21

A nonlinear difference equation

In general a nonlinear difference equation

$$x_{n+1} = f(x_n) \tag{21.1}$$

can have very complicated behaviour. In this chapter we will look at a particular example,

$$x_{n+1} = rx_n(1 - x_n),$$

where r is a parameter that we will vary between 0 and 4.

In order to describe the solution we will use qualitative ideas similar those we used to study autonomous equations ($\dot{x} = f(x)$) in chapter 7, and concentrate on what happens to solutions 'eventually'.

When we have a difference equation the solution is given by a sequence of *iterates of f*,

$$x_0, \quad x_1 = f(x_0), \quad x_2 = f(f(x_0)), \quad x_3 = f(f(f(x_0))), \quad \dots$$

Since these nested fs rapidly become unmanageable we adopt the notation $f^n(x)$ to mean f applied n times to x,

$$f^n(x) = \underbrace{f(f(f(\cdots f}_{n \text{ times}}(x)\cdots)).$$

We can write the 'solution' of (21.1) that has $x_0 = y_0$ as $x_n = f^n(y_0)$, but this is clearly no more descriptive of the solution than (21.1) itself.

21.1 Fixed points and stability

In order to describe the dynamics of solutions we make use of similar concepts as we used for the 1d dynamical systems that arise from autonomous

differential equations. For an *iterated map* like (21.1) a *stationary point* is a point x^* such that

$$f(x^*) = x^*,$$

so that if $x_n = x^*$ then $x_{n+1} = x^*$. [This corresponds to the 'stationary points' of $\dot{x} = f(x)$.]

We describe a stationary point as *stable* if you stay close to it provided that you start sufficiently near: x^* is stable if for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\underbrace{|x_0 - x^*| < \delta}_{\text{start near}} \quad \Rightarrow \quad \underbrace{|f^n(x_0) - x^*| < \epsilon \quad \text{for all} \quad n \in \mathbb{Z}^+}_{\text{stay near}}.$$

Again, we can also introduce the related but distinct concept of being attracting ('start near tend to'): x^* is *attracting* if there is a $\delta > 0$ such that

$$\underbrace{|x_0 - x^*| < \delta}_{\text{start close enough}} \quad \Rightarrow \quad \underbrace{f^n(x_0) \to x^* \quad \text{as} \quad n \to \infty}_{\text{tend to}}.$$

Negating 'stable' we get unstable: x^* is *unstable* if there exists an ϵ such that no matter how small we make δ, we can find an x_0 with

$$|x_0 - x^*| < \delta \qquad \text{but} \qquad |f^n(x_0) - x^*| > \epsilon \quad \text{for some } n \in \mathbb{Z}^+.$$

In order to discover analytically whether or not a stationary point is stable, suppose that $x_n = x^* + \delta_n$ where δ_n is small; then

$$\begin{aligned} x_{n+1} &= f(x^* + \delta_n) \\ &\approx f(x^*) + f'(x^*)\delta_n \\ &= x^* + f'(x^*)\delta_n. \end{aligned}$$

So if we write $x_{n+1} = x^* + \delta_{n+1}$ we have

$$\delta_{n+1} \approx f'(x^*)\delta_n; \tag{21.2}$$

clearly as we iterate (21.2) successive values of δ_j will decrease if $|f'(x^*)| < 1$, and increase if $|f'(x^*)| > 1$. So x^* is stable if $|f'(x^*)| < 1$ and unstable if $|f'(x^*)| > 1$.

21.1.1 'Cobweb' diagrams

In order to work out what happens when we iterate f, i.e. apply it again and again, we can use a graphical method. If we draw the graph of $f(x)$ and of

$y = x$, then given x_n we can find x_{n+1} by drawing a line up to $f(x_n)$, then over to $y = x$, and back down to the x axis to give x_{n+1}. This is illustrated in figure 21.1.
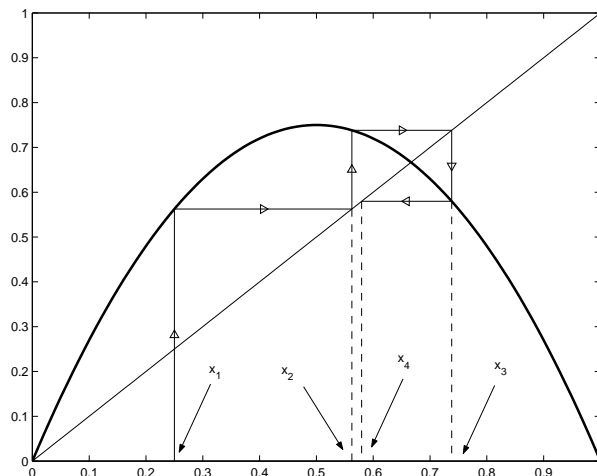

Fig. 21.1. The first few steps of the 'cobweb' method

We apply this method and a little more theory to investigate the chaos theoretician's favourite model, which is very simple but which turns out to behave in an extremely complicated way.

21.2 The logistic equation

For the remainder of this chapter we consider the equation

$$x_{n+1} = rx_n(1 - x_n),$$

with $0 \leq r \leq 4$. We choose this range of values for r since it implies that if $x_n \in [0, 1]$ then $x_{n+1} \in [0, 1]$ (the maximum value of $rx(1 - x)$ occurs when $x = \frac{1}{2}$ and is $r/4$). The graphs of $f(x) = rx(1 - x)$ are shown in figure 21.2 for various values of r.

The equation can be thought of as a discrete model of a population (we saw a very similar differential equation earlier) with limited resources. If the population is small (measured, presumably, in thousands or millions to give an approximately continuous variable) then its size will increase, since when $x_n \approx 0$,

$$x_{n+1} \approx rx_n;$$

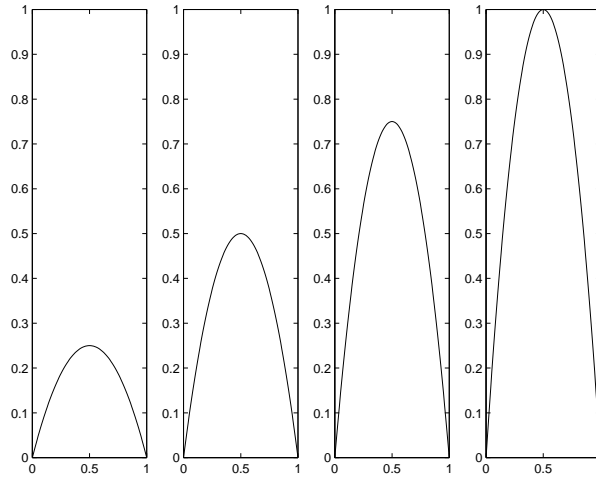but when x_n approaches the maximum size supportable by the resources

Fig. 21.2. Graphs of $rx(1 - x)$ for $r = 1$, 2, 3, and 4

available (which here is 1), the population dies out rapidly: if $x_n = 1 - y_n$ with $y_n \approx 0$ then

$$x_{n+1} \approx ry_n,$$

so that most of the population dies out.

What one would naïvely expect from such a model is that the population size would settle down to some steady state, or perhaps oscillate between two nearby states, and indeed this does happen when r is small. However, as r increases towards 4 this intuition can be very wrong.

We will base our discussion around the existence of stationary points and periodic orbits. However, we will also make use of the idea of the 'attractor'; rather than defining it precisely here, the best way to think of it as the points around which the orbits of $x_{n+1} = f(x_n)$ will move 'eventually'.

21.3 The fixed points and their stability

Note that there are two stationary points, which are the solutions of

$$x = rx(1 - x) :$$

these are $x = 0$ are $x = 1 - (1/r)$. The non-zero stationary point will only be interesting when it is positive (for $r > 1$), since we will only consider positive populations.

If we calculate $f'(x) = r(1 - 2x)$ then

$$f'(0) = r,$$

so the stationary point at $x = 0$ will be stable while $r < 1$, and unstable once $r > 1$.

If $0 < r < 1$ then there is no positive stationary point, and over time the population decreases to zero, since the reproductive rate is not high enough to sustain the population. So in this case the dynamics is very simple, and the attractor is just zero. As remarked above, $x = 0$ is stable. (See figure 21.3.)
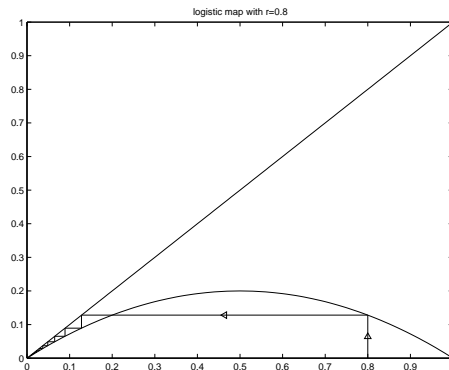

Fig. 21.3. For $0 < r < 1$ the origin is a stable stationary point, and the population dies out

When $r > 1$ the origin is no longer stable, and there is another positive stationary point. Since

$$f'(1 - (1/r)) = r(1 - 2 + (2/r)) = 2 - r,$$

this stationary point is stable while $r < 3$. So for $1 < r < 3$ all orbits are attracted to $1 - (1/r)$, as shown in figure 21.4.

When r increases beyond 3 things become more complicated. The stationary point at $1 - (1/r)$ is now unstable, since the derivative of f there has modulus larger than 1.

21.4 Periodic orbits

When $r > 3$ the solution no longer tends towards a stationary point: there are only two of them, and both are unstable. Instead, if r is just a little larger than three, almost every choice of initial condition (apart from either stationary point) ends up cycling between two different values of x, as shown in figure 21.5.

Here there are values x_1 and x_2 such that

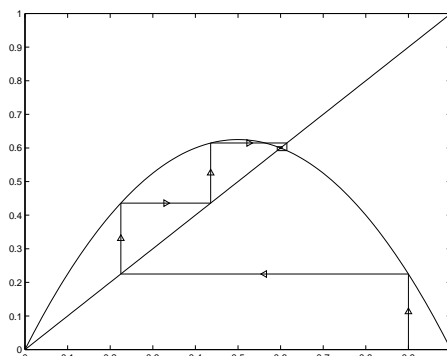$$f(x_1) = x_2 \qquad \text{and} \qquad f(x_2) = x_1,$$

Fig. 21.4. For $1 < r < 3$ the origin is unstable, and there is an attracting non-zero stationary point
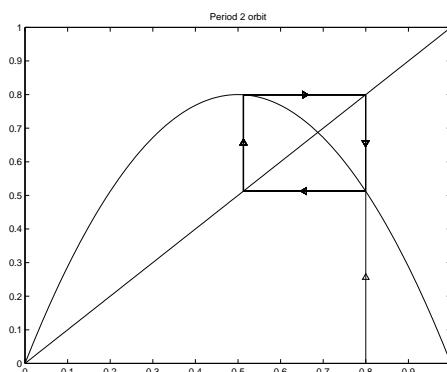

Fig. 21.5. For $r > 3$ there are periodic orbits of period 2. This picture has $r = 1 + \sqrt{5} \approx 3.2361$.

so that $f^2(x_1) = x_1$. Such a pair of points is called a *periodic orbit of period 2* or a *period 2 orbit*.

In general a periodic orbit of period k (or a period k orbit) is a sequence of k values $x_1, \ldots, x_k$ such that

$$f(x_j) = x_{j+1} = f(x_j) \qquad \text{for} \qquad j = 1, \ldots, n-1 \qquad \text{and} \qquad f(x_n) = x_1,$$

so that iterates of x_1 cycle around these k values for ever. [Strictly we also need to make sure that $f(x_j) \neq x_1$ for $j = 1, \ldots, n-1$, so that k is the 'minimal period' of the orbit.]

If we try to find a period 2 orbit analytically then we want to find a value of y such that if $x_n = y$ then $x_{n+2} = y$. So we want

$$y = f^2(y)$$

which is

$$y = r[ry(1 - y)][1 - ry(1 - y)]. \qquad (21.3)$$

This is a quartic (fourth order) equation for y; but since we know that $y = 0$ and $y = 1 - (1/r)$ must be solutions (they are stationary points with $f(y) = y$, so certainly $f(f(y)) = f(y) = y$) we can remove a factor $r(r - [1 - (1/r)])$: if y is a period 2 point it must solve the equation

$$ry^2 - (1 + r)y + \left(1 + \frac{1}{r}\right) = 0.$$

This equation only has real roots if the discriminant is positive ("$b^2 - 4ac > 0$"), i.e. if

$$(1 + r)^2 - 4r\left(1 + \frac{1}{r}\right) = (1 + r)(r - 3) > 0.$$

Since we have restricted to the parameter range to $0 \le r \le 4$ the first factor is positive; for a period 2 orbit we must have $r > 3$.

21.5 Orbits beyond period 2

Insert picture here: r `values:  0.5, 2, 3, 3.01, 3.5, 3.55, 3.565, 3.5701, 3.9, 4`

As r is increased a little further, this attracting periodic orbit of period 2 becomes unstable - two points break off from each point on the orbit, and we end up with an orbit of period 4, as shown in figure 21.6.

In order to see why this might happen, we consider what happens when we apply f twice, i.e. we look at the graph of f^2. When $r < 3$ we have a picture like that shown in figure 21.7 (the graph shows the diagonal, f, and f^2). Both of the boxes are mapped into themselves when we apply f^2. Magnifying the right-hand box we get figure 21.8, which looks very similar to our logistic map when $r < 1$ (no stationary point except the origin).

However, when $r > 3$ the graph of f^2 has changed considerably, as shown in figure 21.9. Blowing up the right-hand box again gives figure 21.10, which now looks the logistic map with a stationary point.

By the time we reach $r = 4$ the graph of f^2 is shown in figure 21.11, and the right-hand box magnified in figure 21.12: the top of the curve has gone right out of the box.

So as we increase r we also increase the height of the bit of f^2 in the right-hand box. So we expect that the dynamics of f^2 restricted to this box will undergo the same changes of behaviour that happen to the dynamics
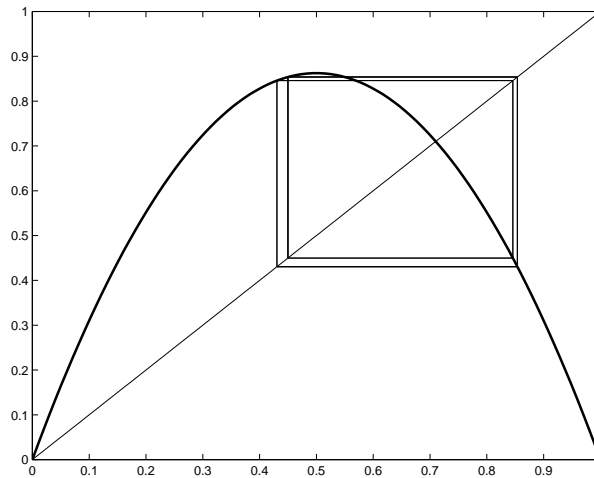
Fig. 21.6. A period 4 orbit when $r = 3.45$
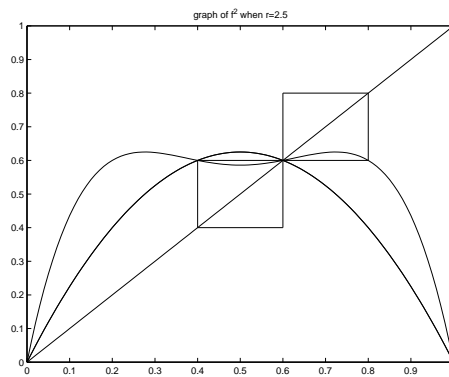
Fig. 21.7. Graph of f^2 when $r = 2.5$

of f as we increase r. This idea, known as 'renormalisation', enables us to understand what happens in great detail, and is also an important idea in theoretical physics when studying phase transitions (liquid/solid, liquid/gas, etc.)

The sketchy version given here goes at least some wasy to explain why we would expect the period 2 orbit to become unstable and turn into an attracting period 4 orbit. But we could then do the same with f^4 and find a period 8 orbit... This *period doubling cascade* continues, creating, faster and faster as we increase parameters, periods of order 2^n, until at a critical
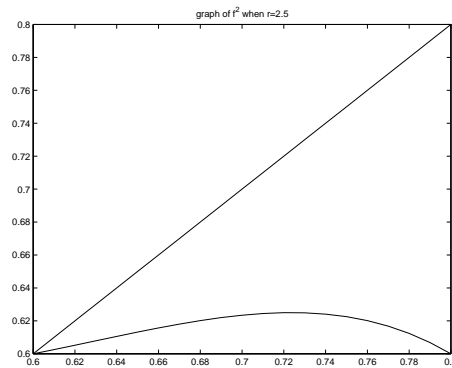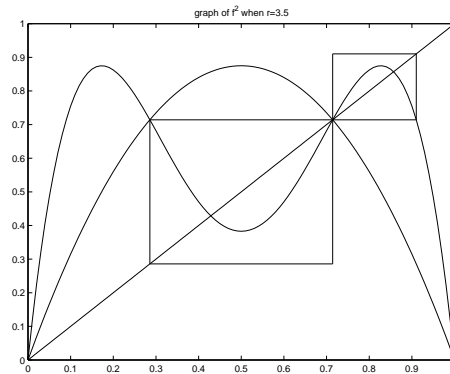
Fig. 21.8. The right-hand box of figure 21.7 magnified

Fig. 21.9. Graph of f^2 when $r = 3.5$

parameter value, $r \approx 3.5701$, there are periodic orbits of orders 2^n for every n.

The universal property that can be obtained from the renormalisation theory relates the convergence of the parameter values r_n at which the orbit of period 2^n becomes unstable,

$$\lim_{n \to \infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.667\ldots,$$

and this number, 'Feigenbaum's constant', is independent of the map, provided that (in a well-defined way) it 'looks like' our map $f(x) = rx(1 - x)$. [In fact it has to be a 'one-hump map with a quadratic maximum'.]
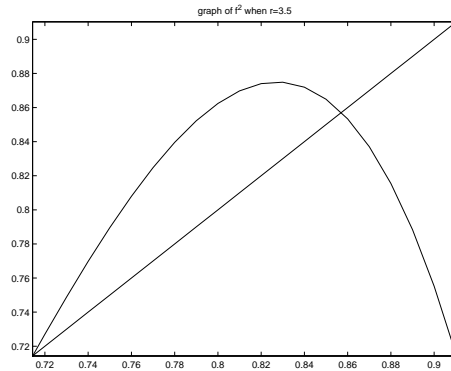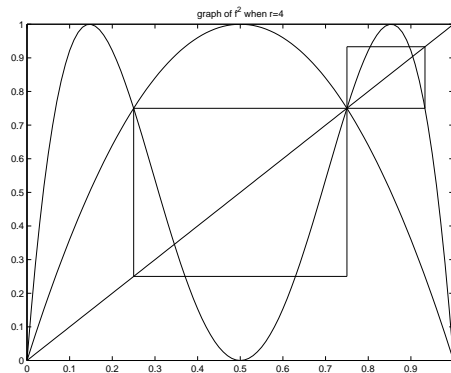
Fig. 21.10. The right-hand box of figure 21.9 magnified

Fig. 21.11. Graph of f^2 when $r = 4$

21.6 The bifurcation diagram and more periodic orbits

Figure 21.13 is the *bifurcation diagram* that plots the attractor (vertically) against the parameter r.

You can see the first few bits of the period doubling cascade before the parameters become too close to distinguish. After $r \approx 3.5701$ everything just starts to look a mess. However, there is still order. For example, you can see that this diagram is 'self-similar' by magnifying a small portion and observing that it looks very similar to the original diagram, as shown in figure 21.14.

Notice also that there are 'windows' in which the solution is more regular again - you can see one in figure 21.15 here where there is a period 6 orbit. By the same mechanism that produces the orbits of period 4, 8, etc. from
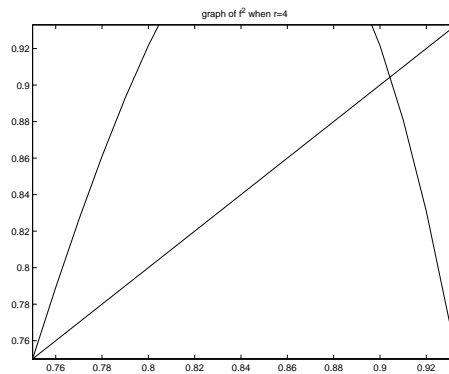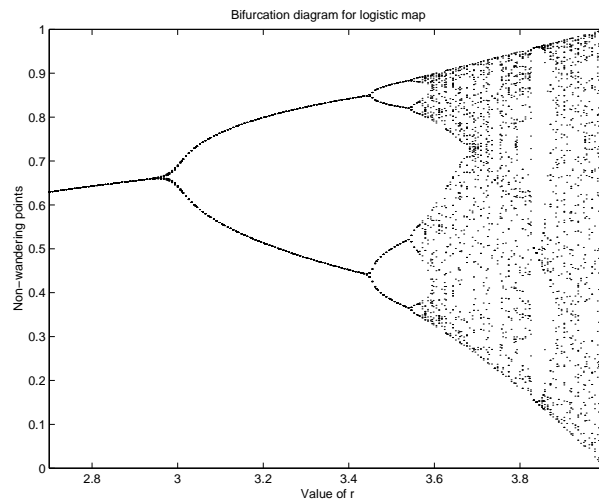
Fig. 21.12. The right-hand box of figure 21.11 magnified

Fig. 21.13. The bifurcation diagram for $0 \leq r \leq 4$

the initial period 2 orbit, this period 6 orbit will period double to 12, 24, 48, and lead to another chaotic region, as you can see from the enlarged picture.

Insert picture here: `bifurcations:  0:4, 2.9:3.6, 3.55:3.58 ylim([0.3 0.4]), 3.6:4, 3.83:3.86 ylim([0.4 0.6])`

21.7 Chaos

Exactly what 'chaos' is is not entirely clear, since the literature contains many definitions. Essentially it is *deterministic* motion in which the motion *appears* to be random - and one auxiliary of this is that small changes to
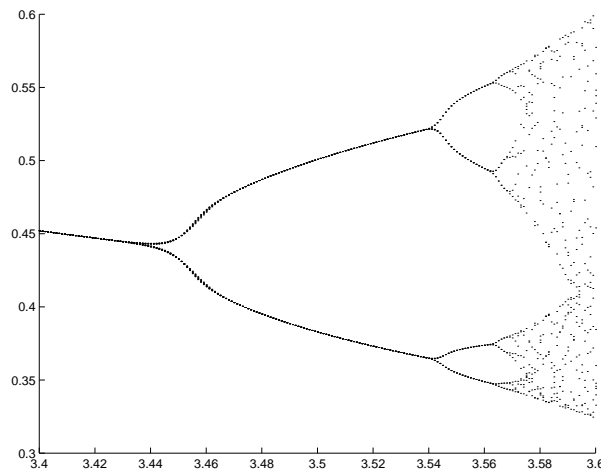
Fig. 21.14. A small portion of figure 21.13, magnified to show its similarity to the whole
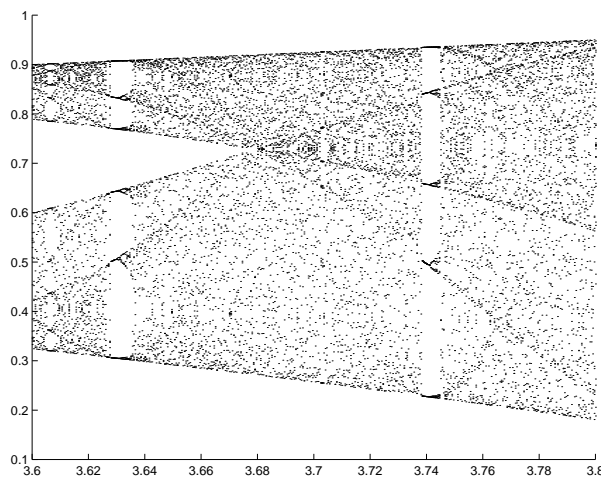

Fig. 21.15. A magnified part of figure 21.13 showing parameter values for which there is an orbit of period 6

the initial condition will produce wildly different behaviour if we wait long enough. If we investigate what happens when we have increased r all the way to 4 the idea should become a little clearer.

In this case the map is

$$x_{n+1} = 4x_n(1 - x_n),$$

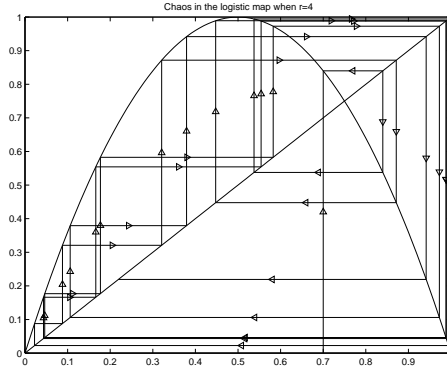and we can simplify the problem by means of a careful substitution. Since

Fig. 21.16. Chaotic trajectories of $x_{n+1} = 4x_n(1 - x_n)$

$x_n \in [0, 1]$, we can set $x_n = \sin^2 \theta_n$, with $\theta_n \in [0, \pi/2]$. The equation for θ_n is then

$$
\begin{aligned}
\sin^2 \theta_{n+1} &= 4 \sin^2 \theta_n \cos^2 \theta_n \\
&= \sin^2(2\theta_n).
\end{aligned}
$$

Since we want $\theta \in [0, \pi/2]$, we can take

$$
\theta_{n+1} = \begin{cases} 2\theta_n & 0 \le \theta_n \le \pi/4 \\ \pi - 2\theta_n & \pi/4 < \theta_n \le \pi/2. \end{cases}
$$

If we rescale, setting $y_n = 2\theta_n/\pi$, we obtain

$$
y_{n+1} = \begin{cases} 2y_n & 0 \le \frac{1}{2} \\ 2(1 - y_n) & \frac{1}{2} < y_n \le 1. \end{cases}
$$

and the new map is shown in figure 21.17.

The easiest way to consider the dynamics of this new map is by writing down the binary decimal expansion of y_n,

$$
y_n = [a_1 a_2 a_3 a_4 \ldots],
$$

with

$$
y_n = \sum a_n 2^{-n}.
$$

Then doubling y_n corresponds to removing the first term,

$$
y_{n+1} = [a_2 a_3 a_4 a_5 \ldots],
$$

and subtracting $2y_n$ from 2 is the same except we then have to switch all the 1s and 0s. If we denote this operation by $\bar{a}$, then if $y_n > 1/2$, i.e. if $a_1 = 1$,

$$
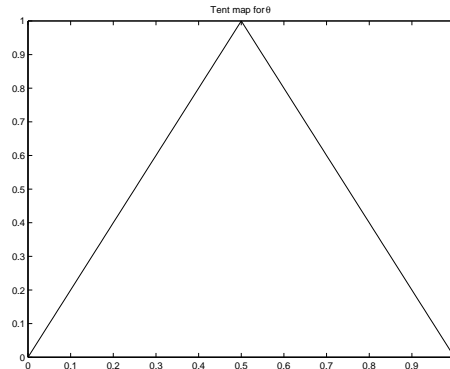y_{n+1} = [\bar{a}_2 \bar{a}_3 \bar{a}_4 \bar{a}_5 \ldots].
$$

Fig. 21.17. The 'tent map' obtained from our original map by a substitution

So we can write

$$\text{if } y_n = [a_1 a_2 a_3 a_4 \ldots] \quad \text{then} \quad y_{n+1} = \begin{cases} [a_2 a_3 a_4 a_5 \ldots] & \text{for } a_1 = 0 \\ [\bar{a}_2 \bar{a}_3 \bar{a}_4 \bar{a}_5 \ldots] & \text{for } a_1 = 1. \end{cases}$$

First observe that there can be no stable periodic motions, since if $y_0 = y^*$ implies that $y_n = y^*$, then if $y_0 = y^* + \epsilon$, assuming $y^* < 1/2$ gives $y_n = y^* + 2^n \epsilon$, so that the trajectory moves *away* from the orbit. However, we can show that all rational points are eventually periodic.

Just as in a decimal expansion, a rational number will have a repeating or finite binary expansion. Thus any rational is eventually periodic, since

$$y_0 = [b_1 b_2 b_3 b_4 b_5 \ldots b_n (a_1 a_2 \ldots a_m)^{\infty}],$$

and so, changing a's to $\bar{a}$'s if necessary,

$$y_{n+1} = [(a_1 a_2 \ldots a_m)^{\infty}],$$

and then either $f^m(y_{n+1}) = y_{n+1}$ or $f^{2m}(y_{n+1}) = y_{n+1}$.

We have a strange situation, then. There are no stable orbits, but all rational numbers are eventually periodic. Any number that starts irrational will have a binary expansion, and hence an orbit, that never repeats itself. One can also show that the distribution of points along an orbit is effectively random, even though the evolution is deterministic. This model is now very well understood, and is one of the standard examples used in dynamical systems.

There are two deductions to be made from this investigation, one straightforward, but the other much more enlightening. The straightforward deduction is that very simple models can have extremely complicated behaviour.

This is not, however, cause to despair at mathematical modelling, but indicates the far more important principle that observed behaviour that is extremely complicated may in fact arise from extremely simple causes. Since the advent of chaos theory, experimental data that was once discarded as spurious and useless has been re-analysed and found, nevertheless, to contain a high degree of order.