Workshop 7: questions for week 8

1. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 \cos x$. This is continuous, so the associated function $g : \mathbb{R} \to \mathbb{R}$,
$$g(x) = \int_0^x f(t)dt$$
is well defined. Compute $g'(\pi)$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be the solution of the differential equation
$$f''(x) = \sin f(x)$$
with initial data $f(0) = 0$, $f'(0) = 1$. You may assume this function exists and is unique.

   (a) Prove that f is strictly increasing. (*Hint: consider the related function* $g(x) = f'(x)^2 + 2\cos f(x)$.)

   (b) Prove that there exists a unique point $b \in (0, \infty)$ such that $f(b) = \pi$.

   (c) Compute
   $$\int_0^b \sqrt{3 - 2\cos f(x)}dx.$$

3. Let $f : \mathbb{R} \to \mathbb{R}$ be Riemann integrable on every closed bounded interval, and define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = \int_0^x f$.

   The First Form of the FTC (Theorem 6.1) implies that **if** f is continuous, then g is differentiable.

   Does the converse hold? If g is differentiable, does it follow that f is continuous?