28/11/2011

Discrete Population Models for a Single Species

$$N_{t+1} = f(N_t) = \underbrace{N_t F(N_t)}_{\sim N_t}$$

Recurrence equations

- linear $\quad N_{t+1} = r N_t$

$$\boxed{N_{t+1} - r N_t = 0}$$

$$N_t = N_0 \cdot r^t$$

$$a N_{t+2} + b N_{t+1} + c N_t = 0$$

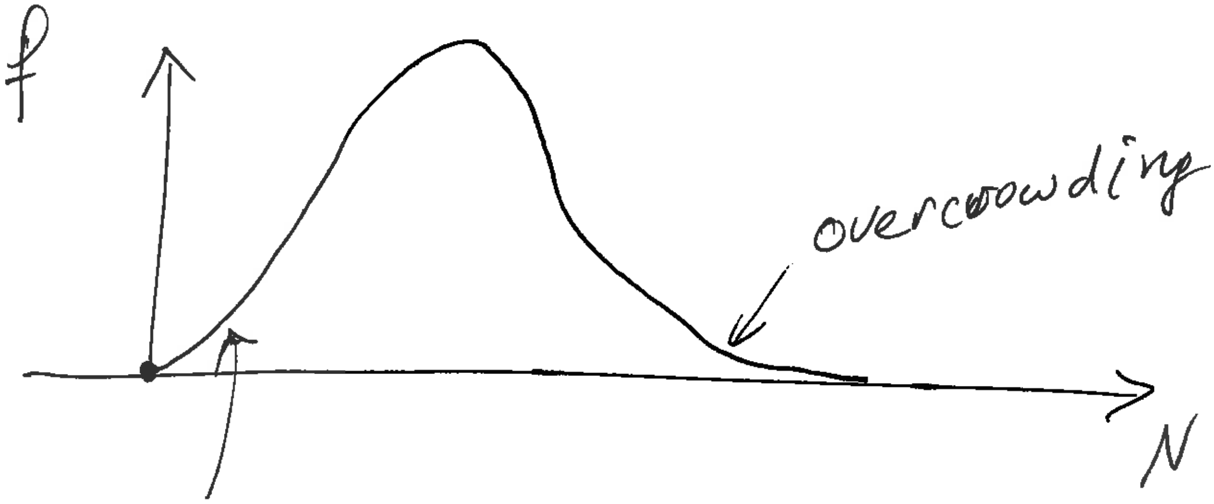
N_0, N_1 given $\quad , \quad N_t = \lambda^t$

$$a \lambda^{t+2} + b \lambda^{t+1} + c \lambda^t = 0 \quad \Big| \cdot \frac{1}{\lambda^t}$$

$\boxed{a\lambda^2 + b\lambda + c = 0}$ characteristic eqn.

$$N_t = A \lambda_1^t + B \lambda_2^t$$

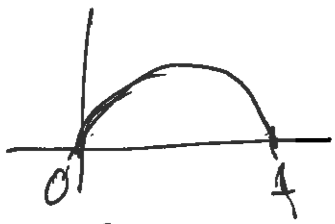
$$N_{t+1} = f(N_t)$$



- $\cancel{u_{t+1} = r' u_t (1 - u_t)}$

$$u_t = \frac{N_t}{K}$$

$r \to \cancel{r'}$ $(1+r)$



- $u_{t+1} = r\, u_t (1 - u_t)$
- $N_{t+1} = N_t\, e^{r\left(1 - \frac{N_t}{K}\right)}$ (Ricker)

$e^{-\frac{N_t}{K}}$ mortality factor

$\boxed{N_t = K}$ constant solution

Equilibrium states

- $u_{t+1} = r u_t (1 - u_t)$

$$u^* = r u^* (1 - u^*)$$

$\boxed{u^* = 0}$

$$1 = r(1 - u^*)$$

$$\frac{1}{r} = 1 - u^* \Rightarrow u^* = 1 - \frac{1}{r} = \frac{r-1}{r}$$

$u^* = \frac{r-1}{r}$ exists if $\boxed{r > 1}$

- $N_{t+1} = N_t \exp\left[r\left(1 - \frac{N_t}{K}\right)\right]$ (Ricker)

N^* - equilibrium, then

$$N^* = N^* \exp\left[r\left(1 - \frac{N^*}{K}\right)\right]$$

$N^* = 0,$

$N^* = K$

$$1 = \exp\left[\underbrace{r\left(1 - \frac{N^*}{K}\right)}_{0}\right]$$

$$f(x) = x\, e^{r\left(1 - \frac{x}{K}\right)}$$

$f(0) = 0$

$f(\infty) = 0$

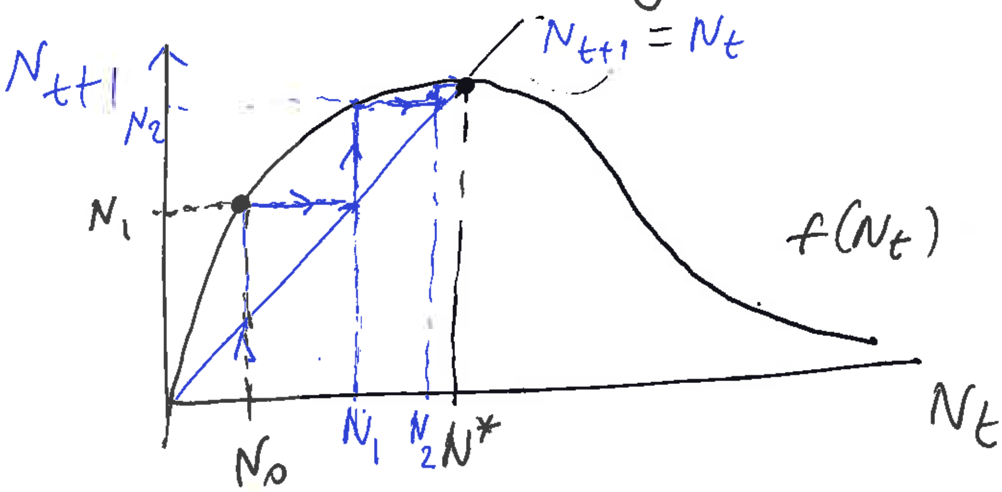
$$f'(x) = e^{r\left(1-\frac{x}{K}\right)} - \frac{xr}{K} e^{r\left(1-\frac{x}{K}\right)} = \left(1 - \frac{xr}{K}\right) e^{r\left(1-\frac{x}{K}\right)}$$

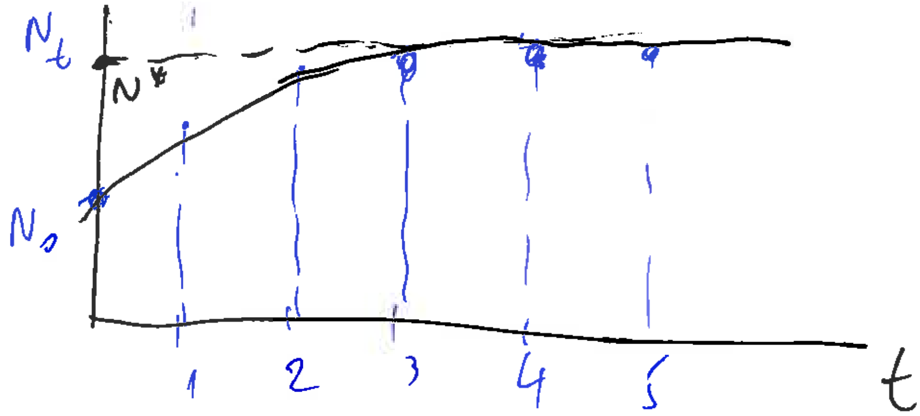
$x_m = \frac{K}{r}$

Graphical Procedure of Solution

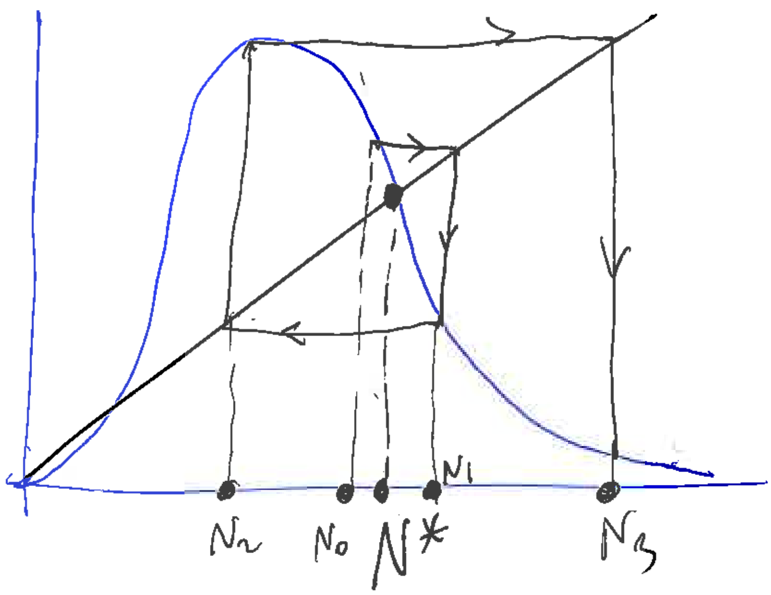
Cobwebbing

$$N_{t+1} = f(N_t)$$

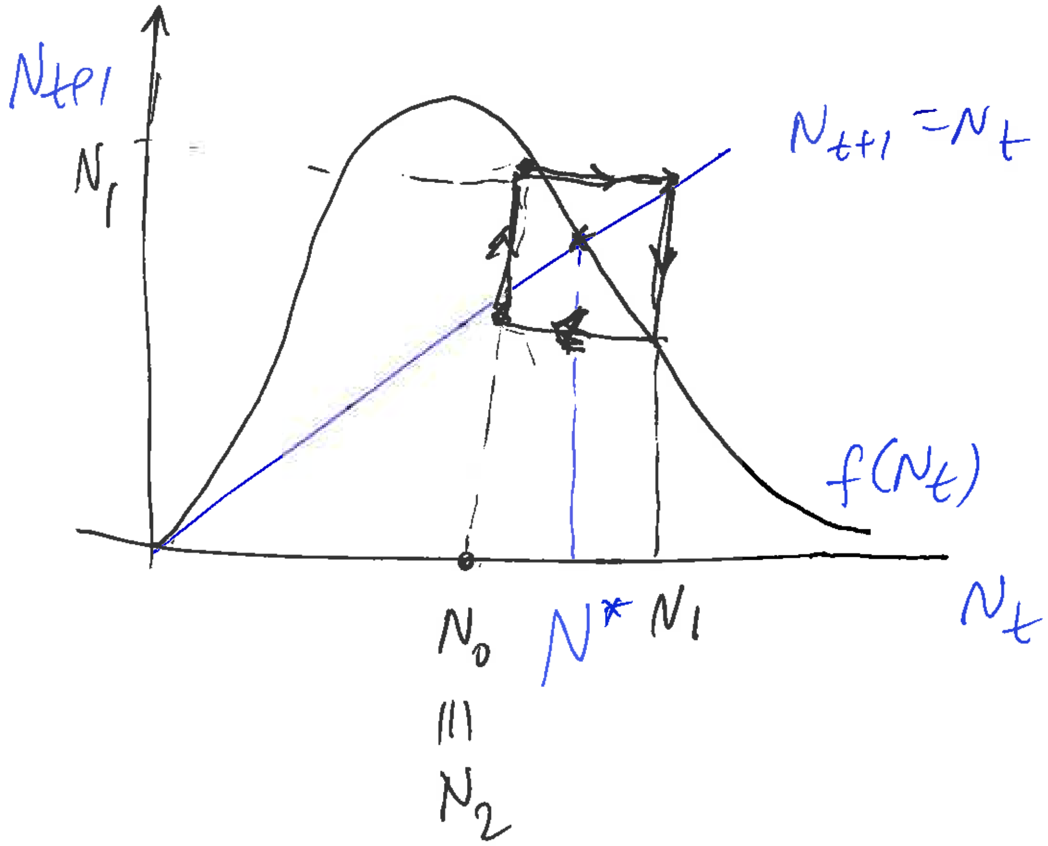




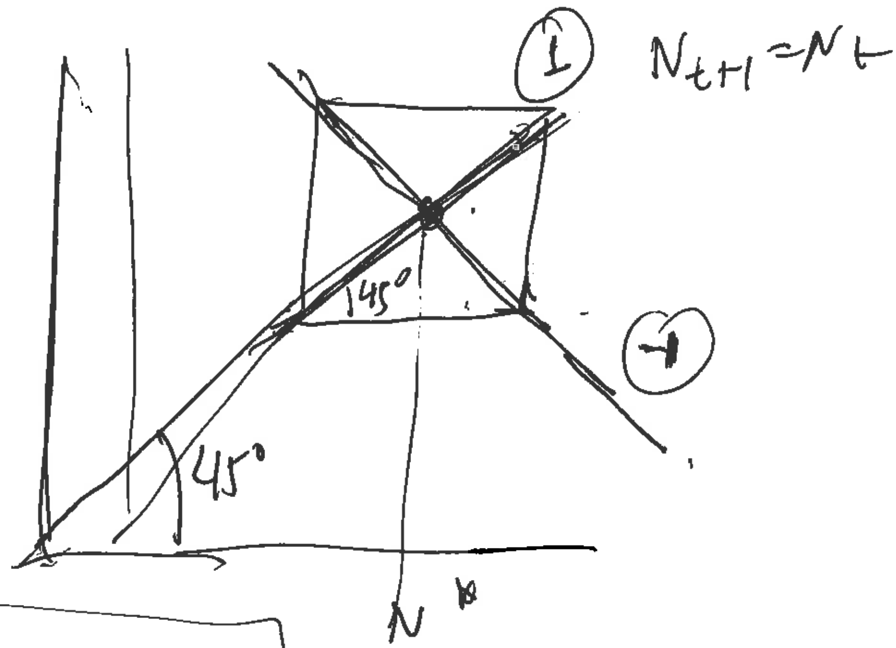
N^* is stable equilibrium



N^* is unstable equilibrium



Example of periodic solution



$$\boxed{\left(\frac{df}{dN}\right)_{N^*} = -1}$$ - periodic

$\left(\frac{df}{dN}\right)_{N^*} < -1$ unstable eq. at N^*

$$\left|\left(\frac{df}{dN}\right)_{N=N^*}\right| < 1 \;, \quad N^* \text{ is stable eq.}$$

$$\left|\left(\frac{df}{dN}\right)_{N=N^*}\right| > 1 \quad N^* \text{ is unstable eq.}$$

$$\left|\left(\frac{df}{dN}\right)_{N=N^*}\right| = 1 \quad \text{periodic solution}$$

(Th) Let $N = N^*$ is a solution of $N = f(N)$ and suppose that $f(N)$ has a continuous derivative in some interval $J \ni N^*$. Then if $|f'(N)| \leq \alpha < 1$ in J

then $\lim_{t \to \infty} N_t = N^*$ for any $N_0 \in J$

where $N_{t+1} = f(N_t)$.

<u>Proof:</u> Mean value theorem, there exists ν between N_t and N^*

$$f(N_t) - f(N^*) = f'(\nu)(N_t - N^*)$$

$$N^* = f(N^*)$$

$$N_{t+1} - N^* = f'(\nu)(N_t - N^*)$$

$$|N_{t+1} - N^*| = |f'(\nu)|\,|N_t - N^*| \leq \alpha\,|N_t - N^*|$$

$$|N_{t+1} - N^*| \leq \alpha |N_t - N^*| \leq \alpha^2 |N_{t-1} - N^*| \leq \dots$$

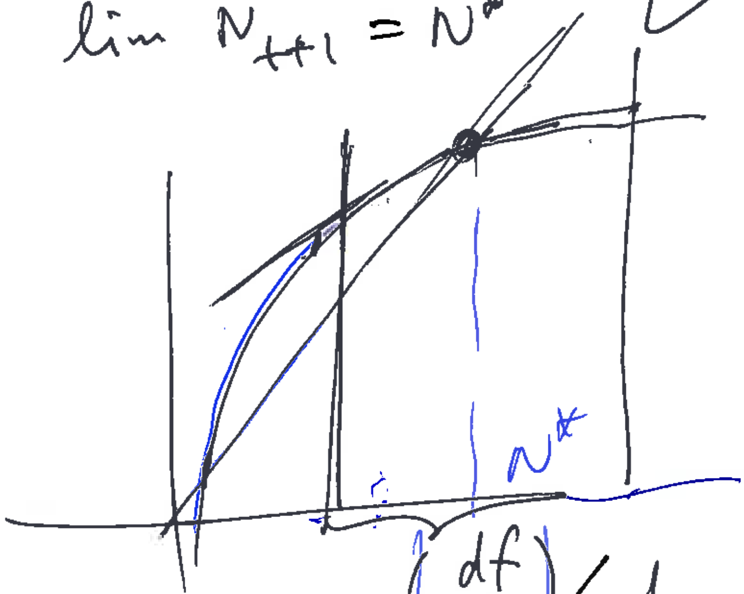
$$\dots \leq \alpha^{t+1} |N_0 - N^*|$$

$$|N_{t+1} - N^*| \leq \alpha^{t+1} |N_0 - N^*|$$

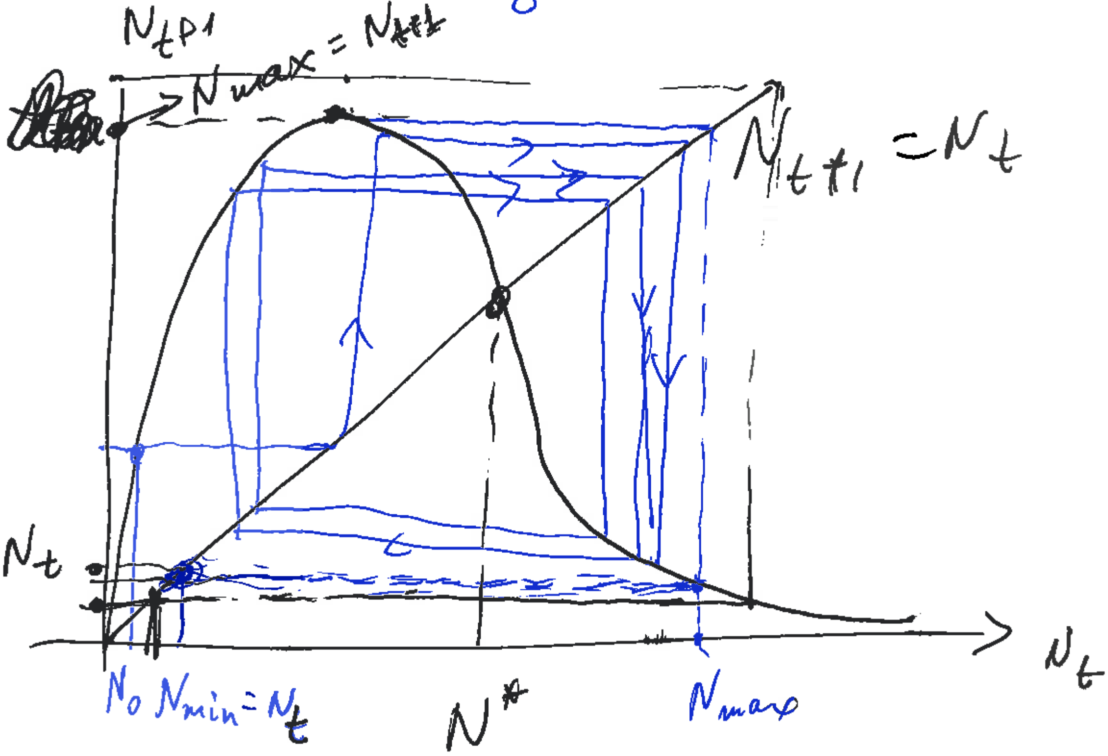
$$\downarrow$$

$$0.$$

$$\lim N_{t+1} = N^* \quad \checkmark$$



$$\left|\left(\frac{df}{dN}\right)\right|_{N=N^*} < 1$$



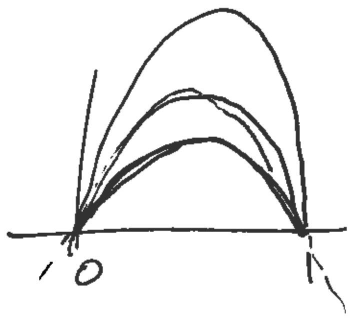
$$\boxed{N_{max} = f(N_{min})}$$

$$N_{min} \leq N_t \leq N_{max}$$

$$u_{t+1} = r u_t (1-u_t) \qquad u_t = \frac{N_t}{K}$$

$$u^* = 0 \qquad u^* = \frac{r-1}{r}$$

$$\underline{f(u) = r u(1-u)}$$

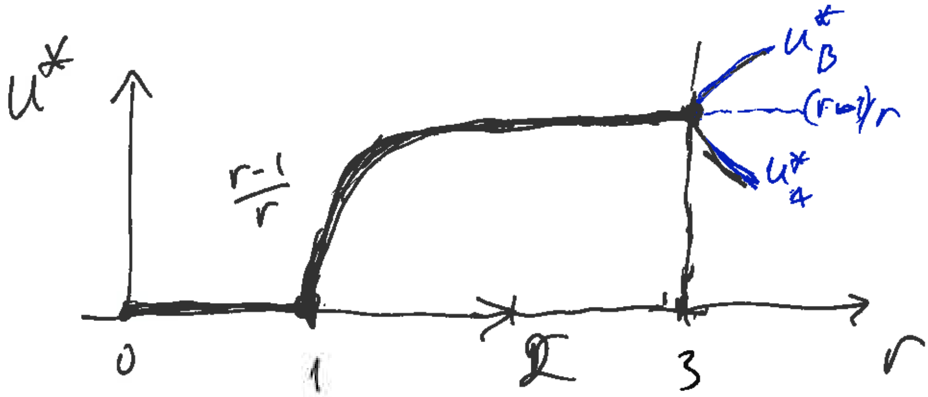


$$f'(u) = r - 2ru$$

'Eigenvalues' $\quad u^* = 0 \Rightarrow \lambda = f'(0) = r$

$$u^* = \frac{r-1}{r} \qquad \lambda = f'\left(\frac{r-1}{r}\right) = r - 2r\frac{r-1}{r}$$

$$= r - 2r + 2 = \underline{\underline{2-r}}$$



$\left.\begin{matrix} r=1 \\ r=3 \end{matrix}\right\}$ bifurcation values $\quad \left|\left(\frac{\partial f}{\partial u}\right)_{u=u^*}\right| = 1$

$$u_{t+2} = r u_{t+1}(1-u_{t+1}) =$$

$$= r[r u_t(1-u_t)][1 - r u_t(1-u_t)]$$

$$u_{t+2} = f(u_t)$$

$$\cancel{u^*}' = r^2 \cancel{u^*}(1-u^*)[1 - r u^* + r(u^*)^2]$$

$u^* = \frac{r-1}{r}$

$u^* = 0$ unstable

$$1 = r^2 (1-u)\left[1 - ru + ru^2\right]$$

Check:

$$0 = 1 - r^2(1-u)\left[1 - ru + ru^2\right] \equiv (ru - r + 1)\left[r^2u^2 - r(r+1)u + r + 1\right]$$

$$\cancel{1} - r^2(\cancel{1} - \cancel{ru} + \cancel{ru^2} - \cancel{u} + ru^2 - \cancel{ru^3}) \overset{?}{\equiv}$$

$$= \cancel{r^3u^3} - r^2(r+1)u^2 + \cancel{r(r+1)u}$$
$$\cancel{-r^3u^2} + r^2(r+1)u - \cancel{r^2 - r}$$
$$+ \cancel{r^2u^2} - \cancel{r(r+1)u} + \cancel{r} + \cancel{1}$$

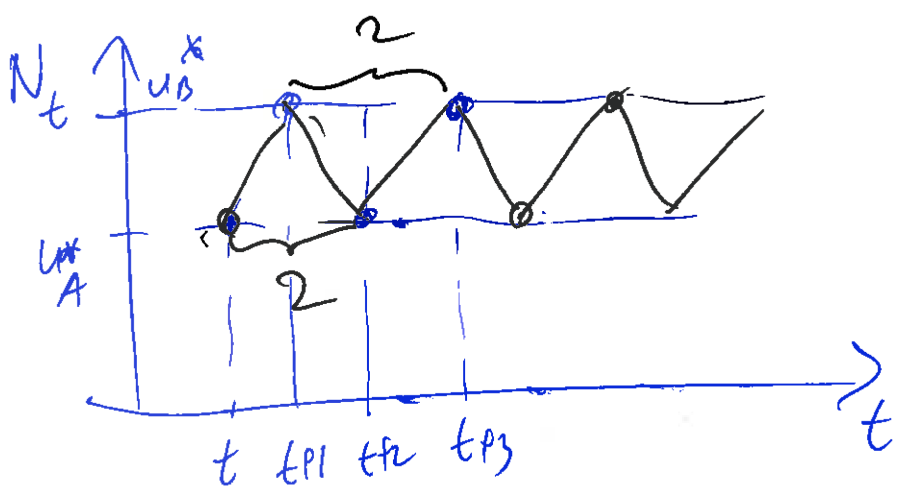
$$\boxed{r^2u^2 - r(r+1)u + r + 1 = 0}$$

$$u^* = \frac{r(r+1) \pm \sqrt{r^2(r+1)^2 - 4r^2(r+1)}}{2r^2}$$

$$u^* = \frac{r+1 \pm \sqrt{r^2 + 2r + 1 - 4r - 4}}{2r} = \frac{r+1 \pm \sqrt{r^2 - 2r - 3}}{2r}$$

$$u^* = \frac{r+1 \pm \sqrt{(r+1)(r-3)}}{2r} > 0 \quad \text{if} \quad \underline{r > 3}$$

$$r + 1 > \sqrt{(r+1)(r-3)}$$
$$(r+1)^{\cancel{2}} > \cancel{(r+1)}(r-3) > 0$$
$$r + 1 > r - 3$$
$$1 > -3$$



periodic solution of period 2

$\cancel{N_{t+4} = f^{(4)}(N_t)}$

$u_{t+4} = f^{(4)}(u_t)$ – 4-periodic

$u_{t+8} = f^{(8)}(u_t)$ – 8-periodic

$u_{t+2^n} = f^{2^n}(u_t)$ 2^n-periodic

$u_{t+3} = f^{(3)}(u_t)$ $r_3 \approx 3.828$

For $r > r_3$ – chaotic solutions

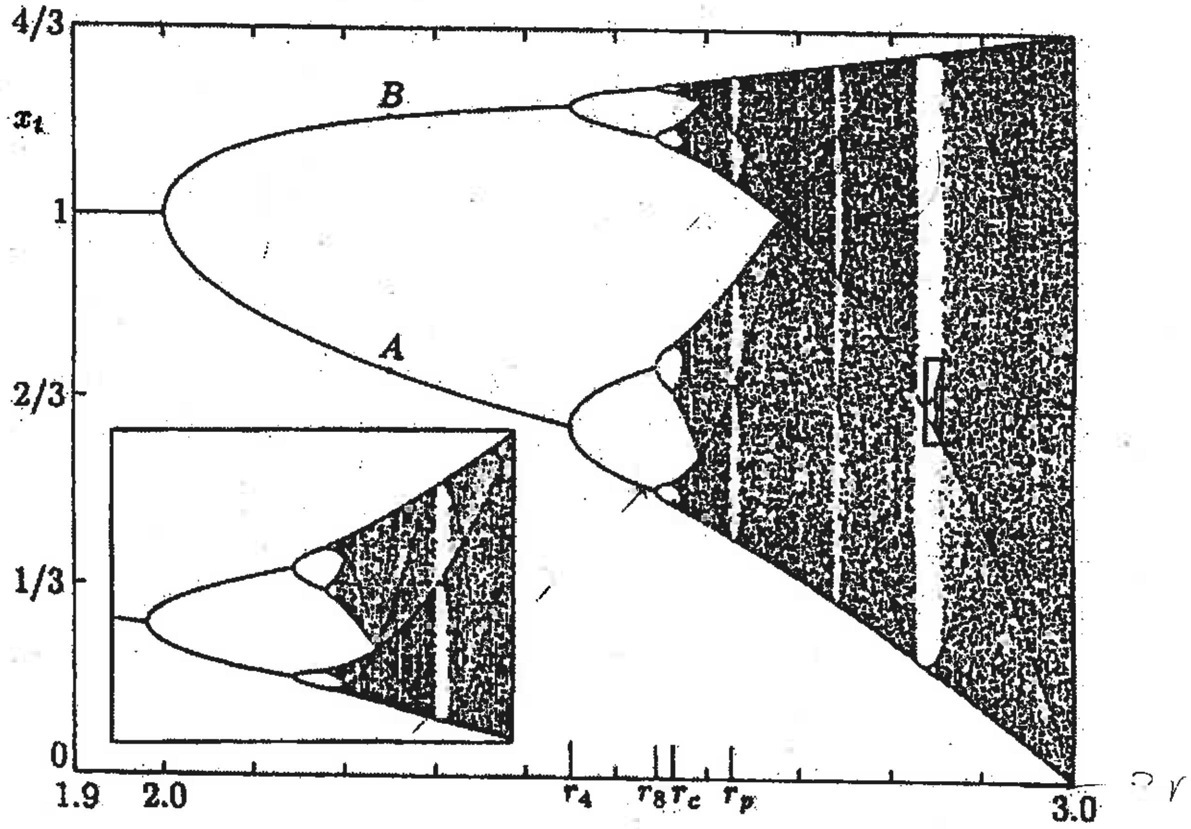


Figure 2.11. Long time asymptotic iterates for the discrete equation $x_{t+1} = x_t + rx_t(1 - x_t)$ for $1.9 < r < 3$. By a suitable rescaling, ($u_t = [r/(r+1)]x_t$, 'r' $= 1 + r$), this can be written in the form (2.11). These are typical of discrete models which exhibit period doubling and eventually chaos and the subsequent path through chaos. Another example is that used in Figure 2.10; see text for a detailed explanation. The enlargement of the small window (with a greater magnification in the r-direction than in the x_t direction) shows the fractal nature of the bifurcation sequences. (Reproduced with permission from Peitgen and Richter 1986; some labelling has been added)