4 Hyperbolic isometries

1. Let $g \in \mathrm{PSL}(2,\mathbb{R})$ and write $\phi \circ g \circ \phi^{-1} \in \mathrm{con}^+(1)$ in standard form. Prove that

$$\mathrm{tr}(g) = \mathrm{tr}(\phi \circ g \circ \phi^{-1}).$$

   Hence, give a classification of the elements of $\mathrm{con}^+(1)$ which relates the trace of an element to the location of its fixed points.

2. Investigate the relationship between the trace of compositions of elements in $\mathrm{PSL}(2,\mathbb{R})$ and the trace of the elements themselves. For example, can one relate $\mathrm{tr}(g^{-1})$ to $\mathrm{tr}(g)$ or $\mathrm{tr}(gh)$ to $\mathrm{tr}(g)$ and $\mathrm{tr}(h)$?

3. (i) Prove that if the fixed points of a hyperbolic element of $\mathrm{PSL}(2,\mathbb{R})$ are 0 and ∞, then it is necessarily of the form $z \mapsto \alpha z$ for a real number α not equal to 0 or 1.

   (ii) Prove that if the fixed point of a parabolic element of $\mathrm{PSL}(2,\mathbb{R})$ is ∞, then it is necessarily of the form $z \mapsto z + \beta$ for a real number $\beta \neq 0$.

4. Write $\mathrm{Fix}(g)$ for the set of fixed points of an element $g \in \mathrm{PSL}(2,\mathbb{R})$. Prove that for any $g, h \in \mathrm{PSL}(2,\mathbb{R})$

$$\mathrm{Fix}(hgh^{-1}) = h\Big(\mathrm{Fix}(g)\Big).$$

   Deduce that conjugation does not change the classification of an element, i.e. g and hgh^{-1} are of the same 'type'.

5. Fix a real number $\lambda > 0$, and let g be defined by

$$g(z) = \frac{z}{\lambda z + 1}.$$

   (i) Verify that $g \in \mathrm{PSL}(2,\mathbb{R})$,
   (ii) classify it as identity, hyperbolic, parabolic, or elliptic,
   (iii) determine its fixed points,
   (iv) draw a picture to demonstrate the action of g on $\mathbb{H}^2$.

6. Let $g \in \mathrm{PSL}(2,\mathbb{R})$ be a parabolic element. Prove that g can either be conjugated to the element $z \mapsto z + 1$ or $z \mapsto z - 1$ via an element of $\mathrm{PSL}(2,\mathbb{R})$.