Chapter 2
Vibration Dynamics

In this chapter, we review the dynamics of vibrations and the methods of deriving the equations of motion of vibrating systems. The Newton–Euler and Lagrange methods are the most common methods of deriving the equations of motion. Having symmetric coefficient matrices is the main advantage of using the Lagrange method in mechanical vibrations.

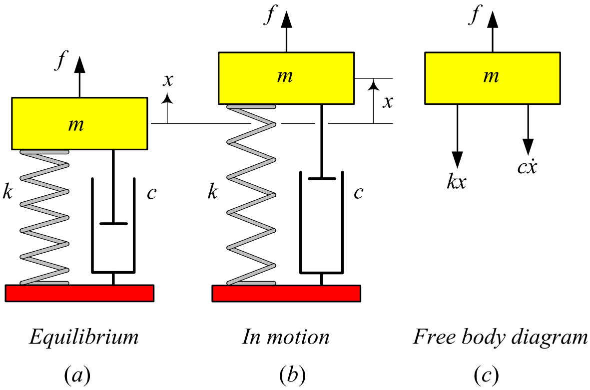

Fig. 2.1 A one *DOF* vibrating systems and its *FBD*

2.1 Newton–Euler Method

When a vibrating system is modeled as a combination of masses m_i, dampers c_i, and springs k_i, it is called a *discrete* or *lumped* model of the system.

To find the equations of motion of a low degree-of-freedom (*DOF*) discrete model of a vibrating system, the Newton–Euler method works very well. We move all the masses m_i out of their equilibria at positions x_i with velocities $\dot{x}_i$. Then a free body diagram (*FBD*) of the lumped masses indicates the total force $\mathbf{F}_i$ on mass m_i.

R.N. Jazar, *Advanced Vibrations*,
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Employing the momentum $\mathbf{p}_i = m_i \mathbf{v}_i$ of the mass m_i, the Newton equation provides us with the equation of motion of the system:

$$\mathbf{F}_i = \frac{\mathrm{d}}{\mathrm{d}t} \mathbf{p}_i = \frac{\mathrm{d}}{\mathrm{d}t} (m_i \mathbf{v}_i) \tag{2.1}$$

When the motion of a massive body with mass moment I_i is rotational, then its equation of motion will be found by Euler equation, in which we employ the moment of momentum $\mathbf{L}_i = I_i \boldsymbol{\omega}$ of the mass m_i:

$$\mathbf{M}_i = \frac{\mathrm{d}}{\mathrm{d}t} \mathbf{L}_i = \frac{\mathrm{d}}{\mathrm{d}t} (I_i \boldsymbol{\omega}) \tag{2.2}$$

For example, Fig. 2.1 illustrates a one degree-of-freedom (*DOF*) vibrating system. Figure 2.1(b) depicts the system when m is out of the equilibrium position at x and moving with velocity $\dot{x}$, both in positive direction. The *FBD* of the system is as shown in Fig. 2.1(c). The Newton equation generates the equations of motion:

$$ma = -cv - kx + f(x, v, t) \tag{2.3}$$

The *equilibrium position* of a vibrating system is where the potential energy of the system, V, is extremum:

$$\frac{\partial V}{\partial x} = 0 \tag{2.4}$$

We usually set $V = 0$ at the equilibrium position. Linear systems with constant stiffness have only one equilibrium or infinity equilibria, while nonlinear systems may have multiple equilibria. An equilibrium is *stable* if

$$\frac{\partial^2 V}{\partial x^2} > 0 \tag{2.5}$$

and is *unstable* if

$$\frac{\partial^2 V}{\partial x^2} < 0 \tag{2.6}$$

The geometric arrangement and the number of employed mechanical elements can be used to classify discrete vibrating systems. The number of masses times the *DOF* of each mass makes the total *DOF* of the vibrating system n. Each independent *DOF* of a mass is indicated by an independent variable, called the *generalized coordinate*. The final set of equations would be n second-order differential equations to be solved for n generalized coordinates. When each moving mass has one *DOF*, then the system's *DOF* is equal to the number of masses. The *DOF* may also be defined as the minimum number of independent coordinates that defines the configuration of a system.

The equation of motion of an n *DOF* linear mechanical vibrating system of can always be arranged as a set of second-order differential equations

$$[m]\ddot{\mathbf{x}} + [c]\dot{\mathbf{x}} + [k]\mathbf{x} = \mathbf{F} \tag{2.7}$$

Fig. 2.2 Two, three, and one *DOF* models for vertical vibrations of vehicles

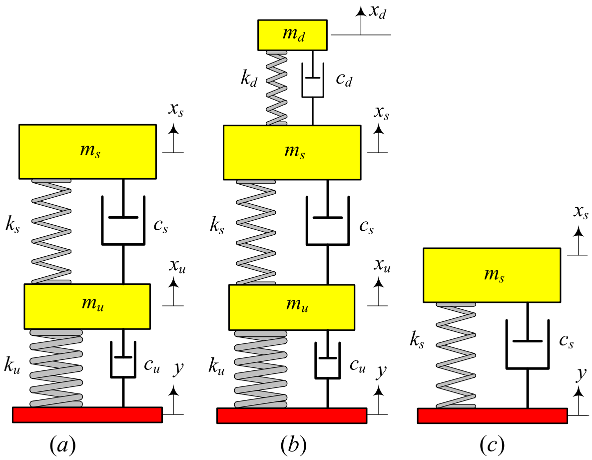
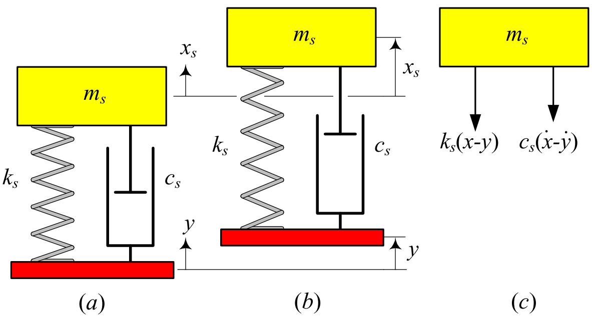


Fig. 2.3 A 1/8 car model and its free body diagram



in which, **x** is a column array of describing coordinates of the system, and **f** is a column array of the associated applied forces. The *square matrices* $[m]$, $[c]$, $[k]$ are the mass, damping, and stiffness matrices.

Example 30 (The one, two, and three *DOF* model of vehicles) The one, two, and three *DOF* model for analysis of vertical vibrations of a vehicle are shown in Fig. 2.2(a)–(c). The system in Fig. 2.2(a) is called the *quarter car* model, in which m_s represents a quarter mass of the body, and m_u represents a wheel. The parameters k_u and c_u are models for tire stiffness and damping. Similarly, k_s and c_u are models for the main suspension of the vehicle. Figure 2.2(c) is called the 1/8 car model, which does not show the wheel of the car, and Fig. 2.2(b) is a quarter car with a driver m_d. The driver's seat is modeled by k_d and c_d.

Example 31 (1/8 car model) Figure 2.3(a) shows the simplest model for vertical vibrations of a vehicle. This model is sometimes called 1/8 car model. The mass m_s represents one quarter of the car's body, which is mounted on a suspension made of a spring k_s and a damper c_s. When m_s is at a position such as shown in Fig. 2.3(b), its free body diagram is as in Fig. 2.3(c).

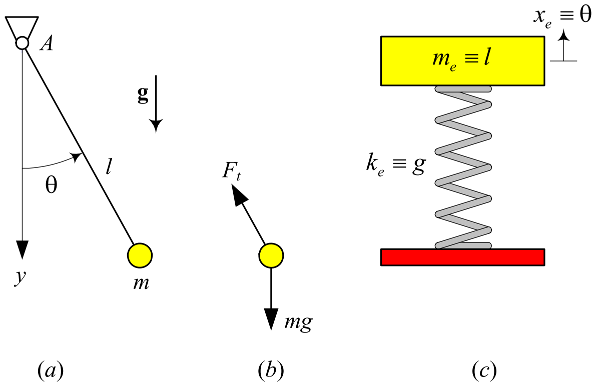


Fig. 2.4 Equivalent mass–spring vibrator for a pendulum

Applying Newton's method, the equation of motion would be

$$m_s\ddot{x} = -k_s(x_s - y) - c_s(\dot{x}_s - \dot{y}) \tag{2.8}$$

which can be simplified to

$$m_s\ddot{x} + c_s\dot{x}_s + k_s x_s = k_s y + c_s\dot{y} \tag{2.9}$$

The coordinate y indicates the input from the road and x indicates the absolute displacement of the body. Absolute displacement refers to displacement with respect to the motionless background.

Example 32 (Equivalent mass and spring) Figure 2.4(a) illustrates a pendulum made by a point mass m attached to a massless bar with length l. The coordinate θ shows the angular position of the bar. The equation of motion for the pendulum can be found by using the Euler equation and employing the *FBD* shown in Fig. 2.4(b):

$$I_A\ddot{\theta} = \sum \mathbf{M}_A \tag{2.10}$$

$$ml^2\ddot{\theta} = -mgl\sin\theta \tag{2.11}$$

Simplifying the equation of motion and assuming a very small swing angle yields

$$l\ddot{\theta} + g\theta = 0 \tag{2.12}$$

This equation is equivalent to an equation of motion of a mass–spring system made by a mass $m_e \equiv l$, and a spring with stiffness $k_e \equiv g$. The displacement of the mass would be $x_e \equiv \theta$. Figure 2.4(c) depicts such an equivalent mass–spring system.

Example 33 (Gravitational force in rectilinear vibrations) When the direction of the gravitational force on a mass m is not varied with respect to the direction of

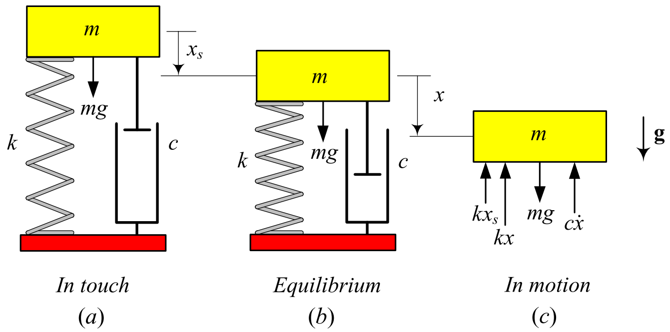


Fig. 2.5 A mass–spring–damper system indicating that the gravitational force in rectilinear vibrations provides us with a static deflection

motion of m, the effect of the weight force can be ignored in deriving the equation of motion. In such a case the equilibrium position of the system is at a point where the gravity is in balance with a deflection in the elastic member. This force–balance equation will not be altered during vibration. Consequently we may ignore both forces; the gravitational force and the static elastic force. It may also be interpreted as an energy balance situation where the work of gravitational force is always equal to the extra stored energy in the elastic member.

Consider a spring k and damper c as is shown in Fig. 2.5(a). A mass m is put on the force free spring and damper. The weight of m compresses the spring a static length x_{s} to bring the system at equilibrium in Fig. 2.5(b). When m is at equilibrium, it is under the balance of two forces, mg and $-kx_{\mathrm{s}}$:

$$mg - kx_{\mathrm{s}} = 0 \tag{2.13}$$

While the mass is in motion, its *FBD* is as shown in Fig. 2.5(c) and its equation of motion is

$$\begin{aligned} m\ddot{x} &= -kx - c\dot{x} + mg - kx_{\mathrm{s}} \\ &= -kx - c\dot{x} \end{aligned} \tag{2.14}$$

It shows that if we examine the motion of the system from equilibrium, we can ignore both the gravitational force and the initial compression of the elastic member of the system.

Example 34 (Force proportionality) The equation of motion for a vibrating system is a balance between four different forces: a force proportional to displacement, $-kx$, a force proportional to velocity, $-cv$, a force proportional to acceleration, ma, and an applied external force $f(x, v, t)$, which can be a function of displacement, velocity, and time. Based on Newton method, the force proportional to acceleration, ma, is always equal to the sum of all the other forces:

$$ma = -cv - kx + f(x, v, t) \tag{2.15}$$

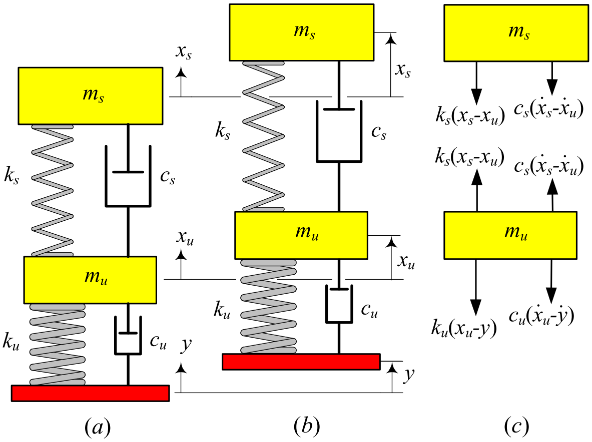


Fig. 2.6 A 1/4 car model and its free body diagram

Example 35 (A two *DOF* base excited system) Figure 2.6(a)–(c) illustrate the equilibrium, motion, and *FBD* of a two *DOF* system. The *FBD* is plotted based on the assumption

$$x_s > x_u > y \tag{2.16}$$

Applying Newton's method provides us with two equations of motion:

$$m_s \ddot{x}_s = -k_s(x_s - x_u) - c_s(\dot{x}_s - \dot{x}_u) \tag{2.17}$$

$$\begin{aligned} m_u \ddot{x}_u = {} & k_s(x_s - x_u) + c_s(\dot{x}_s - \dot{x}_u) \\ & - k_u(x_u - y) - c_u(\dot{x}_u - \dot{y}) \end{aligned} \tag{2.18}$$

The assumption (2.16) is not necessary. We can find the same Eqs. (2.17) and (2.18) using any other assumption, such as $x_s < x_u > y$, $x_s > x_u < y$, or $x_s < x_u < y$. However, having an assumption helps to make a consistent free body diagram.

We usually arrange the equations of motion for a linear system in a matrix form to take advantage of matrix calculus:

$$[M]\dot{\mathbf{x}} + [C]\dot{\mathbf{x}} + [K]\mathbf{x} = \mathbf{F} \tag{2.19}$$

Rearrangement of Eqs. (2.17) and (2.18) yields

$$\begin{aligned} & \begin{bmatrix} m_s & 0 \\ 0 & m_u \end{bmatrix} \begin{bmatrix} \ddot{x}_s \\ \ddot{x}_u \end{bmatrix} + \begin{bmatrix} c_s & -c_s \\ -c_s & c_s + c_u \end{bmatrix} \begin{bmatrix} \dot{x}_s \\ \dot{x}_u \end{bmatrix} \\ & \quad + \begin{bmatrix} k_s & -k_s \\ -k_s & k_s + k_u \end{bmatrix} \begin{bmatrix} x_s \\ x_u \end{bmatrix} = \begin{bmatrix} 0 \\ k_u y + c_u \dot{y} \end{bmatrix} \end{aligned} \tag{2.20}$$

Example 36 ★ (Inverted pendulum and negative stiffness) Figure 2.7(a) illustrates an inverted pendulum with a tip mass m and a length l. The pendulum is supported

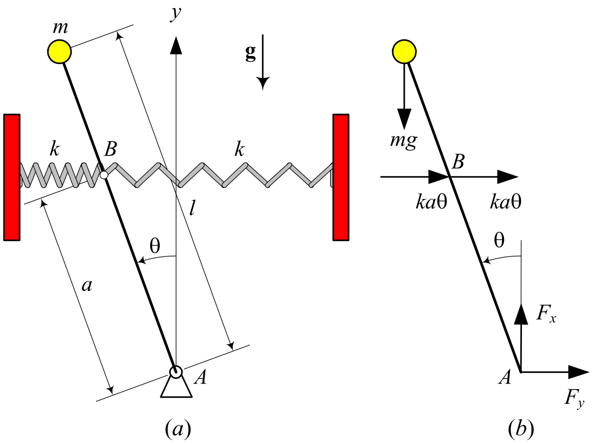


Fig. 2.7 An inverted pendulum with a tip mass m and two supportive springs

by two identical springs attached to point B at a distance $a < l$ from the pivot A. A free body diagram of the pendulum is shown in Fig. 2.7(b). The equation of motion may be found by taking a moment about A:

$$\sum M_A = I_A \ddot{\theta} \tag{2.21}$$

$$mg(l \sin\theta) - 2ka\theta(a\cos\theta) = ml^2\ddot{\theta} \tag{2.22}$$

To derive Eq. (2.22) we assumed that the springs are long enough to remain almost straight when the pendulum oscillates. Rearrangement and assuming a very small θ show that the nonlinear equation of motion (2.22) can be approximated by

$$ml^2\ddot{\theta} + \left(mgl - 2ka^2\right)\theta = 0 \tag{2.23}$$

which is equivalent to a linear oscillator:

$$m_{\mathrm{e}}\ddot{\theta} + k_{\mathrm{e}}\theta = 0 \tag{2.24}$$

with an equivalent mass m_{e} and equivalent stiffness k_{e}:

$$m_{\mathrm{e}} = ml^2 \qquad k_{\mathrm{e}} = mgl - 2ka^2 \tag{2.25}$$

The potential energy of the inverted pendulum is

$$V = -mgl(1 - \cos\theta) + ka^2\theta^2 \tag{2.26}$$

which has a zero value at $\theta = 0$. The potential energy V is approximately equal to the following equation if θ is very small:

$$V \approx -\frac{1}{2}mgl\theta^2 + ka^2\theta^2 \tag{2.27}$$

because

$$\cos\theta \approx 1 - \frac{1}{2}\theta^2 + O\left(\theta^4\right) \tag{2.28}$$

To find the equilibrium positions of the system, we may solve the equation $\partial V/\partial\theta = 0$ for any possible θ:

$$\frac{\partial V}{\partial\theta} = -2mgl\theta + 2ka^2\theta = 0 \tag{2.29}$$

The solution of the equation is

$$\theta = 0 \tag{2.30}$$

which shows that the upright vertical position is the only equilibrium of the inverted pendulum as long as θ is very small. However, if

$$mgl = ka^2 \tag{2.31}$$

then any θ around $\theta = 0$ would be an equilibrium position and, hence, the inverted pendulum would have an infinity of equilibria.

The second derivative of the potential energy

$$\frac{\partial^2 V}{\partial x^2} = -2mgl + 2ka^2 \tag{2.32}$$

indicates that the equilibrium position $\theta = 0$ is stable if

$$ka^2 > mgl \tag{2.33}$$

A stable equilibrium pulls the system back if it deviates from the equilibrium, while an unstable equilibrium repels the system. Vibration happens when the equilibrium is stable.

This example also indicates the fact that having a negative stiffness is possible by geometric arrangement of mechanical components of a vibrating system.

Example 37 ★ (Force function in equation of motion) Qualitatively, force is whatever changes the motion, and quantitatively, force is whatever is equal to mass times acceleration. Mathematically, the equation of motion provides us with a vectorial second-order differential equation

$$m\ddot{\mathbf{r}} = \mathbf{F}(\dot{\mathbf{r}}, \mathbf{r}, t) \tag{2.34}$$

We assume that the force function may generally be a function of time t, position $\mathbf{r}$, and velocity $\dot{\mathbf{r}}$. In other words, the Newton equation of motion is correct as long as we can show that the force is only a function of $\dot{\mathbf{r}}, \mathbf{r}, t$.

If there is a force that depends on the acceleration, jerk, or other variables that cannot be reduced to $\dot{\mathbf{r}}, \mathbf{r}, t$, the system is not Newtonian and we do not know the equation of motion, because

$$\mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}, \ldots, t) \neq m\ddot{\mathbf{r}} \tag{2.35}$$

In Newtonian mechanics, we assume that force can only be a function of $\dot{\mathbf{r}}, \mathbf{r}, t$ and nothing else. In real world, however, force may be a function of everything; however, we always ignore any other variables than $\dot{\mathbf{r}}, \mathbf{r}, t$.

Because Eq. (2.34) is a linear equation for force $\mathbf{F}$, it accepts the superposition principle. When a mass m is affected by several forces $\mathbf{F}_1$, $\mathbf{F}_2, \mathbf{F}_3, \ldots$, we may calculate their summation vectorially

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots \tag{2.36}$$

and apply the resultant force on m. So, if a force $\mathbf{F}_1$ provides us with acceleration $\ddot{\mathbf{r}}_1$, and $\mathbf{F}_2$ provides us with $\ddot{\mathbf{r}}_2$,

$$m\ddot{\mathbf{r}}_1 = \mathbf{F}_1 \qquad m\ddot{\mathbf{r}}_2 = \mathbf{F}_2 \tag{2.37}$$

then the resultant force $\mathbf{F}_3 = \mathbf{F}_1 + \mathbf{F}_2$ provides us with the acceleration $\ddot{\mathbf{r}}_3$ such that

$$\ddot{\mathbf{r}}_3 = \ddot{\mathbf{r}}_1 + \ddot{\mathbf{r}}_2 \tag{2.38}$$

To see that the Newton equation of motion is not correct when the force is not only a function of $\dot{\mathbf{r}}, \mathbf{r}, t$, let us assume that a particle with mass m is under two acceleration dependent forces $F_1(\ddot{x})$ and $F_2(\ddot{x})$ on x-axis:

$$m\ddot{x}_1 = F_1(\ddot{x}_1) \qquad m\ddot{x}_2 = F_2(\ddot{x}_2) \tag{2.39}$$

The acceleration of m under the action of both forces would be $\ddot{x}_3$

$$m\ddot{x}_3 = F_1(\ddot{x}_3) + F_2(\ddot{x}_3) \tag{2.40}$$

however, though we must have

$$\ddot{x}_3 = \ddot{x}_1 + \ddot{x}_2 \tag{2.41}$$

we do have

$$\begin{aligned} m(\ddot{x}_1 + \ddot{x}_2) &= F_1(\ddot{x}_1 + \ddot{x}_2) + F_2(\ddot{x}_1 + \ddot{x}_2) \\ &\neq F_1(\ddot{x}_1) + F_2(\ddot{x}_2) \end{aligned} \tag{2.42}$$

2.2 ★ Energy

In Newtonian mechanics, the acting forces on a system of bodies can be divided into *internal* and *external forces*. Internal forces are acting between bodies of the system, and external forces are acting from outside of the system. External forces and moments are called the *load*. The acting forces and moments on a body are called a *force system*. The *resultant* or *total force* $\mathbf{F}$ is the sum of all the external forces acting on the body, and the *resultant* or *total moment* $\mathbf{M}$ is the sum of all the moments of the external forces about a point, such as the origin of a coordinate frame:

$$\mathbf{F} = \sum_i \mathbf{F}_i \qquad \mathbf{M} = \sum_i \mathbf{M}_i \tag{2.43}$$

The moment $\mathbf{M}$ of a force $\mathbf{F}$, acting at a point P with position vector $\mathbf{r}_P$, about a point Q at $\mathbf{r}_Q$ is

$$\mathbf{M}_Q = (\mathbf{r}_P - \mathbf{r}_Q) \times \mathbf{F} \tag{2.44}$$

and, therefore, the moment of $\mathbf{F}$ about the origin is

$$\mathbf{M} = \mathbf{r}_P \times \mathbf{F} \tag{2.45}$$

The *moment of the force* about a directional line l passing through the origin is

$$\mathbf{M}_l = \hat{u} \cdot (\mathbf{r}_P \times \mathbf{F}) \tag{2.46}$$

where $\hat{u}$ is a unit vector on l. The moment of a force may also be called *torque* or *moment*.

The effect of a force system is equivalent to the effect of the resultant force and resultant moment of the force system. Any two force systems are equivalent if their resultant forces and resultant moments are equal. If the resultant force of a force system is zero, the resultant moment of the force system is independent of the origin of the coordinate frame. Such a resultant moment is called a *couple*.

When a force system is reduced to a resultant $\mathbf{F}_P$ and $\mathbf{M}_P$ with respect to a reference point P, we may change the reference point to another point Q and find the new resultants as

$$\mathbf{F}_Q = \mathbf{F}_P \tag{2.47}$$

$$\mathbf{M}_Q = \mathbf{M}_P + (\mathbf{r}_P - \mathbf{r}_Q) \times \mathbf{F}_P = \mathbf{M}_P + {}_Q\mathbf{r}_P \times \mathbf{F}_P \tag{2.48}$$

The *momentum* of a moving rigid body is a vector quantity equal to the total mass of the body times the translational velocity of the mass center of the body:

$$\mathbf{p} = m\mathbf{v} \tag{2.49}$$

The momentum $\mathbf{p}$ is also called the *translational momentum* or *linear momentum*.

Consider a rigid body with momentum $\mathbf{p}$. The *moment of momentum*, $\mathbf{L}$, about a directional line l passing through the origin is

$$\mathbf{L}_l = \hat{u} \cdot (\mathbf{r}_C \times \mathbf{p}) \tag{2.50}$$

where $\hat{u}$ is a unit vector indicating the direction of the line, and $\mathbf{r}_C$ is the position vector of the mass center C. The moment of momentum about the origin is

$$\mathbf{L} = \mathbf{r}_C \times \mathbf{p} \tag{2.51}$$

The moment of momentum $\mathbf{L}$ is also called *angular momentum*.

Kinetic energy K of a moving body point P with mass m at a position ${}^G\mathbf{r}_P$, and having a velocity ${}^G\mathbf{v}_P$, in the global coordinate frame G is

$$K = \frac{1}{2} m\, {}^G\mathbf{v}_P \cdot {}^G\mathbf{v}_P = \frac{1}{2} m\, {}^G\mathbf{v}_P^2 \tag{2.52}$$

where G indicates the global coordinate frame in which the velocity vector $\mathbf{v}_P$ is expressed. The work done by the applied force ${}^G\mathbf{F}$ on m in moving from point 1 to point 2 on a path, indicated by a vector ${}^G\mathbf{r}$, is

$${}_1W_2 = \int_1^2 {}^G\mathbf{F} \cdot \mathrm{d}\,{}^G\mathbf{r} \tag{2.53}$$

However,

$$\begin{aligned}\int_1^2 {}^G\mathbf{F} \cdot \mathrm{d}\,{}^G\mathbf{r} &= m \int_1^2 \frac{{}^G\mathrm{d}}{\mathrm{d}t}\, {}^G\mathbf{v} \cdot {}^G\mathbf{v}\, \mathrm{d}t = \frac{1}{2} m \int_1^2 \frac{\mathrm{d}}{\mathrm{d}t} v^2\, \mathrm{d}t \\ &= \frac{1}{2} m \left(v_2^2 - v_1^2\right) = K_2 - K_1\end{aligned} \tag{2.54}$$

which shows that ${}_1W_2$ is equal to the difference of the kinetic energy of terminal and initial points:

$${}_1W_2 = K_2 - K_1 \tag{2.55}$$

Equation (2.55) is called the *principle of work and energy*. If there is a scalar *potential field function* $V = V(x, y, z)$ such that

$$\mathbf{F} = -\nabla V = -\frac{\mathrm{d}V}{\mathrm{d}\mathbf{r}} = -\left(\frac{\partial V}{\partial x}\hat{\imath} + \frac{\partial V}{\partial y}\hat{\jmath} + \frac{\partial V}{\partial z}\hat{k}\right) \tag{2.56}$$

then the principle of work and energy (2.55) simplifies to the principle of *conservation of energy*,

$$K_1 + V_1 = K_2 + V_2 \tag{2.57}$$

The value of the potential field function $V = V(x, y, z)$ is the *potential energy* of the system.

Proof Consider the spatial integral of Newton equation of motion

$$\int_1^2 \mathbf{F} \cdot \mathrm{d}\mathbf{r} = m \int_1^2 \mathbf{a} \cdot \mathrm{d}\mathbf{r} \tag{2.58}$$

We can simplify the right-hand side of the integral (2.58) by the change of variable

$$\begin{aligned}\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= m \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{a} \cdot \mathrm{d}\mathbf{r} = m \int_{t_1}^{t_2} \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \cdot \mathbf{v}\, \mathrm{d}t \\ &= m \int_{\mathbf{v}_1}^{\mathbf{v}_2} \mathbf{v} \cdot \mathrm{d}\mathbf{v} = \frac{1}{2} m \left(\mathbf{v}_2^2 - \mathbf{v}_1^2\right)\end{aligned} \tag{2.59}$$

The kinetic energy of a point mass m that is at a position defined by ${}^G\mathbf{r}$ and having a velocity ${}^G\mathbf{v}$ is defined by (2.52). Whenever the global coordinate frame G is the only involved frame, we may drop the superscript G for simplicity. The work done by the applied force ${}^G\mathbf{F}$ on m in going from point $\mathbf{r}_1$ to $\mathbf{r}_2$ is defined by (2.53). Hence the spatial integral of equation of motion (2.58) reduces to the principle of work and energy (2.55):

$$ {}_1W_2 = K_2 - K_1 \tag{2.60} $$

which says that the work ${}_1W_2$ done by the applied force ${}^G\mathbf{F}$ on m during the displacement $\mathbf{r}_2 - \mathbf{r}_1$ is equal to the difference of the kinetic energy of m.

If the force $\mathbf{F}$ is the gradient of a potential function V,

$$ \mathbf{F} = -\nabla V \tag{2.61} $$

then $\mathbf{F} \cdot \mathrm{d}\mathbf{r}$ in Eq. (2.58) is an exact differential and, hence,

$$ \int_1^2 \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_1^2 \mathrm{d}V = -(V_2 - V_1) \tag{2.62} $$

$$ E = K_1 + V_1 = K_2 + V_2 \tag{2.63} $$

In this case the work done by the force is independent of the path of motion between $\mathbf{r}_1$ and $\mathbf{r}_2$ and depends only upon the value of the potential V at start and end points of the path. The function V is called the potential energy; Eq. (2.63) is called the principle of conservation of energy, and the force $\mathbf{F} = -\nabla V$ is called a potential, or a conservative force. The kinetic plus potential energy of the dynamic system is called the mechanical energy of the system and is denoted by $E = K + V$. The mechanical energy E is a constant of motion if all the applied forces are conservative.

A force $\mathbf{F}$ is conservative only if it is the gradient of a stationary scalar function. The components of a conservative force will only be functions of space coordinates:

$$ \mathbf{F} = F_x(x, y, z)\hat{\imath} + F_y(x, y, z)\hat{\jmath} + F_z(x, y, z)\hat{k} \tag{2.64} $$

□

Example 38 (Energy and equation of motion) Whenever there is no loss of energy in a mechanical vibrating system, the sum of kinetic and potential energies is a constant of motion:

$$ E = K + V = \text{const} \tag{2.65} $$

A system with constant energy is called a conservative system. The time derivative of a constant of motion must be zero at all time.

The mass–spring system of Fig. 2.10 is a conservative system with the total mechanical energy of

$$ E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \tag{2.66} $$

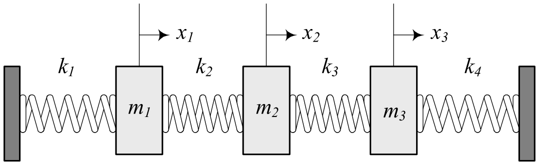


Fig. 2.8 A multi *DOF* conservative vibrating system

Having a zero rate of energy,

$$\dot{E} = m\dot{x}\ddot{x} + kx\dot{x} = \dot{x}(m\ddot{x} + kx) = 0 \tag{2.67}$$

and knowing that $\dot{x}$ cannot be zero at all times provides us with the equation of motion:

$$m\ddot{x} + kx = 0 \tag{2.68}$$

Example 39 ★ (Energy and multi *DOF* systems) We may use energy method and determine the equations of motion of multi *DOF* conservative systems. Consider the system in Fig. 2.8 whose mechanical energy is

$$\begin{aligned} E = K + V = & \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 \\ & + \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_1 - x_2)^2 + \frac{1}{2}k_3(x_2 - x_3)^2 + \frac{1}{2}k_4x_3^2 \end{aligned} \tag{2.69}$$

To find the first equation of motion associated to x_1, we assume x_2 and x_3 are constant and take the time directive:

$$\dot{E} = m_1\dot{x}_1\ddot{x}_1 + k_1x_1\dot{x}_1 + k_2(x_1 - x_2)\dot{x}_1 = 0 \tag{2.70}$$

Because $\dot{x}_1$ cannot be zero at all times, the first equation of motion is

$$m_1\ddot{x}_1 + k_1x_1 + k_2(x_1 - x_2) = 0 \tag{2.71}$$

To find the second equation of motion associated to x_2, we assume that x_1 and x_3 are constant and we take a time directive of E

$$\dot{E} = m_2\dot{x}_2\ddot{x}_2 - k_2(x_1 - x_2)\dot{x}_2 + k_3(x_2 - x_3)\dot{x}_2 = 0 \tag{2.72}$$

which provides us with

$$m_2\ddot{x}_2 - k_2(x_1 - x_2) + k_3(x_2 - x_3) = 0 \tag{2.73}$$

To find the second equation of motion associated to x_2, we assume that x_1 and x_3 are constant and we take the time directive of E

$$\dot{E} = m_3\dot{x}_3\ddot{x}_3 - k_3(x_2 - x_3)\dot{x}_3 + k_4x_3\dot{x}_3 = 0 \tag{2.74}$$

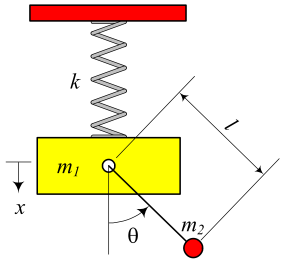


Fig. 2.9 A two *DOF* conservative nonlinear vibrating system

which provides us with

$$m_3\ddot{x}_3 - k_3(x_2 - x_3) + k_4x_3 = 0 \tag{2.75}$$

We may set up the equations in a matrix form:

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \tag{2.76}$$

Example 40 ★ (Energy and nonlinear multi *DOF* systems) The energy method can be applied on every conservative system regardless of linearity of the system. Figure 2.9 illustrates a two *DOF* nonlinear system whose kinetic and potential energies are

$$K = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left(\dot{x}^2 + k^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}\sin\theta\right)^2 \tag{2.77}$$

$$V = \frac{1}{2}kx^2 - m_2g(x - l\cos\theta) \tag{2.78}$$

We assumed that the motionless hanging down position is the equilibrium of interest, and that the gravitational energy is zero at the level of m_1 at the equilibrium.

Example 41 (Maximum energy and frequency of vibrations) The mechanical vibrations is a continuous exchange of energy between kinetic and potential. If there is no waste of energy, their maximum values must be equal.

Consider the simple mass–spring system of Fig. 2.10. The harmonic motion, kinetic energy, and potential energy of the system are

$$x = X\sin\omega t \tag{2.79}$$

$$K = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}mX^2\omega^2\cos^2\omega t \tag{2.80}$$

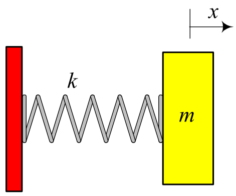


Fig. 2.10 A mass–spring system

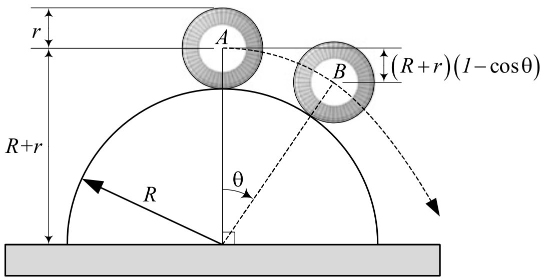


Fig. 2.11 A wheel turning, without slip, over a cylindrical hill

$$V = \frac{1}{2}kx^2 = \frac{1}{2}kX^2 \sin^2 \omega t \tag{2.81}$$

Equating the maximum K and V

$$\frac{1}{2}mX^2\omega^2 = \frac{1}{2}kX^2 \tag{2.82}$$

provides us with the frequency of vibrations:

$$\omega^2 = \frac{k}{m} \tag{2.83}$$

Example 42 ★ (Falling wheel) Figure 2.11 illustrates a wheel turning, without slip, over a cylindrical hill. We may use the conservation of mechanical energy to find the angle at which the wheel leaves the hill.

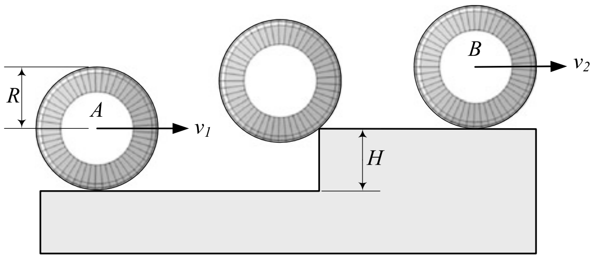
Initially, the wheel is at point A. We assume the initial kinetic and potential, and hence, the mechanical energies $E = K + V$ are zero. When the wheel is turning over the hill, its angular velocity, ω, is

$$\omega = \frac{v}{r} \tag{2.84}$$

where v is the speed at the center of the wheel. At any other point B, the wheel achieves some kinetic energy and loses some potential energy. At a certain angle, where the normal component of the weight cannot provide more centripetal force,

$$mg\cos\theta = \frac{mv^2}{R+r} \tag{2.85}$$

Fig. 2.12 A turning wheel moving up a step



the wheel separates from the surface. Employing the conservation of energy, we have

$$K_A + V_A = K_B + V_B \tag{2.86}$$

The kinetic and potential energy at the separation point B are

$$K_B = \frac{1}{2}mv^2 + \frac{1}{2}I_C\omega^2 \tag{2.87}$$

$$V_B = -mg(R+r)(1-\cos\theta) \tag{2.88}$$

where I_C is the mass moment of inertia for the wheel about its center. Therefore,

$$\frac{1}{2}mv^2 + \frac{1}{2}I_C\omega^2 = mg(R+r)(1-\cos\theta) \tag{2.89}$$

and substituting (2.84) and (2.85) yields

$$\left(1+\frac{I_C}{mr^2}\right)(R+r)g\cos\theta = 2g(R+r)(1-\cos\theta) \tag{2.90}$$

and, therefore, the separation angle is

$$\theta = \cos^{-1}\frac{2mr^2}{I_C+3mr^2} \tag{2.91}$$

Let us examine the equation for a disc wheel with

$$I_C = \frac{1}{2}mr^2 \tag{2.92}$$

and find the separation angle:

$$\theta = \cos^{-1}\frac{4}{7} \approx 0.96 \text{ rad} \approx 55.15 \text{ deg} \tag{2.93}$$

Example 43 ★ (Turning wheel over a step) Figure 2.12 illustrates a wheel of radius R turning with speed v to go over a step with height $H < R$. We may use the principle of energy conservation and find the speed of the wheel after getting across the step. Employing the conservation of energy, we have

$$K_A + V_A = K_B + V_B \tag{2.94}$$

$$\frac{1}{2}mv_1^2 + \frac{1}{2}I_C\omega_1^2 + 0 = \frac{1}{2}mv_2^2 + \frac{1}{2}I_C\omega_2^2 + mgH \tag{2.95}$$

$$\left(m + \frac{I_C}{R^2}\right)v_1^2 = \left(m + \frac{I_C}{R^2}\right)v_2^2 + 2mgH \tag{2.96}$$

and, therefore,

$$v_2 = \sqrt{v_1^2 - \frac{2gH}{1 + \frac{I_C}{mR^2}}} \tag{2.97}$$

The condition for having a real v_2 is

$$v_1 > \sqrt{\frac{2gH}{1 + \frac{I_C}{mR^2}}} \tag{2.98}$$

The second speed (2.97) and the condition (2.98) for a solid disc with $I_C = mR^2/2$ are

$$v_2 = \sqrt{v_1^2 - \frac{4}{3}Hg} \tag{2.99}$$

$$v_1 > \sqrt{\frac{4}{3}Hg} \tag{2.100}$$

Example 44 (Newton equation) The application of a force system is emphasized by Newton's second law of motion, which states that the global rate of change of linear momentum is proportional to the global applied force:

$${}^G\mathbf{F} = \frac{{}^G\mathrm{d}}{\mathrm{d}t}{}^G\mathbf{p} = \frac{{}^G\mathrm{d}}{\mathrm{d}t}\left(m\,{}^G\mathbf{v}\right) \tag{2.101}$$

The second law of motion can be expanded to include rotational motions. Hence, the second law of motion also states that the global rate of change of angular momentum is proportional to the global applied moment:

$${}^G\mathbf{M} = \frac{{}^G\mathrm{d}}{\mathrm{d}t}{}^G\mathbf{L} \tag{2.102}$$

Proof Differentiating the angular momentum (2.51) shows that

$$\begin{aligned}\frac{{}^G\mathrm{d}}{\mathrm{d}t}{}^G\mathbf{L} &= \frac{{}^G\mathrm{d}}{\mathrm{d}t}(\mathbf{r}_C \times \mathbf{p}) = \left(\frac{{}^G\mathrm{d}\mathbf{r}_C}{\mathrm{d}t} \times \mathbf{p} + \mathbf{r}_C \times \frac{{}^G\mathrm{d}\mathbf{p}}{\mathrm{d}t}\right) \\ &= {}^G\mathbf{r}_C \times \frac{{}^G\mathrm{d}\mathbf{p}}{\mathrm{d}t} = {}^G\mathbf{r}_C \times {}^G\mathbf{F} = {}^G\mathbf{M}\end{aligned} \tag{2.103}$$

□

Example 45 ★ (Integral and constant of motion) Any equation of the form

$$f(\mathbf{q}, \dot{\mathbf{q}}, t) = c \tag{2.104}$$

$$c = f(\mathbf{q}_0, \dot{\mathbf{q}}_0, t_0) \tag{2.105}$$

$$q = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} \tag{2.106}$$

with total differential

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial \dot{q}_i}\ddot{q}_i\right) + \frac{\partial f}{\partial t} = 0 \tag{2.107}$$

that the generalized positions $\mathbf{q}$ and velocities $\dot{\mathbf{q}}$ of a dynamic system must satisfy at all times t is called an *integral of motion*. The parameter c, of which the value depends on the initial conditions, is called a *constant of motion*. The maximum number of independent integrals of motion for a dynamic system with n degrees of freedom is $2n$. A constant of motion is a quantity of which the value remains constant during the motion.

Any integral of motion is a result of a conservation principle or a combination of them. There are only three conservation principles for a dynamic system: energy, momentum, and moment of momentum. Every conservation principle is the result of a symmetry in position and time. The conservation of energy indicates the homogeneity of time, the conservation of momentum indicates the homogeneity in position space, and the conservation of moment of momentum indicates the isotropy in position space.

Proof Consider a mechanical system with f_C degrees of freedom. Mathematically, the dynamics of the system is expressed by a set of $n = f_C$ second-order differential equations of n unknown generalized coordinates $q_i(t)$, $i = 1, 2, \ldots, n$:

$$\ddot{q}_i = F_i(q_i, \dot{q}_i, t) \quad i = 1, 2, \ldots, n \tag{2.108}$$

The general solution of the equations contains $2n$ constants of integrals.

$$\dot{q}_i = \dot{q}_i(c_1, c_2, \ldots, c_n, t) \quad i = 1, 2, \ldots, n \tag{2.109}$$

$$q_i = q_i(c_1, c_2, \ldots, c_{2n}, t) \quad i = 1, 2, \ldots, n \tag{2.110}$$

To determine these constants and uniquely identify the motion of the system, it is necessary to know the initial conditions $q_i(t_0)$, $\dot{q}_i(t_0)$, which specify the state of the system at some given instant t_0:

$$c_j = c_j\big(\mathbf{q}(t_0), \dot{\mathbf{q}}(t_0), t_0\big) \quad j = 1, 2, \ldots, 2n \tag{2.111}$$

$$f_j\big(\mathbf{q}(t), \dot{\mathbf{q}}(t), t\big) = c_j\big(\mathbf{q}(t_0), \dot{\mathbf{q}}(t_0), t_0\big) \tag{2.112}$$

Each of these functions f_j is an integral of the motion and each c_i is a constant of the motion. An integral of motion may also be called a *first integral*, and a constant of motion may also be called a *constant of integral*.

When an integral of motion is given,

$$f_1(\mathbf{q}, \dot{\mathbf{q}}, t) = c_1 \tag{2.113}$$

we can substitute one of the equations of (2.108) with the first-order equation of

$$\dot{q}_1 = f(c_1, q_i, \dot{q}_{i+1}, t) \quad i = 1, 2, \ldots, n \tag{2.114}$$

and solve a set of $n-1$ second-order and one first-order differential equations:

$$\begin{cases} \ddot{q}_{i+1} = F_{i+1}(q_i, \dot{q}_i, t) \\ \dot{q}_1 = f(c_1, q_i, \dot{q}_{i+1}, t) \end{cases} \quad i = 1, 2, \ldots, n \tag{2.115}$$

If there exist $2n$ independent first integrals f_j, $j = 1, 2, \ldots, 2n$, then instead of solving n second-order equations of motion (2.108), we can solve a set of $2n$ algebraic equations

$$f_j(\mathbf{q}, \dot{\mathbf{q}}) = c_j\big(\mathbf{q}(t_0), \dot{\mathbf{q}}(t_0), t_0\big) \quad j = 1, 2, \ldots, 2n \tag{2.116}$$

and determine the n generalized coordinates q_i, $i = 1, 2, \ldots, n$:

$$q_i = q_i(c_1, c_2, \ldots, c_{2n}, t) \quad i = 1, 2, \ldots, n \tag{2.117}$$

Generally speaking, an integral of motion f is a function of generalized coordinates $\mathbf{q}$ and velocities $\dot{\mathbf{q}}$ such that its value remains constant. The value of an integral of motion is the constant of motion c, which can be calculated by substituting the given value of the variables $\mathbf{q}(t_0)$, $\dot{\mathbf{q}}(t_0)$ at the associated time t_0. □

Example 46 ★ (A mass–spring–damper vibrator) Consider a mass m attached to a spring with stiffness k and a damper with damping c. The equation of motion of the system and its initial conditions are

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{2.118}$$

$$x(0) = x_0 \qquad \dot{x}(0) = \dot{x}_0 \tag{2.119}$$

Its solution is

$$x = c_1 \exp(s_1 t) + c_2 \exp(s_2 t) \tag{2.120}$$

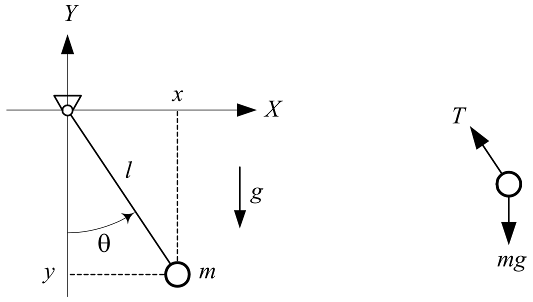


Fig. 2.13 A planar pendulum

$$s_1 = \frac{c - \sqrt{c^2 - 4km}}{-2m} \qquad s_2 = \frac{c + \sqrt{c^2 - 4km}}{-2m} \tag{2.121}$$

Taking the time derivative, we find $\dot{x}$:

$$\dot{x} = c_1 s_1 \exp(s_1 t) + c_2 s_2 \exp(s_2 t) \tag{2.122}$$

Using x and $\dot{x}$, we determine the integrals of motion f_1 and f_2:

$$f_1 = \frac{\dot{x} - x s_2}{(s_1 - s_2)\exp(s_1 t)} = c_1 \tag{2.123}$$

$$f_2 = \frac{\dot{x} - x s_1}{(s_2 - s_1)\exp(s_2 t)} = c_2 \tag{2.124}$$

Because the constants of integral remain constant during the motion, we can calculate their value at any particular time such as $t = 0$:

$$c_1 = \frac{\dot{x}_0 - x_0 s_2}{(s_1 - s_2)} \qquad c_2 = \frac{\dot{x}_0 - x_0 s_1}{(s_2 - s_1)} \tag{2.125}$$

Substituting s_1 and s_2 provides us with the constants of motion c_1 and c_2:

$$c_1 = \frac{\sqrt{c^2 - 4km}\left(c x_0 + x_0\sqrt{c^2 - 4km} + 2m\dot{x}_0\right)}{2(c^2 - 4km)} \tag{2.126}$$

$$c_2 = \frac{\sqrt{c^2 - 4km}\left(c x_0 - x_0\sqrt{c^2 - 4km} + 2m\dot{x}_0\right)}{2(c^2 - 4km)} \tag{2.127}$$

Example 47 ★ (Constraint and first integral of a pendulum) Figure 2.13(a) illustrates a planar pendulum. The free body diagram of Fig. 2.13(b) provides us with two equations of motion:

$$m\ddot{x} = -T\frac{x}{l} \tag{2.128}$$

$$m\ddot{y} = -mg + T\frac{y}{l} \tag{2.129}$$

Eliminating the tension force T, we have one second-order equation of two variables:

$$\ddot{y}x + \ddot{x}y + gx = 0 \tag{2.130}$$

Because of the constant length of the connecting bar we have a constraint equation between x and y:

$$x^2 + y^2 - l^2 = 0 \tag{2.131}$$

Having one constraint indicates that we can express the dynamic of the system by only one generalized coordinate. Choosing θ as the generalized coordinate, we can express x and y by θ, writing the equation of motion (2.130) as

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \tag{2.132}$$

Multiplying the equation by $\dot{\theta}$ and integrating provides us with the integral of energy:

$$f(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^2 - \frac{g}{l}\cos\theta = E \tag{2.133}$$

$$E = \frac{1}{2}\dot{\theta}_0^2 - \frac{g}{l}\cos\theta_0 \tag{2.134}$$

The integral of motion (2.133) is a first-order differential equation:

$$\dot{\theta} = \sqrt{2E + 2\frac{g}{l}\cos\theta} \tag{2.135}$$

This equation expresses the dynamic of the pendulum upon solution.

Let us assume that θ is too small to approximate the equation of motion as

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \tag{2.136}$$

The first integral of this equation is

$$f(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^2 - \frac{g}{l}\theta = E \tag{2.137}$$

$$E = \frac{1}{2}\dot{\theta}_0^2 - \frac{g}{l}\theta_0 \tag{2.138}$$

that provides us with a separated first-order differential equation:

$$\dot{\theta} = \sqrt{2E + 2\frac{g}{l}\theta} \tag{2.139}$$

Its solution is

$$t = \int \frac{\mathrm{d}\theta}{\sqrt{2E + 2\frac{g}{l}\theta}} = \sqrt{2}\frac{l}{g}\sqrt{\frac{g}{l}\theta + E} - p \tag{2.140}$$

where p is the second constant of motion:

$$p = \frac{l}{g}\dot{\theta}_0 \tag{2.141}$$

Now, let us ignore the energy integral and solve the second-order equation of motion (2.136):

$$\theta = c_1 \cos\sqrt{\frac{g}{l}}t + c_2 \sin\sqrt{\frac{g}{l}}t \tag{2.142}$$

The time derivative of the solution

$$\sqrt{\frac{l}{g}}\dot{\theta} = -c_1 \sin\sqrt{\frac{g}{l}}t + c_2 \cos\sqrt{\frac{g}{l}}t \tag{2.143}$$

can be used to determine the integrals and constants of motion:

$$f_1 = \theta \cos\sqrt{\frac{g}{l}}t - \sqrt{\frac{l}{g}}\dot{\theta} \sin\sqrt{\frac{g}{l}}t \tag{2.144}$$

$$f_2 = \theta \sin\sqrt{\frac{g}{l}}t + \sqrt{\frac{l}{g}}\dot{\theta} \cos\sqrt{\frac{g}{l}}t \tag{2.145}$$

Using the initial conditions $\theta(0) = \theta_0$, $\dot{\theta}(0) = \dot{\theta}_0$, we have

$$c_1 = \theta_0 \qquad c_2 = \sqrt{\frac{l}{g}}\dot{\theta}_0 \tag{2.146}$$

A second-order equation has only two constants of integrals. Therefore, we should be able to express E and p in terms of c_1 and c_2 or vice versa:

$$E = \frac{1}{2}\dot{\theta}_0^2 - \frac{g}{l}\theta_0 = \frac{1}{2}\frac{g}{l}c_2^2 - \frac{g}{l}c_1 \tag{2.147}$$

$$p = \frac{l}{g}\dot{\theta}_0 = \frac{l}{g}\sqrt{\frac{g}{l}}c_2 \tag{2.148}$$

$$c_2 = \sqrt{\frac{l}{g}}\dot{\theta}_0 = \frac{g}{l}\sqrt{\frac{l}{g}}p \tag{2.149}$$

$$c_1 = \theta_0 = \frac{1}{2}\frac{g}{l}p^2 - \frac{l}{g}E \tag{2.150}$$

E is the mechanical energy of the pendulum, and p is proportional to its moment of momentum.

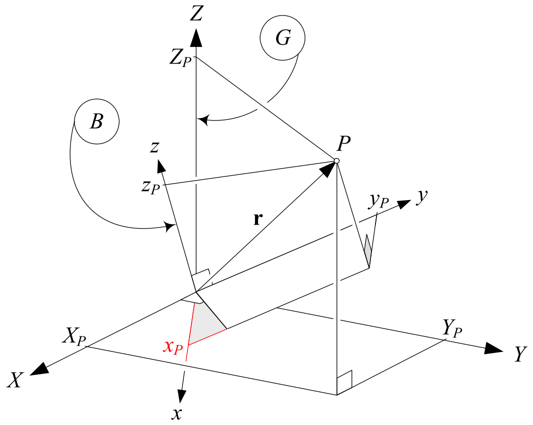


Fig. 2.14 A globally fixed G-frame and a body B-frame with a fixed common origin at O

2.3 ★ Rigid Body Dynamics

A rigid body may have three translational and three rotational DOF. The translational and rotational equations of motion of the rigid body are determined by the Newton–Euler equations.

2.3.1 ★ *Coordinate Frame Transformation*

Consider a rotation of a body coordinate frame $B(Oxyz)$ with respect to a global frame $G(OXYZ)$ about their common origin O as illustrated in Fig. 2.14. The components of any vector $\mathbf{r}$ may be expressed in either frame. There is always a *transformation matrix* ${}^G R_B$ to map the components of $\mathbf{r}$ from the frame $B(Oxyz)$ to the other frame $G(OXYZ)$:

$$ {}^G\mathbf{r} = {}^G R_B \, {}^B\mathbf{r} \tag{2.151}$$

In addition, the inverse map ${}^B\mathbf{r} = {}^G R_B^{-1} \, {}^G\mathbf{r}$ can be done by ${}^B R_G$,

$$ {}^B\mathbf{r} = {}^B R_G \, {}^G\mathbf{r} \tag{2.152}$$

where

$$ \left| {}^G R_B \right| = \left| {}^B R_G \right| = 1 \tag{2.153}$$

and

$$ {}^B R_G = {}^G R_B^{-1} = {}^G R_B^{\mathrm{T}} \tag{2.154}$$

When the coordinate frames B and G are orthogonal, the rotation matrix ${}^G R_B$ is called an *orthogonal matrix*. The transpose R^{T} and inverse R^{-1} of an orthogonal matrix $[R]$ are equal:

$$ R^{\mathrm{T}} = R^{-1} \tag{2.155}$$

Because of the matrix orthogonality condition, only three of the nine elements of ${}^G R_B$ are independent.

Proof Employing the orthogonality condition

$$\mathbf{r} = (\mathbf{r} \cdot \hat{\imath})\hat{\imath} + (\mathbf{r} \cdot \hat{\jmath})\hat{\jmath} + (\mathbf{r} \cdot \hat{k})\hat{k} \tag{2.156}$$

and decomposition of the unit vectors of $G(OXYZ)$ along the axes of $B(Oxyz)$,

$$\hat{I} = (\hat{I} \cdot \hat{\imath})\hat{\imath} + (\hat{I} \cdot \hat{\jmath})\hat{\jmath} + (\hat{I} \cdot \hat{k})\hat{k} \tag{2.157}$$

$$\hat{J} = (\hat{J} \cdot \hat{\imath})\hat{\imath} + (\hat{J} \cdot \hat{\jmath})\hat{\jmath} + (\hat{J} \cdot \hat{k})\hat{k} \tag{2.158}$$

$$\hat{K} = (\hat{K} \cdot \hat{\imath})\hat{\imath} + (\hat{K} \cdot \hat{\jmath})\hat{\jmath} + (\hat{K} \cdot \hat{k})\hat{k} \tag{2.159}$$

introduces the transformation matrix ${}^G R_B$ to map the local axes to the global axes:

$$\begin{bmatrix} \hat{I} \\ \hat{J} \\ \hat{K} \end{bmatrix} = \begin{bmatrix} \hat{I} \cdot \hat{\imath} & \hat{I} \cdot \hat{\jmath} & \hat{I} \cdot \hat{k} \\ \hat{J} \cdot \hat{\imath} & \hat{J} \cdot \hat{\jmath} & \hat{J} \cdot \hat{k} \\ \hat{K} \cdot \hat{\imath} & \hat{K} \cdot \hat{\jmath} & \hat{K} \cdot \hat{k} \end{bmatrix} \begin{bmatrix} \hat{\imath} \\ \hat{\jmath} \\ \hat{k} \end{bmatrix} = {}^G R_B \begin{bmatrix} \hat{\imath} \\ \hat{\jmath} \\ \hat{k} \end{bmatrix} \tag{2.160}$$

where

$$\begin{aligned} {}^G R_B &= \begin{bmatrix} \hat{I} \cdot \hat{\imath} & \hat{I} \cdot \hat{\jmath} & \hat{I} \cdot \hat{k} \\ \hat{J} \cdot \hat{\imath} & \hat{J} \cdot \hat{\jmath} & \hat{J} \cdot \hat{k} \\ \hat{K} \cdot \hat{\imath} & \hat{K} \cdot \hat{\jmath} & \hat{K} \cdot \hat{k} \end{bmatrix} \\ &= \begin{bmatrix} \cos(\hat{I}, \hat{\imath}) & \cos(\hat{I}, \hat{\jmath}) & \cos(\hat{I}, \hat{k}) \\ \cos(\hat{J}, \hat{\imath}) & \cos(\hat{J}, \hat{\jmath}) & \cos(\hat{J}, \hat{k}) \\ \cos(\hat{K}, \hat{\imath}) & \cos(\hat{K}, \hat{\jmath}) & \cos(\hat{K}, \hat{k}) \end{bmatrix} \end{aligned} \tag{2.161}$$

Each column of ${}^G R_B$ is the decomposition of a unit vector of the local frame $B(Oxyz)$ in the global frame $G(OXYZ)$:

$${}^G R_B = \begin{bmatrix} {}^G\hat{\imath} & {}^G\hat{\jmath} & {}^G\hat{k} \end{bmatrix} \tag{2.162}$$

Similarly, each row of ${}^G R_B$ is decomposition of a unit vector of the global frame $G(OXYZ)$ in the local frame $B(Oxyz)$.

$${}^G R_B = \begin{bmatrix} {}^B\hat{I}^{\mathrm{T}} \\ {}^B\hat{J}^{\mathrm{T}} \\ {}^B\hat{K}^{\mathrm{T}} \end{bmatrix} \tag{2.163}$$

so the elements of ${}^G R_B$ are directional cosines of the axes of $G(OXYZ)$ in $B(Oxyz)$ or B in G. This set of nine directional cosines completely specifies the orientation of $B(Oxyz)$ in $G(OXYZ)$ and can be used to map the coordinates of any point (x, y, z) to its corresponding coordinates (X, Y, Z).

Alternatively, using the method of unit-vector decomposition to develop the matrix ${}^{B}R_{G}$ leads to

$$ {}^{B}\mathbf{r} = {}^{B}R_{G}\,{}^{G}\mathbf{r} = {}^{G}R_{B}^{-1}\,{}^{G}\mathbf{r} \tag{2.164} $$

$$ \begin{aligned} {}^{B}R_{G} &= \begin{bmatrix} \hat{\imath}\cdot\hat{I} & \hat{\imath}\cdot\hat{J} & \hat{\imath}\cdot\hat{K} \\ \hat{\jmath}\cdot\hat{I} & \hat{\jmath}\cdot\hat{J} & \hat{\jmath}\cdot\hat{K} \\ \hat{k}\cdot\hat{I} & \hat{k}\cdot\hat{J} & \hat{k}\cdot\hat{K} \end{bmatrix} \\ &= \begin{bmatrix} \cos(\hat{\imath},\hat{I}) & \cos(\hat{\imath},\hat{J}) & \cos(\hat{\imath},\hat{K}) \\ \cos(\hat{\jmath},\hat{I}) & \cos(\hat{\jmath},\hat{J}) & \cos(\hat{\jmath},\hat{K}) \\ \cos(\hat{k},\hat{I}) & \cos(\hat{k},\hat{J}) & \cos(\hat{k},\hat{K}) \end{bmatrix} \end{aligned} \tag{2.165} $$

It shows that the inverse of a transformation matrix is equal to the transpose of the transformation matrix,

$$ {}^{G}R_{B}^{-1} = {}^{G}R_{B}^{\mathrm{T}} \tag{2.166} $$

or

$$ {}^{G}R_{B}\cdot{}^{G}R_{B}^{\mathrm{T}} = \mathbf{I} \tag{2.167} $$

A matrix with condition (2.166) is called an *orthogonal matrix*. Orthogonality of ${}^{G}R_{B}$ comes from the fact that it maps an orthogonal coordinate frame to another orthogonal coordinate frame.

An orthogonal transformation matrix ${}^{G}R_{B}$ has only three *independent* elements. The constraint equations among the elements of ${}^{G}R_{B}$ will be found by applying the matrix orthogonality condition (2.166):

$$ \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.168} $$

Therefore, the inner product of any two different rows of ${}^{G}R_{B}$ is zero, and the inner product of any row of ${}^{G}R_{B}$ by itself is unity:

$$ \begin{aligned} r_{11}^2 + r_{12}^2 + r_{13}^2 &= 1 \\ r_{21}^2 + r_{22}^2 + r_{23}^2 &= 1 \\ r_{31}^2 + r_{32}^2 + r_{33}^2 &= 1 \\ r_{11}r_{21} + r_{12}r_{22} + r_{13}r_{23} &= 0 \\ r_{11}r_{31} + r_{12}r_{32} + r_{13}r_{33} &= 0 \\ r_{21}r_{31} + r_{22}r_{32} + r_{23}r_{33} &= 0 \end{aligned} \tag{2.169} $$

These relations are also true for columns of ${}^{G}R_{B}$ and evidently for rows and columns of ${}^{B}R_{G}$. The orthogonality condition can be summarized by the equation

$$\sum_{i=1}^{3} r_{ij} r_{ik} = \delta_{jk} \qquad j,k = 1,2,3 \tag{2.170}$$

where r_{ij} is the element of row i and column j of the transformation matrix ${}^{G}R_{B}$ and δ_{jk} is the Kronecker delta δ_{ij},

$$\delta_{ij} = \delta_{ji} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{2.171}$$

Equation (2.170) provides us with six independent relations that must be satisfied by the nine directional cosines. Therefore, there are only three independent directional cosines. The independent elements of the matrix ${}^{G}R_{B}$ cannot be in the same row or column or any diagonal.

The determinant of a transformation matrix is equal to unity,

$$\left|{}^{G}R_{B}\right| = 1 \tag{2.172}$$

because of Eq. (2.167) and noting that

$$\left|{}^{G}R_{B} \cdot {}^{G}R_{B}^{\mathrm{T}}\right| = \left|{}^{G}R_{B}\right| \cdot \left|{}^{G}R_{B}^{\mathrm{T}}\right| = \left|{}^{G}R_{B}\right| \cdot \left|{}^{G}R_{B}\right| = \left|{}^{G}R_{B}\right|^{2} = 1 \tag{2.173}$$

Using linear algebra and column vectors ${}^{G}\hat{\imath}$, ${}^{G}\hat{\jmath}$, and ${}^{G}\hat{k}$ of ${}^{G}R_{B}$, we know that

$$\left|{}^{G}R_{B}\right| = {}^{G}\hat{\imath} \cdot \left({}^{G}\hat{\jmath} \times {}^{G}\hat{k}\right) \tag{2.174}$$

and because the coordinate system is right handed, we have ${}^{G}\hat{\jmath} \times {}^{G}\hat{k} = {}^{G}\hat{\imath}$ and, therefore,

$$\left|{}^{G}R_{B}\right| = {}^{G}\hat{\imath}^{\mathrm{T}} \cdot {}^{G}\hat{\imath} = +1 \tag{2.175}$$

□

Example 48 (Global position using ${}^{B}\mathbf{r}$ and ${}^{B}R_{G}$) The position vector $\mathbf{r}$ of a point P may be described in either the $G(OXYZ)$ or the $B(Oxyz)$ frame. If ${}^{B}\mathbf{r} = 10\hat{\imath} - 5\hat{\jmath} + 15\hat{k}$ and the transformation matrix to map ${}^{G}\mathbf{r}$ to ${}^{B}\mathbf{r}$ is

$${}^{B}\mathbf{r} = {}^{B}R_{G}\,{}^{G}\mathbf{r} = \begin{bmatrix} 0.866 & 0 & 0.5 \\ -0.353 & 0.707 & 0.612 \\ 0.353 & 0.707 & -0.612 \end{bmatrix} {}^{G}\mathbf{r} \tag{2.176}$$

then the components of ${}^{G}\mathbf{r}$ in $G(OXYZ)$ would be

$${}^{G}\mathbf{r} = {}^{G}R_{B}\,{}^{B}\mathbf{r} = {}^{B}R_{G}^{\mathrm{T}}\,{}^{B}\mathbf{r} = \begin{bmatrix} 15.72 \\ 7.07 \\ -7.24 \end{bmatrix} \tag{2.177}$$

Example 49 (Two-point transformation matrix) The global position vectors of two points P_1 and P_2, of a rigid body B are

$$
{}^{G}\mathbf{r}_{P_1} = \begin{bmatrix} 1.077 \\ 1.365 \\ 2.666 \end{bmatrix} \qquad {}^{G}\mathbf{r}_{P_2} = \begin{bmatrix} -0.473 \\ 2.239 \\ -0.959 \end{bmatrix} \tag{2.178}
$$

The origin of the body $B(Oxyz)$ is fixed on the origin of $G(OXYZ)$, and the points P_1 and P_2 are lying on the local x- and y-axis, respectively.

To find ${}^{G}R_B$, we use the local unit vectors ${}^{G}\hat{\imath}$ and ${}^{G}\hat{\jmath}$,

$$
{}^{G}\hat{\imath} = \frac{{}^{G}\mathbf{r}_{P_1}}{|{}^{G}\mathbf{r}_{P_1}|} = \begin{bmatrix} 0.338 \\ 0.429 \\ 0.838 \end{bmatrix} \qquad {}^{G}\hat{\jmath} = \frac{{}^{G}\mathbf{r}_{P_2}}{|{}^{G}\mathbf{r}_{P_2}|} = \begin{bmatrix} -0.191 \\ 0.902 \\ -0.387 \end{bmatrix} \tag{2.179}
$$

to obtain ${}^{G}\hat{k}$:

$$
{}^{G}\hat{k} = \hat{\imath} \times \hat{\jmath} = \begin{bmatrix} -0.922 \\ -0.029 \\ 0.387 \end{bmatrix} \tag{2.180}
$$

Hence, the transformation matrix ${}^{G}R_B$ would be

$$
{}^{G}R_B = \begin{bmatrix} {}^{G}\hat{\imath} & {}^{G}\hat{\jmath} & {}^{G}\hat{k} \end{bmatrix} = \begin{bmatrix} 0.338 & -0.191 & -0.922 \\ 0.429 & 0.902 & -0.029 \\ 0.838 & -0.387 & 0.387 \end{bmatrix} \tag{2.181}
$$

Example 50 (Length invariant of a position vector) Expressing a vector in different frames utilizing rotation matrices does not affect the length and direction properties of the vector. Therefore, the length of a vector is an invariant property:

$$
|\mathbf{r}| = \left|{}^{G}\mathbf{r}\right| = \left|{}^{B}\mathbf{r}\right| \tag{2.182}
$$

The length invariant property can be shown as

$$
\begin{aligned}
|\mathbf{r}|^2 &= {}^{G}\mathbf{r}^{\mathrm{T}}\,{}^{G}\mathbf{r} = \left[{}^{G}R_B\,{}^{B}\mathbf{r}\right]^{\mathrm{T}}\,{}^{G}R_B\,{}^{B}\mathbf{r} = {}^{B}\mathbf{r}^{\mathrm{T}}\,{}^{G}R_B^{\mathrm{T}}\,{}^{G}R_B\,{}^{B}\mathbf{r} \\
&= {}^{B}\mathbf{r}^{\mathrm{T}}\,{}^{B}\mathbf{r}
\end{aligned} \tag{2.183}
$$

Example 51 (Multiple rotation about global axes) Consider a globally fixed point P at

$$
{}^{G}\mathbf{r} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \tag{2.184}
$$

The body B will turn 45 deg about the X-axis and then 45 deg about the Y-axis. An observer in B will see P at

$$
\begin{aligned}
{}^{B}\mathbf{r} &= R_{y,-45}\, R_{x,-45}\, {}^{G}\mathbf{r} \\
&= \begin{bmatrix} \cos\frac{-\pi}{4} & 0 & -\sin\frac{-\pi}{4} \\ 0 & 1 & 0 \\ \sin\frac{-\pi}{4} & 0 & \cos\frac{-\pi}{4} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{-\pi}{4} & \sin\frac{-\pi}{4} \\ 0 & -\sin\frac{-\pi}{4} & \cos\frac{-\pi}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 0.707 & 0.5 & 0.5 \\ 0 & 0.707 & -0.707 \\ -0.707 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3.207 \\ -0.707 \\ 1.793 \end{bmatrix}
\end{aligned} \tag{2.185}
$$

To check this result, let us change the role of B and G. So, the body point at

$$
{}^{B}\mathbf{r} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \tag{2.186}
$$

undergoes an active rotation of 45 deg about the x-axis followed by 45 deg about the y-axis. The global coordinates of the point would be

$$
{}^{B}\mathbf{r} = R_{y,45}\, R_{x,45}\, {}^{G}\mathbf{r} \tag{2.187}
$$

so

$$
{}^{G}\mathbf{r} = [R_{y,45}\, R_{x,45}]^{\mathrm{T}}\, {}^{B}\mathbf{r} = R^{\mathrm{T}}_{x,45}\, R^{\mathrm{T}}_{y,45}\, {}^{B}\mathbf{r} \tag{2.188}
$$

Example 52 (Multiple rotations about body axes) Consider a globally fixed point P at

$$
{}^{G}\mathbf{r} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \tag{2.189}
$$

The body B will turn 45 deg about the x-axis and then 45 deg about the y-axis. An observer in B will see P at

$$
\begin{aligned}
{}^{B}\mathbf{r} &= R_{Y,-45}\, R_{X,-45}\, {}^{G}\mathbf{r} \\
&= \begin{bmatrix} \cos\frac{-\pi}{4} & 0 & \sin\frac{-\pi}{4} \\ 0 & 1 & 0 \\ -\sin\frac{-\pi}{4} & 0 & \cos\frac{-\pi}{4} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{-\pi}{4} & -\sin\frac{-\pi}{4} \\ 0 & \sin\frac{-\pi}{4} & \cos\frac{-\pi}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 0.707 & 0.5 & -0.5 \\ 0 & 0.707 & 0.707 \\ 0.707 & -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0.20711 \\ 3.5356 \\ 1.2071 \end{bmatrix}
\end{aligned} \tag{2.190}
$$

Example 53 (Successive rotations about global axes) After a series of sequential rotations R_1, R_2, $R_3, \ldots, R_n$ about the global axes, the final global position of a body point P can be found by

$$
{}^{G}\mathbf{r} = {}^{G}R_{B}\, {}^{B}\mathbf{r} \tag{2.191}
$$

where

$$ {}^{G}R_{B} = R_{n} \cdots R_{3}\, R_{2}\, R_{1} \tag{2.192} $$

The vectors ${}^{G}\mathbf{r}$ and ${}^{B}\mathbf{r}$ indicate the position vectors of the point P in the global and local coordinate frames, respectively. The matrix ${}^{G}R_{B}$, which transforms the local coordinates to their corresponding global coordinates, is called the *global rotation matrix*.

Because matrix multiplications do not commute, the sequence of performing rotations is important and indicates the order of rotations.

Proof Consider a body frame B that undergoes two sequential rotations R_1 and R_2 about the global axes. Assume that the body coordinate frame B is initially coincident with the global coordinate frame G. The rigid body rotates about a global axis, and the global rotation matrix R_1 gives us the new global coordinate ${}^{G}\mathbf{r}_1$ of the body point:

$$ {}^{G}\mathbf{r}_1 = R_1\, {}^{B}\mathbf{r} \tag{2.193} $$

Before the second rotation, the situation is similar to the one before the first rotation. We put the B-frame aside and assume that a new body coordinate frame B_1 is coincident with the global frame. Therefore, the new body coordinate would be ${}^{B_1}\mathbf{r} \equiv {}^{G}\mathbf{r}_1$. The second global rotation matrix R_2 provides us with the new global position ${}^{G}\mathbf{r}_2$ of the body points ${}^{B_1}\mathbf{r}$:

$$ {}^{B_1}\mathbf{r} = R_2\, {}^{B_1}\mathbf{r} \tag{2.194} $$

Substituting (2.193) into (2.194) shows that

$$ {}^{G}\mathbf{r} = R_2\, R_1\, {}^{B}\mathbf{r} \tag{2.195} $$

Following the same procedure we can determine the final global position of a body point after a series of sequential rotations $R_1, R_2, R_3, \ldots, R_n$ as (2.192). □

Example 54 (Successive rotations about local axes) Consider a rigid body B with a local coordinate frame $B(Oxyz)$ that does a series of sequential rotations R_1, R_2, $R_3, \ldots, R_n$ about the local axes. Having the final global position vector ${}^{G}\mathbf{r}$ of a body point P, we can determine its local position vector ${}^{B}\mathbf{r}$ by

$$ {}^{B}\mathbf{r} = {}^{B}R_{G}\, {}^{G}\mathbf{r} \tag{2.196} $$

where

$$ {}^{B}R_{G} = R_{n} \cdots R_{3} R_{2} R_{1} \tag{2.197} $$

The matrix ${}^{B}R_{G}$ is called the *local rotation matrix* and it maps the global coordinates of body points to their local coordinates.

Proof Assume that the body coordinate frame B was initially coincident with the global coordinate frame G. The rigid body rotates about a local axis, and a local rotation matrix R_1 relates the global coordinates of a body point to the associated local coordinates:

$$^{B}\mathbf{r} = R_1 \, {}^{G}\mathbf{r} \tag{2.198}$$

If we introduce an intermediate space-fixed frame G_1 coincident with the new position of the body coordinate frame, then

$$^{G_1}\mathbf{r} \equiv {}^{B}\mathbf{r} \tag{2.199}$$

and we may give the rigid body a second rotation about a local coordinate axis. Now another proper local rotation matrix R_2 relates the coordinates in the intermediate fixed frame to the corresponding local coordinates:

$$^{B}\mathbf{r} = R_2 \, {}^{G_1}\mathbf{r} \tag{2.200}$$

Hence, to relate the final coordinates of the point, we must first transform its global coordinates to the intermediate fixed frame and then transform to the original body frame. Substituting (2.198) in (2.200) shows that

$$^{B}\mathbf{r} = R_2 \, R_1 \, {}^{G}\mathbf{r} \tag{2.201}$$

Following the same procedure we can determine the final global position of a body point after a series of sequential rotations $R_1, R_2, R_3, \ldots, R_n$ as (2.197).

Rotation about the local coordinate axes is conceptually interesting. This is because in a sequence of rotations each rotation is about one of the axes of the local coordinate frame, which has been moved to its new global position during the last rotation. □

2.3.2 ★ *Velocity Kinematics*

Consider a rotating rigid body $B(Oxyz)$ with a fixed point O in a reference frame $G(OXYZ)$, as shown in Fig. 2.15. We express the motion of the body by a time-varying rotation transformation matrix between B and G to transform the instantaneous coordinates of body points to their coordinates in the global frame:

$$^{G}\mathbf{r}(t) = {}^{G}R_B(t) \, {}^{B}\mathbf{r} \tag{2.202}$$

The velocity of a body point in the global frame is

$$^{G}\mathbf{v}(t) = {}^{G}\dot{\mathbf{r}}(t) = {}^{G}\dot{R}_B(t) \, {}^{B}\mathbf{r} = {}_{G}\tilde{\omega}_B \, {}^{G}\mathbf{r}(t) = {}_{G}\boldsymbol{\omega}_B \times {}^{G}\mathbf{r}(t) \tag{2.203}$$

where ${}_{G}\boldsymbol{\omega}_B$ is the *angular velocity vector* of B with respect to G. It is equal to a rotation with *angular speed* $\dot{\phi}$ about an *instantaneous axis of rotation* $\hat{u}$:

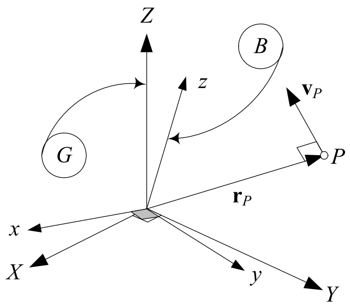


Fig. 2.15 A rotating rigid body $B(Oxyz)$ with a fixed point O in a global frame $G(OXYZ)$

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \dot{\phi}\,\hat{u} \tag{2.204}$$

The angular velocity vector is associated with a skew-symmetric matrix ${}_G\tilde{\omega}_B$ called the *angular velocity matrix*,

$$\tilde{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{2.205}$$

where

$${}_G\tilde{\omega}_B = {}^G\dot{R}_B\,{}^GR_B^{\mathrm{T}} = \dot{\phi}\tilde{u} \tag{2.206}$$

The B-expression of the angular velocity is similarly defined:

$${}^B_G\tilde{\omega}_B = {}^GR_B^{\mathrm{T}}\,{}^G\dot{R}_B \tag{2.207}$$

Employing the global and body expressions of the angular velocity of the body relative to the global coordinate frame, ${}_G\tilde{\omega}_B$ and ${}^B_G\tilde{\omega}_B$, we determine the global and body expressions of the velocity of a body point as

$${}^G_G\mathbf{v}_P = {}^G_G\boldsymbol{\omega}_B \times {}^G\mathbf{r}_P \tag{2.208}$$

$${}^B_G\mathbf{v}_P = {}^B_G\boldsymbol{\omega}_B \times {}^B\mathbf{r}_P \tag{2.209}$$

The G-expression ${}_G\tilde{\omega}_B$ and B-expression ${}^B_G\tilde{\omega}_B$ of the angular velocity matrix can be transformed to each other using the rotation matrix ${}^GR_B$:

$${}_G\tilde{\omega}_B = {}^GR_B\,{}^B_G\tilde{\omega}_B\,{}^GR_B^{\mathrm{T}} \tag{2.210}$$

$${}^B_G\tilde{\omega}_B = {}^GR_B^{\mathrm{T}}\,{}^G_G\tilde{\omega}_B\,{}^GR_B \tag{2.211}$$

They are also related to each other directly by

$${}_G\tilde{\omega}_B\,{}^GR_B = {}^GR_B\,{}^B_G\tilde{\omega}_B^{\mathrm{T}} \tag{2.212}$$

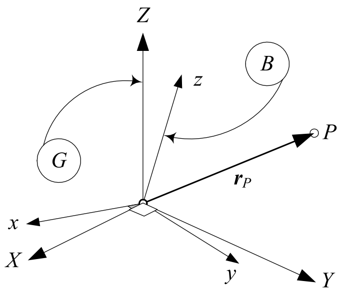


Fig. 2.16 A body fixed point P at ${}^B\mathbf{r}$ in the rotating body frame B

$$
{}^G R_B^{\mathrm{T}} {}_G\tilde{\omega}_B = {}_G^B\tilde{\omega}_B {}^G R_B^{\mathrm{T}} \tag{2.213}
$$

The relative angular velocity vectors of relatively moving rigid bodies can be done only if all the angular velocities are expressed in one coordinate frame:

$$
{}_0\boldsymbol{\omega}_n = {}_0\boldsymbol{\omega}_1 + {}_1^0\boldsymbol{\omega}_2 + {}_2^0\boldsymbol{\omega}_3 + \cdots + {}_{n-1}^{\ \ 0}\boldsymbol{\omega}_n = \sum_{i=1}^{n} {}_{i-1}^{\ \ 0}\boldsymbol{\omega}_i \tag{2.214}
$$

The inverses of the angular velocity matrices ${}_G\tilde{\omega}_B$ and ${}_G^B\tilde{\omega}_B$ are

$$
{}_G\tilde{\omega}_B^{-1} = {}^G R_B {}^G\dot{R}_B^{-1} \tag{2.215}
$$

$$
{}_G^B\tilde{\omega}_B^{-1} = {}^G\dot{R}_B^{-1} {}^G R_B \tag{2.216}
$$

Proof Consider a rigid body with a fixed point O and an attached frame $B(Oxyz)$ as shown in Fig. 2.16. The body frame B is initially coincident with the global frame G. Therefore, the position vector of a body point P at the initial time $t = t_0$ is

$$
{}^G\mathbf{r}(t_0) = {}^B\mathbf{r} \tag{2.217}
$$

and at any other time is found by the associated transformation matrix ${}^G R_B(t)$:

$$
{}^G\mathbf{r}(t) = {}^G R_B(t) {}^B\mathbf{r} = {}^G R_B(t) {}^G\mathbf{r}(t_0) \tag{2.218}
$$

The global time derivative of ${}^G\mathbf{r}$ is

$$
\begin{aligned}
{}^G\mathbf{v} = {}^G\dot{\mathbf{r}} &= \frac{{}^G\mathrm{d}}{\mathrm{d}t} {}^G\mathbf{r}(t) = \frac{{}^G\mathrm{d}}{\mathrm{d}t}\left[{}^G R_B(t) {}^B\mathbf{r}\right] = \frac{{}^G\mathrm{d}}{\mathrm{d}t}\left[{}^G R_B(t) {}^G\mathbf{r}(t_0)\right] \\
&= {}^G\dot{R}_B(t) {}^G\mathbf{r}(t_0) = {}^G\dot{R}_B(t) {}^B\mathbf{r}
\end{aligned} \tag{2.219}
$$

Eliminating ${}^B\mathbf{r}$ between (2.218) and (2.219) determines the velocity of the global point in the global frame:

$$
{}^G\mathbf{v} = {}^G\dot{R}_B(t) {}^G R_B^{\mathrm{T}}(t) {}^G\mathbf{r}(t) \tag{2.220}
$$

We denote the coefficient of ${}^{G}\mathbf{r}(t)$ by ${}_{G}\tilde{\omega}_{B}$

$$ {}_{G}\tilde{\omega}_{B} = {}^{G}\dot{R}_{B}\,{}^{G}R_{B}^{\mathrm{T}} \tag{2.221} $$

and rewrite Eq. (2.220) as

$$ {}^{G}\mathbf{v} = {}_{G}\tilde{\omega}_{B}\,{}^{G}\mathbf{r}(t) \tag{2.222} $$

or equivalently as

$$ {}^{G}\mathbf{v} = {}_{G}\boldsymbol{\omega}_{B} \times {}^{G}\mathbf{r}(t) \tag{2.223} $$

where ${}_{G}\boldsymbol{\omega}_{B}$ is the *instantaneous angular velocity* of the body B relative to the global frame G as seen from the G-frame.

Transforming ${}^{G}\mathbf{v}$ to the body frame provides us with the body expression of the velocity vector:

$$ \begin{aligned} {}_{G}^{B}\mathbf{v}_{P} &= {}^{G}R_{B}^{\mathrm{T}}\,{}^{G}\mathbf{v} = {}^{G}R_{B}^{\mathrm{T}}\,{}_{G}\tilde{\omega}_{B}\,{}^{G}\mathbf{r} = {}^{G}R_{B}^{\mathrm{T}}\,{}^{G}\dot{R}_{B}\,{}^{G}R_{B}^{\mathrm{T}}\,{}^{G}\mathbf{r} \\ &= {}^{G}R_{B}^{\mathrm{T}}\,{}^{G}\dot{R}_{B}\,{}^{B}\mathbf{r} \end{aligned} \tag{2.224} $$

We denote the coefficient of ${}^{B}\mathbf{r}$ by ${}_{G}^{B}\tilde{\omega}_{B}$

$$ {}_{G}^{B}\tilde{\omega}_{B} = {}^{G}R_{B}^{\mathrm{T}}\,{}^{G}\dot{R}_{B} \tag{2.225} $$

and rewrite Eq. (2.224) as

$$ {}_{G}^{B}\mathbf{v}_{P} = {}_{G}^{B}\tilde{\omega}_{B}\,{}^{B}\mathbf{r}_{P} \tag{2.226} $$

or equivalently as

$$ {}_{G}^{B}\mathbf{v}_{P} = {}_{G}^{B}\boldsymbol{\omega}_{B} \times {}^{B}\mathbf{r}_{P} \tag{2.227} $$

where ${}_{G}^{B}\boldsymbol{\omega}_{B}$ is the *instantaneous angular velocity* of B relative to the global frame G as seen from the B-frame.

The time derivative of the orthogonality condition, ${}^{G}R_{B}\,{}^{G}R_{B}^{\mathrm{T}} = \mathbf{I}$, introduces an important identity,

$$ {}^{G}\dot{R}_{B}\,{}^{G}R_{B}^{\mathrm{T}} + {}^{G}R_{B}\,{}^{G}\dot{R}_{B}^{\mathrm{T}} = 0 \tag{2.228} $$

which can be used to show that the angular velocity matrix ${}_{G}\tilde{\omega}_{B} = [{}^{G}\dot{R}_{B}\,{}^{G}R_{B}^{\mathrm{T}}]$ is skew-symmetric:

$$ {}^{G}R_{B}\,{}^{G}\dot{R}_{B}^{\mathrm{T}} = \left[{}^{G}\dot{R}_{B}\,{}^{G}R_{B}^{\mathrm{T}}\right]^{\mathrm{T}} \tag{2.229} $$

Generally speaking, an angular velocity vector is the instantaneous rotation of a coordinate frame A with respect to another frame B that can be expressed in or seen from a third coordinate frame C. We indicate the first coordinate frame A by a right subscript, the second frame B by a left subscript, and the third frame C by a left superscript, ${}_{B}^{C}\boldsymbol{\omega}_{A}$. If the left super and subscripts are the same, we only show the subscript.

We can transform the G-expression of the global velocity of a body point P, ${}^{G}\mathbf{v}_P$, and the B-expression of the global velocity of the point P, ${}^{B}_{G}\mathbf{v}_P$, to each other using a rotation matrix:

$$\begin{aligned}
{}^{B}_{G}\mathbf{v}_P &= {}^{B}R_G\, {}^{G}\mathbf{v}_P = {}^{B}R_G\, {}_{G}\tilde{\omega}_B\, {}^{G}\mathbf{r}_P = {}^{B}R_G\, {}_{G}\tilde{\omega}_B\, {}^{G}R_B\, {}^{B}\mathbf{r}_P \\
&= {}^{B}R_G\, {}^{G}\dot{R}_B\, {}^{G}R_B^{\mathrm{T}}\, {}^{G}R_B\, {}^{B}\mathbf{r}_P = {}^{B}R_G\, {}^{G}\dot{R}_B\, {}^{B}\mathbf{r}_P \\
&= {}^{G}R_B^{\mathrm{T}}\, {}^{G}\dot{R}_B\, {}^{B}\mathbf{r}_P = {}^{B}_{G}\tilde{\omega}_B\, {}^{B}\mathbf{r}_P = {}^{B}_{G}\boldsymbol{\omega}_B \times {}^{B}\mathbf{r}_P
\end{aligned} \tag{2.230}$$

$$\begin{aligned}
{}^{G}\mathbf{v}_P &= {}^{G}R_B\, {}^{B}_{G}\mathbf{v}_P = {}^{G}R_B\, {}^{B}_{G}\tilde{\omega}_B\, {}^{B}\mathbf{r}_P = {}^{G}R_B\, {}^{B}_{G}\tilde{\omega}_B\, {}^{G}R_B^{\mathrm{T}}\, {}^{G}\mathbf{r}_P \\
&= {}^{G}R_B\, {}^{G}R_B^{\mathrm{T}}\, {}^{G}\dot{R}_B\, {}^{G}R_B^{\mathrm{T}}\, {}^{G}\mathbf{r}_P = {}^{G}\dot{R}_B\, {}^{G}R_B^{\mathrm{T}}\, {}^{G}\mathbf{r}_P \\
&= {}_{G}\tilde{\omega}_B\, {}^{G}\mathbf{r}_P = {}_{G}\boldsymbol{\omega}_B \times {}^{G}\mathbf{r}_P = {}^{G}R_B \left({}^{B}_{G}\boldsymbol{\omega}_B \times {}^{B}\mathbf{r}_P \right)
\end{aligned} \tag{2.231}$$

From the definitions of ${}_{G}\tilde{\omega}_B$ and ${}^{B}_{G}\tilde{\omega}_B$ in (2.221) and (2.225) and comparing with (2.230) and (2.231), we are able to transform the two angular velocity matrices by

$${}_{G}\tilde{\omega}_B = {}^{G}R_B\, {}^{B}_{G}\tilde{\omega}_B\, {}^{G}R_B^{\mathrm{T}} \tag{2.232}$$

$${}^{B}_{G}\tilde{\omega}_B = {}^{G}R_B^{\mathrm{T}}\, {}_{G}\tilde{\omega}_B\, {}^{G}R_B \tag{2.233}$$

and derive the following useful equations:

$${}^{G}\dot{R}_B = {}_{G}\tilde{\omega}_B\, {}^{G}R_B \tag{2.234}$$

$${}^{G}\dot{R}_B = {}^{G}R_B\, {}^{B}_{G}\tilde{\omega}_B \tag{2.235}$$

$${}_{G}\tilde{\omega}_B\, {}^{G}R_B = {}^{G}R_B\, {}^{B}_{G}\tilde{\omega}_B \tag{2.236}$$

The angular velocity of B in G is negative of the angular velocity of G in B if both are expressed in the same coordinate frame:

$${}^{G}_{G}\tilde{\omega}_B = -{}^{G}_{B}\tilde{\omega}_G \qquad {}^{G}_{G}\boldsymbol{\omega}_B = -{}^{G}_{B}\boldsymbol{\omega}_G \tag{2.237}$$

$${}^{B}_{G}\tilde{\omega}_B = -{}^{B}_{B}\tilde{\omega}_G \qquad {}^{B}_{G}\boldsymbol{\omega}_B = -{}^{B}_{B}\boldsymbol{\omega}_G \tag{2.238}$$

The vector ${}_{G}\boldsymbol{\omega}_B$ can always be expressed in the natural form

$${}_{G}\boldsymbol{\omega}_B = \omega \hat{u} \tag{2.239}$$

with the magnitude ω and a unit vector $\hat{u}$ parallel to ${}_{G}\boldsymbol{\omega}_B$ that indicates the *instantaneous axis of rotation*.

To show the addition of relative angular velocities in Eq. (2.214), we start from a combination of rotations,

$${}^{0}R_2 = {}^{0}R_1\, {}^{1}R_2 \tag{2.240}$$

and take the time derivative:

$${}^{0}\dot{R}_2 = {}^{0}\dot{R}_1\, {}^{1}R_2 + {}^{0}R_1\, {}^{1}\dot{R}_2 \tag{2.241}$$

Substituting the derivative of the rotation matrices with

$$ {}^{0}\dot{R}_{2} = {}_{0}\tilde{\omega}_{2}\,{}^{0}R_{2} \tag{2.242}$$

$$ {}^{0}\dot{R}_{1} = {}_{0}\tilde{\omega}_{1}\,{}^{0}R_{1} \tag{2.243}$$

$$ {}^{1}\dot{R}_{2} = {}_{1}\tilde{\omega}_{2}\,{}^{1}R_{2} \tag{2.244}$$

results in

$$\begin{aligned} {}_{0}\tilde{\omega}_{2}\,{}^{0}R_{2} &= {}_{0}\tilde{\omega}_{1}\,{}^{0}R_{1}\,{}^{1}R_{2} + {}^{0}R_{1}\,{}_{1}\tilde{\omega}_{2}\,{}^{1}R_{2} \\ &= {}_{0}\tilde{\omega}_{1}\,{}^{0}R_{2} + {}^{0}R_{1}\,{}_{1}\tilde{\omega}_{2}\,{}^{0}R_{1}^{\mathrm{T}}\,{}^{0}R_{1}\,{}^{1}R_{2} \\ &= {}_{0}\tilde{\omega}_{1}\,{}^{0}R_{2} + {}_{1}^{0}\tilde{\omega}_{2}\,{}^{0}R_{2} \end{aligned} \tag{2.245}$$

where

$$ {}^{0}R_{1}\,{}_{1}\tilde{\omega}_{2}\,{}^{0}R_{1}^{\mathrm{T}} = {}_{1}^{0}\tilde{\omega}_{2} \tag{2.246}$$

Therefore, we find

$$ {}_{0}\tilde{\omega}_{2} = {}_{0}\tilde{\omega}_{1} + {}_{1}^{0}\tilde{\omega}_{2} \tag{2.247}$$

which indicates that two angular velocities may be added when they are expressed in the same frame:

$$ {}_{0}\boldsymbol{\omega}_{2} = {}_{0}\boldsymbol{\omega}_{1} + {}_{1}^{0}\boldsymbol{\omega}_{2} \tag{2.248}$$

The expansion of this equation for any number of angular velocities would be Eq. (2.214).

Employing the relative angular velocity formula (2.248), we can find the relative velocity formula of a point P in B_2 at ${}^{0}\mathbf{r}_{P}$:

$$\begin{aligned} {}_{0}\mathbf{v}_{2} &= {}_{0}\boldsymbol{\omega}_{2}\,{}^{0}\mathbf{r}_{P} = \left({}_{0}\boldsymbol{\omega}_{1} + {}_{1}^{0}\boldsymbol{\omega}_{2}\right){}^{0}\mathbf{r}_{P} = {}_{0}\boldsymbol{\omega}_{1}\,{}^{0}\mathbf{r}_{P} + {}_{1}^{0}\boldsymbol{\omega}_{2}\,{}^{0}\mathbf{r}_{P} \\ &= {}_{0}\mathbf{v}_{1} + {}_{1}^{0}\mathbf{v}_{2} \end{aligned} \tag{2.249}$$

The angular velocity matrices ${}_{G}\tilde{\omega}_{B}$ and ${}_{G}^{B}\tilde{\omega}_{B}$ are skew-symmetric and not invertible. However, we can define their inverse by the rules

$$ {}_{G}\tilde{\omega}_{B}^{-1} = {}^{G}R_{B}\,{}^{G}\dot{R}_{B}^{-1} \tag{2.250}$$

$$ {}_{G}^{B}\tilde{\omega}_{B}^{-1} = {}^{G}\dot{R}_{B}^{-1}\,{}^{G}R_{B} \tag{2.251}$$

to get

$$ {}_{G}\tilde{\omega}_{B}^{-1}\,{}_{G}\tilde{\omega}_{B} = {}_{G}\tilde{\omega}_{B}\,{}_{G}\tilde{\omega}_{B}^{-1} = [\mathbf{I}] \tag{2.252}$$

$$ {}_{G}^{B}\tilde{\omega}_{B}^{-1}\,{}_{G}^{B}\tilde{\omega}_{B} = {}_{G}^{B}\tilde{\omega}_{B}\,{}_{G}^{B}\tilde{\omega}_{B}^{-1} = [\mathbf{I}] \tag{2.253}$$

□

Example 55 ★ (Rotation of a body point about a global axis) Consider a rigid body is turning about the Z-axis with a constant angular speed $\dot{\alpha} = 10$ deg/s. The global velocity of a body point at $P(5, 30, 10)$ when the body is at $\alpha = 30$ deg is

$$\begin{aligned}
{}^G\mathbf{v}_P &= {}^G\dot{R}_B(t)\,{}^B\mathbf{r}_P \\
&= \frac{{}^G\mathrm{d}}{\mathrm{d}t}\left(\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} 5 \\ 30 \\ 10 \end{bmatrix} \\
&= \dot{\alpha}\begin{bmatrix} -\sin\alpha & -\cos\alpha & 0 \\ \cos\alpha & -\sin\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 30 \\ 10 \end{bmatrix} \\
&= \frac{10\pi}{180}\begin{bmatrix} -\sin\frac{\pi}{6} & -\cos\frac{\pi}{6} & 0 \\ \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 30 \\ 10 \end{bmatrix} = \begin{bmatrix} -4.97 \\ -1.86 \\ 0 \end{bmatrix}
\end{aligned} \tag{2.254}$$

The point P is now at

$$\begin{aligned}
{}^G\mathbf{r}_P &= {}^GR_B\,{}^B\mathbf{r}_P \\
&= \begin{bmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} & 0 \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 30 \\ 10 \end{bmatrix} = \begin{bmatrix} -10.67 \\ 28.48 \\ 10 \end{bmatrix}
\end{aligned} \tag{2.255}$$

Example 56 ★ (Rotation of a global point about a global axis) A body point P at ${}^B\mathbf{r}_P = [5\ 30\ 10]^{\mathrm{T}}$ is turned $\alpha = 30$ deg about the Z-axis. The global position of P is at

$$\begin{aligned}
{}^G\mathbf{r}_P &= {}^GR_B\,{}^B\mathbf{r}_P \\
&= \begin{bmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} & 0 \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 30 \\ 10 \end{bmatrix} = \begin{bmatrix} -10.67 \\ 28.48 \\ 10 \end{bmatrix}
\end{aligned} \tag{2.256}$$

If the body is turning with a constant angular speed $\dot{\alpha} = 10$ deg/s, the global velocity of the point P would be

$$\begin{aligned}
{}^G\mathbf{v}_P &= {}^G\dot{R}_B\,{}^GR_B^{\mathrm{T}}\,{}^G\mathbf{r}_P \\
&= \frac{10\pi}{180}\begin{bmatrix} -s\frac{\pi}{6} & -c\frac{\pi}{6} & 0 \\ c\frac{\pi}{6} & -s\frac{\pi}{6} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} c\frac{\pi}{6} & -s\frac{\pi}{6} & 0 \\ s\frac{\pi}{6} & c\frac{\pi}{6} & 0 \\ 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} -10.67 \\ 28.48 \\ 10 \end{bmatrix} \\
&= \begin{bmatrix} -4.97 \\ -1.86 \\ 0 \end{bmatrix}
\end{aligned} \tag{2.257}$$

Example 57 ★ (Simple derivative transformation formula) Consider a point P that can move in the body coordinate frame $B(Oxyz)$. The body position vector ${}^B\mathbf{r}_P$ is not constant, and, therefore, the B-expression of the G-velocity of such a point is

$$\frac{{}^G\mathrm{d}}{\mathrm{d}t}{}^B\mathbf{r}_P = {}^B_G\dot{\mathbf{r}}_P = \frac{{}^B\mathrm{d}}{\mathrm{d}t}{}^B\mathbf{r}_P + {}^B_G\boldsymbol{\omega}_B \times {}^B\mathbf{r}_P \tag{2.258}$$

The result of Eq. (2.258) is used to define the transformation of the differential operator on a B-vector ${}^B\Box$ from the body to the global coordinate frame:

$$\frac{{}^G\mathrm{d}}{\mathrm{d}t}{}^B\Box = {}^B_G\dot{\Box} = \frac{{}^B\mathrm{d}}{\mathrm{d}t}{}^B\Box + {}^B_G\boldsymbol{\omega}_B \times {}^B\Box \tag{2.259}$$

However, special attention must be paid to the coordinate frame in which the vector ${}^B\Box$ and the final result are expressed. The final result is ${}^B_G\dot{\Box}$, showing the global (G) time derivative expressed in the body frame (B) or simply the B-expression of the G-derivative of ${}^B\Box$. The vector ${}^B\Box$ may be any vector quantity such as position, velocity, angular velocity, momentum, angular momentum, a time-varying force vector.

Equation (2.259) is called a *simple derivative transformation formula* and relates the derivative of a B-vector as it would be seen from the G-frame to its derivative as seen from the B-frame. The derivative transformation formula (2.259) is more general and can be applied to every vector for a derivative transformation between every two relatively moving coordinate frames.

2.3.3 ★ Acceleration Kinematics

Consider a rotating rigid body $B(Oxyz)$ with a fixed point O in a reference frame $G(OXYZ)$ such as shown in Fig. 2.17. When the body rotates in G, the global acceleration of a body point P is given by

$$\begin{aligned}
{}^G\mathbf{a} = {}^G\dot{\mathbf{v}} = {}^G\ddot{\mathbf{r}} &= {}_GS_B\,{}^G\mathbf{r} &(2.260)\\
&= {}_G\boldsymbol{\alpha}_B \times {}^G\mathbf{r} + {}_G\boldsymbol{\omega}_B \times \left({}_G\boldsymbol{\omega}_B \times {}^G\mathbf{r}\right) &(2.261)\\
&= \left({}_G\tilde{\alpha}_B + {}_G\tilde{\omega}_B^2\right){}^G\mathbf{r} &(2.262)\\
&= {}^G\ddot{R}_B\,{}^GR_B^{\mathrm{T}}\,{}^G\mathbf{r} &(2.263)
\end{aligned}$$

where ${}_G\boldsymbol{\alpha}_B$ is the *angular acceleration vector* of B relative to G,

$${}_G\boldsymbol{\alpha}_B = \frac{{}^G\mathrm{d}}{\mathrm{d}t}{}_G\boldsymbol{\omega}_B \tag{2.264}$$

and ${}_G\tilde{\alpha}_B$ is the *angular acceleration matrix*

$${}_G\tilde{\alpha}_B = {}_G\dot{\tilde{\omega}}_B = {}^G\ddot{R}_B\,{}^GR_B^{\mathrm{T}} + {}^G\dot{R}_B\,{}^G\dot{R}_B^{\mathrm{T}} \tag{2.265}$$

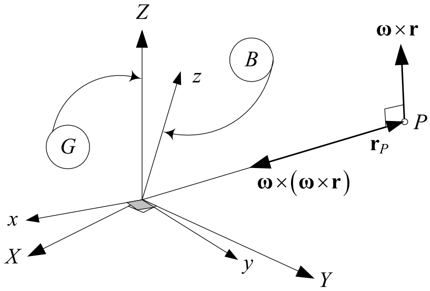


Fig. 2.17 A rotating rigid body $B(Oxyz)$ with a fixed point O in a reference frame $G(OXYZ)$

and ${}_GS_B$ is the *rotational acceleration transformation:*

$$ {}_GS_B = {}^G\ddot{R}_B\,{}^GR_B^{\mathrm{T}} = {}_G\tilde{\alpha}_B + {}_G\tilde{\omega}_B^2 = {}_G\tilde{\alpha}_B - {}_G\tilde{\omega}_B\,{}_G\tilde{\omega}_B^{\mathrm{T}} \tag{2.266}$$

The angular velocity vector ${}_G\boldsymbol{\omega}_B$ and matrix ${}_G\tilde{\omega}_B$ are

$$ {}_G\tilde{\omega}_B = {}^G\dot{R}_B\,{}^GR_B^{\mathrm{T}} \tag{2.267}$$

$$ {}_G\boldsymbol{\omega}_B = \dot{\phi}\hat{u} = \dot{\phi}\hat{u}_\omega \tag{2.268}$$

The relative angular acceleration of two bodies B_1, B_2 in the global frame G can be combined as

$$ {}_G\boldsymbol{\alpha}_2 = \frac{{}^G\mathrm{d}}{\mathrm{d}t}\,{}_G\boldsymbol{\omega}_2 = {}_G\boldsymbol{\alpha}_1 + {}_1^G\alpha_2 \tag{2.269}$$

$$ {}_GS_2 = {}_GS_1 + {}_1^GS_2 + 2\,{}_G\tilde{\omega}_1\,{}_1^G\tilde{\omega}_2 \tag{2.270}$$

The B-expressions of ${}^G\mathbf{a}$ and ${}_GS_B$ are

$$ {}_G^B\mathbf{a} = {}_G^B\boldsymbol{\alpha}_B \times {}^B\mathbf{r} + {}_G^B\boldsymbol{\omega}_B \times \left({}_G^B\boldsymbol{\omega}_B \times {}^B\mathbf{r}\right) \tag{2.271}$$

$$ {}_G^BS_B = {}^BR_G\,{}^G\ddot{R}_B = {}_G^B\tilde{\alpha}_B + {}_G^B\tilde{\omega}_B^2 \tag{2.272}$$

The global and body expressions of the rotational acceleration transformations ${}_GS_B$ and ${}_G^BS_B$ can be transformed to each other by the following rules:

$$ {}_GS_B = {}^GR_B\,{}_G^BS_B\,{}^GR_B^{\mathrm{T}} \tag{2.273}$$

$$ {}_G^BS_B = {}^GR_B^{\mathrm{T}}\,{}_GS_B\,{}^GR_B \tag{2.274}$$

Proof The global position and velocity vectors of the body point P are

$$ {}^G\mathbf{r} = {}^GR_B\,{}^B\mathbf{r} \tag{2.275}$$

$$ {}^G\mathbf{v} = {}^G\dot{\mathbf{r}} = {}^G\dot{R}_B\,{}^B\mathbf{r} = {}_G\tilde{\omega}_B\,{}^G\mathbf{r} = {}_G\boldsymbol{\omega}_B \times {}^G\mathbf{r} \tag{2.276}$$

where ${}_G\tilde{\omega}_B$ is also the rotational velocity transformation because it transforms the global position vector of a point, ${}^G\mathbf{r}$, to its velocity vector ${}^G\mathbf{v}$.

Differentiating Eq. (2.276) and using the notation ${}_G\boldsymbol{\alpha}_B = \frac{{}^G\mathrm{d}}{\mathrm{d}t}{}_G\boldsymbol{\omega}_B$ yield Eq. (2.261):

$$\begin{aligned}{}^G\mathbf{a} = {}^G\ddot{\mathbf{r}} &= {}_G\dot{\omega}_B \times {}^G\mathbf{r} + {}_G\boldsymbol{\omega}_B \times {}^G\dot{\mathbf{r}} \\ &= {}_G\boldsymbol{\alpha}_B \times {}^G\mathbf{r} + {}_G\boldsymbol{\omega}_B \times \left({}_G\boldsymbol{\omega}_B \times {}^G\mathbf{r}\right)\end{aligned} \tag{2.277}$$

We may substitute the matrix expressions of angular velocity and acceleration in (2.277) to derive Eq. (2.262):

$$\begin{aligned}{}^G\ddot{\mathbf{r}} &= {}_G\alpha_B \times {}^G\mathbf{r} + {}_G\boldsymbol{\omega}_B \times \left({}_G\boldsymbol{\omega}_B \times {}^G\mathbf{r}\right) \\ &= {}_G\tilde{\alpha}_B {}^G\mathbf{r} + {}_G\tilde{\omega}_B {}_G\tilde{\omega}_B {}^G\mathbf{r} \\ &= \left({}_G\tilde{\alpha}_B + {}_G\tilde{\omega}_B^2\right){}^G\mathbf{r}\end{aligned} \tag{2.278}$$

Recalling that

$${}_G\tilde{\omega}_B = {}^G\dot{R}_B {}^G R_B^{\mathrm{T}} \tag{2.279}$$

$${}^G\dot{\mathbf{r}}(t) = {}_G\tilde{\omega}_B {}^G\mathbf{r}(t) \tag{2.280}$$

we find Eqs. (2.263) and (2.265):

$$\begin{aligned}{}^G\ddot{\mathbf{r}} &= \frac{{}^G\mathrm{d}}{\mathrm{d}t}\left({}^G\dot{R}_B {}^G R_B^{\mathrm{T}} {}^G\mathbf{r}\right) \\ &= {}^G\ddot{R}_B {}^G R_B^{\mathrm{T}} {}^G\mathbf{r} + {}^G\dot{R}_B {}^G\dot{R}_B^{\mathrm{T}} {}^G\mathbf{r} + \left[{}^G\dot{R}_B {}^G R_B^{\mathrm{T}}\right]\left[{}^G\dot{R}_B {}^G R_B^{\mathrm{T}}\right]{}^G\mathbf{r} \\ &= \left[{}^G\ddot{R}_B {}^G R_B^{\mathrm{T}} + {}^G\dot{R}_B {}^G\dot{R}_B^{\mathrm{T}} + \left[{}^G\dot{R}_B {}^G R_B^{\mathrm{T}}\right]^2\right]{}^G\mathbf{r} \\ &= \left[{}^G\ddot{R}_B {}^G R_B^{\mathrm{T}} - \left[{}^G\dot{R}_B {}^G R_B^{\mathrm{T}}\right]^2 + \left[{}^G\dot{R}_B {}^G R_B^{\mathrm{T}}\right]^2\right]{}^G\mathbf{r} \\ &= {}^G\ddot{R}_B {}^G R_B^{\mathrm{T}} {}^G\mathbf{r}\end{aligned} \tag{2.281}$$

$$\begin{aligned}{}_G\tilde{\alpha}_B = {}_G\dot{\tilde{\omega}}_B &= {}^G\ddot{R}_B {}^G R_B^{\mathrm{T}} + {}^G\dot{R}_B {}^G\dot{R}_B^{\mathrm{T}} \\ &= {}^G\ddot{R}_B {}^G R_B^{\mathrm{T}} + {}^G\dot{R}_B {}^G R_B^{\mathrm{T}} {}^G R_B {}^G\dot{R}_B^{\mathrm{T}} \\ &= {}^G\ddot{R}_B {}^G R_B^{\mathrm{T}} + \left[{}^G\dot{R}_B {}^G R_B^{\mathrm{T}}\right]\left[{}^G\dot{R}_B {}^G R_B^{\mathrm{T}}\right]^{\mathrm{T}} \\ &= {}^G\ddot{R}_B {}^G R_B^{\mathrm{T}} + {}_G\tilde{\omega}_B {}_G\tilde{\omega}_B^{\mathrm{T}} = {}^G\ddot{R}_B {}^G R_B^{\mathrm{T}} - {}_G\tilde{\omega}_B^2\end{aligned} \tag{2.282}$$

which indicates that

$${}^G\ddot{R}_B {}^G R_B^{\mathrm{T}} = {}_G\tilde{\alpha}_B + {}_G\tilde{\omega}_B^2 = {}_G S_B \tag{2.283}$$

The expanded forms of the angular accelerations ${}_G\boldsymbol{\alpha}_B$, ${}_G\tilde{\alpha}_B$ and *rotational acceleration transformation* ${}_GS_B$ are

$$
\begin{aligned}
{}_G\tilde{\alpha}_B = {}_G\dot{\tilde{\omega}}_B = \ddot{\phi}\tilde{u} + \dot{\phi}\dot{\tilde{u}} &= \begin{bmatrix} 0 & -\dot{\omega}_3 & \dot{\omega}_2 \\ \dot{\omega}_3 & 0 & -\dot{\omega}_1 \\ -\dot{\omega}_2 & \dot{\omega}_1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -\dot{u}_3\dot{\phi} - u_3\ddot{\phi} & \dot{u}_2\dot{\phi} + u_2\ddot{\phi} \\ \dot{u}_3\dot{\phi} + u_3\ddot{\phi} & 0 & -\dot{u}_1\dot{\phi} - u_1\ddot{\phi} \\ -\dot{u}_2\dot{\phi} - u_2\ddot{\phi} & \dot{u}_1\dot{\phi} + u_1\ddot{\phi} & 0 \end{bmatrix}
\end{aligned}
\tag{2.284}
$$

$$
{}_G\boldsymbol{\alpha}_B = \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} = \begin{bmatrix} \dot{u}_1\dot{\phi} + u_1\ddot{\phi} \\ \dot{u}_2\dot{\phi} + u_2\ddot{\phi} \\ \dot{u}_3\dot{\phi} + u_3\ddot{\phi} \end{bmatrix}
\tag{2.285}
$$

$$
\begin{aligned}
{}_GS_B &= {}_G\dot{\tilde{\omega}}_B + {}_G\tilde{\omega}_B^2 = {}_G\tilde{\alpha}_B + {}_G\tilde{\omega}_B^2 \\
&= \begin{bmatrix} -\omega_2^2 - \omega_3^2 & \omega_1\omega_2 - \dot{\omega}_3 & \dot{\omega}_2 + \omega_1\omega_3 \\ \dot{\omega}_3 + \omega_1\omega_2 & -\omega_1^2 - \omega_3^2 & \omega_2\omega_3 - \dot{\omega}_1 \\ \omega_1\omega_3 - \dot{\omega}_2 & \dot{\omega}_1 + \omega_2\omega_3 & -\omega_1^2 - \omega_2^2 \end{bmatrix}
\end{aligned}
\tag{2.286}
$$

$$
\begin{aligned}
{}_GS_B &= \ddot{\phi}\tilde{u} + \dot{\phi}\dot{\tilde{u}} + \dot{\phi}^2\tilde{u}^2 \\
&= \begin{bmatrix} -(1-u_1^2)\dot{\phi}^2 & u_1u_2\dot{\phi}^2 - \dot{u}_3\dot{\phi} - u_3\ddot{\phi} & u_1u_3\dot{\phi}^2 + \dot{u}_2\dot{\phi} + u_2\ddot{\phi} \\ u_1u_2\dot{\phi}^2 + \dot{u}_3\dot{\phi} + u_3\ddot{\phi} & -(1-u_2^2)\dot{\phi}^2 & u_2u_3\dot{\phi}^2 - \dot{u}_1\dot{\phi} - u_1\ddot{\phi} \\ u_1u_3\dot{\phi}^2 - \dot{u}_2\dot{\phi} - u_2\ddot{\phi} & u_2u_3\dot{\phi}^2 + \dot{u}_1\dot{\phi} + u_1\ddot{\phi} & -(1-u_3^2)\dot{\phi}^2 \end{bmatrix}
\end{aligned}
\tag{2.287}
$$

The angular velocity of several bodies rotating relative to each other can be related according to (2.214):

$$
{}_0\boldsymbol{\omega}_n = {}_0\boldsymbol{\omega}_1 + {}_1^0\boldsymbol{\omega}_2 + {}_2^0\boldsymbol{\omega}_3 + \cdots + {}_{n-1}^{\ \ 0}\boldsymbol{\omega}_n
\tag{2.288}
$$

The angular accelerations of several relatively rotating rigid bodies follow the same rule:

$$
{}_0\boldsymbol{\alpha}_n = {}_0\alpha_1 + {}_1^0\boldsymbol{\alpha}_2 + {}_2^0\boldsymbol{\alpha}_3 + \cdots + {}_{n-1}^{\ \ 0}\boldsymbol{\alpha}_n
\tag{2.289}
$$

To show this fact and develop the relative acceleration formula, we consider a pair of relatively rotating rigid links in a base coordinate frame B_0 with a fixed point at O. The angular velocities of the links are related as

$$
{}_0\boldsymbol{\omega}_2 = {}_0\boldsymbol{\omega}_1 + {}_1^0\boldsymbol{\omega}_2
\tag{2.290}
$$

So, their angular accelerations are

$$
{}_0\boldsymbol{\alpha}_1 = \frac{{}^0\mathrm{d}}{\mathrm{d}t}{}_0\boldsymbol{\omega}_1
\tag{2.291}
$$

$$ {}_0\boldsymbol{\alpha}_2 = \frac{{}^0\mathrm{d}}{\mathrm{d}t}\,{}_0\boldsymbol{\omega}_2 = {}_0\boldsymbol{\alpha}_1 + {}_1^0\alpha_2 \tag{2.292} $$

and, therefore,

$$ \begin{aligned} {}_0S_2 &= {}_0\tilde{\alpha}_2 + {}_0\tilde{\omega}_2^2 = {}_0\tilde{\alpha}_1 + {}_1^0\tilde{\alpha}_2 + \left({}_0\tilde{\omega}_1 + {}_1^0\tilde{\omega}_2\right)^2 \\ &= {}_0\tilde{\alpha}_1 + {}_1^0\tilde{\alpha}_2 + {}_0\tilde{\omega}_1^2 + {}_1^0\tilde{\omega}_2^2 + 2{}_0\tilde{\omega}_1{}_1^0\tilde{\omega}_2 \\ &= {}_0S_1 + {}_1^0S_2 + 2{}_0\tilde{\omega}_1{}_1^0\tilde{\omega}_2 \end{aligned} \tag{2.293} $$

Equation (2.293) is the required *relative acceleration transformation formula*. It indicates the method of calculation of relative accelerations for a multibody. As a more general case, consider a six-link multibody. The angular acceleration of link (6) in the base frame would be

$$ \begin{aligned} {}_0S_6 = {}& {}_0S_1 + {}_1^0S_2 + {}_2^0S_3 + {}_3^0S_4 + {}_4^0S_5 + {}_5^0S_6 \\ & + 2{}_0\tilde{\omega}_1\left({}_1^0\tilde{\omega}_2 + {}_2^0\tilde{\omega}_3 + {}_3^0\tilde{\omega}_4 + {}_4^0\tilde{\omega}_5 + {}_5^0\tilde{\omega}_6\right) \\ & + 2{}_1^0\tilde{\omega}_2\left({}_2^0\tilde{\omega}_3 + {}_3^0\tilde{\omega}_4 + {}_4^0\tilde{\omega}_5 + {}_5^0\tilde{\omega}_6\right) \\ & \vdots \\ & + 2{}_4^0\tilde{\omega}_5\left({}_5^0\tilde{\omega}_6\right) \end{aligned} \tag{2.294} $$

We can transform the G and B-expressions of the global acceleration of a body point P to each other using a rotation matrix:

$$ \begin{aligned} {}_G^B\mathbf{a}_P &= {}^BR_G\,{}^G\mathbf{a}_P = {}^BR_G\,{}_GS_B\,{}^G\mathbf{r}_P = {}^BR_G\,{}_GS_B\,{}^GR_B\,{}^B\mathbf{r}_P \\ &= {}^BR_G\,{}^G\ddot{R}_B\,{}^GR_B^{\mathrm{T}}\,{}^GR_B\,{}^B\mathbf{r}_P = {}^BR_G\,{}^G\ddot{R}_B\,{}^B\mathbf{r}_P \\ &= {}^GR_B^{\mathrm{T}}\,{}^G\ddot{R}_B\,{}^B\mathbf{r}_P = {}_G^BS_B\,{}^B\mathbf{r}_P = \left({}_G^B\tilde{\alpha}_B + {}_G^B\tilde{\omega}_B^2\right){}^B\mathbf{r}_P \\ &= {}_G^B\boldsymbol{\alpha}_B \times {}^B\mathbf{r} + {}_G^B\boldsymbol{\omega}_B \times \left({}_G^B\boldsymbol{\omega}_B \times {}^B\mathbf{r}\right) \end{aligned} \tag{2.295} $$

$$ \begin{aligned} {}^G\mathbf{a}_P &= {}^GR_B\,{}_G^B\mathbf{a}_P = {}^GR_B\,{}_G^BS_B\,{}^B\mathbf{r}_P = {}^GR_B\,{}_G^BS_B\,{}^GR_B^{\mathrm{T}}\,{}^G\mathbf{r}_P \\ &= {}^GR_B\,{}^GR_B^{\mathrm{T}}\,{}^G\ddot{R}_B\,{}^GR_B^{\mathrm{T}}\,{}^G\mathbf{r}_P = {}^G\ddot{R}_B\,{}^GR_B^{\mathrm{T}}\,{}^G\mathbf{r}_P \\ &= {}_GS_B\,{}^G\mathbf{r}_P = \left({}_G\tilde{\alpha}_B + {}_G\tilde{\omega}_B^2\right){}^G\mathbf{r} \\ &= {}_G\boldsymbol{\alpha}_B \times {}^G\mathbf{r} + {}_G\boldsymbol{\omega}_B \times \left({}_G\boldsymbol{\omega}_B \times {}^G\mathbf{r}\right) \end{aligned} \tag{2.296} $$

From the definitions of ${}_GS_B$ and ${}_G^BS_B$ in (2.266) and (2.272) and comparing with (2.295) and (2.296), we are able to transform the two rotational acceleration transformations by

$$ {}_GS_B = {}^GR_B\,{}_G^BS_B\,{}^GR_B^{\mathrm{T}} \tag{2.297} $$

$$ {}^B_G S_B = {}^G R^{\mathrm{T}}_B \, {}_G S_B \, {}^G R_B \tag{2.298} $$

and derive the useful equations

$$ {}^G \ddot{R}_B = {}_G S_B \, {}^G R_B \tag{2.299} $$
$$ {}^G \ddot{R}_B = {}^G R_B \, {}^B_G S_B \tag{2.300} $$
$$ {}_G S_B \, {}^G R_B = {}^G R_B \, {}^B_G S_B \tag{2.301} $$

The angular acceleration of B in G is negative of the angular acceleration of G in B if both are expressed in the same coordinate frame:

$$ {}_G \tilde{\alpha}_B = -{}^G_B \tilde{\alpha}_G \qquad {}_G \boldsymbol{\alpha}_B = -{}^G_B \boldsymbol{\alpha}_G \tag{2.302} $$
$$ {}^B_G \tilde{\alpha}_B = -{}_B \tilde{\alpha}_G \qquad {}^B_G \boldsymbol{\alpha}_B = -{}_B \boldsymbol{\alpha}_G \tag{2.303} $$

The term ${}_G\boldsymbol{\alpha}_B \times {}^G\mathbf{r}$ in (2.277) is called the *tangential acceleration*, which is a function of the angular acceleration of B in G. The term ${}_G\boldsymbol{\omega}_B \times ({}_G\boldsymbol{\omega}_B \times {}^G\mathbf{r})$ in ${}^G\mathbf{a}$ is called *centripetal acceleration* and is a function of the angular velocity of B in G. □

Example 58 ★ (Rotation of a body point about a global axis) Consider a rigid body is turning about the Z-axis with a constant angular acceleration $\ddot{\alpha} = 2\ \mathrm{rad/s^2}$. The global acceleration of a body point at $P(5, 30, 10)$ cm when the body is at $\dot{\alpha} = 10$ rad/s and $\alpha = 30$ deg is

$$ \begin{aligned} {}^G\mathbf{a}_P &= {}^G\ddot{R}_B(t)\, {}^B\mathbf{r}_P \\ &= \begin{bmatrix} -87.6 & 48.27 & 0 \\ -48.27 & -87.6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 30 \\ 10 \end{bmatrix} = \begin{bmatrix} 1010 \\ -2869.4 \\ 0 \end{bmatrix} \mathrm{cm/s} \end{aligned} \tag{2.304} $$

where

$$ \begin{aligned} {}^G\ddot{R}_B &= \frac{{}^G\mathrm{d}^2}{\mathrm{d}t^2} {}^G R_B = \dot{\alpha} \frac{{}^G\mathrm{d}}{\mathrm{d}\alpha} {}^G R_B = \ddot{\alpha} \frac{{}^G\mathrm{d}}{\mathrm{d}\alpha} {}^G R_B + \dot{\alpha}^2 \frac{{}^G\mathrm{d}^2}{\mathrm{d}\alpha^2} {}^G R_B \\ &= \ddot{\alpha} \begin{bmatrix} -\sin\alpha & -\cos\alpha & 0 \\ \cos\alpha & -\sin\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix} + \dot{\alpha}^2 \begin{bmatrix} -\cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & -\cos\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{2.305} $$

At this moment, the point P is at

$$ \begin{aligned} {}^G\mathbf{r}_P &= {}^G R_B \, {}^B\mathbf{r}_P \\ &= \begin{bmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} & 0 \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 30 \\ 10 \end{bmatrix} = \begin{bmatrix} -10.67 \\ 28.48 \\ 10 \end{bmatrix} \mathrm{cm} \end{aligned} \tag{2.306} $$

Example 59 ★ (Rotation of a global point about a global axis) A body point P at ${}^{B}\mathbf{r}_P = [5\ 30\ 10]^{\mathrm{T}}$ cm is turning with a constant angular acceleration $\ddot{\alpha} = 2\ \mathrm{rad/s^2}$ about the Z-axis. When the body frame is at $\alpha = 30$ deg, its angular speed $\dot{\alpha} = 10$ deg/s.

The transformation matrix ${}^{G}R_B$ between the B- and G-frames is

$$ {}^{G}R_B = \begin{bmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} & 0 \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} & 0 \\ 0 & 0 & 1 \end{bmatrix} \approx \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.307} $$

and, therefore, the acceleration of point P is

$$ {}^{G}\mathbf{a}_P = {}^{G}\ddot{R}_B\, {}^{G}R_B^{\mathrm{T}}\, {}^{G}\mathbf{r}_P = \begin{bmatrix} 1010 \\ -2869.4 \\ 0 \end{bmatrix} \mathrm{cm/s^2} \tag{2.308} $$

where

$$ \frac{{}^{G}\mathrm{d}^2}{\mathrm{d}t^2}\, {}^{G}R_B = \ddot{\alpha}\frac{{}^{G}\mathrm{d}}{\mathrm{d}\alpha}\, {}^{G}R_B - \dot{\alpha}^2 \frac{{}^{G}\mathrm{d}^2}{\mathrm{d}\alpha^2}\, {}^{G}R_B \tag{2.309} $$

is the same as (2.305).

Example 60 ★ (B-expression of angular acceleration) The angular acceleration expressed in the body frame is the body derivative of the angular velocity vector. To show this, we use the derivative transport formula (2.259):

$$ \begin{aligned} {}_{G}^{B}\boldsymbol{\alpha}_B &= {}_{G}^{B}\dot{\boldsymbol{\omega}}_B = \frac{{}^{G}\mathrm{d}}{\mathrm{d}t}\, {}_{G}^{B}\boldsymbol{\omega}_B \\ &= \frac{{}^{B}\mathrm{d}}{\mathrm{d}t}\, {}_{G}^{B}\boldsymbol{\omega}_B + {}_{G}^{B}\boldsymbol{\omega}_B \times {}_{G}^{B}\boldsymbol{\omega}_B = \frac{{}^{B}\mathrm{d}}{\mathrm{d}t}\, {}_{G}^{B}\boldsymbol{\omega}_B \end{aligned} \tag{2.310} $$

Interestingly, the global and body derivatives of ${}_{G}^{B}\boldsymbol{\omega}_B$ are equal:

$$ \frac{{}^{G}\mathrm{d}}{\mathrm{d}t}\, {}_{G}^{B}\boldsymbol{\omega}_B = \frac{{}^{B}\mathrm{d}}{\mathrm{d}t}\, {}_{G}^{B}\boldsymbol{\omega}_B = {}_{G}^{B}\boldsymbol{\alpha}_B \tag{2.311} $$

This is because ${}_{G}\boldsymbol{\omega}_B$ is about an axis $\hat{u}$ that is instantaneously fixed in both B and G.

A vector $\boldsymbol{\alpha}$ can generally indicate the angular acceleration of a coordinate frame A with respect to another frame B. It can be expressed in or seen from a third coordinate frame C. We indicate the first coordinate frame A by a right subscript, the second frame B by a left subscript, and the third frame C by a left superscript, ${}_{B}^{C}\boldsymbol{\alpha}_A$. If the left super and subscripts are the same, we only show the subscript. So, the angular acceleration of A with respect to B as seen from C is the C-expression of ${}_{B}\alpha_A$:

$$ {}_{B}^{C}\boldsymbol{\alpha}_A = {}^{C}R_B\, {}_{B}\alpha_A \tag{2.312} $$

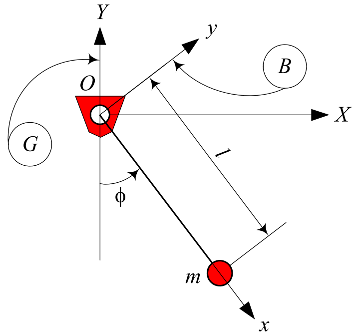


Fig. 2.18 A simple pendulum

Example 61 ★ (B-expression of acceleration) Transforming ${}^{G}\mathbf{a}$ to the body frame provides us with the body expression of the acceleration vector:

$$\begin{aligned} {}^{B}_{G}\mathbf{a}_{P} &= {}^{G}R_{B}^{\mathrm{T}}\,{}^{G}\mathbf{a} = {}^{G}R_{B}^{\mathrm{T}}\,{}_{G}S_{B}\,{}^{G}\mathbf{r} = {}^{G}R_{B}^{\mathrm{T}}\,{}^{G}\ddot{R}_{B}\,{}^{G}R_{B}^{\mathrm{T}}\,{}^{G}\mathbf{r} \\ &= {}^{G}R_{B}^{\mathrm{T}}\,{}^{G}\ddot{R}_{B}\,{}^{B}\mathbf{r} \end{aligned} \tag{2.313}$$

We denote the coefficient of ${}^{B}\mathbf{r}$ by ${}^{B}_{G}S_{B}$

$${}^{B}_{G}S_{B} = {}^{G}R_{B}^{\mathrm{T}}\,{}^{G}\ddot{R}_{B} \tag{2.314}$$

and rewrite Eq. (2.313) as

$${}^{B}_{G}\mathbf{a}_{P} = {}^{B}_{G}S_{B}\,{}^{B}\mathbf{r}_{P} \tag{2.315}$$

where ${}^{B}_{G}S_{B}$ is the rotational acceleration transformation of the B-frame relative to G-frame as seen from the B-frame.

Example 62 (Velocity and acceleration of a simple pendulum) A point mass attached to a massless rod hanging from a revolute joint is what we call a *simple pendulum*. Figure 2.18 illustrates a simple pendulum. A local coordinate frame B is attached to the pendulum, which rotates in a global frame G about the Z-axis. The kinematic information of the mass is given by

$${}^{B}\mathbf{r} = l\hat{\imath} \tag{2.316}$$

$${}^{G}\mathbf{r} = {}^{G}R_{B}\,{}^{B}\mathbf{r} = \begin{bmatrix} l\sin\phi \\ -l\cos\phi \\ 0 \end{bmatrix} \tag{2.317}$$

$${}^{B}_{G}\boldsymbol{\omega}_{B} = \dot{\phi}\hat{k} \tag{2.318}$$

$${}_{G}\boldsymbol{\omega}_{B} = {}^{G}R_{B}^{\mathrm{T}}\,{}^{B}_{G}\boldsymbol{\omega}_{B} = \dot{\phi}\,\hat{K} \tag{2.319}$$

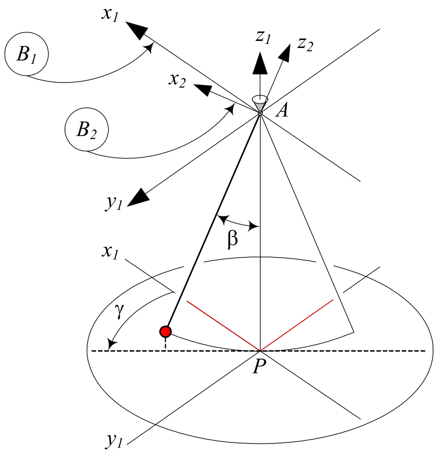


Fig. 2.19 A spherical pendulum

$$
{}^{G}R_{B} = \begin{bmatrix} \cos(\frac{3}{2}\pi+\phi) & -\sin(\frac{3}{2}\pi+\phi) & 0 \\ \sin(\frac{3}{2}\pi+\phi) & \cos(\frac{3}{2}\pi+\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} \sin\phi & \cos\phi & 0 \\ -\cos\phi & \sin\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.320}
$$

Therefore,

$$
{}^{B}_{G}\mathbf{v} = {}^{B}\dot{\mathbf{r}} + {}^{B}_{G}\boldsymbol{\omega}_{B} \times {}^{B}_{G}\mathbf{r} = 0 + \dot{\phi}\hat{k} \times l\hat{\imath} = l\dot{\phi}\hat{\jmath} \tag{2.321}
$$

$$
{}^{G}\mathbf{v} = {}^{G}R_{B}\,{}^{B}\mathbf{v} = \begin{bmatrix} l\,\dot{\phi}\cos\phi \\ l\,\dot{\phi}\sin\phi \\ 0 \end{bmatrix} \tag{2.322}
$$

and

$$
{}^{B}_{G}\mathbf{a} = {}^{B}_{G}\dot{\mathbf{v}} + {}^{B}_{G}\boldsymbol{\omega}_{B} \times {}^{B}_{G}\mathbf{v} = l\,\ddot{\phi}\,\hat{\jmath} + \dot{\phi}\hat{k} \times l\,\dot{\phi}\,\hat{\jmath} = l\,\ddot{\phi}\,\hat{\jmath} - l\,\dot{\phi}^{2}\hat{\imath} \tag{2.323}
$$

$$
{}^{G}\mathbf{a} = {}^{G}R_{B}\,{}^{B}\mathbf{a} = \begin{bmatrix} l\,\ddot{\phi}\cos\phi - l\,\dot{\phi}^{2}\sin\phi \\ l\,\ddot{\phi}\sin\phi + l\,\dot{\phi}^{2}\cos\phi \\ 0 \end{bmatrix} \tag{2.324}
$$

Example 63 (Spherical pendulum) A pendulum free to oscillate in any plane is called a spherical pendulum. This name comes from the codominants that we use to locate the tip mass. Consider a pendulum with a point mass m at the tip point of a long, massless, and straight string with length l. The pendulum is hanging from a point $A(0,0,0)$ in a local coordinate frame $B_1(x_1, y_1, z_1)$.

To indicate the mass m, we attach a coordinate frame $B_2(x_2, y_2, z_2)$ to the pendulum at point A as is shown in Fig. 2.19. The pendulum makes an angle β with the vertical z_1-axis. The pendulum swings in the plane (x_2, z_2) and makes an angle γ with the plane (x_1, z_1). Therefore, the transformation matrix between B_2 and B_1 is

$$\begin{aligned} {}^2R_1 &= R_{y_2,-\beta}\, R_{z_2,\gamma} \\ &= \begin{bmatrix} \cos\gamma\cos\beta & \cos\beta\sin\gamma & \sin\beta \\ -\sin\gamma & \cos\gamma & 0 \\ -\cos\gamma\sin\beta & -\sin\gamma\sin\beta & \cos\beta \end{bmatrix} \end{aligned} \tag{2.325}$$

The position vectors of m are

$${}^2\mathbf{r} = \begin{bmatrix} 0 \\ 0 \\ -l \end{bmatrix} \qquad {}^1\mathbf{r} = {}^1R_2\,{}^2\mathbf{r} = \begin{bmatrix} l\cos\gamma\sin\beta \\ l\sin\beta\sin\gamma \\ -l\cos\beta \end{bmatrix} \tag{2.326}$$

The equation of motion of m is

$${}^1\mathbf{M} = I\,{}_1\boldsymbol{\alpha}_2 \tag{2.327}$$

$${}^1\mathbf{r} \times m\,{}^1\mathbf{g} = ml^2\,{}_1\boldsymbol{\alpha}_2 \tag{2.328}$$

$$\begin{bmatrix} l\cos\gamma\sin\beta \\ l\sin\beta\sin\gamma \\ -l\cos\beta \end{bmatrix} \times m \begin{bmatrix} 0 \\ 0 \\ -g_0 \end{bmatrix} = ml^2\,{}_1\boldsymbol{\alpha}_2 \tag{2.329}$$

Therefore,

$${}_1\boldsymbol{\alpha}_2 = \frac{g_0}{l}\begin{bmatrix} -\sin\beta\sin\gamma \\ \cos\gamma\sin\beta \\ 0 \end{bmatrix} \tag{2.330}$$

To find the angular acceleration of B_2 in B_1, we use ${}^2R_1$:

$$\begin{aligned} {}^1\dot{R}_2 &= \dot{\beta}\frac{\mathrm{d}}{\mathrm{d}\beta}{}^2R_1 + \dot{\gamma}\frac{\mathrm{d}}{\mathrm{d}\gamma}{}^2R_1 \\ &= \begin{bmatrix} -\dot{\beta}c\gamma s\beta - \dot{\gamma}c\beta s\gamma & -\dot{\gamma}c\gamma & \dot{\gamma}s\beta s\gamma - \dot{\beta}c\beta c\gamma \\ \dot{\gamma}c\beta c\gamma - \dot{\beta}s\beta s\gamma & -\dot{\gamma}s\gamma & -\dot{\beta}c\beta s\gamma - \dot{\gamma}c\gamma s\beta \\ \dot{\beta}c\beta & 0 & -\dot{\beta}s\beta \end{bmatrix} \end{aligned} \tag{2.331}$$

$${}_1\tilde{\omega}_2 = {}^1\dot{R}_2\,{}^1R_2^{\mathrm{T}} = \begin{bmatrix} 0 & -\dot{\gamma} & -\dot{\beta}\cos\gamma \\ \dot{\gamma} & 0 & -\dot{\beta}\sin\gamma \\ \dot{\beta}\cos\gamma & \dot{\beta}\sin\gamma & 0 \end{bmatrix} \tag{2.332}$$

$$\begin{aligned} {}^1\ddot{R}_2 &= \ddot{\beta}\frac{\mathrm{d}}{\mathrm{d}\beta}{}^2R_1 + \dot{\beta}^2\frac{\mathrm{d}^2}{\mathrm{d}\beta^2}{}^2R_1 + \dot{\beta}\dot{\gamma}\frac{\mathrm{d}^2}{\mathrm{d}\gamma\,\mathrm{d}\beta}{}^2R_1 \\ &\quad + \ddot{\gamma}\frac{\mathrm{d}}{\mathrm{d}\gamma}{}^2R_1 + \dot{\gamma}\dot{\beta}\frac{\mathrm{d}^2}{\mathrm{d}\beta\,\mathrm{d}\gamma}{}^2R_1 + \dot{\gamma}^2\frac{\mathrm{d}^2}{\mathrm{d}\gamma^2}{}^2R_1 \end{aligned} \tag{2.333}$$

$$
\begin{aligned}
{}_1\tilde{\alpha}_2 &= {}^1\ddot{R}_2\,{}^1R_2^{\mathrm{T}} - {}_1\tilde{\omega}_2^2 \\
&= \begin{bmatrix} 0 & -\ddot{\gamma} & -\ddot{\beta}c\gamma + \dot{\beta}\dot{\gamma}s\gamma \\ \ddot{\gamma} & 0 & -\ddot{\beta}s\gamma - \dot{\beta}\dot{\gamma}c\gamma \\ \ddot{\beta}c\gamma - \dot{\beta}\dot{\gamma}s\gamma & \ddot{\beta}s\gamma + \dot{\beta}\dot{\gamma}c\gamma & 0 \end{bmatrix}
\end{aligned} \tag{2.334}
$$

Therefore, the equation of motion of the pendulum would be

$$
\frac{g_0}{l}\begin{bmatrix} -\sin\beta\sin\gamma \\ \cos\gamma\sin\beta \\ 0 \end{bmatrix} = \begin{bmatrix} \ddot{\beta}\sin\gamma + \dot{\beta}\dot{\gamma}\cos\gamma \\ -\ddot{\beta}\cos\gamma + \dot{\beta}\dot{\gamma}\sin\gamma \\ \ddot{\gamma} \end{bmatrix} \tag{2.335}
$$

The third equation indicates that

$$
\dot{\gamma} = \dot{\gamma}_0 \qquad \gamma = \dot{\gamma}_0 t + \gamma_0 \tag{2.336}
$$

The second and third equations can be combined to form

$$
\ddot{\beta} = -\sqrt{\frac{g_0^2}{l^2}\sin^2\beta + \dot{\beta}^2\dot{\gamma}_0^2} \tag{2.337}
$$

which reduces to the equation of a simple pendulum if $\dot{\gamma}_0 = 0$.

Example 64 ★ (Equation of motion of a spherical pendulum) Consider a particle P of mass m that is suspended by a string of length l from a point A, as shown in Fig. 2.19. If we show the tension of the string by $\mathbf{T}$, then the equation of motion of P is

$$
{}^1\mathbf{T} + m\,{}^1\mathbf{g} = m\,{}^1\ddot{\mathbf{r}} \tag{2.338}
$$

or $-T\,{}^1\mathbf{r} + m\,{}^1\mathbf{g} = m\,{}^1\ddot{\mathbf{r}}$. To eliminate ${}^1\mathbf{T}$, we multiply the equation by ${}^1\mathbf{r}$,

$$
\begin{gathered}
{}^1\mathbf{r} \times {}^1\mathbf{g} = {}^1\mathbf{r} \times {}^1\ddot{\mathbf{r}} \\
\begin{bmatrix} l\cos\gamma\sin\beta \\ l\sin\beta\sin\gamma \\ -l\cos\beta \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ -g_0 \end{bmatrix} = \begin{bmatrix} l\cos\gamma\sin\beta \\ l\sin\beta\sin\gamma \\ -l\cos\beta \end{bmatrix} \times \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix}
\end{gathered} \tag{2.339}
$$

and find

$$
\begin{bmatrix} -lg_0\sin\beta\sin\gamma \\ lg_0\cos\gamma\sin\beta \\ 0 \end{bmatrix} = \begin{bmatrix} l\ddot{y}\cos\beta + l\ddot{z}\sin\beta\sin\gamma \\ -l\ddot{x}\cos\beta - l\ddot{z}\cos\gamma\sin\beta \\ l\ddot{y}\cos\gamma - l\ddot{x}\sin\gamma \end{bmatrix} \tag{2.340}
$$

These are the equations of motion of m. However, we may express the equations only in terms of γ and β. To do so, we may either take time derivatives of ${}^1\mathbf{r}$ or use ${}_1\boldsymbol{\alpha}_2$ from Example 64 and find ${}^1\ddot{\mathbf{r}}$:

$$
{}^1\ddot{\mathbf{r}} = {}_1\boldsymbol{\alpha}_2 \times {}^1\mathbf{r} \tag{2.341}
$$

In either case, Eq. (2.335) would be the equation of motion in terms of γ and β.

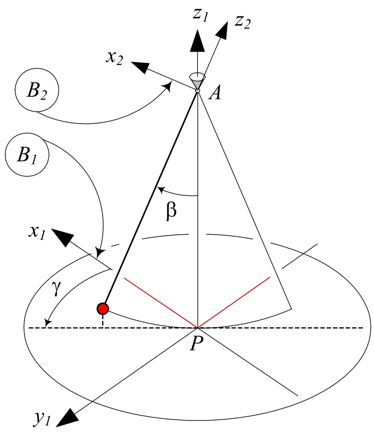


Fig. 2.20 Foucault pendulum is a simple pendulum hanging from a point A above a point P on Earth surface

Example 65 ★ (Foucault pendulum) Consider a pendulum with a point mass m at the tip of a long, massless, and straight string with length l. The pendulum is hanging from a point $A(0,0,l)$ in a local coordinate frame $B_1(x_1,y_1,z_1)$ at a point P on the Earth surface. Point P at longitude φ and latitude λ is indicated by ${}^E\mathbf{d}$ in the Earth frame $E(Oxyz)$. The E-frame is turning in a global frame $G(OXYZ)$ about the Z-axis.

To indicate the mass m, we attach a coordinate frame $B_1(x_1,y_1,z_1)$ to the pendulum at point A as shown in Fig. 2.20. The pendulum makes an angle β with the vertical z_1-axis. The pendulum swings in the plane (x_2,z_2) and makes an angle γ with the plane (x_1,z_1). Therefore, the transformation matrix between B_2 and B_1 is

$$\begin{aligned}{}^1T_2 &= {}^1D_2\,{}^1R_2 \\ &= \begin{bmatrix} \cos\gamma\cos\beta & -\sin\gamma & -\cos\gamma\sin\beta & 0 \\ \cos\beta\sin\gamma & \cos\gamma & -\sin\gamma\sin\beta & 0 \\ \sin\beta & 0 & \cos\beta & l \\ 0 & 0 & 0 & 1 \end{bmatrix}\end{aligned} \tag{2.342}$$

The position vector of m is

$${}^2\mathbf{r} = \begin{bmatrix} 0 \\ 0 \\ -l \end{bmatrix} \tag{2.343}$$

$${}^1\mathbf{r} = {}^1T_2\,{}^2\mathbf{r} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} l\cos\gamma\sin\beta \\ l\sin\beta\sin\gamma \\ l - l\cos\beta \end{bmatrix} \tag{2.344}$$

Employing the acceleration equation,

$${}^1_G\mathbf{a} = {}^1\mathbf{a} + {}^1_G\boldsymbol{\alpha}_1 \times {}^1\mathbf{r} + 2\,{}^1_G\boldsymbol{\omega}_1 \times {}^1\mathbf{v} + {}^1_G\boldsymbol{\omega}_1 \times \left({}^1_G\boldsymbol{\omega}_1 \times {}^1\mathbf{r}\right) \tag{2.345}$$

we can write the equation of motion of m as

$$ {}^1_G\mathbf{F} - m\,{}^1_G\mathbf{g} = m\,{}^1_G\mathbf{a} \tag{2.346} $$

where ${}^1\mathbf{F}$ is the applied nongravitational force on m.

Recalling that

$$ {}^1_G\boldsymbol{\alpha}_1 = 0 \tag{2.347} $$

we find the general equation of motion of a particle in frame B_1 as

$$ {}^1_G\mathbf{F} + m\,{}^1_G\mathbf{g} = m\left({}^1\mathbf{a} + 2\,{}^1_G\boldsymbol{\omega}_1 \times {}^1\mathbf{v} + {}^1_G\boldsymbol{\omega}_1 \times \left({}^1_G\boldsymbol{\omega}_1 \times {}^1\mathbf{r}\right)\right) \tag{2.348} $$

The individual vectors in this equation are

$$ {}^1\mathbf{g} = \begin{bmatrix} 0 \\ 0 \\ -g_0 \end{bmatrix} \qquad {}^1_G\mathbf{F} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} \qquad {}^1_G\boldsymbol{\omega}_1 = \begin{bmatrix} \omega_E \cos\lambda \\ 0 \\ \omega_E \sin\lambda \end{bmatrix} \tag{2.349} $$

$$ {}^1\mathbf{v} = \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{z}_1 \end{bmatrix} = \begin{bmatrix} l\dot{\beta}\cos\beta\cos\gamma - l\dot{\gamma}\sin\beta\sin\gamma \\ l\dot{\beta}\cos\beta\sin\gamma + l\dot{\gamma}\cos\gamma\sin\beta \\ l\dot{\beta}\sin\beta \end{bmatrix} \tag{2.350} $$

$$ {}^1\mathbf{a} = \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} l(\ddot{\beta}\cos\gamma - \dot{\beta}^2\sin\gamma - \dot{\beta}\dot{\gamma}\sin\gamma)\cos\beta \\ \quad - l(\ddot{\gamma}\sin\gamma + \dot{\gamma}^2\cos\gamma + \dot{\beta}\dot{\gamma}\cos\gamma)\sin\beta \\ l(\ddot{\beta}\sin\gamma + \dot{\beta}^2\cos\gamma + \dot{\beta}\dot{\gamma}\cos\gamma)\cos\beta \\ \quad + l(\ddot{\gamma}\cos\gamma - \dot{\gamma}^2\sin\gamma - \dot{\beta}\dot{\gamma}\sin\gamma)\sin\beta \\ l\ddot{\beta}\sin\beta \end{bmatrix} \tag{2.351} $$

In a spherical pendulum, the external force ${}^1\mathbf{F}$ is the tension of the string:

$$ {}^1_G\mathbf{F} = -\frac{F}{l}\,{}^1\mathbf{r} \tag{2.352} $$

Substituting the above vectors in (2.348) provides us with three coupled ordinary differential equations for two angular variables γ and β. One of the equations is not independent and the others may theoretically be integrated to determine $\gamma = \gamma(t)$ and $\beta = \beta(t)$.

For example, let us use

$$ \begin{aligned} \omega_E &\approx 7.292\,1 \times 10^{-5} \text{ rad/s} \\ g_0 &\approx 9.81 \text{ m/s}^2 \\ l &= 100 \text{ m} \\ \lambda &= 28^\circ 58' 30'' \text{N} \approx 28.975 \text{ deg N} \\ \varphi &= 50^\circ 50' 17'' \text{E} \approx 50.838 \text{ deg E} \\ x_0 &= l\cos 10 = 17.365 \text{ m} \end{aligned} \tag{2.353} $$

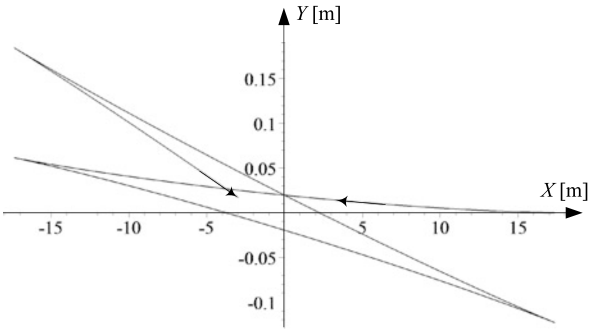


Fig. 2.21 The projection of the path of a pendulum with length $l = 100$ m at latitude $\lambda \approx 28.975$ deg N on Earth for a few oscillations (not to scale)

and find

$$x = 8.6839 \cos(0.313\,16t) + 8.6811 \cos(-0.313\,26t) \tag{2.354}$$

$$y = 8.6839 \sin(0.313\,16t) + 8.6811 \sin(-0.313\,26t) \tag{2.355}$$

At the given latitude, which corresponds to Bushehr, Iran, on the Persian Gulf shore, the plane of oscillation turns about the local **g**-axis with an angular speed $\omega = -3.5325 \times 10^{-5}$ rad/s ≈ -87.437 deg/d. These results are independent of longitude. Therefore, the same phenomena will be seen at Orlando, Florida, or New Delhi, India, which are almost at the same latitude. Figure 2.21 depicts the projection of m on the (x, y)-plane for a few oscillations. It takes $T \approx 49.4$ h for the pendulum to turn 2π:

$$T = \frac{2\pi}{3.5325 \times 10^{-5}} = 1.7787 \times 10^{5} \text{ s} = 49.408 \text{ h} \tag{2.356}$$

However, the pendulum gets back to the (y, x)-plane after $t = T/2 = 24.704$ h. By that time, the pendulum must have oscillated about $n \approx 4433$ times:

$$n = \frac{\omega_n}{2\pi} \frac{T}{2} = \frac{0.313\,21}{2\pi} \frac{1.7787 \times 10^{5}}{2} = 4433.3 \tag{2.357}$$

By shortening the length of the pendulum, say $l = 1$ m, the rotation speed remains the same while the number of oscillations increases to $n \approx 44333$.

2.3.4 ★ Translational Dynamics

Figure 2.22 depicts a moving body B in a global coordinate frame G. Assume that the body frame is attached at the mass center of the body. Point P indicates an infinitesimal sphere of the body, which has a very small mass dm. The point mass dm is acted on by an infinitesimal force $\mathbf{df}$ and has a global velocity ${}^{G}\mathbf{v}_{P}$.

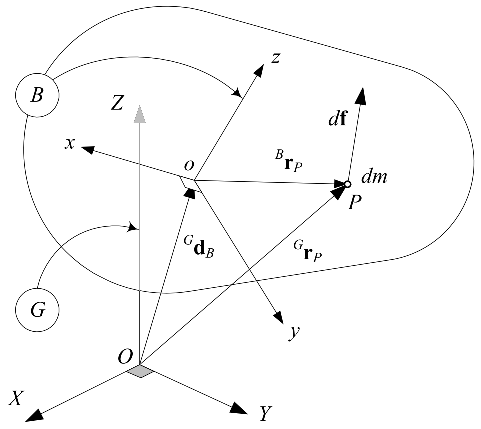


Fig. 2.22 A body point mass moving with velocity ${}^G\mathbf{v}_P$ and acted on by force d$\mathbf{f}$

According to Newton's law of motion

$$\mathrm{d}\mathbf{f} = {}^G\mathbf{a}_P \,\mathrm{d}m \tag{2.358}$$

However, the equation of motion for the whole body in a global coordinate frame is

$$ {}^G\mathbf{F} = m\,{}^G\mathbf{a}_B \tag{2.359}$$

which can be expressed in the body coordinate frame as

$$ {}^B\mathbf{F} = m\,{}^B_G\mathbf{a}_B + m\,{}^B_G\boldsymbol{\omega}_B \times {}^B\mathbf{v}_B \tag{2.360}$$

$$\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} ma_x + m(\omega_y v_z - \omega_z v_y) \\ ma_y - m(\omega_x v_z - \omega_z v_x) \\ ma_z + m(\omega_x v_y - \omega_y v_x) \end{bmatrix} \tag{2.361}$$

In these equations, ${}^G\mathbf{a}_B$ is the G-expression of the acceleration vector of the body mass center C, m is the total mass of the body, and $\mathbf{F}$ is the G-expression of the resultant of the external forces acted on the body at C.

Proof A body coordinate frame at the mass center is called a *central frame*. If frame B is a central frame, then the *center of mass*, C, is defined such that

$$\int_B {}^B\mathbf{r}_{\mathrm{d}m} \,\mathrm{d}m = 0 \tag{2.362}$$

The global position vector of dm is related to its local position vector by

$$ {}^G\mathbf{r}_{\mathrm{d}m} = {}^G\mathbf{d}_B + {}^G R_B \, {}^B\mathbf{r}_{\mathrm{d}m} \tag{2.363}$$

where ${}^{G}\mathbf{d}_{B}$ is the global position vector of the central body frame, and, therefore,

$$\begin{aligned}\int_{B}{}^{G}\mathbf{r}_{dm}\,\mathrm{d}m &= \int_{B}{}^{G}\mathbf{d}_{B}\,\mathrm{d}m + {}^{G}R_{B}\int_{m}{}^{B}\mathbf{r}_{dm}\,\mathrm{d}m \\ &= \int_{B}{}^{G}\mathbf{d}_{B}\,\mathrm{d}m = {}^{G}\mathbf{d}_{B}\int_{B}\mathrm{d}m = m\,{}^{G}\mathbf{d}_{B}\end{aligned} \tag{2.364}$$

The time derivative of both sides shows that

$$m\,{}^{G}\dot{\mathbf{d}}_{B} = m\,{}^{G}\mathbf{v}_{B} = \int_{B}{}^{G}\dot{\mathbf{r}}_{dm}\,\mathrm{d}m = \int_{B}{}^{G}\mathbf{v}_{dm}\,\mathrm{d}m \tag{2.365}$$

and the other derivative is

$$m\,{}^{G}\dot{\mathbf{v}}_{B} = m\,{}^{G}\mathbf{a}_{B} = \int_{B}{}^{G}\dot{\mathbf{v}}_{dm}\,\mathrm{d}m \tag{2.366}$$

However, we have $\mathrm{d}\mathbf{f} = {}^{G}\dot{\mathbf{v}}_{P}\,\mathrm{d}m$ and, therefore,

$$m\,{}^{G}\mathbf{a}_{B} = \int_{B}\mathrm{d}\mathbf{f} \tag{2.367}$$

The integral on the right-hand side collects all the forces acting on the body. The internal forces cancel one another out, so the net result is the vector sum of all the externally applied forces, $\mathbf{F}$, and, therefore,

$${}^{G}\mathbf{F} = m\,{}^{G}\mathbf{a}_{B} = m\,{}^{G}\dot{\mathbf{v}}_{B} \tag{2.368}$$

In the body coordinate frame we have

$$\begin{aligned}{}^{B}\mathbf{F} &= {}^{B}R_{G}\,{}^{G}\mathbf{F} = m\,{}^{B}R_{G}\,{}^{G}\mathbf{a}_{B} = m\,{}^{B}_{G}\mathbf{a}_{B} \\ &= m\,{}^{B}\mathbf{a}_{B} + m\,{}^{B}_{G}\boldsymbol{\omega}_{B}\times{}^{B}\mathbf{v}_{B}\end{aligned} \tag{2.369}$$

The expanded form of Newton's equation in the body coordinate frame is then equal to

$$\begin{aligned}{}^{B}\mathbf{F} &= m\,{}^{B}\mathbf{a}_{B} + m\,{}^{B}_{G}\boldsymbol{\omega}_{B}\times{}^{B}\mathbf{v}_{B} \\ \begin{bmatrix}F_x\\F_y\\F_z\end{bmatrix} &= m\begin{bmatrix}a_x\\a_y\\a_z\end{bmatrix} + m\begin{bmatrix}\omega_x\\\omega_y\\\omega_z\end{bmatrix}\times\begin{bmatrix}v_x\\v_y\\v_z\end{bmatrix} \\ &= \begin{bmatrix}ma_x + m(\omega_y v_z - \omega_z v_y)\\ ma_y - m(\omega_x v_z - \omega_z v_x)\\ ma_z + m(\omega_x v_y - \omega_y v_x)\end{bmatrix}\end{aligned} \tag{2.370}$$

□

2.3.5 ★ Rotational Dynamics

The rigid body rotational equation of motion is the *Euler equation*

$$
\begin{aligned}
{}^{B}\mathbf{M} &= \frac{{}^{G}\mathrm{d}}{\mathrm{d}t}\,{}^{B}\mathbf{L} = {}^{B}\dot{\mathbf{L}} + {}_{G}^{B}\boldsymbol{\omega}_{B} \times {}^{B}\mathbf{L} \\
&= {}^{B}I\,{}_{G}^{B}\dot{\boldsymbol{\omega}}_{B} + {}_{G}^{B}\boldsymbol{\omega}_{B} \times \left({}^{B}I\,{}_{G}^{B}\boldsymbol{\omega}_{B}\right)
\end{aligned}
\tag{2.371}
$$

where $\mathbf{L}$ is the *angular momentum*

$$
{}^{B}\mathbf{L} = {}^{B}I\,{}_{G}^{B}\boldsymbol{\omega}_{B}
\tag{2.372}
$$

and I is the *mass moment* of the rigid body.

$$
I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}
\tag{2.373}
$$

The expanded form of the Euler equation (2.371) is

$$
\begin{aligned}
M_x &= I_{xx}\dot{\omega}_x + I_{xy}\dot{\omega}_y + I_{xz}\dot{\omega}_z - (I_{yy} - I_{zz})\omega_y\omega_z \\
&\quad - I_{yz}\left(\omega_z^2 - \omega_y^2\right) - \omega_x(\omega_z I_{xy} - \omega_y I_{xz})
\end{aligned}
\tag{2.374}
$$

$$
\begin{aligned}
M_y &= I_{yx}\dot{\omega}_x + I_{yy}\dot{\omega}_y + I_{yz}\dot{\omega}_z - (I_{zz} - I_{xx})\omega_z\omega_x \\
&\quad - I_{xz}\left(\omega_x^2 - \omega_z^2\right) - \omega_y(\omega_x I_{yz} - \omega_z I_{xy})
\end{aligned}
\tag{2.375}
$$

$$
\begin{aligned}
M_z &= I_{zx}\dot{\omega}_x + I_{zy}\dot{\omega}_y + I_{zz}\dot{\omega}_z - (I_{xx} - I_{yy})\omega_x\omega_y \\
&\quad - I_{xy}\left(\omega_y^2 - \omega_x^2\right) - \omega_z(\omega_y I_{xz} - \omega_x I_{yz})
\end{aligned}
\tag{2.376}
$$

which can be reduced to

$$
\begin{aligned}
M_1 &= I_1\dot{\omega}_1 - (I_2 - I_2)\omega_2\omega_3 \\
M_2 &= I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 \\
M_3 &= I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2
\end{aligned}
\tag{2.377}
$$

when the body coordinate is the *principal coordinate frame*. The principal coordinate frame is denoted by numbers 123 to indicate the first, second, and third *principal axes*. The parameters I_{ij}, $i \neq j$, are zero in the principal frame. The body and principal coordinate frame sit at the mass center C.

The kinetic energy of a rotating rigid body is

$$
\begin{aligned}
K &= \frac{1}{2}\left(I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2\right) \\
&\quad - I_{xy}\omega_x\omega_y - I_{yz}\omega_y\omega_z - I_{zx}\omega_z\omega_x
\end{aligned}
\tag{2.378}
$$

$$= \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L} = \frac{1}{2}\boldsymbol{\omega}^{\mathrm{T}} I \, \boldsymbol{\omega} \tag{2.379}$$

which in the principal coordinate frame reduces to

$$K = \frac{1}{2}\left(I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2\right) \tag{2.380}$$

Proof Let m_i be the mass of the ith particle of a rigid body B, which is made of n particles and let

$$\mathbf{r}_i = {}^{B}\mathbf{r}_i = \begin{bmatrix} x_i & y_i & z_i \end{bmatrix}^{\mathrm{T}} \tag{2.381}$$

be the Cartesian position vector of m_i in a central body fixed coordinate frame $B(Oxyz)$. Assume that

$$\boldsymbol{\omega} = {}_{G}^{B}\boldsymbol{\omega}_B = \begin{bmatrix} \omega_x & \omega_y & \omega_z \end{bmatrix}^{\mathrm{T}} \tag{2.382}$$

is the angular velocity of the rigid body with respect to the global coordinate frame $G(OXYZ)$, expressed in the body coordinate frame.

The angular momentum of m_i is

$$\begin{aligned} \mathbf{L}_i &= \mathbf{r}_i \times m_i \dot{\mathbf{r}}_i = m_i \left[\mathbf{r}_i \times (\omega \times \mathbf{r}_i)\right] \\ &= m_i \left[(\mathbf{r}_i \cdot \mathbf{r}_i)\boldsymbol{\omega} - (\mathbf{r}_i \cdot \boldsymbol{\omega})\mathbf{r}_i\right] \\ &= m_i r_i^2 \boldsymbol{\omega} - m_i (\mathbf{r}_i \cdot \boldsymbol{\omega})\mathbf{r}_i \end{aligned} \tag{2.383}$$

Hence, the angular momentum of the rigid body would be

$$\mathbf{L} = \boldsymbol{\omega} \sum_{i=1}^{n} m_i r_i^2 - \sum_{i=1}^{n} m_i (\mathbf{r}_i \cdot \boldsymbol{\omega})\mathbf{r}_i \tag{2.384}$$

Substitution for $\mathbf{r}_i$ and $\boldsymbol{\omega}$ gives us

$$\begin{aligned} \mathbf{L} = {} & (\omega_x \hat{\imath} + \omega_y \hat{\jmath} + \omega_z \hat{k}) \sum_{i=1}^{n} m_i \left(x_i^2 + y_i^2 + z_i^2\right) \\ & - \sum_{i=1}^{n} m_i (x_i \omega_x + y_i \omega_y + z_i \omega_z) \cdot (x_i \hat{\imath} + y_i \hat{\jmath} + z_i \hat{k}) \end{aligned} \tag{2.385}$$

which can be rearranged as

$$\begin{aligned} \mathbf{L} = {} & \sum_{i=1}^{n} m_i \left(y_i^2 + z_i^2\right) \omega_x \hat{\imath} + \sum_{i=1}^{n} m_i \left(z_i^2 + x_i^2\right) \omega_y \hat{\jmath} + \sum_{i=1}^{n} m_i \left(x_i^2 + y_i^2\right) \omega_z \hat{k} \\ & - \left(\sum_{i=1}^{n} (m_i x_i y_i) \omega_y + \sum_{i=1}^{n} (m_i x_i z_i) \omega_z\right) \hat{\imath} \end{aligned}$$

$$
- \left(\sum_{i=1}^{n} (m_i y_i z_i) \omega_z + \sum_{i=1}^{n} (m_i y_i x_i) \omega_x \right) \hat{\jmath}
$$
$$
- \left(\sum_{i=1}^{n} (m_i z_i x_i) \omega_x + \sum_{i=1}^{n} (m_i z_i y_i) \omega_y \right) \hat{k} \tag{2.386}
$$

By introducing the mass moment matrix I with the following elements:

$$I_{xx} = \sum_{i=1}^{n} \left[m_i \left(y_i^2 + z_i^2 \right) \right] \tag{2.387}$$

$$I_{yy} = \sum_{i=1}^{n} \left[m_i \left(z_i^2 + x_i^2 \right) \right] \tag{2.388}$$

$$I_{zz} = \sum_{i=1}^{n} \left[m_i \left(x_i^2 + y_i^2 \right) \right] \tag{2.389}$$

$$I_{xy} = I_{yx} = - \sum_{i=1}^{n} (m_i x_i y_i) \tag{2.390}$$

$$I_{yz} = I_{zy} = - \sum_{i=1}^{n} (m_i y_i z_i) \tag{2.391}$$

$$I_{zx} = I_{xz} = - \sum_{i=1}^{n} (m_i z_i x_i) \tag{2.392}$$

we may write the angular momentum $\mathbf{L}$ in concise form:

$$L_x = I_{xx} \omega_x + I_{xy} \omega_y + I_{xz} \omega_z \tag{2.393}$$
$$L_y = I_{yx} \omega_x + I_{yy} \omega_y + I_{yz} \omega_z \tag{2.394}$$
$$L_z = I_{zx} \omega_x + I_{zy} \omega_y + I_{zz} \omega_z \tag{2.395}$$

or in matrix form:

$$\mathbf{L} = I \cdot \boldsymbol{\omega} \tag{2.396}$$

$$
\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{2.397}
$$

For a rigid body that is a continuous solid, the summations must be replaced by integrations over the volume of the body as in Eq. (2.433).

The Euler equation of motion for a rigid body is

$$ {}^{B}\mathbf{M} = \frac{{}^{G}\mathrm{d}}{\mathrm{d}t}\,{}^{B}\mathbf{L} \tag{2.398} $$

where ${}^{B}\mathbf{M}$ is the resultant of the external moments applied on the rigid body. The angular momentum ${}^{B}\mathbf{L}$ is a vector quantity defined in the body coordinate frame. Hence, its time derivative in the global coordinate frame is

$$ \frac{{}^{G}\mathrm{d}\,{}^{B}\mathbf{L}}{\mathrm{d}t} = {}^{B}\dot{\mathbf{L}} + {}^{B}_{G}\boldsymbol{\omega}_{B} \times {}^{B}\mathbf{L} \tag{2.399} $$

Therefore,

$$ {}^{B}\mathbf{M} = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \dot{\mathbf{L}} + \omega \times L = I\dot{\boldsymbol{\omega}} + \omega \times (I\boldsymbol{\omega}) \tag{2.400} $$

or in expanded form

$$ \begin{aligned} {}^{B}\mathbf{M} = {} & (I_{xx}\dot{\omega}_x + I_{xy}\dot{\omega}_y + I_{xz}\dot{\omega}_z)\hat{\imath} + \omega_y(I_{xz}\omega_x + I_{yz}\omega_y + I_{zz}\omega_z)\hat{\imath} \\ & - \omega_z(I_{xy}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z)\hat{\imath} \\ & + (I_{yx}\dot{\omega}_x + I_{yy}\dot{\omega}_y + I_{yz}\dot{\omega}_z)\hat{\jmath} + \omega_z(I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z)\hat{\jmath} \\ & - \omega_x(I_{xz}\omega_x + I_{yz}\omega_y + I_{zz}\omega_z)\hat{\jmath} \\ & + (I_{zx}\dot{\omega}_x + I_{zy}\dot{\omega}_y + I_{zz}\dot{\omega}_z)\hat{k} + \omega_x(I_{xy}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z)\hat{k} \\ & - \omega_y(I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z)\hat{k} \end{aligned} \tag{2.401} $$

and, therefore, the most general form of the Euler equations of motion for a rigid body in a body frame attached to C are

$$ \begin{aligned} M_x = {} & I_{xx}\dot{\omega}_x + I_{xy}\dot{\omega}_y + I_{xz}\dot{\omega}_z - (I_{yy} - I_{zz})\omega_y\omega_z \\ & - I_{yz}\left(\omega_z^2 - \omega_y^2\right) - \omega_x(\omega_z I_{xy} - \omega_y I_{xz}) \end{aligned} \tag{2.402} $$

$$ \begin{aligned} M_y = {} & I_{yx}\dot{\omega}_x + I_{yy}\dot{\omega}_y + I_{yz}\dot{\omega}_z - (I_{zz} - I_{xx})\omega_z\omega_x \\ & - I_{xz}\left(\omega_x^2 - \omega_z^2\right) - \omega_y(\omega_x I_{yz} - \omega_z I_{xy}) \end{aligned} \tag{2.403} $$

$$ \begin{aligned} M_z = {} & I_{zx}\dot{\omega}_x + I_{zy}\dot{\omega}_y + I_{zz}\dot{\omega}_z - (I_{xx} - I_{yy})\omega_x\omega_y \\ & - I_{xy}\left(\omega_y^2 - \omega_x^2\right) - \omega_z(\omega_y I_{xz} - \omega_x I_{yz}) \end{aligned} \tag{2.404} $$

Assume that we are able to rotate the body frame about its origin to find an orientation that makes $I_{ij} = 0$, for $i \neq j$. In such a coordinate frame, which is called a *principal frame*, the Euler equations reduce to

$$ M_1 = I_1\dot{\omega}_1 - (I_2 - I_2)\omega_2\omega_3 \tag{2.405} $$

$$ M_2 = I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 \tag{2.406} $$

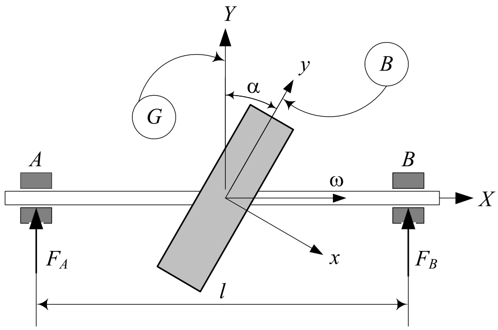


Fig. 2.23 A disc with mass m and radius r, mounted on a massless turning shaft

$$M_3 = I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 \tag{2.407}$$

The kinetic energy of a rigid body may be found by the integral of the kinetic energy of the mass element dm, over the whole body:

$$\begin{aligned}
K &= \frac{1}{2}\int_B \dot{\mathbf{v}}^2 \, \mathrm{d}m = \frac{1}{2}\int_B (\boldsymbol{\omega}\times\mathbf{r})\cdot(\boldsymbol{\omega}\times\mathbf{r})\,\mathrm{d}m \\
&= \frac{\omega_x^2}{2}\int_B \left(y^2+z^2\right)\mathrm{d}m + \frac{\omega_y^2}{2}\int_B \left(z^2+x^2\right)\mathrm{d}m + \frac{\omega_z^2}{2}\int_B \left(x^2+y^2\right)\mathrm{d}m \\
&\quad - \omega_x\omega_y\int_B xy\,\mathrm{d}m - \omega_y\omega_z\int_B yz\,\mathrm{d}m - \omega_z\omega_x\int_B zx\,\mathrm{d}m \\
&= \frac{1}{2}\left(I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2\right) \\
&\quad - I_{xy}\omega_x\omega_y - I_{yz}\omega_y\omega_z - I_{zx}\omega_z\omega_x
\end{aligned} \tag{2.408}$$

The kinetic energy can be rearranged to a matrix multiplication form

$$K = \frac{1}{2}\boldsymbol{\omega}^{\mathrm{T}} I\,\boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L} \tag{2.409}$$

When the body frame is principal, the kinetic energy will simplify to

$$K = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right) \tag{2.410}$$

□

Example 66 ★ (A tilted disc on a massless shaft) Figure 2.23 illustrates a disc with mass m and radius R, mounted on a massless shaft. The shaft is turning with a constant angular speed ω. The disc is attached to the shaft at an angle α. Because of α, the bearings at A and B must support a rotating force.

We attach a principal body coordinate frame at the disc center as shown in the figure. The G-expression of the angular velocity is a constant vector

$${}^{G}\boldsymbol{\omega}_B = \omega\hat{I} \tag{2.411}$$

and the expression of the angular velocity vector in the body frame is

$$ {}_{G}^{B}\boldsymbol{\omega}_{B} = \omega\cos\theta\,\hat{\imath} + \omega\sin\theta\,\hat{\jmath} \tag{2.412}$$

The mass moment of inertia matrix is

$$ {}^{B}I = \begin{bmatrix} mR^2/2 & 0 & 0 \\ 0 & mR^2/4 & 0 \\ 0 & 0 & mR^2/4 \end{bmatrix} \tag{2.413}$$

Substituting (2.412) and (2.413) in (2.405)–(2.407), with $1 \equiv x$, $2 \equiv y$, $3 \equiv z$, yields

$$M_x = 0 \tag{2.414}$$

$$M_y = 0 \tag{2.415}$$

$$M_z = \frac{mr^2}{4}\omega\cos\theta\sin\theta \tag{2.416}$$

Therefore, the bearing reaction forces F_A and F_B are

$$F_A = -F_B = -\frac{M_z}{l} = -\frac{mr^2}{4l}\omega\cos\theta\sin\theta \tag{2.417}$$

Example 67 (Steady rotation of a freely rotating rigid body) Consider a situation in which the resultant applied force and moment on a rigid body are zero:

$$ {}^{G}\mathbf{F} = {}^{B}\mathbf{F} = 0 \tag{2.418}$$

$$ {}^{G}\mathbf{M} = {}^{B}\mathbf{M} = 0 \tag{2.419}$$

Based on Newton's equation,

$$ {}^{G}\mathbf{F} = m\,{}^{G}\dot{\mathbf{v}} \tag{2.420}$$

the velocity of the mass center will be constant in the global coordinate frame. However, the Euler equation

$$ {}^{B}\mathbf{M} = I\,{}_{G}^{B}\dot{\omega}_{B} + {}_{G}^{B}\boldsymbol{\omega}_{B} \times {}^{B}\mathbf{L} \tag{2.421}$$

reduces to

$$\dot{\omega}_1 = \frac{I_2 - I_3}{I_1}\omega_2\omega_3 \tag{2.422}$$

$$\dot{\omega}_2 = \frac{I_3 - I_1}{I_{22}}\omega_3\omega_1 \tag{2.423}$$

$$\dot{\omega}_3 = \frac{I_1 - I_2}{I_3}\omega_1\omega_2 \tag{2.424}$$

which show that the angular velocity can be constant if

$$I_1 = I_2 = I_3 \tag{2.425}$$

or if two principal moments of inertia, say I_1 and I_2, are zero and the third angular velocity, in this case ω_3, is initially zero, or if the angular velocity vector is initially parallel to a principal axis.

2.3.6 ★ Mass Moment Matrix

Two types of integral arise in rigid body dynamics that depend solely on the geometry and mass distribution of the body. The first type defines the center of mass and is important when the translation motion of the body is considered. The second is the *mass moment*, which appears when the rotational motion of the body is considered. The mass moment is also called *moment of inertia*, *centrifugal moments*, or *deviation moment*. Every rigid body has a 3×3 mass moment matrix I, which is denoted by

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \tag{2.426}$$

The diagonal elements I_{ij}, $i = j$, are called *polar mass moments*

$$I_{xx} = I_x = \int_B \left(y^2 + z^2\right) \mathrm{d}m \tag{2.427}$$

$$I_{yy} = I_y = \int_B \left(z^2 + x^2\right) \mathrm{d}m \tag{2.428}$$

$$I_{zz} = I_z = \int_B \left(x^2 + y^2\right) \mathrm{d}m \tag{2.429}$$

and the off-diagonal elements I_{ij}, $i \neq j$, are called *products of inertia*

$$I_{xy} = I_{yx} = -\int_B xy \, \mathrm{d}m \tag{2.430}$$

$$I_{yz} = I_{zy} = -\int_B yz \, \mathrm{d}m \tag{2.431}$$

$$I_{zx} = I_{xz} = -\int_B zx \, \mathrm{d}m \tag{2.432}$$

The elements of I are functions of the mass distribution of the rigid body and may be defined by

$$I_{ij} = \int_B \left(r_i^2 \delta_{mn} - x_{im} x_{jn}\right) \mathrm{d}m \quad i, j = 1, 2, 3 \tag{2.433}$$

where δ_{ij} is Kronecker's delta (2.171),

$$\delta_{mn} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \tag{2.434}$$

The elements of I are calculated in a body coordinate frame attached to the mass center C of the body. Therefore, I is a frame-dependent quantity and must be written with a frame indicator such as ${}^{B}I$ to show the frame in which it is computed:

$${}^{B}I = \int_B \begin{bmatrix} y^2 + z^2 & -xy & -zx \\ -xy & z^2 + x^2 & -yz \\ -zx & -yz & x^2 + y^2 \end{bmatrix} \mathrm{d}m \tag{2.435}$$

$$= \int_B \left(r^2 \mathbf{I} - \mathbf{r}\,\mathbf{r}^{\mathrm{T}}\right) \mathrm{d}m = \int_B -\tilde{r}\,\tilde{r}\,\mathrm{d}m \tag{2.436}$$

where $\tilde{r}$ is the associated skew-symmetric matrix of $\mathbf{r}$:

$$\tilde{r} = \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix} \tag{2.437}$$

The moments of inertia can be transformed from a coordinate frame B_1 to another coordinate frame B_2, both defined at the mass center of the body, according to the rule of the *rotated-axes theorem:*

$${}^{B_2}I = {}^{B_2}R_{B_1}\,{}^{B_1}I\,{}^{B_2}R_{B_1}^{\mathrm{T}} \tag{2.438}$$

Transformation of the moment of inertia from a central frame B_1 located at ${}^{B_2}\mathbf{r}_C$ to another frame B_2, which is parallel to B_1, is, according to the rule of the *parallel-axes theorem:*

$${}^{B_2}I = {}^{B_1}I + m\,\tilde{r}_C\,\tilde{r}_C^{\mathrm{T}} \tag{2.439}$$

If the local coordinate frame $Oxyz$ is located such that the products of inertia vanish, the local coordinate frame is the *principal coordinate frame* and the associated mass moments are *principal mass moments*. Principal axes and principal mass moments can be found by solving the following characteristic equation for I:

$$\begin{vmatrix} I_{xx} - I & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - I & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - I \end{vmatrix} = 0 \tag{2.440}$$

$$\det\bigl([I_{ij}] - I[\delta_{ij}]\bigr) = 0 \tag{2.441}$$

Equation (2.441) is a cubic equation in I, so we obtain three eigenvalues,

$$I_1 = I_x \qquad I_2 = I_y \qquad I_3 = I_z \tag{2.442}$$

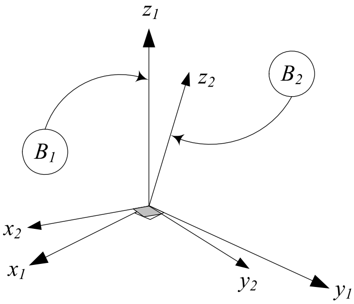


Fig. 2.24 Two coordinate frames with a common origin at the mass center of a rigid body

which are the principal mass moments.

Proof Two coordinate frames with a common origin at the mass center of a rigid body are shown in Fig. 2.24. The angular velocity and angular momentum of a rigid body transform from the frame B_1 to the frame B_2 by the vector transformation rule

$$^2\boldsymbol{\omega} = {}^2R_2{}^1\boldsymbol{\omega} \tag{2.443}$$

$$^2\mathbf{L} = {}^2R_1{}^1\mathbf{L} \tag{2.444}$$

However, $\mathbf{L}$ and $\boldsymbol{\omega}$ are related according to Eq. (2.372)

$$^1\mathbf{L} = {}^1I\,{}^1\boldsymbol{\omega} \tag{2.445}$$

and, therefore,

$$^2\mathbf{L} = {}^2R_1{}^1I\,{}^2R_1^{\mathrm{T}}{}^2\boldsymbol{\omega} = {}^2I\,{}^2\boldsymbol{\omega} \tag{2.446}$$

which shows how to transfer the mass moment from the coordinate frame B_1 to a rotated frame B_2

$$^2I = {}^2R_1{}^1I\,{}^2R_1^{\mathrm{T}} \tag{2.447}$$

Now consider a central frame B_1, shown in Fig. 2.25, at ${}^2\mathbf{r}_C$, which rotates about the origin of a fixed frame B_2 such that their axes remain parallel. The angular velocity and angular momentum of the rigid body transform from frame B_1 to frame B_2 by

$$^2\boldsymbol{\omega} = {}^1\boldsymbol{\omega} \tag{2.448}$$

$$^2\mathbf{L} = {}^1\mathbf{L} + (\mathbf{r}_C \times m\mathbf{v}_C) \tag{2.449}$$

Therefore,

$$\begin{aligned} ^2\mathbf{L} &= {}^1\mathbf{L} + m\,{}^2\mathbf{r}_C \times \left({}^2\boldsymbol{\omega} \times {}^2\mathbf{r}_C\right) \\ &= {}^1\mathbf{L} + \left(m\,{}^2\tilde{r}_C\,{}^2\tilde{r}_C^{\mathrm{T}}\right){}^2\boldsymbol{\omega} \\ &= \left({}^1I + m\,{}^2\tilde{r}_C\,{}^2\tilde{r}_C^{\mathrm{T}}\right){}^2\boldsymbol{\omega} \end{aligned} \tag{2.450}$$

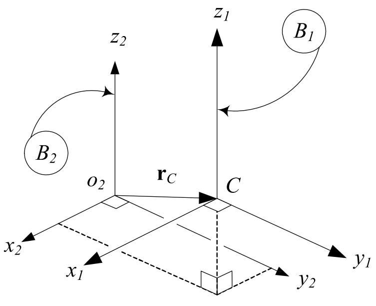


Fig. 2.25 A central coordinate frame B_1 and a translated frame B_2

which shows how to transfer the mass moment from frame B_1 to a parallel frame B_2

$$^2I = {}^1I + m\,\tilde{r}_C\,\tilde{r}_C^{\mathrm{T}} \tag{2.451}$$

The parallel-axes theorem is also called the *Huygens–Steiner theorem*.

Referring to Eq. (2.447) for transformation of the moment of inertia to a rotated frame, we can always find a frame in which ^2I is diagonal. In such a frame, we have

$$^2R_1\,{}^1I = {}^2I\,{}^2R_1 \tag{2.452}$$

or

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \tag{2.453}$$

which shows that I_1, I_2, and I_3 are eigenvalues of ^1I. These eigenvalues can be found by solving the following characteristic equation for λ:

$$\begin{vmatrix} I_{xx}-\lambda & I_{xy} & I_{xz} \\ I_{yx} & I_{yy}-\lambda & I_{yz} \\ I_{zx} & I_{zy} & I_{zz}-\lambda \end{vmatrix} = 0 \tag{2.454}$$

The eigenvalues I_1, I_2, and I_3 are *principal mass moments*, and their associated eigenvectors are *principal directions*. The coordinate frame made by the eigenvectors is the *principal body coordinate frame*. In the principal coordinate frame, the rigid body angular momentum simplifies to

$$\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \tag{2.455}$$

□

Example 68 (Principal moments of inertia) Consider the inertia matrix I

$$I = \begin{bmatrix} 10 & -2 & 0 \\ -2 & 20 & -2 \\ 0 & -2 & 30 \end{bmatrix} \tag{2.456}$$

We set up the determinant (2.441)

$$\begin{vmatrix} 10-\lambda & -2 & 0 \\ -2 & 20-\lambda & -2 \\ 0 & -2 & 30-\lambda \end{vmatrix} = 0 \tag{2.457}$$

which leads to the following characteristic equation:

$$-\lambda^3 + 60\lambda^2 - 1092\lambda + 5840 = 0 \tag{2.458}$$

Three roots of Eq. (2.458) are

$$I_1 = 9.6077 \qquad I_2 = 20 \qquad I_3 = 30.392 \tag{2.459}$$

and, therefore, the principal mass moment matrix is

$$I = \begin{bmatrix} 9.6077 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 30.392 \end{bmatrix} \tag{2.460}$$

Example 69 (Principal coordinate frame) Consider the inertia matrix I

$$I = \begin{bmatrix} 10 & -2 & 0 \\ -2 & 20 & -2 \\ 0 & -2 & 30 \end{bmatrix} \tag{2.461}$$

the direction of a principal axis x_i is established by solving

$$\begin{bmatrix} I_{xx} - I_i & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - I_i & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - I_i \end{bmatrix} \begin{bmatrix} \cos\alpha_i \\ \cos\beta_i \\ \cos\gamma_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.462}$$

The direction cosines $\alpha_i, \beta_i, \gamma_i$ must also satisfy

$$\cos^2\alpha_i + \cos^2\beta_i + \cos^2\gamma_i = 1 \tag{2.463}$$

For the first principal mass moment $I_1 = 9.6077$, we have

$$\begin{bmatrix} 10-9.6 & -2 & 0 \\ -2 & 20-9.6 & -2 \\ 0 & -2 & 30-9.6 \end{bmatrix} \begin{bmatrix} \cos\alpha_1 \\ \cos\beta_1 \\ \cos\gamma_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.464}$$

or

$$0.3923 \cos\alpha_1 - 2\cos\beta_1 + 0 = 0 \tag{2.465}$$

$$10.392 \cos\beta_1 - 2\cos\alpha_1 - 2\cos\gamma_1 = 0 \tag{2.466}$$

$$0 + 20.392 \cos\gamma_1 - 2\cos\beta_1 = 0 \tag{2.467}$$

and we obtain

$$\alpha_1 = 11.148 \text{ deg} \tag{2.468}$$

$$\beta_1 = 78.902 \text{ deg} \tag{2.469}$$

$$\gamma_1 = 88.917 \text{ deg} \tag{2.470}$$

Using $I_2 = 20$ for the second principal axis

$$\begin{bmatrix} 10-20 & -2 & 0 \\ -2 & 20-20 & -2 \\ 0 & -2 & 30-20 \end{bmatrix} \begin{bmatrix} \cos\alpha_2 \\ \cos\beta_2 \\ \cos\gamma_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.471}$$

we obtain

$$\alpha_2 = 101.10 \text{ deg} \tag{2.472}$$

$$\beta_2 = 15.793 \text{ deg} \tag{2.473}$$

$$\gamma_2 = 78.902 \text{ deg} \tag{2.474}$$

The third principal axis is for $I_3 = 30.392$

$$\begin{bmatrix} 10-30.4 & -2 & 0 \\ -2 & 20-30.4 & -2 \\ 0 & -2 & 30-30.4 \end{bmatrix} \begin{bmatrix} \cos\alpha_3 \\ \cos\beta_3 \\ \cos\gamma_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.475}$$

which leads to

$$\alpha_3 = 88.917 \text{ deg} \tag{2.476}$$

$$\beta_3 = 101.10 \text{ deg} \tag{2.477}$$

$$\gamma_3 = 11.139 \text{ deg} \tag{2.478}$$

Example 70 (Mass moment of rectangular bar) Consider a homogeneous rectangular brick with mass m, length l, width w, and height h, as shown in Fig. 2.26.

The local central coordinate frame is attached to the brick at its mass center. The moments of inertia matrix of the brick can be found by the integral method. We

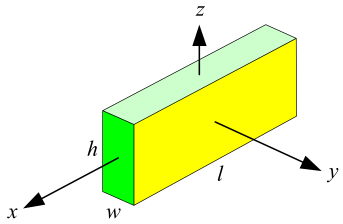


Fig. 2.26 A homogeneous rectangular brick

begin by calculating I_{xx}:

$$
\begin{aligned}
I_{xx} &= \int_B \left(y^2+z^2\right) \mathrm{d}m = \int_v \left(y^2+z^2\right)\rho \,\mathrm{d}v = \frac{m}{lwh}\int_v \left(y^2+z^2\right) \mathrm{d}v \\
&= \frac{m}{lwh}\int_{-h/2}^{h/2}\int_{-w/2}^{w/2}\int_{-l/2}^{l/2}\left(y^2+z^2\right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \frac{m}{12}\left(w^2+h^2\right)
\end{aligned}
\tag{2.479}
$$

The mass moments I_{yy} and I_{zz} can be calculated similarly

$$I_{yy} = \frac{m}{12}\left(h^2+l^2\right) \tag{2.480}$$

$$I_{zz} = \frac{m}{12}\left(l^2+w^2\right) \tag{2.481}$$

The coordinate frame is central and, therefore, the products of inertia must be zero. To show this, we examine I_{xy}:

$$
\begin{aligned}
I_{xy} = I_{yx} &= -\int_B xy\,\mathrm{d}m = \int_v xy\rho\,\mathrm{d}v \\
&= \frac{m}{lwh}\int_{-h/2}^{h/2}\int_{-w/2}^{w/2}\int_{-l/2}^{l/2} xy\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 0
\end{aligned}
\tag{2.482}
$$

Therefore, the moment of inertia matrix for the rigid rectangular brick in its central frame is

$$
I = \begin{bmatrix}
\frac{m}{12}(w^2+h^2) & 0 & 0 \\
0 & \frac{m}{12}(h^2+l^2) & 0 \\
0 & 0 & \frac{m}{12}(l^2+w^2)
\end{bmatrix}
\tag{2.483}
$$

Example 71 (Translation of the inertia matrix) The mass moment matrix of the brick shown in Fig. 2.27, in the principal frame $B(oxyz)$, is given in Eq. (2.483). The mass moment matrix in the non-principal frame $B'(ox'y'z')$ can be found by applying the parallel-axes transformation formula (2.451):

$$^{B'}I = {}^{B}I + m\,{}^{B'}\tilde{r}_C\,{}^{B'}\tilde{r}_C^{\mathrm{T}} \tag{2.484}$$

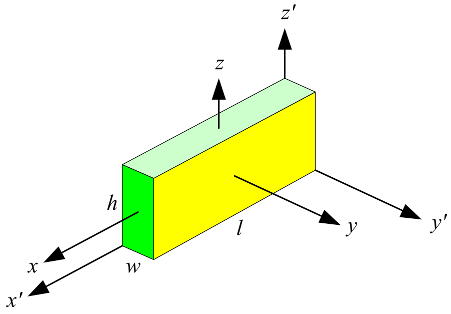


Fig. 2.27 A rigid rectangular brick in the principal and non-principal frames

The mass center is at

$$^{B'}\mathbf{r}_C = \frac{1}{2}\begin{bmatrix} l \\ w \\ h \end{bmatrix} \tag{2.485}$$

and, therefore,

$$^{B'}\tilde{r}_C = \frac{1}{2}\begin{bmatrix} 0 & -h & w \\ h & 0 & -l \\ -w & l & 0 \end{bmatrix} \tag{2.486}$$

which provides

$$^{B'}I = \begin{bmatrix} \frac{1}{3}h^2m + \frac{1}{3}mw^2 & -\frac{1}{4}lmw & -\frac{1}{4}hlm \\ -\frac{1}{4}lmw & \frac{1}{3}h^2m + \frac{1}{3}l^2m & -\frac{1}{4}hmw \\ -\frac{1}{4}hlm & -\frac{1}{4}hmw & \frac{1}{3}l^2m + \frac{1}{3}mw^2 \end{bmatrix} \tag{2.487}$$

Example 72 (Principal rotation matrix) Consider a mass moment matrix in the body frame B_1 as

$$^{1}I = \begin{bmatrix} 2/3 & -1/2 & -1/2 \\ -1/2 & 5/3 & -1/4 \\ -1/2 & -1/4 & 5/3 \end{bmatrix} \tag{2.488}$$

The eigenvalues and eigenvectors of ^{1}I are

$$I_1 = 0.2413 \qquad \begin{bmatrix} 2.351 \\ 1 \\ 1 \end{bmatrix} \tag{2.489}$$

$$I_2 = 1.8421 \qquad \begin{bmatrix} -0.851 \\ 1 \\ 1 \end{bmatrix} \tag{2.490}$$

$$I_3 = 1.9167 \qquad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \tag{2.491}$$

The normalized eigenvector matrix W is the transpose of the required transformation matrix to make the inertia matrix diagonal

$$W = \begin{bmatrix} | & | & | \\ \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 \\ | & | & | \end{bmatrix} = {}^2R_1^{\mathrm{T}}$$
$$= \begin{bmatrix} 0.8569 & -0.5156 & 0.0 \\ 0.36448 & 0.60588 & -0.70711 \\ 0.36448 & 0.60588 & 0.70711 \end{bmatrix} \tag{2.492}$$

We may verify that

$${}^2I = {}^2R_1 {}^1I {}^2R_1^{\mathrm{T}} = W^{\mathrm{T}\,1}IW$$
$$= \begin{bmatrix} 0.2413 & -1\times10^{-4} & 0.0 \\ -1\times10^{-4} & 1.8421 & -1\times10^{-19} \\ 0.0 & 0.0 & 1.9167 \end{bmatrix} \tag{2.493}$$

Example 73 (★ (Relative diagonal moments of inertia) By the definitions for the mass moments (2.427), (2.428), and (2.429) it is seen that the inertia matrix is symmetric, and

$$\int_B \left(x^2 + y^2 + z^2\right) \mathrm{d}m = \frac{1}{2}(I_{xx} + I_{yy} + I_{zz}) \tag{2.494}$$

and also

$$I_{xx} + I_{yy} \geq I_{zz} \qquad I_{yy} + I_{zz} \geq I_{xx} \qquad I_{zz} + I_{xx} \geq I_{yy} \tag{2.495}$$

Noting that $(y-z)^2 \geq 0$, it is evident that $(y^2 + z^2) \geq 2yz$, and therefore

$$I_{xx} \geq 2I_{yz} \tag{2.496}$$

and similarly

$$I_{yy} \geq 2I_{zx} \qquad I_{zz} \geq 2I_{xy} \tag{2.497}$$

Example 74 ★ (Coefficients of the characteristic equation) The determinant (2.454)

$$\begin{vmatrix} I_{xx}-\lambda & I_{xy} & I_{xz} \\ I_{yx} & I_{yy}-\lambda & I_{yz} \\ I_{zx} & I_{zy} & I_{zz}-\lambda \end{vmatrix} = 0 \tag{2.498}$$

for calculating the principal mass moments, leads to a third-degree equation for λ, called the *characteristic equation*:

$$\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0 \tag{2.499}$$

The coefficients of the characteristic equation are called the *principal invariants* of $[I]$. The coefficients of the characteristic equation can directly be found from

$$a_1 = I_{xx} + I_{yy} + I_{zz} = \operatorname{tr}[I] \tag{2.500}$$

$$\begin{aligned} a_2 &= I_{xx}I_{yy} + I_{yy}I_{zz} + I_{zz}I_{xx} - I_{xy}^2 - I_{yz}^2 - I_{zx}^2 \\ &= \begin{vmatrix} I_{xx} & I_{xy} \\ I_{yx} & I_{yy} \end{vmatrix} + \begin{vmatrix} I_{yy} & I_{yz} \\ I_{zy} & I_{zz} \end{vmatrix} + \begin{vmatrix} I_{xx} & I_{xz} \\ I_{zx} & I_{zz} \end{vmatrix} \\ &= \frac{1}{2}\left(a_1^2 - \operatorname{tr}\left[I^2\right]\right) \end{aligned} \tag{2.501}$$

$$\begin{aligned} a_3 &= I_{xx}I_{yy}I_{zz} + I_{xy}I_{yz}I_{zx} + I_{zy}I_{yx}I_{xz} \\ &\quad - (I_{xx}I_{yz}I_{zy} + I_{yy}I_{zx}I_{xz} + I_{zz}I_{xy}I_{yx}) \\ &= I_{xx}I_{yy}I_{zz} + 2I_{xy}I_{yz}I_{zx} - \left(I_{xx}I_{yz}^2 + I_{yy}I_{zx}^2 + I_{zz}I_{xy}^2\right) \\ &= \det[I] \end{aligned} \tag{2.502}$$

Example 75 ★ (The principal mass moments are coordinate invariants) The roots of the inertia characteristic equation are the principal mass moments. They are all real but not necessarily different. The principal mass moments are extreme. That is, the principal mass moments determine the smallest and the largest values of I_{ii}. Since the smallest and largest values of I_{ii} do not depend on the choice of the body coordinate frame, the solution of the characteristic equation is not dependent of the coordinate frame.

In other words, if I_1, I_2, and I_3 are the principal mass moments for ${}^{B_1}I$, the principal mass moments for ${}^{B_2}I$ are also I_1, I_2, and I_3 when

$${}^{B_2}I = {}^{B_2}R_{B_1}\,{}^{B_1}I\,{}^{B_2}R_{B_1}^{\mathrm{T}}$$

We conclude that I_1, I_2, and I_3 are coordinate invariants of the matrix $[I]$, and therefore any quantity that depends on I_1, I_2, and I_3 is also coordinate invariant. Because the mass matrix $[I]$ has rank 3, it has only three independent invariants and every other invariant can be expressed in terms of I_1, I_2, and I_3.

Since I_1, I_2, and I_3 are the solutions of the characteristic equation of $[I]$ given in (2.499), we may write the determinant (2.454) in the form

$$(\lambda - I_1)(\lambda - I_2)(\lambda - I_3) = 0 \tag{2.503}$$

The expanded form of this equation is

$$\lambda^3 - (I_1 + I_2 + I_3)\lambda^2 + (I_1I_2 + I_2I_3 + I_3I_1)a_2\lambda - I_1I_2I_3 = 0 \tag{2.504}$$

By comparing (2.504) and (2.499) we conclude that

$$a_1 = I_{xx} + I_{yy} + I_{zz} = I_1 + I_2 + I_3 \tag{2.505}$$

$$\begin{aligned} a_2 &= I_{xx}I_{yy} + I_{yy}I_{zz} + I_{zz}I_{xx} - I_{xy}^2 - I_{yz}^2 - I_{zx}^2 \\ &= I_1 I_2 + I_2 I_3 + I_3 I_1 \end{aligned} \tag{2.506}$$

$$\begin{aligned} a_3 &= I_{xx}I_{yy}I_{zz} + 2I_{xy}I_{yz}I_{zx} - \left(I_{xx}I_{yz}^2 + I_{yy}I_{zx}^2 + I_{zz}I_{xy}^2\right) \\ &= I_1 I_2 I_3 \end{aligned} \tag{2.507}$$

Being able to express the coefficients a_1, a_2, and a_3 as functions of I_1, I_2, and I_3, we determine the coefficients of the characteristic equation to be coordinate invariant.

Example 76 ★ (Short notation for the elements of inertia matrix) Taking advantage of the *Kronecker delta* δ_{ij} (2.171) we may write the elements of the mass moment matrix I_{ij} in short:

$$I_{ij} = \int_B \left(\left(x_1^2 + x_2^2 + x_3^2\right)\delta_{ij} - x_i x_j\right) \mathrm{d}m \tag{2.508}$$

$$I_{ij} = \int_B \left(r^2 \delta_{ij} - x_i x_j\right) \mathrm{d}m \tag{2.509}$$

$$I_{ij} = \int_B \left(\sum_{k=1}^{3} x_k x_k \delta_{ij} - x_i x_j\right) \mathrm{d}m \tag{2.510}$$

where we utilized the following notation:

$$x_1 = x \qquad x_2 = y \qquad x_3 = z \tag{2.511}$$

Example 77 ★ (Mass moment with respect to a plane, a line, and a point) The mass moment of a system of particles may be defined with respect to a plane, a line, or a point as the sum of the products of the mass of the particles into the square of the perpendicular distance from the particle to the plane, the line, or the point. For a continuous body, the sum would become a definite integral over the volume of the body.

The mass moments with respect to the xy-, yz-, and zx-plane are

$$I_{z^2} = \int_B z^2 \,\mathrm{d}m \tag{2.512}$$

$$I_{y^2} = \int_B y^2 \,\mathrm{d}m \tag{2.513}$$

$$I_{x^2} = \int_B x^2 \,\mathrm{d}m \tag{2.514}$$

The moments of inertia with respect to the x, y, and z axes are

$$I_x = \int_B \left(y^2 + z^2\right) \mathrm{d}m \tag{2.515}$$

$$I_y = \int_B \left(z^2 + x^2\right) \mathrm{d}m \tag{2.516}$$

$$I_z = \int_B \left(x^2 + y^2\right) \mathrm{d}m \tag{2.517}$$

and, therefore,

$$I_x = I_{y^2} + I_{z^2} \tag{2.518}$$

$$I_y = I_{z^2} + I_{x^2} \tag{2.519}$$

$$I_z = I_{x^2} + I_{y^2} \tag{2.520}$$

The moment of inertia with respect to the origin is

$$\begin{aligned} I_{\mathrm{o}} &= \int_B \left(x^2 + y^2 + z^2\right) \mathrm{d}m = I_{x^2} + I_{y^2} + I_{z^2} \\ &= \frac{1}{2}\left(I_x + I_y + I_z\right) \end{aligned} \tag{2.521}$$

Because the choice of the coordinate frame is arbitrary, the mass moment with respect to a line is the sum of the mass moments with respect to any two mutually orthogonal planes that pass through the line. The mass moment with respect to a point has a similar meaning for three mutually orthogonal planes intersecting at the point.

2.4 Lagrange Method

The Lagrange method in deriving the equations of motion of vibrating systems has some advantages over Newton–Euler due to its simplicity and generality, specially for multi *DOF* systems.

2.4.1 ★ Lagrange Form of Newton Equation

Newton's equation of motion can be transformed to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial \dot{q}_r}\right) - \frac{\partial K}{\partial q_r} = F_r \quad r = 1, 2, \ldots, n \tag{2.522}$$

where

$$F_r = \sum_{i=1}^{n} \left(F_{ix}\frac{\partial f_i}{\partial q_1} + F_{iy}\frac{\partial g_i}{\partial q_2} + F_{iz}\frac{\partial h_i}{\partial q_n}\right) \tag{2.523}$$

Equation (2.522) is called the *Lagrange equation of motion*, where K is the *kinetic energy* of the n *DOF* system, q_r, $r = 1, 2, \ldots, n$ are the generalized coordinates of the system, $\mathbf{F}_i = [F_{ix}\ F_{iy}\ F_{iz}]^{\mathrm{T}}$ is the external force acting on the ith particle of the system, and F_r is the generalized force associated to q_r. The functions f_i, g_i, h_i, are the relationships of x_1, y_i, z_i, based on the generalized coordinates of the system $x_i = f_i(q_j, t)$, $y_i = g_i(q_j, t)$, $z_i = h_i(q_j, t)$.

Proof Let m_i be the mass of one of the particles of a system and let (x_i, y_i, z_i) be its Cartesian coordinates in a globally fixed coordinate frame. Assume that the coordinates of every individual particle are functions of another set of coordinates $q_1, q_2, q_3, \ldots, q_n$, and possibly time t:

$$x_i = f_i(q_1, q_2, q_3, \ldots, q_n, t) \tag{2.524}$$

$$y_i = g_i(q_1, q_2, q_3, \ldots, q_n, t) \tag{2.525}$$

$$z_i = h_i(q_1, q_2, q_3, \ldots, q_n, t) \tag{2.526}$$

If F_{xi}, F_{yi}, F_{zi} are components of the total force acting on the particle m_i, then the Newton equations of motion for the particle would be

$$F_{xi} = m_i \ddot{x}_i \tag{2.527}$$

$$F_{yi} = m_i \ddot{y}_i \tag{2.528}$$

$$F_{zi} = m_i \ddot{z}_i \tag{2.529}$$

We multiply both sides of these equations by $\partial f_i/\partial q_r$, $\partial g_i/\partial q_r$, and $\partial h_i/\partial q_r$, respectively, and add them up for all the particles to find

$$\sum_{i=1}^{n} m_i \left(\ddot{x}_i \frac{\partial f_i}{\partial q_r} + \ddot{y}_i \frac{\partial g_i}{\partial q_r} + \ddot{z}_i \frac{\partial h_i}{\partial q_r} \right) = \sum_{i=1}^{n} \left(F_{xi} \frac{\partial f_i}{\partial q_r} + F_{yi} \frac{\partial g_i}{\partial q_r} + F_{zi} \frac{\partial h_i}{\partial q_r} \right) \tag{2.530}$$

where n is the total number of particles.

Taking the time derivative of Eq. (2.524),

$$\dot{x}_i = \frac{\partial f_i}{\partial q_1} \dot{q}_1 + \frac{\partial f_i}{\partial q_2} \dot{q}_2 + \frac{\partial f_i}{\partial q_3} \dot{q}_3 + \cdots + \frac{\partial f_i}{\partial q_n} \dot{q}_n + \frac{\partial f_i}{\partial t} \tag{2.531}$$

we find

$$\frac{\partial \dot{x}_i}{\partial \dot{q}_r} = \frac{\partial}{\partial \dot{q}_r} \left(\frac{\partial f_i}{\partial q_1} \dot{q}_1 + \frac{\partial f_i}{\partial q_2} \dot{q}_2 + \cdots + \frac{\partial f_i}{\partial q_n} \dot{q}_n + \frac{\partial f_i}{\partial t} \right) = \frac{\partial f_i}{\partial q_r} \tag{2.532}$$

and, therefore,

$$\ddot{x}_i \frac{\partial f_i}{\partial q_r} = \ddot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_r} = \frac{\mathrm{d}}{\mathrm{d}t} \left(\dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_r} \right) - \dot{x}_i \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \dot{x}_i}{\partial \dot{q}_r} \right) \tag{2.533}$$

However,

$$\begin{aligned}
\dot{x}_i \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \dot{x}_i}{\partial \dot{q}_r}\right) &= \dot{x}_i \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f_i}{\partial q_r}\right) \\
&= \dot{x}_i \left(\frac{\partial^2 f_i}{\partial q_1 \partial q_r}\dot{q}_1 + \cdots + \frac{\partial^2 f_i}{\partial q_n \partial q_r}\dot{q}_n + \frac{\partial^2 f_i}{\partial t \partial q_r}\right) \\
&= \dot{x}_i \frac{\partial}{\partial q_r}\left(\frac{\partial f_i}{\partial q_1}\dot{q}_1 + \frac{\partial f_i}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial f_i}{\partial q_n}\dot{q}_n + \frac{\partial f_i}{\partial t}\right) \\
&= \dot{x}_i \frac{\partial \dot{x}_i}{\partial q_r}
\end{aligned} \tag{2.534}$$

and we have

$$\ddot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_r} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_r}\right) - \dot{x}_i \frac{\partial \dot{x}_i}{\partial q_r} \tag{2.535}$$

which is equal to

$$\ddot{x}_i \frac{\dot{x}_i}{\dot{q}_r} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial}{\partial \dot{q}_r}\left(\frac{1}{2}\dot{x}_i^2\right)\right] - \frac{\partial}{\partial q_r}\left(\frac{1}{2}\dot{x}_i^2\right) \tag{2.536}$$

Now substituting (2.533) and (2.536) into the left-hand side of (2.530) yields

$$\begin{aligned}
&\sum_{i=1}^{n} m_i \left(\ddot{x}_i \frac{\partial f_i}{\partial q_r} + \ddot{y}_i \frac{\partial g_i}{\partial q_r} + \ddot{z}_i \frac{\partial h_i}{\partial q_r}\right) \\
&\quad = \sum_{i=1}^{n} m_i \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial}{\partial \dot{q}_r}\left(\frac{1}{2}\dot{x}_i^2 + \frac{1}{2}\dot{y}_i^2 + \frac{1}{2}\dot{z}_i^2\right)\right] \\
&\qquad - \sum_{i=1}^{n} m_i \frac{\partial}{\partial q_r}\left(\frac{1}{2}\dot{x}_i^2 + \frac{1}{2}\dot{y}_i^2 + \frac{1}{2}\dot{z}_i^2\right) \\
&\quad = \frac{1}{2}\sum_{i=1}^{n} m_i \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial}{\partial \dot{q}_r}\left(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2\right)\right] \\
&\qquad - \frac{1}{2}\sum_{i=1}^{n} m_i \frac{\partial}{\partial q_r}\left(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2\right)
\end{aligned} \tag{2.537}$$

where

$$\frac{1}{2}\sum_{i=1}^{n} m_i \left(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2\right) = K \tag{2.538}$$

is the *kinetic energy* of the system. Therefore, the Newton equations of motion (2.527), (2.528), and (2.529) are converted to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial \dot{q}_r}\right) - \frac{\partial K}{\partial q_r} = \sum_{i=1}^{n}\left(F_{xi}\frac{\partial f_i}{\partial q_r} + F_{yi}\frac{\partial g_i}{\partial q_r} + F_{zi}\frac{\partial h_i}{\partial q_r}\right) \tag{2.539}$$

Because of (2.524), (2.525), and (2.526), the kinetic energy is a function of $q_1, q_2, q_3, \ldots, q_n$ and time t. The left-hand side of Eq. (2.539) includes the kinetic energy of the whole system and the right-hand side is a generalized force and shows the effect of changing coordinates from x_i to q_j on the external forces. Let us assume that a particular coordinate q_r gets altered to $q_r + \delta q_r$ while the other coordinates $q_1, q_2, q_3, \ldots, q_{r-1}, q_{r+1}, \ldots, q_n$ and time t are unaltered. So, the coordinates of m_i are changed to

$$x_i + \frac{\partial f_i}{\partial q_r}\delta q_r \tag{2.540}$$

$$y_i + \frac{\partial g_i}{\partial q_r}\delta q_r \tag{2.541}$$

$$z_i + \frac{\partial h_i}{\partial q_r}\delta q_r \tag{2.542}$$

Such a displacement is called a *virtual displacement*. The work done in this virtual displacement by all forces acting on the particles of the system is

$$\delta W = \sum_{i=1}^{n}\left(F_{xi}\frac{\partial f_i}{\partial q_r} + F_{yi}\frac{\partial g_i}{\partial q_r} + F_{zi}\frac{\partial h_i}{\partial q_r}\right)\delta q_r \tag{2.543}$$

Because the work done by internal forces appears in opposite pairs, only the work done by external forces remains in Eq. (2.543). Let us denote the virtual work by

$$\delta W = F_r(q_1, q_2, q_3, \ldots, q_n, t)\delta q_r \tag{2.544}$$

Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial \dot{q}_r}\right) - \frac{\partial K}{\partial q_r} = F_r \tag{2.545}$$

where

$$F_r = \sum_{i=1}^{n}\left(F_{xi}\frac{\partial f_i}{\partial q_r} + F_{yi}\frac{\partial g_i}{\partial q_r} + F_{zi}\frac{\partial h_i}{\partial q_r}\right) \tag{2.546}$$

Equation (2.545) is the Lagrange form of the equations of motion. This equation is true for all values of r from 1 to n. We thus have n second-order ordinary differential equations, in which $q_1, q_2, q_3, \ldots, q_n$ are the dependent variables and t is the independent variable. The *generalized coordinates* $q_1, q_2, q_3, \ldots, q_n$ can be any

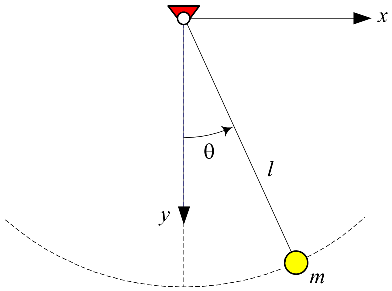


Fig. 2.28 A simple pendulum

measurable parameters to provide the configuration of the system. The number of equations and the number of dependent variables are equal; therefore, the equations are theoretically sufficient to determine the motion of all m_i. □

Example 78 ★ (A simple pendulum) A pendulum is shown in Fig. 2.28. Using x and y for the Cartesian position of m, and using $\theta = q$ as the generalized coordinate, we have

$$x = f(\theta) = l \sin\theta \tag{2.547}$$

$$y = g(\theta) = l \cos\theta \tag{2.548}$$

$$K = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2} m l^2 \dot{\theta}^2 \tag{2.549}$$

and, therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial \dot{\theta}}\right) - \frac{\partial K}{\partial \theta} = \frac{\mathrm{d}}{\mathrm{d}t}\left(m l^2 \dot{\theta}\right) = m l^2 \ddot{\theta} \tag{2.550}$$

The external force components, acting on m, are

$$F_x = 0 \tag{2.551}$$

$$F_y = mg \tag{2.552}$$

and, therefore,

$$F_\theta = F_x \frac{\partial f}{\partial \theta} + F_y \frac{\partial g}{\partial \theta} = -mgl \sin\theta \tag{2.553}$$

Hence, the equation of motion for the pendulum is

$$m l^2 \ddot{\theta} = -mgl \sin\theta \tag{2.554}$$

Example 79 ★ (A pendulum attached to an oscillating mass) Figure 2.29 illustrates a vibrating mass with a hanging pendulum. The pendulum can act as a vibration absorber if designed properly. Starting with coordinate relationships

$$x_M = f_M = x \tag{2.555}$$

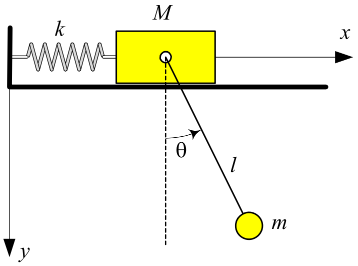


Fig. 2.29 A vibrating mass with a hanging pendulum

$$y_M = g_M = 0 \tag{2.556}$$

$$x_m = f_m = x + l\sin\theta \tag{2.557}$$

$$y_m = g_m = l\cos\theta \tag{2.558}$$

we may find the kinetic energy in terms of the generalized coordinates x and θ:

$$\begin{aligned} K &= \frac{1}{2}M\left(\dot{x}_M^2 + \dot{y}_M^2\right) + \frac{1}{2}m\left(\dot{x}_m^2 + \dot{y}_m^2\right) \\ &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}\cos\theta\right) \end{aligned} \tag{2.559}$$

Then the left-hand sides of the Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial \dot{x}}\right) - \frac{\partial K}{\partial x} = (M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta \tag{2.560}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial \dot{\theta}}\right) - \frac{\partial K}{\partial \theta} = ml^2\ddot{\theta} + ml\ddot{x}\cos\theta \tag{2.561}$$

The external forces acting on M and m are

$$F_{x_M} = -kx \tag{2.562}$$

$$F_{y_M} = 0 \tag{2.563}$$

$$F_{x_m} = 0 \tag{2.564}$$

$$F_{y_m} = mg \tag{2.565}$$

Therefore, the generalized forces are

$$F_x = F_{x_M}\frac{\partial f_M}{\partial x} + F_{y_M}\frac{\partial g_M}{\partial x} + F_{x_m}\frac{\partial f_m}{\partial x} + F_{y_m}\frac{\partial g_m}{\partial x} = -kx \tag{2.566}$$

$$F_\theta = F_{x_M}\frac{\partial f_M}{\partial \theta} + F_{y_M}\frac{\partial g_M}{\partial \theta} + F_{x_m}\frac{\partial f_m}{\partial \theta} + F_{y_m}\frac{\partial g_m}{\partial \theta} = -mgl\sin\theta \tag{2.567}$$

and finally the equations of motion are

$$(M+m)\ddot{x}+ml\ddot{\theta}\cos\theta-ml\dot{\theta}^2\sin\theta=-kx \tag{2.568}$$

$$ml^2\ddot{\theta}+ml\ddot{x}\cos\theta=-mgl\sin\theta \tag{2.569}$$

2.4.2 Lagrangean Mechanics

Let us assume that for some forces $\mathbf{F}=[F_{ix}\ F_{iy}\ F_{iz}]^{\mathrm{T}}$ there is a function V, called *potential energy*, such that the force is derivable from V

$$\mathbf{F}=-\nabla V \tag{2.570}$$

Such a force is called a *potential* or *conservative force*. Then the Lagrange equation of motion can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_r}\right)-\frac{\partial\mathcal{L}}{\partial q_r}=Q_r \quad r=1,2,\ldots,n \tag{2.571}$$

where

$$\mathcal{L}=K-V \tag{2.572}$$

is the *Lagrangean* of the system and Q_r is the nonpotential generalized force

$$Q_r=\sum_{i=1}^{n}\left(F_{ix}\frac{\partial f_i}{\partial q_1}+F_{iy}\frac{\partial g_i}{\partial q_2}+F_{iz}\frac{\partial h_i}{\partial q_n}\right) \tag{2.573}$$

for which there is no potential function.

Proof Assume that the external forces $\mathbf{F}=[F_{xi}\ F_{yi}\ F_{zi}]^{\mathrm{T}}$ acting on the system are conservative:

$$\mathbf{F}=-\nabla V \tag{2.574}$$

The work done by these forces in an arbitrary virtual displacement δq_1, δq_2, $\delta q_3,\ldots,\delta q_n$ is

$$\partial W=-\frac{\partial V}{\partial q_1}\delta q_1-\frac{\partial V}{\partial q_2}\delta q_2-\cdots\frac{\partial V}{\partial q_n}\delta q_n \tag{2.575}$$

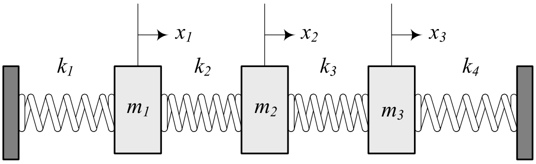
and then the Lagrange equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial\dot{q}_r}\right)-\frac{\partial K}{\partial q_r}=-\frac{\partial V}{\partial q_1} \quad r=1,2,\ldots,n \tag{2.576}$$

Introducing the Lagrangean function $\mathcal{L}=K-V$ converts the Lagrange equation of a conservative system to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_r}\right)-\frac{\partial\mathcal{L}}{\partial q_r}=0 \quad r=1,2,\ldots,n \tag{2.577}$$

Fig. 2.30 An undamped three *DOF* system



If a force is not conservative, then the virtual work done by the force is

$$\delta W = \sum_{i=1}^{n} \left(F_{xi} \frac{\partial f_i}{\partial q_r} + F_{yi} \frac{\partial g_i}{\partial q_r} + F_{zi} \frac{\partial h_i}{\partial q_r} \right) \delta q_r$$
$$= Q_r \, \delta q_r \tag{2.578}$$

and the equation of motion would be

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_r} \right) - \frac{\partial \mathcal{L}}{\partial q_r} = Q_r \quad r = 1, 2, \ldots, n \tag{2.579}$$

where Q_r is the nonpotential generalized force doing work in a virtual displacement of the rth generalized coordinate q_r.

The Lagrangean $\mathcal{L}$ is also called the *kinetic potential*. □

Example 80 (An undamped three *DOF* system) Figure 2.30 illustrates an undamped three *DOF* linear vibrating system. The kinetic and potential energies of the system are

$$K = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_3 \dot{x}_3^2 \tag{2.580}$$

$$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_1 - x_2)^2 + \frac{1}{2} k_3 (x_2 - x_3)^2 + \frac{1}{2} k_4 x_3^2 \tag{2.581}$$

Because there is no damping in the system, we may find the Lagrangean $\mathcal{L}$

$$\mathcal{L} = K - V \tag{2.582}$$

and use Eq. (2.644) with $Q_r = 0$

$$\frac{\partial \mathcal{L}}{\partial x_1} = -k_1 x_1 - k_2 (x_1 - x_2) \tag{2.583}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = k_2 (x_1 - x_2) - k_3 (x_2 - x_3) \tag{2.584}$$

$$\frac{\partial \mathcal{L}}{\partial x_3} = k_3 (x_2 - x_3) - k_4 x_3 \tag{2.585}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_1} = m_1 \dot{x}_1 \tag{2.586}$$

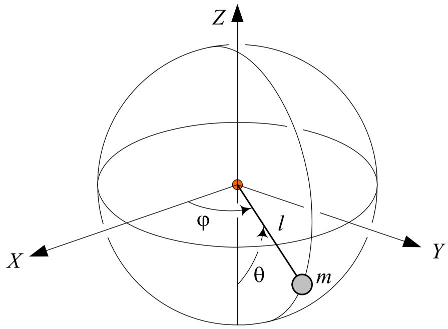


Fig. 2.31 A spherical pendulum

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_2} = m_2 \dot{x}_2 \tag{2.587}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_3} = m_3 \dot{x}_3 \tag{2.588}$$

to find the equations of motion:

$$m_1 \ddot{x}_1 + k_1 x_1 + k_2 (x_1 - x_2) = 0 \tag{2.589}$$

$$m_2 \ddot{x}_2 - k_2 (x_1 - x_2) + k_3 (x_2 - x_3) = 0 \tag{2.590}$$

$$m_3 \ddot{x}_3 - k_3 (x_2 - x_3) + k_4 x_3 = 0 \tag{2.591}$$

These equations can be rewritten in matrix form for simpler calculation:

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \tag{2.592}$$

Example 81 (Spherical pendulum) A pendulum analogy is utilized in modeling of many dynamical problems. Figure 2.31 illustrates a spherical pendulum with mass m and length l. The angles φ and θ may be used as describing coordinates of the system.

The Cartesian coordinates of the mass as a function of the generalized coordinates are

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} r \cos\varphi \sin\theta \\ r \sin\theta \sin\varphi \\ -r \cos\theta \end{bmatrix} \tag{2.593}$$

and, therefore, the kinetic and potential energies of the pendulum are

$$K = \frac{1}{2} m \left(l^2 \dot{\theta}^2 + l^2 \dot{\varphi}^2 \sin^2\theta \right) \tag{2.594}$$

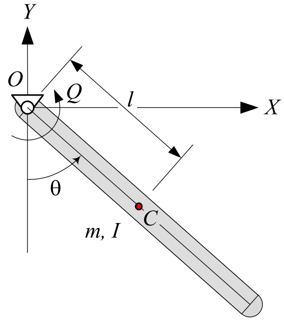


Fig. 2.32 A controlled compound pendulum

$$V = -mgl\cos\theta \tag{2.595}$$

The kinetic potential function of this system is

$$\mathcal{L} = \frac{1}{2}m\left(l^2\dot{\theta}^2 + l^2\dot{\varphi}^2\sin^2\theta\right) + mgl\cos\theta \tag{2.596}$$

which leads to the following equations of motion:

$$\ddot{\theta} - \dot{\varphi}^2\sin\theta\cos\theta + \frac{g}{l}\sin\theta = 0 \tag{2.597}$$

$$\ddot{\varphi}\sin^2\theta + 2\dot{\varphi}\dot{\theta}\sin\theta\cos\theta = 0 \tag{2.598}$$

Example 82 (Controlled compound pendulum) A massive arm is attached to a ceiling at a pin joint O as illustrated in Fig. 2.32. Assume that there is viscous friction in the joint where an ideal motor can apply a torque Q to move the arm. The rotor of an ideal motor has no mass moment by assumption.

The kinetic and potential energies of the manipulator are

$$K = \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}\left(I_C + ml^2\right)\dot{\theta}^2 \tag{2.599}$$

$$V = -mg\cos\theta \tag{2.600}$$

where m is the mass and I is the mass moment of the pendulum about O. The Lagrangean of the manipulator is

$$\mathcal{L} = K - V = \frac{1}{2}I\dot{\theta}^2 + mg\cos\theta \tag{2.601}$$

and, therefore, the equation of motion of the pendulum is

$$M = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = I\ddot{\theta} + mgl\sin\theta \tag{2.602}$$

The generalized force M is the contribution of the motor torque Q and the viscous friction torque $-c\dot{\theta}$. Hence, the equation of motion of the manipulator is

$$Q = I\ddot{\theta} + c\dot{\theta} + mgl\sin\theta \tag{2.603}$$

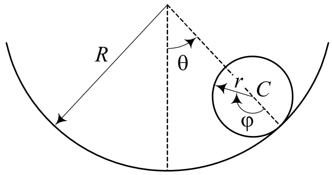


Fig. 2.33 A uniform disc, rolling in a circular path

Example 83 ★ (A rolling disc in a circular path) Figure 2.33 illustrates a uniform disc with mass m and radius r. The disc is rolling without slip in a circular path with radius R. The disc may have a free oscillation around $\theta = 0$.

To find the equation of motion, we employ the Lagrange method. The energies of the system are

$$\begin{aligned} K &= \frac{1}{2} m v_C^2 + \frac{1}{2} I_c \omega^2 \\ &= \frac{1}{2} m (R-r)^2 \dot{\theta}^2 + \frac{1}{2} \left(\frac{1}{2} m r^2 \right) (\dot{\varphi} - \dot{\theta})^2 \end{aligned} \tag{2.604}$$

$$V = -mg(R-r)\cos\theta \tag{2.605}$$

When there is no slip, there is a constraint between θ and φ:

$$R\theta = r\varphi \tag{2.606}$$

which can be used to eliminate φ from K:

$$K = \frac{3}{4} m (R-r)^2 \dot{\theta}^2 \tag{2.607}$$

Based on the partial derivatives

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) = \frac{3}{2} m (R-r)^2 \ddot{\theta} \tag{2.608}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = -mg(R-r)\sin\theta \tag{2.609}$$

we find the equation of motion for the oscillating disc:

$$\frac{3}{2}(R-r)\ddot{\theta} + g\sin\theta = 0 \tag{2.610}$$

When the oscillation is very small, we may substitute the oscillating disc with an equivalent mass–spring system. When θ is very small, this equation is equivalent to a mass–spring system with $m_{\mathrm{e}} = 3(R-r)$ and $k_{\mathrm{e}} = 2g$.

Example 84 ★ (A double pendulum) Figure 2.34 illustrates a double pendulum. There are two massless rods with lengths l_1 and l_2, and two point masses m_1 and m_2. The variables θ_1 and θ_2 can be used as the generalized coordinates to express

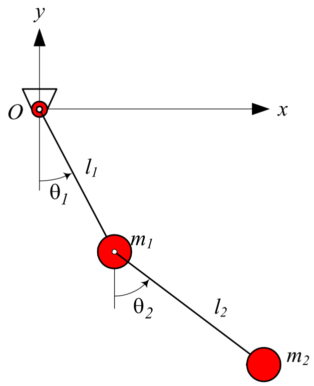


Fig. 2.34 A double pendulum

the system configuration. To calculate the Lagrangean of the system and find the equations of motion, we start by defining the global position of the masses:

$$x_1 = l_1 \sin\theta_1 \tag{2.611}$$

$$y_1 = -l_1 \cos\theta_1 \tag{2.612}$$

$$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2 \tag{2.613}$$

$$y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2 \tag{2.614}$$

The time derivatives of the coordinates are

$$\dot{x}_1 = l_1\dot{\theta}_1 \cos\theta_1 \tag{2.615}$$

$$\dot{y}_1 = l_1\dot{\theta}_1 \sin\theta_1 \tag{2.616}$$

$$\dot{x}_2 = l_1\dot{\theta}_1 \cos\theta_1 + l_2\dot{\theta}_2 \cos\theta_2 \tag{2.617}$$

$$\dot{y}_2 = l_1\dot{\theta}_1 \sin\theta_1 + l_2\dot{\theta}_2 \sin\theta_2 \tag{2.618}$$

and, therefore, the squares of the masses' velocities are

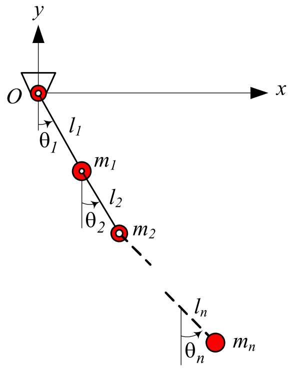
$$v_1^2 = \dot{x}_1^2 + \dot{y}_1^2 = l_1^2\,\dot{\theta}_1^2 \tag{2.619}$$

$$v_2^2 = \dot{x}_2^2 + \dot{y}_2^2 = l_1^2\,\dot{\theta}_1^2 + l_2^2\,\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2) \tag{2.620}$$

The kinetic energy of the pendulum is then equal to

$$\begin{aligned} K &= \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 \\ &= \frac{1}{2}m_1 l_1^2\,\dot{\theta}_1^2 + \frac{1}{2}m_2\big(l_1^2\,\dot{\theta}_1^2 + l_2^2\,\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2)\big) \end{aligned} \tag{2.621}$$

Fig. 2.35 A chain pendulum



The potential energy of the pendulum is equal to the sum of the potentials of each mass:

$$\begin{aligned} V &= m_1 g y_1 + m_2 g y_2 \\ &= -m_1 g l_1 \cos\theta_1 - m_2 g (l_1 \cos\theta_1 + l_2 \cos\theta_2) \end{aligned} \tag{2.622}$$

The kinetic and potential energies constitute the following Lagrangean:

$$\begin{aligned} \mathcal{L} &= K - V \\ &= \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \big(l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \big) \\ &\quad + m_1 g l_1 \cos\theta_1 + m_2 g (l_1 \cos\theta_1 + l_2 \cos\theta_2) \end{aligned} \tag{2.623}$$

Employing Lagrange method (2.644) we find the following equations of motion:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1}\right) - \frac{\partial \mathcal{L}}{\partial \theta_1} &= (m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) \\ &\quad - m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) l_1 g \sin\theta_1 = 0 \end{aligned} \tag{2.624}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) - \frac{\partial \mathcal{L}}{\partial \theta_2} &= m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) \\ &\quad + m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + m_2 l_2 g \sin\theta_2 = 0 \end{aligned} \tag{2.625}$$

Example 85 ★ (Chain pendulum) Consider an n-chain-pendulum as shown in Fig. 2.35. Each pendulum has a massless length l_i with a concentrated point mass m_i, and a generalized angular coordinate θ_i measured from the vertical direction.

The x_i and y_i components of the mass m_i are

$$x_i = \sum_{j=1}^{i} l_j \sin\theta_j \qquad y_i = -\sum_{j=1}^{i} l_j \cos\theta_j \tag{2.626}$$

We find their time derivatives:

$$\dot{x}_i = \sum_{j=1}^{i} l_j \dot{\theta}_j \cos\theta_j \qquad \dot{y}_i = \sum_{j=1}^{i} l_j \dot{\theta}_j \sin\theta_j \tag{2.627}$$

and the square of $\dot{x}_i$ and $\dot{y}_i$:

$$\dot{x}_i^2 = \left(\sum_{j=1}^{i} l_j \dot{\theta}_j \cos\theta_j\right)\left(\sum_{k=1}^{i} l_k \dot{\theta}_k \cos\theta_k\right) = \sum_{j=1}^{n}\sum_{k=1}^{n} l_j l_k \dot{\theta}_j \dot{\theta}_k \cos\theta_j \cos\theta_k \tag{2.628}$$

$$\dot{y}_i^2 = \left(\sum_{j=1}^{i} l_j \dot{\theta}_j \sin\theta_j\right)\left(\sum_{k=1}^{i} l_k \dot{\theta}_k \sin\theta_k\right) = \sum_{j=1}^{i}\sum_{k=1}^{i} l_j l_k \dot{\theta}_j \dot{\theta}_k \sin\theta_j \sin\theta_k \tag{2.629}$$

to calculate the velocity v_i of the mass m_i:

$$\begin{aligned} v_i^2 &= \dot{x}_i^2 + \dot{y}_i^2 \\ &= \sum_{j=1}^{i}\sum_{k=1}^{i} l_j l_k \dot{\theta}_j \dot{\theta}_k (\cos\theta_j \cos\theta_k + \sin\theta_j \sin\theta_k) \\ &= \sum_{j=1}^{i}\sum_{k=1}^{i} l_j l_k \dot{\theta}_j \dot{\theta}_k \cos(\theta_j - \theta_k) \\ &= \sum_{r=1}^{i} l_r^2 \dot{\theta}_r^2 + 2\sum_{j=1}^{i}\sum_{k=j+1}^{i} l_j l_k \dot{\theta}_j \dot{\theta}_k \cos(\theta_j - \theta_k) \end{aligned} \tag{2.630}$$

Now, we calculate the kinetic energy, K, of the chain:

$$\begin{aligned} K &= \frac{1}{2}\sum_{i=1}^{n} m_i v_i^2 \\ &= \frac{1}{2}\sum_{i=1}^{n} m_i \left(\sum_{r=1}^{i} l_r^2 \dot{\theta}_r^2 + 2\sum_{j=1}^{i}\sum_{k=j+1}^{i} l_j l_k \dot{\theta}_j \dot{\theta}_k \cos(\theta_j - \theta_k)\right) \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{r=1}^{i} m_i l_r^2 \dot{\theta}_r^2 + \sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=j+1}^{i} m_i l_j l_k \dot{\theta}_j \dot{\theta}_k \cos(\theta_j - \theta_k) \end{aligned} \tag{2.631}$$

The potential energy of the ith pendulum is related to m_i,

$$V_i = m_i g y_i = -m_i g \sum_{j=1}^{i} l_j \cos\theta_j \tag{2.632}$$

and, therefore, the potential energy of the chain is

$$V = \sum_{i=1}^{n} m_i g y_i = -\sum_{i=1}^{n}\sum_{j=1}^{i} m_i g l_j \cos\theta_j \tag{2.633}$$

To find the equations of motion for the chain, we may use the Lagrangean $\mathcal{L}$

$$\mathcal{L} = K - V \tag{2.634}$$

and apply the Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_s}\right) - \frac{\partial \mathcal{L}}{\partial q_s} = 0 \quad s = 1, 2, \ldots, n \tag{2.635}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial \dot{q}_s}\right) - \frac{\partial K}{\partial q_s} + \frac{\partial V}{\partial q_s} = 0 \quad s = 1, 2, \ldots, n \tag{2.636}$$

Example 86 (Mechanical energy) If a system of masses m_i are moving in a potential force field

$$\mathbf{F}_{m_i} = -\nabla_i V \tag{2.637}$$

their Newton equations of motion will be

$$m_i \ddot{\mathbf{r}}_i = -\nabla_i V \quad i = 1, 2, \ldots, n \tag{2.638}$$

The inner product of equations of motion with $\dot{\mathbf{r}}_i$ and adding the equations

$$\sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i \cdot \ddot{\mathbf{r}}_i = -\sum_{i=1}^{n} \dot{\mathbf{r}}_i \cdot \nabla_i V \tag{2.639}$$

and then integrating over time

$$\frac{1}{2}\sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i = -\int \sum_{i=1}^{n} \mathbf{r}_i \cdot \nabla_i V \tag{2.640}$$

shows that

$$K = -\int \sum_{i=1}^{n}\left(\frac{\partial V}{\partial x_i} x_i + \frac{\partial V}{\partial y_i} y_i + \frac{\partial V}{\partial z_i} z_i\right) = -V + E \tag{2.641}$$

where E is the constant of integration. E is called the *mechanical energy* of the system and is equal to kinetic plus potential energies:

$$E = K + V \tag{2.642}$$

2.5 Dissipation Function

The Lagrange equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial \dot{q}_r}\right) - \frac{\partial K}{\partial q_r} = F_r \quad r = 1, 2, \ldots, n \tag{2.643}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_r}\right) - \frac{\partial \mathcal{L}}{\partial q_r} = Q_r \quad r = 1, 2, \ldots, n \tag{2.644}$$

as introduced in Eqs. (2.522) and (2.571) can both be applied to find the equations of motion of a vibrating system. However, for small and linear vibrations, we may use a simpler and more practical Lagrange equation, thus:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial \dot{q}_r}\right) - \frac{\partial K}{\partial q_r} + \frac{\partial D}{\partial \dot{q}_r} + \frac{\partial V}{\partial q_r} = f_r \quad r = 1, 2, \ldots, n \tag{2.645}$$

where K is the kinetic energy, V is the potential energy, and D is the *dissipation function* of the system

$$K = \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}[m]\dot{\mathbf{q}} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\dot{q}_i m_{ij}\dot{q}_j \tag{2.646}$$

$$V = \frac{1}{2}\mathbf{q}^{\mathrm{T}}[k]\mathbf{q} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}q_i k_{ij} q_j \tag{2.647}$$

$$D = \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}[c]\dot{\mathbf{q}} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\dot{q}_i c_{ij}\dot{q}_j \tag{2.648}$$

and f_r is the applied force on the mass m_r.

Proof Consider a one *DOF* mass–spring–damper vibrating system. When viscous damping is the only type of damping in the system, we may employ a function known as the *Rayleigh dissipation function*

$$D = \frac{1}{2}c\dot{x}^2 \tag{2.649}$$

to find the damping force f_c by differentiation:

$$f_c = -\frac{\partial D}{\partial \dot{x}} \tag{2.650}$$

Remembering that the elastic force f_k can be found from a potential energy V

$$f_k = -\frac{\partial V}{\partial x} \tag{2.651}$$

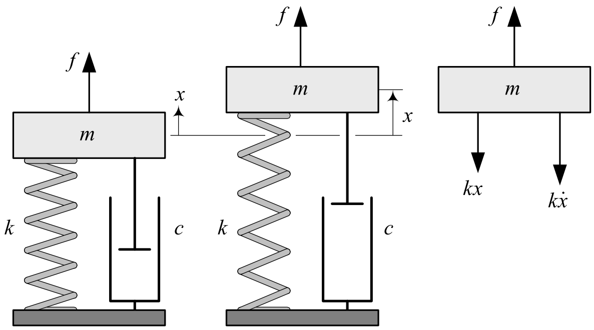


Fig. 2.36 A one *DOF* forced mass–spring–damper system

then the generalized force F can be separated to

$$F = f_c + f_k + f = -\frac{\partial D}{\partial \dot{x}} - \frac{\partial V}{\partial x} + f \tag{2.652}$$

where f is the non-conservative applied force on mass m. Substituting (2.652) in (2.643)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial \dot{x}}\right) - \frac{\partial K}{\partial x} = -\frac{\partial D}{\partial \dot{x}} - \frac{\partial V}{\partial x} + f \tag{2.653}$$

gives us the Lagrange equation for a viscous damped vibrating system.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial \dot{x}}\right) - \frac{\partial K}{\partial x} + \frac{\partial D}{\partial \dot{x}} + \frac{\partial V}{\partial x} = f \tag{2.654}$$

For vibrating systems with n *DOF*, the kinetic energy K, potential energy V, and dissipating function D are as (2.646)–(2.648). Applying the Lagrange equation to the n *DOF* system would result n second-order differential equations (2.645). □

Example 87 (A one *DOF* forced mass–spring–damper system) Figure 2.36 illustrates a single *DOF* mass–spring–damper system with an external force f applied on the mass m. The kinetic and potential energies of the system, when it is in motion, are

$$K = \frac{1}{2} m \dot{x}^2 \tag{2.655}$$

$$V = \frac{1}{2} k x^2 \tag{2.656}$$

and its dissipation function is

$$D = \frac{1}{2} c \dot{x}^2 \tag{2.657}$$

Substituting (2.655)–(2.657) into the Lagrange equation (2.645) provides us with the equation of motion:

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}) + c\dot{x} + kx = f \tag{2.658}$$

because

$$\frac{\partial K}{\partial \dot{x}} = m\dot{x} \qquad \frac{\partial K}{\partial x} = 0 \qquad \frac{\partial D}{\partial \dot{x}} = c\dot{x} \qquad \frac{\partial V}{\partial x} = kx \tag{2.659}$$

Example 88 (An eccentric excited one *DOF* system) An eccentric excited one *DOF* system is shown in Fig. 3.30 with mass m supported by a suspension made up of a spring k and a damper c. There is also a mass m_e at a distance e that is rotating with an angular velocity ω. We may find the equation of motion by applying the Lagrange method.

The kinetic energy of the system is

$$K = \frac{1}{2}(m - m_\mathrm{e})\dot{x}^2 + \frac{1}{2}m_\mathrm{e}(\dot{x} + e\omega\cos\omega t)^2 + \frac{1}{2}m_\mathrm{e}(-e\omega\sin\omega t)^2 \tag{2.660}$$

because the velocity of the main vibrating mass $m - m_\mathrm{e}$ is $\dot{x}$, and the velocity of the eccentric mass m_e has two components $\dot{x} + e\omega\cos\omega t$ and $-e\omega\sin\omega t$. The potential energy and dissipation function of the system are

$$V = \frac{1}{2}kx^2 \qquad D = \frac{1}{2}c\dot{x}^2 \tag{2.661}$$

Applying the Lagrange equation (2.645),

$$\frac{\partial K}{\partial \dot{x}} = m\dot{x} + m_\mathrm{e}e\omega\cos\omega t \tag{2.662}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial \dot{x}}\right) = m\ddot{x} - m_\mathrm{e}e\omega^2\sin\omega t \tag{2.663}$$

$$\frac{\partial D}{\partial \dot{x}} = c\dot{x} \tag{2.664}$$

$$\frac{\partial V}{\partial x} = kx \tag{2.665}$$

provides us with the equation of motion

$$m\ddot{x} + c\dot{x} + kx = m_\mathrm{e}e\omega^2\sin\omega t \tag{2.666}$$

which is the same as Eq. (3.267).

Example 89 (An eccentric base excited vibrating system) Figure 2.37 illustrates a one *DOF* eccentric base excited vibrating system. A mass m is mounted on an eccentric excited base by a spring k and a damper c. The base has a mass m_b with

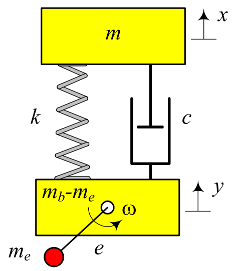


Fig. 2.37 A one *DOF* eccentric base excited vibrating system

an attached unbalance mass m_e at a distance e. The mass m_e is rotating with an angular velocity ω.

We may derive the equation of motion of the system by applying Lagrange method. The required functions are

$$K = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}(m_\mathrm{b} - m_\mathrm{e})\dot{y}^2 + \frac{1}{2}m_\mathrm{e}(\dot{y} - e\omega\cos\omega t)^2 + \frac{1}{2}m_\mathrm{e}(e\omega\sin\omega t)^2 \tag{2.667}$$

$$V = \frac{1}{2}k(x - y)^2 \tag{2.668}$$

$$D = \frac{1}{2}c(\dot{x} - \dot{y})^2 \tag{2.669}$$

Applying the Lagrange method (2.645) provides us with the equations

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0 \tag{2.670}$$

$$m_\mathrm{b}\ddot{y} + m_\mathrm{e}e\omega^2\sin\omega t - c(\dot{x} - \dot{y}) - k(x - y) = 0 \tag{2.671}$$

because

$$\frac{\partial K}{\partial \dot{x}} = m\dot{x} \tag{2.672}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial \dot{x}}\right) = m\ddot{x} \tag{2.673}$$

$$\frac{\partial D}{\partial \dot{x}} = c(\dot{x} - \dot{y}) \tag{2.674}$$

$$\frac{\partial V}{\partial x} = k(x - y) \tag{2.675}$$

$$\frac{\partial K}{\partial \dot{y}} = m_\mathrm{b}\dot{y} - m_\mathrm{e}e\omega\cos\omega t \tag{2.676}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial \dot{y}}\right) = m_\mathrm{b}\ddot{y} + m_\mathrm{e}e\omega^2\sin\omega t \tag{2.677}$$

$$\frac{\partial D}{\partial \dot{y}} = -c(\dot{x} - \dot{y}) \tag{2.678}$$

$$\frac{\partial V}{\partial y} = -k(x - y) \tag{2.679}$$

Using $z = x - y$, we may combine Eqs. (2.670) and (2.671) to find the equation of relative motion

$$\frac{m m_{\mathrm{b}}}{m_{\mathrm{b}} + m}\ddot{z} + c\dot{z} + kz = \frac{m m_{\mathrm{e}}}{m_{\mathrm{b}} + m} e\omega^2 \sin \omega t \tag{2.680}$$

which is equal to

$$\ddot{z} + 2\xi\omega_{\mathrm{n}}\,\dot{z} + \omega_{\mathrm{n}}^2\, z = \varepsilon e\omega^2 \sin \omega t \tag{2.681}$$

$$\varepsilon = \frac{m_{\mathrm{e}}}{m_{\mathrm{b}}} \tag{2.682}$$

Example 90 ★ (Generalized forces)

1. Elastic force:

An elastic force is a recoverable force from an elastic body that has changed its internal energy. An elastic body is a body for which any produced work is stored in the body in the form of internal energy, and is recoverable. Therefore, the variation of the internal potential energy of the body, $V = V(q, t)$ would be

$$\delta V = -\delta W = \sum_{i=1}^{n} Q_i \delta q_i \tag{2.683}$$

where q_i is the generalized coordinate of the particle i of the body, δW is the virtual work of the generalized elastic force Q:

$$Q_i = -\frac{\partial V}{\partial q_i} \tag{2.684}$$

2. Dissipation force:

A dissipative force between two bodies is proportional to and in opposite direction of the relative velocity vector $\mathbf{v}$ between two bodies:

$$Q_i = -c_i f_i(v_i) \frac{\mathbf{v}_i}{v_i} \tag{2.685}$$

The coefficient c_i is a constant, $f_i(v_i)$ is the velocity function of the force, and v_i is the magnitude of the relative velocity:

$$v_i = \sqrt{\sum_{j=1}^{3} v_{ij}^2} \tag{2.686}$$

The virtual work of the dissipation force is

$$\delta W = \sum_{i=1}^{n_1} Q_i \delta q_i \tag{2.687}$$

$$Q_i = -\sum_{k=1}^{n_1} c_k f_k(v_k) \frac{\partial v_k}{\partial \dot{q}_i} \tag{2.688}$$

where n_1 is the total number of dissipation forces. By introducing the dissipation function D as

$$D = \sum_{i=1}^{n_1} \int_0^{v_i} c_k f_k(z_k) \, \mathrm{d}z \tag{2.689}$$

we have

$$Q_i = -\frac{\partial D}{\partial \dot{q}_i} \tag{2.690}$$

The dissipation power P of the dissipation force Q_i is

$$P = \sum_{i=1}^{n} Q_i \dot{q}_i = \sum_{i=1}^{n} \dot{q}_i \frac{\partial D}{\partial \dot{q}_i} \tag{2.691}$$

2.6 ★ Quadratures

If $[m]$ is an $n \times n$ square matrix and $\mathbf{x}$ is an $n \times 1$ vector, then S is a scalar function called *quadrature* and is defined by

$$S = \mathbf{x}^{\mathrm{T}} [m] \mathbf{x} \tag{2.692}$$

The derivative of the quadrature S with respect to the vector $\mathbf{x}$ is

$$\frac{\partial S}{\partial \mathbf{x}} = \left([m] + [m]^{\mathrm{T}} \right) \mathbf{x} \tag{2.693}$$

Kinetic energy K, potential energy V, and dissipation function D are quadratures

$$K = \frac{1}{2} \dot{\mathbf{x}}^{\mathrm{T}} [m] \dot{\mathbf{x}} \tag{2.694}$$

$$V = \frac{1}{2} \mathbf{x}^{\mathrm{T}} [k] \mathbf{x} \tag{2.695}$$

$$D = \frac{1}{2} \dot{\mathbf{x}}^{\mathrm{T}} [c] \dot{\mathbf{x}} \tag{2.696}$$

and, therefore,

$$\frac{\partial K}{\partial \dot{\mathbf{x}}} = \frac{1}{2}\left([m] + [m]^{\mathrm{T}}\right)\dot{\mathbf{x}} \tag{2.697}$$

$$\frac{\partial V}{\partial \mathbf{x}} = \frac{1}{2}\left([k] + [k]^{\mathrm{T}}\right)\mathbf{x} \tag{2.698}$$

$$\frac{\partial D}{\partial \dot{\mathbf{x}}} = \frac{1}{2}\left([c] + [c]^{\mathrm{T}}\right)\dot{\mathbf{x}} \tag{2.699}$$

Employing quadrature derivatives and the Lagrange method,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K}{\partial \dot{\mathbf{x}}} + \frac{\partial K}{\partial \mathbf{x}} + \frac{\partial D}{\partial \dot{\mathbf{x}}} + \frac{\partial V}{\partial \mathbf{x}} = \mathbf{F} \tag{2.700}$$

$$\delta W = \mathbf{F}^{\mathrm{T}} \partial \mathbf{x} \tag{2.701}$$

the equation of motion for a linear n degree-of-freedom vibrating system becomes

$$[\underline{m}]\ddot{\mathbf{x}} + [\underline{c}]\dot{\mathbf{x}} + [\underline{k}]\mathbf{x} = \mathbf{F} \tag{2.702}$$

where $[\underline{m}]$, $[\underline{c}]$, $[\underline{k}]$ are symmetric matrices:

$$[\underline{m}] = \frac{1}{2}\left([m] + [m]^{\mathrm{T}}\right) \tag{2.703}$$

$$[\underline{c}] = \frac{1}{2}\left([c] + [c]^{\mathrm{T}}\right) \tag{2.704}$$

$$[\underline{k}] = \frac{1}{2}\left([k] + [k]^{\mathrm{T}}\right) \tag{2.705}$$

Quadratures are also called *Hermitian forms*.

Proof Let us define a general asymmetric quadrature as

$$S = \mathbf{x}^{\mathrm{T}}[a]\mathbf{y} = \sum_i \sum_j x_i a_{ij} y_j \tag{2.706}$$

If the quadrature is symmetric, then $\mathbf{x} = \mathbf{y}$ and

$$S = \mathbf{x}^{\mathrm{T}}[a]\mathbf{x} = \sum_i \sum_j x_i a_{ij} x_j \tag{2.707}$$

The vectors $\mathbf{x}$ and $\mathbf{y}$ may be functions of n generalized coordinates q_i and time t:

$$\mathbf{x} = \mathbf{x}(q_1, q_2, \ldots, q_n, t) \tag{2.708}$$

$$\mathbf{y} = \mathbf{y}(q_1, q_2, \ldots, q_n, t) \tag{2.709}$$

$$\mathbf{q} = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}^{\mathrm{T}} \tag{2.710}$$

The derivative of **x** with respect to **q** is a square matrix

$$\frac{\partial \mathbf{x}}{\partial \mathbf{q}} = \begin{bmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_2}{\partial q_1} & \cdots & \frac{\partial x_n}{\partial q_1} \\ \frac{\partial x_1}{\partial q_2} & \frac{\partial x_2}{\partial q_2} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x_1}{\partial q_n} & \cdots & \cdots & \frac{\partial x_n}{\partial q_n} \end{bmatrix} \tag{2.711}$$

which can also be expressed by

$$\frac{\partial \mathbf{x}}{\partial \mathbf{q}} = \begin{bmatrix} \frac{\partial \mathbf{x}}{\partial q_1} \\ \frac{\partial \mathbf{x}}{\partial q_2} \\ \cdots \\ \frac{\partial \mathbf{x}}{\partial q_n} \end{bmatrix} \tag{2.712}$$

or

$$\frac{\partial \mathbf{x}}{\partial \mathbf{q}} = \begin{bmatrix} \frac{\partial x_1}{\partial \mathbf{q}} & \frac{\partial x_2}{\partial \mathbf{q}} & \cdots & \frac{\partial x_n}{\partial \mathbf{q}} \end{bmatrix} \tag{2.713}$$

The derivative of S with respect to an element of q_k is

$$\begin{aligned} \frac{\partial S}{\partial q_k} &= \frac{\partial}{\partial q_k} \sum_i \sum_j x_i a_{ij} y_j \\ &= \sum_i \sum_j \frac{\partial x_i}{\partial q_k} a_{ij} y_j + \sum_i \sum_j x_i a_{ij} \frac{\partial y_j}{\partial q_k} \\ &= \sum_j \sum_i \frac{\partial x_i}{\partial q_k} a_{ij} y_j + \sum_i \sum_j \frac{\partial y_j}{\partial q_k} a_{ij} x_i \\ &= \sum_j \sum_i \frac{\partial x_i}{\partial q_k} a_{ij} y_j + \sum_j \sum_i \frac{\partial y_i}{\partial q_k} a_{ji} x_j \end{aligned} \tag{2.714}$$

and, hence, the derivative of S with respect to **q** is

$$\frac{\partial S}{\partial \mathbf{q}} = \frac{\partial \mathbf{x}}{\partial \mathbf{q}} [a] \mathbf{y} + \frac{\partial \mathbf{y}}{\partial \mathbf{q}} [a]^{\mathrm{T}} \mathbf{x} \tag{2.715}$$

If S is a symmetric quadrature, then

$$\frac{\partial S}{\partial \mathbf{q}} = \frac{\partial}{\partial \mathbf{q}} \left(\mathbf{x}^{\mathrm{T}} [a] \mathbf{x} \right) = \frac{\partial \mathbf{x}}{\partial \mathbf{q}} [a] \mathbf{x} + \frac{\partial \mathbf{x}}{\partial \mathbf{q}} [a]^{\mathrm{T}} \mathbf{x} \tag{2.716}$$

and if $\mathbf{q} = \mathbf{x}$, then the derivative of a symmetric S with respect to $\mathbf{x}$ is

$$\begin{aligned}\frac{\partial S}{\partial \mathbf{x}} &= \frac{\partial}{\partial \mathbf{x}}\left(\mathbf{x}^{\mathrm{T}}[a]\mathbf{x}\right) = \frac{\partial \mathbf{x}}{\partial \mathbf{x}}[a]\mathbf{x} + \frac{\partial \mathbf{x}}{\partial \mathbf{x}}[a]^{\mathrm{T}}\mathbf{x} \\ &= [a]\mathbf{x} + [a]^{\mathrm{T}}\mathbf{x} = \left([a] + [a]^{\mathrm{T}}\right)\mathbf{x}\end{aligned} \tag{2.717}$$

If $[a]$ is a symmetric matrix, then

$$[a] + [a]^{\mathrm{T}} = 2[a] \tag{2.718}$$

however, if $[a]$ is not a symmetric matrix, then $[\underline{a}] = [a] + [a]^{\mathrm{T}}$ is a symmetric matrix because

$$\underline{a}_{ij} = a_{ij} + a_{ji} = a_{ji} + a_{ij} = \underline{a}_{ji} \tag{2.719}$$

and, therefore,

$$[\underline{a}] = [\underline{a}]^{\mathrm{T}} \tag{2.720}$$

Kinetic energy K, potential energy V, and dissipation function D can be expressed by quadratures:

$$K = \frac{1}{2}\dot{\mathbf{x}}^{\mathrm{T}}[m]\dot{\mathbf{x}} \tag{2.721}$$

$$V = \frac{1}{2}\mathbf{x}^{\mathrm{T}}[k]\mathbf{x} \tag{2.722}$$

$$D = \frac{1}{2}\dot{\mathbf{x}}^{\mathrm{T}}[c]\dot{\mathbf{x}} \tag{2.723}$$

Substituting K, V, and D in the Lagrange equation provides us with the equations of motion:

$$\begin{aligned}\mathbf{F} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K}{\partial \dot{\mathbf{x}}} + \frac{\partial K}{\partial \mathbf{x}} + \frac{\partial D}{\partial \dot{\mathbf{x}}} + \frac{\partial V}{\partial \mathbf{x}} \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{\mathbf{x}}}\left(\dot{\mathbf{x}}^{\mathrm{T}}[m]\dot{\mathbf{x}}\right) + \frac{1}{2}\frac{\partial}{\partial \dot{\mathbf{x}}}\left(\dot{\mathbf{x}}^{\mathrm{T}}[c]\dot{\mathbf{x}}\right) + \frac{1}{2}\frac{\partial}{\partial \mathbf{x}}\left(\mathbf{x}^{\mathrm{T}}[k]\mathbf{x}\right) \\ &= \frac{1}{2}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\left([m] + [m]^{\mathrm{T}}\right)\dot{\mathbf{x}}\right) + \left([c] + [c]^{\mathrm{T}}\right)\dot{\mathbf{x}} + \left([k] + [k]^{\mathrm{T}}\right)\mathbf{x}\right] \\ &= \frac{1}{2}\left([m] + [m]^{\mathrm{T}}\right)\ddot{\mathbf{x}} + \frac{1}{2}\left([c] + [c]^{\mathrm{T}}\right)\dot{\mathbf{x}} + \frac{1}{2}\left([k] + [k]^{\mathrm{T}}\right)\mathbf{x} \\ &= [\underline{m}]\ddot{\mathbf{x}} + [\underline{c}]\dot{\mathbf{x}} + [\underline{k}]\mathbf{x}\end{aligned} \tag{2.724}$$

where

$$[\underline{m}] = \frac{1}{2}\left([m] + [m]^{\mathrm{T}}\right) \tag{2.725}$$

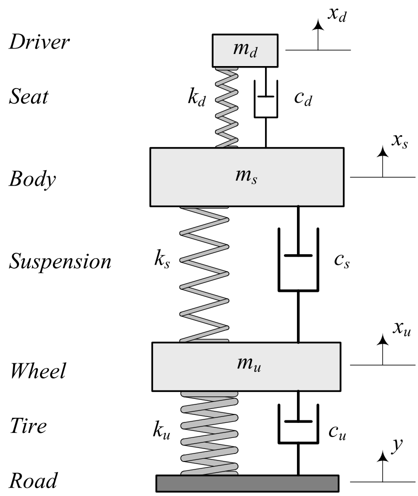


Fig. 2.38 A quarter car model with driver

$$[\underline{c}] = \frac{1}{2}\left([k] + [k]^{\mathrm{T}}\right) \tag{2.726}$$

$$[\underline{k}] = \frac{1}{2}\left([c] + [c]^{\mathrm{T}}\right) \tag{2.727}$$

From now on, we assume that every equation of motion is found from the Lagrange method to have symmetric coefficient matrices. Hence, we show the equations of motion, thus:

$$[m]\ddot{\mathbf{x}} + [c]\dot{\mathbf{x}} + [k]\mathbf{x} = \mathbf{F} \tag{2.728}$$

and use $[m]$, $[c]$, $[k]$ as a substitute for $[\underline{m}]$, $[\underline{c}]$, $[\underline{k}]$:

$$[m] \equiv [\underline{m}] \tag{2.729}$$

$$[c] \equiv [\underline{c}] \tag{2.730}$$

$$[k] \equiv [\underline{k}] \tag{2.731}$$

Symmetric matrices are equal to their transpose:

$$[m] \equiv [m]^{\mathrm{T}} \tag{2.732}$$

$$[c] \equiv [c]^{\mathrm{T}} \tag{2.733}$$

$$[k] \equiv [k]^{\mathrm{T}} \tag{2.734}$$

□

Example 91 ★ (A quarter car model with driver's chair) Figure 2.38 illustrates a quarter car model plus a driver, which is modeled by a mass m_{d} over a linear cushion above the sprung mass m_{s}.

Assuming

$$y = 0 \tag{2.735}$$

we find the free vibration equations of motion. The kinetic energy K of the system can be expressed by

$$\begin{aligned}
K &= \frac{1}{2} m_{\mathrm{u}} \dot{x}_{\mathrm{u}}^2 + \frac{1}{2} m_{\mathrm{s}} \dot{x}_{\mathrm{s}}^2 + \frac{1}{2} m_{\mathrm{d}} \dot{x}_{\mathrm{d}}^2 \\
&= \frac{1}{2} \begin{bmatrix} \dot{x}_{\mathrm{u}} & \dot{x}_{\mathrm{s}} & \dot{x}_{\mathrm{d}} \end{bmatrix} \begin{bmatrix} m_{\mathrm{u}} & 0 & 0 \\ 0 & m_{\mathrm{s}} & 0 \\ 0 & 0 & m_{\mathrm{d}} \end{bmatrix} \begin{bmatrix} \dot{x}_{\mathrm{u}} \\ \dot{x}_{\mathrm{s}} \\ \dot{x}_{\mathrm{d}} \end{bmatrix} \\
&= \frac{1}{2} \dot{\mathbf{x}}^{\mathrm{T}} [m] \dot{\mathbf{x}}
\end{aligned} \tag{2.736}$$

and the potential energy V can be expressed as

$$\begin{aligned}
V &= \frac{1}{2} k_{\mathrm{u}} (x_{\mathrm{u}})^2 + \frac{1}{2} k_{\mathrm{s}} (x_{\mathrm{s}} - x_{\mathrm{u}})^2 + \frac{1}{2} k_{\mathrm{d}} (x_{\mathrm{d}} - x_{\mathrm{s}})^2 \\
&= \frac{1}{2} \begin{bmatrix} x_{\mathrm{u}} & x_{\mathrm{s}} & x_{\mathrm{d}} \end{bmatrix} \begin{bmatrix} k_{\mathrm{u}} + k_{\mathrm{s}} & -k_{\mathrm{s}} & 0 \\ -k_{\mathrm{s}} & k_{\mathrm{s}} + k_{\mathrm{d}} & -k_{\mathrm{d}} \\ 0 & -k_{\mathrm{d}} & k_{\mathrm{d}} \end{bmatrix} \begin{bmatrix} x_{\mathrm{u}} \\ x_{\mathrm{s}} \\ x_{\mathrm{d}} \end{bmatrix} \\
&= \frac{1}{2} \mathbf{x}^{\mathrm{T}} [k] \mathbf{x}
\end{aligned} \tag{2.737}$$

Similarly, the dissipation function D can be expressed by

$$\begin{aligned}
D &= \frac{1}{2} c_{\mathrm{u}} (\dot{x}_{\mathrm{u}})^2 + \frac{1}{2} c_{\mathrm{s}} (\dot{x}_{\mathrm{s}} - \dot{x}_{\mathrm{u}})^2 + \frac{1}{2} c_{\mathrm{d}} (\dot{x}_{\mathrm{d}} - \dot{x}_{\mathrm{s}})^2 \\
&= \frac{1}{2} \begin{bmatrix} \dot{x}_{\mathrm{u}} & \dot{x}_{\mathrm{s}} & \dot{x}_{\mathrm{d}} \end{bmatrix} \begin{bmatrix} c_{\mathrm{u}} + c_{\mathrm{s}} & -c_{\mathrm{s}} & 0 \\ -c_{\mathrm{s}} & c_{\mathrm{s}} + c_{\mathrm{d}} & -c_{\mathrm{d}} \\ 0 & -c_{\mathrm{d}} & c_{\mathrm{d}} \end{bmatrix} \begin{bmatrix} \dot{x}_{\mathrm{u}} \\ \dot{x}_{\mathrm{s}} \\ \dot{x}_{\mathrm{d}} \end{bmatrix} \\
&= \frac{1}{2} \dot{\mathbf{x}}^{\mathrm{T}} [c] \dot{\mathbf{x}}
\end{aligned} \tag{2.738}$$

Employing the quadrature derivative method, we may find the derivatives of K, V, and D with respect to their variable vectors:

$$\begin{aligned}
\frac{\partial K}{\partial \dot{\mathbf{x}}} &= \frac{1}{2} \left([m] + [m]^{\mathrm{T}} \right) \dot{\mathbf{x}} = \frac{1}{2} \left([k] + [k]^{\mathrm{T}} \right) \begin{bmatrix} \dot{x}_{\mathrm{u}} \\ \dot{x}_{\mathrm{s}} \\ \dot{x}_{\mathrm{d}} \end{bmatrix} \\
&= \begin{bmatrix} m_{\mathrm{u}} & 0 & 0 \\ 0 & m_{\mathrm{s}} & 0 \\ 0 & 0 & m_{\mathrm{d}} \end{bmatrix} \begin{bmatrix} \dot{x}_{\mathrm{u}} \\ \dot{x}_{\mathrm{s}} \\ \dot{x}_{\mathrm{d}} \end{bmatrix}
\end{aligned} \tag{2.739}$$

$$\frac{\partial V}{\partial \mathbf{x}} = \frac{1}{2}\left([k]+[k]^{\mathrm{T}}\right)\mathbf{x} = \frac{1}{2}\left([k]+[k]^{\mathrm{T}}\right)\begin{bmatrix} x_{\mathrm{u}} \\ x_{\mathrm{s}} \\ x_{\mathrm{d}} \end{bmatrix}$$
$$= \begin{bmatrix} k_{\mathrm{u}}+k_{\mathrm{s}} & -k_{\mathrm{s}} & 0 \\ -k_{\mathrm{s}} & k_{\mathrm{s}}+k_{\mathrm{d}} & -k_{\mathrm{d}} \\ 0 & -k_{\mathrm{d}} & k_{\mathrm{d}} \end{bmatrix}\begin{bmatrix} x_{\mathrm{u}} \\ x_{\mathrm{s}} \\ x_{\mathrm{d}} \end{bmatrix} \tag{2.740}$$

$$\frac{\partial D}{\partial \dot{\mathbf{x}}} = \frac{1}{2}\left([c]+[c]^{\mathrm{T}}\right)\dot{\mathbf{x}}$$
$$= \frac{1}{2}\left([c]+[c]^{\mathrm{T}}\right)\begin{bmatrix} \dot{x}_{\mathrm{u}} \\ \dot{x}_{\mathrm{s}} \\ \dot{x}_{\mathrm{d}} \end{bmatrix}$$
$$= \begin{bmatrix} c_{\mathrm{u}}+c_{\mathrm{s}} & -c_{\mathrm{s}} & 0 \\ -c_{\mathrm{s}} & c_{\mathrm{s}}+c_{\mathrm{d}} & -c_{\mathrm{d}} \\ 0 & -c_{\mathrm{d}} & c_{\mathrm{d}} \end{bmatrix}\begin{bmatrix} \dot{x}_{\mathrm{u}} \\ \dot{x}_{\mathrm{s}} \\ \dot{x}_{\mathrm{d}} \end{bmatrix} \tag{2.741}$$

Therefore, we find the system's free vibration equations of motion:

$$[m]\ddot{\mathbf{x}} + [c]\dot{\mathbf{x}} + [k]\mathbf{x} = 0 \tag{2.742}$$

$$\begin{bmatrix} m_{\mathrm{u}} & 0 & 0 \\ 0 & m_{\mathrm{s}} & 0 \\ 0 & 0 & m_{\mathrm{d}} \end{bmatrix}\begin{bmatrix} \ddot{x}_{\mathrm{u}} \\ \ddot{x}_{\mathrm{s}} \\ \ddot{x}_{\mathrm{d}} \end{bmatrix} + \begin{bmatrix} c_{\mathrm{u}}+c_{\mathrm{s}} & -c_{\mathrm{s}} & 0 \\ -c_{\mathrm{s}} & c_{\mathrm{s}}+c_{\mathrm{d}} & -c_{\mathrm{d}} \\ 0 & -c_{\mathrm{d}} & c_{\mathrm{d}} \end{bmatrix}\begin{bmatrix} \dot{x}_{\mathrm{u}} \\ \dot{x}_{\mathrm{s}} \\ \dot{x}_{\mathrm{d}} \end{bmatrix}$$
$$+ \begin{bmatrix} k_{\mathrm{u}}+k_{\mathrm{s}} & -k_{\mathrm{s}} & 0 \\ -k_{\mathrm{s}} & k_{\mathrm{s}}+k_{\mathrm{d}} & -k_{\mathrm{d}} \\ 0 & -k_{\mathrm{d}} & k_{\mathrm{d}} \end{bmatrix}\begin{bmatrix} x_{\mathrm{u}} \\ x_{\mathrm{s}} \\ x_{\mathrm{d}} \end{bmatrix} = 0 \tag{2.743}$$

Example 92 ★ (Different $[m]$, $[c]$, and $[k]$ arrangements) Mass, damping, and stiffness matrices $[m]$, $[c]$, $[k]$ for a vibrating system may be arranged in different forms with the same overall kinetic energy K, potential energy V, and dissipation function D. For example, the potential energy V for the quarter car model that is shown in Fig. 2.38 may be expressed by different $[k]$:

$$V = \frac{1}{2}k_{\mathrm{u}}(x_{\mathrm{u}})^2 + \frac{1}{2}k_{\mathrm{s}}(x_{\mathrm{s}}-x_{\mathrm{u}})^2 + \frac{1}{2}k_{\mathrm{d}}(x_{\mathrm{d}}-x_{\mathrm{s}})^2 \tag{2.744}$$

$$V = \frac{1}{2}\mathbf{x}^{\mathrm{T}}\begin{bmatrix} k_{\mathrm{u}}+k_{\mathrm{s}} & -k_{\mathrm{s}} & 0 \\ -k_{\mathrm{s}} & k_{\mathrm{s}}+k_{\mathrm{d}} & -k_{\mathrm{d}} \\ 0 & -k_{\mathrm{d}} & k_{\mathrm{d}} \end{bmatrix}\mathbf{x} \tag{2.745}$$

$$V = \frac{1}{2}\mathbf{x}^{\mathrm{T}}\begin{bmatrix} k_{\mathrm{u}}+k_{\mathrm{s}} & -2k_{\mathrm{s}} & 0 \\ 0 & k_{\mathrm{s}}+k_{\mathrm{d}} & -2k_{\mathrm{d}} \\ 0 & 0 & k_{\mathrm{d}} \end{bmatrix}\mathbf{x} \tag{2.746}$$

$$V = \frac{1}{2}\mathbf{x}^{\mathrm{T}} \begin{bmatrix} k_{\mathrm{u}} + k_{\mathrm{s}} & 0 & 0 \\ -2k_{\mathrm{s}} & k_{\mathrm{s}} + k_{\mathrm{d}} & 0 \\ 0 & -2k_{\mathrm{d}} & k_{\mathrm{d}} \end{bmatrix} \mathbf{x} \tag{2.747}$$

The matrices $[m]$, $[c]$, and $[k]$, in K, D, and V, may not be symmetric; however, the matrices $[\underline{m}]$, $[\underline{c}]$, and $[\underline{k}]$ in $\partial K/\partial \dot{\mathbf{x}}$, $\partial D/\partial \dot{\mathbf{x}}$, $\partial V/\partial \mathbf{x}$ are always symmetric.

When a matrix $[a]$ is diagonal, it is symmetric and

$$[a] = [\underline{a}] \tag{2.748}$$

A diagonal matrix cannot be written in different forms. The matrix $[m]$ in Example 91 is diagonal and, hence, K has only one form, (2.736).

Example 93 (Quadratic form and sum of squares) We can write the sum of x_i^2 in the quadratic form,

$$\sum_{i=1}^{n} x_i^2 = \mathbf{x}^{\mathrm{T}}\mathbf{x} = \mathbf{x}^{\mathrm{T}} I \mathbf{x} \tag{2.749}$$

where

$$\mathbf{x}^{\mathrm{T}} = \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_n \end{bmatrix} \tag{2.750}$$

and I is an $n \times n$ identity matrix. If we are looking for the sum of squares around a mean value x_0, then

$$\begin{aligned} \sum_{i=1}^{n} (x_i - x_0)^2 &= \sum_{i=1}^{n} x_i^2 - n x_0^2 = \mathbf{x}^{\mathrm{T}}\mathbf{x} - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} x_i\right) \\ &= \mathbf{x}^{\mathrm{T}}\mathbf{x} - \frac{1}{n}\left(\mathbf{x}^{\mathrm{T}}\mathbf{1}_n\right)\left(\mathbf{1}_n^{\mathrm{T}}\mathbf{x}\right) \\ &= \mathbf{x}^{\mathrm{T}}\left(\mathbf{I} - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^{\mathrm{T}}\right)\mathbf{x} \end{aligned} \tag{2.751}$$

where

$$\mathbf{1}_n^{\mathrm{T}} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \tag{2.752}$$

Example 94 ★ (Positive definite matrix) A matrix $[a]$ is called *positive definite* if $\mathbf{x}^{\mathrm{T}}[a]\mathbf{x} > 0$ for all $\mathbf{x} \neq 0$. A matrix $[a]$ is called *positive semidefinite* if $\mathbf{x}^{\mathrm{T}}[a]\mathbf{x} \geq 0$ for all $\mathbf{x}$.

The kinetic energy is positive definite, and this means we cannot have $K = 0$ unless $\dot{\mathbf{x}} = 0$. The potential energy is positive semidefinite and this means that we have $V \geq 0$ as long as $\mathbf{x} > 0$; however, it is possible to have a especial $\mathbf{x}_0 > 0$ at which $V = 0$.

A positive definite matrix, such as the mass matrix $[m]$, satisfies Silvester's criterion, which is that the determinant of $[m]$ and determinant of all the diagonal minors must be positive:

$$\Delta_n = \begin{vmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{vmatrix} > 0$$

$$\Delta_{n-1} = \begin{vmatrix} m_{11} & m_{12} & \cdots & m_{1,n-1} \\ m_{21} & m_{22} & \cdots & m_{2,n-1} \\ \vdots & \ddots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{n-1,n-1} \end{vmatrix} > 0$$

$$\cdots \Delta_2 = \begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} > 0 \qquad \Delta_1 = m_{11} > 0 \tag{2.753}$$

Example 95 ★ (Symmetric matrices) Employing the Lagrange method guarantees that the coefficient matrices of equations of motion of linear vibrating systems are symmetric. A matrix $[A]$ is symmetric if the columns and rows of $[A]$ are interchangeable, so $[A]$ is equal to its transpose:

$$[A] = [A]^{\mathrm{T}} \tag{2.754}$$

The characteristic equation of a symmetric matrix $[A]$ is a polynomial for which all the roots are real. Therefore, the eigenvalues of $[A]$ are real and distinct and $[A]$ is diagonalizable.

Any two eigenvectors that come from distinct eigenvalues of the symmetric matrix $[A]$ are orthogonal.

Example 96 (Linearization of energies) The kinetic energy of a system with n particles is

$$K = \frac{1}{2}\sum_{i=1}^{n} m_i \left(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2\right) = \frac{1}{2}\sum_{i=1}^{3n} m_i \dot{u}_i^2 \tag{2.755}$$

Expressing the configuration coordinate u_i in terms of generalized coordinates q_j, we have

$$\dot{u}_i = \sum_{s=1}^{n} \frac{\partial u_i}{\partial q_s}\dot{q}_s + \frac{\partial u_i}{\partial t} \qquad s = 1, 2, \ldots, N \tag{2.756}$$

Therefore, the kinetic energy in terms of generalized coordinates is

$$\begin{aligned} K &= \frac{1}{2}\sum_{i=1}^{N} m_i \left(\sum_{s=1}^{n} \frac{\partial u_i}{\partial q_s}\dot{q}_s + \frac{\partial u_i}{\partial t}\right)^2 \\ &= \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}\dot{q}_j\dot{q}_k + \sum_{j=1}^{n} b_j\dot{q}_j + c \end{aligned} \tag{2.757}$$

where

$$a_{jk} = \sum_{i=1}^{N} m_i \frac{\partial u_i}{\partial q_j}\frac{\partial u_i}{\partial q_k} \tag{2.758}$$

$$b_j = \sum_{i=1}^{N} m_i \frac{\partial u_i}{\partial q_j}\frac{\partial u_i}{\partial t} \tag{2.759}$$

$$c = \frac{1}{2}\sum_{i=1}^{N} m_i \left(\frac{\partial u_i}{\partial t}\right)^2 \tag{2.760}$$

where

$$\begin{aligned} \left(\sum_{s=1}^{n} \frac{\partial u_i}{\partial q_s}\dot{q}_s + \frac{\partial u_i}{\partial t}\right)^2 &= \left(\sum_{j=1}^{n} \frac{\partial u_i}{\partial q_j}\dot{q}_j + \frac{\partial u_i}{\partial t}\right)\left(\sum_{k=1}^{n} \frac{\partial u_i}{\partial q_k}\dot{q}_k + \frac{\partial u_i}{\partial t}\right) \\ &= \sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial u_i}{\partial q_j}\frac{\partial u_i}{\partial q_k}\right)\dot{q}_j\dot{q}_k \\ &\quad + 2\sum_{j=1}^{n} \frac{\partial u_i}{\partial q_j}\frac{\partial u_i}{\partial t}\dot{q}_j + \left(\frac{\partial u_i}{\partial t}\right)^2 \end{aligned} \tag{2.761}$$

Using these expressions, we may show the kinetic energy of the dynamic system:

$$K = K_0 + K_1 + K_2 \tag{2.762}$$

where

$$K_0 = \frac{1}{2}\sum_{i=1}^{N} m_i \left(\frac{\partial u_i}{\partial t}\right)^2 \tag{2.763}$$

$$K_1 = \sum_{j=1}^{n}\sum_{i=1}^{N} m_i \frac{\partial u_i}{\partial q_j}\frac{\partial u_i}{\partial t}\dot{q}_j \tag{2.764}$$

$$K_2 = \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{i=1}^{N} m_i \frac{\partial u_i}{\partial q_j} \frac{\partial u_i}{\partial q_k} \dot{q}_j \dot{q}_k \tag{2.765}$$

where

$$a_{kj} = a_{jk} = \sum_{i=1}^{N} m_i \frac{\partial u_i}{\partial q_k} \frac{\partial u_i}{\partial q_j} = \sum_{i=1}^{N} m_i \frac{\partial \dot{u}_i}{\partial \dot{q}_k} \frac{\partial \dot{u}_i}{\partial \dot{q}_j} \tag{2.766}$$

If the coordinates u_i do not depend explicitly on time t, then $\partial u_i / \partial t = 0$, and we have

$$\begin{aligned} K = K_2 &= \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk} \dot{q}_j \dot{q}_k = \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 K(0)}{\partial q_j \partial q_i} \dot{q}_j \dot{q}_k \\ &= \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} m_{ij} \dot{q}_j \dot{q}_k \end{aligned} \tag{2.767}$$

The kinetic energy is a scalar quantity and, because of (2.755), must be positive definite. The first term of (2.757) is a positive quadratic form. The third term of (2.757) is also a nonnegative quantity, as indicated by (2.760). The second term of (2.755) can be negative for some $\dot{q}_j$ and t. However, because of (2.755), the sum of all three terms of (2.757) must be positive:

$$K = \frac{1}{2} \dot{\mathbf{q}}^{\mathrm{T}} [m] \dot{\mathbf{q}} \tag{2.768}$$

The generalized coordinate q_i represent deviations from equilibrium. The potential energy V is a continuous function of generalized coordinates q_i and, hence, its expansion is

$$V(\mathbf{q}) = V(0) + \sum_{i=1}^{n} \frac{\partial V(0)}{\partial q_i} q_i + \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial^2 V(0)}{\partial q_j \, \partial q_i} q_i q_j + \cdots \tag{2.769}$$

where $\partial V(0)/\partial q_i$ and $\partial^2 V(0)/(\partial q_j \partial q_i)$ are the values of $\partial V/\partial q_i$ and $\partial^2 V/(\partial q_j \partial q_i)$ at $\mathbf{q} = 0$, respectively. By assuming $V(0) = 0$, and knowing that the first derivative of V is zero at equilibrium

$$\frac{\partial V}{\partial q_i} = 0 \quad i = 1, 2, 3, \ldots, n \tag{2.770}$$

we have the second-order approximation

$$V(\mathbf{q}) = \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} k_{ij} q_i q_j \tag{2.771}$$

$$k_{ij} = \frac{\partial^2 V(0)}{\partial q_j \, \partial q_i} \tag{2.772}$$

where k_{ij} are the elastic coefficients. The second-order approximation of V is zero only at $\mathbf{q} = 0$. The expression (2.771) can also be written as a quadrature:

$$V = \frac{1}{2}\mathbf{q}^{\mathrm{T}}[k]\mathbf{q} \tag{2.773}$$

2.7 ★ Variational Dynamics

Consider a function f of $x(t)$, $\dot{x}(t)$, and t:

$$f = f(x, \dot{x}, t) \tag{2.774}$$

The unknown variable $x(t)$, which is a function of the independent variable t, is called a *path*. Let us assume that the path is connecting the fixed points x_0 and x_{f} during a given time $t = t_{\mathrm{f}} - t_0$. So $x = x(t)$ satisfies the boundary conditions

$$x(t_0) = x_0 \qquad x(t_{\mathrm{f}}) = x_{\mathrm{f}} \tag{2.775}$$

The time integral of the function f over $x_0 \le x \le x_{\mathrm{f}}$ is $J(x)$ such that its value depends on the path $x(t)$:

$$J(x) = \int_{t_0}^{t_{\mathrm{f}}} f(x, \dot{x}, t)\, \mathrm{d}t \tag{2.776}$$

where $J(x)$ is called an *objective function* or a *functional*.

The particular path $x(t)$ that minimizes $J(x)$ must satisfy the following equation:

$$\frac{\partial f}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial \dot{x}} = 0 \tag{2.777}$$

This equation is the *Lagrange* or *Euler–Lagrange differential equation* and is in general of second order.

Proof To show that a path $x = x^{\bigstar}(t)$ is a minimizing path for the functional $J(x) = \int_{t_0}^{t_{\mathrm{f}}} f(x, \dot{x}, t)\, \mathrm{d}t$ with boundary conditions (2.775), we need to show that

$$J(x) \ge J\left(x^{\bigstar}\right) \tag{2.778}$$

for all continuous paths $x(t)$. Any path $x(t)$ satisfying the boundary conditions (2.775) is called an *admissible path*. To see that $x^{\bigstar}(t)$ is the optimal path, we may examine the integral J for every admissible path. Let us define an admissible path by superposing another admissible path $y(t)$ onto $x^{\bigstar}$,

$$x(t) = x^{\bigstar} + \epsilon y(t) \tag{2.779}$$

where

$$y(t_0) = y(t_f) = 0 \tag{2.780}$$

and ϵ is a small parameter

$$\epsilon \ll 1 \tag{2.781}$$

Substituting $x(t)$ in J of Eq. (2.776) and subtracting from $J(x^\star)$ provides us with ΔJ:

$$\begin{aligned} \Delta J &= J\big(x^\star + \epsilon y(t)\big) - J\big(x^\star\big) \\ &= \int_{t_0}^{t_f} f\big(x^\star + \epsilon y, \dot{x}^\star + \epsilon \dot{y}, t\big)\,\mathrm{d}t - \int_{t_0}^{t_f} f\big(x^\star, \dot{x}^\star, t\big)\,\mathrm{d}t \end{aligned} \tag{2.782}$$

Let us expand $f(x^\star + \epsilon y, \dot{x}^\star + \epsilon \dot{y}, t)$ about $(x^\star, \dot{x}^\star)$

$$\begin{aligned} &f\big(x^\star + \epsilon y, \dot{x}^\star + \epsilon \dot{y}, t\big) \\ &\quad = f\big(x^\star, \dot{x}^\star, t\big) + \epsilon\left(y\frac{\partial f}{\partial x} + \dot{y}\frac{\partial f}{\partial \dot{x}}\right) \\ &\qquad + \epsilon^2\left(y^2\frac{\partial^2 f}{\partial x^2} + 2y\dot{y}\frac{\partial^2 f}{\partial x\,\partial \dot{x}} + \dot{y}^2\frac{\partial^2 f}{\partial \dot{x}^2}\right)\mathrm{d}t + \mathrm{O}\big(\epsilon^3\big) \end{aligned} \tag{2.783}$$

and find

$$\Delta J = \epsilon V_1 + \epsilon^2 V_2 + \mathrm{O}\big(\epsilon^3\big) \tag{2.784}$$

where

$$V_1 = \int_{t_0}^{t_f}\left(y\frac{\partial f}{\partial x} + \dot{y}\frac{\partial f}{\partial \dot{x}}\right)\mathrm{d}t \tag{2.785}$$

$$V_2 = \int_{t_0}^{t_f}\left(y^2\frac{\partial^2 f}{\partial x^2} + 2y\dot{y}\frac{\partial^2 f}{\partial x\,\partial \dot{x}} + \dot{y}^2\frac{\partial^2 f}{\partial \dot{x}^2}\right) \tag{2.786}$$

The first integral, V_1, is called the *first variation* of J, and the second integral, V_2, is called the *second variation* of J. All the higher variations are combined and shown as $\mathrm{O}(\epsilon^3)$. If $x^\star$ is the minimizing path, then it is necessary that $\Delta J \geq 0$ for every admissible $y(t)$. If we divide ΔJ by ϵ and make $\epsilon \to 0$, then we find a necessary condition for $x^\star$ to be the optimal path as $V_1 = 0$. This condition is equivalent to

$$\int_{t_0}^{t_f}\left(y\frac{\partial f}{\partial x} + \dot{y}\frac{\partial f}{\partial \dot{x}}\right)\mathrm{d}t = 0 \tag{2.787}$$

By integrating by parts, we may write

$$\int_{t_0}^{t_f} \dot{y}\frac{\partial f}{\partial \dot{x}}\,\mathrm{d}t = \left(y\frac{\partial f}{\partial \dot{x}}\right)_{t_0}^{t_f} - \int_{t_0}^{t_f} y\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}}\right)\mathrm{d}t \tag{2.788}$$

Because of $y(t_0) = y(t_f) = 0$, the first term on the right-hand side is zero. Therefore, the minimization integral condition (2.787) for every admissible $y(t)$ reduces to

$$\int_{t_0}^{t_f} y\left(\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}}\right) dt = 0 \tag{2.789}$$

The terms in parentheses are continuous functions of t, evaluated on the optimal path $x^\bigstar$, and they do not involve $y(t)$. Therefore, the only way for the bounded integral of the parentheses, $(\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}})$, multiplied by a nonzero function $y(t)$ from t_0 and t_f, to be zero is

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}} = 0 \tag{2.790}$$

Equation (2.790) is a necessary condition for $x = x^\bigstar(t)$ to be a solution of the minimization problem (2.776). This differential equation is called the *Euler–Lagrange* or *Lagrange* equation. The second necessary condition to have $x = x^\bigstar(t)$ as a minimizing solution is that the second variation, evaluated on $x^\bigstar(t)$, must be negative. □

Example 97 ★ (Basic lemma) Consider two fixed points x_1 and $x_2 (> x_1)$ and $g(x)$ as a continuous function for $x_1 \le x \le x_2$. If

$$\int_{x_1}^{x_2} f(x) g(x)\, dx = 0 \tag{2.791}$$

for every choice of the continuous and differentiable function $f(x)$ for which

$$f(x_1) = f(x_2) = 0 \tag{2.792}$$

then

$$g(x) = 0 \tag{2.793}$$

identically in $x_1 \le x \le x_2$. This result is called the *basic lemma*.

To prove the lemma, let us assume that (2.793) does not hold. Therefore, suppose there is a particular x_0 of x in $x_1 \le x_0 \le x_2$ for which $g(x_0) \ne 0$. At the moment, let us assume that $g(x_0) > 0$. Because $g(x)$ is continuous, there must be an interval around x_0 such as $x_{10} \le x_0 \le x_{20}$ in which $g(x) > 0$ everywhere. However, (2.791) cannot then hold for every permissible choice of $f(x)$. A similar contradiction is reached if we assume $g(x_0) < 0$. Therefore, the lemma is correct.

Example 98 ★ (Lagrange equation for extremizing $J = \int_1^2 \dot{x}^2\, dt$) The Lagrange equation for extremizing the functional J,

$$J = \int_1^2 \dot{x}^2\, dt \tag{2.794}$$

is

$$\frac{\partial f}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial \dot{x}} = -\ddot{x} = 0 \tag{2.795}$$

which shows that the optimal path is

$$x = C_1 t + C_2 \tag{2.796}$$

The boundary conditions $x(1)$, $x(2)$ provide C_1 and C_2. For example, assuming boundary conditions $x(1) = 0$, $x(2) = 3$ provides us with

$$x = 3t - 3 \tag{2.797}$$

Example 99 ★ (Geodesics) The problem of determining the shortest path between two given points at the same level of a quantitative characteristic is called the geodesic problem.

An example of a geodesic problem is: What is the shortest arc lying on the surface of a sphere and connecting two given points? We can generalize the problem as follows.

Given two points on the surface of

$$g(x, y, z) = 0 \tag{2.798}$$

what is the equation of the shortest arc lying on (2.798) and connecting the points? Let us express the equation of the surface in parametric form using parameters u and v:

$$x = x(u, v) \qquad y = (u, v) \qquad z = z(u, v) \tag{2.799}$$

the differential of the arc length may be written as

$$\begin{aligned}(\mathrm{d}s)^2 &= (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2 \\ &= P(u, v)(\mathrm{d}u)^2 + 2Q(u, v)\,\mathrm{d}u\,\mathrm{d}v + R(u, v)(\mathrm{d}v)^2\end{aligned} \tag{2.800}$$

where

$$P(u, v) = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 \tag{2.801}$$

$$R(u, v) = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 \tag{2.802}$$

$$Q(u, v) = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} \tag{2.803}$$

If the curves $u = \text{const}$ are orthogonal to the curves $v = \text{const}$, the quantity Q is zero. If the given fixed points on the surface are (u_1, v_1) and (u_2, v_2) with $u_2 > u_1$ and we express the arcs and points by

$$v = v(u) \qquad v(u_1) = v_1 \qquad v(u_2) = v_2 \tag{2.804}$$

then the length of the arc is given by

$$J = \int_{u_1}^{u_2} \sqrt{P(u,v) + 2Q(u,v)\frac{\mathrm{d}v}{\mathrm{d}u} + R(u,v)\left(\frac{\mathrm{d}v}{\mathrm{d}u}\right)^2} \tag{2.805}$$

Our problem, then, is to find the function $v(u)$ that renders the integral (2.805) a minimum. Employing the Lagrange equation, we find

$$\frac{\frac{\partial P}{\partial v} + 2\frac{\mathrm{d}v}{\mathrm{d}u}\frac{\partial Q}{\partial v} + \left(\frac{\mathrm{d}v}{\mathrm{d}u}\right)^2\frac{\partial R}{\partial v}}{2\sqrt{P + 2Q\frac{\mathrm{d}v}{\mathrm{d}u} + R\left(\frac{\mathrm{d}v}{\mathrm{d}u}\right)^2}} - \frac{\mathrm{d}v}{\mathrm{d}u}\left(\frac{Q + R\frac{\mathrm{d}v}{\mathrm{d}u}}{\sqrt{P + 2Q\frac{\mathrm{d}v}{\mathrm{d}u} + R\left(\frac{\mathrm{d}v}{\mathrm{d}u}\right)^2}}\right) = 0 \tag{2.806}$$

In the special case where P, Q, and R are explicitly functions of u alone, this last result becomes

$$\frac{Q + R\frac{\mathrm{d}v}{\mathrm{d}u}}{\sqrt{P + 2Q\frac{\mathrm{d}v}{\mathrm{d}u} + R\left(\frac{\mathrm{d}v}{\mathrm{d}u}\right)^2}} = C_1 \tag{2.807}$$

If the curves $u = \text{const}$ are orthogonal to the curves $v = \text{const}$, we have

$$v = C_1 \int \frac{\sqrt{P}\,\mathrm{d}u}{\sqrt{R^2 - C_1^2 R}} \tag{2.808}$$

Still supposing that $Q = 0$ but having P and R as explicit functions of v alone, we have

$$u = C_1 \int \frac{\sqrt{R}\,\mathrm{d}v}{\sqrt{P^2 - C_1^2 P}} \tag{2.809}$$

As a particular case let us consider the geodesic connecting two points on a sphere with radius r. The most convenient parameters u and v for describing position on the sphere surface are the colatitude θ and the longitude φ:

$$x = r\cos\theta\sin\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = r\cos\varphi \tag{2.810}$$

where θ is the angle between the positive z-axis and the line drawn from the sphere center to the designated point and φ is the angle between the (x,z)-plane and the half plane bounded by the z-axis and containing the designated point. Therefore,

$$P = r^2\sin^2\theta \qquad R = r \qquad Q = 0 \tag{2.811}$$

$$\varphi = C_1 \int \frac{\mathrm{d}v}{\sqrt{r^2\sin^4\theta - C_1^2\sin^2\theta}} = -\sin^{-1}\frac{\cot\theta}{\sqrt{\left(\frac{r}{C_1}\right)^2 - 1}} + C_2 \tag{2.812}$$

from which it follows that

$$r\sin\theta\cos\varphi\sin C_2 - r\sin\theta\sin\varphi\cos C_2 - \frac{z}{\sqrt{\left(\frac{r}{C_1}\right)^2 - 1}} = 0 \tag{2.813}$$

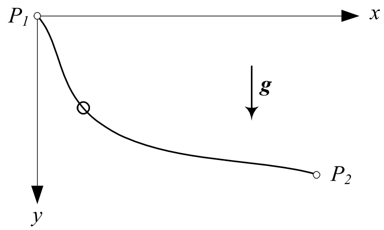


Fig. 2.39 A curve joining points P_1 and P_2 and a frictionless sliding point

Using (2.810) we find that the sphere geodesic lies on the following plane, which passes through the center of the sphere:

$$x \sin C_2 - y \cos C_2 - \frac{z}{\sqrt{\left(\frac{r}{C_1}\right)^2 - 1}} = 0 \tag{2.814}$$

Therefore the shortest arc connecting two points on the surface of a sphere is the intersection of the sphere with the plane containing the given points and the center of the sphere. Such an arc is called a *great-circle* arc.

Example 100 ★ (Brachistochrone problem) We may use the Lagrange equation and find the frictionless curve joining points P_1 and P_2, as shown in Fig. 2.39, along which a particle falling from rest due to gravity travels from the higher to the lower point in minimum time. This is called the *brachistochrone* problem.

If v is the velocity of the falling point along the curve, then the time required to fall an arc length $\mathrm{d}s$ is $\mathrm{d}s/v$. Then the objective function to find the curve of minimum time is

$$J = \int_{s_1}^{s_2} \frac{\mathrm{d}s}{v} \tag{2.815}$$

However,

$$\mathrm{d}s = \sqrt{1 + y'^2}\,\mathrm{d}x \qquad y' = \frac{\mathrm{d}y}{\mathrm{d}x} \tag{2.816}$$

and according to the law of conservation of energy, we have

$$v = \sqrt{2gy} \tag{2.817}$$

Therefore, the objective function simplifies to

$$J = \int_{x_1}^{x_2} \sqrt{\frac{1 + y'^2}{2gy}}\,\mathrm{d}x \tag{2.818}$$

Applying the Lagrange equations, we find

$$y\left(1 + y'^2\right) = 2R \tag{2.819}$$

where R is a constant. The optimal curve starting from $y(0) = 0$ can be expressed by the two parametric equations

$$x = R(\theta - \sin\theta) \qquad y = R(1 - \cos\theta) \tag{2.820}$$

The optimal curve is a *cycloid*.

The name of the problem is derived from the Greek words "$\beta\rho\alpha\chi\iota\sigma\tau o\zeta$," meaning "shortest," and "$\chi\rho o\nu o\zeta$," meaning "time." The brachistochrone problem was originally discussed in 1630 by Galileo Galilei (1564–1642) and was solved in 1696 by Johann and Jacob Bernoulli.

Example 101 ★ (Lagrange multiplier) Assume $f(x)$ is defined on an open interval $x \in (a, b)$ and has continuous first and second-order derivatives in some neighborhood of $x_0 \in (a, b)$. The point x_0 is a local extremum of $f(x)$ if

$$\frac{\mathrm{d}f(x_0)}{\mathrm{d}x} = 0 \tag{2.821}$$

Assume that $f(\mathbf{x}) = 0$, $\mathbf{x} \in \mathbb{R}^n$, and $g_i(\mathbf{x}) = 0$, $i = 1, 2, \ldots, j$, are functions defined on an open region $\mathbb{R}^n$ and have continuous first- and second-order derivatives in $\mathbb{R}^n$. The necessary condition that $\mathbf{x}_0$ is an extremum of $f(\mathbf{x})$ subject to the constraints $g_i(\mathbf{x}) = 0$ is that there exist j *Lagrange multipliers* λ_i, $i = 1, 2, \ldots, j$, such that

$$\nabla\Big(s + \sum \lambda_i g_i\Big) = 0 \tag{2.822}$$

For example, we can find the minimum of f,

$$f = 1 - x_1^2 - x_2^2 \tag{2.823}$$

subject to the constraint g

$$g = x_1^2 + x_2 - 1 = 0 \tag{2.824}$$

by finding the gradient of $f + \lambda g$:

$$\nabla\big(1 - x_1^2 - x_2^2 + \lambda\big(x_1^2 + x_2 - 1\big)\big) = 0 \tag{2.825}$$

which leads to

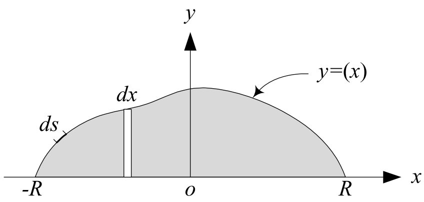
$$\frac{\partial f}{\partial x_1} = -2x_1 + 2\lambda x_1 = 0 \tag{2.826}$$

$$\frac{\partial f}{\partial x_2} = -2x_2 + \lambda = 0 \tag{2.827}$$

To find the three unknowns x_1, x_2, and λ, we employ Eqs. (2.826), (2.827), and (2.824). There are two sets of solutions as follows:

$$\begin{array}{lll} x_1 = 0 & x_2 = 1 & \lambda = 2 \\ x_1 = \pm 1/\sqrt{2} & x_2 = 1/2 & \lambda = 1 \end{array} \tag{2.828}$$

Fig. 2.40 Dido problem is to find a planar curve $y(x)$ with a constant length l to maximize the enclosed area



Example 102 (Dido problem) Consider a planar curve $y(x)$ with a constant length l that connects the points $(-R, 0)$ and $(R, 0)$ as shown in Fig. 2.40. The Dido problem is to find the $y(x)$ that maximized the enclosed area. The objective function of the Dido problem is

$$J = \int_{-R}^{R} y \, \mathrm{d}x \tag{2.829}$$

However, the constant length provides us with a constraint equation:

$$l = \int_{-R}^{R} \mathrm{d}s = \int_{-R}^{R} \sqrt{1 + y'^2} \, \mathrm{d}x \tag{2.830}$$

$$\mathrm{d}s = \sqrt{1 + y'^2} \, \mathrm{d}x \qquad y' = \frac{\mathrm{d}y}{\mathrm{d}x} \tag{2.831}$$

Therefore, using the Lagrange multiplier λ, the objective function with constraint would be

$$J = \int_{-R}^{R} \left(f\left(y, y', x\right) + \lambda g\left(y, y', x\right)\right) \mathrm{d}x \tag{2.832}$$

$$f = y \qquad g = \sqrt{1 + y'^2} \tag{2.833}$$

The Lagrange equation for the constraint objective function (2.832) is

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y'} + \lambda \left(\frac{\partial g}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial g}{\partial y'} \right) = 0 \tag{2.834}$$

Equation (2.834) leads to

$$\frac{1}{\lambda} = \frac{\mathrm{d}}{\mathrm{d}x} \frac{y'}{\sqrt{1 + y'^2}} \tag{2.835}$$

This differential equation must be solved to determine the maximizing curve $y(x)$. First integration provides us with

$$\frac{\lambda y'}{\sqrt{1 + y'^2}} = x + C_1 \tag{2.836}$$

Solving this equation for y' yields

$$y' = \frac{\pm(x + C_1)}{\sqrt{\lambda^2 - (x + C_1)}} \tag{2.837}$$

and a second integration yields

$$y = \pm\sqrt{\lambda^2 - (x + C_1)} + C_2 \tag{2.838}$$

Satisfying the boundary conditions $(-R, 0)$ and $(R, 0)$, we have

$$C_1 = C_2 = 0 \qquad \lambda = R \tag{2.839}$$

which indicates that the function $y(x)$ is

$$x^2 + y^2 = R^2 \tag{2.840}$$

It is a circle with center at O and radius R.

Example 103 ★ (Several independent variables) We now derive the differential equations that must be satisfied by the twice-differentiable functions $q_1(t)$, $q_2(t), \ldots, q_n(t)$ that extremize the integral J:

$$J = \int_{t_1}^{t_2} f(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t)\,\mathrm{d}t \tag{2.841}$$

The functions $q_1(t)$, $q_2(t), \ldots, q_n(t)$ achieve given the values at the fixed limits of integration t_1 and t_2, where $t_1 < t_2$.

Let us show the optimal functions by $q_i^{\bigstar}(t)$, $i = 1, 2, \ldots, n$. We may examine the integral J for every admissible function. An admissible function may be defined by

$$q_i(t) = q_i^{\bigstar} + \epsilon y_i(t) \tag{2.842}$$

where

$$y_i(t_1) = y_i(t_2) = 0 \tag{2.843}$$

and ϵ is a small parameter,

$$\epsilon \ll 1 \tag{2.844}$$

Consider a function $f = f(q_i, \dot{q}_i, t)$. The variables $q_i(t)$ satisfy the boundary conditions

$$q_1(t_1) = q_1 \qquad q(t_2) = q_2 \tag{2.845}$$

Substituting $q_i(t)$ in J and subtracting from (2.841) yields

$$\begin{aligned}\Delta J &= J\left(q_i^{\bigstar} + \epsilon y_i(t)\right) - J\left(q_i^{\bigstar}\right)\\ &= \int_{t_0}^{t_\mathrm{f}} f\left(q_i^{\bigstar} + \epsilon y_i, \dot{q}_i^{\bigstar} + \epsilon \dot{y}_i, t\right) \mathrm{d}t - \int_{t_0}^{t_\mathrm{f}} f\left(q_i^{\bigstar}, \dot{q}_i^{\bigstar}, t\right) \mathrm{d}t \end{aligned} \tag{2.846}$$

Let us expand $f(x^{\bigstar} + \epsilon y, \dot{x}^{\bigstar} + \epsilon \dot{y}, t)$ about $(x^{\bigstar}, \dot{x}^{\bigstar})$:

$$\begin{aligned}& f\left(q_i^{\bigstar} + \epsilon y_i, \dot{q}_i^{\bigstar} + \epsilon \dot{y}_i, t\right)\\ &\quad = f\left(q_i^{\bigstar}, \dot{q}_i^{\bigstar}, t\right) + \epsilon \left(y_i \frac{\partial f}{\partial q_i} + \dot{y}_i \frac{\partial f}{\partial \dot{q}_i} \right)\\ &\qquad + \epsilon^2 \left(y_i^2 \frac{\partial^2 f}{\partial q_i^2} + 2 y_i \dot{y}_j \frac{\partial^2 f}{\partial q_i \, \partial \dot{q}_j} + \dot{y}_i^2 \frac{\partial^2 f}{\partial \dot{q}_i^2} \right) \mathrm{d}t\\ &\qquad + \mathrm{O}\left(\epsilon^3\right)\end{aligned} \tag{2.847}$$

and find

$$\Delta J = \epsilon V_1 + \epsilon^2 V_2 + \mathrm{O}\left(\epsilon^3\right) \tag{2.848}$$

where

$$V_1 = \int_{t_0}^{t_\mathrm{f}} \left(y_i \frac{\partial f}{\partial q_i} + \dot{y}_j \frac{\partial f}{\partial \dot{q}_j} \right) \mathrm{d}t \tag{2.849}$$

$$V_2 = \int_{t_0}^{t_\mathrm{f}} \left(y_i^2 \frac{\partial^2 f}{\partial q_i^2} + 2 y_i \dot{y}_j \frac{\partial^2 f}{\partial q_i \, \partial \dot{q}_j} + \dot{y}_i^2 \frac{\partial^2 f}{\partial \dot{q}_i^2} \right) \tag{2.850}$$

If we divide ΔJ by ϵ and make $\epsilon \to 0$, then we find a necessary condition $V_1 = 0$ for $q_i^{\bigstar}$ to be the optimal path. By integrating V_1 by parts, we may write

$$\int_{t_0}^{t_\mathrm{f}} \dot{y}_1 \frac{\partial f}{\partial \dot{q}_1} \mathrm{d}t = \left(y_1 \frac{\partial f}{\partial \dot{q}_1} \right)_{t_1}^{t_2} - \int_{t_1}^{t_2} y_1 \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial f}{\partial \dot{q}_1} \right) \mathrm{d}t \tag{2.851}$$

Since $y_1(t_1) = y_2(t_2) = 0$, the first term on the right-hand side is zero and the integral of V_1 reduces to

$$\int_{t_1}^{t_2} y_1 \left(\frac{\partial f}{\partial q_1} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial f}{\partial \dot{q}_1} \right) \mathrm{d}t = 0 \tag{2.852}$$

The terms in parentheses are continuous functions of t evaluated on the optimal path $x^{\bigstar}$, and they do not involve $y_1(t)$. So, the only way for the bounded integral of the parentheses, $(\frac{\partial f}{\partial q_1} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial f}{\partial \dot{q}_1})$, multiplied by a nonzero function $y_1(t)$ to be zero is if the parentheses are zero. Therefore, the minimization integral condition for every admissible $y_1(t)$ is

$$\frac{\partial f}{\partial q_1} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial f}{\partial \dot{q}_1} = 0 \tag{2.853}$$

Using a similar treatment of the successive pairs of terms of (2.851), we derive the following n conditions to minimize (2.841):

$$\frac{\partial f}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial \dot{q}_i} = 0 \quad i = 1, 2, \ldots, n \tag{2.854}$$

Therefore, when a definite integral is given which contains n functions to be determined by the condition that the integral be stationary, we can vary these functions independently. So, the Euler–Lagrange equation can be formed for each function separately. This provides us with n differential equations.

2.8 Key Symbols

$a \equiv \ddot{x}$	acceleration
a, b	distance, Fourier series coefficients
a, b, w, h	length
$\mathbf{a}$	acceleration
A, B	weight factor, coefficients for frequency responses
c	damping
$[c]$	damping matrix
c_e	equivalent damping
C	mass center
$\mathbf{d}$	position vector of the body coordinate frame
$\mathrm{d}\mathbf{f}$	infinitesimal force
$\mathrm{d}m$	infinitesimal mass
$\mathrm{d}\mathbf{m}$	infinitesimal moment
E	mechanical energy, Young modulus of elasticity
$f = 1/T$	cyclic frequency [Hz]
$f, F, \mathbf{f}, \mathbf{F}$	force
F_C	Coriolis force
g	gravitational acceleration
H	height
I	moment of inertia matrix
I_1, I_2, I_3	principal moment of inertia
k	stiffness
k_e	equivalent stiffness
k_{ij}	element of row i and column j of a stiffness matrix
$[k]$	stiffness matrix
K	kinetic energy
l	directional line
$\mathbf{L}$	moment of momentum
$\mathcal{L} = K - V$	Lagrangean
m	mass
m_e	eccentric mass, equivalent mass

m_{ij}	element of row i and column j of a mass matrix
m_k	spring mass
m_s	sprung mass
$[m]$	mass matrix
n	number of coils, number of decibels, number of note
N	natural numbers
p	pitch of a coil
P	power
r	frequency ratio
$\mathbf{r}$	position vector
t	time
t_0	initial time
T	period
T_n	natural period
$v \equiv \dot{x}, \mathbf{v}$	velocity
V	potential energy
$x, y, z, \mathbf{x}$	displacement
x_0	initial displacement
$\dot{x}_0$	initial velocity
$\dot{x}, \dot{y}, \dot{z}$	velocity, time derivative of x, y, z
$\ddot{x}$	acceleration
X	amplitude of x
z	relative displacement

Greek

α, β, γ	angle, angle of spring with respect to displacement
δ	deflection, angle
δ_s	static deflection
ε	mass ratio
θ	angular motion coordinate
$\omega, \boldsymbol{\omega}, \Omega$	angular frequency
φ, Φ	phase angle
λ	eigenvalue

2.9 Exercises

1. Equation of motion of vibrating systems.

   Determine the equation of motion of the systems in Figs. 2.41(a) and (b) by

   (a) using the energy method
   (b) using the Newton method
   (c) using the Lagrange method

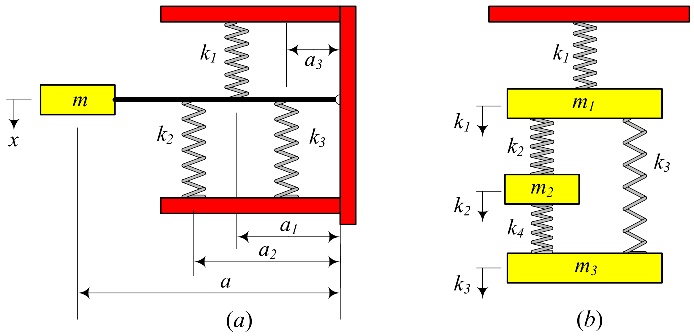


Fig. 2.41 Two undamped discrete vibrating systems

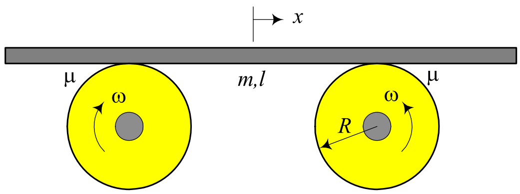


Fig. 2.42 A friction coefficient measurement device

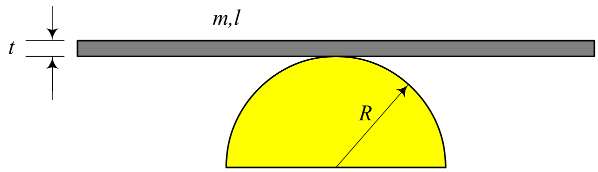


Fig. 2.43 A slab on a semi-cylinder

2. Friction coefficient measurement.

   Figure 2.42 illustrates two rollers that turn in opposite direction with equal angular speed ω. A slab of size $t \times w \times l$ and mass m is put on the rollers. A small disturbance or misplacement will cause the slab to oscillate about the equilibrium position. If the coefficient of friction between the roller and the slab is μ, determine
   (a) the equation of motion of the slab
   (b) the natural frequency of the slab's oscillation for a given μ
   (c) the value of μ for a measured frequency of oscillation ω
3. A slab on a semi-cylinder.

   Determine the equation of motion of the slab $t \times w \times l$ in Figs. 2.43 if
   (a) the thickness of the slab is ignorable, $t = 0$
   (b) ★ the thickness of the slab is not ignorable, $t \neq 0$
4. ★ Velocity dependent friction.

   The device in problem 2 is to measure the friction coefficient between the slab and the rollers.

   Assume the friction force is $f = f_0 - Cv$ where v is the relative velocity of slab and rollers.

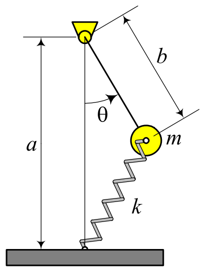


Fig. 2.44 Spring connected pendulum

(a) Determine the equation of motion of the slab if x and $\dot{x}$ are assumed small.
(b) Determine the equation of motion of the slab if $\dot{x}$ is comparable with $R\omega$.
(c) Determine the equation of motion of the slab if one roller is turning with angular speed of 2ω, assuming $\dot{x}$ is very small.

5. Moving on x-axis.
The displacement of a particle moving along the x-axis is given by

$$x = 0.01t^4 - t^3 + 4.5t^2 - 10 \quad t \geq 0$$

(a) Determine t_1 at which x becomes positive.
(b) For how long does x remain positive after $t = t_1$?
(c) How long does it take for x to become positive for the second time?
(d) When and where does the particle reach its maximum acceleration?
(e) Derive an equation to calculate its acceleration when its speed is given.

6. ★ Kinetic energy of a rigid link.
Consider a straight and uniform bar as a rigid link of a manipulator. The link has a mass m. Show that the kinetic energy of the bar can be expressed as

$$K = \frac{1}{6}m(\mathbf{v}_1 \cdot \mathbf{v}_1 + \mathbf{v}_1 \cdot \mathbf{v}_2 + \mathbf{v}_2 \cdot \mathbf{v}_2)$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the velocity vectors of the end points of the link.

7. Ideal spring connected pendulum.
Determine the kinetic and potential energies of the pendulum in Fig. 2.44, at an arbitrary angle θ. The free length of the spring is $l = a - b$.

8. ★ General spring connected pendulum.
Determine the potential energy of the pendulum in Fig. 2.44, at an angle θ, if:
(a) The free length of the spring is $l = a - 1.2b$.
(b) The free length of the spring is $l = a - 0.8b$.

9. ★ Moving on a cycloid.
A particle is moving on a planar curve with the following parametric expression:

$$x = r(\omega t - \sin \omega t) \qquad y = r(1 - \cos \omega t)$$

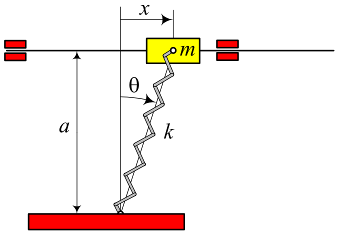


Fig. 2.45 Spring connected rectilinear oscillator

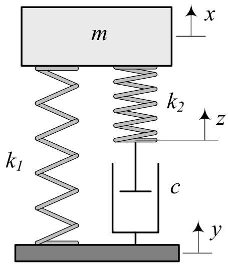


Fig. 2.46 Mathematical model for cushion suspension

(a) Determine the speed of the particle at time t.
(b) Show that the magnitude of acceleration of the particle is constant.
(c) Determine the tangential and normal accelerations of the particle.
(d) Using $\mathrm{d}s = v\,\mathrm{d}t$, determine the length of the path that the particle travels up to time t.
(e) Check if the magnitude of acceleration of the particle is constant for the following path:

$$x = a(\omega t - \sin \omega t) \qquad y = b(1 - \cos \omega t)$$

10. ★ Spring connected rectilinear oscillator.
Determine the kinetic and potential energies of the oscillator shown in Fig. 2.45. The free length of the spring is a.
(a) Express your answers in terms of the variable angle θ.
(b) Express your answers in terms of the variable distance x.
(c) Determine the equation of motion for large and small θ.
(d) Determine the equation of motion for large and small x.
11. ★ Cushion mathematical model.
Figure 2.46 illustrates a mathematical model for cushion suspension. Such a model can be used to analyze the driver's seat, or a rubbery pad suspension.
(a) Derive the equations of motion for the variables x and z and using y as a known input function.
(b) Eliminate z and derive a third-order equation for x.
12. ★ Relative frequency.
Consider a body B that is moving along the x-axis with a constant velocity u and every T seconds emits small particles which move with a constant velocity c along the x-axis. If f denotes the frequency and λ the distance between two

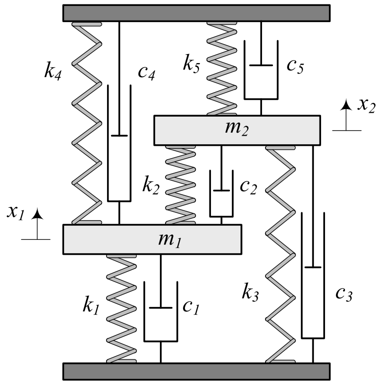


Fig. 2.47 A two *DOF* vibrating system

successively emitted particles, then we have

$$f = \frac{1}{T} = \frac{c-u}{\lambda}$$

Now suppose that an observer moves along the x-axis with velocity v. Let us show the number of particles per second that the observer meets by the relative frequency f' and the time between meeting the two successive particles by the relative period T', where

$$f' = \frac{c-v}{\lambda}$$

Show that

$$f' \approx f\left(1 - \frac{v-u}{c}\right)$$

13. Equation of motion of a multiple *DOF* system.
    Figure 2.47 illustrates a two *DOF* vibrating system.
    (a) Determine the K, V, and D functions.
    (b) Determine the equations of motion using the Lagrange method.
    (c) ★ Rewrite K, V, and D in quadrature form.
    (d) Determine the natural frequencies and mode shapes of the system.
14. Absolute and relative coordinates.
    Figure 2.48 illustrates two similar double pendulums. We express the motion of the left one using absolute coordinates θ_1 and θ_2, and express the motion of the right one with absolute coordinate θ_1 and relative coordinate θ_2.
    (a) Determine the equation of motion of the absolute coordinate double pendulum.
    (b) Determine the equation of motion of the relative coordinate double pendulum.
    (c) Compare their mass and stiffness matrices.
15. Static equilibrium position.
    Figure 2.49 illustrates a combination of pendulums and springs.

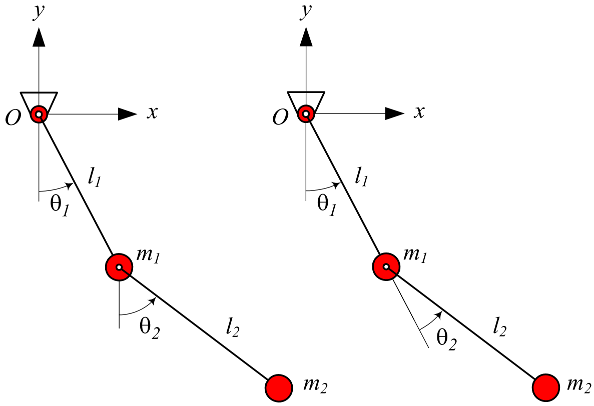


Fig. 2.48 Two similar double pendulums, expressed by absolute and relative coordinates

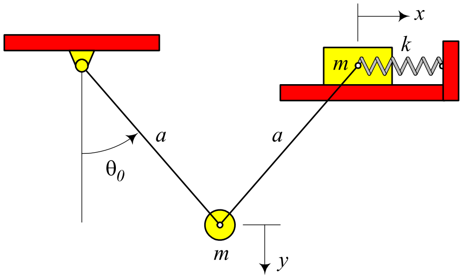


Fig. 2.49 A combination of pendulums and springs

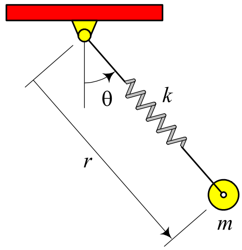


Fig. 2.50 An elastic pendulum

(a) Determine the value of equilibrium θ_0 and static stretch of spring δ_0 if we assemble the system and let it go slowly.
(b) Determine the equation of motion of the system in terms of x measured from the shown equilibrium position.
(c) Determine the equation of motion of the system in terms of y.

16. Elastic pendulum.

Figure 2.50 illustrates an elastic pendulum. Such a pendulum has two *DOF*.

(a) Determine the equations of motion using the energy method.
(b) Determine the equations of motion using the Lagrange method.
(c) Determine the equations of motion using the Newton–Euler method.
(d) Linearize the equations of motion.

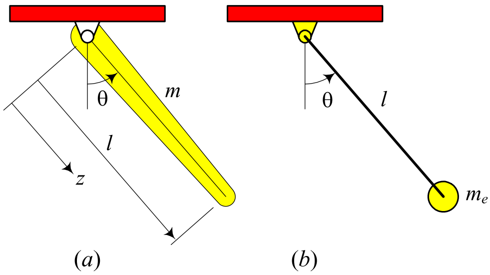


Fig. 2.51 (**a**) A slender as a pendulum with variable density. (**b**) A simple pendulum with the same length

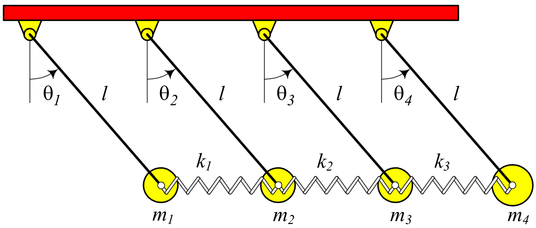


Fig. 2.52 Four connected pendulums

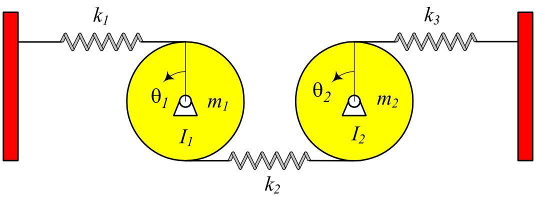


Fig. 2.53 Two spring connected heavy discs

17. Variable density.

    Figure 2.51(a) illustrates a slender as a pendulum with variable density, and Fig. 2.51(b) illustrates a simple pendulum with the same length. Determine the equivalent mass m_e is the mass density $\rho = m/l$ is

    (a) $\rho = C_1 z$
    (b) $\rho = C_2(l - z)$
    (c) $\rho = C_3(z - \frac{l}{2})^2$
    (d) $\rho = C_4(\frac{l}{2} - (z - \frac{l}{2}))^2$

18. Four pendulums are connected as shown in Fig. 2.52.

    (a) Determine the kinetic energy K, linearize the equation and find the mass matrix $[m]$.
    (b) Determine the potential energy V, linearize the equation and find the stiffness matrix $[k]$.
    (c) Determine the equations of motion using K and V and determine the symmetric matrices $[\underline{m}]$ and $[\underline{k}]$.

19. Two spring connected heavy discs.

    The two spring connected disc system of Fig. 2.53 is linear for small θ_1 and θ_2. Find the equations of motion by energy and Lagrange methods.

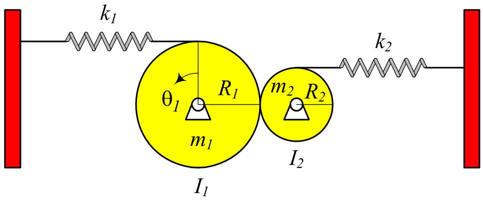


Fig. 2.54 Two connected disc system

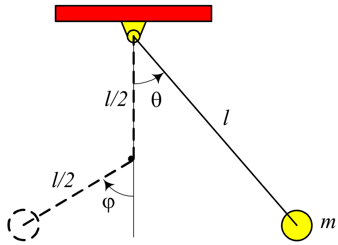


Fig. 2.55 A pendulum and a peg

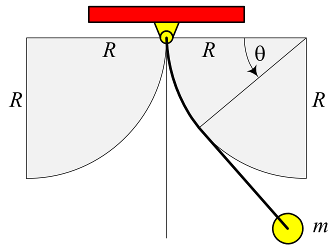


Fig. 2.56 A pendulum on circular wall

20. Two connected heavy discs.

    Determine the equations of motion of the two connected disc system of Fig. 2.54 by energy and Lagrange methods.

21. ★ A pendulum and a peg.

    Determine the equation of motion of the pendulum in Fig. 2.55 using only one variable θ or φ.

22. ★ A pendulum on circular wall.

    Determine the equation of motion of the pendulum in Fig. 2.56 using the variable θ.

23. ★ A wire of the shape $y = f(x)$.

    Consider a wire in an arbitrary shape given by $y = f(x)$ as is shown in Fig. 2.57. Determine the equation of motion of a bid with mass m that is sliding frictionless on the wire.

24. ★ A particle in a cone.

    A particle of mass m slides without friction inside an upside down conical shell of semivertical angle α, as is shown in Fig. 2.58.

    (a) Use the Euler equation $\mathbf{M} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{L}$ to determine the equations of motion of the particle.

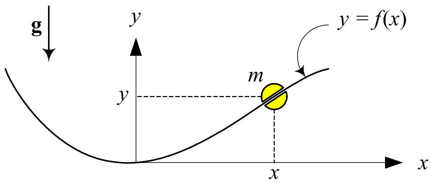


Fig. 2.57 A wire of the shape of $y = f(x)$

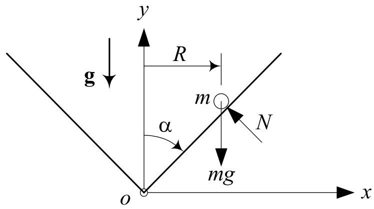


Fig. 2.58 A particle of mass m slides inside a conical shell

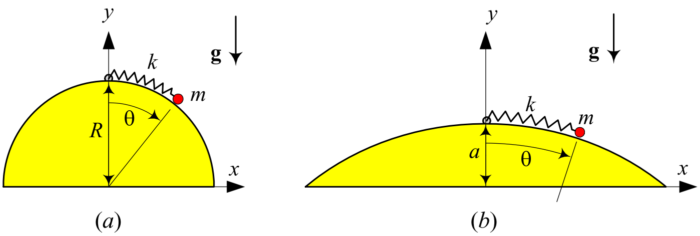


Fig. 2.59 A particle on a circular surface

(b) Show that it is possible for the particle to move such that it is at a constant R with the cone axis.
(c) Determine the angular speed of the particle for a uniform motion of part (b).

25. ★ A particle on a circular surface.
Draw *FBD* of the particle in Figs. 2.59(a) and (b) for $a = cR$, $c < 1$, and determine their equation of motion. The spring is linear and applies a tangential force on m.

26. ★ Falling on a spring.
A solid mass m falls on a spring as shown in Fig. 2.60(a) or with a spring as shown Fig. 2.60(b). The spring exerts a stiffness force F_s. Determine the maximum compression x_{Max} of the springs if:
(a) The restitution coefficient $e = 0$ and $F_s = kx$.
(b) The restitution coefficient $e = 0$ and $F_s = kx^3$.
(c) The restitution coefficient $e = 1$ and $F_s = kx^3$.
We define a *restitution coefficient* e by $v_2' - v_1' = e(v_1 - v_2), 0 \le e \le 1$, where v_1, v_2 are the speed of the two particles before impact and v_1', v_2' are their speeds after impact. The case $e = 1$ indicates an *inelastic collision* in which the particles stick to each other after impact, and the case $e = 0$ is called the *plastic collision* in which the energy conserves in impact.

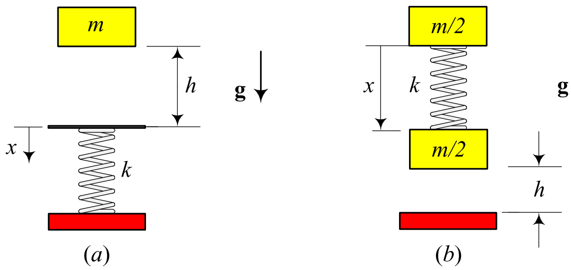


Fig. 2.60 A solid mass m falls on or with a spring

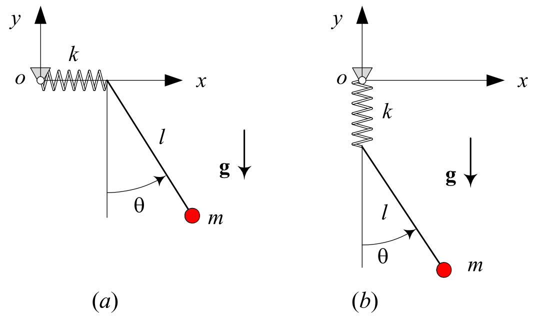


Fig. 2.61 Pendulums with flexible support

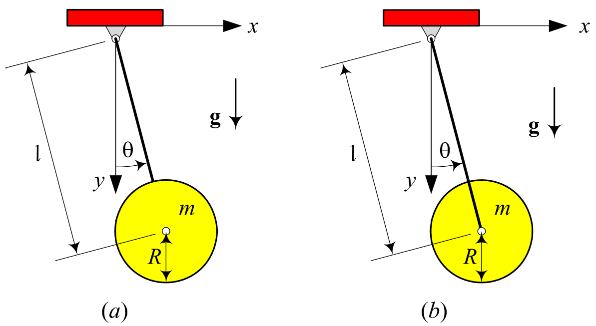


Fig. 2.62 Heavy pendulums, with and without a revolute joint

27. ★ Pendulum with flexible support.

    Figures 2.61(a) and (b) illustrate two pendulums with flexible supports in directions of x and y, respectively. Determine the equations of motion for:
    (a) A pendulum with a flexible support in the x-direction of Fig. 2.61(a).
    (b) A pendulum with a flexible support in the y-direction of Fig. 2.61(b).

28. ★ Heavy pendulum.

    Figure 2.62(a) illustrates a heavy disc with mass m and radius R suspended by a massless rod of length l. Figure 2.62(b) illustrates another heavy disc with mass m and radius R that is attached to a massless rod of length l by a frictionless revolute joint at its center.

(a) Derive the equations of motion for the pendulums in (a) and (b).
(b) Linearize the equations of motion. Is it possible to compare the periods of the oscillations?
(c) ★ Assume that the disc of Fig. 2.62(b) has an angular velocity of ω when $\theta = 0$. Determine the equation of motion, linearize the equation, and determine the period of oscillation.

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