The Parking Problem

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Outline

The Parking Problem

Consider an interval $(0, x)$ upon which we place a segment of unit length at random. We continue by placing a second segment of unit length randomly upon the original interval, discarding the segment if it overlaps with the original one.

We continue in this fashion until we can no longer add unit segments without overlap. At each step the next position within the interval is chosen from a uniform distribution of the remaining locations within the interval.

We are interested in both the expected value of the number of unit segments contained within the interval $(0, x)$, denoted $M(x)$, and the expected filling density of unit segments within the interval, denoted $M(x)/x$.

Applications

- Random Sequential Adsorption
- Genome Sequencing

Approaches

- Wiener's Approach
- Rényi's Approach
- Kinetic Approach
- Generalizations Of The Kinetic Approach

Wiener's Approach

- Elementary Treatment
- Provides Bounds On The Limit Rather Than The Limit Itself
- Uses "Trial-And-Error Method"
- Includes An Error
- Unsatisfying

Rényi's Approach

- Analytic Treatment
- Computes The Limit
- Complicated And Convoluted
- Only Slightly More Satisfying

Rényi's Approach

$$\lim_{x \to \infty} \frac{M(x)}{x} = C_R$$

with

$$C_R = \int_0^{\infty} \exp\left(-2 \int_0^t \frac{1 - e^{-u}}{u} du\right) dt$$

Kinetic Approach

- Time Evolution Of The Distribution Of Gaps
- Models The Process
- Elegant Treatment
- Coverage Function Tends To The Limit
- Very Satisfying

Kinetic Approach

Let X_t be a random variable that represents the length of a gap at time t, with $N(x,t)$ the number of gaps less than or equal to x, and $N(t)$ the total number of gaps. We define the gap length distribution $F(x,t)$ as follows:

$$P(X_t \leq x) = F(x,t) = \frac{N(x,t)}{N(t)}$$

here $P(A)$ represents the probability of event A occurring.

Kinetic Approach

The gap length density function $f(x, t)$ is then:

$$f(x,t) = \frac{\partial F}{\partial x} = \frac{1}{N(t)} \frac{\partial N}{\partial x}(x,t)$$

therefore the probability that a gap has length between a and b is:

$$\int_a^b f(x,t)dx = \frac{1}{N(t)} \int_a^b \frac{\partial N}{\partial x}(x,t)dx$$

which is simply the number of gaps with length between a and b divided by the total number of gaps at time t.

Kinetic Approach

We define the gap density function, $P(x,t)$, as:

$$P(x,t) = \frac{1}{L}\frac{\partial N}{\partial x}(x,t)$$

Kinetic Approach

After some manipulation, the coverage function was shown to be:

$$\theta(t) = \int_0^\infty P(x,t)dx$$

Kinetic Approach

We form a rate equation using our expressions for the gap density function:

$$\frac{\partial P(x,t)}{\partial t} = \begin{cases} 2\int_{x+1}^{\infty} P(y,t)dy & \text{for } x < 1 \\ -(x-1)P(x,t) + 2\int_{x+1}^{\infty} P(y,t)dy & \text{for } x \geq 1 \end{cases}$$

which is composed of creation and destruction terms

Kinetic Approach

The creation term may be be understood by considering a gap of length $y \geq x + 1$, there are exactly two ways to park within this gap that will result in a gap of length x.
The destruction term, which only appears in the case where $x \geq 1$, deals with the case where an existing gap of length x is destroyed by parking a car within it - in this case the remaining length becomes $x - 1$, and this destruction can be accomplished in $P(x, t)$ ways.

Kinetic Approach

$$\lim_{\tau \to \infty} \theta(\tau) = C_R$$

where

$$\theta(\tau) = \int_0^\tau \exp\left(-2\int_0^t \frac{1 - e^{-u}}{u} du\right) dt$$

Kinetic Approach

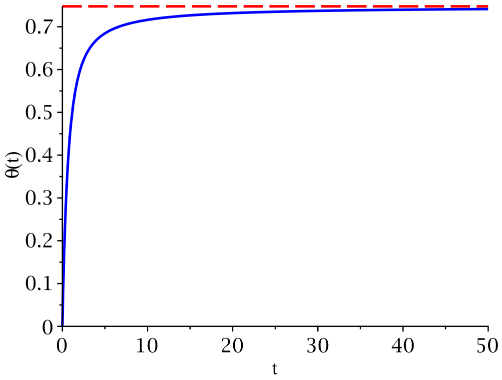

Figure: The coverage function

Kinetic Approach

It should be noted that the approach of reducing the rate equation to an ODE breaks down for finite L.

Generalizations

- Parking With Overlap
- Reversible Parking Problem
- Both Build On Kinetic Approach
- Again Very Satisfying

Generalizations

The coverage by cars for parking with overlap is:

$$\Theta_\phi(t) = \begin{cases} (1-2\phi)\int_0^t F_\phi(\tau)d\tau + \int_0^t F_\phi(\tau)\frac{2}{\tau}(1-e^{-\phi\tau})d\tau & \text{for } \phi < \frac{1}{2} \\ 1 - F_\phi(t)e^{(1-2\phi)t} & \text{for } \phi \geq \frac{1}{2} \end{cases}$$

Generalizations

The the equilibrium coverage for the reversible parking problem is:

$$\begin{aligned} \theta_{eq} &= \frac{\alpha}{1+\alpha} \\ &\approx 1 - \frac{1}{\ln(k_+/k_-)} \end{aligned}$$

Monte Carlo Approach

- define the range of values to be drawn from
- draw values from this range using a distribution
- perform a computation on the results
- record the results of the computation
- repeat as appropriate

Monte Carlo Approach

A Method

- For Evaluating Results
- For Confirming Theories
- Best Applied To Problems With A Probabilistic Element

The Parking Problem Simulation - Discrete Version

- a length L made up of discrete sites
- attempt to park cars that occupy two adjacent sites
- the process is successful if a car tries to park at two adjacent empty sites
- otherwise unsuccessful
- the process continues until there are no more free adjacent empty sites

The Parking Problem Simulation - Discrete Version

The coverage for the discrete parking problem is $\theta_d(t)$, and the jamming coverage is:

$$\lim_{t \to \infty} \theta_d(t) = 1 - e^{-2}$$

which is calculated to be 0.864664 to six decimal places.

The Parking Problem Simulation - Discrete Version

```
    Parking Problem - Discrete Version: results

                                    L:   100000
                           iterations:    10000

                         distribution:
                                 mean: 0.864673
                   standard deviation: 0.000848
```

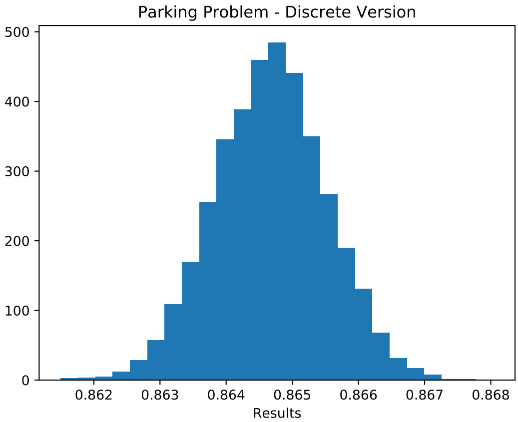

Figure: Histogram of the parking problem simulation - discrete case

The Parking Problem Simulation - Discrete Version

The results of the simulation for the discrete parking problem are consistent with the theory.

The Parking Problem Simulation

```
          Parking Problem: results

                       L:   100000
              iterations:    10000

            distribution:
                    mean: 0.747602
      standard deviation: 0.000617
```

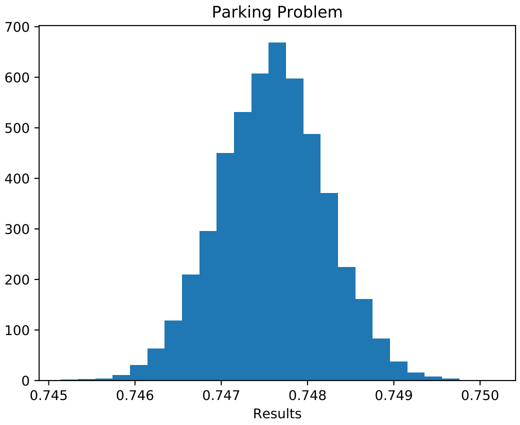

Figure: Histogram of the parking problem simulation

The Parking Problem Simulation - Overlap Version

The coverage by cars for parking with overlap is:

$$\Theta_\phi(t) = \begin{cases} (1-2\phi)\int_0^t F_\phi(\tau)d\tau + \int_0^t F_\phi(\tau)\frac{2}{\tau}(1-e^{-\phi\tau})d\tau & \text{for } \phi < \frac{1}{2} \\ 1 - F_\phi(t)e^{(1-2\phi)t} & \text{for } \phi \geq \frac{1}{2} \end{cases}$$

The jamming coverage for a selection of values of ϕ as $t \to \infty$ is shown below:

ϕ	$\lim_{t\to\infty} \Theta_\phi(t)$
0.1	0.816909
0.2	0.880028
0.3	0.936238
0.4	0.980342
0.5	1

Table: $\lim_{t\to\infty} \Theta_\phi(t)$

The coverage gets closer to full coverage (i.e. 1) as $\phi \to 0.5$.

The Parking Problem Simulation - Overlap Version

```
    Parking Problem - Overlap Version: results

                                  L:   100000
                            overlap:      0.1
                         iterations:    10000

                       distribution:
                               mean: 0.816894
                 standard deviation: 0.000584
```

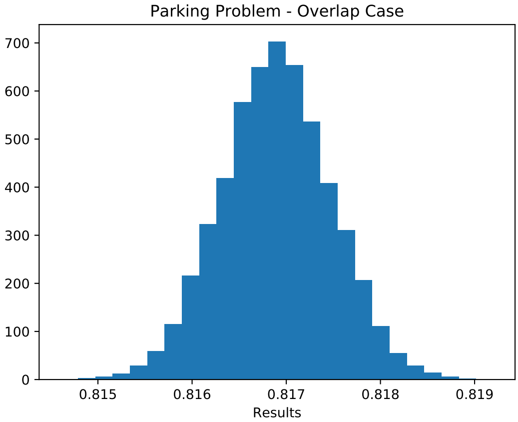

Figure: Histogram of the parking with overlap simulation - $\phi = 0.1$

The Parking Problem Simulation - Overlap Version

```
    Parking Problem - Overlap Version: results

                                L:   100000
                          overlap:      0.2
                       iterations:    10000

                     distribution:
                             mean: 0.880027
               standard deviation: 0.000477
```

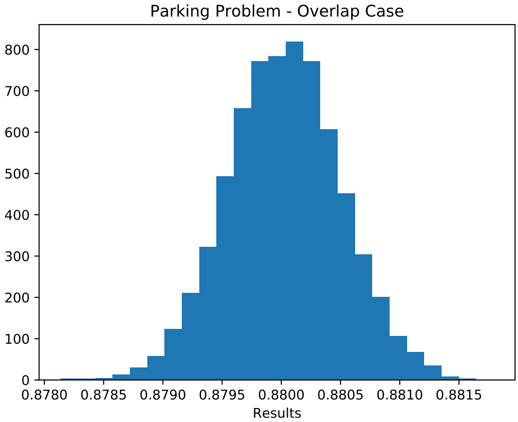

Figure: Histogram of the parking with overlap simulation - $\phi = 0.2$

The Parking Problem Simulation - Overlap Version

```
    Parking Problem - Overlap Version: results

                              L:    100000
                        overlap:       0.3
                     iterations:     10000

                   distribution:
                           mean:  0.936235
             standard deviation:  0.000327
```

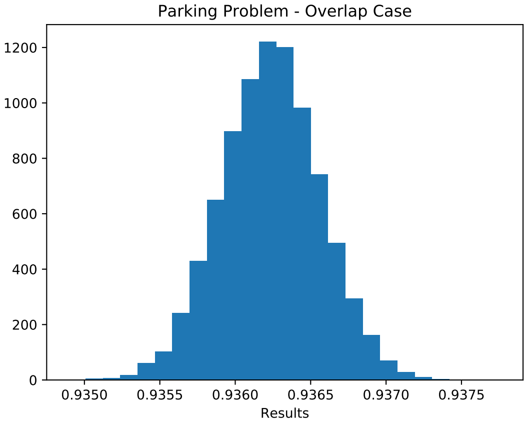

Figure: Histogram of the parking with overlap simulation - $\phi = 0.3$

The Parking Problem Simulation - Overlap Version

```
    Parking Problem - Overlap Version: results

                                   L:   100000
                             overlap:      0.4
                          iterations:    10000

                       distribution:
                                mean: 0.980344
                  standard deviation: 0.000144
```

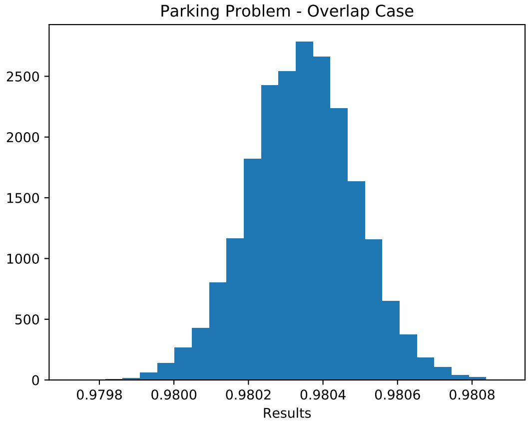

Figure: Histogram of the parking with overlap simulation - $\phi = 0.4$

The Parking Problem Simulation - Overlap Version

```
    Parking Problem - Overlap Version: results

                                  L:    100000
                            overlap:       0.5
                         iterations:     10000

                       distribution:
                               mean: 1.000000
                 standard deviation: 0.000000
```

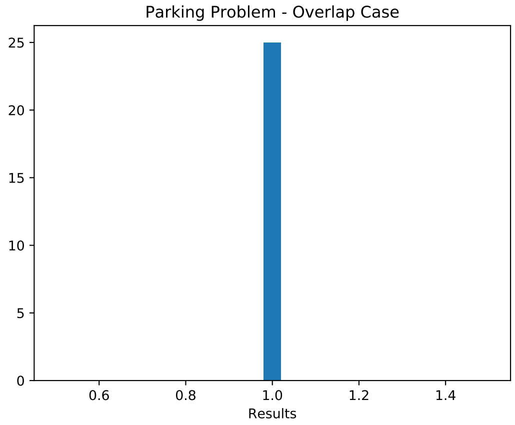

Figure: Histogram of the parking with overlap simulation - $\phi = 0.5$

Simulations - Trade-offs

- high accuracy of outcome - lots of iterations, set L to be as large as possible
- confirmation of theory - fewer iterations, shorter L will suffice
- calculation of a value - recursive approach

Remarks

Is there some underlying conservation law other than the obvious:

$$\int_0^\infty xP(x,t)dx + \int_0^\infty P(x,t)dx = 1$$

i.e that the normalized sum of the car lengths and gap lengths equals 1.

Remarks

Specifically the reversible problem, and it's condition for equilibrium:

$$\frac{\partial P(x,t)}{\partial t} = 0$$

might lead one to believe that there are underlying conservation laws and hidden symmetries at play.

Remarks

When it comes to modelling a problem that does not lend itself easily to empirical verification, such as the parking problem, a simulation can tell you if your theory makes sense very quickly.

Remarks

A simulation can help provide statistical properties relating to the associated constants of a process, which in turn can help establish whether a process is being inhibited by some unknown factor.

Future Research

C_R feels like something universal, akin to e or ϕ. In particular π.

Future Research

Higher dimensional problems, in two and three dimensions.

Investigating what (if any) relationship exists between two and three dimensional packing problems and C_R.

Future Research

Competitive RSA of a binary mixture - where cars of two different lengths compete for parking spaces. A more complicated problem mathematically, and more difficult to simulate.

Finally

Thank you for your time and patience.