Workshop solutions for week 9

1. (a) f is positive and decreasing, so for all $x \in [0,1]$, $0 < f(x) \le f(0) = 1$. Hence

$$\|f\| = \sup\{|f(x)| : x \ge 0\} = 1.$$

   (b) Similarly, g is non-negative and increasing, so for all $x \in [0,1]$, $0 \le g(x) \le g(1) = 2$. Hence

$$\|g\| = \sup\{|g(x)| : x \ge 0\} = g(1) = 2.$$

   (c) Let $h(x) = f(x) - g(x) = (1+x)^{-1} - 2x$. Since f is decreasing and g is increasing, $h = f - g$ is decreasing. Hence, for all $x \in [0,1]$,

$$h(1) = -\frac{3}{2} \le h(x) \le h(0) = 1,$$

   so $|h(x)| \le 3/2$, and $|h(1)| = 3/2$. Hence $\|h\| = \sup\{|h(x)| : x \in [0,1]\} = 3/2$.

2. (a) $f_n(1) = n$ diverges, so (f_n) does not converge pointwise.

   (b) I claim that (f_n) converges uniformly to 0 (and hence converges pointwise):

$$\|f_n - 0\| = \sup\{|f_n(x)| : x \ge 2\} = \sup\{\frac{n}{x^n} : x \ge 2\} = \frac{n}{2^n}$$

   Consider the series whose nth term is $a_n = n/2^n$. Since

$$\frac{a_{n+1}}{a_n} = \left(1 + \frac{1}{n}\right)\frac{1}{2} \to \frac{1}{2} < 1,$$

   $\sum_{n=1}^{\infty} a_n$ converges, by the Ratio Test. Hence $a_n \to 0$ by the Divergence Test. So $\|f_n - 0\| \to 0$, that is, (f_n) converegs uniformly to 0.

   (c) I claim that (f_n) converges pointwise, but not uniformly, to 0. For all $x > 1$, $f_n(x) = \frac{n}{x^n} \to 0$ by a re-run of the argument just advanced:

$$\frac{f_{n+1}(x)}{f_n(x)} = \left(1 + \frac{1}{n}\right)\frac{1}{x} \to \frac{1}{x} < 1$$

   so $\sum_{n=1}^{\infty} f_n(x)$ converges by the Ratio Test, and hence $f_n(x) \to 0$ by the Divergence Test. This holds for all $x \in (1,\infty)$, so f_n converges to 0 pointwise. However

$$\|f_n - 0\| = \sup\{n/x^n : x > 1\} = n.$$

   To see this, note that n is an upper bound on the set, but given any $K < n$, $K = n/\alpha$ with $\alpha > 1$, and there exists $x \in (1,\infty)$ such that $x^n > \alpha$ (e.g. $x = \alpha$ will do). So K is not an upper bound on the set.

   Since $\|f_n - 0\|$ diverges, (f_n) does not converge uniformly to 0 (and since it converges pointwise to 0, it can't converge pointwise to any other function).

3. $f_n(x) = \left\{ \begin{array}{cc} \min\{n, 1/x\} & x \in (0,1] \\ 0 & x = 0 \end{array} \right\}$ will do. To see this, note that, $0 \leq f_n(x) \leq n$ for all n, so each f_n is bounded. For any fixed $x \in (0,1]$, $f_n(x) = 1/x$ for all $n > 1/x$, so the sequence $f_n(x) \to 1/x$. Also, $f_n(0) = 0 \to 0$. So (f_n) converges pointwise to the unbounded function

$$f(x) = \left\{ \begin{array}{cc} 1/x & x \in (0,1] \\ 0 & x = 0. \end{array} \right.$$