MATH2017 Problem Set 4: Uniform convergence

Submit on Gradescope by 17:00, Monday 3 April 2023

1. For each $n \in \mathbb{Z}^+$ let $f_n : \mathbb{R} \to \mathbb{R}$ be the function
$$f_n(x) = \frac{nx}{n|x|+1}.$$
   (a) Prove that (f_n) converges pointwise to some function $f : \mathbb{R} \to \mathbb{R}$.
   (b) Prove that (f_n) does *not* converge uniformly.

2. Construct a sequence of unbounded functions $g_n : [0,1] \to \mathbb{R}$ that converges pointwise to a bounded function $g : [0,1] \to \mathbb{R}$.

3. For each $n \in \mathbb{Z}^+$ let $f_n : [0, \frac{1}{2}] \to \mathbb{R}$, $f_n(x) = 1/(1+x^n)$.
   (a) Prove that (f_n) converges uniformly.
   (b) Compute the limit $\lim\limits_{n\to\infty} \int_0^{1/2} f_n$, rigorously justifying your answer.

4. For each $n \in \mathbb{Z}^+$ let $f_n : \mathbb{R} \to \mathbb{R}$ be the function
$$f_n(x) = \sum_{k=1}^{n} \frac{1}{k^2} \cos(kx).$$
   Prove that (f_n) is uniformly Cauchy, and hence converges uniformly.

5. (a) Let $f, g : D \to \mathbb{R}$ be bounded functions. Prove that $\|fg\| \leq \|f\|\|g\|$.
   (b) Assume $f_n : D \to \mathbb{R}$ converges uniformly to f and $g_n : D \to \mathbb{R}$ converges uniformly to g. Prove that $(f_n g_n)$ converges uniformly to fg.