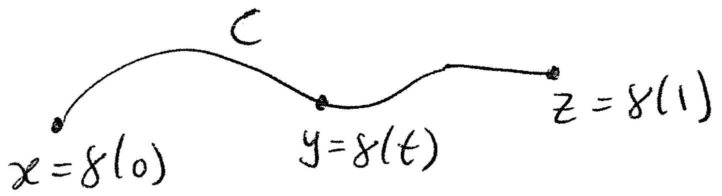
① Suppose $C \subseteq X$ is a geodesic between $x, z \in X$. Therefore $C = \gamma([0,1])$ for a continuous bijection $\gamma : [0,1] \to C$, $\gamma(0) = x$, $\gamma(1) = z$ and $\forall s, t \in [0,1]$ we have

$$d(\gamma(s), \gamma(t)) = |s-t|\, d(x,z).$$



Let $y \in C$ and therefore we may write $y = \gamma(t)$ for some $t \in [0,1]$. It follows that

$$\begin{aligned} d(x,y) + d(y,z) &= d(\gamma(0), \gamma(t)) + d(\gamma(t), \gamma(1)) \\ &= |0-t|\, d(x,z) + |t-1|\, d(x,z) \\ &= t\, d(x,z) + (1-t)\, d(x,z) \\ &= d(x,z) \end{aligned}$$

as required.

(2) Suppose (X, d) is a countable metric space & suppose $C \subseteq X$ is a geodesic. Therefore $C = \gamma([0,1])$ for some bijection $\gamma : [0,1] \to C$. Therefore

$$\mathrm{Card}([0,1]) = \mathrm{Card}(C) \leq \mathrm{Card}(X) = \aleph_0$$
$$= \mathrm{Card}(\mathbb{N})$$

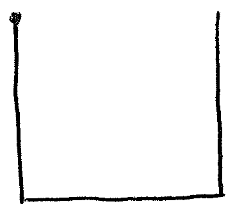
This is a contradiction!
(Here we write $\mathrm{Card}(A)$ for the cardinality of a set A. A set is countable if $\mathrm{Card}(A) = \mathrm{Card}(\mathbb{N}) = \aleph_0$, and we use the fact that $\mathrm{Card}([0,1]) = \aleph_1 = 2^{\aleph_0} > \aleph_0$.)

(3) Let X be a geodesic metric space and let $x, y \in X$ be arbitrary. By definition there exists a geodesic between x and y, ie a curve $C = \gamma([0,1])$ for some continuous function $\gamma : [0,1] \to X$ satisfying $\gamma(0) = x$ and $\gamma(1) = y$. Hence X is path connected.

(A geodesic also satisfies an additional property which we did not use).

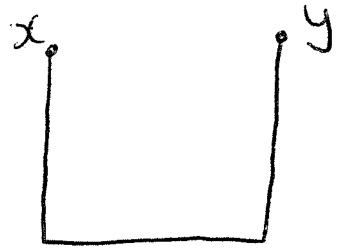
③ cont...

Example of a path connected space which is not geodesic:



← X is the union of 3 side of a 1×1 square, with the Euclidean metric.

- X is clearly path connected
- Suppose X is geodesic. Then there exists a geodesic $c = \gamma([0,1])$ between the points x, y; where $\gamma(0) = x$ & $\gamma(1) = y$.



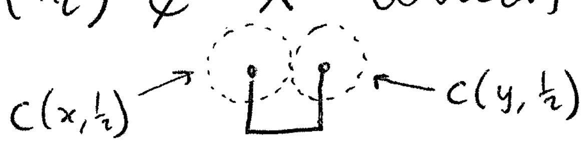
$d(x,y) = 1$ and therefore

$$d(\gamma(0), \gamma(\tfrac{1}{2})) = \tfrac{1}{2} \times d(x,y) = \tfrac{1}{2}$$

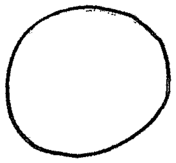
& $$d(\gamma(\tfrac{1}{2}), \gamma(1)) = |1 - \tfrac{1}{2}| \times d(x,y) = \tfrac{1}{2}$$

Therefore $\gamma(\tfrac{1}{2}) \in C(x, \tfrac{1}{2}) \cap C(y, \tfrac{1}{2})$

and so $\gamma(\tfrac{1}{2}) \notin X$ which is a contradiction.



4 (i) $[0,1]$ with the Euclidean metric is uniquely geodesic

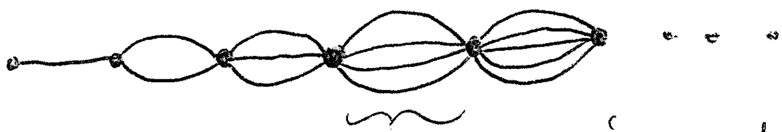
(ii)  S^1 with the Euclidean metric. Geodesics are unique unless the two points are antipodal (in which case there are 2 geodesics).

(iii) $\mathbb{R}^2$ with the "taxi-cab" metric:

$$d_\infty\big((x_1,x_2),(y_1,y_2)\big) = \max\{|x_1-y_1|, |x_2-y_2|\}$$

For any (x_1,x_2), (y_1,y_2) not lying on the same horizontal or vertical line there are multiple geodesics. So this space "almost" does the job. Does a genuine example exist? I'm not sure, but I hope you had fun thinking!

(iv) This infinite chain does the job:

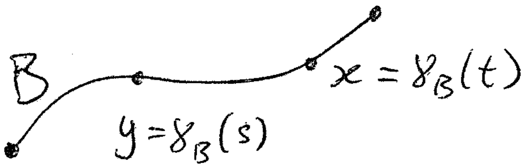


each segment 'normalised' to have length 1.

⑤ Let (X, d) be a metric space and $C \subseteq X$ be a geodesic.

($\Rightarrow$) Suppose $B \subseteq C$ and B is also a geodesic (in X). Then $B = \gamma_B([0,1])$ for a continuous bijection $\gamma_B : [0,1] \to B$. Hence B has the same cardinality as $[0,1]$ and so is not a single point.

It is also path connected: Let $x, y \in B$ which means we can write $x = \gamma_B(t)$ and $y = \gamma_B(s)$ for some $s, t \in [0,1]$



we want to find a path from y to x.

Let $\gamma_{path} : [0,1] \to X$ be defined by

$$\gamma_{path}(u) = \gamma_B(s(1-u) + ut).$$

γ_{path} is continuous since the composition of continuous functions is continuous and

$$\gamma_{path}(0) = \gamma_B(s) = y, \quad \gamma_{path}(1) = \gamma_B(t) = x$$

which proves B is path connected.

(5) cont...

It remains to show that B is closed.
Since $[0,1]$ is compact (closed and bounded)
$B = \gamma_B([0,1])$ is compact since γ_B is
continuous. In particular it is closed.

($\Leftarrow$) Suppose $B \subseteq C$ is closed, path connected, and not equal to a single point.

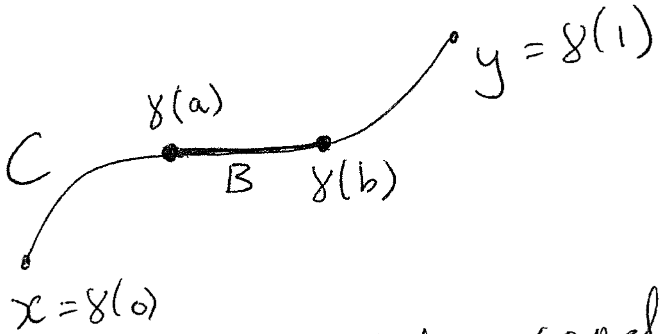
Let $I = \gamma^{-1}(B) = \{ t \in [0,1] : \gamma(t) \in B \}$

Since γ is a continuous bijection and B is closed we can deduce that I is closed.

Claim: I is an interval $[a,b]$ $(a < b)$

Proof of claim: suppose not. This means I can find $c, d \in I$ such that the open interval (c,d) does not intersect I. But then there cannot be a path between $\gamma(c), \gamma(d) \in B$ (within B) which is a contradiction.

(5) cont...



We are now in the situation shown in this picture.

We conclude by showing that B is a geodesic between $\gamma(a)$ and $\gamma(b)$.

Let $\gamma_{geo} : [0,1] \to B$ be defined by

$$\gamma_{geo}(u) = \gamma(ub + (1-u)a)$$

clearly

$$\gamma_{geo}(0) = \gamma(a)$$
$$\gamma_{geo}(1) = \gamma(b)$$
$$\gamma_{geo}([0,1]) = B.$$

Moreover, for any $v, w \in [0,1]$ we have

$$d(\gamma_{geo}(v), \gamma_{geo}(w)) = d(\gamma(vb+(1-v)a), \gamma(wb+(1-w)a))$$

$$\overset{\text{since } C \text{ is a geodesic}}{=} |vb + (1-v)a - wb - (1-w)a| \, d(x,y)$$

$$= |v-w| \, |a-b| \, d(x,y)$$

$$= |v-w| \, d(\gamma(a), \gamma(b)) \quad \text{as required!}$$

(6) Suppose $C \subseteq X$ is a geodesic and $\phi : X \to X$ is an isometry. Then $C = \gamma([0,1])$ for a continuous bijection $\gamma : [0,1] \to C$.

Define $\gamma_\phi : [0,1] \to \phi(C)$ by

$$\gamma_\phi(t) = \phi(\gamma(t)).$$

clearly $\gamma_\phi([0,1]) = \phi(C)$ and γ_ϕ is a continuous bijection. (Here we use that ϕ is a continuous bijection).

Also, for any $s,t \in [0,1]$ we have

$$d(\gamma_\phi(s), \gamma_\phi(t)) = d(\phi(\gamma(s)), \phi(\gamma(t)))$$

$$= d(\gamma(s), \gamma(t)) = |s-t|\, d(\gamma(0), \gamma(1))$$

since ϕ is an isometry

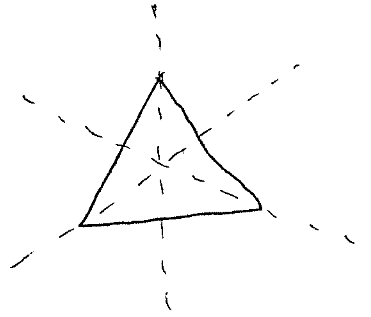
since C is a geodesic

$$= |s-t|\, d(\phi(\gamma(0)), \phi(\gamma(1)))$$

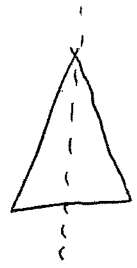
$$= |s-t|\, d(\gamma_\phi(0), \gamma_\phi(1))$$

as required.

(7) (i) $\mathrm{Isom}(\triangle) \cong D_3$ the dihedral group which consists of the identity, rotation by $\frac{2\pi}{3}$, rotation by $\frac{4\pi}{3}$ and three reflections given by



(ii) $\mathrm{Isom}(\triangle) \cong C_2$ the cyclic group of order 2. Geometrically the elements are the identity and one reflection:



(iii) $\mathrm{Isom}(\triangle) \cong \{Id\}$ the trivial group!

$\mathrm{Isom}(\triangle) \leq \mathrm{Isom}(\triangle) \leq \mathrm{Isom}(\triangle)$

(8) arbitrary rotations around a point $z_0 \in \mathbb{C}$ can be written as a composition of

$$z \mapsto z - z_0$$
$$z \mapsto e^{i\theta} z \quad \text{(rotation around 0)}$$
$$z \mapsto z + z_0$$

arbitrary reflections in a line $L = \{ z_0 + r z_1 : r \in \mathbb{R} \}$ are given by

$$\left.\begin{array}{l} z \mapsto z - z_0 \\ z \mapsto e^{-\arg(z_1) i} z \end{array}\right\} \text{these map } L \text{ to } \mathbb{R}$$
$$z \mapsto \bar{z} \quad \text{(reflection in } \mathbb{R}\text{)}$$
$$\left.\begin{array}{l} z \mapsto e^{\arg(z_1) i} z \\ z \mapsto z + z_0 \end{array}\right\} \text{map } \mathbb{R} \text{ back to } L$$

It remains to prove that (i) translations (ii) rotations around 0 & (iii) conjugation are isometries of $\mathbb{C}$.

(i) translations: Let $\phi : \mathbb{C} \to \mathbb{C}$ be given by $\phi(z) = z + t$ for some $t \in \mathbb{C}$. Let $u, v \in \mathbb{C}$ be arbitrary.

$$d(\phi(u), \phi(v)) = |\phi(u) - \phi(v)| = |u + t - v - t| = |u - v| = d(u, v)$$

(8) cont...

(ii) rotations: Let $\phi : \mathbb{C} \to \mathbb{C}$ be given by

$$\phi(z) = e^{i\theta} z \quad \text{for some } \theta \in [0, 2\pi).$$

Again, let $u, v \in \mathbb{C}$ be arbitrary.

$$
\begin{aligned}
d(\phi(u), \phi(v)) &= |\phi(u) - \phi(v)| \\
&= |e^{i\theta} u - e^{i\theta} v| \\
&= |e^{i\theta}(u - v)| \\
&= |e^{i\theta}| \, |u - v| \\
&= d(u, v)
\end{aligned}
$$

(iii) Let $\phi : \mathbb{C} \to \mathbb{C}$ be given by $\phi(z) = \bar{z}$. Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$ be given, $(u_1, u_2, v_1, v_2 \in \mathbb{R})$

$$
\begin{aligned}
d(\phi(u), \phi(v)) &= |(u_1 - iu_2) - (v_1 - iv_2)| \\
&= |(u_1 - v_1) + i(v_2 - u_2)| \\
&= \sqrt{(u_1 - v_1)^2 + (v_2 - u_2)^2} \\
&= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2} \\
&= |u - v|
\end{aligned}
$$