Chapter 5

Taylor Series and stationary points

{chap:5}

5.1 Revision of Taylor Series and Taylor Polynomials for one variable.

{sec:taylorRevis

Suppose

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n (x-x_0)^n , \\ &= a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots + a_n(x-x_0)^n + \cdots , \end{aligned}$$

has a power series expansion about x_0 and a radius of convergence $R > 0$. Then, a power series representation for the derivative, $f'(x)$, may be obtained by differentiating each term in the series for $f(x)$. Hence,

$$\begin{aligned} f'(x) &= a_1 + 2a_2(x-x_0) + 3a_2(x-x_0)^2 + \cdots + na_n(x-x_0)^{n-1} + \cdots \\ &= \sum_{n=1}^{\infty} na_n(x-x_0)^{n-1}. \end{aligned}$$

Differentiating again gives

$$\begin{aligned} f''(x) &= 2a_2 + 6a_2(x-x_0) + \cdots + n(n-1)a_n(x-x_0)^{n-12} + \cdots \\ &= \sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2}. \end{aligned}$$

In a similar manner the third derivative can be expressed as

$$f'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2)a_n(x-x_0)^{n-3},$$

and so on. Each of these series for the various derivatives can be evaluated easily at $x = x_0$ giving

$$\begin{aligned} f(x_0) &= a_0, \\ f'(x_0) &= a_1, \\ f''(x_0) &= 2a_2, \\ f'''(x_0) &= 3 \times 2a_3. \end{aligned}$$

This pattern can be followed to give an expression for the general derivative as

$$f^{(n)}(x_0) = n! a_n,$$

where n factorial is defined by $n! = n(n-1)(n-2)\cdots 3.2.1$. Thus, the coefficients in the power series can be defined by

:1.9}
$$a_n = \frac{1}{n!} f^{(n)}(x_0). \tag{5.1}$$

This is an important result to memorize.

The definition of a Taylor series for $f(x)$: If $f(x)$ has a power series representation

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

with a radius of convergence $R > 0$, then $f^{(k)}(x_0)$ exists for every positive integer k, and

$$a_n = \frac{1}{n!} f^{(n)}(x_0),$$

such that

1.10}
$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \cdots. \tag{5.2}$$

You must learn this result.

Note that the Taylor series with $x_0 = 0$ is called the *Maclaurin Series* (after the Scottish mathematician Colin Maclaurin, 1698 - 1746).

Example 5.32

Calculate the Maclaurin series for e^x.

Solution 5.32

First of all we calculate all the derivatives. This is particularly easy for the exponential function since

$$f(x) = e^x, \qquad f'(x) = e^x, \qquad f''(x) = e^x, \qquad f^{(n)}(x) = e^x.$$

Since $e^0 = 1$, we have $f^{(n)}(0) = 1$ for all $n = 1, 2, 3, \cdots$. Therefore the Maclaurin series is

1.11}
$$f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \tag{5.3}$$

Note that for this function the radius of convergence is given by

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!}{n!} \right| = \lim_{n\to\infty} |n+1| = \infty$$

So the Maclaurin series converges for all values of x.

Example End

Definition of a Taylor Polynomial

Let x_0 be a real number and $f(x)$ be a function that has n derivatives at $x = x_0$. The nth-degree Taylor Polynomial, $P_n(x)$ of $f(x)$ about $x = x_0$ is

{eq:1.12}
$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n. \tag{5.4}$$

Note that is the same as the Taylor series of $f(x)$ about $x = x_0$ except that the series is terminated after the term in $(x - x_0)^n$. Including the constant term, that means that the nth-degree polynomial has $n + 1$ terms.

For the first degree Taylor Polynomial, we have the straight line

$$P_1(x) = f(x_0) + f'(x_0)(x - x_0),$$

and $P_1(x_0) = f(x_0)$ and $P_1'(x_0) = f'(x_0)$. Thus, the first degree Taylor Polynomial has the same function value and the same first derivative at $x = x_0$ as the original function.

For the second degree Taylor Polynomial,

$$P_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2.$$

In this case, not only is the function and the first derivative but also the second derivative has the same value as the original function at $x = x_0$. Thus,

$$\begin{aligned} P_2(x_0) &= f(x_0), \\ P_2'(x_0) &= f'(x_0), \\ P_2''(x_0) &= f''(x_0). \end{aligned}$$

Thus, we can state that $P_n(x)$ mimics a set number of derivatives of $f(x)$ at $x = x_0$. $P_n(x)$ can be thought of as a truncated Taylor series for the first $n + 1$ terms or n powers in $(x - x_0)$. Generally,

$$\frac{d^m}{dx^m}\left(P_n(x)\right)_{x=x_0} = f^{(m)}(x_0), \qquad \text{for } 0 \leq m \leq n.$$

Example 5.33

Construct the Taylor cubic for $(1 - x)^{-1}$ about $x = 0$.

Solution 5.33

Hence, we take $x_0 = 0$ and calculate the first three derivatives of $f(x) = (1 - x)^{-1}$. Thus, we have

$$\begin{aligned} f(x) &= (1 - x)^{-1}, & f(0) &= 1, \\ f'(x) &= (1 - x)^{-2}, & f'(0) &= 1, \\ f''(x) &= 2(1 - x)^{-3}, & f''(0) &= 2, \\ f'''(x) &= 6(1 - x)^{-4}, & f'''(0) &= 6. \end{aligned}$$

So we can now construct $P_3(x)$ for $(1 - x)^{-1}$ about $x = 0$ as

$$\begin{aligned} P_3(x) &= 1 + x + \frac{2}{2!}x^2 + \frac{6}{3!}x^3 \\ &= 1 + x + x^2 + x^3. \end{aligned}$$

$P_3(x)$ is a truncated Taylor series, so it is only an *approximation* to $f(x)$. We denote this by

$$\frac{1}{1-x} \approx 1 + x + x^2 + x^3.$$

However, this simple approximation can be quite accurate if x is small enough.

Example End

Remember that this approximation has been obtained from a truncated Taylor series and only uses information about the function $f(x)$ from one location, namely the approximation only uses the derivatives evaluated at one point, x_0. There are other methods for approximating functions that do not follow this approach. Later on in this course we will use *Fourier Series* to approximate functions in terms of trigonometric functions.

5.1.1 Error associated with $P_n(x)$

Since the truncated Taylor series is only an approximation to the actual function $f(x)$, there will be a difference (or error) between $P_n(x)$ and $f(x)$. This may be estimated using the *Taylor Remainder*, $R_n(x)$.

Suppose $f(x)$ is continuous in $[a, b]$ and has $(n+1)$ derivatives in (a, b). Then, there exists a point c, $a < c < b$, such that

$$f(x) = P_n(x) + R_n(x),$$

where $P_n(x)$ has been defined above as

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

and the remainder is given by

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1},$$

and $x_0 \in (a, b)$, $x \in (a, b)$ and c lies between x_0 and x.

The form $P_n(x) + R_n(x)$ is called the Taylor series for $f(x)$ with a remainder. The proof of the remainder term now follows and it uses Rolle's theorem on a test function g.

Proof of the Remainder Term

Define the remainder term $R_n(x)$ as

.12a}

$$R_n(x) = f(x) - P_n(x). \tag{5.5}$$

Remember that $f(x)$ and $P_n(x)$ have the same n derivatives evaluated at $x = x_0$, the point the series is expanded about. Let g be the function

$$\begin{aligned} g(t) &= f(x) - \Bigg[f(t) + f'(t)(x - t) + \frac{f''(t)}{2!}(x - t)^2 + \cdots \\ &\quad + \frac{f^{(n)}(t)}{n!}(x - t)^n \Bigg] - R_n(x)\frac{(x - t)^{n+1}}{(x - x_0)^{n+1}}. \end{aligned} \tag{5.6}$$

Note that the series inside the square brackets is the Taylor Polynomial of $f(x)$ expanded about $x = t$. This is a rather arbitrary choice of function and you will simply need to memorize this form. We now investigate the properties of $g(t)$ and in particular try to locate the zeros. So if $t = x$, then

$$g(x) = f(x) - [f(x) + 0 + \cdots + 0] - 0 = 0.$$

Thus, $g(x) = 0$. Now try $t = x_0$ and we see that

$$\begin{aligned} g(x_0) &= f(x) - \left[f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots \right. \\ &\left. + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \right] - R_n(x)\frac{(x - x_0)^{n+1}}{(x - x_0)^{n+1}} \\ &= f(x) - P_n(x) - R_n(x) = 0, \end{aligned}$$

on using (5.5). So our function g is zero at both $t = x$ and $t = x_0$. Thus, we can apply Rolle's Theorem to $g(t)$ and deduce that there is a number c, lying between x_0 and x where $g'(c) = 0$. We need to differentiate $g(t)$ with respect to t and evaluate the result at $t = c$. We need to take care in differentiating $g(t)$ and we study how the general term differentiates before considering the complete expression. Consider the term

$$\frac{f^{(n)}(t)}{n!}(x - t)^n.$$

Differentiating the product with respect to t gives

$$\frac{f^{(n+1)}(t)}{n!}(x - t)^n - \frac{f^{(n)}(t)}{n!}n(x - t)^{n-1}.$$

This is firstly simplified into

$$\frac{f^{(n+1)}(t)}{n!}(x - t)^n - \frac{f^{(n)}(t)}{(n-1)!}(x - t)^{n-1}.$$

Now we can calculate $g'(t)$ to get

$$\begin{aligned} g'(t) &= 0 - \left[f'(t) - f'(t) + f''(t)(x - t) - \frac{2}{2!}f''(t)(x - t) + \cdots \right. \\ &\left. - \frac{n}{n!}f^{(n)}(t)(x - t)^{n-1} + \frac{f^{(n+1)}(t)}{n!}(x - t)^n \right] \\ &- R_n(x)(-1)(n + 1)\frac{(x - t)^n}{(x - x_0)^{n+1}}. \end{aligned}$$

Notice all the terms cancel except for the last term in the square brackets and the term involving the remainder term. Thus,

$$g'(t) = -\frac{f^{(n+1)}(t)}{n!}(x - t)^n + R_n(x)(n + 1)\frac{(x - t)^n}{(x - x_0)^{n+1}}.$$

Now we evaluate at $t = c$ where $g'(c) = 0$. Hence,

$$g'(c) = 0 = -\frac{f^{(n+1)}(c)}{n!}(x - c)^n + R_n(x)(n + 1)\frac{(x - c)^n}{(x - x_0)^{n+1}}.$$

Rearranging gives the final answer as

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}, \qquad (5.7) \quad \{\texttt{eq:1.13}\}$$

as required. Notice that c lies between x_0 and x.

Example 5.34

Find the first two non-zero terms of the Maclaurin series for $\sin x$. Use the formula to estimate the first positive zero of $\sin x$. Use the remainder formula to estimate the error when $x \in [0,1]$.

Solution 5.34

So with $f(x) = \sin x$ we have

$$\begin{array}{rclrcl} f(x) &=& \sin x & f(0) &=& 0, \\ f'(x) &=& \cos x & f'(0) &=& 1, \\ f''(x) &=& -\sin x & f''(0) &=& 0, \\ f'''(x) &=& -\cos x & f'''(0) &=& -1, \\ f''''(x) &=& \sin x & f''''(0) &=& 0. \end{array}$$

Hence, the first two non-zero terms gives

$$\sin x = 0 + (1)x + 0 + \frac{-1}{3!}x^3 + 0 + R_4(x) = x - \frac{x^3}{6} + R_4(x).$$

Note that we use $R_4(x)$ instead of $R_3(x)$ since the term involving x^4 is actually zero. Thus,

$$R_4(x) = \frac{f^{(5}(c)}{5!}x^5, \qquad 0 < c < x.$$

Now since, $f^{(5)}(x) = \cos x$, we can state the remainder as

$$R_4(x) = \frac{x^5}{5!}\cos(c).$$

The problem with the remainder term is that the value of c is unknown and the particular value will actually vary when we choose different values for x. What we can do, however, is try to find an upper bound on the value of the remainder term and then we can say that the error is certainly smaller than a particular value. This let us know if the Taylor Polynomial can give a good estimate to the actual value.

Now the first two non-zero terms suggest that we can approximate $\sin x$ by

$$\sin x \approx x - \frac{x^3}{6},$$

and so the first positive zero occurs when

$$x = \frac{x^3}{6} \qquad \Rightarrow \qquad x^2 = 6, \qquad \Rightarrow \qquad x = \sqrt{6}.$$

Obviously the correct value is π and we have estimated this as $\sqrt{6} = 2.45$.

x	$\sin x$	$x - x^3/3!$	error
0.1	0.09983341	0.09983333	8×10^{-8}
0.5	0.47943	0.47917	0.00025
1.0	0.84147	0.8333	0.00814

Table 5.1: Comparison of the exact value of $\sin x$ with the Taylor Polynomial for different values of x. {tab:1.1}

Example End

To use the remainder term to estimate the error, we note that although c is unknown, we have

$$\cos(c) < 1,$$

and so we can state that

$$R_4(x) = \frac{x^5}{120}\cos(c) < \frac{x^5}{120}.$$

Setting $x = \sqrt{6}$ we see that an upper bound on the remainder is

$$\frac{6^{5/2}}{120} = 0.735.$$

The actual error in this case is

$$\pi - \sqrt{6} = 0.692.$$

Note how our estimate of the remainder is slightly larger than the actual error.

Table 5.1 shows how good our simple Taylor Polynomial is for different values of x.

Example 5.35

Find the first three terms of the Taylor series for $f(x) = \log_e(1+x)$ (i.e. $\ln(1+x)$) about $x = 1$.

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + R_2(x).$$

Solution 5.35

Here we have $x_0 = 1$. Thus,

$$\begin{array}{rclrcl} f(x) & = & \ln(1+x), & f(1) & = & \ln(2), \\ f'(x) & = & \frac{1}{1+x}, & f'(1) & = & \frac{1}{2}, \\ f''(x) & = & -\frac{1}{(1+x)^2}, & f''(1) & = & -\frac{1}{4} \\ f'''(x) & = & \frac{2}{(1+x)^3}, & f'''(c) & = & \frac{2}{(1+c)^3}. \end{array}$$

Thus, we have $f(x) = P_2(x) + R_2(x)$ and

$$\ln(1+x) = \ln(2) + \frac{(x-1)}{2} - \frac{(x-1)^2}{8} + \frac{1}{3!}\frac{2}{(1+c)^3}(x-1)^3.$$

Remember that c lies between 1 and x. So if $x > 1$, we have $1 < c < x$ or if $x < 1$, we have $x < c < 1$. Note that if $x < 1$, the remainder term may become very large as $x \to -1$. The Taylor polynomial is unlikely to be accurate in this case.

Example End

Example 5.36

Determine the Maclaurin series for $(1+x)^\alpha$, where α is not an integer. Remember that for the Maclaurin series $x_0 = 0$ and the series is expanded about $x = 0$ in powers of x.

Solution 5.36

$$\begin{array}{rclrcl} f(x) &=& (1+x)^\alpha, & f(0) &=& 1, \\ f'(x) &=& \alpha(1+x)^{\alpha-1}, & f'(0) &=& \alpha, \\ f''(x) &=& \alpha(\alpha-1)(1+x)^{\alpha-2}, & f''(0) &=& \alpha(\alpha-1) \\ f'''(x) &=& \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3}, & f'''(0) &=& \alpha(\alpha-1)(\alpha-2). \end{array}$$

The nth derivative evaluated at $x = 0$ can be expressed as

$$f^{(n)}(0) = \alpha(\alpha-1)\cdots(\alpha-(n-1)).$$

You can easily confirm this is true for $n = 1, 2, 3$ etc. Thus, the Taylor series is

2.10}

$$f(x) = (1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-(n-1))}{n!}x^n + \cdots \tag{5.8}$$

Note that this series terminates if α is a positive integer but does NOT terminate if it is a negative integer. To test convergence we use the ratio test to get

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x}{a_n}\right| < 1.$$

Thus, we have

$$\begin{array}{rl} & \displaystyle\lim_{n\to\infty}\left|\frac{\alpha(\alpha-1)\cdots(\alpha-n)}{(n+1)!}\frac{n!}{\alpha(\alpha-1)\cdots(\alpha-(n-1))}x\right| < 1 \\ \Rightarrow & \displaystyle\lim_{n\to\infty}\left|\frac{(\alpha-n)}{(n+1)}x\right| < 1. \end{array}$$

Note that this implies that

$$|x| < 1.$$

Example End

So the Taylor series only converges for $|x| < 1$. This is expansion is known as the *Binomial expansion* and it is the generalisation of

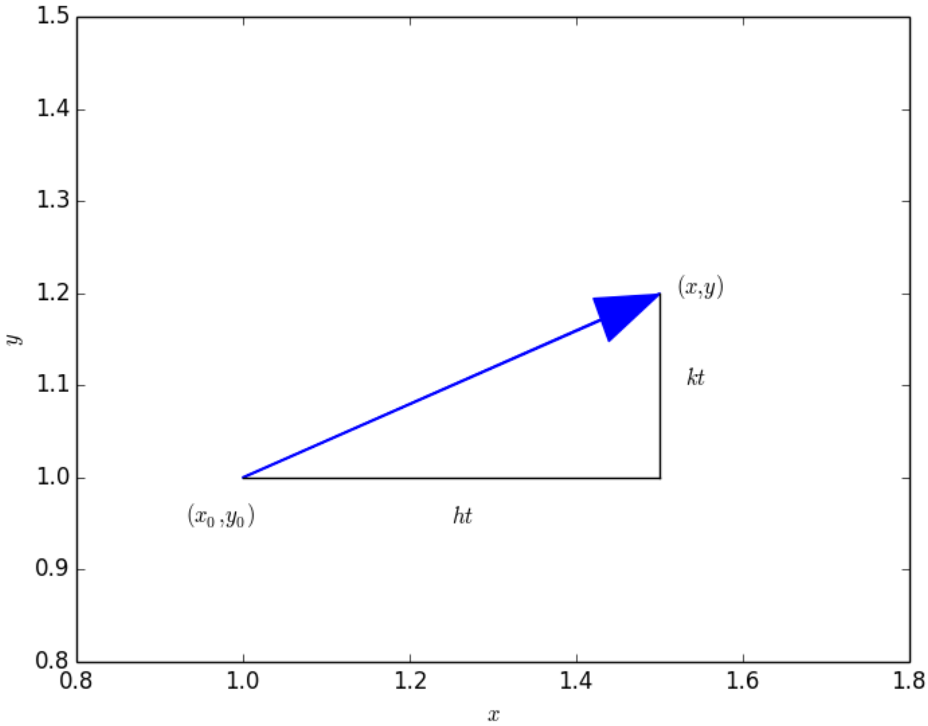
$$(1+x)^n = 1 + nx + \cdots + x^n,$$

where n is a positive integer, to the case of non-integers and negative integers. This is an *extremely important* expansion and is used in many subsequent courses. Learn this formula.

5.2 Taylor Series for functions of 2 variables

{sec:5.1}

The surface $z = f(x, y)$ and its derivatives can give a series approximation for $f(x, y)$ about some point (x_0, y_0) as illustrated in Figure 5.1 Let $x = x_0 + ht$ and $y = y_0 + kt$, where h and k are constants. This



{fig:5.1} Figure 5.1: The general point (x, y) and the point (x_0, y_0) about which the Taylor series is formed.

then gives the parametric form of the straight line joining (x, y) and (x_0, y_0). Note that we can express $x = x(t)$ and $y = y(t)$. Therefore, $z = f(x(t), y(t)) = z(t)$. So they are all functions of the parameter t. We can expand $z(t)$ about $t = 0$, which corresponds to the point (x_0, y_0), in the usual Taylor series as

$$z(t) = z(0) + \left(\frac{dz}{dt}\right)_{t=0} + \frac{t^2}{2!}\left(\frac{d^2z}{dt^2}\right)_{t=0} + \cdots$$

However, the problem lies in calculating the derivatives because we know z as a function of x and y and not explicitly as a function t.

At $t = 0$, $x = x_0$ and $y = y_0$ and so $z(0) = f(x_0, y_0)$. So we are expanding about the point (x_0, y_0) parametrically. To evaluate (dz/dt) we use the chain rule. For a general function $\phi(x(t), y(t))$, we have

$$\frac{d\phi}{dt} = \left(\frac{\partial\phi}{\partial x}\right) \cdot \frac{dx}{dt} + \left(\frac{\partial\phi}{\partial y}\right) \cdot \frac{dy}{dt}, \tag{5.9}$$ {eq:5.1}

where

$$\frac{dx}{dt} = h, \qquad \frac{dy}{dt} = k.$$

Thus,

$$\left(\frac{dz}{dt}\right)_{t=0} t = \left(\frac{\partial f}{\partial x}\right)_{(x_0,y_0)} \cdot th + \left(\frac{\partial f}{\partial y}\right)_{(x_0,y_0)} \cdot tk.$$

Eliminating ht and kt in terms of $x - x_0$ and $y - y_0$, this can be expressed as

$$\left(\frac{dz}{dt}\right)_{t=0} t = \left(\frac{\partial f}{\partial x}\right)_{(x_0,y_0)} (x - x_0) + \left(\frac{\partial f}{\partial y}\right)_{(x_0,y_0)} (y - y_0).$$

Now we need to consider the second derivatives in the next term of the Taylor series.

$$\frac{d^2z}{dt^2} = \frac{d}{dt}\left(\frac{dz}{dt}\right) = \frac{d}{dt}\left(\frac{\partial f}{\partial x}h\right) + \frac{d}{dt}\left(\frac{\partial f}{\partial y}\right)k.$$

To evaluate the right hand side, we replace $\partial f/\partial x \cdot h$ by ϕ and use the previous result. Then we evaluate the second term by replacing $\partial f/\partial y \cdot k$ by ϕ and again use (5.9). Thus, we get

$$\begin{aligned}
\frac{d^2z}{dt^2} &= \left\{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \cdot h \cdot \frac{dx}{dt} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \cdot h \cdot \frac{dy}{dt}\right\} \\
&\quad + \left\{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \cdot k \cdot \frac{dx}{dt} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \cdot k \cdot \frac{dy}{dt}\right\} \\
&= \frac{\partial^2 f}{\partial x^2} \cdot h^2 + \frac{\partial^2 f}{\partial x \partial y} \cdot hk + \frac{\partial^2 f}{\partial x \partial y} \cdot hk + \frac{\partial^2 f}{\partial y^2} k^2 \\
&= h^2 \frac{\partial^2 f}{\partial x^2} + 2hk \frac{\partial^2 f}{\partial x \partial y} + k^2 \frac{\partial^2 f}{\partial y^2}.
\end{aligned}$$

This gives

$$\frac{t^2}{2!}\left(\frac{d^2z}{dt^2}\right)_{t=0} = \frac{1}{2!}\left[t^2h^2\frac{\partial^2 f}{\partial x^2} + 2(ht)(kt)\frac{\partial^2 f}{\partial x \partial y} + t^2k^2\frac{\partial^2 f}{\partial y^2}\right]_{(x_0,y_0)}.$$

This can be written in a shorthand notation as

$$\frac{t^2}{2!}\left(\frac{d^2z}{dt^2}\right)_{t=0} = \frac{t^2}{2!}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^2 f_{(x_0,y_0)}.$$

The square bracket terms can be expanded by the binomial expansion first before operating on the function f to calculate the derivatives. Replacing $ht = (x - x_0)$ and $kt = (y - y_0)$ gives our final expression for the *Taylor series for a function in two variables*, as

$$\begin{aligned}
f(x,y) &= f(x_0,y_0) + \left[(x - x_0)\frac{\partial f}{\partial x} + (y - y_0)\frac{\partial f}{\partial y}\right] \\
&\quad + \frac{1}{2!}\left[(x - x_0)^2\frac{\partial^2 f}{\partial x^2} + 2(x - x_0)(y - y_0)\frac{\partial^2 f}{\partial x \partial y} + (y - y_0)^2\frac{\partial^2 f}{\partial y^2}\right] + \cdots \qquad (5.10)
\end{aligned}$$

	Evaluated at $(0,0)$
$f(x,y)=\sin(x)e^y$	0
$\frac{\partial f}{\partial x}=\cos(x)e^y$	1
$\frac{\partial f}{\partial y}=\sin(x)e^y$	0
$\frac{\partial^2 f}{\partial x^2}=-\sin(x)e^y$	0
$\frac{\partial^2 f}{\partial y^2}=\sin(x)e^y$	0
$\frac{\partial^2 f}{\partial x \partial y}=\cos(x)e^y$	1

Table 5.2: All the first and second order derivatives evaluated at $(0,0)$. {tab:5.1}

All the derivatives are evaluated at (x_0, y_0). Using our shorthand notation we can express this as

$$\begin{aligned} f(x,y) &= f(x_0,y_0) + \left[(x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial y}\right] f + \\ & \frac{1}{2!}\left[(x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial y}\right]^2 f + \cdots \end{aligned}$$

Remember to expand the square term by the binomial expansion before working out the derivatives. All derivatives are only applied to f.

Although we will very rarely continue beyond the second derivative terms, note that the shorthand notation allows us express the higher derivatives in a compact form. Thus, the third derivative gives

$$\frac{t^3}{3!}\frac{d^3 z}{dt^3} = \frac{1}{3!}\left[(x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial y}\right]^3 f.$$

Again we must expand the square brackets by the binomial expansion before evaluating the derivatives.

Example 5.37

Calculate the Taylor series about $(0,0)$ for

$$f(x,y) = \sin(x)e^y.$$

Solution 5.37

This could be done by expanding each factor as a function of one variable and then multiplying the answers together. However, it is our aim to illustrate the method for a function of two variables. Thus, remembering that the derivatives are evaluated at $(0,0)$, we have

$$f(x,y) = f(0,0) + x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + \frac{1}{2!}\left(x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2}{\partial y^2}\right) + \cdots$$

So we need to evaluate all the first and second derivatives. This is done first before substituting the numbers into the equation above.

Thus,

$$\sin(x)e^y \approx 0 + [1\cdot x + 0\cdot y] + \frac{1}{2!}\left[x^2\cdot 0 + 2xy\cdot 1 + y^2\cdot 0\right].$$

Simplifying we get

$$\sin(x)e^y \approx x + xy = x(1+y)$$

Example End

5.3 Stationary Points - Swokowski chapter 12.8

:5.2}

When considering function of one variable, an important application of calculus is to determine the maximum and minimum values. This is done by locating the stationary points where the derivative is zero and then determine the nature of the stationary points by looking at the sign of the second derivatives. The same approach is used in studying functions of two variables. So let us revise the one variable case in detail before progressing to more independent variables.

If $y = f(x)$, then a necessary condition for a maximum or minimum (or more generally stationary points) is

$$\frac{dy}{dx} = f'(x) = 0.$$

For a stationary point at x_0, the Taylor series gives

$$y = f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{1}{2}(x - x_0)^2 f''(x_0) + \cdots$$

Since, $f'(x_0) = 0$ at a stationary point, the second term (linear in $(x - x_0)$) vanishes. Assuming that $f''(x_0) \neq 0$ and $(x - x_0)$ sufficiently small,

$$f(x) - f(x_0) \approx \frac{1}{2}(x - x_0)^2 f''(x_0).$$

If $f''(x_0)$ is positive, then $f(x) > f(x_0)$ in the vicinity of x_0 and so f has a minimum at x_0. Conversely, if $f''(x_0)$ is negative, then $f(x) < f(x_0)$ and f has a local maximum.

In general, the nature of the stationary (or turning) points depends upon the first non-zero derivative. Thus, if $f'(x_0) = 0$ and $f''(x_0) \neq 0$, then turning point is a maximum if $f''(x_0) < 0$ and a minimum if $f''(x_0) > 0$. If $f'(x_0) = f''(x_0) = 0$, then

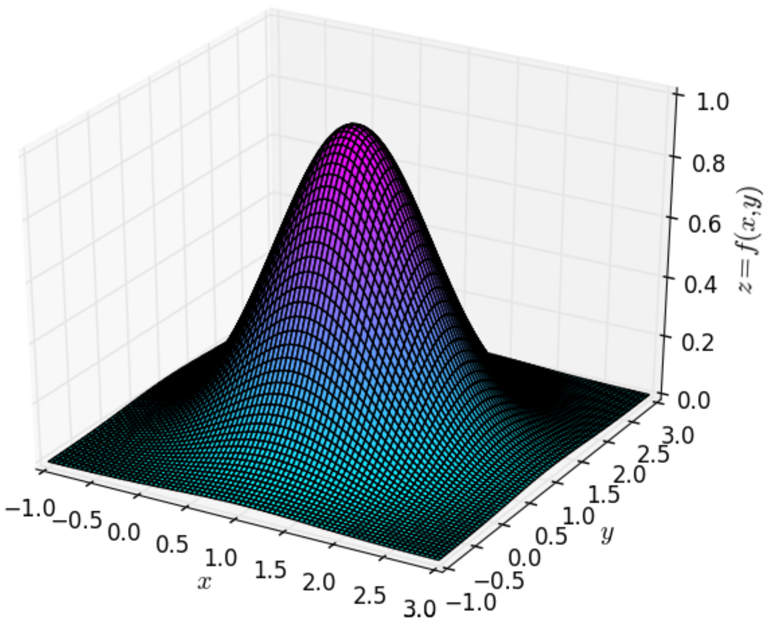
$$f(x) - f(x_0) \approx \frac{1}{3!}(x - x_0)^3 f'''(x_0).$$

Regardless of the sign of the third derivative, $f(x) - f(x_0)$ has a different sign for x either side of x_0. Thus, $f(x)$ has a point of inflexion at x_0.

We can extend this to the case where the first three derivatives are zero and the fourth derivative is non-zero. The same approach will determine the local nature of the turning point.

Similar concepts apply to functions of two variables, $f(x, y)$. $f(x, y)$ has a maximum at (x_0, y_0) if $f(x, y) < f(x_0, y_0)$ in the vicinity of (x_0, y_0). Similarly, if $f(x, y) > f(x_0, y_0)$, then f has a minimum at (x_0, y_0).

Let there be a turning point at (x_0, y_0) as illustrated in Figure 5.2. Along the line $y = y_0$, $f(x, y_0)$



{fig:5.2} Figure 5.2: There is a maximum turning point at $(1, 1)$.

is just a function of one variable and the condition for a turning point is simply

$$\left(\frac{\partial f}{\partial x}\right)_{(x_0,y_0)} = 0.$$

Similarly, along the line $x = x_0$, $f(x_0, y)$ must satisfy

$$\left(\frac{\partial f}{\partial y}\right)_{(x_0,y_0)} = 0.$$

Any point (x_0, y_0) where $\partial f/\partial x = \partial f/\partial y = 0$ is called a *stationary* or *critical* point of the function $f(x, y)$.

Evidently, the tangent plane, T, is horizontal at $(x_0, y_0, z = f(x_0, y_0))$ and ∇f has no horizontal components.

Example 5.38

Let $z = f(x, y) = x^2 + y^2$.

$$\frac{\partial f}{\partial x} = 2x, \qquad \frac{\partial f}{\partial y} = 2y.$$

Therefore, at $(0,0)$ both derivatives are zero and the origin is a stationary point. By inspection, $(0,0)$ is a minimum.

Example 5.3

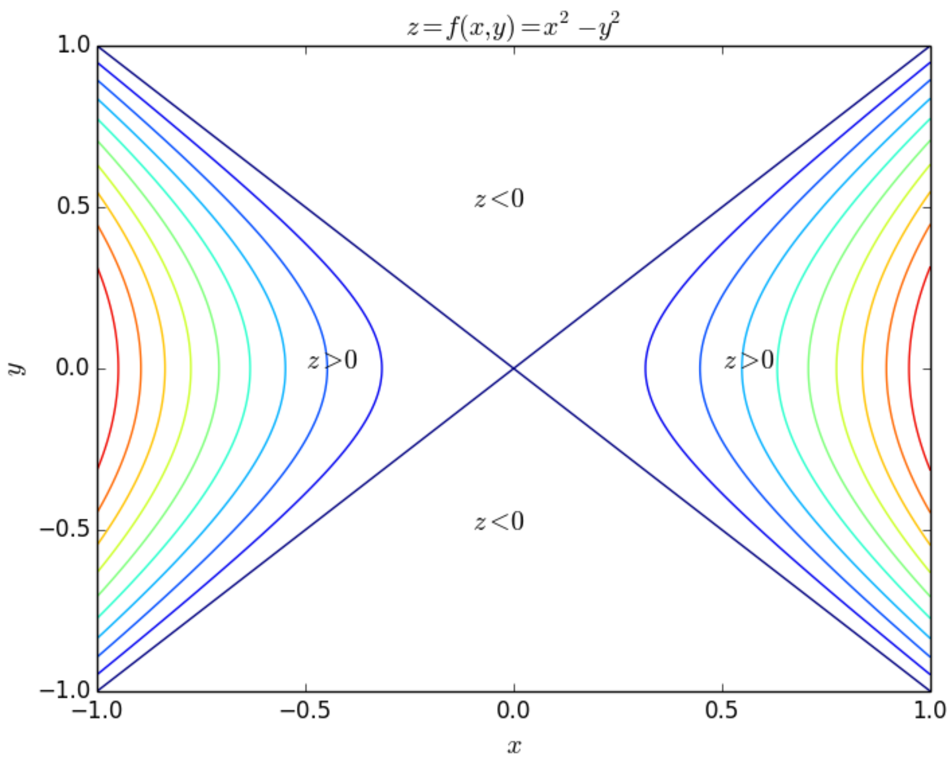
Let $z = f(x,y) = x^2 - y^2$.

$$\frac{\partial f}{\partial x} = 2x, \qquad \frac{\partial f}{\partial y} = -2y.$$

Again the origin is a stationary point. But is it a maximum or a minimum?

Solution 5.38

Obviously, along $y = 0$, $z = x^2$ and so the origin is a minimum. Bur, along $x = 0$, $z = -y^2$ and the origin is a maximum. Note that $z = 0$ everywhere along the two lines $y = x$ and $y = -x$. These lines separate regions where $z < 0$ from regions where $z > 0$. This is shown in Figure 5.3. Thus, if $x^2 > y^2$,



:5.3}

Figure 5.3: Regions where z is positive are indicated by the shading.

then $z > 0$ and if $x^2 < y^2$, $z < 0$. Therefore, the sign of $[f(x,y) - f(x_0,y_0)]$ changes with (x,y) in the vicinity of (x_0,y_0). This is neither a maximum nor a minimum but a *saddle point.*

Example End

5.4 Testing the nature of a stationary point

{sec:5.3}

Recall the Taylor series for $f(x,y)$,

$$f(x,y) = f(x_0,y_0) + t\left[h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right] + \frac{t^2}{2!}\left[h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right] + \cdots, \qquad (5.11)$$ {eq:5.2}

where $ht = x - x_0$ and $kt = y - y_0$. If (x_0, y_0) is a stationary point $\partial f/\partial x = \partial f/\partial y = 0$ and so (5.12) becomes

$$f(x,y) - f(x_0,y_0) = \frac{t^2}{2!}\left[h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right] + \cdots, \qquad (5.12)$$ {eq:5.3}

where the notation

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}, \qquad f_{xy} = \frac{\partial^2 f}{\partial x \partial y}, \qquad f_{yy} = \frac{\partial^2 f}{\partial y^2},$$

is used.

The condition for a *minimum* of f is that $f(x,y).f(x_0,y_0)$ for all (x,y) near (x_0,y_0). This means that the coefficient of t^2 must be positive for all (x,y) or equivalently for all values of h and k. Similarly, the condition for a *maximum* is that the coefficient is negative. If the coefficient is zero, the expansion (5.11) does not help and we need to consider the third derivatives.

Now consider the coefficient of t^2 is more detail.

$$C = h^2 f_{xx} + 2hk f_{xy} = k^2 f_{yy}. \qquad (5.13)$$ {eq:5.4}

Several cases exist and we will assume that $f_x = f_y = 0$ at (x_0,y_0) automatically.

5.4.1 Case 1

{subsec:5.3.1}

$$f_{xx} = f_{yy} = 0, \qquad \text{but} \qquad f_{xy} \neq 0$$

The coefficient reduces to $2hkf_{xy}$ and will change sign as h and k change sign. Thus, f has neither a maximum nor a minimum but a *saddle point*.

5.4.2 Case 2

{subsec:5.3.2}

Suppose $f_{xx} \neq 0$ at (x_0,y_0). We will complete the square in h to obtain a positive term and an extra term. It is the sign of the extra term that determines the nature of the stationary point. Firstly, since $f_{xx} \neq 0$ we express C as

$$C = \left(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right) \cdot \frac{f_{xx}}{f_{xx}}.$$

Hence, we have

$$\begin{aligned} C &= \frac{1}{f_{xx}}\left[h^2 f_{xx}^2 + 2hk f_{xy} f_{xx} + k^2 f_{yy} + \left\{k^2 f_{xy}^2 - k^2 f_{xy}^2\right\}\right] \\ &= \frac{1}{f_{xx}}\left[\left(h f_{xx} + k f_{xy}\right)^2 + k^2\left(f_{xx} f_{yy} - f_{xy}^2\right)\right]. \end{aligned}$$

So we have completed the square and obtained a term

$$\left(h f_{xx} + k f_{xy}\right)^2,$$

that is always positive but we can make it zero by choosing suitable values for h and k. Remember that simply defines the path. The other term

$$k^2\left(f_{xx}f_{yy} - f_{xy}^2\right),$$

is the crucial term for determining the nature of the stationary point. Define

5.4a}
$$\Delta = f_{xx}f_{yy} - f_{xy}^2, \text{ at } (x_0, y_0). \tag{5.14}$$

The sign of Δ determines the nature of the stationary point in a similar way to the use of the second derivative for a function of one variable.

- If $\Delta > 0$, the sign of C is that of f_{xx}. Hence, we have

$$\text{If } \Delta > 0, \text{ and } f_{xx} > 0, \qquad C > 0, \qquad \Rightarrow \qquad (x_0, y_0) \text{ is a minimum.}$$

$$\text{If } \Delta > 0, \text{ and } f_{xx} < 0, \qquad C < 0, \qquad \Rightarrow \qquad (x_0, y_0) \text{ is a maximum.}$$

- If $\Delta < 0$, then

$$C = \frac{1}{f_{xx}}\left[(hf_{xx} + kf_{xy})^2 - k^2|\Delta|\right].$$

  If $k = 0$, then the square bracket is positive. However, if $k \neq 0$, but chosen so that $hf_{xx} + kf_{xy} = 0$, then the square bracket is negative. Thus, C changes sign with the choices of h and k. So we have a *saddle point*.

.3.3}
5.4.3 Case 3

Suppose $f_{yy} \neq 0$ at (x_0, y_0). We can rewrite C as

$$\frac{1}{f_{yy}}\left[(hf_{xy} + kf_{yy})^2 + h^2\Delta\right].$$

This can be verified by expanding to show that it is equivalent to the original expression for C. Hence, we have the same conclusions based upon the sign of Δ as in Case 2.

.3.4}
5.4.4 Summary of Testing

Compute

$$\Delta = \left(\frac{\partial^2 f}{\partial x^2}\right)\cdot\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2, \text{ at } (x_0, y_0).$$

1. $\Delta = 0$ means that the test is inconclusive and we need a higher order Taylor series.

2. $\Delta < 0$ means that the stationary point is a *saddle point*.

3. $\Delta > 0$. In this case it is clear that Δ can only be positive if the product $\partial^2 f/\partial x^2 \cdot \partial^2 f/\partial y^2$ is positive. Therefore, either both $\partial^2 f/\partial x^2$ and $\partial^f/\partial y^2$ are positive or they are both negative.

   - $\Delta > 0$ and $\partial^2 f/\partial x^2 < 0$ means that the stationary point is a maximum.

- $\Delta > 0$ and $\partial^2 f/\partial x^2 > 0$ means that the stationary point is a minimum.

Example 5.39

Find the location and nature of the stationary points of $z = f(x, y) = x^3 - 6xy + y^3$.

Solution 5.39

First the location is given by setting both first derivatives to zero.

$$\frac{\partial f}{\partial x} = 3x^2 - 6y = 0, \qquad \Rightarrow \qquad y = \frac{1}{2}x^2,$$

and

$$\frac{\partial f}{\partial y} = -6x + 3y^2 = 0, \qquad \Rightarrow \qquad -6x + 3\left(\frac{1}{2}x^2\right)^2 = 0.$$

Solving gives

$$-2x + \frac{1}{4}x^4 = 0, \qquad \Rightarrow \qquad x = 0 \text{ or } x = 2.$$

So there are two stationary points at $x = 0, y = 0$ and $x = 2, y = 2$. To determine their nature we calculate all the second derivatives.

$$\frac{\partial^f}{\partial x^2} = 6x, \qquad \frac{\partial^2 f}{\partial y^2} = 6y, \qquad \frac{\partial^2 f}{\partial x \partial y} = -6.$$

Therefore, we have

$$\Delta = f_{xx} \cdot f_{yy} - (f_{xy})^2 = 36xy - 36.$$

- At $(0, 0)$, $\Delta = -36 < 0$ and so this is a saddle point.
- At $(2, 2)$, $\Delta = 36 \times 2 \times 2 - 36 = 108 > 0$ and $f_{xx} = 12 > 0$ (note that f_{yy} is also positive) and so this is a minimum.

Example End

Example 5.40

Determine the location and nature of the stationary points of

$$f(x, y) = x^4 + y^4 - 2(x - y)^2.$$

Solution 5.40

The first derivatives give

$$\frac{\partial f}{\partial x} = 4x^3 - 4(x - y), \qquad \frac{\partial f}{\partial y} = 4y^3 + 4(x - y).$$

Hence, the stationary points satisfy the pair of equations

$$\begin{aligned} 4x^3 - 4x + 4y &= 0, \\ 4y^3 + 4x - 4y &= 0. \end{aligned}$$

Add the two equations together to get

$$x^3 + y^3 = 0, \qquad \Rightarrow \qquad x = -y.$$

Now substitute back into one of the equations to get

$$\begin{aligned} 4x^3 - 8x &= 0, \\ x^3 - 2x &= 0, \\ x(x^2 - 2) &= 0. \end{aligned}$$

This implies there are three stationary points at $(0,0)$, $(+\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, +\sqrt{2})$. Now to test their nature we need the second derivatives.

$$\frac{\partial^2 f}{\partial x^2} = 12x^2 - 4, \qquad \frac{\partial^2 f}{\partial y^2} = 12y^2 - 4, \qquad \frac{\partial^2 f}{\partial x \partial y} = 4.$$

Thus, we calculate $\Delta = f_{xx} \cdot f_{yy} - (f_{xy})^2 = (12x^2 - 4) \cdot (12y^2 - 4) - (-4)^2$ at each of the stationary points.

- At $(+\sqrt{2}, -\sqrt{2})$, $\Delta = (24 - 4) \cdot (24 - 4) - 16 > 0$ and $f_{xx} = 20 > 0$ and so this is a minimum.
- At $(-\sqrt{(2)}, +\sqrt{2})$, $\Delta = (24 - 4) \cdot (24 - 4) - 16 > 0$ and $f_{xx} = 20 > 0$ and so this is again a minimum.
- At the origin $(0,0)$, $\Delta = (-4) \cdot (-4) - 16 = 0$. In this case the test is inconclusive and further analysis is needed. If we assume that x and y are small, then terms like x^4 and y^4 are small and can be neglected when compared to squared terms like $-2(x-y)^2$. Thus, provided that this term does not vanish, namely when $x = y$, we have

  $$f(x,y) - f(0,0) = -2(x-y)^2 < 0.$$

  However, if we choose a path such that $x = y$, then

  $$f(x,y) - f(0,0) = x^4 + x^4 > 0.$$

  Since $f(x,y) - f(0,0)$ changes sign depending on the particular path chosen near the stationary point we conclude that $(0,0)$ is a saddle point.

Example End

5.5 Maxima and Minima subject to a constraint: Lagrange Multipliers - Swokowski chapter 12.9

:5.4}

There are many problems where we need to maximise or minimise a function subject to some kind of constraint. Examples are: maximise the profit for some given costs, maximise the distance travelled for a given amount of fuel, minimise the surface area for a given enclosed volume and so on. Two examples will illustrate these types of problems. Then we will discuss the method of Lagrange Multipliers when the standard approach cannot be used.

Example 5.41

Minimise $f(x,y) = x^2 + y^2$ subject to $y = -x + 1$.

Solution 5.41

Here $y = -x + 1$ is the constraint and so we simply substitute this constraint directly into the function. Hence,

$$f(x, y(x)) = x^2 + (-x+1)^2 = x^2 + x^2 - 2x + 1 = 2x^2 - 2x + 1.$$

This is simply a function of one variable and we may find the turning point by simple differentiation.

$$\left(\frac{df}{dx}\right)_{y=-x+1} = 4x - 2 = 0, \qquad \Rightarrow \qquad x = \frac{1}{2}.$$

From the constraint, $y = -x + 1 = -\frac{1}{2} + 1 = \frac{1}{2}$. The function evaluated at this turning point is

$$f(1/2, 1/2) = (\frac{1}{2})^2 + (\frac{1}{2})^2 = \frac{1}{2}.$$

It is clear (from $f''(x) = 4 > 0$) that this is a minimum.

Example End

Example 5.42

Minimise $f(x,y,z) = x^2 + 2y^2 + z$ subject to the constraint $x + y^2 - z = 0$.

Solution 5.42

To progress we substitute for z to get a function of two variables. Thus,

$$f(x, y, z(x,y)) = f(x,y) = x^2 + 2y^2 + (x + y^2) = x^2 + x + 3y^2.$$

To locate the stationary point, we evaluate the first derivatives,

$$\frac{\partial f}{\partial x} = 2x + 1 = 0, \qquad \frac{\partial f}{\partial y} = 6y = 0.$$

Therefore,

$$x = -\frac{1}{2}, \qquad y = 0.$$

Now we calculate Δ at $(-1/2, 0)$.

$$\Delta = f_{xx} \cdot f_{yy} - (f_{xy})^2,$$

and

$$f_{xx} = 2, \qquad f_{yy} = 6, \qquad f_{xy} = 0.$$

Thus, $\Delta = 2 \times 6 = 12 > 0$ and $\partial^2 f/\partial x^2 = 2 > 0$, so $(-1/2, 0)$ is a minimum.

Example End

Note that substitution is not always possible. Also the constrained functions may be a function of several variables. For such cases we need another method and use a *Lagrange Multiplier* that is denoted by λ.

This method determines the stationary points *but not their nature.*

Suppose we wish to find the stationary points of $f(x,y,z)$ subject to a constraint, $g(x,y,z) = 0$. We define the function $F(x,y,z,\lambda)$,

$$F(x,y,z,\lambda) = f(x,y,z) - \lambda g(x,y,z). \tag{5.15}$$ {eq:5.5}

λ is the Lagrange multiplier. If the constraint $g(x,y,z) = 0$ is satisfied, then the maximum/minimum of F will coincide with those of the constrained f. To locate the stationary points of F we need to locate the zeros of the first derivatives. Thus,

$$\begin{aligned}
\frac{\partial F}{\partial x} &= \frac{\partial f}{\partial x} - \lambda\frac{\partial g}{\partial x} = 0 \\
\frac{\partial F}{\partial y} &= \frac{\partial f}{\partial y} - \lambda\frac{\partial g}{\partial y} \\
\frac{\partial F}{\partial z} &= \frac{\partial f}{\partial z} - \lambda\frac{\partial g}{\partial z} \\
\frac{\partial F}{\partial \lambda} &= -g(x,y,z) = 0
\end{aligned}$$

Note that these equations can also be written in a vector form as

$$\nabla f = \lambda \nabla g, \qquad g(x,y,z) = 0.$$

To apply the method we

1. Add an unknown multiple $(-\lambda)$ of the constraint g to f, then set the partial derivatives of $(f - \lambda g)$ with respect to x, y and z to zero. This gives three equations.

2. Use the above 3 equations together with the constraint $g(x,y,z) = 0$ to find x, y, z and λ.

The method can be generalised for more variables and more constraints.

Example 5.43

Consider our first example again, where $f(x,y) = x^2 + y^2$ and $g(x,y) = x + y - 1$. Define,

$$F = x^2 + y^2 - \lambda(x + y - 1).$$

The first derivatives become

$$\frac{\partial F}{\partial x} = 2x - \lambda, \qquad \frac{\partial F}{\partial y} = 2y - \lambda, \qquad \frac{\partial F}{\partial \lambda} = -(x + y - 1).$$

and setting them equal to zero gives

$$x = \frac{\lambda}{2}, \qquad y = \frac{\lambda}{2}, \qquad x + y = 1.$$

Substituting for x and y in terms of λ in the last equation gives,

$$x + y = 1, \qquad \frac{\lambda}{2} + \frac{\lambda}{2} = \lambda = 1.$$

Hence,

$$x = \frac{1}{2}, \qquad y = \frac{1}{2}, \qquad f = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Consider the two variable case. Why does $\nabla f = \lambda \nabla g$ maximise/minimise $f(x, y)$ along the path $g(x, y) = 0$? This is illustrated in Figures 5.4 and 5.5. If f, subject to $g = 0$ has a max/min at $(x_0), y_0)$,

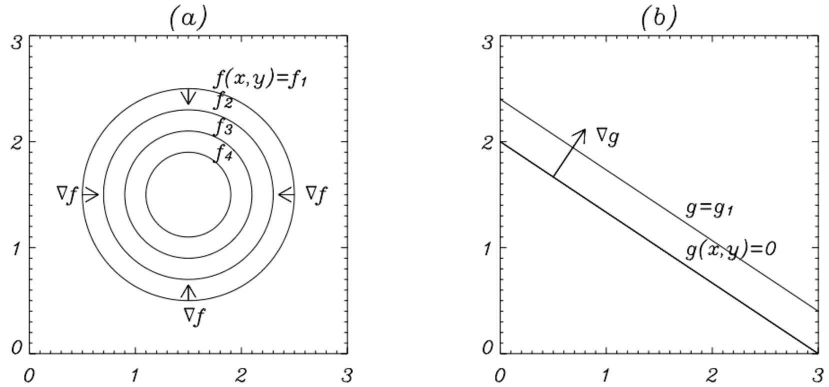


Figure 5.4: (a) Contours of constant f are shown as circles with the maximum value at f_3. (b) The path $g(x, y) = 0$ and the neighbouring contour $g = g_1$.

{fig:5.4}

then $(df/ds)_{x_0,y_0} = 0$ there, where

$$\left(\frac{df}{ds}\right)_{(x_0,y_0)} = [(\nabla f) \cdot \hat{\mathbf{u}}]_{(x_0,y_0)} = 0.$$

Thus, $\hat{\mathbf{u}}(x_0, y_0)$ is perpendicular to $(\nabla f)_{(x_0,y_0)}$. From above, $\hat{\mathbf{u}}$ is perpendicular to ∇g. Therefore, ∇f and ∇g are parallel at (x_0, y_0). Thus,

$$\nabla f = \lambda \nabla g.$$

Physically, following a path defined by $g = 0$ we would walk uphill and then downhill. However, we would walk parallel to a contour at the changeover between up and down.

Example End

Example 5.44

Find the stationary points of

$$f(x, y, z) = x + y + z,$$

on a sphere of radius 5 centred on the origin.

Solution 5.44

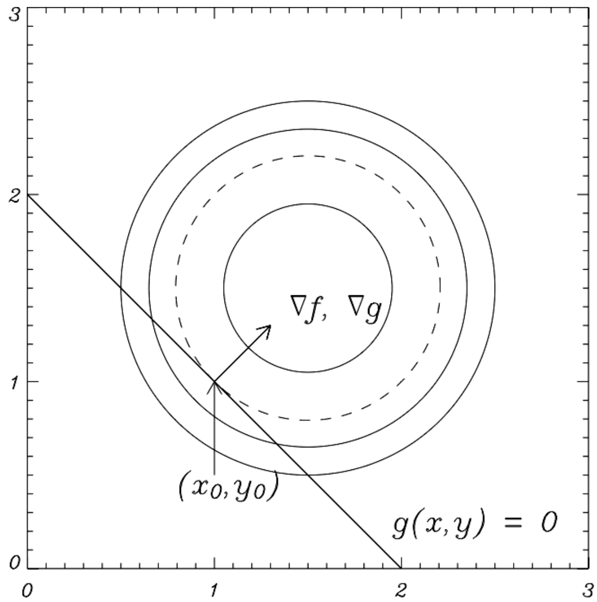


Figure 5.5: Combining the path $g = 0$ with the contours of constant f. The dashed contour gives the max/min contour of the constrained f.

Thus, the constraint is

$$g(x, y, z) = x^2 + y^2 + z^2 - 25 = 0.$$

The modified function with the Lagrange multiplier is

$$F(x, y, z, \lambda) = f - \lambda g = x + y + z - \lambda\left(x^2 + y^2 + z^2 - 25\right).$$

The first derivatives can calculated and set equal to zero to locate the stationary points.

$$\begin{aligned}
\frac{\partial F}{\partial x} &= 1 - 2\lambda x, \qquad \Rightarrow \qquad x = \frac{1}{2\lambda}, \\
\frac{\partial F}{\partial y} &= 1 - 2\lambda y, \qquad \Rightarrow \qquad y = \frac{1}{2\lambda}, \\
\frac{\partial F}{\partial z} &= 1 - 2\lambda z, \qquad \Rightarrow \qquad z = \frac{1}{2\lambda}, \\
\frac{\partial F}{\partial \lambda} &= -g = -\left(x^2 + y^2 + z^2 - 25\right) = 0.
\end{aligned}$$

Substitute in the values for x, y and z into the last equation to get

$$\left(\frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} - 25\right) = \left(\frac{3}{4\lambda^2} - 25\right) = 0.$$

Solving for λ gives

$$\lambda^2 = \frac{3}{100}, \qquad \lambda = \pm\frac{\sqrt{3}}{10}.$$

So there are two stationary points. We have

- $\lambda = +\sqrt{3}/10$, so that $(x, y, z) = (5/\sqrt{3}, 5/\sqrt{3}, 5/\sqrt{3})$ and $f = 5\sqrt{3}$.
- $\lambda = -\sqrt{3}/10$, so that $(x, y, z) = (-5/\sqrt{3}, -5/\sqrt{3}, -5/\sqrt{3})$ and $f = -5\sqrt{3}$.

To determine the nature of the stationary points we need to continue carefully. Setting $x = 5/\sqrt{3} + x1$ and $y = 5/\sqrt{3} + y1$, we can substitute this into the constraint. Then, we find that

$$z = +\sqrt{25 - (\frac{5}{\sqrt{3}} + x1)^2 - (\frac{5}{\sqrt{3}} + y1)^2}.$$

This may be simplified and expanded in a Taylor series of two variables for small values of $x1$ and $y1$ to give

$$\begin{aligned} z &= +\sqrt{25 - \frac{50}{3} - \frac{10}{\sqrt{3}}(x1 + y1) - x1^2 - y1^2} \\ &= +\sqrt{\frac{25}{3} - \frac{10}{\sqrt{3}}(x1 + y1) - x1^2 - y1^2} \\ &= \frac{5}{\sqrt{3}}\sqrt{1 - \frac{2\sqrt{3}}{5}(x1 + y1) - \frac{3}{25}(x1^2 + y1^2)} \\ &= \frac{5}{\sqrt{3}}\left(1 - \frac{\sqrt{3}}{5}(x1 + y1) - \frac{3}{50}(x1^2 + y1^2) - \frac{1}{8}(\frac{2\sqrt{3}}{5})^2(x1 + y1)^2 + \cdots\right). \end{aligned}$$

So we may express z as the value at the stationary point plus a correction due to the value of $x1$ and $y1$. Note that we used the binomial expansion of $\sqrt{1+x} = 1 + x/2 - x^2/8 + \cdots$. Hence, substituting $x = 5/\sqrt{3} + x1$ and $y = 5/\sqrt{3} + y1$ into f gives

$$f = \frac{5}{\sqrt{3}} + x1 + \frac{5}{\sqrt{3}} + y1\frac{5}{\sqrt{3}} - (x1 + y1) - \frac{\sqrt{3}}{10}(x1^2 + y1^2) - \frac{5}{8\sqrt{3}}(x1 + y1)^2.$$

Simplifying gives

$$f = 5\sqrt{3} - \frac{\sqrt{3}}{10}(x1^2 + y1^2) - \frac{5}{8\sqrt{3}}(x1 + y1)^2.$$

Thus, it is clear the this stationary point corresponds to a maximum.

Similarly, we may show that $(x, y, z) = (-5/\sqrt{3}, -5/\sqrt{3}, -5/\sqrt{3})$ corresponds to a minimum.

Example End

Example 5.45

What is the maximum area of a rectangle enclosed by the x and y axes and the curve (an ellipse) $x^2/a^2 + y^2/b^2 = 1$?

Solution 5.45

This is illustrated in Figure 5.6. The area is given by

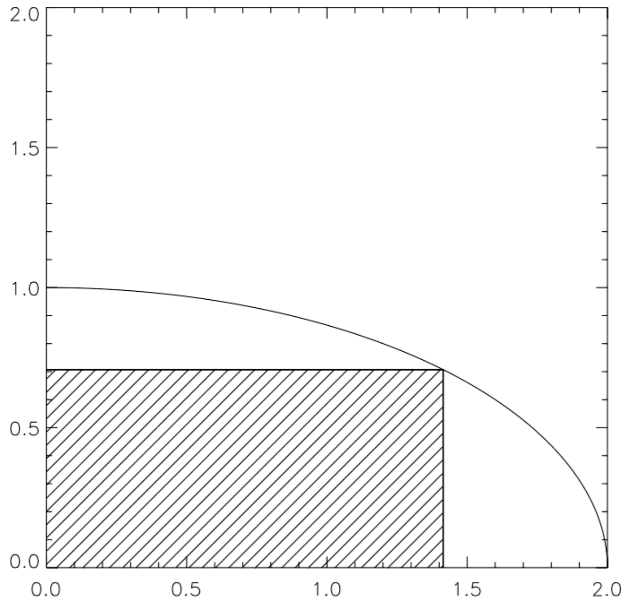


Figure 5.6: The area of the rectangle is shaded. The curve is the ellipse. {fig:5.6}

$$f(X, Y) = XY,$$

and the values of X and Y are constrained by

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1.$$

So we choose

$$g(X, Y) = \frac{X^2}{a^2} + \frac{Y^2}{b^2} - 1.$$

Hence,

$$F(X, Y, \lambda) = XY - \lambda \left(\frac{X^2}{a^2} + \frac{Y^2}{b^2} - 1 \right).$$

Now form all the first derivatives and set equal to zero.

5.5a} $$\frac{\partial F}{\partial X} = Y - \frac{2\lambda X}{a^2} = 0, \qquad \Rightarrow \qquad Y = \frac{2\lambda}{a^2} X \tag{5.16}$$

5.5b} $$\frac{\partial F}{\partial Y} = X - \frac{2\lambda Y}{b^2} = 0, \qquad \Rightarrow \qquad X = \frac{2\lambda}{b^2} Y \tag{5.17}$$

5.5c} $$\frac{\partial F}{\partial \lambda} = -\left(\frac{X^2}{a^2} + \frac{Y^2}{b^2} - 1 \right) = 0. \tag{5.18}$$

(5.16) and (5.17) give

$$Y = \frac{2\lambda}{a^2} \cdot \frac{2\lambda}{b^2} Y.$$

Dividing by Y and solving for λ gives

{eq:5.6}
$$\lambda = \pm\frac{ab}{2}. \tag{5.19}$$

(5.16) and (5.19)

$$Y = \frac{2}{a^2}X\left(\pm\frac{ab}{2}\right).$$

However, both X and Y must be positive and so we discard the negative sign. Thus,

$$Y = \frac{b}{a}X. \tag{5.20}$$
{eq:5.7}

(5.18) and (5.20) give

$$\frac{X^2}{a^2} + \frac{1}{b^2}\left(\frac{b^2X^2}{a^2}\right) = 1, \qquad \Rightarrow \qquad 2\frac{X^2}{a^2} = 1, \qquad \Rightarrow \qquad X = \frac{a}{\sqrt{2}}. \tag{5.21}$$
{eq:5.8}

Finally, (5.20) and (5.21) give

$$Y = \frac{b}{a}\left(\frac{a}{\sqrt{2}}\right), \qquad \Rightarrow \qquad Y = \frac{b}{\sqrt{2}}.$$

Therefore, the maximum area is

$$XY = \frac{ab}{2}.$$

Example End