Chapter 3

Functions of two or more variables

{chap:3}

3.1 Functions and surfaces (Swokowski chapter 12.1)

A real function of two variables $f(x, y)$ takes the *ordered* pair of real numbers (x, y) and returns a single real number, $f(x, y)$. By introducing a third dimension and setting

$$z = f(x, y),$$

the function defines a surface with coordinates,

$$(x, y, z) \equiv (x, y, f(x, y)).$$

The surface $f(x, y) = (1 - x^2)(1 + y^2)$ is shown in Figure 3.1(a). Selecting a value for x and one for y, the height of the surface gives the value of the function $f(x, y)$ at that point. The surface can also be represented as a contour plot. Each contour corresponds to a specific value of f, for example say $f = 1$, and the contour line shows the values of x and y satisfying $f(x, y) = 1$. In our example, the general contours are given by

$$(1 - x^2)(1 + y^2) = C, \qquad \Rightarrow \qquad 1 + y^2 = \frac{C}{1 - x^2}, \qquad \Rightarrow \qquad y = \sqrt{C/(1 - x^2) - 1}.$$

Choosing different values of C gives different contours.

Note that (x, y, z) forms a right-handed coordinate system. This means that you take your right hand and align the x axis with the index finger, the y axis with the middle finger and then the z axis points along the direction of the thumb.

Definitions

1. The *graph* of $f(x, y)$ is the *surface* $z = f(x, y)$.

2. The *domain* of $f(x, y)$ is the area *in the xy-plane* on which f is defined.

3. The *range* of $f(x, y)$ is the set of values that f takes on its domain.

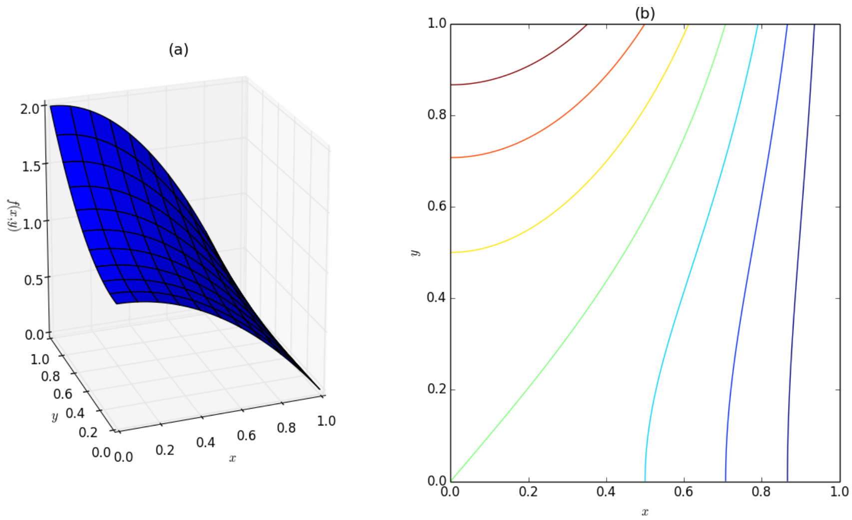


Figure 3.1: (a) The surface defined by $z = f(x, y) = (1 - x^2)(1 + y^2)$. (b) The surface can be illustrated by a contour plot. The contours are given by the same value of f. {fig:3.1}

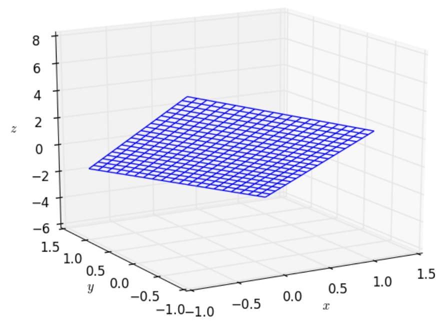
Example 3.7

$f(x, y) = a - \alpha x - \beta y$, defines a plane, since

$$f(x, y) = z = a - \alpha x - \beta y, \qquad \Rightarrow \qquad \alpha x + \beta y + z = a,$$

is the equation of a plane with normal $\mathbf{n} = (\alpha, \beta, 1)$.

$f(x, y)$ is defined for all $(x, y) \Rightarrow$ domain is the whole x, y plane. The range is $-\infty < z < \infty$. The plane intercepts the three coordinate axes at $x = a/\alpha$, $y = a/\beta$ and $z = a$.



:3.2} Figure 3.2: The plane given by $z = a - \alpha x - \beta y$.

Example End

Example 3.8

Consider $f(x, y) = +\sqrt{R^2 - x^2 - y^2}$. Setting $z = f(x, y)$ and squaring both sides gives

$$z^2 = R^2 - x^2 - y^2 \qquad \Rightarrow \qquad x^2 + y^2 + z^2 = R^2.$$

The surface is a sphere of radius R, with $z \geq 0$ centred on the origin. Note that f is only defined for $R^2 - x^2 - y^2 \geq 0$ and so the domain is given by $x^2 + y^2 \leq R^2$. The range (since we are using the positive square root only) is $0 \leq z \leq R$. This is shown in Figure 3.3.

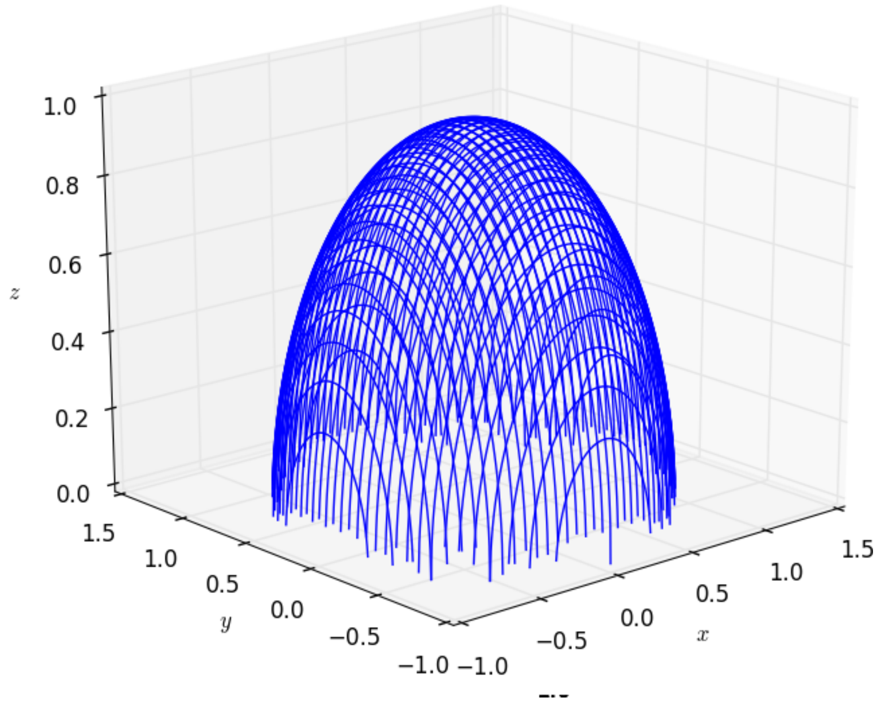


Figure 3.3: The surface $z = +\sqrt{R^2 - x^2 - y^2}$ for $R = 2$. {fig:3.3}

Example End

Example 3.9

Consider $f(x, y) = +\sqrt{\frac{x^2}{2} + y^2}$. Setting $z = f(x, y)$, this defines the surface

$$z = \sqrt{\frac{x^2}{2} + y^2}, \qquad z > 0.$$

Note that when $z = 0$, $x^2/2 + y^2 = 0$ and so both $x = 0$ and $y = 0$. To determine the shape of this surface we first of all eliminate the square root by squaring both sides, but remember that $z > 0$, to give

$$z^2 = \frac{x^2}{2} + y^2.$$

Now look at sections in different planes. Consider the $x - z$ plane, i.e. $y = 0$ so that

$$z^2 = \frac{x^2}{2}, \qquad \Rightarrow \qquad z = \pm\frac{x}{\sqrt{2}}.$$

These are two straight lines with the plus sign used when $x > 0$ and the negative sign when $x < 0$. Similarly in the $y - z$ plane with $x = 0$ we have

$$z^2 = y^2, \qquad \Rightarrow \qquad z = \pm y.$$

Again there are two straight lines. Finally, look at sections in the $x - y$ plane at a constant value of z, say $z = z_0$. Thus,

$$\frac{x^2}{2} + y^2 = z_0^2,$$

and so we have ellipses. Putting all this together, the function $f(x, y)$ describes an elliptical cone. This is shown in Figure 3.4.

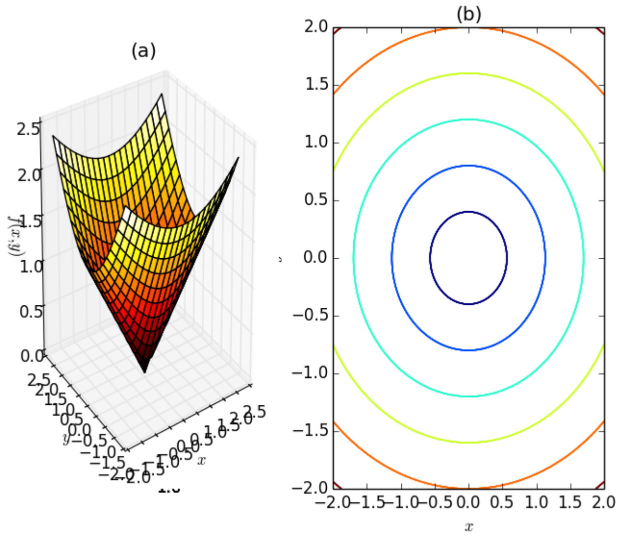


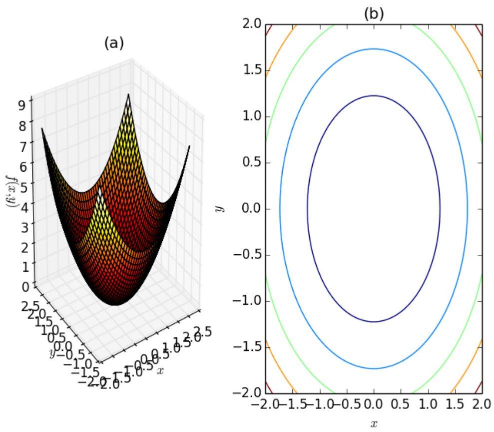
Figure 3.4: (a) A surface plot of $z = \sqrt{x^2/2 + y^2}$. It is clear that it is some kind of cone. (b) A contour
:3.4} plot that shows that the cross-sections are ellipses.

Example End

Example 3.10

Consider $f(x, y) = x^2 + y^2$. Set $z = f(x, y)$ so the surfaces are

$$z = x^2 + y^2, \qquad \text{clearly } z \geq 0.$$



{fig:3.5} Figure 3.5: (a) A surface plot of $z = x^2 + y^2$. It clearly has a parabolic shape. (b) A contour plot shows that the cross-sections are circles.

We can sketch various sections through the surface by selecting different planes. For $y = 0$, $z = x^2$ giving a parabola. For $x = 0$, $z = y^2$ again giving a parabola. Finally, at constant z, say $z = a^2$, we have $x^2 + y^2 = a^2$, namely a circle of radius a in the plane $z = a^2$. The surface is a circular bowl with parabolic sections (a *paraboloid*). This is shown in Figure 3.5

Example End

Example 3.11

Consider $f(x, y) = z = x^2 - y^2$. If $x = 0$, then $z = -y^2$ and we have a parabola with $z \leq 0$. If $y = 0$, then $z = x^2$ and we have a parabola with $z \geq 0$. Finally, if $z = 0$, then $x^2 - y^2 = 0$. Therefore,

$$(x - y)(x + y) = 0, \qquad \Rightarrow \qquad x = y, \qquad x = -y.$$

This is sketched in Figure 3.6. To understand the nature of the surface, we consider two values of z. The sections at constant $z = z_0 > 0$ satisfy

$$x^2 - y^2 = z_0, \qquad \Rightarrow \qquad x = \pm\sqrt{z_0 + y^2}.$$

These are shown in Figure 3.7(a). In a similar manner, if $z = z_0 < 0$, then we have

$$x^2 - y^2 = z_0, \qquad \Rightarrow \qquad y = \pm\sqrt{-z_0 + x^2}.$$

These curves are shown in Figure 3.7(b).

The surface is a hyperbolic paraboloid. The origin is called a *saddle point* of the surface.

Example End

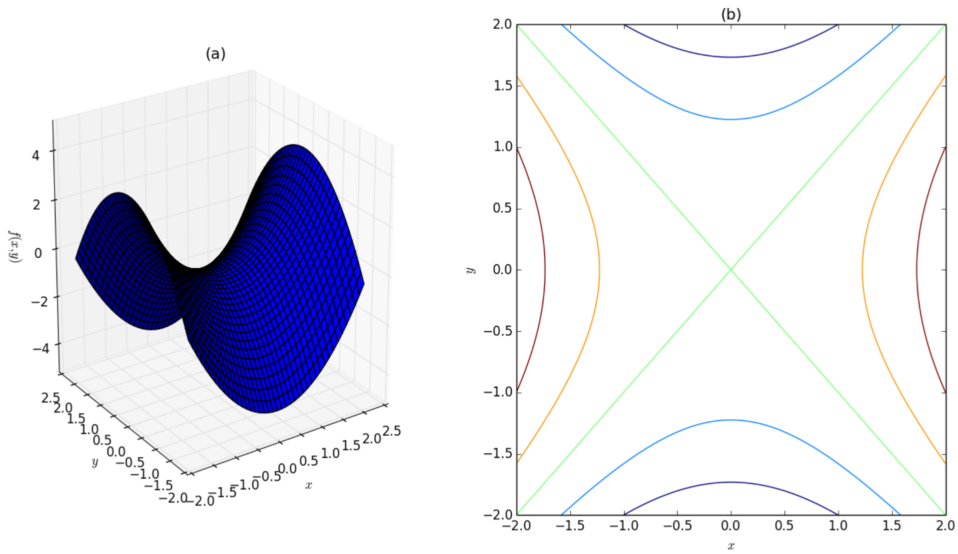


Figure 3.6: (a) A surface plot of $z = x^2 - y^2$. (b) A contour plot of $z = x^2 - y^2$. {fig:3.6}

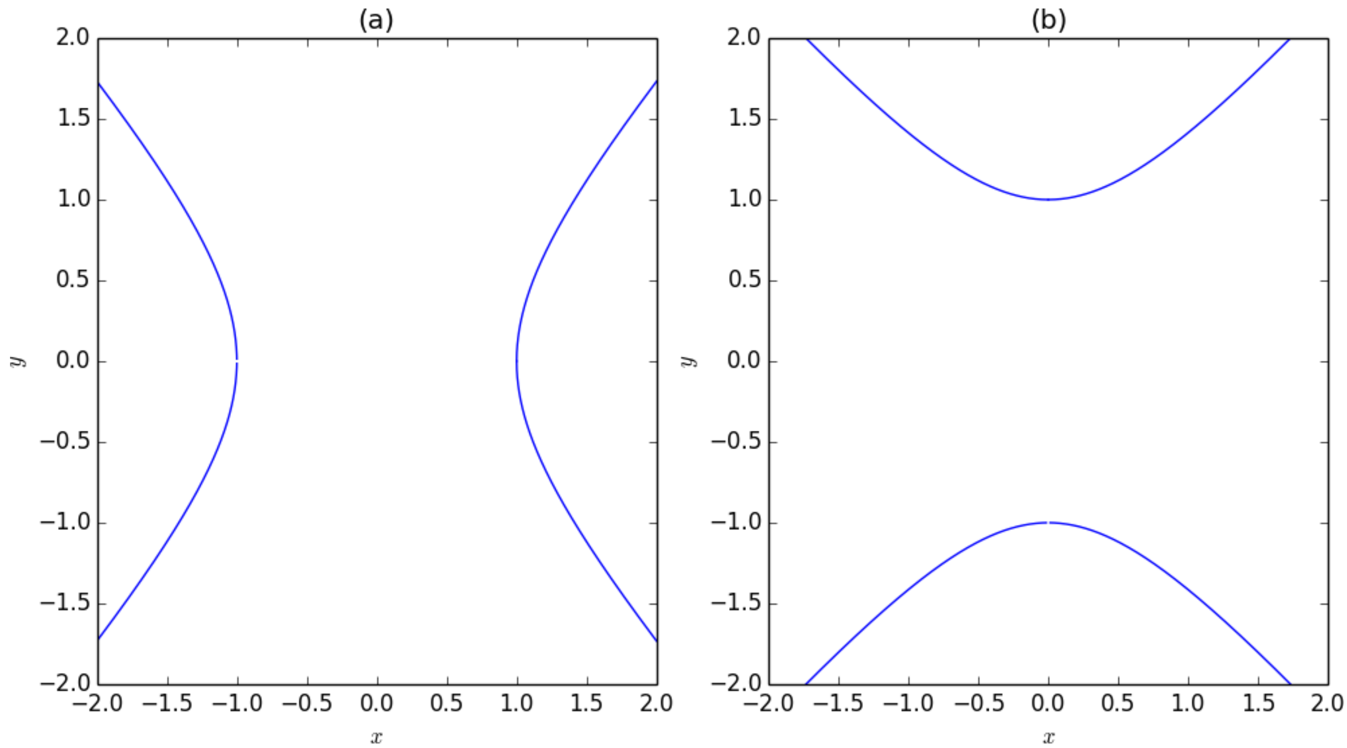


Figure 3.7: (a) $z_0 = -1.0$ and $y = \pm\sqrt{z_0 + x^2}$. (b) $z_0 = +1.0$ and $x = \pm\sqrt{-z_0 + y^2}$ {fig:3.7}

3.2 Limits and Continuity (Swokowski chapter 12.2)

{sec:3.2}

For a function $f(x, y)$ and a point $P = (x_0, y_0)$, we may approach P along several different paths. If the limit of f is the *same* along *every* path (and has value L) we say

$$\lim_{(x,y)\to(x_0,y_0)} \{f(x, y)\} = L,$$

or

$$f(x, y) \to L, \qquad \text{as} \qquad (x, y) \to (x_0, y_0).$$

The limit must be *path independent* - otherwise the limit does not exist and f is not continuous. (This is a generalisation of the case for $f(x)$ where the limit had to be the same from left and right.) The non-existence of a unique limit may be demonstrated by finding two paths where the limits are *not* the same. (Proof of a unique limit is harder.)

Example 3.12

Does

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2},$$

have a limit as $(x, y) \to (0, 0)$?

Solution 3.12

Note that $f(0, 0) = 0/0$ is an indeterminate form. We consider two paths that approach the origin, one along the x axis and one along the y axis. Along $y = 0$,

$$\lim_{x\to 0} f(x, 0) = \lim_{x\to 0} \left\{ \frac{x^2}{x^2} \right\} = 1,$$

and along $x = 0$,

$$\lim_{y\to 0} f(0, y = \lim_{y\to 0} \left\{ \frac{-y^2}{y^2} \right\} = -1.$$

Therefore the limit does not exist.

Example End

A useful strategy is to convert to polar coordinates, as this covers all possible straight line paths as $r \to 0$. Without any loss of generality we may take $x_0 = y_0 = 0$, and consider $(x, y) \to (0, 0)$. We rewrite $f(x, y)$ as $f(r, \theta)$ and consider $r \to 0$.

Example 3.13

Find

$$\lim_{(x,y)\to(0,0)} \left\{ \frac{x^2}{\sqrt{x^2 + y^2}} \right\}.$$

Solution 3.13

Note that it is an indeterminate form at $(0, 0)$.

$$\frac{x^2}{\sqrt{x^2 + y^2}} = \frac{r^2 \cos^2\theta}{\sqrt{r^2}} = r\cos^2\theta \to 0,$$

for any value of θ as $r \to 0$, since $0 \le \cos^2\theta \le 1$. By letting θ increase from 0 to 2π, we can consider all (straight line) approach routes to the origin. Thus, a unique limit exists and $L = 0$.

Example End

If the limit exists, we can consider continuity of $f(x, y)$.

Definition

$f(x, y)$ is continuous at (x_0, y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L = f(x_0, y_0).$$

So we first have to show that the limit exists and then that the limit is equal to $f(x_0, y_0)$ as defined.

Example 3.14

The previous example had indeterminate form at $(0, 0)$. However, by specifying (in a similar way to the one variable case)

$$f(x, y) = \begin{cases} \frac{x^2}{\sqrt{x^2+y^2}}, & (x, y) \neq (0, 0) \\ 0 & x = y = 0 \end{cases}$$

we have a continuous function at $(0, 0)$.

Example End

Example 3.15

Is

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x, y) \neq (0, 0), \\ 0 & x = y = 0, \end{cases}$$

continuous at the origin?

Solution 3.15

Along $x = 0$,

$$\lim_{y\to 0} \left\{ \frac{0}{0 + y^2} \right\} = 0.$$

Along $y = 0$,

$$\lim_{x\to 0} \left\{ \frac{0}{x^2 + 0} \right\} = 0.$$

So far so good. Now consider along $x = y$

$$\lim_{y\to 0} \left\{ \frac{y^2}{y^2 + y^2} \right\} = \lim_{y\to 0} \left\{ \frac{y^2}{2y^2} \right\} = \frac{1}{2}.$$

So we have found one path that does not agree with the other two. So the limit does *not* exist. Consider the more general path using the polar coordinates (r, θ) with

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Thus,

$$f(x,y) = f(r,\theta) = \frac{r^2 \cos\theta \sin\theta}{r^2} = \frac{1}{2}\sin 2\theta,$$

as $r \to 0$. Therefore the limit is dependent upon θ and so is dependent on the path. Hence, a unique limit of f does not exist.

Example End

Example 3.16

Show that

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2}, & (x,y) \neq (0,0), \\ 0, & x = y = 0, \end{cases}$$

is continuous at $(0,0)$.

Solution 3.16

Convert from Cartesian coordinates to polar coordinates so that

$$f(x,y) = \frac{r^3 \cos^2\theta \sin\theta}{r^2} \to 0, \qquad \text{as} \qquad r \to 0,$$

since both $\cos\theta$ and $\sin\theta$ are bounded functions. Since $f(0,0) = 0$ also, we can say that f is continuous at (0,0).

Example End

Cautionary Example

(Swokowski, p 991) Show that

$$\lim_{(x,y)\to(0,0)} \left\{ \frac{x^2 y}{x^4 + y^2} \right\},$$

does not exist. We consider three different paths towards the origin. Along $y = 0$,

$$\lim_{x\to 0} \left\{ \frac{0}{x^4} \right\} = 0.$$

Along $x = 0$,

$$\lim_{y\to 0} \left\{ \frac{0}{y^2} \right\} = 0.$$

Along $y = mx$, where m is a constant,

$$\lim_{x\to} \left\{ \frac{mx^3}{x^4 + m^2 x^2} \right\} = \lim_{x\to 0} \left\{ \frac{mx}{x^2 + m^2} \right\} = 0, \text{ if } m \text{ fixed.}$$

So along all the straight lines $y = mx$, the limit is always 0. However, if we choose the curve defined by $y = x^2$ and let $x \to 0$ we have

$$\lim_{x\to 0} \left\{ \frac{x^2 y}{x^4 + y^2} \right\} = \lim_{x\to 0} \left\{ \frac{x^4}{x^4 + x^4} \right\} = \frac{1}{2}.$$

We can show this in another way by using polar coordinates. Setting $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$f = \frac{r^3\cos^2\theta\sin\theta}{r^4\cos^4\theta + r^2\sin^2\theta} = \frac{r\cos^2\theta\sin\theta}{r^2\cos^4\theta + \sin^2\theta}.$$

Now if we assume that $\theta > 0$ (but in fact it can be made as small as we like) and hold θ fixed, then let $r \to 0$, we have

$$f \approx \frac{r\cos^2\theta}{\sin\theta}.$$

However, $\cos^2\theta/\sin\theta$ cannot be bounded as $\theta \to 0^+$, so the limit is not defined.

3.3 Partial derivatives:

Differentiation of functions of several variables (Swokowski chapter 12.3)

:3.3} Consider the surface formed by plotting $f(x, y) = z$. The slope of the surface at a point P may be

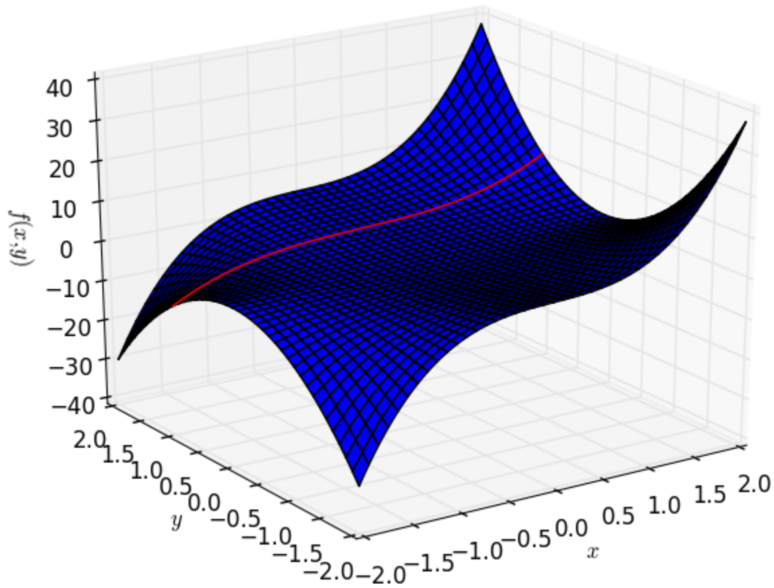


Figure 3.8: The surface $z = f(x, y)$ intersects the plane $y = y_0$, at the curve C. A point P on the curve
;:3.8} is shown.

explained in terms of the derivative of the curve C (in the $y = y_0$ plane. From Figure 3.9, the slope is defined as the partial derivative with respect to x, at constant $y = y_0$. (y_0 is like a parameter.)

$$\left(\frac{\partial f}{\partial x}\right)_y = f_x = \frac{\partial f}{\partial x} = \lim_{h\to 0}\left\{\frac{f(x+h, y) - f(x, y)}{h}\right\}.$$

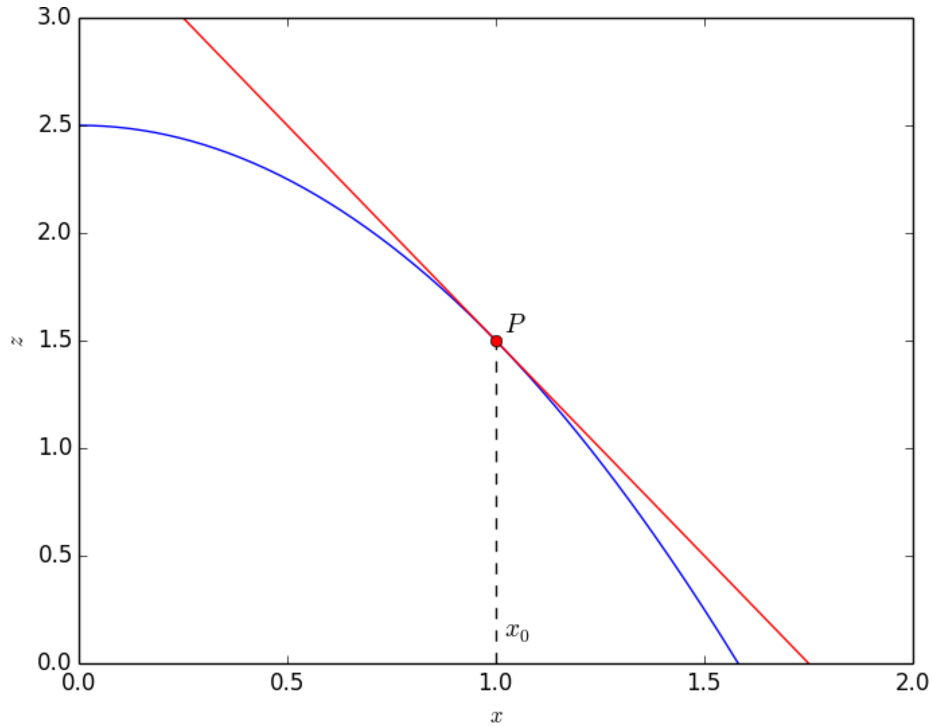


Figure 3.9: The partial derivative with respect to x, at fixed value of $y = y_0$ is obtained by forming the tangent to the curve $f(x, y_0)$ at the point $P = (x_0, y_0)$.

{fig:3.9}

y is treated like a constant parameter while the limit is being taken, but is free to have any value in the domain. Thus, $\partial f/\partial x$ is a function of x and y. Similarly, a partial derivative with respect to y (at constant x) is written

$$\left(\frac{\partial f}{\partial y}\right)_x = f_y = \frac{\partial f}{\partial y} = \lim_{k\to 0}\left\{\frac{f(x, y+k) - f(x, y)}{k}\right\},$$

and is a function of x and y.

In practical terms, the partial derivatives are obtained by applying the usual rules for differentiation. Notation is very important. You must use the 'curly d' for a partial derivative. It tells you to differentiate with respect to the variable indicated, keeping all the others fixed.

Example 3.17

1. $f(x, y) = x^3y^2 + y^2 + x$ thus, keeping y constant we have

$$\frac{\partial f}{\partial x} = 3x^2y^2 + 0 + 1 = 3x^2y^2 + 1.$$

On the other hand, keeping x constant,

$$\frac{\partial f}{\partial y} = 2x^3y + 2y + 0 = 2x^3y + 2y.$$

2. $f(x, y) = \sin(xy)$. This should be compared with $g(x) = \sin(ax)$. Keeping y constant, we have

$$\frac{\partial f}{\partial x} = y\cos(xy).$$

Keeping x constant, we have

$$\frac{\partial f}{\partial y} = x\cos(xy).$$

Example End

There are a couple of points to note here.

1. The previous example shows that $\partial f/\partial x$ and $\partial f/\partial y$ are themselves functions of (x, y). If they are continuous (in a domain), then $f(x, y)$ is *continuously differentiable* in that domain.

2. Higher order derivatives can be constructed. For example,

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} \equiv f_{xx} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \\
\frac{\partial^2 f}{\partial y^2} \equiv f_{yy} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right), \\
\frac{\partial^2 f}{\partial x \partial y} \equiv f_{xy} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right); \qquad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)
\end{aligned}$$

3. If $\partial^2 f/\partial x$ and $\partial f/\partial y$ are continuously differentiable, then the order of the differentiation may be changed. In this case

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \qquad \text{or equivalently} \qquad f_{yx} = f_{xy}.$$

Note how f_{xy} is sometimes used for brevity. Thus,

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}.$$

Example 3.18

If $f(x, y) = \sin(xy) + x^4y^2 + x^3y^3 + 3x$, then

$$\frac{\partial f}{\partial x} = y\cos(xy) + 4x^3y^2 + 3x^2y^3 + 3,$$

$$\frac{\partial f}{\partial y} = x\cos(xy) + 2x^4y + 3x^3y^2.$$

and these are continuous everywhere. Thus,

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = -y^2\sin(xy) + 12x^2y^2 + 6xy^3. \\
\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -x^2\sin(xy) + 2x^4 + 6x^3y. \\
\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) &= \cos(xy) - xy\sin(xy) + 8x^3y + 9x^2y^2. \\
\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) &= \cos(xy) - xy\sin(xy) + 8x^3y + 9x^2y^2.
\end{aligned}$$

Note that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

Example End

Partial differentiation can be applied to functions of any number of variables. For example, we may have

$$f(x, y, z),$$

in terms of the 3 cartesian coordinates. Alternatively, there may be more than 3 independent variables. Thus,

$$f(x_1, x_2, x_3, x_4, \cdots, x_n).$$

Note that we denote the independent variables by a subscript when there are more than three. Then,

$$\frac{\partial f}{\partial x_1}, \qquad \text{keeps } x_2, x_3, \cdots x_n \text{ fixed.}$$

Similarly,

$$\frac{\partial f}{\partial x_2}, \qquad \text{keeps } x_1, x_3, \cdots x_n \text{ fixed.}$$

Higher derivatives also exist such as

$$\frac{\partial^2 f}{\partial x_4 \partial x_2},$$

and so on.

Example 3.19

$f = x_1^3 x_2 + x_2 \sin x_3 + x_1 e^{x_4}$. Here there are 4 independent variables and so there are 4 first order partial derivatives, namely

$$\begin{aligned}
\frac{\partial f}{\partial x_1} &= 3x_1^2 x_2 + e^{x_4}, \\
\frac{\partial f}{\partial x_2} &= x_1^3 + \sin x_3, \\
\frac{\partial f}{\partial x_3} &= x_2 \cos x_3, \\
\frac{\partial f}{\partial x_4} &= x_1 e^{x_4}.
\end{aligned}$$

Note that we have

$$\frac{\partial^2 f}{\partial x_2 \partial x_1} = 3x_1^2 = \frac{\partial^2 f}{\partial x_1 \partial x_2}$$

This is true for any combination of derivatives (when the derivatives are continuously differentiable).

Example End

3.4 The Chain Rule (Swokowski chapter 12.5)

:3.4}

In one variable, suppose that

$$y = y(x), \qquad \text{and} \qquad x = x(t),$$

then

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

Note that we use d and not ∂, since x and y are functions of a single variable. This is called the *chain rule* in one variable.

Example 3.20

If

$$y = \left(t^3 + 4t\right)^{10},$$

find dy/dt.

Solution 3.20

Using the chain rule we let

$$x = t^3 + 4t, \qquad \Rightarrow \qquad y = x^{10}.$$

Hence,

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = (10x^9)(3t^2 + 4).$$

Now substitute back for x in terms of t to get

$$\frac{dy}{dt} = 10\left(t^3 + 4t\right)^9\left(3t^2 + 4\right).$$

Often we use the chain rule automatically.

Example End

The chain rule naturally extends to functions in two variables. Suppose that $f(x, y)$ and $x = x(t), y = y(t)$, then we could substitute and express f as $f(t)$. However, it is sometimes easier to use the chain rule for two variables. Consider the surface defined by $z = f(x, y)$. We give a simple intuitive derivation of the chain rule formula, which can be made much more rigorous.

If t changes by an amount Δt, then x changes by an amount Δx, so that

$$x(t + \Delta t) = x(t) + \Delta x, \qquad \Rightarrow \qquad \frac{\Delta x}{\Delta t} = \frac{x(t + \Delta t) - x(t)}{\Delta t}.$$

Then from the definition of a derivative, we have

$$\Delta x = \frac{dx}{dt}\Delta t,$$

as $\Delta t \to 0$. (Technically the equality only holds as $\Delta t \to 0$. Similarly, the change in y is

$$\Delta y = \frac{dy}{dt}\Delta t.$$

Since both x and y change as t changes, there will be a change in z called Δz where

$$z(x, y) + \Delta z = z(x + \Delta x, y + \Delta y).$$

To proceed, we manipulate the right hand side by adding and subtracting the same term. Thus, we have

$$z(x, y) + \Delta z = [z(x + \Delta x, y + \Delta y) - z(x, y + \Delta y)] + z(x, y + \Delta y).$$

Thus,

$$\Delta z = [z(x + \Delta x, y + \Delta y) - z(x, y + \Delta y)] + [z(x, y + \Delta y) - z(x, y)]\,.$$

The first term on the right hand side is multiplied and divided by Δx and the second term by Δy. Finally, dividing all the terms by Δt gives

$$\begin{aligned}\frac{\Delta z}{\Delta t} &= \frac{z(x + \Delta x, y + \Delta y) - z(x, y + \Delta y)}{\Delta x}\left(\frac{\Delta x}{\Delta t}\right) + \\ &\quad \frac{z(x, y + \Delta y) - z(x, y)}{\Delta y}\left(\frac{\Delta y}{\Delta t}\right)\end{aligned}$$

The first term on the right hand side has a constant value for y and the second term has a constant value for x. Now, if we take the limit $\Delta t, \Delta x, \Delta y, \Delta z \to 0$ and (assuming the limits exist) we have

$$\frac{dz}{dt} = \left(\frac{\partial z}{\partial x}\right)_y \frac{dx}{dt} + \left(\frac{\partial z}{\partial y}\right)_x \frac{dy}{dt} \qquad (3.1)$$ {eq:impdiff}

Note the use of ∂ on the right hand side rather than d because z is a function of two variables and we need to emphasise that the other variable is held constant while we undertake the differentiation. The left hand side is dz/dt, because z is a function of x and y but they are both functions of t. Hence, z is actually just a function of t.

Example 3.21

If $z(x, y) = x^2 + y^2$ and $x = t^2, y = \sin t$, then we have

$$\begin{aligned}\left(\frac{\partial z}{\partial x}\right)_y &= 2x, \\ \left(\frac{\partial z}{\partial y}\right)_x &= 2y, \\ \frac{dx}{dt} &= 2t, \\ \frac{dy}{dt} &= \cos t.\end{aligned}$$

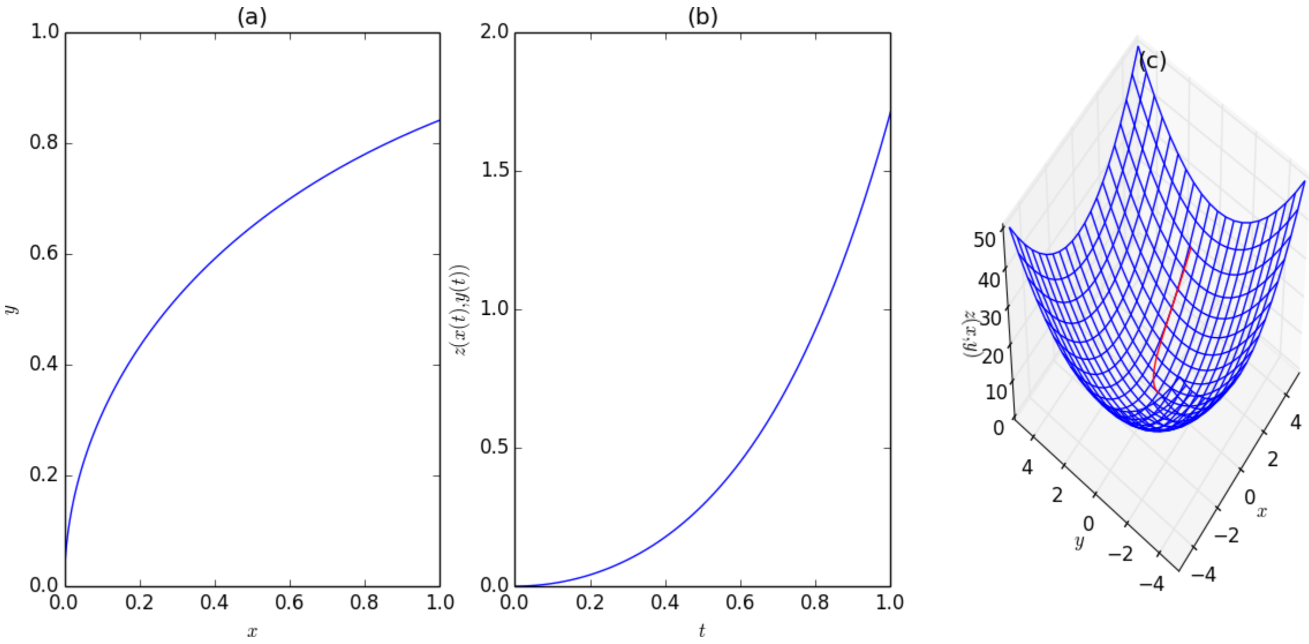


Figure 3.10: Illustration of path in the (x, y) plane and the variation of the height $z = f(x(t), y(t))$
3.10} along that path.

Therefore, the chain rule gives us,

$$\begin{aligned}
\frac{dz}{dt} &= \left(\frac{\partial z}{\partial x}\right)_y \frac{dx}{dt} + \left(\frac{\partial z}{\partial y}\right)_x \frac{dy}{dt} = 2x2t + 2y\cos t, \\
&= 2(t^2)2t + 2(\sin t)\cos t = 4t^3 + 2\sin t\cos t.
\end{aligned}$$

Alternatively, we can express z as a function of t alone.

$$\begin{aligned}
z &= x^2 + y^2 = t^4 + \sin^2 t \\
\frac{dz}{dt} &= 4t^3 + 2\sin t\cos t \\
&= 4t^3 + \sin 2t.
\end{aligned}$$

Example End

3.4.1 Implicit Differentiation Rule

If $x(t)$ and $y(t)$ represent horizontal coordinates, and $z = f(x, y)$ the height of a surface, then $z(x(t), y(t))$ represents the variation of height of the surface along a particular path, whose projection onto the (x, y) plane is $(x(t), y(t))$. This is illustrated in Figure 3.10. The previous use of the chain rule gives dz/dt.

An alternative use of implicit differentiation occurs in the case where the path, $(x(t), y(t))$ is chosen to be a *contour* along which z remains constant (dashed line in Figure 3.10. In this case, we have $dz/dt = 0$, since z does not change. Hence, (3.1), namely

$$\left(\frac{\partial z}{\partial x}\right)\frac{dx}{dt} + \left(\frac{\partial z}{\partial y}\right)\frac{dy}{dt} = 0 \ ,$$

may be written as

$$\left(\frac{\partial z}{\partial x}\right)\frac{dx}{dt} = -\left(\frac{\partial z}{\partial y}\right)\frac{dy}{dt} \ .$$

Hence,

$$\frac{dy}{dx} \equiv \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{(\partial z/\partial x)_y}{(\partial z/\partial y)_x} \equiv -\frac{f_x(x,y)}{f_y(x,y)}.$$

Thus, we have an ordinary differential equation

$$\frac{dy}{dx} = -\frac{f_x(x,y)}{f_y(x,y)},$$

that has solution $y = y(x)$ and so describes the path on the contour $z =$ constant.

If $f(x, y) =$ constant, then partial derivatives f_x and f_y determine dy/dx implicitly. An explicit differentiation of y with respect to x is not required.

Example 3.22
Let $f(x, y) = y^4 + 3y - 4x^3 - 5x - 1 = 0$ define $y = y(x)$ implicitly. Determine dy/dx in terms of x and y.

Solution 3.22

$$\begin{aligned} f_x &= -12x^2 - 5, \qquad f_y = 4y^3 + 3, \\ \frac{dy}{dx} &= -\frac{(-12x-5)}{4y^3+3} = \frac{12x^2+5}{4y^3+3}. \end{aligned}$$

Contrast this with trying to solve for $y(x)$ first and then differentiating with respect to x.
Example End

3.5 Change of Variables

{sec:3.5}

Let $z = f(x, y)$ and $x = x(s, t), y = y(s, t)$, then we can express the function f as

$$f(x, y) = f(x(s, t), y(s, t)) = f(s, t).$$

Thus, we can *change variables* from (x, y) to (s, t). Note that since f is originally expressed in terms of x and y, calculations of $(\partial z/\partial x)_y$ and $(\partial z/\partial y)_x$ are simply done using partial differentiation. However, what happens if we require $(\partial z/\partial s)_t$ and $(\partial z/\partial t)_s$?

First consider changes of t, but keeping s *fixed*. Now, x, y and z become functions of t only and we can use (3.1)

$$\frac{dz}{dt} = \left(\frac{\partial z}{\partial x}\right)_y \frac{dx}{dt} + \left(\frac{\partial z}{\partial y}\right)_x \frac{dy}{dt},$$

where d/dt is representing $(\partial/\partial t)_s$, namely differentiation with respect to t keeping s fixed. Thus, it is better to write

$$\left(\frac{\partial z}{\partial t}\right)_s = \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial t}\right)_s + \left(\frac{\partial z}{\partial y}\right)_x \left(\frac{\partial y}{\partial t}\right)_s.$$

Considering changes at *fixed* t produces the result

$$\left(\frac{\partial z}{\partial s}\right)_t = \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial s}\right)_t + \left(\frac{\partial z}{\partial y}\right)_x \left(\frac{\partial y}{\partial s}\right)_t.$$

Example 3.23

If $z = f(x, y) = xy^2$ and $x = s^2 + t^2$, $y = st$, find $(\partial z/\partial s)_t$ and $(\partial z/\partial t)_s$.

Solution 3.23

Firstly, we calculate the necessary partial derivatives,

$$\begin{aligned}
\left(\frac{\partial z}{\partial x}\right)_y &= y^2,\\
\left(\frac{\partial z}{\partial y}\right)_x &= 2xy,\\
\left(\frac{\partial x}{\partial t}\right)_s &= 2t,\\
\left(\frac{\partial x}{\partial s}\right)_t &= 2s,\\
\left(\frac{\partial y}{\partial t}\right)_s &= s,\\
\left(\frac{\partial y}{\partial s}\right)_t &= t.
\end{aligned}$$

Thus,

$$\begin{aligned}
\left(\frac{\partial z}{\partial t}\right)_s &= (y^2)(2t) + (2xy)(s)\\
&= (s^2t^2)(2t) + (2(s^2+t^2)st)(s) = 4s^2t^3 + 2s^4t,
\end{aligned}$$

and

$$\begin{aligned}
\left(\frac{\partial z}{\partial s}\right)_t &= (y^2)(2s) + (2xy)(t)\\
&= (s^2t^2)(2s) + (2(s^2+t^2)st)(t) = 4s^3t^2 + 2t^4s.
\end{aligned}$$

Example End

3.5.1 Higher Derivatives

The previous result can be thought of as resulting from action of a differential operator, e.g. $(\partial/\partial t)_s$ acting on a general function, say ϕ. For easy of presentation we will omit all subscripts but you must remember that they are still present and that the differentiation with respect to one variable means that the other variable is held fixed.

$$\begin{aligned} \frac{\partial \phi}{\partial t} &= \frac{\partial \phi}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial \phi}{\partial y} \cdot \frac{\partial y}{\partial t}, \qquad (3.2) \quad \{eq{:}3.2\} \\ \frac{\partial \phi}{\partial s} &= \frac{\partial \phi}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial \phi}{\partial y} \cdot \frac{\partial y}{\partial s}. \qquad (3.3) \quad \{eq{:}3.3\} \end{aligned}$$

Letting $\phi = z$ recovers the previous result.

$$\begin{aligned} \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}, \qquad (3.4) \quad \{eq{:}3.4\} \\ \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}. \qquad (3.5) \quad \{eq{:}3.5\} \end{aligned}$$

Now, we can consider the second derivative. For example,

$$\frac{\partial^2 z}{\partial s^2} = \frac{\partial}{\partial s}\left(\frac{\partial z}{\partial s}\right).$$

However, we can replace $\partial z/\partial s$ by (3.5). Hence, we have

$$\frac{\partial}{\partial s}\left(\frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}\right).$$

Treating each of the terms on the right hand side as a product and remembering that the partial derivatives of z with respect to x and y are firstly functions of x and y but that the partial derivatives of x and y with respect to s are functions of s and t, we have

$$\frac{\partial^2 z}{\partial s^2} = \frac{\partial}{\partial s}\left(\frac{\partial z}{\partial x}\right) \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial x} \cdot \frac{\partial^2 x}{\partial s^2} + \frac{\partial}{\partial s}\left(\frac{\partial z}{\partial y}\right) \cdot \frac{\partial y}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial^2 y}{\partial s^2}$$

Now we can use (3.3) with $\phi = \partial z/\partial x$ in the first term on the right hand side and $\phi = \partial z/\partial y$ in the third term. Hence,we have

$$\begin{aligned} \frac{\partial^2 z}{\partial s^2} &= \left\{\frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial x}{\partial s} + \frac{\partial^2 z}{\partial x \partial y} \cdot \frac{\partial y}{\partial s}\right\} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial x} \cdot \frac{\partial^2 x}{\partial s^2} + \\ &\quad \left\{\frac{\partial^2 z}{\partial x \partial y} \cdot \frac{\partial x}{\partial s} + \frac{\partial^2 z}{\partial y^2} \cdot \frac{\partial y}{\partial s}\right\} \cdot \frac{\partial y}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial^2 y}{\partial s^2}. \end{aligned}$$

Expanding the brackets and collecting like terms together we finally have

$$\begin{aligned} \frac{\partial^2 z}{\partial s^2} &= \frac{\partial^2 z}{\partial x^2} \cdot \left(\frac{\partial x}{\partial s}\right)^2 + 2\frac{\partial^2 z}{\partial x \partial y} \cdot \frac{\partial y}{\partial s} \cdot \frac{\partial x}{\partial s} + \frac{\partial^2 z}{\partial y^2} \cdot \left(\frac{\partial y}{\partial s}\right)^2 + \\ &\quad \frac{\partial z}{\partial x} \cdot \frac{\partial^2 x}{\partial s^2} + \frac{\partial z}{\partial y} \cdot \frac{\partial^2 y}{\partial s^2}. \end{aligned}$$

This is quite complicated and need not be memorised. However, it is important to understand the derivation.

There are similar expressions for $\partial^2 z/\partial t^2$ but with s replaced by t. The mixed derivative $\partial^2 z/\partial s\partial t$ can be obtained using the same approach. It just requires a little more care with the algebra.

A common change of variables is from cartesian to polar coordinates, $(x, y) \to (R, \theta)$. The transformation is defined by

$$x = R\cos\theta, \qquad y = R\sin\theta.$$

A general function $f(x, y) \to f(R, \theta)$. How do we change the partial derivatives $\partial/\partial x$ and $\partial/\partial y$ into $\partial/\partial R$ and $\partial/\partial\theta$? Simply use the previous relations but with $s \equiv R$ and $t \equiv \theta$. Thus, we have

$$\begin{aligned}
\left(\frac{\partial f}{\partial R}\right)_\theta &= \left(\frac{\partial f}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial R}\right)_\theta + \left(\frac{\partial f}{\partial y}\right)_x \cdot \left(\frac{\partial y}{\partial R}\right)_\theta \\
\left(\frac{\partial f}{\partial \theta}\right)_R &= \left(\frac{\partial f}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial \theta}\right)_R + \left(\frac{\partial f}{\partial y}\right)_x \cdot \left(\frac{\partial y}{\partial \theta}\right)_R
\end{aligned}$$

Now since $x = R\cos\theta$ and $y = R\sin\theta$, we have

$$\begin{aligned}
\frac{\partial x}{\partial R} &= \cos\theta, \qquad \frac{\partial y}{\partial R} = \sin\theta, \\
\frac{\partial x}{\partial \theta} &= -R\sin\theta, \qquad \frac{\partial y}{\partial \theta} = R\cos\theta.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\left(\frac{\partial f}{\partial R}\right) &= \cos\theta \cdot \left(\frac{\partial f}{\partial x}\right) + \sin\theta \cdot \left(\frac{\partial f}{\partial y}\right), \\
\left(\frac{\partial f}{\partial \theta}\right) &= -R\sin\theta \cdot \left(\frac{\partial f}{\partial x}\right) + R\cos\theta \cdot \left(\frac{\partial f}{\partial y}\right)
\end{aligned}$$

Example 3.24

Let $f = x^2 + y^2$, $\partial f/\partial x = 2x$, $\partial f/\partial y = 2y$. Hence,

$$\frac{\partial f}{\partial R} = \cos\theta \cdot 2x + \sin\theta \cdot 2y = 2R\cos^2\theta + 2R\sin^2\theta = 2R,$$

and

$$\frac{\partial f}{\partial \theta} = -R\sin\theta \cdot 2x + R\cos\theta \cdot 2y = -2R\sin\theta\cos\theta + 2R\cos\theta\sin\theta = 0.$$

We can also get this result directly by substituting for x and y in terms of R and θ before differentiating and confirming the above results. Thus,

$$f = x^2 + y^2 = R^2, \qquad \frac{\partial f}{\partial R} = 2R, \qquad \frac{\partial f}{\partial \theta} = 0.$$

Example End

3.6 Three or more independent variables

{sec:3.6}

Suppose that $W = f(x_1, x_2, x_3, \cdots, x_n)$ and

$$\begin{aligned} x_1 &= x_1(t_1, t_2, \cdots, t_m), \\ x_2 &= x_2(t_1, t_2, \cdots, t_m), \\ \vdots &= \vdots \\ x_n &= x_n(t_1, t_2, \cdots, t_m). \end{aligned}$$

We have already looked at the previous cases,

$$\begin{aligned} n &= 1, \quad m = 1, \quad y = f(x), \quad x = x(t), \\ n &= 2, \quad m = 1, \quad z = f(x, y), \quad x = x(t), \quad y = y(t), \\ n &= 2, \quad m = 2, \quad z = f(x, y), \quad x = x(s, t), \quad y = y(s, t). \end{aligned}$$

The previous result generalises to

$$\begin{aligned} \frac{\partial W}{\partial t_1} &= \frac{\partial W}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_1} + \frac{\partial W}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_1} + \cdots + \frac{\partial W}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_1} \\ \vdots &= \vdots \\ \frac{\partial W}{\partial t_m} &= \frac{\partial W}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_m} + \frac{\partial W}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_m} + \cdots + \frac{\partial W}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_m} \end{aligned}$$

Example 3.25

Consider the case $n = 3$ and $m = 2$. Let $W = W(x, y, z) = xy + yz + xz$. Here we have picked $x \equiv x_1$, $y \equiv x_2$ and $z \equiv x_3$. Using x, y, z saves on writing the subscripts. If

$$x = x(s, t) = s \cos t, \qquad y = y(s, t) = s \sin t, \qquad z = z(s, t) = t,$$

then we may obtain the partial derivatives $\partial W/\partial s$ and $\partial W/\partial t$.

Solution 3.25

$$\frac{\partial W}{\partial s} = \frac{\partial W}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial W}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial W}{\partial z} \cdot \frac{\partial z}{\partial s},$$

and

$$\frac{\partial W}{\partial t} = \frac{\partial W}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial W}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial W}{\partial z} \cdot \frac{\partial z}{\partial t},$$

Now we calculate all the necessary first order partial derivatives.

$$\begin{aligned} \frac{\partial x}{\partial s} &= \cos t, \quad \frac{\partial y}{\partial s} = \sin t, \quad \frac{\partial z}{\partial s} = 0, \\ \frac{\partial x}{\partial t} &= -s \sin t, \quad \frac{\partial y}{\partial t} = s \cos t, \quad \frac{\partial z}{\partial t} = 1. \end{aligned}$$

$$\frac{\partial W}{\partial x} = y + z, \qquad \frac{\partial W}{\partial y} = x + z, \qquad \frac{\partial W}{\partial z} = x + y.$$

Therefore, we may now calculate

$$\frac{\partial W}{\partial s} = (y+z)\cos t + (x+z)\sin t + (x+y)\cdot 0,$$

and

$$\frac{\partial W}{\partial t} = -(y+z)\cdot s\sin t + (x+z)\cdot s\cos t + (x+y)\cdot 1.$$

As an illustration, we can evaluate these derivatives at specific values of s and t. Consider $s = 1$ and $t = 2\pi$. This implies that $x = 1$, $y = 0$ and $z = 2\pi$. Hence,

$$\frac{\partial W}{\partial s} = (0 + 2\pi)\cdot 1 + (1 + 2\pi)\cdot 0 = 2\pi,$$

and

$$\frac{\partial W}{\partial t} = -(0 + 2\pi)\cdot 0 + (1 + 2\pi)\cdot 1 + (1 + 0) = 2(1 + \pi).$$

Example End

Example 3.26

Consider spherical coordinates so that $m = n = 3$. Thus, we have

$$(x, y, z) \to (r, \theta, \phi),$$

where

$$\begin{aligned} x &= r\sin\theta\cos\phi, \\ y &= r\sin\theta\sin\phi, \\ z &= r\cos\theta. \end{aligned}$$

Suppose we have $W = W(x, y, z)$, what are the partial derivatives $\partial W/\partial r$, $\partial W/\partial\theta$ and $\partial W/\partial\phi$?

Solution 3.26

We use the formula already presented but take $x_1 = x$, $x_2 = y$ and $x_3 = z$. Here $n = 3$. In addition, we have $t_1 = r$, $t_2 = \theta$ and $t_3 = \phi$. Thus, $m = 3$. Now we calculate all the first order partial derivatives,

$$\begin{array}{lll} \frac{\partial x}{\partial r} = \sin\theta\cos\phi, & \frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi, & \frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi, \\ \frac{\partial y}{\partial r} = \sin\theta\sin\phi, & \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi, & \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi, \\ \frac{\partial z}{\partial r} = \cos\theta, & \frac{\partial z}{\partial \theta} = -r\sin\theta, & \frac{\partial z}{\partial \phi} = 0. \end{array}$$

Therefore, we have

$$\begin{aligned} \frac{\partial W}{\partial r} &= \frac{\partial W}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial W}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial W}{\partial z}\cdot\frac{\partial z}{\partial r}, \\ &= \frac{\partial W}{\partial x}\cdot\sin\theta\cos\phi + \frac{\partial W}{\partial y}\cdot\sin\theta\sin\phi + \frac{\partial W}{\partial z}\cdot\cos\theta. \end{aligned}$$

Similarly, we have

$$\frac{\partial W}{\partial \theta} = \frac{\partial W}{\partial x} \cdot r\cos\theta\cos\phi + \frac{\partial W}{\partial y} \cdot r\cos\theta\sin\phi + \frac{\partial W}{\partial z} \cdot (-r\sin\theta),$$

and

$$\frac{\partial W}{\partial \phi} = \frac{\partial W}{\partial x} \cdot (-r\sin\theta\sin\phi) + \frac{\partial W}{\partial y} \cdot r\sin\theta\cos\phi.$$

Example End

Implicit differentiation

Assume that z is a function of x and y, namely $z(x, y)$, but that it is defined by, for example,

$$3x^2z - x^2y^2 + 2z^3 + 3yz - 5 = 0.$$

It is not always practical to express z as an explicit function of x and y. Instead we assume that it is defined implicitly by the above equation. Thus, if we wish to calculate the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ we need more care. If we can define

$$W = f(x, y, z),$$

then setting $W = f = 0$ defines $z(x, y)$ implicitly. W is a function of three variables. However, the constraint $W = 0$ means only two of the variables are independent (x and y in this case). Since $W = 0$ we must have

$$\frac{\partial W}{\partial x} = 0, \qquad \frac{\partial W}{\partial y} = 0.$$

Now we use the chain rule, with $z = z(x, y)$, to get

$$\frac{\partial W}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = 0.$$

Rearranging gives,

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z}.$$

Similarly, we have

$$\frac{\partial W}{\partial y} = 0, \qquad \Rightarrow \qquad \frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z}.$$

Example 3.27

Consider, $f(x, y, z) = 3x^2z - x^2y^2 + 2z^3 + 3yz - 5 = 0$. Thus,

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z} = -\frac{6xz - 2xy^2}{3x^2 + 6z^2 + 3y},$$

and

$$\frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z} = -\frac{-2x^2y + 3z}{3x^2 + 6z^2 + 3y}.$$

Example End