Workshop solutions for week 7

1. For each $r \in (0, 1/2)$, let $\mathscr{D}_r = \{0, r, 1-r, 1\}$. Then the upper and lower Riemann sums of f with respect to this dissection are

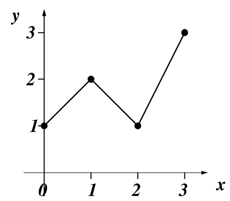
$$\begin{aligned} u_{\mathscr{D}_r}(f) &= 1 \times r + 1 \times (1 - 2r) + 1 \times r = 1 \\ l_{\mathscr{D}_r}(f) &= 0 \times r + 1 \times (1 - 2r) + 0 \times r = 1 - 2r. \end{aligned}$$

Hence the set of all upper Riemann sums contains 1, and so its infimum $u(f) \leq 1$. Further, the set of all lower Riemann sums contains

$$\{1 - 2r : r \in (0, 1/2)\} = (0, 1)$$

so its supremum $l(f) \geq 1$. Hence $l(f) \geq u(f)$. But $l(f) \leq u(f)$ by Lemma 5.13. Hence $l(f) = u(f)$, that is, f is Riemann integrable.

2. The piecewise linear function $f : [0, 3] \to \mathbb{R}$ depicted below has the required properties with respect to the dissections $\mathscr{D} = \{0, 3\}$, $\mathscr{D}' = \{0, 2, 3\}$.



We see that

$$\begin{aligned} l_{\mathscr{D}}(f) &= 3 \times 1 = 3 \\ u_{\mathscr{D}}(f) &= 3 \times 3 = 9 \\ l_{\mathscr{D}'}(f) &= 2 \times 1 + 1 \times 1 = 3 \\ u_{\mathscr{D}'}(f) &= 2 \times 2 + 1 \times 3 = 7. \end{aligned}$$

So in this case, passing from $\mathscr{D}$ to its refinement $\mathscr{D}'$ improves the overestimate (the upper sum), but makes no change to the underestimate (the lower sum).

3. As usual let $\mathscr{D} = \{a_0, a_1, \ldots, a_n\}$ and, for any bounded function h on $[a, b]$ define

$$m_j(h) = \inf\{h(x) : x \in [a_{j-1}, a_j]\}, \qquad M_j(h) = \sup\{h(x) : x \in [a_{j-1}, a_j]\}.$$

Then, for all $x \in [a_{j-1}, a_j]$, $m_j(f) \leq f(x) \leq M_j(f)$ and $m_j(g) \leq g(x) \leq M_j(g)$, so $f(x) + g(x) \leq M_j(f) + M_j(g)$ and $f(x) - g(x) \geq m_j(f) - M_j(g)$. Hence, $M_j(f) + M_j(g)$ is an upper bound on $\{f(x) + g(x) : x \in [a_{j-1}, a_j]\}$ and $M_j(f + g)$

is the *least* upper bound on this set, so $M_j(f+g) \leq M_j(f) + M_j(g)$. Hence

$$\begin{aligned} u_{\mathscr{D}}(f+g) &= \sum_{j=1}^{n} M_j(f+g)(a_j - a_{j-1}) \\ &\leq \sum_{j=1}^{n} (M_j(f) + M_j(g))(a_j - a_{j-1}) \\ &= \sum_{j=1}^{n} M_j(f)(a_j - a_{j-1}) + \sum_{j=1}^{n} M_j(g)(a_j - a_{j-1}) \\ &= u_{\mathscr{D}}(f) + u_{\mathscr{D}}(g). \end{aligned}$$

Similarly, $m_j(f) - M_j(g)$ is a lower bound on $\{f(x) - g(x) : x \in [a_{j-1}, a_j]\}$ and $m_j(f-g)$ is the *greatest* lower bound on this set, so $m_j(f-g) \geq m_j(f) - M_j(g)$. Hence

$$\begin{aligned} l_{\mathscr{D}}(f+g) &= \sum_{j=1}^{n} m_j(f-g)(a_j - a_{j-1}) \\ &\geq \sum_{j=1}^{n} (m_j(f) - M_j(g))(a_j - a_{j-1}) \\ &= \sum_{j=1}^{n} m_j(f)(a_j - a_{j-1}) - \sum_{j=1}^{n} M_j(g)(a_j - a_{j-1}) \\ &= l_{\mathscr{D}}(f) - u_{\mathscr{D}}(g). \end{aligned}$$