Chapter 3

Differentiable functions

3.1 The main definition

Now that we have a rigorous definition of the limit of a function, it is straightforward to define derivatives.

Definition 3.1 Let $f : D \to \mathbb{R}$, where D is some subset of $\mathbb{R}$, and $a \in D$ be a cluster point of D. Then f is **differentiable at** a if the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. In this case, we denote the limit $f'(a)$ and call it the **derivative of** f **at** a. We say that f is **differentiable** if it is differentiable at a for all $a \in D$.

Remarks

- In general, in order to define $\lim_{x \to a} g(x)$, we only need a to be a *cluster point* of the domain of g: it isn't necessary in general for a to be in the domain of g, so $g(a)$ may or may not exist. For example

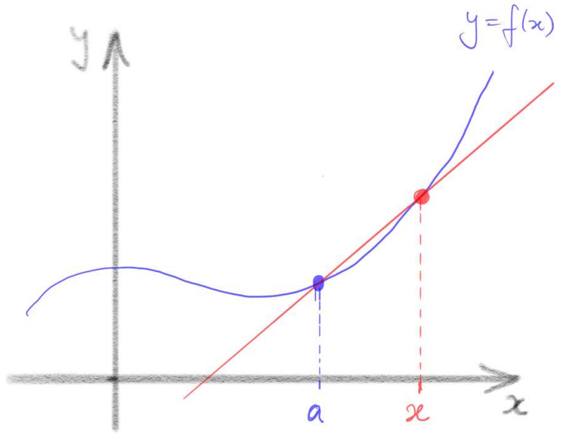
  $$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

  exists, although the function is undefined at $x = 1$. Note, however, that the limit defining $f'(a)$ contains the number $f(a)$, so to be differentiable at a, the point a must be both an *element* and a *cluster point* of the domain of f. For example, a function $f : (0, \infty) \to \mathbb{R}$ cannot be differentiable at 0 (since $0 \notin (0, \infty)$), and a function $f : \mathbb{Z} \to \mathbb{R}$ cannot be differentiable anywhere (since $\mathbb{Z}$ has no cluster points).

- The limit in Definition 3.1 is defined as in Definition 2.8. That is, $f : D \to \mathbb{R}$ is differentiable at a if there exists a real number, denoted $f'(a)$, such that, for each $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x \in D$ with $0 < |x - a| < \delta$,

  $$\left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| < \varepsilon.$$

- Geometrically, the quantity
$$s(x) = \frac{f(x) - f(a)}{x - a}$$
is the slope of the straight line passing through the points $(a, f(a))$ and $(x, f(x))$. Such a straight line is called a **chord** on the graph $y = f(x)$. As x approaches a, the points defining the chord get "arbitrarily close" to one another, and the chords approach a straight line through $(a, f(a))$ with slope $f'(a)$, the so-called **tangent line**. This is why the derivative $f'(a)$ is often interpreted as the slope of the graph $y = f(x)$ at the point $(a, f(a))$. Note, however, that this is just an *interpretation*. The *definition* of $f'(a)$ is in terms of limits which are, in turn, defined precisely in Definition 2.8.



- If $f : D \to \mathbb{R}$ is differentiable, then its derivative is also a function $f' : D \to \mathbb{R}$, mapping D to $\mathbb{R}$. There is a very popular alternative notation for the derivative in this case, namely
$$f'(x) = \frac{df}{dx}.$$
This notation is convenient in some circumstances, but it tends to blur the distinction between a function (in this case f') and the *value* of the function at a particular point (in this case $f'(x)$), so we will tend to avoid it.

Let's verify that some simple, familiar functions have the derivatives we expect. In each case, we will give a direct ε—δ proof that the limit
$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
exists and has the expected value.

Example 3.2 Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = c$, a constant. Then f is differentiable (everywhere) and $f'(a) = 0$ for all $a \in \mathbb{R}$.

Proof: We must show that, for each $\varepsilon > 0$ there is some $\delta > 0$ such that

$$\left| \frac{f(x) - f(a)}{x - a} - 0 \right| < \varepsilon$$

for all x satisfying $0 < |x - a| < \delta$. So, let $\varepsilon > 0$ be given. Let $\delta = 1 > 0$. Then for all $x \in \mathbb{R}$ with $0 < |x - a| < \delta$,

$$\left| \frac{f(x) - f(a)}{x - a} - 0 \right| = \left| \frac{c - c}{x - a} \right| = 0 < \varepsilon.$$

□

So, for a constant function, the same δ will work for every $\varepsilon > 0$. (We chose $\delta = 1$ but any other $\delta > 0$ works equally well.) Here's another example that's equally obliging:

Example 3.3 Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x$. Then f is differentiable (everywhere) and $f'(a) = 1$ for all $a \in \mathbb{R}$.

Proof: We must show that, for each $\varepsilon > 0$ there is some $\delta > 0$ such that

$$\left| \frac{f(x) - f(a)}{x - a} - 1 \right| < \varepsilon$$

for all x satisfying $0 < |x - a| < \delta$. So, let $\varepsilon > 0$ be given. Let $\delta = 1 > 0$. Then for all $x \in \mathbb{R}$ with $0 < |x - a| < \delta$,

$$\left| \frac{f(x) - f(a)}{x - a} - 1 \right| = \left| \frac{x - a}{x - a} - 1 \right| = 0 < \varepsilon.$$

□

The next examples are not quite so straightforward.

Example 3.4 Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = x^2$. Then f is differentiable and $f'(a) = 2a$ for all $a \in \mathbb{R}\backslash\{0\}$.

Proof: Choose and fix $a \in \mathbb{R}$. For any given $\varepsilon > 0$, let $\delta = \varepsilon$. Then for all $x \in \mathbb{R}$ with $0 < |x - a| < \delta$,

$$\left| \frac{f(x) - f(a)}{x - a} - 2a \right| = \left| \frac{x^2 - a^2}{x - a} - 2a \right| = |x + a - 2a| = |x - a| < \delta = \varepsilon.$$

□

Example 3.5 Let $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ such that $f(x) = 1/x$. Then f is differentiable and $f'(a) = -1/a^2$ for all $a \in \mathbb{R}\backslash\{0\}$.

Proof: Choose and fix $a \neq 0$. For any given $\varepsilon > 0$, let

$$\delta =$$

Then, for all $x \in \mathbb{R}\backslash\{0\}$ with $0 < |x - a| < \delta$,

$$\left| \frac{f(x) - f(a)}{x - a} + \frac{1}{a^2} \right| =$$

$$=$$

$$=$$

$$=$$

$$=$$

$$\leq$$

$$<$$

□

Exercise 3.6 Give a direct ε—δ proof that $f : [0, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$ is differentiable at every $a \in (0, \infty)$, and that

$$f'(a) = \frac{1}{2\sqrt{a}}.$$

OK, all is as we expected. What about functions which are *not* differentiable? How do we *prove* they aren't? A function $f : D \to \mathbb{R}$ is *not* differentiable at a precisely if the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

does not exist. We can prove this by thinking carefully about the negation of the statement defining a limit (Definition 2.8), but it's usually easier to exploit Theorem 2.14. Recall that this says that a function, g say, has limit L at a if and only if, for all sequences (x_n) in $D\backslash\{a\}$ converging to a, $g(x_n)$ converges to L. So to prove that $\lim_{x \to a} g(x)$ doesn't exist, it's enough to find just one sequence (x_n) in $D\backslash\{a\}$ converging to a whose image sequence $(g(x_n))$ does not converge.

Example 3.7 Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = |x|$. Then f is not differentiable at 0.

Proof: Assume, towards a contradiction, that f is differentiable at 0 with derivative $f'(0)$. Then

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=f'(0).$$

Let $x_n=(-1)^n/n$. Note that this is a sequence in $\mathbb{R}\backslash\{0\}$ converging to 0. Hence, by Theorem 2.14,

$$\frac{f(x_n)-f(0)}{x_n-0}\to f'(0).$$

But

$$\frac{f(x_n)-f(0)}{x_n-0}=\frac{|(-1)^n/n|}{(-1)^n/n}=\frac{1}{(-1)^n}=(-1)^n$$

which does not converge, a contradiction. Hence f is *not* differentiable at 0. □

Exercise 3.6 revisited Use a sequential argument to prove that $f:[0,\infty)\to\mathbb{R}$, $f(x)=\sqrt{x}$ is **not** differentiable at 0.

There is a link between differentiablity and continuity, as we now show.

Proposition 3.8 *Let $f:D\to\mathbb{R}$ be differentiable at $a\in D$. Then f is continuous at a.*

Proof: By Theorem 2.21, it suffices to show that $\lim_{x\to a}f(x)=f(a)$. Define

$$s:D\backslash\{a\}\to\mathbb{R},\qquad s(x)=\frac{f(x)-f(a)}{x-a}.$$

Then, by assumption, s has a limit (denoted $f'(a)$) at a. But for all $x\in D\backslash\{a\}$,

$$f(x)=f(a)+(x-a)s(x),$$

and hence, by the Algebra of Limits (Theorem 2.13),

$$\lim_{x\to a}f(x)=f(a)+0\times f'(a)=f(a).$$

□

Remark. The converse of Proposition 3.8 is false: if f is continuous at a, a cluster point of its domain, it does not follow that f is differentiable there. We've already seen a counteraxample: $f(x)=|x|$ is continuous at 0 but is not differentiable at 0 (see Example 3.7). In this case, the function fails to be differentiable only at a single isolated point. It is straightforward to construct functions which are *differentiable* only at a single isolated point.

Example 3.9 Let $f:\mathbb{R}\to\mathbb{R}$ be defined such that

$$f(x)=\begin{cases} x & \text{if } x\in\mathbb{Q},\\ x-x^2 & \text{if } x\notin\mathbb{Q}.\end{cases}$$

I claim that f is differentiable at 0, with $f'(0)=1$, but is not differentiable anywhere else.

Proof: Let $a \in \mathbb{R}$ and assume that f is differentiable at a. Then, by Proposition 3.8, f is continuous at a. For each $n \in \mathbb{Z}^+$ there exist $r_n \in \mathbb{Q}$ and $i_n \in \mathbb{R}\backslash\mathbb{Q}$ such that $a < r_n < a + 1/n$ and $a < i_n < a + 1/n$. Clearly $r_n \to a$ and $i_n \to a$ by the Squeeze Rule. Hence, by the definition of continuity, $f(r_n) \to f(a)$ and $f(i_n) \to f(a)$. But r_n is rational, so $f(r_n) = r_n \to a$, and i_n is irrational, so $f(i_n) = i_n - i_n^2 \to a - a^2$. Both these limits equal $f(a)$, so $a = a - a^2$, whence $a = 0$. So *if* f is differentiable at a, then $a = 0$. Equivalently, f is *not* differentiable at any $a \neq 0$.

It remains to show that f is differentiable at 0 with $f'(0) = 1$, that is,

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 1.$$

In this case, it's easier to use Theorem 2.14 than to give a direct ε—δ argument. So, let (x_n) be any sequence in $\mathbb{R}\backslash\{0\}$ converging to 0, and consider

$$s_n = \frac{f(x_n) - f(0)}{x_n - 0} = \frac{f(x_n)}{x_n}.$$

We must show that $s_n \to 1$. If x_n is rational, then $|s_n - 1| = |x_n/x_n - 1| = 0$, whereas if x_n is irrational then

$$|s_n - 1| = \left| \frac{x_n - x_n^2}{x_n} - 1 \right| = |x_n|.$$

Hence, for all n, $0 \leq |s_n - 1| \leq |x_n|$, so $|s_n - 1| \to 0$ by the Squeeze Rule. Hence $s_n \to 1$. It follows that $f'(0) = 1$, as claimed. □

So this (admittedly rather bizarre) function is differentiable at 0, and has a positive derivative there. Naive intuition would suggest, therefore, that the function should be increasing, at least for values of x sufficiently close to 0. In fact, this is **false!**

Example 3.9 continued Claim: the function f is not increasing on any neighbourhood of 0. That is, there does not exist $\varepsilon > 0$ such that f is increasing on $(-\varepsilon, \varepsilon)$.
Proof: Let $\varepsilon > 0$ be given. Then there exists an irrational number x such that $0 < x < \varepsilon$. By definition, $f(x) = x - x^2 < x$. Similarly, there exists a *rational* number y such that $x - x^2 < y < x$. But then $y < x$ and $f(y) = y > x - x^2 = f(x)$. Hence, f is not increasing on the interval $(-\varepsilon, \varepsilon)$. This is true no matter which positive number ε we choose, which establishes the claim. □

Your reaction to this may well be "so what, that's a really crazy function – after all, it's only differentiable at the single point 0." We will see later an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is differentiable *everywhere* and has $f'(0) = 1$, but still is not increasing on any neighbourhood of 0.

3.2 The rules of differentiation

In principle, if we are given an explicit function, like

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = (x^3 + 1)^2,$$

we can show it is differentiable and compute its derivative by directly applying Definition 3.1. This quickly becomes complicated and tedious, however. So we next develop some "derivative theorems" which will allow us to reduce differentiation of many functions to an algorithmic process. You are (I would hope) already familiar with these and quite fluent in applying them. The point of this section is to show that, now we have a mathematically precise definition of the derivative, we can rigorously *prove* them, thus putting differential calculus on a solid foundation.

Proposition 3.10 (Linearity) *Let $f : D \to \mathbb{R}$, $g : D \to \mathbb{R}$ be differentiable at $a \in D$ with derivatives $f'(a)$ and $g'(a)$, respectively, and c be a real constant. Then*

(i) $cf : D \to \mathbb{R}$ is differentiable at a with derivative $cf'(a)$.

(ii) $f + g : D \to \mathbb{R}$ is differentiable at a with derivative $f'(a) + g'(a)$.

Proof: These both follow immediately from the Algebra of Limits (Theorem 2.13):

$$\begin{aligned}
\lim_{x\to a} \frac{cf(x) - cf(a)}{x - a} &= \lim_{x\to a} c\left(\frac{f(x) - f(a)}{x - a}\right) = cf'(a) \\
\lim_{x\to a} \frac{f(x) + g(x) - (f(a) + g(a))}{x - a} &= \lim_{x\to a} \left(\frac{f(x) - f(a)}{x - a} + \frac{g(x) - g(a)}{x - a}\right) = f'(a) + g'(a)
\end{aligned}$$

□

Remark Actually the first part of this proof assumes that $\lim_{x\to a} c = c$. Have we actually proved this? Not directly. (You should compose a direct ε–δ proof in your head now. For a given $\varepsilon > 0$, how small must we take δ?) We *have* proved it indirectly however: constant functions are differentiable (Example 3.2), hence continuous (Proposition 3.8), so equal their limit at each point (Theorem 2.21).
Proposition 3.10 followed easily from the Algebra of Limits. Our next differentiation rule, and its proof, are considerably more subtle. Recall that the *composition* of two functions $f : A \to B$ and $g : B \to C$, denoted $g \circ f : A \to C$ is the function which first "does" f, then feeds the result into g, that is

$$(g \circ f)(x) = g(f(x)).$$

This only makes sense if the range of f is a subset of the domain of g. We've already proved that the composition of two continuous functions is continuous (Theorem 2.23). There's a very famous rule which tells us how to compute the derivative of a composition of two functions, if we know the derivatives of its component pieces.

Theorem 3.11 (Chain Rule) *Let $f : D \to E$ be differentiable at $a \in D$ and $g : E \to \mathbb{R}$ be differentiable at $f(a) \in E$. Then $g \circ f : D \to \mathbb{R}$ is differentiable at a, and*

$$(g \circ f)'(a) = g'(f(a))f'(a).$$

To prove this, we will need to slightly alter the way we think of the derivative. The idea is that, if a function $g : \mathbb{R}\backslash\{a\} \to \mathbb{R}$ has limit L at a, we can extend the function to the domain $\mathbb{R}$ by defining $g(a) = L$, and this extended function is continuous at

a by Theorem 2.21. Conversely, if the function $g : \mathbb{R}\backslash\{a\} \to \mathbb{R}$ has a continuous extension to $\mathbb{R}$, it has a limit at a and that limit is $g(a)$. So we can reinterpret "g has a limit at a" as saying that "g has an extension which is continuous at a." Applying this idea to the difference quotient of a function f,

$$g(x) := \frac{f(x) - f(a)}{x - a},$$

we obtain Carathéodory's criterion for differentiablity:

Proposition 3.12 (Carathéodory's Criterion) *Let $f : D \to \mathbb{R}$ and $a \in D$ be a cluster point of D. Then f is differentiable at $a \in D$ if and only if there exists a function $\phi : D \to \mathbb{R}$ that is continuous at a and satisfies*

$$f(x) - f(a) = \phi(x)(x - a) \qquad (\clubsuit)$$

for all $x \in D$. In this case, $f'(a) = \phi(a)$.

Proof: ($\Rightarrow$): Assume f is differentiable at a with derivative $f'(a)$. Define the function

$$\phi : D \to \mathbb{R}, \qquad \phi(x) = \begin{cases} \frac{f(x)-f(a)}{x-a} & \text{if } x \neq a \\ f'(a) & \text{if } x = a. \end{cases}$$

Clearly ϕ satisifes ($\clubsuit$) for all $x \in D\backslash\{a\}$, and ($\clubsuit$) holds automatically at $x = a$ since both sides equal 0. Furthermore,

$$\lim_{x\to a} \phi(x) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = f'(a) = \phi(a),$$

so ϕ is continuous at a by Theorem 2.21.
($\Leftarrow$): Assume that a function continuous at a satisfying ($\clubsuit$) exists. Then, dividing ($\clubsuit$) by $(x - a)$,

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} \phi(x) = \phi(a)$$

by Theorem 2.21, since ϕ is continuous at a. Hence, f is differentiable at a, with derivative $\phi(a)$. □

Proof of the Chain Rule: By Carathéodory's Criterion, there exist functions $\phi : D \to \mathbb{R}$ and $\psi : E \to \mathbb{R}$ such that ϕ is continuous at a, ψ is continuous at $f(a)$ and, for all $x \in D$, and $y \in E$,

$$\phi(x)(x - a) = f(x) - f(a), \qquad \psi(y)(y - f(a)) = g(y) - g(f(a)).$$

Define the function

$$\Phi : D \to \mathbb{R}, \qquad \Phi(x) = \psi(f(x))\phi(x).$$

By Theorem 2.23 (and the Algebra of Limits), Φ is continuous at a. Furthermore, for all $x \in D$,

$$\begin{aligned} \Phi(x)(x - a) &= \psi(f(x))\phi(x)(x - a) \\ &= \psi(f(x))(f(x) - f(a)) \\ &= g(f(x)) - g(f(a)). \end{aligned}$$

Hence, by Carathéodory's Criterion, $g \circ f$ is differentiable at a, and its derivative is

$$\Phi(a) = \psi(f(a))\phi(a) = g'(f(a))f'(a).$$

$\square$

Example 3.13 Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = (x^2 + 1)^2$. Then f is differentiable and $f'(x) = 4(x^2 + 1)x$.

Proof: $f = h \circ g$ where $h, g : \mathbb{R} \to \mathbb{R}$

$$g(x) =$$

$$h(x) =$$

Now, by Examples 3.4 and 3.2 and the Linearity Property (Proposition 3.10) g, h are differentiable, with derivatives

$$g'(x) =$$

$$h'(x) =$$

Hence, by the Chain Rule, f is differentiable, and

$f'(x) =$

$\square$

Remark You are probably more familiar with the Chain Rule expressed something like the following:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}. \qquad (*)$$

Certainly, this has a look of plausibility about it (don't the "du's" just cancel?) and is arguably more memorable than

$$(g \circ f)'(a) = g'(f(a))f'(a). \qquad (\dagger)$$

Nonetheless, it is vastly inferior. Ask yourself what, exactly, does $(*)$ mean? The derivative of a function y (where? At x?) equals the derivative (where? At u?) of the same (?) function – but with respect to a different variable, whatever that means — times the derivative of some other function u (but isn't u a variable? Can we use the same symbol to denote both a function and a variable?) evaluated somewhere (x? Maybe?). Note that "the function y thought of as a function of u instead of x," where "u is a function of x," is actually a *different* function! And to interpret $(*)$ we're forced to talk about how we *think* about the function y, instead of what it actually *is*. This is the hallmark of imprecise, badly formulated mathematics.

By contrast $(\dagger)$ is a completely unambiguous, precise statement standing on its own. We don't need to explain separately "how we're thinking of" the terms it

contains in order to make sense of it. Be warned: if ever I ask for a statement of the Chain Rule, it is Theorem 3.11 I want, not wooly, ambiguous stuff in the style of (∗).
□

Example 3.14 Let $g : \mathbb{R} \to \mathbb{R}$ be $g(x) = f(\sqrt{2}f(\sqrt{2}x))$, where $f : \mathbb{R} \to \mathbb{R}$ is the bizarre function defined in Example 3.9. Prove that g is differentiable at 0 and compute $g'(0)$.

Proof: Recall that f is differentiable at 0, and $f'(0) = 1$. Now, let $h : \mathbb{R} \to \mathbb{R}$, $h(x) = \sqrt{2}x$. This is differentiable (everywhere, by Linearity) and $h'(x) = \sqrt{2}$. Further, let $k : \mathbb{R} \to \mathbb{R}$, $k = f \circ h$. Then f is differentiable at $h(0) = 0$, and h is differentiable at 0, so, by the Chain Rule, k is differentiable at 0 and

$$k'(0) =$$

Now note that $g = k \circ k$:

$$\begin{aligned} k \circ k(x) &= \\ &= \end{aligned}$$

k is differentiable at $k(0) = 0$, so, by the Chain Rule, g is differentiable at 0 and

$$g'(0) =$$

□

Remark Note that we managed to compute $g'(0)$ without ever writing down a formula for $g(x)$! I invite you to calculate a formula for $g(x)$; it's surprisingly difficult.

You may have expected us to prove the Product Rule before the Chain Rule: it is, after all, considerably simpler. The reason we proved the Chain Rule first is that we can deduce the Product Rule from it.

Proposition 3.15 (Product Rule) *Let $f : D \to \mathbb{R}$, $g : D \to \mathbb{R}$ be differentiable at $a \in D$ with derivatives $f'(a)$ and $g'(a)$, respectively. Then $fg : D \to \mathbb{R}$ is differentiable at a with derivative $f'(a)g(a) + f(a)g'(a)$.*

Proof: Let $s : \mathbb{R} \to \mathbb{R}$ such that $s(x) = x^2$ and recall that this is differentiable with derivative $s'(x) = 2x$ by Example 3.4. Hence, by the Chain Rule and the Linearity Property, $s \circ f$, $s \circ g$ and $s \circ (f + g)$ are all differentiable at a, with derivatives

$$\begin{aligned} (s \circ f)'(a) &= 2f(a)f'(a) \\ (s \circ g)'(a) &= 2g(a)g'(a) \\ (s \circ (f + g))'(a) &= 2(f(a) + g(a))(f'(a) + g'(a)). \end{aligned}$$

Hence, by Proposition 3.10,

$$fg = \frac{1}{2}\left\{(f+g)^2 - f^2 - g^2\right\} = \frac{1}{2}\{s\circ(f+g) - s\circ f - s\circ g\}$$

is differentiable at a with derivative

$$\begin{aligned}(fg)'(a) &= \frac{1}{2}\{2(f(a)+g(a))(f'(a)+g'(a)) - 2f(a)f'(a) - 2g(a)g'(a)\}\\ &= f'(a)g(a) + f(a)g'(a).\end{aligned}$$

□

Exercise 3.16 Give an alternative proof of the Product Rule using the Algebra of Limits for functions (Theorem 2.13).

Having established the Linearity Property and the Product Rule, it's not hard to prove that all polynomial functions are differentiable, with the derivative we expect.

Proposition 3.17 *Every polynomial function $p:\mathbb{R}\to\mathbb{R}$, $p(x) = a_0 + a_1x + \cdots + a_mx^m$ is differentiable, and its derivative is another polynomial function $p':\mathbb{R}\to\mathbb{R}$, namely*

$$p'(x) = a_1 + 2a_2x + \cdots + ma_mx^{m-1}.$$

Proof: Exercise. Try proof by induction (on the degree of p). □

Remark Proposition 3.17 retrospectively justifies our assertion that all polynomials are continuous: since they're differentiable, they're certainly continuous (by Proposition 3.8). □

Example 3.18 Let m be a positive integer and $f:\mathbb{R}\backslash\{0\}\to\mathbb{R}$ be the function $f(x) = 1/x^m$. Then f is differentiable, and $f'(x) = -m/x^{m+1}$.

Proof: $f = h\circ g$ where $g:\mathbb{R}\backslash\{0\}\to\mathbb{R}$, $g(x) = 1/x$ and $h:\mathbb{R}\to\mathbb{R}$, $h(x) = x^m$. By Examples 3.5 and 3.17, both g and h are differentiable with derivatives

$$\begin{aligned}g'(x) &= \\ h'(x) &= \end{aligned}$$

Hence, by the Chain Rule, f is differentiable, and

$$\begin{aligned}f'(x) &= \\ &= \\ &= \\ &= \end{aligned}$$

by Example 3.5 and Proposition 3.17. □

Generalizing the trick used in this proof, we can obtain another useful rule of differentiation:

Proposition 3.19 (Quotient Rule) *Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}\backslash\{0\}$, be differentiable at $a \in D$. Then f/g is differentiable at a and*

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.$$

Proof: Exercise. □

3.3 Open sets and the Localization Lemma

We will often have to deal with functions $f : \mathbb{R} \to \mathbb{R}$ that are defined *piecewise*, by combining different formulae for $f(x)$ valid on different subsets of the domain. A simple example is

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = |x| = \begin{cases} x, & x \in [0, \infty), \\ -x, & x \in (-\infty, 0). \end{cases}$$

It would be handy if we could deduce information about the differentiablity of such functions directly from their constituent pieces. Luckily, we can, because limits, and hence derivatives, are inherently *local* objects. That is, if two functions f, g coincide for all x "close to" a point a, they have the same derivative at a. To formulate this statement precisely (and hence be able to prove it) we need to introduce the concept of *open subsets* of $\mathbb{R}$.

Definition 3.20 $U \subseteq \mathbb{R}$ is **open** if, for each $a \in U$ there exists $\delta > 0$ such that $(a - \delta, a + \delta) \subseteq U$.

Example 3.21 (i) $\mathbb{R}$ itself is open: given any $a \in \mathbb{R}$, $(a - 52, a + 52) \subseteq \mathbb{R}$, for example.

(ii) $(0, 1)$ is open: given any $a \in (0, 1)$ we can take $\delta = \min\{|a|, |1 - a|\} > 0$. Then $(a - \delta, a + \delta) \subseteq (0, 1)$.

(iii) $[0, 1)$ is not open: $0 \in [0, 1)$ but, for all $\delta > 0$, $(-\delta, \delta) \not\subseteq [0, 1)$.

(iv) $\emptyset$ is open (careful!).

Example 3.22 Determine whether the following sets are open.

$$\mathbb{Q}, \qquad \mathbb{R}\backslash\mathbb{Q}, \qquad \mathbb{R}\backslash\mathbb{Z}, \qquad \mathbb{R}\backslash\{1/n \,:\, n \in \mathbb{Z}^+\}.$$

- $\mathbb{Q}$

- $\mathbb{R}\backslash\mathbb{Q}$

- $\mathbb{R}\backslash\mathbb{Z}$

- $\mathbb{R}\backslash\{1/n : n \in \mathbb{Z}^+\}$

We can now give a precise formulation of the idea that derivatives are local.

Lemma 3.23 (Localization Lemma) *Let $f : D \to \mathbb{R}$ coincide with a differentiable function $g : U \to \mathbb{R}$ on some open set $U \subseteq D$. Then f is differentiable, with derivative g', on U.*

Proof: Let $a \in U$. We must show that

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = g'(a).$$

Let $\varepsilon > 0$ be given. Since U is open, there exists $\delta_1 > 0$ such that $(a - \delta_1, a + \delta_1) \subseteq U \subseteq D$. Since g is differentiable at a, there exists $\delta_2 > 0$ such that for all $x \in U$ with $0 < |x - a| < \delta_2$,

$$\left|\frac{g(x) - g(a)}{x - a} - g'(a)\right| < \varepsilon.$$

Let $\delta = \min\{\delta_1, \delta_2\} > 0$. Then for all $x \in D$ with $0 < |x - a| < \delta$, $x \in U$ so $f(x) = g(x)$ (and $f(a) = g(a)$), and $0 < |x - a| < \delta_2$, so

$$\left|\frac{f(x) - f(a)}{x - a} - g'(a)\right| = \left|\frac{g(x) - g(a)}{x - a} - g'(a)\right| < \varepsilon.$$

$\square$

Example 3.24 The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$ coincides with the differentiable function $g(x) = x$ on the open set $(0, \infty)$. Hence, by the Localization Lemma, f is differentiable on $(0, \infty)$ with derivative $f'(x) = 1$. Similarly, f coincides with the differentiable function $h(x) = -x$ on the open set $(-\infty, 0)$. Hence, by the Localization Lemma, f is differentiable on $(-\infty, 0)$ with derivative $f'(x) = -1$.

Warning! The condition that U is an *open* set is crucial for the Localization Lemma. For example, $f(x) = |x|$ coincides with the differentiable function $g(x) = x$ on $U = [0, \infty)$, but f is *not* differentiable on U (it fails to be differentiable at 0).

Summary

- A function $f : D \to \mathbb{R}$ is **differentiable at** $a \in D$ if
$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
exists. In this case, we denote the limit $f'(a)$ and call it the **derivative** of f at a. We say that f is **differentiable** if it is differentiable at a for all $a \in D$.
- The limit in $f'(a)$ is defined as in Definition 2.8.
- If a function is differentiable at a, it is continuous at a. The converse is false.
- Using this definition of $f'(a)$, we can prove that derivatives obey the usual rules of differential calculus, to wit:
$$\begin{array}{lrcl} \text{Linearity:} & (\alpha f + \beta g)'(a) & = & \alpha f'(a) + \beta g'(a), \\ \text{Product Rule:} & (fg)'(a) & = & f'(a)g(a) + f(a)g'(a), \\ \text{Quotient Rule:} & (f/g)'(a) & = & \dfrac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}, \\ \text{Chain Rule:} & (g \circ f)'(a) & = & g'(f(a))f'(a). \end{array}$$
- $U \subseteq \mathbb{R}$ is **open** if for each $a \in U$ there exists $\delta > 0$ such that $(a - \delta, a + \delta) \subseteq U$.
- Derivatives are **local**: if $f : D \to \mathbb{R}$ coincides with a differentiable function $g : U \to \mathbb{R}$ on some open set $U \subseteq D$, then f is differentiable, with derivative g', on U.