Rényi's Parking Problem

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Abstract

Rényi's parking problem is a simple random process. The expected number of cars parked, and the expected density of parked cars can be found in a number of ways. The author has looked at the elementary proof of the problem ([1, 2]), and provided a simulation to evaluate Rényi Parking Constant.

Chapter 1

Background

The problem statement is as follows: consider an interval $(0, x)$ upon which we place a segment of unit length at random. We continue by placing a second segment of unit length randomly upon the original interval, discarding the segment if it overlaps with the original one.

We continue in this fashion until we can no longer add unit segments without overlap. At each step the next position within the interval is chosen from a uniform distribution of the remaining locations within the interval.

We are interested in both the expected value of the number of unit segments contained within the interval $(0, x)$, denoted $M(x)$, and the expected filling density of unit segments within the interval, denoted $M(x)/x$.

Chapter 2

Methods

Before proceeding with the simulations, let us look at the derivation of the equation for $M(x)$. If we initially consider an interval $(0, x+1)$ and upon this place a unit segment $(t, t+1)$. This unit segment partitions the original interval into two smaller intervals - $(0, t)$ and $(t+1, x+1)$. So the expected number of unit segments contained within the original interval is:

$$M(x+1) = M(t) + 1 + M(x-t)$$

where 1 represents the expectation of added segment within the unit interval $(t, t+1)$. Integrating with respect to t we get:

$$\begin{aligned}
\int_0^x M(x+1)dt &= \int_0^x [M(t) + 1 + M(x-t)]dt \\
M(x+1)\int_0^x dt &= \int_0^x dt + \int_0^x [M(t) + M(x-t)]dt \\
M(x+1)\cdot x &= x + \int_0^x [M(t) + M(x-t)]dt \\
M(x+1) &= 1 + \frac{1}{x}\int_0^x [M(t) + M(x-t)]dt
\end{aligned}$$

as the distributions within each of the smaller intervals are uniform, and hence the same, we get:

$$M(x+1) = 1 + \frac{2}{x}\int_0^x M(t)dt$$

changing variables we get:

$$M(x) = 1 + \frac{2}{x-1}\int_0^{x-1} M(t)dt$$

more completely, and because adding a unit segment to an interval of length less than 1 has no meaning, the equation for $M(x)$ can be written as:

$$M(x) = \begin{cases} 0, & \text{for } 0 \leq x < 1 \\ 1 + \dfrac{2}{x-1}\displaystyle\int_0^{x-1} M(t)dt, & \text{for } x \geq 1 \end{cases}$$

Chapter 3

Results

Both the elementary proof and the recurrence form above allows one to write a simulation. We write a function that does the following:

- find a parking spot in the interval using a uniform distribution
- the found parking spot partitions the interval into two
- recursively calls itself on the two partitions
- returns if the partition size is less than 1

Listing 3.1: Parking problem function

```
def parking_problem(length):
    spots = []

    def find_spots(start, end):
        spot = np.random.uniform(start, end - 1.0)
        spots.append(spot)

        if spot - start >= 1.0:
```

```
            find_spots(start, spot)

        if end - spot >= 2.0:
            find_spots(spot + 1.0, end)

    find_spots(0, length)

    return len(spots) / float(length)
```

This function exits when no more candidate spots remain. A second function aggregates the results of the simulation function in order to calculate statistics.

Listing 3.2: Simulation function

```
def simulation(iterations, length):
    results = []

    for i in range(iterations):
        results.append(parking_problem(length))

    return results
```

The simulation produces the following output:

Listing 3.3: Simulation output

```
[ jerry@ponocrates ~ ] $ python3 parking_problem.py 1000 1000000

    Parking Problem: running 1000 simulations...
                   : iteration 1000
```

```
                    : simulations completed...


     Parking Problem: results

        Distribution:

                mean: 0.7476

  standard deviation: 0.0001
```

and a plot of the results can be seen below. As can be seen the estimate for the Rényi number is around 0.7476:

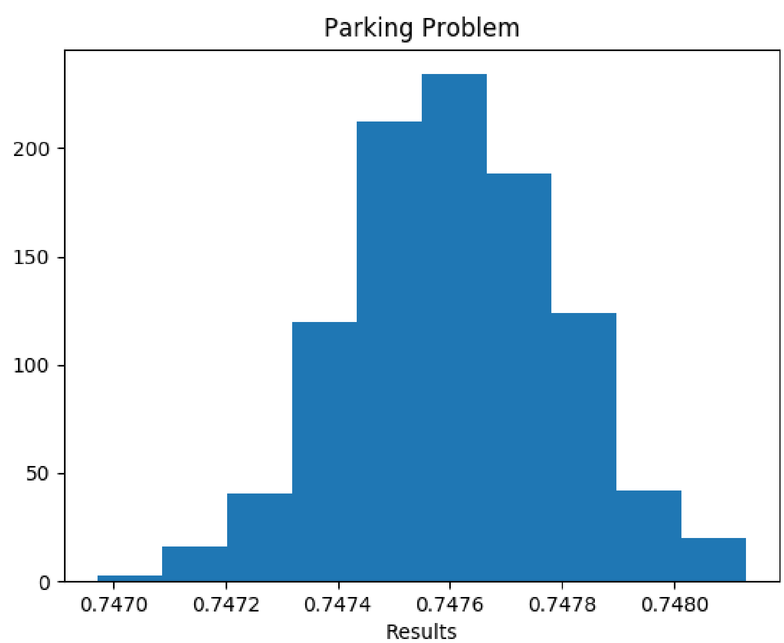

Figure 3.1: The Parking Problem Result Distribution

Chapter 4

Comments

There are many approaches to solving the parking problem, and there are also many variations of this problem:

- the discrete parking problem
- the two dimensional random sequential packing problem
- the three dimensional random sequential packing problem
- the parking problem with segments of different length
- the reversible parking problem

One of the more interesting applications is the theory of Random Sequential Adsorption (RSA). This can describe the adsorption of particles onto a substrate. These particles include proteins, bacteria, gas molecules, polymers, colloids, and more.

Bibliography

[1] Matthew P. Clay. Renyi's parking problem revisited. 2014.

[2] Howard Weiner. Elementary treatment of the parking problem. *The Indian Journal of Statistics*, 31(4):483–486, 1969.