DUBLIN INSTITUTE OF TECHNOLOGY
KEVIN STREET, DUBLIN 8

MSc in Applied Mathematics and Theoretical Physics

DT 238

Autumn Examinations

2010--2011

General Relativity and Cosmology

Dr. Emil M. Prodanov,
Dr. Chris Hills,
Dr. A. Hegarty

Monday 29th August 2011 **9.30 – 1.00 pm**

Answer any three questions. All questions carry equal marks. If more than three questions are attempted, only the best three will be graded.
Latin indexes run from 0 to 3, Greek indexes run from 1 to 3 (over the spatial coordinates).

Log Tables provided

Question One

Consider two observers: A, who is at rest, and B, who is moving away from A with uniform speed.

(a) Define the k–factor. [4 points]

(b) Find the coordinates of an event by bouncing a light signal off it. [4 points]

(c) Express the k–factor in terms of the velocity of B relative to A. [4 points]

(d) Using k–calculus, derive the composition law for velocities. [5 points]

(e) Using k–calculus, derive the special Lorentz transformations (boost in the x–direction). [8 points]

(f) Devise an experiment that shows the relativity of simultaneity and describe this experiment in terms of k–calculus. [9 points]

Question Two

(a) Describe the non-local version of Einstein's Gedanken experiments and explain the significance of the equation of geodesic deviation. [8 points]

(b) Consider a two-dimensional surface S ruled by a congruence of time-like geodesics (with one and only one geodesic through every point on S). Using the following definitions of the Riemann tensor, the Lie derivative and the absolute derivative along a congruence curve with parameter τ and tangent vector v^a :

$$(\nabla_c \nabla_d - \nabla_d \nabla_c) Y^a = R^a{}_{bcd} Y^b,$$
$$L_v \xi^a = v^b \partial_b \xi^a - \xi^b \partial_b v^a,$$
$$\frac{D}{D\tau} T^{a\cdots}_{b\cdots} = \nabla_v T^{a\cdots}_{b\cdots} = v^c \nabla_c T^{a\cdots}_{b\cdots}.$$

derive the equation of geodesic deviation. [13 points]

(c) Using the operator $h^{\alpha}_{\beta}=\delta^{\alpha}_{\beta}-v^{\alpha}v_{\beta}$ that projects tensors onto the three-space orthogonal to v^{α} at any point P of S, re-write the equation of geodesic deviation in terms of the orthogonal vector that connects two neighbouring geodesics in the congruence. [13 points]

Question Three

(a) Show that in the Newtonian limit of a slowly moving weak gravitational field,

$$g_{00}=1+\frac{2\varphi}{c^2}+O\left(\frac{v}{c}\right),$$

where $v \ll c$ is the velocity of the source of the field and φ is the Newtonian gravitational potential. [17 points]

(b) Consider incoherent matter (dust).

(i) Find its energy-momentum tensor. [2 points]

(ii) Give an interpretation of its components. [4 points]

(iii) Show that $\partial_a T^{0a}=0$ is the continuity equation of a perfect fluid. [4 points]

(iv) Show that $\partial_c T^{bc}=0$ is the Navier–Stokes equation of motion for a pressureless perfect fluid, not acted upon by external forces. [7 points]

Question Four

(a) Define a stationary space-time. [3 points]

(b) Consider a congruence of hypersurfaces $f(x^a) = \mu$ and build a Killing vector field orthogonal to these surfaces. [15 points]

(c) Prove that a space-time is static if, and only if, it admits a hypersurface-orthogonal timelike Killing vector field. [16 points]

Question Five

Using the definitions of the Christoffel symbols, Riemann tensor, Ricci tensor, and scalar curvature, respectively:

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} (\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc}),$$

$$R^a{}_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{fc} \Gamma^f_{bd} - \Gamma^a_{fd} \Gamma^f_{bc},$$

$$R_{bd} = R^a{}_{bad},$$

$$R = g^{ab} R_{ab} = R^a{}_a,$$

show that the scalar curvature of a two-dimensional sphere of constant radius R is given by $\frac{2}{R^2}$. [34 points]