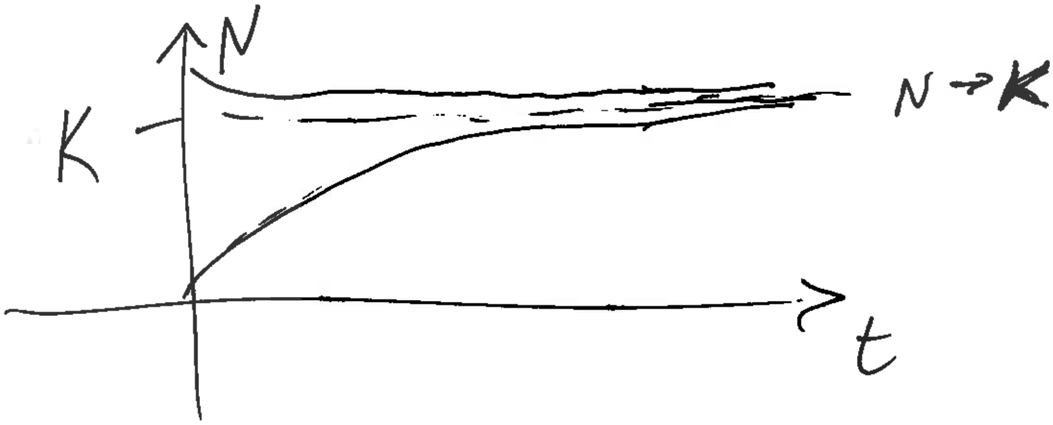
Harvesting a single Natural Population

24/10/2011

$$\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right)$$



$$\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right) - EN = f(N)$$

$E = \text{const}$ (effort)

'Constant effort' $Y = EN$

Steady state $f(N) = 0$

$$rN\left(1-\frac{N}{K}\right) - EN = 0 \quad , \quad \underline{\underline{N_1 = 0}}$$

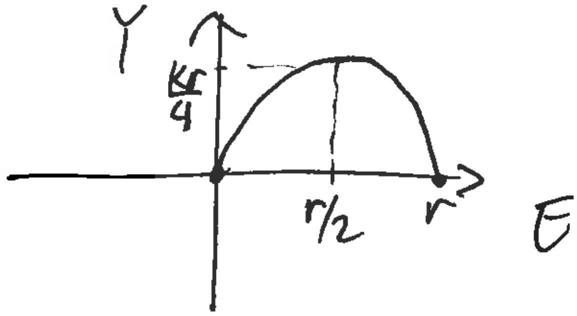
$$r\left(1-\frac{N}{K}\right) - E = 0 \Rightarrow r - E = r\frac{N}{K}$$

$$N_h = K\left(1-\frac{E}{r}\right) > 0 \text{ if } E < r$$

$$0 < E < r$$

'Yield' is $Y = E N_h = E K\left(1-\frac{E}{r}\right)$

$$Y(E) = K E\left(1-\frac{E}{r}\right)$$



$$Y_{max} = Y\left(\frac{r}{2}\right) = K\frac{r}{2}\underbrace{\left(1-\frac{r/2}{r}\right)}_{1/2} = \frac{Kr}{4}$$

$$\cancel{N_h\left(\frac{r}{2}\right) =}$$

$$N_h\left(\frac{r}{2}\right) = K\left(1-\frac{r/2}{r}\right) = \frac{K}{2}$$

Analysis of the behaviour around N_h

$N = N_h + n$ n - small.

$$\frac{dN}{dt} = 0 + \frac{dn}{dt}$$

$$\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right) - EN = f(N)$$

$$\cancel{\frac{dn}{dt} = r(N_h+n)\left(1-\frac{N_h+n}{K}\right) - E(N_h+n)}$$

$$\boxed{f(N_h) = 0}$$ Steady state.

$$\frac{dn}{dt} = \underbrace{f(N_h)}_{=0} + \left(\frac{\partial f}{\partial N}\right)_{N=N_h} \underbrace{(N - N_h)}_{n}$$

$$\frac{dn}{dt} = 0 + \left(\frac{\partial f}{\partial N}\right)_{N=N_h} \cdot n$$

$$\frac{\partial f}{\partial N} = r - \frac{2Nr}{K} - \epsilon, \quad \cancel{\left(\frac{\partial f}{\partial N}\right)_{N_h} = r}$$

$$\left(\frac{\partial f}{\partial N}\right)_{N_h} = r - \frac{2r}{\cancel{K}} \cdot \cancel{K}\left(1 - \frac{\epsilon}{r}\right) - \epsilon = r - 2r\left(1 - \frac{\epsilon}{r}\right) - \epsilon =$$

$$= r - 2r + 2\epsilon - \epsilon = \epsilon - r = -(r - \epsilon) < 0$$

when $\underline{0 < \epsilon < r}$

$$\frac{dn}{dt} = -(r-\epsilon) \cdot n \Rightarrow n = n_0 e^{-(r-\epsilon)t}$$

$$\boxed{N = N_h + n = N_h + n_0 \underbrace{e^{-(r-\epsilon)t}}_{\downarrow 0 \text{ when } \epsilon < r}}$$

$$N = N_h + n_0 e^{-\frac{t}{\tau}}, \quad \tau = \frac{1}{r-\epsilon} > 0$$

Meaning of τ : ('Recovery time')

$$e^{-\frac{t}{\tau}} \longrightarrow t=\tau \Rightarrow e^{-1} \simeq \frac{1}{2.71} \simeq 0.37$$

$$t=2\tau \Rightarrow e^{-2} \simeq 0.14$$

$$t=3\tau \Rightarrow e^{-3} \simeq 0.05$$

$$T_R(E) \simeq \tau(E) = \frac{1}{r-E}$$

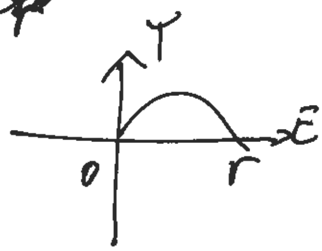
$$\frac{T_R(E)}{T_R(0)} = \frac{\frac{1}{r-E}}{\frac{1}{r-0}} = \frac{r}{r-E} = \frac{1}{1-\frac{E}{r}} \quad (*)$$

'Max yield' when $E = r/2$

$$\frac{T_R(r/2)}{T_R(0)} = \frac{1}{1-\frac{1}{2}} = 2$$

$$T_R(r/2) = 2\,T_R(0)$$

$\frac{T_R(E)}{T_R(0)}$ in terms of Y



$$Y = EK\left(1-\frac{E}{r}\right)$$

$$-\frac{K}{r}E^2 + KE - Y = 0 \quad | r$$

$$-KE^2 + rKE - rY = 0 \Rightarrow E = \frac{-rK \pm \sqrt{r^2K^2 - 4KrY}}{-2K}$$

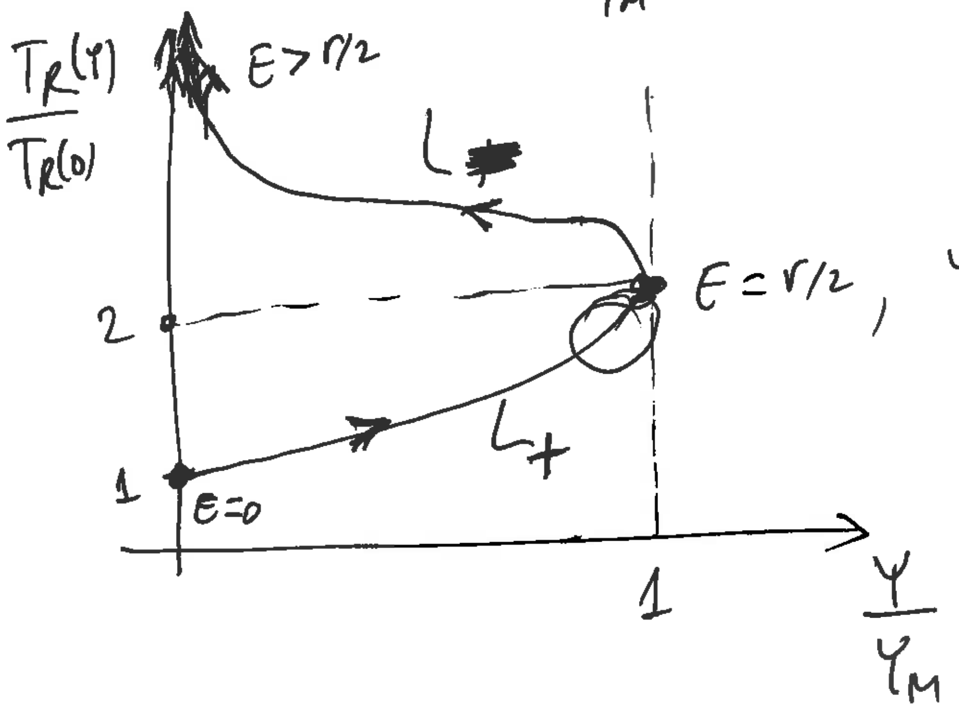
$$E = \frac{r}{2} \pm \frac{1}{2}\sqrt{r^2 - \frac{4Yr}{K}} = \frac{r}{2}\left(1 \pm \sqrt{1-\frac{4Y}{rK}}\right)$$

$$E = \frac{r}{2}\left(1 \pm \sqrt{1 - \frac{4Y}{rK}}\right) \qquad Y_M = Y_{max} = \frac{Kr}{4}$$

$$E = \frac{r}{2}\left(1 \pm \sqrt{1 - \frac{Y}{Y_M}}\right)$$

$$\frac{T_R(Y)}{T_R(0)} = \frac{1}{1 - \frac{E}{r}} = \frac{1}{1 - \frac{1}{\cancel{r}} \cdot \frac{\cancel{r}}{2}\left(1 \pm \sqrt{1 - \frac{Y}{Y_M}}\right)} = \frac{1}{1 - \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{Y}{Y_M}}}$$

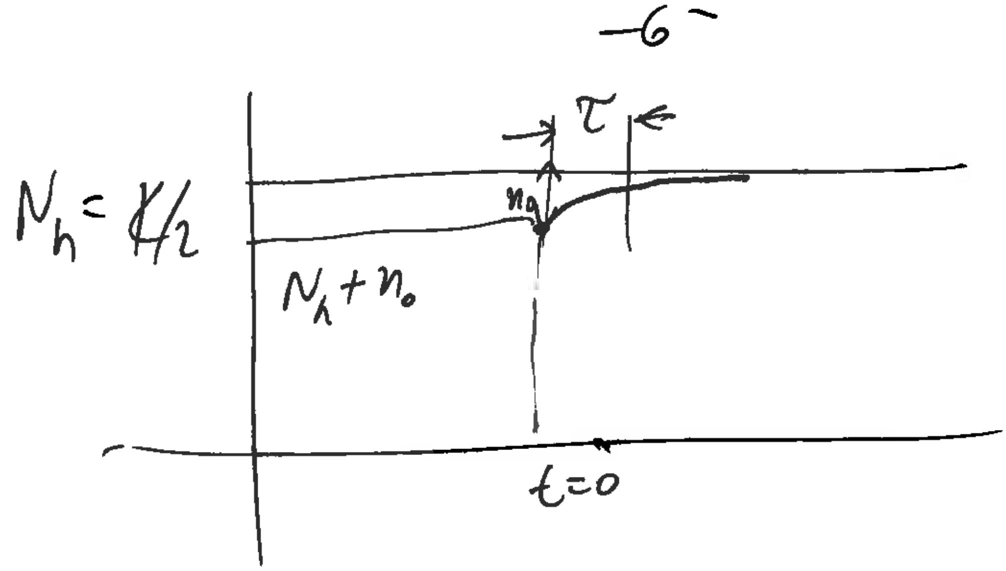
$$\frac{T_R(Y)}{T_R(0)} = \frac{1}{\frac{1}{2} \pm \frac{1}{2}\cdot\sqrt{1 - \frac{Y}{Y_M}}} = \frac{2}{1 \pm \sqrt{1 - \frac{Y}{Y_M}}}$$



$E = r/2$, $Y = Y_M$, $N = K/2$

$$Y = 0 \qquad \frac{T_R(Y)}{T_R(0)} \rightarrow \frac{2}{1 \pm 1} = \begin{cases} \frac{2}{2} = 1 & (E = 0) \\ \frac{2}{0} = \infty & E > r/2 \end{cases}$$

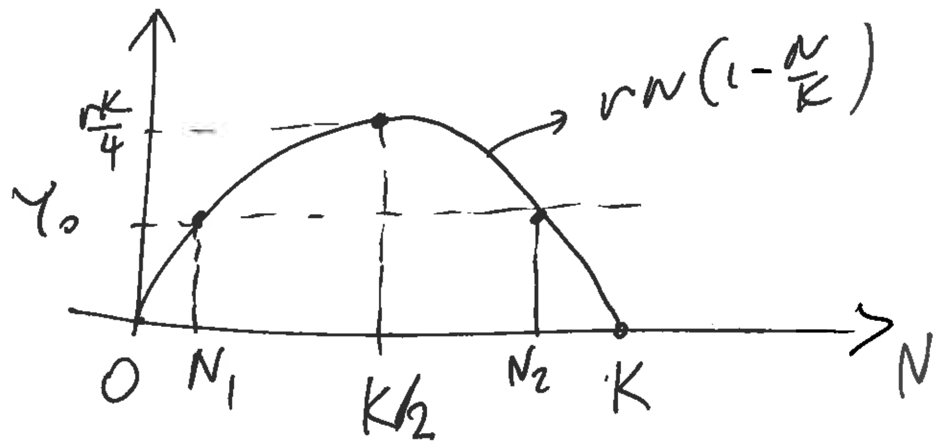
$$Y = Y_M \qquad \frac{T_R(Y_M)}{T_R(0)} = \frac{2}{1 \pm 0} = 2 \qquad \text{when } E = r/2$$



$$N = N_h + n_0 e^{-\frac{t}{\tau}}$$

Harvesting with constant yield

$$\frac{dN}{dt} = \underbrace{rN\left(1-\frac{N}{K}\right) - Y_0}_{f(N)}, \quad Y_0 = \text{const}$$



$$\underbrace{Y_0}_{\text{line}} = \underbrace{rN\left(1-\frac{N}{K}\right)}_{\text{parabola}}$$

At $N = K/2$ the logistic term is $r\frac{K}{2}\left(1-\frac{K/2}{K}\right) = \frac{rK}{4}$

$$(Y_0)_{max} = \frac{rK}{4}, \quad 0 < Y_0 < \frac{rK}{4}$$

$f(N) = 0 \Rightarrow N_1$ & N_2 steady states

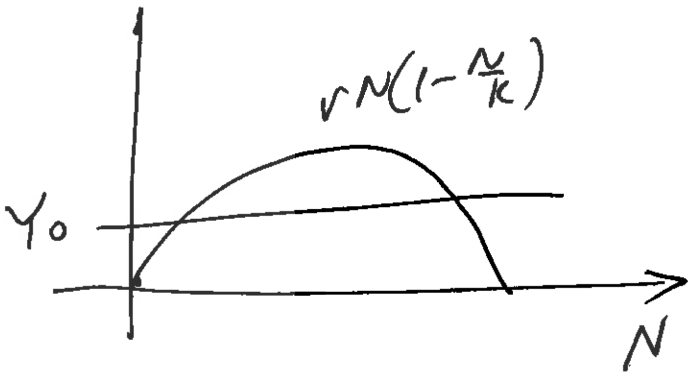
$$rN\left(1-\frac{N}{K}\right) - Y_0 = 0$$

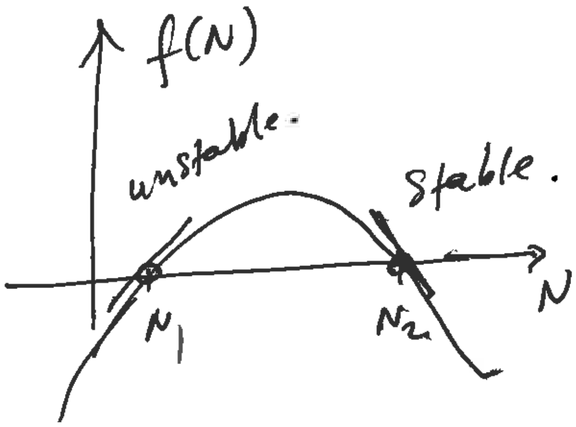
$$rN - \frac{r}{K}N^2 - Y_0 = 0$$

$$-\frac{r}{K}N^2 + rN - Y_0 = 0 \qquad (-K)$$

$$+rN^2 - rKN + Y_0K = 0$$

$$N_{1,2} = \frac{rK \pm \sqrt{r^2K^2 - 4rY_0K}}{2r} = \frac{K}{2}\left(1 \pm \sqrt{1-\frac{4Y_0}{rK}}\right)$$





$$N_2 = \frac{K}{2}\left(1+\sqrt{1-\frac{4Y_0}{rK}}\right) \quad \text{- stable.}$$

$$Y_M = \frac{rK}{4} \;; \qquad N_2 = \frac{K}{2}\left(1+\sqrt{1-\frac{Y_0}{Y_M}}\right)$$

$$N = N_2 + n, \quad n \ll 1$$

$$\frac{dN}{dt} = \frac{dn}{dt} = \underbrace{f(N_2)}_{0} + \left(\frac{\partial f}{\partial N}\right)_{N_2}\underbrace{(N-N_2)}_{n} + \dots$$

$$\frac{dn}{dt} = \left(r - \frac{2Nr}{K}\right)_{N=N_2}\cdot n$$

$$\frac{dn}{dt} = r\left[1 - \frac{\cancel{2}}{\cancel{K}}\cdot\frac{\cancel{K}}{\cancel{2}}\left(1+\sqrt{1-\frac{Y_0}{Y_M}}\right)\right]n = -r\sqrt{1-\frac{Y_0}{Y_M}}\; n$$

$$\frac{dn}{dt} = -r\sqrt{1-\frac{Y_0}{Y_M}}\, n \Rightarrow n = n_0 e^{-r\sqrt{1-\frac{Y_0}{Y_M}}\, t}$$

$$T_R(Y_0) = \frac{1}{r\sqrt{1-\frac{Y_0}{Y_M}}}, \quad T_R(0) = \frac{1}{r}$$

$$\boxed{\frac{T_R(Y_0)}{T_R(0)} = \frac{1}{\sqrt{1-\frac{Y_0}{Y_M}}}}, \quad Y_M = \frac{rK}{4}$$

$$Y_0 \to Y_M \qquad T_R(Y_M) \to \infty$$

$N \to 0$ <u>behaviour</u>:

$$\frac{dN}{dt} = \underbrace{\cancel{rN}}_{\text{small}} - \cancel{\frac{rN^2}{K}} - Y_0$$

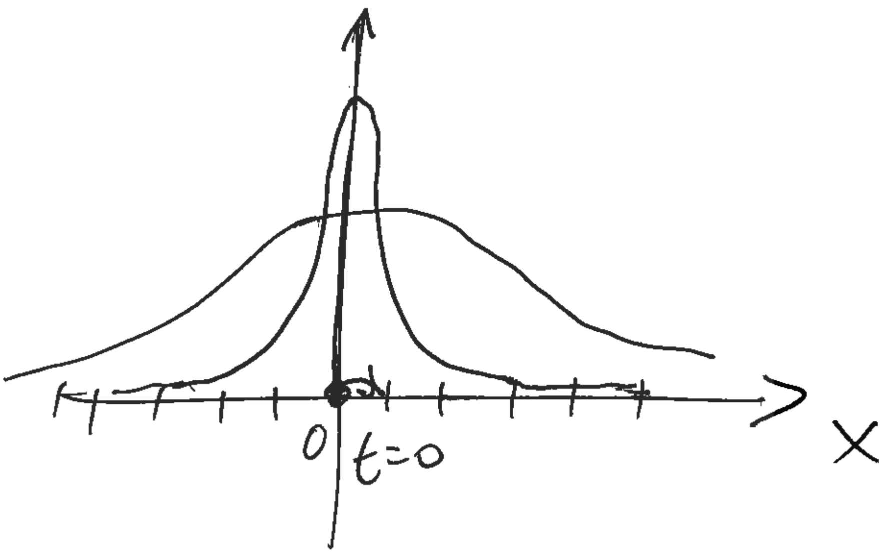
$$\frac{dN}{dt} = -Y_0 \Rightarrow \qquad N(t) \cong \underbrace{N(0)}_{\text{small}} - Y_0 t$$

Fisher-Kolmogoroff Equation

Diffusion process

$$t = n\Delta t$$

$$x = n\Delta x$$



$$p(x,t) = \frac{1}{2}\, p(x-\Delta x,\ t-\Delta t) + \frac{1}{2}\, p(x+\Delta x,\ t-\Delta t)$$

$$= \frac{1}{2}\left[p(x,t) + \cancel{\frac{\partial p}{\partial x}(-\Delta x)} + \frac{\partial p}{\partial t}(-\Delta t) + \frac{1}{2}\frac{\partial^2 p}{\partial x^2}(\Delta x)^2 + \frac{1}{2}\frac{\partial^2 p}{\partial t^2}(\Delta t)^2 + \cancel{\frac{\partial^2 p}{\partial x \partial t}(-\Delta x)(-\Delta t)} \right]$$

$$+ \frac{1}{2}\left[p(x,t) + \cancel{\frac{\partial p}{\partial x}(\Delta x)} + \frac{\partial p}{\partial t}(-\Delta t) + \frac{1}{2}\frac{\partial^2 p}{\partial x^2}(\Delta x)^2 + \frac{1}{2}\frac{\partial^2 p}{\partial t^2}(\Delta t)^2 + \cancel{\frac{\partial^2 p}{\partial x \partial t}(\Delta x)(-\Delta t)} \right]$$

$$\cancel{p(x,t)} = \cancel{p(x,t)} - \frac{\partial p}{\partial t}(\Delta t) + \frac{1}{2}\frac{\partial^2 p}{\partial x^2}(\Delta x)^2 + \frac{1}{2}\frac{\partial^2 p}{\partial t^2}(\Delta t)^2$$

$$\frac{\partial p}{\partial t}\Delta t = \frac{1}{2}\frac{\partial^2 p}{\partial x^2}(\Delta x)^2 + \frac{1}{2}\frac{\partial^2 p}{\partial t^2}(\Delta t)^2$$

$$\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x^2}\frac{(\Delta x)^2}{2\Delta t} + \cancel{\frac{1}{2}\frac{\partial^2 p}{\partial t^2}\Delta t} \qquad \Delta t \to 0$$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \frac{(\Delta x)^2}{2\Delta t} = D = \text{const} \quad (\text{diffusion coefficient})$$

(D) $$\boxed{\frac{\partial p}{\partial t} = D \frac{\partial^2 p}{\partial x^2}}$$ Diffusion equation
Heat equation

$$\frac{\partial N}{\partial t} = D \frac{\partial^2 N}{\partial x^2}$$

$$\frac{\partial N}{\partial t} = D \underbrace{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)}_{\Delta} N = D \Delta N$$

? $$p(x,t) = \frac{c}{2\sqrt{\pi D t}} e^{-\frac{x^2}{4Dt}}$$ $\underline{t > 0}$

Initial conditions $p(x,0) = ?$

$$p(x,0) = \lim_{t \to 0} \frac{c}{2\sqrt{\pi D t}} e^{-\frac{x^2}{4Dt}} = c\,\delta(x)$$

$$\boxed{\frac{\partial N}{\partial t} = rN\left(1 - \frac{N}{K}\right) + D\Delta N}$$ Fisher (1937)

$$N = N(t, x, y, z)$$

Fisher-Kolmogoroff equation.

$$\boxed{\frac{\partial N}{\partial t} = rN\left(1-\frac{N}{K}\right) + D\frac{\partial^2 N}{\partial x^2}}$$

$$u = \frac{N}{K}$$

$$\frac{\partial (N/K)}{\partial t} = r\frac{N}{K}\left(1-\frac{N}{K}\right) + D\frac{\partial^2 (N/K)}{\partial x^2}$$

$$\frac{\partial u}{\partial t} = ru(1-u) + D\frac{\partial^2 u}{\partial x^2} \qquad \Big| \cdot \frac{1}{r}$$

$$\frac{\partial u}{\partial (rt)} = u(1-u) + \frac{D}{r}\frac{\partial^2 u}{\partial x^2}$$

$$\tau = rt \quad , \cancel{X = x\sqrt{\frac{r}{D}}} \quad , \quad x = X\sqrt{\frac{D}{r}}$$

$$\boxed{\frac{\partial u}{\partial \cancel{t}} = u(1-u) + u_{\cancel{XX}}}$$

$$\frac{\partial u}{\partial \tau} = u(1-u) + \frac{\partial^2 u}{\partial X^2}$$

$$\boxed{u_t = u(1-u) + u_{xx}} \qquad u(x,t) = U(x-ct)$$

travelling wave solutions

$$u = U(x - ct) \quad , \quad z = x - ct$$

$$u_t = \frac{dU}{dz} \cdot \frac{\partial z}{\partial t} = U'(-c) = -c\,U'(z)$$

$$u_x = U'(z) \quad , \quad u_{xx} = U''(z) \qquad \left(\frac{\partial z}{\partial x} = 1\right)$$

$$u_t = u(1-u) + u_{xx}$$

$$\boxed{-cU' = U(1-U) + U'' \quad \text{ODE}}$$

c is the wavespeed.