MSc Applied Mathematics Project Notes

DT 9209/9210/9211/9212

Parking problems and random sequential adsorption

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1 Useful Information

1.1 The project module

The Project module is a large component of the MSc Applied Mathematics degree and counts for 25 ECTS. As such, the associated workload is 500 hours, the majority of which is **self-directed**. Note that you should spend approximately 40 hours per week for 13 weeks in order to meet this module's associated workload.

The final submission should be approximately 15,000 to 25,000 words. You should therefore aim to write approximately 1,500 words per week.

1.2 Weekly progress assessment

Students are encouraged to start typing their project immediately (following the templates used in Research Skills) and submit their work weekly so that feedback on their progress can be provided. This could be done through ShareLatex - create a folder called **MScProject_YourName** and share this with the supervisor (dana.mackey@dit.ie). There is a folder called **MScProject_Dana** which is shared with everybody and will contain the notes and assignments for each week as well as all necessary research articles.

1.3 Student handbook

For reference, some of the content from the Student Handbook which describes the Project Module is reproduced below.

1.3.1 Project

As part of the MSc Applied Mathematics students are required to undertake a substantial project. The project is associated with a workload of 25 ECTS and students embark upon the project module upon successful completion of all taught modules and the Mathematical Laboratory and Research Methods modules. In exceptional circumstances students may request permission to proceed to the project having successfully completed at least 50 ECTS of taught modules which include the Mathematical Laboratory and Research Skills modules.

The project is a guided learning project but requires a substantial degree of self-directed learning in association with an allocated supervisor. An important part of the project is developing the ability to interact, communicate and exchange ideas with other students. Projects may be undertaken in parallel with other learners and involve group supervision. However, projects must be solely a students own work and must not be written in conjunction with any other individual. Students are required to include a declaration that the project represents their own individual work and that where other work is utilised to provide necessary background it has been clearly referenced. Projects will be assessed on the original work of the student.

1.3.2 Plagiarism

Plagiarism is deemed to be any material presented by a student that is not the work of the author and has not been fully and accurately referenced and credited. Plagiarism includes the use of others ideas, words, visual or audio material and may be intentional or unintentional. Plagiarism also includes obtaining another individual to produce work on one's own behalf. Plagiarism will automatically result in a candidate failing the module.

More information can be found in the General Assessment Regulations of the Institute:
`http://www.dit.ie/qualityassuranceandacademicprogrammerecords/student-assessment-regulations/`

1.3.3 Project Assignment and Approval

Students will decide upon a provisional project topic during their Research Skills module and as part of the module submit a project proposal and outline workplan. This workplan and a detailed timeline including targets and objectives which will be regularly monitored and updated will form an important part of the ongoing work of the project. The proposal and outline workplan, which include the title of the project, must be approved by the Programme Coordinator in consultation with the Programme Chair. This will take place prior to the end of the semester preceding the start of the project so that preparatory reading and study may be undertaken prior to starting the project with an allocated supervisor. The project will be based upon a topic closely related to modules on the programme and will often include an extensive literature review of a specific module topic.

1.3.4 Role of the supervisor and monitoring

The responsibility for the successful completion of the project module lies with the individual student. The project supervisor will be allocated by the Programme Coordinator in consultation with the Programme Chair prior to the commencement of the project appropriate to the project outline developed as part of the preliminary Research Skills module.

The project supervisor will provide academic input and supervision of a project to students, guiding their academic learning of a topic as stipulated in the contact hours given in the module syllabus. Students will be required to submit regular progress reports to their supervisor. The Stage Tutor will arrange meetings of project students where students will communicate their progress, discuss ideas, and exchange insights with their peers. A students contribution to this group element will contribute to his/her final project module mark. Concerns about an individual students progress or engagement with a project will be notified to the Programme Chair by the supervisor or Stage Tutor. It is the responsibility of the student to submit the final project within the deadlines and requirements of the module and the candidate assumes responsibility for the quality and content of all elements of their submission. If students have any concerns about their progress or supervision they should speak in the first instance to their supervisor and then, if necessary, contact the Stage Tutor immediately.

The project supervisor will primarily assist with the learning and understanding of the academic content of the topic but will also give guidance in terms of the presentation of the project in a formal academic style. However, formatting, referencing and typesetting requirements for the project will be described (together with an introduction to Latex or other appropriate typesetting software) during the Research Skills module. Supervisors are not responsible for correcting large numbers of typographical or formatting errors or rewriting English. Students must proof-read all their work adequately for errors and to ensure that the use of English is acceptable and adequately communicates an understanding before sharing it with their supervisor. Assistance in academic writing is available from the Academic Writing Centre (`http://www.dit.ie/awc/`).

1.3.5 Assessment

Projects are required to be submitted by 5pm on Wednesday of week 14 of the academic calendar. In extraordinary circumstances students may apply for an extension to this deadline. Requests for extensions must be accompanied by documentary evidence of extenuating circumstances and are approved by the Programme Coordinator in consultation with the Programme Chair. The granting of an extension may delay examination of the project and the date of graduation. In some cases additional fees may apply.

The project will be assessed in accordance with the assessment criteria set out in the syllabus based upon students written work, any interim submissions, feedback on their conduct, contributions to group sessions, and an assessment of their understanding and communication of their work. In normal circumstances students will be required to make a presentation of their work and answer questions in a viva voce examination. Candidates must make themselves available during the normal academic year, following the submission of their project, for this examination. The project supervisor and internal examiner(s) nominated by the Programme Coordinator will be responsible for the assessment of the project in consultation with the external examiner of the programme. The external examiner will be invited to attend student presentations.

1.3.6 More information

More information about the Project module can be found in the syllabus in section 5. The Academic Writing Centre (http://www.dit.ie/awc/) is also available to offer assistance to students of the Institute to develop their ability as writers in an academic environment.

1.3.7 Circumstances affecting examinations

Where students experience circumstances that detrimentally affect their preparation for an assessment or performance in an assessment they should complete a Personal Circumstance form
`http://www.dit.ie/qualityassuranceandacademicprogrammerecords/student-assessment-regulations/general/`.
For all students in the School of Mathematical Sciences, the procedure for submission of this form and the deadlines are given in the accompanying document Guidance for Students Studying on Programmes in the School of Mathematical Sciences which can be found on the Webcourses `http://www.dit.ie/lttc/webcourseslogin/` module for your programme or from your Stage Tutor.

Note that, all Personal Circumstances should be accompanied by documentary evidence and that the procedure for circumstances that affect semester-time Continuous Assessments may differ from that for Written Examinations administered by the Examinations Office. Students should familiarise themselves with the document Guidance for Students Studying on Programmes in the School of Mathematical Sciences.

The deadline for submission of Personal Circumstances forms is stipulated by the Institute and forms received after this deadline may not be considered. If unavoidable circumstances arise in advance of an examination session which means that a student is unable to sit his/her examinations they may request a deferral of examinations
`http://www.dit.ie/media/examinations/Deferral%20Applicaton%20Form.pdf`.

For all students in the School of Mathematical Sciences, the procedure for submission of this form and the deadlines are given in the accompanying document Guidance for Students Studying on Programmes in the School of Mathematical Sciences which can be found on the Webcourses `http://www.dit.ie/lttc/webcourseslogin/` module for your programme or from your Stage Tutor.

Requests to defer must be supported by documentary evidence and, if endorsed, students will be required to sit the examination at the next available sitting. Note that students should check the next available sitting of a module. Deferral of an examination, and any subsequent requirement for reassessment, may significantly delay the date of an award and graduation.

Students will receive feedback on their performance in any assessment. In addition, students will be given an opportunity to view their examination scripts during periods designated by the School. Students may only request to view scripts during these designated periods. Where students wish their examination material to be rechecked or remarked they must submit a request via the appropriate form
`http://www.dit.ie/qualityassuranceandacademicprogrammerecords/student-assessment-regulations/general/`
within the deadlines set by the School and Institute. The Stage Tutor can advise students on the procedures associated with rechecks and remarks.

2 Introduction

2.1 Project abstract

The following *parking problem* was proposed in 1958 by the Hungarian mathematician Alfred Rényi in [1]: Cars, modelled as intervals of unit length, are placed (parked) at random, one at a time, along a one-dimensional street of length x until spaces are no longer available. Rényi proved that, if $M(x)$ is the expected number of cars we can place on $(0, x)$ then $\lim_{x\to\infty} M(x)/x \approx 0.7476$. (In other words, when the street is jammed, the expected coverage is approximately 75%.)

This interesting problem has extensive applications in physics and chemistry, for example to the theory of *random sequential adsorption* which describes surface deposition of particles on a solid substrate. (The variable of interest is the jamming density, or the expected area that is covered by particles when no further deposition is possible.)

There are many approaches to solving Rényi's parking problem, such as an elementary solution which gives an estimation of the jamming limit, [2], a recursive approach which solves a delay differential equation for $M(x)$ using Laplace transforms and finally, solving a partial integro-differential equation for $P(x,t)$, a density function describing the evolution of gaps forming between adjacent parked cars. (A numerical code can also be used for simulating the parking process.)

Many interesting extensions of the RSA theory exist such as competitive RSA where the cars, or particles, have two or more different lengths, RSA with overlap, or the so-called cooperative random sequential adsorption, where the parking position depends on the current configuration (as opposed to being chosen from a uniform distribution).

2.2 Recursive approach for the car parking problem

Let $M(x)$ denote the expected number of unit cars (at saturation) parked along an interval (or road) of length x. To calculate $M(x+1)$, we place a car at a random point t, $t \in [0, x]$. This creates two additional intervals, $[0, t]$ and $[t+1, x+1]$ and the expectation becomes $1 + M(t) + M(x - t)$.

Therefore

$$\int_0^x M(x+1)\,dt = \int_0^x \big(1 + M(t) + M(x-t)\big)\,dt$$

and it can be shown that $M(x)$ satisfies the integral equation

$$M(x+1) = 1 + \frac{2}{x}\int_0^x M(t)\,dt. \tag{1}$$

This is Rényi's master equation with initial condition $f(x) = 0$ for $0 \le x < 1$ because the length of the line will not accommodate a unit car. A recursive approach would consist of repeated substitution of $M(x)$ into the right-hand side of Equation (1) resulting in expressions for this function on the intervals [0,1], [1,2], [2,3], etc. See equation (2) in [2]. However, such an approach is not very practical as, after only a few attempts, it becomes impossible to express $M(x)$ in terms of elementary functions.

2.2.1 An elementary treatment to the car parking problem

Using elementary calculus, Weiner proved in [2] that for $x > 4$,

$$0.7432x - 0.2568 \le M(x) \le 0.75x - 0.25. \tag{2}$$

Hence the limit of $\frac{M(x)}{x}$ as $x \to \infty$, if it exists will be bounded by

$$0.7432 \le \lim_{x\to\infty} \frac{M(x)}{x} \le 0.75.$$

In order to show that the above limit exists, Weiner used the argument:

$$\lim_{x\to\infty}\left(\frac{M(x)}{x}\right)' = 0 \Longrightarrow \lim_{x\to\infty}\frac{M(x)}{x} = \text{constant}$$

However, this argument is not correct! It is **not true** that, if the limit of the derivative of a function is zero at infinity then the limit of the function must be a constant. A very simple counterexample is provided by the function

$$f(x) = \sin(\ln(x)), \qquad x > 0 \tag{3}$$

In our project, we will therefore skip this existence proof and just show the inequality (2).

2.2.2 Rényi's original approach using Laplace transforms

Transforming (1) into a delay differential equation, Rényi showed, using non-standard techniques from Laplace transform theory - see [3], that

$$\lim_{x\to\infty}\frac{M(x)}{x} = \int_0^\infty \exp[-2\int_0^\xi \frac{1-e^u}{u}du]d\xi = 0.7475979. \tag{4}$$

2.3 Work objectives

The primary objectives are basic tasks that every student should attempt. The secondary and tertiary objectives will consist of more challenging tasks that should attract a higher mark upon the completion of the project.

Note that the proofs of all results given in the references below must be reproduced in your own words and contain additional details and explanations whenever you feel necessary. Any original contribution to such reading is strongly encouraged. Needless to say, you should have a good understanding of everything that is written in your project and should be prepared to answer questions relating to that topic during your final oral presentation!

2.3.1 Primary objectives ($<=60\%$)

1. Write a section which summarizes the work in [2] (see section 2.2.1). Start by writing down the main theorems and results, attempt to explain what they mean, and only include the proofs if you think you can follow the arguments!

2. Show that the function provided in (3) satisfies $f'(x) \to 0$ as $x \to \infty$ but $\lim_{x\to\infty} f(x)$ is not a constant! This proves that the argument in [2] for the existence of the coverage limit is not correct!

2.3.2 Secondary objectives ($<=80\%$)

1. Calculate $M(x)$ for $4 \leq x < 5$, using the same recursive integration process as before. Can you also obtain an expression on the next interval, $5 \leq x \leq 6$?

2. Graph the function $\frac{M(x)}{x}$ on $[0,5]$ (or $[0,6]$, if you have calculated it) using Maple, and comment on its long term behaviour.

2.3.3 Tertiary objectives ($<=100\%$)

1. Write an additional section which summarizes sections 3.2 and 3.3 ("The solution of the delay differential equation using Laplace transforms" and "The computation of the limit") from reference [3].

2.4 References

[1] A. Rényi, On a one-dimensional problem concerning random space-filling, *Publ. Math. Inst. Hung. Acad. Sci. 3*, 109-127, (1958).

[2] H. Weiner, Elementary treatment of the parking problem, *Sankhya, Indian Journal of Statistics, A*, **31**, no. 4, (1969), 483-486.

[3] M.D. Sikirić, Y.Itoh, Random sequential packing of cubes, World Scientific Publishing, 2011. (Chapter 3: Random interval packing)