MATH2017 Problem Set 1 Solutions:
Limits of functions, differentiability

1. (a) $\exists K \in \mathbb{R}, \forall x \in D, x \leq K$.

   (b) $\exists y \in D, \forall x \in D, x \leq y$.

   (c) $\forall y \in D, \exists x \in D, x > y$.

2. I claim that $a_n \to 3/2$. Proof: Let $\varepsilon > 0$ be given. Then there exists $N \in \mathbb{Z}^+$ such that $N > 5/(2\varepsilon)$. Now, for all $n \geq N$,

$$|a_n - \frac{3}{2}| = \frac{5}{4n-2} \leq \frac{5}{4n-2n} = \frac{5}{2n} \leq \frac{5}{2N} < \varepsilon.$$

3. For all $m > n \geq 2$,

$$\begin{aligned}
|a_m - a_n| &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots + \frac{1}{m!} \\
&= \frac{1}{(n+1)!}\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \cdots + \frac{1}{(n+2)(n+3)\cdots m}\right) \\
&< \frac{1}{(n+1)!}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{m-1}}\right) \\
&\leq \frac{2}{(n+1)!} < \frac{2}{n}.
\end{aligned}$$

   So, let $\varepsilon > 0$ be given. Then choose any $N \in \mathbb{Z}^+$ greater than both 1 and $2/\varepsilon$. Then, for all $m > n \geq N$,

$$|a_m - a_n| < \frac{2}{n} \leq \frac{2}{N} < \varepsilon.$$

   Hence (a_n) is Cauchy.

4. Let $\varepsilon > 0$ be given. Then take $\delta = 4\sqrt{2}\varepsilon > 0$. Then, for all $x \in [0, \infty)$ such that $0 < |x-2| < \delta$,

$$\begin{aligned}
\left|\frac{\sqrt{x}-\sqrt{2}}{x-2} - \frac{1}{2\sqrt{2}}\right| &= \left|\frac{(\sqrt{x}-\sqrt{2})(\sqrt{x}+\sqrt{2})}{(x-2)(\sqrt{x}+\sqrt{2})} - \frac{1}{2\sqrt{2}}\right| = \left|\frac{1}{\sqrt{x}+\sqrt{2}} - \frac{1}{2\sqrt{2}}\right| \\
&= \left|\frac{\sqrt{x}-\sqrt{2}}{2\sqrt{2}(\sqrt{x}+\sqrt{2})}\right| = \frac{1}{2\sqrt{2}}\left|\frac{x-2}{(\sqrt{x}+\sqrt{2})^2}\right| \\
&< \frac{1}{4\sqrt{2}}|x-2| < \frac{1}{4\sqrt{2}}\delta = \varepsilon.
\end{aligned}$$

   This means that the function $f : [0, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$ is differentiable at 2, with derivative $\frac{1}{2\sqrt{2}}$.

5. We must show that

$$\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$

exists. I claim that it does, and equals 4. Proof: Given any $\varepsilon > 0$, let

$$\delta = \min\{1, \varepsilon/11\}.$$

Then for all $x \in \mathbb{R}$ with $0 < |x-1| < \delta$,

$$\begin{aligned}
\left|\frac{f(x)-f(1)}{x-1} - 4\right| &= \left|\frac{x^4-1}{x-1} - 4\right| \\
&= \left|(x+1)(x^2+1) - 4\right| \\
&= |x^3 + x^2 + x - 3| = |(x-1)(x^2+2x+3)| \\
&\leq |x-1|(x^2 + 2|x| + 3) \qquad \text{by the triangle inequality} \\
&\leq |x-1|(4+4+3) \qquad \text{since } |x-1| < 1, \text{ so } |x| < 2 \\
&< 11\delta \leq \varepsilon.
\end{aligned}$$