MATH2017 Problem Set 5 Solutions: Power series

1. Let $b_n = |x^{3n+7}/(2^n+1)|$. Then

$$\frac{b_{n+1}}{b_n} = |x|^3 \frac{2^n+1}{2^{n+1}+1} = |x|^3 \frac{1+2^{-n}}{2+2^{-n}} \to \frac{|x|^3}{2}.$$

Hence, by the Ratio Test, the power series converges absolutely if $|x|^3 < 2$, but does not converge absolutely if $|x|^3 > 2$. So its radius of convergence is $2^{1/3}$.

2. (a) We know that $A_k := \sum_{n=0}^{k} |\alpha_k|$ and $B_k := \sum_{n=0}^{k} |\beta_k|$ both converge, so are bounded above. Hence

$$\begin{aligned} C_k &:= \sum_{n=0}^{k} |\alpha_k \beta_k| = |\alpha_0||\beta_0| + |\alpha_1||\beta_1| + \cdots + |\alpha_k||\beta_k| \\ &\leq (|\alpha_0| + |\alpha_1| + \cdots + |\alpha_k|)(|\beta_0| + |\beta_1| + \cdots + |\beta_k|) = A_k B_k \end{aligned}$$

is also bounded above. Clearly C_k is increasing, so converges by the Monotone Convergence Theorem. Hence $\sum_{n=0}^{\infty} \alpha_n \beta_n$ converges absolutely.

(b) Choose any x with $0 < |x| < R_1 R_2$. Let $\alpha = \ln R_1 / \ln(R_1 R_2) \in (0,1)$ and note that $1 - \alpha = \ln R_2 / \ln(R_1 R_2)$. Now

$$\begin{aligned} \ln |x|^{\alpha} = \alpha \ln |x| &< \alpha \ln(R_1 R_2) = \ln R_1 \\ \ln |x|^{1-\alpha} = (1-\alpha) \ln |x| &< (1-\alpha) \ln(R_1 R_2) = \ln R_2. \end{aligned}$$

Since ln is increasing, we deduce that $|x|^{\alpha} < R_1$ and $|x|^{1-\alpha} < R_2$. Hence $\sum_{n=0}^{\infty} a_n (|x|^{\alpha})^n$ and $\sum_{n=0}^{\infty} b_n (|x|^{1-\alpha})^n$ both converge absolutely. As we just showed, it follows that

$$\sum_{n=0}^{\infty} a_n (|x|^{\alpha})^n b_n (|x|^{1-\alpha})^n = \sum_{n=0}^{\infty} a_n b_n |x|^n$$

converges absolutely, and hence that $\sum_{n=0}^{\infty} a_n b_n x^n$ converges absolutely. Since this holds for any $|x| < R_1 R_2$, we conclude that $R \geq R_1 R_2$.

3. (a) We apply the Weierstrass M Test (Theorem 8.15) with

$$\begin{aligned} D &:= (-\infty, 0], \\ f_n(x) &:= \frac{1}{2^n}\sqrt{1+e^{nx}}, \\ M_n &:= \frac{\sqrt{2}}{2^n}. \end{aligned}$$

(i) For all $x \in (-\infty, 0]$, and all $n \in \mathbb{N}$, $e^{nx} \leq e^0 = 1$, so

$$f_n(x) \leq \frac{1}{2^n}\sqrt{1+1} = M_n.$$

(ii) $M_{n+1}/M_n = 1/2 \to 1/2 < 1$, so the series $\sum_{n=0}^{\infty} M_n$ converges by the Ratio Test.

Hence, by the Weierstrass M Test, the series $\sum_{n=0}^{\infty} f_n(x)$ converges uniformly on $(-\infty, 0]$.

(b) Clearly $(-\infty, 0] \subseteq E$ by part (a). We must determine whether E contains any positive numbers. With $f_n(x)$ as above, and $x > 0$,

$$\frac{f_{n+1}(x)}{f_n(x)} = \frac{1}{2}\sqrt{\frac{1 + e^x e^{nx}}{1 + e^{nx}}} = \frac{1}{2}\sqrt{\frac{e^{-nx} + e^x}{e^{-nx} + 1}} \to \frac{e^{x/2}}{2}$$

by the Algebra of Limits. Hence, by the Ratio Test, the series converges if $e^{x/2} < 2$ and diverges if $e^{x/2} > 2$. So the series converges on $(0, \ln 4)$ and diverges on $(\ln 4, \infty)$. It remains to consider the case $x = \ln 4$. At this point, the series is

$$\sum_{n=0}^{\infty} \frac{1}{2^n}\sqrt{1 + 4^n} = \sum_{n=0}^{\infty} \sqrt{\frac{1}{4^n} + 1}$$

which diverges by the Divergence Test (since its sequence of terms does not converge to 0). Hence $E = (-\infty, \ln 4)$.

4. We must show that, for each $x_0 \in \mathbb{R}$, there exists $\varepsilon > 0$ and a power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ such that, for all $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$, the power series converges to $f(x)$.

Note that

$$f(x) \;=\; x^2 = (x - x_0)^2 + 2x_0 x - x_0^2 = (x - x_0)^2 + 2x_0(x - x_0) + x_0^2.$$

So, for given $x_0 \in \mathbb{R}$, we may choose $\varepsilon = 1$ (or any other $\varepsilon > 0$, actually) and

$$a_n = \begin{cases} x_0^2, & n = 0, \\ 2x_0, & n = 1, \\ 1, & n = 2 \\ 0, & n \geq 3. \end{cases}$$

Then for all $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$,

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n = x^2 = f(x).$$