Module Title: Real Analysis

School of Mathematics **Semester Two 202122**

Calculator instructions:

- You are not allowed to use a calculator in this exam.

Dictionary instructions:

- You are not allowed to use your own dictionary in this exam. A basic English dictionary is available to use. Raise your hand and ask an invigilator if you need it.

Exam information:

- There are 3 pages to this examination.
- There will be **2 hours 30 minutes** to complete this examination.
- Answer all questions.
- The numbers in brackets indicate the marks available for each question.

1. Let D be a subset of $\mathbb{R}$, a be a point in D and $f : D \to \mathbb{R}$. Give precise mathematical definitions of the following terms.

(a) a is a **cluster point** of D.

(b) f is **continuous** at a.

(c) f is **differentiable** at a.

(d) f is **bounded above**.

(e) f **attains a maximum** at a.

(f) f is **unbounded below**.

(g) f is **increasing**.

(h) f is **strictly decreasing**.

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2. Let $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$, $f(x) = 2/x$. Give a direct ε–δ proof that

$$\lim_{x\to 3} f(x) = \frac{2}{3}.$$

[10]

3. (a) State, but do not prove, the **Mean Value Theorem**.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be **twice** differentiable, and assume that

$$f(0) = 1, \qquad f(1) = 0, \qquad f(2) = 3.$$

Prove that there exists $c \in (0, 2)$ such that $f''(c) > 2$.

[10]

4. (a) Let f be the function

$$f : [0, 2] \to \mathbb{R}, \qquad f(x) = \begin{cases} 0, & x \in [0, 1), \\ 1, & x = 1, \\ 2, & x \in (1, 2]. \end{cases}$$

and $\mathscr{D}$ be the dissection

$$\mathscr{D} = \{0, 0.8, 1.2, 2\}.$$

(i) Compute, $u_{\mathscr{D}}(f)$ and $l_{\mathscr{D}}(f)$, the upper and lower Riemann sums of f with respect to $\mathscr{D}$.

(ii) Write down a **refinement** $\mathscr{D}'$ of $\mathscr{D}$ such that

$$l_{\mathscr{D}}(f) = l_{\mathscr{D}'}(f) < u_{\mathscr{D}'}(f) < u_{\mathscr{D}}(f).$$

Verify your claim by computing $u_{\mathscr{D}'}(f)$ and $l_{\mathscr{D}'}(f)$.

(b) Prove that every increasing function $f : [a, b] \to \mathbb{R}$ is Riemann integrable. You may use, without proof, the following theorem taken from the lecture notes:

Theorem 4.21 A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if there exists a sequence $\mathscr{D}_n$ of dissections of $[a, b]$ such that

$$u_{\mathscr{D}_n}(f) - l_{\mathscr{D}_n}(f) \to 0.$$

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5. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be the functions defined by

$$f(x) = x \cos x, \qquad g(x) = \int_0^{x^2} f.$$

Compute $g'(\sqrt{\pi})$. Rigorously justify your answer.

[6]

6. (a) Let $f : D \to \mathbb{R}$ be a bounded function. Define its **sup norm**, denoted $\|f\|$.

(b) Let $f_n : D \to \mathbb{R}$ be a sequence of bounded functions. What does it mean to say that (f_n) **converges uniformly**?

(c) For each $n \in \mathbb{Z}^+$, let $f_n : [0, 1] \to \mathbb{R}$ be the function

$$f_n(x) = \frac{x}{1 + (x/2)^n}.$$

(i). Prove that (f_n) converges uniformly to some function $f : [0, 1] \to \mathbb{R}$, identifying this function.

(ii). Hence or otherwise, compute

$$\lim_{n\to\infty} \int_0^1 f_n.$$

Explain your reasoning.

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7. Exactly sum the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{2^{2n+1}}$. Clearly explain your reasoning.

[10]

8. (a) Let $f : [0, 1] \to [0, 1]$ be increasing. Prove that f has a fixed point, that is, there exists $c \in [0, 1]$ such that $f(c) = c$.
Warning: you may **not** assume that f is continuous.
Hint: Consider the set $A = \{x \in [0, 1] : f(x) \geq x\}$.

(b) Let $g : (0, 1) \to (0, 1)$ be an increasing function. Does it still follow that g has a fixed point? If so, prove it. If not, give a counterexample.

[10]