School of Mathematics and Statistics

MT5836 Galois Theory

Problem Sheet IV: Separability; separable extensions; the Theorem of the Primitive Element

1. Show that $X^3 + 5$ is separable over $\mathbb{F}_7$.

2. Let F be a field of positive characteristic p and let $f(X)$ be an *irreducible* polynomial over F. Show that $f(X)$ is inseparable over F if and only if it has the form

$$f(X) = a_0 + a_1 X^p + a_2 X^{2p} + \cdots + a_k X^{kp}$$

   for some positive integer k and some coefficients $a_0, a_1, \ldots, a_k \in F$.

3. Let $F \subseteq K \subseteq L$ be field extensions such that L is a separable extension of F.
   (a) Show that K is a separable extension of F.
   (b) Show that L is a separable extension of K.

4. Find α such that $\mathbb{Q}(\sqrt{2}, i) = \mathbb{Q}(\alpha)$.

5. Let p be a prime, $F = \mathbb{F}_p(t)$ be the field of rational functions over the finite field $\mathbb{F}_p$, and $f(X)$ be the following polynomial from the polynomial ring $F[X]$:

$$f(X) = X^p - t.$$

   (a) Show that $f(X)$ has no roots in F.
   (b) Let α be a root of an irreducible factor of $f(X)$ in some extension field. Show that $K = F(\alpha)$ is a splitting field for $f(X)$ and that

$$f(X) = (X - \alpha)^p$$

   over the field K.
   (c) By considering the factorization of $g(X)$ over K, or otherwise, show that it is impossible to factorize $f(X)$ as $f(X) = g(X)\,h(X)$ where $g(X), h(X) \in F[X]$ are polynomials over F of smaller degree than $f(X)$.
   (d) Conclude that $f(X)$ is a inseparable polynomial over F.