DT9209/DT9210

Methods for Applied Mathematics, MATH 9951

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Basics

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From module descriptor (see handbook)

The module will be delivered through lectures and tutorials. In addition,students will be required to undertake background reading and self-directed learning. Modules are also typically supported by tutorial sheets, example classes and, laboratory sessions. The self-directed learning hours will be devoted to preparing for lectures, undertaking solutions to example sheets, reflecting upon the lecture material, refining and deepening understanding and consolidating individual learning.

Modules may be supported by online material and delivery and computer laboratory sessions. Where online delivery takes place this may substitute for some contact hours.

Total Teaching Contact Hours 39

Total Self-Directed Learning Hours 61

Assessment: 2 hour exam, 4 questions, answer 3.

Exam may be after Christmas during week starting 02 January (this is not confirmed). **The actual date of the exam is set by the examinations office and information from any other source is provisional only and should not be used.**

`www.dit.ie/academicaffairsandregistrar/calendar/`

The textbook for this module is *Linear Partial Differential Equations* by Tyn Myint-U & Lokenath Debnath (Birkhäuser, Fourth Edition).

Webcourses You need to self-register.

Module ID MATH9951 Methods for Applied Mathematics

enrolment code MATH9951

We will use the notiation $\frac{\partial u}{\partial x}$ and u_x interchageably.

We will use the notiation $\frac{dz}{dx}$ and z' interchageably.

The order of a PDE refers to the highest order derivative present.

In general, unless implied otherwise, we will assume solution functions u as being dependent on two independent variables x and y.

In general, unless implied otherwise: f, g, h, u, v, w, are used to denote functions; t, x, y, s, τ, are used to denote independent variables; a, b, c, A, B, C, are used to denote constants; and i, j, are used for indices.

The compact notation $p = z_x$ and $q = z_y$ may also be used to indicate first derivatives of the variable z (typically).

Note that lower indices may be used to denote components of tensors (eg. τ_{xy}) or particular values of variables (eg. x_0). In these cases, derivatives are indicated by subscripts following a comma. For example

$$\tau_{xy,xz} = \frac{\partial^2 \tau_{xy}}{\partial x \partial z}$$

Introduction

Classification; Linear Operators; Superposition

We say a PDE is linear if it is:

1. linear in the unknown function *u* and all of its derivatives;
2. all coefficients depend only on the independent variables.

We say a PDE is semilinear if it is linear in the highest order derivative (ie. not squared etc.) and the coefficient of this term does not depend on the function itself.

We say a PDE is quasilinear if it is linear in the highest order derivative (ie. not squared etc.) and the coefficient of this term *does* depend on the function itself.

An equation which has a power (other than 1) for the highest order derivative is referred to as nonlinear.

A PDE for which *all* terms contain the unknown function or its derivatives is called homogeneous. Otherwise, it is referred to as inhomogeneous.

General solution to a PDE depends on an arbitrary functions rather than constants, hence has infinite solutions

Typically, we will encounter problems in the form of a PDE with initial conditions and/or boundary conditions. An initial value problem without boundary conditions is called a Cauchy problem.

For systems of equations with supplementary conditions describing physical systems, we will require that the problem is *well-posed*. What this means is that a physically useful solution is available from the model. Formally, this requires:

1. existence
2. uniqueness
3. continuity

An operator is any set of mathematical operations which maps a function to another function, we usually write it as $L[u]$ or just Lu. A *differential* operator consists of derivatives.

A linear operator L satisfies

$$L[c_1 u_1 + c_2 u_2] = c_1 L[u_1] + c_2 L[u_2]$$

Linear differential operators satisfy:

1. $L + M = M + L$ commutative under addition
2. $(L + M) + N = L + (M + N)$ associative under addition
3. $(LM)N = L(MN)$ associative under multiplication
4. $L(c_1 M + c_2 M) = c_1 LM + c_2 LN$ distributive

If the coefficients of linear differential operators are *constant*, then they *commute*, and we also have $LM = ML$.

We frequently use ∇ to denote the operator $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ in 2D, or $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ in 3D.

Suppose we have a linear differential operator L, and n functions w_j which satisfy the homogeneous PDE

$$L[w_j] = 0$$

Now suppose we have n functions v_j which satisfy the n non-homogeneous PDEs

$$L[v_j] = G_j$$

The principle of superposition allows us to put these functions together to find a solution to the composite PDE

$$L[u] = \sum G_j$$

where

$$u = \sum w_j + \sum v_j$$

In fact, we can put together *any* convergent linear combination of solutions to the homogeneous PDE and obtain a further solution.

If solutions to a homogeneous linear PDE are known which are parameterized via a continuous variable k, say, then we can extend the idea of superpositioning solutions to an integral representation.

If $u(x, y; k)$ is a family of solutions with $k \in \mathbb{R}$, and $c(k)$ is any function of k such that

$$U(x, y) \equiv \int c(k)u(x, y; k)dk$$

is convergent, then $U(x, y, k)$ is also a solution.

First order PDEs

Method of Characteristics; Separation of Variables

The method of characteristics offers a means to reduce a PDE to ODEs.

Given a first-order quasilinear PDE

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u),$$

the general solution is obtained using

$$f(\phi,\psi) = 0$$

where f is any function of characteristic curves

$$\phi(x,y,u) = c_1 \quad \psi(x,y,u) = c_2.$$

Consider a three dimensional space with coordinates (x, y, u).

We can define a surface $\mathcal{S}$ by

$$F(x, y, u) = 0$$

where

$$F(x, y, u) \equiv u(x, y) - u$$

with $u(x, y)$ a solution to $a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u)$ - $\mathcal{S}$ is often called the solution surface, or integral surface.

The normal to this surface is

$$\nabla F = (u_x, u_y, -1)$$

Since

$$(\nabla F) \cdot (a, b, c) = 0,$$

we can conclude that (a, b, c) is tangent to the surface $\mathcal{S}$, and F is constant in this direction.

Therefore any curve $(x(\tau), y(\tau), u(\tau))$ (parameterized by τ) with tangent

$$\left(\frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{du}{d\tau}\right) = (a, b, c)$$

lies in the surface $\mathcal{S}$.

We have

$$\frac{dx}{d\tau} = a \quad \frac{dy}{d\tau} = b \quad \frac{du}{d\tau} = c$$

Note that in the homogeneous case ($c = 0$), the value of u along a characteristic curve $(x(\tau), y(\tau), u(\tau))$ is $u(x(\tau), y(\tau))$ and

$$\frac{du}{d\tau} = u_x \frac{dx}{d\tau} + u_y \frac{dy}{d\tau} = au_x + bu_y = 0$$

and hence u is constant along characteristic curves.

The characteristic curves of

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$$

obey the ODEs

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}$$

or, in the homogeneous case,

$$\frac{dx}{a} = \frac{dy}{b} \quad du = 0$$

Since a surface in three dimensions is two dimensional, we expect two independent families of surfaces to be defined by the characteristic equations, which we denote $\phi(x, y, u) = c_1$, $\psi(x, y, u) = c_2$.

The characteristics curves are the lines of intersection between any pair of surfaces, $\phi(x, y, u) = c_1$, $\psi(x, y, u) = c_2$, with one taken from each family.

Therefore, any surface defined by $\phi(x, y, u) = f(\psi(x, y, u))$ must have constant ϕ for constant ψ and is therefore a surface of characteristics, and hence a solution surface.

We can define a surface $\mathcal{S}$ by

$$F(x, y, u) = 0$$

where

$$F(x, y, u) \equiv u(x, y) - u$$

with $u(x, y)$ a solution to $xu_x + yu_y = u$.

(x, y, u) is tangent to the surface $\mathcal{S}$.

The characteristic curves of

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u)$$

obey

$$\phi = \frac{y}{x} = c_1 \quad \psi = \frac{u}{x} = c_2$$

The general solution is

$$u = xg\left(\frac{y}{x}\right)$$

Cauchy problem: PDE with intial data

$$u(x_0(s), y_0(s) = u_0(s)$$

on some interval of a parameter s.

The initial data describes a curve in (x, y, u)

$$\Gamma \ : \ x = x_0(s), \ y = y_0(s), \ u = u_0(s)$$

We seek the solution surface described by characteristics passing through Γ.

Cauchy problem for quasilinear PDE

A unique solution exists for continuously differentiable $x_0(s)$, $y_0(s)$, $u_0(s)$ over $s \in [0, 1]$, and coefficients a, b, c with continuous derivatives if

$$b[x_0(s), y_0(s), u_0(s)]x_0'(s) \neq a[x_0(s), y_0(s), u_0(s)]y_0'(s).$$

(ie. invertible Jacobian)

Example: Cuachy problem QL PDE

$$u_t + uu_x = x \quad u(x,0) = 1$$

Characteristic equations are

$$dt = \frac{dx}{u} = \frac{du}{x}$$

$$dt = \frac{dx}{u} = \frac{du}{x} = \frac{d(x+u)}{x+u}$$

The solutions are

$$(u+x)\mathrm{e}^{-t} = C_1 \quad u^2 - x^2 = C_2$$

The characteristic surface (solution) is given by

$$u^2 - x^2 = f((u+x)\mathrm{e}^{-t})$$

For the given Cauchy data

$$1 + x = c_1 \quad 1 - x^2 = c_2 \implies c_2 = 2c_1 - c_1^2$$

Hence

$$(u^2 - x^2) = 2(u + x)\mathrm{e}^{-t} - (u + x)^2\mathrm{e}^{-2t}$$

The homeneous form of the example considered previously is known as the inviscid Burgers equation. In the following discussion we consider a Caucy problem for this equation given by

$$u_t + uu_x = 0 \quad u(x, t = 0) = 1 - \cos x$$

The characteristic equations are

$$dt = \frac{dx}{u} \quad du = 0$$

with solutions

$$x - ut = c_1 \quad u = c_2$$

Solving these gives

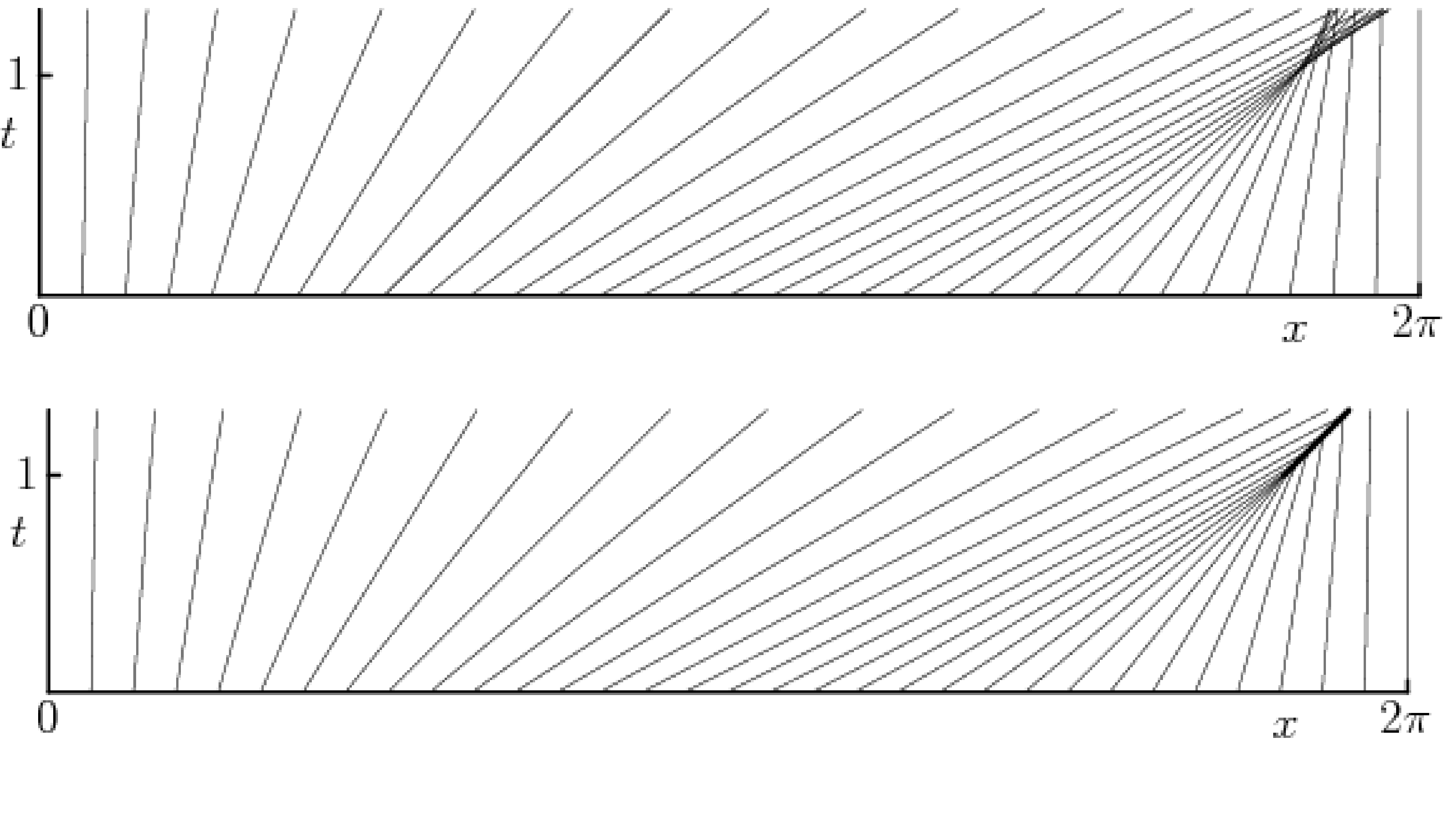
$$x_0 = c_1 \quad u_0 = 1 - \cos x_0 = c_2 \implies x - x_0 = (1 - \cos x_0)t$$

Hence, for the given Cauchy data, characteristics obey

$$t = \frac{x - x_0}{1 - \cos x_0}$$

with constant u along each characteristic.

Note that the characteristics cross - physically this steepening culminates in a *shock*. Can be avoided by adding in a viscous term ϵu_{xx} on RHS.



1
t
0
x
2π
1
t
0
x
2π

Similarly to the method of characteristics, separation of variables presents a way of reducing a PDE to an ODE problem.

If the general solution $u(x, y)$ to a PDE is in the form of a product

$$u(x, y) = X(x)Y(y)$$

or a sum

$$u(x, y) = X(x) + Y(y)$$

we can obtain the solution by solving ODEs. (Otherwise, it doesn't work, so the simple strategy is just to try it.)

$$u_x + 2u_y = 0 \quad u(0, y) = 4e^{-2y}$$

$X'Y + 2XY' = 0$

$\frac{X'}{2X} = -\frac{Y'}{Y} = \lambda$ (say, since LHS depends on x only and RHS depends on y only,

and we disregard the spurious solution $u = 0$.)

Hence, we now have a pair of ODEs

$$X' - 2\lambda X = 0 \quad Y' + \lambda Y = 0$$

with solutions

$$X = Ae^{2\lambda x} \quad Y = Be^{-\lambda y}$$

leading to the general solution

$$u(x, y) = Ce^{\lambda(2x-y)}$$

Appealing to the initial conditions, we finally have

$$u(x, y) = 4e^{2(2x-y)}$$