School of Mathematics and Statistics

MT5836 Galois Theory

Problem Sheet VII: Radical extensions; solution of equations by radicals; soluble groups

1. Find a normal radical extension of $\mathbb{Q}$ that contains $\mathbb{Q}(\sqrt[3]{2})$.

2. Find three radical extensions of $\mathbb{Q}$ all containing $\mathbb{Q}(\sqrt{2})$ such that the Galois groups are distinct.

3. Let $f(X) = X^3 - 3X + 1$ and let K be the splitting field of $f(X)$ over $\mathbb{Q}$. Show that $|K : \mathbb{Q}| = 3$ and find a radical extension of $\mathbb{Q}$ containing K.

   Show that K is not itself a radical extension of $\mathbb{Q}$.

4. Show that $X^5 - 6X + 3$ is not soluble by radicals over $\mathbb{Q}$.

5. Let F be a field of characteristic zero. Show that a polynomial of the form $X^4 + bX^2 + c$ is soluble by radicals over F.

6. Let G be a soluble group with a chain of subgroups

$$G = G_0 \geqslant G_1 \geqslant G_2 \geqslant \ldots \geqslant G_d = \mathbf{1}$$

   where, for $i = 1, 2, \ldots, d$, G_i is a normal subgroup of G_{i-1} and G_{i-1}/G_i is abelian.

   (a) If H is a subgroup of G, show that $H \cap G_i$ is a normal subgroup of $H \cap G_{i-1}$ and that $(H \cap G_{i-1})/(H \cap G_i)$ is isomorphic to a subgroup of G_{i-1}/G_i for each i. [Hint: Second Isomorphism Theorem.]

   Deduce that subgroups of soluble groups are soluble.

   (b) If A, B and C are subgroups of G with $A \leqslant B$, show that $A(B \cap C) = AC \cap B$. [This result is known as the *Modular Law*.]

   (c) If N is a normal subgroup of G, show that G_iN/N is a normal subgroup of $G_{i-1}N/N$ and that $(G_{i-1}N/N)/(G_iN/N)$ is isomorphic to a quotient of G_{i-1}/G_i for each i. [Hint: Use the Second and Third Isomorphism Theorems and the Modular Law.]

   Deduce that quotients of soluble groups are soluble.

7. Let G be a group and N be a normal subgroup of G.

   (a) If G/N is soluble, show that there is a chain of subgroups

$$G = G_0 \geqslant G_1 \geqslant \ldots \geqslant G_k = N$$

   such that G_i is a normal subgroup of G_{i-1} and G_{i-1}/G_i is abelian for $i = 1, 2, \ldots, k$. [Hint: Correspondence Theorem.]

   (b) Deduce that if G/N and N are soluble, then G is soluble.