MT5823 Semigroup theory: Solutions 6 (James D. Mitchell)
Cancellative semigroups, regular semigroups, inverses

Cancellative semigroups

6-1. ($\Leftarrow$) Let S be a cancellative semigroup with no identity and let $a, x, y \in S^1$ such that $ax = ay$. If $a, x, y \in S$, then since S is cancellative $x = y$. If $a = 1$, then since 1 is an identity $x = y$. If $x = 1$ and $a, y \neq 1$, then $a = ay$ and so by Problem **2-3** it follows that y is an identity for S, a contradiction.

($\Rightarrow$) Assume that S has an identity element e. Then $e \cdot 1 = e = e^2$ and cancelling e on the left, we obtain $e = 1$, a contradiction as $1 \notin S$ and $e \in S$. Thus S has no identity. □

6-2. Let $a, b \in S$ such that $a\mathscr{R}b$. Then there exist $u, v \in S^1$ such that $au = b$ and $bv = a$. This implies that $auv = a$ and by Problem **6-1** we can cancel the as to obtain $uv = 1$. Thus $u = v = 1$, as $xy \neq 1$ for all $x, y \in S$. It follows that $a = b$.

Likewise, if $a\mathscr{L}b$, then $a = b$. It follows that $\mathscr{H} = \mathscr{D} = \Delta_S$. □

6-3. Let

$$\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c & 0 \\ d & 1 \end{pmatrix}$$

be arbitrary elements of T. Then

$$\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} c & 0 \\ d & 1 \end{pmatrix} = \begin{pmatrix} ac & 0 \\ bc + d & 1 \end{pmatrix} \in T.$$

It follows that T is a semigroup (associativity follows from the associativity of matrix multiplication).

Let

$$\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}, \begin{pmatrix} x & 0 \\ y & 1 \end{pmatrix}, \begin{pmatrix} z & 0 \\ t & 1 \end{pmatrix} \in T$$

such that

$$\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ t & 1 \end{pmatrix}.$$

Then

$$\begin{pmatrix} ax & 0 \\ bx + y & 1 \end{pmatrix} = \begin{pmatrix} az & 0 \\ bz + t & 1 \end{pmatrix}$$

and so $az = ax$ and $bx + y = bz + t$. The first equality implies that $x = z$ and so $bx + y = bx + t$ and $y = t$. Thus

$$\begin{pmatrix} x & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} z & 0 \\ t & 1 \end{pmatrix}$$

and we have shown that T is cancellative.

By Problem **6-2**, it remains to prove that T has no identity. Assume that

$$\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}.$$

Then $ax = a$ and $bx + y = b$. Thus $x = 1$ and $y = 0$, but in this case

$$\begin{pmatrix} x & 0 \\ y & 1 \end{pmatrix} \notin T.$$

Hence T has no identity and so, by Problem **6-2**, $\mathscr{R} = \mathscr{L} = \mathscr{H} = \mathscr{D} = \Delta_T$.

To show that $\mathscr{J} = T \times T$ take matrices

$$\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}, \begin{pmatrix} x & 0 \\ y & 1 \end{pmatrix} \in T.$$

It is straightforward to verify that

$$\begin{pmatrix} 3cd/ab & 0 \\ b/a & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} d/3b & 0 \\ d/3 & 1 \end{pmatrix} = \begin{pmatrix} c & 0 \\ d & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 3ad/bc & 0 \\ d/c & 1 \end{pmatrix} \begin{pmatrix} c & 0 \\ d & 1 \end{pmatrix} \begin{pmatrix} b/3d & 0 \\ b/3 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}.$$

It follows that $\mathscr{J} = T \times T$. □

Regular semigroups

6-4. By Theorem 11.7, in a regular semigroup every $\mathscr{L}$- and $\mathscr{R}$-class contains an idempotent. Thus S has a single $\mathscr{R}$-class and a single $\mathscr{L}$-class. It follows that S has only one $\mathscr{H}$-class and that this $\mathscr{H}$-class contains an idempotent. Thus by Theorem 10.4, this $\mathscr{H}$-class is a group. □

6-5. Since a/ρ is an idempotent in S/ρ we have that

$$a/\rho = (a/\rho)^2 = a^2/\rho.$$

That is, $(a, a^2) \in \rho$. If x is an inverse of a^2, then $a^2xa^2 = a^2$ and $xa^2x = x$. Now,

$$e^2 = axaaxa = a(xa^2x)a = axa = e$$

and so e is an idempotent. So, $(a, a^2) \in \rho$ implies $(e, a^2xa) = (a{\cdot}xa, a^2{\cdot}xa) \in \rho$ and $(a^2xa, a^2) = (a^2x{\cdot}a, a^2x{\cdot}a^2) \in \rho$ (ρ is a congruence). Hence transitivity of ρ implies that $(e, a^2) \in \rho$, and that $(e, a) \in \rho$, as required. □

6-6. Let $c^ib^j \in B$ (the bicyclic monoid). Then $(c^ib^j)(c^jb^i)(c^ib^j) = c^ib^j$ and B is regular.

Let $(i, \lambda) \in R$ be a rectangular band. Then

$$(i, \lambda)(i, \lambda)(i, \lambda) = (i, \lambda)$$

and R is regular. □

6-7. The $\mathscr{D}$-class containing x^3y has no idempotents and hence it is not regular.

The $\mathscr{R}$-classes of the semigroup defined by

$$\langle a, b | a^3 = a, b^4 = b, ba = a^2b\rangle$$

are

$$\{a, a^2\}, \{b, b^2, b^3\}, \{ab, ab^2, ab^3\}, \{a^2b, a^2b^2, a^2b^3\}.$$

Each of these classes contains an idempotent a^2, b^3, ab^3, a^2b^3. Thus every $\mathscr{D}-$class (being a union of $\mathscr{R}$-classes) contains a regular element and is itself regular. □

Inverses

6-8. (a) As y is an inverse of x, $yxy = x$. Thus

$$f_j^2 = e_j \cdots e_n y e_1 \cdots e_{j-1} e_j \cdots e_n y e_1 \cdots e_{j-1} = e_j \cdots e_n yxy e_1 \cdots e_{j-1} = e_j \cdots e_n x e_1 \cdots e_{j-1} = f_j.$$

and

$$\begin{aligned} yxf_nf_{n-1} \cdots f_2xy &= yx(e_nye_1 \cdots e_{n-1})(e_{n-1}e_nye_1 \cdots e_{n-2}) \cdots (e_2 \cdots e_nye_1)(xy) \\ &= yxe_nyxyx \cdots xye_1xy = y(e_1 \cdots e_n)e_nyxyxy \cdots xye_1(e_1 \cdots e_n)y = yxyx \cdots xyxy = yxy = y. \end{aligned}$$

(b) Recall that $g_j = e_{j+1} \cdots e_{n+1} y e_1 \cdots e_j$ $(j = 1, \ldots, n + 1)$. Then

$$\begin{aligned} g_j^2 = e_{j+1} \cdots c_{n+1} y e_1 \cdots e_j e_{j+1} \cdots e_{n+1} y e_1 \cdots e_j &= e_{j+1} \cdots e_{n+1} yxy e_1 \cdots e_j \\ &= e_{j+1} \cdots e_{n+1} y e_1 \cdots e_j = g_j. \end{aligned}$$

Let $z = g_n \cdots g_1$. Then

$$z = (e_{n+1}ye_1 \cdots e_n)(e_ne_{n+1}ye_1 \cdots e_{n-1}) \cdots (e_2 \cdots e_{n+1}ye_1) = e_{n+1}ye_1.$$

Hence

$$xzx = xe_{n+1}ye_1x = e_1 \cdots e_{n+1}^2 y e_1^2 \cdots e_{n+1} = e_1 \cdots e_{n+1} y e_1 \cdots e_{n+1} = xyx = x$$

and

$$\begin{aligned} zxz = e_{n+1}ye_1xe_{n+1}ye_1 = e_{n+1}ye_1^2e_2 \cdots e_ne_{n+1}^2ye_1 &= e_{n+1}ye_1e_2 \cdots e_ne_{n+1}ye_1 \\ &= e_{n+1}yxye_1 = e_{n+1}ye_1. \end{aligned}$$

Thus $x \in V(g_n \cdots g_1)$.

(c) By part (b) every element in E^{n+1} is the inverse of an element in E^n. So, as $g_n \cdots g_1 \in E^n$ and $x \in E^{n+1}$ we have that $E^{n+1} \subseteq V(E^n)$. On the other hand, if $y \in V(E^n)$, then by part (a),

$$y = \underbrace{yx}_{\text{an idempotent}} \quad \underbrace{f_n \cdots f_2}_{\text{length } n-1} \quad \underbrace{xy}_{\text{an idempotent}} \in E^{n+1}. \quad \square$$

Further problems

6-9. (a) $\Rightarrow$ (c) follows by Problem **6-4**.

(c) $\Rightarrow$ (b) follows as $ax = ay$ implies $a^{-1}ax = a^{-1}ay$ and so $x = y$.

(b) $\Rightarrow$ (a) S is regular and so it contains at least 1 idempotent. If e and f are idempotents, then

$$e^2 f = ef^2 \tag{1}$$

as $e^2 = e$ and $f^2 = f$. Hence $ef = f^2 = f$ by cancelling e from the left in (1). Also $e^2 = ef = e$ by cancelling f from the right in (1). Thus $e = ef = f$ and S has exactly one idempotent. $\square$