Chapter 1

Power series: Revision

{chap:1}

Assume that $f(x)$ is a known function of the independent variable x. Sometimes it is useful to express the function as a power series in term of powers of x. This may be necessary if we wish to approximate the function because it is difficult to evaluate in its original form. It this section we will see how well the function can be approximated by this series.

Definition

A power series in x about the point x_0 has the form,

$$\sum_{n=0}^{\infty} a_n \left(x - x_0\right)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n + \cdots \tag{1.1}$$

where a_0, a_1, a_2 are constant coefficients.

Example 1.1

Consider the geometric series, S, where

$$S = 1 + x + x^2 + x^3 + \cdots x^n.$$

Here there are a finite number of terms and each of the coefficients is equal to one. Multiplying both sides by x gives

$$xS = x + x^2 + x^3 + \cdots x^n + x^{n+1}.$$

Now if we subtract the two equations from each other we obtain

$$\begin{aligned} S - xS &= 1 - x^{n+1} \\ (1 - x)S &= 1 - x^{n+1} \\ S &= \frac{1 - x^{n+1}}{1 - x}. \end{aligned}$$

Note that all the terms cancelled apart from the first and last terms. Now if the absolute value of x is less than unity, namely $|x| < 1$, then the powers of x get smaller and smaller so that as $n \to \infty$ we have $x^{n+1} \to 0$. Thus,

$$\lim_{n\to\infty} S = \frac{1}{1 - x} = (1 - x)^{-1}.$$

Therefore the power series for $(1-x)^{-1}$, about $x = 0$ is

$$1 + x + x^2 + x^3 + \cdots$$

and this is valid for x lying in the range $-1 < x < 1$. This is simply another way of writing $|x| < 1$. Note that $x_0 = 0$ in this example. Now we can never actually use the complete infinite series and so we tend to truncate the series after a finite number of terms. This means that we use the series to *approximate* the value of $(1-x)^{-1}$.

We can obtain a power series expansion for $(1+x)^{-1}$ by simply replacing x by $-x$ in our previous series. Thus,

:1.1}
$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \cdots + (-1)^n x^n + \cdots \tag{1.2}$$

We can compare the truncated series with the exact value to see how good our approximation actually is. This is illustrated in Figure 1.1.

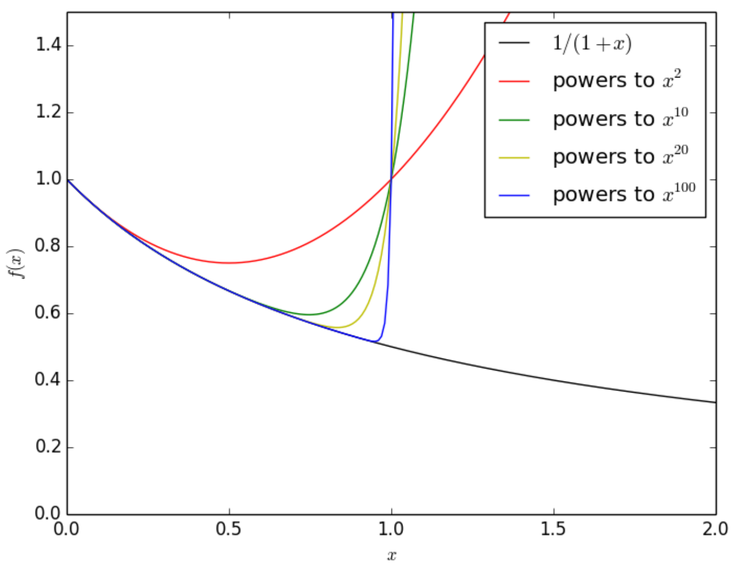

Figure 1.1: The power series approximation for various number of terms in the series is compared with
:1.1} the exact solution.

Figure 1.1 illustrates how the series provides a better and better approximation as we take more and more terms but only for $x < 1$. For $x > 1$, the series becomes worse as more terms are used. This shows graphically how a power series converges when x is less than the Radius of Convergence, R. However, a graphical demonstration of convergence is not the same as a rigorous proof.

Example End

1.1 The Ratio Test

There are several methods for testing whether an infinite series converges or not. A particularly simple test is the *Ratio Test.* The series $\sum_{n=0}^{\infty} b_n$ converges if

{eq:1.2}
$$\lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = L < 1. \tag{1.3}$$

Conversely, the series diverges if $L > 1$.

1.2 Convergence of a power series

We can use the ratio test to investigate the convergence of a power series. Here the n^{th} term in a power series has the form

$$b_n = a_n \left(x - x_0\right)^n .$$

Thus, the series will be convergent if

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x - x_0)^{n+1}}{a_n(x - x_0)^n} \right| = L < 1. \tag{1.4}$$ {eq:1.2a}

Rearranging gives

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n}(x - x_0) \right| = |x - x_0| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1. \tag{1.5}$$ {eq:1.3}

Finally, the power series for all values of x such that

$$|x - x_0| < \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = R. \tag{1.6}$$ {eq:1.4}

R is called the *Radius of Convergence.*

Example 1.2

For the power series for $f(x) = (1 - x)^{-1}$ and $x_0 = 0$, we have

$$f(x) = \frac{1}{1 - x} = 1 + x + x^2 + x^3 + \cdots x^n + \cdots$$

Thus, we have $a_n = 1$. Hence, we find that the series converges for all values of x such that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = |x| < 1.$$

In this case the radius of convergence is 1 and the series converges for $-1 < x < 1$. This is precisely what was observed in Figure 1.1.

The particular case when $x = \pm R$ requires a separate treatment.

Example End

1.3 Differentiating power series

The power series for a function $f(x)$ about the origin is

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots a_n x^n + \cdots$$

The right hand side has a radius of convergence, which I will label by R_f for this series, where

$$R_f = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| .$$

We will now differentiate the power series for $f(x)$ to get the power series approximation for the derivative $f'(x)$. Hence,

$$f'(x) = 0 + a_1 + 2a_2 x + 3a_3 x^2 + \cdots n a_n x^{n-1} + \cdots .$$

We can apply the ratio test again to check the radius of convergence of the series for the derivative. We will label this radius of convergence by $R_{f'}$ so that the series will converge for all $|x| < R_{f'}$. Hence, for convergence we require

$$\lim_{n\to\infty} \left| \frac{(n+1)a_{n+1}x^n}{n a_n x^{n-1}} \right| < 1.$$

Simplifying this (by cancelling the powers of x we have

$$\lim_{n\to\infty} |x| \left| \frac{n+1}{n} \frac{a_{n+1}}{a_n} \right| = |x| \lim_{n\to\infty} \left| \frac{n+1}{n} \right| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Now the last limit is exactly the same limit that we obtained in determining the convergence of the original series. Hence, we have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{R_f},$$

and the first limit can be rearranged to give

$$\lim_{n\to\infty} \left| \frac{n+1}{n} \right| = \lim_{n\to\infty} \left| 1 + \frac{1}{n} \right| = 1$$

Combining the above two limits into the ratio test we have

$$\frac{|x|}{R_f} < 1, \qquad \Rightarrow \qquad |x| < R_f = R_{f'}.$$

Therefore, the important point is that differentiating a convergent power series keeps it convergent with the original radius of convergence, R. The same conclusion is true for integration of a power series.

1.4 Products of power series

The product of two functions is the product of their individual power series. Careful expansion of the product of the two series gives the require power series. Consider the series for e^x (about $x = 0$) is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \frac{x^n}{n!} + \cdots ,$$

and it converges for all values of x (the radius of convergence is infinite). If we wish to calculate the power series of the function defined by

$$\left(\frac{e^{2x}}{1-x}\right),$$

about the origin, then we can multiply the power series for each function together. Thus,

$$e^{2x} = 1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots$$

and

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$

Therefore,

$$\begin{aligned}
\frac{e^{2x}}{1-x} &= \left(1 + 2x + \frac{4}{2}x^2 + \frac{8}{6}x^3 + \cdots\right)\left(1 + x + x^2 + x^3 + \cdots\right) \\
&= 1 + (2x + x) + \left(x^2 + 2x^2 + 2x^2\right) + \left(x^3 + 2x^3 + 2x^3 + \frac{4}{3}x^3\right) + \cdots \\
&= 1 + 3x + 5x^2 + \frac{19}{3}x^3 + \cdots
\end{aligned}$$

Obviously the algebra gets quite complicated and to get more terms in the power series it would be better to use a computer algebra package , such as MAPLE.