University of St Andrews

SPECIMEN EXAM EXAMINATION DIET

SCHOOL OF MATHEMATICS & STATISTICS

MODULE CODE: **MT5830**

MODULE TITLE: Topics in Geometry and Analysis

EXAM DURATION: $2\frac{1}{2}$ hours

EXAM INSTRUCTIONS: Attempt ALL questions.

The number in square brackets shows the maximum marks obtainable for that question or part-question.

Your answers should contain the full working required to justify your solutions.

PERMITTED MATERIALS: Non-programmable calculator

YOU MUST HAND IN THIS EXAM PAPER AT THE END OF THE EXAM

PLEASE DO NOT TURN OVER THIS EXAM PAPER UNTIL YOU ARE INSTRUCTED TO DO SO.

1. (a) Give the definition of a *geodesic* between two distinct points u, w in a metric space. **[2]**

(b) Consider the Poincaré disk model of hyperbolic space $(\mathbb{D}^2, d_{\mathbb{D}^2})$ and let $u, v \in \mathbb{D}^2$ be distinct points. Precisely how many geodesics are there joining u and v? Describe all possible forms of geodesics in $\mathbb{D}^2$. **[2]**

(c) Let $C = C_{\mathbb{D}^2}(i/2, 2) \subseteq \mathbb{D}^2$ be the hyperbolic circle centred at $i/2$ with (hyperbolic) radius 2. Prove that the (hyperbolic) circumference of C is given by

$$\pi(e^2 - e^{-2}).$$ **[5]**

(d) Let u, w be distinct points on the boundary of the circle C from the previous question. Let L_u be a geodesic from u to $i/2$ and L_w be a geodesic from w to $i/2$ and suppose L_u and L_w intersect at $i/2$ at an angle of $\pi/6$. Compute the length of the shortest path between u and v along the boundary of C (i.e. the shortest arc). Note that this is *not* the hyperbolic distance between u and v. You must fully justify your argument. **[3]**

(e) Let $u, w \in \mathbb{D}^2$ be distinct points which are both at hyperbolic distance 2 from the origin and both have strictly positive real and imaginary parts. Let H be the hyperbolic triangle with vertices u, w and 0 and let E be the Euclidean triangle with vertices u, w and 0. Finally, let S be the area enclosed by the hyperbolic geodesic between u and 0, the hyperbolic geodesic between w and 0, and the shortest arc joining u and w on the boundary of the circle $C_{\mathbb{D}^2}(0, 2)$.

Describe precisely the inclusion relationships between the sets H, E and S. You must justify your answer. **[3]**

2. (a) State the *Gauss-Bonnet Theorem.* (You do not need to prove it.) **[1]**

(b) Define a hyperbolic square to be a 4 sided polygon in $\mathbb{D}^2$ where the sides are hyperbolic geodesics meeting at right angles. Using the Gauss-Bonnet Theorem or otherwise, prove that hyperbolic squares do not exist. **[3]**

(c) Let $\Delta \subset \mathbb{D}^2$ be a right-angled hyperbolic triangle with sides of length a and b adjacent to the right-angle and remaining side of length c. Prove that

$$\cosh c = \cosh a \cosh b.$$

You may use (without proof) the identity

$$\cosh d_{\mathbb{D}^2}(z,w) = \frac{|1-z\overline{w}|^2 + |z-w|^2}{|1-z\overline{w}|^2 - |z-w|^2}$$

where $w, z \in \mathbb{D}^2$ are arbitrary distinct points. **[4]**

(d) Prove from the definition of cosh that

$$x - \log 2 \leq \log \cosh x \leq x.$$

for $x \geq 0$. **[1]**

(e) Using parts (c) and (d) above, prove that for the triangle Δ in part (c) we have

$$a + b - 2\log 2 \leq c \leq a + b.$$ **[3]**

(f) Comment briefly on the result of the previous question in the context of the triangle inequality and compare this with the corresponding situation in Euclidean space. **[1]**

3. (a) Give the definition of a *Fuchsian group*. **[2]**

(b) Describe in terms of both the trace and fixed points what it means for an element of $\mathrm{PSL}(2,\mathbb{R})$ to be:
(i) *elliptic*, (ii) *hyperbolic*, (iii) *parabolic*. **[2]**

(c) Let $\Gamma \leq \mathrm{PSL}(2,\mathbb{R})$ be a Fuchsian group and let $z \in \mathbb{H}^2$ be an arbitrary point not fixed by any elliptic elements of Γ. Define the *Dirichlet region* of Γ at the base point z. **[2]**

(d) Let $g, h \in \mathrm{PSL}(2,\mathbb{R})$ be hyperbolic elements which share precisely one fixed point. Prove that the group generated by g and h, $\langle g, h\rangle \leq \mathrm{PSL}(2,\mathbb{R})$, is *not* a Fuchsian group. **[5]**

4. (a) Define the *limit set* $L(\Gamma)$ of a Fuchsian group Γ acting on the Poincaré disk. **[2]**

(b) Let Γ be a Fuchsian group and $g \in \Gamma$. Prove that $g(L(\Gamma)) = L(\Gamma)$. **[3]**

(c) Prove that if Γ is a Fuchsian group such that the limit set consists only of one point $z \in S^1$, then every non-identity element must be parabolic and must fix z. **[3]**

(d) Using the previous question, prove that a Fuchsian group whose limit set is a single point must be cyclic, i.e. generated by a single element. **[3]**

END OF PAPER
