DUBLIN INSTITUTE OF TECHNOLOGY
KEVIN STREET, DUBLIN 8

MSc in Applied Mathematics and Theoretical Physics

DT 238

Modular Examinations for 2011–2012

General Relativity and Cosmology

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Answer any three questions. All questions carry equal marks. If more than three quastions are attempted, only the best three will be graded.

Log tables provided.

1

(a) Assuming totally inelastic collision between two equal particles, show that $E = mc^2$. [28 points]

(b) Show that $E^2 = p^2 c^2 + m_0^2 c^4$, where p is the magnitude of the particle's three-momentum $\vec{p}$ and m_0 is the particle's rest mass. [5 points]

2

(a) Show that in the Newtonian limit of a slowly moving weak gravitational field,

$$g_{00} = 1 + \frac{2\varphi}{c^2} + O(\frac{v}{c}),$$

where $v << c$ is the velocity of the source of the field and φ is the Newtonian gravitational potential. [17 points]

(b) Consider incoherent matter (dust).

(i) Find its energy-momentum tensor. [2 points]

(ii) Give an interpretation of its components. [4 points]

(iii) Show that $\partial_a T^{0a} = 0$ is the continuity equation of a perfect fluid. [4 points]

(iv) Show that $\partial_a T^{\beta a} = 0$ is the Navier–Stokes equation of motion for a pressureless perfect fluid, not acted upon by external forces. [6 points]

3

(a) Define a stationary space-time. [3 points]

(b) Consider a congruence of hypersurfaces $f(x^a) = \mu$ and build a Killing vector field orthogonal to these surfaces. [15 points]

(c) Prove that a space-time is static if, and only if, it admits a hypersurface-orthogonal timelike Killing vector field. [15 points]

4

The formulation of the initial-value problem in General Relativity is as follows: Given the metric tensor g_{ab} and its first derivatives at one time x^0 (i.e. on a space-like hypersurface S), construct the metric which corresponds to a vacuum space-time for all future time.

(a) Develop the metric in a power series in x^0. [5 points]

(b) Using the definitions of the Christoffel symbols, Riemann tensor, and Ricci tensor:

$$
\begin{aligned}
\Gamma^a_{bc} &= \frac{1}{2} g^{ad}(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc}) \, , \\
R^a{}_{bcd} &= \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{fc}\Gamma^f_{bd} - \Gamma^a_{fd}\Gamma^f_{bc} \, , \\
R_{bd} &= R^a{}_{bad} \, ,
\end{aligned}
$$

show that the vacuum field equations can be written as:

$$
\begin{aligned}
R_{00} &= -\frac{1}{2} g^{\alpha\beta} g_{\alpha\beta,00} + M_{00} = 0 \, , \\
R_{0\alpha} &= \frac{1}{2} g^{0\beta} g_{\alpha\beta,00} + M_{0\alpha} = 0 \, , \\
R_{\alpha\beta} &= -\frac{1}{2} g^{00} g_{\alpha\beta,00} + M_{\alpha\beta} = 0 \, ,
\end{aligned}
$$

where the tensor M can be expressed solely in terms of the initial data on S. [18 points]

(c) Discuss the problem of under-determination and show how the coordinate transformation

$$x^a \to x'^a = x^a + \frac{1}{6}(x^0)^3 C^a \, ,$$

with a suitable choice of C^a, resolves this problem.
Next discuss the problem of over-determination and explain which equations are viewed as dynamical equations and which as constraints on the initial data on S. [10 points]

5

Consider a contravariant vector field $Y^a(x)$ evaluated at point Q with coordinates $x^a + \delta x^a$ near to point P with coordinates x^a.

(a) Introduce a vector at Q, parallel to the given vector field $Y^a(x)$, and use it, together with the vector already at Q, to define the covariant derivative of a contravariant vector field. [16 points]

(b) Requiring that the covariant derivative is a tensor, derive the transformation law of the Christoffel symbols Γ^a_{bc} and determine whether Γ^a_{bc} is a tensor. [14 points]

(c) Define an *affine* manifold. [3 points]