Solving The Turnpike

A fast solution using $\max(\Delta_{unknown})$

by Christopher Connor

Solving the Turnpike

Our Core Theorem:

Assume turnpike of Length L,and nodes on the turnpike $N=\{n_0,n_1,..n_k\}$, $n_j>n_{j-1}$ for all $j<=k$. $n_0=0$ and $n_k=L_{total}$

further, you have a collection of found points on the turnpike $N_k=\{0,m_1,m_2,...m_{k-1},L\}$ and you know that $\Delta(N_{unknown})=\Delta(N)-\Delta(N_k)$.

It then follows that if $L_k=\max(\Delta(N_{unknown}))$, at least one of the points $m_{k1}=L_k$ and $m_{k2}=(L_{total}-L_k)$ can be found in N.

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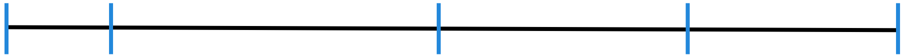
A visual proof:

Suppose that N_4

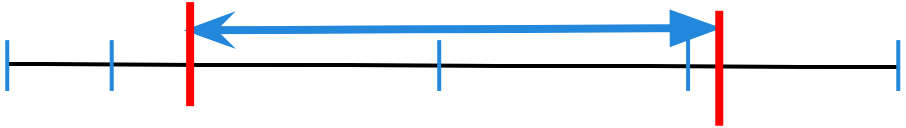


looks like the diagram to the right.

Suppose that the maximum unaccounted for distance

Is this big:

if the distance does not begin at either end of the turnpike, it is generated by points like this:



where at least one of the red nodes is undiscovered.

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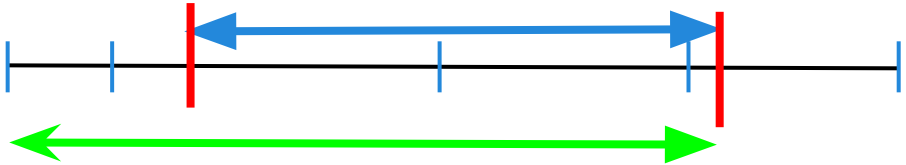
However, that then means that

there is a distance

(pictured in Green)



which is longer than our maximum unaccounted distance, and attached to an undiscovered node.
This then means that new distance is also unaccounted for, leading to a contradiction.

Thus, the maximum unaccounted distance must be attached to one of the endpoints of the turnpike.

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An example of using this theorem to solve the turnpike problem:

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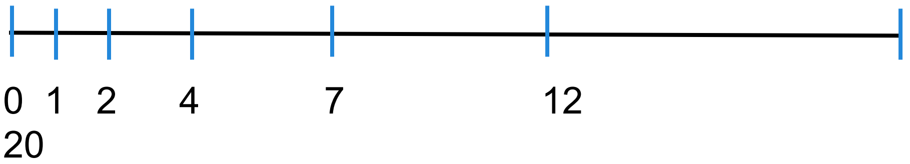
Turnpike=



N={0,1,2,4,7,12,20}

L_{total}=20

Δ(N)={1,1,2,2,3,3,4,5,5,6,7,8,8,10,11,12,13,16,18,19,20}
along with the 0's and the negatives of the above, which we will ignore as being unnecessary.

Solving the Turnpike

Lets start solving:

we will cross out the distances we have found, and circle the maximum undiscovered distance.

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$\text{Turnpike}_1 =$ 

0

20

$N_1=\{0,20\}$

$\max(\Delta_{unknown})=19$

$\Delta(N)=\{1,1,2,2,3,3,4,5,5,6,7,8,8,10,11,12,13,16,18,19,\cancel{20}\}$

Solving the Turnpike

$Turnpike_1 =$

0

20

$N_1=\{0,20\}$

$\max(\Delta_{unknown})=19$

Using the maximum we have discovered, we will now generate our two possible points that are attached to that maximum, Choice1 and Choice2.

$\Delta(N)=\{1,1,2,2,3,3,4,5,5,6,7,8,8,10,11,12,13,16,18,19,\cancel{20}\}$

Solving the Turnpike

$\text{Turnpike}_1 =$

0

20

$N_1=\{0,20\}$

$\max(\Delta_{unknown})=19$

Choice1=(20-19)=1 Choice2=19

$\Delta(N)=\{1,1,2,2,3,3,4,5,5,6,7,8,8,10,11,12,13,16,18,19,\cancel{20}\}$

Solving the Turnpike

$Turnpike_1 =$

0
20

$N_1=\{0,20\}$

$\max(\Delta_{unknown})=19$

Choice1=(20-19)=1 Choice2=19

Finally, we will test Choice1 and Choice2 to see if they are valid possible points in N.

$\Delta(N)=\{1,1,2,2,3,3,4,5,5,6,7,8,8,10,11,12,13,16,18,19,\cancel{20}\}$

Solving the Turnpike

$\text{Turnpike}_1 =$ 

0
20

$N_1=\{0,20\}$

$\max(\Delta_{unknown})=19$

Choice1=(20-19)=1 Choice2=19

$N_{choice1}=\{0,1,20\}$
$\Delta(N_{choice1})=\{0,1,19,20\}$: Okay!

$N_{choice2}=\{0,19,20\}$
$\Delta(N_{choice2})=\{0,1,19,20\}$: Okay!

$$\Delta(N)=\{1,1,2,2,3,3,4,5,5,6,7,8,8,10,11,12,13,16,18,19,\cancel{20}\}$$

Solving the Turnpike

$Turnpike_1=$

0
20

$N_1=\{0,20\}$

$\max(\Delta_{unknown})=19$

Choice1=(20-19)=1 Choice2=19

$N_{choice1}=\{0,1,20\}$
$\Delta(N_{choice1})=\{0,1,19,20\}$: Okay!

$N_{choice2}=\{0,19,20\}$
$\Delta(N_{choice2})=\{0,1,19,20\}$: Okay!

Fact: on the first iteration N_1, both choices will be valid, and lead to solutions that are mirror images of each other.
Because of this, we will only solve for **Choice1**, but will address Choice2 at the end.

$\Delta(N)=\{1,1,2,2,3,3,4,5,5,6,7,8,8,10,11,12,13,16,18,19,\cancel{20}\}$

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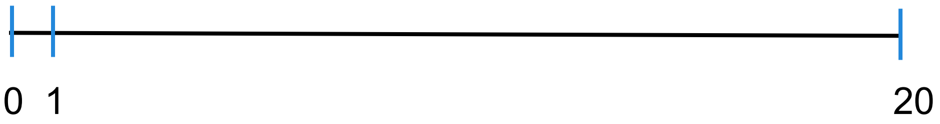
$\text{Turnpike}_2 =$



$N_2=\{0,1,20\}$

$\max(\Delta_{unknown})=18$

Choice1=(20-18)=2

Choice2=18

$N_{choice1}=\{0,1,2,20\}$

$\Delta(N_{choice1})=\{1,1,2,18,19,20\}$: Okay!

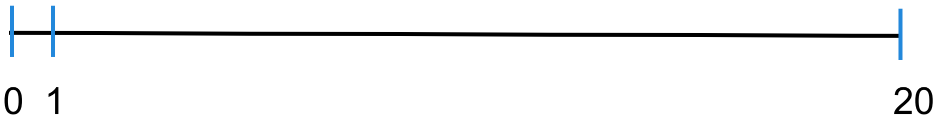
$N_{choice2}=\{0,1,19,20\}$

$\Delta(N_{choice2})=\{1,2,17,18,19,20\}$: No good, too many 17's

$\Delta(N)=\{\cancel{1},1,2,2,3,3,4,5,5,6,7,8,8,10,11,12,13,16,18,\cancel{19},\cancel{20}\}$

Solving the Turnpike

$Turnpike_2$=



$N_2=\{0,1,20\}$

$\max(\Delta_{unknown})=18$

Choice1=(20-18)=2

Choice2=18

$N_{choice1}=\{0,1,2,20\}$

$\Delta(N_{choice1})=\{1,1,2,18,19,20\}$: Okay!

Choice1 is the only valid option. We select it and move on to N_3.

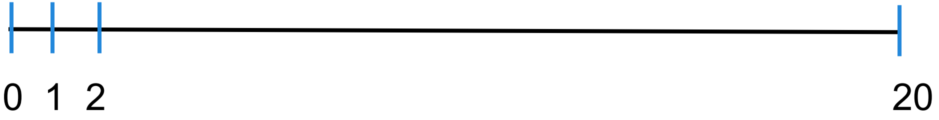
$N_{choice2}=\{0,1,19,20\}$

$\Delta(N_{choice2})=\{1,2,17,18,19,20\}$: No good, too many 17's

$$\Delta(N)=\{\cancel{1},1,2,2,3,3,4,5,5,6,7,8,8,10,11,12,13,16,18,\cancel{19},\cancel{20}\}$$

Solving the Turnpike

$Turnpike_3 =$



$N_3=\{0,1,2,20\}$

$\max(\Delta_{unknown})=16$

$Choice1=(20-16)=4$

$Choice2=16$

$N_{choice1}=\{0,1,2,4,20\}$

$\Delta(N_{choice1})=\{1,1,2,2,3,4,16,18,19,20\}$: Okay!

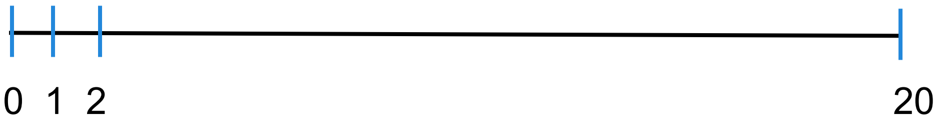
$N_{choice2}=\{0,1,2,16,20\}$

$\Delta(N_{choice2})=\{1,1,2,4,14,15,16,18,19,20\}$: No good, too many 15's

$$\Delta(N)=\{\cancel{1},\cancel{1},\cancel{2},2,3,3,4,5,5,6,7,8,8,10,11,12,13,16,\cancel{18},\cancel{19},\cancel{20}\}$$

Solving the Turnpike

$\text{Turnpike}_3=$



$N_3=\{0,1,2,20\}$

$\max(\Delta_{unknown})=16$

Choice1=(20-16)=4   Choice2=16

$N_{choice1}=\{0,1,2,4,20\}$
$\Delta(N_{choice1})=\{1,1,2,2,3,4,16,18,19,20\}$: Okay!

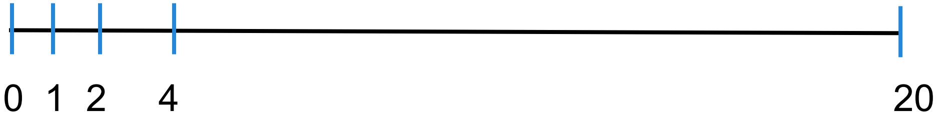
Choice1 is the only valid option. We select it and move on to N_4.

$N_{choice2}=\{0,1,2,16,20\}$
$\Delta(N_{choice2})=\{1,1,2,4,14,15,16,18,19,20\}$: No good, too many 15's

$$\Delta(N)=\{\cancel{1},\cancel{1},\cancel{2},2,3,3,4,5,5,6,7,8,8,10,11,12,13,16,\cancel{18},\cancel{19},\cancel{20}\}$$

Solving the Turnpike

$\text{Turnpike}_4 =$



$N_4=\{0,1,2,4,20\}$ $\max(\Delta_{unknown})=13$ Choice1=(20-13)=7 Choice2=13

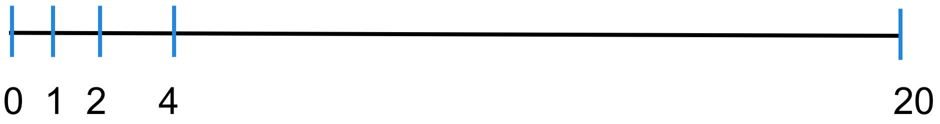
$N_{choice1}=\{0,1,2,4,7,20\}$
$\Delta(N_{choice1})=\{1,1,2,2,3,3,4,5,6,7,13,16,18,19,20\}$: Okay!

$N_{choice2}=\{0,1,2,4,13,20\}$
$\Delta(N_{choice2})=\{1,1,2,2,3,4,9,11,12,13,16,18,19,20\}$: No good, too many 9’s

$\Delta(N)=\{\cancel{1},\cancel{1},\cancel{2},\cancel{2},\cancel{3},3,\cancel{4},5,5,6,7,8,8,10,11,12,13,\cancel{16},\cancel{18},\cancel{19},\cancel{20}\}$

Solving the Turnpike

$\text{Turnpike}_4 =$



$N_4=\{0,1,2,4,20\}$ $\max(\Delta_{unknown})=13$ Choice1=(20-13)=7 Choice2=13

$N_{choice1}=\{0,1,2,4,7,20\}$

$\Delta(N_{choice1})=\{1,1,2,2,3,3,4,5,6,7,13,16,18,19,20\}$: Okay!

$N_{choice2}=\{0,1,2,4,13,20\}$

$\Delta(N_{choice2})=\{1,1,2,2,3,4,9,11,12,13,16,18,19,20\}$: No good, too many 9's

$\Delta(N)=\{\cancel{1},\cancel{1},\cancel{2},\cancel{2},\cancel{3},3,\cancel{4},5,5,6,7,8,8,10,11,12,1$

Choice1 is the only valid option. We select it and move on to N_5.

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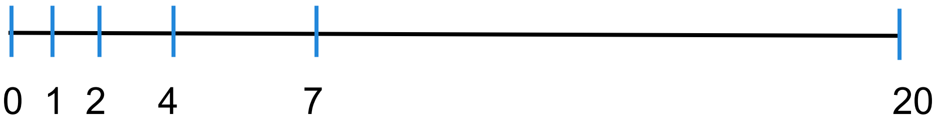
$\text{Turnpike}_5 =$



$N_5=\{0,1,2,4,7,20\}$ $\max(\Delta_{\text{unknown}})=12$ Choice1=(20-12)=8 Choice2=12

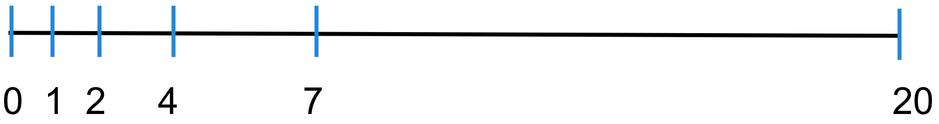
$N_{\text{choice1}}=\{0,1,2,4,7,8,20\}$
$\Delta(N_{\text{choice1}})=\{1,1,1,2,2,3,3,4,4,5,6,6,7,7,8,12,13,16,18,19,20\}$: No good, too many 1's

$N_{\text{choice2}}=\{0,1,2,4,7,12,20\}$
$\Delta(N_{\text{choice2}})=\{1,1,2,2,3,3,4,5,5,6,7,8,8,10,11,12,13,16,18,19,20\}$: Good!

$$\Delta(N)=\{\cancel{1},\cancel{1},\cancel{2},\cancel{2},\cancel{3},\cancel{3},\cancel{4},\cancel{5},5,\cancel{6},\cancel{7},8,8,10,11,12,\cancel{13},\cancel{16},\cancel{18},\cancel{19},\cancel{20}\}$$

Solving the Turnpike

$\text{Turnpike}_5=$



$N_5=\{0,1,2,4,7,20\}$ $\max(\Delta_{\text{unknown}})=12$ Choice1=(20-12)=8 Choice2=12

$N_{\text{choice1}}=\{0,1,2,4,7,8,20\}$
$\Delta(N_{\text{choice1}})=\{1,1,1,2,2,3,3,4,4,5,6,6,7,7,8,12,13,16,18,19,20\}$: No good, too many 1's

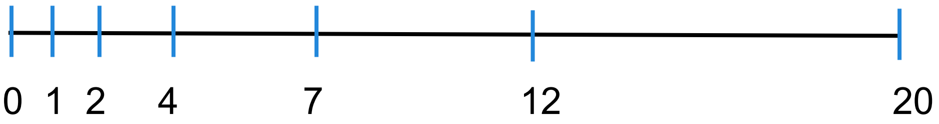
$N_{\text{choice2}}=\{0,1,2,4,7,12,20\}$
$\Delta(N_{\text{choice2}})=\{1,1,2,2,3,3,4,5,5,6,7,8,8,10,11,12,13,16,18,19,20\}$: Good!

$\Delta(N)=\{\cancel{1},\cancel{1},\cancel{2},\cancel{2},\cancel{3},\cancel{3},\cancel{4},\cancel{5},5,\cancel{6},\cancel{7},8,8,10,11,12,$

Choice2 is the only valid option. We select it and move on to N_6.

Solving the Turnpike

$\text{Turnpike}_6 =$



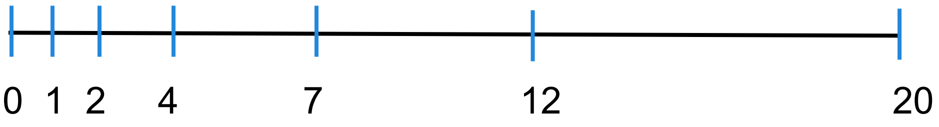
$N_5=\{0,1,2,4,7,12,20\}$

Success!

$\Delta(N)=\{\cancel{1},\cancel{1},\cancel{2},\cancel{2},\cancel{3},\cancel{3},\cancel{4},\cancel{5},5,\cancel{6},\cancel{7},\cancel{8},\cancel{8},\cancel{10},\cancel{11},\cancel{12},\cancel{13},\cancel{16},\cancel{18},\cancel{19},\cancel{20}\}$

Solving the Turnpike

$\text{Turnpike}_6 =$



$N_5=\{0,1,2,4,7,12,20\}$

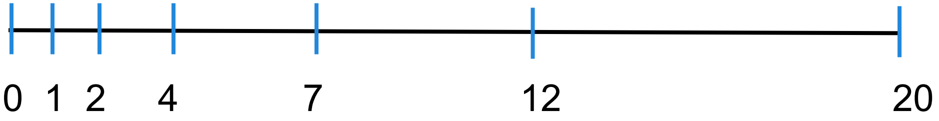
Success!

Our Set of unknown lengths is empty, and $\Delta(n)=\Delta(N_5)$. We have solved this 7-node turnpike problem in 6 steps.

$\Delta(N)=\{\cancel{1},\cancel{1},\cancel{2},\cancel{2},\cancel{3},\cancel{3},\cancel{4},\cancel{5},5,\cancel{6},\cancel{7},\cancel{8},\cancel{8},\cancel{10},\cancel{11},\cancel{12},\cancel{13},\cancel{16},\cancel{18},\cancel{19},\cancel{20}\}$

Solving the Turnpike

$Turnpike_6 =$



Note that $N_6=\{0,8,13,16,18,19,20\}$, the reverse of
$\{0,1,2,4,7,12,20\}$
is also a solution.

Had we selected $N_1=\{0,19,20\}$, this is the solution that we would have found instead.

$\Delta(N)=\{\cancel{1},\cancel{1},\cancel{2},\cancel{2},\cancel{3},\cancel{3},\cancel{4},\cancel{5},5,\cancel{6},\cancel{7},\cancel{8},\cancel{8},\cancel{10},\cancel{11},\cancel{12},\cancel{13},\cancel{16},\cancel{18},\cancel{19},\cancel{20}\}$

Solving the Turnpike

Done.

I am currently working on the documentation to a similar polynomial solution to the Beltway problem, which is like the Turnpike problem if it were a ring instead of a line.

Please contact me at breath.of.zen@gmail.com if you are interested.