MATH2017 Problem Set 3 Solutions: Integration

1. (a) $f'(x) = (1 - x^2)^{-1/2}$ (if you've forgotten why, differentiate $\sin f(x) = x$ with respect to x and use a trig identity). This is enough to construct the 1st Taylor Approximant:

$$p_1(x) \quad = \quad f(1/2) + f'(1/2)(x - \frac{1}{2}) = \frac{\pi}{6} + \frac{2}{\sqrt{3}}(x - \frac{1}{2}).$$

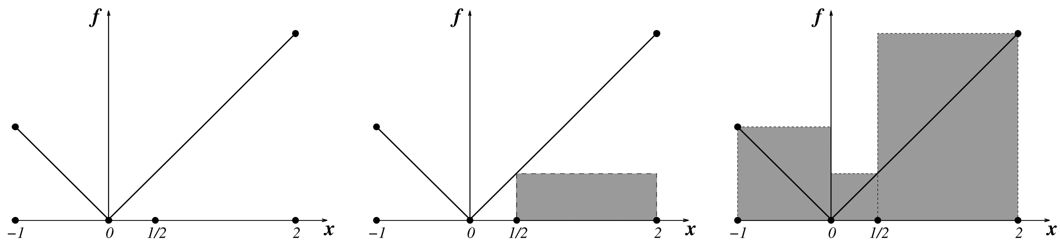
   (b) By Taylor's Theorem, there exists $c \in \left(\frac{1}{2}, \frac{3}{4}\right)$ such that

$$f(3/4) = p_1(3/4) + \frac{f''(c)}{2!}(\frac{3}{4} - \frac{1}{2})^2 = \frac{\pi}{6} + \frac{\sqrt{3}}{6} + \frac{f''(c)}{32}.$$

   Now $f''(x) = x(1-x^2)^{-3/2}$, which is a product of positive increasing functions for $0 < x < 1$, and hence is itself increasing on $(0, 1)$. Since $\frac{1}{2} < c < \frac{3}{4}$, we deduce that $f''(1/2) < f''(c) < f''(3/4)$, and hence

$$\frac{\pi}{6} + \frac{13\sqrt{3}}{72} < \sin^{-1}(3/4) < \frac{\pi}{6} + \frac{\sqrt{3}}{6} + \frac{3\sqrt{7}}{98}.$$

2. (a)



graph of f lower Riemann sum   upper Riemann sum

   (b) Referring to the middle and right-hand diagrams above, we see that

$$\begin{aligned} l_{\mathscr{D}}(f) &= 1 \times 0 + \frac{1}{2} \times 0 + \frac{3}{2} \times \frac{1}{2} = \frac{3}{4}, \\ u_{\mathscr{D}}(f) &= 1 \times 1 + \frac{1}{2} \times \frac{1}{2} + \frac{3}{2} \times 2 = 4\frac{1}{4} \end{aligned}$$

3. (a) We can prove this by induction on n. The formula certainly holds for $n = 1$, since

$$1^3 = \frac{1}{4}(1)^2(2)^2.$$

   Assume it holds for some $n = k \geq 1$. Then

$$\begin{aligned} \sum_{j=1}^{k+1} j^3 &= \sum_{j=1}^{k} j^3 + (k+1)^3 = \frac{1}{4}k^2(k+1)^2 + (k+1)^3 \\ &= \frac{1}{4}(k+1)^2(k^2 + 4k + 4) = \frac{1}{4}(k+1)^2(k+2)^2. \end{aligned}$$

   Hence, the formula holds for $n = k + 1$. Hence, by induction, the formula holds for all $n \in \mathbb{Z}^+$.

(b) For each $n \in \mathbb{Z}^+$, consider the regular dissection of size n,

$$\mathscr{D}_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}.$$

Since x^3 is increasing on $[0,1]$, its lower sum with respect to this dissection is

$$\begin{aligned} l_{\mathscr{D}_n} &= \sum_{j=1}^{n} \frac{1}{n}\left(\frac{j-1}{n}\right)^3 = \frac{1}{n^4}\sum_{j=1}^{n}(j-1)^3 = \frac{1}{n^4}\sum_{k=0}^{n-1} k^3 \\ &= \frac{1}{4n^4}(n-1)^2 n^2 \to \frac{1}{4} \end{aligned}$$

by part (a). Similarly, its upper sum with respect to this dissection is

$$u_{\mathscr{D}_n} = \sum_{j=1}^{n} \frac{1}{n}\left(\frac{j}{n}\right)^3 = \frac{1}{n^4}\sum_{j=1}^{n} j^3 = \frac{1}{4n^4}n^2(n+1)^2 \to \frac{1}{4}.$$

Hence, $u_{\mathscr{D}_n} - l_{\mathscr{D}_n} \to 0$ so, by Theorem 5.16, x^3 is integrable on $[0,1]$ and

$$\int_0^1 x^3 dx = \lim u_{\mathscr{D}_n} = \frac{1}{4}.$$

4. (a) By Theorem 6.1, $g'(x) = f(x) = (1+x^3)^{-3}$.

   (b) $g(x) = g_1(x) + g_2(x)$ where $g_1(x) = \int_0^x f$ and $g_2(x) = \int_{-x}^0 f = -\int_0^{-x} f = -g_1(-x)$. We already showed that $g_1'(x) = f(x)$ and, by the Chain Rule and Theorem 6.1, $g_2'(x) = -(-f(-x)) = f(-x)$. Hence

$$g'(x) = g_1'(x) + g_2'(x) = \frac{1}{(1+x^3)^3} + \frac{1}{(1-x^3)^3}.$$

   (c) Now $g(x) = g_1(f(x))$ so, by the Chain Rule

$$g'(x) = g_1'(f(x))f'(x) = f(f(x))f'(x) = -\frac{9x^2(1+x^3)^5}{1+(1+x^3)^9}.$$

5. Fix any $a \in \mathbb{R}$ and let (x_n) be any sequence in $\mathbb{R}$ such that $x_n \to a$. We must show that $g(x_n) \to g(a)$. So, let $\varepsilon \in (0,\infty)$ be given. We must show that there exists $N \in \mathbb{Z}^+$ such that, ofr all $n \geq N$,

$$|g(x_n) - g(a)| < \varepsilon.$$

Since $x_n \to a$, there exists $N_1 \in \mathbb{Z}^+$ such that, for all $n \geq N_1$, $x_n \in [a-1, a+1]$. Let $K = \sup\{|f(x)| : x \in [a-1, a+1]\}$ which certainly exists, since f is Riemann integrable, and hence bounded, on the interval $[a-1, a+1]$. Now, since $x_n \to a$, there exists $N_2 \in \mathbb{Z}^+$ such that, for all $n \geq N_2$, $|x_n - a| < \varepsilon/(K+1)$. Let

$N = \max\{N_1, N_2\}$. Then , for all $n \geq N$,

$$\begin{aligned}
|g(x_n) - g(a)| &= \left| \int_0^{x_n} f - \int_0^a f \right| = \left| \int_a^{x_n} f \right| \leq \left| \int_a^{x_n} |f| \right| && \text{by Proposition 5.24} \\
&\leq \left| \int_a^{x_n} K \right| && \text{since } n \geq N_1 \text{, so } |x_n - a| < 1 \\
&= K|x_n - a| < K \frac{\varepsilon}{K+1} && \text{since } n \geq N_2 \\
&< \varepsilon.
\end{aligned}$$

Hence, $g(x_n) \to g(a)$. So g is continuous at a. This holds for all $a \in \mathbb{R}$, so g is continuous.

Remark: Had we been allowed to assume that f were continuous, a much quicker proof would have been available: g is differentiable by version one of the Fundamental Theorem of the Calculus, and hence is continuous by Proposition 3.8. But you were explcitly told *not* to assume that f is continuous.