# Section 4

# Diagonalisation of linear transformations

In this section, we seek to discuss the diagonalisation of a linear transformation; that is, to understand when a linear transformation can be represented by a diagonal matrix with respect to some basis. We first need to refer to eigenvectors and eigenvalues.

## Eigenvectors and eigenvalues

**Definition 4.1** Let $V$ be a vector space over a field $F$ and let $T\colon V \to V$ be a linear transformation. (This will be our setup throughout this section.) A *non-zero* vector $v$ is an *eigenvector* for $T$ with *eigenvalue* $\lambda$ (where $\lambda \in F$) if

$$T(v) = \lambda v.$$

**Note:** The condition $v \neq \mathbf{0}$ is important since $T(\mathbf{0}) = \mathbf{0} = \lambda\mathbf{0}$ for every $\lambda \in F$, so considering $\mathbf{0}$ will never provide interesting about our transformation $T$.

Note that $T(v) = \lambda v$ implies that $T(v) - \lambda v = \mathbf{0}$. Consequently, we make the following definition:

**Definition 4.2** Let $V$ be a vector space over a field $F$, let $T\colon V \to V$ be a linear transformation, and $\lambda \in F$. The *eigenspace* corresponding to the eigenvalue $\lambda$ is the subspace

$$\begin{aligned} E_\lambda = \ker(T - \lambda I) &= \{\, v \in V \mid T(v) - \lambda v = \mathbf{0} \,\} \\ &= \{\, v \in V \mid T(v) = \lambda v \,\}. \end{aligned}$$

Recall that $I$ denotes the identity transformation $v \mapsto v$.

Thus $E_\lambda$ consists of all the eigenvectors of $T$ with eigenvalue $\lambda$ together with the zero vector $\mathbf{0}$. Note that $T - \lambda I$ is a linear transformation (since it is built from the linear transformations $T$ and $I$), so $E_\lambda$ is certainly a subspace of $V$.

From now on we assume that $V$ is finite-dimensional over the field $F$.

To see how to find eigenvalues and eigenvectors, note that if $v$ is an eigenvector, then

$$(T - \lambda I)(v) = \mathbf{0} \qquad \text{or} \qquad (\lambda I - T)(v) = \mathbf{0},$$

so $v \in \ker(\lambda I - T)$. Consequently, $\lambda I - T$ is not invertible ($\ker(\lambda I - T) \neq \{\mathbf{0}\}$). If $A$ is the matrix of $T$ with respect to some basis, then $\lambda I - A$ is the matrix of $\lambda I - T$ (where in the former $I$ refers to the identity matrix). We now know that $\lambda I - A$ is not invertible, so $\det(\lambda I - A) = 0$.

**Definition 4.3** Let $T \colon V \to V$ be a linear transformation of the finite-dimensional vector space $V$ (over $F$) and let $A$ be the matrix of $T$ with respect to some basis. The *characteristic polynomial* of $T$ is

$$c_T(x) = \det(xI - A)$$

where $x$ is an indeterminate variable.

We have established one half of the following lemma.

**Lemma 4.4** *Suppose that $T \colon V \to V$ is a linear transformation of the finite-dimensional vector space $V$ over $F$. Then $\lambda$ is an eigenvalue of $T$ if and only if $\lambda$ is a root of the characteristic polynomial of $T$.*

Proof: We have shown that if $\lambda$ is an eigenvalue of $T$, then $\lambda$ is a root of $c_T(x)$ (in the above notation).

For the converse, first note that, for any linear map $S \colon V \to V$, $S$ is invertible if and only if $\ker S = \{\mathbf{0}\}$. (For if $\ker S = \{\mathbf{0}\}$, the Rank-Nullity Theorem says $\operatorname{im} S = V$; that is, $S$ is surjective. If $Sv = Sw$ for some $v$ and $w$, then $S(v - w) = \mathbf{0}$ and the fact that the kernel is zero forces $v = w$. Hence, $S$ is a bijection.)

Now if $\lambda$ is a root of $c_T(x)$, then $\det(\lambda I - A) = 0$, so $\lambda I - A$ is not invertible. The previous observation then tells us $E_\lambda = \ker(\lambda I - T) \neq \{\mathbf{0}\}$ and then any non-zero vector in the eigenspace $E_\lambda$ is an eigenvector for $T$ with eigenvalue $\lambda$. $\square$

**Remarks**

(i) Some authors view it is as easier to calculate $\det(A - xI)$ rather than $\det(xI - A)$ and so define the characteristic polynomial slightly different. Since we are just multiplying every entry in the matrix by $-1$, the resulting determinant will merely be different by a factor of $(-1)^n = \pm 1$ (where $n = \dim V$) and the roots will be unchanged. The reason for defining $c_T(x) = \det(xI - A)$ is that it ensures the highest degree term is *always* $x^n$ (rather than $-x^n$ in the case that $n$ is odd). The latter will be relevant in later discussions.

(ii) On the face of it the characteristic polynomial depends on the choice of basis, since, as we have seen, changing basis changes the matrix of a linear transformation. In fact, we get the same polynomial no matter which basis we use.

**Lemma 4.5** *Let $V$ be a finite-dimensional vector space $V$ over $F$ and $T\colon V \to V$ be a linear transformation. The characteristic polynomial $c_T(x)$ is independent of the choice of basis for $V$.*

Consequently, $c_T(x)$ depends only on $T$.

PROOF: Let $A$ and $A'$ be the matrices of $T$ with respect to two different bases for $V$. Theorem 2.13 tells us that $A' = P^{-1}AP$ for some invertible matrix $P$. Then

$$P^{-1}(xI - A)P = xP^{-1}IP - P^{-1}AP = xI - A',$$

so

$$\begin{aligned}\det(xI - A') &= \det(P^{-1}(xI - A)P) \\ &= \det P^{-1} \cdot \det(xI - A) \cdot \det P \\ &= (\det P)^{-1} \cdot \det(xI - A) \cdot \det P \\ &= \det(xI - A)\end{aligned}$$

(since multiplication in the field $F$ is commutative — see condition (v) of Definition 1.1). Hence we get the same answer for the characteristic polynomial. □

## Diagonalisability

Let us now move onto the diagonalisation of linear transformations.

**Definition 4.6** (i) Let $T\colon V \to V$ be a linear transformation of a finite-dimensional vector space $V$. We say that $T$ is *diagonalisable* if there is a basis with respect to which $T$ is represented by a diagonal matrix.

(ii) A square matrix $A$ is *diagonalisable* if there is an invertible matrix $P$ such that $P^{-1}AP$ is diagonal.

Of course, in (ii) we are simply viewing the $n \times n$ matrix $A$ as a linear transformation $F^n \to F^n$ and forming $P^{-1}AP$ simply corresponds to choosing a (non-standard) basis for $F^n$ with respect to which the transformation is represented by a diagonal matrix (see Theorem 2.13).

**Proposition 4.7** *Let $V$ be a finite-dimensional vector space and $T \colon V \to V$ be a linear transformation. Then $T$ is diagonalisable if and only if there is a basis for $V$ consisting of eigenvectors for $T$.*

PROOF: If $T$ is diagonalisable, there is a basis $\mathscr{B} = \{v_1, v_2, \dots, v_n\}$ with respect to which $T$ is represented by a diagonal matrix, say

$$\mathrm{Mat}_{\mathscr{B},\mathscr{B}}(T) = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

for some $\lambda_1, \lambda_2, \dots, \lambda_n \in F$. Then $T(v_i) = \lambda_i v_i$ for $i = 1, 2, \dots, n$, so each basis vector in $\mathscr{B}$ is an eigenvector.

Conversely, if each vector in a basis $\mathscr{B}$ is an eigenvector, then the matrix $\mathrm{Mat}_{\mathscr{B},\mathscr{B}}(T)$ is diagonal (with each diagonal entry being the corresponding eigenvalue). $\square$

**Example 4.8** *Let $V = \mathbb{R}^3$ and consider the linear transformation $T \colon V \to V$ given by the matrix*

$$A = \begin{pmatrix} 8 & 6 & 0 \\ -9 & -7 & 0 \\ 3 & 3 & 2 \end{pmatrix}$$

*Show that $T$ is diagonalisable, find a matrix $P$ such that $D = P^{-1}AP$ is diagonal and find $D$.*

**Solution:** To say that $T$ is given by the above matrix is to say that this matrix represents $T$ with respect to the standard basis for $\mathbb{R}^3$; i.e., that we obtain $T$ by multiplying vectors on the left by $A$). We first calculate the characteristic polynomial:

$$\begin{aligned} \det(xI - A) &= \det \begin{pmatrix} x-8 & -6 & 0 \\ 9 & x+7 & 0 \\ -3 & -3 & x-2 \end{pmatrix} \\ &= (x-2)\left((x-8)(x+7) + 6 \times 9\right) \\ &= (x-2)\left((x-8)(x+7) + 54\right) \\ &= (x-2)(x^2 - x - 2) \\ &= (x-2)(x+1)(x-2) \end{aligned}$$

$$= (x+1)(x-2)^2,$$

so

$$c_T(x) = (x+1)(x-2)^2$$

and the eigenvalues of $T$ are $-1$ and $2$. (We cannot yet guarantee we have enough eigenvectors to form a basis.)

We now need to go looking for eigenvectors. First we seek $v \in \mathbb{R}^3$ such that $T(v) = -v$; i.e., such that $(T+I)(v) = \mathbf{0}$. Thus we solve

$$\begin{pmatrix} 9 & 6 & 0 \\ -9 & -6 & 0 \\ 3 & 3 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

so

$$9x + 6y = 0, \qquad 3x + 3y + 3z = 0.$$

So given arbitrary $x$, take $y = -\frac{3}{2}x$ and $z = -x - y$. We have one degree of freedom (the choice of $x$) and we determine that

$$\begin{aligned} E_{-1} = \ker(T+I) &= \left\{ \begin{pmatrix} x \\ -\frac{3}{2}x \\ \frac{1}{2}x \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\} \\ &= \operatorname{Span}\left( \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \right). \end{aligned}$$

Hence our eigenspace $E_{-1}$ is one-dimensional and the vector $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ is a suitable eigenvector with eigenvalue $-1$.

Now we seek $v \in \mathbb{R}^3$ such that $T(v) = 2v$; i.e., $(T-2I)(v) = \mathbf{0}$. We therefore solve

$$\begin{pmatrix} 6 & 6 & 0 \\ -9 & -9 & 0 \\ 3 & 3 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

so

$$x + y = 0.$$

Hence our eigenspace is

$$\begin{aligned} E_2 = \ker(T-2I) &= \left\{ \begin{pmatrix} x \\ -x \\ z \end{pmatrix} \;\middle|\; x, z \in \mathbb{R} \right\} \\ &= \operatorname{Span}\left( \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right). \end{aligned}$$

Hence the eigenspace $E_2$ is two-dimensional and we can find two linearly independent eigenvectors with eigenvalue 2, for example

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We conclude that with respect to the basis

$$\mathscr{B} = \left\{ \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

the matrix of $T$ is

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

In particular, $T$ is diagonalisable. It remains to find $P$ such that $D = P^{-1}AP$, but Theorem 2.13 tells us we need simply take the matrix whose entries are the coefficients of the vectors in $\mathscr{B}$ when expressed in terms of the standard basis:

$$P = \begin{pmatrix} 2 & 1 & 0 \\ -3 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

since

$$\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 2\mathbf{e}_1 - 3\mathbf{e}_2 + \mathbf{e}_3,$$

so the entries of the first column are 2, $-3$ and 1, and similarly for the other columns.

Before continuing to develop the theory of diagonalisation, we give an example that illustrates how this can be applied to an applied mathematics setting.

**Example 4.9** *Solve the following system of differential equations involving differentiable functions $x$, $y$ and $z$ of one variable:*

$$\begin{aligned} 8\frac{\mathrm{d}x}{\mathrm{d}t} + 6\frac{\mathrm{d}y}{\mathrm{d}t} \qquad\qquad &= \quad 2 - 2t - 2t^2 \\ -9\frac{\mathrm{d}x}{\mathrm{d}t} - 7\frac{\mathrm{d}y}{\mathrm{d}t} \qquad\qquad &= -2 + 2t + 3t^2 \\ 3\frac{\mathrm{d}x}{\mathrm{d}t} + 3\frac{\mathrm{d}y}{\mathrm{d}t} + 2\frac{\mathrm{d}z}{\mathrm{d}t} &= \quad 2 + 2t - t^2 \end{aligned}$$

**Solution:** To simply notation, we shall use a dash ($'$) to denote differentiation with respect to $t$. Define the vector-valued function $\boldsymbol{v} = \boldsymbol{v}(t)$ by

$$\boldsymbol{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Our system of differential equations then becomes

$$A\boldsymbol{v}' = \begin{pmatrix} 2 - 2t - 2t^2 \\ -2 + 2t + 3t^2 \\ 2 + 2t - t^2 \end{pmatrix}$$

where

$$A = \begin{pmatrix} 8 & 6 & 0 \\ -9 & -7 & 0 \\ 3 & 3 & 2 \end{pmatrix},$$

which happens to be the matrix considered in Example 4.8. As a consequence of that example, we know that $P^{-1}AP = D$, where

$$P = \begin{pmatrix} 2 & 1 & 0 \\ -3 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Multiply our matrix equation on the left by $P^{-1}$ and define $\boldsymbol{w} = P^{-1}\boldsymbol{v}$. We then obtain

$$P^{-1}AP\boldsymbol{w}' = P^{-1}A\boldsymbol{v}' = P^{-1} \begin{pmatrix} 2 - 2t - 2t^2 \\ -2 + 2t + 3t^2 \\ 2 + 2t - t^2 \end{pmatrix}.$$

The usual method [which will be omitted during lectures] finds the inverse of $P$:

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ -3 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 0 & 1 & -2 & 1 & 0 & -2 \\ 0 & -1 & 3 & 0 & 1 & 3 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \qquad \begin{array}{l} r_1 \mapsto r_1 - 2r_3 \\ r_2 \mapsto r_2 + 3r_3 \end{array}$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 0 & 1 & -2 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \qquad r_2 \mapsto r_2 + r_1$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & -1 & -1 & 0 \end{array}\right) \qquad \begin{array}{l} r_1 \mapsto r_1 + 2r_2 \\ r_3 \mapsto r_3 - r_2 \end{array}$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

(finally rearranging the rows). Hence

$$P^{-1} = \begin{pmatrix} -1 & -1 & 0 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Hence our equation becomes

$$D\boldsymbol{w}' = \begin{pmatrix} -1 & -1 & 0 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2-2t-2t^2 \\ -2+2t+3t^2 \\ 2+2t-t^2 \end{pmatrix} = \begin{pmatrix} -t^2 \\ 2-2t \\ 2+2t \end{pmatrix}.$$

Let's write $a$, $b$ and $c$ for the three entries of $\boldsymbol{w}$. So our equation becomes:

$$-\frac{\mathrm{d}a}{\mathrm{d}t} = -t^2, \qquad 2\frac{\mathrm{d}b}{\mathrm{d}t} = 2-2t, \qquad 2\frac{\mathrm{d}c}{\mathrm{d}t} = 2+2t;$$

that is,

$$\frac{\mathrm{d}a}{\mathrm{d}t} = t^2, \qquad \frac{\mathrm{d}b}{\mathrm{d}t} = 1-t, \qquad \frac{\mathrm{d}c}{\mathrm{d}t} = 1+t.$$

Hence

$$a = \tfrac{1}{3}t^3 + c_1, \qquad b = t - \tfrac{1}{2}t^2 + c_2, \qquad c = t + \tfrac{1}{2}t^2 + c_3$$

for some constants $c_1$, $c_2$ and $c_3$; that is,

$$\boldsymbol{w} = \begin{pmatrix} \frac{1}{3}t^3 \\ t - \frac{1}{2}t^2 \\ t + \frac{1}{2}t^2 \end{pmatrix} + \boldsymbol{c}$$

for some constant vector $\boldsymbol{c}$. Therefore

$$\begin{aligned} \boldsymbol{v} = P\boldsymbol{w} &= \begin{pmatrix} 2 & 1 & 0 \\ -3 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \left( \begin{pmatrix} \frac{1}{3}t^3 \\ t - \frac{1}{2}t^2 \\ t + \frac{1}{2}t^2 \end{pmatrix} + \boldsymbol{c} \right) \\ &= \begin{pmatrix} t - \frac{1}{2}t^2 + \frac{2}{3}t^3 \\ -t + \frac{1}{2}t^2 - t^3 \\ t + \frac{1}{2}t^2 + \frac{1}{3}t^3 \end{pmatrix} + \boldsymbol{k} \end{aligned}$$

for some constant vector $\boldsymbol{k}$. Hence our solution is

$$\begin{aligned} x &= \tfrac{2}{3}t^3 - \tfrac{1}{2}t^2 + t + k_1 \\ y &= -t^3 + \tfrac{1}{2}t^2 - t + k_2 \\ z &= \tfrac{1}{3}t^3 + \tfrac{1}{2}t^2 + t + k_3 \end{aligned}$$

for some constants $k_1$, $k_2$ and $k_3$.

## Algebraic and geometric multiplicities

Suppose $T\colon V \to V$ is a diagonalisable linear transformation. Then there is a basis $\mathscr{B}$ for $V$ with respect to which the matrix of $T$ has the form

$$\mathrm{Mat}_{\mathscr{B},\mathscr{B}}(T) = A = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

for some $\lambda_1, \lambda_2, \dots, \lambda_n \in F$ (possibly including repeats). The characteristic polynomial of $T$ does not depend on the choice of basis (Lemma 4.5), so

$$\begin{aligned} c_T(x) = \det(xI - A) &= \det \begin{pmatrix} x-\lambda_1 & & & 0 \\ & x-\lambda_2 & & \\ & & \ddots & \\ 0 & & & x-\lambda_n \end{pmatrix} \\ &= (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n). \end{aligned}$$

So:

**Lemma 4.10** *If the linear transformation $T\colon V \to V$ is diagonalisable, then the characteristic polynomial of $T$ is a product of linear factors.* $\square$

Here, a *linear* polynomial is one of degree 1, so we mean that the characteristic polynomial $c_T(x)$ is a product of factors of the form $\alpha x + \beta$. Of course, as the leading coefficient of $c_T(x)$ is $x^n$, the linear factors can always be arranged to have the form $x + \beta \quad (\beta \in F)$. Then $-\beta$ would be a root of $c_T(x)$ and hence an eigenvalue of $T$ (by Lemma 4.4).

**Note:**

(i) This lemma only gives a necessary condition for diagonalisability. We shall next meet an example where $c_T(x)$ is a product of linear factors but $T$ is not diagonalisable.

(ii) The field $\mathbb{C}$ of complex numbers is *algebraically closed*, i.e., every polynomial factorises as a product of linear factors. So Lemma 4.10 gives no information in that case.

**Example 4.11** *Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by the matrix*

$$B = \begin{pmatrix} 8 & 3 & 0 \\ -18 & -7 & 0 \\ -9 & -4 & 2 \end{pmatrix}.$$

*Determine whether $T$ is diagonalisable.*

**Solution:** Now

$$\begin{aligned}\det(xI - B) &= \det\begin{pmatrix} x-8 & -3 & 0 \\ 18 & x+7 & 0 \\ 9 & 4 & x-2 \end{pmatrix} \\ &= (x-2)\left((x-8)(x+7) + 3 \times 18\right) \\ &= (x-2)\left((x-8)(x+7) + 54\right) \\ &= (x-2)(x^2 - x - 2) \\ &= (x+1)(x-2)^2,\end{aligned}$$

so

$$c_T(x) = (x+1)(x-2)^2.$$

Again we have a product of linear factors, exactly as we did in Example 4.8. This time, however, issues arise when we seek to find eigenvectors with eigenvalue 2. We solve $(T - 2I)(v) = \mathbf{0}$; that is,

$$\begin{pmatrix} 6 & 3 & 0 \\ -18 & -9 & 0 \\ -9 & -4 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

so

$$6x + 3y = -18 - 9y = -9x - 4y = 0;$$

that is,

$$2x + y = 0, \qquad 9x + 4y = 0.$$

The first equation gives $y = -2x$, and when we substitute in the second we obtain $x = 0$ and so $y = 0$. Hence our eigenspace corresponding to eigenvalue 2 is

$$\begin{aligned}E_2 = \ker(T - 2I) &= \left\{ \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \;\middle|\; z \in \mathbb{R} \right\} \\ &= \operatorname{Span}\left( \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right).\end{aligned}$$

Thus $\dim E_2 = 1$ and we cannot find two linearly independent eigenvectors with eigenvalue 2.

I shall omit the calculation, but only a single linearly independent vector can be found with eigenvalue $-1$. We can only find a set of size at most two containing linearly independent eigenvectors. Hence there is no basis for $\mathbb{R}^3$ consisting of eigenvectors for $T$ and $T$ is not diagonalisable.

The basic issue occurring in this example is that not as many linearly independent eigenvectors could be found as there were relevant linear factors in the characteristic polynomial. This relates to the so-called algebraic and geometric multiplicity of the eigenvalue.

**Definition 4.12** Let $V$ be a finite-dimensional vector space over the field $F$ and let $T\colon V \to V$ be a linear transformation of $V$. Let $\lambda \in F$.

(i) The *algebraic multiplicity* of $\lambda$ (as an eigenvalue of $T$) is the largest power $k$ such that $(x-\lambda)^k$ is a factor of the characteristic polynomial $c_T(x)$.

(ii) The *geometric multiplicity* of $\lambda$ is the dimension of the eigenspace $E_\lambda$ corresponding to $\lambda$.

If $\lambda$ is *not* an eigenvalue of $T$, then $\lambda$ is not a root of the characteristic polynomial (by Lemma 4.4), so $(x-\lambda)$ does not divide $c_T(x)$. Equally, there are no eigenvectors with eigenvalue $\lambda$ in this case, so $E_\lambda = \ker(T-\lambda I) = \{\mathbf{0}\}$. Hence the algebraic and geometric multiplicities of $\lambda$ are both 0 when $\lambda$ is not an eigenvalue. Consequently we shall not be at all interested in these two multiplicities in this case.

On the other hand, when $\lambda$ is an eigenvalue, the same sort of argument shows that the algebraic and geometric multiplicities are both at least 1.

We shall shortly discuss how these two multiplicities are linked. We shall make use of the following important fact:

**Proposition 4.13** *Let $T\colon V \to V$ be a linear transformation of a vector space $V$. A set of eigenvectors of $T$ corresponding to* distinct *eigenvalues is linearly independent.*

Proof: Let $\mathscr{A} = \{v_1, v_2, \dots, v_k\}$ be a set of eigenvectors of $T$. Let $\lambda_i$ be the eigenvalue of $v_i$ and assume the $\lambda_i$ are distinct. We proceed by induction on $k$.

Note that if $k=1$, then $\mathscr{A} = \{v_1\}$ consists of a single *non-zero* vector so is linearly independent.

So assume that $k>1$ and that the result holds for smaller sets of eigenvectors for $T$. Suppose

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k = \mathbf{0}. \tag{4.1}$$

Apply $T$:

$$\alpha_1 T(v_1) + \alpha_2 T(v_2) + \dots + \alpha_k T(v_k) = \mathbf{0},$$

so

$$\alpha_1 \lambda_1 v_1 + \alpha_2 \lambda_2 v_2 + \dots + \alpha_k \lambda_k v_k = \mathbf{0}.$$

Multiply (4.1) by $\lambda_k$ and subtract:

$$\alpha_1(\lambda_1 - \lambda_k)v_1 + \alpha_2(\lambda_2 - \lambda_k)v_2 + \dots + \alpha_{k-1}(\lambda_{k-1} - \lambda_k)v_{k-1} = \mathbf{0}.$$

This is an expression of linear dependence involving the eigenvectors $v_1$, $v_2$, …, $v_{k-1}$, so by induction

$$\alpha_i(\lambda_i - \lambda_k) = 0 \qquad \text{for } i = 1, 2, \dots, k-1.$$

By assumption the eigenvalues $\lambda_i$ are distinct, so $\lambda_i - \lambda_k \neq 0$ for $i = 1, 2, \dots, k-1$, and, dividing by this non-zero scalar, we deduce

$$\alpha_i = 0 \qquad \text{for } i = 1, 2, \dots, k-1.$$

Equation 4.1 now yields $\alpha_k v_k = \mathbf{0}$, which forces $\alpha_k = 0$ as the eigenvector $v_k$ is non-zero. This completes the induction step. □

**Theorem 4.14** *Let $V$ be a finite-dimensional vector space over the field $F$ and let $T \colon V \to V$ be a linear transformation of $V$.*

(i) *If the characteristic polynomial $c_T(x)$ is a product of linear factors (as always happens, for example, if $F = \mathbb{C}$), then the sum of the algebraic multiplicities equals* $\dim V$.

(ii) *Let $\lambda \in F$ and let $r_\lambda$ be the algebraic multiplicity and $n_\lambda$ be the geometric multiplicity of $\lambda$. Then*

$$n_\lambda \leqslant r_\lambda.$$

(iii) *The linear transformation $T$ is diagonalisable if and only if $c_T(x)$ is a product of linear factors and $n_\lambda = r_\lambda$ for all eigenvalues $\lambda$.*

Proof: (i) Let $n = \dim V$ and write $c_T(x)$ as a product of linear factors

$$c_T(x) = (x - \lambda_1)^{r_1}(x - \lambda_2)^{r_2} \dots (x - \lambda_k)^{r_k}$$

where $\lambda_1, \lambda_2, \dots, \lambda_k$ are the *distinct* roots of $c_T(x)$ (i.e., the distinct eigenvalues of $T$). Since $c_T(x)$ is the determinant of a specific $n \times n$ matrix, it is a polynomial of degree $n$, so

$$\dim V = n = r_1 + r_2 + \dots + r_k,$$

the sum of the algebraic multiplicities.

(ii) Let $\lambda$ be an eigenvalue of $T$ and let $m = n_\lambda$, the dimension of the eigenspace $E_\lambda = \ker(T - \lambda I)$. Choose a basis $\{v_1, v_2, \dots, v_m\}$ for $E_\lambda$ and extend to a basis $\mathscr{B} = \{v_1, v_2, \dots, v_m, v_{m+1}, \dots, v_n\}$ for $V$. Note then

$$T(v_i) = \lambda v_i \qquad \text{for } i = 1, 2, \dots, m,$$

so the matrix of $T$ with respect to $\mathscr{B}$ has the form

$$A = \mathrm{Mat}_{\mathscr{B},\mathscr{B}}(T) = \begin{pmatrix} \lambda & 0 & \cdots & 0 & * & \cdots & * \\ 0 & \lambda & \ddots & \vdots & * & \cdots & * \\ 0 & 0 & \ddots & 0 & \vdots & & \vdots \\ \vdots & \vdots & \ddots & \lambda & \vdots & & \vdots \\ \vdots & \vdots & & 0 & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & * & \cdots & * \end{pmatrix}$$

Hence

$$\begin{aligned} c_T(x) &= \det \begin{pmatrix} x-\lambda & 0 & \cdots & 0 & * & \cdots & * \\ 0 & x-\lambda & \ddots & \vdots & * & \cdots & * \\ 0 & 0 & \ddots & 0 & \vdots & & \vdots \\ \vdots & \vdots & \ddots & x-\lambda & \vdots & & \vdots \\ \vdots & \vdots & & 0 & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & * & \cdots & * \end{pmatrix} \\ &= (x-\lambda)^m p(x) \end{aligned}$$

for some polynomial $p(x)$. Hence $r_\lambda \geqslant m = n_\lambda$.

(iii) Suppose that

$$c_T(x) = (x-\lambda_1)^{r_1}(x-\lambda_2)^{r_2}\dots(x-\lambda_k)^{r_k}$$

where $\lambda_1$, $\lambda_2$, …, $\lambda_k$ are the distinct eigenvalues of $T$, so

$$r_1 + r_2 + \dots + r_k = n = \dim V$$

(by (i)). Let $n_i = \dim E_{\lambda_i}$ be the geometric multiplicity of $\lambda_i$. We suppose that $n_i = r_i$. Choose a basis $\mathscr{B}_i = \{v_{i1}, v_{i2}, \dots, v_{in_i}\}$ for each $E_{\lambda_i}$ and let

$$\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2 \cup \dots \cup \mathscr{B}_k = \{\, v_{ij} \mid i = 1,2,\dots,k;\ j = 1,2,\dots,n_i \,\}.$$

**Claim:** $\mathscr{B}$ is linear independent.

Suppose

$$\sum_{\substack{1\leqslant i\leqslant k \\ 1\leqslant j\leqslant n_i}} \alpha_{ij} v_{ij} = \mathbf{0}.$$

Let $w_i = \sum_{j=1}^{n_i} \alpha_{ij} v_{ij}$. Then $w_i$ is a linear combination of vectors in the eigenspace $E_{\lambda_i}$, so $w_i \in E_{\lambda_i}$. Now

$$w_1 + w_2 + \cdots + w_k = \mathbf{0}.$$

Proposition 4.13 says that eigenvectors for distinct eigenvalues are linearly independent, so the $w_i$ cannot be eigenvectors. Therefore

$$w_i = \mathbf{0} \qquad \text{for } i = 1, 2, \ldots, k;$$

that is,

$$\sum_{j=1}^{n_i} \alpha_{ij} v_{ij} = \mathbf{0} \qquad \text{for } i = 1, 2, \ldots, k.$$

Since $\mathscr{B}_i$ is a basis for $E_{\lambda_i}$, it is linearly independent and we conclude that $\alpha_{ij} = 0$ for all $i$ and $j$. Hence $\mathscr{B}$ is a linearly independent set.

Now since $n_i = r_i$ by assumption, we have

$$|\mathscr{B}| = n_1 + n_2 + \cdots + n_k = n.$$

Hence $\mathscr{B}$ is a linearly independent set of size equal to the dimension of $V$. Therefore $\mathscr{B}$ is a basis for $V$ and it consists of eigenvectors for $T$. Hence $T$ is diagonalisable.

Conversely, suppose $T$ is diagonalisable. We have already observed that $c_T(x)$ is a product of linear factors (Lemma 4.10). We may therefore maintain the notation of the first part of this proof. Since $T$ is diagonalisable, there is a basis $\mathscr{B}$ for $V$ consisting of eigenvectors for $T$. Let $\mathscr{B}_i = \mathscr{B} \cap E_{\lambda_i}$, that is, $\mathscr{B}_i$ consists of those vectors from $\mathscr{B}$ that have eigenvalue $\lambda_i$. As every vector in $\mathscr{B}$ is an eigenvector,

$$\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2 \cup \cdots \cup \mathscr{B}_k.$$

As $\mathscr{B}$ is linearly independent, so is $\mathscr{B}_i$ and Theorem 1.23 tells us

$$|\mathscr{B}_i| \leqslant \dim E_{\lambda_i} = n_i.$$

Hence

$$n = |\mathscr{B}| = |\mathscr{B}_1| + |\mathscr{B}_2| + \cdots + |\mathscr{B}_k| \leqslant n_1 + n_2 + \cdots + n_k.$$

But $n_i \leqslant r_i$ and $r_1 + r_2 + \cdots + r_k = n$, so we deduce $n_i = r_i$ for all $i$. This completes the proof of (iii). $\square$

This theorem now explains what was actually going on in Example 4.11. It was an example where the characteristic polynomial splits into linear factors but the geometric multiplicity for one of the eigenvalues is smaller than its algebraic multiplicity.

**Example 4A** *Let*

$$A = \begin{pmatrix} -1 & 2 & -1 \\ -4 & 5 & -2 \\ -4 & 3 & 0 \end{pmatrix}.$$

*Show that $A$ is not diagonalisable.*

Solution: The characteristic polynomial of $A$ is

$$\begin{aligned}
c_A(x) &= \det(xI - A) \\
&= \det \begin{pmatrix} x+1 & -2 & 1 \\ 4 & x-5 & 2 \\ 4 & -3 & x \end{pmatrix} \\
&= (x+1)\det\begin{pmatrix} x-5 & 2 \\ -3 & x \end{pmatrix} + 2\det\begin{pmatrix} 4 & 2 \\ 4 & x \end{pmatrix} + \det\begin{pmatrix} 4 & x-5 \\ 4 & -3 \end{pmatrix} \\
&= (x+1)\big(x(x-5)+6\big) + 2(4x-8) + (-12-4x+20) \\
&= (x+1)(x^2-5x+6) + 8(x-2) - 4x + 8 \\
&= (x+1)(x-2)(x-3) + 8(x-2) - 4(x-2) \\
&= (x-2)\big((x+1)(x-3)+8-4\big) \\
&= (x-2)(x^2-2x-3+4) \\
&= (x-2)(x^2-2x+1) \\
&= (x-2)(x-1)^2.
\end{aligned}$$

In particular, the algebraic multiplicity of the eigenvalue 1 is 2.

We now determine the eigenspace for eigenvalue 1. We solve $(A-I)\boldsymbol{v} = \mathbf{0}$; that is,

$$\begin{pmatrix} -2 & 2 & -1 \\ -4 & 4 & -2 \\ -4 & 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.2}$$

We solve this by applying row operations:

$$\left(\begin{array}{ccc|c} -2 & 2 & -1 & 0 \\ -4 & 4 & -2 & 0 \\ -4 & 3 & -1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} -2 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right) \qquad \begin{array}{l} r_2 \mapsto r_2 - 2r_1 \\ r_3 \mapsto r_3 - 2r_1 \end{array}$$

$$\longrightarrow \left(\begin{array}{ccc|c} -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right) \qquad r_1 \mapsto r_1 + 2r_3$$

So Equation (4.2) is equivalent to

$$-2x + z = 0 = -y + z.$$

Hence $z = 2x$ and $y = z = 2x$. Therefore the eigenspace is

$$E_1 = \left\{ \begin{pmatrix} x \\ 2x \\ 2x \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\} = \operatorname{Span}\left( \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right)$$

and we conclude $\dim E_1 = 1$. Thus the geometric multiplicity of 1 is not equal to the algebraic multiplicity, so $A$ is not diagonalisable. □

## Minimum polynomial

To gain further information about diagonalisation of linear transformations, we shall introduce the concept of the minimum polynomial (also called the minimal polynomial).

Let $V$ be a vector space over a field $F$ of dimension $n$. Consider a linear transformation $T \colon V \to V$. We know the following facts about linear transformations:

(i) the composition of two linear transformations is also a linear transformation; in particular, $T^2$, $T^3$, $T^4$, ... are all linear transformations;

(ii) a scalar multiple of a linear transformation is a linear transformation;

(iii) the sum of two linear transformations is a linear transformation.

Consider a polynomial $f(x)$ over the field $F$, say

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k.$$

The facts above ensure that we have a well-defined linear transformation

$$f(T) = a_0 I + a_1 T + a_2 T^2 + \cdots + a_k T^k.$$

This is what we shall mean by substituting the linear transformation $T$ into the polynomial $f(x)$.

Moreover, since $\dim V = n$, the set of linear transformations $V \to V$ also forms a vector space which has dimension $n^2$ (see Problem Sheet II, Question 6). Now if we return to our linear transformation $T \colon V \to V$, then let us consider the following collection of linear transformations

$$I,\ T,\ T^2,\ T^3,\ \ldots,\ T^{n^2}.$$

There are $n^2 + 1$ linear transformations listed, so they must form a linearly dependent set. Hence there exist scalars $\alpha_0, \alpha_1, \ldots, \alpha_{n^2} \in F$ such that

$$\alpha_0 I + \alpha_1 T + \alpha_2 T^2 + \cdots + \alpha_{n^2} T^{n^2} = 0$$

(the latter being the zero map $0\colon v \mapsto \mathbf{0}$ for all $v \in V$). Omitting zero coefficients and dividing by the last non-zero scalar $\alpha_k$ yields an expression of the form

$$T^k + b_{k-1}T^{k-1} + \cdots + b_2T^2 + b_1T + b_0I = 0$$

where $b_i = \alpha_i/\alpha_k$ for $i = 1, 2, \dots, k-1$. Hence there exists a *monic* polynomial (that is, one whose leading coefficient is 1)

$$f(x) = x^k + b_{k-1}x^{k-1} + \cdots + b_2x^2 + b_1x + b_0$$

such that

$$f(T) = 0.$$

We make the following definition:

**Definition 4.15** Let $T\colon V \to V$ be a linear transformation of a finite-dimensional vector space over the field $F$. The *minimum polynomial* $m_T(x)$ is the *monic* polynomial over $F$ of smallest degree such that

$$m_T(T) = 0.$$

**Note:** Our definition of the characteristic polynomial is to ensure that $c_T(x) = \det(xI - A)$ is always a monic polynomial.

We have observed that if $V$ has dimension $n$ and $T\colon V \to V$ is a linear transformation, then there certainly does exist some monic polynomial $f(x)$ such that $f(T) = 0$. Hence it makes sense to speak of *a* monic polynomial of smallest degree such that $f(T) = 0$. However, if

$$f(x) = x^k + \alpha_{k-1}x^{k-1} + \cdots + \alpha_1x + \alpha_0 \qquad \text{and}$$
$$g(x) = x^k + \beta_{k-1}x^{k-1} + \cdots + \beta_1x + \beta_0$$

are two different polynomials of the same degree such that $f(T) = g(T) = 0$, then

$$h(x) = f(x) - g(x) = (\alpha_{k-1} - \beta_{k-1})x^{k-1} + \cdots + (\alpha_1 - \beta_1)x + (\alpha_0 - \beta_0)$$

is a non-zero polynomial of smaller degree satisfying $h(T) = 0$, and some scalar multiple of $h(x)$ is then monic. Hence there is a *unique* monic polynomial $f(x)$ of smallest degree such that $f(T) = 0$.

We have also observed that if $V$ has dimension $n$ and $T\colon V \to V$, then there is a polynomial $f(x)$ of degree at most $n^2$ such that $f(T) = 0$. In fact, there is a major theorem that does considerably better:

**Theorem 4.16 (Cayley–Hamilton Theorem)** *Let $T: V \to V$ be a linear transformation of a finite-dimensional vector space $V$. If $c_T(x)$ is the characteristic polynomial of $T$, then*

$$c_T(T) = 0.$$

This is a difficult theorem to prove, so we omit the proof. One proof can be found in Blyth and Robertson's "Basic" book. We need to prove that when $T$ is substituted into

$$c_T(x) = \det(xI - A) \qquad \text{(where } A = \mathrm{Mat}(T)\text{)}$$

we produce the zero transformation. The main difficulty comes from the fact that $x$ must be treated as a scalar when expanding the determinant and then $T$ is substituted in instead of this scalar variable.

The upshot of the Cayley–Hamilton Theorem is to show that the minimum polynomial of $T$ has degree at most the dimension of $V$ and (as we shall see) to indicate close links between $m_T(x)$ and $c_T(x)$.

To establish the relevance of the minimum polynomial to diagonalisation, we will need some basic properties of polynomials.

**Facts about polynomials:** Let $F$ be a field and recall $F[x]$ denotes the set of polynomials with coefficients from $F$:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \qquad \text{(where } a_i \in F\text{)}.$$

Then $F[x]$ is an example of what is known as a *Euclidean domain* (see *MT4517 Rings and Fields* for full details). A summary of its main properties are:

- We can add, multiply and subtract polynomials;
- if we attempt to divide $f(x)$ by $g(x)$ (where $g(x) \neq 0$), we obtain
$$f(x) = g(x)q(x) + r(x)$$
where either $r(x) = 0$ or the degree of $r(x)$ satisfies $\deg r(x) < \deg g(x)$ (i.e., we can perform long-division with polynomials).
- When the remainder is 0, that is, when $f(x) = g(x)q(x)$ for some polynomial $q(x)$, we say that $g(x)$ *divides* $f(x)$.
- If $f(x)$ and $g(x)$ are non-zero polynomials, their *greatest common divisor* is the polynomial $d(x)$ of largest degree dividing them both. It is uniquely determined up to multiplying by a scalar and can be expressed as
$$d(x) = a(x)f(x) + b(x)g(x)$$
for some polynomials $a(x)$, $b(x)$.

Those familiar with divisibility in the integers $\mathbb{Z}$ (particularly those who have attended *MT1003*) will recognise these facts as being standard properties of $\mathbb{Z}$ (which is also a standard example of a Euclidean domain).

**Example 4.17** *Find the remainder upon dividing $x^3 + 2x^2 + 1$ by $x^2 + 1$ in the Euclidean domain $\mathbb{R}[x]$.*

SOLUTION:

$$\begin{aligned} x^3 + 2x^2 + 1 &= x(x^2 + 1) + 2x^2 - x + 1 \\ &= (x + 2)(x^2 + 1) - x - 1. \end{aligned}$$

Here the degree of the remainder $r(x) = -x - 1$ is less than the degree of $x^2 + 1$, so we have our required form. The quotient is $q(x) = x + 2$ and the remainder $r(x) = -x - 1$. □

**Proposition 4.18** *Let $V$ be a finite-dimensional vector space over a field $F$ and let $T \colon V \to V$ be a linear transformation. If $f(x)$ is any polynomial (over $F$) such that $f(T) = 0$, then the minimum polynomial $m_T(x)$ divides $f(x)$.*

PROOF: Attempt to divide $f(x)$ by the minimum polynomial $m_T(x)$:

$$f(x) = m_T(x)q(x) + r(x)$$

for some polynomials $q(x)$ and $r(x)$ with either $r(x) = 0$ or $\deg r(x) < \deg m_T(x)$. Substituting the transformation $T$ for the variable $x$ gives

$$0 = f(T) = m_T(T)q(T) + r(T) = r(T)$$

since $m_T(T) = 0$ by definition. Since $m_T$ has the smallest degree among non-zero polynomials which vanish when $T$ is substituted, we conclude $r(x) = 0$. Hence

$$f(x) = m_T(x)q(x);$$

that is, $m_T(x)$ divides $f(x)$. □

**Corollary 4.19** *Suppose that $T \colon V \to V$ is a linear transformation of a finite-dimensional vector space $V$. Then the minimum polynomial $m_T(x)$ divides the characteristic polynomial $c_T(x)$.*

PROOF: This follows immediately from the Cayley–Hamilton Theorem and Proposition 4.18. □

This corollary gives one link between the minimum polynomial and the characteristic polynomial. The following gives even stronger information, since it says a linear factor $(x - \lambda)$ occurs in one if and only if it occurs in the other.

**Theorem 4.20** *Let $V$ be a finite-dimensional vector space over a field $F$ and let $T\colon V \to V$ be a linear transformation of $V$. Then the roots of the minimum polynomial $m_T(x)$ and the roots of the characteristic polynomial $c_T(x)$ coincide.*

Recall that the roots of $c_T(x)$ are precisely the eigenvalues of $T$. Thus this theorem has some significance in the context of diagonalisation.

PROOF: Let $\lambda$ be a root of $m_T(x)$, so $(x - \lambda)$ is a factor of $m_T(x)$. It could be deduced from Corollary 4.19 (i.e., from the Cayley–Hamilton Theorem) that $(x - \lambda)$ divides $c_T(x)$, but we will do this directly. Factorise

$$m_T(x) = (x - \lambda) f(x)$$

for some non-zero polynomial $f(x)$. Then $\deg f(x) < \deg m_T(x)$, so $f(T) \neq 0$. Hence there exists a vector $v \in V$ such that $w = f(T)v \neq \mathbf{0}$. Now $m_T(T) = 0$, so

$$\mathbf{0} = m_T(T)v = (T - \lambda I) f(T)v = (T - \lambda I)w,$$

so $T(w) = \lambda w$ and we conclude that $w$ is an eigenvector with eigenvalue $\lambda$. Hence $\lambda$ is a root of $c_T(x)$ (by Lemma 4.4).

Conversely, suppose $\lambda$ is a root of $c_T(x)$, so $\lambda$ is an eigenvalue of $T$. Hence there is an eigenvector $v$ (note $v \neq \mathbf{0}$) with eigenvalue $\lambda$. Then $T(v) = \lambda v$,

$$T^2(v) = T(T(v)) = T(\lambda v) = \lambda T(v) = \lambda^2 v,$$

and in general $T^i(v) = \lambda^i v$ for all $i = 1, 2, \dots$. Suppose

$$m_T(x) = x^k + \alpha_{k-1}x^{k-1} + \dots + \alpha_1 x + \alpha_0.$$

Then

$$\begin{aligned}
\mathbf{0} = m_T(T)v &= (T^k + \alpha_{k-1}T^{k-1} + \dots + \alpha_1 T + \alpha_0 I)v \\
&= T^k(v) + \alpha_{k-1}T^{k-1}(v) + \dots + \alpha_1 T(v) + \alpha_0 v \\
&= \lambda^k v + \alpha_{k-1}\lambda^{k-1} v + \dots + \alpha_1 \lambda v + \alpha_0 v \\
&= (\lambda^k + \alpha_{k-1}\lambda^{k-1} + \dots + \alpha_1 \lambda + \alpha_0)v \\
&= m_T(\lambda)v.
\end{aligned}$$

Since $v \neq \mathbf{0}$, we conclude $m_T(\lambda) = 0$; i.e., $\lambda$ is a root of $m_T(x)$. □

To see the full link to diagonalisability, we finally prove:

**Theorem 4.21** *Let $V$ be a finite-dimensional vector space over the field $F$ and let $T\colon V \to V$ be a linear transformation. Then $T$ is diagonalisable if and only if the minimum polynomial $m_T(x)$ is a product of distinct linear factors.*

PROOF: Suppose there is a basis $\mathscr{B}$ with respect to which $T$ is represented by a diagonal matrix:

$$\mathrm{Mat}_{\mathscr{B},\mathscr{B}}(T) = A = \begin{pmatrix} \lambda_1 & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & \lambda_1 & & & & 0 & & & \\ & & & \lambda_2 & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & \lambda_2 & & & & \\ & & & & & & \ddots & & & \\ & & 0 & & & & & \lambda_k & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & \lambda_k \end{pmatrix}$$

where the $\lambda_i$ are the *distinct* eigenvalues. Then

$$A - \lambda_1 I = \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & \lambda_2 - \lambda_1 & & \\ & & & & \ddots & \\ & & & & & \lambda_k - \lambda_1 \end{pmatrix}$$

(with all non-diagonal entries being 0) and similar expressions apply to $A - \lambda_2 I, \dots, A - \lambda_k I$. Hence

$$(A - \lambda_1 I)(A - \lambda_2 I) \dots (A - \lambda_k I) = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} = 0,$$

so

$$(T - \lambda_1 I)(T - \lambda_2 I) \dots (T - \lambda_k I) = 0$$

Thus $m_T(x)$ divides $(x - \lambda_1)(x - \lambda_2) \dots (x - \lambda_k)$ by Proposition 4.18. (In fact, by Theorem 4.20, it equals this product.) Hence $m_T(x)$ is a product of distinct linear factors.

To prove the converse we make use of the following lemma which exploits the greatest common divisor of polynomials.

**Lemma 4.22** *Let $T \colon V \to V$ be a linear transformation of a vector space over the field $F$ and let $f(x)$ and $g(x)$ be coprime polynomials over $F$. Then*

$$\ker f(T)g(T) = \ker f(T) \oplus \ker g(T).$$

To say that $f(x)$ and $g(x)$ are *coprime* means that the largest degree polynomial that divides both of them is a constant (i.e., has degree 0).

PROOF: The greatest common divisor of $f(x)$ and $g(x)$ is a constant, so there exist polynomials $a(x)$ and $b(x)$ over $F$ such that

$$1 = a(x)f(x) + b(x)g(x).$$

Hence

$$I = a(T)f(T) + b(T)g(T),$$

so

$$v = a(T)f(T)v + b(T)g(T)v \qquad \text{for } v \in V. \tag{4.3}$$

Now if $v \in \ker f(T)g(T)$, then

$$g(T)\,(a(T)f(T)v) = a(T)f(T)g(T)v = a(T)\mathbf{0} = \mathbf{0},$$

so $a(T)f(T)v \in \ker g(T)$. (We are here using the fact that if $f(T)$ and $g(T)$ are polynomial expressions in the linear transformation $T$, then the linear transformations $f(T)$ and $g(T)$ *commute*; that is, $f(T)g(T) = g(T)f(T)$.)

Similarly

$$f(T)\,(b(T)g(T)v) = b(T)f(T)g(T)v = b(T)\mathbf{0} = \mathbf{0},$$

so $b(T)g(T)v \in \ker f(T)$. Therefore

$$\ker f(T)g(T) = \ker f(T) + \ker g(T).$$

(Note that $\ker f(T) \subseteq \ker f(T)g(T)$, since if $v \in \ker f(T)$, then $f(T)g(T)v = g(T)f(T)v = g(T)\mathbf{0} = \mathbf{0}$, and similarly $\ker g(T) \subseteq \ker f(T)g(T)$.)

If $v \in \ker f(T) \cap \ker g(T)$, then, from Equation 4.3,

$$v = a(T)f(T)v + b(T)g(T)v = a(T)\mathbf{0} + b(T)\mathbf{0} = \mathbf{0}.$$

Hence $\ker f(T) \cap \ker g(T) = \{\mathbf{0}\}$ and we deduce

$$\ker f(T)g(T) = \ker f(T) \oplus \ker g(T).$$

□

We now return and complete the proof of Theorem 4.21. Suppose $m(x) = (x-\lambda_1)(x-\lambda_2)\dots(x-\lambda_k)$ where the $\lambda_i$ are distinct scalars. Now $m_T(T) = 0$ and distinct linear polynomials are coprime (as only constants have smaller degree), so

$$\begin{aligned} V &= \ker m_T(T) \\ &= \ker(T - \lambda_1 I)(T - \lambda_2 I)\dots(T - \lambda_k I) \\ &= \ker(T - \lambda_1 I) \oplus \ker(T - \lambda_2 I) \oplus \dots \oplus \ker(T - \lambda_k I) \end{aligned}$$

$$= E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k}$$

by repeated use of 4.22. Pick a basis $\mathscr{B}_i$ for each eigenspace $E_{\lambda_i}$. Then $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2 \cup \cdots \cup \mathscr{B}_k$ is a basis for $V$ consisting of eigenvectors for $T$ (using Proposition 3.12). We deduce that $T$ is diagonalisable. $\square$

**Example 4.23** Let us return to our two earlier examples and discuss them in the context of Theorem 4.21.

*In Examples 4.8 and 4.11, we defined linear transformations $\mathbb{R}^3 \to \mathbb{R}^3$ given by the matrices*

$$A = \begin{pmatrix} 8 & 6 & 0 \\ -9 & -7 & 0 \\ 3 & 3 & 2 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 8 & 3 & 0 \\ -18 & -7 & 0 \\ -9 & -4 & 2 \end{pmatrix},$$

*respectively. Determine the minimum polynomials of $A$ and $B$.*

**Solution:** We observed that the matrices have the same characteristic polynomial

$$c_A(x) = c_B(x) = (x+1)(x-2)^2,$$

but $A$ is diagonalisable while $B$ is not. The minimum polynomial of each divides $(x+1)(x-2)^2$ and certainly has $(x+1)(x-2)$ as a factor (by Theorem 4.20). Now Theorem 4.21 tells us that $m_A(x)$ is a product of distinct linear factors, but $m_B(x)$ is not. Therefore

$$m_A(x) = (x+1)(x-2) \qquad \text{and} \qquad m_B(x) = (x+1)(x-2)^2.$$

[Exercise: Verify $(A+I)(A-2I) = 0$ and $(B+I)(B-2I) \neq 0$.]

**Example 4.24** *Consider the linear transformation $\mathbb{R}^3 \to \mathbb{R}^3$ given by the matrix*

$$D = \begin{pmatrix} 3 & 0 & 1 \\ 2 & 2 & 2 \\ -1 & 0 & 1 \end{pmatrix}.$$

*Calculate the characteristic polynomial of $D$, determine if $D$ is diagonalisable and calculate the minimum polynomial.*

**Solution:** The characteristic polynomial is

$$\begin{aligned} c_D(x) &= \det \begin{pmatrix} x-3 & 0 & -1 \\ -2 & x-2 & -2 \\ 1 & 0 & x-1 \end{pmatrix} \\ &= (x-3)(x-2)(x-1) + (x-2) \end{aligned}$$

$$
\begin{aligned}
&= (x-2)(x^2-4x+3+1) \\
&= (x-2)(x^2-4x+4) \\
&= (x-2)^3.
\end{aligned}
$$

Therefore $D$ is a diagonalisable only if $m_D(x) = x - 2$. But

$$D - 2I = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ -1 & 0 & -1 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so $m_D(x) \neq x - 2$. Thus $D$ is not diagonalisable. Indeed

$$(D-2I)^2 = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ -1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so we deduce $m_D(x) = (x-2)^2$.

**Example 4.25** *Consider the linear transformation $\mathbb{R}^3 \to \mathbb{R}^3$ given by the matrix*

$$E = \begin{pmatrix} -3 & -4 & -12 \\ 0 & -11 & -24 \\ 0 & 4 & 9 \end{pmatrix}.$$

*Calculate the characteristic polynomial of $E$, determine if $E$ is diagonalisable and calculate its minimum polynomial.*

**Solution:**

$$
\begin{aligned}
c_E(x) &= \det \begin{pmatrix} x+3 & 4 & 12 \\ 0 & x+11 & 24 \\ 0 & -4 & x-9 \end{pmatrix} \\
&= (x+3)\left((x+11)(x-9)+96\right) \\
&= (x+3)(x^2+2x-3) \\
&= (x+3)(x-1)(x+3) \\
&= (x-1)(x+3)^2.
\end{aligned}
$$

So the eigenvalues of $E$ are 1 and $-3$. Now $E$ is diagonalisable only if $m_E(x) = (x-1)(x+3)$. We calculate

$$E - I = \begin{pmatrix} -4 & -4 & -12 \\ 0 & -12 & -24 \\ 0 & 4 & 8 \end{pmatrix}, \qquad E + 3I = \begin{pmatrix} 0 & -4 & -12 \\ 0 & -8 & -24 \\ 0 & 4 & 12 \end{pmatrix},$$

so

$$(E-I)(E+3I) = \begin{pmatrix} -4 & -4 & -12 \\ 0 & -12 & -24 \\ 0 & 4 & 8 \end{pmatrix} \begin{pmatrix} 0 & -4 & -12 \\ 0 & -8 & -24 \\ 0 & 4 & 12 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence $m_E(x) = (x-1)(x+3)$ and $E$ is diagonalisable.

**Example 4B** *Let*

$$A = \begin{pmatrix} 0 & -2 & -1 \\ 1 & 5 & 3 \\ -1 & -2 & 0 \end{pmatrix}.$$

*Calculate the characteristic polynomial and the minimum polynomial of $A$. Hence determine whether $A$ is diagonalisable.*

SOLUTION:

$$\begin{aligned}
c_A &= \det(xI - A) \\
&= \det \begin{pmatrix} x & 2 & 1 \\ -1 & x-5 & -3 \\ 1 & 2 & x \end{pmatrix} \\
&= x \det \begin{pmatrix} x-5 & -3 \\ 2 & x \end{pmatrix} - 2 \det \begin{pmatrix} -1 & -3 \\ 1 & x \end{pmatrix} + \det \begin{pmatrix} -1 & x-5 \\ 1 & 2 \end{pmatrix} \\
&= x\big(x(x-5)+6\big) - 2(-x+3) + (-2-x+5) \\
&= x(x^2-5x+6) + 2(x-3) - x + 3 \\
&= x(x-3)(x-2) + 2(x-3) - (x-3) \\
&= (x-3)\big(x(x-2)+2-1\big) \\
&= (x-3)(x^2-2x+1) \\
&= (x-3)(x-1)^2.
\end{aligned}$$

Since the minimum polynomial divides $c_A(x)$ and has the same roots, we deduce

$$m_A(x) = (x-3)(x-1) \quad \text{or} \quad m_A(x) = (x-3)(x-1)^2.$$

We calculate

$$\begin{aligned}
(A-3I)(A-I) &= \begin{pmatrix} -3 & -2 & -1 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{pmatrix} \begin{pmatrix} -1 & -2 & -1 \\ 1 & 4 & 3 \\ -1 & -2 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 2 & 0 & -2 \\ -2 & 0 & 2 \\ 2 & 0 & -2 \end{pmatrix} \neq 0.
\end{aligned}$$

Hence $m_A(x) \neq (x-3)(x-1)$. We conclude

$$m_A(x) = (x-3)(x-1)^2.$$

This is not a product of distinct linear factors, so $A$ is not diagonalisable. $\square$