School of Mathematics and Statistics

MT5836 Galois Theory

Problem Sheet III: Splitting Fields and Normal Extensions

1. For each of the following polynomials $f(X)$ and given base field F, determine the splitting field K of $f(X)$ over F and calculate the degree $|K : F|$ of the extension:
   (a) $X^2 + 1$ over $\mathbb{Q}$;
   (b) $X^2 + 1$ over $\mathbb{R}$;
   (c) $X^2 - 4$ over $\mathbb{Q}$;
   (d) $X^4 + 4$ over $\mathbb{Q}$;
   (e) $X^4 - 1$ over $\mathbb{Q}$;
   (f) $X^4 + 1$ over $\mathbb{Q}$;
   (g) $X^6 - 1$ over $\mathbb{Q}$;
   (h) $X^6 + 1$ over $\mathbb{Q}$;
   (i) $X^6 - 27$ over $\mathbb{Q}$.

2. For each of the following polynomials $f(X)$ and given base field F, determine the degree of the splitting field of $f(X)$ over F:
   (a) $X^3 - 2$ over $\mathbb{F}_5$;
   (b) $X^3 - 3$ over $\mathbb{F}_{13}$.

3. Let p be a prime and $f(X) = X^p - 2$. Find the splitting field of $f(X)$ over $\mathbb{Q}$ and show that the degree of this extension is $p(p-1)$.

4. Let $f(X)$ be a polynomial over a field F and let K be the splitting field of $f(X)$ over F. If L is an intermediate field (that is, $F \subseteq L \subseteq K$), show that K is the splitting field of $f(X)$ over L.

5. Let ϕ be an automorphism of a field F. Show that the set of fixed-points of ϕ,
$$\mathrm{Fix}_F(\phi) = \{\, a \in F \mid a\phi = a \,\},$$
is a subfield of F. Hence deduce that ϕ is a P-isomorphism where P is the prime subfield of F.

6. (a) Determine all automorphisms of $\mathbb{Q}$.
   (b) Determine all automorphisms of $\mathbb{Q}(\sqrt{2})$.
   (c) Determine all $\mathbb{Q}$-automorphisms of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$.
   (d) Show that the only automorphism of $\mathbb{R}$ is the identity.

7. Suppose that $f(X)$ is an arbitrary polynomial over a field F, K is the splitting field for $f(X)$ over F, and α and β are roots of $f(X)$ in K. Does there exist an automorphism of K that maps α to β?

8. Which of the following fields are normal extensions of $\mathbb{Q}$? [As always, justify your answers.]

   (a) $\mathbb{Q}(\sqrt{2})$;

   (b) $\mathbb{Q}(\sqrt[4]{2})$;

   (c) $\mathbb{Q}(\sqrt{2}, \sqrt{3})$;

   (d) $\mathbb{Q}(\theta)$, where $\theta^4 - 10\theta^2 + 1 = 0$.

9. Let $F \subseteq K \subseteq L$ be field extensions where L is a finite extension of F. Prove, or give a counterexample, to each of the following assertions:

   (a) If L is a normal extension of K, then L is a normal extension of F.

   (b) If L is a normal extension of F, then L is a normal extension of K.

   (c) If L is a normal extension of F, then K is a normal extension of F.