5 Fuchsian groups

1. Prove that the modular group, $\mathrm{PSL}(2,\mathbb{Z})$, is Fuchsian, but that $\mathrm{PSL}(2,\mathbb{Q})$ is not.

2. Prove that a Fuchsian group $\Gamma \leqslant \mathrm{PSL}(2,\mathbb{R})$ is closed in $\mathrm{PSL}(2,\mathbb{R})$.

3. Let (X,d) be a metric space. Prove that if $K \subseteq X$ is compact and $P \subseteq X$ is discrete and closed, then $K \cap P$ is at most finite. Give an example to show that this is false if P is discrete but not closed.

   This fact was used in lectures when we proved that Fuchsian groups act properly discontinuously.

4. Let (X,d) be a metric space and say that $K \subseteq X$ is *bounded* if $K \subseteq B(x,r)$ for some $x \in X$ and $r > 0$. Prove that a set $K \subseteq \mathbb{H}^2$ is bounded if and only if there exists a constant $C > 1$ such that for all $z \in K$ we have both

   (i) $|z| \leqslant C$
   (ii) $\mathrm{Im}(z) > 1/C$.

   This fact was used in lectures when we proved that Fuchsian groups act properly discontinuously.

5. Prove that a Fuchsian group $\Gamma \leqslant \mathrm{PSL}(2,\mathbb{R})$ is finite if and only if $\Gamma(z)$ is finite for all $z \in \mathbb{H}^2$.

6. Prove that a group $\Gamma \leqslant \mathrm{PSL}(2,\mathbb{R})$ is Fuchsian if and only if $\Gamma(z)$ is discrete for all $z \in \mathbb{H}^2$.

7. Prove that the elliptic fixed points of a Fuchsian group (i.e. $z \in \mathbb{H}^2$ which are fixed by elliptic elements of the group) form a discrete set.

8. Prove that each of the following monogenic subgroups of $\mathrm{PSL}(2,\mathbb{R})$ are Fuchsian, and describe a typical fundamental domain for its action:

   (i) $\langle g \rangle$ where g is an elliptic element of finite order
   (ii) $\langle g \rangle$ where g is a parabolic element
   (iii) $\langle g \rangle$ where g is a hyperbolic element.

   Describe all monogenic subgroups of $\mathrm{PSL}(2,\mathbb{R})$ which are *not* Fuchsian.

9. Let $g \in \mathrm{PSL}(2,\mathbb{R})$ be the parabolic element given by $g(z) = z+8$ and $h \in \mathrm{PSL}(2,\mathbb{R})$ be the elliptic element given by $h(z) = -4/z$. Find the Dirichlet fundamental domain for the action of $\langle g,h \rangle$ with respect to the base point $4i$.

10. Let $p,h \in \mathrm{PSL}(2,\mathbb{R})$ be a parabolic and a hyperbolic element respectively and suppose they have a common fixed point. Prove that $\langle p,h \rangle \leqslant \mathrm{PSL}(2,\mathbb{R})$ is *not* a Fuchsian group. You may wish to consider elements of the form $h^n p^m h^{-n}$.

11. Consider $g,h \in \mathrm{PSL}(2,\mathbb{R})$ given by $g(z) = z+2$ and $h(z) = z/(2z+1)$. Use Poincaré's Theorem to find a presentation for $\langle g,h \rangle$.