DT9209/DT9210

Methods for Applied Mathematics, MATH 9951

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Semester 1, 2017/2018

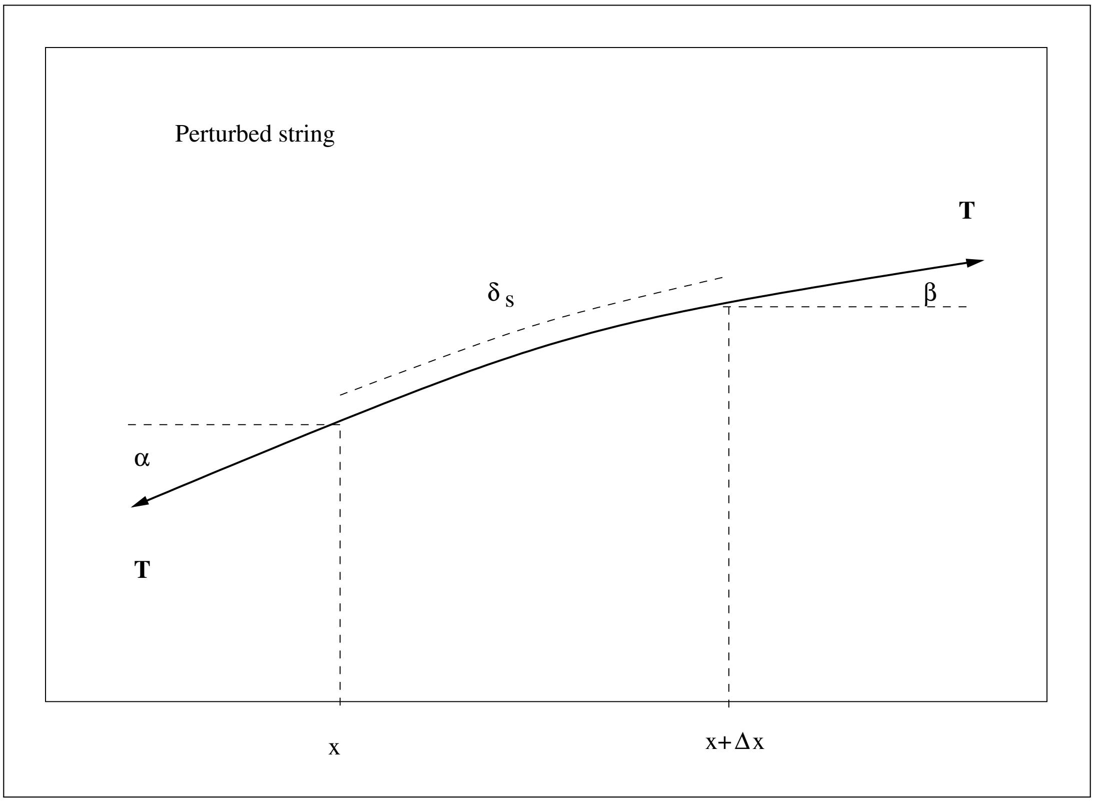
Models

Wave equations; Cauchy stress tensor; Navier equation of motion; Generalized Hooke's Law; Helmholtz Decomposition theorem

We assume the string is long, light, flexible and does not stretch:

1. $\implies$ tension force **T** is constant
2. $\implies$ **T** lies along direction of string

Consider an element of the perturbed string of length δs between x and $x + \Delta x$ as shown



1. Since **T** is constant, the total tension force acting on the element shown is

$$\begin{aligned} T(\sin\beta - \sin\alpha) &\approx T(\tan\beta - \tan\alpha) \\ &= T(u_x(x+\Delta x, t) - u_x(x,t)) \\ &\approx Tu_{xx}(x,t)\Delta x \end{aligned}$$

2. if an external force f per unit length ($f\delta s \approx f\Delta x$) is acting on the is acting on the element then the total force is given by

$$Tu_{xx}(x,t)\Delta x + F\rho\Delta x$$

where $F = f/\rho$

3. And by Newtons second law

$$\begin{aligned} \text{Total Force} &= \rho\delta s u_{tt}(x,t) \\ &\approx \rho\Delta x u_{tt}(x,t) \end{aligned}$$

$$u_{tt}(x,t) = c^2 u_{xx}(x,t) + F \quad \text{where } c^2 \equiv T/\rho$$

The general solution to the wave equation is

$$u(x,t) = F(x - ct) + G(x + ct)$$

since

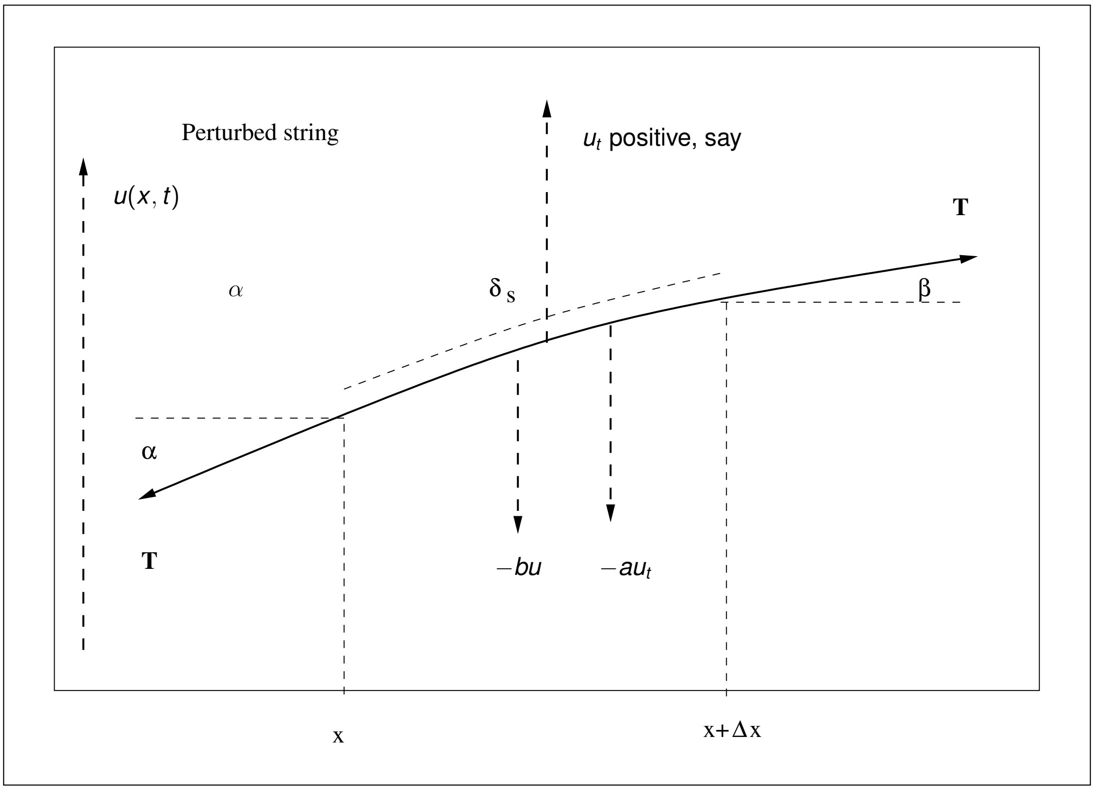
$$u_{xx} = F'' + G'' \quad u_{tt} = c^2 F'' + c^2 G''$$

and the general solution is the sum of any *waveform F* travelling to the right with speed c, and waveform G travelling to the left with speed c.

For a long, light, flexible string which does not stretch as in previous model for wave equation derivation in one dimension.

Assume that a damping force, per unit length, is present proportional to the velocity of the string's displacement with constant of proportionality a.

Assume also that a restoring force, per unit length, proportional to the displacement is present with constant of proportionality b.



The resultant equation is

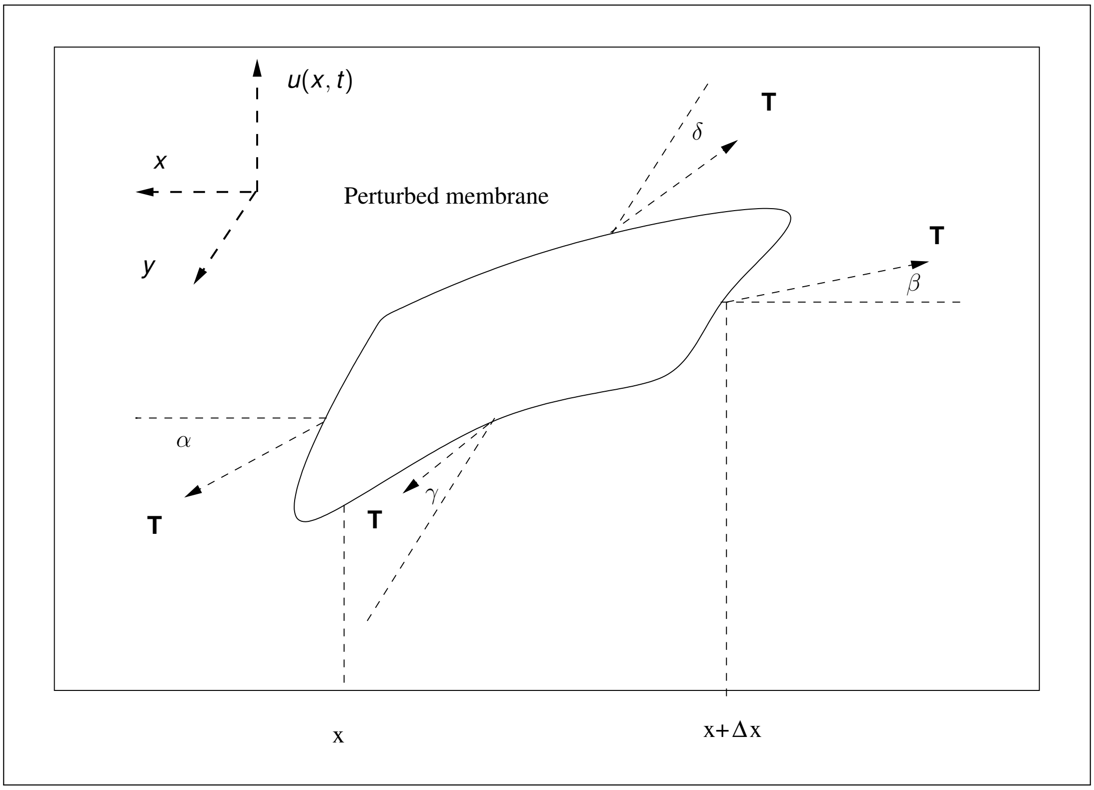
$$u_{tt}(x,t) = c^2 u_{xx}(x,t) - \hat{a} u_t(x,t) - \hat{b} u(x,t)$$

with $\hat{a} = a/\rho$ and $\hat{b} = b/\rho$.

Assuming a membrane which has long dimensions, is light, flexible and does not stretch, and undergoes small deflections.

Consider a small section of the membrane under a small deflection such that the lengths os the edges may be approximated with length Δx and Δy.

The tension force pulling on the edge of the section of length Δl is given by $T\Delta l$ (pulling outwards).



1. The total tension force on the section of membrane in the positive vertical direction is

$$
\begin{aligned}
& T(\sin\beta - \sin\alpha)\Delta y + T(\sin\delta - \sin\gamma)\Delta x \\
\approx & T(\tan\beta - \tan\alpha)\Delta y + T(\tan\delta - \tan\gamma)\Delta x \\
= & T(u_x(x+\Delta x, y, t) - u_x(x, y, t))\Delta y \\
& + T(u_y(x, y+\Delta y, t) - u_y(x, y, t))\Delta x \\
\approx & Tu_{xx}(x, y, t)\Delta x\Delta y + Tu_{yy}(x, y, t)\Delta y\Delta x \\
= & T\Delta A(u_{xx} + u_{yy})
\end{aligned}
$$

2. if an external force f per unit area is acting on the is acting on the element then the total force is given by

$$T\Delta A(u_{xx} + u_{yy}) + F\sigma\Delta A$$

with $F = f/\sigma$

3. And by Newtons second law

$$\text{Total Force} \approx \ \sigma\delta A u_{tt}(x, t, t)$$

$$\boxed{u_{tt}(x, y, t) = c^2\nabla^2 u(x, t) + F \quad \text{where } c^2 \equiv T/\sigma}$$

In continuum mechanics, the Cauchy stress tensor τ defines the stress at a point inside a material. The stress vetor $\mathbf{T}$, acting on a surface element with normal outward unit vector $\hat{\mathbf{n}}$, is obtained via

$$\mathbf{T} = \boldsymbol{\tau} \cdot \hat{\mathbf{n}}$$

where

$$\boldsymbol{\tau} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

Conservation of angular momentum requires that τ is symmmetric.

By considering a small volume element $\Delta V = \Delta x \Delta y \Delta z$ of mass density ρ, undergoing displacement $\mathbf{u} = (u, v, w)$, the equations of motion are easily shown to follow

$$\rho u_{tt} = \tau_{xx,x} + \tau_{xy,y} + \tau_{xz,z}$$

$$\rho v_{tt} = \tau_{yx,x} + \tau_{yy,y} + \tau_{yz,z}$$

$$\rho w_{tt} = \tau_{zx,x} + \tau_{zy,y} + \tau_{zz,z}$$

or

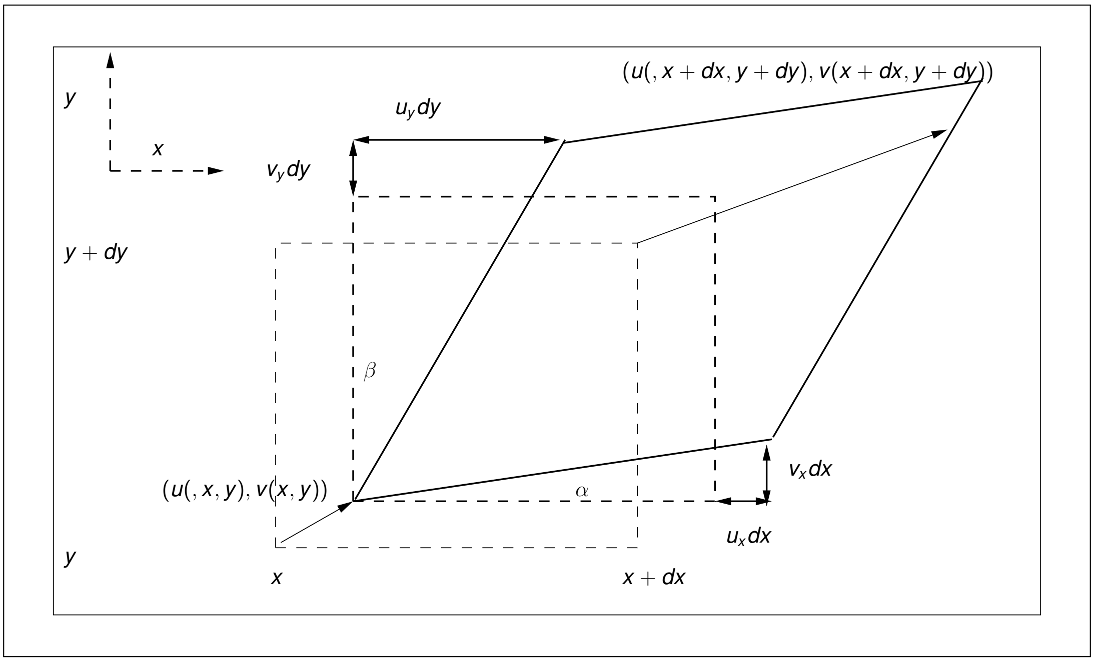
$$\rho \mathbf{u}_{tt} = \nabla \cdot \boldsymbol{\tau}$$

An infinitessimal volume element $dV = dxdydz$ at $t = 0$ is deformed into a parallelpiped with vertices at $\boldsymbol{u}(x, y, z, t)$, $\boldsymbol{u}(x + dx, y, z, t)$, $\boldsymbol{u}(x + dx, y + dy, z, t)$, etc. We assume small displacements.

The change in the length of a face with a dimension in the x-direction is given by $u_x dx$. The normal strain in the x-direction ϵ_{xx} is defined as the proportional change in this length,

$$\epsilon_{xx} = u_x \quad \epsilon_{yy} = v_y \quad \epsilon_{zz} = w_z$$

Orthogonal edge elements of the parallelpiped will be deformed by angles α, β (say) with respect to their original (coordinate) directions.



For the illustrated case, it is clear that

$$\tan\alpha = \frac{v_x dx}{dx + u_x dx} = \frac{v_x}{1 + u_x} \quad \tan\beta = \frac{u_y}{1 + v_y}$$

or, given displacements are small,

$$\alpha \approx v_x \quad \beta \approx u_y$$

and finally, the shear strain tensor component ϵ_{xy}, for an area element in the $x - y$ plane, is defined as the *average* of these angles

$$\epsilon_{xy} = \frac{u_y + v_x}{2} \quad \epsilon_{yz} = \frac{v_z + w_y}{2} \quad \epsilon_{xz} = \frac{u_z + w_x}{2}$$

Hence, we have in matrix form

$$\boldsymbol{\epsilon} = \begin{pmatrix} u_x & \frac{u_y+v_x}{2} & \frac{u_z+w_x}{2} \\ \frac{u_y+v_x}{2} & v_y & \frac{v_z+w_y}{2} \\ \frac{u_z+w_x}{2} & \frac{v_z+w_y}{2} & w_z \end{pmatrix}$$

Recall simple form of Hooke's Law states that the restoring force on a normal displacement in one dimension is proportional to the magnitude of the displacement, or,

$$\tau_{xx} = \kappa \epsilon_{xx}$$

for some constant κ which is intrinsic to the material.

The generalised form of Hook's Law may be written in tensor notation as

$$\boldsymbol{\tau} = \boldsymbol{\kappa} : \boldsymbol{\epsilon}$$

or, in component form,

$$\tau_{ij} = \kappa_{ijkl} \epsilon_{kl}$$

where $\boldsymbol{\kappa}$ is the elasticity tensor whose components are intrinsic.

For an isotropic, homogeneous medium

$$\kappa_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

where λ and μ are the first and second Lamé parameters.

Exercise: Write the Generalized Hooke's Law for an isotropic, homogeneous medium:

1. in matrix form;
2. in tensor form in terms of the dilation $\theta \equiv \mathrm{tr}(\boldsymbol{\epsilon}) = \nabla \cdot \mathbf{u}$.

Infinitesimal strain tensor for an isotropic, homogeneous medium

Recall

$$\boldsymbol{\tau} = \boldsymbol{\kappa} : \boldsymbol{\epsilon}$$

with

$$\boldsymbol{\epsilon} = \begin{pmatrix} u_x & \frac{u_y+v_x}{2} & \frac{u_z+w_x}{2} \\ \frac{u_y+v_x}{2} & v_y & \frac{v_z+w_y}{2} \\ \frac{u_z+w_x}{2} & \frac{v_z+w_y}{2} & w_z \end{pmatrix}$$

and

$$\kappa_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

Hence

$$\boldsymbol{\tau} = \lambda\theta\mathbf{I} + 2\mu\boldsymbol{\epsilon}$$

or in matrix form

$$\boldsymbol{\tau} = \begin{pmatrix} \lambda\theta + 2\mu u_x & \mu(u_y + v_x) & \mu(u_z + w_x) \\ \mu(u_y + v_x) & \lambda\theta + 2\mu v_y & \mu(v_z + w_y) \\ \mu(u_z + w_x) & \mu(v_z + w_y) & \lambda\theta + 2\mu w_z \end{pmatrix}$$

Recall

$$\rho \mathbf{u}_{tt} = \nabla \cdot \boldsymbol{\tau}$$

Show the Navier equation of motion may be obtained directly

$$\rho \mathbf{u}_{tt} = (\lambda + \mu)\nabla\nabla \cdot \mathbf{u} + \mu\nabla^2\mathbf{u}$$

For an*incompressible material*, $\nabla \cdot \mathbf{u} = 0$, and the Navier equation becomes

$$\mathbf{u}_{tt} = c_T^2\nabla^2\mathbf{u} \quad c_T^2 \equiv \mu/\rho$$

For an irrotational matial, $\nabla \times \mathbf{u} = 0$, show $\nabla\nabla \cdot \mathbf{u} = \nabla^2\mathbf{u}$, and hence

$$\mathbf{u}_{tt} = c_L^2\nabla^2\mathbf{u} \quad c_L^2 \equiv (\lambda + 2\mu)/\rho$$

Any continuous vector field $\mathbf{u}$ can be decomposed into

a gradient component $\nabla\phi$ and a curl component $\nabla \times \boldsymbol{\psi}$.

Proof

1. Define

$$\mathbf{a} = -\frac{1}{4\pi}\int \frac{\mathbf{u}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dV'$$

   with $\mathbf{u} \to 0$ assumed faster than $1/r^2$

2. Then

$$\begin{aligned}\nabla^2 a_i &= -\frac{1}{4\pi}\int u_i(\mathbf{r}')\nabla_r^2\frac{1}{|\mathbf{r}-\mathbf{r}'|}dV' \\ &= -\frac{1}{4\pi}\int u_i(\mathbf{r}')(-4\pi\delta(\mathbf{r}-\mathbf{r}'))dV' \\ &= u_i(\mathbf{r})\end{aligned}$$

3. Using

$$\nabla^2\mathbf{a} = \nabla\nabla\cdot\mathbf{a} - \nabla\times(\nabla\times\mathbf{a})$$

   (Show this!)

4. $$\mathbf{u} = -\nabla\phi + \nabla\times\boldsymbol{\psi}$$

   for

$$\phi = -\nabla\cdot\mathbf{a} \quad \boldsymbol{\psi} = -\nabla\times\mathbf{a} \quad \square$$

Using Helmholtz's Decomposition theorem, we write

$$\mathbf{u} = \mathbf{u}_T + \mathbf{u}_L$$

with $\nabla \times \mathbf{u}_L = 0$ and $\nabla \cdot \mathbf{u}_T = 0$.

Hence by inserting $\mathbf{u}$ into the Navier equation, and taking the div and curl,

show that

$$\mathbf{u}_{L\,tt} = c_L^2 \nabla^2 \mathbf{u}_L \quad \mathbf{u}_{T\,tt} = c_T^2 \nabla^2 \mathbf{u}_T$$

Assume time-harmonic solutions in the form

$$\begin{bmatrix} \mathbf{u}_L \\ \mathbf{u}_T \end{bmatrix} = \mathrm{e}^{\mathrm{i}\omega t} \begin{bmatrix} \mathbf{U}_L(x, y, z) \\ \mathbf{U}_T(x, y, z) \end{bmatrix}$$

Hence derive the Helmholtz equations

$$\nabla^2 \mathbf{U}_L + k_L^2 \mathbf{U}_L = 0 \qquad k_L^2 \equiv \omega^2 / c_L^2$$

$$\nabla^2 \mathbf{U}_T + k_T^2 \mathbf{U}_T = 0 \qquad k_T^2 \equiv \omega^2 / c_T^2$$

Maxwell's equations in a charge free vacuum are:

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Hence derive the wave equations.

Gauss' Theorem: For a vector field **v** defined in a volume V bounded by a closed surface S

$$\boxed{\int_S \mathbf{v} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{v} dV}$$

where $d\mathbf{S}$ is directed in the outward sense.

Prove this from first principles by considering a volume element $dV = dx\, dy\, dz$.

Fourier's Law for velocity (**v**) of heat flow (u) in a body with terrmal conductivity K

$$\mathbf{v} = -K\nabla u$$

By Gauss, the total flux of heat leaving a volume bounded by S is

$$\int_S \mathbf{v} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{v} dV = -K \int_V \nabla^2 u dV$$

If the specific heat of the material is assumed to be σ, then this flux must also equal

$$-\frac{\partial}{\partial t} \int_V \sigma\rho u dV = -\int_V \sigma\rho u_t dV$$

hence, we arrive at the heat equation

$$\boxed{u_t = \kappa\nabla^2 u}$$

where $\kappa = K/\sigma\rho$ is the thermal coefficient of the material.

Gravitational potential

$$V(\mathbf{r}) = -\int_{\text{allspace}} \frac{G\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}'$$

Graviational force is given by $\mathbf{F} = -\nabla V$

Gauss' Law of Gravity

$$\begin{aligned}
\nabla^2 V &= -\nabla_r^2 \left(\int_{\text{allspace}} \frac{G\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}' \right) \\
&= -\left(\int_{\text{allspace}} G\rho(\mathbf{r}') \nabla_r^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}' \right) \\
&= -\int_{\text{allspace}} G\rho(\mathbf{r}')(-4\pi\delta(\mathbf{r}-\mathbf{r}')) d^3\mathbf{r}' \\
&= 4\pi G\rho
\end{aligned}$$

Poisson equation for $\rho \neq 0$: $\nabla^2 V = 4\pi G\rho$

Laplace equation for $\rho = 0$: $\nabla^2 V = 0$

Consider density of physical quantity given by $\rho(\mathbf{r}, t)$ and associated flux $\mathbf{q}(\mathbf{r}, t)$

If the quantity ρ is conserved, and we consider a volume V bounded by a surface S, we must have

$$\frac{\partial}{\partial t}\int_V \rho dV = -\int_S \mathbf{q}\cdot d\mathbf{S}$$

where the area element $d\mathbf{S}$ is directed in the outward sense.

Then, by divergence theorem, we obtain the conservation law for ρ

$$\rho_t + \nabla\cdot\mathbf{q} = 0$$

Second-order PDEs

Classification; Characteristics; Canonical Form; General Solution

We consider

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

where coefficients A etc. are functions in x and y.

Assume a change of variables from (x, y) to (ξ, η) such that

$$J \equiv \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0$$

and that under the new coordinate variables, the coefficients of $u_{\xi\xi}$ and $u_{\eta\eta}$ in the second order PDE above vanish identically.

Hence, for ζ equal to either ξ, or η,

$$A\zeta_x^2 + B\zeta_x\zeta_y + C\zeta_y^2 = 0$$

Show this.

Dividing across by ζ_y^2:

$$A\left(\frac{\zeta_x}{\zeta_y}\right)^2 + B\left(\frac{\zeta_x}{\zeta_y}\right) + C = 0$$

Along a curve of constant ζ, have

$$d\zeta = \zeta_x dx + \zeta_y dy = 0$$

or equivalently,

$$\frac{dy}{dx} = -\frac{\zeta_x}{\zeta_y}$$

and

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0$$

The curves defined in this way have the property that ξ =constant and η =constant, these are the *characteristic curves*.

The roots of the above quadratic equation in dy/dx are:

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$$

- If $B^2 - 4AC > 0$ there are two distinct real roots and therefore two real sets of characteristics: hyperbolic.
- If $B^2 - 4AC = 0$ there is only one real root and therefore one real characteristic: parabolic.
- If $B^2 - 4AC < 0$ there are no real roots and therefore no real characteristics: elliptic.

Making the transormation to $\xi(x, y)$ and $\eta(x, y)$ converts the second order linear PDE to canonical form for hyperbolic and parabolic equations.

For elliptic equations, the transformation variables required are

$\alpha(x, y) = (\xi(x, y) + \eta(x, y))/2$ and

$\beta(x, y) = (\xi(x, y) + \eta(x, y))/2\mathrm{i}$.

This is achieved by considering the characteristic equations.

Classify and convert to canonical form:

1. $y^2 u_{xx} - x^2 u_{yy} = 0$
2. $x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = 0$
3. $u_{xx} + x^2 u_{yy} = 0$

Special case of constant coefficients: straight-line characteristics

The roots of the above quadratic equation in dy/dx are:

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} \equiv \lambda_{1,2} \quad \text{(constants)}$$

with straight-line solutions

$$y = \lambda_1 x + c_1 \quad y = \lambda_2 x + c_2$$

Hence, the characteristic coordinates are

$$\xi = y - \lambda_1 x \quad \eta = y - \lambda_2 x$$

Classify and convert to canonical form:

1. $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$
2. $u_{xx} + u_{xy} + u_{yy} + u_x = 0$
3. $u_{tt} - c^2 u_{xx} = 0 \qquad$ (c constant)

The characteristic coordinates are used to

obtain the canonical form.

If the canonical form may be integrated,

the general solution is obtained.

Find the general solution to:

1. $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 0$
2. $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$
3. $3u_{xx} + 10u_{xy} + 3u_{yy} = 0$