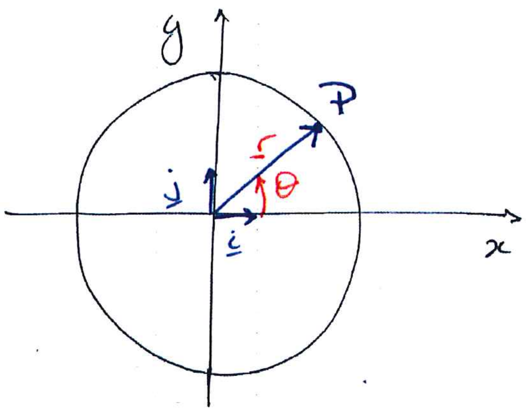
Chapter 2

Coordinate Systems

For many physical applications and problems, the optimal coordinate system will depend on your requirements. So far, we have only considered the Cartesian coordinate system. In this Chapter, we will take a brief look at other coordinate systems.

2.1 2D polar coordinates

Consider a point P=(x,y) in Cartesian coordinates:



$\underline{r} = x\underline{i} + y\underline{j}$ in Cartesian coordinates

From the graph, we see that we can specify the point P by r and θ, where $r = |\mathbf{r}|$ is the distance from the origin to the point P and θ is the polar angle, i.e. the angle between $\mathbf{i}$ (x-axis) and $\mathbf{r}$.

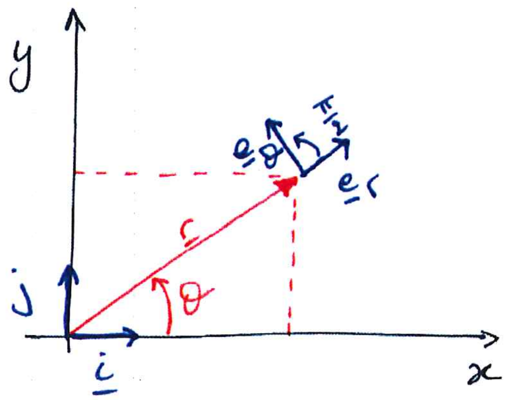
Using geometry of a triangle, we have that

$$r = \sqrt{x^2 + y^2} \qquad \text{and} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right) .$$

x and y can be recovered from r and θ by

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

We now want to associate a basis of orthogonal vectors to this system of coordinates given a position vector $\mathbf{r}$.



We take

$$\mathbf{e}_r = \frac{\mathbf{r}}{|\mathbf{r}|},$$

i.e. a vector in the same direction as $\mathbf{r}$ but with length 1.

We now want $\mathbf{e}_\theta$ normal (perpendicular) to $\mathbf{e}_r$.

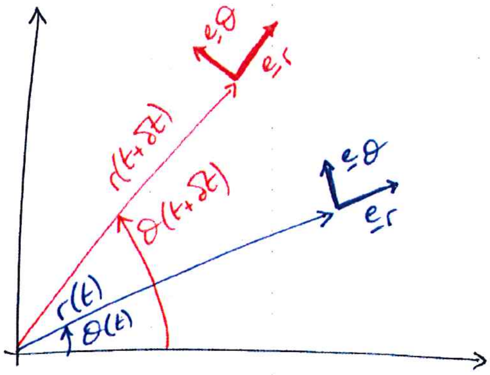
Define

$$\mathbf{e}_r = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad \text{and} \quad \mathbf{e}_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}.$$

Then $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are perpendicular (check $\mathbf{e}_r \cdot \mathbf{e}_\theta = 0$!) and lie in the radial and transverse direction, respectively.

Differentiation of a position vector

What happens when the position vector $\mathbf{r}$ changes with time?



Position vector

$\underline{r} = r(t)\underline{e}_r$

As we see from the figure, the local reference frame defined by $(\mathbf{e}_r, \mathbf{e}_\theta)$ will also change with time. [Note that the basis vectors $\mathbf{i}$ and $\mathbf{j}$ do not change direction with time!]

So

$$\frac{d}{dt}(\mathbf{r}) = \frac{d}{dt}(r\mathbf{e}_r) = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\mathbf{e}_r}{dt}\,.$$

Hence, we need to calculate $\dfrac{d}{dt}(\mathbf{e}_r)$.

The simplest way to do this is to express $\mathbf{e}_r$ in terms of $\mathbf{i}$ and $\mathbf{j}$:

$$\mathbf{e}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}\,,$$

or, more precisely, as θ changes with time:

$$\mathbf{e}_r(t) = \cos\theta(t)\mathbf{i} + \sin\theta(t)\mathbf{j}\,.$$

Then we have that

$$\frac{d\mathbf{e}_r}{dt} = \begin{pmatrix} \dfrac{d}{dt}(\cos\theta(t)) \\ \dfrac{d}{dt}(\sin\theta(t)) \end{pmatrix} = \begin{pmatrix} -\sin\theta\dfrac{d\theta}{dt} \\ \cos\theta\dfrac{d\theta}{dt} \end{pmatrix} = \frac{d\theta}{dt}\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

But

$$\mathbf{e}_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \Rightarrow \frac{d\mathbf{e}_r}{dt} = \frac{d\theta}{dt}\mathbf{e}_\theta$$

Note that $\dfrac{d\mathbf{e}_r}{dt}$ is perpendicular to $\mathbf{e}_r$

Aside: This is in fact true for any arbitrary unit vector.

Proof:

Consider $\mathbf{x}$ *to be a unit vector. That means that* $\mathbf{x} \cdot \mathbf{x} = 1$ *and* $\frac{d}{dt}(\mathbf{x} \cdot \mathbf{x}) = 0 \Rightarrow 2\mathbf{x} \cdot \frac{d\mathbf{x}}{dt} = 0]$.

So we now have $\frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\mathbf{e}_r}{dt}$

$$\Rightarrow \frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta \,.$$

Let us now move on to the second derivative:

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d}{dt}\left(r\frac{d\theta}{dt}\mathbf{e}_\theta + \frac{dr}{dt}\mathbf{e}_r\right) = \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{e}_\theta + r\frac{d^2\theta}{dt^2}\mathbf{e}_\theta + r\frac{d\theta}{dt}\frac{d}{dt}(\mathbf{e}_\theta) + \frac{d^2r}{dt^2}\mathbf{e}_r + \frac{dr}{dt}\frac{d}{dt}(\mathbf{e}_r).$$

We know that $\frac{d}{dt}(\mathbf{e}_r) = \frac{d\theta}{dt}\mathbf{e}_\theta$

What is $\frac{d}{dt}(\mathbf{e}_\theta)$?

$$\mathbf{e}_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

so

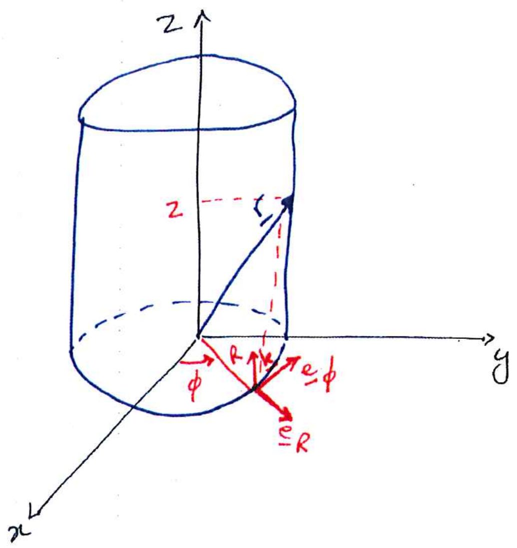
$$\frac{d}{dt}(\mathbf{e}_\theta) = \begin{pmatrix} -\frac{d}{dt}(\sin\theta) \\ \frac{d}{dt}(\cos\theta) \end{pmatrix} = \begin{pmatrix} -\cos\theta\frac{d\theta}{dt} \\ -\sin\theta\frac{d\theta}{dt} \end{pmatrix} = -\frac{d\theta}{dt}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

So we have $\frac{d\mathbf{e}_\theta}{dt} = -\frac{d\theta}{dt}\mathbf{e}_r$.

So

$$\frac{d^2\mathbf{r}}{dt^2} = \left(\frac{d^2r}{dt} - r\left(\frac{d\theta}{dt}\right)^2\right)\mathbf{e}_r + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\mathbf{e}_\theta$$

2.2 Cylindrical Polar Coordinates



Recall that cylindrical polar coordinates work as the 2D polar in the horizontal directions (R, ϕ) and as cartesian in the vertical (z):

$$\mathbf{r}(t) = R\mathbf{e}_R + z\mathbf{k}$$

Note $\begin{cases} \mathbf{e}_R \text{ not fixed, moves with the point} \\ \mathbf{k} \text{ fixed vector, time independent} \end{cases}$

$$\frac{d\mathbf{r}}{dt} = \frac{d}{dt}(R\mathbf{e}_R) + \frac{d}{dt}(z\mathbf{k}) = \frac{dR}{dt}\mathbf{e}_R + R\frac{d\phi}{dt}\mathbf{e}_\phi + \frac{dz}{dt}\mathbf{k}$$

$$\frac{d^2\mathbf{r}}{dt^2} = \left(\frac{d^2R}{dt^2} - R\left(\frac{d\phi}{dt}\right)^2\right)\mathbf{e}_R + \left(R\frac{d^2\phi}{dt^2} + 2\frac{dR}{dt}\frac{d\phi}{dt}\right)\mathbf{e}_\phi + \frac{d^2z}{dt^2}\mathbf{k}$$

2.3 Examples

(i) Suppose that a particle is moving in space along the trajectory

$$\mathbf{r} = \begin{pmatrix} a\cos(2t) \\ a\sin(2t) \\ bt^2 \end{pmatrix} \quad \text{where } a, b \text{ are 2 constants .}$$

We want to calculate $\frac{d\mathbf{r}}{dt}$ and $\frac{d^2\mathbf{r}}{dt^2}$ in cartesian and cylindrical coordinates.

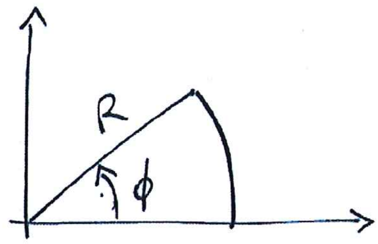
<u>Cartesian</u>

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \begin{pmatrix} -2a\sin(2t) \\ 2a\cos(2t) \\ 2bt \end{pmatrix}$$

$$\mathrm{a} = \frac{d^2\mathbf{r}}{dt^2} = \begin{pmatrix} -4a\cos(2t) \\ -4a\sin(2t) \\ 2b \end{pmatrix}$$

Cylindrical

Top view:



We have

$$R(t) = \sqrt{x^2+y^2} = a \quad \text{(a constant)}$$

and

$$\phi(t) = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{a\sin(2t)}{a\cos(2t)}\right) = \tan^{-1}(\tan(2t)) = 2t\,.$$

$$\Rightarrow \frac{d\phi}{dt} = 2, \quad \frac{d^2\phi}{dt^2} = 0 \text{ and } \frac{dR}{dt} = \frac{da}{dt} = 0$$

So

$$\begin{aligned} \mathbf{r} &= a\mathbf{e}_R + bt^2\mathbf{k} \\ \frac{d\mathbf{r}}{dt} &= 2a\mathbf{e}_\phi + 2bt\mathbf{k} \\ \frac{d^2\mathbf{r}}{dt^2} &= -4a\mathbf{e}_R + 2b\mathbf{k} \end{aligned}$$

(ii) Given $\mathbf{r} = \sin t\mathbf{i} + \cos t\mathbf{j}$, calculate in polar coordinates $\frac{d\mathbf{r}}{dt}$ and $\frac{d^2\mathbf{r}}{dt^2}$.

$$R(t) = \sqrt{\sin^2 t + \cos^2 t} = 1$$

$$\phi(t) = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{\cos t}{\sin t}\right) = \tan^{-1}\left(\frac{\sin(\pi/2-t)}{\cos(\pi/2-t)}\right) = \tan^{-1}\left(\tan\left(\pi/2-t\right)\right)$$

$$\Rightarrow \phi(t) = \pi/2 - t \qquad \text{and} \qquad \frac{d\phi}{dt} = -1$$

As $\mathbf{r} = \mathbf{e}_r$,

$$\frac{d\mathbf{r}}{dt} = \frac{d\phi}{dt}\mathbf{e}_\phi = -\mathbf{e}_\phi$$

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{d\mathbf{e}_\phi}{dt} = -\left(-\frac{d\phi}{dt}\right)\mathbf{e}_r = -\mathbf{e}_r$$

(iii) Given $\mathbf{r}(t) = e^{-t}\cos t\mathbf{i} + e^{-t}\sin t\mathbf{j}$, calculate in polar coordinates $\frac{d\mathbf{r}}{dt}$.

In cylindrical polars

$$R(t) = \sqrt{e^{-2t}\cos^2 t + e^{-2t}\sin^2 t} = e^{-t}$$

$$\phi(t) = \arctan\left(\frac{e^{-t}\sin t}{e^{-t}\cos t}\right) = \arctan(\tan t) = t$$

$$\Rightarrow \frac{d\phi}{dt} = 1$$

As $\mathbf{r} = R\mathbf{e}_R = e^{-t}\mathbf{e}_R$,

$$\frac{d\mathbf{r}}{dt} = -e^{-t}\mathbf{e}_R + e^{-t}\frac{d\phi}{dt}\mathbf{e}_\phi = -e^{-t}\mathbf{e}_R + \mathbf{e}^{-t}\mathbf{e}_\phi$$

2.4 Note on the relation between the unit vectors of the basis

We have seen that, for the cartesian basis

$$\mathbf{i}\times\mathbf{j} = \mathbf{k},\quad \mathbf{j}\times\mathbf{k} = \mathbf{i},\quad \mathbf{k}\times\mathbf{i} = \mathbf{j}.$$

We can also show that for cylindrical polar coordinates,

$$\mathbf{e}_R\times\mathbf{e}_\phi = \mathbf{k},\quad \mathbf{e}_\phi\times\mathbf{k} = \mathbf{e}_R,\quad \mathbf{k}\times\mathbf{e}_R = \mathbf{e}_\phi$$

Proof:

$$\mathbf{e}_R = \begin{pmatrix}\cos\phi\\ \sin\phi\\ 0\end{pmatrix},\quad \mathbf{e}_\phi = \begin{pmatrix}-\sin\phi\\ \cos\phi\\ 0\end{pmatrix},\quad \mathbf{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

So

$$\mathbf{e}_R\times\mathbf{e}_\phi = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\end{vmatrix} = \begin{pmatrix}0-0\\ 0-0\\ \cos^2\phi+\sin^2\phi\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix} = \mathbf{k}$$

$$\mathbf{e}_\phi\times\mathbf{k} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{vmatrix} = \begin{pmatrix}\cos\phi\\ \sin\phi\\ 0\end{pmatrix} = \mathbf{e}_R$$

$$\mathbf{k}\times\mathbf{e}_R = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 0 & 1\\ \cos\phi & \sin\phi & 0\end{vmatrix} = \begin{pmatrix}0-\sin\phi\\ \cos\phi-0\\ 0\end{pmatrix} = \mathbf{e}_\phi$$

□

2.5 Spherical Coordinates

In spherical coordinates, the position of a point can be represented by (r, θ, ϕ), where:

$r =$ the length (norm) of the vector,
$\theta =$ the angle between the z-axis and the vector $(0 \le \theta \le \pi)$,
$\phi =$ the angle between the projection of the vector onto the xy-plane and the x-axis $(0 \le \phi < 2\pi)$.

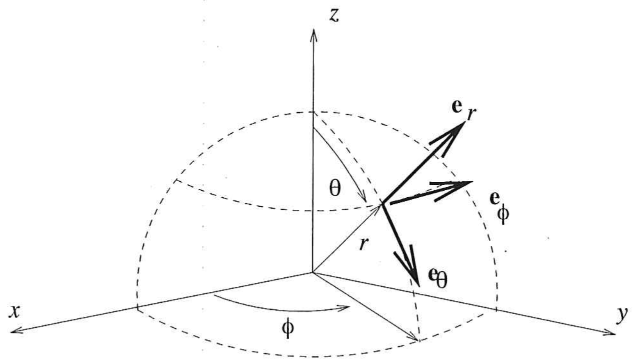


Figure 1.1: Spherical coordinates, (r, θ, ϕ).

We have,

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta\,. \end{aligned}$$

The unit vectors in spherical coordinates $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$ can be expressed in terms of the (fixed) Cartesian basis vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ through

$$\begin{pmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \\ \mathbf{e}_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix}.$$

Let us now again consider the position vector $\mathbf{r} = r\mathbf{e}_r$.

Then

$$\frac{d}{dt}(\mathbf{r}) = \frac{d}{dt}(r\mathbf{e}_r) = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\mathbf{e}_r}{dt}\,.$$

Hence, we again need to calculate $\dfrac{d\mathbf{e}_r}{dt}$, where

$$\mathbf{e}_r = \sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k}\,.$$

$$\Rightarrow \frac{d\mathbf{e}_r}{dt} = \begin{pmatrix} \cos\theta\cos\phi\frac{d\theta}{dt} - \sin\theta\sin\phi\frac{d\phi}{dt} \\ \cos\theta\sin\phi\frac{d\theta}{dt} + \sin\theta\cos\phi\frac{d\phi}{dt} \\ -\sin\theta\frac{d\theta}{dt} \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix}\frac{d\theta}{dt} + \begin{pmatrix} -\sin\theta\sin\phi \\ \sin\theta\cos\phi \\ 0 \end{pmatrix}\frac{d\phi}{dt}$$

$$\Rightarrow \frac{d\mathbf{e}_r}{dt} = \frac{d\theta}{dt}\mathbf{e}_\theta + \frac{d\phi}{dt}\sin\theta\mathbf{e}_\phi\,.$$

Hence, we have

$$\frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\mathbf{e}_r}{dt} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta + r\frac{d\phi}{dt}\sin\theta\mathbf{e}_\phi\,.$$

You can similarly work out that

$$\frac{d\mathbf{e}_\theta}{dt} = -\frac{d\theta}{dt}\mathbf{e}_r + \cos\theta\frac{d\phi}{dt}\mathbf{e}_\phi\,.$$

and

$$\frac{d\mathbf{e}_\phi}{dt} = -\sin\theta\frac{d\phi}{dt}\mathbf{e}_r - \cos\theta\frac{d\phi}{dt}\mathbf{e}_\theta\,.$$

Overview Coordinate Systems

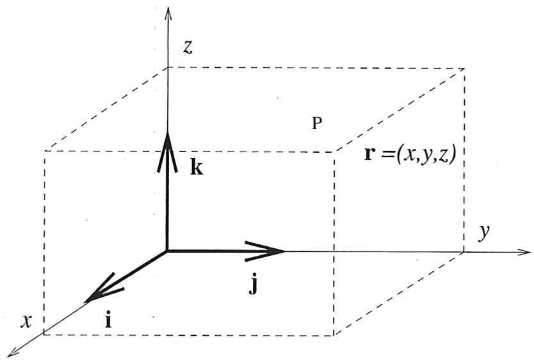


Figure 1.2: Illustration of the Cartesian Coordinates

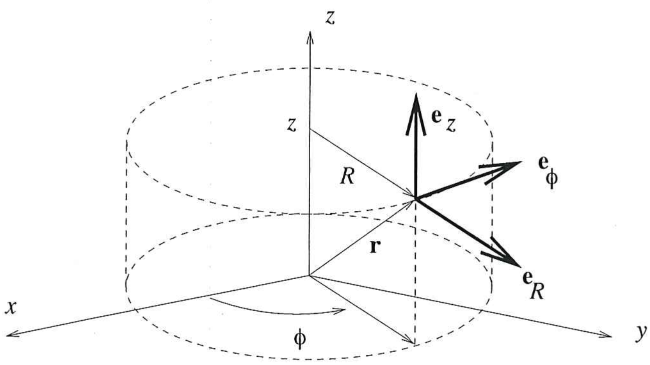


Figure 1.3: Illustration of the Cylindrical Polar Coordinates

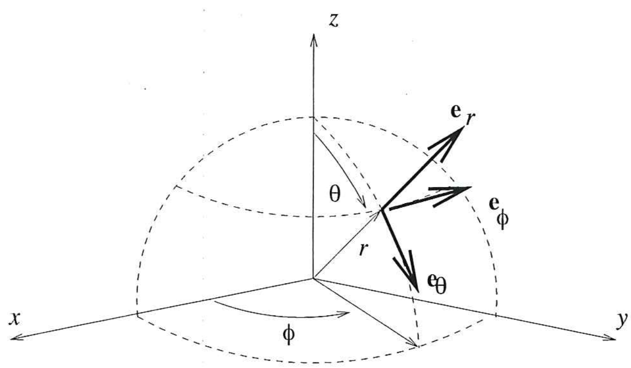


Figure 1.4: Illustration of the Spherical Coordinates