Chapter 5
Traffic Flow

5.1 Introduction

In this chapter we again investigate the movement of objects along a one-dimensional path, but now the motion is directed rather than random. Examples of such situations include:

- Cars moving along a highway (Figure 5.1)
- Blood cells moving along a capillary (Figure 5.2)
- Molecules moving along a carbon nanotube (Figure 5.3)

Although the underlying physics of each of these is quite different they all involve the movement of objects along what is effectively a one-dimensional pathway. We will take advantage of this when developing a mathematical model for the motion, but before doing so we must first decide on how to account for the spatial and temporal variables. For example, for random walks we used discrete steps in space and time. This is also done for traffic models and it is the basis of the cellular automata description presented in Section

Figure 5.1 Aerial view of traffic flow (Google Maps [2007]).

M.H. Holmes, *Introduction to the Foundations of Applied Mathematics*, Texts in Applied Mathematics 56, DOI 10.1007/978-0-387-87765-5_5,

Figure 5.2 Red blood cells flowing in an arteriol (Baskurt [2009]).

5.7. We will start out, however, assuming that the motion is continuous, which is the viewpoint taken when deriving the diffusion equation in Section 4.5.

5.2 Continuum Variables

We are assuming that the objects are numerous enough that it is not necessary to keep track of each one individually, and we can use an averaged value. In deriving the mathematical model, the objects here will be identified as cars and the path as a highway. There are a couple of reasons for using this particular example. One is that most everyone has experience with traffic, and is able to relate the mathematical results with the real-world application. The other reason is that the theory for traffic flow is still not complete, so there are competing ideas that can be explored. However, it should be remembered that all of this material can be applied to other systems, such as the one dimensional motion of blood cells and molecules. In fact, some of the termi-

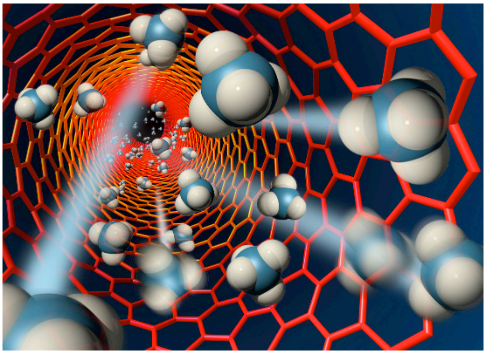

Figure 5.3 Methane molecules flowing through a carbon tubule less than 2 nanometers in diameter (Lawrence Livermore National Laboratory [2009]).

nology that is introduced comes from gas dynamics, because of its early use of the ideas developed here.

5.2.1 Density

The variable that will play a prominent role in our study is the traffic density $\rho(x,t)$. This is the number of cars per unit length, and it is instructive to consider how it might be determined experimentally. To measure ρ at $x = x_0$, for $t = t_0$, one selects a small spatial interval $x_0 - \Delta x < x < x_0 + \Delta x$ on the highway, and then counts the number of cars within this interval (see Figure 5.4). In this case

$$\rho(x_0, t_0) \approx \frac{\text{number of cars from } x_0 - \Delta x \text{ to } x_0 + \Delta x \text{ at } t = t_0}{2\Delta x}. \tag{5.1}$$

The underlying assumption here is that Δx is small enough that only cars in the immediate vicinity of x_0 are used to determine the density at this point. At the same time, Δx cannot be so small that it is on the order of the length of individual cars (and the spacing between them). In the continuum viewpoint, the cars are distributed smoothly over the entire x-axis, and the value of $\rho(x_0, t_0)$ is the limit of the right-hand side of (5.1) as $\Delta x \to 0$.

Example: Uniform Distribution

To illustrate how density is determined suppose the cars all have length ℓ, and they are evenly spaced a distance d apart (see Figure 5.5). Given a sampling interval $2\Delta x$ along the highway then the number of cars in this interval is, approximately, $2\Delta x/(\ell + d)$. Inserting this into (5.1) and letting $\Delta x \to 0$ we find that

$$\rho = \frac{1}{\ell + d}\,. \tag{5.2}$$

One conclusion that comes from this formula is that there is a maximum density. Because $0 \le d < \infty$ then $0 < \rho \le \rho_M$, where $\rho_M = 1/\ell$. For example, if $\ell = 17$ ft (5.2 m) and $d = 12$ ft (3.6 m) then, recalling 1 mi = 5280 ft, $\rho = 182\,\frac{\text{cars}}{\text{mi}}$ $(113\,\frac{\text{cars}}{\text{km}})$. With these dimensions then the maximum density

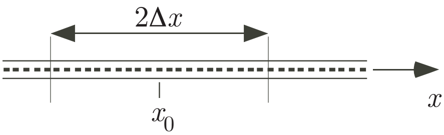

Figure 5.4 The interval along the highway used to calculate an approximate value of the density $\rho(x_0, t_0)$. It is also used to derive the balance law for traffic flow.

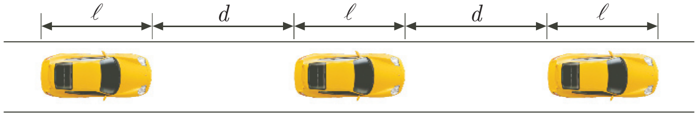

Figure 5.5 For a uniform distribution, the cars are all the same length and are evenly spaced along the highway.

that is possible, which occurs when $d = 0$, is $\rho_M = 310.6\,\frac{\text{cars}}{\text{mi}}$ $(193\,\frac{\text{cars}}{\text{km}})$. When studying traffic flow, it is useful to know the maximum merge density ρ_{mg}, which corresponds to the density that occurs when the spacing is such that exactly one car fits between two cars currently on the highway. This occurs when $d = \ell$ and for this example $\rho_{mg} = 155.3\,\frac{\text{cars}}{\text{mi}}$ $(96.5\,\frac{\text{cars}}{\text{km}})$. ■

5.2.2 Flux

The second variable we need is the flux $J(x,t)$, which has the dimensions of cars per unit time. To measure J at $x = x_0$, for $t = t_0$, one selects a small time interval $t_0 - \Delta t < t < t_0 + \Delta t$ and counts the net number of cars that pass $x = x_0$ during this time period. The convention is that a car moving to the right is counted as $+1$, while one moving to the left is counted as -1. In this case

$$J(x_0, t_0) \approx \frac{\text{net number of cars that pass } x_0 \text{ from } t = t_0 - \Delta t \text{ to } t = t_0 + \Delta t}{2\Delta t}. \tag{5.3}$$

The underlying assumption here is that Δt is small enough that only cars that are passing x_0 at, or near, $t = t_0$ are used to determine the flux at t_0. At the same time, from an experimental standpoint, Δt can not be so small that no cars are able to pass this location during this time interval. In the continuum viewpoint we are taking the cars are distributed smoothly over the entire t-axis and the value of $J(x_0, t_0)$ is the limit of the right hand side of (5.3) as $\Delta t \to 0$.

Example: Uniform Distribution (cont'd)

Returning to the previous example of uniformly distributed cars, shown in Figure 5.5, we now add in the assumption that the cars are moving with a constant positive velocity v. In this case, the cars that start out a distance $2\Delta t v$ from x_0 will pass x_0 in the time interval from $t_0 - \Delta t$ to $t_0 + \Delta t$. The corresponding number of cars is, approximately, $2v\Delta t/(\ell + d)$. Inserting this into (5.3), and letting $\Delta t \to 0$, yields

$$J = \frac{v}{\ell + d}\,. \tag{5.4}$$

For example, if $\ell = 17$ ft, $d = 51$ ft and $v = 70$ mph then $J = 5,435\,\frac{\text{cars}}{\text{hr}}$. Also, note that $J = \rho v$, which is one of the fundamental formulas in traffic flow. ■

5.3 Balance Law

To derive an equation for the density we will use what is known as a control volume argument. For this problem the control volume is a small region on the highway, from $x_0 - \Delta x$ to $x_0 + \Delta x$. This interval is shown in Figure 5.1. During the time period from $t = t_0 - \Delta t$ to $t = t_0 + \Delta$ t it is assumed that the number of cars in this interval can change only due to cars entering or leaving at the left or right ends of the interval. We are therefore assuming cars do not disappear, or pop into existence, on the highway. Actually, this could happen if we were to include an off- or onramp, but this modification will be postponed for the moment (see Exercise 5.21). As stated, our balance law for cars within the highway interval is

$$\begin{aligned}
&\{\text{number of cars in interval at } t = t_0 + \Delta t\} \\
&\qquad - \{\text{number of cars in interval at } t = t_0 - \Delta t\} \\
&\quad = \{\text{net number of cars that cross } x_0 - \Delta x \text{ from } t_0 - \Delta t \text{ to } t_0 + \Delta t\} \\
&\qquad - \{\text{net number of cars that cross } x_0 + \Delta x \text{ from } t_0 - \Delta t \text{ to } t_0 + \Delta t\}.
\end{aligned}$$

Rewriting this using (5.1) and (5.3) yields

$$\begin{aligned}
2\Delta x\,[\rho(x_0, t_0 + \Delta t) - \rho(x_0, t_0 - \Delta t)]& \\
= 2\Delta t\,[J(x_0 - \Delta x, t_0) - J(x_0 + \Delta x, t_0)]&\,.
\end{aligned}$$

Using Taylor's theorem, we have that

$$\begin{aligned}
2\Delta x \Big(&\rho + \Delta t \rho_t + \frac{1}{2}(\Delta t)^2 \rho_{tt} + \frac{1}{6}(\Delta t)^3 \rho_{ttt} + \cdots \\
&-\rho + \Delta t \rho_t - \frac{1}{2}(\Delta t)^2 \rho_{tt} + \frac{1}{6}(\Delta t)^3 \rho_{ttt} + \cdots \Big) \\
= 2\Delta t \Big(&J - \Delta x J_x + \frac{1}{2}(\Delta x)^2 J_{xx} - \frac{1}{6}(\Delta x)^3 J_{xxx} + \cdots \\
&-J - \Delta x J_x - \frac{1}{2}(\Delta x)^2 J_{xx} - \frac{1}{6}(\Delta x)^3 J_{xxx} + \cdots \Big),
\end{aligned}$$

where ρ and J are evaluated at (x_0, t_0). Collecting the terms in the above equation,

$$\rho_t + O\left((\Delta t)^2\right) = -J_x + O\left((\Delta x)^2\right).$$

Letting $\Delta x \to 0$ and $\Delta t \to 0$ we conclude that

$$\frac{\partial \rho}{\partial t} = -\frac{\partial J}{\partial x}\,. \tag{5.5}$$

This is our balance law for motion along the x-axis. It is applicable to any continuous system in which the objects are not created or destroyed. This is why it was also obtained when deriving the model for diffusion (4.48).

5.3.1 Velocity Formulation

It is possible to express the balance law somewhat differently, by introducing the velocity $v(x,t)$ of the cars on the highway. This requires care because the velocity, like the other continuum variables, is an averaged quantity. To explain how this is done, consider a small interval on the highway as shown in Figure 5.4. One measures $v(x_0,t_0)$ experimentally by finding the average velocity of the cars in this interval. Specifically, if there are n cars in the interval, and they have velocities v_1, v_2, ... , v_n, then

$$v(x_0,t_0) \approx \frac{1}{n}\sum_{i=1}^{n} v_i.$$

In the continuum model it is assumed that the limit of this average, when letting $\Delta x \to 0$, exists, and its value is the velocity $v(x_0,t_0)$.

With the above definition, the velocity is assumed to be related to the flux through the equation

$$J = \rho v. \tag{5.6}$$

This equation was derived in the uniform distribution example discussed earlier. It is also possible to derive it for situations where the velocity is not constant (see Exercise 5.26). However, a proof for the general case is not available, and so the above formula is an assumption. Some avoid this difficulty by using (5.6) as the definition of the flux, while others use it as the definition of the velocity.

Introducing (5.6) into (5.5) gives us

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(v\rho) = 0. \tag{5.7}$$

In solving this equation it will be assumed the initial density is known, that is,

$$\rho(x,0) = f(x). \tag{5.8}$$

The equation in (5.7) is the mathematical model for traffic flow that we will investigate in the first part of this chapter. Those working in traffic flow refer to this as the Lighthill-Whitham-Richards (LWR) model, naming it after those who originally derived the equation (Lighthill and Whitham [1955], Richards [1956]). However, the equation has wide applicability, and appears under different banners. For example, in continuum mechanics it is known as the continuity equation, while in electrodynamics (5.7) is the current continuity equation, where ρ is the current density and J is the current volume. Those interested in more mathematical pursuits refer to (5.7) as a scalar conservation law.

It should be kept in mind that, as with most mathematical models, (5.7) is an approximation of the true system. Not unexpectedly, there are limitations on its applicability. As a case in point, it is questionable whether the model provides an accurate description at low densities. If the objects are few and far between then the assumptions made in defining the density and flux are not valid. This will not stop us from using the model in such rarified regimes, but when this is done it should be understood that the continuum model provides more of a qualitative description of the motion. That said, in the regimes where it does apply, the continuum model has proven to be an exceptionally accurate, and mathematical interesting, description.

5.4 Constitutive Laws

Although we have derived the balance law for traffic flow, the mathematical model is incomplete. The issue is the velocity v and how it is related to the density ρ. One possibility is to investigate the physics of the problem a bit more and see if there is another equation relating these variables. This is done in mechanics, and Newton's second law is used to derive a force balance equation that can be used to find the velocity. This option is not easily adaptable to the traffic flow situation so we will take a different approach and postulate how v and ρ are related based on experimental evidence. What we will be doing is specifying a constitutive law relating the velocity and density. To do this the data for several rather different roadways are shown in Figure 5.6. The question is, what function best describes the data in this figure? The answer depends, in part, on what density and velocity intervals are of interest and what applications one has in mind. A few possible constitutive laws are discussed below.

It is worth making a couple of comments about Figure 5.6 that are unrelated to constitutive modeling. The data in the lower two graphs is what was used in the original development of the continuum traffic model, while the data in the upper two graphs is typical of more modern testing. One of the striking differences between the upper and lower graphs is the amount of data shown. This is due to the development of computerized testing systems,

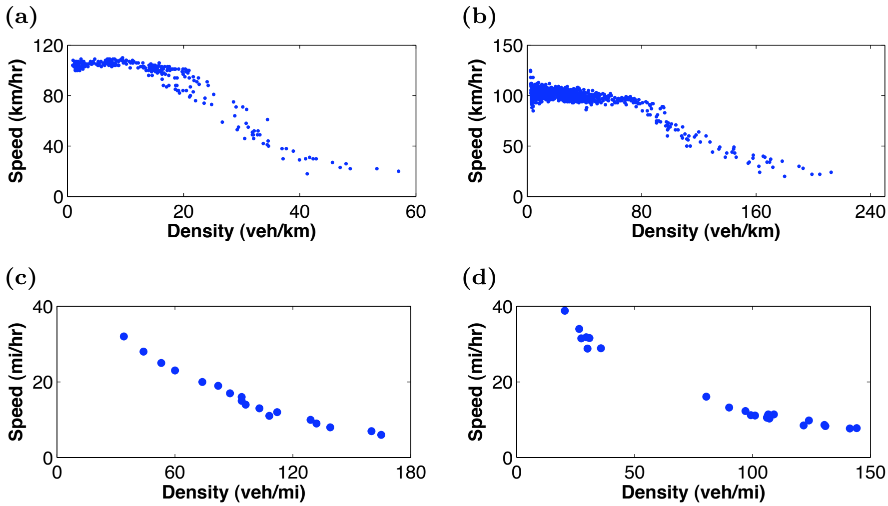

Figure 5.6 The velocity as a function of the density as measured for different roadways. Shown is (a) a highway near Toronto, (b) a freeway near Amsterdam, (c) the Lincoln Tunnel, and (d) the Merritt Parkway. Data for (a) and (b) are from Aerde and Rakha [1995], and (c) and (d) are from Greenberg [1959].

which have been invaluable for modern scientific research. However, what is interesting is the rather tight pattern in the earlier data as compared to the scatter in the more recent results. This begs the question of whether these earlier experimentalists were more careful, or did they force the results. It makes one wonder. A second observation concerns the difference in the densities between Toronto and Amsterdam, which differ by almost a factor of four. Any theory why this happens?

5.4.1 Constant Velocity

The simplest assumption is that v is constant in terms of its dependence on ρ, in other words, $v = a$. In this case the balance law (5.7) reduces to

$$\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} = 0. \tag{5.9}$$

This is known as the advection equation. In looking at the data in Figure 5.6 one might conclude that assuming v is constant borders on delusional. The value of this assumption is not its realistic portrayal of traffic but, rather, what it provides in terms of insights into the type of mathematical problem that arises in traffic flow. The analysis of this problem will provide the foundation needed for solving the more difficult nonlinear problems arising from more realistic velocity functions.

5.4.2 Linear Velocity

The most widely used, and most well known, constitutive laws are linear. For the traffic problem this means we assume $v = a - b\rho$, where a, b are constants. Those working in traffic flow refer to this as the Greenshields model, and the usual way this is written is

$$v = v_M \left(1 - \frac{\rho}{\rho_M}\right), \tag{5.10}$$

where the constants v_M, ρ_M are the maximum velocity and density, respectively. The values of these constants can almost be read off the plot in Figure 5.6. However, a more systematic way to find them is to use a least squares fit. Using the data for the Lincoln Tunnel and Merritt Parkway one finds that $v_M = 36.821$ mph, $\rho_M = 166.4226$ cars/mi and the resulting function is plotted in Figure 5.7 along with the original data. It is seen that even though this function misses the values at the extreme ends, where $\rho = 0$ or $\rho = 180$, it does show the correct monotonic dependence of the velocity on density. This would seem an acceptable approximation, and the traffic flow equation (5.7) reduces to

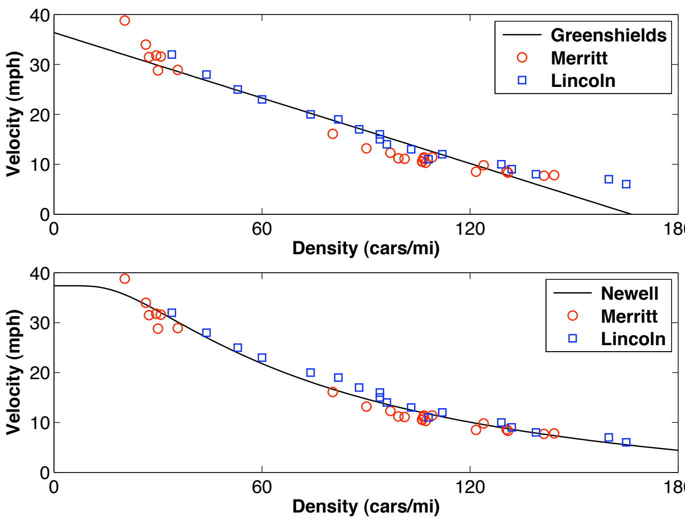

Figure 5.7 Curve fit of the Greenshields law (5.10) and the Newell law (5.17) to traffic data for the Merritt Parkway and the Lincoln Tunnel.

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0, \tag{5.11}$$

where

$$c = v_M \left(1 - \frac{2\rho}{\rho_M}\right). \tag{5.12}$$

This is a nonlinear conservation equation for ρ. It can be solved analytically, but it is certainly more challenging than the linear equation in (5.9). We will return to this problem once we have worked out the constant velocity case later in this chapter.

5.4.3 General Velocity Formulation

It is clear from the data in Figure 5.6 that the relationship between the velocity and density is not linear. In certain applications these differences are considered significant, and a more accurate function is needed. The general version of the constitutive law in this case has the form

$$v = F(\rho). \tag{5.13}$$

With this, the general formula for the flux is $J = \rho F(\rho)$. Assuming that F is a smooth function of ρ then, using the chain rule, it follows that $\frac{\partial}{\partial x}J = J'(\rho)\frac{\partial}{\partial x}\rho$. The general form of the balance law (5.5) now takes the form

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0, \tag{5.14}$$

where

$$c(\rho) = J'(\rho), \tag{5.15}$$

or equivalently

$$c(\rho) = F(\rho) + \rho F'(\rho). \tag{5.16}$$

The function $c(\rho)$ is known as the wave velocity, and it will play a critical role in the solution of the equation. A particular example of this function is given in (5.12), which is the wave velocity associated with the Greenshields constitutive law in (5.10).

It is not possible to use just any function for the constitutive law in (5.13). In particular, there are requirements that are needed to guarantee that (5.14) has a solution. These will become evident once we attempt to solve the problem. For the moment, we will take a more physical viewpoint, and impose conditions on the function $F(\rho)$ that are based on what is known about traffic flow. Interestingly, we will find that these physically based assumptions will overlap with the mathematical requirements needed to guarantee that the problem has a solution.

It has already been assumed that F is a smooth function of ρ. In addition to this, based on the data in Figure 5.6, the following assumptions are made.

NV1. $F'(\rho) \leq 0$ for $0 \leq \rho \leq \rho_M$.
This assumption comes from Figure 5.6 which shows v is a monotonically decreasing function of density. This requirement is consistent with the observation that (most) drivers leave a larger bumper-to-bumper spacing between cars as the speed increases. A consequence of this assumption is that $F(0) = v_M$ is the maximum velocity. This corresponds to the observation that on an uncongested highway, drivers tend to travel at the maximum allowable speed.

NV2. $F(\rho_M) = 0$.
This is based on the assumption that the closer the traffic gets to being bumper-to-bumper the closer the velocity gets to zero.

The list of functions that are capable of satisfying these rather general requirements is endless. It is for this reason that in selecting a particular function one should also consider simplicity. Given the uncertainty in the experimental data, and the approximate nature of the model, it is a waste of time to construct a function that hits every data point exactly. The problem is that the condition of simplicity, like beauty, is difficult to quantify. However, the above conditions require a function containing at least two parameters, namely v_M and ρ_M. The linear relationship in (5.10) is an example of a simple function with two parameters. Another possibility is the function proposed by Newell [1961], given as

$$v = v_M \left(1 - e^{-\lambda(1/\rho - 1/\rho_M)}\right). \tag{5.17}$$

Assuming that $\lambda \geq 0$, this is an example of a three parameter constitutive law that satisfies both NV1 and NV2. Fitting this to the data for the Lincoln Tunnel and Merritt Parkway one finds that $v_M = 37.4$ mph, $\rho_M = 271$ cars/mi, and $\lambda = 67.4$ mi/cars. The resulting function is plotted in Figure 5.7 along with the original data. It is evident that it is better than Greenshields at reproducing the data and, unlike the linear law, this function contains a plateau region near $\rho = 0$ that is seen in the Toronto and Amsterdam data in Figure 5.6. The penalty for this improvement is that the wave velocity, given in (5.16), is

$$c = v_M \left[1 - \left(1 + \frac{\lambda}{\rho}\right) e^{-\lambda(1/\rho - 1/\rho_M)}\right].$$

One therefore has to decide if the resulting complexity in the traffic flow equation (5.14) is worth the improvement in the data fit.

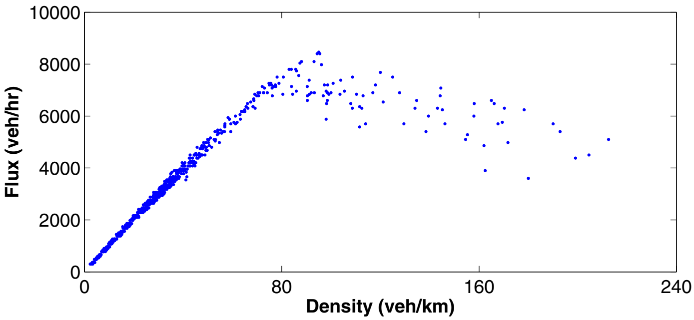

Figure 5.8 The flux as a function of the density measured on a freeway in Amsterdam (Aerde and Rakha [1995]).

5.4.4 Flux and Velocity

Our model has three dependent variables, flux, density, and velocity. Given that the equation of motion is written in terms of density and velocity the conventional approach is to propose a constitutive law that relates these two functions. However, it is worthwhile to consider other possibilities. One alternative is to relate the flux with the density using a constitutive law, and then use the equation $J = \rho v$ to determine the velocity. With this in mind the data in Figure 5.6 for the freeway in Amsterdam is plotted in Figure 5.8, giving the flux as a function of density. This is known as a fundamental diagram, and it is used extensively in developing traffic models. What is striking about this graph is that J has a well-defined dependence on ρ up to about $\rho = 80$ after which there is considerable scatter in the data. This spread is very typical of traffic flow, and it makes formulating a constitutive law for the flux problematic. In contrast, the v, ρ plots in Figure 5.6 show a more well-defined relationship over the entire density range, and for this reason it is more amenable to constitutive modeling.

One conclusion that can be made from Figure 5.8 is that the flux is concave down. From this we obtain an additional general rule for the general constitutive law $v = F(\rho)$, which is

NV3. $J''(\rho) \leq 0$, or equivalently, $2F'(\rho) + \rho F''(\rho) \leq 0$ for $0 \leq \rho \leq \rho_M$.

Recall that a smooth function is concave down if its derivative is monotone decreasing. Consequently, if the function $c(\rho) = J'(\rho)$ is monotone decreasing then the above condition is satisfied.

5.4.5 Reality Check

It is important to understand that even the most complex nonlinear expression relating the velocity and density is still, in the end, an approximation. Inevitably certain aspects of the problem are not accounted for, and many times this is intentional because the goal of the model is to capture the essential mechanisms responsible for the phenomena being studied. This has certainly been the case with the traffic flow problem. We have not included effects of intersections, inclement weather, adverse road conditions, or myriad other things that can influence traffic flow. There is also the problem that the cars are driven by people, who make individual decisions that can have dramatic effects on the traffic pattern. As a simple example, some drivers will speed up if there is lighter traffic ahead. This implies that the velocity depends on the density gradient. This is not accounted for in our model because we are assuming that the law has the form $v = F(\rho)$ and not $v = F(\rho, \rho_x)$. Some of the consequences of this extension are explored in Exercise 5.25. Generally, this sort of application is outside the scope of this textbook. However, a very humorous account of the role of human behavior, and how it affects traffic flow, can be found in Vanderbilt [2008].

A second comment that needs to be made is that the equation of motion (5.7) is general, and in terms of traffic flow can be applied to a multilane freeway or a small farm road. However, once a specific constitutive law for the velocity is introduced then the model becomes more limited in its applicability. For example, the traffic data given in Figure 5.6 measures the velocity on one side of the roadway (e.g., the velocity of the vehicles going east to west). This is reasonable because if both sides are counted, so the measured velocities can be either positive or negative, one could end up concluding that on average the velocity is zero at all density levels. In fact, it is not uncommon in traffic applications to have the constitutive law limited to a particular lane of traffic. For example, some roadways limit trucks to certain lanes of the roadway and this has a significant consequence for the velocity function. The point here is that the equation of motion is general but in applying it to particular problems, which requires the specification of a constitutive law, the model becomes more limited.

All of the above comments are evidence that we are studying a rich problem that has multiple research directions, and our model addresses one of them. Our objective is to understand how traffic flow behaves under the assumed conditions, and our next step is to figure out how to solve the mathematical problem we have produced.

5.5 Constant Velocity

To investigate the properties of the traffic flow problem we will begin with the assumption that the velocity is constant. The problem takes the form

$$\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} = 0, \quad \text{for} \quad \begin{cases} -\infty < x < \infty \\ 0 < t, \end{cases} \tag{5.18}$$

where

$$\rho(x,0) = f(x). \tag{5.19}$$

The partial differential equation (5.18) is known as the advection equation. The solution can be found if one notes that the equation can be written as

$$\left(\frac{\partial}{\partial t} + a\frac{\partial}{\partial x}\right)\rho = 0. \tag{5.20}$$

The idea is to transform x, t to new variables r, s in such a way that the derivatives transform as

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial t} + a\frac{\partial}{\partial x}\,. \tag{5.21}$$

If this is possible then (5.20) becomes $\frac{\partial \rho}{\partial r} = 0$ and this equation is very easy to solve. With this goal in mind let $x = x(r,s), t = t(r,s)$, in which case using the chain rule the r-derivative transforms as

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial t}{\partial r}\frac{\partial}{\partial t}\,. \tag{5.22}$$

Comparing this with (5.20), we require $\frac{\partial x}{\partial r} = a$ and $\frac{\partial t}{\partial r} = 1$. Integrating these equations yields $x = ar + q(s)$ and $t = r + p(s)$. To determine the s dependence recall that the initial condition specifies the solution along the x-axis. To make it easy to apply the initial condition we will ask that the x-axis ($t = 0$) maps onto the s-axis ($r = 0$). In other words, $r = 0$ implies that $t = 0$ and $x = s$. Setting $r = 0$ and $t = 0$ we conclude $q(s) = s$ and $p(s) = 0$, and so, the change of variable we are looking for is

$$x = ar + s, t = r. \tag{5.23}$$

Inverting this transformation one finds that $r = t$ and $s = x - at$. We are now able to write (5.18) as $\frac{\partial \rho}{\partial r} = 0$, which means $\rho = \rho(s) = \rho(x - at)$. With the initial condition we therefore conclude that the solution of the problem is

$$\rho(x,t) = f(x - at). \tag{5.24}$$

Before making general conclusions about this solution we consider an example. This is worked out twice, first as a mathematical problem, and then as a problem in traffic flow.

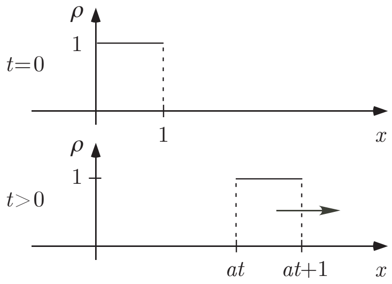

Figure 5.9 Solution of the advection equation (5.18). The top figure is the initial condition, as given in (5.27). The bottom figure is the solution at a later time, as given in (5.24).

Example: Mathematical Version

Suppose the initial condition is the square bump shown in Figure 5.9. In mathematical terms,

$$f(x) = \begin{cases} 1 & \text{if} \quad 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5.25}$$

From (5.24) the solution is

$$\rho(x,t) = \begin{cases} 1 & \text{if} \quad 0 < x - at < 1, \\ 0 & \text{otherwise,} \end{cases}$$

or equivalently,

$$\rho(x,t) = \begin{cases} 1 & \text{if} \quad at < x < 1 + at, \\ 0 & \text{otherwise.} \end{cases} \tag{5.26}$$

A typical solution profile is also shown in Figure 5.9, and it is apparent that at any given time t, the solution is simply the original square bump that has moved over to occupy the interval $at \leq x \leq 1 + at$. ■

Example: Traffic Version

The previous example can be restated in physical terms. Suppose, at $t = 0$, that cars are uniformly spaced over the interval $0 < x < 1$, as shown in Figure 5.10. In this case the density has a constant, positive value for $0 < x < 1$, while the density outside this interval is zero. Also, assuming that each car travels with the same constant velocity a, then they will move as a unit. So, at any given time t, the group of cars will occupy the interval $at < x < at+1$. Because they are traveling at the same velocity, the spacing of the cars has not changed, and therefore the density in this interval is the same as it was at $t = 0$. This is the same result as obtained in the solution (5.26). ■

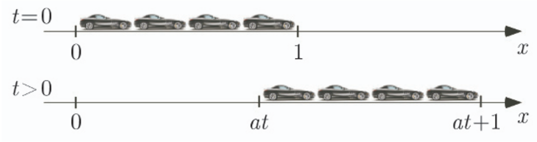

Figure 5.10 A uniformly spaced group of cars moves with constant velocity a along the x-axis.

In the above example, expressing the problem in terms of the motion of the individual cars is analogous to taking a microscopic point of view. In contrast, the macroscopic, or continuum, viewpoint is expressed in the solution given in (5.26). The attractive aspect of the microscopic point of view is that the solution is easy to understand, and it is obtained without having to solve a partial differential equation. Unfortunately, for more realistic problems, where the velocity depends on the density, the micro-scale version loses this advantage and the continuum problem becomes the easier one to solve.

From the above examples, and from the general formula in (5.24), we conclude that the solution is a traveling wave. The wave travels in only one direction, and for this reason (5.18) is sometimes called a one-way wave equation. In the case of when $a > 0$ the wave moves to the right with speed a. What is significant is that it moves at the same velocity as the vehicles, which, if you recall, is $v = a$. It might seem obvious that the wave moves with the vehicle velocity because, after all, the vehicles are responsible for the wave in the first place. However, the answer is not so simple. For example, the waves generated at sporting events by the fans in the audience are obtained not by the fans running around the stadium but, rather, by them periodically standing up and sitting down. Similarly, in heavy traffic if a car's taillights come on you will likely see a wave of taillights come on in the cars that follow. Not only is the wave of taillights not moving with the car's velocity, it is actually moving in the opposite direction. So, the connection between the motion of the constituents and the velocity of the wave requires some consideration. We will return to this point later when solving the problem of nonconstant velocity.

Another observation coming from the above example is that the shape and amplitude do not change as the wave travels along the x-axis. This is in marked contrast to the diffusion equation, where the corners or jumps in the initial condition are immediately smoothed out (see Figure 4.14). Because of this, one might question whether (5.26) is actually a solution since ρ_x is not defined at the jumps located at $x = at, 1 + at$. The short answer is that because there are only a finite number of jumps, everything is fine. What is necessary is to introduce the concept of a weak solution, and the interested reader is referred to Evans [1998] for an extended discussion of this subject. A slightly different approach to justifying the jumps, and understanding some of the difficulties of defining a continuum variable at a jump, are explored in Exercise 5.17.

5.5.1 Characteristics

There is another way to look at this solution that will prove to be particularly worthwhile. It is based on the observation that, from the formula $\rho(x,t) = f(x-at)$, if we hold $x-at$ fixed then the solution is constant. In other words, if $x-at=x_0$ then $\rho = f(x_0)$ along this line (see Figure 5.11). These lines are called characteristics for the equation, and the method we used to find the solution is called the method of characteristics. The observation that the solution is constant along the characteristics can be used to evaluate the solution anywhere in the x,t-plane. The next example illustrates how this is done.

Example

Suppose we want to determine $\rho(0,1)$. To use characteristics to find this value, we need to determine the line $x-at=x_0$ that passes through $(x,t)=(0,1)$ (see Figure 5.12). Plugging $x=0$ and $t=0$ into the equation $x-at=x_0$ we obtain $x_0=-a$. Therefore, $\rho(0,1)=f(x_0)=f(-a)$. As it should, this result agrees with what is obtained from the formula given in (5.24). ■

In general, to determine $\rho(x_1,t_1)$ using characteristics, one first finds the characteristic that passes through (x_1,t_1). The equation for this line is $x-at=x_1-at_1$. The solution is constant along this line, and because the x-intercept is $x_0=x_1-at_1$, it follows that $\rho(x_1,t_1)=f(x_0)$.

Example: Red Light - Green Light

As a second example of how the characteristics can be used to construct the solution, consider the situation of cars waiting at a stoplight. It is assumed that at $t=0$ the light turns from red to green. We will locate the light at $x=0$, and assume that at the start the cars have a constant density to

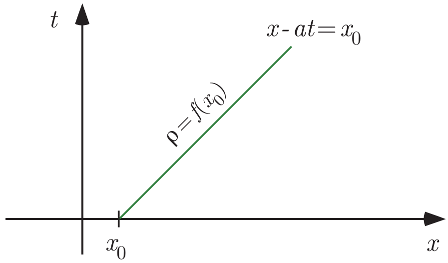

Figure 5.11 The characteristics for (5.18) are the straight lines $x-at=x_0$. Along each line the solution is constant.

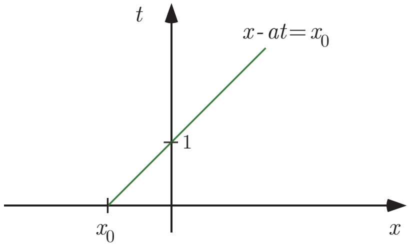

Figure 5.12 The characteristics used in the example to determine the value of $\rho(0, 1)$.

the left of the light. The initial condition that will be used to describe this situation is

$$\rho(x, 0) = \begin{cases} 1 & \text{if} \quad x \le 0 \\ 0 & \text{if} \quad x > 0. \end{cases} \tag{5.27}$$

It is also assumed that $a > 0$. The characteristics for this problem are shown in Figure 5.13(a). Because of where the characteristics intersect the x-axis, the solution in the region covered by the solid lines is $\rho = 1$, while along the dashed lines the solution is $\rho = 0$. The characteristic that separates these two regions is the one that starts at the jump in the initial condition (5.27). Namely, it is the line $x = at$, and it is shown in Figure 5.13(a) using a line with small dots. The resulting solution is shown in Figure 5.13(b), and the corresponding formula is

$$\rho(x, t) = \begin{cases} 1 & \text{if} \quad x \le at \\ 0 & \text{if} \quad x > at. \end{cases} \tag{5.28}$$

A somewhat more traditional view of the solution is given in Figure 5.13(c), where it is apparent that the solution consists of a wave that moves with speed a. ■

The two previous examples were used to introduce how characteristics can be used to find the solution, but in both cases the solution can be determined directly for the formula in (5.24). This is not true for the next, and final, example.

Example: Finite Length Highways

Up to this point our highways have been infinitely long. In the real world this is rather rare, and in this example we consider what happens when the road occupies the interval $0 \le x \le \ell$. This gives rise to the question as to what can, or should, be specified for boundary conditions at $x = 0, \ell$. A mathematically correct choice is to specify a boundary condition at $x = 0$ and not specify one at $x = \ell$. The reason is due to the fact that information in this problem goes

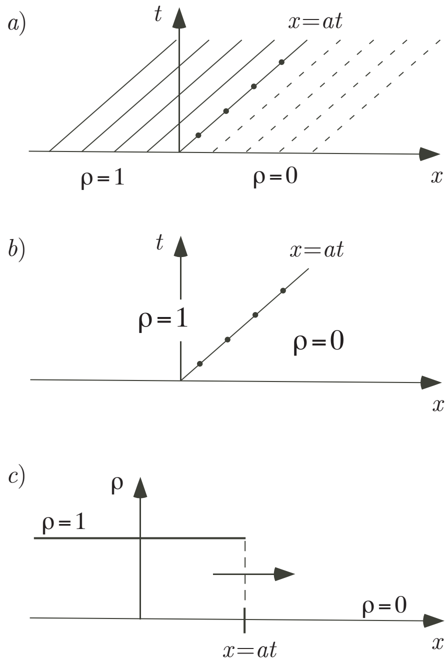

Figure 5.13 The solution of (5.18) when given the initial condition (5.27).

in only one direction, from left to right. Why this is important will become evident once we study the solution in more detail. To this end, we consider solving the equation

$$\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} = 0, \quad \text{for} \quad \begin{cases} 0 < x < \ell \\ 0 < t, \end{cases} \tag{5.29}$$

along with the initial condition

$$\rho(x, 0) = f(x),$$

and the boundary condition

$$\rho(0, t) = g(t).$$

Using characteristics this is not hard to solve. We know that the solution of (5.29) is constant along any line of the form $x - at = const$ and these lines are shown in Figure 5.14. The analysis naturally separates into two components.

Solid Lines: In the region containing the characteristics that are solid lines, the solution is determined by the initial condition. Because the lines in

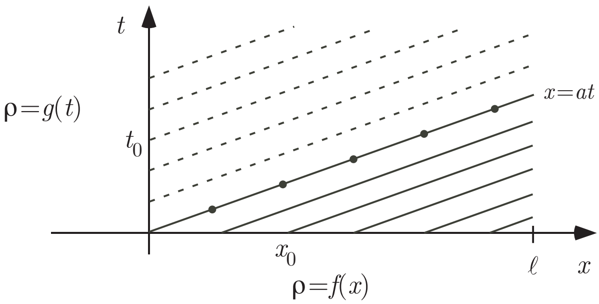

Figure 5.14 Characteristics used in solving the traffic flow problem over a finite interval.

this region have the form $x - at = x_0$, where x_0 is the x-intercept, then in this region the solution is $\rho(x,t) = f(x_0) = f(x - at)$.

Dashed Lines: To find the solution in the region where the characteristics are dashed lines, consider the characteristic shown in Figure 5.14 that has t-intercept t_0. The general form for the equation of this line is $x - at =$ *constant*. Because it must pass through the point $(x,t) = (0,t_0)$, it follows that the equation is $x - at = -at_0$. Because the solution is constant along this line, and we are told that $\rho(0,t_0) = g(t_0)$, then it follows that along this characteristic $\rho(x,t) = g(t_0) = g(t - x/a)$.

Putting this information together, the solution is

$$\rho(x,t) = \begin{cases} f(x-at) & \text{if} \quad 0 \le t < x/a, \\ g(t-x/a) & \text{if} \quad x/a < t. \end{cases}$$

The value at $x = at$ depends on what value the function has at $(x,t) = (0,0)$. If $\rho(0,0) = f(0)$ then $\rho = f(0)$ for $x = at$, while if $\rho(0,0) = g(0)$ then $\rho = g(0)$ for $x = at$. ■

Returning to the question of whether it is possible to impose a boundary condition at $x = \ell$, suppose that $f(x) = 1$. In Figure 5.14, in the region covered with the solid lines the solution is $\rho = 1$. Any boundary condition imposed at $x = \ell$, other than $\rho = 1$, would be in contradiction to the known solution. That is why, in the case of when $a > 0$, it is more natural to impose a boundary condition at the left end of the interval. If one is insistent on specifying a boundary condition at $x = \ell$, it would then be necessary not to include either an initial condition or a boundary condition at $x = 0$. This idea is explored further in Exercise 5.9.

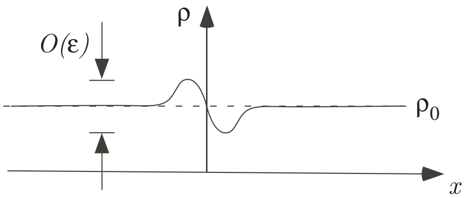

Figure 5.15 Small disturbance imposed onto constant density solution at $t = 0$. The resulting initial condition is given in (5.32)

5.6 Nonconstant Velocity

The linear wave equation studied in the previous section is a valuable source of information about some of the more basic properties of the solution. The fact is, however, the assumption that the velocity is independent of the density is not correct for traffic flow. This is evident in the data given in Figure 5.6. Precisely what constitutive law is used will be left unspecified for the moment other than to assume $v = F(\rho)$, where F is smooth. As shown in Section 5.4.3, the traffic flow equation takes the form

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0, \tag{5.30}$$

where the wave velocity is

$$c(\rho) = F + \rho F'. \tag{5.31}$$

Written this way the equation resembles the constant velocity version in (5.18) we studied earlier. One significant difference is that the wave velocity c can depend on the unknown ρ, and if this happens then (5.30) is nonlinear. Generally nonlinear partial differential equations are very difficult to solve. One option, which works on a wide variety of problems, is to introduce a small disturbance approximation, and this is discussed below. However, for this problem it is possible to solve the fully nonlinear equation using the method of characteristics and this will be considered in Section 5.6.2.

Although the nonlinear traffic flow equation is very general, a couple of restrictions are needed to help guarantee that there is a solution. One is that whatever function $c(\rho)$ is used in this equation, it is a smooth function of ρ. A second condition is related to the observation made in Section 5.4.4 that the flux is concave down. This is equivalent to $c(\rho)$ being a monotonically decreasing function of ρ. Mathematically, what is needed is that $c(\rho)$ is monotonic, either decreasing or increasing, and this is assumed in what follows.

5.6.1 Small Disturbance Approximation

One method for studying nonlinear wave problems is based on a small disturbance approximation. The basic idea is that a particular solution has been determined. This is usually an equilibrium solution, and it is very common that it is a constant. What is investigated is how small perturbations of this particular solution behave. To explain what this entails note that a constant function $\rho = \rho_0$ is a solution of the traffic flow equation (5.30). So, suppose that the traffic is flowing along smoothly with a uniform density $\rho = \rho_0$ and then one or more of the cars change speed slightly and cause a small perturbation in the density. For example if someone applies their brakes then the immediate affect will be to reduce the density in front of their car and to increase the density right behind them. A function that mimics this change in the density is shown in Figure 5.15.

To analyze this situation we will assume the disturbance occurs at $t = 0$. The initial condition that corresponds to this is

$$\rho(x, 0) = \rho_0 + \epsilon g(x). \tag{5.32}$$

The specific form of the function $g(x)$ is not important but we will illustrate the analysis using the example in Figure 5.15. Due to the initial condition the appropriate expansion for the solution is $\rho \sim \rho_0 + \epsilon\rho_1(x, t) + \cdots$. In this case, using Taylor's theorem,

$$\begin{aligned} c(\rho) &\sim c(\rho_0 + \epsilon\rho_1 + \cdots) \\ &\sim c(\rho_0) + (\epsilon\rho_1 + \cdots)c'(\rho_0) + \frac{1}{2}(\epsilon\rho_1 + \cdots)^2 c''(\rho_0) + \cdots \\ &\sim c(\rho_0) + \epsilon\rho_1 c'(\rho_0) + \cdots . \end{aligned}$$

The equation of motion (5.30) takes the form

$$\epsilon\frac{\partial\rho_1}{\partial t} + \cdots + [c(\rho_0) + \epsilon\rho_1 c'(\rho_0) + \cdots]\left(\epsilon\frac{\partial\rho_1}{\partial x} + \cdots\right) = 0, \tag{5.33}$$

where, from (5.32),

$$\rho_0 + \epsilon\rho_1(x, 0) + \cdots = \rho_0 + \epsilon g(x). \tag{5.34}$$

Setting $c_0 = c(\rho_0)$ then the $O(\epsilon)$ problem is

$$\frac{\partial\rho_1}{\partial t} + c_0\frac{\partial\rho_1}{\partial x} = 0, \tag{5.35}$$

where $\rho_1(x, 0) = g(x)$. This is known as the small disturbance equation for the problem and in this case it is a linear wave equation. Using (5.24), the solution is $\rho_1(x, t) = g(x - c_0 t)$. Therefore, the two term small disturbance approximation of the solution is

$$\rho(x,t) \sim \rho_0 + \epsilon g(x - c_0 t). \tag{5.36}$$

It is clear from this that the initial disturbance propagates as a traveling wave, with velocity c_0. We will explore some of the consequences of this in the next example, but it is first necessary to comment on the accuracy of this approximation. If you compare (5.36) with, say, the numerical solution it is found that as time passes the approximation becomes less accurate. This is due to a slow change in the solution that is not accounted for in (5.36), and which over time starts to affect its accuracy. It is possible to use multiple scales, as described in Section 2.6, to improve the approximation. However, later in the chapter, after the nonlinear problem is solved, we will derive an exact solution of the problem.

Example: Phantom Traffic Jams

To investigate the properties of (5.36) we will use the Greenshields constitutive law and assume

$$v = v_M \left(1 - \frac{\rho}{\rho_M}\right). \tag{5.37}$$

In this case, from (5.31),

$$c = v_M \left(1 - \frac{2\rho}{\rho_M}\right), \tag{5.38}$$

and the flux is

$$J = v_M \left(1 - \frac{\rho}{\rho_M}\right)\rho. \tag{5.39}$$

These functions are sketched in Figure 5.16. Note that for a given value of the flux that there are two possible densities. Those satisfying $0 < \rho < \frac{1}{2}\rho_M$ are commonly referred to as light traffic while those satisfying $\frac{1}{2}\rho_M < \rho < \rho_M$ are heavy traffic. Also note that $c = J'$, in other words it equals the slope of the flux function. This means c is negative for lighter traffic and it is positive in heavier traffic.

Based on the above discussion, our conclusion is that in light traffic, where $c > 0$, the disturbance moves forward, and in heavy traffic, where $c < 0$, the disturbance moves backward. Given that $c \leq v$, the disturbance does not move faster than the flow of traffic. In other words, whoever was responsible for generating this disturbance would see it move backward relative to their position, but someone watching from an overpass would see it move forward in light traffic and move backward in heavy traffic. The one exception to this last statement is if the traffic density is $\rho_M/2$, in which case the disturbance would stay in the region where it was generated. Another point to make here is that, unlike the constant velocity example, the wave propagates at

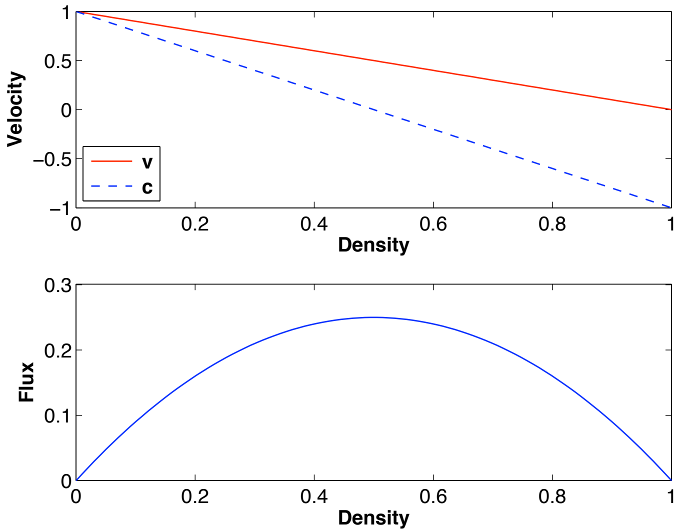

Figure 5.16 Velocities (5.37), (5.38), and the flux (5.39) when using the Greenshields law. In these plots $v_M = 1$ and $\rho_M = 1$.

a velocity that is different from the velocity of the vehicles that form the system.

The solution obtained using a small disturbance approximation provides an explanation of one of the mysteries of driving called the phantom traffic jam. This is the situation when there is no visible reason for a traffic slowdown, as there is no accident, construction, etc. As shown in Figure 5.17 some earlier perturbation in the traffic can result in a density wave propagating backwards along the highway. A driver who enters this region will see no apparent reason for its existence and once through the disturbance will return to the uniform flow they had earlier. One cause of such situations is weaving.

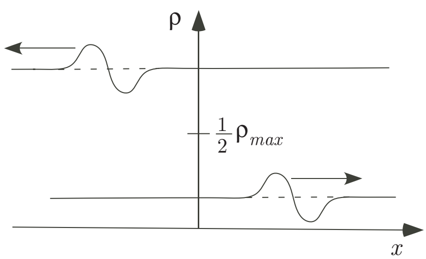

Figure 5.17 Disturbances move to the right if $\rho_0 < \frac{1}{2}\rho_M$ and move toward the left if $\frac{1}{2}\rho_M < \rho_0$. The signal velocity in both cases is $c_0 = c(\rho_0)$.

In heavier traffic drivers who change lanes frequently cause the drivers behind them to slow down or brake to leave room between them and the lane changer. This produces a small disturbance and this propagates along the highway behind the originators of this situation. ■

5.6.2 Method of Characteristics

As it turns out, the method of characteristics we developed to solve the constant velocity problem can be adapted so it also works on the nonlinear equation (5.30). In the constant velocity case, we found that the solution is constant along curves of the form $x = x_0 + at$. So, in a similar manner we will investigate if it is possible to find curves $x = X(t)$ on which the solution of (5.30) is constant. What we are looking for are curves with the property that $\frac{d}{dt}\rho(X(t), t) = 0$. Expanding this using the chain rule it follows that we need to select $X(t)$ in such a way that

$$\rho_t + X'(t)\rho_x = 0. \tag{5.40}$$

To find a function $X(t)$ that works in this equation, recall that ρ satisfies the traffic flow equation

$$\rho_t + c(\rho)\rho_x = 0. \tag{5.41}$$

Comparing this with (5.41) it is evident that $X(t)$ should be selected so that

$$X'(t) = c(\rho). \tag{5.42}$$

Before integrating to find the function $X(t)$, remember that ρ is constant along the curve. Consequently, if the curve begins at $x = x_0$ then at any point along the curve we have $\rho = \rho_0$ where $\rho_0 = f(x_0)$ (see Figure 5.18). Introducing this into (5.42), and integrating, we obtain $X = x_0 + c(\rho_0)t$. Therefore, the characteristic that begins at $x = x_0$ is

$$x = x_0 + c(\rho_0)t, \tag{5.43}$$

and along this characteristic the solution is

$$\rho = \rho_0, \tag{5.44}$$

where $\rho_0 = f(x_0)$. It might seem odd that the characteristics for a nonlinear equation turn out to be linear. However, the nonlinearity does have an affect as it determines the slope of the characteristics and, as we will see, this has major consequences on the solution.

The two expressions (5.43) and (5.44) form the solution of the problem. To explain how they are used, suppose one wants to calculate the value of ρ at a particular point, say at (x_1, t_1). In some cases, the value of $\rho(x_1, t_1)$ is

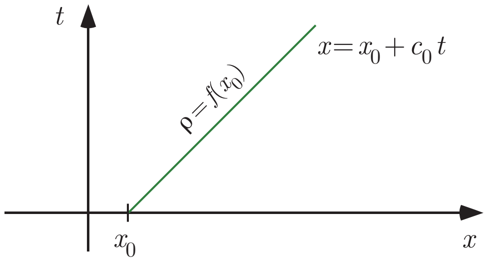

Figure 5.18 The method of characteristics involves finding the curves $x = X(t)$ along which the solution of (5.30) is constant.

easy to determine, and this happens in the next example when $\rho = \rho_L$ and when $\rho = \rho_R$. If the value is not obvious, then it is necessary to calculate the result, and this involves the following steps.

Step 1. Find the characteristic that passes through (x_1, t_1).
Given that the general form of the characteristic is $x - c_0 t = x_0$, then we require that $x_1 - c_0 t_1 = x_0$.

Step 2. Find c_0 in terms of x_0.
From the initial condition, we have that $c_0 = c(f(x_0))$. As an example, using the Greenshields law,

$$c(\rho_0) = v_M \left(1 - 2\frac{f(x_0)}{\rho_M}\right).$$

Step 3. Solve $x_1 = x_0 + c_0 t_1$ for x_0.
In the case of when the Greenshields law is used then the equation to solve is

$$x_1 = x_0 + v_M t_1 \left(1 - 2\frac{f(x_0)}{\rho_M}\right).$$

How difficult this equation is to solve for x_0 depends on $f(x_0)$. We will be using piecewise linear functions, so it is possible to solve the above equation relatively easily.

Once x_0 is known then the solution is $\rho(x_1, t_1) = f(x_0)$. This procedure is not particularly difficult, but it comes with caveats. In particular, it assumes that there is a characteristic passing through (x_1, t_1). As we will see shortly, this might not happen. We will postpone analyzing such difficulties until after we have more experience using the method when all goes according to plan.

Example: Modified Red Light - Green Light

To use the above solution for traffic flow we consider a modified version of the red light - green light problem. It is assumed that the traffic is initially constant to the left of $x = -\epsilon$ and to the right of $x = \epsilon$. Also, there is a transition region, of width 2ϵ, where the density changes linearly between the left and right values. This situation is shown in Figure 5.19. It is assumed the faster cars are in the front, and so, $\rho_L > \rho_R$. The specific function used for the initial condition is

$$\rho(x,0) = \begin{cases} \rho_L & \text{if} \quad x \leq -\epsilon \\ \rho_L + \frac{\rho_R - \rho_L}{2\epsilon}(x+\epsilon) & \text{if} \ -\epsilon < x < \epsilon \\ \rho_R & \text{if} \quad \epsilon \leq x. \end{cases} \tag{5.45}$$

We also need to be specific about what constitutive law is being used for the velocity, and in what follows we use the Greenshields law. Consequently, $v = v_M(1 - \rho/\rho_M)$ and the wave velocity is

$$c(\rho) = v_M \left(1 - \frac{2\rho}{\rho_M}\right). \tag{5.46}$$

To sketch the characteristics, we consider what happens for different starting positions x_0.

- If x_0 is on the left, so $x_0 < -\epsilon$, then ρ_0 has the constant value ρ_L. This means that the characteristics in this region all have the same slope, and this is shown in Figure 5.20(a). Given that the solution is constant along each of these lines it follows that $\rho = \rho_L$ in the region of the x,t-plane to the left of the characteristic $x = -\epsilon + c_L t$, where $c_L = c(\rho_L)$. This is shown in Figure 5.20(b).
- Using a similar argument, the characteristics that start on the right, where $x_0 > \epsilon$, all have the same slope. Because $\rho_L > \rho_R$ then the characteristics on the left have a steeper slope than those on the right, and this is shown in Figure 5.20(a). The solution is constant along each of these lines, and so it follows that $\rho = \rho_R$ in the region of the x,t-plane to the right of the characteristic $x = \epsilon + c_R t$, where $c_R = c(\rho_R)$. This is shown in Figure 5.20(b).
- To determine what happens when $-\epsilon < x_0 < \epsilon$, it is seen in Figure 5.19 that the initial density is continuous over this interval. This means that $c(\rho_0)$ varies continuously from c_L at $x_0 = -\epsilon$, to c_R at $x_0 = \epsilon$. The resulting

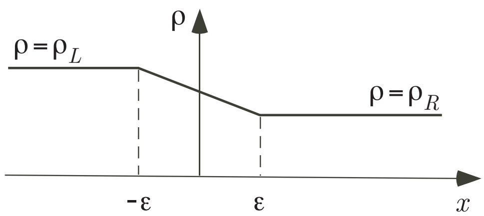

Figure 5.19 Initial density $\rho(x,0)$ for the modified red light - green light problem.

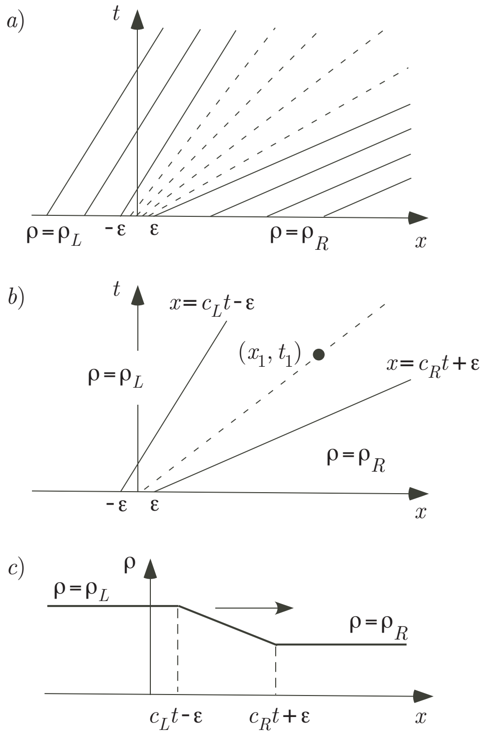

Figure 5.20 The solution of the modified red light - green light problem. The width of the linear transition region between the left and right constant states increases with time because $c_L < c_R$.

characteristics are shown in Figure 5.20(a) using dashed lines. To find the solution at a point (x_1, t_1) in this region, as illustrated in 5.20(b), we need to find the characteristic that passes through this point. This requires finding x_0. Because the density is constant on the characteristic, once x_0 is known then $\rho(x_1, t_1) = \rho(x_0, 0)$. Now, the general formula for the characteristics is $x = x_0 + c_0 t$, and so it is required that $x_1 = x_0 + c_0 t_1$. Given (5.46) and (5.45) we have that

$$\begin{aligned} c_0 &= v_M\left(1 - \frac{2\rho_0}{\rho_M}\right) \\ &= v_M\left[1 - \frac{2}{\rho_M}\left(\rho_L + \frac{\rho_R - \rho_L}{2\epsilon}(x_0 + \epsilon)\right)\right]. \end{aligned}$$

Substituting this into the equation $x_1 = x_0 + c_0 t_1$ and then solving for x_0 one finds that

$$x_0 = \frac{x_1 - t_1(c_L + c_R)/2}{1 + t_1(c_R - c_L)/(2\epsilon)}.$$

With this, and the initial condition in (5.45), the density is

$$\begin{aligned}\rho(x_1, t_1) &= \rho(x_0, 0) \\ &= \rho_L + (\rho_R - \rho_L)\frac{x_1 + \epsilon - c_L t_1}{2\epsilon + (c_R - c_L)t_1}.\end{aligned} \tag{5.47}$$

The formula for the solution is therefore

$$\rho(x,t) = \begin{cases} \rho_L & \text{if} \quad x \le c_L t - \epsilon \\ \rho_L + (\rho_R - \rho_L)\frac{x+\epsilon-c_L t}{2\epsilon+(c_R-c_L)t} & \text{if} \ c_L t - \epsilon < x < c_R t + \epsilon \\ \rho_R & \text{if} \quad c_R t + \epsilon \le x. \end{cases} \tag{5.48}$$

According to this, between the two constant states the density varies linearly, just as it did in the initial condition. There is nothing unusual in this solution as it shows the expected result that the slower group on the left gradually separates from the faster group on the right. This is illustrated in Figure 5.20(c). ■

5.6.3 Rankine-Hugoniot Condition

As will become evident as we study the nonlinear traffic flow equation in Section 5.6.5, the solution has a propensity to evolve into a function with one or more jump discontinuities that move along the x-axis. We studied such a solution with the red light-green light problem for the linear equation, and the result is shown in Figure 5.13. The nonlinear equation is a different animal, and we are going to have to be a bit more careful any time a jump is present. To investigate what happens, suppose we have a situation as shown in Figure 5.21, which consists of a jump that is located at $x = s(t)$. Given that x-derivatives are not defined at such points we will reformulate the problem by integrating over a small spatial interval, $s - \epsilon \le x \le s + \epsilon$, around the jump. So, integrating $\rho_t + J_x = 0$ and remembering that the density is constant on either side of the jump we obtain

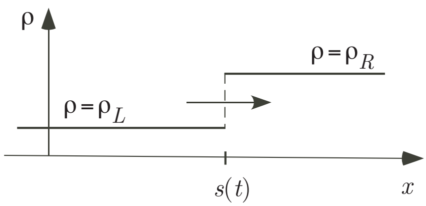

Figure 5.21 A jump discontinuity in the solution, located at $x = s(t)$.

$$\int_{s-\epsilon}^{s+\epsilon} \rho_t dx + J(\rho_R) - J(\rho_L) = 0. \tag{5.49}$$

From the Fundamental Theorem of Calculus recall that

$$\frac{d}{dt}\int_{s-\epsilon}^{s+\epsilon} \rho dx = \int_{s-\epsilon}^{s+\epsilon} \rho_t dx + s'(t)\rho|_{x=s+\epsilon} - s'(t)\rho|_{x=s-\epsilon}.$$

From this and (5.49) it follows that

$$\frac{d}{dt}\int_{s-\epsilon}^{s+\epsilon} \rho dx - \rho_R s'(t) + \rho_L s'(t) + J(\rho_R) - J(\rho_L) = 0. \tag{5.50}$$

Now, using the piecewise constant nature of the density

$$\begin{aligned}\int_{s-\epsilon}^{s+\epsilon} \rho dx &= \int_{s-\epsilon}^{s} \rho dx + \int_{s}^{s+\epsilon} \rho dx \\ &= \epsilon(\rho_L + \rho_R),\end{aligned}$$

and so

$$\frac{d}{dt}\int_{s-\epsilon}^{s+\epsilon} \rho dx = 0.$$

It follows from (5.50) that

$$s'(t) = \frac{J(\rho_R) - J(\rho_L)}{\rho_R - \rho_L}. \tag{5.51}$$

This equation is known as the *Rankine-Hugoniot condition* and it determines the velocity of a jump discontinuity in the solution.

It is useful to express (5.51) in terms of the wave velocity function c. Recalling that $c = J'(\rho)$, and $J(0) = 0$, then

$$J(\rho) = \int_0^{\rho} c(\bar{\rho}) d\bar{\rho}. \tag{5.52}$$

With this, the Rankine-Hugoniot condition takes the form

$$s'(t) = \frac{1}{\rho_R - \rho_L}\int_{\rho_L}^{\rho_R} c(\rho) d\rho. \tag{5.53}$$

This is an interesting result as it shows that any jump in the solution travels at the wave velocity averaged over the given density interval.

There are two types of jumps, and they are determined by what happens to the velocity v at the jump. If ρ has a jump discontinuity at $x = s(t)$, but v is continuous at $x = s(t)$, then the jump is called a contact discontinuity. An example is the red light-green light solution shown in Figure 5.13. The velocity is constant, hence it is continuous no matter where the jumps occur.

Note that because $v = a$ and $J = \rho v$ then the Rankine-Hugoniot condition (5.51) reduces to $s' = a$. In other words, the jumps move with the given constant velocity, and this is what was determined in Figure 5.13.

If v is not continuous at $x = s(t)$ then the jump is called a shock, and the motion of this jump produces a *shock wave*. As shown in the next examples, the velocity of the shock is strongly dependent on the constitutive law.

Examples

1. Greenshields Law. Using the linear law in (5.10), and the fact that $J = \rho v$, then the Rankine-Hugoniot condition (5.51) simplifies to the following

$$\begin{aligned} s'(t) &= \frac{1}{\rho_R - \rho_L}\left[\rho_R v_M \left(1 - \frac{\rho_R}{\rho_M}\right) - \rho_L v_M \left(1 - \frac{\rho_L}{\rho_M}\right)\right] \\ &= v_M \left(1 - \frac{1}{\rho_M}(\rho_R + \rho_L)\right) \\ &= \frac{1}{2}\left(c_R + c_L\right). \end{aligned} \tag{5.54}$$

In other words, when using the Greenshields law, the shock moves at a speed determined by the average of the jump in the wave velocity across the shock. ■

2. Newell Law. Using (5.17) then the Rankine-Hugoniot condition (5.51) is

$$\begin{aligned} s'(t) &= \frac{1}{\rho_R - \rho_L}\left[\rho_R v_M \left(1 - e_R\right) - \rho_L v_M \left(1 - e_L\right)\right] \\ &= v_M \left(1 - \frac{\rho_R e_R - \rho_L e_L}{\rho_R - \rho_L}\right), \end{aligned} \tag{5.55}$$

where

$$\begin{aligned} e_L &= e^{-\lambda(1/\rho_L - 1/\rho_M)}, \\ e_R &= e^{-\lambda(1/\rho_R - 1/\rho_M)}. \quad ■ \end{aligned}$$

When we first started out studying traffic flow, we had only one variable with the dimension of velocity. Now, we have three variables with this dimension. They are:

1. $v(x,t)$. This is the velocity of the car located at x at time t.

2. $c(\rho)$. This is the wave velocity, and it is defined in (5.31). It determines the slopes of the characteristic curves.

3. $s'(t)$. This is the velocity of the jumps in the solution, and it is defined in (5.51).

These velocities all play a critical role in the evolution of the solution and are distinct in the sense that, in most nonlinear problems, they are not simple multiples of each other. This is evident in the definitions of c and s', as well as from the expressions derived in Exercise 5.12. What we conclude from this is that this interesting problem is rich enough that a single velocity is not enough to describe the solution.

5.6.4 Expansion Fan

Now that we have some idea what happens when jumps occur in the solution, we will investigate a problem that starts out with a jump. The initial condition is shown in Figure 5.22, and it is given as

$$\rho(x,0) = \begin{cases} \rho_L & \text{if } x \leq 0, \\ \rho_R & \text{if } 0 < x, \end{cases} \tag{5.56}$$

where $0 < \rho_R < \rho_L$. This piecewise constant function gives rise to what is known as a Riemann problem. This problem is interesting because the solution is not obvious. In fact, it is so unclear that it is possible to produce a plausible argument for at least three different solutions. Before describing what these are we first state what we are certain of about the solution. This comes from the characteristics, and these are shown in Figure 5.23(a). As illustrated in Figure 5.23(b),(c), we conclude that $\rho = \rho_L$ for $x < c_L t$ and $\rho = \rho_R$ for $x > c_R t$. This leaves unresolved what the solution is for $c_L t < x < c_R t$ because there are no characteristics in this region. It is what happens in this sector that produces the three possible solutions.

1. The cars starting on the left, where $x < 0$, travel with velocity v_L while those on the right have velocity v_R. Because $v_L < v_R$ then one might argue, based on physical grounds, that the sector in question is nothing

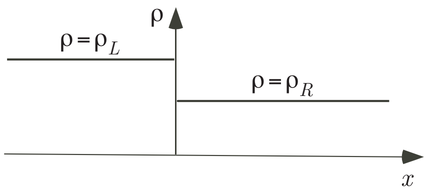

Figure 5.22 Initial density $\rho(x,0)$, where the slow cars start out behind the faster cars (i.e., $\rho_L > \rho_R$).

more than the gap between the slow cars on the left and the fast cars on the right. In other words, for points in this sector the density is just zero and the apparent solution is

$$\rho(x,t) = \begin{cases} \rho_L & \text{if} \quad x \le v_L t, \\ 0 & \text{if } v_L t < x < v_R t, \\ \rho_R & \text{if} \quad v_R t \le x. \end{cases} \tag{5.57}$$

The first indication that there is something wrong with this expression is that the sector is determined by the velocity of the cars, and not the wave velocity. This is a problem because $c(\rho_R) < v(\rho_R)$ and $c(\rho_L) < v(\rho_L)$, so the sector in (5.57) is different from the one shown in Figure 5.23. In other words, the above expression contradicts what we certain of, and that is the solution shown in Figure 5.23(c). Therefore, (5.57) is not the solution. Those who rely on physically motivated arguments to explain what is happening mathematically will almost certainly complain about this result. The reason is that the solution does not agree with what is expected

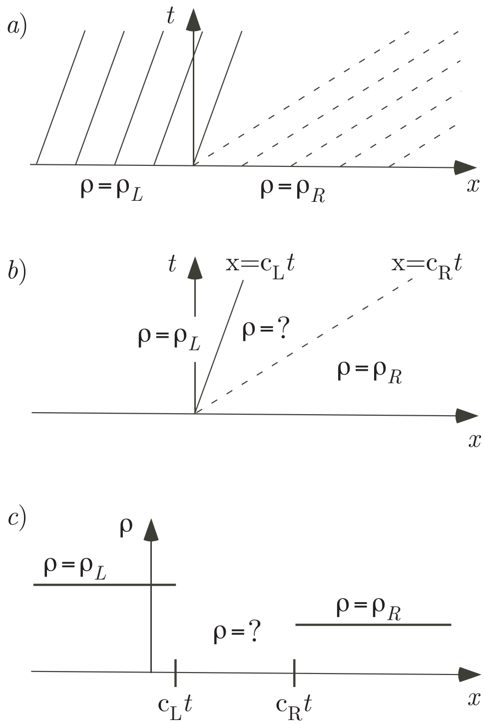

Figure 5.23 The solution obtained using the method of characteristics when the initial density is given in Figure 5.22. As shown in (a) and (b), there are no characteristics in the sector $c_L t < x < c_R t$, and so the solution in that region is unclear.

in the physical problem. More precisely, it does not agree with what might be expected based on a cursory analysis of the situation.

2. As another attempt at finding out what happens in the sector one might argue that the solution of the linear traffic flow equation (5.18), using the initial condition in (5.56), is a traveling wave with a single jump that moves with velocity a. Assuming the nonlinear equation also produces a single jump then the apparent solution is

$$\rho(x,t) = \begin{cases} \rho_L & \text{if } x \le s(t), \\ \rho_R & \text{if } s(t) < x. \end{cases} \tag{5.58}$$

The function $s(t)$ is determined from the Rankine-Hugoniot condition (5.53). Although it is not clear whether (5.58) is the solution, it has promise. For example, it is not hard to show that the line $x = s(t)$ is between $x = c_L t$ and $x = c_R t$. This means that (5.58) agrees with what we already know using characteristics, unlike what happened with (5.57). Moreover, in the special case of when c is constant, (5.58) reduces to the correct solution of the linear problem. These two observations are encouraging, but they do not guarantee that (5.58) is the solution of the Riemann problem we are studying.

3. A third attempt at finding the solution makes use of the modified red-light green-light problem shown in Figure 5.20. The solution of this modified problem should converge to the solution of our Riemann problem when $\epsilon \to 0$. This, in effect, takes the dashed characteristics in Figure 5.20 and pinches them together at the origin with the result shown in Figure 5.23. The radial characteristics form what is known as an *expansion fan*, or rarefaction wave, and it connects the constant states on the left and right. The formula for the solution, which is obtained from (5.48), is

$$\rho(x,t) = \begin{cases} \rho_L & \text{if } \quad x \le c_L t, \\ \rho_L + (\rho_R - \rho_L)\frac{x - c_L t}{(c_R - c_L)t} & \text{if } c_L t < x < c_R t, \\ \rho_R & \text{if } \quad c_R t \le x. \end{cases} \tag{5.59}$$

The resulting solution looks much like the one in Figure 5.20(c) in the sense that the expansion fan is responsible for a linear transition between the constant solutions on the left and right.

From the above discussion we have two contenders for the solution, namely (5.58) and (5.59). The fact that we have multiple possible solutions is because the nonlinear traffic flow problem is ill-posed, which in this case means that the problem is incomplete. What is required is an additional piece of information that will enable us to uniquely determine the solution. Moreover, it must be consistent with the physics of the problem. As an example, equations like the one we are dealing with arise in gas dynamics, and the approach used

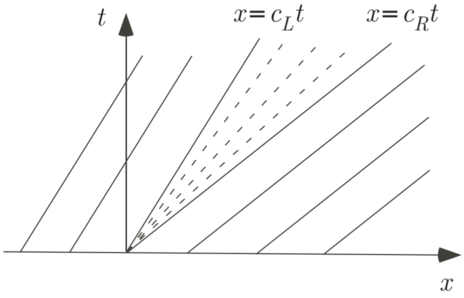

Figure 5.24 By letting $\epsilon \to 0$ the dashed characteristics in Figure 5.20 form an expansion fan between $x = c_L t$ and $x = c_R t$.

there is to introduce entropy and then employ the second law of thermodynamics to derive the needed condition. An effort has been made to define a concept similar to entropy for traffic flow, what is known as "driver's ride impulse," and then use a second law type of argument (Ansorge [1990]). We will take a different tack, and use a more mathematical argument.

The assumption is one of continuity. Namely, the jump appearing in the initial condition is almost impossible to produce physically, and in most experiments there is not a jump, but a small interval where the density changes in a rapid and continuous fashion from ρ_L to ρ_R. In this sense the initial condition containing a jump is simply a mathematical idealization of the true situation. Given that the solution with a continuous, but rapid, transition is known and given in Figure 5.20, the condition we are searching for must be consistent with this result. In other words, the condition must be able to tell us that (5.59) is the solution to this problem.

There are various ways to write the needed condition, and we will use the one introduced by Lax [1973]. The statement is that if the solution contains a jump, at $x = s(t)$, then the wave speed behind the jump is larger than the wave speed in front of it. In other words, the requirement is

$$c(\rho_R) < s' < c(\rho_L). \tag{5.60}$$

This is an example of what is known as an *admissibility condition*, because it provides the necessary information to determine the physically or mathematically admissible solution. In traffic flow it is often called the entropy condition, even though its connection to entropy is not at all clear for traffic problems.

One immediate consequence of the admissibility condition (5.60) is that the solution will only contain a jump if $c_L > c_R$. For our initial condition, given in (5.56), the assumption is that $c_L < c_R$. Therefore, a solution with a jump is not possible, and the solution in the region in question is an expansion

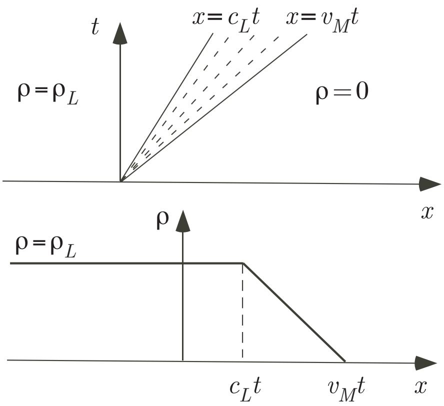

Figure 5.25 The upper plot shows the solution on the left and right, and the characteristics for the expansion fan. The lower plot shows the solution after the light turns green.

fan. In other words, (5.59) is the solution of the stated Riemann problem. The proof of this statement can be found in Lax [1973].

Example: Red Light - Green Light

Suppose a stoplight is located at $x = 0$, and it turns from red to green at $t = 0$. Also, assume that the light was red for so long that there are no cars on the right. In other words, the initial condition is

$$\rho(x, 0) = \begin{cases} \rho_L & \text{if } x \leq 0, \\ 0 & \text{if } 0 < x. \end{cases} \tag{5.61}$$

From (5.59), the solution of the traffic flow equation is

$$\rho(x, t) = \begin{cases} \rho_L & \text{if } \quad x \leq c_L t, \\ \rho_L \frac{v_M t - x}{(v_M - c_L)t} & \text{if } c_L t < x < v_M t, \\ 0 & \text{if } \quad v_M t \leq x. \end{cases} \tag{5.62}$$

The solution is shown in Figure 5.25, along with the associated characteristic curves. This shows that once the light turns green the cars move to the right, with the front moving at the maximum allowable velocity v_M. ■

The exact form of the expansion fan solution (5.59) relies on the specific formula for the wave velocity $c(\rho)$. In general, a fan appears when there is a gap between characteristics as shown in Figure 5.24. This occurs when $f(x)$ has a jump at a point $x = x_0$, with $c_R > c_L$ (see Figure 5.26). The equation for each of the dashed lines making up the fan has the form $x = x_0 + c(\rho)t$, where $c(\rho)$ satisfies $c_L < c < c_R$. There are a couple of methods that can be used to prove this, other than taking a limit as we did earlier, and one is explored in Exercise 5.20. To determine the density at a point (x, t) in the fan, it is necessary to solve the equation $c(\rho) = (x - x_0)/t$ for ρ. This is where the specific form of c affects the solution, and (5.59) is what is obtained when using the Greenshields law. Also, in formulating the nonlinear traffic flow equation in Section 5.6, we made the assumption that $c(\rho)$ is monotonic. This is one of the places where we need that assumption because it guarantees that $c(\rho) = (x - x_0)/t$ has a unique solution.

After reading the above paragraphs one might decide that the best thing to do is avoid using an initial condition with a jump. After all, when using the continuous function in (5.45) the characteristics worked without complication and there was no doubt about the solution. However, as we will see in the next section, this nonlinear equation can take a continuous initial condition and cause it to form jumps. So, even if we do not feed it jumps at the beginning it can easily grow its own and this means there is no avoiding having to consider an admissibility condition.

As a final comment, the admissibility condition that should be used in traffic flow is a topic that continues to receive attention in the research literature. One question is whether the entropy based conditions that are used in gas dynamics are applicable in traffic problems, particularly those that involve unusual flux functions. An example of an unusual function is one that is not convex. Those who are interested in investigating this topic should consult Ansorge [1990], Velan and Florian [2002], Gasser [2003], and Knowles [2008].

5.6.5 Shock Waves

As stated earlier, at a shock wave both the density and velocity are discontinuous. Calling the solution shown in Figure 5.21 a shock wave gives the impression the cars are running into each other. They are not, and what happens when the shock passes over a car is that it immediately undergoes

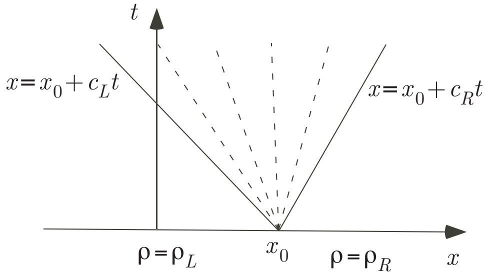

Figure 5.26 An expansion fan is generated at a point x_0 where the initial function $f(x)$ has a jump discontinuity, with $c_R > c_L$.

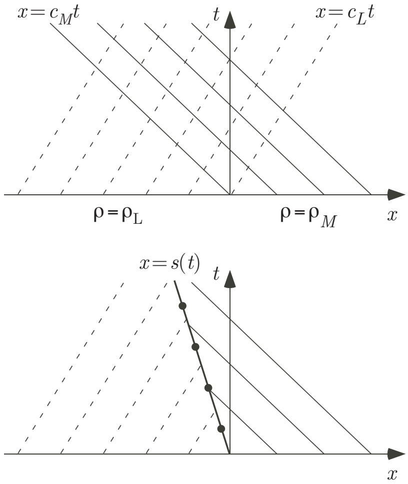

Figure 5.27 The traffic jam problem. The upper plot shows the characteristics associated with the initial condition. The lower plot shows the resulting shock location.

a jump in velocity. This is a bit unrealistic, and we will return to this point later.

Characteristics are used to determine when a shock wave is present in the solution. In contrast to an expansion fan, a shock appears when characteristics overlap, and the values on the characteristics are not equal. The easiest way to explain this is to work through a couple of examples.

Example: Traffic Jam

The first shock solution we will consider involves a traffic jam. Suppose that at the start, for $x \geq 0$, the density is ρ_M. This is the maximum density and means the cars can not move. For the interval $x < 0$ we will assume that the cars have density ρ_L, where $0 < \rho_L < \rho_M$. This means that the cars on the left move right with a constant velocity, in the direction of the traffic jam. Once they reach the jam the cars stop, and the result is that the traffic jam spreads leftward along the negative x-axis. To quantify these statements, the characteristics are shown in Figure 5.27. In the upper graph, along the solid lines the density is $\rho = \rho_M$ while along the dashed lines $\rho = \rho_L$. Clearly, there is a problem in the region where the characteristics overlap. The conclusion is that there is a curve $x = s(t)$ in this overlap region where the solution jumps from ρ_L to ρ_M. The resulting characteristics, and shock curve, are shown in the lower graph in Figure 5.27. The location of the shock, according to (5.53), moves with a velocity determined by an averaged value of the wave speed. Using the Greenshields law, the formula for the velocity is given in (5.55). Given that $c_L = v_M(1 - 2\rho_L/\rho_M)$ and $c_R = -v_M$ then

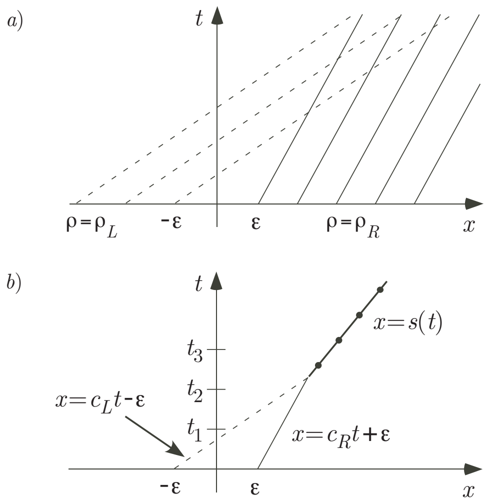

Figure 5.28 Overlapping characteristics are shown in (a), which indicates the existence of a shock wave in this region. The position of the shock is shown in (b), along with the two characteristics that intersect to initiate the formation of the shock at $t = t_s$.

$$s'(t) = -v_M \frac{\rho_L}{\rho_M} \,. \tag{5.63}$$

Integrating this, and using the fact that the shock starts at $(x, t) = (0, 0)$, we have that the position of the shock is given as

$$s(t) = -v_M \frac{\rho_L}{\rho_M} t. \tag{5.64}$$

With this the solution is

$$\rho(x,t) = \begin{cases} \rho_L & \text{if } x < s(t), \\ \rho_M & \text{if } s(t) \leq x. \end{cases} \tag{5.65}$$

As one final comment, it is important to point out that this solution satisfies the admissibility condition (5.60). This is because $c_L = v_M(1 - 2\rho_L/\rho_M)$, $c_R = -v_M$, and $\rho_L < \rho_M$. ■

Example: No Initial Jumps

As a second example suppose the density does not begin with a jump, but is continuous and has the form in (5.48). Now, however, we place the faster cars on the left so $\rho_L < \rho_R$. As usual, we will use the Greenshields law. The characteristics that are produced by these two constant values are shown in Figure 5.28(a). In the region covered by the dashed lines the solution is

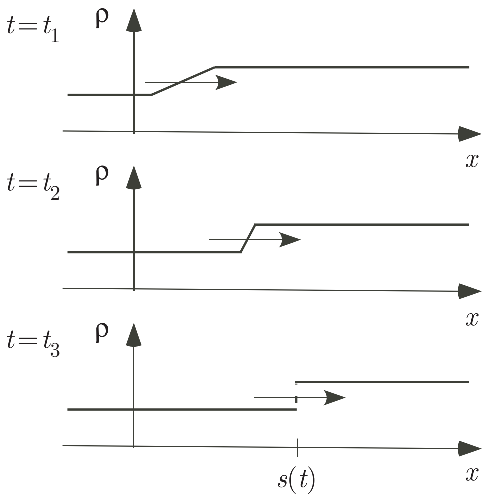

Figure 5.29 The solution of the traffic flow problem at the times shown in Figure 5.20(b). The width of the linear transition region between the left and right groups decreases with time until the left group catches the right group, and that time a shock wave appears.

$\rho = \rho_L$, while in the region covered by the solid lines the solution is $\rho = \rho_R$. The exception to this statement is where the dashed and solid lines overlap. In this region there is a shock wave, that begins where the characteristic $x = -\epsilon + c_L t$ intersects the characteristic $x = \epsilon + c_R t$. This intersection point is (x_s, t_s), where $t_s = 2\epsilon/(c_L - c_R)$ and $x_s = c_R t_s + \epsilon$, and the shock is shown in Figure 5.28(b). To determine the equation of this curve, we have from (5.55) that $s' = \frac{1}{2}(c_L + c_R)$. Integrating this equation yields

$$s(t) = c_s(t - t_s) + x_s, \tag{5.66}$$

where $c_s = \frac{1}{2}(c_L + c_R)$. It remains to determine the solution in the triangular region shown in Figure 5.28(b), which is bounded by the characteristics $x = -\epsilon + c_L t$ and $x = \epsilon + c_R t$. This is the same problem as finding the solution at (x_1, t_1) in Figure 5.20(b), and the solution is given in (5.47). Assembling all of this information, we therefore have that the solution for $t < t_s$ is

$$\rho(x,t) = \begin{cases} \rho_L & \text{if} \quad x \le c_L t - \epsilon, \\ \rho_L + \frac{\rho_R - \rho_L}{2\epsilon}(x + \epsilon) & \text{if } c_L t - \epsilon < x < c_R t + \epsilon, \\ \rho_R & \text{if} \quad c_R t + \epsilon \le x, \end{cases} \tag{5.67}$$

and for $t \ge t_s$ the solution is

$$\rho(x,t) = \begin{cases} \rho_L & \text{if } x < s(t), \\ \rho_R & \text{if } s(t) < x. \end{cases} \tag{5.68}$$

This solution is shown in Figure 5.29 for the three time values shown in Figure 5.20(b). At $t = t_1$ the solution consists of the two constant densities that are connected by a linear function. Because the cars on the left are faster than those on the right, at the later time $t = t_2$ the linear connection between the two densities has been reduced considerably. The effect of this transition region shrinking is to steepen the wave as it moves. The faster cars catch the slower ones in front, at $t = t_s$, at which point a shock forms. This is seen at time $t = t_3$, which shows the solution after the shock has formed. ■

The properties of the solution at a shock wave brings out one of the flaws in the traffic model. Specifically, as a shock passes over a car it immediately undergoes a jump in velocity. This is unrealistic and the reason it happens is that the model does not account for the momentum of the cars. Related to this is the assumption implicit in the constitutive law $v = F(\rho)$. For this to hold, the velocity must instantly adjust to the value of the density. This means that it is impossible to have the cars start from rest unless the density is at its maximum value of ρ_M. There are traffic models that account for the acceleration of the cars, and one is the cellular automata model studied later in the chapter. Also, in the next chapter we will significantly extend the continuum model in such a way that momentum is a central component of the model.

5.6.6 Return of Phantom Traffic Jams

The last example we will work out is the problem that introduced the phenomenon of a phantom traffic jam. The initial condition used here is

$$\rho(x,0) = \begin{cases} \rho_0 & \text{if} \quad x < a \\ \rho_a + \frac{\rho_b - \rho_a}{b-a}(x-a) & \text{if } a \le x \le b \\ \rho_0 & \text{if} \quad b < x, \end{cases} \tag{5.69}$$

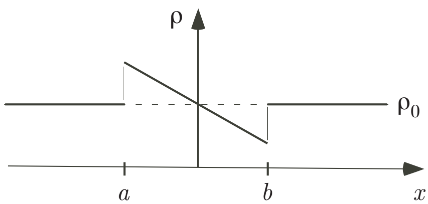

Figure 5.30 Disturbance imposed onto constant density solution at $t = 0$. The resulting initial condition is given in (5.69)

where $\rho_b < \rho_0 < \rho_a$. This function is shown in Figure 5.30. Although it is not continuous, this function is similar to the one in Figure 5.15, and is chosen to make the problem easier to solve. However, unlike what was done in Figure 5.15, we do not assume that the disturbance is necessarily small. Also, as usual, the Greenshields constitutive law is used.

To determine the solution, it is seen in Figure 5.30 that at each jump the faster cars are on the left. This means that a shock wave is going to be generated at each of these points. This is evident if one looks at the characteristics, which are shown in Figure 5.31(a). The characteristics that start to the left of $x = a$ have the form $x = x_0 + c_0 t$, where $c_0 = c(\rho_0)$. Similarly, the one that starts at $x = a$ is $x = a + c_a t$, where $c_a = c(\rho_a)$. Because $\rho_0 < \rho_a$ then $c_0 > c_a$. This means that the characteristic $x = a + c_a t$ is going to overlap with those on the left, as shown in Figure 5.31(a). A similar conclusion applies to the characteristics on the other end, where $x = b$.

The resulting shock waves are shown in Figure 5.31(b). The one on the left end is, from (5.55),

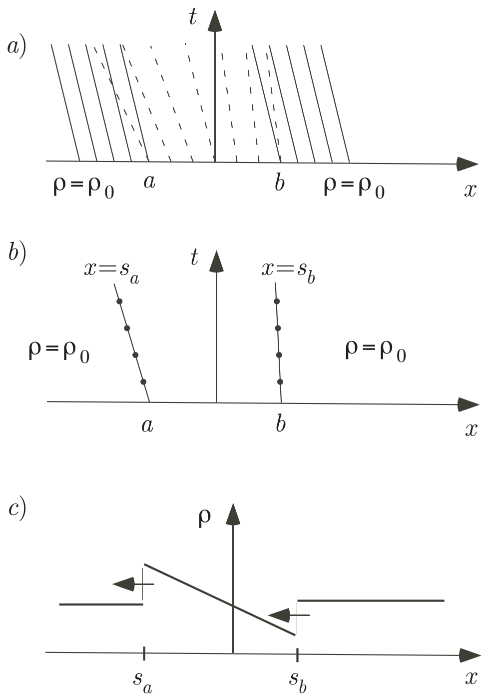

Figure 5.31 The solution of the phantom traffic jam problem, which uses (5.69) as the initial condition. The characteristics, and shock wave, have been drawn for the case of when $c < 0$.

$$s_a(t) = a + \frac{1}{2}(c_0 + c_a)t, \tag{5.70}$$

and the one on the right is

$$s_b(t) = b + \frac{1}{2}(c_0 + c_b)t. \tag{5.71}$$

Carrying out an analysis very similar to the one used for the modified red light - green light example of Section 5.6.2, one finds that the solution is linear in the interval $s_a \leq x \leq s_b$. The resulting solution is, therefore,

$$\rho(x,t) = \begin{cases} \rho_0 & \text{if} \quad x < s_a \\ \rho_a + (\rho_b - \rho_a)\frac{x - s_a}{s_b - s_a} & \text{if } s_a < x < s_b \\ \rho_0 & \text{if} \quad s_b < x. \end{cases} \tag{5.72}$$

This is shown in Figure 5.31(c), and because of its shape it is known as an N-wave. It has the properties mentioned earlier for a phantom traffic jam. Namely, a driver who comes in from the left will be happily driving on a road with a uniform density. When they reach the jam at $x = s_a$ they will have to immediately reduce their speed to adjust for the unexpected increase in the density. They will be able to gradually increase their speed, due to the decrease in the density over the interval $s_a < x < s_b$. However when they reach $x = s_b$ they will have gotten through the disturbance and will need to adjust their speed to match the uniform flow. This is basically the same conclusion reached with the small disturbance approximation in (5.36). What the approximation misses is the change in the width of the disturbance, which it states is constant. The solution in (5.72) shows that the width is $s_b - s_a = b - a + \frac{1}{2}(c_b - c_a)t$, which increases with time.

5.6.7 Summary

The results we have derived for the traffic flow problem are scattered through the preceeding pages, and it is worth finishing up this material by collecting them together. The problem consists of the first-order partial differential equation

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0, \tag{5.73}$$

with the initial condition

$$\rho(x,0) = f(x). \tag{5.74}$$

Assuming $c(\rho)$ is a smooth function, with $c'(\rho) \neq 0$, for $0 \leq \rho \leq \rho_M$, then the solution is constructed using the following information.

a) The solution is constant along the characteristic curves $x = x_0 + c_0 t$ (see Figure 5.18).

b) Characteristics Overlap. In a region containing overlapping characteristic curves the solution contains a shock wave at $x = s(t)$. The velocity of this wave is

$$s'(t) = \frac{\rho_R v_R - \rho_L v_L}{\rho_R - \rho_L} \,. \tag{5.75}$$

On either side of the shock, the respective characteristics determine the solution (as illustrated in Figure 5.27). As an example, if $f(x)$ is piecewise constant with a jump discontinuity at x_0, with $c_R < c_L$, then the solution starts out with a shock wave of the form $x = x_0 + s_0' t$, where s_0' is determined from (5.75).

c) Characteristics Separate. In a region with no characteristics, the solution is an expansion fan. An example is shown in Figure 5.26, where $f(x)$ has a jump discontinuity at x_0, with $c_R > c_L$. In this case, in the region $c_L t < x < c_R t$ the solution is found by solving $c(\rho) = (x - x_0)/t$.

The above conclusions are general in the sense that they apply to traffic flow, where $c' < 0$, but also to the case of when $c' > 0$. The latter occurs, for example, for gas flow, and is the subject of Exercises 5.13 and 5.14.

5.7 Cellular Automata Modeling

The viewpoint of the continuum model derived in Section 5.3 is that the motion of the individual cars can be approximated using an averaging process, giving rise to the density and flux functions. It is interesting to explore how to retain the individuality of the cars, and one approach incorporates ideas from cellular automata. The first step in constructing the model is to divide the road into equal segments, each with length ℓ as shown in Figure 5.32. Conventionally, this distance is taken to be the length of an average car, or vehicle, on the road. Time is also divided into equal segments, producing a time step Δt. The objective of the model is, given the positions of the cars at time t_{old}, to determine their positions at $t_{new} = t_{old} + \Delta t$. With this in mind we introduce an integer variable m that equals the number of road segments the car moves in a time step. For example, $m = 1$ means the car moves one segment, $m = 2$ means it moves two segments, etc. It is assumed that there is a maximum number of segments M that a car is allowed to move in a time step. This is equivalent to assuming there is a maximum velocity v_M on the highway. Given that a car's velocity in this formulation is $v = m\ell/\Delta t$, then $v_M = M\ell/\Delta t$.

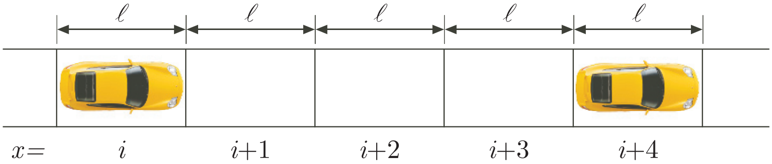

Figure 5.32 In traffic cellular automaton models the roadway is divided into equal segments, and the segments are numbered. For the car on the left, $x = i$ and its gap, which is the number of empty segments in front of it, is $g = 3$.

Example

Taking $\ell = 16$ ft (4.9 m) and $\Delta t = 1$ sec then $m = 1$ corresponds to a velocity of 10.9 mph (17.5 kph), while $m = 6$ corresponds to a velocity of 65.4 mph (105.2 kph). ■

In the model, each car has three integers associated with it, and they are (x, m, g). Here x is the position of the car and its value is determined by the road segment currently occupied by the car. The integer m was defined earlier, and g is called the gap and it is the number of spaces between the car and the one in front of it. For example, for the car on the left in Figure 5.32, $x = i$ and $g = 3$, while for the car on the right $x = i + 4$.

The basic idea in the model is that at time t_{old} we know the values of $(x_{old}, m_{old}, g_{old})$ for each car, and what the model does is to determine their values $(x_{new}, m_{new}, g_{new})$ at time $t_{new} = t_{old} + \Delta t$. This is done by applying the following four rules to each car on the road:

1. *Speedup.*
   If $m_{old} \neq M$, then $m_{new} = m_{old} + 1$.
2. *Do Not Overrun.*
   If $m_{new} > g_{old}$ then $m_{new} = g_{old}$.
3. *Randomization.*
   If $m_{new} \neq 0$ then, with probability p, take $m_{new} = m_{new} - 1$.
4. *Move the Car.*
   Take $x_{new} = x_{old} + m_{new}$.

These four steps constitute what is known as the stochastic traffic cellular automaton (SCTA) model.

The first three steps of the SCTA model contain assumptions, and potential modifications, that need to be discussed.

Speedup. It is assumed that a driver will attempt to drive at the maximum allowed velocity. Assuming the car is not moving at the maximum velocity then in this step the number of segments a car moves is increased by one. It is certainly possible to consider what happens if there are larger accelerations, and increase the movement by two or more segments, but this will not be investigated here.

Do Not Overrun. The idea here is that if there are, say, three empty spaces in front of the car then it cannot move any more than three spaces in the time step. One can argue that the car in front will likely move in the time step and therefore there will be more than three available spaces. This is correct but it is not accounted for in the model.

Randomization. The previous two steps are intuitive, but not so with the randomization. Numerous reasons have been given to justify this assumption, and this includes the statements that it mimics delayed acceleration or that it accounts for an overreaction in braking. These explanations are rather vague, so instead we will concentrate on what effect the probability p has on the motion. If p is close to zero then it is unlikely the velocity is reduced and there will be little noticeable affect on the car's movement. This is not the case for larger values of p. Any time a car slows down there is a potential to effect the motion of those who are following, and the more often this happens the greater the affect on the flow. The extreme case of when $p = 1$ is examined in Exercise 5.28.

Given the recursive nature of how the four rules are used, it is difficult to determine exactly what happens using analytical methods. The approach, therefore, is to use computer simulations and this brings us to the next example.

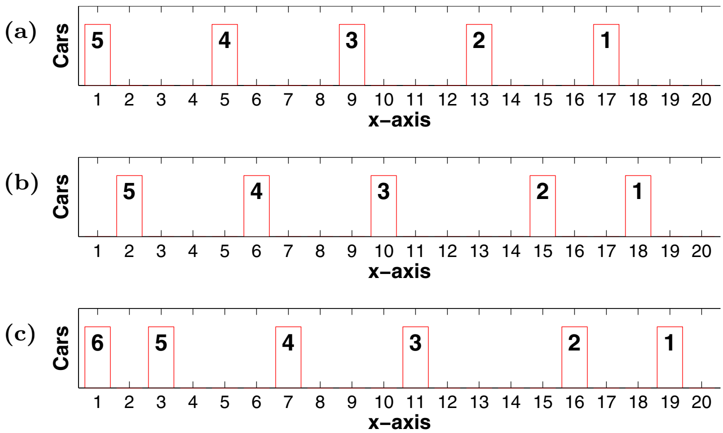

Figure 5.33 In (a) five cars are placed uniformly along a roadway at $t = 0$. Their positions calculated using the SCTA model are shown in (b) at $t = \Delta t$, and in (c) at $t = 2\Delta t$.

Example

Suppose the cars start out uniformly distributed along the highway, where the gap g is the same for all cars. This is shown in Figure 5.33(a), with $n = 20$ and $g = 3$. The lower two graphs are the positions of the cars at the first two time steps, assuming $p = 9/10$, $m = 1$, and $M = 2$. It is seen that after the first time step all cars have moved one space to the right, except for the second that has moved two spaces. To explain this, the speedup step sets $m = 2$ and this means the cars move forward two spaces. However, because of the large value of p it is likely that the velocity of each car in the randomization step is reduced to $m = 1$, and this is borne out in the plot. This one space movement is also seen at the next time step, given in the lower plot. What is new at this time step is the appearance of a sixth car on the left. This is not from the model but, rather, something that is included in the computer code. Specifically, if the first space is empty then the computer adds in a car at this location with a probability equal to the original uniform distribution, namely with probability $1/(g + 1)$. The computer code also removes cars on the right if they pass $n = 20$, although none of the cars in Figure 5.33 have traveled far enough for this to happen. ■

It is possible to take the computer code that produced Figure 5.33 and calculate car positions using a large number of road segments and time steps. Although this generates interesting pictures, little is learned in the process. It is better to study specific situations and compare the results with what is expected on a real roadway. For this we turn to the red light-green light and the green light-red light examples introduced earlier for the continuum model.

Example: Red Light - Green Light

For this the road is divided into 1000 segments, and the stoplight is located at $x = 333$. It is assumed the cars are bumper-to-bumper to the left of the light. This is shown in the lower plot of Figure 5.34(a), where the solid black block on the left are the 333 cars waiting for the light to turn green. Also shown in this plot is the corresponding density, using the continuum model. The solution after a few time steps, for both the SCTA and the continuum models, is shown in Figure 5.34(b). In the calculation $M = 2$ and $p = 1/10$. Both are behaving as expected. In the SCTA model the cars on the right have pulled ahead while those in the block on the left are waiting for room to open up so they can move. For the continuum model we have a expansion fan. From (5.59), in the case of when the light is located at $x = \bar{x}$, the solution is

$$\rho(x,t) = \begin{cases} \rho_L & \text{if} \quad x \le \bar{x} - v_M t, \\ \rho_L \frac{v_M t + \bar{x} - x}{2 v_M t} & \text{if} \quad \bar{x} - v_M t < x < \bar{x} + v_M t, \\ 0 & \text{if} \quad \bar{x} + v_M t \le x, \end{cases} \tag{5.76}$$

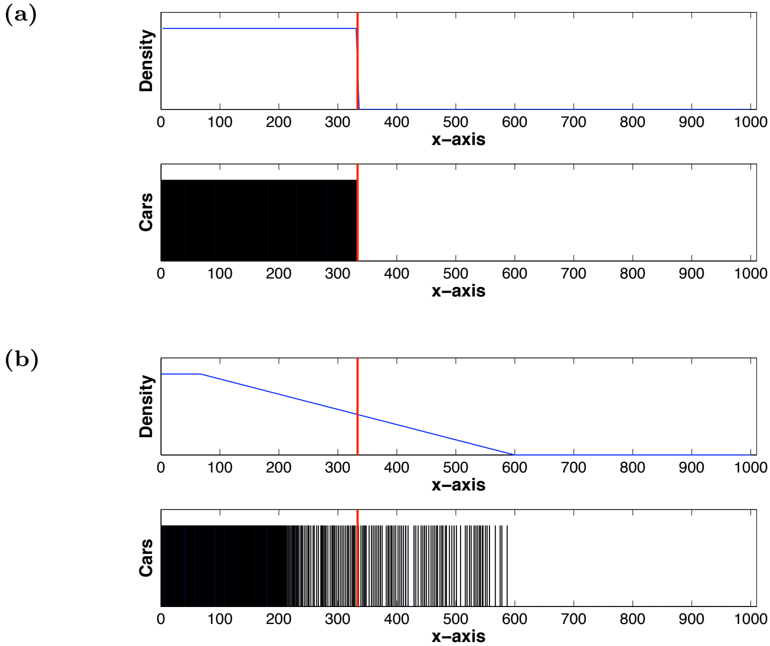

Figure 5.34 Solution of the red light-green light problem. Show are (a) the density and positions of the cars at $t = 0$, and (b) the density and positions after several time steps. The density is computed using (5.76) and the car positions are determined using the SCTA model.

where $\rho_L = 1/\ell$. The linear transition between $\rho = \rho_L$ and $\rho = 0$ is seen in the upper plot of Figure 5.34(b). We are now able to ask the big question, namely, how do the two models compare? To investigate this note that for the SCTA model the car in the front, after N time steps, can move no farther than MN spaces. The actual number will be smaller, depending on how many time steps it takes to accelerate to velocity M and the always present reduction of the velocity due to the randomization step. If p is close to zero, then the first car moves approximately $N_M = M[N - \frac{1}{2}(M-1)]$ spaces, where the $\frac{1}{2}M(M-1)$ term is due to the number of steps it takes the car to reach speed M. In Figure 5.34(b) this car is located at about $x = 600$. The leftmost car that is able to move, for p close to zero, is located approximately N spaces to the left of the light, which in Figure 5.34(b) is near $x = 200$. At the other extreme, the closer p gets to one the closer the number of spaces on either the left or right gets to zero. In this discussion we will assume a linear approximation in p. Therefore, the approximate spatial interval involving the cars that are in motion after N time steps is

$$\bar{x}^* - N\ell(1-p) \leq x^* \leq \bar{x}^* + N_M\ell(1-p). \tag{5.77}$$

In comparison, for the continuum model the interval is

$$\bar{x}^* - NM\Delta t \leq x^* \leq \bar{x}^* + NM\Delta t. \tag{5.78}$$

This result brings out one clear difference between the two models. Specifically, the continuum model predicts the interval is symmetric about the light's position, while for the SCTA model the interval is generally nonsymmetric. This is seen in Figure 5.34(b). The interval is symmetric in the continuum model because of the constitutive law. The speed the interval expands is determined from the slope of the flux function and, as seen in Figure 5.16, this function is symmetric. In real life, as indicated in Figure 5.8, the flux is not symmetric and this means if we were to use a more realistic constitutive law for v then we would obtain a nonsymmetric interval. Can we find a constitutive law that produces the same result as the SCTA model? Well, this rather interesting question will be left for you to think about. ■

Example: Green Light - Red Light

For this example the road is again divided into 1000 segments, with a stoplight located at $x = 1000$. It is assumed that the cars are uniformly spaced with three spaces between them, and each starts out at the maximum velocity $v_M = 2$. The randomization probability in this example is $p = \frac{1}{4}$. This is shown in Figure 5.35(a). Also shown in this plot is the corresponding density $\rho = 1/(4\ell)$ for the continuum model. The solution after a few time steps, for both the SCTA and the continuum models, is shown in Figure 5.35(b). Both are behaving as expected. In the SCTA model the cars coming in from the left stop when they arrive at the traffic jam, which in Figure 5.35(b) is located near $x = 800$. For the continuum model we have a shock wave that moves leftward. Because $\rho_L = 1/(4\ell)$ and $\rho_R = 1/\ell$ then, after N time steps, the shock is located at $s = \bar{x} - MN\Delta t/4$, where $\bar{x}$ is the location of the stoplight. In Figure 5.35(b), where $N = 450$, we have that $s = 780$.

One of the more obvious differences in the two models, when looking at Figure 5.35(b), is the lack of uniformity in the density to the left of the traffic jam in the SCTA description. This is not unexpected and is due to the randomization step. The second difference is the location of the traffic jam. It is difficult to predict where the jam is located using the SCTA model because there is no simple formula as there is in the continuum case. What is interesting in Figure 5.35(b) is that it appears that in the SCTA model the jam affects the motion of the cars before they reach the jam. Specifically, there is an increase in the density as the cars approach the jam. This can be explained by the fact that any time a car slows down this information is sent backwards along the road (Step 2). Therefore, when a car slows down as it arrives at the jam this affects the cars that follow. This is not in the continuum model and consequently represents a fundamental difference between the two descriptions. ■

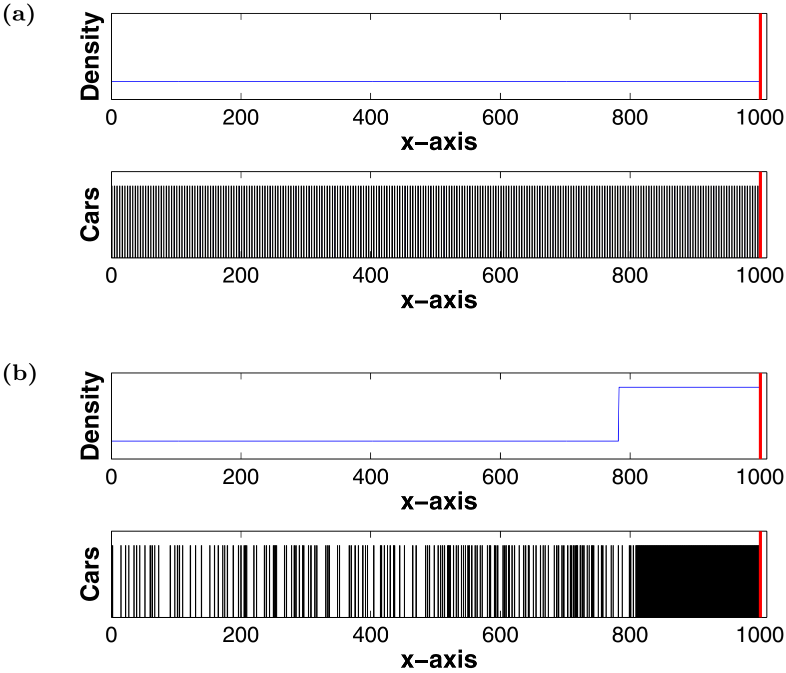

Figure 5.35 Solution of the green light-red light problem. Shown are (a) the density and positions of the cars at $t = 0$, and (b) the density and positions after several time steps. The density is computed using (5.65) and the car positions are determined using the SCTA model.

Exercises

5.1. This problem considers various consequences of the traffic flow equation.

(a) Show that given any two points a and b on the x-axis, with $a < b$,

$$\frac{d}{dt}\int_a^b \rho(x,t)dx = J(\rho_a) - J(\rho_b),$$

where $\rho_a = \rho(a,t)$ and $\rho_b = \rho(b,t)$. Interpret the above equations in physical terms.

(b) Show that

$$\rho(x,t) = f(x) - \frac{\partial}{\partial x}\int_0^t J(x,z)dz.$$

5.2. Consider the situation of when two lanes of traffic merge down to one lane, as shown in Figure 5.36. The densities and velocites at the far left and right are known. Assume a steady flow, so the density and velocity do not depend on time, and assume all variables are non-negative.

(a) Using the result of Exercise 5.1(a), find an equation that relates the values on the right with those on the left.
(b) What does the equation in part (a) reduce to if the Greenshields law is used?
(c) Suppose $\rho_2 = \rho_1$, $v_2 = v_1$, and the Greenshields law is used. Find ρ_3 in terms of ρ_1. Your solution should give $\rho_3 = 0$ if $\rho_1 = 0$. With this, describe what happens to the flow of cars on the right as ρ_1 is increased, starting from $\rho_1 = 0$. Make sure to explain what happens as ρ_1 nears $\frac{1}{4}(2-\sqrt{2})\rho_M$.

5.3. There are various recommendations concerning safe following distances for cars. Below are a few of the more commonly sited rules. Find the resulting constitutive law relating density and velocity if you assume the cars are uniformly spaced according to the given rule. The function $F(\rho)$ must be continuous, and if you need to make additional assumptions to derive the requested constitutive law make sure to state what they are. Finally, determine which, if any, of the three requirements NV1-NV3, given in Sections 5.4.3, 5.4.4 the constitutive law satisfies.

(a) The National Safety Council recommends the 3-second rule. This means that you allow at least 3 seconds between you and the vehicle in front of you.
(b) In the early days of motoring, it was recommended that you keep one car length back (about 20 feet) for each ten miles per hour of speed.
(c) According to an insurance company, you should allow at least 4 seconds between you and the vehicle in front of you, but if traveling more than 50 mph that this time interval should be at least 6 seconds.
(d) According to a motoring society, the minimum safe distance to the car in front is made up of the sum of two terms. One accounts for the distance traveled due to reaction time, which is usually assumed to be 0.7 seconds. The second term is calculated assuming a constant deceleration, and it accounts for the distance the car will travel after the brakes are applied.

5.4. Apparently drivers do not follow the advice given out by insurance companies or motor clubs, and the claim is that they prefer to select their speed according to the rule

$$\rho = \frac{1}{\alpha + \beta v + \gamma v^2} ,$$

where α, β, γ are positive constants (Zhou and Peng [2005]).

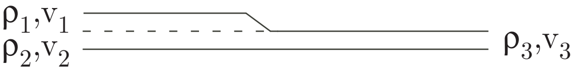

Figure 5.36 Configuration of roadway used for Problem 5.2.

(a) What are the units of α, β, γ?
(b) What is the resulting constitutive law for the velocity? Which of the three conditions on the constitutive law listed in Sections 5.4.3, 5.4.4 does this expression satisfy?
(c) It was found experimentally that in many cases γ is negative. This happens because γ is close to zero, and small variations in the data can cause the curve fitting program to produce a negative value. How does this affect your conclusions in part (b)?

5.5. Assume the flux J is given in Figure 5.37.
(a) Solve the traffic flow equation in the case of when $\rho(x,0) = 4$.
(b) Solve the traffic flow equation when $\rho(x,0) = (5 + x^2)/(1 + x^2)$.
(c) Using the information in the graph, find the velocity in terms of the density.

5.6. In the traffic flow problem suppose the velocity of cars, as a function of the density, is measured on a highway and the data shown in Figure 5.38 are obtained.
(a) Formulate a constitutive law for v as a function of ρ based on these data. Provide an explanation of how you reach your conclusion.
(b) In the traffic model it is assumed that $c'(\rho) \neq 0$ for $0 \leq \rho \leq \rho_M$. Does your constitutive law satisfy this condition?

5.7. This problem explores some of the consequences of the Greenshields model as identified in a typical traffic engineering manual.
(a) Sketch the flux as a function of density. At what density is the flux a maximum?
(b) The constant ρ_M is called the jam density, v_M is called the free-flow velocity, and $\frac{1}{4}v_M\rho_M$ is the capacity. Explain why they are given these names.
(c) The headway is defined as the time interval between a common point on the vehicles (e.g., the front bumper) passing a fixed point in space. How is this related to the flux or velocity?
(d) In the example of Section 5.2 for uniform cars the maximum merge density ρ_{merge} was calculated. Use this and the data in Figure 5.6 to find an approximate value for the maximum merge velocity v_{merge}, which is the velocity corresponding to the maximum merge density.

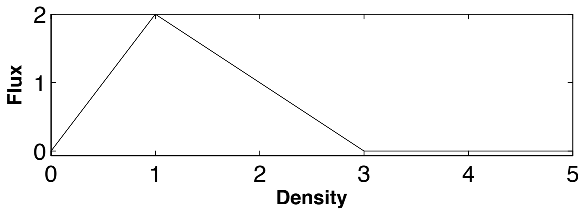

Figure 5.37 Flux-density data used in Exercise 5.5.

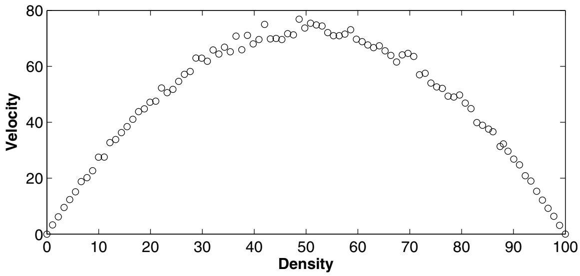

Figure 5.38 Data used for Problem 5.6.

5.8. Solve the following problems by extending the method that was used in Section 5.5 to solve the advection equation.

(a)
$$\frac{\partial \rho}{\partial t} + 2\frac{\partial \rho}{\partial x} = 1,$$
where $\rho(x,0) = f(x)$.

(b)
$$\frac{\partial \rho}{\partial t} - 6\frac{\partial \rho}{\partial x} = \rho,$$
where $\rho(x,0) = f(x)$.

(c)
$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} = \rho^2,$$
where $\rho(x,0) = f(x)$.

(d)
$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} = x, \text{ for } x > 0, t > 0,$$
where $\rho(x,0) = 0$ for $x > 0$, and $\rho(0,t) = 0$ for $t > 0$.

5.9. This problem explores the finite highway problem associated with Figure 5.14.

(a) Suppose it is found that $\rho(\ell, t) = 2$. What is $f(x)$ and what is $g(t)$?
(b) Suppose it is found that $\rho(\ell, t) = e^{-t}$. What is $f(x)$ and what is $g(t)$?

5.10. This problem considers possible solutions of the traffic flow equation when using the Greenshields law. Assume that $\rho(x,0) = f(x)$, where $f(x)$ is piecewise constant. Also, make sure to justify your answers.

(a) Give an example of $f(x)$ that produces a solution with two expansion fans and no shock waves.
(b) Give an example of $f(x)$ that produces a solution that starts out with two shock waves and no expansion fans.
(c) Give an example of $f(x)$ that produces a solution that starts out with one shock wave and one expansion fan.

(d) In part (a), explain why the two expansion fans can not overlap.
(e) In part (b), the two shock waves eventually intersect. Explain why this is expected based on the way the cars are positioned at the very start. When the shocks intersect, a single shock is formed. Find the resulting solution.

5.11. This problem considers what values a solution of the traffic flow equation can have when using the Greenshields law, with $v_M = 1$ and $\rho_M = 10$. Assume that $\rho(x,0) = f(x)$ is piecewise constant.
(a) Assuming that $3 \leq f(x) \leq 4$, explain why it is impossible for $\rho(x,t) = 5$ at any value of (x,t). Is it possible for $\rho(x,t) = 2$?
(b) Suppose that $f(x)$ is piecewise constant, and only takes on the values 1 and 3. Give an example to show that it is possible for $\rho(x,t) = 2$ for one or more points (x,t). For your example, what other values does the solution take on?
(c) Give an example for $f(x)$, so that the only values ρ takes on are 1, 2, and 3.

5.12. This problem explores some of the connections between the velocity functions that arise with nonlinear traffic flow.
(a) Show that

$$v = \frac{1}{\rho} \int_0^\rho c(\bar{\rho}) d\bar{\rho}.$$

(b) Show that

$$s'(t) = \frac{\rho_R v_R - \rho_L v_L}{\rho_R - \rho_L} .$$

(c) Show that if v is a monotonically decreasing function of ρ then $c \leq v$.
(d) Give an example to show that to have c monotonically decreasing, it is not enough to assume v is monotonically decreasing.
(e) Is it possible for a shock wave to stay in one place? You can assume the Greenshields law is used.
(f) Is it possible for the wave velocity c to be independent of ρ without assuming the car velocity v is independent of ρ?

5.13. In fluid dynamics one solves the nonlinear equation (5.30), but the wave velocity is $c(\rho) = \rho$. Using this function, assume that the initial condition is

$$\rho(x,0) = \begin{cases} 1 & \text{if } x \leq -1 \\ \frac{1}{2}(1-x) & \text{if } -1 \leq x \leq 1 \\ 0 & \text{if } 1 \leq x. \end{cases}$$

(a) Sketch the characteristics in the x,t-plane.
(b) Find the solution, and sketch it as a function of x, for $t > 0$.
(c) Show that $v = \frac{1}{2}\rho$.

5.14. As in the previous problem, suppose $c(\rho) = \rho$ but now the initial condition is

$$\rho(x,0) = \begin{cases} 0 & \text{if } x \leq -1 \\ \frac{1}{2}(1+x) & \text{if } -1 \leq x \leq 1 \\ 1 & \text{if } 1 \leq x. \end{cases}$$

(a) Sketch the characteristics in the x,t-plane.
(b) Find the solution, and sketch it as a function of x, for $t > 0$.
(c) Find the points in the x,t-plane where $\rho = \frac{1}{3}$.
(d) Show that $v = \frac{1}{2}\rho$. With this, determine the flux J.

5.15. This problem examines what happens on a finite length highway when the velocity is not constant. The equation is

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0, \quad \text{for} \quad \begin{cases} 0 < x < \ell \\ 0 < t, \end{cases}$$

where

$$\rho(x,0) = f(x), \quad \text{for } 0 \leq x \leq \ell,$$

and

$$\rho(0,t) = g(t), \quad \text{for } 0 < t.$$

Assume the Greenshields law is used, and so the function c is given in (5.38).
(a) Assuming that $2\rho_R < \rho_M$, find the solution when $f(x) = \rho_R(1 - x/\ell)$ and $g(t) = \rho_R$.
(b) For part (a), explain why there is no solution if $2\rho_R > \rho_M$. Which condition, the one at $t = 0$ or the one at $x = 0$, should be dropped so there is a solution?
(c) Find the solution when $f(x) = 0$ and $g(t) = \rho_L$.
(d) Find the solution when $f(x) = \rho_R$ and $g(t) = 0$.

5.16. To investigate just how much influence the constitutive law has on the solution, suppose it is assumed that $v = v_M((1 - (\rho/\rho_M)^2)$. This is a special case of what is known as Drew's constitutive law (Drew [1968]).
(a) What is the wave velocity $c(\rho)$? Is it a monotonic function of ρ?
(b) What is the solution of the modified red light - green light problem? As in Figure 5.20, sketch the solution as a function of x and comment on how the solution differs from the one in Figure 5.20. You can assume in this problem that $\rho_R = 0$.
(c) What is the solution of the red light - green light problem, where $\rho_R = 0$? As in Figure 5.25, sketch the solution as a function of x. Also, comment on how the solution differs from the one in Figure 5.25.
(d) What is the solution of the traffic jam problem? As in Figure 5.27, sketch the solution as a function of x. Also, compare the velocity of the shock with the value obtained using the Greenshields law.

5.17. One way to explain weak solutions is to consider a smooth version of the jump initial condition. Specifically, let $\rho(x,0) = 1/(1 + \alpha e^{x/\epsilon})$, where ϵ and α are positive. Also, this problem considers the linear equation, so $v = a$ is constant.

(a) Find the solution using the above initial condition. Explain why the resulting solution is smooth and sketch it assuming that ϵ is small.
(b) Explain what happens both to the initial condition and solution when $\epsilon \to 0$. Make sure to explain what happens if $x = at$.
(c) Part (b) helps explain why using a jump initial condition is consistent with what is known for smooth solutions, with the exception of what happens at the jump itself. This raises the question of what value the density can have at a jump. Given the definition of the density in (5.1), what should the value of the density be at $x = 0$ and at $x = 1$ in Figure 5.9 when $t = 0$?
(d) Show that the limiting value you found in part (b), when $x = at$, can be obtained for the solution in Figure 5.9 by modifying the averaging interval in (5.1). What this shows is that in a continuum theory, the value at a discontinuity is dependent on the averaging method.

5.18. Suppose one is interested in knowing the position of a particular car when using the continuum model. Assume that at $t = 0$ the car is located at $x = A$, and its position at later times is given by $x = \chi(t)$. This problem is concerned with how to find the function $\chi(t)$. In doing this it is assumed that the traffic flow equation has been solved, so the density $\rho(x, t)$ and the velocity $v(x, t)$ functions are known.
(a) Explain why, to find $\chi(t)$, one solves the differential equation $\chi' = v(\chi, t)$, where $\chi(0) = A$.
(b) For the red light-green light solution given in (5.62), assume $\rho_L = \rho_M$ and $A < 0$. What is the resulting velocity function? With this show that

$$\chi(t) = \begin{cases} A & \text{if } 0 \le t \le -A/v_M \\ v_M t - 2\sqrt{-A v_M t} & \text{if } -A/v_M < t. \end{cases}$$

On the same axes, sketch $\chi(t)$ for $A = -v_M$, $A = -2v_M$, and $A = -3v_M$.
(c) For the red light-green light problem, which cars are able to get through the light if it is green for $0 \le t < t_R$ and turns red at $t = t_R$?
(d) For the traffic jam example studied in Section 5.6.5, find $\chi(t)$ for $A < 0$. On the same axes, sketch $\chi(t)$ for $A = -v_M$, $A = -2v_M$, and $A = -3v_M$.

5.19. It is observed that when a stoplight turns green, the density of traffic passing through the light increases in time up to a constant value ρ_0. Assuming the light is located at $x = 0$, a boundary condition that mimics this observed behavior is

$$\rho(0, t) = \begin{cases} \rho_0 t/t_s & \text{if } 0 \le t \le t_s \\ \rho_0 & \text{if } t > t_s. \end{cases}$$

The domain over which the traffic flow problem is solved is $0 < x$ and $0 < t$. Assume here that $\rho(x, 0) = 0$ and the Greenshields constitutive law is used.
(a) In the case of when $\rho_0 = \frac{1}{3}\rho_M$ find, and then sketch, the solution.
(b) Suppose that $\rho_0 = \frac{2}{3}\rho_M$. Sketch the characteristics, and use this to explain why there is no solution. In fact, explain why there is no solution for any density that satisfies $\rho_0 > \frac{1}{2}\rho_M$.

5.20. This problem investigates how to use similarity variables to find an expansion fan.

(a) Assume $\rho(x,t) = R(\eta)$, where $\eta = x/t$. Show that the traffic flow equation reduces, in the case of when ρ is not constant, to the equation $c(\rho) = x/t$.

(b) Using the Greenshields law, solve $c(\rho) = x/t$ for ρ.

(c) Show that your solution in part (b) is the same as the one given in (5.59).

5.21. This problem examines what happens to the traffic flow problem when cars are allowed to enter or exit the highway. It is assumed this occurs not at discrete locations but continuously along the highway.

(a) Assume that over an interval $x_0 - \Delta x < x < x_0 + \Delta x$ the number that enter (or exit) from $t = t_0 - \Delta t$ to $t = t_0 + \Delta t$ is $4\Delta x \Delta t Q$, where $Q(x,t)$ is the net rate per unit length at which cars are entering or leaving the highway. Show that the resulting balance law for traffic flow is

$$\frac{\partial \rho}{\partial t} = -\frac{\partial J}{\partial x} + Q.$$

(b) One possible constitutive law for this new variable is $Q = \alpha(\rho - \beta)$ where α, β are constants. Can you explain how this assumption could arise for traffic flow? Is there any reason you should assume α is either positive or negative? Any suggestion on how to choose β?

(c) Use the procedure to solve the $\alpha = 0$ case to solve the equation derived in part (a) along with the constitutive assumption in part (b). Assume a constant velocity.

(d) Based on your solution from part (c), what is the effect of Q on the density? Is the solution still a traveling wave? Demonstrate your conclusion using the initial distribution $\rho(x,0) = e^{-x^2}$ by sketching the solution at later times.

5.22. Suppose you had an experimental apparatus that enabled you to measure the velocity of a shock wave. Explain how you could use this to determine a constitutive law for the velocity.

5.23. A variable related to density is the volume fraction $\phi(x,t)$, which is used to determine how much of the highway is taken up by cars (versus empty road). In reference to Figure 5.4,

$$\phi(x_0,t_0) \approx \frac{\text{total length of cars from } x_0 - \Delta x \text{ to } x_0 + \Delta x \text{ at } t = t_0}{2\Delta x}.$$

The value of $\phi(x_0,t_0)$ is the limit of the right-hand side as $\Delta x \to 0$.

(a) For evenly spaced cars as in Example 5.2, show that $\phi(x,t) = \ell/(\ell + d)$, and therefore $\phi = \ell\rho$.

(b) If the cars are not necessarily evenly spaced but still are all of length ℓ show that it is still true that $\phi = \ell\rho$.

(c) Assuming $\phi = \ell\rho$, where ℓ is constant, rewrite the traffic flow equation (5.30) in terms of $\phi(x,t)$.

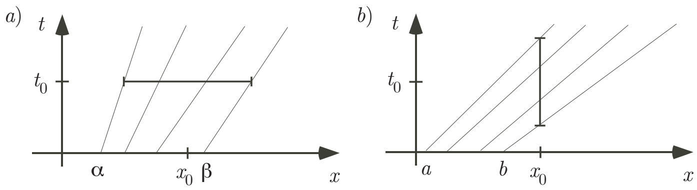

Figure 5.39 Averaging intervals used to define (a) the density, and (b) the flux. The horizontal bar in (a) has length $2\Delta x$, and the vertical bar in (b) has length $2\Delta t$. The four slanted lines shown in each figure are the paths of individual cars. These figures are used in Problem 5.26.

5.24. Suppose the density is given in terms of the velocity, and so, assume $\rho = H(v)$.
(a) Show how the traffic flow equation can be written as $v_t + d(v)v_x = 0$.
(b) Find H for the Greenshields (5.10) and Newell (5.17) functions.
(c) The initial condition used for the small disturbance approximation is $v(x,0) = v_0 + \epsilon h(x)$. Find the resulting two term expansion for the velocity.

5.25. One might argue that if a driver is in a relatively high-density region and sees lower density traffic up ahead that they will speed up with the objective of traveling in the lower-density region.
(a) Explain why an assumption that accounts for this behavior is a constitutive law of the form $v = F(\rho, \rho_x)$.
(b) Write down a simple, three parameter, constitutive law for v that involves ρ and ρ_x.
(c) With the constitutive law from part (b) what is the resulting traffic flow equation?
(d) What is the resulting small disturbance equation and how does it differ from (5.35)?

5.26. This problem examines the averaging used to define the flux and density, and how they relate with the velocity. It is assumed that a car with initial location x_0 has velocity $f(x_0)$. Consequently, the position of this car at any later time t is $x = x_0 + f(x_0)t$. Example paths for the cars are shown in Figure 5.39. Therefore, in this problem, each car has a constant velocity, but different cars can have different velocities. For simplicity, it is assumed that $f(x) = v_0 + w_0 x$, where v_0 and w_0 are constants.
(a) The averaging interval used to define the density in (5.1) is shown in Figure 5.39(a), and it is the same as the one shown in Figure 5.4. Explain why $x_0 - \Delta x = \alpha + f(\alpha)t_0$ and $x_0 + \Delta x = \beta + f(\beta)t_0$. Use these equations to find α and β in terms of x_0 and t_0.
(b) The averaging interval used to define the flux in (5.3) is shown in Figure 5.39(b). Explain why $t_0 - \Delta t = b + f(b)t_0$ and $t_0 + \Delta t = a + f(a)t_0$. Use these equations to find a and b in terms of x_0 and t_0.

(c) Assuming that the cars are continuously distributed, show that the average velocity for the cars in the horizontal bar in Figure 5.39(a) is $v_0 + w_0(x_0 - v_0 t_0)/(1 + w_0 t_0)$.
(d) Assuming that the cars are continuously distributed, find the average velocity for the cars in the vertical bar in Figure 5.39(b). Assuming that Δt is small, show that the average velocity is, approximately, $v_0 + w_0(x_0 - v_0 t_0)/(1 + w_0 t_0)$.
(e) Use the fact that the average velocities in parts (c) and (d) are the same to explain why this provides additional evidence of the validity of the equation $J = \rho v$.

5.27. This problem considers various formulas for the SCTA model.
(a) Show that the first two steps in the SCTA model can be combined into the formula $m_{new} = \min\{m_{old} + 1, M, g_{old}\}$.
(b) Show that the first three steps in the SCTA model can be combined into the formula $m_{new} = \max\{0, \min\{m_{old}+1, M, g_{old}\} - \chi\}$, where $\chi = 1$ with probability p (otherwise, $\chi = 0$).

5.28. Suppose that in the SCTA model the cars start out uniformly spaced with $g = M$. Assume the randomization probability is $p = 1$.
(a) Let $M = 1$. What happens if the cars start out with velocity M? What happens if the cars start out with zero velocity?
(b) Let $M = 3$. What happens if the cars start out with velocity M? What happens if the cars start out with velocity $m = 2$? What happens if the cars start out with velocity $m = 1$? What happens if the cars start out with zero velocity?
(c) Generalize your conclusions from part (b) to describe what happens if $M \geq 2$.

5.29. Suppose there is a stoplight located at $x = 0$. When it turns red assume the cars are uniformly spaced in the region $x < 0$, with three spaces between the cars, and each car has $m = 1$. The maximum movement is $M = 2$. A space here is one car length.
(a) Assuming the Greenshields constitutive law is used, what is the resulting solution of the traffic flow problem? What is the velocity of the shock wave?
(b) In the SCTA model suppose the randomization is turned off (i.e., $p = 0$). Show that the approximate velocity of the shock wave is $-2\Delta x/(3\Delta t)$. How does this compare with the continuum result from part (a)?
(c) In the SCTA model suppose that in the randomization step $p = 1$. Explain why there is a shock-like solution but the jam density is half of what is obtained from the continuum solution. Also show that the shock moves with approximate velocity $-\Delta x/\Delta t$.
(d) Given the solution in part (c) describe, in general terms, what happens when p is close to one.

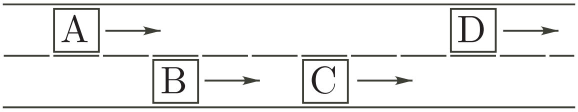

Figure 5.40 Possible car positions when deciding to make a lane change, as considered in Problem 5.30.

(e) Using your results from parts (b) and (d), explain why $-2(1-p)\Delta x/(3\Delta t)$ provides an approximation of the shock velocity for the SCTA model. Given this, what should the randomization probability be so that the SCTA velocity agrees with the continuum model?

5.30. To extend the SCTA model to multilane roads, where individual cars are able to change lanes, consider Figure 5.40. Assume a driver will switch lanes whenever they are able to travel farther in a time step in the other lane. Safe lane changing requires consideration of the backward gap in the other lane, so the driver in car B must consider the position and velocity of car A when deciding to switch. Write down a set of rules for moving the cars along the highway that includes lane changes. Assume in this problem that the randomization step is omitted.