1. Let $g \in PSL(2,\mathbb{Z})$ with standard form
$$g(z) = \frac{az+b}{cz+d}$$

and let $r = \frac{1}{2}$.

Suppose $h \in PSL(2,\mathbb{Z})$ with standard form $h(z) = \dfrac{a'z+b'}{c'z+d'}$ and $h \in B(g, \frac{1}{2})$.

Then $\|(a-a', b-b', c-c', d-d')\| < \frac{1}{2}$

which guarantees $|a-a'|, |b-b'|, |c-c'|, |d-d'| < \frac{1}{2}$.

Therefore $a'=a$, $b'=b$, $c'=c$, $d'=d$

and we can conclude that $g=h$,

$B(g,r) \cap PSL(2,\mathbb{Z}) = \{g\}$.

Thus $PSL(2,\mathbb{Z})$ is discrete and Fuchsian as required.

PSL(2, $\mathbb{Q}$) is not discrete.

Let $g \in PSL(2,\mathbb{Q})$ with standard form $g(z) = \dfrac{az+b}{cz+d}$, $\left(ad-bc=1\right.$ $\left. a,b,c,d \in \mathbb{Q}\right)$;

and let $r>0$ be given.

Consider $g_k \in PSL(2,\mathbb{Q})$ defined by

$$g_k(z) = \frac{\dfrac{(a+\frac{1}{k})z}{\sqrt{1+\frac{d}{k}}} + \dfrac{b}{\sqrt{1+d/k}}}{\dfrac{c}{\sqrt{1+d/k}}\,z + \dfrac{d}{\sqrt{1+d/k}}}$$

(Note that this has been normalised).

Then $\| g - g_k \| \to 0$ as $k \to \infty$ and so if we pick k large enough to guarantee $\|g-g_k\| < r$, then

$$B(g,r) \cap PSL(2,\mathbb{Q}) \supseteq \{g, g_k\}$$

and so $PSL(2,\mathbb{Q})$ is not discrete (since $g \neq g_k$).

Let $\Gamma \leq PSL(2,\mathbb{R})$ be Fuchsian.

2 Suppose $g_k \in \Gamma$ such that

$g_k \to g \in PSL(2,\mathbb{R})$ (in the $\mathbb{R}^4$ metric).

We want to show $g \in \Gamma$.

Proof: It follows that $\overset{\in \Gamma}{g_k g_{k+1}^{-1}} \to Id \in \Gamma$.

Since Γ is discrete, we can find $r > 0$ such that $B(Id, r) \cap \Gamma = \{Id\}$.

Therefore for all sufficiently large k, we must have $g_k g_{k+1}^{-1} = Id$.

Therefore for all sufficiently large k $g_k = g_{k+1}$ and so $g \in \Gamma$, since $g_k \to g$ and the sequence g_k is eventually constant.

Note that subgroups of $PSL(2,\mathbb{R})$ need not be closed, for example $PSL(2,\mathbb{Q})$.

Also, discrete sets need not be closed, for example $\{1/n : n \in \mathbb{Z}\}$.

3 Let (X,d) be a metric space, $K \subseteq X$ be compact & $P \subseteq X$ discrete and closed.

Suppose $K \cap P$ is infinite. We can therefore find a sequence x_k of distinct points with $x_k \in K \cap P$ for all k. Since K is compact, we can find a convergent subsequence $x_{n_k} \to x \in K$. Since P is closed $x \in P$. Therefore

$$B(x, r) \cap P$$

is infinite for all $r > 0$, which contradicts P discrete.

If we do not assume P closed, then $X = \mathbb{R}$, $K = [0,1]$ and $P = \{ \frac{1}{n} : n \in \mathbb{Z}^+ \}$ provides a counter example.

4. Suppose $K \subseteq \mathbb{H}^2$ is bounded. Then $K \subseteq B(z_0, r)$ for some $z_0 \in \mathbb{H}^2$, $r > 0$.

Let $C_1 = \sup\{|z| : z \in B(z_0, r)\} < \infty$

and $C_2 = \inf\{\operatorname{Im}(z) : z \in B(z_0, r)\} > 0$.

Setting $C = \max\{C_1, 1/C_2\}$ does the job.

In the other direction, suppose $K \subseteq \mathbb{H}^2$ is such that for some $C > 1$

(i) $|z| \leq C$, (ii) $\operatorname{Im}(z) > \frac{1}{C}$

for all $z \in K$.

Let $r = \sup\{d_{\mathbb{H}^2}(i, z) : z \in K\} < \infty$.

Then $K \subseteq B(i, r)$ and is therefore bounded as required.

5 ($\Rightarrow$) Suppose $\Gamma \leq PSL(2,\mathbb{R})$ is a finite Fuchsian group. Then for $z \in \mathbb{H}^2$ we have:

$$|\Gamma(z)| = \left|\{ g(z) : g \in \Gamma \}\right| \leq |\Gamma| < \infty .$$

($\Leftarrow$) Suppose $\Gamma \leq PSL(2,\mathbb{R})$ is a Fuchsian group such that for all $z \in \mathbb{H}^2$ the set $\Gamma(z)$ is finite.

Fix $z \neq i$ & supose Γ is infinite. Therefore $g(z) = w$ for some $w \in \mathbb{H}^2$ & infinitely many distinct $g \in \Gamma$. Consider the orbit of i and note that

Since Γ acts properly discontinously

$$\Gamma(i) \cap B_{\mathbb{H}^2}(w, d_{\mathbb{H}^2}(i,z))$$

$\leftarrow$ compact ball

is finite. Since Γ acts by isometry, for every $g \in \Gamma$ with $g(z) = w$, we have $g(i) \in B_{\mathbb{H}^2}(w, d_{\mathbb{H}^2}(i,z))$

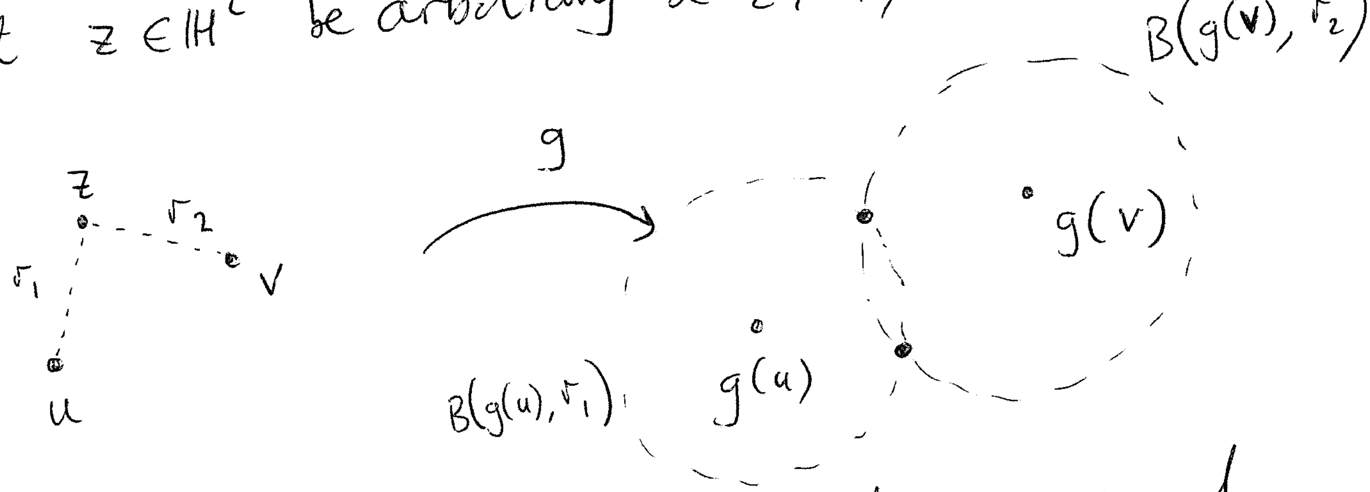
Therefore we can find infinitely many distinct $g \in PSL(2,\mathbb{R})$ such that

$$g(z) = w \quad \text{and} \quad g(i) = w'$$

(since otherwise $\Gamma(i) \cap B_{\mathbb{H}^2}(w, d_{\mathbb{H}^2}(i,z))$ cannot be finite).

This is a contradiction since $g \in PSL(2,\mathbb{R})$ is determined uniquely by its action on 2 distinct points.

(To see this, suppose $g \in PSL(2,\mathbb{R})$ is such that $g(u)$ and $g(v)$ are given, for $u \neq v$.
Let $z \in \mathbb{H}^2$ be arbitrary & $z \neq u$, $z \neq v$.



Then since g is an isometry, there are only 2 possible locations for $g(z)$. However one of these choices is orientation reversing.)

⑥ ($\Rightarrow$) Suppose $\Gamma \leq PSL(2,\mathbb{R})$ is Fuchsian, let $z \in \mathbb{H}^2$ be arbitrary and fix $w \in \Gamma(z)$. Since Γ acts properly discontinuously $\Gamma(z) \cap B_{\mathbb{H}^2}(w, \frac{1}{2})$ is finite. Let

$$r = \frac{1}{2} \min \left\{ d_{\mathbb{H}^2}(w,v) : \begin{matrix} v \neq w \\ v \in B_{\mathbb{H}^2}(w,\frac{1}{2}) \cap \Gamma(z) \end{matrix} \right\}$$

> 0 (think about why this is strictly positive).

Hence $B_{\mathbb{H}^2}(w,r) \cap \Gamma(z) = \{w\}$ and we may conclude $\Gamma(w)$ is discrete.

($\Leftarrow$) Suppose $\Gamma \leq PSL(2,\mathbb{R})$ is such that $\Gamma(z)$ is discrete for all $z \in \mathbb{H}^2$. Suppose Γ is <u>not</u> discrete and so we can find $g_n \in \Gamma$ such that $g_n \neq g$ $\forall n$ and $g_n \to g$ (in $\mathbb{R}^4$ metric). Consider $\Gamma(i)$ and $\Gamma(2i)$, which are both discrete. Since $g_n(i) \to g(i)$ & $g_n(2i) \to g(2i)$ (in $d_{\mathbb{H}^2}$ & $|\cdot|$) we conclude that for large enough n $g_n(i) = g(i)$ & $g_n(2i) = g(2i)$. Since $g \in PSL(2,\mathbb{R})$ is determined by its action on any two points we conclude $g_n = g$ for large n. A contradiction.