Population genetics problem

Genotypes (genes) by alleles

AA   A - dominant

Aa } same
aA

aa   a - recessive

$Aa \times Aa \longrightarrow AA,\ Aa,\ aa$

$p_n(A)\ ,\ p_n(a)$

Mendel's laws of Inheritance

Mating	Offspring AA	Aa	aa
AA x AA	1	0	0
AA x Aa	$\frac{1}{2}$	$\frac{1}{2}$	0
AA x aa	0	1	0
Aa x Aa	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
Aa x aa	0	$\frac{1}{2}$	$\frac{1}{2}$
aa x aa	0	0	1

	A	a
A	AA	Aa
a	Aa	aa

$p_n = P_n(A)$

$q_n = 1 - p_n = P_n(a)$

Proportion of offspring gene pool of type A

$$p_n^2 + p_n q_n = p_n^2 - p_n(1-p_n)$$

$$P_{n+1}(A) = P_n(AA) + \frac{1}{2} P_n(Aa) =$$

$$= p_n \times p_n + \frac{1}{2}\, 2\, p_n \times q_n =$$

$$= p_n^2 + p_n q_n$$

$$P_{n+1}(a) = q_n^2 + p_n q_n$$

$$P_{n+1}(A) + P_{n+1}(a) = p_n^2 + p_n q_n + p_n q_n + q_n^2 = \underbrace{(p_n + q_n)}_{1}^2 = 1$$

$$p_{n+1} = P_{n+1}(A) = p_n^2 + p_n q_n \qquad (1)$$

$$q_{n+1} = P_{n+1}(a) = q_n^2 + p_n q_n \qquad (2)$$

$$p_{n+1} = P_{n+1}(A) \cong W_{AA}\, p_n^2 + W_{Aa}\, p_n q_n$$

$$q_{n+1} = P_{n+1}(a) \cong W_{aa}\, q_n^2 + W_{Aa}\, p_n q_n$$

$$p_{n+1} = p_{n+1}(A) = \frac{W_{AA}\, p_n^2 + W_{Aa}\, p_n q_n}{W_{AA}\, p_n^2 + 2W_{Aa}\, p_n q_n + W_{aa}\, q_n^2}$$

$$q_{n+1} = p_{n+1}(a) = \frac{W_{aa}\, q_n^2 + W_{Aa}\, p_n q_n}{W_{AA}\, p_n^2 + 2W_{Aa}\, p_n q_n + W_{aa}\, q_n^2}$$

$$p_{n+1} + q_{n+1} = 1$$

$$p_n + q_n = 1$$

$$p_{n+1} = \frac{\boxed{W_{AA}}\, p_n^2 + \boxed{W_{Aa}}\, p_n q_n}{\boxed{W_{AA}}\, p_n^2 + 2\boxed{W_{Aa}}\, p_n q_n + \boxed{W_{aa}}\, q_n^2} = f(p_n)$$

$$q_n = 1 - p_n$$

$$W_{AA} = W_{Aa} = W_{aa} = W$$

$$p_{n+1} = \frac{\cancel{W}(p_n^2 + p_n q_n)}{\cancel{W}(p_n^2 + 2p_n q_n + q_n^2)} = \frac{p_n \cancel{(p_n + q_n)}}{(p_n + q_n)^{\cancel{2}}} = p_n$$

W_{AA}, W_{Aa}, W_{aa} - relative fitness of the genotype

$p_n = p = p_{n+1}$ — steady state

$q_n = 1-p$

$$p = \frac{W_{AA}p^2 + W_{Aa}p(1-p)}{W_{AA}p^2 + 2W_{Aa}p(1-p) + W_{aa}(1-p)^2}$$

$W_{Aa}p(1-p)$

$$W_{AA}p^3 + 2W_{Aa}p^2(1-p) + W_{aa}p(1-p)^2 = W_{AA}p^2 + W_{Aa}p(1-p)$$

$$W_{AA}(p^3-p^2) + W_{Aa}\left[2p^2(1-p) - p(1-p)\right] + W_{aa}p(1-p)^2 = 0$$

$$W_{AA}p^2\underbrace{(p-1)}_{-1} + W_{Aa}p(1-p)(2p-1) + W_{aa}p(1-p)^2 = 0$$

$$p(1-p)\left[-W_{AA}p + W_{Aa}(2p-1) + W_{aa}(1-p)\right] = 0$$

$$p(1-p)\left[(-W_{AA}+2W_{Aa}-W_{aa})p + W_{aa} - W_{Aa}\right] = 0$$

$$p^*_{(0)} = 0, \quad p^*_{(1)} = 1, \quad p^*_{(2)} = \frac{W_{aa}-W_{Aa}}{W_{AA}-2W_{Aa}+W_{aa}}$$

$$p^*_{(2)} = \frac{W_{aa}-W_{Aa}}{(W_{aa}-W_{Aa}) + W_{AA}-W_{Aa}} = \frac{1}{1+\frac{W_{AA}-W_{Aa}}{W_{aa}-W_{Aa}}}$$

$p^*_{(2)}$ exists if and only if $\boxed{\frac{W_{AA}-W_{Aa}}{W_{aa}-W_{Aa}} > 0}$, then $0 < p^*_{(2)} < 1$

$$\boxed{(W_{AA}-W_{Aa})(W_{aa}-W_{Aa}) > 0}$$

(a) $W_{AA} > W_{Aa}$ homozigos advantageos

& $W_{aa} > W_{Aa}$

(b) $W_{Aa} > W_{AA}$ heterozygous advantageous

$W_{Aa} > W_{aa}$

$p_{n+1} = f(p_n)$ where

$$f(p) = \frac{W_{AA}p^2 + W_{Aa}\,p(1-p)}{W_{AA}p^2 + 2W_{Aa}\,p(1-p) + W_{aa}(1-p)^2}$$

$$f'(p) = \frac{[2W_{AA}p + W_{Aa}(1-p) - W_{Aa}p]D - [W_{AA}p^2 + W_{Aa}\,p(1-p)][D']}{(W_{AA}p^2 + 2W_{Aa}\,p(1-p) + W_{aa}(1-p)^2)^2}$$

D = Denominator

$$D' = 2pW_{AA} + 2W_{Aa}(1-p) - 2W_{Aa}p - 2W_{aa}(1-p)$$

$$f'(0) = \frac{W_{Aa}\cdot W_{aa} - 0}{(W_{aa})^2} = \frac{W_{Aa}}{W_{aa}}$$

$$\boxed{f'(0) = \frac{W_{Aa}}{W_{aa}}}$$

$$f'(1) = \frac{(2\cancel{W_{AA}} - W_{Aa})\cancel{W_{AA}} - \cancel{W_{AA}}(2\cancel{W_{AA}} - 2W_{Aa})}{(W_{AA})^{\cancel{2}}}$$

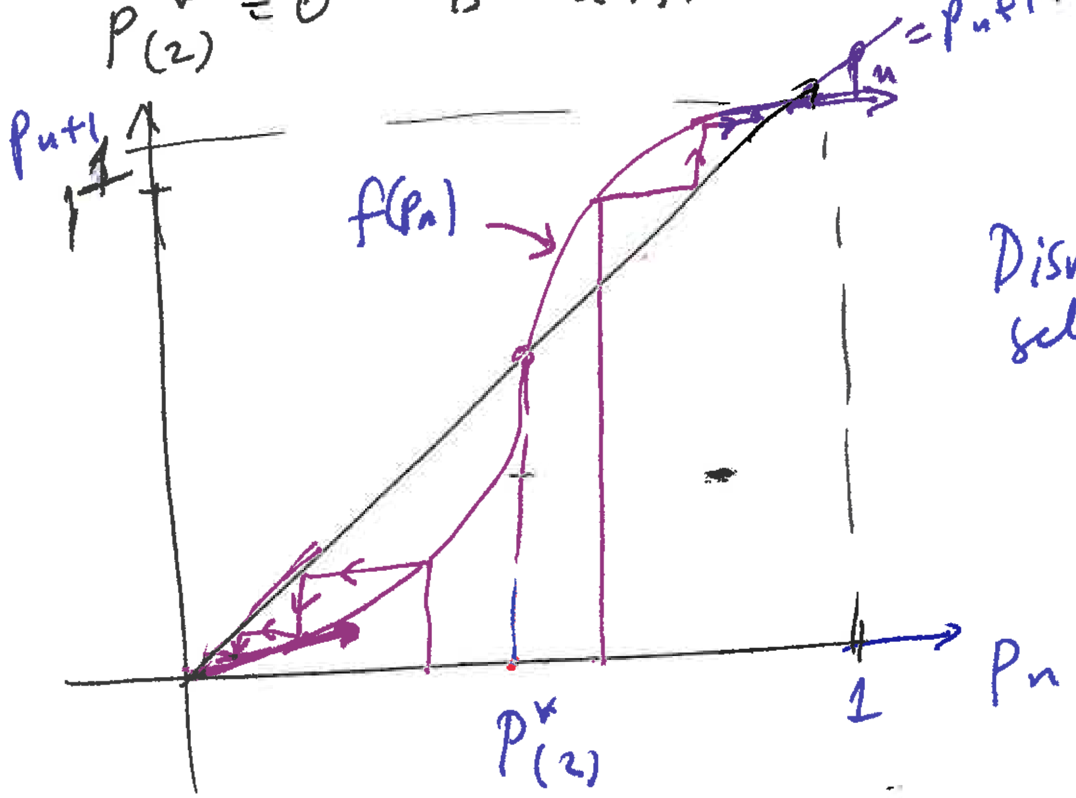
$$\boxed{f'(1) = \frac{W_{Aa}}{W_{AA}}}$$

(a) $W_{AA} > W_{Aa} \longrightarrow 0 < f'(1) < 1$

$W_{aa} > W_{Aa} \qquad 0 < f'(0) < 1$

$p^*_{(0)} = 0$ & $p^*_{(1)} = 1$ are stable

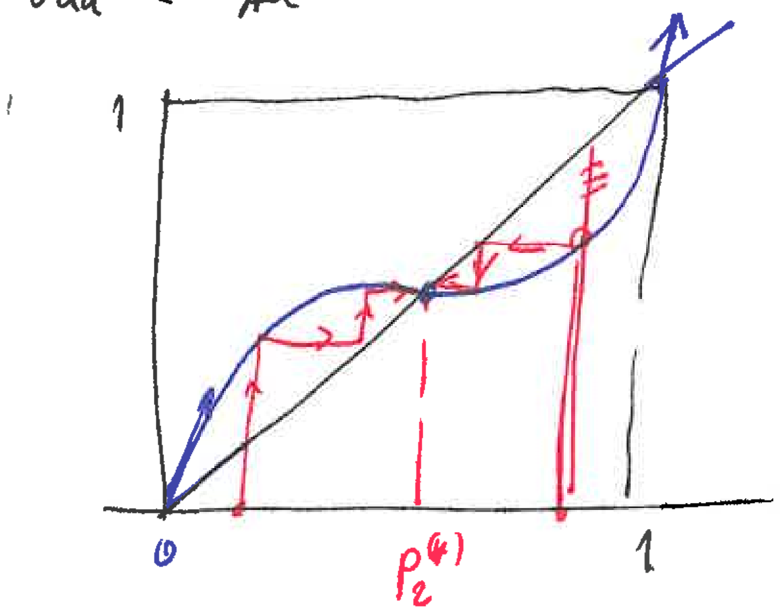
$p^*_{(2)} = 0$ is unstable.



Disruptive selection

(b) $W_{AA} < W_{Aa} \implies f'(1) > 1 \implies p^*_{(1)} = 1$ unstable

$W_{aa} < W_{Aa} \qquad f'(0) > 1 \implies p^*_{(0)} = 0$ unstable

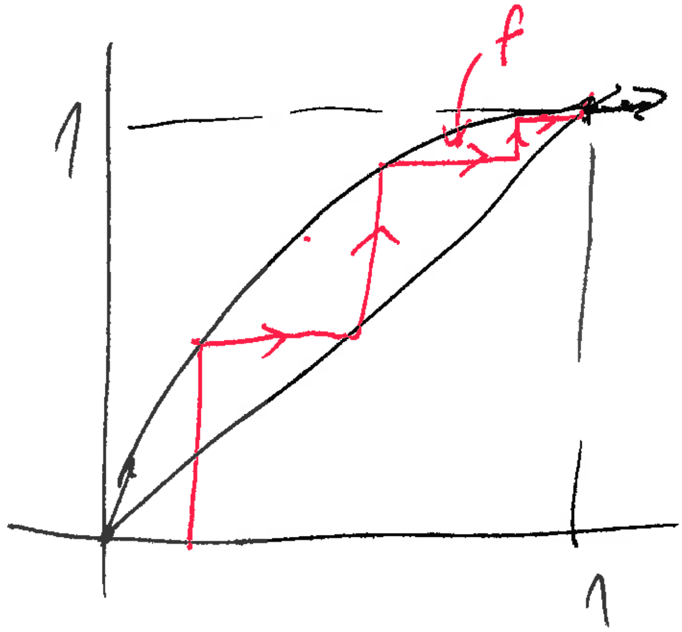


Heterosis

(c) $W_{aa} < W_{Aa} < W_{AA}$

$f(0) = \frac{W_{Aa}}{W_{aa}} > 1$ unstable

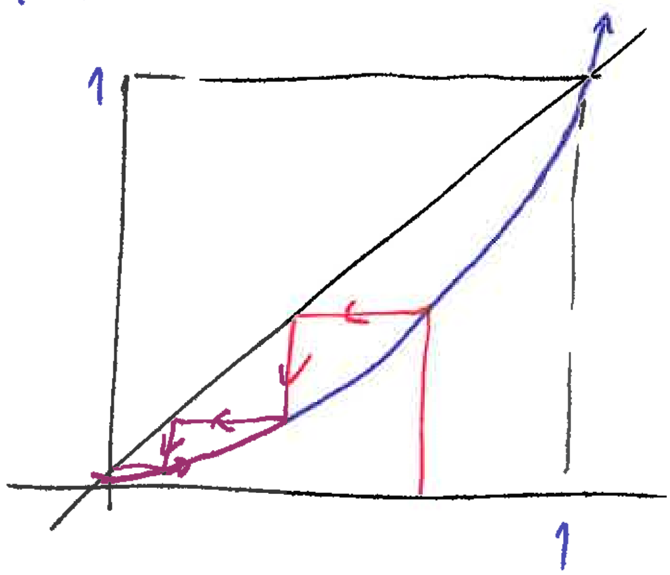
$f(1) = \frac{W_{Aa}}{W_{AA}} < 1$ Stable.



$p_n = p_n(A)$

A dominates

(d) $W_{AA} < W_{Aa} < W_{aa}$



$f'(0) = \frac{W_{Aa}}{W_{aa}} < 1$ stable.

$f'(1) > 1$ unstable.

a dominates.

Slow selection

$$w_{AA} = 1 + \varepsilon w_1 \qquad \varepsilon \ll 1$$

$$w_{Aa} = 1 + \varepsilon w_2$$

$$w_{aa} = 1 + \varepsilon w_3$$

$$p_{n+1} = \frac{(1+\varepsilon w_1)p_n^2 + (1+\varepsilon w_2)p_n(1-p_n)}{(1+\varepsilon w_1)p_n^2 + 2(1+\varepsilon w_2)p_n(1-p_n) + (1+\varepsilon w_3)(1-p_n)^2}$$

$$p_{n+1} = \frac{\cancel{p_n^2} + p_n(1-\cancel{p_n}) + \varepsilon\left[w_1 p_n^2 + w_2 p_n(1-p_n)\right]}{\underbrace{p_n^2 + 2p_n(1-p_n) + (1-p_n)^2}_{1} + \varepsilon\left[w_1 p_n^2 + 2w_2 p_n(1-p_n) + w_3(1-p_n)^2\right]}$$

$$\cancel{p_n^2} + \underline{2p_n} - \cancel{2p_n^2} + 1 \underline{- 2p_n} + \cancel{p_n^2}$$

$$p_{n+1} = \frac{p_n + \varepsilon\left[w_1 p_n^2 + w_2 p_n(1-p_n)\right]}{1 + \varepsilon\left[w_1 p_n^2 + 2w_2 p_n(1-p_n) + w_3(1-p_n)^2\right]}$$

$$\frac{1}{1+\varepsilon q} \approx 1 - \varepsilon q + O(\varepsilon^2)$$

$$p_{n+1} = \frac{p_n + \varepsilon X}{1+\varepsilon Y} \approx (p_n + \varepsilon X)(1 - \varepsilon Y) + O(\varepsilon^2) =$$

$$= p_n + \varepsilon X - \varepsilon Y p_n + O(\varepsilon^2)$$

$$= p_n + \varepsilon(X - p_n Y) + O(\varepsilon^2)$$

$$p_{n+1} = p_n + \varepsilon\Big[\underbrace{w_1 p_n^2 + w_2 p_n (1-p_n)}_{X} - p_n\big(\underbrace{w_1 p_n^2 + 2w_2 p_n(1-p_n) + w_3(1-p_n)^2}_{Y}\big)\Big]$$

$$p_{n+1} = p_n + \varepsilon p_n\big[w_1 p_n + w_2(1-p_n) - w_1 p_n^2 - 2w_2 p_n(1-p_n) - w_3(1-p_n)^2\big]$$

$$w_1 p_n (1-p_n)$$

$$p_{n+1} = p_n + \varepsilon p_n (1-p_n)\big[w_1 p_n + w_2 - 2w_2 p_n - w_3(1-p_n)\big]$$

$$p_{n+1} = p_n + \varepsilon p_n (1-p_n)\big[(w_1 - 2w_2 + w_3)p_n + w_2 - w_3\big]$$

$$\boxed{p_n = p(\varepsilon n)}, \quad \boxed{t = \varepsilon n} \qquad t+\varepsilon = \varepsilon(n+1)$$

$$p(t+\varepsilon) \doteq p(t) + \varepsilon\,[\,\cdots\cdots\,]$$

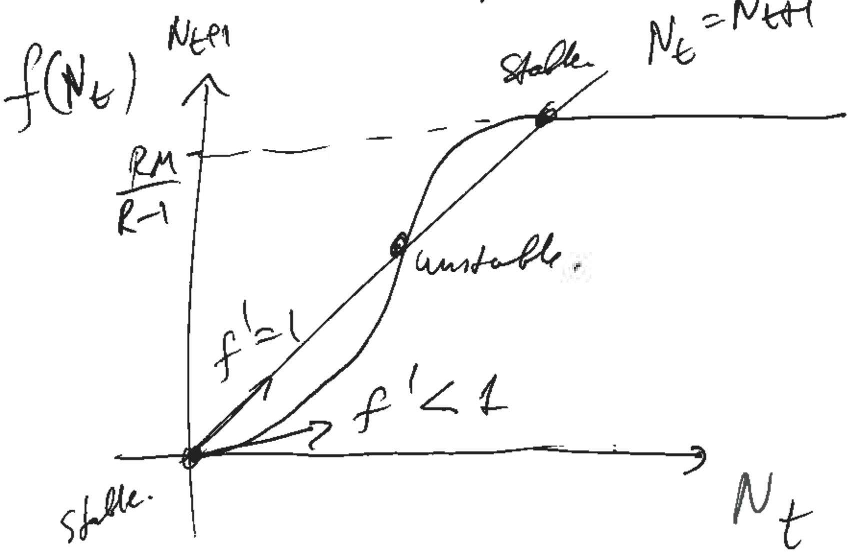
$$\frac{p(t+\varepsilon) - p(t)}{\varepsilon} = [\,\cdots\cdots\,]$$

$$\boxed{p'(t) = p(1-p)\big[(w_1 - 2w_2 + w_3)p + w_2 - w_3\big]}$$

$$p' = p(1-p)(\alpha p + \beta)$$

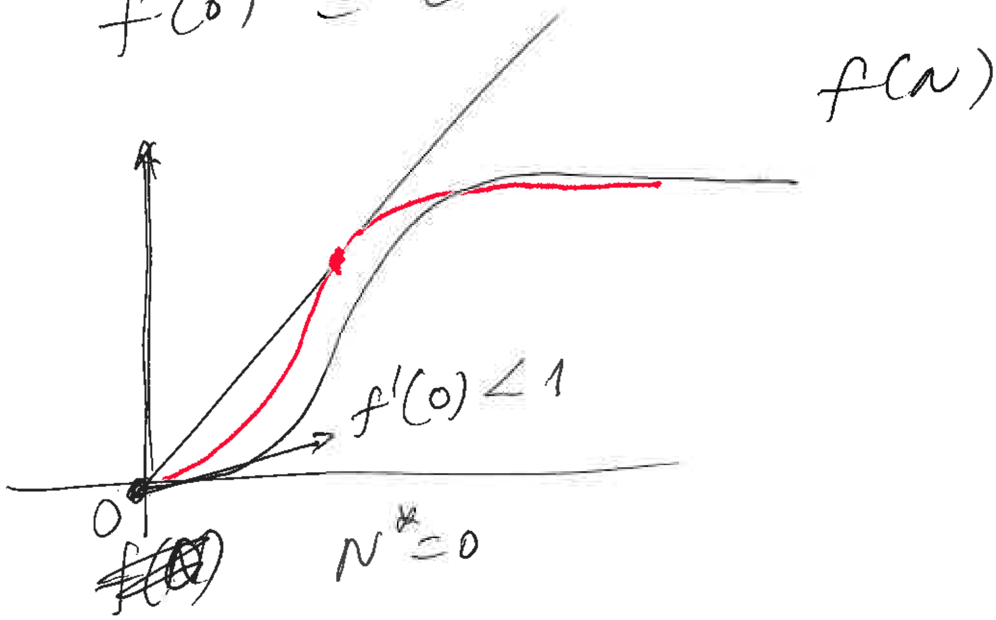
Problem sheet

$$N_{t+1} = \frac{R N_t^2}{\frac{R-1}{M} N_t^2 + N_t + S} = f(N_t)$$



$$\lim_{N \to \infty} f(N_t) \to \frac{R}{\frac{R-1}{M}} = \frac{RM}{R-1}$$

$f(0) = 0$



$$f'(N^*) = \lambda = e^{i\alpha} \sim \lambda^n = e^{i\alpha n} = \underline{\cos \alpha n + i \sin \alpha n}$$

$$1^n = 1$$