CHAPTER 1

Metrics and norms

$\mathbb{R}^n$ has two aspects: it is a vector space, and we will see it is a metric space (the metric comes from a *norm*: a way of giving a *length* to any vector). These two aspects combine to give a very rich structure, on which we can define not only algebraic operations such as addition, but analytic concepts such as limits, continuity and differentiability. We looked at the vector space properties in the last chapter; in this chapter we explore the metric properties and how they are enhanced by the underlying algebraic properties.

1. Metrics

DEFINITION 1.1. Let X be a set. A **metric** on X (also called a **distance function**) is a mapping $d : X \times X \longrightarrow \mathbb{R}$ such that:

M1. (*non-negative*) for all $x, y \in X$, $d(x, y) \geq 0$; and furthermore, $d(x, y) = 0 \iff x = y$;
M2. (*symmetric*) for all $x, y \in X$, $d(x, y) = d(y, x)$;
M3. (*triangle inequality*) for all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$.

A set X equipped with a metric is called a **metric space**.

EXAMPLE 1.2. On $\mathbb{R}$, $d(x, y) := |x-y|$ defines a metric. Here $|\ |$ is the absolute value or modulus function, defined by $|x| = x$ if $x \geq 0$, $|x| = -x$ otherwise. ■

EXAMPLE 1.3. The normed space $\mathbb{R}^n$ has a "natural" *Euclidean* distance on it: the distance between two points x and y is

$$d(x, y) = \|x - y\|_2 = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

■

EXAMPLE 1.4. The Euclidean metric on $\mathbb{R}^n$ is a special case of a *Minkowski metric*:

$$d(x, y) = \|x - y\|_q = ((x_1 - y_1)^q + \cdots + (x_n - y_n)^q)^{1/q},$$

where $q \geq 0$. Clearly when $q = 2$ we have the Euclidean metric. When $q = 1$ we have the *Manhattan* metric, where the shortest path between two points consists of segments parallel to the coordinate axis. ■

EXAMPLE 1.5. Let $\mathcal{C}([0, 1])$ be the set of all continuous functions $f : [0, 1] \longrightarrow \mathbb{R}$. For $f, g \in \mathcal{C}([0, 1])$, define

$$d(f, g) := \max_{x \in [0,1]} |f(x) - g(x)|.$$

Then d is a metric on $\mathcal{C}([0, 1])$. ■

2. Continuity and convergence

DEFINITION 2.1 (Cauchy Sequence and Convergence). Let X be a metric space. A sequence (i.e., a countable list of elements of X) $(x_n)_{n\in\mathbb{N}} = (x_1, x_2, x_3, \ldots)$ in X is called a **Cauchy Sequence** if for each $\varepsilon > 0$ there exists an integer N (depending on ε) such that

$$d(x_n, x_m) < \varepsilon,\ \forall\ n, m > N.$$

A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to **converge to** $x \in X$ if

$$d(x_n, x) \to 0 \text{ as } n \to \infty,$$

that is,

$$\text{for all } \varepsilon > 0 \text{ there exists } N \in \mathbb{N} \text{ such that, for all } n > N, d(x_n, x) < \varepsilon.$$

The element $x \in X$ is called the **limit** of (x_n). We call a sequence **convergent** if it has a limit.

X is called a **complete** metric space if every Cauchy sequence in X converges (to an element of X).

EXAMPLE 2.2. Not every *metric* space is complete: consider the metric space $\mathbb{R}$ and its subspace, the half-open interval $X = (0, 1] \subset \mathbb{R}$.

The sequence $\{x_n\} = \{1/n\}$ is a Cauchy sequence in X but the limit of the sequence is $x = 0$, which is not in X. Thus the space X is not *complete.* ■

LEMMA 2.3 (Sequence Lemma). *A set V in a metric space (X, d) is closed if and only if every convergent sequence of points in V actually has its limit in V.*

DEFINITION 2.4. Let (X, d_1) and (Y, d_2) be metric spaces and let $f : X \longrightarrow Y$. We say that f has **limit** ℓ at $a \in X$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < d_1(x, a) < \delta, x \in X \quad \Rightarrow \quad d_2(f(x), \ell) < \varepsilon.$$

We say that f is **continuous at** a if the limit at a exists and equals $f(a)$. We say that f is **continuous (on X)** if f is continuous at each $x \in X$.

NOTE 2.5. f is continuous at a if and only if *all* of the following are true:
(i) $f(a)$ exists;
(ii) $\lim_{x\to a} f(x)$ exists;
(iii) $\lim_{x\to a} f(x) = f(a)$.

THEOREM 2.6. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be continuous, and let $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, with $f(a) \neq b$. Then there exists an open subset V of $\mathbb{R}^n$ such that $f(x) \neq b$ for all $x \in V$.*

3. Algorithms and Numerical Computation

INTRODUCTION 3.1. You will cover this topic in more detail in the course "Software and Numerical Applications" but we will give a quick outline here. Recall that an **algorithm** is a finite set of instructions that can be carried out in a finite amount of time: that is, it must terminate.

We may view the process of solving numerical problems as a mapping of a point $p \in P$, the *Problem Space*, to a point (or points) $s \in S$, the Solution Space. We regard p as a problem, s a solution to p, and the mapping as the solution process.

The solution process may include all aspects of problem finding, formulation, data collection, algorithm design or selection, and solution generation, interpretation, and implementation.

We will take a more restricted view of this process and limit ourselves to the application of numerical algorithms to problem data, to generate solutions. Hence we use algorithms to map points in the problem data space P to points in the solution data space S. Symbolically we have

$$A : P \to S : p \longmapsto A(p).$$

3.1. Classes of Numerical Algorithms.

There are two broad categories of numerical algorithms: *Iterative* and *Direct or Transformational.*

3.1.1. *Direct or Transformational Algorithms.* These algorithms apply a sequence of transformations to the original problem until it is easy to solve. An example for us is Gaussian elimination.

3.1.2. *Iterative Algorithms.* These algorithms start with an initial guess at the solution and repeatedly refine this guess until a satisfactory answer is obtained.

With an iterative method, we choose an (almost) arbitrary initial approximation to the solution of the problem (a number, a function, etc.) and successively improve this approximate solution (that is, we iterate) in such a way that the sequence of improved solutions converges to the solution of the given problem.

Whereas direct methods (theoretically, at least) converge in a *finite* number of steps, iterative methods require an *infinite* number of steps for convergence. Hence, before using an iterative method, we must be sure that its convergence is rapid enough so that a (reasonably small) finite number of iterations produces an acceptable approximation to a solution, whenever the initial approximation is reasonably close to a solution.

algorithm *Iterate* (p)

```
1.  Choose s_old  {initial guess at solution to p}
    while  s_old is not satisfactory do
2.        s_new := T(s_old)     { refine the guess.}
3.        s_old := s_new        { use the new guess.}
    endwhile
4.  return s_old  { a satisfactory solution to p}
endalg {Iterate}
```

All iterative algorithms have this form, no matter what problem we are solving. The heart of the algorithm is the transformation $T()$, which must be specified for each algorithm and problem type.

Examples for us will be the Jacobi, Gauss-Seidel and SOR iterative methods for solving systems of linear equations.

4. Contraction Mappings and Successive Approximations

4.1. Contraction mappings and successive approximations form the theoretical base on which many numerical algorithms are built. We saw in the last section that the heart of an iterative algorithm is the the step

$$s_{new} := T(s_{old}),$$

where s_{old} is the current guess at the solution and s_{new} is the refined guess. We may view this algorithm as generating a sequence $(x_0, x_1, \ldots, x_n, \ldots)$ in some metric space X. Given x_0, the sequence is generated as follows:

$$x_1 = T(x_0),\ x_2 = T(x_1),\ \ldots,\ x_n = T(x_{n-1}).$$

If this sequence converges to some limit $x \in X$ then we say that this limit is approached by the process of **Successive Approximation**. We sometimes write this sequence as

$$(x_n = T(x_{n-1})) \quad \text{or} \quad (x_n = T^n(x_0)).$$

Here, $T^n(x_0) = T(T(\cdots(x_0)\cdots))$ means T applied n times in succession.

DEFINITION 4.2. Let $T : X \longrightarrow X$ be a function from a set X to itself. A point x for which $x = T(x)$ is called a **fixed point** of T.

EXAMPLE 4.3. Here are some successive approximation sequences that converge. For simplicity we have chosen $X = \mathbb{R}$.

(a) $T(x) = \sqrt{x}$, with $x_0 = 2$, gives the sequence

$$(2.0, 1.4142, 1.1892, 1.0905, \ldots,) \rightarrow 1.0.$$

Note, that at the limit $x = 1$, we have $x = T(x)$.

(b) $T(x) = x + x(1 - ax)$, with $a = 3.0$ and $x_0 = 0.5$, gives the sequence

$$(.500000, .250000, .312500, .3320313, .333328, .333333) \rightarrow 1/3.$$

Note again that at the limit $x = 1/3$, we have $x = T(x)$. Indeed, for any a the mapping T has a fixed point $x = 1/a$ such that $x = T(x)$. Note also that this gives a method for calculating $1/a$ using addition and multiplication only, important for implementation on computers.

(c) $T(x) = 1 + x$, with $x_0 = 0$ gives the sequence

$$(0, 1, 2, 3, 4, \ldots,) \rightarrow \infty.$$

This sequence diverges for all x_0.

■

The convergence of a sequence generated by successive approximation depends on the transformation T.

DEFINITION 4.4. A mapping T of a complete metric space X into itself is called a **contraction mapping** if

$$d(T(x_1), T(x_2)) \leq \lambda d(x_1, x_2), \ \forall \ x_1, x_2 \in X,$$

for some λ, where $0 \leq \lambda < 1$.

In other words, a contraction mapping is such that the distance between a pair of transformed points x_1, x_2 is less than the distance between the points themselves.

Consider the successive approximation sequence $\{x_k = T(x_{k-1})\}$, where T is a contraction mapping. Let x_i, x_{i+1}, x_{i+2} be three successive points in this sequence and let $d_i = d(x_{i+1}, x_i)$ and $d_{i+1} = d(x_{i+2}, x_{i+1})$ be the distances between these points. Then we have

$$d_{i+1} = d(x_{i+2}, x_{i+1}) = d(T(x_{i+1}), T(x_i)) \leq \lambda d(x_{i+1}, x_i) < d_i.$$

This means that the distance between successive pairs of points becomes smaller and $d_k \to 0$ as $k \to \infty$. Thus a point x is approached such that $x = T(x)$. That is, this x is a *fixed point* of T.

We now put all this on a rigorous footing by stating and proving Banach's famous fixed point theorem.

4.1. Banach's Fixed Point Theorem.

THEOREM 4.5. *If T is a contraction mapping $T : X \to X$, where X is a complete metric space, then T has a unique fixed point $x \in X$.*

PROOF. We prove this theorem in three stages:

- (a) Generate the sequence $\{x_k = T(x_{k-1})\}$ and show that it is a Cauchy sequence, i.e., $d(x_k, x_m) = \|x_k - x_m\| < \epsilon, \ \forall \ k, m > N(\epsilon)$, or $d(x_k, x_m) \to 0$ as $k, m \to \infty$.
- (b) Show that the limit of the Cauchy sequence is a fixed point of T, i.e., $x = T(x)$.
- (c) Show that the fixed point x is unique.

We show that the sequence is Cauchy as follows:

$$\begin{aligned} d(x_{k+1}, x_k) &= d(T(x_k), T(x_{k-1})) \\ &\leq \lambda d(x_k, x_{k-1}) \\ &= \lambda d(T(x_{k-1}), T(x_{k-2})) \\ &\leq \lambda^2 d(x_{k-1}, x_{k-2}) \\ &\vdots \\ &\leq \lambda^k d(x_1, x_0). \end{aligned}$$

Hence $d(x_{k+1}, x_k) \leq \lambda^k d(x_1, x_0)$. Now, by the triangle inequality we have, for $m > k$

$$d(x_k, x_m) \leq d(x_k, x_{k+1}) + d(x_{k+1}, x_{k+2}) + \cdots + d(x_{m-1}, x_m).$$

Combining this with the previous result that $d(x_{k+1}, x_k) \leq \lambda^k d(x_1, x_0)$ we get

$$\begin{aligned} d(x_k, x_m) &\leq (\lambda^k + \lambda^{k+1} + \cdots + \lambda^{m-1}) d(x_1, x_0) \\ &= \lambda^k (1 + \lambda + \lambda^2 + \cdots + \lambda^{m-k-1}) d(x_1, x_0) \\ &< \lambda^k \left(\sum_{i=0}^{\infty} \lambda^i \right) d(x_1, x_0) \\ &= \frac{\lambda^k}{1 - \lambda} d(x_1, x_0) \end{aligned}$$

Hence $d(x_k, x_m) \to 0$ as $k, m \to \infty$ and so the sequence is a Cauchy sequence. This sequence has a limit $x \in X$ because X is a complete metric space.

We show that the limit x is a fixed point of T as follows:

$$\begin{aligned} d(x, T(x)) &\leq d(x, x_k) + d(x_k, T(x)) \\ &\leq d(x, x_k) + \lambda d(x_{k-1}, x) \end{aligned}$$

Now, the right-hand side goes to zero as $x_k \to 0$. Hence, $d(x, T(x)) \to 0$ as $x_k \to 0$, which means $x = T(x)$.

This fixed point is unique: assume that it is not unique and that there are two fixed points $x = T(x)$ and $y = T(y)$. Then we have

$$d(x, y) = d(T(x), T(y)) \leq \lambda d(x, y) < d(x, y).$$

This is clearly impossible and so $x = y$. $\square$

REMARK 4.6. We have proved the *existence* and *uniqueness* of the fixed point of any contraction mapping $T : X \to X$. We note that this proof is *constructive* in that we generated a sequence, starting at an arbitrary $x_0 \in X$, that converged to the fixed point $x \in X$. A constructive proof is often more useful for examples than a non-constructive one, such as proof by contradiction.

4.2. Examples of Contraction Mappings.

EXAMPLE 4.7. $T(x) = a + bx$ with $|b| < 1$ is a contraction mapping on $\mathbb{R}$ because, for any $x, y \in \mathbb{R}$ we have

$$\begin{aligned} d(T(x), T(y)) &= |T(x) - T(y)| \\ &= |a + bx - a - by| \\ &= |b||(x - y)| \\ &< |x - y| \\ &= d(x, y). \end{aligned}$$

Let $x_0 = 0$. Then we have the sequence

$$\begin{aligned} x_1 &= a \\ x_2 &= a + ab \\ x_3 &= a + ab + ab^2 \\ \vdots &\ \ \vdots\ \ \vdots \\ x_k &= a + ab + ab^2 + \cdots + ab^{k-1} \\ &= a \sum_{i=0}^{k-1} b^i. \end{aligned}$$

Recall that for a *geometric series* $\sum_{i=0}^{\infty} b^i$ with $b \in \mathbb{R}$, $|b| < 1$ the sum of the series is $\frac{1}{1-b}$. Thus in the limit of our series we have

$$\lim_{k \to \infty} x_k = a \sum_{i=0}^{\infty} b^i = \frac{a}{1 - b},$$

for all $b \in \mathbb{R}$ with $|b| < 1$. This is, of course, the solution of $x = T(x) = a + bx$. ■

5. Normed linear spaces

REMARK 5.1. We now specialise to normed linear spaces. These are vector spaces (i.e., linear spaces) on which the notion of *length* of a vector is defined. These concepts will be used in all subsequent chapters.

DEFINITION 5.2. A **norm** on a vector space V is a mapping $\| \ \| : V \longrightarrow \mathbb{R}$ such that for all $x \in V$,

N1. (*Positivity*) $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$.
N2. (*Scaling*) $\|\alpha x\| = |\alpha| \|x\|$, for all $\alpha \in \mathbb{R}$.
N3. (*Triangle Inequality*) $\|x + y\| \leq \|x\| + \|y\|$.

A **normed (linear) space** is a vector space with a norm, e.g., as in Example 1.3, $\mathbb{R}^n$ with the "usual" Euclidean norm, $\|x - y\| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$.

REMARK 5.3. The norm $\|x\|$ may be viewed as the *length* of a vector x or the distance between the origin 0 and x because $\|x - 0\| = \|x\|$.

Any norm gives rise to a metric. Indeed, let V be a normed linear space. We define the **distance** between two vectors $x, y \in V$ as

$$d(x, y) = \|x - y\|.$$

Then this d is a metric on V. Thus every normed linear spaces is a metric space. A metric space is more general than a normed linear space since the set X in a metric space need not have any algebraic properties like addition, and so may not have a “zero” element (whereas every vector space has a zero vector). For example, consider the metric space $\mathbb{R}$ and its subspace, the half-open interval $X = (0, 1] \subset \mathbb{R}$. Then X is a metric subspace of $\mathbb{R}$, but not a vector subspace, e.g., $1 \in X$ but for the scalar 2 we have $2 \cdot 1 = 2 \notin \mathbb{R}$, so X fails the second property of a subspace, namely, being closed under scalar multiplication. A normed space is a special case of a metric space.

The main examples of metric spaces we are interested in are $\mathbb{R}$, $\mathbb{R}^n$ and spaces of solutions for iterative algorithms (as treated earlier). We are able to use the concept of distance between points in $\mathbb{R}^n$ (derived from the norm $\| \ \|$ on $\mathbb{R}^n$) to define the ε-neighbourhoods (recall an **ε-neighbourhood** is a ball of radius ε about a point). Then the open sets in $\mathbb{R}^n$ are just arbitrary unions of ε-neighbourhoods.

EXAMPLE 5.4. Recall that $\mathcal{C}([0, 1])$ is the metric space of all continuous functions $f : [0, 1] \longrightarrow \mathbb{R}$, with $d(f, g) := \max_{x\in[0,1]} |f(x) - g(x)|$ for all $f, g \in \mathcal{C}([0, 1])$.

Such function spaces arise in Stochastic Calculus, e.g., the Ito integral is a limit of functions in a function space (and depends for its definition on the space being *complete* — see 5.10 below). These function spaces are in fact normed linear spaces, where we define the sum function $f+g$ by $(f+g)(x) := f(x)+g(x)$ and scalar multiple λf by $(\lambda f)(x) := \lambda f(x)$. ■

5.1. Standard Norms. It is possible to define other norms on $\mathbb{R}^n$: the norms used most often are as follows, where $x = (x_1, \ldots, x_n)$:

$$\begin{aligned} \|x\|_1 &= \sum_{i=1}^{n} |x_i| \\ \|x\|_2 &= \left[\sum_{i=1}^{n} |x_i|^2\right]^{1/2} \\ \|x\|_p &= \left[\sum_{i=1}^{n} |x_i|^p\right]^{1/p} \\ \|x\|_\infty &= \max_i |x_i| \end{aligned}$$

These are called the **1-, 2-, p-,** and **infinity norms**, respectively.

EXERCISE 5.5. Show that $\|x\|_1$ and $\|x\|_\infty$ are indeed norms, according to the definition. That is, show that **N1**, **N2**, and **N3** hold.

EXERCISE 5.6. Define $d(x, y) = \|x - y\|$ for each of the standard norms above. What is the distance between $x = (1, 2)$ and $y = (5, 10)$ for the $\| \ \|_2$ norm?

The distances in $\mathbb{R}^n$ change as the norm changes.

5.2. Equivalence of norms.

DEFINITION 5.7. We call the open sets in Definition **??** the **topology induced by the norm** $\| \ \|$. (That is, open sets with respect to the distance function $d(x, y) = \|x - y\|$).

It can be shown that all norms on $\mathbb{R}^n$ are equivalent, in the sense that they all induce the same topology (i.e., the same sets are open with respect to each norm). Formally, this is stated as

THEOREM 5.8 (Equivalence of Norms). *Let $\| \cdot \|_u$ and $\| \cdot \|_v$ be any two norms. Then there exist constants c_1, $c_2 > 0$ for which*

$$c_1\|x\|_u \le \|x\|_v \le c_2\|x\|_u.$$

5.3. Banach spaces. Since every normed space is a metric space, we can rewrite many of our earlier definitions in terms of norms, e.g., Definition 2.4 becomes:

DEFINITION 5.9. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and let $a \in \mathbb{R}^n$. We say f has **limit ℓ at** a if for all $\varepsilon > 0$, there exists $\delta > 0$ such that, if

$$x \in \mathbb{R}^n \text{ with } \|x - a\| < \delta,$$

then

$$\|f(x) - \ell\| < \varepsilon;$$

that is,

$$\forall\, \varepsilon > 0,\ \exists\, \delta > 0 \text{ such that } x \in B_\delta(a) \ \Rightarrow\ f(x) \in B_\varepsilon(\ell).$$

We say f is **continuous at** a if $f(a)$ and the limit ℓ at a both exist and $f(a) = \ell$. Let $U \subseteq \mathbb{R}^n$. We say f is **continuous on** U if for all $a \in U$, f is continuous at a.

DEFINITION 5.10. Let V be a normed linear space. A **Cauchy sequence** in V is a sequence $(a_n)_{n\in\mathbb{N}}$ with the property that

$$\|a_n - a_m\| \to 0 \text{ as } n,\, m \text{ both} \to \infty,$$

that is,

$$\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N} \text{ such that, for all } n, m > N, \|a_n - a_m\| < \varepsilon.$$

A sequence $(a_n)_{n\in\mathbb{N}}$ in V is said to **converge to** $a \in V$ if

$$\|a_n - a\| \to 0 \text{ as } n \to \infty,$$

that is,

$$\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N} \text{ such that, for all } n > N, \|a_n - a\| < \varepsilon.$$

The element $a \in V$ is called the **limit** of (a_n), and we say (a_n) is **convergent**.

V is said to be **complete** if every Cauchy sequence in V converges (to an element of V). A complete normed linear space is called a **Banach space**.

Recall that a mapping T of a complete metric space X into itself is called a **contraction mapping** if

$$d(T(x_1), T(x_2)) \le \lambda d(x_1, x_2),\ \forall\, x_1, x_2 \in X,$$

for some λ, where $0 \le \lambda < 1$. Stated in terms of norms we have that in a Banach space X,

$$\|T(x_1) - T(x_2)\| \le \lambda \|x_1 - x_2\|,\ \forall\, x_1, x_2 \in X$$

for a contraction mapping T. This was the original context of Banach's fixed point theorem.

REMARK 5.11. It can be shown that $\mathbb{R}$ and $\mathbb{R}^n$ are complete (and thus Banach spaces). This is one of the fundamental facts about the real numbers and is often called the *Completeness Axiom* or *Supremum Axiom.* Not every normed linear space is complete. For example, $\mathbb{Q}$ is not. Consider the sequence (a_n) in $\mathbb{Q}$ where we define

$$a_n = \left(1 + \frac{1}{n}\right)^n.$$

Each a_n is a rational number, and it can be shown that (a_n) is a Cauchy sequence (in $\mathbb{Q}$). But (a_n) converges to the *irrational* number $e \in \mathbb{R}$, i.e., does not converge in $\mathbb{Q}$, so $\mathbb{Q}$ is not complete.

5.4. Matrix Norms. Recall from Chapter 2 that we may view a square matrix A as a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ or as a vector $A \in M_n\mathbb{R} \cong \mathbb{R}^{n\times n}$. It is thus reasonable to investigate assigning norms to matrices.

Matrix norms, to be useful, have to have the following properties:

M1. $\|A\|_m \ge 0$, for any non-zero A.
M2. $\|\alpha A\|_m = |\alpha|\ \|A\|_m$, for any scalar α.
M3. $\|A + B\|_m \le \|A\|_m + \|B\|_m$.
M4. $\|AB\|_m \le \|A\|_m \|B\|_m$
M5. $\|Ax\|_v \le \|A\|_m\ \|x\|_v$

The first 3 properties are the standard properties of any norm. The fourth property is called the triangle inequality for matrix multiplication, while the fifth requires the matrix norm $\|\ \|_m$ to be compatible with the vector norm $\|\ \|_v$.

All these properties hold for the *standard definition of matrix norm*:

$$\|A\| = \max_{\|x\|_v = 1} \|Ax\|_v = \max_{x \neq 0} \frac{\|Ax\|_v}{\|x\|_v}$$

This definition gives a matrix norm corresponding to each vector norm because the right-hand side above is in terms of vector norms only.

In fact one can define a norm for a linear operator (i.e., linear map) $T : V \longrightarrow W$ from one normed space V with norm $\| \|_V$ to another normed space W with norm $\| \|_W$ in the same way:

$$\|T\| = \max_{\|x\|_V=1} \|T(x)\|_W = \max_{x\neq 0} \frac{\|T(x)\|_W}{\|x\|_V}.$$

This is often called an **operator norm**.

Here are the matrix norms corresponding to the 1-, 2-, and ∞-norms:

- $\|x\|_1 = \sum_{i=1}^n |x_i| \qquad \|A\|_1 = \max_j \sum_{i=1}^n |a_{ij}|$
- $\|x\|_2 = [\sum_{i=1}^n |x_i|^2]^{\frac{1}{2}} \qquad \|A\|_2 = [r_\sigma(A^tA)]^{\frac{1}{2}}$
- $\|x\|_\infty = \max_i |x_i| \qquad \|A\|_\infty = \max_i \sum_{j=1}^n |a_{ij}|$

where $r_\sigma(A^tA)$ is the *spectral radius* of A^tA, i.e., the magnitude of the largest eigenvalue of A^tA.

THEOREM 5.12. *Let A be an arbitrary square matrix. Then for any matrix norm*

(*a*) $r_\sigma(A) \leq \|A\|_m$
(*b*) *Given $\epsilon > 0$ there exists a matrix norm $\| \|_m$ for which*

$$\|A\|_m \leq r_\sigma(A) + \epsilon$$

This says the spectral radius is a *lower bound* for matrix norms, in fact it is an infimum since there is a matrix norm arbitrarily close to the spectral radius. Thus the spectral radius is a "good" measure of "length/size" of a matrix.

COROLLARY 5.13. *For any square matrix A*

$$r_\sigma(A) < 1 \Longleftrightarrow \|A\|_m < 1 \textit{ for some matrix norm } \| \|_m.$$

Thus if the spectral radius of A is less than 1, iterating by multiplication by A will converge — see later.

THEOREM 5.14 (Matrix Geometric Series). *Let A be a square matrix. If $r_\sigma(A) < 1$ then $(I - A)^{-1}$ exists and can be expressed as*

$$(I - A)^{-1} = I + A + A^2 + \cdots + A^k + \cdots = \sum_{k=0}^{\infty} A^k$$

EXAMPLE 5.15. This example shows that the result of Example 4.7 can be generalized to $X = \mathbb{R}^n$.

Let $T(x) = a + Bx$, where $x, a \in \mathbb{R}^n$, and $B \in M_n\mathbb{R}$, i.e., a $n \times n$ real matrix. Assume that some suitable norm $\| \|$ has been defined for matrices and that $\|B\| < 1$.

$T(x)$ is a contraction mapping on $\mathbb{R}^n$ because, for any $x, y \in \mathbb{R}^n$ we have

$$\begin{aligned} d(T(x), T(y)) &= \|T(x) - T(y)\| = \|a + Bx - a - By\| \\ &\leq \|B\| \, \|(x - y)\| < \|x - y\| = d(x, y). \end{aligned}$$

Let $x_0 = 0$, the zero vector. Then we have the sequence

$$\begin{aligned} x_1 &= a \\ x_2 &= a + Ba \\ x_3 &= a + Ba + B^2a \\ \vdots &\ \ \vdots \quad \vdots \\ x_k &= Ia + Ba + B^2a + \cdots + B^{k-1}a \\ &= \left(\sum_{i=0}^{k-1} B^i\right) a. \end{aligned}$$

In the limit we have

$$\lim_{k\to\infty} x_k = \left(\sum_{i=0}^{\infty} B^i\right) a = (I - B)^{-1}a, \ \forall \ \|B\| < 1.$$

This is, of course, the solution of $x = T(x) = a + Bx$. Note that

$$\sum_{i=0}^{\infty} B^i = (I - B)^{-1}, \text{ for all } B \text{ with } \|B\| < 1,$$

is the n-dimensional equivalent of the geometric series, $\sum_{i=0}^{\infty} b^i = 1/(1-b),\ \forall\ |b| < 1$. The matrix equivalent is called the **Neumann Series**. ■

Remark 5.16. The iterative solution of the system of linear equations $Ax = b$ requires the equation to be re-arranged into *fixed point form* as follows:

$$x = T(x) \quad \text{where} \quad T(x) := Cx + d.$$

If $T(x)$ is a contraction mapping then by Banach's Fixed Point Theorem it has a unique solution which is the limit of the sequence (x_k) generated by $x^{k+1} := T(x^k)$.