MT5823 Semigroup theory: Solutions 3 (James D. Mitchell)
Binary relations, equivalences, homomorphisms, and isomorphisms

## Binary relations and equivalences

**3-1**. The answers are:

$$\begin{aligned}
\rho &= \{(1,4),(4,1),(2,3),(2,6),(3,6),(3,2),(6,2),(6,3)\}\cup\Delta_6\\
\sigma &= \{(3,5),(3,1),(3,6),(4,5),(4,1),(4,6),(2,6),(5,1),(5,6)\}\cup\Delta_6\\
\rho\cap\sigma &= \sigma\cap\rho \;=\; \{(4,1),(3,6),(2,6)\}\cup\Delta_6\\
\rho\cup\sigma &= \{(1,4),(4,1),(2,3),(2,6),(3,6),(3,2),(6,2),(6,3),(3,5),(3,1),\\
&\quad (4,5),(4,6),(5,1),(5,6)\}\cup\Delta_6\\
\sigma^{-1} &= \{(5,3),(1,3),(6,3),(5,4),(1,4),(6,4),(6,2),(1,5),(6,5)\}\cup\Delta_6\\
\rho\circ\sigma &= \{(3,5),(3,1),(3,6),(4,5),(4,1),(4,6),(2,6),(1,4),(4,1),(2,3),(2,6),\\
&\quad (3,6),(3,2),(6,2),(6,3),(1,5),(1,6),(2,5),(6,5),(6,1)\}\cup\Delta_6\\
\sigma\circ\rho &= \sigma\cup\rho\cup\Delta_6\cup\{(3,4),(3,2),(4,2),(4,3)\},
\end{aligned}$$

where $\Delta_6 = \{\ (x,x)\ :\ x\in\{1,\ldots,6\}\ \}$. □

**3-2**. (a) $\rho$ is reflexive if and only if $(x,x)\in\rho$ for all $x\in X$ if and only if $\{\ (x,x)\ :\ x\in X\ \}\subseteq\rho$;

(b) $\rho$ is symmetric if and only if $(x,y)\in\rho$ implies $(y,x)\in\rho$ if and only if $\rho^{-1}\subseteq\rho$;

(c) $\rho$ is transitive if and only if $(x,y),(y,z)\in\rho$ implies $(x,z)\in\rho$ if and only if $\rho\circ\rho=\{\ (x,z)\ :\ \exists y\in X \text{ with } (x,y),(y,z)\in\rho\ \}\subseteq\rho$. □

**3-3**. It suffices to prove that $\rho\cap\sigma$ is reflexive, symmetric and transitive.

**Reflexive:** by Problem **3-2**(a) $\Delta_X\subseteq\rho$ and $\Delta_X\subseteq\sigma$. It follows that $\Delta_X\subseteq\rho\cap\sigma$ and so $\rho\cap\sigma$ is reflexive.

**Symmetric:** $(x,y)\in\rho\cap\sigma$ implies that $(y,x)\in\rho$ and $(y,x)\in\sigma$. It follows that $(y,x)\in\rho\cap\sigma$.

**Transitive:** $(x,y),(y,z)\in\rho\cap\sigma$ implies that $(x,y),(y,z)\in\rho$ and $(x,y),(y,z)\in\sigma$. Thus $(x,z)\in\rho$ and $(x,z)\in\sigma$ and so $(x,z)\rho\cap\sigma$.

The classes of $\rho\cap\sigma$ are intersections of the classes of $\rho$ and $\sigma$. For example, if $\rho=\{\{1,2,3,4\},\{5,6\}\}$ and $\sigma=\{\{1,2,3\},\{4,5\},\{6\}\}$, then $\rho\cap\sigma=\{\{1,2,3\},\{4\},\{5\},\{6\}\}$. □

**3-4**. Let $\sigma=\{(1,2),(2,1)\}\cup\Delta_2$ and $\rho=\{(2,3),(3,2)\}\cup\Delta_3$. Then $(1,2),(2,3)\in\rho\cup\sigma$ but $(1,3)\notin\rho\cup\sigma$ and so $\rho\cup\sigma$ is not transitive and hence not an equivalence relation.

If $\sigma$ is the relation with classes $\{\{1,2\},\{3\}\}$, and $\rho$ is the equivalence relation with classes $\{\{1\},\{2,3\}\}$, then $(1,3)\in\sigma\circ\rho$ but $(3,1)\notin\sigma\circ\rho$. Hence $\sigma\circ\rho$ is not symmetric and hence not an equivalence relation. □

**3-5**. ($\Leftarrow$) Since $(x,x)\in\alpha$ and $(x,x)\in\beta$, for all $x\in X$, it follows that $(x,x)\in\alpha\circ\beta$ and so $\alpha\circ\beta$ is reflexive.

From the definition and Problem **3-2**(b), $(\alpha\circ\beta)^{-1}=\beta^{-1}\circ\alpha^{-1}\subseteq\beta\circ\alpha=\alpha\circ\beta$. It follows that $\alpha\circ\beta$ is symmetric.

By Problem **3-2**(c) it suffices to show that $(\alpha\circ\beta)^2\subseteq\alpha\circ\beta$. But $\alpha\circ\beta\circ\alpha\circ\beta=\alpha^2\circ\beta^2=\alpha\circ\beta$. It follows that $\alpha\circ\beta$ is transitive.

($\Rightarrow$) $\alpha\circ\beta$ is an equivalence relation implies that $\alpha\circ\beta$ is symmetric and so by Problem **3-2**(b) $(\alpha\circ\beta)^{-1}\subseteq\alpha\circ\beta$. But $(\alpha\circ\beta)^{-1}=\beta^{-1}\circ\alpha^{-1}=\beta\circ\alpha$. Thus $\beta\circ\alpha\subseteq\alpha\circ\beta$.

Let $(x,y)\in\alpha\circ\beta$. Then $(y,x)\in\alpha\circ\beta$ and so there exists $z$ such that $(y,z)\in\alpha$ and $(z,x)\in\beta$. Thus $(x,z)\in\beta$ and $(z,y)\in\alpha$. It follows that $(x,y)\in\beta\circ\alpha$. □

**3-6**. Recall from the definition that a binary relation is just a subset of $X\times X$. There are $2^{n^2}$ subsets of an $n^2$ element set. □

**3-7**. Clearly, the only partitions of the set $\{1,\ldots,n\}$ with a single part is $\{\{1,\ldots,n\}\}$ and the only partition with $n$ parts is $\{\{1\},\ldots,\{n\}\}$. Hence $S(n,1)=S(n,n)=1$.

In any partition of $\{1,\ldots,n\}$, either $\{n\}$ is a part, or $n$ belongs to a part of size at least 2.

The number of partitions of $\{1,\ldots,n\}$ with $r$ parts where $\{n\}$ is a part equals the number of partitions of $\{1,\ldots,n-1\}$ into $r-1$ parts. In other words, the number of partitions of $\{1,\ldots,n\}$ with $r$ parts is $S(n-1,r-1)$.

The number of partitions of $\{1,\ldots,n\}$ with $r$ parts where $n$ belongs to a part of size at least 2 can be determined by first partitioning $\{1,\ldots,n-1\}$ into $r$ parts, and then adding $n$ to one of those parts. There are $S(n-1,r)$

| $n \setminus r$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 1 | - | - | - | - | - |
| 2 | 1 | 1 | - | - | - | - |
| 3 | 1 | 3 | 1 | - | - | - |
| 4 | 1 | 7 | 6 | 1 | - | - |
| 5 | 1 | 15 | 25 | 10 | 1 | - |
| 6 | 1 | 31 | 90 | 65 | 15 | 1 |

Figure 1: The first few values of the Stirling numbers of the second kind.

partitions of $\{1, \ldots, n-1\}$ into $r$ parts, and given such a partition, there are $r$ distinct partitions arising from adding $n$ to any of the parts. Hence there are $rS(n-1, r)$ such partitions in total.

Therefore $S(n, r) = S(n-1, r-1) + rS(n-1, r)$, as required.

The values of $S(n, r)$ when $1 \leq r \leq n \leq 6$ are displayed in Figure 1.

## Homomorphisms and isomorphisms

**3-8**. Let $x \in S$ be an idempotent. Then $x^2 = x$ and so

$$(x)f = (x^2)f = (x)f\ (x)f.$$

Thus $(x)f$ is an idempotent.

Let $S$ be a monoid and $T$ be a monoid with zero element 0. Then define a mapping $f : S \longrightarrow T$ by $(s)f = 0$ for all $s \in S$. Since 0 is an idempotent, $f$ is a homomorphism and $(1_S)f = 0$ is not the identity of $T$.

Since $f$ is onto, for all $t \in T$ there exists $s \in S$ such that $(s)f = t$. Now, if $x$ is the identity of $S$, then for any $t \in T$

$$t\ (x)f = (s)f\ (x)f = (sx)f = (s)f = t$$

and

$$(x)ft = (x)f\ (s)f = (xs)f = (s)f = t.$$

Hence $(x)f$ is the identity of $T$.

To see that $(P)f$ is a subsemigroup it suffices to prove that it is closed. Let $(x)f, (y)f \in Pf$. Then $(x)f\ (y)f = (xy)f \in Pf$ since $f$ is a homomorphism and so $xy \in P$, as required. □

**3-9**. Suppose that $S$ is a semigroup such that $x^2 = x$ and $xyz = xz$ for all $x, y, z \in S$, and let $a \in S$ be arbitrary. We will show that $f : S \longrightarrow Sa \times aS$ defined by $(s)f = (sa, as)$ is an isomorphism.

**Injective:** Suppose that $(x)f = (y)f$ for some $x, y \in S$. Then $(xa, ax) = (ya, ay)$ and so $xa = ya$ and $ax = ay$. It follows that

$$x = x^2 = xax = yax = yay = y^2 = y,$$

and so $f$ is injective.

**Surjective:** Trivial.

**Homomorphism:** If $x, y \in S$, then $(x)f\ (y)f = (xa, ax)(ya, ay) = (xa, ay)$ and $(xy)f = (xya, axy) = (xa, ay) = (x)f\ (y)f$. Hence $f$ is a homomorphism.

## Further problems

**3-10**. By Problems **2-1** and **3-9**, $S$ is a rectangular band if and only if $xyz = xz$ and $x^2 = x$ for all $x, y, z \in S$.

($\Rightarrow$)

[**First proof.**] If $S$ is a rectangular band, then we may assume without loss of generality that $S = I \times \Lambda$ for some $I$ and $\Lambda$. If $a = (i, \lambda)$ and $b = (j, \mu)$, then $ab = ba$ implies that $(i, \mu) = (i, \lambda)(j, \mu) = (j, \mu)(i, \lambda) = (j, \lambda)$ and so $i = j$ and $\lambda = \mu$. Thus $a = b$.

[**Second proof.**] Suppose $a, b \in S$ are such that $ab = ba$. Then

$$a = a^2 = aba = ba^2 = ba = ab = ab^2 = bab = b^2 = b.$$

($\Leftarrow$) It suffices to show that $xyz = xz$ and $x^2 = x$ for all $x, y, z \in S$.

If $x \in S$ is arbitrary, then since $x^2 \cdot x = x \cdot x^2$, it follows by the assumption of this implication that $x^2 = x$.

If $x, y \in S$ are arbitrary, then

$$xyx \cdot x = xyx^2 = xyx = x^2yx = x \cdot xyx$$

and so $xyx = x$. Hence

$$xyz \cdot xz = xyz = xz \cdot xyz$$

and so $xyz = xz$, as required. □

**3-11**. It suffices, by Problem **2-3**, to show that there exist $e, a \in S$ such that $ea = a$. But $S$ is finite, and so it contains an idempotent $e$ by Problem **2-9**. In particular, $ee = e$ and so $e$ is the identity of $S$, and so $S$ is a monoid. □