MATH2017 Problem Set 5:
Power series

1. Compute the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{x^{3n+7}}{2^n + 1}$.

2. (a) Prove that if $\sum_{n=0}^{\infty} \alpha_n$ and $\sum_{n=0}^{\infty} \beta_n$ converge absolutely, so does $\sum_{n=0}^{\infty} \alpha_n \beta_n$.

    (b) Prove that if $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R_1 > 1$ and $\sum_{n=0}^{\infty} b_n x^n$ has radius of convergence $R_2 > 1$, then the radius of convergence R of $\sum_{n=0}^{\infty} a_n b_n x^n$ is at least $R_1 R_2$.

3. (a) Prove that the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} \sqrt{1 + e^{nx}}$ converges uniformly on $(-\infty, 0]$. *(Hint: Weierstrass M Test!)*

    (b) Determine the subset $E \subseteq \mathbb{R}$ on which the series converges. Rigorously justify your answer.

4. Show, directly from Definition 8.32, that the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is analytic.