28/11/2011

Discrete Population Models for a Single Species

$$N_{t+1} = f(N_t) = \underbrace{N_t F(N_t)}_{\sim N_t}$$

Recurrence equations

- linear $\quad N_{t+1} = r N_t$

$$\boxed{N_{t+1} - r N_t = 0}$$

$$N_t = N_0 \cdot r^t$$

$$a N_{t+2} + b N_{t+1} + c N_t = 0$$

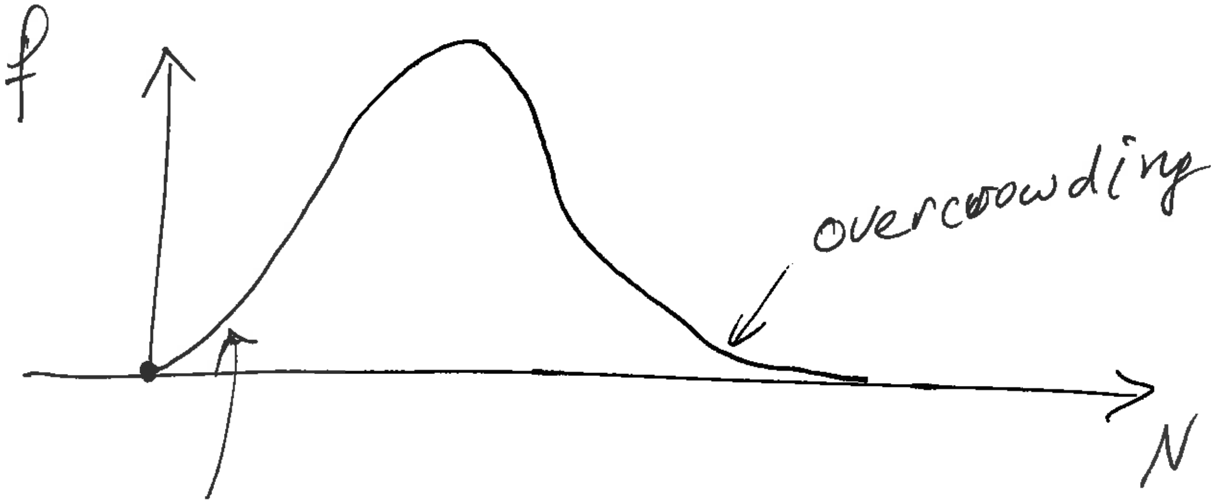
N_0, N_1 given , $N_t = \lambda^t$

$$a\lambda^{t+2} + b\lambda^{t+1} + c\lambda^t = 0 \quad \Big| \cdot \frac{1}{\lambda^t}$$

$\boxed{a\lambda^2 + b\lambda + c = 0}$ characteristic eqn.

$$N_t = A\lambda_1^t + B\lambda_2^t$$

$$N_{t+1} = f(N_t)$$

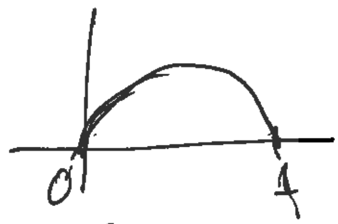


- $\cancel{u_{t+1} = r' u_t (1 - u_t)}$

$$u_t = \frac{N_t}{K}$$

$r \to \cancel{r'}$ $1+r$

- $u_{t+1} = r\, u_t (1 - u_t)$



- $N_{t+1} = N_t \, e^{r\left(1 - \frac{N_t}{K}\right)}$ (Ricker)

$e^{-\frac{N_t}{K}}$ mortality factor

$\boxed{N_t = K}$ constant solution

Equilibrium states

- $u_{t+1} = r u_t (1-u_t)$

$$u^* = r u^* (1-u^*)$$

$\boxed{u^*=0}$ $\qquad 1 = r(1-u^*)$

$$\frac{1}{r} = 1-u^* \Rightarrow u^* = 1-\frac{1}{r} = \frac{r-1}{r}$$

$u^* = \frac{r-1}{r}$ exists if $\boxed{r>1}$

- $N_{t+1} = N_t \exp\left[r\left(1-\frac{N_t}{K}\right)\right]$ (Ricker)

N^* - equilibrium, then

$$N^* = N^* \exp\left[r\left(1-\frac{N^*}{K}\right)\right]$$

$N^* = 0,$ $\qquad 1 = \exp\left[\underbrace{r\left(1-\frac{N^*}{K}\right)}_{0}\right]$

$N^* = K$

$$f(x) = x\, e^{r(1-\frac{x}{K})}$$

$f(0) = 0$

$f(\infty) = 0$

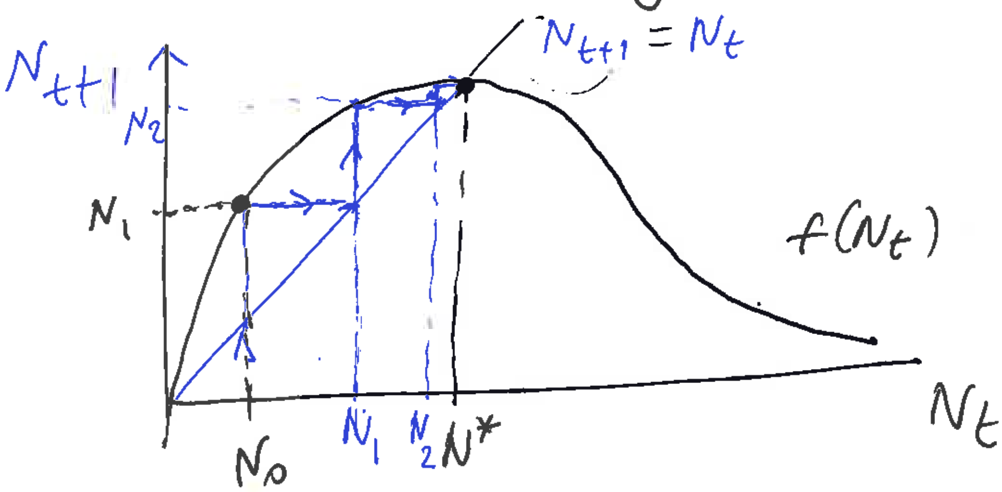
$$f'(x) = e^{r(1-\frac{x}{K})} - \frac{xr}{K} e^{r(1-\frac{x}{K})} = \left(1-\frac{xr}{K}\right) e^{r(1-\frac{x}{K})}$$

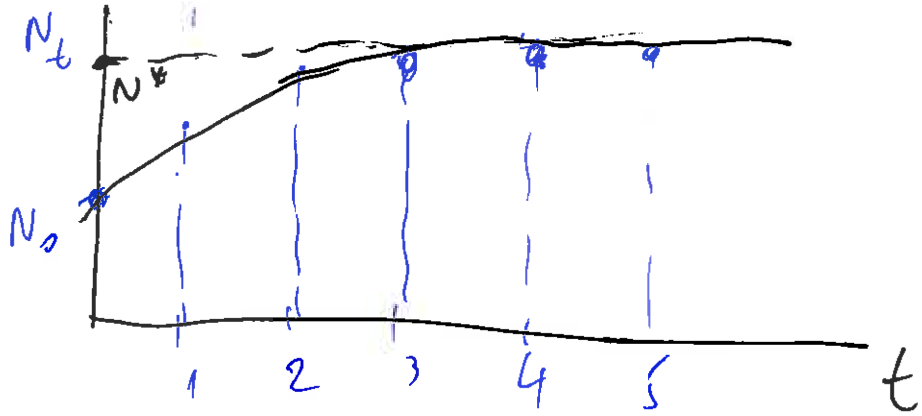
$x_m = \frac{K}{r}$

Graphical Procedure of Solution

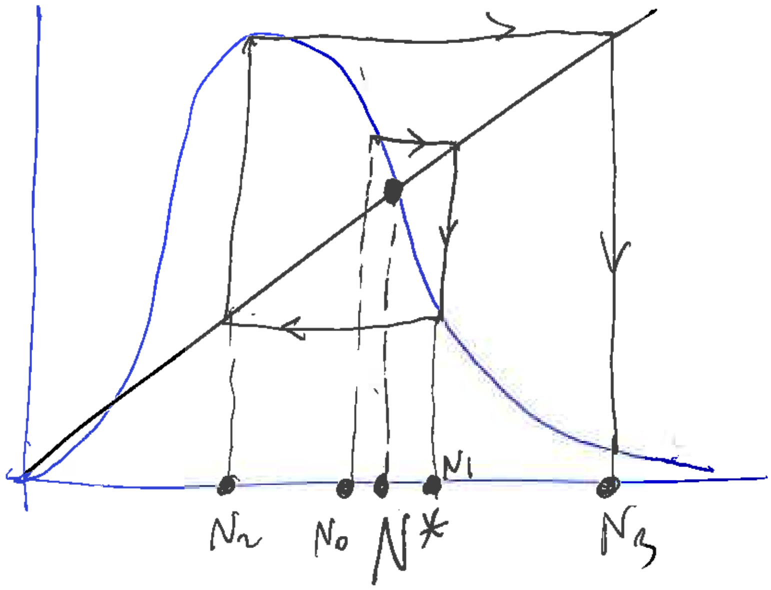
Cobwebbing

$$N_{t+1} = f(N_t)$$

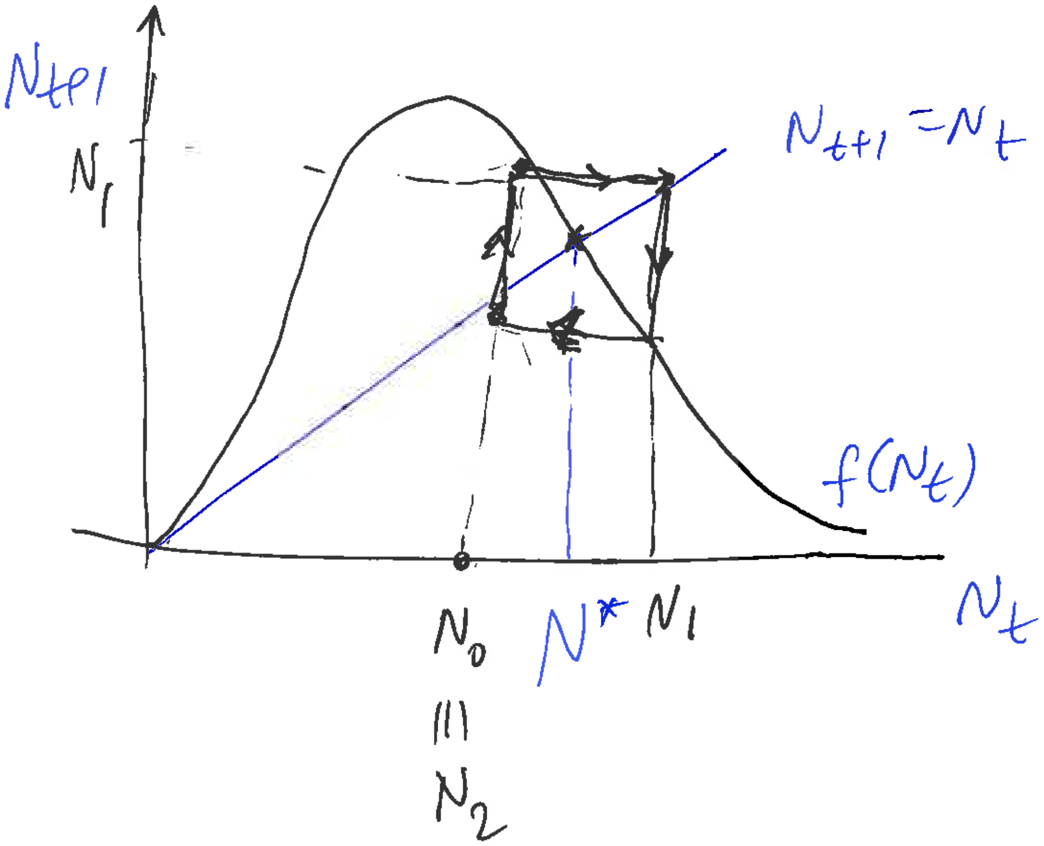




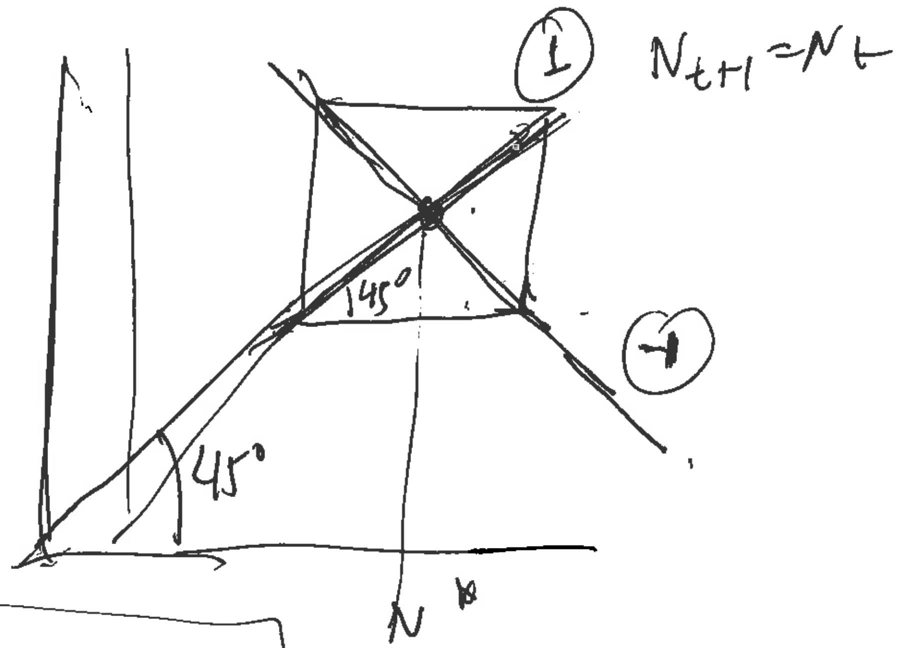
N^* is stable equilibrium



N^* is unstable equilibrium



Example of periodic solution



$$\boxed{\left(\frac{df}{dN}\right)_{N^*} = -1}$$ - periodic

$$\left(\frac{df}{dN}\right)_{N^*} < -1$$ unstable eq. at N^*

$$\left|\left(\frac{df}{dN}\right)_{N=N^*}\right| < 1 , \quad N^* \text{ is stable eq.}$$

$$\left|\left(\frac{df}{dN}\right)_{N=N^*}\right| > 1 \quad N^* \text{ is unstable eq.}$$

$$\left|\left(\frac{df}{dN}\right)_{N=N^*}\right| = 1 \quad \text{periodic solution}$$

Th Let $N = N^*$ is a solution of $N = f(N)$ and suppose that $f(N)$ has a continuous derivative in some interval $J \ni N^*$. Then if $|f'(N)| \leq \alpha < 1$ in J

then $\lim_{t \to \infty} N_t = N^*$ for any $N_0 \in J$

where $N_{t+1} = f(N_t)$.

<u>Proof</u>: Mean value theorem, there exists ν between N_t and N^*

$$f(N_t) - f(N^*) = f'(\nu)(N_t - N^*)$$

$$N^* = f(N^*)$$

$$N_{t+1} - N^* = f'(\nu)(N_t - N^*)$$

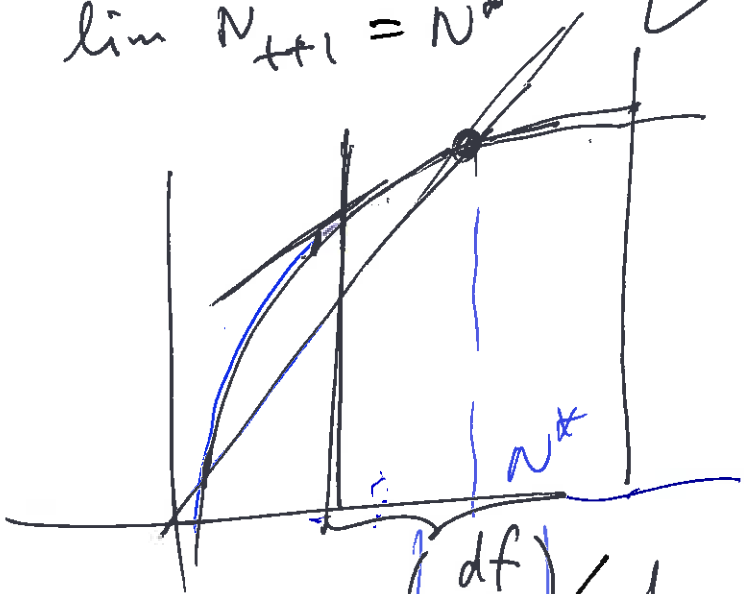
$$|N_{t+1} - N^*| = |f'(\nu)|\,|N_t - N^*| \leq \alpha\,|N_t - N^*|$$

$$|N_{t+1} - N^*| \leq \alpha |N_t - N^*| \leq \alpha^2 |N_{t-1} - N^*| \leq \cdots$$

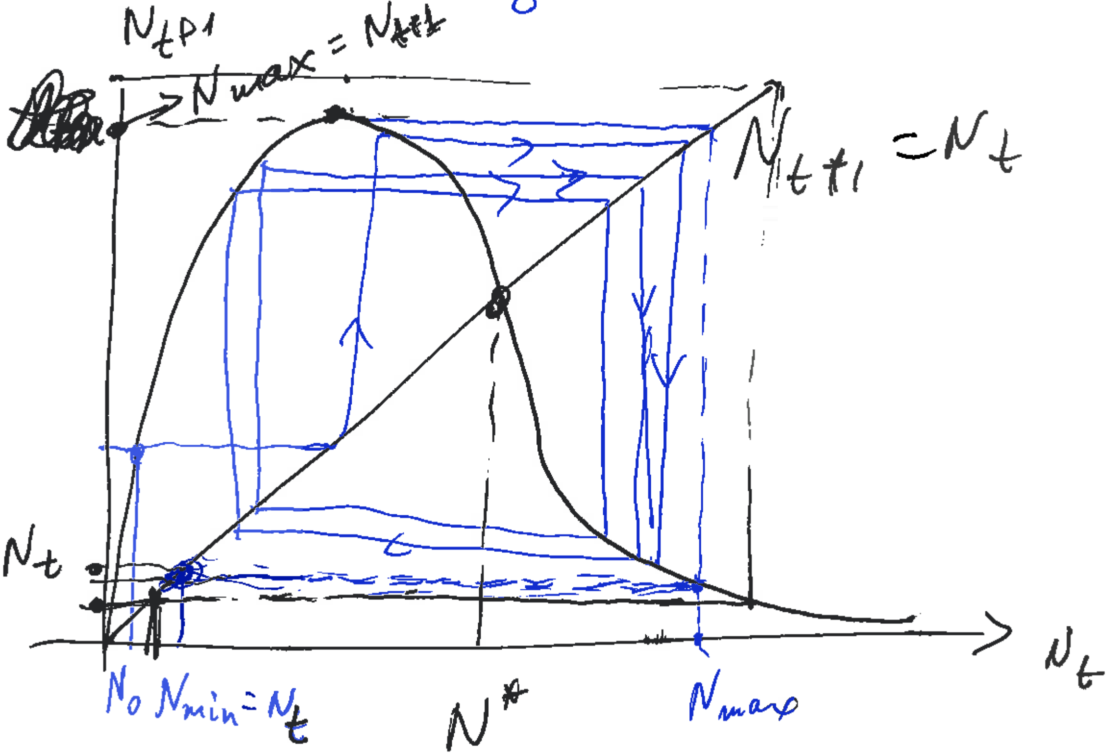
$$\cdots \leq \alpha^{t+1} |N_0 - N^*|$$

$$|N_{t+1} - N^*| \leq \underset{\downarrow \atop 0.}{\alpha^{t+1}} |N_0 - N^*|$$

$$\lim N_{t+1} = N^*$$



$$\left|\frac{df}{dN}\right|_{N=N^*} < 1$$



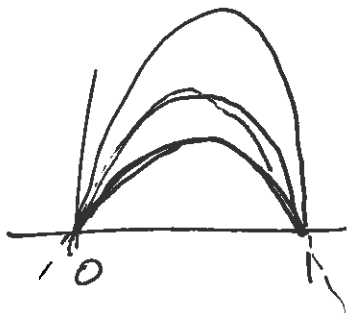
$$\boxed{N_{max} = f(N_{min})}$$

$$N_{min} \leq N_t \leq N_{max}$$

$$u_{t+1} = r u_t (1-u_t) \qquad u_t = \frac{N_t}{K}$$

$$u^* = 0 \qquad u^* = \frac{r-1}{r}$$

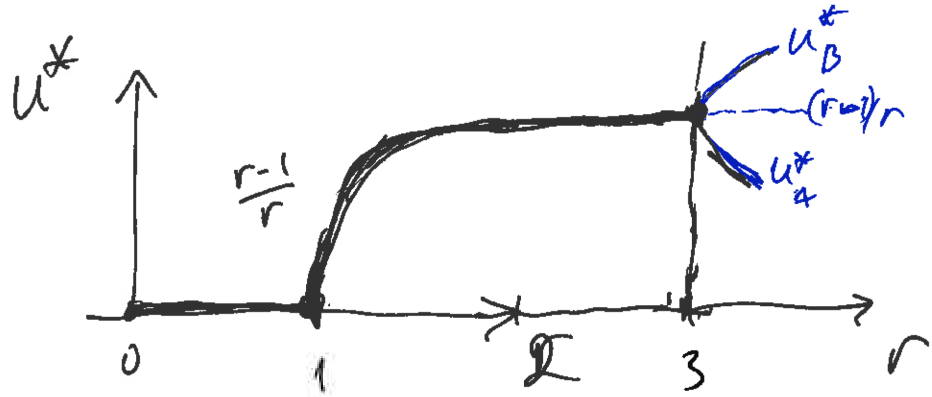
$$\underline{f(u) = ru(1-u)}$$



$$f'(u) = r - 2ru$$

'Eigenvalues' $\quad u^* = 0 \Rightarrow \lambda = f'(0) = r$

$$u^* = \frac{r-1}{r} \qquad \lambda = f'\left(\frac{r-1}{r}\right) = r - 2r\frac{r-1}{r}$$
$$= r - 2r + 2 = \underline{\underline{2-r}}$$



$\left.\begin{matrix} r=1 \\ r=3 \end{matrix}\right\}$ bifurcation values $\left|\left(\frac{\partial f}{\partial u}\right)_{u=u^*}\right| = 1$

$$u_{t+2} = r u_{t+1}(1-u_{t+1}) =$$
$$= r[ru_t(1-u_t)][1 - ru_t(1-u_t)]$$

$$u_{t+2} = f(u_t)$$

$$u^{*\prime} = r^2 \cancel{u^*}(1-u^*)[1 - ru^* + r(u^*)^2]$$

$$u^* = \frac{r-1}{r}$$

$u^* = 0$ unstable

$$1 = r^2(1-u)[1-ru+ru^2]$$

Check:

$$0 = 1 - r^2(1-u)[1-ru+ru^2] \equiv \underbrace{(ru - r+1)[r^2u^2 - r(r+1)u + r+1]}$$

$$\cancel{1} - r^2(\cancel{1} - \cancel{ru} + \cancel{ru^2} - \cancel{u} + ru^2 - \cancel{ru^3}) \stackrel{?}{\equiv}$$

$$= \cancel{r^3u^3} - r^2(r+1)u^2 + \cancel{r(r+1)u}$$
$$\cancel{-r^3u^2} + r^2(\cancel{r}+\cancel{1})u - \cancel{r^2(-r)}$$
$$+ \cancel{r^2u^2} - \cancel{r(r+1)u} + \cancel{r} + \cancel{1}$$

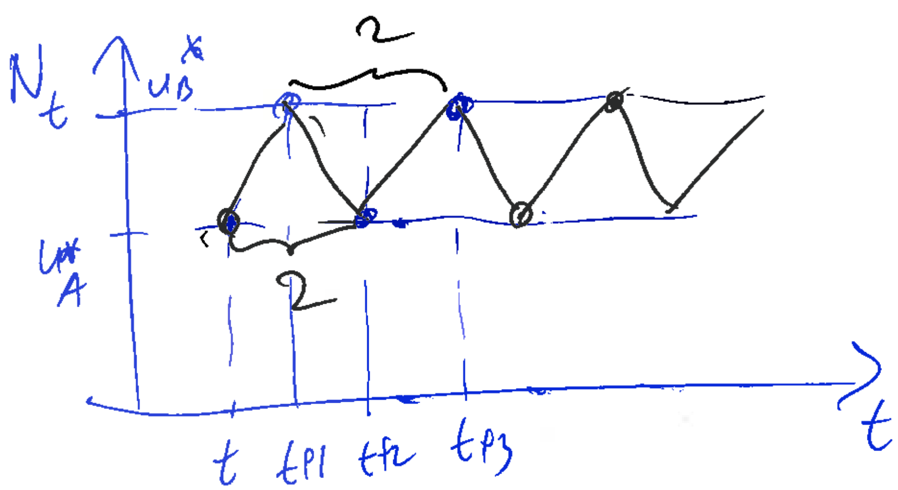
$$\boxed{r^2u^2 - r(r+1)u + r+1 = 0}$$

$$u^* = \frac{r(r+1) \pm \sqrt{r^2(r+1)^2 - 4r^2(r+1)}}{2r^2}$$

$$u^* = \frac{r+1 \pm \sqrt{r^2+2r+1-4r-4}}{2r} = \frac{r+1 \pm \sqrt{r^2-2r-3}}{2r}$$

$$u^* = \frac{r+1 \pm \sqrt{(r+1)(r-3)}}{2r} > 0 \quad \text{if} \quad \underline{r > 3}$$

$$r+1 > \sqrt{(r+1)(r-3)}$$
$$(r+1)^{\cancel{2}} > \cancel{(r+1)}(r-3) > 0$$
$$r+1 > r-3$$
$$1 > -3$$



periodic solution of period 2

~~$N_{t+4} = f^{(4)}(N_t)$~~

$u_{t+4} = f^{(4)}(u_t)$ - 4-periodic

$u_{t+8} = f^{(8)}(u_t)$ - 8-periodic

$u_{t+2^n} = f^{2^n}(u_t)$ 2^n-periodic.

$u_{t+3} = f^{(3)}(u_t)$ $\underline{r_3 \approx 3.828}$

For $r > r_3$ — chaotic solutions

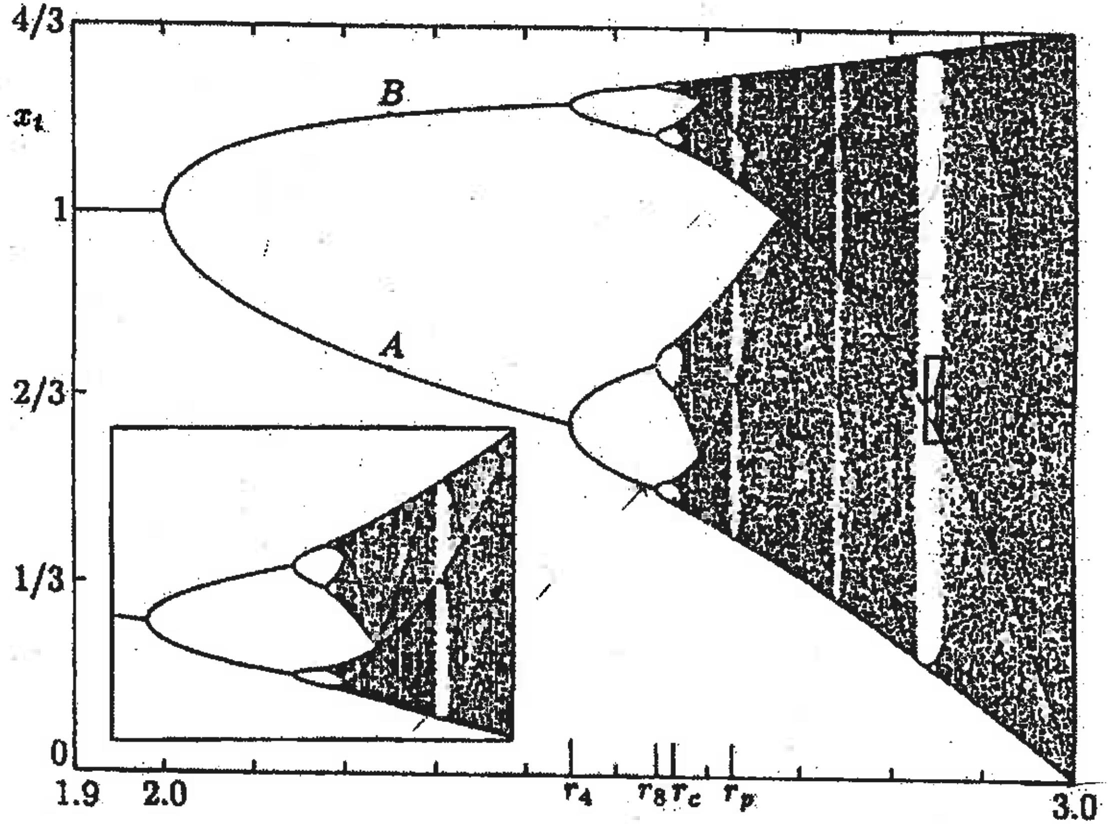


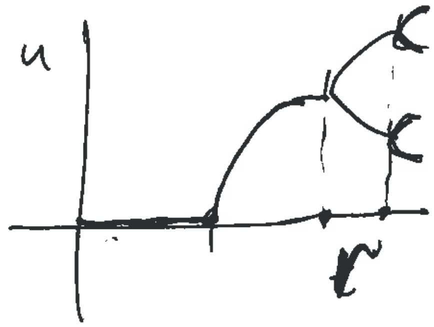
Figure 2.11. Long time asymptotic iterates for the discrete equation $x_{t+1} = x_t + rx_t(1 - x_t)$ for $1.9 < r < 3$. By a suitable rescaling, ($u_t = [r/(r+1)]x_t$, 'r' $= 1 + r$), this can be written in the form (2.11). These are typical of discrete models which exhibit period doubling and eventually chaos and the subsequent path through chaos. Another example is that used in Figure 2.10; see text for a detailed explanation. The enlargement of the small window (with a greater magnification in the r-direction than in the x_t direction) shows the fractal nature of the bifurcation sequences. (Reproduced with permission from Peitgen and Richter 1986; some labelling has been added)

Lecture 10

Stability, Periodic Solutions and Bifurcations

$$u_{t+1} = f(u_t; r)$$

$r = r_c$ - bifurcation



$$u^* = f(u^*; r) \longrightarrow u^*(r)$$

$$u_t = u^* + v_t, \quad |v_t| \ll 1$$

$$\cancel{u^*} + v_{t+1} = f(u^* + v_t) = \cancel{f(u^*)} + \left(\frac{\partial f}{\partial u}\right)_{u=u^*} \cdot v_t$$

$$v_{t+1} = f'(u^*)\, v_t \qquad \boxed{\lambda = f'(u^*), \text{ eigenvalue}}$$

$$v_t = [f'(u^*)]^t\, v_0 = \lambda^t v_0 \to \begin{cases} 0 & \text{if } |\lambda| < 1 \\ \pm\infty & \text{if } |\lambda| > 1 \end{cases}$$

$$u^* \text{ is } \begin{cases} \text{stable} \\ \text{unstable} \end{cases} \text{ if } \begin{cases} -1 < f'(u^*) < 1 \\ |f'(u^*)| > 1 \end{cases}$$

$$u_t = u^* + [f'(u^*)]^t c_0$$

$$\begin{cases} x_{t+1} = \sqrt{2+x_t} \\ x_0 = \sqrt{2} \end{cases}$$

$$x_1 = \sqrt{2+x_0} = \sqrt{2+\sqrt{2}}$$

$$x_2 = \sqrt{2+x_1} = \sqrt{2+\sqrt{2+\sqrt{2}}}$$

$$\vdots$$

$$x_t = \sqrt{2+\sqrt{2+\sqrt{2+\cdots}}} \quad (t+1 \text{ radicals})$$

If x^* is a stable steady state, then

$$\lim_{t\to\infty} x_t = x^*$$

$$x^* = \sqrt{2+x^*} \Rightarrow (x^*)^2 = 2+x^*$$

$$(x^*)^2 - x^* - 2 = 0 \Rightarrow \begin{array}{l} x^* = 2 \\ \cancel{x^* = -1} \end{array}$$

$$x^* = \underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}}}_{\text{infinitely many}} = 2$$

$f(x) = \sqrt{2+x}$, Eig. is $f'(2) = \left(\frac{1}{2\sqrt{2+x}}\right)_{x=2} = \frac{1}{4} < 1$

$|f'(2)| < 1$ – stable.

$$u_{t+1} = u_t \, e^{r(1-u_t)} \quad , \quad \underline{r > 0}$$

$$u_t = \frac{N_t}{K} \quad , \quad \underline{u^* = 0} \qquad \underline{u^* = 1}$$

$$f(u) = u e^{r(1-u)}$$

$$f'(u) = e^{r(1-u)} + u \, e^{r(1-u)} (-r)$$

$$f'(0) = e^r > 1 \quad \text{for } \underline{r>0} \quad \text{unstable}$$

$$f'(1) = e^0 + (-r)(1)e^0 = \underline{1-r}$$

$$f'(1) = 1-r \Rightarrow$$

$$-1 < 1-r < 1$$

$$\cancel{-2<} \quad \underline{\underline{r<2}} \qquad \underline{\underline{0<r}}$$

$\Rightarrow$ Stability condition is $\underline{\underline{0<r<2}}$

$\Rightarrow$ The first bifurcation value at $\underline{r=2}$

$$u_t = 1 + v_t \quad , \quad |v_t| \ll 1$$

$$\underbrace{\cancel{v_{t+1}}}_{u_{t+1}} = (\underbrace{\cancel{1+v_t}}_{u_t}) \, e^{r(\cancel{1} - \underbrace{\cancel{1} - v_t}_{-u_t})} = (\overset{u_t}{\cancel{1+v_t}}) \underline{\underline{e^{-r v_t}}}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots \approx 1 + x$$

$$u_{t+1} = u_t \, e^{-r v_t} = u_t (1 - r v_t) \qquad (v_t = u_t - 1)$$

$$u_{t+1} = u_t \left[1 + r \underbrace{(1 - u_t)}_{-v_t}\right]$$

$$u_{t+1} \approx u_t \left[1 + r(1 - u_t)\right]$$

$$U_t = \frac{r u_t}{1+r} \Rightarrow u_t = \frac{1+r}{r} U_t$$

$$\cancel{\frac{1+r}{r}} U_{t+1} = \cancel{\frac{1+r}{r}} U_t \left[1 + r\left(1 - \frac{1+r}{r} U_t\right)\right]$$

$$U_{t+1} = U_t \left[1 + r - (1+r) U_t\right]$$

$$\boxed{U_{t+1} = (1+r) U_t \left[1 - U_t\right]}$$

Logistic <u>model</u> with $r \to r+1$

$r = 2$ is a bifurcation to 2-periodic Solution

4-periodic solution $\quad r_4 = 2.45$

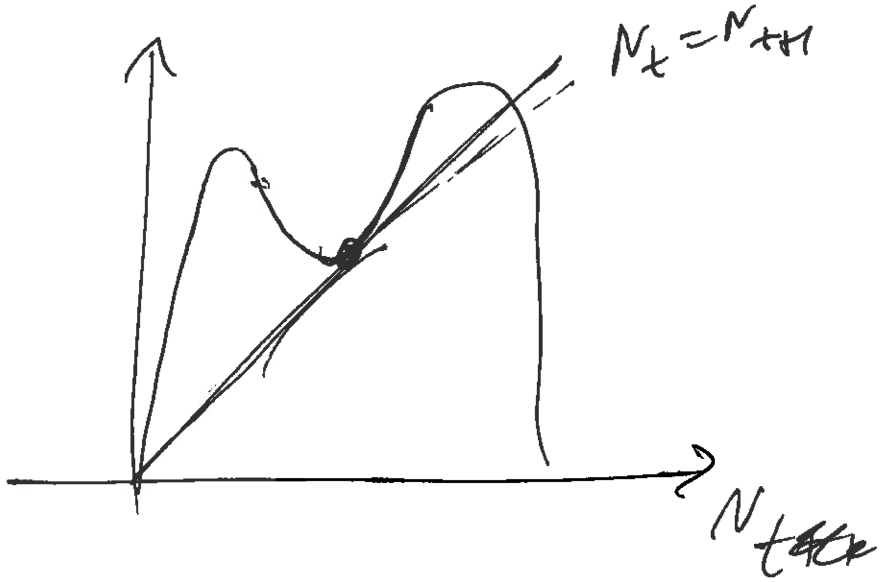
6-periodic $\quad r_6 \approx 2.54$

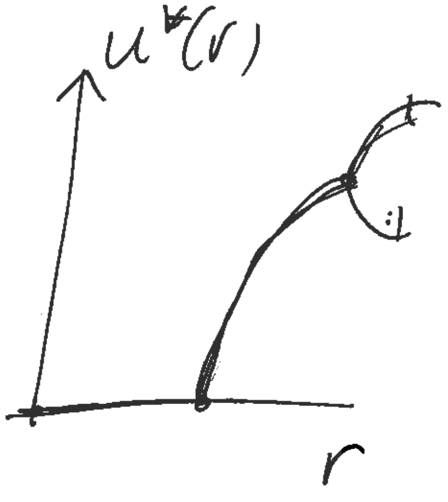
chaotic $\quad r > r_c = 2.57$

The bifurcation occurs at parameter value r_0, if there is qualitative change of the dynamics for $r < r_0$ & $r > r_0$

Bifurcations with $\lambda = -1$ are called 'period doubling'

with $\lambda = 1$ are called 'tangent'





Sarkovskii theorem: Chaos appears when odd-periodic solutions are possible.

$$u_{t+3} = f(\underbrace{f(\underbrace{f(u_t)}_{u_{t+1}})}_{u_{t+2}})$$

$$u^* = f^{(3)}(u^*)$$

Discrete delay models

$$u_{t+1} = f(u_t, u_{t-1}, \dots u_{t-T})$$

$$u_0, u_1, u_2, \dots u_T$$

$$u_{T+1} = f(u_T, u_{T-1}, \dots u_0)$$

Example:

$$u_{t+1} = u_t e^{r(1-u_{t-1})},$$

$$\underline{u_0, u_1}$$

Delay version of the Ricker's model.

$$\underbrace{u^* = 0}_{\text{unstable}} \quad \& \quad \underbrace{u^* = 1}_{?}$$

$$u_t = 1 + v_t, \quad |v_t| \ll 1$$

$$(1+v_{t+1}) = (1+v_t) e^{r(\cancel{1}-\cancel{1}-v_{t-1})} \approx (1+v_t)(1-rv_{t-1})$$

$$e^{-rv_{t-1}} \approx 1 - rv_{t-1}$$

$$(1+v_{t+1}) = (1+v_t)(1-rv_{t-1})$$

$$\cancel{1}+v_{t+1} = \cancel{1}+v_t - r v_{t-1} - \cancel{r v_t v_{t-1}}$$

$$v_{t+1} = v_t - r v_{t-1}$$

$$\boxed{v_{t+1} - v_t + r v_{t-1} = 0}$$ linear

$$v_t = z^t$$

$$z^{t+1} - z^t + r z^{t-1} = 0 \quad \Big| \cdot \frac{1}{z^{t-1}}$$

$$\boxed{z^2 - z + r = 0}$$ Characteristic equ.

$$z_{1,2} = \frac{1}{2}\left[1 \pm \sqrt{1-4r}\right] \qquad 0<r<\frac{1}{4}$$

$$z_{1,2} = \frac{1}{2}\left[1 \pm i\sqrt{4r-1}\right] \qquad r>\frac{1}{4}$$

I $\quad 0<r<\frac{1}{4}, \quad 0 < z_{1,2} < 1 \Rightarrow$

$$v_t = c_1 z_1^t + c_2 z^t \to 0, \quad u_t \to u^* = 1$$

$$\boxed{u^* = 1 \text{ is stable when } 0<r<\frac{1}{4}}$$

II $\quad r>\frac{1}{4}, \quad \rho = |z_{1,2}| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{4r-1}}{2}\right)^2} =$

$$\rho = \sqrt{\frac{\cancel{1}}{4} + \frac{4r-\cancel{1}}{4}} = \sqrt{\frac{4r}{4}} = \sqrt{r}$$

$$\theta = \tan^{-1}\sqrt{4r-1}, \quad z_{1,2} = \sqrt{r}\, e^{\pm i\theta}$$

General solution

$$v_t = A z_1^t + B z_2^t = A z_1^t + \bar{A} \bar{z}_1^t$$

$$A = |A| e^{i\gamma}$$

$$v_t = |A| e^{i\gamma} (\sqrt{r} e^{i\theta})^t + |A| e^{-i\gamma} (\sqrt{r} e^{-i\theta})^t$$

$$= 2|A| r^{t/2} \frac{\left(e^{i\theta t + i\gamma} + e^{-i\theta t - i\gamma} \right)}{2}$$

$$v_t = 2|A| \underline{\underline{r^{\frac{t}{2}}}} \cos(\theta t + \gamma)$$

Stable as $|\sqrt{r}| < 1$, $\boxed{\frac{1}{4} < r < 1}$

$\boxed{r_c = 1}$

$$v_t = 2|A| 1^{t/2} \cos(t \cdot \theta_1 + \gamma)$$

$$\theta_1 = \tan^{-1} \sqrt{4r - 1} = \tan^{-1} \sqrt{3} = \pi/3$$

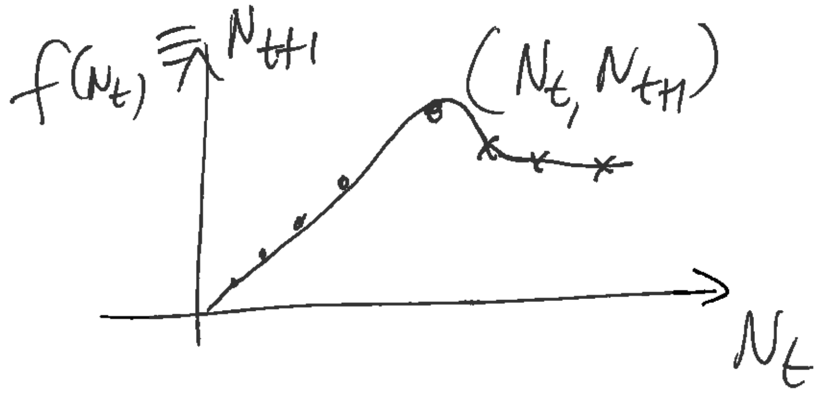
$$v_t = 2|A| \cos\left(\frac{\pi}{3} t + \gamma\right)$$

$$\frac{\cancel{\pi}}{3} t_p = 2\cancel{\pi} \Rightarrow t_p = 6$$

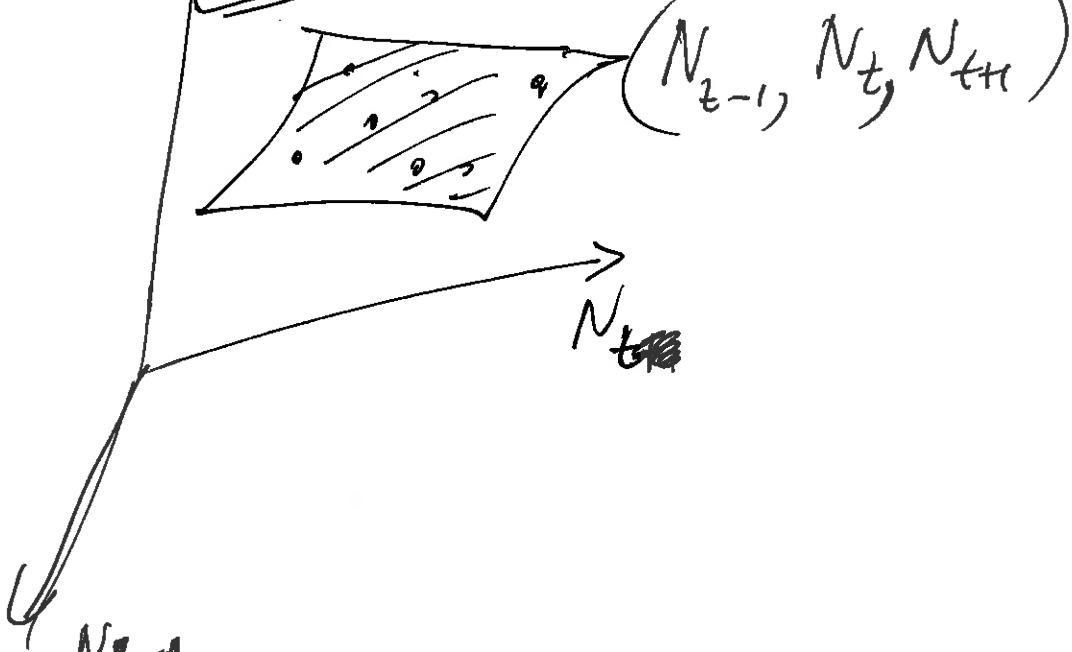
$$v_{t+6} = v_t$$

$N_1, N_2, N_3, \ldots$

1) $\boxed{N_{t+1} = f(N_t)}$



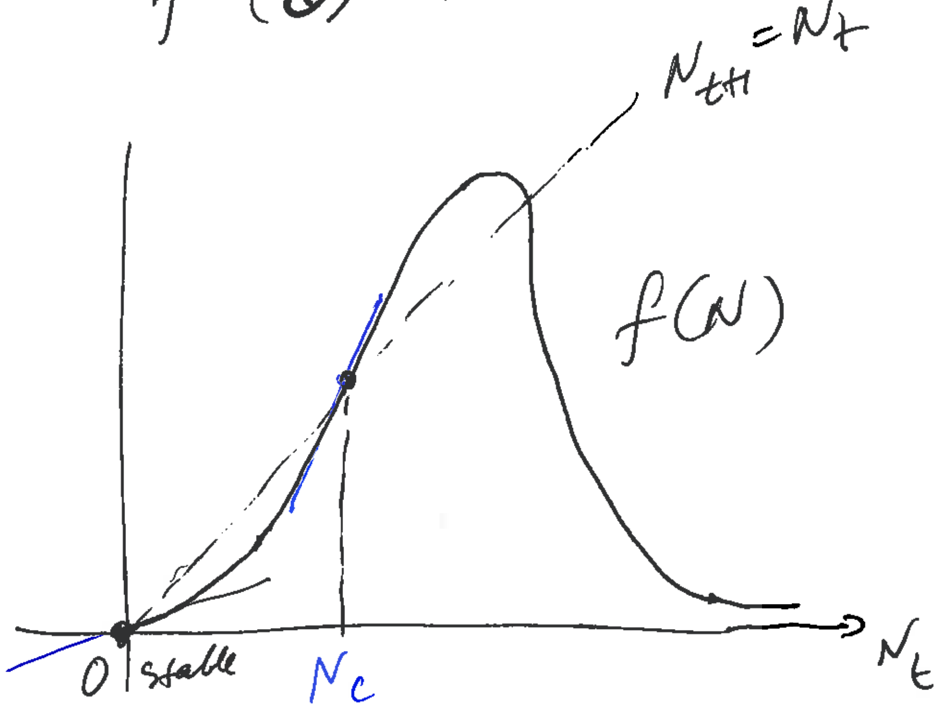
2) $\boxed{N_{t+1} = f(N_t, N_{t-1})}$

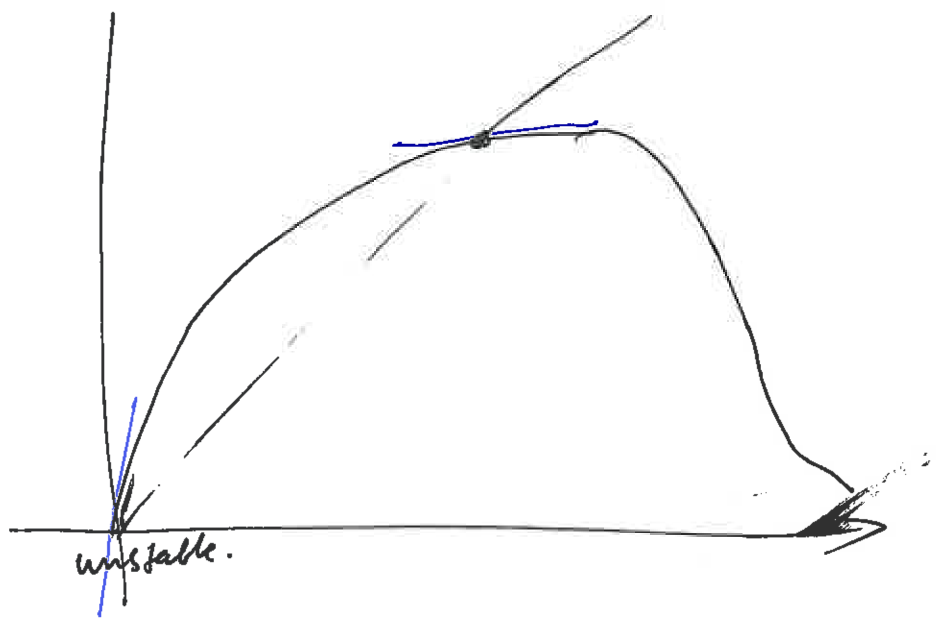


Allee effect :

$N^* = 0$ is a stable steady state

$f'(0) \leq 1$





$$\frac{df}{dN} = 0 \Leftrightarrow N_m, \quad N_{max} = f(N_m)$$

$$N_{min} \equiv f(N_{max}) = f(f(N_m))$$

$\boxed{N_{min} < 1}$ extinction of species

$$N_{t+1} = N_t e^{r\left(1 - \frac{N_t}{K}\right)}$$

$$f(N) = N e^{r\left(1 - \frac{N}{K}\right)}$$

$$f'(N) = e^{r\left(1 - \frac{N}{K}\right)} + N e^{r\left(1 - \frac{N}{K}\right)}\left(-\frac{r}{K}\right)$$

$$f'(N) = e^{r\left(1 - \frac{N}{K}\right)}\left(1 - \frac{Nr}{K}\right) = 0$$

$$\boxed{N_m = \frac{K}{r}}$$

$$N_{max} = f(N_m) = f\left(\frac{K}{r}\right) = \frac{K}{r} e^{r\left(1 - \frac{1}{K}\frac{K}{r}\right)}$$

$$\boxed{N_{max} = \frac{K}{r} e^{r-1}}$$

$$N_{min} = f(N_{max}) = f\left(\frac{K}{r} e^{r-1}\right)$$

$$N_{min} = \frac{K}{r} e^{r-1} \cdot e^{r\left(1 - \frac{1}{\cancel{K}}\frac{\cancel{K}}{r} e^{r-1}\right)} = \frac{K}{r} e^{r-1+r-e^{r-1}}$$

$$N_{min} = \frac{K}{r} e^{2r-1-e^{r-1}} = \frac{K}{r} \exp\left(2r-1-e^{r-1}\right)$$

Extinction: $N_{min} \leq 1$

$$\frac{K}{r} \exp\left(2r-1-e^{r-1}\right) \leq 1$$

May happen if $\underline{r = 3.5 \ \& \ K < 1600}$

$$N_{t+1} = N_t \underbrace{\exp\left[r\left(1-\frac{N_t}{K}\right)\right]}_{\text{effective birth rate}}$$

Harvesting

$$u_{t+1} = \frac{b u_t^2}{1+u_t^2} - E u_t$$

Phase Plane Analysis

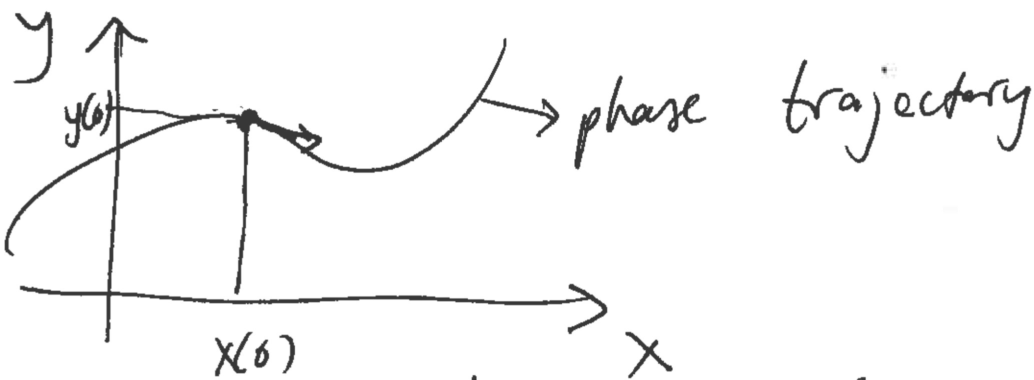
Appendix A, p. 501

$$\begin{cases} \dfrac{dx}{dt} = f(x,y) \\[2ex] \dfrac{dy}{dt} = g(x,y) \end{cases}$$

$x = x(t)$, Initial data $x(0), y(0)$
$y = y(t)$

$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)} = F(x,y)$$

$$\frac{dy}{dx} = F(x,y)$$



phase portrait

$$\boxed{\frac{dN}{dt} = F(N)}$$

$$N = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$F = \begin{pmatrix} f \\ g \end{pmatrix}$$

Equilibrium points $F(x_0, y_0) = 0$

- $f(x_0, y_0) = 0$
  $g(x_0, y_0) = 0$

~~$\Rightarrow x_0$ & y_0 are~~

(x_0, y_0) is a constant solution of the system.

$$\frac{dx}{dt} = \underbrace{f(x_0,y_0)}_{0} + \left(\frac{\partial f}{\partial x}\right)_{(x_0,y_0)} (x-x_0) + \left(\frac{\partial f}{\partial y}\right)_{(x_0,y_0)} (y-y_0)$$

$$\frac{dy}{dt} = \underbrace{g(x_0,y_0)}_{0} + \left(\frac{\partial g}{\partial x}\right)_{(x_0,y_0)} (x-x_0) + \left(\frac{\partial g}{\partial y}\right)_{(x_0,y_0)} (y-y_0)$$

$$X = x - x_0, \quad Y = y - y_0$$

$$\frac{dx}{dt} = \frac{dX}{dt} \ ; \quad \frac{dy}{dt} = \frac{dY}{dt}$$

$$\left\{\begin{aligned} \frac{dX}{dt} &= \left(\frac{\partial f}{\partial x}\right)_0 \cdot X + \left(\frac{\partial f}{\partial y}\right)_0 Y \\ \frac{dY}{dt} &= \left(\frac{\partial g}{\partial x}\right)_0 X + \left(\frac{\partial g}{\partial y}\right)_0 Y \end{aligned}\right.$$

$$\frac{d}{dt}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_0 \begin{pmatrix} X \\ Y \end{pmatrix}$$

Linear system with constant coefficients

$$A(x_0, y_0) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{\substack{x = x_0 \\ y = y_0}}$$

Jacobian at x_0, y_0

$$\frac{d}{dt}\begin{pmatrix} X \\ Y \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix} \; ; \quad \boxed{A = U \Lambda U^{-1}}$$

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} X \\ Y \end{pmatrix} = U \Lambda U^{-1} \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$\frac{d}{dt} \underbrace{\left(U^{-1} \begin{pmatrix} X \\ Y \end{pmatrix} \right)}_{Z} = \Lambda \underbrace{\left(U^{-1} \begin{pmatrix} X \\ Y \end{pmatrix} \right)}_{Z}$$

$$z = U^{-1}\begin{pmatrix} x \\ y \end{pmatrix}; \quad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\frac{d}{dt} z = \Lambda z \iff \frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\frac{d}{dt} z_1 = \lambda_1 z_1 \implies z_1 = c_1 e^{\lambda_1 t}$$

$$\frac{d}{dt} z_2 = \lambda_2 z_2 \implies z_2 = c_2 e^{\lambda_2 t}$$

$$\frac{dz_1}{z_1} = \lambda_1 \implies \ln z_1 = \lambda_1 t + \alpha_1$$

$$z_1 = e^{\lambda_1 t + \alpha_1} = c_1 e^{\lambda_1 t}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = Uz = \begin{bmatrix} v_1^{(1)} & v_1^{(2)} \\ v_2^{(1)} & v_2^{(2)} \end{bmatrix}\begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{bmatrix}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$p(\lambda) = \det(A - \lambda \mathbb{1}) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix}$$

$$p(\lambda) = (a-\lambda)(d-\lambda) - bc = \lambda^2 - \lambda(a+d) + ad - bc$$

$p(\lambda) = \lambda^2 - (\mathrm{tr} A)\lambda + \det A$ characteristic polynomial

$p(\lambda) = 0 \implies \lambda_1, \lambda_2 \to$ eigenvalues

$$Av^{(1)} = \lambda_1 v^{(1)}$$

$$Av^{(2)} = \lambda_2 v^{(2)} \Rightarrow \cancel{A\left[\boxed{v^{(1)}}\ \boxed{v^{(2)}}\right] = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\left[v^{(1)} \ \cdots\right]}$$

$$A\underbrace{\left[\boxed{v^{(1)}}\ \boxed{v^{(2)}}\right]}_{U} = \left[v^{(1)}\ \ v^{(2)}\right]\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$$

$$\begin{bmatrix} v_1^{(1)} & v_1^{(2)} \\ v_2^{(1)} & v_2^{(2)} \end{bmatrix}\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$$

$$\left[\underline{\underline{Av^{(1)}}},\ \underline{\underline{Av^{(2)}}}\right]$$

1st col. 2nd col.

$$\begin{pmatrix} \lambda_1 v_1^{(1)} & \lambda_2 v_1^{(2)} \\ \lambda_1 v_2^{(1)} & \lambda_2 v_2^{(2)} \end{pmatrix}$$

$$\left[\boxed{\lambda_1 \begin{pmatrix} v^{(1)} \end{pmatrix}},\ \boxed{\lambda_2 v^{(2)}}\right]$$

first column second colum

$$AU = U\Lambda$$

$$A = U\Lambda U^{-1}$$

$$A v^{(i)} = \lambda_i v^{(i)}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} v_1^{(i)} \\ v_2^{(i)} \end{pmatrix} = \lambda_i \begin{pmatrix} v_1^{(i)} \\ v_2^{(i)} \end{pmatrix}$$

$$a v_1^{(i)} + b v_2^{(i)} = \lambda_i v_1^{(i)} \qquad i = 1, 2$$

$$b v_2^{(i)} = (\lambda_i - a) v_1^{(i)}$$

$$v_2^{(i)} = \frac{\lambda_i - a}{b} v_1^{(i)} \qquad i = 1, 2$$

Define $\boxed{p_i = \frac{\lambda_i - a}{b}} \Rightarrow v_2^{(i)} = p_i v_1^{(i)}$

$$v^{(i)} = \begin{pmatrix} v_1^{(i)} \\ p_i \cdot v_1^{(i)} \end{pmatrix} = v_1^{(i)} \begin{pmatrix} 1 \\ p_i \end{pmatrix}$$

$$v^{(i)} = \begin{pmatrix} 1 \\ p_i \end{pmatrix}$$

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ p_1 & p_2 \end{pmatrix} \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix} = \begin{bmatrix} c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ c_1 p_1 e^{\lambda_1 t} + c_2 p_2 e^{\lambda_2 t} \end{bmatrix}$$

$$\| \cdot \quad X = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \longrightarrow 0, \ t \to \infty$$

$$Y = c_1 p_1 e^{\lambda_1 t} + c_2 p_2 e^{\lambda_2 t} \longrightarrow 0, \ t \to \infty$$

$$e^{\lambda t} = e^{\mu t + i\omega t} = e^{\mu t} \cdot e^{i\omega t} \to 0, \ (\mu < 0)$$

$$|e^{i\omega t}| = |\underbrace{\cos \omega t}_{} + i \underbrace{\sin \omega t}_{}| \, I$$

$$= \sqrt{\cos^2 + \sin^2} = \sqrt{1} = 1$$

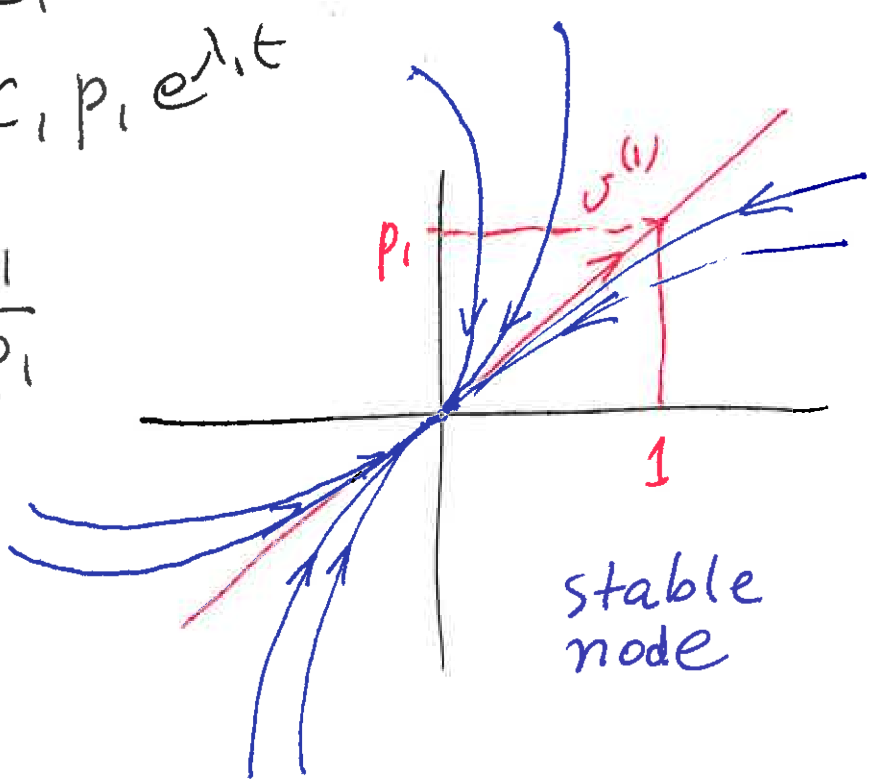
$\lambda_2 < \lambda_1 < 0$ two real negative

$$\begin{pmatrix} X \\ Y \end{pmatrix} = c_1 v^{(1)} e^{\lambda_1 t} + c_2 v^{(2)} e^{\lambda_2 t}, \quad t \to \infty$$

$$\to c_1 v^{(1)} e^{\lambda_1 t} \qquad v^{(1)} = \begin{pmatrix} 1 \\ p_1 \end{pmatrix}$$

$$X = c_1 e^{\lambda_1 t}$$
$$Y = c_1 p_1 e^{\lambda_1 t}$$

$$\frac{X}{Y} = \frac{1}{p_1}$$



Two real & positive eigenvalues

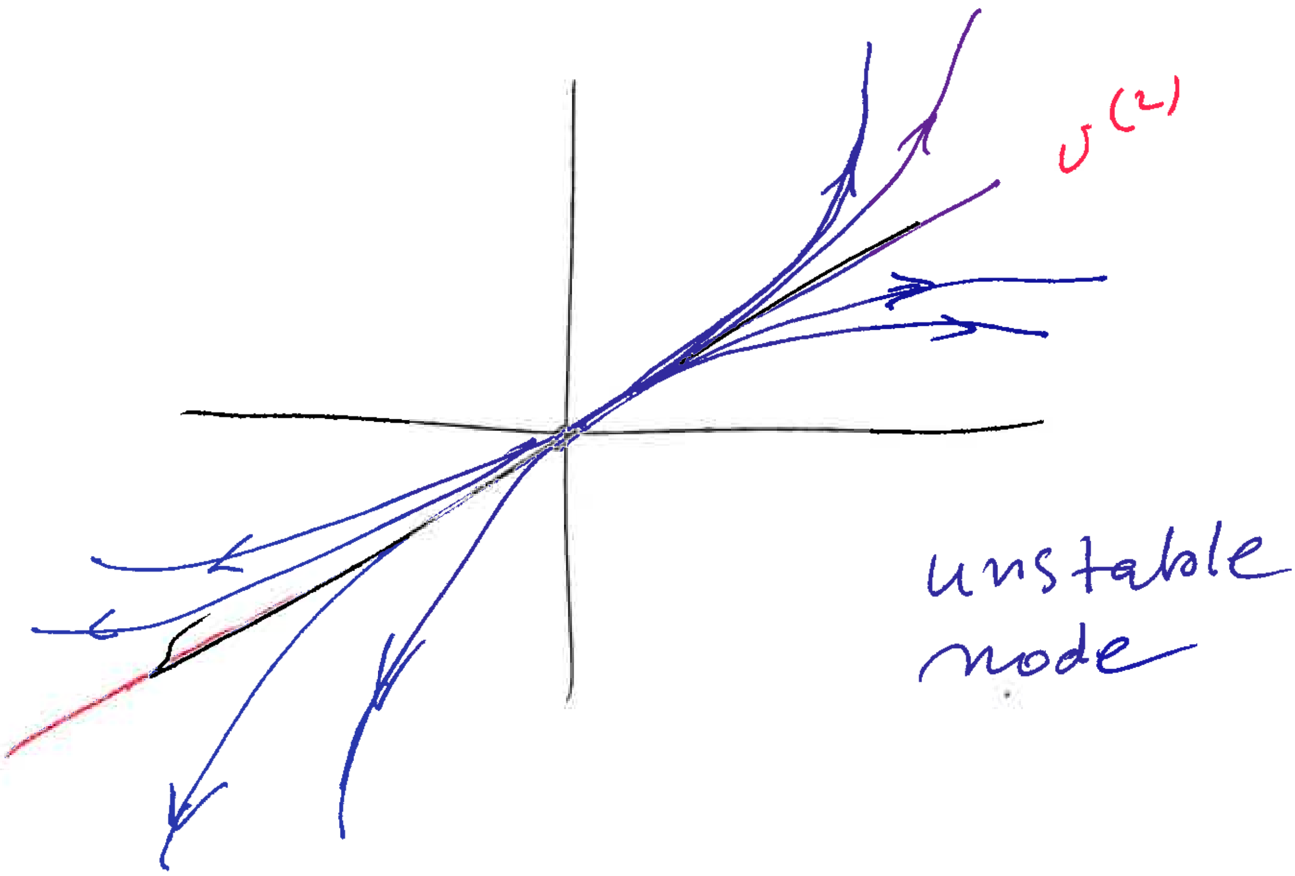
$\lambda_1 > \lambda_2 > 0$

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 v^{(1)} e^{\lambda_1 t} + c_2 v^{(2)} e^{\lambda_2 t}$$

If $t \to \infty$ $\quad x \to \infty$, $y \to \infty$

$t \to -\infty$ $\quad x \to 0 \quad y \to 0$

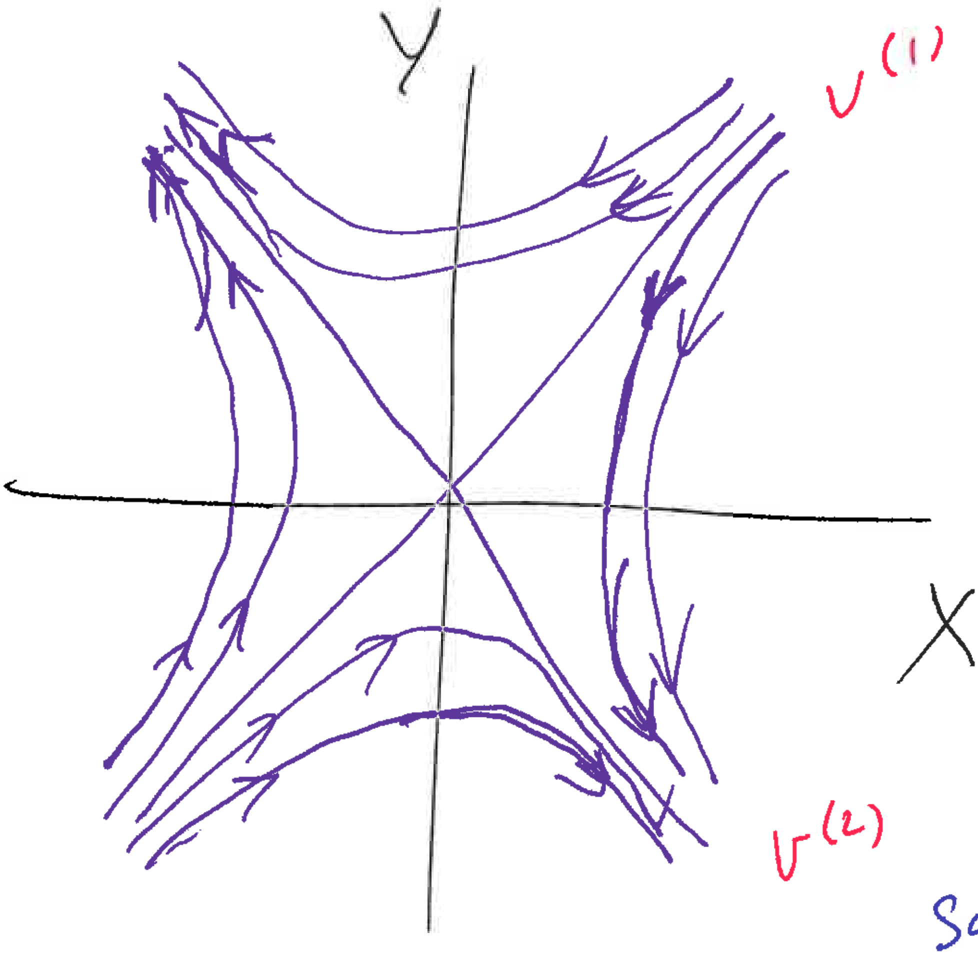
$$\begin{pmatrix} x \\ y \end{pmatrix} = c_2 v^{(2)} e^{\lambda_2 t} \qquad t \to -\infty$$



Real with different signs

$$\lambda_1 < 0 < \lambda_2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 v^{(1)} e^{\lambda_1 t} + c_2 v^{(2)} e^{\lambda_2 t}$$



Saddle point

(unstable)

λ_1, λ_2 — complex

$$\lambda_{1,2} = \mu \pm i\omega$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 v^{(1)} e^{\mu t + i\omega t} + \bar{c}_1 \bar{v}^{(1)} e^{\mu t - i\omega t}$$

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \underline{\underline{e^{\mu t}}} \Big[\underbrace{c_1 v^{(1)} e^{i\omega t} + \bar{c}_1 \bar{v}^{(1)} e^{-i\omega t}}_{2\pi\text{-periodic, bounded}} \Big]$$

$\mu < 0$ stable

$\mu > 0$ unstable

$\mu = 0$ periodic

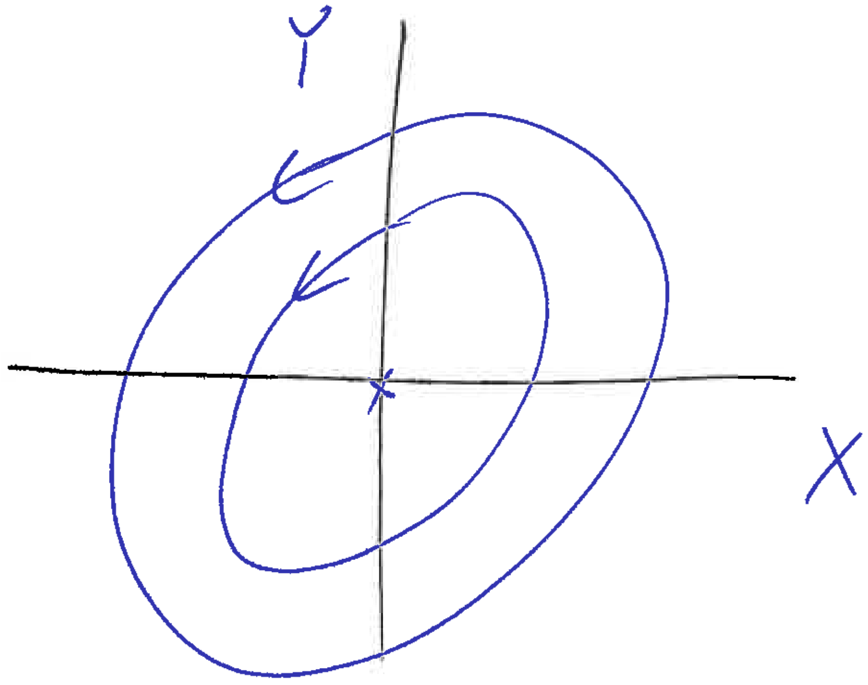
$$\cancel{\begin{pmatrix} X \\ Y \end{pmatrix}} =$$

Using $\cos\omega t = \dfrac{e^{i\omega t} + e^{-i\omega t}}{2}$ $\quad \sin\omega t = \dfrac{e^{i\omega t} - e^{-i\omega t}}{2i}$

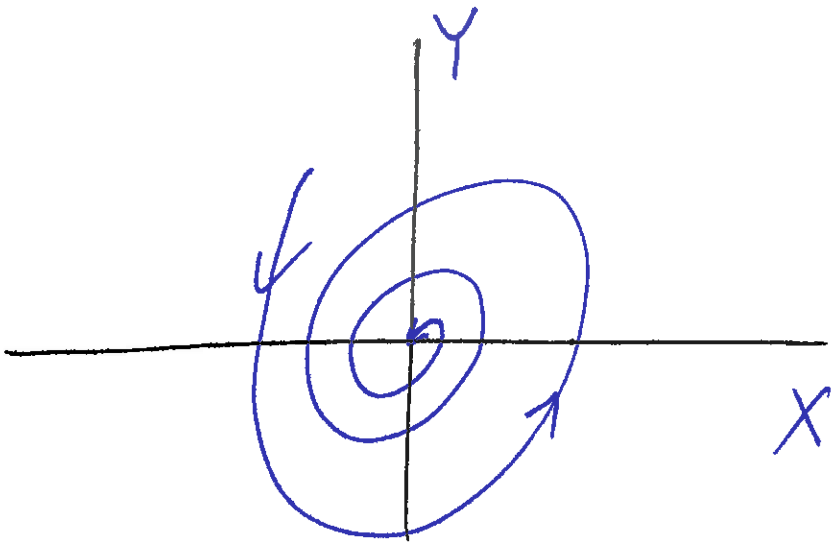
If $\mu = 0$ $\quad \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \sin\omega t \\ \cos\omega t \end{pmatrix}$

periodic $\quad X(t) = X\left(t + \frac{2\pi}{\omega}\right)$

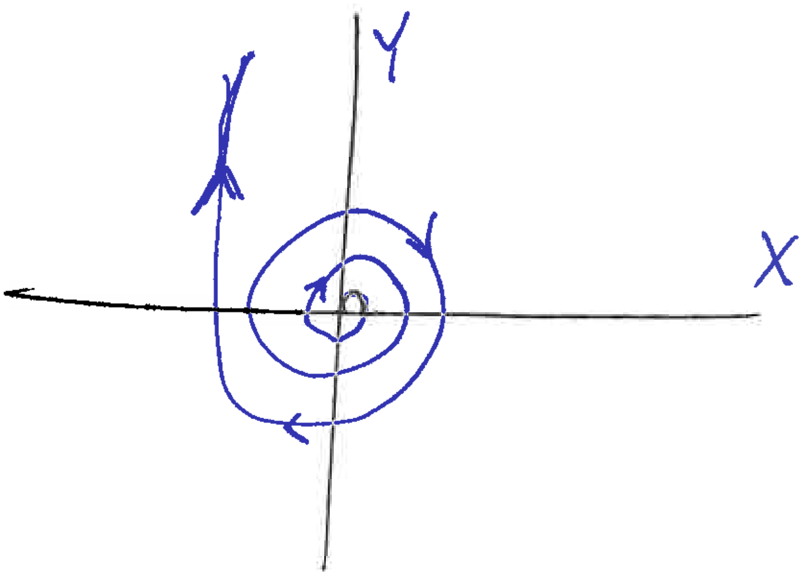
$\quad\quad\quad\quad Y(t) = Y\left(t + \frac{2\pi}{\omega}\right)$



Closed phase trajectories ($\mu = 0$) 'centre'



$\mu < 0$ stable spiral



$\mu > 0$

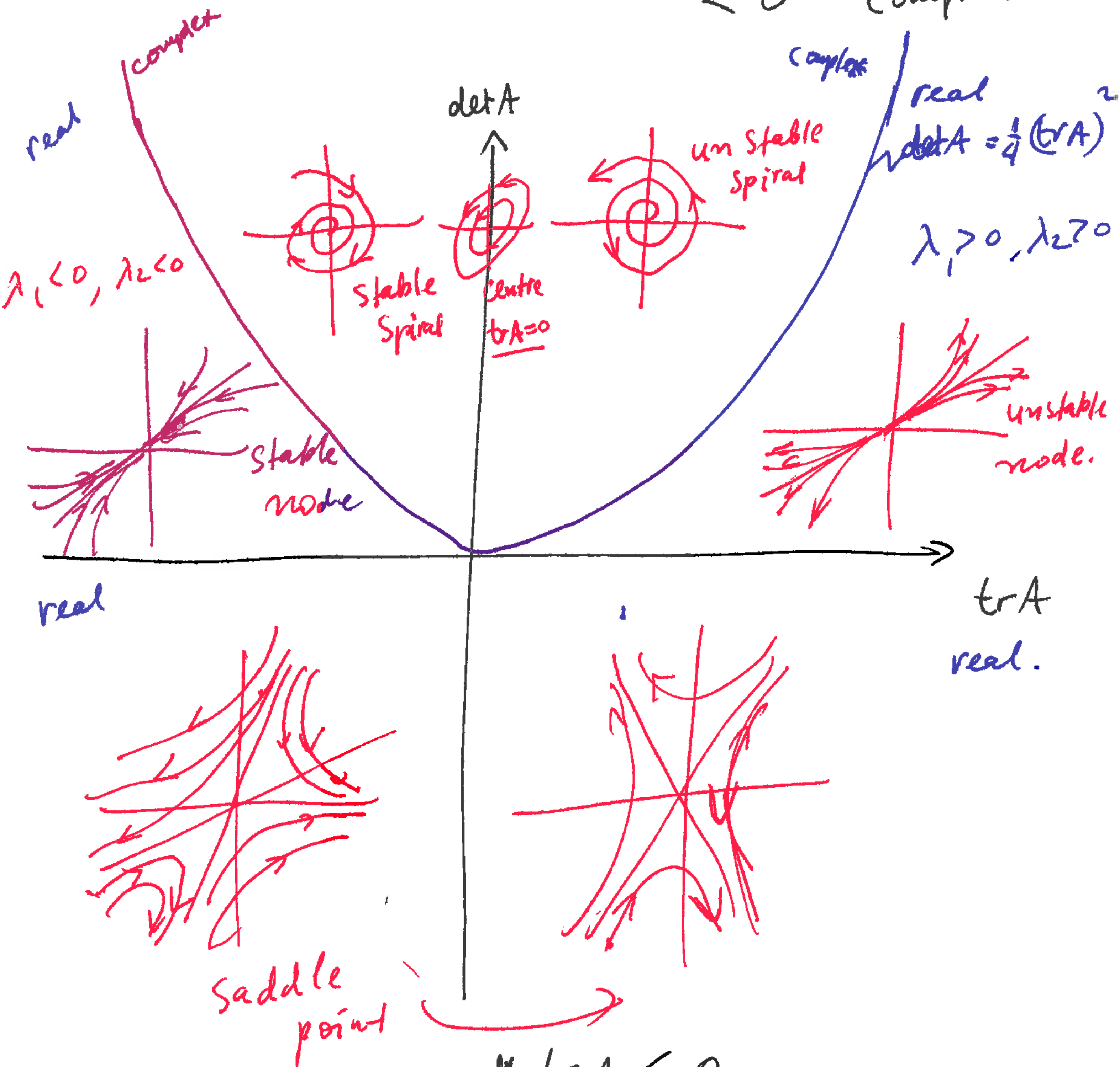
unstable spiral

$$p(\lambda) = \lambda^2 - (\mathrm{tr}A)\lambda + \det A$$

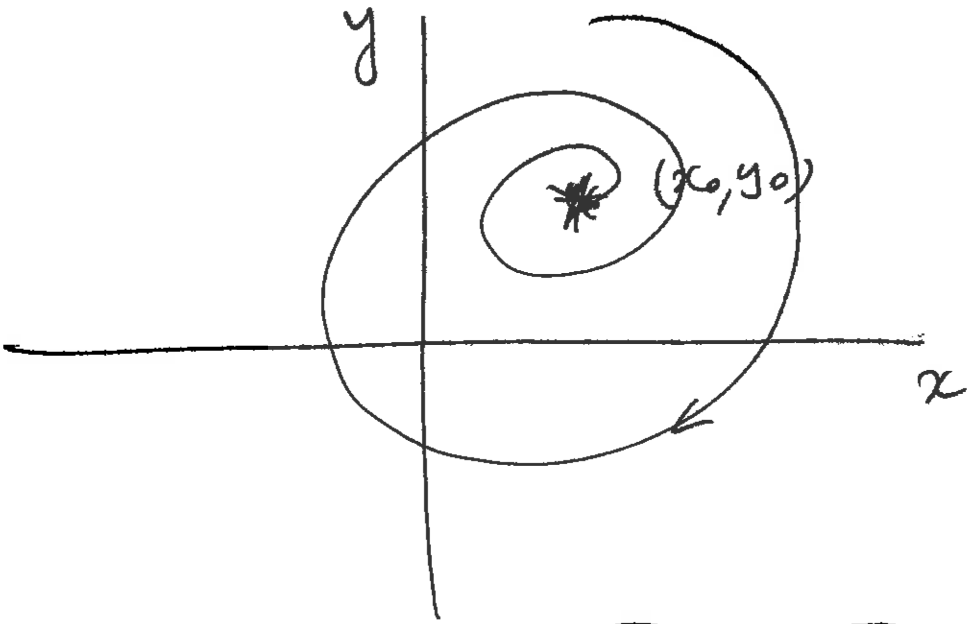
$$\lambda_1 + \lambda_2 = \mathrm{tr}A$$

$$\lambda_1 \lambda_2 = \det A$$

$$D = (\mathrm{tr}A)^2 - 4\det A \quad \begin{array}{ll} > 0 & \text{(real)} \\ < 0 & \text{complex} \end{array}$$



For stability : $\left\| \begin{array}{l} \mathrm{tr}A < 0 \\ \det A > 0 \end{array} \right.$



Not possible if
$x(t) > 0$
$y(t) > 0$
is required.

Finally $D = 0$ $\lambda_1 = \lambda_2 = \lambda$
(double root)

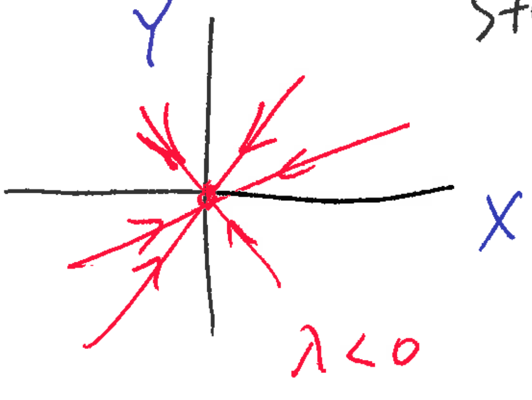
I Two eigenvectors $v^{(1)} \neq v^{(2)}$

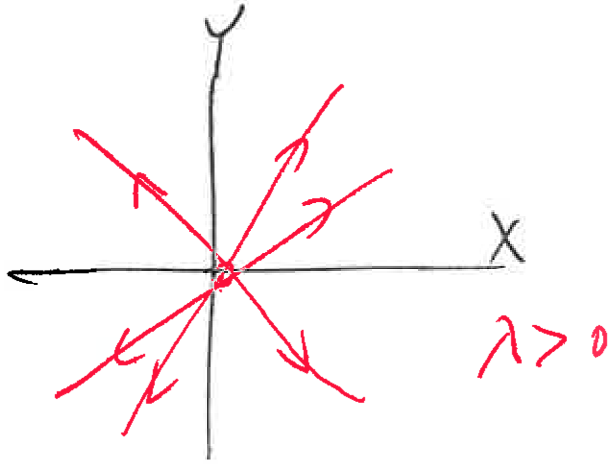
$\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ is diagonal

$X = \alpha_1 e^{\lambda t}$
$Y = \alpha_2 e^{\lambda t}$

$\dfrac{X}{Y} = \dfrac{\alpha_1}{\alpha_2} = \text{const}$

'star'





$\lambda_1 = \lambda_2 = \lambda$

A can not be diagonalized, λ

$A v^{(1)} = \lambda v^{(1)} \to t e^{\lambda t}$

Example: $y'' + 2ay' + a^2 y = 0$

$a = \text{const}$, $y = e^{\lambda t}$

$\lambda^2 + 2a\lambda + a^2 = 0 \qquad (\lambda + a)^2 = 0$

$\boxed{\lambda = -a}$

$y = (A + Bt)e^{-at}$, A, B – arbitrary

$x = y'$

$x' + 2ax + a^2 y = 0$

$$\begin{cases} x' = -2ax - a^2 y \\ y' = x \end{cases} = \begin{pmatrix} -2a & -a^2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$\lambda_{1,2} = -a$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y' \\ y \end{pmatrix} = \begin{pmatrix} (B-aA)e^{-at} - aBte^{-at} \\ (A+Bt)e^{-at} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \left\{ \begin{pmatrix} B-aA \\ A \end{pmatrix} + \begin{pmatrix} -aB \\ B \end{pmatrix} t \right\} e^{-at}$$

$$\cancel{v_1 + v_2 t)}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \left(v^{(1)} + v^{(2)} \underline{\underline{t}} \right) e^{-at}$$

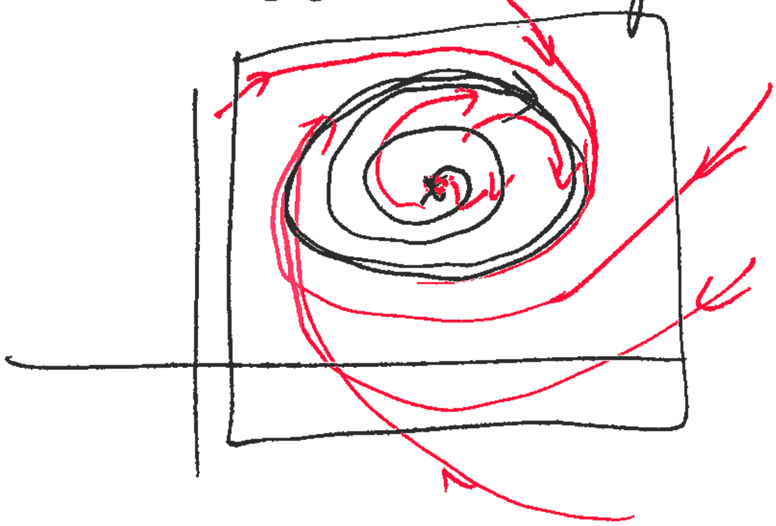
IN GENERAL,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \left(v^{(1)} + v^{(2)} t \right) e^{\lambda t}$$

$\lambda > 0$ unstable

$\lambda < 0$ stable.

Limit cycles



Poincaré – Bendixson

6/02/2012

Phase plane analysis (2)

$$\begin{cases} \dot{x} = 2x - \frac{3}{2}y \\ \dot{y} = -\frac{2}{3}x + 2y \end{cases} ; \qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} \overset{a}{2} & \overset{b}{-\frac{3}{2}} \\ \underset{c}{-\frac{2}{3}} & \underset{d}{2} \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

$(0,0)$

Eigenvalues of A

$$\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0$$

$$\lambda^2 - 4\lambda + 3 = 0 \iff (\lambda - 1)(\lambda - 3) = 0$$

$$\lambda_1 = 1 , \quad \lambda_2 = 3 , \quad \text{unstable node}$$

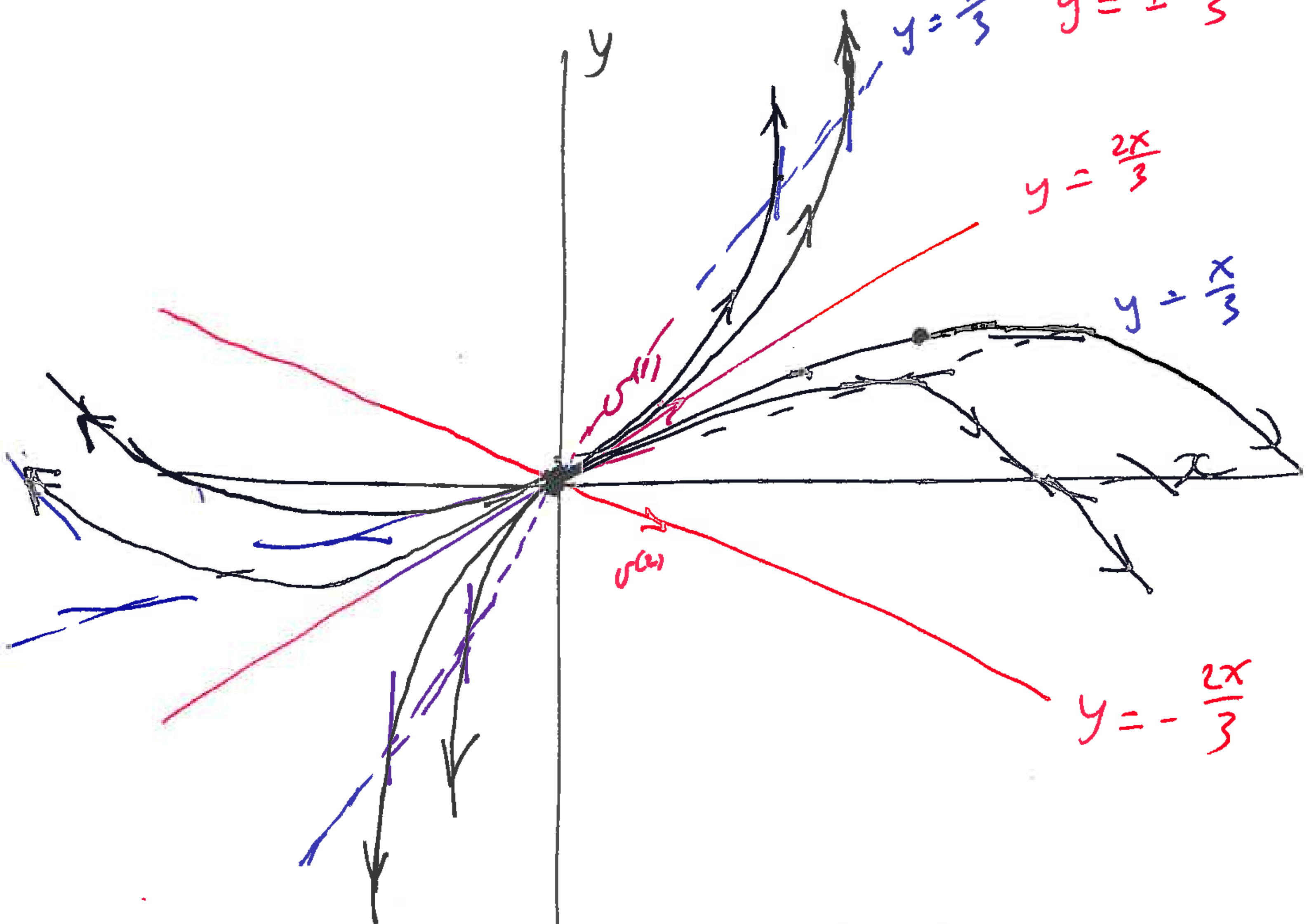
$$v^{(1)} = \begin{pmatrix} 1 \\ p_1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{\lambda_1 - a}{b} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1-2}{-\frac{3}{2}} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{2}{3} \end{pmatrix}$$

$$v^{(2)} = \begin{pmatrix} 1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{\lambda_2 - a}{b} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{3-2}{-\frac{3}{2}} \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{2}{3} \end{pmatrix}$$

$v^{(1)}$ points along the line $\frac{x}{y} = \frac{1}{2/3} = \frac{3}{2}$

$v^{(2)}$ —"— $\qquad \frac{x}{y} = \frac{1}{-2/3} = -\frac{3}{2}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 v^{(1)} \underline{\underline{e^{\lambda_1 t}}} + c_2 v^{(2)} \underline{\underline{e^{\lambda_2 t}}} \qquad \frac{y}{x} = \pm\frac{2}{3} \Rightarrow y = \frac{\pm 2}{3}x$$



$$\left.\begin{aligned}\frac{dy}{dt} &= -\frac{2}{3}x + 2y \\ \frac{dx}{dt} &= 2x - \frac{3}{2}y\end{aligned}\right\} \Rightarrow \quad \frac{dy}{dx} = \frac{-\frac{2}{3}x + 2y}{2x - \frac{3}{2}y}$$

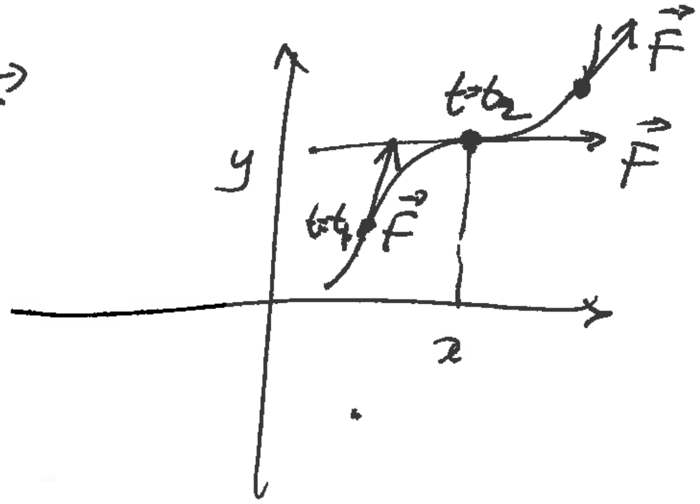
$\frac{dy}{dx} = \infty$ when $2x = \frac{3}{2}y$, ~~w~~ i.e. $y = \frac{4x}{3}$

$\frac{dy}{dx} = 0$ when $+\frac{2}{3}x = 2y \Rightarrow y = \frac{x}{3}$

when $y = 0$ $\quad \frac{dy}{dx} = \frac{-\frac{2}{3}}{2} = -\frac{1}{3} < 0$

$x = 0$ $\quad \frac{dy}{dx} = \frac{2}{-\frac{3}{2}} = -\frac{4}{3} < 0$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \vec{F}$$



$$\begin{cases} \dot{x} = y \\ \dot{y} = 4x \end{cases} \qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \overset{a}{0} & \overset{b}{1} \\ \underset{c}{4} & \underset{d}{0} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\lambda^2 - \operatorname{tr}A\,\lambda + \det A = 0$$

$$\lambda^2 - 4 = 0, \quad \lambda_1 = 2, \quad \lambda_2 = -2 \qquad \text{saddle}$$

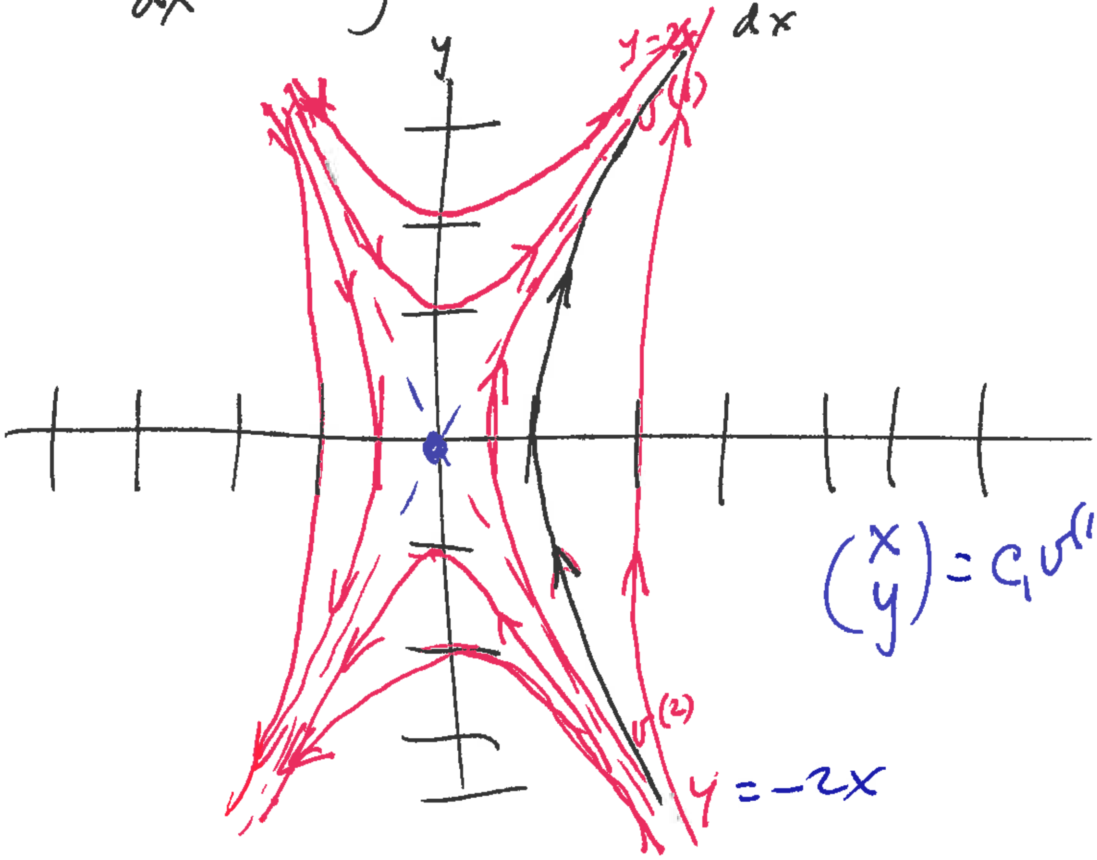
$$v^{(1)} = \begin{pmatrix} 1 \\ p_1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{\lambda_1 - a}{b} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{2-0}{1} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$v^{(2)} = \begin{pmatrix} 1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{\lambda_2 - a}{b} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{-2-0}{1} \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$v^{(1)}: \quad \frac{y}{x} = \frac{2}{1} = 2 \quad \Rightarrow \quad y = 2x$$

$$v^{(2)}: \quad \frac{y}{x} = -2 \quad \Rightarrow \quad y = -2x$$

$$\frac{dy}{dx} = \frac{4x}{y} \quad \Rightarrow \quad \begin{array}{ll} \frac{dy}{dx} = \infty & \text{when } y = 0 \\ \frac{dy}{dx} = 0 & \text{when } x = 0 \end{array}$$



$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 v^{(1)} e^{\lambda_1 t} + c_2 v^{(2)} e^{\lambda_2 t}$$

$$\begin{cases} \dot{x} = y \\ \dot{y} = -2x - 2y \end{cases} \qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\lambda^2 - \mathrm{tr}A + \det A = 0$$

$$\lambda^2 + 2\lambda + 2 = 0 \qquad (\lambda+1)^2 + 1 = 0$$

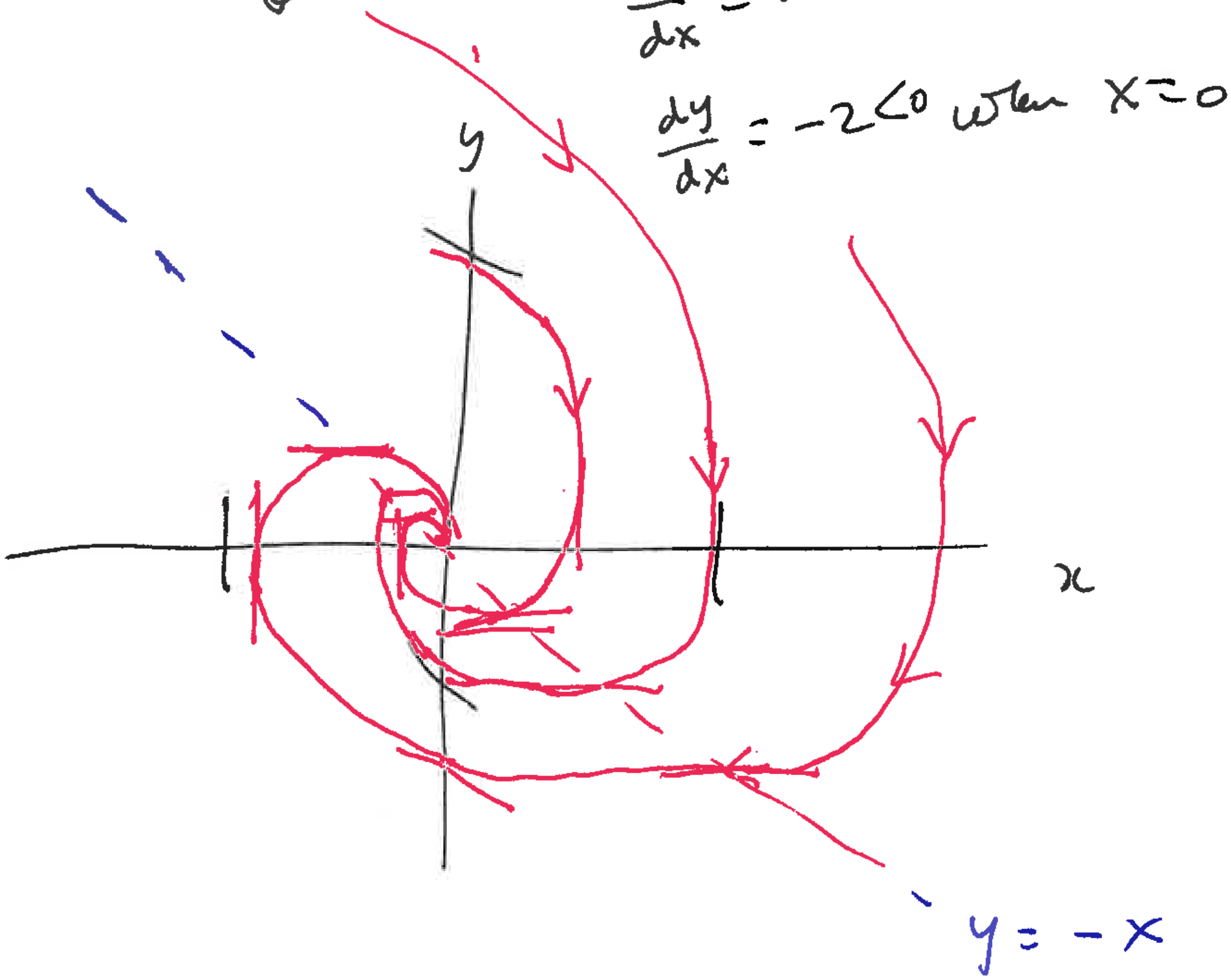
$$\lambda + 1 = \pm i \implies \boxed{\lambda_{1,2} = -1 \pm i}$$

$$\frac{dy}{dx} = \frac{-2x-2y}{y};$$

$\frac{dy}{dx} = 0$ on $y = -x$

$\frac{dy}{dx} = \infty$ on $y = 0$

$\frac{dy}{dx} = -2 < 0$ when $x = 0$



$$\left|\begin{array}{l} \dot{x} = x(y-1) = f(x,y) \\ \dot{y} = y(x-1) = g(x,y) \end{array}\right.$$

$y=0$ $x=1$

$x=0$, $y=1$

Equilibrium $\left|\begin{array}{l} f(x,y)=0 \\ g(x,y)=0 \end{array}\right.$ $\left\|\begin{array}{l} x(y-1)=0 \\ y(x-1)=0 \end{array}\right.$

$(0,0)$ $(1,1)$

Jacobian $A(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} y-1 & x \\ y & x-1 \end{pmatrix}$

$\left\|\begin{array}{l} x_0 = 0 \\ y_0 = 0 \end{array}\right.$ $\quad 0 = \dot{x}_0 = f(x_0,y_0) = 0$ ✓
$\quad 0 = \dot{y}_0 = f(x_0,y_0) = 0$ ✓

$A(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, Eig. $\lambda_1 = \lambda_2 = -1$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are eigenvectors

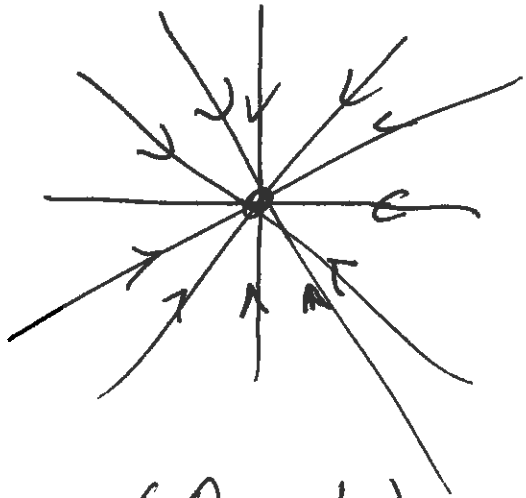
$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} v^{(1)} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (-1)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 v^{(1)} e^{\lambda_1 t} + c_2 v^{(2)} e^{\lambda_2 t}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t} = \begin{pmatrix} c_1 e^{-t} \\ c_2 e^{-t} \end{pmatrix}$$

$$\left.\begin{aligned} x &= c_1 e^{-t} \\ y &= c_2 e^{-t} \end{aligned}\right\} \quad \frac{y}{x} = \frac{c_2}{c_1} = \text{const}$$

$$y = (\text{const})\, x \qquad \text{'star'}$$



stable

$$A(1,1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{Eig. } \lambda_1 = 1 \quad \lambda_2 = -1$$

$$\lambda^2 + (-1) = 0$$

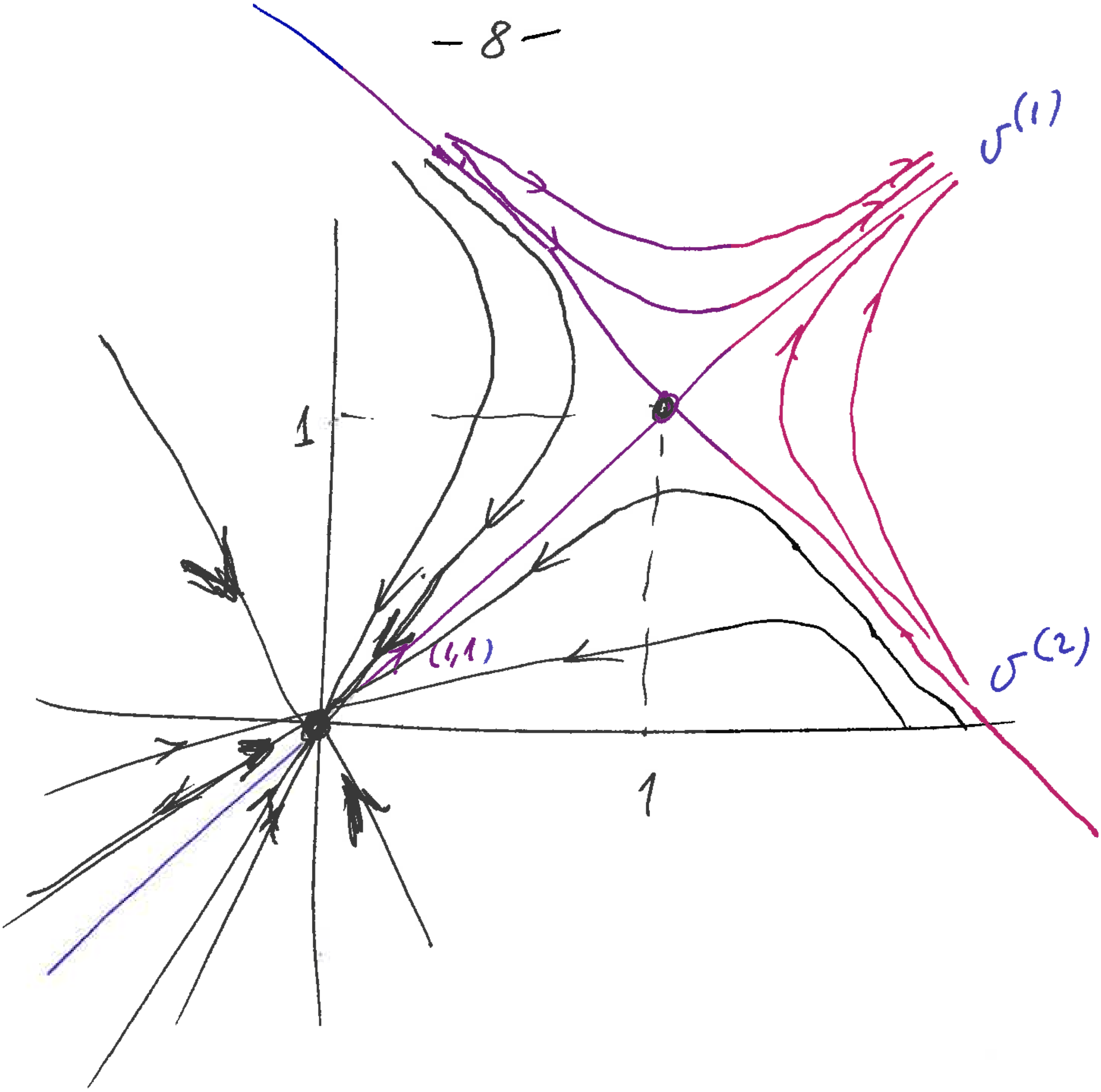
Eigenvectors

$\lambda_1 = 1 \rightarrow$ $$v^{(1)} = \begin{pmatrix} 1 \\ p_1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{\lambda_1 - a}{b} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1-0}{1} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\lambda_2 = -1 \rightarrow$ $$v^{(2)} = \begin{pmatrix} 1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{\lambda_2 - a}{b} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{-1-0}{1} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

'saddle'

$$\begin{pmatrix} x-1 \\ y-1 \end{pmatrix} = c_1 \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{v^{(1)}} e^{t} + c_2 \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{v^{(2)}} e^{-t}$$



$v^{(1)}$
1
(1,1)
1
$v^{(2)}$

$$\frac{dx}{dt} = -y + x\left(1-x^2-y^2\right)$$

$$\frac{dy}{dt} = x + y\left(1-x^2-y^2\right)$$

$$x = r\cos\varphi \qquad \boxed{x^2+y^2=r^2}$$
$$y = r\sin\varphi$$

$$\dot{x} = \dot{r}\cos\varphi - r\sin\varphi\,\dot{\varphi}$$
$$\dot{y} = \dot{r}\sin\varphi + r\cos\varphi\,\dot{\varphi}$$

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}\dot{r}\\ r\dot{\varphi}\end{pmatrix}$$

$$\underbrace{\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}}_{R}\begin{pmatrix}\dot{r}\\ r\dot{\varphi}\end{pmatrix} = \begin{pmatrix}-r\sin\varphi + r\cos\varphi\left(1-r^2\right)\\ r\cos\varphi + r\sin\varphi\left(1-r^2\right)\end{pmatrix}$$

$$\begin{pmatrix}\dot{r}\\ r\dot{\varphi}\end{pmatrix} = \underbrace{\begin{pmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}}_{R^{-1}}\begin{pmatrix}-r\sin\varphi + r\left(1-r^2\right)\cos\varphi\\ r\cos\varphi + r\left(1-r^2\right)\sin\varphi\end{pmatrix}$$

$$\begin{pmatrix}\dot{r}\\ r\dot{\varphi}\end{pmatrix} = \begin{pmatrix}\cancel{-r\cos\varphi\sin\varphi} + r\left(1-r^2\right)\cos^2\varphi + \cancel{r\sin\varphi\cos\varphi} + r\left(1-r^2\right)\sin^2\varphi\\ r\sin^2\varphi \cancel{- r\left(1-r^2\right)\sin\varphi\cos\varphi} + r\cos^2\varphi + \cancel{r\left(1-r^2\right)\sin\varphi\cos\varphi}\end{pmatrix}$$

$$\begin{pmatrix}\dot{r}\\ r\dot{\varphi}\end{pmatrix} = \begin{pmatrix}r\left(1-r^2\right)\\ r\end{pmatrix} \Rightarrow \left|\begin{array}{l}\dot{r} = r\left(1-r^2\right)\\ r\dot{\varphi} = r\\ \dot{\varphi} = 1\end{array}\right.$$

$$\begin{cases} \dot{r} = r(1-r^2) \\ \dot{\varphi} = 1 \end{cases} \Rightarrow \quad \varphi = t + \text{const} \;/\bmod 2\pi$$

$$\boxed{\dot{r} = r(1-r^2)}$$

$$\frac{dN}{dt} = \underline{\underline{N(1-N^2)}} = f(N)$$

$$\boxed{r = 0 \quad \& \quad r = 1}$$

$r = 0$ unstable

$r = 1$ stable.

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$\frac{dx}{dt} = -y + x\left(1 - x^2 - y^2\right) = f_1$$

$$\frac{dy}{dt} = x + y\left(1 - x^2 - y^2\right) = f_2$$

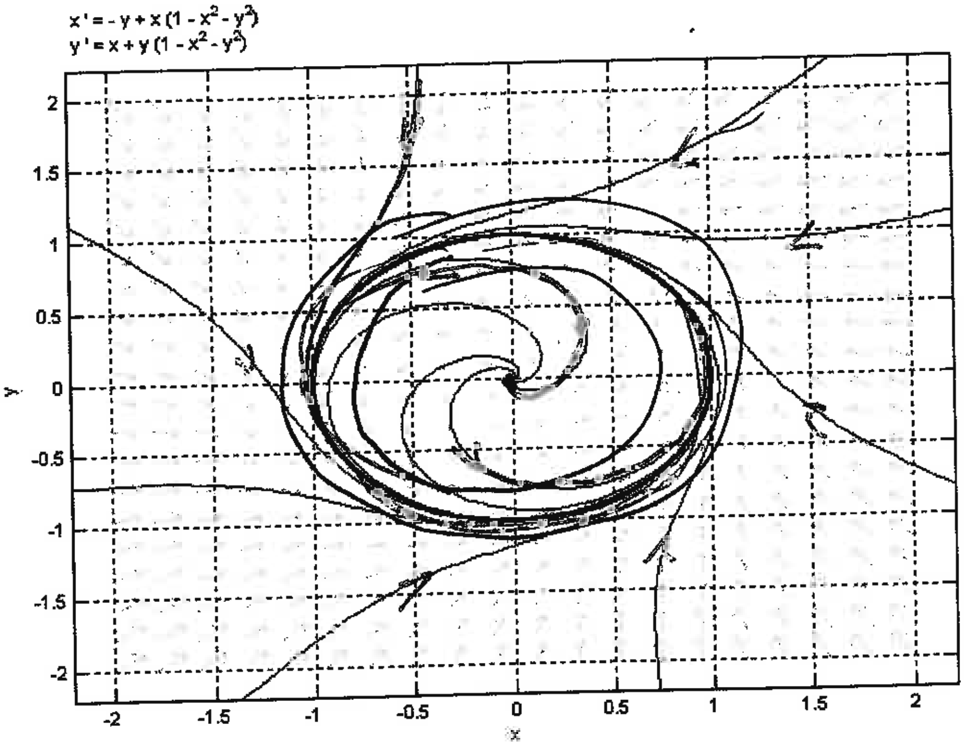


Figure A.13: a limit cycle

$$\dot{x} = x - y - x\,(x^2 + 2y)$$

$$\dot{y} = x + y - y\,(x^2 + y^2)$$

I. Change the coordinates:

$x = r\cos\varphi$ polar form

$y = r\sin\varphi$

$$\begin{cases} \dot{r} = r - r^3\left(1 + \frac{1}{4}\sin^2 2\varphi\right) \\ \dot{\varphi} = 1 + \frac{1}{2} r^2 \sin^2\varphi \sin 2\varphi \end{cases}$$

$\dot{r} > 0$

$$\cancel{r} > r^{\cancel{3}\,2}\left(1 + \frac{1}{4}\sin^2 2\varphi\right) \quad \text{all } \varphi$$

1

$$r^2 < \frac{1}{1 + \frac{1}{4}\sin^2 2\varphi} \quad \text{all } \varphi$$

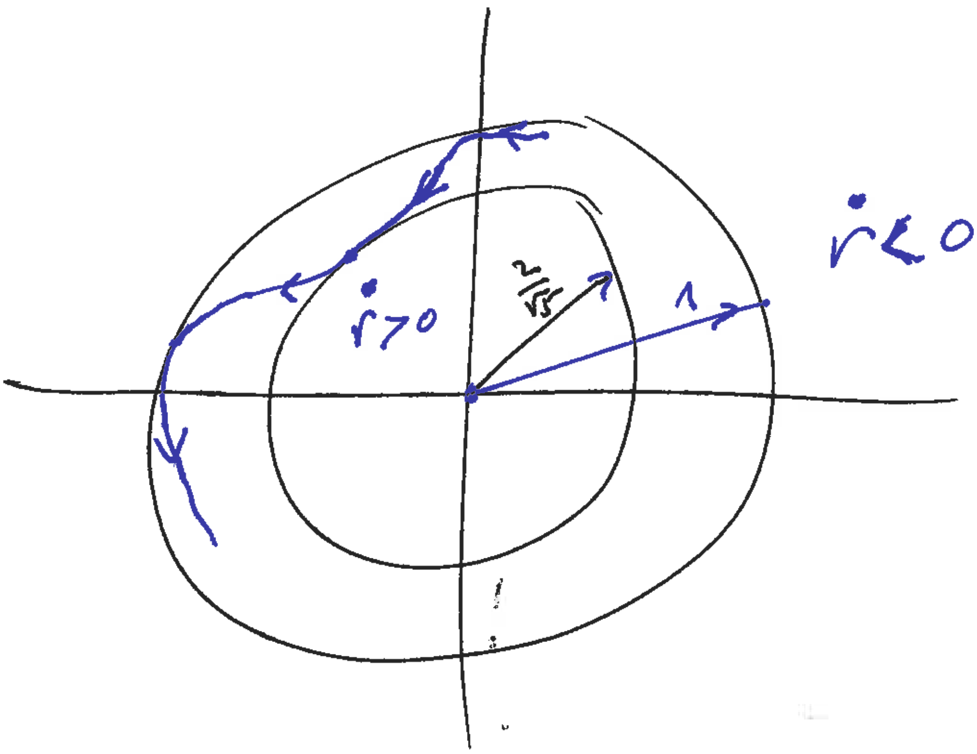
$$r^2 < \frac{1}{1 + \frac{1}{4}\cdot 1} = \frac{1}{5/4} = \frac{4}{5} \Rightarrow \dot{r} > 0$$

$\dot{r} < 0$

$$r^2 > \frac{1}{1 + \frac{1}{4}\sin^2 2\varphi} \quad \text{all } \varphi$$

$$r^2 > \frac{1}{1+0} = 1 \quad \text{all } \varphi. \Rightarrow \dot{r} < 0$$

$\frac{2}{\sqrt{5}} < r < 1$



'trapping region'

+ (0, 0) unstable (check?)

=> limit cycle exists

13/02/2012

$$\frac{du}{d\tau} = u(1-v) = f$$

$$\frac{dv}{d\tau} = \alpha v(u-1) = g, \quad \alpha > 0, \text{ constant}$$

<u>Lotka – Volterra Model</u>

Lotka (1920)
Volterra (1926)

Equilibrium states $(0,0)$
$(1,1)$

$$A = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix} = \begin{pmatrix} 1-v & -u \\ \alpha v & \alpha(u-1) \end{pmatrix}$$

$$A(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -\alpha \end{pmatrix}, \quad \lambda_1 = 1, \quad \lambda_2 = -\alpha$$

$= b = 0!$

Saddle point.

$$v^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad v^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Indeed: $Av^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & -\alpha \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda_1 v^{(1)}$

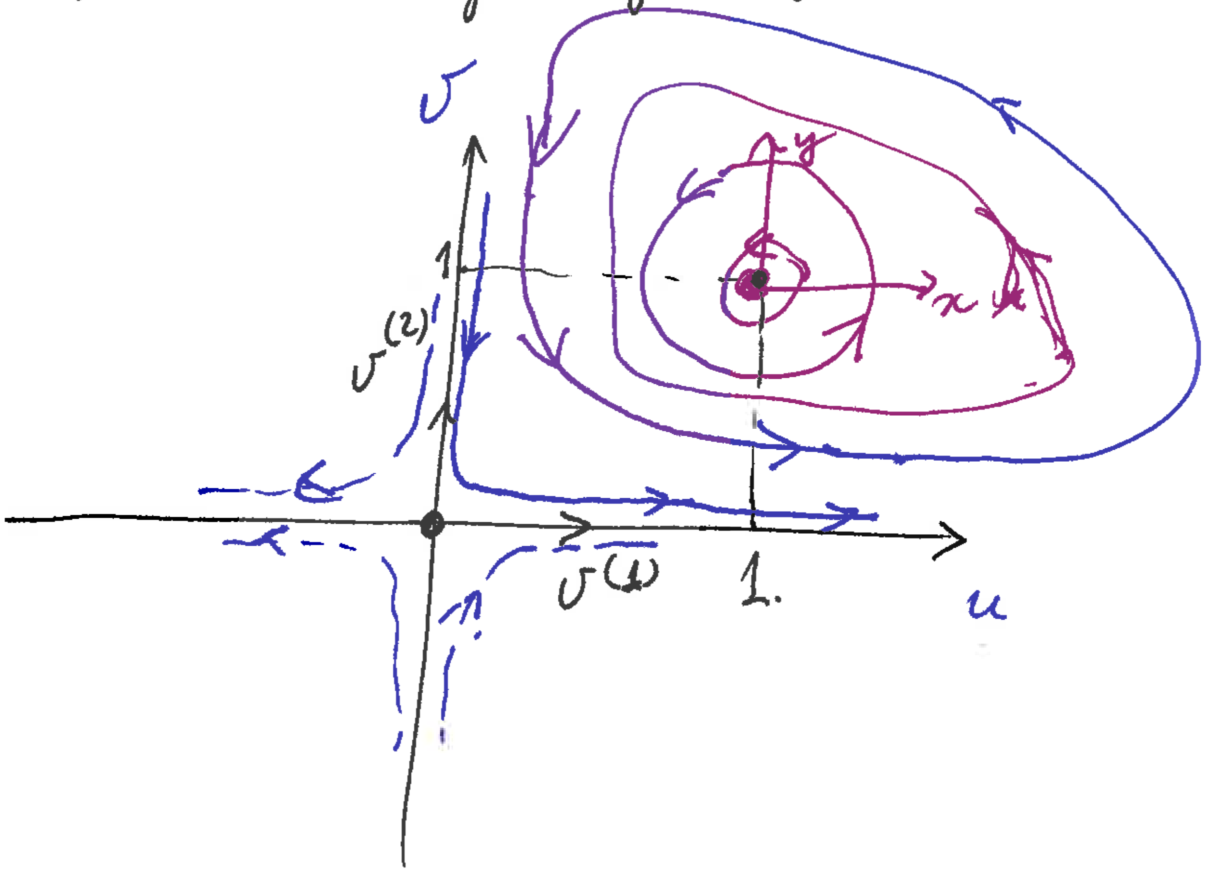
$Av^{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & -\alpha \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\alpha\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \lambda_2 v^{(2)}$

$$A(1,1) = \begin{pmatrix} 1-1 & -1 \\ \alpha(1) & \alpha(1-1) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ \alpha & 0 \end{pmatrix}$$

characteristic equation $\lambda^2 + \alpha = 0$

$$\lambda_{1,2} = \pm i\sqrt{\alpha}$$

Two imaginary eigenvalues



$u = 1 + x$
$v = 1 + y$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A(1,1) \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 v^{(1)} e^{i\sqrt{\alpha}\tau} + \bar{c}_1 \bar{v}^{(1)} e^{-i\sqrt{\alpha}\tau}$$

$$v^{(1)} = \begin{pmatrix} 1 \\ p_1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{\lambda_1 - a}{b} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{i\sqrt{\alpha} - 0}{-1} \end{pmatrix} = \begin{pmatrix} 1 \\ -i\sqrt{\alpha} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -i\sqrt{\alpha} \end{pmatrix} e^{i\sqrt{\alpha}\tau} + \bar{c}_1 \begin{pmatrix} 1 \\ i\sqrt{\alpha} \end{pmatrix} e^{-i\sqrt{\alpha}\tau}$$

$$x = c_1 e^{i\sqrt{\alpha}\tau} + \bar{c}_1 e^{-i\sqrt{\alpha}\tau}$$

$$\boxed{c_1 = |c_1| e^{i\gamma}}$$ in polar form

$$x = |c_1| \left(e^{i\sqrt{\alpha}\tau + i\gamma} + e^{-i\sqrt{\alpha}\tau - i\gamma} \right)$$

Recall: $\cos\phi = \dfrac{e^{i\phi} + e^{-i\phi}}{2}$

$$\boxed{x = 2|c_1| \cos(\sqrt{\alpha}\tau + \gamma)}$$

$$y = c_1(-i\sqrt{\alpha}) e^{i\sqrt{\alpha}\tau} + \bar{c}_1 i\sqrt{\alpha} e^{-i\sqrt{\alpha}\tau}$$

$$y = |c_1|\sqrt{\alpha}(i) \left[-e^{i\sqrt{\alpha}\tau + \gamma} + e^{-i\sqrt{\alpha}\tau - \gamma} \right]$$

$$\boxed{\frac{e^{i\phi} - e^{-i\phi}}{2i} = \sin\phi}$$

$$y = \sqrt{\alpha}|c_1| i \left[-2i \sin(\sqrt{\alpha}\tau + \gamma) \right]$$

$$\boxed{y = 2\sqrt{\alpha}|c_1| \sin(\sqrt{\alpha}\tau + \gamma)}$$

$$\begin{cases} x = 2|C_1| \cos(\sqrt{\alpha}\,\tau + \gamma) \\ y = 2\sqrt{\alpha}\,|C_1| \sin(\sqrt{\alpha}\,\tau + \gamma) \end{cases}$$

$$\frac{x^2}{4|C_1|^2} + \frac{y^2}{4\alpha|C_1|^2} = \cos^2(\sqrt{\alpha}\,\tau+\gamma) + \sin^2(\sqrt{\alpha}\,\tau+\gamma) = 1$$

$$\left(\frac{x}{2|C_1|}\right)^2 + \left(\frac{y}{2\sqrt{\alpha}|C_1|}\right)^2 = 1$$

$$u = 1 + x$$
$$v = 1 + y$$

$$\left(\frac{u-1}{2|C_1|}\right)^2 + \left(\frac{v-1}{2\sqrt{\alpha}|C_1|}\right)^2 = 1$$

<u>Period</u> : $\boxed{T = \frac{2\pi}{\sqrt{\alpha}}}$

Models for Interacting Populations

$$\frac{dN}{dt} = N(a - bP)$$

$$\frac{dP}{dt} = P(cN - d)$$

$\underline{P = 0}$ $\qquad \frac{dN}{dt} = Na \Rightarrow N = N_0 e^{at}$

$\underline{N = 0}$ $\qquad \frac{dP}{dt} = -dP \Rightarrow P = P_0 e^{-dt}$

Predator – Pray <u>model</u>

N – pray population

P – predators population

$$\frac{d\left(N\frac{c}{d}\right)}{d(at)} = \left(\frac{c}{d}N\right)\Big(1 - \underbrace{\frac{b}{a}P}_{v}\Big)$$

$$\frac{d\left(P\frac{b}{a}\right)}{d(at)} = \frac{\left(\frac{b}{a}P\right)d}{a}\Big(\underbrace{\frac{c}{d}N}_{u} - 1\Big)$$

$\tau = at$

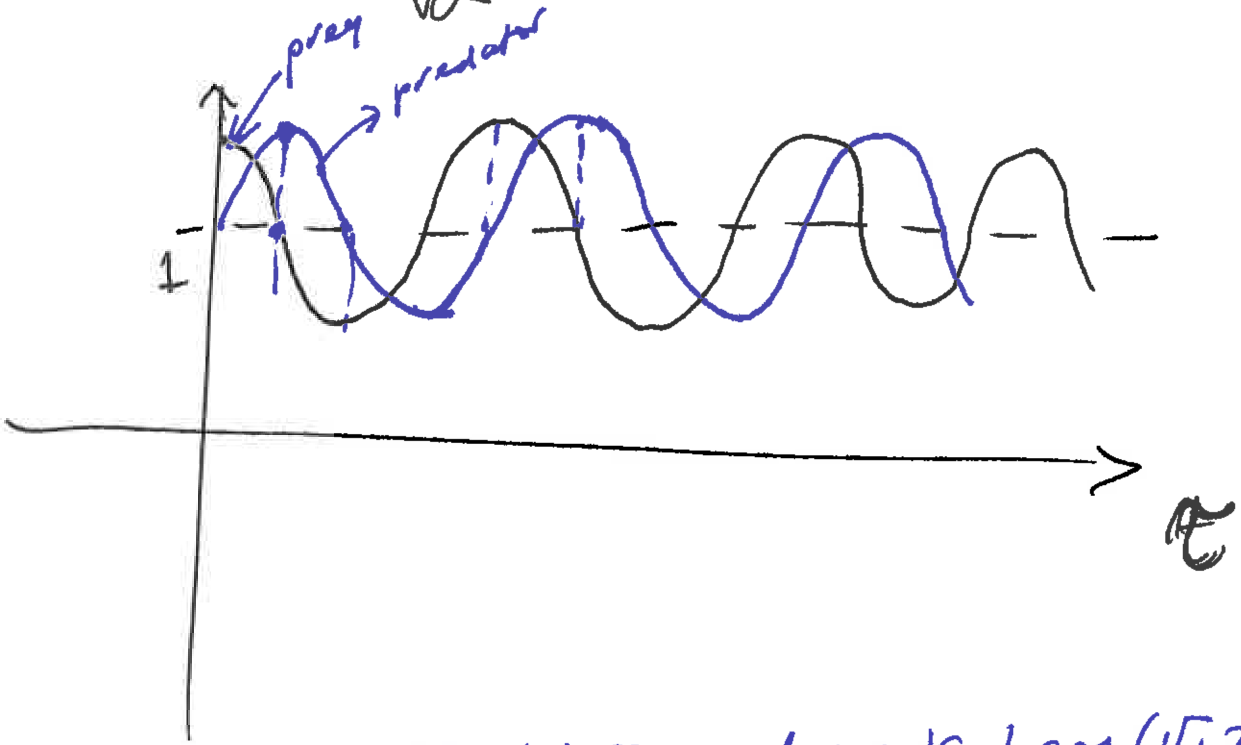
$u = \frac{c}{d}N$

$v = \frac{b}{a}P$

$\alpha = \frac{d}{a}$

$$\begin{cases} \dfrac{du}{d\tau} = u(1-v) \\ \dfrac{dv}{d\tau} = \alpha v(u-1) \end{cases}$$

$$T = \frac{2\pi}{\sqrt{\alpha}} = 2\pi\sqrt{\frac{a}{d}}$$



$$u = 1 + x = 1 + 2|c_1|\cos(\sqrt{\alpha}\tau + \gamma)$$
$$v = 1 + y = 1 + 2\sqrt{\alpha}|c_1|\sin(\sqrt{\alpha}\tau + \delta)$$

$$\frac{dv}{du} = \alpha\frac{v(u-1)}{u(1-v)}$$

$$\frac{1-v}{v}\,dv = \alpha\frac{(u-1)}{u}\,du$$

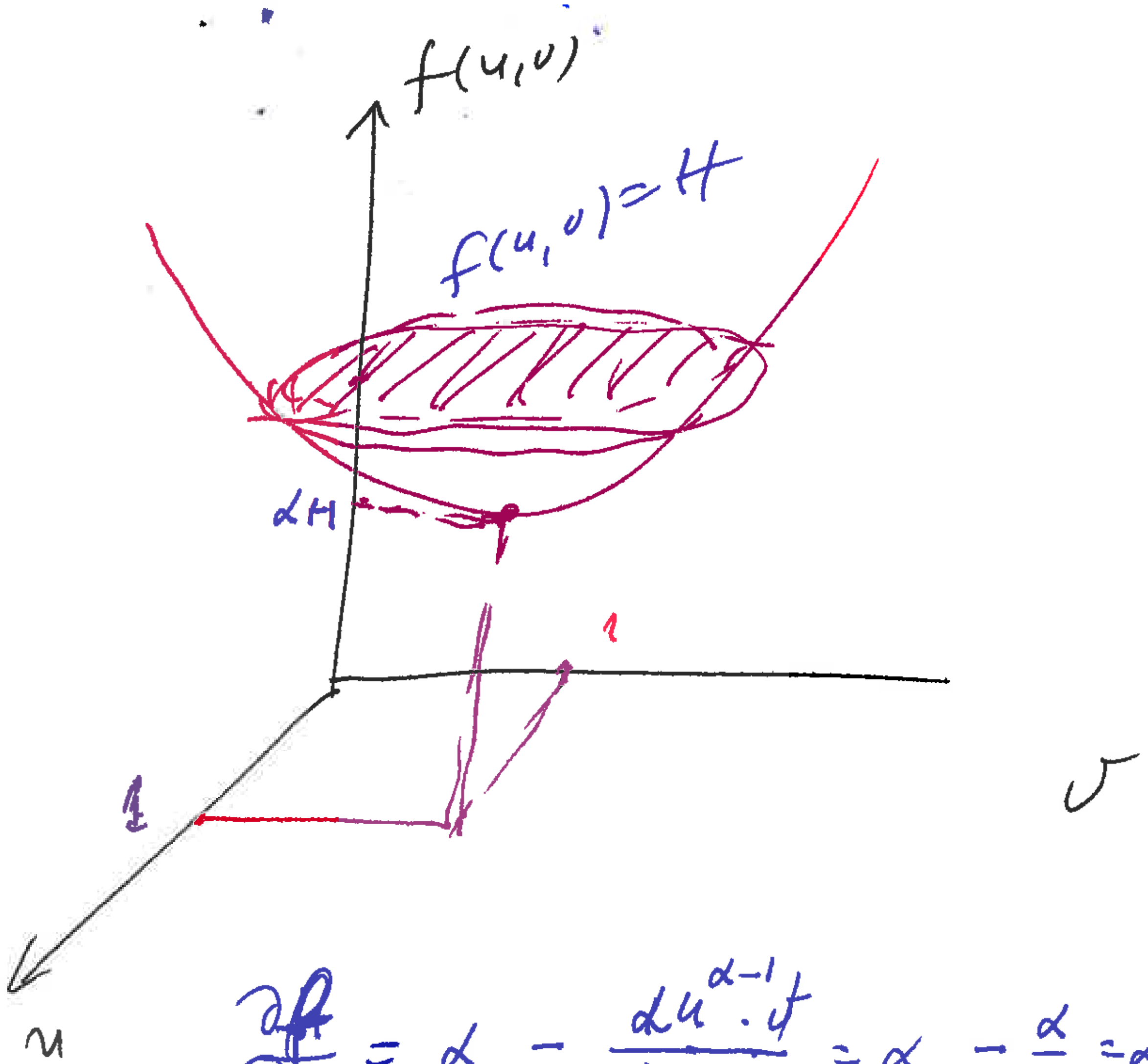
$$\left(\frac{1}{v}-1\right)dv = \alpha\left(1-\frac{1}{u}\right)du$$

$$\ln v - v = \alpha u - \alpha \ln u + \text{const}$$

$$-\text{const} = \alpha u + v - \ln v - \alpha \ln u$$

(3.6) $$\boxed{H = \alpha u + v - \ln u^{\alpha} v = \text{const.}}$$

$$f(u,v) = \alpha u + v - \ln u^{\alpha} v = H = \text{const.}$$



$$\frac{\partial f}{\partial u} = \alpha - \frac{\alpha u^{\alpha-1}\cdot v}{u^{\alpha} v} = \alpha - \frac{\alpha}{u} = \alpha\left(1-\frac{1}{u}\right) = 0$$

$$\frac{\partial f}{\partial v} = 1 - \frac{\cancel{u^{\alpha}}\cdot 1}{\cancel{u^{\alpha}} v} = 1 - \frac{1}{v} = 1 - \frac{1}{v} = 0$$

$H_{min} = \alpha + 1$ $\qquad u = v = 1$

1845 - 1930's

fur cath records

lynx

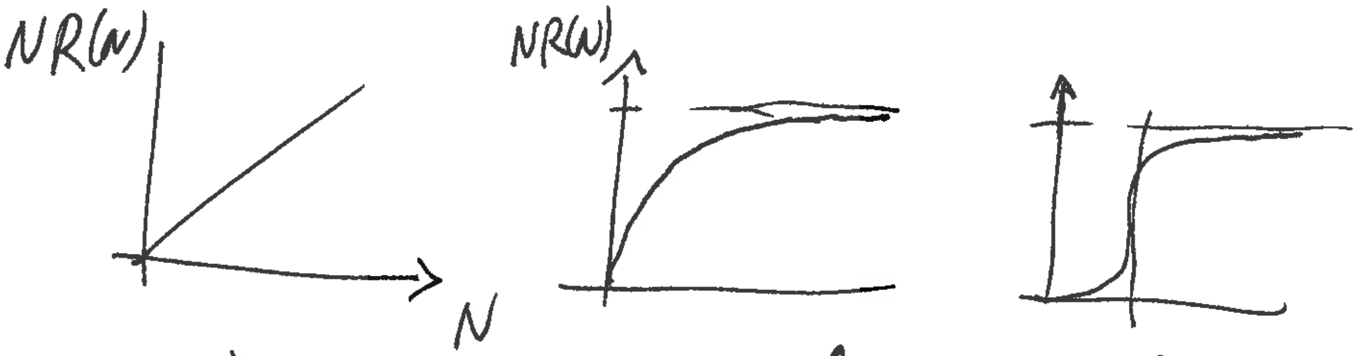
hare

Realistic Predator – Prey Models

$$\begin{cases} \dfrac{dN}{dt} = N\,F(N,P) \\ \dfrac{dP}{dt} = P\,G(N,P) \end{cases}$$

$$\begin{cases} \dfrac{dN}{dt} = rN\left(1-\dfrac{N}{K}\right) - P\boxed{NR(N)} \\ \dfrac{dP}{dt} = kP\left(1-\dfrac{hP}{N}\right) \end{cases}$$

r, k, K, h – positive constants



$R(N) = A$
(as in Lotka–Volterra case)

$R(N) = \dfrac{A}{N+B}$

$R = \dfrac{AN}{N^2+B^2}$

$$\begin{cases} \dfrac{dN}{dt} = N\left[r\left(1-\dfrac{N}{K}\right) - \dfrac{kP}{N+D}\right] \\[2ex] \dfrac{dP}{dt} = P\left[s\left(1-\dfrac{hP}{N}\right)\right] \end{cases}$$

r, K, k, s, h, D, 6 positive constants

$$u(\tau) = \frac{N}{K} \qquad v(\tau) = \frac{hP}{K}, \qquad \tau = rt$$

$$\frac{d\,(N/K)}{d\,tr} = \frac{N}{K}\left[\cancel{\frac{r}{r}}(1-u) - \frac{\frac{k}{hr}\,hP/K}{\frac{N}{K} + \frac{D}{K}}\right]$$

$$d = \frac{D}{K} \qquad a = \frac{k}{hr}$$

$$\boxed{\frac{du}{d\tau} = u\left[(1-u) - \frac{a \cdot v}{u+d}\right]}$$

$$\frac{d\,(hP/K)}{d\,rt} = \frac{hP}{K}\,\frac{s}{r}\left(1 - \frac{hP/K}{N/K}\right), \qquad b = \frac{s}{r}$$

$$\boxed{\frac{dv}{d\tau} = bv\left(1 - \frac{v}{u}\right)}$$

$$\boxed{a, b, d}$$

$$\frac{du}{d\tau} = u(1-u) - \frac{auv}{u+d} = f(u,v)$$

$$\frac{dv}{d\tau} = bv\left(1-\frac{v}{u}\right) = g(u,v)$$

$$\begin{cases} f(u^*,v^*) = 0 \\ g(u^*,v^*) = 0 \end{cases}$$

$$v=0 \implies u(1-u)=0$$

$\cancel{(0,0)}$.

$(1,0)$; $\quad \underline{u^* = v^* > 0}$

$$\cancel{u^*}(1-u^*) - \frac{a(u^*)^{\cancel{2}}}{u^*+d} = 0$$

$$(1-u^*)(u^*+d) = a u^*$$

$$(u^*)^2 + (a+d-1)u^* - d = 0$$

$$\begin{cases} u^* = \dfrac{1-a-d+\sqrt{(1-a-d)^2+4d}}{2} \\ v^* = u^* \end{cases}$$

$$\mathcal{A} = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix} = \begin{pmatrix} 1-2u - \frac{av}{u+d} + \frac{auv}{(u+d)^2} & \frac{-au}{u+d} \\ + \frac{bv^2}{u^2} & b - \frac{2bv}{u} \end{pmatrix}$$

$$\boxed{\left(\frac{uv}{u+d}\right)' = \left((uv)\cdot\frac{1}{u+d}\right)' = v\cdot\frac{1}{u+d} - \frac{(uv)}{(u+d)^2}}$$

$$\mathcal{A}(1,0) = \begin{pmatrix} 1-2 & -\frac{a}{1+d} \\ 0 & b \end{pmatrix} = \begin{pmatrix} -1 & -\frac{a}{1+d} \\ 0 & b \end{pmatrix}$$

$$\begin{vmatrix} -1-\lambda & -\frac{a}{1+d} \\ 0 & b-\lambda \end{vmatrix} = 0 \qquad (-1-\lambda)(b-\lambda)=0$$

$$\underline{\lambda = -1, \quad \lambda = b}$$

'saddle point' => unstable.

$$\mathcal{A}(u^*,u^*) = \begin{bmatrix} u^*\left[\frac{au^*}{(u^*+d)^2} - 1\right] & -\frac{au^*}{u^*+d} \\ +b & -b \end{bmatrix}$$

$$1 - 2u^* - \frac{au^*}{u^*+d} + \frac{a(u^*)^2}{(u^*+d)^2} = \text{ but } \boxed{1-u^* = \frac{au^*}{u^*+d}}$$

$$\overset{\shortparallel}{1-u^*}$$

$$= \cancel{1} - 2u^* - (\cancel{1} - u^*) + \frac{a(u^*)^2}{(u^*+d)^2} = -u^* + \frac{a(u^*)^2}{(u^*+d)^2} =$$

$$= u^*\left[\frac{au^*}{u^*+d} - 1\right]$$

Characteristic equation

$$\lambda^2 - (\mathrm{tr}\, A)\lambda + \det A = 0$$

For stability $\boxed{\det A > 0}$, $\underline{\mathrm{tr}\, A < 0}$

$\mathrm{tr}\, A \leq 0$ means

$$\boxed{u^* \left[\frac{a u^*}{(u^*+d)^2} - 1\right] < b \qquad (1)}$$?

$$\det A = -b\, u^* \left[\frac{a u^*}{(u^*+d)^2} - 1\right] + b \frac{a u^*}{u^*+d} =$$

$$= b u^* \left[1 + \frac{a}{u^*+d} - \frac{a u^*}{(u^*+d)^2}\right] =$$

$$= b u^* \left[1 + \frac{a(\cancel{u^*}+d) - \cancel{a u^*}}{(u^*+d)^2}\right]$$

$$= b u^* \left[1 + \frac{ad}{(u^*+d)^2}\right] > 0 \quad \checkmark$$

But $u^* = \dfrac{1-a-d+\sqrt{(1-a-d)^2+4d}}{2}$

$$\boxed{b > \left[a - \sqrt{(1-a-d)^2+4d}\,\right] \frac{\left[1+a+d - \sqrt{(1-a-d)^2+4d}\,\right]}{2a}}$$

Predator-Prey Model (continuation)

$$\frac{du}{d\tau} = u(1-u) - \frac{auv}{u+d} = f(u,v)$$

$$\frac{dv}{d\tau} = bv\left(1 - \frac{v}{u}\right) = g(u,v)$$

a, b, d - positive parameters

$$u^* = \frac{1-a-d+\sqrt{(1-a-d)^2+4d}}{2} \qquad v^* = u^*$$

$$1 - u^* = \frac{au^*}{u^*+d}, \quad D = (1-a-d)^2 + 4d$$

$$A(u^*,u^*) = \begin{pmatrix} u^*\left[\frac{au^*}{(u^*+d)^2} - 1\right] & -\frac{au^*}{u^*+d} \\ b & -b \end{pmatrix}$$

For stability it is necessary and sufficient

$$\begin{cases} \operatorname{tr} A < 0 & (?) \\ \det A > 0 & \checkmark \text{ always} \end{cases}$$

$$u^*\left[\frac{au^*}{(u^*+d)^2} - 1\right] < b$$

$$u^*\left[\frac{au^*}{(u^*+d)^2} - 1\right] = u^*\left[\frac{1-u^*}{u^*+d} - 1\right] = u^* \cdot \frac{1-u^*-u^*-d}{u^*+d} =$$

$$= \frac{u^*}{u^*+d}\cdot(1-d-2u^*) = \frac{1-u^*}{a}\cdot\left[\cancel{1}-\cancel{d}-(\cancel{1}-a-\cancel{d})-\sqrt{D}\right]$$

$$= \frac{1}{a}\left(\frac{2-1+a+d-\sqrt{D}}{2}\right)(a-\sqrt{D}) < b$$

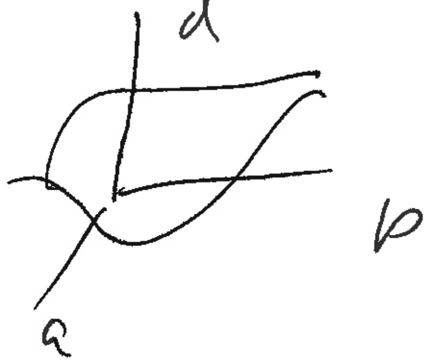
$$b > (a-\sqrt{D}) \cdot \frac{1+a+d-\sqrt{D}}{2a}$$

$$b > \left[a-\sqrt{(1-a-d)^2+4d}\right] \cdot \frac{\left[1+a+d-\sqrt{(1-a-d)^2+4d}\right]}{2a}$$

(3.4)

It defines 3-dimensional surface in (a, b, d)-space

$a>0,\ b>0,\ d>0$



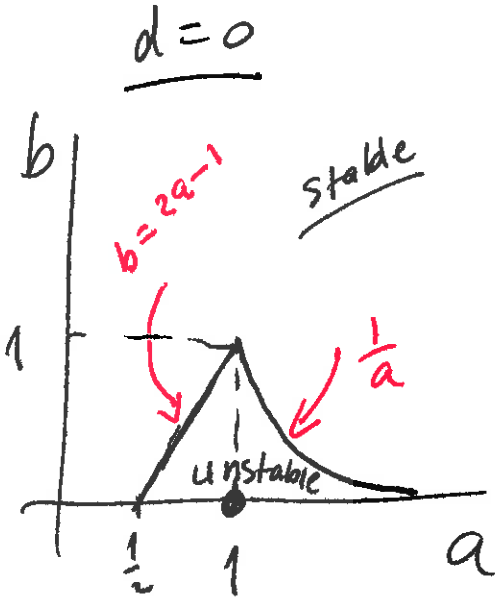
Observations:

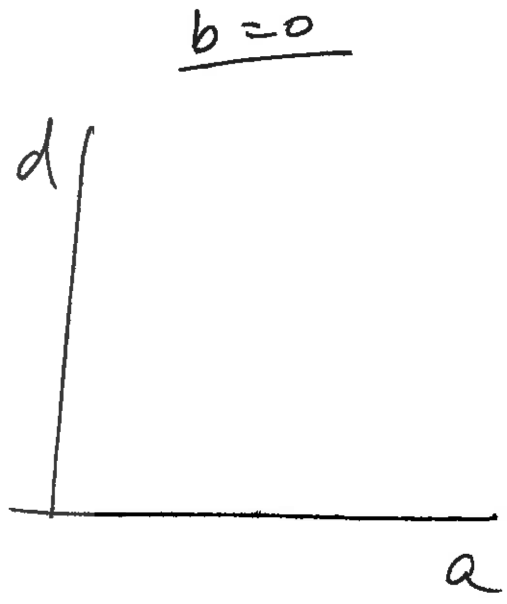
① $1+a+d-\sqrt{(1-a-d)^2+4d}$ monotonic & decreasing function of d, with max at $d=0$.

$$1+a+d-\sqrt{(1-a-d)^2+4d} = \frac{(1+a+d)^2-(1-a-d)^2-4d}{1+a+d+\sqrt{(1-a-d)^2+4d}} =$$

$$= \cdots = \frac{4a}{1+a+d+\sqrt{1+a^2+d^2+2d+2ad-2a}}$$

$$(x+y+z)^2 = x^2+y^2+z^2+2xy+2xz+2yz$$





I $\underline{d=0}$; $b > \left[a - \sqrt{(1-a)^2}\right] \dfrac{1+a-\sqrt{(1-a)^2}}{2a}$

$$b > \left(a - |1-a|\right) \frac{1+a-|1-a|}{2a}$$

when $0 < a < 1$

$$b > (a-1+a) \underbrace{\frac{\cancel{1}+a-\cancel{1}+a}{2a}}_{1} \quad ; \quad b > 2a-1$$

$$\boxed{b > 2a-1 \quad \text{when} \quad 0 < a < 1} \qquad (1)$$

when $a > 1$

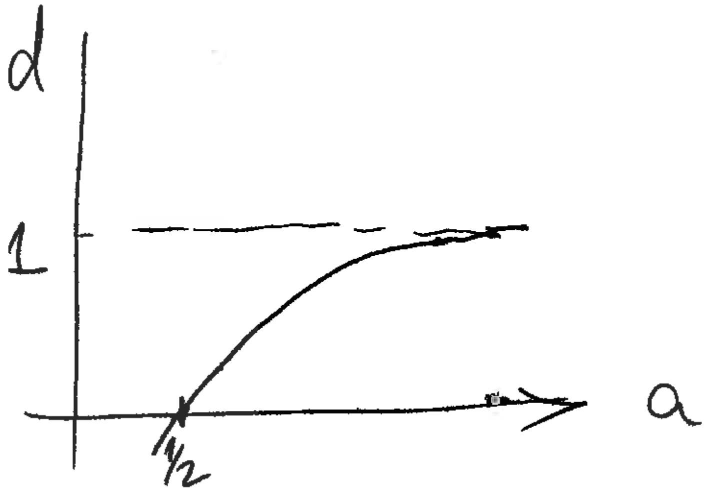
$$b > (\underbrace{\cancel{a}+1-\cancel{a}}_{1}) \frac{1+\cancel{a}+1-\cancel{a}}{2a} = \frac{2}{2a} = \frac{1}{a}$$

$$\boxed{b > \frac{1}{a} \quad \text{when} \quad a > 1} \qquad (2)$$

$a=1, \quad b \overset{-4}{=} d=0$

$0 > [1-0]\cdot\frac{1}{2}[2-0] \qquad 0>1 \quad \times$

$b=0$ case $\alpha \quad a<\sqrt{(1-a-d)^2+4d}$



$a^2=(1-a-d)^2+4d \Rightarrow d(a)$

$\cancel{a^2}=1+\cancel{a^2}+d^2-2a-2d+2ad+4d$

$d^2+(2+2a)d+1-2a=0$

$d^2 + 2(a+1)d+(1-2a)=0$

$d=-(a+1)\pm\sqrt{(a+1)^2-(1-2a)}$

$d=\sqrt{a^2+2a+\cancel{1}-\cancel{1}+2a}-(a+1)$

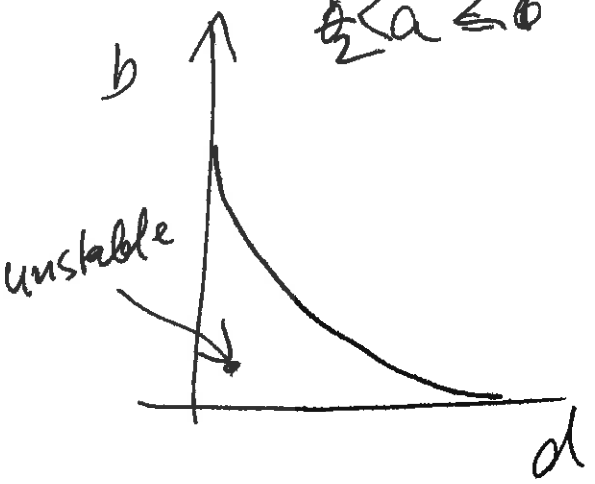
$\boxed{d(a)=\sqrt{a^2+4a}-(a+1)}$

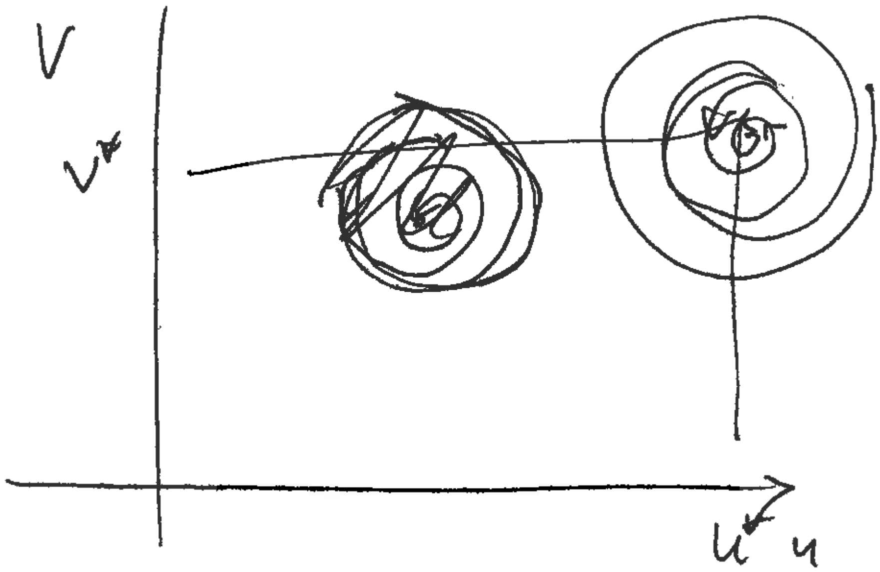
$d(\frac{1}{2})=\sqrt{\frac{1}{4}+2}-\frac{3}{2}=\frac{3}{2}-\frac{3}{2}=0$

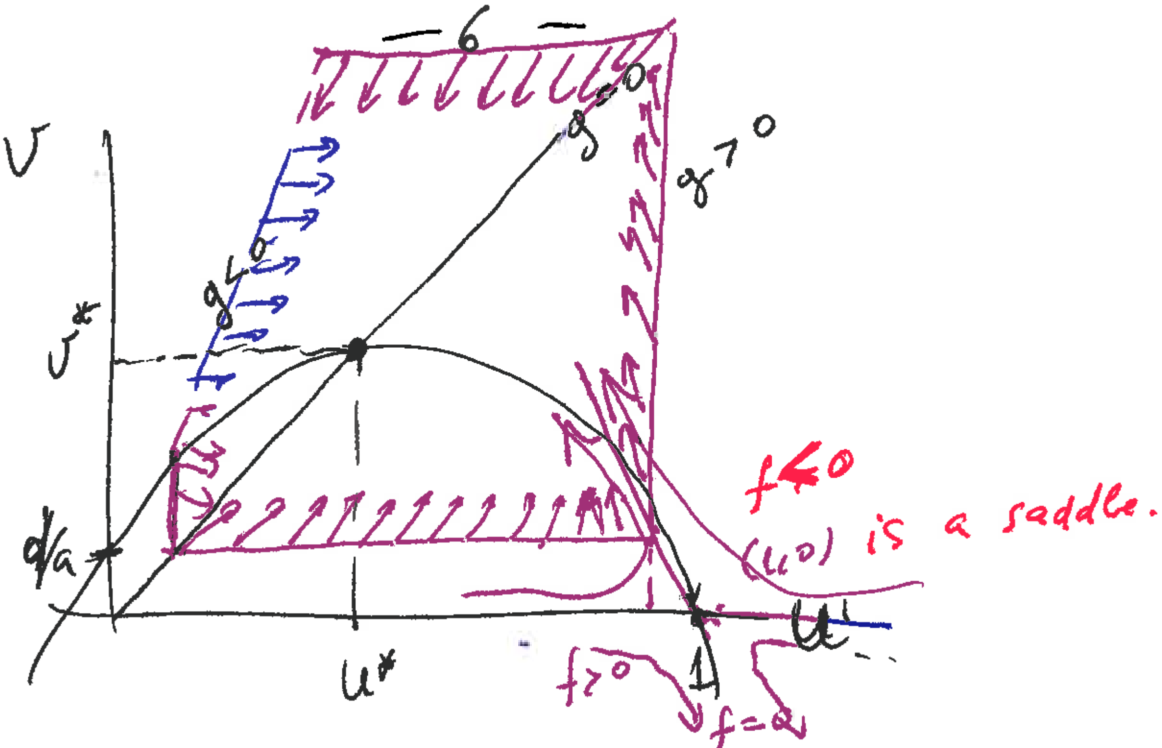
$$d(a) = \frac{a^2+4a-(a+1)^2}{\sqrt{a^2+4a}+(a+1)} = \frac{\cancel{a^2}+4a-\cancel{a^2}-2a-1}{\sqrt{a^2+4a}+a+1}$$

$$d(a) = \frac{2a-1}{\sqrt{a^2+4a}+a+1} \xrightarrow{a\to\infty} \frac{2a}{a+a} = 1$$

$\frac{1}{2} < a \leq 6$ $\qquad$ $1 < a < \infty$



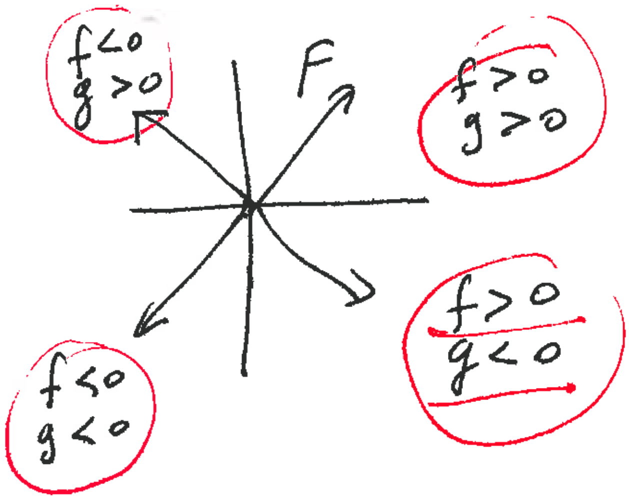


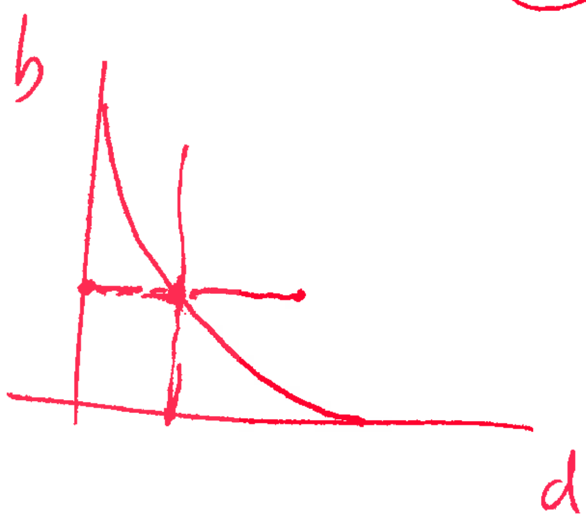


$$\left.\begin{array}{l} f(u,v)=0 \\ g(u,v)=0 \end{array}\right\} \text{null-clines}$$

$$f=0 \iff \cancel{u}(1-u) = \frac{a\cancel{u}v}{u+d} \Rightarrow v = \frac{1}{a}(1-u)(u+d)$$

$$F = \begin{pmatrix} f \\ g \end{pmatrix}$$





$$b = \frac{s}{r}$$

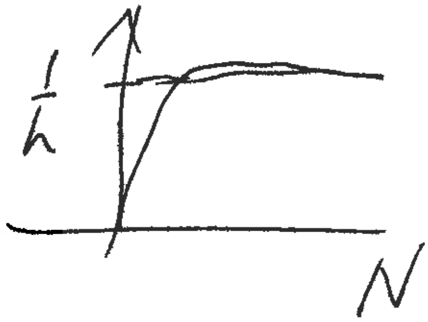
Rosenzweig - MacArthur Model

$$\begin{cases} \dot u = a u(1-u) - \dfrac{auv}{1+bu} \\ \dot v = v\left(\dfrac{bu}{1+bu} - c\right) \end{cases}$$

parameters: $a, b, c > 0$

Originally

$$\frac{dN}{dt} = \underbrace{rN\left(1-\frac{N}{K}\right)}_{\text{logistic}} - P\underbrace{\frac{sN}{1+shN}}_{\text{predation}}$$



$$\frac{dP}{dt} = eP\frac{sN}{1+shN} - mP$$

Parameters: r, K, s, h, e, m

$$u = \frac{N}{K}; \quad v = \frac{sP}{r}; \quad \tau = \frac{et}{h}; \quad a = \frac{rh}{e}, \quad b = shK, \quad c = \frac{mh}{e}$$

Steady states: $(0,0)$, $(1,0)$, (u^*, v^*)

$$u^* = \frac{c}{b(1-c)} \qquad v^* = \frac{b - c(1+b)}{b(1-c)^2}$$

~~Steady~~

$$1 > \frac{b}{1+b} > c > 0$$

in order to ensure $u^* > 0$, $v^* > 0$

$$f = au - au^2 - \frac{auv}{1+bu}, \qquad g = \frac{buv}{1+bu} - vc$$

$$A = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix} = \begin{pmatrix} a - 2au - \frac{av}{1+bu} + \frac{abuv}{(1+bu)^2} & -\frac{au}{1+bu} \\ \frac{bv}{1+bu} - \frac{b^2uv}{(1+bu)^2} & \frac{bu}{1+bu} - c \end{pmatrix}$$

$$A(0,0) = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}$$ saddle pt. $v^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\lambda_1 = a$, $\lambda_2 = -c$

$$A(1,0) = \begin{pmatrix} -a & -\frac{a}{1+b} \\ 0 & \frac{b}{1+b} - c \end{pmatrix}$$ saddle

positive

$\lambda_1 = -a$, $v^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\lambda_2 = \frac{b}{1+b} - c > 0$

$$v^{(2)} = \begin{pmatrix} 1 \\ -\frac{1}{a}(1+b)(\lambda_2 + a) \end{pmatrix}$$

$$A(u^*, v^*) = \begin{pmatrix} \dfrac{ac\,[b(1-c) - (c+1)]}{b(1-c)} & -\dfrac{ac}{b} \\ b(1-c) - c & 0 \end{pmatrix}$$

$$\left.\begin{array}{l} \det A > 0 \ \checkmark \\ \operatorname{tr} A < 0 \ ? \end{array}\right\} \text{stability}$$

If $\dfrac{1+c}{1-c} > \boxed{b \geq \dfrac{c}{1+c}} > 0$ the solution (u^*, v^*) exists and is stable.

$$\frac{b}{1+b} > c$$

$$b(1-c) > c$$

$$\cancel{b} = c \qquad \frac{b}{1+b} > c$$

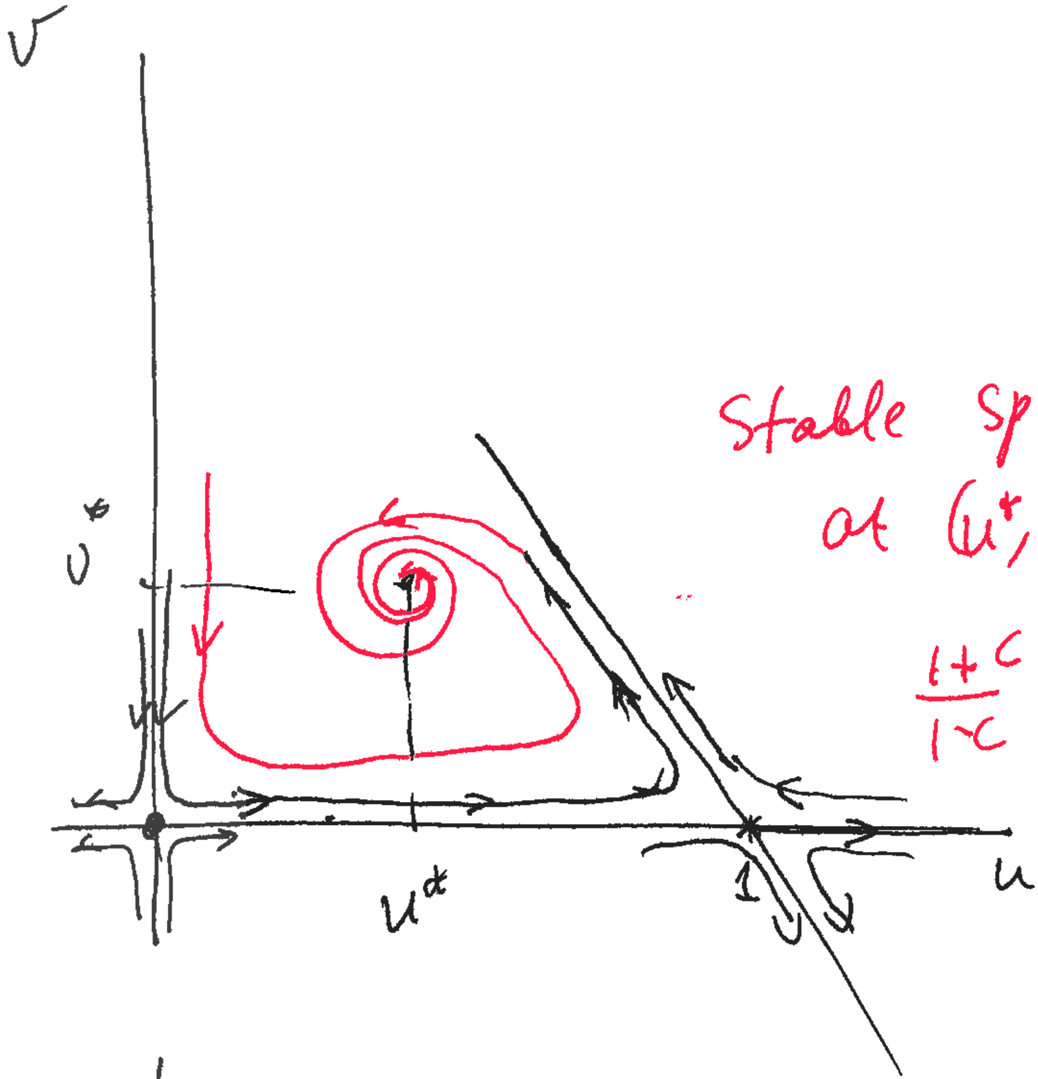
$$b > c + bc$$

$$b - bc > c$$

Bifurcation value is $b^* = \dfrac{1+c}{1-c}$

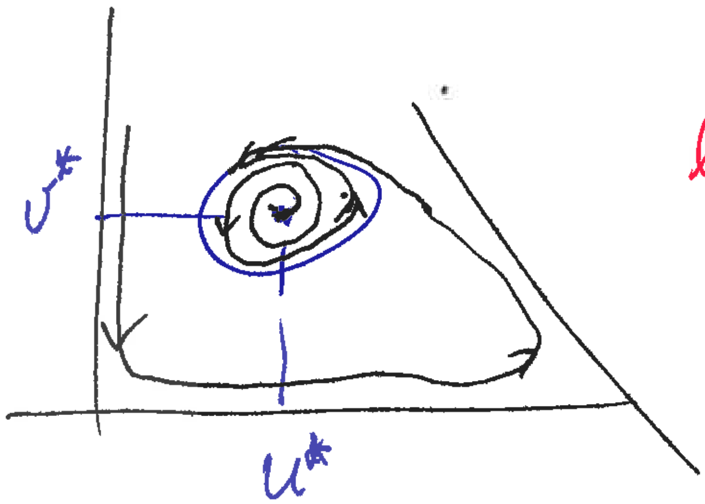
Charact. equation $\lambda^2 + \dfrac{ac(1-c)}{1+c} = 0$

$$\lambda_{1,2} = \pm i \sqrt{\frac{ac(1-c)}{1+c}} \quad \text{imaginary}$$



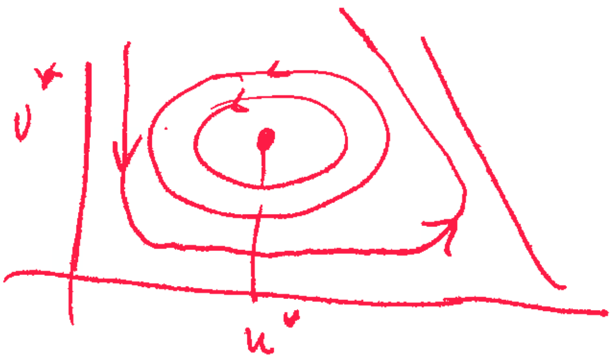
Stable spiral
at (u^*, v^*)

$$\frac{1+c}{1-c} > b > \frac{c}{1-c}$$



limit cycle

$$b > \frac{1+c}{1-c} = b^*$$



Limiting case
(bifurcation)

$$b = \frac{1+c}{1-c}$$

'center'

1. Predator- pray
2. Competition
3. Mutualism / Simbiosis