DUBLIN INSTITUTE OF TECHNOLOGY

School of Mathematical Sciences

Structured PhD

Assignment 2 2016/2017

MATH 9973: NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS

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External examiner missing!

Due: 12, March 2017

Duration:

Attempt all questions

All questions carry equal marks

Approved calculators may be used

Mathematical tables are provided

New Cambridge Statistical Tables are NOT permitted

1. **a)** Use the central difference formula for the second derivative

$$f''(x_0) = \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} + \mathcal{O}(h^2)$$

to derive the Crank Nicholson numerical scheme for the Heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

on the rectangular domain

$$\Omega = \{(x,y) |\ a \le x \le b, c \le y \le d\}.$$

b) Consider the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

on the rectangular domain

$$\Omega = \{(x,y) |\ a \le x \le b, c \le y \le d\},$$

with the boundary conditions

$$u(0,t) = 0,\ u(1,t) = 0,$$

and initial condition

$$u(x,0) = 2\sin(2\pi x)$$

Taking $h = \frac{1}{6}$ in the x-direction and $k = \frac{1}{216}$ in the t-direction, set up and write in matrix form the corresponding systems of finite difference equations for one time step.

2. **a)** Use the central difference formula for the second derivative

$$f''(x_0) = \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} + \mathcal{O}(h^2)$$

to derive the fully implicit numerical scheme for the Heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

on the rectangular domain

$$\Omega = \{(x,y) |\ a \le x \le b, c \le y \le d\}.$$

b) Consider the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

on the rectangular domain

$$\Omega = \{(x,y) |\ a \le x \le b, c \le y \le d\},$$

with the boundary conditions

$$u(0,t) = 1,\ u(1,t) = 1,$$

and initial condition

$$u(x,0) = 4x^2 - 4x + 1.$$

Taking $h = \frac{1}{4}$ in the x-direction and $k = \frac{1}{32}$ in the t-direction, set up and solve the corresponding systems of finite difference equations for one time step.

c) Show that the method is unconditionally stable using von Neumann's method.

3. a) Use the central difference formula for the second derivative

$$f''(x_0) = \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} + \mathcal{O}(h^2)$$

to derive a numerical scheme for the Elliptic equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$$

on the rectangular domain

$$\Omega = \{(x,y) |\ a \le x \le b, c \le y \le d\}.$$

b) Consider the problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = y^2$$

on the rectangular domain

$$\Omega = \{(x,y) |\ 0 \le x \le 1, 0 \le y \le 1\},$$

with the boundary conditions

$$u(x,0) = x,\ u(x,1) = x,$$

$$u(0,y) = 0,\ u(1,y) = 1.$$

Taking $N = 4$ steps in the x-direction and $M = 4$ steps in the y-direction, set up and write in matrix form (but do not solve) the corresponding systems of finite difference equations.

4. a) Use the central difference formula for the second derivative

$$f^{''}(x_0) = \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} + \mathcal{O}(h^2)$$

and central difference formula for the second derivative

$$f^{'}(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h} + \mathcal{O}(h^2)$$

to derive the explicit numerical scheme for the Elliptic equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} = f(x,y)$$

on the rectangular domain

$$\Omega = \{(x,y) | \ a \le x \le b, c \le y \le d\}.$$

b) Consider the problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} = 0$$

on the rectangular domain

$$\Omega = \{(x,y) | \ 0 \le x \le 1, 0 \le y \le 1\},$$

with the boundary conditions

$$u(x,0) = 4x^2 - 4x + 1, \ u(x,1) = 4x^2 - 4x + 1,$$

$$u(0,y) = 4y^2 - 4y + 1, \ u(1,y) = 4y^2 - 4y + 1.$$

Taking $N = 4$ steps in the x-direction and $M = 4$ steps in the y-direction, set up and write in matrix form the corresponding systems of finite difference equations and solve.