Workshop 1: solutions for week 2

1. (a) $\exists K \in \mathbb{R},\ \forall x \in D,\ x \geq K.$
   (b) $\forall K \in \mathbb{R},\ \exists x \in D,\ f(x) > K.$
   (c) $\forall y \in \mathbb{R},\ \exists x \in D,\ f(x) = y.$
   (d) $\exists y \in \mathbb{R},\ \forall x \in D,\ f(x) \neq y.$

2. I claim that $a_n \to 1$. Proof: let $\varepsilon > 0$ be given. Then there exists $N \in \mathbb{Z}^+$ such that $N > \sqrt{3/\varepsilon}$. Now, for all $n \geq N$,

$$|a_n - 1| \;=\; \left|\frac{2-(-1)^n}{n^2+2}\right| \leq \frac{3}{n^2+1} < \frac{3}{n^2} \leq \frac{3}{N^2} < \varepsilon.$$

   Hence $a_n \to 1$.

3. Let $\varepsilon \in (0,\infty)$. Since $a_n \to A$ and $\varepsilon/2 \in (0,\infty)$, there exists $N_1 \in \mathbb{Z}^+$ such that, for all $n \geq N_1$, $|a_n - A| < \varepsilon/2$. Similarly, since $b_n \to B$, and $\varepsilon/2 \in (0,\infty)$, there exists $N_2 \in \mathbb{Z}^+$ such that, for all $n \geq N_2$, $|b_n - B| < \varepsilon/2$. Let $N = \max\{N_1, N_2\}$. Then, for all $n \geq N$,

$$\begin{aligned} |(a_n+b_n)-(A+B)| &= |(a_n-A)+(b_n-B)| \\ &\leq |a_n-A|+|b_n-B| \qquad \text{(by the Triangle Inequality)} \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} \qquad \text{(since } n \geq N_1 \text{ and } n \geq N_2\text{)} \\ &= \varepsilon. \end{aligned}$$

   Hence $a_n + b_n \to A + B$.

4. (a) We first note that $0 < a_n \leq 1$ for all n (if this isn't clear, prove it by induction). By definition, $a_{n+1} = 1/(2+a_n^2)$ and $a_n = 1/(2+a_{n-1}^2)$, so

$$\begin{aligned} |a_{n+1}-a_n| &= \left|\frac{1}{2+a_n^2} - \frac{1}{2+a_{n-1}^2}\right| = \left|\frac{a_n^2-a_{n-1}^2}{(2+a_n^2)(2+a_{n-1}^2)}\right| \leq \left|\frac{a_n^2-a_{n-1}^2}{2\times 2}\right| \\ &= \frac{1}{4}|a_n+a_{n-1}||a_n-a_{n-1}| \leq \frac{1}{4}\times 2|a_n-a_{n-1}|. \end{aligned}$$

   (b) We first show that, for all $n \in \mathbb{Z}^+$, $|a_n - a_{n+1}| \leq \frac{1}{2^{n-1}}|a_1 - a_2|$. The claim clearly holds for $n = 1$. Assume it holds for $n = k$. Then, by part (a),

$$|a_{k+1}-a_{k+2}| \leq \frac{1}{2}|a_k - a_{k+1}| \leq \frac{1}{2}\frac{1}{2^{k-1}}|a_1-a_2| = \frac{1}{2^k}|a_1-a_2|,$$

   so the claim also holds for $n = k+1$. Hence, by induction, the claim holds for all n.

   Let $\varepsilon > 0$ be given. Then there exists $N \in \mathbb{Z}^+$ such that $2^N > 4|a_1 - a_2|/\varepsilon$.

Now, for all $m > n \geq N$,

$$\begin{aligned}
|a_n - a_m| &= |a_n - a_{n+1} + a_{n+1} - a_{n+2} + \cdots + a_{m-1} - a_m| \\
&\leq |a_n - a_{n+1}| + |a_{n+1} + a_{n+2}| + \cdots + |a_{m-1} - a_m| \\
&\leq \left(\frac{1}{2^{n-1}} + \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{m-1}} \right) |a_1 - a_2| \\
&= \frac{1}{2^{n-1}} \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{m-n-1}} \right) |a_1 - a_2| \\
&\leq \frac{1}{2^{n-1}} \times 2 \times |a_1 - a_2| \qquad (*) \\
&= \frac{4|a_1 - a_2|}{2^n} \leq \frac{4|a_1 - a_2|}{2^N} < \varepsilon
\end{aligned}$$

where we have used, to obtain line $(*)$, the fact that the geometric series $\sum_{k=0}^{\infty} (1/2)^k$ is increasing and converges to 2.