MAGIC assessment cover sheet

Course:	**MAGIC063**
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Student:	**Charlotte Satchwell**
University:	**University of Essex**
Serial number:	**8544**

① a) No - If $x_1, x_2 \in X$ s.t $x_1 = 0$ and $x_2 \neq 0$ then the only open set s.t $x_1 \in U_1 \subset \mathcal{T}$ is $U_1 = \mathbb{R}$ so $x_2 \in U_2 \subset \mathcal{T}$ will have $U_1 \cap U_2 = \mathbb{R} \cap U_2 = U_2 \neq \emptyset$

b) No - For open set $U = \{x\}$ where $x \in \mathbb{R}/\{0\}$, if $\mathcal{T}$ generated by $\mathcal{B}$ $\exists B \in \mathcal{B}$ s.t ~~x~~ $x \in B$ and $B \subseteq U = \{x\}$ so $B = \{x\}$ ~~~~ and there are $\mathbb{R}/\{0\}$ different such $\{x\}$ and $\mathbb{R}/\{0\}$ not countable

c) Yes - Open cover $\bigcup_{\alpha \in A} U_\alpha = X$ then must have an $\alpha \in A$ s.t $U_\alpha = \mathbb{R}$ as $\mathbb{R}$ only open set containing 0 therefore has finite subcover $\mathbb{R}$

d) No - For example $\{1\} \in \mathcal{T}$ but $f^{-1}(\{1\}) = \{0\}$ and $\{0\} \notin \mathcal{T}$

e) No - as $\mathbb{R} \in \mathcal{T}$ but $f^{-1}(\mathbb{R}) = [-1,1] \notin \mathcal{T}$

f) Yes - If $X_1, X_2 \in \mathcal{T}$ s.t $X = X_1 \cup X_2$ then one of X_1 or X_2 is $\{\mathbb{R}\}$ as only open set containing 0 so if $X_1 = \mathbb{R}$ then X_2 cannot be disjoint to X_1

② a) $x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$

$$x^T J x = J \Rightarrow \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & x_1x_4 - x_2x_3 \\ x_2x_3 - x_1x_4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

which means M matrices s.t $\det(x) = 1$

The map $\det : N \longrightarrow \mathbb{R}$ is smooth and $1 \in \mathbb{R}$ is a regular value of det as for every $a \in f^{-1}(1)$ we must have at least one of $x_1, x_2, x_3, x_4 \neq 0$ so ~~~~ $d(\det)_a = (x_4 \ -x_3 \ -x_2 \ x_1)$ has rank 1. Therefore by the regular value theorem $M = \det^{-1}(1)$ is a submanifold of ~~~~ N ~~of dimension~~ It is of dimension 3 since $4 = \dim N = \dim \mathbb{R} + 3$

b) M is not compact as it is not bounded so by Heine-Borel cannot be compact

② c) Can have open covering $U_1 = \mathbb{R}^4 / \{x \in \mathbb{R}^4 \text{ s.t } \cancel{x}\, x_4 = 0\}$

$U_2 = \mathbb{R}^4 / \{x \in \mathbb{R}^4 \text{ s.t } x_3 = 0\}$

$\phi_1(x_1, x_2, x_3, x_4) = (x_2, x_3, x_4)$ $\quad \phi_1 : U_1 \to \mathbb{R}^3$

$\phi_2(x_1, x_2, x_3, x_4) = (x_1, x_3, x_4)$ $\quad \phi_2 : U_2 \to \mathbb{R}^3$

$$\phi_1^{-1}(y_1, y_2, y_3) = \left(\frac{1+y_1y_2}{y_3}, y_1, y_2, y_3\right)$$

$$\phi_2^{-1}(y_1, y_2, y_3) = \left(y_1, \frac{y_1y_3 - 1}{y_2}, y_2, y_3\right)$$

$U_1 \cap U_2 = \mathbb{R}^4 / \{\cancel{xx}\, x \in M \text{ s.t } x_4 = 0 \text{ or } x_3 = 0\}$

$$\phi_2 \circ \phi_1^{-1}(y_1, y_2, y_3) = \left(\frac{1+y_1y_2}{y_3}, y_2, y_3\right)$$ smooth on

$\phi_1(U_1 \cap U_2)$ as $x_4 \neq 0$ so $y_3 \neq 0$

$$\phi_1 \circ \phi_2^{-1}(y_1, y_2, y_3) = \left(\frac{y_1y_3 - 1}{y_2}, y_2, y_3\right)$$ smooth $\phi_2(U_1 \cap U_2)$

as $x_3 \neq 0$ so $y_2 \neq 0$

$$T_{12} = \phi_1 \circ \phi_2^{-1} \text{ so } dT_{12} = \begin{pmatrix} y_3/y_2 & -\frac{y_1y_3 - 1}{y_2^2} & y_1/y_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$dT_{21} = \begin{pmatrix} y_2'/y_3' & y_1'/y_3' & -\frac{1+y_1'y_2'}{y_3'^2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det[dT_{12}] = y_3/y_2$$

$$\det[dT_{21}] = y_2'/y_3'$$

oth Negative at for example $(1, 2, -1)$ so not an orientation

② d) ~~$x \in M$~~ For $x \in M$ $\quad x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$

$$h(x) = x^2 = \begin{pmatrix} x_1^2 + x_2x_3 & x_1x_2 + x_2x_4 \\ x_3x_1 + x_3x_4 & x_3x_2 + x_4^2 \end{pmatrix}$$

$$\begin{aligned} \det h(x) &= x_1^2x_2x_3 + x_1^2x_4^2 + x_2^2x_3^2 + x_2x_3x_4^2 \\ &\quad - x_1^2x_2x_3 - x_1x_2x_3x_4 - x_3x_2x_1x_4 \\ &\quad - x_2x_3x_4^2 \\ &= x_1x_4(x_1x_4 - x_2x_3) + x_2x_3(x_2x_3 - x_1x_4) \end{aligned}$$

$x \in M$ so $x_1x_4 - x_2x_3 = 1$

$$\begin{aligned} &= x_1x_4(1) + x_2x_3(-1) \\ &= x_1x_4 - x_2x_3 = 1 \end{aligned}$$

so $h(x) \in M$

② e) ~~$\det : M \to \mathbb{R}$ submersion~~

~~$\det(x) = x_1$~~

$f : M \longrightarrow \mathbb{R} \quad x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in M$

$f(x) = x_1$

$df_x = (1 \; 0 \; 0 \; 0)$ has maximal rank for all $x \in M$ so df_x surjection

② d) $h(x) = (x_1^2 + x_2x_3 \quad x_1x_2 + x_2x_4 \quad x_3x_1 + x_3x_4 \quad x_3x_2 + x_4^2)$

$$dh_p = \begin{pmatrix} 2x_1 & x_3 & x_2 & 0 \\ x_2 & x_1+x_4 & 0 & x_2 \\ x_3 & 0 & x_1+x_4 & x_3 \\ 0 & x_3 & x_2 & 2x_4 \end{pmatrix}$$

for $p = (0,1,1,0) \in M$

$$dh_p = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

has rank 2 so dh_p not injective therefore not an immersion so also can't be an embedding

③ a) ~~$j^*\eta = x_i \xi\, dx^i \otimes dx^i$~~

$j^* dx^1 = y\,dx + x\,dy$

$j^* dx^2 = z\,dx + x\,dz$

so $j^*\eta = y^2\,dx \otimes dx + xy\,dx \otimes dy + xy\,dy \otimes dx$
$+ x^2\,dy \otimes dy + z^2\,dx \otimes dx + xz\,dx \otimes dz$
$+ xz\,dz \otimes dx + x^2\,dz \otimes dz$

b) Need $\eta(dj(X), dj(Y)) = 0$

$dj(X) = yX^1 + xX^2$

$dj(Y) = zY^1 + xY^3$

$X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}$

(4) 1 is a section of E s

∇ ~~...~~ e^{x^1} is the section ~~function~~

$(x_1, x_2, x_3) \longmapsto e^{x^1}$

for $(0, x_2, x_3)$, e^{x^1}

on $\{x \in M : x = (0, x_2, x_3)\}$

$\nabla 1 = dx_3$

3 c) $X = x\frac{\partial}{\partial x}$ ...

$\dot{x}(t) = x(t)$ $\quad x(0) = x_0$

$\dot{y}(t) = -y(t)$ $\quad y(0) = y_0$

$\dot{z}(t) = \ldots$ $\quad z(0) = z_0$

⑤ a) Torsion $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i} = 0$$

so $T\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} - \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}$

For $i=1$ $j=2$ $\quad T = \nabla_{\frac{\partial}{\partial x^1}} \frac{\partial}{\partial x^2} - \nabla_{\frac{\partial}{\partial x^2}} \frac{\partial}{\partial x^1}$

$$= x^2 \frac{\partial}{\partial x^1}$$

For $i=2$ $j=2$ $\quad T = -x^2 \frac{\partial}{\partial x^1}$

For $i=j=1$ or 2 $\quad T = 0$

For curvature

$$(R_{ij})^a{}_b = \partial_i \Gamma^a_{jb} - \partial_j \Gamma^a_{ib} + \Gamma^c_{jb}\Gamma^a_{ic} - \Gamma^c_{ib}\Gamma^a_{jc}$$

so $(R_{12}) = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$

$(R_{21}) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$(R_{11}) = (R_{22}) = 0$

b) From the notes for a closed loop square γ of length $L > 0$ the holonomy of ∇ around γ is

$$Tr_\gamma = \mathbb{I}_2 + L^2 (R_{ij}) + o(L^2)$$

where (R_{ij}) is the curvature given above

$\dot{x}(t) = x(t)$ $\quad x(0) = x_0$

$\dot{y}(t) = -y(t)$ $\quad y(0) = y_0$

$z(0) = z_0$

⑥ a) $\dot{\alpha}(t) = [(1-t^2) - 2t^2](\cos t, \sin t)$

$\qquad + t(1-t^2)(-\sin t, \cos t)$

$\qquad = (1-3t^2)(\cos t, \sin t) + t(1-t^2)(-\sin t, \cos t)$

From problem set 6 question 1 α a geodesic if α a critical point of E where

$$E = \frac{1}{2}\int_0^1 g(\dot\gamma(t), \dot\gamma(t))\,dt$$

Let $\dot\alpha(t) = (\dot x_1(t), \dot x_2(t))$ and $d_s|_{s=0}(x_1^s(t), x_2^s(t)) = (X_1(t), X_2(t))$

$$\frac{d}{ds}E[\gamma_s]\Big|_{s=0} = \frac{1}{2}\int_0^1 2\dot x_1(t)X_1(t) + 2\dot x_2(t)X_2(t)\,dt$$

$$= \int_0^1 [(1-3t^2)\cos(t) - t(1-t^2)\sin(t)]X_1(t)\,dt$$

$$+ \int_0^1 [(1-3t^2)\sin(t) + t(1-t^2)\cos(t)]X_2(t)\,dt$$

which is not 0 for all variations so not a geodesic

b) β geodesic so $\nabla^{\beta}_{\frac{d}{dt}}\dot\beta = 0$

③ c) $X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}$

$\dot{x}(t) = x(t) \qquad x(0) = x_0$

$\dot{y}(t) = -y(t) \qquad y(0) = y_0$

$\dot{z}(t) = -z(t) \qquad z(0) = z_0$

$\cancel{x(t) =}\ \ln|x(t)| = t + k = t + \ln|x_0|$

$x(t) = e^{t + \ln|x_0|}$

Hence $\theta_x(t) = e^{t+\ln|x|}$

$y(t) = e^{t - \ln|x_0|} \qquad z(t) = e^{t - \ln|z_0|}$

$\theta_{(x,y,z)}(t) = (e^{t+\ln|x|}, e^{t-\ln|y|}, e^{t-\ln|z|})$

$\theta_1(x,y,z) = (e^{1+\ln|x|}, e^{1-\ln|y|}, e^{1-\ln|z|})$

d) $bX = a(x,y,z)\,dx + b(x,y,z)\,dy + c(x,y,z)\,dz$

$\cancel{a(x,y,z) = bX(\frac{\partial}{\partial x}) = g(X, \frac{\partial}{\partial x}) = \theta_1^* g_{Euc}(X, \frac{\partial}{\partial x})}$

$\cancel{= g_{Euc}(d\theta_p \frac{\partial}{\partial x}, d\theta_p \frac{\partial}{\partial x})}$

$\cancel{= g_{Euc}(e \frac{\partial}{\partial x}, e \frac{\partial}{\partial x})}$

$a(x,y,z) = bX(\frac{\partial}{\partial x}) = \theta_1^* g_{Euc}(X, \frac{\partial}{\partial x})$

$= g_{Euc}(d\theta_1 X, d\theta_1 \frac{\partial}{\partial x})$

(7) a) $k(X_1 \times X_2, Y_1 \times Y_2) = g(X_1, Y_1) + f(p) h(X_2, Y_2)$

for $X_1, Y_1 \in T_pM$ $X_2, Y_2 \in T_pN$ $p \in M$

b) Both g and h ~~sym~~ symmetric so

~~k~~ $k(Y_1 \times Y_2, X_1 \times X_2) = g(Y_1, X_1) + f(p) h(\cancel{X} Y_2, X_2)$

$= g(X_1, Y_1) + f(p) h(X_2, Y_2)$

so k symmetric

Both g and h positive definite and $f(p) \in (0, \infty)$

so $k(X_1 \times X_2, Y_1 \times Y_2) = g(X_1, Y_1) + f(p) h(X_2, Y_2) \geq 0$

~~$g(X_1, Y_1) = 0$ only if X_2~~

$k(X_1 \times X_2, X_1 \times X_2) = g(X_1, X_1) + f(p) h(X_2, X_2)$

~~$g(X_1, X_1) = 0$ only if $X_1 = 0$~~

~~$f(p) \neq 0$~~

$k = 0$ only if $g(X_1, X_1) = 0$ and

$h(X_2, X_2) = 0$ as ~~$f(p) \neq 0$~~ and $f(p) > 0$ both g, h pos. def

so $k(X, X) = 0$ only if $X = 0$

⑦c) (M,g) and (N,h) complete, so all geodesics exist for all time for connexion $(\nabla_X g)$ $(\nabla_X h)$

$$(\nabla_X k)(Y,Z) = X[k(Y,Z)] - k(\nabla_X Y, Z) - k(Y, \nabla_X Z)$$

so $\nabla^{\gamma}_{\frac{d}{dt}} k(\dot\gamma) = 0$

$$\nabla^{\gamma}_{\frac{d}{dt}} k(\dot\gamma) = \frac{d}{dt}[k(\dot\gamma,\dot\gamma)] - k(\nabla^{\gamma}_{\frac{d}{dt}}\dot\gamma, \dot\gamma) - k(\dot\gamma, \nabla_{\frac{d}{dt}}\dot\gamma)$$

$$= \frac{d}{dt}[(g+fh)(\dot\gamma\cdot\dot\gamma)] - (g+fh)(\nabla^{\gamma}_{\frac{d}{dt}}\dot\gamma,\dot\gamma) - (g+fh)(\dot\gamma, \nabla_{\frac{d}{dt}}\dot\gamma)$$

$$= \frac{d}{dt} g(\dot\gamma,\dot\gamma) + \frac{d}{dt}[fh(\dot\gamma,\dot\gamma)]$$

$$- g(\nabla^{\gamma}_{\frac{d}{dt}}\dot\gamma,\dot\gamma) - fh(\nabla^{\gamma}_{\frac{d}{dt}}\dot\gamma,\dot\gamma) - g(\nabla^{\gamma}_{\frac{d}{dt}}\dot\gamma,\dot\gamma)$$

$$- fh(\dot\gamma, \nabla_{\frac{d}{dt}}\dot\gamma)$$

since $f(p) \geq c$ we can see that a curve c is a geodesic iff it is a geodesic for g and h since (M,g) and (N,h) complete we then see that the geodesic exist for all time