Dynamics of infectious diseases (continuation)

S - I - R model

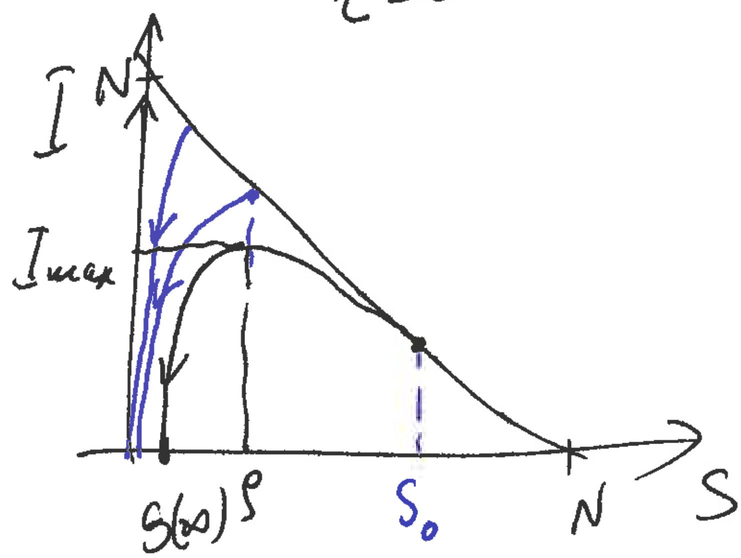
$$\frac{dS}{dt} = -rSI$$

$$\frac{dI}{dt} = rSI - aI$$

$$\frac{dR}{dt} = aI$$

$$S + I + R = N = \text{const}$$

$t=0$ $$\underbrace{S_0 + I_0 + 0}_{t=0} = N = \underbrace{S(\infty) + 0 + R(\infty)}_{t=\infty}$$



$$S + I = N$$

$$\frac{dI}{dt} = I(rS - a) = 0$$

$$S = \frac{a}{r} \equiv \rho$$

Reproduction rate $R_0 = \frac{rS_0}{a} = \frac{S_0}{\rho}$

If $R_0 > 1$ then clearly an epidemic ensues.

$$\frac{dI}{dS} = \frac{rS\cancel{I} - a\cancel{I}}{-rS\cancel{I}} = -\frac{rS-a}{rS} = -1 + \frac{\rho}{S}$$

$$\left(\rho = \frac{a}{r}\right)$$

$$dI = \left(-1 + \frac{\rho}{S}\right)dS \Rightarrow \boxed{I = -S + \rho \ln S + const}$$

$$I + S - \rho \ln S = const = \underbrace{I_0 + S_0}_{N} - \rho \ln S_0$$

$$I(S) = N - S + \rho \ln S - \rho \ln S_0$$

$$\boxed{I(S) = N - S + \rho \ln\left(\frac{S}{S_0}\right)}$$

$$I_{max} = I(\rho) = N - \rho + \rho \ln \frac{\rho}{S_0}, \quad S_0 > \rho$$

$$I_{max} = I(S_0) = N - S_0 + 0 = N - S_0 \qquad S_0 < \rho$$

$$\frac{dS}{dR} = \frac{-rS\cancel{I}}{a\cancel{I}} = -\frac{1}{\rho}S \Rightarrow \frac{dS}{dR} = -\rho S$$

$$S(t) = S_0 e^{-\frac{1}{\rho}R(t)}$$

$$t \to \infty \qquad \boxed{S(\infty) = S_0 e^{-\frac{1}{\rho}[N - S(\infty)]}}$$

S_∞ is the positive root $0 < S(\infty) < \rho$

$$\frac{dR}{dt} = a I = a(N - S - R)$$

$$\frac{dR}{dt} = a\left(N - S_0 e^{-\frac{1}{\rho}R} - R\right) \rightarrow \text{ODE for } \underline{\underline{R}}$$

$$\frac{dR}{dt} = a\left(N - R - S_0 e^{-\frac{R}{\rho}}\right) \rightarrow \underline{\underline{R = R(t)}}$$

$$S(t) = S_0 \exp\left(-\frac{R(t)}{\rho}\right)$$

$$I(t) = N - S(t) + \rho \ln \frac{S(t)}{S_0}$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots$$

Assumption : $\boxed{\frac{R}{\rho} \ll 1}$

$$e^{-\frac{R}{\rho}} \approx 1 - \frac{R}{\rho} + \frac{1}{2}\frac{R^2}{\rho^2}$$

$$\frac{dR}{dt} = a\left[N - R - S_0\left(1 - \frac{R}{\rho} + \frac{1}{2}\frac{R^2}{\rho^2}\right)\right]$$

$$\frac{dR}{dt} = a\left(N - R - S_0 + \frac{S_0 R}{\rho} - \frac{S_0}{2\rho^2}R^2\right)$$

$$\frac{dR}{dt} = a\left[N - S_0 + \left(\frac{S_0}{\rho} - 1\right)R - \frac{S_0}{2\rho^2}R^2\right]$$

$$\boxed{x = \frac{S_0 R}{\rho^2}} \quad . \quad R = \frac{\rho^2 x}{S_0}$$

$$\frac{\rho^2}{S_0}\frac{dx}{dt} = a\left[N - S_0 + \left(\frac{S_0}{\rho} - 1\right)\frac{\rho^2 x}{S_0} - \frac{\cancel{S_0}}{2\cancel{\rho^2}}\frac{\rho^{\cancel{4}\,2} x^2}{S_0^{\cancel{2}}}\right]$$

$$\frac{\rho^2}{S_0}\cdot\frac{dx}{dt} = a\left[N - S_0 + \left(\frac{S_0}{\rho} - 1\right)\frac{\rho^2}{S_0}x - \frac{\rho^2 x^2}{2S_0}\right]$$

$$\frac{\cancel{\rho^2}}{\cancel{S_0}}\frac{dx}{dt} = -a\frac{\cancel{\rho^2}}{2\cancel{S_0}}\left[(N - S_0)\left(-\frac{2S_0}{\rho^2}\right) - \frac{2\cancel{S_0}}{\cancel{\rho^2}}\left(\frac{S_0}{\rho} - 1\right)\frac{\cancel{\rho^2}}{\cancel{S_0}}x + x^2\right]$$

$$\frac{dx}{dt} = -\frac{a}{2}\left[x^2 - 2x\boxed{\left(\frac{S_0}{\rho} - 1\right)} - \frac{2S_0}{\rho^2}(N - S_0)\right]$$

$$\frac{dx}{dt} = -\frac{a}{2}\left[\left(x - \frac{S_0}{\rho} + 1\right)^2 - \left(\frac{S_0}{\rho} - 1\right)^2 - \frac{2S_0}{\rho^2}(N - S_0)\right]$$

$$\boxed{\alpha^2 = \left(\frac{S_0}{\rho} - 1\right)^2 + \frac{2S_0(N - S_0)}{\rho^2}} \qquad (10.14)$$

Define: $y = x - \frac{S_0}{\rho} + 1 \qquad dx = dy$

$$\frac{dy}{dt} = -\frac{a}{2}\left(y^2 - \alpha^2\right)$$

$$\int \frac{dy}{y^2 - \alpha^2} = \int -\frac{a}{2}\, dt$$

$$\int \frac{dy}{y^2-\alpha^2} = -\frac{1}{\alpha}\operatorname{arctanh}\frac{y}{\alpha}$$

$$\tanh y = \frac{\sinh y}{\cosh y} = \frac{e^{y}-e^{-y}}{e^{y}+e^{-y}}$$

$$-\frac{1}{\alpha}\operatorname{arctanh}\frac{y}{\alpha} = -\frac{a}{2}t + c$$

$$\operatorname{arctanh}\frac{y}{\alpha} = \frac{\alpha a}{2}t + c_1$$

$$y = \alpha\tanh\left(\frac{\alpha a t}{2}+c_1\right)$$

$$y = x - \frac{S_0}{\rho} + 1 = \frac{S_0 R}{\rho^2} - \frac{S_0}{\rho} + 1.$$

$$\underbrace{\frac{S_0 R}{\rho^2} - \frac{S_0}{\rho} + 1}_{y} = \alpha\tanh\left(\frac{\alpha a t}{2} \cancel{+} \underset{\text{const.}}{\phi}\right)$$

$t=0$ then $\underline{R(0)=0}$

$$1-\frac{S_0}{\rho} = \alpha\tanh(\cancel{c_1}\,\phi)$$

$$\phi = -\tanh^{-1}\left[\frac{1}{\alpha}\left(1-\frac{S_0}{\rho}\right)\right]$$

$$\boxed{\phi = \tanh^{-1}\left[\frac{1}{\alpha}\left(\frac{S_0}{\rho}-1\right)\right]}$$

$$\underline{\underline{R(t)}} = \frac{\rho^2}{S_0}\left[\frac{S_0}{\rho} - 1 + \alpha \tanh\left(\frac{\alpha a t}{2} - \phi\right)\right]$$

$$\phi = \tanh^{-1}\left[\frac{1}{\alpha}\left(\frac{S_0}{\rho} - 1\right)\right] \quad = \text{const.}$$

$$\alpha = \sqrt{\left(\frac{S_0}{\rho} - 1\right)^2 + \frac{2S_0}{\rho^2}(N - S_0)} \quad = \text{const.}$$

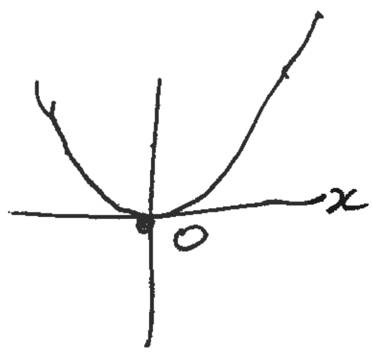
'Removal rate', $\dfrac{dR}{dt}$

$$\frac{dR}{dt} = \frac{\rho^2}{S_0}\alpha \operatorname{sech}^2\left(\frac{\alpha a t}{2} - \phi\right) \cdot \left(\frac{\alpha a}{2}\right)$$

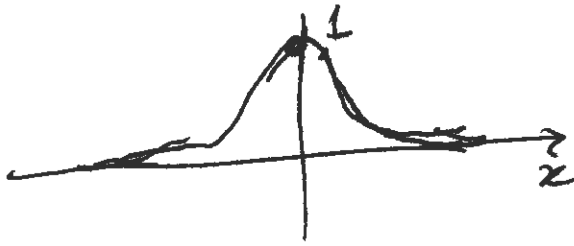
$$\frac{dR}{dt} = \frac{a\rho^2\alpha^2}{2S_0}\operatorname{sech}^2\left(\frac{\alpha a t}{2} - \phi\right) \qquad (10.15)$$

$$\frac{dR}{dt} = \frac{a\rho^2\alpha^2}{2S_0\cosh^2\left(\frac{\alpha a t}{2} - \phi\right)}$$

$$\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$



$$\frac{1}{\cosh^2 x}$$



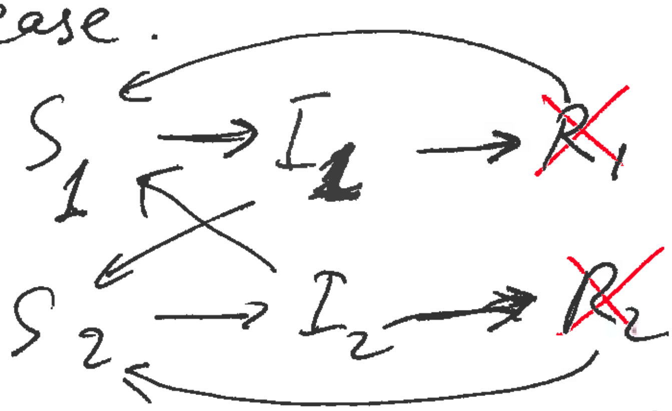
Modelling Veneral Diseases

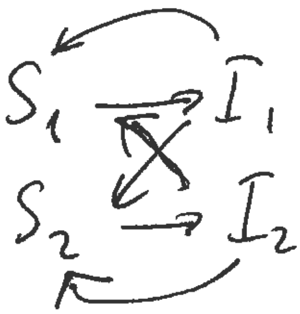
S.T.D. characteristics

1. Restricted to sexually active community.
2. Carrier is asymptomatic.
3. No acquired immunity.

gonorrhea, syphilis, chlamidia

4. Two interacting classes - males & females
5. Passed from member of one class to the other 'criss-cross' type of disease.





$$\frac{dS_1}{dt} = -r_1 S_1 I_2 + a_1 I_1$$

$$\frac{dI_1}{dt} = r_1 S_1 I_2 - a_1 I_1$$

$S_1(t) + I_1(t) = N_1 = \text{const}$

$$1 \Longleftrightarrow 2$$

$$S_1(0) = S_{1,0} \qquad I_1(0) = I_{1,0}$$

$$S_2(0) = S_{2,0} \qquad I_2(0) = I_{2,0}$$

$$S_{1,2} = \underset{\substack{\uparrow \\ \text{const}}}{N_{1,2}} - I_{1,2}$$

$$\left|\begin{aligned} \frac{dI_1}{dt} &= r_1 \underbrace{(N_1 - I_1)}_{S_1} I_2 - a_1 I_1 = f_1(I_1, I_2) \\ \frac{dI_2}{dt} &= r_2 (N_2 - I_2) I_1 - a_2 I_2 = f_2(I_1, I_2) \end{aligned}\right.$$

$$(I_1, I_2) = (0, 0)$$

$$r_1(N_1 - I_1) I_2 = a_1 I_1 \Rightarrow I_2 = \frac{a_1 I_1}{r_1 (N_1 - I_1)}$$

$$\rho_1 = \frac{a_1}{r_1} \qquad \rho_2 = \frac{a_2}{r_2} \qquad \Rightarrow I_2 = \frac{\rho_1 I_1}{N_1 - I_1}$$

$$\underset{\substack{\| \\ 1}}{\frac{r_2}{r_2}} \left(N_2 - \frac{\rho_1 I_1}{N_1 - I_1}\right) \cancel{I_1} = \underset{\substack{\| \\ \rho_2}}{\frac{a_2}{r_2}} \frac{\rho_1 \cancel{I_1}}{N_1 - I_1} \Longleftrightarrow f_2 = 0$$

$$\frac{N_2(N_1 - I_1) - \rho_1 I_1}{\cancel{N_1 - I_1}} = \frac{\rho_1 \rho_2}{\cancel{N_1 - I_1}}$$

$$N_1 N_2 - N_2 I_1 - \rho_1 I_1 = \rho_1 \rho_2$$

$$N_1 N_2 - \rho_1 \rho_2 = I_1 (\rho_1 + N_2)$$

$$\cancel{I_1^* = \frac{\rho_1 A N_2}{\rho_1 + N_2}}$$

$$\boxed{I_1^* = \frac{N_1 N_2 - \rho_1 \rho_2}{\rho_1 + N_2}}$$

$1 \leftrightarrow 2$

$$\boxed{I_2^* = \frac{N_1 N_2 - \rho_1 \rho_2}{\rho_2 + N_1}}$$

$$(0,0) \quad \& \quad \underbrace{(I_1^*, I_2^*)}_{\text{exists only if } N_1 N_2 > \rho_1 \rho_2}$$

'treshold condition'

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial I_1} & \frac{\partial f_1}{\partial I_2} \\ \frac{\partial f_2}{\partial I_1} & \frac{\partial f_2}{\partial I_2} \end{pmatrix} = \begin{pmatrix} -r_1 I_2 - a_1 & r_1 (N_1 - I_1) \\ r_2 (N_2 - I_2) & -r_2 I_1 - a_2 \end{pmatrix}$$

$$A(0,0) = \begin{pmatrix} -a_1 & r_1 N_1 \\ r_2 N_2 & -a_2 \end{pmatrix}$$

$$\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0$$

$$\lambda^2 + (a_1 + a_2)\lambda + a_1 a_2 - r_1 r_2 N_1 N_2 = 0$$

$$\lambda_{1,2} = \frac{-(a_1+a_2) \pm \sqrt{(a_1+a_2)^2 - 4(a_1a_2 - r_1r_2N_1N_2)}}{2}$$

$\rho_k = \frac{a_k}{r_k}$

$$\lambda_{1,2} = \frac{-(a_1+a_2) \pm \sqrt{(a_1+a_2)^2 + 4a_1a_2\left(\frac{N_1N_2}{\rho_1\rho_2} - 1\right)}}{2}$$

I. $N_1N_2 > \rho_1\rho_2$ then $D > 0$,

$\lambda_1 > 0$, $\lambda_2 < 0$ saddle pt. unstable.

II. $N_1N_2 < \rho_1\rho_2$ then $\lambda_1 < 0$, $\lambda_2 < 0$

i.e. $(0,0)$ is the only stable steady state, (I_1^*, I_2^*) does not exist.

→ Stability of (I_1^*, I_2^*) when $N_1N_2 > \rho_1\rho_2$

$$\mathcal{A}(I_1^*, I_2^*) = \begin{pmatrix} -a_1 - r_1 I_2^* & r_1(N_1 - I_1^*) \\ r_2(N_2 - I_2^*) & -a_2 - r_2 I_1^* \end{pmatrix}$$

$$\lambda^2 - \mathrm{tr}A\,\lambda + \det A = 0$$

$$\lambda^2 + (a_1 + a_2 + r_1 I_1^* + r_2 I_2^*)\lambda + (a_1 + r_1 I_2^*)(a_2 + r_2 I_1^*)$$

$$- r_1 r_2 (N_1 - I_1^*)(N_2 - I_2^*) = 0$$

$$\underline{\mathrm{tr}A = -(a_1 + a_2) - r_1 I_2^* - r_2 I_1^* < 0}$$

always.

$$\det A = a_1 a_2 + a_1 r_2 I_1^* + r_1 a_2 I_2^* + \cancel{r_1 r_2 I_1^* I_2^*}$$

$$- r_1 r_2 (N_1 N_2 - I_1^* N_2 - N_1 I_2^* + \cancel{I_1^* I_2^*})$$

$$= a_1 a_2 - r_1 r_2 N_1 N_2 + a_1 r_2 I_1^* + r_1 a_2 I_2^*$$

$$a_1 a_2 \left(1 - \frac{N_1 N_2}{\rho_1 \rho_2}\right) \qquad + r_1 r_2 I_1^* N_2 + r_1 r_2 I_2^* N_1$$

$$\rho_k = \frac{a_k}{r_k}$$

$$\boxed{\rho_k = \frac{a_k}{r_k}}$$

$$\det A = a_1 a_2\left(1 - \frac{N_1 N_2}{\rho_1\rho_2}\right) + (a_1 r_2 + r_1 r_2 N_2)\, I_1^* + (r_1 a_2 + r_1 r_2 N_1)\, I_2^*$$

$$= a_1 a_2 \left(\frac{\rho_1\rho_2 - N_1 N_2}{\rho_1\rho_2}\right) + r_2 a_1 \left(1 + \frac{N_2}{\rho_1}\right) \frac{N_1 N_2 - \rho_1\rho_2}{\rho_1 + N_2} + r_1 a_2 \left(1 + \frac{N_1}{\rho_2}\right) \frac{N_1 N_2 - \rho_1\rho_2}{\rho_2 + N_1} =$$

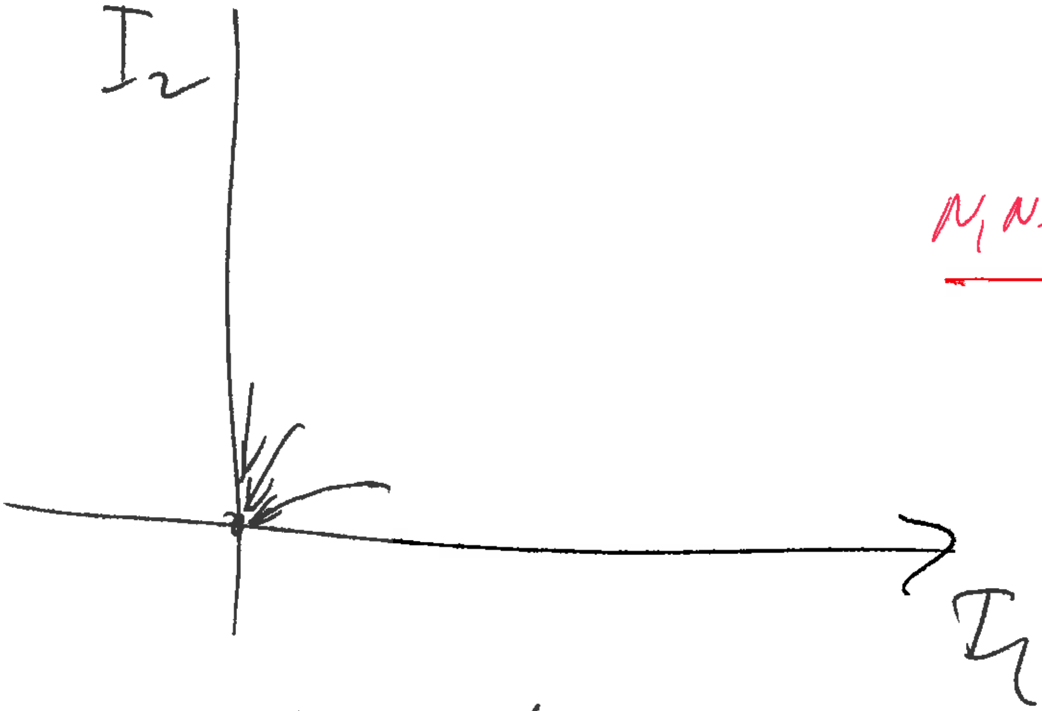
$$= \underbrace{(N_1 N_2 - \rho_1\rho_2)}_{+} \Bigg\{ \underbrace{\frac{r_2 a_1 \cancel{(\rho_1 + N_2)}}{\rho_1 \cancel{(\rho_1 + N_2)}} + \frac{r_1 a_2 \cancel{(\rho_2 + N_1)}}{\rho_2 \cancel{(\rho_2 + N_1)}}}_{+(?)} - \underbrace{\frac{a_1 a_2}{\rho_1\rho_2}}_{= r_1 r_2} \Bigg\}$$

$$= (N_1 N_2 - \rho_1\rho_2)(r_1 r_2 + r_1 r_2 - r_1 r_2) = r_1 r_2 (N_1 N_2 - \rho_1\rho_2) > 0$$

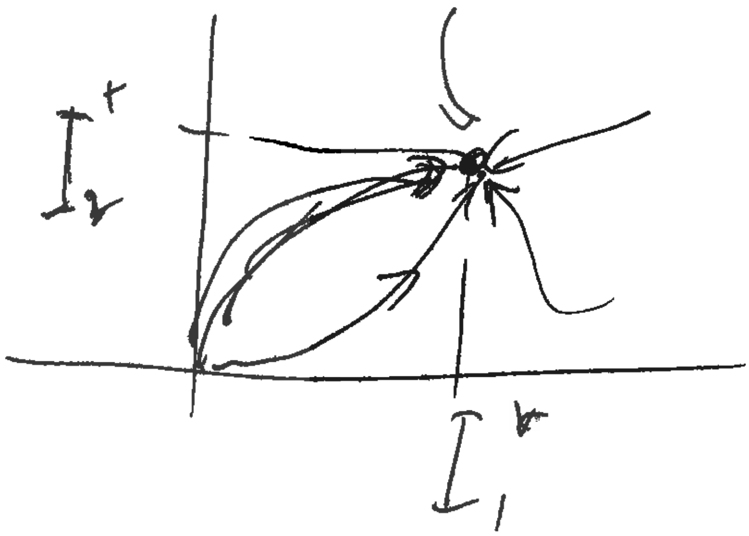
$$A = U \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} U^{-1}$$

$\det A > 0$!

$\Rightarrow (I_1^*, I_2^*)$ is a stable steady state!



$N_1 N_2 < \rho_1 \rho_2$



$N_1 N_2 > \rho_1 \rho_2$