MT4514: Graph Theory

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Contents

Chapter 1

Introduction

1. About the course

The course will not be solely based on a single book. Therefore, the best study source will be the lecture notes. Some useful texts are:

1. Robin Wilson, *Introduction to Graph Theory*
2. Robin Wilson and John Watkins, *Graphs – an Introductory Approach.*
3. Frank Harary, *Graph Theory.*
4. Norman Biggs, *Discrete Mathematics*

All these books, as well as all tutorial sheets and solutions, will be available in Mathematics/Physics library on short loan. Also, any other book containing in its title the words such as 'graph theory', 'discrete mathematics', 'combinatorics' is likely to contain material relevant to the course.

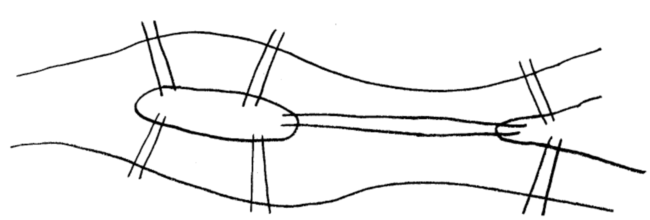
It will be useful to bring coloured pens or pencils to lectures, although I've had to do these notes in black and white.

2. Some introductory examples

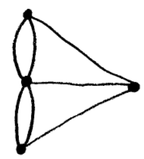
Graphs are a very useful way of recording information about objects and relationships between pairs of objects. A graph consists of a set V of *vertices* or *points* and a collection E of *edges* joining some of the pairs of points.

In this introduction we'll look at a range of examples to indicate the flavour and variety of the topic. Graph theory goes back to ancient Greek times, with the study of the 5 regular platonic solids, but it really started with the following problem.

Example 2.1. Königsberg Bridge problem. Consider this town plan of Königsberg (now Kaliningrad) in East Prussia. The river Pregel runs through the middle of town, and a set of 7 bridges links the banks and two islands in the river. The problem is to find a walk through the town that begins and ends at the same place, crossing each bridge exactly once.



Euler in 1736 realised that we can simplify the problem by representing it with this graph, with one edge for each bridge, and vertices representing each island and each bank.

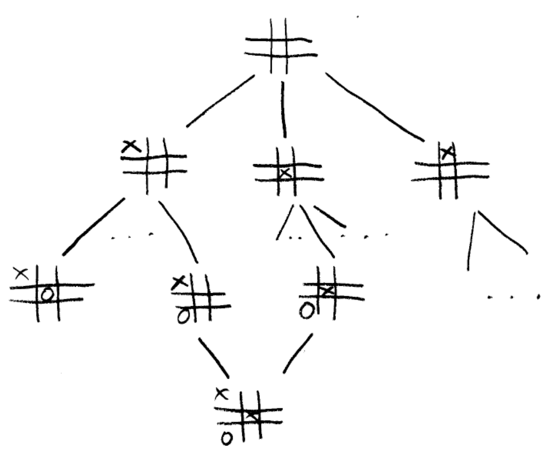


The problem now is to find a path around the graph which begins and ends at the same vertex and follows each edge exactly once. Looking at this graph, Euler was able to show that the problem has no solution.

Example 2.2. Knight's tour. Given an $n \times m$ chessboard, we ask whether it's possible to find a route for a knight that visits each square on the board exactly once. We represent the 3×4 chessboard with a graph like this, with one vertex for each square, and edges linking squares that a knight can move between in a single go:

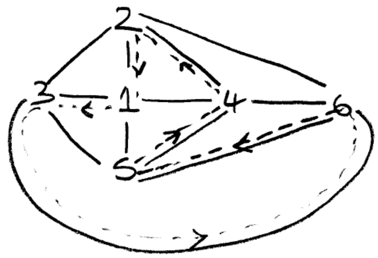
A knight's tour corresponds to finding a path on the graph that visits each vertex exactly once. Euler considered the problem of the 8×8 board.

Example 2.3. Logical games. Consider a game such as noughts and crosses, where there is no element of chance. We can draw a graph whose vertices represent all possible positions during a game. The edges of the graph connect each position to those positions which can be reached by a single move.



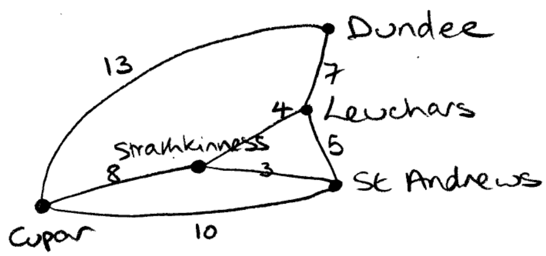
A strategic analysis of the game is possible by looking at the paths in the graph.

Example 2.4. Paths on a die. To find a circuit on a die visiting each face exactly once we consider this graph:



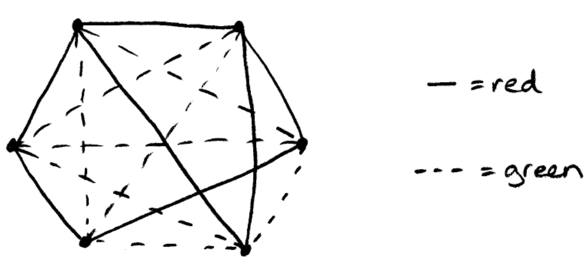
There is one vertex for each face, and an edge connecting adjacent faces. We want to find a closed circuit on the graph visiting each vertex exactly once. One solution is $1, 3, 6, 5, 4, 2, 1$.
Problem: How many such paths are there?

Example 2.5. Maps. Information on maps can often be presented with a graph, where edges may also be labelled with distances. An example with no distances is the London underground map. Here we give a more local map with distances:



A famous problem connected with graphs representing maps is the **Travelling Salesman Problem**: find the shortest path that visits each town at least once and returns to the start.

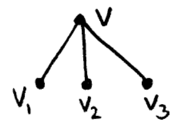
Example 2.6. People at a party. Given a group of six people, we can always find either three who all know each other or three who don't know each other. We can represent this with this graph, with one vertex for each person. We put a red edge between two vertices if the two people know each other, and a green edge if they do not.



The previous example is equivalent to the following:

Theorem 2.7. *Let (V, E) be a graph with six vertices and an edge between every pair of vertices that is either red or green. Then (V, E) contains either a red triangle or a green triangle.*

Proof. Let $v \in V$ be any vertex. Then v has five adjacent edges, so either at least three red edges or at least three green edges leave v. Without loss of generality we may assume that at least three red edges leave v, and that they join v to vertices v_1, v_2, v_3:

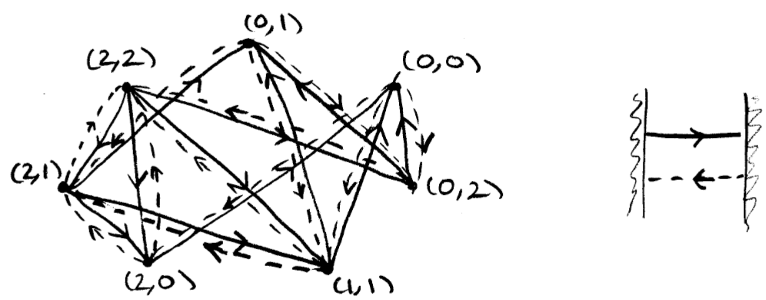


Now consider the edges v_1v_2, v_1v_3 and v_2v_3. If any of them, say v_iv_j, is red then we have found a red triangle vv_iv_j. If none of them is red then we have found a green triangle $v_1v_2v_3$. ■

Similarly, in a group of 18 people there are 4 mutual acquaintances or non-acquaintances, and we know that in a group of 17 people there may not be such a group of four people. In a group of 49 people there are 5 mutual acquaintances or non-acquaintances, but we **do not know** if 49 is the smallest number that guarantees the existence of 5 acquaintances or non-acquaintances: the smallest such number may be as low as 43.

Example 2.8. A puzzle: missionaries and cannibals. Two missionaries and two cannibals need to cross a river from west to east in a boat holding at most two people. To avoid being eaten, if there are any missionaries on a bank then there must be at least as many missionaries as cannibals.

We make a graph whose vertices are pairs (M, C) describing how many missionaries and cannibals are on the west bank of the river. We put arrows of one type indicating boat trips from west to east, and of another type to indicate boat trips from east to west.



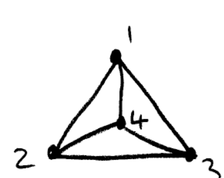
To find a solution we need to find a path of alternating types of arrows (as we must always bring the boat back from the other side), that starts with the vertex $(2,2)$ and ends with the vertex $(0,0)$. A possible solution is:

$$(2,2) \to (1,1) \to (2,1) \to (0,1) \to (0,2) \to (0,0).$$

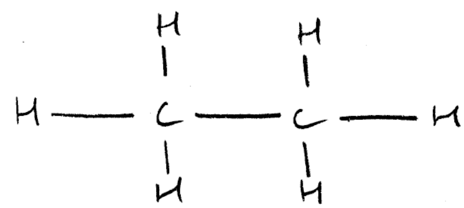
That is, one missionary and one cannibal cross, the missionary comes back, then two missionaries cross, then the cannibal comes back, then both cannibals cross.

Example 2.9. Further examples include

- The internet: vertices for computers and edges for connections, maybe labelled by their bandwidth.
- Graphs of groups, for example the automorphism group of the following graph is S_4:



- Chemistry: diagrams of saturated hydrocarbons.



- Management structures.
- Network flows, such as the national grid, the water supply.

Chapter 2

Basic definitions and isomorphism

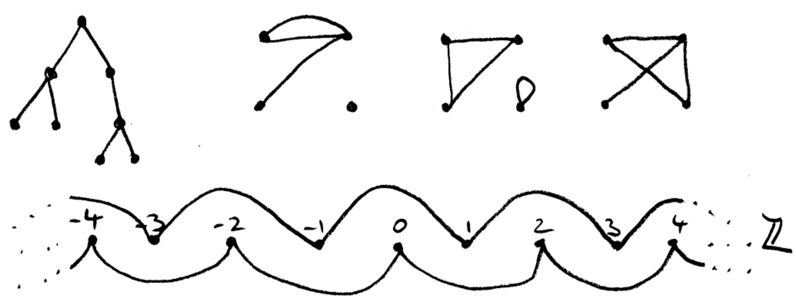
In this chapter we give some of the formal definitions of graph theory, and establish some elementary properties of graphs.

1. Some definitions

Definition 1.1. A *simple graph* G consists of a set V of *vertices* (also called *points* or *nodes*) and a set E of *edges* such that each edge joins a pair of vertices in V. More formally, a simple graph is a pair $G = (V, E)$ where V is a set and E is a set of subsets of V, each of size 2.

When V and E are finite sets we write $|G|$ or $|V|$ to denote the number of vertices of G. The graph G is *finite* when V and E are finite. We denote the edge joining v_i and v_j by v_iv_j. Note that $v_iv_j = v_jv_i$ as the edges of a graph do not have a direction.

Example 1.2. Here are some finite and infinite graphs:



Definition 1.3. A *loop* is an edge joining a vertex to itself. A graph has *multiple edges* if there exist two vertices with more than one edge between them. In general, a graph may have loops and multiple edges: a simple graph is a finite graph with no loops and no multiple edges.

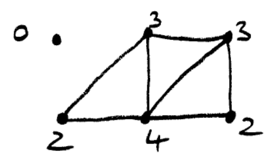
Often we will simply say "graph" for "finite graph" when the context is clear.

Definition 1.4. A graph is *connected* if it is possible to go between any pair of vertices by traversing a sequence of edges. If a graph G is not connected then we may consider G as a disjoint union of connected graphs, known as the *connected components* of G.

Example 1.5. The graph G is connected, the graph H is unconnected and has two connected components.

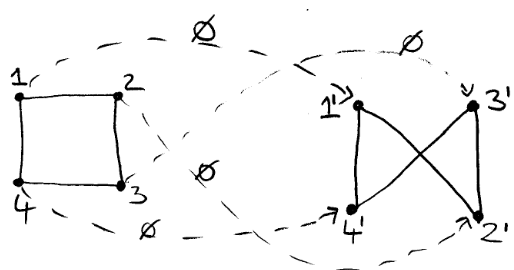
Definition 1.6. Two vertices $v_1, v_2 \in V$ are *adjacent* if there is an edge joining v_1 to v_2. The *degree* or *valency* of a vertex v is the number of edges that have one end at v: each loop at a vertex v contributes 2 to the valency of v. The graph is *regular* if all vertices have the same degree.

Example 1.7. This graph has vertices labelled by their valencies.



2. Isomorphism

The way that we draw a graph, or the nature of the set V, does not matter when we are studying graphs. Consider the bijection ϕ between the following graphs, which shows that they are essentially the same:



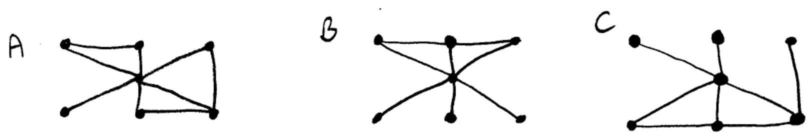
Definition 2.1. The simple graphs $G = (V, E)$ and $G' = (V', E')$ are *isomorphic* if there exists a bijection $\phi : V \to V'$ which *preserves edges*, i.e. such that $v_1 v_2$ is an edge of E if and only if $\phi(v_1)\phi(v_2)$ is an edge of E'. If an isomorphism exists then we write $G \cong G'$.

Isomorphic graphs are indistinguishable from the point of view of graph theory, i.e. they share any graph-theoretic properties. In particular:

- They have the same number of vertices.
- They have the same number of edges.
- Corresponding vertices have equal valencies.

Turning this around, to prove that two graphs are *not* isomorphic, it suffices to find a property that they do not share.

Example 2.2. Here are three non-isomorphic graphs on 7 vertices.



To see that they are pairwise nonisomorphic, note that the graph A has more edges than B or C. The graph B has a vertex of degree 6 whereas the graph C does not.

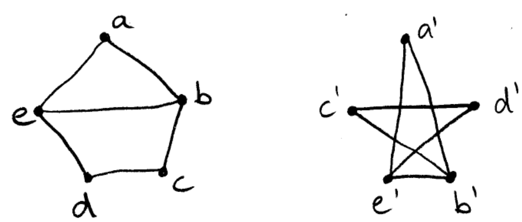
Example 2.3. Problem: Find a complete set of non-isomorphic simple graphs with a given number of vertices. For instance with 3 vertices we find:

The following information is known:

$\|G\|$	1	2	3	4	5	6	7	8	...	25
# of simple graphs	1	2	4	11	34	156	1044	12346	...	$\sim 1.3 \times 10^{66}$

Two prove that two graphs *are* isomorphic one must exhibit a requisite bijection. This is best done by re-drawing one of the graphs, to look exactly like the other, while keeping track of the labels of the vertices.

Example 2.4. Two isomorphic graphs:

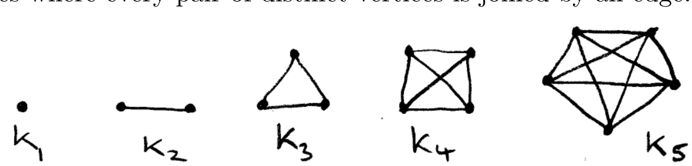


The mapping $\begin{pmatrix} a & b & c & d & e \\ a' & b' & c' & d' & e' \end{pmatrix}$ is an isomorphism (check this by re-drawing the second graph). Two non-isomorphic graphs:

These graphs both have 9 vertices, 12 edges, one vertex of degree 4, four vertices of degree 3 and four vertices of degree 2. However, they are *not* isomorphic as the vertex of degree 4 is joined to vertices of degree 3 in the first graph, and vertices of degree 2 in the second graph. (Try to find some other properties that these two graphs don't share.)

3. Some families of graphs

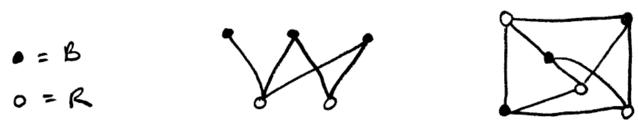
Definition 3.1. The *complete graph* on n vertices is denoted K_n, and is the graph on n vertices where every pair of distinct vertices is joined by an edge.



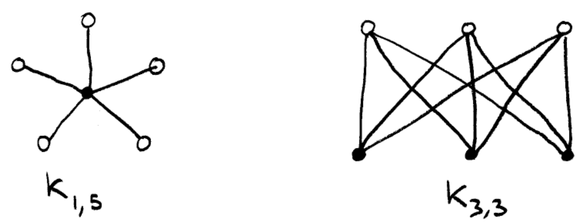
Every vertex of K_n has degree $(n-1)$ so there are $n(n-1)/2$ edges.

Definition 3.2. The *empy graph* (or *null graph*) and is the graph on n vertices with no edges.

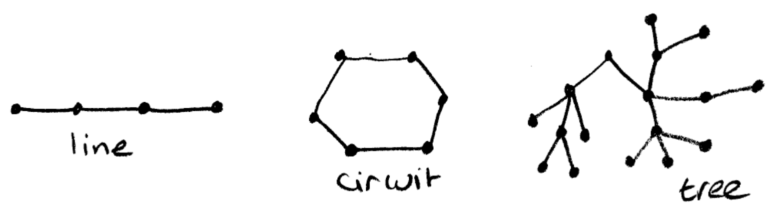
Definition 3.3. A *bipartite* graph is one where the vertices are divided into two classes, R and B say, with the property that every edge has one end in R and the other end in B. (R and B are often interpreted as colours, e.g. red and blue, and then the bipartite condition says that all vertices are coloured red or blue, and there are no edges between vertices of the same colour.) For example:



We denote by $K_{m,n}$ the *complete bipartite graph* with $|R| = m$ and $|B| = n$ and every vertex of R joined to every vertex of B.



Here are some other kinds of graph.

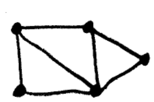


A *tree* is a connected graph which contains no circuits.

4. Degree sequences

Definition 4.1. Let G be a simple graph. The *degree sequence* of G is the list of degrees of the vertices of G, usually in decreasing order.

For instance the following graph has degree sequence $4, 3, 3, 2, 2$:



Given a sequence of non-negative integers, how can we tell whether there is a graph with that as a degree sequence? For example, $6, 6, 5, 5, 3, 3, 3, 3$ is the degree sequence of a simple graph, but $6, 6, 5, 5, 2, 2, 2, 2$ is not.

Here are some necessary conditions:

Theorem 4.2. *The following are necessary conditions for $d_1, \ldots, d_v$ to be a degree sequence of a simple graph.*

1. *We have* $d_i \leq v - 1$ *for* $1 \leq i \leq v$.
2. *At least two vertices have equal degree.*

Proof.

1. In a simple graph, each vertex can be joined to at most $v - 1$ others.
2. Suppose not, then by (2) the degrees must be $v-1, v-2, \ldots, 1, 0$. Therefore the vertex of degree $v - 1$ must be joined to all other vertices, including that of degree 0.

■

Lemma 4.3 (Handshaking lemma) *If G is a finite graph with v vertices and e edges, and if $d_1, \ldots, d_v$ is its degree sequence, then*

$$d_1 + d_2 + \cdots + d_v = 2e.$$

In particular, the sum of the degrees of all vertices of a finite graph is even.

Proof. Each edge contributes 1 to the degree of two vertices, or 2 to a single one. ■

Remark 4.4. Nonisomorphic graphs can have the same degree sequences. For example consider the following two graphs:

They both have degree sequence $3, 3, 2, 2, 2$ but are nonisomorphic since the first contains a triangle but the other does not.

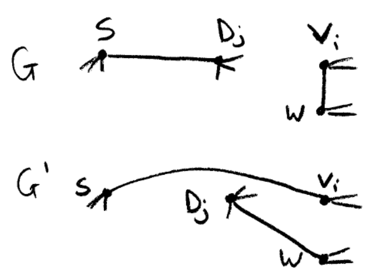
Lemma 4.5. *Let a graph G have the degree sequence*

$$s \geq v_1 \geq v_2 \geq \cdots \geq v_s \geq d_1 \geq \cdots \geq d_k$$

and vertices S, V_i and D_j of corresponding degrees. Then there exists a graph H with the same degree sequence such that S is incident with $V_1, V_2, \ldots, V_s$.

Proof. Suppose that in G the vertex S is *not* incident with at least one of the V_i. Then S must be joined to at least one of the D_j. Recall that $v_i \geq d_j$. If $v_i = d_j$ then we may swap the labels V_i and D_j on the graph to get a graph with S incident with one more of the V_i and one less of the D_j vertices.

Otherwise $v_i > d_j$. Then there is some vertex W joined to V_i but not to D_j. Consider the graph G' obtained from G by removing edges SD_j and WV_i and replacing them by edges SV_i and WD_j:



None of the vertex degrees have changed, but S is joined to one more of the V_i and one fewer of the D_j vertices. In particular, G' has the same degree sequence, so we may replace G by G'.

We may then repeat these two processes to get a graph H with S joined to all of the V_i, proving the lemma. ∎

Theorem 4.6. *Consider the sequences*

$$s, v_1, v_2, \ldots, v_s, d_1, \ldots, d_k \qquad (1)$$

and

$$v_1 - 1, v_2 - 1, \ldots, v_s - 1, d_1, d_2, \ldots, d_k \qquad (2),$$

where (1) *is written in decreasing order,* $v_i \geq 1$*, and* $v_i, d_i \in \mathbb{N}$ *for all* i*. Then* (1) *is a degree sequence of a simple graph if and only if* (2) *is the degree sequence of a simple graph.*

Proof.
$\Leftarrow$: If (2) is the degree sequence of some graph then by adding one further vertex S joined to the vertices of degree $v_1 - 1, v_2 - 1, \ldots, v_s - 1$ we get a graph with degree sequence (1).
$\Rightarrow$: Let G have degree sequence (1) and write S, V_i, D_j for the vertices with degrees s, v_i and d_j respectively. The graph H given by Lemma 4.5 has the same degree sequence as G, and in H the vertex S is joined to $V_1, \ldots, V_s$. By removing the vertex S and edges $SV_1, SV_2, \ldots, SV_s$ we get a graph with degree sequence (2). ∎

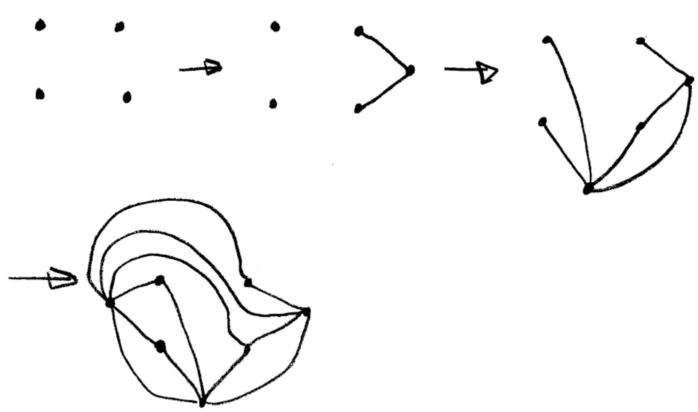
Example 4.7. For example $6, 6, 5, 5, 2, 2, 2, 2$ is a degree sequence if and only if $5, 4, 4, 2, 1, 1, 1$ is a degree sequence, if and only if $3, 3, 1, 1, 0, 0$ is a degree sequence, if and only if $2, 0, 0, 0, 0$ is a degree sequence. Since this final sequence is clearly not the degree sequence of a simple graph, none of the preceding ones are.

Example 4.8.

1. The sequence $4, 3, 3, 2, 1$ is not a degree sequence since the sum is odd.

2. Consider the sequence $6, 5, 4, 3, 2, 2, 2$. Applying the theorem we get a sequence $4, 3, 2, 1, 1, 1$, and hence a sequence $2, 1, 1, 0, 0$. From this we obtain $0, 0, 0, 0$ which is obviously a degree sequence.

   We use the theorem in reverse to construct a graph with the original degree sequence:



3. Consider the sequence $7, 7, 7, 6, 6, 5, 2, 2, 1, 1$. Applying the theorem we get a sequence $6, 6, 5, 5, 4, 1, 1, 1, 1$, then $5, 4, 4, 3, 1, 1, 0, 0$, and then $3, 3, 2, 0, 0, 0, 0$ ($*$). There is no graph with degree sequence $*$, as in order for a vertex to have degree 3 there must be at least four vertices of nonzero degree. Hence by the theorem there is no graph with the original degree sequence.

4. Consider the sequence $7, 7, 6, 5, 4, 4, 4, 2, 1$. Applying the theorem we get a sequence $6, 5, 4, 3, 3, 3, 1, 1$, which yields $4, 3, 2, 2, 2, 1, 0$. In turn we now get $2, 1, 1, 1, 1, 0$ and then $1, 1, 0, 0, 0$. This last sequence is the degree sequence of a graph with 5 vertices and a single edge, so all of these sequences are degree sequences of simple graphs.

Chapter 3

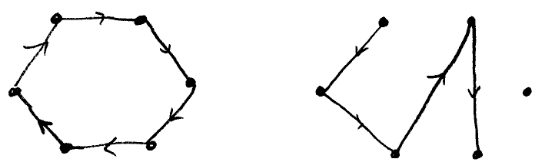
Paths on graphs

1. Definitions

Throughout this chapter, G will be a finite connected graph.

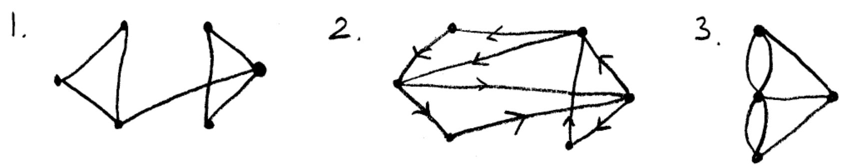
Definition 1.1. A *path* on G is a sequence of vertices $v_1, v_2, \dots, v_n$ such that $v_i v_{i+1}$ is an edge of G for $1 \leq i \leq n-1$. A *circuit* is a path $v_1, \dots, v_n$ such that $v_n v_1$ is also an edge.

Example 1.2. In the first graph a circuit is marked, in the second a path is marked.



Definition 1.3. A connected graph is *Eulerian* if there exists a circuit using every edge exactly once. It is *semi-Eulerian* if there is a path including every edge exactly once. It is *Hamiltonian* if there exists a circuit including every vertex exactly once, and *semi-Hamiltonian* if there exists a path including every vertex exactly once.

Example 1.4. Graph 1 is semi-Eulerian and semi-Hamiltonian. Graph 2 is Eulerian and Hamiltonian (follow the arrows). Graph 3 is Hamiltonian but not Eulerian or semi-Eulerian. Do you recognise graph 3?



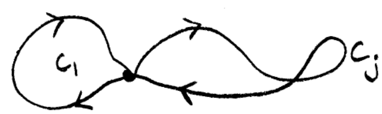
2. Eulerian circuits and paths

Theorem 2.1 (Euler's Theorem) *A connected graph is Eulerian if and only if every vertex has even degree.*

Proof.
$\Rightarrow$: A circuit goes into and then out of each vertex a certain number of times, each time using two edges (one to get in and then one to get out). There is a slight difference for the start vertex of the circuit: here we leave it first, then perhaps

re-visit it at certain points while traversing the circuit, taking two edges as above, and then finally we use one more edge to return to it at the end of the circuit. Thus there are an even number of edges incident to each vertex, and if circuit is eulerian this will exhaust all the edges.
$\Leftarrow$: Start at any vertex v and walk around the graph using no edge more than once. Since each vertex has even degree, we will always have a way out of each vertex until we return to vertex v. Call this circuit C_1. If C_1 contains all of the edges then we are done. If not, the remaining edges are such that an even number of them are incident to each vertex, so we may repeat our initial process to get further circuits $C_2, C_3, \ldots, C_k$, until all of the edges are used in exactly one of the C_i.

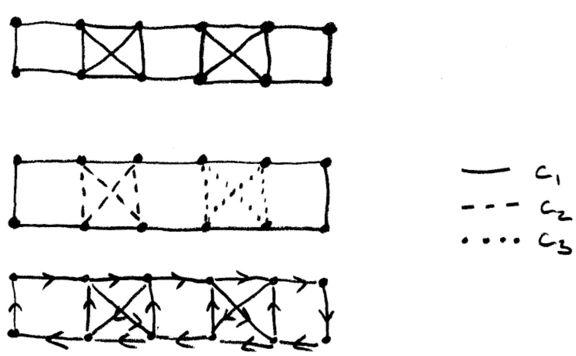
If $k > 1$ we now join up the circuits to form a single circuit. Since the graph is connected, C_1 must have a vertex in common with at least one C_j for $j > 1$. We may amalgamate C_1 and C_j to form a single circuit by rerouting at the common vertex:



Repeating this we may combine $C_1, \ldots, C_k$ into a single path including each edge exactly once.

An alternative way of executing essentially the same proof is to let C be the longest circuit without repeated edges in the graph. Suppose that C does not traverse all the edges in the graph. Removing the edges of C from the graph results in another graph in which all the vertices have even degrees. Connectedness of the graph implies that at least one circuit C_j in the new graph shares a vertex with C. Then gluing C and C_j as in the picture produces a circuit that is longer than C, and yields a contradiction. ■

Example 2.2. In the following graph, we note that all vertices have even degree, so we may find an Eulerian circuit.



In the final graph we amalgamate C_1, C_2, C_3 to make a single circuit.

Corollary 2.3. *A connected graph is semi-Eulerian but not Eulerian if and only if it has exactly two vertices of odd degree.*

Proof. We start by noting that any semi-Eulerian path must begin at one odd degree vertex and end at another.
$\Rightarrow$: Except for the vertices at the start and finish of the path, every time we go into a vertex along one edge, we go out along another edge. Therefore each time that we visit an intermediate vertex we account for two edges. Thus every vertex other than the first and last must have even degree.

$\Leftarrow$: Let v_0, v_1 be the two vertices of odd degree in a graph G. Add an edge joining v_0 and v_1 to get a graph G' such that every vertex of G' has even degree. By Euler's theorem, G' has an Eulerian circuit. Removing the edge v_0v_1 leaves an Eulerian path on G which begins at v_0 and ends at v_1, as required. ■

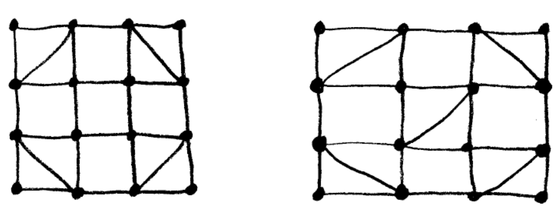
Example 2.4. It is immediate from Euler's theorem that the Bridges of Königsberg graph from Chapter 1, Example 2.1 has no Eulerian path.

Example 2.5. In the complete graph K_n every vertext has degree $n-1$. Hence K_n is eulerian if and only if n is odd. Consider the complete graph K_7, with vertices labelled $0, 1, \ldots, 6$. An eulerian path is:

$$01, 12, 23, 34, 45, 56, 60, 02, 24, 46, 61, 13, 35, 50, 03, 36, 62, 25, 51, 24, 40.$$

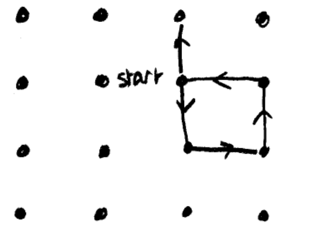
Interpreting edges as domino tiles, you obtain a circuit including all the domino tiles except for the doubles ii. These can be added at any appropriate points, resulting in a circuit of all dominos.

Example 2.6. The first of these graphs is Eulerian. The second, with one extra edge, is semi-Eulerian.

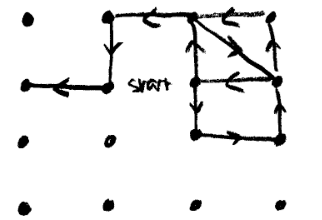


Algorithm 2.7. Fleury's algorithm This is an algorithm for finding Eulerian paths or circuits in a graph. Given a graph with either two or no vertices of odd degree, start at any vertex in the Eulerian case and at one of the two odd degree vertices in the semi-Eulerian case. Walk around the graph choosing new edges with the proviso that you never take an edge if it would split the remaining edges into two disconnected graphs, and that you don't use the last edge into the start vertex too soon.

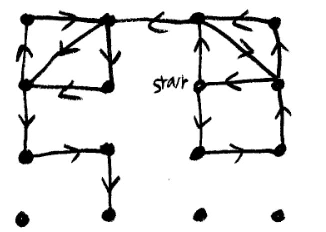
Example 2.8. Consider the first graph of Example 2.6:



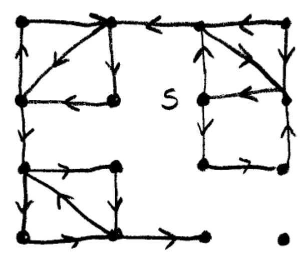
now must go right, or
diagonally down



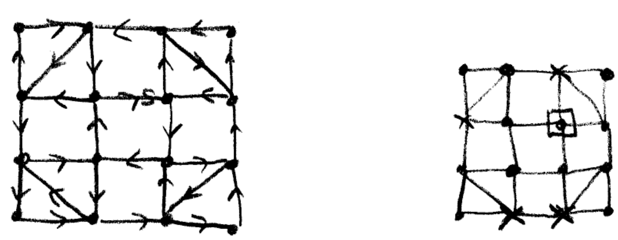
now must go up, or
diagonally up.



after this must go left,
or diagonally up.



next we must complete
the last corner



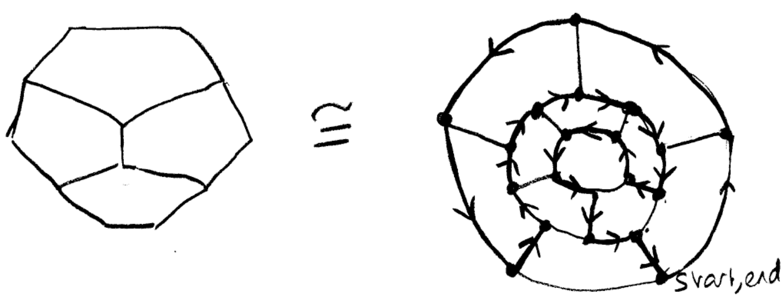
X = vertices where we need to check Fleury's
algorithm
▣ = start; end point

Remark 2.9. Fleury's algorithm is common sense, in that if we end up with two disconnected blocks of edges we clearly cannot complete a circuit.

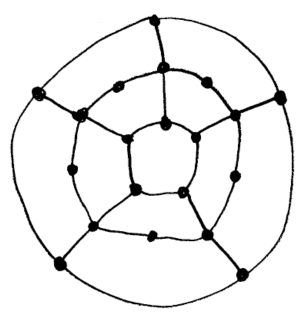
3. Hamiltonian Circuits

Unlike the Eulerian case, where Euler's Theorem 2.1 completely classifies the Eulerian graphs, there is no simple criterion which guarantees the existence of a Hamiltonian circuit. Many puzzles and problems require one to find a Hamiltonian path, for instance the Knight's Tour of Chapter 1, Example 2.2 and the Travelling Salesman problem.

Example 3.1. Hamilton's Round the World Problem
Can you find a Hamiltonian circuit visiting all vertices of a dodecahedron? First we sketch a dodecahedron, which has 20 vertices, 12 pentagonal faces, and 30 edges. Then we give an isomorphic graph to consider:



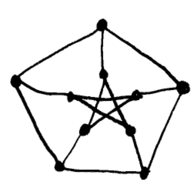
One can see a Hamiltonian circuit for this graph relatively easily. However, a small change gives the following graph, which has no Hamiltonian path or circuit:



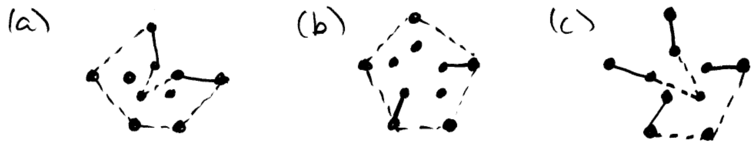
To see this, note that although this graph has the same number of vertices and edges as the previous one, to visit the vertices of degree two we must go around the middle ring of edges. But now any circuit cannot include both the inner and outer rings.

Remark 3.2. In general, one has to argue in a systematic way to show that a graph is not Hamiltonian, or to find a Hamiltonian circuit.

Example 3.3. The Petersen Graph
The Petersen graph is *not* Hamiltonian.



To prove this, note that up to symmetry there are three ways of passing from the inner to the outer pentagon. In picture (a) we pass at adjacent vertices, in (b) we pass at non-adjacent vertices, and in (c) we pass twice from the inner to the outer pentagon. The passing edges are the non-dotted ones.



In each case, we are forced to traverse certain other edges to guarantee visiting each vertex. These are the dotted edges. In each case it follows that we cannot complete the circuit.

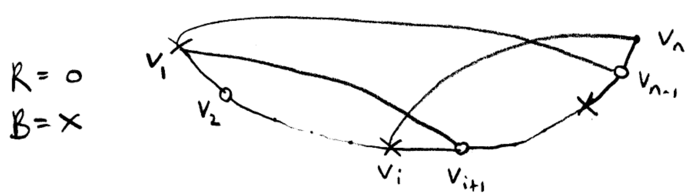
Intuitively, given a graph, if it has many edges then it is more likely to have a Hamiltonian circuit. We formalise this as follows:

Theorem 3.4 (Dirac's theorem) *Let G be a simple graph with $n \geq 3$ vertices such that every vertex has degree at least $n/2$. Then G has a Hamiltonian circuit.*

Proof. We assume that G is a simple graph with every vertex of degree at least $n/2$ that is *not* Hamiltonian, and derive a contradiction.

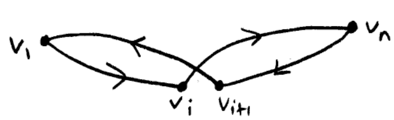
By adding as many edges as possible without making the graph Hamiltonian, we may assume that G is maximal, in the sense that it is non-Hamiltonian but the addition of any further edges will make it Hamiltonian. Since adding edges doesn't alter the number of vertices, and the complete graph is Hamiltonian, this assumption is safe.

Let v, w be any pair of non-adjacent vertices of G. There must be a path from v to w which visits all vertices once, since adding the edge vw would create a Hamiltonian circuit, from which we can remove the edge vw. Label the vertices in this path $v = v_1, v_2, \ldots, v_n = w$.



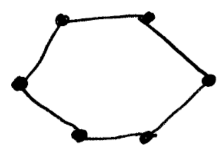
Let R be the set of vertices joined to v_1, so $|R| \geq n/2$, and let B be the set of vertices that are one before the vertices in R as we go along the path $v_1, v_2, \ldots, v_n$. Note that some vertices may be in both B and R. Then $|B| = |R| \geq n/2$ and $v_1 \in B$ (as v_1 is one before v_2, and v_1v_2 is an edge in the path).

The vertex v_n is joined to at least $n/2$ of the $n-2$ vertices $v_2, v_3, \ldots, v_{n-1}$, as v_n is not joined to v_1 or itself. Since $v_1 \in B$ and $v_n \notin B$, at least $n/2 - 1$ of the vertices $v_2, v_3, \ldots, v_{n-1}$ are in B. Since $n/2 + (n/2 - 1) > n - 2$ there exists at least one vertex v_i in $v_2, v_3, \ldots, v_{n-1}$ that is both in B and joined to v_n.



Therefore we have a Hamiltonian circuit $v_1, v_2, \ldots, v_i, v_n, v_{n-1}, v_{n-2}, \ldots, v_{i+1}, v_1$, a contradiction. ■

Remark 3.5. Dirac's theorem does *not* give a necessary condition. For instance the cycle with 6 vertices has all vertices of degree 2 (which is less than $6/2$) and yet has a Hamiltonian circuit.



A minor modification of the proof of Dirac's theorem allows one to prove the following:

Theorem 3.6 (Ore's Theorem) *Let G be a simple graph with $n \geq 3$ vertices such that for all non-adjacent vertices v and w we have* $\mathrm{Degree}(v) + \mathrm{Degree}(w) \geq n$. *Then G has a Hamiltonian circuit.*

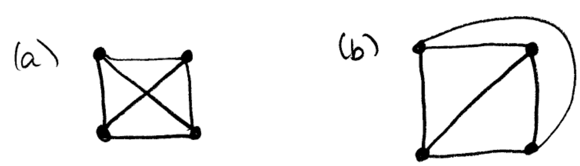
Chapter 4

Planarity and Colouring

1. Euler's Formula and Kuratowski's Theorem

Definition 1.1. A graph G is *planar* if it can be drawn in the plane without edges crossing.

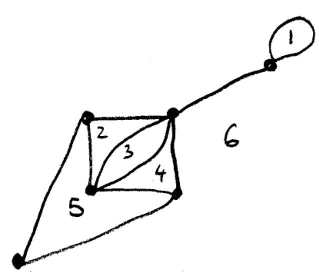
Example 1.2. The graph K_4 (a) is planar because it is isomorphic to the graph (b) and this second graph has no crossing edges.



We call both graphs planar.

Definition 1.3. The *faces* of a planar graph are the regions into which the edges divide the plane, when the graph is drawn with no crossing edges. We always include the *unbounded* or *exterior* face.

Example 1.4. The following graph has six faces:



Given a planar graph, we write v, e, f for the number of vertices, edges and faces respectively. So this graph has $v = 6$, $e = 10$ and $f = 6$.

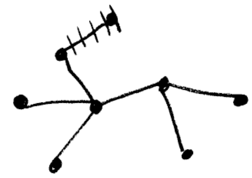
Theorem 1.5 (Euler's formula) *In every connected planar graph* $v - e + f = 2$.

Proof. We prove this by induction on e.

Base case: The connected graph with no edges consists of a single vertex. It has $v = 1, e = 0$ and $f = 1$ so the theorem holds.

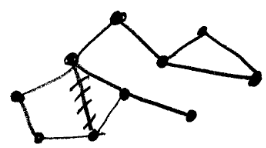
Inductive step: Assume that the theorem holds for every planar connected graph with at most n edges. Let G have $n+1$ edges. Then either G is a tree or G contains a circuit. We consider the two cases separately.

If G is a tree then by removing any edge at the end of a "branch", along with its terminal vertex, we get a connected graph with n edges.



The value of v is decreased by 1, the value of e is decreased by 1, and the value of f is unchanged. So $v - e + f$ is unchanged, and so $v - e + f = 2$ by the inductive hypothesis and the theorem holds for G.

If G is not a tree, and so contains a circuit, we consider the effect of removing an edge that lies in a circuit, to get a connected graph with n edges.



The value of v is unchanged, e is decreased by 1, and f is decreased by 1 (since two faces have been merged). So $v - e + f$ is unchanged, and so $v - e + f = 2$ by the inductive hypothesis and the theorem holds for G.

Therefore the result follows for all e by induction. ■

Remark 1.6. Euler's formula is also valid for graphs drawn on a sphere. In fact a generalisation of Euler's formula holds for graphs drawn on *any* surface, and it reads $v - e + f = 2 - 2g$, where g is an invariant of the surface called *genus*.

Corollary 1.7. *The complete graph K_5 is not planar.*

Proof. The complete graph K_5 has

$$v = 5 \text{ and } e = \binom{5}{2} = 10.$$

Therefore by Theorem 1.5 we must have $f = 2 - v + e = 7$ if K_5 is planar.

The boundary of any face is a circuit. The circuit cannot have a single vertex as K_5 is simple so has no loops. It also cannot have precisely two vertices as K_5 is simple so has no multiple edges. Therefore the boundary of each face contains at least three vertices, and hence at least three edges.

Since each edge separates two faces we have $e \geq 3f/2$, that is $10 \geq 21/2$, a contradiction. ■

Corollary 1.8. *The complete bipartite graph $K_{3,3}$ is not planar.*

Proof. The graph $K_{3,3}$ has

$$v = 6 \text{ and } e = 3 \times 3 = 9.$$

Therefore if $K_{3,3}$ can be drawn in the plane then Theorem 1.5 gives $f = 2 - v + e = 5$. However, since $K_{3,3}$ is bipartite each face is bounded by at least four edges. Each edge separates 2 faces, giving $e \geq 4f/2$, which gives $9 \geq 10$, a contradiction. ■

Corollary 1.9. *Every planar connected simple graph G with $v \geq 3$ satisfies $e \leq 3v - 6$.*

Proof. If G has a single face, then by Euler's formula we have $e = v - 1$. (Alternatively, G is a tree, and then use Theorem 1.2 (3).) Otherwise, every face of G is bounded by at least three edges, and every edge is adjacent to at most 2 faces. Therefore $e \geq 3f/2$. We have

$$2 = v - e + f \leq v - e + 2e/3 = v - e/3 \Rightarrow e \leq 3v - 6.$$

■

Corollary 1.10. *Every planar, connected simple graph has at least one vertex of degree less than* 6*.*

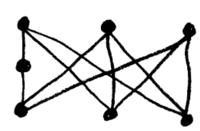
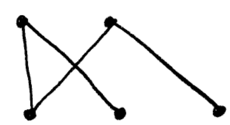
Proof. Suppose all vertices have degree at least 6. Each edge ends in two vertices, so $e \geq 6v/2 = 3v$. However, by Corollary 1.9 we have $e \leq 3v - 6$ so $3v \leq 3v - 6$, a contradiction. ■

Exercise 1.11. In fact, every such graph with at least four vertices has at least four vertices of degree at most 5. Prove this.

The graphs $K_{3,3}$ and K_5 turn out to be the two "basic" non-planar graphs. To make this formal, we need the following notions.

Definition 1.12. A *subgraph* of a graph G is any graph that can be produced from G by removing edges and vertices. Two graphs G_1 and G_2 are *homeomorphic* if there exists a graph G such that both G_1 and G_2 may be produced from G by putting additional vertices of degree 2 along the edges of G. Note that any graph is homeomorphic to itself.

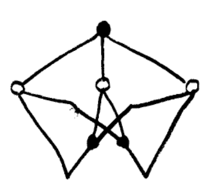
Example 1.13. The first graph is homeomorphic to $K_{3,3}$, the second is a subgraph of $K_{3,3}$.

(a)

(b)


Theorem 1.14 (Kuratowski's Theorem) *A finite connected graph is planar if and only if it contains no subgraph that is homeomorphic to* $K_{3,3}$ *or* K_5*.*

Less formally: A finite connected graph G is non-planar if and only if we can find 5 or 6 vertices in G that are connected together, possibly via other vertices, as $K_{3,3}$ or K_5.
Non–proof: One direction follows from Corollaries 1.7 and 1.8. The other direction is too long to be included in the course – see Chapter 11 of the book by Harary.

Example 1.15. The Petersen graph is non-planar because it has a subgraph that is homeomorphic to $K_{3,3}$.



2. Regular Polyhedra

We may use Euler's formula 1.5 to show that there are just 5 regular polyhedra, or platonic solids. This method using Euler's formula is due to Cauchy in 1813.

Definition 2.1. A polyhedron is *convex* if it has no inward-bending faces. A convex polyhedron is *regular* if every face is a congruent regular polygon and the arrangement of faces at each vertex is the same.

Theorem 2.2. *There are exactly* 5 *regular polyhedra.*

Proof. Let P be a regular polyhedron with all faces regular p-gons (that is, regular p-sided shapes). Assume that d edges meet at each vertex. We can project the edges and vertices of P on to the plane to get a corresponding planar graph G. Each vertex of G has degree d and each face of G has p edges. Then $e = vd/2$ and $e = fp/2$, as each edge bounds two faces. Therefore $v = 2e/d$ and $f = 2e/p$.

By Euler's formula we have

$$2 = v - e + f = \frac{2e}{d} - e + \frac{2e}{p} \Rightarrow 2 + e = \frac{2e}{d} + \frac{2e}{p}.$$

Dividing by $2e$ we get

$$\frac{1}{2} + \frac{1}{e} = \frac{1}{d} + \frac{1}{p}.$$

Therefore, $1/2 < 1/d + 1/p$, where d and p are integers with $d \geq 3$, $p \geq 3$. We have

$$1/2 < 1/d + 1/p \leq 1/3 + 1/p$$

and so $1/6 < 1/p$ and so $p < 6$. Similarly, $d < 6$. Thus

$$(d, p) \in \{(3,3), (3,4), (4,3), (3,5), (5,3)\}.$$

Case 1 $d = p = 3$. Then we have $1/e = 1/3 + 1/3 - 1/2$, so $e = 6$. Hence $v = 2e/d = 4$ and $f = 2e/p = 4$. This is the *tetrahedron*, or triangle-based pyramid.

Case 2 $d = 3, p = 4$. We have $1/e = 1/3 + 1/4 - 1/2$ so $e = 12$. Hence $v = 2e/d = 8$ and $f = 2e/p = 6$. This is the *cube*.

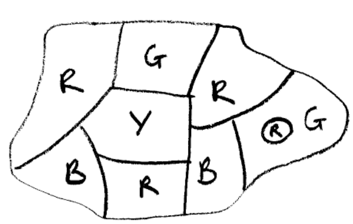
Case 3 $d = 4, p = 3$. This time we have $e = 12$, $v = 6$ and $f = 8$. This is the *octahedron*, made by gluing the bottoms of two square-based pyramids together.

Case 4 $d = 3, p = 5$. Now we have $e = 30$, $v = 20$ and $f = 12$. This is the *dodecahedron*; a solid whose surface is made by gluing 12 identical regular pentagons together.

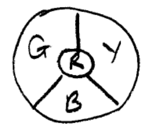
Case 5 $d = 5, p = 3$. This gives $e = 30$, $v = 12$, $f = 20$, corresponding to the *icosahedron*; a solid whose surface is made by gluing 20 identical equilateral triangles together. ■

3. Colouring Planar Graphs

In 1852 Guthrie asked whether it is possible to colour every map in the plane with at most four colours so that no two countries with a common boundary are the same colour.



Remark 3.1. Some maps cannot be coloured with less than four colours, for instance



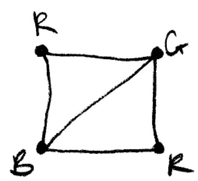
The theorem was first announced as proved true by Kempe in 1879. However, Heawood found an error in the proof in 1890. The theorem was finally proved true in 1976 by Appel and Haken, by one of the first lengthy proofs to use a computer.

The problem can easily be rephrased as a problem of colouring the vertices of a graph.

Definition 3.2. A graph G is *k-colourable* if we may label its vertices with k colours so that no two adjacent vertices are of the same colour.

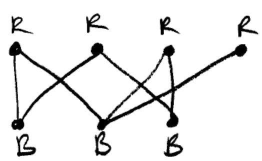
Definition 3.3. The *chromatic number* of a graph G is the least k for which G is k-colourable. We write $\chi(G)$ for the chromatic number of G.

Example 3.4. The following graph is 3-colourable:



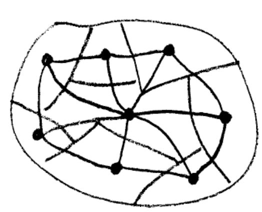
It is not 2-colourable because it contains triangles.

Example 3.5. A nontrivial bipartite graph G has $\chi(G) = 2$.



Colouring maps is directly related to colouring graphs.

Definition 3.6. Given a map in the plane, the *graph of the map* is a graph with a vertex in each region, and an edge joining a pair of vertices if and only if the regions have a common boundary.



Property: Every map in the plane has a simple planar graph.

The four colour map theorem is therefore equivalent to:

Theorem 3.7 (The four colour theorem) *Every planar simple graph is four-colourable.*

The proof of this theorem is very long, instead we prove the following.

Theorem 3.8 (The five colour theorem) *Every planar simple graph is five-colourable.*

Proof. We prove this by induction on the number n of vertices.

For $n \leq 5$ the result is trivial as we can simply colour each vertex a distinct colour. So assume that the result holds for all graphs with fewer than n vertices.

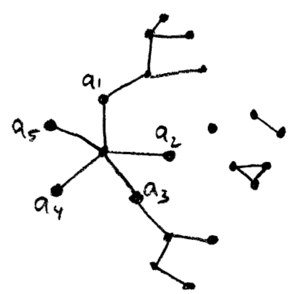
Let G be a planar simple graph with n vertices. By Corollary 1.10 we have that G has a vertex A of degree at most 5. Remove A (and all edges ending at A) to get a new graph G'. Since G' has $n-1$ vertices, we can colour G' with at most 5 colours by induction.

If the degree of A is at most 4, then there is a colour not assigned to any vertex that is adjacent to A in G, so we can take the colouring for G' and use this remaining colour on A to finish the colouring of G.

Therefore we need only consider the case where A has degree 5. Let the neighbours of A be a_1, a_2, a_3, a_4, a_5, numbered to go in a clockwise direction around A. If any two of the a_i have the same colour, then there is a spare colour available for A and we are done.

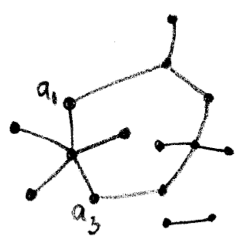
Let i denote the colour of the vertex a_i, and write G_{ij} for the subgraph of $G \setminus \{A\}$ consisting of the vertices with colours i and j, and any edges between them that are in G. We divide into two cases:

Case 1. a_1 and a_3 are not joined within $G_{1,3}$.



Then we may interchange colours 1 and 3 on the part of $G_{1,3}$ connected to a_1 to give a 5-colouring of $G \setminus \{A\}$ such that a_1 now has colour 3. We now can colour A with colour 1, as none of its neighbours is coloured 1.

Case 2. a_1 and a_3 are joined by a path in $G_{1,3}$.



Then a_2 and a_4 cannot be joined by a path in $G_{2,4}$ as G is planar so $G_{2,4}$ cannot cross $G_{1,3}$ (and cannot share a vertex with $G_{1,3}$). Therefore we may proceed as in case 1 to recolour a_2 with colour 4, and then colour A with colour 2. ∎

Remark 3.9. 1. Since K_4 is planar there are planar graphs which require 4 colours.

2. Colouring graphs on more complicated surfaces is easier. Suppose that we start with a sphere, and glue n "handles" onto it. That is, gluing one handle produces a torus, gluing two produces a double torus and so on. Then any graph that can be drawn on a surface with n handles can be coloured with at most N colours, where
$$N = \lfloor \frac{7 + \sqrt{1 + 48n}}{2} \rfloor,$$
and $\lfloor x \rfloor$ is the largest integer that is less than or equal to x. Thus for a torus we require $N = 7$ colours. This is best possible as K_7 can be drawn on the surface of the torus without crossings.

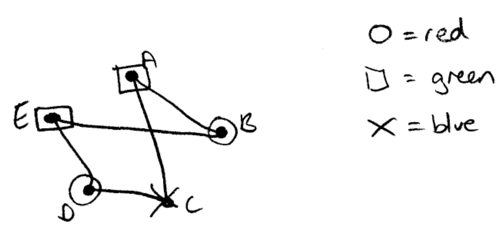
4. Colouring non-planar graphs

Colouring questions are also of interest for non-planar graphs, even though these do not correspond to maps. We define a *k-colouring* of a graph G and its *chromatic number* $\chi(G)$ as in Definitions 3.2 and 3.3.

Example 4.1. We show how to solve a timetabling problem using graph-colouring. Suppose we have the following requirements:

Student	Courses
1	A, B
2	A, C
3	C, D
4	B, E
5	D, E

We wish to schedule classes so that no student has to be in two places at once. To do so we draw the following graph G, with one node per course and edges joining two courses if they are taken by the same student.



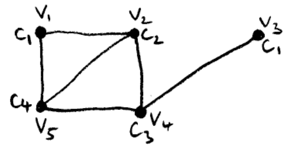
We see that $\chi(G) = 3$. Colouring the graph G with 3 colours gives a timetabling arrangement with each colour corresponding to a timetabling slot.

We now consider some more general estimates on χ.

Proposition 4.2. *If G is a graph with every vertex of degree at most n for some n, then $\chi(G) \leq n + 1$.*

Proof. Label the vertices of G with $v_1, \ldots, v_k$ in any order. Let the colours be $c_1, \ldots, c_{n+1}$. We then use the *greedy algorithm* to colour the graph: consider the vertices in order $v_1, \ldots, v_k$, and for each vertex v_i use the smallest value of j such that c_j is not the colour of a neighbouring vertex. Since no vertex has more than n neighbours, there will always be at least one colour available. ■

Example 4.3. Here's a worked example:



Here is a more powerful theorem, whose proof is too hard for this course:

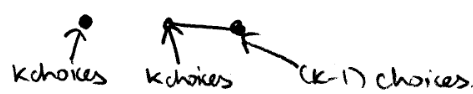
Theorem 4.4 (Brooke's Theorem) *Let G be a graph with all vertices of degree at most n. Then $\chi(G) \leq n$ unless one of the following holds*

1. *A connected component of G is K_{n+1} where $\chi(G) = n + 1$.*
2. *We have $n = 2$ and a connected component of G is a circuit with an odd number of vertices, where $\chi(G) = 3$.*

5. The chromatic polynomial

Definition 5.1. Let G be a graph with no loops. We write $P_G(k)$ for the number of different ways of colouring G using k colours so that adjacent vertices are different colours.

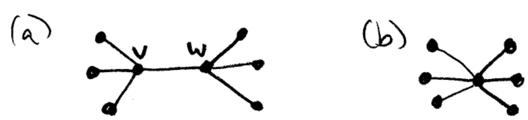
Example 5.2. Consider the following graph:



We have $P_G(k) = k \cdot k \cdot (k-1) = k^2(k-1)$.

Definition 5.3. Let G be a graph, and let vw be an edge in G. The *contraction* of the edge vw (or *amalgamation* of v and w) is an operation resulting in a new graph as follows: we remove the vertices v and w and all the edges incident with them, add a new vertex u, and connect it to every vertex z such that at least one of vz or wz was originally an edge.

Example 5.4. Graph (a) has two labelled vertices v and w. Graph (b) has been produced from graph (a) by amalgamating v and w.



We use amalgamation to show that $P_G(k)$ is a polynomial with integer coefficients.

Lemma 5.5 (The Deletion/Contraction Lemma) *Let vw be an edge of the simple graph G. Let G_1 be the graph obtained by deleting the edge vw and let G_2 be the graph obtained by amalgamating the vertices v and w. Then*

$$P_G(k) = P_{G_1}(k) - P_{G_2}(k). \tag{4.1}$$

Consequently, $P_G(k)$ is a polynomial in k.

Proof. Consider the possible ways of colouring G_1 with k colours. In a given colouring, one of the following holds:

1. v and w are the same colour, in which case there are $P_{G_2}(k)$ ways of colouring G_1;

2. v and w are different colours, in which case the colouring of G_1 is a colouring of G, so there are $P_G(k)$ ways of doing this.

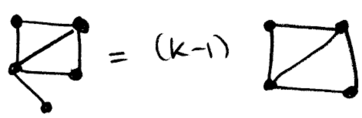
Therefore we have $P_{G_1}(k) = P_{G_2}(k) + P_G(k)$, so $P_G(k) = P_{G_1}(k) - P_{G_2}(k)$, as required.

Since G_1 and G_2 have strictly fewer edges than G we may apply these deletion and amalgamation processes repeatedly until we reach graphs with no edges (which we must do eventually since there are only finitely many edges in G). If there are n vertices and no edges in a graph H, then $P_H(k) = k^n$. Substituting these values back into equation 4.1 shows that $P_G(k)$ is a polynomial in k. ■

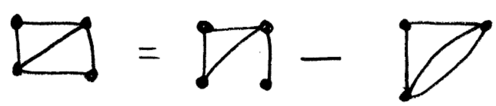
Remark 5.6. The chromatic number $\chi(G)$ of a graph is the least positive integer k such that $P_G(k) \neq 0$. The deletion/contraction lemma 5.5 gives an algorithm for chromatic polynomials. We usually just draw diagrams rather than writing equations in $P_G(k)$.

Example 5.7. Consider the following graph G, with labelled vertex v.

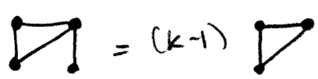
There are $(k-1)$ ways of colouring v once the other vertices have been coloured, so we write



By the Deletion/Contraction Lemma 5.5 we have

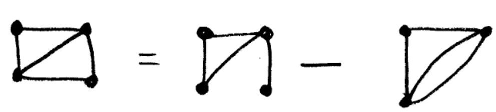


Once again, considering the isolated vertex we have



There are $k(k-1)(k-2)$ ways of colouring a triangle since all vertices are joined. Similarly, there are $k(k-1)(k-2)$ ways of colouring

since all vertices are joined. So the number of ways of k-colouring



is

$$(k-1)^2k(k-2) - k(k-1)(k-2) = k(k-1)(k-2)[k-1-1] = k(k-1)(k-2)^2$$

Therefore the number of ways of colouring G is

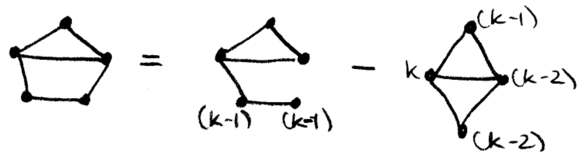
$$k(k-1)^2(k-2)^2$$

and hence $\chi(G) = 3$.

Proposition 5.8. *For a simple graph G with v vertices and e edges, $P_G(k) = k^v - ek^{v-1} +$ lower order terms. Moreover, the signs of the terms alternate.*

Exercise 5.9. Prove this using the Deletion/Contraction Lemma 5.5

Example 5.10. Consider the following graph G, and let us use the deletion/contraction lemma to find $P_k(G)$. We have



So the number of ways of colouring G is

$$\begin{aligned} k(k-1)(k-2)(k-1)^2 - k(k-1)(k-2)^2 &= k(k-1)(k-2)[(k-1)^2-(k-2)] \\ &= k(k-1)(k-2)[k^2-2k+1-k+2] \\ &= k(k-1)(k-2)[k^2-3k+3] \end{aligned}$$

Therefore $\chi(G) = 3$.

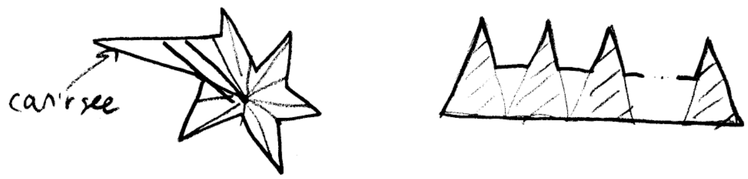
Definition 5.11. We will use the term *polygon* to mean a circuit C_n drawn in the plane, with edges non-crossing straight line segments.

Lemma 5.12. *Let G be a graph consisting of the vertices and edges of some polygon, together with a set of non-crossing diagonals of the polygon as further edges. Then G is* 3*-colourable.*

Proof. We prove this by induction on the number of vertices. It clearly holds for $n = 3$ vertices:

We assume inductively that the result holds for all graphs of this type with less than n vertices. Let G be such a graph with n vertices. If G is itself a polygon then G is clearly 3-colourable, so assume that G contains at least one diagonal v_1v_2. Split G into two smaller graphs along v_1v_2. Each of the two graphs to either side of the diagonal satisfies the hypotheses of the lemma, and has less than n vertices. Therefore each part may be 3-coloured by induction. By permuting the colours on one of the parts so that v_1 and v_2 are assigned the same colour on both parts, we get a 3-colouring of G. ■

Example 5.13. Guarding an Art Gallery Let an art gallery be in the shape of a polygon with n sides. What is the least number of guards that can be placed in the gallery so that they can see every point in the gallery?

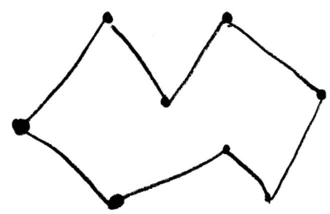


The first of these graphs requires at least two guards, the second requires at least one guard in each shaded region.

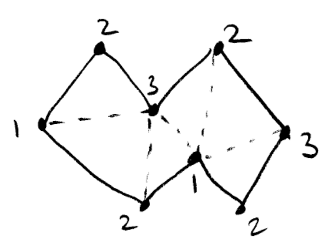
Proposition 5.14. *A polygonal art gallery with n sides may be guarded by $\lfloor n/3 \rfloor$ guards.*

Proof. Break the polygon up into triangles by adding diagonals. A guard standing at the vertex of a triangle can see all of the triangle. By Lemma 5.12 we may 3-colour the graph. One of the colours is used at most $\lfloor n/3 \rfloor$ times. Put a guard at each vertex of that colour. Since each triangle has one vertex of each colour, these guards cover the whole region. ∎

Example 5.15. Consider the following art gallery:



We triangulate it, then colour it as as follows:



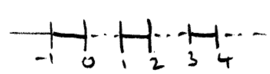
We have eight vertices, and used colours 1 and 3 both no more than $\lfloor 8/3 \rfloor = 2$ times. So placing guards at vertices coloured 1 will do, a total of 2 guards.

6. The chromatic number of the line and plane

Question 6.1. Let G be a graph whose vertices are the points of either $\mathbb{R}$ or $\mathbb{R}^2$, with a pair of points x, y joined if and only if $|x - y| = 1$. What is $\chi(G)$?

We may rephrase the question by asking how many colours are needed to colour $\mathbb{R}$ (or $\mathbb{R}^2$) such that no pair of points which are distance 1 apart are of the same colour?

Here is a 2-colouring of $\mathbb{R}$, where each interval is $[n, n+1)$ for n an integer:

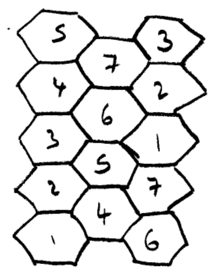


Therefore, $\chi(\mathbb{R}) = 2$: since some points are distance 1 apart we cannot 1-colour $\mathbb{R}$.

The chromatic number of $\mathbb{R}^2$ is harder. We prove some bounds.

Lemma 6.2. *With this adjacency relation we have* $\chi(\mathbb{R}^2) \leq 7$.

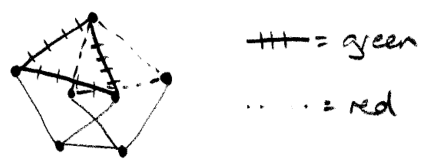
Proof. Tile the plane with hexagons of diameter slightly less than 1.



Colour them in a periodic manner to get a 7-colouring. It is clear that any two tiles of the same colour are distance more than 1 apart. ■

Lemma 6.3. *With these edges we have* $\chi(\mathbb{R}^2) \geq 4$.

Proof. We may draw the following graph in the plane with all edges of length 1.



In any 3-colouring of this graph, the bottom two vertices have the same colour as the top one because of the red and green triangles, contradicting the fact that the bottom two vertices are distance 1 apart. Hence this graph has chromatic number 4 and so $\chi(\mathbb{R}^2) \geq 4$. ■

It is unknown whether $\chi(\mathbb{R}^2)$ is $4, 5, 6$ or 7.

Chapter 5

Matchings and Connectivity

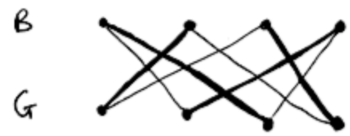
1. Hall's Marriage Theorem

Many problems involve matching objects from one set with objects in another.

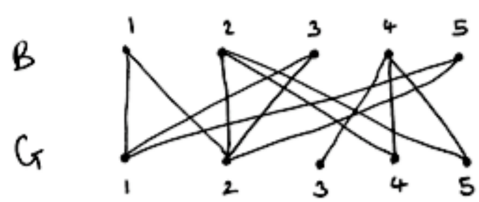
Problem 1.1. [Hall's Marriage Problem] Given two (disjoint) groups of people B and G some of whom know each other, is it possible to pair them up in such a way that each $x \in B$ is paired with a $y \in G$ that b knows?

We can represent this problem with a bipartite graph with vertex sets B and G, where we put an edge joining two people from different groups if they know each other.

Example 1.2. In this graph a matching is possible.



In this graph a matching is not possible, as boys $1, 3, 5 \in B$ only know 2 people from G.



Definition 1.3. Two edges of a (not necessarily bipartite) graph are called *independent* if they do not share a vertex. Let $P = B \cup G$ be a bipartite graph. A *matching from B to G* is a set of independent edges incident to every vertex in B. A *matching from G to B* is defined by analogy. A *perfect matching* is set of edges that is a matching both ways.

Definition 1.4. Let $P = B \cup G$ be a bipartite graph. The *neighbourhood* of a set $X \subset B$ is the set

$$N(X) = \{y \in G \,:\, xy \text{ is an edge}\} \subseteq G.$$

The neighbourhood of a set $Y \subset G$ is defined analogously.

Thus Hall's Marriage Problem asks if there is a matching from B to G. Clearly, if there is a subset X of B with k elements with a neighbourhood smaller than k, no matching is possible. Hall's Marriage Theorem says (surprisingly) that this is the only situation where a matching is not possible.

Theorem 1.5 (Hall's Marriage Theorem) *Let $P = B \cup G$ be a bipartite graph. There is a matching from B to G if and only if for every $X \subset B$ we have*

$$|N(X)| \geq |X|.$$

Proof. ($\Rightarrow$): Obvious, as indicated above.
($\Leftarrow$): Induction on $b = |B|$. The case $b = 1$ is obvious. Suppose that $b > 1$ and that the assertion holds for smaller values.

Case 1: For every proper subset $X \subset B$ we have $|N(X)| > |X|$. Take an arbitrary $x \in B$, and let xy be an edge. Remove x, y from the graph, yielding a new bipartite graph $P' = B' \cup G'$, where $B'B \setminus \{x\}$ and $G' = G \setminus \{y\}$. In this new graph, for every $X' subseteq B'$ we have $|N(X')| \geq |X'|$, because only one element has been removed from G. Also, $|B'| = b-1 < b$, and induction applies: there exists a matching from B' to G'. Add xy to this matching, and we obtain a matching from B to G.

Case 2: There exists a proper subset $B_1 \subset B$ with $|N(B_1)| = |B_1|$. Let $k = |B_1|$ and note $k < b$. By induction there is a matching from B_1 to $N(B_1) = G_1$. Remove $B_1 \cup G_1$ from P, yielding a new bipartite graph $P' = B_2 \cup G_2$. Let $X \subset B_2$. If $|N(X)| < |X|$ then

$$|N(B_1 \cup X)| = |N(B_1) \cup N(X)| \leq |N(B_1)| + |N(X)| < |B_1| + |X|,$$

a contradiction in P. Hence $|N(X)| \geq |X|$ for all $X \subset B_2$, and induction applies to P', yielding a matching from B_2 to G_2. Joining the two matching together yields a matching from B to G. ■

Corollary 1.6. *If P is a regular bipartite graph with parts B and G, and degree $r > 0$, then $|B| = |G|$ and P has a perfect matching.*

Proof. Let $X \subseteq B$ be a set of k vertices. There are kr edges that are incident with X, and each of them is also incident with $N(A)$. Since each vertex of $N(A)$ has degree r, it receives at most r of these edges (it may have neghbours outside of X!) Hence there are at least $kr/r = k$ vertices in $N(A)$. Thus Hall's condition is satisfied, and there is a matching from B to G. By symmetry, there is also a matching from G to B, so $|B| = |G|$ and the matching is perfect. ■

We now consider the case where Hall's condition is not satisfied, but we would like to match as many elements of B as possible.

Corollary 1.7. *Suppose the bipartite graph $P = B \cup G$ with $|B| = b$, satisfies the condition that any set $X \subseteq B$ of vertices has at least $|X| - d$ neighbours. Then P contains $b - d$ independent edges.*

Proof. Add d vertices to G, and join them to each vertex in B. Then the new graph P^* satisfies Hall's condition, and so has a matching from B to the new G. At least $m - d$ of the edges in this matching belong to P. ■

2. Edge colourings of graphs

Definition 2.1. An *edge colouring* of a graph is an assignment of colours to edges such that edges that meet at a common vertex are of different colours. A graph is

k-edge-colourable if it has an edge colouring with k colours. If G is k-edge-colourable but not $(k-1)$-edge colourable, then the *edge-chromatic number* of G is k, and we write $\chi_e(G) = k$.

Here is a result about edge colourings that uses Hall's Marriage Theorem 1.5.

Theorem 2.2. *A bipartite graph with all vertices of degree at most r can be edge coloured with r colours.*

Proof. We prove this by induction. If $r = 1$ then no edges ever meet at a vertex, and the graph can be edge coloured with a single colour.

Assume that the result holds for all bipartite graphs with maximum vertex degree less than r, and let G be a bipartite graph with maximum vertex degree r.

If any of the vertices of G have degree less than r, add new vertices and edges to make a regular bipartite graph G_1 of degree r. If we can edge colour G_1 then we can edge colour G. By Corollary 1.6, the graph G_1 has a perfect matching, which is a set F of edges. Colour all of the edges in F with colour r, and then delete them. This produces a bipartite graph with maximum vertex degree $r-1$, which can be coloured with colours $\{1, \ldots, r-1\}$ by induction. ■

The following result, whose proof is omitted, shows that something similar is true in general.

Theorem 2.3 (Vizing's theorem) *Let G be a graph without loops, with all vertices of degree at most r. Then $r \leq \chi_e(G) \leq r+1$.*

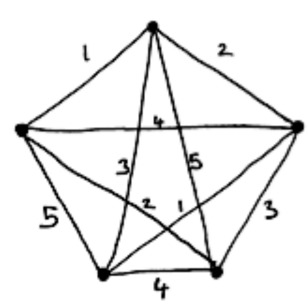
For graphs it is straightforward to decide whether the edge-chromatic number is r or $r+1$. For example, the cycle of length n has edge-chromatic number 2 if n is even and 3 if n is odd.

Theorem 2.4. *For $n > 1$ the number $\chi_e(K_n) = n$ if n is odd, and $\chi_e(K_n) = n-1$ if n is even.*

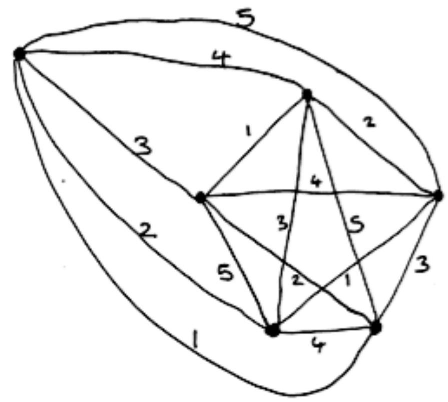
Proof. If n is odd, then the edges of K_n can be n-coloured by placing the vertices of K_n in the form of a regular n-gon, colouring the edges around the outside with a different colour for each edge. Then colour each remaining edge with the same colour as the one that you used on the outside edge that is parallel to it. The largest possible number of edges of the same colour is $(n-1)/2$, as each edge touches two vertices and they must all be distinct. There are $n(n-1)/2$ edges in K_n, so n colours are definitely required and hence $\chi_e(K_n) = n$.

If n is even then think of K_n as the sum of a complete graph K_{n-1} and a single vertex. Colour the edges of K_{n-1} using the method above, using $n-1$ colours. Since each vertex has $n-2$ neighbours within the K_{n-1}, there will be one colour missing at it. This colour will be the colour of the outside edge that is opposite it, so all of the missing colours are distinct. Thus the colouring of the edges of K_n can therefore be completed by colouring the remaining edges with these missing colours. It is clear that this colouring is minimal, so $\chi_e(K_n) = n-1$. ■

Example 2.5. Here is an edge colouring of K_5 using the method of Theorem 2.4.



Here is an edge colouring of K_6 using the method of Theorem 2.4.



3. Factorisation of graphs

When considering Hall's Marriage Theorem we were interested in *bipartite* graphs. In this section we consider sets of independent edges of graphs that need not be bipartite.

Definition 3.1. A *factor* of a graph G is a subgraph of G which uses all of the vertices of G and has at least one edge. We say that G can be *factorised* if it is the union of factors $G_1, \ldots, G_k$, and they have no edges in common. An *n-factor* of G is a factor of G that is regular of degree n.

Example 3.2. Here is a graph and a 1-factorisation.

Lemma 3.3. *If G has a 1-factor H then $|V(G)|$ is even.*

Proof. Every vertex in H has degree 1, so H is a set of independent edges. Thus H has an even number of vertices. Since H uses all vertices of G, the graph G must also have an even number of vertices. ■

Theorem 3.4. *1. The complete graph K_{2n+1} is not 1-factorable.*

2. The complete graph K_{2n} is 1-factorable.

Proof. 1. It is an immediate consequence of Lemma 3.3 that K_{2n+1} does not have any 1-factors, and so is not 1-factorable.
2. To prove this we need to divide up the edges of K_{2n} into $(2n-1)$ 1-factors, each containing n edges, since K_{2n} has $2n$ vertices and $n(2n-1)$ edges.

Let the vertices of K_{2n} be $\{v_0, \ldots, v_{2n-1}\}$. For $0 \leq i \leq 2n-2$ we define a set of edges:

$$X_i = \{v_i v_{2n-1}\} \cup \{v_{i-j} v_{i+j} \;:\; 1 \leq j \leq n-1\}$$

where the subscripts in the second set are taken modulo $2n-1$.

It is clear that X_i contains n edges, and that no vertex occurs more than once in X_i, so that $(\{v_0, \ldots, v_{2n-1}\}, X_i)$ is a 1-factor of K_{2n}.

We check that no edge occurs in both X_i and X_k, for $0 \leq i < k \leq 2n-2$. Suppose to the contrary that $e = v_a v_b$ is an edge in both X_i and X_k. If a or b is $2n-1$ then the other vertex is v_i (in X_i) and v_k (in X_k), a contradiction since $i \neq k$. Thus we may assume without loss of generality that in X_i we have

$$a \equiv i - j \pmod{2n-1} \text{ and } b \equiv i + j \pmod{2n-1}$$

for some j. Considering X_j we have two cases. If

$$i - j \equiv k - l \pmod{2n-1} \text{ and } i + j \equiv k + l \pmod{2n-1}$$

then we deduce that

$$2i = 2k \pmod{2n-1}$$

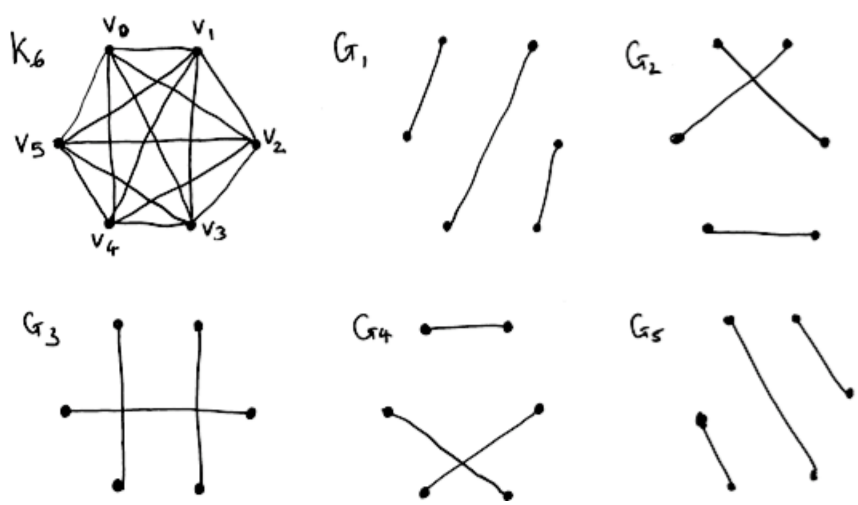
which implies $i = k$ (because 2 and $2n-1$ are coprime, a contradiction. If, on the other hand,

$$i - j \equiv k + l \pmod{2n-1} \text{ and } i + j \equiv k - l \pmod{2n-1},$$

then similarly we deduce that $i = l$, and then it follows that $2k \equiv 0 \pmod{2n-1}$, also a contradiction.

Thus K_{2n} can be 1-factorised as a sum of $X_0, X_1, \ldots, X_{2n-2}$. ■

Example 3.5. Here is a 1-factorisation of K_6 using the method of Theorem 3.4

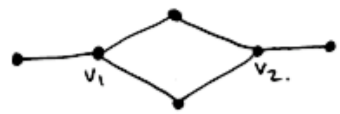


By an *odd component* of a graph, we mean a connected component with an odd number of vertices. A proof of the following can be found in Harary, Theorem 9.4.

Theorem 3.6 (Tutte, 1947) *A graph G has a 1-factor if and only if $|V(G)|$ is even and there is no set S of vertices such that the number of odd components of $G \setminus S$ is greater than $|S|$.*

It can be awkward to use this theorem to prove that a 1-factor exists, as one must consider all possible sets of vertices. However, the following example shows that we can easily use it to show that a 1-factor does not exist.

Example 3.7. The following graph has an even number of vertices. If the set $S = \{v_1, v_2\}$ is removed from G then four isolated vertices (and hence four odd components) are left over, so G does not have a 1-factor.



What about 2-factorisations? If a graph is 2-factorable, then each factor must be a union of disjoint cycles, as these are the connected graphs that are regular of degree 2.

We have seen that the complete graph K_{2n} is 1-factorable but that K_{2n+1} is not 1-factorable.

Theorem 3.8. *1. The graph K_{2n} is not 2-factorable.*

2. The graph K_{2n+1} is 2-factorable.

Proof. 1. In K_{2n} all vertices have odd degree, but all vertices of a 2-factorable graph must have even degree.
2. Label the vertices of K_{2n+1} as $v_0, v_1, \ldots, v_{2n}$. For $0 \leq i \leq n-1$ we construct a path P_i as follows:

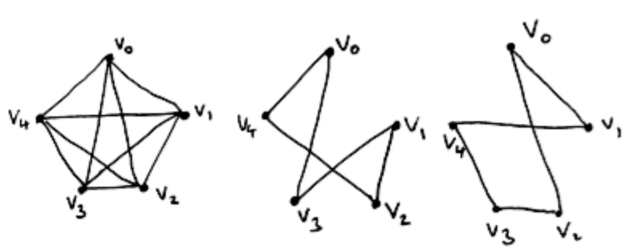
$$P_i = v_i v_{i-1} v_{i+1} v_{i-2} v_{i+2} \cdots v_{i+n-1} v_{i-n},$$

where the subscripts are taken modulo $2n$. We then join v_{2n} to the two endpoints of P_i to make a cycle, Z_i.

We claim that Z_i is a 2-factor, and that every edge lies in exactly one of the Z_i. It is clear that P_i contains $2n$ different vertices, none of which are v_{2n} since the subscripts are taken modulo $2n$, therefore Z_i is a 2-factor.

Let $v_a v_b$ be an edge in Z_i and Z_k, and assume (by way of contradiction) that $i \neq k$. If either of a or b is $2n$ then the other is either i or $i-n$ in Z_i, and either k or $k-n$ in Z_k. Since $i \neq k$ we get $k \equiv i-n \pmod{2n}$, a contradiction. Thus without loss of generality we may assume that for some j the pair $(a,b) = (v_{i-j}, v_{i+j})$ or (v_{i+j}, v_{i-j-1}) and simultaneously for some l the pair $(a,b) = (v_{k-l}, v_{k+l})$ or (v_{k+l}, v_{k-l-1}). If $i-j \equiv k-l \pmod{2n}$ and $i+j \equiv k+l \pmod{2n}$, then $i=k$, a contradiction. If $i-j \equiv k+l \pmod{2n}$ and $i+j \equiv k-l-1 \pmod{2n}$, then $2l \equiv 1pmod2n$, a contradiction. If $i+j \equiv k-l \pmod{2n}$ and $i-j-1 \equiv k+l \pmod{2n}$, then $2l+2j \equiv 1 \pmod{2n}$, a contradiction. If $i+j \equiv k+l \pmod{2n}$ and $i-j-1 \equiv k-l-1 \pmod{2n}$, then $i=k$, a contradiction. Thus Z_i and Z_k have no edge in common. ■

Example 3.9. Here is a 2-factorisation of K_5, using the techniques of Theorem 3.8.

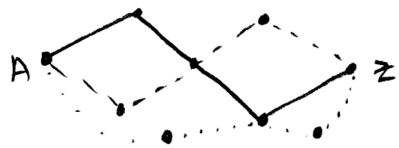


4. Connectivity of graphs

We now discuss a theorem which implies Hall's Marriage Theorem 1.5 and which has very far-reaching practical applications. It concerns the number of paths connecting two given vertices v and w in a graph G.

Definition 4.1. Let v and w be vertices of a graph. Several paths from v to w are *edge disjoint* if different paths have no edges in common. They are *vertex disjoint* if they have no vertices other than v and w in common.

Example 4.2. Here are three edge-disjoint paths from A to Z:



In general we will be interested in finding the maximum number of edge-disjoint paths between two vertices.

Definition 4.3. Let G be a connnected graph, and let v and w be distinct vertices of G. A *disconnecting set* of G is a set of edges whose removal breaks G into more than one component. A *vw-disconnecting set* of G is a set S of edges of G with the property that any path from v to w includes an edge of S: note that it must be a disconnecting set for G. A *vw-separating set* of G is a set S of vertices (not including v or w) such that any path from v to w includes at least one vertex in S.

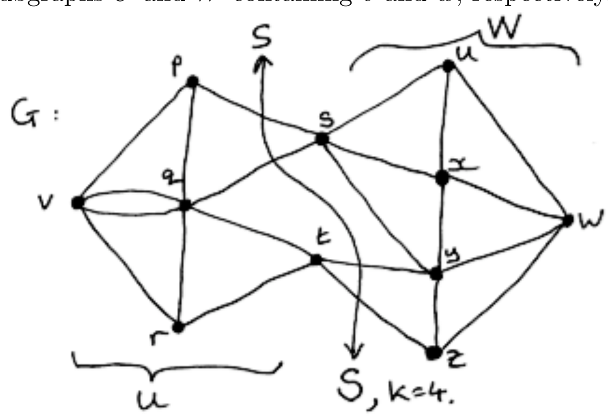
Theorem 4.4 (Menger's Theorem (Edge version)) *Let v and w be distinct vertices of a connected graph G. The maximum number of edge-disjoint paths between v and w is equal to the size of the smallest vw-disconnecting set.*

Proof. The maximum number of edge-disjoint paths from v to w certainly cannot exceed the size of the smallest vw-disconnecting set, as every such path must use a distinct edge from each vw-disconnecting set.

We use induction on the number e of edges of the graph G to show that these numbers are equal. If $e = 1$ then v and w are the only vertices, since G is connected. There is therefore a single path between them, and a single edge in the only vw-disconnecting set.

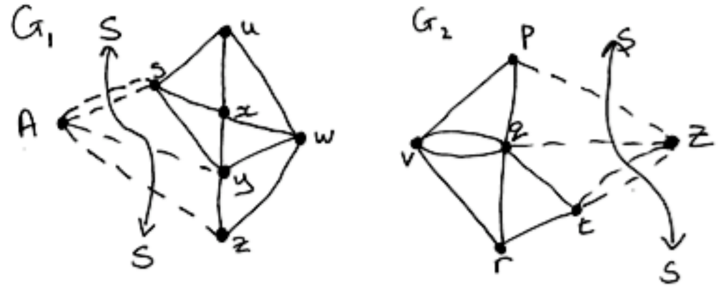
So suppose that the result holds for all graphs with fewer than e edges, and let G be a graph with e edges and a vw-disconnecting set S of size k, where S is as small as possible. We must show that the maximum number of edge-disjoint paths from v to w is k. There are two cases to consider.

Case 1: *Suppose first that S contains at least one edge not incident with v and at least one edge not incident with w.* Say, the edge $ab, cd \in S$ have $a, b \neq v$, $c, d \neq w$. The removal from G of the edges in S (bot not their end-point vertices) results in two disjoint subgraphs U and W containing v and w, respectively.



We now define two new graphs, G_1 and G_2. The graph G_1 is obtained from G by shrinking down all of the vertices in U to a single vertex A, that is incident to all vertices that were incident to vertices of U in G: note that if there was more than one edge from a vertex in U to a vertex in W then there will be a multiple edge from A to W in G_1.

The graph G_2 is obtained similarly, by shrinking all of the vertices in W down to a single vertex Z, which is incident to all vertices that were incident to vertices of W in G.



The graphs G_1 and G_2 have fewer edges than G, because there is at least one edge in U that is not in S, and at least one edge in W that is not in S. To see this, consider the edge ab (renaming if necessary to have $a \in U$, $b \in W$); since $a \neq v$ there must exist edges in U connecting these two vertices. Since S is a minimal Aw-disconnecting set in G_1 and vZ-disconnecting set in G_2, the induction hypothesis tells us that there are k edge-disjoint paths in G_1 from A to w and in G_2 from v to Z. The required k edge-disjoint paths in G are then obtained by combining these paths in the obvious way.

Case 2: *Suppose that in every vw-disconnecting set S of size k all edges are incident with v or all edges are incident with w.* We can assume by induction that every edge of G is contained in one of these minimum vw-disconnecting sets, since otherwise its removal would not affect the value of k and we could use the induction hypothesis to obtain k edge-disjoint paths. It follows that every edge in G is incident with at least one of v or w; hence any path P from v to w consists of at most two edges, and therefore contains at most one edge of any vw-disconnecting set of size k. By removing all edges in P from G, we obtain a graph which contains at least $k-1$ edge-disjoint paths (by the induction hypothesis). These $(k-1)$ paths, together with P, give the required k paths in G. ■

Theorem 4.5 (Menger's theorem (vertex version)) *Let v and w be distinct nonadjacent vertices of a graph G. The maximum number of vertex disjoint paths from v to w is equal to the minimum number of vertices in a vw-separating set.*

Exercise 4.6. The proof of the vertex version of Menger's theorem is very similar to that of the edge version, try to write it down.

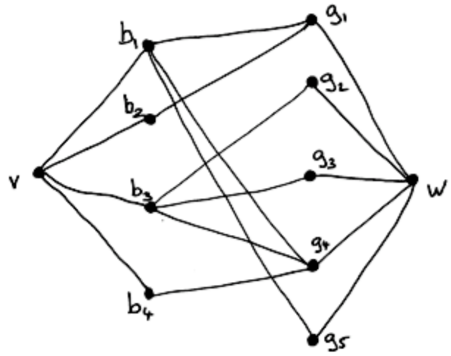
We finish by showing that Hall's Marriage Theorem 1.5 can be deduced from the vertex version of Menger's theorem.

Theorem 4.7 (Hall's Marriage Theorem, Take 2) *Let $P = B \cup G$ be a bipartite graph. There is a matching from B to G if and only if for every $X \subset B$ we have*

$$|N(X)| \geq |X|.$$

Proof. [Proof of Hall's Marriage Theorem using Menger's Theorem] We have to show that it follows from Theorem 4.5 that if each set $X \subset B$ of vertices has at least $|X|$ neighbours then there exists a matching from B to G. Make a new graph

P_1 by adjoining to P a vertex v that is adjacent to every vertex in B and a vertex w that is adjacent to every vertex in G.



A matching from VB to G exists if and only if the number of vertex-disjoint paths from v to w is equal to the number of vertices in B. Let $|B| = k$. By Theorem 4.5, it is therefore enough to show that every vw-separating set contains at least k vertices.

Let S be a vw-separating set, consisting of a subset X of V_1 and a subset Y of V_2. Since $S = X \cup Y$ is a vw-separating set, there are no edges joining vertices in $B \setminus X$ to vertices in $G \setminus Y$, otherwise there would be paths using these edges between v and w. Therefore, the neighbours of $B \setminus X$ that lie in G must all be in Y. By assumption, the set $B \setminus X$ of vertices has at least $|B \setminus X|$ neighbours in G. We deduce that

$$|B \setminus X| \leq |Y|,$$

so $|S| = |X| + |Y| \geq |X| + |B \setminus X| = |X| = k$, as required. ■

Chapter 6

Trees

1. Characterisation of Trees

In this section we study some properties of trees.

Definition 1.1. A graph with no circuits is a *forest*. Recall that a *tree* is a connected graph that contains no circuits.

In many ways a tree is the simplest type of graph. They have many nice properties which often enable one to prove results about trees which are too hard to prove about graphs in general.

There are many equivalent ways of defining trees, the following theorem collects some of them.

Theorem 1.2. *Let G be a graph with v vertices and e edges. The following are equivalent:*

1. *G is a tree;*
2. *G contains no circuits and $e = v - 1$;*
3. *G is connected and $e = v - 1$;*
4. *G is connected but the removal of any edge disconnects G;*
5. *Any two vertices of G are connected by exactly one path;*
6. *G contains no circuits, but the addition of any edge creates exactly one circuit.*

Proof. If $v = 1$ then all of these results are trivial, so we assume throughout that $v \geq 2$.

(1) $\Rightarrow$ (2): By definition G contains no circuits. We prove by induction that $e = v - 1$. For $v = 2$ the only tree has 1 edge, so the result holds. Assume that the result is true for all trees with at most $v - 1$ edges. Let ab be an edge in G, and consider what happens when we remove ab. If there were still a path from a to b then there would originally have been more than one way of getting from a to b, so G would have contained a cycle. Therefore the removal of ab disconnects G into two components, which are trees on v_1 and v_2 vertices, with $v_1 + v_2 = v$. Inductively these trees have $v_1 - 1$ and $v_2 - 1$ edges, so the total number of edges in G is $(v_1 - 1) + (v_2 - 1) + 1 = v - 1$.

(2) $\Rightarrow$ (3): Assume that the connected components of G are $C_1, \ldots, C_k$, on $v_1, \ldots, v_k$ vertices with $v = v_1 + v_2 + \cdots + v_k$. Since G contains no circuits, each C_i is connected and contains no circuits, so is a tree. Therefore by (1) $\Rightarrow$ (2) we have $e = \sum_{i=1}^{k}(v_i - 1) = v - k$. Since $e = v - 1$ we have $k = 1$, so G is connected.

(3) $\Rightarrow$ (4): The removal of an edge from G produces a graph H with $v-2$ edges. We prove by induction that any graph with $v-2$ edges is disconnected. If $v=2$ then the result is clear, so assume inductively that the result holds for all graphs on less than v vertices. The sum of the vertex degrees in H is $2(v-2)$. If H has any vertices of degree 0 then H is disconnected and we are done, so all vertices in H are incident to at least one edge. Since $2v > 2(v-2)$ there exists at least one vertex, w say, of H that has degree 1. Consider the graph H' produced by removing w and the edge ending at w. This is a graph with $v-1$ vertices and $v-3$ edges, so by the inductive hypothesis H' is disconnected. Therefore H is disconnected.

(4) $\Rightarrow$ (5): Since G is connected there exists at least one path from any vertex to any other. If two vertices were connected by more than one path, then it would be possible to remove an edge from G, lying on one of the paths, without disconnecting G. Therefore any two vertices are connected by exactly one path.

(5) $\Rightarrow$ (6): If G contained a circuit, then any pair of vertices on the circuit would be connected by at least two chains. If an edge ab is added to G then a circuit will be created, since there is already a path from a to b. If more than one circuit is created then there must be more than one path from a to b in G, a contradiction.

(6) $\Rightarrow$ (1): If G is disconnected, then the addition of an edge between one component of G and another would not create a circuit, a contradiction. Therefore G is connected and contains no circuits, so is a tree. ■

A vertex of degree 1 is called an *endpoint*.

Corollary 1.3. *1. Every nontrivial tree has at least two endpoints.*

2. Let G be a forest with n vertices and k components, then G has $n-k$ edges.

Exercise 1.4. Prove this corollary.

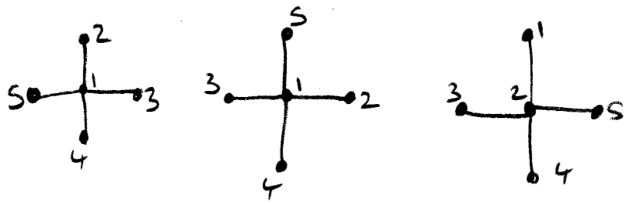
2. Counting Trees

We have already seen that people are interested in counting graphs. Whilst in general graph counting problems can be very hard, if we restrict ourselves to counting trees then there are strong results. In this section we present a famous result by Cayley in 1889 on the number of labelled trees.

Definition 2.1. A *labelled graph* is a pair (G, ϕ) where G is a graph on v vertices and ϕ is a bijection from the vertices of G to the set $\{1, \dots, v\}$. Less formally, we associate to each vertex of G a distinct number from $\{1, \dots, v\}$. Two labelled graphs (G_1, ϕ_1) and (G_2, ϕ_2) are *isomorphic* if there is a graph isomorphism from G_1 to G_2 that preserves the labelling of the vertices.

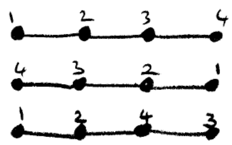
Example 2.2. The first of these graphs is labelled, the second is not.

The first two of these labelled graphs are isomorphic as labelled graphs. The third graph is isomorphic as a graph to the first two, but not as a labelled graph since the vertex of degree 4 has a different label.



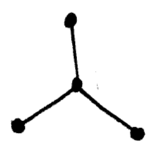
We now consider counting isomorphism classes of labelled graphs.

Example 2.3. This picture shows various ways of labelling a tree with four vertices:



The second tree is the reverse of the first one, and so is isomorphic to it. Neither of them is isomorphic to the third tree, as it has a vertex of degree 2 and label 4. It follows that there are $(4!)/2 = 12$ ways of labelling this tree, since the reverse of any labelling does not result in a new one.

Similarly, there are four ways of labelling this tree:



since the labelling is completely determined by the label of the middle vertex.

Since these are the only trees on four vertices, there are 16 isomorphism classes of labelled trees on four vertices.

We now consider the general case of labelled trees on n vertices. We will count the labelled trees on n vertices by establishing a bijection between labelled trees with n vertices and certain sequences of $n-2$ numbers. We assume that $n \geq 3$.

Definition 2.4. A *Prüfer sequence* is a sequence of the form $(a_1, a_2, \ldots, a_{n-2})$, where each $a_i \in \{1, \ldots, n\}$ and repetitions are allowed.

Here is an algorithm which constructs a Prüfer sequence from a labelled tree:

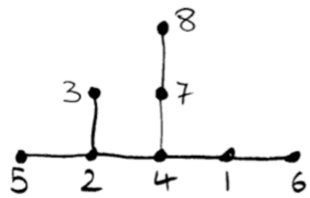
1. Look at the vertices of degree 1 and choose the one, w say, with the smallest label.

2. Find the vertex adjacent to w and place its label in the first available position in the sequence.

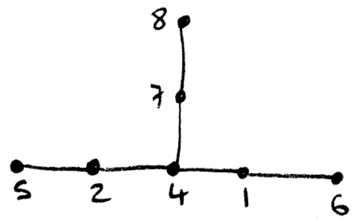
3. Remove w and its incident edge, leaving a smaller tree.

4. Repeat steps 1 to 3 until there are only two vertices left, at which point a sequence of length $n-2$ will have been constructed.

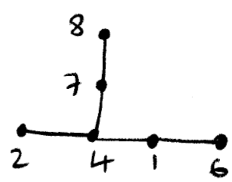
Example 2.5. Consider the following labelled tree:



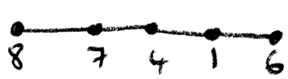
1. The vertices of degree 1 are vertices $3, 5, 6$ and 8, the one with the smallest label is vertex 3.
2. The vertex adjacent to vertex 3 is vertex 2, so the sequence starts with **2**.
3. Removing vertex 3 and its incident edge gives the following tree:



4. The vertices of degree 1 are vertices $5, 6$ and 8, the smallest label is vertex 5.
5. The vertex adjacent to vertex 5 is vertex 2, so the next number in the sequence is **2**.
6. Removing vertex 5 and its incident edge gives the following tree:



7. The vertices of degree 1 are vertices $2, 6$ and 8, the smallest label is vertex 2.
8. The vertex adjacent to vertex 2 is vertex 4, so the next number in the sequence is **4**.
9. Removing vertex 2 and its incident edge gives the following tree:



10. Continue in the way, removing the edges $(6, 1)$, $(1, 4)$, $(4, 7)$ to get the Prüfer sequence $(2, 2, 4, 1, 4, 7)$.

In order to reverse the operation, we take a Prüfer sequence and apply the following three steps:

1. Draw the n vertices, labelling them from 1 to n, and make a list L of the numbers from 1 to n.

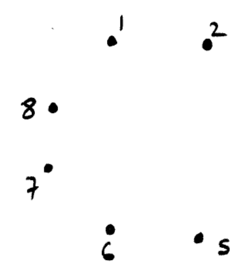
2. Find the smallest number a that is in L but *not* in the Prüfer sequence, and also find the first number b in the Prüfer sequence. Draw an edge joining the vertices with these labels.

3. Remove a from L, and b from the Prüfer sequence, leaving a smaller list and sequence.

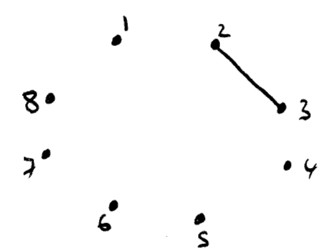
4. Repeat steps 2 and 3 for the remaining list and sequence until there are only two numbers left in the list. Finish by joining the vertices with these labels.

Example 2.6. Consider the Prüfer sequence $(2, 2, 4, 1, 4, 7)$.

1. Since our Prüfer sequence contains $8 - 2 = 6$ numbers we start with the list $[1, 2, 3, 4, 5, 6, 7, 8]$ and draw the vertices 1 to 8 as shown:

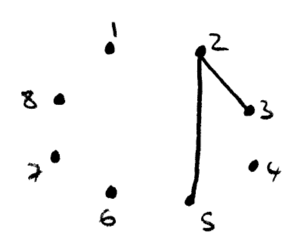


2. The smallest number in the list but not in the Prüfer sequence is 3, and the first number in the Prüfer sequence is 2, so we add an edge joining vertices 2 and 3, as shown:



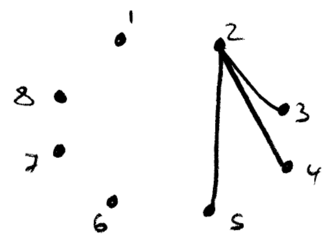
3. Our new list is $[1, 2, 4, 5, 6, 7, 8]$ and our new sequence is $(2, 4, 1, 4, 7)$.

4. The smallest number in the list but not the sequence is 5, and the first number in the sequence is 2, so we add an edge joining vertices 2 and 5, as shown:

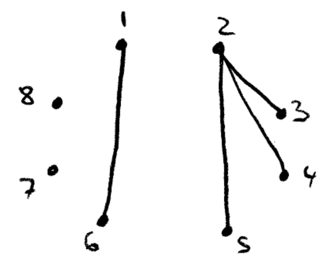


5. Our new list is $[1, 2, 4, 6, 7, 8]$ and our new sequence is $(4, 1, 4, 7)$.

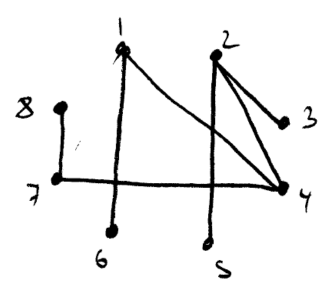
6. The smallest number in the list but not in the sequence is 2, and the first number in the sequence is 4, so we add an edge joining vertices 2 and 4, as shown:



7. Our new list is $[1, 4, 6, 7, 8]$ and our new sequence is $(1, 4, 7)$.
8. The smallest number in the list but not in the sequence is 6, and the first number in the sequence is 1, so we add an edge joining vertices 1 and 6, as shown:



9. Our new list is $[1, 4, 7, 8]$ and our new sequence is $(4, 7)$.
10. We then add an edge between vertices 1 and 4, then one between 4 and 7. Finally our list consists of the two numbers $[7, 8]$, so we join the vertices with these labels. This gives the following labelled tree:



Note that the labelled tree of this example is isomorphic to the tree of the previous example. This is true in general: if you start with a tree, find the corresponding Prüfer sequence, then construct a tree from the sequence, you get back an isomorphic labelled tree.

Theorem 2.7 (Cayley's Theorem, 1889) *There are n^{n-2} distinct labelled trees on n vertices.*

Proof. We consider the above bijection between the set of labelled trees with n vertices and the set of all sequences of the form $(a_1, \ldots, a_{n-2})$ with $a_i \in \{1, \ldots, n\}$. There are exactly n possible values for each number a_i, so the total number of possible sequences is n^{n-2}, and the same holds for labelled trees. ∎

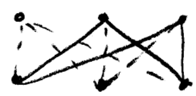
Chapter 7

Ramsey Theory

1. Graphs with no triangles

Given a graph, we can consider whether it contains triangles, quadrilaterals, tetrahedra (triangle-based pyramids) like K_4, or other geometric objects.

Example 1.1. The graph $K_{3,3}$, or any other bipartite graph, contains no triangles, but $K_{3,3}$ contains lots of quadrilaterals.



Question 1.2. How many edges can a simple graph on n vertices have without containing a triangle?

Answer: For n even, $K_{n/2,n/2}$ has $n^2/4$ edges but no triangles. For n odd, $K_{(n+1)/2,(n-1)/2}$ has $(n^2-1)/4$ edges but no triangles. In fact, these bipartite graphs give the most edges on n vertices with no triangles.

We prove this claim in the following theorem. Recall that $\lfloor x \rfloor$ means the largest integer less than or equal to x, known as the *floor* of x.

Theorem 1.3 (Turan's Theorem) *Let G be a simple graph on n vertices. If G contains no triangles then G has at most $\lfloor n^2/4 \rfloor$ edges.*

Proof. Let $n = 2k$ be even. We prove the result for even n by induction on k: the proof for n odd is similar.

When $k = 1$ we have $n = 2$. The only simple graphs on 2 vertices are the graph with 2 vertices and no edges and the graph with 2 vertices and one edge. These have no triangles, and at most $2^2/4 = 1$ edge, which starts the induction.

Assume inductively that the conclusion holds for graphs with at most $n = 2k$ vertices. Let G have $2(k+1)$ vertices and contain no triangle. Let vw be any edge of G, and define G' to be the graph obtained from G by removing the vertices v and w along with all edges ending at v or w.

The graph G' has $2k$ vertices and contains no triangles, so by the inductive hypothesis G' has at most $\lfloor n^2/4 \rfloor = \lfloor (2k)^2/4 \rfloor = k^2$ edges. Now consider replacing the removed edges and vertices to recover G. As G contains no triangles, v and w cannot both be joined to the same vertex z of G' as this would give a triangle. The number of edges we replace is therefore at most the number of vertices in G', plus the edge vw. Thus the number of edges in G is the number of edges of G' plus the number of edges we replace, which is at most

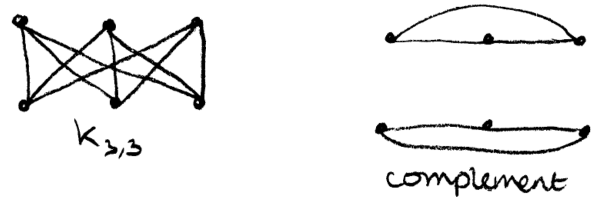
$$k^2 + 2k + 1 = (k+1)^2 = \lfloor (2(k+1))^2/4 \rfloor$$

This completes the inductive step, so the number of edges is at most k^2 for all k.

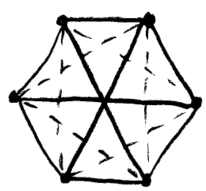
When n is odd the induction is similar, starting with the graph with one vertex, no edges, and no triangles. ■

Definition 1.4. The *complement* of a graph G has the same vertices as G, but whenever G has an edge between two vertices the complement has a non-edge between those vertices, and whenever G has no edge between two vertices the complement has an edge.

We have seen that it is possible for a graph to have many edges and no triangles. However, in this situation the complement will have many triangles.



Example 1.5. Consider the following graph.



Here we have 6 vertices, 9 edges and no triangle. The complementary graph (with dashed edges) has two triangles.

Ramsey Theory. We proved in Chapter 1, Theorem 2.7 that in a group of six people there are either three who mutually know each other or three who mutually don't know each other. In the previous sentence, we can replace $(6, 3)$ by $(18, 4)$ and it will still be true. These are the prototype Ramsey Theorems. We represent people by vertices of a complete graph, and colour the edge vw red if v and w know each other, and blue otherwise.

Note that we are now talking about colouring the *edges* of a graph, rather than the vertices as in Chapter 4.

Proposition 1.6. *If the edges of K_{18} are all coloured either red or blue, then there is either a red K_4 or a blue K_4.*

This proposition will follow from a more general result given in the next section.

2. Ramsey Numbers

Definition 2.1. By the *Ramsey number* $R(m, n)$ we denote the least integer such that when the edges of the complete graph $K_{R(m,n)}$ are all coloured either red or blue then the graph contains either a red K_m or a blue K_n.

For example, we have seen that $R(3, 3) = 6$ and claimed that $R(4, 4) = 18$.

Proposition 2.2. *For all $m \geq 2$ and $n \geq 2$ we have*

$$R(m, n) \leq R(m, n - 1) + R(m - 1, n).$$

In particular, $R(m, n)$ is always finite.

Proof. Consider a complete graph with $R(m,n-1)+R(m-1,n)$ vertices, and all edges coloured either red or blue. Each vertex has degree $R(m,n-1)+R(m-1,n)-1$. So for any vertex v, either at least $R(m,n-1)$ of the edges incident to v are blue or at least $R(m-1,n)$ of them are red.

Assume first that at least $R(m,n-1)$ of the edges are blue, joining v to vertices $v_1, v_2, \ldots, v_{R(m,n-1)}$ (and possibly some other vertices too). Consider the edges joining these v_i to one another. They form $K_{R(m,n-1)}$ and so either these vertices contain a red K_m, in which case we are done, or they form a blue K_{n-1}. In this second case v and all of the edges that are joined in this K_{n-1} form a blue K_n.

By a similar argument, if at least $R(m-1,n)$ of the edges at v are red, we again get either a red K_m containing v, or a blue K_n amongst the vertices that are joined to v by red edges. ■

Corollary 2.3. *We have*

$$R(m,n) \leq \frac{(m+n-2)!}{(m-1)!(n-1)!} = \binom{m+n-2}{m-1}.$$

Proof. We prove this by double induction using Proposition 2.2. The graph K_1 (a single isolated vertex) always contains K_1, so we have $R(m,1) = R(1,n) = 1$. Therefore

$$1 = R(m,1) \leq \frac{(m+1-2)!}{(m-1)!0!} = \frac{(m-1)!}{(m-1)!} = 1.$$

A similar calculation shows that $R(1,n)$ also satisfies the inequality, so the corollary is true when either m or n is 1.

The graph K_m either contains a red K_m or has at least one blue edge, giving a blue K_2, so $R(m,2) = m$, and similarly $R(2,n) = n$. Therefore

$$m = R(m,2) \leq \frac{(m+2-2)!}{(m-1)!1!} = \frac{m!}{(m-1)!} = m,$$

and similarly for $R(2,n)$. These cases found the induction.

Assume inductively the corollary holds for $R(i,j)$ with $i \leq m-1$ and $j \leq n$, and for $R(i,j)$ with $i \leq m$ and $j \leq n-1$. By Proposition 2.2 we have

$$\begin{aligned} R(m,n) &\leq R(m,n-1)+R(m-1,n) \\ &\leq \tfrac{(m+n-3)!}{(m-1)!(n-2)!} + \tfrac{(m+n-3)!}{(m-2)!(n-1)!} \text{ by the inductive hypothesis} \\ &= \tfrac{(m+n-3)!}{(m-1)!(n-1)!}((n-1)+(m-1)) \\ &= \tfrac{(m+n-2)!}{(m-1)!(n-1)!} \end{aligned}$$

The result therefore holds for $R(m,n)$ and so for all m and n by induction. ■

Remark 2.4. Proposition 2.2 does not always give best results. For example,

$$9 = R(3,4) < R(3,3) + R(2,4) = 6 + 4 = 10.$$

Theorem 2.5. *The following are values for $R(m,n)$:*

$R(m,n)$		m									
		1	2	3	4	5	6	7	8	9	10	…
	1	1	1	1	1	1	1	1	1	1	1	…
	2	1	2	3	4	5	6	7	8	9	10	…
	3	1	3	6	9	14	18	23	28	36	40–43	
n	4	1	4	9	18	25	35–41	49–61	56–84			
	5	1	5	14	25	43–49						
	6	1	6	18								
	⋮	⋮										

Challenges Find $R(3, 10), R(3, 11), R(4, 6), R(5, 5), \ldots$.

3. Ramsey numbers for several colours

We write $R_k(3) = R(3, 3, \ldots, 3)$ (with k threes) for the smallest number such that if the edges of the complete graph on $R_k(3)$ vertices are coloured with k colours then we can find a monochromatic triangle.

For example, $R_2(3) = R(3, 3) = 6$. We have $R_k(3) \leq (k + 1)!$.

Theorem 3.1 (An application to number theory) *Let $m = R_k(3) - 1$. If the integers $1, 2, \ldots, m$ are coloured with k colours then the equation $x + y = z$ has a solution with x, y and z all the same colour.*

For example, suppose $k = 2$ and $m = 5$ (recall that $R_2(3) = 6$), and that we colour 1 and 3 black and $2, 4$ and 5 red. Then $2 + 2 = 4$ is a red solution.

Proof. Fix a colouring of the numbers $1, \ldots, m$. Label the vertices of $K_{R_k(3)}$ with the integers $1, 2, \ldots, m, m+1$ and colour the edge rs the colour of the number $|r - s|$. Then by definition of $R_k(3)$ we may find a monochromatic triangle in this graph, with vertices a, b, c such that

$$1 \leq a < b < c \leq m + 1.$$

Let $x = b - a$, $y = c - b$ and $z = c - a$. Then the numbers x, y, z represent edges in the triangle, so they all have the same colour. Also,

$$x + y = (b - a) + (c - b) = c - a = z.$$

■

Remark 3.2. This theorem also holds for all $m \geq R_k(3) - 1$.

Next we consider infinite graphs.

Theorem 3.3 (Ramsey's Theorem) *Let V be a countably infinite set of vertices with each distinct pair joined by a red or blue edge. Then V has an infinite subset of vertices $\{a_0, a_1, \ldots\}$ joined by edges all of the same colour.*

Proof. We divide the proof into two cases

Case (i): There is an infinite subset A of V such that just finitely many vertices of A are red-joined to infinitely many other vertices of A.

Delete the vertices of A which have infinite red degree to get an infinite set A' with all vertices red-joined to only finitely many other vertices of A. We now select inductively a sequence of vertices a_k all blue joined.

Take $a_0 \in A'$. Suppose we have selected $a_0, a_1, \ldots, a_k \in A'$ all blue joined. Since these vertices are joined to only finitely many points of A' by red edges they are joined to an infinite number of points of A' by only blue edges. Therefore we may pick a new point $a_{k+1} \in A'$ that is blue joined to all of $a_0, a_1, \ldots, a_k$. Continuing in this way we get an infinite sequence of vertices $a_0, a_1, \ldots$ all blue joined.

Case (ii): Every infinite subset A of V contains infinitely many vertices red-joined to infinitely many other vertices of A.

In this case we select inductively a sequence of vertices a_k that are all red-joined. Take $a_0 \in V$ with infinite red degree and let A_0 be the infinite set of vertices red joined to a_0. Suppose inductively that there are points $a_0, a_1, \ldots, a_k$ all red-joined and an infinite set of vertices A_k all red-joined to all of $a_0, a_1, \ldots, a_k$. Take a new point $a_{k+1} \in A_k$ that is red-joined to infinitely many other vertices of A_k (since we are in case (ii)), and let A_{k+1} be the infinite subset of A_k of points red joined to a_{k+1}. Then $a_0, a_1, \ldots, a_{k+1}$ are all red-joined to each other and to all points of A_{k+1}. Proceeding in this way gives an infinite sequence of vertices $a_0, a_1, \ldots$ all red-joined. ■

Corollary 3.4. *Let V be an infinite set of vertices with each pair joined by an edge coloured by one of k colours. Then V has an infinite subset of vertices $\{a_0, a_1, \ldots\}$ joined by edges all of the same colour.*

Proof. Let the colours be $c_1, c_2, \ldots, c_k$. By Ramsey's Theorem 3.3, there is either an infinite set of vertices with all joining edges coloured c_1 or an infinite set with joining edges coloured only by colours $c_2, \ldots, c_k$. In the latter case, continue to apply Ramsey's Theorem up to $(k-2)$ more times until an infinite set with joining edges of one colour is reached. ■

Corollary 3.5. *There is an infinite subset S of $\mathbb{N}$ such that for all distinct $m, n \in S$ the number $m+n$ has an even number of prime factors (including multiplicity).*

Proof. Consider a graph with vertices the natural numbers. Join m to n by a red edge if $m+n$ has an even number of prime factors. Join m to n by a blue edge if $m+n$ has an odd number of prime factors. By Ramsey's theorem *either* there is an infinite subset of integers all red-joined, in which case the result follows, *or* there is an infinite subset $T = \{a_1, a_2, \ldots\}$ that are all blue-joined. In this case $a_i + a_j$ always has an odd number of prime factors, so the set $S = \{2a_1, 2a_2, 2a_3, \ldots\}$ gives the result. ■

Consider a set S with $|S| = 3$, say $S = \{3, 7, 18\}$. Here $3 + 7 = 10 = 2 \cdot 5$, $3 + 18 = 21 = 3 \cdot 7$, $7 + 18 = 25 = 5^2$, so all sums have an even number of factors.
Challenge Find a specific infinite set S with the property in the corollary.

Corollary 3.6. *Let $a_1, a_2, \ldots$ be an infinite sequence of real numbers. Then this sequence either has an infinite increasing subsequence $a'_1 \leq a'_2 \leq \ldots$ or a decreasing subsequence $a'_1 \geq a'_2 \geq \ldots$.*

Proof. Consider a graph whose vertices are the natural numbers. If $i > j$ then we red-join i to j if $a_i \geq a_j$ and blue-join i to j if $a_i < a_j$. By Ramsey's theorem there is an infinite subset $a'_1, a'_2, \ldots$ (in that order) such that *either* all corresponding integers are red-joined, in which case $a'_i \geq a'_j$ whenever $i > j$ so the sequence is increasing, *or* they are all blue-joined in which case $a'_i < a'_j$ whenever $i > j$ so the sequence is decreasing. ■

Chapter 8

Adjacency matrices

Aim: To use results from Linear Algebra to study the structure of graphs.

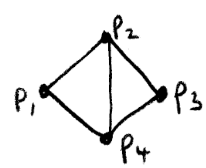
1. Definitions and basic results

Definition 1.1. Let G be a simple (undirected) graph with v vertices $p_1, \dots, p_v$. The *adjacency matrix* A of G is the $v \times v$ square matrix such that

$$a_{ij} = [A]_{ij} = \begin{cases} 1 & \text{if } p_i p_j \text{ is an edge} \\ 0 & \text{otherwise} \end{cases}$$

Thus A records which pairs of vertices are joined by an edge.

Example 1.2. Consider the following graph:



The corresponding adjacency matrix is:

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

We collect some straightforward facts in the following lemma.

Lemma 1.3. *Let A be the adjacency matrix of a simple graph with vertices $p_1, \dots, p_v$.*

1. *A is a symmetric matrix with 0s on the diagonal and 0 or 1 elsewhere.*
2. *For each i, the sum of the entries in the ith row, $\sum_{j=1}^{v} a_{ij}$, equals the degree of p_i.*
3. *A^2 has (i,j) entry $\sum_{k=1}^{v} a_{ik}a_{kj}$. This sum counts 1 for every vertex p_k that is joined to both p_i and p_j. This is the number of different paths of length two from p_i to p_j. In particular the diagonal entries of A^2 are the degrees of the vertices.*
4. *A^m gives the number of paths of length m between vertices p_i and p_j, where we allow paths to traverse edges several times.*

Remark 1.4. We may define adjacency matrices in the same way for non-simple graphs, with $a_{ij} = x$ if p_i and p_j are joined by x edges, and for directed graphs (in which case A is not necessarily symmetric).

Example 1.5. In Example 1.2 we have

$$A^2 = \begin{pmatrix} 2 & 1 & 2 & 1 \\ 1 & 3 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}$$

Since $[A^2]_{1,3} = 2$ there are two paths of length two from p_1 to p_3 (via p_2 or p_4). Similarly,

$$A^3 = \begin{pmatrix} 2 & 5 & 2 & 5 \\ 5 & 4 & 5 & 5 \\ 2 & 5 & 2 & 5 \\ 5 & 5 & 5 & 4 \end{pmatrix}.$$

Note that the sum of the diagonal entries gives the number of triangles in the graph, with each triangle counted six times. So the number of triangles in the graph of Example 1.2 is $(2+4+2+4)/6 = 2$.

2. Eigenvalues

Recall that λ is an *eigenvalue* of a square matrix A if

$$\det(A - \lambda I) = 0$$

and that this holds if and only if there exists a vector $x \neq 0$ such that

$$Ax = \lambda x.$$

Such an x is called an *eigenvector*.

Note that a $v \times v$ real matrix A has v real or complex eigenvalues, which are not necessarily all distinct, since $\det(A - \lambda I)$ is a polynomial of degree v in λ.

Lemma 2.1. *Let A be a square real matrix and suppose that A satisfies a polynomial equation. Then every eigenvalue of A satisfies the same polynomial equation.*

Proof. Suppose $b_n A^n + b_{n-1} A^{n-1} + \cdots + b_1 A + b_0 I = 0$. Let λ be any eigenvalue of A. Then $Ax = \lambda x$ for some vector $x \neq 0$. Therefore, $A^2 x = \lambda A x = \lambda^2 x$, and similarly $A^k x = \lambda^k x$ for $2 \leq k \leq n$. Therefore

$$\begin{aligned} 0 &= 0x \\ &= (b_n A^n + \cdots + b_1 A + b_0 I)x \\ &= (b_n \lambda^n + \cdots + b_1 \lambda + b_0)x \end{aligned}$$

Since $b_n \lambda^n + \cdots + b_1 \lambda + b_0$ is a real or complex number and x is a nonzero vector, this implies that $b_n \lambda^n + \cdots + b_1 \lambda + b_0 = 0$. ■

Recall that the *trace* of a square matrix is the sum of its diagonal entries.

Lemma 2.2. *For any $v \times v$ matrix A, the sum of the eigenvalues of A (counted with multiplicities) is equal to the trace of A.*

Proof. Denote the eigenvalues by $\lambda_1, \lambda_2, \dots, \lambda_v$, so that some values may be repeated. We have

$$\begin{array}{rcl} \det(A-\lambda I) & = & \det\begin{pmatrix} a_{11}-\lambda & a_{12} & a_{13} & \dots \\ a_{21} & a_{22}-\lambda & a_{23} & \dots \\ \vdots & \vdots & & \end{pmatrix} \\ & = & (-\lambda)^v + (-\lambda)^{v-1}(a_{11}+a_{22}+\dots+a_{vv}) + \text{ terms involving lower powers of } \lambda \\ \det(A-\lambda I) & = & (\lambda_1-\lambda)(\lambda_2-\lambda)\cdots(\lambda_v-\lambda) \text{ by definition of an eigenvalue} \\ & = & (-\lambda)^v + (-\lambda)^{v-1}(\lambda_1+\lambda_2+\dots+\lambda_v) + \text{ terms involving lower powers of } \lambda \end{array}$$

Comparing coefficients we see that the sum of the eigenvalues is equal to the trace of A. ■

Lemma 2.3. *Let A be the adjacency matrix of a simple connected graph with every vertex of degree d, so that every row and column of A sums to d. Then d is an eigenvalue of A with eigenvector $(1, 1, \dots, 1)^T$ and multiplicity 1.*

Proof. We have

$$A\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} d \\ d \\ \vdots \\ d \end{pmatrix} = d\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

This shows that d is an eigenvalue and that $(1, 1, \dots, 1)^T$ is a corresponding eigenvector.

To see that the multiplicity of this eigenvalue is 1, first note that because A is symmetric real, the multiplicity of each eigenvalue λ is equal to the so called geometric multiplicity, i.e. the dimension of the eigenspace corresponding to λ; this is a fact from Linear Algebra, and will not be (re)proved here. Now, suppose that $(e_1, e_2, \dots, e_v)^T$ is another eigenvector with eigenvalue d. By reordering the basis vectors if necessary, and remembering that eigenvectors are only defined up to scalar multiplication, we may assume without loss of generality that $e_1 = 1$ and $|e_i| \leq 1$ for $1 \leq i \leq v$. Then

$$d\begin{pmatrix} 1 \\ e_2 \\ \vdots \\ e_v \end{pmatrix} = A\begin{pmatrix} 1 \\ e_2 \\ \vdots \\ e_v \end{pmatrix} = \begin{pmatrix} a_{11}+a_{12}e_2+\dots+a_{1v}e_v \\ \vdots \\ \vdots \\ \vdots \end{pmatrix}$$

Therefore $d = a_{11} + a_{12}e_2 + \dots + a_{1v}e_v \leq a_1 1 + a_{12}|e_2| + \dots + a_{1v}|e_v| \leq a_{11} + a_{12} + \dots + a_{1v} = d$. Thus we must have equality throughout and so $e_j = 1$ for all nonzero a_{1j}, that is for every vertex p_j that is adjacent to p_1. Using the fact that G is connected, we may consider other coefficients of the eigenvector to show that $e_j = 1$ for all j. Hence we do not have a new eigenvector. ■

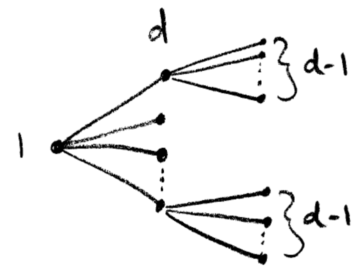
3. The eigenvalue method

Idea: Given a graph G satisfying certain conditions we find a polynomial equation satisfied by the adjacency matrix A. We study the eigenvalues which are solutions of the polynomial equation and use these to deduce information about G.

Theorem 3.1. *Let G be a connected simple graph with all vertices of degree d. Suppose that G has no triangles and that every pair of non-adjacent vertices have exactly one common neighbour. Then $d = 0, 1, 2, 3, 7$ or 57 and the graph has $v = d^2 + 1$ vertices.*

Proof. Every vertex is distance 1 or 2 from any given vertex and so examining the figure we see that

$$v = 1 + d + d(d-1) = 1 + d^2$$



Let A denote the $v \times v$ adjacency matrix of G, let B be the $v \times v$ matrix with all entries 1, let I be the identity matrix and 0 the zero matrix. We consider the entries of A and A^2.

In A	In A^2	
0 on diagonal	d on diagonal	(degree of each vertex is d)
1 off diagonal	0 in corresponding entry	(as there are no $\triangle$s)
0 off diagonal	1 in corresponding entry	(1 common neighbour)

Hence, considering the 3 types of entry of A and A^2 we have

$$A^2 + A = \begin{pmatrix} d & 1 & \dots & 1 \\ 1 & d & \dots & 1 \\ \vdots & & \ddots & \vdots \\ 1 & \dots & 1 & d \end{pmatrix} = B + (d-1)I$$

or

$$A^2 + A - (d-1)I = B.$$

The row sums of A equal d, so $AB = dB$ and therefore

$$A^3 + A^2 - (d-1)A = AB = dB = d(A^2 + A - (d-1)I)$$

We rearrange this to get the cubic equation

$$0 = A^3 - (d-1)A^2 - (2d-1)A + d(d-1)I.$$

Thus the eigenvalues λ of A satisfy the equation

$$\begin{aligned} 0 &= \lambda^3 - (d-1)\lambda^2 - (2d-1)\lambda + d(d-1) \\ &= (\lambda - d)(\lambda^2 + \lambda - (d-1)) \end{aligned}$$

(noting that d must be an eigenvalue by Lemma 2.3). Solving this for the eigenvalues we get

$$\begin{aligned} \lambda &= d && \text{with multiplicity } 1 \\ \lambda &= \frac{-1-\sqrt{4d-3}}{2} && \text{with multiplicity } r \in \mathbb{N} \\ \text{or } \lambda &= \frac{-1+\sqrt{4d-3}}{2} && \text{with multiplicity } s \in \mathbb{N}. \end{aligned}$$

Moreover, a $v \times v$ matrix has v eigenvalues so $r + s + 1 = v$. Also we know that $0 = \text{Trace} A =$ sum of eigenvalues so

$$r\left(\frac{-1-\sqrt{4d-3}}{2}\right) + s\left(\frac{-1+\sqrt{4d-3}}{2}\right) + d = 0$$

Solving this pair of simultaneous linear equations for r and s gives

$$r, s = \frac{1}{2}(v-1) \pm \frac{1}{2}\frac{(2d-(v-1))}{\sqrt{4d-3}} = \frac{1}{2}d^2 \pm \frac{1}{2}\frac{(2d-d^2)}{\sqrt{4d-3}} \qquad (1)$$

since $v = 1 + d^2$.

But r and s are integers, so from (1) *either* (i) $(2d - d^2) = 0$ *or* (ii) $4d - 3$ is a perfect square (otherwise $\sqrt{4d-3}$, and thus r and s, would be irrational).

In case (i), $d = 0$ (trivial) or $d = 2$ and $v = 5$ (giving a pentagon).

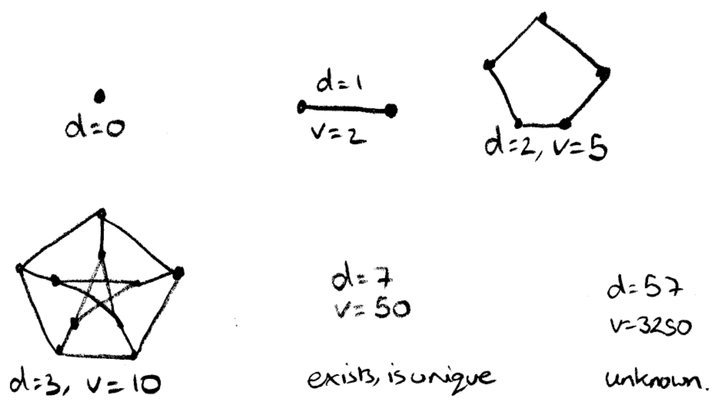
In case (ii), setting $t = \sqrt{4d-3}$ for some integer t, we get $d = \frac{1}{4}(t^2+3)$, so from (1)

$$s \quad = \quad \frac{1}{2}\left[d^2 + \frac{(2d-d^2)}{t}\right] = \frac{1}{2}\left[\frac{1}{16}(t^2+3)^2 + \frac{2 \times \frac{1}{4}(t^2+3) - \frac{1}{16}(t^2+3)^2}{t}\right],$$

simplifying to

$$t^5 - t^4 + 6t^3 + 2t^2 + (9 - 32s)t + 15 \quad = \quad 0.$$

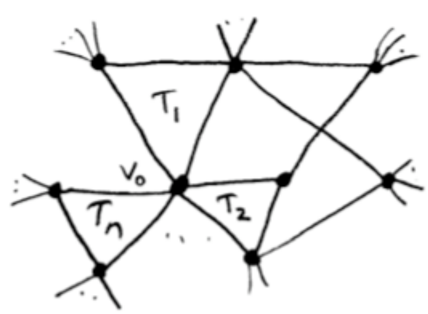
Thus t divides 15 (the product of the roots of this equation), so $\pm t = 1, 3, 5$ or 15, giving $d = \frac{1}{4}(t^2+3) = 1, 3, 7$ or 57. ■



Now we prove a second, similar result.

Theorem 3.2. *Let G be a connected simple graph. Suppose that every edge of G belongs to exactly one triangle, and every two non-adjacent vertices have exactly two common neighbours (i.e. form the (missing) diagonal of exactly one quadrilateral). Then every vertex has the same degree d, where $d = 0, 2, 4, 14, 22, 112$ or 974, with the number of vertices $v = 1 + \frac{1}{2}d^2$.*

Proof. Let v_0 be some vertex of G. Then v_0 must be the common vertex of n edge-disjoint triangles, $T_1, \ldots, T_n$, for some integer n.



Moreover, every pair of vertices (other than v_0) from different triangles T_i has one further common neighbour. This accounts for all the vertices, as no vertices are distance more than 2 apart. Counting these vertices

$$v = 1 + 2n + \frac{1}{2}2n(2n-2) = 1 + 2n^2.$$

In particular (since we can count the vertices in this way starting with any vertex v_0) every vertex has degree $d = 2n$ for the same integer n.

Let A denote the $v \times v$ adjacency matrix of G, let B be the $v \times v$ matrix with all entries 1, let I be the identity matrix and 0 be the zero matrix. We collect some facts about A and A^2.

In A	In A^2	
0 on diagonal	$2n$ on diagonal	(degree of each vertex is $2n$)
1 off diagonal	1 in corresponding entry	(every edge belongs to 1 $\triangle$)
0 off diagonal	2 in corresponding entry	(every non-edge is diagonal of 1 quadrilateral)

Hence, considering the 3 types of entry of A and A^2, we have

$$A^2 + A = 2B + (2n-2)I$$

or

$$A^2 + A - (2n-2)I = 2B.$$

We note that $AB = 2nB$ since the row sums of A equal $2n$, so

$$A^3 + A^2 - (2n-2)A = 2AB = 4nB = 2n(A^2 + A - (2n-2)I).$$

Rearranging, we get the cubic

$$0 = A^3 + (1-2n)A^2 + (2-4n)A + 4n(n-1)I.$$

Thus the eigenvalues λ of A satisfy the equation

$$\begin{aligned} 0 &= \lambda^3 + (1-2n)\lambda^2 + (2-4n)\lambda + 4n(n-1) \\ &= (\lambda - 2n)(\lambda^2 + \lambda - 2(n-1)) \end{aligned}$$

(noting that $d = 2n$ must be an eigenvalue, so that we can factorise out $(\lambda - 2n)$). Solving we find eigenvalues

$$\begin{array}{rrcll} & \lambda & = & 2n & \text{with multiplicity } 1 \\ & \lambda & = & \dfrac{-1-\sqrt{8n-7}}{2} & \text{with multiplicity } r \in \mathbb{N} \\ \text{or} & \lambda & = & \dfrac{-1+\sqrt{8n-7}}{2} & \text{with multiplicity } s \in \mathbb{N}. \end{array}$$

Moreover, a $v \times v$ matrix has v eigenvalues so $r + s + 1 = v = 1 + 2n^2$. Also $0 = \text{Trace}A$ = sum of eigenvalues so

$$r\left(\frac{-1-\sqrt{8n-7}}{2}\right) + s\left(\frac{-1+\sqrt{8n-7}}{2}\right) + 2n = 0.$$

Solving this pair of simultaneous linear equations for r and s gives

$$r, s \quad = \quad n^2 \pm \frac{(n^2 - 2n)}{\sqrt{8n-7}}.$$

But r and s are integers, so *either* (i) $n(n-2) = 0$ *or* (ii) $\sqrt{8n-7}$ is an integer that divides $n(n-2)$ (since the square root of an integer is either an integer or is irrational).

In case (i), $n = 0$ (trivial) or $n = 2$ and $v = 9$.

In case (ii), set $t = \sqrt{8n-7}$ for some integer t so $8n = t^2+7$, and let $n(n-2) = kt$ for some integer k. Then

$$8n(8n-16) = 64kt,$$

so

$$(t^2+7)(t^2-9) = 64kt,$$

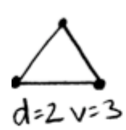
that is

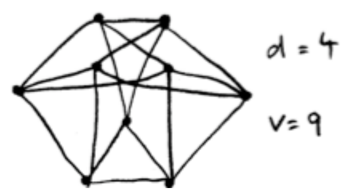
$$t^4 - 2t^2 - 64kt - 63 \quad = \quad 0.$$

Thus t divides 63 (the product of the roots of this equation), so we have the following possibilities:

$t = \sqrt{8n-7}$:	1	3	7	9	21	63
$n = (t^2+7)/8$:	1	2	7	11	56	487
$d = 2n$:	2	4	14	22	112	974
$v = 1 + 2n^2$:	3	9	99	243	6273	494019

Graphs of this form with $d = 2, 4, 22$ are known to exist. ∎





Theorem 3.3. *Let G be a connected simple graph with v vertices. Suppose that G contains no triangles, but that every pair of non-adjacent vertices have two common neighbours. Then every vertex has the same degree $d = n^2 + 1$ for some $n \in \mathbb{N}$, where n is not a multiple of 4, and $v = (n^4 + 3n^2 + 4)/2$.*

Proof. See tutorial sheet. ∎