DUBLIN INSTITUTE OF TECHNOLOGY
KEVIN STREET, DUBLIN 8

MSc in Applied Mathematics and Theoretical Physics
DT238

Supplemental Examinations
2011

INTRODUCTION TO BIOMATHEMATICS

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Tuesday, 18 January 2011 9:30am – 1 pm

Answer any FOUR questions. All questions carry 25 marks. If more than four questions are attempted, only the best four will be graded.

Dept. of Education Tables allowed

Question 1. (i) Find the steady states of the logistic population growth model and determine their stability:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN\Big(1 - \frac{N}{K}\Big), \qquad N(0) > 0$$

with r and K positive constants.

[7 marks]

(ii) Solve explicitly the model and compute the limit of $N(t)$ when $t \to \infty$. Compare the obtained limit with the results from (i) and explain your findings.

[9 marks]

(iii) Find the steady states and determine their stability for the logistic model with harvesting

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN\Big(1 - \frac{N}{K}\Big) - EN,$$

where the positive constant E measures the harvesting effort. Comment on the ecological implications of the results in the two cases: $E > r$ and $E < r$.

[9 marks]

Question 2. A population growth model is described by a differential equation with delay $T > 0$:

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = rN(t)\Big(1 - \frac{N(t-T)}{K}\Big),$$

with $r > 0$, $K > 0$ and $N(0) > 0$.

(i) Show that in the new nondimensional variables $u = N/K$, $\tau = rt$ and $\Theta = rT$ the equation has the form

$$\frac{\mathrm{d}u(\tau)}{\mathrm{d}\tau} = u(\tau)[1 - u(\tau - \Theta)],$$

[2 marks]

(ii) Show that $u^* = 1$ is a steady state and linearize the equation about the steady state by writing $u(\tau) = u^* + n(\tau)$, where $|n(\tau)| \ll 1$.

[5 marks]

(iii) Look for solutions of the form $n(\tau) = ce^{\lambda\tau}$, where c is a constant, and write the corresponding equation for λ.

[3 marks]

(iv) Decompose λ into real and imaginary parts ($\lambda = \mu + i\omega$) and write separately the real and imaginary parts of the equation for λ, obtained in (iii) in terms of μ and ω.

[5 marks]

(v) Analyze the equations obtained in (iv) when the delay Θ increases from 0 to $\frac{\pi}{2}$, having in mind that μ and ω depend on Θ. Show that the steady state solution u^* is stable when $0 \leq \Theta < \pi/2$. Demonstrate that the first bifurcation value of Θ for which the steady state u^* becomes unstable and the solution becomes oscillatory corresponds to $\Theta = \pi/2$, (and $\mu = 0$, $\omega = \pm 1$) or, in dimensional terms, $rT = \pi/2$. Show that this bifurcation occurs when μ, being negative when $\Theta = 0$ reaches the bifurcation value $\mu = 0$ when $\Theta = \pi/2$.

[5 marks]

(vi) Show that the period of oscillations (in nondimensional units) at the bifurcation value is 4Θ.

[5 marks]

Question 3. Consider the effect of regularly harvesting the population of a species for which the model equation is

$$N_{t+1} = \frac{bN_t^2}{1+N_t^2} - EN_t \equiv f(N_t; E), \qquad b > 2, \qquad E > 0,$$

where E is a measure of the effort expended in obtaining the harvest EN_t.

(i) Determine the steady states and hence show that if the effort $E > E_m = (b-2)/2$ no harvest is obtained.

[9 marks]

(ii) If $E < E_m$ show that the model is realistic only if the population N_t always lies between two positive values which you should determine analytically.

[9 marks]

(iii) With $E < E_m$ evaluate the eigenvalue of the largest positive steady state. Demonstrate that a bifurcation exists as $E \to E_m$.

[7 marks]

Question 4. Consider the discrete population model with delay

$$U_{t+1} = U_t e^{r(1-U_{t-1})},$$

where t is the discrete time and r is a positive parameter.

(i) Show that $U^* = 1$ is a steady state of the model. Linearize the equation about $U^* = 1$ by writing $U_t = U^* + v_t$, where $|v_t| \ll 1$. Obtain the linearized equation

$$v_{t+1} - v_t + rv_{t-1} = 0.$$

[7 marks]

(ii) Prove that, for $0 < r < 1$, $U^* = 1$ is a stable steady state.

[6 marks]

(iii) Show that $r = 1$ is a bifurcation value for which the steady state bifurcates to a periodic solution of period 6.

[6 marks]

(iv) Prove that, for $r > 1$ the steady state $U^* = 1$ is unstable.

[6 marks]

Question 5. (i) Describe the interaction between two species with populations N_1 and N_2 that is implied by the model

$$\begin{aligned}\frac{\mathrm{d}N_1}{\mathrm{d}t} &= r_1 N_1\Big(1 - \frac{N_1}{K_1 + b_{12}N_2}\Big),\\ \frac{\mathrm{d}N_2}{\mathrm{d}t} &= r_2 N_2\Big(1 - \frac{N_2}{K_2 + b_{21}N_1}\Big),\end{aligned}$$

where all parameters, b_{12}, b_{21}, K_1, K_2, r_1, r_2 are positive.

[5 marks]

(ii) Determine the steady states and their stability.

[15 marks]

(iii) Briefly describe the ecological implications of the results of the analysis.

[5 marks]

Question 6. An epidemic model involves only two interacting populations, infectives with density $i(\mathbf{x}, t)$, and susceptibles with density $s(\mathbf{x}, t)$, both functions of the space variable $\mathbf{x} = (x_1, x_2, x_3)$ as well as time t. The model is described by the system

$$\begin{aligned}\frac{\partial s}{\partial t} &= -rsi + D\Delta s,\\ \frac{\partial i}{\partial t} &= rsi - ai + D\Delta i,\end{aligned}$$

where r, a and D are positive constants and $\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$.

(i) Explain briefly the rationale of the model and the meaning of the constants r, a and D.

[3 marks]

(ii) Consider the one-dimensional problem, $i = i(x_1, t)$, $s = s(x_1, t)$. Nondimensionalise the system by writing

$$I = \frac{i}{s_0}, \quad S = \frac{s}{s_0}, \quad x = \sqrt{\frac{rs_0}{D}}x_1, \quad \tau = rs_0 t, \quad \lambda = \frac{a}{rs_0},$$

where s_0 is the population density in the absence of epidemic. Show that the system takes the form

$$\begin{aligned} \frac{\partial S}{\partial \tau} &= -SI + \frac{\partial^2 S}{\partial x^2}, \\ \frac{\partial I}{\partial \tau} &= SI - \lambda I + \frac{\partial^2 I}{\partial x^2}. \end{aligned}$$

[2 marks]

(iii) Look for travelling wave solutions by setting $I(x,\tau) = I(z)$, $S(x,\tau) = S(z)$, where $z = x - c\tau$ and c is the travelling speed. Obtain the system of ordinary differential equations:

$$I'' + cI' + I(S - \lambda) = 0, \qquad S'' + cS' - IS = 0,$$

where the prime denotes differentiation with respect to z.

[5 marks]

(iv) Consider solutions, such that $I(-\infty) = I(\infty) = 0$, $I(z) > 0$, $S(z) > 0$, $0 \le \sigma = S(-\infty) < S(\infty) = 1$, where σ is a constant. Explain the meaning of σ. Prove that for all finite z, $\sigma < S(z) < 1$. Prove that $S(z)$ is monotonic for all z with $S'(z) > 0$.

[5 marks]

(v) Prove that

$$\int_{-\infty}^{\infty} I(z)\mathrm{d}z > \int_{-\infty}^{\infty} I(z)S(z)\mathrm{d}z = \lambda \int_{-\infty}^{\infty} I(z)\mathrm{d}z$$

and hence deduce that the threshold criterion for travelling epidemic wave solution to exist is $\lambda < 1$.

[5 marks]

(vi) Linearize the equation for I in the region $z \to \infty$ where $S \to 1$, $I \to 0$. Demonstrate that if the travelling wave solution exists, the wavespeed c must satisfy $c \ge 2\sqrt{1-\lambda}$.

[5 marks]

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SOLUTION SHEET

COURSE / YEAR: DT 238

EXAMINATION SITTING: SUMMER ☐ SUPPLEMENTAL ☑ AUTUMN ☐ WINTER ☐

SUBJECT: INTRODUCTION TO BIOMATHEMATICS

EXPECTED SOLUTION TO QUESTION NO: 1

EXAMINER: ROSSEN IVANOV

PROPOSED MARK ALLOCATION: 25

(i) The two steady states are $N^*=0$ and $N^*=K$

$f(N)=rN\left(1-\frac{N}{K}\right)=rN-\frac{r}{K}N^2$

$f'(N)=r-\frac{2r}{K}N$ [3 marks]

$f'(0)=r>0 \Rightarrow N_1^*=0$ is an unstable steady state [2 marks]

$f'(K)=r-2r=-r<0 \Rightarrow N_2^*=K$ is a stable steady state

[2 marks]

(ii) $\frac{dN}{N\left(1-\frac{N}{K}\right)}=r\,dt$, $\frac{1}{N\left(1-\frac{N}{K}\right)}=\frac{1}{N}+\frac{1}{K-N}$

$\Rightarrow \ln N-\ln(K-N)=rt+\text{const}$

[3 marks]

$\Rightarrow$

$\frac{N}{K-N}=Ce^{rt}$, $C=\text{const}$ At $t=0$ $N(0)=N_0$

$\frac{N_0}{K-N_0}=Ce^0 \Rightarrow C=\frac{N_0}{K-N_0}$ [3 marks]

$N=K\frac{N_0}{(K-N_0)e^{-rt}+N_0}\xrightarrow{t\to\infty}K=N_2^*$ – the second steady state, which is stable [3 marks]

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COURSE / YEAR ______

SUBJECT: ______

EXAMINER: ______

EXAMINATION SITTING: SUMMER ☐ SUPPLEMENTAL ☐ AUTUMN ☐ WINTER ☐

EXPECTED SOLUTION TO QUESTION NO: 1 - ctd.

PROPOSED MARK ALLOCATION: 25

(iii) $f(N) = rN(1-\frac{N}{K}) - EN$

$f(N)=0$ has two roots: $N_1^* = 0$, $N_2^* = K(1-\frac{E}{r})$, which exists if $E<r$!

$f'(N) = r - E - \frac{2r}{K}N$ [3 marks]

$f'(0) = r-E \begin{cases} >0 & \text{if } r>E,\ N_1^*=0 \text{ is unstable} \\ <0 & \text{if } r<E,\ N_1^*=0 \text{ is stable} \end{cases}$

[2 marks]

$f'(N_2^*) = -(r-E) < 0$ when N_2^* exists

$\Rightarrow$ when N_2^* exists it is stable (i.e. $E<r$)

[2 marks]

$\Rightarrow$ ① If $E>r$ only $N_1^*=0$ exists and is stable

$\Rightarrow$ harvesting destroys the population!

[1 mark]

② If $E<r$ $N_2^* = K(1-\frac{E}{r})$ is a stable steady state — the population number approaches N_2^*

[1 mark]

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SOLUTION SHEET

COURSE / YEAR: DT 238

EXAMINATION SITTING: SUMMER ☐ SUPPLEMENTAL ☑ AUTUMN ☐ WINTER ☐

SUBJECT: INTRODUCTION TO BIOMATHEMATICS

EXPECTED SOLUTION TO QUESTION NO: 2

EXAMINER: ROSSEN IVANOV

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PROPOSED MARK ALLOCATION: 25

(i) $$\frac{d(N/K)}{d(rt)} = \left(\frac{N}{K}\right)\left(1-\frac{N(t-T)}{K}\right) \Rightarrow u=\frac{N}{K},\ \tau=rt,\ \theta=rT$$

$$\frac{du(\tau)}{d\tau} = u(\tau)\left[1-u(\tau-\theta)\right]$$ [2 marks]

(ii) The steady state satisfies $u^*(1-u^*)=0 \Rightarrow$
$u^*=1$ is a steady state; if $u=1+n(\tau)$

[2 marks]

$$\frac{dn(\tau)}{d\tau} = (1+n(\tau))(\cancel{1}-\cancel{1}-n(\tau-\theta)) = -n(\tau-\theta)$$

$$\frac{dn(\tau)}{d\tau} = -n(\tau-\theta)$$ [3 marks]

(iii) $n(\tau)=ce^{\lambda\tau} \Rightarrow c\lambda e^{\lambda\tau} = -ce^{\lambda(\tau-\theta)}$

$$\Rightarrow \boxed{\lambda = -e^{-\lambda\theta}}$$ [3 marks]

(iv) $\lambda=\mu+i\omega$, $\mu+i\omega = -e^{-(\mu+i\omega)\theta}$

$$\mu+i\omega = -e^{-\mu\theta}(\cos\omega\theta - i\sin\omega\theta)$$ [1 mark]

$$\mu = -e^{-\mu\theta}\cos\omega\theta$$ [2 marks]

$$\omega = e^{-\mu\theta}\sin\omega\theta$$ [2 marks]

(v) Suppose $\omega = 0 \Rightarrow \mu = -e^{-\mu\theta} \Rightarrow$ there are no positive roots for μ, (solution $u^* = 1$ is stable)

We also notice that if ω is a solution, so is $-\omega$, so we need only to consider $\omega > 0$.

When $\theta = 0$, $\mu = -1$, $\omega = 0$ and the solution $u^* = 1$ is stable, $n(\tau) \to 0$. The first bifurcation occurs when μ reaches $\mu = 0$ and then $u^* = 1$ becomes unstable, since $|n(\tau)|$ does not approach 0.
[1 mark]

If $\mu = 0$ we have $\begin{cases} 0 = -\cos\omega\theta \\ \omega = \sin\omega\theta \end{cases}$

$\Rightarrow \sin\omega\theta = \pm 1 \Rightarrow \omega = \pm 1$ but we consider

$\omega > 0 \Rightarrow \omega = 1$ [2 marks]

$\theta = \frac{\pi}{2}$ [2 marks]

(vi) When $\theta = \frac{\pi}{2}$ $\lambda = \mu + i\omega = 0 \pm i$ [1 mark]

$n(\tau) = ce^{\lambda\tau} = ce^{\pm i\tau}$ and has period

$t_p = 2\pi$ [2 marks]

$\Rightarrow \dfrac{t_p}{\theta} = \dfrac{2\pi}{\pi/2} = 4 \Rightarrow t_p = 4\theta$ [2 marks]

(i) The steady states solve the equation

$$u = \frac{bu^2}{1+u^2} - Eu \Rightarrow u_1 = 0 \qquad [1 \text{ mark}]$$

$$u_{2,3} = \frac{b \pm \sqrt{b^2 - 4(1+E)^2}}{2(1+E)} \qquad [3 \text{ marks}]$$

If there is no real steady state $\Rightarrow$ there is no stable steady state $\Rightarrow$ no harvest [2 marks]

$$\Rightarrow b^2 \geq 4(1+E)^2 \Rightarrow b \geq 2(1+E),$$

$$E \leq \frac{b-2}{2} \Rightarrow E_m = \frac{b-2}{2} \qquad [3 \text{ marks}]$$

(ii) $0 \leq u_{t+1} \Rightarrow \dfrac{bu_t^2}{1+u_t^2} - Eu_t \geq 0, \quad u_t \geq 0$ [3 marks]

$$\Rightarrow Eu_t^2 - bu_t + E < 0 \qquad [3 \text{ marks}]$$

$$\Rightarrow \frac{b - \sqrt{b^2 - 4E^2}}{2E} < u_t < \frac{b + \sqrt{b^2 - 4E^2}}{2E} \qquad [3 \text{ marks}]$$

(iii) $u_{t+1} = f(u_t)$ where $f(u) = \dfrac{bu^2}{1+u^2} - Eu$

$$f'(u) = \frac{2bu}{(1+u^2)^2} - E$$

At the steady state $(1+E)(1+u^2) = bu$

(iii) -ctd.

$$f'(u) = \frac{2}{bu}\left(\frac{bu}{1+u^2}\right)^2 - E = \frac{2}{bu}(1+E)^2 - (1+E) + 1 =$$

$$= 1 - (1+E)\left[1 - \frac{2}{bu}(1+E)\right]$$ [3 marks]

$$f'(u_{max}) = 1 + (1+E)\left[\frac{2(1+E)}{bu_{max}} - 1\right] =$$

$$= 1 + (1+E)\left[\frac{2(1+E)}{b} u_{min} - 1\right] =$$

$$= 1 + (1+E)\left[\frac{\cancel{2(1+E)}}{b} \frac{b - \sqrt{b^2 - 4(1+E)^2}}{\cancel{2(1+E)}} - 1\right] =$$

$$= 1 - \frac{(1+E)}{b}\sqrt{b^2 - 4(1+E)^2} < 1 \Rightarrow u_{max} \text{ is stable.}$$

Since $u_{min} u_{max} = 1$ [2 marks]

The bifurcation happens when $f'(u_{max}) = 1$, i.e.

$b^2 = 4(1+E)^2$, i.e. $E = E_m$ [2 marks]

(i) $u^* = 1$ is a steady state since $u^* = u^* e^{r(1-u^*)}$

[3 marks]

$$u_t = u^* + v_t = 1 + v_t \qquad |v_t| \ll 1$$

$$1 + v_{t+1} = (1+v_t)\, e^{r(1-1-v_{t-1})} = (1+v_t)\, e^{-r v_{t-1}}$$

$$1 + v_{t+1} \approx (1+v_t)(1 - r v_{t-1}) \approx 1 + v_t - r v_{t-1}$$ [2 marks]

$$\Rightarrow \quad v_{t+1} - v_t + r v_{t-1} = 0$$ [2 marks]

(ii) The linearised equation has solution $v_t = z^t$ where $z^2 - z + r = 0$, i.e. [2 marks]

$$z_{1,2} = \frac{1}{2}\left[1 \pm \sqrt{1-4r}\right]$$

If $0 < r < \frac{1}{4}$ z_1 and z_2 are real, $0 < z_{1,2} < 1$ and therefore $v_t \to 0$ as $t \to \infty$ and hence $u^* = 1$ is a linearly stable equilibrium state.

[2 marks]

If $1 > r > \frac{1}{4}$ z_1 and z_2 are complex,

$$z_{1,2} = \frac{1}{2}\left[1 \pm i\sqrt{4r-1}\right], \quad |z_{1,2}| = \frac{1}{2}\sqrt{1+4r-1} = \sqrt{r} < 1$$

$\Rightarrow$ again $v_t \to 0$ as $t \to \infty$ $\left(v_t = A z_1^t + \bar{A}_1 \bar{z}_1^t\right)$ [2 marks]

(iii) When $r=1$ $z_{1,2} = \frac{1}{2}(1 \pm i\sqrt{3}) = e^{\pm \pi i/3}$

$|z_{1,2}| = 1$ and V_t does not approach 0 as $t \to \infty$.

$$V_t = A e^{\frac{\pi i}{3}t} + \bar{A} e^{-\frac{\pi i}{3}t}$$ [3 marks]

$$V_{t+6} = e^{2\pi i}\left(A e^{\frac{\pi i}{3}t} + \bar{A} e^{-\frac{\pi i}{3}t}\right) = e^{2\pi i} V_t = V_t$$

$\Rightarrow$ when $r=1$ V_t is periodic with period 6

[3 marks]

(iv) When $r>1$ $|z_{1,2}| = \sqrt{r} > 1$ [2 marks]

$V_t \to \infty$ as $t \to \infty$ [2 marks]

$\Rightarrow$ V_t grows unboundedly and $u^* = 1$ is unstable.

[2 marks]

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SOLUTION SHEET

COURSE / YEAR: DT 238

EXAMINATION SITTING: SUMMER ☐ SUPPLEMENTAL ☑ AUTUMN ☐ WINTER ☐

SUBJECT: INTRODUCTION TO BIOMATHEMATICS

EXPECTED SOLUTION TO QUESTION NO: 5

EXAMINER: ROSSEN IVANOV

PROPOSED MARK ALLOCATION: 25

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(i) In comparisson to the logistic model $\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right)$ we see that the carrying capacities, e.g $K_1' = K_1 + b_{12}N_2 > K_1$ are increased due to the interaction $\Rightarrow$ the model describes symbiosis [5 marks]

(ii) The obvious steady states are $(0,0)$, $(0,K_2)$, $(K_1,0)$ and the last one satisfies

$$\begin{cases} N_1 = K_1 + b_{12}N_2 \\ N_2 = K_2 + b_{21}N_1 \end{cases}$$

$$\begin{cases} N_1 - b_{12}N_2 = K_1 \\ -b_{21}N_1 + N_2 = K_2 \end{cases}$$

$$N_1^* = \frac{\begin{vmatrix} K_1 & -b_{12} \\ K_2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -b_{12} \\ -b_{21} & 1 \end{vmatrix}} = \frac{K_1 + K_2 b_{12}}{1 - b_{12}b_{21}} \quad \text{exists if } b_{12}b_{21} < 1$$

$$N_2^* = \frac{\begin{vmatrix} 1 & K_1 \\ -b_{21} & K_2 \end{vmatrix}}{1 - b_{12}b_{21}} = \frac{K_2 + K_1 b_{21}}{1 - b_{12}b_{21}} \quad \text{exists if } b_{12}b_{21} < 1.$$

[3 marks]

$\Rightarrow$ the fourth steady state exists only if $b_{12} b_{21} < 1$.

Let
$$\frac{dN_1}{dt} = r_1 N_1 \left(1 - \frac{N_1}{K_1 + b_{12} N_2}\right) \equiv f_1(N_1 N_2)$$
$$\frac{dN_2}{dt} = r_2 N_2 \left(1 - \frac{N_2}{K_2 + b_{21} N_1}\right) \equiv f_2(N_1 N_2)$$

$$A(N_1, N_2) = \left\{ \frac{\partial f_i}{\partial N_j} \right\} = \begin{bmatrix} r_1 \left(1 - \frac{2N_1}{K_1 + b_{12} N_2}\right) & \frac{r_1 b_{12} N_1^2}{(K_1 + b_{12} N_2)^2} \\ \frac{r_2 b_{21} N_2^2}{(K_2 + b_{21} N_1)^2} & r_2 \left(1 - \frac{2N_2}{K_2 + b_{21} N_1}\right) \end{bmatrix}$$

[3 marks]

$$A(0,0) = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \quad \lambda_{1,2} = r_{1,2} > 0 \Rightarrow \text{unstable}$$

[2 marks]

$$A(K_1, 0) = \begin{bmatrix} -r_1 & r_1 b_{12} \\ 0 & r_2 \end{bmatrix} \quad \begin{matrix} \lambda_1 = -r_1 < 0 \\ \lambda_2 = r_2 > 0 \end{matrix}$$

saddle point unstable

[2 marks]

$$A(0, K_2) = \begin{bmatrix} r_1 & 0 \\ r_2 b_{21} & -r_2 \end{bmatrix} \quad \begin{matrix} \lambda_1 = r_1 > 0 \\ \lambda_2 = -r_2 < 0 \end{matrix}$$

saddle point unstable

[2 marks]

$$A(N_1^*, N_2^*) = \begin{bmatrix} -r_1 & r_1 b_{12} \\ r_2 b_{21} & -r_2 \end{bmatrix}$$

$$\lambda^2 + \lambda(r_1 + r_2) + r_1 r_2 (1 - b_{12} b_{21}) = 0$$

$\lambda_1 + \lambda_2 = -(r_1 + r_2) < 0$

$\lambda_1 \lambda_2 = r_1 r_2 (1 - b_{12} b_{21}) > 0$ if the steady state exists

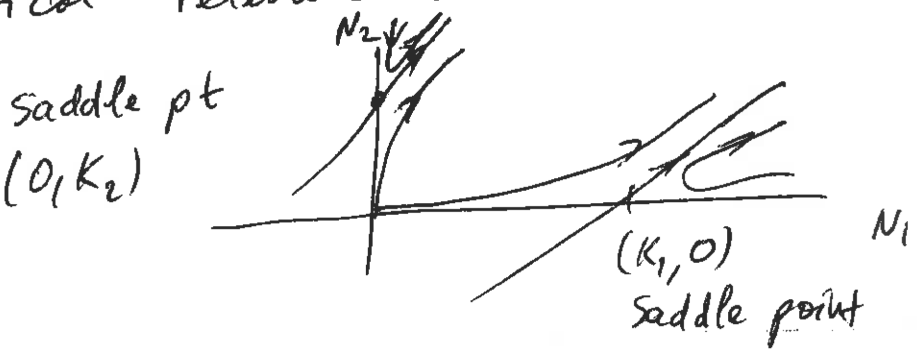
$\Rightarrow$ If the steady state (N_1^*, N_2^*) exists, i.e. when

$1 - b_{12} b_{21} > 0$ it is stable.

[3 marks]

(iii) In the case of small interactions $b_{12} b_{21} < 1$ the population approaches the steady state (N_1^*, N_2^*).
In the case of intensive interactions $b_{12} b_{21} > 1$ there is no stable steady state $\Rightarrow$ unlimited growth of both populations, which seems to be of no practical relevance:

$b_{12} b_{21} > 1$



[5 marks]

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SOLUTION SHEET

COURSE / YEAR: DT238

EXAMINATION SITTING: SUMMER ☐ SUPPLEMENTAL ☑ AUTUMN ☐ WINTER ☐

SUBJECT: INTRODUCTION TO BIOMATHEMATICS

EXPECTED SOLUTION TO QUESTION NO: 6

EXAMINER: ROSSEN IVANOV

PROPOSED MARK ALLOCATION: 25

(i) The model describes the spatial dispersal of i and s by simple diffusion with coefficient D. The transition from susceptibles to infectives is proportional to rsi where r is a parameter, measuring the transmission efficiency of the disease from infectives to susceptibles. The infectives have a disease-induced mortality rate ai, $\frac{1}{a}$ is the life-expectancy of an infective.

[3 marks]

(ii) $$\frac{\partial s}{\partial t} = -rsi + D\frac{\partial^2 s}{\partial x_1^2}$$

$$\frac{s_0\,\partial S}{\partial(\tau/rs_0)} = -rs_0^2\, SI + D\frac{\partial^2(s_0 S)}{\left(\sqrt{\frac{D}{rs_0}}\right)^2 \partial x^2}$$

$$rs_0\frac{\partial S}{\partial \tau} = -rs_0\, SI + D\frac{rs_0}{D}\frac{\partial^2 S}{\partial x^2} \Rightarrow$$

$$\Rightarrow \quad \frac{\partial S}{\partial \tau} = -SI + \frac{\partial^2 S}{\partial x^2}$$

[1 mark]

$$\frac{\partial i}{\partial t} = rsi - ai + D\frac{\partial^2 i}{\partial x_1^2}$$

$$\frac{s_0\,\partial I}{\partial\left(\frac{\tau}{rs_0}\right)} = rs_0^2\, SI - as_0 I + D\frac{s_0\,\partial^2 I}{\left(\sqrt{\frac{D}{rs_0}}\right)^2 \partial x^2}, \quad \boxed{a = \lambda r s_0}$$

EXAMINATION SITTING: SUMMER ☐ SUPPLEMENTAL ☑ AUTUMN ☐ WINTER ☐

EXPECTED SOLUTION TO QUESTION NO: 6 - ctd.

PROPOSED MARK ALLOCATION: 25

$$rS_0 \frac{\partial I}{\partial \tau} = rS_0 SI - \lambda r S_0 I + D\frac{rS_0}{D}\frac{\partial^2 I}{\partial x^2}$$

$$\Rightarrow \frac{\partial I}{\partial \tau} = SI - \lambda I + \frac{\partial^2 I}{\partial x^2}$$ [1 mark]

(iii) $z = x - c\tau$, $I_t = \frac{dI}{dz}\frac{\partial z}{\partial t} = -cI'$

$I_x = \frac{dI}{dz}\frac{\partial z}{\partial x} = I'$ etc. [3 marks]

$$\Rightarrow -cI' = SI - \lambda I + I'' \Rightarrow I'' + cI' + I(S-\lambda) = 0$$

$$-cS' = -SI + S'' \Rightarrow S'' + cS' - SI = 0$$ [2 marks]

(iv) $t \to -\infty \Leftrightarrow z \to \infty$

$S(\infty) = 1 \Leftrightarrow S(+\infty) = S_0$ - the population density before the epidemic

$S(-\infty) = \sigma < 1$, $S(-\infty) = \sigma S_0$ - the population density after the epidemic $\Rightarrow \sigma$ represents the fraction that survives the epidemic. [1 mark]

Suppose that $S(z)$ has a maximum. Then $S'(z) = 0$

$\Rightarrow S'' = SI > 0$ wich is condition for a minimum.

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EXPECTED SOLUTION TO QUESTION NO: 6 -cont.

PROPOSED MARK ALLOCATION: 25

$$S'' + cS' = IS \quad \Big| \; e^{cz}$$

$$\frac{d}{dz}\left(e^{cz}S'\right) = e^{cz} IS$$

$$e^{cz}S'(z) = \int_{-\infty}^{z} e^{cz'} I(z') S(z')\, dz' > 0$$

[2 marks]

$\Rightarrow$ $S'(z) > 0$ if $I(z) \not\equiv 0$

$\Rightarrow$ $S(z)$ increases monotonically from σ to 1

$$\sigma \leq S(z) < 1$$

[2 marks]

(v) Since $S(z) < 1$, $\int_{-\infty}^{\infty} I(z')\, dz' > \int_{-\infty}^{\infty} I(z') S(z')\, dz'$

Integrating $I'' + cI' + I(S - \lambda) = 0$ gives

$$I'(\infty) - I'(-\infty) + c\left[I(\infty) - I(-\infty)\right] + \int_{-\infty}^{\infty} I(z')S(z')\, dz' -$$

[3 marks]

$$-\lambda \int_{-\infty}^{\infty} I(z')\, dz' = 0 \quad \Rightarrow$$

$$\int_{-\infty}^{\infty} I(z')S(z')\, dz' = \lambda \int_{-\infty}^{\infty} I(z')\, dz' \Rightarrow$$

$$\int_{-\infty}^{\infty} I(z')\, dz' > \lambda \int_{-\infty}^{\infty} I(z')\, dz' \quad \Rightarrow \quad \lambda < 1$$

[2 marks]

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EXPECTED SOLUTION TO QUESTION NO: 6 ctd.

PROPOSED MARK ALLOCATION: 25

(vi) If $S \to 1$, $I \to 0$ we have

$$I'' + cI' + (1-\lambda)I = 0$$ with solution

$$I = Ae^{m_1 z} + Be^{m_2 z}, \quad A, B = \text{const}$$

$$m^2 + cm + (1-\lambda) = 0$$

$$m_{1,2} = \frac{-c \pm \sqrt{c^2 - 4(1-\lambda)}}{2}$$

[3 marks]

Since at $z \to \infty$ $I(\infty) = 0$ there are no oscillatory solutions, that would give $I(z) < 0$

$\Rightarrow c^2 - 4(1-\lambda) \geq 0 \Rightarrow c \geq 2\sqrt{1-\lambda}$.

[2 marks]