Workshop 3: solutions for week 4

1. I claim that $f'(1) = 5$. Proof: Given any $\varepsilon > 0$, let $\delta = \varepsilon/2$. Then for all $x \in \mathbb{R}$ with $0 < |x - 1| < \delta$,

$$\begin{aligned}\left|\frac{f(x) - f(1)}{x - 1} - 5\right| &= \left|\frac{2x^2 + x - 3}{x - 1} - 5\right| \\ &= |(2x + 3) - 5| \\ &= 2|x - 1| \\ &< 2\delta \\ &= \varepsilon.\end{aligned}$$

Hence

$$\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = 5,$$

so f is differentiable at 1 (with $f'(1) = 5$).

2. (a) $\forall x \in D$, $f(x) \le f(c)$.
   D could be any set. In particular, it need not be a subset of $\mathbb{R}$.

   (b) We can deduce that $f'(b) \ge 0$.
   Proof: Let x_n be any sequence in $[a, b)$ converging to b. Since

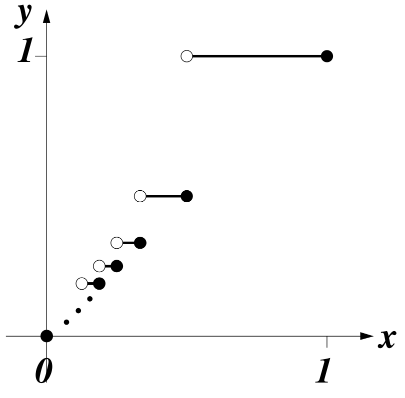
$$\lim_{x \to b} \frac{f(x) - f(b)}{x - b} = f'(b),$$

   the sequence

$$z_n = \frac{f(x_n) - f(b)}{x_n - b} \to f'(b)$$

   (Theorem 2.14). But f attains a maximum at b, so for all $n \in \mathbb{Z}^+$, $f(x_n) \le f(b)$. Further, $x_n \in [a, b)$ so $x_n < b$. Hence, for all n, $f(x_n) - f(b) \le 0$ and $x_n - b < 0$, so $z_n \ge 0$. Hence, $f'(b) = \lim z_n \ge 0$ (Proposition 1.7). $\square$

3. (a)



   (b) I claim that f **is** differentiable at 0, and $f'(0) = 1$.

Proof: By Theorem 2.14, it suffices to show that, for all sequences (x_n) in $(0,1]$ such that $x_n \to 0$,

$$\frac{f(x_n) - f(0)}{x_n - 0} = \frac{f(x_n)}{x_n} \to 1.$$

To prove this, note that, for all $x \in (0,1)$, there exists some (unique) $n \in \mathbb{Z}^+$ such that $x \in (1/(n+1), 1/n]$. Then

$$f(x) = \frac{1}{n} \geq x,$$

and

$$\begin{aligned} && x &> \frac{1}{n+1} \\ &\Rightarrow & \frac{1}{x} &< n+1 \\ &\Rightarrow & n &> \frac{1}{x} - 1 = \frac{1-x}{x} \\ \Rightarrow \quad f(x) = && \frac{1}{n} &< \frac{x}{1-x}. \end{aligned}$$

Hence, for all $x \in (0,1)$,

$$x \leq f(x) < \frac{x}{1-x}.$$

[Aside: can you see this on your graph? Add the curves $y = x$ and $y = x/(1-x)$.] Hence

$$1 \leq \frac{f(x_n)}{x_n} < \frac{1}{1-x_n}$$

and so $f(x_n)/x_n \to 1$ by the Squeeze Rule and the Algebra of Limits.

(c)
- Bounded? Yes, above by 1, below by 0
- Differentiable? No. for example, it's discontinuous at $1/2$, so can't be differentiable at $1/2$.
- Continuous? No, as already observed.
- Surjective? No. Since it's bounded and has codomain $\mathbb{R}$, it can't be surjective. For example, it never attains the value -2.
- Injective? No. For example $f(3/4) = f(1) = 1$.
- Monotonic? Yes, it's increasing. Note that this means that for all $x, y \in [0,1]$, if $x < y$ then $f(x) \leq f(y)$. (Note the non-strict inequality.)