## MATH2017 Problem Set 2: Differentiability on an interval

*Submit on Gradescope by 17:00, Monday 6 March 2023*

1. Prove that, for all $x \in \mathbb{R}$, $|\sin x| \leq |x|$.
2. Prove that the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 4x^7 - 14x^4 + 30x - 17$, is injective.
3. Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 2x^4 + x^4 \sin(1/x)$ if $x \neq 0$, and $f(0) = 0$.
   (a) Prove that $f$ attains a minimum at 0. *(Hint: no need for calculus!)*
   (b) Prove that $f$ is differentiable (everywhere), and that $f'(0) = 0$. *(Hint: you can reduce the work involved using the Localization Lemma, Lemma 3.23.)*
   (c) Prove that, for all $\varepsilon > 0$, $f'$ takes both positive and negative values on $(-\varepsilon, \varepsilon)$.
4. In each of the following cases, either write down a function with the specified properties, or explain why no such function exists.
   (a) An unbounded function $f : [0, 1] \to \mathbb{R}$.
   (b) An unbounded differentiable function $f : [0, 1] \to \mathbb{R}$.
   (c) A bounded differentiable function $f : \mathbb{R} \to \mathbb{R}$ whose derivative $f' : \mathbb{R} \to \mathbb{R}$ is unbounded.
   (d) A differentiable function $f : (0, 1) \to \mathbb{R}$ which is unbounded above but whose derivative is bounded.