Chapter 5

Integration

5.1 Suprema and infima

In this section we will develop a mathematcally rigorous theory of integration. This will make frequent use of the idea of the *supremum* and *infimum* of a bounded set $A \subset \mathbb{R}$. Recall that A is **bounded above** if there exists $K \in \mathbb{R}$ such that, for all $x \in A$, $x \leq K$. Any such K is called an **upper bound** on A. "Bounded below" and "lower bound" are defined analogously: $A \subset \mathbb{R}$ is **bounded below** if there exists $L \in \mathbb{R}$ such that, for all $x \in A$, $x \geq L$, and any such L is called a **lower bound** on A. A set is **bounded** if it is bounded above and below.

Definition 5.1 Let A be a nonempty subset of $\mathbb{R}$. The **supremum** of A, denoted $\sup A$, is the least upper bound on A, if this exists. That is, $\sup A$ is a real number with two properties:

(i) $\sup A$ is an upper bound on A (i.e. for all $x \in A$, $x \leq \sup A$), and

(ii) no number less than $\sup A$ is an upper bound on A.

Similarly, the **infimum** of A, denoted $\inf A$, is the greatest lower bound on A, if this exists. That is, $\inf A$ is a real number with two properties:

(i) $\inf A$ is a lower bound on A (i.e. for all $x \in A$, $x \geq \inf A$), and

(ii) no number greater than $\inf A$ is a lower bound on A.

One of the defining properties of $\mathbb{R}$ is:

The Axiom of Completeness Every nonempty subset of $\mathbb{R}$ which is bounded above has a supremum in $\mathbb{R}$.

Example 5.2 Let $A = (-\infty, 1)$. This is nonempty and bounded above, by 1, for example. Hence, by the Axiom of Completeness, it must have a supremum. In fact, $\sup A = 1$.

Proof: We've already observed that 1 is an upper bound on A. Let $K < 1$. Then $x = (K+1)/2 < (1+1)/2 = 1$ so $x \in A$. But $x > (K+K)/2 = K$, so K is not an upper bound on A. Hence 1 is the *least* upper bound on A. $\square$

Exercise 5.3 Let $B = \{1/n : n \in \mathbb{Z}^+\}$. Find $\sup B$ and $\inf B$. Rigorously justify your answers (i.e. *prove* they satisfy Definition 5.1).

Warning! Do not confuse "supremum" with "maximum" or "infimum" with "minimum". Note that, in the Example 5.2, $A = (-\infty, 1)$ does *not* have a maximum element! It does have a supremum, however, $1 \notin A$. This set is unbounded below, so has no infimum. In Example 5.3, B has an infimum, but does *not* have a minimum element: given any $1/n \in B$ there is always some other element $1/m \in B$ which is smaller (e.g. $1/m = 1/(n+1)$). Do we know that every (nonempty) set which is bounded below has an infimum (greatest lower bound)? Yes!

Theorem 5.4 *Let $A \subset \mathbb{R}$ be nonempty and bounded below. Then A has an infimum in $\mathbb{R}$.*

Proof: Let B be the set of lower bounds on A. Then B is nonempty (since A is bounded below) and is bounded above (by any element of A – such an element exists since A is nonempty). Hence B has a supremum, L say. I claim that $L = \inf A$. First note that L is an upper bound on B, so if $K > L$ it isn't in B, so is *not* a lower bound on A. Second, note that every element x of A is an upper bound on B, and L is the *least* upper bound on B, so $L \leq x$. Hence L is a lower bound on A. $\square$

Although "supremum" and "maximum" are not the same thing, they behave similarly in many respects. Here's an example. If sets A and B both have maxima, and A is a subset of B, then clearly $\max A \leq \max B$. The same holds for suprema. If B is bounded above and $A \subseteq B$, then any upper bound on B is also an upper bound on A. In particular, $\sup B$ is an upper bound on A. Since $\sup A$ is the *least* upper bound on A, it follows that $\sup A \leq \sup B$. That is

$$A \subseteq B \qquad \Rightarrow \qquad \sup A \leq \sup B.$$

By similar reasoning,

$$A \subseteq B \qquad \Rightarrow \qquad \inf A \geq \inf B.$$

5.2 Dissections and Riemann sums

How do we define the area of a bounded subset of the plane $\mathbb{R}^2$? If the subset is a rectangle, the answer is easy: its area is its length times its width. Similarly, if the subset is a union of non-overlapping rectangles it's easy: we just add up the areas of all the constituent rectangles. But what if the subset is more complicated: the region bounded by the x-axis, the vertical lines $x = a$ and $x = b > a$ and the graph $y = f(x)$ of some non-constant function, for example? One approach is to define the area of such a region to be the unique real number (if it exists) which is no bigger than the total area of any collection of rectangles which covers the region, and no smaller than the total area of any collection of rectangles which is covered by the region. This is the underlying idea that leads to the *Riemann integral*.

We begin by identifying the collections of rectangles we will use. These are determined by *dissecting* the interval $[a, b]$ into a finite collection of subintervals.

Definition 5.5 A **dissection** of a closed bounded interval $[a, b]$ is a finite subset $\mathscr{D}$ of $[a, b]$ containing both a and b. By convention, if $\mathscr{D}$ has $n + 1$ elements, we label these $a_0, a_1, \ldots, a_n$, so that

$$a = a_0 < a_1 < a_2 < \cdots < a_n = b,$$

and say that $\mathscr{D}$ is a dissection of **size** n. We say that $\mathscr{D}$ is a **regular** dissection if $a_j - a_{j-1} = (b - a)/n$ for all j, that is, if the points in the dissection are regularly spaced.

Remark The size of a dissection is one *less* than the number of elements it contains. This is the number of *subintervals* $[a_{j-1}, a_j]$ into which the dissection divides the interval $[a, b]$.

Definition 5.6 Let $f : [a, b] \to \mathbb{R}$ be a bounded function and $\mathscr{D}$ be a dissection of size n of $[a, b]$. For each $j \in \{1, 2, \ldots, n\}$, let

$$\begin{aligned} m_j &= \inf\{f(x) \,:\, a_{j-1} \le x \le a_j\}, \\ \text{and} \quad M_j &= \sup\{f(x) \,:\, a_{j-1} \le x \le a_j\}. \end{aligned}$$

Note that these numbers exist, since f is bounded. The **lower Riemann sum** of f with respect to $\mathscr{D}$ is

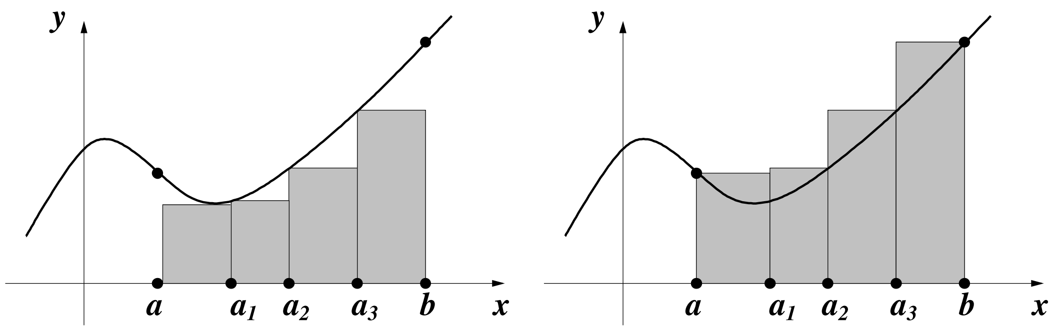
$$l_{\mathscr{D}}(f) = \sum_{j=1}^{n} m_j(a_j - a_{j-1}),$$

and the **upper Riemann sum** of f with respect to $\mathscr{D}$ is

$$u_{\mathscr{D}}(f) = \sum_{j=1}^{n} M_j(a_j - a_{j-1}).$$

□

The idea is that a dissection of size n divides $[a, b]$ into n subintervals, $[a_{j-1}, a_j]$ for $j = 1, 2, \ldots, n$. If $f(x) \ge 0$ for all x, then the lower Riemann sum can be visualized as the total area of the tallest rectangles with bases $[a_{j-1}, a_j]$ which fit under the graph $y = f(x)$ between $x = a$ and $x = b$. So $l_{\mathscr{D}}(f)$ is an underestimate of the area under the graph. Similarly, the upper Riemann sum can be visualized as the total area of the shortest rectangles with bases $[a_{j-1}, a_j]$ which the graph $y = f(x)$ between $x = a$ and $x = b$ fits under and is thus an overestimate of the area under the graph. This is illustrated below



Example 5.7 $\mathscr{D}_1 = \{0, 1\}$, $\mathscr{D}_2 = \{0, \frac{1}{2}, 1\}$, and $\mathscr{D}_3 = \{0, \frac{1}{2}, \frac{3}{4}, 1\}$ are dissections of $[0, 1]$. The function $f : [0, 1] \to \mathbb{R}$, $f(x) = x^2$ is bounded, and its lower and upper Riemann sums with respect to these dissections are:

$$\begin{aligned}
l_{\mathscr{D}_1}(f) &= \\
u_{\mathscr{D}_1}(f) &= \\
l_{\mathscr{D}_2}(f) &= \\
u_{\mathscr{D}_2}(f) &= \\
l_{\mathscr{D}_3}(f) &= \\
u_{\mathscr{D}_3}(f) &=
\end{aligned}$$

Note that

$$l_{\mathscr{D}_1} \leq l_{\mathscr{D}_2} \leq l_{\mathscr{D}_3} \leq u_{\mathscr{D}_3} \leq u_{\mathscr{D}_2} \leq u_{\mathscr{D}_1}.$$

We will see shortly that this is no accident.

Proposition 5.8 *Let $f : [a, b] \to \mathbb{R}$ be bounded above by M and below by m, and let $\mathscr{D}$ be any dissection of $[a, b]$. Then*

$$m(b - a) \leq l_{\mathscr{D}}(f) \leq u_{\mathscr{D}}(f) \leq M(b - a).$$

Proof: Let $\mathscr{D} = \{a_0, a_1, \ldots, a_n\}$ and m_j, M_j be defined as in Definition 5.6. Then, for all j, $m \leq m_j \leq M_j \leq M$, so

$$l_{\mathscr{D}}(f) = \sum_{j=1}^{n} m_j(a_j - a_{j-1}) \leq \sum_{j=1}^{n} M_j(a_j - a_{j-1}) = u_{\mathscr{D}}(f)$$

and

$$\begin{aligned}
l_{\mathscr{D}}(f) &\geq \sum_{j=1}^{n} m(a_j - a_{j-1}) \\
&= m\left\{(a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_n - a_{n-1})\right\} = m(a_n - a_0) \\
&= m(b - a)
\end{aligned}$$

and

$$\begin{aligned}
u_{\mathscr{D}}(f) &\leq \sum_{j=1}^{n} M(a_j - a_{j-1}) \\
&= M\left\{(a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_n - a_{n-1})\right\} = M(a_n - a_0) \\
&= M(b - a)
\end{aligned}$$

since the sums "telescope." □

5.3 Definition of the Riemann integral

It follows from Proposition 5.8 that the set of all lower Riemann sums

$$\{l_{\mathscr{D}}(f) \ : \ \mathscr{D} \text{ any dissection of } [a,b]\}$$

is bounded above, by $M(b-a)$, and the set of all upper Riemann sums

$$\{u_{\mathscr{D}}(f) \ : \ \mathscr{D} \text{ any dissection of } [a,b]\}$$

is bounded below, by $m(b-a)$, where

$$m = \inf\{f(x) \ : \ x \in [a,b]\}, \qquad M = \sup\{f(x) \ : \ x \in [a,b]\}.$$

Hence, by the Axiom of Completeness and Theorem 5.4, the following definition makes sense.

Definition 5.9 Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then its **lower Riemann integral** is

$$l(f) = \sup\{l_{\mathscr{D}}(f) \ : \ \mathscr{D} \text{ any dissection of } [a,b]\},$$

and its **upper Riemann integral** is

$$u(f) = \inf\{u_{\mathscr{D}}(f) \ : \ \mathscr{D} \text{ any dissection of } [a,b]\}.$$

We say that f is **Riemann integrable** (on $[a,b]$) if $l(f) = u(f)$. In that case, we denote this commmon value by

$$\int_a^b f \quad \text{or} \quad \int_a^b f(x)dx,$$

and call this number the **Riemann integral** of f (over $[a,b]$). □

Geometrically, we can think of $l(f)$ as the least upper bound on the collection of all underestimates of the area under the curve $y = f(x)$, and $u(f)$ as the greatest lower bound on the collection of overestimates of the area under the curve. Then $\int_a^b f$ is, loosely, the unique number (where this exists) which is smaller than every overestimate and larger than every underestimate. As it stands, to compute this number, or even show that it exists, we need to consider the collection of all possible dissections of $[a,b]$. This is a very large and complicated set, so, before proceeding further, we need to develop some tools for handling $l(f)$ and $u(f)$ which allow us to avoid considering all possible dissections. The key idea is that of *refinement* of a dissection.

Definition 5.10 Given dissections $\mathscr{D}, \mathscr{D}'$ of $[a,b]$, we say that $\mathscr{D}'$ is a **refinement** of $\mathscr{D}$ if $\mathscr{D} \subseteq \mathscr{D}'$. If $\mathscr{D}' \backslash \mathscr{D}$ contains k points, we say that $\mathscr{D}'$ is a **k-point refinement** of $\mathscr{D}$. Note that $\mathscr{D}$ is the unique 0-point refinement of itself.

The idea is that a refinement $\mathscr{D}'$ of $\mathscr{D}$ contains all the points in $\mathscr{D}$ and (unless $\mathscr{D}' = \mathscr{D}$) some more points too, so it splits $[a,b]$ up into more subintervals, at least some of which are narrower. Intuitively, one expects that passing from $\mathscr{D}$ to a refinement of $\mathscr{D}$ can only improve, that is, increase, the underestimate $l_{\mathscr{D}}(f)$. Similarly, passing to a refinement of $\mathscr{D}$, one expects, can only reduce the overestimate $u_{\mathscr{D}}(f)$. This expectation turns out to be essentially correct and is fundamental.

Lemma 5.11 (Refinement Lemma) *Let $f : [a,b] \to \mathbb{R}$ be bounded, $\mathscr{D}, \mathscr{D}'$ be dissections of $[a,b]$, and $\mathscr{D}'$ be a refinement of $\mathscr{D}$. Then*

$$l_{\mathscr{D}}(f) \le l_{\mathscr{D}'}(f) \le u_{\mathscr{D}'}(f) \le u_{\mathscr{D}}(f).$$

Proof: We first prove this in the special case where $\mathscr{D}'$ is a 1-point refinement of $\mathscr{D}$. So, let $\mathscr{D} = \{a = a_0, a_1, \ldots, a_n = b\}$ and $\mathscr{D}' = \mathscr{D} \cup \{z\}$ where $z \notin \mathscr{D}$. Then there exists $k \in \{1, \ldots, n\}$ such that $z \in (a_{k-1}, a_k)$. Let

$$\begin{aligned}
m_j &= \inf\{f(x) : a_j \le x \le a_{j-1}\} \\
m' &= \inf\{f(x) : a_{k-1} \le x \le z\} \\
m'' &= \inf\{f(x) : z \le x \le a_k\} \\
M_j &= \sup\{f(x) : a_j \le x \le a_{j-1}\} \\
M' &= \sup\{f(x) : a_{k-1} \le x \le z\} \\
M'' &= \sup\{f(x) : z \le x \le a_k\}
\end{aligned}$$

and note that, since $[a_{k-1}, z]$ and $[z, a_k]$ are subsets of $[a_{k-1}, a_k]$, we know immediately that $m', m'' \ge m_k$, and $M', M'' \le M_k$. Now

$$\begin{aligned}
l_{\mathscr{D}'}(f) &= \sum_{j \in \{1,2,\ldots,n\} \setminus \{k\}} m_j(a_j - a_{j-1}) + m'(z - a_{k-1}) + m''(a_k - z) \\
&= l_{\mathscr{D}}(f) - m_k(a_k - a_{k-1}) + m'(z - a_{k-1}) + m''(a_k - z) \\
&\ge l_{\mathscr{D}}(f) - m_k(a_k - a_{k-1}) + m_k(z - a_{k-1}) + m_k(a_k - z) \\
&= l_{\mathscr{D}}(f),
\end{aligned}$$

and

$$\begin{aligned}
u_{\mathscr{D}'}(f) &= \sum_{j \in \{1,2,\ldots,n\} \setminus \{k\}} M_j(a_j - a_{j-1}) + M'(z - a_{k-1}) + M''(a_k - z) \\
&= u_{\mathscr{D}}(f) - M_k(a_k - a_{k-1}) + M'(z - a_{k-1}) + M''(a_k - z) \\
&\le u_{\mathscr{D}}(f) - M_k(a_k - a_{k-1}) + M_k(z - a_{k-1}) + M_k(a_k - z) \\
&= u_{\mathscr{D}}(f).
\end{aligned}$$

Hence, by Proposition 5.8,

$$l_{\mathscr{D}}(f) \le l_{\mathscr{D}'}(f) \le u_{\mathscr{D}'}(f) \le u_{\mathscr{D}}(f),$$

that is, the Refinement Lemma holds for every 1-point refinement of every dissection.

Consider now the case where $\mathscr{D}' = \mathscr{D} \cup \{z_1, z_2, \ldots, z_p\}$ is a p-point refinement of $\mathscr{D}$. Then we can define a chain of dissections $\mathscr{D}_i$, $i = 0, \ldots, p$, by $\mathscr{D}_0 = \mathscr{D}$ and

$\mathscr{D}_i = \mathscr{D}_{i-1} \cup \{z_i\}$ for each $i = 1, \ldots, p$. Note that $\mathscr{D}_p = \mathscr{D}'$, and that each $\mathscr{D}_i$ is a 1-point refinement of $\mathscr{D}_{i-1}$. Hence, by the result just proved,

$$l_{\mathscr{D}}(f) = l_{\mathscr{D}_0}(f) \leq l_{\mathscr{D}_1}(f) \leq l_{\mathscr{D}_2(f)} \leq \cdots \leq l_{\mathscr{D}_p}(f) = l_{\mathscr{D}'}(f)$$

and

$$u_{\mathscr{D}}(f) = u_{\mathscr{D}_0}(f) \geq u_{\mathscr{D}_1}(f) \geq u_{\mathscr{D}_2(f)} \geq \cdots \geq u_{\mathscr{D}_p}(f) = u_{\mathscr{D}'}(f).$$

By Proposition 5.8, $l_{\mathscr{D}'}(f) \leq u_{\mathscr{D}'}(f)$, and so

$$l_{\mathscr{D}}(f) \leq l_{\mathscr{D}'}(f) \leq u_{\mathscr{D}'}(f) \leq u_{\mathscr{D}}(f),$$

as was to be proved. □

It follows immediately from the Refinement Lemma that *every* upper Riemann sum is at least as large as *every* lower Riemann sum, whatever (possibly different) dissections we use to compute them:

Lemma 5.12 *Let $\mathscr{D}, \widehat{\mathscr{D}}$ be two dissections of $[a, b]$ and $f : [a, b] \to \mathbb{R}$ be bounded. Then $l_{\mathscr{D}}(f) \leq u_{\widehat{\mathscr{D}}}(f)$.*

Proof: $\mathscr{D}' = \mathscr{D} \cup \widehat{\mathscr{D}}$ is a refinement of both $\mathscr{D}$ and $\widehat{\mathscr{D}}$, so by the Refinement Lemma,

$$l_{\mathscr{D}}(f) \leq l_{\mathscr{D}'}(f) \leq u_{\mathscr{D}'}(f) \leq u_{\widehat{\mathscr{D}}}(f).$$

□

From this, it follows that the upper Riemann integral is no less than the lower Riemann integral.

Lemma 5.13 *Let $f : [a, b] \to \mathbb{R}$ be bounded. Then $l(f) \leq u(f)$.*

Proof: Assume, towards a contradiction, that $l(f) > u(f)$. Then $l(f)$ is not a lower bound on the set of upper Riemann sums of f (since $u(f)$ is, by definition, the *greatest* lower bound on this set). Hence, there exists a dissection $\mathscr{D}$ such that $u_{\mathscr{D}}(f) < l(f)$. Hence, $u_{\mathscr{D}}(f)$ is not an upper bound on the set of lower Riemann sums of f (since $l(f)$ is, by definition, the *least* upper bound on this set), so there exists $\mathscr{D}'$ such that $l_{\mathscr{D}'}(f) > u_{\mathscr{D}}(f)$. But this contradicts Lemma 5.12. □

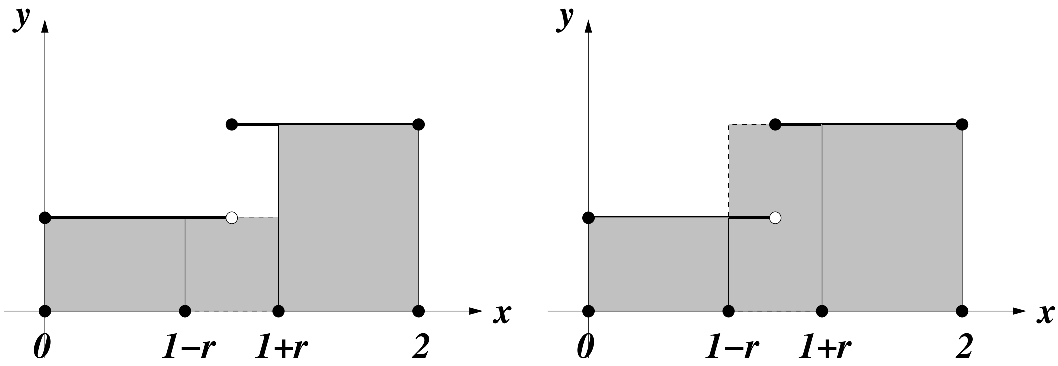
We are now, finally, in a position to compute some nontrivial Riemann integrals.

Example 5.14 Let $f : [0, 2] \to \mathbb{R}$ be the "step" function

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1, \\ 2 & \text{if } 1 \leq x \leq 2. \end{cases}$$

Show that f is Riemann integrable on $[0, 2]$ and compute $\int_0^2 f$.

Solution For each $r \in (0, 1)$ define the dissection $\mathscr{D}_r = \{0, 1 - r, 1 + r, 2\}$ of $\mathscr{D}_r$



Then the lower and upper Riemann sums of f with respect to this dissection are

$$\begin{aligned}
l_{\mathscr{D}_r}(f) &= 1(1-r-0)+1((1+r)-(1-r))+2(2-(1+r)) = 3-r \\
u_{\mathscr{D}_r}(f) &= 1(1-r-0)+2((1+r)-(1-r))+2(2-(1+r)) = 3+r.
\end{aligned}$$

Hence, $l(f)$ is the supremum of a set (the set of all lower sums) which contains $\{3-r\ :\ 0<r<1\}=(2,3)$, so $l(f)\geq 3$. Similarly, $u(f)$ is the infimum of a set (the set of all upper sums) which contains $\{3+r\ :\ 0<r<1\}=(3,4)$, so $u(f)\leq 3$. Hence $l(f)\geq u(f)$. But by Lemma 5.13, $l(f)\leq u(f)$, so $l(f)=u(f)$, that is, f is Riemann integrable. Furthermore

$$\int_0^2 f = l(f) \geq 3 \quad \text{and} \quad \int_0^2 f = u(f) \leq 3$$

so we conclude that $\displaystyle\int_0^2 f = 3.$ □

Example 5.15 Let $f:[0,1]\to\mathbb{R}$ be the function $f(x)=x^2$. Show that f is Riemann integrable on $[0,1]$ and compute $\int_0^1 f$.

Solution For each integer $n\geq 1$ let $\mathscr{D}_n$ be the *regular* dissection of $[0,1]$ of size n, that is, the dissection that divides $[0,1]$ into n subintervals of equal width,

$$\mathscr{D}_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right\}.$$

Each subinterval $[a_{j-1},a_j]=[(j-1)/n, j/n]$ has width $1/n$, and, since the function f is increasing,

$$\begin{aligned}
m_j &= \inf\left\{f(x)\ :\ \frac{j-1}{n}\leq x\leq \frac{j}{n}\right\} \\
&= f\left(\frac{j-1}{n}\right) = \frac{(j-1)^2}{n^2}, \\
M_j &= \sup\left\{f(x)\ :\ \frac{j-1}{n}\leq x\leq \frac{j}{n}\right\} \\
&= f\left(\frac{j}{n}\right) = \frac{j^2}{n^2},
\end{aligned}$$

so the lower and upper Riemann sums with respect to $\mathscr{D}_n$ are

$$\begin{aligned} l_{\mathscr{D}_n}(f) &= \frac{1}{n}\sum_{j=1}^{n}\frac{(j-1)^2}{n^2} = \frac{1}{n^3}\sum_{k=1}^{n-1}k^2, \\ u_{\mathscr{D}_n}(f) &= \frac{1}{n}\sum_{j=1}^{n}\frac{j^2}{n^2} = \frac{1}{n^3}\sum_{j=1}^{n}j^2. \end{aligned}$$

I claim that, for all $n \in \mathbb{Z}^+$,

$$\sum_{j=1}^{n} j^2 = \frac{1}{6}n(n+1)(2n+1),$$

and leave the proof of this as an exercise (hint: use induction!). Hence, for each $n \in \mathbb{Z}^+$,

$$\begin{aligned} l_{\mathscr{D}_n}(f) &= \frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right), \\ u_{\mathscr{D}_n}(f) &= \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right). \end{aligned}$$

By definition, $l(f) \geq l_{\mathscr{D}_n}(f)$ for all n (it is an upper bound on a set containing every $l_{\mathscr{D}_n}(f)$), and $l_{\mathscr{D}_n}(f) \to \frac{1}{3}$. Hence $l(f) \geq \frac{1}{3}$ (Theorem 1.7).

Similarly, $u(f) \leq u_{\mathscr{D}_n}(f)$ for all n (it is a lower bound on a set containing every $u_{\mathscr{D}_n}(f)$), and $u_{\mathscr{D}_n}(f) \to \frac{1}{3}$. Hence $u(f) \leq \frac{1}{3}$ (Proposition 1.7).

Hence $u(f) \leq \frac{1}{3} \leq l(f)$. But, by Lemma 5.13, $l(f) \leq u(f)$, so we conclude that $l(f) = u(f)$, that is, f is Riemann integrable. Furthermore,

$$\int_0^1 f = l(f) \geq \frac{1}{3} \quad \text{and} \quad \int_0^1 f = u(f) \leq \frac{1}{3},$$

whence $\int_0^1 f = \frac{1}{3}$. $\square$

It is convenient to extend the definition of dissection to include the case where the interval is $[a, a] = \{a\}$. The only dissection of $[a, a]$ is the singleton set $\mathscr{D} = \{a\}$ (a dissection of size 0). Every function $f : [a, a] \to \mathbb{R}$ is bounded, above and below by $f(a)$, so the one and only lower Riemann sum is

$$l_{\{a\}}(f) = f(a)(a-a) = 0,$$

and the one and only upper Riemann sum is

$$u_{\{a\}}(f) = f(a)(a-a) = 0.$$

It follows that every function is integrable on $[a, a]$, and

$$\int_a^a f = 0.$$

Given a Riemann integrable function f on $[a,b]$ where $a \le b$, it is also convenient to define

$$\int_b^a f = -\int_a^b f.$$

I leave it as an exercise to verify that all the results we prove about $\int_a^b f$ trivially extend to the case $a \ge b$ with these conventions.

5.4 A sequential characterization of integrability

We can generalize the argument used in Example 5.15 to give a rather elegant (and useful) characterization of Riemann integrability in terms of *sequences* of dissections.

Theorem 5.16 *Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then f is Riemann integrable if and only if there exists a sequence $(\mathscr{D}_n)$ of dissections of $[a,b]$ such that*

$$u_{\mathscr{D}_n}(f) - l_{\mathscr{D}_n}(f) \to 0.$$

In this case,

$$\int_a^b f = \lim l_{\mathscr{D}_n}(f) = \lim u_{\mathscr{D}_n}(f).$$

Proof: Let $\mathcal{D}$ be the set of all dissections of $[a,b]$, $\mathcal{L} = \{l_{\mathscr{D}}(f) : \mathscr{D} \in \mathcal{D}\}$ the set of all lower Riemann sums of f, and $\mathcal{U} = \{u_{\mathscr{D}}(f) : \mathscr{D} \in \mathcal{D}\}$ the set of all upper Riemann sums of f, so $l(f) = \sup \mathcal{L}$ and $u(f) = \inf \mathcal{U}$.

We first prove the "only if" ($\Rightarrow$) direction. So assume that $u(f) = l(f)$ (that is, f is Riemann integrable). For each $n \in \mathbb{Z}^+$, $u(f) + 1/n > u(f)$ the infimum of $\mathcal{U}$, so is *not* a lower bound on $\mathcal{U}$. Hence there exists $\mathscr{D}'_n \in \mathcal{D}$ such that $u_{\mathscr{D}'_n}(f) < u(f) + 1/n$. Similarly, $l(f) - 1/n < l(f) = \sup \mathcal{L}$, so is not an upper bound on $\mathcal{L}$. Hence there exists $\mathscr{D}''_n \in \mathscr{D}$ such that $l_{\mathscr{D}''_n}(f) > l(f) - 1/n$. Let $\mathscr{D}_n = \mathscr{D}'_n \cup \mathscr{D}''_n$ and note this is a refinement of both $\mathscr{D}'_n$ and $\mathscr{D}''_n$. Hence, by the Refinement Lemma,

$$\begin{aligned} u(f) \le {} & u_{\mathscr{D}_n}(f) & \le u_{\mathscr{D}'_n}(f) < u(f) + \frac{1}{n} \\ l(f) - \frac{1}{n} < l_{\mathscr{D}''_n}(f) \le {} & l_{\mathscr{D}_n}(f) & \le l(f). \end{aligned}$$

Hence, by the Squeeze Rule, $u_{\mathscr{D}_n}(f) \to u(f)$ and $l_{\mathscr{D}_n}(f) \to l(f)$. But $u(f) = l(f)$, so $u_{\mathscr{D}_n}(f) - l_{\mathscr{D}_n}(f) \to 0$ by the Algebra of Limits. Furthermore

$$\int_a^b f = u(f) = \lim u_{\mathscr{D}_n}(f) = \lim l_{\mathscr{D}_n}(f).$$

We now prove the "if" ($\Leftarrow$) direction. So assume that a sequence of dissections $\mathscr{D}_n$ exists such that $u_{\mathscr{D}_n}(f) - l_{\mathscr{D}_n}(f) \to 0$. For all n, $u_{\mathscr{D}_n}(f) \ge u(f)$ and $l_{\mathscr{D}_n}(f) \le l(f)$, so $u_{\mathscr{D}_n}(f) - l_{\mathscr{D}_n}(f) \ge u(f) - l(f)$. Hence, by Proposition 1.7

$$0 = \lim(u_{\mathscr{D}_n}(f) - l_{\mathscr{D}_n}(f)) \ge u(f) - l(f),$$

that is, $l(f) \geq u(f)$. But $l(f) \leq u(f)$ by Proposition 5.12, so $l(f) = u(f)$, that is, f is Riemann integrable. It remains to show that $u(f) = \lim u_{\mathscr{D}_n}(f)$ (from which it follows, by the Algebra of Limits that $l(f) = \lim l_{\mathscr{D}_n}(f)$). Now

$$0 \leq u_{\mathscr{D}_n}(f) - u(f) \leq u_{\mathscr{D}_n}(f) - u(f) + l(f) - l_{\mathscr{D}_n}(f) = u_{\mathscr{D}_n}(f) - l_{\mathscr{D}_n}(f) \to 0$$

so, by the Squeeze Rule, $u_{\mathscr{D}_n}(f) - u(f) \to 0$. Hence $u_{\mathscr{D}_n}(f) \to u(f)$.

□

To illustrate the power of Theorem 5.16, let's use it to prove that *every* monotonic (i.e. increasing or decreasing) function is Riemann integrable.

Theorem 5.17 *Let $f : [a, b] \to \mathbb{R}$ be monotonic (increasing or decreasing). Then f is Riemann integrable.*

Proof: I will prove the theorem in the case where f is increasing, and leave the case where f is decreasing as an exercise(just modify the argument below in the obvious way).

So, assume f is increasing. The f is bounded (below by $f(a)$ and above by $f(b)$), so $l(f)$ and $u(f)$ exist. For each $n \in \mathbb{Z}^+$ let $\mathscr{D}_n$ be the regular dissection of $[a, b]$ of size n, that is, the dissection which divides $[a, b]$ into n subintervals $[a_{j-1}, a_j]$ each of width $(b - a)/n$.

Then, for each $j = 1, \ldots, n$,

$$\begin{aligned} m_j &= \inf\{f(x) : a_{j-1} \leq x \leq a_j\} = f(a_{j-1}) \\ M_j &= \sup\{f(x) : a_{j-1} \leq x \leq a_j\} = f(a_j), \end{aligned}$$

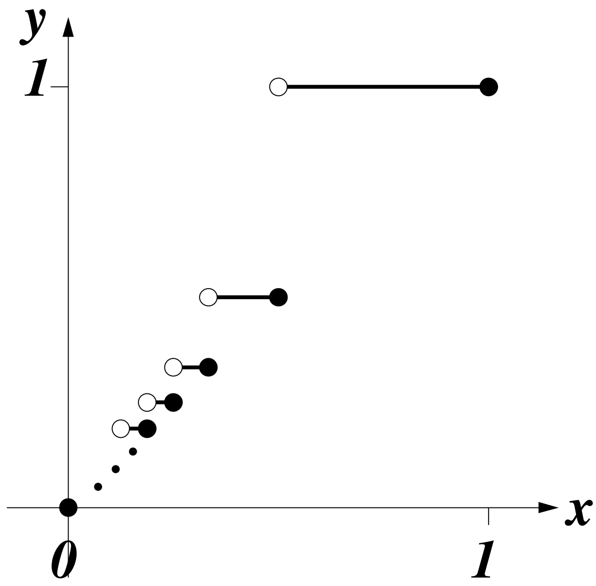
and so

$$\begin{aligned} u_{\mathscr{D}_n}(f) - l_{\mathscr{D}_n}(f) &= \frac{b-a}{n} \sum_{j=1}^{n} (f(a_j) - f(a_{j-1})) \\ &= \frac{b-a}{n}(f(b) - f(a)) \end{aligned}$$

since the sum telescopes. Hence $u_{\mathscr{D}_n}(f) - l_{\mathscr{D}_n}(f) \to 0$, so f is Riemann integrable on $[a, b]$ by Theorem 5.16. □

Note that Theorem 5.17 partially covers both Examples 5.14 and 5.15: in both these examples the function in question is increasing so Theorem 5.17 implies that they're both Riemann integrable. It does not, however, tell us anything about the *value* of $\int_a^b f$, only that it exists. Note also that the function in Example 5.14 is discontinuous, at a single point ($x = 1$), which illustrates that while a discontinuous function cannot be differentiable (Proposition 3.8), it certainly can be integrable. In fact, we can construct examples of functions which are discontinuous at infinitely many points in $[a, b]$ and yet are still Riemann integrable on $[a, b]$.

Example 5.18 Consider the function $f : [0, 1] \to \mathbb{R}$ defined so that, $f(0) = 0$ and, for all $x \in (\frac{1}{n+1}, \frac{1}{n}]$, where $n \in \mathbb{Z}^+$, $f(x) = \frac{1}{n}$. By construction, f is increasing and so is Riemann integrable on $[0,1]$ by Theorem 5.17. Note, however, that f is discontinuous at every point $\frac{1}{n}$ for $n \geq 2$.



If even such an extreme example as this is Riemann integrable, you might wonder whether there are any bounded functions which *fail* to be integrable. Fear not: such functions certainly do exist. Here is an example.

Example 5.19 Let $f : [0,1] \to \mathbb{R}$ such that $f(x) = 0$ if $x \in \mathbb{Q}$ and $f(x) = 1$ if $x \notin \mathbb{Q}$. I claim that f is not Riemann integrable.

Proof: Let $\mathscr{D} = \{a_0, \ldots, a_n\}$ be any dissection of $[0,1]$. Then, every subinterval $[a_{j-1}, a_j]$ contains both rational and irrational members, so $m_j = 0$ and $M_j = 1$ for all j. Hence

$$\begin{aligned} l_{\mathscr{D}}(f) &= \sum_{j=1}^{n} 0(a_j - a_{j-1}) = 0 \\ u_{\mathscr{D}}(f) &= \sum_{j=1}^{n} 1(a_j - a_{j-1}) = a_n - a_0 = 1. \end{aligned}$$

Since this is true for *all* dissections of $[0,1]$, $l(f) = \sup\{0\} = 0$ and $u(f) = \inf\{1\} = 1$, so $l(f) \neq u(f)$, that is, f is not Riemann integrable. □

Theorem 5.17 gives us one interesting class of functions $f : [a,b] \to \mathbb{R}$ that are Riemann integrable: those that are monotonic. Our next theorem gives us another rather more useful class: continuous functions.

Theorem 5.20 *Let $f : [a,b] \to \mathbb{R}$ be continuous. Then f is Riemann integrable.*

Proof: First note that, since f is continuous, it is certainly bounded (by the Extreme Value Theorem), so $l(f)$ and $u(f)$ exist.

Consider the sequence $(\mathscr{D}_n)$ of regular dissections of $[a, b]$ of size 2^n. I claim that $u_{\mathscr{D}_n}(f) - l_{\mathscr{D}_n}(f) \to 0$, so f is Riemann integrable by Theorem 5.16. Note that $\mathscr{D}_n \subset \mathscr{D}_{n+1}$ for all n, so by the Refinement Lemma, $l_{\mathscr{D}_n}(f)$ is an increasing sequence and $u_{\mathscr{D}_n}(f)$ is a decreasing sequence. Hence $l_{\mathscr{D}_n}(f) \to K$ and $u_{\mathscr{D}_n}(f) \to L$ for some numbers K and L by the Monotone Convergence Theorem, so $u_{\mathscr{D}_n}(f) - l_{\mathscr{D}_n}(f) \to K - L \geq 0$ by the Algebra of Limits and Proposition 1.7. We must show that $K - L = 0$.

Assume, towards a contradiction, that $\varepsilon = K - L > 0$. As usual, let

$$\begin{aligned} m_j &= \inf\{f(x) \,:\, a_{j-1} \leq x \leq a_j\} \\ \text{and} \quad M_j &= \sup\{f(x) \,:\, a_{j-1} \leq x \leq a_j\}. \end{aligned}$$

Then

$$u_{\mathscr{D}_n}(f) - l_{\mathscr{D}_n}(f) = \sum_{j=1}^{2^n} \frac{b-a}{2^n}(M_j - m_j) \geq \varepsilon.$$

This sum consists of 2^n terms, each non-negative, so at least one term must be greater than or equal to $\varepsilon/2^n$. That is, there must exist $j \in \{1, 2, 3, \ldots, 2^n\}$ such that

$$M_j - m_j \geq \frac{\varepsilon}{b-a}.$$

But $f : [a_{j-1}, a_j] \to \mathbb{R}$ is continuous so, by the Extreme Value Theorem, f attains both a maximum and minimum value on $[a_{j-1}, a_j]$, that is, there are points, x_n and y_n say, in $[a_{j-1}, a_j]$ such that $f(x_n) = M_j$ and $f(y_n) = m_j$. Hence, for each $n \in \mathbb{Z}^+$ there exist $x_n, y_n \in [a, b]$ such that

$$|x_n - y_n| \leq \frac{b-a}{2^n}, \tag{5.1}$$

since both lie in an interval of this width, and

$$f(x_n) - f(y_n) \geq \frac{\varepsilon}{b-a}. \tag{5.2}$$

Consider the sequence (x_n). It is bounded, so, by the Bolzano–Weierstrass Theorem (Theorem 1.15), it has a convergent subsequence $x_{n_k} \to c \in [a, b]$. By (5.1),

$$x_{n_k} - \frac{b-a}{2^{n_k}} \leq y_{n_k} \leq x_{n_k} + \frac{b-a}{2^{n_k}}$$

so $y_{n_k} \to c$ also, by the Squeeze Rule. Now f is continuous, so $f(x_{n_k}) \to f(c)$ and $f(y_{n_k}) \to f(c)$, and hence $f(x_{n_k}) - f(y_{n_k}) \to f(c) - f(c) = 0$. But this contradicts (5.2) and Proposition 1.7. □

5.5 Elementary properties of the Riemann integral

Theorem 5.16 is a convenient tool for establishing many useful properties of the Riemann integral.

We will first prove that the Riemann integral is *linear*, that is,

$$\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g,$$

for any constants α, β and any integrable functions f, g. To do this, we need the following:

Lemma 5.21 *Let $f, g : [a, b] \to \mathbb{R}$ be bounded functions and $\mathscr{D}$ be a dissection of $[a, b]$. Then*

$$l_{\mathscr{D}}(f+g) \geq l_{\mathscr{D}}(f) + l_{\mathscr{D}}(g) \quad \textit{and} \quad u_{\mathscr{D}}(f+g) \leq u_{\mathscr{D}}(f) + u_{\mathscr{D}}(g).$$

Proof: I will prove the first inequality, and leave the second as an exercise.
Let $\mathscr{D} = \{a_0, a_1, \ldots, a_n\}$ and $I_j = [a_{j-1}, a_j]$. Let

$$\begin{aligned} m_j(f) &= \inf\{f(x) : x \in I_j\}, \\ M_j(f) &= \sup\{f(x) : x \in I_j\}, \end{aligned}$$

and $m_j(g), M_j(g), m_j(g+f)$, and $M_j(g+f)$ be defined similarly. Then, for all $x \in I_j$, $f(x) + g(x) \geq m_j(f) + m_j(g)$, so $m_j(f) + m_j(g)$ is certainly a lower bound on $\{f(x) + g(x) : x \in I_j\}$. Since $m_j(f+g)$ is the *greatest* lower bound on this set, it follows that $m_j(f+g) \geq m_j(f) + m_j(g)$. Hence

$$\begin{aligned} l_{\mathscr{D}}(f+g) &= \sum_{j=1}^{n} m_j(f+g)(a_j - a_{j-1}) \\ &\geq \sum_{j=1}^{n} (m_j(f) + m_j(g))(a_j - a_{j-1}) = l_{\mathscr{D}}(f) + l_{\mathscr{D}}(g). \end{aligned}$$

□

Theorem 5.22 (Linearity of the Riemann Integral) *Let f, g be Riemann integrable on $[a, b]$, and $\alpha \in \mathbb{R}$ be a constant. Then*

(i) αf is Riemann integrable on $[a, b]$, and $\int_a^b \alpha f = \alpha \int_a^b f$,

(ii) $f + g$ is Riemann integrable on $[a, b]$, and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

Proof: (i) By Theorem 5.16, there is a sequence of dissections $(\mathscr{D}_n)$ such that $l_{\mathscr{D}_n}(f) \to \int_a^b f$ and $u_{\mathscr{D}_n}(f) \to \int_a^b f$. It follows directly from Definition 5.6 that $l_{\mathscr{D}_n}(\alpha f) = \alpha l_{\mathscr{D}_n}(f)$ and $u_{\mathscr{D}_n}(\alpha f) = \alpha u_{\mathscr{D}_n}(f)$ if $\alpha \geq 0$, and $l_{\mathscr{D}_n}(\alpha f) = \alpha u_{\mathscr{D}_n}(f)$ and $u_{\mathscr{D}_n}(\alpha f) = \alpha l_{\mathscr{D}_n}(f)$ if $\alpha < 0$. In either case, $l_{\mathscr{D}_n}(\alpha f) \to \alpha \int_a^b f$ and $u_{\mathscr{D}_n}(\alpha f) \to \alpha \int_a^b f$ by the Algebra of Limits, and the claim follows from Theorem 5.16.
(ii) Let $(\mathscr{D}_n)$ be as defined above. By Theorem 5.16, there is also a sequence of dissections $(\mathscr{D}'_n)$ such that $l_{\mathscr{D}'_n}(g) \to \int_a^b g$ and $u_{\mathscr{D}'_n}(g) \to \int_a^b g$. Let $\mathscr{D}''_n = \mathscr{D}_n \cup \mathscr{D}'_n$.

This is a refinement of both $\mathscr{D}_n$ and $\mathscr{D}'_n$ and so, by the Refinement Lemma

$$\begin{aligned} l_{\mathscr{D}_n}(f) &\le l_{\mathscr{D}''_n}(f) \le \int_a^b f, \\ l_{\mathscr{D}'_n}(g) &\le l_{\mathscr{D}''_n}(g) \le \int_a^b g, \\ \int_a^b f &\le u_{\mathscr{D}''_n}(f) \le u_{\mathscr{D}_n}(f), \\ \int_a^b f &\le u_{\mathscr{D}''_n}(g) \le u_{\mathscr{D}'_n}(g). \end{aligned}$$

Hence, by the Squeeze Rule, $l_{\mathscr{D}''_n}(f) \to \int_a^b f$, $l_{\mathscr{D}''_n}(g) \to \int_a^b g$, $u_{\mathscr{D}''_n}(f) \to \int_a^b f$ and $u_{\mathscr{D}''_n}(f) \to \int_a^b g$. Now, by Lemma 5.21,

$$l_{\mathscr{D}''_n}(f) + l_{\mathscr{D}''_n}(g) \le l_{\mathscr{D}''_n}(f+g) \le l(f+g),$$

and

$$u(f+g) \le u_{\mathscr{D}''_n}(f+g) \le u_{\mathscr{D}''_n}(f) + u_{\mathscr{D}''_n}(g).$$

But both $l_{\mathscr{D}''_n}(f) + l_{\mathscr{D}''_n}(g)$ and $u_{\mathscr{D}''_n}(f) + u_{\mathscr{D}''_n}(g)$ converge to $\int_a^b f + \int_a^b g$ so, by Proposition 1.7, $\int_a^b f + \int_a^b g \le l(f+g)$ and $u(f+g) \le \int_a^b f + \int_a^b g$. It follows that $l(f+g) \ge u(f+g)$ and hence, by Lemma 5.13, that $l(f+g) = u(f+g)$, that is, $f+g$ is Riemann integrable. Furthermore

$$\int_a^b (f+g) = l(f+g) \ge \int_a^b f + \int_a^b g$$

and

$$\int_a^b (f+g) = u(f+g) \le \int_a^b f + \int_a^b g,$$

so $\int_a^b (f+g) = \int_a^b f + \int_a^b g$, as claimed. □

Let's denote by $L([a,b])$ the set of Riemann integrable functions on $[a,b]$. Theorem 5.22 establishes that $L([a,b])$ is in fact a **vector space** over $\mathbb{R}$: it is closed under the obvious (vector) addition and scalar multiplication operations. Better still, it shows that the map

$$L([a,b]) \to \mathbb{R}, \qquad f \mapsto \int_a^b f$$

is a **linear map** between vector spaces.

We next prove that Riemann integration preserves inequalities.

Proposition 5.23 *Let $f, g : [a,b] \to \mathbb{R}$ be Riemann integrable and assume $f(x) \le g(x)$ for all $x \in [a,b]$. Then*

$$\int_a^b f \le \int_a^b g.$$

Proof: By Theorem 5.16 there exist sequences of dissections $\mathscr{D}'_n$, $\mathscr{D}''_n$ such that $l_{\mathscr{D}'_n}(f) \to \int_a^b f$ and $l_{\mathscr{D}''_n} \to \int_a^b g$. Let $\mathscr{D}_n = \mathscr{D}'_n \cup \mathscr{D}''_n$. Then, by the Refinement Lemma (and the Squeeze Rule) $l_{\mathscr{D}_n}(f) \to \int_a^b f$ and $l_{\mathscr{D}_n} \to \int_a^b g$ also. But, for each n, $l_{\mathscr{D}_n}(g) \geq l_{\mathscr{D}_n}(f)$ since $\inf g \geq \inf f$ on any subset of $[a,b]$. Hence, $\int_a^b g - \int_a^b f$, being the limit of a convergent non-negative sequence, is ≥ 0 (Proposition 1.7). □

From this we can deduce a kind of "triangle inequality" for Riemann integrals.

Proposition 5.24 (Integral Triangle Inequality) *Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable.*

$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$

Before proving this, we should first establish that $|f|$ is Riemann integrable (so we can be sure the number on the right of the claimed inequality exists).

Lemma 5.25 *Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable. Then $|f| : [a,b] \to \mathbb{R}$ is Riemann integrable.*

Proof: Let $f_+ : [a,b] \to \mathbb{R}$ denote the "positive part" of the function f, that is

$$f_+(x) = \begin{cases} f(x), & \text{if } f(x) \geq 0 \\ 0, \text{if } f(x) < 0. \end{cases}$$

We will show that f_+ is Riemann integrable.

Let $\mathscr{D} = \{a = a_0, a_, \ldots, a_n = b\}$ be any dissection of $[a,b]$ and consider, for a particular subinterval $I_j = [a_{j-1}, a_j]$ the numbers $m_j(f), m_j(f_+), M_j(f)$ and $M_j(f_+)$. There are 3 possible cases:

- If $f \geq 0$ on I_j, then $f_+ = f$ (on I_j), so
$$M_j(f_+) - m_j(f_+) = M_j(f) - m_j(f).$$
- If $f < 0$ on I_j then $f_+ = 0$ (on I_j), so
$$M_j(f_+) - m_j(f_+) = 0 - 0 = 0 \leq M_j(f) - m_j(f).$$
- If f takes both negative and non-negative values on I_j, then $m_j(f_+) = 0 > m_j(f)$ while $M_j(f_+) = M_j(f)$, so
$$M_j(f_+) - m_j(f_+) < M_j(f) - m_j(f).$$

In all cases, we see that, for all j,

$$M_j(f_+) - m_j(f_+) \leq M_j(f) - m_j(f).$$

Hence, for any dissection $\mathscr{D}$,

$$\begin{aligned} u_{\mathscr{D}}(f_+) - l_{\mathscr{D}}(f_+) &= \sum_{j=1}^n (M_j(f_+) - m_j(f_+))(a_j - a_{j-1}) \\ &\leq \sum_{j=1}^n (M_j(f) - m_j(f))(a_j - a_{j-1}) = u_{\mathscr{D}}(f) - l_{\mathscr{D}}(f). \end{aligned} \quad (5.3)$$

Since f is Riemann integrable, there exists, by Theorem 5.16, a sequence $(\mathscr{D}_n)$ of dissections of $[a,b]$ such that

$$u_{\mathscr{D}_n}(f) - l_{\mathscr{D}_n}(f) \to 0.$$

By inequality (5.3) and the Squeeze Rule, $u_{\mathscr{D}_n}(f_+) - l_{\mathscr{D}_n}(f_+) \to 0$ also, so, again by Theorem 5.16, f_+ is also Riemann integrable.

But $|f| = 2f_+ - f$,

$$2f_+(x) - f(x) = \begin{cases} 2f(x) - f(x) = f(x), & \text{if } f(x) \geq 0 \\ 0 - f(x) = -f(x), & \text{if } f(x) < 0, \end{cases}$$

so $|f|$ is also Riemann integrable by Theorem 5.22 . □

Proof of Proposition 5.24: Note that $-|f(x)| \leq f(x) \leq |f(x)|$ for all $x \in [a,b]$, so

$$-\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f|$$

by Proposition 5.23. The claimed inequality immediately follows. □

We conclude this section by proving the "join rule" for integration:

Proposition 5.26 (Join Rule) *Let f be Riemann integrable on $[a,b]$ and on $[b,c]$. Then f is Riemann integrable on $[a,c]$ and*

$$\int_a^c f = \int_a^b f + \int_b^c f.$$

Proof: By Theorem 5.16, there exist sequences of dissections $(\mathscr{D}'_n)$ of $[a,b]$ and $(\mathscr{D}''_n)$ of $[b,c]$ such that $l_{\mathscr{D}'_n}(f) \to \int_a^b f$, $u_{\mathscr{D}'_n}(f) \to \int_a^b f$, $l_{\mathscr{D}''_n}(f) \to \int_b^c f$, and $u_{\mathscr{D}''_n}(f) \to \int_b^c f$. Let $\mathscr{D}_n = \mathscr{D}'_n \cup \mathscr{D}''_n$. Then $\mathscr{D}_n$ is a dissection of $[a,c]$ and it follows directly from Definition 5.6 that $l_{\mathscr{D}_n}(f) = l_{\mathscr{D}'_n}(f) + l_{\mathscr{D}''_n}(f)$ and $u_{\mathscr{D}_n}(f) = u_{\mathscr{D}'_n}(f) + u_{\mathscr{D}''_n}(f)$. Hence, by the Algebra of Limits, $l_{\mathscr{D}_n}(f) \to \int_a^b f + \int_b^c f$ and $u_{\mathscr{D}_n}(f) \to \int_a^b f + \int_b^c f$, so the claim follows from Theorem 5.16. □

Summary

- A **dissection** of $[a, b]$ is a finite subset $\mathscr{D} = \{a_0, a_1, \ldots, a_n\}$ of $[a, b]$ such that
$$a = a_0 < a_1 < a_2 < \cdots < a_n = b.$$
- Given a bounded function $f : [a, b] \to \mathbb{R}$, and a dissection $\mathscr{D}$, the **lower Riemann sum** is
$$l_{\mathscr{D}}(f) = \sum_{j=1}^{n} \inf\{f(x) \,:\, a_{j-1} \leq x \leq a_j\}(a_j - a_{j-1}),$$
and the **upper Riemann sum** is
$$u_{\mathscr{D}}(f) = \sum_{j=1}^{n} \sup\{f(x) \,:\, a_{j-1} \leq x \leq a_j\}(a_j - a_{j-1}).$$
- The **lower Riemann integral** of f is
$$l(f) = \sup\{l_{\mathscr{D}}(f) \,:\, \mathscr{D} \text{ any dissection of } [a, b]\},$$
and the **upper Riemann integral** of f is
$$u(f) = \inf\{u_{\mathscr{D}}(f) \,:\, \mathscr{D} \text{ any dissection of } [a, b]\}.$$
- f is **Riemann integrable** if $l(f) = u(f)$, and in this case we denote their common value by
$$\int_a^b f$$
and call it the **Riemann integral** of f (on, or over, $[a, b]$).
- A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if there exists some sequence $(\mathscr{D}_n)$ of dissections of $[a, b]$ such that $l_{\mathscr{D}_n}(f) - u_{\mathscr{D}_n}(f) \to 0$ (and, in this case, $\int_a^b f = \lim l_{\mathscr{D}_n}(f) = \lim u_{\mathscr{D}_n}(f)$).
- We used this theorem to prove that all **continuous** functions, and all **monotonic** functions are Riemann integrable. We also used it to prove that
$$\int_a^b f + \int_b^c f = \int_a^c f, \qquad \int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g, \qquad \text{and} \qquad \left| \int_a^b f \right| \leq \int_a^b |f|$$
where α, β are constants.