Static contract checking for Haskell

1. Introduction

Our approach to static contract-checking is to translate source code to a first-order logic theory and then use an automated theorem prover to check the consistency of the theory.

Consider:

```
data List a = Nil | Cons a (List a)

notnull x = case x of
  Nil -> False
  Cons(x,y) -> True

head ::: (CF && {x | notnull x }) -> CF
head xs = case xs of
  Nil -> BAD
  Cons(x,y) -> x
```

First, we need to encode the List structure.

We start by stating that Nil and $Cons$ can never be equal:

$$\forall a, b.\ Cons(a, b) \neq Nil$$

Then, we must state that Nil never crashes (ie cannot be evaluated to an exception) and that $Cons(x, y)$ crashes iff either x or y crashes. The statement x crashes is encoded by the term $\texttt{CF}(x)$.

$$\texttt{CF}(Nil)$$

$$\forall a, b.\ \texttt{CF}(a) \wedge \texttt{CF}(b) \iff \texttt{CF}(Cons(a, b)) \tag{1}$$

We also say some stuff about unreachibility but I can't think of a good way to explain it right now.

$$\forall y, ys.\ Cons(y, ys) \neq \texttt{UNR}$$

$$Nil \neq \texttt{UNR}$$

Finally, we define projections for $Cons$. It is not strictly necessary, but it will be handy: $\forall xs, y, ys.\ sel_{1,Cons}(Cons(y, ys)) = xs \implies xs = y$

$$\forall xs, y, ys.\ sel_{2,Cons}(Cons(y, ys)) = xs \implies xs = ys$$

Now we translate the $null$ function. Note that the symbols $true$ and $false$ are the representation of the data constructors $True$ and $False$ in Haskell, not the boolean values $\top$ and $\bot$ in our logic.

$$\forall xs.\ xs = Nil \implies notnull(xs) = false$$

$$\forall xs, y, ys.\ x = Cons(y, ys) \implies notnull(xs) = true \tag{2}$$

We also need to specify the translation of calls to $notnull$ with $\texttt{BAD}$ values (to encode the fact that $notnull(\texttt{BAD}) = \texttt{BAD}$.

$$\forall xs.\ xs = \texttt{BAD} \rightarrow notnull(xs) = \texttt{BAD}$$

Finally, we say that one call $notnull$ with a an argument which is not Nil or $Cons$ or $\texttt{BAD}$ then the result is $\texttt{UNR}$.

$$\forall xs.\ xs \neq \texttt{BAD} \wedge xs \neq Nil \wedge xs \neq Cons(sel_{1,Cons}(xs), sel_{2,Cons}(xs)) \rightarrow notnull(X) = \texttt{UNR} \tag{3}$$

The translation of $head$ follows the same pattern:

$$\forall xs.\ x = Nil \implies head(xs) = \texttt{BAD} \tag{4}$$

$$\forall xs, y, ys.\ x = Cons(y, ys) \implies head(xs) = y \tag{5}$$

$$\forall xs.\ xs = \texttt{BAD} \rightarrow head(xs) = \texttt{BAD}$$

$$\forall xs.\ xs \neq \texttt{BAD} \wedge xs \neq Nil \wedge xs \neq Cons(sel_{1,Cons}(xs), sel_{2,Cons}(xs)) \rightarrow head(X) = \texttt{UNR}$$

We now have translated the source code. Let us call all those formulae the theory T. We translate separatly the contract:

$$\phi := \forall xs.\ \texttt{CF}(xs) \wedge notnull(xs) = true \rightarrow \texttt{CF}(head(xs))$$

Now that we translated everything to first-order logic, we can ask the theorem prover if the theory formed by those formulae is consistent, ie if $T \vdash \phi$.

Intuitively, T is consistent (ie $T \nvdash \bot$), because each formula serves a specific purpose. Now, assume that xs satisfies $\texttt{CF}(xs)$ and $notnull(xs) = true$. We can derive that $xs \neq \texttt{BAD}$ because we have $\texttt{CF}(xs)$ and $\neg\texttt{CF}(\texttt{BAD})$. The constraint $notnull(xs) = true$ doesn't directly imply that $xs = Cons(y, ys)$ for some y and ys. But $notnull$ is totally defined, because of (3). This implies (by (2)) that there exist y and ys such that $xs = Cons(y, ys)$. Recalling $\texttt{CF}(xs)$, we can now derive $\texttt{CF}(y)$ and $\texttt{CF}(ys)$ (by (1)). But $head(xs) = y$ because of (4), and y is crash-free, so we can finally derive $\texttt{CF}(head(xs))$. QED.

2. Languages

2.1 $\mathcal{H}'$: λ-calculus variant

The syntax of $\mathcal{H}$ is defined in 1. A module is a list of toplevel definitions, claims that functions statisfy contracts and data definitions.

- There's no λ-abstraction, because we can always lift them to toplevel declaration.
- We do not allow nested case expressions, because once again, we can always lift them to the toplevel.
- Until we will only consider full application of functions ($f(x, y)$), in order to remove clutter. Dealing with partial application is not hard but a bit cumbersome.

2.2 Contracts

Contract syntax is described in figure 2.2. The predicates we use in our contracts can be any boolean $\mathcal{H}'$ expression. We only consider pairs of contract for simplicity, although there is no issue with generalisation to arbitrary tuples.

$$
\begin{array}{rcl}
mod & := & def_1, \ldots, def_n \\
def & \in & \text{Definition} \\
def & := & \texttt{data}\ T = K_1 \mid \cdots \mid K_n \\
 & \mid & f \in c \\
 & \mid & f\ \vec{x}\ =\ e \\
 & \mid & f\ \vec{x}\ =\ \texttt{case}\ e\ \texttt{of} \\
 & & \qquad K_1(\vec{x}) \to e_1 \mid \cdots \mid K_n(\vec{x}) \to e_n \\
x, y, f, g, a, b & \in & \text{Variables} \\
T & \in & \text{Type Constructors} \\
K & \in & \text{Data Constructors} \\
e & \in & \text{Expressions} \\
e & ::= & x \\
 & \mid & \texttt{BAD} \\
 & \mid & e\ e \\
 & \mid & f(e, \ldots, e) \\
 & \mid & K(e, \ldots, e)
\end{array}
$$

Figure 1. Syntax of the language $\mathcal{H}'$

$$
\begin{array}{rcl}
c & := & x : c \to c \\
 & \mid & (c, c) \\
 & \mid & c \wedge c \\
 & \mid & c \vee c \\
 & \mid & \{x \mid p\} \\
 & \mid & \texttt{CF}
\end{array}
$$

Figure 2. Contract syntax

We give the semantics of contract by defining "e satisfies t", written $e \in t$ in figure 2.2. Note that this definition doesn't yield any operative way to check that an expression actually meets the specification given by its contract.

$$
\begin{array}{rcl}
e \in \{x \mid p\} & \iff & e \text{ diverges or } p[e/x] \not\to^\star \{\texttt{BAD}, False\} \\
e \in x : c_1 \to c_2 & \iff & \forall e_1 \in c_1, (e\ e_1) \in c_2[e_1/x] \\
e \in (c_1, c_2) & \iff & e \text{ diverges or} \\
 & & (e \to^\star (e_1, e_2) \text{ and } e_1 \in c_1, e_2 \in c_2) \\
e \in c_1 \wedge c_2 & \iff & e \in c_1 \text{ and } e \in c_2 \\
e \in c_1 \vee c_2 & \iff & e \in c_1 \text{ or } e \in c_2 \\
e \in \texttt{CF} & \iff & e \text{ is crash-free}
\end{array}
$$

Figure 3. Semantics of contract satisfaction

2.3 Crash-freeness

Note that CF represents two things: it can be a contract, as in $f \in \texttt{CF}$ or a special formula in first-order logic $\texttt{CF}(f)$.

We use BAD to signal that something has gone wrong in the program : it has crashed.

[Crash] A closed term e crashed iff $e \to^* \texttt{BAD}$.

[Diverges] A closed expression e diverges iff either $e \to^* \texttt{UNR}$ or there is no value val such that $e \to^* val$

[Syntactic safety] A (possibly open) expression e is syntactically safe iff $\texttt{BAD} \not\in_s e$. Similarly a context C is syntactically safe iff $\texttt{BAD} \not\in_s \mathcal{C}$.

The notation $\texttt{BAD} \not\in e$ means that BAD does not appear anywhere in e, similarly for $\texttt{BAD} \not\in_s \mathcal{C}$. For example, $Just3$ is syntactically safe whereas $Just\texttt{BAD}$ is not.

[Crash-free] An expression e is said to be crash-free iff

$$\forall \mathcal{C}.\ \texttt{BAD} \not\in_s \mathcal{C} \text{ and } \vdash \mathcal{C}[\![e]\!] :: ()\mathcal{C} \not\to^* \texttt{BAD}$$

The notation $\mathcal{C}[\![e]\!] :: ()$ means that $\mathcal{C}[\![e]\!]$ is closed and well-typed. Note that there are crash-free expression that are not syntactically safe, for example $fst\ (1, \texttt{BAD})$.

$$
\begin{array}{rcl}
v, w, s, t & := & x \mid K(t, \ldots, t) \mid f(t, \ldots, t) \mid app(t, t) \\
 & & \mid \texttt{BAD} \mid \texttt{UNR} \\
\phi & := & \forall x.\phi \mid \neg\phi \mid \phi \vee \phi \mid \top \mid \bot \mid t = t \mid \text{CF}(t) \\
 & & \mid \phi \wedge \phi \mid \phi \implies \phi \mid \phi \to \phi \mid \phi \leftrightarrow \phi \\
\Phi & := & \epsilon \mid \phi \mid \Phi \cup \Phi
\end{array}
$$

Figure 4. First-order logic syntax

$$
\begin{array}{rcl}
\mathcal{E}[\![x]\!] & = & x \\
\mathcal{E}[\![f(e_1, \ldots, e_n)]\!] & = & f(\mathcal{E}[\![e_1]\!], \ldots, \mathcal{E}[\![e_n]\!]) \\
\mathcal{E}[\![K(e_1, \ldots, e_n)]\!] & = & K(\mathcal{E}[\![e_1]\!], \ldots, \mathcal{E}[\![e_n]\!]) \\
\mathcal{E}[\![\texttt{BAD}]\!] & = & \texttt{BAD}
\end{array}
$$

Figure 5. $\mathcal{E}[\![]\!]$ – Expression translation

2.4 BAD and UNR

Consider the following piece of code:

```
a = 0 + True

b ::: CF
b = undefined

c = error "foo"
```

- a is ill-typed
- b's implementation is not correct wrt its contract
- c goes through the whole toolchain (compiler, typechecker, contractchecker)

One thing to notice is that a and b are things that "sould not happen" but are caught statically whereas c should not happen but can only be dealt with dynamically.

We can now define two types of problematic expressions: those that cannot happen during a run of the program and those that can. Expressions of the first type are called unreachable (and equated to the special value UNR in our first-order theory), whilst expressions of the latter type are called bad (and equated to the special value BAD).

We said earlier that we only considered syntactically correct and well-typed programs as input. That implies that the "a" case cannot happen. But given that our first-order logic is not typed, the theorem prover may decide to instanciate a variable with an ill-typed value! In order to prevent this, we will need to encode some basic type-checking mecanism directly in our first-order theory.

2.5 First-order logic with equality

We use first-order logic with equality, defined in figure 2.5. There are two different symbols for implication : $\to$ and $\implies$. Altough they share the same semantics, their operational behaviour differs, as explained in **??**.

3. Translations

3.1 $\mathcal{E}[\![]\!]$ – Expressions

Our most basic translation is from expressions in $\mathcal{H}'$ to terms in first-order logic. Given this translation we will be able to translate definitions, data types and contracts to first-order formulae. It is described in 3.1.

$$\mathcal{D}[\![f(x_1,\ldots,x_n)=e]\!] = Y = f(X_1,\ldots,X_n) \implies Y = \mathcal{D}[\![e]\!]$$

Figure 6. $\mathcal{D}[\![]\!]$ – Regular definitions

3.2 $\mathcal{D}[\![]\!]$ – Definitions

We give in 3.2 and 3.2 the two translations of function definitions.

3.2 gives the translation of function not defined by pattern matching, which is really easy: we just have to state the equality between le left-hand side and the right-hand side.

Translating definitions that use pattern-matching is more challenging and is described in 3.2.

The first line says that when applied to an argument that matches a pattern of the case expression, we should equate the function call to the corresponding expression.

The second line states that if the pattern-matching failed or if we pattern-matched on `BAD` then the result should be `UNR`.

3.3 $\mathcal{T}[\![]\!]$ – Datatypes

We break down the translation for datatypes in four parts, described in 3.3

$$\mathcal{T}[\![\text{data } T = K_1,\ldots,K_n]\!] = S_1 \cup S_2 \cup S_3 \cup S_4$$

- (S_1) For each K_i of arity a_i we introduce selectors sel_{k,K_i}, which are the projection of $K_i(x_1,\ldots,x_{n_i})$ on its k-th component.
- (S_2) For each pair of constructors K_i, K_j, we state that they can never map to the same value.
- (S_3) Then, we have to give crash-freeness conditions for each K_i: Notice that we have a equivalence.
  - $\leftarrow$: if we pack crash-free values in a data constructor, the resulting value is crash-free.
  - $\rightarrow$: a value t of type T is crash-free implies that every value packed in it is crash-free. Recall that one can define projection on any argument of a value of type t. So if the k-th argument of t is not crash-free, then the k-th projection is a crash-free context that throws an expression that is not crash-free.

    Note that this is not true for functions: a function is not required to use all of its arguments. `fst` is crash-free if and only if the first argument of the pair is crash-free. The second argument being crash-free or not doesn't matter.
- (S_4) None of the K_i is unreachable.
- One may want to also state that if $\vec{x} \neq \texttt{BAD}$ then $K_i(\vec{x}) \neq \texttt{BAD}$. It is already implied by the fact that $\texttt{CF}(\vec{x}) \rightarrow \texttt{CF}(K_i(\vec{x}))$.

3.4 $\mathcal{S}[\![]\!]$ – Contracts

We give in 9 the translation of contract satisfaction.

4. $[\![]\!]$ – Checking a module

4.1 Prelude

There are some formulae that should always be included in our FO theory.

We need to state that `BAD` is not crash-free with the formula: $\neg\texttt{CF}(\texttt{BAD})$.

Plus we need to give formulae for the boolean datatype and for unreachability. Strictly speaking, we can omit them and just add the following lines to source files:

```
data UNR = UNR
data Bool = True | False
```

But given that those datatypes are used by our translation, we can just directly include their translation every time we translate a module.

4.2 Contract checking – Non-recursive case

Input: a module M that consists of a list of definitions, datatypes, contracts and a contract c for a non-recursive function f this is defined in M.

We say that the function implementation is correct wrt to its contract iff

$$[\![M]\!] \vdash \mathcal{S}[\![f \in c]\!]$$

4.3 Contract checking – Recursive case

If the function $f = e$ is recursive, then we ask the theorem prover the following:

$$[\![M - f]\!], \mathcal{D}[\![f = e[f/f_p]]\!], \mathcal{S}[\![f_p \in c]\!] \vdash \mathcal{S}[\![f \in c]\!]$$

Where $M - f$ means the content of the module M without f's definition and f's contract. TODO Stress that it's not always enough and that we may have to unroll several times!

4.4 Module checking

A module is a collection of function definitions, data definitions and contracts. What we want to do is to check that functions satisfy their contract(s).

4.4.1 Naive example

Here is a little example showing that we should be careful about which formulae should belong to a theory.

Assume that we have a module that contains two functions defintion f and g and two contracts : $f \in \texttt{CF}$ and $g \in \texttt{CF}$. We assume that those contracts do not hold, for example if f is $head$ and g is $last$.

First, we want to check f's contract. So we ask the theorem prover if

$$\mathcal{D}[\![f]\!], \mathcal{D}[\![g]\!], \mathcal{S}[\![g]\!] \vdash \mathcal{S}[\![f]\!]$$

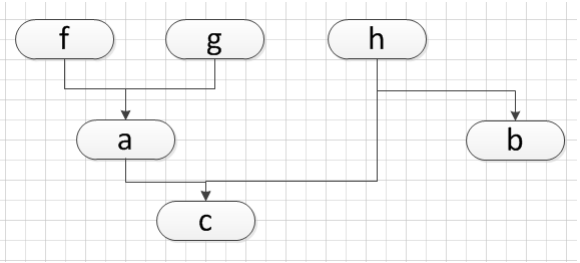
But, given that g's contract does not hold, we can derive $\bot$ and then prove that f's contract hold.

For the same reason, we can prove that g contract's holds, when in fact it doesn't.

Finally, the user thinks he's done, but in fact he has proven nothing.

4.4.2 The proper way to check a module

Consider the following situation, where a's definition relies on f and g.



We should only include formulae that belongs to functions that are actually used. For example, to prove a's contract, we should only include f and g translations, and we would ask equinox:

$$\mathcal{D}[\![f]\!], \mathcal{D}[\![g]\!], \mathcal{D}[\![a]\!], \mathcal{S}[\![f]\!], \mathcal{S}[\![g]\!] \vdash \mathcal{S}[\![a]\!]$$

$$\mathcal{D}[\![f(x_1,\ldots,x_n) = \text{case } e \text{ of } [K_i(\overline{z})] = e_i]\!] = \bigwedge_{1\leq i\leq n} Y = f(K_i(x_1,\ldots,x_{K_i})) \implies Y = \mathcal{E}[\![e_i]\!]$$

$$\cup f(X_1,\ldots,X_n) = Y \implies Y = \texttt{UNR} \vee \bigvee_{1\leq i\leq n} \mathcal{E}[\![e]\!] = K_i(sel_{1,K_i}(\mathcal{E}[\![e]\!]),\ldots,sel_{n_K,K_i}(\mathcal{E}[\![e]\!])) \vee \mathcal{E}[\![e]\!] = \texttt{BAD}$$

Figure 7. $\mathcal{D}[\![]\!]$ – Case definitions

$$S_1 := \bigwedge_{1\leq i\leq n} sel_{i,K_j}(K_j(X_1,\ldots,X_n)) = Y \implies Y = X_i \mid 1 \leq j \leq n$$

$$S_2 := \forall x_1,\ldots,x_{n_i}\ \forall y_1,\ldots,y_{n_j}.K_i(x_1,\ldots,x_{n_i}) \neq K_j(y_1,\ldots,y_{n_j}) \mid 1 \leq i < j \leq n$$

$$S_3 := \forall x_1,\ldots,x_{n_i}.(\texttt{CF}(x_1) \wedge \cdots \wedge \texttt{CF}(x_{n_i}) \leftrightarrow \texttt{CF}(K_i(x_1,\ldots,x_{n_i}))) \mid 1 \leq i \leq n$$

$$S_4 := \forall x_1,\ldots,x_n.\ K_i(x_1,\ldots,x_n) \neq \texttt{UNR} \mid 1 \leq i \leq n$$

Figure 8. $\mathcal{T}[\![]\!]$ – Data type translation

$$\mathcal{S}[\![f \in c]\!] = \top \to f \in c \vee \bot$$

Given r a fresh predicate and a a fresh variable,

$$\begin{aligned}
p \to f \in \{x \mid b(x)\} \vee q &= p \wedge f = X \implies b(X) = true \vee b(X) = \texttt{UNR} \vee q \\
p \to f \in x : c_1 \to c_2(x) \vee q &= p \to f(X) \in c_2(X) \vee r(X), r(X) \to X \notin c_1 \vee q \\
p \to f \in \texttt{CF} \vee q &= p \wedge \forall x_1,\ldots,x_n. \bigwedge_i \texttt{CF}(x_i) \to \texttt{CF}(f(x_1,\ldots,x_n)) \vee q
\end{aligned}$$

$$\begin{aligned}
p \to f \notin \{x \mid b(x)\} \vee q &= p \wedge f = X \implies b(X) = false \vee b(X) = \texttt{BAD} \vee q \\
p \to f \notin x : c_1 \to c_2(x) \vee q &= p \to a \in c_1 \vee q, p \to f(a) \notin c_2(a) \vee q \\
p \to f \notin \texttt{CF} \vee q &= p \wedge \forall x_1,\ldots,x_n.\neg \bigwedge_i \texttt{CF}(x_i) \to \texttt{CF}(f(x_1,\ldots,x_n)) \vee q
\end{aligned}$$

Figure 9. $\mathcal{S}[\![]\!]$ – Contract translation

5. Correctness of the translation

For the translation to be useful, $[\![M]\!] \vdash \mathcal{S}[\![f \in c]\!]$ should imply that $f \in c$.

6. Guarded patterns

It is our first encounter with the guarded-pattern operator ($\implies$). It syntax is:

$$s_1 = t_1 \wedge \cdots \wedge s_k = t_k \implies v_1 = w_1 \vee \cdots \vee v_n = w_n$$

s_i, t_i, v_i, w_i are terms, and all the variables on the right-hand side should appear somewhere on the left-hand side.

The semantics are: if during the proof by the theorem prover, we have equations that match the LHS than the theorem prover should strongly consider the equations on the RHS. Those semantics are sound (we can interpret $\implies$ as normal implication) but if modifies the operational behabviour of Equinox, because it tells it very finely when to use equations and how to instantiate them.

7. Higher-orderness

Our current translation only considers fully applied functions and first-order functions. For example, we cannot translate any contract for map because it would involve quantifying over a function, a thing that is not first-order.

There is a possible workaround, which involves the "app" function in first-order logic. Assume we have a function f that is not fully applied somewhere in a module. We create the term f_ptr which relates to f by the equations given in 7

This way, we can emulate quantification over function by quantification on their ptr counterpart.

8. Experiments

That's how we roll.

$$\mathcal{E}[\![e_1\ e_2]\!] = app(e_1, e_2)$$

$$\forall x_1, \ldots, x_n.\ f(x_1, \ldots, x_n) = app(app(\ldots app(f\_ptr, x_1), x_2), \ldots, x_n)$$

$$\texttt{CF}(f\_ptr) \iff \forall x_1, \ldots, x_n.\ \texttt{CF}(x_1) \wedge \cdots \wedge \texttt{CF}(x_n) \implies \texttt{CF}(f(x_1, \ldots, x_n))$$

$$\forall f\_ptr, x\ \texttt{CF}(f\_ptr) \wedge \texttt{CF}(x) \implies \texttt{CF}(app(f\_ptr, x))$$

Figure 10. Encoding of higher-orderness

Problem	Equinox	Equinox (+ weak)	SPASS	Vampire	E
Add.hs	0.25	0.08	0.04	0.12	0.05
BinaryTree.hs	0.45	0.2	0.04	0.01	0.04
Branch.hs	0.27	0.40	0.04	0.01	0.03
Copy.hs	0.86	0.09	0.03	0.01	184.3
Head.hs	0.32	0.29	0.03	0.03	4.2
Implies.hs	3.24	0.32	0.06	0.02	0.11
Map.hs	2.47	0.14	0.92	1.02	>300
Mult.hs	>300	0.41	0.05	0.22	11.71
Multgt.hs	>300	1.24	0.62	1.31	>300
NatEq.hs	203.12	0.33	0.02	0.03	0.343
Odd.hs	0.42	1.17	0.06	0.03	>300
Reverse.hs	72.32	0.12	0.05	0.02	0.038
Simple.hs	0.07	0.04	0.01	0.01	0.022
Test.hs	7.76	2.86	0.08	0.05	>300
Test2.hs	5.63	0.09	0.07	0.01	1.02

Figure 11. Comparison (in seconds) with other theorem provers