Module Title: Real Analysis

School of Mathematics **Semester Two 201819**

Calculator instructions:

- You are allowed to use a calculator which has had an approval sticker issued by the School of Mathematics.

Dictionaries:

- You are not allowed to use your own dictionary in this exam. A basic English dictionary is available to use: raise your hand and ask an invigilator, if you need it.

Exam information:

- There are 5 pages to this exam.
- There will be **2 hours 30 minutes** to complete this exam.
- Answer all questions.
- The numbers in brackets indicate the marks available for each question.

1. Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$. Write down precise mathematical formulations of the following statements, using quantifiers:

(a) D is bounded above. [1]

(b) f is unbounded below. [1]

(c) f is increasing. [1]

(d) f is not increasing. [1]

(e) f attains a maximum. [1]

2. (a) Let $D \subseteq \mathbb{R}$ and $a \in \mathbb{R}$. State precisely what is meant by the phrase "*a is a cluster point of D.*" [1]

(b) Write down a nonempty set that has no cluster points. [1]

(c) Assume a is a cluster point of D. Prove that there exists a sequence (x_n) in $D\backslash\{a\}$ which converges to a. [2]

(d) Let $f : D \to \mathbb{R}$, a be a cluster point of D and $L \in \mathbb{R}$. State precisely what is meant by the phrase "*f has limit L at a.*" [1]

(e) Show, directly from the definition, that $\lim_{x\to 1} x^2 = 1$. [3]

(f) Assume $f : D \to \mathbb{R}$ and $\lim_{x\to a} f(x) = L$. Let (x_n) be any sequence in $D\backslash\{a\}$ converging to a. Prove that $f(x_n)$ converges to L. [3]

3. (a) Let $f : D \to \mathbb{R}$ and $a \in D$ be a cluster point of D. State precisely what is meant by the phrase "*f is differentiable at a*". [1]

(b) Prove directly from this definition that the function

$$f : \mathbb{R}\backslash\{0\} \to \mathbb{R}, \qquad f(x) = x^{-2},$$

is differentiable at 1. [3]

4. (a) State, but do not prove, the *Chain Rule*. [2]

(b) Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable and satisfy $f(1) = -1$, $f'(1) = 2$, $g(-1) = 1$, $g'(-1) = 7$. Compute:

(i) $(g \circ f)'(1)$. [1]

(ii) $h'(1)$ where $h(x) = f(f(x)^2)$. [1]

5. (a) State, but do not prove, the *Extreme Value Theorem.* [1]

(b) State, but do not prove, the *Interior Extremum Theorem.* [1]

(c) State and prove *Rolle's Theorem.* [3]

(d) State and prove the *Mean Value Theorem.* [2]

6. In each of the following cases, either write down a function with the specified properties, or explain why no such function exists.

(a) An unbounded function $f : [0,1] \to \mathbb{R}$. [1]

(b) A continuous unbounded function $f : [0,1] \to \mathbb{R}$. [1]

(c) A bounded but discontinuous function $f : [0,1] \to \mathbb{R}$. [1]

(d) An unbounded differentiable function $f : [0,1] \to \mathbb{R}$. [1]

(e) A bounded differentiable function $f : \mathbb{R} \to \mathbb{R}$ whose derivative $f' : \mathbb{R} \to \mathbb{R}$ is unbounded. [1]

(f) A differentiable function $f : (0,1) \to \mathbb{R}$ which is unbounded above but whose derivative is bounded. [3]

7. (a) State, but do not prove, *Taylor's Theorem*. [2]

(b) Consider the function $f : (0,\infty) \to \mathbb{R}$, $f(x) = x^{1/4}$.

(i) Construct $p_2(x)$, the second Taylor approximant to f about $a = 1$. [2]

(ii) Use p_2 to approximate $1.1^{1/4}$. Find an upper bound on the error in your approximation. [2]

8. (a) Let A be a subset of $\mathbb{R}$. Define the terms *supremum* and *infimum* of A. [2]

(b) State the *Axiom of Completeness of* $\mathbb{R}$. [1]

(c) Let $A = \{1/(1+|x|) \ : \ x \in \mathbb{R}\}$. Prove that $\sup A = 1$ and $\inf A = 0$. [4]

9. (a) Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Define the following terms:

(i) A *dissection* $\mathcal{D}$ of $[a,b]$. [1]

(ii) The *upper Riemann sum* $u_{\mathcal{D}}(f)$ and *lower Riemann sum* $l_{\mathcal{D}}(f)$. [2]

(iii) The *upper Riemann integral* $u(f)$ and *lower Riemann integral* $l(f)$. [2]

(iv) The *Riemann integral* $\int_a^b f$. [1]

(b) Consider the specific function

$$f : [0,2] \to \mathbb{R}, \qquad f(x) = \begin{cases} 3, & x = 0 \\ 1, & 0 < x < 2 \\ 2, & x = 2. \end{cases}$$

(i) Compute the lower and upper Riemann sums of f with respect to the following dissections:

$$\mathcal{D}_1 = \{0,2\}, \quad \mathcal{D}_2 = \{0,1,2\}, \quad \mathcal{D}_3 = \{0, 0.1, 1.9, 2\}.$$

[3]

(ii) Prove that f is Riemann integrable and $\int_0^2 f = 2$. You may assume that $l(f) \leq u(f)$. [3]

10. (a) Let $\mathcal{D}$ be a dissection of $[a, b]$. Define the term *refinement* of $\mathcal{D}$. [1]

(b) State, but do not prove, the *Refinement Lemma*. [1]

(c) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Prove that f is Riemann integrable if and only if there exists a sequence $(\mathcal{D}_n)$ of dissections of $[a, b]$ such that

$$u_{\mathcal{D}_n}(f) - l_{\mathcal{D}_n}(f) \to 0.$$

(You may assume that $l(f) \leq u(f)$.) [4]

(d) Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be Riemann integrable functions. Prove that $f + g$ is Riemann integrable. You may assume that, for any dissection $\mathcal{D}$ of $[a, b]$,

$$l_{\mathcal{D}}(f + g) \geq l_{\mathcal{D}}(f) + l_{\mathcal{D}}(g) \qquad \text{and} \qquad u_{\mathcal{D}}(f + g) \leq u_{\mathcal{D}}(f) + u_{\mathcal{D}}(g).$$

[4]

11. (a) State, but do not prove, the *First Form of the Fundamental Theorem of the Calculus*. [2]

(b) Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \displaystyle\int_{-\log 5}^{x} e^t \cos(\pi e^t) dt$. Compute $f'(\log 3)$. [1]

(c) State, but do not prove, the *Second Form of the Fundamental Theorem of the Calculus*. [2]

(d) Compute $\displaystyle\int_0^1 \frac{x}{(1 + x^2)^2} dx$, rigorously justifying your answer. [2]

12. (a) Let $f : D \to \mathbb{R}$ be a bounded function. Define $\|f\|$, its *sup norm*. [1]

(b) Let $f_n : D \to \mathbb{R}$ be a sequence of bounded functions. What precisely does it mean to say that:

(i) (f_n) converges to f *pointwise*. [1]

(ii) (f_n) converges to f *uniformly*. [1]

What relationship, if any, exists between these two types of convergence? [1]

(c) Write down a sequence of bounded functions $f_n : [0, 1] \to \mathbb{R}$ that converges pointwise to an unbounded function. [2]

(d) Write down a sequence of unbounded functions $f_n : [0, 1] \to \mathbb{R}$ that converges pointwise to a bounded function. [2]

(e) Consider the sequence $f_n : [0, \pi] \to \mathbb{R}$, $f_n(x) = \left(x + \dfrac{\sin x}{n}\right)^2$.

(i) Prove that (f_n) converges uniformly to some function $f : [0, \pi] \to \mathbb{R}$. [2]

(ii) Compute $\displaystyle\lim_{n\to\infty} \int_0^{\pi} f_n$. Explain you reasoning. [2]

13. (a) Define the term *open subset* of $\mathbb{R}$. [1]

(b) Determine whether the following subsets of $\mathbb{R}$ are open. Rigorously justify your answers.

(i) $A = (12, \infty)$. [2]

(ii) $B = (-12, 20]$. [2]

(c) Let $U \subseteq \mathbb{R}$ be open and $f : U \to \mathbb{R}$ be smooth.

(i) State precisely what is meant by the phrase "f is *analytic*". [1]

(ii) Write down a smooth function $f : \mathbb{R} \to \mathbb{R}$ which is *not* analytic. [1]

Solutions

1. (a) $\exists K \in \mathbb{R}, \forall x \in D, x \leq K$.[seen similar, 1]

(b) $\forall K \in \mathbb{R}, \exists x \in D, f(x) < K$.[seen similar, 1]

(c) $\forall x, y \in D, x < y \Rightarrow f(x) \leq f(y)$.[seen similar, 1]

(d) $\exists x, y \in D, x < y$ but $f(x) > f(y)$.[seen similar, 1]

(e) $\exists x \in D, \forall y \in D, f(y) \leq f(x)$.[seen similar, 1]

2. (a) For each $\delta > 0$ there exists $x \in D$ such that $0 < |x - a| < \delta$.[bookwork, 1]

(b) Any finite set will do, as will $\mathbb{Z}$ or any subset thereof.[unseen, 1]

(c) Since a is a cluster point of D, for each $n \in \mathbb{Z}^+$, there exists $x_n \in D$ such that $0 < |x_n - a| < 1/n$. The sequence (x_n) converges to a by the Squeeze Rule (or by a direct ε–N argument).[bookwork, 2]

(d) For each $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in D$ with $0 < |x - a| < \delta$, $|f(x) - L| < \varepsilon$.[bookwork, 1]

(e) Given any $\varepsilon > 0$, let $\delta = \min\{1, \varepsilon/3\}$. Then for all $x \in \mathbb{R}$ satisfying $0 < |x-1| < \delta$,

$$|x^2 - 1| = |x+1||x-1| = |(x-1)+2||x-1| \leq (|x-1|+2)|x-1| \leq 3|x-1| < 3\delta \leq \varepsilon.$$

[seen similar, 3]

(f) Let $\varepsilon > 0$ be given. Since $\lim_{x \to a} f(x) = L$, there exists $\delta > 0$ such that for all $x \in D$ with $0 < |x - a| < \delta$, $|f(x) - L| < \varepsilon$. Since $x_n \to a$, there exists $N \in \mathbb{Z}^+$ such that for all $n \geq N$, $|x_n - a| < \delta$. But $x_n \in D \backslash \{a\}$, so $0 < |x_n - a|$ for all n. Hence, for all $n \geq N$, $x_n \in D$ and $0 < |x_n - a| < \delta$, so $|f(x_n) - L| < \varepsilon$.[bookwork, 3]

3. (a) $\displaystyle\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists.[bookwork, 1]

(b) Given $\varepsilon > 0$, let $\delta = \min\{1/2, \varepsilon/8\}$. Then for all $x \in \mathbb{R}\backslash\{0\}$ with $0 < |x-1| < \delta$,

$$\begin{aligned}\left|\frac{f(x) - f(1)}{x - 1} + 2\right| &= \left|\frac{1 - x^2}{x^2(x-1)} + 2\right| = \left|-\frac{1+x}{x^2} + 2\right| = \frac{|2x^2 - x - 1|}{x^2} \\ &= \frac{|2x+1||x-1|}{x^2} \leq \left(\frac{2}{|x|} + \frac{1}{x^2}\right)|x-1| \leq (4+4)|x-1| < 8\delta \leq \varepsilon.\end{aligned}$$

[seen similar, 3]

4. (a) Let $f : D \to E$ be differentiable at a and $g : E \to \mathbb{R}$ be differentiable at $f(a)$. Then $g \circ f$ is differentiable at a and $(g \circ f)'(a) = g'(f(a))f'(a)$. [No credit for $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ or similar.][bookwork, 2]

(b) (i) $(g \circ f)'(1) = g'(-1)f'(1) = 14$.[seen similar, 1]

(ii) $h'(1) = f'(f(1)^2) \times 2f(1)f'(1) = -8$.[seen similar, 1]

5. (a) Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is bounded and attains both a maximum and a minimum value.[bookwork, 1]

(b) Let $f : (a, b) \to \mathbb{R}$ be differentiable and attain an extremum at $c \in (a, b)$. Then $f'(c) = 0$.[bookwork, 1]

(c) Let f be continuous on $[a, b]$, differentiable on (a, b) and have $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = 0$.
Proof: By the Extreme Value Theorem, f attain both a max and a min. If both occur at the endpoints then, since $f(a) = f(b)$, the max value equals the min value, so f is constant. But then $f'(c) = 0$ for any $c \in (a, b)$. So we may assume that either the max or the min does *not* occur at an endpoint. But then f has an extremum at some $c \in (a, b)$ and $f'(c) = 0$ by the Interior Extremum Theorem.[bookwork, 3]

(d) Let f be continuous on $[a, b]$, differentiable on (a, b). Then there exists $c \in (a, b)$ such that $f'(c) = (f(b) - f(a))/(b - a)$.
Proof: Let $k = (f(b)-f(a))/(b-a)$ and $g(x) = f(x)-k(x-a)$. Then g satisfies the conditions of Rolle's Theorem, so there exists $c \in (a, b)$ such that $g'(c) = 0$, whence $f'(c) = k$.[bookwork, 2]

6. (a) $f(x) = \begin{cases} 1/x, & x > 0, \\ 0, & x = 0. \end{cases}$ [unseen, 1]

(b) Does not exist, by the Extreme Value Theorem.[seen similar, 1]

(c) $f(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0. \end{cases}$ [unseen, 1]

(d) Does not exist, by the Extreme Value Theorem, since every differentiable function is continuous.[seen similar, 1]

(e) $f(x) = \sin(x^2)$ will do.[unseen, 1]

(f) This does not exist, by the Mean Value Theorem. Assume towards a contradiction, that f has the required properties. Since f is unbounded above, for each $n \in \mathbb{Z}^+$ there exists $x_n \in (0, 1)$ such that $f(x_n) > f(1/2) + n$. Applying the MVT to f on the interval from $1/2$ to x_n, there exists $y_n \in (0, 1)$ such that

$$|f'(y_n)| = \left| \frac{f(x_n) - f(1/2)}{x_n - 1/2} \right| > \frac{n}{1/2}.$$

Clearly $|f'(y_n)|$ is unbounded above.[unseen, 3]

7. (a) If f is $n + 1$ times differentiable on an interval I, then for each $a, x \in I$, there exists c between a and x such that

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \frac{1}{2!} f''(a)(x-a)^2 + \cdots \\ &\quad \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n + \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}. \end{aligned}$$

[bookwork, 2]

(b) (i)

$$\begin{aligned} f(1) &= 1 \\ f'(x) &= \frac{1}{4}x^{-3/4} \quad \Rightarrow \quad f'(1) = \frac{1}{4} \\ f''(x) &= -\frac{3}{16}x^{-7/4} \quad \Rightarrow \quad f''(1) = -\frac{3}{16} \\ f'''(x) &= \frac{21}{64}x^{-11/4}. \end{aligned}$$

Hence $p_2(x) = 1 + \frac{1}{4}(x-1) - \frac{3}{32}(x-1)^2$. [seen similar, 2]

(ii) $1.1^{1/3} = f(1.1) \approx p_2(1.1) = \frac{3277}{3200} = 1.0240625$.
By Taylor's Theorem, there exists $c \in (1, 1.1)$ such that

$$f(1.1) = p_2(1.1) + \frac{f'''(c)}{3!}(1.1-1)^3 = p_2(1.1) + \frac{7c^{-11/4}}{128} \times 10^{-3}.$$

Hence the error in the approximation is

$$|f(1.1) - p_2(1.1)| = \frac{7c^{-11/4}}{128} \times 10^{-3} < \frac{7}{128,000} < 5.47 \times 10^{-5}.$$

[seen similar, 2]

8. (a) $\sup A$ is its least upper bound, if this exists, and $\inf A$ is its greatest lower bound, if this exists.[bookwork, 2]

(b) Every nonempty subset of $\mathbb{R}$ which is bounded above has a supremum.[bookwork, 1]

(c) For all $x \in \mathbb{R}$, $1/(1+|x|) \leq 1$ so 1 is an upper bound on A. Every $K < 1$ is not an upper bound on A since $1 = 1/(1+|0|) \in A$. Hence $\sup A = 1$.
For all $x \in \mathbb{R}$, $1/(1+|x|) > 0$ so 0 is an lower bound on A. Given any $K > 0$, $1/(1+|1/K|) \in A$ and

$$\frac{1}{1+|1/K|} < \frac{1}{|1/K|} = K,$$

so K is not a lower bound on A. Hence $\inf A = 0$.[seen similar, 4]

9. (a) (i) $\mathcal{D}$ is a finite subset of $[a, b]$ containing both a and b.[bookwork, 1]

(ii)

$$\begin{aligned} u_{\mathcal{D}}(f) &= \sum_{j=1}^{n} \sup\{f(x) : a_{j-1} \leq x \leq a_j\}(a_j - a_{j-1}) \\ l_{\mathcal{D}}(f) &= \sum_{j=1}^{n} \inf\{f(x) : a_{j-1} \leq x \leq a_j\}(a_j - a_{j-1}) \end{aligned}$$

where $\mathcal{D} = \{a_0, \ldots, a_n\}$ and $a_0 < a_1 < \cdots < a_n$. [bookwork, 2]

(iii)

$$\begin{aligned} u(f) &= \inf\{u_{\mathcal{D}}(f) : \mathcal{D} \text{ a dissection of } [a,b]\} \\ l(f) &= \sup\{l_{\mathcal{D}}(f) : \mathcal{D} \text{ a dissection of } [a,b]\} \end{aligned}$$

[bookwork, 2]

(iv) $\int_a^b f = l(f) = u(f)$, assuming these are equal. [bookwork, 1]

(b) (i)

$$\begin{aligned} l_{\mathcal{D}_1}(f) &= 1 \times 2 = 2 \\ u_{\mathcal{D}_1}(f) &= 3 \times 2 = 6 \\ l_{\mathcal{D}_2}(f) &= 1 \times 1 + 1 \times 1 = 2 \\ u_{\mathcal{D}_2}(f) &= 1 \times 3 + 1 \times 2 = 5 \\ l_{\mathcal{D}_3}(f) &= 1 \times 0.1 + 1 \times 1.8 + 1 \times 0.1 = 2 \\ u_{\mathcal{D}_3}(f) &= 3 \times 0.1 + 1 \times 1.8 + 2 \times 0.1 = 2.3 \end{aligned}$$

[seen similar, 3]

(ii) We've already observed that the set of lower Riemann sums contains 2, so its supremum, $l(f) \geq 2$. For each $r \in (0,1)$, let $\mathcal{D}_r = \{0, r, 2-r, 2\}$. Then

$$u_{\mathcal{D}_r}(f) = 3r + 1(2-2r) + 2r = 2 + 3r.$$

Hence $u(f)$ is the infimum of a set containing $(2,5)$, so $u(f) \leq 2$. Hence $u(f) \leq l(f)$. But $u(f) \geq l(f)$ always, so $u(f) = l(f)$, that is, f is Riemann integrable, and $\int_0^2 = l(f) \geq 2$ and $\int_0^2 f = u(f) \leq 2$, so $\int_0^2 f = 2$. [seen similar, 3]

10. (a) A refinement of $\mathcal{D}$ is any dissection $\mathcal{D}'$ of $[a,b]$ such that $\mathcal{D} \subseteq \mathcal{D}'$[bookwork, 1]

(b) If $\mathcal{D}'$ is a refinement of $\mathcal{D}$, then for any bounded function f,

$$l_{\mathcal{D}}(f) \leq l_{\mathcal{D}'}(f) \leq u_{\mathcal{D}'}(f) \leq u_{\mathcal{D}}(f).$$

[bookwork, 1]

(c) Let $\mathbb{D}$ be the set of dissections of $[a,b]$, $\mathbb{L} = \{l_{\mathcal{D}}(f) : \mathcal{D} \in \mathbb{D}\}$, $\mathbb{U} = \{u_{\mathcal{D}}(f) : \mathcal{D} \in \mathbb{D}\}$.

Assume f is Riemann integrable, so $u(f) = l(f)$. For each $n \in \mathbb{Z}^+$ there $u(f)+1/n$ is not a lower bound on $\mathbb{U}$, so there exists $\mathcal{D}_n' \in \mathbb{D}$ such that $u_{\mathcal{D}_n'}(f) < u(f) + 1/n$. Similarly, $l(f) - 1/n$ is not an upper bound on $\mathbb{L}$ so there exists $\mathcal{D}_n'' \in \mathbb{D}$ such that $l_{\mathcal{D}_n''}(f) > l(f) - 1/n$. Let $\mathcal{D}_n = \mathcal{D}_n' \cup \mathcal{D}_n''$. Since $\mathcal{D}_n$ is a refinement of both $\mathcal{D}_n'$ and $\mathcal{D}_n''$,

$$l(f) - \frac{1}{n} < l_{\mathcal{D}_n''}(f) \leq l_{\mathcal{D}_n}(f) \leq u_{\mathcal{D}_n}(f) \leq u_{\mathcal{D}_n'}(f) < u(f) + \frac{1}{n}$$

by the Refinement Lemma. But $l(f) = u(f)$, so $l_{\mathcal{D}_n}(f) \to u(f)$ and $u_{\mathcal{D}_n}(f) \to u(f)$ by the Squeeze Rule. Hence $u_{\mathcal{D}_n}(f) - l_{\mathcal{D}_n}(f) \to 0$.

Conversely, assume a sequence $\mathcal{D}_n \in \mathbb{D}$ exists such that $u_{\mathcal{D}_n}(f) - l_{\mathcal{D}_n}(f) \to 0$. Then, for all $n \in \mathbb{Z}^+$, $u_{\mathcal{D}_n}(f) \geq u(f)$ and $l_{\mathcal{D}_n}(f) \leq l(f)$, so $u_{\mathcal{D}_n} - l_{\mathcal{D}_n}(f) \geq u(f) - l(f)$. Hence $0 = \lim(u_{\mathcal{D}_n}(f) - l_{\mathcal{D}_n}(f)) \geq u(f) - l(f)$, that is, $l(f) \geq u(f)$. But, for any bounded function, $l(f) \leq u(f)$, so $l(f) = u(f)$, that is, f is Riemann integrable.[bookwork, 4]

(d) By the Sequential Criterion proved above, there exist sequences $\mathcal{D}'_n$, $\mathcal{D}''_n$ of dissections such that $u_{\mathcal{D}'_n}(f)-l_{\mathcal{D}'_n}(f) \to 0$ and $u_{\mathcal{D}''_n}(g)-l_{\mathcal{D}''_n}(g) \to 0$. Let $\mathcal{D}_n = \mathcal{D}'_n \cup \mathcal{D}''_n$, by the Refinement Lemma and the facts given,

$$\begin{aligned} l_{\mathcal{D}_n}(f+g) &\geq l_{\mathcal{D}_n}(f) + l_{\mathcal{D}_n}(g) \geq l_{\mathcal{D}'_n}(f) + l_{\mathcal{D}''_n}(g) \\ u_{\mathcal{D}_n}(f+g) &\leq u_{\mathcal{D}_n}(f) + u_{\mathcal{D}_n}(g) \leq u_{\mathcal{D}'_n}(f) + u_{\mathcal{D}''_n}(g) \\ \Rightarrow \quad 0 &\leq u_{\mathcal{D}_n}(f+g) - l_{\mathcal{D}_n}(f+g) \leq u_{\mathcal{D}'_n}(f) - l_{\mathcal{D}'_n}(f) + u_{\mathcal{D}''_n}(g) - l_{\mathcal{D}''_n}(g) \to 0 \end{aligned}$$

so $u_{\mathcal{D}_n}(f+g) - l_{\mathcal{D}_n}(f+g) \to 0$ by the Squeeze Rule. Hence, by the Sequential Criterion, $f+g$ is Riemann integrable.[bookwork, 4]

11. (a) Let $f : [a,b] \to \mathbb{R}$ be continuous and $F : [a,b] \to \mathbb{R}$, $F(x) = \int_a^x f$. Then F is differentiable and $F'(x) = f(x)$. [bookwork, 2]

(b) By FTC1, $f'(\log 3) = e^{\log 3}\cos(\pi e^{\log 3}) = 3\cos 3\pi = -3$. [seen similar, 1]

(c) Let $f : [a,b] \to \mathbb{R}$ be continuous and $F : [a,b] \to \mathbb{R}$ be a differentiable function such that $F' = f$. Then $\int_a^b f = F(b) - F(a)$. [bookwork, 2]

(d) $f : [0,1] \to \mathbb{R}$, $f(x) = x(1+x^2)^{-2}$ is continuous, and $F : [0,1] \to \mathbb{R}$, $F(x) = -\frac{1}{2}(1+x^2)^{-1}$ is a function such that $F' = f$. Hence, by FTC2,

$$\int_0^1 f = F(1) - F(0) = \frac{1}{4}.$$

[seen similar, 2]

12. (a) $\|f\| = \sup\{|f(x)| : x \in D\}$ [bookwork, 1]

(b) (i) For each $x \in D$, $f_n(x) \to f(x)$. [bookwork, 1]

(ii) $\|f_n - f\| \to 0$. [bookwork, 1]

Uniform convergence implies pointwise convergence, but not the converse. [bookwork, 1]

(c) $f_n(x) = \begin{cases} \min\{n, 1/x\}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is a sequence of bounded functions converging pointwise to $g(x) = \begin{cases} 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$, which is unbounded.[seen similar, 2]

(d) $f_n(x) = \begin{cases} 1/(nx), & x \neq 0 \\ 0, & x = 0 \end{cases}$ is a sequence of unbounded functions converging pointwise to $g(x) = 0$, which is bounded.[seen similar, 2]

(e) (i) Let $f(x) = x^2$. Then

$$\begin{aligned} 0 \leq \|f_n - f\| &= \sup\{|\frac{2}{n}x\sin x + \frac{1}{n^2}\sin^2 x| : 0 \leq x \leq \pi\} \\ &\leq \sup\{\frac{2x}{n} + \frac{1}{n^2} : 0 \leq x \leq \pi\} = \frac{2\pi}{n} + \frac{1}{n^2} \end{aligned}$$

so $\|f_n - f\| \to 0$ by the Squeeze Rule. [seen similar, 2]

(ii) Since f is Riemann integrable and the convergence $f_n \to f$ is uniform, $\int_0^\pi f_n \to \int_0^\pi f = \frac{\pi^3}{3}$ by FTC2. [seen similar, 2]

13. (a) $U \subseteq \mathbb{R}$ is open if for each $a \in U$ there exists $\varepsilon > 0$ such that $(a-\varepsilon, a+\varepsilon) \subseteq U$.[bookwork, 1]

(b) (i) Open.For each $a > 12$, let $\varepsilon = a - 12 > 0$. Then $(a-\varepsilon, a+\varepsilon) = (12, 2a-12) \subseteq (12, \infty)$.[seen similar, 2]

(ii) Not open.$20 \in B$ but for any $\varepsilon > 0$, $(20-\varepsilon, 20+\varepsilon)$ contains $x = 20 + \varepsilon/2 \notin B$.[seen similar, 2]

(c) (i) For each $x_0 \in U$ there exists $\varepsilon > 0$ and a power series $\sum_{n=0}^\infty a_n(x-x_0)^n$ such that $(x_0-\varepsilon, x_0+\varepsilon) \subseteq U$ and $f(x) = \sum_{n=0}^\infty a_n(x-x_0)^n$ for all $x \in (x_0-\varepsilon, x_0+\varepsilon)$.[bookwork, 1]

(ii) $f(x) = \begin{cases} 0, & x \leq 0 \\ e^{-1/x}, & x > 0. \end{cases}$ [bookwork, 1]

Total score = 100

Check Sheet

1. (a) $\exists K \in \mathbb{R}, \forall x \in D, x \leq K$.
   (b) $\forall K \in \mathbb{R}, \exists x \in D, f(x) < K$.
   (c) $\forall x, y \in D, x < y \Rightarrow f(x) \leq f(y)$.
   (d) $\exists x, y \in D, x < y$ but $f(x) > f(y)$.
   (e) $\exists x \in D, \forall y \in D, f(y) \leq f(x)$.

2. (a)
   (b)
   (c)
   (d)
   (e)
   (f)

3. (a)
   (b)

4. (a)
   (b) (i) $(g \circ f)'(1) = g'(-1)f'(1) = 14$.
   (ii) $h'(1) = f'(f(1)^2) \times 2f(1)f'(1) = -8$.

5. (a)
   (b)
   (c)
   (d)

6. (a)
   (b) Does not exist
   (c)
   (d) Does not exist
   (e)
   (f) This does not exist

7. (a)
   (b) (i) $p_2(x) = 1 + \frac{1}{4}(x-1) - \frac{3}{32}(x-1)^2$.
   (ii) $1.1^{1/3} = f(1.1) \approx p_2(1.1) = \frac{3277}{3200} = 1.0240625$.

$$|f(1.1) - p_2(1.1)| = \frac{7c^{-11/4}}{128} \times 10^{-3} < \frac{7}{128,000} < 5.47 \times 10^{-5}.$$

8. (a)

(b)

(c)

9. (a) (i)

(ii)

(iii)

(iv)

(b) (i)

$$\begin{aligned} l_{\mathcal{D}_1}(f) &= 1 \times 2 = 2 \\ u_{\mathcal{D}_1}(f) &= 3 \times 2 = 6 \\ l_{\mathcal{D}_2}(f) &= 1 \times 1 + 1 \times 1 = 2 \\ u_{\mathcal{D}_2}(f) &= 1 \times 3 + 1 \times 2 = 5 \\ l_{\mathcal{D}_3}(f) &= 1 \times 0.1 + 1 \times 1.8 + 1 \times 0.1 = 2 \\ u_{\mathcal{D}_3}(f) &= 3 \times 0.1 + 1 \times 1.8 + 2 \times 0.1 = 2.3 \end{aligned}$$

(ii)

10. (a)

(b)

(c)

(d)

11. (a)

(b) By FTC1, $f'(\log 3) = e^{\log 3}\cos(\pi e^{\log 3}) = 3\cos 3\pi = -3.$

(c)

(d) $f : [0,1] \to \mathbb{R}$, $f(x) = x(1+x^2)^{-2}$ is continuous, and $F : [0,1] \to \mathbb{R}$, $F(x) = -\frac{1}{2}(1+x^2)^{-1}$ is a function such that $F' = f$. Hence, by FTC2,

$$\int_0^1 f = F(1) - F(0) = \frac{1}{4}.$$

12. (a)

(b) (i)

(ii)

Uniform convergence implies pointwise convergence, but not the converse.

(c)

(d)

(e) (i)

(ii) $\int_0^\pi f_n \to \int_0^\pi f = \frac{\pi^3}{3}$

13. (a)

(b) (i) Open.

(ii) Not open.

(c) (i)

(ii)