

Recurrence Relations
(Difference Equations)

Pure mathematics is the subject in which
we do not know what we are talking about,
or whether what we are saying is true.
— Bertrand Russell (1872–1970)

Recurrence Relations Arise Naturally

- When a problem has a recursive nature, recurrence relations often arise.
 - A problem can be solved by solving 2 subproblems of the same nature.
- When an algorithm is of the divide-and-conquer type, a recurrence relation describes its running time.
 - Sorting, fast Fourier transform, etc.
- Certain combinatorial objects are constructed recursively such as hypercubes (p. 610).

First-Order Linear Homogeneous Recurrence Relations

- Consider the recurrence relation

$$a_{n+1} = da_n,$$

where $n \geq 0$ and d is a constant.

- The **general solution** is given by

$$a_n = Cd^n$$

for any constant C .

- It satisfies the relation: $Cd^{n+1} = dCd^n$.
- There are infinitely many solutions, one for each choice of C .

First-Order Linear Homogeneous Recurrence Relations (concluded)

- Now suppose we impose the **initial condition** $a_0 = A$.
- Then the (unique) **particular solution** is $a_n = Ad^n$.
 - Because $A = a_0 = Cd^0 = C$.
- Note that $a_n = na_{n-1}$ is *not* a first-order linear homogeneous recurrence relation.
 - Its solution is $n!$ when $a_0 = 1$.

First-Order Linear Nonhomogeneous Recurrence Relations

- Consider the recurrence relation

$$a_{n+1} + da_n = f(n).$$

- $n \geq 0$.
 - d is a constant.
 - $f(n) : \mathbb{N} \rightarrow \mathbb{N}$.
- A general solution no longer exists.

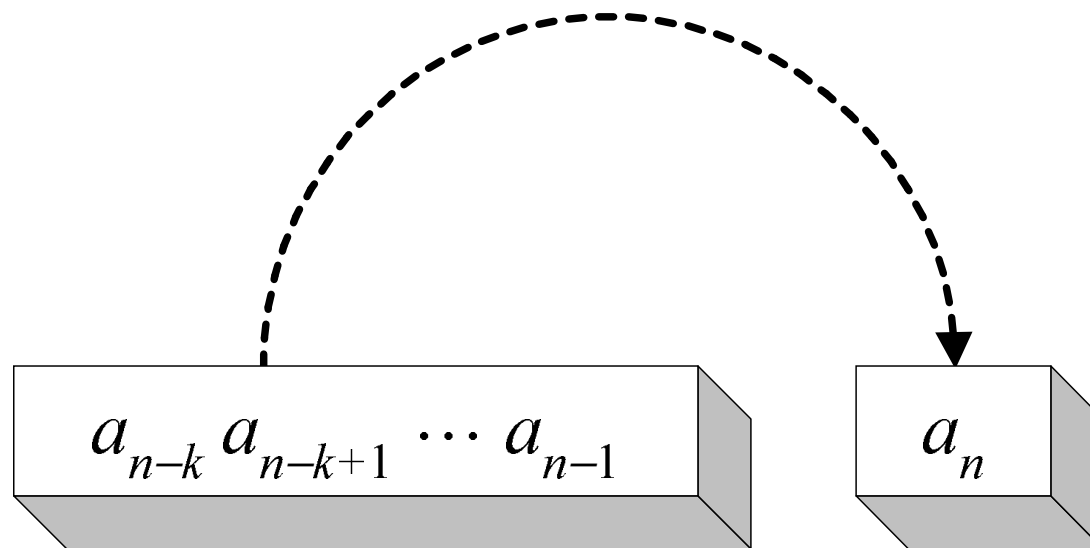
k th-Order Linear Homogeneous Recurrence Relations with Constant Coefficients

- Consider the k th-order recurrence relation

$$C_n a_n + C_{n-1} a_{n-1} + \cdots + C_{n-k} a_{n-k} = 0, \quad (68)$$

where $C_n, C_{n-1}, \dots, C_{n-k} \in \mathbb{R}$, $C_n \neq 0$, and $C_{n-k} \neq 0$.

- Add k initial conditions for a_0, a_1, \dots, a_{k-1} .
- Clearly, a_n is well-defined for each $n = k, k+1, \dots$.
- Indeed, a_n can be calculated with $O(kn)$ operations.



k th-Order Linear Homogeneous Recurrence Relations with Constant Coefficients (concluded)

- A solution y for a_n is general if for any particular solution y^* , the undetermined coefficients of y can be found so that y is identical to y^* .
- Any general solution for a_n that satisfies the k initial conditions and Eq. (68) is a particular solution.
- In fact, it is the *unique* particular solution because any solution agreeing at $n = 0, 1, \dots, k - 1$ must agree for all $n \geq 0$.

Conditions for the General Solution

Theorem 83 *Let $a_n^{(1)}, a_n^{(2)}, \dots, a_n^{(k)}$ be k particular solutions of Eq. (68). If*

$$\begin{vmatrix} a_0^{(1)} & a_0^{(2)} & \cdots & a_0^{(k)} \\ a_1^{(1)} & a_1^{(2)} & \cdots & a_1^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k-1}^{(1)} & a_{k-1}^{(2)} & \cdots & a_{k-1}^{(k)} \end{vmatrix} \neq 0, \quad (69)$$

then $a_n = c_1 a_n^{(1)} + c_2 a_n^{(2)} + \cdots + c_k a_n^{(k)}$ is the general solution, where c_1, c_2, \dots, c_k are arbitrary constants.^a

^aSamuel Goldberg, *Introduction to Difference Equations* (1986).

Fundamental Sets

- The particular solutions of Eq. (68) on p. 510,

$$a_n^{(1)}, a_n^{(2)}, \dots, a_n^{(k)},$$

that also satisfy inequality (69) in Theorem 83 (p. 513) are said to form a **fundamental set of solutions**.

- Solving a linear homogeneous recurrence equation thus reduces to finding a fundamental set!

k th-Order Linear Homogeneous Recurrence Relations with Constant Coefficients: Distinct Roots

- Let r_1, r_2, \dots, r_k be the (characteristic) roots of the **characteristic equation**

$$C_n x^k + C_{n-1} x^{k-1} + \dots + C_{n-k} = 0. \quad (70)$$

- If r_1, r_2, \dots, r_k are distinct, then the general solution has the form

$$a_n = c_1 r_1^n + c_2 r_2^n + \dots + c_k r_k^n,$$

for constants c_1, c_2, \dots, c_k determined by the initial conditions.

The Proof

- Assume a_n has the form cr^n for nonzero c and r .
- After substitution into recurrence equation (68) on p. 510, r satisfies characteristic equation (70).
- Let r_1, r_2, \dots, r_k be the k distinct (nonzero) roots.
- Hence $a_n = r_i^n$ is a solution for $1 \leq i \leq k$.
- Solutions r_i^n form a fundamental set because

The Proof (continued)

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_k \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{k-1} & r_2^{k-1} & \cdots & r_k^{k-1} \end{vmatrix} \neq 0.$$

- The $k \times k$ matrix is called a **Vandermonde matrix**, which is nonsingular whenever r_1, r_2, \dots, r_k are distinct.^a

^aThis is a standard result in linear algebra.

The Proof (concluded)

- Hence

$$a_n = c_1 r_1^n + c_2 r_2^n + \cdots + c_k r_k^n$$

is the general solution.

- The k coefficients c_1, c_2, \dots, c_k are determined uniquely by the k initial conditions a_0, a_1, \dots, a_{k-1} :

$$\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_k \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{k-1} & r_2^{k-1} & \cdots & r_k^{k-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}. \quad (71)$$

The Fibonacci Relation

- Consider $a_{n+2} = a_{n+1} + a_n$.
- The initial conditions are $a_0 = 0$ and $a_1 = 1$.^a
- The characteristic equation is $r^2 - r - 1 = 0$, with two roots $(1 \pm \sqrt{5})/2$.
- The fundamental set is hence

$$\left\{ \left(\frac{1 + \sqrt{5}}{2} \right)^n, \left(\frac{1 - \sqrt{5}}{2} \right)^n \right\}.$$

^aSo a_n can be calculated with $O(n)$ operations.

The Fibonacci Relation (continued)

- For example, $\left(\frac{1+\sqrt{5}}{2}\right)^n$ satisfies the Fibonacci relation, as

$$\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} = \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} + \left(\frac{1+\sqrt{5}}{2}\right)^n.$$

- The general solution is hence

$$a_n = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n. \quad (72)$$

The Fibonacci Relation (concluded)

- Solve

$$0 = a_0 = c_1 + c_2$$

$$1 = a_1 = c_1 \frac{1 + \sqrt{5}}{2} + c_2 \frac{1 - \sqrt{5}}{2}$$

for $c_1 = 1/\sqrt{5}$ and $c_2 = -1/\sqrt{5}$.

- The particular solution is finally

$$a_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n,$$

known as the **Binet formula**.^a

^aSo a_n can now be calculated with $O(\log n)$ operations!

Don't Believe It?

$$\begin{aligned}a_2 &= \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^2 - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^2 \\&= \frac{1}{\sqrt{5}} \frac{1 + 2\sqrt{5} + 5}{4} - \frac{1}{\sqrt{5}} \frac{1 - 2\sqrt{5} + 5}{4} = 1.\end{aligned}$$

$$\begin{aligned}a_3 &= \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^3 - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^3 \\&= \frac{1}{\sqrt{5}} \frac{1 + 3\sqrt{5} + 15 + 5\sqrt{5}}{8} - \frac{1}{\sqrt{5}} \frac{1 - 3\sqrt{5} + 15 - 5\sqrt{5}}{8} \\&= 2.\end{aligned}$$

Initial Conditions

- Different initial conditions give rise to different solutions.
- Suppose $a_0 = 1$ and $a_1 = 2$.
- Then solve

$$\begin{aligned} 1 &= a_0 = c_1 + c_2, \\ 2 &= a_1 = c_1 \frac{1 + \sqrt{5}}{2} + c_2 \frac{1 - \sqrt{5}}{2}, \end{aligned}$$

for $c_1 = [(1 + \sqrt{5})/2]^2/\sqrt{5}$ and
 $c_2 = -[(1 - \sqrt{5})/2]^2/\sqrt{5}$ to obtain

$$a_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^{n+2} - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^{n+2}. \quad (73)$$

Initial Conditions (concluded)

- Suppose $a_0 = a_1 = 1$ instead.
- Then solve

$$\begin{aligned}1 &= a_0 = c_1 + c_2, \\1 &= a_1 = c_1 \frac{1 + \sqrt{5}}{2} + c_2 \frac{1 - \sqrt{5}}{2},\end{aligned}$$

for $c_1 = [(1 + \sqrt{5})/2]/\sqrt{5}$ and $c_2 = -[(1 - \sqrt{5})/2]/\sqrt{5}$ to obtain

$$a_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^{n+1} - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^{n+1}. \quad (74)$$

Generating Function for the Fibonacci Numbers

- From $a_{n+2} = a_{n+1} + a_n$, we obtain

$$\sum_{n=0}^{\infty} a_{n+2} x^{n+2} = \sum_{n=0}^{\infty} (a_{n+1} x^{n+2} + a_n x^{n+2}).$$

- Let $f(x)$ be the generating function for $\{a_n\}_{n=0,1,2,\dots}$.
- Then

$$f(x) - a_0 - a_1 x = x[f(x) - a_0] + x^2 f(x).$$

- Hence

$$f(x) = \frac{-a_0 x + a_0 + a_1 x}{1 - x - x^2}. \quad (75)$$

A Formula for the Fibonacci Numbers

$$a_n = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots + \binom{n - \lceil n/2 \rceil}{\lceil n/2 \rceil - 1}.$$

- From Eq. (75) on p. 525, the generating function is^a

$$\begin{aligned} & \frac{-a_0x + a_0 + a_1x}{1 - x - x^2} \\ = & \frac{x}{1 - x(1 + x)} \\ = & x + x^2(1 + x) + x^3(1 + x)^2 + \cdots \\ & + x^{n-1}(1 + x)^{n-2} + x^n(1 + x)^{n-1} + \cdots \\ = & \cdots + \left[\binom{n - \lceil n/2 \rceil}{\lceil n/2 \rceil - 1} + \cdots + \binom{n-2}{1} + \binom{n-1}{0} \right] x^n + \cdots . \end{aligned}$$

^aRecall that $a_0 = 0$ and $a_1 = 1$.

Number of Binary Sequences without Consecutive 0s

- Let a_n denote the number of binary sequences of length n without consecutive 0s.
- There are a_{n-1} valid sequences with the n th symbol being 1.
- There are a_{n-2} valid sequences with the n th symbol being 0 because any such sequence must end with 10.
- Hence $a_n = a_{n-1} + a_{n-2}$, a Fibonacci sequence.
- Because $a_1 = 2$ and $a_2 = 3$, we must have $a_0 = 1$ to **retrofit** the Fibonacci sequence.
- The formula is Eq. (73) on p. 523.

Number of Subsets without Consecutive Numbers

- A binary sequences $b_1 b_2 \cdots b_n$ of length n can be interpreted as the set $\{i : b_i = 0\} \subseteq \{1, 2, \dots, n\}$.
- Hence there are a_n subsets of $\{1, 2, \dots, n\}$ that contain no 2 consecutive integers, where
 - a_n is the Fibonacci number with $a_0 = 1$ and $a_1 = 2$ (formula is Eq. (73) on p. 523).

- It can be shown that

$$a_n = \binom{n+1}{0} + \binom{n}{1} + \cdots + \binom{n - \lceil n/2 \rceil + 1}{\lceil n/2 \rceil}.$$

- This formula can also be proved by Eq. (11) on p. 77.

Number of Subsets without Consecutive Numbers (continued)

- How many subsets of $\{1, 2, \dots, n\}$ contain no 2 consecutive integers when 1 and n are considered consecutive?
- Let a_n be the solution on p. 528.
- So a_n is the Fibonacci number with $a_0 = 1$ and $a_1 = 2$ (formula is Eq. (73) on p. 523).
- Now assume $n \geq 3$.
- There are a_{n-1} acceptable subsets that do not contain n .

Number of Subsets without Consecutive Numbers (continued)

- If n is included, an acceptable subset cannot contain 1 or $n - 1$.
- Hence there are a_{n-3} such subsets.
- The total is therefore $L_n \equiv a_{n-1} + a_{n-3}$, the **Lucas number**.^a
- It can be easily checked that

$$\begin{aligned} L_n &= a_{n-1} + a_{n-3} \\ &= a_{n-2} + a_{n-3} + a_{n-4} + a_{n-5} \\ &= L_{n-1} + L_{n-2}. \end{aligned}$$

^aCorrected by Mr. Gong-Ching Lin (B00703082) on May 19, 2012.

Number of Subsets without Consecutive Numbers (continued)

- Furthermore, $L_0 = 2$ and $L_1 = 1$.
 - $L_3 = a_2 + a_0 = 3 + 1 = 4$ and
 $L_4 = a_3 + a_1 = 5 + 2 = 7$.
 - So

$$L_2 = L_4 - L_3 = 3,$$

$$L_1 = L_3 - L_2 = 1,$$

$$L_0 = L_2 - L_1 = 2.$$

Number of Subsets without Consecutive Numbers (continued)

- The general solution is

$$L_n = c_1 \left(\frac{1 + \sqrt{5}}{2} \right)^n + c_2 \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

by Eq. (72) on p. 520.

- Solve

$$2 = L_0 = c_1 + c_2,$$

$$1 = L_1 = c_1 \frac{1 + \sqrt{5}}{2} + c_2 \frac{1 - \sqrt{5}}{2},$$

for $c_1 = 1$ and $c_2 = 1$.

Number of Subsets without Consecutive Numbers (concluded)

- The solution is finally

$$L_n = \left(\frac{1 + \sqrt{5}}{2} \right)^n + \left(\frac{1 - \sqrt{5}}{2} \right)^n .$$

Number of Palindromes Revisited

- A palindrome is a composition for $m \in \mathbb{Z}^+$ that reads the same left to right as right to left (p. 84).
- Let a_n denote the number of palindromes for n .
- Clearly, $a_1 = 1$ and $a_2 = 2$.
- Given each palindrome for $n - 2$, we can do two things.
 - Add 1 to the first and last summands to obtain a palindrome for n .
 - Insert summand 1 to the start and end to obtain a palindrome for n .
- Hence $a_{n+2} = 2a_n$, $n \geq 1$.

The Proof (continued)

- The characteristic equation $r^2 - 2 = 0$ has two roots $\pm\sqrt{2}$.
- The general solution is hence

$$a_n = c_1(\sqrt{2})^n + c_2(-\sqrt{2})^n.$$

- Solve^a

$$1 = a_1 = \sqrt{2}(c_1 - c_2),$$

$$2 = a_2 = 2(c_1 + c_2),$$

for $c_1 = (1 + \frac{1}{\sqrt{2}})/2$ and $c_2 = (1 - \frac{1}{\sqrt{2}})/2$.

^aThis time, we are not retrofitting.

The Proof (concluded)

- The number of palindromes for n therefore equals

$$\begin{aligned} a_n &= \frac{1 + \frac{1}{\sqrt{2}}}{2} (\sqrt{2})^n + \frac{1 - \frac{1}{\sqrt{2}}}{2} (-\sqrt{2})^n \\ &= \begin{cases} \frac{1 + \frac{1}{\sqrt{2}}}{2} 2^{n/2} + \frac{1 - \frac{1}{\sqrt{2}}}{2} 2^{n/2} & \text{if } n \text{ is even,} \\ \frac{1 + \frac{1}{\sqrt{2}}}{2} \sqrt{2} 2^{(n-1)/2} - \frac{1 - \frac{1}{\sqrt{2}}}{2} \sqrt{2} 2^{(n-1)/2} & \text{if } n \text{ is odd,} \end{cases} \\ &= \begin{cases} 2^{n/2} & \text{if } n \text{ is even,} \\ 2^{(n-1)/2} & \text{if } n \text{ is odd,} \end{cases} \\ &= 2^{\lfloor n/2 \rfloor}. \end{aligned}$$

- This matches Theorem 22 (p. 85).

An Example: A Third-Order Relation

- Consider

$$2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n$$

with $a_0 = 0$, $a_1 = 1$, and $a_2 = 2$.

- The characteristic equation $2r^3 - r^2 - 2r + 1 = 0$ has three distinct real roots: 1, -1 , and 0.5 .
- The general solution is

$$\begin{aligned} a_n &= c_1 1^n + c_2 (-1)^n + c_3 (1/2)^n \\ &= c_1 + c_2 (-1)^n + c_3 (1/2)^n. \end{aligned}$$

An Example: A Third-Order Relation (concluded)

- Solve the three initial conditions with Eq. (71) on p. 518,

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0.5 \\ 1^2 & (-1)^2 & 0.5^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

- The solutions are $c_1 = 2.5$, $c_2 = 1/6$, and $c_3 = -8/3$.

The Case of Complex Roots

- Consider

$$a_n = 2(a_{n-1} - a_{n-2})$$

with $a_0 = 1$ and $a_1 = 2$.

- The characteristic equation $r^2 - 2r + 2 = 0$ has two distinct complex roots $1 \pm i$.
- The general solution is

$$a_n = c_1(1 + i)^n + c_2(1 - i)^n.$$

The Case of Complex Roots (concluded)

- Solve the two initial conditions for $c_1 = (1 - i)/2$ and $c_2 = (1 + i)/2$.
- The particular solution becomes^a

$$\begin{aligned}a_n &= (1 + i)^{n-1} + (1 - i)^{n-1} \\ &= (\sqrt{2})^n [\cos(n\pi/4) + \sin(n\pi/4)].\end{aligned}$$

^aAn equivalent one is $a_n = (\sqrt{2})^{n+1} \cos((n - 1)\pi/4)$ by Mr. Tunglin Wu (B00902040) on May 17, 2012.

k th-Order Linear Homogeneous Recurrence Relations with Constant Coefficients: Repeated Real Roots

- Consider the recurrence relation

$$C_n a_n + C_{n-1} a_{n-1} + \cdots + C_{n-k} a_{n-k} = 0,$$

where C_n, C_{n-1}, \dots are real constants, $C_n \neq 0$, $C_{n-k} \neq 0$.

- Let r be a characteristic root of **multiplicity** m , where $2 \leq m \leq k$, of the characteristic equation

$$f(x) = C_n x^k + C_{n-1} x^{k-1} + \cdots + C_{n-k} = 0.$$

- The general solution that involves r has the form

$$(A_0 + A_1 n + A_2 n^2 + \cdots + A_{m-1} n^{m-1}) r^n,$$

with A_0, A_1, \dots, A_{m-1} are constants to be determined.

The Proof

- If $f(x)$ has a root r of multiplicity m , then
 $f(r) = f'(r) = \cdots = f^{(m-1)}(r) = 0$.
- Because $r \neq 0$ is a root of multiplicity m ,

$$0 = r^{n-k} f(r),$$

$$0 = r(r^{n-k} f(r))',$$

$$0 = r(r(r^{n-k} f(r)))',$$

$$\vdots$$

$$0 = \overbrace{r(\cdots r(r(r^{n-k} f(r)))')'}^{m-1} \cdots'.$$

The Proof (continued)

- We differentiate and then multiply by r before iterating.
- These give

$$0 = C_n r^n + C_{n-1} r^{n-1} + \cdots + C_{n-k} r^{n-k},$$

$$0 = C_n n r^n + C_{n-1} (n-1) r^{n-1} + \cdots + C_{n-k} (n-k) r^{n-k},$$

$$0 = C_n n^2 r^n + C_{n-1} (n-1)^2 r^{n-1} + \cdots + C_{n-k} (n-k)^2 r^{n-k},$$

$$\vdots$$

- Now, $n^k r^n$ for $0 \leq k \leq m-1$ is indeed a solution for the k th row above:

$$0 = C_n n^k r^n + C_{n-1} (n-1)^k r^{n-1} + \cdots + C_{n-k} (n-k)^k r^{n-k}.$$

The Proof (continued)

- $r^n, nr^n, n^2r^n, \dots, n^{m-1}r^n$ form a fundamental set if

$$\begin{vmatrix} 1 & 0 & \dots & 0 \\ r & r & \dots & r \\ r^2 & 2r^2 & \dots & 2^{m-1}r^2 \\ \vdots & \vdots & \ddots & \vdots \\ r^{m-1} & (m-1)r^{m-1} & \dots & (m-1)^{m-1}r^{m-1} \end{vmatrix} \neq 0.$$

- But it is a Vandermonde matrix in disguise.

The Proof (concluded)

- Specifically, the determinant equals

$$(m-1)! r^{1+2+\dots+(m-1)} \\ \times \begin{vmatrix} 1 & 1 & \dots & 1 \\ 1 & 2 & \dots & 2^{m-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & (m-1) & \dots & (m-1)^{m-2} \end{vmatrix} \neq 0.$$

Nonhomogeneous Recurrence Relations

- Consider

$$C_n a_n + C_{n-1} a_{n-1} + \cdots + C_{n-k} a_{n-k} = f(n). \quad (76)$$

- If $a_n = a_{n-1} + f(n)$, then the solution is $a_n = a_0 + \sum_{i=1}^n f(i)$.
 - A closed-form formula exists if one for $\sum_{i=1}^n f(i)$ does.
- In general, no failure-free methods exist except for specific $f(n)$ s.
 - Consult pp. 441–2 of the textbook (4th ed.).

Examples (c, c_1, c_2, \dots Are Arbitrary Constants)

$a_{n+1} - a_n = 0$	$a_n = c$
$a_{n+1} - a_n = 1$	$a_n = n + c$
$a_{n+1} - a_n = n$	$a_n = n(n-1)/2 + c$
$a_{n+2} - 3a_{n+1} + 2a_n = 0$	$a_n = c_1 + c_2 2^n$
$a_{n+2} - 3a_{n+1} + 2a_n = 1$	$a_n = c_1 + c_2 2^n - n$
$a_{n+2} - a_n = 0$	$a_n = c_1 + c_2 (-1)^n$
$a_{n+1} = a_n / (1 + a_n)$	$a_n = c / (1 + cn)$

Trial and Error

- Consider $a_{n+1} = 2a_n + 2^n$ with $a_1 = 1$.
- Calculations show that $a_2 = 4$ and $a_3 = 12$.
- Conjecture:

$$a_n = n2^{n-1}. \quad (77)$$

- Verify that, indeed,

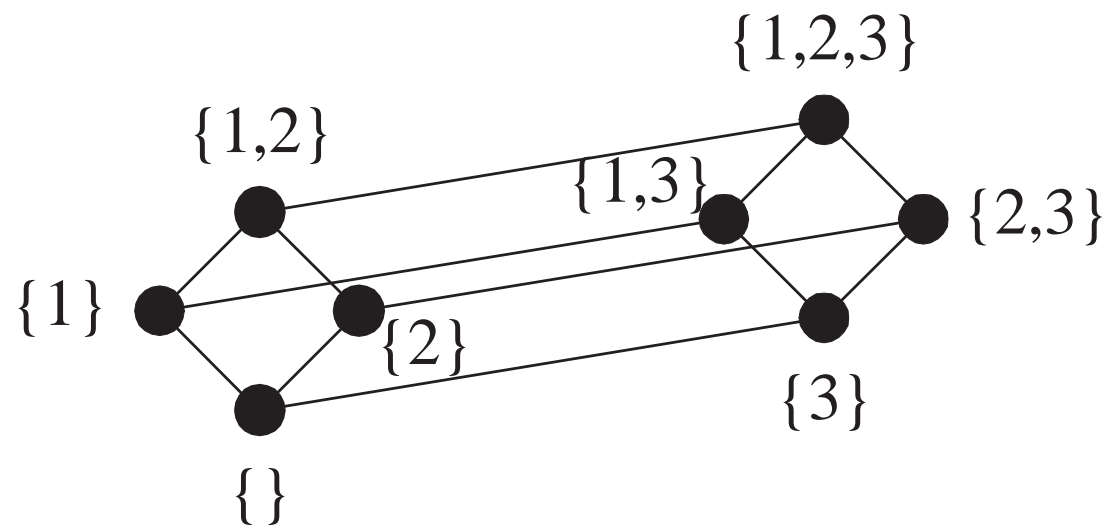
$$(n+1)2^n = 2(n2^{n-1}) + 2^n,$$

and $a_1 = 1$.

Application: Number of Edges of a Hasse Diagram

- Let a_n be the number of edges of the Hasse diagram for the partial order $(2^{\{1,2,\dots,n\}}, \subseteq)$.
- Consider the Hasse diagrams H_1 for $(2^{\{1,2,\dots,n\}}, \subseteq)$ and H_2 for $(\{T \cup \{n+1\} : T \subseteq \{1,2,\dots,n\}\}, \subseteq)$.
 - H_1 and H_2 are “isomorphic.”
- The Hasse diagram for $(2^{\{1,2,\dots,n+1\}}, \subseteq)$ is constructed by adding an edge from each node T of H_1 to node $T \cup \{n+1\}$ of H_2 .
- Hence $a_{n+1} = 2a_n + 2^n$ with $a_1 = 1$.
- The desired number has been solved in Eq. (77) on p. 548.

Illustration with $(2^{\{1,2,3\}}, \subseteq)$



Trial and Error Again

- Consider $a_{n+1} - Aa_n = B$.
- Calculations show that

$$a_1 = Aa_0 + B,$$

$$a_2 = Aa_1 + B = A^2a_0 + B(A + 1),$$

$$a_3 = Aa_2 + B = A^3a_0 + B(A^2 + A + 1).$$

- Conjecture (easily verified by substitution):

$$a_n = \begin{cases} A^n a_0 + B \frac{A^n - 1}{A - 1}, & \text{if } A \neq 1 \\ a_0 + Bn, & \text{if } A = 1 \end{cases}. \quad (78)$$

Financial Application: Compound Interest^a

- Consider $a_{n+1} = (1 + r) a_n$.
 - Deposit grows at a period interest rate of $r > 0$.
 - The initial deposit is a_0 dollars.
- By Eq. (78) on p. 551, the solution is

$$a_n = (1 + r)^n a_0.$$

- The deposit therefore grows exponentially with time.

^a“In the fifteenth century mathematics was mainly concerned with questions of commercial arithmetic and the problems of the architect,” wrote Joseph Alois Schumpeter (1883–1950) in *Capitalism, Socialism and Democracy*.

Financial Application: Amortization

- Consider $a_{n+1} = (1 + r) a_n - M$.
 - The initial loan amount is a_0 dollars.
 - The monthly payment is M dollars.
 - The outstanding loan principal after the n th payment is a_{n+1} .
- By Eq. (78) on p. 551, the solution is

$$a_n = (1 + r)^n a_0 - M \frac{(1 + r)^n - 1}{r}.$$

The Proof (concluded)

- What is the unique monthly payment M for the loan to be closed after k months?
- Set $a_k = 0$ to obtain

$$a_k = (1 + r)^k a_0 - M \frac{(1 + r)^k - 1}{r} = 0.$$

- Hence

$$M = \frac{(1 + r)^k a_0 r}{(1 + r)^k - 1}.$$

- This is standard calculation for home mortgages and annuities.^a

^aLyyu (2002).

Trial and Error a Third Time

- Consider the more general $a_{n+1} - Aa_n = BC^n$.
- Calculations show that

$$a_1 = Aa_0 + B,$$

$$a_2 = Aa_1 + BC = A^2a_0 + B(A + C),$$

$$a_3 = Aa_2 + BC^2 = A^3a_0 + B(A^2 + AC + C^2).$$

- Conjecture (easily verified by substitution):

$$a_n = \begin{cases} A^n a_0 + B \frac{A^n - C^n}{A - C} & \text{if } A \neq C \\ A^n a_0 + BA^{n-1}n & \text{if } A = C \end{cases}. \quad (79)$$

Application: Runs of Binary Strings

- A run is a maximal consecutive list of identical objects (p. 87).
 - Binary string “0 0 1 1 1 0” has 3 runs.
- Let r_n denote the total number of runs determined by the 2^n binary strings of length n .
- First, $r_1 = 2$.
 - Each of “0” and “1” has 1 run.
- In general, suppose we append a bit to an $(n - 1)$ -bit string $b_1b_2 \cdots b_{n-1}$ to make $b_1b_2 \cdots b_{n-1}b_n$.

The Proof (continued)

- For those with $b_{n-1} = b_n$, the total number of runs does not change.
 - The total number of runs remains r_{n-1} .
- For those with $b_{n-1} \neq b_n$, the total number of runs *increases* by 1 for each $(n-1)$ -bit string.
 - There are 2^{n-1} of them.
 - The total number of runs becomes $r_{n-1} + 2^{n-1}$.
- Hence

$$r_n = 2r_{n-1} + 2^{n-1}, n \geq 2.$$

The Proof (concluded)

- By Eq. (79) on p. 555,

$$r_n = 2^n r_0 + 2^{n-1} n.$$

- To make sure that $r_1 = 2$, it is easy to see that $r_0 = 1/2$.
- Hence

$$r_n = 2^{n-1} + 2^{n-1} n = 2^{n-1} (n + 1).$$

- The recurrence is identical to that for the number of edges of a Hasse diagram (p. 549) except for the initial condition, whose solution is in Eq. (77) on p. 548, $a_n = n2^{n-1}$.

Method of Undetermined Coefficients

- Recall Eq. (76) on p. 546:

$$C_n a_n + C_{n-1} a_{n-1} + \cdots + C_{n-k} a_{n-k} = f(n).$$

- Let $a_n^{(h)}$ denote the general solution of the associated homogeneous relation (with $f(n) = 0$).
- Let $a_n^{(p)}$ denote a particular solution of the nonhomogeneous relation.
- Then

$$a_n = a_n^{(h)} + a_n^{(p)}.$$

- All the entries in the table on p. 547 fit the claim.