Chapter 8

Power series

8.1 Convergence tests for series

We begin with a brisk review of the basics concerning convergence of series. All this material was covered in detail in MATH1026. (The main ideas were covered in MATH1055, albeit with some of the proofs omitted.)

Recall that a **series** is a formal infinite sum

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots ,$$

that its k^{th} **partial sum** is the real number

$$s_k := \sum_{n=0}^{k} a_n = a_0 + a_1 + \cdots + a_k,$$

and that the series **converges** precisely if the sequence (s_k) converges (in the sense of Definition 1.1). In this case, we also use $\sum\limits_{n=0}^{\infty} a_n$ to denote its limit.

The series **converges absolutely** if

$$\sum_{n=0}^{\infty} |a_n|$$

converges. Absolute convergence implies convergence, but convergence does *not* imply absolute convergence. For example, the **alternating harmonic series**

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges, but it does not converge absolutely since the **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

If a series $\sum_{n=0}^{\infty} a_n$ converges then its sequence of terms (a_n) converges to 0. This fact is often called the **Divergence Test** since we can apply its contrapositive: if $a_n \not\to 0$ then $\sum a_n$ diverges. Note that the *converse is false*, as illustrated by the harmonic series: the sequence of terms $a_n = 1/n \to 0$, but the corresponding *series* $\sum_{n=1}^{\infty} a_n$ diverges.

If all the terms of a series are positive, $a_n > 0$, then the sequence of partial sums $s_k = \sum_{n=0}^{k} a_n$ is an increasing sequence,

$$s_{k+1} = s_k + a_{k+1} > s_k,$$

so, by the Monotone Convergence Theorem, the series converges if (and only if) s_k is bounded above. This observation allows one to establish several useful convergence tests applicable to the case $a_n > 0$. The first allows us to prove convergence (or divergence) of a series $\sum a_n$ by comparing it with a (simpler) series $\sum b_n$ that we already know converges (or diverges).

Theorem 8.1 (The Comparison Test) *Let $a_n > 0$ and $b_n > 0$.*

(i) If a_n/b_n is bounded above and $\sum b_n$ converges, then $\sum a_n$ converges.

(ii) If b_n/a_n is bounded above and $\sum b_n$ diverges, then $\sum a_n$ diverges.

To use the comparison test effectively, you need a stock of simple example series whose convergence/divergence you've already established. Here's a useful and simple family of examples:

Example 8.2 Claim: $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where $p \in (0, \infty)$, converges if $p > 1$ and diverges if $0 < p \leq 1$.

Proof: Exercise. The idea is to bound

$$s_k = \sum_{n=1}^{k} \frac{1}{n^p}$$

in terms of

$$\int_1^k \frac{1}{x^p} dx.$$

□

Example 8.3 Claim: $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges.

Proof: Let $a_n = n/(n^2+1)$ and $b_n = 1/n$. Then

$$\frac{b_n}{a_n} = \frac{n^2+1}{n^2} = 1 + \frac{1}{n^2} \leq 2,$$

so the sequence (b_n/a_n) is bounded above. Since $\sum_{n=1}^{\infty} b_n$ diverges, we conclude from part (ii) of the Comparison Test that $sum_{n=1}^{\infty} a_n$ also diverges. $\square$

Exercise 8.4 Determine whether $\sum_{n=1}^{\infty} \frac{n \sin n}{n^3+2}$ converges.

Our next convergence test is perhaps the most useful – as well as being the easiest to use!

Theorem 8.5 (The Ratio Test) *Let $a_n > 0$ for all n and $a_{n+1}/a_n \to L$. Then*

(i) if $L < 1$, the series $\sum a_n$ converges,

(ii) if $L > 1$, the series $\sum a_n$ diverges.

Example 8.6 Claim: $\sum_{n=1}^{\infty} n^{50} \left(\frac{49}{50}\right)^n$ converges.

Proof: Let $a_n = n^{50}(49/50)^n > 0$, and note that

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{50}}{n^{50}} \frac{49}{50} = \left(1 + \frac{1}{n}\right)^{50} \frac{49}{50} \to \frac{49}{50} < 1.$$

Hence, by the Ratio Test, $\sum_{n=1}^{\infty} a_n$ converges. $\square$

Note that if the terms a_n of your series aren't all positive, you can always try applying the Comparison or Ratio Test to $\sum |a_n|$ to determine whether $\sum a_n$ converge absolutely. If it does, it certainly converges. Discarding all information about the sign of a_n does lose us potentially vital knowledge, however. If the terms *alternate* in sign, we have a particularly neat convergence test:

Theorem 8.7 (The Alternating Series Test) *Let (a_n) be a positive, decreasing sequence that converges to 0. Then*

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converges.

Example 8.8 The series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$ converges, by the Alternating Series Test, since $a_n = 1/\sqrt{n}$ is positive, decreasing and converges to 0. As usual with convergence tests, we've proved that the series converges but have no idea what its limit is.

8.2 Power series and their radius of convergence

A **power series** is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n$$

where x is interpreted as a real *variable*. The series depends on the choice of value for $x \in \mathbb{R}$. The terms of the series are $a_n x^n$, and the k^{th} partial sum is the polynomial function

$$f_k(x) = \sum_{n=0}^{k} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k,$$

just as for series in general. Note that, if $x = 0$, then every partial sum is $f_k(0) = a_0$, so the series certainly converges (to a_0) in that case. In general, a power series may converge for some values of x but diverge for others. If the series converges for a particular choice of x, its limit will, in general, depend on x. So a power series defines a real-valued function on some subset (perhaps all) of $\mathbb{R}$. The following example is fundamental (and, hopefully, familiar).

Example 8.9 (Geometric series) Claim: The series

$$\sum_{n=0}^{\infty} x^n$$

converges to $1/(1-x)$ if $|x| < 1$, and diverges otherwise.

Proof: If $x = 1$, the k-th partial sum is $s_k = k+1$, and this sequence clearly diverges. If $x \neq 1$ then

$$\begin{aligned} s_k &= 1 + x + x^2 + \cdots x^k \\ \Rightarrow \quad x s_k &= x + x^2 + x^3 + \cdots + x^{k+1} \\ \Rightarrow \quad (1-x)s_k &= 1 - x^{k+1} \\ \Rightarrow \quad s_k &= \frac{1 - x^{k+1}}{1-x}. \end{aligned}$$

The sequence x^k converges to 0 if $|x| < 1$ and diverges if $x = -1$. Hence, $s_k \to 1/(1-x)$ if $|x| < 1$ and diverges otherwise. □

So if we use the power series to define a function

$$f(x) = \sum_{n=0}^{\infty} x^n$$

then $f : D \to \mathbb{R}$ where $D = (-1, 1)$ and for all $x \in D$,

$$f(x) = \frac{1}{1-x}.$$

Definition 8.10 The **radius of convergence** of a power series $\sum_{n=0}^{\infty} a_n x^n$ is

$$R := \sup\{|x| \ : \ \textstyle\sum_{n=0}^{\infty} |a_n x^n| \text{ converges}\}.$$

If this set is unbounded above, we say that $R = \infty$.

The radius of convergence of a power series tells us almost everything about the subset of $\mathbb{R}$ on which it converges:

Theorem 8.11 *Let a power series have radius of convergence $R > 0$. Then the series* **converges absolutely** *for $|x| < R$ and* **diverges** *for $|x| > R$.*

So the series converges on $(-R, R)$ and diverges on $(-\infty, -R) \cup (R, \infty)$. The only information not revealed by Theorem 8.11 is whether the series converges at $x = \pm R$. So, it's important to be able to compute radii of convergence. Luckily this can usually be achieved with a simple application of the Ratio Test.

Example 8.12 Claim: $\displaystyle\sum_{n=0}^{\infty} \frac{nx^{3n+1}}{n^2+1}$ has radius of convergence $R = 1$.

Proof: Let $b_n = |nx^{3n+1}/(n^2+1)| > 0$ and note that

$$\begin{aligned}
\frac{b_{n+1}}{b_n} &= \frac{|(n+1)x^{3n+4}|}{n^2+2n+3} \times \frac{n^2+1}{|nx^{3n+1}|} \\
&= \frac{(n+1)(n^2+1)|x|^3}{(n^2+2n+3)n} \\
&= \frac{(1+1/n)(1+1/n^2)}{1+2/n+3/n^2}|x|^3 \\
&\to |x|^3.
\end{aligned}$$

Hence, by the Ratio Test, the power series converges absolutely if $|x| < 1$, but not if $|x| > 1$. Comparing with Definition 8.10, we see that $R = 1$. $\square$

It follows that this series converges absolutely on $(-1, 1)$ and diverges on $\mathbb{R}\backslash[-1, 1]$ (Theorem 8.11).

Question What precisely is the set of values of x in $\mathbb{R}$ for which the power series in Example 8.12 converges?

Many functions are most conveniently *defined* by power series:

Definition 8.13

$$\begin{aligned}
\exp : \mathbb{R} \to \mathbb{R}, \qquad \exp(x) &= \sum_{n=0}^{\infty} \frac{x^n}{n!}, \\
\sin : \mathbb{R} \to \mathbb{R}, \qquad \sin(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \\
\cos : \mathbb{R} \to \mathbb{R}, \qquad \cos(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.
\end{aligned}$$

Exercise 8.14 Show that these functions are well defined (in fact, the series defining them converge absolutely) for all $x \in \mathbb{R}$. This amounts to showing that each of them has radius of convergence $R = \infty$.

So we can use power series to define functions like exp, sin, and cos. But what do we know about such functions? Are they continuous? Differentiable? Smooth? To answer these questions we'll need to apply the theory of uniform convergence developed in the previous chapter.

8.3 Uniform convergence of series of functions

In the language of chapter 7, Theorem 8.11 establishes that the sequence of polynomial functions

$$f_k(x) = \sum_{n=0}^{k} a_n x^n$$

converges **pointwise** on $(-R, R)$ to some function $f(x)$, which we denote

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

As we learned, pointwise convergence is a rather weak property, from which we can't usually deduce much about the limit function (e.g. it may be discontinuous, even if each function in the sequence is continuous). We can deduce much more about the limit function if we can establish that the convergence is actually **uniform**. Recall also that we can show that a sequence of functions converges uniformly even if we have no information about its limit function by showing that the sequence is **uniformly Cauchy**.

In this section we will develop a test which allows us to establish quickly and easily that a *series* of functions is uniformly Cauchy, and hence uniformly convergent. We will state the test for general series of functions, not just polynomials, because there are many other useful choices, for example, **Fourier series**

$$\sum_{n=1}^{\infty} a_n \sin nx.$$

Theorem 8.15 (The Weierstrass M Test) *Let $g_n : D \to \mathbb{R}$ be a sequence of functions and, M_n a sequence of non-negative real numbers such that*

(i) for all $x \in D$, $|g_n(x)| \leq M_n$, and

(ii) the series $\sum_{n=0}^{\infty} M_n$ converges.

Then the sequence of functions

$$f_k : D \to \mathbb{R}, \qquad f_k(x) = \sum_{n=0}^{k} g_n(x)$$

converges uniformly.

Proof: We will prove that (f_k) is uniformly Cauchy. Let $\varepsilon > 0$ be given. By assumption, the sequence

$$s_k = \sum_{n=0}^{k} M_n$$

converges, so is Cauchy (Theorem 1.20). Hence, there exists $N \in \mathbb{Z}^+$ such that for all $k > l \geq N$, $|s_k - s_l| < \varepsilon/2$. Since $M_n \geq 0$ for all n,

$$|s_k - s_l| = s_k - s_l = \sum_{n=l+1}^{k} M_n.$$

Hence, for all $k > l \geq N$,

$$\sum_{n=l+1}^{k} M_n < \frac{\varepsilon}{2}.$$

But, for all $x \in D$, and all n, $|g_n(x)| \leq M_n$. Hence, for all $x \in D$, and all $k > l \geq N$

$$\begin{aligned} |f_k(x) - f_l(x)| &= \left| \sum_{n=0}^{k} g_n(x) - \sum_{n=0}^{l} g_n(x) \right| \\ &= \left| \sum_{n=l+1}^{k} g_n(x) \right| \\ &\leq \sum_{n=l+1}^{k} |g_n(x)| \\ &\leq \sum_{n=l+1}^{k} M_n \\ &< \frac{\varepsilon}{2}. \end{aligned}$$

Hence, for all $k > l \geq N$,

$$\|f_k - f_l\| = \sup\{|f_k(x) - f_l(x)| : x \in D\} \leq \frac{\varepsilon}{2} < \varepsilon.$$

So (f_k) is uniformly Cauchy and hence uniformly convergent by Theorem 7.23. $\square$

Exercise 8.16 Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series. Prove that the Fourier series

$$\sum_{n=1}^{\infty} a_n \sin nx$$

converges uniformly to a continuous function $f : \mathbb{R} \to \mathbb{R}$.

Solution:

Returning to our main focus of study, it would be nice if we could use the M test to prove that power series converge uniformly on $(-R, R)$, where R is the radius of convergence. In fact, that's a bit too much to hope for, as we can deduce by thinking about the geometric series

$$\sum_{n=1}^{\infty} x^n.$$

This has $R = 1$ (as we showed directly in Example 8.9). If it converged *uniformly* its limit function $f : (-1, 1) \to \mathbb{R}$ would perforce be bounded. But we showed that

$$f(x) = \frac{1}{1-x}$$

which grows unbounded above as x approaches 1. So we deduce that this series does *not* converge uniformly on $(-1, 1)$. Nil desperandum! We can prove something almost as good as uniform convergence on $(-R, R)$:

Theorem 8.17 *Let the power series* $\sum_{n=1}^{\infty} a_n x^n$ *have radius of convergence* $R > 0$ *and* ρ *be any constant in* $(0, R)$*. Then the series converges uniformly on* $[-\rho, \rho]$*.*

Proof: We apply the Weierstrass M Test with $g_n(x) = a_n x^n$, $D = [-\rho, \rho]$ and $M_n = |a_n|\rho^n$. Certainly, for all $x \in [-\rho, \rho]$

$$|g_n(x)| = |a_n||x|^n \leq |a_n|\rho^n = M_n$$

so condition (i) is satisfied. Since $|\rho| < R$, the series converges absolutely at $x = \rho$, that is

$$\sum_{n=0}^{\infty} M_n = \sum_{n=0}^{\infty} |a_n|\rho^n$$

converges, by Theorem 8.11. Hence condition (ii) is satisfied also. So the series converges uniformly on $[-\rho, \rho]$ by the Weierstrass M Test (Theorem 8.15). $\square$

Each of the partial sums of a power series is a polynomial, and hence is continuous. It follows from Theorems 7.11 and 8.17 that the limit function

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

is continuous on $[-\rho, \rho]$ for every $\rho \in (0, R)$. But that implies that f is continuous on $(-R, R)$ since, at every point $a \in (-R, R)$ we can choose $\rho = (a + R)/2$, say (or any other $\rho \in (a, R)$). So we immediately deduce:

Corollary 8.18 *Let the power series* $\sum_{n=1}^{\infty} a_n x^n$ *have radius of convergence* $R > 0$. *Then this series converges to a continuous function* $f : (-R, R) \to \mathbb{R}$.

Excellent! But we want (much) more: we're actually going to prove that the limit function is *smooth* (that is, infinitely differentiable) on $(-R, R)$.

8.4 Differentiability of power series

Given a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ of radius of convergence R, we know that f is a continuous function on $(-R, R)$. Is f differentiable on $(-R, R)$? If it is, what is f'? Your first guess is probably

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1},$$

the series obtained by differentiating f term by term. Actually, this turns out to be absolutely correct, but *proving* it is not entirely trivial. Note that, in writing down our formula for f' we have swapped the order of two limits (the limit defining the derivative, and the limit defining the series). We need to justify this. Also, how do we know that the proposed power series for $f'(x)$ even converges on $(-R, R)$? Maybe its radius of convergence is less than R? Given the growth given by the extra factor of n, this seems like a genuine worry.

So, our first task will be to prove that a power series and the obvious candidate for its derivative always have the same radius of convergence. Since we will use this "obvious candidate" series a lot, it's helpful to develop some notation and a name for it.

Definition 8.19 Given a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$, its **termwise derivative** is the power series

$$\widehat{f}(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n.$$

Careful! At the moment, termwise **derivative** is just a name: we haven't (yet) established that $\widehat{f}$ is the derivative of f.

Lemma 8.20 *The power series* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *and its termwise derivative* $\widehat{f}(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ *have the same radius of convergence.*

Proof: We define the following subsets of $[0, \infty)$,

$$A = \{|x| \,:\, \sum |a_n x^n| \text{ converges}\}, \qquad B = \{|x| \,:\, \sum |n a_n x^{n-1}| \text{ converges}\},$$

and recall that $\sup A$ and $\sup B$ are the radii of convergence of f and $\widehat{f}$, respectively. We will prove that

(i) for all x with $|x| < \sup B$, $f(x)$ converges absolutely (so $|x| \in A$, and hence $\sup A \geq \sup B$), and

(ii) for all x with $|x| < \sup A$, $\widehat{f}(x)$ converges absolutely (so $|x| \in B$, and hence $\sup B \geq \sup A$).

It follows immediately that $\sup A = \sup B$, which is what we seek to prove.
(i) Let $x \in \mathbb{R}$ with $|x| < \sup B$. Then $s_k = \sum_{n=1}^{k} |n a_n x^{n-1}|$ converges (Theorem 8.11), and hence is bounded above, and

$$t_k = \sum_{n=0}^{k} |a_n x^n| = |a_0| + |x| \sum_{n=1}^{k} |a_n x^{n-1}| \leq |a_0| + |x| s_k.$$

So (t_k) is increasing and bounded above, and hence converges, by the Monotone Convergence Theorem. Hence, $|x| \in A$.
(ii) Let $x \in \mathbb{R}$ with $|x| < \sup A$. If $x = 0$ then $\widehat{f}(x)$ certainly converges, so we may assume $|x| > 0$. Choose $\rho \in (|x|, \sup A)$. Then $t_k = \sum_{n=0}^{k} |a_n| \rho^n$ converges (Theorem 8.11), and hence is bounded above. Now

$$s_k = \sum_{n=1}^{k} |n a_n x^{n-1}| = |x|^{-1} \sum_{n=1}^{k} n \left(\frac{|x|}{\rho} \right)^n |a_n| \rho^n.$$

We know that the sequence $n(|x|/\rho)^n \to 0$, since $(|x|/\rho) < 1$ (exponentials beat polynomials), so it must be bounded, that is, there exists $K > 0$ such that, for all n, $n(|x|/\rho)^n \leq K$. But then

$$s_k \leq |x|^{-1} \sum_{n=0}^{k} K |a_n| \rho^n = \frac{K}{|x|} t_k.$$

Hence, s_k is increasing and bounded above, so converges, by the Monotone Convergence Theorem. Hence, $|x| \in B$. $\square$

We now show that the termwise derivative $\widehat{f}(x)$ really is the derivative of $f(x)$. We do this by applying Theorem 7.16 to the sequence of partial sums of $f(x)$. It's a good idea to read the statement of Theorem 7.16 carefully before attempting to understand the next proof.

Theorem 8.21 *Suppose that* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *has radius of convergence* $R > 0$. *Then* f *is differentiable on* $(-R, R)$, *and*

$$f'(x) = \widehat{f}(x) := \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Proof: Choose and fix $\rho \in (0, R)$, and consider the sequence of functions

$$f_k : [-\rho, \rho] \to \mathbb{R}, \qquad f_k(x) = \sum_{n=0}^{k} a_n x^n.$$

This is a sequence of continuously differentiable functions which converges pointwise to f (Theorem 8.11), and its sequence of derivatives

$$f_k' : [-\rho, \rho] \to \mathbb{R}, \qquad f_k'(x) = \sum_{n=1}^{k} n a_n x^{n-1}$$

converges uniformly to $\widehat{f}$ (Lemma 8.20 and Theorem 8.17). Hence, by Theorem 7.16, f is differentiable and $f' = \widehat{f}$. This holds for all $x \in [-\rho, \rho]$, but $\rho \in (0, R)$ was arbitrary, so holds for all $x \in (-R, R)$. $\square$

This is the most important theorem in this chapter. It says that the derivative of a power series exists on its open interval of convergence and is just the power series obtained by termwise differentiation.

Since $f'(x)$ is also a power series with radius of convergence R, we can apply Theorem 8.21 to $f'(x)$ and deduce that $f : (-R, R) \to \mathbb{R}$ is actually *twice* differentiable. Further, $f''(x)$ also has radius of convergence R, so is differentiable on $(-R, R)$, that is, f is *three times* differentiable. In fact, we can keep applying Theorem 8.21 as often as we like, and we conclude that f is *smooth*:

Corollary 8.22 *Let* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *have radius convergence* $R > 0$. *Then* $f : (-R, R) \to \mathbb{R}$ *is a smooth function, and its* k^{th} *derivative is given by the power series*

$$f^{(k)}(x) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k},$$

which also has radius of convergence R. *In particular,*

$$a_k = \frac{f^{(k)}(0)}{k!}$$

for all $k \geq 0$.

Proof: For each integer $k \geq 0$, define

$$g_k(x) = \sum_{n=0}^{\infty} \frac{(n+k)!}{n!} a_{n+k} x^n.$$

I claim that each power series $g_k(x)$ has radius of convergence R and that, for all $x \in (-R, R)$, $f^{(k)}(x) = g_k(x)$. We prove this by induction on k.

Certainly the claim holds for $k = 0$, since $g_0 = f$. So, assume that the claim holds for some value $k \geq 0$, and consider g_{k+1}. Defining coefficients $b_n = \frac{(n+k)!}{n!} a_{n+k}$ so that $g_k(x) = \sum_{n=0}^{\infty} b_n x^n$, then

$$\begin{aligned} g_{k+1}(x) &= \sum_{n=0}^{\infty} \frac{(n+k+1)!}{n!} a_{n+k+1} x^n = \sum_{m=1}^{\infty} \frac{(m+k)!}{(m-1)!} a_{m+k} x^{m-1} \\ &= \sum_{m=1}^{\infty} m b_m x^{m-1}. \end{aligned}$$

Hence, by Lemma 8.20, g_{k+1} has the same radius of convergence as g_k, and by Theorem 8.21, $g_{k+1}(x)$ coincides with the derivative of g_k. But, by our induction hypothesis, g_k has radius of convergence R and coincides with $f^{(k)}$, so g_{k+1} has radius of convergence R and coincides with $f^{(k+1)}$. Hence, if the claim holds for some $k \geq 0$, it also holds for $k + 1$. Hence, by induction, the claim holds for all integers $k \geq 0$.

It follows that $f^{(k)}(0) = g_k(0) = k! a_k$, which completes the proof. $\square$

Corollary 8.22 applies to any power series with non-zero radius of convergence so, in particular, it applies to the series defining exp, sin, and cos.

Proposition 8.23 *The functions* $\exp : \mathbb{R} \to \mathbb{R}$, $\sin : \mathbb{R} \to \mathbb{R}$, $\cos : \mathbb{R} \to \mathbb{R}$ *defined in Definition 8.13 are smooth, and their derivatives are*

$$\exp' = \exp, \qquad \sin' = \cos, \qquad \cos' = -\sin.$$

Proof: That these functions are smooth follows immediately from Corollary 8.22. Furthermore, by Theorem 8.21

$$\exp'(x) = \sum_{n=1}^{\infty} n \left(\frac{1}{n!}\right) x^{n-1} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{m=0}^{\infty} \frac{x^m}{m!} = \exp(x)$$

for all $x \in \mathbb{R}$, so $\exp' = \exp$. The formulae for $\sin'$ and $\cos'$ follow from similar arguments. $\square$

With a bit of ingenuity, we can often use power series to exactly sum (that is, exactly compute the limit of) a given series. Here's a simple example.

Example 8.24 Claim: $\displaystyle\sum_{n=1}^{\infty} \frac{n}{7^n} = \frac{7}{36}$.

Proof: The geometric series $f(x) = \sum_{n=0}^{\infty} x^n$ has radius of convergence $R = 1$ and, for all $x \in (-1, 1)$,

$$f(x) = \frac{1}{1-x}.$$

Hence, by Theorem 8.21, the power series

$$\sum_{n=1}^{\infty} nx^{n-1}$$

also has radius of convergence $R = 1$ and, for all $x \in (-1, 1)$ converges to

$$f'(x) = \frac{1}{(1-x)^2}.$$

Evaluating this at $x = 1/7$ yields

$$\sum_{n=1}^{\infty} \frac{n}{7^{n-1}} = f'(1/7) = \left(\frac{7}{6}\right)^2.$$

Hence

$$\sum_{n=1}^{\infty} \frac{n}{7^n} = \frac{1}{7} \sum_{n=1}^{\infty} \frac{n}{7^{n-1}} = \frac{7}{36}.$$

□

Exercise 8.25 Exactly sum the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$. Rigorously justify your answer (of course!).

8.5 Properties of the exponential function

In this section we will prove that the function $\exp : \mathbb{R} \to \mathbb{R}$, defined as a power series, has all the properties you're familiar with. We begin with the following fundamental identity.

Lemma 8.26 *For all $x, y \in \mathbb{R}$, $\exp(x + y) = \exp(x)\exp(y)$.*

Proof: For each fixed number $b \in \mathbb{R}$, define the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \exp(x)\exp(b - x).$$

This is a product of differentiable functions, so is differentiable, and by the Product and Chain Rules, and Proposition 8.23,

$$f'(x) = \exp'(x)\exp(b - x) - \exp(x)\exp'(b - x) = 0.$$

Since $f'(x) = 0$ for all x in the interval $\mathbb{R}$, it follows that f is constant (Proposition 4.11). Hence, for all $x \in \mathbb{R}$ and any $b \in \mathbb{R}$, $f(x) = f(b)$, that is,

$$\exp(x)\exp(b - x) = \exp(b).$$

Applying this in the case where $b = x + y$ establishes the claim. $\square$

Lemma 8.26 explains why $\exp(x)$ is often denoted e^x and thought of as the constant $e = \exp(1) = \sum_{n=0}^{\infty} \frac{1}{n!}$ "raised to the power x". The point is that, written in this alternative notation, Lemma 8.26 looks like one of the standard algebraic rules of integer exponents:

$$e^{x+y} = e^x \times e^y \qquad \text{in analogy with} \qquad a^{n+m} = a^n \times a^m.$$

This analogy, and the associated notation, are a useful mnemonic, but it's important to realize that they are just that, an analogy. We don't literally *define* $\exp(x)$ to be the irrational number e "raised to the power x", for obvious reasons: what on earth does it mean to raise e to the power $\sqrt{2}$, for example? In fact, the logic works in the opposite direction. Having proved that the function exp, defined as a power series, satisfies a rule analogous to the behaviour of integer (and rational) exponents, we *define* e^x, for any exponent x (rational or irrational) to be $\exp(x)$.

Proposition 8.27 *For all $x \in \mathbb{R}$, $\exp(x) > 0$.*

Proof: We first note that, for all $x \in \mathbb{R}$, $\exp(x) \neq 0$ since, if there exists $x \in \mathbb{R}$ such that $\exp(x) = 0$, then Lemma 8.26 in the case $y = -x$ implies that

$$1 = \exp(0) = 0 \times \exp(-x) = 0.$$

Now $\exp(0) = 1 > 0$, and exp is differentiable, hence continuous, so it follows from the Intermediate Value Theorem that $\exp(x) > 0$ for all $x \in \mathbb{R}$ (if $\exp(x) < 0$ then there exists c between x and 0 such that $\exp(c) = 0$). $\square$

So $\exp : \mathbb{R} \to (0, \infty)$. I claim that exp is bijective and that $\ln : (0, \infty) \to \mathbb{R}$ is its inverse function. Recall (Definition 6.6) that

$$\ln : (0, \infty) \to \mathbb{R}, \qquad \ln x := \int_1^x \frac{1}{t} dt.$$

Proposition 8.28 $\exp : \mathbb{R} \to (0, \infty)$ *is bijective and* $\ln : (0, \infty) \to \mathbb{R}$ *is its inverse function.*

Proof: We must show that, for all $x \in \mathbb{R}$, $\ln(\exp(x)) = x$, and for all $y \in (0, \infty)$, $\exp(\ln y) = y$. Consider the function

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = \ln(\exp(x)) - x.$$

By the Chain Rule,

$$f'(x) = \frac{\exp(x)}{\exp(x)} - 1 = 0$$

for all x, so f is constant (Corollary 4.11). Hence, for all x, $f(x) = f(0) = \ln 1 - 0 = 0$, that is, $\ln(\exp(x)) = x$. Now, for all $y \in (0, \infty)$,

$$\ln(\exp(\ln y)) = \ln y,$$

as we have just shown. But ln is injective, so if $\ln z = \ln y$ then $z = y$. Hence $\exp(\ln y) = y$, as was to be shown.

It follows immediately that exp is surjective:

$$\forall y \in (0, \infty), \quad \exp(\ln(y)) = y,$$

and injective:

$$\exp(x) = \exp(x') \quad \Rightarrow \quad \ln(\exp(x)) = \ln(\exp(x')) \quad \Rightarrow \quad x = x'.$$

$\square$

So, for each positive number y, $\ln y$ is the real number whose exponential is y. Recall that *Euler's number* is

$$e = \exp(1) = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

It follows that $\ln e = 1$. There is an alternative way to define e, as the limit of the sequence $(1 + \frac{1}{n})^n$. You are now in a position to prove this.

Exercise 8.29 Prove that the sequence $x_n = \left(1 + \dfrac{1}{n}\right)^n$ converges to e.

Solution:

Proposition 8.28 implies that, for all $a > 0$, $\ln a$ is the unique real number y such that $\exp(y) = a$. This motivates the following definition:

Definition 8.30 Let $a > 0$ and $x \in \mathbb{R}$. Then a **to the power** x is

$$a^x = \exp(x \ln a).$$

It follows from Proposition 6.8 that this definition of a^x coincides with the more obvious definition of a^x in the case where x is an integer.

So now we can make sense of the function $f(x) = x^r$ for all $x > 0$ and any constant $r \in \mathbb{R}$ (even if r is irrational!). It is not hard to prove that $f : (0, \infty) \to \mathbb{R}$ is differentiable, and has the derivative we expect.

Proposition 8.31 *Let $f : (0, \infty) \to \mathbb{R}$ be the function $f(x) = x^r$, where r is any real constant. Then f is differentiable, and for all $x \in (0, \infty)$,*

$$f'(x) = rx^{r-1}.$$

Proof: Exercise. (Use the Chain Rule on $f(x) = \exp(r \ln x)$.) □

8.6 Analyticity versus smoothness

So far we have only considered power series based at 0, but the whole theory extends immediately to power series with some other base point $x_0 \in \mathbb{R}$, that is, series of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

The point is that $f(x) = g(x - x_0)$, where $g(x) = \sum_{n=0}^{\infty} a_n x^n$, so (by the Chain Rule) f is smooth on $(x_0 - R, x_0 + R)$ where R is the radius of convergence of g, and the derivative of f is again its termwise derivative. We will refer to R as the radius of convergence of f.

A function f is **analytic** if it locally coincides with some convergent power series, everywhere in its domain. More precisely:

Definition 8.32 Let $U \subseteq \mathbb{R}$ be open. Then $f : U \to \mathbb{R}$ is **analytic** if, for each $x_0 \in U$, there exists $\varepsilon > 0$ and a real sequnce (a_n) such that, for all $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$,

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

Example 8.33 $\exp : \mathbb{R} \to \mathbb{R}$ is analytic since

$$\exp(x) = \exp(x_0) \exp(x - x_0) = \sum_{n=0}^{\infty} \frac{\exp(x_0)}{n!} (x - x_0)^n$$

for all $x, x_0 \in \mathbb{R}$ (so for any x_0 we can take any $\varepsilon > 0$ in the definition above).

Exercise 8.34 Prove that $f : \mathbb{R} \backslash \{0\} \to \mathbb{R}$, $f(x) = 1/x$, is analytic.

If $f : U \to \mathbb{R}$ is analytic, it is certainly smooth, by Corollary 8.22 and the Localization Lemma, since it coincides on an open set $(x_0 - \varepsilon, x_0 + \varepsilon)$ with a convergent power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

So all analytic functions are smooth. Furthermore, by Corollary 8.22, we know that the coefficients of this series are

$$a_n = \frac{f^{(n)}(x_0)}{n!}.$$

That is, on $(x_0 - \varepsilon, x_0 + \varepsilon)$, the function f coincides with its **Taylor series**

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Now, given *any* smooth function on an open set U and a point x_0, we can compute all the derivative $f^{(n)}(x_0)$, and hence construct its Taylor series. Does it follow that all smooth functions are analytic? Rather remarkably, the answer is **no!** The rest of this section is devoted to constructing a class of functions $f : \mathbb{R} \to \mathbb{R}$ which are smooth but **not** analytic.

Definition 8.35 Given a polynomial $p(x) = a_0 + a_1 x + \cdots + a_k x^k$, we define the function $f_p : \mathbb{R} \to \mathbb{R}$ such that

$$f_p(x) = \begin{cases} 0, & x \leq 0 \\ p(1/x)\exp(-1/x), & x > 0. \end{cases}$$

We will prove that the function f_p is smooth but not (except in the trivial case $p(x) = 0$) analytic. This follows fairly quickly once we establish that $\lim_{x \to 0} f_p(x) = 0$.

Lemma 8.36 (Exponentials beat powers – sequences) *For any $k \in \mathbb{N}$ the sequence*

$$a_n = n^k \exp(-n) \to 0.$$

Proof:

$$\frac{a_{n+1}}{a_n} = \left(1 + \frac{1}{n}\right)^k \exp(-1) \to \frac{1}{e} < 1$$

so $\sum_{n=1}^{\infty} a_n$ converges by the Ratio Test. Hence $a_n \to 0$ by the Divergence Test. □

Lemma 8.37 (Exponentials beat powers – functions) *For any $k \in \mathbb{N}$,*

$$\lim_{x \to \infty} x^k \exp(-x) = 0.$$

Proof: Let $g(x) = x^k \exp(-x)$. We must show that, for each $\varepsilon > 0$, there exists $K \in \mathbb{R}$ such that, for all $x > K$, $|g(x) - 0| < \varepsilon$.

So, let $\varepsilon > 0$ be given. Since the sequence $g(n) = n^k \exp(-n) \to 0$ (Lemma 8.36), there exists $N \in \mathbb{Z}^+$ such that, for all $n \geq N$, $|g(n) - 0| < \varepsilon$, and hence $0 < g(n) < \varepsilon$. Let $K = \max\{k, N\}$. Then for all $x > K$,

$$g'(x) = (kx^{k-1} - x^k)\exp(-x) = -x^{k-1}(x - k)\exp(-x) < 0,$$

(by the Chain and Product Rules), so g is strictly decreasing (by Proposition 4.11). Hence, for all $x > K$, $0 < g(x) < g(K) < \varepsilon$ (since $K \geq N$). □

Lemma 8.38 *Let $f_p : \mathbb{R} \to \mathbb{R}$ be the function associated to a polynomial p, as in Definition 8.35. Then*

$$\lim_{x\to 0} f_p(x) = 0.$$

Proof: Let $p(x) = a_0 + a_1x + \cdots + a_kx^k$, and for each $j \in \mathbb{N}$, denote by m_j the monomial of degree j (so $m_j(x) = x^j$). Then

$$f_p = a_0 f_{m_0} + a_1 f_{m_1} + \cdots + a_k f_{m_k}$$

so, by the Algebra of Limits, it suffices to prove the claim in the case where $p(x) = m_k(x) = x^k$, a general monomial.

Let $\varepsilon > 0$ be given. By Lemma 8.37, there exists $K > 0$ such that, for all $x > K$, $|x^k \exp(-x) - 0| < \varepsilon$. Let $\delta = 1/K > 0$. Then, for all $x \in (0, \delta)$,

$$0 \leq f_p(x) = (1/x)^k \exp(-1/x) < \varepsilon,$$

since $1/x > 1/\delta = K$, and for all $x \in (-\delta, 0)$,

$$0 = f_p(x) < \varepsilon.$$

Hence, for all $x \in \mathbb{R}$ with $0 < |x - 0| < \delta$, $|f_p(x) - 0| < \varepsilon$. □

Theorem 8.39 *Let $f_p : \mathbb{R} \to \mathbb{R}$ be the function associated to a polynomial p, as in Definition 8.35. Then f_p is differentiable and $f_p' = f_q$ where $q(x) = x^2(p(x) - p'(x))$.*

Proof: f_p coincides with the differentiable function $p(1/x)\exp(-1/x)$ on the open set $(0, \infty)$, so is differentiable on $(0, \infty)$ with derivative

$$f'(x) = p'(x^{-1})(-x^{-2})\exp(-x^{-1}) + p(x^{-1})\exp(-x^{-1})(x^{-2}) = q(x^{-1})\exp(-x^{-1})$$

by the Chain and Product Rules, and the Localization Lemma (Lemma 3.23). Similarly, f_p coincides with the differentiable function 0 on the open set $(-\infty, 0)$, so is differentiable on $(-\infty, 0)$ with derivative 0 by the Localization Lemma. It remains to show that f_p is differentiable at 0 and $f_p'(0) = 0$. That is, we must prove that

$$\lim_{x\to 0} \frac{f_p(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f_p(x)}{x} = 0.$$

But $f_p(x)/x = f_P(x)$ where $P(x) = xp(x)$, so this follows immediately from Lemma 8.38. □

Corollary 8.40 *The function f_p defined in Definition 8.35 is smooth and has $f_p^{(n)}(0) = 0$ for all $n \geq 0$.*

Proof: Denote by X the set of functions $\{f_p : p \text{ a polynomial}\}$. Then for all $f_p \in X$, $f_p(0) = 0$ (by definition), f_p is differentiable, and $f_p' \in X$ (Theorem 8.39). Hence every element of X is infinitely differentiable and has all its derivatives zero at 0. $\square$

It follows that almost every function f_p is smooth, but not analytic. (The only exception is f_0, where we take $p(x)$ to be the trivial polynomial $p = 0$. Clearly $f_0 \equiv 0$ which is trivially analytic.) Let's consider the simplest example.

Example 8.41 Consider the special case $p(x) = 1$. The corresponding function is

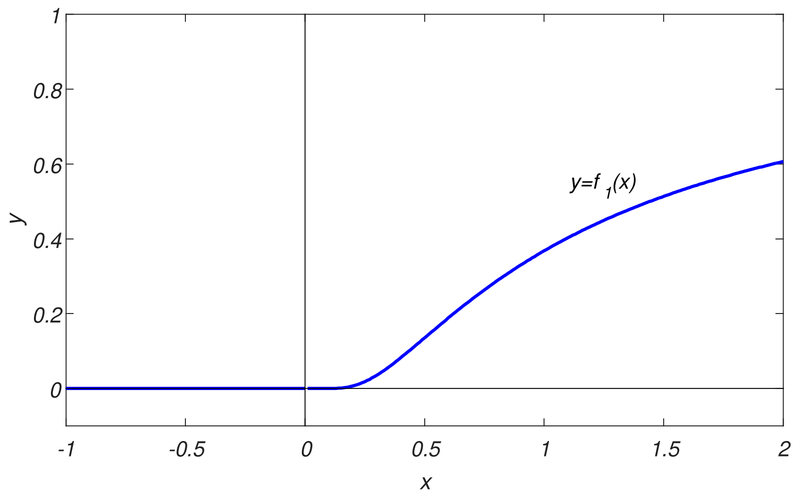
$$f(x) = \begin{cases} \exp(-1/x), & x > 0 \\ 0, & x \leq 0. \end{cases}$$

By Corollary 8.40, this is smooth and all its derivatives at 0 are 0. I claim that f is **not** analytic.

Assume, towards a contradiction, that f *is* analytic. Then there exist $\varepsilon > 0$ and $\{a_n : n \in \mathbb{N}\}$ such that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

for all $x \in (-\varepsilon, \varepsilon)$. But then, by Corollary 8.22, $a_n = f^{(n)}(0)/n! = 0$. Hence $f(x) = 0$ for all $x \in (-\varepsilon, \varepsilon)$. But $f(\varepsilon/2) = \exp(-2/\varepsilon) > 0$, a contradiction.



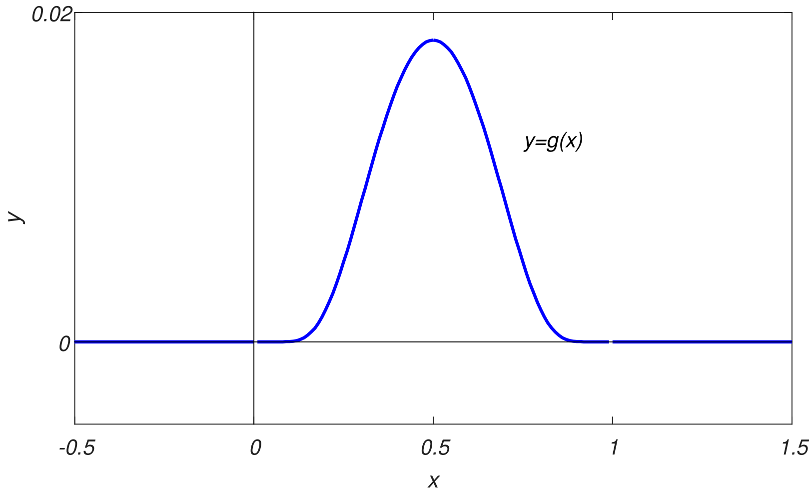
$\square$

It's not hard to construct from Example 8.41 smooth functions which vanish exactly outside some bounded interval (a, b) but are strictly positive on (a, b). Such functions are often called *bump functions*.

Example 8.42 Given an interval $[a, b]$ define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = f(x - a)f(b - x)$$

where f is the smooth function of Example 8.41. Then g is smooth by the product and chain rules. If $x \leq a$, $x - a \leq 0$ so $f(x - a) = 0$. If $x \geq b$, $b - x \leq 0$ so $f(b - x) = 0$. Hence, $g(x) = 0$ for all $x \notin (a, b)$. If $x \in (a, b)$, both $x - a$ and $b - x$ are positive, so $f(x - a) > 0$ and $f(b - x) > 0$, whence $g(x) > 0$. So g is strictly positive on (a, b). Here's a plot of g in the case $[a, b] = [0, 1]$:



Summary

- A **power series** is a series of the form $\sum_{n=0}^{\infty} a_n x^n$ where x is a real variable. Its **radius of convergence** is

$$R = \sup\{|x| \,:\, \textstyle\sum_{n=0}^{\infty} a_n x^n \text{ converges absolutely}\}.$$

- If a power series has radius of convergence R, it converges absolutely for all $|x| < R$ and diverges for all $|x| > R$.

- The **Weierstrass M Test**: if $|g_n(x)| \leq M_n$ for all $x \in D$ and the series $\sum_{n=0}^{\infty} M_n$ converges, then the series $\sum_{n=0}^{\infty} g_n(x)$ converges uniformly on D.

- Using the M test, we can prove that a power series with radius of convergence $R > 0$ converges **uniformly** on every interval $[-\rho, \rho]$, where $\rho \in (0, R)$.

- A power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with radius of convergence $R > 0$ is **differentiable** on $(-R, R)$ and its derivative is

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

  This power series also has radius of convergence R.

- Convergent power series are, in fact, **smooth**.

- A function $f : U \to \mathbb{R}$, where $U \subseteq \mathbb{R}$ is an open set, is **analytic** if for each $x_0 \in U$ there is $\varepsilon > 0$ and a power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ which converges to $f(x)$ for all $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$.

- All analytic functions are smooth, but **not all smooth functions are analytic**. An example of a smooth but non-analytic function is

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = \begin{cases} \exp(-1/x), & x > 0 \\ 0, & x \leq 0. \end{cases}$$