Workshop 2: solutions for week 3

1. Assume a is a cluster point of D. Then, for each $\varepsilon > 0$, there exists $x \in D$ such that $0 < |x - a| < \varepsilon$, and hence $x \in D \backslash \{a\}$ such that $|x - a| < \varepsilon$. This is true, in particular, if $\varepsilon = 1/n$, where $n \in \mathbb{Z}^+$. That is, for each $n \in \mathbb{Z}^+$, there exists $x_n \in D \backslash \{a\}$ such that $|x_n - a| < 1/n$. The sequence (x_n) lies in $D \backslash \{a\}$ and converges to a by the Squeeze Rule.

   Conversely, assume that a sequence (x_n) in $D \backslash \{a\}$ exists such that $x_n \to a$. Then, given any $\varepsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that for all $n \geq N$, $|x_n - a| < \varepsilon$. In particular, $|x_N - a| < \varepsilon$. But $x_N \neq a$ (since (x_n) is a sequence in $D \backslash \{a\}$), so $|x_N - a| > 0$. Hence, x_N is a point in D satisfying $0 < |x_N - a| < \varepsilon$. Since such a point exists for any $\varepsilon > 0$, a is a cluster point of D.

2. First note that the maximal domain of the function $f(x) = (x + 2)/(x^3 + 8)$ is $D = \mathbb{R} \backslash \{-2\}$, and -2 is a cluster point of D. Let $\varepsilon > 0$ be given. Then let $\delta = \min\{1, \varepsilon\}$. Then for all $x \in D$ such that $0 < |x + 2| < \delta$,

$$\begin{aligned}
\left| f(x) - \frac{1}{12} \right| &= \left| \frac{1}{x^2 - 2x + 4} - \frac{1}{12} \right| \\
&= \left| \frac{x^2 - 2x - 8}{12(x^2 - 2 + 4)} \right| \\
&= \frac{|x-4||x+2|}{12((x-1)^2 + 3)} \\
&\leq \frac{|x-4|}{36}|x+2| \\
&< \frac{7}{36}|x+2| \qquad \text{(since } |x+2| < 1\text{, so } x - 4 \in (-7, -5)) \\
&\leq |x+2| \\
&< \varepsilon \qquad \text{(since } |x+2| < \delta \leq \varepsilon).
\end{aligned}$$

3. Assume, towards a contradiction, that $\lim_{x \to 0} \frac{1}{x} = L$ for some $L \in \mathbb{R}$. Then, for each $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in \mathbb{R} \backslash \{0\}$ with $0 < |x - 0| < \delta$, $|1/x - L| < \varepsilon$. This is true, in particular, for $\varepsilon = |L| + 1$: there exists $\delta > 0$ such that for all $x \in \mathbb{R} \backslash \{0\}$ with $0 < |x| < \delta$, $|1/x - L| < |L| + 1$, and hence $1/x < L + |L| + 1 \leq 2|L| + 1$. Consider $x_* = \min\{\delta/2, 1/(2|L| + 1)\}$. Note that $x_* \in \mathbb{R} \backslash \{0\}$ and $0 < |x_*| < \delta$. Hence (by the definition of δ), $1/x_* < 2|L| + 1$. But $x_* \leq 1/(2|L| + 1)$, so $1/x_* \geq 2|L| + 1$, a contradiction.

4. (a) $x_n = 50 + 1/n \to 50$, but $f(x_n) = 7 \to 7 \neq f(50) = 26$. Hence $f(x_n) \not\to f(50)$, so f is discontinuous at 50.

   (b) Let x_n be any sequence that converges to 49.9. We must prove that $f(x_n) \to f(49.9) = 26$. So, let $\varepsilon > 0$ be given. Since $x_n \to 49.9$, there exists $N \in \mathbb{Z}^+$ such that, for all $n \geq N$, $|x_n - 49.9| < 0.1$, and hence $x_n < 49.9 + 0.1 = 50$. Hence, for all $n \geq N$, $f(x_n) = 26$. So, for all $n \geq N$, $|f(x_n) - 26| = 0 < \varepsilon$. Hence $f(x_n) \to f(49.9)$.