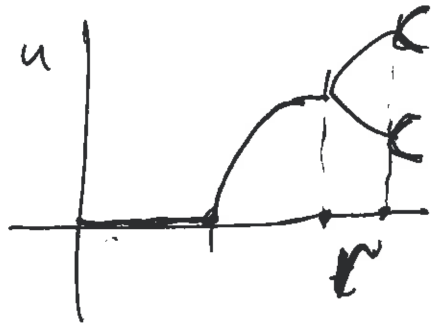
Lecture 10

Stability, Periodic Solutions and Bifurcations

$$u_{t+1} = f(u_t; r)$$

$r = r_c$ - bifurcation



$$u^* = f(u^*; r) \longrightarrow u^*(r)$$

$$u_t = u^* + v_t, \quad |v_t| \ll 1$$

$$\cancel{u^*} + v_{t+1} = f(u^* + v_t) = \cancel{f(u^*)} + \left(\frac{\partial f}{\partial u}\right)_{u=u^*} \cdot v_t$$

$$v_{t+1} = f'(u^*)\, v_t \qquad \boxed{\lambda = f'(u^*), \text{ eigenvalue}}$$

$$v_t = [f'(u^*)]^t v_0 = \lambda^t v_0 \to \begin{cases} 0 & \text{if } |\lambda| < 1 \\ \pm\infty & \text{if } |\lambda| > 1 \end{cases}$$

$$u^* \text{ is } \begin{cases} \text{stable} \\ \text{unstable} \end{cases} \text{ if } \begin{cases} -1 < f'(u^*) < 1 \\ |f'(u^*)| > 1 \end{cases}$$

$$u_t = u^* + [f'(u^*)]^t c_0$$

$$\left|\begin{array}{l} x_{t+1} = \sqrt{2+x_t} \\ x_0 = \sqrt{2} \end{array}\right.$$

$$x_1 = \sqrt{2+x_0} = \sqrt{2+\sqrt{2}}$$

$$x_2 = \sqrt{2+x_1} = \sqrt{2+\sqrt{2+\sqrt{2}}}$$

$$\vdots$$

$$x_t = \sqrt{2+\sqrt{2+\sqrt{2+\cdots}}} \quad \text{(t+1 radicals)}$$

If x^* is a stable steady state, then

$$\lim_{t\to\infty} x_t = x^*$$

$$x^* = \sqrt{2+x^*} \Rightarrow (x^*)^2 = 2+x^*$$

$$(x^*)^2 - x^* - 2 = 0 \Rightarrow \begin{array}{l} x^* = 2 \\ \cancel{x^* = -1} \end{array}$$

$$x^* = \sqrt{\underbrace{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}}_{\text{infinitely many}}} = 2$$

$f(x) = \sqrt{2+x}$, Eig. is $f'(2) = \left(\frac{1}{2\sqrt{2+x}}\right)_{x=2} = \frac{1}{4} < 1$

$|f'(2)| < 1$ – stable.

$$u_{t+1} = u_t\, e^{r(1-u_t)} \quad , \quad \underline{r>0}$$

$$u_t = \frac{N_t}{K} \quad , \quad \underline{u^* = 0} \qquad \underline{u^* = 1}$$

$$f(u) = u e^{r(1-u)}$$

$$f'(u) = e^{r(1-u)} + u\, e^{r(1-u)} (-r)$$

$$f'(0) = e^{r} > \cancel{1} \quad \text{for } \underline{r>0} \quad \text{unstable}$$

$$f'(1) = e^{0} + (-r)(1) e^{0} = \underline{1-r}$$

$$f'(1) = 1-r \Rightarrow$$

$$-1 < 1-r < 1$$

$$\cancel{-2<} \quad \underline{\underline{r<2}} \qquad \underline{\underline{0<r}}$$

$\Rightarrow$ Stability condition is $\underline{\underline{0<r<2}}$

$\Rightarrow$ The first bifurcation value at $\underline{\underline{r=2}}$

$$u_t = 1 + v_t \quad , \quad |v_t| \ll 1$$

$$\underbrace{\cancel{v_{t+1}}}_{u_{t+1}} = \underbrace{\cancel{(1+v_t)}}_{u_t} e^{r(\cancel{1}-\underbrace{\cancel{1}-v_t}_{-u_t})} = \overset{u_t}{(\cancel{1+v_t})}\, \underline{\underline{e^{-r v_t}}}$$

$$e^{x} = 1 + x + \frac{x^2}{2!} + \dots \approx 1 + x$$

$$\left(v_t = u_t - 1\right)$$

$$u_{t+1} = u_t e^{-rv_t} = u_t(1 - rv_t)$$

$$u_{t+1} = u_t \left[1 + r\underbrace{(1-u_t)}_{-v_t}\right]$$

$$u_{t+1} \approx u_t \left[1 + r(1-u_t)\right]$$

$$U_t = \frac{ru_t}{1+r} \Rightarrow u_t = \frac{1+r}{r} U_t$$

$$\cancel{\frac{1+r}{r}} U_{t+1} = \cancel{\frac{1+r}{r}} U_t \left[1 + r\left(1 - \frac{1+r}{r} U_t\right)\right]$$

$$U_{t+1} = U_t \left[1 + r - (1+r) U_t\right]$$

$$\boxed{U_{t+1} = (1+r) U_t \left[1 - U_t\right]}$$

Logistic <u>model</u> with $r \to r+1$

$r = 2$ is a bifurcation to 2-periodic Solution

4-periodic solution $r_4 = 2.~~4~~5$

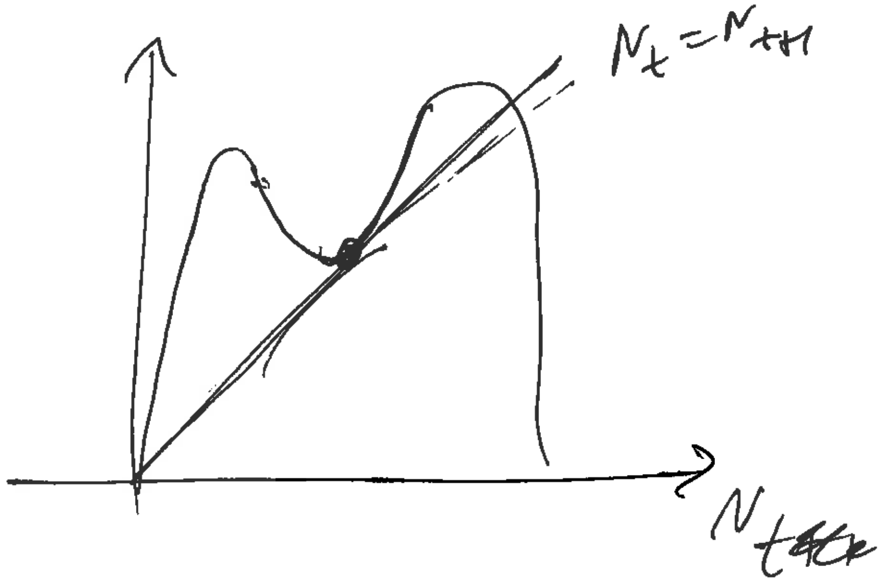
6-periodic $r_6 \approx 2.54$

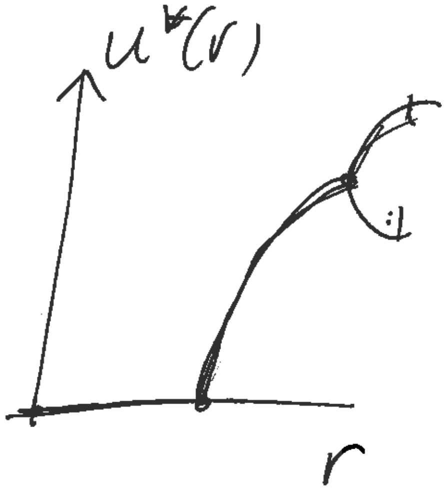
chaotic $r > r_c = 2.57$

The bifurcation occurs at parameter value r_0, if there is qualitative change of the dynamics for $r < r_0$ & $r > r_0$

Bifurcations with $\lambda = -1$ are called 'period doubling'

with $\lambda = 1$ are called 'tangent'





Sarkovskii theorem: Chaos appears when odd-periodic solutions are possible.

$$u_{t+3} = f(\underbrace{f(\underbrace{f(u_t)}_{u_{t+1}})}_{u_{t+2}})$$

$$u^* = f^{(3)}(u^*)$$

Discrete delay models

$$u_{t+1} = f(u_t, u_{t-1}, \dots u_{t-T})$$

$$u_0, u_1, u_2, \dots u_T$$

$$u_{T+1} = f(u_T, u_{T-1}, \dots u_0)$$

Example:

$$u_{t+1} = u_t e^{r(1-u_{t-1})},$$

$$\underline{u_0, u_1}$$

Delay version of the Ricker's model.

$$\underbrace{u^* = 0}_{\text{unstable}} \quad \& \quad \underbrace{u^* = 1}_{?}$$

$$u_t = 1 + v_t, \quad |v_t| \ll 1$$

$$(1+v_{t+1}) = (1+v_t) e^{r(\cancel{1}-\cancel{1}-v_{t-1})} \approx (1+v_t)(1-rv_{t-1})$$

$$e^{-rv_{t-1}} \approx 1 - rv_{t-1}$$

$$(1+v_{t+1}) = (1+v_t)(1-rv_{t-1})$$

$$\cancel{1}+v_{t+1} = \cancel{1}+v_t - r v_{t-1} - \cancel{r v_t v_{t-1}}$$

$$v_{t+1} = v_t - r v_{t-1}$$

$$\boxed{v_{t+1} - v_t + r v_{t-1} = 0}$$ linear

$$v_t = z^t$$

$$z^{t+1} - z^t + r z^{t-1} = 0 \quad \Big| \cdot \frac{1}{z^{t-1}}$$

$$\boxed{z^2 - z + r = 0}$$ Characteristic equ.

$$z_{1,2} = \frac{1}{2}\left[1 \pm \sqrt{1-4r}\right] \qquad \underline{0<r<\frac{1}{4}}$$

$$z_{1,2} = \frac{1}{2}\left[1 \pm i\sqrt{4r-1}\right] \qquad \underline{r>\frac{1}{4}}$$

$\underline{\text{I}}$ $0<r<\frac{1}{4}$, $0 < z_{1,2} < 1 \Rightarrow$

$$v_t = c_1 z_1^t + c_2 z_2^t \to 0, \quad u_t \to u^* = 1$$

$$\boxed{u^* = 1 \text{ is stable when } 0<r<\frac{1}{4}}$$

$\underline{\text{II}}$ $r>\frac{1}{4}$, $\rho = |z_{1,2}| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{4r-1}}{2}\right)^2} =$

$$\rho = \sqrt{\frac{\cancel{1}}{4} + \frac{4r-\cancel{1}}{4}} = \sqrt{\frac{4r}{4}} = \sqrt{r}$$

$$\theta = \tan^{-1}\sqrt{4r-1}, \quad z_{1,2} = \sqrt{r}\, e^{\pm i\theta}$$

General solution

$$v_t = A z_1^t + B z_2^t = A z_1^t + \bar{A} \bar{z}_1^t$$

$$A = |A| e^{i\gamma}$$

$$v_t = |A| e^{i\gamma} (\sqrt{r} e^{i\theta})^t + |A| e^{-i\gamma} (\sqrt{r} e^{-i\theta})^t$$

$$= 2|A| r^{t/2} \frac{\left(e^{i\theta t + i\gamma} + e^{-i\theta t - i\gamma} \right)}{2}$$

$$v_t = 2|A| \underline{\underline{r^{\frac{t}{2}}}} \cos(\theta t + \gamma)$$

Stable as $|\sqrt{r}| < 1$, $\boxed{\frac{1}{4} < r < 1}$

$\boxed{r_c = 1}$

$$v_t = 2|A| 1^{t/2} \cos(t \cdot \theta_1 + \gamma)$$

$$\theta_1 = \tan^{-1} \sqrt{4r - 1} = \tan^{-1} \sqrt{3} = \pi/3$$

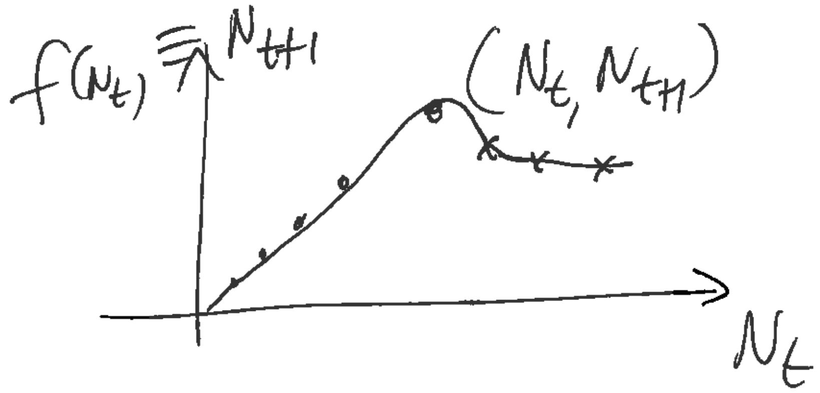
$$v_t = 2|A| \cos\left(\frac{\pi}{3} t + \gamma\right)$$

$$\frac{\cancel{\pi}}{3} t_p = 2\cancel{\pi} \Rightarrow t_p = 6$$

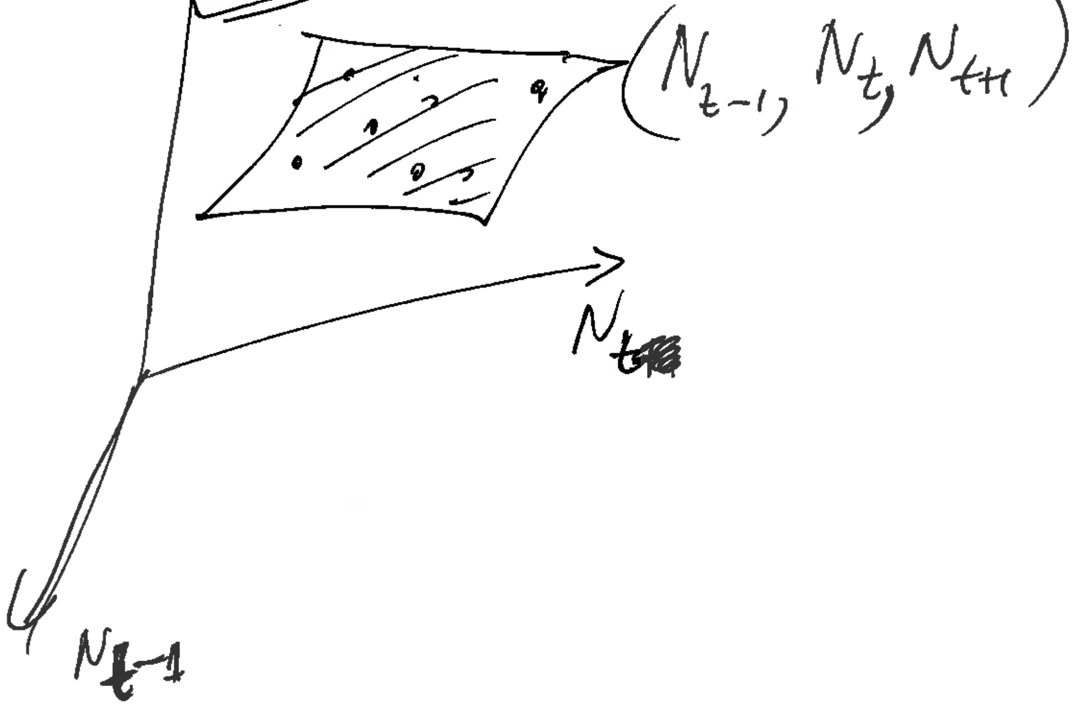
$$v_{t+6} = v_t$$

$N_1, N_2, N_3, \ldots$

1) $\boxed{N_{t+1} = f(N_t)}$



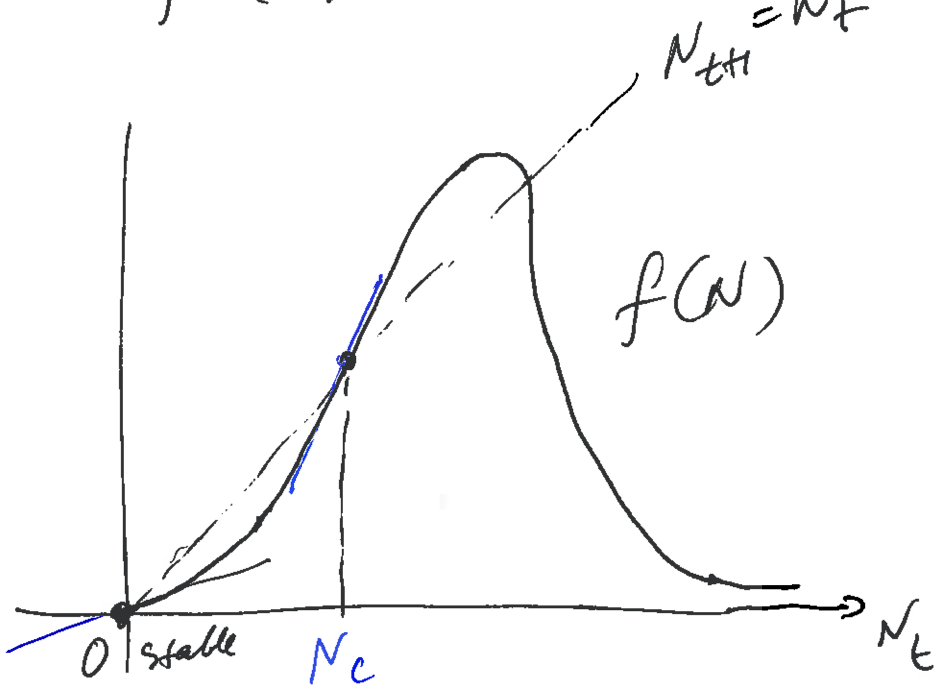
2) $\boxed{N_{t+1} = f(N_t, N_{t-1})}$

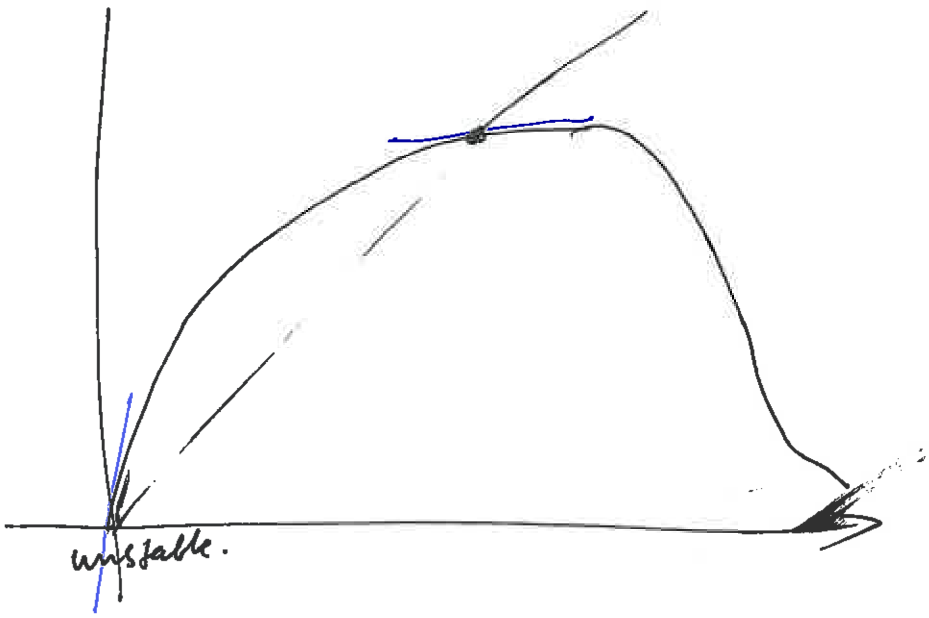


Allee effect:

$N^* = 0$ is a stable steady state

$f'(0) \leq 1$





$$\frac{df}{dN} = 0 \iff N_m, \quad N_{max} = f(N_m)$$

$$N_{min} \equiv f(N_{max}) = f(f(N_m))$$

$\boxed{N_{min} < 1}$ extinction of species

$$N_{t+1} = N_t e^{r(1-\frac{N_t}{K})}$$

$$f(N) = N e^{r(1-\frac{N}{K})}$$

$$f'(N) = e^{r(1-\frac{N}{K})} + N e^{r(1-\frac{N}{K})} \left(-\frac{r}{K}\right)$$

$$f'(N) = e^{r(1-\frac{N}{K})} \left(1 - \frac{Nr}{K}\right) = 0$$

$$\boxed{N_m = \frac{K}{r}}$$

$$N_{max} = f(N_m) = f\left(\frac{K}{r}\right) = \frac{K}{r} e^{r(1-\frac{1}{K}\frac{K}{r})}$$

$$\boxed{N_{max} = \frac{K}{r} e^{r-1}}$$

$$N_{min} = f(N_{max}) = f\left(\frac{K}{r} e^{r-1}\right)$$

$$N_{min} = \frac{K}{r} e^{r-1} \cdot e^{r(1-\frac{1}{\cancel{K}}\frac{\cancel{K}}{r} e^{r-1})} = \frac{K}{r} e^{r-1+r-e^{r-1}}$$

$$N_{min} = \frac{K}{r} e^{2r-1-e^{r-1}} = \frac{K}{r} \exp\left(2r-1-e^{r-1}\right)$$

Extinction: $N_{min} \leq 1$

$$\frac{K}{r} \exp\left(2r-1-e^{r-1}\right) \leq 1$$

May happen if $\underline{r=3.5 \ \& \ K < 1600}$

$$N_{t+1} = N_t \underbrace{\exp\left[r\left(1-\frac{N_t}{K}\right)\right]}_{\text{effective birth rate}}$$

Harvesting

$$u_{t+1} = \frac{b u_t^2}{1+u_t^2} - E u_t$$