W9209/9210/9211/9212/104C

DUBLIN INSTITUTE OF TECHNOLOGY

School of Mathematical Sciences

DT9209 MSc Applied Mathematics
DT9210 MSc Applied Mathematics
DT9211 MSc Applied Mathematics
DT9212 MSc Applied Mathematics

WINTER EXAMINATIONS 2016/2017

MATH 9974: BIOMATHEMATICS

DR R. IVANOV

DR C HILLS

PROFESSOR E O'RIORDAN

9.30 – 12.30 pm, Friday, 13 January 2017

Duration: 3 hours

Answer three questions

All questions carry equal marks

Approved calculators may be used

Mathematical tables are provided

New Cambridge Statistical Tables are NOT permitted

1. a) Find the steady states of the logistic population growth model and determine their stability:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN\Big(1 - \frac{N}{K}\Big), \qquad N(0) > 0$$

with r and K positive constants.

(6)

b) Solve explicitly the model and compute the limit of $N(t)$ when $t \to \infty$. Compare the obtained limit with the results from part **a)** and explain your findings. (12)

c) Find the steady states and determine their stability for the logistic model with harvesting

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN\Big(1 - \frac{N}{K}\Big) - EN,$$

where the positive constant E measures the harvesting effort. Comment on the ecological implications of the results in the two cases: $E > r$ and $E < r$. (15)

[33]

2. A population growth model is described by a differential equation with delay $T > 0$:

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = rN(t)\Big(1 - \frac{N(t-T)}{K}\Big),$$

with $r > 0$, $K > 0$ and $N(0) > 0$.

a) Show that in the new non-dimensional variables $u = N/K$, $\tau = rt$ and $\Theta = rT$ the equation has the form

$$\frac{\mathrm{d}u(\tau)}{\mathrm{d}\tau} = u(\tau)[1 - u(\tau - \Theta)],$$

(3)

b) Show that $u^* = 1$ is a steady state and linearize the equation about the steady state by writing $u(\tau) = u^* + n(\tau)$, where $|n(\tau)| \ll 1$.

(5)

c) Look for solutions of the form $n(\tau) = ce^{\lambda\tau}$, where c is a constant, and write the corresponding equation for λ.

(5)

d) Decompose λ into real and imaginary parts ($\lambda = \mu + i\omega$) and write separately the real and imaginary parts of the equation for λ, obtained in part **c)** in terms of μ and ω.

(5)

e) Analyze the equations obtained in (iv) when the delay Θ increases from 0 to $\frac{\pi}{2}$, having in mind that μ and ω depend on Θ. Show that the steady state solution u^* is stable when $0 \leq \Theta < \pi/2$. Demonstrate that the first bifurcation value of Θ for which the steady state u^* becomes unstable and the solution becomes oscillatory corresponds to $\Theta = \pi/2$, (and $\mu = 0$, $\omega = \pm 1$) or, in dimensional terms, $rT = \pi/2$.

(10)

f) Show that the period of oscillations (in non-dimensional units) at the bifurcation value is 4Θ.

(5)

[33]

3. Consider the discrete population model with delay

$$U_{t+1} = U_t e^{r(1-U_{t-1})},$$

where t is the discrete time and r is a positive parameter.

a) Show that $U^* = 1$ is a steady state of the model. Linearize the equation about $U^* = 1$ by writing $U_t = U^* + v_t$, where $|v_t| \ll 1$. Obtain the linearized equation

$$v_{t+1} - v_t + rv_{t-1} = 0.$$

(10)

b) Prove that, for $0 < r < 1$, $U^* = 1$ is a stable steady state.

(11)

c) Show that $r = 1$ is a bifurcation value for which the steady state bifurcates to a periodic solution of period 6.

(7)

d) Prove that, for $r > 1$ the steady state $U^* = 1$ is unstable.

(5)

[33]

4. The Lotka-Volterra predator-pray model is given by the following system of equations for the non-dimensional variables $u(t)$, $v(t)$:

$$\begin{aligned}
\frac{du}{dt} &= u(1 - v) \\
\frac{dv}{dt} &= \alpha v(u - 1)
\end{aligned}$$

where α is a positive parameter.

a) Show that $u^* = 1$, $v^* = 1$ is a steady state of the model. Linearise the equations about $u^* = 1$, $v^* = 1$ and show that the obtained linear model possess periodic solutions. Find the period of these solutions in terms of α. (12)

b) Show that the full nonlinear system has a conservation law

$$H(u, v) = \alpha u + v - \ln(u^\alpha v).$$

(7)

c) Using the conservation law from part b) (or otherwise) demonstrate that the full nonlinear system has periodic solutions. (14)

[33]

Summary Sheet for Examinations

Title of Exam: Biomathematics
Programme(s)/stage: MSc Applied Mathematics

	Part (a)				Part (b)				Part (c)				Part (d)			
	(i)	(ii)	(iii)	(iv)	(i)	(ii)	(iii)	(iv)	(i)	(ii)	(iii)	(iv)	(i)	(ii)	(iii)	(iv)
Question 1	A				A				B							
Question 2	A				A				A				A			
Question 3	A				A				B				B			
Question 4	A				B				B							
Question 5																

(add rows/columns as necessary)

Key:
Blank where that part is not used

A previously-seen material
e.g. theorem/example discussed in lectures

B partly-unseen
e.g. similar structure to an example seen in lectures

C completely unseen

SOLUTION SHEET

COURSE / YEAR: DT9209, DT9210 (MSc)

EXAMINATION SITTING: SUMMER ☐ SUPPLEMENTAL ☐ AUTUMN ☐ WINTER ☑

SUBJECT: BIOMATHEMATICS

EXAMINER: ROSSEN IVANOV

EXPECTED SOLUTION TO QUESTION NO: 1

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PROPOSED MARK ALLOCATION: 33

a) The two steady states are $N^* = 0$ and $N^* = K$

$$f(N) = rN\left(1 - \frac{N}{K}\right) = rN - \frac{r}{K}N^2$$

$$f'(N) = r - \frac{2r}{K}N \qquad [2 \text{ marks}]$$

$f'(0) = r > 0 \Rightarrow N_1^* = 0$ is an <u>unstable</u> steady state [2 marks]

$f'(K) = r - 2r = -r < 0 \Rightarrow N_2^* = K$ is a <u>stable</u> —"— [2 marks]

b) $$\frac{dN}{N(1-\frac{N}{K})} = r\,dt, \quad \frac{1}{N(1-\frac{N}{K})} = \frac{1}{N} + \frac{1}{K-N}$$

$\Rightarrow \ln N - \ln(K-N) = rt + \text{const}$ [4 marks]

$\frac{N}{K-N} = ce^{rt}$, $c = \text{const}$, at $t=0$ $N(0) = N_0$

$\frac{N_0}{K-N_0} = ce^0 = c \Rightarrow c = \frac{N_0}{K-N_0}$ [4 marks]

$N = K\frac{N_0}{(K-N_0)e^{-rt} + N_0} \xrightarrow{t\to\infty} K = N_2^*$ - the second steady state, which is stable [4 marks]

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SOLUTION SHEET

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SUBJECT: ____________________

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EXAMINATION SITTING: SUMMER ☐ SUPPLEMENTAL ☐ AUTUMN ☐ WINTER ☐

EXPECTED SOLUTION TO QUESTION NO: 1 - cont.

PROPOSED MARK ALLOCATION: ____

c) $f(N) = rN\left(1-\frac{N}{K}\right) - EN$

$f(N)=0$ has two roots: $N_1^*=0$, $N_2^*=K\left(1-\frac{E}{r}\right)$, which exists iff $E<r$!

[4 marks]

$f'(N) = r-E-\frac{2r}{K}N$

$$f'(0) = r-E \begin{cases} >0 \text{ if } r>E, & N_1^*=0 \text{ is unstable} \\ <0 \text{ if } r<E, & N_1^*=0 \text{ is stable} \end{cases}$$

[4 marks]

$f'(N_2^*) = -(r-E) < 0$ when N_2^* exists

$\Leftrightarrow$ when N_2^* exists it is stable, i.e. $(E<r)$

[3 marks]

$\Rightarrow$ ① If $E>r$ only $N_1^*=0$ exists and is stable

$\Rightarrow$ harvesting destroys the population!

[2 marks]

② If $E<r$, $N_2^*=K\left(1-\frac{E}{r}\right)$ is a stable steady state - the population number approaches N_2^*

[2 marks]

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COURSE / YEAR		**EXAMINATION SITTING:**	**SUMMER** ☐ **SUPPLEMENTAL** ☐ **AUTUMN** ☐ **WINTER** ☐
SUBJECT:		**EXPECTED SOLUTION TO QUESTION NO:**	2
EXAMINER:			
PAGE 3 **OF** 10		**PROPOSED MARK ALLOCATION:**	33

(a) $\dfrac{d(N/K)}{d(rt)} = \left(\dfrac{N}{K}\right)\left(1 - \dfrac{N(t-T)}{K}\right) \Rightarrow u = \dfrac{N}{K},\ \tau = rt$

$\theta = rT$

$\dfrac{du(\tau)}{d\tau} = u(\tau)\left[1 - u(\tau-\theta)\right]$ .... [3 marks]

(b) The steady state satisfies $u^*(1-u^*) = 0$

$\Rightarrow u^* = 1$ is a steady state. [2 marks]

If $u = 1 + n(\tau)$

$\dfrac{dn(\tau)}{d\tau} = (1+n(\tau))\left(\cancel{1}-\cancel{1}-n(\tau-\theta)\right) = -n(\tau-\theta)$

$\dfrac{dn(\tau)}{d\tau} = -n(\tau-\theta)$ ... [3 marks]

(c) $n(\tau) = ce^{\lambda\tau} \Rightarrow c\lambda e^{\lambda\tau} = -ce^{\lambda(\tau-\theta)}$

$\Rightarrow \lambda = -e^{-\lambda\theta}$ [5 marks]

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EXAMINATION SITTING: SUMMER ☐ SUPPLEMENTAL ☐ AUTUMN ☐ WINTER ☐

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EXPECTED SOLUTION TO QUESTION NO: ______

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PROPOSED MARK ALLOCATION: ______

d) $\lambda = \mu + i\omega$

$$\mu + i\omega = -e^{-\mu\theta} e^{-i\omega\theta} = -e^{-\mu\theta}(\cos\omega\theta - i\sin\omega\theta)$$

$$\mu = -e^{-\mu\theta}\cos\omega\theta \qquad (*)$$
$$\omega = e^{-\mu\theta}\sin\omega\theta$$

[5 marks]

e) Suppose $\omega = 0 \Rightarrow \mu = -e^{-\mu\theta} \Rightarrow$ there are no positive roots for μ, (solution $u^* = 1$ is stable)
We also notice that if ω is a solution of $(*)$ above so is $-\omega$, so we need only to consider $\omega > 0$.

When $\theta = 0$, $\mu = -1$ $\omega = 0$ and the solution $u^* = 1$ is stable; $n(T) \to 0$. The first bifurcation occurs when μ reaches $\mu = 0$ and then $u^* = 1$ becomes unstable, since $|n(T)|$ does not approach 0.

[5 marks]

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EXPECTED SOLUTION TO QUESTION NO:

PROPOSED MARK ALLOCATION:

If $\mu = 0$ we have $0 = -\cos\omega\theta$

$\omega = \sin\omega\theta$

$\Rightarrow \sin\omega\theta = \pm 1 \Rightarrow \omega = \pm 1$ but we consider only $\omega > 0 \Rightarrow \omega = 1$

$\Rightarrow \theta = \frac{\pi}{2}$

in dimensional units $\theta = r\cancel{T} \Rightarrow rT = \frac{\pi}{2}$.

[5 marks]

f) When $\theta = \frac{\pi}{2}$ $\lambda = \mu + i\omega = 0 \pm i$

$n(\tau) = ce^{\lambda\tau} = ce^{\pm i\tau}$ and has period

$t_p = 2\pi$

$\frac{t_p}{\theta} = \frac{2\pi}{\pi/2} = 4 \Rightarrow t_p = 4\theta$ [5 marks]

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COURSE / YEAR		EXAMINATION SITTING:	SUMMER	☐
			SUPPLEMENTAL	☐
			AUTUMN	☐
SUBJECT:			WINTER	☐
EXAMINER:		EXPECTED SOLUTION TO QUESTION NO:	3	
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a) $U^* = 1$ is a steady state since $U^* = U^* e^{r(1-U^*)}$

[3 marks]

$U_t = U^* + V_t = 1 + V_t \ ; \ |V_t| \ll 1$

$1 + V_{t+1} = (1+V_t)\, e^{r(1 - \cancel{1} - V_{t-1})} = (1+V_t)\, e^{-rV_{t-1}}$

$1 + V_{t+1} \approx (1+V_t)(1 - rV_{t-1}) \approx 1 + V_t - rV_{t-1}$

$$\boxed{V_{t+1} - V_t + rV_{t-1} = 0.}$$

[7 marks]

b) The linearised equation has solution $V_t = z^t$ where $z^2 - z + r = 0$, i.e.

$z_{1,2} = \frac{1}{2}\left[1 \pm \sqrt{1-4r}\,\right]$

[5 marks]

if $0 < r < \frac{1}{4}$, z_1 and z_2 are real, $0 < z_{12} < 1$ and therefore $V_t \to 0$ as $t \to \infty$ and hence $U^* = 1$ is a linearly stable equilibrium state

If $1 > r > \frac{1}{4}$ $z_{1,2}$ are complex, $z_{12} = \frac{1}{2}\left[1 \pm i\sqrt{4r-1}\,\right]$

$|z_{12}| = \frac{1}{2}\sqrt{1 + 4r - 1} = \sqrt{r} < 1 \Rightarrow$ again $V_t \to 0$ as $t \to \infty$

[6 marks]

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EXPECTED SOLUTION TO QUESTION NO: 3 - cont.

EXAMINER: ____________

PROPOSED MARK ALLOCATION: ____________

c) $V_t = A z_1^t + \bar{A}_1 \bar{z}_1^t$ in the case $z_{1,2}$ - complex.

when $r=1$ $z_{1,2} = \frac{1}{2}(1 \pm i\sqrt{3}) = e^{\pm \pi i/3}$, $|z_{12}|=1$

and V_t does not approach 0 as $t \to \infty$.

$$V_t = A e^{\frac{\pi i}{3} t} + \bar{A} e^{-\pi i/3\, t}$$

$$V_{t+6} = e^{2\pi i}\left(A e^{\pi i/3\, t} + \bar{A} e^{-\frac{\pi i t}{3}}\right) = e^{2\pi i} V_t = V_t$$

$\Rightarrow$ when $r=1$, V_t is periodic with period 6

[7 marks]

d) When $r > 1$ $|z_{1,2}| = \sqrt{r} > 1$

since when $t \to \infty$ ~~$|z_{12}|$~~ $V_t \sim (\sqrt{r})^t \to \infty$

i.e. V_t grows unboundedly and $V_t^* = 1$ is unstable.

[5 marks]

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EXPECTED SOLUTION TO QUESTION NO: 4

PROPOSED MARK ALLOCATION: 33

$$\frac{du}{dt} = u(1-v) = f(u,v) \qquad = u - uv$$

$$\frac{dv}{dt} = \alpha v(u-1) = g(u,v) = \alpha uv - \alpha v$$

a) when $u^* = v^* = 1$ we have $f(u^*, v^*) = g(u^*, v^*) = 0$ [2 marks]

$\Rightarrow$ thus $(1,1)$ is a steady state.

The linearised eq. are $\left(U = u - u^* = u-1, \; V = v - v^* = v - 1 \right)$

$$\begin{pmatrix} U' \\ V' \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix}_{u^*=v^*=1} \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 1-v & -u \\ \alpha v & \alpha(u-1) \end{pmatrix}_{(1,1)} \begin{pmatrix} U \\ V \end{pmatrix}.$$

$$\begin{pmatrix} U' \\ V' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ \alpha & 0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}$$ [5 marks]

$$\Rightarrow \quad U' = -V, \quad V' = \alpha U$$

$$U'' = -V' = -\alpha U \Rightarrow \boxed{U'' + \alpha U = 0}$$

$U = A\cos(\sqrt{\alpha}\, t) + B\sin(\sqrt{\alpha}\, t) \; (\equiv u - 1)$ is periodic [4 marks]

with period $T = \frac{2\pi}{\sqrt{\alpha}}$ [1 mark]

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EXPECTED SOLUTION TO QUESTION NO: 4 - cont.

EXAMINER: ____________

PROPOSED MARK ALLOCATION: 33

$\boxed{H = \alpha u + v - \alpha \ln u - \ln v}$

b) $\frac{dH}{dt} = \alpha \frac{du}{dt} + \frac{dv}{dt} - \alpha \frac{1}{u}\frac{du}{dt} - \frac{1}{v}\frac{dv}{dt} =$

$= \alpha(u)(1-v) + \alpha v(u-1) - \alpha \frac{1}{\cancel{u}} \cancel{u}(1-v) - \frac{1}{\cancel{v}} \alpha \cancel{v}(u-1)$

$= \cancel{\alpha u} - \cancel{\alpha uv} + \cancel{\alpha uv} - \cancel{\alpha v} - \cancel{\alpha} + \cancel{\alpha v} - \cancel{\alpha u} + \cancel{\alpha} = 0$

(7 marks)

c) The trajectories are an integral curve

$H(u,v) = c$ const

H has a local min at $u = v = 1$:

$\frac{\partial H}{\partial u} = \alpha - \alpha \frac{1}{u} = 0 \qquad \frac{\partial H}{\partial v} = 1 - \frac{1}{v} = 0$ (4 marks)

$H(1,1) = \alpha + 1 \Rightarrow$ if $c \geq \alpha + 1$ the trajectories are the curve of intersection of the 'bowl' with a "bottom" at $u = v = 1$ and the plane $H = c$:

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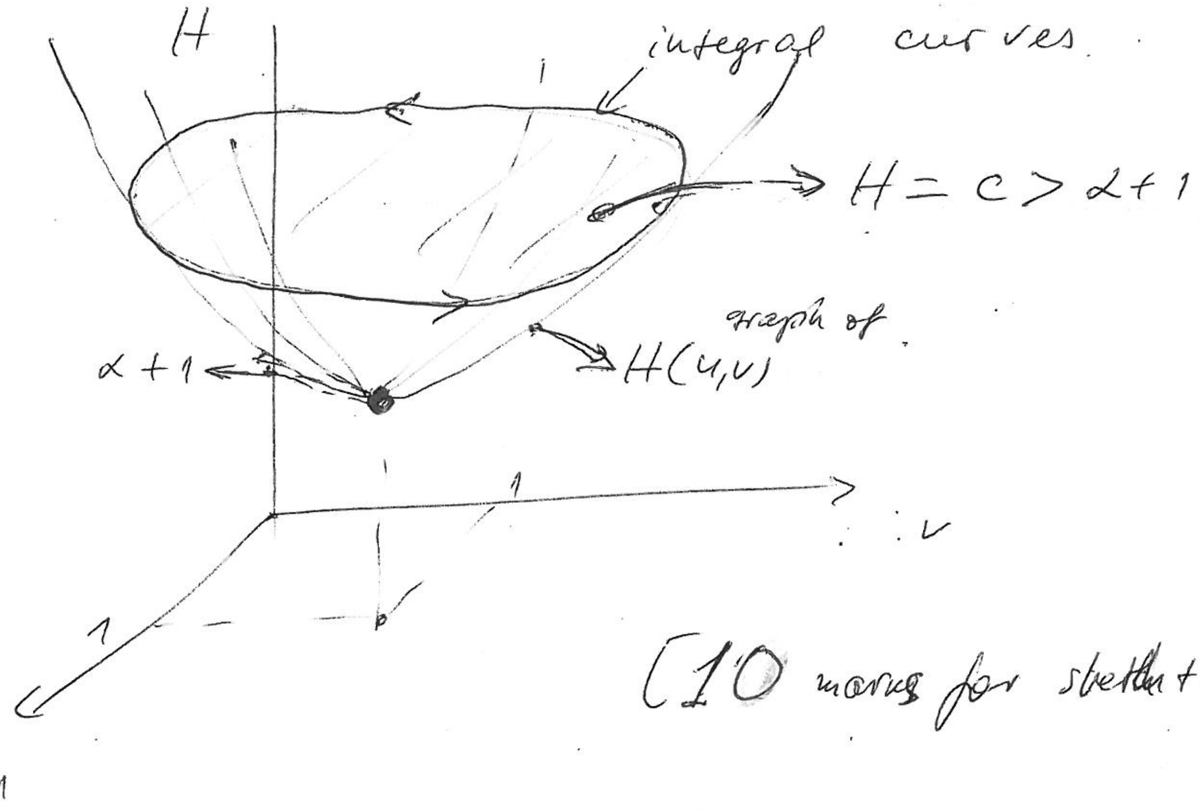
EXAMINATION SITTING: SUMMER ☐ SUPPLEMENTAL ☐ AUTUMN ☐ WINTER ☐

EXPECTED SOLUTION TO QUESTION NO: 4-cont.

PROPOSED MARK ALLOCATION:



[10 marks for sketch + expl.]

(Hessian matrix $\begin{pmatrix} \frac{\partial^2 H}{\partial u^2} & \frac{\partial^2 H}{\partial u \partial v} \\ \frac{\partial^2 H}{\partial v \partial u} & \frac{\partial^2 H}{\partial v^2} \end{pmatrix} = \begin{pmatrix} +\frac{\alpha}{u^2} & 0 \\ 0 & \frac{1}{v^2} \end{pmatrix}$

$\Rightarrow$ min.]