MATH2017 Problem Set 4 Solutions: Uniform convergence

1. (a) $f_n(0) = 0 \to 0$. For all $x > 0$,

$$f_n(x) = \frac{nx}{nx+1} = \frac{x}{x+1/n} \to \frac{x}{x+0} = 1$$

by the Algebra of Limits. For all $x < 0$,

$$f_n(x) = \frac{nx}{-nx+1} = \frac{x}{-x+1/n} \to \frac{x}{-x+0} = -1$$

by the Algebra of Limits. Hence, (f_n) converges pointwise to the discontinuous function

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0, \end{cases}$$

(b) Assume, towards a contradiction, that (f_n) converges uniformly. Then its limit is f (Theorem 7.9). Since each f_n is continuous, f is continuous (Theorem 7.11). But f is discontinuous at 0.

2. The sequence

$$f_n(x) = \begin{cases} 1/(nx) & x \in (0,1] \\ 0 & x = 0 \end{cases}$$

will do. Each f_n is unbounded above, but the sequence (f_n) converges pointwise to 0.

3. (a) Let $f : [0, 1/2] \to \mathbb{R}$, $f(x) = 1$. Then

$$\|f_n - f\| = \sup\{\frac{|x^n|}{|1+x^n|} : 0 \leq x \leq 1/2\} \leq \sup\{x^n : 0 \leq x \leq 1/2\} = \frac{1}{2^n} \to 0.$$

Hence $\|f_n - f\| \to 0$ by the Squeeze Rule.

(b) Since each f_n is continuous, Theorem 7.13 implies that

$$\lim_{n\to\infty} \int_0^{1/2} f_n = \int_0^{1/2} f = \frac{1}{2}.$$

4. Let $\varepsilon \in (0, \infty)$ be given. The sequence

$$a_n = \sum_{k=1}^{n} \frac{1}{k^2}$$

converges, since it is the sequence of partial sums of a convergent series. Hence, (a_n) is Cauchy (Theorem 1.20), so there exists $N \in \mathbb{Z}^+$ such that, for all $n, m \geq$

N, $|a_n - a_m| < \varepsilon$. Now, for all $x \in \mathbb{R}$,

$$\begin{aligned}
|f_n(x) - f_m(x)| &= \left| \sum_{k=\min\{n,m\}+1}^{\max\{n,m\}} \frac{1}{k^2}\cos(kx) \right| \\
&\le \sum_{k=\min\{n,m\}+1}^{\max\{n,m\}} \frac{1}{k^2}|\cos(kx)| \quad \text{(Triangle inequality)} \\
&\le \sum_{k=\min\{n,m\}+1}^{\max\{n,m\}} \frac{1}{k^2} \\
&= |a_n - a_m|.
\end{aligned}$$

Hence

$$\|f_n - f_m\| = \sup\{|f_n(x) - f_m(x)| : x \in \mathbb{R}\} \le |a_n - a_m|,$$

so, for all $n, m \ge N$, $\|f_n - f_m| < \varepsilon$. That is, (f_n) is uniformly Cauchy. It follows that (f_n) is uniformly convergent (Theorem 7.23). *[Remark: note that we proved this despite having no idea what the limit function*

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k^2}\cos(kx)$$

really is! We can conclude immediately that f, whatever it is, is certainly continuous (since each f_n is continuous).]

5. (a) For all $x \in D$,

$$0 \le |f(x)| \le \|f\| \qquad \text{and} \qquad 0 \le |g(x)| \le \|g\|,$$

and so

$$|f(x)g(x)| = |f(x)||g(x)| \le \|f\|\|g\|.$$

Hence $\|f\|\|g\|$ is an upper bound on $\{|f(x)g(x)| : x \in D\}$. Since $\|fg\|$, is the *least* upper bound on this set, $\|fg\| \le \|f\|\|g\|$.

(b) It follows from Lemma 7.22 that both f and g are bounded. Now

$$\|g_n\| = \|g_n - g + g\| \le \|g_n - g\| + \|g\|$$

by Lemma 7.20, and $\|g_n - g\| \to 0$, so the real sequence $\|g_n\|$ is bounded above: there exists $K > 0$ such that $\|g_n\| \le K$ for all n. Hence, for all n,

$$\begin{aligned}
0 \le \|f_n g_n - fg\| &= \|(f_n - f)g_n + f(g_n - g)\| \\
&\le \|(f_n - f)g_n\| + \|f(g_n - g)\| \qquad \text{(by Lemma 7.20)} \\
&\le \|f_n - f\|\|g_n\| + \|f\|\|g_n - g\| \qquad \text{(by part (a))} \\
&\le K\|f_n - f\| + \|f\|\|g_n - g\| =: s_n.
\end{aligned}$$

Now $\|f_n - f\| \to 0$ and $\|g_n - g\| \to 0$, so $s_n \to 0$ by the Algebra of Limits. Hence $\|f_n g_n - fg\| \to 0$ by the Squeeze Rule.