L^2 geometry of vortices

degree n at (L,h)
$\downarrow$
(Σ, g_Σ)

$$E(\varphi, A) = \frac{1}{2}\|d_A\varphi\|^2_{L^2} + \frac{1}{2}\|F_A\|^2_{L^2} + \frac{1}{2}\left\|\frac{1}{2}(\tau - |\varphi|^2)\right\|^2_{L^2}$$

vortices = minimizers of E.

EL eqns:
$$\Delta_A\varphi - \frac{1}{2}(\tau - |\varphi|^2)\varphi = 0$$

$$\delta F_A = h(i\varphi, d_A\varphi)$$
" curl B = j "

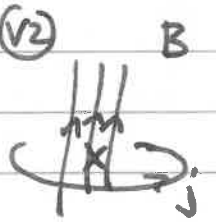
Bogomol'nyi bound (1976)

$$E = \frac{1}{2}\left\| F_A - \frac{1}{2}(\tau - |\varphi|^2)\omega_\Sigma\right\|^2_{L^2} + \|\bar\partial_A\varphi\|^2_{L^2} + \frac{\tau}{2}\int_\Sigma F_A$$

(because $\langle \varphi\omega_\Sigma, F_A\varphi\rangle_{L^2} \equiv \|\partial_A\varphi\|^2_{L^2} - \|\bar\partial_A\varphi\|^2_{L^2}$)

So $E \geq \tau\pi n$ with equality $\iff$

$$\bar\partial_A\varphi = 0 \quad \text{— (V1)}$$
$$\& \; F_A = \frac{1}{2}(\tau - |\varphi|^2) \quad \text{— (V2)}$$



"Vortices" located at zeroes of φ:
[DEFINE MODULI SPACE]

Bradlow bound (1990)

$\int_\Sigma$ (V2) :
$$2\pi n = \frac{\tau}{2}|\Sigma| - \frac{1}{2}\|\varphi\|^2_{L^2}$$

i.e. $\|\varphi\|^2_{L^2} = \tau|\Sigma| - 4\pi n =: \varepsilon \geq 0.$

So if $\varepsilon < 0$, (V1), (V2) have no solutions. $\boxed{M_n = \emptyset}$

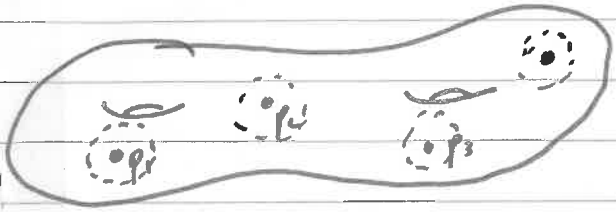
$\boxed{\text{"Moduli space": } M_n = \{\text{solutions of (V1), (V2)}\}/\text{Gauge transformations}}$

$\varepsilon = 0$: $\quad \varphi = 0, \quad *F_A = \text{const} \quad M_n \equiv T^{2g}$

<u>Theorem</u> (Bradlow 1990, Garcia-Prada 1991) If $\varepsilon > 0$,

$$M_n \equiv \mathrm{Div}_n(\Sigma) = S^n\Sigma = (\Sigma^n)/S_n.$$

"Proof":



$D = n_1 p_1 + n_2 p_2 + \dots + n_k p_k$

$\Rightarrow$ holomorphic structure on L

$\tau_{j0} = z^{n_j}$

and holomorphic section φ_0 with $\varphi_0^{-1}(0) = D$:

$$\varphi_0 = 1 \text{ on } U_0, \quad \varphi_0 = z^{n_j} \text{ on } U_j$$

$A_0 =$ Chern connection on (L_D, h). $\quad u : \Sigma \to \mathbb{R}$ smooth

$$\varphi = e^{u/2}\varphi_0, \quad A = A_0 - \tfrac{1}{2} * du \quad \text{satisfies (V1)}$$

$$\text{(V2)} \iff \Delta u + (2 * F_{A_0} - \tau) + |\varphi_0|^2 e^u = 0$$

c.f. Kazdan-Warner 1974

$\exists!$ smooth solution. $\square$

M_n is <u>smooth</u>! $\quad U \subset \Sigma = \bar{U}$

$$D \iff p(z) = (z - z_1)(z - z_2)\cdots(z - z_n) = z^n + a_1 z^{n-1} + \dots + a_n$$

local coords on $S^n\Sigma$

<u>Dynamics</u>: $\Sigma \to \mathbb{R} \times \Sigma, \quad g_\Sigma \to dt^2 - g_\Sigma$

$EL \to$ coupled nonlinear <u>wave</u> equations.

Cauchy problem $\Rightarrow$ dynamics.

Adiabatic approximation (Manton 1982)

$(\varphi(0), A(0)) \in M_n$, $(\dot\varphi(0), \dot A(0)) \in T_{(\varphi(0),A(0))} M_n$ "small"

$\Rightarrow$ $(\varphi(t), A(t))$ is approximately the geodesic in M_n w.r.t. L^2 metric

Consider curve $(\varphi(t), A(t)) \in V_n$ $\quad (M_n = V_n/\mathcal{G})$
project $(\dot\varphi(0), \dot A(0))$ $\perp_{L^2}$ $\mathcal{G}$ orbit through $(\varphi(0), A(0))$
Then $\quad g_{L^2}(v,v) = \| P(\dot\varphi(0), \dot A(0)) \|_{L^2}^2$.

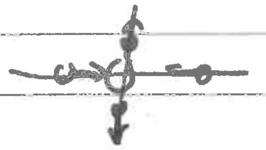
Stuart (1994) proved pointwise convergence in small velocity limit $(\Sigma = \mathbb{C}\,!)$

Motivates study of g_{L^2}

Strachan (1992) Explicit formula for M_2, $\Sigma = \mathbb{H}$

Samols (1992) Careful numerics for M_2, $\Sigma = \mathbb{C}$

$z^2 - t$



~~Both~~ All used localization formula... $\pi : \Sigma^n \longrightarrow S^n\Sigma$

$$|\varphi(z)|^2 = \log|z - z_i|^2 + a_i + \tfrac{1}{2} b_i (z - z_i) + \text{c.c.} + \dots$$

$$b = \sum_i b_i \, dz_i$$

$$\pi^* \omega_{L^2} = \tau\pi(\omega_{\Sigma_1} + \omega_{\Sigma_2} + \dots + \omega_{\Sigma_n}) - i\bar\partial b$$

Manton–Nasir (1999) Computed $|M_n|$.

Point particle limit $\tau \to \infty$ (Nagy, 2017)

On the complement of any nbhd of $\Delta_{fat} \subset S^n\Sigma$,

$$\frac{g_{L^2}}{\tau\pi} \xrightarrow{C^0} \text{product metric.}$$

Dissolving limit $\varepsilon \to 0$ $\left(\tau \searrow \frac{4\pi n}{|\Sigma|} \right)$

Pseudovortex (Manton, Baptista 2003)

Given $D \in \mathrm{Div}_n(\Sigma)$ choose $\hat{A}$ s.t. $\bar\partial_{\hat{A}} = \bar\partial_{L_D}$

and $* F_{\hat{A}} = \frac{2\pi n}{|\Sigma|}$

$\exists!$ line in $H^0(\Sigma, L_D)$ s.t. $\hat\varphi^{-1}(0) = D$. Normalise s.t. $\|\hat\varphi\|_{L^2} = 1$.

Pseudovortex with divisor $D = (\sqrt{\varepsilon}\,\hat\varphi, \hat{A})$.

Satisfies (V1): $\bar\partial_{\hat{A}} \hat\varphi = 0$

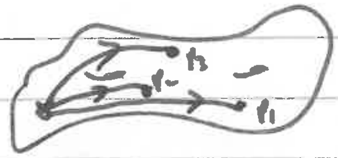
$\int_\Sigma$ (V2): $\int_\Sigma * F_{\hat{A}} = \int_\Sigma \frac{1}{2}(\tau - |\varphi|^2)$

Conjecture: for $\varepsilon > 0$ small, pseudovortices are "a good approximation" to vortices

In particular $g_{L^2} \approx g_{L^2}^{\text{Pseudo}}$.

Fibration $M_n = S^n\Sigma \longrightarrow \mathrm{Pic}_n(\Sigma)$

$[(\varphi, A)] \longmapsto [\bar\partial_A]$



AJ: $S^n\Sigma \longrightarrow \mathbb{C}^g / \Lambda_{Abel}$

$\{p_1, \dots, p_n\} \longmapsto \left(\int_{p_0}^{p_1} v_1 + \int_{p_0}^{p_2} v_1 + \dots + \int_{p_0}^{p_n} v_1 \right.$

$$F_{[D]} = AJ^{-1}(v) \cong \mathbb{P}H^0(\Sigma, L_{[D]}) \quad \leftarrow \text{has a canonical Fubini–Study metric from } L^2.$$

<u>Theorem</u> (JMS, Chaudhuri, Harland, 2027)

$\exists\, C > 0$ (depending on $[D]$ and g_Σ) s.t. $\forall D \in [D]$,

$$\left\| \frac{1}{\sqrt{\varepsilon}}|\varphi| - |\hat{\varphi}| \right\|_{C^0} + \left\| F_A - F_{\hat{A}} \right\|_{C^0} \leq C\varepsilon.$$

Furthermore,

$$\left\| \frac{1}{\varepsilon}\, g_{L^2}\Big|_{F_{[D]}} - g_{FS} \right\|_{C^1} \leq C\varepsilon.$$

Induced metric on fibres of AJ converges in C^1 to g_{FS} as $\varepsilon \searrow 0$. [IDEA OF PROOF: (*)]

Two situations in which this is useful:

(1) $\Sigma = S^2$ (i.e. $g=0$). Then $\mathrm{Pic}_n(\Sigma) = \{pt\}$

$$M_n \equiv \text{Fibre} \equiv \mathbb{P}^n$$

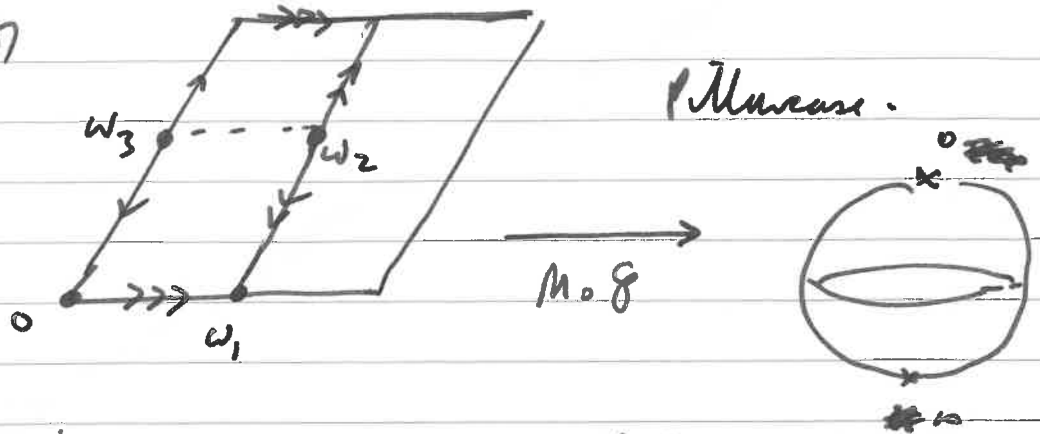
(Special case proved by JMS, García-Lara 2023?)

(2) $\Sigma = T^2 = \mathbb{C}/\Lambda$, $n = 2$.

$$AJ : S^2\Sigma \to \Sigma \qquad \{z_1, z_2\} \mapsto z_1 + z_2$$

All fibres are isometric and totally geodesic (w.r.t. g_{L^2}!)

$\Delta J^{-1}(0)$



Symmetries $z \mapsto z + \omega_i$ determine M (up to $SU(2,1)$)

(*) $\quad \varphi = \sqrt{\varepsilon}\, e^{u/2} \hat{\varphi} \quad , \quad A = \hat{A} - \frac{1}{2} * du.$

(v1) $\quad \bar{\partial}_A \varphi = 0 \quad \checkmark$

(v2) $\quad \Delta u - \frac{\varepsilon}{|\Sigma|} + \varepsilon |\hat{\varphi}|^2 e^u = 0$

Part (i): show $\|u\|_{C^0} \le C\varepsilon$.

Part (ii) $\quad D(i) \in [\boxed{D}] \Rightarrow \hat{A}$ constant in (t).

~~show~~ show $\dot{u} = O(\varepsilon)$

$$\Delta \dot{u} + \varepsilon |\hat{\varphi}|^2 e^u \dot{u} = -2\varepsilon h(\hat{\varphi}, \dot{\hat{\varphi}})$$

Lax-Milgram