Mathematical descriptions of physical systems

1. For a long, light, flexible string which does not stretch, assume that a damping force, per unit length, is present proportional to the velocity of the string's displacement with constant of proportionality a. Assume also that a restoring force per unit length proportional to the displacement is present with constant of proportionality b. Hence derive the telegraph equation
$$u_{tt}(x,t) = c^2 u_{xx}(x,t) - \hat{a}u_t(x,t) - \hat{b}u(x,t)$$

2. Consider the transverse vibration of a uniform beam. Adopting Eulers beam theory, the moment M at a point can be written as
$$M = EIu_{xx},$$
where EI is called the flexural rigidity, E is the elastic modulus, and I is the moment of inertia of the cross section of the beam. Show that the transverse motion of the beam may be described by
$$utt + c^2 u_{xxxx} = 0,$$
where $c^2 = EI/\rho A$, ρ is the density, and A is the cross-sectional area of the beam.

3. By considering a small volume element $\Delta V = \Delta x \Delta y \Delta z$ of mass density ρ, undergoing displacement $\mathbf{u} = (u, v, w)$, the equations of motion are easily shown to follow
$$\begin{aligned}\rho u_{tt} &= \tau_{xx,x} + \tau_{xy,y} + \tau_{xz,z}\\ \rho v_{tt} &= \tau_{yx,x} + \tau_{yy,y} + \tau_{yz,z}\\ \rho w_{tt} &= \tau_{zx,x} + \tau_{zy,y} + \tau_{zz,z}\end{aligned}$$

4. Derive the Navier equation of motion
$$\rho\mathbf{u}_{tt} = (\lambda+\mu)\nabla\nabla\cdot\mathbf{u} + \mu\nabla^2\mathbf{u}$$

5. For an irrotational vector field $\mathbf{u}$ show
$$\nabla\nabla\cdot\mathbf{u} = \nabla^2\mathbf{u}$$

6. (a) Show
$$\nabla^2\mathbf{u} = \nabla\nabla\cdot\mathbf{u} - \nabla\times(\nabla\times\mathbf{u})$$
   (b) State and prove Helmholtz's Theorem.
   (c) Decompose $\mathbf{u}$ via Helmholtz's Theorem, and hence show
$$\mathbf{u}_{L\,tt} = c_L^2\nabla^2\mathbf{u}_L \quad \mathbf{u}_{T\,tt} = c_T^2\nabla^2\mathbf{u}_T$$
   for the Navier equation of motion.