DUBLIN INSTITUTE OF TECHNOLOGY

KEVIN STREET, DUBLIN 8

Taught Master in Applied Mathematics and Theoretical Physics

Exam 2010

Introduction to Biomathematics

Answer any FOUR Questions

Dept. of Education Tables allowed

Question 1. Consider the following population growth model:

$$\frac{dN}{dt} = \frac{rN}{\alpha}\left[1 - \left(\frac{N}{K}\right)^{\alpha}\right], \tag{1}$$

with $r > 0$, $\alpha > 0$, $K > 0$ and $N(0) > 0$.

(i) Analyze the steady states of the model for stability. What is the limit of $N(t)$ when $t \to \infty$?

[7 marks]

(ii) Consider the limit $\alpha \to 0$ in the equation (1). Write the obtained model equation and solve it explicitly for $N(t)$. Compute $\lim_{t\to\infty} N(t)$ for your solution.

[11 marks]

(iii) Prove that (1) does not have any periodic solutions.

[7 marks]

Question 2. (i) Show that an exact travelling wave solution exists for the Fisher-Kolmogoroff-type equation

$$\frac{\partial u}{\partial t} = u(1 - u^q) + \frac{\partial^2 u}{\partial x^2}, \tag{2}$$

where $q > 0$, by looking for solutions in the form

$$u(x,t) = U(z) = \frac{1}{(1 + ae^{bz})^s}, \qquad z = x - ct, \tag{3}$$

where $c > 0$ is the wavespeed and b and s are positive constants. Determine the unique values for c, b and s in terms of q. Choose a value for a such that $U(0) = 1/2$.

[20 marks]

(ii) Take for simplicity $q = 1$ and sketch the solution (3). Explain briefly the relevance of (2) in modelling population dynamics and the meaning of the solution (3) in particular.

[5 marks]

Question 3. The frequency p_n of the allele A of a gene in the n-th generation satisfies the equation of a selection model

$$p_{n+1} = \frac{w_{AA}p_n^2 + w_{Aa}p_n q_n}{w_{AA}p_n^2 + 2w_{Aa}p_n q_n + w_{aa}q_n^2}$$

where $q_n = 1 - p_n$ is the frequency of the other allele a of the same gene and w_{AA}, w_{Aa} and w_{aa} are constant positive coefficients called *relative fitness* of the corresponding genotype.

(i) Determine all steady states, their existence and stability in dependence on w_{AA}, w_{Aa} and w_{aa} (assume that these are all different from each other).

[20 marks]

(ii) Explain the biological meaning of the case $w_{AA} = w_{Aa} = w_{aa}$.

[5 marks]

Question 4. Consider the discrete population model

$$N_{t+1} = \frac{rN_t}{1+bN_t^2} \equiv f(N_t),$$

where t is the discrete time and r and b are positive parameters.

(i) Show that after a long time the population is bounded by

$$N_{\min} = \frac{2r^2}{(4+r^2)\sqrt{b}} \leq N_t \leq \frac{r}{2\sqrt{b}}$$

[5 marks]

(ii) Determine the steady states and their eigenvalues and hence show that $r = 1$ is a bifurcation value.

[5 marks]

(iii) What do r and b represent in this model? Prove that, for any r, the population will become extinct if $b > 4$.

[5 marks]

(iv) Consider a delay version of the model, given by

$$N_{t+1} = \frac{rN_t}{1+bN_{t-1}^2} \equiv f(N_t), \qquad r > 1.$$

Investigate the linear stability about the positive steady state N^* by setting $N_t = N^* + n_t$. Show that the linearized equation is

$$n_{t+1} - n_t + 2\frac{r-1}{r}n_{t-1} = 0. \tag{4}$$

[5 marks]

(v) Show that $r = 2$ is a bifurcation value for (4) and that as $r \to 2$ the steady state bifurcates to a periodic solution of period 6.

[5 marks]

Question 5. A predator-pray interaction is described by the following system of non-dimensional variables

$$\frac{du}{dt} = a\Big(u(1-u) - \frac{uv}{1+bu}\Big),$$
$$\frac{dv}{dt} = v\Big(\frac{bu}{1+bu} - c\Big)$$

where a, b and $c < 1$ are positive parameters (Rosenzweig-MacArthur model).

(i) Determine the steady states and the parameter values for which they are biologically relevant.

[5 marks]

(ii) Investigate their stability if $\frac{c}{1-c} < b < \frac{1+c}{1-c}$ and sketch the phase plane trajectories,

[15 marks]

(iii) Explain briefly how the phase plane trajectories will change if $b > \frac{1+c}{1-c}$.

[5 marks]

Question 6. Consider the 'SIR' epidemic model with $S(t)$, $I(t)$ and $R(t)$ as the number of individuals of the three classes: susceptibles, infectives and the removed class correspondingly. The model is described by the system

$$\frac{dS}{dt} = -rSI,$$
$$\frac{dI}{dt} = rSI - aI,$$
$$\frac{dR}{dt} = aI,$$

where $r > 0$ is the infection rate and $a > 0$ is the removal rate of infectives. The initial data is $S(0) = S_0 > \rho$, $(\rho = a/r)$, $I(0) = I_0 > 0$, $R(0) = 0$.

(i) Show that if $S_0 > \rho$ there is an epidemic outbreak.

[3 marks]

(ii) Prove that $S(t) + I(t) + R(t) = N =$ constant. What is the meaning of the constant N?

[2 marks]

(iii) Demonstrate that $R(t)$ satisfies the equation

$$\frac{dR}{dt} = a(N - R - S_0 e^{-R/\rho}), \qquad R(0) = 0. \tag{5}$$

[5 marks]

(iv) Show that when R/ρ is small, (5) can be approximated with

$$\frac{\mathrm{d}R}{\mathrm{d}t} = a\left[N - S_0 + \left(\frac{S_0}{\rho} - 1\right)R - \frac{S_0 R^2}{2\rho^2}\right], \qquad R(0) = 0. \tag{6}$$

[5 marks]

(v) Without solving (6), find $R(\infty) = \lim_{t\to\infty} R(t)$. To this end look at the steady states of this equation. Explain the meaning of $R(\infty)$.

[5 marks]

(vi) Sketch the phase portrait of the system using I (vertical) and S (horizontal) coordinates.

[5 marks]

END OF PAPER

Question 1

(i) The steady states are $N* = 0$ and $N* = K$, at the zeros of

$$f(N) = \frac{rN}{\alpha}\left[1-\left(\frac{N}{K}\right)^{\alpha}\right]$$

$$f'(N) = \frac{r}{\alpha}\left[1-\left(\frac{N}{K}\right)^{\alpha}\right] - \frac{rN}{\alpha}\frac{\alpha N^{\alpha-1}}{K^{\alpha}}$$

$$f'(0) = \frac{r}{\alpha} > 0 \qquad \Longrightarrow N* = 0 \text{ is an } \underline{\text{unstable}} \text{ steady state.}$$

$$f'(K) = \frac{r}{\alpha}(1-\alpha-1)$$

$$= -r < 0 \qquad \Longrightarrow N* = K \text{ is a } \underline{\text{stable}} \text{ steady state.}$$

$$\Longrightarrow \lim_{t\to 0} N(t) = K$$

(ii) The L'Hospital's rule gives

$$\lim_{\alpha\to 0}\frac{rN}{\alpha}\left[1-\left(\frac{N}{K}\right)^{\alpha}\right]$$

$$= rN\left[\frac{\frac{\partial}{\partial\alpha}\left(1-\left(\frac{N}{K}\right)^{\alpha}\right)}{\frac{\partial}{\partial\alpha}(\alpha)}\right]$$

$$= rN\left[\frac{-\left(\frac{N}{K}\right)^{\alpha}\ln\frac{N}{K}}{1}\right]$$

$$= -rN\ln\frac{N}{K}$$

The equation is:

$$\frac{dN}{dt} = -rN\ln\frac{N}{K}$$

$$\frac{1}{N}\frac{dN}{dt} = -r\ln\left(\frac{N}{K}\right)$$

$$\frac{d}{dt}\ln\left(\frac{N}{K}\right) = -r\ln\left(\frac{N}{K}\right)$$

$$\ln\frac{N}{K} = ce^{-rt}$$

$$\ln\frac{N(0)}{K} = ce^{-rt}$$

$$c = \ln\left[\frac{N(0)}{K}\right]$$

$$
\begin{aligned}
N(t) &= K\exp\left[\ln\frac{N(0)}{K}e^{-rt}\right] \\
\lim_{t\to\infty} N(t) &= K\exp\left[\ln\frac{N(0)}{K}\cdot 0\right] \\
&= Ke^{0} \\
&= K \qquad \text{as in (i)}
\end{aligned}
$$

(iii) Suppose that the equation has a periodic solution with period T, i.e. $N(t+T) = N(T)$. Then we consider the integral

$$
\int_t^{t+T}\left(\frac{dN}{dt}^2\right)dt = \int_t^{t+T} f(N)\frac{dN}{dt}\,dt = \int_{N(t)}^{N(t+T)} f(N)\;dN = \int_{N(t)}^{N(t)} f(N)\,dN = 0
$$

But the left-hand integral is positive, since $\left(\frac{dN}{dt}\right)^2$ is not identically zero, so we have a contradiction. So, the simple scalar equation $\frac{dN}{dt} = f(N)$ cannot have periodic solutions.

Question 2

(i) We notice that the solution $U(z) = (1 + ae^{bz})^{-s}$ automatically satisfies the boundary conditions

$$U(\infty) = 0 \qquad \text{and } U(-\infty) = 1$$

The corresponding ODE for $U(z)$ is

$$L(U) = U'' + cU' + U(1 - U^q) = 0$$

Since the solution is translational-invariant[1], $z \to z + \text{const}$, then it is clear that a is an arbitrary constant, i.e. b and s should not depend on a. We compute:

$$U'(z) = -s\frac{abe^{bz}}{(1 + ae^{bz})^{s+1}}$$

$$U''(t) = -sb^2ae^{bt}\left(1 + ae^{bz}\right)^{-s-2}\left(1 - ase^{bz}\right)$$

$$L(U) = \frac{1}{(1 + ae^{bz})^{s+2}}\left[a^2\left(s^2b^2 - seb + 1\right)e^{2bz} + \left(2 - scb - sb^2\right)ae^{bz} + 1 - \left(1 + ae^{bt}\right)^{2-sq}\right]$$

We want $L(U) \equiv 0$ for all $z \implies$ all coefficients of e^b, e^{bt} and e^{2bt} must be all 0.

$$\implies 2 - sq = 0 \qquad ,1 \text{ or } 2$$

$$2 - sq = 0 \qquad \implies sq = 2$$

$$2 - sq = 1 \qquad \implies sq = 1$$

$$2 - sq = 2 \qquad \implies sq = 0 (\text{not possibie since } s > 0, q > 0.)$$

$\implies$ Two possibilities: $s = \frac{1}{q}$ and $s = \frac{2}{q}$. Let

$$s = \frac{1}{q} \qquad (sq = 1)$$

$$2 - sq = 2 - 1 = 1$$

$$L(U) = \left(1 + ae^{bz}\right)^{-s-2}\left[a^2\left(s^2b^2 - scb + 1\right)e^{2bz} + \left(2 - scb - sb^2\right)ae^{bz} - ae^{bz}\right] \equiv 0$$

$$\implies sb(sb - c) + 1 = 0$$

$$2 - scb - sb^2 - 1 = 0 \qquad \Leftrightarrow 1 = sb(b + c)$$

[1] Analogously an operator A on functions is said to be translation invariant with respect to a translation operator T_δ if the result after applying A doesn't change if the argument function is translated. More precisely it must hold that $\forall \delta Af = A(T_\delta f)$

$$sb\,(sb-c)+sb\,(b+c)=0$$
$$sb\,(sb+b)=0$$
$$sb^2\,(s+1)=0 \qquad \text{not possible, } s>0 \quad \& \quad (s+1)>0$$

The only remaining possibility is

$$s=\frac{2}{q} \qquad \text{or } sq=2$$

This leads to

$$s^2b^2-scb+1=0 \qquad \Longrightarrow\ sb\,(c-sb)=1 \qquad (1)$$
$$2-scb-sb^2=0 \qquad \Longrightarrow\ sb\,(b+c)=2 \qquad (2)$$
$$(1) \text{ and } (2) \implies b+c=2\,(c-sb)$$
$$\implies c=b\,(1+2s)$$
$$sb\,[b+b\,(1+2s)]=2 \qquad \Longrightarrow\ sb^2\,(2+2s)=2$$
$$\implies b=\frac{1}{\sqrt{s\,(s+1)}} \qquad c=\frac{1+2s}{\sqrt{s\,(s+1)}}$$

But $s=\frac{2}{q}\implies$

$$b=\frac{q}{\sqrt{2\,(q+2)}} \qquad c=\frac{q+4}{\sqrt{2\,(2+q)}}$$

If $U\,(0)=\frac{1}{2}$,

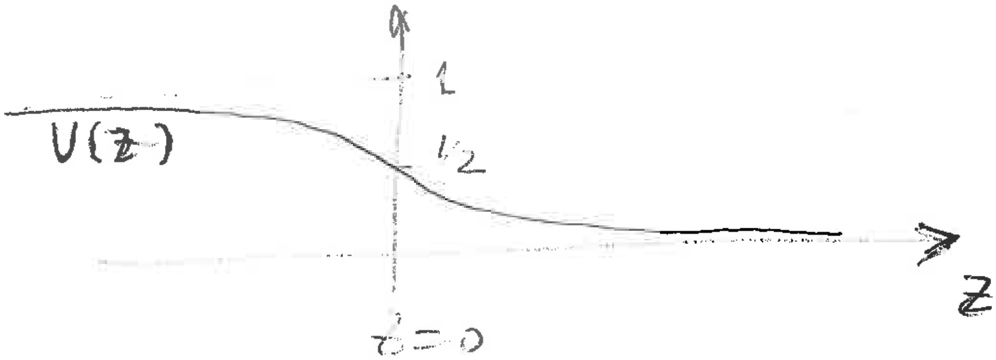
$$\frac{1}{(1+a)^s}=\frac{1}{2}$$
$$(1+a)^2=2$$
$$1+a=2^{1/s}$$
$$a=2^{1/s}-1$$
$$\implies a=\boxed{2^{q/2}-1}$$

(ii) When $q=1$, the equation is known as "Fisher-Kolmogorov" equation

$$S=2 \qquad b=\frac{1}{\sqrt{6}} \qquad c=\frac{5}{\sqrt{6}}\approx 2.04 \qquad a=\sqrt{2}-1$$

$$U\,(z)=\frac{1}{\left[1+\left(\sqrt{2}-1\right)e^{z/\sqrt{6}}\right]^2}$$

The equation describes 'geographic' spread of population that grows logistically since $z = x - ct$ initially $(t = -\infty), z = \infty$ and $U(\infty) = 0$, there is no population. As it grows it spreads as a travelling wave until it reaches the 'carrying capacity' $U = 1$ i.e $U(-\infty) = 1$ that corresponds to $t = \infty$. The term $\frac{\partial^2 U}{\partial x^2}$ describes the spatial spread of the population (i.e. diffusion).



Question 3

(i) Let $p_n = p$ be the equilibrium (steady) state, then

$$p = \frac{W_{AA}p^2 + W_{Aa}p\,(1-p)}{W_{AA}p^2 + 2W_{Aa}p\,(1-p) + W_{aa}\,(1-p)^2} \qquad \text{or}$$

$$W_{AA}p^3 + 2W_{Aa}p^2\,(1-p) + W_{aa}p\,(1-p)^2 = W_{AA}p^2 + W_{Aa}p\,(1-p)$$

$$\Longrightarrow p\,(1-p)\left[(-W_{AA} + 2W_{Aa} - W_{aa})\,p + W_{aa} - W_{Aa}\right] = 0$$

$$\Longrightarrow p^*_{(0)} = 0 \qquad p^*_{(1)} = 1 \qquad p^*_{(2)} = \frac{1}{1 + \frac{W_{AA}-W_{Aa}}{W_{aa}-W_{Aa}}}$$

The last one only exists if

$$\frac{W_{AA} - W_{Aa}}{W_{aa} - W_{Aa}} > 0 \qquad (0 \le p^* \le 1)$$

$$\text{if } (W_{aa} - W_{Aa})\,(W_{AA} - W_{Aa}) > 0$$

There are two possibilities

(a)

$$W_{AA} > W_{Aa}$$
$$W_{aa} > W_{Aa} \qquad \text{(homozygate advantageous)}$$

(b)

$$W_{AA} < W_{Aa}$$
$$W_{aa} < W_{Aa} \qquad \text{(hetrozygate advantageous)}$$

Denote

$$f(p) = \frac{W_{AA}p^2 + W_{Aa}p\,(1-p)}{W_{AA}p^2 + 2W_{Aa}p\,(1-p) + W_{aa}\,(1-p)^2}$$

$$f'(0) = \frac{W_{Aa}}{W_{aa}} \qquad f(1) = \frac{W_{Aa}}{W_{AA}} \qquad \text{(after some computation)}$$

(a)

$f'(0) < 1$ $\qquad$ $f'(1) < 1$ $\qquad$ $p^*_{(0)} = 1$ and $p^*_{(1)} = 1$ are <u>stable</u>

$p^*_{(2)}$ is an <u>unstable</u> steady state

(b)

$$f'(0) > 1 \qquad f'(1) > 1 \qquad p^*_{(0)} = 1 \text{ and } p^*_{(1)} = 1 \text{ are } \underline{\text{unstable}}$$

$p^*_{(2)}$ is an <u>stable</u> steady state

If (a) and (b) are not satisfied, then $p^*_{(2)}$ <u>does not</u> exist. Then only $p^*_{(0)} = 0$ and $p^*_{(1)} = 1$ exist and

(c)

$$W_{aa} < W_{Aa} < W_{AA} \qquad \Longrightarrow p^*_{(0)} = 0 \ \underline{\text{unstable}} \qquad p^*_{(1)} = 1 \ \underline{\text{stable}}$$

(d)

$$W_{AA} < W_{Aa} < W_{aa} \qquad \Longrightarrow p^*_{(0)} = 0 \ \underline{\text{stable}} \qquad p^*_{(1)} = 1 \ \underline{\text{unstable}}$$

(ii) If $W_{aa} = W_{Aa} = W_{AA}$ then $P_{n+1} = P_n = P_0, q_n = q_0$, i.e. if there is no relation and the mating is random the frequencies are unchanged. This is the so-called Hardy-Weinberg equilibrium.

Question 4

(i)

$$f(N) = \frac{rN}{1+bN^2}$$

$$f'(N) = \frac{r(1+bN^2) - rN2bN}{(1+bN^2)^2}$$

$$= r\frac{1-bN^2}{(1+bN^2)^2}$$

$$f'(N) = 0 \text{ if } N = \frac{1}{\sqrt{b}}$$

$$N_{max} = f\left(\frac{1}{\sqrt{b}}\right) = \frac{r}{2\sqrt{b}}$$

$$M_{min} = f\left(M_{max}\right) = \frac{2r^2}{\left(4+r^2\right)\sqrt{b}}$$

(ii) The steady states are $N^* = 0$ and $1 = \frac{r}{1+b(N^*)^2}$

$$r = b\left(N^*\right)^2 + 1$$

$$\Rightarrow \quad N^* = \sqrt{\frac{r-1}{b}} (\text{ exists iff } r > 1)$$

$$f'(0) = r \text{ and}$$

$$f'\left(\sqrt{\frac{r-1}{b}}\right) = r\frac{1-(r-1)}{r^2} = \frac{2-r}{r}$$

Bifurcation values:

$$f'(0) = 1 \implies r = 1$$

$$f'\left(\sqrt{\frac{r-1}{b}}\right) = 1 \implies \frac{2-r}{r} = 1 \implies 2-r = r \implies r = 1$$

$$\text{if } 0 < r < 1,\ N^* = 0 \text{ is stable, } N^* = \sqrt{\frac{r-1}{b}} \text{ \underline{unstable}}$$

$$\text{if } r > 1, N^* = 0 \text{ is unstable, } N^* = \sqrt{\frac{r-1}{b}} \text{ is \underline{stable}}$$

(iii) Usually the steady state is N^* = carrying capacity $= \frac{r-1}{b}$, r clearly has the meaning of a birth rate, and $\rightarrow b \sim \frac{1}{(\text{carrying capacity})^2}$ is a parameter related to the carrying capacity of the model.

The population becomes extinct when $N_{min} < 1$. i.e.

$$\frac{2r^2}{(4+r^2)\sqrt{b}} < 1$$

$$\sqrt{b} > \frac{2r^2}{4+r^2} = 2 - \frac{8}{r^2+4} \tag{3}$$

if $b > 4$ then

$$\sqrt{b} = 2 > 2 - \frac{8}{r^2+4}$$

and the above condition (3) is satisfied.

(iv)

$$N_t = \sqrt{\frac{r-1}{b}} + n_t, \qquad n_t \ll 1$$

$$N_t\left(1+bN_t^2\right) = rN_t$$

$$\left(\sqrt{\frac{r-1}{b}} + n_{t+1}\right)\left[1 + b\left(\sqrt{\frac{r-1}{b}} + n_{t-1}\right)^2\right] = r\sqrt{\frac{r-1}{b}} + rn_t$$

$$\left(\sqrt{\frac{r-1}{b}} + n_{t+1}\right)\left[1 + b\frac{r-1}{b} + b\sqrt{\frac{r-1}{b}}n_{t-1} + \underbrace{bn_t^2}_{quadratic}\right] = r\sqrt{\frac{r-1}{b}} + rn_t$$

$$\left(\sqrt{\frac{r-1}{b}} + n_{t+1}\right)\left(r + 2\sqrt{b(r-1)}\,n_{t-1}\right) = r\sqrt{\frac{r-1}{b}} + rn_t$$

$$r\sqrt{\frac{r-1}{b}} + rn_{t+1} + \sqrt{\frac{r-1}{b}}\,2\sqrt{b(r-1)}\,n_{t-1} + quadratic = r\sqrt{\frac{r-1}{b}} + rn_t$$

$$rn_{t+1} + 2(r-1)n_{t-1} - rn_t = 0 \implies n_{t+1} - n_t + 2\frac{r-1}{r}n_{t-1} = 0 \quad \blacksquare$$

(v) Looking for solutions $n = n_0\lambda^t$, we have a characteristic equation

$$\lambda^2 - \lambda + 2\frac{r-1}{r} = 0$$

$$\lambda_{1,2} = \frac{1 \pm \sqrt{1 - 8\frac{r-1}{r}}}{2},$$

$$\text{if } r \to 2, \quad \lambda_{1,2} = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm i\sqrt{3}}{2}$$

$$|\lambda_{1,2}| = \frac{1}{2}|1 \pm i\sqrt{3}|$$

$$= \frac{1}{2}\sqrt{1^2 + (\sqrt{3})^2} = 1 \implies r = 2 \text{ is a bifurcation value.}$$

$$\lambda_{1,2} = e^{(XX\pi i/3)} \implies n_t = n_0 e^{\pi l/3} + \overline{n}_0 e^{\pi lt/3}, if n_0 = |n_0| e^{i\gamma}$$

$$\implies n_t = |n_0| \left[e^{i(\pi t/3 + \gamma)} + e^{-i(\pi 3/t + \gamma)} \right]$$

$$= 2|n_0| \cos\left(\frac{\pi}{3}\, t + \gamma\right)$$

n_0, γ are constants, real, this is a periodic solution and the period p is

$$\frac{\pi}{3}\, p = 2\pi$$

$$\implies p = 6$$

Question 5

(i)

$$\dot{u} = a\left[u(1-u) - \frac{uv}{1+bu}\right] = f(u,v)$$
$$\dot{v} = v\left[\frac{bu}{1+bu} - c\right] = g(u,v)$$

The steady states are the solution of

$$\left| \begin{array}{ll} f(u^*,v^*) & = 0 \\ f(u^*,v^*) & = 0 \end{array} \right. \qquad \text{i.e. } (0,0),\ (1,0)\ \&\ (u^*,v^*)$$

where

$$u^* = \frac{c}{b(1-c)} \qquad\qquad v^* = \frac{b-c(1+b)}{b(1-c)}$$

$u^* > 0$ and meaningful if $u^* = c(1+b)$, i.e. $b > \dfrac{c}{1-c}$.

(ii)

$$\mathcal{A} = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{pmatrix}$$
$$= \begin{pmatrix} a\left(1-2a-\frac{v}{1+bu}+\frac{uvb}{(1+bu)^2}\right) & -\frac{au}{1+bu} \\ \frac{bv}{1+bu}-\frac{b^2uv}{(1+bu)^2} & \frac{bu}{1+bu}-c \end{pmatrix}$$

$\mathcal{A}(0,0) = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}$ with eigenvalues $\lambda_1 = a > 0,\ \lambda_2 = -c < 0 \implies (0,0)$ is a saddle point.

Eigenvectors $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\mathcal{A}(1,0) = \begin{pmatrix} -a & -\frac{a}{1+b} \\ 0 & \frac{b}{1+b}-c \end{pmatrix} \implies \lambda_1 = -a < 0, \qquad \lambda_2 = \frac{b}{1+b} - c > 0$$

since $\lambda_2 = \dfrac{b-c(1+b)}{1+b} = \dfrac{b(1-c)-c}{1+b} > \dfrac{\frac{c}{1-c}(1-c)-c}{1+b} = 0$.

$\implies$ $(1,0)$ is a saddle point, eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} \frac{c(1-b)-b-a(1+b)}{a} \end{pmatrix}$, second entry is negative.

We notice also that $a^* = \frac{\frac{c}{1-c}}{b} < 1$.

$$\mathcal{A}(u^*, v^*) = \begin{pmatrix} \frac{ac[b(1-c)-(c+1)]}{b(1-c)} & -\frac{ac}{b} \\ b(1-c)-c & 0 \end{pmatrix}$$

Characteristic equation

$$\lambda^2 - (\mathrm{tr}A)\,\lambda + \det A = 0$$

$$\lambda^2 - ac\frac{b(1-c)-(c+1)}{b(1-c)}\lambda + \frac{ac}{b}\underbrace{[b(1-c)-c]}_{\text{always positive}} = 0$$

Since $\det A > 0$, for stability, we have $\mathrm{tr}\, A < 0$ i.e.

$$b < \frac{c+1}{1-c} \implies \text{if } \frac{1+c}{1-c} > b > \frac{c}{1-c}$$

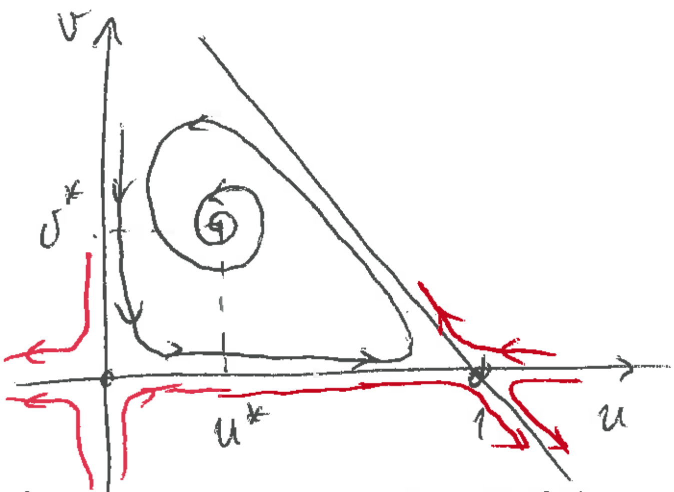
the solution (u^*, v^*) exists and is stable.

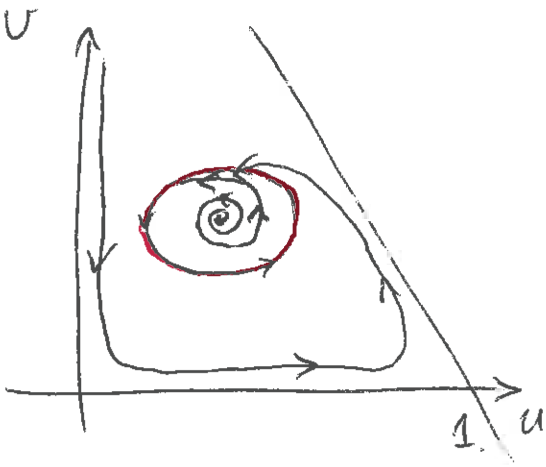
(iii) If b approaches the bifurcation value $b = \dfrac{1+c}{1-c}$, the characteristic equation is

$$\lambda^2 + \frac{ac(1-c)}{1+c}\left[\frac{1+c}{1-c}(1-c)\right] = 0$$

$$\lambda^2 + \frac{ac(1-c)}{1+c} = 0$$

with two imaginary roots $\implies$ (u^*, v^*) becomes centre-type equilibrium. Since the other two equilbina are unstable, we expect that (u^*, v^*) is a stable limit cycle, i.e. (u, v) is approaching a stable periodic solution.





Question 6

(i)

$$\left(\frac{dI}{dt}_{t=0}\right) = rI\,(0)\,[S\,(0) - \rho] > 0$$

So I is increasing if $S_0 > \rho$ which means an epidemic outbreak.

(ii)

$$\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = (-rSI) + (rSI - aI) + (aI) = 0$$

$$\implies N = S + I + R = \text{const} \quad \text{(total number of population)}.$$

(iii)

$$\frac{dS}{dR} = -\frac{S}{\rho} \qquad \implies S = S_0 e^{-R/\rho}$$

$$\frac{dR}{dt} = aI = a\,(N - S - R) = a\left(N - R - S_0 e^{-R/\rho}\right)$$

$$R\,(0) = 0$$

(iv) If $\dfrac{R}{\rho} \ll 1$, we have $e^{-R/\rho} \simeq 1 - \dfrac{R}{\rho} + \dfrac{1}{2}\left(\dfrac{R}{\rho}\right)^2$ and then

$$\frac{dR}{dt} = a\left[N - R - S_0 + \frac{S_0 R}{\rho} - \frac{S_0 R^2}{2\rho^2}\right]$$

$$\frac{dR}{dt} = a\left[N - S_0 + \left(\frac{S_0}{\rho} - 1\right) R - \frac{S_0 R^2}{2\rho^2}\right]$$

$$R\,(0) = 0$$

(v)

$$\frac{dR}{dt} = f\,(N)$$

with roots

$$R_{1,2} = \frac{-\left(\frac{S_0}{\rho} - 1\right) \pm \sqrt{\left(\frac{S_0}{\rho} - 1\right)^2 + 9\,(N - S_0)\,\frac{S_0}{2\rho^2}}}{-2\frac{S_0}{2\rho^2}}$$

The only positive one is

$$R^* = \rho^2 \frac{\sqrt{\left(\frac{S_0}{\rho} - 1\right)^2 + 2\frac{S_0(NS_0)}{\rho^2}} + \left(\frac{S_0}{\rho} - 1\right)}{2S_0}$$

and thus we expect $\lim_{t\to\infty} R(t) = R^* \equiv R(\infty)$. $R(\infty)$ is the number of the removed class after the epidemic outbreak.

(vi) We have an integral of motion $I + S - \varphi \ln S =$ const i.e.

$$I = -S + \varphi \ln S + I(0) + S(0) - \varphi \ln S(0)$$

The maximum $I_{\max}$ occurs at $S = \varphi$ where $\frac{dI}{dt} = 0$.

$$I_{\max} = X - \varphi + \varphi \ln\left(\frac{\varphi}{S_0}\right)$$

Thus for initial values $I_0 > 0$ and $S_0 > \varphi$, the phase trajectory starts with $S > \varphi$ and I increases from I_0 and an epidemic ensues If $S_0 < \varphi$ decreases from I_0 and an epidemic occurs. All trajectories start from the line $S + I = N$ since initially $R(0) = 0$.