Testing Bipartiteness: An Application of Breadth-First Search

Section 3.4

Bipartite Graphs

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  - **V can be partitioned into sets X and Y in such a way that every edge has one end-point in X and other end-point in Y**

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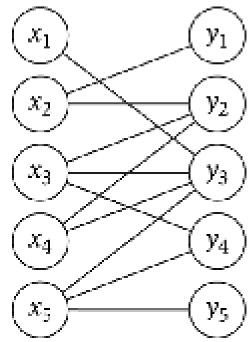

Figure 1.5 A bipartite graph.

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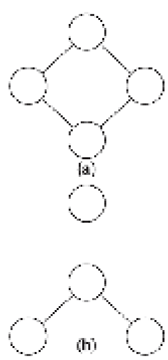

Figure 1.3 Each of (a) and (b) depicts a graph on four nodes.

Bipartite Graphs: Examples

- X = set of students
- Y = set of classes
- Edge (u,v): if and only if student u is enrolled in class v

Bipartite Graphs: Examples

- X = set of TA applicants
- Y = set of classes
- Edge (u,v): if and only if applicant u is eligible for teaching class v

Bipartite Graphs: Examples

- X = set of TA applicants
- Y = set of classes
- Edge (u,v): if and only if applicant u is eligible for teaching class v
- Our graduate secretary handles this graph every semester

Bipartite Graphs

- X = set of red nodes
- Y = set of blue nodes
- Edges are between red and blue nodes
  - No edge between red nodes
  - No edge between blue nodes

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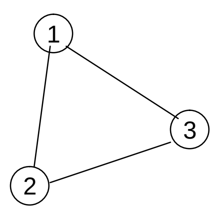

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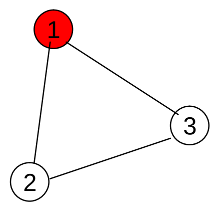

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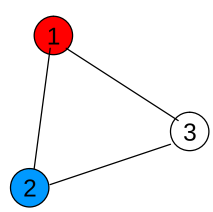

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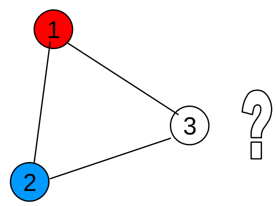

What graphs are not bipartite?

- More generally, odd cycles have problems
  - Consider cycle with nodes 1, 2, … 2k + 1
  - Node 1 = red
  - Node 2 = blue
  - Node 3 = red
  - …
  - Node 2k = blue
  - Node 2k + 1 = ?

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  - **If no odd cycle, is there a valid red/blue coloring of the nodes?**
    - **i.e., every edge has a red and blue end-point.**

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- **[3.14] If a graph G is bipartite, then it cannot contain an odd cycle**
  - **But is odd cycle the only obstacle?**
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How can we know if a given graph is bipartite?

An algorithm for testing "bipartiteness"

- Choose a vertex s and color it red
- Color neighbors of s blue
- Color neighbors of *these* nodes red
- … and so on until all nodes are colored

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- Do we have a valid coloring?
  - Yes: bipartite
  - No: not bipartite

BFS Coloring

- **Start exploring from *s***
- **Color *s* red**
- **Color layer L_1 blue, L_2 red, and so on…**
- **Color odd layers blue and even layers red**
- **Easy to implement on top of BFS!**

BFS Coloring

```
BFS(s):
  Set Discovered[s] = true and Discovered[v] = false for all other v
  Initialize L[0] to consist of the single element s
  Set the layer counter i = 0
  Set the current BFS tree T = ∅
  While L[i] is not empty
    Initialize an empty list L[i + 1]
    For each node u ∈ L[i]
      Consider each edge (u, v) incident to u
      If Discovered[v] = false then
        Set Discovered[v] = true
        Add edge (u, v) to the tree T
        Add v to the list L[i + 1]
      Endif
    Endfor
    Increment the layer counter i by one
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```

Set color[v] here

Analysis of BFS coloring

- [3.15] Let *G* be a connected graph, and let L_1, L_2, … be the layers produced by BFS starting at node s. Then exactly one of the following two things hold.
  1. There is no edge of *G* joining two nodes of the same layer. In this case *G* is a bipartite graph in which the nodes in even-numbered layers can be colored red, and the nodes in odd-numbered layers can be colored blue
  2. There is an edge of *G* joining two nodes of the same layer. In this case, *G* contains an odd-cycle, and so it cannot be bipartite.