The Weighted Random Graph Model

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Abstract

The author introduced the weighted random graph (WRG) model, which represents "the weighted counter-part of the Erdos-Rényi random graph model", and which provides "fundamental insights into more complicated weighted networks".

Chapter 1

Background

A graph $G(V, E)$ may be considered a collection of vertices and edges. The vertices, or the nodes, are the things we are modelling - computers, people, towns, etc. The edges are the relationships between the things we are modelling - network cables, relationships, roads, etc.

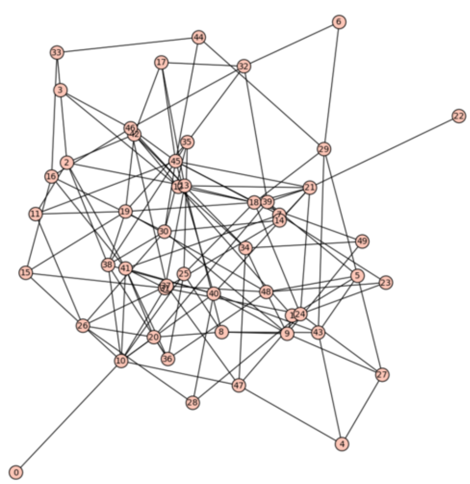

Figure 1.1: A Random Graph

A graph can be *directed* or *undirected*. For example, in an undirected graph representing a network, two nodes a and b (representing computers) connected by an edge (a network cable) can be represented by an unordered pair $\{a, b\}$ - unordered because the connection goes in both directions. In a directed graph representing the world wide web, two nodes a and b (representing pages) connected by and edge (a hyperlink) can be represented by an ordered pair (a, b) - ordered because the connection goes in one direction.

Also a graph can be weighted or unweighted - where edges between vertices can have an associated weight. For example, in a graph representing towns in a country, two nodes a and b representing towns would have an edge $\{a, b\}$ with an associated weight representing the distance between the two towns.

Chapter 2

Methods

When we talk about *Random Graphs* ([1, 2]) we refer to probability distributions on graphs. These graphs can be directed, undirected, weighted, unweighted, etc. Generating ramdom graphs can be done in a number of different ways. For example:

- fix the number of nodes n and the number of edges m
- number the nodes 1 to n
- select two nodes at random forming an edge
- continue m times

another approach:

- fix the number of nodes n and a probability threshold p
- number the nodes 1 to n
- for each pair of nodes generate a random number
- form an edge if the random number is greater than p

There are many statistics of interest around random graphs:

- the degree distribution of the vertices (popularity, centre)
- the number of edges (relationships)
- the number of sub-graphs (triangles, etc.)
- the number of connected components

Chapter 3

Results

Draw an edge between two nodes with weight w with probability:

$$q_{ij}(w) = (y_1 y_j)^w (1 - y_1 y_j)$$

making a few assumptions we can generalise this as:

$$q(w) = p^w (1 - p)$$

it is easy to show that the likelihood maximising choice is:

$$p^* = \frac{2W}{N(N-1) + 2W}$$

the expected weight of any edge ([3]) is:

$$\langle w \rangle = \sum_{w=0}^{+\infty} q(w) w = \frac{p}{1-p}$$

the variance is:

$$\langle w^2 \rangle - \langle w \rangle^2 = \frac{p}{(1-p)^2}$$

the degree distribution of the whole network is:

$$P(k) = \binom{N-1}{k} p^k (1-p)^{N-1-k}$$

the strength distribution is:

$$P(s) = \binom{N-2+s}{N-2} p^s (1-p)^{N-1}$$

If all edges with weight smaller than w are removed, it is clear that the remaining edges form an unweighted projection equivalent to an ER random graph with probability:

$$p_w^+ \equiv \sum_{v=w}^{+\infty} q(v) = p^w$$

similarly, if all edges greater than w are removed, we have:

$$p_w^- \equiv \sum_{v=1}^{w} q(v) = p - p^{w+1}$$

and so on. The Critical weight:

$$w_c^+ \equiv -\frac{\ln N}{\ln p}$$

removing edges of weight greater than or equal to w_c^+ the network will fragment.

$$\lim_{N \to \infty} w_c^+ = \frac{1}{\alpha} \qquad \text{if} \qquad p \sim N^{-\alpha}$$

for strong link removal the critical weight is:

$$w_c^+ \equiv -\frac{\ln (p - 1/N)}{\ln p} - 1 \simeq 0$$

Chapter 4

Comments

The study of random graphs has many applications. For example, the study of the passage of infectious diseases can be helped by the study of random graphs - modelling with a random graph may be the only possibility when the actual graph is not known in advance, or where the actual graph may change regularly.

Also, when studying a graph, by comparing it to known properties of a random graph we can easily decide if an observed property is worth studying - in other words, if an observed property of a graph is present within a random graph, then it is likely not anomalous, and does not warrant further research.

Weighted random graphs provide a more complete description of graphs used to model many different phenomena, and arise naturally when describing networks that model transport for example.

Bibliography

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