(6) Let $z \in \mathbb{D}^2$. Then we know from lectures that

$$r := d_{\mathbb{D}^2}(0,z) = \log \frac{1+|z|}{1-|z|}$$

and so
$$e^r = \frac{1+|z|}{1-|z|}.$$

Solving for $|z|$ yields

$$|z| = \frac{e^r - 1}{e^r + 1} = \frac{(e^{r/2} - e^{-r/2})/2}{(e^{r/2} + e^{-r/2})/2}$$

$$= \frac{\sinh \frac{r}{2}}{\cosh \frac{r}{2}} = \tanh \frac{r}{2} \qquad \square$$

(7) By applying rotation by $\arg(z)$ (clockwise) we may assume $w, z \in \mathbb{R}$ with $1 > z > w > 0$. Therefore

$$d_{\mathbb{D}^2}(0,w) = d_{\mathbb{D}^2}(w,z)$$

...

(7) cont...

$$\iff \log\frac{1+w}{1-w} = \log\frac{(1-zw)+(z-w)}{(1-zw)-(z-w)}$$

$$\iff (1+w)\Big((1-zw)-(z-w)\Big)$$

$$= (1-w)\Big((1-zw)+(z-w)\Big)$$

$$\iff \quad w(1-zw) = z-w \qquad \left(\text{multiply out and cancel}\right).$$

This gives us a quadratic in w:

$$zw^2 - 2w + z = 0$$

and so

$$w = \frac{2 \pm \sqrt{4-4z^2}}{2z}$$

$$= \frac{2 \pm 2\sqrt{1-z^2}}{2z}$$

and since we can ignore the ~~negative~~ positive root since $w<1$ we get

$$w = \frac{1-\sqrt{1-z^2}}{z} \sim z \text{ as } z \to 1$$

7 cont...

So for general $z \in \mathbb{D}^2$, the midpoint of the geodesic joining 0 and z is given by:

$$\frac{1 - \sqrt{1 - |z|^2}}{|z|} e^{i \arg(z)}$$

Note that in the Euclidean setting, the midpoint is $\frac{|z|}{2} e^{i \arg(z)}$

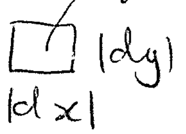
and $\frac{|z|}{2} \sim \frac{|z|}{2}$ as $|z| \to 1$ (or as $|z| \to \infty$).

(8) We will prove this for $g \in \mathrm{Con}^+(1)$, and $F \subseteq \mathbb{D}^2$. We have

$$A_{\mathbb{D}^2}(g(F)) = \int_{g(F)} \frac{4}{(1-|z|^2)^2} |dz|$$

(Remember that in this case $|dz|$ is an "area" infinitesimal:

$|dz| = |dx|\,|dy|$



As before we make the substitution $z = g(w)$, but this time

$$|dz| = |D_g(w)|\,|dw|$$

where $|D_g(w)|$ is the determinant of the Jacobian derivative of g (at w).

This is because g is viewed as a 2-dimensional map:

$$g(z) = g(x+iy) = g_1(x,y) + i\,g_2(x,y)$$

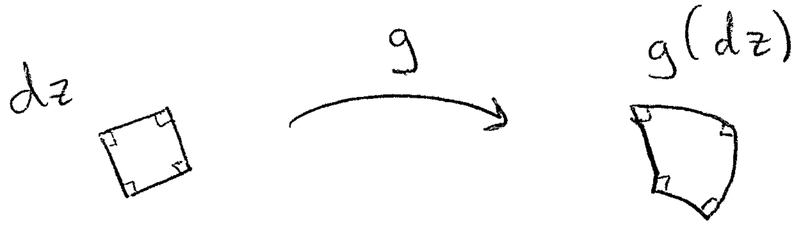
(8) ... cont...

A lot of algebra yields

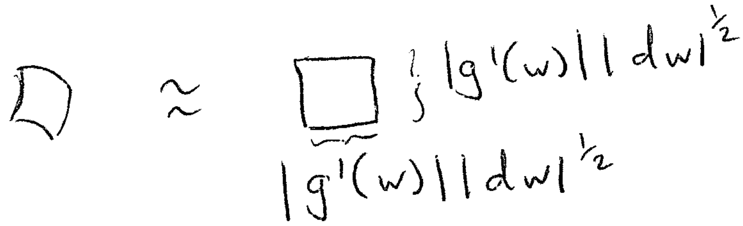
$$|D_g(w)| = |g'(w)|^2 = \frac{(1-|g(w)|^2)^2}{(1-|w|^2)^2}$$

Lemma from notes.

but this can be seen in an easier way since g is conformal!



since g is angle preserving, it maps small squares to small squares and so



giving $|D_g(w)| = |g'(w)|^2$.

8 ... cont...

So we have:

$$A_{\mathbb{D}^2}(g(F)) = \int_F \frac{4\,|D_g(w)|\,|dw|}{(1-|g(w)|^2)^2}$$

$$= \int_F \frac{4}{(1-|g(w)|^2)^2} \frac{(1-|g(w)|^2)^2}{(1-|w|^2)^2} |dw|$$

$$= \int_F \frac{4}{(1-|w|^2)^2} |dw|$$

$$= A_{\mathbb{D}^2}(F)$$

as required.

(9) By using con$^+(1)$, we may assume C is centered at the origin and therefore C has the following parameterisation

$$C = \left\{ \tanh(\tfrac{r}{2})\, e^{i\theta} : 0 \le \theta < 2\pi \right\}.$$

Therefore, using $z = \tanh(\frac{r}{2})\, e^{i\theta}$ $\left(|dz| = \tanh\frac{r}{2}\, d\theta \right)$

$$L_{\mathbb{D}^2}(C) = \int_C \frac{2\,|dz|}{(1-|z|^2)}$$

$$= \int_0^{2\pi} \frac{2\tanh\frac{r}{2}\, d\theta}{1-\tanh^2\frac{r}{2}}$$

$$= 4\pi \sinh\tfrac{r}{2}\cosh\tfrac{r}{2}$$ (multiply top & bottom by $\cosh^2\frac{r}{2}$ & use $\cosh^2\frac{r}{2} - \sinh^2\frac{r}{2} = 1$)

$$= 2\pi \sinh r$$

as required. Here we used the 'hyperbolic double angle formula':

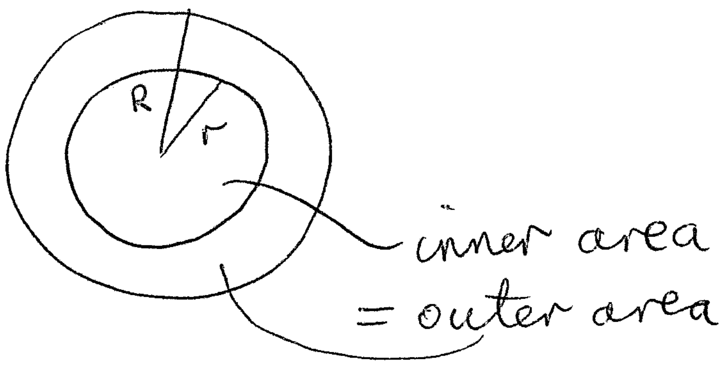
$$\sinh(2\theta) = 2\sinh\theta\cosh\theta$$

Check from the definitions that this holds!

(10) We want r such that

$$A_{\mathbb{D}^2}\left(B_{\mathbb{D}^2}(0,r)\right) = A_{\mathbb{D}^2}\left(B_{\mathbb{D}^2}(0,R) \setminus B_{\mathbb{D}^2}(0,r)\right)$$

ie



This is equivalent to solving for $r > 0$ such that

$$4\pi \sinh^2 \frac{r}{2} = 4\pi \sinh^2 \frac{R}{2} - 4\pi \sinh^2 \frac{r}{2}$$

$$\Leftrightarrow \quad 2 \sinh^2 \frac{r}{2} = \sinh^2 \frac{R}{2}$$

$$\Leftrightarrow \quad \sinh \frac{r}{2} = \frac{1}{\sqrt{2}} \sinh \frac{R}{2}$$

$$\Leftrightarrow \quad r = 2 \operatorname{arcsinh}\left(\frac{1}{\sqrt{2}} \sinh \frac{R}{2}\right)$$

$$\sim R \quad \text{as } R \to \infty.$$

10 cont...

In Euclidean space we want $r > 0$ such that

$$\pi r^2 = \pi R^2 - \pi r^2$$

$$\Leftrightarrow \quad r^2 = \frac{R^2}{2}$$

$$\Leftrightarrow \quad r = \frac{1}{\sqrt{2}} R$$

In hyperbolic space, all the area is at the boundary (asymptotically).