MT5823 Semigroup theory: Solutions 7 (James D. Mitchell)
More Green's relations, Simple semigroups, Rees matrix semigroups

More Green's relations

7-1. (a) If $a, b \in T_8$, then $f\mathscr{R}a$ if and only if $\ker(f) = \ker(a)$; and $f\mathscr{L}b$ if and only if $\mathrm{im}(f) = \mathrm{im}(b)$. Hence we require idempotents $a, b \in T_8$ such that

$$\ker(a) = \ker(f) = \{\{1, 4, 6\}, \{2, 7\}, \{3\}, \{5, 8\}\}$$

and

$$\mathrm{im}(b) = \mathrm{im}(f) = \{1, 3, 4, 6\}.$$

One of possible choice is:

$$\begin{aligned} a &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 1 & 5 & 1 & 2 & 5 \end{pmatrix} \\ b &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 1 & 3 & 4 & 1 & 6 & 1 & 1 \end{pmatrix}. \end{aligned}$$

Recall that $e \in T_8$ is idempotent if and only if $i \in (i)e^{-1}$ for all $i \in \mathrm{im}(e)$. Hence if $a \in T_8$ is an idempotent such that $\ker(a) = \ker(f)$, then $\mathrm{im}(a)$ is one of the sets in $\{1, 4, 6\} \times \{2, 7\} \times \{3\} \times \{5, 8\}$, i.e. one of:

$$\begin{array}{llllll} \{1,\ 2,\ 3,\ 5\}, & \{1,\ 2,\ 3,\ 8\}, & \{1,\ 7,\ 3,\ 5\}, & \{1,\ 7,\ 3,\ 8\}, & \{4,\ 2,\ 3,\ 5\}, & \{4,\ 2,\ 3,\ 8\}, \\ \{4,\ 7,\ 3,\ 5\}, & \{4,\ 7,\ 3,\ 8\}, & \{6,\ 2,\ 3,\ 5\}, & \{6,\ 2,\ 3,\ 8\}, & \{6,\ 7,\ 3,\ 5\}, & \{6,\ 7,\ 3,\ 8\}. \end{array}$$

It follows that there are 12 choices for a.

Recall that $e \in T_8$ is idempotent if and only if $(i)e = i$ for all $i \in \mathrm{im}(e)$. Similarly, if $b \in T_8$ is an idempotent such that $\mathrm{im}(b) = \mathrm{im}(f)$, then $(i)b = i$ for all $i \in \{1, 3, 4, 6\}$. The value of $(i)b$ for any $i \in \{2, 5, 7, 8\}$ must be in $\{1, 3, 4, 6\}$ and any such choice gives an idempotent. Hence there are $4^4 = 256$ choices for b. $\square$

(b) By Theorem 11.6(b), if f' is an inverse of f such that $ff' = a$ and $f'f = b$, then $f'\mathscr{L}a$ and $f'\mathscr{R}b$. That is,

$$\mathrm{im}(f') = \mathrm{im}(a) = \{1, 2, 3, 5\} \quad \text{and} \quad \ker(f') = \ker(b) = \{\{1, 2, 5, 7, 8\}, \{4\}, \{6\}, \{7\}\}.$$

One choice for f' is:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 3 & 5 & 1 & 3 & 2 & 3 & 3 \end{pmatrix}.$$

Again by Theorem 11.6, for every choice of idempotents a, b from part (a), there exists precisely one inverse f' of f such that $ff' = a$ and $f'f = b$. Since the distinct choices of a and b correspond to f' with distinct kernel and image, it follows that no two choices for a and b give rise to the same value of f'. Hence there are $3072 = 256 \times 12$ inverses $f' \in T_8$ for f. $\square$

7-2. An $\mathscr{L}$-class is characterised by the common image of its elements, which is a subset of $\{1, 2, \ldots, n\}$ of size r. There are $\binom{n}{r}$ such subsets.

An $\mathscr{R}$-class is characterised by the common kernel of its elements, which is an equivalence relation with r classes. There are $S(n, r)$ (the Stirling number of the second kind) such equivalences (see Problem **3-7**).

Every $\mathscr{L}$-class and every $\mathscr{R}$-class in D_r intersects to give an $\mathscr{H}$-class. Hence the number of $\mathscr{H}$-classes in D_r is $\binom{n}{r}\ S(n, r)$.

Every $\mathscr{H}$-class of D_r has size $r!$ and so

$$|D_r| = \binom{n}{r}\ S(n, r)\ r!. \quad \square$$

7-3. ($\Rightarrow$) Since H_f is a group, it is closed under multiplication and so $f^2 \in H_f$. Hence $(f, f^2) \in \mathscr{H} \subseteq \mathscr{L}$ and so $\mathrm{im}(f) = \mathrm{im}(f^2)$. It follows that $\mathrm{rank}(f) = |\mathrm{im}(f)| = |\mathrm{im}(f^2)| = \mathrm{rank}(f^2)$.

($\Leftarrow$) Suppose that $\mathrm{rank}(f^2) = \mathrm{rank}(f)$. Since $\mathrm{im}(f^2) = \mathrm{im}(f)f \subseteq \mathrm{im}(f)$ and since $\mathrm{im}(f)$ is a finite set, it follows that $\mathrm{im}(f^2) = \mathrm{im}(f)$. Hence $f^2\mathscr{L}f$ and, in particular, $(\mathrm{im}(f))f = \mathrm{im}(f)$. This implies that f is injective on $\mathrm{im}(f)$ and so $(x, y) \in \ker(f^2)$ if and only $(x)f^2 = (y)f^2$ if and only if $((x)f)f = ((y)f)f$ if and only if $(x)f = (y)f$ if and only if $(x, y) \in \ker(f)$. Therefore $\ker(f) = \ker(f^2)$ and so $f\mathscr{R}f^2$.

We have shown that $f^2\mathscr{L}f$ and $f\mathscr{R}f^2$ and so $f\mathscr{H}f^2$. Thus $f^2 \in H_f^2 \cap H_f$ and so $H_f^2 \cap H_f \neq \varnothing$ and so, by Theorem 10.7, H_f is a group. $\square$

7-4. (a) Let $f \in P_n$ be arbitrary and for every $i \in \text{im}(f)$ choose $j_i \in (i)f^{-1}$. We define $g \in P_n$ by

$$(i)g = \begin{cases} j_i & i \in \text{im}(f) \\ - & \text{otherwise.} \end{cases}$$

Then $(i)fgf = (j_{(i)f})f = (i)f$ for all $i \in \{1, 2, \ldots, n\}$ and so $fgf = f$. It follows that f is regular and hence P_n is too.

(b) If $f \in P_n$, then define $f^* \in T_{n+1}$ by

$$(x)f^* = \begin{cases} (x)f & x \in \{1, 2, \ldots, n\} \text{ and } x \in \text{dom}(f) \\ n+1 & \text{otherwise.} \end{cases}$$

It is straightforward to verify that the mapping $\phi : P_n \longrightarrow T_{n+1}$ defined by

$$(f)\phi = f^*$$

is a monomorphism and so $P_n \cong (P_n)\phi \leq T_{n+1}$.

(c) Since P_n is regular by part (a), $(P_n)\phi$ is a regular subsemigroup of T_{n+1}. Thus by Theorem **??**

$$f^*\mathscr{R}^{(P_n)\phi}g^* \iff \ker(f^*) = \ker(g^*) \iff \ker(f) = \ker(g) \iff f\mathscr{R}^{P_n}.$$

The remaining assertions are proved in a similar way. □

7-5. To prove that $L_eR_e \subseteq D_e$, let $a \in L_eR_e$. There exist $x \in L_e$ and $y \in R_e$ such that $a = xy$. Hence $x\mathscr{L}e$ and $y\mathscr{R}e$. By Problem **5-8**, e is a left identity for its $\mathscr{R}$-class and, by Theorem 9.5, $\mathscr{L}$ is a right congruence. Therefore

$$a = xy\mathscr{L}ey = y\mathscr{R}e$$

and so $a \in D_e$.

To prove that $D_e \subseteq L_eR_e$, let $a \in D_e$. Then there exists $b \in S$ such that $a\mathscr{R}b\mathscr{L}e$, and so there are $s_1, s_2, t_1, t_2 \in S^1$ such that

$$a = bs_1, \quad b = t_1e, \quad b = as_2, \quad e = t_2b.$$

We set $c = es_1$. Then

$$cs_2 = es_1s_2 = t_2bs_1s_2 = t_2as_2 = t_2b = e$$

and so $c\mathscr{R}e$. Finally, $a = bs_1 = t_1es_1 = (t_1e)(es_1) = bc \in L_eR_e$. □

7-6. (a) ($\Rightarrow$) Since a is regular, there exists an inverse $a' \in S$ for a. By Theorem 11.6(a), a' is an element of the $\mathscr{D}$-class D_a of a and $a'a \in L_a \cap R_{a'}$. In particular, $a'a\mathscr{L}a\mathscr{L}b$ and $a'a$ is an idempotent. On the other hand, by Theorem 11.7, the $\mathscr{R}$-class of b contains an idempotent e. Therefore, by Theorem 11.6(b), there exists an inverse b' of b in $L_e \cap R_{a'a}$ such that $b'b = a'a$ as required.

($\Leftarrow$) If $a'a = b'b$, then $a = aa'a = ab'b$ and $b = bb'b = ba'a$, and so $a\mathscr{L}b$.

(b) The proof of this part is analogous to that of part (a).

(c) This is just a combination of parts (a) and (b).

Inverse semigroups

7-7. The function $x \mapsto x^{-1}$ is a bijection since $(x^{-1})^{-1} = x$ for all $x \in S$. By Problem **7-6**, applied to the regular semigroup S,

(a) $a\mathscr{L}b$ if and only if $a^{-1}a = b^{-1}b$;

(b) $a\mathscr{R}b$ if and only if $aa^{-1} = bb^{-1}$.

Hence $a\mathscr{L}b$ if and only if $a^{-1}a = b^{-1}b$ if and only if $(a^{-1})(a^{-1})^{-1} = (b^{-1})(b^{-1})^{-1}$ if and only if $a^{-1}\mathscr{R}b^{-1}$. In particular, $a\mathscr{L}b$ if and only if $(a)\phi\mathscr{R}(b)\phi$.

By Theorem 11.6, $a^{-1}\mathscr{D}a$ for all $a \in S$. Hence, if $a\mathscr{D}b$, then

$$(a)\phi = a^{-1}\mathscr{D}a\mathscr{D}b\mathscr{D}b^{-1} = (b)\phi$$

and so ϕ preserves $\mathscr{D}$-classes.

We have shown that the mapping ϕ is a bijection from the $\mathscr{L}$-classes in a $\mathscr{D}$-class to the $\mathscr{R}$-classes in that $\mathscr{D}$-class. Hence the number of $\mathscr{L}$-classes must equal the number of $\mathscr{R}$-classes.

The less technical argument that the numbers of $\mathscr{L}$-classes and $\mathscr{R}$-classes in a single $\mathscr{D}$-class are equal follows from the fact that each $\mathscr{L}$-class and $\mathscr{R}$-class has exactly 1 idempotent in it and the pigeonhole principle.

More specifically, suppose that l is the number $\mathscr{L}$-classes, and that r is the number of $\mathscr{R}$-class, in a $\mathscr{D}$-class D of S. If $l < r$, then the r idempotents in the $\mathscr{R}$-classes of D must be belong to some $\mathscr{L}$-class of D. But $l < r$ and so two idempotents must belong to the same $\mathscr{L}$-class, a contradiction. A similar contradiction is obtained when $r < l$, and so we conclude that $l = r$. $\square$

7-8. For any $a/\rho \in S/\rho$,

$$(a/\rho)(a^{-1}/\rho)(a/\rho) = (aa^{-1}a)/\rho = a/\rho$$

and so S/ρ is regular.

Let $x/\rho, y/\rho \in S/\rho$ be idempotents. By Lallement's Lemma (Problem **6-5**) there are idempotents $e, f \in S$ such that $x/\rho = e/\rho$ and $y/\rho = e/\rho$. But then

$$(x/\rho)(y/\rho) = (e/\rho)(f/\rho) = (ef)/\rho = (fe)/\rho = (f/\rho)(e/\rho) = (y/\rho)(x/\rho)$$

and hence S/ρ is a regular semigroup whose idempotents commute, i.e. it is an inverse semigroup.

Since

$$(a^{-1}/\rho)(a/\rho)(a^{-1}/\rho) = (a^{-1}aa^{-1})/\rho = a^{-1}/\rho,$$

and from the above, a^{-1}/ρ is an inverse of a/ρ. By uniqueness of inverses, it follows that $(a/\rho)^{-1} = a^{-1}/\rho$. $\square$

7-9. We start by showing that if $a\rho b$, then $a^{-1}\rho b^{-1}$. By Problem **7-8**, $(a/\rho)^{-1} = a^{-1}/\rho$, and so if $a\rho b$, then $a/\rho = b/\rho$ and so

$$(b^{-1}/\rho) = (b/\rho)^{-1} = (a/\rho)^{-1} = (a^{-1}/\rho),$$

or in other words, $a^{-1}\rho b^{-1}$.

($\Rightarrow$) Since $a\rho b$ and ρ is a right congruence, $ab^{-1}\rho bb^{-1}$. Since $a\rho b$ implies $a^{-1}\rho b^{-1}$, and since ρ is a left congruence, $aa^{-1}\rho ab^{-1}$. But ρ is transitive and so $aa^{-1}\rho bb^{-1}$.

($\Leftarrow$) By assumption $aa^{-1}\rho ab^{-1}$ and so $aa^{-1} = (aa^{-1})^{-1}\rho(ab^{-1})^{-1} = ba^{-1}$. Hence

$$a = aa^{-1}a \ \rho\ ba^{-1}a = bb^{-1}ba^{-1}a = ba^{-1}ab^{-1}b \ \rho\ aa^{-1}ab^{-1}b = ab^{-1}b \ \rho\ bb^{-1}b = b. \quad \square$$

7-10. Note that $T = \{\, t \in S \,:\, \exists e \in E,\ e\rho t \,\}$. If $s, t \in T$ with $s\rho e$ and $t\rho f$ (e, f idempotents), then $st\rho ef$. But ef is also an idempotent ($efef = eeff = ef$), and so $st \in T$. Also for any $s \in S$ we have that $s^{-1}ss\rho s^{-1}es$. Since $s^{-1}ess^{-1}es = s^{-1}ss^{-1}ees = s^{-1}es$, $s^{-1}es$ is an idempotent and hence $s^{-1}ss \in T$. $\square$

7-11. Let $f \in I_n$ be any element with $|\operatorname{dom}(f)| = r$. Then since f is a bijection, it follows that $|\operatorname{im}(f)| = r$. Hence there are

$$\binom{n}{r}$$

choices for $\operatorname{dom}(f)$ corresponding to subsets of $\{1, \ldots, n\}$ of size r. Similarly, there are $\binom{n}{r}$ choices for $\operatorname{im}(f)$. The choice of $\operatorname{dom}(f)$ and $\operatorname{im}(f)$ are obviously independent of each other. Given two subsets A and B of $\{1, \ldots, n\}$ of size r, there are $r!$ bijections from A to B. Hence there are:

$$\binom{n}{r}^2 r!$$

elements in I_n with $|\operatorname{dom}(f)| = r$. Summing over all possible values of r, from 0 to n, yields

$$|I_n| = \sum_{r=0}^{n} \binom{n}{r}^2 r!.$$

We apply the usual algorithm to compute the elements of the subsemigroup S generated by:

$$a = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 2 \\ 1 & - \end{pmatrix}.$$

...TODO It is clear that S is a subsemigroup of I_2 and

$$|I_2| = \binom{2}{0}^2 0! + \binom{2}{1}^2 1! + \binom{2}{2}^2 2! = 1 + 4 + 2 = 7 = |S|$$

and so $S = I_2$.

7-12. Let $a \in S$ and let $e \in E$ be arbitrary. Then $ae = aa^{-1}ae = aea^{-1}a$ and

$$(aea^{-1})^2 = aea^{-1}aea^{-1} = aa^{-1}ae^2a^{-1} = aea^{-1}.$$

Hence $ae = (aea^{-1})a \in Ea$. Similarly, $ea \in aE$ and so $Ea = aE$.

It is not true that $ae = ea$ for all $a \in S$ and for all $e \in E$. For example, if

$$a = \begin{pmatrix} 1 & 2 \\ 2 & - \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 \\ 1 & - \end{pmatrix},$$

then $ea = a$ but $ae = \varnothing \neq a$.