S238/106

DUBLIN INSTITUTE OF TECHNOLOGY

School of Mathematical Sciences

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# DT238 MSc Applied Mathematics & Theoretical Physics

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SUMMER EXAMINATIONS 2015/2016

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## MATH 1860: INTRODUCTION TO BIOMATHEMATICS

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DR R IVANOV

DR C HILLS

PROFESSOR E O'RIORDAN

1.00 – 4.30 pm, Friday, 13 May 2016

Duration: 3½ hours

Full marks may be obtained by answering 4 questions. Candidate's four best questions will contribute to their final mark.

All questions carry equal marks

Approved calculators may be used

Mathematical tables are provided

New Cambridge Statistical Tables are NOT permitted

1. a) Solve explicitly the logistic model with a given time-dependent intrinsic growth rate $r(t)$:

$$\frac{dN(t)}{dt} = r(t)N(t)\left(1 - \frac{N(t)}{K}\right), \qquad K > 0, \qquad N(0) > 0.$$

(10)

b) Find the steady states of the following model

$$\frac{dN}{dt} = RN\left(1 - \frac{N}{K}\right)\left(\frac{N}{K_0} - 1\right), \qquad 0 < K_0 < K$$

and determine their linear stability. $R$, $K_0$ and $K$ are positive constants. (10)

c) For the model in part **b)** find

$$\lim_{t\to\infty} N(t)$$

if $K_0 < N(0) < K$.

(5)

[25]

2. A single population model with a very "fast" predation is given by the equation

$$\frac{dN}{dt} = RN\left(1 - \frac{N}{K}\right) - P\left(1 - \exp\left(-\frac{N^2}{\varepsilon A^2}\right)\right), \qquad 0 < \varepsilon \ll 1,$$

where $R$, $K$, $P$ and $A$ are positive constants.

a) By an appropriate nondimensionalisation show that the equation is equivalent to

$$\frac{du}{d\tau} = ru\left(1 - \frac{u}{q}\right) - \left(1 - \exp\left(-\frac{u^2}{\varepsilon}\right)\right),$$

where $r$ and $q$ are positive parameters.

(5)

b) Sketch the graphs of the functions $g(u) = ru\left(1 - \frac{u}{q}\right)$ and $h(u) = 1 - \exp\left(-\frac{u^2}{\varepsilon}\right)$ for positive $u$ remembering that $\varepsilon$ is small, i.e. $0 < \varepsilon \ll 1$.

(8)

c) Demonstrate that there are three possible nonzero steady states if $r$ and $q$ lie in a domain in $r$, $q$ space given approximately by $rq > 4$.

(7)

d) Could this model exhibit hysteresis?

(5)

[25]

3. It has been suggested that a means of controlling insect numbers is to introduce and maintain a number of sterile insects in the population. One such model for the resulting population dynamics is

$$N_{t+1} = \frac{rN_t^2}{\frac{r-1}{M}N_t^2 + N_t + S},$$

where $r > 1$ and $M > 0$ are constant parameters, and $S$ is the constant sterile insect population.

a) Determine the steady states and discuss their linear stability, noting whether any type of bifurcation is possible.

(9)

b) Find the critical value $S_c$ of the sterile population in terms of $r$ and $M$ so that if $S > S_c$ the insect population is eradicated.

(9)

c) Construct a cobweb map and draw a graph of the steady state population density against $S$, and hence determine the possible solution behavior if $0 < S < S_c$.

(7)

[25]

4. Show that an exact travelling wave solution exists for the scalar reaction-diffusion equation

$$\frac{\partial u}{\partial t} = u^{q+1}(1 - u^q) + \frac{\partial^2 u}{\partial x^2}, \qquad q > 0,$$

by looking for solution in the form

$$u(x,t) = U(z) = \frac{1}{(1 + ae^{bz})^s}, \qquad z = x - ct,$$

where $c$ is the wave speed and $s$ and $b$ are positive constants. Determine the unique values for $c$, $s$ and $b$ in terms of $q$. Choose the value for $a$ such that the magnitude of the wave's gradient is at its maximum at $z = 0$.

[25]

5. Flores (1998) proposed the following model for competition between Neanderthal man ($N$) and Early Modern man ($E$).

$$\begin{aligned} \frac{dN}{dt} &= N\left(A - D(N+E) - B\right), \\ \frac{dE}{dt} &= E\left(A - D(N+E) - sB\right), \end{aligned}$$

where $A$, $B$, $D$ are positive constants and $0 < s < 1$ is a measure of the difference in mortality of the two species.

a) Explain the model briefly, nondimensionalise the system and perform the phase plane analysis.

(15)

b) Show that for large values of $t$ the population $N(t)$ decays according to the law

$$N(t) = C \exp[-B(1-s)t],$$

where $C$ is some constant. Hence give the order of magnitude of the time for Neanderthal extinction if $s = 0.995$ and if the lifetime of an individual is roughly 40 years.

(10)

[25]

6. A model for venereal diseases includes the following classes: male and female infectives ($I_1$ and $I_2$ respectively) and male and female susceptibles ($S_1$ and $S_2$ respectively). It assumes that once infectives have recovered they rejoin the susceptibles:

$$\begin{aligned}
\frac{dS_1}{dt} &= -r_1 S_1 I_2 + a_1 I_1, \\
\frac{dI_1}{dt} &= r_1 S_1 I_2 - a_1 I_1, \\
\frac{dS_2}{dt} &= -r_2 S_2 I_1 + a_2 I_2, \\
\frac{dI_2}{dt} &= r_2 S_2 I_1 - a_2 I_2,
\end{aligned}$$

$r_1$, $r_2$, $a_1$ and $a_2$ are positive constant parameters.

a) Briefly explain all terms and parameters in these equations and show that

$$I_k + S_k = N_k,$$

$k = 1, 2$ where $N_k$ are constants.

(5)

b) Using the result from (i) exclude $S_1$ and $S_2$ and reduce the system to two nonlinear equations for $I_1$ and $I_2$. Find the possible steady states and examine their linear stability for all possible values of the parameters.

(20)

[25]

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EXPECTED SOLUTION TO QUESTION NO: 1

EXAMINER: ROSSEN IVANOV

PROPOSED MARK ALLOCATION: 25

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(i) $\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right) \Rightarrow \frac{dN}{N\left(1-\frac{N}{K}\right)} = r(t)\,dt$ (separation of variables)

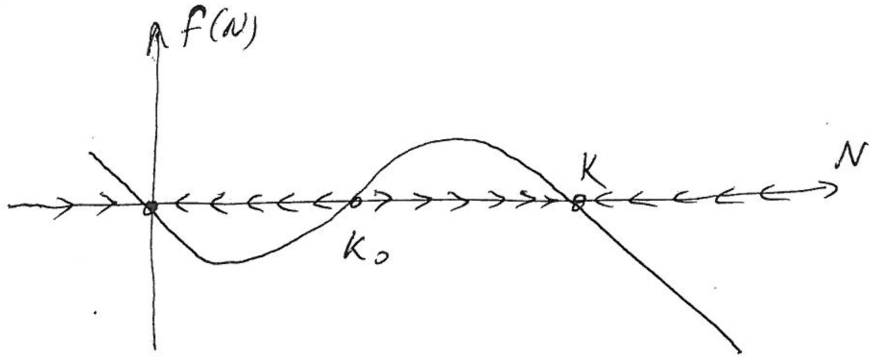
$\Rightarrow \int_{N(0)}^{N} \frac{d\tilde{N}}{\tilde{N}\left(1-\frac{\tilde{N}}{K}\right)} = \int_0^t r(s)\,ds$, the first integral can be [5 marks]

computed via partial fractions; the integration gives

$$N(t) = \frac{KN(0)}{N(0) + (K-N(0))\exp\left(-\int_0^t r(s)\,ds\right)}$$ [5 marks]

(ii) $\frac{dN}{dt} = RN\left(1-\frac{N}{K}\right)\left(\frac{N}{K_0}-1\right) = f(N);\ 0 < K_0 < K$

$f(N)$ is cubic function with zeroes at $N=0$, $K_0$ and $K$:




[5 marks]

Since $f(-\infty) > 0$, $f(\infty) < 0$ the graph is as above

$\Rightarrow f'(0) < 0 \Rightarrow N^* = 0$ is a stable steady state

$f'(K_0) > 0 \Rightarrow N^* = K_0$ is an unstable steady state

$f'(K) < 0 \Rightarrow N^* = K$ is a stable steady state

[5 marks]

(iii) From the figure it is clear that all initial data $N(0) \in [K_0, K]$ approach the steady state $N^* = K$ ~~for~~ a long time asymptotic value;

$$\lim_{t \to \infty} N(t) = K$$

[5 marks]

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EXPECTED SOLUTION TO QUESTION NO: 2

PROPOSED MARK ALLOCATION: 25

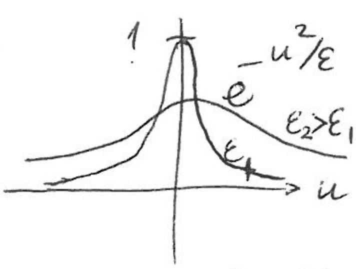
a) Clearly, $u = \frac{N}{A}$. then we have (division by $P$)

$$\frac{d\left(\frac{N}{A}\right)}{d\left(\frac{Pt}{A}\right)} = \frac{RA}{P}\left(\frac{N}{A}\right)\left(1 - \frac{N/A}{K/A}\right) - \left(1 - \exp\left(-\frac{u^2}{\varepsilon}\right)\right)$$

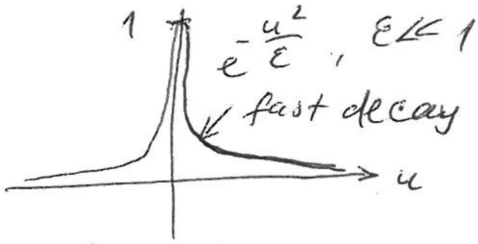
Thus $\tau = \frac{Pt}{A}$, $r = \frac{RA}{P}$, $q = K/A$ [5 marks]

$$\Rightarrow \frac{du}{d\tau} = ru\left(1 - \frac{u}{q}\right) - \left(1 - e^{-u^2/\varepsilon}\right) = f(u)$$

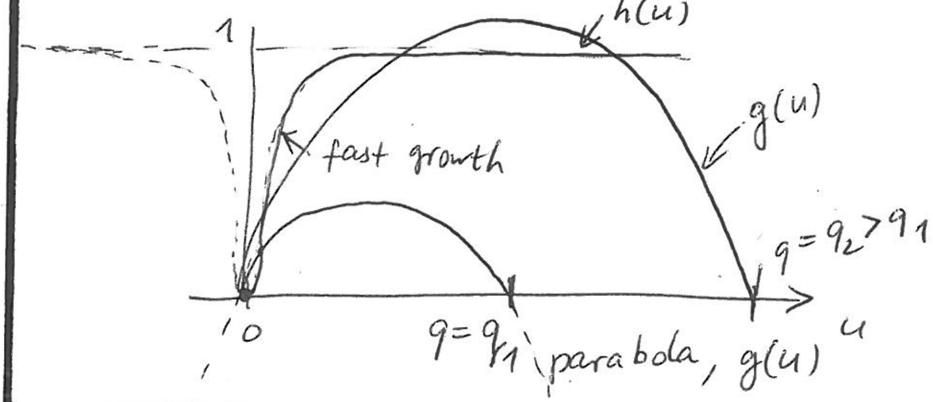
b) Recall that the graph of $e^{-\frac{u^2}{\varepsilon}}$ is the Gauss curve.




and for $\varepsilon \ll 1$




Thus both graphs are




[5 marks for $h(u)$]
[3 marks for $g(u)$]

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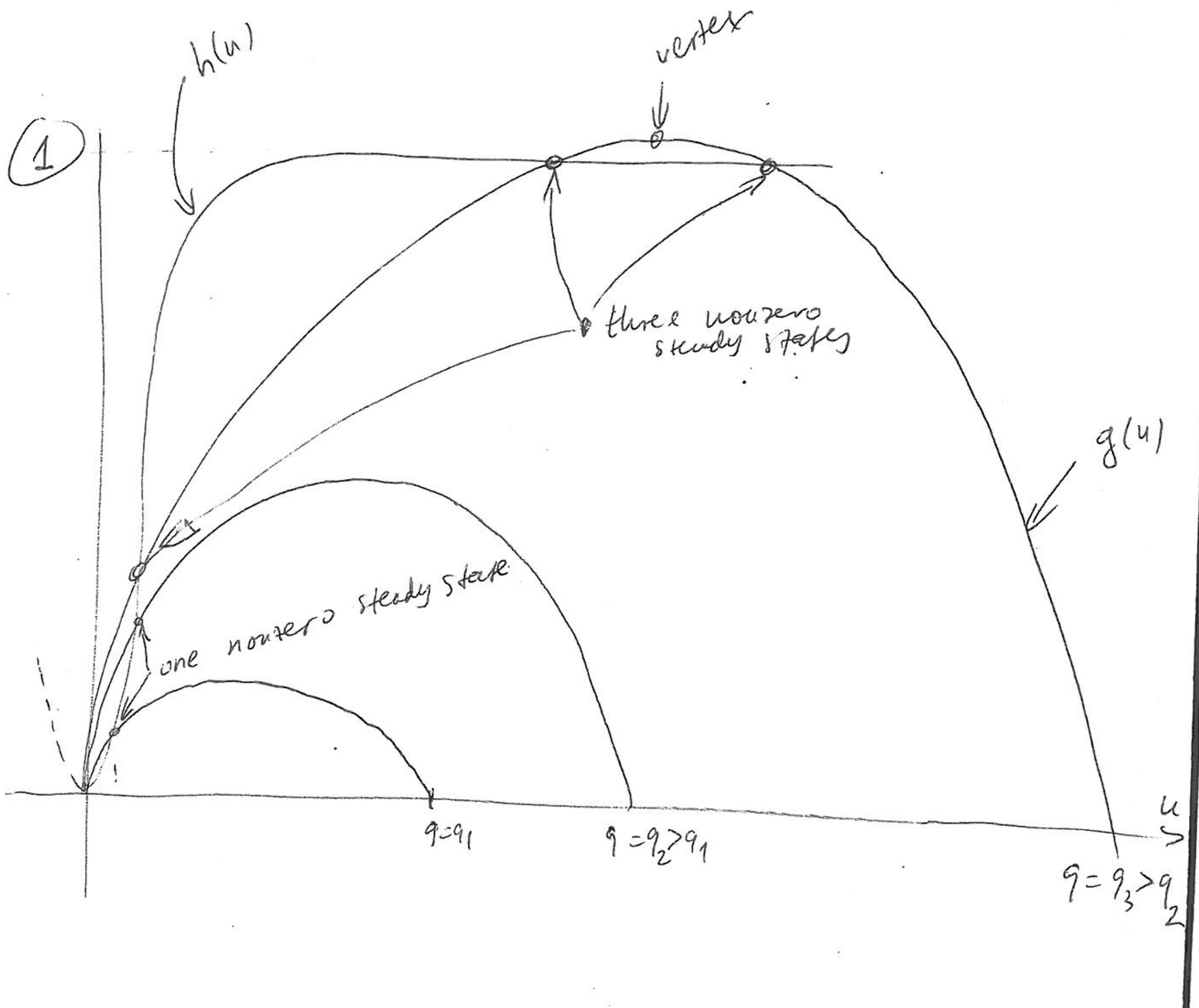
EXPECTED SOLUTION TO QUESTION NO: 2

PROPOSED MARK ALLOCATION: cont.

Graph: Larger scale



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EXPECTED SOLUTION TO QUESTION NO: 2

EXAMINER: ______________________

PROPOSED MARK ALLOCATION: ctd.

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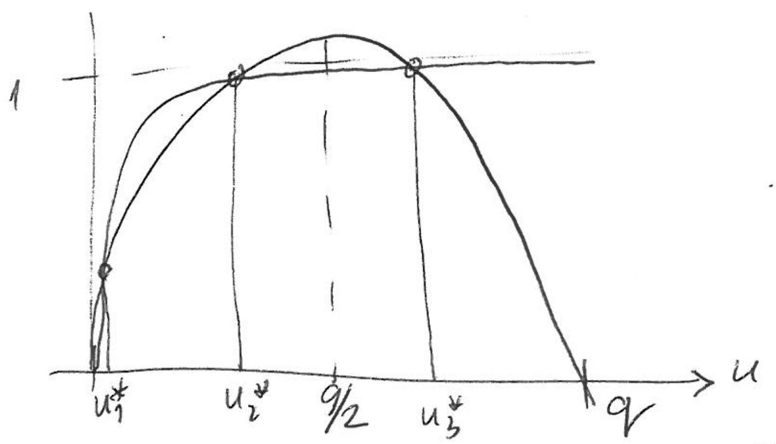
c)




If the vertex of the parabola $h(9/2) > 1$ there are 3 steady states $u_1^* < u_2^* < u_3^*$ and, of course $u_0^* = 0$.

The condition is $r\frac{9}{2}\left(1 - \frac{9/2}{9}\right) > 1 \Rightarrow r\frac{9}{2}\cdot\frac{1}{2} > 1$ or

$\boxed{r9 > 4}$ [5 marks]

$f'(u) = r - \frac{2u}{9} \cancel{+} e^{-u^2/2}\cdot\frac{2u}{e} \Rightarrow f'(0) = r > 0$

$\Rightarrow u_0^* = 0$ is unstable, $u_1^*$ - stable, $u_2^*$ - unstable, $u_3^*$ - stable [2 marks]

d) In the case of 3 stable steady states $(r9 > 4)$ we have $u_2^*$ - unstable, and two stable levels: $u_1^*$ (low) $u_1^* < 9/2$ and $u_3^* > 9/2$ (high), i.e. hysteresis is possible.

[5 marks]

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EXPECTED SOLUTION TO QUESTION NO: 3

EXAMINER: ROSSEN IVANOV

PROPOSED MARK ALLOCATION: 25



$$N_{t+1} = \frac{r N_t^2}{\frac{r-1}{M} N_t^2 + N_t + S} \equiv f(N_t)$$

(i) The steady states from $N^* = f(N^*)$ are

$$N^*_{\pm} = \frac{M}{2}\left(1 \pm \sqrt{1 - \frac{4S}{M(r-1)}}\right)$$

$N^*_{\pm}$ exist iff $4S \leq M(r-1)$, i.e. $S < S_c = \frac{M}{4}(r-1)$

$$f'(N) = \frac{rN(N+2S)}{\left(\frac{r-1}{M}N^2 + N + S\right)^2}$$ is everywhere positive.

$f'(0) = 0$, $|f'(0)| < 1 \Rightarrow N^* = 0$ is stable.

[So if $S > S_c$ only $N^* = 0$ is a stable steady state & population is eradicated]

Since $f'(N) > 0 \Rightarrow f(N)$ is monotonic:

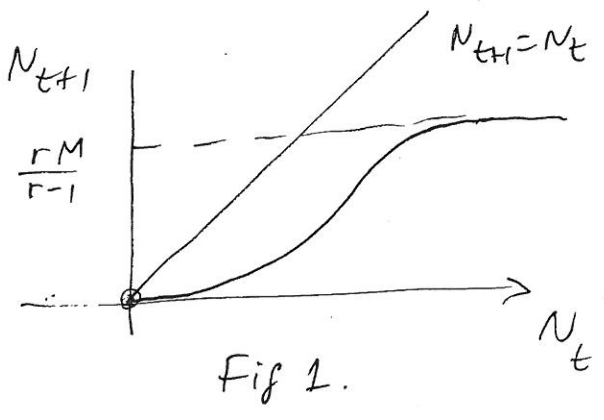


Fig 1.

or

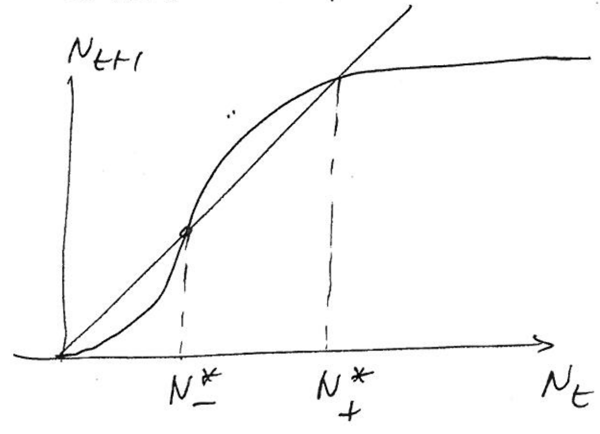


Fig (2)

$$\lim_{N \to \infty} f(N) = \frac{rM}{r-1} = \text{const}$$

[5 marks]

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EXPECTED SOLUTION TO QUESTION NO: 3 – cont.

EXAMINER: ROSSEN IVANOV



PROPOSED MARK ALLOCATION:

Fig (1) corresponds to complex roots, $S > S_c$
only $N^* = 0$ is a stable steady state

Fig (2) corresponds to two real roots giving $N^*_\pm$
$(S < S_c)$

$$f'(N^*_\pm) = \frac{r N^*_\pm (N^*_\pm + 2S)}{\left(\frac{r-1}{M} N^{*2}_\pm + N^*_\pm + S\right)^2} \equiv \frac{r N^*_\pm (N^*_\pm + 2S)}{(r N^*_\pm)^2} = \frac{1}{r} + \frac{2S}{N_\pm}$$

$$= \frac{1}{r} + 2\cancel{S}\left(\frac{r-1}{\cancel{S}M} N^*_\mp\right) = \frac{1}{r} + \frac{r-1}{\cancel{M}}\frac{\cancel{M}}{2}\left(1 \mp \sqrt{1 - \frac{4S}{M(r-1)}}\right)$$

Since $N^*_+ N^*_- = \dfrac{SM}{r-1}$

$$f'(N^*_\pm) = \frac{1}{r} + r - 1 \mp \sqrt{(r-1)^2 - \frac{4S}{M}(r-1)}$$

Note that $\frac{1}{r} + r \geq 2$

$$\Rightarrow f'(N^*_-) = \frac{1}{r} + r - 1 + \sqrt{(r-1)^2 - \frac{4S}{M}(r-1)} \geq 2 - 1 + \sqrt{(r-1)^2 - \frac{4S}{M}(r-1)} > 1$$

$f'(N^*_-) > 1 \Rightarrow N^*_-$ is <u>unstable</u>
Similarly $\Rightarrow N^*_+$ is <u>stable</u> (5 marks)

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EXPECTED SOLUTION TO QUESTION NO: 3 cont.

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PROPOSED MARK ALLOCATION: 25

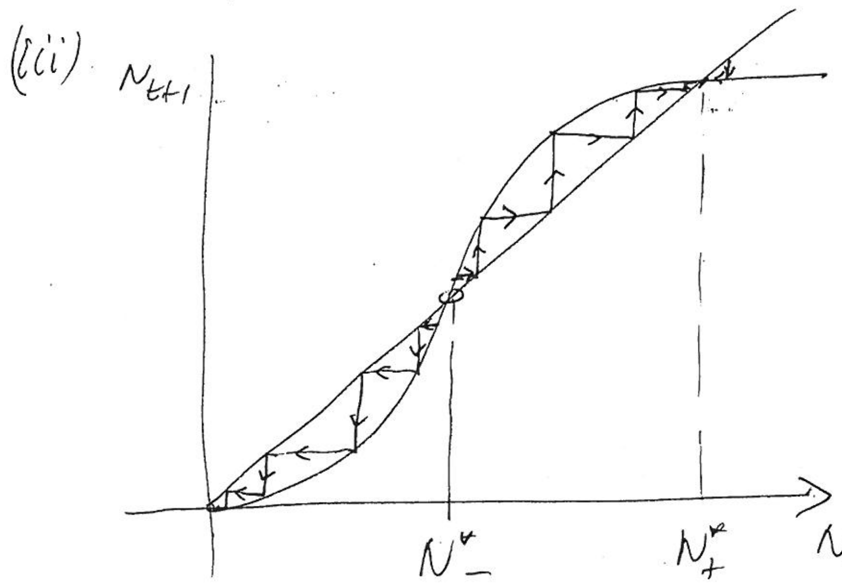
(ii) If $S > S_c$ there is only one stable steady state $N^* = 0$ and the population is eradicated, ($N^*_\pm$ are complex.)

If $S \cancel{<} 0$ there are two stable steady states, $N^* = 0$ & outbreak level steady state $N^*_+$.

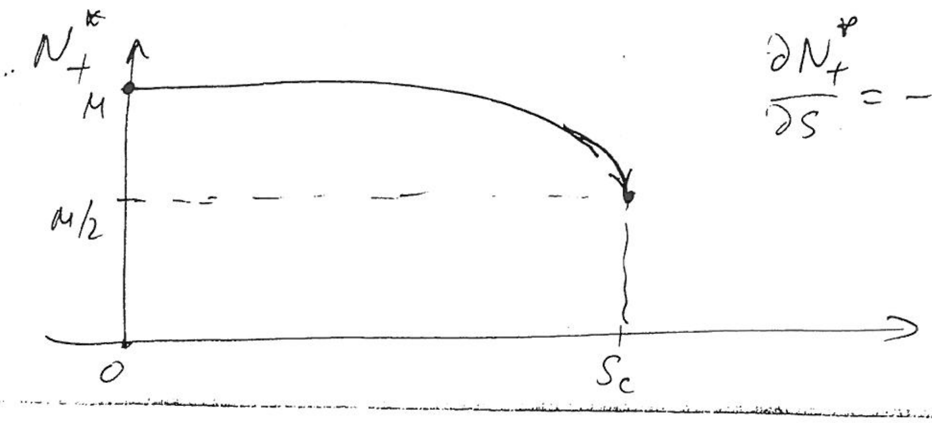
$S_c = \frac{M}{4}(r-1)$ – see previous pages.

[9 marks]

(iii)




[3 marks]




$$\frac{\partial N^*_+}{\partial S} = -\frac{4}{(r-1)} \cdot \frac{1}{\sqrt{1 - \frac{4S}{M(r-1)}}} < 0$$

$$\frac{\partial N^*_+}{\partial S} = -\infty \text{ at } S = S_c$$

$N^*_+$ decreases monotonically with $S$ from $M$ to $M/2$.

[? marks]

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EXPECTED SOLUTION TO QUESTION NO: 4

EXAMINER: ROSSEN IVANOV



PROPOSED MARK ALLOCATION: 25

$$\frac{\partial u}{\partial t} = u^{q+1}(1-u^q) + u_{xx}\ ; \quad u = U(z) = \frac{1}{(1+ae^{bz})^s} \qquad q>0, \quad z = x-ct$$

$$-cU' = +U^{q+1} - U^{2q+1} + U'' \qquad (*)$$

$$U' = -\frac{sabe^{bz}}{(1+ae^{bz})^{s+1}}$$

$$U'' = -sab^2\frac{e^{bz}(1-ase^{bz})}{(1+ae^{bz})^{s+2}}$$

[5 marks]

The substitution of $U, U'$ and $U''$ in $(*)$ gives

$$csab\frac{e^{bz}}{(1+ae^{bz})^{s+1}} = \frac{1}{(1+ae^{bz})^{s(q+1)}} - \frac{1}{(1+ae^{bz})^{s(2q+1)}} - \frac{asb^2e^{bz}(1-ase^{bz})}{(1+ae^{bz})^{s+2}}$$

Note that always $s+1 < s+2$ and $q+1 < 2q+1$

In order to be able to match the denominators, we need

$$s+1 = s(q+1) \quad \Rightarrow \quad 1 = sq$$
$$s+2 = s(2q+1) \qquad 2 = 2sq \quad \Rightarrow \boxed{sq = 1}$$

$\Rightarrow \boxed{s = \frac{1}{q}}$ $\Rightarrow$ the common denom. is $(1+ae^{bz})^{s+2}$ [8 marks]

$$\frac{csabe^{bz}(1+ae^{bz})}{(1+ae^{bz})^{s+2}} = \frac{1+ae^{bz} - 1 - sab^2e^{bz}(1-ase^{bz})}{(1+ae^{bz})^{s+2}}$$

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PROPOSED MARK ALLOCATION: 25

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The comparison of the nominators gives

$$csab(1+ae^{bz}) = a - sab^2(1-ase^{bz})$$

This leads to

$$csab = a - sab^2 \Rightarrow scb = 1 - sb^2$$

$$csa^2b = sab^2 as \Rightarrow c = sb \Rightarrow c = \frac{b}{9}$$

$$\Rightarrow \frac{b}{9}\cdot\frac{1}{9}\cdot b = 1 - \frac{1}{9}\cdot b^2 \Rightarrow b = \frac{9}{\sqrt{1+9}}$$

$$c = \frac{1}{9} b = \frac{1}{\sqrt{1+9}}$$

$$s = \frac{1}{9}$$

[7 marks]

The gradient is $U'$, max gradient when $U''=0$

$$\Rightarrow 1 = ase^{b\cdot 0} \quad \text{or} \quad 1 = a\cdot s \Rightarrow a = \frac{1}{s} = 9$$

$$\Rightarrow U = \frac{1}{\left(1+9e^{\frac{9}{\sqrt{9+1}}z}\right)^{\frac{1}{2}}}$$

[5 marks]

$$z = x - ct = x - \frac{1}{\sqrt{9+1}}t$$

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EXPECTED SOLUTION TO QUESTION NO: 5

EXAMINER: ROSSEN IVANOV

PROPOSED MARK ALLOCATION: 25



$$\begin{cases} \dfrac{dN}{dt} = N\left(A - B - DN - DE\right) \\[2mm] \dfrac{dE}{dt} = E\left(A - SB - DN - DE\right) \end{cases}$$

(i) $$\frac{dN}{dt} = (A-B)\,N\left[1 - \frac{D}{A-B}N - \frac{D}{A-B}E\right]$$

$$\frac{dE}{dt} = (A-SB)\,E\left[1 - \frac{D}{A-SB}E - \frac{D}{A-SB}N\right]$$

This is competition model, $N\left(1-\frac{N}{(A-B)/D}\right)$ is logistic term, $-\frac{DE}{A-B}$ is term describing competition, similarly for the second eq.

Nondimensionalization, $u = \dfrac{D}{A-B}N$, $v = \dfrac{D}{A-SB}E$

$$\tau = t(A-B), \quad \rho = \frac{A-SB}{A-B}$$

$$\Rightarrow \begin{cases} \dfrac{du}{d\tau} = u\left[1 - u - \dfrac{A-SB}{A-B}v\right] \\[2mm] \dfrac{dv}{d\tau} = \rho v\left[1 - v - \dfrac{A-B}{A-SB}u\right] \end{cases}$$

$$\frac{du}{d\tau} = u\left[1 - u - a_{12}v\right] = f; \quad a_{12} = \frac{A-SB}{A-B} > 1$$

$$\frac{du}{dt} = \rho v\left[1 - v - a_{21}u\right] = g; \quad a_{21} = \frac{A-B}{A-SB} < 1$$

(5 marks)

The only steady states are $(0,0)$ $(0,1)$ & $(1,0)$

$$A = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix} = \begin{pmatrix} 1-2u-a_{12}v & -a_{12}u \\ -\rho a_{21}v & \rho(1-2v-a_{21}u) \end{pmatrix}$$

[2 marks]

$$A(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix}$$, Eigenvalues $\lambda_1 = 1 > 0$ & $\lambda_2 = \rho > 0$

$\Rightarrow$ unstable node (always)

[2 marks]

$$A(1,0) = \begin{pmatrix} 1-2 & -a_{12} \\ 0 & \rho(1-a_{21}) \end{pmatrix} = \begin{pmatrix} -1 & -a_{12} \\ 0 & \rho(1-a_{21}) \end{pmatrix},$$

$\lambda_1 = -1 < 0$ $\quad \lambda_2 = \rho(1-a_{21}) > 0$ $\Rightarrow$ saddle pt.

[2 marks]

$$A(0,1) = \begin{pmatrix} 1-a_{12} & 0 \\ -\rho a_{21} & \rho(1-2) \end{pmatrix} = \begin{pmatrix} 1-a_{12} & 0 \\ -\rho a_{21} & -\rho \end{pmatrix}$$

$\lambda_1 = 1-a_{12} < 0$ & $\lambda_2 = -\rho < 0$ $\Rightarrow$ stable node

$\Rightarrow$ the only stable steady state is $(0,1)$ thus

[2 marks]

$\lim_{t\to\infty} u = 0$, $\lim_{t\to\infty} v = 1$ $\Rightarrow$ $\frac{D}{A-sB}E \to 1$ $\Rightarrow$

$\lim_{t\to\infty} N \to 0$ $\qquad \lim_{t\to\infty} E \Rightarrow \frac{A-sB}{D} = \text{const.}$

[2 marks]

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EXPECTED SOLUTION TO QUESTION NO: 5

EXAMINER: ROSSEN IVANOV

PROPOSED MARK ALLOCATION: 25



(ii) When $N \to 0$, $E \to$ const for large $t$, we can linearize the first equation

$$\frac{dN}{dt} = N(A-B) - \underbrace{\cancel{D}N^2}_{\text{neglect}} - DEN$$

$$\frac{dN}{dt} = N(A-B-DE) = N\left(A-B-\cancel{D}\frac{A-sB}{\cancel{D}}\right)$$

$$\frac{dN}{dt} = N(\cancel{A}-B-\cancel{A}+sB) \Rightarrow \frac{dN}{dt} = N(s-1)B$$

$$\Rightarrow N = \underbrace{N(0)}_{\text{const}} e^{-(1-s)Bt} \rightarrow \text{exponential decay.}$$

[5 marks]

The timescale of the extinction is

$\sim \frac{1}{(1-s)B}$ where $\frac{1}{B} = 40y.$ is the lifetime of the individual $\Rightarrow$ timescale $\sim \frac{1}{1-0.995} \times 40$ years $= \frac{40y}{0.005} = \underline{8000 \text{ year}}$

[5 marks]

(i) $r_1 S_1 I_2$ describes the rate of infection of men as a result of contacts with infected women.

$r_1$ is a measure for the intensity of these contacts etc.

$\frac{1}{a_1}$ is the lifetime of the disease for men.

adding the first two eqns gives $\frac{d}{dt}(S_1+I_1)=0$

$\Rightarrow S_1+I_1=N_1$ = number of men population etc.

[5 marks]

(ii) The system is

$$\frac{dI_1}{dt} = r_1(N_1-I_1)I_2 - a_1 I_1 = f_1(I_1,I_2)$$

$$\frac{dI_2}{dt} = r_2(N_2-I_2)I_1 - a_2 I_2 = f_2(I_1,I_2)$$

One steady state is $(0,0)$.

The other is a solution of $f_1=0$ & $f_2=0$

$$I_1^* = \frac{N_1N_2-\rho_1\rho_2}{\rho_1+N_2}, \quad I_2^* = \frac{N_1N_2-\rho_1\rho_2}{\rho_2+N_1}, \quad \rho_k = \frac{a_k}{r_k}$$

making sense only if $N_1N_2>\rho_1\rho_2$

(5 marks)

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial I_1} & \frac{\partial f_1}{\partial I_2} \\ \frac{\partial f_2}{\partial I_1} & \frac{\partial f_2}{\partial I_2} \end{pmatrix} = \begin{pmatrix} -r_1 I_2 - a_1 & r_1(N_1 - I_1) \\ r_2(N_2 - I_2) & -r_2 I_1 - a_2 \end{pmatrix}$$

$$A(0,0) = \begin{pmatrix} -a_1 & r_1 N_1 \\ r_2 N_2 & -a_2 \end{pmatrix}$$

Characteristic equation $\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0$

$$\lambda^2 + (a_1 + a_2)\lambda + a_1 a_2 - r_1 r_2 N_1 N_2 = 0, \qquad \rho_k = \frac{a_k}{r_k}$$

$$\lambda_{1,2} = \frac{-(a_1+a_2) \pm \sqrt{(a_1+a_2)^2 + 4a_1a_2\left(\frac{N_1N_2}{\rho_1\rho_2} - 1\right)}}{2}$$

If $N_1 N_2 < \rho_1 \rho_2$ then $\det A > 0$, $\operatorname{tr} A < 0$ and $(0,0)$ is the only stable steady state; $(I_1^*, I_2^*)$ does not exist then. [5 marks]

If $N_1 N_2 > \rho_1 \rho_2$ $\det A > 0$ and $(0,0)$ is unstable

In this case $(I_1^*, I_2^*)$ exists and is stable

The proof of the stability is as follows:

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**SOLUTION SHEET**

COURSE / YEAR: DT238

EXAMINATION SITTING: SUMMER ☐ SUPPLEMENTAL ☐ AUTUMN ☐ WINTER ☐

SUBJECT: INTRODUCTION TO BIOMATHEMATICS

EXAMINER: ROSSEN IVANOV

EXPECTED SOLUTION TO QUESTION NO: 6 - cont.



PROPOSED MARK ALLOCATION: 25

$$A(I_1^*, I_2^*) = \begin{pmatrix} -a_1 - r_1 I_2^* & r_1(N_1 - I_1^*) \\ r_2(N_2 - I_2^*) & -a_2 - r_2 I_1^* \end{pmatrix}$$

Characteristic equation

$$\lambda^2 - \mathrm{tr}A\,\lambda + \det A = 0$$

$$\lambda^2 + (a_1 + a_2 + r_1 I_1^* + r_2 I_2^*)\lambda + \det A = 0$$

$$\mathrm{tr}A = -(a_1 + a_2) - r_1 I_2^* - r_2 I_1^* < 0 \quad \text{always} \checkmark$$

$$\det A = a_1 a_2 + a_1 r_2 I_1^* + r_1 a_2 I_2^* + r_1 r_2 I_1^* I_2^* - r_1 r_2 (N_1 - I_1^*)(N_2 - I_2^*)$$

after substitution of $I_1^*$ & $I_2^*$ we get

$$\det A = r_1 r_2 (N_1 N_2 - \rho_1 \rho_2) > 0$$

with $\mathrm{tr}A < 0$ and $\det A > 0 \Rightarrow \lambda_1 < 0$ & $\lambda_2 < 0$

$\Rightarrow (I_1^*, I_2^*)$ is stable steady state.

[10 marks]

Thus:

1) When $N_1 N_2 < \rho_1 \rho_2$ the only steady state is $(0,0)$ – stable
2) When $N_1 N_2 > \rho_1 \rho_2$ $(I_1^*, I_2^*)$ is stable, $(0,0)$ – unstable

In case ① the epidemic dies out with time.

In case ② there is a constant fraction of population always infected.