What are we going to study? Why?

Differential calculus is arguably the single most powerful and influential mathematical idea ever invented. Its applications pervade the natural sciences, and it is essential to most applied mathematics and much pure mathematics too (for example, in my own research specialism, Differential Geometry). For scientists in general, it's OK to use calculus as a computational method, without worrying overmuch about what the objects being calculated (derivatives, integrals, limits) really mean. We, however, are *mathematicians*, and we hold ourselves to a much higher standard. For us, it is essential that every object we use has a precise *definition*, and that general facts about, and relationships between, these objects are *proved*. It is not good enough simply to appeal to "common sense," or claim that a given assertion is "obvious": **just because something is "obvious" doesn't mean it's true!**

Example 0.1 Say we have a function $f : \mathbb{R} \to \mathbb{R}$ which is differentiable everywhere and has $f'(a) > 0$ at some particular point a. Then obviously, f is a strictly increasing function on some neighbourhood of a, right? That is, there exists $\delta > 0$ (possibly extremely small) such that, for all $x, y \in (a - \delta, a + \delta)$, if $x < y$ then $f(x) < f(y)$. Well, actually, no: this doesn't follow, and we will construct a counterexample shortly (Example 4.13 if you're impatient).

If something as "obvious" as this isn't necessarily true, how do we know that *any* of the rules of calculus, which you've informally derived and used profusely (the chain rule, product rule, quotient rule, fundamental theorem of the calculus etc. etc.), are reliable? We don't, until we have *proved* them, and we can't *prove* theorems about derivatives and integrals until we have properly *defined* derivatives and integrals.

The object of this module is to do just this: provide a rigorous development of calculus for functions $f : \mathbb{R} \to \mathbb{R}$, with all terms precisely defined, and all assertions proved. The central object in calculus is the *derivative*. By now, you should be well aware that this is actually a *limit*:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

So our first task will be to precisely define limits of functions. Luckily, you've already thought carefully about limits, at least in one context: limits of convergent *sequences*. We will begin, therefore, by reminding ourselves what exactly it means to say that a real sequence *converges* to a limit, then see how the basic underlying idea can be adapted to deal with limits of functions.

Chapter 1

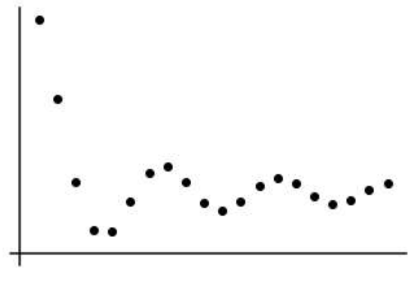
Limits of sequences

1.1 The limit of a convergent sequence

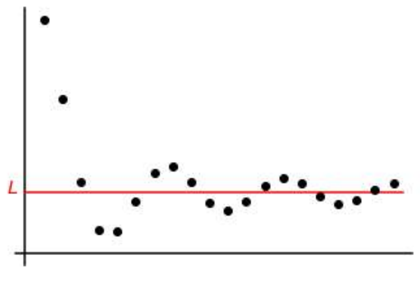
Recall the key definition from MATH1026 Sets, Sequences and Series (also covered in MATH1055 Numbers and Vectors), that of the *limit* of a *convergent* real sequence:

Definition 1.1 A real sequence (a_n) **converges to the limit** L if, for each $\varepsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that, for all $n \geq N$, $|a_n - L| < \varepsilon$. The shorthand for this is $a_n \to L$ or $\lim_{n\to\infty} a_n = L$.

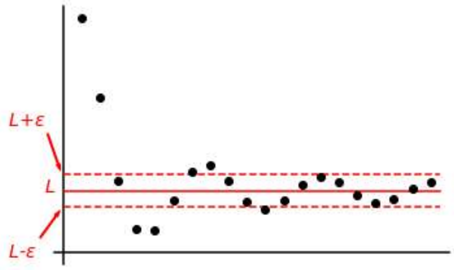
We can give Definition 1.1 a geometric interpretation. Imagine plotting the graph of the sequence (a_n):



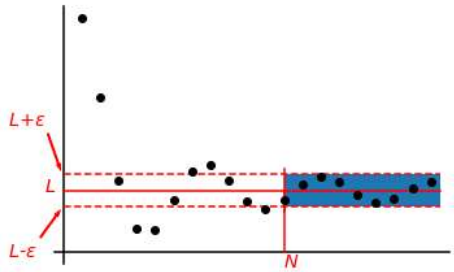
Now draw the horizontal line $y = L$, where L is our claimed limit:



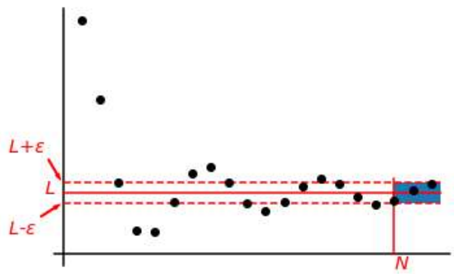
Then the statement that (a_n) converges to L means that, if we draw a horizontal strip of width 2ε centred on the line $y = L$, then, no matter how small we choose $\varepsilon > 0$, there is a vertical line $x = N$ to the right of which all points on the graph lie in the horizontal strip. For example, if we choose this value of ε:



then this value of N (or any N larger than this) will do:

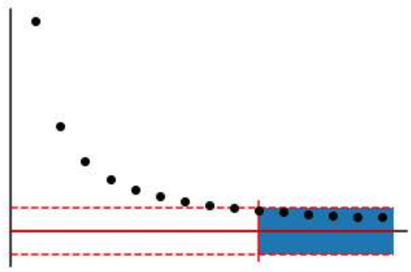


If we choose a smaller value of ε, we need to make N larger:



But no matter how small we make ε, there is always a point on the graph to the right of which all points lie in the narrow strip.

Pictures can be a powerful aid to understanding in Real Analysis, and I strongly encourage you to develop the habit of drawing them to illustrate definitions or to organize your thoughts at the beginning of a problem or proof. It's not hard, for example, to draw a picture that convinces us that the simple sequence $1/n$ converges to 0:



Of course, it's also vital that you can convert the picture into a proof:

Example 1.2 Claim: $a_n = 1/n \to 0$.

Proof: Let $\varepsilon \in (0, \infty)$ be given. Then there exists $N \in \mathbb{Z}^+$ such that $N > 1/\varepsilon$ (because the set $\mathbb{Z}^+$ is *unbounded above*). But then, for all $n \geq N$,

$$|a_n - 0| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

□

In the proof above, we showed that, for any prescribed $\varepsilon > 0$, there does indeed exist a positive integer N such that, for all $n \geq N$, $|a_n - 0| < \varepsilon$ (namely, any $N > 1/\varepsilon$ will do). A proof like this is called a proof "from first principles", or an "ε—N proof." The key step was to estimate (that is, bound above) the quantity $|a_n - L|$.

Example 1.3 Give a direct ε—N proof that $x_n = 2n/(n+1) \to 2$.

Solution Rough work: we begin by examining the quantity $|x_n - 2|$:

$$|x_n - 2| =$$

$$=$$

$$<$$

OK. So, for a given constant $\varepsilon > 0$, how big should we make n to ensure that $|x_n - 2| < \varepsilon$? Answer: making n bigger than $\boxed{}$ will do. We're now ready to write out our proof.

Proof: Let $\varepsilon \in (0, \infty)$ be given. Then there exists $N \in \mathbb{Z}^+$ such that

$$N >$$

Now, for all $n \geq N$,

$$\begin{aligned}
|x_n - 2| &= \\
&= \\
&< \\
&\leq \\
&< \varepsilon.
\end{aligned}$$

□

Many fundamental properties of convergent sequences follow quickly from Definition 1.1.

Proposition 1.4 (Uniqueness) *The limit of a convergent sequence is unique.*

Proof: Assume, towards a contradiction, that $x_n \to L_1$ and $x_n \to L_2$ where $L_1 \neq L_2$. Let $\varepsilon = |L_1 - L_2|/2 > 0$. Since $x_n \to L_1$, there exists $N_1 \in \mathbb{Z}^+$ such that, for all $n \geq N_1$, $|x_n - L_1| < \varepsilon$. Since $x_n \to L_2$, there exists $N_2 \in \mathbb{Z}^+$ such that, for all $n \geq N_2$, $|x_n - L_2| < \varepsilon$. Now let n be any positive integer greater than both N_1 and N_2 (for example, $n = N_1 + N_2$ will do). Then

$$\begin{aligned}
2\varepsilon &= |L_1 - L_2| = |(x_n - L_2) - (x_n - L_1)| \\
&\leq |x_n - L_2| + |x_n - L_1| \qquad \text{(by the triangle inequality)} \\
&< |x_n - L_2| + \varepsilon \qquad \text{(since } n > N_1\text{)} \\
&< \varepsilon + \varepsilon \qquad \text{(since } n > N_2\text{)}
\end{aligned}$$

Hence, $2\varepsilon < 2\varepsilon$, a contradiction. It follows that our original assumption that x_n converges to two different limits is false. □

Definition 1.5 A sequence (a_n) is **bounded above** if there exists $M \in \mathbb{R}$ such that, for all $n \in \mathbb{Z}^+$, $a_n \leq M$. Any such M is called an **upper bound** on (a_n). A sequence (a_n) is **bounded below** if there exists $K \in \mathbb{R}$ such that, for all $n \in \mathbb{Z}^+$, $a_n \geq K$. Any such K is called a **lower bound** on (a_n). The sequence is **bounded** if it bounded above and below.

Proposition 1.6 (Boundedness) *If (a_n) converges then (a_n) is bounded (that is, there exists $K \in \mathbb{R}$ such that, for all $n \in \mathbb{Z}^+$, $|a_n| \leq K$).*

Proof: Exercise. □

Note the converse of Proposition 1.6 is *false*. E.g. $a_n = (-1)^n$ is bounded, but not convergent. If a sequence does not converge, we say it **diverges** (or is **divergent**). So $(-1)^n$ diverges. Don't confuse the concepts of divergence and unboundedness.

Proposition 1.7 (Limits preserve non-strict inequalities) *If $a_n \to L$ and, for all $n \in \mathbb{Z}^+$, $K \leq a_n \leq M$, then $K \leq L \leq M$.*

Proof: Assume, towards a contradiction, that $L \notin [K, M]$. Then $L < K$ or $L > M$. We handle each of these two cases separately.

If $L < K$, then let $\varepsilon = K - L > 0$. Since $a_n \to L$, there exists $N \in \mathbb{Z}^+$ such that, for all $n \geq N$, $|a_n - L| < \varepsilon$, and hence $L - \varepsilon < a_n < L + \varepsilon$. In particular, $a_N < L + \varepsilon = L + (K - L) = K$. But *all* terms a_n of the sequence satisfy $a_n \geq K$, a contradiction. Hence $L \geq K$.

I leave the case where $L > M$ as an exercise. □

Another way of stating this is that, if $K \leq a_n \leq M$ for all $n \in \mathbb{Z}^+$ then

$$K \leq \lim_{n\to\infty} a_n \leq M,$$

assuming the limit exists. So limits of sequences preserve non-strict inequalities. We will make very frequent use of this fact, so it's important you understand and can prove it.

What if we know that a_n actually obeys *strict* inequalities,

$$K < a_n < M \quad ?$$

Can we conclude that

$$K < \lim_{n\to\infty} a_n < M \quad ?$$

Sadly, no, and Example 1.2 provides a counterexample: $0 < 1/n < 2$ for all $n \in \mathbb{Z}^+$, but its limit $L = 0$ does not satisfy $0 < L < 2$.

Proposition 1.8 (The Squeeze Rule) *Assume $a_n \leq b_n \leq c_n$, $a_n \to L$ and $c_n \to L$. Then $b_n \to L$.*

Proof: Let $\varepsilon \in (0, \infty)$ be given. We must show that there exists $N \in \mathbb{Z}^+$ such that, for all $n \geq N$, $|b_n - L| < \varepsilon$.

Since $a_n \to L$, there exists $N_1 \in \mathbb{Z}^+$ such that, for all $n \geq N_1$, $|a_n - L| < \varepsilon$, and hence $a_n > L - \varepsilon$.

Since $c_n \to L$, there exists $N_2 \in \mathbb{Z}^+$ such that, for all $n \geq N_2$, $|c_n - L| < \varepsilon$, and hence $c_n < L + \varepsilon$.

Hence, for all $n \geq N = \max\{N_1, N_2\}$,

$$b_n \geq a_n > L - \varepsilon,$$

since $n \geq N_1$, and

$$b_n \leq c_n < L + \varepsilon,$$

since $n \geq N_2$. So, for all $n \geq N$, $|b_n - L| < \varepsilon$. □

So if a sequence (b_n) is "squeezed" between two sequences which both converge to a common limit L, (b_n) converges to L also. We will use this fact very frequently.

Proposition 1.9 (The Algebra of Limits) *If $a_n \to A$ and $b_n \to B$ then*

(i) $a_n + b_n \to A + B$.

(ii) $a_n b_n \to AB$.

(iii) $a_n/b_n \to A/B$, *provided* $B \neq 0$ *and, for all* $n \in \mathbb{Z}^+$, $b_n \neq 0$.

Proof: I leave the proofs of parts 1 and 3 as exercises, and present a proof of part 2.

Let $\varepsilon \in (0, \infty)$ be given. We must show that there exists $N \in \mathbb{Z}^+$ such that, for all $n \geq N$,

$$|a_n b_n - AB| < \varepsilon.$$

We begin by noting that

$$|a_n b_n - AB| =$$

$$\leq$$

Since $a_n \to A$, there exists $N_1 \in \mathbb{Z}^+$ such that, for all $n \geq N_1$, $|a_n - A| < 1$, and hence, $a_n < A + 1 \leq |A| + 1$ and $a_n > A - 1 \geq -|A| - 1$, so $-a_n \leq |A| + 1$. Hence, for all $n \geq N_1$, $|a_n| < |A| + 1$.

Since $b_n \to B$, there exists $N_2 \in \mathbb{Z}^+$ such that, for all $n \geq N_2$,

$$|b_n - B| <$$

(Here we're just applying the definition of convergence to A for the particular choice of positive real number $\varepsilon' = \varepsilon/(2|A| + 2)$.)

Since $a_n \to A$, there exists $N_3 \in \mathbb{Z}^+$ such that, for all $n \geq N_3$,

$$|a_n - A| <$$

(Here we're just applying the definition of convergence to B for the particular choice of positive real number $\varepsilon'' = \varepsilon/(2|B| + 2)$.)

Now, let $N = \max\{N_1, N_2, N_3\}$. Then, for all $n \geq N$,

$$
\begin{aligned}
|a_n b_n - AB| &\leq \\
&\leq \\
&< \\
&< \\
&< \varepsilon
\end{aligned}
$$

□

Informally speaking, a subsequence (b_k) of a sequence (a_n) is a sequence all of whose terms occur in (a_n), in the correct order. More precisely:

Definition 1.10 (b_k) is a **subsequence** of (a_n) if there exists a strictly increasing sequence of positive integers (n_k) such that $b_k = a_{n_k}$.

For example $b_k = 1/2^k$ is a subsequence of $a_n = 1/n$: take $n_k = 2^k$. So is $c_k = 1/(k+50)$: take $n_k = k + 50$. However $d_k = k/(k+1)$ is not a subsequence of (a_n) because, for example, $b_2 = 2/3$ is not a term in (a_n). Neither is the sequence $(e_k) = (1, 1/2, 1, 1/3, 1, 1/4, \ldots)$. All the terms of (e_k) do appear in (a_n), but not in the correct order.

The key fact about subsequences is the following:

Proposition 1.11 *If $a_n \to L$ and (b_k) is a subsequence of (a_n), then $b_k \to L$.*

Proof: Exercise. □

So to prove that a sequence *diverges* it's enough to show that it has a subsequence that diverges, or to find two subsequences which converge to different limits (why? Look at Proposition 1.4).

Example 1.12 The sequence $x_n = (-1)^n$ diverges.

Proof: Assume, towards a contradiction, that $x_n \to L$. The even subsequence $y_n = x_{2n} = 1$ which converges to 1. But, by Proposition 1.11, $y_n \to L$. Limits are unique, by Proposition 1.4, so $L = 1$. But the odd subsequence $z_n = x_{2n-1} = -1$ converges to -1. Again, by Proposition 1.11, $z_n \to L$, and limits are unique, by Proposition 1.4, so $L = -1$. Hence $1 = -1$, a contradiction. □

So far, all the facts we've established about limits of sequences follow very directly from Definition 1.1. The following two results are rather deeper. To prove them, one needs to appeal to a fundamental defining property of the set of real numbers called *completeness*. We will return to this later in the module, when we study the theory of integration, but for the time being you should just take them on trust (or, if you studied MATH1026 last year, consult your notes from there). The first result is called the Montone Convergence Theorem, and to state it we need a preliminary definition:

Definition 1.13 A sequence (a_n) is **increasing** if $a_{n+1} \geq a_n$ for all $n \in \mathbb{Z}^+$. It is **decreasing** if $a_{n+1} \leq a_n$ for all $n \in \mathbb{Z}^+$. It is **monotonic** (or **monotone**) if it is increasing or decreasing.

So $(1/n)$ is decreasing, $(2n+1)$ is increasing, and hence both are monotonic. The constant sequence $(1, 1, 1, \ldots)$ is both increasing and decreasing , hence monotonic. The sequence $(-1, 1, -1, 1, -1, 1, \ldots)$ is neither increasing nor decreasing, so is not monotonic.

Theorem 1.14 (Montone Convergence Theorem) *If (a_n) is bounded and monotonic, then (a_n) converges.*

Finally we have the wonderfully named Bolzano-Weierstrass[1] Theorem:

Theorem 1.15 (Bolzano-Weierstrass Theorem) *Every bounded real sequence has a convergent subsequence.*

In fact, the Bolzano-Weierstrass Theorem follows quite easily from the Monontone Convergence theorem: we just show that *every* sequence has a monotonic subsequence. If the sequence is bounded, so is its monotonic subsequence, which much converge by Theorem 1.14.

1.2 Convergence of sequences and the Cauchy property

One slightly inconvenient thing about Definition 1.1 is that, to prove a sequence converges from first principles, we must first know what its limit is. In this section we will study a criterion which turns out to be *equivalent* to convergence, but which makes no mention of limits. This will turn out to be extremely useful later on, when we come to analyze the properties of functions defined by power series.

Definition 1.16 A real sequence (a_n) is **Cauchy** (or **has the Cauchy property**) if, for each $\varepsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that, for all $n, m \geq N$, $|a_n - a_m| < \varepsilon$.

Remark This looks very similar to the definition of convergence. It says that, given any $\varepsilon > 0$ (no matter how small), there is a point in the sequence beyond which *all terms lie closer than distance ε from one another*. Note that it makes no mention of any limit.

We will show that real sequences converge if and only if they have the Cauchy property. Once again, this is revision if you studied MATH1026, but is probably new if not.

Lemma 1.17 *If (a_n) converges then (a_n) is Cauchy.*

[1] Weiertstraß wenn du Deutsch bist.

Proof: Assume $a_n \to L$. Let $\varepsilon \in (0, \infty)$ be given. Then $\varepsilon/2 > 0$, so there exists $N \in \mathbb{Z}^+$ such that, for all $n \geq N$,

$$|a_n - L| < \frac{\varepsilon}{2}.$$

But then, for all $n, m \geq N$,

$$\begin{aligned} |a_n - a_m| &= \\ &\leq \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

by the Triangle Inequality. Hence (a_n) is Cauchy. □

Lemma 1.18 *If (a_n) is Cauchy then (a_n) is bounded.*

Proof: Since (a_n) is Cauchy, there exists $N \in \mathbb{Z}^+$ such that, for all $n, m \geq N$, $|a_n - a_m| < 1$. In particular, for all $n \geq N$, $|a_n - a_N| < 1$, so $|a_n| < |a_N| + 1$. Consider the finite set

$$A = \{|a_1|, |a_2|, \ldots, |a_{N-1}|, |a_N| + 1\}.$$

Since A is finite, it has a maximum element, K say. Then, for all $n < N$, $|a_n| \leq K$ (since each of these numbers appears in the set A) and for all $n \geq N$, $|a_n| < |a_N| + 1 \leq K$. Hence, for all $n \in \mathbb{Z}^+$, $|a_n| \leq K$. □

Remark Since every convergent sequence is Cauchy, this gives a sneaky proof of Proposition 1.6 (every convergent sequence is bounded).

Recall that if a sequence converges to a limit L, every subsequence of that sequence also converges to L (Proposition 1.11). In general, knowing that a single subsequence converges to L tells us nothing about convergence of the original sequence, however. For example $a_n = (1 + (-1)^n)n$ has the subsequence $b_k = a_{2k+1} = (1-1)(2k+1) = 0$ which converges to 0. The sequence (a_n), however, does not converge to 0: indeed, it isn't even bounded.

However, if we know that (a_n) is *Cauchy*, and it has a convergent subsequence, we *can* conclude that (a_n) converges:

Lemma 1.19 *Let (a_n) be Cauchy, and assume some subsequence of (a_n) converges to L. Then (a_n) converges to L.*

Proof: Let (a_{n_k}) be the subsequence which converges to L. Let $\varepsilon > 0$ be given. Since $a_{n_k} \to L$, there exists $N_1 \in \mathbb{Z}^+$ such that, for all $k \geq N_1$, $|a_{n_k} - L| < \varepsilon/2$. Further, since (a_n) is Cauchy, there exists $N_2 \in \mathbb{Z}^+$ such that, for all $n, m \geq N_2$,

$|a_n - a_m| < \varepsilon/2$. Let $N = \max\{N_1, N_2\}$. Then, for all $n \geq N$,

$$\begin{aligned}
|a_n - L| &= |(a_n - a_{n_N}) + (a_{n_N} - L)| \\
&\leq |a_n - a_{n_N}| + |a_{n_N} - L| && \text{(Triangle Inequality)} \\
&< \frac{\varepsilon}{2} + |a_{n_N} - L| && \text{(since } n \geq N_2\text{, and } n_N \geq N \geq N_2\text{)} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} && \text{(since } N \geq N_1\text{).}
\end{aligned}$$

Hence, $a_n \to L$. □

Our target theorem (a sequence converges if and only if it is Cauchy) now follows immediately from the Bolzano-Weierstrass Theorem (Theorem 1.15).

Theorem 1.20 *A real sequence converges if and only if it is Cauchy.*

Proof: If (a_n) converges, it is Cauchy, by Lemma 1.17. Conversely, if (a_n) is Cauchy, it is bounded (Lemma 1.18), so has a convergent subsequence (by the Bolzano-Weierstrass Theorem) and so converges (Lemma 1.19). □

Let's put this theory to some use.

Example 1.21 Let (a_n) be the sequence defined so that $a_1 = 1$, $a_2 = 2$, and for all $n \geq 3$,

$$a_n = \frac{1}{2}(a_{n-1} + a_{n-2}),$$

so each subsequent term is generated by averaging the previous two.
Claim: (a_n) converges.

Proof: It's not at all obvious what the limit of this sequence is, so a direct ε—N proof would be a tall order (how do we estimate $|a_n - L|$ when we don't know L?). A nice idea would be try to use the Monotone Convergence Theorem, that is, show that (a_n) is monotone (increasing or decreasing) and bounded. Unfortunately, this sequence isn't monotone, as you can check by computing the first few terms, so that strategy fails too. Instead, we'll show that the sequence is Cauchy.

We first show (by induction) that, for all $n \in \mathbb{Z}^+$,

$$|a_n - a_{n+1}| = \frac{1}{2^{n-1}}. \tag{1.1}$$

This holds for $n = 1$:

$$|a_1 - a_2| = |1 - 2| = 1 = \frac{1}{2^{1-1}}$$

Assume it holds for some $n = k \in \mathbb{Z}^+$. Then, by the definition of (a_n)

$$\begin{aligned}
|a_{k+1} - a_{k+2}| &= \\
&= \\
&=
\end{aligned}$$

by the induction hypothesis. Hence if (1.1) holds for $n = k$, it holds for $n = k + 1$. Hence, by induction, (1.1) holds for all $n \in \mathbb{Z}^+$.

We now show that (1.1) implies that (a_n) is Cauchy. Let $\varepsilon \in (0, \infty)$ be given. We must show that there exists $N \in \mathbb{Z}^+$ such that, for all $n, m \geq N$, $|a_n - a_m| < \varepsilon$. Note that this inequality holds trivially if $n = m$, and is symmetric under interchange of n and m, so it's enough to show that there exists $N \in \mathbb{Z}^+$ such that, for all $m > n \geq N$, $|a_n - a_m| < \varepsilon$. Since the sequence 2^n is unbounded above, there exists $N \in \mathbb{Z}^+$ such that, $2^N > 4/\varepsilon$. I claim that this N is the one we want. Assume $m > n$. Then,

$$
\begin{aligned}
|a_n - a_m| &= \\
&\leq \\
&= \\
&= \\
&< \frac{1}{2^{n-2}}
\end{aligned}
$$

since the geometric series $\sum_{n=0}^{\infty}(1/2)^n$ converges to 2 (if you've forgotten why, I'll remind you in a few weeks). Hence, for all $m > n \geq N$, with $2^N > 4/\varepsilon$,

$$|a_n - a_m| < \frac{1}{2^{n-2}} \leq \frac{1}{2^{N-2}} = \frac{4}{2^N} < \varepsilon.$$

Since (a_n) is Cauchy, it converges by Theorem 1.20. $\square$

Exercise Show that $a_n \to \frac{5}{3}$. (Hint: think about the subsequence (a_{2n-1}) of odd numbered terms.)

Remark We proved that (a_n) was Cauchy by exploiting the fact that the sequence of distances between consecutive terms, $|a_n - a_{n+1}|$, converges to 0 very fast (exponentially fast, in fact). It's tempting to think that the Cauchy property is equivalent to the property that $|a_n - a_{n+1}| \to 0$, but this is **false**!

Counterexample 1.22 (the Harmonic Series) Consider the sequence

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

This is the sequence of *partial sums* of the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$, which, as you proved in year 1, **diverges**. Hence, by Theorem 1.20 it is **not** Cauchy. Note that

$$|a_n - a_{n+1}| = \frac{1}{n+1} \to 0$$

however.

Summary

- A real sequence (a_n) **converges** to a **limit** L (written $a_n \to L$, or $\lim_{n\to\infty} a_n = L$) if, for each $\varepsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that, for all $n \geq N$, $|a_n - L| < \varepsilon$.
- We can use this definition to prove many basic facts about convergent sequences:
  - If a sequence converges, its limit is unique.
  - Convergent sequences are bounded.
  - Limits preserve non-strict inequalities.
  - The **Squeeze Rule**: If $a_n \leq b_n \leq c_n$, $a_n \to L$ and $c_n \to L$, then $b_n \to L$.
  - The **Algebra of Limits**: If $a_n \to A$ and $b_n \to B$ then $a_n + b_n \to A + B$, $a_n b_n \to AB$ and $a_n/b_n \to A/B$ (provided both sides make sense).
  - Any subsequence of a convergent sequence converges (to the same limit).
  - The **Monotone Convergence Theorem**: If a_n is bounded and monotonic (increasing or decreasing) then it converges.
  - The **Bolzano-Weierstrass Theorem**: Every bounded sequence has a convergent subsequence.
- A real sequence (a_n) is **Cauchy** if, for each $\varepsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that, for all $n, m \geq N$, $|a_n - a_m| < \varepsilon$.
- A real sequence converges if and only if it is Cauchy. This allows us to prove that some sequences converge even if we don't know their limit.