Problem Sheet 2

PhD students: solutions are to be handed in by Tuesday 14th of March

(1) Consider $\mathbb{R}^n$ with the norm $\|x\|_1 = \sum_{i=1}^{n} |x_i|$. Prove that the matrix norm corresponding to it is calculated by the formula $\|A\|_1 = \max_{j=1..n} \left(\sum_{i=1}^{n} |a_{ij}| \right)$.

(2) Consider the system of linear equations

$$Ax = b$$

where

$$A = \begin{bmatrix} 7 & 1 & 0 \\ 1 & -7 & 1 \\ 0 & 1 & 8 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$ is a vector of unknowns.

(a) Explain why it is valid to apply the Jacobi method to this system of linear equations. Carry out two iterations of the Jacobi method on this system, taking the initial x^0 to be the zero vector.

(b) Explain why it is valid to apply the Gauss-Seidel method to this system of linear equations. Carry out two iterations of the Gauss-Seidel method on this system, taking the initial x^0 to be the zero vector.

(3) Given the matrix:

$$B = \begin{bmatrix} 10 & 1 \\ 3 & -12 \end{bmatrix},$$

(a) calculate $\|B\|_\infty$, $\|B\|_1$ and $\|B\|_2$.

(b) Calculate the condition number of the matrix B using the norm $\|B\|_\infty = \max_i(|b_{i1}| + |b_{i2}|)$. Is B ill-conditioned?

(4) Let $Ax = b$ be a system of n linear equations with n variables, suppose A is invertible and let x^* be the solution of the system. If x^n is the nth iteration of a certain iterative method applied to the system, $e^n = x^* - x^n$ is the error at step n, $k(A)$ is the condition number of A and $r^n = b - Ax^n$ is the residual vector, show that

$$\frac{1}{k(A)} \frac{\|r^n\|}{\|b\|} \leq \frac{\|e^n\|}{\|x^*\|}$$