

10.4 APPLICATIONS OF NUMERICAL METHODS

Applications of Gaussian Elimination with Pivoting

In Section 2.5 you used least squares regression analysis to find *linear* mathematical models that best fit a set of n points in the plane. This procedure can be extended to cover polynomial models of any degree as follows.

Regression Analysis for Polynomials

The least squares regression polynomial of degree m for the points $\{(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)\}$ is given by

$$y = a_mx^m + a_{m-1}x^{m-1} + \dots + a_2x^2 + a_1x + a_0,$$

where the coefficients are determined by the following system of $m + 1$ linear equations.

$$\begin{aligned} na_0 + (\Sigma x_i)a_1 + (\Sigma x_i^2)a_2 + \dots + (\Sigma x_i^m)a_m &= \Sigma y_i \\ (\Sigma x_i)a_0 + (\Sigma x_i^2)a_1 + (\Sigma x_i^3)a_2 + \dots + (\Sigma x_i^{m+1})a_m &= \Sigma x_i y_i \\ (\Sigma x_i^2)a_0 + (\Sigma x_i^3)a_1 + (\Sigma x_i^4)a_2 + \dots + (\Sigma x_i^{m+2})a_m &= \Sigma x_i^2 y_i \\ &\vdots \\ (\Sigma x_i^m)a_0 + (\Sigma x_i^{m+1})a_1 + (\Sigma x_i^{m+2})a_2 + \dots + (\Sigma x_i^{2m})a_m &= \Sigma x_i^m y_i \end{aligned}$$

Note that if $m = 1$ this system of equations reduces to

$$\begin{aligned} na_0 + (\Sigma x_i)a_1 &= \Sigma y_i \\ (\Sigma x_i)a_0 + (\Sigma x_i^2)a_1 &= \Sigma x_i y_i, \end{aligned}$$

which has a solution of

$$a_1 = \frac{n \sum x_i y_i - (\sum x_i)(\sum y_i)}{n \sum x_i^2 - (\sum x_i)^2} \quad \text{and} \quad a_0 = \frac{\sum y_i}{n} - a_1 \frac{\sum x_i}{n}.$$

Exercise 16 asks you to show that this formula is equivalent to the matrix formula for linear regression that was presented in Section 2.5.

Example 1 illustrates the use of regression analysis to find a second-degree polynomial model.

EXAMPLE 1 *Least Squares Regression Analysis*

The world population in billions for the years between 1965 and 2000, is shown in Table 10.9. (Source: U.S. Census Bureau)

TABLE 10.9

Year	1965	1970	1975	1980	1985	1990	1995	2000
Population	3.36	3.72	4.10	4.46	4.86	5.28	5.69	6.08

Find the second-degree least squares regression polynomial for these data and use the resulting model to predict the world population for 2005 and 2010.

Solution Begin by letting $x = -4$ represent 1965, $x = -3$ represent 1970, and so on. So the collection of points is given by $\{(-4, 3.36), (-3, 3.72), (-2, 4.10), (-1, 4.46), (0, 4.86), (1, 5.28), (2, 5.69), (3, 6.08)\}$, which yields

$$\begin{aligned} n = 8, \quad \sum_{i=1}^8 x_i = -4, \quad \sum_{i=1}^8 x_i^2 = 44, \quad \sum_{i=1}^8 x_i^3 = -64, \\ \sum_{i=1}^8 x_i^4 = 452, \quad \sum_{i=1}^8 y_i = 37.55, \quad \sum_{i=1}^8 x_i y_i = -2.36, \quad \sum_{i=1}^8 x_i^2 y_i = 190.86. \end{aligned}$$

So the system of linear equations giving the coefficients of the quadratic model $y = a_2 x^2 + a_1 x + a_0$ is

$$\begin{aligned} 8a_0 - 4a_1 + 44a_2 &= 37.55 \\ -4a_0 + 44a_1 - 64a_2 &= -2.36 \\ 44a_0 - 64a_1 + 452a_2 &= 190.86. \end{aligned}$$

Gaussian elimination with pivoting on the matrix

$$\begin{bmatrix} 8 & -4 & 44 & 37.55 \\ -4 & 44 & -64 & -2.36 \\ 44 & -64 & 452 & 190.86 \end{bmatrix}$$

produces

$$\begin{bmatrix} 1 & -1.4545 & 10.2727 & 4.3377 \\ 0 & 1 & -0.6000 & 0.3926 \\ 0 & 0 & 1 & 0.0045 \end{bmatrix}.$$

So by back substitution you find the solution to be

$$a_2 = 0.0045, \quad a_1 = 0.3953, \quad a_0 = 4.8667,$$

and the regression quadratic is

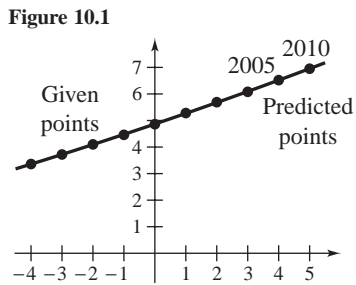
$$y = 0.0045x^2 + 0.3953x + 4.8667.$$

Figure 10.1 compares this model with the given points. To predict the world population for 2005, let $x = 4$, obtaining

$$y = 0.0045(4^2) + 0.3953(4) + 4.8667 = 6.52 \text{ billion.}$$

Similarly, the prediction for 2010 ($x = 5$) is

$$y = 0.0045(5^2) + 0.3953(5) + 4.8667 = 6.96 \text{ billion.}$$



EXAMPLE 2 *Least Squares Regression Analysis*

Find the third-degree least squares regression polynomial

$$y = a_3x^3 + a_2x^2 + a_1x + a_0$$

for the points

$$\{(0, 0), (1, 2), (2, 3), (3, 2), (4, 1), (5, 2), (6, 4)\}.$$

Solution For this set of points the linear system

$$\begin{aligned} na_0 + (\sum x_i)a_1 + (\sum x_i^2)a_2 + (\sum x_i^3)a_3 &= \sum y_i \\ (\sum x_i)a_0 + (\sum x_i^2)a_1 + (\sum x_i^3)a_2 + (\sum x_i^4)a_3 &= \sum x_i y_i \\ (\sum x_i^2)a_0 + (\sum x_i^3)a_1 + (\sum x_i^4)a_2 + (\sum x_i^5)a_3 &= \sum x_i^2 y_i \\ (\sum x_i^3)a_0 + (\sum x_i^4)a_1 + (\sum x_i^5)a_2 + (\sum x_i^6)a_3 &= \sum x_i^3 y_i \end{aligned}$$

becomes

$$\begin{aligned} 7a_0 + 21a_1 + 91a_2 + 441a_3 &= 14 \\ 21a_0 + 91a_1 + 441a_2 + 2275a_3 &= 52 \\ 91a_0 + 441a_1 + 2275a_2 + 12,201a_3 &= 242 \\ 441a_0 + 2275a_1 + 12,201a_2 + 67,171a_3 &= 1258. \end{aligned}$$

Using Gaussian elimination with pivoting on the matrix

$$\begin{bmatrix} 7 & 21 & 91 & 441 & 14 \\ 21 & 91 & 441 & 2275 & 52 \\ 91 & 441 & 2275 & 12,201 & 242 \\ 441 & 2275 & 12,201 & 67,171 & 1258 \end{bmatrix}$$

produces

$$\begin{bmatrix} 1.0000 & 5.1587 & 27.6667 & 152.3150 & 2.8526 \\ 0.0000 & 1.0000 & 8.5312 & 58.3482 & 0.6183 \\ 0.0000 & 0.0000 & 1.0000 & 9.7714 & 0.1286 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 & 0.1667 \end{bmatrix},$$

which implies that

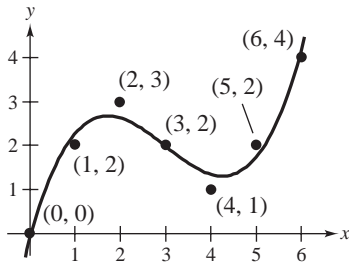
$$a_3 = 0.1667, \quad a_2 = -1.5000, \quad a_1 = 3.6905, \quad a_0 = -0.0714.$$

So the cubic model is

$$y = 0.1667x^3 - 1.5000x^2 + 3.6905x - 0.0714.$$

Figure 10.2 compares this model with the given points.

Figure 10.2

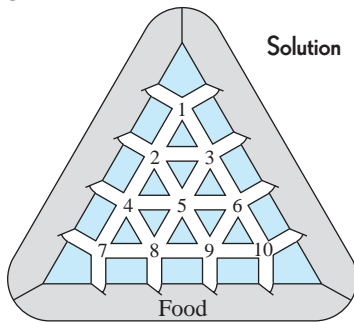


Applications of the Gauss-Seidel Method

EXAMPLE 3 An Application to Probability

Figure 10.3 is a diagram of a maze used in a laboratory experiment. The experiment begins by placing a mouse at one of the ten interior intersections of the maze. Once the mouse emerges in the outer corridor, it cannot return to the maze. When the mouse is at an interior intersection, its choice of paths is assumed to be random. What is the probability that the mouse will emerge in the “food corridor” when it begins at the i th intersection?

Figure 10.3



Solution

Let the probability of winning (getting food) by starting at the i th intersection be represented by p_i . Then form a linear equation involving p_i and the probabilities associated with the intersections bordering the i th intersection. For instance, at the first intersection the mouse has a probability of $\frac{1}{4}$ of choosing the upper right path and losing, a probability of $\frac{1}{4}$ of choosing the upper left path and losing, a probability of $\frac{1}{4}$ of choosing the lower right-path (at which point it has a probability of p_3 of winning), and a probability of $\frac{1}{4}$ of choosing the lower left path (at which point it has a probability of p_2 of winning). So

$$p_1 = \underbrace{\frac{1}{4}(0)}_{\text{Upper right}} + \underbrace{\frac{1}{4}(0)}_{\text{Upper left}} + \underbrace{\frac{1}{4}p_2}_{\text{Lower left}} + \underbrace{\frac{1}{4}p_3}_{\text{Lower right}}.$$

Using similar reasoning, the other nine probabilities can be represented by the following equations.

$$\begin{aligned}
 p_2 &= \frac{1}{5}(0) + \frac{1}{5}p_1 + \frac{1}{5}p_3 + \frac{1}{5}p_4 + \frac{1}{5}p_5 \\
 p_3 &= \frac{1}{5}(0) + \frac{1}{5}p_1 + \frac{1}{5}p_2 + \frac{1}{5}p_5 + \frac{1}{5}p_6 \\
 p_4 &= \frac{1}{5}(0) + \frac{1}{5}p_2 + \frac{1}{5}p_5 + \frac{1}{5}p_7 + \frac{1}{5}p_8 \\
 p_5 &= \frac{1}{6}p_2 + \frac{1}{6}p_3 + \frac{1}{6}p_4 + \frac{1}{6}p_6 + \frac{1}{6}p_8 + \frac{1}{6}p_9 \\
 p_6 &= \frac{1}{5}(0) + \frac{1}{5}p_3 + \frac{1}{5}p_5 + \frac{1}{5}p_9 + \frac{1}{5}p_{10} \\
 p_7 &= \frac{1}{4}(0) + \frac{1}{4}(1) + \frac{1}{4}p_4 + \frac{1}{4}p_8 \\
 p_8 &= \frac{1}{5}(1) + \frac{1}{5}p_4 + \frac{1}{5}p_5 + \frac{1}{5}p_7 + \frac{1}{5}p_9 \\
 p_9 &= \frac{1}{5}(1) + \frac{1}{5}p_5 + \frac{1}{5}p_6 + \frac{1}{5}p_8 + \frac{1}{5}p_{10} \\
 p_{10} &= \frac{1}{4}(0) + \frac{1}{4}(1) + \frac{1}{4}p_6 + \frac{1}{4}p_9
 \end{aligned}$$

Rewriting these equations in standard form produces the following system of ten linear equations in ten variables.

$$\begin{aligned}
 4p_1 - p_2 - p_3 &= 0 \\
 -p_1 + 5p_2 - p_3 - p_4 - p_5 &= 0 \\
 -p_1 - p_2 + 5p_3 - p_5 - p_6 &= 0 \\
 -p_2 + 5p_4 - p_5 - p_7 - p_8 &= 0 \\
 -p_2 - p_3 - p_4 + 6p_5 - p_6 - p_8 - p_9 &= 0 \\
 -p_3 - p_5 + 5p_6 - p_9 - p_{10} &= 0 \\
 -p_4 + 4p_7 - p_8 &= 1 \\
 -p_4 - p_5 - p_7 + 5p_8 - p_9 &= 1 \\
 -p_5 - p_6 - p_8 + 5p_9 - p_{10} &= 1 \\
 -p_6 - p_9 + 4p_{10} &= 1
 \end{aligned}$$

The augmented matrix for this system is

$$\left[\begin{array}{cccccccccccc}
 4 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 -1 & 5 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 -1 & -1 & 5 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
 0 & -1 & 0 & 5 & -1 & 0 & -1 & -1 & 0 & 0 & 0 \\
 0 & -1 & -1 & -1 & 6 & -1 & 0 & -1 & -1 & 0 & 0 \\
 0 & 0 & -1 & 0 & -1 & 5 & 0 & 0 & -1 & -1 & 0 \\
 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 & 0 & 1 \\
 0 & 0 & 0 & -1 & -1 & 0 & -1 & 5 & -1 & 0 & 1 \\
 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 5 & -1 & 1 \\
 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & 1
 \end{array} \right].$$

Using the Gauss-Seidel method with an initial approximation of $p_1 = p_2 = \dots = p_{10} = 0$ produces (after 18 iterations) an approximation of

$$\begin{array}{ll} p_1 = 0.090, & p_2 = 0.180 \\ p_3 = 0.180, & p_4 = 0.298 \\ p_5 = 0.333, & p_6 = 0.298 \\ p_7 = 0.455, & p_8 = 0.522 \\ p_9 = 0.522, & p_{10} = 0.455. \end{array}$$

The structure of the probability problem described in Example 3 is related to a technique called **finite element analysis**, which is used in many engineering problems.

Note that the matrix developed in Example 3 has mostly zero entries. Such matrices are called **sparse**. For solving systems of equations with sparse coefficient matrices, the Jacobi and Gauss-Seidel methods are much more efficient than Gaussian elimination.

Applications of the Power Method

Section 7.4 introduced the idea of an *age transition matrix* as a model for population growth. Recall that this model was developed by grouping the population into n age classes of equal duration. So for a maximum life span of L years, the age classes are given by the following intervals.

$$\begin{array}{ccc} \text{First age class} & \text{Second age class} & \text{nth age class} \\ \left[0, \frac{L}{n}\right), & \left[\frac{L}{n}, \frac{2L}{n}\right), & \dots, \left[\frac{(n-1)L}{n}, L\right] \end{array}$$

The number of population members in each age class is then represented by the age distribution vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \begin{array}{l} \text{Number in first age class} \\ \text{Number in second age class} \\ \text{Number in } n\text{th age class} \end{array}$$

Over a period of L/n years, the *probability* that a member of the i th age class will survive to become a member of the $(i+1)$ th age class is given by p_i , where $0 \leq p_i \leq 1$, $i = 1, 2, \dots, n-1$. The *average number* of offspring produced by a member of the i th age class is given by b_i , where $0 \leq b_i$, $i = 1, 2, \dots, n$. These numbers can be written in matrix form as follows.

$$A = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_{n-1} & b_n \\ p_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & p_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_{n-1} & 0 \end{bmatrix}$$

Multiplying this age transition matrix by the age distribution vector for a given period of time produces the age distribution vector for the next period of time. That is,

$$A\mathbf{x}_i = \mathbf{x}_{i+1}.$$

In Section 7.4 you saw that the growth pattern for a population is *stable* if the same percentage of the total population is in each age class each year. That is,

$$A\mathbf{x}_i = \mathbf{x}_{i+1} = \lambda\mathbf{x}_i.$$

For populations with many age classes, the solution to this eigenvalue problem can be found with the power method, as illustrated in Example 4.

EXAMPLE 4 *A Population Growth Model*

Assume that a population of human females has the following characteristics.

<i>Age Class (in years)</i>	<i>Average Number of Female Children During Ten-Year Period</i>	<i>Probability of Surviving to Next Age Class</i>
0 ≤ age < 10	0.000	0.985
10 ≤ age < 20	0.174	0.996
20 ≤ age < 30	0.782	0.994
30 ≤ age < 40	0.263	0.990
40 ≤ age < 50	0.022	0.975
50 ≤ age < 60	0.000	0.940
60 ≤ age < 70	0.000	0.866
70 ≤ age < 80	0.000	0.680
80 ≤ age < 90	0.000	0.361
90 ≤ age < 100	0.000	0.000

Find a stable age distribution for this population.

Solution The age transition matrix for this population is

[illegible]

To apply the power method with scaling to find an eigenvector for this matrix, use an initial approximation of $\mathbf{x}_0 = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$. The following is an approximation for an eigenvector of A , with the percentage of each age in the total population.

	<i>Eigenvector</i>	<i>Age Class</i>	<i>Percentage in Age Class</i>
$\mathbf{x} =$	1.000	$0 \leq \text{age} < 10$	15.27
	0.925	$10 \leq \text{age} < 20$	14.13
	0.864	$20 \leq \text{age} < 30$	13.20
	0.806	$30 \leq \text{age} < 40$	12.31
	0.749	$40 \leq \text{age} < 50$	11.44
	0.686	$50 \leq \text{age} < 60$	10.48
	0.605	$60 \leq \text{age} < 70$	9.24
	0.492	$70 \leq \text{age} < 80$	7.51
	0.314	$80 \leq \text{age} < 90$	4.80
	0.106	$90 \leq \text{age} < 100$	1.62

The eigenvalue corresponding to the eigenvector \mathbf{x} in Example 4 is $\lambda \approx 1.065$. That is,


$$A\mathbf{x} = A \begin{bmatrix} 1.000 \\ 0.925 \\ 0.864 \\ 0.806 \\ 0.749 \\ 0.686 \\ 0.605 \\ 0.492 \\ 0.314 \\ 0.106 \end{bmatrix} \approx \begin{bmatrix} 1.065 \\ 0.985 \\ 0.921 \\ 0.859 \\ 0.798 \\ 0.731 \\ 0.645 \\ 0.524 \\ 0.334 \\ 0.113 \end{bmatrix} \approx 1.065 \begin{bmatrix} 1.000 \\ 0.925 \\ 0.864 \\ 0.806 \\ 0.749 \\ 0.686 \\ 0.605 \\ 0.492 \\ 0.314 \\ 0.106 \end{bmatrix}.$$

This means that the population in Example 4 increases by 6.5% every ten years.


REMARK: Should you try duplicating the results of Example 4, you would notice that the convergence of the power method for this problem is very slow. The reason is that the dominant eigenvalue of $\lambda \approx 1.065$ is only slightly larger in absolute value than the next largest eigenvalue.

SECTION 10.4 EXERCISES


Applications of Gaussian Elimination with Pivoting

 In Exercises 1–4, find the second-degree least squares regression polynomial for the given data. Then graphically compare the model to the given points.

- $(-2, 1), (-1, 0), (0, 0), (1, 1), (3, 2)$
- $(0, 4), (1, 2), (2, -1), (3, 0), (4, 1), (5, 4)$
- $(-2, 1), (-1, 2), (0, 6), (1, 3), (2, 0), (3, -1)$
- $(1, 1), (2, 1), (3, 0), (4, -1), (5, -4)$


 In Exercises 5–8, find the third-degree least squares regression polynomial for the given data. Then graphically compare the model to the given points.

- $(0, 0), (1, 2), (2, 4), (3, 1), (4, 0), (5, 1)$
- $(1, 1), (2, 4), (3, 4), (5, 1), (6, 2)$
- $(-3, 4), (-1, 1), (0, 0), (1, 2), (2, 5)$
- $(-7, 2), (-3, 0), (1, -1), (2, 3), (4, 6)$

 9. Find the second-degree least squares regression polynomial for the points


$$\left(-\frac{\pi}{2}, 0\right), \left(-\frac{\pi}{3}, \frac{1}{2}\right), (0, 1), \left(\frac{\pi}{3}, \frac{1}{2}\right), \left(\frac{\pi}{2}, 0\right).$$

Then use the results to approximate $\cos(\pi/4)$. Compare the approximation with the exact value.

 10. Find the third-degree least squares regression polynomial for the points

$$\left(-\frac{\pi}{4}, -1\right), \left(-\frac{\pi}{3}, -\sqrt{3}\right), (0, 0), \left(\frac{\pi}{3}, \sqrt{3}\right), \left(\frac{\pi}{4}, 1\right).$$


Then use the result to approximate $\tan(\pi/6)$. Compare the approximation with the exact value.

 11. The number of minutes a scuba diver can stay at a particular depth without acquiring decompression sickness is shown in the table. (Source: United States Navy's Standard Air Decompression Tables)

Depth (in feet)	35	40	50	60	70
Time (in minutes)	310	200	100	60	50

Depth (in feet)	80	90	100	110
Time (in minutes)	40	30	25	20


- Find the least squares regression line for these data.
- Find the second-degree least squares regression polynomial for these data.
- Sketch the graphs of the models found in parts (a) and (b).
- Use the models found in parts (a) and (b) to approximate the maximum number of minutes a diver should stay at a depth of 120 feet. (The value given in the Navy's tables is 15 minutes.)

 12. The life expectancy for additional years of life for females in the United States as of 1998 is shown in the table. (Source: U.S. Census Bureau)

Current Age	10	20	30	40
Life Expectancy	70.6	60.8	51.0	41.4


Current Age	50	60	70	80
Life Expectancy	32.0	23.3	15.6	9.1

- Find the second-degree least squares regression polynomial for these data.
- Use the result of part (a) to predict the life expectancy of a newborn female and a female of age 100 years.

 13. Total sales in billions of dollars of cellular phones in the United States from 1992 to 1999 are shown in the table. (Source: Electronic Market Data Book).

Year	1992	1993	1994	1995	1996	1997	1998	1999
Sales	1.15	1.26	1.28	2.57	2.66	2.75	2.78	2.81

- Find the second degree least squares regression polynomial for the data.
- Use the result of part (a) to predict the total cellular phone sales in 2005 and 2010.
- Are your predictions from part (b) realistic? Explain.

 14. Total new domestic truck unit sales in hundreds of thousands in the United States from 1993 to 2000 are shown in the table. (Source: Ward's Auto info bank)

Year	1993	1994	1995	1996	1997	1998	1999	2000
Trucks	5.29	6.00	6.06	6.48	6.63	7.51	7.92	8.09

- Find the second degree least squares regression polynomial for the data.

(b) Use the result of part (a) to predict the total new domestic truck sales in 2005 and 2010.

(c) Are your predictions from part (b) realistic? Explain.

15. Find the least squares regression line for the population data given in Example 1. Then use the model to predict the world population in 2005 and 2010, and compare the results with the predictions obtained in Example 1.
16. Show that the formula for the least squares regression line presented in Section 2.5 is equivalent to the formula presented in this section. That is, if

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}, \quad A = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix},$$

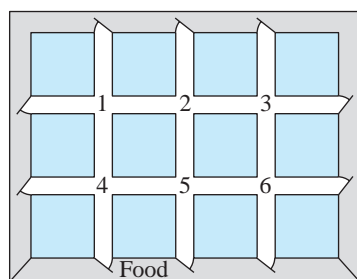
then the matrix equation $A = (X^T X)^{-1} X^T Y$ is equivalent to

$$a_1 = \frac{n \sum x_i y_i - (\sum x_i)(\sum y_i)}{n \sum x_i^2 - (\sum x_i)^2} \quad \text{and} \quad a_0 = \frac{\sum y_i}{n} - a_1 \frac{\sum x_i}{n}.$$

Applications of the Gauss-Seidel Method

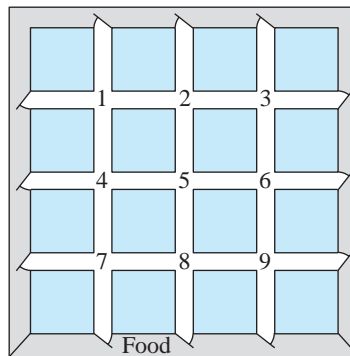
17. Suppose that the experiment in Example 3 is performed with the maze shown in Figure 10.4. Find the probability that the mouse will emerge in the food corridor when it begins in the i th intersection.

Figure 10.4



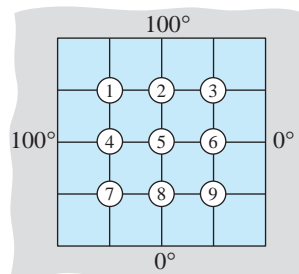
18. Suppose that the experiment in Example 3 is performed with the maze shown in Figure 10.5. Find the probability that the mouse will emerge in the food corridor when it begins in the i th intersection.

Figure 10.5



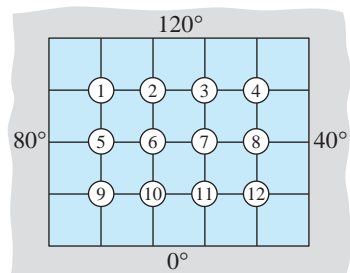
19. A square metal plate has a constant temperature on each of its four boundaries, as shown in Figure 10.6. Use a 4×4 grid to approximate the temperature distribution in the interior of the plate. Assume that the temperature at each interior point is the average of the temperatures at the four closest neighboring points.

Figure 10.6



20. A rectangular metal plate has a constant temperature on each of its four boundaries, as shown in Figure 10.7. Use a 4×5 grid to approximate the temperature distribution in the interior of the plate. Assume that the temperature at each interior point is the average of the temperatures at the four closest neighboring points.

Figure 10.7





Applications of the Power Method


In Exercises 21–24, the matrix represents the age transition matrix for a population. Use the power method with scaling to find a stable age distribution.

21. $A = \begin{bmatrix} 1 & 4 \\ \frac{1}{2} & 0 \end{bmatrix}$

22. $A = \begin{bmatrix} 1 & 2 \\ \frac{1}{4} & 0 \end{bmatrix}$


 23. $A = \begin{bmatrix} 0 & 1 & 2 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \end{bmatrix}$

 24. $A = \begin{bmatrix} 1 & 2 & 2 \\ \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}$

-  25. In Example 1 of Section 7.4, a laboratory population of rabbits is described. The age transition matrix for the population is

$$A = \begin{bmatrix} 0 & 6 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}.$$

Find a stable age distribution for this population.

-  26. A population has the following characteristics.

- (a) A total of 75% of the population survives its first year. Of that 75%, 25% survives its second year. The maximum life span is three years.
- (b) The average number of offspring for each member of the population is 2 the first year, 4 the second year, and 2 the third year.

Find a stable age distribution for this population. (See Exercise 9, Section 7.4.)

27. Apply the power method to the matrix


$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

discussed in Chapter 7 (Fibonacci sequence). Use the power method to approximate the dominant eigenvalue of A . (The dominant eigenvalue is $\lambda = (1 + \sqrt{5})/2$.)

28. **Writing** In Example 2 of Section 2.5, the stochastic matrix

$$P = \begin{bmatrix} 0.70 & 0.15 & 0.15 \\ 0.20 & 0.80 & 0.15 \\ 0.10 & 0.05 & 0.70 \end{bmatrix}$$

represents the transition probabilities for a consumer preference model. Use the power method to approximate a dominant eigenvector for this matrix. How does the approximation relate to the steady-state matrix described in the discussion following Example 3 in Section 2.5?

-  29. In Exercise 9 of Section 2.5, a population of 10,000 is divided into nonsmokers, moderate smokers, and heavy smokers. Use the power method to approximate a dominant eigenvector for this matrix.