Section 2

Linear transformations (Revision)

Definition and basic properties

Linear transformations are functions between vector spaces that interact well with the vector space structure and probably the most important thing we study in linear algebra.

Definition 2.1 Let V and W be vector spaces over the same field F. A *linear mapping* (also called a *linear transformation*) from V to W is a function $T \colon V \to W$ such that

(i) $T(u + v) = T(u) + T(v)$ for all $u, v \in V$, and

(ii) $T(\alpha v) = \alpha T(v)$ for all $v \in V$ and $\alpha \in F$.

Comment: Sometimes we shall write Tv for the image of the vector v under the linear transformation T (instead of $T(v)$). We shall particularly do this when v is a column vector, so already possesses its own pair of brackets already.

Linear transformations were discussed in great detail during the MT2501 module. We recall below some of these facts but omit the proofs in the lectures. (The proofs do appear in the lecture notes however.)

Lemma 2.2 *Let $T \colon V \to W$ be a linear mapping between two vector spaces over the field F. Then*

(i) $T(\mathbf{0}) = \mathbf{0}$*;*

(ii) $T(-v) = -T(v)$ *for all $v \in V$;*

(iii) *if* $v_1, v_2, \dots, v_k \in V$ *and* $\alpha_1, \alpha_2, \dots, \alpha_k \in F$, *then*

$$T\left(\sum_{i=1}^{k} \alpha_i v_i\right) = \sum_{i=1}^{k} \alpha_i T(v_i).$$

PROOF: [Omitted in lectures — appears in MT2501]

(i) Since $0 \cdot \mathbf{0} = \mathbf{0}$, we have

$$T(\mathbf{0}) = T(0 \cdot \mathbf{0}) = 0 \cdot T(\mathbf{0}) = \mathbf{0}$$

(using Proposition 1.5(ii)).

(ii) Proposition 1.5(iv) tells us $(-1)v = -1v = -v$, so

$$T(-v) = T\left((-1)v\right) = (-1)T(v) = -T(v).$$

(iii) Using the two conditions of Definition 2.1, we see

$$T\left(\sum_{i=1}^{k} \alpha_i v_i\right) = \sum_{i=1}^{k} T(\alpha_i v_i) = \sum_{i=1}^{k} \alpha_i T(v_i).$$

□

Definition 2.3 Let $T \colon V \to W$ be a linear transformation between vector spaces over a field F.

(i) The *image* of T is

$$T(V) = \operatorname{im} T = \{\, T(v) \mid v \in V \,\}.$$

(ii) The *kernel* or *null space* of T is

$$\ker T = \{\, v \in V \mid T(v) = \mathbf{0}_W \,\}.$$

Note here that we are working with two vector spaces, each of which will possess its own zero vector. For emphasis in the definition, we are writing $\mathbf{0}_W$ for the zero vector belonging to the vector space W, so the kernel consists of those vectors in V which are mapped by T to the zero vector of W.

Of course, $T(v)$ has to be a vector in W, so actually there is little harm in writing simply $T(v) = \mathbf{0}$. For this equation to make any sense, the zero vector referred to must be that belonging to W, so confusion should not arise. Nevertheless to start with we shall write $\mathbf{0}_W$ just to be completely careful and clear.

Warning: The previous version of the MT3501 lecture course used $\mathcal{R}(T)$ to denote the image (and called it the range) and used $\mathcal{N}(T)$ to denote the null space. This will be observed when consulting previous exam papers and appropriate care should be taken.

Proposition 2.4 *Let $T \colon V \to W$ be a linear transformation between vector spaces V and W over the field F. The image and kernel of T are subspaces of W and V, respectively.*

Proof: [Omitted in lectures — appears in MT2501]

Certainly $\operatorname{im} T$ is non-empty since it contains all the images of vectors under the application of T. Let $x, y \in \operatorname{im} T$. Then $x = T(u)$ and $y = T(v)$ for some $u, v \in V$. Hence

$$x + y = T(u) + T(v) = T(u + v) \in \operatorname{im} T$$

and

$$\alpha x = \alpha T(v) = T(\alpha v) \in \operatorname{im} T$$

for any $\alpha \in F$. Hence $\operatorname{im} T$ is a subspace of W.

Note that $T(\mathbf{0}_V) = \mathbf{0}_W$, so we see $\mathbf{0}_V \in \ker T$. So to start with $\ker T$ is non-empty. Now let $u, v \in \ker T$. Then

$$T(u + v) = T(u) + T(v) = \mathbf{0}_W + \mathbf{0}_W = \mathbf{0}_W$$

and

$$T(\alpha v) = \alpha T(v) = \alpha \cdot \mathbf{0}_W = \mathbf{0}_W.$$

Hence $u + v, \alpha v \in \ker T$ (for all $\alpha \in F$) and we deduce that $\ker T$ is a subspace of V. $\square$

Definition 2.5 Let $T \colon V \to W$ be a linear transformation between vector spaces over the field F.

(i) The *rank* of T, which we shall denote $\operatorname{rank} T$, is the dimension of the image of T.

(ii) The *nullity* of T, which we shall denote $\operatorname{null} T$, is the dimension of the kernel of T.

Comment: The notations here are not uniformly established and I have simply selected a convenient notation rather than a definitive one. Many authors use different notation or, more often, no specific notation whatsoever for these two concepts.

Theorem 2.6 (Rank-Nullity Theorem) *Let V and W be vector spaces over the field F with V finite-dimensional and let $T \colon V \to W$ be a linear transformation. Then*

$$\operatorname{rank} T + \operatorname{null} T = \dim V.$$

Comment: [For those who have done MT2505] This can be viewed as an analogue of the First Isomorphism Theorem for groups within the world of vector spaces. Rearranging gives

$$\dim V - \dim \ker T = \dim \operatorname{im} T$$

and since (as we shall see in Problem Sheet II, Question 2) dimension essentially determines vector spaces we conclude

$$V/\ker T \cong \operatorname{im} T.$$

Of course, I have not specified what is meant by a quotients and isomorphism (and I will not address the former at all in this course), but this does give some context for the theorem.

PROOF: [Omitted in lectures — appears in MT2501]

Let $\mathscr{B} = \{v_1, v_2, \dots, v_n\}$ be a basis for $\ker T$ (so that $n = \operatorname{null} T$) and extend this (by Proposition 1.26) to a basis $\mathscr{C} = \{v_1, v_2, \dots, v_n, v_{n+1}, \dots, v_{n+k}\}$ for V (so that $\dim V = n + k$). We now seek to find a basis for $\operatorname{im} T$.

If $w \in \operatorname{im} T$, then $w = T(v)$ for some $v \in V$. We can write v as a linear combination of the vectors in the basis $\mathscr{C}$, say

$$v = \sum_{i=1}^{n+k} \alpha_i v_i.$$

Then, applying T and using linearity,

$$w = T(v) = T\left(\sum_{i=1}^{n+k} \alpha_i v_i\right) = \sum_{i=1}^{n+k} \alpha_i T(v_i) = \sum_{j=1}^{k} \alpha_{n+j} T(v_{n+j})$$

since $T(v_1) = \cdots = T(v_n) = \mathbf{0}$ as $v_1, \dots, v_n \in \ker T$. This shows that $\mathscr{D} = \{T(v_{n+1}), \dots, T(v_{n+k})\}$ spans $\operatorname{im} T$.

Now suppose that

$$\sum_{j=1}^{k} \beta_j T(v_{n+j}) = \mathbf{0};$$

that is,

$$T\left(\sum_{j=1}^{k} \beta_j v_{n+j}\right) = \mathbf{0}.$$

Hence $\sum_{j=1}^{k} \beta_j v_{n+j} \in \ker T$, so as $\mathscr{B} = \{v_1, \dots, v_n\}$ is a basis for $\ker T$, we have

$$\sum_{j=1}^{k} \beta_j v_{n+j} = \sum_{i=1}^{n} \gamma_i v_i$$

for some $\gamma_1, \gamma_2, \dots, \gamma_n \in F$. We now have an expression

$$(-\gamma_1)v_1 + \cdots + (-\gamma_n)v_n + \beta_1 v_{n+1} + \cdots + \beta_k v_{n+k} = \mathbf{0}$$

involving the vectors in the basis $\mathscr{C}$ for V. Since $\mathscr{C}$ is linearly independent, we conclude all the coefficients occurring here are zero. In particular, $\beta_1 = \beta_2 = \cdots = \beta_k = 0$. This shows that $\mathscr{D} = \{T(v_{n+1}), \dots, T(v_{n+k})\}$ is a linearly independent set and consequently a basis for $\operatorname{im} T$. Thus

$$\operatorname{rank}(T) = \dim \operatorname{im} T = k = \dim V - \operatorname{null} T$$

and this establishes the theorem. □

Constructing linear transformations

These have described the basic facts about linear transformations. When it comes to giving examples, it is possible to describe various different ones, some of which can seem natural, some more esoteric. Instead, what we shall do is to describe a standard method for defining linear transformations.

Let V and W be vector spaces over a field F. Suppose that V is finite-dimensional and that $\mathscr{B} = \{v_1, v_2, \dots, v_n\}$ is a basis for V. We shall define a linear transformation $T \colon V \to W$ by specifying its effect on each of the basis vectors v_i.

Pick any vectors $y_1, y_2, \dots, y_n \in W$. (These can be completely arbitrary, we do not need them to be linearly independent nor to span W; some can be the zero vector of W and we can even repeat the same vector over and over again if we want.) We intend to show that there is a linear transformation $T \colon V \to W$ satisfying $T(v_i) = y_i$ for $i = 1, 2, \dots, n$. If such a T does exist, consider the effect T has on an arbitrary vector v in V. Since $\mathscr{B}$ is a basis for V, the vector v can be *uniquely* expressed as

$$v = \sum_{i=1}^{n} \alpha_i v_i$$

for scalars α_i in F. Then linearity of T implies

$$T(v) = T\left(\sum_{i=1}^{n} \alpha_i v_i\right) = \sum_{i=1}^{n} \alpha_i T(v_i) = \sum_{i=1}^{n} \alpha_i y_i. \tag{2.1}$$

Hence T, if it exists, is uniquely specified by this formula (2.1).

We now claim that this formula does indeed define a linear transformation $T \colon V \to W$. If $u, v \in V$, say

$$u = \sum_{i=1}^{n} \alpha_i v_i \qquad \text{and} \qquad v = \sum_{i=1}^{n} \beta_i v_i$$

for some uniquely determined $\alpha_i, \beta_i \in F$. Then $u+v = \sum_{i=1}^{n}(\alpha_i+\beta_i)v_i$ and this must be the unique expression for $u+v$ in terms of the basis $\mathscr{B}$. So

$$T(u+v) = \sum_{i=1}^{n}(\alpha_i+\beta_i)y_i = \sum_{i=1}^{n}\alpha_i y_i + \sum_{i=1}^{n}\beta_i y_i = T(u)+T(v).$$

Similarly $\gamma u = \sum_{i=1}^{n}(\gamma\alpha_i)v_i$, so

$$T(\gamma u) = \sum_{i=1}^{n}(\gamma\alpha_i)y_i = \gamma\sum_{i=1}^{n}\alpha_i y_i = \gamma T(u) \qquad \text{for any } \gamma \in F.$$

This shows T is a linear transformation.

We have now established the following result:

Proposition 2.7 *Let V be a finite-dimensional vector space over the field F with basis $\{v_1, v_2, \dots, v_n\}$ and let W be any vector space over F. If y_1, y_2, $\dots$, y_n are arbitrary vectors in W, there is a* unique *linear transformation $T\colon V \to W$ such that*

$$T(v_i) = y_i \qquad \text{for } i = 1, 2, \dots, n.$$

□

Moreover, we have shown this transformation T is given by

$$T\left(\sum_{i=1}^{n}\alpha_i v_i\right) = \sum_{i=1}^{n}\alpha_i y_i$$

for an arbitrary linear combination $\sum_{i=1}^{n}\alpha_i v_i$ in V.

We now give an example of our method of creating linear transformations.

Example 2.8 *Define a linear transformation $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ in terms of the standard basis $\mathscr{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ by*

$$T(\mathbf{e}_1) = \boldsymbol{y}_1 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \qquad T(\mathbf{e}_2) = \boldsymbol{y}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},$$
$$T(\mathbf{e}_3) = \boldsymbol{y}_3 = \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}, \qquad T(\mathbf{e}_4) = \boldsymbol{y}_4 = \begin{pmatrix} -5 \\ -2 \\ -5 \end{pmatrix}.$$

Calculate the linear transformation T and its rank and nullity.

Solution: The effect of T on an arbitrary vector of $\mathbb{R}^4$ can be calculated by the linearity property:

$$\begin{aligned} T\begin{pmatrix}\alpha\\\beta\\\gamma\\\delta\end{pmatrix} &= T(\alpha\mathbf{e}_1+\beta\mathbf{e}_2+\gamma\mathbf{e}_3+\delta\mathbf{e}_4) \\ &= \alpha T(\mathbf{e}_1)+\beta T(\mathbf{e}_2)+\gamma T(\mathbf{e}_3)+\delta T(\mathbf{e}_4) \\ &= \alpha\begin{pmatrix}2\\1\\3\end{pmatrix}+\beta\begin{pmatrix}-1\\0\\1\end{pmatrix}+\gamma\begin{pmatrix}0\\1\\5\end{pmatrix}+\delta\begin{pmatrix}-5\\-2\\-5\end{pmatrix} \\ &= \begin{pmatrix}2\alpha-\beta-5\delta\\ \alpha+\gamma-2\delta\\ 3\alpha+\beta+5\gamma-5\delta\end{pmatrix}. \end{aligned}$$

[Exercise: Check by hand that this formula does really define a linear transformation $T\colon \mathbb{R}^4 \to \mathbb{R}^3$.]

Now let us determine the kernel of this transformation T. Suppose $v \in \ker T$. Here $\boldsymbol{v}$ is some vector in $\mathbb{R}^4$, say

$$\boldsymbol{v} = \begin{pmatrix}\alpha\\\beta\\\gamma\\\delta\end{pmatrix} = \alpha\mathbf{e}_1+\beta\mathbf{e}_2+\gamma\mathbf{e}_3+\delta\mathbf{e}_4$$

where

$$T(\boldsymbol{v}) = \begin{pmatrix}2\alpha-\beta-5\delta\\ \alpha+\gamma-2\delta\\ 3\alpha+\beta+5\gamma-5\delta\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

We have here three simultaneous equations in four variables which we convert to the matrix equation

$$\begin{pmatrix}2&-1&0&-5\\1&0&1&-2\\3&1&5&-5\end{pmatrix}\begin{pmatrix}\alpha\\\beta\\\gamma\\\delta\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

We solve this by performing the usual row operations used in Gaussian elimination [see MT1002]:

$$\begin{aligned} \left(\begin{array}{cccc|c}2&-1&0&-5&0\\1&0&1&-2&0\\3&1&5&-5&0\end{array}\right) &\longrightarrow \left(\begin{array}{cccc|c}1&0&1&-2&0\\2&-1&0&-5&0\\3&1&5&-5&0\end{array}\right) && (r_1 \leftrightarrow r_2) \\ &\longrightarrow \left(\begin{array}{cccc|c}1&0&1&-2&0\\0&-1&-2&-1&0\\0&1&2&1&0\end{array}\right) && \begin{array}{l}(r_2 \mapsto r_2-2r_1,\\ \ r_3 \mapsto r_3-3r_1)\end{array} \end{aligned}$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{array}{l} (r_3 \mapsto r_3 + r_2, \\ \ r_2 \mapsto -r_2) \end{array}$$

So given arbitrary γ and δ, we require

$$\alpha + \gamma - 2\delta = 0 \qquad \text{and} \qquad \beta + 2\gamma + \delta = 0.$$

We remain with two degrees of freedom (the free choice of γ and δ) and so $\ker T$ is 2-dimensional:

$$\begin{aligned} \ker T &= \left\{ \begin{pmatrix} -\gamma + 2\delta \\ -2\gamma - \delta \\ \gamma \\ \delta \end{pmatrix} \;\middle|\; \gamma, \delta \in \mathbb{R} \right\} \\ &= \left\{ \gamma \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \delta \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; \gamma, \delta \in \mathbb{R} \right\} \\ &= \operatorname{Span}\left(\begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right). \end{aligned}$$

It is easy to check these two spanning vectors are linearly independent, so $\operatorname{null} T = \dim \ker T = 2$. The Rank-Nullity Theorem then says

$$\operatorname{rank} T = \dim \mathbb{R}^4 - \operatorname{null} T = 4 - 2 = 2.$$

Essentially this boils down to the four image vectors $\boldsymbol{y}_1$, $\boldsymbol{y}_2$, $\boldsymbol{y}_3$, $\boldsymbol{y}_4$ spanning a 2-dimensional space. Indeed, note that they are not linearly independent because

$$\begin{aligned} \boldsymbol{y}_3 &= \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + 2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \boldsymbol{y}_1 + 2\boldsymbol{y}_2 \\ \boldsymbol{y}_4 &= \begin{pmatrix} -5 \\ -2 \\ -5 \end{pmatrix} = -2 \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = -2\boldsymbol{y}_1 + \boldsymbol{y}_2. \end{aligned}$$

The full explanation behind this lies in the following result. □

Proposition 2.9 *Let V be a finite-dimensional vector space over the field F with basis $\{v_1, v_2, \dots, v_n\}$ and let W be a vector space over F. Fix vectors y_1, y_2, …, y_n in W and let $T \colon V \to W$ be the unique linear transformation given by $T(v_i) = y_i$ for $i = 1$, 2, …, n. Then*

(i) $\operatorname{im} T = \operatorname{Span}(y_1, y_2, \dots, y_n)$.

(ii) $\ker T = \{\mathbf{0}\}$ *if and only if* $\{y_1, y_2, \dots, y_n\}$ *is a linearly independent set.*

PROOF: (i) If $x \in \operatorname{im} T$, then $x = T(v)$ for some $v \in V$. We can write $v = \sum_{i=1}^{n} \alpha_i v_i$ for some $\alpha_i \in F$. Then

$$x = T(v) = T\left(\sum_{i=1}^{n} \alpha_i v_i\right) = \sum_{i=1}^{n} \alpha_i T(v_i) = \sum_{i=1}^{n} \alpha_i y_i.$$

Thus, $\operatorname{im} T$ consists of all linear combinations of the vectors y_1, y_2, $\dots$, y_n; that is,

$$\operatorname{im} T = \operatorname{Span}(y_1, y_2, \dots, y_n).$$

(ii) Consider a vector $v = \sum_{i=1}^{n} \alpha_i v_i$ expressed in terms of the basis vectors of V. If v lies in $\ker T$, then $T(v) = \sum_{i=1}^{n} \alpha_i y_i$ equals $\mathbf{0}$. If the y_i are linearly independent, this forces $\alpha_i = 0$ for all i and we deduce $v = \mathbf{0}$. So linear independence of the w_i implies $\ker T = \{\mathbf{0}\}$.

Conversely, if $\ker T = \{\mathbf{0}\}$, consider an equation $\sum_{i=1}^{n} \alpha_i y_i = \mathbf{0}$ involving the y_i. Set $v = \sum_{i=1}^{n} \alpha_i v_i$. Our assumption forces $v \in \ker T$, so $v = \mathbf{0}$ by hypothesis. Thus $\sum_{i=1}^{n} \alpha_i v_i = \mathbf{0}$ and, since the v_i are linearly independent, we deduce $\alpha_i = 0$ for all i. Hence $\{y_1, y_2, \dots, y_n\}$ is linearly independent. $\square$

Comment: Part (ii) can also be deduced (pretty much immediately) from part (i) using the Rank-Nullity Theorem.

Example 2A *Define a linear transformation* $T \colon \mathbb{R}^3 \to \mathbb{R}^3$ *in terms of the standard basis* $\mathscr{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ *by*

$$T(\mathbf{e}_1) = \boldsymbol{y}_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \quad T(\mathbf{e}_2) = \boldsymbol{y}_2 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, \quad T(\mathbf{e}_3) = \boldsymbol{y}_3 = \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}.$$

Show that $\ker T = \{\mathbf{0}\}$ *and* $\operatorname{im} T = \mathbb{R}^3$*.*

SOLUTION: We check whether $\{\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3\}$ is linearly independent. Solve

$$\alpha \boldsymbol{y}_1 + \beta \boldsymbol{y}_2 + \gamma \boldsymbol{y}_3 = \mathbf{0};$$

that is,

$$\begin{aligned} 2\alpha - \beta \phantom{{}+4\gamma} &= 0 \\ \alpha \phantom{{}+2\beta} - \gamma &= 0 \\ -\alpha + 2\beta + 4\gamma &= 0. \end{aligned}$$

The second equation tells us that $\gamma = \alpha$ while the first says $\beta = 2\alpha$. Substituting for β and γ in the third equation gives

$$-\alpha + 4\alpha + 4\alpha = 7\alpha = 0.$$

Hence $\alpha = 0$ and consequently $\beta = \gamma = 0$.

This shows $\{\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3\}$ is linearly independent. Consequently, $\ker T = \{\mathbf{0}\}$ by Proposition 2.9. The Rank-Nullity Theorem now says

$$\dim \operatorname{im} T = \dim \mathbb{R}^3 - \dim \ker T = 3 - 0 = 3.$$

Therefore $\operatorname{im} T = \mathbb{R}^3$ as it has the same dimension.

[Alternatively, since $\dim \mathbb{R}^3 = 3$ and $\{\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3\}$ is linearly independent, this set must be a basis for $\mathbb{R}^3$ (see Corollary 1.27). Therefore, by Proposition 2.9(i),

$$\operatorname{im} T = \operatorname{Span}(\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3) = \mathbb{R}^3,$$

once again.] $\square$

Proposition 2.9(i) tells us that if $\{v_1, v_2, \dots, v_n\}$ is a basis for V and $T\colon V \to W$ is a linear transformation, then $\operatorname{im} T$ is spanned by the n vectors

$$y_i = T(v_i) \quad \text{for } i = 1, 2, \dots, n.$$

There is, however, no expectation that this set is a basis for $\operatorname{im} T$. (Indeed, it is only linearly independent when $\ker T = \{\mathbf{0}\}$ by part (ii) of the proposition.) In such a situation, we apply Theorem 1.19 to tell us that there is a basis

$$\mathscr{B} \subseteq \{y_1, y_2, \dots, y_n\}$$

for $\operatorname{im} T$ and the method presented in Example 1.25 can be used to find this basis.

Example 2.10 *Let $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation defined in terms of the standard basis $\mathscr{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ by*

$$T(\mathbf{e}_1) = \boldsymbol{y}_1 = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \qquad T(\mathbf{e}_2) = \boldsymbol{y}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$T(\mathbf{e}_3) = \boldsymbol{y}_3 = \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}, \qquad T(\mathbf{e}_4) = \boldsymbol{y}_4 = \begin{pmatrix} -5 \\ -2 \\ -5 \end{pmatrix}.$$

Find a basis for the image of T.

SOLUTION: This is the linear transformation considered in Example 2.8. We observed there that $\dim \operatorname{im} T = \operatorname{rank} T = 2$. We also know from Proposition 2.9 that

$$\operatorname{im} T = \operatorname{Span}(\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3, \boldsymbol{y}_4),$$

so we conclude that $\operatorname{im} T$ has a basis $\mathscr{C}$ containing 2 vectors and satisfying $\mathscr{C} \subseteq \{\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3, \boldsymbol{y}_4\}$. Note that

$$\{\boldsymbol{y}_1, \boldsymbol{y}_2\} = \left\{ \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is linearly independent. Indeed if

$$\alpha \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

then we deduce straightaway $\alpha = 0$ and then $\beta = 0$. We now have a linearly independent subset of $\operatorname{im} T$ of the right size to be a basis. Hence $\mathscr{C} = \{\boldsymbol{y}_1, \boldsymbol{y}_2\}$ is a basis for $\operatorname{im} T$. $\square$

The matrix of a linear transformation

We have attempted to describe an arbitrary linear transformation $T \colon V \to W$. Given a basis $\{v_1, v_2, \dots, v_n\}$ for V, we have observed that T is uniquely determined by specifying the images $T(v_1)$, $T(v_2)$, ..., $T(v_n)$ of the basis vectors. If we are also given a basis for W, we can then express these image vectors as a linear combination of the basis vectors of W and hence completely specify them.

Definition 2.11 Let V and W be finite-dimensional vector spaces over the field F and let $\mathscr{B} = \{v_1, v_2, \dots, v_n\}$ and $\mathscr{C} = \{w_1, w_2, \dots, w_m\}$ be bases for V and W, respectively. If $T \colon V \to W$ is a linear transformation, let

$$T(v_j) = \sum_{i=1}^{m} \alpha_{ij} w_i$$

express the image of the vector v_j under T as a linear combination of the basis $\mathscr{C}$ (for $j = 1, 2, \dots, n$). The $m \times n$ matrix $[\alpha_{ij}]$ is called the *matrix of T with respect to the bases $\mathscr{B}$ and $\mathscr{C}$*. We shall denote this by $\operatorname{Mat}(T)$ or, when we wish to be explicit about the dependence upon the bases $\mathscr{B}$ and $\mathscr{C}$, by $\operatorname{Mat}_{\mathscr{B},\mathscr{C}}(T)$.

In the special case of a linear transformation $T \colon V \to V$, we shall speak of the *matrix of T with respect to the basis $\mathscr{B}$* to mean $\operatorname{Mat}_{\mathscr{B},\mathscr{B}}(T)$.

Note that the entries of the jth column of the matrix of T are:

$$\begin{matrix} \alpha_{1j} \\ \alpha_{2j} \\ \vdots \\ \alpha_{mj} \end{matrix}$$

i.e., the jth column specifies the image of $T(v_j)$ by listing the coefficients when it is expressed as a linear combination of the vectors in $\mathscr{C}$.

It should be noted that the matrix of a linear transformation does very much depend upon the choices of bases. Accordingly it is much safer to employ the notation $\mathrm{Mat}_{\mathscr{B},\mathscr{C}}(T)$ and retain reference to the bases involved.

What does the matrix of a linear transformation actually represent? This question could be answered at great length and can get as complicated and subtle as one wants. The short answer is that if V and W are m and n-dimensional vector spaces over a field F, then they "look like" F^m and F^n (formally, are *isomorphic* to these spaces, see Problem Sheet II for details). Then T maps vectors from V into W in the same way that the matrix $\mathrm{Mat}(T)$ maps vectors from F^m into F^n. (There is a technical formulation of what "in the same way" means here, but that goes way beyond the requirements of this course. It will result in the kernels of the two linear maps being of the same dimension, similarly for the images, etc.)

Example 2.12 *Define a linear transformation* $T \colon \mathbb{R}^4 \to \mathbb{R}^4$ *by the following formula:*

$$T \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} x+4y \\ y \\ 2z+t \\ z+2t \end{pmatrix}.$$

Let $\mathscr{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ *denote the standard basis for* $\mathbb{R}^4$ *and let* $\mathscr{C}$ *be the basis*

$$\mathscr{C} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\} = \left\{ \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Determine the matrices $\mathrm{Mat}_{\mathscr{B},\mathscr{B}}(T)$, $\mathrm{Mat}_{\mathscr{C},\mathscr{B}}(T)$ *and* $\mathrm{Mat}_{\mathscr{C},\mathscr{C}}(T)$.

Solution: We calculate

$$T(\mathbf{e}_1) = T \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \mathbf{e}_1$$

$$T(\mathbf{e}_2) = T\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}4\\1\\0\\0\end{pmatrix} = 4\mathbf{e}_1 + \mathbf{e}_2$$

$$T(\mathbf{e}_3) = T\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\2\\1\end{pmatrix} = 2\mathbf{e}_3 + \mathbf{e}_4$$

$$T(\mathbf{e}_4) = T\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\\2\end{pmatrix} = \mathbf{e}_3 + 2\mathbf{e}_4.$$

So the matrix of T with respect to the basis $\mathscr{B}$ is

$$\mathrm{Mat}_{\mathscr{B},\mathscr{B}}(T) = \begin{pmatrix}1 & 4 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 2 & 1\\0 & 0 & 1 & 2\end{pmatrix}.$$

[We leave it as an exercise for the reader to check that $\mathscr{C}$ is indeed a basis for $\mathbb{R}^4$. Do this by showing it is linearly independent, i.e., the only solution to

$$\begin{pmatrix}2 & 0 & 0 & 3\\0 & 1 & 0 & 0\\2 & -1 & 1 & 0\\0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\alpha\\\beta\\\gamma\\\delta\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

is $\alpha = \beta = \gamma = \delta = 0$.]

We shall calculate the matrices $\mathrm{Mat}_{\mathscr{C},\mathscr{B}}(T)$ and $\mathrm{Mat}_{\mathscr{C},\mathscr{C}}(T)$.

$$T(\boldsymbol{v}_1) = T\begin{pmatrix}2\\0\\2\\0\end{pmatrix} = \begin{pmatrix}2\\0\\4\\2\end{pmatrix} = 2\mathbf{e}_1 + 4\mathbf{e}_3 + 2\mathbf{e}_4$$

$$T(\boldsymbol{v}_2) = T\begin{pmatrix}0\\1\\-1\\0\end{pmatrix} = \begin{pmatrix}4\\1\\-2\\-1\end{pmatrix} = 4\mathbf{e}_1 + \mathbf{e}_2 - 2\mathbf{e}_3 - \mathbf{e}_4$$

$$T(\boldsymbol{v}_3) = T\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\2\\1\end{pmatrix} = 2\mathbf{e}_3 + \mathbf{e}_4$$

$$T(\boldsymbol{v}_4) = T\begin{pmatrix}3\\0\\0\\1\end{pmatrix} = \begin{pmatrix}3\\0\\1\\2\end{pmatrix} = 3\mathbf{e}_1 + \mathbf{e}_3 + 2\mathbf{e}_4.$$

Hence

$$\mathrm{Mat}_{\mathscr{C},\mathscr{B}}(T) = \begin{pmatrix}2 & 4 & 0 & 3\\ 0 & 1 & 0 & 0\\ 4 & -2 & 2 & 1\\ 2 & -1 & 1 & 2\end{pmatrix}.$$

To find $\mathrm{Mat}_{\mathscr{C},\mathscr{C}}(T)$, we need to express each $T(\boldsymbol{v}_j)$ in terms of the basis $\mathscr{C}$.

$$\begin{aligned}
T(\boldsymbol{v}_1) = \begin{pmatrix}2\\0\\4\\2\end{pmatrix} &= -2\begin{pmatrix}2\\0\\2\\0\end{pmatrix} + 8\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + 2\begin{pmatrix}3\\0\\0\\1\end{pmatrix}\\
&= -2\boldsymbol{v}_1 + 8\boldsymbol{v}_3 + 2\boldsymbol{v}_4\\
T(\boldsymbol{v}_2) = \begin{pmatrix}4\\1\\-2\\-1\end{pmatrix} &= \frac{7}{2}\begin{pmatrix}2\\0\\2\\0\end{pmatrix} + \begin{pmatrix}0\\1\\-1\\0\end{pmatrix} - 8\begin{pmatrix}0\\0\\1\\0\end{pmatrix} - \begin{pmatrix}3\\0\\0\\1\end{pmatrix}\\
&= \tfrac{7}{2}\boldsymbol{v}_1 + \boldsymbol{v}_2 - 8\boldsymbol{v}_3 - \boldsymbol{v}_4\\
T(\boldsymbol{v}_3) = \begin{pmatrix}0\\0\\2\\1\end{pmatrix} &= -\frac{3}{2}\begin{pmatrix}2\\0\\2\\0\end{pmatrix} + 5\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + \begin{pmatrix}3\\0\\0\\1\end{pmatrix}\\
&= -\tfrac{3}{2}\boldsymbol{v}_1 + 5\boldsymbol{v}_3 + \boldsymbol{v}_4\\
T(\boldsymbol{v}_4) = \begin{pmatrix}3\\0\\1\\2\end{pmatrix} &= -\frac{3}{2}\begin{pmatrix}2\\0\\2\\0\end{pmatrix} + 4\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + 2\begin{pmatrix}3\\0\\0\\1\end{pmatrix}\\
&= -\tfrac{3}{2}\boldsymbol{v}_1 + 4\boldsymbol{v}_3 + 2\boldsymbol{v}_4.
\end{aligned}$$

Hence

$$\mathrm{Mat}_{\mathscr{C},\mathscr{C}}(T) = \begin{pmatrix}-2 & 3\frac{1}{2} & -1\frac{1}{2} & -1\frac{1}{2}\\ 0 & 1 & 0 & 0\\ 8 & -8 & 5 & 4\\ 2 & -1 & 1 & 2\end{pmatrix}.$$

□

Change of basis

Suppose we are given two bases $\mathscr{B}$ and $\mathscr{C}$ for the same vector space V. We shall now describe how $\mathrm{Mat}_{\mathscr{B},\mathscr{B}}(T)$ and $\mathrm{Mat}_{\mathscr{C},\mathscr{C}}(T)$ are related for some linear transformation $T \colon V \to V$. (A similar description can be given for a linear transformation $V \to W$ with two bases $\mathscr{B}, \mathscr{B}'$ for V and two bases $\mathscr{C}, \mathscr{C}'$ for W. This would be more complicated, but essentially the same ideas apply.) [The following discussion also appeared in MT2501. Only a brief summary will appear presented during lectures.]

Let $\mathscr{B} = \{v_1, v_2, \dots, v_n\}$ and $\mathscr{C} = \{w_1, w_2, \dots, w_n\}$ be our two bases for V. Since they are both bases, we can write a vector in one as a linear combination of the vectors in the other and *vice versa*. Say

$$w_j = \sum_{k=1}^{n} \lambda_{kj} v_k \tag{2.2}$$

and

$$v_\ell = \sum_{i=1}^{n} \mu_{i\ell} w_i \tag{2.3}$$

for some scalars $\lambda_{kj}, \mu_{i\ell} \in F$. Let $P = [\lambda_{ij}]$ be the matrix whose entries are the coefficients from (2.2). We call P the *change of basis matrix* from $\mathscr{B}$ to $\mathscr{C}$. Note that we write the coefficients appearing when w_j is expressed in terms of the basis $\mathscr{B}$ *down* the *jth column* of P. Similarly $Q = [\mu_{ij}]$ is the change of basis matrix from $\mathscr{C}$ to $\mathscr{B}$.

Let us substitute (2.2) into (2.3):

$$v_\ell = \sum_{i=1}^{n} \mu_{i\ell} \sum_{k=1}^{n} \lambda_{ki} v_k = \sum_{k=1}^{n} \left(\sum_{i=1}^{n} \lambda_{ki} \mu_{i\ell} \right) v_k.$$

This must be the unique way of writing v_ℓ as a linear combination of the vectors in $\mathscr{B} = \{v_1, v_2, \dots, v_n\}$. Thus

$$\sum_{i=1}^{n} \lambda_{ki} \mu_{i\ell} = \delta_{k\ell} = \begin{cases} 1 & \text{if } k = \ell \\ 0 & \text{if } k \neq \ell. \end{cases}$$

(This $\delta_{k\ell}$ is called the *Kronecker delta*.) The left-hand side is the formula for matrix multiplication, so

$$PQ = I,$$

the $n \times n$ identity matrix. Similarly, substituting (2.3) into (2.2) gives

$$QP = I.$$

We conclude that $Q = P^{-1}$.

Now suppose that $T\colon V \to V$ is a linear transformation whose matrix is $\mathrm{Mat}_{\mathscr{B},\mathscr{B}}(T) = A = [\alpha_{ij}]$. This means

$$T(v_j) = \sum_{i=1}^{n} \alpha_{ij} v_i \qquad \text{for } j = 1,\ 2,\ \dots,\ n. \tag{2.4}$$

To find $\mathrm{Mat}_{\mathscr{C},\mathscr{C}}(T)$, apply T to (2.2):

$$\begin{aligned}
T(w_j) &= T\biggl(\sum_{k=1}^{n} \lambda_{kj} v_k\biggr) \\
&= \sum_{k=1}^{n} \lambda_{kj} T(v_k) \\
&= \sum_{k=1}^{n} \lambda_{kj} \sum_{\ell=1}^{n} \alpha_{\ell k} v_\ell && \text{(from (2.4))} \\
&= \sum_{\ell=1}^{n} \sum_{k=1}^{n} \alpha_{\ell k} \lambda_{kj} \sum_{i=1}^{n} \mu_{i\ell} w_i && \text{(from (2.3))} \\
&= \sum_{i=1}^{n} \biggl(\sum_{\ell=1}^{n} \sum_{k=1}^{n} \mu_{i\ell} \alpha_{\ell k} \lambda_{kj}\biggr) w_i.
\end{aligned}$$

Hence $\mathrm{Mat}_{\mathscr{C},\mathscr{C}}(T) = B = [\beta_{ij}]$ where

$$\beta_{ij} = \sum_{\ell=1}^{n} \sum_{k=1}^{n} \mu_{i\ell} \alpha_{\ell k} \lambda_{kj};$$

that is,

$$B = QAP = P^{-1}AP.$$

We have proved:

Theorem 2.13 *Let V be a vector space of dimension n over a field F and let $T\colon V \to V$ be a linear transformation. Let $\mathscr{B} = \{v_1, v_2, \dots, v_n\}$ and $\mathscr{C} = \{w_1, w_2, \dots, w_n\}$ be bases for V and let A and B be the matrices of T with respect to $\mathscr{B}$ and $\mathscr{C}$, respectively. Then there is an invertible matrix P such that*

$$B = P^{-1}AP.$$

Specifically, the (i,j)th entry of P is the coefficient of v_i when w_j is expressed as a linear combination of the basis vectors in $\mathscr{B}$. □

Let us illustrate what we have just done with an example. This happens to be the first part of Question 1 on the January 2005 exam paper. It features a 2-dimensional vector space, principally chosen because calculating the inverse of a 2×2 matrix is much easier than doing so with one of larger dimension. However, for larger dimension exactly the same method should be used.

Example 2.14 *Let V be a 2-dimensional vector space over $\mathbb{R}$ with basis $\mathscr{B} = \{v_1, v_2\}$. Let*

$$w_1 = 3v_1 - 5v_2, \qquad w_2 = -v_1 + 2v_2 \tag{2.5}$$

and $\mathscr{C} = \{w_1, w_2\}$. Define the linear transformation $T\colon V \to V$ by

$$\begin{aligned} T(v_1) &= 16v_1 - 30v_2 \\ T(v_2) &= 9v_1 - 17v_2. \end{aligned}$$

Find the matrix $\operatorname{Mat}_{\mathscr{C},\mathscr{C}}(T)$.

Solution: The formula for T tells us that the matrix of T in terms of the basis $\mathscr{B}$ is

$$A = \operatorname{Mat}_{\mathscr{B},\mathscr{B}}(T) = \begin{pmatrix} 16 & 9 \\ -30 & -17 \end{pmatrix}.$$

The formula (2.5) expresses the w_j in terms of the v_i. Hence, our change of basis matrix is

$$P = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}.$$

Then

$$\det P = 3 \times 2 - (-1 \times -5) = 6 - 5 = 1,$$

so

$$P^{-1} = \frac{1}{\det P} \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}.$$

So

$$\begin{aligned} \operatorname{Mat}_{\mathscr{C},\mathscr{C}}(T) &= P^{-1}AP \\ &= \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 16 & 9 \\ -30 & -17 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 \\ -10 & -6 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}. \end{aligned}$$

We have diagonalised our linear transformation T. We shall discuss this topic in more detail later in these notes.

As a check, observe

$$\begin{aligned} T(w_2) &= T(-v_1 + 2v_2) \\ &= -T(v_1) + 2T(v_2) \\ &= -(16v_1 - 30v_2) + 2(9v_1 - 17v_2) \\ &= 2v_1 - 4v_2 \\ &= -2(-v_1 + 2v_2) = -2w_2, \end{aligned}$$

and similarly for $T(w_1)$. □

Example 2B *Let*

$$\mathscr{B} = \left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right\}.$$

(i) *Show that $\mathscr{B}$ is a basis for $\mathbb{R}^3$.*

(ii) *Write down the change of basis matrix from the standard basis $\mathscr{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to $\mathscr{B}$.*

(iii) *Let*

$$A = \begin{pmatrix} -2 & -2 & -3 \\ 1 & 1 & 2 \\ -1 & -2 & -2 \end{pmatrix}$$

and view A as a linear transformation $\mathbb{R}^3 \to \mathbb{R}^3$. Find the matrix of A with respect to the basis $\mathscr{B}$.

Solution: (i) We first establish that $\mathscr{B}$ is linearly independent. Solve

$$\alpha \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \gamma \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix};$$

that is,

$$\begin{aligned} \beta + 2\gamma &= 0 \\ \alpha \qquad - \gamma &= 0 \\ -\alpha - \beta \qquad &= 0. \end{aligned}$$

Thus $\gamma = \alpha$ and the first equation yields $2\alpha + \beta = 0$. Adding the third equation now gives $\alpha = 0$ and hence $\beta = \gamma = 0$. This show $\mathscr{B}$ is linearly independent and it is therefore a basis for $\mathbb{R}^3$ since $\dim \mathbb{R}^3 = 3 = |\mathscr{B}|$.

(ii) We write each vector in $\mathscr{B}$ in terms of the standard basis

$$\begin{aligned} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} &= \qquad \mathbf{e}_2 - \mathbf{e}_3 \\ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} &= \mathbf{e}_1 \qquad - \mathbf{e}_3 \\ \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} &= 2\mathbf{e}_1 - \mathbf{e}_2 \end{aligned}$$

and write the coefficients appearing down the columns of the change of basis matrix:

$$P = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}.$$

(iii) Theorem 2.13 says $\mathrm{Mat}_{\mathscr{B},\mathscr{B}}(A) = P^{-1}AP$ (as the matrix of A with respect to the standard basis is A itself). We first calculate the inverse of P via the usual row operation method:

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 \end{array}\right) \qquad r_3 \mapsto r_3 + r_1$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 \end{array}\right) \qquad r_1 \leftrightarrow r_2$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right) \qquad r_3 \mapsto r_3 + r_2$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & -1 & -2 & -2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right) \qquad \begin{array}{l} r_1 \mapsto r_1 + r_3 \\ r_2 \mapsto r_2 - 2r_3 \end{array}$$

Hence

$$P^{-1} = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & -2 \\ 1 & 1 & 1 \end{pmatrix}$$

and so

$$\begin{aligned} \mathrm{Mat}_{\mathscr{B},\mathscr{B}}(A) &= P^{-1}AP \\ &= \begin{pmatrix} 1 & 2 & 1 \\ -1 & -2 & -2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & -2 & -3 \\ 1 & 1 & 2 \\ -1 & -2 & -2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 & -2 & -1 \\ 2 & 4 & 3 \\ -2 & -3 & -3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}. \end{aligned}$$

$\square$