Notes on Moving Boundary problems, Voller U of M, volle001@umn.edu

1. Filling an initially porous tube under a constant head imposed at x =0

Governing equation is based on calculating the water volume flux by the Darcy law, i.e.

$$q = -K\frac{\partial h}{\partial x}$$

Such that a volume (flux) balance at a point in the wet portion gives the following governing equation

$$\frac{\partial^2 h}{\partial x^2} = 0, \quad 0 \le x \le s(t) \qquad (1)$$

with boundary conditions

$$h(0) = h_o ; h(s) = 0 \qquad (2)$$

And the additional volume balance at the wetting front $s(t)$ which states that the speed of this front is equal to the speed of the water flow at that point, i.e.,

$$-k^* \left.\frac{\partial h}{\partial x}\right|_s = \frac{ds}{dt}, \quad s(0) = 0 \qquad (3)$$

where k^* is a hydraulic conductivity adjusted to account for porosity.

The Solution of (1) that satisfies (2) is

$$h = h_0(1 - \frac{x}{s}) \qquad (4)$$

This provides the linear head profile in the wet fraction of the tube. When (4) is used in (3)it gives the following ODE for movement of the wetting front $s(t)$

$$k^* h_0 = s\frac{ds}{dt}, \quad s(0) = 0$$

The solution

$$s = \sqrt{2k^* h_0 t}$$

Gives the movement of the wetting front

2. One-d melting of ice initially at temperature $T = T_m$

Governing equation (assuming heat conduction

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}, \quad 0 \le x \le s(t) \quad (1)$$

Where $\alpha = K / \rho c$ is the thermal diffusivity. The main boundary conditions are

$$T(0) = T_0, \quad T(s) = T_m \qquad (2)$$

the initial condition is

$$T(x,0) = T_m \qquad (3)$$

The moving boundary condition (the Stefan condition) expressing the heat balance at the front (see full derivation below) is

$$-K \left. \frac{\partial T}{\partial x} \right|_{x=s} = \rho \Delta H \frac{ds}{dt} \quad (4)$$

Where ΔH is the latent heat of fusion.

We seek a similarity solution using the similarity variable

$$\xi = \frac{x}{2\sqrt{\alpha} t^{\frac{1}{2}}} \quad (5)$$

And assuming the melt front moves as

$$s = 2\lambda \sqrt{\alpha} t^{\frac{1}{2}} \quad (6)$$

Where λ is a constant .

With (5) and (6) it is observed that

$$\frac{\partial T}{\partial x} = \frac{1}{2\sqrt{\alpha} t^{\frac{1}{2}}} \frac{\partial T}{\partial \xi}, \quad \frac{\partial^2 T}{\partial x^2} = \frac{1}{4\alpha t} \frac{\partial^2 T}{\partial \xi^2}, \quad \frac{\partial T}{\partial t} = \frac{-x}{4\sqrt{\alpha} t^{\frac{3}{2}}} \frac{\partial T}{\partial \xi} = -\frac{\xi}{2t} \frac{\partial T}{\partial \xi}, \quad \frac{ds}{dt} = \lambda \sqrt{\alpha} t^{-\frac{1}{2}} \quad (7)$$

With (7) equation (1) and (2) becomes

$$\frac{1}{2}\frac{\partial^2 T}{\partial \xi^2}+\xi\frac{\partial T}{\partial \xi}=0, \quad 0\le \xi \le \lambda \qquad (8)$$

$$T(0)=T_0, \quad T(\lambda)=T_m \qquad (9)$$

And the moving boundary condition becomes

$$-\frac{c}{\Delta H}\frac{\partial T}{\partial \xi}\bigg|_{\lambda}=2\lambda \qquad (10)$$

The solution of (8) and (9) can be shown to be

$$T=\frac{T_m-T_0}{erf(\lambda)}erf(\xi)+T_0 \quad (11)$$

Where the error function, its derivative and second derivative are given by

$$erf(\xi)=\frac{2}{\sqrt{\pi}}\int_0^{\xi} e^{-\beta^2}d\beta, \quad \frac{d}{d\xi}erf(\xi)=\frac{2}{\sqrt{\pi}}e^{-\xi^2}, \quad \frac{d^2}{d^2\xi}erf(\xi)=-\frac{4}{\sqrt{\pi}}\xi e^{-\xi^2} \quad (12)$$

Substitution of (11) into (10) provides an equation for calculating the unknown value of λ

$$\sqrt{\pi}\lambda e^{\lambda^2}erf(\lambda)=S_t \quad (13)$$

where

$$S_t=\frac{c(T_0-T_m)}{\Delta H} \quad (14)$$

Is the dimensionless Stefan number

3. Derivation of the one-domain Stefan problem with constant properties

Consider a 3-D domain Ω^D with surface Γ^D, initially solid at the constant phase change temperature $T = 0$. At time $t \geq 0$ melting is induced by setting and fixing part of the domain surface to a temperature $T_{fix} > 0$. Such that at some later time the domain is segmented into a liquid region and a solid region. The liquid region has a temperature $T_{fix} \geq T \geq 0$ varying in space and time, while the solid region remains at the initial constant temperature of $T = 0$. These regions are separated by a melt interface Γ^f continuously moving into the solid. To arrive at a governing equation, at a given instant in time, we consider an arbitrary closed, fully liquid domain Ω where part of the domains surface Γ^f coincides with the melt front. The heat content of this domain is given by

$$H^{\Omega} = \int_{\Omega} \rho c T + \rho \Delta H \, dV \qquad (1)$$

Where ΔH is the latent heat of fusion, c is the specific heat, and ρ, assumed constant in both the solid and liquid phase, is the density. The surface of out domain is made of two segments one entirely in the liquid phase Γ and the other Γ^f on the moving melt interface. In considering the melt interface it is often helpful to consider separately the surface of material points on the liquid side of this interface, Γ^{f-} from the surface made up of material points on the solid of this interface Γ^{f+}. In particular it is noted that on the liquid side the temperature gradient $\nabla T > 0$, whereas on the solid side the gradient is $\nabla T = 0$ (due to the constant solid temperture); a discontinuity in temperature gradient required to provide the latent heat of fusion for the solid to change phase.

From the definition of heat in (1) a transient heat balance on our arbitrary volume can be written as

$$\frac{d}{dt}\int_{\Omega} \rho c T + \rho \Delta H \, dV = \int_{\Gamma} K \nabla T . \underline{n} \, dA + \int_{\Gamma^{f+}} K \nabla T . \underline{n} \, dA \qquad (2)$$

Or since we have zero gradient

$$\frac{d}{dt}\int_{\Omega} \rho c T + \rho \Delta H \, dV = \int_{\Gamma} K \nabla T . \underline{n} \, dA \qquad (3)$$

Where the unit outward normal points out from the volume .On using the Reynolds transport formula and noting that $T = 0$ on the melt interface, this becomes

$$\int_{\Omega} \rho c \frac{\partial T}{\partial t} \, dV + \int_{\Gamma^{f-}} \rho \Delta H \, \underline{v} . \underline{n} \, dA = \int_{\Gamma} K \nabla T . \underline{n} \, dA \qquad (4)$$

Where, without loss of exactness, the second term on the left is evaluated on material points on the liquid side of the melt interface and $\underline{v}$ is the velocity of the material points on this interface. On adding to both sides the flux on the liquid side of the melt face (3) is reformed as

$$\int_{\Omega} \rho c \frac{\partial T}{\partial t}\, dV + \int_{\Gamma^{f-}} \rho \Delta H \underline{v}.\underline{n} + K \nabla T.\underline{n}\, dA = \int_{\Gamma+\Gamma^{f-}} K \nabla T.\underline{n}\, dA \qquad (5)$$

On recognizing that the term on the left is over a surfaces that encloses our arbitrary volume and that the term $K\nabla T$ is continuous in this volume we can use the divergence theorem to arrive at

$$\int_{\Omega} \rho c \frac{\partial T}{\partial t} + K \nabla^2 T\, dV + \int_{\Gamma^{f-}} \rho \Delta H \underline{v}.\underline{n} + K \nabla T.\underline{n}\, dA = 0 \qquad (6)$$

But since our volume choice is arbitrary both arguments in the above integral must be identically zero at every point in the domain of integration, i.e,

$$\rho c \frac{\partial T}{\partial t} = K \nabla^2 T, \quad in \quad \Omega^D, \qquad (7)$$

the appropriate governing equation and

$$-K \nabla T = \rho \Delta H \underline{v}, \quad on \quad \Gamma^{f-} \qquad (8)$$

which is seen to be the Stefan condition on the moving boundary. Note in one dimension (7) and (8) become as expected the previously used one-phase Stefan problem formulization

$$\begin{aligned} &\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}, \quad 0 \le x \le s(t) \\ &T(0) = T_{fix}, \quad T(s) = 0 \end{aligned} \qquad (9)$$

$$-K \frac{\partial T}{\partial x} = \rho \Delta H \frac{ds}{dt} \qquad (10)$$

4. An Enthalpy Formulization

We first extend the previous melting problem in a way that assumes that the transition from liquid to full solid takes place over a narrow and arbitrarily small temperature range $\varepsilon \geq T \geq 0$. Within this setting we define the enthalpy (total heat) at any point (liquid or solid) in Ω^D to be given by

$$H = cT + f\Delta H \qquad (1)$$

where f is a liquid fraction that changes smoothly and monotonically from a value of $f = 1$ for temperatures $T \geq \varepsilon$ to a value of $f = 0$ for temperatures $T \leq 0$. In this way, the latent heat is evolved smoothly across a temperature range and as such no temperature gradient discontinuities are required to drive the phase change. This type of phase change system is referred to as a diffusive phase change system to indicate that the phase transition takes place over a spatial region of finite thickness as opposed to the sharp transition defined in the Stefan problem. The assumption is made (which can be supported by rigorous analysis) that as the region of transition decreases in thickness—i.e., as the temperature range $\varepsilon \to 0$—this diffusive interface model becomes an increasingly more accurate representation of the sharp interface Stefan model.

With (1) and the diffuse interface assumption the transient heat balance of the entire domain can be formed as

$$\frac{d}{dt}\int_{\Omega^D} \rho H \, dV = \int_{\Gamma^D} K\nabla T . \underline{n} \, dA \qquad (2)$$

And since temperature gradients are continuous in the domain we can use the divergence theorem and arbitrary volume observation to arrive at the governing equation

$$\rho \frac{\partial H}{\partial t} = K\nabla^2 T, \quad in \quad \Omega^D \qquad (3)$$

An equation which is applicable in the entire domain (liquid and solid).

In a one-phase one-dimensional system this equation becomes

$$\rho \frac{\partial H}{\partial t} = K \frac{\partial^2 T}{\partial x^2}, \quad x \geq 0 \qquad (4)$$

Where is seeking a numerical solution we can assume an arbitrarily thin temperature range over which the phase can occurs such that with no loss of accuracy we can essentially assume a sharp interface that sets

$$T = \begin{cases} \dfrac{H - \Delta H}{c}, & H \geq \Delta H \\ 0 & H < \Delta H \end{cases} \tag{5}$$

An example problem—to follow