Workshop 5: solutions for week 6

1. f is differentiable and $f'(x) = 1 + \cos x \geq 0$ for all $x \in \mathbb{R}$. Hence, f is increasing on $\mathbb{R}$ (Proposition 4.11). Assume, towards a contradiction, that f is *not* injective. Then there exists $a, b \in \mathbb{R}$ with $a < b$ such that $f(a) = f(b)$. But f is increasing, so it follows that f is constant on $[a, b]$, and hence $f'(x) = 0$ for all $x \in [a, b]$ (Proposition 4.11 again). But the set of critical points of f consists of the odd multiples of π ($f'(x) = 0$ iff $x = (2k+1)\pi$, $k \in \mathbb{Z}$) and this set contains no intervals, so cannot contain $[a, b]$. Hence, f is injective.

2. $I = (0, \infty)$, $a = 1$, $f : I \to \mathbb{R}$, $f(x) = \sin \pi x$, $g : I \to \mathbb{R}$, $g(x) = 1 - x^3$. Then $f.g$ are differentiable, $f(a) = g(a) = 0$, for all $x \in I$, $g'(x) \neq 0$, and for all $x \in I \backslash \{a\}$, $g(x) \neq 0$. Hence, I, a, f, g satisfy the hypotheses of L'Hospital's Rule. Furthermore
$$\lim_{x \to 1} \frac{f'(x)}{g'(x)} = \lim_{x \to 1} \frac{\pi \cos \pi x}{-3x^2} = \frac{\pi}{3}.$$
Hence, by L'Hospital's Rule
$$\lim_{x \to 1} \frac{f(x)}{g(x)} = \frac{\pi}{3}$$
also.

   (Careful: if you chose $I = \mathbb{R}$, strictly speaking you made a mistake since $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 1 - x^3$ does *not* satisfy the hypotheses of L'Hospital's Rule; there's a point in $\mathbb{R}$ where $g' = 0$.)

3. (a) I claim that the n-th derivative of $f(x) = \ln x$ is
$$f^{(n)}(x) = (-1)^{n+1} \frac{(n-1)!}{x^n}.$$
This is easily proven by induction. Hence $f^{(n)}(1) = (-1)^{n+1}(n-1)!$ and so
$$\begin{aligned} p_n(x) &= f(1) + f'(1)(x-1) + \frac{1}{2!} f''(1)(x-1)^2 + \cdots \frac{1}{n!} f^{(n)}(1)(x-1)^n \\ &= 0 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots + (-1)^{n+1} \frac{1}{n}(x-1)^n. \end{aligned}$$

   (b) By Taylor's Theorem, there exists $c \in (1, 2)$ such that
$$\begin{aligned} f(2) &= p_n(2) + \frac{f^{(n+1)}(c)}{(n+1)!}(2-1)^{n+1} = p_n(2) + (-1)^{n+2} \frac{n!}{(n+1)! c^{n+1}} \\ \Rightarrow \quad |p_n(2) - f(2)| &= \frac{1}{(n+1)c} < \frac{1}{n+1}. \end{aligned}$$
Hence, by the Squeeze Rule, $p_n(2) \to f(2)$. But
$$p_n(2) = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + (-1)^{n+1} \frac{1}{n}$$
is precisely the n-th partial sum of the alternating harmonic series. Hence this series converges to $f(2) = \ln 2$.