Chapter 2
Homotopy Theory: Elementary Basic Concepts

This chapter opens with a study of homotopy theory by introducing its elementary basic concepts such as homotopy of continuous maps, homotopy equivalence, H-group, H-cogroup, contractible space, retraction, deformation with illustrative geometrical examples and applications. The study of homotopy theory continues explicitly up to Chap. 9 of the present book. Its many key concepts are also applied to other chapters. The basic aim of homotopy theory is to investigate 'algebraic principles' latent in homotopy equivalent spaces. Such principles are also important in the study of topology and geometry as well as in many other subjects such as algebra, algebraic geometry, number theory, theoretical physics, chemistry, computer science, economics, bioscience, medical science, and some other subjects.

Algebraic topology flows mainly through two channels: one is the homotopy theory and other one is the homology theory. The concept of homotopy is a mathematical formulation of the intuitive idea of a continuous deformation from one geometrical configuration to other in the sense that this concept formalizes the naive idea of continuous deformation of a continuous map. On the other hand, the concept of homology is a mathematical precision to the intuitive idea of a curve bounding an area or a surface bounding a volume. Cohomology theory which is a dual concept of homology theory is also closely related to homotopy theory. The idea guiding the development of mathematical theory of homotopy, homology, and cohomology is described nowadays in the language of category theory by constructing certain functors.

Algebraic topology is one of the most important creations in mathematics which uses algebraic tools to study topological spaces. The basic goal is to find algebraic invariants that classify topological spaces up to homeomorphism (though usually classify up to homotopy equivalence). The most important of these invariants are homotopy, homology, and cohomology groups. This subject is an interplay between topology and algebra and studies algebraic invariants provided by homotopy and homology theories. The twentieth century witnessed its greatest development.

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A basic problem in homotopy theory is to classify continuous maps up to homotopy: two continuous maps from one topological space to other are homotopic if one map can be continuously deformed into the other map. On the other hand, the basic problem in algebraic topology is to devise ways to assign various algebraic objects such as groups, rings, modules to topological spaces and homomorphisms to the corresponding algebraic structures in a functorial way. More precisely, although the ultimate aim of topology is to classify topological spaces up to homeomorphism, the main problem of algebraic topology is the 'classification' of topological spaces up to homotopy equivalence, the concept introduced by W. Hurewicz (1904–1956) in 1935. So in algebraic topology a homotopy equivalence plays a more influential role than a homeomorphism, because the basic tools of algebraic topology such as homotopy groups, and homology & cohomology groups are invariants with respect to homotopy equivalence.

Homotopy theory constitutes a basic part of algebraic topology and studies topological spaces up to homotopy equivalence which is a weaker relation than topological equivalence in the sense that homotopy classes of spaces are larger than homeomorphic classes. The concept of the homotopy equivalence gives rise to the classification of topological spaces according to their homotopy properties. The basic idea of this classification is to assign to each topological space 'invariants', which may be integers, or algebraic objects in such a way that homotopy equivalent spaces have the same invariants (up to isomorphism), called homotopy invariants, which characterize homotopy equivalent spaces completely. The main numerical invariants of homotopy equivalent spaces are dimensions and degrees of connectedness.

Historically, the idea of homotopy for the continuous maps of unit interval was originated by C. Jordan (1838–1922) in 1866 and that of for loops was introduced by H. Poincaré (1854–1912) in 1895 to define an algebraic invariant called the fundamental group, which is studied, in Chap. 3. The monumental work of Poincaré in 'Analysis situs', Paris, 1895, organized the subject for the first time. This work explained the difference between curves deformable to one another and curves bounding a larger space. The first one led to the concepts of homotopy and fundamental group; the second one led to the concept of homology. Poincaré is the first mathematician who systemically attacked the problems of assigning algebraic (topological) invariants to topological spaces. His vision of the key role of topology in all mathematical theories began to materialize from 1920. Of course, many of the ideas he developed had their origins prior to him, with L. Euler (1707–1783), and B. Riemann (1826–1866) above all. H. Hopf (1894–1971) introduced the concept of H-spaces and H-groups from the viewpoint of homotopy theory. Some of his amazing results have made a strong foundation of algebraic topology. Many topologists regard H. Poincaré as the founder and regard H. Hopf and W. Hurewicz as cofounders of many key concepts in algebraic topology.

Throughout this book a space means a topological space and a map means a continuous function between topological spaces; the terms: map (or continuous map) and continuous function will be used interchangeably in the context of topological spaces, unless specified otherwise.

For this chapter the books Eilenberg and Steenrod (1952), Hatcher (2002), Maunder (1970), Spanier (1966) and some others are referred in Bibliography

2.1 Homotopy: Introductory Concepts and Examples

This section is devoted to the study of the concept of homotopy formalizing the intuitive idea of continuous deformation of a continuous map between two topological spaces and presents introductory basic concepts of homotopy with illustrative examples. Homeomorphism generates equivalence classes whose members are topological spaces. On the other hand, homotopy generates equivalence classes whose members are continuous maps. The term homotopy was first given by Max Dehn (1878–1952) and Poul Heegaard (1871–1948) in 1907. It is sometimes replaced by a complicated function between two topological spaces by another simpler function sharing some important properties of the original function. An allied concept is the notion of deformation. This leads to the concept of homotopy of functions.

The relation between topological spaces of being homeomorphic is an equivalence relation. So it divides any set of topological spaces into disjoint classes. The main problem of topology is the classification of topological spaces. Given two topological spaces X and Y, are they homeomorphic? This is a very difficult problem. Algebraic topology transforms such topological problems into algebraic problems which may have a better chance for solution. The algebraic techniques are usually not delicate enough to classify topological spaces up to homeomorphism. The notion of homotopy of continuous functions defines somewhat coarser classification. This leads to the concept of a continuous deformation. The relation of homotopy of continuous functions generalizes path connectedness of a point, which is a fundamental concept of homotopy theory.

2.1.1 Concept of Homotopy

The intuitive concept of a continuous deformation is now explained with the concept of homotopy. Moreover the concept of 'flow' which is also known as one parameter group of homeomorphisms is conveyed through homotopy. Let $I = [0, 1]$ be the closed unit interval with topology induced by the natural topology on the real line $\mathbf{R}$ (sometimes written as $\mathbf{R}^1$).

Definition 2.1.1 Let X and Y be topological spaces. Two continuous maps $f, g : X \to Y$ are said to be homotopic (or f is said to be homotopic to g), if there exists a continuous map $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$, $\forall\ x \in X$. The map F is said to be a homotopy between f and g, written $F : f \simeq g$.

Remark 2.1.2 Geometrically, two continuous maps $f, g : X \to Y$ are said to be homotopic if f can be continuously deformed into g by a continuous family of maps $F_t : X \to Y$ defined by $F_t(x) = F(x, t)$ such that $F_0 = f$ and $F_1 = g$, $\forall\ x \in X$, $\forall\ t \in I$. By saying that the maps F_t form a continuous family, we mean that F is continuous with respect to both x and t as a function from the product space $X \times I$ to Y. Clearly, a homotopy F between two continuous maps $f, g : X \to Y$ can be considered as a special case of extension:
consider in the topological space $X \times I$ the subspace

$$A = (X \times \{0\}) \cup (X \times \{1\}) \subset X \times I$$

and consider the continuous map

$$G : A \to Y, (x, 0) \mapsto f(x), (x, 1) \mapsto g(x);$$

then a homotopy F from f to g is an extension of G from A to $X \times I$.

Definition 2.1.3 Let X be a topological space. A continuous map $f : I \to X$ such that $f(0) = x_0$ and $f(1) = x_1$, is called a path in X from x_0 to x_1. The point x_0 is called the initial point and the point x_1 is called the final or terminal point of the path f.

Definition 2.1.4 Two paths $f, g : I \to X$ are said to be homotopic if they have the same initial point x_0, the same final point x_1 and there exists a continuous map $F : I \times I \to X$ such that

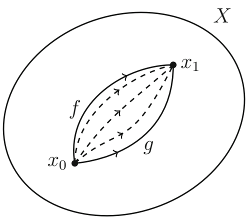
$$F(t, 0) = f(t), F(t, 1) = g(t), \forall\ t \in I \tag{2.1}$$

$$\text{and } F(0, s) = x_0 \text{ and } F(1, s) = x_1, \forall\ s \in I \tag{2.2}$$

We call F a path homotopy between f and g as shown in Fig. 2.1 and is written $F : f \underset{p}{\simeq} g$.

Remark 2.1.5 The condition (2.1) says that F is a homotopy between f and g and the condition (2.2) says that for each $t \in I$, the path $t \mapsto F(t, s)$ is a path in X from x_0 to x_1. In other words, (2.1) shows that F represents a continuous way of deforming

Fig. 2.1 Path homotopy



the path f to the path g and (2.2) shows that the end points of the path remain fixed during the deformation.

We now prove the following two lemmas of point set topology which will be used throughout the book.

Lemma 2.1.6 (Pasting or Gluing lemma) *Let X be a topological space and A, B be closed subsets of X such that $X = A \cup B$. Let Y be a topological space and $f : A \to Y$ and $g : B \to Y$ be continuous maps. If $f(x) = g(x),\ \forall\ x \in A \cap B$, then the function $h : X \to Y$ defined by*

$$h(x) = \begin{cases} f(x), & \forall\ x \in A \\ g(x), & \forall\ x \in B \end{cases}$$

is continuous.

Proof h defined in the lemma is the unique well-defined function $X \to Y$ such that $h|_A = f$ and $h|_B = g$. We now show that h is continuous. Let C be a closed set in Y. Then $h^{-1}(C) = X \cap h^{-1}(C) = (A \cup B) \cap h^{-1}(C) = (A \cap h^{-1}(C)) \cup (B \cap h^{-1}(C)) = (A \cap f^{-1}(C)) \cup (B \cap g^{-1}(C)) = f^{-1}(C) \cup g^{-1}(C)$. Since each of f and g is continuous, $f^{-1}(C)$ and $g^{-1}(C)$ are both closed in X. Hence $h^{-1}(C)$ is closed in X. Consequently, h is continuous. ❑

This lemma can be generalized as follows:

Lemma 2.1.7 (Generalized Pasting or Gluing lemma) *Let a topological space X be a finite union of closed subsets $X_i : X = \bigcup_{i=1}^{n} X_i$. If for some topological space Y, there are continuous maps $f_i : X_i \to Y$ that agree on overlaps (i.e., $f_i|_{X_i \cap X_j} = f_j|_{X_i \cap X_j},\ \forall\ i, j$), then $\exists$ a unique continuous function $f : X \to Y$ with $f|_{X_i} = f_i,\ \forall\ i$.*

Proof The proof is similar to proof of Lemma 2.1.6. ❑

Theorem 2.1.8 *Let $P(X)$ denote the set of all paths in a space X having the same initial point x_0 and the same final point x_1. Then the path homotopy relation '$\underset{p}{\simeq}$' is an equivalence relation on $P(X)$.*

Proof Let $f, g, h \in P(X)$. Then $f(0) = g(0) = h(0) = x_0$ and $f(1) = g(1) = h(1) = x_1$. Let a map $F : I \times I \to X$ be defined by $F(t, s) = f(t),\ \forall\ t, s \in I$. Then F is continuous, because it is the composite of the projection map onto the first factor and the continuous map f. Hence F is a continuous map such that $F(t, 0) = f(t)$, $F(t, 1) = f(t)$, $\forall\ t \in I$ and $F(0, s) = x_0$, $F(1, s) = x_1$, $\forall\ s \in I$. Thus $F : f \underset{p}{\simeq} f$, $\forall\ f \in P(X)$. Next, let $f \underset{p}{\simeq} g$ and $F : f \underset{p}{\simeq} g$. Then $F : I \times I \to X$ is a continuous map such that $F(t, 0) = f(t)$, $F(t, 1) = g(t)$, $\forall\ t \in I$ and $F(0, s) = x_0$, $F(1, s) = x_1$, $\forall\ s \in I$. Let $G : I \times I \to X$ be the map defined by $G(t, s) =$

$F(t, 1-s)$. Since the maps $I \to I, t \mapsto t$ and $s \mapsto 1-s$ are both continuous, G is continuous. Now $G(t, 0) = F(t, 1) = g(t)$, $G(t, 1) = F(t, 0) = f(t)$, $\forall\ t \in I$ and $G(0, s) = F(0, 1-s) = x_0$, $G(1, s) = F(1, 1-s) = x_1$. Hence $G : g \underset{p}{\simeq} f$.

Finally, let $f \underset{p}{\simeq} g$ and $g \underset{p}{\simeq} h$. Then $\exists$ continuous maps $F, G : I \times I \to X$ such that $F : f \underset{p}{\simeq} g$ and $G : g \underset{p}{\simeq} h$. Consequently, for all $t, s \in I$, $F(t, 0) = f(t)$, $F(t, 1) = g(t)$, $F(0, s) = x_0$, $F(1, s) = x_1$, $G(t, 0) = g(t)$, $G(t, 1) = h(t)$, $G(0, s) = x_0$ and $G(1, s) = x_1$. We now define a map $H : I \times I \to X$ by the equations

$$H(t, s) = \begin{cases} F(t, 2s), & 0 \leq s \leq 1/2 \\ G(t, 2s-1), & 1/2 \leq s \leq 1 \end{cases}$$

At $s = \frac{1}{2}$, $F(t, 2s) = F(t, 1) = g(t)$ and $G(t, 2s-1) = G(t, 0) = g(t)$, $\forall\ t \in I$ show that F and G agree at $t \times \frac{1}{2}$. Moreover, F is continuous on $I \times [0, \frac{1}{2}]$ and G is continuous on $I \times [\frac{1}{2}, 1]$. Hence by Pasting lemma, H is continuous. Now, $H(t, 0) = F(t, 0) = f(t)$, $\forall\ t \in I$, $H(t, 1) = G(t, 1) = h(t)$, $\forall\ t \in I$,

$$\begin{aligned} H(0, s) &= \begin{cases} F(0, 2s), & 0 \leq s \leq 1/2 \\ G(0, 2s-1), & 1/2 \leq s \leq 1 \end{cases} \\ &= x_0, \quad \forall\ s \in I \end{aligned}$$

and

$$\begin{aligned} H(1, s) &= \begin{cases} F(1, 2s), & 0 \leq s \leq 1/2 \\ G(1, 2s-1), & 1/2 \leq s \leq 1 \end{cases} \\ &= x_1, \quad \forall\ s \in I \end{aligned}$$

Hence $H : f \underset{p}{\simeq} h$. Consequently, '$\underset{p}{\simeq}$' is an equivalence relation on $P(X)$. ❑

Definition 2.1.9 The quotient set $P(X)/\underset{p}{\simeq}$ is called the set of path homotopy classes of paths in X.

Example 2.1.10 Let $f, g : X \to \mathbf{R}^2$ be two continuous maps. Define $F : X \times I \to \mathbf{R}^2$ by the rule $F(x, t) = (1-t)f(x) + tg(x)$, $\forall\ x \in X$, $\forall\ t \in I$. Then $F : f \simeq g$. In this example, F shifts the point $f(x)$ to the point $g(x)$ along the straight line segment joining $f(x)$ and $g(x)$, as shown in Fig. 2.2. The map F is called a straight line homotopy.

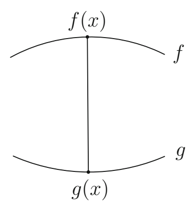


Fig. 2.2 Straight line homotopy

Example 2.1.11 If $X = Y = \mathbf{R}^n$ and $f(x) = x$ and $g(x) = 0 \equiv (0, \dots, 0) \in \mathbf{R}^n$, $\forall\ x \in \mathbf{R}^n$, i.e., if $f = 1_X$ (identity map on X) and g is the constant map at 0, then $F : X \times I \to X$, defined by $F(x, t) = (1 - t)x$ is a homotopy from f to g, i.e., $F : f \simeq g$. Again $G : X \times I \to X$, defined by $G(x, t) = (1 - t^2)x$ is also a homotopy from f to g. These examples show that homotopy between two maps is not unique.

Remark 2.1.12 As there are many homotopies between two maps, we can deform a map f into a given map g in different ways.

Example 2.1.13 Let X denote the punctured plane $X = \mathbf{R}^2 - \{0\}$. Then the paths $f(t) = (\cos \pi t, \sin \pi t)$, $g(t) = (\cos \pi t, 2 \sin \pi t)$ are path homotopic; the straight line homotopy between them is an acceptable path homotopy.

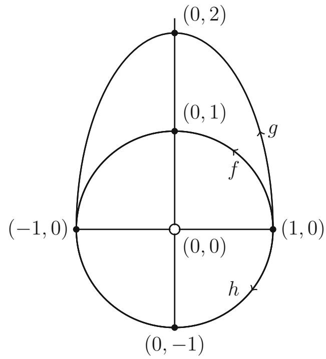
On the other hand, the straight line homotopy between the paths $f(t) = (\cos \pi t, \sin \pi t)$ and $h(t) = (\cos \pi t, - \sin \pi t)$ is not acceptable, because it passes through 0 and hence it does not entirely lie in the space $X = \mathbf{R}^2 - \{0\}$, as shown in the Fig. 2.3. There does not exist any path homotopy in X between the paths f and h, because one cannot deform f into g continuously passing through the hole at 0.

Example 2.1.14 Let $\mathbf{D}^n = \{x \in \mathbf{R}^n : ||x|| \leq 1\}$ be the n-disk. If $f(x) = x$ and $g(x) = 0$, $\forall\ x \in \mathbf{D}^n$, then $g \simeq f$. Define $F : \mathbf{D}^n \times I \to \mathbf{D}^n$ by $F(x, t) = tx$, $\forall\ x \in \mathbf{D}^n$, $\forall\ t \in I$. Now $||tx|| = |t| \cdot ||x|| \leq 1 \Rightarrow tx \in \mathbf{D}^n$, $\forall\ t \in I$ and $\forall\ x \in \mathbf{D}^n \Rightarrow F$ is well defined. Clearly, F is continuous and $F : g \simeq f$. Similarly, $G : \mathbf{D}^n \times I \to \mathbf{D}^n$ defined by $G(x, t) = (1 - t)x$ is a continuous map such that $G : f \simeq g$.

Example 2.1.15 Let $f, g : I \to I$ be defined by $f(t) = t$ and $g(t) = 0$, $\forall\ t \in I$. Then $F : I \times I \to I$ defined by $F(t, s) = (1 - s)t$ is a continuous map such that $F : f \simeq g$.

Example 2.1.16 Let Y be a subspace of $\mathbf{R}^n$ and $f, g : X \to Y$ be two continuous maps such that for every $x \in X$, $f(x)$ and $g(x)$ can be joined by a straight line segment in Y, then $F : f \simeq g$, where $F : X \times I \to Y$ is defined by

Fig. 2.3 Path homotopy



$F(x,t) = (1-t)f(x) + tg(x)$. Since $f(x)$ and $g(x)$ can be joined by a line segment in Y by hypothesis, F is well defined. To prove the continuity of F, we take $x, u \in X$ and $t, s \in I$. Then $F(u,s) = (1-s)f(u) + sg(u)$. Now $F(u,s) - F(x,t) = (s-t)(g(u) - f(u)) + (1-t)(f(u) - f(x)) + t(g(u) - g(x))$. Let $\epsilon > 0$ be an arbitrary small positive number. Then

$$||F(u,s) - F(x,t)|| \leq |(s-t)|||g(u) - f(u)|| + |(1-t)|||f(u) - f(x)|| + |t|||g(u) - g(x)|| \tag{2.3}$$

Again, f and g being continuous, $\exists$ open neighborhoods U_1 and U_2 of x in X such that for $u \in U_1 \cap U_2$, $||f(u) - f(x)|| < \epsilon/3$, $||g(u) - g(x)|| < \epsilon/3$. Then for $u \in U_1 \cap U_2$, $||g(u) - f(u)|| \leq ||g(u) - g(x)|| + ||g(x) - f(x)|| + ||f(x) - f(u)|| < c$, where c is the positive constant $||g(x) - f(x)|| + 2\epsilon/3$. Thus if $|s-t| < \epsilon/3c$, then from (2.3) it follows that

$$||F(u,s) - F(x,t)|| < \epsilon \tag{2.4}$$

Since the set $(U_1 \cap U_2) \times (t - \epsilon/3c, t + \epsilon/3c)$ is open in $X \times I$, this shows that F is continuous. Finally, $F(x,0) = f(x)$ and $F(x,1) = g(x)$, $\forall\ x \in X$. Consequently, $F : f \simeq g$.

Remark 2.1.17 Geometrically, the above homotopy F deforms f into g along the straight line segment in Y joining the points $f(x)$ and $g(x)$ for every $x \in X$. The function F is called a straight line homotopy.

Example 2.1.18 Let X be a topological space and S^n be the n-sphere in $\mathbf{R}^{n+1}$. If $f, g : X \to S^n$ are two continuous maps such that $f(x) \neq -g(x)$ for any $x \in X$, then $f \simeq g$. To show this define the map $F : X \times I \to S^n$ by $F(x,t) = \frac{(1-t)f(x)+tg(x)}{||(1-t)f(x)+tg(x)||}$. For each $x \in S^n$ and $t \in I$, $(1-t)f(x) + tg(x) \in \mathbf{R}^{n+1}$. The given condition $f(x) \neq -g(x)$ for any $x \in X$ shows that the line segment joining $f(x)$ and $g(x)$ cannot pass through the origin $0 = (0, 0, \ldots, 0) \in \mathbf{R}^{n+1}$. In other words, $(1-t)f(x) + tg(x) \neq 0$ for any $t \in I$ and any $x \in X$. Hence $F(x,t) \in S^n$, $\forall\ (x,t) \in X \times I$ and F is well defined. We now consider f, g as $f, g : X \to \mathbf{R}^{n+1} - \{0\}$. Then by Example 2.1.16, $\exists$ a straight line homotopy $G : X \times I \to \mathbf{R}^{n+1} - \{0\}$ defined by $G(x,t) = (1-t)f(x) + tg(x)$, i.e., $G : f \simeq g$. Again consider the map $h : \mathbf{R}^{n+1} - \{0\} \to S^n$ defined by $h(x) = \frac{x}{||x||}$. Then $F = h \circ G$ is the composite of two continuous maps h and G and hence F is continuous. Finally, $F(x,0) = \frac{f(x)}{||f(x)||} = f(x)$, because $f(x) \in S^n$, $\forall\ x \in X$ and $F(x,1) = \frac{g(x)}{||g(x)||} = g(x)$, because $g(x) \in S^n$, $\forall\ x \in X$. Consequently, $F : f \simeq g$.

Example 2.1.19 Let X be a topological space and $f : X \to S^n$ be a continuous non-surjective map. Then f is homotopic to a constant map $c : X \to S^n$. By hypothesis $f(X) \subsetneqq S^n \Rightarrow \exists$ a point $s_0 \in S^n$ such that $s_0 \notin f(X)$. Define a constant map $c : X \to S^n$ by $c(x) = -s_0$, $\forall\ x \in X$. Then $f(x) \neq -c(x)$ for any $x \in X$. Hence $f \simeq c$ by Example 2.1.18.

Example 2.1.20 Let $S^1 = \{z \in \mathbf{C} : |z| = 1\} = \{e^{i\theta} : 0 \leq \theta \leq 2\pi\}$ be the unit circle in $\mathbf{C}$. Then the maps $f, g : S^1 \to S^1$ defined by $f(z) = z$ and $g(z) = -z$ are homotopic. Consider the map $F : S^1 \times I \to S^1$ defined by $F(e^{i\theta}, t) = e^{i(\theta + t\pi)}$. Clearly, F is the composite of the maps

$$S^1 \times I \to S^1 \times S^1 \to S^1, \ (e^{i\theta}, t) \mapsto (e^{i\theta}, e^{it\theta}) \mapsto e^{i(\theta+t\pi)},$$

where the second map is the usual multiplication of complex numbers. Consequently, F is a continuous map such that $F : f \simeq g$.

We now extend Theorem 2.1.8 to the set $C(X, Y)$ of all continuous maps from X to Y by extending the concept of path homotopy (obtained by replacing I by any topological space X).

Theorem 2.1.21 *Given topological spaces X and Y, the relation '$\simeq$' (of being homotopic) is an equivalence relation on the set $C(X, Y)$.*

Proof Each $f \in C(X, Y)$ is homotopic to itself by a homotopy $H : X \times I \to Y$ defined by $H(x, t) = f(x)$. Thus $f \simeq f$, $\forall \ f \in C(X, Y)$. Next suppose $H : f \simeq g$, $f, g \in C(X, Y)$. Define $G : X \times I \to Y$ by $G(x, t) = H(x, 1 - t)$. Then G is continuous, because G is the composite of continuous maps

$$X \times I \to X \times I \to Y, \ (x, t) \mapsto (x, 1 - t) \mapsto H(x, 1 - t),$$

where the first map is continuous, because the projection maps $(x, t) \mapsto x$ and $(x, t) \mapsto (1 - t)$ are continuous and the second map is H. Then $G(x, 0) = H(x, 1) = g(x)$ and $G(x, 1) = H(x, 0) = f(x)$, $\forall \ \ x \in X$. Thus $G : g \simeq f$. Finally, let $f, g, h \in C(X, Y)$ be such that $F : f \simeq g$ and $G : g \simeq h$. Define a map $H : X \times I \to Y$ by

$$H(x, t) = \begin{cases} F(x, 2t), & 0 \leq t \leq 1/2 \\ G(x, 2t - 1), & 1/2 \leq t \leq 1 \end{cases}$$

Then H is continuous by Pasting lemma. Finally, $H(x, 0) = F(x, 0) = f(x)$ and $H(x, 1) = G(x, 1) = h(x)$, $\forall \ x \in X \Rightarrow H : f \simeq h$. Consequently, '$\simeq$' is an equivalence relation on $C(X, Y)$. ❑

Definition 2.1.22 The quotient set $C(X, Y)/\simeq$ is called the set of all homotopy classes of maps $f \in C(X, Y)$, denoted by $[X, Y]$ and for $f \in C(X, Y)$, $[f] \in [X, Y]$ is called the homotopy class of f.

Remark 2.1.23 The set $[X, Y]$ was first systemically studied by M.G. Barratt in 1955 in his paper (Barratt 1955). This set plays the central role in algebraic topology and is used throughout the book. Some of its properties are displayed in Sect. 2.3.

We now show that composites of homotopic maps are homotopic.

Theorem 2.1.24 *Let $f_1, g_1 \in C(X, Y)$ and $f_2, g_2 \in C(Y, Z)$ be maps such that $f_1 \simeq g_1$ and $f_2 \simeq g_2$. Then the composite maps $f_2 \circ f_1$ and $g_2 \circ g_1 : X \to Z$ are homotopic.*

Proof Let $F : f_1 \simeq g_1$ and $G : f_2 \simeq g_2$. Then $f_2 \circ F : X \times I \to Z$ is a continuous map such that $(f_2 \circ F)(x, 0) = f_2(F(x, 0)) = f_2(f_1(x)) = (f_2 \circ f_1)(x),\ \forall\ x \in X$ and $(f_2 \circ F)(x, 1) = f_2(F(x, 1)) = f_2(g_1(x)) = (f_2 \circ g_1)(x),\ \forall\ x \in X$. Consequently,

$$f_2 \circ F : f_2 \circ f_1 \simeq f_2 \circ g_1 \tag{2.5}$$

Again we define $H : X \times I \to Z$ by $H(x, t) = G(g_1(x), t)$. Thus H is the composite

$$X \times I \xrightarrow{g_1 \times 1_d} Y \times I \xrightarrow{G} Z,$$

$$(x, t) \mapsto (g_1(x), t) \mapsto G(g_1(x), t).$$

Then H is a continuous map such that $H(x, 0) = G(g_1(x), 0) = f_2(g_1(x)) = (f_2 \circ g_1)(x),\ \forall\ \ x \in X$ and $H(x, 1) = G(g_1(x), 1) = g_2(g_1(x)) = (g_2 \circ g_1)(x)$, $\forall\ x \in X$. Hence

$$H : f_2 \circ g_1 \simeq g_2 \circ g_1 \tag{2.6}$$

Consequently, by transitive property of homotopy relation, it follows from (2.5) and (2.6) that $f_2 \circ f_1 \simeq g_2 \circ g_1$. ❑

Remark 2.1.25 Theorem 2.1.24 asserts in the language of category theory that topological spaces and homotopy classes of continuous maps form a category denoted by $\mathcal{Htp}$ called homotopy category of topological spaces (see Appendix B). Thus $\mathcal{Htp}$ is the category whose objects are topological spaces and mor (X, Y) consists of homotopy classes of continuous maps from X to Y, where the composition of maps is consistent with homotopies (see Theorem 2.1.24).

Given a topological space X the concept of a flow $\psi_t : X \to X$ $(t \in \mathbf{R})$ is closely related to homotopy.

Definition 2.1.26 A continuous family $\psi_t : X \to X$ $(t \in \mathbf{R})$ of maps is called a flow if

(i) $\psi_0 = 1_d$;
(ii) ψ_t is a homeomorphism for all $t \in \mathbf{R}$;
(iii) $\psi_{t+s} = \psi_t \circ \psi_s$.

Remark 2.1.27 It is sometimes convenient to consider a flow ψ_t as a continuous map

$$\psi : X \times \mathbf{R} \to \mathbf{R},\ (x, t) \mapsto \psi_t(x).$$

A flow is also known as one parameter group of homeomorphisms.

Proposition 2.1.28 *$\psi_t : X \to X$ is homotopic to I_X.*

Proof Consider the map

$$F : X \times I \to X, (x, s) \mapsto \psi(x, (1 - s)t).$$

This shows that every ψ_t is homotopic to 1_X. ❑

We now extend the Definition 2.1.1 of homotopy of continuous maps for pairs of topological spaces.

Definition 2.1.29 A topological pair (X, A) consists of a topological space X and a subspace A of X. If $A = \emptyset$, the empty set, we shall not distinguish between the pair $(X, \emptyset)$ and the space X. A subpair (X', A') of (X, A) is a pair such that $X' \subset X$ and $A' \subset A$.

Definition 2.1.30 A continuous map $f : (X, A) \to (Y, B)$ is a continuous function $f : X \to Y$ such that $f(A) \subset B$.

Given a topological pair (X, A), $(X, A) \times I$ represents the pair $(X \times I, A \times I)$.

Definition 2.1.31 Given pairs of topological spaces (X, A) and (Y, B), two continuous maps $f, g : (X, A) \to (Y, B)$ are said to be homotopic if $\exists$ a continuous map $F : (X \times I, A \times I) \to (Y, B)$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$, $\forall\ x \in X$. Then the map F is called a homotopy from f to g and written $F : f \simeq g$.

We now consider a more restricted type of homotopy of continuous maps between pairs of topological spaces, which extends the concept of path homotopy obtained by replacing I by any topological space X and $\{0, 1\}$ by a subspace of X under consideration.

Definition 2.1.32 Let $f, g : (X, A) \to (Y, B)$ be two continuous maps of pairs of topological spaces and $X' \subset X$ be such that $f|_{X'} = g|_{X'}$ (i.e., $f(x') = g(x')$, $\forall\ x' \in X'$, which implies that f and g agree at x', $\forall\ x' \in X'$). Then f and g are said to be homotopic relative to X' if there exists a continuous map $F : (X \times I, A \times I) \to (Y, B)$ such that $F(x, 0) = f(x)$, $F(x, 1) = g(x)$, $\forall\ x \in X$ and $F(x', t) = f(x') = g(x')$, $\forall\ x' \in X'$ and $\forall\ t \in I$, and written $F : f \simeq g$ rel X'.

If $X' = \emptyset$, we omit the phrase relative to X'.

Remark 2.1.33 $f \simeq g$ rel $X' \Rightarrow f \simeq g$ rel X'' for any subspace $X'' \subset X'$.

Geometrical Interpretation: For $t \in I$, if we define $h_t : (X, A) \to (X \times I, A \times I)$ by $h_t(x) = (x, t)$, then $h_0(x) = (x, 0)$ and $h_1(x) = (x, 1)$. Thus $F : f \simeq g$ rel $X' \Rightarrow F \circ h_0 = f$, $F \circ h_1 = g$, and $F \circ h_t|_{X'} = f|_{X'} = g|_{X'}$, $\forall\ t \in I \Rightarrow$ the collection $\{F \circ h_t\}_{t \in I}$ is a continuous one parameter family of maps from (X, A) to (Y, B) agreeing on X' and satisfying the relations $f = F \circ h_0$ and $g = F \circ h_1$. Thus $f \simeq g$ rel X' represents geometrically a continuous deformation deforming f into g by maps all of which agree on X'. For example, $f \simeq g$ rel $\{0\}$ in Example 2.1.14.

Example 2.1.34 Consider $D^2 = \{z \in \mathbf{C} : z = re^{i\theta}, 0 \le r \le 1\}$ and $S^1 = \{z \in \mathbf{C} : z = e^{i\theta}, 0 \le \theta \le 2\pi\}$. Then $S^1 \subset D^2$. Let $f : (D^2, S^1) \to (D^2, S^1)$ be the identity map and $g : (D^2, S^1) \to (D^2, S^1)$ be the reflection in the origin, i.e., $g(re^{i\theta}) = re^{i(\theta+\pi)}$. Then $f \simeq g \;\text{rel}\,\{0\}$ under the homotopy $F : (D^2, S^1) \times I \to (D^2, S^1)$ defined by $F(re^{i\theta}, t) = re^{i(\theta+t\pi)}$. Moreover, $G : (D^2, S^1) \times I \to (D^2, S^1)$ defined by $G(re^{i\theta}, t) = e^{i(\theta-t\pi)}$ is also a homotopy $G : f \simeq g \;\text{rel}\,\{0\}$ (compare Example 2.1.14).

Remark 2.1.35 There may exist different homotopies from f to g relative to a subspace and thus homotopy from f to g is not unique.

Example 2.1.36 Let X be a topological space, $A \subset X$ and Y be any convex subspace of $\mathbf{R}^n$. If $f, g : X \to Y$ are two continuous maps such that $f|_A = g|_A$, then $f \simeq g \;\text{rel}\, A$ by a homotopy $G : X \times I \to Y$ defined by $G(x, t) = (1-t)f(x) + tg(x)$.

We now generalize Theorem 2.1.21.

Theorem 2.1.37 *The relation between continuous maps from (X, A) to (Y, B) of being homotopic relative to a subspace $X' \subset X$ is an equivalence relation.*

Proof **Reflexivity** Let $f : (X, A) \to (Y, B)$ be a continuous map. Define $F : (X \times I, A \times I) \to (Y, B)$ by the rule $F(x, t) = f(x)$, $\forall\; x \in X$, $\forall\; t \in I$. Then $F : f \simeq f \;\text{rel}\, X'$.
Symmetry Let $F : f \simeq g \;\text{rel}\, X'$. Define $G : (X \times I, A \times I) \to (Y, B)$ by the rule $G(x, t) = F(x, 1-t)$, $\forall\; x \in X$, $\forall\; t \in I$. For continuity of G see Theorem 2.1.21. Then $G : g \simeq f \;\text{rel}\, X'$.
Transitivity Let $f, g, h : (X, A) \to (Y, B)$ be three continuous maps such that $F : f \simeq g \;\text{rel}\, X'$ and $G : g \simeq h \;\text{rel}\, X'$. Define $H : (X \times I, A \times I) \to (Y, B)$ by the rule

$$H(x, t) = \begin{cases} F(x, 2t), & 0 \le t \le 1/2 \\ G(x, 2t-1), & 1/2 \le t \le 1 \end{cases}$$

Then H is continuous by Pasting Lemma 2.1.6. Moreover, $H : f \simeq h \;\text{rel}\, X'$ ❑

Remark 2.1.38 It follows from Theorem 2.1.37 that the set of continuous maps from (X, A) to (Y, B) is partitioned into disjoint equivalence classes by the relation of homotopy relative to X' denoted by $[X, A; Y, B]$. This set is very important in the study of algebraic topology. Given a continuous map $f : (X, A) \to (Y, B)$, $[f|_{X'}]$ represents the homotopy class in $[X, A; Y, B]$ determined by f.

We now generalize Theorem 2.1.24 for homotopies relative to a subspace.

Theorem 2.1.39 *Let $f_0, f_1 : (X, A) \to (Y, B)$ be homotopies relative to $X' \subset X$ and $g_0, g_1 : (Y, B) \to (Z, C)$ be homotopies relative to Y', where $f_1(X') \subset Y' \subset Y$. Then the composites $g_0 \circ f_0, g_1 \circ f_1 : (X, A) \to (Z, C)$ are homotopic relative to X', i.e., composites of homotopic maps are homotopic.*

Proof Let $F : f_0 \simeq f_1 \text{ rel } X'$ and $G : g_0 \simeq g_1 \text{ rel } Y'$. Then the composite mapping

$$(X \times I, A \times I) \xrightarrow{F} (Y, B) \xrightarrow{g_0} (Z, C)$$

is a homotopy relative to X' from $g_0 \circ f_0$ to $g_0 \circ f_1$, i.e.,

$$g_0 \circ F : g_0 \circ f_0 \simeq g_0 \circ f_1 \text{ rel } X' \tag{2.7}$$

Again the composite mapping

$$(X \times I, A \times I) \xrightarrow{f_1 \times 1_d} (Y \times I, B \times I) \xrightarrow{G} (Z, C)$$

is a homotopy relative to

$$f_1^{-1}(Y') \text{ from } g_0 \circ f_1 \text{ to } g_1 \circ f_1 \tag{2.8}$$

Since $X' \subset f_1^{-1}(Y)$, (2.7) and (2.8) show that $g_0 \circ f_0 \simeq g_0 \circ f_1 \text{ rel } X'$ and $g_0 \circ f_1 \simeq g_1 \circ f_1 \text{ rel } X'$ and hence $g_0 \circ f_0 \simeq g_1 \circ f_1 \text{ rel } X'$ by transitivity of the relation $\simeq$.❑

2.1.2 Functorial Representation

This subsection summarizes the earlier discussion in the basic result from the viewpoint of category theory which gives important examples of categories, functors and natural transformations, the concepts defined in Appendix B.

Theorem 2.1.39 shows that there is a category, called the homotopy category of pairs of spaces whose objects are topological pairs and whose morphisms are homotopy classes relative to a subspace. This category contains as full subcategories the homotopy category $\mathcal{H}tp$ of topological spaces and also the homotopy category $\mathcal{H}tp_*$ of pointed topological spaces.

Theorem 2.1.40 *There is a covariant functor from the category of pairs of topological spaces and their continuous maps to the homotopy category whose object function is the identity function and whose morphism function sends a continuous map f to its homotopy class $[f]$. Moreover, for any pair (P, Q) of topological spaces there is a covariant functor $\pi_{(P,Q)}$ from the homotopy category of pairs to the category of sets and functions defined by $\pi_{(P,Q)}(X, A) = [P, Q; X, A]$ and if $f : (X, A) \to (Y, B)$ is continuous, then $f_* = \pi_{(P,Q)}([f]) : [P, Q; X, A] \to [P, Q; Y, B]$ is defined by $f_*([g]) = [f \circ g]$ for $g : (P, Q) \to (X, A)$.*

If $\alpha : (P, Q) \to (P', Q')$, then there is a natural transformation $\alpha^* : \pi_{(P',Q')} \to \pi_{(P,Q)}$. Similarly, we can define a contravariant functor $\pi^{(P,Q)}$ for a given (P, Q) of pair of topological spaces and a natural transformation $\alpha_* : \pi^{(P,Q)} \to \pi^{(P',Q')}$.

2.2 Homotopy Equivalence

This section studies the concept of homotopy equivalence introduced by W. Hurewicz (1935) to establish a connection between homotopy and homology groups of a certain class of topological spaces. The problem of classification of continuous maps from one topological space to other is closely related to the problem of classification of topological spaces according to their homotopy properties. This problem led to the concept of homotopy equivalence which is not only a generalization of the concept of homeomorphism but also gives a new foundation for the development of the combinatorial invariants of topological spaces and manifolds. The higher homotopy groups and the homology groups are invariants of the homotopy equivalent class of a topological space.

Classification of topological spaces up to homotopy equivalences is the main problem of algebraic topology. This is a weaker relation than a topological equivalence in the sense that homotopy classes of continuous maps of topological spaces are larger than their homeomorphism classes. Although the main aim of topology is to classify topological spaces up to homeomorphism; in algebraic topology, a homotopy equivalence plays a more important role than a homeomorphism. Because the basic tools of algebraic topology such as homotopy and homology groups are invariants with respect to homotopy equivalences.

Definition 2.2.1 A continuous map $f : (X, A) \to (Y, B)$ is called a homotopy equivalence if $[f]$ is an equivalence in the homotopy category of pairs. In particular, a map $f \in C(X, Y)$ is said to be a homotopy equivalence if $\exists$ a map $g \in C(Y, X)$ such that $g \circ f \simeq 1_X$ (existence of left homotopy inverse of f) and $f \circ g \simeq 1_Y$ (existence of right homotopy inverse of f). In such a situation g is unique and the map g is called a homotopy inverse of f.

Remark 2.2.2 Let f be a homotopy equivalence with g as its homotopy inverse. Then $[g] = [f]^{-1}$ in the homotopy category $\mathcal{H}tp$ which has the same objects in the category $\mathcal{T}op$ of topological spaces and their continuous maps but the morphism in $\mathcal{H}tp$ are the homotopy classes of continuous maps, so that their morphisms $\text{mor}_{\mathcal{H}tp}(X, Y) = [X, Y]$. The isomorphisms in the category $\mathcal{T}op$ are homeomorphisms and in the category $\mathcal{H}tp$ are homotopy equivalences.

Example 2.2.3 Let Y be the $(n-1)$-sphere $S^{n-1} \subset \mathbf{R}^n \subset \mathbf{R}^{n+q}$ and X be the subset of $\mathbf{R}^{n+q}$ of points not lying on the plane $x_1 = \cdots = x_n = 0$. Then the inclusion map $i : Y \hookrightarrow X$ is a homotopy equivalence.

Define $f : X \to Y$ by $f(x_1, x_2, \ldots, x_{n+q}) = (rx_1, \ldots, rx_n, 0, \ldots, 0)$, where $r = (x_1^2 + x_2^2 + \cdots + x_n^2)^{-1/2}$. Then $f \circ i = 1_Y$. Again define

$$H : X \times I \to X, (x_1, x_2, \ldots, x_{n+q}, t) \mapsto (r^{1-t}x_1, \ldots, r^{1-t}x_n, tx_{n+1}, \ldots, tx_{n+q})$$

is a homotopy from $i \circ f$ to 1_X. Consequently, i is a homotopy equivalence.

Example 2.2.4 Let D^2 be the unit disk in $\mathbf{R}^2$ and $p \in D^2$. Let $i : P = \{p\} \hookrightarrow D^2$ be the inclusion map and $c : D^2 \to P$ be the constant map. Then $c \circ i = 1_P$. Again the map $H : D^2 \times I \to D^2$ defined by $H(x, t) = (1 - t)x + tp$, being a homotopy from 1_{D^2} to $i \circ c$, i.e., $i \circ c \simeq 1_{D^2}$. Consequently, i is a homotopy equivalence.

Definition 2.2.5 If $f \in C(X, Y)$ is a homotopy equivalence, then X and Y are said to be homotopy equivalent spaces, denoted by $X \simeq Y$.

We now extend Definition 2.2.5 for pairs of topological spaces.

Definition 2.2.6 Two pairs of topological spaces (X, A) and (Y, B) are said to be homotopy equivalent, written $(X, A) \simeq (Y, B)$, if $\exists$ continuous maps $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (X, A)$ such that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$, the homotopy being the homotopy of pairs.

Remark 2.2.7 Homeomorphic spaces are homotopy equivalent but its converse is not true in general.

Consider the following example:

Example 2.2.8 Let X be the unit circle S^1 in $\mathbf{R}^2$ and Y be the topological space S^1, together with the line segment I_1 joining the points $(1, 0)$ and $(2, 0)$ in $\mathbf{R}^2$, i.e., $I_1 = \{(r, 0) \in \mathbf{R}^2 : 1 \leq r \leq 2\}$. Then X and Y are of the same homotopy type but they are not homeomorphic (Fig. 2.4).

X and Y cannot be homeomorphic, because removal of the point $(1, 0)$ from Y makes Y disconnected. On the other hand, removal of any point from X leaves X connected. We claim that $X \simeq Y$. We take $f : X \hookrightarrow Y$ to be the inclusion map and $g : Y \to X$ defined by

$$g(y) = \begin{cases} y, & \text{if } y \in X \\ (1, 0), & \text{if } y \in I_1 \end{cases}$$

The continuity of g follows from Pasting lemma. Then $f \circ g, 1_Y : Y \to Y$ are two continuous maps such that $(f \circ g)(y) = f(g(y)) = g(y),\ \forall\ y \in Y$ and

$$1_Y(y) = \begin{cases} y, & \text{if } y \in X \\ (r, 0), & \text{if } y = (r, 0) \in I_1. \end{cases}$$

Since for every $y \in Y$, $(f \circ g)(y)$ and $1_Y(y)$ can be joined by a straight line segment in Y, it follows that $f \circ g \simeq 1_Y$ (see Example 2.1.16). Again $g \circ f = 1_X \simeq 1_X$. Thus f

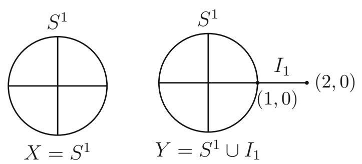


Fig. 2.4 Homotopy equivalent but non-homeomorphic spaces

and g are two continuous maps such that $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$. Consequently, $X \simeq Y$.

This example shows that $X \simeq Y$ does not imply $X \approx Y$.

Example 2.2.9 The topological spaces X and Y in Example 2.2.8 are homotopy equivalent. The unit disk D^2 is homotopy equivalent to a one-point topological space $\{p\} \subset D^2$ (see Example 2.2.4).

As the name suggests, the relation of being homotopy equivalent is an equivalence relation on the set of topological spaces (or pairs of topological spaces).

Theorem 2.2.10 *The relation '$\simeq$' between topological spaces (or pairs of topological spaces) of being homotopy equivalent is an equivalence relation.*

Proof **Reflexivity**: $1_X : X \to X$ is a homotopy equivalence $\Rightarrow X \simeq X$ for all X.
Symmetry: Let $X \simeq Y$. Then $\exists$ continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y \Rightarrow g : Y \to X$ is a homotopy equivalence with homotopy inverse $f : X \to Y$. Consequently, $Y \simeq X$. Thus $X \simeq Y \Rightarrow Y \simeq X$.
Transitivity: Let $X \simeq Y$ and $Y \simeq Z$. Then $\exists$ continuous maps $f : X \to Y, g : Y \to X, h : Y \to Z$ and $k : Z \to Y$ such that $g \circ f \simeq 1_X$, $f \circ g \simeq 1_Y$, $h \circ k \simeq 1_Z$ and $k \circ h \simeq 1_Y$. Now $h \circ f : X \to Z$ and $g \circ k : Z \to X$ are continuous maps such that $(h \circ f) \circ (g \circ k) = h \circ (f \circ g) \circ k \simeq h \circ 1_Y \circ k = h \circ k \simeq 1_Z$ and $(g \circ k) \circ (h \circ f) = g \circ (k \circ h) \circ f \simeq g \circ 1_Y \circ f = g \circ f \simeq 1_X$. Consequently, $h \circ f$ is a homotopy equivalence with $g \circ k$ homotopy inverse $\Rightarrow X \simeq Z$. The equivalence relation $\simeq$ divides the set of spaces up to homotopy equivalent classes. ❑

Definition 2.2.11 The homotopy equivalent class containing X is called the homotopy type of X.

Remark 2.2.12 Two topological spaces X and Y are homotopy equivalent or of the same homotopy type if there exists a homotopy equivalence $f \in C(X, Y)$. For example, D^2 and $\{p\}$ in Example 2.2.9 are homotopy equivalent spaces. The homeomorphic spaces are said to have the same topological type. On the other hand, the homotopy equivalent spaces are said to have the same homotopy type.

Proposition 2.2.13 *Two homeomorphic spaces have the same homotopy type.*

Proof Let X and Y be two homeomorphic spaces and $f : X \to Y$ be a homeomorphism. Then its inverse $g = f^{-1} : Y \to X$ is continuous and satisfies the conditions: $g \circ f = f^{-1} \circ f = 1_X \simeq 1_X$ and $f \circ g = f \circ f^{-1} = 1_Y \simeq 1_Y$ by reflexivity of the relation $\simeq$. Consequently, f is a homotopy equivalence. Hence X and Y are of the same homotopy type. ❑

Remark 2.2.14 The converse of Proposition 2.2.13 is not true. For example, the disk D^n is of the same homotopy type of a single point $\{p\} \subset D^n$ but D^n is not homeomorphic to $\{p\}$.

Proposition 2.2.15 *Any continuous map homotopic to a homotopy equivalence is a homotopy equivalence.*

Proof Let $C(X, Y)$ denote the set of all continuous maps from X to Y and $f \in C(X, Y)$ be a homotopy equivalence. Suppose $g \in C(X, Y)$ is such that $f \simeq g$. Now f is a homotopy equivalence $\Rightarrow \exists\, h \in C(Y, X)$ such that $h \circ f \simeq 1_X$ and $f \circ h \simeq 1_Y$. Again $f \simeq g \Rightarrow f \circ h \simeq g \circ h \Rightarrow 1_Y \simeq g \circ h \Rightarrow g \circ h \simeq 1_Y$. Similarly, $h \circ g \simeq 1_X$. Consequently, g is a homotopy equivalence. ❑

Definition 2.2.16 A continuous map $f : X \to S^n$ is called inessential if f is homotopic to a continuous map of X into a single point of S^n (i.e., if f is homotopic to a constant map). Otherwise f is called essential. In general, a map $f \in C(X, Y)$ is said to be nullhomotopic or inessential if it is homotopic to some constant map.

Example 2.2.17 Let $X = Y = I$. Define $f, g : I \to I$ by $f(t) = t$ and $g(t) = 0$, $\forall\, t \in I$. Then f is the identity map and g is a constant map. Define $F : I \times I \to I$ by $F(t, s) = (1 - s)t$. Then $F : f \simeq g \Rightarrow f$ is nullhomotopic.

Remark 2.2.18 Two nullhomotopic maps may not be homotopic.

Example 2.2.19 Let X be a connected space and Y be not a connected space. Let y_0 and y_1 be points in distinct components of Y. Let $f_0(x) = y_0$ and $f_1(x) = y_1$, $\forall\, x \in X$ be two constant maps from X to Y. If possible, let $f_0 \simeq f_1$. Then $\exists$ a continuous map $F : X \times I \to Y$ such that $F : f_0 \simeq f_1$. Since $X \times I$ is connected and F is continuous, $F(X \times I)$ must be connected, which contradicts the fact that Y is not connected.

Proposition 2.2.20 *Let $f, g : (X, A) \to (Y, B)$ be pairs of continuous maps such that $f \simeq g$ as maps of pairs. Then the induced maps $\tilde{f}, \tilde{g} : X/A \to Y/B$ (corresponding quotient spaces) are also homotopic.*

Proof Let $H : (X \times I, A \times I) \to (Y, B)$ be a homotopy between f and g. Then H induces a function $\tilde{H} : (X/A) \times I \to Y/B$ such that the diagram in Fig. 2.5 is commutative, where p and q are the identification maps. Since $\tilde{H} \circ (p \times 1_d) = q \circ H$ is continuous, I is locally compact and Hausdorff, $p \times 1_d$ is an identification map, it follows that $\tilde{H}$ is a (based) homotopy between $\tilde{f}$ and $\tilde{g}$, where base points of X/A and Y/B are respectively the points to which A and B are identified. ❑

Corollary 2.2.21 *Let $f : (X, A) \to (Y, B)$ be a homotopy equivalence of pairs. Then $\tilde{f} : X/A \to Y/B$ is a (based) homotopy equivalence.*

Proof As $f : (X, A) \to (Y, B)$ is a homotopy equivalence, there exists a map $g : (Y, B) \to (X, A)$ such that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$. Hence the corollary follows from the Proposition 2.2.20. ❑

Fig. 2.5 Diagram for identification map

$$\begin{tikzcd} X \times I \arrow[r, "H"] \arrow[d, "p \times 1_d"'] & Y \arrow[d, "q"] \\ X/A \times I \arrow[r, "\tilde{H}"'] & Y/B \end{tikzcd}$$

2.3 Homotopy Classes of Maps

This section continues the study of homotopy classes of continuous maps given in Sect. 2.2. These classes play an important role in the study of algebraic topology as depicted throughout the book. Homotopy theory studies those properties of topological spaces and continuous maps which are invariants under homotopic maps, called homotopy invariants.

Let $[X, Y]$ be the set of homotopy classes of continuous maps from X to Y: by keeping X fixed and varying Y, this set is an invariant of the homotopy type of Y, in the sense that if $Y \simeq Z$, then there exists a bijective correspondence between the sets $[X, Y]$ and $[X, Z]$. Similar result holds for pairs of topological spaces and hence for pointed topological spaces. Many homotopy invariants can be obtained from the sets $[X, Y]$ on which some short of algebraic structure is often given.for particular choice of X and Y. Most of the classical invariants of algebraic topology are homotopy invariants. Many homotopy invariants can be obtained by specializing the sets $[X, Y]$.

The following two natural problems are posed in this section but solved in Sect. 2.4.

(i) Given a pointed topological space Y, does there exist a natural product defined in $[X, Y]$ admitting the set $[X, Y]$ a group structure for all pointed topological spaces X?

(ii) Given a pointed topological space X, does there exist a natural product defined in $[X, Y]$ admitting the set $[X, Y]$ a group structure for all pointed topological space Y?

In this section we work in the homotopy category $\mathcal{H}tp_*$ of pointed topological spaces and their base point preserving continuous maps. Thus $[X, Y]$ is the set of morphisms from X to Y in the homotopy category $\mathcal{H}tp_*$ of pointed topological spaces. This set $[X, Y]$ only depends on homotopy types of X and Y. Given two continuous maps $f : X \to Y$ and $g : Y \to Z$, we can compose them and obtain $g \circ f : X \to Z$. The homotopy class of $g \circ f$ depends only on the homotopy classes of f and g. So the composition with g gives a function

$$g_* : [X, Y] \to [X, Z]$$

and the composition with f gives a function

$$f^* : [Y, Z] \to [X, Z].$$

Theorem 2.3.1 *Let X, Y, Z be pointed topological spaces and $f : Y \to Z$ be a base point preserving continuous map. Then f induces a function. $f_* : [X, Y] \to [X, Z]$ satisfying the following properties:*

(a) *If $f \simeq h : Y \to Z$, then $f_* = h_*$;*

(b) *If $1_Y : Y \to Y$ is the identity map, then 1_{Y*} is the identity function;*

(c) *If $g : Z \to W$ is another base point preserving continuous map, then $(g \circ f)_* = g_* \circ f_*$.*

Proof Define $f_* : [X, Y] \to [X, Z]$ by the rule $f_*([\alpha]) = [f \circ \alpha]$, $\forall\ [\alpha] \in [X, Y]$. Since $\alpha \simeq \beta \Rightarrow f \circ \alpha \simeq f \circ \beta \Rightarrow f_*([\alpha]) = f_*[\beta] \Rightarrow f_*$ is independent of the choice of the representatives of the classes. Hence f_* is well defined.

(a) Consider the functions $f_*, h_* : [X, Y] \to [X, Z]$. Then $h_*([\alpha]) = [h \circ \alpha] = [f \circ \alpha] = f_*([\alpha])$, since $f \simeq h \Rightarrow f \circ \alpha \simeq h \circ \alpha = f_*([\alpha])$, $\forall\ [\alpha] \in [X, Y]$. Hence $h_* = f_*$.
(b) $1_{Y^*} : [X, Y] \to [X, Y]$ is given by $1_{Y^*}([\alpha]) = [1_Y \circ \alpha] = [\alpha]$, $\forall\ [\alpha] \in [X, Y]$. Hence 1_{Y^*} is the identity function.
(c) $(g \circ f)_* : [X, Y] \to [X, W]$ is given by $(g \circ f) * ([\alpha]) = [(g \circ f) \circ \alpha] = [g \circ (f \circ \alpha)] = (g_* \circ f_*)[\alpha]$, $\forall\ [\alpha] \in [X, Y]$. Hence $(g \circ f)_* = g_* \circ f_*$.

❑

Corollary 2.3.2 *If $f : Y \to Z$ is a homotopy equivalence, then $f_* : [X, Y] \to [X, Z]$ is a bijection for every topological space X.*

Proof If $f \in C(Y, Z)$ is a homotopy equivalence, then $\exists\ g \in C(Z, Y)$ such that $g \circ f \simeq 1_Y$ and $f \circ g \simeq 1_Z$. Hence $(g \circ f)_* = g_* \circ f_*$ is the identity function and $(f \circ g)_* = f_* \circ g_*$ is also identity function $\Rightarrow f_*$ is a bijection with g_* as its inverse. ❑

Corollary 2.3.3 *If $Y \simeq Z$, then there exists a bijection $\psi : [X, Y] \to [X, Z]$ for every topological space X.*

Proof If $Y \simeq Z$, then $\exists$ a homotopy equivalence $f \in C(Y, Z)$. Hence $f_* = \psi : [X, Y] \to [X, Z]$ is a bijection for every X by Corollary 2.3.2. ❑

Corollary 2.3.4 *Given a pointed topological space X, there exists a covariant functor π_X from the homotopy category of pointed topological spaces to the category of sets and functions defined by $\pi_X(Y) = [X, Y]$ and if $f : Y \to Z$ is a base point preserving continuous map, then $\pi_X(f) = f_* : [X, Y] \to [X, Z]$ is defined by $f_*([\alpha]) = [f \circ \alpha]$.*

Proof The Corollary follows from Theorem 2.3.1. ❑

We obtain the corresponding dual results.

Theorem 2.3.5 *A base point preserving continuous map $f : Y \to Z$ induces a function $f^* : [Z, X] \to [Y, X]$ for every pointed space X, satisfying the following properties:*

(a) *$f \simeq h : Y \to Z \Rightarrow f^* = h^*$;*
(b) *$1_Y : Y \to Y$ is the identity map $\Rightarrow 1_{Y^*}$ is the identity function;*
(c) *If $g : Z \to W$ is another base point preserving continuous map, then $(g \circ f)^* = f^* \circ g^*$.*

Proof Similar to the proof of Theorem 2.3.1. ❑

Corollary 2.3.6 *If $f : Y \to Z$ is a homotopy equivalence, then $f^* : [Z, X] \to [Y, X]$ is a bijection for every pointed topological space X.*

Corollary 2.3.7 *If $Y \simeq Z$, then $\exists$ a bijection $\psi : [Z, X] \to [Y, X]$ for every pointed topological space X.*

Corollary 2.3.8 *Given a pointed topological space X, there exists a contravariant functor π^X from the homotopy category of pointed topological spaces to the category of sets and functions.*

Converses of Corollaries 2.3.2 and 2.3.6 are also true.

Theorem 2.3.9 *If $f : Y \to Z$ is a base point preserving continuous map such that*

(a) *$f_* : [X, Y] \to [X, Z]$ is a bijection for all pointed topological spaces X, then f is a homotopy equivalence.*
(b) *$f^* : [Z, X] \to [Y, X]$ is a bijection for all pointed topological spaces X, then f is a homotopy equivalence.*

Proof **(a)** In particular, $f_* : [Z, Y] \to [Z, Z]$ is a bijection (by hypothesis) $\Rightarrow \exists$ a continuous map $g : Z \to Y$ such that $f_*([g]) = [1_Z] \Rightarrow f \circ g \simeq 1_Z$. Similarly, $g_*([f]) = [1_Y] \Rightarrow g \circ f \simeq 1_Y$. Consequently, f is a homotopy equivalence.
(b) Similar to **(a)**.

❑

2.4 *H*-Groups and *H*-Cogroups

This section conveys the concept of a grouplike space, called an H-group and its dual concept, called an H-cogroup as a continuation of the study of the set $[X, Y]$ by considering the problem: when is the set $[X, Y]$ a group for every pointed topological space X (or for every pointed topological space Y)? The concepts of H-groups and H-cogroups arose through the study of such problems. These concepts develop homotopy theory. The loop spaces and suspension spaces of pointed topological spaces play an important role in the study of homotopy theory. Loop spaces of pointed spaces provide an extensive class of H-groups. On the other hand suspension spaces of pointed topological spaces form an extensive class of H-cogroups, a dual concept of H-group.

We consider topological spaces Y such that $[X, Y]$ admits a group structure for all X. There is a close relation between the natural group structures on $[X, Y]$ for all X and 'grouplike' structure on Y. Before systematic study of the homotopy sets $[X, Y]$ or $[(X, x_0), (Y, y_0)]$ by M.G. Barratt (1955) in his paper, the concept of an H-space introduced by H. Hopf in 1933 arose as a generalization of a topological group which is used to solve the above problem.

2.4.1 H-Groups and Loop Spaces

This subsection continues to study H-groups by specializing the the sets $[X, Y]$ and presents loop spaces which form an important class of H-groups. Given pointed topological spaces X and Y, we often give the set $[X, Y]$ some sort of algebraic structure. With this objective this subsection studies a grouplike space which is a group up to homotopy, called an H-group. More precisely, this subsection introduces the concepts of H-groups to obtain algebraic structures on the set of certain homotopy classes of continuous maps and introduces the concept of an H-group with loop spaces as illustrative examples. An H-group is a generalization of a topological group. Such groups were first introduced by H. Hopf in 1941 and they are named in his honor. Loop spaces of pointed topological spaces constitute an extensive class of H-groups.

The motivation of this study is to describe an additional structure needed on a pointed space P so that $\pi^P(X) = [X, P]$ is a group and for $f : X \to Y$, $f^* = \pi^P(f) : [Y, P] \to [X, P]$ is a group homomorphism. If $f : X \to Y$ and $g : X \to Z$ are continuous maps, we define $(f, g) : X \to Y \times Z$ to be the map $(f, g)(x) = (f(x), g(x))$, $\forall\ x \in X$.

If Y is a topological group, then $[X, Y]$ admits a group structure by Theorem 2.4.1.

Now the following two natural questions arise:

(i) Given a pointed topological space Y, does there exist a natural product defined in $[X, Y]$ for all pointed topological spaces X?

(ii) Given a pointed topological space X, does there exist a natural product defined in $[X, Y]$ for all pointed topological spaces Y?

We start with a topological group P (see Appendix A) followed by H-groups and H-cogroups. The essential feature which is retained in an H space is a continuous multiplication with a unit. There is a significant class of topological spaces which are H-spaces but not topological groups.

Theorem 2.4.1 *Let X be any pointed topological space and P be a topological group with identity element as base point. Then $[X, P]$ can be given the structure of a group.*

Proof Given two base point preserving continuous maps $f, g : X \to P$, let their product $f \cdot g$ be defined by pointwise multiplication, i.e., $f \cdot g : X \to P$ is defined by $(f \cdot g)(x) = f(x)g(x)$, where the right side is the group multiplication μ in P. Thus $f \cdot g = \mu \circ (f \times g) \circ \Delta$, where $\Delta(x) = (x, x)$ is the diagonal map, i.e., $f \cdot g$ is the composite

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} P \times P \xrightarrow{\mu} P$$

Then $f \cdot g$ is another continuous map from X to P. Moreover, given further maps $f', g' : X \to P$ such that $f \simeq f'$ and $g \simeq g'$ then $f \cdot g \simeq f' \cdot g'$ (by Ex. 2 of Sect. 2.11)

$\Rightarrow [f]\cdot[g] = [f\cdot g] \Rightarrow$ the law of composition $f\cdot g$ carries over to give an operation '$\circ$' on $[X, P]$. Then the group structure on $[X, P]$ follows from the corresponding properties of the topological group. Consequently, $([X, P], \circ)$ is a group. ❑

Corollary 2.4.2 *If P is a topological group and $f : X \to Y$ is a base point preserving continuous map, then f induces a group homomorphism $f^* : [Y, P] \to [X, P]$ defined by $f^*([\alpha]) = [\alpha \circ f]$, $\forall\ [\alpha] \in [Y, P]$.*

Theorem 2.4.3 *Given a topological group P, there exists a contravariant functor π^P from the homotopy category of pointed topological spaces to the category of groups and homomorphisms.*

Proof It follows from Theorem 2.4.1 and Corollaries 2.4.2 and 2.3.8. ❑

Remark 2.4.4 Given a topological group P, the group structure on $[X, P]$ is endowed from the group structure on the set of base point preserving continuous maps from X to P. We come across some situations in which $[X, P]$ admits a natural group structure, but the set of base point preserving continuous maps from X to P has no group structure. If P is a pointed topological space having the same homotopy type as some topological group P', then π^P is naturally equivalent to $\pi^{P'}$. Hence π^P can be regarded as a functor to the category of groups.

Example 2.4.5 $S^1 = \{z \in \mathbf{C} : |z| = 1\}$ is an abelian topological group under usual multiplication of complex numbers. Then $[X, S^1]$ is an abelian group and if $f : X \to Y$, then $f^* : [Y, S^1] \to [X, S^1]$ is a homomorphism of groups.

Example 2.4.6 S^3 is a topological group (the multiplicative group of quaternions of norm 1). Then $[X, S^3]$ is a group and if $f : X \to Y$, then $f^* : [Y, S^3] \to [X, S^3]$ is group homomorphism.

Remark 2.4.7 If Y is a topological group, then a product $f\cdot g : X \to Y$ is given by $(f\cdot g)(x) = f(x)\cdot g(x)\,\forall\, x \in X$. To solve the problems (i) and (ii) we search for certain other classes of pointed topological spaces, called Hopf spaces (H-spaces) and Hopf groups (H-groups).

Definition 2.4.8 A pointed topological space P with a base point p_0, is called an H-space if there exists a continuous multiplication $\mu : P \times P \to P$, $(p, p') \mapsto pp'$ for which the (unique) constant map $c : P \to p_0 \in P$ is a homotopy identity, i.e., each composite

$$P \xrightarrow{(c,1_P)} P \times P \xrightarrow{\mu} P \text{ and } P \xrightarrow{(1_P,c)} P \times P \xrightarrow{\mu} P$$

is homotopic to 1_P, i.e., if each of the triangles in Fig. 2.6 is homotopy commutative; sometimes it is written as an ordered pair (P, μ).

Fig. 2.6 H-space

$$\begin{tikzcd} P \arrow[r, "(c,1_P)"] \arrow[dr, "1_P"'] & P \times P \arrow[d, "\mu"] & P \arrow[l, "(1_P,c)"'] \arrow[dl, "1_P"] \\ & P & \end{tikzcd}$$

Remark 2.4.9 The above homotopy commutativity means that the maps $P \to P$, $p \mapsto p_0p, pp_0$ are homotopic to 1_P rel $\{p_0\}$. In other words, $\exists$ homotopies L and R from $P \times I \to P$ such that $L(p, 0) = p_0p$, $L(p, 1) = p$, $L(p_0, t) = p_0$, $R(p, 0) = pp_0$, $R(p, 1) = p$ and $R(p_0, t) = p_0$, $\forall\ p \in P$ and $\forall\ t \in I$.

Definition 2.4.10 An H-space (P, μ) is said to be homotopy associative if the square in Fig. 2.7 is homotopy commutative, i.e., $\mu \circ (\mu \times 1_P) \simeq \mu \circ (1_P \times \mu)$, i.e., the two maps

$$P \times P \times P \to P, (p_1, p_2, p_3) \mapsto (p_1p_2)p_3, p_1(p_2p_3) \text{ are homotopic rel } \{p_0\}.$$

Definition 2.4.11 A continuous map $\phi : P \to P$ is said to be homotopy inverse for P and μ if each of the composites $P \xrightarrow{(1_P,\phi)} P \times P \xrightarrow{\mu} P$ and $P \xrightarrow{(\phi,1_P)} P \times P \xrightarrow{\mu} P$ is homotopic to the constant map $c : P \to P$, $p \mapsto p_0 \in P$, i.e., each of the maps $P \to P$, $p \mapsto p\phi(p), \phi(p)p$ is homotopic to c rel $\{p_0\}$.

Definition 2.4.12 An associative H-space P with an inverse is called an H-group or a generalized topological group. The point $p_0 \in P$ is called the homotopy unit of (P, μ).

Definition 2.4.13 A multiplicative μ on an H-space P is said to be homotopy abelian if the triangle in Fig. 2.8 is homotopy commutative, where $T(p, p') = (p', p)$, i.e., the two maps $P \times P \to P$, $(p, p') \mapsto pp', p'p$ are homotopic rel $\{p_0\}$.

Example 2.4.14 **(i)** Every topological group is an H-space with homotopy inverse. But its converse is not true in general (see (ii)). In particular, Lie groups (for

Fig. 2.7 Homotopy associative H-space

$$\begin{tikzcd} P \times P \times P \arrow[r, "\mu \times 1_P"] \arrow[d, "1_P \times \mu"'] & P \times P \arrow[d, "\mu"] \\ P \times P \arrow[r, "\mu"] & P \end{tikzcd}$$

Fig. 2.8 Abelian H-space

$$\begin{tikzcd} P \times P \arrow[rr, "T"] \arrow[dr, "\mu"'] & & P \times P \arrow[dl, "\mu"] \\ & P & \end{tikzcd}$$

Fig. 2.9 Homotopy homomorphism

$$\begin{tikzcd} P \times P \arrow[r, "\mu"] \arrow[d, "\alpha\times\alpha"'] & P \arrow[d, "\alpha"] \\ P' \times P' \arrow[r, "\mu'"'] & P' \end{tikzcd}$$

example, the general linear group, $GL(n, \mathbf{R})$ or the orthogonal group $O(n, \mathbf{R})$, see Appendix A) are H-spaces.

(ii) The infinite real projective space $\mathbf{R}P^\infty = \bigcup\limits_{n\geq 0} \mathbf{R}P^n$ and infinite complex projective space $\mathbf{C}P^\infty = \bigcup\limits_{n\geq 0} \mathbf{C}P^n$ are H-spaces but not topological groups.

Definition 2.4.15 Let (P, μ) and (P', μ') be two H-spaces. Then a continuous map $\alpha : P \to P'$ is called a homotopy homomorphism if the square in Fig. 2.9 is homotopy commutative.

Clearly, H-groups (H-spaces) and homotopy homomorphisms form a category.

Theorem 2.4.16 *A pointed topological space having the same homotopy type of an H-space (or an H-group) is itself an H-space (or H-group) in such a way that the homotopy equivalence is a homotopy homomorphism.*

Proof Let (P, μ) be an H-space and P' be a pointed topological space having the homotopy type of the space P. Then there exist continuous maps $f : P \to P'$ and $g : P' \to P$ such that $g \circ f \simeq 1_P$ and $f \circ g \simeq 1_{P'}$. Define $\mu' : P' \times P' \to P'$ to be the composite

$$P' \times P' \xrightarrow{g\times g} P \times P \xrightarrow{\mu} P \xrightarrow{f} P' \text{ i.e., } \mu' = f \circ \mu \circ (g \times g).$$

Then μ' is a continuous multiplication in P'. Moreover, the composites

$$P' \xrightarrow{(1,c')} P' \times P' \xrightarrow{\mu'} P' \tag{2.9}$$

$$\text{and } P' \xrightarrow{g} P \xrightarrow{(1,c)} P \times P \xrightarrow{\mu} P \xrightarrow{f} P' \tag{2.10}$$

are equal. As P is an H-space, the composite in (2.10) is homotopic to the composite $f \circ g$, because $f \circ \mu \circ (1, c) \circ g \simeq f \circ 1_P \circ g \simeq f \circ g$. Again, $f \circ g \simeq 1_{P'} \Rightarrow \mu' \circ (1, c') \simeq 1_{P'}$ by (2.9) and (2.10). Similarly, $\mu' \circ (c', 1) \simeq 1_{P'}$. Consequently, P' is an H-space with continuous multiplication μ'. Since the diagram in Fig. 2.10 is homotopy commutative, g is a homotopy homomorphism and so is f. If μ is homotopy associative or homotopy abelian, and if $\phi : P \to P$ is a homotopy inverse for P, then the composite map $f \circ \phi \circ g : P' \to P'$ is a homotopy inverse for P'. ❑

Generalizing the Theorem 2.4.1 we have the following theorem:

Fig. 2.10 Homotopy commutative square

$$\begin{tikzcd} P' \times P' \arrow[r, "\mu'"] \arrow[d, "g\times g"'] & P' \arrow[d, "g"] \\ P \times P \arrow[r, "\mu"] & P \end{tikzcd}$$

Theorem 2.4.17 *If X is any pointed topological space and P is an H-group, then $[X, P]$ can be given the structure of a group.*

Proof Similar to the proof of Theorem 2.4.1. ❑

Theorem 2.4.18 *Let P be a pointed space with base point p_0 and $p_1, p_2 : P \times P \to P$ be the projections from the first and the second factors respectively. If $i_1, i_2 : P \to P \times P$ are inclusions defined by $i_1(p) = (p, p_0), i_2(p) = (p_0, p)$ for all $p \in P$, then the pointed space P is an H-space iff there exists a map $\mu : P \times P \to P$ such that $\mu \circ i_1 \simeq \mu \circ i_2$. Moreover, this map μ satisfies the condition $[\mu] = [p_1]\cdot[p_2]$ and if $f_1, f_2 : X \to P$ are maps then $[f_1]\cdot[f_2]$ is the homotopy class of the composite*

$$X \xrightarrow{\triangle} X \times X \xrightarrow{f_1\times f_2} P \times P \xrightarrow{\mu} P$$

Proof It follows from hypothesis that $p_1 \circ i_1 = p_2 \circ i_2 = 1_P$ and $p_1 \circ i_2 = p_2 \circ i_1 = c$, where $c : P \to p_0$ is the constant map. If $\mu = P \times P \to P$ is a map such that $\mu \circ i_1 \simeq 1_P \simeq \mu \circ i_2$, then this μ is a multiplication admitting P the structure of an H-space.

Conversely suppose P is a pointed space and $\mu : P \times P \to P$ is a map such that $\mu \circ i_1 \simeq 1_P \simeq \mu \circ i_2$. Then given maps $f_1, f_2 : X \to P$ define $f_1 \cdot f_2 = \mu \circ (f_1 \times f_2) \circ \triangle$. This composition is compatible with homotopy, and induces a natural product in $[X, P]$. Hence P is an H-space and $p_1 \cdot p_2 = \mu \circ (p_1 \times p_2) \circ \triangle = \mu \circ 1_P = \mu$. Consequently, P is an H-space. ❑

Remark 2.4.19 For an arbitrary H-space P, the multiplication defined in $[X, P]$ may not be associative. If $f_1, f_2, f_3 : X \to P$ are maps, then $(f_1 \cdot f_2) \cdot f_3$ and $f_1 \cdot (f_2 \cdot f_3)$ are by definition the homotopy classes of the composites

$$X \xrightarrow{\triangle_3} X \times X \times X \xrightarrow{f_1\times f_2\times f_3} P \times P \times P \xrightarrow{\mu\times 1_P} P \times P \xrightarrow{\mu} P,$$

$$X \xrightarrow{\triangle_3} X \times X \times X \xrightarrow{f_1\times f_2\times f_3} P \times P \times P \xrightarrow{1_P\times\mu} P \times P \xrightarrow{\mu} P,$$

where $\triangle_3 = (\triangle \times 1_X) \circ \triangle = (1_X \times \triangle) \circ \triangle : X \to X \times X \times X$ is the diagonal map. Hence the condition$\mu \circ (\mu \times 1_P) \simeq \mu \circ (1_P \times \mu P)$ is sufficient for associativity. It is also necessary that if $X = P \times P \times P$ and f_1, f_2, f_3 are projections of $P \times P \times P$ into P, then $(f_1 \times f_2 \times f_3) \circ \triangle_3$ is the identity map.

The above discussion can be stated in the form of the following interesting result:

Theorem 2.4.20 *The set $[X, P]$ admits a monoid structure natural with respect to X iff P is a homotopy associative H-space.*

Theorem 2.4.21 *Let (P, p_0) be an H-group with multiplication μ and homotopy inverse ϕ. Then for every pointed topological space (X, x_0), the set $[(X, x_0), (P, p_0)]$ denoted $[X, P]$ can be given the structure of a group if we define the product $[f]\cdot[g]$ to be the homotopy class of the composite map*

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f\times g} P \times P \xrightarrow{\mu} P,$$

where Δ is the diagonal map given by $\Delta(x) = (x, x)$. The identity element of the group is the class $[c]$ of the constant map $c : X \to p_0$ and the inverse of $[f]$ is given by $[f]^{-1} = [\phi \circ f]$. If μ is homotopy commutative, then $[X, P]$ is abelian.

Proof Define the product $[f]\cdot[g] = [\mu \circ (f \times g) \circ \Delta]$. We claim that $[f]\cdot[g]$ is well defined. To show this, let $H : X \times I \to P$ be a homotopy between f and f' and $G : X \times I \to P$ a homotopy between g and g'. Define a homotopy $M : X \times I \to P$ by $M_t(x) = M(x, t) = \mu(H(x, t), G(x, t))$. Then $M_0 = \mu \circ (f \times g) \circ \Delta$ and $M_1 = \mu \circ (f' \times g') \circ \Delta \Rightarrow \mu \circ (f \times g) \circ \Delta \simeq \mu \circ (f' \times g') \circ \Delta \Rightarrow [f]\cdot$ $[g] = [f']\cdot[g'] \Rightarrow$ the multiplication is independent of the choice of representatives of the classes. $\Rightarrow$ the multiplication is well defined.

We now prove the associativity of the multiplication. Let $h : (X, x_0) \to (P, p_0)$ be a third map. Then

$$\begin{aligned}
[f]\cdot([g]\cdot[h]) &= [\mu \circ (f \times \{\mu \circ (g \times h) \circ \Delta\}) \circ \Delta] \\
&= [\mu \circ (1 \times \mu) \circ (f \times g \times h) \circ (1 \times \Delta) \circ \Delta] \\
&= [\mu \circ (\mu \times 1) \circ (f \times g \times h) \\
&\qquad \circ (\Delta \times 1) \circ \Delta] \text{ by homotopy associativity of } \mu \\
&= [\mu \circ (\{\mu \circ (f \times g) \circ \Delta\} \times h) \circ \Delta] \\
&= ([f]\cdot[g])\cdot[h].
\end{aligned}$$

Again $[f]\cdot[c] = [\mu \circ (f \times c) \circ \Delta] = [\mu \circ (1, c) \circ f] = [1_P \circ f] = [f]$ and similarly, $[c]\cdot[f] = [f]$, $\forall\ [f] \in [X, P]] \Rightarrow [c]$ is an identity element for $[X, P]$. Finally, $[\phi \circ f]\cdot[f] = [\mu \circ ((\phi \circ f) \times f) \circ \Delta] = [\mu \circ (\phi, 1) \circ f] = [c]$ and $[f]\cdot$ $[\phi \circ f] = [c] \Rightarrow [f]^{-1} = [\phi \circ f]$. Consequently, if (P, p_0) is an H-group, then $[X, P]$ is a group. If μ is homotopy commutative, the last part follows immediately ❑

The converse of the Theorem 2.4.21 is also true.

Theorem 2.4.22 *The set $[X, P]$ admits a group structure natural with respect to X iff P is an H-group.*

Proof It follows from the Theorems 2.4.20 and 2.4.21. ❑

Remark 2.4.23 The set $[X, P]$ can be endowed with a monoid structure natural with respect to X iff P is a homotopy associative H-space.

Theorem 2.4.24 *If $g : X \to Y$ is a base point preserving continuous map and P is an H-group, then the induced function $g^* : [Y, P] \to [X, P]$ defined by $g^*([\alpha]) = [\alpha \circ g]$ is a group homomorphism. In particular, if g is a homotopy equivalence, then g^* is an isomorphism.*

Proof Given continuous maps $f_1, f_2 : Y \to P$, we have $(f_1 \cdot f_2) \circ g = \mu \circ (f_1 \times f_2) \circ \Delta \circ g = \mu \circ (f_1 \times f_2) \circ (g \times g) \circ \Delta = \mu \circ (f_1 \circ g \times f_2 \circ g) \circ \Delta = (f_1 \circ g) \cdot (f_2 \cdot g)$. Then $g^*([f_1] \cdot [f_2]) = g^*[f_1] \cdot g^*[f_2] \Rightarrow g^*$ is a homomorphism. If g is a homotopy equivalence, then $\exists$ a continuous map $f : Y \to X$ such that $f \circ g \simeq 1_X$ and $g \circ f = 1_Y$. Then $(f \circ g)^* = g^* \circ f^* = 1_d$ and $(g \circ f)^* = f^* \circ g^* = 1_d \Rightarrow g^*$ is an isomorphism. ❑

Theorem 2.4.25 *If P is an H-group, π^P is a contravariant functor from the homotopy category of pointed topological spaces to the category of groups and homomorphisms. If P is an abelian H-group, then the functor π^P takes values in the category of abelian groups.*

Proof Define the object function by $\pi^P(X) = [X, P]$, which is a group for every pointed space X by Theorem 2.4.22. If $g : X \to Y$ is a base point preserving map, define the morphism function by $\pi^P(g) = g^*$ by Theorem 2.4.24. Then the theorem follows. ❑

The converse of the Theorem 2.4.25 is also true.

Theorem 2.4.26 *If P is a pointed topological space such that π^P takes values in the category of groups, then P is an H-group (abelian if π^P takes values in the category of abelian groups). Moreover, for any pointed space X, the group structures on $\pi^P(X)$ and on $[X, P]$ given in the Theorem 2.4.21 coincide.*

Proof Let $p_1 : P \times P \to P$ and $p_2 : P \times P \to P$ be the projections on the first and second factor respectively. Let $\mu : P \times P \to P$ be a map such that $[\mu] = [p_1] \cdot [p_2]$, where $\cdot$ is the product in the group $[P \times P, P]$. For any continuous maps $f, g : X \to P$, the induced map $(f, g)^* : [P \times P, P] \to [X, P]$ is a homomorphism and

$$[\mu \circ (f, g)] = (f, g)^*[\mu] = (f, g)^*([p_1] \cdot [p_2]) = (f, g)^*[p_1] \cdot (f, g)^*[p_2] = [f] \cdot [g]$$

implies that the multiplication in $[X, P]$ is induced by the multiplication map μ. Let X be a one-point space. The unique map $X \to P$ represents the identity element of the group $[X, P]$. Since the unique map $P \to X$ induces a homomorphism $[X, P] \to [P, P]$, it follows that the composite $P \to X \to P$, which is the constant map $c : P \to P$ represents the identity element of $[P, P]$. Hence it follows that $\mu \circ (1_P, c) \simeq 1_P$ and $\mu \circ (c, 1_P) \simeq I_P$. Consequently, P is an H-space. To prove that μ is homotopy associative, let $q_1, q_2, q_3 : P \times P \times P \to P$ be the projections. Then $[\mu \circ (1_P \times \mu)] = (1_P \times \mu)^*[\mu] = (1_P \times \mu)^*[p_1] \cdot (1_P \times \mu)^*[p_2] = [q_1] \cdot [\mu \circ (q_2, q_3)] = [q_1] \cdot ([q_2] \cdot [q_3])$. Similarly, $[\mu \circ (\mu \times 1_P)] = ([q_1] \circ [q_2]) \circ [q_3]$. Since $[P \times P \times P, P]$ has an associative multiplication, it follows that

$$\mu \circ (1_P \times \mu) \simeq \mu \circ (\mu \times 1_P).$$

Finally, we show that P has a homotopy inverse. Let $\phi : P \to P$ be the map such that $[1_P]\cdot[\phi] = [c]$. Then $\mu \circ (1_P, \phi) \simeq c$. Similarly, $\mu \circ (\phi, 1_P) \simeq c$. Hence, ϕ is a homotopy inverse for P and μ.

Consequently, P is an H-group. Moreover, if $[P \times P, P]$ is an abelian group, a similar argument shows that P is an abelian H-group. ❑

Given two H-groups P and P', we now compare between the contravariant functors π^P and $\pi^{P'}$.

Theorem 2.4.27 *Let $\alpha : P \to P'$, be a continuous map between H-groups. Then α induces a natural transformation $\alpha_* : \pi^P \to \pi^{P'}$ in the category of H-groups iff α is a homomorphism.*

Proof For each pointed topological space X, define $\alpha_*(X) : \pi^P(X) \to \pi^{P'}(X)$ by the rule $\alpha_*(X)[h] = [\alpha \circ h]$, $\forall\, [h] \in \pi^P(X)$. Then diagram in Fig. 2.11 is commutative, for every $f : Y \to X$, because, $(\pi^{P'}(f) \circ \alpha_*(X))[h] = \pi^{P'}(f)([\alpha \circ h]) = [(\alpha \circ h) \circ f]$ and $(\alpha_*(Y) \circ \pi^P(f))[h] = \alpha_*(Y)[h \circ f] = [\alpha \circ (h \circ f)]$, which are equal. Hence α_* is a natural transformation.
The converse part is left as an exercise. ❑

We now investigate the question of existence of homotopy inverses for a homotopy associative H-space.

Theorem 2.4.28 *If P is a homotopy associative H-space, then P is an H-group if and only if the shear map $\psi : P \times P \to P \times P$, given by $\psi(x, y) = (x, xy)$ is a homotopy equivalence.*

Proof Case I. First we consider the particular case when P is a topological group. Then the map ψ is a homeomorphism with inverse $\psi^{-1} : P \times P \to P \times P$ defined by $\psi^{-1}(u, v) = (u, u^{-1}v)$. Let $j = p_2 \circ \psi^{-1} \circ i_1$, where $i_1 : P \to P \times P$ is the inclusion, defined by $i_1(y) = (y, y_0)$, where y_0 is the base point of P and $p_1, p_2 : P \times P \to P$ be the projections on the first and the second factor respectively. Then $[j]$ is the inverse of the homotopy class of the identity map $1_P \in [P, P]$, so that the composites

$$P \xrightarrow{\triangle} P \times P \xrightarrow{j \times 1_P} P \times P \xrightarrow{\mu} P,$$

$$P \xrightarrow{\triangle} P \times P \xrightarrow{1_P \times j} P \times P \xrightarrow{\mu} P$$

are each nullhomotopic.

Fig. 2.11 Natural transformation α_*

$$\begin{array}{ccc} \pi^P(X) & \xrightarrow{\alpha_*(X)} & \pi^{P'}(X) \\ {\scriptstyle \pi^P(f)=f^*}\downarrow & & \downarrow{\scriptstyle f^*=\pi^{P'}(f)} \\ \pi^P(Y) & \xrightarrow{\alpha_*(Y)} & \pi^{P'}(Y) \end{array}$$

Case II. We now consider the general case. Let ψ be a homotopy equivalence with homotopy inverse ϕ. Define $j \in [P, P]$ by $j = p_2 \circ \phi \circ i_1$. Then

$$\begin{aligned}(p_1 \circ \psi)(x, y) &= p_1(\psi(x, y)) = p_1(x, xy) = x \\ &= p_1(x, y), \ \forall \ (x, y) \in P \times P \Rightarrow p_1 \circ \psi = p_1.\end{aligned}$$

Again, $(p_2 \circ \psi)(x, y) = p_2(x, xy) = xy = \mu(x, y), \ \forall \quad (x, y) \in P \times P \Rightarrow p_2 \circ \psi = \mu$.

Hence

$$p_1 \simeq p_1 \circ \overbrace{\psi \circ \phi}^{1_d} = (p_1 \circ \psi) \circ \phi = p_1 \circ \phi,$$
$$p_2 \simeq p_2 \circ \overbrace{\psi \circ \phi}^{1_d} = (p_2 \circ \psi) \circ \phi = \mu \circ \phi.$$

In particular, $p_1 \circ \phi \circ i_1 \simeq p_1 \circ i_1 = 1_P$, since $(p_1 \circ i_1)(y) = p_1(y, y_0) = y = 1_P(y), \ \forall \ y \in P$.

Hence

$$\begin{aligned}\mu \circ (1_P \times j) \circ \Delta &= \mu \circ (p_1 \circ \phi \circ i_1 \times p_2 \circ \phi \circ i_1) \circ \Delta \\ &= \mu \circ (p_1 \times p_2) \circ (\phi \circ i_1 \times \phi \circ i_1) \circ \Delta \\ &= \mu \circ (p_1 \times p_2) \circ \Delta \circ \phi \circ i_1 \\ &= \mu \circ \phi \circ i_1 = p_2 \circ \overbrace{\psi \circ \phi}^{1_d} \circ i_1 \simeq p_2 \circ i_1 \simeq c,\end{aligned}$$

where $c : P \to p_0 \in P$ is the constant map.

Hence j is a right inverse of the identity map.

It follows from the above argument that every element of $[X, P]$ has a left inverse, and hence $[X, P]$ is a group. Conversely, if P is an H-group, then the map $\phi : (u, v) \mapsto (u, j(u)v)$ is a homotopy inverse of the shear map ψ, because $(\psi \circ \phi)(u, v) = \psi(\phi(u, v)) = \psi(u, j(u)v) = (u, uj(u)v) = (u, p_0 v)$ and since p_0 is a homotopy unit, it follows that $\psi \circ \phi \simeq 1_d$. Similarly, $\phi \circ \psi \simeq 1_d$. Consequently, ψ is a homotopy equivalence. ❑

Remark 2.4.29 Some of the techniques which apply to topological groups can be applied to H-spaces, but not all. From the viewpoint of homotopy theory, it is not the existence of a continuous inverse which is the important distinguishing feature, but rather the associativity of multiplication. If we consider S^1, S^3 and S^7 as the complex, quaternionic and Cayley numbers of unit norm, these spaces have continuous multiplication. The multiplication in the first two cases are associative but not associative in the last case. $S^!$ and S^3 are topological groups. The spheres S^1, S^3 and S^7 are the only spheres that are H-spaces proved by J.F. Adams (1930–1989) in (1962).

Remark 2.4.30 Every topological group is an H-group.

We now describe another important example of an H-group. Loop spaces form an important class of grouplike spaces, called H-groups.

Definition 2.4.31 (*Loop Space*) Let Y be a pointed topological space with base point y_0. The loop space of Y(based at y_0) denoted ΩY(or $\Omega(Y, y_0)$), is defined to be the space of continuous functions $\alpha : (I, \dot{I}) \to (Y, y_0)$, topologized by the compact open topology. Then $\Omega(Y, y_0)$ is considered as a pointed space with base point α_0 equals to the constant map $c : I \to y_0$.

The elements of ΩY are called loops in Y.

Theorem 2.4.32 *$\Omega(Y, y_0)$ is an H-group.*

Proof Define a map $\mu : \Omega Y \times \Omega Y \to \Omega Y$ by

$$\mu(\alpha, \beta)(t) = \begin{cases} \alpha(2t), 0 \leq t \leq 1/2 \\ \beta(2t-1), 1/2 \leq t \leq 1. \end{cases}$$

To show that μ is continuous, consider the evaluation map $E : \Omega Y \times I \to Y$ defined by $E(\alpha, t) = \alpha(t)$. Since I is locally compact, by Theorem of exponential correspondence (see Theorem 1.14.2 of Chap. 1) it is sufficient to show that the composite map

$$\Omega Y \times \Omega Y \times I \xrightarrow{\mu \times 1_d} \Omega Y \times I \xrightarrow{E} Y$$

is continuous.

Then the theorem of exponential correspondence and the Pasting lemma show the continuity of μ, since the above composite is continuous on each of the closed sets $\Omega Y \times \Omega Y \times [0, \frac{1}{2}]$ and $\Omega Y \times \Omega Y \times [\frac{1}{2}, 1]$.

μ **is associative**: To show this define $G : \Omega Y \times \Omega Y \times \Omega Y \times I \to \Omega Y$ by the rule

$$G(\alpha, \beta, \gamma, s)(t) = \begin{cases} \alpha(\frac{4t}{1+s}), & 0 \leq t \leq (1+s)/4 \\ \beta(4t-1-s), & (1+s)/4 \leq t \leq (2+s)/4 \\ \gamma(\frac{4t-2-s}{2-s}), & (2+s)/4 \leq t \leq 1. \end{cases}$$

The continuity of G follows from the Pasting lemma. Clearly, $G : \mu \circ (\mu \times 1_d) \simeq \mu \circ (1_d \times \mu)$

Existence of homotopy unit: If $c : \Omega Y \to \Omega Y$ is the constant map whose value is the constant loop $c : I \to Y, c(t) = y_0$, then $\mu \circ (\beta, c) \simeq \beta$ and $\mu \circ (c, \beta) \simeq \beta$ for every loop β.

The first homotopy is given by $F : \Omega Y \times I \to \Omega Y$, where

$$F(\beta, s)(t) = \begin{cases} \beta(\frac{2t}{1+s}), & 0 \leq t \leq (1+s)/2 \\ y_0, & (1+s)/2 \leq t \leq 1. \end{cases}$$

The continuity of F follows from Pasting lemma. The second homotopy is defined in an analogous manner.

Existence of homotopy inverse: Let $\phi : \Omega Y \to \Omega Y$ be a map such that $\phi(\alpha)(t) = \alpha(1-t)$. Then ϕ determines homotopy inverses. The homotopy $H : \Omega Y \times I \to \Omega Y$, where

$$H(\alpha, s)(t) = \begin{cases} \alpha(2(1-s)t), & 0 \leq t \leq 1/2 \\ \alpha(2(1-s)(1-t)), & 1/2 \leq t \leq 1, \end{cases}$$

begins at $\mu \circ (\alpha, \phi(\alpha))$ and ends at c. The second homotopy is given in an analogous manner.

Consequently, ΩY is an H-group. □ ❑

Definition 2.4.33 Given a pointed space X, iterated loop spaces $\Omega^n X$ are defined inductively: $\Omega^n X = \Omega(\Omega^{n-1} X)$ for ≥ 1 and $\Omega^0 X$ is taken to be X.

Corollary 2.4.34 *For $n \geq 1$, $\Omega^n X$ is an H-group for every pointed space X.*

Theorem 2.4.35 *Ω is a covariant functor from the category of pointed topological spaces and continuous maps to the category of H-groups (Hopf groups) and their continuous homomorphisms.*

Proof If $f : X \to Y$ is base point preserving continuous maps, then $\Omega f : \Omega(X) \to \Omega(Y)$ defined by $(\Omega f)(\alpha)(t) = f(\alpha(t))$ is a homomorphisms of H-groups. The object function is given by $X \mapsto \Omega(X)$ and the morphism function is given by $\Omega f : \Omega(X) \to \Omega(X)$. Then Ω is a covariant functor. ❑

Theorem 2.4.36 *For every pointed topological space Y, ΩY is an H-group and for every pointed space X, $[X, \Omega Y]$ is a group. If $f : X \to X'$ is a base point preserving continuous map, then $f^* : [X', \Omega Y] \to [X, \Omega Y]$ is a group homomorphism.*

Proof The theorem follows from Theorems 2.4.32 and 2.4.24. ❑

2.4.2 H-Cogroups and Suspension Spaces

This subsection conveys the dual concepts of H-groups, called H-cogroups introduced by Beno Eckmann (1917–2008) and Peter John Hilton (1923–2010) in 1958. It involves wedge products of pointed topological spaces. Suspension spaces of pointed topological spaces form an extensive class of H-cogroups.

Recall that the wedge $X \vee X$ is viewed as the subspace $X \times \{x_0\} \cup \{x_0\} \times X$ of the product space $X \times X$. If $p_i : X \times X \to X$, for $i = 1, 2$ are the usual projections onto the first or second coordinate respectively, then define 'projections' $q_i : X \vee X \to X$, for $i = 1, 2$ by $q_i = p_i|_{X \vee X}$; each q_i sends the appropriate copy of $x \in X$, namely, (x, x_0) or (x_0, x) into itself.

Definition 2.4.37 A pointed topological space (X, x_0) is called an H-cogroup if there exists a base point preserving continuous map $\mu : X \to X \vee X$, called comultiplication, such that $q_1 \circ \mu \simeq 1_X \simeq q_2 \circ \mu$, $(1_X \vee \mu) \circ \mu \simeq (\mu \vee 1_X) \circ \mu$ (co-associativity) and there exists a base point preserving continuous map $h : X \to X$ such that $(1_X, h) \circ \mu \simeq c \simeq (h, 1_X) \circ \mu$, ($h$ is called an inverse), where $c : X \to X$ is the constant map at x_0.

Remark 2.4.38 In an H-cogroup given maps $f : X \to Z$ and $g : Y \to Z$ in $\mathcal{Top}_*$, the map $(f, g) : X \vee Y \to Z$ is defined by the characteristic property:
$(f, g)|_X = f$ and $(f, g)|_Y = g$.

H-cogroup is now defined more explicitly keeping similarity with the definition of H-group.

Definition 2.4.39 A pointed topological space X with base point x_0 is called an H-cogroup if there exists a base point preserving continuous map

$$\mu : X \to X \vee X,$$

called H-comultiplication such that the following conditions hold:
Existence of homotopy identity. If $c : X \to X$ is the (unique) constant map at x_0, then each composite map

$$X \xrightarrow{\mu} X \vee X \xrightarrow{(c,1_X)} X \text{ and } X \xrightarrow{\mu} X \vee X \xrightarrow{(1_X,c)} X$$

is homotopic to 1_X.
Homotopy associativity. The diagram in Fig. 2.12 is commutative up to homotopy, i.e.,

$$(1_X \vee \mu) \circ \mu \simeq (\mu \vee 1_X) \circ \mu.$$

Existence of homotopy inverse. There exists a map $h : X \to X$ such that each composite map

$$X \xrightarrow{\mu} X \vee X \xrightarrow{(1_X,h)} X \text{ and } X \xrightarrow{\mu} X \vee X \xrightarrow{(h,1_X)} X$$

is homotopic to $c : X \to X$.

$$\begin{tikzcd} X \arrow[r, "\mu"] \arrow[d, "\mu"'] & X \vee X \arrow[d, "(1_X \vee \mu)"] \\ X \vee X \arrow[r, "(\mu \vee 1_X)"] & X \vee X \vee X \end{tikzcd}$$

Fig. 2.12 Homotopy associativity of H-cogroups

Fig. 2.13 Diagram for abelian H-cogroup

$$\begin{tikzcd} & X \arrow[dl, "\mu"'] \arrow[dr, "\mu"] & \\ X \vee X \arrow[rr, "T"] & & X \vee X \end{tikzcd}$$

Fig. 2.14 Homotopy homomorphism of H-cogroups

$$\begin{tikzcd} X \arrow[r, "\mu"] \arrow[d, "\alpha"'] & X \vee X \arrow[d, "\alpha \vee \alpha"] \\ Y \arrow[r, "\gamma"] & Y \vee Y \end{tikzcd}$$

Definition 2.4.40 An H-cogroup X is said to be abelian if the triangle in Fig. 2.13 is homotopy commutative, where $T(x, x') = (x', x)$, $\forall\ x, x' \in X$.

Definition 2.4.41 Let X and Y be H-cogroups with comultiplications μ and γ respectively. Then the map $\alpha : X \to Y$ is said to be a homomorphism of H-cogroups if diagram in Fig. 2.14 is homotopy commutative, i.e., $\alpha \vee \alpha \circ \mu \simeq \gamma \circ \alpha$.

Remark 2.4.42 The definition of an H-cogroup closely resembles to that of an H-group. We merely turn all the maps round and use the one-point union instead of the product.

Theorem 2.4.43 *If X is an H-cogroup and Y is any pointed space, then $[X, Y]$ can be given the structure of a group. Moreover, if $g : Y \to Z$ is a base point preserving continuous map, then the induced function $g_* : [X, Y] \to [X, Z]$ is a homomorphism in general and it is an isomorphism if g is a homotopy equivalence.*

Proof Given $f_1, f_2 : X \to Y$, define a product in $[X, Y]$ by the rule $f_1 \cdot f_2 = \bigtriangledown \circ (f \vee g) \circ \mu$, where $\bigtriangledown : X \vee X \to X$ is the folding map, defined by $\bigtriangledown(x_0, x) = \bigtriangledown(x, x_0) = x$. Proceed as in proofs of Theorems 2.4.22 and 2.4.24. ❑

Dualizing the Theorems 2.4.16, 2.4.25–2.4.27, following theorems are proved.

Theorem 2.4.44 *A pointed space having the same homotopy type of an H-cogroup is itself an H-cogroup in such a way that the homotopy equivalence is a homomorphism.*

Theorem 2.4.45 *If Q is a H-cogroup, then π_Q is a covariant functor from the homotopy category of pointed spaces with values in the category of groups and homomorphisms. If Q is an abelian H-cogroup, this functor takes values in the category of abelian groups.*

Theorem 2.4.46 *If Q is a pointed topological space such that π_Q takes values in the category of groups, then Q is an H-cogroup (abelian if π_Q takes values in the category of abelian groups). Furthermore, for a pointed topological space X the group structure on $\pi_Q(X)$ is identical with that determined by the H-cogroup Q as in Theorem 2.4.43.*

Theorem 2.4.47 *If $\alpha : Q \to Q'$ is a continuous map between H-cogroups, then there is a natural transformation from $\pi_{Q'}$ to π_Q in the category of groups if and only if α is a homomorphism.*

We now describe suspension spaces which are dual to loop spaces. Suspension spaces give an extensive class of H-cogroups which are dual to H-groups. The impact of suspension operator is realized from a classical theorem of H. Freudenthal (1905–1990) known as Freudenthal suspension theorem (see Chap. 7).

Example 2.4.48 (*Suspension Space*) Let X be a pointed topological space with base point x_0. The suspension space of X, denoted by ΣX, is defined to be the quotient space of $X \times I$ in which $(X \times 0) \cup (x_0 \times I) \cup (X \times 1)$ has been identified to a single point. This is sometimes called the reduced suspension. If $(x, t) \in X \times I$, we use $[x, t]$ to denote the corresponding point of ΣX under the quotient map $X \times I \to \Sigma X$. Then $[x_0, 0] = [x_0, t] = [x', 1]$, $\forall\ x, x' \in X$ and $\forall\ t \in I$. The point $[x_0, 0] \in \Sigma X$ is also denoted by x_0 and ΣX is a pointed space with base point x_0. Moreover, if $f : X \to Y$ is a base point preserving continuous map, then $\Sigma f : \Sigma X \to \Sigma Y$ is defined by

$$\Sigma f([x, t]) = [f(x), t].$$

Consequently, Σ is a covariant functor from the category $\mathcal{T}op_*$ of pointed spaces and continuous maps to itself.

Remark 2.4.49 If $f \simeq g : X \to Y$, then $\Sigma f \simeq \Sigma g : \Sigma X \to \Sigma Y$.

We now show that Σ is also a covariant functor from the category $\mathcal{T}op_*$ to the category of H-cogroups and homomorphisms. We define a comultiplication $\gamma : \Sigma X \to \Sigma X \vee \Sigma X$ by the formula as shown in Fig. 2.15

$$\gamma([x, t]) = \begin{cases} ([x, 2t], x_0), & 0 \le t \le 1/2 \\ (x_0, [x, 2t - 1]), & 1/2 \le t \le 1. \end{cases}$$

Clearly, γ is continuous and makes ΣX an H-cogroup.

Theorem 2.4.50 *For any pointed topological space X, ΣX is an H-cogroup. Moreover, if $f : X \to Y$ is a base point preserving continuous map, then $\Sigma f : \Sigma X \to \Sigma Y$, $[x, t] \mapsto [f(x), t]$ is a homomorphism of H-cogroups.*

Proof As ΣX and ΣY are both H-cogroups, the proof follows from the definition of Σf. ❑

Theorem 2.4.51 *For any pair of pointed spaces X and Y, $[\Sigma X \to Y]$ is a group.*

Proof As ΣX is an H-cogroup, the theorem follows from Theorem 2.4.43. ❑

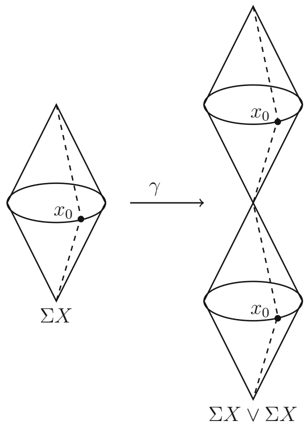


Fig. 2.15 Comultiplication γ

2.5 Adjoint Functors

This section provides an example of a special pair of functors, called adjoint functors in the language of category theory. This categorical notion of adjoint functors was introduced by Daniel Kan (1927–2013) in 1958. There is a close relation between the loop functor Ω and the suspension functor Σ in the category $\mathcal{T}op_*$.

Proposition 2.5.1 *The functors Ω and Σ defined from the category $\mathcal{T}op_*$ of pointed spaces and continuous maps to itself form a pair of adjoint functors in the sense that for pointed topological spaces X and Y in $\mathcal{T}op_*$ there is an equivalence* $\text{mor}\,(\Sigma X, Y) \approx \text{mor}\,(X, \Omega Y)$*, where both sides are the set of morphisms in the category $\mathcal{T}op_*$.*

Proof If $g : X \to \Omega Y$ is in $\mathcal{T}op_*$, then the corresponding morphism $g' : \Sigma X \to Y$ is defined by $g'[x, t] = g(x)(t)$, $\forall\ x \in X$ and $\forall\ t \in I$. Thus if $h : Y \to Y'$, then $(\Omega h \circ g)' = h \circ g' : \Sigma X \to Y'$, and if $f : X' \to X$, then $(g \circ f)' = g' \circ \Sigma f : \Sigma X' \to Y$. Then the correspondence $g \leftrightarrow g'$ gives a natural equivalence from the functor $(\Sigma -, -)$ to the functor $(-, \Omega -)$ on the category $\mathcal{T}op_*$. ❑

This natural equivalence plays an important role in the homotopy category $\mathcal{H}tp_*$ of pointed topological spaces.

Theorem 2.5.2 *There exists a natural equivalence from the functor* $[\Sigma -, -]$ *to the functor* $[-, \Omega -]$ *on the category $\mathcal{H}tp_*$.*

Proof For pointed spaces X and Y, a homotopy $G : X \times I \to Y$ maps $x_0 \times I$ into y_0. Therefore it defines a map $F : X \times I/x_0 \times I \to Y$. Since $\Sigma(X \times I/x_0 \times I)$ can be identified with $\Sigma X \times I/x_0 \times I$ by the homeomorphism $[(x, t), t'] \leftrightarrow ([x, t'], t)$, $\forall\ x \in X, t, t' \in I$, it follows that homotopies $F : X \times I/x_0 \times I \to \Omega Y$ correspond bijectively to homotopies $F' : \Sigma X \times I/x_0 \times I \to Y$. Consequently, the equivalence defined in Proposition 2.5.1 gives rise to an equivalence $[\Sigma X, Y] \approx [X, \Omega Y]$ such that if $g : X \to \Omega Y$ and $g' : \Sigma X \to Y$ are related by $g'[x, t] = g(x)(t)$, then $[g']$ corresponds to $[g]$. Hence there is a natural equivalence from the functor $[\Sigma -, -]$ to the functor $[-, \Omega -]$. ❑

Definition 2.5.3 In the language of category theory, the functors Ω and Σ are called adjoint functors in the sense of Theorem 2.5.2.

The above results are summarized in the basic Theorem 2.5.4.

Theorem 2.5.4 *The suspension functor Σ is a covariant functor from the category $\mathcal{T}op_*$ of pointed topological spaces and continuous maps to the category of H-cogroups and continuous homomorphisms. Moreover, the functor Σ preserves homotopies, i.e., if $f_0, f_1 : X \to Y_0$ are homotopic by the homotopy F_t, then $\Sigma f_0, \Sigma f_1$ are homotopic by the homotopy ΣF_t, which is a continuous homomorphism for each $t \in I$.*

Corollary 2.5.5 *The suspension functor Σ is a covariant functor from the homotopy category of pointed topological spaces and homotopy classes of continuous maps to the category of H-cogroups and continuous homomorphisms.*

We now show that for $n \geq 1$, the sphere S^n admits an extensive family of H-cogroups.

Proposition 2.5.6 *For $n \geq 0$, S^{n+1} is an H-cogroup.*

Proof To show this it is sufficient to prove that $\Sigma S^n \approx S^{n+1}$. Let $p_0 = (1, 0, \ldots, 0)$ be the base point of S^n. We consider $\mathbf{R}^{n+1}$ as embedded in $\mathbf{R}^{n+2}$ as the set of points in $\mathbf{R}^{n+2}$ whose $(n+2)$nd coordinate is 0. Then S^n is embedded as an equator in S^{n+1}. Again $S^n = \{x \in \mathbf{R}^{n+2} : ||x|| = 1 \text{ and } x_{n+2} = 0\}$ and D^{n+1} is also embedded in D^{n+2}, where $D^{n+1} = \{x \in \mathbf{R}^{n+2} : ||x|| \leq 1 \text{ and } x_{n+2} = 0\}$. Let H_+ and H_- be two closed hemispheres of S^{n+1} defined by the equator S^n. Then $H_+ = \{x \in S^{n+1} : x_{n+2} \geq 0\}$, called upper hemisphere, $H_- = \{x \in S^{n+1} : x_{n+2} \leq 0\}$, called lower hemisphere are such that $S^{n+1} = H_+ \cup H_-$ and $S^n = H_+ \cap H_-$.

The maps

$$p_+ : (D^{n+1}, S^n) \to (H_+^{n+1}, S^n), (x_1, x_2, \ldots, x_{n+1}, 0)$$
$$\mapsto \left(x_1, x_2, \ldots, x_{n+1}, \sqrt{1 - \sum_{i=1}^{n+1} x_i^2}\right)$$
$$\text{and } p_- : (D^{n+1}, S^n) \to (H_-^{n+1}, S^n), (x_1, x_2, \ldots, x_{n+1}, 0)$$
$$\mapsto \left(x_1, x_2, \ldots, x_{n+1}, -\sqrt{1 - \sum_{i=1}^{n+1} x_i^2}\right)$$

are homeomorphisms. Again for $t \in I$, $x \in S^n$, the point $tx + (1-t)p_0 \in D^{n+1}$.

Clearly, the map $f : \Sigma(S^n) \to S^{n+1}$ defined by

$$f([x,t]) = \begin{cases} p_-^{-1}(2tx + (1-2t)p_0), & 0 \le t \le 1/2 \\ p_+^{-1}((2-2t)x + (2t-1)p_0), & 1/2 \le t \le 1 \end{cases}$$

is well defined and bijective. It is a homeomorphism, since $S^1 \wedge S^n$ and S^{n+1} are both compact.

Hence $f : \Sigma(S^n) \approx S^{n+1} \Rightarrow S^{n+1}$ is an H-cogroup, since $\Sigma(S^n)$ is an H-cogroup. ❑

Definition 2.5.7 Given a pointed topological space X, its iterated suspension spaces are defined inductively:

$$\Sigma^n X = \Sigma(\Sigma^{n-1}X) \text{ for } n \ge 1, \text{ and } \Sigma^0 X \text{ is taken to be } X.$$

Remark 2.5.8 The groups $[\Sigma X, \Omega Y]$, $[X, \Omega^2 Y]$ and $[\Sigma^2 X, Y]$ are isomorphic for any pointed space Y.

Corollary 2.5.9 *For every integer $n \ge 0$, any pointed topological space X, $\Sigma^n X$ is homeomorphic to $S^n \wedge X$.*

Proof Since $S^0 = \{-1, 1\}$, it follows that $S^0 \wedge X \approx X \approx \Sigma^0 X$. Suppose $\Sigma^n X = S^n \wedge X$. Then

$$\Sigma^{n+1} X = \Sigma(\Sigma^n X) \approx \Sigma(S^n \wedge X) = S^1 \wedge (S^n \wedge X) \approx (S^1 \wedge S^n) \wedge X \approx S^{n+1} \wedge X.$$

Hence the corollary follows by induction on n. ❑

Corollary 2.5.10 *For every integer $n \ge 0$, the $(n+1)$-space S^{n+1} is an H-cogroup.*

Remark 2.5.11 The Corollary 2.5.10 shows that for $n \ge 1$, the space S^n admits an extensive family of H-cogroups.

Definition 2.5.12 (*Adjoint functors*) In the language of category theory the equivalence between $[\Sigma X, Y]$ and $[\Sigma X, \Omega Y]$ is expressed by saying that in the homotopy subcategory of pointed Hausdorff spaces of the homotopy category $\mathcal{H}tp_*$ of pointed topological spaces, the functors Ω and Σ are adjoint.

Remark 2.5.13 Recall that given a pointed topological space X, we have formed iterated loop spaces $\Omega^n X$ inductively: $\Omega^n X = \Omega(\Omega^{n-1} X)$ for ≥ 1 and $\Omega^0 X$ is taken to be X. and we have similarly formed iterated suspension spaces inductively:

$$\Sigma^n X = \Sigma(\Sigma^{n-1}X) \text{ for } n \ge 1, \text{ and } \Sigma^0 X \text{ is taken to be } X.$$

Then the groups $[\Sigma X, \Omega Y]$, $[X, \Omega^2 Y]$ and $[\Sigma^2 X, Y]$ are isomorphic for any pointed topological space Y.

Theorem 2.5.14 **(i)** *$\Omega^n X$ is an abelian H-group for $n \geq 2$ for all pointed topological spaces X.*

(ii) *$\Sigma^n X$ is an abelian H-cogroup for $n \geq 2$ for all pointed topological spaces X.*

(iii) *For any pair of pointed Hausdorff spaces X and Y, the adjoint functors Σ and Ω give an isomorphism*

$$\psi : [\Sigma X, Y] \to [X, \Omega Y]$$

of groups. For $n \geq 2$, the isomorphisms

$$\psi : [\Sigma X, \Omega^{n-1} Y] \to [X, \Omega^n Y]$$

are of abelian groups;

Proof **(i)** Let X be a pointed topological space with base point x_0. Then for any pointed topological space Y,

$$[X, \Omega^n Y] \cong [\Sigma X, \Omega^{n-1} Y] = [\Sigma X, \Omega(\Omega^{n-2} Y)]$$

is an abelian group. Hence if $[f], [g] \in [X, \Omega^n Y]$, then $[f]\cdot[g] = [g]\cdot[f]$. This shows that

$$\mu \circ (f \times g) \circ \triangle \simeq \mu \circ (g \times f) \circ \triangle.$$

In particular, if $X = \Omega^n Y \times \Omega^n Y$, $f = p_1$, the projection on the first factor and $g = p_2$, the projection on the second factor, then we have

$$(f \times g) \circ \triangle(x, y) = (p_1 \times p_2)((x, y), (x, y)) = (x, y).$$

This implies that $(f \times g) \circ \triangle = 1_d$. On the other hand,

$$(g \times f) \circ \triangle(x, y) = (p_2 \times p_1)((x, y), (x, y)) = (y, x) = T(x, y).$$

This implies that $(g \times f) \circ \triangle = T$. Hence $\mu \simeq \mu \circ T$ shows that μ is homotopy commutative. Consequently, (i) follows from Theorem 2.4.36.

(ii) Similarly, $\Sigma^n X$ is homotopy commutative. Hence (ii) follows from Theorem 2.4.50.

(iii) It follows from Ex. 29 of Sect. 2.11. ❑

Remark 2.5.15 If X is an H-cogroup and Y is an H-group, the products available in $[X, Y]$ determine isomorphic groups which are abelian.

2.6 Contractible Spaces

This section studies a special class of topological spaces, called contractible spaces, for each of which there exists a homotopy that starts with the identity map and ends with some constant map. This introduces the concept of contractible spaces. The concept of contractible spaces is very important. Contractible spaces are in a natural sense, the trivial objects from the view point of homotopy theory, because all contractible spaces have the homotopy type of a space reduced to a single point. Such spaces are connected topological objects having no 'holes' or 'cycles' and have nice intrinsic properties. The simplest nonempty space is one-point space. We characterize the homotopy type of such spaces.

2.6.1 Introductory Concepts

This subsection opens with introductory concepts of contractible spaces.

Definition 2.6.1 A topological space X is said to be contractible if the identity map $1_X : X \to X$ is homotopic to some constant map of X to itself. If $c : X \to X$ defined by $c(x) = x_0 \in X$ is such that $1_X \simeq c$, then a homotopy $F : 1_X \simeq c$ is called a contraction of the space X to the point x_0.

Example 2.6.2 Any convex subspace X of $\mathbf{R}^n$ is contractible. Because, any continuous map $H : X \times I \to X$ defined by $H(x, t) = (1 - t)x + tx_0, x, x_0 \in X, t \in I$ is such that $H(x, 0) = x = 1_X(x), \ \forall \ x \in X$ and $H(x, 1) = x_0 = c(x), \ \forall \ x \in X$. Hence $H : 1_X \simeq c \Rightarrow X$ is contractible and H is a contraction of X to the point $x_0 \in X$. In particular, $\mathbf{R}^n$, D^n, I are contractible spaces.

Geometrical meaning: A contraction $H : 1_X \simeq c$ can be interpreted geometrically as a continuous deformation of the space X which ultimately shrinks the whole space X into the point $x_0 \in X$ and hence X can be contracted to a point of X.

Can we contract a topological space X to an arbitrary point $x_0 \in X$?

To answer this question we need the following Proposition:

Proposition 2.6.3 *A topological space X is contractible if and only if an arbitrary continuous map $f : Y \to X$ from any topological space Y to X is homotopic to a constant map.*

Proof Let X be contractible. Then the identity map $1_X : X \to X$ is homotopic to some constant map $c : X \to X, x \mapsto x_0$ (say). Let $f : Y \to X$ be any continuous map. Now $1_X \simeq c \Rightarrow 1_X \circ f \simeq c \circ f$. But $c \circ f : Y \to X, y \mapsto x_0$ is a constant map. Thus f is homotopic to a constant map. For the converse, we take $Y = X$ and $f = 1_X : X \to X$. Then by hypothesis, $1_X : X \to X$ is a constant map. Hence X is contractible. ❑

Corollary 2.6.4 *Any two continuous maps from an arbitrary space to a contractible space are homotopic.*

Proof Let X be a contractible space and $f, g : Y \to X$ be two continuous maps from an arbitrary space Y to the space X. Now $1_X \simeq c$, where $c : X \to X$ is defined by $x \mapsto x_0 \in X \Rightarrow 1_X \circ f \simeq c \circ f$ and $1_X \circ g \simeq c \circ g \Rightarrow f = 1_X \circ f \simeq c \circ f = c \circ g \simeq 1_X \circ g = g \Rightarrow f \simeq g$. ❑

Corollary 2.6.5 *If X is contractible, then the identity map $1_X : X \to X$ is homotopic to any constant map of X to itself.*

Proof If X is contractible, then by Corollary 2.6.4 it follows in particular that 1_X is homotopic to any constant map of X to itself. ❑

Remark 2.6.6 In absence of the condition of contractibility of X, the Corollary 2.6.4 fails.

Example 2.6.7 Let X be a connected topological space and $Y = \{y_0, y_1\}$, $(y_0 \neq y_1)$ with discrete topology, i.e., Y is a discrete space consisting of two distinct elements. Consider the constant maps $f, g : X \to Y$ defined by constant $f(x) = y_0$ and $g(x) = y_1$, $\forall\ x \in X$. Then f and g are not homotopic (see Example 2.2.19)

We now characterize contractible spaces.

Theorem 2.6.8 *A topological space X is contractible if and only if X is of the same homotopy type of a one-point space $P = \{p\}$.*

Proof Suppose X is contractible. Then $1_X : X \to X$ is homotopic to some constant map $c_o : X \to X, x \mapsto x_0 \in X$. Let $H : 1_X \simeq c_0$. Define maps $i : P \to X$ and $c : X \to P$ by $i(P) = x_0$ and $c(x) = p$, $\forall\ x \in X$. Then $c \circ i = 1_P$. Moreover, $H : 1_X \simeq i \circ c$, because $H(x, 0) = x$ and $H(x, 1) = c_0$. Hence $X \simeq P$. Conversely, let $X \simeq P$. then $\exists$ continuous maps $f : X \to P$ and $g : P \to X$ such that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_P$. Leth $g(p) = x_0 \in X$ and $H : 1_X \simeq g \circ f$. Since $(g \circ f)(x) = g(f(x)) = g(p) = x_0$, $\forall\ x \in X, g \circ f : X \to X, x \mapsto x_0$, is the constant map c_0. Thus $1_X \simeq c_0 \Rightarrow X$ is contractible. ❑

Corollary 2.6.9 *Two contractible spaces have the same homotopy type, and any continuous map between contractible spaces is a homotopy equivalence.*

Proof Let X and Y be two contractible spaces and P be a one-point space. Then $X \simeq P$ and $Y \simeq P \Rightarrow X \simeq Y$ by symmetry and transitivity of the relation $\simeq$. Hence $\exists$ a homotopy equivalence $f \in C(X, Y)$. Let $g : X \to Y$ be an arbitrary continuous map. Then $f \simeq g$ by Corollary 2.6.4 $\Rightarrow g$ is a homotopy equivalence by Proposition 2.2.15. ❑

Remark 2.6.10 Contractible spaces are precisely those spaces which are homotopy equivalent to a point space. Thus all contractible spaces have the homotopy type of a space reduced to a single point.

Definition 2.6.11 A topological space X is said to be contractible to a point $a \in X$ relative to the subset $A = \{a\}$ if $\exists$ a homotopy $H : X \times I \to X$ such that $H : 1_X \simeq c$ rel A, where $c : X \to X, x \mapsto a$ is a constant map.

Theorem 2.6.12 *If a topological space X is contractible to a point $a \in X$ relative to the subset $A = \{a\}$, then for each neighborhood U of a in X, $\exists$ a neighborhood V of a contained in U such that any point of V can be joined to a by a path lying entirely inside U.*

Proof Let the space X be contractible to a point $a \in X$ relative to the subset $A = \{a\}$. Then there exists a continuous map F such that $F : 1_X \simeq c$ rel $A \Rightarrow$ the line $\{a\} \times I$ is mapped by F to the point $a \in X$. We now take a neighborhood U of a. Then the continuity of $F \Rightarrow$ for each $t \in I$, neighborhoods $V_t(a)$ of a in X and $W(t)$ of t in I are such that $F(V_t(a) \times W(t)) \subset U$. Since I is compact, the open covering $\{W(t) : t \in I\}$ of I has a finite subcovering $W(t_1), W(t_2), \ldots, W(t_n)$ (say) such that $F(V_{t_i}(a) \times W(t_i)) \subset U$, for $i = 1, 2, \ldots, n$. Thus $V(a) = \bigcap_{i=1}^{n} V_{t_i}(a)$ is a neighborhood of a in X such that $F(V(a) \times I) \subset U$. Now, if $x \in V(a)$, then considering the image $F(V(a) \times I)$ in U, it follows that the point x can be joined to the point a by a path which lies inside U. ❑

Proposition 2.6.13 *Every contractible space is path-connected.*

Proof Let X be contractible to a point $x_0 \in X$ and $H : 1_X \simeq c$, where $c : X \to X, x \mapsto x_0 \in X$ is a constant map. Now $H(x, 0) = 1_X(x) = x$ and $H(x, 1) = c(x) = x_0$, $\forall\ x \in X$. Given $a \in X$ define a path $f : I \to X \times I$ by $f(t) = (a, t)$. Then $\alpha = H \circ f : I \to X$ is a continuous map such that $\alpha(0) = H(f(0)) = H(a, 0) = a$ and $\alpha(1) = H(f(1)) = H(a, 1) = x_0 \Rightarrow \alpha$ is a path from a to x_0. In other words, X is path-connected. ❑

2.6.2 Infinite-Dimensional Sphere and Comb Space

We now examine the contractibility of the infinite-dimensional sphere S^∞. We also study comb space which is contractible in absolute sense but not contractible in relative sense. First we describe $\mathbf{R}^\infty$, $\mathbf{C}^\infty$ and S^∞.

Definition 2.6.14 The set of all sequences $x = (x_1, x_2, \ldots, x_n, \ldots)$ of real numbers such that $\sum_1^\infty |x_n|^2$ converges, is denoted by $\mathbf{R}^\infty$. Under coordinatewise addition and scalar multiplication, $\mathbf{R}^\infty$ is a vector space over $\mathbf{R}$. Moreover, $\mathbf{R}^\infty$ endowed with a norm function defined by $\|x\| = (\sum_1^\infty |x_n|^2)^{1/2}$ is called a real Banach space. The space $\mathbf{R}^\infty$ is called infinite-dimensional Euclidean space. Similarly, the infinite-dimensional unitary space $\mathbf{C}^\infty$ is defined.

Remark 2.6.15 The space $\mathbf{C}^\infty$ is a complex Banach space. Clearly, as a topological space $\mathbf{C}^n$ is homeomorphic to $\mathbf{R}^{2n}$ and $\mathbf{C}^\infty$ is homeomorphic to $\mathbf{R}^\infty$. The space S^∞ is now defined.

Definition 2.6.16 The infinite-dimensional sphere S^∞ is the subspace of $\mathbf{R}^\infty$ (under weak topology) consisting of all real sequences $(x_1, x_2, x_3, \ldots)$ such that $x_1^2 + x_2^2 + x_3^2 + \cdots = 1$ (i.e., $S^\infty = \{(x_1, x_2, x_3, \ldots) \in \mathbf{R}^\infty : x_1^2 + x_2^2 + x_3^2 + \cdots = 1\}$).

As the diagram in Fig. 2.16 is commutative, we may consider S^∞ as the subspace of $\mathbf{C}^\infty$ consisting of the sequences $(z_1, z_2, \ldots)$ over $\mathbf{C}$ such that $|z_1|^2 + |z_2|^2 + \cdots = 1$. We are now in a position to prove the contractibility of S^∞.

Proposition 2.6.17 *The infinite-dimensional sphere S^∞ is contractible.*

Proof Consider the map

$$\begin{aligned} F : S^\infty \times I \to S^\infty, &\ (x_1, x_2, x_3, \ldots, t) \\ &\mapsto ((1-t)x_1, tx_1 + (1-t)x_2, tx_2 + (1-t)x_3, \ldots)/N_t, \end{aligned}$$

where $N_t = [((1-t)x_1)^2 + (tx_1 + (1-t)x_2)^2 + (tx_2 + (1-t)x_3)^2 + \cdots]^{1/2}$, which is the norm of the nonzero vector of the numerator. We may parametrize F as $F_t(x_1, x_2, x_3, \ldots) = F(x_1, x_2, x_3, \ldots, t)$.

Then $F_0(x_1, x_2, x_3, \ldots) = (x_1, x_2, x_3, \ldots)$, since $N_0 = 1$ and $F_1(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots)$, since $N_1 = 1$. Consequently, F_0 is the identity map $1_d : S^\infty \to S^\infty$, the image of F_1 is the set $X = \{x \in S^\infty : x_1 = 0\}$ and $F : F_0 \simeq F_1$.

Consider another homotopy

$$H : X \times I \to S^\infty, H(x_1 = 0, x_2, x_3, \ldots, t) \mapsto (t, (1-t)x_2, (1-t)x_3, \ldots)/N'_t,$$

where $N'_t = [t^2 + ((1-t)x_2)^2 + ((1-t)x_3)^2 + \cdots]^{1/2}$.

If $i : X \hookrightarrow S^\infty$ is the inclusion map, then $H : i \simeq c$, where c is a constant map.

Let $H * F : S^\infty \times I \to S^\infty$ be defined by

$$(H * F)(t) = \begin{cases} F(x, 2t), 0 \le t \le 1/2 \\ H(x, 2t-1), 1/2 \le t \le 1. \end{cases}$$

where $x = (x_1, x_2, x_3, \ldots) \in S^\infty$.

Then $H * F$ is a contraction. Consequently, S^∞ is a contractible space. ❑

$$\begin{tikzcd} S^{2n-1} \arrow[r, hook] \arrow[d, hook] & \mathbf{C}^n \arrow[d, hook] \\ S^{2n+1} \arrow[r, hook] & \mathbf{C}^{\mathbf{n+1}} \end{tikzcd}$$

Fig. 2.16 Commutative diagram involving $\mathbf{C}^n$ and $\mathbf{C}^{n+1}$

Remark 2.6.18 The infinite-dimensional sphere S^∞ is contractible. On the other hand, the n-sphere S^n is not contractible for any integer $n \geq 0$ (see Proposition 14.1.13 of Chap. 14).

Corollary 2.6.19 *The inclusion map $i : S^{n-1} \hookrightarrow S^n$ is nullhomotopic.*

Remark 2.6.20 We now search for a topological space which is contractible in absolute sense but not contractible in relative sense.

Example 2.6.21 (*Comb Space*) The subspace Y of the plane $\mathbf{R}^2$ defined by

$$Y = \left\{(x, y) \in \mathbf{R}^2 : 0 \leq y \leq 1, x = 0, \frac{1}{n}(n \in \mathbf{N}) \text{ or } y = 0, 0 \leq x \leq 1\right\}$$

is called the comb space, i.e., Y consists of the horizontal line segment L joining $(0, 0)$ to $(1, 0)$ and vertical unit closed line segments standing on points $(1/n, 0)$ for each $n \in \mathbf{N}$, together with the line segment joining $(0, 0)$ with $(0, 1)$ as shown in Fig. 2.17. It is an important example of a contractible space.

Proposition 2.6.22 *Comb space Y is contractible but not contractible relative to $\{(0, 1)\}$.*

Proof **First part**: First we show that $L \simeq Y$. Let $p : Y \to L$, $(x, y) \mapsto (x, 0)$ be the projection map and $i : L \hookrightarrow Y$ be the inclusion map. Then $(p \circ i)(x, 0) = p(x, 0) = (x, 0)$, $\forall\ (x, 0) \in L \Rightarrow p \circ i = 1_L$(identity map on L). Define $F : Y \times I \to Y$ by the rule $F((x, y), t) = (x, (1 - t)y)$. Then $F((x, y), 0) = (x, y) = 1_Y(hx, y)$, $\forall\ (x, y) \in Y$ and $F((x, y), 1) = (x, 0) = (i \circ p)(x, y)$, $\forall\ (x, y) \in Y$ show that $F : 1_Y \simeq i \circ p \Rightarrow p \in C(Y, L)$ is a homotopy equivalence $\Rightarrow Y \simeq L$. Again $L \approx I \Rightarrow L \simeq I$. Moreover, I being a contractible space, I is of the same homotopy type of a one-point space and hence L is of the same homotopy type of one-point space. Consequently, Y is of the same homotopy type of one-point space. In other words, Y is contractible by Theorem 2.6.8.

Second part: Any small neighborhood V of $(0, 1)$ has infinite number of path components. Let D be the open disk around $(0, 1)$ of radius $\frac{1}{2}$. Then the neighborhood $U = D \cap Y$ of $(0, 1)$ in Y cannot have any neighborhood V each of whose points can be joined to $(0, 1)$ by a path entirely lying in $U \Rightarrow Y$ is not contractible relative to $\{(0, 1)\}$ by Theorem 2.6.12, otherwise we would reach a contradiction by the same theorem. ❑

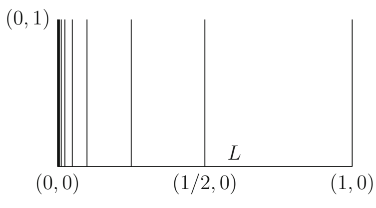


Fig. 2.17 Comb space

Remark 2.6.23 The concept of relative homotopy is stronger than the concept of homotopy.

Let A be a subspace of X and $f, g : X \to Y$ be two continuous maps such that $f \simeq g$ rel A. Then $f \simeq g$. But its converse is not true.

Example 2.6.24 Let Y be the comb space, $1_Y : Y \to Y$ be the identity map and $c : Y \to Y$ be the constant map defined by $c(x, y) = (0, 1)$, $\forall\ (x, y) \in Y$. Then I_Y and c agree on $\{(0, 1)\}$ and hence $1_Y \simeq c$ by Corollary 2.6.5, since Y is contractible. But the comb space Y is not contractible relative to $\{(0, 1)\}$ (see Proposition 2.6.22).

2.7 Retraction and Deformation

This section mainly studies inclusion maps from the viewpoint of homotopy theory. We consider whether an inclusion map $i : A \hookrightarrow X$ has a left inverse or a right inverse or a left homotopy inverse or a right homotopy inverse or two sided inverse or two-sided homotopy inverse. More precisely, the concepts of retraction and weak retraction are introduced and it is proved that these two concepts coincide under suitable homotopy extension property (HEP).

Let A be a subspace of a topological space X and $i : A \hookrightarrow X$ be the inclusion map. Then a continuous map $f : A \to Y$ from A to a subspace Y is said to have a continuous extension over X if $\exists$ a continuous map $F : X \to Y$ such that the diagram in Fig. 2.18 is commutative, i.e., $F \circ i = f$. Thus F is said to be a continuous extension of f over X if $F|_A = f$.

Definition 2.7.1 A subspace A of X is called a retract of X if there exists a continuous map $r : X \to A$ such that $r \circ i = 1_A$, i.e., if i has a left inverse in the category $\mathcal{Top}$ of topological spaces and continuous maps, i.e., if $r(x) = x$, $\forall\ x \in A$. Such a map r is called a retraction of X to A. On the other hand, if $i \circ r \simeq 1_X$, A is called a deformation retract of X and r is called a deformation retraction.

Thus A is a retract of X if $\exists$ a continuous map $r : X \to A$ making the diagram in Fig. 2.19 is commutative.

$$\begin{tikzcd} A \arrow[rr, hook, "i"] \arrow[dr, "f"'] & & X \arrow[dl, dashed, "F"] \\ & Y & \end{tikzcd}$$

Fig. 2.18 Continuous extension of f

$$\begin{tikzcd} A \arrow[rr, hook, "i"] \arrow[dr, "1_A"'] & & X \arrow[dl, dashed, "r"] \\ & A & \end{tikzcd}$$

Fig. 2.19 Retraction and retract

Remark 2.7.2 The main property of a retract A of X is that any continuous map $f : A \to Y$ has at least one continuous extension $\tilde{f} : X \to Y$, namely, $\tilde{f} = f \circ r$, where $r : X \to A$ is a retraction.

Example 2.7.3 The circle A as shown in Fig. 2.20 is a retract of the annulus X. The arrows indicate the action of the retraction. The whole of X is mapped onto A keeping points in A fixed.

Example 2.7.4 Consider the inclusion map $i : D^n \hookrightarrow \mathbf{R}^n$. Define a map $r : \mathbf{R}^n \to D^n$ by the rule

$$r(x) = \begin{cases} \frac{x}{||x||}, & \text{if } ||x|| > 1 \\ x, & \text{if } ||x|| \leq 1. \end{cases}$$

Then r is a retraction and hence D^n is a retract of $\mathbf{R}^n$. Geometrically, the map r fixes points in D^n and shifts points x outside of D^n along a straight line from the origin to x onto the boundary S^{n-1} of D^n.

Definition 2.7.5 A subspace A of a topological space X is called a weak retract of X if there exists a continuous map $r : X \to A$ such that $r \circ i \simeq 1_A$, i.e., if i has a left homotopy inverse, i.e., if i has a left inverse in the homotopy category $\mathcal{H}tp$ of topological spaces and continuous maps.

Thus A is a weak retract of X if $\exists$ a continuous map $r : X \to A$ making the diagram in Fig. 2.21 homotopy commutative. Such a map r is called a weak retraction of X to A.

Remark 2.7.6 A is retract of $X \Rightarrow A$ is a weak retract of X, because $r \circ i = 1_A \simeq 1_A$. But its converse is not true.

Example 2.7.7 Consider $X = I^2$ and $A =$comb space (see Example 2.6.21). Then $A \subsetneqq X$. As A and X are both contractible spaces, the inclusion map $i : A \hookrightarrow X$ is

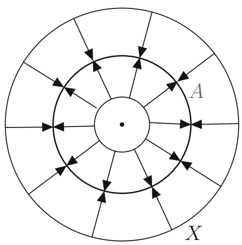


Fig. 2.20 The circle A (*deep black*) is a retract of the annulus X

$$\begin{tikzcd} A \arrow[rr, hook, "i"] \arrow[dr, "1_A"'] & & X \arrow[dl, dashed, "r"] \\ & A & \end{tikzcd}$$

Fig. 2.21 Weak retraction and weak retract a homotopy equivalence by Corollary 2.6.9. Hence $\exists$ a continuous map $r : X \to A$ such that $r \circ i \simeq 1_A$. This shows that r is a weak homotopy equivalence. Clearly $r \circ i \neq 1_A$. Consequently, A is a weak retract of X but not a retract of X.

We now search conditions under which the concepts of retraction and weak retraction coincide. For this purpose we introduce the concept of Homotopy Extension Property for the pair of spaces (X, A).

Definition 2.7.8 Let (X, A) be pair of topological spaces and Y be an arbitrary topological space. Then the pair (X, A) is said to have the Homotopy Extension Property (HEP) with respect to the space Y if given continuous maps $g : X \to Y$ and $G : A \times I \to Y$ such that $g(x) = G(x, 0)$, $\forall\ x \in A$, there is a continuous map $F : X \times I \to Y$ with the property $F(x, 0) = g(x)$, $\forall\ x \in X$ and $F|_{A\times I} = G$.

If $h_0(x) = (x, 0)$, $\forall\ x \in X$, the existence of F is equivalent to the existence of a continuous map represented by the dotted arrow which makes the diagram in Fig. 2.22 is commutative.

Thus (X, A) has the HEP with respect to Y if $\exists$ a continuous map $F : X \times I \to Y$ such that the square and the two triangles in the diagram in Fig. 2.22 are commutative.

Proposition 2.7.9 *If (X, A) has the HEP with respect to Y and if $f_0, f_1 : A \to Y$ are homotopic, then f_0 has a continuous extension over X iff f_1 has also a continuous extension over X.*

Proof Let $f_0, f_1 : A \to Y$ be two continuous maps such that $f_0 \simeq f_1$. Then $\exists$ a homotopy $G : A \times I \to Y$ such that $G(x, 0) = f_0(x)$ and $G(x, 1) = f_1(x)$, $\forall\ x \in A$. Let $f_0 : A \to Y$ have a continuous extension $g_0 : X \to Y$. Then $G(x, 0) = f_0(x) = g_0(x)$, $\forall\ x \in A$. As (X, A) has the HEP with respect to Y, $\exists$ a map $F : X \times I \to Y$ extending $G : A \times I \to Y$ and therefore the diagram in Fig. 2.23 is commutative. The existence of F follows from the HEP of (X, A) with respect to Y. Define a map $g_1 : X \times Y$ by $g_1(x) = F(x, 1)$, $\forall\ x \in X$. Then g_1 is an extension

Fig. 2.22 Homotopy extension property (HEP)

$$\begin{tikzcd}
 & A \arrow[rr, "h_0"] \arrow[dl, hook, "i"'] & & A \times I \arrow[dl, "G"'] \arrow[dd, "i\times 1_d"] \\
X \arrow[r, "g"] \arrow[drrr, "h_0"'] & & Y & \\
 & & & X \times I \arrow[ul, dashed, "F"']
\end{tikzcd}$$

Fig. 2.23 Homotopy extension property of (X, A) w.r.t. Y

$$\begin{tikzcd}
 & A \arrow[rr, "h_0"] \arrow[dl, hook, "i"'] & & A \times I \arrow[dl, "G"'] \arrow[dd, "i\times 1_d"] \\
X \arrow[r, "g"] \arrow[drrr, "h_0"'] & & Y & \\
 & & & X \times I \arrow[ul, dashed, "F"']
\end{tikzcd}$$

of f_1 over X, because, $g_1(a) = F(a, 1) = G(a, 1) = f_1(a)$, $\forall\ a \in A \Rightarrow g_1|_A = f_1$. Moreover g_1 is continuous and hence f_1 has a continuous extension g_1 over X. ❑

Remark 2.7.10 A continuous map $f : A \to Y$ can or cannot be extended over X is a property of the homotopy class of that map. Thus the homotopy extension property implies that the extension problem for continuous maps $A \to Y$ is a problem in the homotopy category.

The map $r : \mathbf{R}^n \to D^n$ defined in Example 2.7.4 is a deformation retraction. To show this define a homotopy $F : \mathbf{R}^n \times I \to \mathbf{R}^n$ by the rule

$$F(x,t) = \begin{cases} (1-t)x + tx/||x||, & \text{if } ||x|| \geq 1 \\ x, & \text{if } ||x|| < 1 \end{cases}$$

Then $F : 1_d \simeq i \circ r$ shows that r is a deformation retraction.

Geometrically, F fixes points in D^n and shifts points x outside of D^n linearly from x to $r(x)$ along the straight line determined by x and the origin $0 = (0, 0, \ldots, 0)$.

Theorem 2.7.11 *If (X, A) has the HEP with respect to A, then A is a weak retract of X iff A is a retract of X.*

Proof Let $A \subset X$ be a retract of X and $r : X \to A$ be a retraction. Then $r \circ i = 1_A \Rightarrow r \circ i \simeq 1_A \Rightarrow A$ is a weak retract of X. Conversely, let $r : X \to A$ be a weak retraction. Then $r \circ i \simeq 1_A$, where $i : A \hookrightarrow X$ is the inclusion map. Then $\exists$ a homotopy $G : A \times I \to Y$ such that $G(x, 0) = r(x)$, $G(x, 1) = 1_A(x) = x$, $\forall\ x \in A$. As (X, A) has the HEP with respect ot A, $\exists$ a continuous map $F : X \times I \to A$ extending $G : A \times I \to A$. Hence $F(x, 0) = r(x)$, $\forall\ x \in X$ and $F|_{A\times I} = G$. Define a map $r' : X \to A$ by the rule $r'(x) = F(x, 1)$, $\forall\ x \in X$. Now, for all $a \in A$, $(r' \circ i)(a) = r'(i(a)) = F(a, 1) = G(a, 1) = a \Rightarrow r' \circ i = 1_A \Rightarrow A$ is a retract of X and r' is a retraction of X into A. ❑

We can as well consider inclusion maps with right homotopy inverses.

Definition 2.7.12 Given a subspace $X' \subset X$, a deformation D of X' in X is a homotopy $D : X' \times I \to X$ such that $D(x', 0) = x'$, $\forall\ x' \in X'$. If moreover, $D(X' \times 1) \subset A \subset X'$, D is said to a deformation of X' into A and X' is said to be deformable in X into A. If $X = X'$, then a space X is said to be deformable into a subspace A of X if it is deformable in itself into A.

Theorem 2.7.13 *A topological space X is deformable into a subspace A of X iff the inclusion map $i : A \hookrightarrow X$ has a right homotopy inverse.*

Proof Let X be deformable into a subspace A of X. Then $\exists$ a continuous map $D : X \times I \to X$ such that $D(x, 0) = x$ and $D(x, 1) \in A \subset X$, $\forall\ x \in X$. Let $f : X \to A$ be defined by the equation $(i \circ f)(x) = D(x, 1)$, $\forall\ x \in X$. Then $D : 1_X \simeq i \circ f \Rightarrow i$ has a right homotopy inverse. Conversely, let $i : A \subset X$ has a right homotopy inverse $f : X \to A$. Then $1_X \simeq i \circ f$. Let $F : X \times I \to X$ be such that $F : 1_X \simeq i \circ f$. Then $F(x, 0) = 1_X(x) = x$, $\forall\ x \in X$ and $F(X \times 1) = (i \circ f)(X) \subset A$(i.e., $F(x, 1) = i(f(x)) = f(x)$, $\forall\ x \in X) \Rightarrow X$ is deformable into A. ❑

Remark 2.7.14 The homotopy D which starts with identity map $1_X : X \to X$, simply moves each point of X continuously, including the points of A and finally, pushes every point into a point of A. In particular, if X is deformable into a point $x_0 \in X$, then X is contractible and vice verse (see Ex. 4 of Sect. 2.11).

Remark 2.7.15 An inclusion map $i : A \hookrightarrow X$ has never a right inverse in the category of topological spaces and continuous maps in the trivial case when $A = X$.

We now consider inclusion maps which are homotopy equivalences.

Definition 2.7.16 A subspace A of a topological space X is called a weak deformation retract of X if the inclusion map $i : A \hookrightarrow X$ is a homotopy equivalence.

Theorem 2.7.17 *A subspace A of a topological space X is a weak deformation retract of X iff A is a weak retract of X and X is deformable into A.*

Proof Let A be a weak deformation retract of X. Then $i : A \hookrightarrow X$ is a homotopy equivalence. Hence $\exists$ a continuous map $r : X \to A$ such that $r \circ i \simeq 1_A$ and $i \circ r \simeq 1_X$. Thus i has a right homotopy inverse and also a left homotopy inverse. Consequently, A is a weak retract of X and X is deformable into A.

Conversely, let A be a weak retract of X and X be deformable into A. Then $i : A \hookrightarrow X$ has a left homotopy inverse f(say) and a right homotopy inverse g(say). Now $f \circ i \simeq 1_A$ and $i \circ g \simeq 1_X \Rightarrow f = f \circ 1_X \simeq f \circ (i \circ g) = (f \circ i) \circ g \simeq 1_A \circ g = g \Rightarrow f \simeq g$. Hence $f \circ i \simeq 1_A$ and $i \circ f \simeq 1_X \Rightarrow i$ is a homotopy equivalence $\Rightarrow A$ is a weak deformation retract of X. ❑

We now consider a deformation D which deforms X into A, but the points of A do not move at all.

This led to the concept of strong deformation retract introduced by Borsuk in 1933.

Definition 2.7.18 A subspace A of a topological space X is called a strong deformation retract of X if there exists a retraction $r : X \to A$ such that $1_X \simeq i \circ r$ rel A. If $F : 1_X \simeq i \circ r$ rel A, then F is called a strong deformation retract of X to A.

There is an intermediate concept between the concepts of weak deformation retraction and strong deformation retraction.

Definition 2.7.19 A subspace A of a topological space X is called a deformation retract of X if $\exists$ a retraction $r : X \to A$ such that $1_X \simeq i \circ r$. If $F : 1_X \simeq i \circ r$, then F is called a deformation retraction of X to A.

Remark 2.7.20 A homotopy $F : X \times I \to X$ is a deformation retraction of X to A iff $F(x, 0) = 1_X(x) = x$, $\forall\ x \in X$, $F(x, 1) = x$, $\forall\ x \in A$ and $F(X \times 1) \subset A$. A map F is called a strong deformation retraction iff it also satisfies the condition $F(x, t) = x$, $\forall\ x \in A$ and $\forall\ t \in I$. Thus F is a strong deformation retraction of X to A iff F satisfies the conditions:

(i) $F(x, 0) = x$, $\forall\ x \in X$;

(ii) $F(x, 1) = x,\ \forall\ x \in A$;
(iii) $F(X \times 1) \subset A \subset X$;
(iv) $F(x, t) = x,\ \forall\ x \in A$ and $\forall\ t \in I$.

Example 2.7.21 Let $X = \mathbf{R}^{n+1} - \{0\}$ and S^n be the n-sphere ($n \geq 1$). Then $S^n \subset X$ is a strong deformation retract of X. Let $i : S^n \hookrightarrow X$ be the inclusion. Define a retraction $r : X \to S^n$ by $r(x) = \frac{x}{||x||}$. Geometrically, this map shifts points $x \in X$ to the boundary S^n along a straight line from the origin. Define a continuous map $F : X \times I \to X$ by $F(x, t) = (1 - t)x + \frac{tx}{||x||},\ \forall\ x \in X,\ \forall\ t \in I$. Then

(i) $F(x, 0) = x,\ \forall\ x \in X$;
(ii) $F(x, 1) = x,\ \forall\ x \in S^n$;
(iii) $F(X \times 1) \subset S^n \subset X$ and
(iv) $F(x, t) = x,\ \forall\ x \in S^n$ and $\forall\ t \in I$.

Clearly, $r \circ i$ is the identity map on S^n and $i \circ r$ is homotopic to the identity map on X. Hence i is a homotopy equivalence. Geometrically, this homotopy F moves linearly along the straight path defined above from x to $r(x)$. Hence F is a strong deformation retraction of X to S^n. Therefore S^n is a strong deformation retract of $X = \mathbf{R}^{n+1} - \{0\}$.

Remark 2.7.22 We now explain Example 2.7.21 geometrically for $n = 1$ as shown in Fig. 2.24.

Let l be an arbitrary half-line starting from the origin $0 = (0, 0)$. Then it intersects the circle S^1 at exactly one point l_P(say). Since $0 = (0, 0)$ is not a point of $\mathbf{R}^2 - \{0\}$, the lines $l - \{0\}$ are disjoint and their union is $\mathbf{R}^2 - \{0\}$. Define a map $r : \mathbf{R}^2 - \{0\} \to S^1$ by $r(x) = l_P,\ \forall\ x \in l$. Then r is a retraction and S^1 is a retract of $\mathbf{R}^2 - \{0\}$. Define a deformation $D : (\mathbf{R}^2 - \{0\}) \times I \to \mathbf{R}^2 - \{0\}$ by $D(x, t) = (1 - t)x + t\frac{x}{||x||}$. Then as before D is a strong deformation retraction of $\mathbf{R}^2 - \{0\}$ relative to S^1 into S^1.

Example 2.7.23 Consider the product space $X = D^n \times I$ and the subspace $A = (S^{n-1} \times I) \cup (D^n \times \{0\})$. If P is the point $(0, 2)$ in $\mathbf{R}^n \times \mathbf{R}$, a retraction $r : X \to A$ is defined by taking $r(x)$ to be the point where the line joining P and x meets A. Consequently, the map

$$F : X \times I \to X,\ (x, t) \mapsto F(x, t) = t\, r(x) + (1 - t)x$$

is such that $F(x, 0) = x$ and $F(x, 1) = r(x)$. This shows that F is a strong deformation retraction.

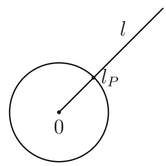
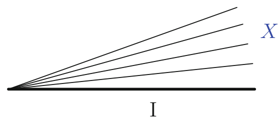


Fig. 2.24 Half line starting from the origin and intersecting the circle

Fig. 2.25 Construction of X



Example 2.7.24 Let X be the topological space given by I together with a family of segments approaching it as shown in Fig. 2.25.

Then I is a deformation retract of X but not a strong deformation retract.

Proposition 2.7.25 *A topological space X is deformable into a retract $A \subset X$ if and only if A is a deformation retract of X.*

Proof Let X be deformable into a retract $A \subset X$ and $i : A \hookrightarrow X$ be the inclusion map. Then there exists a retraction $r : X \to A$ such that $r \circ i = 1_A$. Hence r is a left inverse of i and thus r is a left homotopy inverse of i. Again as X is deformable into A, i has a right homotopy inverse by Theorem 2.7.13, which is also r, i.e., $1_X \simeq i \circ r$. Consequently, A is a deformation retract of X. Conversely, let A be a deformation retract of X. Then $\exists$ a retraction $r : X \to A$ such that $1_X \simeq i \circ r$ and $r \circ i = 1_A$. Consequently, X is deformable into a retract A. ❑

We now show that if (X, A) has the HEP with respect to A, then the concepts of weak deformation retraction and deformation retraction coincide.

Theorem 2.7.26 *If (X, A) has the HEP with respect to A, then A is a weak deformation retract of X if and only if A is a deformation retract of X.*

Proof Suppose (X, A) has the HEP with respect to A. Then A is a weak retract of X and X is deformable into A if and only if A is a retract of X and X is deformable into A by Theorem 2.7.11. Then the theorem follows from Proposition 2.7.25. ❑

We now show that under suitable HEP the concepts of strong deformation retraction and deformation retraction coincide.

Theorem 2.7.27 *If $A \subset X$ and $(X \times I, (X \times \{0\}) \cup (A \times I) \cup (X \times 1))$ has the HEP with respect to X and A is closed in X, then A is a deformation retract of X if and only if A is a strong deformation retract of X.*

Proof Let $X' = X \times I$ and $A' = (X \times \{0\}) \cup (A \times I) \cup (X \times 1)$. Then $A' \subset X'$. Let (X', A') has the HEP with respect to X and A be closed in X. Suppose $i : A \hookrightarrow X$ and A is a deformation retract of X. We claim that A is also a strong deformation retract of X. Since A is a deformation retract of X, $\exists$ a retraction $r : X \to A$ such that $1_X \simeq i \circ r$. Let $F : X \times I \to X$ be a continuous map such that $F : 1_X \simeq i \circ r$. Then $F(x, 0) = 1_X(x) = x$, $\forall\ x \in X$ and $F(x, 1) = (i \circ r)(x) = r(x)$, $\forall\ x \in X$. We now define a map $G : A' \times I \to X$ by the equations

$$\begin{tikzcd}
& A' \arrow[r, "h_0"] \arrow[dl, hook', "i"'] & A' \times I \arrow[d, "G"'] \arrow[ddr, "i \times 1_d"] & \\
X' \arrow[rr, "F"] \arrow[drrr, "h_0"'] & & X & \\
& & & X' \times I \arrow[ul, dashed, "G'"]
\end{tikzcd}$$

Fig. 2.26 Homotopy extension property of (X', A') w.r.t. X

$$G((x, 0), t') = x, \ \forall\ x \in X, \ \forall\ t' \in I \tag{2.11}$$

$$G((x, t), t') = F(x, (1 - t')t), \ \forall\ x \in A, \ \forall\ t, t' \in I \tag{2.12}$$

$$G((x, t), t') = F(r(x), 1 - t'), \ \forall\ x \in X, \ \forall\ t, t' \in I \tag{2.13}$$

Then G is well defined, because for $x \in A$, $G((x, 0), t') = x = F(x, 0)$ by the first two Eqs. (2.11) and (2.12) and $G((x, 1), t') = F(x, 1 - t') = F(r(x), 1 - t')$ by the last two Eqs. (2.12) and (2.13). Again G is continuous, because its restriction to each of the closed sets $(X \times \{0\}) \times I$, $(A \times I) \times I$ and $(X \times 1) \times I$ is continuous. For $(x, t) \in A'$, $G((x, t), 0) = F(x, t)$, because $F(x, 0) = x$ and since $r : X \to A$ is a retraction.

$F(r(x), 1) = (i \circ r)(r(x)) = r(x) = F(x, 1)$. Therefore G restricted to $A' \times \{0\}$ can be extended to $(X \times I) \times \{0\}$. Then by HEP of (X', A') w.r.t X in the hypothesis, $\exists$ a homotopy $G' : X' \times I \to X$ extending $G : A' \times I \to X$ (see Fig. 2.26). Define $H : X \times I \to X$ by $H(x, t) = G'((x, t), 1)$. Then we have the equations
$H(x, 0) = G'((x, 0), 1) = G((x, 0), 1) = x, \ \forall\ x \in X$;
$H(x, 1) = G'((x, 1), 1) = F(r(x), 0) = r(x), \ \forall\ x \in X$;
and $H(x, t) = G'((x, t), 1) = G((x, t), 1) = F(x, 0) = x, \ \forall\ x \in A, \ \forall\ t \in I$.

Therefore $H : 1_X \simeq i \circ r$ rel A. Hence A is a strong deformation retract of X. Conversely, if A is a strong deformation retract of X, then A is automatically a deformation retract of X. ❑

2.8 NDR and DR Pairs

This section defines the concepts of NDR-pair and DR-pair which are closely related to the concepts of retraction and homotopy extension property for compactly generated spaces (see Sect. B.4 of Appendix B). N. Steenrod (1910–1971) proved in 1967 the equivalence between the NDR condition and the homotopy extension property (Steenrod 1967).

We now use the concept of compactly generated space defined in Appendix B.

Definition 2.8.1 Let X be a compactly generated topological space and $A \subset X$ be a subspace. Then (X, A) is said to be an NDR-pair (NDR stands for 'neighborhood deformation retract') if there exist continuous maps $u : X \to I$ and $h : X \times I \to X$ such that

NDR(i) $A = u^{-1}(0)$;
NDR(ii) $h(x, 0) = x, \ \forall\, x \in X$;
NDR(iii) $h(a, t) = a, \ \forall\, t \in I, a \in A$;
NDR(iv) $h(x, 1) \in A$ for all $x \in X$ such that $u(x) < 1$.

In particular, A is a retract of its neighborhood $U = \{x \in X : u(x) < 1\}$, and hence is a neighborhood retract of X.

Definition 2.8.2 A pair (X, A) is called a DR-pair (DR stands for "deformation retract") if in addition to **NDR(i)–NDR(iii),** another condition **NDR(v)**: $h(x, 1) \in A$ (instead of **NDR(iv)**) holds for all $x \in X$.

Remark 2.8.3 The concepts of DR-pair and NDR-pair are closely related to the concepts of retraction and HEP (see Ex. 32 of Sect. 2.11).

2.9 Homotopy Properties of Infinite Symmetric Product Spaces

This section conveys homotopy properties of infinite symmetric product spaces defined for spaces in $\mathcal{T}op_*$ (see Sect. B.2.5 of Appendix B). These spaces link homotopy theory with homology theory via Elienberg–MacLane spaces (see Chaps. 11 and 17) and form an important class of topological spaces in the study of algebraic topology. So it has become necessary to study such spaces from homotopy viewpoint.

We have constructed in Sect. B.2.5 of Appendix B the finite symmetric product SP^nX and infinite symmetric product $SP^\infty X$ of a pointed topological space X. Both SP^n and SP^∞ are functors from the category $\mathcal{T}op_*$ to itself (see Sect. B.2.5 of Appendix B). A continuous map $f : X \to Y$ in $\mathcal{T}op_*$ induces maps $f^n : X^n \to Y^n$, $(x_1, x_2, \ldots, x_n) \mapsto (f(x_1), f(x_2), \ldots, f(x_n))$. These maps are compatible with the action of the symmetric group S_n of the set $\{1, 2, \ldots, n\}$ and hence induce maps $SP^n(f) : SP^nX \to SP^nY$ between the corresponding orbit spaces and also induce maps $SP^\infty(f) = f_* : SP^\infty X \to SP^\infty Y$ (see Sect. B.2.5 of Appendix B).

Theorem 2.9.1 *If $f, g : X \to Y$ are in $\mathcal{T}op_*$ and $f \simeq g$, then $SP^\infty(f) \simeq SP^\infty(g)$.*

Proof Let $F : X \times I \to Y$ be a map such that $F : f \simeq g$. For all $n \geq 1$, define $F^n : X^n \times I \to Y^n$, $(x_1, x_2, \ldots, x_n, t) \mapsto (F(x, t), F(x_2, t), \ldots, F(x_n, t))$. Then F^n is continuous, because its projection onto each coordinate is continuous. Since S_n acts on $X^n \times I$ by permuting the coordinate of X^n and fixing I, and F^n respects this action, F^n induces maps $SP^n(F) : SP^nX \to SP^nY$, which passing to the limit induces a map $SP^\infty(F) : SP^\infty X \times I \to SP^\infty Y$. Define

$$h_t : X \to Y, x \mapsto F(x, t) \text{ and } SP^\infty(F) : SP^\infty X \times I \to SP^\infty Y, (x, t) \mapsto SP^\infty(h_t)(x).$$

Hence $SP^\infty(F) : SP^\infty(f) \simeq SP^\infty(g)$. ❑

Corollary 2.9.2 *If spaces X and Y in $\mathcal{T}op_*$ are homotopy equivalent, then the spaces $SP^\infty X$ and $SP^\infty Y$ are also homotopy equivalent. In particular, if X is contractible, then $SP^\infty X$ is contractible.*

Proof Let $f : X \to Y$ be a homotopy equivalence with homotopy inverse $g : Y \to X$. Then $SP^\infty(g)$ is a homotopy inverse of $SP^\infty(f)$. Consequently, the spaces $SP^\infty X$ and $SP^\infty Y$ are homotopy equivalent. Again $SP^\infty\{*\} = \{*\}$ proves the second part. ❑

Theorem 2.9.3 $SP^\infty : \mathcal{H}tp_* \to \mathcal{H}tp_*$ *is a covariant functor.*

Proof It follows from Theorem 2.9.1 and Proposition B.2.18. ❑

2.10 Applications

This section presents some interesting immediate applications of homotopy. It deals with some extension problems and proves 'Fundamental Theorem of Algebra' by using homotopic concepts.

2.10.1 Extension Problems

This subsection solves some extensions problems with the help of homotopy.

Theorem 2.10.1 *A continuous map $f : S^n \to Y$ from S^n to any space Y can be continuously extended over D^{n+1} if and only if f is nullhomotopic, i.e., iff f is homotopic to a constant map.*

Proof Let $c : S^n \to Y$ be a constant map defined by $c(S^n) = y_0 \in Y$ such that $f \simeq c$. Then *exists* a homotopy $H : S^n \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = c(x) = y_0$, $\forall\ x \in S^n$. We now construct a map $F : D^{n+1} \to Y$ by the rule

$$F(x) = \begin{cases} y_0, & 0 \le ||x|| \le 1/2 \\ H\left(\frac{x}{||x||}, 2 - 2||x||\right), & 1/2 \le ||x|| \le 1. \end{cases}$$

Since at $||x|| = \frac{1}{2}$, $H(\frac{x}{||x||}, 1) = y_0$, F is well defined. Again, since its retraction to each of the closed sets $C_1 = \{x \in D^{n+1} : 0 \le ||x|| \le 1/2\}$ and $C_2 = \{x \in D^{n+1} : 1/2 \le ||x|| \le 1\}$ is continuous, F agrees on $C_1 \cap C_2$ and $D^{n+1} = C_1 \cup C_2$, F is continuous by Pasting lemma. Moreover, $\forall\ x \in S^n$, $||x|| = 1$ and hence $F(x) = H(x, 0) = f(x) \Rightarrow F$ is a continuous extension of f over D^{n+1}. Thus $f \simeq c \Rightarrow f$ has a continuous extension over D^{n+1}. Conversely, let $F : D^{n+1} \to Y$ be a continuous extension of $f : S^n \to Y$. Then $\forall\ x \in S^n$, $F(x) = f(x)$. Suppose $p_0 \in S^n$ and $f(p_0) = y_0 \in Y$. We now define a mapping $H : S^n \times I \to Y$ by $H(x, t) =$

$F((1-t)x + tp_0)$. H is well defined, because D^{n+1} is a convex set. Moreover, H is continuous and $H : f \simeq c$. ❑

Theorem 2.10.2 *Any continuous map from S^n to a contractible space has a continuous extension over D^{n+1}.*

Proof Let $c : S^n \to Y$ be a constant map from S^n to a contractible space Y and $f : S^n \to Y$ be an arbitrary continuous map. Then $f \simeq c$ by Corollary 2.6.4. Hence it follows by Theorem 2.10.1 that f has a continuous extension over D^{n+1}. ❑

Theorem 2.10.3 *Let p_0 be an arbitrary point of S^n and let $f : S^n \to Y$ be continuous. Then the following statements are equivalent.*

(a) *f is nullhomotopic.*
(b) *f can be continuously extended over D^{n+1}.*
(c) *f is nullhomotopic relative to $\{p_0\}$.*

Proof **(a)** $\Rightarrow$ **(b)** follows from Theorem 2.10.1.

(b) $\Rightarrow$ **(c)** Let $F : D^{n+1} \to Y$ be a continuous extension of f over D^{n+1}. Suppose $f(p_0) = y_0 \in Y$. Define a map $H : S^n \times I \to Y$ by $H(x,t) = F((1-t)x + tp_0)$ as shown in Fig. 2.27.
Then $\forall\ x \in S^n$, $H(x,0) = F(x) = f(x)$, $H(x,1) = F(p_0) = f(p_0) = y_0 = c(x)$ and $H(p_0,t) = F(p_0) = f(p_0) = y_0 = c(p_0)$, $\forall\ t \in I$. Hence $H : f \simeq c$ rel $\{p_0\}$.

(c) $\Rightarrow$ **(a)** It follows trivially.

❑

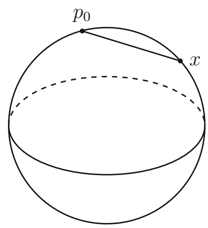
Proposition 2.10.4 *There exists a continuous map $f : D^n \to S^{n-1}$ with $f \circ i = 1_d$ iff the identity map $1_d : S^{n-1} \to S^{n-1}$ is nullhomotopic.*

Proof Suppose there exists such a map $f : D^n \to S^{n-1}$. Define a homotopy

$$H : S^{n-1} \times I \to S^{n-1}, (x,t) \mapsto f(tx).$$

Then $H(x,1) = x$, $\forall x \in S^{n-1}$ and $H(x,0) = f(0)$, $\forall x \in S^{n-1}$, i.e., $H(x,0)$ is independent of x. Hence 1_d is homotopic to a constant map. Conversely, let there exist $H : S^{n-1} \times I \to S^{n-1}$ such that $H(x,0) = c$ and $H(x,1) = x$. Define

Fig. 2.27 Construction of H



$$f : D^n \to S^{n-1}, x \mapsto \begin{cases} H\left(\frac{x}{||x||}, ||x||\right), & \text{if } x \neq 0 \\ c, & \text{if } x = 0 \end{cases}$$

Since S^{n-1} is compact, H is uniformly continuous. Hence for every $\epsilon > 0$, e a $\delta > 0$ (depends on ϵ but not on x) such that $||H(x,t) - c|| < \delta$ if $t < \epsilon$. This shows that f is continuous at $0 \in D^n$. ❑

Proposition 2.10.5 *Let (X, A) be a normal pair (i.e., X is normal and A is closed in X) such that $X \times I$ is normal and $f : X \to S^n$ be a continuous function. Then every homotopy of $f|_A$ can be extended to a homotopy of f.*

Proof Let $H : A \times I \to S^n$ be a homotopy of $f|_A$. Then H can be extended to a continuous map $\tilde{H} : X \times \{0\} \cup A \times I$ be setting $H(x, 0) = f(x)$ for all $x \in X$. Then by using Ex. 22 of Sect. 1.16 of Chap. 1 it follows that there exists an extension $F : X \times I \to S^n$, which is required homotopy. ❑

Proposition 2.10.6 *If a continuous map $f : X \to S^n$ is essential, then $f(X) = S^n$ (i.e., f is a surjection).*

Proof If $f(X) \neq S^n$, then there exists an element $y \in S^n - f(X)$. Since $S^n - y$ is contractible to a point and $f(X) \subset S^n - y$, it follows that f is inessential. This is a contradiction. ❑

Proposition 2.10.7 *Let (X, A) be a normal pair such that $X \times I$ is normal. Then every inessential map $f : A \to S^n$ admits an inessential extension $\tilde{f} : X \to S^n$.*

Proof Let $g : X \to S^n$ be a map of X into a single point of S^n. Then $g|_A$ is homotopic to f. Hence the existence of $\tilde{f}$ follows from Proposition 2.10.5. ❑

Definition 2.10.8 A topological space X is said to be aspherical if every continuous map $f : S^n \to X$ extends to a continuous map $\tilde{f} : D^{n+1} \to X$.

Example 2.10.9 Every convex subspace of Euclidean space and every contractible space are aspherical.

2.10.2 Fundamental Theorem of Algebra

This subsection applies the tools of homotopy to prove the celebrated fundamental theorem of algebra which shows that the field of complex numbers is algebraically closed. There are several methods to prove the fundamental theorem of algebra. We now present a proof by homotopy. For an alternative proof see Theorem 3.8.1 of Chap. 3.

Theorem 2.10.10 *Let $\mathbf{C}$ denote the field of complex numbers, and $C_\rho \subset \mathbf{C} \approx \mathbf{R}^2$ denote the circle at the origin and of radius ρ. Let $f_\rho^n : C_\rho \to \mathbf{C} - \{0\}$ be the restriction to C_ρ to the map $z \mapsto z^n$. If none of the maps f_ρ^n is nullhomotopic ($n \geq 1$ and $\rho > 0$), then every nonconstant polynomial over $\mathbf{C}$ has a root in $\mathbf{C}$.*

Proof of Fundamental Theorem of Algebra by Homotopy:

Proof Without loss of generality we consider the polynomial $g(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + z^n$, $a_i \in \mathbf{C}$. We choose

$$\rho > \max\left\{1, \sum_{i=0}^{n-1} |a_i|\right\} \tag{2.14}$$

We define a map $F : C_\rho \times I \to \mathbf{C}$ by $F(z, t) = z^n + \sum_{i=0}^{n-1}(1-t)a_i z^i$. Then $F(z, t) \neq 0$ for any $(z, t) \in C_\rho \times I$. Otherwise, $F(z, t) = 0$ for some $z \in C_\rho$ and $t \in I$ would imply $z^n = -\sum_{i=0}^{n-1}(1-t)a_i z^i$. This implies $\rho^n \leq \sum_{i=0}^{n-1}(1-t)|a_i|\rho^i \leq \sum_{i=0}^{n-1}|a_i|\rho^i \leq \sum_{i=0}^{n-1}|a_i|\rho^{n-1}$ for $\rho > 1$, because $\rho^i \leq \rho^{n-1}$ for $\rho > 1$.

Hence $\rho \leq \sum_{i=0}^{n-1}|a_i|$, by canceling ρ^{n-1}
$\Rightarrow$ a contradiction to the relation (2.14).

In other words, $F(z, t) \neq 0$ for any z with $|z| = 1$ and for any $t \in I$. We now assume that g has a root in $\mathbf{C}$. We define $G : C_\rho \times I \to \mathbf{C} - \{0\}$ by $G(z, t) = g((1-t)z)$. Since g has no root in $\mathbf{C}$, $G(z, t) \neq 0$ and hence the values of G must lie in $\mathbf{C} - \{0\}$. Now $G : g|_{C_\rho} \simeq k$, where k is the constant map $z \mapsto g(0) = a_0$ at a_0. Hence $g|_{C_\rho}$ is nullhomotopic. Again $g|_{C_\rho} \simeq f_\rho^n$. Thus f_ρ^n is nullhomotopic by symmetric and transitive properties of the relation $\simeq$. This contradicts the hypothesis. Consequently, g has a complex root. ❑

2.11 Exercises

1. For all $n \geq 0$, show that the topological spaces $S^1 \wedge S^n$ and S^{n+1} are homeomorphic.
   [Hint: Let S^{n+1} be the $(n+1)$-sphere in $\mathbf{R}^{n+2}$, S^n be equator, D^{n+1} be the $n+1$-disk embedded in D^{n+2}, H_+^{n+1} be upper hemisphere, H_-^{n+1} be lower hemisphere and $s_0 = (1, 0, 0, \ldots, 0)$ be base point. Now proceed as in Proposition 2.5.6.]
2. Given a collection of pointed topological spaces X_α, $Y_\alpha (\alpha \in A)$, and maps $f_\alpha \simeq g_\alpha : X_\alpha \to Y_\alpha$, show that $\times f_\alpha \simeq \times g_\alpha$.
   [Hint: Let $F_\alpha : X_\alpha \times I \to Y_\alpha$ be a homotopy between f_α and g_α. Then $F : (\times X_\alpha) \times I \to \times Y_\alpha$, defined by $F((x_\alpha), t) = (F_\alpha(x_\alpha, t))$, $\forall\, t \in I$ is continuous and a homotopy between $\times f_\alpha$ and $\times g_\alpha$ (relative to base point).]

3. Consider the homotopy set $[A, X]$, where A is a fixed space. Show that a continuous map $f : X \to Y$ induces a function $f_* : [A, X] \to [A, Y]$ satisfying the following properties:

(i) If $f \simeq g$, then $f_* = g_*$;
(ii) If $1_X : X \to X$ is the identity map, then $1_{X^*} : [A, X] \to [A, X]$ is the identity function;
(iii) If $g : Y \to Z$ is another continuous map, then $(g \circ f)_* = g_* \circ f_*$.
Deduce that if $X \simeq Y$, then $\exists$ a bijection between the sets $[A, X]$ and $[A, Y]$. What are the corresponding results for the sets $[X, A]$ for a fixed space A? [See Theorems 2.3.1 and 2.3.5 and their corollaries.]

4. Show that

(a) $S^1 = \{z \in \mathbf{C} : |z| = 1\}$ is a topological group under usual multiplication of complex numbers.
(b) For any space X, pointwise multiplication endows the set of continuous maps $X \to S^1$ with the structure of an abelian group. It is compatible with homotopy and then the set $[X, S^1]$ acquires the structure of a group.
(c) If $f : Y \to X$ is continuous then $f^* : [X, S^1] \to [Y, S^1]$ is a homomorphism.

5. Show that a space X is contractible iff it is deformable into one of its points.
6. Show that if A is a deformation retract of X, then A and X have the same homotopy type.
7. Show that any one-point subset of a convex subspace Y of $\mathbf{R}^n$ is a strong deformation retract of Y.
8. Let X be the closed unit square and A be the comb space. Show that A is weak deformation retract of X but not a deformation retract of X.
9. Show that the point $(0, 1)$ of the comb space X is a deformation retract of X but not a strong deformation retract of X.
10. Let X be a Hausdorff space and $A \subset X$ be a retract of X. Prove that A is closed in X. Hence show that an open interval $(0, 1)$ cannot be a retract of any closed subset of the real line $\mathbf{R}^1$.
11. Show that

(a) A continuous map $f : X \to Y$ is nullhomotopic iff it has a continuous extension over the cone $CX = (X \times I)/X \times \{1\}$.
(b) Given a continuous map $f : X \to Y$, its mapping cylinder $M_f = (X \times I) \cup Y = (X \times I) \cup Y/\sim$, where for all $x \in X$, $\sim$ identifies $(x, 1)$ with $f(x)$.
(c) $S^1 = \{z \in \mathbf{C} - \{0\} : |z| = 1\}$ is a strong deformation retract of $\mathbf{C} - \{0\}$.
(d) S^1 and $\mathbf{C} - \{0\}$ have the same homotopy type.
(e) For all $f : X \to Y$, the space Y is a deformation retract of the its mapping cylinder M_f.
(f) Any continuous map from a closed subset of $\mathbf{R}^n$ into a sphere is extendable over the whole of $\mathbf{R}^m$ iff f is essential.

(g) Two constant maps $k_i : X \to Y, x \mapsto y_i, i = 0, 1$ are homotopic iff $\exists$ a continuous curve $\gamma : I \to Y$ from y_0 to y_1.

12. Let $[X, Y]$ denote the set of homotopy classes of maps $f : X \to Y$. Show that

(i) for any space X, $[X, I]$ has a single element;
(ii) if X is path-connected, then $[I, X]$ has a single element;
(iii) a contractible space is path-connected;
(iv) if X is contractible, then for any space Y, $[Y, X]$ has a single element;
(v) if X is contractible, and Y is path-connected, then $[X, Y]$ has a single element.

13. Show that a retract of a contractible space is contractible.
14. Show that $\mathbf{R}^{n+1} - \{0\}$ is homotopy equivalent to S^n.
15. Show that the space $X = \{(x, y, z) \in \mathbf{R}^3 : y^2 > xz\}$ is homotopy equivalent to a circle. Interpret this result by considering the roots of the equation $ax^2 + 2hxy + by^2 = 0$.
16. Let $X = \{(p, q) \in S^n \times S^n : p \neq -q\}$. Show that the map $f : S^n \to X$ defined by $f(p) = (p, p)$ is a homotopy equivalence.
17. In $\mathbf{R}^2$, define $A_1 = \{(x_1, x_2) \in \mathbf{R}^2 : (x_1 - 1)^2 + x_2^2 = 1\}$, $A_2 = \{(x_1, x_2) \in \mathbf{R}^2 : (x_1 + 1)^2 + x_2^2 = 1\}$. Suppose $Y = A_1 \cup A_2$, $X = Y \setminus \{(2, 0), (-2, 0)\}$, $A = 0 = \{(0, 0)\}$. Show that A is a strong deformation retract of X.
18. **(a)** Let X and Y be pointed topological spaces. Show that there exists a bijection $\psi : [\Sigma X, Y] \to [X, \Omega Y]$ such that it is natural in X and in Y in the sense that if $f : X' \to X$ and $g : Y \to Y'$ are base point preserving continuous maps, then the diagrams in the Fig. 2.28 and in Fig. 2.29 are commutative, where the horizontal arrows represents the corresponding isomorphism.
(b) Show that for $n \geq 2$ and any pointed Hausdorff space X the iterated loop spaces $\Omega^n X (= \Omega(\Omega^{n-1} X) = (\Omega^{n-1} X)^{S^1})$ are homotopy commutative H-groups. Hence prove that for $n \geq 2$ and pointed spaces X, Y, the groups $[X, \Omega^n Y]$ are abelian.
[Hint: See Theorem 2.5.14.]
19. Let (X, A) have the absolute homotopy extension property (AHEP) (in the sense that A has HEP in X with respect to every space Y) and A be contractible. Show that the identification map $p : X \to X/A$ is a homotopy equivalence.

Fig. 2.28 Naturality of ψ in X

$$\begin{tikzcd} {[\Sigma X, Y]} \arrow[r, "\cong"] \arrow[d, "(\Sigma f)^*"'] & {[X, \Omega Y]} \arrow[d, "f^*"] \\ {[\Sigma X', Y]_*} \arrow[r, "\cong"] & {[X', \Omega Y]} \end{tikzcd}$$

Fig. 2.29 Naturality of ψ in Y

$$\begin{tikzcd} {[\Sigma X, Y]} \arrow[r, "\cong"] \arrow[d, "g_*"'] & {[X, \Omega Y]} \arrow[d, "(\Omega g)_*"] \\ {[\Sigma X, Y']_*} \arrow[r, "\cong"] & {[X, \Omega Y']} \end{tikzcd}$$

20. **(a)** Let G be a fixed H-group with base point e with continuous multiplication $\mu : G \times G \to G$ and homotopy inverse $\phi : G \to G$. Show that there exists a contravariant functor $\pi^G : \mathcal{H}tp_* \to \mathcal{G}rp$.

(b) For each homotopy associative H-space K, show that π^K is a contravariant from $\mathcal{H}tp_*$ to the category of monoids and their homomorphisms.

(c) Show that π^G is homotopy type invariant for each H-group G.

(d) Let G be a pointed topological space such that π^G assumes values in $\mathcal{G}rp$. Show that G is an H-group. Moreover, for any pointed space X, show that the group structure on $\pi^G(X)$ and $[X, G]$ coincide.

(e) Let $\alpha : G \to H$ be a homomorphism of H-groups. Show that α induces a natural transformation $N(\alpha) : \pi^G \to \pi^H$, where $N(\alpha)(X) : [X, G] \to [X, H]$ is defined by $N(\alpha)(X)([f]) = [\alpha \circ f]$, $\forall [f] \in [X, G]$.

21. Given a closed curve C in the plane $\mathbf{R}^2 \times \{0\}$, show that there exists a continuous deformation deforming C into a spherical closed curve $\tilde{C}$ and conversely given a spherical curve $\tilde{C}$, show that there exists a continuous deformation deforming $\tilde{C}$ into a closed curve C in the plane $\mathbf{R}^2 \times \{0\}$ such that total normal twists of C and $\tilde{C}$ remain the same.

22. (M. Fuchs) Prove that two topological spaces X and Y have the same homotopy type iff they are homeomorphic to a strong deformation retract of a space Z.

23. Using the notation of Theorem 2.4.18, show that a pointed space P is an H-space iff there is a continuous map $\mu : P \times P \to P$ such that $\mu \circ i_1 = \mu \circ i_2 = c$. The map μ satisfies the condition $[\mu] = [p_1]\cdot[p_2]$ and if $f, g : X \to P$ are base point preserving continuous maps, then $[f]\cdot[g]$ is the homotopy class of the composite

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} P \times P \xrightarrow{\mu} P.$$

24. Let X and Y be topological spaces and $f : X \to Y$ be a continuous map. Show that Y is a strong deformation retract of its mapping cylinder M_f.

25. Show that a continuous map $f : X \to Y$ has a left homotopy inverse iff X is a retract of its mapping cylinder M_f.

26. Show that a continuous map $f : X \to Y$ has a right homotopy inverse iff the mapping cylinder M_f deforms into X.

27. Show that

(a) A continuous map $f : X \to Y$ is a homotopy equivalence iff X is a deformation retract of the mapping cylinder M_f;

(b) If D is such a deformation retraction, then $D|_{Y \times \{1\}}$ is a homotopy inverse to f and for any homotopy inverse g, there is a deformation retract of M_f into X which gives g.

(c) Let X be a normal space. If $A \subset X$ is the set of zeros of a continuous map $f : X \to I$, and if A is a strong deformation retract of a neighborhood U of A in X, then $(X \times \{0\} \cup A \times I)$ is a strong deformation retract of $X \times I$.

(d) S^1 is a deformation retract of $\mathbf{R}^2 \setminus \{0\} (= \mathbf{R}^2 - \{0\})$.

(e) Möbius strip is homotopy equivalent to S^1.

28. Show that the suspension

$$\Sigma : \mathcal{H}tp_* \to \mathcal{H}tp_*$$

is an endofunctor (i.e., a functor from $\mathcal{H}tp_*$ to itself).

29. Let X and Y be pointed Hausdorff spaces. Show that

(i) Both $[\Sigma X, Y]$ and $[X, \Omega Y]$ are groups.
(ii) The groups $[\Sigma X, Y]$ and $[X, \Omega Y]$ are isomorphic.
(iii) If X is an H-cogroup and Y is an H-group, then the products available on $[X, Y]$ determine isomorphic groups which are abelian;

30. Let A be a closed (or open) subspace of X in $\mathcal{T}op_*$. Then the inclusion $i : A \hookrightarrow X$ induces closed (or open) inclusions $SP^\infty(i) : SP^\infty A \hookrightarrow SP^\infty X$.

31. Let (X, A) be a pair of topological spaces such that X is a compact Hausdorff space, A is closed in X and A is a strong deformation retract of X. Let $p : X \to X/A$ be the identification map and $p(A) = y \in X/A$. Show that $\{y\}$ is a strong deformation retract of X/A.

32. (Steenrod) Let the space X be compactly generated and A be closed in X. Show that the following statements are equivalent:

(i) (X, A) is an NDR-pair;
(ii) $(X \times I, X \times \{0\} \cup A \times I)$ is a DR-pair;
(iii) $X \times \{0\} \cup A \times I$ is a retract of $X \times I$;
(iv) (X, A) has the homotopy extension property (HEP) with respect to arbitrary topological spaces.

33. **(i)** Let $f, g : (X, A) \to (Y, B)$ be two continuous maps of pair of spaces such that $f \simeq g$. Show that that their induced maps $\tilde{f}, \tilde{g} : X/A \to Y/B$ are also homotopic.
(ii) Let $f : (X, A) \to (Y, B)$ be a homotopy equivalence. Show that the induced map $\tilde{f} : X/A \to Y/B$ is a (based) homotopy equivalence.

34. Let (X, A) be a pair of topological spaces such that A is closed in X and $X \times I$ is a normal space. If there is a neighborhood U such that U is a retract of $(X \times \{0\} \cup (A \times I))$, show that any continuous map $G : (X \times \{0\}) \cup (A \times I) \to Y$ has a continuous extension over $X \times I$.

2.12 Additional Reading

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