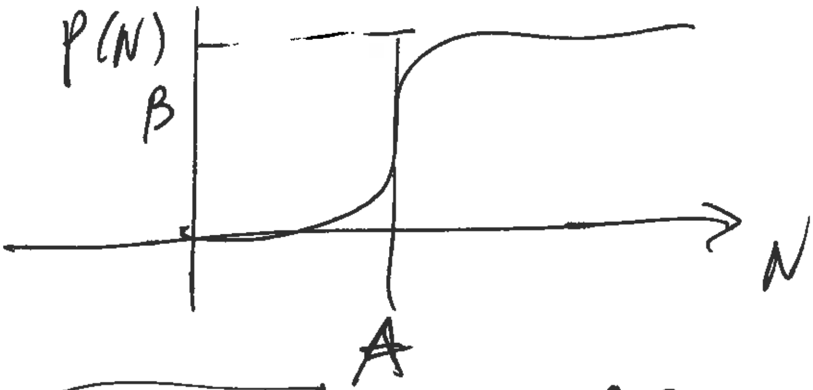
03/10/2011

Insect Outbreak Model

$$\frac{dN}{dt} = \underbrace{r_B N\left(1 - \frac{N}{K_B}\right)}_{\text{logistic term}} - \underbrace{p(N)}_{\text{predation term}}$$

Spruce Budworm



$$p(N) = \frac{BN^2}{A^2+N^2}$$

$$\boxed{u = \frac{N}{A}}, \quad \underbrace{r = \frac{Ar_B}{B}, \quad q = \frac{K_B}{A}}_{\text{parameters}}, \quad \boxed{\tau = \frac{Bt}{A}}$$

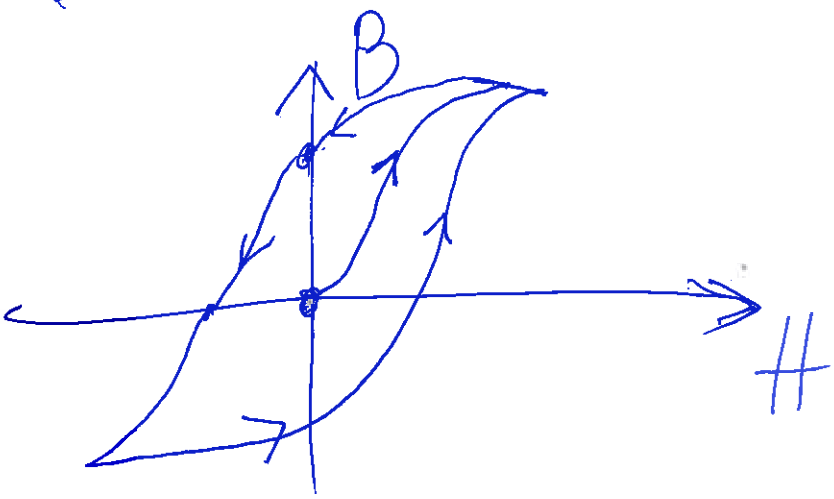
$$\frac{du}{d\tau} = ru\left(1 - \frac{u}{q}\right) - \frac{u^2}{1+u^2} = f(u; q, r)$$

Equilibrium states $\quad \underline{f(u^*; q, r) = 0}$

$$ru\left(1 - \frac{u}{q}\right) = \frac{u^2}{1+u^2} \Rightarrow u^* = 0$$

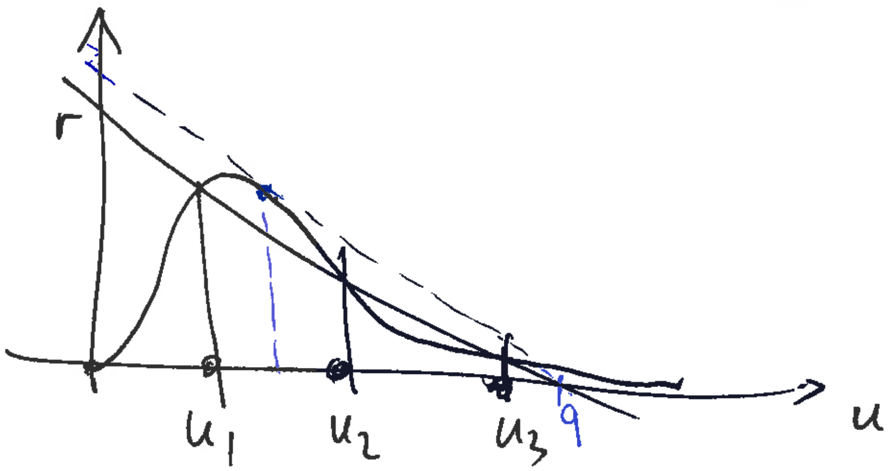
$$r\left(1-\frac{u}{q}\right)=\frac{u}{1+u^2} \qquad \underbrace{r(1+u^2)\left(1-\frac{u}{q}\right)=u}_{\text{cubic eq. for } u}$$

hysteresis - feromagnetic materials



(?) . For which values of (q, r) we have 1 or 3 real solutions?

$$f(u^*) = 0 \iff \boxed{r\left(1-\frac{u}{q}\right) = \frac{u}{1+u^2}} \quad (1)$$



$$\text{Slope} = -\frac{r}{q} = \frac{d}{du}\left(\frac{u}{1+u^2}\right) = \frac{1.(1+u^2) - u(2u)}{(1+u^2)^2}$$

$$-\frac{r}{q} = \frac{1-u^2}{(1+u^2)^2} \qquad \frac{r}{q} = \frac{u^2-1}{(u^2+1)^2} \quad (2)$$

$$r - \left(\frac{r}{q}\right)u = \frac{u}{1+u^2} \Rightarrow r = \left(\frac{r}{q}\right)u + \frac{u}{1+u^2}$$

$$r = u\frac{u^2-1}{(u^2+1)^2} + \frac{u}{1+u^2} = \frac{u[u^2 \cancel{-1} + u^2 \cancel{+1}]}{(u^2+1)^2} = \frac{2u^3}{(u^2+1)^2}$$

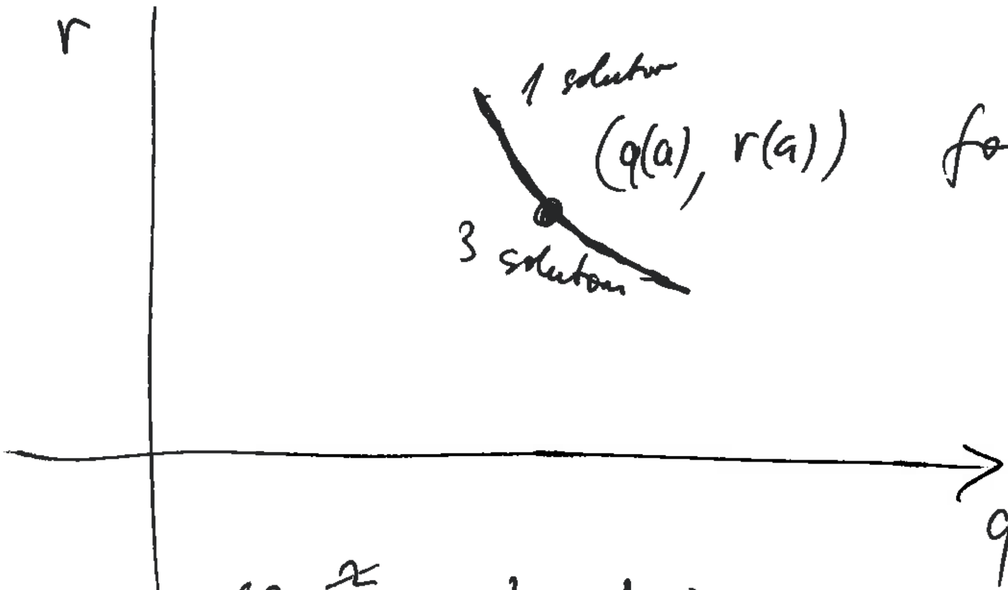
$$\boxed{r = \frac{2u^3}{(u^2+1)^2}} \quad q = r\frac{(u^2+1)^2}{u^2-1} = \frac{2u^3}{\cancel{(u^2+1)^2}} \cdot \frac{\cancel{(u^2+1)^2}}{u^2-1}$$

$$\boxed{q = \frac{2u^3}{u^2-1}}$$

$u = a$ is a parameter

$$r(a) = \frac{2a^3}{(a^2+1)^2} > 0 \qquad q(a) = \frac{2a^3}{a^2-1} > 0$$

$$\boxed{a > 1}$$



$$\frac{dr}{da} = \frac{6a^2(a^2+1)^{\cancel{2}} - 2a^3 \cdot 2\cancel{(a^2+1)} \cdot 2a}{(a^2+1)^{\cancel{4}\,3}} = \frac{a^2(6a^2+6-8a^2)}{(1+a^2)^3}$$

$$\frac{dr}{da} = \frac{a^2(6-2a^2)}{(1+a^2)^3} = \frac{2a^2(3-a^2)}{(1+a^2)^3}$$

$$\frac{dq}{da} = \frac{d}{da}\left(\frac{2a^3}{a^2-1}\right) = \frac{6a^2(a^2-1) - 2a^3(2a)}{(a^2-1)^2} = \frac{2a^2(3a^2-3-2a^2)}{(a^2-1)^2}$$

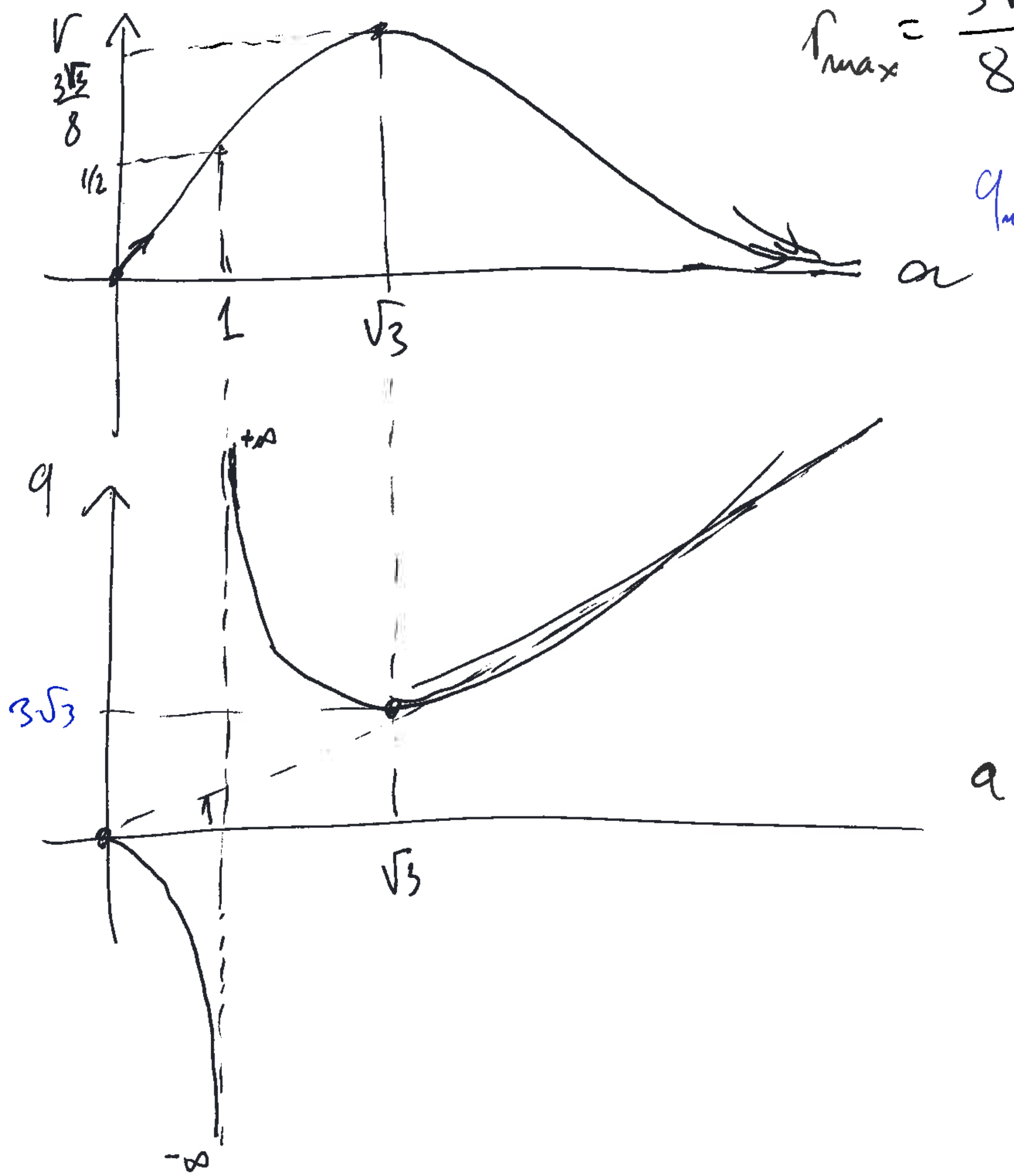
$$\boxed{\frac{dq}{da} = \frac{2a^2(a^2-3)}{(a^2-1)^2}}$$

$$\frac{dr}{da} = \frac{2a^2(3-a^2)}{(1+a^2)^3} \qquad \frac{dq}{da} = \frac{2a^2(a^2-3)}{(a^2-1)^2}$$

$$r(a) = \frac{2a^3}{(a^2+1)^2} \qquad q(a) = \frac{2a^3}{a^2-1}$$

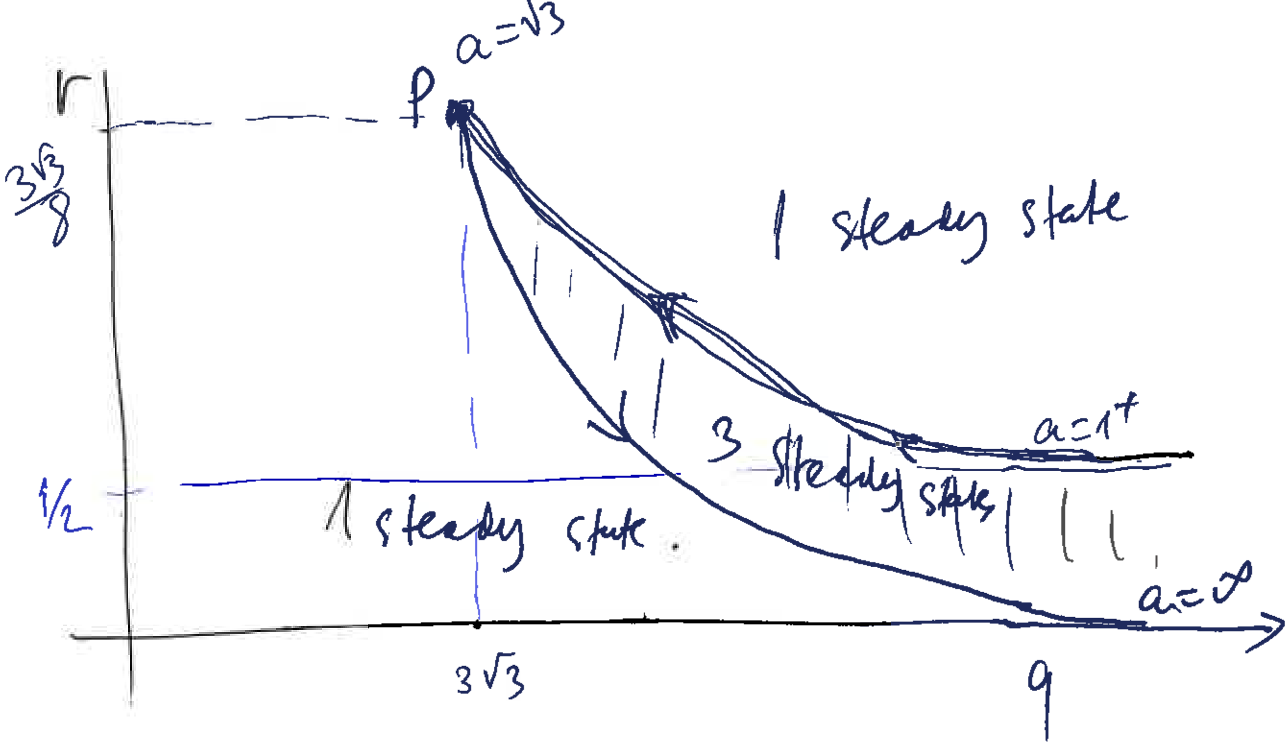
$r_{max} = \dfrac{3\sqrt{3}}{8}$ when $a = \sqrt{3}$

$q_{min} = \dfrac{2 \cdot 3\sqrt{3}}{2} = 3\sqrt{3}$



$a > 1$

$a \to 1^+$



P – 'cusp point'

Delay Models

$$\frac{dN(t)}{dt} = f(N(t))$$

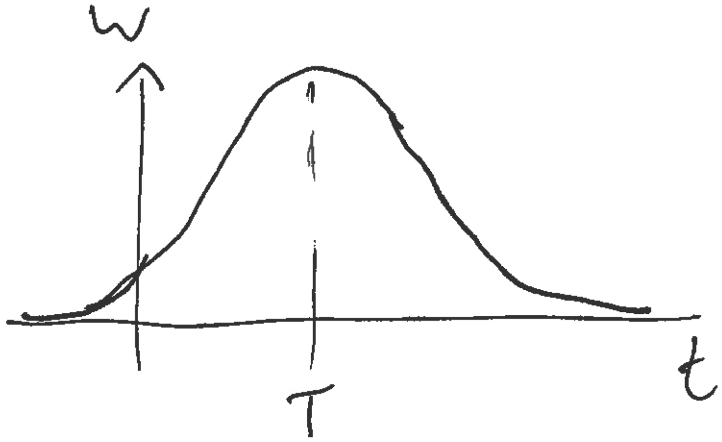
$$\frac{dN(t)}{dt} = f(N(t), N(t-T))$$

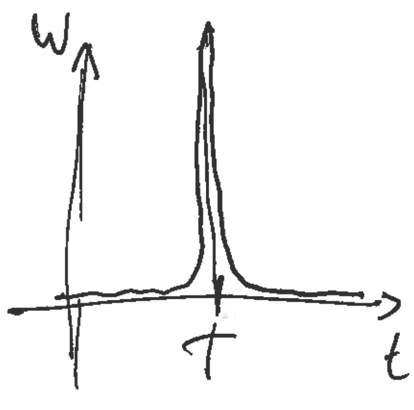
T - 'Delay'

$$\frac{dN}{dt} = rN(t)\underbrace{\left[1 - \frac{N(t-T)}{K}\right]}_{} \qquad (1)$$

regulatory effect which depends on the population at earlier time $t-T$, rather than t.

$$\frac{dN}{dt} = rN(t)\left[1 - \frac{1}{K}\int_{-\infty}^{t} w(t-t')N(t')dt'\right] \qquad (2)$$





Let $w(t) = \delta(t-T)$.

Then $\int_{-\infty}^{t} w(t-t')\, N(t')\, dt' \longrightarrow$

$$\longrightarrow \int_{-\infty}^{t} \delta(t-t'-T)\, \underline{N(t')}\, dt' = N(t-T)$$

$$t' = t - T$$

Delta function

φ, g defined on $(-\infty, \infty)$

$(\varphi, g) = \int_{-\infty}^{\infty} f(x) g(x)\, dx$ – scalar product.

$(\varphi, \delta) = \varphi(0)$

$$\int_{-\infty}^{\infty} \varphi(x)\, \delta(x)\, dx = \varphi(0)$$

$$\int_{-\infty}^{\infty} \varphi(x)\, \delta(x-x_0)\, dx = \varphi(x_0)$$

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1.$$

$$\delta(x) = \frac{1}{\pi}\lim_{\varepsilon \to 0} \frac{\varepsilon}{x^2+\varepsilon^2}$$

$x \neq 0 \quad \delta(x) = 0$

$x = 0 \quad \frac{1}{\varepsilon} \to \infty$

$$\int_{-\infty}^{\infty} \delta(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\varepsilon\, dx}{x^2+\varepsilon^2} = \frac{1}{\pi}\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \frac{d\left(\frac{x}{\varepsilon}\right)}{1+\left(\frac{x}{\varepsilon}\right)^2}$$

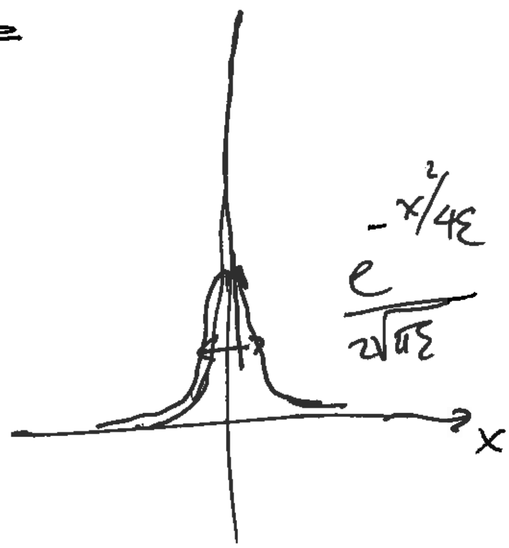
$$= \frac{1}{\pi}\lim_{\varepsilon \to 0} \left(\tan^{-1}\left(\frac{x}{\varepsilon}\right)\Big|_{-\infty}^{\infty} \right) = \frac{1}{\pi}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 1.$$

$$\tan \pi/2 = \frac{\sin \pi/2}{\cos \pi/2} = \infty$$

$$\tan(-\pi/2) = \underline{\underline{-\infty}}$$

$$\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi x} \sin\left(\frac{x}{\varepsilon}\right)$$

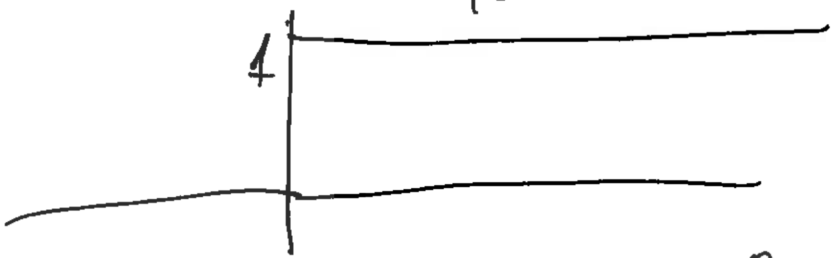
$$\delta(x) = \lim_{\varepsilon \to 0^+} \frac{1}{2\sqrt{\pi\varepsilon}}\, e^{-\frac{x^2}{4\varepsilon}}$$



$$\delta(\alpha x) = \frac{\delta(x)}{|\alpha|}$$

$$\delta(-x) = \delta(x)$$

$$\delta(g(x)) = \frac{\delta(x-x_0)}{|g'(x_0)|} \qquad g(x_0) = 0,\ g'(x_0) \neq 0$$



$\theta(t)$ 'step'

$$\theta(t) = \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases}$$

$$\delta(t) = \frac{d\,\theta(t)}{dt}$$

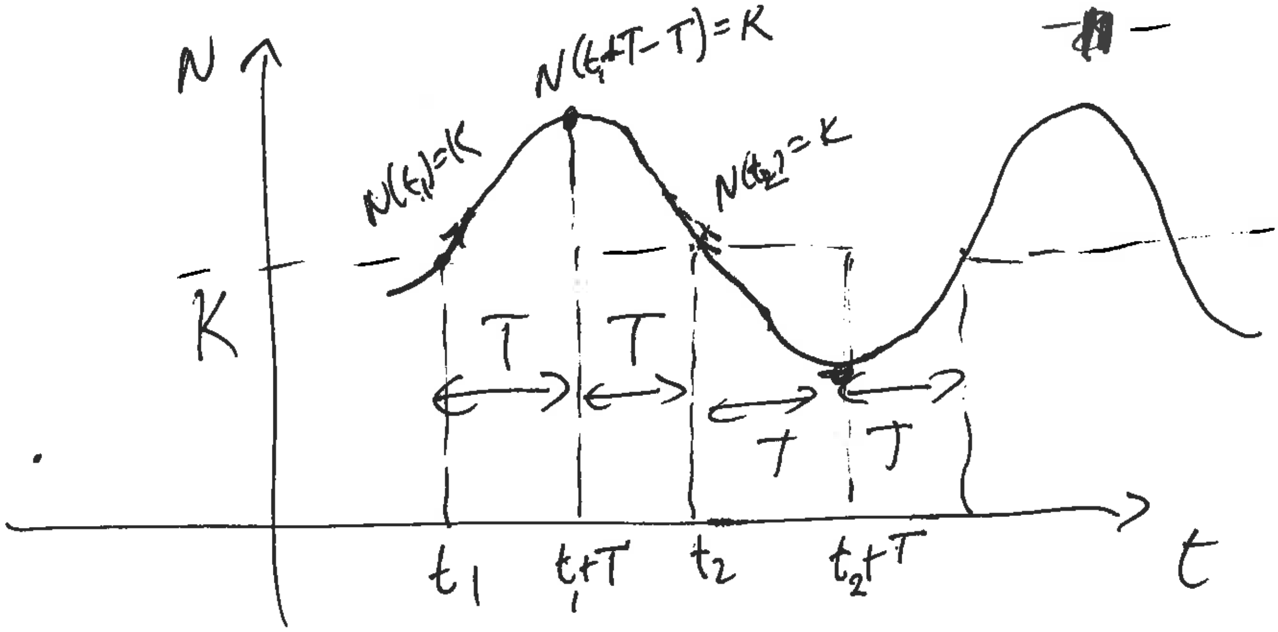
$$\int_{-\infty}^{\infty} \delta(t)\,dt = \int_{-\infty}^{\infty} \frac{d\theta(t)}{dt}\,dt = \underbrace{\theta(\infty)}_{=1} - \underbrace{\theta(-\infty)}_{=0} = 1.$$

$$\boxed{x\,\delta(x) = 0} \qquad x^n\,\delta(x) = 0$$

$$\cancel{\int \varphi(x)\,\delta(x-x_0) = \varphi(x_0)}$$

$\cancel{x_0 = 0}$

$$f(x)\,\delta(x) = f(0)\,\delta(x)$$



$$\frac{dN}{dt} = rN(t)\left(1 - \frac{N(t-T)}{K}\right)$$

Suppose that $t=t_1 \Rightarrow N(t_1)=K$

? periodic behaviour?

Period $= 4T = 4 \times$ Delay ?

Example:

$$\frac{dN}{dt} = -\frac{\pi}{2T}N(t-T) \quad , \quad N(t) = A\cos\frac{\pi t}{2T}$$

$$\frac{dN}{dt} = -A\left(\frac{\pi}{2T}\right)\sin\frac{\pi t}{2T} \overset{?}{=} \underbrace{\left(-\frac{\pi}{2T}\right)A\cos\left(\frac{\pi(t-T)}{2T}\right)}_{-\pi/2\, N(t-T)}$$

$$\sin\frac{\pi t}{2T} = \cos\left(\frac{\pi}{2T} - \frac{\pi \cancel{T}}{2\cancel{T}}\right) = \cos\left(\frac{\pi t}{2T} - \frac{\pi}{2}\right) =$$

$$= \cos\frac{\pi t}{2T}\cos\frac{\pi}{2} + \sin\left(\frac{\pi t}{2T}\right)\sin\left(\frac{\pi}{2}\right) = \sin\frac{\pi t}{2T} \checkmark$$

Period of the solution t_p

$$\frac{\cancel{\pi}\, t_p}{2T} = 2\cancel{\pi} \quad \Rightarrow \cos \text{ is } 2\pi\text{-periodic}$$

$$\boxed{t_p = 4T}$$

<u>stable limit cycle</u> periodic solutions

Remark: periodic solutions are not possible for models without delay.

Proof! $\frac{dN}{dt} = f(N)$, $\quad \boxed{N(t) = N(t \pm t_p)}$.

$$0 < \int_0^{t_p} \left(\frac{dN}{dt}\right)^2 dt = \int_0^{t_p} \underbrace{\frac{dN}{dt}}_{= f(N)} \cdot \frac{dN}{dt}\, dt = \int_{N(0)}^{N(t_p)} f(N)\, dN$$

$$= \int_{N(0)}^{N(t_p)} f(N)\, dN = 0$$

Contradiction!