Chapter 4

Integral Vector Calculus: Line Integrals

4.1 Introduction: Integration of a vector function

The integration of a vector function of a single variable, say t, referred to the cartesian basis $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ is performed by integrating the components.

Let

$$\mathbf{F}(t) = F_x(t)\mathbf{i} + F_y(t)\mathbf{j} + F_z(t)\mathbf{k}$$

then

$$\int \mathbf{F}(t)dt = \mathbf{i}\int F_x(t)dt + \mathbf{j}\int F_y(t)dt + \mathbf{k}\int F_z(t)dt$$

for the indefinite integral. Note that for the indefinite integral, the answer will contain a constant vector.

For a definite integral, we would have

$$\int_{t_1}^{t_2} \mathbf{F}(t)dt = \mathbf{i}\int_{t_1}^{t_2} F_x(t)dt + \mathbf{j}\int_{t_1}^{t_2} F_y(t)dt + \mathbf{k}\int_{t_1}^{t_2} F_z(t)dt$$

Examples:

$$\mathbf{F}(t) = \cos t\mathbf{i} + t\mathbf{j} + \frac{1}{t}\mathbf{k}$$

$$\Rightarrow \int \mathbf{F}(t)dt = \sin t\mathbf{i} + \frac{t^2}{2}\mathbf{j} + \ln t\mathbf{k} + \mathbf{c}\,, \quad \text{(where } \mathbf{c} \text{ is a constant vector)}$$

$$\Rightarrow \int_1^2 \mathbf{F}(t)dt = [\sin 2 - \sin 1]\mathbf{i} + \frac{3}{2}\mathbf{j} + \ln 2\mathbf{k}$$

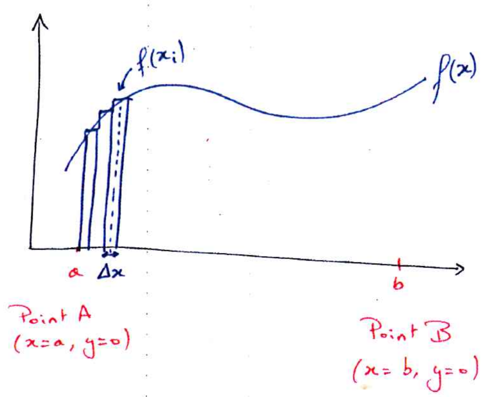
4.2 Line integral & Parameterisation

Line integrals are the generalisation of the definite integral

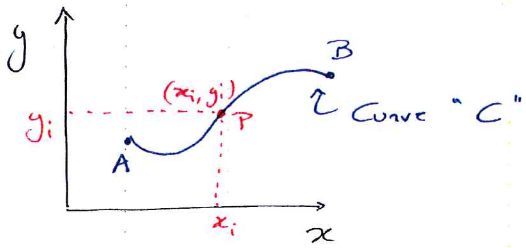
$$\int_a^b f(x)dx$$

in one variable. Here the "line" is the straight portion of the x-axis between $x = a$ and $x = b$.

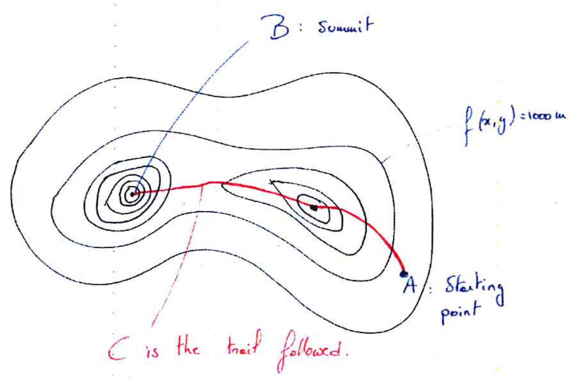
$$\lim_{\nabla x \to 0} \sum_i f(x_i) \nabla x = \int_a^b f(x) dx.$$



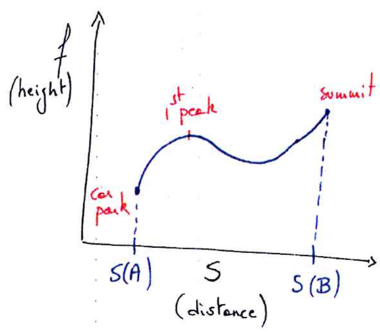
We want to generalise this idea to a general "line" (curve) connecting two points A and B in the $x - y$ plane.



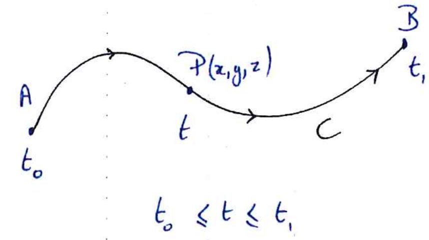
For each point (x, y) we associate the value of a function $f(x, y)$, say e.g. the altitude (height) at the point (x, y).



We can now represent the height (i.e the function f) as a function of the distance travelled (say s) along the trail (i.e. the curve C) from the starting point (A) to the summit (B).



In general, any curve in space can be represented by specifying any point $P(x, y, z)$ along it in terms of a single parameter, say t.



In other words, we parameterise the curve C in such a way that it can be traced by only one parameter, say t.

Example

Say the curve C is given by

$$y = x^2, \quad 0 \leq x \leq 1,$$

then let

$$x = t, y = t^2, \quad 0 \leq t \leq 1.$$

Now any point on C only depends on t ($x = x(t), y = y(t)$) and along the curve C, the function f only depends on t:

$$f(x, y) = f(x(t), y(t)) \qquad \text{the } \textit{parametric} \text{ form of } f$$

and

$$\int_A^B f(x, y) dx = \int_{t(A)}^{t(B)} f(x(t), y(t)) dt .$$

Example

Evaluate $\int_C xy^2 dy$ where C is the portion of the parabola $y = x^2$ from $(0, 0)$ to $(2, 4)$.

A convenient parameterisation is

$$x = t, y = t^2, \quad 0 \leq t \leq 2.$$

As $y = t^2$, we have that

$$dy = 2tdt$$

so

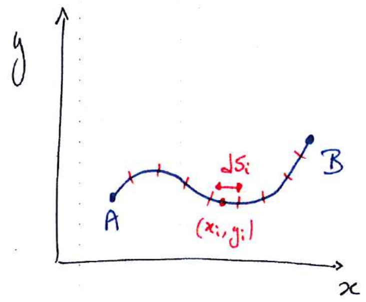
$$\int_C xy^2 dy = \int_0^2 t(t^2)^2 2tdt = 2\int_0^2 t^6 dt = \frac{2}{7}\left[t^7\right]_0^2 = \frac{256}{7}.$$

Note that as the function f depends on x and y, we could have also evaluated $\int_C xy^2 dx$:

$$\int_C xy^2 dx = \int_0^2 t(t^2)^2 dt = \int_0^2 t^5 dt = \frac{1}{6}\left[t^6\right]_0^2 = \frac{32}{3}.$$

4.2.1 Arclength

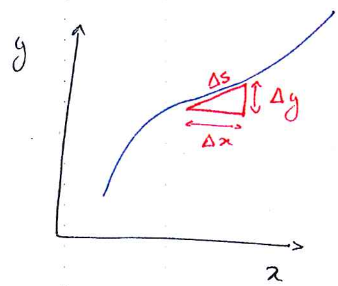
With each point $P = (x_i, y_i)$ on the curve C between A and B, we associate an incremental distance ds_i (arc length).



Let us have a closer look at the arc length.

By Pythagoras, we have

$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}\,.$$



Now taking the limit $\Delta s \to 0$ we have

$$ds = \sqrt{(dx)^2 + (dy)^2}\,.$$

In general, when the curve is parameterised by a parameter t, we have

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

and

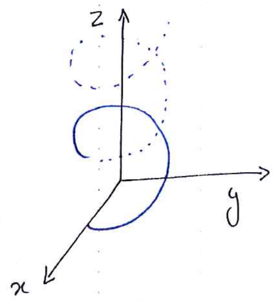
$$dx = \left(\frac{dx}{dt}\right) dt \text{ and } dy = \left(\frac{dy}{dt}\right) dt\,.$$

This can easily be extended to 3 dimensions as

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

where the term $\left(\frac{dz}{dt}\right)$ represents the obvious extension to 3 variables.

Example
Find the length of one turn of the helix:



$x(t) = a \cos t$
$y(t) = a \sin t$
$z(t) = bt$
$0 \leq t \leq 2\pi$

$$x(t) = a \cos t$$
$$y(t) = a \sin t$$
$$z(t) = bt$$

where $0 \leq t \leq 2\pi$ and where a and b are constants.

The length along the curve is given by:

$$L = \int_C ds$$

where ds is the arc length given by

$$ds = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + b^2 dt} = \sqrt{a^2 + b^2 dt}\,.$$

Hence, we have

$$L = \int_C ds = \int_0^{2\pi} \sqrt{a^2 + b^2 dt} = 2\pi \sqrt{a^2 + b^2}\,.$$

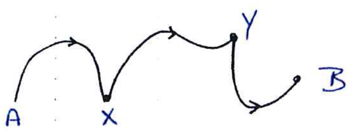
Of course, for $b = 0$, this simply reduces to $2\pi a$, i.e. the circumference of a circle of radius a.

4.3 Basic Properties of Line Integrals

The curve C along which we integrate must be piecewise smooth and continuous.

(i) The integral will be

$$\int_A^B = \int_A^X + \int_X^Y + \int_Y^B$$



It can have "corners".

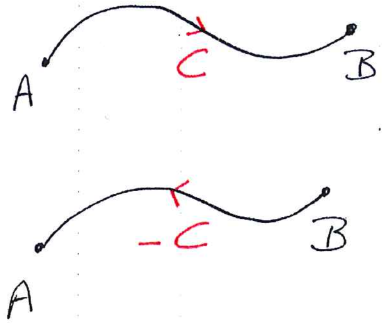
(ii) Reversing the direction of integration reverses the sign of the integral:

$$\int_A^B = -\int_B^A \quad \text{or} \quad \int_A^B + \int_B^A = 0\,.$$

Different notation:

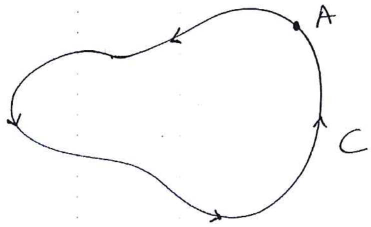
$$\int_{-C} = -\int_C$$

where the curve $-C$ denotes the same curve as C but traced in the opposite direction.



(iii) We can also integrate from a point A to itself along a non-trivial curve.

Integrals along a <u>closed curve</u> are denoted as $\oint_C$.



In general a curve will have many parameterisations. The integral will yield the same result, irrespective of the parameterisation, as long as it is traced in the same direction.

Say the curve C is the graph of $y = f(x)$ on the interval $a \leq x \leq b$. Then C has the parameterisation

$$x = t, \quad y = f(t), \qquad a \leq t \leq b\,.$$

Moreover, traced in the opposite direction, the curve $-C$ has parameterisation

$$x = b - t, \quad y = f(b-t), \qquad 0 \leq t \leq b - a\,.$$

Obviously some parameterisations will be easier to work with than others!

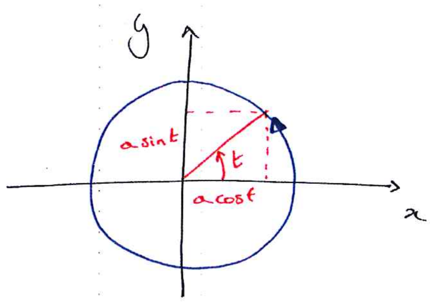
4.4 Examples of Parametric Form of Line Integrals

(i) Circle of radius a

$$x^2 + y^2 = a^2$$

$$x(t) = a\cos t$$
$$y(t) = a\sin t$$

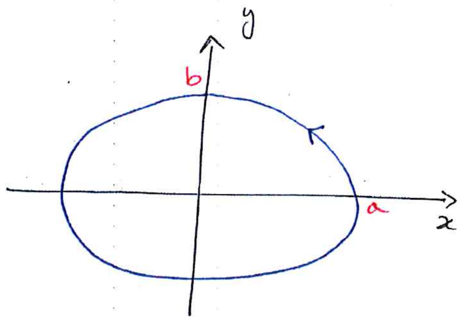
where $0 \leq t \leq 2\pi$ and where a is a constant.
Here t is the polar angle.



(ii) Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$x(t) = a\cos t$$
$$y(t) = b\sin t$$



where $0 \le t \le 2\pi$ and where a and b are constants.

Example

Evaluate $\int_C x\,dy$ where C is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,.$$

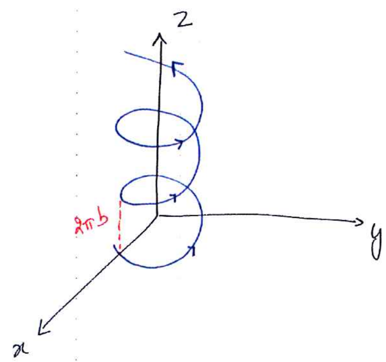
Using the parameterisation above, we have that

$$dy = \frac{dy}{dt}\,dt = b\cos t\,dt\,.$$

$$\begin{aligned}
\oint_C x\,dy &= \int_0^{2\pi} (a\cos t)(b\cos t)dt \\
&= ab\int_0^{2\pi} \cos^2 t\,dt \\
&= ab\int_0^{\pi} \frac{1+\cos 2t}{2}\,dt \\
&= ab\left\{\pi + \int_0^{\pi} \frac{\cos 2t}{2}\,dt\right\} \\
&= ab\pi
\end{aligned}$$

(iii) A helix

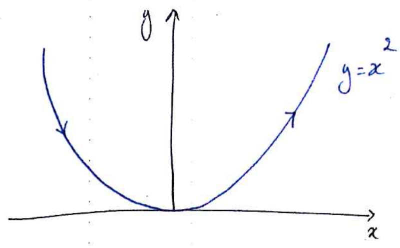
$$x(t) = a\cos t$$
$$y(t) = a\sin t$$
$$z(t) = bt$$



where $0 \le t \le 2\pi$ and where a and b are constants.

(iv) Parabola

$$x(t) = t$$
$$y(t) = t^2$$



(v) A square of side 1

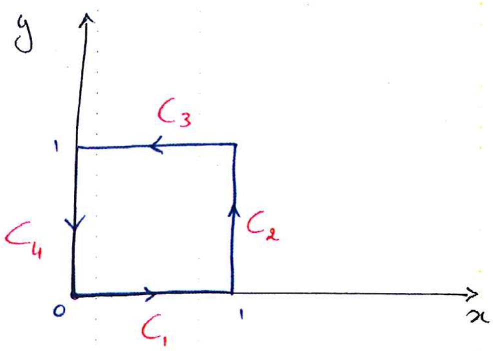
C is composed of 4 sides.

$$C_1 : \begin{cases} x \text{ from } 0 \text{ to } 1 \\ y = 0 \end{cases} \Rightarrow \begin{cases} x(t) = t & 0 \le t \le 1 \\ y(t) = 0 \end{cases}$$

$$C_2 : \begin{cases} x = 1 \\ y \text{ from } 0 \text{ to } 1 \end{cases} \Rightarrow \begin{cases} x(t) = 1 \\ y(t) = t & 0 \le t \le 1 \end{cases}$$

$$C_3 : \begin{cases} x \text{ from } 1 \text{ to } 0 \\ y = 1 \end{cases} \Rightarrow \begin{cases} x(t) = 1 - t & 0 \le t \le 1 \\ y(t) = 1 \end{cases}$$

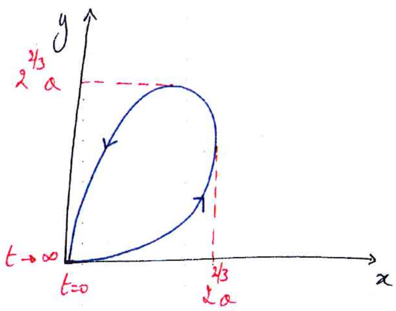
$$C_4 : \begin{cases} x = 0 \\ y \text{ from } 1 \text{ to } 0 \end{cases} \Rightarrow \begin{cases} x(t) = 0 \\ y(t) = 1 - t & 0 \le t \le 1 \end{cases}$$



(vi) Folium of Descartes

$$x(t) = \frac{3at}{1+t^3}$$
$$y(t) = \frac{3at^2}{1+t^3}$$
$$z(t) = 0$$

where $0 \le t < \infty$ and where a is a constant.



Example

Evaluate $\int_C xydx + x^2dy$ if

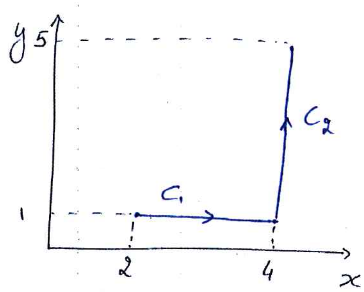
(i) C consists of line segments from $(2,1)$ to $(4,1)$ and from $(4,1)$ to $(4,5)$,

(ii) C is the line segment from $(2,1)$ to $(4,5)$,

(iii) parametric equations for C are $x = 3t-1$, $y = 3t^2 - 2t$; $1 \leq t \leq 5/3$.

(i) Here the curve C consists of two parts C_1 and C_2 and the line integral along C can be written as the sum of two line integrals, the first one along C_1 and the second one along C_2:

$$\int_C xydx+x^2dy = \int_{C_1} xydx+x^2dy+\int_{C_2} xydx+x^2dy\,.$$



The parametric equations for these two curves are given by

$$\begin{aligned} C_1: &\quad x = t,\ y = 1; &\quad 2 \leq t \leq 4 \\ C_2: &\quad x = 4,\ y = t; &\quad 1 \leq t \leq 5 \end{aligned}$$

On C_1 we have $dy = 0$ and $dx = dt$ so we find:

$$\int_{C_1} xydx + x^2dy = \int_2^4 t\,1\,dt + 0 = \left[\frac{1}{2}t^2\right]_2^4 = 6\,.$$

On C_2 we have $dx = 0$ and $dy = dt$ so:

$$\int_{C_2} xydx + x^2dy = \int_1^5 0 + 16dt = 16\,[t]_1^5 = 64\,.$$

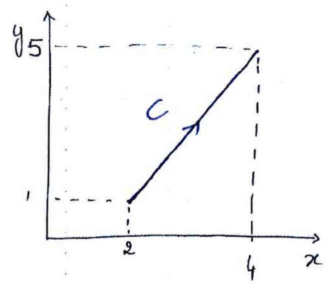
Hence, we have that

$$\int_C xydx + x^2dy = 6 + 64 = 70\,.$$

(ii) In this case, an equation for the curve C (straight line!) is given by $y = 2x - 3$ with $2 \leq x \leq 4$.

Using $x = t$ and $y = 2t - 3$ ($2 \leq t \leq 4$), we have that $dx = dt$ and $dy = 2dt$ so:

$$\begin{aligned} \int_C xydx + x^2dy &= \int_2^4 t(2t-3)dt + t^2 2dt \\ &= \int_2^4 (4t^2 - 3t)dt \\ &= \left[\frac{4}{3}t^3 - \frac{3}{2}t^2\right]_2^4 = \frac{170}{3} \end{aligned}$$



(iii) Using $x = 3t - 1$, $y = 3t^2 - 2t$ $(1 \le t \le 5/3)$, we have

$$dx = 3dt \qquad \text{and} dy = (6t-2)dt$$

so we find

$$\int_C xydx + x^2dy = \int_1^{5/3} (3t-1)(3t^2-2t)3dt + (3t-1)^2(6t-2)dt$$
$$= \ldots$$
$$= \int_1^{5/3} (81t^3 - 81t^2 + 24t - 2)dt$$
$$= \left[\frac{81}{4}t^4 - 27t^3 + 12t^2 - 2t\right]_1^{5/3} = 58$$

From the above example, it is clear that we get a different answer for different paths (i.e. curves C).

4.5 Line Integrals of Vector Functions

Let us now consider a vector function again, say

$$\mathbf{F}(x,y,z) = \begin{pmatrix} f_1(x,y,z) \\ f_2(x,y,z) \\ f_3(x,y,z) \end{pmatrix}.$$

A combination of integrals which occurs commonly is

$$I = \int_C f_1(x,y,z)dx + \int_C f_2(x,y,z)dy + \int_C f_3(x,y,z)dz = \int_C f_1(x,y,z)dx + f_2(x,y,z)dy + f_3(x,y,z)dz,$$

where it is understood that x, y, z all vary along a specified curve C.

Now let

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad d\mathbf{r} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}.$$

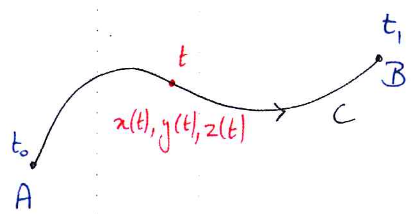
then we can write that

$$I = \int_C f_1(x,y,z)dx + f_2(x,y,z)dy + f_3(x,y,z)dz = \int_C \mathbf{F} \cdot d\mathbf{r}\,.$$

We will now again use the parametric representation of the curve C between points A and B.

Given $x(t)$, $y(t)$, $z(t)$ we have

$$dx = \frac{dx}{dt}dt, \quad dy = \frac{dy}{dt}dt, \quad dz = \frac{dz}{dt}dt.$$



So in vector form we have:

$$d\mathbf{r} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} dx/dt \\ dy/dt \\ dz/dt \end{pmatrix} dt = \frac{d\mathbf{r}}{dt}dt.$$

Hence, the line integral can be written as

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \underbrace{\left(\mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\right)}_{\text{scalar function!}} dt = \int_{t_0}^{t_1} f(t)dt.$$

4.5.1 Line integrals in practice

Here we outline the general method to calculate

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

before considering some more examples.

Step 1: Given the curve C, write down a parametric representation of C in terms of $\mathbf{r}(t) = (x(t), y(t), z(t))$.

Step 2: Calculate $\dfrac{d\mathbf{r}}{dt}$.

Step 3: Express $\mathbf{F}$ along C, i.e. write down $\mathbf{F}(\mathbf{r}(t))$ using the parametric representation of C.

Step 4: Calculate $\mathbf{F} \cdot \dfrac{d\mathbf{r}}{dt}$ ($= f(t)$, a scalar function of t!)

Step 5: Integrate $\displaystyle\int_C \left(\mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\right) dt = \int_{t_0}^{t_1} f(t)dt$.

Example 1

For $\mathbf{F}(\mathbf{r}) = x\mathbf{i} + 2y\mathbf{j} + 3z\mathbf{k}$, calculate

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where C is the curve defined by

$$\begin{cases} x(t) = t \\ y(t) = t^2 \\ z(t) = t^3 \end{cases} \qquad \text{between } t = 0 \text{ and } t = 1\,.$$

Step 1: This is basically done for us, as we are given C in its parametric representation:

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$$

Step 2: $\dfrac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$

Step 3: $\mathbf{F}(\mathbf{r}(t)) = t\mathbf{i} + 2t^2\mathbf{j} + 3t^3\mathbf{k}$

Step 4: $\mathbf{F} \cdot \dfrac{d\mathbf{r}}{dt} = (t\mathbf{i} + 2t^2\mathbf{j} + 3t^3\mathbf{k}) \cdot (\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}) = t + 4t^3 + 9t^5$

Step 5: Integrate:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (t + 4t^3 + 9t^5)dt = \left[\frac{t^2}{2} + t^4 + \frac{3}{2}t^6\right]_0^1 = \frac{1}{2} + 1 + \frac{3}{2} = \frac{6}{2} = 3$$

Example 2

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$ and where the curve C is the portion of the parabola $y = x^2$ between the points $(0,0)$ and $(3,9)$.

Step 1: As before, use the parametersation $x = t$ and $y = t^2$, then

$$\mathbf{r} = (x, y, z) = (t, t^2, 0) = t\mathbf{i} + t^2\mathbf{j}\,.$$

Step 2: $\dfrac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j}$

Step 3: $\mathbf{F}(\mathbf{r}(t)) = -t^2\mathbf{i} + t\mathbf{j}$

Step 4: $\mathbf{F} \cdot \dfrac{d\mathbf{r}}{dt} = (-t^2\mathbf{i} + t\mathbf{j}) \cdot (\mathbf{i} + 2t\mathbf{j}) = -t^2 + 2t^2 = t^2$

Step 5: Integrate:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 t^2 dt = \left[\frac{1}{3}t^3\right]_0^3 = 9$$

Example 3

Evaluate $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F} = yz\mathbf{i} + zx\mathbf{j} + xy\mathbf{k} = \begin{pmatrix} yz \\ zx \\ xy \end{pmatrix}$$

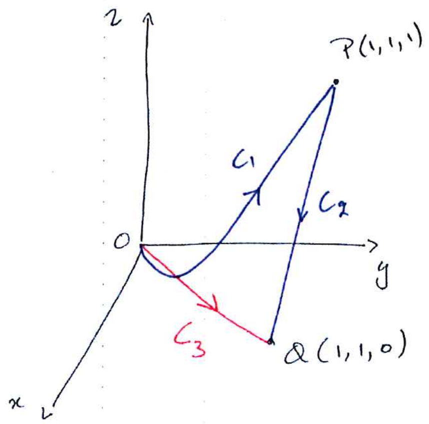
and the curve C consists of two segments, $C = C_1 + C_2$ where

C_1 is the "twisted" cubic

$$\begin{aligned} x(t) &= t \\ y(t) &= t^2 \qquad 0 \le t \le 1 \\ z(t) &= t^3 \end{aligned}$$

between the origin and $P = (1,1,1)$.

C_2 is the vertical line between P and $Q = (1,1,0)$.



Along C_1 we have:

$$\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k} \qquad \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$$

and

$$\mathbf{F} = yz\mathbf{i} + zx\mathbf{j} + xy\mathbf{k} = t^5\mathbf{i} + t^4\mathbf{j} + t^3\mathbf{k}$$

such that

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t^5 + 2t^5 + 3t^5 = 6t^5 \,.$$

Integrating gives:

$$\int_{C_1} \left(\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} \right) dt = \int_0^1 6t^5 dt = \left[t^6 \right]_0^1 = 1$$

Along C_2, we can use the parameterisation $x = 1$, $y = 1$ and $z = 1 - t$ $(0 \leq t \leq 1)$ such that

$$\mathbf{r} = \mathbf{i} + \mathbf{j} + (1 - t)\mathbf{k} \qquad \Rightarrow \frac{d\mathbf{r}}{dt} = -\mathbf{k}$$

and

$$\mathbf{F} = yz\mathbf{i} + zx\mathbf{j} + xy\mathbf{k} = (1 - t)\mathbf{i} + (1 - t)\mathbf{j} + \mathbf{k}$$

such that

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -1 \,.$$

Integrating gives:

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (-1) dt = [-t]_0^1 = -1.$$

Hence

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 1 - 1 = 0$$

Let us consider a different path (C_3), namely a straight line from the origin to the point $Q = (1, 1, 0)$. We can use the parameterisation $x = t$, $y = t$ and $z = 0$ $(0 \leq t \leq 1)$ such that

$$\mathbf{r} = t\mathbf{i} + t\mathbf{j} \qquad \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j}$$

and

$$\mathbf{F} = yz\mathbf{i} + zx\mathbf{j} + xy\mathbf{k} = 0\mathbf{i} + 0\mathbf{j} + t^2\mathbf{k}$$

such that

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 0 \,.$$

Integrating of course gives:

$$\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = 0.$$

In fact, it turns out that any curve between the origin and the point Q will give the same answer for this particular function $\mathbf{F}$.

Let us try a different example

Example 4

Let $\mathbf{F} = y\cos z\mathbf{i} + x\cos z\mathbf{j} - xy\sin z\mathbf{k}$.

Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ over the following paths from $(0,0,0)$ to $(1,1,0)$:

C_1: the line segment $x = t$, $y = t$, $z = 0$ $(0 \le t \le 1)$

C_2: the parabola $x = t$, $y = t^2$, $z = 0$ $(0 \le t \le 1)$

C_3: the cubic $x = t$, $y = t^3$, $z = 0$ $(0 \le t \le 1)$

$\underline{C_1}$:

$$\begin{aligned}
\mathbf{r} &= t\mathbf{i} + t\mathbf{j} && \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} \\
\mathbf{F} &= t\mathbf{i} + t\mathbf{j} && \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = t + t = 2t \\
\int_{C_1} \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 2t\,dt = [t^2]_0^1 = 1
\end{aligned}$$

$\underline{C_2}$:

$$\begin{aligned}
\mathbf{r} &= t\mathbf{i} + t^2\mathbf{j} && \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \\
\mathbf{F} &= t^2\mathbf{i} + t\mathbf{j} && \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = t^2 + 2t^2 = 3t^2 \\
\int_{C_1} \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 3t^2\,dt = [t^3]_0^1 = 1
\end{aligned}$$

$\underline{C_3}$:

$$\begin{aligned}
\mathbf{r} &= t\mathbf{i} + t^3\mathbf{j} && \Rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{i} + 3t^2\mathbf{j} \\
\mathbf{F} &= t^3\mathbf{i} + t\mathbf{j} && \Rightarrow \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = t^3 + 3t^3 = 4t^3 \\
\int_{C_1} \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 4t^3\,dt = [t^4]_0^1 = 1
\end{aligned}$$

Again we see that the answer does not depend on the path. In fact, any curve joining $(0,0,0)$ and $(1,1,0)$ will again give the same answer. In other words, for the last two examples, $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of the path.

Why? For both these last two examples, we have that $\boldsymbol{\nabla}\times\mathbf{F} = \mathbf{0}$. (Verify!)

$$\boldsymbol{\nabla}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & zx & xy \end{vmatrix} = \begin{pmatrix} x - x \\ -y + y \\ z - z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y\cos z & x\cos z & -xy\sin z \end{vmatrix} = \begin{pmatrix} -x\sin z - (-x\sin z) \\ y\sin z - y\sin z \\ \cos z - \cos z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We have seen that if a vector function $\mathbf{F}$ satisfies $\nabla \times \mathbf{F} = 0 \Rightarrow \exists$ a scalar function f such that $\mathbf{F} = \nabla f$. (Recall, the vector identity $\nabla \times (\nabla f) = \mathbf{0}$. Remember that the scalar function f is not necessarily unique.)

For example (3) with $\mathbf{F} = (yz, zx, xy)$, try $f = xyz$ and you will see that $\mathbf{F} = \nabla f$.

For example (4), $\mathbf{F} = y\cos z, x\cos z, -xy\sin z$, you can use $f = xy\cos z$.

Both the vector function in the last two examples belong to a class of function with a property: their line integrals are independent of the path.(See also Section 3.7)

As the line integrals in the above examples are independent of the path, only the value of the scalar function f at the end points matters and $\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)$.

In example (3), we found $f = xyz$ and hence we have indeed that $\int_C \mathbf{F} \cdot d\mathbf{r} = f(1,1,0) - f(0,0,0) = 0 - 0 = 0$.

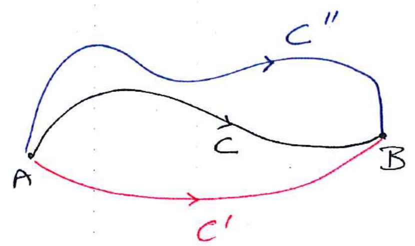
For example (4), we have that $f = xy\cos z$ and hence $\int_C \mathbf{F} \cdot d\mathbf{r} = f(1,1,0) - f(0,0,0) = 1 - 0 = 1$.

4.6 Line integrals independent of the path

The integral over an open curve C connecting two points A and B will be independent of the path if

$$\mathbf{F} = \nabla \phi$$

(which is equivalent to say $\nabla \times \mathbf{F} = 0$).



Fundamental Theorem of Line Integrals:

Let $\mathbf{F}$ be a vector function such that $\mathbf{F} = \nabla \phi$ (which is equivalent to $\nabla \times \mathbf{F} = \mathbf{0}$). Let C be a curve connecting the points A (starting point) and B (end point). Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \phi(B) - \phi(A).$$

Proof:

Suppose

$$\mathbf{F} = \nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right).$$

With $\mathbf{r} = (x, y, z)$ we have

$$\mathbf{dr} = (dx, dy, dz) \ ,$$

and hence

$$\mathbf{F} \cdot \mathbf{dr} = \nabla\phi \cdot \mathbf{dr} = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz .$$

Now let us write $\mathbf{r}$ in parametric form along the curve C:

$$\mathbf{r} = \mathbf{r}(t) = (x(t), y(t), z(t)) \qquad \text{for } t_0 \le t \le t_1 ,$$

and hence we have

$$\mathbf{F} \cdot \mathbf{dr} = \underbrace{\left[\frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt}\right]}_{= \frac{d}{dt}(\phi(\mathbf{r}(t)))} dt .$$

Note here that $\frac{d}{dt}\phi(\mathbf{r}(t)) = \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial t}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt}$ from the chainrule.

Therefore

$$\mathbf{F} \cdot \mathbf{dr} = \frac{d\phi}{dt}dt = d\phi$$

and thus

$$\int_C \mathbf{F} \cdot \mathbf{dr} = \int_C d\phi = \phi(B) - \phi(A) .$$

This implies that $\int_C \mathbf{F} \cdot \mathbf{dr}$ only depends on the value of ϕ at the points A and B – it does not depend on the path joining A to B. □

A vector field $\mathbf{F}$ is called conservative if $\mathbf{F} = \nabla\phi$ for some scalar function ϕ (i.e. $\nabla \times \mathbf{F} = \mathbf{0}$). The function ϕ is a potential function for $\mathbf{F}$.

Example

Consider

$$\mathbf{F} = \mathbf{k} .$$

Then $\nabla \times \mathbf{F} = 0$ and $\mathbf{F} = \nabla\phi$ with $\phi = z$ (+ arbitrary constant). As $\phi = z$, you can think of ϕ as the altitude. Clearly

$$\int_A^B \mathbf{F} \cdot \mathbf{dt} = \phi(B) - \phi(A)$$

is just the difference in altitude between B and A and does not depend on the path joining A and B.

Example

Consider

$$\mathbf{F} = \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\,.$$

$(\boldsymbol{\nabla} \times \mathbf{r} = 0)$

Then

$$\mathbf{F} \cdot d\mathbf{r} = \mathbf{r} \cdot d\mathbf{r} = x\,dx + y\,dy + z\,dz = d\left(\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}\right)$$

$$\Rightarrow \phi = \frac{1}{2}\left(x^2 + y^2 + z^2\right) \qquad (+\text{ arbitrary constant})\,.$$

Let us verify this line integral explicitly for curves connecting the point $A = (0,0,0)$ and $B = (1,1,1)$. As $\mathbf{F}$ is conservative with $\phi = \frac{1}{2}(x^2 + y^2 + z^2)$, we know that

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \phi(B) - \phi(A) = \frac{1}{2}(1 + 1 + 1) - 0 = \frac{3}{2}\,.$$

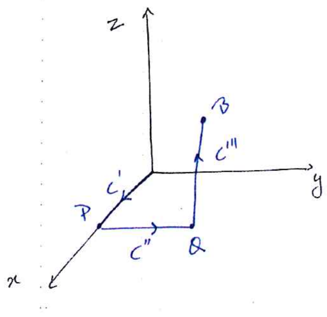
Let us now verify this result explicitly for 2 different curves C_1 and C_2 connecting the point $A = (0,0,0)$ and $B = (1,1,1)$:

(i) Let C_1 be the "twisted cubic"

$$x(t) = t,\; y(t) = t^2,\; z(t) = t^3 \qquad (0 \leq t \leq 1).$$

(ii) Let C_2 be the piecewise - continuous curve made up of three straight line segments:

$$C' : A(0,0,0) \to P(1,0,0)$$
$$C'' : P(1,0,0) \to Q(1,1,0)$$
$$C''' : Q(1,1,0) \to B(1,1,1)$$



(i) C_1:

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 2t \quad \frac{dz}{dt} = 3t^2$$

$$F_x = x = t,\; F_y = y = t^2,\; F_z = z = t^3$$

$$\Rightarrow \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left(\mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\right)\,dt = \int_0^1 \left(t + 2t^3 + 3t^5\right)\,dt = \left[\frac{1}{2}t^2 + \frac{1}{2}t^4 + \frac{1}{2}t^6\right]_0^1 = \frac{3}{2}$$

(ii) C_2:

C' $(A \to P)$

$$\begin{aligned}
& x = t,\ y = 0,\ z = 0 \quad (0 \le t \le 1) \\
\Rightarrow\ & \mathbf{r}(t) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = t\mathbf{i} \qquad \Rightarrow \quad \frac{d\mathbf{r}}{dt} = \mathbf{i} \\
\Rightarrow\ & \mathbf{F}(\mathbf{r}(t)) = \mathbf{r}(t) = t\mathbf{i} \qquad \Rightarrow \quad \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t \\
\Rightarrow\ & \int_{C'} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 t\, dt = \frac{1}{2}
\end{aligned}$$

C'' $(P \to Q)$

$$\begin{aligned}
& x = 1,\ y = t,\ z = 0 \quad (0 \le t \le 1) \\
\Rightarrow\ & \mathbf{r}(t) = \mathbf{i} + t\mathbf{j} \qquad \Rightarrow \quad \frac{d\mathbf{r}}{dt} = \mathbf{j} \\
\Rightarrow\ & \mathbf{F}(\mathbf{r}(t)) = \mathbf{r}(t) = \mathbf{i} + t\mathbf{j} \qquad \Rightarrow \quad \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t \\
\Rightarrow\ & \int_{C''} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 t\, dt = \frac{1}{2}
\end{aligned}$$

C''' $(Q \to B)$

$$\begin{aligned}
& x = 1,\ y = 1,\ z = t \quad (0 \le t \le 1) \\
\Rightarrow\ & \mathbf{r}(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k} \qquad \Rightarrow \quad \frac{d\mathbf{r}}{dt} = \mathbf{k} \\
\Rightarrow\ & \mathbf{F}(\mathbf{r}(t)) = \mathbf{r}(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k} \qquad \Rightarrow \quad \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = t \\
\Rightarrow\ & \int_{C'''} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 t\, dt = \frac{1}{2}
\end{aligned}$$

They all add up to

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_C^{\prime} + \int_C^{\prime\prime} + \int_C^{\prime\prime\prime} = \frac{3}{2} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \phi(B) - \phi(A).$$

Example

Consider the vector function

$$\mathbf{F} = xz^3\mathbf{i} - yz^3\mathbf{j} + \frac{3}{2}\left(x^2 - y^2\right) z^2\mathbf{k}.$$

Line integrals of this function will be independent of the path as

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz^3 & -yz^3 & \frac{3}{2}\left(x^2 - y^2\right) z^2 \end{vmatrix} = \mathbf{i}\left(-3yz^2 + 3yz^2\right) + \left(-3xz^2 + 3xz^2\right)\mathbf{j} + 0\mathbf{k} = 0\,.$$

To compute the integral $\int_C \mathbf{F} \cdot \mathbf{dr}$ we may either:

(i) consider the simplest path C joining the end points A and B or,

(ii) find f such that $\mathbf{F} = \nabla f$ as $\displaystyle\int_A^B \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)$.

To find the potential function ϕ, we have

$$\begin{cases} \dfrac{\partial \phi}{\partial x} = xz^3 & (1) \\ \dfrac{\partial \phi}{\partial y} = -yz^3 & (2) \\ \dfrac{\partial \phi}{\partial z} = \dfrac{3}{2}\left(x^2 - y^2\right) z^2 & (3) \end{cases}$$

We start by integrating (1) with respect to x, holding y and z fixed:

$$\phi(x, y, z) = \frac{1}{2} x^2 z^3 + \phi_1(y, z) \,.$$

Note that $\phi_1(y, z)$ is constant as far as the $x-$integration goes.

We now need to find $\phi_1(y, z)$.

If we differentiate this solution with respect to y, we have

$$\frac{\partial \phi}{\partial y} = \frac{\partial \phi_1}{\partial y}$$

but according to (2)

$$\frac{\partial \phi}{\partial y} = -yz^3 \Rightarrow \frac{\partial \phi_1}{\partial y} = -yz^3 \Rightarrow \phi_1 = -\frac{1}{2} y^2 z^3 + \phi_2(z) \,.$$

Note that ϕ_2 cannot depend on x as ϕ_1 is only a function of y and z.

$$\Rightarrow \phi = \frac{1}{2} x^2 z^3 + \phi_1(y, z) = \frac{1}{2} x^2 z^3 - \frac{1}{2} y^2 z^3 + \phi_2(z)$$

Now, to determine ϕ_2 we differentiate the solution with respect to z:

$$\phi = \frac{3}{2}\left(x^2 - y^2\right) z^2 + \frac{\partial \phi_2}{\partial z} \,.$$

Comparing with (3),

$$\frac{\partial \phi_2}{\partial z} = 0 \quad \Rightarrow \phi_2 = \text{constant.}$$

We can assume that this constant $= 0$ without any loss of generality.

Hence we have

$$\phi = \frac{1}{2}\left(x^2 - y^2\right) z^3.$$

(Check that $\boldsymbol{\nabla}\phi = \mathbf{F}$!)

Now $\displaystyle\int_A^B \mathbf{F}\cdot d\mathbf{r} = \phi(B) - \phi(A)$, for all points A and B.

Example

Check that

$$\int_{(0,0,0)}^{(1,1,1)} \mathbf{F}\cdot d\mathbf{r}$$

is independent of the path, where $\mathbf{F} = (yz+1)\mathbf{i} + (xz+1)\mathbf{j} + (xy+1)\mathbf{k}$.

Evaluate the integral by finding a suitable potential function ϕ.

Firstly,

$$\boldsymbol{\nabla}\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz+1 & xz+1 & xy+1 \end{vmatrix} = \begin{pmatrix} x-x \\ -y+y \\ z-z \end{pmatrix} = \mathbf{0}.$$

Hence, $\displaystyle\int_{(0,0,0)}^{(1,1,1)} \mathbf{F}\cdot d\mathbf{r}$ will indeed be independent of the path.

Now we have to find a scalar (potential) function ϕ such that $\mathbf{F} = \boldsymbol{\nabla}\phi$. We have that

$$\begin{cases} \dfrac{\partial\phi}{\partial x} = yz+1 & (1) \\[2ex] \dfrac{\partial\phi}{\partial y} = xz+1 & (2) \\[2ex] \dfrac{\partial\phi}{\partial z} = xy+1 & (3) \end{cases}$$

First, integrate (1) with respect to x:

$$\phi(x,y,z) = xyz + x + \underbrace{\phi_1(y,z)}_{\text{constant wrt } x}\,. \qquad (4)$$

Now differentiate (4) with respect to y and equate with (2):

$$\frac{\partial\phi}{\partial y} = xz + \frac{\partial\phi_1}{\partial y} \overset{(2)}{=} xz+1 \quad \Rightarrow \quad \frac{\partial\phi_1}{\partial y} = 1\,.$$

Hence,

$$\phi_1(y,z) = y + \phi_2(z) \qquad \Rightarrow \phi(x,y,z) = xyz + x + y + \phi_2(z)\,. \qquad (5)$$

Finally, differentiate (5) with respect to z and equate with (3):

$$\frac{\partial\phi}{\partial z} = xy + \frac{\partial\phi_2}{\partial z} = xy+1 \qquad \Rightarrow \phi_2(z) = z + C.$$

We can again take $C = 0$ without loss of generality so we have

$$\phi(x, y, z) = xyz + x + y + z\,.$$

(Check that we do indeed have $\mathbf{F} = \boldsymbol{\nabla}\phi$!)

Using this potential function ϕ, we find that

$$\int_{(0,0,0)}^{(1,1,1)} \mathbf{F} \cdot d\mathbf{r} = \phi(1,1,1) - \phi(0,0,0) = 4 - 0 = 4\,.$$

As an alternative solution, consider a simple path from $(0, 0, 0)$ to $(1, 1, 1)$, namely:

$$x = y = z = t \quad (0 \le t \le 1)$$

$$\Rightarrow \mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k} \qquad \Rightarrow \quad \frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

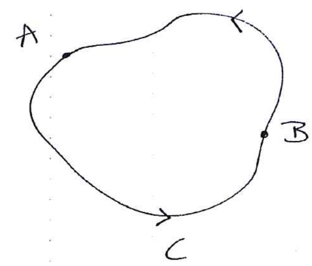
$$\Rightarrow \mathbf{F}(\mathbf{r}(t)) = (t^2+1)\mathbf{i} + (t^2+1)\mathbf{j} + (t^2+1)\mathbf{k} \qquad \Rightarrow \quad \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 3(t^2+1)$$

$$\Rightarrow \int_{(0,0,0)}^{(1,1,1)} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (3t^2+3)dt = \left[t^3 + 3t\right]_0^1 = 4\,.$$

The same answer, of course!

4.7 Application to Closed Integrals

For closed curves, we have of course that the starting point equals the end point.

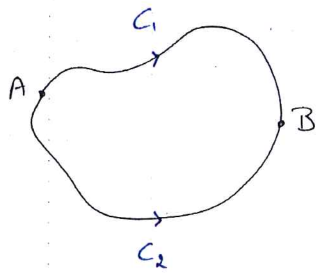


Consider

$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

where $\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}$.

Take any 2 points A and B on C:



C_1: upper path from A to B (open curve)

C_2: lower path from A to B

Since $\boldsymbol{\nabla} \times \mathbf{F} = 0$, the line integrals are independent of the path, hence

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}\,.$$

On the other hand,

$$\oint_C \mathbf{F} \cdot \mathbf{dr} = \underbrace{\int_{C_1} \mathbf{F} \cdot \mathbf{dr}}_{\text{from A to B}} - \underbrace{\int_{C_2} \mathbf{F} \cdot \mathbf{dr}}_{\text{from B to A}} = 0\,.$$

The value of the line integral of a conservative vector function around a closed curve C is zero.

Note that the result is in fact trivial. Since $\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}$, we know that there exists a scalar (potential) function ϕ such that

$$\mathbf{F} = \boldsymbol{\nabla}\phi \Rightarrow \oint_C \mathbf{F} \cdot \mathbf{dr} = \int_A^A \mathbf{F} \cdot \mathbf{dr} = \phi(A) - \phi(A) = 0.$$

Summary

The line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of the path between two fixed points A and B if $\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}$.

For such a (conservative) vector function $\mathbf{F}$, there exists a scalar (potential) function ϕ such that $\mathbf{F} = \boldsymbol{\nabla}\phi$ and

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \boldsymbol{\nabla}\phi \cdot \mathbf{dr} = \int_C d\phi = \phi(B) - \phi(A)\,.$$

If C is a closed curve, then $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ (for all closed curves C).