July 21, 2003

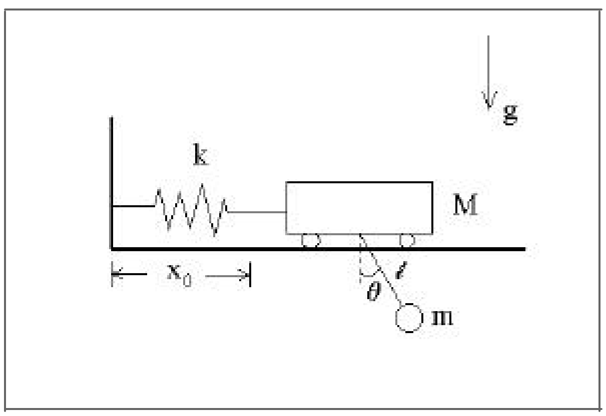
PHYSICS 44 MECHANICS

Homework Assignment II

SOLUTION

Problem 1

A cart of mass M is placed on rails and attached to a wall with the help of a massless spring with constant k (as shown in the Figure below); the spring is in its equilibrium state when the cart is at a distance x_0 from the wall. A pendulum of mass m and length ℓ is attached to the cart (as shown).



(a) Write the Lagrangian $L(x, \dot{x}, \theta, \dot{\theta})$ for the cart-pendulum system, where x denotes the position of the cart (as measured from a suitable origin) and θ denotes the angular position of the pendulum.

(b) From your Lagrangian, write the Euler-Lagrange equations for the generalized coordinates x and θ.

Solution

(a) Using the generalized coordinates x (the displacement of the cart from the equilibrium point of the spring) and θ (the displacement of the pendulum from the vertical) shown in the Figure, the coordinates of the cart are $x_M = x$ and $y_M = 0$ while the coordinates of the pendulum are $x_m = x + \ell \sin\theta$ and $y_m = -\,\ell \cos\theta$ and thus the squared velocities are

$$\begin{aligned}
v_M^2 &= \dot{x}_M^2 + \dot{y}_M^2 = \dot{x}^2 \\
v_m^2 &= \dot{x}_m^2 + \dot{y}_m^2 = (\dot{x} + \ell\dot{\theta}\cos\theta)^2 + (\ell\dot{\theta}\sin\theta)^2 \\
&= \dot{x}^2 + \ell^2\dot{\theta}^2 + 2\ell\,\dot{x}\dot{\theta}\cos\theta.
\end{aligned}$$

The expression for the kinetic energy of the cart-pendulum system is therefore

$$K \;=\; (m \,+\, M)\,\frac{\dot{x}^2}{2} \;+\; m\,\ell^2\,\frac{\dot{\theta}^2}{2} \;+\; m\,\ell\,\dot{x}\,\dot{\theta}\,\cos\theta.$$

The potential energy U of the cart-pendulum system is broken into two parts: the gravitational potential energy $-\,mg\ell\,\cos\theta$ of the pendulum and the elastic potential energy $k\,x^2/2$ stored in the spring. The Lagrangian $L = K - U$ of the cart-pendulum system is therefore

$$L(x, \dot{x},\, \theta, \dot{\theta}) \;=\; (m \,+\, M)\,\frac{\dot{x}^2}{2} \;+\; m\,\ell^2\,\frac{\dot{\theta}^2}{2} \;+\; m\,\ell\,\cos\theta\left(\dot{x}\,\dot{\theta} \;+\; g\right) \;-\; \frac{k}{2}\,x^2.$$

(b) The Euler-Lagrange equation for x is

$$\begin{aligned}
\frac{\partial L}{\partial \dot{x}} &= (m + M)\,\dot{x} \;+\; m\ell\,\dot{\theta}\,\cos\theta \;\rightarrow \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) &= (m + M)\,\ddot{x} \;+\; m\ell\left(\ddot{\theta}\,\cos\theta \;-\; \dot{\theta}^2\,\sin\theta\right) \\
\frac{\partial L}{\partial x} &= -\,k\,x
\end{aligned}$$

or

$$\boxed{(m + M)\,\ddot{x} \;+\; m\ell\left(\ddot{\theta}\,\cos\theta \;-\; \dot{\theta}^2\,\sin\theta\right) \;+\; k\,x \;=\; 0}$$

The Euler-Lagrange equation for θ is

$$\begin{aligned}
\frac{\partial L}{\partial \dot{\theta}} &= m\,\ell\left(\ell\,\dot{\theta} \;+\; \dot{x}\,\cos\theta\right) \;\rightarrow \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= m\,\ell\left(\ell\,\ddot{\theta} \;+\; \ddot{x}\,\cos\theta \;-\; \dot{x}\dot{\theta}\,\sin\theta\right) \\
\frac{\partial L}{\partial \theta} &= -\,m\,\ell\,\dot{x}\dot{\theta}\,\sin\theta \;-\; mg\ell\,\sin\theta
\end{aligned}$$

or

$$\boxed{\ell\,\ddot{\theta} \;+\; \ddot{x}\,\cos\theta \;+\; g\,\sin\theta \;=\; 0}$$

Problem 2

Show that the two Lagrangians

$$L(\mathbf{q}, \dot{\mathbf{q}}; t) \quad \text{and} \quad L'(\mathbf{q}, \dot{\mathbf{q}}; t) \; = \; L(\mathbf{q}, \dot{\mathbf{q}}; t) \; + \; \frac{dF(\mathbf{q}, t)}{dt},$$

where $F(\mathbf{q}, t)$ is an arbitrary function of the generalized coordinates $\mathbf{q}(t)$, yield the same Euler-Lagrange equations. Hence, two Lagrangians which differ only by an exact time derivative are said to be *equivalent*.

Solution

We call $L' = L + dF/dt$ the new Lagrangian and L the old Lagrangian. The Euler-Lagrange equations for the new Lagrangian are

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}^i}\right) \; = \; \frac{\partial L'}{\partial q^i},$$

where

$$\frac{dF(\mathbf{q}, t)}{dt} \; = \; \frac{\partial F}{\partial t} \; + \; \sum_j \dot{q}^j \, \frac{\partial F}{\partial q^j}.$$

Let us begin with

$$\frac{\partial L'}{\partial \dot{q}^i} \; = \; \frac{\partial}{\partial \dot{q}^i}\left(L \; + \; \frac{\partial F}{\partial t} \; + \; \sum_j \dot{q}^j \, \frac{\partial F}{\partial q^j}\right) \; = \; \frac{\partial L}{\partial \dot{q}^i} \; + \; \frac{\partial F}{\partial q^i},$$

so that

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}^i}\right) \; = \; \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) \; + \; \frac{\partial^2 F}{\partial t \partial q^i} \; + \; \sum_k \dot{q}^k \, \frac{\partial^2 F}{\partial q^k \partial q^i}.$$

Next, we find

$$\frac{\partial L'}{\partial q^i} \; = \; \frac{\partial}{\partial q^i}\left(L \; + \; \frac{\partial F}{\partial t} \; + \; \sum_j \dot{q}^j \, \frac{\partial F}{\partial q^j}\right) \; = \; \frac{\partial L}{\partial q^i} \; + \; \frac{\partial^2 F}{\partial q^i \partial t} \; + \; \sum_j \dot{q}^j \, \frac{\partial^2 F}{\partial q^i \partial q^j}.$$

Using the symmetry properties

$$\dot{q}^j \, \frac{\partial^2 F}{\partial q^i \partial q^j} \; = \; \dot{q}^j \, \frac{\partial^2 F}{\partial q^j \partial q^i} \quad \text{and} \quad \frac{\partial^2 F}{\partial t \partial q^i} \; = \; \frac{\partial^2 F}{\partial q^i \partial t},$$

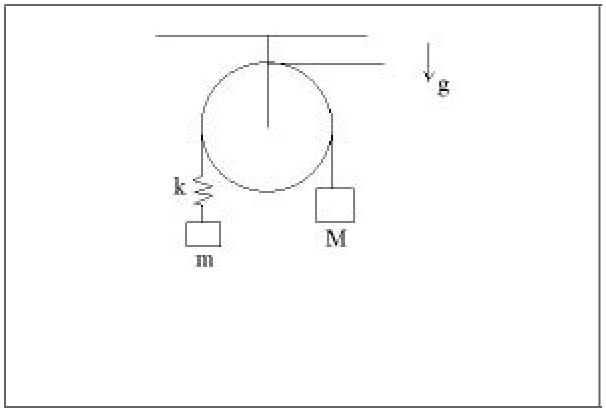
we easily verify

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}^i}\right) \; - \; \frac{\partial L'}{\partial q^i} \; = \; \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) \; - \; \frac{\partial L}{\partial q^i} \; = \; 0,$$

and thus since L and $L' = L + dF/dt$ lead to the same Euler-Lagrange equations, they are said to be equivalent.

Problem 3

An Atwood machine is composed of two masses m and M attached by means of a massless rope into which a massless spring (with constant k) is inserted (as shown in the Figure below). When the spring is in a relaxed state, the spring-rope length is ℓ.

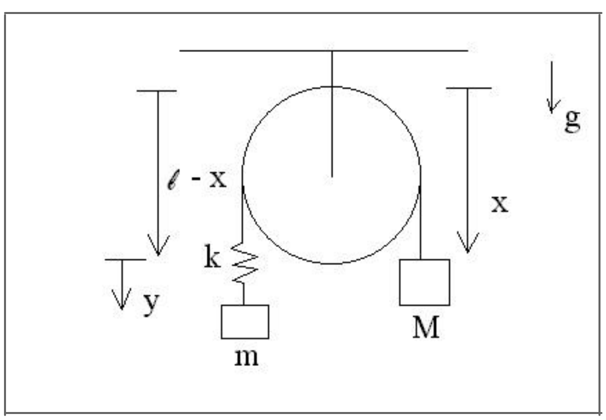


(a) Find suitable generalized coordinates to describe the motion of the two masses (allowing for elongation or compression of the spring).

(b) Using these generalized coordinates, construct the Lagrangian and derive the appropriate Euler-Lagrange equations.

Solution

(a) The Figure below shows a suitable set of generalized coordinates



where x denotes the distance of mass M from the top of the pulley, $\ell - x$ denotes the distance of the equilibrium point of the spring from the top of the pulley, and y denotes the distance of mass m from the equilibrium point of the spring (i.e., its elongation).

(b) Using the generalized coordinates (x, y), the coordinate of mass M is $x_M = x$ while the coordinate of mass m is $x_m = \ell - x + y$ and thus the squared velocities are

$$v_M^2 \;=\; \dot{x}^2 \quad \text{and} \quad v_m^2 \;=\; (\dot{y} - \dot{x})^2 .$$

The expression for the kinetic energy of the system is therefore

$$K \;=\; (m \;+\; M)\,\frac{\dot{x}^2}{2} \;+\; m\,\frac{\dot{y}^2}{2} \;-\; m\,\dot{x}\,\dot{y}.$$

The potential energy U of the system is broken into two parts: the gravitational potential energy $-\,Mgx -\, mg(y - x)$ and the elastic potential energy $k\,y^2/2$ stored in the spring. The Lagrangian $L = K - U$ of the system is therefore

$$L(x, \dot{x},\; y, \dot{y}) \;=\; (m \;+\; M)\,\frac{\dot{x}^2}{2} \;+\; m\,\frac{\dot{y}^2}{2} \;-\; m\,\dot{x}\,\dot{y} \;+\; (M - m)g\,x \;+\; mg\,y \;-\; \frac{k}{2}\,y^2.$$

(c) The Euler-Lagrange equation for x is

$$\frac{\partial L}{\partial \dot{x}} \;=\; (m + M)\,\dot{x} \;-\; m\,\dot{y} \;\rightarrow\; \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) \;=\; (m + M)\,\ddot{x} \;-\; m\,\ddot{y}$$

$$\frac{\partial L}{\partial x} \;=\; (M - m)\,g$$

or

$$\boxed{(m + M)\,\ddot{x} \;-\; m\,\ddot{y} \;=\; (M - m)\,g}$$

The Euler-Lagrange equation for y is

$$\frac{\partial L}{\partial \dot{y}} \;=\; m\;(\dot{y} \;-\; \dot{x}) \;\rightarrow\; \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) \;=\; m\;(\ddot{y} \;-\; \ddot{x})$$

$$\frac{\partial L}{\partial x} \;=\; m\,g \;-\; k\,y$$

or

$$\boxed{m\;(\ddot{y} \;-\; \ddot{x}) \;+\; k\,y \;=\; m\,g}$$