MT5823 Semigroup theory: Solutions 2 (James D. Mitchell)
Rectangular bands, cancellative semigroups, subsemigroups, monogenic semigroups, and idempotents

Rectangular bands

2-1. Let $(i, \lambda) \in S$ be arbitrary. Then $(i, \lambda)^2 = (i, \lambda)(i, \lambda) = (i, \lambda)$ and so (i, λ) is an idempotent.

Let $(i, \lambda), (j, \mu), (k, \nu) \in S$. Then $[(i, \lambda)(j, \mu)](k, \nu) = (i, \mu)(k, \nu) = (i, \nu) = (i, \lambda)(k, \nu)$. □

2-2. Assume that $(i, \lambda) \in S$ is a left zero. Then for all $(j, \mu) \in S$ we have $(i, \lambda) = (i, \lambda)(j, \mu) = (i, \mu)$. It follows that $\lambda = \mu$ and since (j, μ) was arbitrary $|\Lambda| = 1$.

To prove the additional statement let (i, λ) and $(j, \lambda) \in S$. Then $(j, \lambda)(i, \lambda) = (j, \lambda)$ and so (j, λ) is a left zero. □

Cancellative semigroups

2-3. Since $ea = a$, it follows that $e(ea) = ea$ and cancelling a, we obtain $e^2 = e$ and e is an idempotent.

Let $b \in S$ be arbitrary. Then since e is an idempotent $e(eb) = eb$ and so cancelling e, $eb = b$, and e is a left identity. On the other hand, $(be)e = be$ and so $be = b$. Therefore e is the identity of S. □

2-4. The free semigroup A^+ is cancellative but has no identity. □

Subsemigroups

2-5. Applying the algorithm from lectures:

$$t_1 = x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & - & - \end{pmatrix} \ \textbf{(new)} \qquad t_2 = y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 4 & 5 & 4 \end{pmatrix} \ \textbf{(new)}$$

$$t_1x = x^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & - & - \end{pmatrix} = t_3 \ \textbf{(new)} \qquad t_1y = xy = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 4 & - & - \end{pmatrix} = t_4 \ \textbf{(new)}$$

$$t_2x = yx = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ - & - & - & - & - \end{pmatrix} = t_5 \ \textbf{(new)} \qquad t_2y = y^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 5 & 4 & 5 \end{pmatrix} = t_6 \ \textbf{(new)}$$

$$t_3x = x^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & - & - \end{pmatrix} = t_7 \ \textbf{(new)} \qquad t_3y = x^2y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 4 & 5 & - & - \end{pmatrix} = t_8 \ \textbf{(new)}$$

$$t_4x = xyx = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ - & - & - & - & - \end{pmatrix} = t_5 \ \textbf{(old)} \qquad t_4y = xy^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 5 & - & - \end{pmatrix} = t_9 \ \textbf{(new)}$$

$$t_5x = yx^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ - & - & - & - & - \end{pmatrix} = t_5 \ \textbf{(old)} \qquad t_5y = yxy = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ - & - & - & - & - \end{pmatrix} = t_5 \ \textbf{(old)}$$

$$t_6x = y^2x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ - & - & - & - & - \end{pmatrix} = t_5 \ \textbf{(old)} \qquad t_6y = y^3 = y = t_2 \ \textbf{(old)}$$

$$t_7x = x^4 = x = t_1 \ \textbf{(old)} \qquad t_7y = x^3y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 4 & - & - \end{pmatrix} = t_{10} \ \textbf{(new)}$$

$$t_8x = x^2yx = t_5 \ \textbf{(old)} \qquad t_8y = x^2y^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 5 & 4 & - & - \end{pmatrix} = t_{11} \ \textbf{(new)}$$

$$t_9x = xy^2x = t_5 \ \textbf{(old)} \qquad t_9y = xy^3 = xy = t_4 \ \textbf{(old)}$$

$$t_{10}x = x^3yx = t_5 \ \textbf{(old)} \qquad t_{10}y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 5 & - & - \end{pmatrix} = t_{12} \ \textbf{(new)}$$

$$t_{11}x = x^2y^2x = t_5 \ \textbf{(old)} \qquad t_{11}y = x^2y^3 = x^2y = t_8 \ \textbf{(old)}$$

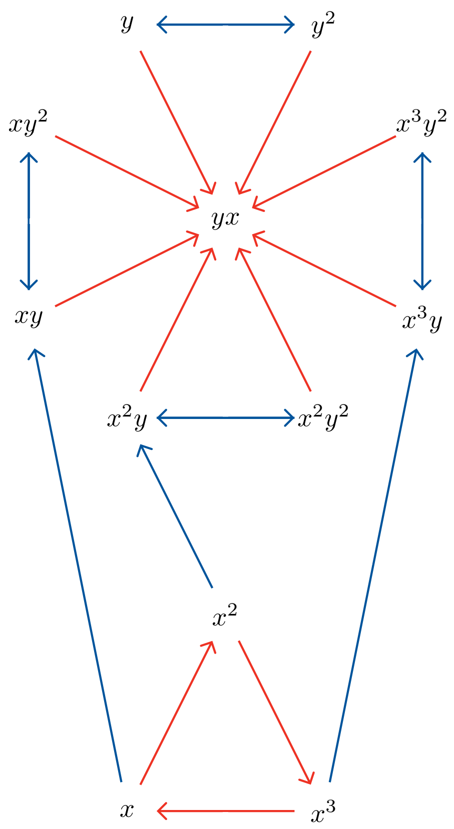

Figure 1: The right Cayley graph of the semigroup in Problem **2-5** (red is for x and blue is for y, loops are omitted).

$$t_{12}x = x^3y^2x = t_5 \ \ (\textbf{old}) \qquad\qquad t_{12}y = x^3y^3 = x^3y = t_{10} \ \ (\textbf{old}).$$

The semigroup S has 12 elements:

$$x, y, x^2, xy, yx, y^2, x^3, x^2y, xy^2, x^3y, x^2y^2, x^3y^2.$$

The right Cayley graph of S can be seen in Figure 1. □

2-6. Let S be a semigroup of right zeros and let $T \subseteq S$. Then $xy = y$ for all $x, y \in T$ and so T is a subsemigroup. Adjoining an identity or a zero to S gives another semigroup with the same property.

Let $U = \{1, 2, \ldots, n\}$ and define multiplication by

$$x \cdot y = \min(x, y).$$

Then U has the same property as S.

A subset X of a zero semigroup S is a subsemigroup if and only if $0 \in X$. There are 2^{n-1} such subsets of an n-element set. □

2-7. Let S be any subsemigroup of a finite group G. It suffices to show that if $x \in S$, then $x^{-1} \in S$. Since G is finite, there exists $n \in \mathbb{N}$ such that $x^n = 1_G$ where 1_G is the identity of G. Hence $x^{-1} = x^{n-1} \in S$.

If $G = \mathbb{Z}$ under addition, then $\mathbb{N}$ is a subsemigroup of $\mathbb{Z}$ that is not a subgroup. □

Monogenic semigroups and idempotents

2-8. The fact that S is finite implies that not all the powers of a are distinct. Hence $a^m = a^n$ for some $m \leq n$. It follows that $n = m + r$ for some r and so $a^{m+r} = a^m$.

To prove that some power of a is an idempotent, let $i \in \mathbb{N}$, $i > 0$, be any number such that $ir \geq m$. Then $2ir \geq ir + m \geq m + r$ and so repeatedly applying the equality $a^{m+r} = a^m$ a total of i times, we obtain

$$(a^{ir})^2 = a^{2ir} = a^{2ir-r} = \cdots = a^{2ir-ir} = a^{ir}$$

and so a^{ir} is an idempotent. □

2-9. Every finite semigroup contains at least one element a. The monogenic (sub)semigroup generated by a contains an idempotent by Problem **2-8**. $\square$

2-10. Every group has exactly one idempotent. The semigroup of non-zero natural numbers $\mathbb{N} \setminus \{0\}$ under addition has no idempotents. $\square$

Further problems

2-11. Showing that $\mathbb{N} \times \mathbb{N}$ is a semigroup is trivial.

Suppose that $X \subseteq \mathbb{N} \times \mathbb{N}$ is a generating set for $\mathbb{N} \times \mathbb{N}$. If $(1, m) \notin X$ for some $m \in \mathbb{N}$, then there exits $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n) \in X$, for some $n \in \mathbb{N}$, $n > 1$, such that

$$(1, m) = (x_1, y_1) + (x_2, y_2) + \cdots + (x_n, y_n)$$

and so $1 = x_1 + x_2 + \cdots + x_n$, which is a contradiction. Therefore $(1, m) \in X$ for all $m \in \mathbb{N}$ and so X is infinite. We have shown that $\mathbb{N} \times \mathbb{N}$ is not finitely generated.

2-12. Suppose that $\max\{|I|, |\Lambda|\} = |I|$ and let J be any subset of I such that $|J| = |\Lambda|$. Let $\phi : J \longrightarrow \Lambda$ be any bijection, and let $\lambda_0 \in \Lambda$ be arbitrary. Then we will show that

$$X = \{\ (j, (j)\phi)\ :\ j \in J\ \} \cup \{\ (i, \lambda_0)\ :\ i \in I \setminus J\ \}$$

is a generating set for $I \times \Lambda$.

Let $(i, \lambda) \in I \times \Lambda$ be arbitrary. Since ϕ is a bijection, there exists $j \in J$ such that $(j)\phi = \lambda$. If $i \in J$, then

$$(i, \lambda) = (i, (i)\phi)(j, (j)\phi) \in \langle\ X\ \rangle.$$

If $i \in I \setminus J$, then

$$(i, \lambda) = (i, \lambda_0)(j, (j)\phi) \in \langle\ X\ \rangle.$$

Since,

$$|X| = |J| + |I \setminus J| = |I| = \max\{|I|, |\Lambda|\}$$

the result follows. $\square$

2-13. It suffices to prove that S is 1-generated as every infinite 1-generated semigroup is isomorphic to the natural numbers without zero under addition. Seeking a contradiction assume that S is not 1-generated. Let $s_0 \in S$ be arbitrary. Then $\langle\ s_0\ \rangle \neq S$ and so there exists $u \in S$ such that $u \notin \langle\ s_0\ \rangle$. But every countable subset of S is contained in a monogenic semigroup and so there exists $s_1 \in S$ such that $\langle\ s_0\ \rangle \lneq \langle\ s_0, u\ \rangle \leq \langle\ s_1\ \rangle$. Continuing in this way there exist $s_0, s_1, \ldots \in S$ such that

$$\langle\ s_0\ \rangle \lneq \langle\ s_1\ \rangle \lneq \cdots.$$

Since every countable subset of S is contained in a monogenic subsemigroup, there exists $t \in S$ such that $s_0, s_1, \ldots \in \langle\ t\ \rangle$. In particular, for all $i \in \mathbb{N}$ there exists $m_i > 0$ such that $t^{m_i} = s_i$. Hence for all $i \in \mathbb{N}$

$$\{\ t^{qm_i}\ :\ q \geq 1\ \} = \langle\ s_i\ \rangle \lneq \langle\ s_{i+1}\ \rangle = \{\ t^{qm_{i+1}}\ :\ q \geq 1\ \}.$$

Thus for all $i \in \mathbb{N}$ we have that m_{i+1} divides m_i and $m_{i+1} \neq m_i$. It follows that $m_0 > m_1 > \cdots$, a contradiction. $\square$