W9209/9210/104C

DUBLIN INSTITUTE OF TECHNOLOGY

School of Mathematical Sciences

DT9209 MSc Applied Mathematics
DT9210 MSc Applied Mathematics

WINTER EXAMINATIONS 2016/2017

MATH 9974: BIOMATHEMATICS

DR R. IVANOV

DR C HILLS

PROFESSOR E O'RIORDAN

9.30 – 12.30 pm, Friday, 13 January 2017

Duration: 3 hours

Answer three questions

All questions carry equal marks

Approved calculators may be used

Mathematical tables are provided

New Cambridge Statistical Tables are NOT permitted

1. a) Find the steady states of the logistic population growth model and determine their stability:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), \qquad N(0) > 0$$

with r and K positive constants.

(6)

b) Solve explicitly the model and compute the limit of $N(t)$ when $t \to \infty$. Compare the obtained limit with the results from part **a)** and explain your findings. (12)

c) Find the steady states and determine their stability for the logistic model with harvesting

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - EN,$$

where the positive constant E measures the harvesting effort. Comment on the ecological implications of the results in the two cases: $E > r$ and $E < r$. (15)

[33]

2. A population growth model is described by a differential equation with delay $T > 0$:

$$\frac{dN(t)}{dt} = rN(t)\left(1 - \frac{N(t-T)}{K}\right),$$

with $r > 0$, $K > 0$ and $N(0) > 0$.

a) Show that in the new non-dimensional variables $u = N/K$, $\tau = rt$ and $\Theta = rT$ the equation has the form

$$\frac{du(\tau)}{d\tau} = u(\tau)[1 - u(\tau - \Theta)],$$

(3)

b) Show that $u^* = 1$ is a steady state and linearize the equation about the steady state by writing $u(\tau) = u^* + n(\tau)$, where $|n(\tau)| \ll 1$.

(5)

c) Look for solutions of the form $n(\tau) = ce^{\lambda\tau}$, where c is a constant, and write the corresponding equation for λ.

(5)

d) Decompose λ into real and imaginary parts ($\lambda = \mu + i\omega$) and write separately the real and imaginary parts of the equation for λ, obtained in part **c)** in terms of μ and ω.

(5)

e) Analyze the equations obtained in part **d)** when the delay Θ increases from 0 to $\frac{\pi}{2}$, having in mind that μ and ω depend on Θ. Show that the steady state solution u^* is stable when $0 \le \Theta < \pi/2$. Demonstrate that the first bifurcation value of Θ for which the steady state u^* becomes unstable and the solution becomes oscillatory corresponds to $\Theta = \pi/2$, (and $\mu = 0$, $\omega = \pm 1$) or, in dimensional terms, $rT = \pi/2$.

(10)

f) Show that the period of oscillations (in non-dimensional units) at the bifurcation value is 4Θ.

(5)

[33]

3. Consider the discrete population model with delay

$$U_{t+1} = U_t e^{r(1-U_{t-1})},$$

where t is the discrete time and r is a positive parameter.

a) Show that $U^* = 1$ is a steady state of the model. Linearize the equation about $U^* = 1$ by writing $U_t = U^* + v_t$, where $|v_t| \ll 1$. Obtain the linearized equation

$$v_{t+1} - v_t + rv_{t-1} = 0.$$

(10)

b) Prove that, for $0 < r < 1$, $U^* = 1$ is a stable steady state.

(11)

c) Show that $r = 1$ is a bifurcation value for which the steady state bifurcates to a periodic solution of period 6.

(7)

d) Prove that, for $r > 1$, the steady state $U^* = 1$ is unstable.

(5)

[33]

4. The Lotka-Volterra predator-prey model is given by the following system of equations for the non-dimensional variables $u(t)$, $v(t)$:

$$\begin{aligned}\frac{du}{dt} &= u(1-v)\\ \frac{dv}{dt} &= \alpha v(u-1)\end{aligned}$$

where α is a positive parameter.

a) Show that $u^* = 1$, $v^* = 1$ is a steady state of the model. Linearise the equations about $u^* = 1$, $v^* = 1$ and show that the obtained linear model possess periodic solutions. Find the period of these solutions in terms of α.

(12)

b) Show that the full nonlinear system has a conservation law

$$H(u, v) = \alpha u + v - \ln(u^\alpha v).$$

(7)

c) Using the conservation law from part **b)** (or otherwise) demonstrate that the full nonlinear system has periodic solutions.

(14)

[33]