Chapter 7

Uniform convergence

7.1 Pointwise convergence versus uniform convergence

It is often convenient (or even necessary) to define a function $f : \mathbb{R} \to \mathbb{R}$ as a limit of a sequence of other functions $f_n : \mathbb{R} \to \mathbb{R}$, for example, using a power series,

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

or a Fourier series,

$$g(x) = \sum_{k=1}^{\infty} \frac{4}{(2k+1)\pi} \sin(2k+1)x.$$

These are limits of the sequences of functions

$$f_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} \qquad \text{and} \qquad g_n(x) = \sum_{k=0}^{n} \frac{4}{(2k+1)\pi} \sin kx$$

respectively. We would like to be able to tell which properties of the terms of the sequence are shared by the limit function. For example, all of the functions f_n, g_n defined above are smooth (that is infinitely differentiable) on the whole of $\mathbb{R}$. Can we conclude that f and g are also smooth? The answer is no, not immediately: it turns out that f *is* smooth (it's the exponential function) but g isn't even continuous. To address this kind of question we will need to develop some new tools.

First, let's ask, given a sequence of functions, $f_n : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}$ is some fixed set, what does it *mean* to say that (f_n) converges to $f : D \to \mathbb{R}$? The obvious answer (implicitly used abvove) is:

Definition 7.1 A sequence of functions $f_n : D \to \mathbb{R}$ **converges pointwise** to a function $f : D \to \mathbb{R}$ if, for each fixed $x \in D$, the real sequence $(f_n(x))$ converges (in the sense of Definition 1.1) to the real number $f(x)$.

Here's an instructive example:

Example 7.2 Consider the sequence $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = x^n$. If $0 \leq x < 1$ then

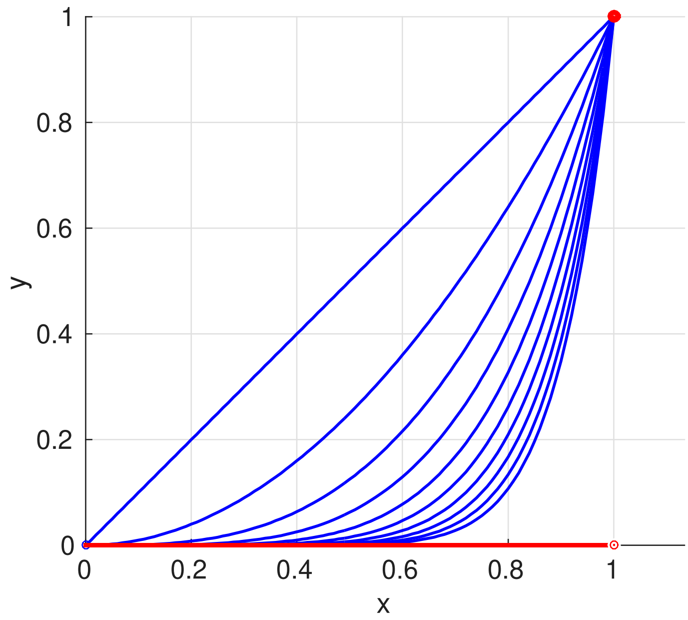
$$f_n(x) = x^n \to 0.$$

If $x = 1$, however,

$$f_n(1) = 1^n = 1 \to 1.$$

Hence (f_n) converges pointwise to the discontinuous function

$$f(x) = \begin{cases} 0, & 0 \le x < 1, \\ 1, & x = 1. \end{cases} \tag{7.1}$$



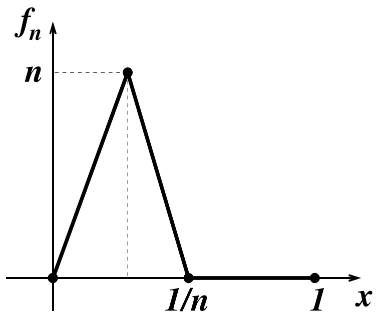
This illustrates a key weakness of pointwise convergence: it doesn't necessarily preserve continuity. Another weakness is that pointwise convergence doesn't interact well with integration. That is, a sequence of Riemann integrable functions $f_n : [a, b] \to \mathbb{R}$ may converge pointwise to a function $f : [a, b] \to \mathbb{R}$ which is not Riemann integrable or, even if it is, one may find that

$$\lim_{n\to\infty} \int_a^b f \neq \int_a^b f.$$

Example 7.3 For each $n \in \mathbb{Z}^+$, let $f_n : [0, 1] \to \mathbb{R}$ be the function

$$f_n(x) = \begin{cases} 2n^2 x, & 0 \le x \le \frac{1}{2n}, \\ 2n - 2n^2 x, & \frac{1}{2n} \le x \le \frac{1}{n}, \\ 0, & \frac{1}{n} \le x \le 1 \end{cases}$$

whose graph is depicted below:



Choose and fix any $x \in (0, 1]$. Then, for all $n > 1/x$, $x > 1/n$ so $f_n(x) = 0$. Hence, for each fixed $x \in (0, 1]$, the real sequence $f_n(x)$ is eventually 0. Hence $f_n(x) \to 0$. Furthermore $f_n(0) = 0$ for all n, so this sequence also converges to 0. So, if we define $f : [0, 1] \to \mathbb{R}$ to be the constant function $f(x) = 0$, then for all $x \in [0, 1]$, $\lim_{n\to\infty} f_n(x) = f(x)$.

Note that each f_n is continuous, hence Riemann integrable, and clearly

$$\int_0^1 f_n = \frac{1}{2} \times \frac{1}{n} \times n = \frac{1}{2},$$

a (constant) sequence that converges to $1/2$. But

$$\int_0^1 f = 0.$$

□

Nonetheless, pointwise limits do have some nice properties.

Proposition 7.4 *If (f_n) converges pointwise, its limit is unique.*

Proof: Assume, towards a contradiction, that (f_n) converges to both f and g, where $f \neq g$. Then, by assumption, there exists $c \in D$ such that $f(c) \neq g(c)$. But $f_n(c) \to f(c)$ and $f_n(c) \to g(c)$, which contradicts the uniqueness of the limit of a convergent real sequence. □

Similarly, we can show that pointwise limits have an Algebra of Limits property.

Proposition 7.5 *Let (f_n) converge to f pointwise and (g_n) to g pointwise on D. Then*

(i) $(f_n + g_n)$ converges to $f + g$ pointwise on D and

(ii) $(f_n g_n)$ converges to fg pointwise on D.

Proof: Exercise. Just apply the usual Algebra of Limits for sequences at each fixed $x \in D$. □

A much stronger, and more useful, notion of convergence is that of *uniform* convergence. To define this, it's convenient to make a preliminary definition.

Definition 7.6 The **sup norm** of a bounded function $f : D \to \mathbb{R}$ is

$$\|f\| := \sup\{|f(x)| \ : \ x \in D\}.$$

Example 7.7 Compute the sup norm of $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \dfrac{x^2}{1+x^2}$.
Solution: For all $x \in \mathbb{R}$,

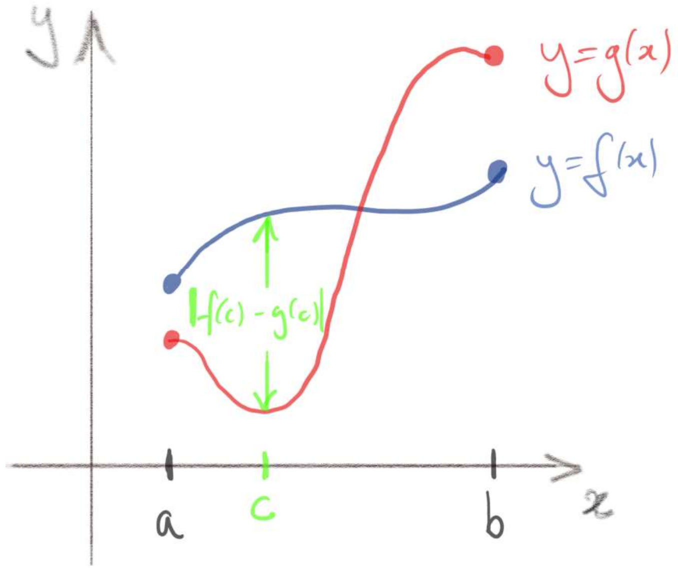
$$0 \leq f(x) = \frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2} < 1$$

and we see that the range of f is $[0, 1)$. Hence,

$$\|f\| = \sup\{|f(x)| : x \in \mathbb{R}\} = 1.$$

□

In the case where $f, g : [a, b] \to \mathbb{R}$ are continuous, we can give $\|f - g\|$ a geometric interpretation: it's the maximum separation between $(x, f(x))$ and $(x, g(x))$ as x ranges over $[a, b]$. We may think of this as a "distance" between the functions f and g.



So $\|f - g\|$ defines a sort of "distance" between bounded functions, in the same way that $|a - b|$ defines a distance between real numbers a and b. This line of thought leads to the main definition of this chapter:

Definition 7.8 A sequence of bounded functions $f_n : D \to \mathbb{R}$ **converges uniformly** to a function $f : D \to \mathbb{R}$ if the real sequence $\|f_n - f\| \to 0$.

Example 7.2 revisited Claim: The sequence $f_n : [0, 1] \to \mathbb{R}$, $f_n(x) = x^n$ does *not* converge uniformly to f, the function defined in (7.1).

Proof: For each fixed n,

$$|f_n(x) - f(x)| =$$

Hence

$$\{|f_n(x) - f(x)| \, : \, x \in [0,1]\} =$$

$$=$$

and so

$$\|f_n - f\| =$$

$$=$$

for all n. This sequence clearly doesn't converge to 0. □

So pointwise convergence certainly does *not* imply uniform convergence. However, the converse holds:

Theorem 7.9 *If a sequence $f_n : D \to \mathbb{R}$ converges uniformly to $f : D \to \mathbb{R}$, it converges pointwise to f.*

Proof: Assume the sequence of functions $f_n : D \to \mathbb{R}$ converges uniformly to $f : D \to \mathbb{R}$. Then, for each fixed $x \in D$,

$$0 \leq |f_n(x) - f(x)| \leq \sup\{|f_n(y) - f(y)| \, : \, y \in D\} = \|f_n - f\| \to 0,$$

so $|f_n(x) - f(x)| \to 0$ by the Squeeze Rule. Hence $f_n(x) \to f(x)$, Since this holds for each $x \in D$, f_n converges pointwise to f. □

Corollary 7.10 *If (f_n) converges uniformly, its limit is unique.*

Proof: Assume f_n converges uniformly to both f and g. Then f_n converges pointwise to f and f_n converges pointwise to g (Theorem 7.9). Hence $f = g$ (Proposition 7.4). □

We have seen that pointwise convergence doesn't always preserve continuity. That is, a sequence of continuous functions can converge pointwise to a discontinuous function (Example 7.2). A key advantage of *uniform* convergence is that it *does* preserve continuity.

Theorem 7.11 *Let $f_n : D \to \mathbb{R}$ be a sequence of continuous functions converging uniformly to $f : D \to \mathbb{R}$. Then f is continuous.*

Before diving into the proof of this, let's outline the strategy. We wish to show that the limit function f is continuous at a for each $a \in D$. Using the ε—δ criterion for continuity (Theorem 2.21), this amounts (roughly speaking) to showing that $|f(x) - f(a)|$ can be made as small as we like by taking $|x-a|$ sufficiently small. The key observation is that, for any fixed $n \in \mathbb{Z}^+$,

$$\begin{aligned}|f(x) - f(a)| &= |f(x) - f_n(x) + f_n(x) - f_n(a) + f_n(a) - f(a)| \\ &\leq |f_n(x) - f(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)| \\ &\leq \|f_n - f\| + |f_n(x) - f_n(a)| + \|f_n - f\|.\end{aligned}$$

Now $\|f_n - f\|$ can be made as small as we like by choosing n suitably large (since f_n converges to f uniformly, meaning $\|f_n - f\| \to 0$). Having chosen n suitably large, f_n is a fixed continuous function, so $|f_n(x) - f_n(a)|$ can be made as small as we like by taking $|x-a|$ sufficiently small. OK, let's write it up:

Proof of Theorem 7.11: We must prove that f is continuous at a for all $a \in D$. By Theorem 2.21, it suffices to show that, for each $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x \in D$ with $|x-a| < \delta$, $|f(x) - f(a)| < \varepsilon$.

So let a fixed $\varepsilon > 0$ be given. Since f_n converges to f uniformly, $\|f_n - f\| \to 0$. Hence, there exists $N \in \mathbb{Z}^+$ such that, for all $n \geq N$, $\|f_n - f\| < \varepsilon/3$. In particular,

$$\|f_N - f\| = \sup\{|f_N(x) - f(x)| \,:\, x \in D\} < \frac{\varepsilon}{3},$$

and hence

$$|f_N(x) - f(x)| < \frac{\varepsilon}{3} \qquad \text{for all } x \in D. \tag{7.2}$$

Furthermore, f_N is, by assumption, continuous, so, by Theorem 2.21, there exists $\delta > 0$ such that, for all $x \in D$ with $|x-a| < \delta$,

$$|f_N(x) - f_N(a)| < \frac{\varepsilon}{3}.$$

Hence, for all $x \in D$ with $|x-a| < \delta$,

$$\begin{aligned}|f(x) - f(a)| &= |f(x) - f_N(x) + f_N(x) - f_N(a) + f_N(a) - f(a)| \\ &\leq |f_N(x) - f(x)| + |f_N(x) - f_N(a)| + |f_N(a) - f(a)| \\ &< \frac{\varepsilon}{3} + |f_N(x) - f_N(a)| + \frac{\varepsilon}{3} \qquad \text{by (7.2)} \\ &< \varepsilon.\end{aligned}$$

□

This gives an indirect way of showing that $f_n(x) = x^n$ does not converge uniformly on $[0,1]$. Assume, towards a contradiction, that f_n converges uniformly to some function f. Then f_n converges pointwise to f (Theorem 7.9) and hence f must be the discontinuous function defined in equation (7.1) (by Proposition 7.4). But this contradicts Theorem 7.11 (because each f_n is continuous on $[0,1]$).

Exercise 7.12 Determine whether the following sequences of functions converge pointwise. For those that do, determine whether they converge uniformly.

(i) $f_n : [0, 1] \to \mathbb{R},\ f_n(x) = x/n.$

(ii) $f_n : [0, \pi] \to \mathbb{R},\ f_n(x) = \sin(x + x^2/n).$

Solution:

(i)

(ii)

7.2 Uniform convergence and calculus

We saw in Example 7.3 that a sequence of continuous, and hence Riemann integrable, functions $f_n : [0,1] \to \mathbb{R}$ which converge pointwise to a continuous function f, may have integrals which do *not* converge to the integral of f. So pointwise convergence does not justify a manoeuvre like

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b \lim_{n\to\infty} f_n,$$

even if both sides of the equation exist. Things are much nicer if we know the convergence $f_n \to f$ is uniform, however:

Theorem 7.13 *Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of continuous functions converging uniformly to $f : [a,b] \to \mathbb{R}$. Then*

$$\int_a^b f_n \to \int_a^b f.$$

Proof: First note that f is necessarily continuous (Theorem 7.11) and hence Riemann integrable (Theorem 5.20), so the proposed limit $\int_a^b f$ certainly exists. Now, for all n,

$$\begin{aligned} 0 \le \left| \int_a^b f_n - \int_a^b f \right| &= \left| \int_a^b (f_n - f) \right| && \text{(Theorem 5.22)} \\ &\le \int_a^b |f_n - f| && \text{(Theorem 5.24)} \\ &\le (b-a)\|f_n - f\| && \text{(Theorem 5.23).} \end{aligned}$$

Hence, $\int_a^b f_n \to \int_a^b f$ by the Squeeze Rule. □

We deduce from this that the sequence $f_n : [0,1] \to \mathbb{R}$, of "triangle" functions defined in Example 7.3 does *not* converge uniformly. Note that its pointwise limit (the constant function 0) is continuous, so we couldn't deduce this from Theorem 7.11.

Example 7.14 Claim: $\lim_{n\to\infty}\int_0^\pi \sin(x+\frac{x^2}{n})dx = 2.$

Proof: We saw in Exercise 7.12 that $f_n : [0,\pi] \to \mathbb{R}$, $f_n(x) = \sin(x+x^2/n)$, converges uniformly to sin. Hence, by Theorem 7.13,

$$\begin{aligned}\int_0^\pi f_n &\to \int_0^\pi \sin x\, dx \\ &= [-\cos x]_0^\pi = 2\end{aligned}$$

by the Fundamental Theorem of the Calculus (version 2). □

Note we have no hope of computing the integrals $\int_0^\pi f_n$ exactly (I invite you to try to write down an antiderivative for f_n), so this limit can't be computed directly.

Exercise 7.15 Compute

$$\lim_{n\to\infty}\int_0^1 \frac{n}{n+x^n}dx,$$

rigorously justifying your answer.

Solution:

Our next goal is to prove an analogue of Theorem 7.13 for *derivatives*. Under what circumstances can we be sure that a sequence f_n of differentiable functions converges to a differentiable function? The functions f_n in the following theorem are assumed to be *continuously differentiable*, meaning that each function $f_n : [a, b] \to \mathbb{R}$ is differentiable and that its derivative $f_n' : [a, b] \to \mathbb{R}$ is continuous. (Recall that every differentiable function is continuous, but its *derivative* isn't necessarily continuous: see Example 4.13 for a counterexample.)

Theorem 7.16 *Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of continuously differentiable functions which converge pointwise to $f : [a, b] \to \mathbb{R}$, and whose derivatives $f_n' : [a, b] \to \mathbb{R}$ converge uniformly to $g : [a, b] \to \mathbb{R}$. Then f is continuously differentiable and $f' = g$.*

Proof: For each $n \in \mathbb{Z}^+$ let $F_n : [a, b] \to \mathbb{R}$ be defined by

$$F_n(x) = \int_a^x f_n'.$$

Note that this exists since f_n' is, by assumption, continuous. By version 2 of the Fundamental Theorem of the Calculus,

$$F_n(x) = f_n(x) - f_n(a).$$

By Theorem 7.13, for each fixed x,

$$f_n(x) - f_n(a) = F_n(x) \to \int_a^x g,$$

since f_n' converges uniformly to g on $[a, x]$. But f_n converges pointwise to f, so $f_n(x) \to f(x)$ and $f_n(a) \to f(a)$. Hence, $f_n(x) - f_n(a) \to f(x) - f(a)$ (by the Algebra of Limits). But limits are unique, so

$$f(x) - f(a) = \int_a^x g.$$

Hence, by version 1 of the Fundamental Theorem of the Calculus, f is differentiable and $f'(x) = g(x)$. □

At the moment, it's hard to see why Theorem 7.16 is useful, because it's hard to imagine a situation where we can check that f_n' converges uniformly to a continuous function g without already knowing that the limit f of f_n is differentiable. For example, if we apply this Theorem to the sequence of functions in Exercise 7.15 we can deduce (at some effort) that the constant function $f(x) = 1$ is differentiable and has derivative 0 – hardly a startling revelation!

We will see in the next chapter that Theorem 7.16 can be extremely powerful when applied to the sequence of partial sums of a convergent power series. But before we can explore that application, we need to develop a way to prove that a sequence f_n of functions converges uniformly to *something* f even when we can't really say explicitly what f is (except tautologically: $f(x) = \lim_{n\to\infty} f_n(x)$).

7.3 Completeness of the set of bounded functions

Recall that a real sequence (a_n) is **Cauchy** if, for each $\varepsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that, for all $n, m \geq N$, $|a_n - a_m| < \varepsilon$ (Definition 1.16), and that this condition turns out to be *equivalent* to being convergent, that is, (a_n) converges if and only if it is Cauchy (Theorem 1.20). This Theorem allows us to prove that a real sequence converges to *something* $A \in \mathbb{R}$ even when we have no idea what A actually is. The aim of this section is to repeat this trick for uniform convergence of sequences of functions.

For a sequence of *functions* $f_n : D \to \mathbb{R}$, where D is some fixed subset of $\mathbb{R}$, we can define what it means to be *uniformly* Cauchy, in much the same way that we defined uniform convergence (Definition 7.8):

Definition 7.17 A sequence of bounded functions $f_n : D \to \mathbb{R}$ is **uniformly Cauchy** if, for each $\varepsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that, for all $n, m \geq N$, $\|f_n - f_m\| < \varepsilon$.

As usual, $\|\cdot\|$ denotes the sup norm here. Note that this is only well defined for *bounded* functions, which is why Definition 7.17 specifies that each $f_n : D \to \mathbb{R}$ is bounded.

Exercise 7.18 Show, directly from Definition 7.17, that the sequence

$$f_n : [-1, 1] \to \mathbb{R}, \qquad f_n(x) = \frac{x^3}{n}$$

is uniformly Cauchy.
Solution:

Our goal in this section is to prove that a sequence of bounded functions is uniformly convergent if and only if it is uniformly Cauchy. The "only if" direction of this statement is much easier to prove than the "if" direction, so let's start with that.

Theorem 7.19 *Let $f_n : D \to \mathbb{R}$ be a sequence of bounded functions converging uniformly to a bounded function $f : D \to \mathbb{R}$. Then (f_n) is uniformly Cauchy.*

The proof of this follows the proof of Lemma 1.17 (the analogous statement for real sequences) almost word for word. To make this work, we need a "triangle inequality" for the sup norm.

Lemma 7.20 *For all bounded functions $f, g : D \to \mathbb{R}$,*

$$\|f + g\| \leq \|f\| + \|g\|.$$

Proof: For all $x \in D$, $|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \|f\| + \|g\|$, since $\|f\|$ is an upper bound on $\{|f(x)| : x \in D\}$ and $\|g\|$ is an upper bound on $\{|g(x)| : x \in D\}$. Hence, $\|f\| + \|g\|$ is an upper bound on $\{|f(x) + g(x)| : x \in D\}$. But $\|f + g\|$ is, by definition, the *least* upper bound on this set. Hence $\|f + g\| \leq \|f\| + \|g\|$. $\square$

Proof of Theorem 7.19: Let $\varepsilon > 0$ be given. Since (f_n) converges to f uniformly, there exists $N \in \mathbb{Z}^+$ such that, for all $n \geq N$, $\|f_n - f\| < \varepsilon/2$. Hence, for all $n, m \geq N$,

$$\|f_n - f_m\| = \|f_n - f - (f_m - f)\| \leq \|f_n - f\| - \|f_m - f\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

by Lemma 7.20. $\square$

OK, now for the hard(er) part. We first note that, if (f_n) is *uniformly* Cauchy, it is certainly *pointwise* Cauchy, hence pointwise convergent. That is:

Lemma 7.21 *Let $f_n : D \to \mathbb{R}$ be a uniformly Cauchy sequence of bounded functions. Then (f_n) converges pointwise to some function $f : D \to \mathbb{R}$.*

Proof: Choose and fix $x \in D$. I claim that the real sequence $(f_n(x))$ is Cauchy. To see this, let $\varepsilon > 0$ by given. Since (f_n) is uniformly Cauchy, there exists $N \in \mathbb{Z}^+$ such that, for all $n, m \geq N$, $\|f_n - f_m\| < \varepsilon$. Hence, for all $n, m \geq N$,

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\| < \varepsilon.$$

Since the real sequence $(f_n(x))$ is Cauchy, it converges, by Theorem 1.20, to some limit $f(x) \in \mathbb{R}$. Allowing x to vary in D, we obtain a function $f : D \to \mathbb{R}$ to which (f_n) converges pointwise. $\square$

Note that we don't (yet) know that the pointwise limit $f : D \to \mathbb{R}$ of our uniformly Cauchy sequence of bounded functions is itself a bounded function! We need to prove this.

Lemma 7.22 *Let a uniformly Cauchy sequence of bounded functions $f_n : D \to \mathbb{R}$ converge pointwise to $f : D \to \mathbb{R}$. Then f is bounded.*

Proof: We must show that there exists $M > 0$ such that, for all $x \in D$, $|f(x)| \leq M$.

Since (f_n) is uniformly Cauchy, there exists $N \in \mathbb{Z}^+$ such that, for all $n, m \geq N$, $\|f_n - f_m\| < 1$. Furthermore, $f_N : D \to \mathbb{R}$ is a bounded function, so there exists $K > 0$ such that, for all $x \in D$, $|f_N(x)| \leq K$. I claim that $|f|$ is bounded above by $K + 1$.

To see this, choose and fix $x \in D$. Then, for all $n \in \mathbb{Z}^+$,

$$\begin{aligned} |f(x)| &= |f(x) - f_n(x) + f_n(x) - f_N(x) + f_N(x)| \\ &\leq |f(x) - f_n(x)| + |f_n(x) - f_N(x)| + |f_N(x)| \\ &\leq |f(x) - f_n(x)| + \|f_n - f_N\| + K \end{aligned}$$

Hence, for all $n \geq N$,

$$|f(x)| \leq |f(x) - f_n(x)| + 1 + K. \tag{7.3}$$

But, by assumption, $f_n(x) \to f(x)$, so the sequence on the right hand side of (7.3) converges to $1 + K$. Since this sequence is bounded below by $|f(x)|$, it follows that $1 + K \geq |f(x)|$ (Proposition 1.7). This is true whatever $x \in D$ we choose, so $|f|$ is bounded above by $M = K + 1$. $\square$

So we now know that every uniformly Cauchy sequence of bounded functions converges *pointwise* to a *bounded* function. The last job is to show that this convergence is actually *uniform* (recall uniform convergence implies pointwise convergence, but not vice versa, so this is not automatic).

Theorem 7.23 *A sequence $f_n : D \to \mathbb{R}$ of bounded functions converges uniformly if and only if it is uniformly Cauchy.*

Proof: We have already proved the "only if" part (Theorem 7.19).

Assume that (f_n) is uniformly Cauchy. Then (f_n) converges pointwise to some bounded function $f : D \to \mathbb{R}$ by Lemmas 7.21 and 7.22. We must show that $\|f_n - f\| \to 0$.

Let $\varepsilon > 0$ be given. Since (f_n) is uniformly Cauchy, there exists $N \in \mathbb{Z}^+$ such that, for all $n, m \geq N$, $\|f_n - f_m\| < \varepsilon/4$. I claim that, for all $n \geq N$,

$$\|f_n - f\| < \varepsilon.$$

To see this, choose and fix $x \in D$. Then, for all $n, m \geq N$,

$$\begin{aligned} |f_n(x) - f(x)| &= |f_n(x) - f_m(x) + f_m(x) - f(x)| \\ &\leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| \\ &\leq \|f_n - f_m\| + |f_m(x) - f(x)| \\ &< \frac{\varepsilon}{4} + |f_m(x) - f(x)|. \qquad (7.4) \end{aligned}$$

Note this inequality holds for *all* $m \geq N$. Now (f_n) converges pointwise to f, so $f_n(x) \to f(x)$. Hence, there exists $N_1 \in \mathbb{Z}^+$ (depending on x) such that, for all $m \geq N_1$, $|f_m(x) - f(x)| < \varepsilon/4$. Applying (7.4) in the case $m = \max\{N, N_1\}$, we see that, for all $n \geq N$,

$$|f_n(x) - f(x)| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

This inequality holds for all $x \in D$, and N is independent of x, so, for all $n \geq N$,

$$\|f_n - f\| = \sup\{|f_n(x) - f(x)| \,:\, x \in D\} \leq \frac{\varepsilon}{2} < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $\|f_n - f\| \to 0$. □

Let us denote the set of bounded functions on D

$$B(D) := \{f : D \to \mathbb{R} \,:\, f \text{ is bounded}\}$$

and the set of continuous functions on D

$$C(D) := \{f : D \to \mathbb{R} \,:\, f \text{ is continuous}\}$$

For general sets D, neither of these is a subset of the other. For example $f(x) = x^2$ is in $C(\mathbb{R})$ but not $B(\mathbb{R})$, and $g(x) = 1$ for $x \geq 0$, $g(x) = 0$ for $x < 0$ is in $B(\mathbb{R})$ but not $C(\mathbb{R})$. In the case where $D = [a, b]$, a closed bounded interval, however, every continuous function is bounded (by the Extreme Value Theorem), so $C([a, b]) \subset B([a, b])$.

We have just shown that the set $B(D)$ enjoys (with respect to the sup norm) a fundamental property of the set $\mathbb{R}$ (with respect to absolute value): every Cauchy sequence in the set converges to a limit in the set. Mathematicians call this property **completeness**. So what we have just shown is that $B(D)$ is *complete* with respect to the sup norm. In the case where $D = [a, b]$, it's not hard to deduce that $C(D)$ is also complete with respect to the sup norm.

Theorem 7.24 *If a sequence of continuous functions $f_n : [a, b] \to \mathbb{R}$ is uniformly Cauchy, then it converges uniformly to some continuous function $f : [a, b] \to \mathbb{R}$.*

Proof: Each f_n is a bounded function (by the Extreme Value Theorem), so (f_n) converges uniformly to some bounded function $f : [a, b] \to \mathbb{R}$ by Theorem 7.23. But f is continuous by Theorem 7.11. $\square$

The neat thing about Theorems 7.23 and 7.24 is that they allow us to prove that a sequence of functions converges *uniformly* without knowing what its limit is. As we will see, this is extremely useful when we come to consider functions defined by power series.

Summary

- A sequence of functions $f_n : D \to \mathbb{R}$ **converges pointwise** to $f : D \to \mathbb{R}$ if, for each $x \in D$, $f_n(x) \to f(x)$.
- The **sup norm** of a function $f : D \to \mathbb{R}$ is $\|f\| = \sup\{|f(x)| : x \in D\}$.
- A sequence of functions $f_n : D \to \mathbb{R}$ **converges uniformly** to $f : D \to \mathbb{R}$ if the real sequence $\|f_n - f\|$ converges to 0.
- If f_n converges uniformly to f, it converges pointwise to f. The converse is **false**.
- If a sequence of continuous functions $f_n : D \to \mathbb{R}$ converges uniformly to f then f is continuous.
- If a sequence of continuous functions $f_n : [a, b] \to \mathbb{R}$ converges uniformly to f then $\displaystyle\int_a^b f_n \to \int_a^b f$.
- If a sequence of continuously differentiable functions $f_n : [a, b] \to \mathbb{R}$ converges pointwise to f, and its sequence of derivatives $f_n' : [a, b] \to \mathbb{R}$ converges uniformly to g, then f is differentiable and $f' = g$.
- A sequence $f_n : D \to \mathbb{R}$ of bounded functions is **uniformly Cauchy** if, for each $\varepsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that, for all $n, m \geq N$, $\|f_n - f_m\| < \varepsilon$.
- A sequence of bounded functions is uniformly convergent if and only if it is uniformly Cauchy.