Competition Models (Continuation)

p. 115 - 2 exercises

(2)
$$\begin{cases} \dfrac{dN}{dt} = N[A - D(N+E) - B] \\ \dfrac{dE}{dt} = E[A - D(N+E) - sB] \end{cases}$$

$0 < s < 1$, A, B, D - constants, $\underline{A > B}$,

$$\begin{cases} \dfrac{dN}{dt} = N(A-B)\left[1 - \dfrac{D}{A-B}(N) - \dfrac{D}{A-B}E\right] \\ \dfrac{dE}{dt} = N(A-sB)\left[1 - \dfrac{D}{A-sB}E - \dfrac{D}{A-sB}N\right] \end{cases}$$

$r_1 = A-B$, $K_1 = \dfrac{A-B}{D}$, $b_{12} = 1$

$r_2 = A-sB$, $K_2 = \dfrac{A-sB}{D}$, $b_{21} = 1$

$a_{12} = b_{12}\dfrac{K_2}{K_1} = \dfrac{K_2}{K_1} > 1$, $a_{21} = b_{21}\dfrac{K_1}{K_2} = \dfrac{K_1}{K_2} < 1$

$a_{12} > 1$, $a_{21} < 1$

$$\boxed{\frac{N(t)}{E(t)} \sim e^{-B(1-s)t}} \quad ?$$

Stable steady state $N^* = 0$, $E^* = E^*$

$$E = E^* = K_2 = \frac{A - sB}{D}$$

$$N = N^* + n = n$$
$$E = E^* + \varepsilon \qquad , \; n, \varepsilon \ll 1$$

$$\frac{dN}{dt} = N \left[\underline{\underline{A}} - D(N+E) - \underline{\underline{B}} \right]$$

Linearise:

$$\frac{dn}{dt} = n \left(A - B - D(n + E^* + \varepsilon) \right)$$

$$\frac{dn}{dt} = \underline{\underline{n}}(A-B) - \cancel{Dn^2} - \underline{\underline{n}} DE^* - \cancel{n\varepsilon D}$$

$$\frac{dn}{dt} = n \left(A - B - D \cdot E^* \right)$$

$$\frac{dn}{dt} = n \left(A - B - \cancel{D} \cdot \frac{A - sB}{\cancel{D}} \right) =$$

$$\frac{dn}{dt} = -n \underline{\underline{B(1-s)}} \quad \Rightarrow \quad \underline{n(t) = n(0) e^{-B(1-s)t}}$$

$$N = N(0) e^{-B(1-s)t}$$

$$\frac{N(t)}{E(t)} = \frac{N(0)\, e^{-B(1-s)t}}{E^* + \cancel{\text{decay}}} = \left[\frac{N(0)}{E^*}\right] \cdot e^{-B(1-s)t} \mapsto 0$$

lifetime of an individual is 30 years
time to extinction 5000 years

$s = ?$

$$B = \frac{1}{30} \text{ years}^{-1}$$

$$B(1-s) = \frac{1}{5000} \text{ years}^{-1}$$

$$1 - s = \frac{30}{5000} = 0.006$$

$$\boxed{s = 0.994}$$

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$$\frac{dN_1}{dt} = r_1 N_1 \left(1 - \frac{N_1}{K_1} - b_{12} \frac{N_2}{K_1}\right)$$

$$\frac{dN_2}{dt} = r_2 N_2 \left(1 - b_{21} \frac{N_1}{K_2}\right)$$

$$u_1 = \frac{N_1}{K_1}, \quad u_2 = \frac{N_2}{K_2}, \quad \rho = \frac{r_1}{r_2}, \quad \tau = r_1 t$$

$$a_{12} = b_{12} \frac{K_2}{K_1} \qquad a_{21} = b_{21} \frac{K_1}{K_2}$$

$$\begin{cases} \dfrac{du}{d\tau} = u(1-u-a_{12}v) & = f \\ \dfrac{dv}{d\tau} = \rho v(1-a_{21}u) & = g \end{cases}$$

$$A = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix} = \begin{pmatrix} 1-2u-a_{12}v & -a_{12}u \\ -\rho a_{21}v & \rho(1-a_{21}u) \end{pmatrix}$$

Steady states

$$(0,0), \quad (1,0), \quad \left(\frac{1}{a_{21}}, \frac{a_{21}-1}{a_{12}a_{21}}\right)$$

$$(u^*, v^*)$$

$\lambda_1 = 1$
$\lambda_2 = \rho$
unstable node

$\lambda_1 = -1$
$\lambda_2 = \rho(1-a_{21})$
stable node if $a_{21} > 1$
saddle if $0 < a_{21} < 1$

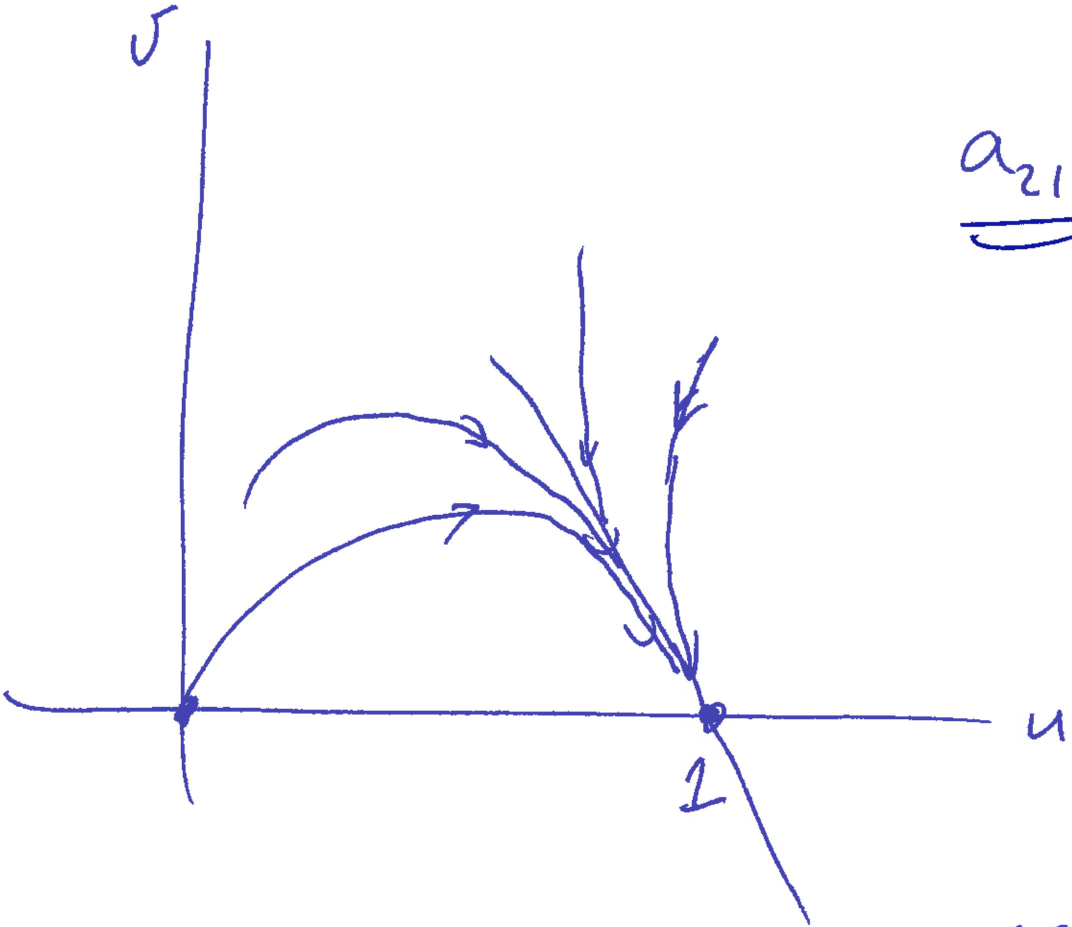
$\mathrm{tr} A = -\frac{1}{a_{21}} < 0$
$\det A = \rho \frac{1-a_{21}}{a_{12}}$
always unstable node if exists

I $a_{21} > 1$
(0,0) unstable
(1,0) stable node
(u^*, v^*) unstable.

II $0 < a_{21} < 1$
(0,0) unstable node
(1,0) saddle pt.
~~(u^*, v^*) does not exist~~

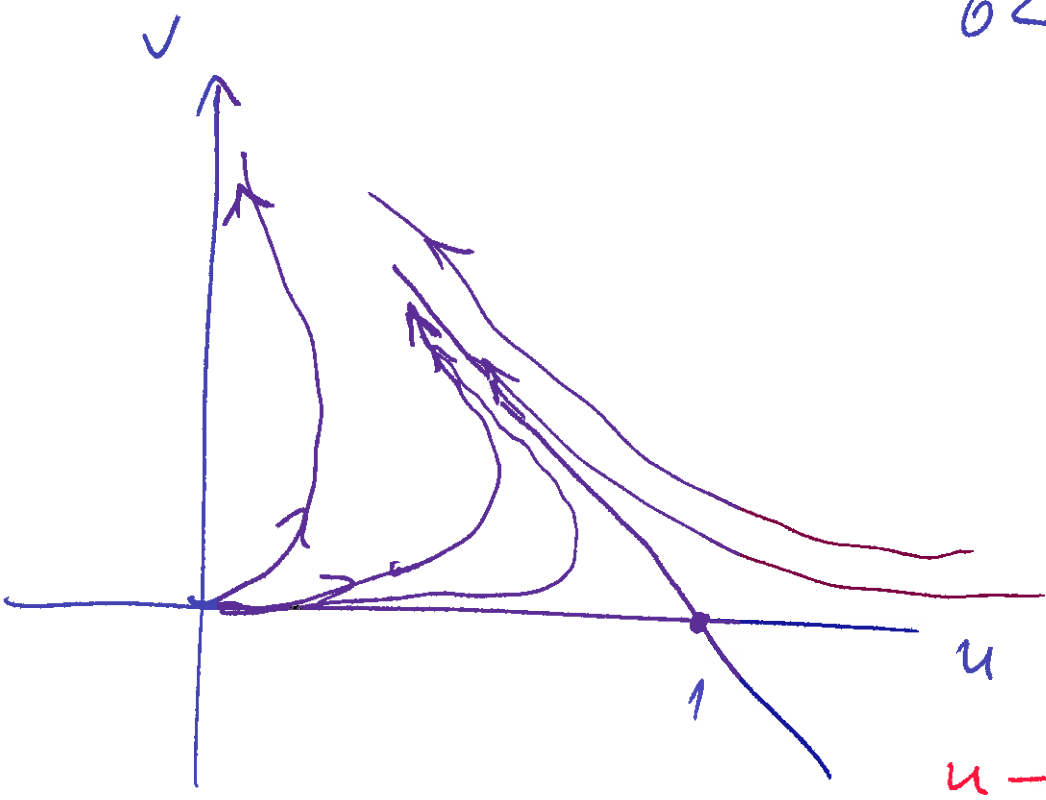
I

$\underline{a_{21} > 1}$



II

$0 < a_{21} < 1$



$u \to 0$
$v \to \infty$

Mutualism or Symbiosis

$$\begin{cases} \dfrac{dN_1}{dt} = r_1 N_1 \left(1 - \dfrac{N_1}{K_1} + b_{12} \dfrac{N_2}{K_1}\right) \\[2ex] \dfrac{dN_2}{dt} = r_2 N_2 \left(1 - \dfrac{N_2}{K_2} + b_{21} \dfrac{N_1}{K_2}\right) \end{cases}$$

$$u_1 = \frac{N_1}{K_1}, \quad u_2 = \frac{N_2}{K_2}, \quad \tau = r_1 t, \quad \rho = \frac{r_2}{r_1}$$

$$a_{12} = b_{12} \frac{K_2}{K_1} \qquad a_{21} = b_{21} \frac{K_1}{K_2}$$

$$\begin{cases} \dfrac{du_1}{d\tau} = u_1 (1 - u_1 + a_{12} u_2) \\[2ex] \dfrac{du_2}{d\tau} = \rho u_2 (1 - u_2 + a_{21} u_1) \end{cases}$$

Steady states: $(0,0)$, $(1,0)$, $(0,1)$,

$$u_1^* = \frac{1 + a_{12}}{1 - a_{12} a_{21}} \qquad u_2^* = \frac{1 + a_{21}}{1 - a_{12} a_{21}}$$

Positive if $1 - a_{12} a_{21} > 0$

$(0,0)$	$(1,0)$	$(0,1)$	(u_1^*, u_2^*)
$\lambda_1 = 1$	$\lambda_1 = -1$	$\lambda_1 = -\rho$	$\lambda_1 < 0$
$\lambda_2 = \rho$	$\lambda_2 = \rho(1 + a_{21})$	$\lambda_2 = 1 + a_{12}$	$\lambda_2 < 0$
unst.	saddle.	saddle.	stable node if exists.

$(u_1^*, u_2^*) \longrightarrow$

$$\lambda_{1,2} = \frac{-a_{12}-1+\rho(-a_{21}-1) \pm \sqrt{(+a_{12}+1+\rho(+a_{21}+1))^2 - 4\rho(1-a_{12}a_{21})(a_{21}+1)(a_{12}+1)}}{2(1-a_{12}a_{21})}$$

$$\lambda_{1,2} = \frac{-[a_{12}+1+\rho(a_{21}+1)] \pm \sqrt{[a_{12}+1+\rho(a_{21}+1)]^2 - 4\rho(1-a_{12}a_{21})\underline{(1+a_{12})}\,\underline{(1+a_{21})}}}{2(1-a_{12}a_{21})}$$

<u>Observations:</u>

less than 1

$D>0$: $\quad [A+B]^2 \geq 4AB \geq 4(1-a_{12}a_{21})AB$

$A = a_{12}+1$

$B = \rho(a_{21}+1)$

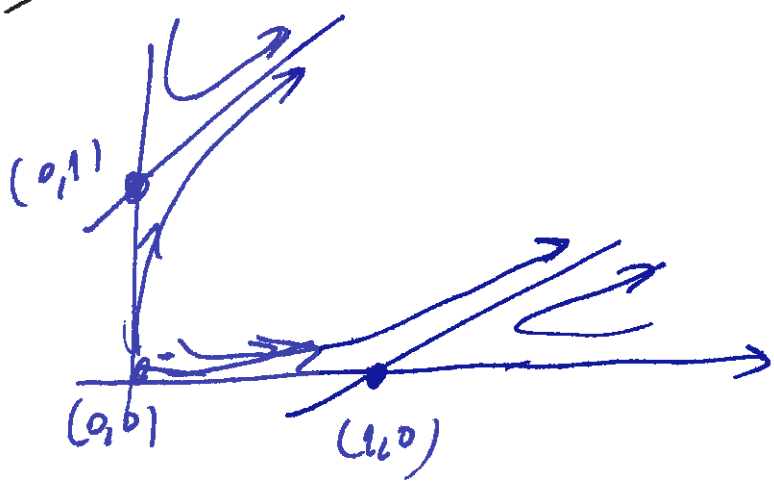
$A^2+2AB+B^2 \geq 4AB$

$A^2-2AB+B^2 \geq 0$

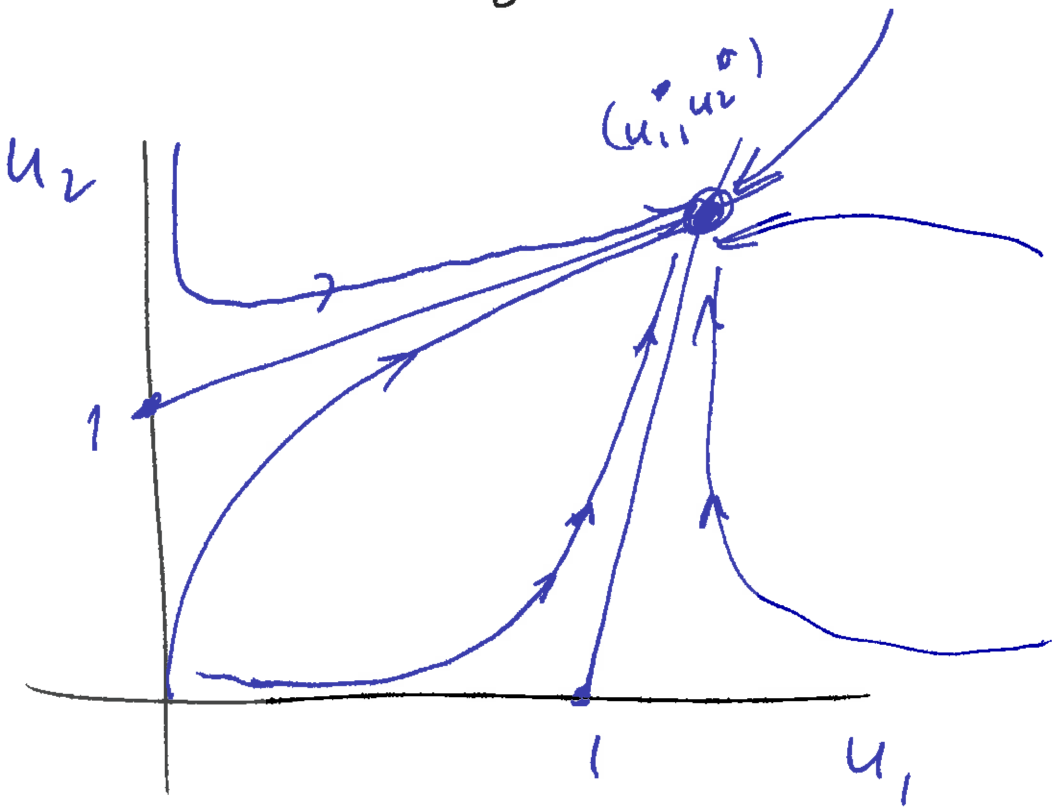
$(A-B)^2 \geq 0$ ✓

$\left| \begin{array}{l} \lambda_1 < 0 \\ \lambda_2 < 0 \end{array} \right.$ stable node.

<u>I</u> (u_1^*, u_2^*) does not exist $\quad \underline{1-a_{12}a_{21} < 0}$



unbounded growth of both u_1, u_2.



Homework, ex. # 3, page 116

Dynamics of Infectious Diseases

(p. 315) vol. 1.

Historical facts

14th sentury 'Black Death'

1/3 of 85 mln. have died

Influenza (1918-1919)

bacterial	viral	parasitic	fungal
chlamidia gonorrhea syphilis TB cholera	HIV herpes influenza SARS chickenpox small pox measles (rubeola)	scabis	candidiasis (thrush)
antibiotiq	vacsines		

S – susceptibles class
I – infectious class
R – removed class

'SIR' – models

$$S \longrightarrow I \longrightarrow R.$$

$$\begin{cases} \dfrac{dS}{dt} = -rSI \\ \dfrac{dI}{dt} = rSI - aI \\ \dfrac{dR}{dt} = aI \end{cases}$$

$\frac{1}{a}$ – 'lifespan' of the infectious state

$r > 0$
$a > 0$

$$\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = -rSI + rSI - aI + aI = 0$$

$S + I + R = N$ = total size of the population.

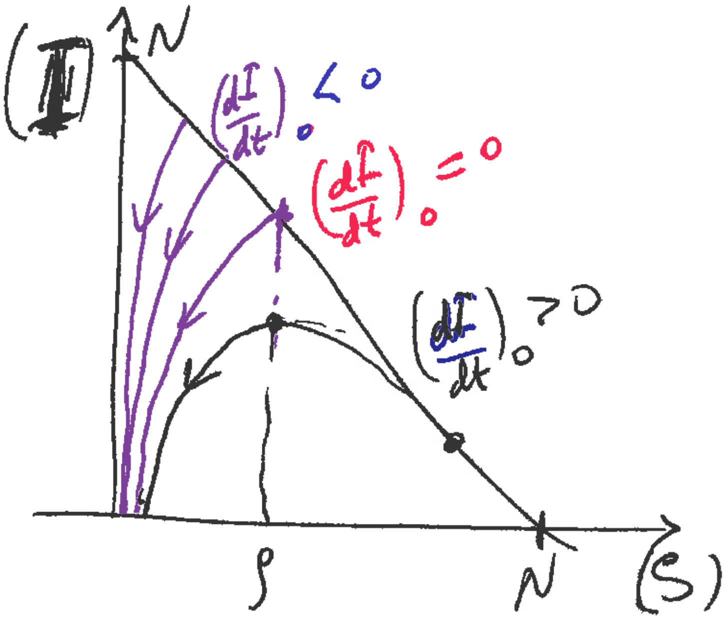
$S(0) = S_0$

$I(0) = I_0$

$R(0) = 0$

$\boxed{N = S_0 + I_0}$

$t=0: \quad \boxed{S + I = N}$



$$\left(\frac{dI}{dt}\right)_{t=0} = r I_0 S_0 - a I_0 = I_0 (r S_0 - a)$$

$$r S_0 - a \begin{cases} > 0 & \text{if } S_0 > \rho = \frac{a}{r} \\ < 0 & \quad\;\; S_0 < \rho = \frac{a}{\rho} \end{cases}$$

$$\frac{dI}{dt} = I(rS - a) = Ir(S - \rho)$$

$S_0 = \rho$ is a 'threshold'

$t \to \infty \quad \boxed{S(\infty) < \rho} \quad I(\infty) = 0,$

$R(\infty) = N - I(\infty) - S(\infty) = 1 - S(\infty)$