Problem Sheet 1

(1) Approximate $\sqrt[3]{13}$ to three decimal places by applying the bisection method to the equation $x^3 - 13 = 0$.

(2) Consider the function $g(x) = e^{-x^2}$.

(a) Prove that this function has a unique fixed point on the interval $[0, 1]$.

(b) Consider the iteration scheme $x_{n+1} = g(x_n)$. Use the theoretical error bound:

$$|x_n - x| \leq \frac{\alpha^n}{1-\alpha}|x_1 - x_0|$$

to obtain an estimate for the number of iterations needed to approximate the fixed point within 10^{-6}.

(3) Consider the function $g(x) = \cos(x)$.

(a) Graphically verify that this function has a unique fixed point on the real line.

(b) Can we prove that the fixed point is unique using Banach's Fixed Point Theorem for $X = \mathbb{R}$? Explain your answer.

(c) By restricting the interval on which g is defined, demonstrate that for suitable X Banach's Fixed Point Theorem can be applied.

(4) The function $g(x) = \sin(x)$ has a zero on the interval $[3, 4]$, namely $x = \pi$.

(a) Perform three iterations of Newton's method to approximate this zero, starting at $x_0 = 4$.

(b) Determine the error in each of the approximations. What is the order of convergence?