① Let $g \in PSL(2,\mathbb{R})$ be written as

$$g(z) = \frac{az+b}{cz+d}$$ where $a,b,c,d \in \mathbb{R}$ with $ad-bc$.

We know from tutorial sheet 3, question 3a) that

$$\phi g \phi^{-1}(z) = \frac{\frac{1}{2}(a+d+i(b-c))z + \overline{\frac{1}{2}(a-d+i(b+c))}}{\frac{1}{2}(a-d+i(b+c))z + \overline{\frac{1}{2}(a+d+i(b-c))}}$$

$\in$ ~~PSL~~ $Con^+(1)$

and that this is in standard form.

Hence

$$tr(\phi g \phi^{-1}) = \frac{1}{2}(a+d) + \frac{1}{2}(a+d)$$

$$= a+d$$

$$= tr(g)$$

as required.

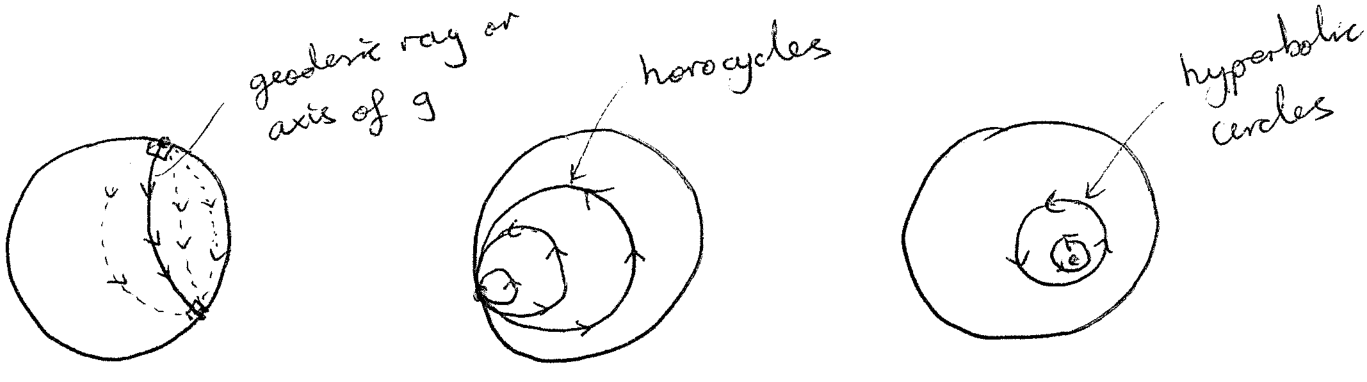
① continued... therefore we have the following classification:

a non-identity element $g \in \mathrm{con}^{+}(1)$ is

1) Hyperbolic $\Leftrightarrow tr(g)^2 > 4 \Leftrightarrow$ g has precisely 2 fixed points, both in S^1.

2) parabolic $\Leftrightarrow tr(g)^2 = 4 \Leftrightarrow$ g has precisely 1 fixed point, which lies in S^1.

3) elliptic $\Leftrightarrow tr(g)^2 < 4 \Leftrightarrow$ g has precisely 2 fixed points, one in $\mathbb{D}^2$ and one in $\hat{\mathbb{C}} \setminus \{\mathbb{D}^2 \cup S^1\}$.



hyperbolic parabolic elliptic

(2) This question is open ended and designed just to make you think and explore.

Important relations you may have come across ~~are~~ include

$$tr(ghg^{-1}) = \pm tr(h), \quad tr(gh) = \pm tr(hg)$$

$$tr(g) = \pm tr(g^{-1}).$$

In particular, if we write elements in "standard form" such that $tr(g) \geq 0$, then

$$tr(ghg^{-1}) = tr(h), \quad tr(gh) = tr(hg).$$

$$tr(g) = tr(g^{-1})$$

Therefore, conjugation & inversion do not alter the "type" of an element.

③ (i) Suppose $g \in PSL(2,\mathbb{R})$ is given by $g(z) = \dfrac{az+b}{cz+d}$ and

$$g(\infty) = \frac{a}{c} = \infty, \quad g(0) = \frac{b}{d} = 0.$$

We deduce that $c = b = 0$ and so

$$g(z) = \frac{a}{d} z.$$

Moreover, $\dfrac{a}{d} = a^2 \geq 0$ since $ad - bc = ad = 1$

and $a^2 = 1 \iff g$ is the identity.

The result follows by setting $\alpha = \dfrac{a}{d}$

(ii) Suppose parabolic $g \in PSL(2,\mathbb{R})$ is given by $g(z) = \dfrac{az+b}{cz+d}$ and

$g(\infty) = \infty$. We deduce that $c = 0$ as before and so

$g(z) = \dfrac{a}{d} z + \dfrac{b}{d}$. Suppose $\dfrac{a}{d} \neq 1$. Then

$\dfrac{-b/d}{a/d - 1} \neq \infty$ is a second fixed point, which contradicts g being parabolic.

(4) (i) $\underline{\text{Fix}(hgh^{-1}) \subseteq h(\text{Fix}(g))}$

Let $z \in \text{Fix}(hgh^{-1})$. Then $hgh^{-1}(z) = z$ and so $gh^{-1}(z) = h^{-1}(z)$ and so

$h^{-1}(z) \in \text{Fix}(g)$ and therefore

$$z = h(h^{-1}(z)) \in h(\text{Fix}(g)).$$

(ii) $\underline{h(\text{Fix}(g)) \subseteq \text{Fix}(hgh^{-1})}$

Let $z \in \text{Fix}(g)$. Then $g(z) = z$ and

$$hgh^{-1}(h(z)) = hg(z)$$
$$= h(z)$$

and so $h(z) \in \text{Fix}(hgh^{-1})$.

Since every element of $h(\text{Fix}(g))$ can be expressed as $h(z)$ for some $z \in \text{Fix}(g)$, the result follows.

Thus we have given a second proof that conjugation does not change the 'type' of an element.

(4) cont...

In particular

(i) g hyperbolic $\iff |\mathrm{Fix}(g) \cap \mathbb{R} \cup \{\infty\}| = 2$

and since $h \in PSL(2, \mathbb{R})$ preserves the boundary

$$|\mathrm{Fix}(g) \cap \mathbb{R} \cup \{\infty\}|$$
$$= |\mathrm{Fix}(hgh^{-1}) \cap \mathbb{R} \cup \{\infty\}|$$

(ii) g parabolic $\iff |\mathrm{Fix}(g)| = 1$

(iii) g elliptic $\iff |\mathrm{Fix}(g) \cap \mathbb{H}^2| = 1$

and since $h \in PSL(2, \mathbb{R})$ preserves $\mathbb{H}^2$

$$|\mathrm{Fix}(g) \cap \mathbb{H}^2| = |\mathrm{Fix}(hgh^{-1}) \cap \mathbb{H}^2|.$$

(5) Let $\lambda > 0$ and g be defined by

$$g(z) = \frac{z}{\lambda z + 1}$$

(i) $$g(z) = \frac{1 \times z + 0}{\lambda \times z + 1} \in PSL(2, \mathbb{R})$$

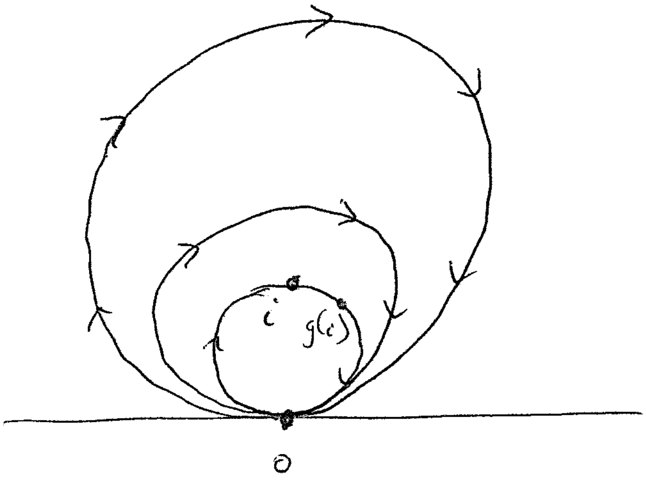
since $1 \times 1 - 0 \times \lambda = 1$

(ii) $tr(g)^2 = (1+1)^2 = 4$

and so g is parabolic

(iii) Suppose $g(z) = \dfrac{z}{\lambda z + 1} = z$.

$z = 0$ is a solution & since g is parabolic it is the only solution.



to test the 'direction' of the action

$$g(i) = \frac{i}{\lambda i + 1} = \frac{\lambda}{1+\lambda^2} + \frac{i}{1+\lambda^2}$$

and so i is moved clockwise round the horocycle!

(6) Let $g \in PSL(2,\mathbb{R})$ be parabolic. We have already seen in lectures that we can conjugate g to the map $z \mapsto z+\beta$. That is, for some $h_1 \in PSL(2,\mathbb{R})$

$$h_1 g h_1^{-1}(z) = z+\beta \text{ for some } \beta \in \mathbb{R} \ (\beta \neq 0).$$

Let $h_2 \in PSL(2,\mathbb{R})$ be given by

$$h_2(z) = \alpha z \quad \text{for some} \quad \alpha > 0.$$

Then

$$\begin{aligned} & h_2(h_1 g h_1^{-1}) h_2^{-1}(z) \\ = \ & h_2\left(h_1 g h_1^{-1}\left(\frac{z}{\alpha}\right)\right) \\ = \ & h_2\left(\frac{z}{\alpha}+\beta\right) \\ = \ & z + \alpha\beta. \end{aligned}$$

Therefore if we choose $\alpha = \frac{1}{|\beta|}$ then

$$(h_2 h_1) g (h_2 h_1)^{-1} = h_2 h g h_1 h_2^{-1}$$

does the job!