Chapter 4

Sturm-Liouville problems

4.1 Motivating example

Consider a streched string between two end-points at $x = 0$ and $x = 1$. The motion of free vibrations of the string is captured to good approximation by the partial differential equation

$$c^2 u_{xx} = u_{tt} \tag{4.1}$$

where $u(x,t)$ represents small displacements of the string from its resting position, and where c is a constant. Since the end points are fixed we have $u(a,t) = u(b,t) = 0$ for all times t. Without loss of generality we can take $c = 1$. Equation (4.1) is *separable* in the sense that we can write $u(x,t) = X(x)T(t)$, giving

$$X''T = XT''$$

so

$$X''/X = T''/T.$$

Since the left-hand side is a function of x only and the right-hand side is a function of t only, both sides must be constant, and so

$$X''/X = T''/T = -\lambda,$$

where λ is a *separation constant*. The time equation becomes

$$T'' + \lambda T = 0,$$

with solutions $T = e^{\pm\sqrt{-\lambda}t}$, and to avoid solutions that grow or decay exponentially, we must take $\lambda > 0$, giving oscillatory solutions $T = e^{\pm i\sqrt{\lambda}t}$.

The X equation is then

$$X'' + \lambda X = 0 \tag{4.2}$$

with solution

$$X = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x.$$

Since $u(0,t) = u(1,t) = 0$ for all t, we must have $X(0) = X(1) = 0$, which then imply

$$A = 0 \qquad \text{and} \qquad B\sin\sqrt{\lambda} = 0.$$

Nontrivial solutions with $B \neq 0$ are then obtained only for the special values of λ for which $\sin\sqrt{\lambda} = 0$, namely for

$$\sqrt{\lambda} = \pi, 2\pi, 3\pi, \ldots$$

that is,

$$\lambda = \lambda_n = n^2\pi^2, \qquad n = 1, 2, \ldots.$$

The corresponding solutions are

$$X_n(x) = B \sin n\pi x, \qquad n = 1, 2, \ldots.$$

These are orthogonal in the sense that

$$\int_0^1 X_n(x) X_m(x) dx = 0 \qquad \text{for } n \neq m.$$

Equation (4.2) may be written in the form

$$L[X] = \lambda X \tag{4.3}$$

where the linear differential operator L is defined by $L = -\frac{d^2}{dx^2}$. Values of λ for which there are nontrivial solutions to (4.3) (together with the relevant boundary conditions) are called *eigenvalues*; the corresponding solutions are called *eigenfunctions*. Note that in our example $\lambda = 0$ is not an eigenvalue since it only gives rise to the trivial solution, $X(x) \equiv 0$. The eigenvalues and eigenfunctions in our example have the following properties:

1. There are infinitely many discrete eigenvalues λ_n, $n = 1, 2, \ldots$
2. The corresponding eigenfunctions $X_n(x)$ are unique and orthogonal on $[a, b]$.
3. The eigenfunctions are *complete* and form a basis for the infinite dimensional vector space of twice-differentiable functions satisfying the boundary conditions (the vector space on which L acts). In our example, the representation of any function in terms of this basis is called a Fourier series.
4. The eigenvalues grow asymptotically like n^2 for large n.

These properties turn out to be relevant to a much wider class of second order boundary value problems that we will explore in the rest of this chapter.

4.2 The Sturm-Liouville problem

A *Sturm-Liouville* problem is an eigenvalue problem for a self-adjoint operator, L.

We will consider functions $y(x)$ twice differentiable on an interval $[a, b]$ and a second order linear differential operator, L, of the form

$$L = -\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x) \tag{4.4}$$

with $p > 0$ on $[a, b]$. Whether or not this operator is self-adjoint will depend on the details of the boundary conditions at $x = a$ and $x = b$, which we consider in more detail in sections 4.3 and 4.5 below. Note that any second order linear differential operator may be written in the form (4.4). Suppose we have

$$L_2 = a_2(x)\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_0(x)$$

and want to solve $L_2[y] = f(x)$. Multiplying through by an integrating factor $\mu(x)$ gives

$$\mu(x)a_2(x)y'' + \mu(x)a_1(x)y' + \mu(x)a_0(x)y = \mu(x)f(x).$$

This is equivalent to the problem

$$-(p(x)y')' + q(x)y = g(x)$$

if we identify

$$\begin{aligned}
\mu(x)a_2(x) &= -p(x) \\
\mu(x)a_1(x) &= -p'(x) \\
\mu(x)a_0(x) &= q(x) \\
\mu(x)f(x) &= g(x)
\end{aligned}$$

The top two equations give $p'/p = a_1/a_2$ which integrates to

$$\log p = \int \frac{a_1}{a_2} dx$$

or

$$p = e^{\int \frac{a_1}{a_2} dx} > 0$$

from which μ can be found, and hence $q(x)$ and $g(x)$.

The Sturm-Liouville problem is to obtain the solutions to the eigenvalue equation

$$L[y] = \lambda w(x)y \tag{4.5}$$

where L is of the form (4.4), where $w(x)$ is a given *weight function* with $w(x) > 0$ on $[a, b]$, and where $y(x)$ is subject to appropriate boundary conditions at $x = a$ and $x = b$. The real number λ is an eigenvalue of the operator L, and nontrivial solutions $y(x)$ are eigenfunctions. The eigenvalue problem (4.5), with appropriate boundary conditions, has the following properties, which should be compared with those of the example in section 4.1:

1. There exist nontrivial solutions of (4.5) (with boundary conditions) if and only if λ is an eigenvalue of the operator L; if L is regular then the eigenvalues are discrete and may be ordered

$$\lambda_1 < \lambda_2 < \lambda_3 < \dots$$

2. The corresponding eigenfunctions $y_n(x)$ are unique and orthogonal on $[a, b]$ with respect to the weight function $w(x)$:

$$\int_a^b w(x)y_n(x)y_m(x)dx = 0 \quad \text{for } n \neq m.$$

3. The eigenfunctions are *complete* and form a basis for the infinite dimensional vector space of twice-differentiable functions satisfying the boundary conditions.

4. The eigenvalues grow asymptotically like n^2 for large n.

In the example of the previous section, our simple boundary value problem is a Sturm-Liouville problem, with operator $L = -\frac{d^2}{dx^2}$ (so $p(x) = 1$ and $q(x) = 0$), boundary conditions $y(0) = y(1) = 0$, and eigenvalues $\lambda_n = n^2\pi^2$ and eigenfunctions $y_n(x) = \sin n\pi x$, for $n = 1, 2, \ldots$.

Example 4.2 Write the following equations in the form (4.5):

$$\begin{aligned} y'' - 2xy' + \lambda y =& 0, && \text{(Hermite's equation)} \\ xy'' + (1-x)y' + \lambda y =& 0, && \text{(Laguerre's equation)} \\ (1-x^2)y'' - xy' + \alpha^2 y =& 0, && \text{(Chebyshev's equation)} \end{aligned}$$

4.3 Boundary conditions

We require the boundary conditions to be such that

$$\int_a^b (y_1 L[y_2] - y_2 L[y_1])\, dx = 0, \tag{4.6}$$

is satisfied for all functions y_1 and y_2 in the domain of L. Using (4.4), we have

$$\begin{aligned} \int_a^b (y_1 L[y_2] - y_2 L[y_1])\, dx =& \int_a^b \big(y_1(-(py_2')' + qy_2) - y_2(-(py_1')' + qy_1)\big)\, dx \\ =& -\int_a^b y_1(py_2')'dx + \int_a^b y_2(py_1')'dx \\ =& -\big[y_1py_2'\big]_a^b + \int_a^b y_1'py_2'dx + \big[y_2py_1'\big]_a^b - \int_a^b y_2'py_1'dx \\ =& -\big[p(y_1y_2' - y_1'y_2)\big]_a^b. \end{aligned} \tag{4.7}$$

This relation is known as Lagrange's identity. The requirement (4.6) is therefore equivalent to the requirement that

$$[p(y_1y_2' - y_1'y_2)]_a^b = 0. \tag{4.8}$$

This can acheived in one of three ways:

(i) Regular unmixed boundary conditions:

$$\begin{aligned} & y_1(a)y_2'(a) - y_1'(a)y_2(a) = 0 \\ \text{and} \quad & y_1(b)y_2'(b) - y_1'(b)y_2(b) = 0 \end{aligned}$$

These are equivalent to the conditions:

$$\begin{aligned} & \alpha_1 y(a) + \alpha_2 y'(a) = 0 \\ \text{and} \quad & \beta_1 y(b) + \beta_2 y'(b) = 0 \end{aligned}$$

with at least one of α_1, α_2 and one of β_1, β_2 nonzero.

(ii) Singular boundary conditions: $p(a) = p(b) = 0$. In this case, we relax the condition $p > 0$ on $[a, b]$ to $p > 0$ on (a, b). At singular boundary points we typically require y to be bounded.

(iii) Periodic boundary conditions: $y(a) = y(b)$ and $y'(a) = y'(b)$.

We may have situations with a regular boundary condition at one boundary point and a singular boundary condition at the other. In this course, we will not consider case (iii); the case may arise, for example, in problems of electron motion in a regular crystal lattice.

Problems also arise in which either $a = -\infty$ or $b = \infty$, or both.

The problem in section 4.1 had regular boundary conditions of the simplest form $y(a) = y(b) = 0$. We will also encounter boundary conditions like $y'(a) = y'(b) = 0$, $y(a) + y'(a) = y(b) = 0$, etc.

When the boundary conditions are such that (4.8) holds, we have that

$$\int_a^b y_1 L[y_2] dx = \int_a^b y_2 L[y_1] dx$$

and we say that the operator L is self-adjoint. The integrals are examples of an inner product on an infinite dimensional vector space, and we will return to this idea in section 4.5

Example 4.3 Boundary conditions for Hermite, Laguerre, Chebyshev.

4.4 Properties of the eigenvalues and eigenfunctions

We consider the Sturm-Liouville problem (4.5) with boundary conditions as in section 4.3, such that (4.8), and hence (4.6), holds.

Theorem 4.1: The eigenvalues λ for which there are nontrivial solutions of (4.5) are real.

Proof. Let y satisfy $L[y] = \lambda w y$. Then, since p, q, w are all real, the complex conjugate, y^* satisfies $L[y^*] = \lambda^* w y^*$. Then, by (4.6) applied to y and y^*,

$$\begin{aligned} 0 &= \int_a^b (y^* L[y] - y L[y^*]) dx \\ &= \int_a^b (y^* \lambda w y - y \lambda^* w y^*) dx \\ &= (\lambda - \lambda^*) \int_a^b w |y|^2 dx. \end{aligned}$$

Since $w > 0$ on $[a, b]$, the integral is positive and so $\lambda = \lambda^*$, i.e. $\lambda \in \mathbb{R}$. □

Theorem 4.2: The eigenfunction y_n and y_m corresponding to distinct eigenvalues $\lambda_n \neq \lambda_m$ are orthogonal with respect to the weight $w(x)$.

Proof. Let y_n and y_m satisfy $L[y_n] = \lambda_n w y_n$ and $L[y_m] = \lambda_m w y_m$. Then, by (4.6) applied to y_n and y_m,

$$\begin{aligned} 0 &= \int_a^b (y_n L[y_m] - y_m L[y_n]) dx \\ &= \int_a^b (y_n \lambda_m w y_m - y_m \lambda_n w y_n) dx \\ &= (\lambda_n - \lambda_m) \int_a^b w y_n y_m dx. \end{aligned}$$

Since the eigenvalues are distinct, we have that

$$\int_a^b w y_n y_m dx = 0$$

□

Example 4.4 Orthogonality of Chebyshev polynomials.

4.5 Vector space formulation

We consider $f, g \in C^2[a, b]$ satisfying boundary conditions at $x = a$ and $x = b$ as elements of a (infinite-dimensional) vector space V, and define an *inner product* on V by

$$(f, g) = \int_a^b w(x)f(x)g(x)dx \tag{4.9}$$

for all $f, g \in V$. If $y_n(x), n = 1, 2, \ldots$ are solutions to $L[y_n] = \lambda_n w(x) y_n$ and the boundary conditions with corresponding eigenvalues λ_n, then we have seen that the y_n are orthogonal with respect to this inner product:

$$(y_n, y_m) = 0 \qquad \text{for } n \neq m.$$

We can normalize these by defining

$$\phi_n = y_n/(y_n, y_n)^{1/2}$$

so that

$$(\phi_n, \phi_m) = \delta_{nm}$$

where δ_{nm} is the Kronecker delta: $\delta_{nm} = 1$ if $n = m$ and $\delta_{nm} = 0$ if $n \neq m$. The functions ϕ_n are *orthonormal* with respect to this inner product.

If V is also spanned by the y_n then the y_n, or equivalently the ϕ_n, form a basis for V and the vector space V is said to be complete. This turns out to be the case for all regular Sturm-Liouville problems (this will be demonstrated in MT3506). It means that any (square integrable) $f \in V$ can be expressed as an infinite sum of the orthonormal basis functions $\phi_n(x)$:

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x).$$

The coefficients c_n are computed by taking the inner product with respect to ϕ_m:

$$\begin{aligned}
(\phi_m, f) &= \sum_{n=1}^{\infty} c_n (\phi_m, \phi_n) \qquad \text{by linearity of } (,) \\
&= \sum_{n=1}^{\infty} c_n \delta_{nm} \\
&= c_m.
\end{aligned}$$

So given an f, we can write

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x),$$

where

$$c_m = (\phi_m, f) = \int_a^b w(x)\phi_n(x)f(x)dx.$$

In the case where ϕ_n are sines or cosines and $w(x) = 1$, this is the usual Fourier series representation for $f(x)$. If we substitute the expression for the c_m back into the series for $f(x)$ we obtain:

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \int_a^b w(s)\phi_n(s)f(s)ds\phi_n(x) \\ &= \int_a^b f(s)\left\{ w(s)\sum_{n=1}^{\infty} \phi_n(x)\phi_n(s) \right\} ds. \end{aligned}$$

By comparing this with definition (ii) from section 3.4 we can identify,

$$w(s)\sum_{n=1}^{\infty} \phi_n(x)\phi_n(s) = \delta(x-s). \tag{4.10}$$

Alternatively, we can say that if $\delta(x-s)$ has an eigenfunction expansion for $a < s < b$, that is if

$$\delta(x-s) = \sum_{n=1}^{\infty} c_n\phi_n(x),$$

then

$$c_n = \int_a^b w(x)y_n(x)\delta(x-s) \tag{4.11}$$

$$= w(s)y_n(s) \tag{4.12}$$

giving (4.10) again.

Note on adjointness

Usual definition of a self-adjoint operator on an inner product space is that A is self-adjoint if

$$(f, Ag) = (Af, g)$$

for all $f, g \in V$. The natural inner product for the Sturm-Liouville problem is (4.9). But we, and Sturm-Liouville theory in general, call L self-adjoint if

$$\int_a^b y_1 L[y_2]dx = \int_a^b L[y_1]y_2 dx. \tag{4.13}$$

This is only the same as

$$(y_1, L[y_2]) = (L[y_1], y_2)$$

for the special case of $w(x) = 1$. We will basically ignore this distinction and call L self adjoint when it satisfies (4.13).

Example 4.5 Orthonormality and completeness: $\sin n\pi x$ on $[0, 1]$.

4.6 The inhomogeneous problem: Green's function, part 2.

Consider the Sturm-Liouville problem: $L[y] = \lambda w y$, with boundary conditions, and suppose we have found the eigenvalues λ_n and orthonormal eigenfunctions ϕ_n, with $(\phi_n, \phi_m) = \delta_{nm}$.

Now suppose we want to find the solution to the inhomogeneous problem

$$(L - \lambda w)[y] = f(x) \tag{4.14}$$

with y subject to the same boundary conditions, where here λ is some fixed real number, and $f(x)$ some function given on $[a, b]$. To solve this we basically write both sides of (4.14) as a series expansion in terms of the functions ϕ_n. First define $h(x) = f(x)/w(x)$, which is always possible since $w(x) > 0$, and expand:

$$h(x) = \sum_{n=1}^{\infty} h_n \phi_n(x)$$

where the coefficients h_n are given by $h_n = (\phi_n, h)$, as described in the previous section. We seek solutions to (4.14) in the form

$$y(x) = \sum_{n=1}^{\infty} a_n \phi_n(x)$$

for some coefficients a_n to be determined. The left-hand side of (4.14) becomes

$$\begin{aligned}(L - \lambda w)y &= (L - \lambda w) \sum_{n=1}^{\infty} a_n \phi_n \\ &= \sum_{n=1}^{\infty} a_n (L - \lambda w)\phi_n \\ &= \sum_{n=1}^{\infty} a_n (\lambda_n w - \lambda w)\phi_n \\ &= \sum_{n=1}^{\infty} a_n (\lambda_n - \lambda) w \phi_n.\end{aligned}$$

The right-hand side of (4.14) is

$$f(x) = w \sum_{n=1}^{\infty} h_n \phi_n.$$

Equating these gives

$$\sum_{n=1}^{\infty} a_n (\lambda_n - \lambda) w \phi_n = \sum_{n=1}^{\infty} h_n w \phi_n.$$

Now take the inner product with ϕ_m (i.e. multiply by $\phi_m(x)$ and integrate over $[a, b]$):

$$\begin{aligned}\sum_{n=1}^{\infty} a_n (\lambda_n - \lambda)(\phi_m, \phi_n) &= \sum_{n=1}^{\infty} h_n (\phi_m, \phi_n) \\ \implies a_m(\lambda_m - \lambda) &= h_m \qquad m = 1, 2, \ldots\end{aligned}$$

since $(\phi_m, \phi_n) = \delta_{mn}$. Therefore, if $\lambda \neq \lambda_n$, for any n, we have

$$a_n = \frac{h_n}{\lambda_n - \lambda}$$

and the solution is

$$y(x) = \sum_{n=1}^{\infty} \frac{h_n}{\lambda_n - \lambda} \phi_n.$$

Finally, writing out the expression for the coefficients h_n,

$$\begin{aligned} h_n = (\phi_n, h) &= \int_a^b w(s)\phi_n(s)h(s)ds \\ &= \int_a^b f(s)\phi_n(s)ds, \end{aligned}$$

we obtain

$$\begin{aligned} y(x) &= \sum_{n=1}^{\infty} \frac{1}{\lambda_n - \lambda} \int_a^b f(s)\phi_n(s)ds\phi_n(x) \\ &= \int_a^b \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(s)}{\lambda_n - \lambda} f(s)ds. \end{aligned}$$

We therefore identify the Green's function:

$$G(x,s) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(s)}{\lambda_n - \lambda}$$

so that the solution is given, as in section 3.2, by

$$y(x) = \int_a^b G(x,s)f(s)ds.$$

From section 3.3, we know that this Green's function should satisfy

$$(L_x - \lambda w(x))[G(x,s)] = \delta(x-s)$$

together with the appropriate boundary conditions. Here we have,

$$\begin{aligned} (L_x - \lambda w(x))[G(x,s)] =& (L_x - \lambda w(x)) \left[\sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(s)}{\lambda_n - \lambda} \right] \\ =& \sum_{n=1}^{\infty} \frac{(L_x - \lambda w(x))[\phi_n(x)]\phi_n(s)}{\lambda_n - \lambda} \\ =& \sum_{n=1}^{\infty} \frac{(\lambda_n w - \lambda w)\phi_n(x)\phi_n(s)}{\lambda_n - \lambda} \\ =& \sum_{n=1}^{\infty} w(x)\phi_n(x)\phi_n(s) \\ =& \delta(x-s) \end{aligned}$$

as required.

(Note that since $w(s)$ is just a scale factor and that $\delta(x-s)$ is only non-zero when $x = s$, we can replace $w(s)$ with $w(x)$ in the eigenfunction expansion for $\delta(x-s)$.)

Example 4.6 Green's function for $-y'' - \lambda y = -g(x)$; relation to chapter 3.

4.7 Solubility

We have made various remarks about boundary value problems that may have no or infinitely many solutions, and mentioned some conditions on the boundary values that must be satisfied (e.g., section 3.2) for solutions to be obtained. We close this chapter with a few more definite statements.

In the previous section, we saw that the solution to the inhomogeneous equation, $(L-\lambda w)[y] = f(x)$, can be expressed as an expansion of basis functions $\phi_n(x)$:

$$y(x) = \sum_{n=1}^{\infty} a_n \phi_n(x)$$

with the a_n given by

$$a_n = \frac{h_n}{\lambda_n - \lambda}$$

provided $\lambda \neq \lambda_n$ for any n, that is, provided λ is not an eigenvalue of the homogeneous Sturm-Liouville problem $(L - \lambda w)[y] = 0$. If λ is equal to an eigenvalue, say $\lambda = \lambda_k$ for some k, then we must have instead that the coefficient $h_k = 0$ and in that case a_k is undefined. The condition $h_k = 0$ means that $h(x)$ is orthogonal to $\phi_k(x)$, since

$$\begin{aligned}
(h, \phi_k) &= \left(\sum_{n=1}^{\infty} h_n \phi_n, \phi_k \right) \\
&= \sum_{n=1}^{\infty} h_n (\phi_n, \phi_k) \\
&= h_k \\
&= 0
\end{aligned}$$

Therefore, since $f = wh$ we must have that

$$\int_a^b f(x)\phi_k(x)dx = 0. \tag{4.15}$$

In summary, if λ is an eigenvalue of the homogenuous problem, $(L - \lambda w)[y] = 0$, then the inhomogeneous problem $(L - \lambda w)[y] = f(x)$, can only have solutions if $f(x)$ is such that condition (4.15) holds.

If this condition does hold then there are infinitely many solutions to the inhomogeneous equation. To see this, note that if y is a solution to $(L-\lambda w)[y] = f(x)$, and since ϕ_k is a solution to $(L-\lambda w)[y] = 0$, we have that

$$\begin{aligned}
(L - \lambda w)[y + A\phi_k] &= (L - \lambda w)[y] + (L - \lambda w)[A\phi_k] \\
&= f(x) + 0
\end{aligned}$$

and so $y + A\phi_k$ is a solution to $(L - \lambda w)[y] = f(x)$ for any constant A.

The two possibilities λ equal or not equal to an eigenvalue give rise to the two possibilities of an important result known as the **Fredholm Alternative**, which states that *either*

(A) the homogeneous equation $(L - \lambda w)y = 0$ has non-trivial solutions; *or*

(B) there is a unique solution to the inhomogeneous equation $(L - \lambda w)y = f(x)$.

Case (A) corresponds to the case where λ is an eigenvalue, and hence there are nontrivial solutions to the homogeneouse equation (the eigenfunctions). Case (B) corresponds to the case where λ is not an eigenvalue, and the solution to the inhomogeneous equation is as given in section 4.6. As we have just seen, in the case (A) there are either no or infinitely many solutions to the inhomogeneous equation depending on whether condition (4.15) holds.

Example 4.7: $y'' + y = g(x)$, $y(0) = y(\pi) = 0$