MT5823 Semigroup theory: Solutions 1 (James D. Mitchell)
Definition and basic properties

1-1. (a) Since f is a right identity, $ef = e$ and $ef = f$ since e is a left identity. Hence $e = f$, and e is a left and right identity, which is therefore an identity.

(b) Suppose $z, z' \in S$ are zeros. Then $zz' = z$ since z is a zero, and so z times any element of S is z. Likewise, $zz' = z'$, because z' is a zero. Thus $z' = z$.

(c) **YES!** See part (b). ■

1-2. The semigroup T_n consists of all mappings from $\{1, 2, \ldots, n\}$ to itself. Such a mapping is uniquely determined by the images of the elements $1, 2, \ldots, n$. There are precisely n choices for each image and so $|T_n| = n^n$. ■

1-3. ($\Leftarrow$) Let c_j denote the constant mapping with value j, that is,

$$c_j = \begin{pmatrix} 1 & 2 & \cdots & n \\ j & j & \cdots & j \end{pmatrix}$$

where $1 \leq j \leq n$. If $f \in T_n$ is arbitrary, then $(if)c_j = j$ for all i. It follows that $fc_j = c_j$ for all $f \in T_n$ and so c_j is a right zero.

($\Rightarrow$) Seeking a contradiction, suppose that $f \in T_n$ is a right zero but not a constant map. It follows that there exist $i, j \in \{1, 2, \ldots, n\}$ such that $if \neq jf$. But then $(ic_j)f = jf \neq if$ and so $c_jf \neq f$. Thus f is not a right zero.

Recall that a semigroup has at most one (2-sided) zero element by Problem **1-1**(b). Moreover recall that if the semigroup has a left zero and a right zero, then they are equal. If $n \geq 2$ then, the semigroup T_n has more than one right zero, and so it has no left zeros (or 2-sided zeros).

The identity of T_n is

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}.$$

■

1-4. ($\Rightarrow$) Recall that $f \in T_n$ is an idempotent if $f^2 = f$. So, if $y \in \{1, 2, \ldots, n\}$, then $yf = x \in \operatorname{im}(f)$. This implies that

$$xf = (yf)f = yf^2 = yf = x,$$

as required.

($\Leftarrow$) If $y \in \{1, 2, \ldots, n\}$, then $yf = x \in \operatorname{im}(f)$. This implies that

$$yf^2 = (yf)f = xf = x = yf$$

and so $f^2 = f$. ■

1-5. To specify an idempotent f with image size k ($1 \leq k \leq n$) proceed as follows. First choose the image I of f from $\{1, 2, \ldots, n\}$. There are $\binom{n}{k}$ choices for I. We must now determine where f maps each $i \in \{1, 2, \ldots, n\}$. If $i \in I$, then from Problem **1-4**, $if = i$. If $i \notin I$, then there are k choices for if (the elements of I). There are $n - k$ elements not in I, yielding k^{n-k} possible idempotents with image I. Summing over k give the desired result. ■

1-6. Since $\operatorname{im}(fg) \subseteq \operatorname{im}(g)$ it follows that $\operatorname{rank}(fg) \leq \operatorname{rank}(g)$. On the other hand, since g is a mapping $|\operatorname{im}(f)g| \leq |\operatorname{im}(f)|$. It follows that

$$\operatorname{rank}(fg) \leq \min\{\operatorname{rank}(f), \operatorname{rank}(g)\}.$$

If f and g are both permutations, then

$$\operatorname{rank}(fg) = n = \operatorname{rank}(f) = \operatorname{rank}(g).$$

If

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 4 & 2 & 5 \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 1 & 5 \end{pmatrix},$$

then $\operatorname{rank}(f) = \operatorname{rank}(g) = 4$. But

$$f \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 2 & 5 \end{pmatrix}$$

and $\operatorname{rank}(fg) = 3$. ■

1-7. Let $a \in G$ be arbitrary. Since G is a group, it follows that $ag \in G$ for all $g \in G$, and so $aG \subseteq G$. If $g \in G$, then $g = a(a^{-1}g) \in aG$ and so $G \subseteq aG$. Hence $G = aG$.

The proof that $Ga = G$ follows by a symmetric argument. ∎

1-8. (a) If $b \in S = bS$ is arbitrary, then there exists $e \in S$ such that $b = be \in bS$.

(b) If $x \in S$ is arbitrary, then $x \in Sb$ and so there exists $y \in S$ such that $x = yb$. Hence $xe = ybe = yb = x$, and so e is a right identity for S.

(c) The proof of this part is similar to parts (a) and (b), and that S is a monoid follows by Problem **1-1**(a).

(d) Let e denote the identity of S. Since S is a monoid, it suffices to show that for all $x \in S$ there exists $y \in S$ such that $xy = yx = e$. If $x \in S$ is arbitrary, then $e \in S = xS$ and so there exists $y \in S$ such that $xy = e$. Similarly, there exists $z \in S$ such that $zx = e$, and so

$$z = ze = z(xy) = (zx)y = ey = y,$$

as required. ∎