Chapter 6

The Fundamental Theorem of the Calculus

So far we have developed some powerful theoretical tools to show that a given function is integrable and to establish relationships between the integrals of related integrable functions, but we don't have any really convenient techniques for actually computing Riemann integrals. To compute

$$\int_0^1 x^2\,dx,$$

for example, we had to resort to exhibiting a sequence of dissections whose upper and lower sums converge to a common limit, in this case, $\frac{1}{3}$ (see Example 5.15). Theorem 5.16 tells us that, in principle, we can always compute Riemann integrals like this, but in practice this method of computing integrals is very onerous and, unless the integrand (the function to be integrated) is fairly simple, is likely to be intractable. Consider, for example, attempting to compute

$$\int_0^\pi \sin x\,dx$$

using this method. We know that this integral exists (Theorem 5.20) because we know that sin is continuous, but we have no hope of computing it directly using Riemann sums. In this chapter, we establish a fundamental connexion between Riemann integration and differentiation which will give us a convenient means of computing $\int_a^b f$ whenever we can dream up a function whose *derivative* equals f. Once we have done this we will rarely have to resort to computing sequences of Riemann sums to compute integrals.

6.1 The first form

It's important to realize that the Riemann integral of a function $f : [a,b] \to \mathbb{R}$ is a single *number* $\int_a^b f$, not a *function*. If $f : \mathbb{R} \to \mathbb{R}$ is continuous, it is Riemann integrable on any interval $[a,b] \subset \mathbb{R}$. If we allow (say) the right endpoint of the interval $[a,b]$ to vary while keeping a fixed, we can think of $\int_a^b f$ as a function of b. It's natural to ask what analytic properties this new function has. The (first form of) the Fundamental Theorem of the Calculus says that $b \mapsto \int_a^b f$ is differentiable, and its derivative is f.

Theorem 6.1 (Fundamental Theorem of the Calculus version 1) *Let $f : I \to \mathbb{R}$ be continuous, where $I \subseteq \mathbb{R}$ is an interval. Choose $a \in I$ and define $F : I \to \mathbb{R}$ by*

$$F(x) = \int_a^x f.$$

Then F is differentiable, and $F' = f$.

Proof: First note that f is continuous on $[a, x] \subseteq I$ (or $[x, a]$ if $x < a$) for all $x \in [a, b]$, so F is well-defined by Theorem 5.20. We wish to compute

$$\lim_{y \to x} \frac{F(y) - F(x)}{y - x}.$$

We will do this using the sequential criterion for limits (Theorem 2.14). So, let (y_n) be any sequence in $I \backslash \{x\}$ such that $y_n \to x$, and

$$s_n = \frac{F(y_n) - F(x)}{y_n - x}.$$

We must show that $s_n \to f(x)$.

For each n, either $y_n > x$ or $y_n < x$. If $y_n > x$ then, by Theorem 5.26,

$$s_n = \frac{1}{y_n - x} \int_x^{y_n} f,$$

whereas if $y_n < x$,

$$s_n = \frac{1}{x - y_n} \int_{y_n}^x f.$$

In either case, by the Extreme Value Theorem, there exist w_n and z_n between x and y_n such that $f(w_n)$ is the minimum value of f on the closed interval with endpoints x and y_n, and $f(z_n)$ is the maximum value of f on this interval. Hence, by Proposition 5.8,

$$\begin{aligned} & \frac{1}{y_n - x} f(w_n)(y_n - x) \leq s_n \leq \frac{1}{y_n - x} f(z_n)(y_n - x) \qquad \text{if } y_n > x, \\ \text{and} \quad & \frac{1}{x - y_n} f(w_n)(x - y_n) \leq s_n \leq \frac{1}{x - y_n} f(z_n)(x - y_n) \qquad \text{if } y_n < x. \end{aligned}$$

In either case, we conclude that

$$f(w_n) \leq s_n \leq f(z_n).$$

Now, $y_n \to x$ so $w_n \to x$ and $z_n \to x$ also, by the Squeeze Rule, and f is continuous, so $f(w_n) \to f(x)$ and $f(z_n) \to f(x)$. Hence, by the Squeeze Rule again, $s_n \to f(x)$, which completes the proof. $\square$

This theorem tells us something interesting and far from obvious: any function which is continuous on an interval is the derivative of some differentiable function on that interval. Note that the converse of this is **false**: the derivative of a function may fail to be a continuous function (see Example 4.13 for a counterexample).

Theorem 6.1 says that (provided f is continuous!) $\int_a^b f$ is a differentiable function of the upper integration limit b, and its derivative is f. What if we fix b and think of it as a function of a? Is this function also differentiable? If so, what is *its* derivative?

Corollary 6.2 *Let $f : I \to \mathbb{R}$ be continuous, where $I \subseteq \mathbb{R}$ is an interval. Choose $b \in I$ and define $F : I \to \mathbb{R}$ by*

$$F(x) = \int_x^b f.$$

Then F is differentiable, and $F' = -f$.

Proof: Choose any $a \in I$ and define $G : I \to \mathbb{R}$, $G(x) = \int_a^x f$. Then by Proposition 5.26

$$G(x) + F(x) = \int_a^x f + \int_x^b f = \int_a^b f = C$$

a constant. G is differentiable and $G' = f$ by Theorem 6.1, so $F(x) = C - G(x)$ is differentiable and $F'(x) = 0 - F'(x) = -f(x)$. $\square$

Of course, we can apply all the usual rules of differentiation to functions defined in this way (as integrals of other functions).

Exercise 6.3 Let $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \int_x^{x^2+1} \frac{t}{t^2+1} dt.$$

Compute $f'(x)$.

□

6.2 The second form and a practical method for computing integrals

Theorem 6.1 immediately implies a second theorem, also called the Fundamental Theorem of the Calculus, which renders the job of computing many Riemann integrals almost trivial:

Theorem 6.4 (Fundamental Theorem of the Calculus version 2) *Let $f : [a, b] \to \mathbb{R}$ be continuous and $F : [a, b] \to \mathbb{R}$ be any differentiable function such that $F' = f$. Then*

$$\int_a^b f = F(b) - F(a).$$

Proof: Define the function $g : [a, b] \to \mathbb{R}$,

$$g(x) = \left(\int_a^x f\right) - F(x).$$

Then, by Theorem 6.1 and the definition of F,

$$g'(x) = f(x) - f(x) = 0$$

for all $x \in [a, b]$. Hence, by Proposition 4.11, g is constant, so $g(b) = g(a)$, that is

$$\left(\int_a^b f\right) - F(b) = \left(\int_a^a f\right) - F(a) = 0 - F(a),$$

and the result immediately follows. □

Example 6.5 Compute the Riemann integrals

(i) $\displaystyle\int_{-1}^{1} x^2\,dx,$

(ii) $\displaystyle\int_{0}^{\pi} \sin x\,dx,$

Solution (i) Let $f(x) = x^2$ and $F(x) = x^3/3$. Then f is continuous and $F' = f$ on $[-1, 1]$ so, by Theorem 6.4,

$$\int_{-1}^{1} f = F(1) - F(-1) = \frac{2}{3}.$$

(ii) Let $f(x) = \sin x$ and $F(x) = -\cos x$. Then f is continuous and $F' = f$ on $[0, \pi]$ so, by Theorem 6.4,

$$\int_{0}^{\pi} f = F(\pi) - F(0) = -\cos\pi + \cos 0 = 2.$$

Having established Theorem 6.4, we see that we can explicitly compute

$$\int_a^b f(x)\,dx$$

if we can think of an **antiderivative** of f, that is, any function whose *derivative* is $f(x)$. This trick is so pervasive in integral calculus that it leads the unwary to identify *integration* of f (that is, the process of computing a Riemann integral $\int_a^b f$) with the process of writing down an antiderivative of f. Indeed, it is common practice to call an antiderivative of f an *indefinite integral* of f and to denote any such function by the symbol

$$\int f(x)\,dx.$$

This notation has some practical advantages, but it also, unfortunately, generates a huge amount of confusion. Note that

$$\int_a^b f(x)\,dx$$

is a single number and that its *definition* has nothing whatsoever to do with antiderivatives of f: it is the unique real number which is no greater than any upper Riemann sum of f on $[a, b]$ and no less than any lower Riemann sum of f on $[a, b]$. In particular, it is *not* a function of the "variable" x. Indeed, we could equally well have written it $\int_a^b f(y)\,dy$ or $\int_a^b f(\Gamma_{\widehat{\aleph}})\,d\Gamma_{\widehat{\aleph}}$, which is one reason to prefer the simpler notation $\int_a^b f$. It *is* a function of a and b, however.

By contrast, $\int f(x)\,dx$ is just a (slightly ambiguous) symbol denoting any function whose derivative (at x) is $f(x)$. It *is* a function of x, not a single number, and its *definition* has nothing to do with Riemann sums. That these two things turn out to be closely related is (as its name suggests) a very important theorem: the Fundamental Theorem of the Calculus. To understand calculus properly it is important to maintain a clear conceptual distinction between the Riemann integral $\int_a^b f$ and any antiderivative that one might use to compute it.

6.3 The natural logarithm

Definition 6.6 The **(natural) logarithm** is the function

$$\ln : (0,\infty) \to \mathbb{R}, \qquad \ln x = \int_1^x \frac{1}{t}dt.$$

Remarks

- The function $f(t) = 1/t$ is continuous on $(0,\infty)$, and hence Riemann integrable on $[1,x]$ (if $x \geq 1$) or $[x,1]$ (if $0 < x < 1$), so the function ln is well-defined (Theorem 5.20).
- ln is differentiable, by the Fundamental Theorem of the Calculus version 1 (Theorem 6.1), and
$$\ln'(x) = \frac{1}{x}.$$
- It follows immediately from the definition that
$$\ln 1 = \int_1^1 \frac{1}{t}dt = 0.$$

The logarithm function obeys a very useful identity.

Proposition 6.7 *For all $x, y \in (0,\infty)$, $\ln(xy) = \ln x + \ln y$.*

Proof: Choose and fix $y \in (0,\infty)$ and consider the function

$$f : (0,\infty) \to \mathbb{R}, \qquad f(x) = \ln(xy) - \ln x - \ln y.$$

By the Chain Rule,
$$f'(x) = \frac{y}{xy} - \frac{1}{x} = 0$$
so f is constant (Proposition 4.11). Hence, for all x, $f(x) = f(1) = \ln y - 0 - \ln y = 0$.
□

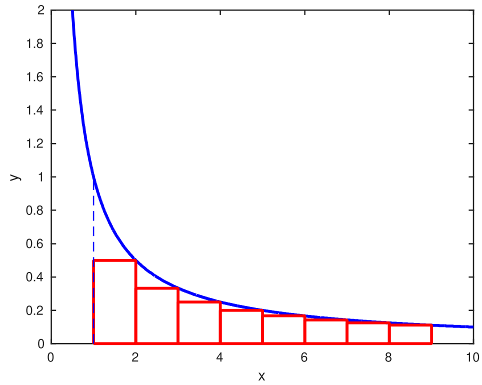
A second useful identity quickly follows from this.

Proposition 6.8 *For all $x \in (0,\infty)$ and $n \in \mathbb{Z}$, $\ln x^n = n \ln x$.*

Proof: Exercise. (*Hint*: proof by induction.) □

Proposition 6.9 *The function $\ln : (0,\infty) \to \mathbb{R}$ is smooth, strictly increasing, and bijective.*

Proof: Let $f : (0,\infty) \to \mathbb{R}$, $f(x) = 1/x$. We have noted that ln is differentiable with derivative f. But f is smooth, so ln is smooth. Also, for all $x \in (0,\infty)$, $f(x) > 0$, so ln is strictly increasing, and hence injective (Proposition 4.11). It remains to show that ln is surjective.



For each $n \in \mathbb{Z}^+$, $n \geq 2$, let $\mathscr{D}_n$ be the regular dissection of $[1, n]$ of size $(n-1)$, that is

$$\mathscr{D}_n = \{1, 2, \ldots, n\}.$$

Then

$$\ln n = \int_1^n f \quad \geq \quad l_{\mathscr{D}_n}(f) = \sum_{j=1}^{n-1} (1) f(j+1) = \sum_{k=2}^{n} \frac{1}{k} =: s_n,$$

where we have used the fact that f is monotonically decreasing (the case $n = 8$ is depicted above). The sequence (s_n) is unbounded above (Example 1.22), so the sequence $\ln n$ is also unbounded above. Hence, given any $K \geq 0$, there exists $n \in \mathbb{Z}^+$ such that $\ln n > K$. But $\ln 1 = 0$, and ln is continuous (since it is differentiable) so, by the Intermediate Value Theorem, there exists $x \in [1, n]$ such that $\ln x = K$. Hence, ln takes all non-negative values. Let $L < 0$. As we just showed, there exists $x \in [1, \infty)$ such that $\ln x = -L$. But then, by Proposition 6.8, $\ln(1/x) = -\ln x = L$. So ln also takes all negative values, and we conclude that $\ln : (0, \infty) \to \mathbb{R}$ is surjective, hence bijective. $\square$

It is common to define *natural logarithm* to mean the inverse function to the exponential function. Definition 6.6 has no obvious connexion with the exponential function (which as we will see, is defined using a convergent power series), so this coincidence of terminology will need to be justified.

Summary

- There is an important link between Riemann integrals and derivatives, given by the **Fundamental Theorem of the Calculus**:
  - Version 1: if f is continuous and $F(x) = \int_a^x f$, then $F' = f$.
  - Version 2: if f is continuous and F is some antiderivative of f (that is, a function satisfying $F' = f$), then $\int_a^b f = F(b) - F(a)$.
- Version 2 of the Fundamental Theorem of the Calculus provides a flexible and convenient method for computing many Riemann integrals.
- The **natural logarithm** function is
$$\ln : (0, \infty) \to \mathbb{R}, \qquad \ln x = \int_1^x \frac{1}{t} dt.$$
- ln is smooth, increasing, and bijective.
- For all $x, y \in (0, \infty)$,
$$\ln(xy) = \ln x + \ln y.$$
  We can prove this using version 1 of the FTC.