S9209/9210/103C

DUBLIN INSTITUTE OF TECHNOLOGY

School of Mathematical Sciences

DT9205 MSc Mathematical Physics
DT9206 MSc Mathematical Physics
DT9209 MSc Applied Mathematics
DT9210 MSc Applied Mathematics

SUMMER EXAMINATIONS 2015/2016

MATH9973: NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS

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PROFESSOR E O'RIORDAN

9:30 – 11:30, Monday, 16 May 2016

Duration: 2 hour

Full marks may be obtained by answering three questions. Candidate's three best questions will contribute to their final mark.

All questions carry equal marks

Approved calculators may be used

Mathematical tables are provided

New Cambridge Statistical Tables are NOT permitted

1. a) Derive the midpoint (Runge-Kutta) method:

$$w_0 = y_0,$$

$$w_{i+1} = w_i + hf(x_i + \frac{h}{2}, w_i + \frac{h}{2}f(x_i, w_i)) \text{ for each } i = 0, 1, ...N-1,$$

and its truncation error, where h is the step size in the x direction for the ordinary differential equation

$$\frac{dy}{dx} = f(x, y),$$

with initial condition

$$y(a) = \alpha.$$

using the following Theorem:
Suppose $f(x, y)$ and all its partial derivatives of order less than or equal to $n+1$ are continuous on $D = \{(x, y)|a \leq x \leq b, c \leq y \leq d\}$ and let $(x_0, y_0) \in D$ for every $(x, y) \in D$, then there exists $\xi \in (x, x_0)$ and $\mu \in (y, y_0)$ with

$$f(x, y) = P_n(x, y) + R_n(x, y)$$

where

$$\begin{aligned} P_n(x,y) &= f(x_0, y_0) + \left[(x-x_0)\frac{\partial f}{\partial x}(x_0, y_0) + (y-y_0)\frac{\partial f}{\partial y}(x_0, y_0)\right] \\ &+ \left[\frac{(x-x_0)^2}{2}\frac{\partial^2 f}{\partial x^2}(x_0, y_0) + (y-y_0)(x-x_0)\frac{\partial f}{\partial y \partial x}(x_0, y_0) + \frac{(y-y_0)^2}{2}\frac{\partial^2 f}{\partial y^2}(x_0, y_0)\right] \\ &+ \ldots + \\ &+ \left[\frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}(x-x_0)^{n-j}(y-y_0)^j\frac{\partial^n f}{\partial y^j \partial x^{n-j}}(x_0, y_0)\right], \end{aligned}$$

and

$$R_n(x, y) = \left[\frac{1}{(n+1)!}\sum_{j=0}^{n+1}\binom{n+1}{j}(x-x_0)^{n+1-j}(y-y_0)^j\frac{\partial^{n+1} f}{\partial y^j \partial x^{n+1-j}}(\xi, \mu)\right]. \tag{12}$$

b) The initial value problem is defined

$$\frac{dy}{dx} = f(x, y),$$

$$y(x_0) = y_0.$$

Definition: The function $f(x, y)$ satisfies a Lipschitz Condition in the variable y,

$$|f(x, y_1) - f(x, y_2)| < L|y_1 - y_2|$$

whenever $(x, y_1), (x, y_2) \in D = \{(x, y)|a \leq x \leq b, c \leq y \leq d\}$.
Show that the midpoint method satisfies the Lipschitz Condition. (9)

c) Consider the differential equation

$$\frac{dy}{dx} - xy + x^2 = 0, \quad 0 \leq x \leq 1, \quad y(0) = 1.$$

Apply the midpoint method to approximate the solution at $y(0.6)$ using $h = 0.2$.
How would you improve on this result? (12)

[33]

2. a) State the 3 classes and conditions of 2nd order Partial Differential Equations defined by the characteristic curves. (5)

b) Given the non-dimensional form of the heat equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2},$$

on the domain $(0, L) \times (0, T]$.
Supply sample boundary conditions to specify this problem.
Write an explicit scheme to solve this partial differential equation. (8)

c) Derive the local truncation error for the explicit method, for the heat equation.

(10)

d) Show that the method is consistent, convergent and stable for $\frac{k}{h^2} < \frac{1}{2}$, where k is the step-size in the t direction and h is the step-size in the x direction. (10)

[33]

3. a) Approximate the Poisson equation

$$-\nabla^2 U(x,y) = f(x,y), \qquad (x,y) \in \Omega = (0,1)\times(0,1),$$

with boundary conditions

$$U(x,y) = g(x,y), \qquad (x,y) \in \delta\Omega$$

using the five point method ∇_h^2. Sketch how the finite difference scheme may be rewritten in the form $Ax = b$, where A is a sparse $N^2 \times N^2$ matrix, and b , x are an N^2 component vectors. (Assume your 2d discretised grid contains N components in the x and y direction).

(10)

b) Prove (DISCRETE MAXIMUM PRINCIPLE). If $\nabla_h^2 V_{ij} \geq 0$ for all points $(x_i, y_j) \in \Omega_h$, then

$$\max_{(x_i,y_j)\in\Omega_h} V_{ij} \leq \max_{(x_i,y_j)\in\partial\Omega_h} V_{ij}.$$

If $\nabla_h^2 V_{ij} \leq 0$ for all points $(x_i, y_j) \in \Omega_h$, then

$$\min_{(x_i,y_j)\in\Omega_h} V_{ij} \geq \min_{(x_i,y_j)\in\partial\Omega_h} V_{ij},$$

where Ω_h is the discrete grid of the area Ω, ∇_h^2 is the five point approximation of ∇^2 and h is the step-size in the x and y direction.

(12)

c) Hence prove: Let U be a solution to the Poisson equation and let w be the grid function that satisfies the discrete form

$$-\nabla_h^2 w_{ij} = f_{ij} \quad \text{for } (x_i, y_j) \in \Omega_h,$$

$$w_{ij} = g_{ij} \quad \text{for } (x_i, y_j) \in \partial\Omega_h.$$

Then there exists a positive constant K such that

$$||U - w||_\Omega \leq KMh^2,$$

where

$$M = \max\left\{ \left\|\frac{\partial^4 U}{\partial x^4}\right\|_\infty, \left\|\frac{\partial^4 U}{\partial x^3 \partial y}\right\|_\infty, \left\|\frac{\partial^4 U}{\partial x^2 \partial y^2}\right\|_\infty, \left\|\frac{\partial^4 U}{\partial x \partial y^3}\right\|_\infty, \left\|\frac{\partial^4 U}{\partial y^4}\right\|_\infty \right\}.$$

(11)

You may assume the **Lemma:** If the grid function $V : \Omega_h \bigcup \partial\Omega_h \to R$ satisfies the boundary condition $V_{ij} = 0$ for $(x_i, y_j) \in \partial\Omega_h$, then

$$||V||_\Omega \leq \frac{1}{8} ||\nabla_h^2 V||_\Omega.$$

[33]

4. **a)** For a finite difference scheme approximating a partial differential equation of the form

$$\frac{\partial U}{\partial t} = -a\frac{\partial U}{\partial x} + f(x,t), \quad x \in R, \quad t > 0$$
$$U(x,0) = U_0(x), \quad x \in R$$

define what is meant by:

i) convergence,

ii) consistency,

iii) stability.

(10)

b) Describe the forward Euler/centered difference method for the transport equation and derive the local truncation error.

(6)

c) For the numerical method in (b) define the Courant Friedrichs Lewy condition and state how it is related to stability.

(5)

d) Show that the numerical method in (b) is unstable under the Courant Friedrichs Lewy condition using Von Neumann analysis, you may assume $f(x,t) = 0$. (12)

[33]