Solved Exercises of chapter 5 from D'Inverno
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1. In Euclidean 3-space $\mathbb{R}^3$:

   (i) Write down the equation of a circle of radius a lying in the (x,y)-plane centred at the origin in (a) parametric form and (b) constraint form.
   Solution:
   Circle equation: $x^2 + y^2 = a^2$
   Parametric form: $\begin{cases} x = t \\ y = \pm\sqrt{a^2 - t^2} \end{cases}$
   Constraint form: $y = \pm\sqrt{a^2 - x^2}$

   (ii) Write down the equation of a hypersurface consisting of a sphere of radius a centred at the origin in (a) parametric form and (b) constraint form. Eliminate the parameters in form (a) to obtain form (b).
   Solution:
   Sphere equation: $x^2 + y^2 + z^2 = a^2$
   Parametric form: $\begin{cases} x = t \\ y = u \\ z = \pm\sqrt{a^2 - t^2 - u^2} \end{cases}$
   Constraint form: $z = \pm\sqrt{a^2 - x^2 - y^2}$

2. Write down the change of coordinates from Cartesian coordinates $(x^a) = (x, y, z)$ to spherical polar coordinates $(x'^a) = (r, \theta, \phi)$ in $\mathbb{R}^3$. Obtain the transformation matrices $[\partial x^a / \partial x'^b]$ and $[\partial x'^a / \partial x^b]$ expressing them both in terms of the primed coordinates. Obtain the Jacobians J and J'. Where is J' zero or infinite?

   Solution:

   The spherical polar coordinates in terms of Cartesian ones:

   $\mathrm{r}\,(x, y, z) := \sqrt{x^2 + y^2 + z^2}$

   $\theta\,(x, y, z) := \mathrm{atan}\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$

   $\phi\,(x, y, z) := \mathrm{atan}\left(\frac{y}{x}\right)$

   Producing the transformation matrix:

$$\left[\frac{\partial x'^a}{\partial x^b}\right] = \begin{pmatrix} \frac{x}{\sqrt{z^2+y^2+x^2}} & \frac{y}{\sqrt{z^2+y^2+x^2}} & \frac{z}{\sqrt{z^2+y^2+x^2}} \\ \frac{x\,z}{\sqrt{y^2+x^2}\,(z^2+y^2+x^2)} & \frac{y\,z}{\sqrt{y^2+x^2}\,(z^2+y^2+x^2)} & -\frac{\sqrt{y^2+x^2}}{z^2+y^2+x^2} \\ -\frac{y}{y^2+x^2} & \frac{x}{y^2+x^2} & 0 \end{pmatrix}$$

   and the Jacobian of the transformation is

$$J' = \left|\frac{\partial x'^a}{\partial x^b}\right| = \frac{1}{\sqrt{y^2 + x^2}\,\sqrt{z^2 + y^2 + x^2}} = \frac{1}{r^2 \sin(\theta)}$$

   And the Cartesian coordinates in terms of spherical polar ones:

   $\mathrm{x}\,(r, \theta, \phi) := r \sin(\theta)\, \cos(\phi)$

   $\mathrm{y}\,(r, \theta, \phi) := r \sin(\theta)\, \sin(\phi)$

   $\mathrm{z}\,(r, \theta, \phi) := r \cos(\theta)$

   Producing the transformation matrix:

$$\left[\frac{\partial x^a}{\partial x'^b}\right] = \begin{pmatrix} \cos(\phi)\,\sin(\theta) & \cos(\phi)\,r\cos(\theta) & -\sin(\phi)\,r\sin(\theta) \\ \sin(\phi)\,\sin(\theta) & \sin(\phi)\,r\cos(\theta) & \cos(\phi)\,r\sin(\theta) \\ \cos(\theta) & -r\sin(\theta) & 0 \end{pmatrix}$$

and the Jacobian of the transformation is

$$J = \left|\frac{\partial x^a}{\partial x'^b}\right| = r^2\sin(\theta)$$

Observe that $J' \to 0$ when $r \to \infty$ and it diverges when $r \to 0$. But when $\theta \to 0$ or $\theta \to n\pi$, that is, on the poles, J' also diverges.

As we see, we confirm the product rule for the determinants that the Jacobian of the inverse transformation where $J = 1/J'$

3. Show by manipulating the dummy indices that

$$(Z_{abc} + Z_{cab} + Z_{bca})\,X^aX^bX^c = 3Z_{abc}X^aX^bX^c.$$

Solution:

$$\begin{aligned} (Z_{abc} + Z_{cab} + Z_{bca})\,X^aX^bX^c &= Z_{abc}X^aX^bX^c + Z_{cab}X^aX^bX^c + Z_{bca}X^aX^bX^c & (1)\\ &= Z_{abc}X^aX^bX^c + Z_{cab}X^aX^bX^c + Z_{bca}X^aX^bX^c & (2)\\ &= Z_{abc}X^aX^bX^c + Z_{abc}X^aX^bX^c + Z_{abc}X^aX^bX^c & (3)\\ &= 3Z_{abc}X^aX^bX^c & (4) \end{aligned}$$

PS: From line 2 to 3 we switch in the second term $c \to a$, $a \to b$ and $b \to c$, in the third term $b \to a$, $c \to b$ and $a \to c$.

4. Show that:

(i) $\delta^b_a X^a = X^b$,
Solution:

$$\delta^b_a X^a = \frac{\partial x^b}{\partial x^a}X^a = \sum_{a=1}^{n}\frac{\partial x^b}{\partial x^a}X^a = X^b$$

because

$$\sum_{a=1}^{n}\frac{\partial x^b}{\partial x^a} = \begin{cases} 0 & \text{when } a \neq b \\ 1 & \text{when } a = b \end{cases}$$

(ii) $\delta^b_a X_b = X_a$,
Solution:

$$\delta^b_a X_b = \frac{\partial x^b}{\partial x^a}X_b = \sum_{b=1}^{n}\frac{\partial x^b}{\partial x^a}X_b = X_a$$

because

$$\sum_{b=1}^{n}\frac{\partial x^b}{\partial x^a} = \begin{cases} 0 & \text{when } b \neq a \\ 1 & \text{when } b = a \end{cases}$$

(iii) $\delta^b_a\delta^c_b\delta^d_c = \delta^d_a$.
Solution:

$$\delta^b_a\delta^c_b\delta^d_c = \frac{\partial x^b}{\partial x^a}\frac{\partial x^c}{\partial x^b}\frac{\partial x^d}{\partial x^c} = \sum_{b=1}^{n}\sum_{c=1}^{n}\frac{\partial x^b}{\partial x^a}\frac{\partial x^c}{\partial x^b}\frac{\partial x^d}{\partial x^c} = \sum_{b=1}^{n}\frac{\partial x^b}{\partial x^a}\frac{\partial x^d}{\partial x^b} = \frac{\partial x^d}{\partial x^a} = \delta^d_a$$

because

$$\sum_{c=1}^{n}\frac{\partial x^c}{\partial x^b} = \begin{cases} 0 & \text{when } c \neq b \\ 1 & \text{when } c = b \end{cases}$$

and

$$\sum_{b=1}^{n} \frac{\partial x^b}{\partial x^a} = \begin{cases} 0 \text{ when } b \neq a \\ 1 \text{ when } b = a \end{cases}$$

5. If Y^a and Z^a are contravariant vectors, then show that $Y^a Z^b$ is a contravariant vector of rank 2.

   Solution:

$$Y'^a Z'^b = \frac{\partial x'^a}{\partial x^c} Y^c \, \frac{\partial x'^b}{\partial x^d} Z^d = \left(\frac{\partial x'^a}{\partial x^c} \frac{\partial x'^b}{\partial x^d} \right) Y^c Z^d$$

   then, as we see above, $Y^a Z^b$ transform according to a contravariant tensor of rank 2.

   PS: In term of components, we had have,

$$Y^a Z^b = \begin{pmatrix} Y^1 Z^1 & Y^1 Z^2 & \cdots & Y^1 Z^n \\ Y^2 Z^1 & Y^2 Z^2 & \cdots & Y^2 Z^n \\ \vdots & \vdots & \ddots & \vdots \\ Y^n Z^1 & Y^n Z^2 & \cdots & Y^n Z^n \end{pmatrix}$$

   with n^2 components, as expected.

6. Write down the change of coordinates from Cartesian coordinates $(x^a) = (x, y)$ to plane polar coordinates $(x'^a) = (R, \phi)$ in $\mathbb{R}^2$ and obtain the transformation matrix $\left[\partial x'^a / \partial x^b\right]$ expressed as a function of the primed coordinates. Find the components of the tangent vector to the curve consisting of a circle of radius a centred at origin with the standard parametrization (see exercise 5.1(i)) and use $X'^a = \partial x'^a / \partial x^b \, X^b$ to find its components in the primed coordinate system.

   Solution:

   The change of coordinates are:

$$R = \sqrt{x^2 + y^2} \tag{5}$$

$$\phi = \operatorname{atan}\left(\frac{y}{x}\right) \tag{6}$$

   The transformation matrix is

$$\left[\frac{\partial x'^a}{\partial x^b}\right] = \begin{pmatrix} \frac{x}{\sqrt{y^2+x^2}} & \frac{y}{\sqrt{y^2+x^2}} \\ -\frac{y}{y^2+x^2} & \frac{x}{y^2+x^2} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\frac{\sin\phi}{R} & \frac{\cos\phi}{R} \end{pmatrix} \tag{7}$$

   as a function of the primed coordinates.

   To find the components of the tangent vector to the curve:

   Circle equation: $x^2 + y^2 = a^2$

   Parametric form: $\begin{cases} x = t \\ y = \pm\sqrt{a^2 - t^2} \end{cases}$

   thus $x^1 = x$ and $x^2 = y$ we represent the curve as $x^a(t)$. So, the tangent vector,

$$\frac{dx^a}{dt} = \begin{pmatrix} \frac{dx^1}{dt} \\ \frac{dx^2}{dt} \end{pmatrix} = \begin{pmatrix} 1 \\ \mp\frac{t}{\sqrt{a^2-t^2}} \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{x}{y} \end{pmatrix}$$

Using $X'^a = (\partial x'^a / \partial x^b)\ X^b$:

$$
\begin{aligned}
X'^a &= \begin{pmatrix} \frac{x}{\sqrt{y^2+x^2}} & \frac{y}{\sqrt{y^2+x^2}} \\ -\frac{y}{y^2+x^2} & \frac{x}{y^2+x^2} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -\frac{x}{y} \end{pmatrix} && (8) \\
&= \begin{pmatrix} \frac{x}{\sqrt{y^2+x^2}} - \frac{x}{\sqrt{y^2+x^2}} \\ -\frac{y}{y^2+x^2} - \frac{x^2}{y(y^2+x^2)} \end{pmatrix} && (9) \\
&= \begin{pmatrix} 0 \\ -\frac{y^2}{y(y^2+x^2)} - \frac{x^2}{y(y^2+x^2)} \end{pmatrix} && (10) \\
&= \begin{pmatrix} 0 \\ -\frac{1}{y} \end{pmatrix} && (11) \\
&= \begin{pmatrix} 0 \\ -\frac{1}{a\ \sin\phi} \end{pmatrix} && (12)
\end{aligned}
$$

that are the components of the tangent vector in primed coordinate system.