Section 3 Exercise 2

$$\frac{dQ}{dt} = 2t f_\varphi e^{-(\bar{x}+\varphi)t} \qquad \bar{x} < 1-2\varphi$$

$$= 2t f_\varphi e^{-\varphi t} e^{-\bar{x}t}$$

$$\frac{dQ}{dt} = -\int_0^{1-2\varphi} \bar{x} \frac{\partial Q}{\partial t} d\bar{x} - \int_{1-2\varphi}^{\infty} \bar{x} \frac{\partial Q}{\partial t} d\bar{x}$$

① $$-\int_0^{1-2\varphi} 2t f_\varphi e^{-\varphi t} \bar{x} e^{-\bar{x}t} d\bar{x}$$

$$= -2t f_\varphi e^{-\varphi t}\left[-\frac{(1-2\varphi)e^{-(1-2\varphi)t}}{t} - \frac{e^{-(1-2\varphi)t}}{t^2} + \frac{1}{t^2}\right]$$

$$= +2(1-2\varphi)\cancel{f_\varphi e^{-\varphi t}} e^{-(1-2\varphi)t} + \cancel{\frac{2}{t} f_\varphi e^{-\varphi t}} e^{-(1-2\varphi)t} - \frac{2}{t} f_\varphi e^{-\varphi t}$$

② $$-\int_{1-2\varphi}^{\infty} 2t f_\varphi e^{(1-2\varphi)t} \bar{x} e^{-\bar{x}t} d\bar{x}$$

$$= -2t f_\varphi e^{(1-2\varphi)t}\left[\frac{(1-2\varphi)e^{-(1-2\varphi)t}}{t} + \frac{e^{-(1-2\varphi)t}}{t^2}\right]$$

$$= -2\cancel{f_\varphi(1-2\varphi)} - \cancel{\frac{2}{t}f_\varphi}$$

$$-\int_{1-2\varphi}^{\infty} t^2 f'_\varphi\, e^{(1-2\varphi)t}\, \bar{x}\, e^{-\bar{x}t}\, d\bar{x}$$

$$= -t^2 f'_\varphi\, e^{(1-2\varphi)t}\left[\frac{(1-2\varphi)e^{-(1-2\varphi)t}}{t} + \frac{e^{-(1-2\varphi)t}}{t^2}\right]$$

$$= -t f'_\varphi (1-2\varphi) - f'_\varphi$$

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$$-\int_{1-2\varphi}^{\infty} -t^2 f_\varphi\, e^{(1-2\varphi)t}\, \bar{x}^2 e^{-\bar{x}t}\, d\bar{x}$$

$$= t^2 f_\varphi\, e^{(1-2\varphi)t}\left[\frac{(1-2\varphi)^2 e^{-(1-2\varphi)t}}{t} + \frac{2(1-2\varphi)e^{-(1-2\varphi)t}}{t^2} + \frac{2e^{-(1-2\varphi)t}}{t^3}\right]$$

$$= t\, \cancel{f_\varphi (1-2\varphi)^2} + 2\cancel{f_\varphi(1-2\varphi)} + \cancel{\frac{2}{t} f_\varphi}$$

$$-\int_{1-2\varphi}^{\infty} t^2 f_\varphi (1-2\varphi)\, e^{(1-2\varphi)t}\, \bar{x}\, e^{-\bar{x}t}\, d\bar{x}$$

$$= -t^2 f_\varphi (1-2\varphi)\, e^{(1-2\varphi)t} \int_{1-2\varphi}^{\infty} \bar{x}\, e^{-\bar{x}t}\, d\bar{x}$$

$$= -t^2 f_\varphi (1-2\varphi)\, e^{(1-2\varphi)t}\left[\frac{(1-2\varphi)e^{-(1-2\varphi)t}}{t} + \frac{e^{(1-2\varphi)t}}{t^2}\right]$$

$$= -t\, \cancel{f_\varphi (1-2\varphi)^2} - \cancel{f_\varphi (1-2\varphi)}$$

$$-t f_\varphi' (1-2\varphi) - f_\varphi'$$

$$= -t f_\varphi (1-2\varphi)\left(-2\left(\frac{1-e^{-(1-\varphi)t}}{t}\right)\right) - f_\varphi\left(-2\left(\frac{1-e^{-(1-\varphi)t}}{t}\right)\right)$$

$$= 2f_\varphi \cancel{(1-2\varphi)} - 2f_\varphi \cancel{(1-2\varphi)}\, e^{-(1-\varphi)t} + \frac{2f_\varphi}{t} - \cancel{\frac{2f_\varphi}{t}}\, e^{-(1-\varphi)t}$$

$$= f_\varphi(1-2\varphi) + \frac{2}{t} f_\varphi - \frac{2}{t} f_\varphi e^{-\varphi t}$$

$$= f_\varphi\left[\frac{2}{t}\left(1-e^{-\varphi t}\right) + (1-2\varphi)\right]$$

$$\boxed{\text{SECTION 3 \quad EXERCISE 3}}$$

$$
\begin{aligned}
\frac{dQ}{dt} = {} & 2t \, f_\varphi \, e^{(1-2\varphi)t} \, e^{-\bar{x}t} \\
& + t^2 f_\varphi' \, e^{(1-2\varphi)t} \, e^{-\bar{x}t} \\
& - t^2 f_\varphi \, e^{(1-2\varphi)t} \, \bar{x} \, e^{-\bar{x}t} \\
& + t^2 f_\varphi (1-2\varphi) \, e^{(1-2\varphi)t} \, e^{-\bar{x}t}
\end{aligned}
$$

$$\frac{dQ}{dt} = -\int_0^\infty \frac{\partial Q}{\partial t} \, \bar{x} \, d\bar{x}$$

$$
\begin{aligned}
= {} & -2t f_\varphi \, e^{(1-2\varphi)t} \int_0^\infty \bar{x} e^{-\bar{x}t} \, d\bar{x} \\
& - t^2 f_\varphi' \, e^{(1-2\varphi)t} \int_0^\infty \bar{x} e^{-\bar{x}t} \, d\bar{x} \\
& + t^2 f_\varphi \, e^{(1-2\varphi)t} \int_0^\infty \bar{x}^2 e^{-\bar{x}t} \, d\bar{x} \\
& - t^2 f_\varphi (1-2\varphi) \, e^{(1-2\varphi)t} \int_0^\infty \bar{x} e^{-\bar{x}t} \, d\bar{x}
\end{aligned}
$$

$$
\begin{aligned}
= {} & \cancel{-\frac{2}{t} f_\varphi e^{(1-2\varphi)t}} - f_\varphi' e^{(1-2\varphi)t} + \cancel{\frac{2}{t} f_\varphi e^{(1-2\varphi)t}} \\
& - f_\varphi (1-2\varphi) \, e^{(1-2\varphi)t}
\end{aligned}
$$

$$= - f_\varphi' e^{(1-2\varphi)t} - f_\varphi (1-2\varphi) e^{(1-2\varphi)t}$$

$$= -\frac{d}{dt}\left(f_\varphi e^{(1-2\varphi)t}\right)$$

$$\therefore \; Q = -\int_0^t d\left(f_\varphi e^{(1-2\varphi)t}\right)$$

$$= -f_\varphi e^{(1-2\varphi)t}\Big|_0^t$$

$$= -f_\varphi e^{(1-2\varphi)t} + 1$$

$$= 1 - f_\varphi e^{(1-2\varphi)t}$$

SECTION 3 EXERCISE 5

$$Q_\varphi(t) = 1 - F_\varphi(t)\, e^{-t(2\varphi-1)}$$

$$F_\varphi(t) = \exp\left(-2\int_0^t \frac{1-e^{-(1-\varphi)\tau}}{\tau}\, d\tau\right)$$

$$\varphi = 1, \quad F_1(t) = \exp\left(-2\int_0^t \frac{1-1}{\tau}\, d\tau\right)$$

$$= \exp\left(-2\int_0^t 0 \, d\tau\right)$$

$$= \exp(0) = 1$$

$$\therefore \quad Q_\varphi(t) = 1 - F_\varphi(t)\, e^{-t(2\varphi-1)}$$

$$\varphi = 1 : \quad Q_{\varphi=1}(t) = 1 - e^{-t}$$

$$\lim_{t\to\infty} Q_{\varphi=1}(t) = 1 - \lim_{t\to\infty} e^{-t}$$

$$= 1$$

SECTION 3 EXERCISE 6

$$Q_\varphi = 1 - f_\varphi e^{-t(2\varphi-1)}$$

for $1 > \varphi \geq \frac{1}{2}$ (or $\frac{1}{2} \leq \varphi < 1$)

$$\boxed{0 \leq F_\varphi \leq 1}$$

$$\lim_{t\to\infty} Q_\varphi = 1 - \lim_{t\to\infty} F\varphi \, e^{-t(2\varphi-1)}$$

$$= 1 - f\varphi \cdot 0$$

$$= 1$$

for $\varphi = \frac{1}{2}$

$$\lim_{t\to\infty} Q_\varphi = 1 - \lim_{t\to\infty} F_\varphi \quad \ldots \quad e^{-t(2\varphi-1)} = 1, \varphi = \tfrac{1}{2}$$

$$\lim_{t\to\infty} F_\varphi = \lim_{t\to\infty} \exp\left(-2\int_0^t \frac{1-e^{-(1-\varphi)\tau}}{\tau} d\tau\right)$$

$$\leq \lim_{t\to\infty} \exp\left(-2\int_0^t \frac{1}{\tau} d\tau\right)$$

$$\leq \lim_{t\to\infty} \exp\left(-2 \left[\ln \tau\right|_0^t\right)$$

$$\leq \lim_{a\to 0} \lim_{t\to\infty} \exp\left(-2 \ln \tau \Big|_a^t\right)$$

$$\leq \lim_{a\to 0} \lim_{t\to\infty} \exp\left(\ln\left(\frac{a}{t}\right)^2 \right)$$

$$\leq \lim_{a\to 0} \lim_{t\to\infty} \left(\frac{a}{t}\right)^2$$

$$\leq 0$$

$$\therefore \lim_{t\to\infty} Q_\varphi = 1 - \lim_{t\to\infty} F_\varphi$$

$$= 1 - 0$$

$$= 1$$