S238/106

DUBLIN INSTITUTE OF TECHNOLOGY

School of Mathematical Sciences

DT238 MSc Applied Mathematics & Theoretical Physics

SUMMER EXAMINATIONS 2015/2016

MATH 1860: INTRODUCTION TO BIOMATHEMATICS

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PROFESSOR E O'RIORDAN

1.00 – 4.30 pm, Friday, 13 May 2016

Duration: 3½ hours

Full marks may be obtained by answering 4 questions. Candidate's four best questions will contribute to their final mark.

All questions carry equal marks

Approved calculators may be used

Mathematical tables are provided

New Cambridge Statistical Tables are NOT permitted

1. a) Solve explicitly the logistic model with a given time-dependent intrinsic growth rate $r(t)$:

$$\frac{dN(t)}{dt} = r(t)N(t)\left(1 - \frac{N(t)}{K}\right), \qquad K > 0, \qquad N(0) > 0.$$

(10)

b) Find the steady states of the following model

$$\frac{dN}{dt} = RN\left(1 - \frac{N}{K}\right)\left(\frac{N}{K_0} - 1\right), \qquad 0 < K_0 < K$$

and determine their linear stability. R, K_0 and K are positive constants. (10)

c) For the model in part **b)** find

$$\lim_{t\to\infty} N(t)$$

if $K_0 < N(0) < K$.

(5)

[25]

2. A single population model with a very "fast" predation is given by the equation

$$\frac{dN}{dt} = RN\left(1 - \frac{N}{K}\right) - P\left(1 - \exp\left(-\frac{N^2}{\varepsilon A^2}\right)\right), \qquad 0 < \varepsilon \ll 1,$$

where R, K, P and A are positive constants.

a) By an appropriate nondimensionalisation show that the equation is equivalent to

$$\frac{du}{d\tau} = ru\left(1 - \frac{u}{q}\right) - \left(1 - \exp\left(-\frac{u^2}{\varepsilon}\right)\right),$$

where r and q are positive parameters.

(5)

b) Sketch the graphs of the functions $g(u) = ru\left(1 - \frac{u}{q}\right)$ and $h(u) = 1 - \exp\left(-\frac{u^2}{\varepsilon}\right)$ for positive u remembering that ε is small, i.e. $0 < \varepsilon \ll 1$.

(8)

c) Demonstrate that there are three possible nonzero steady states if r and q lie in a domain in r, q space given approximately by $rq > 4$.

(7)

d) Could this model exhibit hysteresis?

(5)

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3. It has been suggested that a means of controlling insect numbers is to introduce and maintain a number of sterile insects in the population. One such model for the resulting population dynamics is

$$N_{t+1} = \frac{rN_t^2}{\frac{r-1}{M}N_t^2 + N_t + S},$$

where $r > 1$ and $M > 0$ are constant parameters, and S is the constant sterile insect population.

a) Determine the steady states and discuss their linear stability, noting whether any type of bifurcation is possible. (9)

b) Find the critical value S_c of the sterile population in terms of r and M so that if $S > S_c$ the insect population is eradicated. (9)

c) Construct a cobweb map and draw a graph of the steady state population density against S, and hence determine the possible solution behavior if $0 < S < S_c$. (7)

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4. Show that an exact travelling wave solution exists for the scalar reaction-diffusion equation

$$\frac{\partial u}{\partial t} = u^{q+1}\left(1 - u^q\right) + \frac{\partial^2 u}{\partial x^2}, \qquad q > 0,$$

by looking for solution in the form

$$u(x,t) = U(z) = \frac{1}{(1 + ae^{bz})^s}, \qquad z = x - ct,$$

where c is the wave speed and s and b are positive constants. Determine the unique values for c, s and b in terms of q. Choose the value for a such that the magnitude of the wave's gradient is at its maximum at $z = 0$.

[25]

5. Flores (1998) proposed the following model for competition between Neanderthal man (N) and Early Modern man (E).

$$\begin{aligned}
\frac{dN}{dt} &= N\left(A - D(N+E) - B\right), \\
\frac{dE}{dt} &= E\left(A - D(N+E) - sB\right),
\end{aligned}$$

where A, B, D are positive constants and $0 < s < 1$ is a measure of the difference in mortality of the two species.

a) Explain the model briefly, nondimensionalise the system and perform the phase plane analysis. (15)

b) Show that for large values of t the population $N(t)$ decays according to the law

$$N(t) = C\exp[-B(1-s)t],$$

where C is some constant. Hence give the order of magnitude of the time for Neanderthal extinction if $s = 0.995$ and if the lifetime of an individual is roughly 40 years. (10)

[25]

6. A model for venereal diseases includes the following classes: male and female infectives (I_1 and I_2 respectively) and male and female susceptibles (S_1 and S_2 respectively). It assumes that once infectives have recovered they rejoin the susceptibles:

$$\begin{aligned}
\frac{dS_1}{dt} &= -r_1 S_1 I_2 + a_1 I_1, \\
\frac{dI_1}{dt} &= r_1 S_1 I_2 - a_1 I_1, \\
\frac{dS_2}{dt} &= -r_2 S_2 I_1 + a_2 I_2, \\
\frac{dI_2}{dt} &= r_2 S_2 I_1 - a_2 I_2,
\end{aligned}$$

r_1, r_2, a_1 and a_2 are positive constant parameters.

a) Briefly explain all terms and parameters in these equations and show that

$$I_k + S_k = N_k,$$

$k = 1, 2$ where N_k are constants. (5)

b) Using the result from (i) exclude S_1 and S_2 and reduce the system to two nonlinear equations for I_1 and I_2. Find the possible steady states and examine their linear stability for all possible values of the parameters. (20)

[25]