S234/238/107C

Dublin Institute of Technology

School of Mathematical Sciences

MATH 9853: Numerical Analysis

DT234 MSc Applied Mathematics & Theoretical Physics

DT238 MSc Applied Mathematics & Theoretical Physics

Stage 1

Summer Examination Session

2013/2014

Examiners: Dr M. Venkova

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Date: Wednesday, 21 May 2014 | Time: 1.00 – 4.30 pm

Answer five questions

All questions carry equal marks

Approved calculators may be used

Mathematical tables will be provided

1. a) Let (X, d) be a metric space, define what is meant by saying that the mapping $T : X \to X$ is a *contraction*.

(3 marks)

b) Show that the function $g(x) = \frac{1}{3}e^{-x^3}$ is a contraction on the interval $[0, 1]$, and hence prove that $g(x)$ has a unique fixed point on this interval.

(6 marks)

c) With a starting approximation of $x_0 = 1$, use the iteration scheme $x_{n+1} = g(x_n)$ to approximate the fixed point to within 10^{-3} (use at least 5 decimal places in your calculations).

(5 marks)

d) Find the rate of convergence of the fixed point iteration scheme $x_{n+1} = g(x_n)$.

(6 marks)

[20]

2. a) Define what is meant by the *condition number* $k(A)$ of a matrix A.

(2 marks)

b) If $\tilde{x}$ is the approximate solution of the system $Ax = b$, x is the exact solution, $r = A\tilde{x} - b$ is the residual vector and $e = \tilde{x} - x$ is the error, show that

$$\frac{1}{k(A)} \frac{||r||}{||b||} \leq \frac{||e||}{||x||} \leq k(A) \frac{||r||}{||b||}.$$

(8 marks)

c) Calculate the condition number of the matrix B:

$$B = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix},$$

using the norm $||B||_\infty = \max_i(|b_{i1}| + |b_{i2}|)$. Is B ill-conditioned?

(5 marks)

d) Carry out three iterations of the Power Iteration Method to approximate the dominant eigenvalue of the matrix B.

(5 marks)

[20]

3. a) Derive the Gauss-Seidel method for solving the system of linear equations

$$Ax = b.$$

(7 marks)

b) Consider the system of linear equations

$$Ax = b$$

where

$$A = \begin{bmatrix} 5 & 1 & 0 \\ 1 & -5 & 1 \\ 0 & 1 & 6 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ is a vector of unknowns.}$$

Explain why it is valid to apply the Gauss-Seidel method to this system of linear equations. Carry out two iterations of the Gauss-Seidel method on this system, taking the initial x^0 to be the zero vector.

(8 marks)

c) Find the rate of convergence of the Gauss-Seidel method.

(5 marks)

[20]

4. a) Explain what it means for a finite-dimensional subspace Y of $C[a,b]$ to satisfy the *Haar condition*. Does the space $P_n[a,b]$ of all polynomials of degree n satisfy the Haar condition?

(5 marks)

b) Let Y be a finite-dimensional subspace of $C[a,b]$, and suppose Y satisfies the Haar condition. Given x in $C[a,b]$, let $y \in C[a,b]$ be such that for some $a \leq t_0 < t_1 \ldots, t_n \leq b$ the value of $x(t_j) - y(t_j)$ alternates between $\|x - y\|$ and $-\|x - y\|$. Show that y is the unique best approximation of x out of Y.

(5 marks)

c) Show that the Chebyshev polynomials (see appendix at the end of the paper) satisfy the recursion formula:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

(5 marks)

d) In $C[-1,1]$ find the best approximation to $f(x) = x^3 - x^2$ by a quadratic polynomial.

(5 marks)

[20]

5. a) Let V be an inner product space with an orthonormal basis $(e_n)_{n=1}^{\infty}$. If x is an element of V and Y is the subspace of V generated by $(e_n)_{n=1}^{m}$, define what is meant by the *least square approximation* of x out of Y.

(3 marks)

b) Prove that the least square approximation of x is the best approximation of x out of Y.

(7 marks)

c) Consider the data $(-1, 3)$, $(0, 2)$ and $(1, 3)$. Construct a natural cubic spline fitting this data.

(10 marks)

[20]

6. The linear multistep method for solving the initial value problem:

$$y' = f(t, y), \; y(t_0) = y_0$$

is given by the scheme:

$$y_{n+1} = 4y_n - 3y_{n-1} - 2hf(t_{n-1}, y_{n-1}). \tag{1}$$

a) Determine the order and the truncation error term for this method.

(12 marks)

b) Explain what it means for a linear multistep method to be stable.

(3 marks)

c) Explain what it means for a linear multistep method to satisfy the Root Condition. Find the characteristic polynomial of the method (1) and determine whether or not the method is stable.

(5 marks)

[20]

7. a) Define what is meant for a difference method to be

   i) consistent;

   ii) convergent.

   State the Lax Equivalence Theorem.

(6 marks)

b) Derive the numerical scheme

$$u_{j-1,k} + u_{j+1,k} + u_{j,k-1} + u_{j,k+1} - 4u_{j,k} = h^2 f_{j,k}$$

for the Poisson equation $u_{xx} + u_{yy} = f(x, y)$ on the rectangular domain $R = \{(x, y)|a < x < b, c < y < d\}$.

(5 marks)

c) Consider the problem

$$u_{xx} + u_{yy} = 6x + 8$$

on $R = \{(x, y)|0 \leq x \leq 1, 0 \leq y \leq 0.5\}$, with boundary conditions

$$u(x, 0) = x^3, \quad u(x, 0.5) = x^3 + 1, \quad u(0, y) = 4y^2, \quad u(1, y) = 4y^2 + 1.$$

Taking $N = 4$ steps in the x-direction and $M = 2$ steps in the y-direction , set up and write in matrix form (but do not solve) the corresponding system of finite difference equations.

(9 marks)

[20]

Appendix

- Chebyshev polynomials:

$$T_n(x) = \cos(n \cos^{-1}(x)).$$

- Central difference formula for the second derivative:

$$f''(x_0) = \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} + \mathcal{O}(h^2).$$