Workshop 10: solutions for week 11

1. (a) Let
$$g_n(x) = \frac{2n+1}{n(n+1)^2}\frac{x^{2n-1}}{1+x^{2n}}.$$
Then, if $|x| \leq 1$,
$$|g_n(x)| \leq \frac{2n+1}{n(n+1)^2}\frac{1^{2n-1}}{1+0^{2n}} = \frac{2n+1}{n(n+1)^2},$$
while, if $|x| > 1$,
$$|g_n(x)| \leq \frac{2n+1}{n(n+1)^2}\frac{|x|^{2n-1}}{0+x^{2n}} = \frac{2n+1}{n(n+1)^2|x|} \leq \frac{2n+1}{n(n+1)^2}.$$
Hence, for all $x \in \mathbb{R}$,
$$|g_n(x)| \leq M_n := \frac{2n+1}{n(n+1)^2}.$$
I claim that $\sum_{n=1}^{\infty} M_n$ converges. To see this, define $b_n = 1/n^2$ and note that
$$\frac{M_n}{b_n} = \frac{(2n+1)n}{(n+1)^2} < \frac{2n+1}{n+1} < 2$$
and $\sum_{n=1}^{\infty} b_n$ converges, so $\sum_{n=1}^{\infty} M_n$ converges by the Comparison Test.
Hence, by the Weierstrass M Test, the series $f(x)$ converges uniformly on $\mathbb{R}$ (Theorem 8.15).

(b) Since each function g_n is continuous on $\mathbb{R}$, so is each partial sum
$$f_k(x) = \sum_{n=1}^{k} g_n(x)$$
of the series. Hence, by Theorem 7.13,
$$\int_0^1 f = \int_0^1 \lim_{k\to\infty} f_k = \lim_{k\to\infty}\int_0^1 f_k.$$
But
$$\int_0^1 f_k = \int_0^1 \sum_{n=1}^{k} g_n = \sum_{n=1}^{k}\int_0^1 g_n$$
by Theorem 5.22. Hence,
$$\int_0^1 f = \lim_{k\to\infty}\sum_{n=1}^{k}\int_0^1 g_n = \sum_{n=1}^{\infty}\int_0^1 g_n.$$

Now,

$$\begin{aligned}
\int_0^1 g_n &= \int_0^1 \frac{2n+1}{n(n+1)^2}\frac{x^{2n-1}}{1+x^{2n}}dx \\
&= \frac{2n+1}{2n^2(n+1)^2}\int_0^1 \frac{d}{dx}\ln(1+x^{2n})\,dx \\
&= \frac{2n+1}{2n^2(n+1)^2}\left[\ln(1+x^n)\right]_0^1 \\
&= \frac{2n+1}{n^2(n+1)^2}\frac{1}{2}\ln 2 \\
&= \left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right)\frac{1}{2}\ln 2.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_0^1 f &= \lim_{k\to\infty}\sum_{n=1}^{k}\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right)\frac{1}{2}\ln 2 \\
&= \lim_{k\to\infty}\left(1-\frac{1}{(k+1)^2}\right)\frac{1}{2}\ln 2 \\
&= \frac{\ln 2}{2}
\end{aligned}$$

2. Let $f(x) = 1/x^2$ and $g(x) = \sum_{n=0}^{\infty} a_n x^n$. Then, by assumption, for all $x \in (-2, 2)$, $g(x)$ converges to $f(x+2)$. Hence by Corollary 8.22,

$$a_n = \frac{g^{(n)}(0)}{n!} = \frac{f^{(n)}(2)}{n!}$$

Now

$$\begin{aligned}
f(x) &= \frac{1}{x^2} \\
f'(x) &= \frac{-2}{x^3} \\
f''(x) &= \frac{2\cdot 3}{x^4} \\
f'''(x) &= \frac{-2\cdot 3\cdot 4}{x^5}
\end{aligned}$$

which suggests that, for all $n \in \mathbb{N}$,

$$f^{(n)}(x) = (-1)^n\frac{(n+1)!}{x^{n+2}}.$$

We can prove this by induction: it certainly holds for $n = 0$ and, if it holds for $n = k$, then

$$f^{(k+1)}(x) = \frac{d}{dx}(-1)^k\frac{(k+1)!}{x^{k+2}} = -(-1)^k\frac{(k+1)!(k+2)}{x^{k+3}} = (-1)^{k+1}\frac{(k+2)!}{x^{k+3}}$$

so it also hold for $n = k + 1$. Hence, by induction, the claim holds for all $n \in \mathbb{N}$.

Hence

$$a_n = (-1)^n \frac{(n+1)!}{n!2^{n+2}} = (-1)^n \frac{(n+1)}{2^{n+2}}.$$

3. f is not analytic, since it is not smooth: it fails to be differentiable at 0. Recall that, since analytic functions coincide locally with power series, and power series are smooth, analytic functions must be smooth.

   According to our definition, g *is* analytic. To see this, note that its domain $\mathbb{R}\backslash\{0\}$ is open and that, for any $x_0 > 0$, $g(x)$ coincides with the convergent power series

   $$g(x) = x_0 + (x - x_0)$$

   for all $x \in (0, 2x_0)$, and for any $x_0 < 0$, $g(x)$ coincides with the convergent power series

   $$g(x) = -x_0 - (x - x_0)$$

   for all $x \in (2x_0, 0)$.