W9209/9210/9211/9212

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**School of Mathematical Sciences**

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# DT9209 MSc Applied Mathematics
# DT9210 MSc Applied Mathematics
# DT9211 MSc Applied Mathematics
# DT9212 MSc Applied Mathematics

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**WINTER EXAMINATIONS 2017/2018**

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## MATH 9971: Classical Mechanics and Thermodynamics

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Dr R. Ivanov

Dr C Hills

Dr S Mitchell


18:30 – 20:30 pm, Tuesday, 9 January 2018

Duration: 2 hours

Answer three questions

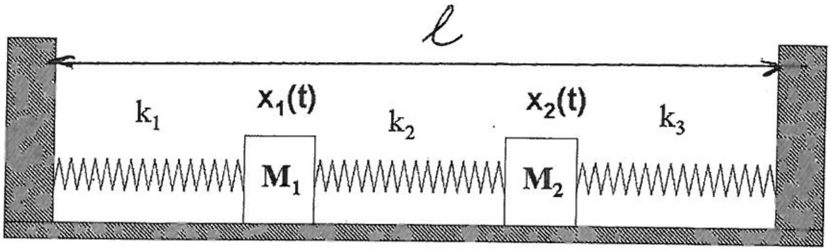
All questions carry equal marks

Approved calculators may be used

Mathematical tables are provided

New Cambridge Statistical Tables are NOT permitted

1. Two point masses $M_1$ and $M_2$ are connected with three ideal springs with elasticities $k_1$, $k_2$ and $k_3$, as shown to the figure below. The distance $l$ is constant. There is no friction.




a) Introducing appropriate generalised coordinates, write down the Lagrangian of the system. (6)

b) Write down the Lagrangian equations of motion. (8)

c) Find the equilibrium positions of the two masses. (9)

d) Find the possible frequencies of the oscillations about the equilibrium found in part c). (10)

[33]

2. A point mass $m$ moves in the field with a scalar potential $V(\mathbf{x})$, where $\mathbf{x} = (x_1, x_2, x_3)$ is the position vector in Cartesian coordinates.

a) Find the Hamiltonian and write down the Hamiltonian equations of motion for the point mass in cylindrical coordinates $(r, \theta, z)$ where

$$x_1 = r\cos\theta, \qquad x_2 = r\sin\theta, \qquad x_3 = z.$$

(17)

b) Simplify the equations in the case of a central force $V = V(r)$ where $r$ is introduced in part a). Do not solve these equations. Find the conserved momenta.

(6)

c) Using the conservation of the Hamiltonian and the conserved momenta from part b) (or otherwise) demonstrate that in the case of a potential $V = \alpha r^2$ with $\alpha$ being a positive constant, the trajectory is a closed orbit. (10)

[33]

**3.** a) Show that for an ideal gas the thermal capacities (per 1 mol) satisfy the relation

$$C_p - C_V = R$$

where $R = 8.3144598$ J/(mol$\cdot$ K) is the gas constant.

(10)

b) In a polytropic process the thermal capacity $C$ (per 1 mol) is constant. Show that an ideal gas undergoing a polytropic process satisfies the relation $pV^n$ =constant with a constant index $n$. Express $n$ through $C$, $C_p$ and $C_V$.

(18)

c) Show that any adiabatic process for an ideal gas is also polytropic and find $n$ for an adiabatic process.

(5)

**[33]**

**4.** a) Two bodies with initial temperatures $T_1$ and $T_2$ and constant thermal capacities $C_1$ and $C_2$ are put in contact until they reach equilibrium and their temperatures equalise. Find the equilibrium temperature, the entropy change and demonstrate that this process is irreversible. Take for simplicity $C_1 = C_2$.

(13)

b) Two bodies with initial temperatures $T_1$ and $T_2$ and constant thermal capacities $C_1$ and $C_2$ are used as a heater and a cooler of a heat engine. Find the maximal work that can be produced by the heat engine. Solve the problem for arbitrary $C_1$ and $C_2$ first and then obtain the solution for $C_1 = C_2$ as a particular case.

(20)

**[33]**

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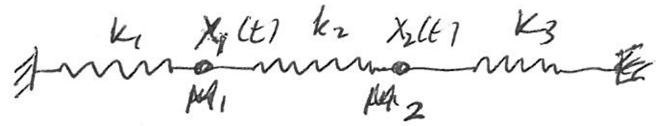
Sample Examination Solutions

Qn part (e.g. 4 a) ii) ) | Mark

a) For generalised coordinates $x_{1,2}(t)$ like on the figure




The kinetic energy is $T = \frac{M_1 \dot{x}_1^2}{2} + \frac{M_2 \dot{x}_2^2}{2}$ — **2**

the potential energy is

$$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_2 - x_1)^2 + \frac{1}{2} k_3 (\ell - x_2)^2$$ — **2**

$$L = T - V = \frac{M_1 \dot{x}_1^2}{2} + \frac{M_2 \dot{x}_2^2}{2} - \frac{k_1}{2} x_1^2 - \frac{k_2}{2} (x_2 - x_1)^2 - \frac{k_3}{2} (\ell - x_2)^2$$ — **2**

b)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$$

$$\frac{d}{dt}(M_1 \dot{x}_1) - (-k_1 x_1 + k_2 (x_2 - x_1)) = 0$$

$$M_1 \ddot{x}_1 + k_1 x_1 - k_2 (x_2 - x_1) = 0$$

$$\boxed{M_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = 0} \quad \text{Eq. 1.}$$ — **4**

Question continued on next page ✓ OR Total Final Mark Allocation (check)

Sample Examination Solutions

Qn part [e.g. 4 a) ii) ] | Mark

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0$$

$$\frac{d}{dt}\left(M_2 \dot{x}_2\right) - \left(-k_2(x_2 - x_1) + k_3(\ell - x_2)\right) = 0$$

$$M_2 \ddot{x}_2 + k_2(x_2 - x_1) - k_3(\ell - x_2) = 0$$

$$\boxed{M_2 \ddot{x}_2 + (k_2+k_3)x_2 - k_2 x_1 = k_3 \ell} \quad Eq. 2.$$ 4

(c) Assume that $x_k = \ell_k + q_k$ where $\ell_k$ is the distance at equiliberium, $\ell_k = \text{const}$.

From eq. 1 => From eq. 2
$$\begin{cases} (k_1+k_2)\ell_1 - k_2 \ell_2 = 0 \\ -k_2 \ell_1 + (k_2+k_3)\ell_2 = k_3 \ell \end{cases}$$ 3

$$\ell_1 = \frac{\begin{vmatrix} 0 & -k_2 \\ k_3\ell & k_2+k_3 \end{vmatrix}}{\begin{vmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{vmatrix}} = \frac{k_2 k_3 \ell}{(k_1+k_2)(k_2+k_3) - k_2^2} = \frac{k_2 k_3 \ell}{k_1k_2 + k_1k_3 + k_2k_3}$$ 3

$$\ell_2 = \frac{\begin{vmatrix} k_1+k_2 & 0 \\ -k_2 & k_3\ell \end{vmatrix}}{\begin{vmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{vmatrix}} = \frac{(k_1+k_2)k_3 \ell}{k_1k_2 + k_1k_3 + k_2k_3}$$ 3

Also $\ell_2 = \ell_1 + \dfrac{k_1 k_3 \ell}{k_1k_2 + k_1k_3 + k_2k_3}$.

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Sample Examination Solutions

| Qn part (e.g. 4 a) ii) ) | | Mark |
|---|---|---|
| (d) | Assuming $q_K = q_{K,0} e^{i\omega t}$, thus $\ddot{q}_K = -\omega^2 q_K$ | 2 |
| | From eq(1) and eq. 2 we have | |
| | $\begin{cases} -M_1\omega^2 q_1 + (K_1+K_2) q_1 - K_2 q_2 = 0 \\ -M_2\omega^2 q_2 + (K_2+K_3) q_2 - K_2 q_1 = 0 \end{cases}$ | 2 |
| | or | |
| | $\begin{cases} (-M_1\omega^2 + K_1+K_2) q_1 - K_2 q_2 = 0 \\ -K_2 q_1 + (K_2+K_3 - M_2\omega^2) q_2 = 0 \end{cases}$ | |
| | the compatibility condition is | |
| | $\Rightarrow (K_1+K_2-M_1\omega^2)(K_2+K_3-M_2\omega^2) - K_2^2 = 0$ | 3 |
| | $M_1M_2\omega^4 - [M_1(K_2+K_3) + M_2(K_1+K_2)]\omega^2 + K_1K_2+K_1K_3+K_2K_3 = 0$ | 2 |
| | $\omega_\pm^2 = \dfrac{M_1(K_2+K_3)+M_2(K_1+K_2) \pm \sqrt{[M_1(K_2+K_3)+M_2(K_1+K_2)]^2 - 4M_1M_2(K_1K_2+K_1K_3+K_2K_3)}}{2M_1M_2}$ | 1 |
| | two possible frequences. | |
| not required | Further examination shows that the discriminant of the above eq. is positive, i.e. $\omega_\pm^2 > 0$ – two possible frequences | |

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Sample Examination Solutions

| Qn part | | Mark |
|---|---|---|
| (a) | The Lagrangian is | |
| | $L = T - V = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - V(\vec{x})$ | |
| | $\dot{x}_1 = \dot{r}\cos\theta + r(-\sin\theta)\dot{\theta}$ | |
| | $\dot{x}_2 = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta}$ | |
| | $\dot{x}_1^2 + \dot{x}_2^2 = \dot{r}^2 + r^2\dot{\theta}^2$ | |
| | $L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) - V(r,\theta,z)$ | 5 |
| | $p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad \Rightarrow \quad \dot{r} = \frac{p_r}{m}$ | |
| | $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \quad \Rightarrow \quad \dot{\theta} = \frac{p_\theta}{mr^2}$ | |
| | $p_z = m\dot{z} \quad \Rightarrow \quad \dot{z} = \frac{p_z}{m}$ | 3 |
| | $H = p_r\dot{r} + p_\theta\dot{\theta} + p_z\dot{z} - L =$ | |
| | $= p_r\left(\frac{p_r}{m}\right) + p_\theta\left(\frac{p_\theta}{mr^2}\right) + p_z\left(\frac{p_z}{m}\right) -$ | |
| | $- \frac{m}{2}\left[\left(\frac{p_r}{m}\right)^2 + r^2\left(\frac{p_\theta}{mr^2}\right)^2 + \left(\frac{p_z}{m}\right)^2\right] + V(r,\theta,z)$ | |
| | $H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + p_z^2\right) + V(r,\theta,z)$ | 3 |

Question continued on next page ✓ OR Total Final Mark Allocation

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Sample Examination Solutions

Qn part [e.g. 4 a) ii) ] | Mark

$$\dot{p}_r = -\frac{\partial H}{\partial r} = -\frac{p_\theta^2}{2m}\left(-\frac{2}{r^3}\right) - \frac{\partial V}{\partial r} = \frac{p_\theta^2}{mr^3} - \frac{\partial V}{\partial r}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -\frac{\partial V}{\partial \theta}$$

$$\dot{p}_z = -\frac{\partial H}{\partial z} = -\frac{\partial V}{\partial z}$$

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}$$

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m}$$

6

b) When $V = V(r) \Rightarrow \frac{\partial V}{\partial r} \neq 0$ but $\frac{\partial V}{\partial \theta} = 0$, $\frac{\partial V}{\partial z} = 0$

$\Rightarrow p_\theta = \text{const}$ (Angular momentum conserv.)
$p_z = \text{const}$ (center of mass motion) — 2

$$\dot{p}_r = \frac{p_\theta^2}{mr^3} - \frac{dV(r)}{dr}$$

$$\dot{r} = \frac{p_r}{m}, \quad \dot{\theta} = \frac{p_\theta}{mr^2}, \quad \dot{z} = \frac{p_z}{m}$$

— 4

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Sample Examination Solutions

Qn part | Mark

c) The trajectory is a line, which represents the intersection of 2 surfaces in the $(r, p_r, H)$ space

(i) $H = \text{const}$, $H(r, p_r) = \text{const}$ — 2

(ii) $H = H(r, p_r)$

Note that $H(r, p_r) = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + \alpha r^2 + \frac{p_z^2}{2m} \geq 0$ — 2

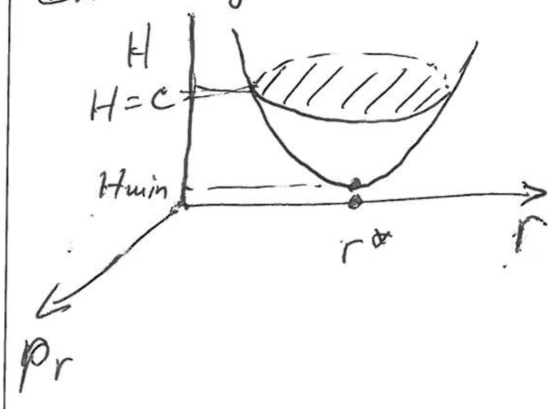
const.

and $\Rightarrow$ $H(r, p_r)$ has a minimum at

$\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - 2\alpha r = 0 \Rightarrow (r^*)^4 = \frac{p_\theta^2}{2\alpha m} = \text{const.}$ — 2

$\frac{\partial H}{\partial p_r} = \frac{p_r}{m} = 0 \Leftrightarrow \dot{r} = 0 \Rightarrow r = r^* = \text{const.}$

In every direction $(r, p_r) \to \infty$, $H \to \infty$




The section $H = c$ is a closed orbit. — 4

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Sample Examination Solutions

| Qn part | | Mark |
|---|---|---|
| (a) | From the first principle | |
| | $\delta Q = dU + dW$ | 3 |
| | For a process with $p = \text{const} \Rightarrow$ | |
| | $C_p\,dT = C_v\,dT + p\,dV \quad ; \quad pV = RT$ | 3 |
| | $\Rightarrow p\,dV = R\,dT \quad (p = \text{const})$ | 1 |
| | $\Rightarrow C_p\,dT = C_v\,dT + R\,dT \Rightarrow \boxed{C_p - C_v = R}$ | 3 |
| (b) | $c\,dT = C_v\,dT + p\,dV$ | 3 |
| | $d(pV) = R\,dT \Rightarrow dT = \frac{1}{R}\,d(pV)$ | 3 |
| | $\Rightarrow (c - C_v)\frac{1}{R}\,d(pV) = p\,dV$ | |
| | $\frac{c - C_v}{R}(p\,dV + V\,dp) = p\,dV$ | |
| | $\frac{c - C_v}{C_p - C_v}\,p\,dV - p\,dV + \frac{c - C_v}{C_p - C_v}\,V\,dp = 0$ | |
| | $\frac{c - \cancel{C_v} - C_p + \cancel{C_v}}{C_p - C_v}\,p\,dV + \frac{c - C_v}{C_p - C_v}\,V\,dp = 0$ | |

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Sample Examination Solutions

| Qn part [e.g. 4 a) ii) ] | | Mark |
|---|---|---|
| | $$\frac{C-C_p}{C-C_v}\cdot\frac{dV}{V}+\frac{dp}{p}=0$$ | 8 |
| | $$\Rightarrow \frac{C-C_p}{C-C_v}\ln V+\ln p=\text{const}$$ | |
| | $$pV^{\frac{C-C_p}{C-C_v}}=\text{const}$$ | 3 |
| | $$n=\frac{C-C_p}{C-C_v}$$ | 1 |
| (c) | For an adiabatic process $\delta Q=0 \Rightarrow C=0$ | 4 |
| | $\Rightarrow n=\frac{C_p}{C_v}$. | 1 |

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Sample Examination Solutions

Qn part (e.g. 4 a) ii) ) | Mark

(a) Let us suppose that $T_1 > T_2$

then, if $T$ is the equilibrium temperature we have

$$C_1(T_1 - T) = C_2(T - T_2)$$

$$C_1T_1 + C_2T_2 = (C_1 + C_2)T \Rightarrow T = \frac{C_1T_1 + C_2T_2}{C_1 + C_2}$$

$$T = \frac{1}{2}(T_1 + T_2) \quad \text{when } C_1 = C_2$$ 5

$$\Delta S = \Delta S_1 + \Delta S_2 = \int_{T_1}^{T} \frac{\delta Q_1}{T} + \int_{T_2}^{T} \frac{\delta Q_2}{T} = \int_{T_1}^{T} \frac{C_1 dT}{T} + \int_{T_2}^{T} \frac{C_2 dT}{T}$$

$$= C_1 \ln\frac{T}{T_1} + C_2 \ln\frac{T}{T_2} = \ln\frac{T^{C_1+C_2}}{T_1^{C_1} T_2^{C_2}}$$

when $C_1 = C_2 = C$

$$\Delta S = C \ln\frac{T^2}{T_1T_2} = C \ln\frac{(T_1+T_2)^2}{4T_1T_2} > 0$$ 4

because $(T_1+T_2)^2 > 4T_1T_2 \Leftrightarrow (T_1-T_2)^2 > 0$ 2

Since $\Delta S > 0$ the process is irreversible 2

Question continued on next page ✓ OR Total Final Mark Allocation (check)

Sample Examination Solutions

| Qn part | | Mark |
|---|---|---|
| b) | The maximal work is achieved when the process is quasistatic, that is, when the process is reversible. Then $\underline{\Delta S = 0}$ | 4 |
| | Recall $\Rightarrow \Delta S = \ln \dfrac{T^{C_1+C_2}}{T_1^{C_1} T_2^{C_2}} = 0 \Rightarrow$ | 4 |
| | $T^{C_1+C_2} = T_1^{C_1} T_2^{C_2} \Rightarrow T = T_1^{\frac{C_1}{C_1+C_2}} T_2^{\frac{C_2}{C_1+C_2}}$ | 2 |
| | Then $C_1(T_1 - T) = C_2(T - T_2) + W_{max}$ | 4 |
| | $W_{max} = C_1T_1 + C_2T_2 - (C_1+C_2)T$ <br> $W_{max} = C_1T_1 + C_2T_2 - (C_1+C_2) T_1^{\frac{C_1}{C_1+C_2}} T_2^{\frac{C_2}{C_1+C_2}}$ | 2 |
| | when $C_1 = C_2 = C$ <br> $W_{max} = C(T_1 + T_2 - 2\sqrt{T_1T_2}) = C(\sqrt{T_1} - \sqrt{T_2})^2 > 0$ | 4 |
| | when $C_1 \neq C_2$ $W_{max} > 0$ as well, this can be proven, but not required | |

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