MT5823 Semigroup theory: Problem sheet 7 (James D. Mitchell)
More Green's relations, simple semigroups, Rees matrix semigroups

More Green's relations

7-1. Let
$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 6 & 1 & 4 & 3 & 4 & 6 & 3 \end{pmatrix} \in T_8.$$

(a) Find idempotents $a, b \in T_8$ such that $f \mathscr{R} a$ and $f \mathscr{L} b$. How many different choices for a and b are there?

(b) Find an inverse $f' \in T_8$ of f such that $ff' = a$ and $f'f = b$. How many inverses does f have?

7-2. Let D_r be the $\mathscr{D}$-class of the full transformation semigroup T_n ($n \geq r$) consisting of all the mappings with rank r. How many $\mathscr{L}$-classes, $\mathscr{R}$-classes and $\mathscr{H}$-classes does D_r contain? What is the size of D_r?

7-3. Prove that the $\mathscr{H}$-class H_f of a mapping $f \in T_n$ is a group if and only if $\operatorname{rank}(f) = \operatorname{rank}(f^2)$.

7-4. (a) Prove that the monoid of all partial transformations P_n is regular.

(b) Prove that there is a monomorphism from P_n to T_{n+1}. (Hint: replace '$-$' by $n+1$.)

(c) Prove that the following hold in P_n:

$$\begin{array}{lll} f \mathscr{L} g & \text{if and only if} & \operatorname{im}(f) = \operatorname{im}(g) \\ f \mathscr{R} g & \text{if and only if} & \ker(f) = \ker(g) \\ f \mathscr{H} g & \text{if and only if} & \operatorname{im}(f) = \operatorname{im}(g) \ \text{ and } \ \ker(f) = \ker(g) \\ f \mathscr{D} g & \text{if and only if} & \operatorname{rank}(f) = \operatorname{rank}(g) \\ \mathscr{J} = \mathscr{D}. & & \end{array}$$

7-5. Let S be a semigroup and let $e \in S$ be an idempotent. Suppose that L_e, R_e and D_e are the $\mathscr{L}$-, $\mathscr{R}$-, and $\mathscr{D}$-class of e, respectively. Prove that $L_e R_e = D_e$.

7-6. Let a and b be regular elements of a semigroup S. Prove that

(a) $a \mathscr{L} b$ if and only if there are inverses $a', b' \in S$ of a and b, respectively, such that $a'a = b'b$;

(b) $a \mathscr{R} b$ if and only if there are inverses a' and b' of a and b, respectively, such that $aa' = bb'$;

(c) $a \mathscr{H} b$ if and only if there are inverses a' and b' of a and b, respectively, such that $a'a = b'b$ and $aa' = bb'$.

Inverse semigroups

7-7. Let S be an inverse semigroup. Prove that the mapping $\phi : S \longrightarrow S$, $x \mapsto x^{-1}$ is a bijection. Prove that ϕ maps $\mathscr{L}$-classes onto $\mathscr{R}$-classes. (Hint: prove that $a \mathscr{L} b$ if and only if $a^{-1} \mathscr{R} b^{-1}$.) Prove that ϕ preserves $\mathscr{D}$-classes. Conclude that the numbers of $\mathscr{L}$-classes and $\mathscr{R}$-classes in a single $\mathscr{D}$-class are equal. (Can you think of a less technical argument for the same result?)

7-8. Prove that a quotient S/ρ (and hence a homomorphic image as well) of an inverse semigroup S is again inverse. (Hint: prove that S/ρ is regular and that the idempotents commute; remember Lallement's Lemma from Problem **6-5**.) Prove that $(a/\rho)^{-1} = a^{-1}/\rho$.

7-9. Prove that a congruence on an inverse semigroup S is uniquely determined by the classes of idempotents. (Hint: for a congruence ρ prove that $a \rho b \iff aa^{-1} \rho ab^{-1} \rho bb^{-1}$.)

7-10. Let S be an inverse semigroup, let E be the set of idempotents of S, and let ρ be a congruence on S. Prove that the set $T = \bigcup_{e \in E} e/\rho$ is a subsemigroup of S. Also prove that for every $s \in S$, $s^{-1}Ts \subseteq T$.

7-11. Prove that I_n (the symmetric inverse semigroup on the set $\{1, 2, \ldots, n\}$) has
$$\sum_{r=0}^{n} \binom{n}{r}^2 r!.$$

List the elements of I_2 and prove that this semigroup is generated by the elements
$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 \\ 1 & - \end{pmatrix}.$$

7-12. Let S be an inverse semigroup, and let E be the set of idempotents of S. Prove that $aE = Ea$ for all $a \in S$. Is it true that $ae = ea$ for all $a \in S$ and all $e \in E$?