S9209/9210/9211/9212/101c

DUBLIN INSTITUTE OF TECHNOLOGY

School of Mathematical Sciences

DT9209 MSc Applied Mathematics
DT9210 MSc Applied Mathematics

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SUMMER EXAMINATIONS 2015/2016

MATH9953: Algorithms and Approximation Theory

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Dr C Hills

Professor E O'Riordan

9.30 – 11.30 , Monday, 9 May 2016

Duration: 2 hours

Attempt three questions only

All questions carry equal marks

Approved calculators may be used

Mathematical tables are provided

New Cambridge Statistical Tables are NOT permitted

1. a) State Banach's Fixed Point Theorem.

(5 marks)

b) Prove that if X is a complete metric space and $T : X \to X$ is a contraction, then the sequence of iterations $\{x_n = T(x_{n-1})\}$ is convergent.

(9 marks)

c) Derive the iteration scheme for the Secant rootfinding method:

$$x_{n+1} = x_n - f(x_n)\frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}.$$

(8 marks)

d) Define what is meant by the *rate of convergence* of an iterative method.

(3 marks)

e) If e_k is the error of the Secant method at step k and given that

$$\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k e_{k-1}|} = \frac{|f''(x^*)|}{2|f'(x^*)|},$$

show that the rate of convergence of the Secant method is $\dfrac{1+\sqrt{5}}{2}$.

(8 marks)

[33]

2. a) Let $(V, \|.\|)$ be a normed space, x be a vector in V and Y be a subset of V. Define what is meant by *a best approximation* of x out of Y.

(4 marks)

b) Consider the space $(\mathbb{R}^2, \|(x,y)\|_1)$ of ordered pairs of real numbers (x, y) with the norm $\|(x,y)\|_1 = |x| + |y|$.

i) Define what is meant by the norm in a normed space being *strictly convex*. Show that $\|(x,y)\|_1 = |x| + |y|$ is not strictly convex. (7 marks)

ii) Let $a = (2,2)$, find all points which are best approximation of a out of the unit ball of $(\mathbb{R}^2, \|(x,y)\|_1)$. Explain your findings. (8 marks)

iii) Suppose $\mathbb{R}^2$ has norm $\|(x,y)\|_2 = \sqrt{x^2+y^2}$. Find the best approximation of $b = (0,3)$ out of the unit ball of this space. (8 marks)

iv) In $C[0,1]$ consider the subspace Y spanned by $x_1(t) = 1$ and $x_2(t) = t$. Find the best approximation of $x(t) = e^t$ out of Y. (6 marks)

[33]

3. a) Let A be a square matrix. Define what is meant by its *spectral radius* $r(A)$. Prove that $r(A) \leq \|A\|$ for any matrix norm of A.

(7 marks)

b) Suppose the matrix equation $Ax = b$ is re-written in a fixed-point form:

$$x = Cx + d.$$

Explain why is $r(C) < 1$ a sufficient condition for the iterative method $x^{(n+1)} = Cx^{(n)} + d$ to converge to the solution of $Ax = b$.

(7 marks)

c) Derive the Jacobi method for solving the system of linear equations

$$Ax = b.$$

(7 marks)

d) Consider the system of linear equations

$$Ax = b$$

where

$$A = \begin{bmatrix} 6 & 1 & 2 \\ -3 & 8 & 1 \\ 1 & 2 & -8 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ is a vector of unknowns.}$$

Carry out two iterations of the Jacobi method on this system, taking the initial x^0 to be the zero vector.

(7 marks)

e) Find the rate of convergence of the Jacobi method.

(5 marks)

[33]

4. a) Let V be an inner product space with an orthonormal basis $(e_n)_{n=1}^{\infty}$. If x is an element of V and Y is the subspace of V generated by $(e_n)_{n=1}^{m}$, define what is meant by the *least square approximation* of x out of Y.

(5 marks)

b) Let M be a complete convex subset of an inner product space V. Prove that for every x in V the best approximation of x out of M is unique.

(8 marks)

c) Show that the subset

$$M = \{(x_1, \dots x_n) : \sum_{i=1}^{n} x_i = 1\}$$

of the Euclidean space $\mathbb{R}^n$ is convex. Find the vector of minimum norm in M.

(9 marks)

d) Consider the data $(-1,-1)$, $(0,3)$, $(2,11)$ and $(3,27)$. Construct an interpolating polynomial $p(x)$ for this data by using:

i) Lagrange's formula; (5 marks)

ii) the method of undetermined coefficients. (6 marks)

[33]