MT5823 Semigroup theory: Problem sheet 6 (James D. Mitchell)
Cancellative semigroups, regular semigroups, inverses

Cancellative semigroups

Recall that a semigroup S is ***cancellative*** if $ax = bx$ and $yc = yd$ implies that $a = b$ and $c = d$ for all $a, b, c, d, x, y \in S$.

6-1. Let S be a cancellative semigroup. Prove that the semigroup S^1 (S with an identity adjoined) is cancellative if and only if S has no identity element.

6-2. Let S be a cancellative semigroup without identity. Prove that in S we have $\mathscr{R} = \mathscr{L} = \mathscr{H} = \mathscr{D} = \Delta_S$.

6-3. Let T be the semigroup consisting of matrices of the form

$$\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}$$

where $a, b \in \mathbb{R}$ and $a, b > 0$. Prove that T is a semigroup with respect to the usual matrix multiplication. Prove that T is cancellative, and deduce that $\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$ and $\mathscr{D}$ are all trivial. Prove that T has no proper two-sided ideals, so that $\mathscr{J} = T \times T$. (This is an example where $\mathscr{J} \neq \mathscr{D}$.)

Regular semigroups

6-4. Let S be a regular semigroup with a single idempotent. How many $\mathscr{R}$-classes and $\mathscr{L}$-classes does S have? Prove that S is a group.

6-5. Let S be a regular semigroup and let ρ be a congruence on S. Assume that $a \in S$ is such that a/ρ is an idempotent in S/ρ. If x is any inverse of a^2, then prove that $e = axa$ is an idempotent, and that it belongs to a/ρ. (This result is known as ***Lallement's Lemma***.)

6-6. Prove that the bicyclic monoid is regular. Prove that every rectangular band is regular.

6-7. Prove that the semigroup generated by the mappings

$$x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & - & - \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 4 & 5 & 4 \end{pmatrix}$$

is not regular. Prove that the semigroup defined by the presentation $\langle a, b \mid a^3 = a,\ b^4 = b,\ ba = a^2 b\rangle$ is regular.

Inverses

6-8.* Let S be a regular semigroup, and let E be the set of idempotents of S. For a set $X \subseteq S$ and $n \geq 1$ let $X^n = \{x_1 x_2 \dots x_n : x_1, \dots, x_n \in X\}$. For an element $x \in S$ denote by $V(x)$ the set of all inverses of x. Finally, for $X \subseteq S$ let $V(X) = \bigcup_{x \in X} V(x)$.

(a) For $x = e_1 e_2 \dots e_n \in E^n$ and $y \in V(x)$ define

$$f_j = e_j \dots e_n y e_1 \dots e_{j-1} \quad j = 1, \dots, n.$$

Prove that all f_j are idempotents and that $yxf_n f_{n-1} \dots f_2 xy = y$.

(b) For $x = e_1 e_2 \dots e_{n+1} \in E^{n+1}$ and any $y \in V(x)$ let

$$g_j = e_{j+1} \dots e_{n+1} y e_1 \dots e_j \quad j = 1, \dots, n+1.$$

Prove that all g_j are idempotents and that $x \in V(g_n g_{n-1} \dots g_1)$.

(c) Prove that $V(E^n) = E^{n+1}$ for all $n \geq 1$.

Further problems

6-9. Let S be a regular semigroup. Show that the following are equivalent:

(a) S has exactly one idempotent;

(b) S is cancellative;

(c) S is a group.