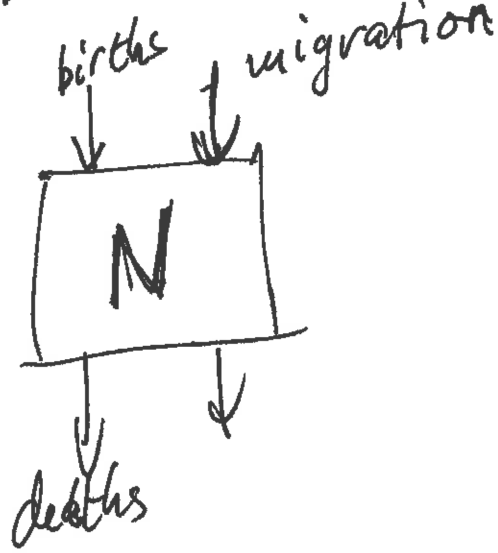
CONTINUOUS population models for single species

N $\quad \underline{N(t)}$

$$\frac{dN}{dt} = \text{births} - \text{deaths} + \text{migration}$$



$$\frac{dN}{dt} = rN - dN = (r-d)\,N$$

$$N(t) = N_0\, e^{+(r-d)t}$$

$$N_0 = N(0)$$

$r > d$ — exponential growth

$r < d$ — exponential decay

Malthus 1798

Logistic growth model
Verhulst 1838

$$\frac{dN}{dt} = \underbrace{r\left(1-\frac{N}{K}\right)}_{\text{effective birth rate}} \cdot N$$

r, K - positive constants

K - carrying capacity of the environment

Equilibrium states (constant solutions)

$\underline{N^* = 0}$ $\quad$ $\underline{N^* = K}$

$N \to 0$.

$$\frac{dN}{dt} = rN - \underbrace{\cancel{\frac{rN^2}{K}}}_{\text{small}} \approx rN$$

$\underline{N = N_0 e^{rt}}$, - grows exponentially

$N^* = 0$ is <u>unstable</u> solution

$$N^* = K$$

$$N = N^* + n = K + n \quad \text{where } |n| \ll 1$$

$$\frac{dN}{dt} = \frac{dK}{dt} + \frac{dn}{dt} = 0 + \frac{dn}{dt} = \frac{dn}{dt}$$

$$\parallel$$

$$r\left(1-\frac{N}{K}\right)\cdot N = r\,\frac{K-N}{K}\cdot N = r\,\frac{(-n)}{K}(K+n) =$$

$$= -rn - \cancel{\frac{r}{K}n^2} = -rn$$

$$\frac{dn}{dt} = -rn \Rightarrow n = n_0 e^{-rt} \to 0$$

$$N(t) = K + n_0 e^{-rt} \to K \quad \text{stable.}$$

$\frac{1}{r}$ timescale

K - size of the population at equilibrium

$$\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right), \quad \underbrace{\int \frac{dN}{N\left(1-\frac{N}{K}\right)}}_{?} = \int r\,dt$$

$$\parallel$$

$$rt$$

$$\frac{1}{N\left(1-\frac{N}{K}\right)} = \frac{A}{N} + \frac{B}{1-\frac{N}{K}} = \frac{A\left(1-\frac{N}{K}\right)+BN}{N\left(1-\frac{N}{K}\right)}$$

$$1 = A\left(1-\frac{N}{K}\right)+BN$$

$$N=0 \Rightarrow A=1.$$

$$N=K \Rightarrow B=1/K$$

$$\int \frac{dN}{N\left(1-\frac{N}{K}\right)} = \int\frac{dN}{N} + \int \frac{d\,N/K}{1-\frac{N}{K}} = \ln N - \ln\left(1-\frac{N}{K}\right)$$

$$\boxed{x=\frac{N}{K} \qquad \int\frac{dx}{1-x} = -\int\frac{d(1-x)}{1-x} = -\ln(1-x)}$$

$$= \ln\frac{N}{1-\frac{N}{K}} + \text{const}$$

$$\ln\frac{N}{1-\frac{N}{K}} = rt + c \qquad e^{rt+c} = e^{rt}\,\underset{\overset{\parallel}{C'}}{\boxed{e^{c}}}$$

$$\frac{N}{1-\frac{N}{K}} = Ce^{rt}$$

$$N = \left(1-\frac{N}{K}\right)Ce^{rt}$$

$$N + \frac{N}{K} c e^{rt} = c e^{rt}$$

$$N\left(1 + \frac{c}{K} e^{rt}\right) = c e^{rt}$$

$t = 0$

$N = N_0 = N(0)$

$$N_0 \left(1 + \frac{c}{K}\right) = c$$

$$N_0 = c - \frac{N_0}{K} c = \left(1 - \frac{N_0}{K}\right) c$$

$$\boxed{c = \frac{N_0}{1 - \frac{N_0}{K}}}$$

$$\Rightarrow N\left(1 + \frac{N_0}{K} \cdot \frac{e^{rt}}{1 - \frac{N_0}{K}}\right) = \frac{N_0}{1 - \frac{N_0}{K}} e^{rt}$$

$\cdot \left(1 - \frac{N_0}{K}\right)$

$$N\left(1-\frac{N_0}{K}+\frac{N_0}{K}e^{rt}\right)=N_0e^{rt}$$

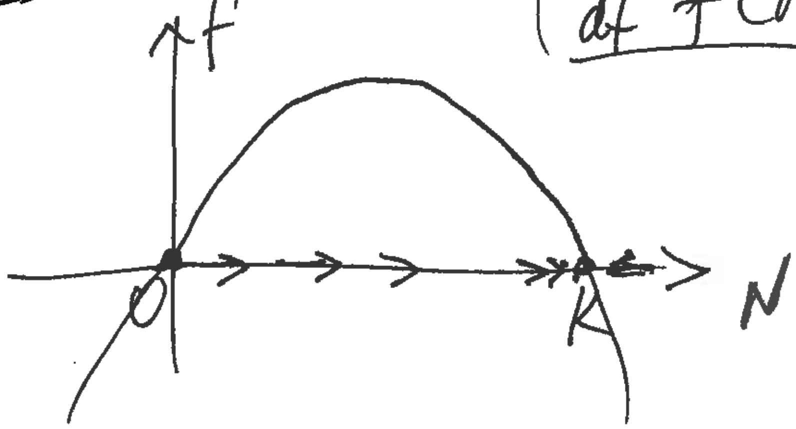
$$\boxed{N=N_0\frac{e^{rt}}{1+\frac{N_0}{K}(e^{rt}-1)}}\quad \frac{\times e^{-rt}}{\times e^{-rt}}$$

$$N=N_0\frac{1}{e^{-rt}+\frac{N_0}{K}(1-e^{-rt})}\xrightarrow{t\to\infty}$$

$$\Rightarrow \cancel{N_0}\frac{1}{\frac{\cancel{N_0}}{K}}=K$$

1925 Pearl.

$$\boxed{\frac{dN}{dt}=f(N)}=rN\left(1-\frac{N}{K}\right)$$



$$\frac{dN}{dt}=f(N)$$

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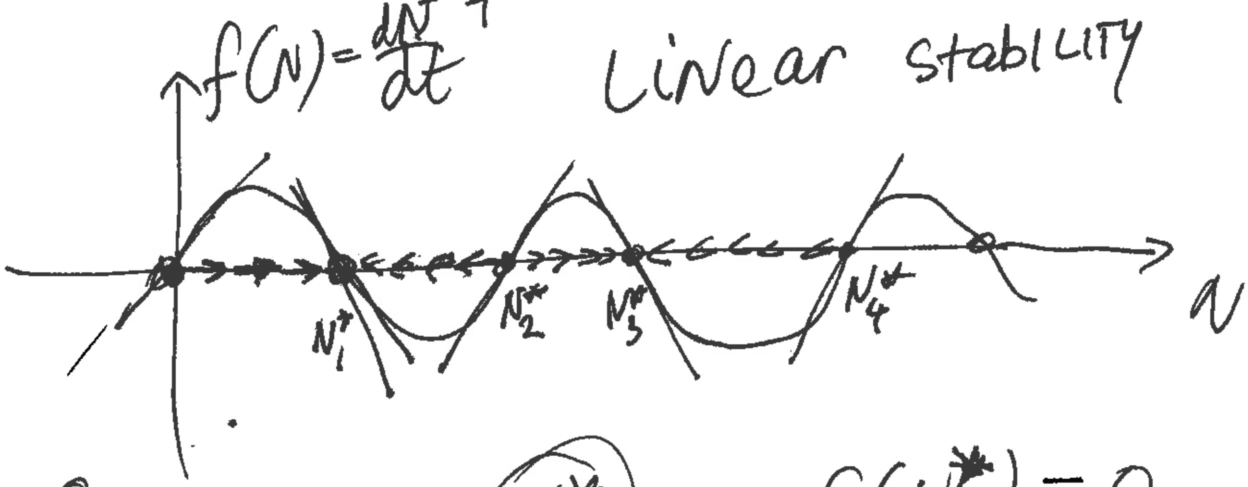
Linear stability



$f(0)=0 \to N^* \qquad f(N^*)=0$

$$f(N) = \underset{\parallel\, 0}{f(N^*)} + f'(N^*)(N-N^*) + \underbrace{\tfrac{1}{2}f''(N^*)(N-N^*)^2 + \dots}_{\text{neglect}}$$

$$n = N - N^* \ll 1$$

$$f(N) = f'(N^*)\, n$$

$$\frac{dn}{dt} = \frac{dN}{dt} - \underset{=0}{\frac{dN^*}{dt}} = \frac{dN}{dt}$$

$$\frac{dN}{dt} = f(N) \Longrightarrow \frac{dn}{dt} \approx \underbrace{f'(N^*)}_{\text{const.}} \cdot n$$

$$n(t) = n_0 e^{f'(N^*)t}$$

$$N = N^* + n_0 e^{f'(N^*)t}$$

If $f'(N^*) > 0 \Rightarrow N^*$ unstable equilibrium

If $f'(N^*) < 0 \Rightarrow N^*$ is stable

Insect outbreak model

spruce budworm

Ludwig et al. 1978

$$\frac{dN}{dt} = r_B N\left(1-\frac{N}{K_B}\right) - p(N)$$

p(N) - predation

$$p(N) = \frac{BN^2}{A^2+N^2}$$

$$\frac{dN/A}{dt/A} = (Ar_B)\,\frac{N}{A}\left(1-\frac{N/A}{K_B/A}\right) - \frac{B\,N^2/A^2}{A^2/A^2+N^2/A^2}$$

$r_B,\ K_B,\ A,\ B$ - 4 parameters

NONDIMENSIONAL terms

$$u=\frac{N}{A},\quad r=\frac{Ar_B}{B},\quad q=\frac{K_B}{A},\quad \tau=\frac{Bt}{A}$$

$$\frac{du}{d\left(\frac{t}{A}\right)B} = \frac{Ar_B}{B}\,u\left(1-\frac{u}{q}\right) - \frac{\cancel{B}u^2}{(1+u^2)\cancel{B}}$$

$1\cdot\frac{1}{B}$

$$\boxed{\frac{du}{d\tau} = r\left(1-\frac{u}{q}\right) - \frac{u^2}{1+u^2}}$$

$u(\tau; r, q)$

$$ma = F_{el}$$

$$m\ddot{x} = -Kx$$

$$\ddot{x} + \frac{K}{m}x = 0 \qquad \frac{K}{m} = \omega_0^2$$

$$\ddot{x} + \omega_0^2 x = 0$$

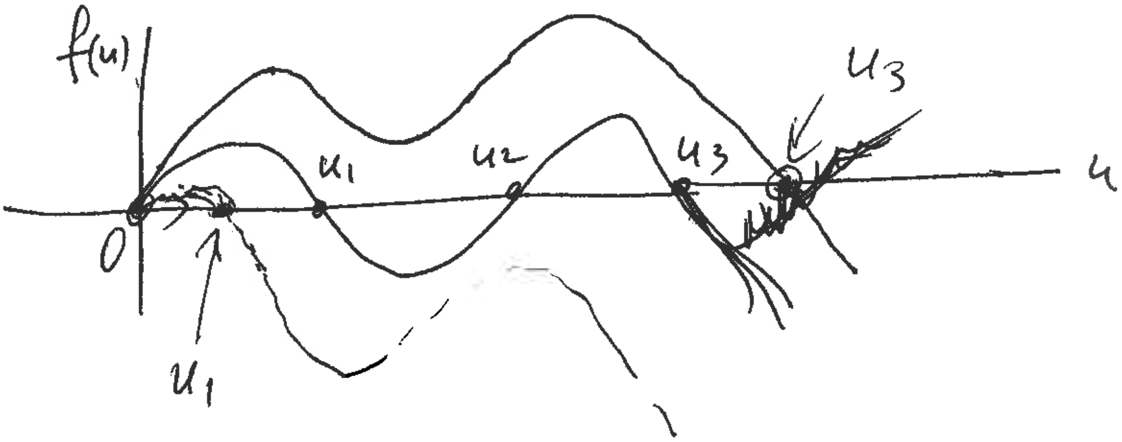
$$\frac{du}{d\tau} = f(u) = \underbrace{ru\left(1-\frac{u}{q}\right) - \frac{u^2}{1+u^2}}_{f(u^*;\, q, r) = 0}$$

$$ru\left(1 - \frac{u}{q}\right) = \frac{u^2}{1+u^2}$$

1) $u = 0$ is always a solution

2) $\boxed{r\left(1-\frac{u}{q}\right) = \frac{u}{1+u^2}}$

$r(1+u^2)\left(1-\frac{u}{q}\right) = u$ - qubic eqn.



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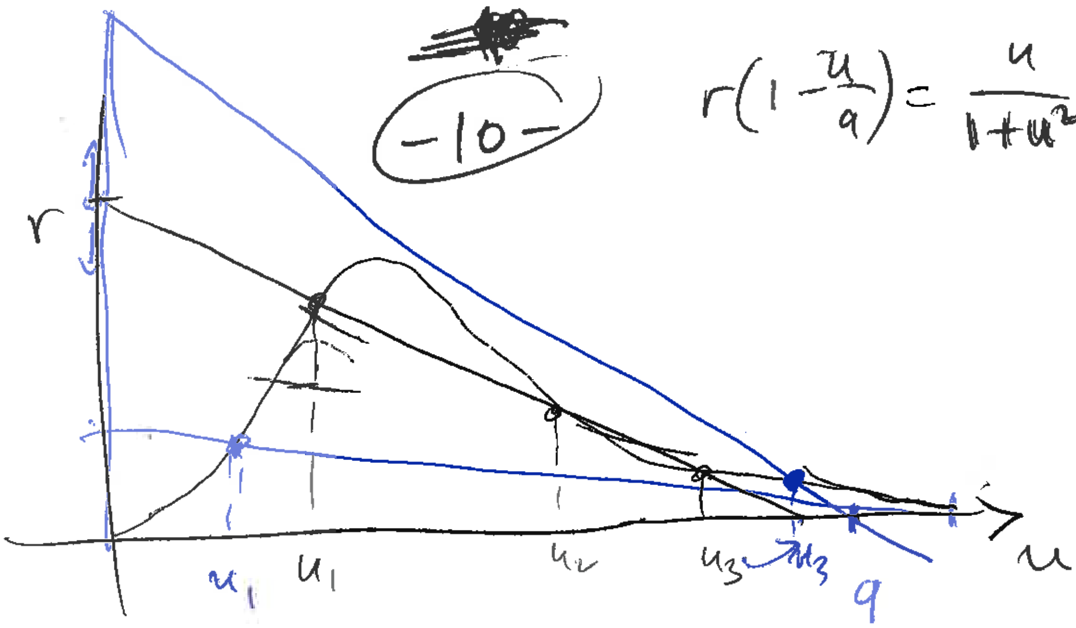
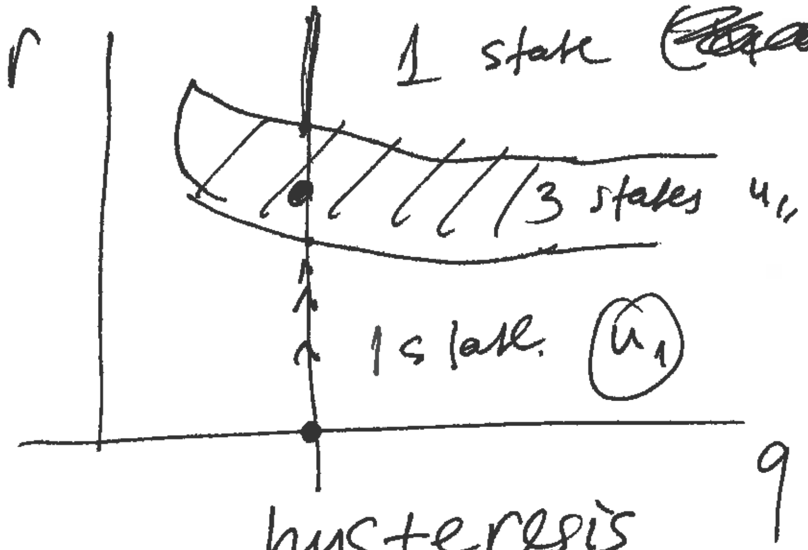
$$r\left(1-\frac{u}{q}\right)=\frac{u}{1+u^2}$$





$\underline{u^*(q,r)}$

<u>hysteresis</u>