M368K
Homework #11

Burden and Faires.

Section 11.5 (#12[1]). Section 12.1 (#2, 8[2]).

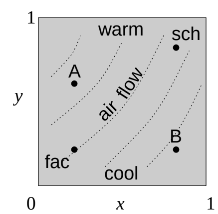
[1] Assume $p(x) > 0$ and $q(x) \geq 0$. Show $c^T Ac = \int_a^b p[v']^2 + qv^2\, dx \geq 0$, where $v(x) = \sum_{i=1}^N c_i\phi_i(x)$. Moreover, explain each of the implications $c^T Ac = 0 \Rightarrow v'(x) \equiv 0$ in each subinterval $(x_i, x_{i+1}) \Rightarrow v(x) \equiv 0$ in $[a, b] \Rightarrow c = 0$, and conclude that A is positive-definite.

[2] Use two interior nodes in each direction instead of the given spacings. Explicitly write the central-diff eqns and solve.

Programming mini-project.

A city consultant is studying the impact that an additional factory would have on air pollution in a given region; two proposed locations A and B are being considered. A simple model for the steady-state concentration $u(x, y)$ of pollutants in the air near the surface is the advection-diffusion equation

$$
\begin{aligned}
&\alpha\Delta u - v \cdot \nabla u + \phi = 0, \qquad a \leq x \leq b, \quad c \leq y \leq d\\
&u(a, y) = 0, \quad u(b, y) = 0, \quad c \leq y \leq d\\
&u(x, c) = 0, \quad u(x, d) = 0, \quad a \leq x \leq b
\end{aligned}
$$



where $\Delta u = u_{xx} + u_{yy}$ is the Laplacian of u, $\nabla u = (u_x, u_y)$ is the gradient of u, α is a diffusion coefficient that depends on the local air temperature, $v = (v_1, v_2)$ is the local air velocity and ϕ is the pollution source function. Zero boundary conditions are assumed for simplicity. Here we approximate solutions of this BVP using the central-difference method. For concreteness, we take $[a, b] = [c, d] = [0, 1]$, and consider the conditions $\alpha(x, y) = 0.3 + 0.05y$, $v_1(x, y) = 10 - 10x$ and $v_2(x, y) = 5y$.

(a) Download the C++ files `program11.cpp`, `linearcd2D.cpp` and `gauss_elim.cpp` from the course webpage. Complete the file `linearcd2D.cpp` so that it implements the central-difference method for boundary-value problems of the type $Pu_{xx} + Qu_{yy} + pu_x + qu_y + ru = f$ on rectangular domains. Check the code by running it on Section 12.1(#8) as above; compare the program output with your hand-worked solution.

(b) For reference, consider the case of the existing factory at $(x_0, y_0) = (0.2, 0.2)$ described by the source function $\phi(x, y) = 10e^{-30(x-x_0)^2-30(y-y_0)^2}$. After rewriting the PDE in standard form, use `program11.cpp` with $N = M = 29$ to solve the BVP. Use the supplementary file `matlab11.m` to produce a contour map of the solution. Approximately where does the maximum concentration of pollutants occur and what is its value? What is the concentration at the school location $(x_s, y_s) = (0.8, 0.8)$?

(c) Suppose that an additional, smaller factory is located at (x_1, y_1), and that the combined source function is now $\phi(x, y) = 10e^{-30(x-x_0)^2-30(y-y_0)^2} + 8e^{-30(x-x_1)^2-30(y-y_1)^2}$. For each proposed location $(x_1, y_1) = (0.2, 0.6)$ and $(0.8, 0.2)$, use `program11.cpp` and `matlab11.m` to solve the BVP and make a contour map of the solution as before, and determine the percentage increase in pollution level at the school. Which of the two proposed locations would have the lowest impact on the school under the assumed conditions?

Turn in: versions of `linearcd2D.cpp` and `program11.cpp` used for this problem and responses to (b),(c).