Chapter 8

Workshop problems and solutions

8.1 Workshop questions for week 2

1. Let $f : D \to \mathbb{R}$ (read as "let f map D to $\mathbb{R}$"), where $D \subseteq \mathbb{R}$. Write down precise mathematical formulations of the following statements, using quantifiers ($\forall$, $\exists$):
   (a) D is bounded below.
   (b) f is unbounded above.
   (c) f is surjective.
   (d) f is not surjective.
2. Prove from first principles (i.e. give a direct ε–N proof) that the following sequence converges:
$$a_n = \frac{n^2 + (-1)^n}{n^2 + 2}$$
3. Prove from first principles (i.e. give a direct ε–K proof) that
$$\lim_{x\to\infty} \frac{2x^2 + 3x + 4}{x^2 + 1} = 2.$$
4. Give a direct ε–δ proof that $\lim_{x\to -2} \dfrac{x+2}{x^3+8} = \dfrac{1}{12}$.

8.2 Workshop solutions for week 2

A video of me solving these problems is available, see [VIDEO].

1. (a) $\exists K \in \mathbb{R}, \forall x \in D, x \geq K$.

   (b) $\forall K \in \mathbb{R}, \exists x \in D, f(x) > K$.

   (c) $\forall y \in \mathbb{R}, \exists x \in D, f(x) = y$.

   (d) $\exists y \in \mathbb{R}, \forall x \in D, f(x) \neq y$.

2. I claim that $a_n \to 1$. Proof: let $\varepsilon > 0$ be given. Then there exists $N \in \mathbb{Z}^+$ such that $N > \sqrt{3/\varepsilon}$. Now, for all $n \geq N$,

$$|a_n - 1| = \left|\frac{2-(-1)^n}{n^2+2}\right| \leq \frac{3}{n^2+1} < \frac{3}{n^2} \leq \frac{3}{N^2} < \varepsilon.$$

   Hence $a_n \to 1$.

3. For all $x \in (0, \infty)$,

$$\begin{aligned} \left|\frac{2x^2+3x+4}{x^2+1} - 2\right| &= \left|\frac{3x+2}{x^2+1}\right| \\ &\leq \frac{3x}{x^2+1} + \frac{2}{x^2+1} \\ &< \frac{3x}{x^2} + \frac{2}{x^2} \\ &= \frac{3}{x} + \frac{2}{x^2}. \end{aligned}$$

   Let $\varepsilon > 0$ be given. Let $K = \max\{1, 5/\varepsilon\}$. Then, for all $x > K$,

$$\begin{aligned} \left|\frac{2x^2+3x+4}{x^2+1} - 2\right| &< \frac{3}{x} + \frac{2}{x^2} \\ &< \frac{3}{K} + \frac{2}{K^2} \\ &\leq \frac{3}{K} + \frac{2}{K} \qquad \text{(since } K \geq 1\text{)} \\ &\leq \varepsilon. \end{aligned}$$

4. First note that the maximal domain of the function $f(x) = (x+2)/(x^3+8)$ is $D = \mathbb{R}\backslash\{-2\}$, and -2 is a cluster point of D. Let $\varepsilon > 0$ be given. Then let

$\delta = \min\{1, \varepsilon\}$. Then for all $x \in D$ such that $0 < |x+2| < \delta$,

$$\begin{aligned}
\left| f(x) - \frac{1}{12} \right| &= \left| \frac{1}{x^2 - 2x + 4} - \frac{1}{12} \right| \\
&= \left| \frac{x^2 - 2x - 8}{12(x^2 - 2 + 4)} \right| \\
&= \frac{|x-4||x+2|}{12((x-1)^2 + 3)} \\
&\leq \frac{|x-4|}{36}|x+2| \\
&< \frac{7}{36}|x+2| \qquad \text{(since } |x+2| < 1\text{, so } x - 4 \in (-7, -5)) \\
&\leq |x+2| \\
&< \varepsilon \qquad \text{(since } |x+2| < \delta \leq \varepsilon).
\end{aligned}$$

8.3 Workshop questions for week 3

1. Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = \begin{cases} 7 & x > 50 \\ 26 & x \leq 50 \end{cases}$.
   (a) Prove that f is discontinuous at 50.
   (b) Prove that f is continuous at 49.9.
2. Give a direct ε–δ proof that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x^2 + x$, is differentiable at 1.
3. Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable and satisfy $f(1) = -1$, $f'(1) = 2$, $g(-1) = 1$, $g'(-1) = 7$. Compute:
   (a) $(g \circ f)'(1)$.
   (b) $h'(1)$ where $h(x) = f(f(x)^2)$.
4. Determine whether the following sets are open:
$$[0,1), \quad \mathbb{R}\backslash[0,1), \quad \mathbb{R}\backslash[0,1], \quad \mathbb{R}\backslash\{2^n : n \in \mathbb{Z}\}.$$

8.4 Workshop solutions for week 3

A video of me solving these problems is available, see [VIDEO]

1. (a) $x_n = 50 + 1/n \to 50$, but $f(x_n) = 7 \to 7 \neq f(50) = 26$. Hence $f(x_n) \not\to f(50)$, so f is discontinuous at 50.

   (b) Let x_n be any sequence that converges to 49.9. We must prove that $f(x_n) \to f(49.9) = 26$. So, let $\varepsilon > 0$ be given. Since $x_n \to 49.9$, there exists $N \in \mathbb{Z}^+$ such that, for all $n \geq N$, $|x_n - 49.9| < 0.1$, and hence $x_n < 49.9 + 0.1 = 50$. Hence, for all $n \geq N$, $f(x_n) = 26$. So, for all $n \geq N$, $|f(x_n) - 26| = 0 < \varepsilon$. Hence $f(x_n) \to f(49.9)$.

2. I claim that $f'(1) = 5$. Proof: Given any $\varepsilon > 0$, let $\delta = \varepsilon/2$. Then for all $x \in \mathbb{R}$ with $0 < |x - 1| < \delta$,

$$\begin{aligned}
\left|\frac{f(x) - f(1)}{x - 1} - 5\right| &= \left|\frac{2x^2 + x - 3}{x - 1} - 5\right| \\
&= |(2x + 3) - 5| \\
&= 2|x - 1| \\
&< 2\delta \\
&= \varepsilon.
\end{aligned}$$

   Hence

$$\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = 5,$$

   so f is differentiable at 1 (with $f'(1) = 5$).

3. (a) $(g \circ f)'(1) = g'(-1)f'(1) = 14$.

   (b) $h'(1) = f'(f(1)^2) \times 2f(1)f'(1) = -8$.

4. (a) $[0, 1)$ is not open since it contains 0, but for all $\varepsilon > 0$, $(0 - \varepsilon, 0 + \varepsilon)$ is not a subset of $[0, 1)$ (since it contains $-\varepsilon/2$, for example).

   (b) $\mathbb{R}\backslash[0, 1) = (-\infty, 0) \cup [1, \infty)$ is not open since it contains 1, but for all $\varepsilon > 0$, $(1 - \varepsilon, 1 + \varepsilon)$ is not a subset of $\mathbb{R}\backslash[0, 1)$.

   (c) $\mathbb{R}\backslash[0, 1] = (-\infty, 0) \cup (1, \infty)$ is open. If $x > 1$ we may choose $\varepsilon = x - 1 > 0$. If $x < 0$, we may choose $\varepsilon = -x > 0$. In either case, $(x - \varepsilon, x + \varepsilon) \subset \mathbb{R}\backslash[0, 1]$.

   (d) $\mathbb{R}\backslash\{2^n : n \in \mathbb{Z}\}$ is not open since it contains 0, but for all $\varepsilon > 0$ there exists $N \in \mathbb{Z}^+$ such that $2^N > 1/\varepsilon$, whence $0 < 2^{-N} < \varepsilon$. So $2^{-N} \in (0 - \varepsilon, 0 + \varepsilon)$ but $2^{-N} \notin \mathbb{R}\backslash\{2^n : n \in \mathbb{Z}\}$.

8.5 Workshop questions for week 4

1. (a) What precisely does it mean to say that a function $f : D \to \mathbb{R}$ **attains a maximum** at $c \in D$? Write your answer using quantifiers. What, if any, restrictions must one place on the domain D of f for this definition to make sense?

   (b) Assume that $f : [a, b] \to \mathbb{R}$ is differentiable, and attains a maximum at b. What can you deduce about $f'(b)$? Prove your assertion.

2. Consider the function $f : [0, 1] \to \mathbb{R}$ defined so that $f(0) = 0$, and, for all $x \in (1/(n + 1), 1/n]$, $f(x) = 1/n$, where n is any positive integer.

   (a) Draw the graph of the function f.

   (b) Is f differentiable at 0? If so, what is $f'(0)$? Prove your assertion.

   (c) What properties does the function f have? (Bounded? Differentiable? Continuous? Surjective? Injective? Monotonic?)

3. Say that $f : \mathbb{Q} \to \mathbb{R}$ is differentiable at $2/3$ and $f'(2/3) = -4$.

   (a) Reinterpret this information using Carathéodory's Criterion.

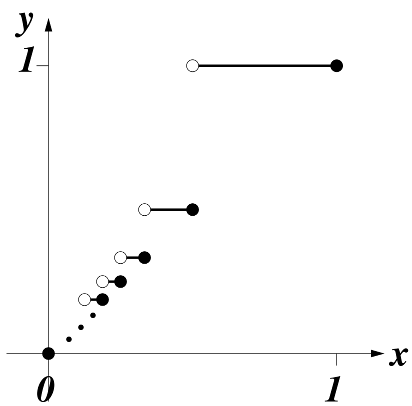
   (b) Compute $h'(1)$ where $h : \mathbb{Q} \to \mathbb{R}$,

$$h(x) = f\left(\frac{x + 1}{x^2 + 2}\right).$$

8.6 Workshop solutions for week 4

1. (a) $\forall x \in D,\ f(x) \leq f(c)$.
   D could be any set. In particular, it need not be a subset of $\mathbb{R}$.

   (b) We can deduce that $f'(b) \geq 0$.
   Proof: Let x_n be any sequence in $[a, b)$ converging to b. Since
   $$\lim_{x \to b} \frac{f(x) - f(b)}{x - b} = f'(b),$$
   the sequence
   $$z_n = \frac{f(x_n) - f(b)}{x_n - b} \to f'(b)$$
   (Theorem 1.35). But f attains a maximum at b, so for all $n \in \mathbb{Z}^+$, $f(x_n) \leq f(b)$. Further, $x_n \in [a, b)$ so $x_n < b$. Hence, for all n, $f(x_n) - f(b) \leq 0$ and $x_n - b < 0$, so $z_n \geq 0$. Hence, $f'(b) = \lim z_n \geq 0$ (Proposition 1.7). $\square$

2. (a) 

   (b) I claim that f **is** differentiable at 0, and $f'(0) = 1$.
   Proof: By Theorem 1.35, it suffices to show that, for all sequences (x_n) in $(0, 1]$ such that $x_n \to 0$,
   $$\frac{f(x_n) - f(0)}{x_n - 0} = \frac{f(x_n)}{x_n} \to 1.$$
   To prove this, note that, for all $x \in (0, 1)$, there exists some (unique) $n \in \mathbb{Z}^+$ such that $x \in (1/(n+1), 1/n]$. Then
   $$f(x) = \frac{1}{n} \geq x,$$
   and
   $$\begin{aligned} x &> \frac{1}{n+1} \\ \Rightarrow \quad \frac{1}{x} &< n+1 \\ \Rightarrow \quad n &> \frac{1}{x} - 1 = \frac{1-x}{x} \\ \Rightarrow \quad f(x) = \frac{1}{n} &< \frac{x}{1-x}. \end{aligned}$$

Hence, for all $x \in (0, 1)$,

$$x \leq f(x) < \frac{x}{1-x}.$$

[Aside: can you see this on your graph? Add the curves $y = x$ and $y = x/(1-x)$.] Hence

$$1 \leq \frac{f(x_n)}{x_n} < \frac{1}{1-x_n}$$

and so $f(x_n)/x_n \to 1$ by the Squeeze Rule and the Algebra of Limits.

(c)
- Bounded? Yes, above by 1, below by 0
- Differentiable? No. for example, it's discontinuous at $1/2$, so can't be differentiable at $1/2$.
- Continuous? No, as already observed.
- Surjective? No. Since it's bounded and has codomain $\mathbb{R}$, it can't be surjective. For example, it never attains the value -2.
- Injective? No. For example $f(3/4) = f(1) = 1$.
- Monotonic? Yes, it's increasing. Note that this means that for all $x, y \in [0, 1]$, if $x < y$ then $f(x) \leq f(y)$. (Note the non-strict inequality.)

3. (a) According to Carathéodory's Criterion (Proposition 2.12), this is precisely equivalent to the following: there exists a function $\phi : \mathbb{Q} \to \mathbb{R}$ which is continuous at $2/3$, has $\phi(2/3) = -4$ and, for all $x \in \mathbb{Q}$,

$$f(x) - f(2/3) = \phi(x)(x - \frac{2}{3}).$$

(b) $h = f \circ g$ where $g : \mathbb{Q} \to \mathbb{Q}$, $g(x) = \frac{x+1}{x^2+2}$. g is differentiable at 1 and, by the Quotient Rule,

$$g'(1) = \frac{1(3) - 2(2)}{3^2} = -\frac{1}{9},$$

so

$$h'(1) = f'(g(1))g'(1) = f'(2/3)g'(1) = \frac{4}{9}.$$

8.7 Workshop questions for week 5

1. (a) Prove that, for all $x, y \in \mathbb{R}$, $\frac{1}{2}(x^2 + y^2) \geq |xy|$.

   (b) Prove that, for all $x, y \in \mathbb{R}$,
$$\left| \ln \frac{4 + x^2}{4 + y^2} \right| \leq \frac{1}{2}|x - y|.$$

2. Assume $f : \mathbb{R} \to \mathbb{R}$ is smooth (that is, infinitely differentiable), that $f(2) = 1$, $f'(2) = 3$ and, for all $x \in \mathbb{R}$, $f''(x) < x$. What can we deduce about $f(1)$?

3. Let $f : [0, 3] \to \mathbb{R}$, $f(x) = |x - 1|$ and $\mathscr{D} = \{0, 1/2, 2, 3\}$. Compute the upper and lower Riemann sums $u_{\mathscr{D}}(f)$, $l_{\mathscr{D}}(f)$. [See Definition 4.11 for an explanation of the terminology.]

8.8 Workshop solutions for week 5

1. (a) Assume, towards a contradiction, that there exist $x, y \in \mathbb{R}$ such that $|xy| > \frac{1}{2}(x^2 + y^2)$. Since both sides of this inequality are non-negative, it follows that

$$\begin{aligned} 4|xy|^2 &> (x^2+y^2)^2 \\ \Rightarrow 0 &> x^4 - 2x^2y^2 + y^4 = (x^2-y^2)^2. \end{aligned}$$

But the square of a real number cannot be negative.

(b) The claim holds trivially if $x = y$, and is symmetric under interchange of x, y, so it suffices to prove it in the case where $x > y$. The function $f : [y, x] \to \mathbb{R}$, $f(t) = \ln(4+t^2)$ is differentiable. Hence, by the MVT, there exists $c \in (y, x)$ such that

$$\begin{aligned} \frac{f(x)-f(y)}{x-y} &= f'(c) = \frac{2c}{4+c^2} \\ \Rightarrow \quad |f(x)-f(y)| &= \frac{2|c|}{4+c^2}|x-y| \leq \frac{\frac{1}{2}(2^2+c^2)}{4+c^2}|x-y| = \frac{1}{2}|x-y| \end{aligned}$$

by part (a). The claim immediately follows.

2. By Taylor's Theorem, for each $x \in \mathbb{R}$, there exists c between x and 2 such that

$$f(x) = f(2) + f'(2)(x-2) + \frac{1}{2}f''(c)(x-2)^2.$$

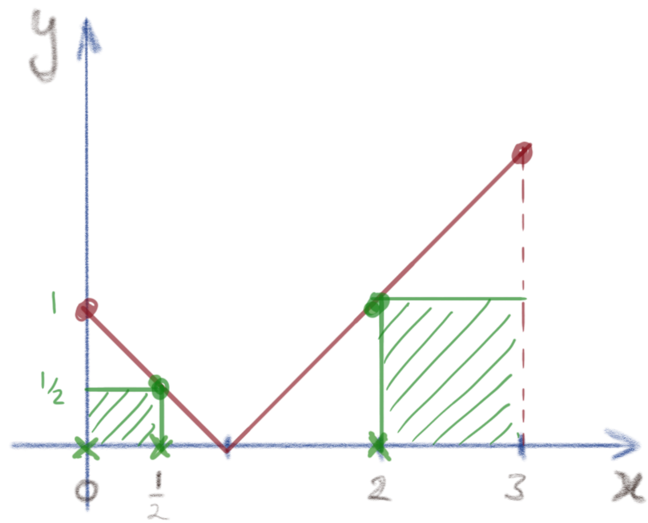
Hence, there exists $c \in (1, 2)$ such that

$$f(1) = 1 + 3(1-2) + \frac{1}{2}f''(c)(1-2)^2 = -2 + \frac{1}{2}f''(c).$$

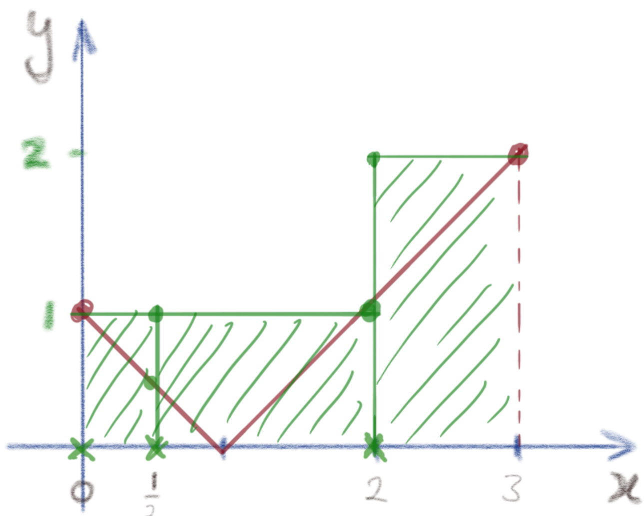
We are told that, for all $x \in \mathbb{R}$, $f''(x) < x$, so $f''(c) < c < 2$. Hence, we can deduce that

$$f(1) < -2 + \frac{1}{2}(2) = -1.$$

3.



$$\ell_D(f) = \left(\tfrac{1}{2} \times \tfrac{1}{2}\right) + \left(\tfrac{3}{2} \times 0\right) + (1 \times 1) = \underline{\underline{\tfrac{5}{4}}}$$



$$u_D(f) = \left(\tfrac{1}{2} \times 1\right) + \left(\tfrac{3}{2} \times 1\right) + (1 \times 2) = \underline{\underline{4}}$$

8.9 Workshop questions for week 6

1. Write down (or draw the graph of) a function $f : [a, b] \to \mathbb{R}$ and a pair of dissections $\mathscr{D}, \mathscr{D}'$ of $[a, b]$ such that $\mathscr{D}'$ is a refinement of $\mathscr{D}$, $u_{\mathscr{D}'}(f) < u_{\mathscr{D}}(f)$, but $l_{\mathscr{D}'}(f)$ is not greater than $l_{\mathscr{D}}(f)$.

2. Write down a function $f : [0, 1] \to \mathbb{R}$ which is *not* Riemann integrable but whose square, $f^2 : [0, 1] \to \mathbb{R}$, *is*. Rigorously justify your answer.

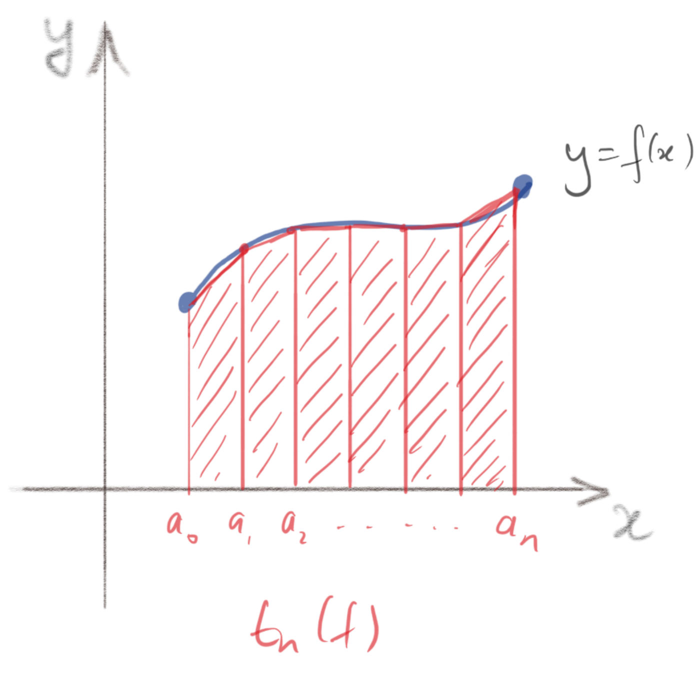
3. One might be tempted to define the integral of a function f on $[a, b]$ as follows. For each $n \in \mathbb{Z}^+$, let $\mathscr{D}_n$ be the regular dissection of $[a, b]$ of size n, and define

$$t_n(f) := \sum_{j=1}^{n} \frac{1}{2}(f(a_{j-1}) + f(a_j))(a_j - a_{j-1}) = \frac{b-a}{n} \sum_{j=1}^{n} \frac{1}{2}(f(a_{j-1}) + f(a_j)).$$

[This is the approximation to the integral of f obtained by using the "trapezium rule".] Then let us define the "trapezium integral" of f to be

$$\mathop{\mathtt{T}}_a^b f := \lim_{n\to\infty} t_n(f)$$

if this limit exists.



(a) Write down a function $f : [0, 1] \to \mathbb{R}$ which is "trapezium integrable" (meaning $\mathtt{T}_a^b f$ exists) but not Riemann integrable.

(b) Write down a function $f : [0, 2] \to \mathbb{R}$ which is trapezium integrable on $[0, \sqrt{2}]$, trapezium integrable on $[\sqrt{2}, 2]$ and trapezium integrable on $[0, 2]$ for which

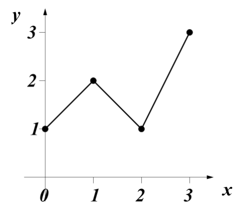
$$\mathop{\mathtt{T}}_0^2 f \neq \mathop{\mathtt{T}}_0^{\sqrt{2}} f + \mathop{\mathtt{T}}_{\sqrt{2}}^2 f.$$

(c) Does the "trapezium integral" provide a satisfactory definition of integration?

8.10 Workshop solutions for week 6

A video of me solving these problems is available, see [VIDEO].

1. The piecewise linear function $f : [0,3] \to \mathbb{R}$ depicted below has the required properties with respect to the dissections $\mathscr{D} = \{0,3\}$, $\mathscr{D}' = \{0,2,3\}$.



We see that

$$\begin{aligned} l_{\mathscr{D}}(f) &= 3 \times 1 = 3 \\ u_{\mathscr{D}}(f) &= 3 \times 3 = 9 \\ l_{\mathscr{D}'}(f) &= 2 \times 1 + 1 \times 1 = 3 \\ u_{\mathscr{D}'}(f) &= 2 \times 2 + 1 \times 3 = 7. \end{aligned}$$

So in this case, passing from $\mathscr{D}$ to its refinement $\mathscr{D}'$ improves the overestimate (the upper sum), but makes no change to the underestimate (the lower sum).

2. The function $f : [0,1] \to \mathbb{R}$ defined so that

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is not Riemann integrable. To see this note that for *any* dissection $\mathscr{D} = \{a_0, \dots, a_n\}$ of $[0,1]$,

$$l_{\mathscr{D}}(f) = \sum_{j=1}^{n} (a_j - a_{j-1})(-1) = -\sum_{j=1}^{n} (a_j - a_{j-1}) = -1$$

since every interval $[a_{j-1}, a_j]$ contains an irrational number, and

$$u_{\mathscr{D}}(f) = \sum_{j=1}^{n} (a_j - a_{j-1})(+1) = \sum_{j=1}^{n} (a_j - a_{j-1}) = 1$$

since every interval $[a_{j-1}, a_j]$ contains a rational number. Hence $l(f) = \sup\{-1\} = -1$ and $u(f) = \inf\{1\} = 1$, so $l(f) \neq u(f)$.

However, f^2 is the constant function, $f^2(x) = 1$, which is certainly integrable (since it's continuous, for example).

3. (a) The same function from question 2 will work. We already observed that it isn't Riemann integrable. Let $\mathscr{D}_n$ be the regular dissection of $[0,1]$ of size n, and note that every point $a_j \in \mathscr{D}_n$ is *rational* (since $a_j = j/n$). Hence $f(a_j) = 1$ for all j, and so $t_n(f) = \frac{1}{n} \sum j = 1^n \frac{1}{2}(1+1) = 1 \to 1$. So f is trapezium integrable and

$$\mathop{\mathrm{T}}_{0}^{1} f = 1.$$

(b) Again, we can use a similar function, $f : [0, 2] \to \mathbb{R}$ with $f(x) = 1$ if $x \in \mathbb{Q}$ and $f(x) = 0$ if $x \notin \mathbb{Q}$. Let $\mathscr{D}_n$ be the regular dissection of $[0, \sqrt{2}]$ of size n and $\mathscr{D}'_n$ be the regular dissection of $[\sqrt{2}, 2]$ of size n. Almost all points in these dissections are *irrational*:

$$\begin{aligned} \mathscr{D}_n &= \{a_j : j = 0, \dots, n\} = \{\sqrt{2}\frac{j}{n} : j = 0, \dots, n\} \\ \mathscr{D}'_n &= \{a'_j : j = 0, \dots, n\} = \{\sqrt{2} + (2 - \sqrt{2})\frac{j}{n} : j = 0, \dots, n\} \end{aligned}$$

so all points except $a_0 = 0$ and $a'_n = 2$ are irrational. It follows that $f(a_j) = 0$ for all j except 1, and $f(a'_j) = 0$ for all j except n. Hence

$$t_n(f) = \frac{\sqrt{2}}{n} \sum_{j=1}^{n} \frac{1}{2}(f(a_{j-1}) + f(a_j)) = \frac{\sqrt{2}}{n}\frac{1}{2} \to 0 \tag{8.1}$$

$$t'_n(f) = \frac{2 - \sqrt{2}}{n} \sum_{j=1}^{n} \frac{1}{2}(f(a'_{j-1}) + f(a'_j)) = \frac{2 - \sqrt{2}}{n}\frac{1}{2} \to 0, \tag{8.2}$$

and so

$$\overset{\sqrt{2}}{\underset{0}{\mathrm{T}}} f + \overset{2}{\underset{\sqrt{2}}{\mathrm{T}}} f = 0 + 0 = 0.$$

Now let $\mathscr{D}''_n$ be the regular dissection of $[0, 2]$ of size n

$$\mathscr{D}''_n = \{a''_j : j = 0, \dots, n\} = \{2\frac{j}{n} : j = 0, \dots, n\}.$$

Every a_j is rational, so $f(a_j) = 1$. Hence

$$t''_n(f) = \frac{2}{n} \sum_{j=1}^{n} \frac{1}{2}(f(a''_{j-1}) + f(a''_j)) = \frac{2}{n} \sum_{j=1}^{n} 1 = 2 \to 2,$$

and so

$$\overset{2}{\underset{0}{\mathrm{T}}} f = 2.$$

(c) No, certainly not. The "join rule" of integration,

$$\int_a^b f + \int_b^c f = \int_a^c f,$$

is a fundamental property that any satisfactory theory of integration should possess.

8.11 Workshop questions for week 7

1. Let $f, g : [a, b] \to \mathbb{R}$ be bounded functions and $\mathscr{D}$ be a dissection of $[a, b]$. Prove that
$$u_{\mathscr{D}}(f + g) \leq u_{\mathscr{D}}(f) + u_{\mathscr{D}}(g)$$
and
$$l_{\mathscr{D}}(f - g) \geq l_{\mathscr{D}}(f) - u_{\mathscr{D}}(g).$$

2. Given a function $f : [a, b] \to \mathbb{R}$ define its "non-negative part" to be
$$f^+ : [a, b] \to \mathbb{R}, \qquad f^+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0, \\ 0 & \text{if } f(x) < 0. \end{cases}$$
   (a) Prove that if f is Riemann integrable, so is f^+.
   (b) Is the converse of the above "theorem" true?
   (c) How would you define the "non-positive part" $f^- : [a, b] \to \mathbb{R}$ of the function f?
   (d) If f is Riemann integrable, is f^- necessarily Riemann integrable?
   (e) Assume both f^+ and f^- are Riemann integrable. Does it follow that f is Riemann integrable?

3. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 \cos x$. This is continuous, so the associated function $g : \mathbb{R} \to \mathbb{R}$,
$$g(x) = \int_0^x f(t)dt$$
is well defined. Compute $g'(\pi)$.

8.12 Workshop solutions for week 7

1. As usual let $\mathscr{D} = \{a_0, a_1, \dots, a_n\}$ and, for any bounded function h on $[a,b]$ define

$$m_j(h) = \inf\{h(x) : x \in [a_{j-1}, a_j]\}, \qquad M_j(h) = \sup\{h(x) : x \in [a_{j-1}, a_j]\}.$$

Then, for all $x \in [a_{j-1}, a_j]$, $m_j(f) \le f(x) \le M_j(f)$ and $m_j(g) \le g(x) \le M_j(g)$, so $f(x) + g(x) \le M_j(f) + M_j(g)$ and $f(x) - g(x) \ge m_j(f) - M_j(g)$. Hence, $M_j(f)+M_j(g)$ is an upper bound on $\{f(x)+g(x) : x \in [a_{j-1}, a_j]\}$ and $M_j(f+g)$ is the *least* upper bound on this set, so $M_j(f+g) \le M_j(f) + M_j(g)$. Hence

$$\begin{aligned} u_{\mathscr{D}}(f+g) &= \sum_{j=1}^{n} M_j(f+g)(a_j - a_{j-1}) \\ &\le \sum_{j=1}^{n} (M_j(f) + M_j(g))(a_j - a_{j-1}) \\ &= \sum_{j=1}^{n} M_j(f)(a_j - a_{j-1}) + \sum_{j=1}^{n} M_j(g)(a_j - a_{j-1}) \\ &= u_{\mathscr{D}}(f) + u_{\mathscr{D}}(g). \end{aligned}$$

Similarly, $m_j(f) - M_j(g)$ is a lower bound on $\{f(x) - g(x) : x \in [a_{j-1}, a_j]\}$ and $m_j(f-g)$ is the *greatest* lower bound on this set, so $m_j(f-g) \ge m_j(f) - M_j(g)$. Hence

$$\begin{aligned} l_{\mathscr{D}}(f+g) &= \sum_{j=1}^{n} m_j(f-g)(a_j - a_{j-1}) \\ &\ge \sum_{j=1}^{n} (m_j(f) - M_j(g))(a_j - a_{j-1}) \\ &= \sum_{j=1}^{n} m_j(f)(a_j - a_{j-1}) - \sum_{j=1}^{n} M_j(g)(a_j - a_{j-1}) \\ &= l_{\mathscr{D}}(f) - u_{\mathscr{D}}(g). \end{aligned}$$

2. (a) Note that $f^+ = \frac{1}{2}(f + |f|)$. Check: if $f(x) \ge 0$ then

$$\frac{1}{2}(f(x) + |f(x|) = \frac{1}{2}(f(x) + f(x)) = f(x) = f^+(x),$$

and if $f(x) < 0$ then

$$\frac{1}{2}(f(x) + |f(x|) = \frac{1}{2}(f(x) - f(x)) = 0 = f^+(x).$$

If f is Riemann integrable then $|f|$ is Riemann integrable (Proposition 4.29) so f^+ is a linear combination of integrable functions, which is integrable by Theorem 4.31 (linearity).

(b) No. For example, the function $f:[0,1]\to\mathbb{R}$, $f(x)=-1/x$ if $x\neq 0$, $f(0)=0$ is not Riemann integrable (it's not even bounded), but its non-negative part $f^+(x)=0$ is.

(c) $f^-(x)=f(x)$ if $f(x)\leq 0$, $f^-(x)=0$ if $f(x)>0$.

(d) Yes. We could argue similarly to part (a), noting that $f^-=\frac{1}{2}(f-|f|)$, or we could note that $f=f^++f_-$, so $f^-=f-f^+$. Either way, f^- is a linear combination of integrable functions, so is integrable by Theorem 4.31.

(e) Yes. Since $f=f^++f^-$, if both f^+ and f^- are integrable, so is f, by Theorem 4.31.

3. By FTC1 (Theorem 5.1), $g'(\pi)=f(\pi)=-\pi^2$. You didn't try to compute the integral did you?

8.13 Workshop questions for week 8

Hint/solution [VIDEO]

1. Let $f : [0,1] \to \mathbb{R}$, $f(x) = 1/(1+x)$, and $g : [0,1] \to \mathbb{R}$, $g(x) = 2x$. Compute

$$(a) \quad \|f\|, \qquad (b) \quad \|g\|, \qquad \text{and} \quad (c) \quad \|f-g\|.$$

   The **sup norm** of a function $\|f\|$ is defined in Definition 6.9.

2. Determine whether each of the following sequences converges pointwise (Definition 6.4). If it converges pointwise determine whether it converges uniformly (Definition 6.11).

   (a) $f_n : [1,\infty) \to \mathbb{R}$, $f_n(x) = nx^{-n}$.

   (b) $f_n : [2,\infty) \to \mathbb{R}$, $f_n(x) = nx^{-n}$.

   (c) $f_n : (1,\infty) \to \mathbb{R}$, $f_n(x) = nx^{-n}$.

3. Construct a sequence of bounded functions $f_n : [0,1] \to \mathbb{R}$ that converges pointwise to an unbounded function $f : [0,1] \to \mathbb{R}$.

8.14 Workshop solutions for week 8

1. (a) f is positive and decreasing, so for all $x \in [0,1]$, $0 < f(x) \leq f(0) = 1$. Hence

$$\|f\| = \sup\{|f(x)| : x \geq 0\} = 1.$$

(b) Similarly, g is non-negative and increasing, so for all $x \in [0,1]$, $0 \leq g(x) \leq g(1) = 2$. Hence

$$\|g\| = \sup\{|g(x)| : x \geq 0\} = g(1) = 2.$$

(c) Let $h(x) = f(x) - g(x) = (1+x)^{-1} - 2x$. Since f is decreasing and g is increasing, $h = f - g$ is decreasing. Hence, for all $x \in [0,1]$,

$$h(1) = -\frac{3}{2} \leq h(x) \leq h(0) = 1,$$

so $|h(x)| \leq 3/2$, and $|h(1)| = 3/2$. Hence $\|h\| = \sup\{|h(x)| : x \in [0,1]\} = 3/2$.

2. (a) $f_n(1) = n$ diverges, so (f_n) does not converge pointwise.

(b) I claim that (f_n) converges uniformly to 0 (and hence converges pointwise):

$$\|f_n - 0\| = \sup\{|f_n(x)| : x \geq 2\} = \sup\{\frac{n}{x^n} : x \geq 2\} = \frac{n}{2^n}$$

Consider the series whose nth term is $a_n = n/2^n$. Since

$$\frac{a_{n+1}}{a_n} = \left(1 + \frac{1}{n}\right)\frac{1}{2} \to \frac{1}{2} < 1,$$

$\sum_{n=1}^{\infty} a_n$ converges, by the Ratio Test. Hence $a_n \to 0$ by the Divergence Test. So $\|f_n - 0\| \to 0$, that is, (f_n) converegs uniformly to 0.

(c) I claim that (f_n) converges pointwise, but not uniformly, to 0. For all $x > 1$, $f_n(x) = \frac{n}{x^n} \to 0$ by a re-run of the argument just advanced:

$$\frac{f_{n+1}(x)}{f_n(x)} = \left(1 + \frac{1}{n}\right)\frac{1}{x} \to \frac{1}{x} < 1$$

so $\sum_{n=1}^{\infty} f_n(x)$ converges by the Ratio Test, and hence $f_n(x) \to 0$ by the Divergence Test. This holds for all $x \in (1, \infty)$, so f_n converges to 0 pointwise. However

$$\|f_n - 0\| = \sup\{n/x^n : x > 1\} = n.$$

To see this, note that n is an upper bound on the set, but given any $K < n$, $K = n/\alpha$ with $\alpha > 1$, and there exists $x \in (1, \infty)$ such that $x^n > \alpha$ (e.g. $x = \alpha$ will do). So K is not an upper bound on the set.

Since $\|f_n - 0\|$ diverges, (f_n) does not converge uniformly to 0 (and since it converges pointwise to 0, it can't converge pointwise to any other function).

3. $f_n(x) = \left\{ \begin{array}{cc} \min\{n, 1/x\} & x \in (0,1] \\ 0 & x = 0 \end{array} \right\}$ will do. To see this, note that, $0 \leq f_n(x) \leq n$ for all n, so each f_n is bounded. For any fixed $x \in (0,1]$, $f_n(x) = 1/x$ for all $n > 1/x$, so the sequence $f_n(x) \to 1/x$. Also, $f_n(0) = 0 \to 0$. So (f_n) converges pointwise to the unbounded function

$$f(x) = \left\{ \begin{array}{cc} 1/x & x \in (0,1] \\ 0 & x = 0. \end{array} \right.$$

8.15 Workshop questions for week 9

1. (a) Write down a sequence of continuous functions that converges pointwise, but not uniformly, to $\sin : \mathbb{R} \to \mathbb{R}$.

   (b) Write down a sequence of discontinuous functions that converges uniformly to $\sin : \mathbb{R} \to \mathbb{R}$.

2. (a) Write down a sequence of bounded functions $f_n : \mathbb{R} \to \mathbb{R}$ whose sequence of sup norms $(\|f_n\|)$ is unbounded.

   (b) Let $f_n : D \to \mathbb{R}$ be a uniformly convergent sequence of bounded functions. Prove that the sequence $(\|f_n\|)$ is bounded. *[Hint: you will find Lemma 6.23 useful here.]*

3. For each $n \in \mathbb{Z}^+$, let $f_n : \mathbb{R} \to \mathbb{R}$ be the function

$$f_n(x) = \sum_{k=1}^{n} \frac{1}{2^k} \sin(x^k).$$

   (a) Prove that each of these functions is bounded.

   (b) Prove that the sequence (f_n) is uniformly Cauchy (Definition 6.20).

   (c) Deduce that (f_n) converges uniformly to some bounded function $f : \mathbb{R} \to \mathbb{R}$ and that f is continuous.

8.16 Workshop solutions for week 9

1. (a) $f_n(x) = \sin x + \frac{x}{n}$ will do. Note that each f_n is unbounded, so it is clear that the convergence cannot be uniform (since $\|f_n - \sin\|$ does not exist).

   (b) $f_n(x) = \begin{cases} 1/n, & x = 0, \\ \sin x, & x \neq 0 \end{cases}$ will do. Note that $\|f_n - \sin\| = 1/n \to 0$.

2. (a) $f_n(x) = n$ will do. Each f_n is bounded, but $\|f_n\| = n$ which is an unbounded sequence.

   (b) By assumption f_n converges uniformly to some bounded function $f : D \to \mathbb{R}$. Hence, for all $n \in \mathbb{Z}^+$,

$$\begin{aligned} \|f_n\| &= \|f_n - f + f\| \\ &\leq \|f_n - f\| + \|f\| \end{aligned}$$

   by Lemma 6.23. By assumption, the sequence $\|f_n - f\| \to 0$, so is bounded above, by K say. Hence $(\|f_n\|)$ is bounded above by $K + \|f\|$ (and below by 0).

3. (a) For all $x \in \mathbb{R}$,

$$|f_n(x)| \leq \sum_{k=1}^{n} \frac{1}{2^k} |\sin(x^k)| \leq \sum_{k=1}^{n} \frac{1}{2^k} = 1 - \frac{1}{2^n} < 1.$$

   Hence, each function f_n is bounded.

   (b) Note that, for all $x \in \mathbb{R}$, and all $n, m \in \mathbb{Z}^+$ with $n < m$,

$$\begin{aligned} |f_n(x) - f_m(x)| &= \left| \sum_{k=n+1}^{m} \frac{1}{2^k} \sin(x^k) \right| \\ &\leq \sum_{k=n+1}^{m} \frac{1}{2^k} |\sin(x^k)| \\ &\leq \sum_{k=n+1}^{m} \frac{1}{2^k} \\ &= \frac{1}{2^n} \sum_{k=1}^{m-n} \frac{1}{2^k} \\ &= \frac{1}{2^n} \left(1 - \frac{1}{2^{m-n}} \right) \\ &< \frac{1}{2^n}. \end{aligned}$$

   Similarly, for all $x \in \mathbb{R}$, and all $n, m \in \mathbb{Z}^+$ with $m < n$,

$$|f_n(x) - f_m(x)| < \frac{1}{2^m}.$$

So, given any $\varepsilon > 0$, let N be any positive integer such that $1/2^N < \varepsilon$ (such a positive integer exists since the sequence $1/2^n \to 0$). Then for all $n, m \geq N$,

$$\begin{aligned} \|f_n - f_m\| &= \sup\{|f_n(x) - f_m(x)| : x \in \mathbb{R}\} \\ &< \frac{1}{2^N} \\ &< \varepsilon. \end{aligned}$$

Hence, (f_n) is uniformly Cauchy.

(c) It follows from Theorem 6.26 that (f_n) converges uniformly to some bounded function $f : \mathbb{R} \to \mathbb{R}$, and from Theorem 6.14 that f is continuous.

8.17 Workshop questions for week 10

1. Let $a_n = 1$ if n is prime and $a_n = 0$ otherwise. Compute the radius of convergence of the power series
$$\sum_{n=2}^{\infty} a_n x^n.$$

2. Prove that
$$f(x) = \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)^2} \frac{x^{2n-1}}{1+x^{2n}}$$
converges uniformly on $\mathbb{R}$. *[Hint: Weierstrass M Test!]*

3. Assume that, for all $x \in (0,4)$,
$$\sum_{n=0}^{\infty} a_n (x-2)^n = \frac{1}{x^2}.$$
Find a formula for a_n.

8.18 Workshop solutions for week 10

1. I claim the the radius of convergence is $R = 1$.

   Proof: For all $|x| < 1$, and all $k \geq 2$,

   $$\sum_{n=2}^{k} |a_n x^n| \leq \sum_{n=0}^{k} |x|^n \leq \frac{1}{1-|x|}$$

   so the sequence $\sum_{n=2}^{k} |a_n x^n|$ is increasing and bounded above, and hence converges by the Monotone Convergence Theorem (MCT). It follows that $R \geq 1$ (it is the supremum of a set which contains $[0,1)$). On the other hand, at $x = 1$ the k-th partial sum of the power series is precisely the number of prime numbers less than or equal to k. Since the set of primes is infinite, this sequence is unbounded above, so the power series diverges at $x = 1$. Hence $R \leq 1$ (since $R > 1$ would contradict Theorem 7.11). $\square$

2. Let

   $$g_n(x) = \frac{2n+1}{n(n+1)^2} \frac{x^{2n-1}}{1+x^{2n}}.$$

   Then, if $|x| \leq 1$,

   $$|g_n(x)| \leq \frac{2n+1}{n(n+1)^2} \frac{1^{2n-1}}{1+0^{2n}} = \frac{2n+1}{n(n+1)^2},$$

   while, if $|x| > 1$,

   $$|g_n(x)| \leq \frac{2n+1}{n(n+1)^2} \frac{|x|^{2n-1}}{0+x^{2n}} = \frac{2n+1}{n(n+1)^2|x|} \leq \frac{2n+1}{n(n+1)^2}.$$

   Hence, for all $x \in \mathbb{R}$,

   $$|g_n(x)| \leq M_n := \frac{2n+1}{n(n+1)^2}.$$

   I claim that $\displaystyle\sum_{n=1}^{\infty} M_n$ converges. To see this, define $b_n = 1/n^2$ and note that

   $$\frac{M_n}{b_n} = \frac{(2n+1)n}{(n+1)^2} < \frac{2n+1}{n+1} < 2$$

   and $\displaystyle\sum_{n=1}^{\infty} b_n$ converges, so $\displaystyle\sum_{n=1}^{\infty} M_n$ converges by the Comparison Test.

   Hence, by the Weierstrass M Test, the series $f(x)$ converges uniformly on $\mathbb{R}$ (Theorem 7.16).

3. Let $f(x) = 1/x^2$ and $g(x) = \sum_{n=0}^{\infty} a_n x^n$. Then, by assumption, for all $x \in (-2,2)$, $g(x)$ converges to $f(x+2)$. Hence by Corollary 7.23,

   $$a_n = \frac{g^{(n)}(0)}{n!} = \frac{f^{(n)}(2)}{n!}$$

Now

$$\begin{aligned} f(x) &= \frac{1}{x^2} \\ f'(x) &= \frac{-2}{x^3} \\ f''(x) &= \frac{2\cdot 3}{x^4} \\ f'''(x) &= \frac{-2\cdot 3\cdot 4}{x^5} \end{aligned}$$

which suggests that, for all $n \in \mathbb{N}$,

$$f^{(n)}(x) = (-1)^n \frac{(n+1)!}{x^{n+2}}.$$

We can prove this by induction: it certainly holds for $n = 0$ and, if it holds for $n = k$, then

$$f^{(k+1)}(x) = \frac{d}{dx}(-1)^k \frac{(k+1)!}{x^{k+2}} = -(-1)^k \frac{(k+1)!(k+2)}{x^{k+3}} = (-1)^{k+1} \frac{(k+2)!}{x^{k+3}}$$

so it also hold for $n = k+1$. Hence, by induction, the claim holds for all $n \in \mathbb{N}$. Hence

$$a_n = (-1)^n \frac{(n+1)!}{n!2^n} = (-1)^n \frac{(n+1)}{2^n}.$$