MT3503 Complex Analysis[1]

1 Revision

1.1 Complex Numbers

In practice, recall that a complex number can be written in the form $a + ib$ where a and b are real numbers and i is the 'imaginary' number such that $i^2 = -1$. The quantity a is called the *real part* of the complex number and b is called the *imaginary part.* We operate on complex numbers in much the same way as we do on real numbers.

- Addition $(a + ib) + (a' + ib') = (a + a') + i(b + b')$.
- Multiplication$(a + ib)(a' + ib') = (aa' - bb') + i(ab' + ba')$.
- Division $\dfrac{a + ib}{a' + ib'} = \dfrac{(a + ib)(a' - ib')}{(a' + ib')(a' - ib')} = \dfrac{(aa' + bb') + i(ba' - ab')}{a'^2 + b'^2}$.

Each of the above arithmetic operations (addition, multiplication and division) produces another complex number. Note that $(a + ib)(a - ib) = a^2 + b^2$, which is real and non-negative. The number $a - ib$ is called the *conjugate* of $a + ib$.

Let z be a complex number where

$$z = x + iy,$$

then the conjugate of z is

$$\bar{z} = x - iy$$

and

$$z\bar{z} = x^2 + y^2 = |z|^2,$$

where $|z|$ is called the *modulus* of z and $|z| > 0$ unless $z \equiv 0$. The following notation is also used:

$$x = \mathrm{Re}(z) = (z + \bar{z})/2 \quad \text{and} \quad y = \mathrm{Im}(z) = (z - \bar{z})/2i.$$

It should be noted that if z_1, and z_2 are two complex number then

$$|z_1 z_2| = |z_1||z_2|.$$

The corollaries are that $|z|^2 = |z^2|$. However, $|z|^2 \neq z^2$, unless z is real. The results generalises itself naturally, and for complex numbers $z_1,\ z_2, \cdots, z_n$, we have

$$|z_1 z_2 \cdots z_n| = |z_1||z_2| \cdots |z_n|.$$

Two complex numbers z_1 and z_2 are equal when

$$\mathrm{Re}(z_1) = \mathrm{Re}(z_2) \quad \text{and} \quad \mathrm{Im}(z_1) = \mathrm{Im}(z_2).$$

[1]Original notes written by Dr C.V. Tran 2014/2015. Several minor modifications by JNR 2015/2016.

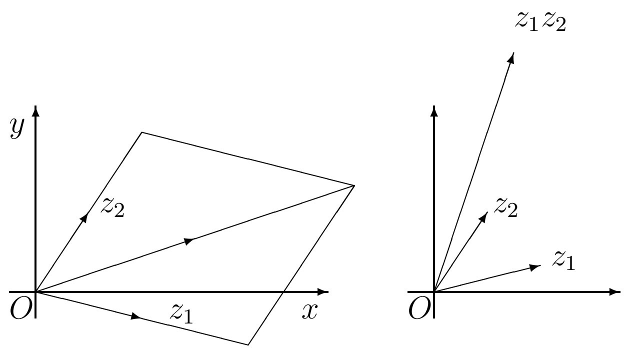

Figure 1: addition and multiplication of complex numbers.

We can identify z with a point or a vector on a plane (the Argand diagram). With this identification, complex addition is like vector addition, while multiplication has the effect of rotating and scaling a vector. From a geometric point of view, an obvious result is the triangle inequality (see figure 1)

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

Here the equality occurs iff $z_1 = \alpha z_2$, where $\alpha \geq 0$. We also have $|z_1 + z_2| \geq |z_1| - |z_2|$.

We will refer to the plane, upon which $z = x + iy$ is a point with co-ordinates (x, y), as the "complex plane". The $x-$axis is called the "real axis" while the $y-$axis is called the "imaginary axis". This visualisation is important, allowing us to express complex numbers in *modulus-argument form* and to perform certain calculations with ease.

1.2 Modulus-argument form

Recall that we can represent a point (x, y) using polar coordinates (r, θ):

$$(x, y) = r(\cos\theta, \sin\theta).$$

This allows us to write z in modulus-argument form (sometimes polar form for short)

$$z = x + iy = r(\cos\theta + i\sin\theta).$$

Note that

$$|z|^2 = x^2 + y^2 = r^2,$$

and remember $r > 0$ unless $z \equiv 0$. The argument θ satisfies

$$\tan\theta = \frac{y}{x}.$$

When expressed in modulus-argument form, the signs of the real and imaginary part of z are taken care of by the value of the argument θ (see figure 2).

Example 1.2.1 The real number -1 has modulus 1 and argument π. Note that the same point in the complex plane is identified when the argument is $-\pi$.

Hence, the value of θ is not unique. In fact, we can always add (or subtract) a multiple of 2π and arrive at the same point in the complex plane. It is therefore convenient to adopt the convention that the value of θ lies in the range $(-\pi, \pi]$. This is called the *Principal Value* of the argument and is denoted by $\text{Arg}(z)$.

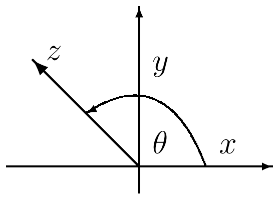

Figure 2: A complex number z in the second quadrant with modulus r and principal argument θ.

Example 1.2.2 Determine the principal value of the argument of $(-\sqrt{3}+i)$

Solution

$$\tan\theta = \frac{y}{x} = \frac{1}{-\sqrt{3}} = -\frac{1}{\sqrt{3}}.$$

Therefore, the two solution within $(-\pi, \pi]$ are

$$\theta = -\frac{\pi}{6} \quad \text{and} \quad \theta = \frac{5\pi}{6}.$$

Observe that the given complex number lies in the second quadrant of the complex plane (see figure 2). So the argument is $5\pi/6$. Thus

$$\text{Arg}(-\sqrt{3}+i) = \frac{5\pi}{6}.$$

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1.3 Introducting the complex exponential

For real x, recall the power series of e^x:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots.$$

We can formally define the exponential of a complex number z by replacing x in the above series by z. We will see the complex exponential in more details later in the course, but the following result should be very familiar. When z is purely imaginary, i.e. $z = i\theta$ (for real θ), we have (by definition)

$$\begin{aligned} e^{i\theta} &= 1 + i\theta - \frac{\theta^2}{2} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \cdots \\ &= 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + \cdots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right). \end{aligned}$$

The two series appear above are those of $\cos\theta$ and $\sin\theta$. So we have the identity

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Replacing θ by $-\theta$ yields

$$e^{-i\theta} = \cos\theta - i\sin\theta.$$

It follows that

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta},$$
$$\bar{z} = x - iy = r(\cos\theta - i\sin\theta) = re^{-i\theta}$$

and

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

The modulus-argument form $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ provides a very powerful means of manipulating complex numbers. For example, let

$$z_1 = r_1 e^{i\theta_1} \quad \text{and} \quad z_2 = r_2 e^{i\theta_2},$$

then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

and

$$z_1^\alpha = \left(r_1 e^{i\theta_1}\right)^\alpha = r_1^\alpha e^{i\alpha\theta_1} = r_1^\alpha \left(\cos\alpha\theta_1 + i\sin\alpha\theta_1\right).$$

Note that $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$. This is not necessarily true for Arg. In a similar way

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}.$$

The modulus-argument representation allows us to contemplate strange things such as i^i:

$$i = e^{i\pi/2}, \quad \text{so} \quad i^i = \left(e^{i\pi/2}\right)^i = e^{-\pi/2}.$$

This may seem rather fanciful but shortly we will consider complex functions such as e^z and $\log(z)$ and wish to make sense of such mathematical forms. Some more routine illustrative examples follow.

Example 1.3.1 Evaluate $(1 + 2i)(3 - i)$.

Solution Direct calculation gives

$$(1 + 2i)(3 - i) = 3 - i + 6i - 2i^2 = 3 + 5i + 2 = 5(1 + i).$$

Now observe that this implies that

$$\tan^{-1}(2) + \tan^{-1}(-1/3) = \pi/4.$$

Compute the values (in radians) to get

$$1.107148718 + (-0.3217505544) = \pi/4.$$

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Example 1.3.2 Determine the real and imaginary parts of $z^n = (x+iy)^n$, where n is a positive integer.

Solution We write

$$z = re^{i\theta}, \quad \text{where} \quad r = (x^2+y^2)^{1/2} \quad \text{and} \quad \tan\theta = \frac{y}{x}.$$

Then

$$z^n = r^n e^{in\theta} = r^n \cos(n\theta) + ir^n \sin(n\theta).$$

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otherwise $z^n = (x+iy)^n$ leads to a cumbersome binomial expansion for large value of n.

Example 1.3.3 Determine the modulus and argument of $(1+i\sqrt{3})^5$.

Solution Let $z = 1+i\sqrt{3}$, then

$$|z| = \sqrt{1+3} = 2 = r \quad \text{and} \quad \tan\theta = \sqrt{3}.$$

Since z is in the first quadrant we choose $\theta = \pi/3$. Therefore,

$$z = 2e^{i\pi/3} \quad \text{and} \quad z^5 = 32e^{5\pi i/3}$$

giving a modulus of 32 and an argument of $5\pi/3$. The principal value of the argument is $\mathrm{Arg}(z^5) = -\pi/3$. Writing z^5 in the form $a+ib$ we get

$$z^5 = 32(\cos 5\pi/3 + i\sin 5\pi/3) = 32(\cos\pi/3 - i\sin\pi/3) = 16(1-i\sqrt{3}).$$

You can check this by direct evaluation:

$$\begin{aligned}(1+i\sqrt{3})^5 &= 1 + 5i\sqrt{3} + 10(i\sqrt{3})^2 + 10(i\sqrt{3})^3 + 5(i\sqrt{3})^4 + (i\sqrt{3})^5 \\ &= 1 + i5\sqrt{3} - 30 - i30\sqrt{3} + 45 + i9\sqrt{3} = 16 - i16\sqrt{3}.\end{aligned}$$

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Likewise we can determine the roots of a complex number.

Example 1.3.4 Determine $(1+i\sqrt{3})^{1/2}$.

Solution Observe that this exercise is equivalent to finding the solutions of the quadratic equation:

$$z^2 = 1+i\sqrt{3},$$

so we should expect 2 solutions. This is a straightforward exercise using the modulus-argument form. From above we have

$$(1+i\sqrt{3}) = 2e^{i\pi/3} = 2e^{i\pi/3+2ik\pi},$$

where k is an integer. Therefore,

$$(1+i\sqrt{3})^{1/2} = \sqrt{2}e^{i\pi/6+ik\pi}\ , \quad \text{where} \quad k = 0, 1$$

and we have two roots of modulus $\sqrt{2}$ with arguments $\pi/6$ and $7\pi/6$ (or principal arguments $\pi/6$ and $-5\pi/6$). In the form of $a+ib$ we get

$$\frac{\sqrt{3}}{\sqrt{2}} + \frac{i}{\sqrt{2}} \quad \text{and} \quad -\frac{\sqrt{3}}{\sqrt{2}} - \frac{i}{\sqrt{2}}.$$

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1.4 Geometric interpretation of some complex equations

Equations involving the modulus $|z|$ identify curves and regions of the complex plane. Below are popular examples, many of which appear in subsequent sections.

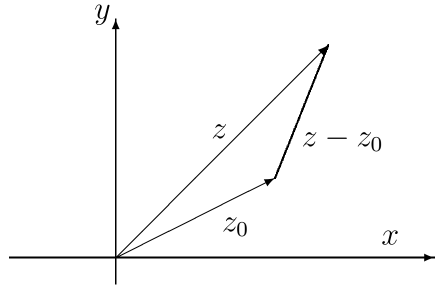

Figure 3: The points z satisfying $|z - z_0| = r$ form a circle centred at z_0 with radius r.

Example 1.4.1 $|z| = 1$ is the circle with centre $(0, 0)$ and radius unity. It is equivalent to $\sqrt{x^2 + y^2} = 1$ or $x^2 + y^2 = 1$. This is usually called the unit circle and can also be described by $z = e^{i\theta}$, for $-\pi < \theta \leq \pi$.

Example 1.4.2 $|z - z_0| = r$ is the circle with centre z_0 and radius r (see figure 3). This circle can also be described by $z - z_0 = re^{i\theta}$, for $-\pi < \theta \leq \pi$.

Example 1.4.3 $|z| < 1$ is an open circular region with centre $(0, 0)$ and radius unity. This is usually called the (open) unit disk.

Example 1.4.4 $|z - z_0| \leq a$ is the closed disk centred on the point z_0 with radius a.

Example 1.4.5 $|z - z_0| > a$ are all point outside the circle centred on z_0 with radius a.

Example 1.4.6 $a \leq |z - z_0| \leq b$, where a and b are real, is the closed annulus centred on the point z_0 with inner radius a and outer radius b.

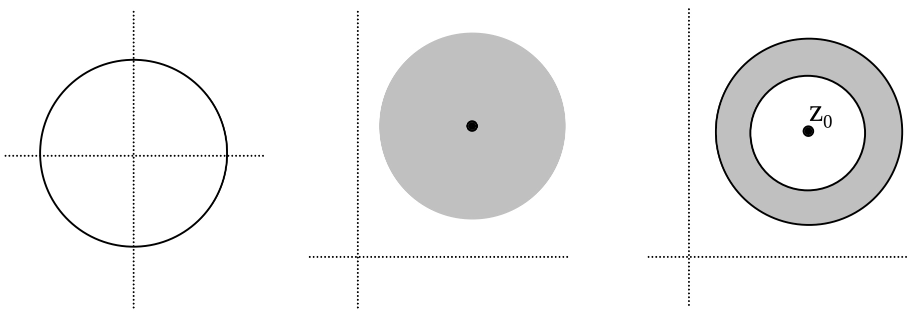

Figure 4: circle, open circular and closed annular regions

Example 1.4.7 $\mathrm{Im}(z) \geq 0$ is the upper half plane, including the real axis.

Example 1.4.8 $|z - \bar{z}| < a$ is the (open) infinite horizontal strip $-a/2 < y < a/2$.

Example 1.4.9 $|z + \bar{z}| < a$ is the (open) infinite vertical strip $-a/2 < x < a/2$.