MT5823 Semigroup theory: Problem sheet 4 (James D. Mitchell)
Congruences and presentations

Congruences

4-1. Prove that every equivalence relation on a semigroup of left zeros is a congruence. Find an equivalence relation on a rectangular band which is not a congruence.

4-2. Let S be a semigroup, and let ρ be a congruence on S. Prove that if $e \in S$ is an idempotent, then its equivalence class e/ρ is a subsemigroup of S, and e/ρ is an idempotent in the quotient S/ρ. Also, prove that if S is finite and x/ρ is an idempotent of S/ρ, then x/ρ contains an idempotent.

4-3. Let $S = I \times \Lambda$ be a rectangular band where $|I| = |\Lambda| = 2$.

(a) Show that if ρ is a congruence on S and $(1,1)\rho(2,2)$, then $\rho = S \times S$;

(b) Describe the least congruence ρ on S such that $(1,1)\rho(1,2)$.

(c) Prove that S has 4 distinct congruences, and describe the quotient of S by each of these congruences.

4-4. Let ρ and σ be congruences on a semigroup S such that $\rho \subseteq \sigma$. Prove that

$$\sigma/\rho = \{\, (x/\rho, y/\rho) \in S/\rho \times S/\rho \ :\ (x,y) \in \sigma \,\}$$

is a congruence on S/ρ and that $(S/\rho)/(\sigma/\rho) \cong S/\sigma$.

Presentations

4-5. Let S be the semigroup defined by the presentation

$$\langle a, b \mid a^3 = a,\ b^4 = b,\ ba = a^2 b\rangle.$$

Prove that S has order 11. Find the idempotents of S. Draw the right Cayley graph of S.

4-6. Let S be the semigroup defined by the presentation

$$\langle a, b, 0 \mid a^2 = b^2 = 0,\ aba = a,\ bab = b,\ 0^2 = 0, 0a = a0 = b0 = 0b = 0\rangle.$$

Prove that S has order 5. Write down the multiplication table for S. Draw the left and right Cayley graphs of S.

4-7. Let S be the semigroup defined by the presentation

$$\langle a_1, \ldots, a_n \mid a_1a_2 = a_1,\ a_2a_3 = a_2, \ldots, a_{n-1}a_n = a_{n-1}, a_na_1 = a_n\rangle.$$

Prove that $a_ia_j = a_i$ for any i and j, and so S is the semigroup of left zeros of order n.

4-8. Consider the monoid S defined by the presentation $\langle x, y \mid xyx = 1\rangle$. Prove that $xy = yx$ holds in S. Prove that every element of S is equal to one of x^i, y^j, xy^j ($i \geq 0$, $j \geq 1$). Find two integers which generate the additive (semi)group $\mathbb{Z}$ and satisfy $2p + q = 0$. Prove that $S \cong \mathbb{Z}$.