Chapter 1

Existence and uniqueness of solutions to initial-value problems

1.1 Notation, terminology, etc.

We will typically use y to denote the dependent variable, x or t to denote the independent variable; thus $y = y(x)$, or $y = y(t)$ etc.

We will often (not always) use y', y'', ... $y^{(n)}$ to denote derivatives; sometimes also $\dot{y}$, etc, when $y = y(t)$.

In its most general form, an ordinary differential equation (ODE) is a relation between x, y, and the derivatives of y:

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0.$$

The ODE is *linear* if F is linear in $y, y', y'', \ldots, y^{(n)}$. There is no requirement that F be linear in x. So, for example:

$$\begin{aligned} x^2y'' + xy' + x^2y = 0 \qquad & \text{is linear} \\ y' + xy^2 = 0 \qquad & \text{is nonlinear} \end{aligned}$$

The *order* of an ODE is the degree of the highest derivative appearing in F.

In Part I, we will consider the general first order ODE:

$$y' = f(x, y) \tag{1.1}$$

where $f(x, y)$ is some function. There is no general method to obtain solutions to this equation for arbitrary $f(x, y)$.

A first order ODE together with an initial condition $y(x_0) = y_0$ is called an *initial value problem* (IVP):

$$y' = f(x, y) \qquad \text{with} \qquad y(x_0) = y_0. \tag{1.2}$$

1.2 Direction field and integral curves

A simple graphical representation of solutions to (1.1) may be obtained by observing that $y' = dy/dx = f(x, y)$ is the slope of the curve $y = y(x)$ in the xy-plane. In this context, the xy-plane is called the *phase plane* and solution curves whose slope is equal to $f(x, y)$ at each point are called *integral curves*. We construct the *direction field* of a given ODE by drawing short line segments at many points (x, y) each having slope equal to $f(x, y)$ at that point. The integral curves are obtained by joining the segments smoothly. This gives an infinite family of integral curves; each curve is a solution of the ODE corresponding to a different constant of integration. The solution to the initial value problem is then the particular integral curve that passes through the point (x_0, y_0), that is, the particular curve for which $y(x_0) = y_0$.

1.3 Solution of linear 1st order ODE by integrating factor

When $f(x, y)$ is linear in y then (1.1) is linear and can be written in the form

$$y' + p(x)y = q(x). \tag{1.3}$$

It can be solved using an integrating factor

$$\mu(x) = \exp\left[\int p(x)dx\right]$$

with solution

$$y(x) = \frac{1}{\mu(x)} \int \mu(x)q(x)\,dx.$$

In the case of the initial value problem (1.2), the constant of integration is determined by the initial condition.

Since we construct the solution explicitly, we know that the solution must exist, provided that $\int p(x)dx$ is well-defined (so that $\mu(x)$ is strictly positive). Is it the only solution?

We will do this properly for the general case in the next section, but as an example, suppose there are two solutions, $y_1(x)$ and $y_2(x)$ to the linear initial value problem:

$$y' + p(x)y = q(x) \qquad \text{with} \qquad y(x_0) = y_0. \tag{1.4}$$

Then the difference $u(x) = y_1(x) - y_2(x)$ satisfies

$$u' + p(x)u = 0 \qquad \text{with} \qquad u(x_0) = 0.$$

This is a linear and separable equation (see MT1002; also section 2.1 below) and can be integrated

$$\int \frac{du}{u} = -\int p(x)dx$$

to give

$$u(x) = \exp\left[-\int p(x)dx\right] \qquad\qquad = A\mu^{-1}$$

where the factor A is an arbitrary constant of integration. The initial condition $u(x_0) = 0$ then implies that $A = 0$ and so $u(x) \equiv 0$. Hence $y_1 = y_2$.

1.4 Existence and uniqueness

We will prove existence and uniqueness of solutions to the initial value problem (1.2) for suitable $f(x, y)$.

Theorem 1.1: Let f and $\dfrac{\partial f}{\partial y}$ be continuous on the closed rectangle R in the xy-plane defined by

$$x_0 - a \leq x \leq x_0 + a \qquad y_0 - b \leq y \leq y_0 + b.$$

Then for some $\alpha \leq a$ there exists a solution $y(x)$ satisfying $y' = f(x, y)$ and $y(x_0) = y_0$ for all $|x - x_0| < \alpha$.

Remarks

(i) The conditions are sufficient, not necessary.

(ii) This is a local result: the solution exists locally in a neighbourhood of x_0. Global results are much harder.

Strategy

1. We will prove existence of solutions to the equivalent integral equation, obtained by integrating (1.2) from x_0 to x:

$$y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))ds \tag{1.2$'$}$$

Note from (1.2$'$) we have $y(x_0) = y_0 + 0$, which is the initial condition, and, by the fundamental theorem of calculus,

$$\frac{d}{dx}y = 0 + f(x, y(x)),$$

which is the original ODE. In other words, (1.2) and (1.2$'$) are completely equivalent formulations of the same initial value problem. We now generate sequence of *successive approximations*, $y_0(x), y_1(x), \ldots$, to the solution of (1.2$'$):

$$\begin{aligned}
y_0(x) &= y_0 \\
y_1(x) &= y_0 + \int_{x_0}^{x} f(s, y_0(s))ds \\
y_2(x) &= y_0 + \int_{x_0}^{x} f(s, y_1(s))ds \\
&\vdots \\
y_{n+1}(x) &= y_0 + \int_{x_0}^{x} f(s, y_n(s))ds
\end{aligned}$$

2. We need to show that the sequence is well-defined on an appropriate subset of R_α of R; essentially we require that $(s, y_n(s)) \in R_\alpha \subset R$ so that f is bounded for every n.

3. We then show that the sequence converges to a limit and that the limit satisfies (1.2$'$).

Example 1.1. Find the y_n for the IVP $y' = y$ with $y(0) = 2$. (See separate examples sheet.)

In the proof of Theorem 1.1, we will need the following preliminary results:

P1. A function everywhere continuous on a closed domain is bounded.

Proof. See MT2502.

Thus if f and $\dfrac{\partial f}{\partial y}$ are continuous on the closed rectangle R then we have that $|f(x, y)| \leq K$ and $\dfrac{\partial f}{\partial y}(x, y) \leq L$ for all $(x, y) \in R$, for some real numbers K and L.

P2. Consider $f(x, y)$ as a function of y at fixed x, and assume that f is continuous for all $y \in [y_0 - b, y_0 + b]$, i.e. for $|y - y_0| \leq b$. Then by the mean value theorem (see MT2502), for any $y_1, y_2 \in [y_0 - b, y_0 + b]$ (with $y_1 < y_2$, say) there exists a $c \in [y_1, y_2]$ for which

$$f(x, y_2) - f(x, y_1) = (y_2 - y_1) \left. \frac{\partial f}{\partial y} \right|_{y=c}.$$

If $\left| \dfrac{\partial f}{\partial y} \right|$ is bounded by L then it follows that

$$|f(x, y_2) - f(x, y_1)| \leq L|y_2 - y_1| \tag{1.5}$$

for all $y_1, y_2 \in [y_0 - b, y_0 + b]$. This inequality is called a *Lipschitz condition* and we say that f is Lipschitz in y on R. Thus

$$\left| \frac{\partial f}{\partial y} \right| \leq L \text{ on } R \implies f \text{ is Lipschitz in } y \text{ on } R.$$

Proof of Theorem 1.1.

By assumption, f and $\dfrac{\partial f}{\partial y}$ are continuous on the closed domain R and so, by (P1), there exist real numbers K and L with $|f(x, y)| \leq K$ and $\left| \dfrac{\partial f}{\partial y}(x, y) \right| \leq L$ for all $(x, y) \in R$.

1. Define y_n as above by

$$\begin{aligned} y_0(x) &= y_0 \\ y_n(x) &= y_0 + \int_{x_0}^{x} f(s, y_{n-1}(s)) ds \end{aligned} \tag{1.6}$$

2. We need to show that $(s, y_n(s) \in R$ for all n, so that the sequence is well-defined. We use a proof by induction.

First, define $\alpha_1 = \min(a, b/K)$ and let $I_{\alpha_1} = [x_0 - \alpha_1, x_0 + \alpha_1]$. By definition $y_0(x) = y_0 =$ constant and so $y_0(x) \in [y_0 - b, y_0 + b]$ for all $x \in I_{\alpha_1}$, i.e. $|y_0(x) - y_0| \leq b$.

Now suppose $|y_n(x) - y_0| \leq b$ for $x \in I_{\alpha_1}$. Then

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(s, y_n(s)) ds$$

$$\begin{aligned}
\implies |y_{n+1}(x) - y_0| &= \left| \int_{x_0}^{x} f(s, y_n(s)) ds \right| \\
&\leq \int_{x_0}^{x} |f(s, y_n(s))| ds \\
&\leq K \int_{x_0}^{x} ds \\
&\leq K|x_0 - x| \\
&\leq b \qquad \text{when } x \in I_{\alpha_1}.
\end{aligned}$$

Hence by induction $|y_n(x) - y_0| \leq b$ for all n and $x \in I_{\alpha_1}$. Therefore, for $s \in I_{\alpha_1}$ we have $(s, y_n(s)) \in R_{\alpha_1} \subset R$ and so $f(s, y_n(s))$ is defined and bounded by K, for all n.

3. To show that the sequence converges, consider $y_{n+1}(x) - y_n(x)$:

$$\begin{aligned}
|y_{n+1}(x) - y_n(x)| &= \left| \int_{x_0}^{x} \left[f(s, y_n(s)) - f(s, y_{n-1}(s)) \right] ds \right| \\
&\leq \int_{x_0}^{x} |f(s, y_n(s)) - f(s, y_{n-1}(s))| ds \\
&\leq \int_{x_0}^{x} L|y_n(s) - y_{n-1}(s)| ds \qquad \text{(by P2)}
\end{aligned}$$

Now suppose $|x - x_0| \leq \min(\alpha_1, \frac{1}{2L}) = \alpha_2$. Then

$$\begin{aligned}
|y_{n+1}(x) - y_n(x)| &\leq L \frac{1}{2L} \max_{x \in I_{\alpha_2}} |y_n(x) - y_{n-1}(x)| \\
&\leq \frac{1}{2} \max_{x \in I_{\alpha_2}} |y_n(x) - y_{n-1}(x)|
\end{aligned}$$

Similarly

$$|y_n(x) - y_{n-1}(x)| \leq \frac{1}{2} \max_{x \in I_{\alpha_2}} |y_{n-1}(x) - y_{n-2}(x)|$$

Continuing to lower n and combining all inequalities eventually gives:

$$|y_{n+1}(x) - y_n(x)| \leq \frac{1}{2^n} \max_{x \in I_{\alpha_2}} |y_1(x) - y_0(x)|.$$

Now write

$$\begin{aligned}
y_N(x) &= y_0(x) + (y_1(x) - y_0(x)) + (y_2(x) - y_1(x)) + \ldots + (y_N(x) - y_{N-1}(x)) \\
&= y_0(x) + \sum_{n=1}^{N} (y_n(x) - y_{n-1}(x))
\end{aligned}$$

The partial sums converge uniformly on I_{α_2} as $N \to \infty$ by the comparison test (since $\sum 2^{-n}$ converges), and so the sequence $y_n(x)$ converges uniformly to the limit

$$y_*(x) = y_0(x) + \sum_{n=1}^{\infty} (y_n(x) - y_{n-1}(x))$$

Since $y_*(x)$ is the uniform limit of continuous functions it is also continuous.

To show that $y_*(x)$ satisfies ($1.2'$), note that by (P2) we have

$$|f(x, y_*(x)) - f(x, y_n(x))| \leq L|y_*(x) - y_n(x)|$$

for all n and for all $x \in I_{\alpha_2}$ and so $f(x, y_n(x))$ also converges uniformly to $f(x, y_*(x))$ for $x \in I_{\alpha_2}$. Hence we can exchange limit and integral and conclude that

$$y_*(x) = y_0 + \int_{x_0}^{x} f(s, y_*(s))ds$$

as required. $\square$

Theorem 1.2. Let f and $\dfrac{\partial f}{\partial y}$ be continuous on the closed rectangle R and let α_2 be as defined in Theorem 1.1. Then the solution to the initial value problem ($1.2'$) is *unique* for all $x \in I_{\alpha_2}$.

Proof. Let $y_1(x)$ and $y_2(x)$ be two solutions to ($1.2'$) defined on I_{α_2} and let $u(x) = y_1(x) - y_2(x)$. Then $u(x)$ satisfies $u' = f(x, y_1) - f(x, y_1)$ and the initial condition $u(x_0) = 0$. Integrating from x_0 to x gives

$$u(x) - 0 = \int_{x_0}^{x} [f(x, y_1(s)) - f(x, y_2(s))]ds$$

and so

$$\begin{aligned}
|u(x)| &\leq \int_{x_0}^{x} |f(s, y_1(s)) - f(s, y_2(s))|ds \\
&\leq L \int_{x_0}^{x} |y_1(s) - y_2(s)|ds \qquad \text{(by P2)} \\
&\leq L \max_{x \in I_{\alpha_2}} |u(x)||x - x_0| \\
&\leq L \max_{x \in I_{\alpha_2}} |u(x)|\alpha_2 \\
&\leq \frac{1}{2} \max_{x \in I_{\alpha_2}} |u(x)|.
\end{aligned}$$

The only way the inequality can be satisfied for all $x \in I_{\alpha_2}$ is if $u(x) \equiv 0$ on I_{α_2}, in other words if $y_1(x) = y_2(x)$. $\square$

Remark. These theorems are not sharp. With a little more work it is possible to prove existence and uniqueness over the larger interval I_{α_1}, where $\alpha_1 = \min(a, b/K)$. This involves a refinement to step 3 of the proof of Theorem 1.1 and the use of *Gronwall's Inequality* in the proof of Theorem 1.2.

Example 1.2. $y' = \sin(xy)$ with $y(0) = 1$.

Example 1.3. $y' = (x + y)x^2y^2$ with $y(0) = 1$.

Example 1.4. $y' = e^x + x/y$ with $y(0) = 1$.

Example 1.5. $y' = y^{1/3}$ with $y(0) = 0$.