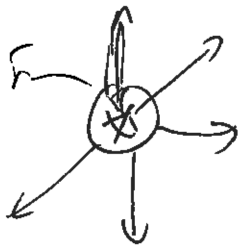
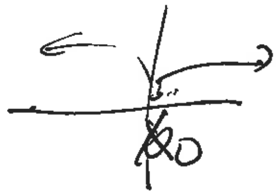
14/11/2011

$$\frac{\partial u}{\partial t} = u(1-u) + \frac{\partial^2 u}{\partial x^2} \qquad (1)$$

$u = 0$ steady state stable

$u = 1$ <u>unstable</u>





$z = x - ct$

$$\underline{U'' + cU' + U(1-U) = 0} \qquad \underline{U = U(z)}$$

$$\underline{c \geq 2}$$

$\varepsilon = \frac{1}{c^2} < \frac{1}{4} = 0.25$ is small parameter

$$U(z) = \frac{1}{1+e^{z/c}} + \frac{1}{c^2} \cdot \frac{e^{z/c}}{(1+e^{z/c})^2} \ln\left[\frac{4e^{z/c}}{(1+e^{z/c})^2}\right] + O\left(\frac{1}{c^4}\right)$$

$u(\infty) = 0$, $u(-\infty) = 1$, $U(0) = \frac{1}{2}$

$$\frac{\partial u}{\partial t} = u\,(1-u^q) + \frac{\partial^2 u}{\partial x^2} \qquad \underline{q > 0}$$

$$u(x,t) = U(z) \qquad z = x - ct$$

$$U(-\infty) = 1, \quad U(\infty) = 0$$

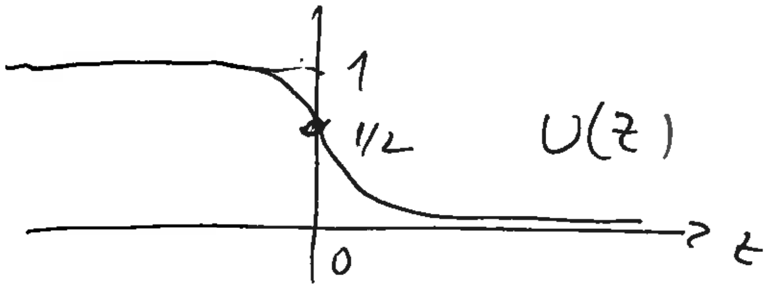
Exact solution $U(z) = \dfrac{1}{(1+ae^{bz})^s}$

$$s = \frac{2}{q}, \quad b = \frac{1}{\sqrt{2(q+2)}}, \quad c = \frac{q+4}{\sqrt{2(q+2)}}$$

When $q = 1$ Fisher – Kolmogoroff

$$s = 2 \quad b = \frac{1}{\sqrt{6}} \quad c = \frac{5}{\sqrt{6}} = 2.04 > 2$$

$\underline{U(0) = \frac{1}{2}}$



$$U(0) = \frac{1}{2} = \frac{1}{(1+a\cdot 1)^2} \Rightarrow \sqrt{2} = 1 + a$$

$$\boxed{a = \sqrt{2} - 1}$$

$$U(z) = \frac{1}{\left(1 + (\sqrt{2}-1)e^{z/\sqrt{6}}\right)^2}, \quad \boxed{z = x - \frac{5}{\sqrt{6}}t}$$

$$\frac{\partial u}{\partial t} = ru(1-u) + \frac{\partial}{\partial x}\left(D(u)\frac{\partial u}{\partial x}\right)$$

$$D = D_0 u^m, \quad D_0 > 0 \text{ is a constant}$$

$$\frac{\partial u}{\partial t} = ru(1-u) + D_0 \frac{\partial}{\partial x}\left(u^m \frac{\partial u}{\partial x}\right)$$

$$t \to rt \qquad x \to x\sqrt{\frac{r}{D_0}}$$

$$\frac{\partial u}{\partial t} = u(1-u) + \frac{\partial}{\partial x}\left(u^m \frac{\partial u}{\partial x}\right)$$

$m > 0$ is a constant

$\boxed{\text{Example: } m=1}$

$$\frac{\partial u}{\partial t} = u(1-u) + \frac{\partial}{\partial x}\left(u \frac{\partial u}{\partial x}\right)$$

two nonlinear terms.

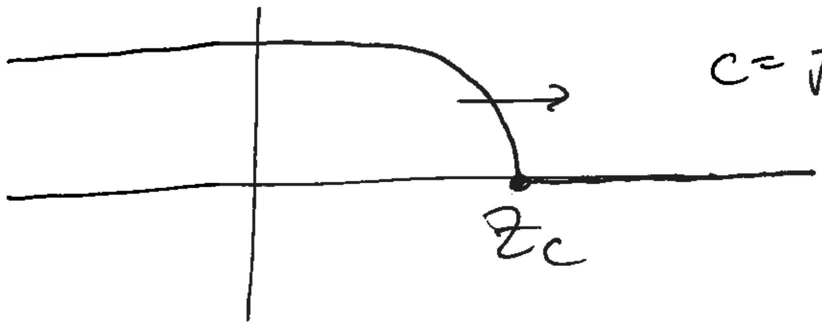
Travelling wave solution $U = U(z) = u(x,t)$

$$\frac{\partial u}{\partial t} = -cU' \qquad \frac{\partial u}{\partial x} = U', \quad z = x - ct$$

$$\boxed{(UU')' + cU' + U(1-U) = 0}$$

Exact travelling wave solution

$$U(z) = \left[1 - e^{\frac{z-z_c}{\sqrt{2}}}\right]\theta(z_c - z)$$



$$c = \frac{1}{\sqrt{2}} = 0.7071$$

$$\cancel{u =}\quad u(x,t) = \cancel{U_1(x-ct)} + U_2(x-ct)$$

$$U' = -\frac{1}{\sqrt{2}} e^{\frac{z-z_c}{\sqrt{2}}}\theta(z_c - z) + \left(1 - e^{\frac{z-z_c}{\sqrt{2}}}\right)\delta(z_c - z)(-1)$$

$$\underbrace{(1-1)}_{0} \leftarrow z = z_c$$

$$\boxed{U' = -\frac{1}{\sqrt{2}} e^{\frac{z-z_c}{\sqrt{2}}}\theta(z_c - z)}$$

$$UU' = \left(1 - e^{\frac{z-z_c}{\sqrt{2}}}\right)\theta(z_c - z)\left(-\frac{1}{\sqrt{2}}\right) e^{\frac{z-z_c}{\sqrt{2}}}\,\theta(z_c - c)$$

$$= \left(-\frac{1}{\sqrt{2}} e^{\frac{z-z_c}{\sqrt{2}}} + \frac{1}{\sqrt{2}} e^{2\frac{z-z_c}{\sqrt{2}}}\right)\theta(z_c - z)$$

$$(UU')' = \left(-\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} e^{\frac{z-z_c}{\sqrt{2}}} + \frac{1}{\sqrt{2}}\cdot\frac{2}{\sqrt{2}} e^{2\cdot\frac{z-z_c}{\sqrt{2}}}\right)\theta(z_c - z) +$$

$$\underbrace{\left(-\frac{1}{\sqrt{2}} e^{\frac{z-z_c}{\sqrt{2}}} + \frac{1}{\sqrt{2}} e^{2\left(\frac{z-z_c}{\sqrt{2}}\right)}\right)}_{0}\delta(z_c - z)(-1)$$

$$z = z_c$$

$$(uu')' = \left(-\frac{1}{2}e^{\frac{z-z_c}{\sqrt{2}}} + e^{2\frac{z-z_c}{\sqrt{2}}}\right)\theta(z_c - z)$$

$$c\,u' = -\frac{c}{\sqrt{2}}\,e^{\frac{z-z_c}{\sqrt{2}}}\,\theta(z_c - z)$$

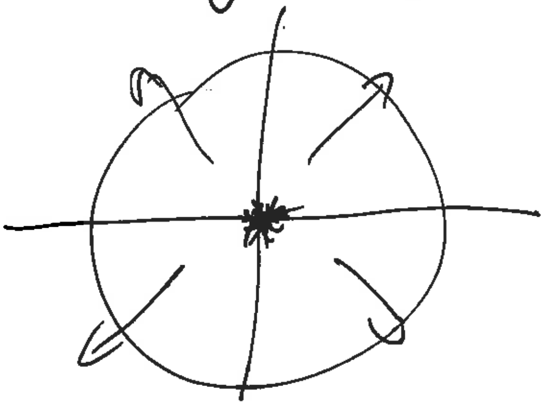
$$u(1-u) = \left(1 - e^{\frac{z-z_c}{\sqrt{2}}}\right)e^{\frac{z-z_c}{\sqrt{2}}}\,\theta(z_c - z)$$

$$(uu')' + cu' + u(1-u) =$$

$$\theta(z_c - z)\left\{-\frac{1}{2}e^{\frac{z-z_c}{\sqrt{2}}} + \cancel{e^{2\frac{z-z_c}{\sqrt{2}}}} - \frac{c}{\sqrt{2}}e^{\frac{z-z_c}{\sqrt{2}}} + e^{\frac{z-z_c}{\sqrt{2}}} - \cancel{e^{2\frac{z-z_c}{\sqrt{2}}}}\right\}$$

$$= \theta(z_c - z)\left(\frac{1}{2} - \frac{c}{\sqrt{2}}\right)e^{\frac{z-z_c}{\sqrt{2}}} = 0 \quad \text{when } c = \frac{1}{\sqrt{2}}$$

Radial symmetry



$$\frac{\partial u}{\partial t} = \underline{\underline{\varepsilon u}} + D\Delta u$$

$\boxed{\text{Linear!}}$

$$\Delta f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}}$$

$$\boxed{u(r,t) = \frac{N_0}{4\pi D t}\, e^{\varepsilon t - \frac{r^2}{4Dt}}}$$

$$\lim_{t\to 0} u(r,t) = N_0\,\delta(r) = \underline{u(r,0)}$$

$$\frac{\partial u}{\partial t} = -\frac{N_0}{4\pi D t^2}\, e^{\varepsilon t - \frac{r^2}{4Dt}} + \frac{N_0\left(\varepsilon + \frac{r^2}{4Dt^2}\right)}{4\pi D t}\, e^{\varepsilon t - \frac{r^2}{4Dt}}$$

$$\frac{\partial u}{\partial t} = \left(-\frac{1}{t} + \varepsilon + \frac{r^2}{4Dt^2}\right) u(x,t)$$

$$D\Delta u = D\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{D}{r}\frac{\partial}{\partial r}\left[r\,\frac{N}{4\pi D t}\, e^{\varepsilon t - \frac{r^2}{4Dt}}\left(-\frac{2r}{4Dt}\right)\right]$$

$$= \frac{N_0}{4\pi D t}\cdot\frac{D}{r}\frac{\partial}{\partial r}\left(-\frac{2r^2}{4Dt}\, e^{\varepsilon t - \frac{r^2}{4Dt}}\right) =$$

$$= \frac{N_0}{4\pi D t} \cdot \frac{D}{r}\left[-\frac{\cancel{4}r}{\cancel{4}Dt} - \frac{2r^2}{4Dt}\left(\frac{-2r}{4Dt}\right)\right] e^{\varepsilon t - \frac{r^2}{4Dt}}$$

$$= \left(-\frac{1}{t} + \frac{\cancel{D}}{\cancel{x}} \frac{\cancel{4} r^{\cancel{3}\,2}}{4^{\cancel{2}} D^{\cancel{2}} t^2}\right) u(x,t) =$$

$$= \left(-\frac{1}{t} + \frac{r^2}{4Dt^2}\right) u(x,t) = D\Delta u$$

$$\frac{\partial u}{\partial t} \overset{?}{=} \varepsilon u + D\Delta u \quad \text{- equation}$$

$$\left(\cancel{-\frac{1}{t}} \cancel{+\varepsilon} + \frac{r^2}{4Dt^2}\right) u \overset{?}{=} \cancel{\varepsilon u} + \left(\cancel{-\frac{1}{t}} + \frac{r^2}{4Dt^2}\right) \checkmark$$

u is <u>~~detectable~~</u> if $\boxed{u > u^*}$

$$u(r,t) = u^* \Rightarrow r(t)$$

$$u^* = \frac{N_0}{4\pi D t} \exp\left(\varepsilon t - \frac{r^2}{4Dt}\right)$$

$$\ln \frac{4\pi D t u^*}{N_0} = \varepsilon t - \frac{r^2}{4Dt}$$

$$r^2 = \varepsilon t \cdot 4Dt - 4Dt \ln \frac{4\pi D t u^*}{N_0}$$

$$r^2 = 4D\varepsilon t^2 - 4Dt \ln \frac{4\pi D u^* t}{N_0}$$

$$r(t) = 2t\left(D\varepsilon - \frac{D}{t}\ln\frac{4\pi D u^* t}{N_0}\right)^{1/2}$$

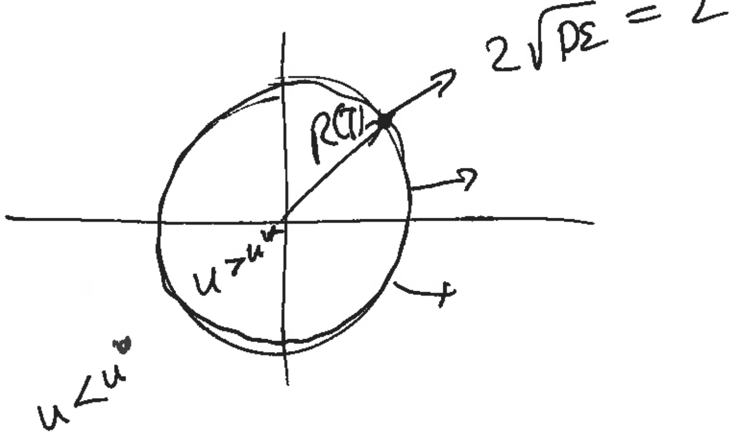
$$R(T) = 2T\left(D\varepsilon + \frac{D}{T}\cancel{\ln}\frac{N_0}{4\pi D u^* T}\right)^{1/2}$$

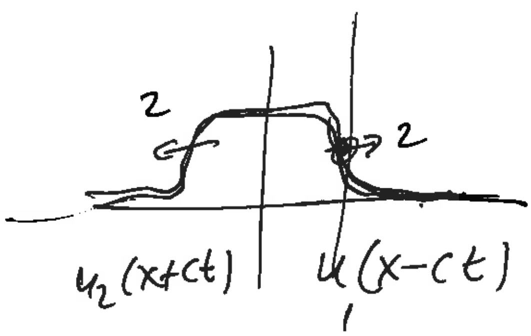
$$T \to \infty \qquad \frac{1}{T}\ln\frac{\gamma}{T} \to 0$$

$$\lim_{T\to\infty}\frac{\ln\frac{\gamma}{T}}{T} = \lim\frac{\left[\ln\left(\frac{\gamma}{T}\right)\right]'}{(T)'} = \frac{\frac{T}{\gamma}\left(-\frac{\gamma}{T^2}\right)}{1} = -\frac{1}{T} = 0$$

$$R(T) = 2T\sqrt{D\varepsilon}$$

$$C = \frac{R(T)}{T} \simeq 2\sqrt{D\varepsilon} \quad \text{wavespeed.}$$

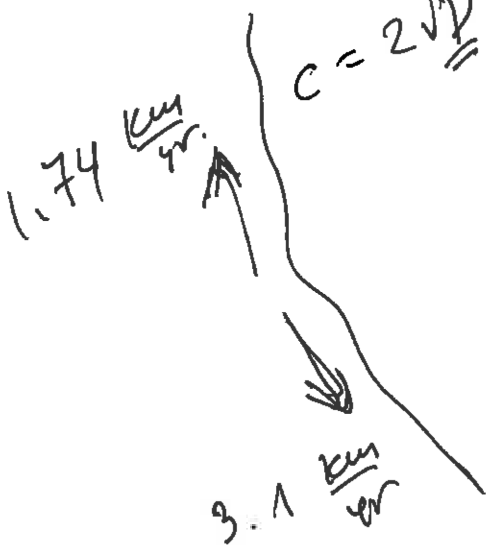




Ottep population central coast of California

'Big Sur'

$c = 2\sqrt{Dr}$ $\quad D = 13.5\ \mathrm{km}^2/\mathrm{yr}$



$D = 54.7\ \mathrm{km}^2/\mathrm{yr}, \quad r = 0.056\ \mathrm{yr}^{-1}$

<u>1938 – 1972</u>

Discrete Population Models for a Single Species (Chapter 2, vol. 1)

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

$\underline{\underline{N(t)}} \rightarrow N(1),\ N(2),\ N(3),\ \ldots$

$N_1,\ N_2,\ N_3,\ \ldots$

$$\frac{N(t+\Delta t) - N(t)}{\Delta t} \approx \frac{N_{t+1} - N_t}{1} \simeq N_{t+1} - N_t$$

$$N_{t+1} = f(N_t) = N_t F(N_t)$$

$$N_{t+1} - N_t = rN_t(1 - N_t)$$

$$N_{t+1} = N_t(1 + r - rN_t)$$

$$N_{t+1} = (1+r)N_t\Big(1 - \underbrace{\frac{r}{r+1}N_t}_{x_t}\Big)$$

$$x_t = \frac{r}{r+1}N_t, \quad r' = 1+r$$

$$\boxed{x_{t+1} = r' x_t (1 - x_t)}$$

$$N_{t+1} = r N_t$$ Simplest difference eq.

$$N_t = r^t N_0$$

$$N_{t+1} = r N_t = r(r N_{t-1}) = r^2 N_{t-1} = r^3 N_{t-2}$$

$|r| > 1$ growth
$|r| < 1$ decay

Fibonacci Sequence

(*) $$N_{t+1} = N_t + N_{t-1}$$ $N_0 = N_1 = 1$

1, 1, 2, 3, 5, 8, 13, . . .

linear difference equation

$$N_t = N_0 \lambda^t$$

$$\cancel{N_0} \lambda^{t+1} = \cancel{N_0} \lambda^t + \cancel{N_0} \lambda^{t-1} \quad \Big| \frac{1}{\lambda^{t-1}}$$

$$\boxed{\lambda^2 = \lambda + 1}$$ Characteristic equation

$$\lambda_{1,2} = \frac{1}{2}(1 \pm \sqrt{5})$$

$$\boxed{N_t = C_1 \lambda_1^t + C_2 \lambda_2^t} \qquad \lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}$$

$$N_0 = 1, \quad N_1 = 1$$

$$\boxed{t=0} \qquad 1 = C_1 \lambda_1^0 + C_2 \lambda_2^0 \Rightarrow \qquad C_1 + C_2 = 1$$

$$t = 1 \qquad 1 = C_1 \lambda_1 + C_2 \lambda_2 \qquad C_1 \frac{1+\sqrt{5}}{2} + C_2 \frac{1-\sqrt{5}}{2} = 1$$

$$\frac{C_1 + C_2}{2} + \frac{\sqrt{5}(C_1 - C_2)}{2} = 1$$

$$\frac{1}{2} + \frac{\sqrt{5}}{2}(C_1 - C_2) = 1 \Rightarrow \qquad C_1 - C_2 = \frac{1}{2} \cdot \frac{2}{\sqrt{5}} = \frac{1}{\sqrt{5}}$$

$$\left.\begin{aligned} C_1 + C_2 &= 1 \\ C_1 - C_2 &= \frac{1}{\sqrt{5}} \end{aligned}\right| \qquad 2C_1 = 1 + \frac{1}{\sqrt{5}}$$

$$C_1 = \frac{1}{2}\left(1 + \frac{1}{\sqrt{5}}\right), \qquad \underline{C_2 = \frac{1}{2}\left(1 - \frac{1}{\sqrt{5}}\right)}$$

$$N_t = \frac{1}{2}\left(1 + \frac{1}{\sqrt{5}}\right)\lambda_1^t + \frac{1}{2}\left(1 - \frac{1}{\sqrt{5}}\right)\lambda_2^t, \quad \lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}$$

$$\lambda_1 = \frac{1+\sqrt{5}}{2} > 1 \qquad |\lambda_2| = \left|\frac{1-\sqrt{5}}{2}\right| < 1, \quad \lambda_2^t \to \underline{\underline{0}}$$

For large t, $\boxed{N_t \approx \frac{1}{2}\left(1 + \frac{1}{\sqrt{5}}\right)\lambda_1^t}$

$$\lim \frac{N_{t+1}}{N_t} = \frac{\cancel{\frac{1}{2}\left(1+\frac{1}{\sqrt{5}}\right)} \lambda_1^{t+1}}{\cancel{\frac{1}{2}\left(1+\frac{1}{\sqrt{5}}\right)} \lambda_1^{t}} = \lambda_1 = \frac{1+\sqrt{5}}{2}$$

$$\frac{N_t}{N_{t+1}} \to \frac{1}{\lambda_1} = \frac{2}{1+\sqrt{5}} \cdot \frac{(\sqrt{5}-1)}{(\sqrt{5}-1)} = \frac{2(\sqrt{5}-1)}{5-1} =$$

$$\lim_{t\to\infty} \frac{N_t}{N_{t+1}} \longrightarrow \frac{\sqrt{5}-1}{2}$$

'golden mean'