DUBLIN INSTITUTE OF TECHNOLOGY

SCHOOL OF MATHEMATICAL SCIENCES

MATH 9851: FUNCTIONAL ANALYSIS AND SPECTRAL THEORY

DT234 MSc APPLIED MATHEMATICS & THEORETICAL PHYSICS

DT238 MSc APPLIED MATHEMATICS & THEORETICAL PHYSICS

SUMMER EXAMINATION SESSION

2014/2015

Examiners: Dr M Venkova
Professor E O'Riordan
Dr C Hills

Date: Friday, 22 May 2015 Time: 9.30 – 1.00 pm

Answer five questions

All questions carry equal marks

Approved calculators may be used

Mathematical tables will be provided

New Cambridge Statistical Tables are NOT permitted

1. a) Show that the mapping

$$d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + \ldots + |x_n - y_n|,$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$, defines a metric on $\mathbb{R}^n$. (3 marks)

b) Let (X, d) be a metric space and $B \subset X$. Define what is meant by the *closure* $\overline{B}$ of B. Show that $x \in \overline{B}$ if and only if every open ball with centre at x contains a point of B. (6 marks)

c) Let Y be a closed subspace of the Hilbert space H, define what is meant by the *orthogonal complement* $Y^\perp$ of Y. Show that $Y^\perp$ is a closed subspace of H. (6 marks)

d) Let (X, d) be a metric space and $A \subset X$. Define what is meant by the set A being *dense* in X. Show that A is dense in X if and only if for every $x \in X$ there exists $(x_n)_n \subset A$ such that $x_n \to x$ as $n \to \infty$. (5 marks)

[20]

2. a) Let V be a vector space. What properties does a mapping $\|\cdot\| : V \to \mathbb{R}$ need to have in order to be a norm on V? (3 marks)

b) Let E and F be normed spaces, and let $\mathcal{L}(E, F)$ be the vector space of all bounded linear operators from E into F. Show that $\mathcal{L}(E, F)$ is a normed space with a norm defined by:

$$\|A\| = \sup_{\|x\|=1} \|Ax\|,$$

where $A \in \mathcal{L}(E, F)$ and $x \in E$. (8 marks)

c) On $C[-1, 1]$ with norm $\|f\| = max_{x\in[-1,1]}|f(x)|$, define an operator F by:

$$F(f) = t \int_{-1}^{1} f(s)ds,$$

where $t \in [-1, 1]$ is fixed. Show that F is linear and bounded, and find $\|F\|$. (9 marks)

[20]

3. a) State what is meant by saying that a subset of a vector space is *convex*. (2 marks)

   b) Let E be a normed space and C be a convex subset of E. Prove that the closure of C is a convex set. (6 marks)

   c) Let E be a normed space, define what is meant by the *dual space* E' and specify the norm on E'. If $E = \mathbb{R}^n$ with norm $\|x\| = \max_{i=1\ldots n} |x_i|$, find the corresponding norm on the dual space E'. (9 marks)

   d) State the Separating (or Geometric) Hahn-Banach Theorem. (3 marks)

[20]

4. a) Show that the real vector space l^2 is an inner product space with

$$\langle x, y \rangle = \sum_{n=1}^{\infty} x_n y_n.$$

(4 marks)

   b) Show that in an inner product space the inner product is continuous. (4 marks)

   c) Suppose $(v_n)_n$ is an orthonormal sequence in the inner product space V and let $m \in \mathbb{N}$.

   i) Prove that $v_1, \ldots, v_m$ are linearly independent. (4 marks)

   ii) Let $x \in V$ be any fixed element and $y = a_1 v_1 + \ldots + a_m v_m$, show that $\|x - y\|$ is minimal if and only if $a_i = \langle x, v_i \rangle$ where $i = 1 \ldots m$. Hence prove the *Bessel inequality*. (8 marks)

[20]

5. a) Let E and F be vector spaces and $h : E \times F \to \mathbb{C}$. What is meant by saying that h is a *sesquilinear form*?

(4 marks)

b) Let E and F be Hilbert spaces and $Q : E \to F$ a bounded linear operator. Show that $Q = 0$ if and only if $\langle Qx, y \rangle = 0$ for all $x \in X$ and all $y \in Y$.

(5 marks)

c) Let E and F be Hilbert spaces and $A : E \to F$ a bounded linear operator. Define what is meant by the *Hilbert-adjoint* operator A^*. Show that

i) $(A^*)^* = A$. (5 marks)

ii) If $(T_n)_n$ is a sequence of bounded linear operators and T_n converges to T, then T_n^* converges to T^*. (6 marks)

[20]

6. a) Let E be a complex normed space and $T : E \to E$ a linear operator. Define what is meant by the *spectrum* of T, $\sigma(T)$, and the *resolvent set* of T, $\rho(T)$.

(4 marks)

b) Let E be a Banach space.

i) Prove that if $T : E \to E$ is a bounded linear operator and $\|T\| < 1$, then $(I - T)^{-1}$ is a bounded linear operator and

$$(I - T)^{-1} = \sum_{j=0}^{\infty} T^j.$$

(8 marks)

ii) Show that if $A : E \to E$ is a bounded linear operator then the spectrum of A is bounded: if $\lambda \in \sigma(A)$ then $|\lambda| \leq \|A\|$. (8 marks)

[20]

7. Let H be a complex Hilbert space and $A : H \to H$ be a bounded linear operator.

a) Define what is meant by A being *self-adjoint*. (2 marks)

b) Prove that if A is self-adjoint then the eigenvalues of A are real. (5 marks)

c) Let $(a_n)_n$ be a bounded sequence in $\mathbb{R}$, consider the operator $T : l^2 \to l^2$ defined by

$$T(x_1, x_2, \ldots, x_n, \ldots) = (a_1 x_1, a_2 x_2, \ldots, a_n x_n, \ldots).$$

i) Show that T is linear and bounded. (5 marks)

ii) Show that T is self-adjoint. (3 marks)

iii) Show that each a_n is an eigenvalue of T. (5 marks)

[20]