

1. Introduction

Our approach to static contract-checking is to translate source code to a first-order logic theory and then use an automated theorem prover to check the consistency of the theory.

Consider:

```
data List a = Nil | Cons a (List a)
```

```
notnull x = case x of
  Nil -> False
  Cons(x,y) -> True
```

```
head :: (CF && {x | notnull x }) -> CF
head xs = case xs of
  Nil -> BAD
  Cons(x,y) -> x
```

First, we need to encode the List structure.

We start by stating that *Nil* and *Cons* can never be equal:

$$\forall a, b. \text{Cons}(a, b) \neq \text{Nil}$$

Then, we must state that *Nil* never crashes (ie cannot be evaluated to an exception) and that *Cons*(*x*, *y*) crashes iff either *x* or *y* crashes. The statement *x* crashes is encoded by the term *CF*(*x*).

$$\text{CF}(\text{Nil})$$

$$\forall a, b. \text{CF}(a) \wedge \text{CF}(b) \iff \text{CF}(\text{Cons}(a, b)) \quad (1)$$

We also say some stuff about unreachability but I can't think of a good way to explain it right now.

$$\forall y, ys. \text{Cons}(y, ys) \neq \text{UNR}$$

$$\text{Nil} \neq \text{UNR}$$

Finally, we define projections for *Cons*. It is not strictly necessary, but it will be handy:

$$\forall xs, y, ys. \text{sel}_{1, \text{Cons}}(\text{Cons}(y, ys)) = xs \implies xs = y$$

$$\forall xs, y, ys. \text{sel}_{2, \text{Cons}}(\text{Cons}(y, ys)) = xs \implies xs = ys$$

Now we translate the *null* function. Note that the symbols *true* and *false* are the representation of the data constructors *True* and *False* in Haskell, not the boolean values \top and \perp in our logic.

$$\forall xs. xs = \text{Nil} \implies \text{notnull}(xs) = \text{false}$$

$$\forall xs, y, ys. x = \text{Cons}(y, ys) \implies \text{notnull}(xs) = \text{true} \quad (2)$$

We also need to specify the translation of calls to *notnull* with BAD values (to encode the fact that *notnull*(BAD) = BAD).

$$\forall xs. xs = \text{BAD} \rightarrow \text{notnull}(xs) = \text{BAD}$$

Finally, we say that one call *notnull* with a an argument which is not *Nil* or *Cons* or BAD then the result is UNR.

$$\begin{aligned} &\forall xs. xs \neq \text{BAD} \wedge xs \neq \text{Nil} \\ &\wedge xs \neq \text{Cons}(\text{sel}_{1, \text{Cons}}(xs), \text{sel}_{2, \text{Cons}}(xs)) \rightarrow \\ &\quad \text{notnull}(X) = \text{UNR} \end{aligned} \quad (3)$$

The translation of *head* follows the same pattern:

$$\begin{aligned} &\forall xs. x = \text{Nil} \implies \text{head}(xs) = \text{BAD} \\ &\forall xs, y, ys. x = \text{Cons}(y, ys) \implies \text{head}(xs) = y \\ &\forall xs. xs = \text{BAD} \rightarrow \text{head}(xs) = \text{BAD} \end{aligned} \quad (4)$$

$$\begin{aligned} &\forall xs. xs \neq \text{BAD} \wedge xs \neq \text{Nil} \\ &\wedge xs \neq \text{Cons}(\text{sel}_{1, \text{Cons}}(xs), \text{sel}_{2, \text{Cons}}(xs)) \rightarrow \\ &\quad \text{head}(X) = \text{UNR} \end{aligned}$$

We now have translated the source code. Let us call all those formulae the theory *T*. We translate separately the contract:

$$\phi := \forall xs. \text{CF}(xs) \wedge \text{notnull}(xs) = \text{true} \rightarrow \text{CF}(\text{head}(xs))$$

Now that we translated everything to first-order logic, we can ask the theorem prover if the theory formed by those formulae is consistent, ie if $T \vdash \phi$.

Intuitively, *T* is consistent (ie $T \not\vdash \perp$), because each formula serves a specific purpose. Now, assume that *xs* satisfies *CF*(*xs*) and *notnull*(*xs*) = *true*. We can derive that *xs* \neq BAD because we have *CF*(*xs*) and $\neg \text{CF}(\text{BAD})$. The constraint *notnull*(*xs*) = *true* doesn't directly imply that *xs* = *Cons*(*y*, *ys*) for some *y* and *ys*. But *notnull* is totally defined, because of (3). This implies (by (2)) that there exist *y* and *ys* such that *xs* = *Cons*(*y*, *ys*). Recalling *CF*(*xs*), we can now derive *CF*(*y*) and *CF*(*ys*) (by (1)). But *head*(*xs*) = *y* because of (4), and *y* is crash-free, so we can finally derive *CF*(*head*(*xs*)). QED.

2. Languages

2.1 \mathcal{H}' : λ -calculus variant

The syntax of \mathcal{H} is defined in figure 1. A module is a list of toplevel definitions, claims that functions satisfy contracts and data definitions.

- There's no λ -abstraction, because we can always lift them to toplevel declaration.
- We do not allow nested case expressions, because once again, we can always lift them to the toplevel.
- Until section 6 we will only consider full application of functions (*f*(*x*, *y*)), in order to remove clutter. Dealing with partial application is not hard but a bit cumbersome.

```
mod  := def1, ..., defn
def  ∈ Definition
def  := data T = K1 | ... | Kn
      | f ∈ c
      | f  $\vec{x}$  = e
      | f  $\vec{x}$  = case e of
                  K1( $\vec{x}$ ) → e1 | ... | Kn( $\vec{x}$ ) → en
```

```
x, y, f, g, a, b ∈ Variables
T                ∈ Type Constructors
K                ∈ Data Constructors
```

```
e                ∈ Expressions
e ::= x
    | BAD
    | e e
    | f(e, ..., e)
    | K(e, ..., e)
```

Figure 1. Syntax of the language \mathcal{H}'

2.2 Contracts

Contract syntax is described in figure 2. The predicates we use in our contracts can be any boolean \mathcal{H}' expression. We only consider

pairs of contract for simplicity, although there is no issue with generalisation to arbitrary tuples.

$$c := \begin{array}{l} x : c \rightarrow c \\ (c, c) \\ c \wedge c \\ c \vee c \\ \{x \mid p\} \\ \text{CF} \end{array}$$

Figure 2. Contract syntax

We give the semantics of contract by defining “ e satisfies t ”, written $e \in t$ in figure 3. Note that this definition doesn’t yield any operative way to check that an expression actually meets the specification given by its contract.

$$\begin{array}{ll} e \in \{x \mid p\} & \iff e \text{ diverges or } p[e/x] \not\rightarrow^* \{\text{BAD}, \text{False}\} \\ e \in x : c_1 \rightarrow c_2 & \iff \forall e_1 \in c_1, (e e_1) \in c_2[e_1/x] \\ e \in (c_1, c_2) & \iff e \text{ diverges or } (e \rightarrow^* (e_1, e_2) \text{ and } e_1 \in c_1, e_2 \in c_2) \\ e \in c_1 \wedge c_2 & \iff e \in c_1 \text{ and } e \in c_2 \\ e \in c_1 \vee c_2 & \iff e \in c_1 \text{ or } e \in c_2 \\ e \in \text{CF} & \iff e \text{ is crash-free} \end{array}$$

Figure 3. Semantics of contract satisfaction

2.3 Crash-freeness

Note that CF represents two things: it can be a contract, as in $f \in \text{CF}$ or a special formula in first-order logic $\text{CF}(f)$.

We use BAD to signal that something has gone wrong in the program : it has crashed.

Definition 1 (Crash). *A closed term e crashed iff $e \rightarrow^* \text{BAD}$.*

Definition 2 (Diverges). *A closed expression e diverges iff either $e \rightarrow^* \text{UNR}$ or there is no value val such that $e \rightarrow^* val$*

Definition 3 (Syntactic safety). *A (possibly open) expression e is syntactically safe iff $\text{BAD} \notin_s e$. Similarly a context C is syntactically safe iff $\text{BAD} \notin_s C$.*

The notation $\text{BAD} \notin_s e$ means that BAD does not appear anywhere in e , similarly for $\text{BAD} \notin_s C$. For example, *Just3* is syntactically safe whereas *JustBAD* is not.

Definition 4 (Crash-free). *An expression e is said to be crash-free iff*

$$\forall C. \text{BAD} \notin_s C \text{ and } \vdash C[e] :: () \not\rightarrow^* \text{BAD}$$

The notation $C[e] :: ()$ means that $C[e]$ is closed and well-typed. Note that there are crash-free expression that are not syntactically safe, for example *fst* (1, BAD).

2.4 BAD and UNR

Consider the following piece of code:

`a = 0 + True`

`b :: CF`

`b = undefined`

`c = error "foo"`

- a is ill-typed
- b ’s implementation is not correct wrt its contract

$$\begin{array}{lcl} v, w, s, t & := & x \mid K(t, \dots, t) \mid f(t, \dots, t) \mid \text{app}(t, t) \\ & & \mid \text{BAD} \mid \text{UNR} \\ \phi & := & \forall x. \phi \mid \neg \phi \mid \phi \vee \phi \mid \top \mid \perp \mid t = t \mid \text{CF}(t) \\ & & \mid \phi \wedge \phi \mid \phi \rightarrow \phi \mid \phi \leftrightarrow \phi \\ \Phi & := & \epsilon \mid \phi \mid \Phi \cup \Phi \end{array}$$

Figure 4. First-order logic syntax

$$\begin{array}{ll} \mathcal{E}[\text{expression}] & \rightarrow t \text{ (Term)} \\ \mathcal{D}[f \vec{x} = \dots] & \rightarrow \Phi \text{ (Set of formulae)} \\ \mathcal{K}[\text{data } T = \dots] & \rightarrow \Phi \text{ (Set of formulae)} \\ \mathcal{C}[f \in c] & \rightarrow \Phi \text{ (Set of formulae)} \end{array}$$

Figure 5. Translations

$$\begin{array}{ll} \boxed{\mathcal{E}[e] = t} & \\ \mathcal{E}[x] & = x \\ \mathcal{E}[f(e_1, \dots, e_n)] & = f(\mathcal{E}[e_1], \dots, \mathcal{E}[e_n]) \\ \mathcal{E}[K(e_1, \dots, e_n)] & = K(\mathcal{E}[e_1], \dots, \mathcal{E}[e_n]) \\ \mathcal{E}[\text{BAD}] & = \text{BAD} \end{array}$$

Figure 6. $\mathcal{E}[\]$ – Expression translation

- c goes through the whole toolchain (compiler, typechecker, contractchecker)

One thing to notice is that a and b are things that “sould not happen” but are caught statically whereas c should not happen but can only be dealt with dynamically.

We can now define two types of problematic expressions: those that cannot happen during a run of the program and those that can. Expressions of the first type are called unreachable (and equated to the special value UNR in our first-order theory), whilst expressions of the latter type are called bad (and equated to the special value BAD).

We said earlier that we only considered syntactically correct and well-typed programs as input. That implies that the “ a ” case cannot happen. But given that our first-order logic is not typed, the theorem prover may decide to instantiate a variable with an ill-typed value! In order to prevent this, we will need to encode some basic type-checking mechanism directly in our first-order theory.

2.5 First-order logic with equality

We use first-order logic with equality, defined in figure 4.

3. Translations

For an overview of the different translations we define, see figure 5

3.1 $\mathcal{E}[\]$ – Expressions

Our most basic translation is from expressions in \mathcal{H}' to terms in first-order logic. Given this translation we will be able to translate definitions, data types and contracts to first-order formulae. It is described in figure 6.

3.2 $\mathcal{D}[\]$ – Definitions

We give in figure 7 and figure 8 the two translations of function definitions.

figure 7 gives the translation of function not defined by pattern matching, which is really easy: we just have to state the equality between the left-hand side and the right-hand side.

$$\frac{\mathcal{D}[\![f(\vec{x}) = e]\!]\quad \forall \vec{x}, y. \min(y) \wedge y = f(\vec{x}) \rightarrow y = \mathcal{E}[e]}{\text{Figure 7. } \mathcal{D}[\![\cdot]\!] \text{ -- Regular definitions}}$$

Figure 7. $\mathcal{D}[\![\cdot]\!]$ – Regular definitions

Translating definitions that use pattern-matching is more challenging and is described in figure 8.

The first line says that when applied to an argument that matches a pattern of the case expression, we should equate the function call to the corresponding expression.

The second line states that if the pattern-matching failed or if we pattern-matched on BAD then the result should be UNR.

3.3 $\mathcal{K}[\![\cdot]\!]$ – Datatypes

We break down the translation for datatypes in four parts, described in figure 9

$$\mathcal{K}[\![\text{data } T = K_1, \dots, K_n]\!] = \Phi_1 \cup \Phi_2 \cup \Phi_3 \cup \Phi_4$$

- (Φ_1) For each K_i of arity a_i we introduce selectors sel_{k, K_i} , which are the projection of $K_i(x_1, \dots, x_{a_i})$ on its k -th component.
- (Φ_2) For each pair of constructors K_i, K_j , we state that they can never map to the same value.
- (Φ_3) Then, we have to give crash-freeness conditions for each K_i : Notice that we have an equivalence.
 - \leftarrow : if we pack crash-free values in a data constructor, the resulting value is crash-free.
 - \rightarrow : a value t of type T is crash-free implies that every value packed in it is crash-free. Recall that one can define projection on any argument of a value of type t . So if the k -th argument of t is not crash-free, then the k -th projection is a crash-free context that throws an expression that is not crash-free.

Note that this is not true for functions: a function is not required to use all of its arguments. `fst` is crash-free if and only if the first argument of the pair is crash-free. The second argument being crash-free or not doesn't matter.
- (Φ_4) None of the K_i is unreachable.
- One may want to also state that if $\vec{x} \neq \text{BAD}$ then $K_i(\vec{x}) \neq \text{BAD}$. It is already implied by the fact that $\text{CF}(\vec{x}) \rightarrow \text{CF}(K_i(\vec{x}))$.

3.4 $\mathcal{C}[\![\cdot]\!]$ – Contracts

We give in figure 10 the translation of contract satisfaction. *true* refers to the translation to a term of the data constructor `True` in \mathcal{H}' , not to the actual true value.

Note that we define the translation of $f \in c$ and of $f \notin c$. We have to do that because one is not the negation of the other, even though $\neg(f \notin c)$ implies $f \in c$.

4. $\mathcal{M}[\![\cdot]\!]$ – Checking a module

4.1 Prelude

There are some formulae that should always be included in our FO theory.

We need to state that BAD is not crash-free with the formula: $\neg \text{CF}(\text{BAD})$.

Plus we need to give formulae for the boolean datatype and for unreachability. Strictly speaking, we can omit them and just add the following lines to source files:

```
data UNR = UNR
```

```
data Bool = True | False
```

But given that those datatypes are used by our translation, we can just directly include their translation every time we translate a module.

4.2 Contract checking – Non-recursive case

Input: a module M that consists of a list of definitions, datatypes, contracts and a contract c for a non-recursive function f this is defined in M .

We say that the function implementation is correct wrt to its contract iff

$$\mathcal{M} \vdash \mathcal{C}[f \in c]$$

4.3 Contract checking – Recursive case

If the function $f = e$ is recursive, then we ask the theorem prover the following:

$$\mathcal{M} - f, \mathcal{D}[f = e[f/f_p]], \mathcal{C}[f_p \in c] \vdash \mathcal{C}[f \in c]$$

Where $M - f$ means the content of the module M without f 's definition and f 's contract. TODO Stress that it's not always enough and that we may have to unroll several times!

4.4 Module checking

A module is a collection of function definitions, data definitions and contracts. What we want to do is to check that functions satisfy their contract(s).

4.4.1 Naive example

Here is a little example showing that we should be careful about which formulae should belong to a theory.

Assume that we have a module that contains two functions definition f and g and two contracts : $f \in \text{CF}$ and $g \in \text{CF}$. We assume that those contracts do not hold, for example if f is *head* and g is *last*.

First, we want to check f 's contract. So we ask the theorem prover if

$$\mathcal{D}[f], \mathcal{D}[g], \mathcal{C}[g] \vdash \mathcal{C}[f]$$

But, given that g 's contract does not hold, we can derive \perp and then prove that f 's contract hold.

For the same reason, we can prove that g contract's holds, when in fact it doesn't.

Finally, the user thinks he's done, but in fact he has proven nothing.

4.4.2 The proper way to check a module

Consider the following situation, where a 's definition relies on f and g .

$$\mathcal{D}[\![f(x_1, \dots, x_n) = \text{case } e \text{ of } [K_1(\vec{x}_1) = e_1, \dots, K_n(\vec{x}_n) = e_n]]\!] =$$

$$\begin{aligned} \forall \vec{a}, y. \min(y) \wedge f(\vec{a}) = y &\rightarrow \min(e) \wedge (e = \text{BAD} \vee \bigvee_i \forall \vec{x}. e = K_i(\vec{x}) \vee y = \text{UNR}) \\ &\wedge \forall \vec{x}_1. \mathcal{E}[\![e]\!] = K_1(\vec{x}_1) \rightarrow y = \mathcal{E}[\![e_1]\!] \\ &\wedge \dots \\ &\wedge \forall \vec{x}_n. \mathcal{E}[\![e]\!] = K_n(\vec{x}_n) \rightarrow y = \mathcal{E}[\![e_n]\!] \end{aligned}$$

Figure 8. $\mathcal{D}[\![]$ – Case definitions

$$\mathcal{K}[\![\text{data } T = K_1, \dots, K_n]\!] = \Phi_1 \cup \Phi_2 \cup \Phi_3 \cup \Phi_4$$

where

$$\begin{aligned} \Phi_1 &= \bigcup_{1 \leq i \leq n} \forall \vec{x}, a. (\min(a) \wedge K_i(\vec{x}) = a) \rightarrow \bigwedge_{1 \leq j \leq k} x_j = \text{sel}_{j, K_i}(a) \\ \Phi_2 &= \bigcup_{1 \leq i < j \leq n} \forall \vec{x}, \vec{y}, a. \neg(\min(a) \wedge K_i(\vec{x}) = a \wedge K_j(\vec{y}) = a) \\ \Phi_3 &= \bigcup_{1 \leq i \leq n} \forall \vec{x}, a. \min(a) \wedge a = K_i(\vec{x}) \rightarrow ((\bigwedge_{1 \leq j \leq k} \text{CF}(x_j)) \leftrightarrow \text{CF}(K_i(\vec{x}))) \\ \Phi_4 &= \bigcup_{1 \leq i \leq n} \forall \vec{x}, a. (\min(a) \wedge a = K_i(\vec{x})) \rightarrow a \neq \text{UNR} \wedge a \neq \text{BAD} \end{aligned}$$

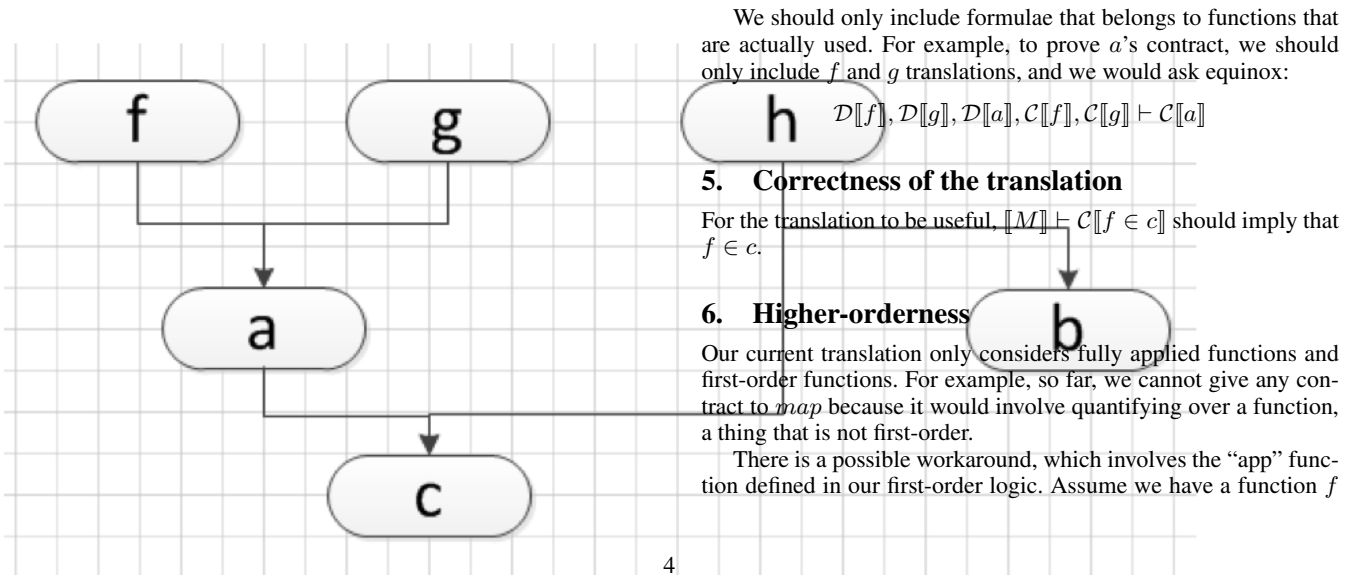
Figure 9. $\mathcal{K}[\![]$ – Data type translation

$$\begin{aligned} \mathcal{C}[\![e \in \{x \mid b(x)\}]\!] &= \min(\mathcal{E}[\![b(e)]\!]) \wedge (\mathcal{E}[\![b(e)]\!] = \text{true} \vee \mathcal{E}[\![e]\!] = \text{UNR}) \\ \mathcal{C}[\![e \in x : c_1 \rightarrow c_2(x)]\!] &= \forall x. \min(\mathcal{E}[\![e(x)]\!]) \rightarrow (\mathcal{C}[\![x \notin c_1]\!] \vee \mathcal{C}[\![\mathcal{E}[\![e(x)]\!] \in c_2(x)]\!]) \\ \mathcal{C}[\![e \in \text{CF}]\!] &= \text{CF}(\mathcal{E}[\![e]\!]) \end{aligned}$$

$$\begin{aligned} \mathcal{C}[\![e \notin \{x \mid b(x)\}]\!] &= \min(\mathcal{E}[\![b(e)]\!]) \wedge (\mathcal{E}[\![b(e)]\!] = \text{false} \vee \mathcal{E}[\![e]\!] = \text{BAD}) \\ \mathcal{C}[\![e \notin x : c_1 \rightarrow c_2(x)]\!] &= \exists x. \mathcal{C}[\![x \in c_1]\!] \wedge \mathcal{C}[\![\mathcal{E}[\![e(x)]\!] \notin c_2(x)]\!] \\ \mathcal{C}[\![e \notin \text{CF}]\!] &= \neg \text{CF}(\mathcal{E}[\![e]\!]) \end{aligned}$$

$$\begin{aligned} \mathcal{C}[\![e \in c_1 \&\& c_2]\!] &= \mathcal{C}[\![e \in c_1]\!] \wedge \mathcal{C}[\![e \in c_2]\!] \\ \mathcal{C}[\![e \notin c_1 \&\& c_2]\!] &= \mathcal{C}[\![e \notin c_1]\!] \vee \mathcal{C}[\![e \notin c_2]\!] \\ \mathcal{C}[\![e \in c_1 \mid c_2]\!] &= \mathcal{C}[\![e \in c_1]\!] \vee \mathcal{C}[\![e \in c_2]\!] \\ \mathcal{C}[\![e \notin c_1 \mid c_2]\!] &= \mathcal{C}[\![e \notin c_1]\!] \wedge \mathcal{C}[\![e \notin c_2]\!] \\ \mathcal{C}[\![a, b \in (c_1, c_2)]\!] &= \mathcal{C}[\![a \in c_1]\!] \wedge \mathcal{C}[\![b \in c_2]\!] \\ \mathcal{C}[\![a, b \notin (c_1, c_2)]\!] &= \mathcal{C}[\![a \notin c_1]\!] \vee \mathcal{C}[\![b \notin c_2]\!] \end{aligned}$$

Figure 10. $\mathcal{C}[\![]$ – Contract translation



that is not fully applied somewhere in a module. We create the term f_ptr which relates to f by the equations given in figure 11

This way, we can emulate quantification over function by quantifying on their ptr counterpart.

7. Experiments

That's how we roll.

$$\begin{aligned}
\mathcal{E}[\![e_1\ e_2]\!] &= app(e_1, e_2) \\
\forall x_1, \dots, x_n. f(x_1, \dots, x_n) &= app(app(\dots app(f_ptr, x_1), x_2), \dots, x_n) \\
CF(f_ptr) &\leftrightarrow \forall x_1, \dots, x_n. CF(x_1) \wedge \dots \wedge CF(x_n) \rightarrow CF(f(x_1, \dots, x_n)) \\
\forall f_ptr, x. CF(f_ptr) \wedge CF(x) &\rightarrow CF(app(f_ptr, x))
\end{aligned}$$

Figure 11. Encoding of higher-orderness

Problem	Equinox	Equinox (+ weak)	SPASS	Vampire	E
Add.hs	0.25	0.08	0.04	0.12	0.05
BinaryTree.hs	0.45	0.2	0.04	0.01	0.04
Branch.hs	0.27	0.40	0.04	0.01	0.03
Copy.hs	0.86	0.09	0.03	0.01	184.3
Head.hs	0.32	0.29	0.03	0.03	4.2
Implies.hs	3.24	0.32	0.06	0.02	0.11
Map.hs	2.47	0.14	0.92	1.02	>300
Mult.hs	>300	0.41	0.05	0.22	11.71
Multgt.hs	>300	1.24	0.62	1.31	>300
NatEq.hs	203.12	0.33	0.02	0.03	0.343
Odd.hs	0.42	1.17	0.06	0.03	>300
Reverse.hs	72.32	0.12	0.05	0.02	0.038
Simple.hs	0.07	0.04	0.01	0.01	0.022
Test.hs	7.76	2.86	0.08	0.05	>300
Test2.hs	5.63	0.09	0.07	0.01	1.02

Figure 12. Comparison (in seconds) with other theorem provers