W9209/9210/102C

DUBLIN INSTITUTE OF TECHNOLOGY

School of Mathematical Sciences

DT9209 MSc Applied Mathematics

DT9210 MSc Applied Mathematics

WINTER EXAMINATIONS 2015/2016

MATH 9951: Methods for Applied Mathematics

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09.30 – 11.30 am, Wednesday, 06 January 2016

Duration: 2 hours

Attempt three questions only

All questions carry equal marks

Approved calculators may be used

Mathematical tables are provided

New Cambridge Statistical Tables are NOT permitted

1. a) Assume a membrane in two-dimensions x and y, which is large, light, and undergoes small deflections $u(x, y)$ from the (x, y) plane. By considering the tension forces on a small section of the membrane of dimensions Δx and Δy and mass per unit area σ, show that u obeys the two-dimensional wave equation

$$u_{tt} = \frac{T}{\sigma}\nabla^2 u.$$

(11)

b) Using the method of characteristics, find the solution to

$$(y - u)u_x + (u - x)u_y = x - y, \text{ with } u = 0 \text{ on } xy = 1.$$

(11)

c) Find the integral surfaces for

$$x(y^2 + u)u_x - y(x^2 + u)u_y = (x^2 - y^2)u,$$

under the condition

$$u = 1 \text{ on } x + y = 0.$$

(11)

[33]

2. a) For the second-order partial differential equation describing $u(x,y)$,

$$A(x,y)u_{xx}+B(x,y)u_{xy}+C(x,y)u_{yy}+D(x,y)u_x+E(x,y)u_y+F(x,y)u=G(x,y),$$

derive the conditions for the equation to have the form

$$B^*(x,y)u_{\xi\eta}+D^*(x,y)u_\xi+E^*(x,y)u_\eta+F^*(x,y)u=G^*(x,y),$$

under the change of variables from x,y to ξ,η. (12)

b) Hence, express ordinary differential equations for the characteristics and state the conditions for classification of the equations as hyperbolic, parabolic, or elliptic. (8)

c) Classify and convert to canonical form the equation

$$y^2u_{xx}+x^2u_{yy}=0.$$

(13)

[33]

3. The wave equation for small displacements in a light string with line density ρ under tension T is

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < l, \quad t > 0,$$

where $c^2 \equiv T/\rho$. The initial conditions are

$$u(x,0) = f(x), \quad u_t(x,0) = g(x),$$

and the boundary conditions are

$$u(0,t) = 0, \quad u(l,t) = 0.$$

Assume $u(x,t)$ is twice-differentiable with respect to both x and t.

a) Prove that at most one solution exists to the given wave equation. (13)

b) By separation of variables, derive the infinite series solution to the given wave equation.
You may use the identity below to express the coefficients of the series in terms of integrals involving $f(x)$ and $g(x)$:

$$\int_0^l \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)dx = \begin{cases} \frac{l}{2} & \textit{if } n = m \\ 0 & \textit{if } n \neq m \end{cases}$$

(12)

c) Obtain the solution for the particular case

$$f(x) = \begin{cases} hx/a, & 0 \le x \le a, \\ \frac{h(l-x)}{l-a}, & a \le x \le l \end{cases}$$

and

$$g(x) = 0.$$

(8)

[33]

4. The Fourier transform of a function $u(x,t)$, with respect to x, is given by $U(k,t) = \mathcal{F}\{u(x,t)\}$ where

$$\mathcal{F}\{u(x,t)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx} u(x,t)dx.$$

The inverse Fourier transform of $U(k,t)$ is given by $u(x,t) = \mathcal{F}^{-1}\{U(k,t)\}$ where

$$\mathcal{F}^{-1}\{U(k,t)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx} U(k,t)dk.$$

a) Given $F(k)$ and $G(k)$, the Fourier transforms of $f(x)$ and $g(x)$, prove that

$$\mathcal{F}\{(f * g)(x)\} = F(k)G(k),$$

and hence,

$$\mathcal{F}^{-1}\{F(k)G(k)\} = (f * g)(x),$$

where $f * g$ is the convolution of f and g defined by

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-\xi)g(\xi)d\xi.$$

(10)

b) Prove that

$$\sin x * \mathrm{e}^{-a|x|} = \sqrt{\frac{2}{\pi}}\frac{a\sin x}{1+a^2}.$$

You may wish to use the identities

$$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v.$$

(6)

c) i) By using Fourier transformation methods, solve the diffusion equation

$$u_t = \kappa u_{xx}, \quad -\infty < x < \infty, \quad t > 0,$$

with initial and boundary conditions

$$\begin{aligned} u(x,0) &= f(x), \quad -\infty < x < \infty, \\ u(x,t) &\to 0, \quad \text{as } |x| \to \infty, \end{aligned}$$

where f is some function and κ is a constant.
You may use the identity

$$\int_{-\infty}^{\infty} \mathrm{e}^{-ax^2} dx = \sqrt{\frac{\pi}{a}}.$$

(13)

ii) Hence, obtain the solution $u(x,t)$ for the case $f(x) = \delta(x-a)$. (4)

[33]