Chapter 9

Homework problems

9.1 Problem Set 1: Limits of functions, differentiability

Submit on Gradescope by 17:00, Friday 11 February 2022

1. Let $D \subseteq \mathbb{R}$. Write down precise mathematical formulations of the following statements, using quantifiers ($\forall$, $\exists$):
   (a) D is bounded above.
   (b) D has a maximum element.
   (c) D does not have a maximum element.
2. Prove from first principles (i.e. give a direct ε–N proof) that the following sequence converges:
$$a_n = \frac{3n+1}{2n-1}.$$
3. Prove Proposition 1.28: $a \in \mathbb{R}$ is a cluster point of a set $D \subseteq \mathbb{R}$ if and only if there exists a sequence (x_n) in $D\backslash\{a\}$ which converges to a.
4. Give a direct ε–δ proof that
$$\lim_{x\to 2} \frac{\sqrt{x}-\sqrt{2}}{x-2} = \frac{1}{2\sqrt{2}}.$$
   Interpret this result in terms of differentiability.
5. Give a direct ε–δ proof that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist. Interpret this result in terms of differentiability.
6. Give a direct ε–δ proof that the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^4$ is differentiable at 1.

9.2 Problem Set 2: Differentiability on an interval

Submit on Gradescope by 17:00, Friday 4 March 2022

1. Prove that, for all $x \in \mathbb{R}$, $|\sin x| \leq |x|$.
2. Prove that the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 4x^7 - 14x^4 + 30x - 17$, is injective.
3. Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 2x^4 + x^4 \sin(1/x)$ if $x \neq 0$, and $f(0) = 0$.
   (a) Prove that f attains a minimum at 0. *(Hint: no need for calculus!)*
   (b) Prove that f is differentiable (everywhere), and that $f'(0) = 0$. *(Hint: you can reduce the work involved using the Localization Lemma, Lemma 2.24.)*
   (c) Prove that, for all $\varepsilon > 0$, f' takes both positive and negative values on $(-\varepsilon, \varepsilon)$.
4. (a) Construct the first Taylor approximant to $f : (-1, 1) \to \mathbb{R}$, $f(x) = \sin^{-1} x$ about $a = \frac{1}{2}$.
   (b) Use this approximant, and Taylor's Theorem, to find upper and lower bounds on $\sin^{-1}(3/4)$.
5. In each of the following cases, either write down a function with the specified properties, or explain why no such function exists.
   (a) An unbounded function $f : [0, 1] \to \mathbb{R}$.
   (b) An unbounded differentiable function $f : [0, 1] \to \mathbb{R}$.
   (c) A bounded differentiable function $f : \mathbb{R} \to \mathbb{R}$ whose derivative $f' : \mathbb{R} \to \mathbb{R}$ is unbounded.
   (d) A differentiable function $f : (0, 1) \to \mathbb{R}$ which is unbounded above but whose derivative is bounded.

9.3 Problem Set 3: Integration

Submit on Gradescope by 17:00, Friday 18 March 2022

1. Let $A = \{x + \frac{1}{n} \ : \ x \in (0,1),\ n \in \mathbb{Z}^+\}$. Determine the supremum and infimum of A, if they exist. Rigorously justify your answers.

2. Let $f : [-1,2] \to \mathbb{R}$, $f(x) = |x|$, and $\mathscr{D} = \{-1, 0, 1/2, 2\}$.
   (a) Draw the graph of f.
   (b) Compute the lower and upper Riemann sums $l_{\mathscr{D}}(f)$, $u_{\mathscr{D}}(f)$.

3. (a) Prove that $\displaystyle\sum_{j=1}^{n} j^3 = \frac{1}{4}n^2(n+1)^2$.
   (b) Evaluate $\int_0^1 x^3\,dx$ directly from the definition, without using the Fundamental Theorem of the Calculus. *(Hint: use Lemma 4.18 or Theorem 4.21.)*

4. Let $f : (-1,1) \to \mathbb{R}$, $f(x) = (1+x^3)^{-3}$. Compute $g' : (0,1) \to \mathbb{R}$ where
   (a) $g(x) = \displaystyle\int_0^x f$.
   (b) $g(x) = \displaystyle\int_{-x}^x f$.
   (c) $g(x) = \displaystyle\int_0^{f(x)} f$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be Riemann integrable on every interval $[a,b] \subset \mathbb{R}$ and define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = \int_0^x f$. Prove that g is continuous. (Warning: you may **not** assume that f is continuous.)

9.4 Problem Set 4: Uniform convergence

Submit on Gradescope by 17:00, Friday 29 April 2022

1. For each $n \in \mathbb{Z}^+$ let $f_n : \mathbb{R} \to \mathbb{R}$ be the function
$$f_n(x) = \frac{nx}{n|x| + 1}.$$
   (a) Prove that (f_n) converges pointwise to some function $f : \mathbb{R} \to \mathbb{R}$.
   (b) Prove that (f_n) does *not* converge uniformly.
2. Construct a sequence of unbounded functions $g_n : [0, 1] \to \mathbb{R}$ that converges pointwise to a bounded function $g : [0, 1] \to \mathbb{R}$.
3. For each $n \in \mathbb{Z}^+$ let $f_n : [0, \frac{1}{2}] \to \mathbb{R}$, $f_n(x) = 1/(1 + x^n)$.
   (a) Prove that (f_n) converges uniformly.
   (b) Compute the limit $\lim_{n\to\infty} \int_0^{1/2} f_n$, rigorously justifying your answer.
4. For each $n \in \mathbb{Z}^+$ let $f_n : \mathbb{R} \to \mathbb{R}$ be the function
$$f_n(x) = \sum_{k=1}^{n} \frac{1}{k^2} \cos(kx).$$
   Prove that (f_n) is uniformly Cauchy, and hence converges uniformly.
5. (a) Let $f, g : D \to \mathbb{R}$ be bounded functions. Prove that $\|fg\| \leq \|f\|\|g\|$.
   (b) Assume $f_n : D \to \mathbb{R}$ converges uniformly to f and $g_n : D \to \mathbb{R}$ converges uniformly to g. Prove that $(f_n g_n)$ converges uniformly to fg.

9.5 Problem Set 5: Power series

Submit on Gradescope by 17:00, Friday 6 May 2022

1. Compute the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{x^{3n+7}}{2^n+1}$.

2. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n n} = \ln 2$. *[Hint: integrate the geometric series.]*

3. (a) Prove that the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n}\sqrt{1+e^{nx}}$ converges uniformly on $(-\infty, 0]$. *(Hint: Weierstrass M Test!)*

   (b) Determine the subset $E \subseteq \mathbb{R}$ on which the series converges. Rigorously justify your answer.