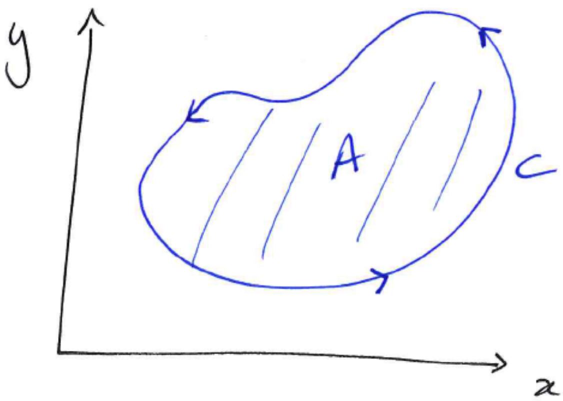
Chapter 6

Fundamental Integral Theorems

6.1 Green's Theorem

6.1.1 Theorem

George Green (1793-1841) discovered an important connection between closed line integrals and double (area) integrals in a plane $z = 0$.



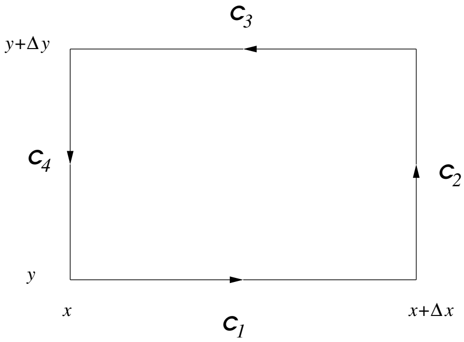
Let φ be a "right-handed" curve bounding an area A of the x-y place. "Right-handed" means that φ is followed counter-clockwise, i.e. the region A is <u>on the left</u>. This is also called the positive direction.

Consider 2 scalar functions P and Q defined over A, including Ci. Green's theorem states:

$$\oint_{\varphi} P(x,y)dx + Q(x,y)dy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

for P, Q smooth functions. This is the classical form of Green's theorem. We can however rewrite this using vector notations.

6.1.2 Elements of proof



We have first going to prove that

$$\iint_S -\frac{\partial P}{\partial y'} dy' dx' = \oint_C P dx'$$

First consider the rectangular area S bounded by the contour $C = C_1 \cup C_2 \cup C_3 \cup C_4$. Consider the differentiable function $P(x', y')$. Then we have

$$\iint_S -\frac{\partial P}{\partial y'} dy' dx' = \int_x^{x+\Delta x} -\left(\int_y^{y+\Delta y} \frac{\partial P}{\partial y'} dy' \right) dx'$$

Using the fundamental theorem of Calculus, the inner integral gives

$$\int_x^{x+\Delta x} \big(-(P(x', y+\Delta y) - P(x', y)\big) \, dx'$$

On the other hand, the contour integral gives

$$\int_{C_1} P dx' + \int_{C_2} P dx' + \int_{C_3} P dx' + \int_{C_4} P dx' = \int_{C_1} P dx' + \int_{C_3} P dx'$$

because $dx' = 0$ along $C_2 \& C_4$. Moreover, along C_1, $y = y$, and $y = y + \Delta y$ along C_3. With the orientation of C_1, x' goes from x to $x + \Delta x$ while it goes from $x + \Delta x$ along C_3 so

$$\int_{C_1} P dx' = \int_x^{x+\Delta x} P(y, x') dx'$$

$$\int_{C_3} P dx' = \int_{x+\Delta x}^{x} P(y+\Delta y, x') dx' = -\int_x^{x+\Delta x} P(y+\Delta y, x') dx'$$

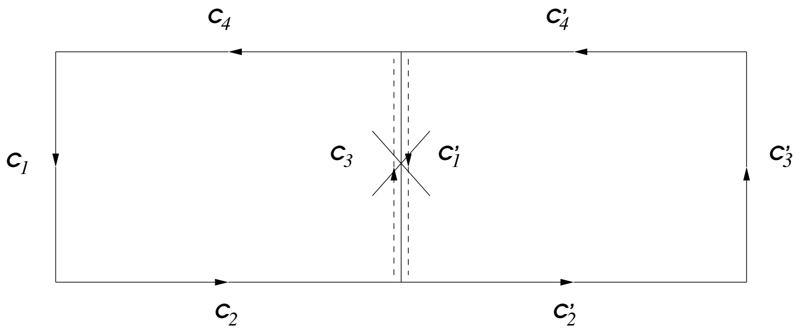
So the line integral is

$$\int_x^{x+\Delta x} \big(-(P(x', y+\Delta y) - P(x', y)\big) \, dx'$$

In a similar way, you can prove that

$$\iint_S \frac{\partial Q}{\partial x} dx dy = \oint_C Q dy$$

The difference in sign comes from the orientation of the contour. So over all Green's theorem holds, and it holds for any rectangle, even when taking the limit of the rectangle to a point with local area $dxdy$. Now, following that logic, that would mean that the integral over any surface described as the sum of infinitesimal rectangles can be seen as the sum of the line integrals over the curves bounding all the infinitesimal rectangles. We are just one step away from the final result. All the line integrals inside the surface cancel out! Only the ones at the external boundary remain!



6.1.3 Green's theorem in vector form

Consider the vector function $\mathbf{F}$

$$\mathbf{F} = \begin{pmatrix} P \\ Q \\ 0 \end{pmatrix} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$$

$$d\mathbf{r} = \begin{pmatrix} dx \\ dy \\ 0 \end{pmatrix} = \mathbf{i}dx + \mathbf{j}dy$$

It follows that

$$\oint_C Pdx + Qdy = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

where C lies on the x-y plane.

Then

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ \partial Q/\partial x - \partial P/\partial y \end{pmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

The surface element associated with the x-y plane is

$$\mathbf{ds} = \mathbf{n}dxdy = \mathbf{k}dxdy$$

Hence

$$\iint_A (\nabla \times \mathbf{F}) \cdot \mathbf{ds} = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy\mathbf{k} \cdot \mathbf{k} = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

So finally, in vector notation Green's theorem reads

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_A (\nabla \times \mathbf{F}) \cdot d\mathbf{s}$$

6.2 Examples

6.2.1 Calculating surfaces

When calculating

$$\iint_A ds = \iint_A dxdy$$

we have

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

Possible choices

$$\begin{cases} Q = x, \;\; P = 0 \\ Q = 0, \;\; P = -y \\ P = \frac{1}{2}x, \;\; Q = -\frac{1}{2}y \end{cases}$$

Application: the planimeter used in the past by engineers to integrate.

6.2.2 First example

We are going to verify Green's theorem on another example (which includes some funny integrations!)

Question

Consider the two scalar functions P and Q defined as

$$P(x, y) = y^3 \;\text{ and }\; Q(x, y) = 0,$$

and a region of the x-y-plane consisting of a disk A of radius 1 centred at the origin $(0, 0)$ bounded by the circle C of equation $x^2 + y^2 = 1$.

Verify for this example Green's theorem, i.e. verify that

$$I = \oint_C Pdx + Qdy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy.$$

Answer

-Surface integral:

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 - 3y^2$$

so

$$I = -3 \iint_A y^2 dxdy = -3 \iint_A y^2 dS.$$

Obviously, here using Cartesian coordinates is a bit silly! Let us use polars! Although the problem is 2D, I will use the notation convention of 3D cylindrical polar coordinates as we have rarely if ever, used 2D polars in the course.

$$dS = RdRd\phi \;\text{ with }\; - \leq R \leq 1 \;\text{ and }\; 0 \leq \phi \leq 2\pi,$$

$$y = R\sin\phi$$

So

$$I = -3\int_0^1\int_0^{2\pi} R^2\sin^2\phi R d\phi dR = -3\int_0^1\int_0^{2\pi} R^3\sin^2\phi d\phi dR$$

$$I = -3\int_0^1 R^3 dR\int_0^{2\pi}\sin^2\phi d\phi$$

$$I = -3\left[\frac{R^4}{4}\right]_0^1\int_0^{2\pi}\left(\frac{1-\cos 2\phi}{2}\right)d\phi$$

$$I = -\frac{3}{4}\left[\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right]_0^{2\pi} = -\frac{3\pi}{4}.$$

Note from above that

$$\int_0^{2\pi}\sin^2\phi d\phi = \pi.$$

-Line integral:

As $Q = 0$

$$\oint_C Pdx + Qdy = \oint_C Pdx = \oint_C y^3 dx.$$

The parametric representation of the circle of radius 1 can be written as

$$y = \cos\phi,$$

$$y = \sin\phi,$$

with $0 \leq \phi \leq 2\pi$.

Hence

$$P = \sin^3\phi,$$

and

$$dx = -\sin\phi d\phi.$$

Then

$$I = \oint Pdx = -\int_0^{2\pi}\sin^4\phi d\phi = -\int_0^{2\pi}\sin^2\phi(1-\cos^2\phi)d\phi = -\int_0^{2\pi}\sin^2\phi d\phi + \int_0^{2\pi}(\sin\phi\cos\phi)^2 d\phi,$$

using $\sin^2\phi + \cos^2\phi = 1$.

We have seen that

$$\int_0^{2\pi}\sin^2\phi d\phi = \pi.$$

Let us evaluate the second integral

$$\int_0^{2\pi} (\sin\phi\cos\phi)^2 d\phi = \int_0^{2\pi} \left(\frac{\sin 2\phi}{2}\right)^2 d\phi = \frac{1}{4}\int_0^{2\pi} \sin^2 2\phi d\phi$$

Taking the change of variable $u = 2\phi$, this integral becomes

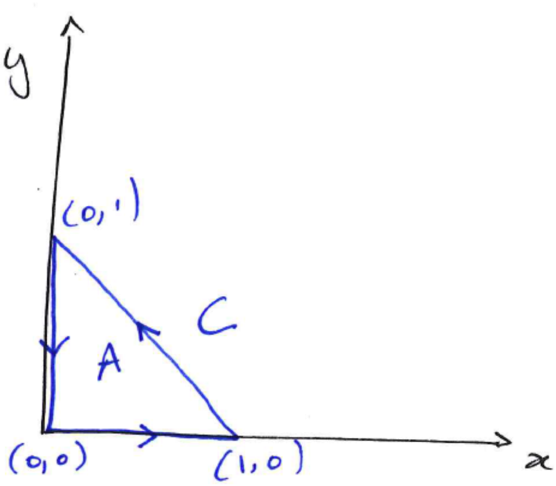
$$\frac{1}{4}\int_0^{2\pi} \sin^2 2\phi d\phi = \frac{1}{8}\int_0^{4\pi} \sin^2 u du = \frac{1}{8}\int_0^{4\pi} \left(\frac{1-\cos u}{2}\right) du = \frac{1}{16}\left[u - \frac{\sin 2u}{2}\right]_0^{4\pi} = \frac{4\pi}{16} = \frac{\pi}{4}.$$

Finally,

$$I = \oint P dx = -\pi + \frac{\pi}{4} = -\frac{3\pi}{4}$$

6.2.3 Extra example

Let $P = x^2 y,\ Q = 3xy^2$ and let φ be the closed curve (contour) bounding the triangle of vertices $(0,0),\ (1,0)$ and $(0,1)$.

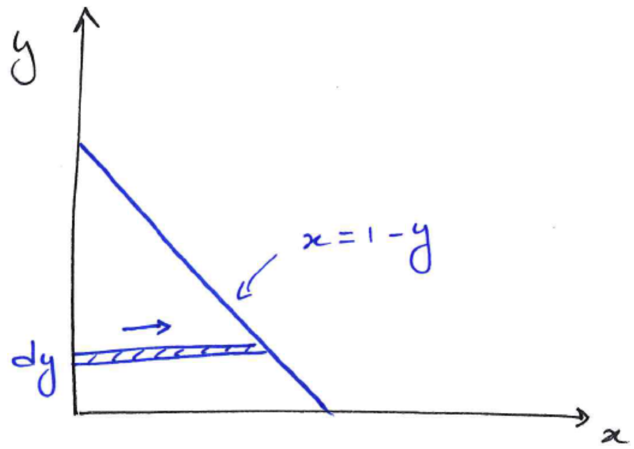


Let's verify Green's theorem:

Surface integral

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 3y^2 - x^2$$

$$\oint_\varphi P dx + Q dy = \iint_A (3y^2 - x^2) dx dy = I$$



$$I = \int_0^1 \int_0^{1-y} (3y^2 - x^2) dx dy$$

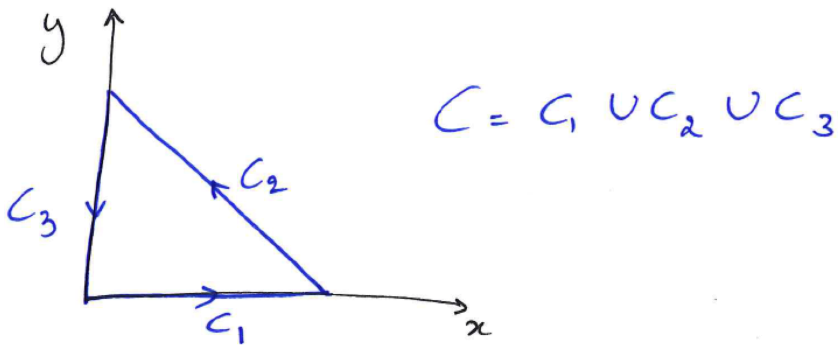
$$= \int_0^1 \left[3y^2 x - \frac{x^3}{3} \right]_0^{1-y} dy$$

$$= \int_0^1 \left[3y^2(1-y) - \frac{1}{3}(1-y)^3 \right] dy$$

$$= \left[y^3 - \frac{3}{4}y^4 + \frac{1}{12}(1-y)^4 \right]_0^1$$

$$= 1 - \frac{3}{4} - \frac{1}{12} = \frac{1}{4} - \frac{1}{12} = \frac{3}{12} - \frac{1}{12} = \frac{2}{12} = \frac{1}{6}$$

Compare with the line integral



Along C_1 $\quad y = 0 \Rightarrow dy = 0$ and $P = 0$

$$\int_{C_1} P dx + Q dy = \int_{C_1} P dx = 0$$

Along C_3 $\quad x = 0 \Rightarrow dx = 0$ and $Q = 0$

$$\int_{C_3} Pdx + Qdy = \int_{C_3} Qdy = 0$$

Along C_2: Parametric representation of C_2

$$\begin{cases} x(t) = 1 - t \\ y(t) = t \end{cases} \qquad 0 \leq r \leq 1$$

Check at

$$\begin{cases} t = 0 & (x, y) = (1, 0) \\ t = 1 & (x, y) = (0, 1) \end{cases}$$

So

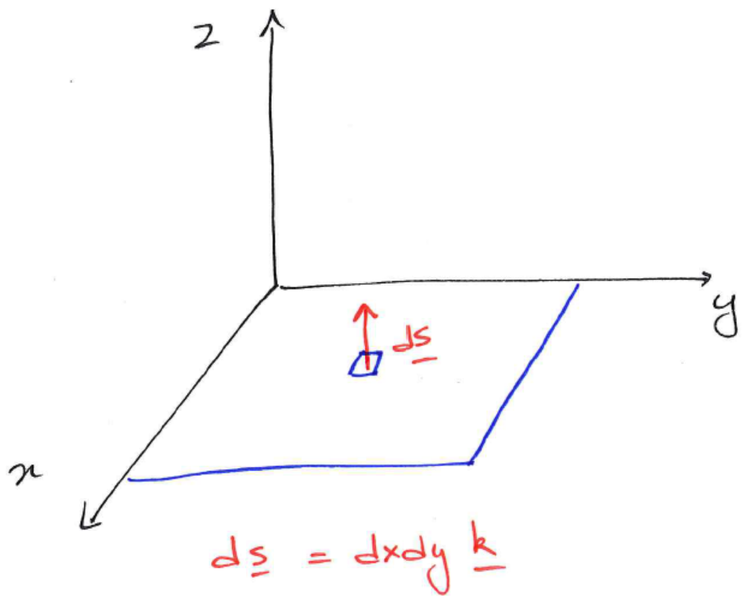
$$\begin{cases} dx = -dt & P = x^2 y = (1-t)^2 t \\ dy = dt & Q = 3xy^2 = 3(1-t)t^2 \end{cases}$$

$$\begin{aligned} \textstyle\int_{C_3} Pdx + Qdy &= \textstyle\int_0^1 (1-t)^2 t(-dt) + 3(1-t)t^2 dt \\ &= \textstyle\int_0^1 \left(-t^2 + 2t^2 - t + 3t^2 - 3t^3\right) dt \\ &= \left[\frac{-t^4}{4} + \frac{2t^3}{3} - \frac{t^2}{2} + t^3 - \frac{3t^4}{4}\right]_0^1 \\ & -\tfrac{1}{4} + \tfrac{2}{3} - \tfrac{1}{2} + 1 - \tfrac{3}{4} = \tfrac{2}{3} - \tfrac{1}{2} = \tfrac{4-3}{6} = \tfrac{1}{6} \end{aligned}$$

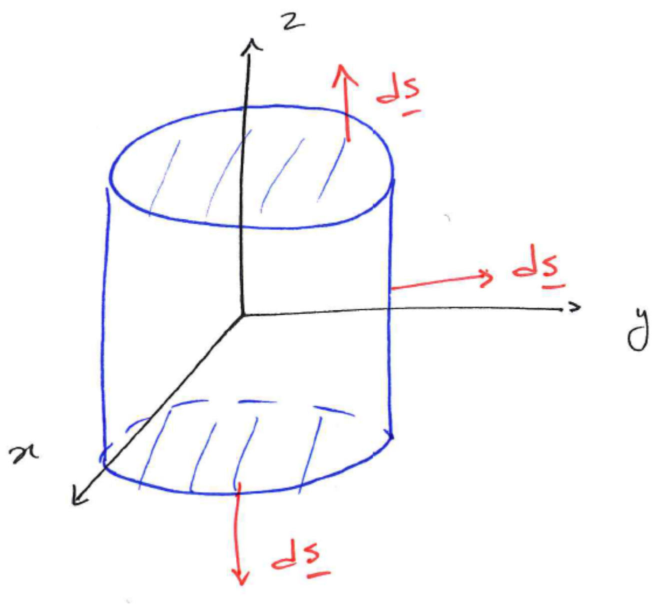
Compare with the line integral: This works!

6.3 Typical vector surface elements – mostly reminders

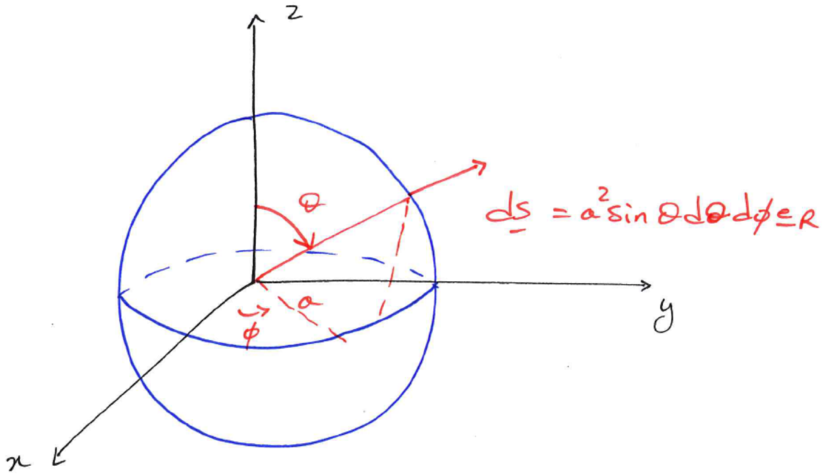
(i) xy-plane



(ii) Cylinders



(iii) Sphere



(iv) Method of projection

Consider the surface defined by $z = g(x, y)$, that is $f(x, y, z) = z - g(x, y) = 0$, and the vector function $\mathbf{F}$ defined on it. The flux of $\mathbf{F}$ across S is

$$\iint_S \mathbf{F} \cdot \mathbf{dS} = \iint_S \mathbf{F} \cdot \mathbf{n} dS$$

with

$$\mathbf{n} = \frac{\boldsymbol{\nabla} f}{|\boldsymbol{\nabla} f|} = \frac{\boldsymbol{\nabla} f}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2}} \frac{\boldsymbol{\nabla} f}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}}$$

and, following the method of projection

$$dS = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}$$

So *in this particular case* where the surface S is defined $f(x, y, z) = z - g(x, y) = 0$

$$\iint_S \mathbf{F} \cdot \mathbf{dS} = \iint_S \mathbf{F} \cdot \mathbf{n} dS = \iint_S \mathbf{F} \cdot \boldsymbol{\nabla} f dx dy$$

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6.4 Stokes' theorem

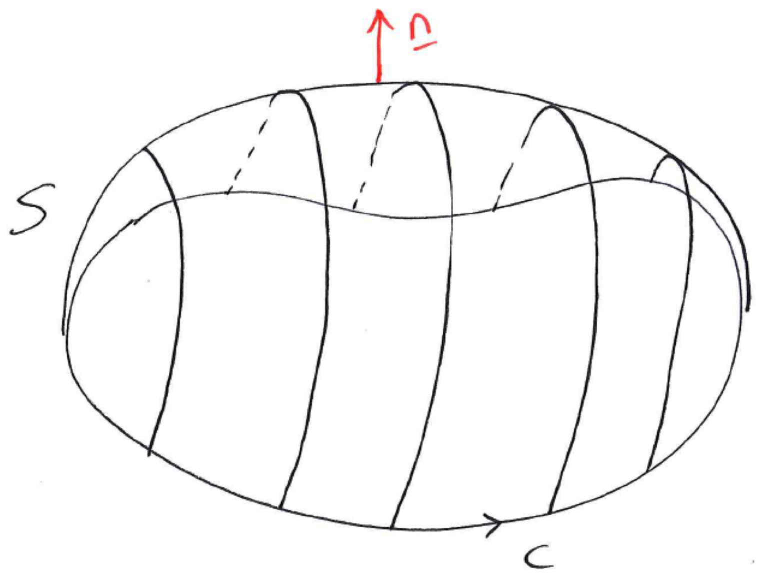
George Gabriel Stokes (1819-1903) knew of Green's work and generalised his theorem to any arbitrary surface S, i.e. not just planar surface A.

Green's theorem (in vector form).

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_A (\boldsymbol{\nabla} \times \mathbf{F}) d\mathbf{s} = \mathbf{k} dxdy$$

Stokes' theorem

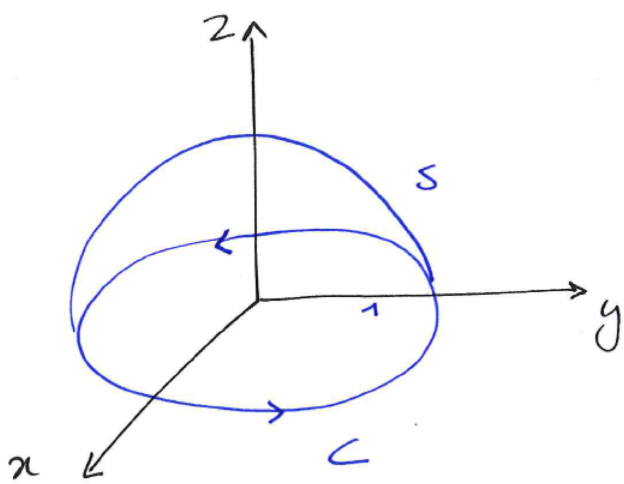
$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot d\mathbf{s}$$



The convention is simply that S is always to the left of C.

6.4.1 Applications of Stokes' theorem

Consider the vector function $\mathbf{F} = y^3\mathbf{i}$ over the hemisphere of radius 1 for $z \geq 0$, bounded by the circle of radius 1:



$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C y^3 dx = -\frac{3\pi}{4}$ as "seen" in a previous example.

$\iint_S (\mathbf{F})\cdot d\mathbf{s}$? $\quad \mathbf{ds} = a^2 \sin\theta d\theta d\phi \mathbf{e}_r \quad \begin{array}{l} 0 \le \phi \le 2\pi \\ 0 \le \theta \le \frac{\pi}{2} \end{array}$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^3 & 0 & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ -3y^2 \end{pmatrix} = -3y^2\mathbf{k}$$

$$d\mathbf{s} = a^2 \sin\theta d\theta d\phi \mathbf{e}_r \qquad \begin{array}{l} 0 \le \phi \le 2\pi \\ 0 \le \theta \le \pi/2 \end{array}$$

$$\nabla \times \mathbf{F} = -3\sin^2\theta \sin^2\phi \mathbf{k} \ \text{ as } \ y = r\sin\theta\sin\phi$$

$$\mathbf{e}_r = \sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k}$$

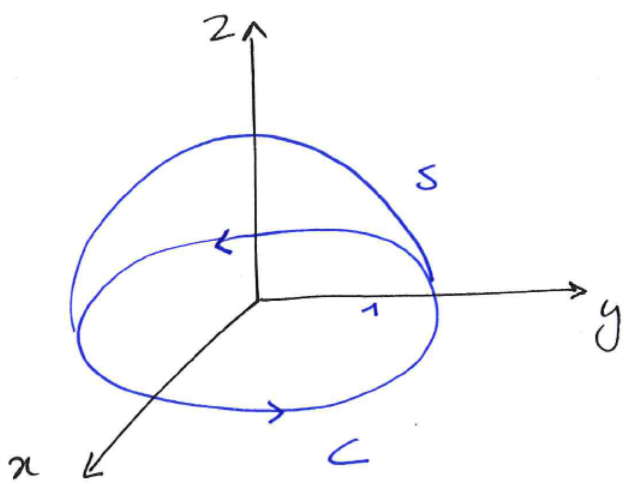
$$\mathbf{e}_r \cdot \mathbf{k} = \cos\theta$$

$$\nabla \times \mathbf{F}\cdot d\mathbf{s} = -3\sin^2\theta\sin^2\phi\cos\theta\sin\theta d\theta d\phi$$

$$-3\int_0^{2\pi}\int_0^{\pi/2} \sin^3\theta\cos\theta\sin^2\phi d\theta d\phi = -3\int_0^{2\pi}\sin^2\phi d\phi \int_0^{\pi/2}\sin^3\theta d(\sin\theta) = -3\pi\left[\frac{\sin^4\theta}{4}\right]_0^{\pi/2} = -\frac{3\pi}{4}$$

Example

$\mathbf{F} = -3y\mathbf{i} + 3x\mathbf{j} + z^4\mathbf{k}$ on the hemisphere $x^2+y^2+z^2 = 1, \quad z \ge 0$



$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -3y & 3x & z^4 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$$

$$I = \iint \nabla \times \mathbf{F} \cdot d\mathbf{s} = \int_0^{\pi/2} \int_0^{2\pi} 6\mathbf{k} \cdot \sin\theta d\phi d\theta \mathbf{e}_r$$

$$\mathbf{k} \cdot \mathbf{e}_r = \cos\theta$$

$$= 12\pi \int_0^{\pi/2} \sin\theta \cos\theta d\theta = 12\pi \left[\frac{\sin^2\theta}{2} \right]_0^1 = 6\pi$$

Line integral:

$$I = \oint_{\varphi} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \begin{pmatrix} -3\sin\theta \\ 3\cos\theta \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} d\theta = \int_0^{2\pi} 3\left(\sin^2\theta + \cos^2\theta\right) d\theta = 6\pi$$

Example - Verify Stoke's theorem

Consider $\mathbf{F}(x, y, z) = 2z\mathbf{i} + 3x\mathbf{j} - 2y\mathbf{k}$ on the paraboloid $z = 9 - x^2 - y^2$ for $z \geq 0$.

Line integral: The contour C is the circle $x^2 + y^2 = 9$ in the x, y-plane.

$$d\mathbf{r} = (dx, dy, 0) = (-3\sin\phi d\phi, 3\cos\phi d\phi, 0)$$

and along C

$$\mathbf{F} = (0, 9\cos\phi,\ 6\sin\phi)$$

Therefore

$$\mathbf{F} \cdot d\mathbf{r} = 27\cos^2\phi d\phi$$

And the integral is

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} 2727\cos^2\phi d\phi = 27\int_0^{2pi} \left(\frac{1+\cos(2\phi)}{2}\right) = 27\pi$$

Surface integral: Using the method of projection seen before, and using $f(x, y, z) = z - 9 + x^2 + y^2 = 0$ to describe the surface, we have seen that the integral becomes

$$\iint_A (\nabla \times \mathbf{F}) \cdot \nabla f \, dxdy$$

where A is the area of the projection of S on the plane, i.e. the disk $x^2 + y^2 = 9$.

$$\nabla \times \mathbf{F} = \begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix}$$

and

$$\boldsymbol{\nabla} f = \begin{pmatrix} 2x \\ 2y \\ 1 \end{pmatrix}$$

such that $(\boldsymbol{\nabla} \times \mathbf{F}) \cdot \boldsymbol{\nabla} f = -2x + 2y + 3$

the integral becomes

$$\int_0^2 \int_0^{2\pi} (-2R\cos\phi + 2R\sin\phi + 3) R dR d\phi = 3 \int_0^{2pi} d\phi \int_0^3 R dR = 3 * 2\pi * \frac{3^2}{2} = 27\pi$$

.

Example where the line integral is simple and the surface integral is not

Consider

$$\mathbf{F} = 2y\mathbf{i} + xz^2\mathbf{j} + x^2ye^z\mathbf{k}$$

We want to calculate the flux of the curl of F over the surface of the upper hemisphere $z = \sqrt{4 - x^2 - y^2}$.

$$\boldsymbol{\nabla} \times \mathbf{F} = \begin{pmatrix} x^2e^z - 2xz \\ -2xye^z \\ z^2 - 2 \end{pmatrix}$$

... I'm out! I don't even want to consider taking the scalar project of *that* with $\mathbf{e}_r$ and try to integrate over a sphere! On the over hand, the surface is bounded but the circle $x^2 + y^2 = 4$ on the plane $z = 0$!

On the circle

$$x = 2\cos\phi,\ y = 2\sin\phi,\ z = 0$$

$$\mathbf{F} = 2(2\sin\phi)\mathbf{i} + (2\cos\phi)^2(2\sin\phi)\mathbf{k}$$

$$d\mathbf{r} = -2\sin\phi\mathbf{i} + 2\cos\phi\mathbf{j}$$

So

$$\mathbf{F} \cdot d\mathbf{r} = -8\sin^2\phi d\phi$$

The flux of the curl of $\mathbf{F}$ is

$$-8\int_0^{2\pi} \sin^2\phi d\phi = -8\pi.$$

Example where the surface integral is simpler than the line integral

Consider the circulation of the vector function $\mathbf{F}$ along C

$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

where $\mathbf{F} = (2xz + tan^{-1}(x^2))\mathbf{i} + e^{xy}\mathbf{j} + (2xy + e^{z^2})\mathbf{k}$, and C is the intersection of the cylinder $x^2 + z^2 = 1$ and the plane $y = 3$.

$$\boldsymbol{\nabla} \times \mathbf{F} = \begin{pmatrix} 2x - e^{xy} \\ 3x - 2y \\ 2x \end{pmatrix} = \begin{pmatrix} 2x - e^{3x} \\ 3x - 6 \\ 2x \end{pmatrix}$$

on the surface bounded by C (y=3) which is just a disk of radius $1!$ $d\mathbf{s} = dxdy\mathbf{j}$ so $\boldsymbol{\nabla} \times \mathbf{F} \cdot d\mathbf{s} = (3x - 6)dxdz$

Using equivalent polars in the x, z-plane ($x = r\cos\theta,\ z = r\sin\theta$)

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 rdr \int_0^{2\pi} (3\cos\theta - 6) = -6\pi$$

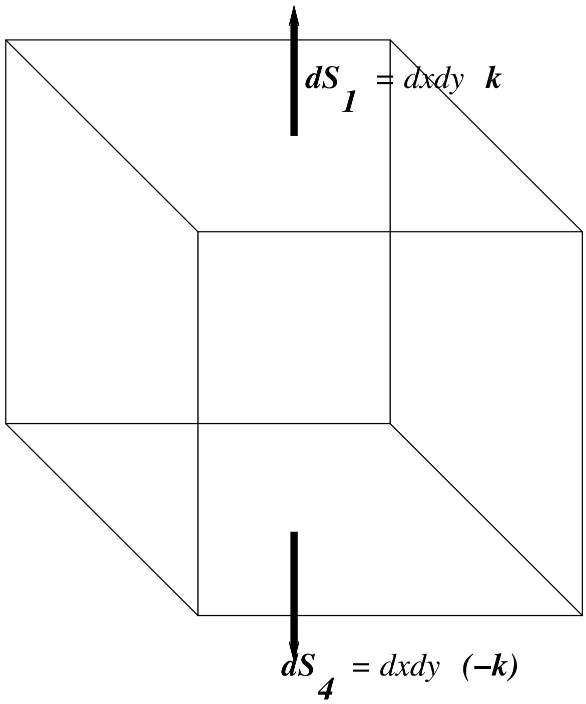
6.5 Gauss' divergence theorem

6.5.1 Theorem

This theorem converts a surface integral over a closed surface S bounding a volume V into a volume integral over the volume V.

$$\iint_S \mathbf{F} \cdot d\mathbf{s} = \iint_V (\boldsymbol{\nabla} \cdot \mathbf{F}) dV$$

6.5.2 Notion of proof



Consider the cube of sides Δx, Δy and Δz, see figure above. Consider the vector function $\mathbf{F} = F_x i + F_y j + F_z k$, the flux of F over $\mathbf{dS_1}$ and $\mathbf{dS_4}$, i.e. the top and buttom lids is

$$\iint_{dS_1} F \cdot \mathbf{dS_1} + \iint_{dS_4} F \cdot \mathbf{dS_4}$$

$$\iint_{dS_1} F_z(x, y, z + \Delta z/2) dxdy - \iint_{dS_4} F_z(x, y, z - \Delta z/2) dxdy$$

dS_1 and dS_4 have the same limits, end we can regroup the integrals

$$\iint_{dS} \left(F_z(x, yz + \Delta z/2) - F_z(x, y, z - \Delta z/2) \right) dydz$$

$$\iint_{dS} \left(\int_{z-\Delta z/2}^{z+\Delta z/2} \frac{\partial F_z}{\partial z} dz \right) dxdy = \iiint \frac{\partial F_z}{\partial z} dV$$

The same works for the 2 other components with the fluxes along i and j.

Chopping a volume V into infinitesimal volumes dV whose size goes to 0, each bounded by the surface dS, we can transform the volume integral into a sum of local integrals which transform themselves into surface integrals. But all internal surfaces appear twice in the sum (in a similar way lines integrals were appears twice in Green's and Stokes' theorems). Because of the orientation of the vector surface elements, they all cancel one another. Only the external surface integrals (which only appear once) remains. Hence the result.

6.5.3 Examples

Let $\mathbf{F} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and V the volume of a sphere of radius 3.

Volume integral

$$\nabla \cdot \mathbf{F} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3$$

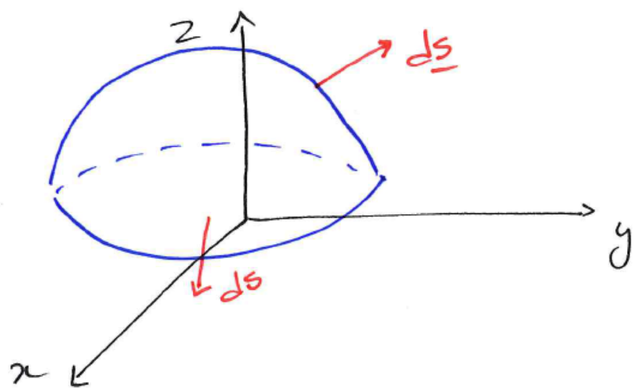
Hence $\iiint_V \nabla \cdot \mathbf{F} dV \cdot 3 \iiint_V dV = 3x$ volume of the sphere of radius a $\left(= \frac{4\pi a^3}{3} \right) = 4\pi a^3$

Surface integral

$$\begin{cases} d\mathbf{s} = a^2 \sin\theta d\theta d\phi \mathbf{e}_r \\ \mathbf{F} = \mathbf{r} = r\mathbf{e}_s = a\mathbf{e}_r \ \text{ at the surface of the sphere} \end{cases}$$

$$\begin{aligned} \iint \mathbf{F} \cdot d\mathbf{s} &= \int_0^{2\pi} \int_0^{\pi} a\mathbf{e}_r a^2 \cdot \sin\theta d\theta d\phi \mathbf{e}_r \\ &= a^3 \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \\ &= 2\pi a^3 [-\cos\theta]_0^{\pi} = 4\pi a^3 \end{aligned}$$

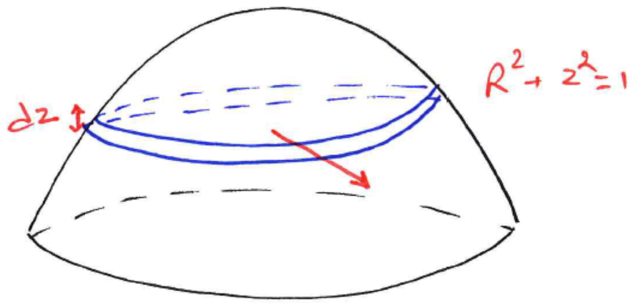
Example 2: Verify the divergence theorem for $\mathbf{F} = z^2\mathbf{k}$ over the "complete" hemisphere $z \geq 0$, of radius 1.



Volume integral

$$\nabla \cdot \mathbf{F} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ z^2 \end{pmatrix} = 2z.$$

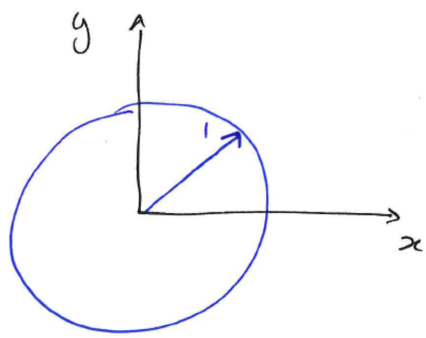
Here it is easiest to use cylindrical coordinates



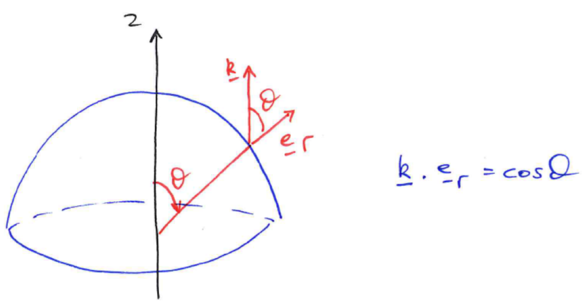
$$
\begin{aligned}
\iiint_V \boldsymbol{\nabla}\cdot\mathbf{F}dV &= \int_0^{2\pi}\int_0^1\int_0^{\sqrt{1-z^2}} 2zRdRdzd\phi \\
&= \int_0^{2\pi} d\phi \int_0^1 2z\left[\frac{R^2}{2}\right]_0^{\sqrt{1-z^2}} \\
&= 2\pi\int_0^1 2z\frac{1}{2}(1-z^2)dz \\
&= 2\pi\int_0^1 z - z^3 dz \\
&= 2\pi\left[\frac{z^2}{2}-\frac{z^4}{4}\right]_0^1 = 2\pi\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{\pi}{2}
\end{aligned}
$$

Surface integral

Bottom (S_1):



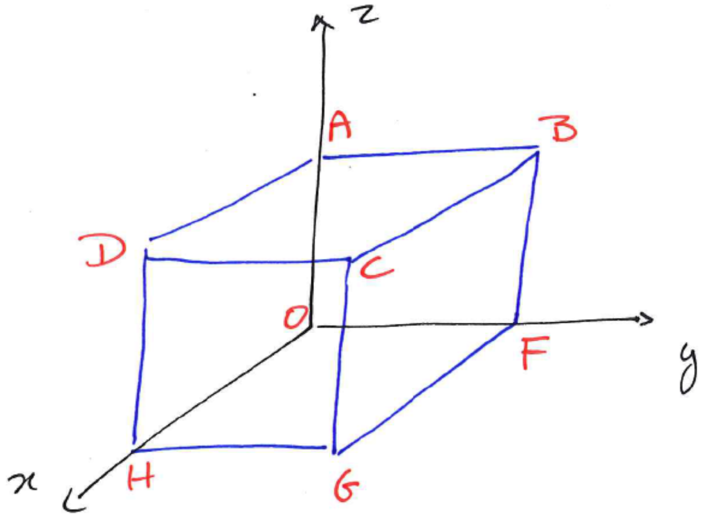
Dome (S_2):



On S_2, $z = \cos\phi$, so

$$\iint_{S_2} \mathbf{F}\cdot d\mathbf{s} = \int_0^{2\pi} d\phi \int_0^{\pi/2} \sin\phi\cos^3\phi d\phi$$
$$= 2\pi\left[-\frac{1}{4}\cos^4\phi\right]_0^{\pi/2}$$
$$= 2\pi\left[0+\frac{1}{4}\right] = \frac{\pi}{2}$$

Example 3



Verify Gauss' theorem for

$$\mathbf{F} = 4xz\mathbf{i} - y^2\mathbf{j} + yz\mathbf{k}$$

over the cube of side 1 with $0 < x < 1,\ 0 < y < 1,\ 0 < z < 1.$

$$\boldsymbol{\nabla} \cdot \mathbf{F} = 4z - 2y + y = 4z - y.$$

The volume integral is

$$\iiint_V \boldsymbol{\nabla} \cdot \mathbf{F} dV = \int_0^1 \int_0^1 \int_0^1 (4z - y) dxdydz = \int_0^1 dx \int_0^1 \int_0^1 (4z - y) dydz$$

$$\int_0^1 \left[4zy - \frac{y^2}{2}\right]_{y=0}^{y=1} dz = \int_0^1 \left(4z - \frac{1}{2}\right) dz = \left[2z^2 - \frac{z}{2}\right]_0^1 = \frac{3}{2}$$

The surface integrals are :

- on $OFAB$, $x = 0$, and $d\mathbf{s} = -dydz\mathbf{i}$ so $\mathbf{F} \cdot d\mathbf{s} = 0$

$$\iint_{OFAB} \mathbf{F} \cdot d\mathbf{s} = 0$$

- on $HGCD$, $x = 1$, and $d\mathbf{s} = dydz\mathbf{i}$ so $\mathbf{F} \cdot d\mathbf{s} = 4zdydz$

$$\iint_{HGCD} \mathbf{F} \cdot d\mathbf{s} = 4 \int_0^1 dy \int_0^1 zdz = 2$$

- on $HOAD$, $y = 0$, and $d\mathbf{s} = -dxdz\mathbf{j}$ so $\mathbf{F} \cdot d\mathbf{s} = 0$

$$\iint_{HOAD} \mathbf{F} \cdot d\mathbf{s} = 0$$

- on $GFBC$, $y = 1$, and $d\mathbf{s} = dxdz\mathbf{j}$ so $\mathbf{F} \cdot d\mathbf{s} = -dxdz$

$$\iint_{GFBC} \mathbf{F} \cdot d\mathbf{s} = -\int_0^1 dx \int_0^1 dz = -1$$

$$\iint_{HOAD} \mathbf{F} \cdot d\mathbf{s} = 0$$

- on $HGFO$, $z = 0$, and $d\mathbf{s} = -dxdy\mathbf{k}$ so $\mathbf{F} \cdot d\mathbf{s} = 0$

$$\iint_{HGF0} \mathbf{F} \cdot d\mathbf{s} = 0$$

- on $DCBA$, $z = 1$, and $d\mathbf{s} = dxdy\mathbf{k}$ so $\mathbf{F} \cdot d\mathbf{s} = ydxdy$

$$s \iint_{DCBA} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 dx \int_0^1 ydy = \frac{1}{2}$$

Overall the integral is $I = 0 + 2 + 0 - 1 + 0 + \dfrac{1}{2} = \dfrac{3}{2}$

6.5.4 The continuity equation

This equation is the first equation of the system of equation which governs the evolution of fluids (gas, liquids, plasmas) and is therefore on the the fundamental equation used in aerodynamics, meteorology, climate science, etc... anytime a fluid is involved! It expresses the fact that in classical mechanics mass is conserved (i.e. is not created nor destroyed). To express this fact mathematically, we need... vector calculus!

Consider a volume V fixed in space. The fact 'mass is conserved' means that the variation of mass of fluid inside the volume has to be equal to the amount of fluid which has got into the volume, minus the amount of fluid which has left the volume. i.e. the net flux of fluid across the boundary of the volume V

$$\frac{d}{dt}\iint_V \rho dV = \iiint_V \frac{\partial \rho}{\partial t} dV = -\iint_S \rho \mathbf{u} \cdot d\mathbf{s} = -\iiint_V \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) dV$$

$$\iiint_V \left(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) \right) dV = 0$$

Since it has to be true $\forall V$,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0$$

6.6 Appendice