The L^2 geometry of the moduli space of vortices on the two-sphere in the dissolving limit

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Rene $\longrightarrow$ 

Vortices on the sphere

- Hermitian line bundle L over $\Sigma = (S^2, g_\Sigma)$, degree n

$$E(\phi, A) = \frac{1}{2}\|\mathrm{d}_A\phi\|^2_{L^2} + \frac{1}{2}\|F_A\|^2_{L^2} + \frac{1}{8}\|\tau - |\phi|^2\|^2_{L^2}$$

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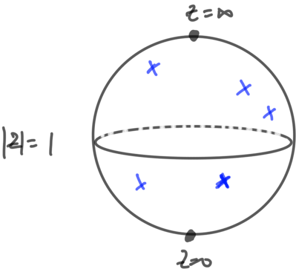
- $\varepsilon > 0$: $[(\phi, A)]$ uniquely determined by **divisor** (ϕ)



$$\leftrightarrow p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$$
$$\leftrightarrow [a_0, a_1, \ldots, a_n]$$

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- Baptista-Manton conjecture: $\lim_{\varepsilon \to 0} g_\varepsilon =$ Fubini-Study metric

The conjecture

- Equip L with hol structure $\overline{\partial}_L = \overline{\partial}_{\widehat{A}}$

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$$\pi : S \to (\mathbb{P}(H^0(L)), g_{FS}), \qquad \widehat{\phi} \mapsto \{c\widehat{\phi}\}$$

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- Baptista-Manton conjecture: $\lim\limits_{\varepsilon\to 0} g_\varepsilon = g_0$
- Surprising? Massive gain in symmetry

The theorem

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More precisely:

There exists $C > 0$ such that, for all $v \in TM_n$ and all $\varepsilon \in (0, 1)$

$$|g_\varepsilon(v, v) - g_0(v, v)| \leq C\varepsilon g_0(v, v)$$

The proof

- Given divisor D, exists $\widehat{\phi} \in S \subset H^0(L)$ with $(\widehat{\phi}) = D$
  Unique up to $\widehat{\phi} \mapsto e^{ic}\widehat{\phi}$

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- Energy estimate, elliptic estimate, Sobolev $\Rightarrow$

$$\|u\|_{C^0} \leq C\varepsilon.$$

  Vortices are uniformly well approximated by pseudovortices (for small ε)

Unpacking that a little. . .

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- Vortices are uniformly well approximated by pseudovortices (for small ε)

Unpacking that a little...

- So... $\|u\|_{C^0} \leq C$
- But

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- Vortices are uniformly well approximated by pseudovortices (for small ε)
- Now we need to estimate the *metric*

The proof (cont)

- Take a curve of vortex solutions

$$(\phi(t), A(t)) = (\sqrt{\varepsilon}\widehat{\phi}(t)e^{u(t)/2}, \widehat{A} - \frac{1}{2} * \mathrm{d}u(t))$$

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- "SEE for $\Delta + \varepsilon|\widehat{\phi}|^2 e^u$" $\Rightarrow \|\dot{u}\|_{H^2} \leq C\varepsilon\|\dot{\widehat{\phi}}\|_{L^2}$
- Suffices to get bound on $|g_\varepsilon - g_0|$.

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$$0 = \lambda_0(g) < \lambda_1(g) \leq \lambda_2(g) \leq \lambda_3(g) \leq \cdots$$

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- **Corollary** (JMS,RGL): There exists $C, \varepsilon_* > 0$ such that, for all $k \in \mathbb{Z}^+$ and all $\varepsilon \in (0, \varepsilon_*)$,

$$\left|\frac{\lambda_k(g_\varepsilon)}{\lambda_k(g_0)} - 1\right| \leq C\varepsilon$$

- Spectrum of M_n converges uniformly to spectrum of FS!
- Surprising this follows from only C^0 convergence!

Open questions

- Urakawa-Bando (1983): for any finite dimensional subspace $V \subset C^\infty(M)$

$$\Lambda_g(V) := \sup \left\{ \frac{\|\mathrm{d}f\|_{L^2}^2}{\|f\|_{L^2}^2} : f \in V \backslash \{0\} \right\}.$$

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- Corollary easily follows

Open questions

- Convergence of geodesics? Need $g_\varepsilon \to g_0$ in C^1
- Convergence of curvature? Need $g_\varepsilon \to g_0$ in C^2
- n-dependence of the bounds?
- Leading correction to g_0?
- Higher genus? Much more subtle (Manton, Romao)