MAGIC assessment cover sheet

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MAGIC063

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Differentiable Manifolds — Adam Keyes

1. Let $X = (\mathbb{R}, \mathcal{T})$ where

$$\mathcal{T} = \{U \subset \mathbb{R} : 0 \notin U\} \cup \{\mathbb{R}\}$$

Then

(a) X is not Hausdorff - for $x_1 = 0$, $x_2 \in \mathbb{R}$ arbitrary, we cannot find $U_1, U_2 \in \mathcal{T}$ w/ $x_1 \in U_1$, $x_2 \in U_2$ w/ $U_1 \cap U_2 = \emptyset$, as the only such U_1 is $\mathbb{R}$, & $\mathbb{R} \cap U_2 \neq \emptyset$ $\forall\, U_2 \in \mathcal{T}$ as $U_2 \subset \mathbb{R}$ by definition ($U_2 \neq \emptyset$ as $x_2 \in U_2$).

(b) X is not second countable - Suppose we have a base $\mathcal{B}$. Then for any $U \in \mathcal{T}$ & any $x \in U$, $\exists\, B \in \mathcal{B}$ w/ $x \in B \subseteq U$. But for any $x \in \mathbb{R}\setminus\{0\}$, $\{x\} \in \mathcal{T}$. Thus we must have $\{x\} \in \mathcal{B}$ $\forall\, x \in \mathbb{R}\setminus\{0\}$, & there are uncountably many of these sets so any such base is uncountable.

(c) X is compact - suppose we have an open cover $\{U_\alpha : \alpha \in A\}$. Then $X = \bigcup_{\alpha \in A} U_\alpha$, in particular $0 \in \bigcup_{\alpha \in A} U_\alpha$. But the only open set containing 0 is $\mathbb{R}$. Thus we must have $\mathbb{R} \in \{U_\alpha : \alpha \in A\}$ meaning we always have a finite subcover given simply by $\{\mathbb{R}\}$.

(d) $f(x) = e^x$ is not continuous - Consider $U = \{1\} \in \mathcal{T}$. Then $f^{-1}(U) = \{0\} \notin \mathcal{T}$.

(e) $f(x) = \sin x$ is continuous - take $U \neq \mathbb{R} \in \mathcal{T}$. Then $0 \notin U$. But as $f(0) = 0$, then $0 \notin f^{-1}(U)$, so $f^{-1}(U) \in \mathcal{T}$. Also, $f^{-1}(\mathbb{R}) = \mathbb{R}$. Thus the preimage of any open set is open.

(f) X is connected - suppose it were not, then there would exist disjoint $X_1, X_2 \in \mathcal{T}$ s.t. $X = X_1 \cup X_2$. But then $0 \in X_1 \cup X_2$, meaning $0 \in X_1$ or X_2. WLOG take $0 \in X_1$. Then we must have $X_1 = \mathbb{R}$ as $X_1 \in \mathcal{T}$. But then $X_1 \cap X_2 \neq \emptyset$ i.e. the sets are not disjoint, a contradiction.

2. Let $N = \mathrm{Mat}_2\mathbb{R}$ &

$$M = \{x \in N : x^T J x = J\}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(a) First observe that

$$(x^T J x)^T = x^T J^T x$$

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$$= -x^T J x$$

& so we can consider the map

$$f : N \to \mathrm{Skew}_2 \mathbb{R}$$

$$x \mapsto f(x) = x^T J x.$$

We can clearly identify $\mathrm{Skew}_2 \mathbb{R} \cong \mathbb{R}$ (such matrices are specified by a single off-diagonal element). Now noting that

$$M = f^{-1}(J),$$

& using $N \cong \mathbb{R}^4$, $\mathrm{Skew}_2 \mathbb{R} \cong \mathbb{R}$, the regular value theorem will give that M is a 3-dimensional submanifold of N provided J is a regular value of f. To check this, first let

$$x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}.$$

Then

$$f(x) = \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix} \begin{pmatrix} x_3 & x_4 \\ -x_1 & -x_2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & x_1 x_4 - x_2 x_3 \\ x_2 x_3 - x_1 x_4 & 0 \end{pmatrix}$$

or viewing f as a map $\mathbb{R}^4 \to \mathbb{R}$, we have

$$f(x) = x_1 x_4 - x_2 x_3.$$

The differential is then

$$df_x = (x_4 \quad -x_3 \quad -x_2 \quad x_1).$$

Now let $f(c) = J$, then we must have

$$c_1 c_4 - c_2 c_3 = 1,$$

thus not all the c_i can be 0. WLOG suppose $c_1 \neq 0$. Then

$$c_4 = \frac{1 + c_2 c_3}{\quad}$$

Then

$$c_4 = \frac{1+c_2c_3}{c_1},$$

and

$$df_c = \left(\frac{1+c_2c_3}{c_1} \quad -c_3 \quad -c_2 \quad c_1\right).$$

This has maximal rank meaning J is a regular value of f, so we can conclude by the regular value theorem that $M=f^{-1}(J)$ is a 3d submanifold of N.

(b) No - if we again view N as $\mathbb{R}^4$, then M is a subset of $\mathbb{R}^4$, and hence is compact iff it is closed & bounded. The eq^{n}s defining M constrain one of the coordinates while the others are arbitrary - in particular, they can be chosen to be arbitrarily far from one another (in terms of the Euclidean metric). Thus M is not bounded & so is not compact.

(c) To construct an atlas, inspired by the above we write $x \in M$ as

$$x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$$

& using that $x_1x_4 - x_3x_2 = 1$ (i.e. $\det x = 1$) we define charts with open sets

$$U_i = \{x \in M : x_i \neq 0\}$$

which allows us to eliminate one of the x_i using the constraint & so we can define maps

$$\phi_1 : U_1 \to \Omega_1 = \{(x_1,x_2,x_3) \subset \mathbb{R}^3 : x_1 \neq 0\}$$

$$\phi_1(x_1,\dots,x_4) = (x_1,x_2,x_3)$$

$$\phi_2 : U_2 \to \Omega_2 = \{(x_1,x_2,x_4) \subset \mathbb{R}^3 : x_2 \neq 0\}$$

$$\phi_2(x_1,\dots,x_4) = (x_1,x_2,x_4)$$

& vice-versa for ϕ_3 & ϕ_4 (i.e. we eliminate x_2 & x_1 respectively). The transition functions then have form, on $U_1 \cap U_2$ say,

x_i respectively). The transition functions then have form, on $U_1 \cap U_2$ say,

$$T_{21} = \phi_2 \circ \phi_1^{-1}(x_1, x_2, x_3) = \phi_2\left(x_1, x_2, x_3, \frac{1+x_2x_3}{x_1}\right)$$

$$= \left(x_1, x_2, \frac{1+x_2x_3}{x_1}\right)$$

which are smooth as $x_1 \neq 0$ on these sets. To check if this is an orientation atlas, we must compute the Jacobian of mappings of the above form:

$$dT_{21} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{(1+x_2x_3)}{x_1^2} & \frac{1+x_3}{x_1} & \frac{1+x_2}{x_1} \end{pmatrix}.$$

To check the sign of the determinant, it suffices to look at only one point (as T_{21} is a diffeo & $\mathbb{R}^3 \setminus \{0\}$ is connected), we take $(x_1, x_2, x_3) = (1, 1, 0)$, so

$$dT_{21(1,1,0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 2 \end{pmatrix}, \text{ so}$$

$$\det dT_{21(1,1,0)} = 2 > 0$$

The computation will go through in the same way for the other charts. Thus this is an orientation atlas.

(d) Consider $h(x) = x^2$. Then $h: M \to M$, as if $x \in M$, then

$$(x^2)^T J x^2 = x^T \cdot x^T \cdot J \cdot x \cdot x$$

$$= x^T \cdot J \cdot x \qquad (x \in M)$$

$$= J.$$

To check if h is an immersion, we must compute its differential. We first would like to compute the coordinate expression of h. Observe that

$$h(x) = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}^2 = \begin{pmatrix} x_1^2 + x_2x_3 & x_2(x_1+x_4) \\ x_3(x_1+x_4) & x_2x_3 + x_4^2 \end{pmatrix},$$

so on (U_1, ϕ_1) for example,

$$\hat{h}(y_1, y_2, y_3) = \phi_1 \circ h \circ \phi_1^{-1}(y_1, y_2, y_3)$$

$$= \phi_1 \circ h\left(y_1, y_2, y_3, \frac{1+y_2y_3}{y_1}\right)$$

$$\phi_1 \begin{pmatrix} y_1^2 + y_2y_3 & y_2\left(y_1 + \frac{1+y_2y_3}{y_1}\right) \\ y_3\left(y_1 + \frac{1+y_2y_3}{y_1}\right) & y_2y_3 + \left(\frac{1+y_2y_3}{y_1}\right)^2 \end{pmatrix}$$

$$= \left(y_1^2 + y_2y_3,\ y_2\left(\frac{y_1^2 + y_2y_3 + 1}{y_1}\right),\ y_3\left(\frac{y_1^2 + y_2y_3 + 1}{y_1}\right)\right)$$

$$= \left(y_1^2 + y_2y_3,\ y_1^2y_2 + \frac{y_2^2y_3 + y_2}{y_1},\ y_1^2y_3 + \frac{y_3^2y_2 + y_3}{y_1}\right),$$

meaning

$$d\hat{h}_{(y_1,y_2,y_3)} = \begin{pmatrix} 2y_1 & y_3 & y_2 \\ y_2\left(1 - \frac{y_2y_3+1}{y_1^2}\right) & y_1 + \frac{1+2y_2y_3}{y_1} & y_2^2/y_1 \\ y_3\left(1 - \frac{y_2y_3+1}{y_1^2}\right) & y_3^2/y_1 & y_1 + \frac{1+2y_2y_3}{y_1} \end{pmatrix}$$

which has maximal rank (3) on $\phi_1(U_1)$ & similar for other charts so this is an immersion. It is not an embedding however as it is not a homeomorphism onto its image - it fails injectivity as $\forall x \in M, -x \in M$, and $h(-x) = (-x)^2 = x^2 = h(x)$.

(e) Consider $g: M \to \mathbb{R}$ given by

$$g(x) = x_1 + x_2 + x_3 + x_4$$

On U_1, we have

$$\hat{g}(y_1, y_2, y_3) = g \circ \phi_1^{-1}(y_1, y_2, y_3)$$

$$= y_1 + y_2 + y_3 + \frac{1+y_2y_3}{y_1}$$

so

$$d\hat{g}_{(y_1,y_2,y_3)} = \left(1 - \frac{1+y_2y_3}{y_1^2},\ 1 + \frac{1+y_3}{y_1},\ 1 + \frac{1+y_2}{y_1}\right).$$

This has maximal rank, as for the first component to vanish we require $1+y_2y_3 = y_1^2$, & the second & third $1+y_3 = -y_1$ & $1+y_2 = -y_1$, but then we must have

$$(1+y_2)(1+y_3) = 1 + y_2 + y_3 + y_2y_3 = y_1^2$$

so compatibility w/ the first eq$^{\underline{n}}$ requires $y_2 = -y_3$, but then the first reads

$$y_1^2 = 1 - y_3^2 \leq 0$$

which is not possible as $y_1 \neq 0$ on $\phi_1(U_1)$. Similar arguments hold for the other charts, so all components of $d\hat{g}$ can't vanish simultaneously, meaning it has arguments hold for the other charts, so all components of $d\hat{g}$ can't vanish simultaneously meaning it has maximal rank making g a submersion.

3 Let $M = \mathbb{R}^3$, $N = \mathbb{R}^2$, $f: M \to N$ be given by $f(x,y,z) = (xy, xz)$ & $\eta \in \Gamma(T^{(0,2)}N)$ be given by
$$\eta = dx^1 \otimes dx^1 + dx^2 \otimes dx^2.$$

(a) Then given that $f^*\eta \in \Gamma(T^{(0,2)}M)$, we can write
$$f^*\eta = a_{xx}\, dx \otimes dx + a_{xy}\, dx \otimes dy + a_{xz}\, dx \otimes dz$$
$$+ \dots + a_{zz}\, dz \otimes dz$$
where $a_{ij} = (f^*\eta)_{ij}$ &
$$(f^*\eta)_{ij} = f^*\eta(\partial_i, \partial_j)$$
$$= \eta(df(\partial_i), df(\partial_j)), \quad i = 1,2,3.$$

Now
$$df = \begin{pmatrix} y & x & 0 \\ z & 0 & x \end{pmatrix}$$
so
$$(f^*\eta)_{xx} = \eta(df(\partial_x), df(\partial_x))$$
$$= \eta(y\partial_1 + z\partial_2, y\partial_1 + z\partial_2)$$
$$= y^2 + z^2$$
$$(f^*\eta)_{xy} = \eta(df(\partial_x), df(\partial_y))$$
$$= \eta(y\partial_1 + z\partial_2, x\partial_1)$$
$$= xy = (f^*\eta)_{yx} \quad \text{(by symmetry of } \eta)$$
$$(f^*\eta)_{xz} = \eta(df(\partial_x), df(\partial_z))$$
$$= \eta(y\partial_1 + z\partial_2, x\partial_2)$$

$$= xz = (f^*\eta)_{zx}$$

$$(f^*\eta)_{yy} = \eta(x\partial_1, x\partial_1)$$

$$= x^2$$

$$(f^*\eta)_{yz} = \eta(x\partial_1, x\partial_2)$$

$$= 0 = (f^*\eta)_{zy}$$

$$(f^*\eta)_{zz} = \eta(x\partial_2, x\partial_2)$$

$$= x^2$$

meaning

$$f^*\eta = (y^2+z^2)\,dx\otimes dx + x^2(dy\otimes dy + dz\otimes dz)$$

$$+ xy(dx\otimes dy + dy\otimes dx) + xz(dx\otimes dz + dz\otimes dx)$$

(b) Now we will have, in components

$$f^*\eta(X,Y) = (y^2+z^2)X^xY^x + x^2(X^yY^y + X^zY^z)$$

$$+ xy(X^xY^y + X^yY^x) + xz(X^xY^z + X^zY^x)$$

$$= Y^x\left[(y^2+z^2)X^x + xyX^y + xzX^z\right] +$$

$$Y^y\left[x^2X^y + xyX^x\right] + Y^z\left[x^2X^z + xzX^x\right].$$

Thus if we take

$$X = -x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$$

then $f^*\eta(X,Y) = 0 \;\; \forall\; Y \in \Gamma(TM)$.

(c) Let us compute the flow of X - the flow eq^{n}s are

$$\dot{x}(t) = -x(t), \quad \dot{y}(t) = y(t), \quad \dot{z}(t) = z(t),$$

$\vec{x}(0) = \vec{x}_0$. This has sol^{n}

$$x(t) = x_0e^{-t}, \quad y(t) = y_0e^{t}, \quad z(t) = z_0e^{t}.$$

Thus the diffeomorphism induced by the flow

Thus the diffeomorphism induced by the flow

$$\theta_t : M \to M$$

is given by

$$\theta_t(x,y,z) = (xe^{-t}, ye^{t}, ze^{t})$$

meaning

$$\theta_1(x,y,z) = \left(\tfrac{x}{e}, ye, ze\right).$$

(d) Using that

$$d\theta_1 = \begin{pmatrix} \frac{1}{e} & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{pmatrix},$$

we clearly have

$$\theta_1^* g_{Euc} = \frac{1}{e^2} dx \otimes dx + e^2 dy \otimes dy + e^2 dz \otimes dz.$$

Now

$$\flat X = a_x dx + a_y dy + a_z dz$$

where

$$a_x = \flat X\left(\tfrac{\partial}{\partial x}\right) = \theta_1^* g_{Euc}\left(X, \tfrac{\partial}{\partial x}\right)$$

$$= -\frac{1}{e^2} x$$

$$a_y = \flat X\left(\tfrac{\partial}{\partial y}\right) = \theta_1^* g_{Euc}\left(X, \tfrac{\partial}{\partial y}\right)$$

$$= e^2 y$$

$$a_z = \dots = e^2 z.$$

Thus using $g = \theta_1^* g_{Euc}$, we have

$$\flat X = -\frac{1}{e^2}\frac{\partial}{\partial x} + e^2 \frac{\partial}{\partial y} + e^2 \frac{\partial}{\partial z}.$$

4. Let $M = \mathbb{R}^3$ & $E = M \times \mathbb{R}$ be the trivial line bundle over M. Equip E w/ a connection ∇ s.t. $\nabla e^{x^1} = dx^3$.

Then

$$\nabla_{\partial/\partial x^i} e^{x^1} = \nabla e^{x^1}\left(\tfrac{\partial}{\partial x^i}\right) = dx^3\left(\tfrac{\partial}{\partial x^i}\right) = \delta_i^3$$

$$\nabla_{\partial/\partial x^i} e^{x^1} = \nabla e^{x^1}\left(\tfrac{\partial}{\partial x^i}\right) = dx^3\left(\tfrac{\partial}{\partial x^i}\right) = \delta_i^{\,3}$$

so as $e^{-x^1} \in C^\infty(M)$, we can write

$$\nabla_{\partial/\partial x^i} 1 = \nabla_{\partial/\partial x^i} e^{-x^1} e^{x^1}$$

$$= \tfrac{\partial}{\partial x^i}[e^{-x^1}]\, e^{x^1} + e^{-x^1} \nabla_{\partial/\partial x^i} e^{x^1} \quad \left(\text{Property (iv)}\right)$$

$$= -\delta_i^{\,1} + e^{-x^1}\delta_i^{\,3}$$

meaning

$$\nabla 1 = -dx^1 + e^{-x^1} dx^3.$$

5. Let $M = \mathbb{R}^2$ & take the connection on TM given by

$$\nabla_{\partial/\partial x^1} \tfrac{\partial}{\partial x^2} = x^2 \tfrac{\partial}{\partial x^1}, \qquad \nabla_{\partial/\partial x^i} \tfrac{\partial}{\partial x^j} = 0 \text{ otherwise.}$$

This gives the connection coefficients

$$\Gamma_{12}^{\,1} = x^2, \qquad \Gamma_{jk}^{\,i} = 0 \quad \text{otherwise.}$$

We can then use the component expression for T in a coordinate basis:

$$T_{ij}^{\,k} = \Gamma_{ij}^{\,k} - \Gamma_{ji}^{\,k}$$

and similarly for R:

$$(R_{ij})^k{}_l = \partial_i \Gamma_{jl}^{\,k} - \partial_j \Gamma_{il}^{\,k} + \Gamma_{jl}^{\,m}\Gamma_{im}^{\,k} - \Gamma_{il}^{\,m}\Gamma_{jm}^{\,k}.$$

Computing T:

$$T_{12}^{\,1} = \Gamma_{12}^{\,1} - \cancel{\Gamma_{21}^{\,1}}$$

$$= x^2$$

$$T_{21}^{\,1} = \Gamma_{21}^{\,1} - \Gamma_{12}^{\,1}$$

$$= -x^2,$$

with all other components vanishing. Computing R:

$$(R_{12})^1{}_1 = \partial_1 \Gamma_{21}^{\,1} - \partial_2 \Gamma_{11}^{\,1} + \Gamma_{21}^{\,m}\Gamma_{1m}^{\,1} - \Gamma_{11}^{\,m}\Gamma_{2m}^{\,1}$$

$$= 0.$$

$$(R_{12})^1{}_2 = \cancel{\partial_1 \Gamma^1_{22}} - \partial_2 \Gamma^1_{12} + \cancel{\Gamma^m_{22}} \Gamma^1_{1m} - \cancel{\Gamma^m_{12}} \Gamma^1_{2m}$$

$$= -\partial_2 x^2$$

$$= -1.$$

$$(R_{12})^2{}_1 = 0 \quad (\Gamma^2_{jk} = 0)$$

$$(R_{12})^2{}_2 = 0 \quad (\text{same reasoning}).$$

Note by antisymmetry $R_{21} = -R_{12}$ & $R_{11} = R_{22} = 0$ so this completely specifies R.

(b) We can solve the parallel transport equations w/ general initial data & then consider each side of the square separately. We must consider 2 cases:

① Horizontals:

Let $\alpha : [0,1] \to \mathbb{R}^2$, $\alpha(t) = (at, b)$. The equations are

$$\dot{V}^1 + \Gamma^1_{12} \dot{\alpha}^1 V^2 = 0$$

$$\dot{V}^2 = 0.$$

The second gives $V^2(t) = V^2(0)$, so the first becomes

$$0 = \dot{V}^1 + \Gamma^1_{12}(\alpha(t)) \dot{\alpha}^1 V^2(0)$$

$$= \dot{V}^1 + ab V^2(0),$$

so

$$V^1(t) = ab V^2(0) t + V^1(0)$$

meaning

$$\begin{pmatrix} V^1(t) \\ V^2(t) \end{pmatrix} = \begin{pmatrix} 1 & abt \\ 0 & 1 \end{pmatrix} \begin{pmatrix} V^1(0) \\ V^2(0) \end{pmatrix}$$

so the map $Tr_\alpha : T_{(0,b)}M \to T_{(a,b)}M$ is given by

$$Tr_\alpha = \begin{pmatrix} 1 & ab \\ 0 & 1 \end{pmatrix}$$

$\begin{matrix} 0 & 1 \end{matrix}\Big)$

② Verticals:

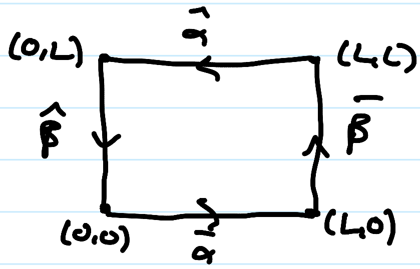
Let $\beta:[0,1]\to M$, $\beta(t)=(a,b+vt)$. The equations are

$$0=\dot V^1+\cancel{\Gamma^1_{12}\dot\beta^1 V^2}=\dot V^1 \quad (\dot\beta^1=0)$$

$$\dot V^2=0,$$

thus the map $Tr_\beta: T_{(a,b)}M\to T_{(a,b+v)}M$ is just the identity

$$Tr_\beta=\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$



The holonomy around γ is then

$$Tr_\gamma=(Tr_{\hat\beta})^{-1}\circ(Tr_{\hat\alpha})^{-1}\circ Tr_{\bar\beta}\circ Tr_{\bar\alpha}$$

$$=(Tr_{\hat\alpha})^{-1}\circ Tr_{\bar\alpha} \qquad (Tr_\beta=I)$$

where $\bar\alpha$ is α w/ $a=L$, $b=0$, so $Tr_{\bar\alpha}=I$, and $\hat\alpha$ is α w/ $a=L$, $b=L$, so

$$Tr_{\hat\alpha}=\begin{pmatrix}1&L^2\\0&1\end{pmatrix},$$

so

$$Tr_\gamma=Tr_{\hat\alpha}^{-1}=\begin{pmatrix}1&-L^2\\0&1\end{pmatrix}$$

(c) No - we have by a proposition in the notes that on an orientable Riemannian manifold, for ∇ a metric connection on TM & $x\in M$, that the based holonomy group at x is isomorphic to a subgroup of $SO(n)$. The above computation gives the form of $H_{(0,0)}$ on $\mathbb{R}^2$ (which is trivially a 2D orientable manifold) & it is clearly not isomorphic to a subgroup of $SO(2)$ as it is not orthogonal ($Tr_\gamma^T Tr_\gamma\neq I$). Thus ∇ cannot be a metric compatible connection for any Riemannian metric on M a metric compatible connection for any Riemannian metric on M.

6. Let $M=\mathbb{R}^2$ w/ a Riemannian metric g that at $(0,0)$ takes value $dx^1 \otimes dx^1 + dx^2 \otimes dx^2$.

(a) Consider $\alpha:[0,1]\to M$, $\alpha(t)=t(1-t^2)(\cos t, \sin t)$. For α to be a geodesic in (M,g), it must be traversed at constant speed i.e.

$$\frac{d}{dt}\|\dot\alpha\|^2 = 0$$

$\forall\, t\in[0,1]$. Clearly both $t=0$ & $t=1$ correspond to the point $(0,0)$, therefore we must have by the above

$$\|\dot\alpha(0)\|^2 = \|\dot\alpha(1)\|^2.$$

First observe that

$$\dot\alpha(t) = t(1-t^2)(-\sin t, \cos t) + (1-3t^2)(\cos t, \sin t),$$

meaning

$$\dot\alpha(0) = (1,0), \qquad \dot\alpha(1) = (-2\cos(1), -2\sin(1)).$$

Importantly then, as we know the value of the metric at $t=0,1$ (it is just g_{Euc}) we can compute the speed at these values:

$$\|\dot\alpha(0)\|^2 = g_{Euc}(\dot\alpha(0), \dot\alpha(0))$$
$$= 1$$

$$\|\dot\alpha(1)\|^2 = g_{Euc}(\dot\alpha(1), \dot\alpha(1))$$
$$= 2^2\cos^2(1) + 2^2\sin^2(1)$$
$$= 4 \neq \|\dot\alpha(0)\|^2,$$

thus α is not a geodesic.

(b) β being a geodesic means it parallel transports its own tangent vector, which is given by

$$\dot\beta(t) = \frac{\pi}{3}t(1-t)\left(-\sin\frac{\pi t}{3}, \cos\frac{\pi t}{3}\right) + (1-2t)\left(\cos\frac{\pi t}{3}, \sin\frac{\pi t}{3}\right),$$

so

so

$$\dot\beta(0) = (1,0), \qquad \dot\beta(1) = \left(-\tfrac{1}{2}, -\tfrac{\sqrt{3}}{2}\right).$$

Thus using that the based holonomy at $(0,0)$ is given by the transformation of a parallely transported vector field around a closed loop based at $(0,0)$, we have upon choosing this loop to be the geodesic β,

$$\begin{pmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} = Tr_\beta \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

so the holonomy is given by

$$Tr_\beta = \begin{pmatrix} -\frac{1}{2} & a \\ -\frac{\sqrt{3}}{2} & b \end{pmatrix}$$

where a & $b \in \mathbb{R}$ are arbitrary.

7. Let (M,g) & (N,h) be Riemannian & $f: M \to (0,\infty)$ be smooth.

(a) The claim

'$k = g + fh$ is a Riemannian metric on $M \times N$'

contains an abuse of notation - for k to be a metric on $M \times N$ it must take as it's arguments elements of $T_{(p,q)} M \times N$ where $p \in M$ & $q \in N$, while g has arguments in T_pM & h in T_qN. Also, f is a function on M. Here however, it is stated as if all of these quantities take the same argument. To remedy this, we can use the natural projection maps

$$\pi_M : M \times N \to M, \quad \pi_N : M \times N \to N$$

to pullback the metrics on M & N to one on $M \times N$. Explicitly, we can write $\forall$ $(p,q) \in M \times N$, $X, Y \in T_{(p,q)} M \times N$,

$$k(X,Y) = (\pi_M^* g)(X,Y) + f(p)(\pi_N^* h)(X,Y).$$

(b) It then follows that k defined like so is a Riemannian metric on $M \times N$ - it is symmetric, as $\forall\ (p,q) \in M \times N$, $X, Y \in T_{(p,q)} M \times N$

$$k(Y,X) = (\pi_M^* g)(Y,X) + f(p)\,(\pi_N^* h)(Y,X)$$

$$= g(d\pi_M(Y), d\pi_M(X)) +$$

$$f(p)\, h(d\pi_N(Y), d\pi_N(X))$$

$$= g(d\pi_M(X), d\pi_M(Y)) +$$

$$f(p)\, h(d\pi_N(X), d\pi_N(Y))$$

(g & h Riemannian & so themselves are symmetric)

$$= (\pi_M^* g)(X,Y) + f(p)\,(\pi_N^* h)(X,Y)$$

$$= k(X,Y)$$

and positive definite, as $\forall\ (p,q) \in M \times N$, $X \in T_{(p,q)} M \times N$,

$$k(X,X) = (\pi_M^* g)(X,X) + f(p)\,(\pi_N^* h)(X,X)$$

$$= g(d\pi_M(X), d\pi_M(X)) + f(p)\, h(d\pi_N(Y), d\pi_N(Y))$$

which is ≥ 0 w/ equality iff $X = 0$, as g & h are both Riemannian & $f(p) > 0$. Thus k is a Riemannian metric on $M \times N$.

(c) Now suppose (M,g) & (N,h) are complete, and $\exists\ c > 0$ s.t. $\forall\ p \in M$, $f(p) \geq c$. Completeness means that every Cauchy sequence on these manifolds is convergent wrt the metric given by, for M,

$$d_M(p_1,p_2) = \inf_\gamma \int_a^b \sqrt{g(\dot\gamma,\dot\gamma)}\, dt = \inf_\gamma \int_a^b \|\dot\gamma\|\, dt$$

where $\gamma: [a,b] \to M$, $\gamma(a) = p_1$, $\gamma(b) = p_2$ is piecewise C^1. A similar definition holds for N. Now suppose (p_n, q_n) is a Cauchy sequence on $M \times N$, i.e. $\forall \epsilon > 0\ \exists\ N > 0$ s.t. $\forall\ m, n > N$,

$$d_{M \times N}((p_n,q_n),(p_m,q_m)) < \epsilon, \text{ where}$$

$$d_{M \times N}((p_1,q_1),(p_2,q_2)) = \inf_\gamma \int_a^b \sqrt{k(\dot\gamma,\dot\gamma)}\, dt,$$

$$\int^b \quad \overline{\qquad\qquad\qquad}$$

$$d_{M\times N}((p_1,q_1),(p_2,q_2)) = \inf_{\gamma} \int_a \sqrt{k(\dot\gamma,\dot\gamma)}\, dt,$$

$$= \inf_{\gamma} \int_a^b \sqrt{\pi_M^* g(\dot\gamma,\dot\gamma) + f(\gamma_1(t))\pi_N^* h(\dot\gamma,\dot\gamma)}\, dt$$

where $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ w/ γ_1 & γ_2 curves in M & N respectively satisfying the obvious requirements. Observe then that using $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for $a,b \geq 0$, we have

$$d_{M\times N}((p_1,q_1),(p_2,q_2)) \leq \inf_{\gamma} \int_a^b \|\dot\gamma_1\| + \sqrt{f(\gamma_1(t))}\|\dot\gamma_2\|\, dt$$

$$= d_M(p_1,p_2) + K d_N(q_1,q_2), \quad (*)$$

where $K = \inf_{\gamma} \sqrt{f(\gamma_1(t))} \geq \sqrt{c} > 0$. Observe also that

$$d_{M\times N}((p_1,q_1),(p_2,q_2)) \geq d_M(p_1,p_2), \text{ and}$$

$$d_{M\times N}((p_1,q_1),(p_2,q_2)) \geq \sqrt{c}\, d_N(q_1,q_2)$$

Thus (p_n,q_n) Cauchy $\Rightarrow$ p_n & q_n are Cauchy as if the former holds, then $\forall \epsilon > 0$ $\exists$ $N \in \mathbb{N}$ s.t. $\forall m,n > N$,

$$\epsilon > d_{M\times N}((p_n,q_n),(p_m,q_m)) \geq d(p_n,p_m)$$

and also $\exists$ $M \in \mathbb{N}$ s.t. $\forall m,n > M$,

$$\sqrt{c}\,\epsilon > d_{M\times N}((p_n,q_n),(p_m,q_m)) \geq \sqrt{c}\, d_N(q_n,q_m).$$

Therefore as (M,d_M) & (N,d_N) are complete, p_n & q_n are convergent to p & q say. Then $\forall \epsilon > 0$, we can choose $N > 0$ s.t. $\forall n > N$, $d_M(p_n,p) < \frac{\epsilon}{2}$, $d_N(q_n,q) < \frac{\epsilon}{2K}$, so by $(*)$

$$d_{M\times N}((p_n,q_n),(p,q)) \leq d_M(p_n,p) + K d_N(q_n,q)$$

$$< \frac{\epsilon}{2} + K \cdot \frac{\epsilon}{2K}$$

$$= \epsilon.$$

Thus Cauchy sequences are convergent on $(M\times N, k)$ meaning it is complete.