W9209/9210/102C

DUBLIN INSTITUTE OF TECHNOLOGY

School of Mathematical Sciences

DT9209 MSc Applied Mathematics
DT9210 MSc Applied Mathematics

WINTER EXAMINATIONS 2016/2017

MATH 9951: METHODS FOR APPLIED MATHEMATICS

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09.30 – 11.30 am, Friday, 6 January 2017

Duration: 2 hours

Answer three questions

All questions carry equal marks

Approved calculators may be used

Mathematical tables are provided

New Cambridge Statistical Tables are NOT permitted

1. a) Assuming $u(x, y)$ follows the form $f(\lambda x+y)$, where f is a general twice-differentiable function and λ is an unknown parameter, find the general solution to

$$4u_{xx} - 4u_{xy} + 3u_{yy} = 0.$$

(9)

b) Assume a string in one dimension x with mass per unit length ρ, which is long, light, and undergoes small deflections $u(x)$ as illustrated in Figure 1b. Assume that a damping force is present proportional to the velocity of the string's displacement with constant of proportionality a. Assume that a restoring force proportional to the displacement is present with constant of proportionality b.

By considering the tension $\mathbf{T}$, damping and restoring forces on a small section of the string of length Δx, show that u obeys the one-dimensional telegraph equation

$$u_{tt}(x,t) = \frac{T}{\rho}u_{xx}(x,t) - au_t(x,t) - bu(x,t).$$

(9)

c) Find the integral surface and general solution for $u(x,y)$ where

$$(y-u)u_x + (u-x)u_y = x - y$$

with the Cauchy data

$$u = 0 \qquad \text{on } xy = 1.$$

(15)

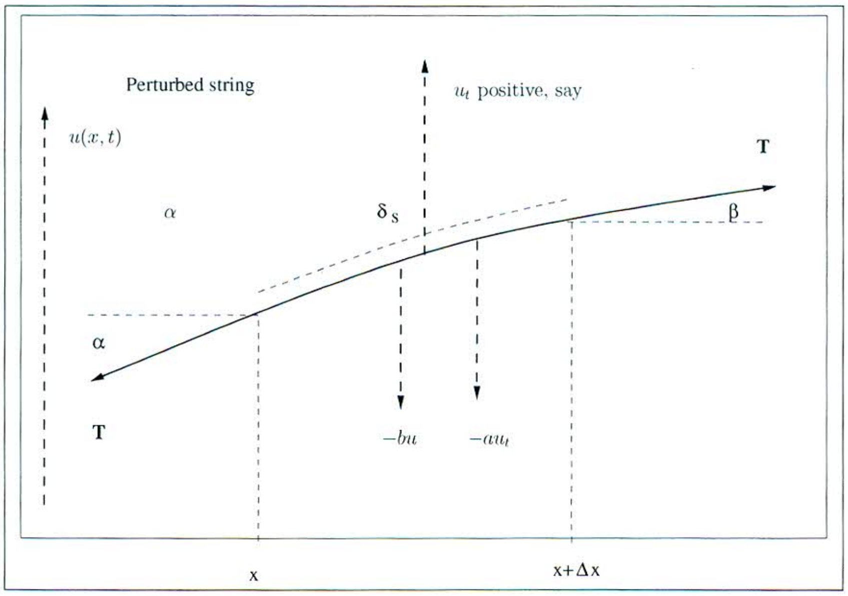

Figure 1: Figure for Question 1b

[33]

2. **a)** Gauss' Divergence Theorem states that given a vector field $\mathbf{a}$, and a volume V bounded by a surface S

$$\int_V \nabla \cdot \mathbf{a} dV = \int_S \mathbf{a} \cdot d\mathbf{A}$$

where $d\mathbf{A}$ is in the direction of the outward normal.
State defining properties of $\delta^3(\mathbf{r} - \mathbf{r}')$, the delta function in three dimensions. Hence, considering a spherical volume V, use Gauss' Divergence Theorem to show that

$$\nabla_r^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} = -4\pi\delta^3(\mathbf{r} - \mathbf{r}').$$

(10)

b) Prove

$$\nabla^2 \mathbf{u} = \nabla\nabla \cdot \mathbf{u} - \nabla \times (\nabla \times \mathbf{u}).$$

(6)

c) State and prove Helmholtz's Decomposition Theorem.
Hint: Assume a vector field of the form

$$\mathbf{a} = -\frac{1}{4\pi} \int \frac{\mathbf{u}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dV'.$$

(10)

d) The Navier equation of motion is given by

$$\rho \mathbf{u}_{tt} = (\lambda + \mu)\nabla\nabla \cdot \mathbf{u} + \mu \nabla^2 \mathbf{u}$$

where λ, μ and ρ may be assumed constant.
Decompose $\mathbf{u}$ into a divergence-free component $\mathbf{u}_T$, and an irrotational component $\mathbf{u}_L$, and hence show

$$\mathbf{u}_{L,tt} = c_T^2 \nabla^2 \mathbf{u}_L, \quad \mathbf{u}_{T,tt} = c_L^2 \nabla^2 \mathbf{u}_T$$

for $c_T^2 = \mu/\rho$, $c_L^2 = (\lambda + 2\mu)/\rho$.

(7)

[33]

3. Assume $u(x,t)$ satisfies the heat equation

$$u_t = \kappa u_{xx}, \quad 0 < x < l, \quad t > 0,$$

under initial conditions

$$u(x,0) = f(x),$$

and boundary conditions

$$u(0,t) = 0, \quad u(l,t) = 0.$$

a) By separation of variables, derive the infinite series solution to the given heat equation.
You may use the identity below to express the coefficients of the series in terms of integrals involving $f(x)$ and $g(x)$:

$$\int_0^l \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)dx = \begin{cases} \frac{l}{2} & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

(13)

b) Prove that $u(x,t)$ is unique. (10)

c) Obtain the solution for the particular case

$$f(x) = x(l - x).$$

(10)

[33]

4. The Laplace transform of a function $u(x,t)$, with respect to t, is given by $\overline{u}(x,s) = \mathcal{L}\{u(x,t)\}$ where

$$\mathcal{L}\{u(x,t)\} = \int_0^\infty e^{-st} u(x,t) dx.$$

a) i) Let f be continuous and f' piecewise continuous, for $0 \le t \le T$ for all $T > 0$. Let f be of exponential order as $t \to \infty$.
Prove the transform of $f'(t)$ exists and is given by

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0).$$

(6)

ii) Extend the above result to obtain an expression for $\mathcal{L}\{f^{(n)}(t)\}$ (where $f^{(n)}$ denotes the n-th derivative of f). (6)

iii) Given $\overline{f}(s) = \mathcal{L}\{f(t)\}$, show that

$$\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = \frac{\overline{f}(s)}{s}.$$

(4)

iv) Derive an expression for

$$\mathcal{L}\left\{e^{-t^2}\right\}$$

in terms of the complementary error function erfc where

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-z^2} dz$$

.

(4)

b) By using Laplace transformation methods, solve the diffusion equation

$$u_t = \kappa u_{xx}, \quad 0 < x < a, \quad t > 0,$$

with initial and boundary conditions

$$\begin{aligned} u(x,0) &= 0, \quad 0 < x < a, \\ u(a,t) &= U, \quad (U \text{constant}) \\ u_x(0,t) &= 0, \end{aligned}$$

where U and κ are constants.
You may use the result:

$$\mathcal{L}\left\{\operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right)\right\} = \frac{1}{s} e^{-a\sqrt{s}}, \quad a \ge 0.$$

(13)

[33]