Tutorial: Cobwebbing and population growth Models **MATH 1310**

SOLUTIONS IN COLOR

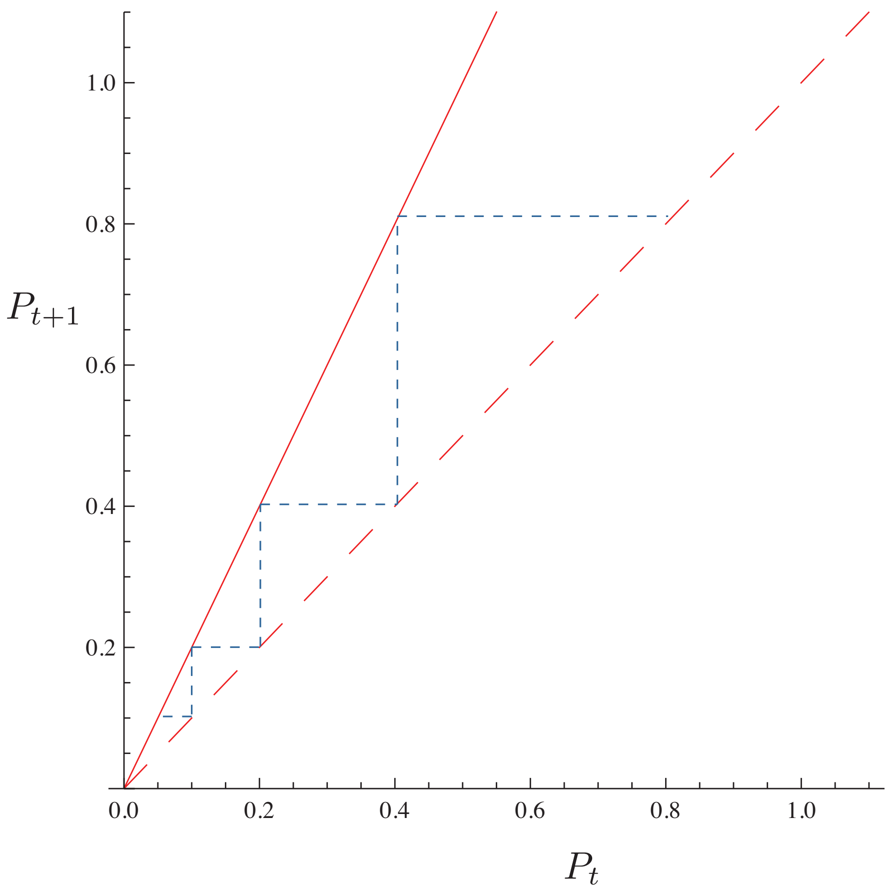
Goal: To explore various models of population growth through updating functions and cobwebbing

The most basic, “unchecked” model of population growth assumes that population increases by a constant factor per unit time. Let’s say, for the sake of argument, that our population doubles in each time unit:

$$P_{t+1} = 2P_t.$$

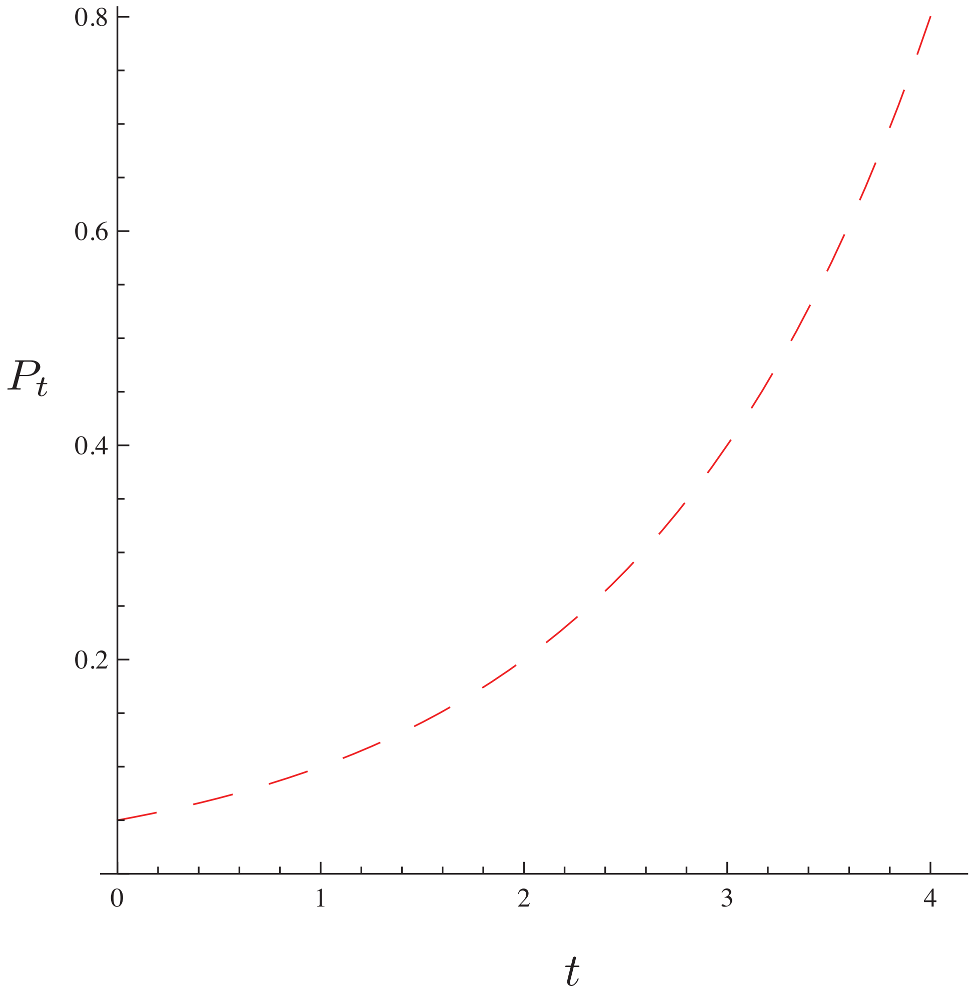
Let’s also say, for the sake of argument, that we’re measuring P_t in millions (of bacteria, or whatever).

1. Sketch the graph of this updating function on the axes below, along with the graph of the diagonal line $P_{t+1} = P_t$.



2. Now, cobweb your graph above, to find successive values $P_1, P_2, P_3, \ldots,$ assuming $P_0 = 0.05$. See above Cobweb in BLUE.

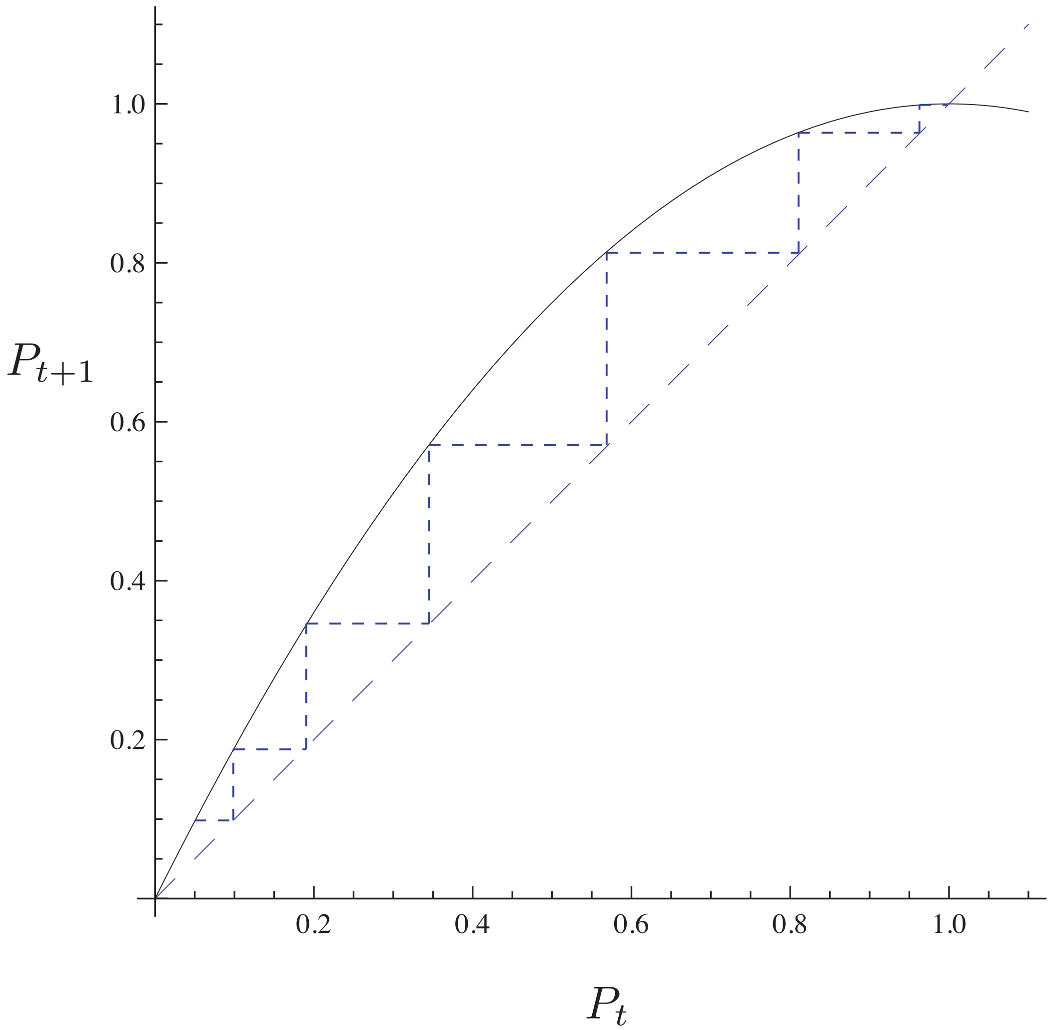
3. Sketch the graph of P_t versus t on the axes on the next page.



4. How would you describe the growth of P_t as a function of t? The growth is exponential.

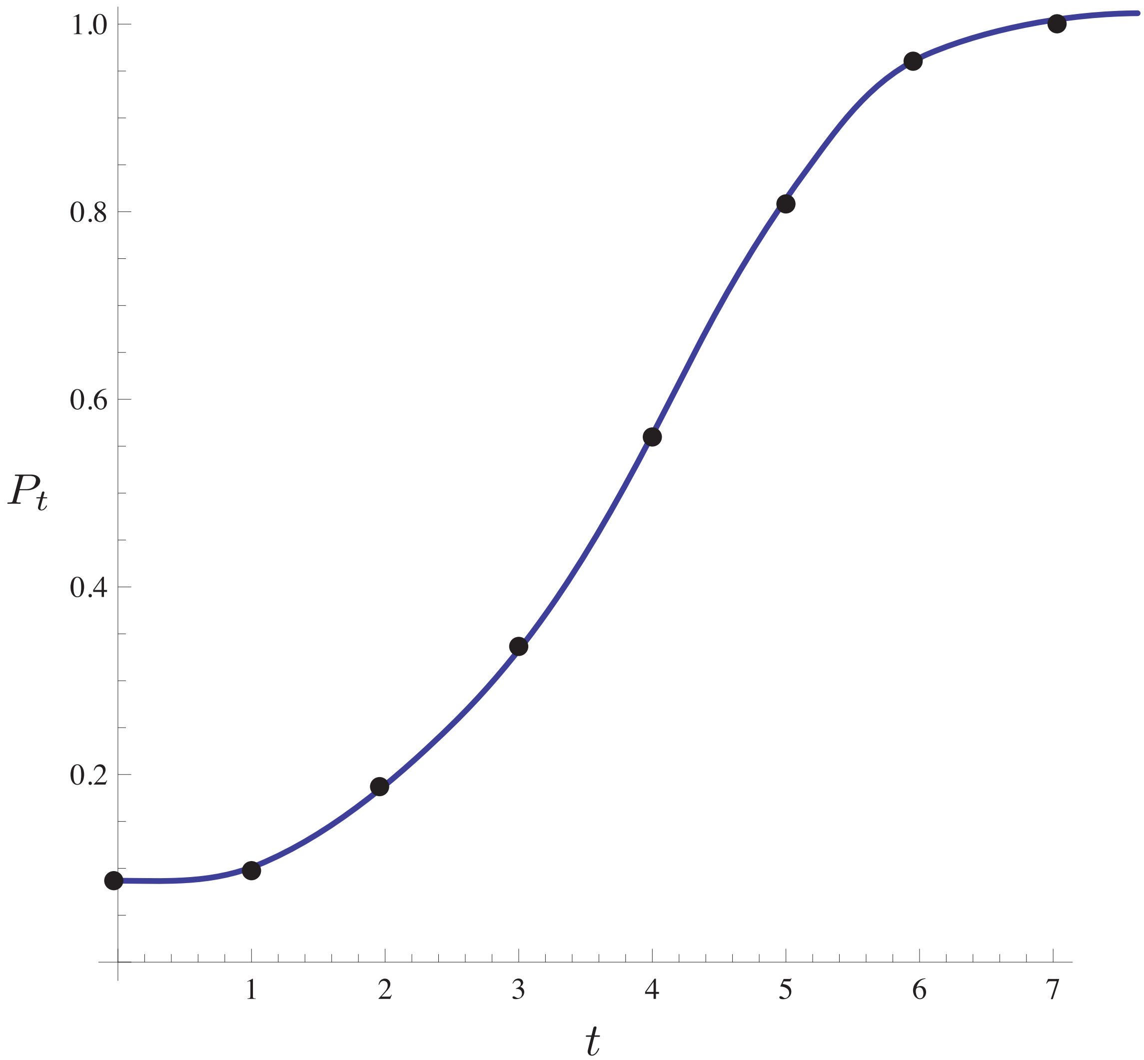
A more realistic model of population growth (in some situations) is a *logistic* model. In logistic growth, the "per capita reproduction rate" levels off as the population size increases.

On the axes below we have drawn the graph of a possible updating function f for logistic growth. We have also dashed in the diagonal line $P_{t+1} = P_t$.



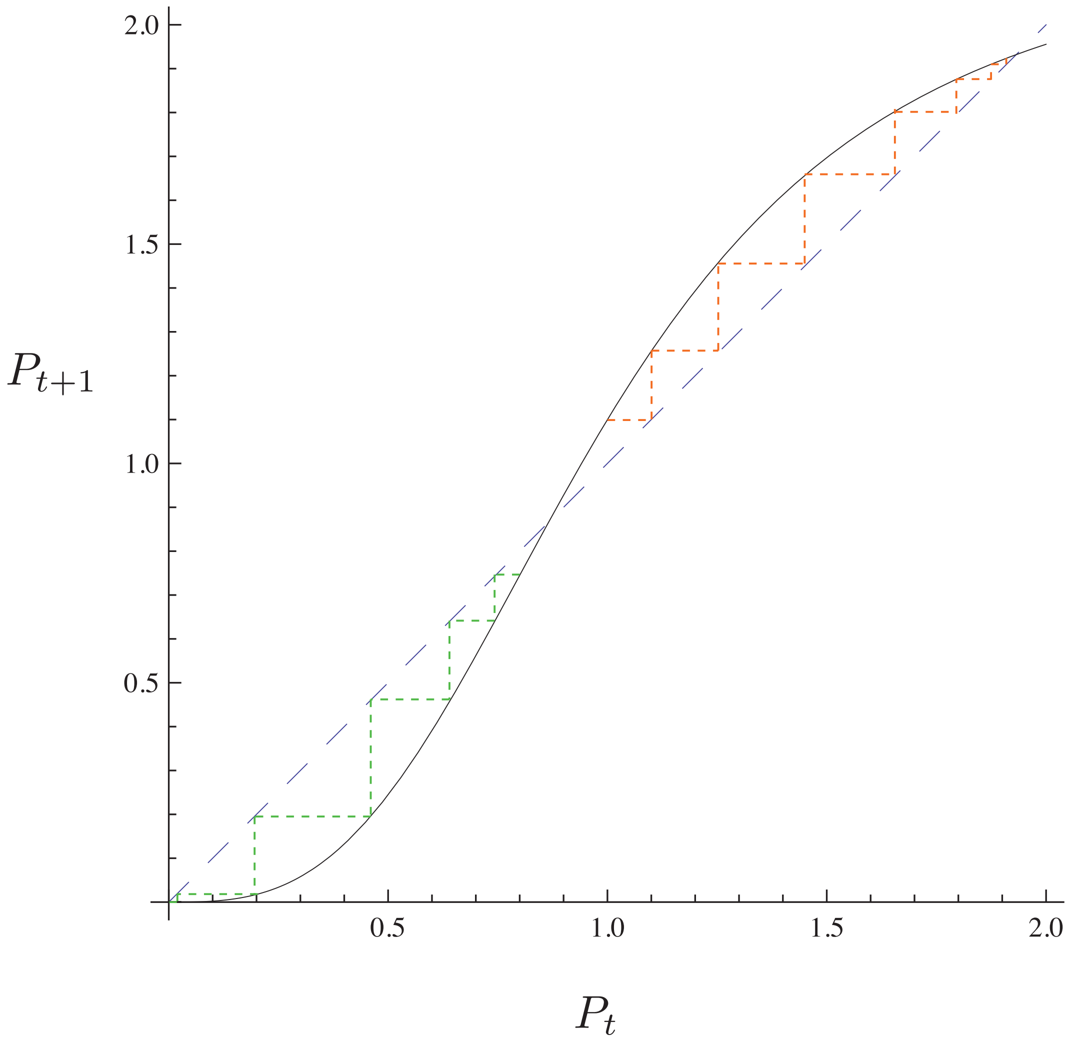
5. Cobweb your graph above, to find successive values $P_1, P_2, P_3, \ldots$, assuming $P_0 = 0.05$.

6. Sketch the graph of P_t versus t on the axes on the next page.



7. How would you describe the growth of P_t as a function of t? The population grows slowly at first, then more quickly, then more slowly. The population growth follows an "S" curve.

Other models of population growth have updating functions $P_{t+1} = f(P_t)$ that cross the line $P_{t+1} = P_t$ several times. One example is sketched below.

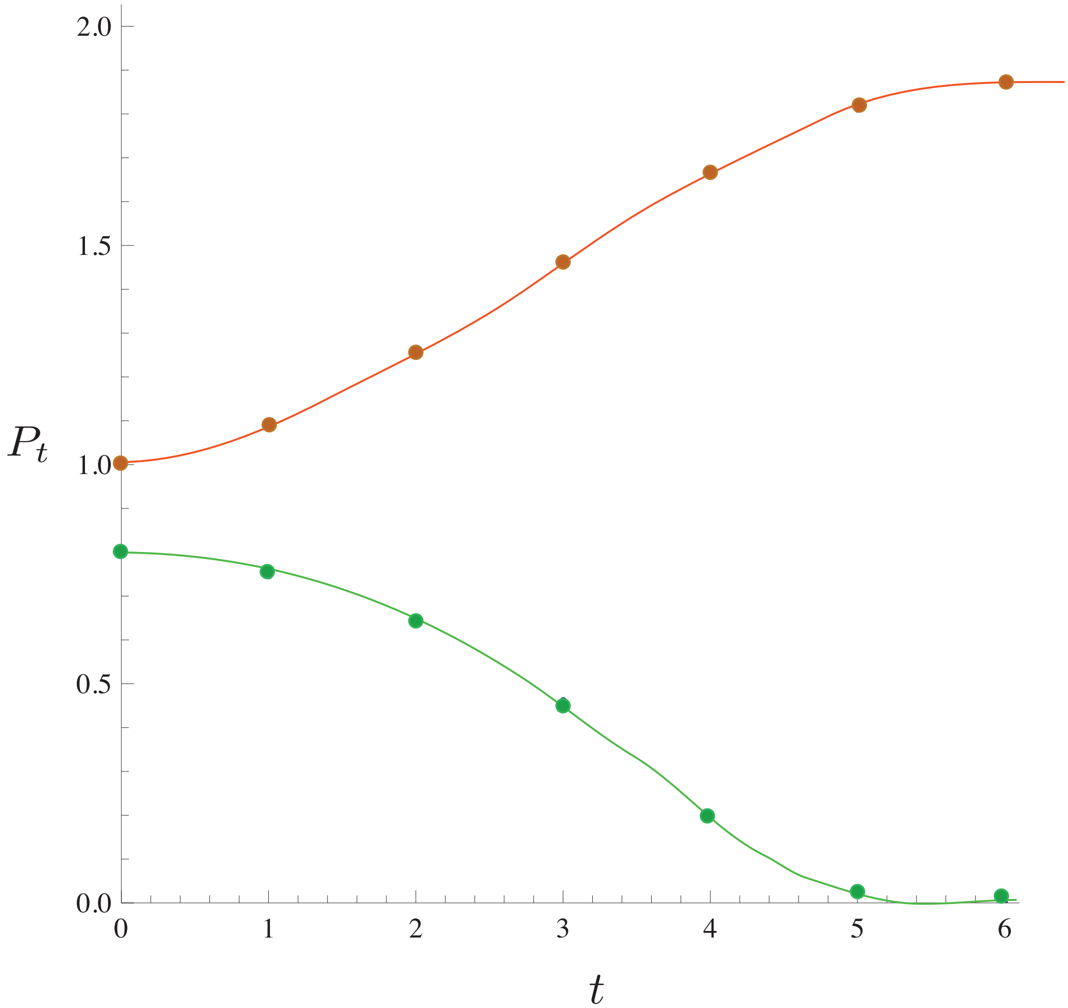


8. What (approximately) are the equilibrium points P^* for the above discrete time dynamical system? The eq uilibrium points are the values, call them P^*, of $_t$ where the graph of the updating function crosses the graph of the diagonal line $P_{t+1} = P_t$. From the above graph, we see that this happens roughly at $P^* = 0$, $P^* = 0.9$, and $P^* = 2.0$.

9. Cobweb your graph above, to find successive values $P_1, P_2, P_3, \ldots,$ assuming $P_0 = 0.8$. See above GREEN cobweb.

10. Repeat problem 9 above, but this time assuming $P_0 = 1.0$. See above ORANGE cobweb.

11. On the axes below, sketch the graphs of P_t versus t, for *each* of the two initial values considered in problems 9 and 10 above.



12. Provide some qualitative discussion of why and under what conditions a population might look like the one in problems 8–11 above. In particular, note that the behavior of the solution depends rather dramatically on the initial conditions. What kind of phenomenon might this be reflecting? There seems to be some sort of "critical mass" that ones needs from one's initial population, in order to insure survival. If one starts below this critical mass (that is, below the *unstable* equilibrium point at about $P^* = 0.9$), then there is not enough interaction in the population to create a birth rate that exceeds the death rate. So eventually, in this case, the population dies off to zero.

If, on the other hand, initial population exceeds the critical mass, then the population thrives. However, even in this case, it does not go unchecked; it levels off at about $P_t = 2.0$. (Once the population gets large enough, factors like disease and competition for resources will act to stabilize the population.)