Workshop 9: solutions for week 10

1. (a) This series diverges, by the Divergence Test, since its sequence of terms $a_n = n/(n+1)$, does not converge to 0.

   (b) This series converges, by the Ratio Test: if $a_n = n!/n^n$, then

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{n!}\frac{n^n}{(n+1)^{n+1}} = \frac{n^n}{(n+1)^n} = \frac{1}{(1+1/n)^n} \to \frac{1}{e} < 1.$$

2. I claim the the radius of convergence is $R = 1$.

   Proof: For all $|x| < 1$, and all $k \geq 2$,

$$\sum_{n=2}^{k} |a_n x^n| \leq \sum_{n=0}^{k} |x|^n \leq \frac{1}{1-|x|}$$

   so the sequence $\sum_{n=2}^{k} |a_n x^n|$ is increasing and bounded above, and hence converges by the Monotone Convergence Theorem (MCT). It follows that $R \geq 1$ (it is the supremum of a set which contains $[0,1)$). On the other hand, at $x = 1$ the k-th partial sum of the power series is precisely the number of prime numbers less than or equal to k. Since the set of primes is infinite, this sequence is unbounded above, so the power series diverges at $x = 1$. Hence $R \leq 1$ (since $R > 1$ would contradict Theorem 8.11). $\square$

3. We can conclude that $R \geq \min\{R_1, R_2\}$.

   Proof: Let x have $|x| < \min\{R_1, R_2\}$. Then $|x| < R_1$ and $|x| < R_2$ so, by Theorem 8.11, both $f(x)$ and $g(x)$ converge absolutely. But

$$s_k := \sum_{n=0}^{k} |(a_n + b_n)x^n| \leq \sum_{n=0}^{k} |a_n x^n| + \sum_{n=0}^{k} |b_n x^n|$$

   by the Triangle Inequality, so (s_k) is bounded above by a sum of two convergent sequences. Hence (s_k) is bounded above, and increasing, so converges by the MCT. So $h(x) = \sum_{n=0}^{\infty}(a_n + b_n)x^n$ converges absolutely. Since this holds for all x with $|x| < \min\{R_1, R_2\}$, R is the supremum of a set containing $[0, \min\{R_1, R_2\})$, and hence $R \geq \min\{R_1, R_2\}$. $\square$

   If $R_1 \neq R_2$ we can conclude further that $R = \min\{R_1, R_2\}$.

   Proof: We can assume, without loss of generality, that $R_1 < R_2$. Assume, towards a contradiction, that $R > R_1$. Then there exists $x \in \mathbb{R}$ with $|x| > R_1$, $|x| < R$ and $|x| < R_2$. Since $|x| < R$ and $|x| < R_2$, but $\sum_{n=0}^{\infty}(a_n + b_n)x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ converge. Hence, by the Algebra of Limits,

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}(a_n + b_n)x^n - \sum_{n=0}^{\infty} b_n x^n$$

converges. But $|x| > R_1$, so this contradicts Theorem 8.11. $\square$

Note we certainly can't conclude that $R = \min\{R_1, R_2\}$ if $R_1 = R_2$. Cheap counterexample: $a_n = 1$, $b_n = -1$. Then $R_1 = R_2 = 1$ ($f(x)$ is the geometric series, and $g(x) = -f(x)$) but $h(x) = f(x) + g(x) = 0$, which has radius of convergence $R = \infty$.