6/02/2012

Phase plane analysis (2)

$$\begin{cases} \dot{x} = 2x - \frac{3}{2}y \\ \dot{y} = -\frac{2}{3}x + 2y \end{cases}; \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & -\frac{3}{2} \\ -\frac{2}{3} & 2 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

a b c d

(0, 0)

Eigenvalues of A

$$\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0$$

$$\lambda^2 - 4\lambda + 3 = 0 \iff (\lambda-1)(\lambda-3) = 0$$

$\lambda_1 = 1$, $\lambda_2 = 3$, unstable node

$$v^{(1)} = \begin{pmatrix} 1 \\ p_1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{\lambda_1 - a}{b} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1-2}{-\frac{3}{2}} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{2}{3} \end{pmatrix}$$

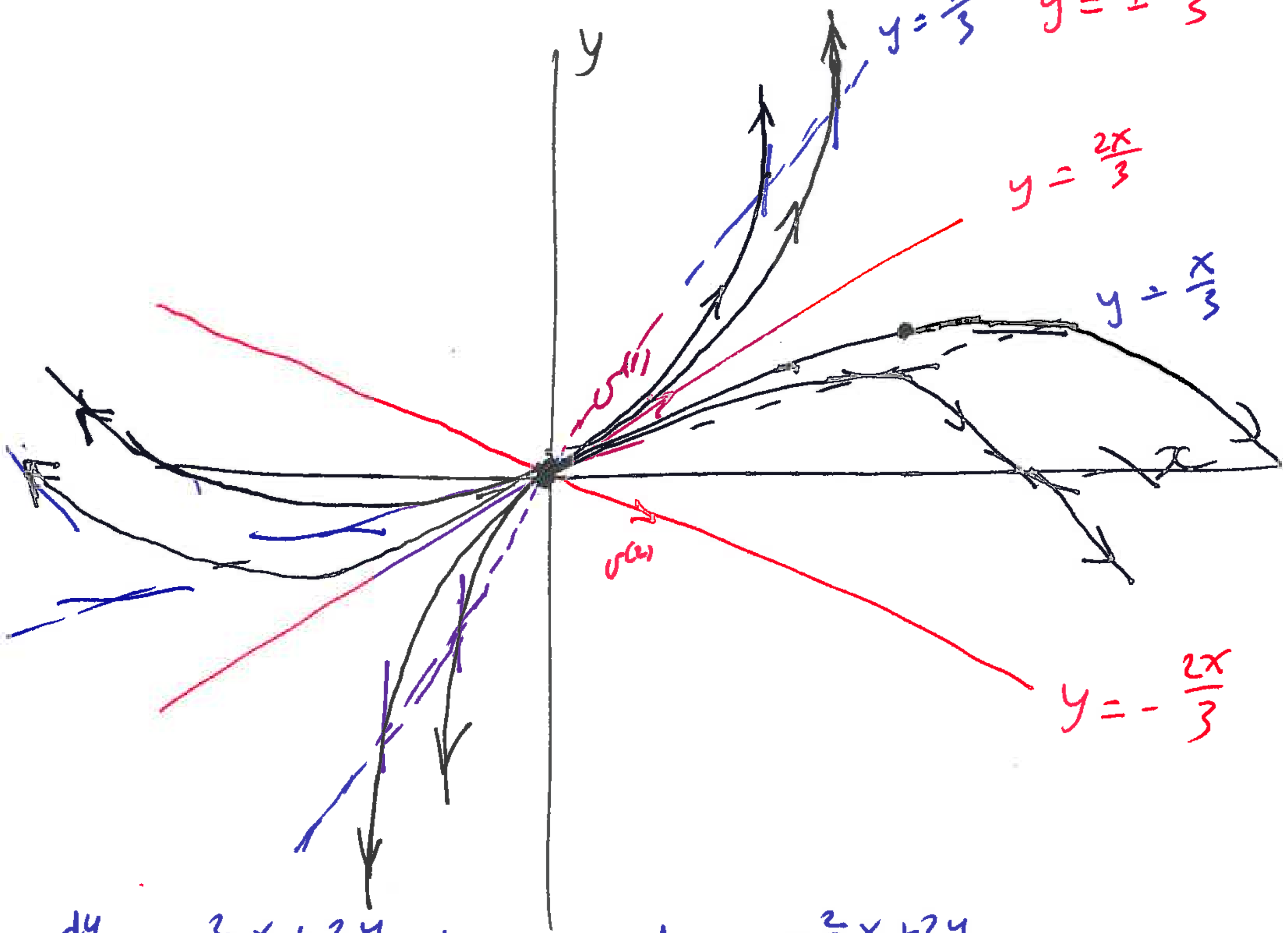
$$v^{(2)} = \begin{pmatrix} 1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{\lambda_2 - a}{b} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{3-2}{-\frac{3}{2}} \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{2}{3} \end{pmatrix}$$

$v^{(1)}$ points along the line $\frac{x}{y} = \frac{1}{2/3} = \frac{3}{2}$

$v^{(2)}$ — " — $\frac{x}{y} = \frac{1}{-2/3} = -\frac{3}{2}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 v^{(1)} \underline{\underline{e^{\lambda_1 t}}} + c_2 v^{(2)} \underline{\underline{e^{\lambda_2 t}}}$$

$$\frac{y}{x} = \pm\frac{2}{3} \Rightarrow y = \frac{\pm 2}{3}x$$



$$\left.\begin{aligned}\frac{dy}{dt} &= -\tfrac{2}{3}x + 2y \\ \frac{dx}{dt} &= 2x - \tfrac{3}{2}y\end{aligned}\right\} \Rightarrow \quad \frac{dy}{dx} = \frac{-\frac{2}{3}x + 2y}{2x - \frac{3}{2}y}$$

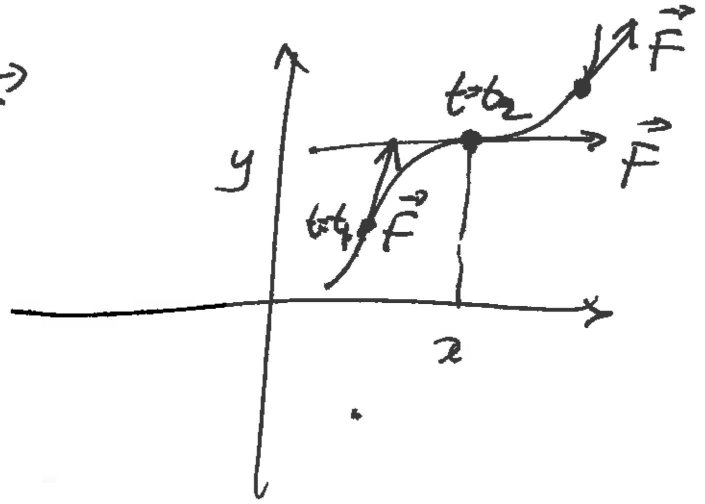
$\frac{dy}{dx} = \infty$ when $2x = \frac{3}{2}y$, i.e. $y = \frac{4x}{3}$

$\frac{dy}{dx} = 0$ when $+\frac{2}{3}x = 2y \Rightarrow y = \frac{x}{3}$

when $y = 0$ $\quad \frac{dy}{dx} = \frac{-\frac{2}{3}}{2} = -\frac{1}{3} < 0$

$x = 0$ $\quad \frac{dy}{dx} = \frac{2}{-\frac{3}{2}} = -\frac{4}{3} < 0$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \vec{F}$$



$$\begin{cases} \dot{x} = y \\ \dot{y} = 4x \end{cases} \qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \overset{a}{0} & \overset{b}{1} \\ \underset{c}{4} & \underset{d}{0} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\lambda^2 - \mathrm{tr}A\,\lambda + \det A = 0$$

$$\lambda^2 - 4 = 0, \quad \lambda_1 = 2, \quad \lambda_2 = -2$$ saddle

$$v^{(1)} = \begin{pmatrix} 1 \\ p_1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{\lambda_1 - a}{b} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{2-0}{1} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$v^{(2)} = \begin{pmatrix} 1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{\lambda_2 - a}{b} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{-2-0}{1} \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

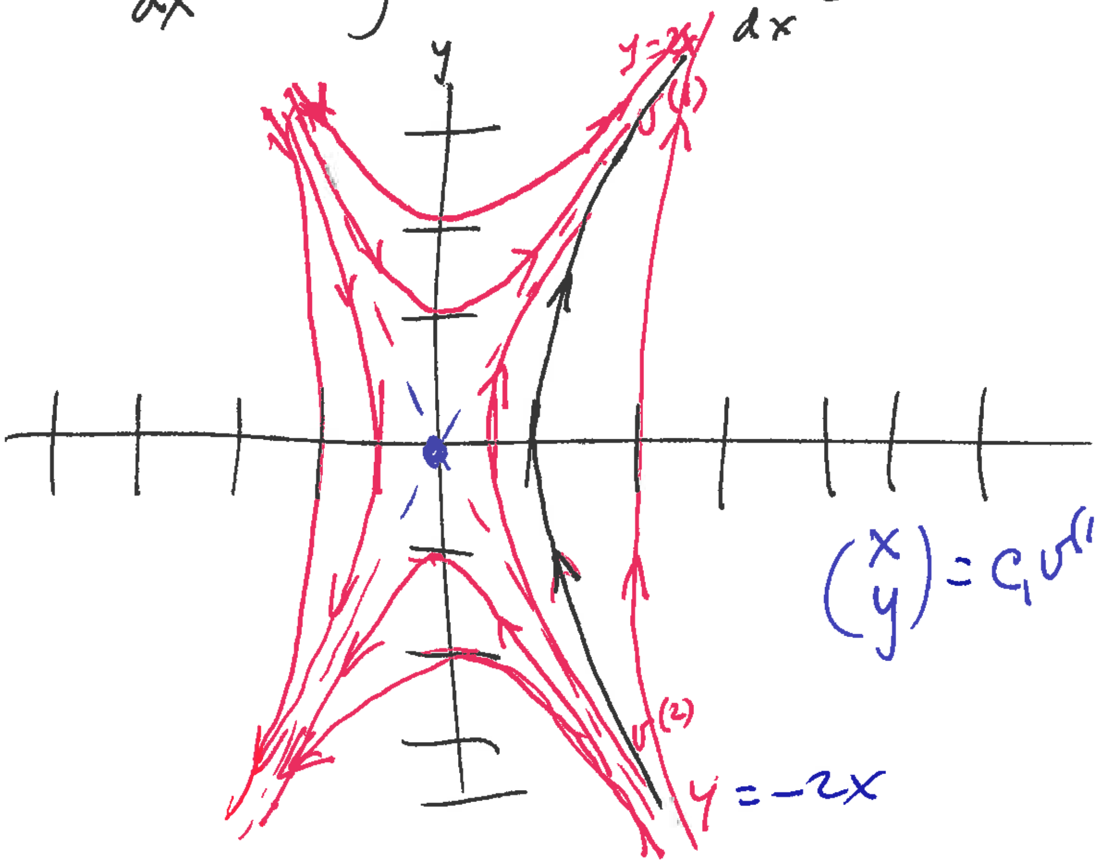
$v^{(1)}$: $\frac{y}{x} = \frac{2}{1} = 2 \Rightarrow y = 2x$

$v^{(2)}$: $\frac{y}{x} = -2 \Rightarrow y = -2x$

$$\frac{dy}{dx} = \frac{4x}{y} \Rightarrow$$

$\frac{dy}{dx} = \infty$ when $y = 0$

$\frac{dy}{dx} = 0$ when $x = 0$



$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 v^{(1)} e^{\lambda_1 t} + c_2 v^{(2)} e^{\lambda_2 t}$$

$$\begin{cases} \dot{x} = y \\ \dot{y} = -2x - 2y \end{cases} \qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\lambda^2 - \operatorname{tr} A + \det A = 0$$

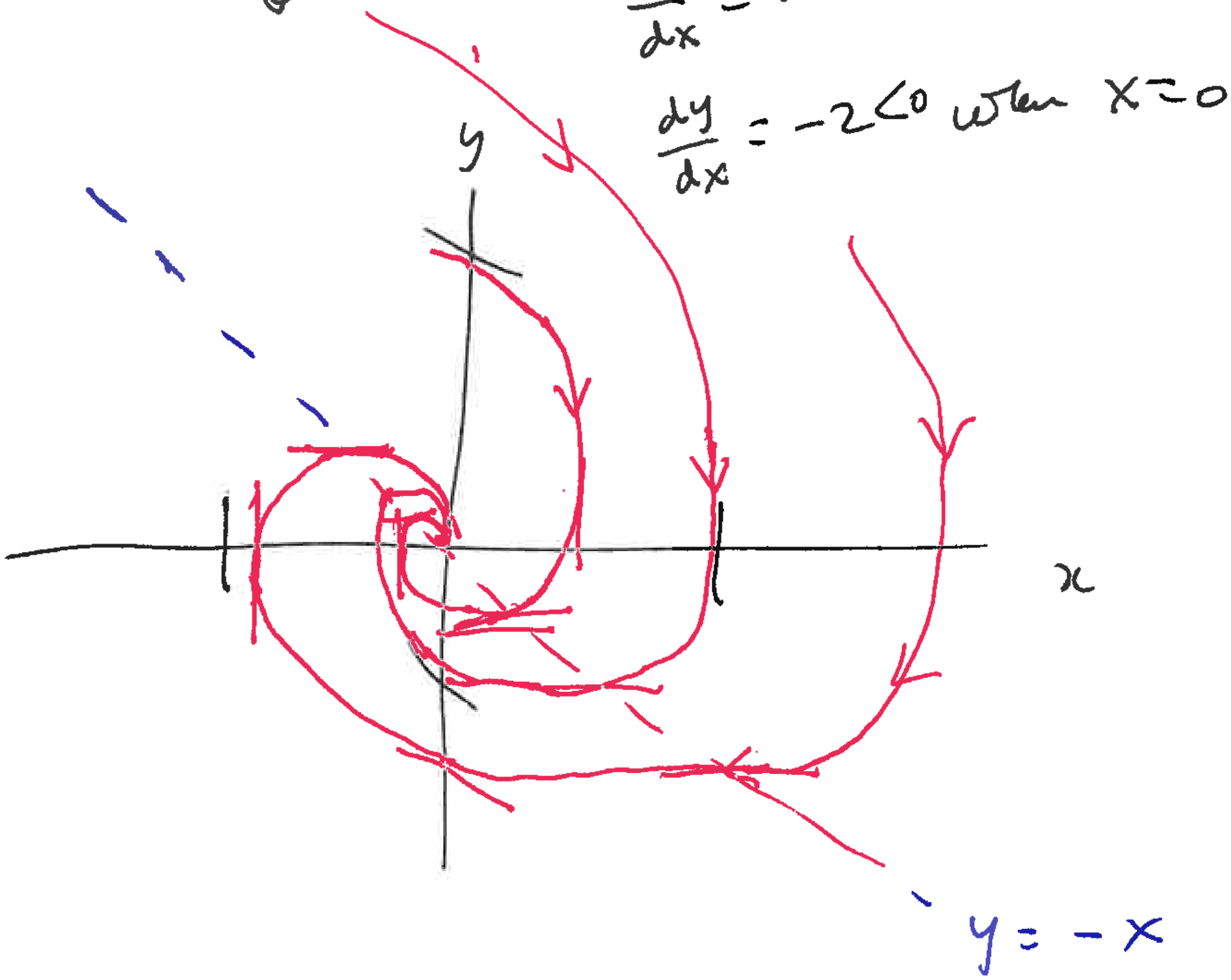
$$\lambda^2 + 2\lambda + 2 = 0 \qquad (\lambda+1)^2 + 1 = 0$$

$$\lambda + 1 = \pm i \quad \Rightarrow \quad \boxed{\lambda_{1,2} = -1 \pm i}$$

$$\frac{dy}{dx} = \frac{-2x - 2y}{y} ; \qquad \frac{dy}{dx} = 0 \text{ on } y = -x$$

$$\frac{dy}{dx} = \infty \text{ on } y = 0$$

$$\frac{dy}{dx} = -2 < 0 \text{ when } x = 0$$



$$\left|\begin{array}{l} \dot{x} = x(y-1) = f(x,y) \\ \dot{y} = y(x-1) = g(x,y) \end{array}\right.$$

$y=0$ $x=1$

$x=0$, $y=1$

Equilibrium $\left|\begin{array}{l} f(x,y)=0 \\ g(x,y)=0 \end{array}\right.$ $\left\|\begin{array}{l} x(y-1)=0 \\ y(x-1)=0 \end{array}\right.$

$(0,0)$ $(1,1)$

Jacobian $A(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} y-1 & x \\ y & x-1 \end{pmatrix}$

$\left\|\begin{array}{l} x_0 = 0 \\ y_0 = 0 \end{array}\right.$ $\begin{array}{l} 0 = \dot{x}_0 = f(x_0, y_0) = 0 \\ 0 = \dot{y}_0 = f(x_0, y_0) = 0 \end{array}$

$A(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, Eig. $\lambda_1 = \lambda_2 = -1$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are eigenvectors

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} v^{(1)} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (-1)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

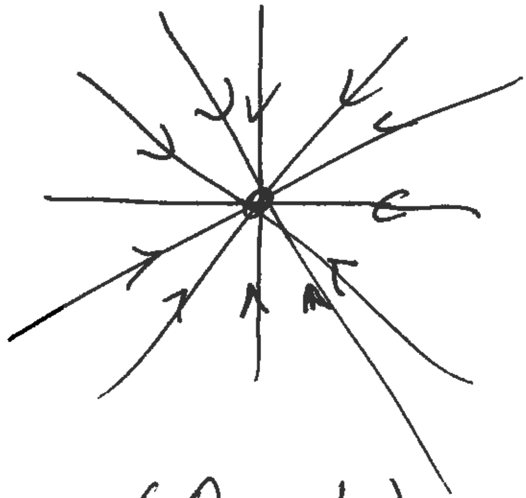
$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 v^{(1)} e^{\lambda_1 t} + c_2 v^{(2)} e^{\lambda_2 t}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t} = \begin{pmatrix} c_1 e^{-t} \\ c_2 e^{-t} \end{pmatrix}$$

$$\left.\begin{aligned} x &= c_1 e^{-t} \\ y &= c_2 e^{-t} \end{aligned}\right\} \quad \frac{y}{x} = \frac{c_2}{c_1} = \text{const}$$

$$y = (\text{const})\, x$$

'star'



stable

$$A(1,1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{Eig. } \lambda_1 = 1 \quad \lambda_2 = -1$$

$$\lambda^2 + (-1) = 0$$

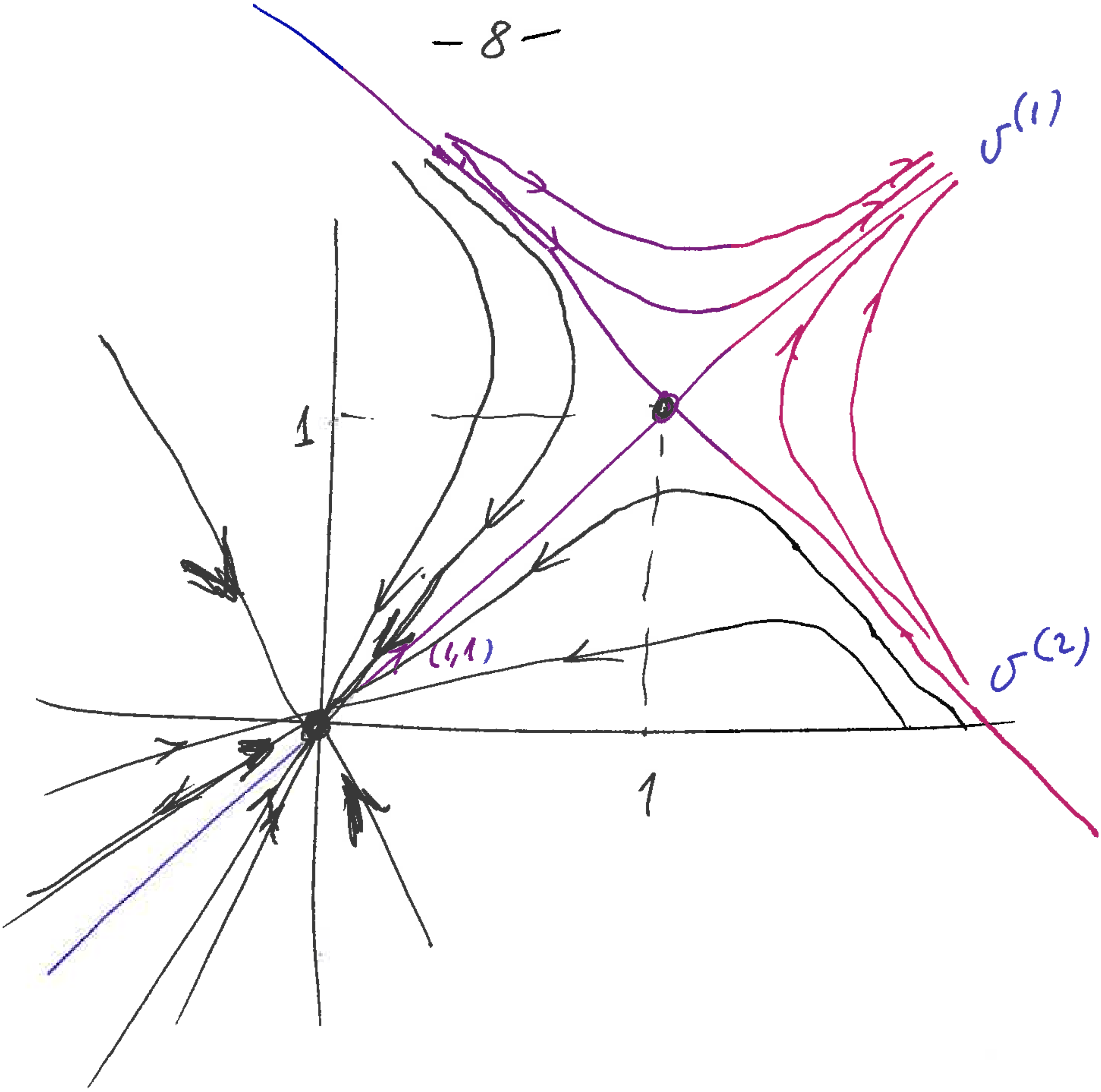
Eigenvectors

$\lambda_1 = 1 \rightarrow$ $$v^{(1)} = \begin{pmatrix} 1 \\ p_1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{\lambda_1 - a}{b} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1-0}{1} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\lambda_2 = -1 \rightarrow$ $$v^{(2)} = \begin{pmatrix} 1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{\lambda_2 - a}{b} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{-1-0}{1} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

'saddle'

$$\begin{pmatrix} x - \cancel{1} \\ y - 1 \end{pmatrix} = c_1 \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{v^{(1)}} e^{t} + c_2 \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{v^{(2)}} e^{-t}$$



$v^{(1)}$
1
$(1,1)$
$v^{(2)}$
1

$$\frac{dx}{dt} = -y + x(1-x^2-y^2)$$

$$\frac{dy}{dt} = x + y(1-x^2-y^2)$$

$$x = r\cos\varphi \qquad \boxed{x^2+y^2=r^2}$$
$$y = r\sin\varphi$$

$$\dot{x} = \dot{r}\cos\varphi - r\sin\varphi\,\dot{\varphi}$$
$$\dot{y} = \dot{r}\sin\varphi + r\cos\varphi\,\dot{\varphi}$$

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}\dot{r}\\ r\dot{\varphi}\end{pmatrix}$$

$$\underbrace{\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}}_{R}\begin{pmatrix}\dot{r}\\ r\dot{\varphi}\end{pmatrix} = \begin{pmatrix}-r\sin\varphi + r\cos\varphi(1-r^2)\\ r\cos\varphi + r\sin\varphi(1-r^2)\end{pmatrix}$$

$$\begin{pmatrix}\dot{r}\\ r\dot{\varphi}\end{pmatrix} = \underbrace{\begin{pmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}}_{R^{-1}}\begin{pmatrix}-r\sin\varphi + r(1-r^2)\cos\varphi\\ r\cos\varphi + r(1-r^2)\sin\varphi\end{pmatrix}$$

$$\begin{pmatrix}\dot{r}\\ r\dot{\varphi}\end{pmatrix} = \begin{pmatrix}\cancel{-r\cos\varphi\sin\varphi} + r(1-r^2)\cos^2\varphi + \cancel{r\sin\varphi\cos\varphi} + r(1-r^2)\sin^2\varphi\\ r\sin^2\varphi - \cancel{r(1-r^2)\sin\varphi\cos\varphi} + r\cos^2\varphi + \cancel{r(1-r^2)\sin\varphi\cos\varphi}\end{pmatrix}$$

$$\begin{pmatrix}\dot{r}\\ r\dot{\varphi}\end{pmatrix} = \begin{pmatrix}r(1-r^2)\\ r\end{pmatrix} \Rightarrow \left|\begin{array}{l}\dot{r} = r(1-r^2)\\ r\dot{\varphi} = r\\ \dot{\varphi} = 1\end{array}\right.$$

$$\begin{cases} \dot{r} = r(1-r^2) \\ \dot{\varphi} = 1 \end{cases} \Rightarrow \quad \varphi = t + \text{const} \ / \bmod 2\pi$$

$$\boxed{\dot{r} = r(1-r^2)}$$

$$\frac{dN}{dt} = \underline{\underline{N(1-N^2)}} = f(N)$$

$$\boxed{r = 0 \quad \& \quad r = 1}$$

$r = 0$ unstable

$r = 1$ stable.

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$\frac{dx}{dt} = -y + x\left(1 - x^2 - y^2\right) = f_1$$

$$\frac{dy}{dt} = x + y\left(1 - x^2 - y^2\right) = f_2$$

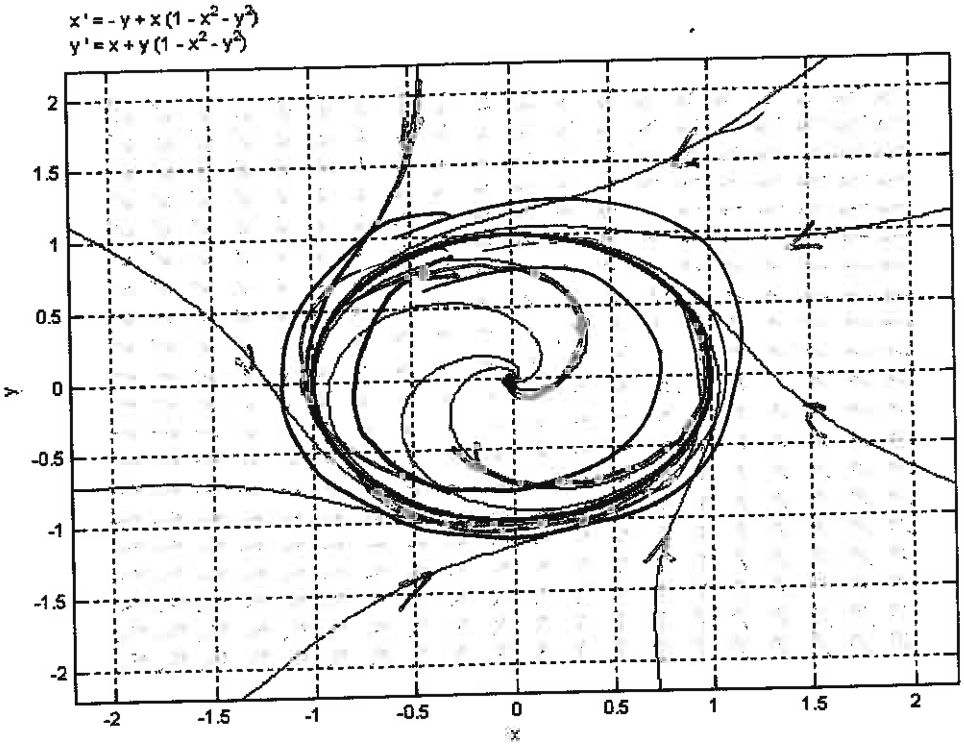


Figure A.13: a limit cycle

$$\dot{x} = x - y - x\,(x^2 + 2y)$$

$$\dot{y} = x + y - y\,(x^2 + y^2)$$

I. Change the coordinates:

$x = r\cos\varphi$ polar form

$y = r\sin\varphi$

$$\begin{cases} \dot{r} = r - r^3\left(1 + \frac{1}{4}\sin^2 2\varphi\right) \\ \dot{\varphi} = 1 + \frac{1}{2} r^2 \sin^2\varphi \sin 2\varphi \end{cases}$$

$\underline{\underline{\dot{r} > 0}}$

$$\underset{1}{\cancel{r}} > r^{\cancel{3}\,2}\left(1 + \frac{1}{4}\sin^2 2\varphi\right) \quad \text{all } \varphi$$

$$r^2 < \frac{1}{1 + \frac{1}{4}\sin^2 2\varphi} \quad \text{all } \varphi$$

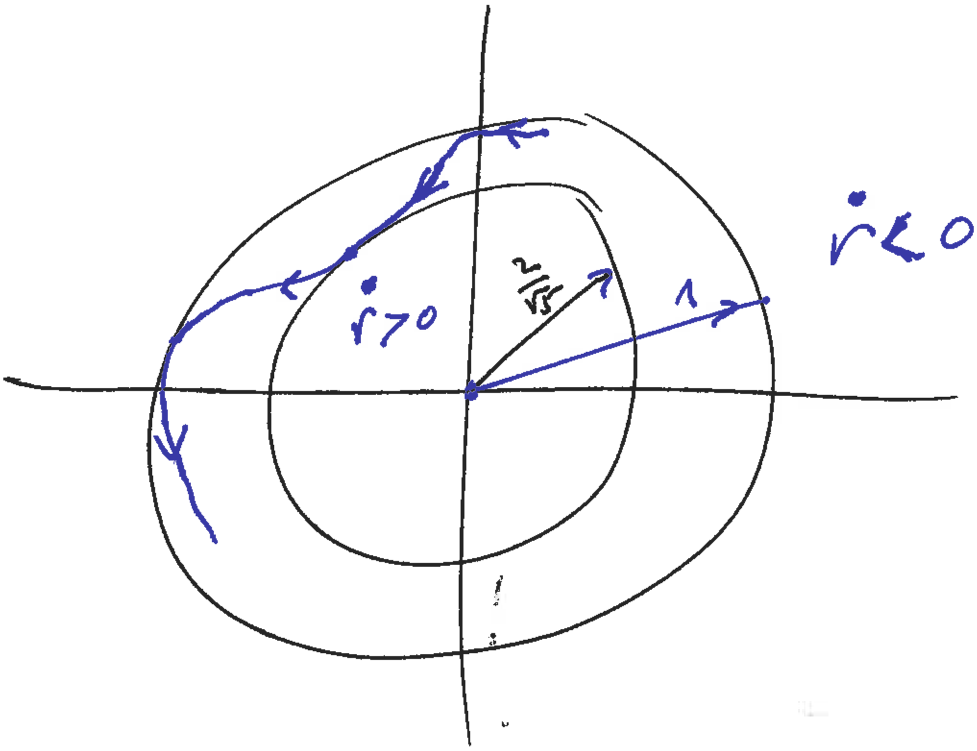
$$r^2 < \frac{1}{1 + \frac{1}{4}\cdot 1} = \frac{1}{5/4} = \frac{4}{5} \Rightarrow \dot{r} > 0$$

$\dot{r} < 0$

$$r^2 > \frac{1}{1 + \frac{1}{4}\sin^2 2\varphi} \quad \text{all } \varphi$$

$$r^2 > \frac{1}{1+0} = 1 \quad \text{all } \varphi. \Rightarrow \dot{r} < 0$$

$\frac{2}{\sqrt{5}} < r < 1$



'trapping region'

+ (0, 0) unstable (check ?)

=> limit cycle exists