MT5823 SUMMARY

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Semigroups

- **The empty semigroup:** the semigroup with 0 element;
- **The trivial semigroup:** the semigroup with 1 element;
- **Full transformation monoids:** the set of all transformations from a set X to X with composition of mappings; denoted by T_X or T_n if $|X| = n$:
  - $|T_n| = n^n$ (Problem **1-2**);
  - T_n can be generated by 3 mappings (Proposition 2.12) ;
  - $S_n \leq T_n$ where S_n denotes the symmetric group on $\{1, \ldots, n\}$;
  - every semigroup is isomorphic to a subsemigroup of some T_X (Theorem 3.4, analogous to Cayley's Theorem for groups);
  - $f \mathscr{L} g$ if and only if $\operatorname{im}(f) = \operatorname{im}(g)$ (Theorem 9.4);
  - $f \mathscr{R} g$ if and only if $\ker(f) = \ker(g)$ (Theorem 9.4);
  - $f \mathscr{H} g$ if and only if $\operatorname{im}(f) = \operatorname{im}(g)$ and $\ker(f) = \ker(g)$;
  - $f \mathscr{D} g$ if and only if $\operatorname{rank}(f) = \operatorname{rank}(g)$ (Theorem 9.17);
  - let D_r denote the $\mathscr{D}$-class of T_n of all those elements with rank r. The number of $\mathscr{L}$-classes in D_r is $\binom{n}{r}$, the number of $\mathscr{R}$-classes is $S(n, r)$ and the size of an $\mathscr{H}$-class is $r!$ (Problem **7-2**);
  - $|D_r| = \binom{n}{r} S(n, r)\, r!$ (Problem **7-2**);
  - T_n is regular but not inverse.
- **Partial transformation monoids:** the set of all partial mappings from a set X to X with composition of mappings; denoted by P_X or P_n if $|X| = n$:
  - $|P_n| = n^{n+1}$;
  - $T_n \leq P_n$;
  - $f \mathscr{L} g$ if and only if $\operatorname{im}(f) = \operatorname{im}(g)$ (Problem **7-4**);
  - $f \mathscr{R} g$ if and only if $\ker(f) = \ker(g)$ (Problem **7-4**);
  - $f \mathscr{H} g$ if and only if $\operatorname{im}(f) = \operatorname{im}(g)$ and $\ker(f) = \ker(g)$ (Problem **7-4**);
  - $f \mathscr{D} g$ if and only if $\operatorname{rank}(f) = \operatorname{rank}(g)$ (Problem **7-4**).
- **Free semigroups:** the set of all non-empty words over an alphabet A with concatenation is called the ***free semigroup*** and denoted by A^+:
  - every semigroup is a homomorphic image of a free semigroup (Theorem 3.9);
  - every semigroup is isomorphic to a quotient of a free semigroup (Theorem 3.9 and the First Isomorphism Theorem (Theorem 5.4));
  - the free semigroup over the alphabet A is defined by the presentation $\langle A | \, \rangle$.
- **Semigroup of left zeros:** any set X with multiplication $xy = x$ for all $x, y \in X$;
- **Semigroup of right zeros:** any set X with multiplication $xy = y$ for all $x, y \in X$;
- **Zero semigroup:** the set $X \cup \{0\}$ where $xy = 0$ for all $x, y \in X \cup \{0\}$;
- **Monoid:** a semigroup with an identity;
- **Group:** a monoid in which every element has a unique (group) inverse:
  - a semigroup S is a group if and only if it is non-empty and $aS = Sa = S$ for all $a \in S$ (Problems **1-7** and **1-8**).
- **Subsemigroup:** any nonempty subset T of a semigroup S closed under multiplication;
- **Rectangular bands:** the set of pairs $I \times \Lambda$ with multiplication $(i, \lambda)(j, \mu) = (i, \mu)$:
  - S is a rectangular band if and only if $x^2 = x$ and $xyz = xz$ holds for all $x, y \in S$ (Problem **2-1** and Problem **3-9**);
  - $I \times \Lambda$ is a left zero semigroup if and only if $|\Lambda| = 1$ (Problem **2-2**);
  - $I \times \Lambda$ is a right zero semigroup if and only if $|I| = 1$;
  - every subsemigroup and every quotient of a rectangular band is a rectangular band.

- **Monoids of binary relations:** the set of all binary relations on a set X with composition of relations, denoted B_X and it has $2^{(n^2)}$ elements (Problem **3-6**);
- **Monogenic semigroup:** a semigroup S generated by a single element a, in fewer words, $S = \langle a \rangle = \{a^i : i \in \mathbb{N},\ i > 0\}$;
- **Bicyclic monoid:** the monoid B defined by the presentation $\langle b, c | bc = 1\rangle$:
  - the elements of B are $\{c^ib^j : i, j \geq 0\}$ (Example 6.8);
  - c^ib^j is an idempotent if and only if $i = j$ (Problem **5-1**);
  - $c^ib^j \mathscr{R} c^kb^l$ if and only if $j = l$; (Problem **5-7**)
  - $c^ib^j \mathscr{L} c^kb^l$ if and only if $i = k$ (Problem **5-7**);
  - $c^ib^j \mathscr{H} c^kb^l$ if and only if $c^ib^j = c^kb^l$ (Problem **5-10**);
  - $\mathscr{J} = \mathscr{D} = B \times B$ (Problem **5-10**).
  - B is an inverse monoid.
- **Commutative:** a semigroup S where $xy = yx$ for all $x, y \in S$;
- **Semilattices:** a commutative semigroup of idempotents;
- **Free semilattice:** the set of subsets of a set X with the usual union of sets;
- **Cancellative semigroup:** a semigroup S satisfying $ax = ay \Rightarrow x = y$ and $xa = ya \Rightarrow x = y$ for all $a, x, y \in S$;
- **Periodic semigroup:** a semigroup S where $\langle x \rangle$ is finite for all $x \in S$
- **Regular semigroup:** a semigroup where every element is regular:
  - every rectangular band is regular, and so it the bicyclic monoid;
  - a semigroup is regular if every $\mathscr{D}$-class contains a regular element;
  - a semigroup is regular if every $\mathscr{D}$-class contains an idempotent.
- **Inverse semigroup:** a semigroup S where every element has a unique (semigroup) inverse (Definition 12.1)
  - every group is an inverse semigroup;
  - every semilattice is an inverse semigroup;
  - if $x \in S$ and x^{-1} denotes its inverse, then $xx^{-1}x = x$, $x^{-1}xx^{-1} = x^{-1}$, $(x^{-1})^{-1} = x$, $x^2 = x$ implies that $x = x^{-1}$ and $(xx^{-1})^2 = xx^{-1}$;
  - S is inverse if and only if it is regular and its idempotents commute (Theorem 12.2(b));
  - S is inverse if and only if every $\mathscr{L}$-class and every $\mathscr{R}$-class of S contains exactly 1 idempotent (Theorem 12.2(iii)).
- **The symmetric inverse monoid:** the set of all partial bijections on a set X with composition of mappings; denoted by I_X or I_n if $|X| = n$:
  - I_X is an inverse semigroup (Example 12.5);
  - every inverse semigroup is isomorphic to an inverse subsemigroup of a symmetric inverse semigroup I_X (the Vagner-Preston Representation Theorem (Theorem 13.1));
  - not every subsemigroup of I_X is an inverse semigroup.

Distinguished elements of a semigroup

Throughout assume that S is a semigroup.

- **Left zero:** an element $s \in S$ such that $st = s$ for all $t \in S$ (Definition 1.6);
- **Right zero:** an element $s \in S$ such that $ts = s$ for all $t \in S$ (Definition 1.6);
- **Zero:** both a left and a right zero (Definition 1.6):
  - a semigroup has at most one zero (Problem **1-1**(b)).
- **Idempotent:** an element $e \in S$ satisfying $e^2 = e$ (Definition 1.19):
  - if S is finite, then every $s \in S$ has an idempotent power (Problem **2-10**);
  - an idempotent is a left identity in its $\mathscr{R}$-class and a right identity in its $\mathscr{L}$-class (Problem **5-8**);
  - if an $\mathscr{H}$-class H contains an idempotent, then H is a group (Theorem 10.7);
  - if a $\mathscr{D}$-class D contains an idempotent, then D is regular;
  - in a regular D-class every $\mathscr{L}$-class and every $\mathscr{R}$-class contains at least one idempotent (Theorem 12.2(c));
  - a transformation $f \in T_n$ is an idempotent if and only if $xf = x$ for all $x \in \mathrm{im}(f)$;
- **Left identity:** an element $e \in S$ such that $es = s$ for all $s \in S$ (Definition 1.12);
- **Right identity:** an element $e \in S$ such that $se = s$ for all $s \in S$ (Definition 1.12);
- **Identity:** an element that is both a left identity and a right identity (Definition 1.12)

  - a semigroup has at most one identity (Lemma 1.16);
  - if a semigroup has both a left and a right identity, then it is a monoid (Problem **1-1**(a)).
- **Regular:** an element $x \in S$ where there exists $y \in S$ such that $xyx = x$ (Definition 11.1):
  - if x is regular and $x\mathscr{D}y$, then y is regular (Theorem 11.3);
- **Inverse:** elements $x, y \in S$ are inverse if $xyx = x$ and $yxy = y$ (Definition 11.4):
  - $x \in S$ has an inverse if and only if x is regular (Theorem 11.5);
  - inverses are not necessarily unique (Problem **7-1**(b)).

Green's relations

- **Green's $\mathscr{L}$-relation:** $x, y \in S$ are $\mathscr{L}$-related if and only if $S^1x = S^1y$ (Definition 9.1)
  - denoted $x\mathscr{L}y$;
  - $x\mathscr{L}y$ if and only if there exist $s, t \in S^1$ such that $sx = y$ and $ty = x$ (Lemma 9.2);
  - $\mathscr{L}$ is a right congruence over S (Theorem 9.5) but not always a left congruence (Example 9.6);
  - $x\mathscr{L}y$ if and only if x and y are in the same strongly connected component of the left Cayley graph of S (Theorem 9.10);
  - $\mathscr{L} \subseteq \mathscr{D}$;
  - if L_1 and L_2 are $\mathscr{L}$-classes in the same $\mathscr{D}$-class, then $|L_1| = |L_2|$ (Corollary 10.4).
- **Green's $\mathscr{R}$-relation:** $x, y \in S$ are $\mathscr{R}$-related if and only if $xS^1 = yS^1$ (Definition 9.1)
  - denoted $x\mathscr{R}y$;
  - $x\mathscr{R}y$ if and only if there exist $s, t \in S^1$ such that $xs = y$ and $yt = x$ (Lemma 9.2);
  - $\mathscr{R}$ is a left congruence over S (Theorem 9.5) but not always a right congruence;
  - $x\mathscr{R}y$ if and only if x and y are in the same strongly connected component of the right Cayley graph of S (Theorem 9.10);
  - $\mathscr{R} \subseteq \mathscr{D}$;
  - if R_1 and R_2 are $\mathscr{R}$-classes in the same $\mathscr{D}$-class, then $|R_1| = |R_2|$ (Corollary 10.4).
- **Green's $\mathscr{H}$-relation:** $\mathscr{H} = \mathscr{L} \cap \mathscr{R}$ (Definition 9.12)
  - denoted $x\mathscr{H}y$;
  - $\mathscr{H}$ is neither a left nor a right congruence in general;
  - $x\mathscr{H}y$ if and only if x and y are in the same strongly connected component of the right and left Cayley graphs of S;
  - $\mathscr{H} \subseteq \mathscr{R}, \mathscr{L}$;
  - if H_1 and H_2 are $\mathscr{H}$-classes in the same $\mathscr{D}$-class, then $|H_1| = |H_2|$ (Corollary 10.6).
  - if H is an $\mathscr{H}$-class, then either $H^2 \cap H = \emptyset$ or H is a group (Theorem 10.7);
  - if H is an $\mathscr{H}$-class and $e \in H$ is an idempotent, then H is a group;
  - if H and K are group $\mathscr{H}$-classes of the same $\mathscr{D}$-class, then $H \cong K$ (Theorem 11.8).
- **Green's $\mathscr{D}$-relation:** $\mathscr{D} = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}$ (Definition 9.14 and Theorem 9.15)
  - denoted $x\mathscr{D}y$;
  - $x\mathscr{D}y$ if and only if there exists $z \in S$ (not S^1) such that $x\mathscr{L}z$ and $z\mathscr{R}y$;
  - if S is periodic, then $\mathscr{D} = \mathscr{J}$. In particular, if S is finite, then $\mathscr{D} = \mathscr{J}$;
  - any two $\mathscr{L}$-, $\mathscr{R}$-, or $\mathscr{H}$-classes of a $\mathscr{D}$-class have the same size (Corollaries 10.4 and 10.6);
  - a $\mathscr{D}$-class D is regular if every element in D is regular;
  - If a is a regular element of a semigroup S and $a\mathscr{D}b$, then b is regular. (Theorem 11.3);
  - if D is a regular $\mathscr{D}$-class, then every $\mathscr{R}$-class and $\mathscr{L}$-class of D contains at least one idempotent (Theorem 11.7).

Miscellaneous

- **Homomorphism:** If S and T are semigroup, then $\phi : S \to T$ is a ***homomorphism*** if $(x)\phi(y)\phi = (xy)\phi$ for all $x, y \in S$;
- **Isomorphism:** a bijective homomorphism;
- **Homomorphic image:** T is a homomorphic image of S if there exists a surjective homomorphism $\phi : S \to T$ (Definition 3.1) ;
- **Equivalence relation:** a reflexive, symmetric, and transitive binary relation (Definition 4.6);
- **Partial order:** a reflexive, antisymmetric and transitive binary relation (Definition 4.6);
- **Left congruence:** An equivalence relation ρ over S such that $(sx, sy) \in \rho$ whenever $(x, y) \in \rho$ for all $s \in S$ (Definition 5.1);

- **Right congruence:** An equivalence relation ρ over S such that $(xs, ys) \in \rho$ whenever $(x, y) \in \rho$ for all $s \in S$ (Definition 5.1);
- **Congruence:** both a left and a right congruence (Definition 5.1);
- **Quotient:** the set of equivalence classes of a congruence ρ with multiplication $(x/\rho)(y/\rho) = (xy)/\rho$ (Theorem 5.3);
- **Presentation:** A (semigroup) presentation is a pair $\langle A|R\rangle$ where A is an alphabet and $R \subseteq A^+ \times A^+$ is a set of relations on A^+ (Definition 6.2);
- **Semigroup defined by a presentation $\langle A|R\rangle$:** is any semigroup isomorphic to A^+/ρ where ρ is the least congruence containing R (Definition 6.2);
- **Left ideal:** a subset I of a semigroup S such that $si \in I$ for all $s \in S$ and $i \in I$ (Definition 8.1);
- **Right ideal:** a subset I of a semigroup S such that $is \in I$ for all $s \in S$ and $i \in I$ (Definition 8.1):
  - the intersection of a non-empty right and a non-empty left ideal is always non-empty (Problem **5-3**);
  - the intersection of two left ideals can be empty (Problem **5-3**).
- **(2-sided) ideal:** both a left and a right ideal (Definition 8.1);
- **Rees equivalence:** Let S be a semigroup and I be an ideal of S. Then the relation $\rho_I = (I \times I) \cup \Delta_S = \{(x, y) : x, y \in I \text{ or } x = y\}$ is the Rees equivalence of I
  - ρ_I is a congruence (Theorem 8.4);
  - ρ_I has equivalence classes I and $\{s\}$ for all $s \in S \setminus I$;
  - ρ_I has $|S \setminus I| + 1$ equivalence classes.
- **Rees quotient of S by I:** the quotient of S by ρ_I where I is an ideal; denoted S/I (Definition 8.5);
  - S/I is isomorphic to the semigroup with elements $(S/I) \cup \{0\}$ and multiplication $*$ defined by

$$x * y = \begin{cases} xy & xy \notin I \\ 0 & xy \in I; \end{cases}$$

- **Cayley graph:** The ***left Cayley graph*** of a semigroup $S = \langle X\rangle$ is a graph with vertices S and directed edge (a, b) labelled x if $xa = b$. The ***right Cayley graph*** is defined analogously.