Workshop 4: solutions for week 5

1. (a) $[0,1)$ is not open since it contains 0, but for all $\varepsilon > 0$, $(0-\varepsilon, 0+\varepsilon)$ is not a subset of $[0,1)$ (since it contains $-\varepsilon/2$, for example).
   (b) $\mathbb{R}\backslash[0,1) = (-\infty,0)\cup[1,\infty)$ is not open since it contains 1, but for all $\varepsilon > 0$, $(1-\varepsilon, 1+\varepsilon)$ is not a subset of $\mathbb{R}\backslash[0,1)$.
   (c) $\mathbb{R}\backslash[0,1] = (-\infty,0)\cup(1,\infty)$ is open. If $x > 1$ we may choose $\varepsilon = 1-x > 0$. If $x < 0$, we may choose $\varepsilon = -x > 0$. In either case, $(x-\varepsilon, x+\varepsilon) \subset \mathbb{R}\backslash[0,1]$.
   (d) $\mathbb{R}\backslash\{2^n : n \in \mathbb{Z}\}$ is not open since it contains 0, but for all $\varepsilon > 0$ there exists $N \in \mathbb{Z}^+$ such that $2^N > 1/\varepsilon$, whence $0 < 2^{-N} < \varepsilon$. So $2^{-N} \in (0-\varepsilon, 0+\varepsilon)$ but $2^{-N} \notin \mathbb{R}\backslash\{2^n : n \in \mathbb{Z}\}$.

2. (a) $(g\circ f)'(1) = g'(-1)f'(1) = 14$.
   (b) $h'(1) = f'(f(1)^2)\times 2f(1)f'(1) = -8$.

3. (a) Assume, towards a contradiction, that there exist $x, y \in \mathbb{R}$ such that $|xy| > \frac{1}{2}(x^2+y^2)$. Since both sides of this inequality are non-negative, it follows that

$$\begin{aligned} 4|xy|^2 &> (x^2+y^2)^2 \\ \Rightarrow 0 &> x^4 - 2x^2y^2 + y^4 = (x^2-y^2)^2. \end{aligned}$$

   But the square of a real number cannot be negative.

   (b) The claim holds trivially if $x = y$, and is symmetric under interchange of x, y, so it suffices to prove it in the case where $x > y$. The function $f : [y,x] \to \mathbb{R}$, $f(t) = \ln(4+t^2)$ is differentiable. Hence, by the MVT, there exists $c \in (y,x)$ such that

$$\begin{aligned} \frac{f(x)-f(y)}{x-y} &= f'(c) = \frac{2c}{4+c^2} \\ \Rightarrow \quad |f(x)-f(y)| &= \frac{2|c|}{4+c^2}|x-y| \leq \frac{\frac{1}{2}(2^2+c^2)}{4+c^2}|x-y| = \frac{1}{2}|x-y| \end{aligned}$$

   by part (a). The claim immediately follows.