University of St Andrews

MAY 2010 EXAMINATION DIET
SCHOOL OF MATHEMATICS & STATISTICS

MODULE CODE:	**MT5823**
MODULE TITLE:	Semigroup Theory
EXAM DURATION:	2 hours
EXAM INSTRUCTIONS	Attempt ALL questions.

The number in square brackets shows the maximum marks obtainable for that question or part-question.

Your answers should contain the full working required to justify your solutions.

PLEASE DO NOT TURN OVER THIS EXAM PAPER UNTIL YOU ARE INSTRUCTED TO DO SO.

1. Let S be the semigroup generated by the transformations

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 6 & 7 & 4 & 4 & 6 & 7 & 5 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 1 & 4 & 5 & 6 & 7 & 3 \end{pmatrix}.$$

(a) List the elements of S. Prove that S is not a monoid and that S has 4 idempotents. [4]

(b) Prove that the set of idempotents of S forms a subsemigroup of S. [3]

(c) State (without proof) the Vagner representation theorem for inverse semigroups. Is it true that every subsemigroup of the semigroup I_X of all partial bijections on X is inverse? [3]

(d) Is S an inverse semigroup? How many $\mathcal{R}$-classes does S have? How many $\mathcal{L}$-classes does S have? Justify your answers. [5]

(e) Define a simple semigroup and a Clifford semigroup. Prove that S is neither simple nor Clifford. [3]

2. Let $S = \mathcal{M}[T; I, \Lambda; P]$ be a Rees matrix semigroup over a semigroup T. Recall that this means that S is the set $I \times T \times \Lambda$ with multiplication

$$(i, t, \lambda)(j, u, \mu) = (i, tp_{\lambda j}u, \mu),$$

where $P = (p_{\lambda i})_{\lambda \in \Lambda, i \in I}$ is a matrix with entries from T.

(a) Prove that if T is simple, then S is simple as well. Conclude that S is simple when T is a group. [4]

(b) Prove that if S is regular, then so is T. [4]

(c) Let $T = \{a, b, c, d\}$ be the semigroup with multiplication table

	a	b	c	d
a	a	a	a	a
b	a	b	a	d
c	c	c	c	c
d	d	d	d	d

Prove that T is regular. Find a sandwich matrix $P = (p_{\lambda i})_{\lambda \in \Lambda, i \in I}$ with entries in T such that $\mathcal{M}[T; I, \Lambda; P]$ is not regular. [Hint: Try a 1×1 matrix.] [4]

(d) Prove that if T is regular, then the element $(i, x, \lambda) \in S$ is regular if and only if there exist $j \in I$ and $\mu \in \Lambda$ such that the set $p_{\lambda j} T p_{\mu i}$ contains an inverse of x. [4]

3. Let S be a band. Recall that this means that every element of S is an idempotent, that is, $x^2 = x$ for all $x \in S$.

You may use the following facts about S without proof: if $x, y \in S$, then

$$x\mathcal{L}y \text{ if and only if } Sx = Sy,$$

$$x\mathcal{R}y \text{ if and only if } xS = yS,$$

$$x\mathcal{D}y \text{ if and only if } SxS = SyS.$$

(a) Prove that every $\mathcal{H}$-class of S has precisely one element. [2]

(b) Show that $\mathcal{D} = \mathcal{J}$ on S. [2]

(c) Prove that the following are equivalent:

(i) $sts = st$ for all $s, t \in S$;
(ii) $xS = xSx$ for all $x \in S$;
(iii) $Sx = SxS$ for all $x \in S$;
(iv) $\mathcal{D} = \mathcal{L}$ on S;
(v) every $\mathcal{R}$-class of S has 1 element;

(vi) the binary relation ρ defined by

$$(s, t) \in \rho \text{ if and only if } st = t$$

is a partial order.

[Hints: (v) $\Rightarrow$ (vi) Recall that ρ is a partial order if it is reflexive, antisymmetric, and transitive.
(vi) $\Rightarrow$ (i) Use the antisymmetry of ρ.] **[12]**