W9205/9206/9209/9210/104C

DUBLIN INSTITUTE OF TECHNOLOGY

School of Mathematical Sciences

DT9205 MSc Mathematical Physics
DT9206 MSc Mathematical Physics
DT9209 MSc Applied Mathematics
DT9210 MSc Applied Mathematics

WINTER EXAMINATIONS 2015/2016

MATH 9974: BIOMATHEMATICS

DR R IVANOV

DR C HILLS

PROFESSOR E O'RIORDAN

9:30 – 12:30 pm, Monday, 11 January 2016

Duration: 3 hours

Attempt three questions only

All questions carry equal marks

Approved calculators may be used

Mathematical tables are provided

New Cambridge Statistical Tables are NOT permitted

1. A model for population growth is given in non-dimensional units in the form

$$\frac{du}{dt} = u(1-u^2), \qquad u(0) > 0.$$

a) Sketch the graph of the function $f(u) = u(1-u^2)$ against u for all real values of u.

(5)

b) Determine all steady states of the model (including the negative ones if any) and their stability. From this analysis find $\lim_{t\to\infty} u(t)$. Sketch the behaviour of the solution $u(t)$.

(14)

c) Solve the model explicitly for $u(t)$ and from the solution find again $\lim_{t\to\infty} u(t)$.

(14)

[33]

2. A model for the spruce budworm population $u(t)$ (in non-dimensional units) is governed by the equation

$$\frac{du}{dt} = ru\left(1 - \frac{u}{q}\right) - \frac{u^2}{1+u^2},$$

where r and q are positive parameters.

a) Show that if $q \leq 1$ there is only one nonzero steady state $u^* < 1$ for all positive values of r. Determine the stability of that steady state.

(16)

b) Determine the number of nonzero steady states and the number of the stable ones if $r = 0.5$ and $q = 20$. Explain if this choice of parameters allows for an insect outbreak.

(17)

[33]

3. a) For the following discrete population model

$$N_{t+1} = \frac{rN_t}{1+N_t},$$

where t is the discrete time and r is a positive parameter, find all steady states of the model, their existence and stability for all positive values of r.

(9)

b) A delay version of the model is given by

$$N_{t+1} = \frac{rN_t}{1+N_{t-1}}.$$

Show that for $r > 1$ there is a unique positive steady state. Linearise the equation about the positive steady state and write the obtained linear equation.

(12)

c) Analyse the stability of the positive steady state of the delay model, using the obtained linear equation from part **b)**. Explain if there is a bifurcation from monotonic to oscillatory solution and if so, find the parameter value for which the bifurcation occurs.

(12)

[33]

4. Leslie's population model is given by the system (in non-dimensional units)

$$\frac{du}{dt} = u(1-u) - \alpha uv, \qquad \frac{dv}{dt} = \rho v\left(1 - \frac{v}{u}\right),$$

where α and ρ are positive parameters.

a) Determine the kind of behavior between the two species that is implied by the model and explain briefly the role of the terms that appear in the equations.

(6)

b) Determine the steady states and their stability in dependence on the parameter values.

(10)

c) Sketch the phase portrait of the system and briefly describe the ecological implications of the results of the analysis.

(17)

[33]

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SUBJECT: BIOMATHEMATICS

EXAMINER: ROSSEN IVANOV

EXPECTED SOLUTION TO QUESTION NO: 1

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PROPOSED MARK ALLOCATION: 33

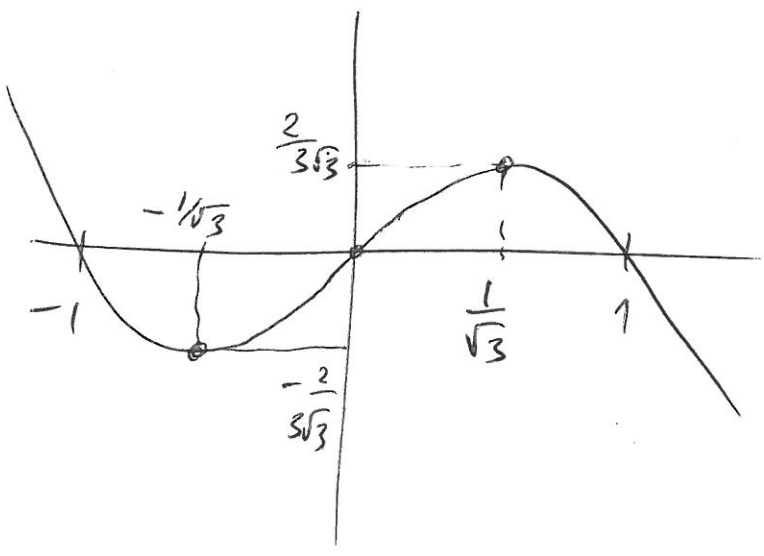
a) $f(u) = u(1-u^2) = u - u^3$

Zeroes at $u=0$, $u=\pm 1$

$f(\infty) = -\infty \qquad f(-\infty) = \infty$

$f'(u) = 1 - 3u^2 \Rightarrow u^* = \pm \frac{1}{\sqrt{3}}$ possible max / min

$f(\pm \frac{1}{\sqrt{3}}) = \pm \frac{1}{\sqrt{3}} \left(1 - \frac{1}{3}\right) = \pm \frac{2}{3\sqrt{3}}$



[5 marks]

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EXPECTED SOLUTION TO QUESTION NO: 1- cont.

PROPOSED MARK ALLOCATION:

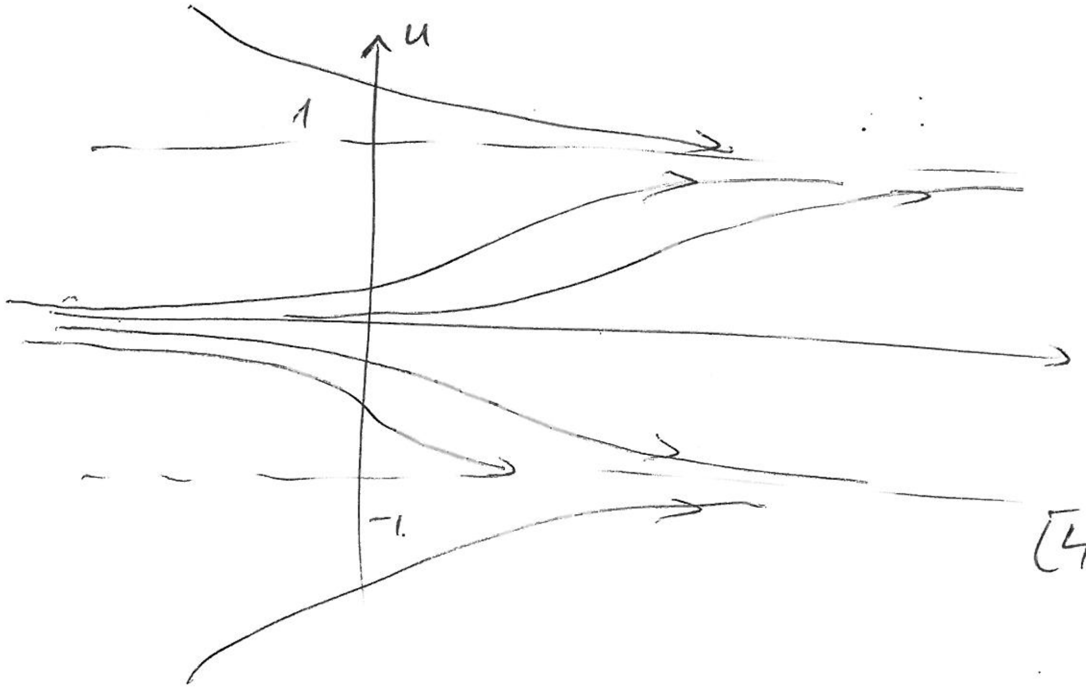
b) from the graph or directly

$f'(-1) < 0 \quad \Rightarrow u^* = -1$ is stable

$f'(0) > 0 \quad \Rightarrow u^* = 0$ is unstable

$f'(1) < 0 \quad \Rightarrow u^* = 1$ is stable

[6 marks]



[4 marks]

Since $u(0) > 0 \Rightarrow \lim_{t\to\infty} u(t) = 1$

[4 marks]

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EXPECTED SOLUTION TO QUESTION NO: 1 - cont.

PROPOSED MARK ALLOCATION:

c) By separation of variables

$$\int_{u_0}^{u} \frac{du}{u(1-u^2)} = \int_{t_0}^{t} dt \Rightarrow$$ [3 marks]

$$\int_{u_0}^{u} \left[\frac{A}{u} + \frac{B}{1-u} + \frac{C}{1+u} \right] du = (t-t_0)$$

$$A = 1 \qquad B = \frac{1}{2} \qquad C = -\frac{1}{2} \Rightarrow$$ [3 marks]

$$\left[\ln u - \frac{1}{2}\ln(1-u) - \frac{1}{2}\ln(1+u)\right]_{u_0}^{u} = t - t_0$$ 1.2

$$\ln u^2 - \ln(1-u^2) = 2(t-t_0) + \text{const.}$$

$$\frac{u^2}{1-u^2} = c\, e^{2(t-t_0)} \qquad c = \text{const.}$$ [4 marks]

$$u^2(1 + c\, e^{2(t-t_0)}) = c\, e^{2(t-t_0)}$$

$$u^2 = \frac{c\, e^{2(t-t_0)}}{1 + c\, e^{2(t-t_0)}} \to 1 \text{ as } t \to \infty$$

$\Rightarrow u(t) \to \pm 1$ and since $u(0) > 0 \Rightarrow \lim_{t\to\infty} u(t) = 1$

[4 marks]

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EXPECTED SOLUTION TO QUESTION NO: 2

PROPOSED MARK ALLOCATION: 33

a) $$\frac{du}{dt} = ru\left(1-\frac{u}{q}\right) - \frac{u^2}{1+u^2} = f(u)$$

One steady state is $u^* = 0$, the other possible steady states can be found as solutions of

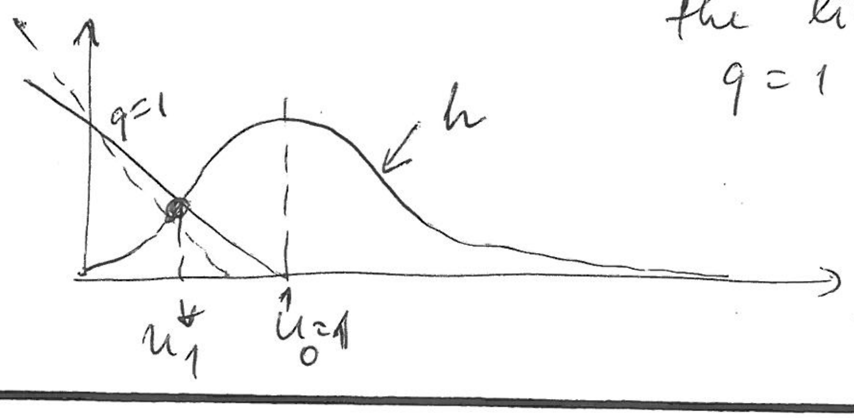
$$g(u) = r\left(1-\frac{u}{q}\right) = \frac{u}{1+u^2} \equiv h(u)$$

Graphically we can solve $g(u) = h(u)$:

$h(u) \to 0$ as $u \to 0$ & $h(u) \to 0$ as $u \to \infty$

$h(u)$ has a max $h(1) = \frac{1}{2}$ (easy to check) (6 marks)

$g(u) = r\left(1-\frac{u}{q}\right)$ is linear:



the line $g = r(1-u)$ when $q=1$ has only one intersection point $u_1^* < 1$

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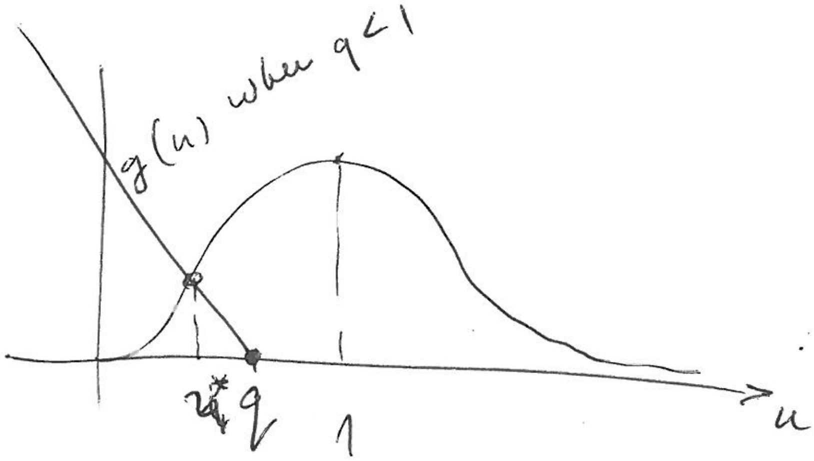
EXPECTED SOLUTION TO QUESTION NO: 2 - cont.

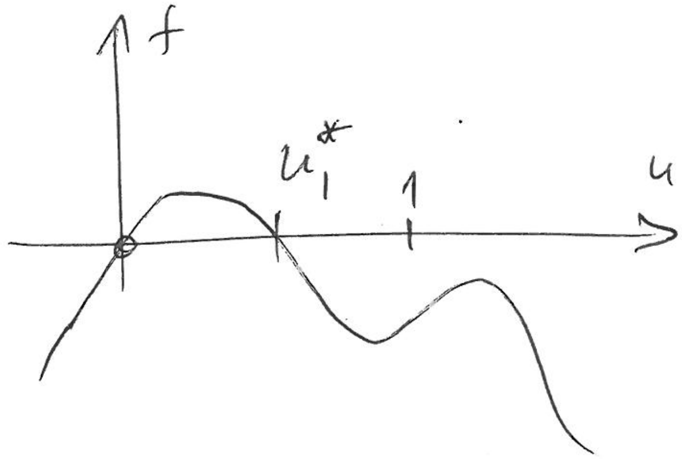
PROPOSED MARK ALLOCATION:

if $q < 1$ then the intersection is clearly again at a single point $u^* < 1$.



[5 marks]

Then the graph of $f(u)$ will be as follows, since



$\lim_{u \to \infty} f(u) = -\infty$

$\Rightarrow f'(u^*) < 0$

$\Rightarrow u^*$ is a <u>stable</u> steady state

(5 marks)

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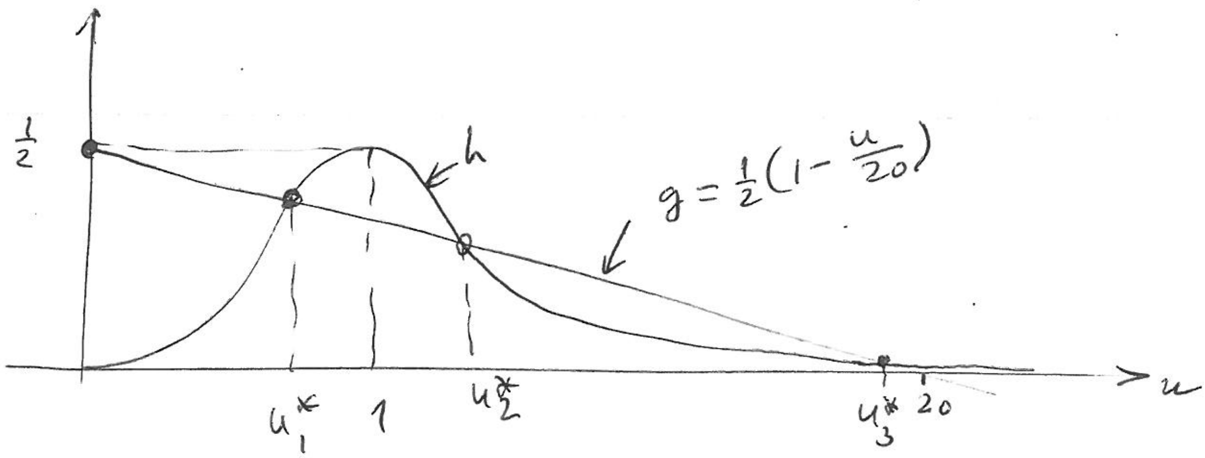
EXPECTED SOLUTION TO QUESTION NO: 2

PROPOSED MARK ALLOCATION:

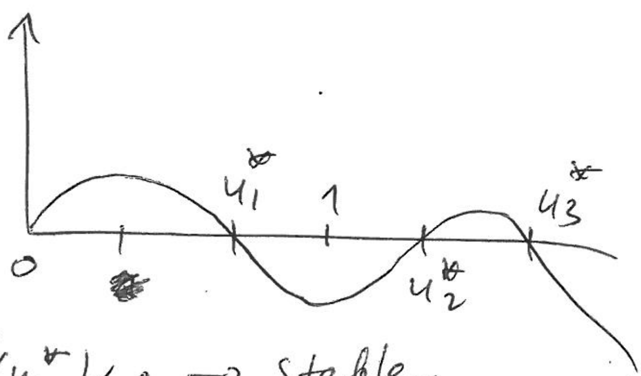
b)



In this case there are clearly 3 solutions

$0 < u_1^* < 1 < u_2^* < u_3^* < 20$ [6 marks]

The graph of $f(u)$ is now



[6 marks]

$\Rightarrow f'(u_1^*) < 0 \rightarrow$ stable

$f'(u_2^*) > 0 \rightarrow$ unstable

$f'(u_3^*) < 0 \rightarrow$ stable

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EXPECTED SOLUTION TO QUESTION NO: 2

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PROPOSED MARK ALLOCATION:

There are two stable steady states:
$u_1^* < 1$ and $u_2^* \simeq 20$, thus, an outbreak is possible between the low (u_1^*) and the high (u_2^*) steady state.

[5 marks]

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EXPECTED SOLUTION TO QUESTION NO: 3

EXAMINER: ____________________

PROPOSED MARK ALLOCATION: 33

a) $N^* = \frac{rN^*}{1+N^*}$ if $N_t^* = \text{const} = N^*$

$\underline{N^* = 0}$ or $1+N^* = r \Rightarrow \underline{N^* = r-1}$ [3 marks]

The second solution is physical only if $r>1$.

$f(N) = \frac{rN}{1+N}$, $f'(N) = \left(r - \frac{r}{1+N}\right)' = \frac{r}{(1+N)^2}$

$f'(0) = r$ $\quad f'(r-1) = \frac{r}{(1+r-1)^2} = \frac{1}{r}$ [2 marks]

N^* \ r	$0<r<1$	$r>1$
0	stable	unstable
$r-1$	does not exist	exists & stable

[4 marks]

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EXPECTED SOLUTION TO QUESTION NO: 3 - cont.

PROPOSED MARK ALLOCATION:

b) We linearise around $N^* = r-1$, $r>1$ [3 marks]

$N_t = r-1+n_t \quad |n_t| \ll 1 \Rightarrow$

$$r-1+n_{t+1} = \frac{r(r-1+n_t)}{\cancel{1}+(r-\cancel{1}+n_{t-1})}$$

$$(r-1+n_{t+1})(r+n_{t-1}) = r(r-1+n_t), \quad |n_t| \ll 1$$

$$\cancel{r(r-1)} + rn_{t+1} + (r-1)n_{t-1} + O(n^2) = \cancel{r(r-1)} + rn_t$$

$$\boxed{n_{t+1} - n_t + \frac{r-1}{r}n_{t-1} = 0}$$ [9 marks]

c) Looking for solution of the form

$n_t \sim z^t \Rightarrow z^2 - z + \frac{r-1}{r} = 0$

$$z_{1,2} = \frac{1}{2}\left[1 \pm \sqrt{1-4\frac{r-1}{r}}\right] = \frac{1}{2}\left[1 \pm \sqrt{\frac{4}{r}-3}\right]$$

$z_{1,2}$ are real for $1<r<\frac{4}{3}$ with $\frac{1}{2}<|z|<1$

$\Rightarrow$ monotonically stable solution [4 points]

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EXPECTED SOLUTION TO QUESTION NO: 3 - cont.

PROPOSED MARK ALLOCATION:

• $\boxed{r = \frac{4}{3}}$ is a bifurcation value $\Rightarrow$ the ~~solutions~~ solution becomes oscillatory. [4 marks]

$$z_{1,2} = \frac{1}{2}\left[1 \pm i\sqrt{3 - \frac{4}{r}}\right] \quad \text{for} \quad r > \frac{4}{3}$$

Since $|z_{12}| = \frac{1}{2}\sqrt{(1)^2 + \left(\sqrt{3 - \frac{4}{r}}\right)^2} = \frac{1}{2}\sqrt{1 + 3 - \frac{4}{r}}$

$|z_{1,2}| = \frac{1}{2}\sqrt{4\left(1 - \frac{1}{r}\right)} = \sqrt{1 - \frac{1}{r}} < 1$, again the ~~solutions is~~ solution stable. [4 marks]

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EXPECTED SOLUTION TO QUESTION NO: 4

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PROPOSED MARK ALLOCATION: 33

a) Predator-pray model

Predator (v) & pray (u)

The Carrying capacity for v is given by u & v evolves according to the logistic model → in the second equation;

u is ~~also~~ satisfying modified logistic equation – the $-\alpha uv$ term is further reduction of u due to interaction with predator v.

[6 marks]

b) Two steady states: $(u,v) = (1,0)$

and $\left(\frac{1}{\alpha+1}, \frac{1}{\alpha+1}\right)$

$$\text{Jacobian} = \begin{pmatrix} 1-2u+\alpha v & -\alpha u \\ \frac{\rho v^2}{u^2} & \rho\left(1-\frac{2v}{u}\right) \end{pmatrix} = A(u,v)$$

$$A(1,0) = \begin{pmatrix} -1 & \alpha \\ 0 & \rho \end{pmatrix} \Rightarrow \lambda_1 = -1 < 0, \quad \lambda_2 = \rho > 0,$$

saddle point ⇒ unstable

[4 marks]

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EXPECTED SOLUTION TO QUESTION NO: 4 - cont

PROPOSED MARK ALLOCATION:

$$A\left(\frac{1}{\alpha+1}, \frac{1}{\alpha+1}\right) = \begin{pmatrix} -\frac{1}{\alpha+1} & \frac{-\alpha}{\alpha+1} \\ \rho & -\rho \end{pmatrix}$$

$$\operatorname{tr} A = -\frac{1}{\alpha+1} - \rho < 0$$

$$\det A = \frac{\rho}{\alpha+1} + \frac{\rho\alpha}{\alpha+1} = \frac{\rho(\alpha+1)}{\alpha+1} = \rho > 0$$

$\Rightarrow \quad \lambda_1\lambda_2 = \det A > 0$

$\lambda_1 + \lambda_2 = \operatorname{tr} A < 0 \Rightarrow$ || $\underline{\text{stable}}$ steady state

$$\lambda^2 + \left(\frac{1}{\alpha+1} + \rho\right)\lambda + \rho = 0$$

$$\lambda_{1,2} = \frac{1}{2}\left[-\left(\frac{1}{\alpha+1}+\rho\right) \pm \sqrt{\left(\frac{1}{\alpha+1}+\rho\right)^2 - 4\rho}\right]$$

$\lambda_{1,2}$ - negative real, or with negative real part.

[6 marks]

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EXPECTED SOLUTION TO QUESTION NO: 4 cont.

PROPOSED MARK ALLOCATION:

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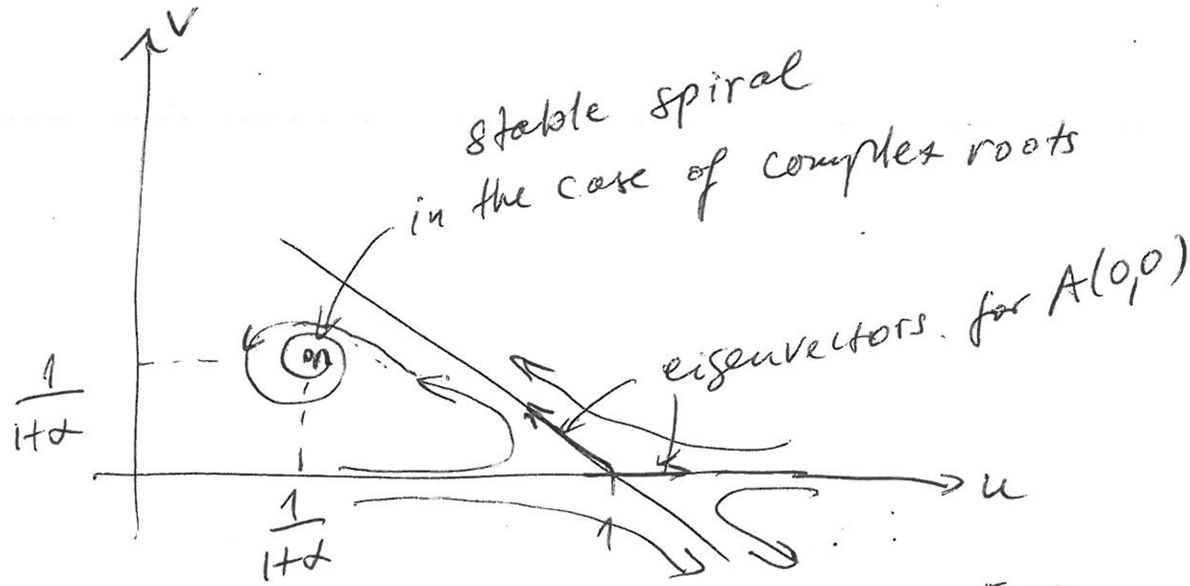

[10 marks]

If $\alpha = 0$ $u^* \to 1$ (logistic model)

now $u^* \to \frac{1}{1+\alpha} < 1$ – reduction due to predation.

Predators & pray coexist at a fixed rate $u = v = \frac{1}{1+\alpha}$.

[7 marks]