DUBLIN INSTITUTE OF TECHNOLOGY, KEVIN STREET, DUBLIN 8

SOLUTION SHEET

COURSE / YEAR: MSc in Math. Physics, MSc. in Appl. Math.

SUBJECT: SPECIAL RELATIVITY AND TENSOR CALCULUS

EXAMINER: ROSSEN IVANOV

EXAMINATION SITTING: SUMMER ☐ SUPPLEMENTAL ☐ AUTUMN ☐ WINTER ☑

EXPECTED SOLUTION TO QUESTION NO: 1

PROPOSED MARK ALLOCATION: 33

a)

The projections of the rod on the horizontal and 'vertical' directions in S' are

$$\Delta x' = l_0 \cos\theta_0$$
$$\Delta y' = l_0 \sin\theta_0$$

[5 marks]

in S : $\Delta y' = \Delta y$ since $y' = y$

$$\Delta y = l_0 \sin\theta_0$$
$$\Delta X = \sqrt{1-\beta^2}\,\Delta X' \;, \quad \beta = \frac{v}{c} \quad \therefore \quad \Delta X = \sqrt{1-\beta^2}\, l_0 \cos\theta_0$$

[5 marks]

$\Rightarrow$ in S $\quad l = \sqrt{(\Delta X)^2 + (\Delta Y)^2} = \sqrt{l_0^2(1-\beta^2)\cos^2\theta_0 + l_0^2 \sin^2\theta_0}$

$$l = l_0\sqrt{1-\beta^2\cos^2\theta_0} \;, \quad l = l_0\sqrt{1-\left(\frac{v}{c}\right)^2\cos^2\theta_0}$$

[5 marks]

b) $\tan\theta = \dfrac{\Delta Y}{\Delta X} = \dfrac{l_0 \sin\theta_0}{\sqrt{1-\beta^2}\, l_0 \cos\theta_0} = \dfrac{\tan\theta_0}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$

[8 marks]

c) $\tan 45^\circ = \dfrac{\tan 30^\circ}{\sqrt{1-\beta^2}} \Rightarrow \dfrac{1}{\sqrt{3}} = \sqrt{1-\beta^2} \Rightarrow \beta = \sqrt{\dfrac{2}{3}}$

$$v = \sqrt{\frac{2}{3}}\, c$$

[10 marks]

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EXPECTED SOLUTION TO QUESTION NO: 2

PROPOSED MARK ALLOCATION: 33

a) The energy of the electron is

$$E_1 = m_0c^2 + K$$ [4 marks]

its momentum is $pc = \sqrt{E_1^2 - m_0^2c^4}$

$$pc = \sqrt{(K+m_0c^2)^2 - m_0^2c^4} = \sqrt{K^2 + 2Km_0c^2} = \sqrt{K(K+2m_0c^2)}$$

$$\Rightarrow pc = 1.422 \text{ MeV}$$ [6 marks]

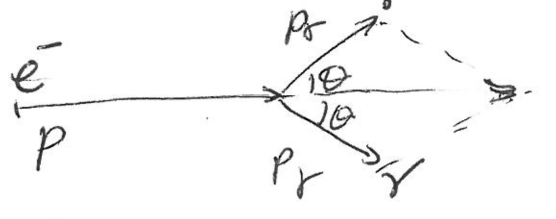
The total energy of the two electrons is

$$E = E_1 + m_0c^2 = K + 2m_0c^2 = 2.022 \text{ MeV}$$ [4 marks]

the two photons emerge from the collision, each with energy

$$E_\gamma = \tfrac{1}{2}E = 1.011 \text{ MeV}, \quad P_\gamma = \frac{E_\gamma}{c} = 1.011 \frac{\text{MeV}}{c}$$ [6 marks]

b)



$$\Rightarrow p = 2p_\gamma \cos\theta$$

(Momentum conservation) [7 marks]

$$\cos\theta = \frac{p}{2p_\gamma} = \frac{pc}{2p_\gamma c} = \frac{1.422}{2.022} = 0.703 \Rightarrow \theta = 45.3^\circ$$

[6 marks]

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EXPECTED SOLUTION TO QUESTION NO: 3

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PROPOSED MARK ALLOCATION: 33

a) $L^a{}_b = \dfrac{\partial x'^a}{\partial x^b}$ - transformation matrix

$$A'^{ij} = L^i{}_k L^j{}_q A^{kq}$$

If $A^{kq} = A^{qk}$ is symmetric then

$$\Rightarrow A'^{ji} \equiv L^j{}_q L^i{}_k A^{qk} = L^j{}_q L^i{}_k A^{kq} = L^i{}_k L^j{}_q A^{kq} \equiv A'^{ij}$$

$\Rightarrow A'^{ji} = A'^{ij}$ is symmetric. [9 marks]

b) Lorentz transformations can be determined from the relation between the coordinates $(x'^a) = (t', x', y', z') \longleftrightarrow (x^a) = (t, x, y, z)$

as they transform as contravariant vector:

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}, \quad x' = \frac{x - vt}{\sqrt{1-(\frac{v}{c})^2}}, \quad y' = y, \quad z' = z$$

$$L = \begin{pmatrix} \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} & \frac{-\frac{v}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} & 0 & 0 \\ \frac{-v}{\sqrt{1-\frac{v^2}{c^2}}} & \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

[4 marks]

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PROPOSED MARK ALLOCATION: ____

Symmetric tensor transforms the same way as $x^i x^j$

i.e. $A^{ij} \leftrightarrow x^i x^j$

$$\Rightarrow A'^{00} \leftrightarrow (x'^0)^2 = (t')^2 = \frac{t^2 - 2\frac{vxt}{c^2} + \frac{v^2 x^2}{c^2}}{1-\frac{v^2}{c^2}} \leftrightarrow \frac{A^{00} - 2\frac{v}{c^2}A^{01} + \frac{v^2}{c^2}A^{11}}{\left(\sqrt{1-\frac{v^2}{c^2}}\right)^2}$$

$$\boxed{(A')^{00} = \frac{1}{1-\frac{v^2}{c^2}}\left(A^{00} - 2\frac{v}{c^2}A^{01} + \frac{v^2}{c^2}A^{11}\right)}$$

[2 marks]

$$(A')^{11} \leftrightarrow (x'^1)^2 = \left(\frac{x-vt}{\sqrt{1-\left(\frac{v}{c}\right)^2}}\right)^2 = \frac{x^2 - 2vxt + v^2t^2}{1-\frac{v^2}{c^2}} \leftrightarrow \frac{A^{11} - 2vA^{01} + \frac{v^2}{c^2}A^{00}}{\left(\sqrt{1-\frac{v^2}{c^2}}\right)^2}$$

$$\boxed{(A')^{11} = \frac{1}{1-\frac{v^2}{c^2}}\left(A^{11} - 2vA^{01} + \frac{v^2}{c^2}A^{00}\right)}$$

[2 marks]

$$(A')^{22} \leftrightarrow (x'^2)^2 = y^2 \leftrightarrow A^{22}$$

$$\boxed{\begin{aligned}(A')^{22} &= A^{22} \\ (A')^{33} &= A^{33}\end{aligned}}$$ similarly

[4 marks]

$$(A')^{01} \leftrightarrow (x')^0 (x')^1 = t' \cdot x' = \frac{\left(t-\frac{vx}{c^2}\right)(x-vt)}{1-\frac{v^2}{c^2}} = \frac{\left(1+\frac{v^2}{c^2}\right)xt - \frac{vx^2}{c^2} - vt^2}{1-\frac{v^2}{c^2}}$$

$$\boxed{(A')^{01} = \frac{1}{1-\frac{v^2}{c^2}}\left(-vA^{00} + \left(1+\frac{v^2}{c^2}\right)A^{01} - \frac{v}{c^2}A^{11}\right)}$$

[2 marks]

$$(A')^{12} \leftrightarrow (x')^1 (x')^2 = x' \cdot y = \frac{(x-vt)y}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{xy - vty}{\sqrt{1-\frac{v^2}{c^2}}} \leftrightarrow \frac{A^{12} - vA^{02}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\boxed{(A')^{12} = \frac{A^{12} - vA^{02}}{\sqrt{1-\frac{v^2}{c^2}}}}$$

[2 marks]

$$A'^{02} \leftrightarrow (x')^0 (x')^2 = t' \cdot y = \frac{\left(t - \frac{vx}{c^2}\right)y}{\sqrt{1-\frac{v^2}{c^2}}} \leftrightarrow \frac{A^{02} - \frac{v}{c^2}A^{12}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\boxed{A'^{02} = \frac{A^{02} - \frac{v}{c^2}A^{12}}{\sqrt{1-\frac{v^2}{c^2}}}}$$

[2 marks]

$$(A')^{23} \leftrightarrow (x')^2 (x')^3 = y'z' = yz \leftrightarrow A^{23}$$

[2 marks]

$$\boxed{(A')^{23} = A^{23}}$$

$$(A')^{13} \leftrightarrow (x')^1 (x')^3 = x' \cdot z' = \frac{(x-vt)z}{\sqrt{1-\frac{v^2}{c^2}}} \leftrightarrow \frac{A^{13} - vA^{03}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\boxed{(A')^{13} = \frac{A^{13} - vA^{03}}{\sqrt{1-\frac{v^2}{c^2}}}}$$

[2 marks]

$$A'^{03} \leftrightarrow (t')(z') = \frac{\left(t - \frac{vx}{c^2}\right)z}{\sqrt{1-\frac{v^2}{c^2}}} \leftrightarrow \frac{A^{03} - \frac{v}{c^2}A^{13}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\boxed{(A')^{03} = \frac{A^{03} - \frac{v}{c^2}A^{13}}{\sqrt{1-\frac{v^2}{c^2}}}}$$

[2 marks]

Another possible solution: from $\boxed{A' = LAL^T}$

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		EXPECTED SOLUTION TO QUESTION NO:	4
EXAMINER:			
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The coordinates are $x^1=r$, $x^2=\theta$ then

a) $g_{11}=1+\alpha^2r^2$, $g_{12}=0$, $g_{22}=r^2$

$g^{11}=(1+\alpha^2r^2)^{-1}$, $g^{12}=0$, $g^{22}=r^{-2}$ (5 marks)

b) $g_{11,1}=2\alpha^2 r$, $g_{22,1}=2r$, $g_{12,1}=g_{12,2}=g_{11,2}=g_{22,2}=0$

$\Gamma_{i,mn}=\frac{1}{2}(g_{im,n}+g_{in,m}-g_{mn,i}) \Rightarrow$

$\Gamma_{1,11}=\alpha^2 r$, $\Gamma_{1,22}=-r$, $\Gamma_{2,12}=\Gamma_{2,21}=r$

$\Gamma_{1,12}=\Gamma_{1,21}=\Gamma_{2,11}=\Gamma_{2,12}=\Gamma_{2,22}=0$

$\Gamma^1_{11}=\frac{\alpha^2 r}{1+\alpha^2r^2}$, $\Gamma^2_{12}=\Gamma^2_{21}=\frac{1}{r}$, $\Gamma^1_{22}=-\frac{r}{1+\alpha^2r^2}$, $\Gamma^2_{11}=\Gamma^1_{12}=\Gamma^1_{21}=\Gamma^2_{22}=0$

[10 marks]

c) The only nonzero components of $R^a{}_{bcd}$ are

$R^1{}_{212}=\frac{\alpha^2r^2}{(1+\alpha^2r^2)^2}$, $R^2{}_{212}=\frac{1}{r^2}$

$\Rightarrow R_{1212}=g_{11}R^1{}_{212}+g_{12}R^2{}_{212}=\frac{\alpha^2r^2}{1+\alpha^2r^2}$ is the only essential component (others are obtained by symmetry properties)

[11 marks]

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EXPECTED SOLUTION TO QUESTION NO:

PROPOSED MARK ALLOCATION:

d) The Scalar curvature is

$$R = g^{ij} g^{lm} R_{iljm}$$

Nonzero terms will be

$$R = g^{11} g^{22} R_{1212} + g^{22} g^{11} R_{\underbrace{21}\,\underbrace{21}} =$$

$$= 2 g^{11} g^{22} R_{1212} = 2\left(1+\alpha^2 r^2\right)^{-1} r^{-2} \cdot \frac{\alpha^2 r^2}{1+\alpha^2 r^2}$$

$$R = \frac{2\alpha^2}{(1+\alpha^2 r^2)^2}$$

(7 marks)