Vectors in 3-d space

A *vector* is a quantity which possesses both magnitude and direction. A *scalar* possesses magnitude only.

In the Cartesian coordinate system the unit vectors are $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, all of unit length, directed along the positive x, y and z axes respectively.

Throughout let $\mathbf{a}$=$a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, $\mathbf{b}$=$b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ and $\mathbf{c}$=$c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}$.

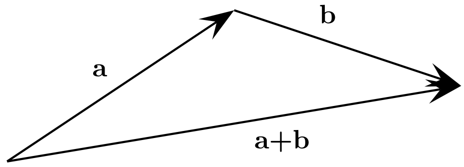
The *magnitude* of $\mathbf{a}$ is: $|\mathbf{a}|$=$\sqrt{a_1^2+a_2^2+a_3^2}$.

The *unit vector* in the direction of $\mathbf{a}$ is

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}.$$

Scalar multiplication: $\lambda\mathbf{a} = \lambda a_1\mathbf{i}+\lambda a_2\mathbf{j}+\lambda a_3\mathbf{k}$ (where λ – "*lambda*" is a scalar).

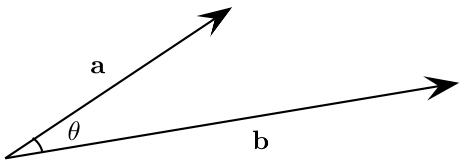
Vector addition: $\mathbf{a}$+$\mathbf{b}$=$(a_1+b_1)\mathbf{i}+(a_2+b_2)\mathbf{j}+(a_3+b_3)\mathbf{k}$.



Vectors can be viewed as directed displacements. If $\mathbf{a}$ and $\mathbf{b}$ are displacements then the net result of these two displacements is $\mathbf{a}$+$\mathbf{b}$.

Scalar / dot product

The *scalar* or *dot product* is defined as $\mathbf{a}.\mathbf{b}$=$|\mathbf{a}||\mathbf{b}|\cos\theta$ where θ denotes the smallest angle between the two vectors.



Alternatively, $\mathbf{a}.\mathbf{b}$=$a_1b_1+a_2b_2+a_3b_3$.

If $\mathbf{a}$ and $\mathbf{b}$ are perpendicular then $\mathbf{a}.\mathbf{b}$=0 (since $\theta=\pi/2$).

A simple rearrangement gives a formula for the angle between two vectors:

$$\cos\theta = \frac{\mathbf{a}.\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}.$$

Note that $\mathbf{i}.\mathbf{j}=\mathbf{j}.\mathbf{k}$=$\mathbf{k}.\mathbf{i}$=0. *(why?)*

Vector / cross product

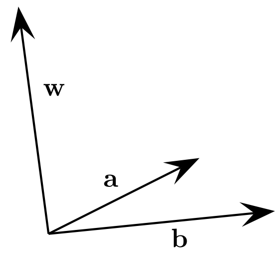
The *vector* or *cross product* is defined as

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$$

where

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

If **w**=**a**×**b** then **w** is perpendicular to both **a** and **b**. *(how would you check this?)* Note that **a**×**b**=-**b**×**a**.



It can be easily seen that **i**×**j**=**k**, **j**×**k**=**i** and **k**×**i**=**j**.

Another way of defining the cross product is $\mathbf{a}\times\mathbf{b}=|\mathbf{a}||\mathbf{b}|\sin\theta\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is the unit vector normal to both **a** and **b**. It follows that **a**×**b**=**0** if and only if **a** is parallel to **b**. *(why?)*

Triple scalar product

a×**b**.**c** = **a**.**b**×**c** = **b**.**c**×**a** = **b**×**c**.**a** = etc

Once the cyclic order $\mathbf{a} \to \mathbf{b} \to \mathbf{c} \to \mathbf{a}$ etc is maintained, × and . can be interchanged.

Triple vector product

a×(**b**×**c**) = (**a**.**c**)**b**-(**a**.**b**)**c**

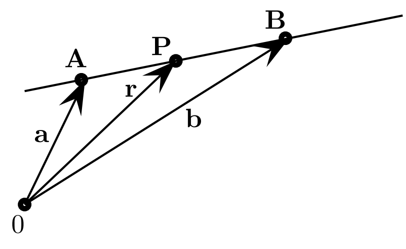
Notice that the order of the vectors *is* important here.

Vector equation of a line

Given two points A and B on a line. Let P be any point on the line. Let **a**, **b** and **r** be the position vectors for A, B and P respectively. Then

$$\mathbf{r} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$$

for suitable t.



Cartesian equation of a line

From above **r**=**a**+t(**b**-**a**). If $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, then equating coefficients yields

$$x = a_1 + t(b_1 - a_1), y = a_2 + t(b_2 - a_2), z = a_3 + t(b_3 - a_3),$$

which gives us the Cartesian equation of the line.

Equation of a plane

Let (x_0, y_0, z_0) be a given point on a plane and $\mathbf{n}$=$n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$ the normal to the plane at that point. If (x, y, z) is any point on the plane then

$$(x - x_0)n_1 + (y - y_0)n_2 + (z - z_0)n_3 = 0$$

or equivalently

$$n_1x + n_2y + n_3z = d$$

where

$$d = n_1x_0 + n_2y_0 + n_3z_0.$$

Note Given 3 points on a plane, the normal can be constructed using the cross product. *(how?)*