University of St Andrews

HONOURS M. A. AND HONOURS M. Math. EXAMINATION

MATHEMATICS AND STATISTICS

Paper MT5823 Advanced Semigroup Theory

May 2008

Time allowed : Two hours

Attempt all THREE questions

1. Let S be the semigroup defined by the following multiplication table

$\cdot$	a	b	c	d	e	f	g
a	a	b	c	d	e	f	g
b	b	b	d	d	g	g	g
c	c	e	a	f	b	d	g
d	d	g	b	g	b	d	g
e	e	e	f	f	g	g	g
f	f	g	e	g	e	f	g
g	g	g	g	g	g	g	g

(a) Prove that the elements b and c generate S. Is S generated by a single element? Fully justify your answer. [3]

(b) Draw the left and right Cayley graphs of S with respect to the generating set $\{b, c\}$. [2]

(c) Determine Green's $\mathcal{L}$-, $\mathcal{R}$-, $\mathcal{H}$-, and $\mathcal{D}$-relations on S and draw the eggbox diagrams of the $\mathcal{D}$-classes. [4]

[See over

(d) Define what it means for a semigroup to be inverse. Prove that S is an inverse semigroup. [2]

(e) Let T be the semigroup defined by the presentation

$$\langle\, x, y \mid x^2 = x, xy^2 = x, y^2x = x, y^3 = y, (xy)^2 = xyx, (yx)^2 = xyx \,\rangle.$$

Find the elements of T. Prove that $S \cong T$. [3]

(f) Prove that S is isomorphic to the symmetric inverse semigroup on the set $\{1, 2\}$. [4]

2. (a) Give the definition of a completely simple semigroup and a Rees matrix semigroup. [2]

(b) Let $\mathcal{M}[G; I, \Lambda; P]$ be a Rees matrix semigroup. Prove that $(i, g, \lambda)\mathcal{R}(j, h, \mu)$ if and only if $i = j$. Prove that $(i, g, \lambda)\mathcal{L}(j, h, \mu)$ if and only if $\lambda = \mu$. Deduce that $(i, g, \lambda)\mathcal{H}(j, h, \mu)$ if and only if $i = j$ and $\lambda = \mu$. [3]

(c) Let S be an arbitrary semigroup and let ρ and σ be congruences on S. Prove that $\rho \cap \sigma$ is a congruence on S. [1]

(d) Prove that $\mathcal{R}$ and $\mathcal{L}$ are two-sided congruences on $\mathcal{M}[G; I, \Lambda; P]$. Deduce that $\mathcal{H}$ is a congruence on $\mathcal{M}[G; I, \Lambda; P]$. [2]

(e) Prove that $\mathcal{M}[G; I, \Lambda; P]/\mathcal{R}$ is a semigroup of left zeros. Prove that $\mathcal{M}[G; I, \Lambda; P]/\mathcal{H}$ is a rectangular band. [4]

(f) Give the definition of a Clifford semigroup. [1]

(g) Let S be a finite simple semigroup. Prove that the following are equivalent:

(i) S is an inverse semigroup;

(ii) S is a Clifford semigroup;

(iii) S is a group.

(Hint: use the Rees Theorem to represent S as a Rees matrix semigroup.) [3]

3. Let S be the semigroup generated by the mappings

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 4 & 1 & 5 \end{pmatrix} \;\&\; g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 3 & 5 \end{pmatrix}.$$

Then the elements of S are

$$f^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 5 \end{pmatrix}, f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 4 & 1 & 5 \end{pmatrix}, gf^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 5 & 1 & 5 \end{pmatrix},$$

$$gf = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 5 & 4 & 5 \end{pmatrix}, g^2f^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 5 & 5 & 5 \end{pmatrix}, f^2g^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 3 & 3 & 3 & 5 \end{pmatrix},$$

$$fg^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 3 & 5 & 3 & 5 \end{pmatrix}, g^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 5 & 5 & 5 \end{pmatrix}, g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 3 & 5 \end{pmatrix},$$

$$fg = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 4 & 3 & 4 & 5 \end{pmatrix}, f^2g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 4 & 4 & 4 & 5 \end{pmatrix}, fg^2f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 4 & 5 & 4 & 5 \end{pmatrix},$$

$$g^2f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 5 & 5 & 5 \end{pmatrix}, g^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 5 & 5 & 5 & 5 \end{pmatrix}$$

and the $\mathcal{D}$-classes of S are

$$D_f = \{f, gf, fg, g\}, D_{f^2} = \{f^2, gf^2, g^2f^2, f^2g, fg^2f, g^2f, f^2g^2, fg^2, g^2\},$$
$$\& \ D_{g^3} = \{g^3\}.$$

(You do **not** need to prove that the above are the elements and $\mathcal{D}$-classes of S.)

(a) Find the idempotent elements of S. [2]

(b) Give the definition of a regular semigroup. [1]

(c) Prove that S is a regular semigroup. [2]

Let T be an arbitrary semigroup. Recall that if $x \in T$, then $y \in T$ is an *inverse* for x if $xyx = x$ and $yxy = y$.

(d) Prove that $x \in T$ has an inverse if and only if $x \in T$ is regular. [3]

(e) Find inverses of the elements f, f^2, and g^3. [3]

(f) How many $\mathcal{L}$- and $\mathcal{R}$-classes do the $\mathcal{D}$-classes D_f, D_{f^2}, and D_{g^3} have? [3]

(g) Prove that S is not an inverse semigroup. [2]