Biometrika (1976), **63**, 2, *p.* 361–6
With 2 text-figures
Printed in Great Britain

On random sequential packing in two and three dimensions

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SUMMARY

The two- and three-dimensional random packing problems in Palásti's (1960) sense are studied by means of Monte Carlo simulations and extrapolation techniques. In the two-dimensional case, a 95 % confidence interval of $0{\cdot}5629 \pm 0{\cdot}0006$ is obtained for the limit of mean random packing density. This estimate is slightly but significantly higher than the value conjectured by Palásti, $c^2 \simeq 0{\cdot}5589$. For the three-dimensional case, we calculate only a lower bound of the limiting density and give a 95 % confidence interval of $0{\cdot}4148 \pm 0{\cdot}0025$, which is also fairly near to the conjectured value ($c^3 \simeq 0{\cdot}4178$).

Some key words: Car-parking problem; Monte Carlo simulation; Palásti's conjecture; Random packing density.

1. INTRODUCTION

The problem of one-dimensional random packing, sometimes called the car-parking problem, has been solved analytically by several investigators; see, for example, Rényi (1958), Bánkövi (1962), Ney (1962), Mannion (1964) and Dvoretzky & Robbins (1964). However, higher-dimensional analogues of the parking problem are difficult to treat rigorously, so that we are compelled to employ Monte Carlo simulation techniques. Palásti (1960) conjectured that the random packing density in n dimensions is equal to the nth power of that in one dimension. In the two-dimensional case, previous results of computer simulations (Solomon, 1967; Akeda & Hori, 1975) are apparently consistent with her conjecture. On the other hand, Blaisdell & Solomon (1970) studied a lattice model corresponding to the car-parking problem, and claimed that her conjecture is incorrect, although the error is small in two dimensions. In the three-dimensional case, no available Monte Carlo data have so far been reported. In order to test her conjecture more stringently, therefore, we have repeated the computer calculations in both two and three dimensions. On the basis of the new Monte Carlo data, we shall reexamine the validity of Palásti's conjecture.

2. TWO-DIMENSIONAL CASE

The procedure of two-dimensional random packing in Palásti's sense can be described as follows. Consider a rectangle T_{xy} with side lengths $x, y > 1$. Fill the rectangle sequentially at random by unit squares whose sides are parallel to those of T_{xy}. The centre of a filling square is positioned according to the uniform distribution over the unfilled portion of the rectangle. Continue the process until no additional squares can be placed without overlapping. Let $M(x, y)$ denote the total number of unit squares placed in T_{xy} and put

$$D(x, y) = M(x, y)/(xy).$$

Then the mean packing density $d(x,y)$ is defined as the expectation of the random variable $D(x,y)$. Palásti's conjecture in two dimensions asserts that

$$d(\infty,\infty) = \lim_{x\to\infty,\, y\to\infty} d(x,y) = c^2 \simeq 0{\cdot}5589, \tag{1}$$

where

$$c = \int_0^\infty \exp\left\{-2\int_0^t (1-e^{-u})\,u^{-1}du\right\}dt \simeq 0{\cdot}7476. \tag{2}$$

This is just the limit of the mean random packing density in one dimension (Rényi, 1958).

The original statement of Palásti mentioned above is concerned with the limit for large rectangles, but Blaisdell & Solomon (1970) remarked that similar arguments apply equally well to finite rectangles. If $d(x)$ stands for the expected packing density of unit intervals on a line segment of length x, we have

$$d(x,y) = d(x)\,d(y). \tag{3}$$

In one dimension, the random packing density $D(x)$ has asymptotically a normal distribution (Mannion, 1964; Dvoretzky & Robbins, 1964), and its mean becomes

$$d(x) = c-(1-c)/x+o(1/x) \tag{4}$$

as $x\to\infty$ (Rényi, 1958; Ney, 1962). Substitution of (4) into (3) yields

$$d(x,y) = c^2-c(1-c)\,(1/x+1/y)+o(1/x)+o(1/y), \tag{5}$$

which is of course an extension of (1) to T_{xy} with finite size.

Our previous calculations (Akeda & Hori, 1975) were performed for eight cases of x and y ranging from 5×15 to 100×100. Making use of a special simulation algorithm, we determined upper bounds, lower bounds, and accurate estimates of the random variable $D(x,y)$, and then computed its sample means, sample standard deviations, and confidence intervals. Especially when $x\times y = 100\times 100$, we obtained a 95% confidence interval of $0{\cdot}5593 \pm 0{\cdot}0013$ for the mean packing density. Identifying this value of $d(100, 100)$ with the limit $d(\infty,\infty)$, we arrived at the conclusion that $d(\infty,\infty)$ does not differ significantly from c^2. Strictly speaking, however, there is no evidence to show that the discrepancy between $d(100, 100)$ and $d(\infty,\infty)$ is negligible. In order to estimate this limit with high accuracy, it would be necessary to employ an adequate extrapolation technique, because Monte Carlo experiments on larger rectangles require an enormous amount of computer time.

In the present work, we have used the same procedure as in the previous simulations. The samplings were replicated forty times on the computer for $x\times y = 5\times 15$, 10×15, 15×15, 20×15, 20×30, 40×40, 60×60, 100×100. Furthermore, we added a larger rectangle with size 200×200 on which ten trials were made. Table 1 summarizes the results of our Monte Carlo calculations together with the data from other sources (Palásti, 1960; Solomon, 1967; Akeda & Hori, 1975). All the results except those of Palásti seem to be in good agreement with one another. For the sake of comparison, we also list numerical values predicted from (3) or (5), which are found to be systematically and significantly lower than the experimental data. Thus, it is shown that Palásti's conjecture does not necessarily hold for finite rectangles. In the case of $x\times y = 200\times 200$, we have a 95% confidence interval of $0{\cdot}5610 \pm 0{\cdot}0009$ for $d(x,y)$. This suggests that $d(200, 200)$ is a little more than c^2.

The extrapolation procedure used here is based upon the following three assumptions: (i) $D(x,y)$ is normally distributed; (ii) the expectation of $D(x,y)$ is expressed as

$$d(x,y) = d(\infty,\infty) - \alpha u^{-1}, \tag{6}$$

where α is a positive constant and $u^{-1} = x^{-1}+y^{-1}$; and (iii) the standard deviation of $D(x,y)$ is proportional to u^{-1}. The first assumption of asymptotic normality is made by analogy with the one-dimensional problem. The extrapolation formula (6) in the second assumption has the same form as the conjectured relation (5). The factor u^{-1} entering in the second term on the right-hand side of (6) may be regarded as proportional to the perimeter-to-area ratio. In general, boundary effects on an n-dimensional rectangular parallelepiped with edge lengths x_i $(i = 1, \ldots, n)$ fall off proportionately to

$$u^{-1} = x_1^{-1} + \ldots + x_n^{-1}.$$

Consequently, the formula (6) would make sense even though the extended Palásti conjecture (5) is not exactly valid. The last assumption is derived empirically from the actual data. Note that this hypothesis contradicts the requirement of the one-dimensional theory, since the variance of $D(x)$ proves to be inversely proportional to x.

Table 1. *Computer simulation results for two-dimensional random packing density*

	Present work			Previous work, mean			
Rectangle	Number of trials	Mean	Standard deviation	Palásti (1960)	Solomon (1967)	Akeda & Hori (1975)	$d(x)\,d(y)$ Eq. (3)
5×15	40	0·5060	0·0312	0·56	0·50270	0·5214	0·5094
10×15	40	0·5328	0·0219	0·56	0·53933	0·5293	0·5279
15×15	40	0·5382	0·0173	0·56	0·53644	0·5422	0·5340
20×15	40	0·5400	0·0140	0·56	0·53700	0·5430	0·5371
20×30	40	0·5478	0·0096	—	0·54833	0·5465	0·5433
40×40	40	0·5534	0·0043	—	—	0·5536	0·5495
60×60	40	0·5567	0·0040	—	—	0·5571	0·5526
100×100	40	0·5592	0·0023	—	—	0·5593	0·5551
200×200	10	0·5610	0·0012	—	—	—	0·5570

Figure 1 illustrates the relationship observed between $d(x,y)$ and u^{-1}, the linearity of which is found to be quite satisfactory. The dependence of the standard deviation of $D(x,y)$ on u^{-1} is also approximately linear. Thus, the assumptions (i), (ii) and (iii) enable us to apply an ordinary weighted least-squares analysis. The resulting regression line gives a 95 % confidence interval of $0{\cdot}5629 \pm 0{\cdot}0006$ for $d(\infty,\infty)$. This is in excellent agreement with the value obtained by extrapolation from the lattice data; for such a case, indeed, Blaisdell & Solomon (1970) evaluated $d(\infty,\infty)$ as $(c+0{\cdot}0025)^2 \simeq 0{\cdot}5627$. It is concluded that the estimate of $d(\infty,\infty)$ is significantly larger than the conjectured value $c^2 \simeq 0{\cdot}5589$, although it is certain that they are very close to each other. As for the slope of the regression line, we obtain 95 % confidence limits of $\alpha = 0{\cdot}186 \pm 0{\cdot}020$. Hence α is not significantly different from the first-order coefficient of (5), namely, $c(1-c) \simeq 0{\cdot}1887$. It is interesting to notice that $d(x,y)$ and $d(x)\,d(y)$ have almost the same slope.

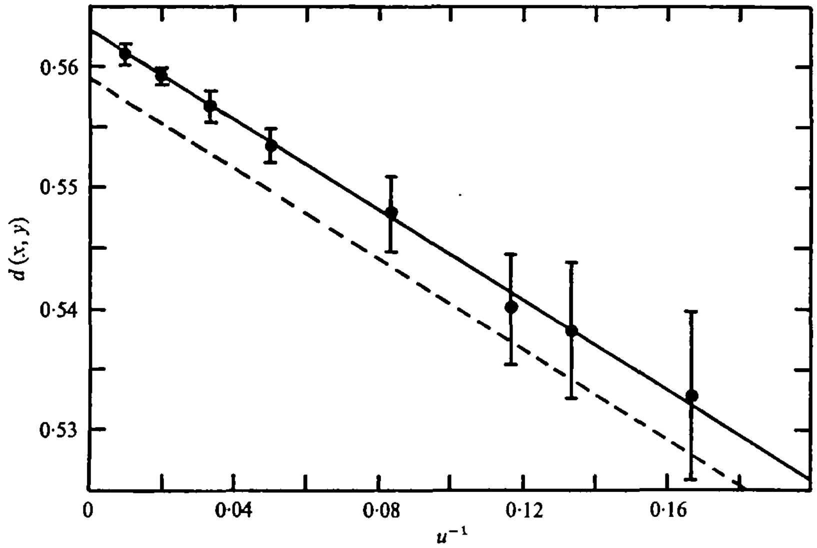

Fig. 1. Plot of $d(x, y)$ as a function of u^{-1}. Data points indicate sample means of $D(x, y)$ and error bars represent 95 % confidence intervals for $d(x, y)$. The solid straight line obtained by a linear regression analysis gives 95 % confidence limits of $0{\cdot}5629 \pm 0{\cdot}0006$ for $d(\infty, \infty)$ and of $0{\cdot}186 \pm 0{\cdot}020$ for α. The dashed line expresses the relation $d(x, y) = d(x)\, d(y)$ determined by Palásti's conjecture.

3. THREE-DIMENSIONAL CASE

In the three-dimensional car-parking problem, we deal with the process of random sequential packing of unit cubes in a rectangular parallelepiped T_{xyz} with edge lengths $x, y, z > 1$. As in the two-dimensional case, the sides of unit cubes are always oriented appropriately. Palásti's conjecture in three dimensions means that

$$\lim_{x\to\infty,\, y\to\infty,\, z\to\infty} d(x,y,z) = c^3 \simeq 0{\cdot}4178. \tag{7}$$

The equations corresponding to (3) and (5) are

$$d(x,y,z) = d(x)\, d(y)\, d(z), \tag{8}$$

$$d(x,y,z) = c^3 - c^2(1-c)\,(1/x + 1/y + 1/z) + o(1/x) + o(1/y) + o(1/z). \tag{9}$$

Similarly, the extrapolation formula (6) should be replaced by

$$d(x,y,z) = d(\infty,\infty,\infty) - \beta u^{-1}, \tag{10}$$

where β is a positive constant and $u^{-1} = x^{-1} + y^{-1} + z^{-1}$.

The computer program used for the two-dimensional packing is not adequate for the present purpose. As a matter of fact, the difference between upper and lower bounds of $D(x, y, z)$ obtained thereby is considerably greater than in two dimensions and it is difficult to determine a sample value of $D(x, y, z)$ with high accuracy. Accordingly, we have developed a new technique by which the lower bound $\hat{D}(x, y, z)$ can be improved. The average of $\hat{D}(x, y, z)$ thus calculated, $\hat{d}(x, y, z)$, yields a good estimate of $d(x, y, z)$ itself. Table 2 shows sample means and standard deviations of the improved lower bound $\hat{D}(x, y, z)$ along with the values of $d(x,y,z)$ predicted from (8). The sizes of rectangular parallelepipeds were $8\times8\times8$, $10\times10\times10$, $12\times12\times12$, $15\times15\times15$, $20\times20\times20$ and $30\times30\times30$. Each sampling was replicated twenty times on the computer except for the case of $x\times y\times z = 30\times30\times30$, where only three trials were made.

Table 2. *Computer simulation results for three-dimensional random packing density*

Rectangular parallelepiped	Number of trials	Mean	Standard deviation	$d(x)\,d(y)\,d(z)$ Eq. (8)
$8\times8\times8$	20	0·3700	0·00866	0·3671
$10\times10\times10$	20	0·3783	0·00572	0·3769
$12\times12\times12$	20	0·3847	0·00572	0·3836
$15\times15\times15$	20	0·3899	0·00513	0·3903
$20\times20\times20$	20	0·3967	0·00286	0·3970
$30\times30\times30$	3	0·4031	0·00095	0·4039

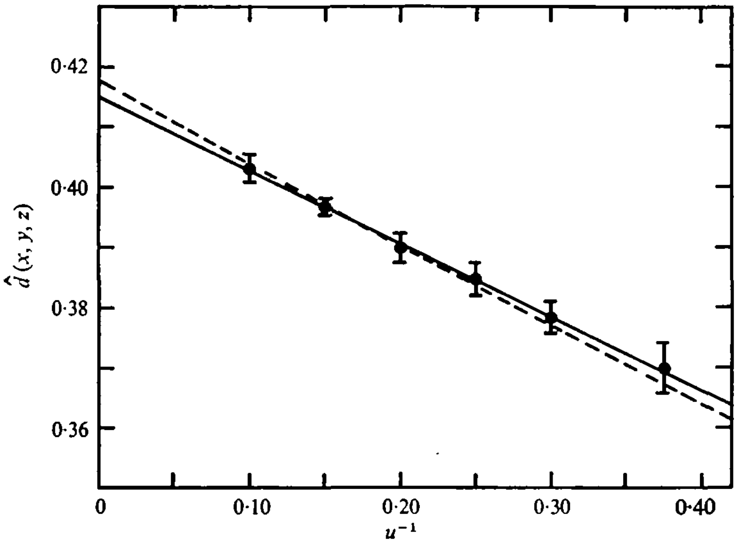

Fig. 2. Plot of $\hat{d}(x, y, z)$ as a function of u^{-1}. Data points indicate sample means of $\hat{D}(x, y, z)$ and error bars represent 95 % confidence intervals for $\hat{d}(x, y)$. The solid straight line obtained by a linear regression analysis gives 95 % confidence limits of $0{\cdot}4148\pm0{\cdot}0025$ for $\hat{d}(\infty, \infty, \infty)$ and of $0{\cdot}121\pm0{\cdot}012$ for β. The dashed line expresses the relation $d(x, y, z) = d(x)\,d(y)\,d(z)$ determined by Palásti's conjecture.

In Fig. 2 we plot the empirical values of $(\hat{d}x,y,z)$ against u^{-1}; the relation is effectively linear. Besides, the standard deviation of $\hat{D}(x,y,z)$ depends roughly linearly on u^{-1}. An ordinary regression analysis based on (10) gives a 95 % confidence interval of $0{\cdot}4148\pm0{\cdot}0025$ for $\hat{d}(\infty,\infty,\infty)$. Considering that $\hat{d}(\infty,\infty,\infty)$ is a lower bound of $d(\infty,\infty,\infty)$, it might be expected that $d(\infty,\infty,\infty)$ is approximately equal to or somewhat higher than $c^3 \simeq 0{\cdot}4178$. In contrast to the two-dimensional case, the slope β differs significantly from the first-order coefficient in (9). In fact, the 95 % confidence interval for β is $0{\cdot}121\pm0{\cdot}012$, while $c^2(1-c)$ is about 0·1411. This discrepancy is probably due to the fact that the accuracy of the lower bound $\hat{d}(x,y,z)$ varies with u^{-1}.

The authors would like to thank the referee for helpful comments.

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[*Received September* 1975. *Revised December* 1975]