Competition Models

2-species competition models

$$\frac{dN_1}{dt} = r_1 N_1 \left(1 - \frac{N_1}{K_1} - b_{12}\frac{N_2}{K_1}\right)$$

$$\frac{dN_2}{dt} = r_2 N_2 \left(1 - \frac{N_2}{K_2} - b_{21}\frac{N_1}{K_2}\right)$$

r_1, r_2, K_1, K_2, b_{12}, b_{21} . 6 parameters

$$u_1 = \frac{N_1}{K_1}, \quad u_2 = \frac{N_2}{K_2}, \quad \tau = r_1 t$$

$$\frac{du_1}{d\tau} = u_1\left(1 - u_1 - \overbrace{b_{12}\frac{K_2}{K_1}}^{= a_{12}} u_2\right)$$

$$\frac{du_2}{d\tau} = \underbrace{\frac{r_2}{r_1}}_{= \rho} u_2\left(1 - u_2 - \underbrace{b_{21}\frac{K_1}{K_2}}_{= a_{21}} u_1\right)$$

$$\frac{du_1}{d\tau} = u_1(1 - u_1 - a_{12}u_2) = f_1(u_1, u_2)$$

$$\frac{du_2}{d\tau} = \rho u_2(1 - u_2 - a_{21}u_1) = f_2(u_1, u_2)$$

$$\rho = \frac{r_2}{r_1}; \quad a_{12} = b_{12}\frac{K_2}{K_1}; \quad a_{21} = b_{21}\frac{K_1}{K_2}$$

$$f_1(u_1,u_2)=0$$
$$f_2(u_1,u_2)=0$$

$$(0,0),\ (0,1),\ (1,0),\ (u_1^*,u_2^*)$$

$$\begin{array}{l|l} f_1 & 1-u_1-a_{12}u_2=0 \\ f_2 & 1-u_2-a_{21}u_1=0 \end{array} \qquad \left| \begin{array}{l} u_1+a_{12}u_2=1 \\ u_2+a_{21}u_1=1 \end{array}\right.$$

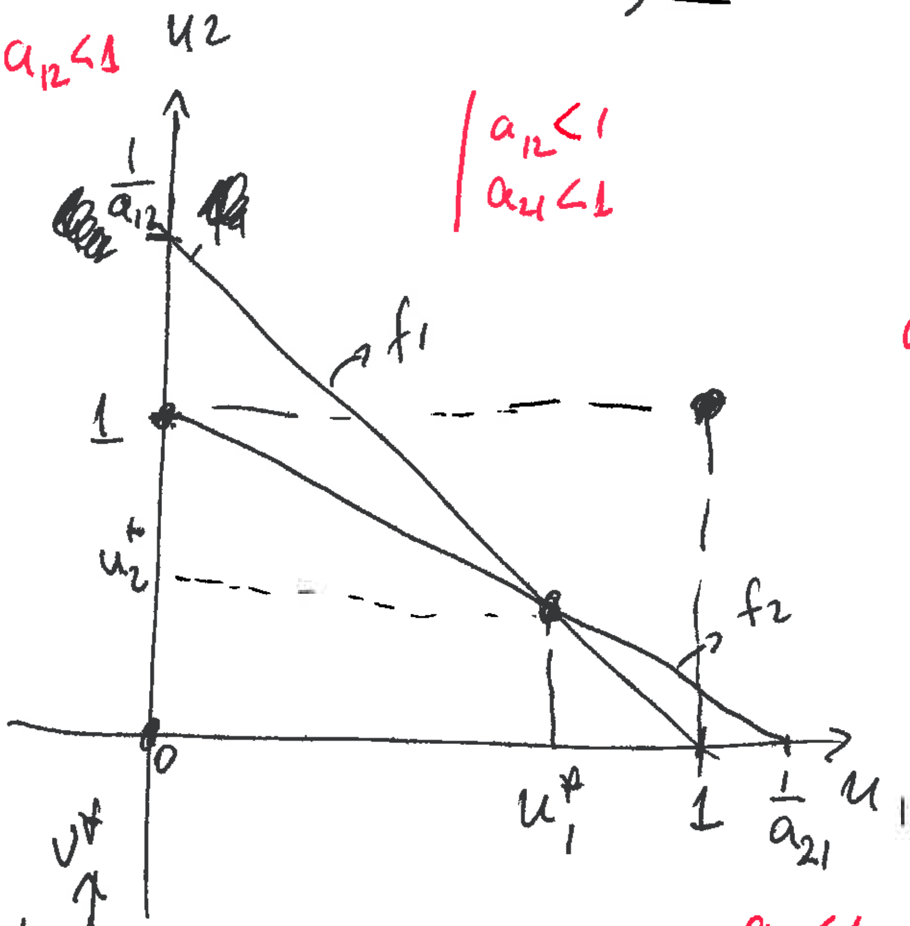
$$\underbrace{\begin{pmatrix} 1 & a_{12} \\ a_{21} & 1 \end{pmatrix}}_{A}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}=\underbrace{\frac{1}{1-a_{12}a_{21}}\begin{pmatrix} 1 & -a_{12} \\ -a_{21} & 1 \end{pmatrix}}_{A^{-1}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

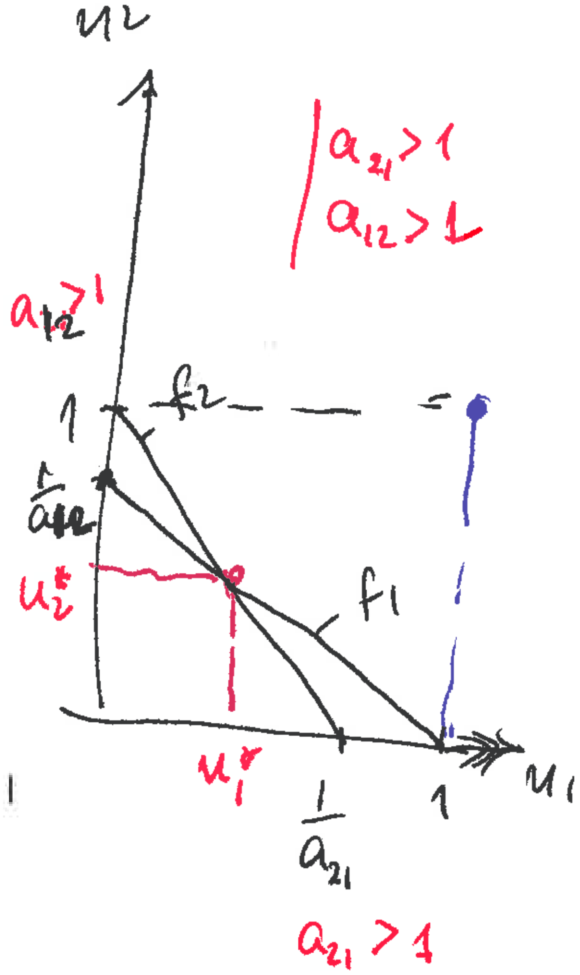
$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}=\begin{bmatrix} \dfrac{1-a_{12}}{1-a_{12}a_{21}} \\[2ex] \dfrac{1-a_{21}}{1-a_{12}a_{21}} \end{bmatrix}$$

$$u_1^*=\frac{1-a_{12}}{1-a_{12}a_{21}},\quad u_2^*=\frac{1-a_{21}}{1-a_{12}a_{21}}$$

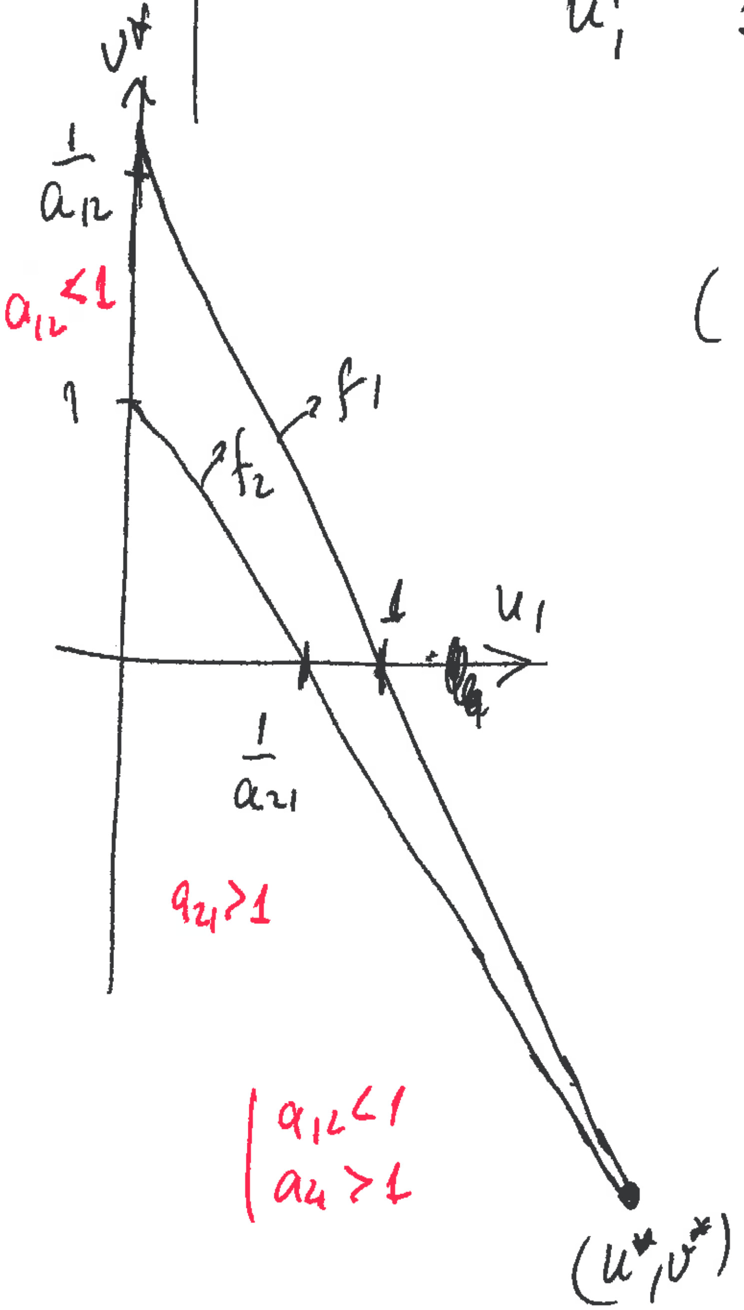
$$u_1^*\geq 0,\quad u_2^*\geq 0,\quad a_{12}a_{21}\neq 0$$



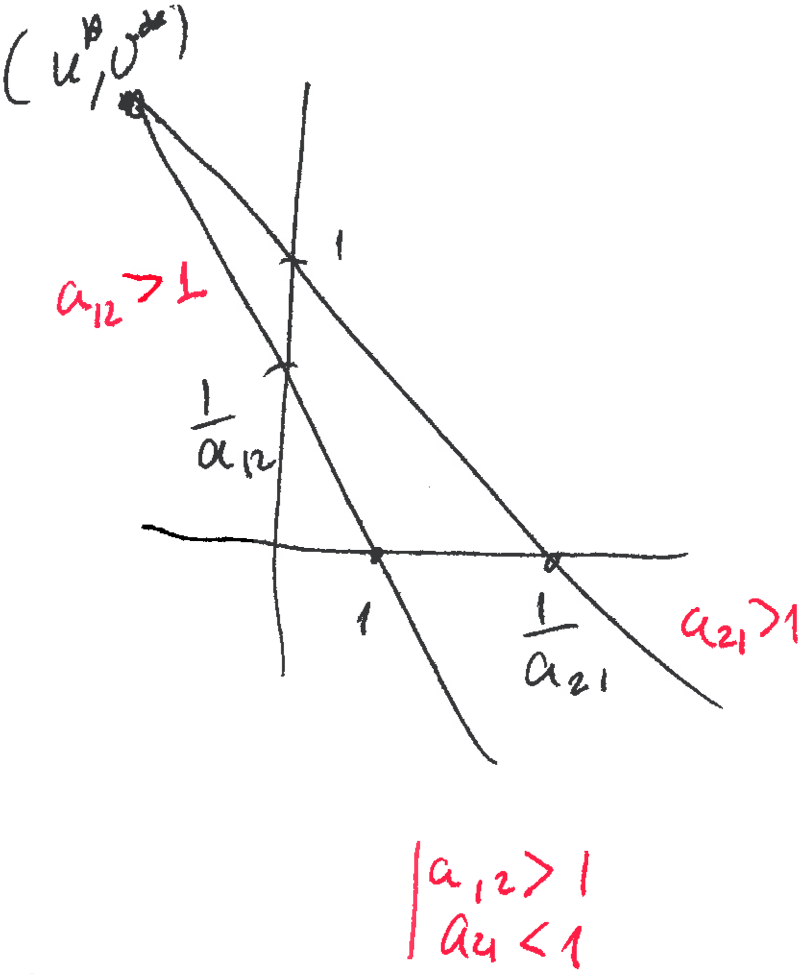
$a_{12}<1$
u_2
$a_{12}<1$
$a_{21}<1$
$\frac{1}{a_{12}}$
f_1
1
u_2^*
f_2
0
u_1^*
1
$\frac{1}{a_{21}}$
u_1
$a_{21}<1$



u_2
$a_{21}>1$
$a_{12}>1$
$a_{12}>1$
1
f_2
$\frac{1}{a_{12}}$
u_2^*
f_1
u_1^*
$\frac{1}{a_{21}}$
1
u_1
$a_{21}>1$



v^*
$\frac{1}{a_{12}}$
$a_{12}<1$
1
f_1
f_2
1
u_1
$\frac{1}{a_{21}}$
$a_{21}>1$
$a_{12}<1$
$a_{21}>1$
(u^*, v^*)



(u_1^*, v^*)
1
$a_{12}>1$
$\frac{1}{a_{12}}$
1
$\frac{1}{a_{21}}$
$a_{21}>1$
$a_{12}>1$
$a_{21}<1$

$$\mathcal{A} = \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \end{pmatrix} = \begin{pmatrix} 1-2u_1 - a_{12}u_2 & -a_{12}u_1 \\ -\rho a_{21} u_2 & \rho(1-2u_2-a_{21}u_1) \end{pmatrix}$$

$$\mathcal{A}(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix} \qquad \lambda_1 = 1 > 0 \, ; \quad \lambda_2 = \rho > 0$$

$\boxed{(0,0) \text{ unstable node (always)}}$

$$\mathcal{A}(1,0) = \begin{pmatrix} \overset{-1}{1-2-0} & -a_{12} \\ 0 & \rho(1-a_{21}) \end{pmatrix}, \qquad \begin{aligned} \lambda_1 &= -1 < 0 \\ \lambda_2 &= \rho(1-a_{21}) \end{aligned}$$

$$\boxed{u_1^* = 1, \; u_2^* = 0 \quad \text{is} \quad \begin{cases} \text{stable (node)} & \text{if} \quad a_{21} > 1 \\ \text{unstable (saddle)} & \text{if} \quad a_{21} < 1 \end{cases}}$$

$$\mathcal{A}(0,1) = \begin{pmatrix} 1-a_{12} & 0 \\ -\rho a_{21} & -\rho \end{pmatrix} \qquad \begin{aligned} \lambda_1 &= 1-a_{12} \\ \lambda_2 &= -\rho < 0 \end{aligned}$$

$$\boxed{u_1^* = 0, \; u_2^* = 1 \quad \text{is} \quad \begin{cases} \text{stable (node)} & \text{if} \quad a_{12} > 1 \\ \text{unstable (saddle)} & \text{if} \quad a_{12} < 1 \end{cases}}$$

$$A = \begin{pmatrix} 1-2u_1 - a_{12}u_2 & -a_{12}u_1 \\ -\rho a_{21} u_2 & \rho(1-2u_2 - a_{21}u_1) \end{pmatrix}, \qquad u_1^* = \frac{1-a_{12}}{1-a_{12}a_{21}}, \quad u_2^* = \frac{1-a_{21}}{1-a_{12}a_{21}}$$

$$A = \frac{1}{1-a_{12}a_{21}} \begin{pmatrix} 1-a_{12}a_{21} - 2(1-a_{12}) - a_{12}(1-a_{21}) & -a_{12}(1-a_{12}) \\ -\rho a_{21}(1-a_{21}) & \rho\left(1-a_{12}a_{21} - 2(1-a_{21}) - a_{21}(1-a_{12})\right) \end{pmatrix}$$

$$A = \frac{1}{1-a_{12}a_{21}} \begin{pmatrix} 1-\cancel{a_{12}a_{21}} - 2 + 2a_{12} - a_{12} + \cancel{a_{12}a_{21}} & a_{12}(a_{12}-1) \\ \rho a_{21}(a_{21}-1) & \rho\left(1-\cancel{a_{12}a_{21}} - 2 + 2a_{21} - a_{21} + \cancel{a_{21}a_{12}}\right) \end{pmatrix}$$

$$A = \frac{1}{1-a_{12}a_{21}} \begin{pmatrix} \underline{a_{12}-1} & a_{12}(\underline{a_{12}-1}) \\ \underline{\rho}\, a_{21}(\underline{a_{21}-1}) & \underline{\rho}(\underline{a_{21}-1}) \end{pmatrix}$$

$$\lambda^2 - \frac{a_{12}-1+\rho(a_{21}-1)}{1-a_{12}a_{21}}\lambda + \underbrace{\frac{\rho(a_{12}-1)(a_{21}-1) - \rho a_{12}a_{21}(a_{21}-1)(a_{12}-1)}{(1-a_{12}a_{21})^2}}_{\det A .} = 0$$

$$\det A = \frac{1}{(1-a_{12}a_{21})^2}\rho(a_{21}-1)(a_{12}-1)\cancel{(1-a_{12}a_{21})} = \frac{\rho(a_{21}-1)(a_{12}-1)}{1-a_{12}a_{21}}$$

$$\lambda^2 - \frac{a_{12}-1+\rho(a_{21}-1)}{1-a_{12}a_{21}}\lambda + \frac{\rho(a_{21}-1)(a_{12}-1)}{1-a_{12}a_{21}} = 0$$

$$\boxed{(1-a_{12}a_{21})\lambda^2 - [a_{12}-1+\rho(a_{21}-1)]\lambda + \rho(a_{21}-1)(a_{12}-1)=0}$$

Discriminant is

$$D = \sqrt{[a_{12}-1+\rho(a_{21}-1)]^2 - 4\rho(1-a_{12}a_{21})(a_{21}-1)(a_{12}-1)}$$

$$\lambda_{1,2} = \frac{\underbrace{a_{12}-1}_{+} + \rho\underbrace{(a_{21}-1)}_{+} \pm \sqrt{[a_{12}-1+\rho(a_{21}-1)]^2 - 4\rho\underbrace{(1-a_{12}a_{21})}_{+\ (-)}\underbrace{(a_{21}-1)}_{-\ (+)}\underbrace{(a_{12}-1)}_{-\ (+)}}}{2\underbrace{(1-a_{12}a_{21})}_{+(-)}}$$

are the eigenvalues at (u^*, v^*) which makes sense if either $a_{21} > 1$, $a_{12} > 1$ or $a_{21} < 1$, $a_{12} < 1$

I $a_{12} < 1$, $a_{21} < 1$

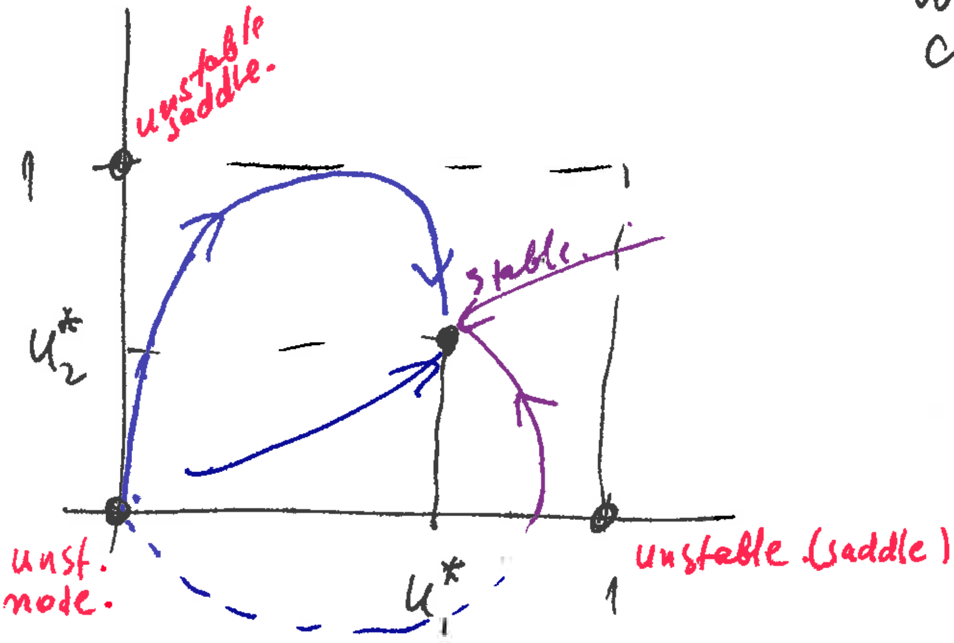
1) if complex, stable
2) if real, $\lambda_1 < 0$, $\lambda_2 < 0$, stable.

II $a_{12} > 1$, $a_{21} > 1$

1) can not be complex!
2) if real $\lambda_1 < 0$
$\lambda_2 > 0$
saddle point.

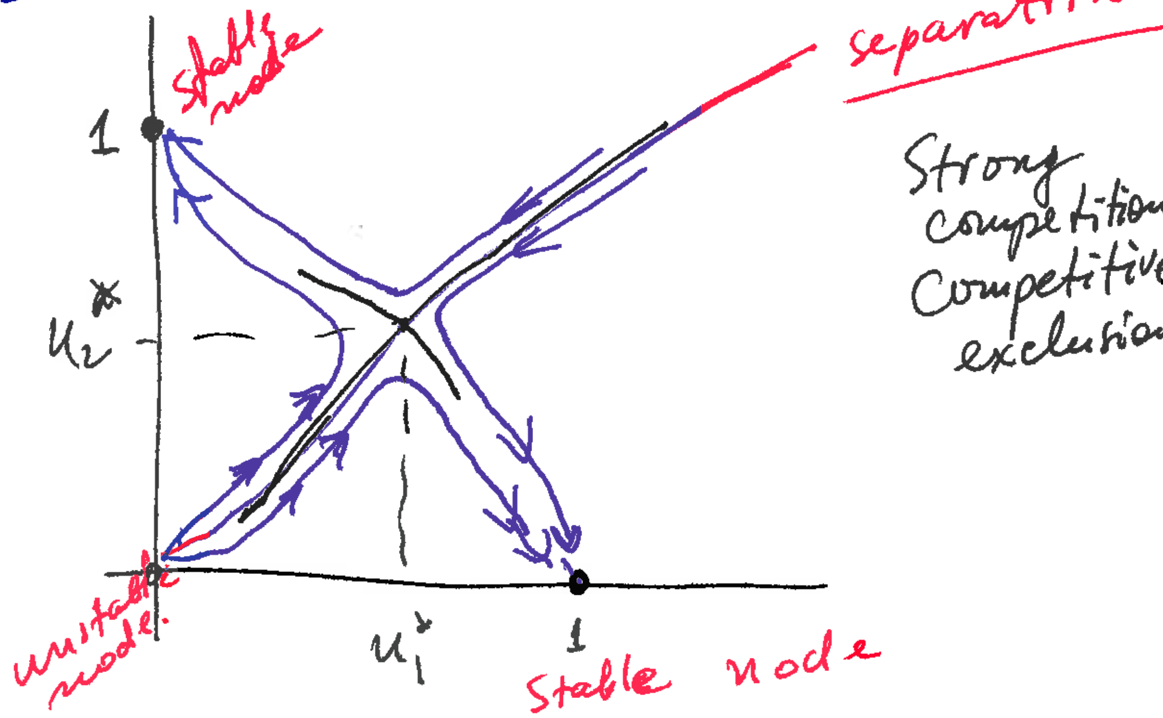
I $a_{12} < 1$ & $a_{21} < 1$

weak competition



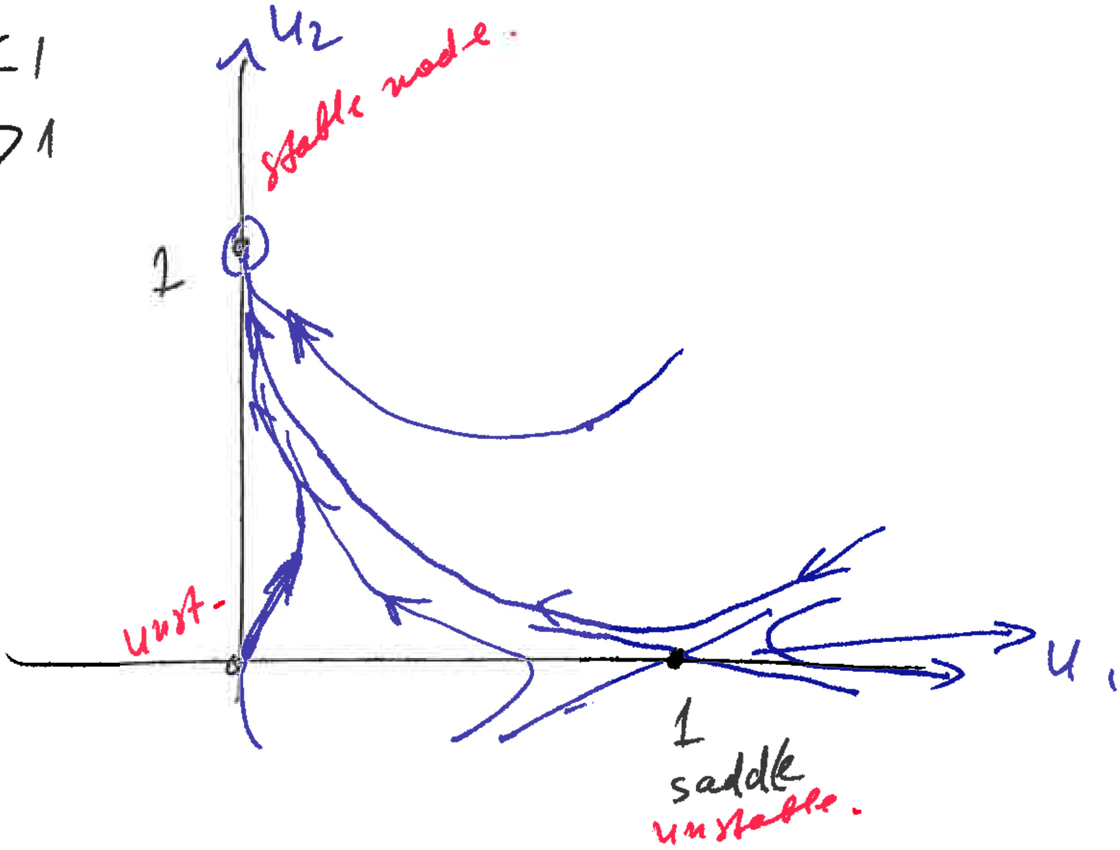
long-term population site is $\underline{\underline{(u_1^*, u_2^*)}}$

II $a_{12} > 1$, $a_{21} > 1$

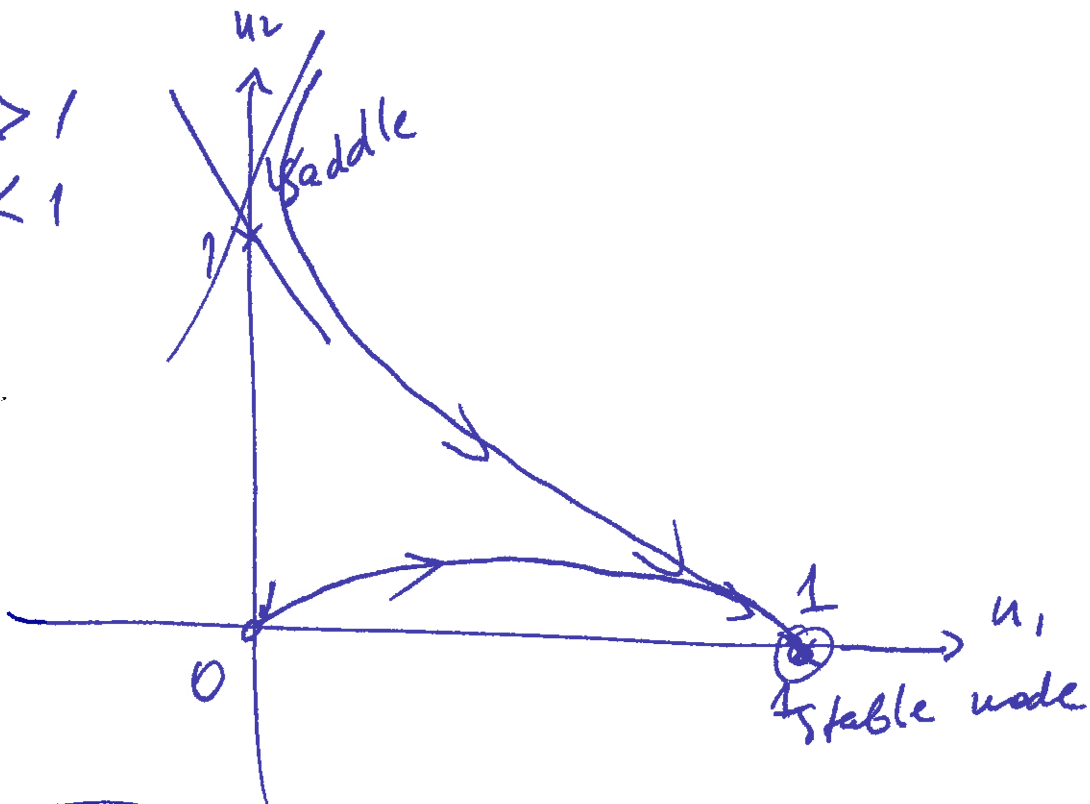


Strong competition. Competitive exclusion

$a_{12} < 1$
$a_{21} > 1$



$a_{12} > 1$
$a_{21} < 1$



$$\boxed{a_{12} = b_{12} \frac{K_2}{K_1}}$$

Mutualism or Symbiosis

Lotka - Volterra Version

$$\frac{dN_1}{dt} = r_1 N_1 + \underline{a_1 N_1 N_2}$$

$$\frac{dN_2}{dt} = r_2 N_2 + \underline{a_2 N_2 N_1}$$

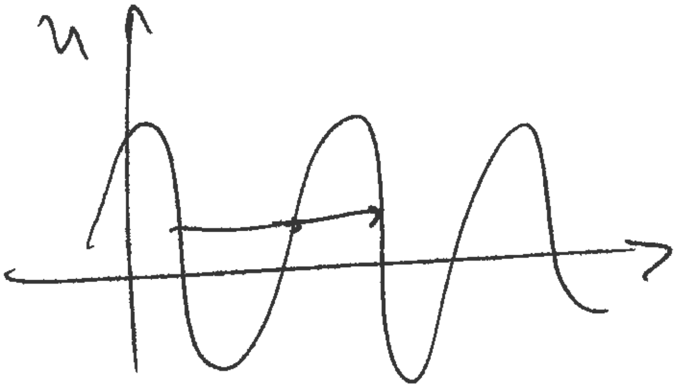
$$\frac{dN_1}{dt} = r_1 N_1 \left(1 - \frac{N_1}{K_1} + b_{12} \frac{N_2}{K_1}\right)$$

$$\frac{dN_2}{dt} = r_2 N_2 \left(1 - \frac{N_2}{K_2} + b_{21} \frac{N_1}{K_2}\right)$$

Lorentz (1963)

$$\begin{cases} \dfrac{du}{dt} = a(v-u) \\[2ex] \dfrac{dv}{dt} = -uw + bu - v \\[2ex] \dfrac{dw}{dt} = uv - cw \end{cases}$$

~~cha~~ apperiodic (chaotic) behaviour



$$y'' + ay' + by = 0$$

$$x = y' \qquad y' = x$$

$$x' + ax + by = 0$$

$$\begin{cases} y' = x \\ x' = -ax + by \end{cases}$$