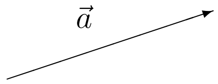
Vectors, Gradient, Divergence and Curl.

1 Introduction

A vector is determined by its length and direction. They are usually denoted with letters with arrows on the top $\vec{a}$ or in bold letter **a**. We will use arrows.



Two vectors are equal if they have the same length and the same direction:

$$\vec{v} = \vec{w}$$

If we are given two points in the space (p_1, p_2, p_3) and (q_1, q_2, q_3) then we can compute the vector that goes from p to q as follows:

$$\vec{pq} = [q_1 - p_1, q_2 - p_2, q_3 - p_3]$$

For example if you are given the point $p = (1, 0, 1)$ and $q = (3, 2, 4)$ then the vector joining p and q is:

$$\vec{pq} = [3 - 1, 2 - 0, 4 - 1] = [2, 2, 3].$$

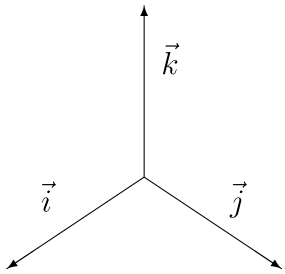
The three principal directions (unitary vectors, vectors of length one) in the space are

$$\vec{i} = [1, 0, 0],$$

$$\vec{j} = [0, 1, 0]$$

and

$$\vec{k} = [0, 0, 1]$$



The length of a vector with coordinates $[a_1, a_2, a_3]$ is

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

The sum of two vectors $\vec{a} = [a_1, a_2, a_3]$ and $\vec{b} = [b_1, b_2, b_3]$ is obtained by adding the corresponding components $\vec{a} + \vec{b} = [a_1 + b_1, a_2 + b_2, a_3 + b_3]$.

We have some properties:

1. Commutative: $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

2. Associative: $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$

3. There exists $\vec{0}$ such that:

$$\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}$$

and

$$\vec{a} - \vec{a} = \vec{0}$$

We can multiply a vector by a real number, often called scalar $\beta \in \mathbb{R}$:

$$\beta\vec{a} = \beta[a_1, a_2, a_3] = [\beta a_1, \beta a_2 \beta a_3]$$

And we have some properties:

1. Distributive 1: $\beta(\vec{a}+\vec{b}) = \beta\vec{a}+\beta\vec{b}$

2. Distributive 2: $(\beta+\gamma)\vec{a} = \beta\vec{a}+\gamma\vec{a}$

3. $1\vec{a} = \vec{a}, \quad (-1)\vec{a} = -\vec{a}, \quad \text{and} \quad 0\vec{a} = \vec{0}.$

Example 1 If $\vec{a} = [1,2,3]$ and $\vec{b} = [0,1,0]$, then compute $|3\vec{a}-\vec{b}|$*:*

$$\begin{aligned} 3\vec{a}-\vec{b} &= 3[1,2,3]-[0,1,0] \\ &= [3,6,9-[0,1,0] \\ &= [3,5,9] \end{aligned}$$

Then:

$$|3\vec{a}-\vec{b}| = \sqrt{3^2+5^2+9^2} = \sqrt{115}.$$

1.1 Inner product (dot product)

The *inner product* of the vectors $\vec{a}$ and $\vec{b}$ is a scalar (real number) defined as follows:

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\alpha,$$

where α is the angle between $\vec{a}$ and $\vec{b}$. If $\vec{a} = [a_1, a_2, a_3]$ and $\vec{b} = [b_1, b_2, b_3]$, then the inner product is:

$$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

Observe that

$$|\vec{a}| = \sqrt{\vec{a}\cdot\vec{a}} = a_1^2+a_2^2+a_3^2,$$

and then we can compute the angle between $\vec{a}$ and $\vec{b}$ with the formula:

$$\cos\alpha = \frac{\vec{a}\vec{b}}{|\vec{a}||\vec{b}|} = \frac{a_1b_1+a_2b_2+a_3b_3}{\left(\sqrt{a_1^2+a_2^2+a_3^2}\right)\left(\sqrt{b_1^2+b_2^2+b_3^2}\right)}.$$

Two vectors are *orthogonal* if and only if its inner product is 0:

$$\vec{a} \text{ is orthogonal to } \vec{b} \Leftrightarrow \vec{a}\cdot\vec{b} = 0$$

The inner product has some properties:

1. Commutative: $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
2. Distributive: $(\beta\vec{a} + \gamma\vec{b}) \cdot \vec{c} = \beta\vec{a} \cdot \vec{c} + \gamma\vec{b} \cdot \vec{c}$
3. $\vec{a} \cdot \vec{a}$ and $\vec{a} \cdot \vec{a} = 0$ if and only if $\vec{a} = 0$.

Recall that there are two important inequalities with vectors:

- **Cauchy-Schwarz**: $|\vec{a} \cdot \vec{b}| \leq |\vec{a}||\vec{b}|$
- **Triangle Inequality:** $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$

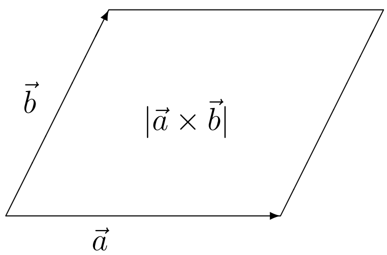
1.2 Vector product (cross-product)

It is denoted by $\vec{v} = \vec{a} \times \vec{b}$ and it is a vector with length

$$|\vec{v}| = |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \alpha,$$

where α is the angle between $\vec{a}$ and $\vec{b}$, and the direction of $\vec{v}$ is perpendicular to both $\vec{a}$ and $\vec{b}$ and such $\vec{a}, \vec{b}, \vec{v}$, in this order, form a right-handed triple. (This means that the determinant of $\vec{a}, \vec{b}, \vec{v}$ is positive).

Geometrically speaking $|\vec{a} \times \vec{b}|$ is the area of the parallelogram with sides $\vec{a}$ and $\vec{b}$.



And we have some properties:

1. If $\beta \in \mathbb{R}$ then $\beta(\vec{a} \times \vec{b}) = (\beta\vec{a}) \times \vec{b} = \vec{a} \times (\beta\vec{b})$
2. Distributives: $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ and $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
3. Anticommutative: $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

Remark: The vector product is NOT associative. Thus, in general:

$$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$$

Let us see this with an example:

$$\vec{a} = [1, 0, 0], \vec{b} = [1, 0, 0], \vec{c} = [0, 1, 0]$$

Then

$$\vec{b} \times \vec{c} = [0, 0, 1] \Rightarrow \vec{a} \times (\vec{b} \times \vec{c}) = [0, -1, 0],$$

but

$$\vec{a} \times \vec{b} = [1, 0, 0] \times [1, 0, 0] = [0, 0, 0] \Rightarrow (\vec{a} \times \vec{b}) \times \vec{c} = [0, 0, 0]$$

1.3 Scalar triple product

It is a scalar (real number) defined as follows:

$$(\vec{a}\ \vec{b}\ \vec{c}) = \vec{a} \cdot (\vec{b} \times \vec{c})$$

(Notice that there are no commas in $(\vec{a}\ \vec{b}\ \vec{c})$). We have:

$$(\vec{a}\ \vec{b}\ \vec{c}) = \vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

It is an immediate consequence of the properties of the determinants that:

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}.$$

Geometric interpretation: The absolute value of $(\vec{a}\ \vec{b}\ \vec{c})$ is the volume of the parallelepiped (oblique box) with $\vec{a}, \vec{b}$ and $\vec{c}$ as edges.

Example 2 Find the area of the parallelogram with vertices $(2,2,0)$, $(9,2,0)$,$(10,3,0)$,$(3,3,0)$.

First, we need to find the vectors joining the vertices $(2,2,0)$, $(9,2,0)$ and $(2,2,0)$, $(3,3,0)$ ($(10,3,0)$ is not needed!). We will denote by $\vec{a}$ the vector from $(2,2,0)$ to $(9,2,0)$ and $\vec{b}$ the vector from $(2,2,0)$ to $(3,3,0)$. Then:

$$\vec{a} = [9-2, 2-2, 0-0] = [7,0,0]$$

$$\vec{b} = [3-2, 3-2, 0] = [1,1,0].$$

We know that the area of the parallelogram with edges $\vec{a}$ and $\vec{b}$ is the module of its vector product:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 7 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 7\vec{k} = [0,0,7],$$

And therefore:

$$|\vec{a} \times \vec{b}| = 7.$$

Example 3 Find the area of the triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

To do this, we first compute the area of the the parallelogram that has as edges the vector joining $(1,0,0)$, $(0,1,0)$ and the vector joining $(1,0,0)$, $(0,0,1)$ and then we divide by 2.

$$\vec{a} = [0-1, 1-0, 0-0] = [-1,1,0] \quad \text{and} \quad \vec{b} = [0-1, 0-0, 1-0] = [-1,0,1].$$

Then:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \vec{i} + \vec{j} + \vec{k} = [1,1,1],$$

so the area of the parallelogram is:

$$|\vec{a} \times \vec{b}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}.$$

Hence the area of the triangle is $\frac{\sqrt{3}}{2}$.

Example 4 Find the volume of the parallelepiped determined by the vertices $(1,1,1)$,$(4,7,2)$,$(3,2,1)$ and $(5,4,3)$

The volume of the parallelepiped is the absolute value of $\vec{a}\cdot(\vec{b}\times\vec{c})$, where $\vec{a}$,$\vec{b}$ and $\vec{c}$ are the vectors joining the points $(1,1,1)$ and $(4,7,2)$, $(1,1,1)$ and $(3,2,1)$, and $(1,1,1)$ and $(5,4,3)$ respectively:

$$\vec{a} = [4-1, 7-1, 2-1] = [3,6,1]$$

$$\vec{b} = [3-1, 2-1, 3-1] = [2,1,0]$$

$$\vec{c} = [5-1, 4-1, 3-1] = [4,3,2].$$

Then:

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} 3 & 6 & 1 \\ 2 & 1 & 0 \\ 4 & 3 & 2 \end{vmatrix} = -16.$$

Hence the volume is $|\vec{a}\cdot(\vec{b}\times\vec{c})| = 16$.

2 Gradient of a Scalar Field

A *vector field* F is a function that takes any point in space and assign a vector to it:

$$\begin{aligned} F: \text{From points in the space} &\longrightarrow \text{To vectors in the space} \\ (x,y,z) &\longrightarrow [F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)] \end{aligned}$$

A *scalar field* f is a function that takes a point in space and assigns a number to it:

$$\begin{aligned} f: \text{From points in the space} &\longrightarrow \text{To real numbers} \\ (x,y,z) &\longrightarrow f(x,y,z) \end{aligned}$$

Example 5 The function

$$f(x,y,z) = xyz$$

is a scalar field that assigns to the point in the space (x,y,z) the real number xyz but the function

$$F(x,y,z) = [\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}]$$

is a vector field that assigns to the point in the space (x,y,z) the unit position vector $[\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}]$.

Definition 1 *The* gradient *of a given scalar field $f(x,y,z)$ is a vector field denoted by* grad f *or* $\vec{\nabla} f$ *and it is defined as follows:*

$$\vec{\nabla} f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

This notation means that if we have a point in the space (x_0, y_0, z_0) then

$$\vec{\nabla} f(x_0,y_0,z_0) = \frac{\partial f}{\partial x}|_{(x_0,y_0,z_0)}\vec{i} + \frac{\partial f}{\partial y}|_{(x_0,y_0,z_0)}\vec{j} + \frac{\partial f}{\partial z}|_{(x_0,y_0,z_0)}\vec{k}$$

We can think $\vec{\nabla}$ as a linear operator that acts on scalar fields giving vectors fields:

$$\begin{aligned} \vec{\nabla} : \text{Smooth Scalar Fields} &\longrightarrow \text{Vector fields} \\ f &\longrightarrow \vec{\nabla} f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] \end{aligned}$$

From Vector Calculus we know that the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ give the rates of change of $f(x,y,z)$ in the directions of $\vec{i}, \vec{j}, \vec{k}$ respectively. It seems natural to ask what is the rate of change in any other direction. This is what the directional derivatives represent.

Suppose that $\vec{b} = [b_1, b_2, b_3]$ is a *unit vector*, thus it has length 1:

$$|\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2} = 1$$

Then the *directional derivative of $f(x,y,z)$ in the direction of* a unit vector $\vec{b}$ is:

$$D_{\vec{b}} f = \vec{b} \cdot \vec{\nabla} f, \tag{1}$$

where $\cdot$ is the scalar product.

If $\vec{a} = (a_1, a_2, a_3)$ is a vector which is not unit then a unit vector with the same direction than $\vec{a}$ is $\frac{\vec{a}}{|\vec{a}|}$:

$$\left|\frac{\vec{a}}{|\vec{a}|}\right| = \frac{1}{|\vec{a}|}\sqrt{a_1^2 + a_2^2 + a_3^2} = \frac{1}{|\vec{a}|}|\vec{a}| = 1.$$

We define the directional derivative of $f(x,y,z)$ in the direction of $\vec{a}$ as

$$D_{\vec{a}} f = \frac{\vec{a}}{|\vec{a}|} \cdot \vec{\nabla} f. \tag{2}$$

Please note that equation (**??**) is equivalent to equation (**??**) when $\vec{a}$ is a unit vector. (The two definitions of directional derivative are the same.)

Example 6 Given $f(x,y,z) = 2x^2 + 3y^2 + z^2$ and $\vec{a} = [1, 0, -2]$, evaluate the directional derivative of f in the direction of $\vec{a}$ at the point $(2, 1, 3)$.

First we compute the gradient of f:

$$\vec{\nabla} f = [4x, 6y, 2z]$$

Since $\vec{a}$ is not a unit vector the directional derivative is:

$$D_{\vec{a}} f = \frac{[1, 0, -2]}{\sqrt{1^2 + 0^2 + (-2)^2}} \cdot \begin{bmatrix} 4x \\ 6y \\ 2z \end{bmatrix} = \frac{[1, 0, -2]}{\sqrt{5}} \begin{bmatrix} 4x \\ 6y \\ 2z \end{bmatrix} = \frac{1}{\sqrt{5}}(4x - 4z)$$

And evaluating at the point $(2, 1, 3)$ we get:

$$D_{\vec{a}} f(2, 1, 3) = \frac{8 - 12}{\sqrt{5}} = \frac{-4}{\sqrt{5}},$$

the minus sign indicates that the function f decreases in the direction of $\vec{a}$.

Remark: If the gradient of f at a point P is not zero $\vec{\nabla} f(P) = \text{grad } f(P) \neq 0$, then it is a vector in the direction of maximum increase of f at P.

Definition 2 *A surface is all points in $(x, y, z) \in \mathbb{R}^3$ that verifies $f(x, y, z) = c$, for some smooth scalar field f and constant c.*

Consider the unit sphere in $\mathbb{R}^3$, S^2:

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

In this case $f(x, y, z) = x^2 + y^2 + z^2$ and $c = 1$.

The cone with center at $(0, 0, 0)$ is another surface with $f(x, y, z) = x^2 + y^2 - z^2$ and $c = 0$:

$$C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 0\}.$$

Theorem 1 *Let f be differentiable and $S = \{(x, y, z) : f(x, y, z) = c\}$ be a surface. Then the gradient of f, $\vec{\nabla} f$ at a point P of the surface S is a normal vector to S at P. (Provided $\vec{\nabla} f(P) \neq 0$).*

Example 7 Find a unit normal vector of the cone of the revolution $z^2 = 4(x^2 + y^2)$ at the point $P = (1, 0, 2)$.

In this case:

$$f(x, y, z) = 4(x^2 + y^2) - z^2$$

The theorem ensures that the gradient of f at the point $(1, 0, 2)$ is normal to the cone:

$$\vec{\nabla} f = [8x, 8y, -2z],$$

and evaluating it at P

$$\vec{\nabla} f(1, 0, 2) = [8, 0, -4].$$

But we observe that $\vec{\nabla} f(1,0,2)$ is not unit vector, so we calculate the unit vector in the direction of $\vec{\nabla} f(1,0,2)$:

$$\frac{\vec{\nabla} f(1,0,2)}{|\vec{\nabla} f(1,0,2)|} = \frac{[8,0,-4]}{\sqrt{8^2+(-2)^2}} = \frac{[8,0,-4]}{\sqrt{80}} = \frac{[2,0,-1]}{\sqrt{5}}$$

And more examples (FOR YOU TO TRY):

1. Find the directional derivative of f at P in the direction of $\vec{a}$ in the following cases:

   - $f(x,y,z) = x^2 + y^2 - z, P = (1,1,-2), \vec{a} = [1,1,2]$
   - $f(x,y,z) = x^2 + y^2 + z^2, P = (2,2,-1), \vec{a} = [-1,-1,0]$
   - $f(x,y,z) = xyz, P = (-1,1,3), \vec{a} = [1,-2,2]$

2. Compute the normal to the surface:

   - $S = \{(x,y,z) \in \mathbb{R}^3 : ax + by + cz = d\}$ for any P.
   - $S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + 3y^2 + z^2 = 28\}$ and $P = (4,1,3)$
   - $S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 25\}$ and $P = (4,3,8)$

Proposition 1 *For all f, g smooth scalar fields, the $\vec{\nabla}$ operator verifies:*

1. $\vec{\nabla}(fg) = g\vec{\nabla} f + f\vec{\nabla} g$,
2. $\vec{\nabla} f^n = nf^{n-1}\vec{\nabla} f$, *for any* $n \in \mathbb{N}$,
3. $\vec{\nabla}(\frac{f}{g}) = \frac{1}{g^2}(g\vec{\nabla} f - f\vec{\nabla} g)$,
4. $\vec{\nabla}^2(fg) = g\vec{\nabla}^2 f + 2\vec{\nabla} f\vec{\nabla} g + f\vec{\nabla}^2 g$.

Proof:

1. By definition of the gradient:

$$\begin{aligned}\vec{\nabla}(fg) &= [\frac{\partial(fg)}{\partial x}, \frac{\partial(fg)}{\partial y}, \frac{\partial(fg)}{\partial z}] \\ &= [\frac{\partial f}{\partial x}g + f\frac{\partial g}{\partial x}, \frac{\partial f}{\partial y}g + f\frac{\partial g}{\partial y}, \frac{\partial f}{\partial z}g + f\frac{\partial g}{\partial z}] \\ &\quad \text{using the product rule for partial derivatives} \\ &= [\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}]g + f[\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}] \\ &\quad \text{using linearity of vectors} \\ &= g\vec{\nabla}f + f\vec{\nabla}g,\end{aligned}$$

2. It is a consequence of the chain rule for partial derivatives:

$$\begin{aligned}\vec{\nabla}f^n &= [\frac{\partial(f^n)}{\partial x}, \frac{\partial(f^n)}{\partial y}, \frac{\partial(f^n)}{\partial z}] \\ &= [nf^{n-1}\frac{\partial f}{\partial x}, nf^{n-1}\frac{\partial f}{\partial y}, nf^{n-1}\frac{\partial f}{\partial z}] = nf^{n-1}\vec{\nabla}f, \text{ for any } n \in \mathbb{N},\end{aligned}$$

3. For $\vec{\nabla}(\frac{f}{g}) = \frac{1}{g^2}(g\vec{\nabla}f - f\vec{\nabla}g)$, use part 1 with f and $\frac{1}{g}$ and recall that $\frac{\partial}{\partial x}\frac{1}{g} = \frac{-1}{g^2}\frac{\partial g}{\partial x}$. (Similarly with the partial derivatives with respect to to y and z).

4. Notice that

$$\vec{\nabla}^2(fg) = \vec{\nabla} \cdot \vec{\nabla}(fg) = \frac{\partial^2(fg)}{\partial x^2} + \frac{\partial^2(fg)}{\partial y^2} + \frac{\partial^2(fg)}{\partial z^2}$$

Furthermore:

$$\frac{\partial^2(fg)}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial(fg)}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}g + f\frac{\partial g}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}g + 2\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + f\frac{\partial g}{\partial x}$$

Similarly:

$$\frac{\partial^2(fg)}{\partial y^2} = \frac{\partial^2 f}{\partial y^2}g + 2\frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + f\frac{\partial g}{\partial y}$$

and

$$\frac{\partial^2(fg)}{\partial z^2} = \frac{\partial^2 f}{\partial z^2}g + 2\frac{\partial f}{\partial z}\frac{\partial g}{\partial z} + f\frac{\partial g}{\partial z}$$

The result follows on adding these three equations together and observing that

$$\vec{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

and

$$\vec{f} \cdot \vec{g} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z}$$

(Important) Remark: Some vector fields are gradients of scalar fields. This means that for a vector field $\vec{F}$ we can find a scalar field f called *potential of* $\vec{F}$ such that

$$\vec{F} = \vec{\nabla} f = [\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}].$$

In this case we say that $\vec{F}$ is a *gradient field.*

In the exercises you will be given a vector field and you will be asked to find a potential for it:

Example 8 Is $\vec{F}(x, y, z) = [yz, xz, xy]$ a gradient field?

If we find a scalar field $f(x, y, z)$ such that:

$$F_1(x, y, z) = yz = \frac{\partial f}{\partial x} \tag{3}$$

$$F_2(x, y, z) = xz = \frac{\partial f}{\partial y} \tag{4}$$

and

$$F_3(x, y, z) = xy = \frac{\partial f}{\partial z} \tag{5}$$

then $\vec{F}$ is a gradient field.

If we consider y and z as constants and we integrate with respect to to x equation (**??**), we obtain:

$$\int yz dx = \int \frac{\partial f}{\partial x} dx \tag{6}$$

$$xyz + C(y, z) = f(x, y, z) \tag{7}$$

This means that if the potential function for $\vec{F}$ exists then it has the shape of $xyz + C(y, z)$ where $C(y, z)$ is the constant of integration that we have to determine using equations (**??**) and (**??**). (The constant of integration must depend on y and z because to integrate with respect to x (**??**) we thought y and z as constants).

We use now equation (**??**) to equate with the differential of equation (**??**) with respect to y :

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(xyz + C(y, z)) = xz$$

or

$$xz + \frac{\partial}{\partial y}C(y, z) = xz,$$

and we conclude that for this equality to hold we need

$$\frac{\partial}{\partial y}C(y, z) = 0$$

That is, $C(y, z)$ does not depend on y or $C(y, z)$ could be written as $C(z)$. Now we use equation (**??**) and we equate to the derivative with respect to z of equation (**??**):

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial y}(xyz + C(z)) = xy$$

or

$$xy + \frac{\partial}{\partial z}C(z) = xy,$$

which implies that the constant $C(z)$ does not depend on z so it is just a constant, C say. We conclude that the potential for $\vec{F}$ is

$$f(x, y, z) = xyz + C,$$

C any constant in $\mathbb{R}$.

You can double check that $f(x, y, z) = xyz + C$ verifies that $\vec{F} = \vec{\nabla} f$.

Example 9 Find a potential for $\vec{F} = \left[\frac{x}{x^2+y^2+z^2}, \frac{y}{x^2+y^2+z^2}, \frac{z}{x^2+y^2+z^2}\right]$
We need to find a scalar field $f(x, y, z)$ such that:

$$F_1(x,y,z) = \frac{x}{x^2+y^2+z^2} = \frac{\partial f}{\partial x} \tag{8}$$

$$F_2(x,y,z) = \frac{y}{x^2+y^2+z^2} = \frac{\partial f}{\partial y} \tag{9}$$

and

$$F_3(x,y,z) = \frac{z}{x^2+y^2+z^2} = \frac{\partial f}{\partial z} \tag{10}$$

Integrating with respect to x equation (**??**) we get that if the potential exists must it be as follows:

$$f(x,y,z) = \frac{1}{2}\ln(x^2+y^2+z^2) + C(y,z). \tag{11}$$

Differentiating equation (**??**) with respect to y and equating to equation (**??**):

$$\frac{y}{x^2+y^2+z^2} + \frac{\partial}{\partial y}C(y,z) = \frac{y}{x^2+y^2+z^2}$$

which implies that the constant $C(y, z)$ does not depend on y. Similarly differentiating equation (**??**) with respect to z and equating it to equation (**??**) we conclude that the constant does not depend on z either and therefore the potential is:

$$f(x,y,z) = \frac{1}{2}\ln(x^2+y^2+z^2) + C, \quad C \in \mathbb{R}.$$

Give a try to find a potential for:

1. $[3x, 5y, -4z]$

2. $[4x^3, 3y^2, -6z]$

(The solutions are $\frac{3}{2}x^2 + \frac{5}{2}y^2 - 2z^2 + C$ and $x^4 + y^3 - 3z^2 + C$ respectively).

The differential operator

$$\vec{\nabla}^2 f = \triangle f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

is called *Laplace Operator*.

Notice

$$\begin{aligned} \triangle : \text{Smooth Scalar Fields} &\longrightarrow \text{Scalar Fields} \\ f &\longrightarrow \triangle f \end{aligned}$$

Example 10 If $f(x,y,z) = x^3 + y^2 + 3xz^5$ then:

$$\frac{\partial f}{\partial x} = 3x^2 + 3z^5 \Rightarrow \frac{\partial^2 f}{\partial x^2} = 6x$$

$$\frac{\partial f}{\partial y} = 2y \Rightarrow \frac{\partial^2 f}{\partial y^2} = 2$$

$$\frac{\partial f}{\partial z} = 15xz^4 \Rightarrow \frac{\partial^2 f}{\partial z^2} = 60xz^3$$

Hence:

$$\triangle f = 6x + 2 + 60xz^3,$$

for example:

$$\triangle f(1,1,1) = 6 + 2 + 60 = 68.$$

3 Divergence of a Vector Field

The *divergence of a vector field* $\vec{F} = [F_1, F_2, F_3]$ is a scalar field div $\vec{F}$ defined as follows:

$$\text{div } \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

We can understand the divergence as a differential operator acting on smooth vector fields that produces scalar fields:

$$\begin{aligned} \text{div: Smooth Vector Fields} &\longrightarrow \text{Scalar Vector Fields} \\ \vec{F} &\longrightarrow \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \end{aligned}$$

Sometimes div $\vec{F}$ is denoted by

$$\vec{\nabla} \cdot \vec{F} = \text{div } \vec{F},$$

the dot is the scalar product dot because $\vec{\nabla}$ can be considered as the 'vector'

$$\vec{\nabla} = [\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}],$$

so we have:

$$\vec{\nabla} \cdot \vec{F} = [\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}] \cdot [F_1, F_2, F_3] = \frac{\partial}{\partial x}F_1 + \frac{\partial}{\partial y}F_2 + \frac{\partial}{\partial z}F_3,$$

and the 'products' $\frac{\partial}{\partial x}F_1$, $\frac{\partial}{\partial y}F_2$ and $\frac{\partial}{\partial z}F_3$ mean that we have to differentiate F_1 with respect to x, F_2 with respect to y and F_3 with respect to z:

$$\begin{aligned} \frac{\partial}{\partial x}F_1 &= \frac{\partial F_1}{\partial x} \\ \frac{\partial}{\partial y}F_2 &= \frac{\partial F_2}{\partial y} \\ \frac{\partial}{\partial z}F_3 &= \frac{\partial F_3}{\partial z} \end{aligned}$$

Remark: If f is a scalar field then we have:

$$\text{div } (\text{grad} f) = \triangle f$$

or equivalently

$$\vec{\nabla} \cdot \vec{\nabla} f = \triangle f$$

Example 11 Compute the divergence of the following vector field:

$$\vec{F}(x, y, z) = [x^3 + y^3, 3x^2, 3zy^2].$$

By definition:

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial(x^3 + y^3)}{\partial x} + \frac{\partial(3x^2)}{\partial y} + \frac{\partial(3zy^2)}{\partial z} = 3x^2 + 3y^2$$

4 The Curl of a Vector Field

The Curl of a vector field $\vec{F}$ is another vector defined as the following determinant:

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}$$

Suppose that $\vec{F}$ represents the velocity of a rotating body then curl $\vec{F}$ has the direction of the axis of rotation and its module is twice the angular speed of the rotation.

Definition 3 *If the curl of a vector field is zero we say that the vector field is* irrotational*.*

Remark: The definition of *irrotational* vector field $\vec{F} = [F_1, F_2, F_3]$ is equivalent to have the following three conditions in the partial derivatives of F_1, F_2 and F_3:

$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

$$\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$$

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}$$

These three equations can be a short way to prove that something is not irrotational without computing the whole determinant:

Example 12 Consider $\vec{F} = [-\omega y, \omega x, 0]$, with $\omega > 0$ since $\dfrac{\partial F_2}{\partial x} = \omega$ and $\dfrac{\partial F_1}{\partial y} = -\omega$ then $\vec{F}$ is not irrotational.

However if we compute the determinant we obtain curl $\vec{F} = [0, 0, 2\omega]$ which means that the body rotates around the z axis with angular speed ω.

Lemma 1 *Gradient fields are irrotationals. That is, if $\vec{F} = \vec{\nabla} f$ for some smooth scalar field f then* curl $\vec{F} = 0$.

Proof: Using the denition of curl:

$$\text{curl } \vec{\nabla} f = \text{curl } \left([\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}]\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial y}\right)\vec{i} + \left(\frac{\partial}{\partial z}\frac{\partial f}{\partial x} - \frac{\partial}{\partial x}\frac{\partial f}{\partial z}\right)\vec{j} + \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}\frac{\partial f}{\partial x}\right)\vec{k}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\vec{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\vec{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\vec{k},$$

The result follows since the mixed partial derivatives of a smooth scalar field are equal.

□

Example 13 The gravitational field $\vec{p}$ is the force of attraction between two particles at points $P_0 = (x_0, y_0, z_0)$ and $P = (x, y, z)$. It is defined by

$$\vec{p} = -\frac{c}{r^3}\vec{r} = -\frac{c}{r^3}[x - x_0, y - y_0, z - z_0]$$

where $r = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$ and it is an irrotational field because the scalar field $f(x, y, z) = \frac{c}{r}$ is a potential for it. (Check it!)

Lemma 2 *Let $\vec{F}$ be a smooth vector field then*

$$\text{div (curl } \vec{F}) = 0$$

Proof:

$$\text{div (curl } \vec{F}) = \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y} = 0$$

Because if $\vec{F}$ is smooth then F_1, F_2, F_3 are smooth (scalar fields) and the mixed second partial derivatives of F_1, F_2, F_3 are equal.

$\square$

Example 14 Compute the curl of the following vector field $\vec{F}(x, y, z) = [e^x \cos y, e^x \sin y, 0]$

$$\begin{aligned}
curl\ \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x \cos y & e^x \sin y & 0 \end{vmatrix} \\
&= \frac{\partial(e^x \sin y)}{\partial x}\vec{k} + \frac{\partial(e^x \cos y)}{\partial z}\vec{j} - \frac{(\partial e^x \cos y)}{\partial y}\vec{k} - \frac{(\partial e^x \sin y)}{\partial z}\vec{i} \\
&= (e^x \sin y - e^x(-\sin y))\vec{k} = [0, 0, 2e^x \sin y]
\end{aligned}$$