S9209/9210/103C

DUBLIN INSTITUTE OF TECHNOLOGY

School of Mathematical Sciences

DT9205 MSc Mathematical Physics
DT9206 MSc Mathematical Physics
DT9209 MSc Applied Mathematics
DT9210 MSc Applied Mathematics

SAMPLE EXAM 2 2015/2016

MATH9973: NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS

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Duration: 2 hour

Full marks may be obtained by answering three questions. Candidate's three best questions will contribute to their final mark.

All questions carry equal marks

Approved calculators may be used

Mathematical tables are provided

New Cambridge Statistical Tables are NOT permitted

1. a) Derive the midpoint (Runge-Kutta) method:

$$w_0 = y_0,$$

$$w_{i+1} = w_i + hf(x_i + \frac{h}{2}, w_i + \frac{h}{2}f(x_i, w_i)) \text{ for each } i = 0, 1, \ldots N-1,$$

and its truncation error, where h is the step size in the x direction for the ordinary differential equation

$$\frac{dy}{dx} = f(x, y),$$

with initial condition

$$y(a) = \alpha.$$

using the following Theorem:
Suppose $f(x, y)$ and all its partial derivatives of order less than or equal to $n+1$ are continuous on $D = \{(x, y)|a \le x \le b, c \le y \le d\}$ and let $(x_0, y_0) \in D$ for every $(x, y) \in D$, then there exists $\xi \in (x, x_0)$ and $\mu \in (y, y_0)$ with

$$f(x, y) = P_n(x, y) + R_n(x, y)$$

where

$$\begin{aligned}
P_n(x, y) \quad = \quad & f(x_0, y_0) + \left[(x - x_0)\frac{\partial f}{\partial x}(x_0, y_0) + (y - y_0)\frac{\partial f}{\partial y}(x_0, y_0)\right] \\
& + \left[\frac{(x - x_0)^2}{2}\frac{\partial^2 f}{\partial x^2}(x_0, y_0) + (y - y_0)(x - x_0)\frac{\partial f}{\partial y \partial x}(x_0, y_0) + \frac{(y - y_0)^2}{2}\frac{\partial^2 f}{\partial y^2}(x_0, y_0)\right] \\
& + \ldots + \\
& + \left[\frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}(x - x_0)^{n-j}(y - y_0)^j \frac{\partial^n f}{\partial y^j \partial x^{n-j}}(x_0, y_0)\right],
\end{aligned}$$

and

$$R_n(x, y) = \left[\frac{1}{(n+1)!}\sum_{j=0}^{n+1}\binom{n+1}{j}(x - x_0)^{n+1-j}(y - y_0)^j \frac{\partial^{n+1} f}{\partial y^j \partial x^{n+1-j}}(\xi, \mu)\right]. \tag{12}$$

b) The initial value problem is defined

$$\frac{dy}{dx} = f(x, y),$$

$$y(x_0) = y_0.$$

Definition: The function $f(x, y)$ satisfies a Lipschitz Condition in the variable y,

$$|f(x, y_1) - f(x, y_2)| < L|y_1 - y_2|$$

whenever $(x, y_1), (x, y_2) \in D = \{(x, y)|a \leq x \leq b, c \leq y \leq d\}$.
Show that the midpoint method satisfies the Lipschitz Condition. (9)

c) Consider the differential equation

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + y + x, \quad 0 \leq x \leq 1, \quad y(0) = 1 \quad y'(0) = -1.$$

as a system of two first order initial value problems and using $h = 0.25$, estimate the value of the solution at $x = 0.5$ (12)

[33]

2. **a)** For the Ordinary Differential Equation

$$\frac{dy}{dx} = f(x, y), \ \ a < x \le b$$

with initial condition

$$y(a) = \alpha,$$

derive the Adams-Moulton two step method and its truncation error, which is of the form

$$w_0 = \alpha_0, \quad w_1 = \alpha_1,$$
$$w_{i+1} = w_i + \frac{h}{2}[f(x_{i+1}, w_{i+1}) + f(x_i, w_i)].$$

(10)

b) Define the terms consistent and convergent methods for a multistep method.

(5)

c) Define the terms strongly stable, weakly stable and unstable with respect to the characteristic equation.

(6)

d) Show that the Adams-Moulton two step method is strongly stable. (4)

e) Use an Adams-Moulton method of your choice to approximate the solution to the initial value problem

$$\frac{dy}{dx} = y^2 x^2, \ \ 1 \le x \le 2, \quad y(1) = 1$$

with $h = 0.1$ to approximate $y(0.2)$. (8)

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3. **a)** State the 3 classes and conditions of 2nd order Partial Differential Equations defined by the characteristic curves.

(5)

b) Given the non-dimensional form of the heat equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$$

supply sample boundary and initial conditions to specify a well-posed problem problem.

Write a fully explicit scheme to solve this partial differential equation. (8)

c) Derive the local truncation error for the fully explicit method, for the heat equation.

(10)

d) Show that the method is unconditionally stable using von Neumann's method. (10)

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4. **a)** Approximate the Poisson equation

$$-\nabla^2 U(x,y) = f(x,y), \qquad (x,y) \in \Omega = (0,1)\times(0,1),$$

with boundary conditions

$$U(x,y) = g(x,y), \qquad (x,y) \in \delta\Omega$$

using the five point method ∇_h^2. Sketch how the finite difference scheme may be rewritten in the form $Ax = b$, where A is a sparse $N^2 \times N^2$ matrix, and b , x are an N^2 component vectors. (Assume your 2d discretised grid contains N components in the x and y direction).

(10)

b) Prove (DISCRETE MAXIMUM PRINCIPLE). If $\nabla_h^2 V_{ij} \geq 0$ for all points $(x_i, y_j) \in \Omega_h$, then

$$\max_{(x_i,y_j)\in\Omega_h} V_{ij} \leq \max_{(x_i,y_j)\in\partial\Omega_h} V_{ij}.$$

If $\nabla_h^2 V_{ij} \leq 0$ for all points $(x_i, y_j) \in \Omega_h$, then

$$\min_{(x_i,y_j)\in\Omega_h} V_{ij} \geq \min_{(x_i,y_j)\in\partial\Omega_h} V_{ij},$$

where Ω_h is the discrete grid of the area Ω, ∇_h^2 is the five point approximation of ∇^2 and h is the step-size in the x and y direction.

(12)

c) Hence prove:

Let U be a solution to the Poisson equation and let w be the grid function that satisfies the discrete form

$$-\nabla_h^2 w_{ij} = f_{ij} \quad \text{for } (x_i, y_j) \in \Omega_h,$$

$$w_{ij} = g_{ij} \quad \text{for } (x_i, y_j) \in \partial\Omega_h.$$

Then there exists a positive constant K such that

$$||U - w||_\Omega \leq KMh^2,$$

where

$$M = max\left\{\left\|\frac{\partial^4 U}{\partial x^4}\right\|_\infty, \left\|\frac{\partial^4 U}{\partial x^3 \partial y}\right\|_\infty, \ldots, \left\|\frac{\partial^4 U}{\partial y^4}\right\|_\infty\right\}.$$

(11)

You may assume: If the grid function $V : \Omega_h \bigcup \partial\Omega_h \to R$ satisfies the boundary condition $V_{ij} = 0$ for $(x_i, y_j) \in \partial\Omega_h$, then

$$||V||_\Omega \leq \frac{1}{8} ||\nabla_h^2 V||_\Omega.$$

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