J. Appl. Math. & Computing Vol. **14**(2004), No. 1 - 2, pp. 97 - 113

# AN APPROACH FOR SOLVING OF A MOVING BOUNDARY PROBLEM

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Abstract. In this paper we shall study moving boundary problems, and we introduce an approach for solving a wide range of them by using calculus of variations and optimization. First, we transform the problem equivalently into an optimal control problem by defining an objective function and artificial control functions. By using measure theory, the new problem is modified into one consisting of the minimization of a linear functional over a set of Radon measures; then we obtain an optimal measure which is then approximated by a finite combination of atomic measures and the problem converted to an infinite-dimensional linear programming. We approximate the infinite linear programming to a finite-dimensional linear programming. Then by using the solution of the latter problem we obtain an approximate solution for moving boundary function on specific time. Furthermore, we show the path of moving boundary from initial state to final state.

AMS Mathematics Subject Classification : 34B15, 35K55, 35R35, 49J20, 80A22.
*Key words and phrases* : Nonlinear boundary value problems, nonlinear PDE, free and moving boundary problems, measure theory, optimal control, Stefan problems.

## 1. Introduction

Moving boundary problems in which the boundary of the domain is not known are very important cases in many practical problems. Moving boundaries are associated with time dependent problems and the position of the boundary has to be determined as a function of time and space. Moving boundary problems are often called Stefan problems, with reference to the work of J. Stefan who was interested in the melting of the polar ice cap [6]. A wide ranging application to problems in physical and biological sciences, engineering, metallurgy,

Received July 16, 2002. Revised July 23, 2003.
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soil mechanics, decision and control theory, have been written by many authors who were mentioned by Crank [6], for example, Boley, Furzeland, Fox, Solomon, Magenes, Hoffman, Rubinstein and Fasano.
A mathematical model for the growth of tumors have studied by A. Friedman, and F. Reitich [13] and also Byrne and Chaplain [4].
In this paper, we are going to introduce a new approach for solving the free and moving boundary problems, by using calculus of variations and measure theory. Many authors have used measure theory to solve optimal control problems, we just mention Rubio [21]-[23], Wilson et al. [26], V. Kamyad et al. [17]-[19], Farahi et al. [12].
Measure theory has recently been used to solve optimal shape design problems as well [10]-[11], and nonlinear ODE's and PDE's, and infinite-horizon optimal control problems [7]-[9], and Stokes problem [14]-[15], and also to solve nonlinear programming problems [1].
Here we apply this theory to solve a wide range of free and moving boundary problems, by using artificial controls and measure theory. First we define artificial controls and then proceed to transform the problem into an optimal control problem, then we convert the problem to another in which the goal is the minimization of a linear form over a subset of Radon measures. By using the solution of the finite linear programming we obtain the suboptimal measures. Finally, the approximate optimal control will be constructed, and then we obtain an approximate solution for moving boundary function on specific time.

## 2. Mathematical formulation of Stefan problems

The following examples describe the Stefan problems as free and moving boundary problems. A simple mathematical model of a Stefan problem is the freezing of a liquid, the other is the melting of a semi-infinite sheet of ice as follows :

$$\begin{gathered}
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < s(t), \quad t > 0, \\
u = 1, \quad x = 0, \quad t > 0, \\
u = 0, \ x > 0, \ t = 0, \quad s(0) = 0, \\
u = 0, \ x = s(t), \ t > 0, \\
-\frac{\partial u}{\partial x} = \lambda \frac{ds}{dt}, \ x = s(t), \ t > 0,
\end{gathered}$$

where $\lambda$ is a dimension-less ' latent heat' and $1/\lambda$ is the Stefan number and $s(t)$ denotes the thickness of the water phase at time $t$.
The analytical properties of more general Stefan problems, which include super-heated solids and super-cooled liquids, have been examined by Sherman[6].
The following equations describe the melting of a super-heated solid:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \ 0 < x < s(t), \ t > 0$$

$$\frac{\partial u}{\partial x} = 0,\ x = 0,\ t > 0,$$
$$u = \phi(x) \geq 0,\ \phi(a) = 0,,\ 0 < x < s(0) = a,\ t = 0,$$
$$u = 0,\ \frac{ds}{dt} = \frac{\partial u}{\partial x},\ x = s(t),\ t > 0.$$

The solid initially occupies the region $0 < x < a$, and melting starts at $x = a$. The boundary condition stipulates that $ds/dt$ is negative which corresponds to physical situation that the melting front proceeds in the direction of $x$ decrasing. The same system of equations can describe the freezing of a liquid which has a negative latent heat. Mathematically the problem of freezing with negative latent heat is the same as the melting of a superheated solid. The solidification or melting of a supercooled liquid or a superheated solid respectively, with their possible reformulation as Stefan problems with negative latent heat, are of wider interest because their mathematical formulation and solution may be relevant to other physical situations [6].

A mathematical model for the growth of tumors is in the form of a free-boundary problem, which its boundary is an unknown function. A. Friedman, and F. Reitich [13], obtained a necessary condition for the existence of a unique stationary solution of a model proposed by Byrne and Chaplain [4], for the growth of a tumor. Byrne and Chaplain represent the tumors evolution in the form of a free-boundary problem. They shall assume the tumor to be spherically symmetric and to occupy a region $\{r < s(t), r = |x|, x = (x_1, x_2, x_3)\}$ at each time $t$, the boundary of the tumor is given by $r = s(t)$, an unknown funtion of $t$. The following equations describe the growing of tumors:

$$c\frac{\partial u}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) - \lambda u,\ r < s(t),\ t > 0,$$
$$\frac{1}{3}s^2(t)\frac{ds(t)}{dt} = \int_0^{s(t)} (u - \hat{u})r^2 dr,$$
$$u = \bar{u} \ \text{ on } \ r = s(t),$$
$$\bar{u} > \hat{u} > \lambda,$$
$$u(r,0) = u_0(r) \ \text{ if } \ 0 < r < s(0),\ \frac{\partial u_0}{\partial r}(0,0) = 0,$$

where $s(0)$ is given, and $u(r,t)$ denote the nutrient concentration. Here the external nutrient concentration is assumed to be a constant $\bar{u}$, $\hat{u}$ is positive constant, the term $\lambda u$ is the nutrient consumption rate, and $c = T_{diffusion}/T_{growth}$ is the ratio of the nutrient diffusion time scale to the tumor growth (e.g. tumor doubling) time scale.

## 3. Definition of the problem

We shall study moving boundary problem, and we try to solve the Stefan problem by using calculus of variations and optimization.
let us consider the moving boundary problem of the form:

$$V_t(x,t) = V_{xx}(x,t)\ ,\ 0 < x < s(t),\ t > 0, \tag{1}$$

$$V_x(0,t) = g(t)\ ,\ t > 0, \tag{2}$$

$$-\frac{\partial V}{\partial x} = \frac{dx}{dt},\ V = 0,\ x = s(t), \tag{3}$$

$$V = 0,\ x > s(t),\ s(0) = a, \tag{4}$$

where $V(x,t)$ is the temperature distribution in a point $x$ at time $t$, and $s(t)$ is moving boundary. Equation (3) is known as ' Stefan condition' and it expresses the heat balance on the interface [20].

Now we try to transform the Stefan problem (1)-(4) into an optimal control form which satisfies the conditions as follows:

(i) Transform the moving boundary problem into an integral form of calculus of variations,
(ii) Define an objective function and artificial control functions,
(iii) Transform the problem equivalently into an optimal control problem,
(iv) Modify the optimal control problem into a linear functional form,
(v) Use measure theory, the new problem is modified into one consisting of the minimization of a linear functional over a set of Radon measures,
(vi) Obtain an optimal measure which is then approximated by a finite combination of atomic measures and the problem converted to an infinite-dimensional linear programming,
(vii) Approximate the infinite linear programming to a finite-dimensional linear programming,
(viii) By using the solution of the latter problem we obtain an approximate solution for moving boundary function on specific time,
(ix) Finally, plotting the path of moving boundary from initial state to final state.

## 4. Transformation the problem into an integral form

Let us assume that the moving boundary $x \in A$ be a function of time variable $t \in J$, where $A = [0, s(t)]$ and $J = [0, T]$.

**Lemma 1.** *An alternative integral form of problem (1) and boundary conditions (2)-(4) is as follows :*

$$\int_0^{s(t)} V(x,t)dx = s(0) - s(T) - \int_0^T g(t)dt.$$

*Proof.* By integrating (1) with respect to $x$ and then $t$ and using (2),(3),(4), we have :

$$\int_0^T \int_0^{s(t)} V_t(x,t)dxdt = \int_0^T \int_0^{s(t)} V_{xx}(x,t)dxdt, \tag{5}$$

where the left hand side of equation (5) is

$$\int_0^T \int_0^{s(t)} V_t(x,t)dxdt = \int_0^{s(t)} V(x,t)dx$$

and the right hand side of equation (5) is

$$\begin{aligned} \int_0^T \int_0^{s(t)} V_{xx}(x,t)dxdt &= \int_0^T (V_x(s(t),t) - V_x(0,t))dt \\ &= \int_0^T (V_x(s(t),t) - g(t))dt \\ &= \int_0^T \left(-\frac{ds(t)}{dt} - g(t)\right) dt \\ &= -(s(T) - s(0)) - \int_0^T g(t)dt. \end{aligned} \tag{6}$$

Therefore an equivalent form of problem (1)-(4) is as follows:

$$\int_0^{s(t)} V(x,t)dx = s(0) - s(T) - \int_0^T g(t)dt. \tag{7}$$

□

Now we define an objective function such as $H$ which is the square error function on the whole domain of the variables for our problem as follows: Minimize

$$H = \int_0^T \int_0^{s(t)} (V_{xx}(x,t) - V_t(x,t))^2 dxdt \tag{8}$$

subject to (2)-(4) but the upper bound of the second integral $s(t)$ is unknown, which it makes many difficulties when we want to use measure theory to minimize it as a linear functional over a set of Radon measures. Here if the optimal value of $H$ be zero, it is equivalent to problem (1)-(4). We re-scale the interval $[0, s(t)]$ of the variable $x$, where $s(t)$ is unknown, by setting a new variable $y(.) = x/s(t)$ which it's domain is fixed, so $y \in [0,1]$ when $x \in [0, s(t)]$, therefore we have following alternative formulas:

$$V(x,t) = Z(y,t), \quad y = \frac{x}{s(t)},$$

$$x = ys(t), \; dy = \frac{dx}{s(t)}, \quad s(t)dy = dx,$$

$$
\begin{aligned}
\frac{\partial V}{\partial t} &= \frac{\partial Z}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial Z}{\partial t} = \frac{\partial Z}{\partial y}\left(-x\frac{s'(t)}{s^2(t)}\right) + \frac{\partial Z}{\partial t} \\
&= \frac{\partial Z}{\partial y}\left(-s(t)y\frac{s'(t)}{s^2(t)}\right) + \frac{\partial Z}{\partial t} = -\frac{\partial Z}{\partial y}y\frac{s'(t)}{s(t)} + \frac{\partial Z}{\partial t}, \qquad (9) \\
\frac{\partial V}{\partial x} &= \frac{\partial Z}{\partial y}\frac{\partial y}{\partial x} = \frac{1}{s(t)}\frac{\partial Z}{\partial y}. \\
\frac{\partial^2 V}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial Z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{1}{s(t)}\frac{\partial Z}{\partial y}\right) = \frac{1}{s(t)}\frac{\partial}{\partial x}\left(\frac{\partial Z}{\partial y}\right) \\
&= \frac{1}{s(t)}\frac{\partial^2 Z}{\partial y^2}\frac{\partial y}{\partial x} = \frac{1}{s^2(t)}\frac{\partial^2 Z}{\partial y^2}. \qquad (10)
\end{aligned}
$$

Thus by using (9), (10) we can rewrite the problem (1)-(4) with respect to $y$ and $t$ as follows:

$$
\begin{aligned}
\frac{\partial^2 Z}{\partial y^2} &= -\frac{\partial Z}{\partial y}ys'(t)s(t) + s^2(t)\frac{\partial Z}{\partial t}, 0 < y < 1, t > 0, \qquad (11) \\
Z_y(0,t) &= s(t)g(t), t > 0, \qquad (12) \\
-\frac{\partial Z}{\partial y} &= s(t)\frac{ds}{dt}, Z = 0, y = 1, \qquad (13) \\
Z &= 0, \quad y > 1; \quad s(0) = a, \qquad (14)
\end{aligned}
$$

where $y \in Y = [0,1]$ and $t \in J = [0,T]$, and $T$ is an arbitrary positive fixed real number.
Now we intend to obtain a performance function as double integral with fixed boundaries. Thus by using (9), (10) we can rewrite the objective function $H$ in (8) with respect to $y$ and $t$ as follows:

$$
\int_0^T \int_0^1 \left(\frac{\partial^2 Z}{\partial y^2} + \frac{\partial Z}{\partial y}ys'(t)s(t) - s^2(t)\frac{\partial Z}{\partial t}\right)^2 s(t)dydt
$$

$$
\int_0^T \int_0^1 (\sqrt{s}(t)\frac{\partial^2 Z}{\partial y^2} + \sqrt{s}(t)\frac{\partial Z}{\partial y}ys'(t)s(t) - \sqrt{s}(t)\left(s^2(t)\frac{\partial Z}{\partial t})\right)^2 dydt
$$

$$
\int_0^T \int_0^1 \left(\sqrt{s}(t)\frac{\partial^2 Z}{\partial y^2} + s^{3/2}(t)\frac{\partial Z}{\partial y}ys'(t) - s^{5/2}(t)\frac{\partial Z}{\partial t}\right)^2 dydt.
$$

Here, we define the following artificial control functions $u, v_1, v_2, v_3, v_4$:

$$
\left.\begin{array}{l}
u(t) = s'(t) \\
v_1(y,t) = Z_y(y,t) \\
v_2(y,t) = Z_{yy}(y,t) \\
v_3(y,t) = Z_t(y,t) \\
v_4(y,t) = Z_{yt}(y,t).
\end{array}\right\} \qquad (15)
$$

By these notations we can rewrite (8) as follows :

$$\int_0^T \int_0^1 (\sqrt{s}(t)v_2 + s^{3/2}(t)v_1 ys'(t) - s^{5/2}(t)v_3)^2 dydt.$$

Now we convert the problem (1)-(4) into an optimal control problem as follows: Minimize

$$H = \int_0^T \int_0^1 (\sqrt{s}(t)v_2 + s^{3/2}(t)v_1 ys'(t) - s^{5/2}(t)v_3)^2 dydt, \tag{16}$$

subject to

$$s'(t) = u(t) \tag{17}$$

and boundary conditions (12)-(14).
Therefore we claim the following proposition :

**Proposition.** *The optimal control problem (15)-(16) with boundary conditions (12)-(14) is an equivalent alternative form of problem (1) and boundary conditions (2)-(4).*

## 5. Solving of the optimal control problem

For using measure theory to solve the optimal control problem (16)-(17) with boundary conditions (12)-(14), we need some considerations to define functions which satisfy differential equation (17) and boundary conditions (12)-(14).
A triple $w = (Z, u, v)$ is said to be *admissible* if the following conditions hold:

(i) $Z(y,t) \in Q = Y \times J,\ t \in J = [0,T], y \in Y = [0,1]$,
(ii) $u(t) \in U,\ t \in J$, $U$ is closed and bounded set in $R^1$,
(iii) $v_i \in V_i, i = 1,2,3,4,\ v = (v_1, v_2, v_3, v_4) \in V = V_1 \times V_2 \times V_3 \times V_4$ is a closed and bounded set in $R^4$.
(iv) The boundary conditions (11)-(14) are satisfied,
(v) The triple $w$ satisfies (16)-(17) and (12)-(14) a.e. on $J^0 = (0,T)$.

We assume that the set of all admissible triple defined above is non-empty and denote it by $W$. Let $w = (Z, u, v)$ be an admissible triple, and $B$ be an open ball in $\mathbb{R}^6$ containing $Y \times J \times V$; and $C'(B)$ be the space of all real-valued twice continuously differentiable functions on it such that they and their first and second partial derivatives are bounded on $B$. Let $\phi_{1i} \in C'(B)$, which we can consider them as $\phi_{1i} = h_i(t)Z(y,t), i = 1,2,3,\cdots$ , where we can consider $h_i$ as a suitable function of $t$ or $s(t)$, and define functions ${\phi_1}^{(u,v)} : \Omega \subset R^6 \to \mathbb{R}$ as $d^2\phi_{1i}/dydt$ or $d^2\phi_{1i}/dy^2$. For example if we choose $\phi_1 = (1/s(t))Z(y,t)$ then

$$\frac{d^2\phi_1}{dy^2} = \frac{1}{s(t)}\frac{\partial^2 Z}{\partial y^2}.$$

Now by using Lemma (1) and by integrating with respect to $y$ and then $t$ and using (12)-(14), we have:

$$\begin{aligned}
{\phi_1}^{(u,v)}(y,t,u,Z,v) &= \int_0^T \int_0^1 \frac{d^2\phi_1}{dy^2} dydt \\
&= \int_0^T \int_0^1 \frac{1}{s(t)} \frac{\partial^2 Z}{\partial y^2} dydt \\
&= \int_0^T \frac{1}{s(t)} \left( \frac{\partial Z}{\partial y}(1,t) - \frac{\partial Z}{\partial y}(0,t) \right) dt \\
&= \int_0^T \frac{1}{s(t)} \left( -s(t)\frac{ds}{dt} - s(t)g(t) \right) dt \\
&= \int_0^T \left( -\frac{ds}{dt} - g(t) \right) dt \\
&= -\int_0^T \frac{ds}{dt} dt + \int_0^T g(t)dt.
\end{aligned}$$

By choosing $g(t) = -1$ we have

$$\int_0^T \int_0^1 \frac{d^2\phi_1}{dy^2} dydt = -\int_0^T \frac{ds}{dt} dt - \int_0^T g(t)dt = -s(T) + T. \tag{18}$$

And by replacing (11) in (18), we can write

$$\int_0^T \int_0^1 \left( -\frac{\partial Z}{\partial y} ys'(t) + s(t)\frac{\partial Z}{\partial t} \right) dydt = -s(T) + T, \tag{19}$$

for all $(y,t,u,Z,v) \in \Omega = Y \times J \times U \times Q \times V \subset R^8$. The functions ${\phi_1}^{(u,v)}$ are in the space $C(\Omega)$ on the compact set $\Omega$.

Let $D(G^0)$ be the space of all infinitely differentiable real-valued functions with compact support in $G^0$, the interior of $G$, where $G = A \times J$ (see [5] and [21]). Now we consider $\phi_{2j}$ either as the type $\phi_{2j} = p(y,t)q(y,t)$ which the functions $q(y,t)$ are considered as

$$\sin(2\pi rt/T)\sin(2\pi ry), \quad (1 - \cos(2\pi rt/T))\sin(2\pi ry),$$

$$(1 - \cos(2\pi rt/T))(1 - \cos(2\pi ry)), \quad (\sin(2\pi rt/T))(1 - \cos(2\pi ry),$$

$r = 1, 2, \cdots$, for all $q \in D(G^0)$ and the functions $p(y,t)$ are considered as $s(t)$ or $Z(y,t)$ or $\partial Z(y,t)/\partial y$.

Define functions ${\phi_2}^{(u,v)}$ as follows:

$$\begin{aligned} {\phi_2}^{(u,v)}(y,t,u,Z,v) &= \frac{d^2\phi_{2j}}{dydt} \\ &= \frac{\partial^2\phi_{2j}}{\partial Z^2}\frac{\partial Z}{\partial y}\frac{\partial Z}{\partial t} + \frac{\partial^2\phi_{2j}}{\partial Z\partial y}\frac{\partial Z}{\partial t} + \frac{\partial^2\phi_{2j}}{\partial Z\partial t}\frac{\partial Z}{\partial y} \\ &\quad + \frac{\partial\phi_{2j}}{\partial Z}\frac{\partial^2 Z}{\partial y\partial t} + \frac{\partial^2\phi_{2j}}{\partial y\partial t}. \end{aligned} \tag{20}$$

For example if we choose the functions $\phi_{2j}$ as the type $\phi_{2j} = s(t)q(y,t)$ then we have

$$ {\phi_2}^{(u,v)}(y,t,u,Z,v) = \frac{d^2\phi_{2j}}{dydt} = \frac{\partial q(y,t)}{\partial y}\frac{\partial s(t)}{\partial t} + s(t)\frac{\partial^2 q(y,t)}{\partial y\partial t},$$

and if $w = (Z,u,v)$ be an admissible triple, we have, for $j = 1,2,...,n$ and $q \in D(G^0)$,

$$\begin{aligned} \int_0^T\int_0^1 \frac{d^2\phi_{2j}}{dydt}dydt &= \int_0^T\int_0^1 \left(\frac{\partial q(y,t)}{\partial y}\frac{\partial s(t)}{\partial t} + s(t)\frac{\partial^2 q(y,t)}{\partial y\partial t}\right)dydt \\ &= s(T)q(1,T) - s(0)q(1,0) - s(T)q(0,T) + s(0)q(0,0) = 0. \end{aligned}$$

Since the function $q(y,t)$ has compact support in $G^0, q(1,T) = q(1,0) = q(0,T) = q(0,0) = 0$.
With the choice of functions which depend only on the time variable $t$ or depend only on state variable $y$, we have

$$\int_0^T\int_0^1 k(y,t)dydt = a_k,\ k \in C_1(G),$$

where $a_k$ is the Lebesgue integral of $k(y,t)$ on $G = Y \times J$ and $C_1(G)$ is subspace of the space $C(G)$, of the continuous functions depends on the time variable $t$ or state $y$.

## 6. Linear functional form of problem

Now we modify the problem into a linear functional form consisting of the minimization of a linear functional over a set of Radon measures and consider,
(1) The mapping

$$\Lambda_w : F \to \int_{Y\times J} F(y,t,u(t),Z,v)dydt,\ F \in C(\Omega),$$

defines a positive linear functional on $C(\Omega)$.
(2) By the Riesz representation theorem (see [5], [24], [25]), there exists a unique positive Radon measure $\mu$ on $\Omega$ such that

$$\Lambda_w(F) = \int_{Y\times J} F(y,t,u(t),Z,v))dydt = \int_\Omega F d\mu \equiv \mu(F),\ F \in C(\Omega).$$

Thus, the minimization of the functional (16) is equivalent to the minimization of

$$E[w] = \Lambda_w(\sqrt{s}v_2 + s^{3/2}v_1 y s'(t) - s^{5/2}v_3)^2, \tag{21}$$

over the following constraints:

$$\left.\begin{array}{l} \Lambda_w(\phi_{1i}^{(u,v)}) = \delta\phi_{1i},\ i = 1,2,\cdots,m,\ \phi_{1i} \in C'(B) \\ \Lambda_w(\phi_{2j}^{(u,v)}) = 0,\ j = 1,2,\cdots,n,\ \phi_{2j} \in D(G^0) \\ \Lambda_w(k) = a_k,\ k \in C_1(G) \\ \Lambda_w(\sqrt{s}v_2 + s^{3/2}v_1 y s'(t) - s^{5/2}v_3)^2 < \epsilon \end{array}\right\}. \tag{22}$$

Now, suppose that the space of all positive Radon measures on $\Omega$ will be denoted by $M^+(\Omega)$. By the Riesz representation theorem, the positive linear functionals above will be replaced by their representing measures, thus we seek a measure in $M^+(\Omega)$, to be normally denoted by $\mu^*$ which minimizes the functional (21). Thus, the minimization of the functional $E$ in (21) over $W$ is equivalent to the minimization of

$$\begin{aligned} E[\mu] &= \int_\Omega \left(\sqrt{s}v_2 + s^{3/2}v_1 y s'(t) - s^{5/2}v_3\right)^2 d\mu \qquad (23) \\ &\equiv \mu\left(\sqrt{s}v_2 + s^{3/2}v_1 y s'(t) - s^{5/2}v_3\right)^2 \in I\!R \end{aligned}$$

over the set of all positive measures $\mu$ corresponding to admissible triples $w$, which satisfy

$$\left.\begin{array}{l} \mu(\phi_{1i}^{(u,v)}) = \delta\phi_{1i},\ i = 1,2,\cdots,m,\ \phi_{1i} \in C'(B) \\ \mu(\phi_{2j}^{(u,v)}) = 0,\ j = 1,2,\cdots,n,\ \phi_{2j} \in D(G^0) \\ \mu(k) = a_k,\ k \in C_1(G) \\ \mu(\sqrt{s}v_2 + s^{3/2}v_1 y s'(t) - s^{5/2}v_3)^2 < \epsilon \end{array}\right\}. \tag{24}$$

We shall consider the minimization of (23) over the set $Q$ of all positive Radon measures on $\Omega$ satisfying (24). Now if we ' *topologize*' the space $M^+(\Omega)$ by the weak*-topology, it can be seen from (see [21]) that $Q$ is compact. The functional $E : Q \to I\!R$, defined by

$$\begin{aligned} E[\mu] &= \int_\Omega \left(\sqrt{s}v_2 - s^{3/2}v_1 y s'(t) - s^{5/2}v_3\right)^2 d\mu \\ &\equiv \mu\left(\sqrt{s}v_2 + s^{3/2}v_1 y s'(t) - s^{5/2}v_3\right)^2 \in I\!R,\ \mu \in Q \end{aligned}$$

is a linear continuous functional on a compact set $Q$; so attains its minimum on $Q$ (see [21]), thus, the measure-theoretical problem, which consists of finding the minimum of the functional (23) over the subset $Q$ of $M^+(\Omega)$, possesses a minimizing solution $\mu^*$, say, in $Q$.

## 7. Approximation of the optimal measure

For the estimation by the nearly optimal piece-wise constant control, consider the minimization of the functional (23) not over the set $Q$ but over a subset of $M^+(\Omega)$ which is defined by requiring that only a finite number of the constraints in (24) to be satisfied. This will be achieved by choosing countable sets of functions whose linear combinations are dense in the appropriate spaces, and then selecting a finite number of them. In the first step, we obtain an approximation of the optimal measure $\mu^*$ by a finite combination of atomic measures, that is, from the Theorem ( see [21], Appendix), $\mu^*$ has the form $\mu^* = \sum_{i=1}^{N} \alpha_i^* \delta(z_i^*)$ where $\alpha_i^* \geq 0$ and $z_i^* \in \Omega$ for $i$=1,2,$\cdots$,$N$ (here $\delta(z)$ is a unitary atomic measure, characterized by $\delta(z)(F) = F(z)$ where $F \in C(\Omega)$ and $z \in \Omega$). Then, we construct a piecewise- constant control function corresponding to the finite-dimensional problem. Therefore in the infinite-dimensional linear programming problem (23) with restriction defined by (24), we shall consider only a finite number $M_1$ of functions $\phi_{1i}$'s of the type $h_i(t)Z(y,t), i = 1, 2, 3, \cdots$ , where we can consider $h_i$ as a suitable function of $t$ or $s(t)$. Also, only a finite number of functions $\phi_{2j},\ j = 1, 2, \cdots, M_2$, either as the type $p(y,t)q(y,t)$ which the functions $q(y,t)$ are considered as $\sin(\frac{2\pi rt}{T})\sin(2\pi ry)$, $(1-\cos(\frac{2\pi rt}{T}))\sin(2\pi ry)$, $(1-\cos(\frac{2\pi rt}{T}))(1-\cos(2\pi ry))$, $(\sin(\frac{2\pi rt}{T}))(1-\cos(2\pi ry)$ $r = 1, 2, \cdots$. for all $q \in D(G^0)$ and the functions $p(y,t)$ are considered as $s(t)$ or $Z(y,t)$ or $\partial Z(y,t)/\partial y$. Also, only a finite number $M_3$ of functions $k$ of the type

$$k_{ss'}(y,t) = \begin{cases} 1 & \text{if } (y,t) \in J'_s \times J_s \\ 0 & \text{otherwise} \end{cases}$$

will be considered, where

$$\begin{aligned} J'_s &= ((s'-1)d', s'd'), \quad d' = \frac{1}{L'}, s' = 1, \cdots, L', \\ J_s &= ((s-1)d, sd), \quad d = \frac{T}{L}, s = 1, \cdots, L, \\ J'_s &= ((s'-1)d', s'd'). \end{aligned}$$

The set $\Omega = Y \times J \times U \times Q \times V$ is covered by a partition, where the partition is defined by taking all points in $\Omega$ as $z_j = (y_j, t_j, u_j, Z_j, v_{1j}, v_{2j}, v_{3j}, v_{4j})$. Of course, we only need to construct the control function $u(.)$, since the trajectory is then simply the corresponding solution of the system of differential equations (1), with conditions (2)-(4), which can be estimated numerically. The infinite-dimensional linear programming problem with objective function $E$ in (23) with restriction defined by (24) can be approximated by the following problem, which $z_j$ for $j = 1, \cdots, N$ belong to a dense subset of $\Omega$ [2]-[3].

Minimize

$$\sum_{j=1}^{N} \alpha_j E(z_j) \tag{25}$$

subject to

$$\left.\begin{array}{l}
\sum_{j=1}^{N} \alpha_j \phi_{1i}^{(u,v)}(z_j) = \delta\phi_i,\ i = 1, \cdots, M_1 \\
\sum_{j=1}^{N} \alpha_j \phi_{2h}^{(u,v)}(z_j) = 0,\ h = 1, \cdots, M_2 \\
\sum_{j=1}^{N} \alpha_j k_s(y_j, t_j) = a_k,\ s = 1, \cdots, M_3 \\
\sum_{j=1}^{N} \alpha_j E(z_j) < \epsilon \\
\alpha_j \geq 0\ ,\ j = 1, \cdots, N.
\end{array}\right\} \tag{26}$$

Note that the elements $z_j$, $j = 1, 2, \cdots, N$ are fixed, the only unknowns are the numbers $\alpha_j$, $j = 1, 2, \cdots, N$. The procedure to construct the piecewise constant control function approximating the action of the optimal measure is based on the analysis [21].

## 8. Numerical examples

Some numerical examples are considered below to illustrate the procedure.

**Example 8.1.** Solve the moving boundary problem of the following form when $T = 12$:

$$\begin{aligned}
V_t(x,t) &= V_{xx}(x,t), \quad 0 < x < s(t), \quad t > 0, && (27)\\
V_x(0,t) &= g(t) = -1, \quad t > 0, && (28)\\
-\frac{\partial V}{\partial x} &= \frac{dx}{dt}, \quad V = 0, \quad x = s(t), && (29)\\
V &= 0, \quad x > s(t); \quad s(0) = 0, && (30)
\end{aligned}$$

where $V(x,t)$ is the temperature distribution in a point $x$ at time $t$, and $s(t)$ is moving boundary which is unknown. Thus by using (11)-(14) we can rewrite the problem (27)-(30) with respect to $y$ and $t$ as follows:

$$\begin{aligned}
Z_t(y,t) &= Z_{yy}(y,t), \quad 0 < y < 1, \quad t > 0, && (31)\\
Z_y(0,t) &= s(t)g(t) = -s(t), \quad t > 0, && (32)\\
-\frac{\partial Z}{\partial y} &= s(t)\frac{ds}{dt}, \quad Z = 0, \quad y = 1, && (33)\\
Z &= 0, \quad y > 1; \quad s(0) = 0, && (34)
\end{aligned}$$

where $y \in Y = [0,1]$ and $t \in J = [0,T]$, when $T = 12$.
Now define the artificial control functions $u, v_1, v_2, v_3, v_4$ like as (15) and then

convert the problem (31)-(34) into an optimal control problem as follows:
Minimize

$$H = \int_0^{12} \int_0^1 \left(\sqrt{s}(t)v_2 + s^{3/2}(t)v_1 y s'(t) - s^{5/2}(t)v_3\right)^2 dydt, \tag{35}$$

Subject to

$$s'(t) = u(t) \tag{36}$$

and boundary conditions (32)-(34).
Define

$$\phi_1 = \frac{1}{s(t)} Z(y,t).$$

Then

$$\frac{d^2\phi_1}{dy^2} = \frac{1}{s(t)} \frac{\partial^2 Z}{\partial y^2}.$$

Now by integrating with respect to $y$ and then $t$ and using boundaries, we have:

$$\begin{aligned}
{\phi_1}^{(u,v)}(y,t,u,Z,v) &= \int_0^{12} \int_0^1 \frac{d^2\phi_1}{dy^2} dydt \\
&= \int_0^{12} \int_0^1 \frac{1}{s(t)} \frac{\partial^2 Z}{\partial y^2} dydt \\
&= \int_0^{12} \frac{1}{s(t)} \left( \frac{\partial Z}{\partial y}(1,t) - \frac{\partial Z}{\partial y}(0,t) \right) dt \\
&= \int_0^{12} \frac{1}{s(t)} \left( -s(t)\frac{ds}{dt} + s(t) \right) dt \\
&= \int_0^{12} \left( -\frac{ds}{dt} + 1) \right) dt \\
&= -\int_0^{12} \frac{ds}{dt} dt + \int_0^{12} dt.
\end{aligned}$$

Thus,

$$\int_0^{12} \int_0^1 \frac{d^2\phi_1}{dy^2} dydt = -\int_0^{12} \frac{ds}{dt} dt - \int_0^{12} dt = -s(12) + 12.$$

And by replacing $T = 12$ in (19), we can write

$$\int_0^{12} \int_0^1 \left( -\frac{\partial Z}{\partial y} y s'(t) + s(t) \frac{\partial Z}{\partial t} \right) dydt = -s(12) + 12, \tag{37}$$

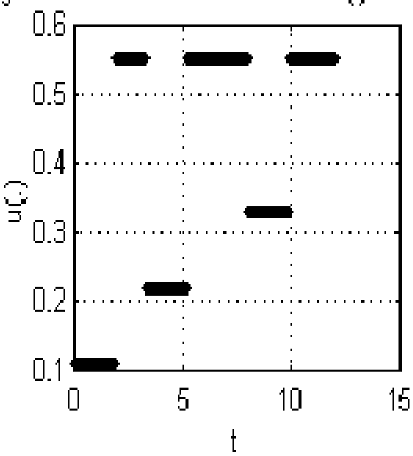
Now consider $\phi_{2j}$ either as the type $\phi_{2j}$ as the type $\phi_{2j} = s(t)q(y,t)$ which the functions $q_r(y,t)$ are considered as $\sin(\frac{2\pi r t}{12})\sin(2\pi r y)$, $r = 1, 2, \cdots$, Define functions ${\phi_2}^{u,v}$ as follows:

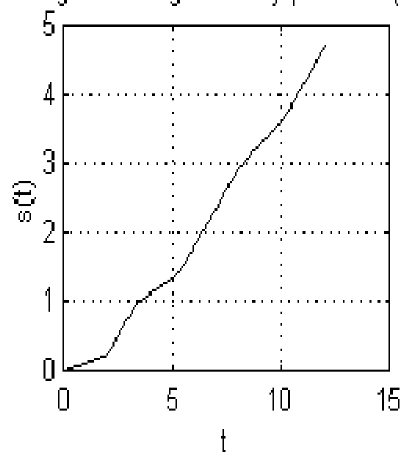
$${\phi_2}^{u,v}(y,t,u,Z,v) = \frac{d^2\phi_{2j}}{dydt} = \frac{\partial q(y,t)}{\partial y} \frac{ds(t)}{dt} + s(t) \frac{\partial^2 q(y,t)}{\partial y \partial t},$$

so we have

$$\begin{aligned}
\int_0^{12}\int_0^1 \frac{d^2\phi_{2j}}{dydt}dydt &= \int_0^{12}\int_0^1 \left(\frac{\partial q(y,t)}{\partial y}\frac{ds(t)}{dt} + s(t)\frac{\partial^2 q(y,t)}{\partial y \partial t}\right) dydt,\\
&= s(12)q(1,12) - s(0)q(1,0) - s(12)q(0,12)\\
&\quad +s(0)q(0,0)\\
&= 0.
\end{aligned}$$

Fig. 1.Piecewise constant control u(.) when T=12 Fig. 2. Moving boundary position s(t)




Since the function $q(y,t)$ has compact support in $G^0$, $q(1,12) = q(1,0) = q(0,12) = q(0,0) = 0$. Let $t \in J = [0,12]$, and $y \in A = [0,1]$, and $u \in U = [0, 0.55]$, $v_1 \in V_1 = [-1,0]$ ,$v_2 \in V_2 = [-1,0]$ and $v_3 \in V_3 = [-1,0]$. Let the sets $J$ and $A$ and $U$ and $V_1$ are divided into 5 subintervals, and the set $V_2$, $V_3$ and $V_4$ are divided into 4 subintervals so that $\Omega = J \times A \times U \times V$ is divided into 40000 subintervals. Now if $M_1 = 1, M_2 = 4, L = 5$ and $\epsilon = 10^{-3}$, then we have a linear programming problem as follows:

Minimize

$$\sum_{j=1}^{40000} \alpha_j \left(\sqrt{s}(t)v_2 + s^{3/2}(t)v_1 y s'(t) - s^{5/2}(t)v_3\right)^2$$

Subject to

$$\sum_{j=1}^{40000} \alpha_j \left(-\frac{\partial Z}{\partial y} y s'(t) + s(t)\frac{\partial Z}{\partial t}\right) + \alpha_{40001} - \alpha_{40002} = 12$$

Fig. 3.Piecewise constant control u(.) when T=16

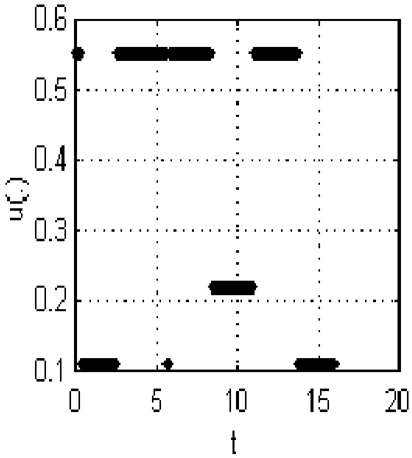



Fig. 4. Moving boundary position s(t)

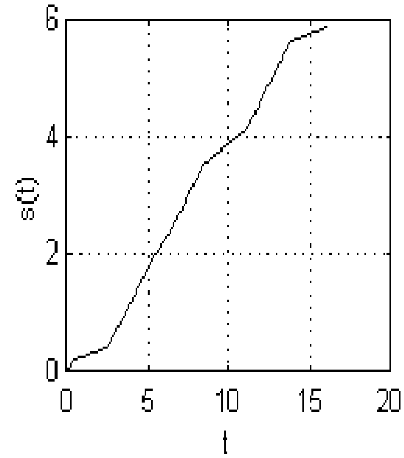



Fig. 5.Piecewise constant control u(.) when T=28

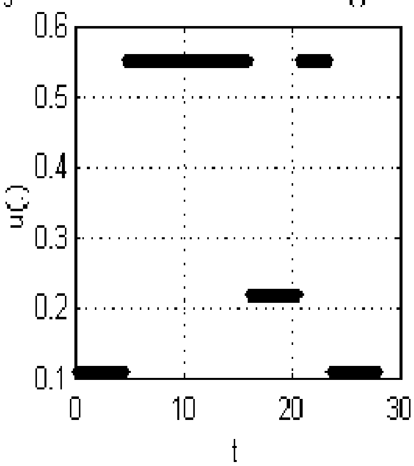
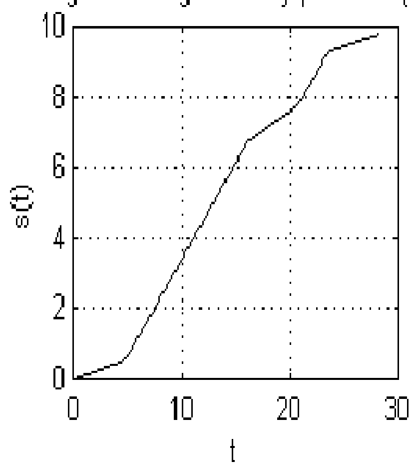



Fig. 6. Moving boundary position s(t)




$$\sum_{j=1}^{40000} \alpha_j \left( \frac{\partial q_r(y,t)}{\partial y} \frac{ds(t)}{dt} + s(t) \frac{\partial^2 q_r(y,t)}{\partial y \partial t} \right) = 0, \ r = 1, 2, 3, 4$$

$$\sum_{j=1}^{40000} \alpha_j \left( \sqrt{s}(t) v_2 + s^{3/2}(t) v_1 y s'(t) - s^{5/2}(t) v_3 \right)^2 < \epsilon$$

$$\alpha_{1+4000(i-1)} + \cdots + \alpha_{4000+4000(i-1)} = \frac{12}{5}, \quad i = 1, \cdots, 5.$$

The position of moving boundary in $T = 12$ is 4.6929, and the optimal value of the cost function $f$ in $T = 12$ is 1.3010e-018. The graphs of the piecewise constant control functions and the trajectory functions when $T = 12$, $T = 16$, $T = 28$ are shown in figures 1-6, respectively.

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