M368K
Homework #3

Burden and Faires.

Section 8.1 (#4[1], 6de, 14[2]).   Section 8.2 (#4f[3], 11[4], 12d[4]).

[1] Consider degree 2 only.

[2] Assuming x_k are distinct, show that $A \in \mathbb{R}^{(n+1)\times(n+1)}$ is actually symmetric, positive-definite for any $0 \le n \le m-1$. To do the latter, establish the identity $\alpha^T A\alpha = \sum_{k=1}^m [P_n(x_k)]^2 \ge 0$, where $P_n(x)$ is the polynomial of degree n defined by the coefficient vector α, and explain why $\alpha^T A\alpha > 0$ for any $\alpha \ne 0$ when $n \le m-1$. What happens when $n > m-1$?

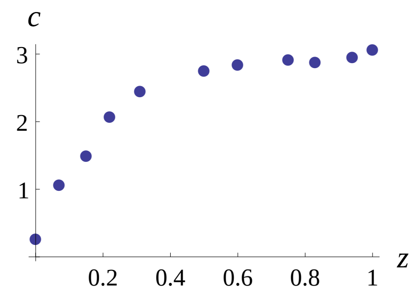
[3] Refer to Exercise 2 (not 3).

[4] For Exercise 11 find L_1 and L_2 only, and for Exercise 12 find the least-squares polynomial of degree 2 only.

Programming mini-project.

The concentration c of algae at depth z is measured near the center of a lake yielding data as shown below:

z	c
0	0.2817
0.07	1.0845
0.15	1.5160
0.22	2.0952
0.31	2.4731
0.50	2.7769
0.60	2.8647
0.75	2.9361
0.83	2.8972
0.94	2.9753
1.00	3.0817



Here we fit the data with various low-degree polynomials and identify an appropriate degree.

(a) Download the C++ program file `program3.cpp` and the function file `gauss_elim.cpp` from the course webpage. Complete the program file so that it builds and solves (using Gauss Elimination) the normal equations for the least-squares polynomial $P_n(x) = a_n x^n + \cdots + a_0$ for any given data (x_k, y_k) $(k = 1, \ldots, m)$ and degree $0 \le n \le m-1$. Recall that the normal equations take the form $A\alpha = b$ where

$$A = \begin{bmatrix} \sum_k 1 & \sum_k x_k & \cdots & \sum_k x_k^n \\ \sum_k x_k & \sum_k x_k^2 & \cdots & \sum_k x_k^{n+1} \\ \vdots & & & \\ \sum_k x_k^n & \sum_k x_k^{n+1} & \cdots & \sum_k x_k^{2n} \end{bmatrix}, \quad \alpha = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}, \quad b = \begin{bmatrix} \sum_k y_k \\ \sum_k y_k x_k \\ \vdots \\ \sum_k y_k x_k^n \end{bmatrix}.$$

(b) Find the least-squares polynomial of degree $n = 1, 2, 3$ for the above data. For each n, report the polynomial $P_n(x)$ and the fitting error $E_n = \sum_k [y_k - P_n(x_k)]^2$. Does E_n decrease as n is increased? Briefly explain why this should happen. One simple way to identify an appropriate fitting degree is to increase n until the relative change $|E_n - E_{n-1}|/E_{n-1}$ is moderately small, say $< 5\%$. Use this criterion to find an appropriate degree. Make a single plot of the data and all polynomials up to the fitting degree to illustrate your results.

Turn in: completed copy of `program3.cpp` and response to (b).