Workshop solutions for week 8

1. By FTC1 (Theorem 6.1) $g'(\pi) = f(\pi) = -\pi^2$. You didn't compute the integral, did you?

2. (a) Following the hint, let $g(x) = f'(x)^2 + 2\cos f(x)$. Then, by the Chain Rule, g is differentiable and

$$g'(x) = 2f'(x)f''(x) - 2\sin f(x)f'(x) = 2f'(x)(f''(x) - \sin f(x)) = 0$$

by the equation defining f. Hence, by Proposition 4.11, g is constant, that is, for all $x \in \mathbb{R}$,

$$g(x) = f'(x)^2 + 2\cos f(x) = g(0) = 1 + 2\cos 0 = 3.$$

Hence

$$f'(x)^2 = 3 - 2\cos f(x) \geq 1,$$

and, in particular f' is never 0. Now f' is continuous (since it is differentiable) and $f'(0) > 0$, so it follows that $f'(x) > 0$ for all x by the Intermediate Value Theorem. Hence f is strictly increasing by Proposition 4.11.

(b) First note that, since f is strictly increasing, it is injective, so if b exists, it is unique. By the argument above, we know that $f'(x) \geq 1$ for all x. Hence, by the Mean Value Theorem, for all $x > 0$ there exists $c \in (0, x)$ such that

$$f(x) - f(0) = f'(c)(x - 0) \geq x.$$

It follows that $f(\pi) \geq \pi$. Now $f(0) = 0 < \pi$ and f is continuous on $[0, \pi]$, so by the Intermediate Value Theorem, there exists $b \in [0, \pi]$ such that $f(b) = \pi$.

(c) By above work $f'(x)^2 = 3 - 2\cos f(x)$ and $f'(x) > 0$, so

$$f'(x) = \sqrt{3 - 2\cos f(x)}.$$

Hence, by FTC2 (Theorem 6.4),

$$\int_0^b \sqrt{3 - 2\cos f(x)}dx = \int_0^b f'(x)dx = f(b) - f(0) = \pi.$$

3. **No!** Counterexample: let $f(x) = 1$ for all $x \in \mathbb{R}\backslash\{0\}$ and $f(0) = 0$. This is Riemann integrable on every closed bounded interval $[a, b]$: if $0 \notin (a, b)$, f is monotonic on $[a, b]$, hence f is integrable on $[a, b]$ by Theorem 5.17; if $0 \in (a, b)$, f is decreasing on $[a, 0]$, hence integrable on $[a, 0]$ by Theorem 5.17, and is increasing on $[0, b]$, hence integrable by Theorem 5.17, and hence is integrable on $[a, b]$ by the Join Rule. Clearly f is **not** continuous. I claim that g is differentiable nonetheless.

If $x > 0$, consider the sequence $\mathscr{D}_n = \{0, x/n, x\}$ of dissections of $[0, x]$. Clearly

$$\begin{aligned} l_{\mathscr{D}_n}(f) &= 0 \times \frac{x}{n} + 1 \times x\left(1 - \frac{1}{n}\right)) \to x \\ u_{\mathscr{D}_n}(f) &= 1 \times \frac{x}{n} + 1 \times x\left(1 - \frac{1}{n}\right) = x \to x. \end{aligned}$$

Hence, by Theorem 5.16, $g(x) = \int_0^x f = x$.

If $x < 0$, consider the sequence $\mathscr{D}_n = \{x, x/n, 0\}$ of dissections of $[x, 0]$. Clearly

$$\begin{aligned}
l_{\mathscr{D}_n}(f) &= 1 \times x\left(\frac{1}{n} - 1\right) + 0 \times \left(0 - \frac{x}{n}\right) \to -x \\
u_{\mathscr{D}_n}(f) &= 1 \times x\left(\frac{1}{n} - 1\right) + 1 \times \left(0 - \frac{x}{n}\right) = -x \to -x.
\end{aligned}$$

Hence, by Theorem 5.16, $g(x) = -\int_x^0 f = x$.

Clearly $g(0) = \int_0^0 f = 0$. Hence, for all $x \in \mathbb{R}$, $g(x) = x$. This function is certainly differentiable.