MT5823 Semigroup theory: Problem sheet 1 (James D. Mitchell)
Definition and basic properties

Let S be a semigroup, and let $e, z, u \in S$. Then:

(i) e is a ***left identity*** if $ex = x$ for all $x \in S$;

(ii) e is a ***right identity*** if $xe = x$ for all $x \in S$;

(iii) z is a ***left zero*** if $zx = z$ for all $x \in S$;

(iv) z is a ***right zero*** if $xz = z$ for all $x \in S$;

(v) z is a ***zero*** if it is both a left zero and a right zero;

(vi) e is an ***idempotent*** if $e^2 = e$.

1-1. Let S be a semigroup.

(a) Suppose that S has a left identity e and a right identity f. Show that $e = f$ and S has a 2-sided identity.
(b) Prove that if S has a zero element, then it is unique.
(c) Is it true that if a semigroup S has left zero and right zero, then they are equal and S has a zero element?

1-2. Prove that the order of the full transformation semigroup T_n is n^n.

1-3. Prove that a mapping $f \in T_n$ is a right zero if and only if it is a constant mapping. Does T_n have left zeros? Does it have a zero? Does T_n have an identity?

Let $f \in T_X$. Then denote the ***image*** of f by $\mathrm{im}(f)$, that is the set

$$\mathrm{im}(f) = \{\, xf \;:\; x \in X \,\}.$$

The size $|\mathrm{im} f|$ of the image is called the ***rank*** of f, and is denoted by $\mathrm{rank}(f)$. For example, if

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 1 & 1 & 5 \end{pmatrix},$$

then $\mathrm{im}(f) = \{1, 3, 5\}$ and $\mathrm{rank}(f) = 3$.

1-4. Prove that $f \in T_n$ is an idempotent if and only if $(x)f = x$ for all $x \in \mathrm{im}(f)$.

1-5. Prove that T_n has precisely

$$\sum_{k=1}^{n} \binom{n}{k} k^{n-k}$$

idempotents.

1-6. Prove that for any $f, g \in T_n$

$$\mathrm{rank}(fg) \leq \min(\mathrm{rank}(f), \mathrm{rank}\, g).$$

Find examples which show that both the equality and the strict inequality may occur.

1-7. Let G be a group and $a \in G$. Then define

$$aG = \{\, ag \;:\; g \in G \,\} \quad \text{and} \quad Ga = \{\, ga \;:\; g \in G \,\}.$$

Prove that $aG = Ga = G$ for all $a \in G$.

1-8.* Let S be a semigroup such that $aS = Sa = S$ for all $a \in S$.

(a) If $b \in S$ is arbitrary, then prove that there exists an element $e \in S$ such that $be = b$.
(b) Prove that e is a right identity for S.
(c) In a similar way prove that S has a left identity too. Conclude that S is a monoid.
(d) Prove that S is a group.