Spatial Spread of Rabies among Foxes

$$\begin{cases} \dfrac{\partial S}{\partial t} = -rIS + BS\left(1 - \dfrac{S}{S_0}\right) \\[2em] \dfrac{\partial I}{\partial t} = rIS - aI + D\dfrac{\partial^2 I}{\partial X^2} \end{cases}$$

$$U = \frac{S}{S_0}, \quad V = \frac{rI}{BS_0}, \quad t^* = Bt$$

$$x^* = \sqrt{\frac{B}{D}}\, x, \quad \lambda = \frac{a}{rS_0}, \quad \alpha = \frac{rS_0}{B}$$

$$\begin{cases} u_t = u(1 - u - v) \\ v_t = \alpha v(u - \lambda) + v_{xx} \end{cases}$$

Looking for a travelling wave solution

$$z = x - ct \Rightarrow \quad u_t = -c\,u'(z)$$

$$u_x = u'(z)$$

$$\begin{cases} -cu' = u(1-u-v) \\ -cv' = \alpha v(u-\lambda) + v'' \quad w' \end{cases}$$

$v' = w$ (new variable)

$$\begin{cases} u' = -\frac{1}{c} u(1-u-v) & = f_1(u,v,w) \\ v' = w & = f_2(u,v,w) \\ w' = -cw - \alpha v(u-\lambda) & = f_3(u,v,w) \end{cases}$$

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}^* = c_1 \vec{v}_1 e^{\lambda_1 z} + c_2 \vec{v}_2 e^{\lambda_2 z} + c_3 \vec{v}_3 e^{\lambda_3 z}$$

Equilibrium points: $w^* = 0$

1) $v = 0$ $\nearrow u = 0 \rightarrow (0, 0, 0)$; $\searrow u = 1 \rightarrow (1, 0, 0)$

2) $u = \lambda$, $v = 1-\lambda \rightarrow (\lambda, 1-\lambda, 0)$

The last equilibrium exists only if $\underline{0<\lambda<1}$

$$\mathcal{A} = \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{pmatrix} = \begin{pmatrix} -\frac{1}{c} + \frac{2u}{c} + \frac{v}{c} & \frac{u}{c} & 0 \\ 0 & 0 & 1 \\ -\alpha v & -\alpha(u-\lambda) & -c \end{pmatrix}$$

$$\mathcal{A}(0,0,0) = \begin{pmatrix} -\frac{1}{c} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \alpha\lambda & -c \end{pmatrix}$$

$\underline{E = \text{eigenvalue}}$, $z = x - ct$

$t \to \infty$, $z \to -\infty$

$$\left(-\frac{1}{c} - E\right) \begin{vmatrix} -E & 1 \\ \alpha\lambda & -c-E \end{vmatrix} = \left(-\frac{1}{c} - E\right)\left[E(E+c) - \alpha\lambda\right]$$

$$E^2 + Ec - \alpha\lambda = 0$$

$$E_1 = -\frac{1}{c} \ , \ E_{2,3} = \frac{-c \pm \sqrt{c^2 + 4\alpha\lambda}}{2}$$

$$\underline{u^* = \lambda, \quad v^* = 1-\lambda, \quad w^* = 0}$$

$$A(\lambda, 1-\lambda, 0) = \begin{pmatrix} -\frac{\cancel{1}}{c} + \frac{2\lambda}{c} + \frac{\cancel{1}-\lambda}{c} & \frac{\lambda}{c} & 0 \\ 0 & 0 & 1 \\ -\alpha(1-\lambda) & & -c \end{pmatrix} =$$

$$= \begin{bmatrix} \frac{\lambda}{c} & \frac{\lambda}{c} & 0 \\ 0 & 0 & 1 \\ -\alpha(1-\lambda) & 0 & -c \end{bmatrix}$$

$$p(\lambda) = \begin{vmatrix} \frac{\lambda}{c} - E & \frac{\lambda}{c} & 0 \\ 0 & -E & 1 \\ -\alpha(1-\lambda) & 0 & -c-E \end{vmatrix} =$$

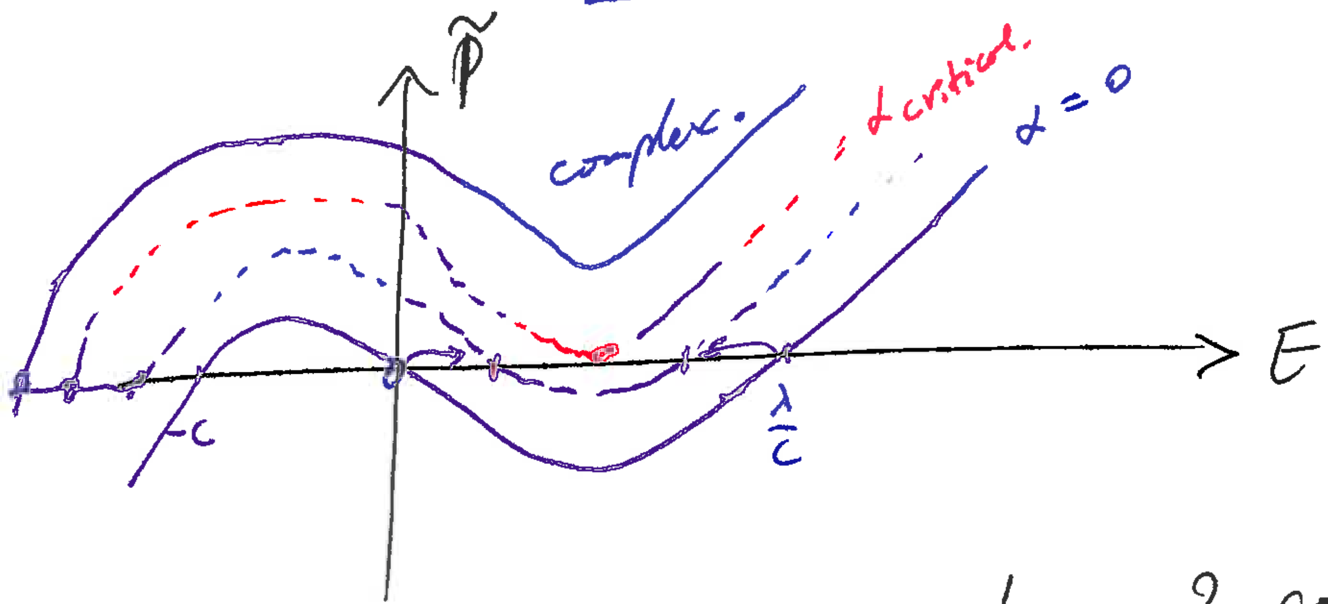
$$= \left(\frac{\lambda}{c} - E\right) \begin{vmatrix} -E & 1 \\ 0 & -c-E \end{vmatrix} - \frac{\lambda}{c} \begin{vmatrix} 0 & 1 \\ -\alpha(1-\lambda) & -c-E \end{vmatrix}$$

$$= \left(\frac{\lambda}{c} - E\right) E (E+c) - \frac{\lambda}{c} \alpha (1-\lambda)$$

$$p(\lambda) = E(E+c)\left(\frac{\lambda}{c} - E\right) - \frac{\alpha}{c}\lambda(1-\lambda)$$

$$\tilde{p}(\lambda) = -p(\lambda) = E(E+c)\left(E - \frac{\lambda}{c}\right) + \frac{\alpha}{c}\lambda(1-\lambda)$$

$$\boxed{\alpha = 0}$$



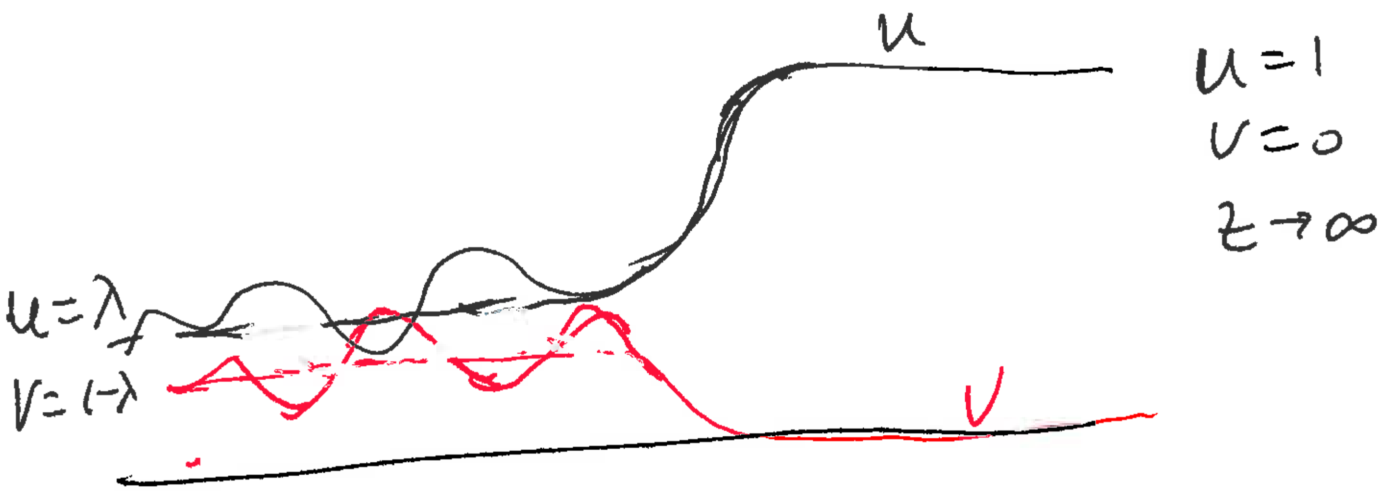
When $\alpha > \alpha_{\text{critical}}$ we have 2 complex roots with positive real part.

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = C_1 \vec{V}_1 e^{E_1 z} + C_2 \vec{V}_2 e^{E_2 z} + \cancel{C_3 \vec{V}_3 e^{E_3 z}}$$

$$E_1 = E_2^*, \quad C_2 = C_1^* \quad \vec{V}_2 = \vec{V}_2^*$$

If $\alpha < \alpha_{\text{critical}}$ $\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} \lambda \\ 1-\lambda \end{pmatrix}$ monotonically

— " — $\alpha > \alpha_{\text{critical}}$ $\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} \lambda \\ 1-\lambda \end{pmatrix}$ oscillatory



Biological Waves

$$\frac{\partial u}{\partial t} = ku(1-u) + D\frac{\partial^2 u}{\partial x^2}$$

$$z = x - ct, \quad t^* = kt \ \& \ x^* = x\sqrt{\frac{k}{D}}$$

$$u = U(z), \quad u_t = -cU'(z), \quad u_x = U'$$

$$\frac{\partial u}{\partial t} = u(1-u) + u_{xx}$$

$$-cU' = U(1-U) + U''$$

$V = U'$,

$$\begin{cases} U' = V & = f_1(U, V) \\ V' = -cV - U(1-U) & = f_2(U, V) \end{cases}$$

$V = 0 \rightarrow U = 0 \quad (0, 0)$

$ \searrow U = 1 \quad (1, 0)$

$u(-\infty) = 1$
$u(\infty) = 0$

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1+2u & -c \end{pmatrix}$$

$$A(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & -c \end{pmatrix}$$

$$\lambda^2 - \operatorname{tr} A + \det A = 0$$

$$\lambda^2 + c\lambda + 1 = 0$$

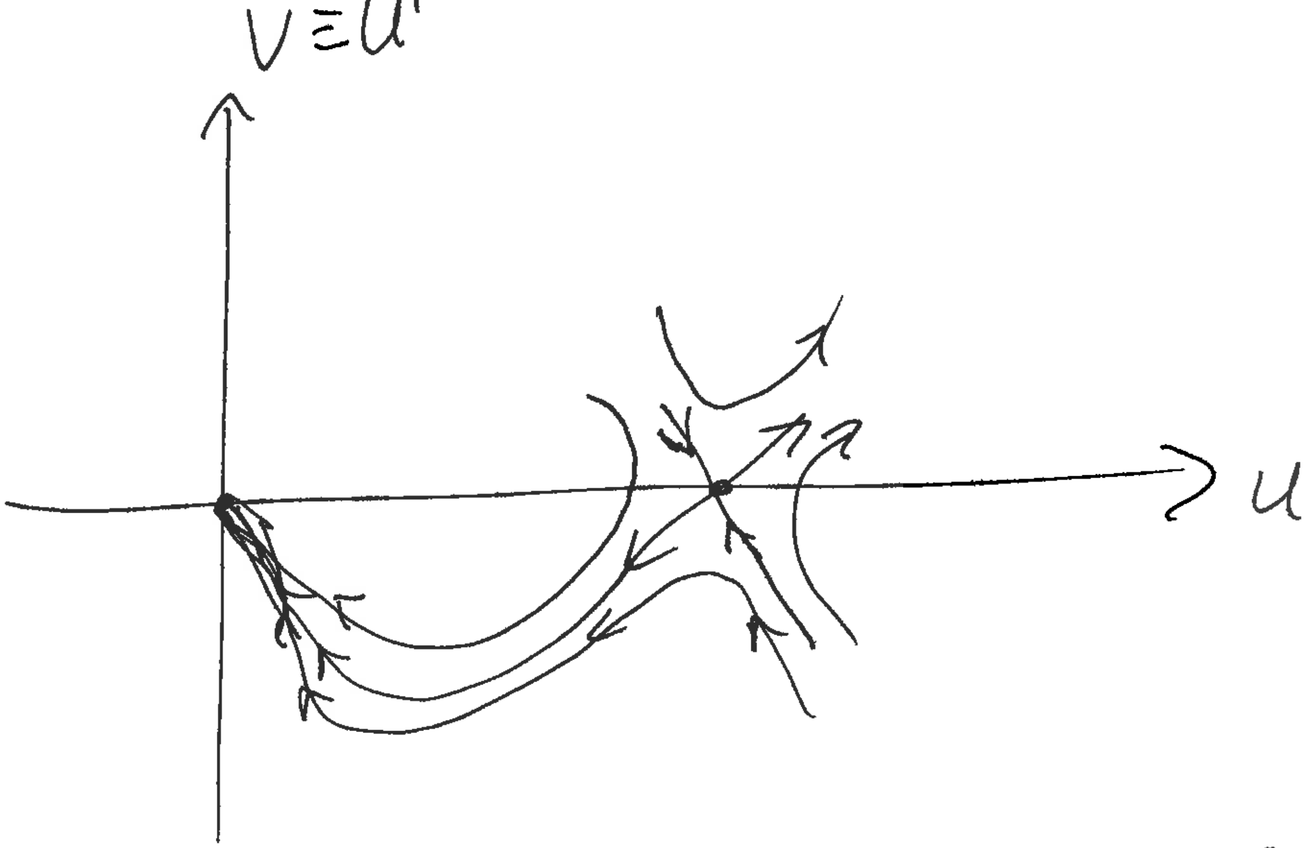
$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2-4}}{2} \Rightarrow \begin{cases} \text{stable node} & c^2 > 4 \\ \cancel{\text{saddle spiral} \quad c^2 < 4} \end{cases}$$

$$c^2 > 4 \qquad \boxed{c > 2}$$

$$A(1,0) = \begin{pmatrix} 0 & 1 \\ 1 & -c \end{pmatrix}$$

$$\lambda^2 + c\lambda - 1 = 0, \quad \lambda_{1,2} = \frac{-c \pm \sqrt{c^2+4}}{2}$$

saddle point; unst.



$$\frac{\partial u}{\partial t} = f(u) + \frac{\partial}{\partial x}\left[D(u)\frac{\partial u}{\partial x}\right]$$

$$\frac{\partial u}{\partial t} = u^p(1-u^q) + \frac{\partial}{\partial x}\left[u^m \frac{\partial u}{\partial x}\right]$$

$m = 0, \quad p = 1$

$$\frac{\partial u}{\partial t} = u(1-u^q) + \frac{\partial^2 u}{\partial x^2}$$

$$\cancel{u = U(z) = \frac{1}{1+(\sqrt{2}-1)e}}$$

$$u = U(z) = \frac{1}{(1+a e^{bz})^s}$$

$$s = \frac{2}{q} \qquad b = \frac{q}{[2(q+2)]^{1/2}} \qquad c = \frac{q+4}{[2(q+2)]^{1/2}}$$

$$\boxed{\frac{\partial u}{\partial t} = u(1-u) + \frac{\partial}{\partial x}\left[u \frac{\partial u}{\partial x}\right]}$$

$$p = q = m = 1$$

$$u = U(z) = U(x - ct)$$

$$-cU' = U(1-U) + [UU']'$$

$$\begin{cases} U' = V \\ UV' = -cV - V^2 - U(1-U) \end{cases}$$

$$V^* = 0 \quad \begin{matrix} \nearrow U^* = 0 & (0, 0) \\ \searrow U^* = 1 & (1, 0) \end{matrix}$$

$$U \frac{d}{dz} = \frac{d}{d\xi}$$

$$\cancel{U \frac{dU}{dz} = V} \rightarrow U \frac{dU}{dz} = U \cdot V$$

$$\frac{dU}{d\xi} = U \cdot V$$

$$\left\|\begin{aligned} \frac{du}{d\xi} &= uv \quad = f_1 \\ \frac{dv}{d\xi} &= -cv - v^2 - u(1-u) \quad = f_2 \end{aligned}\right.$$

$(0, 0)$

$(1, 0)$

$(0, -c)$

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{pmatrix}$$

$$A = \begin{pmatrix} v & u \\ -1+2u & -c-2v \end{pmatrix}$$

$$A(0,0) = \begin{pmatrix} 0 & 0 \\ -1 & -c \end{pmatrix} \qquad \left.\begin{aligned} \lambda &= 0 \\ \lambda &= -c \end{aligned}\right\} \text{ stable node}$$

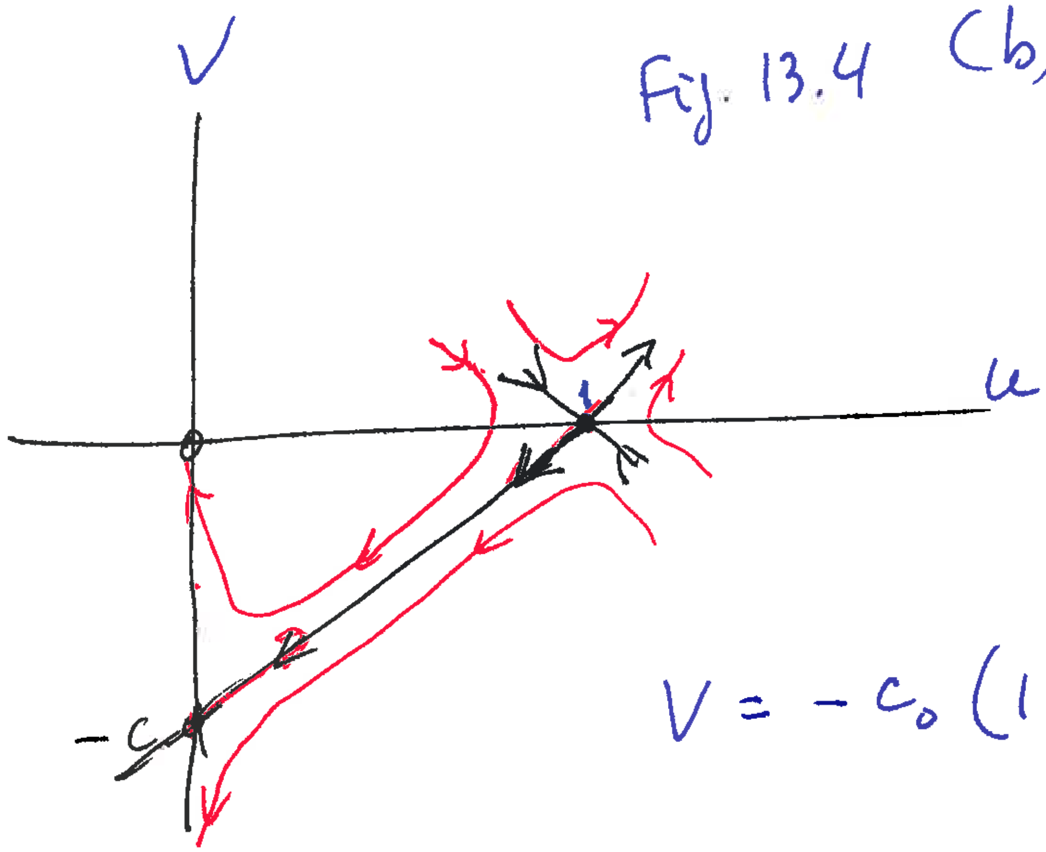
$$A(1,0) = \begin{pmatrix} 0 & 1 \\ 1 & -c \end{pmatrix}, \qquad \lambda^2 + c\lambda - 1 = 0$$

$$\lambda_\pm = \frac{-c \pm \sqrt{c^2+4}}{2} \qquad \text{(saddle point)}$$

$$A(0,-c) = \begin{pmatrix} -c & 0 \\ -1 & c \end{pmatrix}, \qquad \lambda_\pm = \pm c$$

saddle point

Fig. 13.4 (b) p. 453



$$V = -c_0(1-u) \quad ?$$

$$\frac{dV}{du} = \frac{-cV - V^2 - u(1-u)}{uV}$$

$$c_0 = \frac{+c_0^2\cancel{(1-u)} - c_0^2(1-u)^{\cancel{2}} - u\cancel{(1-u)}}{u(-c_0)\cancel{(1-u)}}$$

$$-c_0^2 u = +c_0^2 - c_0^2(1-u) - u$$

$$-c_0^2 u = \cancel{+c_0^2} - \cancel{c_0^2} + c_0^2 u - u$$

$$-2c_0^2\cancel{u} = -\cancel{u}$$

$$\boxed{2c_0^2 = 1} \Rightarrow \boxed{c_0 = \frac{1}{\sqrt{2}}}$$

$$\frac{du}{dz} = V = -\frac{1}{\sqrt{2}}(1-u)$$

$$\frac{du}{dz} = -\frac{1}{\sqrt{2}}(1-u)$$

$$\frac{du}{dz} - \frac{1}{\sqrt{2}}u = -\frac{1}{\sqrt{2}} \qquad \Big| \times e^{-\frac{z}{\sqrt{2}}}$$

$$e^{-\frac{z}{\sqrt{2}}}u' - \frac{1}{\sqrt{2}}e^{-\frac{z}{\sqrt{2}}}u = -\frac{1}{\sqrt{2}}e^{-\frac{z}{\sqrt{2}}}$$

$$\left(e^{-\frac{z}{\sqrt{2}}}u\right)' = -\frac{1}{\sqrt{2}}e^{-\frac{z}{\sqrt{2}}} = \left(e^{-\frac{z}{\sqrt{2}}}\right)'$$

$$u e^{-\frac{z}{\sqrt{2}}} = -k + e^{-\frac{z}{\sqrt{2}}}, \qquad k = \text{const}$$

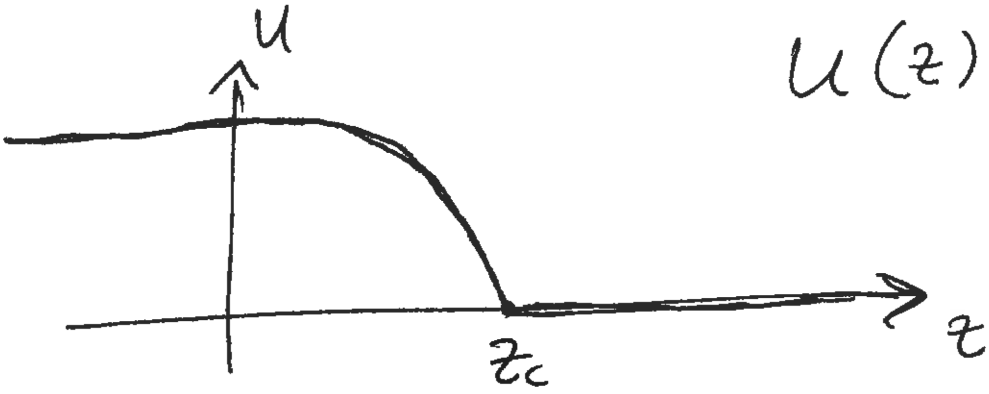
$$u = 1 - k e^{+z/\sqrt{2}}$$

$$\boxed{u(z) = 1 - e^{\frac{z-z_c}{\sqrt{2}}}}, \quad \underline{\underline{z<z_c}} \quad z_c = \text{const.}$$

If $z \to -\infty$ $\quad u(-\infty) = 1$

$u(z_c) = 0$

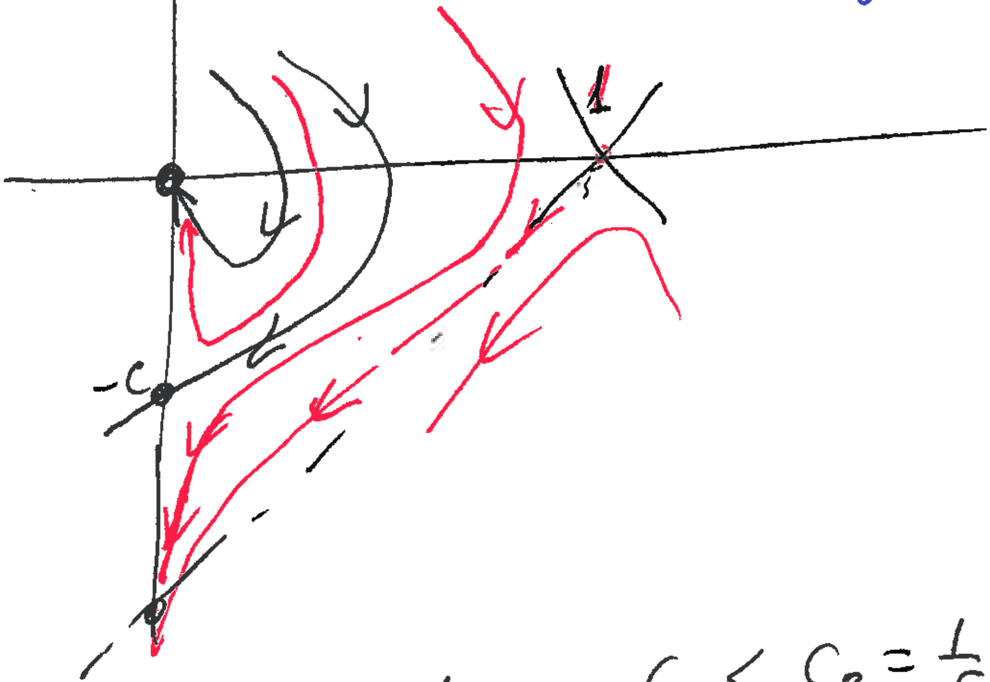
$u(z) = 0, \quad z > z_c$



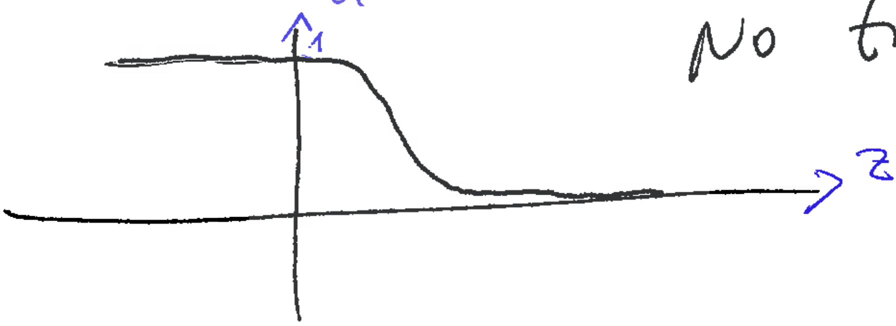
$$u(z) = \left(1 - e^{\frac{z-z_c}{\sqrt{2}}}\right)\theta(z_c - z)$$

$\boxed{c < c_0}$

Fig 13.4 (a) p. 453



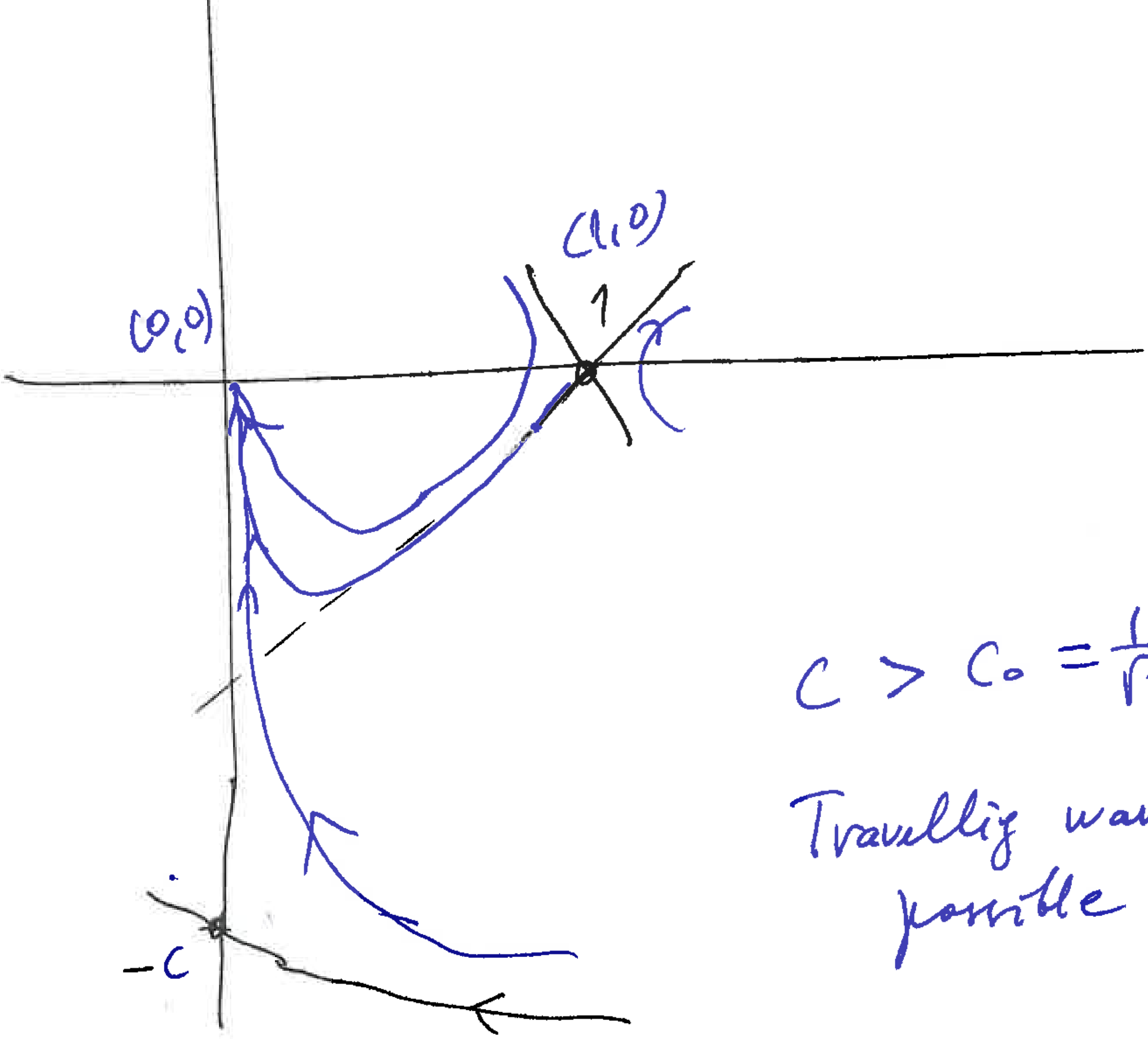
when $c < c_0 = \frac{1}{\sqrt{2}}$ the (0,0) ~~(0,0)~~ equilibrium can not be reached from (1,0) equilibrium



No travelling wave

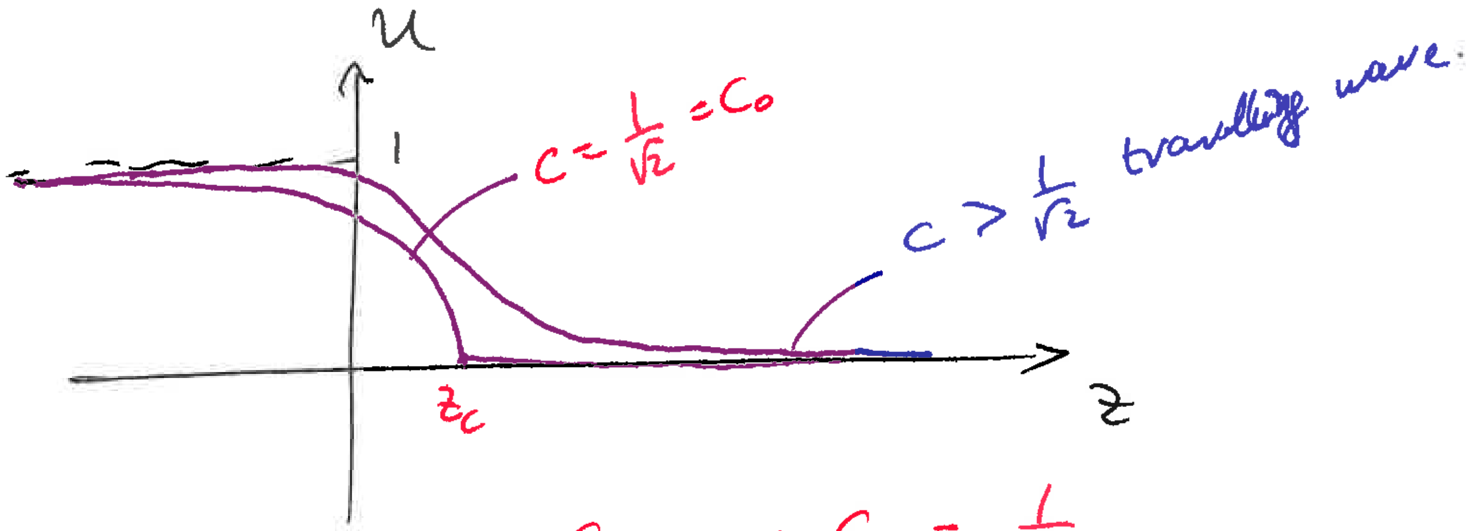
$\Rightarrow c > \frac{1}{\sqrt{2}} \equiv c_0$ threshold for travelling wave solution

Fig B.4(c) p. 413



$c > c_0 = \frac{1}{\sqrt{2}}$

Travelling waves are possible



$$C_{min} = C_0 = \frac{1}{\sqrt{2}}$$