The Inverse of a Partitioned Matrix

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Consider a pair A, B of $n \times n$ matrices, partitioned as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \ B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

where A_{11} and B_{11} are $k \times k$ matrices. Suppose that A is nonsingular and $B = A^{-1}$. In this note it will be shown how to derive the B_{ij}'s in terms of the A_{ij}'s, given that

$$\det(A_{11}) \neq 0 \text{ and } \det(A_{22}) \neq 0. \tag{1}$$

The latter conditions are sufficient for the nonsingularity of A. However, in general they are not necessary conditions. For example, consider the case

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

On the other hand, if A is positive definite then the conditions (1) are necessary as well.

If $B = A^{-1}$ then

$$\begin{aligned} AB &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \\ &= \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix} \\ &= \begin{pmatrix} I_k & O_{k,n-k} \\ O_{n-k,k} & I_{n-k} \end{pmatrix}, \end{aligned} \tag{2}$$

where as usual I denotes the unit matrix and O a zero matrix, with sizes indicated by the subscripts involved.

To solve (2), we need to solve four matrix equations:

$$\begin{aligned} A_{11}B_{11} + A_{12}B_{21} &= I_k && (3) \\ A_{11}B_{12} + A_{12}B_{22} &= O_{k,n-k} && (4) \\ A_{21}B_{11} + A_{22}B_{21} &= O_{n-k,k} && (5) \\ A_{21}B_{12} + A_{22}B_{22} &= I_{n-k} && (6) \end{aligned}$$

It follows from (4) and (5) that

$$\begin{aligned} B_{12} &= -A_{11}^{-1}A_{12}B_{22}, && (7) \\ B_{21} &= -A_{22}^{-1}A_{21}B_{11}, && (8) \end{aligned}$$

so that (3) and (6) become

$$\begin{aligned} \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)B_{11} &= I_k \\ \left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)B_{22} &= I_{n-k} \end{aligned}$$

Hence

$$\begin{aligned} B_{11} &= \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1} \\ B_{22} &= \left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)^{-1} \end{aligned}$$

Substituting these solutions in (7) and (8) it follows that

$$\begin{aligned} B_{12} &= -A_{11}^{-1}A_{12}\left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)^{-1} \\ B_{21} &= -A_{22}^{-1}A_{21}\left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1} \end{aligned}$$

Thus,

$$A^{-1} = \begin{pmatrix} \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1} & -A_{11}^{-1}A_{12}\left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)^{-1} \\ -A_{22}^{-1}A_{21}\left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1} & \left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)^{-1} \end{pmatrix}$$

Moreover, since $A.A^{-1} = I_n$ implies $A^{-1}A = I_n$, we also have

$$A^{-1} = \begin{pmatrix} \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1} & -\left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1}A_{12}A_{22}^{-1} \\ -\left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)^{-1}A_{21}A_{11}^{-1} & \left(A_{22} - A_{21}A_{11}^{-1}A_{12}\right)^{-1} \end{pmatrix}$$