

Geometry of vortices on the sphere in the dissolving limit

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Rene →



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- Hermitian line bundle L over $\Sigma = (S^2, g_\Sigma)$, degree n

$$E(\phi, A) = \frac{1}{2} \|d_A \phi\|_{L^2}^2 + \frac{1}{2} \|F_A\|_{L^2}^2 + \frac{1}{8} \|\tau - |\phi|^2\|_{L^2}^2$$

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$$\bar{\partial}_A \phi = 0 \quad (V1)$$

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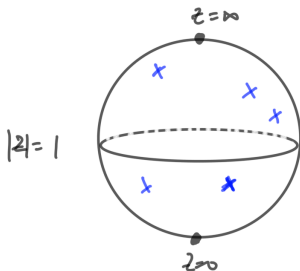
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- ▶ $\varepsilon > 0$: $[(\phi, A)]$ uniquely determined by **divisor** (ϕ)



$$\Leftrightarrow p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$$

$$\Leftrightarrow [a_0, a_1, \dots, a_n]$$

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- ▶ Baptista-Manton conjecture: $\lim_{\varepsilon \rightarrow 0} g_\varepsilon = \text{Fubini-Study metric}$

The conjecture

- Equip L with hol struc $\bar{\partial}_L = \bar{\partial}_{\hat{A}}$

$$H^0(L) = \{\phi \in \Gamma(L) : \bar{\partial}_{\hat{A}}\phi = 0\} \equiv \mathbb{C}^{n+1}$$

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- ▶ $S = \{\hat{\phi} \in H^0(L) : \|\hat{\phi}\|_{L^2} = 1\}$, unit sphere

$$\pi : S \rightarrow (\mathbb{P}(H^0(L)), g_{FS}), \quad \hat{\phi} \mapsto \{c\hat{\phi}\}$$

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- ▶ “The” Fubini-Study metric: $g_0 := f^*g_{FS}$
- ▶ Baptista-Manton conjecture: $\lim_{\varepsilon \rightarrow 0} g_\varepsilon = g_0$
- ▶ Surprising? Massive gain in symmetry

The theorem

Theorem (JMS, RGL 2022) In the limit $\varepsilon \searrow 0$, g_ε converges in C^0 topology to g_0 .

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More precisely:

There exists $C > 0$ such that, for all $v \in TM_n$ and all $\varepsilon \in (0, 1)$

$$|g_\varepsilon(v, v) - g_0(v, v)| \leq C\varepsilon g_0(v, v)$$

The proof

- ▶ Given divisor D , exists $\hat{\phi} \in S \subset H^0(L)$ with $(\hat{\phi}) = D$
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$$(\phi, A) = (\sqrt{\varepsilon} \widehat{\phi} e^{u/2}, \widehat{A} - \frac{1}{2} * du)$$

for some smooth $u : \Sigma \rightarrow \mathbb{R}$.

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- ▶ Energy estimate, elliptic estimate, Sobolev \Rightarrow

$$\|u\|_{C^0} \leq C\varepsilon.$$

Vortices are uniformly well approximated by pseudovortices
(for small ε)

The proof (cont)

- Take a curve of vortex solutions

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$$\Delta \dot{u} + \varepsilon |\hat{\phi}|^2 e^u \dot{u} = -2\varepsilon e^u h(\hat{\phi}, \dot{\hat{\phi}})$$

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- ▶ Lax-Milgram Lemma \Rightarrow estimate $\|\dot{u}\|_{H^1} \leq C\varepsilon \|\dot{\hat{\phi}}\|_{L^2}$

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- ▶ Lax-Milgram Lemma \Rightarrow estimate $\|\dot{u}\|_{H^1} \leq C\varepsilon \|\dot{\hat{\phi}}\|_{L^2}$
- ▶ Just enough to get bound on $|g_\varepsilon - g_0|$.

A corollary

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$$0 = \lambda_0(g) < \lambda_1(g) \leq \lambda_2(g) \leq \lambda_3(g) \leq \dots$$

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Then

$$\lambda_k(g) = \inf \{ \Lambda_g(V) : \dim V = k + 1 \}$$

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- **Corollary** (JMS,RGL): There exists $C > 0$ such that, for all $k \in \mathbb{Z}^+$

$$\left| \frac{\lambda_k(g_\varepsilon)}{\lambda_k(g_0)} - 1 \right| \leq C\varepsilon$$

- Spectrum of M_n converges uniformly to spectrum of FS

Open questions

- ▶ Convergence of geodesics? Need $g_\varepsilon \rightarrow g_0$ in C^1
- ▶ Convergence of curvature? Need $g_\varepsilon \rightarrow g_0$ in C^2
- ▶ n -dependence of the bounds?
- ▶ Leading correction to g_0 ?
- ▶ Higher genus? Much more subtle (Manton, Romao)