Tensors for Beginners

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1 Tensor Notations

The velocity of the wind at the top of Eiffel's tower, at a given moment, can be represented by a *vector* $\mathbf{v}$ with components, in some local, given, basis, $\{v^i\}$ $(i = 1, 2, 3)$. The velocity of the wind is defined at any point $\mathbf{x}$ of the atmosphere at any time t : we have a *vector field* $v^i(\mathbf{x}, t)$.

The water's temperature at some point in the ocean, at a given moment, can be represented by a *scalar* T . The field $T(\mathbf{x}, t)$ is a *scalar field*.

The state of stress at a given point of the Earth's crust, at a given moment, is represented by a *second order tensor* $\boldsymbol{\sigma}$ with components $\{\sigma^{ij}\}$ $(i = 1, 2, 3;\ j = 1, 2, 3)$. In a general model of continuous media, where it is not assumed that the stress tensor is symmetric, this means that we need 9 scalar quantities to characterize the state of stress. In more particular models, the stress tensor is symmetric, $\sigma^{ij} = \sigma^{ji}$, and only six scalar quantities are needed. The stress field $\sigma^{ij}(\mathbf{x}, t)$ is a *second order tensor field*.

Tensor fields can be combined, to give other fields. For instance, if n_i is a unit vector considered at a point inside a medium, the vector

$$\tau^i(\mathbf{x}, t) \;=\; \sum_{j=1}^{3} \sigma^{ij}(\mathbf{x}, t)\, n_j(\mathbf{x}) \;=\; \sigma^{ij}(\mathbf{x}, t)\, n_j(\mathbf{x}) \tag{1}$$

represents the traction that the medium at one side of the surface defined by the normal n_i exerts the medium at the other side, at the considered point.

As a further example, if the deformations of an elastic solid are small enough, the stress tensor is related linearly to the strain tensor (Hooke's law). A linear relation between two second order tensors means that each component of one tensor can be computed as a linear combination of all the components of the other tensor:

$$\sigma^{ij}(\mathbf{x}, t) \;=\; \sum_{k=1}^{3} \sum_{\ell=1}^{3} c^{ijk\ell}(\mathbf{x})\, \varepsilon_{k\ell}(\mathbf{x}, t) \;=\; c^{ijk\ell}(\mathbf{x})\, \varepsilon_{k\ell}(\mathbf{x}, t)\,. \tag{2}$$

The *fourth order tensor* c^{ijkl} represents a property of an elastic medium: its elastic stiffness. As each index takes 3 values, there are $3 \times 3 \times 3 \times 3 = 81$ scalars to define the elastic stiffness of a solid at a point (assuming some symmetries we may reduce this number to 21, and asuming isotropy of the medium, to 2).

We are not yet interested in the physical meaning of the equations above, but in their structure. First, tensor notations are such that they are independent on the coordinates being used. This is not obvious, as changing the coordinates implies changing the local basis where the components of vectors and tensors are expressed. That the two equalities equalities above hold for any coordinate system, means that all the components of all tensors will change if we change the coordinate system being used (for instance, from Cartesian to spherical coordinates), but still the two sides of the expression will take equal values.

The mechanics of the notation, once understood, are such that it is only possible to write expressions that make sense (see a list of rules at the end of this section).

For reasons about to be discussed, indices may come in upper or lower positions, like in v^i , f_i or $T_i{}^j$. The definitions will be such that in all tensor expression (i.e., in all expressions that will be valid for all coordinate systems), the sums over indices will always concern an index in lower position an one index on upper position. For instance, we may encounter expressions like

$$\varphi \;=\; \sum_{i=1}^{3} A_i B^i \;=\; A_i B^i \tag{3}$$

or

$$A_i \;=\; \sum_{j=1}^{3}\sum_{k=1}^{3} D_{ijk} E^{jk} \;=\; D_{ijk} E^{jk} \quad . \tag{4}$$

These two equations (as equations 1 and 2) have been written in two versions, one with the sums over the indices explicitly indicated, and another where this sum is implicitly assumed. This implicit notation is useful as one easily forgets that one is dealing with sums, and that it happens that, with respect to the usual tensor operations (sum with another tensor field, multiplication with another tensor field, and derivation), a sum of such terms is handled as one single term of the sum could be handled.

In an expression like $A_i = D_{ijk} E^{jk}$ it is said that the indices j and k have been *contracted* (or are "dummy indices"), while the index i is a *free index*. A tensor equation is assumed to hold for all possible values of the free indices.

In some spaces, like our physical 3-D space, it is posible to define the distance between two points, and in such a way that, in a local system of coordinates, approximately Cartesian, the distance has approximately the Euclidean form (square root of a sum of squares). These spaces are called *metric spaces*. A mathematically convenient manner to introduce a metric is by defining the length of an arc Γ by $S = \int_\Gamma ds$, where, for instance, in Cartesian coordinates, $ds^2 = dx^2 + dy^2 + dz^2$ or, in spherical coordinates, $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2$. In general, we write $ds^2 = g_{ij} dx^i dx^j$, and we call $g_{ij}(\mathbf{x})$ the *metric field* or, simply, the *metric*.

The components of a vector $\mathbf{v}$ are associated to a given basis (the vector will have different components on different basis). If a basis $\mathbf{e}_i$ is given, then, the components v^i are defined through $\mathbf{v} = v^i \mathbf{e}_i$ (implicit sum). The *dual basis* of the basis $\{\mathbf{e}_i\}$ is denoted $\{\mathbf{e}^i\}$ and is defined by the equation $\mathbf{e}_i \mathbf{e}^j = \delta_i{}^j$ (equal to 1 if i are the same index and to 0 if not). When there is a metric, this equation can be interpreted as a scalar vector product, and the dual basis is just another basis (identical to the first one when working with Cartesian coordinates in Euclidena spaces, but different in general). The properties of the dual basis will be analyzed later in the chapter. Here we just need to recall that if v^i are the components of the vector $\mathbf{v}$ on the basis $\{\mathbf{e}_i\}$ (remember the expression $\mathbf{v} = v^i \mathbf{e}_i$), we will denote by v_i are the components of the vector $\mathbf{v}$ on the basis $\{\mathbf{e}^i\}$: $\mathbf{v} = v_i \mathbf{e}^i$. In that case (metric spaces) the components on the two basis are related by $v_i = g_{ij} v^i$: It is said that "the metric tensor ascends (or descends) the indices".

Here is a list with some rules helping to recognize tensor equations:

- A tensor expression must have the same *free* indices, at the top and at the bottom, of the two sides of an equality. For instance, the expressions

$$\begin{aligned} \varphi &= A_i\, B^i \\ \varphi &= g_{ij}\, B^i\, C^j \\ A_i &= D_{ijk}\, E^{jk} \\ D_{ijk} &= \nabla_i F_{jk} \end{aligned} \tag{5}$$

  are valid, but the expressions

$$\begin{aligned} A_i &= F_{ij}\, B^i \\ B^i &= A_j\, C^j \\ A_i &= B^i \end{aligned} \tag{6}$$

  are not.

- Sum and multiplication of tensors (with eventual "contraction" of indices) gives tensors. For instance, if D_{ijk} , G_{ijk} and $H_i{}^j$ are tensors, then

$$\begin{aligned} J_{ijk} &= D_{ijk} + G_{ijk} \\ K_{ijk\ell}{}^m &= D_{ijk}\, H_\ell{}^m \\ L_{ik\ell} &= D_{ijk}\, H_\ell{}^j \end{aligned} \tag{7}$$

  also are tensors.

- True (or "covariant") derivatives of tensor felds give tensor fields. For instance, if E^{ij} is a tensor field, then

$$\begin{aligned} M_i{}^{jk} &= \nabla_i E^{jk} \\ B^j &= \nabla_i E^{ij} \end{aligned} \tag{8}$$

  also are tensor fields. But partial derivatives of tensors do not define, in general, tensors. For instance, if E^{ij} is a tensor field, then

$$\begin{aligned} M_i{}^{jk} &= \partial_i V^{jk} \\ B^j &= \partial_i V^{ij} \end{aligned} \tag{9}$$

  are not tensors, in general.

- All “objects with indices” that are normally introduced are tensors, with four notable exceptions. The first exception are the coordinates $\{x^i\}$ (to see that it makes no sense to add coordinates, think, for instance, in adding the spherical coordinates of two points). But the differentials dx^i appearing in an expression like $ds^2 = g_{ij}\, dx^i\, dx^j$ do correspond to the components on a vector $d\mathbf{r} = dx^i \mathbf{e}_i$. Another notable exception is the “symbol” ∂_i mentioned above. The third exception is the “connection” $\Gamma_{ij}{}^k$ to be introduced later in the chapter. In fact, it is because both of the symbols ∂_i and $\Gamma_{ij}{}^k$ are not tensors than an expression like

$$\nabla_i V^j \;=\; \partial_i V^j + \Gamma_{ik}{}^j\, V^k \tag{10}$$

can have a tensorial sense: if one of the terms at right was a tensor and not the other, their sum could never give a tensor. The objects ∂_i and $\Gamma_{ij}{}^k$ are both non tensors, and “what one term misses, the other term has”. The fourth and last case of “objects with indices” which are not tensors are the Jacobian matrices arising in coordinate changes $\mathbf{x} \rightleftharpoons \mathbf{y}$,

$$J^i{}_I \;=\; \frac{\partial x^i}{\partial y^I} \, . \tag{11}$$

That this is not a tensor is obvious when considering that, contrarily to a tensor, the Jacobian matrix is not defined per se, but it is only defined when two different coordinate systems have been chosen. A tensor exists even if no coordinate system at all has been defined.

2 Differentiable Manifolds

A manifold is a continuous space of points. In an n-dimensional manifold it is always possible to “draw” *coordinate lines* in such a way that to any point $\mathcal{P}$ of the manifold correspond coordinates $\{x^1, x^2, \dots, x^n\}$ and vice versa.

Saying that the manifold is a continuous space of points is equivalent to say that the coordinates themselves are “continuous”, i.e., if they are, in fact, a part of $\mathcal{R}^n$. On such manifolds we define physical fields, and the continuity of the manifold will allow to define the derivatives of the considered fields. When derivatives of fields on a manifold can be defined, the manifold is then called a *differentiable manifold*.

Obvious examples of differentiable manifolds are the lines and surfaces of ordinary geometry. Our 3-D physical space (with, possibly, curvature and torsion) is also represented by a differentiable manifold. The space-time of general relativity is a four dimensional differentiable manifold.

A coordinate system may not “cover” all the manifold. For instance, the poles of a sphere are as ordinary as any other point in the sphere, but the coordinates are singular there (the coordinate φ is not defined). Changing the coordinate system around the poles will make any problem related to the coordinate choice to vanish there. A more serious difficulty appears when at some point, not the coordinates, but the manifold itself is singular (the linear tangent space is not defined at this point), as for instance, in the example shown in figure 1. Those ane named “essential singularities”. No effort will be made on this book to classify them.

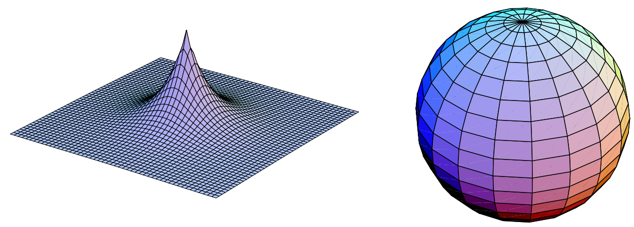


Figure 1: The surface at left has an essential singularity that will cause trouble for whatever system of coordinates we may choose (the tangent linear space is not defined at the singular point). The sphere at rigth has no essential singularity, but the coordinate system chosen is singular at the two poles. Other coordinate systems will be singular at different points.

3 Tangent Linear Space, Tensors.

Consider, for instance, in classical dynamics, a trajectory $x^i(t)$ on a space which may not be flat, as the surface of a sphere. The trajectory is "on" the sphere. If we define now the velocity at some point,

$$v^i = \frac{dx^i}{dt} , \tag{12}$$

we get a vector which is not "on" the sphere, but *tangent* to it. It belongs to what is called the *tangent linear space* to the considered point. At that point, we will have a basis for vectors. At another point, we will another tangent linear space, and another vector basis.

More generally, at every point of a differential manifold, we can consider different vector or tensor quantities, like the *forces*, *velocities*, or *stresses* of mechanics of continuous media. As suggested by figure 2, those tensorial objects do not belong to the nonlinear manifold, but to the *tangent linear space* to the manifold at the considered point (that will only be introduced intuitively here).

At every point of an space, tensors can be added, multiplied by scalars, contracted, etc. This means that at every point of the manifold we have to consider a different vector space (in general, a tensor space). It is important to understand that two tensors at two different points of the space belong to two different tangent spaces, and can not be added as such (see figure 2). This is why we will later need to introduce the concept of "parallel transport of tensors".

All through this book, the two names *linear space* and *vector space* will be used as completely equivalent.

The structure of vector space is too narrow to be of any use in physics. What is needed is the structure where equations like

$$\begin{aligned} \lambda &= R_i \, S^i \\ T^j &= U_i \, V^{ij} + \mu \, W^j \\ X^{ij} &= Y^i \, Z^j \end{aligned} \tag{13}$$

make sense. This structure is that of a *tensor space*. In short, a tensor space is a collection of vector spaces and rules of multiplication and differentiation that use elements of the vector spaces considered to get other elements of other vector spaces.

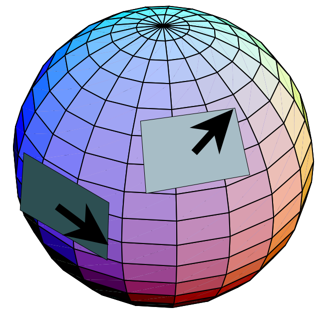


Figure 2: Surface with two planes tangent at two points, and a vector drawn at each point. As the vectors belong to two different vector spaces, their sum is not defined. Should we need to add them, for instance, to define true (or "covariant") derivatives of the vector field, then, we would need to transport them (by "parallel transportation") to a common point.

4 Vectors and Forms

When we introduce some vector space, with elements denoted, for instance, $\mathbf{V}$, $\mathbf{v}' \ldots$, it often happens that a new, different, vector space is needed, with elements denoted, for instance $\mathbf{F}$, $\mathbf{F}' \ldots$, and such that when taking an element of each space, we can "multiply" them and get a scalar,

$$\lambda = \langle\, \mathbf{F} \,,\ \mathbf{V} \,\rangle \,. \tag{14}$$

In terms of components, this will be written

$$\lambda = F_i\, V^i \,. \tag{15}$$

The product in 14–15, is called a *duality product*, and it has to be clearly distinguished from an inner (or scalar) product: in an inner product, we multiply two elements of a vector space; in a duality product, we multiply an element of a vector space by an element of a "dual space".

This operation can always be defined, including the case where the do not have a metric (and, therefore, a scalar product). As an example, imagine that we work with pieces of metal and we need to consider the two parameters "electric conductivity" σ and "temperature" T . We may need to consider some (possibly nonlinear) function of σ and T , say $S(\sigma, T)$. For instance, $S(\sigma, T)$ may represent a "misfit function" on the (σ, T) space of those encountered when solving inverse problems in physics if we are measuring the parameters σ and T using indirect means. In this case, S is adimensional[1]. We may wish to know by which amont will S change when passing from point (σ_0, T_0) to a neighbouring point $(\sigma_0 + \Delta\sigma, T_0 + \Delta T)$. Writing only the first order term, and using matrix notations,

$$S(\sigma_0 + \Delta\sigma, T_0 + \Delta T) = S(\sigma_0, T_0) + \begin{pmatrix} \frac{\partial S}{\partial \sigma} \\ \frac{\partial S}{\partial T} \end{pmatrix}^T \begin{pmatrix} \Delta\sigma \\ \Delta T \end{pmatrix} + \ldots \,, \tag{16}$$

where the partial derivatives are taken at point (σ_0, T_0) . Using tensor notations, setting $\mathbf{x} = (x^1, x^2) = (\sigma, T)$, we can write

$$\begin{aligned} S(\mathbf{x} + \Delta\mathbf{x}) &= S(\mathbf{x}) + \sum_i \frac{\partial S}{\partial x^i}\, \Delta x^i \\ &= S(\mathbf{x}) + \gamma_i\, \Delta x^i \\ &= S(\mathbf{x}) + \langle\, \boldsymbol{\gamma} \,,\ \Delta\mathbf{x} \,\rangle \,, \end{aligned} \tag{17}$$

[1] For instance, one could have the simple expression $S(\sigma, T) = \frac{|\sigma - \sigma_0|}{s_P} + \frac{|T - T_0|}{s_T}$, where s_P and s_T are standard deviations (or mean deviations) of some probability distribution.

where the notation introduced in equations 14–15 is used. As above, the partial derivatives are taken at point $\mathbf{x}_0 = (x_0^1, x_0^2) = (\sigma_0, T_0)$.

Note: say that figure 3 illustrates the definition of gradient as a tangent linear application. Say that the "mille-feuilles" are the "level-lines" of that tangent linear application.

Note: I have to explain somewhere the reason for putting an index in lower position to represent $\partial/\partial x^i$, i.e., to use the notation

$$\partial_i = \frac{\partial}{\partial x^i} \ .$$

Note: I have also to explain in spite of the fact that we have here partial derivatives, we have defined a tensorial object: the partial derivative of a scalar equals its true (covariant) derivative.

It is important that we realize that there is no "scalar product" involved in equations 17. Here are the arguments:

- The components of γ_i are **not** the components of a vector in the (σ, T) space. This can directly be seen by an inspection of their physical dimensions. As the function S is adimensional (see footnote 1), the components of $\boldsymbol{\gamma}$ have as dimensions the **inverse** of the physical dimensions of the components of the vector $\Delta\mathbf{x} = (\Delta x^1, \Delta x^2) = (\Delta\sigma, \Delta T)$. This clearly means that $\Delta\mathbf{x}$ and $\boldsymbol{\gamma}$ are "objects" that do not belong to the same space.

- If equations 17 involved a scalar product we could define the norm of $\mathbf{x}$, the norm of $\boldsymbol{\gamma}$ and the angle between $\mathbf{x}$ and $\boldsymbol{\gamma}$. But these norms and angle are not defined. For instance, what could be the norm of $\mathbf{x} = (\Delta\sigma, \Delta T)$? Should we choose an L_2 norm? Or, as suggested by footnote 1, an L_1 norm? And, in any case, how could we make consistent such a definition of a norm with a change of variables where, instead of electric conductivity we use electric resistivity? (Note: make an appendix where the solution to this problem is given).

The product in equations 17 is not a scalar product (i.e., it is not the "product" of two elements belonging to the same space): it is a "duality product", multiplying an element of a vector space and one element of a "dual space".

Why this discussion is needed? Because of the tendency of imagining the gradient of a function $S(\sigma, T)$ as a vector (an "arrow") in the $S(\sigma, T)$ space. If the gradient is not an arrow, then, what it is? Note: say here that figures 4 and 5 answer this by showing that an element of a dual space can be represented as a "mille-feuilles".

Up to here we have only considered a vector space and its dual. But the notion generalizes to more general tensor spaces, i.e., to the case where "we have more than one index". For instance, instead of equation 15 we could use an equation like

$$\lambda = F_{ij}{}^k \; V^{ij}{}_k \tag{18}$$

to define scalars, consider that we are doing a duality product, and also use the notation of equation 14 to denote it. But this is not very useful, as, from a given "tensor" $F_{ij}{}^k$ we can obtain scalar by operations like

$$\lambda = F_{ij}{}^k \; V^i \; W^j{}_k \ . \tag{19}$$

It is better, in general, to just write explicitly the indices to indocate which sort of "product" we consider.

Sometimes (like in quantum mechanics), a "bra-ket" notation is used, where the name stands for the *bra* "$\langle\, |$" and the *ket* "$|\, \rangle$". Then, instead of $\lambda = \langle\, \mathbf{F}\, ,\, \mathbf{V}\, \rangle$ one writes

$$\lambda = \langle\, \mathbf{F} \mid \mathbf{V}\, \rangle = F_i\, V^i\, . \tag{20}$$

Then, the bra-ket notation is also used for the expression

$$\lambda = \langle\, \mathbf{V} \mid \mathbf{H} \mid \mathbf{W}\, \rangle = H_{ij}\, V^i\, W^j\, . \tag{21}$$

Note: say that the general rules for the change of component values in a change of coordinates, allow us to talk about "tensors" for "generalized vectors" as well as for "generalized forms".

The "number of indices" that have to be used to represent the components of a tensor is called the *rank*, or the *order* of the tensor. Thus the tensors $\mathbf{F}$ and $\mathbf{V}$ just introduced are second rank, or second order. A tensor object with components $R_{ijk}{}^{\ell}$ could be called, in all rigor, a "(third-rank-form)-(first-rank-vector)" will we wil not try to usew this heavy terminology, the simple writing of the indices being explicit.

Note: say that if there is a metric, there is a trivial identification between a vector space and its dual, through equations like $F_i = g_{ij} V^j$, or $S^{ijk}{}_{\ell} = g^{ip} g^{jq} g^{kr} g_{\ell s} R_{pqr}{}^{s}$, and in that case, the same letter is used to designate one vector and *its* dual element, as in $V_i = g_{ij} V^j$, and $R^{ijk}{}_{\ell} = g^{ip} g^{jq} g^{kr} g_{\ell s} R_{pqr}{}^{s}$. But in *non metric* spaces (i.e., spaces without metric), there is usually a big difference between an space and its dual.

Gradient and Hessian Explain somewhere that if $\phi(\mathbf{x})$ is a scalar function, the Taylor development

$$\phi(\mathbf{x} + \Delta\mathbf{x}) = \phi(\mathbf{x}) + \langle\, \mathbf{g}\, | \Delta\mathbf{x}\, \rangle + \frac{1}{2!}\, \langle\, \Delta\mathbf{x} \,|\, \mathbf{H} \,|\, \Delta\mathbf{x}\, \rangle \tag{22}$$

defines the gradien $\mathbf{g}$ and the Hessian $\mathbf{H}$.

Old text We may want the gradient to be "perpendicular" at the level lines of φ at $\mathcal{O}$, but there is no *natural* way to define a scalar product in the $\{P, T\}$ space, so we can not naturally define what "perpendicularity" is. That there is no natural way to define a scalar product does not mean that we can not define one: we can define many. For any symmetric, positive-definite matrix with the right physical dimensions (i.e., for any covariance matrix), the expression

$$\left(\begin{bmatrix} \delta P_1 \\ \delta T_1 \end{bmatrix} , \begin{bmatrix} \delta P_2 \\ \delta T_2 \end{bmatrix} \right) = \begin{bmatrix} \delta P_1 \\ \delta T_1 \end{bmatrix}^T \begin{bmatrix} C_{PP} & C_{PT} \\ C_{TP} & C_{TT} \end{bmatrix}^{-1} \begin{bmatrix} \delta P_2 \\ \delta T_2 \end{bmatrix}$$

defines a scalar product. By an appropriate choice of the covariance matrix, we can make any of the two lines in figure 4 (or any other line) to be perpendicular to the level lines at the considered point: the gradient at a given point is something univocally defined, even in the absence of any scalar product; the "direction of steepest descent" is not, and there are as many as we may choose different scalar products. The gradient is not an arrow, i.e, it is not a *vector*. So, then, how to draw the gradient? Roughly speaking, the gradient is the *linear tangent application* at the considered point. It is represented in figure 5. As, by definition, it is a linear application, the level lines are straight lines, and the spacing of the level lines in the tangent linear application corresponds to the spacing of the level lines in the original function around the point where the gradient is computed. Speaking more technically, it is the development

$$\begin{aligned} \varphi(\mathbf{x}+\delta\mathbf{x}) &= \varphi(\mathbf{x}) + \langle\, \mathbf{g} \,,\, \delta\mathbf{x} \,\rangle + \dots \\ &= \varphi(\mathbf{x}) + g_i\, \delta x^i + \dots\,, \end{aligned}$$

when limited to its first order, that defines the tangent linear application. The *gradient* of φ is then $\mathbf{g}$. The gradient $\mathbf{g} = \{g_i\}$ at $\mathcal{O}$ allows to associate a scalar to any vector $\mathbf{V} = \{V^i\}$ (also at $\mathcal{O}$): $\lambda = g_i\, V^i = \langle\, \mathbf{g} \,,\, \mathbf{V} \,\rangle$. This scalar is the difference of the values at the top and the bottom of the arrow representing the vector $\mathbf{V}$ on the local tangent linear application to φ at $\mathcal{O}$. The index on the gradient can be a lower index, as the gradient is not a vector.

Note: say that figure 6 illustrates the fact that an element of the dual space can be represented as a "mille-feuilles" in the "primal" space or a an "arrow" un the dual space. And reciprocally.

Note: say that figure 7 illustrates the sum of arrows and the sum of "mille-feuilles".

Note: say that figure 8 illutrates the sum of "mille-feuilles" in 3-D.

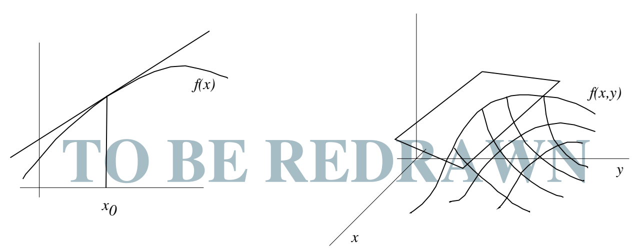


Figure 3: The gradient of a function (i.e., of an application) at a point x_0 is the tangent linear application at the given point. Let $x \mapsto f(x)$ represent the original (possibly nonlinear) application. The tangent linear application could be considered as mapping x into the values given by the linearized approximation fo $f(x)$: $x \mapsto F(x) = \alpha + \beta\, x$. (Note: explain better). Rather, it is mathematically simpler to consider that the gradient maps *increments* of the independent variable x, $\Delta x = x - x_0$ into increments of the linearized dependent variable: $\Delta y = y - f(x_0)$: $\Delta x \mapsto \Delta y = \beta\, \Delta x$. (Note: explain this MUCH better).

5 Natural Basis

A coordinate system associates to any point of the space, its coordinates. Each individual coordinate can be seen as a function associating, to any point of the space, the particular coordinate. We can define the gradient of this scalar function. We will have as many gradients f^i as coordinates x^i. As a gradient, we have seen, is a form, we will have as many forms as coordinates. The usual requirements that coordinate systems have to fulfill (different points

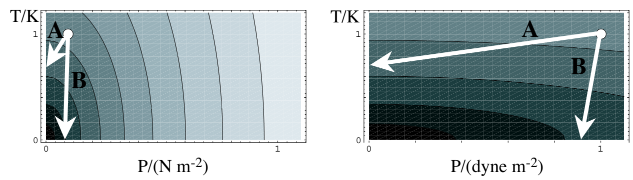


Figure 4: A scalar function $\varphi(P,T)$ depends on pressure and temperature. From a given point, two directions in the $\{P,T\}$ space are drawn. Which one corresponds to the gradient of $\varphi(P,T)$? In the figure at left, the pressure is indicated in International Units (m, kg, s), while in the figure at right, the c.g.s. units (cm, g, s) are used (remember that 1 Pa = 10 dyne/cm^{-2}). From the left figure, we may think that the gradient is direction A , while from the figure at right we may think it is B . It is none: the right definition of gradient (see text) only allows, as graphic representation, the result shown in figure 5.

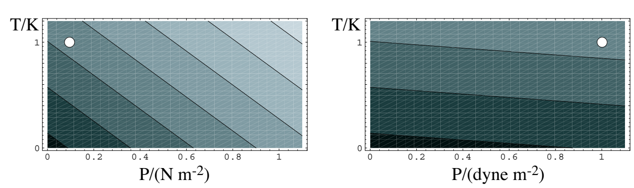


Figure 5: Gradient of the function displayed in figure 4, at the considered point. As the gradient is the linear tangent application at the given point, it is a linear application, and its level lines are stright lines. The value of the gradient at the considered point equals the value of the original function at that point. The spacing of the level lines in the gradient corresponds to the spacing of the level lines in the original function around the point where the gradient is computed. The two figures shown here are perflectly equivalent, as it should.

Figure 6: A point, at the left of the figure, may serve as the origin point for any vector we may want to represent. As usual, we may represent a vector $\mathbf{V}$ by an arrow. Then, a form $\mathbf{F}$ is represented by an oriented pattern of lines (or by an oriented pattern of surfaces in 3-D) with the line of zero value passing through the origin point. Each line has a value, that is the number that the form associates to any vector whose end point is on the line. Here, $\mathbf{V}$ and $\mathbf{F}$ are such that $\langle \mathbf{F}, \mathbf{V} \rangle = 2$. But a form is an element of the dual space, wich is also a linear space. In the dual space, then, the form $\mathbf{F}$ can be represented by an arrow (figure at right). In turn, V is represented, in the dual space, by a pattern of lines.

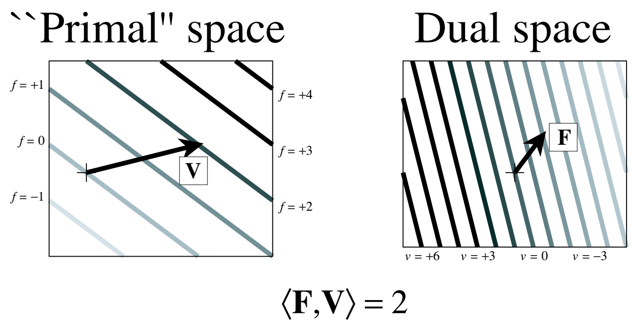


Figure 7: When representing vectors by arrows, the sum of two vectors is given by the main diagonal of the "parallelogram" drawn by two arrows. Then, a form is represented by a pattern of lines. The sum of two forms can be geometrically obtained using the "parallelogram" defined by the principal lozenge (containing the origin and with positive sense for both forms): the secondary diagonal of the lozenge is a line of the sum of the two forms. Note: explain this better.

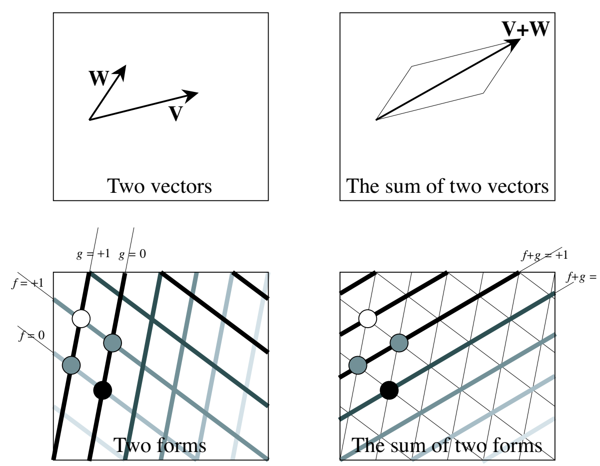
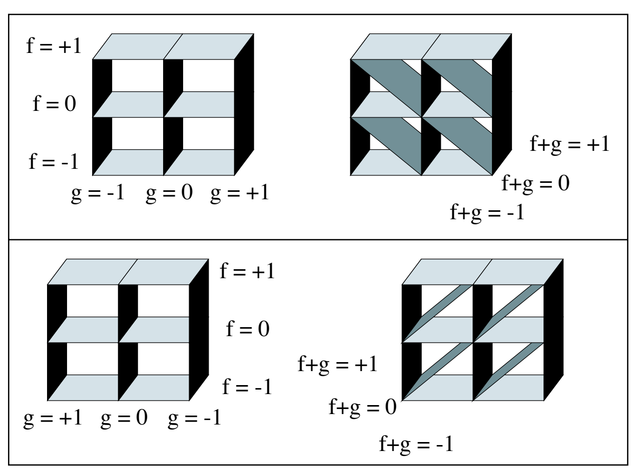


Figure 8: Sum of two forms, like in the previous figure, but here in 3-D. Note: explain that this figure can be "sheared" an one wants (we do not need to have a metric). Note: explain this better.



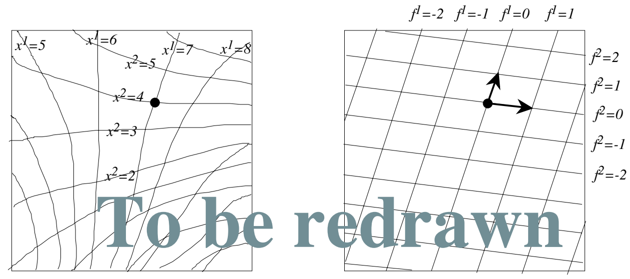


Figure 9: A system of coordinates, at left, and their gradients, at right. These gradient are forms. When in an n-dimensional space we have n forms, we can define n associate vertors by $\langle\, \mathbf{f}^i \,,\, \mathbf{e}_j \,\rangle = \delta^j{}_i$.

of the space have different coordinates, and vice versa) gives n linearly independent forms (we can not obtain one of them by linear combination of the others), i.e., a *basis* for the forms.

If we have a basis $\mathbf{f}^i$ of forms, then we can introduce a basis $\mathbf{e}_i$ of vectors, through

$$\langle\, \mathbf{f}^i \,,\, \mathbf{e}_j \,\rangle = \delta^j{}_i \ . \tag{23}$$

If we define the components V^i of a vector $\mathbf{V}$ by

$$\mathbf{V} = V^i \, \mathbf{e}_i \ , \tag{24}$$

then, we can compute the components V^i by the formula

$$V^i = \langle\, \mathbf{f}^i \,,\, \mathbf{V} \,\rangle \ , \tag{25}$$

as we have

$$\langle\, \mathbf{f}^i \,,\, \mathbf{V} \,\rangle = \langle\, \mathbf{f}^i \,,\, V^j \, \mathbf{e}_j \,\rangle = \langle\, \mathbf{f}^i \,,\, \sum_j V^j \, \mathbf{e}_j \,\rangle = \sum_i V^j \, \langle\, \mathbf{f}^i \,,\, \mathbf{e}_j \,\rangle = \sum_i V^j \, \delta^i{}_j = V^j \, \delta^i{}_j = V^i \ . \tag{26}$$

Note that the computation of the components of a vector does not involve a scalar product, but a duality product.

To find the equivalent of equations 24 and 25 for forms, one defines the components F_i of a form $\mathbf{F}$ by

$$\mathbf{F} = F_i \, \mathbf{f}^i \ , \tag{27}$$

and one easily gets

$$F_i = \langle\, \mathbf{F} \,,\, \mathbf{e}_i \,\rangle \ . \tag{28}$$

The notation $\mathbf{e}_i$ for the basis of vectors is quite universal. Although the notation $\mathbf{f}^i$ seems well adapted for a basis of forms, it is quite common to use the same letter for the basis of forms and for the basis of vectors. In what follows, we will use the notation

$$\mathbf{e}^i \equiv \mathbf{f}^i \ . \tag{29}$$

whose dangerousness vanishes only if we have a metric, i.e., when we can give sense to an expression like $\mathbf{e}_i = g_{ij} \mathbf{e}^j$. Using this notation the expressions

$$\mathbf{V} = V^i \, \mathbf{e}_i \quad \Longleftrightarrow \quad V^i = \langle\, \mathbf{f}^i \,,\, \mathbf{V} \,\rangle \qquad ; \qquad \mathbf{F} = F_i \, \mathbf{f}^i \ , \quad \Longleftrightarrow \quad F_i = \langle\, \mathbf{F} \,,\, \mathbf{e}_i \,\rangle \tag{30}$$

become

$$\mathbf{V} = V^i \, \mathbf{e}_i \quad \Longleftrightarrow \quad V^i = \langle\, \mathbf{e}^i \,,\, \mathbf{V} \,\rangle \qquad ; \qquad \mathbf{F} = F_i \, \mathbf{e}^i \ , \quad \Longleftrightarrow \quad F_i = \langle\, \mathbf{F} \,,\, \mathbf{e}_i \,\rangle \ . \tag{31}$$

We have now basis for vectors and forms, so wa can write expressions like $\mathbf{V} = V^i\,\mathbf{e}_i$ and $\mathbf{F} = F_i\,\mathbf{e}^i$. We need basis for objects "with more than one index", so we can write expressions like

$$\mathbf{B} = B^{ij}\,\mathbf{e}_{ij} \quad ; \quad \mathbf{C} = C_{ij}\,\mathbf{e}^{ij} \quad ; \quad \mathbf{D} = C_i{}^j\,\mathbf{e}^i{}_j \quad ; \quad \mathbf{E} = E_{ijk\ldots}{}^{\ell mn\ldots}\,\mathbf{e}^{ijk\ldots}{}_{\ell mn\ldots} \tag{32}$$

The introduction of these basis raises a difficulty. While we have an immediate intuitive representation for vectors (as "arrows") and for forms (as "millefeuilles"), tensor objects of higher rank are more difficult to represent. If a symmetric 2-tensor, like the stress tensor σ^{ij} of mechanics, can be viewed as an ellipsoid, how could we view a tensor $T_{ijk}{}^{\ell m}$? It is the power of mathematics to suggest analogies, so we can work even without geometric interpretations. But this absence of intuitive interpreation of high-rank tensors tells us that we will have to introduce the basis for these objects in a non-intuitive way. Essentially, what we want is that the basis for high rank tensors is not independent for the basis of vectors and forms. We want, in fact, more than this. Given two vectors U^i and V^i , we understand what we mean when we define a 2-tensor $\mathbf{W}$ by $W^{ij} = U^i\,V^j$. The basis for 2-tensors is perfectly defined by the condition that we wish that the components of $\mathbf{W}$ are precisely $U^i\,V^j$ and not, for instance, the values obtained after some rotation or change of coordinates.

This is enough, and we could directly use the notations introduced by equations 32. Instead, common mathematical developments introduce the notion of "tensor product", and, instead of notations like $\mathbf{e}_{ij}$, $\mathbf{e}^{ij}$, $\mathbf{e}_i{}^j$, or $\mathbf{e}^{ijk\ldots}{}_{\ell mn\ldots}$, introduce the notations $\mathbf{e}_i \otimes \mathbf{e}_i$, $\mathbf{e}^i \otimes \mathbf{e}^j$, $\mathbf{e}_i \otimes \mathbf{e}^j$, or $\mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^k \otimes \ldots \mathbf{e}_\ell \otimes \mathbf{e}_m \otimes \mathbf{e}_n \otimes \ldots$. Then, equations 32 are written

$$\mathbf{B} = B^{ij}\,\mathbf{e}_i \otimes \mathbf{e}_i \quad ; \quad \mathbf{C} = C_{ij}\,\mathbf{e}^i \otimes \mathbf{e}^j \quad ; \quad \mathbf{D} = C_i{}^j\,\mathbf{e}^i \otimes \mathbf{e}^j$$

$$\mathbf{E} = E_{ijk\ldots}{}^{\ell mn\ldots}\,\mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^k \otimes \ldots \mathbf{e}_\ell \otimes \mathbf{e}_m \otimes \mathbf{e}_n \otimes \ldots \; . \tag{33}$$

What follows is an old text, to be updated.

The metric tensor has been introduced in section **??**. Let us show here that if the space into consideration has a scalar product, then, the metric can be computed. Here, the scalar product of two vectors $\mathbf{V}$ and $\mathbf{W}$ is denoted $\mathbf{V} \cdot \mathbf{W}$. Then, defining

$$d\mathbf{r} = dx^i\,\mathbf{e}_i \tag{34}$$

and

$$ds^2 = d\mathbf{r} \cdot d\mathbf{r} \tag{35}$$

gives

$$ds^2 = d\mathbf{r} \cdot d\mathbf{r} = (dx^i\,\mathbf{e}_i) \cdot (dx^j\,\mathbf{e}_j) = (\mathbf{e}_i \cdot \mathbf{e}_j)\,dx^i\,dx^j \; . \tag{36}$$

Defining the metric tensor

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j \tag{37}$$

gives then

$$ds^2 = g_{ij}\,dx^i\,dx^j \; . \tag{38}$$

To emphasize that at every point of the manifold we have a different tensor space, and a different basis, we can always write explicitly the dependence of the basis vectors on the coordinates, as in $\mathbf{e}_i(\mathbf{x})$. Equation 24 is then just a short notation for

$$\mathbf{V}(\mathbf{x}) = V^i(\mathbf{x})\,\mathbf{e}_i(\mathbf{x}) \, , \tag{39}$$

while equation 27 is a short notation for

$$\mathbf{F}(\mathbf{x}) = F_i(\mathbf{x})\, \mathbf{e}^i(\mathbf{x})\,. \tag{40}$$

Here and in most places of the book, the notation $\mathbf{x}$ is a short-cut notation for $\{x^1, x^2, \dots\}$. The reader should just remember that $\mathbf{x}$ represents a point in the space, but it is not a vector.

It is important to realize that, when dealing with tensor mathematics, a single basis is a basis for all the vector spaces at the considered point. For instance, the vector $\mathbf{V}$ may be a velocity, and the vector $\mathbf{E}$ may be an electric field. The two vectors belong to different vector spaces, but the are obtained as "linear combinations" of the same basis vectors:

$$\begin{aligned} \mathbf{V} &= V^i\, \mathbf{e}_i \\ \mathbf{E} &= E^i\, \mathbf{e}_i\,, \end{aligned} \tag{41}$$

but, of course, the components are not pure real numbers: they have dimensions.

Let us examine the components of the basis vectors (on the basis they define). Obviously,

$$(\mathbf{e}_i)^j = \delta_i{}^j \qquad (\mathbf{e}^j)_i = \delta_i{}^j\,, \tag{42}$$

or, explicitly,

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \qquad \dots\,. \tag{43}$$

Equivalently, for the basis of 2-tensors we have

$$(\mathbf{e}_i \otimes \mathbf{e}_j)^{kl} = \delta_i{}^k\, \delta_j{}^l \tag{44}$$

$$\begin{array}{ccc} \mathbf{e}_1 \otimes \mathbf{e}_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \ddots \end{pmatrix} & \mathbf{e}_1 \otimes \mathbf{e}_2 = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \ddots \end{pmatrix} & \cdots \\ \mathbf{e}_2 \otimes \mathbf{e}_1 = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \ddots \end{pmatrix} & \mathbf{e}_2 \otimes \mathbf{e}_2 = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \ddots \end{pmatrix} & \cdots \\ \cdots & \cdots & \cdots \end{array} \tag{45}$$

and similar formular for other basis.

Note: say somewhere that the definition of basis vectors given above imposes that the vectors of the natural basis are, at any point, tangent to the coordinate lines at that point. The notion of tangency is independent of the existence, or not, of a metric, i.e., of the possibility of measuring distances in the space. This is not so for the notion of perpendicularity, that makes sense only if we can measure distances (and, therefore, angles). In, general, then, the vectors of the natural basis are tangent to the coordinate lines. When a metric has been introduced, the vectors in the natural basis at a given point will be mutually perpendicular only if the coordinate lines themselves are mutually perpendicular at that point. Ordinary coordinates in the Euclidean 3-D space (Cartesian, cylindrical, spherical,...) define coordinate lines that are orthogonal at every point. Then, the vectors of the natural basis will also be mutually orthogonal at all points. *But the vectors of the natural basis are* **not**, *in general, normed to 1*. For instance, figure XXX illustrates the fact that the norm of the vectors of the natural basis in polar coordinates are, at point (r, φ) , $\|\mathbf{e}_r\| = 1$ and $\|\mathbf{e}_\varphi\| = r$.

6 Tensor Components

Consider, over an n-dimensional manifold $\mathcal{X}$. At any point $\mathcal{P}$ of the manifold, one can consider the linear space $\mathcal{L}$ that is tangent to the manifold at that point, and its dual $\widehat{\mathcal{L}}$. One can also consider, at point $\mathcal{P}$, the 'tensor product' of spaces $\mathcal{L}(p,q) = \underbrace{\mathcal{L}\otimes\mathcal{L}\otimes\cdots\otimes\mathcal{L}}_{p \text{ times}}\otimes\underbrace{\widehat{\mathcal{L}}\otimes\widehat{\mathcal{L}}\otimes\cdots\otimes\widehat{\mathcal{L}}}_{q \text{ times}}$. A 'p-times contravariant, q-times covariant tensor' at point $\mathcal{P}$ of the manifold is an element of $\mathcal{L}(p,q)$.

When a coordinate system $\mathbf{x} = \{x^1, \dots, x^n\}$ is chosen over $\mathcal{X}$, one has, at point $\mathcal{P}$, the 'natural basis' for the linear tangent space $\mathcal{L}$, say $\{\mathbf{e}_i\}$, and, by virtue of the tensor product, also a basis for the space $\mathcal{L}(p,q)$, say $\{\mathbf{e}_{i_1}\otimes\mathbf{e}_{i_2}\otimes\cdots\otimes\mathbf{e}_{i_p}\otimes\mathbf{e}^{j_1}\otimes\mathbf{e}^{j_2}\otimes\cdots\otimes\mathbf{e}^{j_q}\}$. Any tensor $\mathbf{T}$ at point $\mathcal{P}$ of $\mathcal{X}$ can then be developed on this basis,

$$\mathbf{T} \;=\; T_{\mathbf{x}}{}^{i_1 i_2 \dots i_p}{}_{j_1 j_2 \dots j_q}\, \mathbf{e}_{i_1}\otimes\mathbf{e}_{i_2}\otimes\cdots\otimes\mathbf{e}_{i_p}\otimes\mathbf{e}^{j_1}\otimes\mathbf{e}^{j_2}\otimes\cdots\otimes\mathbf{e}^{j_q} \quad , \tag{46}$$

to define the natural components of the tensor, $T_{\mathbf{x}}{}^{i_1 i_2 \dots i_p}{}_{j_1 j_2 \dots j_q}$. They are intimately linked to the coordinate system chosen over $\mathcal{X}$, as this coordinate system has induced the natural basis $\{\mathbf{e}_i\}$ at the considered point $\mathcal{P}$. The index $\mathbf{x}$ in the components is there to recall this fact. It is essential when different coordinates are going to be simultaneously considered, but it can be droped when there is no possible confusion about the coordinate system being used. Its lower or upper position may be chosen for typographical clarity, and, of course, has no special variance meaning.

7 Tensors in Metric Spaces

Comment: explain here that it is possible to give a lot of structure to a manifold (tangent linear space, (covariant) derivation, etc.) without the need of a metric. It is introduced here to simplify the text, as, if not, we would have need to come bak to most of the results to add the particular properties arising when there is a metric. But, in all rigor, it would be preferable to introduce the metric after, for instance, the definition of covariant differentiaition, that does not need it.

Having a metric in a differential manifold means being able to define the length of a line. This will then imply that we can define a scalar product at every local tangent linear space (and, thus, the angle between two crossing lines).

The metric will also allow to define a natural bijection between vectors and forms, and between tensors densities and capacities.

A metric is defined when a second rank symetric form $\mathbf{g}$ with components g_{ij} is given. The length L of a line $x^i(\lambda)$ is then defined by the line integral

$$L = \int_\lambda ds \, , \tag{47}$$

where

$$ds^2 = g_{ij} dx^i dx^j \, . \tag{48}$$

Once we have a metric, it is possible to define a bijection between forms and vectors. For, to the vector $\mathbf{V}$ with components V^i we can associate the form $\mathbf{F}$ with components

$$F_i = g_{ij} \, V^j \, . \tag{49}$$

Then, it is customary to use the same letter to designate a vector and a form that are linked by this natural bijection, as in

$$V_i = g_{ij} \, V^j \, . \tag{50}$$

The inverse of the previous equation is written

$$V^i = g^{ij} \, V_j \, , \tag{51}$$

where

$$g_{ij} g^{jk} = \delta_i{}^k \, . \tag{52}$$

The reader will easily give sense to the expression

$$\mathbf{e}_i = g_{ij} \mathbf{e}^j \, . \tag{53}$$

The equations above, and equations like

$$T_{ij\ldots}{}^{kl\ldots} = g_{ip} \, g_{jq} \ldots g^{kr} g^{ls} \ldots T^{pq\ldots}{}_{rs\ldots} \, , \tag{54}$$

are summarized by saying that “the metric tensor allows to raise and lower indices”.

The value of the metric at a particular point of the manifold allows to define a scalar product for the vectors in the local tangent linear space. Denoting the scalar product of two vectors $\mathbf{V}$ and $\mathbf{W}$ by $\mathbf{V} \cdot \mathbf{W}$, we can use any of the definitions

$$\mathbf{V} \cdot \mathbf{W} = g_{ij} \, V^i \, W^j = V_i \, W^j = V^i \, W_j \, . \tag{55}$$

To define parallel transportation of tensors, we have introduced a connection $\Gamma_{ij}{}^k$. Now that we have a metric we may wonder if when parallel-transporting a vector, it conserves constant length. It is easy to show (see demonstration in [Comment: where?]) that this is true if we have the *compatibility condition*

$$\nabla_i g_{jk} = 0 \, , \tag{56}$$

i.e.,

$$\partial_i g_{jk} = g_{sk} \Gamma_{ij}{}^s + g_{js} \Gamma_{ik}{}^s \, . \tag{57}$$

The compatibility condition 56 implies that the metric tensor and the nabla symbol commute:

$$\nabla_i (g_{jk} T^{pq\dots}{}_{rs\dots}) = g_{jk} (\nabla_i T^{pq\dots}{}_{rs\dots}) , \tag{58}$$

which, in fact, means that it is equivalent to take a covariant derivative, then raise or lower an index, or first raise or lower an index, then take the covariant derivative.

Note: introduce somewhere the notation

$$\Gamma_{ijk} = g_{ks} \Gamma_{ij}{}^{s} , \tag{59}$$

warn the reader that this is just a *notation*: the connection coefficients are not the components of a tensor. and say that if the condition 56 holds, then, it is possible to compute the connection coefficients from the metric and the torsion:

$$\Gamma_{ijk} = \frac{1}{2} \left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} \right) + \frac{1}{2} \left(S_{ijk} + S_{kij} + S_{kji} \right) . \tag{60}$$

As the basis vectors have components

$$(\mathbf{e}_i)^j = \delta_i{}^j , \tag{61}$$

we have

$$\mathbf{e}_i \cdot \mathbf{e}_j = g_{ij} . \tag{62}$$

Defining

$$d\mathbf{r} = dx^i \, \mathbf{e}_i \tag{63}$$

gives then

$$d\mathbf{r} \cdot d\mathbf{r} = ds^2 . \tag{64}$$

We have seen that the metric can be used to define a natural bijection between forms and vectors. Let us now see that it can also be used to define a natural bijection between tensors, densities, and capacities.

We denote by $\overline{\overline{g}}$ the determinant of g_{ij} :

$$\overline{\overline{g}} = \det(\{g_{ij}\}) = \frac{1}{n!} \, \overline{\varepsilon}^{ijk\dots} \, \overline{\varepsilon}^{pqr\dots} \, g_{ip} \, g_{jq} \, g_{kr} \dots . \tag{65}$$

The two upper bars recall that $\overline{\overline{g}}$ is a second order density, as there is the product of two densities at the right-hand side.

For a reason that will become obvious soon, the square root of $\overline{\overline{g}}$ is denoted $\overline{g}$:

$$\overline{\overline{g}} = \overline{g} \, \overline{g} . \tag{66}$$

In (Comment: where?) we demonstrate that we have

$$\partial_i \, \overline{g} = \overline{g} \, \Gamma_{is}{}^{s} . \tag{67}$$

Using expression (Comment: which one?) for the (covariant) derivative of a scalar density, this simply gives

$$\nabla_i \, \overline{g} = \partial_i \, \overline{g} - \overline{g} \, \Gamma_{is}{}^{s} = 0 , \tag{68}$$

which is consistent with the fact that

$$\nabla_i \, g_{jk} = 0 \, . \tag{69}$$

We can also define the determinant of g^{ij} :

$$\underline{\underline{g}} = \det(\{g^{ij}\}) = \frac{1}{n!} \, \underline{\varepsilon}_{ijk\ldots} \, \underline{\varepsilon}_{pqr\ldots} \, g^{ip} \, g^{jq} \, g^{kr} \ldots \, , \tag{70}$$

and its square root $\underline{g}$:

$$\underline{\underline{g}} = \underline{g} \, \underline{g} \, . \tag{71}$$

As the matrices g_{ij} and g^{ij} are mutually inverses, we have

$$\overline{g} \, \underline{g} = 1 \, . \tag{72}$$

Using the scalar density $\overline{g}$ and the scalar capacity $\underline{g}$ we can associate tensor densities, pure tensors, and tensor capacities. Using the same letter to designate the objects related through this natural bijection, we will write expressions like

$$\overline{\rho} = \overline{g} \, \rho \, , \tag{73}$$

$$\overline{V}^i = \overline{g} \, V^i \, , \tag{74}$$

or

$$T_{ij\ldots}{}^{kl\ldots} = \underline{g} \, \overline{T}_{ij\ldots}{}^{kl\ldots} \, . \tag{75}$$

So, if g_{ij} and g^{ij} can be used to “lower and raise indices”, $\overline{g}$ and $\underline{g}$ can be used to “put and remove bars”.

Comment: say somewhere that $\overline{g}$ is the *density of volumetric content*, as the volume element of a metric space is given by

$$dV = \overline{g} \, d\underline{\tau} \, , \tag{76}$$

where $d\underline{\tau}$ is the *capacity element* defined in (Comment: where?), and which, when we take an element along the coordinate lines, equals $dx^1 dx^2 dx^3 \ldots$.

Comment: Say that we can demonstrate that, in an Euclidean space, the matrix representing the metric equals the product of the Jacobian matrix times the transposed matrix:

$$\{g_{ij}\} = \begin{pmatrix} g_{11} & g_{12} & \cdots \\ g_{21} & g_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} \frac{\partial X^1}{\partial x^1} & \frac{\partial X^1}{\partial x^2} & \cdots \\ \frac{\partial X^2}{\partial x^1} & \frac{\partial X^2}{\partial x^2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \times \begin{pmatrix} \frac{\partial X^1}{\partial x^1} & \frac{\partial X^2}{\partial x^1} & \cdots \\ \frac{\partial X^1}{\partial x^2} & \frac{\partial X^2}{\partial x^2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \, . \tag{77}$$

In short,

$$g_{ij} = \sum_K \frac{\partial X^K}{\partial x^i} \, \frac{\partial X^K}{\partial x^j} \, . \tag{78}$$

This follows directly from the general equation

$$g_{ij} = \frac{\partial X^I}{\partial x^i} \, \frac{\partial X^J}{\partial x^j} \, g_{IJ} \tag{79}$$

using the fact that, if the $\{X^I\}$ are Cartesian coordinates,

$$\{g_{IJ}\} = \begin{pmatrix} g_{11} & g_{12} & \cdots \\ g_{21} & g_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots \\ 0 & 1 & \dots \\ 0 & 0 & \dots \end{pmatrix} . \tag{80}$$

Comment: explain here that the metric introduces a bijection between forms and vectors:

$$V_i = g_{ij} V^j . \tag{81}$$

Comment: introduce here the notation

$$(\mathbf{V}, \mathbf{W}) = g_{ij} V^i W^j = V_i W^i = W_i V^i . \tag{82}$$