DUBLIN INSTITUTE OF TECHNOLOGY
KEVIN STREET, DUBLIN 8

MSc in Applied Mathematics and Theoretical Physics
DT234 and DT 238

Summer Examinations
2012

INTRODUCTION TO BIOMATHEMATICS

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Dr. Chris Hills
Dr. Alan Hegarty

Monday, 14 May 2012

1pm – 4:30 pm

Answer any FOUR questions. All questions carry 25 marks. If more than four questions are attempted, only the best four will be graded.

Dept. of Education Tables allowed

Question 1. (i) Solve explicitly the logistic model with a time-dependent intrinsic growth rate $r(t)$:

$$\frac{dN(t)}{dt} = r(t)N(t)\left(1 - \frac{N(t)}{K}\right), \qquad K > 0, \qquad N(0) > 0.$$

[**10 marks**]

(ii) Find the steady states of the following model

$$\frac{dN}{dt} = RN\left(1 - \frac{N}{K}\right)\left(\frac{N}{K_0} - 1\right), \qquad 0 < K_0 < K$$

and determine their linear stability. R, K_0 and K are positive constants.

[**10 marks**]

(iii) For the model in (ii) find

$$\lim_{t\to\infty} N(t)$$

if $K_0 < N(0) < K$.

[**5 marks**]

Question 2. A model for the spruce budworm population $u(t)$ (in dimensionless units) is governed by the equation

$$\frac{du}{dt} = ru\left(1 - \frac{u}{q}\right) - \frac{u^2}{1+u^2}, \qquad r, q > 0.$$

(a) Determine the number of the steady states and the number of the stable steady states in each of the following two cases (it is not necessary to compute the values of these steady states):

(i) $r = q = 1$, [**10 marks**]

(ii) $r = 0.5$, $q = 20$. [**10 marks**]

(b) Does any of the above parameter choices allow for an insect outbreak?

[**5 marks**]

Question 3. It has been suggested that a means of controlling insect numbers is to introduce and maintain a number of sterile insects in the population. One such model for the resulting population dynamics is

$$N_{t+1} = \frac{rN_t^2}{\frac{r-1}{M}N_t^2 + N_t + S},$$

where $r > 1$ and $M > 0$ are constant parameters, and S is the constant sterile insect population.

(i) Determine the steady states and discuss their linear stability, noting whether any type of bifurcation is possible.

[9 marks]

(ii) Find the critical value S_c of the sterile population in terms of r and M so that if $S > S_c$ the insect population is eradicated.

[9 marks]

(iii) Construct a cobweb map and draw a graph of the steady state population density against S, and hence determine the possible solution behavior if $0 < S < S_c$.

[7 marks]

Question 4. Show that an exact travelling wave solution exists for the scalar reaction-diffusion equation

$$\frac{\partial u}{\partial t} = u^{q+1}(1 - u^q) + \frac{\partial^2 u}{\partial x^2}, \qquad q > 0,$$

by looking for solution in the form

$$u(x,t) = U(z) = \frac{1}{(1 + ae^{bz})^s}, \qquad z = x - ct,$$

where c is wavespeed and s and b are positive constants. Determine the unique values for c, s and b in terms of q. Choose the value for a such that the magnitude of the wave's gradient is at its maximum at $z = 0$.

[25 marks]

Question 5. Flores (1998) proposed the following model for competition between Neanderthal man (N) and Early Modern man (E).

$$\begin{aligned} \frac{dN}{dt} &= N\left(A - D(N+E) - B\right), \\ \frac{dE}{dt} &= = E\left(A - D(N+E) - sB\right), \end{aligned}$$

where A, B, D are positive constants and $0 < s < 1$ is a measure of the difference in mortality of the two species.

(i) Explain the model briefly, nondimensionalise the system and perform the phase plane analysis.

[15 marks]

(ii) Show that for large values of t the population $N(t)$ decays according to the law

$$N(t) = C\exp[-B(1-s)t],$$

where C is some constant. Hence give the order of magnitude of the time for Neanderthal extinction if $s = 0.995$ and if the lifetime of an individual is roughly 40 years.

[10 marks]

Question 6. A model for venereal diseases includes the following classses: male and female infectives (I_1 and I_2) and male and female susceptibles (S_1 and S_2). It assumes that once infectives have recovered they rejoin the susceptibles:

$$\begin{aligned}
\frac{dS_1}{dt} &= -r_1 S_1 I_2 + a_1 I_1,\\
\frac{dI_1}{dt} &= r_1 S_1 I_2 - a_1 I_1,\\
\frac{dS_2}{dt} &= -r_2 S_2 I_1 + a_2 I_2,\\
\frac{dI_2}{dt} &= r_2 S_2 I_1 - a_2 I_2,
\end{aligned}$$

r_1, r_2, a_1 and a_2 are positive constant parameters.

(i) Briefly explain all terms and parameters in these equations and show that

$$I_k + S_k = N_k,$$

$k = 1, 2$ where N_k are constants.

[5 marks]

(ii) Using the result from (i) exclude S_1 and S_2 and reduce the system to two nonlinear equations for I_1 and I_2. Find the possible steady states and examine their linear stability for all possible values of the parameters.

[20 marks]

END OF PAPER

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EXPECTED SOLUTION TO QUESTION NO: 1

EXAMINER: ROSSEN IVANOV

PROPOSED MARK ALLOCATION: 25

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(i) $\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right) \Rightarrow \frac{dN}{N\left(1-\frac{N}{K}\right)} = r(t)\,dt$ (separation of variables)

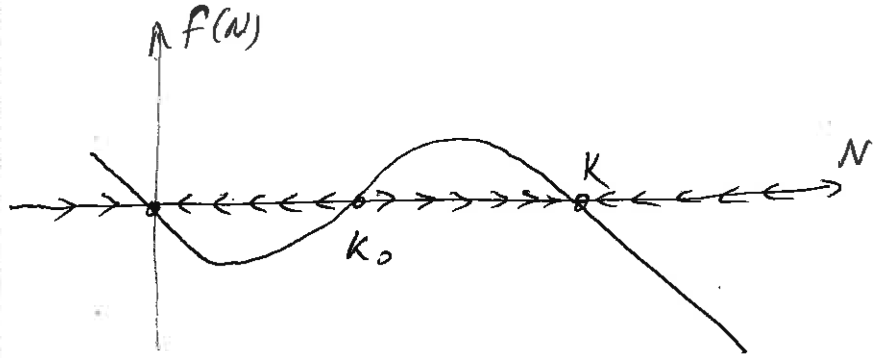
$\Rightarrow \int_{N(0)}^{N} \frac{d\tilde{N}}{\tilde{N}\left(1-\frac{\tilde{N}}{K}\right)} = \int_0^t r(s)\,ds$; the first integral can be [5 marks]

computed via partial fractions; the integration gives

$$N(t) = \frac{K N(0)}{N(0) + (K-N(0))\exp\left(-\int_0^t r(s)\,ds\right)}$$ [5 marks]

(ii) $\frac{dN}{dt} = RN\left(1-\frac{N}{K}\right)\left(\frac{N}{K_0}-1\right) = f(N); 0<K_0<K$

$f(N)$ is cubic function with zeroes at $N=0$, K_0 and K:



[5 marks]

Since $f(-\infty)>0$, $f(\infty)<0$ the graph is as above

$\Rightarrow f'(0)<0 \Rightarrow N^*=0$ is a stable steady state

$f'(K_0)>0 \Rightarrow N^*=K_0$ is an unstable steady state

$f'(K)<0 \Rightarrow N^*=K$ is a stable steady state [5 marks]

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EXPECTED SOLUTION TO QUESTION NO: 1 (cont)

PROPOSED MARK ALLOCATION: 25

(iii) From the figure it is clear that all initial data $N(0) \in [K_0, K]$ approach the steady state $N^* = K$ ~~for~~ a long time asymptotic value;

$$\lim_{t \to \infty} N(t) = K$$

[5 marks]

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EXPECTED SOLUTION TO QUESTION NO: 2

EXAMINER: ROSSEN IVANOV

PROPOSED MARK ALLOCATION: 25

$$\frac{du}{dt} = ru\left(1-\frac{u}{9}\right) - \frac{u^2}{1+u^2} \equiv f(u)$$

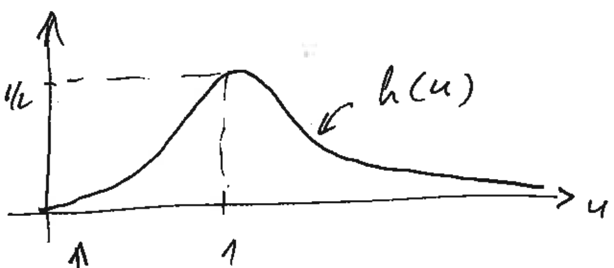
(a) $f'(u) = r - \frac{2ru}{9} - \frac{2u}{(1+u^2)^2} \Rightarrow f'(0) = r > 0$

$\Rightarrow u^* = 0$ is always an unstable steady state

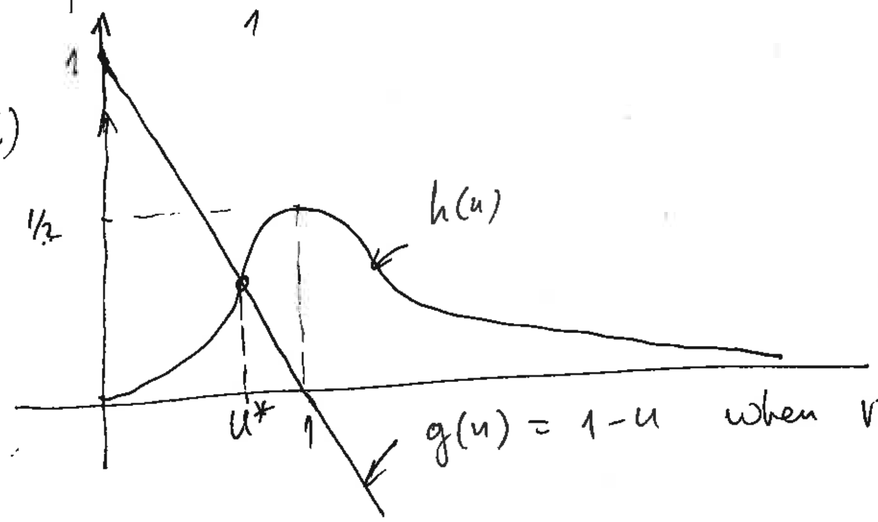
In order to find the other steady states we solve

$$g(u) \equiv r\left(1-\frac{u}{9}\right) = \frac{u}{1+u^2} = h(u)$$

$h'(u) = \frac{1-u^2}{(1+u^2)^2}$ has a maximum at $u=1$: $h(1) = \frac{1}{2}$



(i)

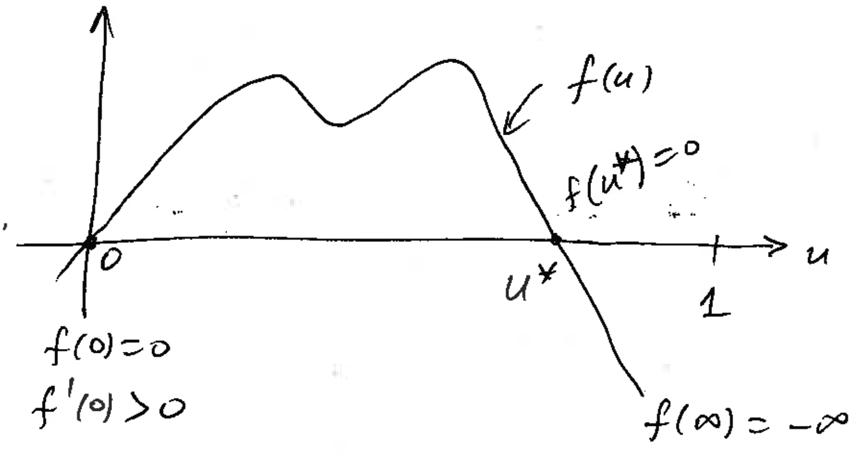


$g(u) = 1-u$ when $r=9=1$

[5 marks]

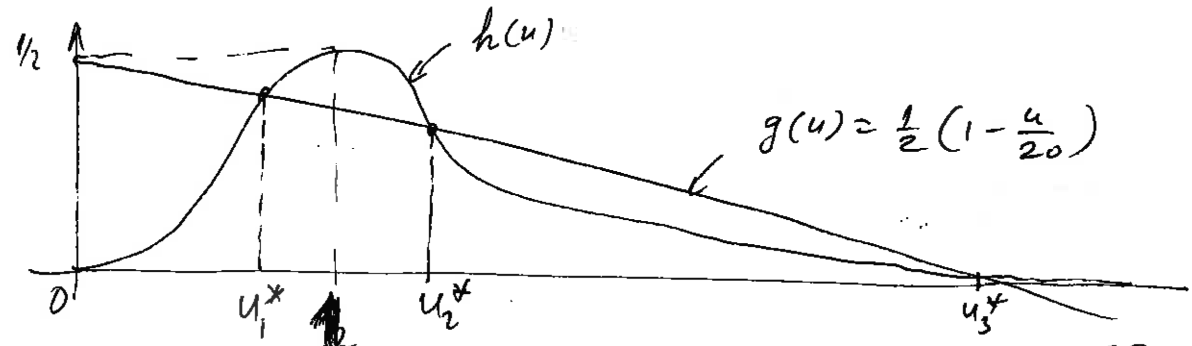
In case (i) $h(u)$ and $g(u)=1-u$ have only 1 intersection point $\Rightarrow$ one more steady state $0<u^*<1$

since $f(0)=0$, $f'(0)>0$ and $f(\infty)=-\infty$ we have the following graph of f:



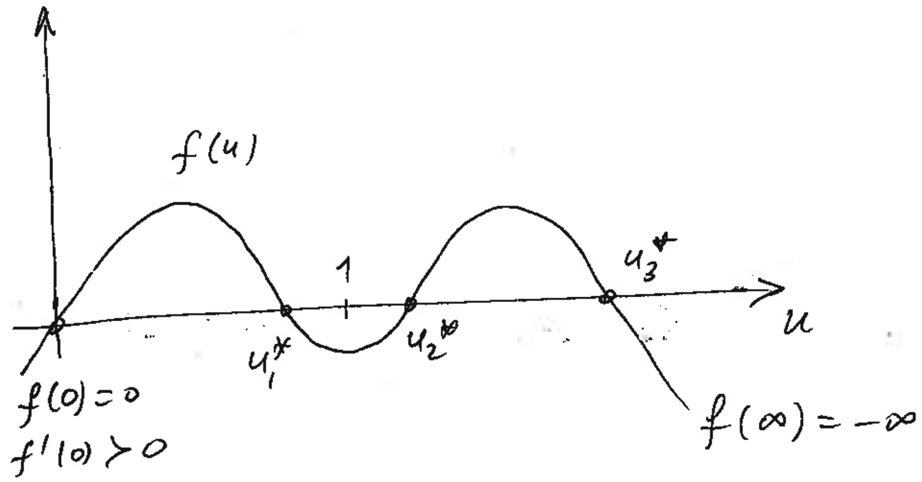
From this graph $\Rightarrow$ $f'(u^*)<0$ $\Rightarrow$ u^* is a stable steady state, i.e. in case (i) we have one stable steady state [5 marks]

(ii).



[5 marks]

The graphs of $h(u)$ & $g(u)$ intersect in 3 points $0 < u_1^* < 1 < u_2^* < u_3^*$ $\Rightarrow$ the graph of $f(u)$ is



thus $f'(u_1^*) < 0 \Rightarrow u_1^*$ & u_3^* are stable

$f'(u_3^*) < 0$

$f'(u_2^*) > 0 \Rightarrow u_2^*$ is unstable [5 marks]

$\Rightarrow$ in case (ii) there are two stable steady states.

(b) (ii) allows for an outbreak: u_3^* is much bigger than u_1^* and the insect population can switch from the low level u_1^* to the other stable level u_3^*.

[5 marks]

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EXPECTED SOLUTION TO QUESTION NO: 3

EXAMINER: ROSSEN IVANOV

PROPOSED MARK ALLOCATION: 25

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$$N_{t+1} = \frac{r N_t^2}{\frac{r-1}{M} N_t^2 + N_t + S} \equiv f(N_t)$$

(i) The steady states from $N^* = f(N^*)$ are

$$N^*_{\pm} = \frac{M}{2}\left(1 \pm \sqrt{1 - \frac{4S}{M(r-1)}}\right)$$

$N^*_{\pm}$ exist iff $4S \leq M(r-1)$, i.e. $S < S_c = \frac{M}{4}(r-1)$

$$f'(N) = \frac{rN(N+2S)}{\left(\frac{r-1}{M}N^2 + N + S\right)^2}$$ is everywhere positive.

$f'(0) = 0$, $|f'(0)| < 1 \Rightarrow N^* = 0$ is <u>stable</u>

[So if $S > S_c$ only $N^* = 0$ is a stable steady state & population is eradicated]

Since $f'(N) > 0 \Rightarrow f(N)$ is monotonic;

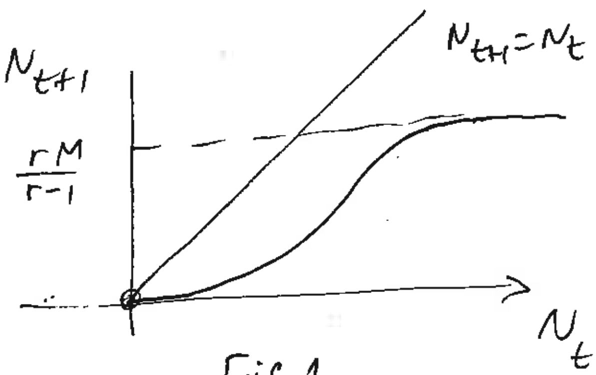

Fig 1.

or

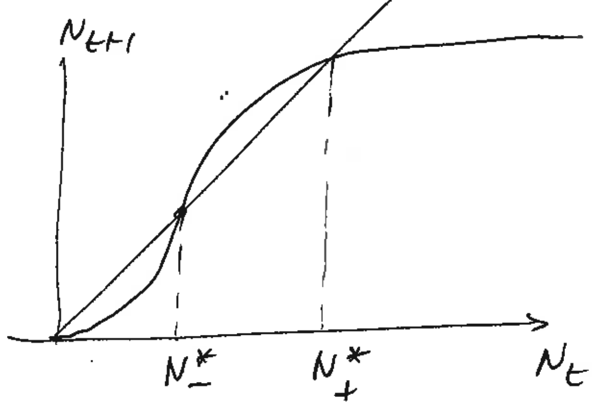

Fig (2)

$$\lim_{N \to \infty} f(N) = \frac{rM}{r-1} = \text{const}$$

[5 marks]

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EXPECTED SOLUTION TO QUESTION NO: 3 – cont.

EXAMINER: ROSSEN IVANOV

PROPOSED MARK ALLOCATION:

Fig (1) corresponds to complex roots, $S > S_c$
only $N^* = 0$ is a stable steady state

Fig (2) corresponds to two real roots giving $N^*_\pm$
$(S < S_c)$

$$f'(N^*_\pm) = \frac{r N^*_\pm (N^*_\pm + 2S)}{\left(\frac{r-1}{M} N^{*2}_\pm + N^*_\pm + S\right)^2} \equiv \frac{r N^*_\pm (N^*_\pm + 2S)}{(r N^*_\pm)^2} = \frac{1}{r} + \frac{2S}{N_\pm}$$

$$= \frac{1}{r} + 2\cancel{S}\left(\frac{r-1}{\cancel{S}M} N^*_\mp\right) = \frac{1}{r} + \frac{r-1}{\cancel{M}}\frac{\cancel{M}}{2}\left(1 \mp \sqrt{1 - \frac{4S}{M(r-1)}}\right)$$

Since $N^*_+ N^*_- = \frac{SM}{r-1}$

$$f'(N^*_\pm) = \frac{1}{r} + r - 1 \mp \sqrt{(r-1)^2 - \frac{4S}{M}(r-1)}$$

Note that $\frac{1}{r} + r \geq 2$

$$\Rightarrow f'(N^*_-) = \frac{1}{r} + r - 1 + \sqrt{(r-1)^2 - \frac{4S}{M}(r-1)} \geq 2 - 1 + \sqrt{(r-1)^2 - \frac{4S}{M}(r-1)} > 1$$

$f'(N^*_-) > 1 \Rightarrow N^*_-$ is <u>unstable</u>
Similarly $\Rightarrow N^*_+$ is <u>stable</u> (5 marks)

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EXPECTED SOLUTION TO QUESTION NO: 3 cont.

PROPOSED MARK ALLOCATION: 25

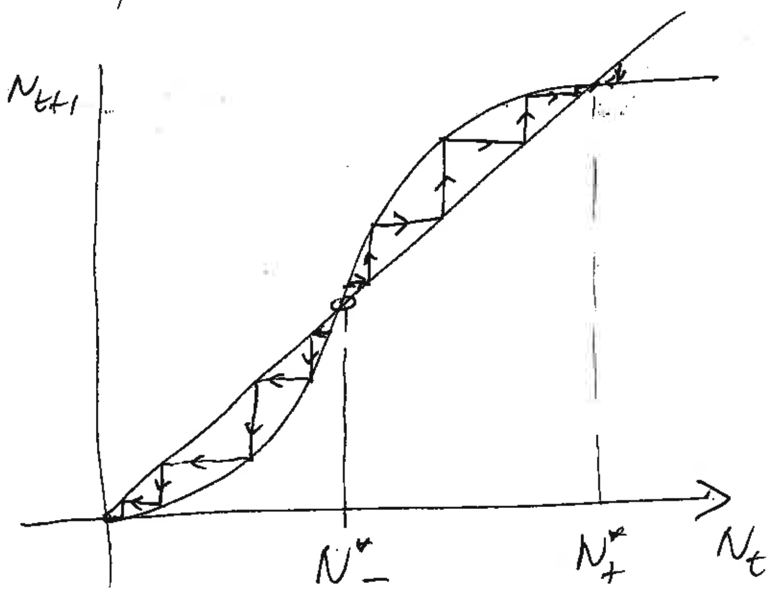
(ii) If $S>S_c$ there is only one stable steady state $N^*=0$ and the population is eradicated, ($N^*_\pm$ are complex.)

If $S \not< 0$ there are two stable steady states, $N^*=0$ & outbreak level steady state N^*_+

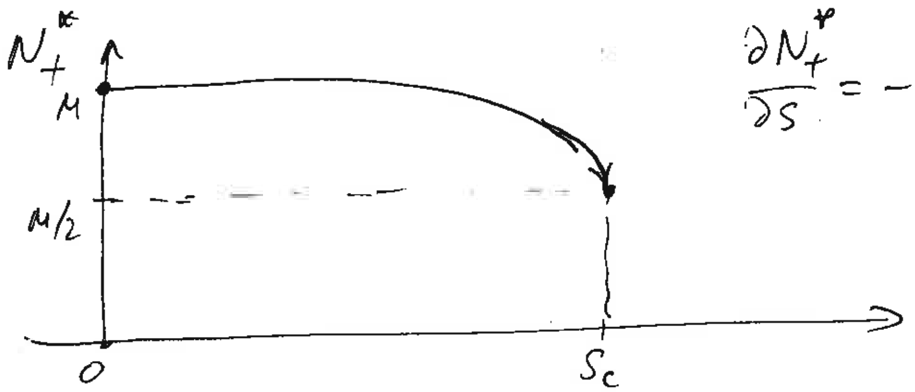
$S_c = \frac{M}{4}(r-1)$ – see previous pages.

[9 marks]

(iii)



[3 marks]



$$\frac{\partial N^*_+}{\partial S} = -\frac{4}{(r-1)}\cdot\frac{1}{\sqrt{1-\frac{4S}{M(r-1)}}} < 0$$

$\frac{\partial N^*_+}{\partial S} = -\infty$ at $S=S_c$

N^*_+ decreases monotonically with S from M to $M/2$

[3 marks]

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EXPECTED SOLUTION TO QUESTION NO: 4

EXAMINER: ROSSEN IVANOV

PROPOSED MARK ALLOCATION: 25

$$\frac{\partial u}{\partial t} = u^{q+1}(1-u^q) + u_{xx}\ ; \quad u = U(z) = \frac{1}{(1+ae^{bz})^s} \qquad q>0, \quad z = x-ct$$

$$-cU' = +U^{q+1} - U^{2q+1} + U'' \qquad (*)$$

$$U' = -\frac{sabe^{bz}}{(1+ae^{bz})^{s+1}}$$

$$U'' = -sab^2\frac{e^{bz}(1-ase^{bz})}{(1+ae^{bz})^{s+2}}$$

The substitution of U, U' and U'' in $(*)$ gives

$$csab\frac{e^{bz}}{(1+ae^{bz})^{s+1}} = \frac{1}{(1+ae^{bz})^{s(q+1)}} - \frac{1}{(1+ae^{bz})^{s(2q+1)}} - \frac{asb^2e^{bz}(1-ase^{bz})}{(1+ae^{bz})^{s+2}}$$

Note that always $s+1 < s+2$ and $q+1 < 2q+1$

In order to be able to match the denominators, we need

$$s+1 = s(q+1) \rightarrow 1 = sq$$
$$s+2 = s(2q+1) \quad 2 = 2sq \Rightarrow \boxed{sq=1}$$

$\Rightarrow \boxed{s = \frac{1}{q}}$ $\Rightarrow$ the common denom. is $(1+ae^{bz})^{s+2}$

$$\frac{csabe^{bz}(1+ae^{bz})}{(1+ae^{bz})^{s+2}} = \frac{1+ae^{bz} - 1 - sab^2e^{bz}(1-ase^{bz})}{(1+ae^{bz})^{s+2}}$$

The comparison of the nominators gives

$$csab(1+ae^{bz}) = a - sab^2(1-ase^{bz})$$

This leads to

$$csab = a - sab^2 \Rightarrow scb = 1 - sb^2$$

$$csa^2b = sab^2as \Rightarrow c = sb \Rightarrow c = \frac{b}{9}$$

$$\Rightarrow \frac{b}{9}\cdot\frac{1}{9}\cdot b = 1 - \frac{1}{9}\cdot b^2 \Rightarrow b = \frac{9}{\sqrt{1+9}}$$

$$c = \frac{1}{9}b = \frac{1}{\sqrt{1+9}}$$

$$s = \frac{1}{9}$$

The gradient is U', max gradient when $U''=0$

$$\Rightarrow 1 = ase^{b\cdot 0} \quad \text{or} \quad 1 = a\cdot s \Rightarrow a = \frac{1}{s} = 9$$

$$\Rightarrow U = \frac{1}{\left(1+9e^{\frac{9}{\sqrt{9+1}}z}\right)^{\frac{1}{2}}}$$

$$z = x - ct = x - \frac{1}{\sqrt{9+1}}t$$

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EXPECTED SOLUTION TO QUESTION NO: 5

EXAMINER: ROSSEN IVANOV

PROPOSED MARK ALLOCATION: 25

$$\left|\begin{array}{l} \dfrac{dN}{dt} = N(A - B - DN \cancel{+} DE) \\[2ex] \dfrac{dE}{dt} = E(A - SB - DN - DE) \end{array}\right.$$

(i) $$\frac{dN}{dt} = (A-B)N\left[1 - \frac{D}{A-B}N - \frac{D}{A-B}E\right]$$

$$\frac{dE}{dt} = (A-SB)E\left[1 - \frac{D}{A-SB}E - \frac{D}{A-SB}N\right]$$

This is competition model, $N\left(1 - \frac{N}{(A-B)/D}\right)$ is logistic term, $-\frac{DE}{A-B}$ is term describing competition; similarly for the second eq.

Nondimensionalization $u = \frac{D}{A-B}N$, $v = \frac{D}{A-SB}E$

$$\tau = t(A-B), \quad \rho = \frac{A-SB}{A-B}$$

$$\Rightarrow \left|\begin{array}{l} \dfrac{du}{d\tau} = u\left[1 - u - \dfrac{A-SB}{A-B}v\right] \\[2ex] \dfrac{dv}{d\tau} = \rho v\left[1 - v - \dfrac{A-B}{A-SB}u\right] \end{array}\right.$$

$$\frac{du}{d\tau} = u\left[1 - u - a_{12}v\right] = f; \quad a_{12} = \frac{A-SB}{A-B} > 1$$

$$\frac{du}{d\tau} = \rho v\left[1 - v - a_{21}u\right] = g; \quad a_{21} = \frac{A-B}{A-SB} < 1$$

(5 marks)

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EXPECTED SOLUTION TO QUESTION NO: 5 - cont.

EXAMINER: ROSSEN IVANOV

PROPOSED MARK ALLOCATION: 25

The only steady states are $(0,0)$ $(0,1)$ & $(1,0)$

$$\mathcal{A} = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix} = \begin{pmatrix} 1-2u-a_{12}v & -a_{12}u \\ -\rho a_{21}v & \rho(1-2v-a_{21}u) \end{pmatrix}$$

[2 marks]

$$\mathcal{A}(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix},$$ Eigenvalues $\lambda_1 = 1 > 0$ & $\lambda_2 = \rho > 0$ $\Rightarrow$ unstable node (always) [2 marks]

$$\mathcal{A}(1,0) = \begin{pmatrix} 1-2 & -a_{12} \\ 0 & \rho(1-a_{21}) \end{pmatrix} = \begin{pmatrix} -1 & -a_{12} \\ 0 & \rho(1-a_{21}) \end{pmatrix},$$

$\lambda_1 = -1 < 0$ $\quad \lambda_2 = \rho(1-a_{21}) > 0$ $\Rightarrow$ saddle pt. [2 marks]

$$\mathcal{A}(0,1) = \begin{pmatrix} 1-a_{12} & 0 \\ -\rho a_{21} & \rho(1-2) \end{pmatrix} = \begin{pmatrix} 1-a_{12} & 0 \\ -\rho a_{21} & -\rho \end{pmatrix}$$

$\lambda_1 = 1-a_{12} < 0$ & $\lambda_2 = -\rho < 0$ $\Rightarrow$ stable node

$\Rightarrow$ the only stable steady state is $(0,1)$ thus [2 marks]

$\lim\limits_{t\to\infty} u = 0$, $\lim\limits_{t\to\infty} v = 1$ $\Rightarrow$ $\frac{D}{A-sB}E \to 1$ $\Rightarrow$

$\lim\limits_{t\to\infty} N \to 0$ $\quad \lim\limits_{t\to\infty} E \to \frac{A-sB}{D} = \text{const.}$ [2 marks]

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EXPECTED SOLUTION TO QUESTION NO: 5

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PROPOSED MARK ALLOCATION: 25

(ii) When $N \to 0$, $E \to$ const for large t, we can linearize the first equation

$$\frac{dN}{dt} = N(A-B) - \underbrace{\cancel{D}N^2}_{\text{neglect}} - DEN$$

$$\frac{dN}{dt} = N(A-B-DE) = N\left(A-B-\cancel{D}\frac{A-sB}{\cancel{D}}\right)$$

$$\frac{dN}{dt} = N(\cancel{A}-B-\cancel{A}+sB) \Rightarrow \frac{dN}{dt} = N(s-1)B$$

$$\Rightarrow N = \underbrace{N(0)}_{\text{const}} e^{-(1-s)Bt} \quad \to \text{exponential decay.}$$

[5 marks]

The timescale of the extinction is

$\sim \frac{1}{(1-s)B}$ where $\frac{1}{B} = 40y$. is the lifetime of the individual $\Rightarrow$ timescale $\sim \frac{1}{1-0.995} \times 40$ years $= \frac{40y}{0.005} = \underline{8000 \text{ years}}$

[5 marks]

(i) $r_1 S_1 I_2$ describes the rate of infection of men as a result of contacts with infected women.

r_1 is a measure for the intensity of these contacts etc.

$\frac{1}{a_1}$ is the lifetime of the disease for men

adding the first two eqns gives $\frac{d}{dt}(S_1+I_1)=0$

$\Rightarrow S_1+I_1=N_1$ = number of men population etc.

[5 marks]

(ii) The system is

$$\frac{dI_1}{dt} = r_1(N_1-I_1)I_2 - a_1 I_1 = f_1(I_1,I_2)$$

$$\frac{dI_2}{dt} = r_2(N_2-I_2)I_1 - a_2 I_2 = f_2(I_1,I_2)$$

One steady state is $(0,0)$

The other is a solution of $f_1=0$ & $f_2=0$

$$I_1^* = \frac{N_1N_2-\rho_1\rho_2}{\rho_1+N_2}, \quad I_2^* = \frac{N_1N_2-\rho_1\rho_2}{\rho_2+N_1}, \quad \rho_k = \frac{a_k}{r_k}$$

making sense only if $N_1N_2>\rho_1\rho_2$

(5 marks)

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial I_1} & \frac{\partial f_1}{\partial I_2} \\ \frac{\partial f_2}{\partial I_1} & \frac{\partial f_2}{\partial I_2} \end{pmatrix} = \begin{pmatrix} -r_1 I_2 - a_1 & r_1 (N_1 - I_1) \\ r_2 (N_2 - I_2) & -r_2 I_1 - a_2 \end{pmatrix}$$

$$A(0,0) = \begin{pmatrix} -a_1 & r_1 N_1 \\ r_2 N_2 & -a_2 \end{pmatrix}$$

Characteristic equation $\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0$

$$\lambda^2 + (a_1 + a_2)\lambda + a_1 a_2 - r_1 r_2 N_1 N_2 = 0, \qquad \rho_k = \frac{a_k}{r_k}$$

$$\lambda_{1,2} = \frac{-(a_1 + a_2) \pm \sqrt{(a_1 + a_2)^2 + 4a_1 a_2 \left(\frac{N_1 N_2}{\rho_1 \rho_2} - 1\right)}}{2}$$

If $N_1 N_2 < \rho_1 \rho_2$ then $\det A > 0$, $\operatorname{tr} A < 0$ and $(0,0)$ is the only stable steady state; (I_1^*, I_2^*) does not exist then. [5 marks]

If $N_1 N_2 > \rho_1 \rho_2$ $\det A > 0$ and $(0,0)$ is unstable

In this case (I_1^*, I_2^*) exists and is stable

The proof of the stability is as follows:

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SOLUTION SHEET

COURSE / YEAR DT238

EXAMINATION SITTING: SUMMER ☐ SUPPLEMENTAL ☐ AUTUMN ☐ WINTER ☐

SUBJECT: INTRODUCTION TO BIOMATHEMATICS

EXPECTED SOLUTION TO QUESTION NO: 6 - cont.

EXAMINER: ROSSEN IVANOV

PROPOSED MARK ALLOCATION: 25

$$A(I_1^*, I_2^*) = \begin{pmatrix} -a_1 - r_1 I_2^* & r_1 (N_1 - I_1^*) \\ r_2 (N_2 - I_2^*) & -a_2 - r_2 I_1^* \end{pmatrix}$$

characteristic equation

$$\lambda^2 - \mathrm{tr} A \, \lambda + \det A = 0$$

$$\lambda^2 + (a_1 + a_2 + r_1 I_1^* + r_2 I_2^*)\lambda + \det A = 0$$

$$\mathrm{tr} A = -(a_1 + a_2) - r_1 I_2^* - r_2 I_1^* < 0 \quad \text{always} \checkmark$$

$$\det A = a_1 a_2 + a_1 r_2 I_1^* + r_1 a_2 I_2^* + r_1 r_2 I_1^* I_2^* - $$
$$- r_1 r_2 (N_1 - I_1^*)(N_2 - I_2^*)$$

after substitution of I_1^* & I_2^* we get

$$\det A = r_1 r_2 (N_1 N_2 - \rho_1 \rho_2) > 0$$

with $\mathrm{tr} A < 0$ and $\det A > 0 \Rightarrow \lambda_1 < 0$ & $\lambda_2 < 0$

$\Rightarrow (I_1^*, I_2^*)$ is stable steady state.

[10 marks]

Thus:

① When $N_1 N_2 < \rho_1 \rho_2$ the only steady state is $(0,0)$ - stable

② When $N_1 N_2 > \rho_1 \rho_2$ (I_1^*, I_2^*) is stable, $(0,0)$ - unstable

In case ① the epidemic dies out with time.

In case ② there is a constant fraction of population always infected.