University of St Andrews

SAMPLE EXAMINATION DIET

SCHOOL OF MATHEMATICS & STATISTICS

MODULE CODE:	**MT5836**
MODULE TITLE:	Galois Theory
EXAM DURATION:	$2\frac{1}{2}$ hours
EXAM INSTRUCTIONS:	Attempt ALL questions. The number in square brackets shows the maximum marks obtainable for that question or part-question. Your answers should contain the full working required to justify your solutions.
PERMITTED MATERIALS:	Non-programmable calculator

YOU MUST HAND IN THIS EXAM PAPER AT THE END OF THE EXAM.

PLEASE DO NOT TURN OVER THIS EXAM PAPER UNTIL YOU ARE INSTRUCTED TO DO SO.

1. (a) State Eistenstein's criterion. **[2]**

(b) Is the converse of Eisenstein's criterion true? Justify your answer. **[2]**

(c) Are the following polynomials irreducible over $\mathbb{Q}$? Justify your answer in each case. You may use standard combinatorial results without proof, provided that they are clearly stated.

(i) $f(X) = 3X^5 + 6X^4 + 18X^3 + 12X + 2$.

(ii) $g(X) = 3X^5 + 6X^4 + 18X^3 + 12X + 4$.

(iii) $h(X) = X^{p-1} + X^{p-2} + \cdots + 1$, where p is an odd prime.

[5]

2. (a) Let F be a field and $f \in F[X]$ be a polynomial with coefficients in F with $\deg f \geq 1$. Define what is meant by a *splitting field* of f over F. **[2]**

(b) Assume that $\deg f = 2$ and that f has a root α in some extension K of F. Prove that $F(\alpha)$ is a splitting field for f over F. **[2]**

(c) Let $f \in F[X]$. Define what is meant by the *Galois group* of f. **[1]**

(d) Now consider the polynomials $f = (X^2 - 2)(X^2 + 1)$ and $g = X^4 + 1$ in $\mathbb{Q}[X]$. Prove that the Galois groups of f and of g are isomorphic. **[5]**

3. (a) State the Fundamental Theorem of Galois Theory (a proof is not required). **[4]**

(b) Let E be the splitting field over $\mathbb{Q}$ of the polynomial $f = X^4 - 5$, and let $G = \mathrm{Gal}(E : Q)$.

(i) Show that $E = \mathbb{Q}(\alpha, i)$, where $\alpha = \sqrt[4]{5}$. **[3]**

(ii) State the Tower Law. Hence, or otherwise, show that $[E : \mathbb{Q}] = 8$, and deduce that $|G| = 8$. **[5]**

(iii) Show that G contains an automorphism σ such that $\sigma(\alpha) = i\alpha$ and $\sigma(i) = i$, and an automorphism τ such that $\tau(\alpha) = \alpha$ and $\tau(i) = -i$. **[2]**

(iv) Prove that every element of G can be written as $\sigma^k \tau^l$, where $k \in \{0, 1, 2, 3\}$ and $l \in \{0, 1\}$. **[4]**

(v) Find the values of k and l such that $\tau\sigma\tau^{-1} = \sigma^k \tau^l$. **[2]**

(vi) Prove that the extension $E : \mathbb{Q}$ has exactly five intermediate fields B with $[B : \mathbb{Q}] = 4$, and that for exactly one of these intermediate fields the extension $B : \mathbb{Q}$ is a normal extension. **[6]**

(vii) Find the subgroup of G that corresponds to the intermediate field $\mathbb{Q}(i\sqrt{5})$ under the Galois correspondence for the extension $E : \mathbb{Q}$. **[4]**

4. (a) Define what is meant by saying that a group G is *solvable*. **[2]**

(b) Define what is meant by saying that a polynomial is *solvable by radicals*. **[2]**

(c) State (without proof) Galois' Great Theorem. **[1]**

(d) Let n be a natural number, and let $\epsilon = e^{2\pi i/n}$ be a primitive nth root of unity. Let $K = \mathbb{Q}(\epsilon)$. Prove that $\mathrm{Gal}(K : \mathbb{Q})$ is abelian. **[2]**

(e) Now let $\alpha = \sqrt[n]{2}$ be the positive real nth root of 2, and let $E = \mathbb{Q}(\epsilon, \alpha)$. Prove that $\mathrm{Gal}(E : K)$ is abelian, and that $\mathrm{Gal}(E : \mathbb{Q})$ is solvable. You may use both the Fundamental Theorem and results from group theory, without proof, provided that the results you are using are clearly stated. **[4]**

END OF PAPER
