Phase Plane Analysis

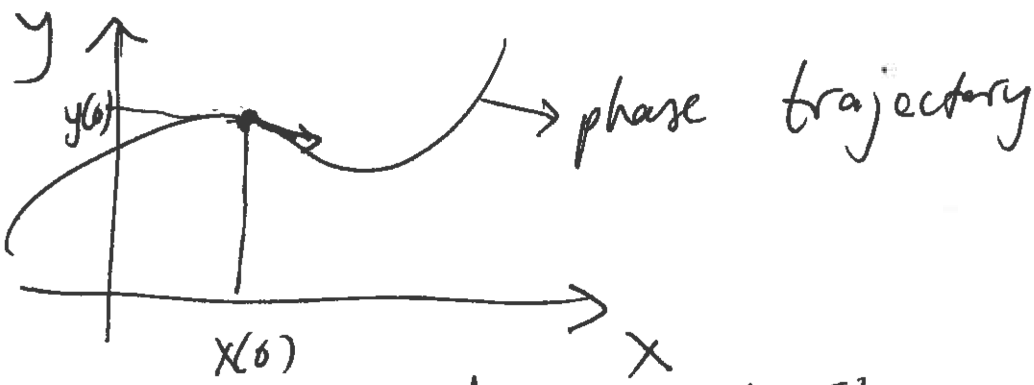
Appendix A, p. 501

$$\begin{cases} \dfrac{dx}{dt} = f(x,y) \\[2mm] \dfrac{dy}{dt} = g(x,y) \end{cases}$$

$x = x(t)$, Initial data $x(0), y(0)$
$y = y(t)$

$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)} = F(x,y)$$

$$\frac{dy}{dx} = F(x,y)$$



phase portrait

$$\boxed{\frac{dN}{dt} = \mathbb{F}(N)}$$

~~N~~ $N = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\mathbb{F} = \begin{pmatrix} f \\ g \end{pmatrix}$$

Equilibrium points $F(x_0, y_0) = 0$

- $f(x_0, y_0) = 0$

  $g(x_0, y_0) = 0$

~~$\Rightarrow x_0$ & y_0 are~~

(x_0, y_0) is a constant solution of the system.

$$\frac{dx}{dt} = \underbrace{f(x_0,y_0)}_{0} + \left(\frac{\partial f}{\partial x}\right)_{(x_0,y_0)} (x - x_0) + \left(\frac{\partial f}{\partial y}\right)_{(x_0,y_0)} (y - y_0)$$

$$\frac{dy}{dt} = \underbrace{g(x_0,y_0)}_{0} + \left(\frac{\partial g}{\partial x}\right)_{(x_0,y_0)} (x - x_0) + \left(\frac{\partial g}{\partial y}\right)_{(x_0,y_0)} (y - y_0)$$

$X = x - x_0$, $Y = y - y_0$

$$\frac{dx}{dt} = \frac{dX}{dt} \;;\; \frac{dy}{dt} = \frac{dY}{dt}$$

$$\left|\begin{aligned} \frac{dX}{dt} &= \left(\frac{\partial f}{\partial x}\right)_0 \cdot X + \left(\frac{\partial f}{\partial y}\right)_0 Y \\ \frac{dY}{dt} &= \left(\frac{\partial g}{\partial x}\right)_0 X + \left(\frac{\partial g}{\partial y}\right)_0 Y \end{aligned}\right.$$

$$\frac{d}{dt}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_0 \begin{pmatrix} X \\ Y \end{pmatrix}$$

Linear system with constant coefficients

$$A(x_0, y_0) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{\substack{x = x_0 \\ y = y_0}}$$

Jacobian at x_0, y_0

$$\frac{d}{dt}\begin{pmatrix} X \\ Y \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix} ; \quad \boxed{A = U \Lambda U^{-1}}$$

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} X \\ Y \end{pmatrix} = U \Lambda U^{-1} \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$\frac{d}{dt}\underbrace{\left(U^{-1}\begin{pmatrix} X \\ Y \end{pmatrix}\right)}_{Z} = \Lambda \underbrace{\left(U^{-1}\begin{pmatrix} X \\ Y \end{pmatrix}\right)}_{Z}$$

$$Z = U^{-1}\begin{pmatrix} x \\ y \end{pmatrix}; \quad Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\frac{d}{dt} Z = \Lambda Z \iff \frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\frac{d}{dt} z_1 = \lambda_1 z_1 \implies z_1 = c_1 e^{\lambda_1 t}$$

$$\frac{d}{dt} z_2 = \lambda_2 z_2 \implies z_2 = c_2 e^{\lambda_2 t}$$

$$\frac{dz_1}{z_1} = \lambda_1 \implies \ln z_1 = \lambda_1 t + \alpha_1$$

$$z_1 = e^{\lambda_1 t + \alpha_1} = c_1 e^{\lambda_1 t}$$

$$\begin{pmatrix} X \\ Y \end{pmatrix} = UZ = \begin{bmatrix} v_1^{(1)} & v_1^{(2)} \\ v_2^{(1)} & v_2^{(2)} \end{bmatrix}\begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{bmatrix}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$p(\lambda) = \det(A - \lambda \mathbb{1}) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix}$$

$$p(\lambda) = (a-\lambda)(d-\lambda) - bc = \lambda^2 - \lambda(a+d) + ad - bc$$

$$p(\lambda) = \lambda^2 - (\mathrm{tr} A)\lambda + \det A$$ characteristic polynomial

$$p(\lambda) = 0 \implies \lambda_1, \lambda_2 \to$$ eigenvalues

$$A v^{(1)} = \lambda_1 v^{(1)}$$
$$A v^{(2)} = \lambda_2 v^{(2)} \Rightarrow \cancel{A \left[\boxed{v^{(1)}} \boxed{v^{(2)}} \right] = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} [v^{(1)}}$$

$$A \underbrace{\left[\boxed{v^{(1)}} \; \boxed{v^{(2)}} \right]}_{U} = \left[v^{(1)} \; v^{(2)} \right] \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$\begin{bmatrix} v_1^{(1)} & v_1^{(2)} \\ v_2^{(1)} & v_2^{(2)} \end{bmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$\left[\underset{1^{st}\text{ col.}}{\underline{A v^{(1)}}}, \; \underset{2^{nd}\text{ col.}}{\underline{A v^{(2)}}} \right]$$

$$\begin{pmatrix} \lambda_1 v_1^{(1)} & \lambda_2 v_1^{(2)} \\ \lambda_1 v_2^{(1)} & \lambda_2 v_2^{(2)} \end{pmatrix}$$

$$\left[\underset{\text{first column}}{\boxed{\lambda_1 \left(v^{(1)} \right)}}, \; \underset{\text{second colm}}{\boxed{\lambda_2 v^{(2)}}} \right]$$

$$AU = U\Lambda$$

$$A = U \Lambda U^{-1}$$

$$A v^{(i)} = \lambda_i v^{(i)}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} v_1^{(i)} \\ v_2^{(i)} \end{pmatrix} = \lambda_i \begin{pmatrix} v_1^{(i)} \\ v_2^{(i)} \end{pmatrix}$$

$$a v_1^{(i)} + b v_2^{(i)} = \lambda_i v_1^{(i)} \qquad i = 1, 2$$

$$b v_2^{(i)} = (\lambda_i - a) v_1^{(i)}$$

$$v_2^{(i)} = \frac{\lambda_i - a}{b} v_1^{(i)} \qquad i = 1, 2$$

Define $\boxed{p_i = \frac{\lambda_i - a}{b}} \Rightarrow v_2^{(i)} = p_i v_1^{(i)}$

$$v^{(i)} = \begin{pmatrix} v_1^{(i)} \\ p_i \cdot v_1^{(i)} \end{pmatrix} = v_1^{(i)} \begin{pmatrix} 1 \\ p_i \end{pmatrix}$$

$$v^{(i)} = \begin{pmatrix} 1 \\ p_i \end{pmatrix}$$

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ p_1 & p_2 \end{pmatrix} \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix} = \begin{bmatrix} c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ c_1 p_1 e^{\lambda_1 t} + c_2 p_2 e^{\lambda_2 t} \end{bmatrix}$$

$$X = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \longrightarrow 0, \ t \to \infty$$

$$Y = c_1 p_1 e^{\lambda_1 t} + c_2 p_2 e^{\lambda_2 t} \longrightarrow 0, \ t \to \infty$$

$$e^{\lambda t} = e^{\mu t + i\omega t} = e^{\mu t} \cdot e^{i\omega t} \to 0, \ (\mu < 0)$$

$$|e^{i\omega t}| = |\underbrace{\cos \omega t} + i \underbrace{\sin \omega t}| \, I$$

$$= \sqrt{\cos^2 + \sin^2} = \sqrt{1} = 1$$

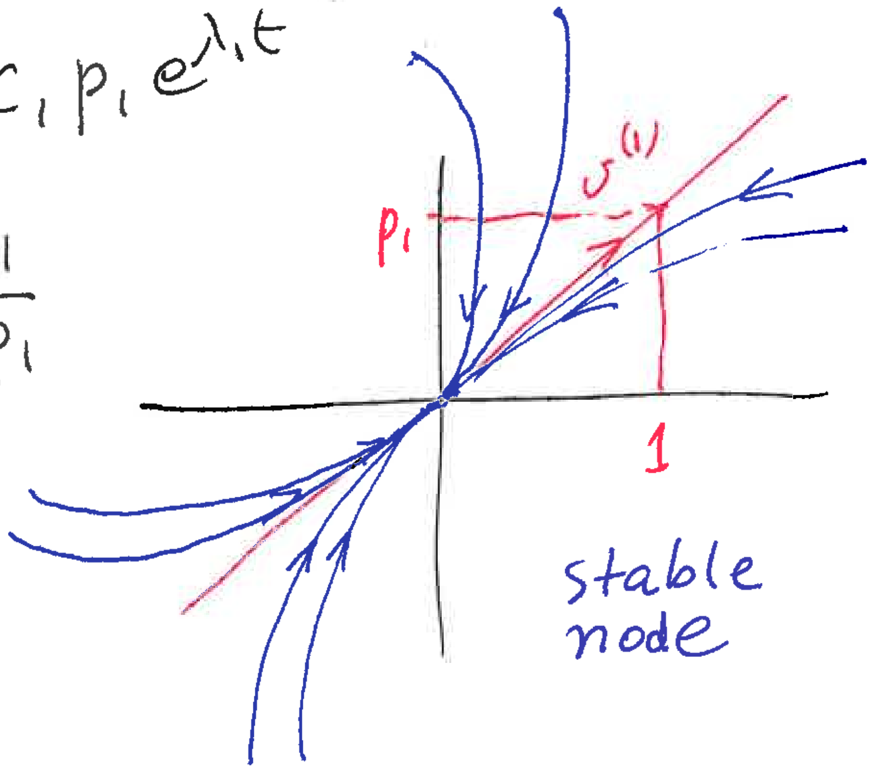
$\lambda_2 < \lambda_1 < 0$ two real negative

$$\begin{pmatrix} X \\ Y \end{pmatrix} = c_1 v^{(1)} e^{\lambda_1 t} + c_2 v^{(2)} e^{\lambda_2 t}, \quad t \to \infty$$

$$\to c_1 v^{(1)} e^{\lambda_1 t} \qquad v^{(1)} = \begin{pmatrix} 1 \\ p_1 \end{pmatrix}$$

$$X = c_1 e^{\lambda_1 t}$$
$$Y = c_1 p_1 e^{\lambda_1 t}$$

$$\frac{X}{Y} = \frac{1}{p_1}$$



Two real & positive eigenvalues

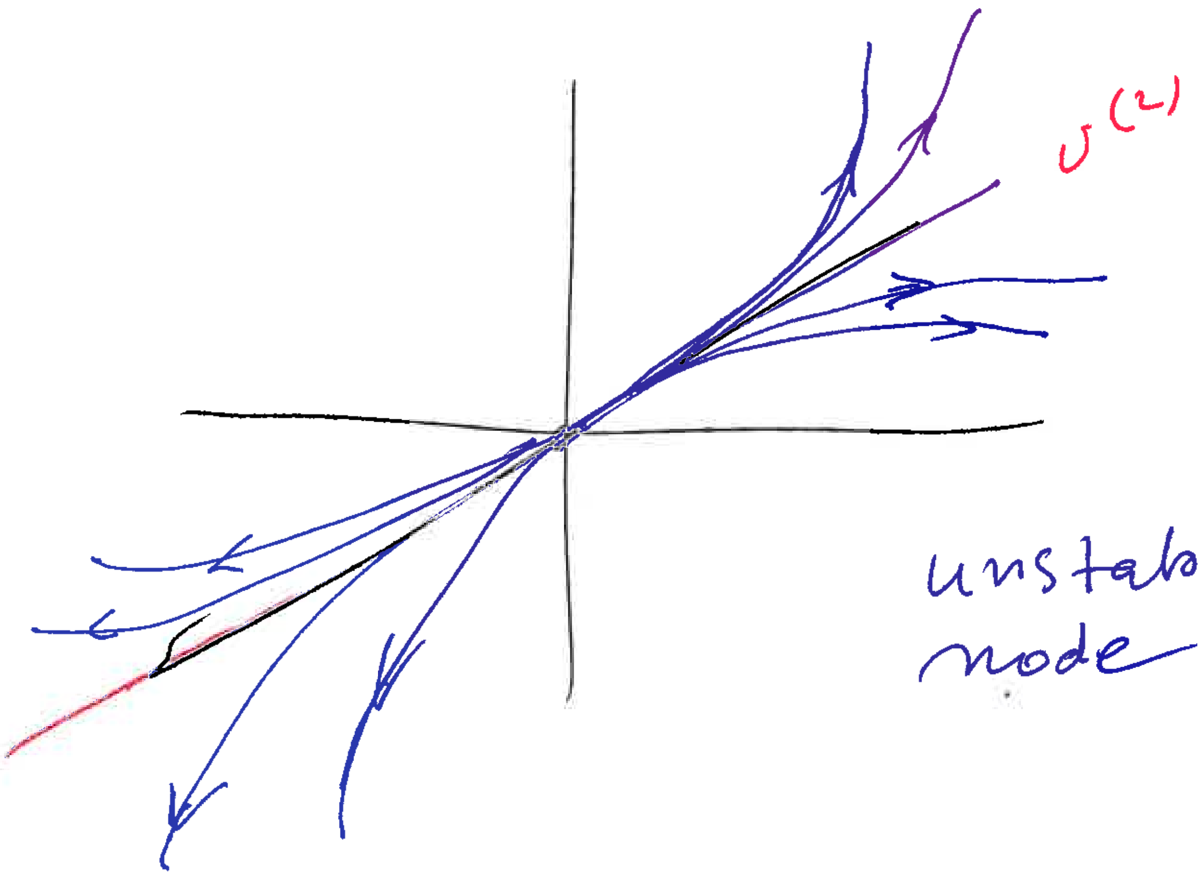
$\lambda_1 > \lambda_2 > 0$

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 v^{(1)} e^{\lambda_1 t} + c_2 v^{(2)} e^{\lambda_2 t}$$

If $t \to \infty$ $\quad x \to \infty$, $y \to \infty$

$t \to -\infty$ $\quad x \to 0$ $\quad y \to 0$

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_2 v^{(2)} e^{\lambda_2 t} \qquad t \to -\infty$$

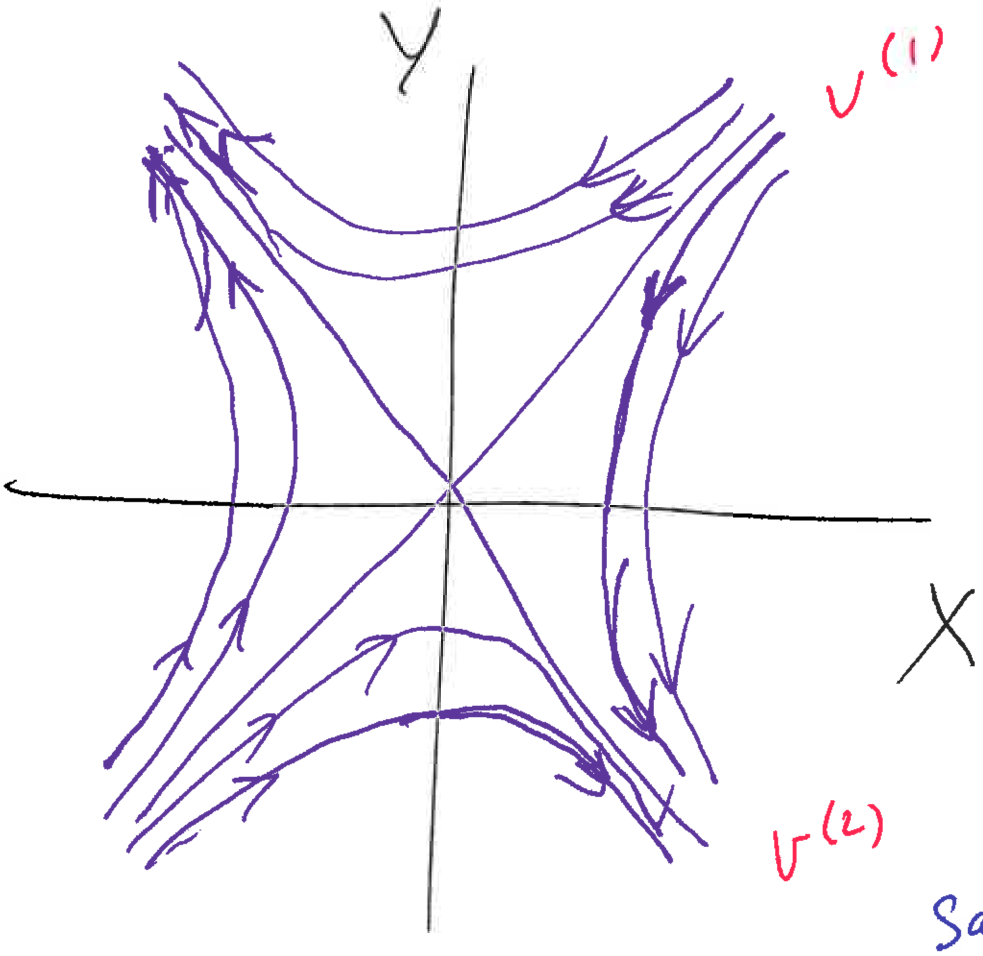


unstable node

Real with different signs

$\lambda_1 < 0 < \lambda_2$

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 v^{(1)} e^{\lambda_1 t} + c_2 v^{(2)} e^{\lambda_2 t}$$



Saddle point

(unstable)

λ_1, λ_2 — complex

$$\lambda_{1,2} = \mu \pm i\omega$$

$$\begin{pmatrix} X \\ Y \end{pmatrix} = c_1 v^{(1)} e^{\mu t + i\omega t} + \bar{c}_1 \overline{v^{(1)}} e^{\mu t - i\omega t}$$

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \underline{\underline{e^{\mu t}}} \left[\underbrace{c_1 v^{(1)} e^{i\omega t} + \bar{c}_1 \overline{v^{(1)}} e^{-i\omega t}}_{2\pi\text{-periodic, bounded}} \right]$$

$\mu < 0$ stable

$\mu > 0$ unstable

$\mu = 0$ periodic

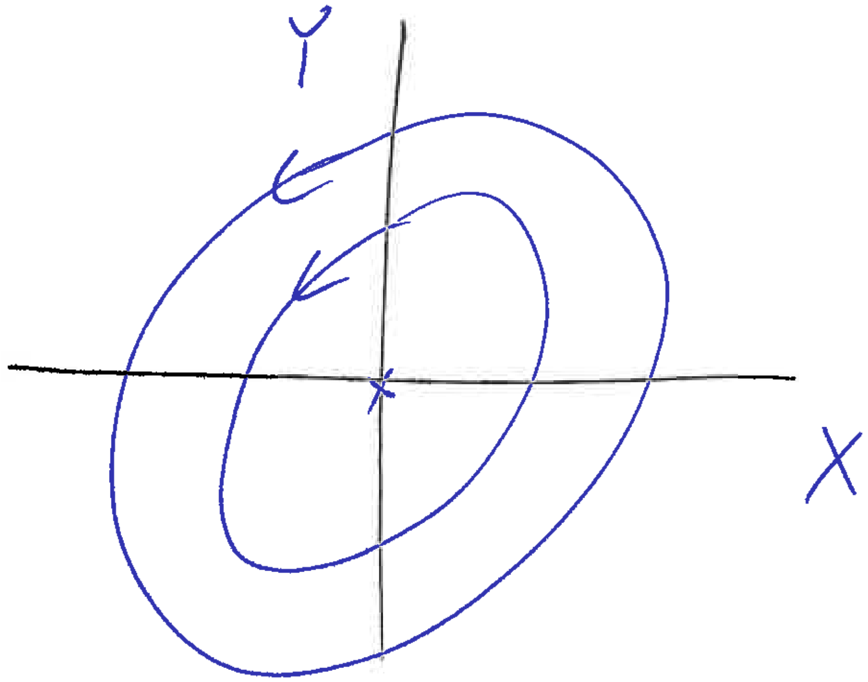
$$\cancel{\begin{pmatrix} X \\ Y \end{pmatrix}} =$$

Using $\cos\omega t = \dfrac{e^{i\omega t} + e^{-i\omega t}}{2}$ $\sin\omega t = \dfrac{e^{i\omega t} - e^{-i\omega t}}{2i}$

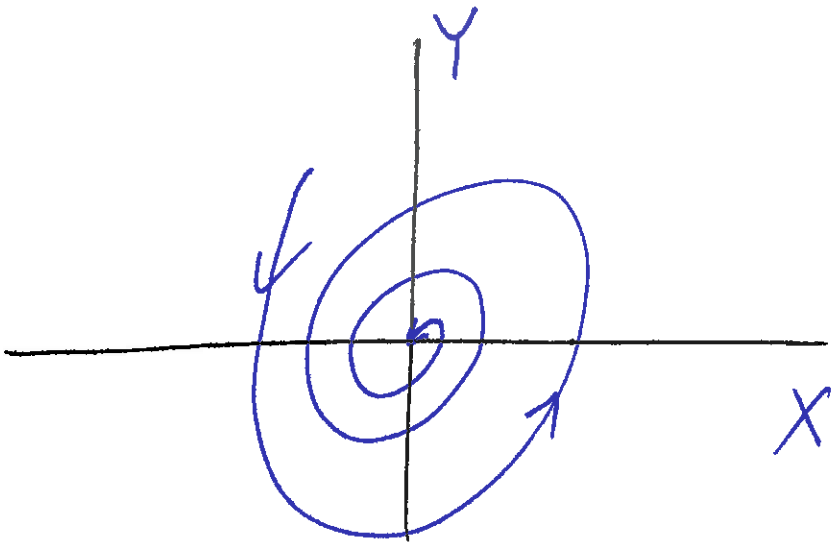
If $\mu = 0$ $\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \sin\omega t \\ \cos\omega t \end{pmatrix}$

periodic $X(t) = X(t + \frac{2\pi}{\omega})$

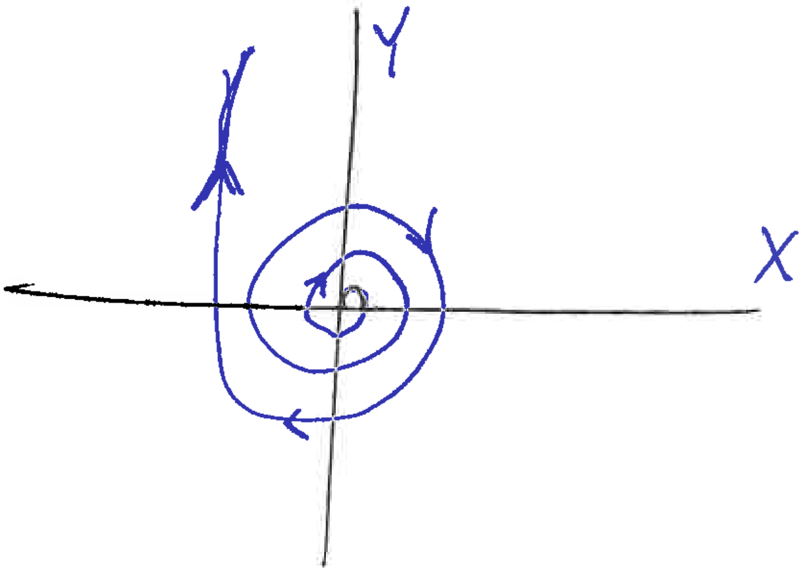
$Y(t) = Y(t + \frac{2\pi}{\omega})$



Closed phase trajectories ($\mu = 0$) 'centre'



$\mu < 0$
stable spiral



$\mu > 0$

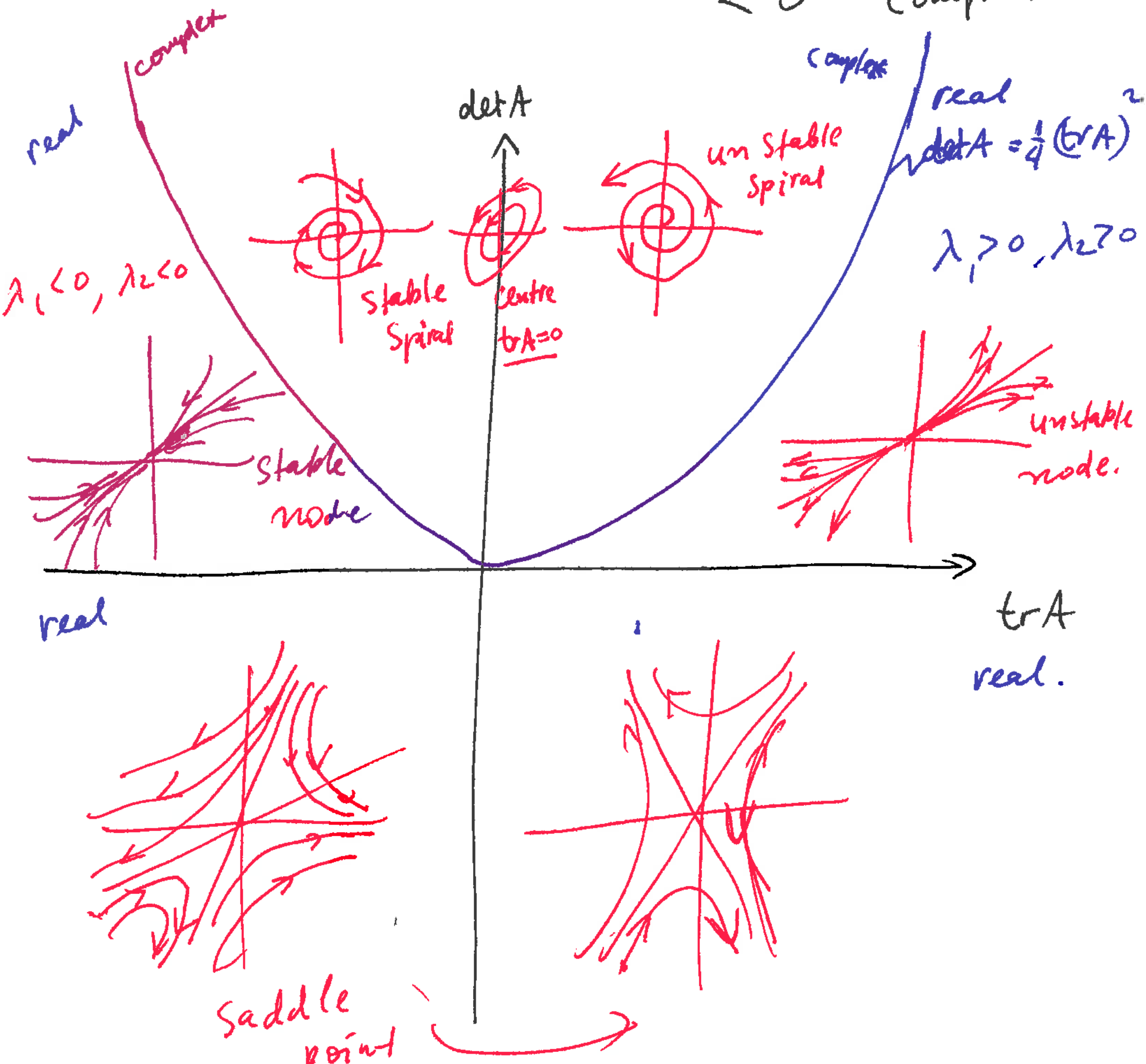
unstable spiral

$$p(\lambda) = \lambda^2 - (\mathrm{tr}A)\lambda + \det A$$

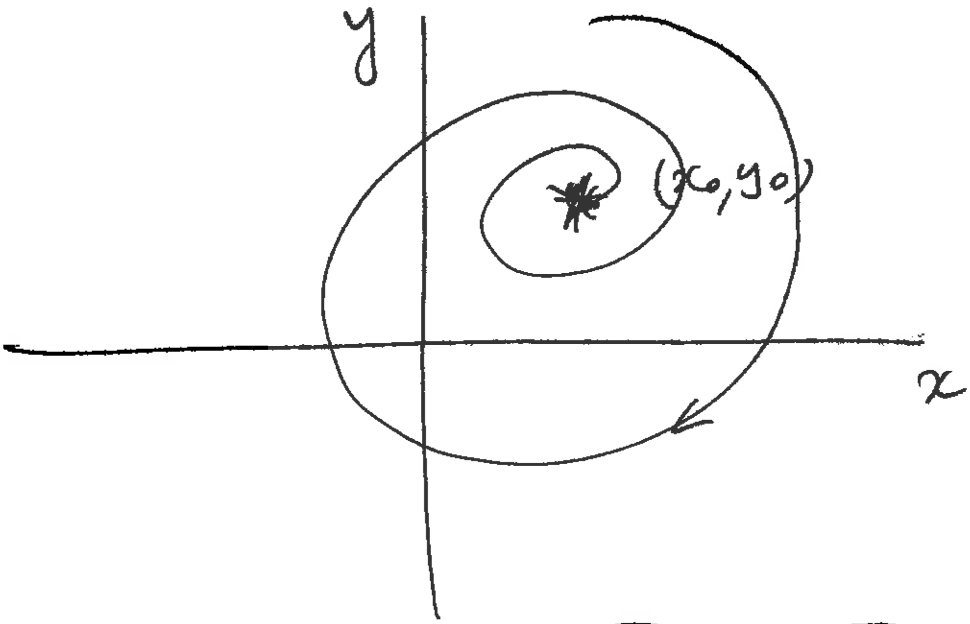
$$\lambda_1 + \lambda_2 = \mathrm{tr}A$$

$$\lambda_1 \lambda_2 = \det A$$

$$D = (\mathrm{tr}A)^2 - 4\det A \quad \begin{matrix} > 0 & \text{(real)} \\ < 0 & \text{complex} \end{matrix}$$



For stability: $\Big\Vert \begin{matrix} \mathrm{tr}A < 0 \\ \det A > 0 \end{matrix}$



Not possible if
$x(t) > 0$
$y(t) > 0$
is required.

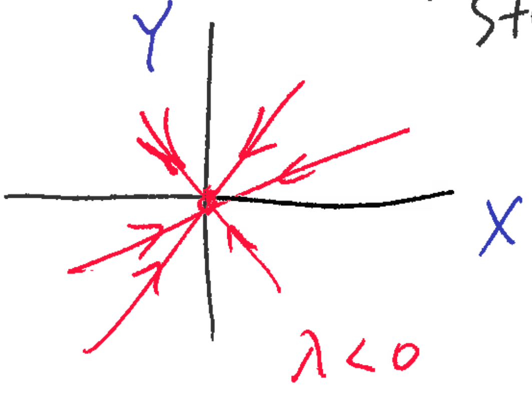
Finally $D = 0$ $\quad \lambda_1 = \lambda_2 = \lambda$
(double root)

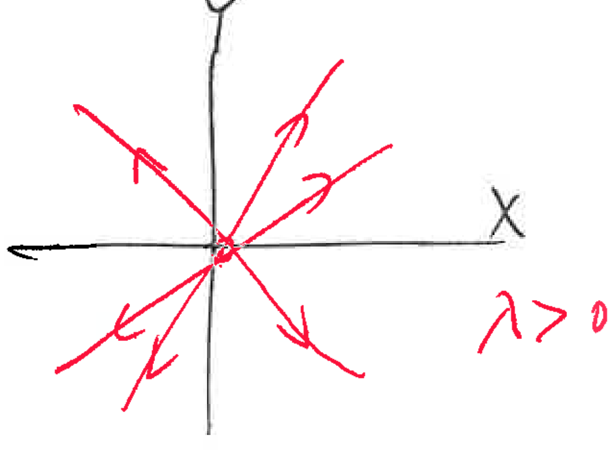
I Two eigenvectors $v^{(1)} \neq v^{(2)}$

$$\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \text{ is diagonal}$$

$$X = \alpha_1 e^{\lambda t} \qquad \frac{X}{Y} = \frac{\alpha_1}{\alpha_2} = \text{const}$$
$$Y = \alpha_2 e^{\lambda t}$$

'star'





$\lambda_1 = \lambda_2 = \lambda$

A can not be diagonalized, Λ

$A v^{(1)} = \lambda v^{(1)} \quad \to \quad t e^{\lambda t}$

Example:

$$y'' + 2ay' + a^2 y = 0$$

$a = \text{const}, \quad y = e^{\lambda t}$

$$\lambda^2 + 2a\lambda + a^2 = 0 \qquad (\lambda + a)^2 = 0$$

$$\boxed{\lambda = -a}$$

$$y = (A + Bt)e^{-at}, \quad A, B \text{ arbitrary}$$

$x = y'$

$x' + 2ax + a^2 y = 0$

$$\left|\begin{array}{l} x' = -2ax - a^2 y \\ y' = x \end{array}\right. = \begin{pmatrix} -2a & -a^2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$\lambda_{1,2} = -a$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y' \\ y \end{pmatrix} = \begin{pmatrix} (B - aA)e^{-at} - aBte^{-at} \\ (A + Bt)e^{-at} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \left\{ \begin{pmatrix} B - aA \\ A \end{pmatrix} + \begin{pmatrix} -aB \\ B \end{pmatrix} t \right\} e^{-at}$$

$$\cancel{v_1 + v_2 t)}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \left(v^{(1)} + v^{(2)} \underline{\underline{t}} \right) e^{-at}$$

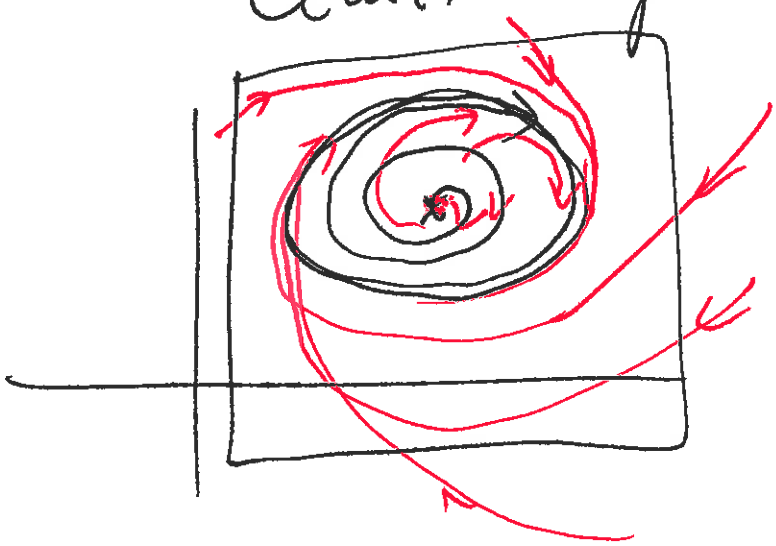
IN GENERAL,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \left(v^{(1)} + v^{(2)} t \right) e^{\lambda t}$$

$\lambda > 0$ unstable

$\lambda < 0$ stable.

Limit cycles



Poincaré — Bendixson