Workshop 6: questions for week 7

1. Prove that the function

$$f : [0,1] \to \mathbb{R}, \qquad f(x) = \begin{cases} 0, & x = 0 \\ 1, & 0 < x < 1 \\ 0, & x = 1 \end{cases}$$

   is Riemann integrable.

2. Write down (or draw the graph of) a function $f : [a,b] \to \mathbb{R}$ and a pair of dissections $\mathscr{D}, \mathscr{D}'$ of $[a,b]$ such that $\mathscr{D}'$ is a refinement of $\mathscr{D}$, $u_{\mathscr{D}'}(f) < u_{\mathscr{D}}(f)$, but $l_{\mathscr{D}'}(f)$ is not greater than $l_{\mathscr{D}}(f)$.

3. Let $f, g : [a,b] \to \mathbb{R}$ be bounded functions and $\mathscr{D}$ be a dissection of $[a,b]$. Prove that

$$u_{\mathscr{D}}(f+g) \leq u_{\mathscr{D}}(f) + u_{\mathscr{D}}(g)$$

   and

$$l_{\mathscr{D}}(f-g) \geq l_{\mathscr{D}}(f) - u_{\mathscr{D}}(g).$$