Workshop 1: questions for week 2

1. Let $f : D \to \mathbb{R}$ (read as "let f map D to $\mathbb{R}$"), where $D \subseteq \mathbb{R}$. Write down precise mathematical formulations of the following statements, using quantifiers ($\forall$, $\exists$):
   (a) D is bounded below.
   (b) f is unbounded above.
   (c) f is surjective.
   (d) f is not surjective.
2. Prove from first principles (i.e. give a direct ε–N proof) that the following sequence converges:
$$a_n = \frac{n^2 + (-1)^n}{n^2 + 2}$$
3. Give a direct ε–N proof of part (i) of Proposition 1: if $a_n \to A$ and $b_n \to B$ then $a_n + b_n \to A + B$.
4. Let the sequence (a_n) be defined so that $a_1 = 1$ and $a_{n+1} = 1/(2 + a_n^2)$ for all $n \geq 1$.
   (a) Show that, for all $n \geq 2$, $|a_{n+1} - a_n| \leq \frac{1}{2}|a_n - a_{n-1}|$.
   (b) Deduce that (a_n) is Cauchy, hence convergent. Any idea what its limit is?