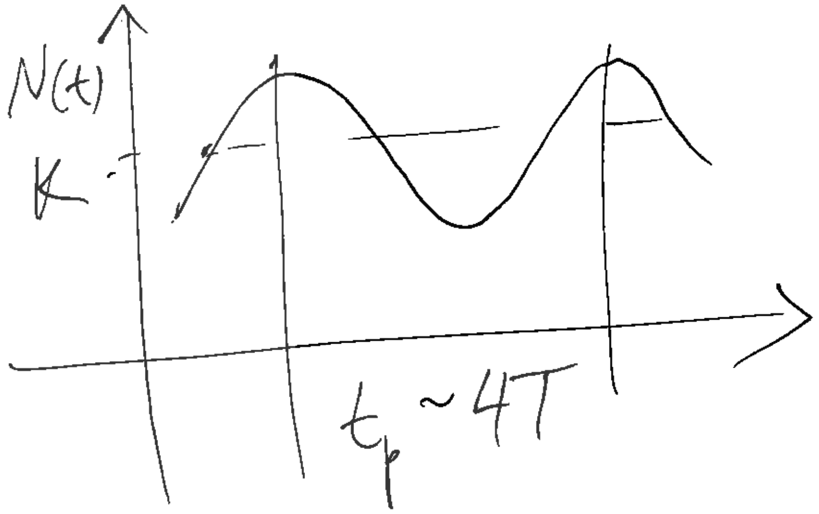
Lecture 4
10/10/2011

Models with delay

$$\frac{dN(t)}{dt} = r\,N(t)\underbrace{\left(1 - \frac{N(t-T)}{K}\right)}$$

T – 'delay'



$$u(t) = \frac{N(t)}{K} \qquad \tau = rt$$

$$\frac{dN/K}{d(tr)} = \left(\frac{N}{K}\right)\left(1 - \frac{N(t-T)}{K}\right)$$

$$\cancel{\frac{du}{d\tau} = u(t)\left(1 - \frac{u(t-T)}{\ }\right)}$$

$$\frac{du}{d\tau} = u(\tau)\left[1 - u(\tau - T^*)\right] \qquad \tau = rt \quad T^* = rT$$

$T^* = rT$	$\frac{N_{max}}{N_{min}} = \frac{u_{max}}{u_{min}}$	t_p
1.6	2.56	4.03T
2.1	42.3	4.54T
2.5	2930	5.36T

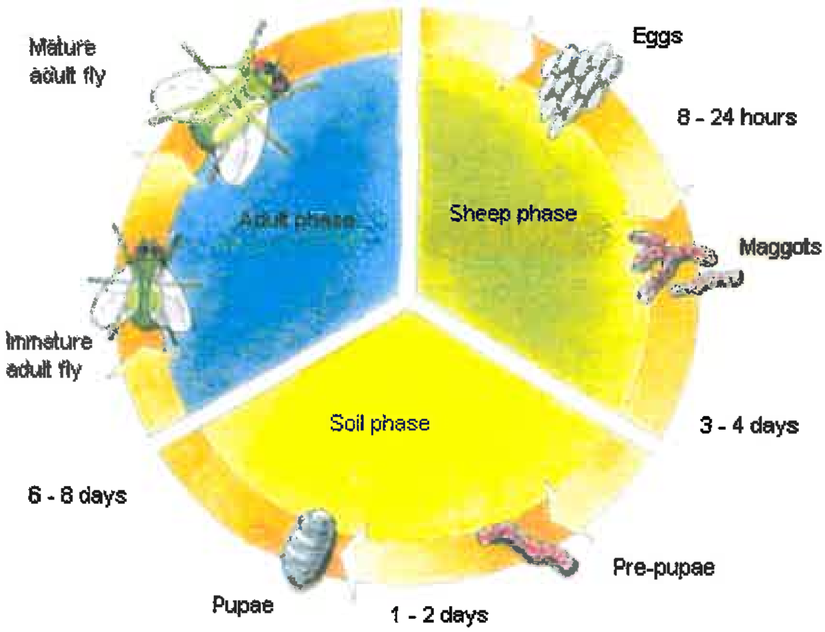
sheep - blowfly

$t_p = 40$ days

$$40 = 4.54\,T$$

$$\boxed{T \simeq 9 \text{ days}}$$

Models with $T = 0$ ~~can not~~ do not have periodic solutions.


Mature
adult fly
Eggs
8 - 24 hours
Adult phase
Sheep phase
Maggots
Immature
adult fly
Soil phase
3 - 4 days
6 - 8 days
Pre-pupae
Pupae
1 - 2 days

again approximately ... the effect of predation is incorporated into the single equation for the vole population.

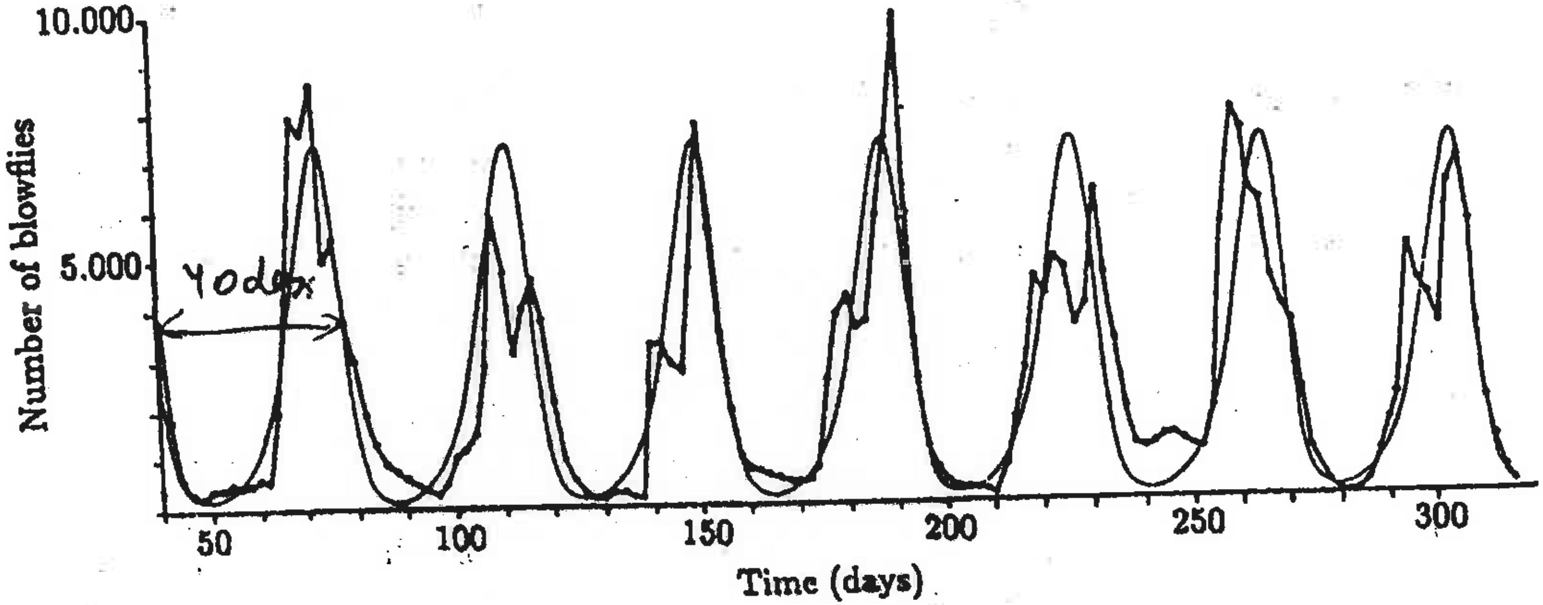
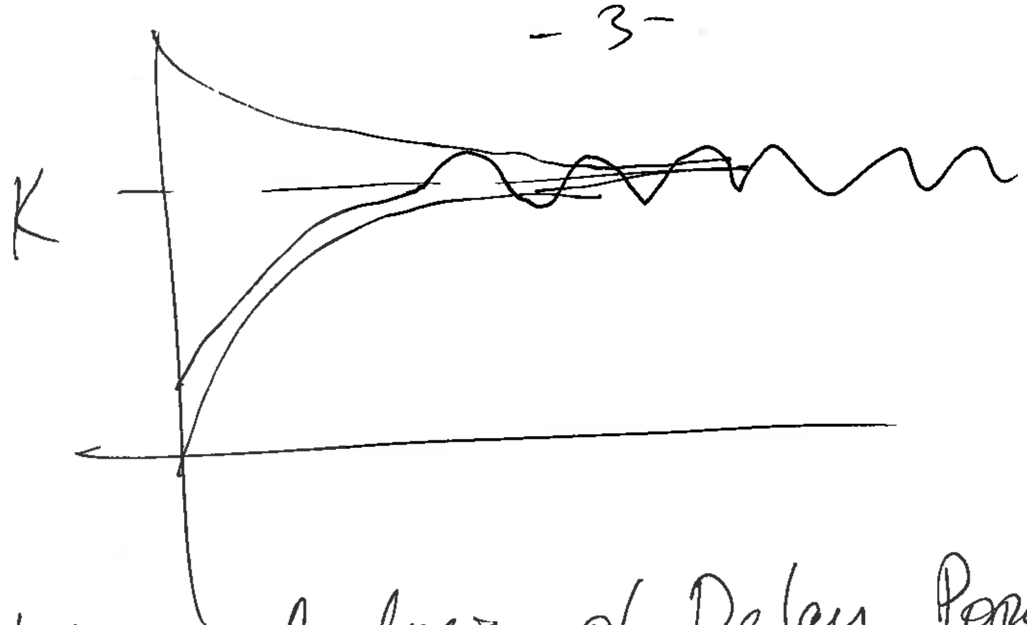


Figure 1.12. Comparison of Nicholson's (1957) experimental data for the population of the Australian sheep-blowfly and the model solution from (1.12) with $rT = 2.1$. (From May 1975).



Linear Analysis of Delay Population Models: Periodic Solutions

Instead of $u = \frac{N}{K} \rightarrow N$

$$\frac{dN}{dt} = N(t)\left[1 - N(t-T)\right]$$

Steady states $N=0$, $N=1$

Investigate $N = 1 + n(t)$, $|n| \ll 1$

$$\frac{dN}{dt} = \frac{dn}{dt}$$

$$\frac{dn}{dt} = \left[1 + n(t)\right]\left[\cancel{1} - (\cancel{1} + n(t-T))\right]$$

$$\frac{dn}{dt} = (1 + \underbrace{n(t))(-n}_{n^2 \approx 0}(t-T)) \approx -n(t-T)$$

$$\boxed{\frac{dn(t)}{dt} = -n(t-T)}$$ Linearised

$$n(t) = c e^{\lambda t} \qquad c, \lambda \text{ - constants}$$

$$\frac{dn}{dt} = c \lambda e^{\lambda t}$$

$$n(t-T) = c e^{\lambda(t-T)}$$

$$\frac{dn}{dt} = -n(t-T) \iff \cancel{c} \lambda e^{\lambda t} = -\cancel{c} e^{\lambda(t-T)}$$

$$\lambda \cancel{e^{\lambda t}} = -\cancel{e^{\lambda t}} e^{-\lambda T}$$

$$(*) \quad \boxed{\lambda = -e^{-\lambda T}} \qquad \text{transcendental eqn for } \lambda$$

$$z = \frac{1}{\lambda}$$

$$w(z) = 1 \cancel{+} \frac{1}{\lambda} e^{-T/(1/\lambda)}$$

$$w(z) = 1 + z e^{-T/z} = 0$$

$\frac{1}{z}$ is 'essential' singularity for $w(z)$

Picard's theorem $\Rightarrow$ $w(z)$ infinitely many complex roots.

$$\lambda = \mu + i\omega \qquad \boxed{e^{ix} = \cos x + i \sin x}$$

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$i = \sqrt{-1} \quad i^2 = -1 \quad i^3 = -i, \quad i^4 = 1, \quad i^5 = i$$

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

$$\boxed{e^{ix} = \cos x + i \sin x} \qquad (1)$$

$$e^{-ix} = \cos(-x) + i \sin(-x) = \cos x - i \sin x$$

$$\boxed{e^{-ix} = \cos x - i \sin x} \qquad (2)$$

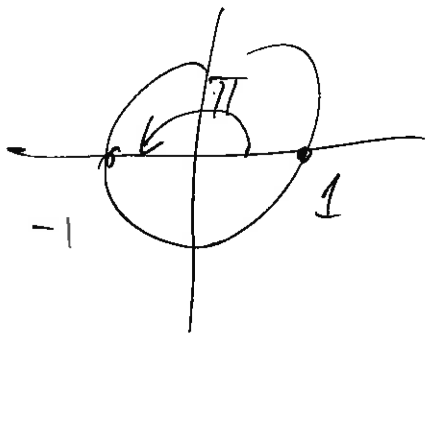
$$(1) + (2) \Rightarrow \quad e^{ix} + e^{-ix} = 2 \cos x$$

$$\boxed{\cos x = \frac{e^{ix} + e^{-ix}}{2}} \qquad \cosh x \equiv \frac{e^{x} + e^{-x}}{2}$$

$$(1) - (2) \Rightarrow \quad e^{ix} - e^{-ix} = 2i \sin x$$

$$\boxed{\sin x = \frac{e^{ix} - e^{-ix}}{2i}} \qquad \sinh x = \frac{e^{x} - e^{-x}}{2}$$

Take $x = \pi$ $\quad e^{i\pi} = \underbrace{\cos \pi}_{-1} + i \underbrace{\sin \pi}_{0}$



$$e^{i\pi} = -1$$

$$\boxed{e^{i\pi} + 1 = 0}$$

$i, \pi, e, 1, 0$

$$\lambda = -e^{-\lambda T} \qquad (1.17)$$

$$\lambda = \mu + i\omega$$

$$\mu + i\omega = -e^{-(\mu+i\omega)T} = -e^{-\mu T} \cdot e^{-i\omega T}$$

$$\mu + i\omega = -e^{-\mu T}\left[\cos(-\omega T) + i\sin(-\omega T)\right]$$

$$\mu + i\omega = -e^{-\mu T}(\cos\omega T - i\sin\omega T)$$

$$\underline{\mu} + i\omega = \underline{-e^{-\mu T}\cos\omega T} + i e^{-\mu T}\sin\omega T$$

$$\left|\begin{array}{l} \mu = -e^{-\mu T}\cos\omega T \\ \omega = e^{-\mu T}\sin\omega T \end{array}\right. \qquad (1.18)$$

$$\mu(T), \quad \omega(T)$$

$$\mu(T_c) = 0,$$

Observation $\omega \to -\omega$, $\underline{\omega \geq 0}$

$$\underline{\mu = 0}$$

$$\left\|\begin{array}{l} 0 = -\cos\omega T_c \\ \omega = \sin\omega T_c \end{array}\right.$$

smallest solution $\omega T_c = \pi/2$

$\omega = \sin\pi/2 = 1$, $\underline{\omega = 1}$ $\Rightarrow$ $1 . T_c = \frac{\pi}{2}$, $\boxed{T_c = \frac{\pi}{2}}$

$T_c = \frac{\pi}{2}$ is a bifurcation value.

~~when~~ $0 < T < \frac{\pi}{2}$ condition for stability

Back to physical variables $T^* = r\pi$

$$0 < rT < \frac{\pi}{2}$$

at the critical point ~~$\mu = 0$~~

$$\lambda = \mu + i\omega = 0 + i1 = i$$

$$N = 1 + n = 1 + ce^{\lambda t} = 1 + c\, e^{it} =$$

$$\boxed{N = 1 + c \cos t}$$ take real part.

$$t_p = 2\pi = 4\left(\frac{\pi}{2}\right) = 4T_c.$$

$$1 + |c| e^{i\gamma} e^{it} +$$

$$N = 1 + ce^{it} + (\text{c.c.})$$

$$T_c = \frac{\pi}{2}, \quad T = \frac{\pi}{2} + \varepsilon \qquad 0 < \varepsilon \ll 1$$

$$\mu = \mu(T), \quad \omega = \omega(T)$$

$$\mu = 0 + \delta, \quad \omega = 1 + \sigma, \quad 0 < \delta \ll 1, \quad |\sigma| \ll 1$$

$$e^x = 1 + \underbrace{x + \frac{x^2}{2!} + \cdots}_{\text{neglect}} \approx 1 + x \quad , \quad x \ll 1$$

$$\sin x = x \underbrace{- \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots}_{\text{neglect}} \approx x \quad , \quad x \ll 1$$

$$\cos x = 1 \underbrace{- \frac{x^2}{2!} + \frac{x^4}{4!}}_{\text{small}} \approx 1 \quad , \quad x \ll 1$$

$$\omega = e^{-\mu T} \sin \omega T$$

$$1 + \sigma = e^{-\delta\left(\frac{\pi}{2} + \varepsilon\right)} \sin\left[(1+\sigma)\left(\frac{\pi}{2} + \varepsilon\right)\right] \quad , \quad \delta\varepsilon \approx 0$$

$$1 + \sigma = e^{-\delta\frac{\pi}{2}} \sin\left(\frac{\pi}{2} + \sigma\frac{\pi}{2} + \varepsilon + \cancel{\sigma\varepsilon}\right)$$

$$1 + \sigma = \left(1 - \delta\frac{\pi}{2}\right)\left[\underset{\underset{1}{\parallel}}{\sin\frac{\pi}{2}} \underbrace{\cos\left(\sigma\frac{\pi}{2} + \varepsilon\right)}_{1} + \underset{\underset{0}{\parallel}}{\cos\frac{\pi}{2}} \sin\left(\sigma\frac{\pi}{2} + \varepsilon\right)\right]$$

$$\cancel{1} + \sigma = \cancel{1} - \delta\frac{\pi}{2} \qquad \boxed{\sigma = -\delta\frac{\pi}{2}}$$

$$\mu = -e^{-\mu T} \cos \omega T$$

$$0 + \delta = -e^{-(0+\delta)\left(\frac{\pi}{2} + \varepsilon\right)} \cos\left(\frac{\pi}{2} + \sigma\frac{\pi}{2} + \varepsilon\right)$$

$$\delta = -e^{-\delta\frac{\pi}{2}}\left[\underset{\underset{0}{\parallel}}{\cos\frac{\pi}{2}} \cos\left(\sigma\frac{\pi}{2} + \varepsilon\right) - \underset{\underset{1}{\parallel}}{\sin\frac{\pi}{2}} \sin\left(\sigma\frac{\pi}{2} + \varepsilon\right)\right]$$

$$\delta = -\left(1-\delta\frac{\pi}{2}\right)\left[0 - \sin\left(\sigma\frac{\pi}{2}+\varepsilon\right)\right]$$

$$\delta = \left(1-\delta\frac{\pi}{2}\right)\left(\sigma\frac{\pi}{2}+\varepsilon\right) = \sigma\frac{\pi}{2}+\varepsilon + \cancel{\text{higher order}}$$

$$\boxed{\delta \approx \varepsilon + \frac{\sigma\pi}{2}}$$

$$\boxed{\sigma = -\delta\frac{\pi}{2}}$$

$$\delta = \varepsilon + \left(-\delta\frac{\pi}{2}\right)\frac{\pi}{2} \Rightarrow \delta\left(1+\frac{\pi^2}{4}\right) = \varepsilon$$

$$\boxed{\delta = \frac{\varepsilon}{1+\frac{\pi^2}{4}}} \qquad \boxed{\sigma = -\frac{\varepsilon\pi}{2\left(1+\frac{\pi^2}{4}\right)}}$$

$$T = \frac{\pi}{2}+\varepsilon$$

$$\mu = 0 + \frac{\varepsilon}{1+\frac{\pi^2}{4}} \qquad \omega = 1+\sigma = 1 - \frac{\varepsilon\pi}{2\left(1+\frac{\pi^2}{4}\right)}$$

$$N(t) = 1 + \text{Real}\left[c\, e^{(\mu+i\omega)t}\right] =$$

$$= 1 + \text{Real}\left[c\, e^{\frac{\varepsilon t}{1+\frac{\pi^2}{4}}}\, e^{it\left(1-\frac{\varepsilon\pi}{2\left(1+\frac{\pi^2}{4}\right)}\right)}\right]$$

$$= 1 + c\, e^{\frac{\varepsilon t}{1+\frac{\pi^2}{4}}} \cos\left[t\left(1-\frac{\varepsilon\pi}{2\left(1+\frac{\pi^2}{4}\right)}\right)\right]$$

$$t_p\left(1-\frac{\varepsilon\pi}{2\left(1+\frac{\pi^2}{4}\right)}\right) = 2\pi$$

$$t_p = \frac{2\pi}{1 - \frac{\varepsilon\pi}{2\left(1+\frac{\pi^2}{4}\right)}} \longrightarrow 2\pi \quad \text{when } \varepsilon = 0$$

Recall: $T = \frac{\pi}{2} + \varepsilon$

$T = 1.6$ $\qquad \boxed{t_p = 4.03\,T}$

$$\varepsilon = 1.6 - \frac{\pi}{2} = 0.029 \ll 1.$$

$$t_p = \frac{2\pi}{1 - \frac{0.029\pi}{2\left(1+\frac{\pi^2}{4}\right)}} \simeq (4.05)\underbrace{(1.6)}_{T}$$

If $T = 2.1$, $\varepsilon = 0.53$ (not small)

$$t_p = (5.26) \times \underbrace{(2.1)}_{T} = \underline{5.26}\ T$$

Numerically $\quad t_p = \underline{4.54T}$

$$f(x+y) = f(x)f(y) \qquad f(x) = e^x$$

$$f'(x+y) = f(x)f(y)$$

Liapunov function technique for necessary conditions for stability

If y_s is a steady state then $L[y]$ is a Liapunov function if

1) $L[y(t)] > 0$ for all $y(t) \neq y_s$
2) $L[y_s] = 0$
3) $\frac{dL[y(t)]}{dt} < 0$ for all $y(t) \neq y_s$

Then y_s is globally asymptotically stable.

~~And two~~

$$\boxed{\frac{dy}{dt} = a y(t) + b y(t-\tau)}, \quad t > 0$$

$$L[y(t)] = y^2(t) + |b| \int_{t-\tau}^{t} y^2(s)\, ds$$

$$\frac{dL}{dt} = 2y(t)\frac{dy}{dt} + |b|\left[y^2(t) - y^2(t-\tau)\right] =$$

$$= 2y(t)\left[a y(t) + b y(t-\tau)\right] + |b|\left[y^2(t) - y^2(t-\tau)\right]$$

$$= 2a y^2(t) + \underline{2b y(t) y(t-\tau)} + |b|\left(y^2(t) - y^2(t-\tau)\right)$$

(?) $\underline{2b y(t) y(t-\tau)} \leq |b| \left(y^2(t) + y^2(t-\tau)\right)$

$0 \leq y^2(t) \pm 2y(t)y(t-\tau) + y^2(t-\tau)$

$0 \leq \left[y(t) \pm y(t-\tau)\right]^2$ ✓

$$\frac{dL}{dt} \leqq 2a y^2(t) + |b| \left(y^2(t) + \cancel{y^2(t-\tau)} \right) + |b| \left(y^2(t) - \cancel{y^2(t-\tau)} \right)$$

$$= 2(a+|b|) y^2(t) < 0$$

If $\underline{a < -|b|}$ $y_s = 0$ is stable.