ELEMENTARY TREATMENT OF THE PARKING PROBLEM

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SUMMARY. Line segments of unit length are placed at random on an interval of length $x > 1$ until ν_x fit without overlap and no more can be placed without overlap. It is shown that $E[\nu_x] \equiv M(x) \sim Cx$ as $x \to \infty$. Bounds for C are obtained by approximating $M(x)$ by straight lines ultimately above and below it, respectively.

1. Introduction

Consider the following problem, known as the parking problem, as described by Renyi (1963) : Let us place on the interval $(0, x)$ a segment of unit length at random. By this we mean that, denoting the segment by I_1, the starting point ξ of I_1 is a uniformly distributed probability variable in the interval $(0, x-1)$. Let us place a second unit segment at random (in the same sense) independently of the first, on the interval $(0, x)$. If the second segment has a point in common with the first, we discard it, and repeat the choice until the two segments have no points in common. Having already chosen the first k segments $I_1, I_2, \ldots, I_k$ so that they are mutually exclusive, we retain the next randomly chosen segment I_{k+1} only if it has no point in common with either one of the set $\{I_1, I_2, \ldots, I_k\}$. The process is finished when there is no further possibility of placing still one unit segment so that it will have no point in common with any of the previously placed ones. Let us suppose that this will come about after placing the segment I_{v_x} on the interval. The problem is the determination of the probability variable $M(x) = M\{v_x\}$.

Renyi (1963) solved this problem and it has been attacked by others in the same way (Flatto and Konheim, 1962) and on a modified version (Solomon, 1965). Results on asymptotic normality have been obtained in Mannion (1964), Dvoretzky and Robbins (1964). (See Mannion (1964), Solomon (1965) for further references to work done on this problem. For simulation results in 2 and 3 dimensions, see Solomon (1965)). The method of solution in Renyi (1963), Flatto and Konheim (1962) consists in taking the Laplace transforms of a differential—difference equation for $M(x)$ and then to show by standard Abelian and Tauberian theorems, that $\lim_{x\to\infty} M(x)/x = C$. The expression for C is a 2-fold definite integral which must be numerically computed. To 3 decimal places, $C = .748$ according to Renyi (1963).

It is the purpose of this note to show by means of elementary calculus that $\lim_{x\to\infty} M(x)/x =$ constant, and to give upper and lower bounds for this constant which differ by less than .007. The method of obtaining the bounds is to exhibit straight lines satisfying the same integral equation as $M(x)$, and which ultimately lie above $M(x)$ or below it.

2. Integral equation and limiting properties

Following Renyi (1963) we will first indicate that $M(x)$ satisfies the integral equation

$$\left.\begin{aligned} M(x+1) &= \frac{2}{x}\int_0^x M(t)dt+1 \qquad \text{for } x > 0, \\ \text{where clearly} \qquad M(x) &= 0, \quad 0 \leqslant x < 1. \end{aligned}\right\} \qquad \ldots \quad (1)$$

We arrive at this equation by the following consideration : Having placed a unit segment $(t, t+1)$ on the interval $(0, x+1)$, $0 \leqslant t \leqslant x$, then the average number of segments to the left of $(t, t+1)$ is clearly $M(t)$, while their number to the right of it is $M(x-t)$; and thus, since t is uniformly distributed in $(0, x)$,

$$M(x+1) = \frac{1}{x} \int_0^x (M(t)+M(x-t))dt+1 = \frac{2}{x}\int_0^x M(t)dt+1.$$

Writing (1) as

$$\frac{M(x+1)}{x} = \frac{2}{x^2}\int_0^x M(t)dt+\frac{1}{x},$$

it follows that

$$\left|\left(\frac{M(x+1)}{x}\right)'\right| \leqslant \frac{2}{x^3}\int_0^x M(t)dt+\frac{1}{x^3}+\frac{2M(x)}{x^3} \leqslant \frac{1}{x}+\frac{1}{x^3}+\frac{2}{x^2} = O\left(\frac{1}{x}\right)$$

since $0 \leqslant M(x) \leqslant x$. Therefore $\lim_{x \to \infty} (M(x)/x)' = 0$ and $\lim_{x \to \infty} M(x)/x =$ constant. The constant will be denoted by C.

An immediate lower bound for C may be obtained by noting that for $x \geqslant 1$, the only linear function $L(x) = ax+b$ satisfying the integral equation

$$L(x+1) = \frac{2}{x}\int_1^x L(t)dt+1$$

is the function

$$L_{2/3}(x) \equiv \frac{2}{3}x-\frac{1}{3}.$$

Since $\qquad L_{2/3}(x) \leqslant M(x) = 1 \quad \text{for } 1 \leqslant x < 2,$

it follows from (1) that $\qquad M(x) \geqslant L_{2/3}(x),\ x \geqslant 1.$

Hence

$$C \geqslant \frac{2}{3}.$$

3. Upper and lower bounds

We will need the following result in order to obtain the bounds.

Theorem 1 : *Define* $L_a(x) \equiv ax+a-1$. *If for some* $t > 0$, $L_a(x) \leqslant M(x)$ $(L_a(x) \geqslant M(x))$ *for* $t \leqslant x \leqslant t+1$, *then* $L_a(x) \leqslant M(x)$ $(L_a(x) \geqslant M(x))$ *for all* $x \geqslant t$.

Proof : First note that $L_a(x)$ satisfies the integral equation

$$L_a(x+1) = \frac{2}{x}\int_0^x L_a(t)dt+1.$$

Suppose that $L_a(x) \leqslant M(x)$, $t \leqslant x \leqslant t+1$, since the proof for the reverse inequality is the same. Assume by induction that $L_a(x) \leqslant M(x)$ for $t+k-1 \leqslant x \leqslant t+k$, where $k > 0$ is some integer. Note that the induction hypothesis may be wirtten as

$$M(x)-L_a(x) = \frac{2}{x-1}\left[\int_0^{x-1}[M(t)-L_a(t)]\,dt\right] \geqslant 0$$

for $t+k-1 \leqslant x \leqslant t+k$. Then for $t+k-1 \leqslant x \leqslant t+k$,

$$M(x+1)-L(x+1) = \frac{2}{x}\left[\int_0^x [M(t)-L_a(t)]\,dt\right]$$

$$= \frac{2}{x}\left[\int_0^{x-1} [M(t)-L_a(t)]dt + \int_{x-1}^{x} [M(t)-L_a(t)]\,dt\right] \geqslant 0,$$

using both forms of the induction hypothesis, which completes the proof.

To obtain the bounds it is necessary to recursively compute $M(x)$, $1 \leqslant x < 4$. It is obtained from (1) that

$$M(x) = \begin{cases} 1 & 1 \leqslant x < 2 \\ 3-\dfrac{2}{x-1}, & 2 \leqslant x < 3 \\ 7-\dfrac{10}{x-1}-\dfrac{4\ln(x-2)}{x-1}, & 3 \leqslant x < 4. \end{cases} \qquad \ldots\; (2)$$

We also need the following.

Lemma : $M'(x) > 0$ *for* $x \geqslant 2$.

Proof : From (2) it may be seen that $M'(x) > 0$ for $2 \leqslant x < 4$. Assume $M'(t) > 0$ for $2 \leqslant t \leqslant x$. Differentiating (1), we easily obtain

$$M'(x+1) = \frac{2}{x^2}\int_1^x M(t)dt + \frac{2}{x}M(x)$$

$$\geqslant M(x) - \frac{2M(x)(x-1)}{x^2}$$

$$= \frac{2M(x)}{x^2} > 0$$

using the fact that $M(x) = 0$, $0 \leqslant x \leqslant 1$, and the induction hypothesis. This suffices for the proof.

An upper bound for C is obtained as follows :

Let

$$R(x) \equiv L_{\cdot 75}(x) - M(x) = \frac{3}{4}x - \frac{1}{4} - 7 + \frac{10}{x-1} + \frac{4\ln(x-2)}{x-1}, \quad \text{for } 3 \leqslant x < 4.$$

Note that $R(3) = 0$, $R'(3) > 0$, $R(4) > 0$, $R'(4) < 0$, by direct computation. Since $\frac{3}{4}x$ is increasing, while $\frac{10+4\ln(x-2)}{x-1}$ is decreasing in x, for $3 < x < 4$, $R'(x)$ has at most one sign change in the interval $3 < x < 4$. From this and the conditions given above at $x = 3$ and $x = 4$, it follows that $R(x) > 0$ for $3 \leqslant x \leqslant 4$. By the theorem, $L_{\cdot 75}(x) \geqslant M(x)$ for $x > 3$ and hence $C \leqslant .75$.

To obtain a lower bound for C, note that $M''(3+) > 0$, hence by the lemma, $M(x)$ is concave in a neighbourhood to the right of $x = 3$. We first attempt to find a line $L_{a_0}(x)$ which is tangent to $M(x)$ to the right of $x = 3$ in the region of concavity.

Hence we solve for a_0 and x_0 in

$$L_{a_0}(x_0) = M(x_0) \quad 3 < x_0 < 4,\ 0 < a_0 < \frac{3}{4} \qquad \dots (3)$$

$$L'_{a_0}(x_0) = M'(x_0). \qquad \dots (4)$$

This is clearly

$$a_0x_0-1+a_0 = 7-\frac{10}{x_0-1}-\frac{4\ln(x_0-2)}{x_0-1} \qquad \dots (5)$$

and

$$a_0 = \frac{10}{(x_0-1)^2}+\frac{4\ln(x_0-2)}{(x_0-1)^2}-\frac{4}{(x_0-2)(x_0-1)}. \qquad \dots (6)$$

Solving (5), (6) simultaneously by a trial-and-error method using a desk computer, one obtains $x_0 = 3.2805$ and $a_0 = .7432$ to 4 decimal place accuracy. Next, it is checked that $M''(x_0) > 0$ and that

$$L_{a_0}(x_0-1) \leqslant M(x_0-1), \qquad \dots (7)$$

that is that $a_0x_0-1 \leqslant 3-\frac{2}{x_0-2}$, which checks by computation using 5 decimal place accuracy.

Since $M''(x) < 0$ for $2 < x < 3$, $M''(x_0) > 0$, and $M(x)$ is strictly increasing for $x > 2$, (3), (4) (7) establish that $L_{a_0}(x) \leqslant M(x)$ for $x_0-1 < x \leqslant x_0$. By the theorem, $L_{a_0}(x) \leqslant M(x)$ for all $x \geqslant x_0 = 3.2805$.

We have thus established the following result.

Theorem 2 : *For* $x > 4$, $.7432x-.2568 \leqslant M(x) \leqslant .75x-.25$. *Hence as* $x \to \infty$, $M(x) \sim Cx$, *where* $.7432 \leqslant C \leqslant .75$.

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