$.

**Example 2.3** $\frac{dy}{dx} = \frac{4y}{x} + x\sqrt{y}$

**Example 2.4** $\frac{dy}{dx} - \frac{y}{2x} = y^3 \sin(x^2)$

## 2.3 Homogeneous equations

A first order ODE is *homogeneous* if it can be written in the form

$$y' = F\left(\frac{y}{x}\right).$$

This equation can be solved by the substitution $v(x) = \dfrac{y}{x}$:

$$\frac{dv}{dx} = \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = \frac{1}{x}[F(v) - v]$$

i.e. $v' = [F(v) - v]/x$, which is a separable ODE for $v$. Hence

$$\int \frac{dv}{F(v) - v} = \int \frac{dx}{x}$$

can be integrated directly to find $v$, from which $y$ can be obtained from $y = xv$.

**Example 2.5** $\frac{dy}{dx} = \frac{y^2+2xy}{x^2}$

**Example 2.6** $x^2 + 3xy + y^2 - x^2\frac{dy}{dx} = 0$

## 2.4 Exact equations

A first order ODE is *exact* if it can be written in the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

where $M(x,y)$ and $N(x,y)$ satisfy

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

To solve this, we make use of the following theorem.

**Theorem 2.1** Two functions $M(x,y)$ and $N(x,y)$ satisfy $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ if and only if there exists a function $\psi(x,y)$ that satisfies $\frac{\partial \psi}{\partial x} = M$ and $\frac{\partial \psi}{\partial y} = N$.

*Proof.* $\Longleftarrow$ (easy):

$$M = \frac{\partial \psi}{\partial x}, N = \frac{\partial \psi}{\partial y} \implies \frac{\partial M}{\partial y} = \frac{\partial^2 \psi}{\partial y \partial x} = \frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial N}{\partial x}$$

as required.

$\Longrightarrow$ (harder): Start by defining $\phi(x,y)$ by the indefinite integral,

$$\phi = \int^x M dx$$

so that $\frac{\partial \phi}{\partial x} = M$. Differentiate with respect to $y$:

$$\frac{\partial \phi}{\partial y} = \int^x \frac{\partial M}{\partial y} dx = \int^x \frac{\partial N}{\partial x} dx = N(x,y) + f(y)$$

where $f(y)$ is some arbitrary function of integration. Now define $\psi(x, y)$ by

$$\psi = \phi - \int^{y} f(\eta) d\eta.$$

The function $\psi$ has the required properties, since

$$\begin{aligned} \text{(i)} \qquad & \frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial x} = M \\ \text{(ii)} \qquad & \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial y} - f(y) = N. \end{aligned}$$

□

To solve an exact equation, we seek the $\psi(x, y)$ that satisfies $M = \dfrac{\partial \psi}{\partial x}$ and $N = \dfrac{\partial \psi}{\partial y}$. We do this by integrating this pair of equations with respect to $x$ and $y$, respectively:

$$\begin{aligned} \psi &= \int M dx + f(y) \\ \psi &= \int N dy + g(x) \end{aligned}$$

where $f(y)$ and $g(x)$ are arbitrary functions of integration which are then determined by the requirement that the two expressions for $\psi$ have the same functional form. Having found such a $\psi$, the ODE may now be rewritten as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} = 0.$$

But by the chain rule, and considering $\psi(x, y(x))$ as a function of $x$, this is equal to the total derivative:

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx}.$$

Hence the ODE is equivalent to

$$\frac{d\psi}{dx} = 0$$

and so the solution is given by $\psi = C$ where $C$ is a constant of integration that may be determined from the initial conditions as usual.

**Example 2.7** $3x^2 + y^2 + (2xy + 1)\frac{dy}{dx} = 0$

**Example 2.8** $2x + y^2 + 2xy\frac{dy}{dx} = 0$ with $y(1) = 0$

**Example 2.9** $y \cos x + 2xe^y + (\sin x + x^2 e^y - 1)\frac{dy}{dx} = 0$