S9209/9210/103C

DUBLIN INSTITUTE OF TECHNOLOGY

School of Mathematical Sciences

DT9205 MSc Mathematical Physics
DT9206 MSc Mathematical Physics
DT9209 MSc Applied Mathematics
DT9210 MSc Applied Mathematics

SAMPLE EXAM 1 2015/2016

MATH9973: NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS

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Duration: 2 hour

Full marks may be obtained by answering three questions. Candidate's three best questions will contribute to their final mark.

All questions carry equal marks

Approved calculators may be used

Mathematical tables are provided

New Cambridge Statistical Tables are NOT permitted

1. **a)** For the Ordinary Differential Equation

$$\frac{dy}{dx} = f(x, y), \quad a < x \leq b$$

with initial condition

$$y(a) = \alpha,$$

derive the Adams-Bashforth two step method and its truncation error, which is of the form

$$w_0 = \alpha_0, \quad w_1 = \alpha_1,$$

$$w_{i+1} = w_i + \frac{h}{2}[3f(x_i, w_i) - f(x_{i-1}, w_{i-1})].$$

(10)

b) Define the terms consistent and convergent methods for a multistep method.

(5)

c) Define the terms strongly stable, weakly stable and unstable with respect to the characteristic equation.

(6)

d) Show that the Adams-Bashforth two step method is strongly stable.

(4)

e) Use an Adams-Bashforth method of your choice to approximate the solution to the initial value problem

$$\frac{dy}{dx} = 1 + e^x + y^2, \quad 1 \leq x \leq 2, \quad y(1) = 1$$

with $h = 0.1$ to approximate $y(0.2)$.

(8)

[33]

2. a) Approximate the Poisson equation

$$-\nabla^2 U(x,y) = f(x,y), \qquad (x,y) \in \Omega = (0,1)\times(0,1),$$

with boundary conditions

$$U(x,y) = g(x,y), \qquad (x,y) \in \delta\Omega$$

using the five point method ∇_h^2. Sketch how the finite difference scheme may be rewritten in the form $Ax = b$, where A is a sparse $N^2 \times N^2$ matrix, and b , x are an N^2 component vectors. (Assume your 2d discretised grid contains N components in the x and y direction).

(10)

b) Prove (DISCRETE MAXIMUM PRINCIPLE). If $\nabla_h^2 V_{ij} \geq 0$ for all points $(x_i, y_j) \in \Omega_h$, then

$$\max_{(x_i,y_j)\in\Omega_h} V_{ij} \leq \max_{(x_i,y_j)\in\partial\Omega_h} V_{ij}.$$

If $\nabla_h^2 V_{ij} \leq 0$ for all points $(x_i, y_j) \in \Omega_h$, then

$$\min_{(x_i,y_j)\in\Omega_h} V_{ij} \geq \min_{(x_i,y_j)\in\partial\Omega_h} V_{ij},$$

where Ω_h is the discrete grid of the area Ω, ∇_h^2 is the five point approximation of ∇^2 and h is the step-size in the x and y direction.

(12)

c) Hence prove:

Let U be a solution to the Poisson equation and let w be the grid function that satisfies the discrete form

$$-\nabla_h^2 w_{ij} = f_{ij} \quad \text{for } (x_i, y_j) \in \Omega_h,$$

$$w_{ij} = g_{ij} \quad \text{for } (x_i, y_j) \in \partial\Omega_h.$$

Then there exists a positive constant K such that

$$||U - w||_\Omega \leq K M h^2,$$

where

$$M = max\left\{ \left\|\frac{\partial^4 U}{\partial x^4}\right\|_\infty, \left\|\frac{\partial^4 U}{\partial x^3 \partial y}\right\|_\infty, \ldots, \left\|\frac{\partial^4 U}{\partial y^4}\right\|_\infty \right\}.$$

(11)

You may assume: If the grid function $V : \Omega_h \bigcup \partial\Omega_h \rightarrow R$ satisfies the boundary condition $V_{ij} = 0$ for $(x_i, y_j) \in \partial\Omega_h$, then

$$||V||_\Omega \leq \frac{1}{8} ||\nabla_h^2 V||_\Omega.$$

[33]

3. a) State the 3 classes and conditions of 2nd order Partial Differential Equations defined by the characteristic curves. (5)

b) Given the non-dimensional form of the heat equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$$

supply sample boundary and initial conditions to specify a well-posed problem problem.
Write a fully implicit scheme to solve this partial differential equation. (8)

c) Derive the local truncation error for the fully implicit method, for the heat equation. (10)

d) Show that the method is unconditionally stable using von Neumann's method. (10)

[33]

4. a) For a finite difference scheme approximating a partial differential equation of the form

$$\frac{\partial U}{\partial t} = -a\frac{\partial U}{\partial x} + f(x,t), \quad x \in R, \quad t > 0$$
$$U(x,0) = U_0(x), \quad x \in R$$

define what is meant by:

i) convergence,

ii) consistency,

iii) stability.

(10)

b) Describe the forward Euler/centered difference method for the transport equation and derive the local truncation error. (6)

c) For the numerical method in (b) define the Courant Friedrichs Lewy condition and state how it is related to stability. (5)

d) Show that the numerical method in (b) is stable under the Courant Friedrichs Lewy condition using Von Neumann analysis, you may assume $f(x,t) = 0$. (12)

[33]