Introduction to Biomathematics

Problem sheet

1. A model for population growth is given in non-dimensional units in the form

$$\frac{du}{dt} = u(1-u^2), \qquad u(0) > 0.$$

(a) Sketch the graph of the function $f(u) = u(1-u^2)$ against u for all real values of u.

(b) Determine all steady states of the model (including the negative ones if any) and their stability. From this analysis find $\lim_{t\to\infty} u(t)$. Sketch the behaviour of the solution $u(t)$.

(c) Solve the model explicitly for $u(t)$ and from the solution find again $\lim_{t\to\infty} u(t)$.

2. (Growth of bacteria) Let us consider the growth of a mass of bacteria in a Perti dish. The mass grows uniformly in every direction. Only the bacteria on the surface of the mass reproduce. Let $N(t)$ be the number of bacteria at time t. Justify the following model and solve it:

$$\frac{dN(t)}{dt} = rN^{2/3}(t), \qquad N(0) > 0, \qquad r > 0.$$

3. For the given models find all equilibria and determine their stability:

$$(i) \qquad \frac{dN(t)}{dt} = rN\frac{K-N}{K+aN} \qquad r, K, a > 0.$$

$$(ii) \qquad \frac{dN(t)}{dt} = N(re^{1-\frac{N}{K}} - a) \qquad r, K, a > 0.$$

4. (Gompertz model) Study the one species model given by:

$$\frac{dN(t)}{dt} = -rN(t)\ln\left(\frac{N(t)}{K}\right), \qquad r, K > 0, \qquad N(0) > 0.$$

5. (Demographic models) Let us consider the following model:

$$\frac{dN(t)}{dt} = r\frac{N(t)}{\alpha}\left(1 - \left(\frac{N(t)}{K}\right)^{\alpha}\right), \qquad r, K, \alpha > 0, \qquad N(0) > 0.$$

(i) Recognize this model when $\alpha = 1$ and $\alpha \to 0$.
(ii) What is the time limit of $N(t)$ when $t \to \infty$?

6. Show that for a population which satisfies the logistic model, the maximum rate of growth of population size is $rK/4$, attained when population size is $K/2$.

7. Solve the logistic model with a time-dependent intrinsic growth rate:

$$\frac{dN(t)}{dt} = r(t)N(t)\left(1 - \frac{N(t)}{K}\right), \qquad K > 0, \qquad N(0) > 0.$$

Answer:

$$N(t) = \frac{KN(0)}{N(0) + (K - N(0))\exp(-\int_0^t r(s)ds)}$$

8. (Seasonal capacity model) A species N is subject to a seasonal periodic constraint that changes carrying capacity. The proposed model is:

$$\frac{dN(t)}{dt} = rN(t)\left(1 - N(t)\frac{1 + \beta\cos(\gamma t)}{K}\right), \qquad r, K, \gamma > 0, \qquad 0 < \beta < 1.$$

(i) Solve the equation (Hint: set $y = 1/N$).
(ii) Compare with the behavior of the logistic model.

9. Discuss the model:

$$\frac{dN(t)}{dt} = rN\left(1 - \frac{N}{K}\right)\left(\frac{N}{K_0} - 1\right), \qquad 0 < K_0 < K.$$

Find all limits of solutions with $N(0) > 0$ as $t \to \infty$ and find the set of initial values corresponding to each limit.

10. Recall that we showed that it was not possible to have a periodic solution for a simple first order nondelay equation

$$\frac{dN(t)}{dt} = f(N(t)).$$

A seeming counterexample is $N(t) = 2 + \sin t$. Determine $f(N)$ for which this is a solution of the differential equation and explain why it is not a counterexample.

11. A model for the spruce budworm population $u(t)$ is governed by

$$\frac{du}{dt} = ru\left(1 - \frac{u}{q}\right) - \frac{u^2}{1+u^2}, \qquad r, q > 0.$$

Determine the number of the steady states and their stability if:
(a) $r = q = 1$,
(b) $r = 0.5$, $q = 20$.

12. A single population model with a very "fast" predation is given by the equation

$$\frac{dN}{dt} = RN\left(1 - \frac{N}{K}\right) - P\left(1 - \exp\left(-\frac{N^2}{\varepsilon A^2}\right)\right), \qquad 0 < \varepsilon \ll 1,$$

where R, K, P and A are positive constants.

(a) By an appropriate nondimensionalisation show that the equation is equivalent to

$$\frac{du}{d\tau} = ru\left(1 - \frac{u}{q}\right) - \left(1 - \exp\left(-\frac{u^2}{\varepsilon}\right)\right),$$

where r and q are positive parameters.

(b) Sketch the graphs of the functions $g(u) = ru\left(1 - \frac{u}{q}\right)$ and $h(u) = 1 - \exp\left(-\frac{u^2}{\varepsilon}\right)$ for positive u remembering that ε is small, i.e. $0 < \varepsilon \ll 1$. (c) Demonstrate that there are three possible nonzero steady states if r and q lie in a domain in r, q space given approximately by $rq > 4$. Without evaluating these steady states comment on their stability.

(d) Could this model exhibit hysteresis?

13. (Model with delay) A population growth model is described by a differential equation with delay $T > 0$:

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = rN(t)\Big(1 - \frac{N(t-T)}{K}\Big),$$

with $r > 0$, $K > 0$ and $N(0) > 0$.

(i) Show that in the new nondimensional variables $u = N/K$, $\tau = rt$ and $\Theta = rT$ the equation has the form

$$\frac{\mathrm{d}u(\tau)}{\mathrm{d}\tau} = u(\tau)[1 - u(\tau - \Theta)],$$

(ii) Show that $u^* = 1$ is a steady state and linearize the equation about the steady state by writing $u(\tau) = u^* + n(\tau)$, where $|n(\tau)| \ll 1$.

(iii) Look for solutions of the form $n(\tau) = ce^{\lambda\tau}$, where c is a constant, and write the corresponding equation for λ.

(iv) Decompose λ into real and imaginary parts ($\lambda = \mu + i\omega$) and write separately the real and imaginary parts of the equation for λ, obtained in (iii) in terms of μ and ω.

(v) Analyze the equations obtained in (iv) when the delay Θ increases from 0 to $\frac{\pi}{2}$, having in mind that μ and ω depend on Θ. Show that the steady state solution u^* is stable when $0 \le \Theta < \pi/2$. Demonstrate that the first bifurcation value of Θ for which the steady state u^* becomes unstable and the solution becomes oscillatory corresponds to $\Theta = \pi/2$, (and $\mu = 0$, $\omega = \pm 1$) or, in dimensional terms, $rT = \pi/2$. Show that this bifurcation occurs when μ, being negative when $\Theta = 0$ reaches the bifurcation value $\mu = 0$ when $\Theta = \pi/2$.

(vi) Show that the period of oscillations (in nondimensional units) at the bifurcation value is 4Θ.

14. Show that an exact travelling wave solution exists for the scalar reaction-diffusion equation

$$\frac{\partial u}{\partial t} = u^{q+1}\left(1 - u^q\right) + \frac{\partial^2 u}{\partial x^2}, \qquad q > 0,$$

by looking for solution in the form

$$u(x,t) = U(z) = \frac{1}{(1 + ae^{bz})^s}, \qquad z = x - ct,$$

where c is wavespeed and s and b are positive constants. Determine the unique values for c, s and b in terms of q. Choose the value for a such that the magnitude of the wave's gradient is at its maximum at $z = 0$.

15. Indicate whether each of the following equations is linear or nonlinear. If linear, determine the solution, if nonlinear, find any steady states of the equation.

(a) $x_n = (1-\alpha)x_{n-1} + \beta x_n, \qquad \alpha$ and β are constants,

(b) $x_{n+1} = \dfrac{x_n}{1+x_n},$

(c) $x_{n+1} = x_n e^{-ax_n}, \qquad \alpha$ is a constant,

(d) $(x_{n+1}-\alpha)^2 = \alpha^2(x_n^2 - 2x_n + 1), \qquad \alpha$ is a constant,

(e) $x_{n+1} = \dfrac{K}{k_1 + k_2/x_n}, \qquad k_1,\ k_2$ and K are constants.

16. Determine when the following steady states are stable:

(a) $x_{n+1} = rx_n(1-x_n), \qquad x^* = 0,$

(b) $x_{n+1} = -x_n^2(1-x_n), \qquad x^* = \dfrac{1+\sqrt{5}}{2},$

(c) $x_{n+1} = \dfrac{1}{2+x_n}, \qquad x^* = \sqrt{2}-1,$

(d) $x_{n+1} = x_n \ln x_n^2, \qquad x^* = \sqrt{e}.$

Sketch the functions $f(x)$ given in this problem. Use the cobwebbing method to sketch the approximate behaviourof solutions to the equationsfrom some initial starting value of x_0.

17. Find the number $\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}$ with infinitely many radicals. To this end you can show that the above number is the stable steady state of the equation

$$x_{n+1} = \sqrt{2+x_n}$$

and find this steady state.

18. It has been suggested that a means of controlling insect numbers is to introduce and maintain a number of sterile insects in the population. One such model for the resulting population dynamics is

$$N_{t+1} = \frac{rN_t^2}{\frac{r-1}{M}N_t^2 + N_t + S},$$

where $r > 1$ and $M > 0$ are constant parameters, and S is the constant sterile insect population.

(i) Determine the steady states and discuss their linear stability, noting whether any type of bifurcation is possible.

(ii) Find the critical value S_c of the sterile population in terms of r and M so that if $S > S_c$ the insect population is eradicated.

(iii) Construct a cobweb map and draw a graph of S against the steady state population density, and hence determine the possible solution behaviour if $0 < S < S_c$.

19. Flores (1998) proposed the following model for competition between Neanderthal man (N) and Early Modern man (E).

$$\begin{aligned} \frac{dN}{dt} &= N\left(A - D(N+E) - B\right), \\ \frac{dE}{dt} &= = E\left(A - D(N+E) - sB\right), \end{aligned}$$

where A, B, D are positive constants and $s < 1$ is a measure of the difference in mortality of the two species.

Nondimensionalise the system and describe the meaning of any dimensionless parameters.

Show that the populations N and E are related by

$$\frac{N(t)}{E(t)} = C\exp[-B(1-s)t],$$

where C is some constant. Hence give the order of magnitude of the time for Neanderthal extinction. If the lifetime of an individual is roughly 30 to 40 years and the time to extinction is (from the palaeontological data) 5000 to 10,000 years, determine the range of the mortality difference parameter s. [An independent estimate (Flores 1998) is of s = 0.995.]

20.(i) Describe the type of interaction between two species with populations u and v that is implied by the model

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} &= u(1-u) - \frac{uv}{u+\alpha}, \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= \beta v\Big(1 - \frac{v}{u}\Big), \end{aligned}$$

where the parameters α and β are positive.

(ii) Determine the steady states and their stability for all possible positive values of the parameters α and β. Briefly describe the ecological implications of the results of the analysis. Support your explanations by a sketch of the phase portrait of the system.

21. Leslie's population model is given by the system (in non-dimensional units)

$$\frac{du}{dt} = u(1-u) - \alpha uv, \qquad \frac{dv}{dt} = \rho v\left(1 - \frac{v}{u}\right),$$

where α and ρ are positive parameters.

(i) Determine the kind of behavior between the two species that is implied by the model and explain briefly the role of the terms that appear in the equations.

(ii) Determine the steady states and their stability in dependence on the parameter values.

(iii) Sketch the phase portrait of the system and briefly describe the ecological implications of the results of the analysis.