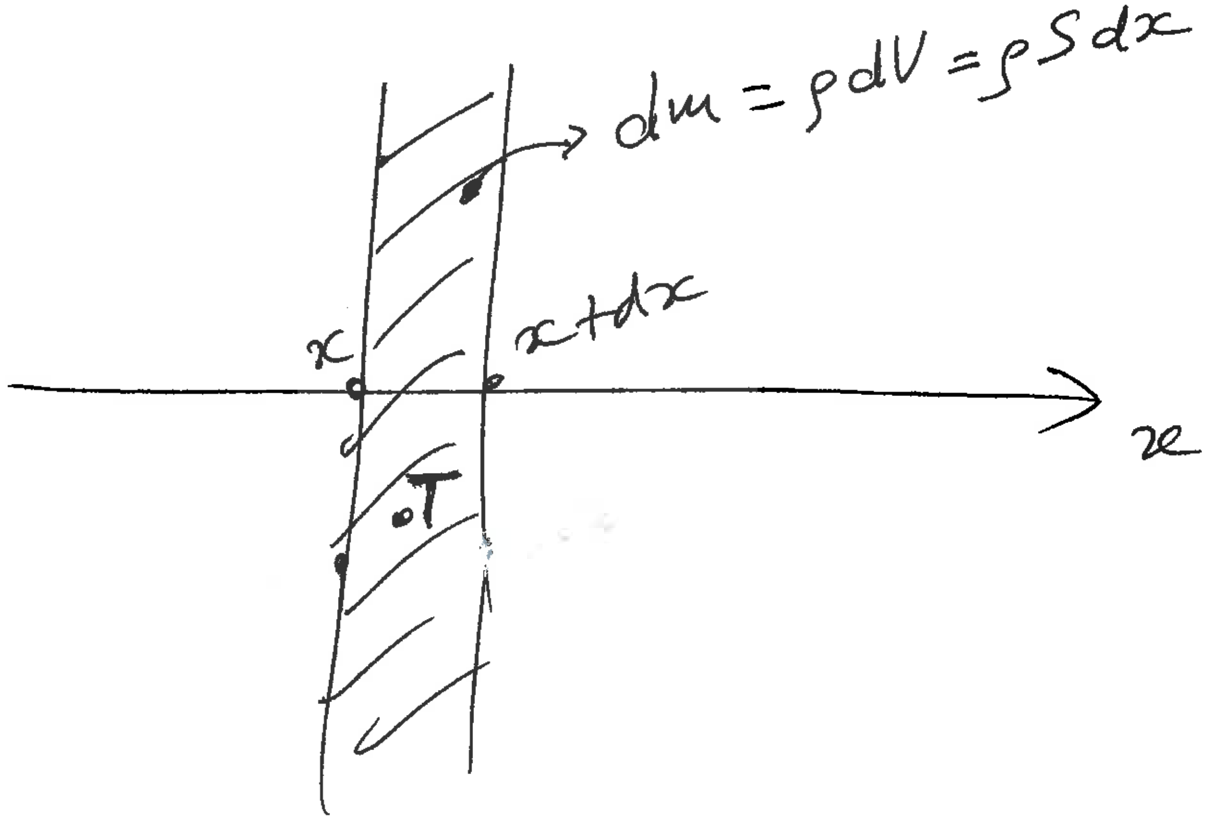
Geographic Spread of Epidemics.



$$\frac{\partial Q}{S\,dt} = (flow)_{x+dx} - (flow)_x = d_x(flow(x))$$

$$\frac{c\,m\,dT}{S\,dt} = d\left(-\varkappa \,grad\, T\right) = d_x\left(-\varkappa\frac{dT}{dx}\right)$$

$$\frac{c\rho \cancel{S}\, dx\, \partial T}{\cancel{S}\, dt} = -\varkappa\frac{\partial^2 T}{\partial x}\, dx$$

$$c\rho\frac{\partial T}{\partial t} + \varkappa\frac{\partial^2 T}{\partial x^2} = 0$$

$$\frac{\partial T}{\partial t} + \frac{\varkappa}{c\rho}\frac{\partial^2 T}{\partial x^2} = 0$$

$D = \frac{\varkappa}{c\rho}$

Heat equation:

$$\frac{\partial T}{\partial t} + D \frac{\partial^2 T}{\partial x^2} = 0$$

in 3-Dim

$$\frac{\partial T}{\partial t} + D \Delta T = 0$$

$$\nabla^2 T = \Delta T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

$S(\vec{x}, t) \rightarrow$ population density of susceptobles

$I(\vec{x}, t) \rightarrow$ population density of infectives.

$$\frac{\partial S}{\partial t} = -rIS + D\nabla^2 S$$

$$\frac{\partial I}{\partial t} = rIS + D\nabla^2 I - aI$$

$\frac{1}{a}$ - 'life' expectancy of an infective

r, a, D - positive constant parameters.

$$\frac{\partial S}{\partial t} = -rIS + D\frac{\partial^2 S}{\partial x^2}$$

$$\frac{\partial I}{\partial t} = rIS - aI + D\frac{\partial^2 I}{\partial x^2}$$

$S, I \rightarrow S_0$ - population density in absence of epidemics

$$I^* = \frac{I}{S_0} \qquad S^* = \frac{S}{S_0}$$

$$\frac{\partial (S/S_0)}{\partial t S_0 r} = -\cancel{\frac{S_0 r}{S_0 r}} \frac{I}{S_0}\left(\frac{S}{S_0}\right) + \frac{D}{S_0 r}\frac{\partial^2 (S/S_0)}{\partial x^2}$$

$$\boxed{t^* = t S_0 r}$$

$$x^* = \frac{\sqrt{S_0 r}}{\sqrt{D}}\, x$$

$$\boxed{\frac{\partial S^*}{\partial t^*} = -I^* S^* + \frac{\partial^2 S^*}{\partial x^{*2}}}$$

$$\frac{\partial (I/S_0)}{\partial (t S_0 r)} = \frac{\cancel{r}}{\cancel{r}} \frac{I}{S_0}\left(\frac{S}{S_0}\right) - \frac{a}{S_0 r}\left(\frac{I}{S_0}\right) + \frac{D}{S_0 r}\frac{\partial^2 (I/S_0)}{\partial x^2}$$

$$\frac{\partial I^*}{\partial t^*} = I^* S^* - \left(\frac{a}{S_0 r}\right) I^* + \frac{\partial^2 I^*}{\partial x^{*2}} ; \quad \boxed{\lambda = \frac{a}{S_0 r}}$$

$$\frac{\partial S}{\partial t} = -IS + \frac{\partial^2 S}{\partial x^2}$$

$$\frac{\partial I}{\partial t} = IS - \lambda I + \frac{\partial^2 I}{\partial x^2}, \quad \lambda = \frac{a}{S_0 r}$$

Travelling wave solution: $z = x - ct$

$$I(x,t) = I(z) = I(x-ct)$$

$$S(x,t) = S(z) = S(x-ct)$$

$$\frac{\partial S}{\partial t} = \frac{dS}{dz} \cdot \frac{\partial z}{\partial t} = S'(z)(-c) = -cS'(z)$$

$$\frac{\partial S}{\partial x} = \frac{dS}{dz} \cdot \frac{\partial z}{\partial x} = S'(z) \cdot 1 = S'(z)$$

$$-cS' = -IS + S''$$

$$-cI' = IS - \lambda I + I''$$

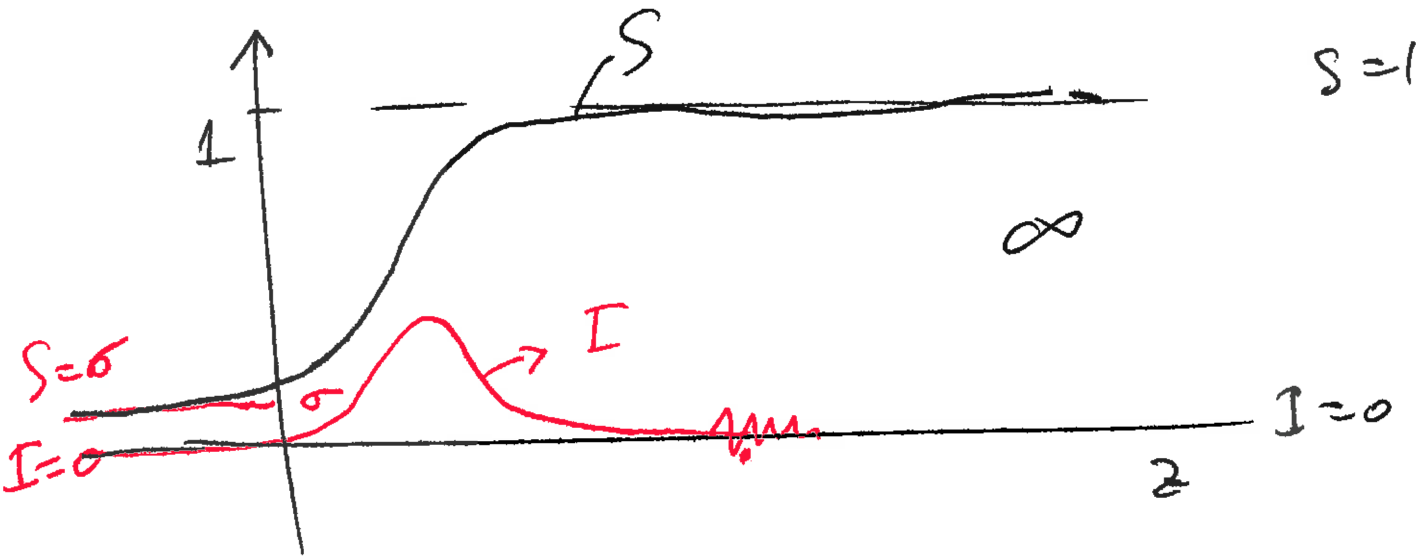
$$I'' + cI' + I(S - \lambda) = 0$$

$$S'' + cS' - IS = 0$$

before $z = \infty$ after $z = -\infty$

$I(\infty) = 0$ $I(-\infty) = 0$

$S(\infty) = 1$ $\sigma = S(-\infty) < 1$



There are 'steady states'

$S = 1$ & $I = 0$ at $z = \infty$

$S = \sigma$ & $I = 0$ at $z = -\infty$

$I \to 0 \quad S \to 1$

$$I'' + cI' + I(1-\lambda) = 0$$

$$I = ae^{mz} \qquad m = \text{const.}$$

$$m^2 e^{mz} + cme^{mz} + e^{mz}(1-\lambda) = 0$$

Characteristic equation

$$\boxed{m^2 + cm + (1-\lambda) = 0}$$

$$m_\pm = \tfrac{1}{2}\left(-c \pm \sqrt{c^2 - 4(1-\lambda)}\right)$$

$$I(z) = a_1 e^{\frac{1}{2}\left(-c+\sqrt{c^2-4(1-\lambda)}\right)z} + a_2 e^{\frac{1}{2}\left(-c-\sqrt{c^2-4(1-\lambda)}\right)z}$$

$z \to \infty \qquad \lambda > 1 ?$ ✗ $\Rightarrow$ $\underline{\lambda < 1}$ & $\underline{c^2 - 4(1-\lambda) \geq 0}$

$0 < \lambda < 1$

$c^2 \geq 4(1-\lambda)$

$\boxed{c \geq 2\sqrt{1-\lambda}}$

treshold condition for the propagation of the epidemic wave

$0 < \frac{a}{rS_0} = \lambda < 1$

$\boxed{S_0 > \frac{a}{r}}$

$c = 2\sqrt{1-\lambda}$

$$[v] = \frac{[x]}{[t]} = \frac{\sqrt{\frac{D}{rS_0}}}{\frac{1}{rS_0}} = \sqrt{rS_0 D}$$

$$\boxed{c = 2\sqrt{rS_0 D} \cdot \sqrt{1 - \frac{a}{rS_0}}}$$

S – asymptotic when $z \to \infty$, $S \to 1$

$S = 1 + s$, $|s| \ll 1$,

$$I \to a_{1,2}\, e^{\frac{1}{2}\left(-c \pm \sqrt{c^2 - 4(1-\lambda)}\right) z}$$

$$S'' + c_1 S' - IS = 0$$

$$s'' + cs' - I(1+\cancel{x}) = 0$$

$$s'' + cs' - I = 0 \qquad \Big| \; e^{cz}$$

$$e^{cz}s'' + \underbrace{ce^{cz}}_{\frac{d}{dz}e^{cz}} s' - e^{cz}\cdot I = 0$$

$$\frac{d}{dz}\left(e^{cz}\cdot s'\right) = e^{cz}\cdot I(z)$$

$$\frac{d}{dz}\left(e^{cz}\cdot s'\right) = e^{\underline{\underline{cz}}}\left(a_{1,2}\, e^{\frac{1}{2}\left(\underline{-c}\pm\sqrt{c^2-4(1-\lambda)}\right)z}\right)$$

$$\frac{d}{dz}\left(e^{cz}\cdot s'\right) = a_{1,2}\, e^{\frac{1}{2}\left(c\pm\sqrt{c^2-4(1-\lambda)}\right)z}$$

$$e^{cz}\cdot s' = b_{1,2}\, e^{\frac{1}{2}\left(c\pm\sqrt{c^2-4(1-\lambda)}\right)z}$$

$$s' = b_{1,2}\, e^{\frac{1}{2}\left(-c\pm\sqrt{c^2-4(1-\lambda)}\right)z}$$

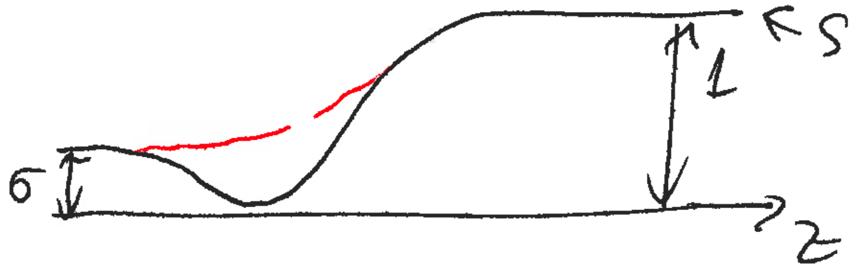
$$s = d_{1,2}\, e^{\frac{1}{2}\left(-c\pm\sqrt{c^2-4(1-\lambda)}\right)z}$$

$$S = 1 - s = $$
$$= 1 - d_1\, e^{\frac{1}{2}\left(-c+\sqrt{c^2-4(1-\lambda)}\right)z} - d_2\, e^{\frac{1}{2}\left(-c-\sqrt{}\right)z}$$

$$S'' + cS' - IS = 0 \qquad \Big| \, e^{cz}$$

For a max/min $\quad S' = 0$

$$S'' = IS > 0 \qquad \underline{\text{no max}}\,!$$



$$\frac{d}{dz}\left(e^{cz} S'\right) = e^{cz} I S$$

$$e^{cz} S'(z) = \int_{-\infty}^{z} e^{cz'} I(z') S(z')\, dz' + \underset{\downarrow\, 0}{e^{c(-\infty)}}\, \underset{\downarrow\, 0}{S'(-\infty)}$$

$$S'(z) = \underbrace{e^{-cz}}_{+} \int_{-\infty}^{z} \underbrace{e^{cz'}}_{+} \underbrace{I(z') S(z')}_{+}\, dz' > 0$$

$$0 < \sigma < S < 1$$

$c \sim 200 - 400$ miles/year

The Spatial Spread of Rabies among Foxes

$$\frac{\partial S}{\partial t} = -rIS$$

$$\frac{\partial I}{\partial t} = rIS - aI + D\frac{\partial^2 I}{\partial x^2}$$

$$\lambda = \frac{a}{rS_0}$$

<u>Nondimensional variables</u>

$$\frac{\partial S}{\partial t} = -IS$$

$$\frac{\partial I}{\partial t} = IS - \lambda I + \frac{\partial^2 I}{\partial x^2}$$

$$S(-\infty) = \sigma < 1, \quad S(\infty) = 1$$
$$I(-\infty) = 0 \qquad I(\infty) = 0$$

Travelling wave solution $z = x - ct$

$$\begin{cases} -cS' = -IS \\ -cI' = IS - \lambda I + I'' \end{cases}$$

$$\begin{cases} cS' = IS \\ I'' + cI' + I(S - \lambda) = 0 \end{cases}$$

Again $c \geq 2\sqrt{1-\lambda}$, $\lambda < 1$

$$\cancel{I} = \frac{cS'}{\cancel{S}}$$

$$\cancel{I'' + cI' + I\left(\frac{cS'}{I} - \lambda\right) = 0}$$

$$I'' + cI' + \underbrace{\frac{cS'}{S}}_{I}(S - \lambda) = 0$$

$$I'' + cI' + cS' - c\lambda\frac{S'}{S} = 0$$

$$\frac{d}{dz}\left(I' + cI + cS - c\lambda \ln S\right) = 0$$

$$I' + cI + cS - c\lambda \ln S = \text{const}$$

$$\underset{0}{\underbrace{I'(\infty)}} + c\underset{0}{\underbrace{I(\infty)}} + c\underset{1}{\underbrace{S(\infty)}} - c\lambda \ln \underset{1}{\underbrace{S(\infty)}} =$$

$$= \underbrace{I'(-\infty)}_{0} + c\underbrace{I(-\infty)}_{0} + c\underbrace{S(-\infty)}_{\sigma} - c\lambda \ln \underbrace{S(-\infty)}_{\sigma}$$

$$c = c\sigma - c\lambda \ln \sigma \qquad \times \frac{1}{c}$$

$$\boxed{1 = \sigma - \lambda \ln \sigma, \quad \lambda < 1, \quad \sigma = S(-\infty)}$$

$$\frac{\sigma - 1}{\ln \sigma} = \lambda; \qquad 0 < \sigma < \lambda < 1$$

$$\begin{cases} \dfrac{\partial S}{\partial t} = -rIS + BS\left(1 - \dfrac{S}{S_0}\right) \\ \dfrac{\partial I}{\partial t} = rIS - aI + D\dfrac{\partial^2 I}{\partial x^2} \end{cases}$$