13/02/2012

$$\frac{du}{d\tau} = u(1-v) = f$$

$$\frac{dv}{d\tau} = \alpha v(u-1) = g, \quad \alpha > 0, \text{ constant}$$

Lotka – Volterra Model

Lotka (1920)
Volterra (1926)

Equilibrium states $(0,0)$
$(1,1)$

$$A = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix} = \begin{pmatrix} 1-v & -u \\ \alpha v & \alpha(u-1) \end{pmatrix}$$

$$A(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -\alpha \end{pmatrix}, \quad \lambda_1 = 1, \quad \lambda_2 = -\alpha$$

$= b = 0!$

Saddle point.

$$v^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad v^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Indeed: $Av^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & -\alpha \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda_1 v^{(1)}$

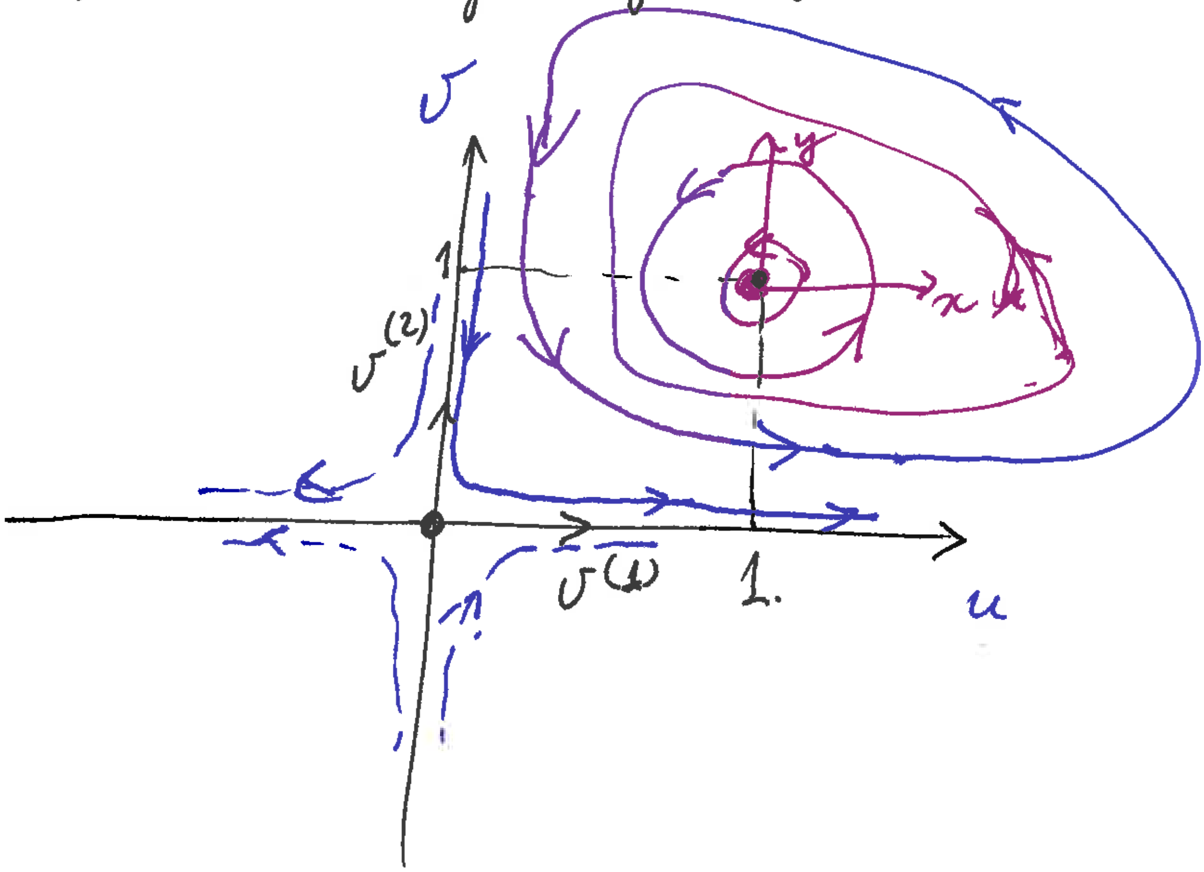
$Av^{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & -\alpha \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\alpha\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \lambda_2 v^{(2)}$

$$A(1,1) = \begin{pmatrix} 1-1 & -1 \\ \alpha(1) & \alpha(1-1) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ \alpha & 0 \end{pmatrix}$$

characteristic equation $\lambda^2 + \alpha = 0$

$$\lambda_{1,2} = \pm i\sqrt{\alpha}$$

Two imaginary eigenvalues



$u = 1 + x$
$v = 1 + y$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A(1,1) \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 v^{(1)} e^{i\sqrt{\alpha}\tau} + \bar{c}_1 \bar{v}^{(1)} e^{-i\sqrt{\alpha}\tau}$$

$$v^{(1)} = \begin{pmatrix} 1 \\ p_1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{\lambda_1 - a}{b} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{i\sqrt{\alpha} - 0}{-1} \end{pmatrix} = \begin{pmatrix} 1 \\ -i\sqrt{\alpha} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -i\sqrt{\alpha} \end{pmatrix} e^{i\sqrt{\alpha}\tau} + \bar{c}_1 \begin{pmatrix} 1 \\ i\sqrt{\alpha} \end{pmatrix} e^{-i\sqrt{\alpha}\tau}$$

$$x = c_1 e^{i\sqrt{\alpha}\tau} + \bar{c}_1 e^{-i\sqrt{\alpha}\tau}$$

$$\boxed{c_1 = |c_1| e^{i\gamma}}$$ in polar form

$$x = |c_1| \left(e^{i\sqrt{\alpha}\tau + i\gamma} + e^{-i\sqrt{\alpha}\tau - i\gamma} \right)$$

Recall: $\cos\phi = \dfrac{e^{i\phi} + e^{-i\phi}}{2}$

$$\boxed{x = 2|c_1| \cos(\sqrt{\alpha}\tau + \gamma)}$$

$$y = c_1(-i\sqrt{\alpha}) e^{i\sqrt{\alpha}\tau} + \bar{c}_1 i\sqrt{\alpha}\, e^{-i\sqrt{\alpha}\tau}$$

$$y = |c_1|\sqrt{\alpha}(i) \left[-e^{i\sqrt{\alpha}\tau + \gamma} + e^{-i\sqrt{\alpha}\tau - \gamma} \right]$$

$$\boxed{\frac{e^{i\phi} - e^{-i\phi}}{2i} = \sin\phi}$$

$$y = \sqrt{\alpha}|c_1|\, i \left[-2i \sin(\sqrt{\alpha}\tau + \gamma) \right]$$

$$\boxed{y = 2\sqrt{\alpha}|c_1| \sin(\sqrt{\alpha}\tau + \gamma)}$$

$$\begin{cases} x = 2|C_1| \cos(\sqrt{\alpha}\tau + \gamma) \\ y = 2\sqrt{\alpha}|C_1| \sin(\sqrt{\alpha}\tau + \gamma) \end{cases}$$

$$\frac{x^2}{4|C_1|^2} + \frac{y^2}{4\alpha|C_1|^2} = \cos^2(\sqrt{\alpha}\tau+\gamma) + \sin^2(\sqrt{\alpha}\tau+\gamma) = 1$$

$$\left(\frac{x}{2|C_1|}\right)^2 + \left(\frac{y}{2\sqrt{\alpha}|C_1|}\right)^2 = 1$$

$$u = 1 + x$$
$$v = 1 + y$$

$$\left(\frac{u-1}{2|C_1|}\right)^2 + \left(\frac{v-1}{2\sqrt{\alpha}|C_1|}\right)^2 = 1$$

<u>Period</u> : $\boxed{T = \frac{2\pi}{\sqrt{\alpha}}}$

Models for Interacting Populations

$$\frac{dN}{dt} = N(a - bP)$$

$$\frac{dP}{dt} = P(cN - d)$$

$\underline{P = 0}$ $\qquad \frac{dN}{dt} = Na \Rightarrow N = N_0 e^{at}$

$\underline{N = 0}$ $\qquad \frac{dP}{dt} = -dP \Rightarrow P = P_0 e^{-dt}$

Predator - Pray $\underline{\text{model}}$

N - pray population
P - predators population

$$\frac{d\left(N\frac{c}{d}\right)}{d(at)} = \left(\frac{c}{d}N\right)\left(1 - \underbrace{\frac{b}{a}P}_{v}\right)$$

$$\frac{d\left(P\frac{b}{a}\right)}{d(at)} = \frac{\left(\frac{b}{a}P\right)d}{a}\left(\underbrace{\frac{c}{d}N}_{u} - 1\right)$$

$\tau = at$

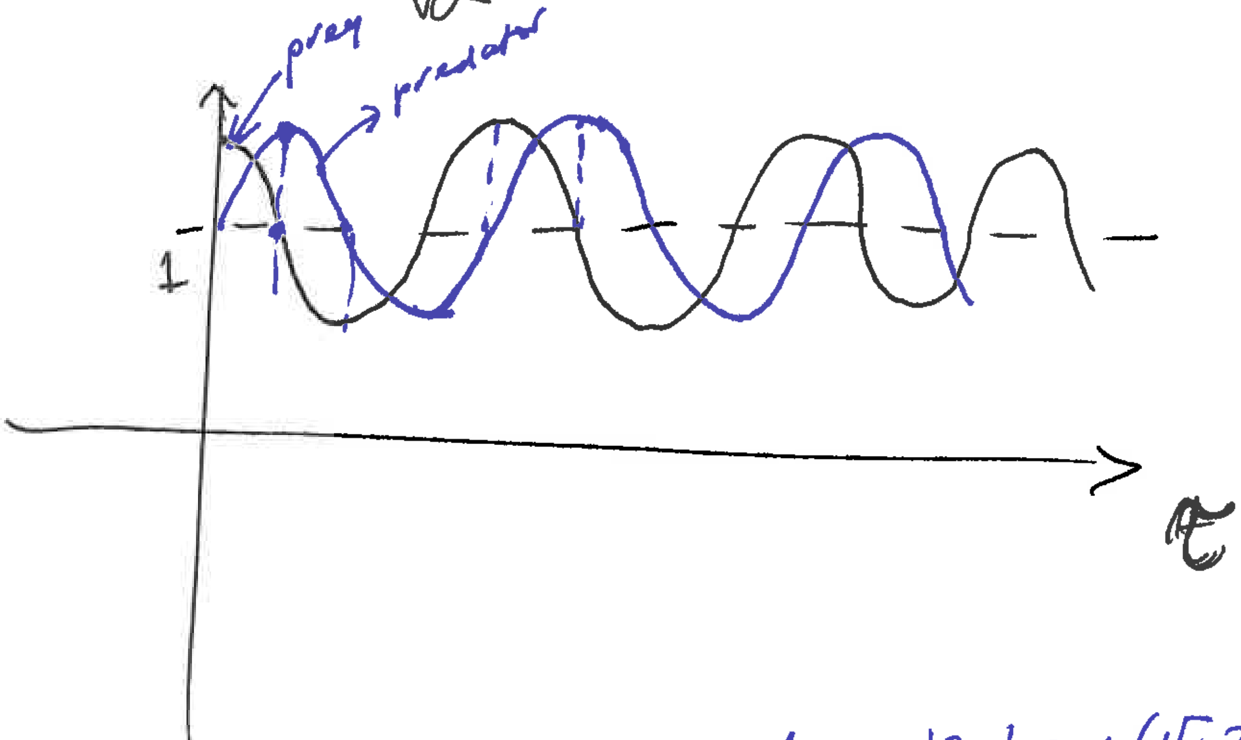
$u = \frac{c}{d}N$

$v = \frac{b}{a}P$

$\alpha = \frac{d}{a}$

$$\begin{cases} \dfrac{du}{d\tau} = u(1-v) \\[2ex] \dfrac{dv}{d\tau} = \alpha v(u-1) \end{cases}$$

$$T = \frac{2\pi}{\sqrt{\alpha}} = 2\pi\sqrt{\frac{a}{d}}$$



$$u = 1 + x = 1 + 2|c_1|\cos(\sqrt{\alpha}\tau + \gamma)$$

$$v = 1 + y = 1 + 2\sqrt{\alpha}|c_1|\sin(\sqrt{\alpha}\tau + \gamma)$$

$$\frac{dv}{du} = \alpha\frac{v(u-1)}{u(1-v)}$$

$$\frac{1-v}{v}\,dv = \alpha\frac{(u-1)}{u}\,du$$

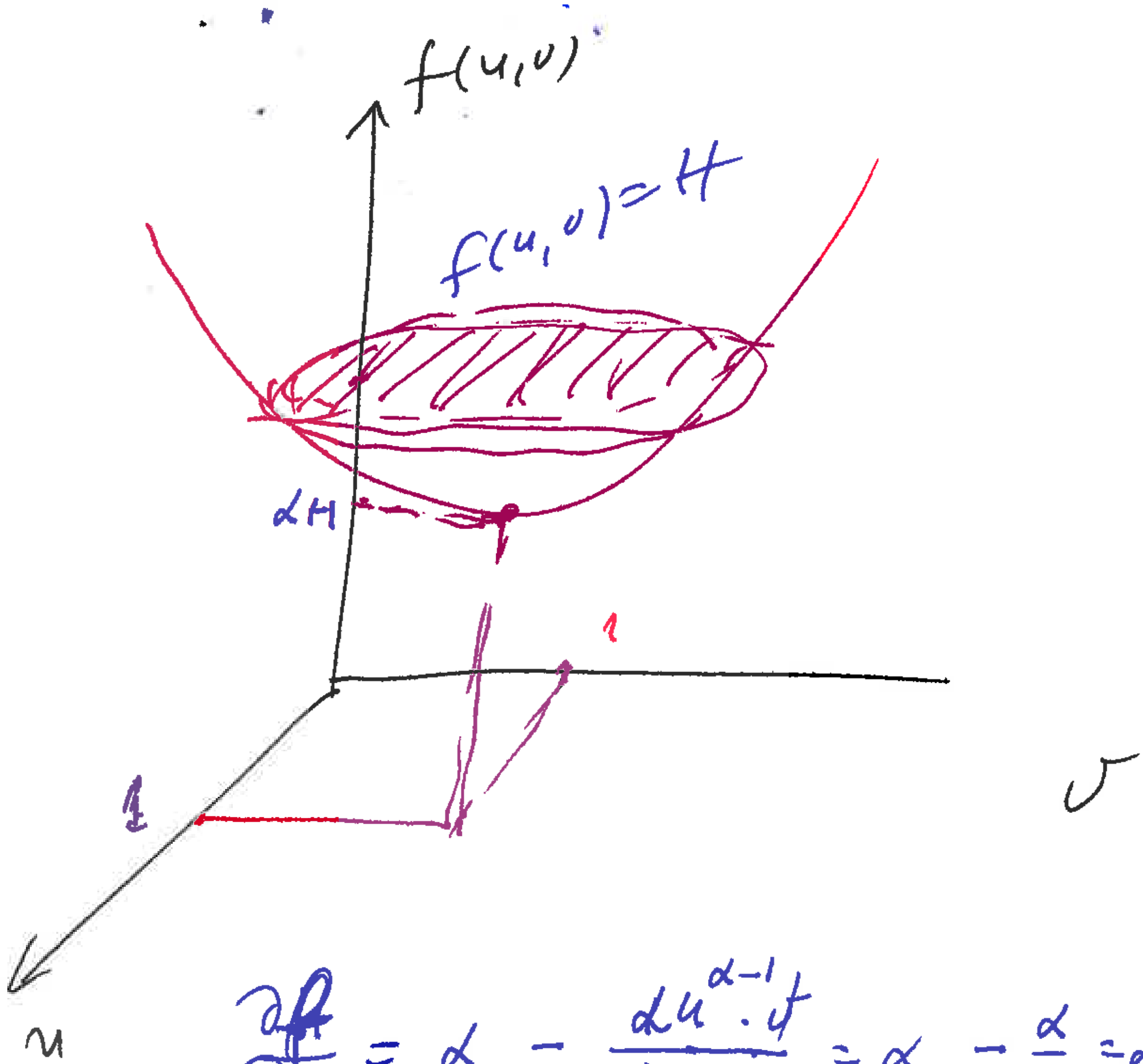
$$\left(\frac{1}{v}-1\right) dv = \alpha\left(1-\frac{1}{u}\right)du$$

$$\ln v - v = \alpha u - \alpha \ln u + \text{const}$$

$$-\text{const} = \alpha u + v - \ln v - \alpha \ln u$$

(3.6) $$\boxed{H = \alpha u + v - \ln u^{\alpha} v = \text{const.}}$$

$$f(u,v) = \alpha u + v - \ln u^{\alpha} v = H = \text{const.}$$



$$\frac{\partial f}{\partial u} = \alpha - \frac{\alpha u^{\alpha-1}\cdot v}{u^{\alpha} v} = \alpha - \frac{\alpha}{u} = \alpha\left(1-\frac{1}{u}\right) = 0$$

$$\frac{\partial f}{\partial v} = 1 - \frac{\cancel{u^{\alpha}}\cdot 1}{\cancel{u^{\alpha}} v} = 1 - \frac{1}{v} = 1 - \frac{1}{v} = 0$$

$H_{min} = \alpha + 1$ $\qquad u = v = 1$

1845 - 1930's

fur cath records

lynx

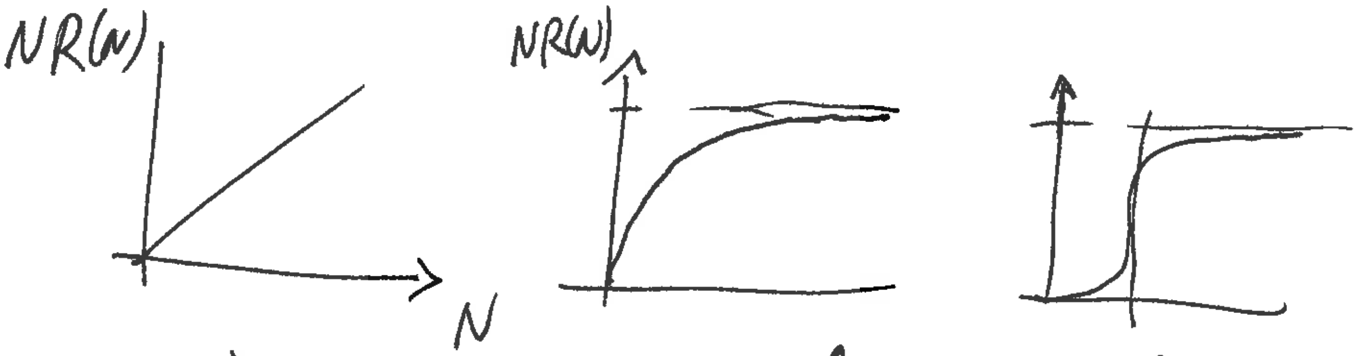
hare

Realistic Predator - Prey Models

$$\begin{cases} \dfrac{dN}{dt} = N\,F(N,P) \\[2ex] \dfrac{dP}{dt} = P\,G(N,P) \end{cases}$$

$$\begin{cases} \dfrac{dN}{dt} = rN\left(1 - \dfrac{N}{K}\right) - PN\boxed{R(N)} \\[2ex] \dfrac{dP}{dt} = kP\left(1 - \dfrac{hP}{N}\right) \end{cases}$$

r, k, K, h - positive constants



$R(N) = A$ (as in Lotka-Volterra case)

$R(N) = \dfrac{A}{N+B}$

$R = \dfrac{AN}{N^2+B^2}$

$$\begin{cases} \dfrac{dN}{dt} = N\left[r\left(1-\dfrac{N}{K}\right) - \dfrac{kP}{N+D}\right] \\[2ex] \dfrac{dP}{dt} = P\left[s\left(1-\dfrac{hP}{N}\right)\right] \end{cases}$$

r, K, k, s, h, D, 6 positive constants

$$u(\tau) = \frac{N}{K} \qquad v(\tau) = \frac{hP}{K}, \qquad \tau = rt$$

$$\frac{d\,N/K}{d\,tr} = \frac{N}{K}\left[\frac{\cancel{r}}{\cancel{r}}(1-u) - \frac{\frac{k}{hr}\,hP/K}{\frac{N}{K}+\frac{D}{K}}\right]$$

$d = \dfrac{D}{K}$ $\qquad a = \dfrac{k}{hr}$

$$\boxed{\frac{du}{d\tau} = u\left[(1-u) - \frac{a\cdot v}{u+d}\right]}$$

$$\frac{d\,hP/K}{d\,rt} = \frac{hP}{K}\,\frac{s}{r}\left(1 - \frac{h\,P/K}{N/K}\right), \qquad b = \frac{s}{r}$$

$$\boxed{\frac{dv}{d\tau} = bv\left(1-\frac{v}{u}\right)}$$

$\boxed{a, b, d}$

$$\frac{du}{d\tau} = u(1-u) - \frac{auv}{u+d} = f(u,v)$$

$$\frac{dv}{d\tau} = bv\left(1-\frac{v}{u}\right) = g(u,v)$$

$$\begin{cases} f(u^*, v^*) = 0 \\ g(u^*, v^*) = 0 \end{cases}$$

$$v = 0 \implies u(1-u) = 0$$

$\cancel{(0,0)}$.

$(1, 0)$; $\quad \underline{u^* = v^* > 0}$

$$\cancel{u^*}(1-u^*) - \frac{a(u^*)^{\cancel{2}}}{u^*+d} = 0$$

$$(1-u^*)(u^*+d) = au^*$$

$$(u^*)^2 + (a+d-1)u^* - d = 0$$

$$\begin{cases} u^* = \dfrac{1-a-d+\sqrt{(1-a-d)^2+4d}}{2} \\ v^* = u^* \end{cases}$$

$$\mathcal{J} = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix} = \begin{pmatrix} 1-2u-\frac{av}{u+d}+\frac{auv}{(u+d)^2} & -\frac{au}{u+d} \\ +\frac{bv^2}{u^2} & b-\frac{2bv}{u} \end{pmatrix}$$

$$\boxed{\left(\frac{uv}{u+d}\right)' = \left((uv)\cdot\frac{1}{u+d}\right)' = v\cdot\frac{1}{u+d} - \frac{(uv)}{(u+d)^2}}$$

$$\mathcal{A}(1,0) = \begin{pmatrix} 1-2 & -\frac{a}{1+d} \\ 0 & b \end{pmatrix} = \begin{pmatrix} -1 & -\frac{a}{1+d} \\ 0 & b \end{pmatrix}$$

$$\begin{vmatrix} -1-\lambda & -\frac{a}{1+d} \\ 0 & b-\lambda \end{vmatrix} = 0 \qquad (-1-\lambda)(b-\lambda) = 0$$

$$\lambda = -1, \quad \lambda = b$$

'saddle point' => unstable.

$$\mathcal{A}(u^*,u^*) = \begin{bmatrix} u^*\left[\frac{au^*}{(u^*+d)^2}-1\right] & -\frac{au^*}{u^*+d} \\ +b & -b \end{bmatrix}$$

$$1-2u^* - \underbrace{\frac{au^*}{u^*+d}}_{=1-u^*} + \frac{a(u^*)^2}{(u^*+d)^2} = \text{ but } \boxed{1-u^* = \frac{au^*}{u^*+d}}$$

$$= \cancel{1}-2u^* - (\cancel{1}-u^*) + \frac{a(u^*)^2}{(u^*+d)^2} = -u^* + \frac{a(u^*)^2}{(u^*+d)^2} =$$

$$= u^*\left[\frac{au^*}{u^*+d}-1\right]$$

Characteristic equation

$$\lambda^2 - (\mathrm{tr}\,A)\,\lambda + \det A = 0$$

For stability $\boxed{\det A > 0}$, $\mathrm{tr}\,A < 0$

$\mathrm{tr}\,A \leq 0$ means

$$\boxed{u^*\left[\frac{au^*}{(u^*+d)^2} - 1\right] < b \qquad (1)}$$?

$$\det A = -b\,u^*\left[\frac{au^*}{(u^*+d)^2} - 1\right] + b\,\frac{au^*}{u^*+d} =$$

$$= bu^*\left[1 + \frac{a}{u^*+d} - \frac{au^*}{(u^*+d)^2}\right] =$$

$$= bu^*\left[1 + \frac{a(\cancel{u^*}+d) - \cancel{au^*}}{(u^*+d)^2}\right]$$

$$= bu^*\left[1 + \frac{ad}{(u^*+d)^2}\right] > 0 \quad \checkmark$$

But $u^* = \dfrac{1-a-d+\sqrt{(1-a-d)^2+4d}}{2}$

$$\boxed{b > \left[a - \sqrt{(1-a-d)^2+4d}\,\right]\frac{\left[1+a+d-\sqrt{(1-a-d)^2+4d}\,\right]}{2a}}$$