DUBLIN INSTITUTE OF TECHNOLOGY
KEVIN STREET, DUBLIN 8

MSc in Applied Mathematics and Theoretical Physics
DT234 and DT 238

Summer Examinations
2014

INTRODUCTION TO BIOMATHEMATICS

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Answer any FOUR questions. All questions carry 25 marks. If more than four questions are attempted, only the best four will be graded.

Dept. of Education Tables allowed

Question 1. The logistic population growth model including harvesting is described by the equation

$$\frac{dN}{dt} = rN\Big(1 - \frac{N}{K}\Big) - EN, \qquad N(0) > 0$$

with r, E and K positive constants.

(i) Explain briefly the terms appearing in the equation and the meaning of r, E and K.

[5 marks]

(ii) Find the steady states and determine their stability in terms of the possible values of r, E and K. Comment on the ecological implications of the results in the two cases: $E > r$ and $E < r$.

[8 marks]

(iii) Solve explicitly the model and compute the limit of $N(t)$ when $t \to \infty$. Compare the obtained limit with the results from (ii) and explain your findings.

[12 marks]

Question 2. A population growth model is described by a differential equation with delay $T > 0$:

$$\frac{dN(t)}{dt} = rN(t)\Big(1 - \frac{N(t-T)}{K}\Big),$$

with $r > 0$, $K > 0$ and $N(0) > 0$.

(i) Show that in the new nondimensional variables $u = N/K$, $\tau = rt$ and $\Theta = rT$ the equation has the form

$$\frac{du(\tau)}{d\tau} = u(\tau)[1 - u(\tau - \Theta)],$$

[2 marks]

(ii) Show that $u^* = 1$ is a steady state and linearize the equation about the steady state by writing $u(\tau) = u^* + n(\tau)$, where $|n(\tau)| \ll 1$.

[5 marks]

(iii) Look for solutions of the form $n(\tau) = ce^{\lambda\tau}$, where c is a constant, and write the corresponding equation for λ.

[3 marks]

(iv) Decompose λ into real and imaginary parts ($\lambda = \mu + i\omega$) and write separately the real and imaginary parts of the equation for λ, obtained in (iii) in terms of μ and ω.

[5 marks]

(v) Analyze the equations obtained in (iv) when the delay Θ increases from 0 to $\frac{\pi}{2}$, bearing in mind that μ and ω depend on Θ. Show that the steady state solution u^* is stable when $0 \leq \Theta < \pi/2$. Demonstrate that the first bifurcation value of Θ for which the steady state u^* becomes unstable and the solution becomes oscillatory corresponds to $\Theta = \pi/2$, (and $\mu = 0$, $\omega = \pm 1$) or, in dimensional terms, $rT = \pi/2$. Show that this bifurcation occurs when μ (being negative when $\Theta = 0$) reaches the bifurcation value $\mu = 0$ when $\Theta = \pi/2$.

[5 marks]

(vi) Show that the period of oscillations (in nondimensional units) at the bifurcation value is 4Θ.

[5 marks]

Question 3. (i) Show that an exact traveling wave solution exists for the Fisher-Kolmogoroff type equation

$$\frac{\partial u}{\partial t} = u(1-u^q) + \frac{\partial^2 u}{\partial x^2}$$

where $q > 0$, by looking for solutions in the form

$$u(x,t) = U(z) = \frac{1}{(1+ae^{bz})^s}, \qquad z = x - ct$$

where $c > 0$ is the wavespeed and b and s are positive constants. Determine the unique values for c, b and s in terms of q. Choose a value for a such that $U(0) = 1/2$.

[20 marks]

(ii) Take for simplicity $q = 1$ and sketch the solution from (i). Explain briefly the relevance of the equation in modeling population dynamics in this case and the meaning of the above solution in particular.

[5 marks]

Question 4. Consider the discrete population model with delay

$$U_{t+1} = U_t e^{r(1-U_{t-1})},$$

where t is the discrete time and r is a positive parameter.

(i) Show that $U^* = 1$ is a steady state of the model. Linearize the equation about $U^* = 1$ by writing $U_t = U^* + v_t$, where $|v_t| \ll 1$. Obtain the linearized equation

$$v_{t+1} - v_t + rv_{t-1} = 0.$$

[7 marks]

(ii) Prove that, for $0 < r < 1$, $U^* = 1$ is a stable steady state.

[6 marks]

(iii) Show that $r = 1$ is a bifurcation value for which the steady state bifurcates to a periodic solution of period 6.

[6 marks]

(iv) Prove that, for $r > 1$ the steady state $U^* = 1$ is unstable.

[6 marks]

Question 5. (i) Describe the type of interaction between two species with populations u and v that is implied by the model

$$\begin{aligned} \frac{du}{dt} &= u(1-u) - \frac{uv}{u+\alpha}, \\ \frac{dv}{dt} &= \beta v\left(1 - \frac{v}{u}\right), \end{aligned}$$

where the parameters α and β are positive.

[5 marks]

(ii) Determine the steady states and their stability for all possible positive values of the parameters α and β. Briefly describe the ecological implications of the results of the analysis. Support your explanations by a sketch of the phase portrait of the system.

[20 marks]

Question 6. An epidemic model of the spread of rabies among foxes involves two interacting populations: infectives with density $i(\mathbf{x}, t)$ and susceptibles with density $s(\mathbf{x}, t)$, both functions of the space variable $\mathbf{x} = (x_1, x_2, x_3)$ as well as time t. The model is described by the system

$$\begin{aligned} \frac{\partial s}{\partial t} &= -rsi, \\ \frac{\partial i}{\partial t} &= rsi - ai + D\Delta i, \end{aligned}$$

where r, a and D are positive constants and $\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$.

(i) Explain briefly the rationale of the model and the meaning of the constants r, a and D.

[3 marks]

(ii) Consider the one-dimensional problem, $i = i(x_1, t)$, $s = s(x_1, t)$. Nondimensionalise the system by writing

$$I = \frac{i}{s_0}, \quad S = \frac{s}{s_0}, \quad x = \sqrt{\frac{rs_0}{D}}x_1, \quad \tau = rs_0 t, \quad \lambda = \frac{a}{rs_0},$$

where s_0 is the population density in the absence of epidemic. Show that the system takes the form

$$\frac{\partial S}{\partial \tau} = -SI,$$
$$\frac{\partial I}{\partial \tau} = SI - \lambda I + \frac{\partial^2 I}{\partial x^2}.$$

[2 marks]

(iii) Look for traveling wave solutions by setting $I(x,\tau) = I(z)$, $S(x,\tau) = S(z)$, where $z = x - c\tau$ and c is the traveling speed. Obtain the system of ordinary differential equations:

$$I'' + cI' + I(S-\lambda) = 0, \qquad cS' - IS = 0,$$

where the prime denotes differentiation with respect to z.

[4 marks]

(iv) Consider solutions, such that $I(-\infty) = I(\infty) = 0$, $I(z) > 0$, $S(z) > 0$, $0 \leq \sigma = S(-\infty) < S(\infty) = 1$, where σ is a positive constant. Explain the meaning of σ. Prove that the system has a conservation law

$$I' + cI + cS - c\lambda \ln S = \text{constant}$$

and show that

$$\frac{\sigma - 1}{\ln \sigma} = \lambda.$$

[6 marks]

(v) Prove that $\sigma < \lambda < 1$.

[7 marks]

(vi) Linearize the equation for I in the region $z \to \infty$ where $S \to 1$, $I \to 0$. Demonstrate that if the traveling wave solution exists, the wavespeed c must satisfy $c \geq 2\sqrt{1-\lambda}$.

[3 marks]

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EXAMINATION SITTING: SUMMER ☑ SUPPLEMENTAL ☐ AUTUMN ☐ WINTER ☐

SUBJECT: Introduction to Biomathematics

EXAMINER: ROSSEN IVANOV

EXPECTED SOLUTION TO QUESTION NO: 1

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PROPOSED MARK ALLOCATION: 25

(i) $rN\left(1-\frac{N}{K}\right)$ → logistic growth term;

r → birth rate

K → carrying capacity of the ecosystem

EN → harvesting term (yield); E measures the constant effort of the harvesting.

[5 marks]

(ii) The steady states satisfy

$$f(N) = rN\left(1-\frac{N}{K}\right) - EN = 0$$

$$\Rightarrow N^* = 0 \quad \text{or} \quad N^* = \frac{r-E}{r}K,$$

the second exists only if $E<r$. If $E>r$ only $N^*=0$ exists.

$$f'(N) = (r-E) - \frac{2rN}{K}$$

$$f'(0) = r-E$$

$$f'\left(N^* = \frac{r-E}{r}K\right) = r-E - \frac{2\cancel{K}(r-E)\cancel{r}}{\cancel{r}\cancel{K}} = -(r-E)$$

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SUBJECT:

EXPECTED SOLUTION TO QUESTION NO: 1 – cont.

EXAMINER:

PROPOSED MARK ALLOCATION:

$\Rightarrow$ we have 2 situations

	$E<r$	$E>r$
$N^*=0$	unstable	stable
$N^*=\frac{r-E}{r}K$	stable	does not exist

[5 marks]

$\Rightarrow$ when $E>r$ the harvesting effort is too big and the harvesting brings the population to extinction ($N^*=0$ is stable)

when $E<r$ the harvesting reduces the equilibrium population size from $N^*=K$ (when $E=0$) to $N^*=\frac{r-E}{r}K<K$.

(iii) Separation of variables: [3 marks]

$$\frac{dN}{N(r-E-\frac{rN}{K})}=dt \quad ; \quad \text{Partial fractions:}$$

$$\frac{1}{r-E}\left\{\frac{dN/N^*}{N/N^*}+\frac{dN/N^*}{1-\frac{N}{N^*}}\right\}=dt \quad , \quad \underline{N^*=\frac{r-E}{r}K}$$

$$\left\{ \ln\left(\frac{N}{N^*}\right) - \ln\left(1 - \frac{N}{N^*}\right) \right\} = (r-E)t + \text{const.}$$

$$\frac{\frac{N}{N^*}}{1 - \frac{N}{N^*}} = c\, e^{(r-E)t} \qquad \text{where } c = \text{const.}$$

$$\frac{N}{N^*} = c\, e^{(r-E)t} \left[1 - \frac{N}{N^*}\right]$$

$$N = N^* \frac{1}{1 + c^{-1} e^{-(r-E)t}} \qquad [8 \text{ marks}]$$

1) $r - E > 0$ $\quad e^{-(r-E)t} \to 0$ as $t \to \infty$

$$\Rightarrow \lim_{t\to\infty} N = N^* = \frac{r-E}{r} K \quad (\text{stable equilibrium})$$

2) $r - E < 0$ $\quad e^{-(r-E)t} \to \infty$, $\Rightarrow \lim_{t\to\infty} N = 0$

(extinction in this case, as described in (ii).

[2+2 marks]

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EXPECTED SOLUTION TO QUESTION NO: 2

EXAMINER: ____________

PROPOSED MARK ALLOCATION: 25

(i) $\frac{d(N/K)}{d(rt)} = \left(\frac{N}{K}\right)\left(1 - \frac{N(t-T)}{K}\right) \Rightarrow u = \frac{N}{K},\ \tau = rt,\ \theta = rT$

$$\frac{du(\tau)}{d\tau} = u(\tau)\left[1 - u(\tau-\theta)\right].$$

(ii) the steady state satisfies $u^*(1-u^*) = 0 \Rightarrow$ $u^* = 1$ is a steady state, if $u = 1 + n(\tau)$

$$\frac{dn(\tau)}{d\tau} = (1+n(\tau))(\cancel{1} - \cancel{1} - n(\tau-\theta)) \simeq -n(\tau-\theta)$$

$$\frac{dn(\tau)}{d\tau} = -n(\tau-\theta)$$

(iii) $n(\tau) = ce^{\lambda\tau} \Rightarrow \quad c\lambda e^{\lambda\tau} = -ce^{\lambda(\tau-\theta)}$

$\Rightarrow \boxed{\lambda = -e^{-\lambda\theta}}$

(iv) $\lambda = \mu + i\omega \Rightarrow \mu + i\omega = -e^{-(\mu+i\omega)\theta}$

$$\mu + i\omega = -e^{-\mu\theta}(\cos\omega\theta - i\sin\omega\theta)$$

$$\mu = -e^{-\mu\theta}\cos\omega\theta, \quad \omega = e^{-\mu\theta}\sin\omega\theta$$

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EXPECTED SOLUTION TO QUESTION NO: 2 - cont.

PROPOSED MARK ALLOCATION:

(v) Suppose $\omega = 0 \Rightarrow \mu = -e^{-\mu\Theta} \Rightarrow$ there are no positive roots for μ, (solution $u^*=1$ is stable)

We also notice that if ω is a solution, so is $-\omega$, so we need only to consider $\omega > 0$

When $\Theta = 0$ $\quad \mu = -1 \quad \omega = 0$ and the solution $u^* = 1$ is stable, $n(\tau) \to 0$. The first bifurcation occurs when μ reaches $\mu = 0$ and then $u^* = 1$ becomes unstable, since $|n(\tau)|$ does not approach 0.

If $\mu = 0$, we have
$$\begin{cases} 0 = -\cos\omega\Theta \\ \omega = \sin\omega\Theta \end{cases}$$

$\Rightarrow \sin\omega\Theta = \pm 1 \Rightarrow \omega = \pm 1$ but we consider $\omega > 0 \Rightarrow \omega = 1 \Rightarrow \Theta = \frac{\pi}{2}$

(vi) When $\Theta = \pi/2$ $\quad \lambda = \mu + i\omega = 0 \pm i$

$$n(\tau) = ce^{\lambda\tau} = ce^{\pm i\tau}$$ and has period $t_p = 2\pi$

$$\frac{t_p}{\Theta} = \frac{2\pi}{\pi/2} = 4 \Rightarrow t_p = 4\Theta.$$

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EXAMINER:			
PAGE 6 **OF** 20		**PROPOSED MARK ALLOCATION:**	25

(i) We notice that the solution $U(z) = (1+ae^{bz})^{-s}$ automatically satisfies the boundary conditions

$$u(\infty) = 0 \quad \& \quad u(-\infty) = 1$$

The corresponding ODE for $U(z)$ is

$$L(U) = U'' + cU' + U(1 - U^q) = 0$$

Since the solution is translational-invariant $z \to z + \text{const}$ then it is clear that a is an arbitrary constant, i.e. b & s should not depend on a.

We compute $U'(z) = -s \dfrac{ab\, e^{bz}}{(1+ae^{bz})^{s+1}}$

$$U''(z) = -sb^2 a e^{bz} (1+ae^{bz})^{-s-2} (1 - as\, e^{bz})$$

$$L(U) = \frac{1}{(1+ae^{bz})^{s+2}} \left[a^2(s^2b^2 - scb + 1) e^{2bz} + (2 - scb - sb^2) a e^{bz} + 1 - (1 + abe^{bz})^{2-sq} \right]$$

We want $L(U) = 0$ for all z $\Rightarrow$ all coefficients of e^0, e^{bz} and e^{2bz} must be 0.

$\Rightarrow 2 - sq = 0, 1, \text{ or } 2;$

$2 - sq = 0 \Rightarrow sq = 2$

$2 - sq = 1 \Rightarrow sq = 1$

$2 - sq = 2 \Rightarrow sq = 0$ (not possible: $s > 0$ & $q > 0$)

Two possibilities $\Rightarrow$ $s = \frac{1}{q}$ and $s = \frac{2}{q}$ [10 marks]

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EXPECTED SOLUTION TO QUESTION NO: 3 - cont

PROPOSED MARK ALLOCATION:

Let $s = \frac{1}{q}$, $sq = 1$; $2 - sq = 2 - 1 = 1$

$$L(u) = (1+ae^{bz})^{-s-2}\left[a^2(s^2b^2 - scb + 1)e^{2bz} + (2 - scb - sb^2)ae^{bz} - ae^{bz}\right] = 0$$

$$\Rightarrow \quad sb(sb-c) + 1 = 0$$

$$2 - scb - sb^2 - 1 = 0 \iff 1 = sb(b+c)$$

$$sb(sb-c) + 1 = sb(sb-c) + sb(b+c) = 0$$

$$\Rightarrow sb(sb - \cancel{c} + b + \cancel{c}) = 0 \Rightarrow sb^2(s+1) = 0 \quad \times$$

Not possible, $s > 0$ & $s+1 > 0$.

The only remaining possibility is $s = \frac{2}{q}$ or $sq = 2$

This leads to

$$s^2b^2 - scb + 1 = 0 \Rightarrow sb(c - sb) = 1$$

$$2 - scb - sb^2 = 0 \Rightarrow sb(b+c) = 2$$

$$\Big\} \; b + c = 2(c - sb)$$

$$\Rightarrow c = b(1+2s), \quad sb[b + b(1+2s)] = 2 \Rightarrow sb^2(2+2s) = 2$$

$$\Rightarrow b = \frac{1}{\sqrt{s(s+1)}} \Rightarrow c = \frac{1+2s}{\sqrt{s(s+1)}} \quad \text{but } s = \frac{2}{q} \Rightarrow$$

$$b = \frac{q}{\sqrt{2(q+2)}}, \quad c = \frac{q+4}{\sqrt{2(q+2)}}, \quad s = \frac{2}{q}$$

If $u(0) = \frac{1}{2} \Rightarrow \frac{1}{(1+a)^s} = \frac{1}{2} \Rightarrow (1+a)^s = 2 \Rightarrow a = 2^{q/2} - 1$

[10 marks]

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EXPECTED SOLUTION TO QUESTION NO: 3 - cont.

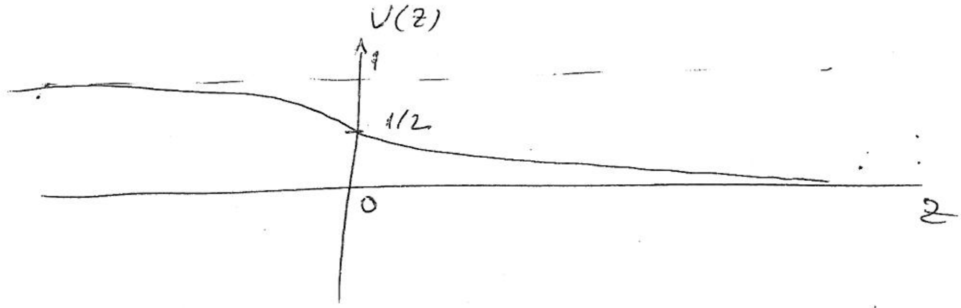
EXAMINER:

PROPOSED MARK ALLOCATION:

(ii) When $q=1$ the equation is known as Fisher - Kolmogoroff equation;

$s=2$, $b=\frac{1}{\sqrt{6}}$, $c=\frac{5}{\sqrt{6}}\simeq 2.04$, $a=\sqrt{2}-1$

$$U(z)=\frac{1}{[1+(\sqrt{2}-1)e^{z/\sqrt{6}}]^2}$$



The equation describes 'geographic' spread of population that grows logistically since $z=x-ct$, initially ($t=-\infty$) $z=\infty$ and $U(\infty)=0$, there is no population. As it grows it spreads as a traveling wave until it reaches the carrying capacity $U=1$ at $z=-\infty$ ($t\to\infty$). The term $\frac{\partial^2 u}{\partial x^2}$ describes the spatial spread of the population (i.e. diffusion).

[5 marks]

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EXAMINATION SITTING: SUMMER ☐ SUPPLEMENTAL ☐ AUTUMN ☐ WINTER ☐

EXPECTED SOLUTION TO QUESTION NO: 4

PROPOSED MARK ALLOCATION: 25

(i) $U^* = 1$ is a steady state since $U^* = U^* e^{r(1-U^*)}$.

$U_t = U^* + v_t = 1 + v_t$, $|v_t| \ll 1$:

$1 + v_{t+1} = (1+v_t)\, e^{r(1-\cancel{1}-v_{t-1})} = (1+v_t)\, e^{-rv_{t-1}}$.

$1 + v_{t+1} \approx (1+v_t)(1 - r v_{t-1}) \simeq 1 + v_t - r v_{t-1}$

$\Rightarrow \quad v_{t+1} - v_t + r v_{t-1} = 0$. [7 marks]

(ii) The linearized equation has solutions $v_t = z^t$ where $z^2 - z + r = 0$, i.e.

$$z_{1,2} = \frac{1}{2}\left[1 \pm \sqrt{1-4r}\,\right]$$

If $0 < r < \frac{1}{4}$ z_1 and z_2 are real, $0 < z_{1,2} < 1$ and therefore $v_t \to 0$ as $t \to \infty$ and hence $U^* = 1$ is a linearly stable equilibrium state. [3 marks]

if $1 > r > 1/4$ z_1 and z_2 are complex,

$$z_{1,2} = \frac{1}{2}\left[1 \pm i\sqrt{4r-1}\,\right], \quad |z_{1,2}| = \frac{1}{2}\sqrt{1^2 + 4r - 1} = \sqrt{r} < 1$$

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EXPECTED SOLUTION TO QUESTION NO: 4 - cont.

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PROPOSED MARK ALLOCATION: ______

$\Rightarrow$ again $v_t \to 0$ as $t \to \infty$ $\left(v_t = A z_1^t + B z_2^t \right)$

[3 marks]

(iii) When $r = 1$, $z_{1,2} = \frac{1}{2}(1 \pm i\sqrt{3}) = e^{\pm \pi i/3}$

$|z_{1,2}| = 1$ and v_t does not approach 0 for $t \to \infty$

$$v_t = A e^{(\pi i/3)t} + A^* e^{-(\pi i/3)t}$$

$$v_{t+6} = e^{2\pi i}\left(A e^{(\pi i/3)t} + A^* e^{-(\pi i/3)t} \right) = e^{2\pi i} v_t = v_t$$

$\Rightarrow$ when $r = 1$ v_t is periodic with period 6.

[6 marks]

(iv) when $r > 1$ $|z_{1,2}| = \sqrt{r} > 1$ and

$v_t \to \infty$ as $t \to \infty$, i.e. v_t grows unboundedly

and $u^* = 1$ is unstable. (6 marks)

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EXPECTED SOLUTION TO QUESTION NO: 5

PROPOSED MARK ALLOCATION: 25

(i) $\beta v\left(1-\frac{v}{u}\right)$ resembles logistic growth term for v with carrying capacity u, determined by the other population u; this suggests that v is 'eating' u
$\Rightarrow$ predator - pray model [5 marks]

(ii) The steady states are $u=1$ & $v=0$ and another one, determined by (note: $u \neq 0$)

$$1-u^* = \frac{v^*}{u^*+\alpha} \quad \& \quad u^* = v^* \quad ; \text{ i.e.}$$

$$1-u^* = \frac{u^*}{u^*+\alpha} \rightarrow \text{quadratic equation:}$$

$$(1-u^*)(u^*+\alpha) = u^*$$

$$-(u^*)^2 - \alpha u^* + \cancel{u^*} + \alpha = \cancel{u^*} ; \quad (u^*)^2 + \alpha u^* - \alpha = 0$$

$$u^* = \frac{-\alpha + \sqrt{\alpha^2+4\alpha}}{2} \rightarrow \text{the positive root}$$

$$\Rightarrow \quad u^* = v^* = \frac{1}{2}\left(-\alpha + \sqrt{\alpha^2+4\alpha}\right) \qquad \text{(2 marks)}$$

$$\left| \begin{array}{l} u' = u(1-u) - \dfrac{uv}{u+\alpha} = f(u,v) \\ v' = \beta v\left(1-\dfrac{v}{u}\right) = g(u,v) \end{array} \right.$$

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EXPECTED SOLUTION TO QUESTION NO: 5 - cont.

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PROPOSED MARK ALLOCATION:

Linearization around (u^*, v^*) : Jacobian matrix

$$A(u,v) = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix} = \begin{pmatrix} 1-2u - \frac{v}{u+\alpha} + \frac{uv}{(u+\alpha)^2} & -\frac{u}{u+\alpha} \\ +\beta\frac{v^2}{u^2} & \beta - \frac{2\beta v}{u} \end{pmatrix}$$

Two steady states

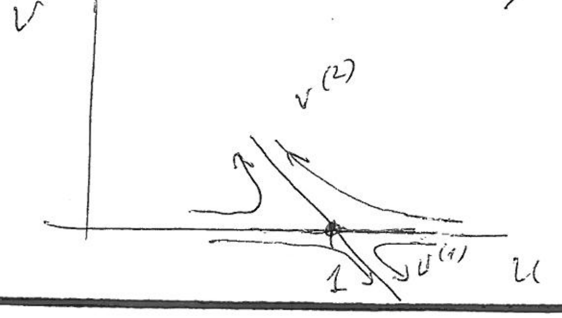
$$A(1,0) = \begin{pmatrix} 1-2 & -\frac{1}{1+\alpha} \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} -1 & -\frac{1}{1+\alpha} \\ 0 & \beta \end{pmatrix}$$

$\lambda_1 = -1 < 0$ & $\lambda_2 = \beta > 0 \rightarrow$ saddle point

As $t \to \infty$ the trajectories align along $v^{(2)}$:

$$\begin{pmatrix} -1-\beta & -\frac{1}{1+\alpha} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1^{(2)} \\ v_2^{(2)} \end{pmatrix} = 0 \Rightarrow -(1+\beta) v_1^{(2)} = \frac{1}{1+\alpha} v_2^{(2)}$$

$$v^{(2)} \cong \begin{pmatrix} -(1+\alpha)(1+\beta) \\ 1 \end{pmatrix} \quad ; \quad v^{(1)} \cong \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



[5 marks]

$$A(u^*, u^*) = \begin{pmatrix} u^*\left[\frac{u^*}{(u^*+\alpha)^2} - 1\right] & -\frac{u^*}{u^*+\alpha} \\ \beta & -\beta \end{pmatrix}$$

$$1 - 2u^* - \frac{u^*}{u^*+\alpha} + \frac{u^{*2}}{(u^*+\alpha)^2} = \cancel{1} - 2u^* - (\cancel{1} - u^*) + \frac{u^{*2}}{(u^*+\alpha)^2} = u^*\left[\frac{u^*}{(u^*+\alpha)^2} - 1\right]$$

The characteristic equation is $\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0$

The stability condition is

$\operatorname{tr} A < 0$

$\det A > 0$

$$\det A = \beta u^*\left[1 - \frac{u^*}{(u^*+\alpha)^2}\right] + \beta\frac{u^*}{u^*+\alpha} = \beta u^*\left[1 - \frac{1-u^*}{u^*+\alpha} + \frac{1}{u^*+\alpha}\right] =$$

$$= \beta u^* \frac{u^*+\alpha \cancel{-1} + u^* + \cancel{1}}{u^*+\alpha} = \frac{\beta u^*(2u^*+\alpha)}{u^*+\alpha} > 0 \text{ always}$$

when $u^* > 0$ exists. [3 marks]

Thus, the stability condition is $\operatorname{tr} A < 0$ or

$\beta > u^*\left[\frac{u^*}{(u^*+\alpha)^2} - 1\right] = \varphi(\alpha)$ which can be written also as

$\beta > u^*\left[\frac{1-u^*}{u^*+\alpha} - 1\right] = \varphi(\alpha)$

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EXPECTED SOLUTION TO QUESTION NO: 5 - cont

PROPOSED MARK ALLOCATION:

$$\beta > \frac{u^*}{u^*+\alpha}\left[1-u^*-u^*-\alpha\right] = \varphi(\alpha)$$

$$\beta > (1-u^*)(1-\alpha-2u^*) = \varphi(\alpha)$$

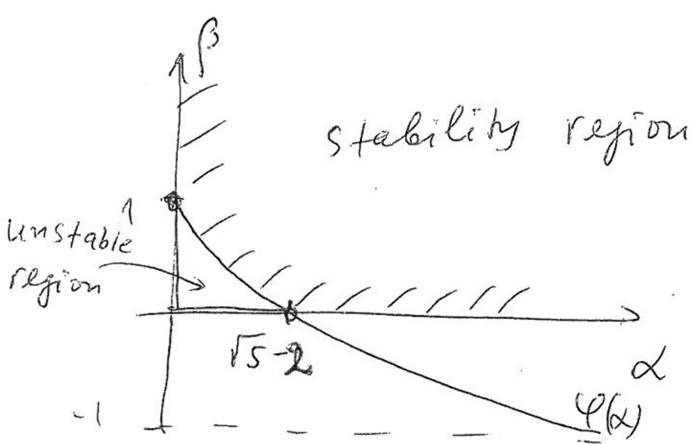
$$\beta > \left(1-\frac{\sqrt{\alpha^2+4\alpha}-\alpha}{2}\right)\left(1-\cancel{\alpha}+\cancel{\alpha}-\sqrt{\alpha^2+4\alpha}\right) = \varphi(\alpha)$$

$$\beta > \left(1-\sqrt{\alpha^2+4\alpha}\right)\left(\frac{2+\alpha-\sqrt{\alpha^2+4\alpha}}{2}\right) = \varphi(\alpha)$$

$$\beta > \left(1-\sqrt{\alpha^2+4\alpha}\right)\frac{(2+\alpha)^2-(\alpha^2+4\alpha)}{2\left(2+\alpha+\sqrt{\alpha^2+4\alpha}\right)} = \varphi(\alpha)$$

$$\beta > \left(1-\sqrt{\alpha^2+4\alpha}\right)\frac{2}{2+\alpha+\sqrt{\alpha^2+4\alpha}} = \varphi(\alpha)$$

$\varphi(0) = 1$; $\varphi(\infty) = -1$; $\varphi(\sqrt{5}-2) = 0$



There are values of α & β where the inequality is satisfied (stability region) & values when the inequality is NOT satisfied (region of instability)

[5 marks]

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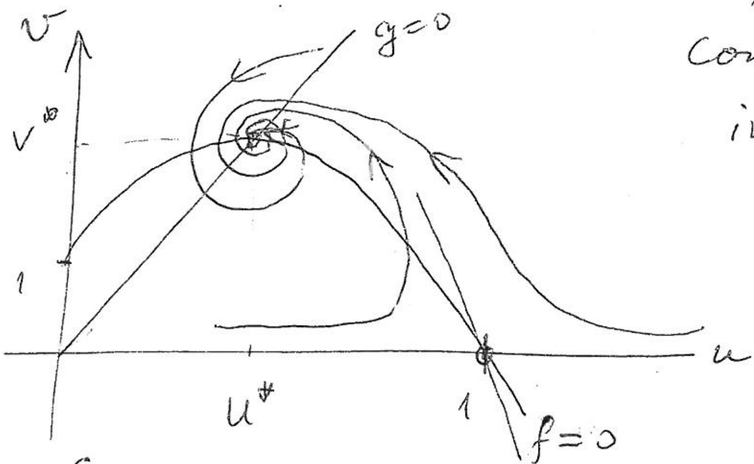
EXPECTED SOLUTION TO QUESTION NO: 5 - cont.

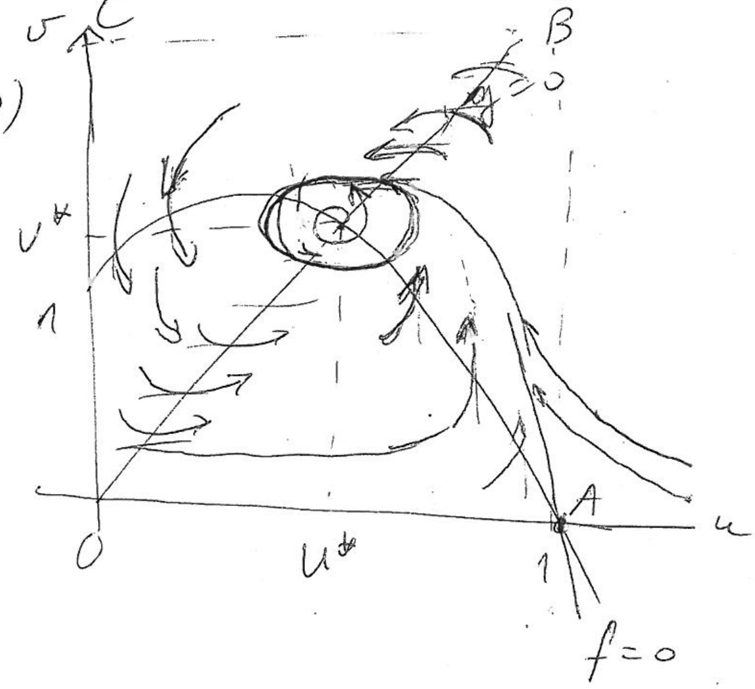
PROPOSED MARK ALLOCATION: ______

(a) When $u^* = v^*$ is stable, all trajectories



converge to $u^* = v^*$ and in long-term behaviour is $u \to u^* = \text{const.}$ $v \to u^* = \text{const.}$ as $t \to \infty$.

(b)



When $u^* = v^*$ is unstable the trajectories are still trapped in the trapping region OABC as shown on the picture. Since they can not escape, there is a limit cycle stable periodic solution around the point (u^*, u^*)

(Poincaré-Bendixson Theorem)

[5 marks]

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PAGE 16 OF 20		PROPOSED MARK ALLOCATION:	25

(i) The healthy foxes are territorial animals, so they do not change their teritory. the infected are not, so they move and enter the neighbouring teritories according to the Ds_i term.
rsi term is the rate of the transmission of the infection, ai is the death rate of the infective state.
r is a coefficient measuring the transmission efficiency of the disease from the infectives to the susceptibles.
$\frac{1}{a}$ is the life-expectancy of an infective
D is the diffusion coefficient, describing the spatial dispersal properties of the infectives

[3 marks]

(ii) $\frac{\partial S}{\partial t} = -rSi \Rightarrow$

$$\frac{\partial(\phi_0 S)}{\partial\left(\frac{\tau}{r\phi_0}\right)} = -r(\phi_0 S)(\phi_0 I) \Rightarrow \boxed{\frac{\partial S}{\partial \tau} = -SI}$$

[1 mark]

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EXPECTED SOLUTION TO QUESTION NO: 6 – cont.

PROPOSED MARK ALLOCATION:

$$\frac{\partial(s_0 I)}{\partial\left(\frac{\tau}{rs_0}\right)} = r(s_0 S)(s_0 I) - a s_0 I + D\frac{\partial^2(s_0 I)}{\partial\left[\left(\sqrt{\frac{D}{rs_0}}\right)x\right]^2}$$

$$\cancel{rs_0^2}\frac{\partial I}{\partial \tau} = \cancel{rs_0^2}\, SI - \frac{a}{rs_0} I + \frac{\cancel{D}\, s_0}{\frac{\cancel{D}}{\cancel{rs_0}}}\frac{\partial^2 I}{\partial x^2} \Rightarrow$$

$$\boxed{\frac{\partial I}{\partial \tau} = SI - \lambda I + \frac{\partial^2 I}{\partial x^2}}$$ (1 mark)

(iii) $$\frac{\partial I}{\partial \tau} = \frac{dI}{dz}\frac{\partial z}{\partial \tau} = \frac{dI}{dz}(-c) = -c\frac{dI}{dz} = -cI'$$

$$\frac{\partial I}{\partial x} = \frac{dI}{dz}\cdot\frac{dz}{dx} = I'(1) = I' \quad \text{etc.}$$

$$\Rightarrow \left|\begin{array}{l} I'' + cI' + I(S-\lambda) = 0 \\ cS' - IS = 0 \end{array}\right.$$ [4 marks]

(iv) $\sigma = \lim_{z \to -\infty} S = \lim_{t \to +\infty} S$ i.e. the population unaffected by the disease after the epidemic outbreak

[1 mark]

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EXPECTED SOLUTION TO QUESTION NO: 6 - Cont.

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PROPOSED MARK ALLOCATION:

$$I'' + cI' + \underbrace{IS}_{cS'} - \lambda \underbrace{I}_{cS'/S} = 0$$

$$I'' + cI' + cS' - \lambda c \frac{S'}{S} = 0 \longrightarrow \text{integration gives}$$

$$I' + cI + cS - \lambda c \ln S = \text{const}$$

[3 marks]

$$\underbrace{I'(\infty)}_{0} + c\underbrace{I(\infty)}_{0} + c\underbrace{S(\infty)}_{1} - \lambda c \ln \underbrace{S(\infty)}_{1} = \underbrace{I'(-\infty)}_{0} + c\underbrace{I(-\infty)}_{0} + c\underbrace{S(-\infty)}_{\sigma} - \lambda c \ln \underbrace{S(-\infty)}_{\sigma}$$

$$\cancel{c}(1) = \cancel{c}(\sigma) - \lambda \cancel{c} \ln \sigma \Rightarrow \lambda \ln \sigma = \sigma - 1$$

$$\boxed{\lambda = \frac{\sigma - 1}{\ln \sigma}}$$

[2 marks]

(v) Since $0 < \sigma < 1$ we examine the function

$$\varphi(\sigma) = \lambda - \sigma = \frac{\sigma - 1}{\ln \sigma} - \sigma$$

$$\lim_{\sigma \to 1} (\varphi) = \lim_{\sigma \to 1} \frac{(\sigma - 1)'}{(\ln \sigma)'} - 1 = \lim_{\sigma \to 1} \frac{1}{\frac{1}{\sigma}} - 1 = 0, \quad \lim_{\sigma \to 0} \varphi(\sigma) = 0

$$\varphi(\sigma) = \frac{\sigma - 1 - \sigma \ln \sigma}{\ln \sigma} = \frac{\xi(\sigma)}{\ln \sigma}$$

$\xi(\sigma) = \sigma - 1 - \sigma \ln \sigma$

$\ln \sigma < 0$ when $0 < \sigma < 1$.

We can show that $\xi(\sigma) < 0$ when $0 < \sigma < 1$ and then it follows that $\varphi(\sigma) > 0$

$\xi'(\sigma) = 1 - 0 - \ln \sigma - \sigma\left(\frac{1}{\sigma}\right) = -\ln \sigma > 0$ ✓

$\Rightarrow$ $\xi(\sigma)$ is monotonically increasing;

$\xi(1) = 1 - 1 - 1 \ln 1 = 0 \Rightarrow$ when $0 < \sigma < 1$

$\xi(\sigma) < 0$ and $\xi(1) = 0$.

[4 marks]

$\Rightarrow \varphi(\sigma) > 0 \Rightarrow \lambda > \sigma$.

Since $S' = \frac{IS}{c} > 0 \Rightarrow S$ is strictly monotonically increasing from $S(-\infty) = \sigma < 1$ to $S(\infty) = 1$

$\Rightarrow \quad \sigma < S(z) < 1$

Integrating $I'' + cI' + I(S - \lambda) = 0$ we have

$$\int_{-\infty}^{\infty} IS\,dz - \lambda \int_{-\infty}^{\infty} I\,dz = 0 \Rightarrow \lambda \int_{-\infty}^{\infty} I\,dz = \int_{-\infty}^{\infty} IS\,dz < \int_{-\infty}^{\infty} I\,dz \text{ or}$$

$\lambda < 1 \Rightarrow \sigma < \lambda < 1$ [3 marks]

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EXPECTED SOLUTION TO QUESTION NO: 6 – cont

PROPOSED MARK ALLOCATION:

(vi) Looking for a solution $I \sim e^{mz}$ of the linearised equation when $S \to 1$

$$-m^2 e^{mz} + c m e^{mz} + (1-\lambda) e^{mz} = 0$$

or $\quad m^2 + c m + (1-\lambda) = 0$

$$m_{1,2} = \frac{-c \pm \sqrt{c^2 - 4(1-\lambda)}}{2}$$

Since at $z \to \infty$ $I(\infty) = 0$, there are no oscillatory solutions (otherwise $I(z)$ will have negative values)

$$\Rightarrow \quad c^2 \geq 4(1-\lambda) \quad \Rightarrow \quad \underline{c \geq 2\sqrt{1-\lambda}}$$

[3 marks]

There is always exponentially decaying solution at $z \to \infty$