MATH2017 Problem Set 2 Solutions: Differentiability on an interval

1. The inequality holds trivially if $x = 0$, so assume $x \neq 0$. Then, by the Mean Value Theorem applied to sin on $[0, x]$ (if $x > 0$) or $[x, 0]$ (if $x < 0$), there exists c between x and 0 such that

$$\begin{aligned} \frac{\sin x - \sin 0}{x - 0} &= \sin' c = \cos c \\ \Rightarrow \quad \left| \frac{\sin x}{x} \right| &= |\cos c| \leq 1. \end{aligned}$$

   Hence $|\sin x| \leq |x|$.

2. For all $x \in \mathbb{R}$,

$$f'(x) = 28x^6 - 56x^3 + 30 = 28(x^3 - 1)^2 + 2 > 0,$$

   so f is strictly increasing (Theorem 4.11) and hence injective.

3. (a) For all $x \in \mathbb{R}$, $f(x) \geq 2x^4 - x^4 = x^4 \geq 0 = f(0)$. Hence, f attains a minimum at 0.

   (b) f coincides on the open set $U = \mathbb{R} \backslash \{0\}$ with the function

$$g : U \to \mathbb{R}, \qquad g(x) = 2x^4 + x^4 \sin(1/x).$$

   This is differentiable, by the Chain and Product Rules, so, by the Localization Lemma, f is differentiable on U with derivative

$$f'(a) = g'(a) = 8a^3 + 4a^3 \sin(1/a) - a^2 \cos(1/a).$$

   Consider now the case $a = 0$. Given any $\varepsilon > 0$, let $\delta = \min\{1, \varepsilon/3\} > 0$. Then for all $x \in (-\delta, 0) \cup (0, \delta)$,

$$\begin{aligned} \left| \frac{f(x) - f(0)}{x - 0} - 0 \right| &= \left| 2x^3 + x^3 \sin(1/x) \right| \leq 2|x|^3 + |x|^3 \\ &< 2|x| + |x| \qquad (\text{since } |x| < \delta \leq 1) \\ &< \varepsilon \qquad (\text{since } |x| < \delta \leq \varepsilon/3). \end{aligned}$$

   Hence f is differentiable at 0 also, and $f'(0) = 0$.

   (c) Let $\varepsilon > 0$ be given. Then there exists $n \in \mathbb{Z}^+$ such that $n > 1/(2\pi\varepsilon)$, and hence $2\pi(n + \frac{1}{4}) > 1/\varepsilon$. Hence, $x = \frac{1}{2\pi(n+1/4)} \in (0, \varepsilon)$, and

$$f'(x) = 8x^3 + 4x^3 \sin\frac{\pi}{2} - x^2 \cos\frac{\pi}{2} = 12x^3 > 0$$

   while

$$f'(-x) = -8x^3 - 4x^3 \sin(-\frac{\pi}{2}) - x^2 \cos(-\frac{\pi}{2}) = -4x^3 < 0.$$

4. (a) Infinitely many correct answers are possible. Here's one:

$$f(x) = \begin{cases} 1/x, & x > 0, \\ 0, & x = 0. \end{cases}$$

(b) No such function exists, by the Extreme Value Theorem, since every differentiable function is continuous.

(c) Again, infinitely many correct answers are possible. The one that springs to my mind is

$$f(x) = \sin(x^2).$$

(d) No such function exists.

Proof: Assume towards a contradiction, that f has the required properties. Since f is unbounded above, for each $n \in \mathbb{Z}^+$ there exists $x_n \in (0, 1)$ such that $f(x_n) > f(1/2) + n$. Applying the MVT to f on the interval from $1/2$ to x_n, there exists $y_n \in (0, 1)$ such that

$$|f'(y_n)| = \left| \frac{f(x_n) - f(1/2)}{x_n - 1/2} \right| > \frac{n}{1/2}.$$

Clearly $|f'(y_n)|$ is unbounded above, which contradicts the assumption that f has bounded derivative.