2 Functions of a Complex Variable

2.1 Introduction

We shall denote by $\mathbb{C}$ the set of all complex numbers. Just as we define functions of a real variable we can define functions of a complex variable:

$$w = f(z),$$

where z is allowed to take values from some domain $D \subset \mathbb{C}$ and f is a rule which assigns to each z a unique complex number w. As all complex numbers can be expressed in the form $x + iy$ we can write

$$w = f(x + iy) = u(x, y) + iv(x, y),$$

where u and v are the real and imaginary parts of w. Note that u and v are each a real-valued function of two real variables.

Example 2.1.1. Determine the real and imaginary parts of $f(z) = z^2$, for $z \in \mathbb{C}$.

Solution

$$w = (x + iy)^2 = x^2 + 2ixy + (iy)^2 = (x^2 - y^2) + i(2xy) = u + iv,$$

thus

$$u = x^2 - y^2 \quad \text{and} \quad v = 2xy.$$

For each value of z there corresponds a w. •

Example 2.1.2. In example (2.1.1) find w when $z = 1 + i$.

Solution From above (where $x = y = 1$), we have

$$u = x^2 - y^2 = 1 - 1 = 0 \quad \text{and} \quad v = 2xy = 2,$$

thus $w = 2i$. •

Unlike the case of real variable, a plot of $w = f(z)$ is not possible as that would require 4 dimensions. However, we can still have a pictorial description of $w = f(z)$ by thinking of f as mapping points from the z-plane to points on a corresponding w-plane (or to points on the z-plane itself).

Example 2.1.3. Show that points on the unit circle (circle centred at the origin with unit radius) in the $z-$plane is mapped by $f(z) = z^2$ onto the unit circle in the $w-$plane.

Solution Using the results of example (2.1.1) with $x^2 + y^2 = 1$ we have

$$\begin{aligned} u^2 + v^2 &= (x^2 - y^2)^2 + 4x^2y^2 = (x^4 - 2x^2y^2 + y^4) + 4x^2y^2 \\ &= x^4 + 2x^2y^2 + y^4 = (x^2 + y^2)^2 = 1. \end{aligned}$$

Therefore, $x^2 + y^2 = 1$ maps to $u^2 + v^2 = 1$ (unit circle in the $w-$plane). Alternatively, we can write

$$|w| = |f(z)| = |z^2| = |z|^2.$$

So if $|z| = 1$ then $|w| = 1$, if $|z| < 1$ then $|w| < 1$ and if $|z| > 1$ then $|w| > 1$. This means that points on the unit circle in the $z-$plane map onto the unit circle in the $w-$plane and that points inside and outside the unit circle in the $z-$plane map to points inside and outside the unit circle in the $w-$plane, respectively. Finally, using the modulus-argument representation $z = re^{i\theta}$ we can write

$$w = f(re^{i\theta}) = r^2 e^{2i\theta}.$$

Apparently, under the effect of f, θ and r becomes 2θ and r^2, respectively. •

Example 2.1.4. Show that $f(z) = z^2$ maps points on the straight line $x = 1$ in the z-plane onto the parabola $v^2 = 4(1 - u)$ in the $w-$plane.

Solution The straight line $x = 1$ is given by $z = 1 + iy$. So

$$f(z) = (1 + iy)^2 = 1 - y^2 + 2iy$$

and therefore

$$u = 1 - y^2 \quad \text{and} \quad v = 2y.$$

This leads to (see figure 5)

$$v^2 = 4y^2 = 4(1 - u).$$

•

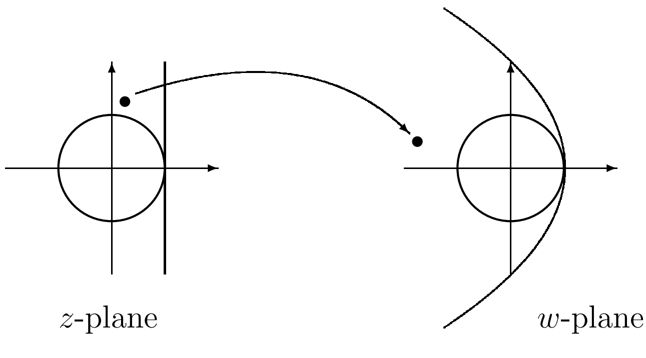

Figure 5: $w = z^2$.

In the above simple example it is quite easy to see how points in one plane map to points in the other. But in general this can be quite a difficult exercise. As an illustration consider the following example.

Example 2.1.5. Consider the complex quadratic function $f(z) = 2z^2 - iz$ and the unit circle $|z| = 1$. With the aid of MAPLE, the image of $|z| = 1$ is depicted by figure 6.

Example 2.1.6. Consider the function

$$f(z) = 1/z, \quad \text{for} \quad z \neq 0.$$

Determine the image of the line $y = x$ (with the origin removed).

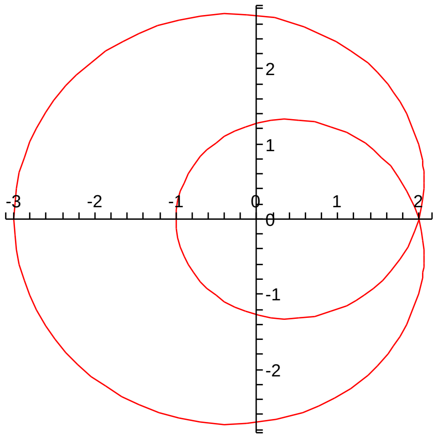

Figure 6: $w-$plane for $w = 2z^2 - iz$ where $|z| = 1$.

Solution

$$w = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2}$$

giving

$$u = \frac{x}{x^2+y^2} \quad \text{and} \quad v = \frac{-y}{x^2+y^2}.$$

For $y = x \neq 0$, we have $v = -u$. So the image of the line $y = x$ with the origin removed is the line $v = -u$ (of course with the origin removed). This image, together with further detail of the map, can be derived without having to calculate u and v. Indeed using the polar form $z = re^{i\theta}$, we have $f(z) = r^{-1}e^{-i\theta}$. It is then easy to see that the map is a combination of a reflection through the real axis ($\theta \to -\theta$) and an inverse of length ($r \to 1/r$). •

Example 2.1.7. Joukowsky's airfoil:

One of the best known example of complex function is

$$w = z + \frac{1}{z},$$

defined $\forall z \in \mathbb{C}$, $z \neq 0$. when applied to a circle whose centre is not the origin (N.E. Joukowsky, 1847-1921).

2.2 Limits and Continuity

The usual idea of a limit can easily be extended to a function of a complex variable. Suppose $f(z)$ is defined in a domain D and suppose further that by taking z sufficiently close to z_0, a point within D, we can make $f(z)$ as close as we wish to some complex number L. Then we say that

$$\lim_{z \to z_0} f(z) = L,$$

i.e. $f(z)$ has limit L as z tends to z_0.

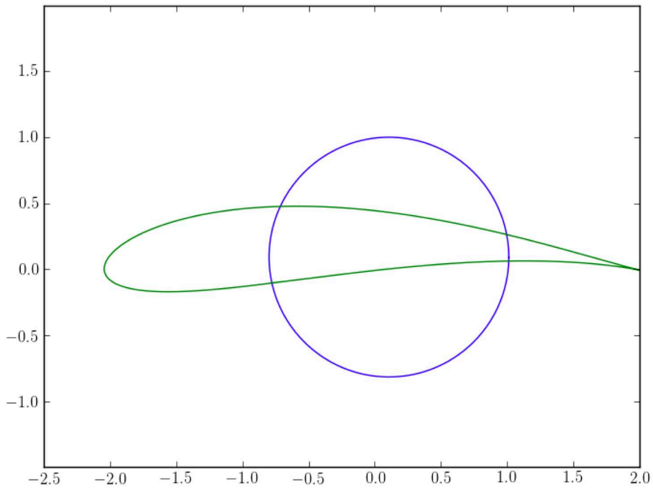

Figure 7: Joukowsky's Airfoil: Circle centred at $z = 0.1(1 + i)$ passing through the point (1,0) and its transform through $w = z + 1/z$.

There is an important assumption in the above definition of limit. The point z_0 is a point in the z-plane and we can approach this point by different routes. Therefore the value L must be independent of the path of approach - the limit must be unique! This is analogous to the situation for limits of real-valued functions of two real variables.

Formally, you should realise that the definition of the limit we have seen in section 0 still works!

$$\forall \epsilon \in (0, \infty), \exists \alpha \in (0, \infty) \text{ such that } |z - z_0| < \alpha \Rightarrow |f(z) - L| < \epsilon,$$

but now $z \in \mathbb{C}$! And this guaranties it has to work for z going to z_0 from any direction as the condition $|z - z_0| < \alpha$ covers an open disk around z_0.

Example 2.2.1. The limit

$$\lim_{z \to 0} \frac{\bar{z}}{z}$$

does <u>not exist</u> since

$$\lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{z \to 0} \frac{x - iy}{x + iy} = \begin{cases} 1 & \text{along } y = 0 \\ -1 & \text{along } x = 0. \end{cases}$$

Example 2.2.2. Similar to the above example, the limit

$$\lim_{z \to 0} \frac{z^2}{|z|^2}$$

is path-dependent, hence does <u>not exist</u>. Indeed

$$\lim_{z \to 0} \frac{z^2}{|z|^2} = \lim_{z \to 0} \frac{x^2 - y^2 + 2ixy}{x^2 + y^2} = \begin{cases} 1 & \text{along } y = 0 \\ -1 & \text{along } x = 0 \\ i & \text{along } x = y. \end{cases}$$

The function $f(z)$ is said to be **continuous** at z_0 if

$$\lim_{z \to z_0} f(z) = L = f(z_0).$$

In the above we require

- the limit to exist and be uniquely defined;
- the value of $f(z_0)$ to exist;
- and for both these complex numbers to be the same.

If any of these conditions are not met then the function is said to be discontinuous.

Again, the similarity to the two variable situation should be noted. The complex function $f(z)$ can be expressed as $u(x, y) + iv(x, y)$ and we require the continuity of the functions $u(x, y)$ and $v(x, y)$ at the point (x_0, y_0) for the continuity of $f(z)$. Hence we require

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u(x_0, y_0)$$

and

$$\lim_{(x,y)\to(x_0,y_0)} v(x, y) = v(x_0, y_0).$$

Example 2.2.3. $f(z) = z^2$ is continuous everywhere.

Example 2.2.4. $f(z) = 1/z$ is continuous everywhere except at the origin.

Example 2.2.5. $f(z) = \lim_{n\to\infty}(z^n - 1)/(z^n + 1)$ defines what could be called a "step function", for the following reason

$$f(z) = \lim_{n\to\infty} \frac{z^n - 1}{z^n + 1} = \begin{cases} 1 & \text{for } |z| > 1 \\ -1 & \text{for } |z| < 1. \end{cases}$$

Note that $f(z)$ is undefined for $|z| = 1$ as the limit does not exist. The function is continuous everywhere, except on the unit circle.

2.3 Differentiability

Definition:

The definition of the derivative of a complex function looks much the same as that for a real function. The function $f(z)$ is said to be **differentiable** at z_0 if

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{(z - z_0)} \quad \text{exists (and by implication, is unique).}$$

The derivative at z_0 is denoted by $f'(z_0)$ and all the usual rules of differentiation apply.

Example 2.3.1. $\dfrac{d}{dz}z^2 = 2z$ since

$$\lim_{z\to z_0} \frac{z^2 - z_0^2}{z - z_0} = \lim_{z\to z_0} \frac{(z - z_0)(z + z_0)}{z - z_0} = \lim_{z\to z_0} (z + z_0) = 2z_0 \text{ for all } z_0.$$

Example 2.3.2. $\frac{d}{dz}z^n = nz^{n-1}$, integer n, since

$$\begin{aligned}\lim_{z\to z_0}\frac{z^n - z_0^n}{z - z_0} &= \lim_{z\to z_0}\frac{(z-z_0)(z^{n-1}+z^{n-2}z_0+z^{n-3}z_0^2+\cdots+z_0^{n-1})}{z-z_0}\\ &= (z_0^{n-1}+z_0^{n-2}z_0+z_0^{n-3}z_0^2+\cdots+z_0^{n-1}) = nz_0^{n-1},\end{aligned}$$

for all z_0 if $n \geq 0$ and for $z_0 \neq 0$ if $n < 0$.

Example 2.3.3. The exponential function e^z is formally defined by

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{n=0}^{\infty}\frac{z^n}{n!}$$

where $0! = 1$, and $z^0 = 1\forall z$.

From the derivatives of power functions we have

$$\frac{d}{dz}e^z = \frac{d}{dz}\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = e^z.$$

Note that we recover that the exponential function satisfy $e^{w+z} = e^w e^z$.

Proof: Consider $f(z) = e^{w+z}/e^z$, for $w \in \mathbb{C}$, a complex constant. Using the quotient rule, we can show that $f'(z) = 0$. Therefore $f(z) = const = f(0) = e^w$. So $e^{w+z}/e^z = e^w$, hence the result.

2.4 Holomorphic functions and analytic functions

Defintions:

Consider D an open set in $\mathbb{C}$. A function f of the variable $z \in D \subset \mathbb{C}$ with values in $\mathbb{C}$ is **holomorphic** in the domain D if f in differentiable in a neighbourhood[1] of all points of D.

We will see later that because of the restrictions of notion of differentiability in $\mathbb{C}$, associated with the existence of the limit that if a function f is holomorphic in D, it is infinitely differentiable at all points z of D (such a thing is obviously not true in real analysis).

More than this, we can write down a Taylor expansion (in the same way you would to in Real analysis) about all $a \in D$

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^2$$

where

$$a_n = \frac{f^{(n)}(a)}{n!}$$

.

[1] A neighbourhood of z is or includes an open disk centred at z.

So not only $f^{(n)}(a)$ exists $\forall n$ (infinitely differentiable), but the Taylor series converges. This means in fact that all **holomorphic** functions in $\mathbb{C}$ are **analytic**[2]. Since it is obvious that all analytic functions are differentiable, the two terms are equivalent in Complex Analysis. Talking about a holomorphic function or an analytic function is the same. As a consequence, we will use both terms in this course, without distinction. **This is not true in real analysis, where the two definitions are not equivalent**.

** There is no magic in $\mathbb{C}$. This result is due to the fact that differentiability in $\mathbb{C}$ is much more restrictive than in $\mathbb{R}$.

The examples we have seen are simple examples of holomorphic functions, i..e analytic functions - in fact, they are analytic throughout the complex plane and as such are called **entire functions**.

In this module, we are particularly interested in analytic functions, except possibly at isolated points.

Example 2.4.1. $f(z) = 1/z$ is analytic, with derivative $-1/z^2$, everywhere except at $z = 0$.

The theory, so far, looks much like that for real functions of one variable and, as the above examples illustrate, **all the standard results and techniques concerning limits and derivatives apply**. However, the definition of a derivative does have one important consequence. Given

$$f(z) = u(x, y) + iv(x, y), \quad \text{where} \quad z = x + iy,$$

we have

$$f'(z) = \lim_{\delta z \to 0} \frac{u(x + \delta x, y + \delta y) + iv(x + \delta x, y + \delta y) - u(x, y) - iv(x, y)}{(\delta x + i\delta y)}.$$

The uniqueness of this limit with respect to all directions of approach imposes stiff constraints on $u(x, y)$ and $v(x, y)$. Taking the limit in the y-direction with x fixed ($\delta x = 0$) yields

$$\begin{aligned} f'(z) &= \lim_{\delta y \to 0} \frac{u(x, y + \delta y) + iv(x, y + \delta y) - u(x, y) - iv(x, y)}{i\delta y} \\ &= \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}. \end{aligned}$$

Now repeating the process but with y fixed and $\delta x \to 0$ yields

$$f'(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

For an analytic function these limits must be the same. Hence, we obtain the **Cauchy–Riemann** equations:

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.}$$

[2]By definition, an analytic function is a function which has a Taylor expansion.

Thus, analyticity implies Cauchy–Riemann equations. If we require the partial derivatives of u and v to be continuous throughout D then it can be shown that the Cauchy-Riemann equations imply analyticity in D.

Example 2.4.2. Verify that the Cauchy–Riemann equations hold for $f(z) = z^2$.

Solution From example (2.1.1) we have

$$u = x^2 - y^2 \quad \text{and} \quad v = 2xy.$$

So that

$$\frac{\partial u}{\partial x} = 2x,\ \frac{\partial v}{\partial y} = 2x,\ \frac{\partial u}{\partial y} = -2y \quad \text{and} \quad \frac{\partial v}{\partial x} = 2y$$

hence result. •

Example 2.4.3. Show that $f(z) = \bar{z}$ is not analytic.

Solution Here $f(z) = x - iy = u + iv$ so that

$$u = x,\ v = -y \text{ and } \frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y}.$$

•

Example 2.4.4. Show that $f(x + iy) = x + iy^2$ is not an analytic function.

Solution In this example $u = x$ and $v = y^2$, so

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = 2y$$

and in general

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}.$$

•

Example 2.4.5. Verify that the Cauchy–Riemann equations hold everywhere for the entire function $f(z) = 2z^2 - iz$ (example 2.1.5).

Solution

$$2z^2 - iz = 2(x^2 - y^2) + 4ixy - ix + y,$$

so

$$u = 2(x^2 - y^2) + y, \quad v = 4xy - x.$$

It follows that

$$\frac{\partial u}{\partial x} = 4x, \quad \frac{\partial v}{\partial y} = 4x \checkmark$$

and

$$\frac{\partial u}{\partial y} = -4y + 1\ , \quad \frac{\partial v}{\partial x} = 4y - 1. \checkmark$$

•

Given two functions $u(x, y)$ and $v(x, y)$, it is important to realise that the chance for these functions to be the real and imaginary parts of an analytic function is quite remote. The real and imaginary parts of an analytic function are special and related to each other through the Cauchy–Riemann equations. Further properties of u and v can be derived by differentiating these equations:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)$$

and

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right).$$

So

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2},$$

which requires u to satisfy Laplace's equation

$$\boxed{\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.}$$

Likewise, we can show that v must also satisfy Laplace's equation

$$\boxed{\Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.}$$

Solutions to Laplace's equation are said to be **harmonic** functions and u and v are said to be **harmonic conjugates**.

Example 2.4.6. Does there exist an analytic function $f(x + iy) = x^2 + iv(x, y)$?

Solution

$$\frac{\partial^2 u}{\partial x^2} = 2 \quad \text{but} \quad \frac{\partial^2 u}{\partial y^2} = 0.$$

So $u(x, y) = x^2$ does not satisfy Laplace's equation (it is not a harmonic function). Therefore it cannot be the real part of an analytic function - no such analytic function exists! •

Example 2.4.7. Given $u = e^x \cos y$, construct an analytic function $f(z)$ with u as its real part.

Solution We have

$$\frac{\partial u}{\partial x} = e^x \cos y,\ \frac{\partial u}{\partial y} = -e^x \sin y,\ \frac{\partial^2 u}{\partial x^2} = e^x \cos y,\ \frac{\partial^2 u}{\partial y^2} = -e^x \cos y.$$

So $\Delta u = 0$: u is harmonic and can be the real part of an analytic function. To determine the corresponding v, we resort to the Cauchy–Riemann equations. First

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^x \cos y \implies v = e^x \sin y + g(x).$$

Second

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = e^x \sin y \quad \Longrightarrow \quad v = e^x \sin y + h(y).$$

Comparing these results yields

$$g(x) = h(y),$$

which implies $g(x) = h(y) =$ constant. By ignoring this constant we have

$$v = e^x \sin y.$$

Hence,

$$f(z) = u + iv = e^x \cos y + ie^x \sin y = e^x(\cos y + i \sin y) = e^{x+iy} = e^z.$$

•

It should be noted that given a harmonic function u of x, y ($\Delta u = 0$) we can always construct a holomorphic function f with $Re(f) = u$. We can find the imaginary part of f, v from the Cauchy conditions in a similar way to what was done in the example above.

2.5 Elementary Analytic Functions

We have already seen that z^n, n integer, is analytic everywhere (an entire function) and consequently so are polynomials in z. Likewise, e^z (as defined earlier via power series) is an entire function. The real and imaginary parts of e^z satisfy Laplace's equation (see example 2.4.7). By the same argument the real and imaginary parts of z^n are solutions of Laplace's equation.

Let $z = x + iy$, then

$$e^z = e^x e^{iy} = e^x(\cos y + i \sin y).$$

Hence

$$|e^z| = e^x > 0 \quad \text{and} \quad \arg(e^z) = y$$

and e^z is periodic (in y):

$$e^z = e^{z+2n\pi i}, \quad n \text{ integer}.$$

Thus, the infinite strip $-\infty < x < \infty$, $-\pi < y \leq \pi$ is mapped by $w = e^z$ to all points on the w-plane, with the exception of $w = 0$.

It is natural to define the **hyperbolic functions** $\sinh z$ and $\cosh z$ in terms of e^z:

$$\boxed{\cosh z = \frac{e^z + e^{-z}}{2} \quad \text{and} \quad \sinh z = \frac{e^z - e^{-z}}{2}}$$

and they are also analytic everywhere (i.e. entire functions). The derivatives follow the usual pattern.

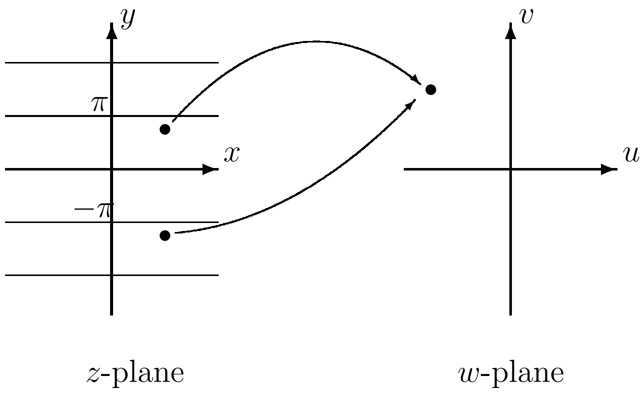

Figure 8: The periodic nature of $w = e^z$.

Recall the identities

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

In accordance with these identities, we define the **trigonometric functions**

$$\boxed{\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.}$$

All the usual trigonometric formulae apply and it is easy to see from the definition that

$$\frac{d}{dz}\cos z = -\sin z \quad \text{and} \quad \frac{d}{dz}\sin z = \cos z.$$

It is worth noting the following relationships:

$$\boxed{\begin{array}{ll} \cos iz = \cosh z\,, & \sin iz = i\sinh z\,, \\ \cosh iz = \cos z\,, & \sinh iz = i\sin z. \end{array}}$$

Example 2.5.1. Determine the real and imaginary parts of $\cos z$.

Solution Let $z = x + iy$, so that

$$\cos z = \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2}$$

$$\cos z = \frac{e^{-y+ix} + e^{y-ix}}{2} = \frac{e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)}{2}$$

$$\cos z = (\cos x\cosh y) - i(\sin x\sinh y) = u + iv.$$

Hence

$$u = \cos x\cosh y \quad \text{and} \quad v = -\sin x\sinh y$$

and each is a solution of Laplace's equation. •

2.6 The logarithm of a complex variable

Recall that the exponential function is periodic, with period $2\pi i$. Indeed, we have

$$f(z) = e^z = e^{z+2n\pi i}\ , \quad n = 0, \pm1, \pm2, \cdots .$$

Hence, in order to define a logarithm as an inverse function, we need to restrict the domain of $f(z) = e^z$ to make it one-to-one. This can be accomplished by considering the argument θ of z in a range of 2π. Different ranges define different branches of the inverse function. For the principal range $\theta \in (-\pi, \pi]$, we define the *principal logarithm*, denoted by $\operatorname{Log} z$, by

$$\operatorname{Log} z = \operatorname{Log}(|z|e^{i\operatorname{Arg}(z)}) = \ln|z| + i\operatorname{Arg}(z).$$

Apparently, for each $z \neq 0$, the image $\operatorname{Log} z$ lies in the infinite strip $-\infty < u < \infty$, $-\pi < v \leq \pi$.

It can be seen that the function $\operatorname{Log} z$ defined above is discontinuous on the negative real axis. Indeed, consider a point $(-a, 0)$ on the negative real axis. As z approaches $(-a, 0)$ from above and below the axis, $\operatorname{Log} z$ approaches $\ln a + i\pi$ and $\ln a - i\pi$, respectively. Therefore, in order to make $\operatorname{Log} z$ a continuous function, we remove the negative real line (including the origin) from the z-plane. The resulting plane is often called the **cut plane**, as we imagine cutting the z-plane along the negative real axis to the origin.

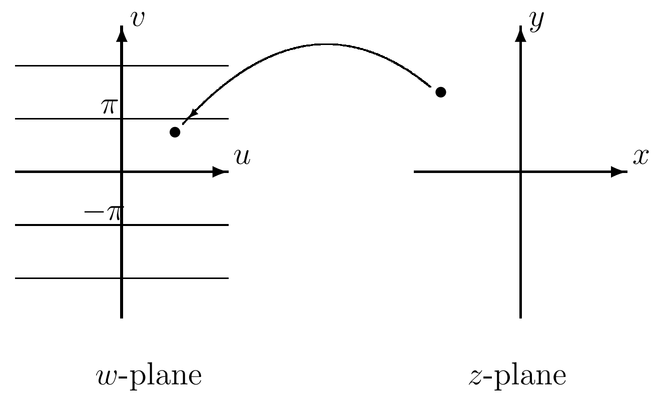

Figure 9: $\operatorname{Log} z$

In the above the point $z = 0$ is called a **branch point** and the negative real axis a **branch cut**. On the cut plane, the function $\operatorname{Log} z$ is analytic.

It is straightforward to define a "general" logarithm, denoted by $\log z$, for a general range of the argument of z other than the principal range. Just as the function $\operatorname{Log} z$ is analytic when z remains within the cut-plane, the function $\log z$ is also analytic within each branch (for each value of n in $z = re^{i(\theta+2n\pi)}$) for z restricted to the cut-plane. Thus, we can contemplate differentiation of $\log z$ at points in the cut plane and the outcome is valid within each branch of $\log z$. Write $w = \log z$ as $z = e^w$ and diffrentiate we obtain

$$1 = \frac{d}{dz}e^w = \frac{d}{dw}e^w\,\frac{dw}{dz} = e^w\,\frac{dw}{dz}.$$

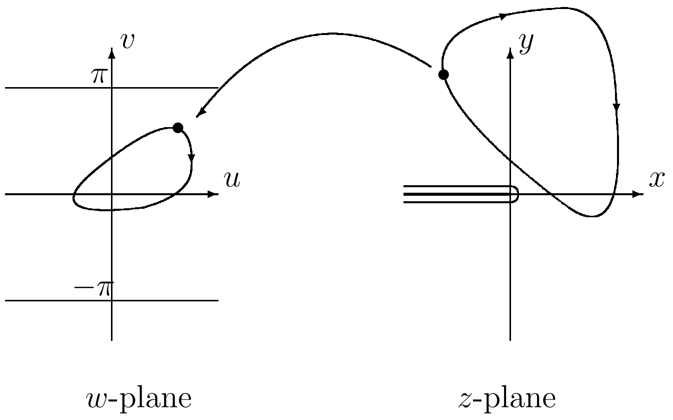

Figure 10: Log z

It follows that

$$\frac{dw}{dz} = \frac{1}{e^w} = \frac{1}{z}.$$

Alternatively, we may calculate the derivative of $\log z$ in the usual way. Indeed, from the definition of derivative we have

$$\begin{aligned}
\frac{dw}{dz} &= \lim_{\delta z \to 0} \frac{\log(z+\delta z) - \log z}{\delta z} = \lim_{\delta r, \delta\theta \to 0} \frac{\log\left[(r+\delta r)e^{i(\theta+\delta\theta)}\right] - \log(re^{i\theta})}{(r+\delta r)e^{i(\theta+\delta\theta)} - re^{i\theta}} \\
&= \lim_{\delta r, \delta\theta \to 0} \frac{\ln(r+\delta r) + i(\theta+\delta\theta) - \ln r - i\theta}{(r+\delta r)e^{i(\theta+\delta\theta)} - re^{i\theta}} = \lim_{\delta r, \delta\theta \to 0} \frac{\ln(1+\delta r/r) + i\delta\theta}{re^{i\theta}\left[(1+\delta r/r)e^{i\delta\theta} - 1\right]} \\
&= \lim_{\delta r, \delta\theta \to 0} \frac{\delta r/r - (\delta r/r)^2 + \cdots + i\delta\theta}{re^{i\theta}\left[(1+\delta r/r)(1 + i\delta\theta - (\delta\theta)^2/2 + \cdots) - 1\right]} \\
&= \lim_{\delta r, \delta\theta \to 0} \frac{\delta r/r - (\delta r/r)^2 + \cdots + i\delta\theta}{re^{i\theta}\left[i\delta\theta - (\delta\theta)^2/2 + \delta r/r + i\delta\theta\delta r/r - \delta r(\delta\theta)^2/2r + \cdots\right]} \\
&= \frac{1}{re^{i\theta}} = \frac{1}{z},
\end{aligned}$$

hence recovering the earlier result obtained by implicit differentiation.

A consequence of the nature of $\log z$ is that we need to consider with care functions such as z^α, where α is non-integer, which is defined in terms of $\log z$. For example, let us define

$$f(z) = z^\alpha = e^{\alpha \log z},$$

for non-integer α. In general, this function is multi-valued. By way of illustration, consider

$$w = f(z) = z^{1/2} = \left(re^{i\theta + 2ni\pi}\right)^{\frac{1}{2}}$$

$$w = r^{1/2}e^{i\theta/2 + in\pi}$$

which gives two branches for $n = 0$ and $n = 1$. All other values of n reproduce one of these two. If we define $n = 0$ as the **principal value** of the square root function the z-plane maps to the half plane $-\pi/2 < \mathrm{Arg}(w) < \pi/2$ and we again encounter a discontinuity as z approaches the negative real axis. Thus, as before, the origin is a branch point and we introduce a cut along the negative real axis to ensure that the principal value is analytic within any domain contained wholly within the cut plane. Within such a domain differentiation can take place.

Within the cut plane (see figure 11), the function

$$w = f(z) = z^{\alpha}$$

is analytic and we can differentiate:

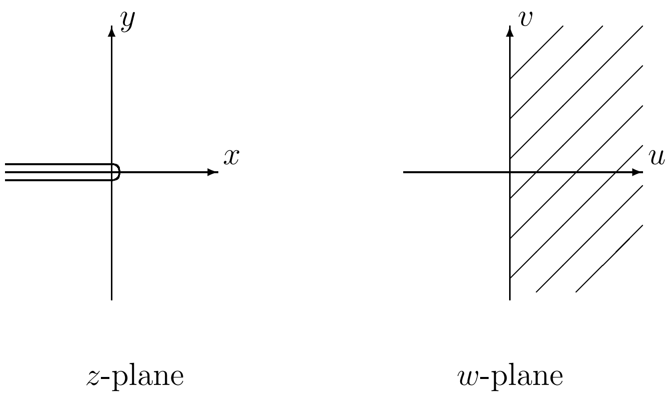

Figure 11: $w = \sqrt{z}$

$$\frac{d}{dz}w = \frac{d}{dz}e^{\alpha \log z} = \alpha\,\frac{1}{z}\,e^{\alpha \log z} = \alpha z^{\alpha-1}.$$

Therefore, all the usual rules and results concerning differentiation translate to the complex variable case. It is merely necessary to interpret some of the results produced with a little care.

Identifying branch points and determining how to cut the plane to create a single-valued, continuous function can be very complicated. The difficulties arise due to the nature of the argument of $f(z)$ and the basic idea is to try to ensure that the arguments involved do not exceed a range of 2π on a closed path lying within the cut plane.

Example 2.6.1. Consider the complex function

$$f(z) = \sqrt{z^2-1} = \sqrt{z-1}\sqrt{z+1}.$$

Just as the function $\sqrt{z}$ has a branch point at $z = 0$, $f(z)$ has branch points at $z = 1$ and $z = -1$. Let

$$z - 1 = r_1 e^{i\theta_1} \qquad z + 1 = r_2 e^{i\theta_2},$$

then
$$f(z) = \sqrt{r_1}\sqrt{r_2}\, e^{i(\theta_1+\theta_2)/2}.$$
To ensure that $f(z)$ is single-valued, we make each of $\sqrt{z-1}$ and $\sqrt{z+1}$ single-valued. This can be done by requiring
$$0 < \theta_1 \leq 2\pi \quad \text{and} \quad 0 < \theta_2 \leq 2\pi.$$
With this choice, it is easy to see that $f(z)$ is discontinuous on the real line segment connecting -1 and 1. Hence, this line segment can be chosen as a branch cut.

On the other hand, if we impose the restriction
$$0 < \theta_1 \leq 2\pi \quad -\pi < \theta_2 \leq \pi,$$
then $f(z)$ is discontinuous on $(-\infty, -1] \cup [1, +\infty)$. Hence, another choice of branch cut is $(-\infty, -1] \cup [1, +\infty)$.

2.7 Solving equations with complex unknown

All tricks and techniques for equations of a real unknown are applicable to the present case. Here we need to excercise some extra care due to greater scope for solutions. For illustration, consider the following two simple examples.

Example 2.7.1. Solve $\cosh z = 0$.

Solution Note that for real x, $\cosh x \geq 1$ for all x. So there are no real solutions to this equation. However, for complex z, we have
$$\cosh z = \frac{e^z + e^{-z}}{2} = 0 \quad \Longrightarrow \quad e^{2z} = -1 = e^{i\pi + i2n\pi},$$
for $n = 0, \pm 1, \pm 2, \cdots$. Hence
$$2z = i(\pi + 2n\pi).$$
So
$$z = i\pi\left(n + \frac{1}{2}\right).$$
There are infinitely many imaginary solutions. Alternatively, observe that $\cosh z = \cos iz$ and hence result. •

Example 2.7.2. Solve $\cos z = 2$.

Solution Again note that for x real $\cos x \leq 1$ for all x. So the equation has no real solutions. In the complex case we have
$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = 2.$$
Rearranging terms yields
$$e^{2iz} - 4e^{iz} + 1 = 0.$$
Solving as a quadratic yields
$$e^{iz} = 2 \pm \sqrt{4-1} = 2 \pm \sqrt{3} = (2 \pm \sqrt{3})e^{i2n\pi} = e^{\ln(2\pm\sqrt{3}) + i2n\pi},$$

which further yields

$$iz = \ln(2 \pm \sqrt{3}) + 2n\pi i,$$

for $n = 0, \pm 1, \pm 2, \cdots$. So

$$z = 2n\pi - i\ln(2 \pm \sqrt{3}) = 2n\pi \mp i\ln(2 + \sqrt{3}).$$

Thus, we have a doubly infinite number of complex solutions (see figure 11). •

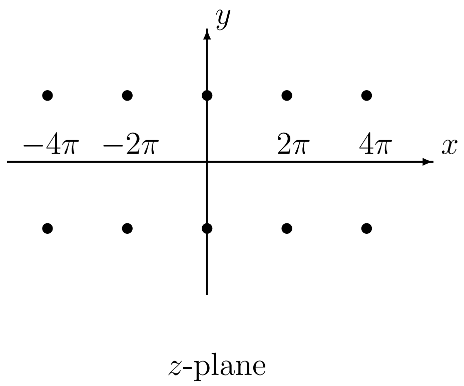

Figure 12: Solutions of $\cos z = 2$.

2.8 Singular Points

A point at which $f(z)$ fails to be analytic is called a **Singular point**. We have already met one type of singular point - namely a branch point. At a branch point and along the associated branch cut the function is not analytic.

Here we will consider what are called **isolated singularities**. The point $z = z_0$ is called an isolated singularity of $f(z)$ if we can surround it by a circle which contains no other singular point. A branch point is not isolated in this way. Singularities play an important role in integration theory so it is essential to be able to recognise, locate and classify different types of singularities.

1. Poles If $f(z_0)$ is undefined but we can find a positive integer n such that

$$\lim_{z \to z_0} \{(z - z_0)^n f(z)\} = a \neq 0$$

then $z = z_0$ is called a **pole** of order n. If $n = 1$ z_0 is called a **simple pole**.

Example 2.8.1. $f(z) = 1/z$ has a simple pole at the origin.

Example 2.8.2. $f(z) = \dfrac{1}{(z-3)^4}$ has a pole of order 4 at the point $z = 3$.

Example 2.8.3. $f(z) = \dfrac{3z - 2}{(z-1)^2(z+1)(z-4)}$ has simple poles at $z = -1$, $z = 4$ and a pole of order 2 (double pole) at $z = 1$.

2. Removable singularities An isolated singular point z_0 is called a **removable** singularity of $f(z)$ if $f(z)$ is undefined at z_0 but

$$\lim_{z \to z_0} f(z) = L.$$

Such a singularity can be "removed" by defining $f(z_0) = L$. The modified function (with $f(z_0) = L$) then becomes analytic at z_0, thus the term "removable singularity".

Example 2.8.4. The function

$$f(z) = \frac{e^z - 1}{z}$$

has a removable singularity at $z = 0$, since

$$\lim_{z \to 0} \frac{e^z - 1}{z} = \lim_{z \to 0} \frac{1 + z + z^2/2 + \cdots - 1}{z} = \lim_{z \to 0}(1 + z/2 + \cdots) = 1.$$

Hence define $f(0) = 1$.

Example 2.8.5. Consider

$$f(z) = \frac{\cos z - 1}{\sin z}.$$

The origin is an apparent singularity since $f(z)$ is not defined there. But we have

$$\lim_{z \to 0} \frac{\cos z - 1}{\sin z} = \lim_{z \to 0} \frac{1 - z^2/2 + z^4/4! + \cdots - 1}{z - z^3/3! + z^5/5! + \cdots} = \lim_{z \to 0} \frac{-z/2 + z^3/4! + \cdots}{1 - z^2/3! + z^4/5! + \cdots} = 0.$$

Hence by defining $f(0) = 0$, one can remove the apparent singularity.

3. Essential singularities The function $f(z)$ has an essential singularity at $z = z_0$ when

$$\lim_{z \to z_0} \{(z - z_0)^n f(z)\}$$

does not exist for any n. In other words, an expansion of $f(z)$ about z_0 involves an infinite series of negative powers of $(z - z_0)$. We will discuss power series of a complex function later in the course.

Example 2.8.6. $f(z) = e^{1/z}$ has an essential singularity at $z = 0$.

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$$

for all $z \neq 0$ and no power of z will create a finite limit for

$$\lim_{z \to 0} z^n e^{1/z}.$$

Finally we may consider $z = \infty$ as a point and examine the behaviour of a function at this point. This is quite natural if we think of a one-to-one correspondence between points on the complex plane and those on a "touching" sphere through *stereographic projection*, whereby the north pole of the sphere corresponds to $z = \infty$. The behaviour of a function $f(z)$ at $z = \infty$ can be studied via that of $f(1/\xi)$ at $\xi = 0$. Suppose that $f(z)$ is analytic in the exterior of a disk centred at the origin with radius R, i.e. in the region $\{z : |z| > R\}$. Then $F(\xi) = f(1/\xi)$ has an isolated singularity at $\xi = 0$. For this reason, we say that $f(z)$ has an isolated singularity at infinity. This singularity may be removable, of order $m \geq 1$ or essential, and a formal classification is as follows. Let $z = \infty$ be an isolated singularity of $f(z)$. Then the singularity is (i) removable if $f(1/z)$ has a removable singularity at $z = 0$, (ii) a pole of order $m \geq 1$ if $f(1/z)$ has a pole of order $m \geq 1$ at $z = 0$ and (iii) essential if $f(1/z)$ has an essential singularity at $z = 0$.

Example 2.8.7. The function $f(z) = z^3$ has a pole of order 3 at $z = \infty$. Indeed, let

$$z = \frac{1}{\xi}$$

so that

$$F(\xi) = f(1/\xi) = \frac{1}{\xi^3}.$$

$F(\xi)$ has a pole of order 3 at $\xi = 0$, i.e. $f(z)$ has a pole of order 3 at $z = \infty$.