S9209/9210/103C

**DUBLIN INSTITUTE OF TECHNOLOGY**

**School of Mathematical Sciences**

# DT9205 MSc Mathematical Physics
# DT9206 MSc Mathematical Physics
# DT9209 MSc Applied Mathematics
# DT9210 MSc Applied Mathematics

**AUTUMN EXAMINATIONS 2015/2016**

## MATH9973: NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS

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Duration: 2 hour

Full marks may be obtained by answering three questions. Candidate's three best questions will contribute to their final mark.

All questions carry equal marks

Approved calculators may be used

Mathematical tables are provided

New Cambridge Statistical Tables are NOT permitted

**1. a)** Derive the Euler's method and show that it has a local truncation error of $O(h)$ for the Ordinary Differential Equation

$$\frac{dy}{dx} = f(x, y), \;\; a < x \le b$$

with initial condition

$$y(a) = \alpha.$$

(7)

**b)** Suppose $f$ is a continuous and satisfies a Lipschitz condition with constant L on $D = \{(x, y)|a \le x \le b, -\infty < y < \infty\}$ and that a constant M exists with the property that

$$\left|\frac{d^2y}{dx^2}\right| \le M.$$

Let $y(x)$ denote the unique solution of the Initial Value Problem

$$\frac{dy}{dx} = f(x, y), \quad a < x \le b, \quad y(a) = \alpha,$$

let $x_i = a + ih$, and $w_0, w_1, ..., w_N$ be the approximation generated by the Euler method for some positive integer N.
Then show for $i = 0, 1, ..., N$

$$|y(x_i) - w_i| \le \frac{Mh}{2L}|e^{L(x_i - a)} - 1|.$$

(14)

You may assume the two lemmas:
**Lemma 1.** If s and t are positive real numbers $\{a_i\}_{i=0}^{N}$ is a sequence satisfying $a_0 \ge \frac{-t}{s}$ and $a_{i+1} \le (1+s)a_i + t$ then

$$a_{i+1} \le e^{(i+1)s}\left(a_0 + \frac{t}{s}\right) - \frac{t}{s}.$$

**Lemma 2.** For all $x \ge 0.1$ and any positive m we have

$$0 \le (1+x)^m \le e^{mx}.$$

**c)** Use the Euler's method to estimate the solution of

$$\frac{d^2y}{dx^2} = -2x\left(\frac{dy}{dx}\right)^2, \;\; y(0) = 1, \;\; \frac{dy}{dx}(0) = -1,$$

as a system of two first order initial value problems and using $h = 0.25$, estimate the value of the solution at $x = 0.5$ .

(12)

**[33]**

**2.** **a)** For the Ordinary Differential Equation

$$\frac{dy}{dx} = f(x, y), \quad a < x \leq b$$

with initial condition

$$y(a) = \alpha,$$

derive the Adams-Bashforth two step method and its truncation error, which is of the form

$$w_0 = \alpha_0, \quad w_1 = \alpha_1,$$

$$w_{i+1} = w_i + \frac{h}{2}[3f(x_i, w_i) - f(x_{i-1}, w_{i-1})].$$

(10)

**b)** Define the terms consistent and convergent methods for a multistep method.

(5)

**c)** Define the terms strongly stable, weakly stable and unstable with respect to the characteristic equation.

(6)

**d)** Show that the Adams-Bashforth two step method is stongly stable.

(4)

**e)** Use an Adams-Bashforth method of your choice to approximate the solution to the initial value problem

$$\frac{dy}{dx} = 1 + (x - y)^2, \quad 2 \leq x \leq 3, \quad y(2) = 1$$

with $h = 0.2$ to approximate $y(0.4)$.

(8)

**[33]**

**3. a)** Approximate the Poisson equation

$$-\nabla^2 U(x,y) = f(x,y), \qquad (x,y) \in \Omega = (0,1)\times(0,1),$$

with boundary conditions

$$U(x,y) = g(x,y), \qquad (x,y) \in \delta\Omega$$

using the five point method $\nabla_h^2$. Sketch how the finite difference scheme may be rewritten in the form $Ax = b$, where A is a sparse $N^2 \times N^2$ matrix, and $b$ , $x$ are an $N^2$ component vectors. (Assume your 2d discretised grid contains $N$ components in the $x$ and $y$ direction).

(10)

**b)** Prove (DISCRETE MAXIMUM PRINCIPLE). If $\nabla_h^2 V_{ij} \geq 0$ for all points $(x_i, y_j) \in \Omega_h$, then

$$\max_{(x_i,y_j)\in\Omega_h} V_{ij} \leq \max_{(x_i,y_j)\in\partial\Omega_h} V_{ij}.$$

If $\nabla_h^2 V_{ij} \leq 0$ for all points $(x_i, y_j) \in \Omega_h$, then

$$\min_{(x_i,y_j)\in\Omega_h} V_{ij} \geq \min_{(x_i,y_j)\in\partial\Omega_h} V_{ij},$$

where $\Omega_h$ is the discrete grid of the area $\Omega$, $\nabla_h^2$ is the five point approximation of $\nabla^2$ and $h$ is the step-size in the $x$ and $y$ direction.

(12)

**c)** Hence prove:
Let $U$ be a solution to the Poisson equation and let $w$ be the grid function that satisfies the discrete form

$$-\nabla_h^2 w_{ij} = f_{ij} \quad \text{for } (x_i, y_j) \in \Omega_h,$$

$$w_{ij} = g_{ij} \quad \text{for } (x_i, y_j) \in \partial\Omega_h.$$

Then there exists a positive constant $K$ such that

$$\|U - w\|_\Omega \leq K M h^2,$$

where

$$M = max\left\{ \left\|\frac{\partial^4 U}{\partial x^4}\right\|_\infty, \left\|\frac{\partial^4 U}{\partial x^3 \partial y}\right\|_\infty, \ldots, \left\|\frac{\partial^4 U}{\partial y^4}\right\|_\infty \right\}.$$

(11)

You may assume: If the grid function $V : \Omega_h \bigcup \partial\Omega_h \to R$ satisfies the boundary condition $V_{ij} = 0$ for $(x_i, y_j) \in \partial\Omega_h$, then

$$\|V\|_\Omega \leq \frac{1}{8}\|\nabla_h^2 V\|_\Omega.$$

**[33]**

**4. a)** State the 3 classes and conditions of 2nd order Partial Differential Equations defined by the characteristic curves.
(5)

**b)** Given the non-dimensional form of the heat equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$$

supply sample boundary and initial conditions to specify a well-posed problem problem.
Write a fully implicit scheme to solve this partial differential equation.
(8)

**c)** Derive the local truncation error for the fully implicit method, for the heat equation.
(10)

**d)** Show that the method is unconditionally stable using von Neumann's method.
(10)

**[33]**