Predator-Prey Model (continuation)

$$\frac{du}{d\tau} = u(1-u) - \frac{auv}{u+d} = f(u,v)$$

$$\frac{dv}{d\tau} = bv\left(1-\frac{v}{u}\right) = g(u,v)$$

a, b, d - positive parameters

$$u^* = \frac{1-a-d+\sqrt{(1-a-d)^2+4d}}{2} \qquad v^* = u^*$$

$$1-u^* = \frac{au^*}{u^*+d}, \quad D = (1-a-d)^2 + 4d$$

$$A(u^*,u^*) = \begin{pmatrix} u^*\left[\frac{au^*}{(u^*+d)^2} - 1\right] & -\frac{au^*}{u^*+d} \\ b & -b \end{pmatrix}$$

For stability it is necessary and sufficient

$$\begin{cases} \operatorname{tr} A < 0 \quad (?) \\ \det A > 0 \quad \checkmark \text{ always} \end{cases}$$

$$u^*\left[\frac{au^*}{(u^*+d)^2} - 1\right] < \underline{\underline{b}}$$

$$u^*\left[\frac{au^*}{(u^*+d)^2} - 1\right] = u^*\left[\frac{1-u^*}{u^*+d} - 1\right] = u^* \cdot \frac{1-u^*-u^*-d}{u^*+d} =$$

$$= \frac{u^*}{u^*+d} \cdot (1-d-2u^*) = \frac{1-u^*}{a} \cdot \left[\cancel{1}-\cancel{d} - (\cancel{1}-a-\cancel{d}) - \sqrt{D}\right]$$

$$= \frac{1}{a}\left(\frac{2-1+a+d-\sqrt{D}}{2}\right)(a-\sqrt{D}) < b$$

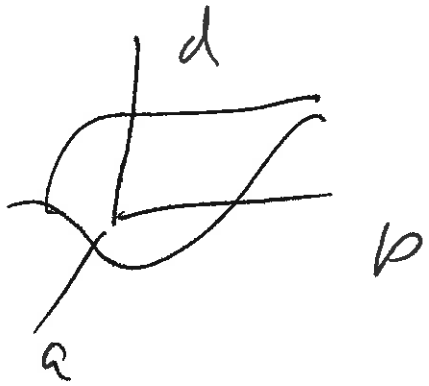
$$b > (a - \sqrt{D}) \cdot \frac{1+a+d-\sqrt{D}}{2a}$$

$$b > \left[a - \sqrt{(1-a-d)^2+4d}\right] \cdot \frac{\left[1+a+d-\sqrt{(1-a-d)^2+4d}\right]}{2a}$$

(3. 4)

It defines 3-dimensional surface in (a, b, d) - space

$a > 0, \; b > 0, \; d > 0$



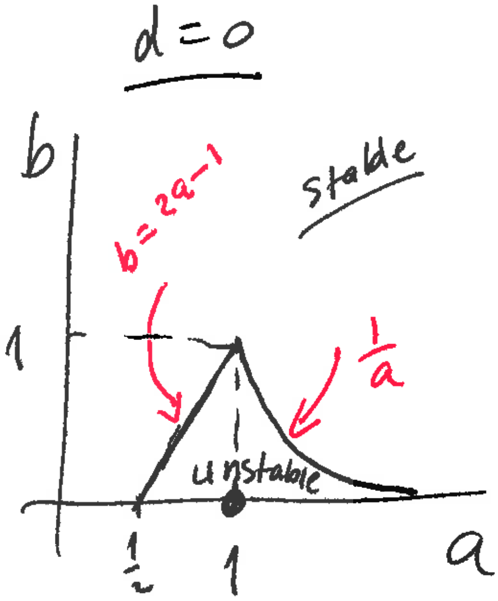
Observations:

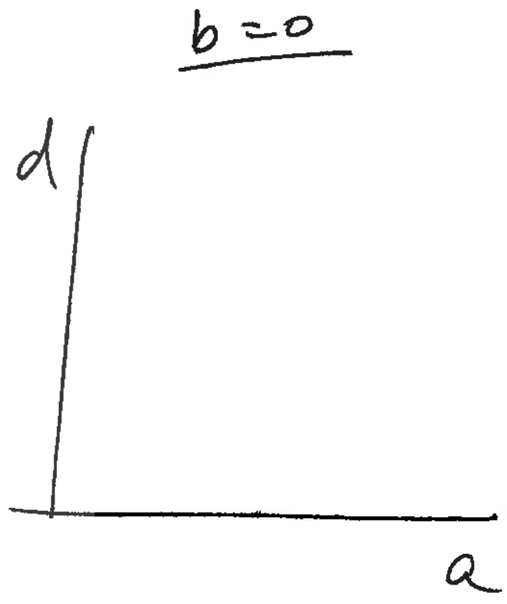
① $1+a+d-\sqrt{(1-a-d)^2+4d}$ monotonic & decreasing function of d, with max at $d=0$.

$$1+a+d-\sqrt{(1-a-d)^2+4d} = \frac{(1+a+d)^2-(1-a-d)^2-4d}{1+a+d+\sqrt{(1-a-d)^2+4d}} =$$

$$= \cdots = \frac{4a}{1+a+d+\sqrt{1+a^2+d^2+2d+2ad-2a}}$$

$(x+y+z)^2 = x^2+y^2+z^2+2xy+2xz+2yz$





I $\underline{d=0}$; $b > \left[a - \sqrt{(1-a)^2}\right] \dfrac{1+a-\sqrt{(1-a)^2}}{2a}$

$$b > (a - |1-a|) \frac{1+a-|1-a|}{2a}$$

when $0 < a < 1$

$$b > (a-1+a) \underbrace{\frac{\cancel{1}+a-\cancel{1}+a}{2a}}_{1} \;;\; b > 2a-1$$

$$\boxed{b > 2a-1 \text{ when } 0 < a < 1} \quad (1)$$

when $a > 1$

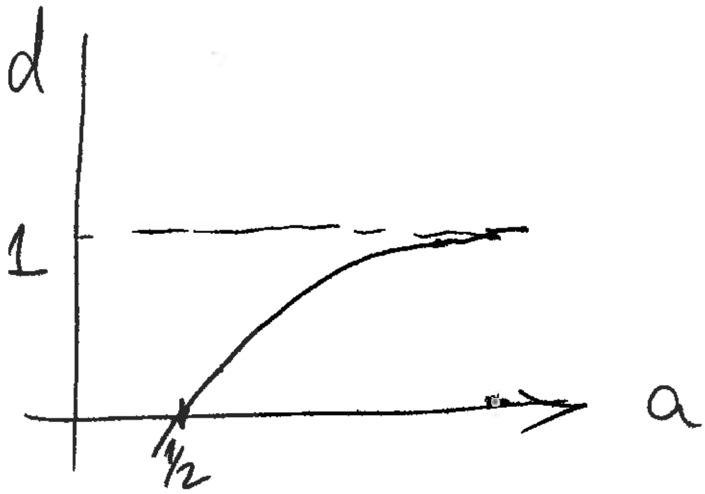
$$b > (\underbrace{\cancel{a}+1-\cancel{a}}_{1}) \frac{1+\cancel{a}+1-\cancel{a}}{2a} = \frac{2}{2a} = \frac{1}{a}$$

$$\boxed{b > \frac{1}{a} \text{ when } a > 1} \quad (2)$$

$$a = 1, \quad b \overset{-4}{=} d = 0$$

$$0 > [1-0]\cdot\frac{1}{2}[2-0] \qquad 0 > 1 \quad \times$$

$b = 0$ case $\quad \alpha \; a < \sqrt{(1-a-d)^2 + 4d}$



$$a^2 = (1-a-d)^2 + 4d \Rightarrow d(a)$$

$$\cancel{a^2} = 1 + \cancel{a^2} + d^2 - 2a - 2d + 2ad + 4d$$

$$d^2 + (2+2a)d + 1 - 2a = 0$$

$$d^2 + 2(a+1)d + (1-2a) = 0$$

$$d = -(a+1) \pm \sqrt{(a+1)^2 - (1-2a)}$$

$$d = \sqrt{a^2 + 2a + \cancel{1} - \cancel{1} + 2a} - (a+1)$$

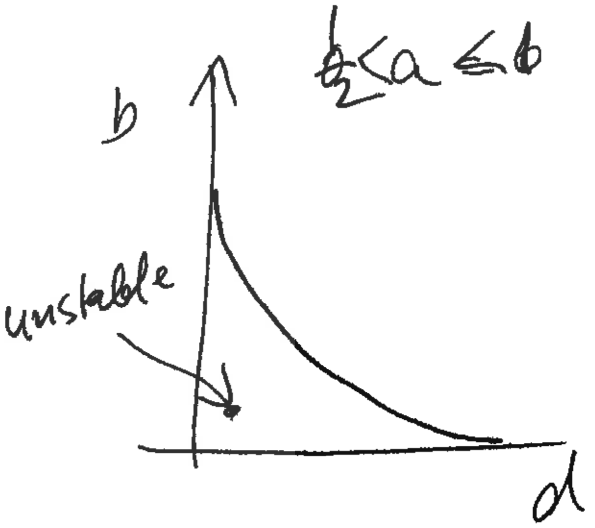
$$\boxed{d(a) = \sqrt{a^2 + 4a} - (a+1)}$$

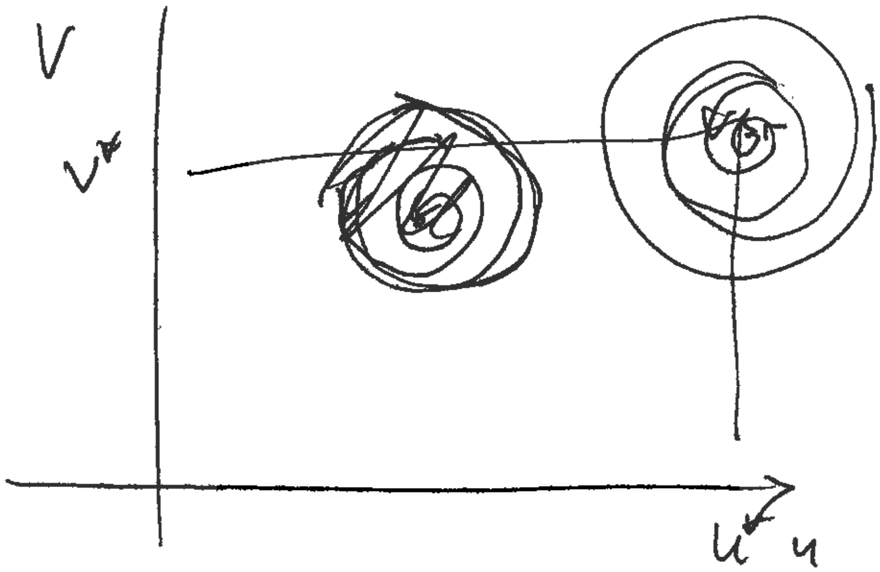
$$d\left(\frac{1}{2}\right) = \sqrt{\frac{1}{4} + 2} - \frac{3}{2} = \frac{3}{2} - \frac{3}{2} = 0$$

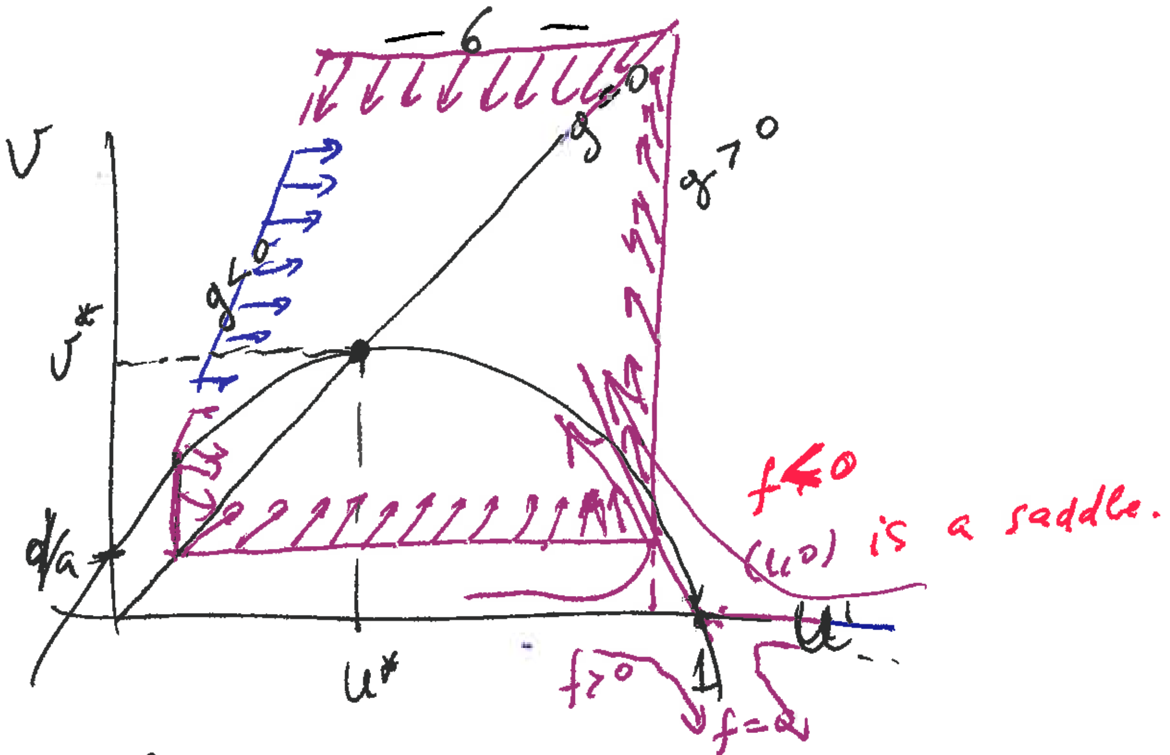
$$d(a) = \frac{a^2+4a-(a+1)^2}{\sqrt{a^2+4a}+(a+1)} = \frac{\cancel{a^2}+4a-\cancel{a^2}-2a-1}{\sqrt{a^2+4a}+a+1}$$

$$d(a) = \frac{2a-1}{\sqrt{a^2+4a}+a+1} \xrightarrow{a\to\infty} \frac{2a}{a+a} = 1$$

$\frac{1}{2} < a \leq 1$ $\qquad\qquad$ $1 < a < \infty$



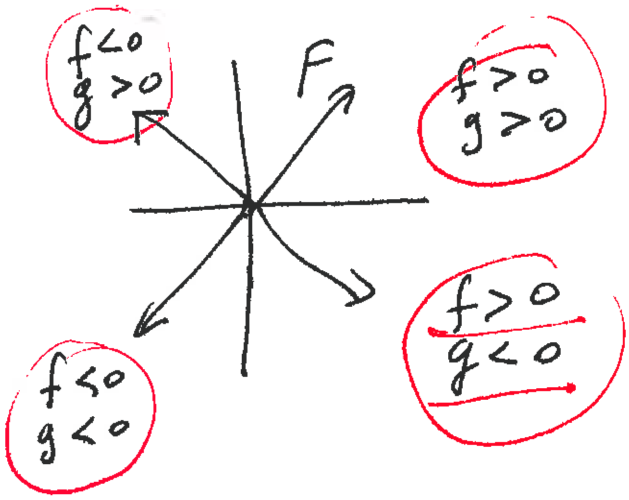


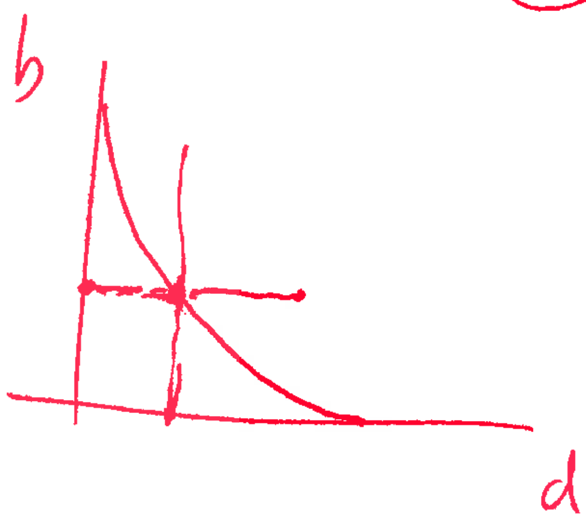


$$\left.\begin{aligned} f(u,v)&=0\\ g(u,v)&=0 \end{aligned}\right\} \text{ null-clines}$$

$$f=0 \iff \cancel{u}(1-u) = \frac{a\cancel{u}v}{u+d} \Rightarrow v = \frac{1}{a}(1-u)(u+d)$$

$$F = \begin{pmatrix} f \\ g \end{pmatrix}$$





$$b = \frac{s}{r}$$

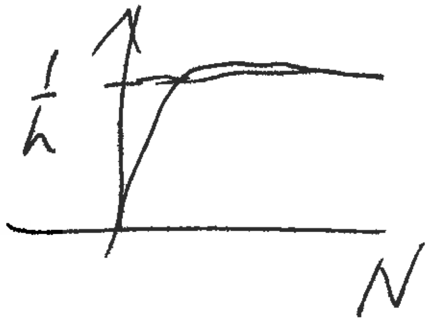
Rosenzweig – MacArthur Model

$$\begin{cases} \dot{u} = a u(1-u) - \dfrac{auv}{1+bu} \\ \dot{v} = v\left(\dfrac{bu}{1+bu} - c\right) \end{cases}$$

parameters: $a, b, c > 0$

Originally

$$\frac{dN}{dt} = \underbrace{rN\left(1-\frac{N}{K}\right)}_{\text{logistic}} - P\underbrace{\frac{sN}{1+shN}}_{\text{predation}}$$



$$\frac{dP}{dt} = eP\frac{sN}{1+shN} - mP$$

Parameters: r, K, s, h, e, m

$$u = \frac{N}{K}; \quad v = \frac{sP}{r}; \quad \tau = \frac{et}{h}; \quad a = \frac{rh}{e}, \quad b = shK, \quad c = \frac{mh}{e}$$

Steady states: $(0,0)$, $(1,0)$, (u^*, v^*)

$$u^* = \frac{c}{b(1-c)} \qquad v^* = \frac{b-c(1+b)}{b(1-c)^2}$$

~~Steady~~

$$1 > \frac{b}{1+b} > c > 0$$

in order to ensure $u^* > 0$, $v^* > 0$

$$f = au - au^2 - \frac{auv}{1+bu}, \qquad g = \frac{buv}{1+bu} - vc$$

$$A = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix} = \begin{pmatrix} a - 2au - \frac{av}{1+bu} + \frac{abuv}{(1+bu)^2} & -\frac{au}{1+bu} \\ \frac{bv}{1+bu} - \frac{b^2uv}{(1+bu)^2} & \frac{bu}{1+bu} - c \end{pmatrix}$$

$$A(0,0) = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}$$

saddle pt. $v^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\lambda_1 = a$, $\lambda_2 = -c$

$$A(1,0) = \begin{pmatrix} -a & -\frac{a}{1+b} \\ 0 & \frac{b}{1+b} - c \end{pmatrix}$$

saddle

positive

$\lambda_1 = -a$, $v^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\lambda_2 = \frac{b}{1+b} - c > 0$

$$v^{(2)} = \begin{pmatrix} 1 \\ -\frac{1}{a}(1+b)(\lambda_2 + a) \end{pmatrix}$$

$$A(u^*, v^*) = \begin{pmatrix} \dfrac{ac[b(1-c)-(c+1)]}{b(1-c)} & -\dfrac{ac}{b} \\ b(1-c)-c & 0 \end{pmatrix}$$

$$\left.\begin{array}{l} \det A > 0 \checkmark \\ \operatorname{tr} A < 0 \ ? \end{array}\right\} \text{stability}$$

If $\dfrac{1+c}{1-c} > \boxed{b \geqslant \dfrac{c}{1+c}} > 0$ the solution (u^*, v^*) exists and is stable.

$$\frac{b}{1+b} > c$$

$$\cancel{b > c + bc}$$

$$\cancel{b(1+c) > c}$$

$$b = c \qquad \frac{b}{1+b} > c$$

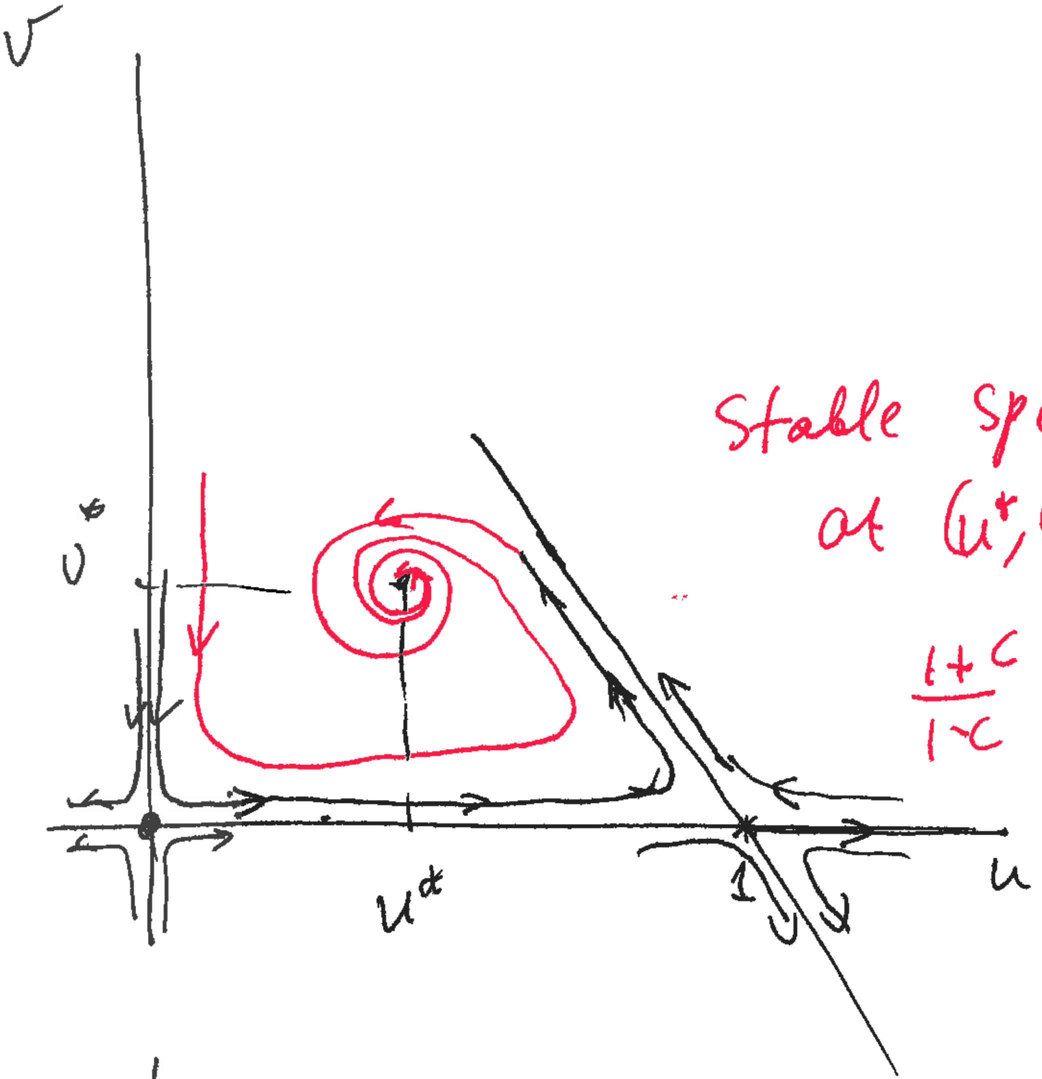
$$b > c + bc$$

$$b - bc > c$$

Bifurcation value is $b^* = \dfrac{1+c}{1-c}$

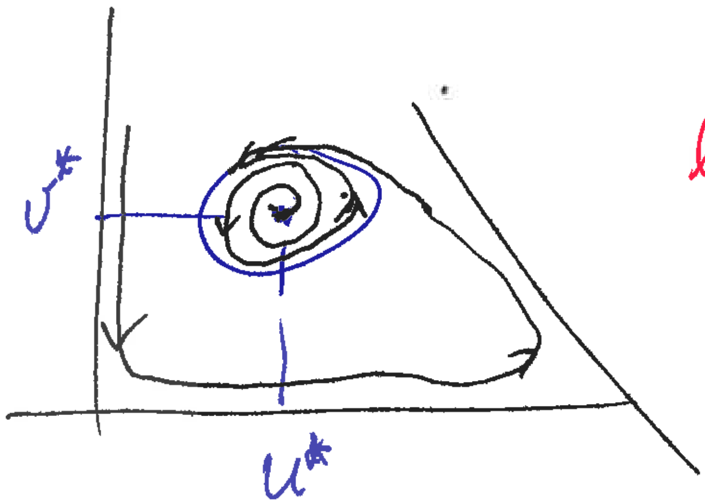
Charact. equation $\lambda^2 + \dfrac{ac(1-c)}{1+c} = 0$

$$\lambda_{1,2} = \pm i \sqrt{\frac{ac(1-c)}{1+c}} \quad \text{imaginary}$$



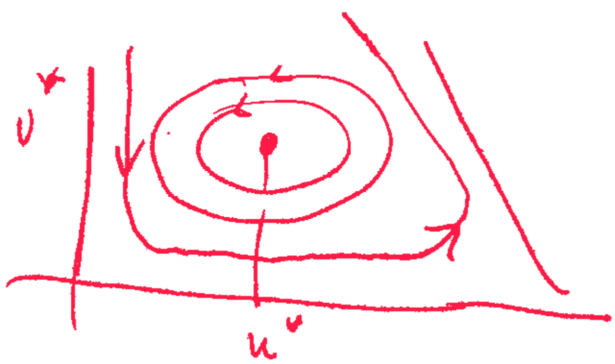
Stable spiral at (u^*, v^*)

$$\frac{1+c}{1-c} > b > \frac{c}{1-c}$$



limit cycle

$$b > \frac{1+c}{1-c} = b^*$$



Limiting case (bifurcation)

$$b = \frac{1+c}{1-c}$$

'center'

1. Predator - pray
2. Competition
3. Mutualism / Simbiosis