(1)

Separation of Variables

Sometimes it is possible to write a solution function as a sum or product of functions of one variable.

e.g.

$$u(x,y) = X(x)\,Y(y)$$

or $$u(x,y) = X(x) + Y(y)$$

Not always possible!

To obtain conditions for $u(x,y) = X(x)Y(y)$ to be a solution of

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0$$

Assume $u(x,y) = X(x)Y(y)$ an insert into the above equation

Using $u_x = X'Y$ $\quad u_y = XY'$ etc

we obtain:

$$AX''Y + BX'Y' + CXY'' + DX'Y + EXY' + FXY = 0$$

Divide by XY

$$\Rightarrow A\frac{X''}{X} + B\frac{X'}{X}\frac{Y'}{Y} + C\frac{Y''}{Y} + D\frac{X'}{X} + E\frac{Y'}{Y} + F = 0 \quad (*)$$

Now differentiate across by x

$$\Rightarrow A\left(\frac{X''}{X}\right)' + B\left(\frac{X'}{X}\right)'\frac{Y'}{Y} + D\left(\frac{X'}{X}\right)' = 0$$

Hence moving terms in y to R.H.S

$$A\left(\frac{X''}{X}\right)' + D\left(\frac{X'}{X}\right)' = -B\left(\frac{X'}{X}\right)'\frac{y'}{y}$$

$$\Rightarrow \frac{A\left(\frac{X''}{X}\right)'}{B\left(\frac{X'}{X}\right)'} + \frac{D}{B} = -\frac{y'}{y}$$

L.H.S. depends only on x, RHS only on y
$\Rightarrow$ For them to be equal, they must both equal the same constant (λ say)
Hence

$$\frac{y'}{y} = -\lambda \quad \text{or} \quad y' + \lambda y = 0 \qquad (1)$$

and

$$\frac{A\left(\frac{X''}{X}\right)'}{B\left(\frac{X'}{X}\right)'} + \frac{D}{B} = \lambda$$

$$\Rightarrow \left(\frac{X''}{X'}\right)' + \left(\frac{D}{B} - \lambda\right)\frac{B}{A}\left(\frac{X'}{X}\right)' = 0$$

Integrating this w.r.t to x gives

$$\frac{X''}{X'} + \left(\frac{D}{B} - \lambda\right)\frac{B}{A}\left(\frac{X'}{X}\right) = -\beta$$

β - some constant.

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Now we substitute 1 into * to obtain

$$(2) \quad X'' + \left(\frac{D}{B} - \lambda\right)\left(\frac{B}{A}\right)X' + \left(\lambda^2 - \frac{E}{C}\lambda + \frac{F}{C}\right)\frac{C}{A}X = 0$$

For these to be consistent we require

$$\boxed{\beta = \left(\lambda^2 - \frac{E}{C}\lambda + \frac{F}{C}\right)\frac{C}{A}}$$

So $u(x,y) = X(x)Y(y)$ is a solution if $X(x)$ and $Y(x)$ satisfy 2 and 1 respectively

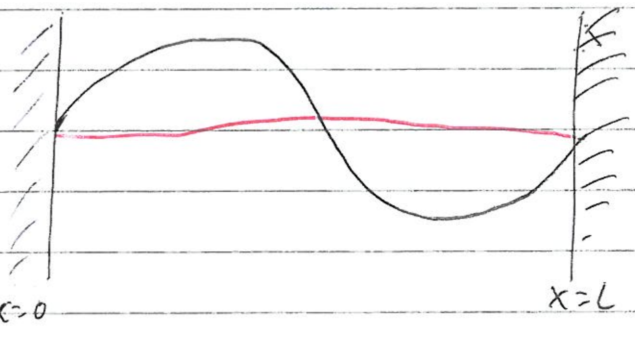
We may now obtain the solution to the 1-d wave equation

$$(*) \quad u_{tt} - c^2 u_{xx} = 0 \qquad 0 < x < L \qquad t > 0$$

$u(x,0) = f(x)$ Initial displacement

$u_t(x,0) = g(x)$ Initial velocity

$u(0,t) = u(L,t) = 0 \qquad t \geq 0$



Assume $u(x,t) = X(x)T(t) \neq 0$

Sub into * to obtain

$$XT'' = c^2 X'' T$$

Separation of Variables

Rearranging to obtain

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = \lambda$$

λ - constant

$\Rightarrow \frac{X''}{X} = \lambda$ and $\frac{1}{c^2}\frac{T''}{T} = \lambda$

or

A $X'' - \lambda X = 0$ B $T'' - c^2 \lambda T = 0$

We consider A first, using BCs

$u(0,t) = X(0)\,T(t) = 0 \Rightarrow X(0) = 0$

$u(L,t) = X(L)\,T(t) = 0 \Rightarrow X(L) = 0$

$X'' - \lambda x = 0$

IF $\lambda > 0 \Rightarrow A e^{\sqrt{\lambda}x} + B e^{-\sqrt{\lambda}x} = X(x)$

but $X(0) = 0$ and $X(L) = 0 \Rightarrow B = A = 0$

If $\lambda < 0 \Rightarrow X(x) = A\cos(\sqrt{-\lambda}\,x) + B\sin(\sqrt{-\lambda}\,x)$

$X(0) = 0 \Rightarrow A = 0$

$X(L) = 0 \Rightarrow \sin(\sqrt{-\lambda}\,L) = 0 \Rightarrow -\lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1,2,3\ldots$

Hence only non-trivial solutions are for $\lambda < 0$ with $\lambda_n = -\left(\frac{n\pi}{L}\right)^2$ with $n = 1,2,3\ldots$

Hence : $\boxed{X_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right) \quad n = 1,2,3}$

Separation of Variables

Returning to equation B, this is in the same form as A with $c^2 \lambda$ appearing in place of λ

$$\Rightarrow T(t) = C\cos(\sqrt{-\lambda c^2}\,t) + D\sin(\sqrt{-\lambda c^2}\,t)$$

but $\lambda_n = -\left(\frac{n\pi}{L}\right)^2 \quad n = 1, 2, 3 \ldots$

$$\Rightarrow T_n(t) = C_n\cos\left(\frac{n\pi ct}{L}\right) + D_n\sin\left(\frac{n\pi ct}{L}\right) \quad n = 1, 2, 3 \ldots$$

~~$\Rightarrow u_n(x,t) =$~~

General Solution $u(x,t)$ is sum of all possible n values

$$\Rightarrow \boxed{u(x,t) = \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi ct}{L}\right) + b_n\sin\left(\frac{n\pi ct}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right)}$$

with $a_n = C_n B_n$ and $b_n = D_n B_n$.

Hence using Initial conditions

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \qquad \boxed{C}$$

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} b_n\left(\frac{n\pi c}{L}\right)\sin\left(\frac{n\pi x}{L}\right) \qquad \boxed{D}$$

To proceed we make use of the identity

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} \frac{L}{2} & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

Hence ~~integ~~ multiplying both sides of C, D by

Separation of Variables:

$\sin\left(\frac{n\pi x}{L}\right)$ and integrating from 0 to L

gives:

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

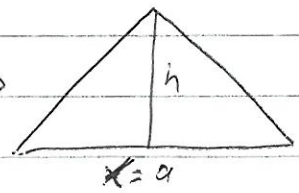
Example

Consider

$$f(x) = \begin{cases} hx/a & 0 \le x \le a \\ \frac{h(L-x)}{L-a} & a \le x \le L \end{cases}$$

$$g(x) = 0$$

"plucked string" $\longrightarrow$



$g(x) = 0 \Rightarrow b_n = 0$

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\left[f(x)\left(\frac{-L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)\right]_0^L$$

$f(0) = 0$

$f(L) = 0$

$$+ \frac{L}{n\pi}\int_0^L f'(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$f'(x) = \begin{cases} h/a & 0 \le x \le a \\ -h/L-a & a \le x \le L \end{cases}$$

$$\Rightarrow a_n = \frac{2}{n\pi}\int_0^a \frac{h}{a}\cos\left(\frac{n\pi x}{L}\right)dx + \frac{2}{n\pi}\int_a^L \frac{-h}{L-a}\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{n\pi}\left\{\left[\frac{h}{a}\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right]_0^a + \left[\frac{h}{a-L}\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right]_a^L\right.$$

$$\sin(0) = \sin(n\pi) = 0$$

$$\underline{a_n} = \frac{2hL^2}{(n\pi)^2 a(L-a)}\sin\left(\frac{n\pi a}{L}\right)$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty}\frac{2hL^2}{(n\pi)^2 a(L-a)}\sin\left(\frac{n\pi a}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

Separation of Variables
Uniqueness

Now that we have found a solution how do we know that it is unique:
Suppose we have two solutions u_1 and u_2 which solve

$$u_{tt} = c^2 u_{xx} \qquad 0 < x < L \qquad t > 0$$

with $\quad u(x,0) = f(x) \qquad u_t(x,0) = g(x) \qquad 0 \leq x \leq L$
and $\quad u(0,t) = 0 \qquad u(L,t) = 0 \qquad t \geq 0$

The define $V = u_1 - u_2$, it remains to show that $V(x,t) = 0$.

$$V_{xx} = u_{1xx} - u_{2xx} \qquad V_{tt} = u_{1tt} - u_{2tt}$$

$$\Rightarrow V_{tt} - c^2 V_{xx} = (u_{1tt} - c^2 u_{1xx}) + (u_{2tt} - c^2 u_{2xx}) = 0$$

Also $\quad V(x,0) = f(x) - f(x) = 0$
$\qquad V_t(x,0) = g(x) - g(x) = 0$

Similarly $\quad V(0,t) = V(L,t) = 0$

Now we consider the Integral

$$E(t) = \frac{1}{2} \int_{x=0}^{L} (\rho V_t^2 + T V_x^2)\, dx$$

which corresponds to total energy in the string.

$$E(t) = \frac{\rho}{2} \int_{x=0}^{L} (c^2 V_x^2 + V_t^2)\, dx$$

$$\Rightarrow \frac{dE}{dt} = \frac{\rho}{\cancel{2}} \int_0^L (c^2 V_x V_{xt} + V_t V_{tt})\, dx$$

I.B.P $\Rightarrow$
$$\int_0^L c^2 V_x V_{xt}\,dx = \left[c^2 V_x V_t\right]_0^L - c^2\int_0^L V_t V_{xx}\,dx$$

but $V(0,t) = 0 \Rightarrow V_t(0,t) = 0 \Rightarrow \left[c^2 V_x V_t\right]_0^L = 0$
$V(L,t) = 0 \Rightarrow V_t(L,t) = 0$

$$\Rightarrow \frac{dE}{dt} = \int_0^L V_t \underbrace{(V_{tt} - c^2 V_{xx})}_{=0}\,dx = 0$$

$\Rightarrow E(t) = C$ (constant)

Since $V(x,0) = 0 \Rightarrow V_x(x,0) = 0$ and $V_t(x,0) = 0$

$$\Rightarrow E(0) = \frac{\rho}{2}\int_0^L \left(c^2 V_x^2(x,0) + V_t^2(x,0)\right) dx = 0$$

but $E(t) = $ Constant $\Rightarrow E(t) = 0$ $\forall$ time.

$\Rightarrow V_x = 0$ and $V_t = 0$
$\Rightarrow V(x,t) = D$ - constant

but $V(x,0) = 0 \Rightarrow D = 0 \Rightarrow V(x,t) = 0$
$\Rightarrow u_1(x,t) = u_2(x,t)$

and the solution is unique.