

Øksendal: Stochastic Differential Equations

Solutions Manual
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March 16, 2021

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Chapter 1

Introduction

This is a solutions manual for Stochastic Differential Equations by Bernt Øksendal. This is a working document last updated March 16, 2021. Progress to date:

- Chapter 2: Problems #1-17
- Chapter 3: Problems #1-13, 15-17 (omitted: #14)
- Chapter 4: Problems #1-15
- Chapter 5: Problems #1-17
- Chapters 6–12: none so far

Chapter 2

Some Mathematical Preliminaries

1. Suppose $X : \Omega \rightarrow \mathbb{R}$ is a function that assumes countably many values $\{a_j\}$ in \mathbb{R} .

(a) Note that X is a random variable if and only if it is measurable. If $X : \Omega \rightarrow \mathbb{R}$ is measurable, then $U = X^{-1}(\mathbb{R} \setminus a_k) \in \mathcal{F}$ and thus $X^{-1}(a_k) = \Omega \setminus U \in \mathcal{F}$, $\forall k$. On the other hand, if $X^{-1}(a_k) \in \mathcal{F}$, $\forall k$, then Borel set $V \subseteq \mathbb{R}$, $X^{-1}(V) = \bigcup_{a_k \in V} X^{-1}(a_k) \in \mathcal{F}$ and thus X is measurable.

(b) Compute $\mathbb{E}(|X|) = \int_{\mathbb{R}} |x| d\mathbb{P}_X = \int_{\bigcup_{k=1}^{\infty} \{a_k\}} |x| d\mathbb{P}_X = \sum_{k=1}^{\infty} |a_k| \mathbb{P}(X = a_k)$.

(c) If $\mathbb{E}(|X|) < \infty$, then the series

$$\mathbb{E}(X) = \int_{\mathbb{R}} x d\mathbb{P}_X = \int_{\bigcup_{k=1}^{\infty} \{a_k\}} x d\mathbb{P}_X = \sum_{k=1}^{\infty} a_k \mathbb{P}(X = a_k)$$

is absolutely convergent and therefore converges.

(d) If f is measurable and $|f|$ is bounded by M , then

$$\mathbb{E}(|f(X)|) = \int_{\mathbb{R}} |f(x)| d\mathbb{P}_X \leq \int_{\mathbb{R}} M d\mathbb{P}_X = M \int_{\mathbb{R}} d\mathbb{P}_X = M < \infty.$$

Hence,

$$\mathbb{E}(f(X)) = \int_{\mathbb{R}} f(x) d\mathbb{P}_X = \int_{\bigcup_{k=1}^{\infty} \{a_k\}} f(x) d\mathbb{P}_X = \sum_{k=1}^{\infty} f(a_k) \mathbb{P}(X = a_k)$$

is absolutely convergent and therefore converges.

2. Let $F(x) = \mathbb{P}(X \leq x)$ be the distribution function of X .

(a) By monotonicity of \mathbb{P} , $0 = \mathbb{P}(\emptyset) \leq \mathbb{P}(X \leq x) \leq \mathbb{P}(\mathbb{R}) = 1$. Now, by the Monotone Convergence Theorem,

$$\lim_{n \rightarrow \infty} F(n) = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \chi_{(-\infty, n]} d\mathbb{P}(x) = \int_{\mathbb{R}} d\mathbb{P}(x) = 1.$$

Similarly, for $G(n) := 1 - F(-n)$, we have

$$\lim_{n \rightarrow \infty} G(n) = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} (1 - \chi_{(-\infty, -n]}) dP_X(x) = 1.$$

Moreover, F is increasing by monotonicity of P and finally, again by Monotone Convergence,

$$\lim_{h \rightarrow 0^+} 1 - F(x+h) + F(x) = \lim_{h \rightarrow 0^+} \int_{\mathbb{R}} (1 - \chi_{(x, x+h]}) d\mathbb{P}(x) = \int_{\mathbb{R}} d\mathbb{P}(x) = 1$$

and so $\lim_{h \rightarrow 0^+} F(x+h) = F(x)$, i.e. F is right-continuous.

(b) Compute the expectation

$$\mathbb{E}(g(X)) = \int_{\mathbb{R}} g(x) d\mathbb{P}(x) = \int_{\mathbb{R}} g(x) \chi_{(-\infty, x]} d\mathbb{P}(x) = \int_{\mathbb{R}} g(x) dF(x).$$

(c) Compute the density of B_t^2

$$\begin{aligned} F(u) &:= \mathbb{P}(B_t^2 \leq u) = \mathbb{P}(-\sqrt{u} \leq B_t \leq \sqrt{u}) \\ &= 2 \int_{[0, \sqrt{u}]} p(y) dy \\ &= 2 \int_{[0, u]} \frac{p(\sqrt{u})}{2\sqrt{u}} du \\ &= \int_{(-\infty, u]} \chi_{[0, \infty)} \frac{p(\sqrt{u})}{\sqrt{u}} du. \end{aligned}$$

and so $p(u) = \chi_{[0, \infty)} \frac{p(\sqrt{u})}{\sqrt{u}}$ where $p(u)$ is the density of B_t .

3. Since \mathcal{H}_i is a σ -algebra, $\emptyset \in \mathcal{H}_i$, $\forall i \in I$. So $\emptyset \in \mathcal{H} = \cap_{i \in I} \mathcal{H}_i$. If $\{U_j\}_{j \in \mathbb{N}} \subseteq \mathcal{H}$, then $\{U_j\}_{j \in \mathbb{N}} \subseteq \mathcal{H}_i$ for each $i \in I$ and so $\Omega \setminus U_j \in \mathcal{H}_i$ and $\cup_{j \in \mathbb{A}} U_j \in \mathcal{H}_i$, $\forall i \in I$. Conclude that $\Omega \setminus U_j \in \mathcal{H}$ and $\cup_{j \in \mathbb{A}} U_j \in \mathcal{H}$ and $\mathcal{H} = \cap_{i \in I} \mathcal{H}_i$ is also a σ -algebra.

4. Let $X : \Omega \mapsto \mathbb{R}$ be a random variable with $\mathbb{E}(|X|^p) < \infty$.

(a) Let $A = \{\omega \in \Omega \mid |X| \geq \lambda > 0\}$ and compute

$$\mathbb{E}(|X|^p) = \int_{\Omega} |X|^p d\mathbb{P} \geq \int_A |X|^p d\mathbb{P} \geq \lambda^p \int_A d\mathbb{P} = \lambda^p \mathbb{P}(|X| \geq \lambda).$$

(b) By Chebychev, $\mathbb{P}(|X| \geq \lambda) = \mathbb{P}(e^{|X|} \geq e^\lambda) \leq \frac{1}{e^{k\lambda}} \mathbb{E}(e^{k|X|}) = Me^{-k\lambda}$.

5. Since the measures are σ -finite, $f(x, y) = xy$ is $\mathbb{P}_X \otimes \mathbb{P}_Y$ measurable and $\mathbb{E}(|XY|) < \infty$, apply Fubini-Tonelli and compute

$$\begin{aligned}
 \mathbb{E}(XY) &= \int_{\mathbb{R}^2} xy \, d\mathbb{P}_{XY}(x, y) \\
 &= \int_{\mathbb{R}^2} xy \, d\mathbb{P}_X(x) \otimes d\mathbb{P}_Y(y) \\
 &= \int_{\mathbb{R}} y \left(\int_{\mathbb{R}} x \, d\mathbb{P}_X(x) \right) d\mathbb{P}_Y(y) \\
 &= \mathbb{E}(X) \int_{\mathbb{R}} y \, d\mathbb{P}_Y(y) \\
 &= \mathbb{E}(X)\mathbb{E}(Y).
 \end{aligned}$$

6. (Borel-Cantelli) Let $\{A_k\}_{k=1}^{\infty} \subseteq \mathcal{F}$ and suppose $\sum_{k=1}^{\infty} \mathbb{P}(A_k) < \infty$. Then

$$\mathbb{P}(\cap_{m=1}^{\infty} \cup_{k=m}^{\infty} A_k) \leq \lim_{m \rightarrow \infty} \sup_{k \geq m} \mathbb{P}(A_k) = 0$$

by dominated convergence.

7. Let $\Omega = \bigsqcup_{i=1}^n G_i$.

- (a) Note $\emptyset \in \mathcal{G}$ and \mathcal{G} is closed under unions by construction. It is also closed under complements as $\Omega \setminus G_i = \cup_{j \neq i} G_j \in \mathcal{G}$.
- (b) Write a new sequence defined by $F_i = G_i \setminus \cup_{j \leq i} F_j$ and $\{F_i\}$ will satisfy (a).
- (c) Note that $\{X^{-1}(x \in \mathbb{R})\} \subseteq \mathcal{F}$ is disjoint. So, by (a) and (b), \mathcal{F} is finite if and only if all but finitely many $X^{-1}(x \in \mathbb{R})$ are empty.

8. Let B_t be a 1-dimensional Wiener process.

- (a) By Equation 2.2.3, since $B_t \sim N(0, t)$,

$$\mathbb{E}(e^{iuB_t}) = \exp\left(-\frac{u^2}{2}\mathbb{V}(B_t) + iu\mathbb{E}(B_t)\right) = e^{-\frac{u^2}{2}}.$$

- (b) Comparing power series coefficients, we deduce that

$$\frac{(iu)^{2n}}{(2n)!} \mathbb{E}(B_t^{2n}) = \frac{1}{n!} \left(-\frac{u^2 t}{2}\right)^n,$$

and so $\mathbb{E}(B_t^{2n}) = \frac{(2n)!}{2^n n!} t^n$.

(c) Integrating by parts, compute the n^{th} moment of B_t

$$\begin{aligned}
 \mathbb{E}(B_t^{2k}) &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} x^{2k} e^{-\frac{x^2}{2t}} dx \\
 &= x^{2k-1} \sqrt{\frac{2t}{\pi}} \int_{\frac{x}{\sqrt{2t}}} u e^{-u^2} du \Big|_{x=-\infty}^{x=\infty} - \int_{\mathbb{R}} (2k-1) x^{2k-2} \sqrt{\frac{2t}{\pi}} \int_{\frac{x}{\sqrt{2t}}} u e^{-u^2} du \\
 &= -(2k-1) \sqrt{\frac{2t}{\pi}} \int_{\mathbb{R}} x^{2k-2} \left(\frac{-1}{2} e^{-\frac{x^2}{2t}} \right) dx \\
 &= (2k-1)t \cdot \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} x^{2k-2} e^{-\frac{x^2}{2t}} dx \\
 &= (2k-1)t \mathbb{E}(B_t^{2k-2}).
 \end{aligned}$$

As $\mathbb{E}(B_t^2) = t$, we have that $\mathbb{E}(B_t^{2k}) = \frac{(2k)!t^{k-1}}{2^k k!} \cdot t = \frac{(2k)!t^k}{2^k k!}$.

(d) Check the base case, $n = 2k = 2$, where $\mathbb{E}(B_t^2) = \frac{2!t}{2 \cdot 1!} = t$. If the claim is true for $n = 2k$, then

$$\mathbb{E}(B_t^{2k+2}) = (2k-1)t \mathbb{E}(B_t^{2k}) = (2k+1)t \cdot \frac{(2k)!t^k}{2^k k!} = \frac{(2k+2)!t^{k+1}}{2^{k+1}(k+1)!},$$

and so it is also true for $n = 2(k+1) = 2k+2$, thus completing the induction step.

9. Note that $\{X_t\}$ and $\{Y_t\}$ have the same distributions since neither distribution has any atoms and they agree except on a zero set $\forall t \geq 0$. Yet $t \mapsto X_t$ is discontinuous while $t \mapsto Y_t$ is continuous.
10. As B_t is Brownian, $B_{t+h} - B_t \sim N(0, h)$. Since h is fixed, $\{B_{t+h} - B_t\}_{h \geq 0}$ have the same distributions $\forall t \geq 0$.
11. As $B_0 = (B_0^{(1)}, B_0^{(2)}, \dots, B_0^{(n)}) = 0$, $B_0^{(j)} = 0$ for all $j \in \{1, \dots, n\}$. B_t is almost surely continuous only if its components are almost surely continuous. Each component is normally distributed with $\mathbb{E}(B_t^{(j)}) = 0$ as $\mathbb{E}(B_t) = \vec{0}$ and $\text{Cov}(B_t^{(i)}, B_t^{(j)}) = t\delta_{ij}$ as $\text{Cov}(B_t) = tI$.
12. Let $W_t := B_{s+t} - B_s$ where $s \geq 0$ is fixed. Then $W_0 = B_s - B_s = 0$ and W_t is almost surely continuous as the sum of two almost surely continuous stochastic processes. Noting $W_{t_2} - W_{t_1} = B_{s+t_2} - B_{s+t_1}$ is independent of both B_{s+t_1} and B_s , deduce that $W_{t_2} - W_{t_1}$ is independent of $W_{t_1} = B_{s+t_1} - B_s$. The expected value is $\mathbb{E}(W_t) = \mathbb{E}(B_{s+t}) - \mathbb{E}(B_s) = 0$ and the variance is

$$\begin{aligned}
 \mathbb{V}(W_t) &= \mathbb{E}((B_{s+t} - B_s)^2) \\
 &= \mathbb{E}(B_{s+t}^2) - 2\mathbb{E}(B_s B_{s+t}) + \mathbb{E}(B_s^2) \\
 &= \mathbb{E}(B_{s+t}^2) - 2\mathbb{E}(B_s(B_{s+t} - B_s)) - \mathbb{E}(B_s^2) \\
 &= \mathbb{E}(B_{s+t}^2) - 2\mathbb{E}(B_s)\mathbb{E}(B_{s+t} - B_s) - \mathbb{E}(B_s^2) \\
 &= (s+t) - 0 - s \\
 &= t.
 \end{aligned}$$

Since W_t is the sum of two normal distributions, it is also normal and $W_t \sim N(0, t)$.

13. Compute

$$\mathbb{P}_0(B_t \in D_\rho) = \int_{|x| < \rho} \frac{1}{2\pi t} e^{-\frac{|x|^2}{2t}} d^2 \vec{x} = \frac{2\pi}{2\pi t} \int_0^\rho r e^{-\frac{r^2}{2t}} dr = \int_0^{\frac{\rho^2}{2t}} e^{-u} du = 1 - e^{-\frac{\rho^2}{2t}}.$$

14. Compute

$$\begin{aligned} \mathbb{E}_x \left(\int_{[0, \infty]} \chi_K(B_t) dt \right) &= \int_{[0, \infty]} \mathbb{P}(B_t \in K) dt \\ &= \int_{[0, \infty]} \left(\int_K \frac{1}{(2\pi t)^{n/2}} e^{-\frac{|\vec{x}-\vec{y}|^2}{2t}} d^n \vec{x} \right) dt \\ &\leq \int_{[0, \infty]} \left\| \frac{1}{(2\pi t)^{n/2}} e^{-\frac{|\vec{x}-\vec{y}|^2}{2t}} \right\|_\infty \mu(K) dt \\ &= 0 \end{aligned}$$

and deduce that the expected total time spent in K is 0.

15. Note that $UU^T = I$, whence $|\det U| = 1$ and the probability measures are identical by change of variables. It follows that both are Brownian.

16. Let $W_t = \frac{1}{c} B_{c^2 t}$. We have $W_0 = B_0 = 0$ and that W_t is absolutely continuous as a scaling of absolutely continuous B_t . Finally,

$$\begin{aligned} \mathbb{P}_0(W_t \in U) &= \mathbb{P}_0(B_{c^2 t} \in cU) \\ &= \int_{cU} p(c^2 t, 0, y) dy \\ &= \int_{cU} \frac{1}{c} p(t, 0, y/c) dy \\ &= \int_U \frac{1}{c} p(t, 0, y')(cdy') \\ &= \mathbb{P}_0(B_t \in U), \end{aligned}$$

and so W_t is also a Brownian motion.

17. Let $X_t(\cdot)$ be a continuous stochastic process.

(a) Recall that $\mathbb{E}(B_t) = 0$, $\mathbb{E}(B_t^2) = t$ and $\mathbb{E}(B_t^4) = 3t^2$. Then

$$\begin{aligned}
 \mathbb{E} \left(\left(\sum_k (\Delta B_k^2 - \Delta t_k) \right)^2 \right) &= \mathbb{E} \left(\left(\sum_k (\Delta B_k^2 - \Delta t_k)^2 \right) \right) \\
 &= \sum_k \left(\mathbb{E}(\Delta B_k^4) - 2\Delta t_k \mathbb{E}(\Delta B_k^2) + \Delta t_k^2 \right) \\
 &= \sum_k (3\Delta t_k^2 - 2\Delta t_k^2 + \Delta t_k^2) \\
 &= 2 \sum_k \Delta t_k^2.
 \end{aligned}$$

So $\langle B, B \rangle_t^{(2)}(w) = t$.

(b) Note that the Brownian motion has positive quadratic variation t on $[0, t]$. So

$$\langle B, B \rangle_t^{(1)}(w) \geq \lim_{\|\Delta B_k\| \rightarrow 0^+} \frac{\langle B, B \rangle_t^{(2)}(w)}{\|\Delta B_k\|} = \infty.$$

Chapter 3

Itô Integrals

1. Compute

$$\begin{aligned}
 \int_0^t s dB_s &= \lim_{n \rightarrow \infty} \sum_{j=0}^{\frac{[nt]}{t}-1} \frac{jt}{n} (B_{\frac{(j+1)t}{n}} - B_{\frac{jt}{n}}) \\
 &= \lim_{n \rightarrow \infty} \frac{[nt]}{n} B_{\frac{[nt]}{n}} - \lim_{n \rightarrow \infty} \frac{t}{n} \sum_{j=0}^{\frac{[nt]}{t}-1} B_{\frac{jt}{n}} + \lim_{n \rightarrow \infty} \frac{t}{n} (B_0 - B_{\frac{[nt]}{n}}) \\
 &= tB_t - \int_0^t B_s ds.
 \end{aligned}$$

2. Compute

$$\begin{aligned}
 \int_0^t B_s^2 dB_s &= \lim_{n \rightarrow \infty} \sum_{j=0}^{\frac{[nt]}{t}-1} B_{\frac{jt}{n}}^2 (B_{\frac{(j+1)t}{n}} - B_{\frac{jt}{n}}) \\
 &= \lim_{n \rightarrow \infty} \sum_{j=0}^{\frac{[nt]}{t}-1} \left(\frac{1}{3} B_{\frac{(j+1)t}{n}}^3 - \frac{1}{3} B_{\frac{jt}{n}}^3 - B_{\frac{jt}{n}} (B_{\frac{(j+1)t}{n}} - B_{\frac{jt}{n}})^2 - \frac{1}{3} (B_{\frac{(j+1)t}{n}} - B_{\frac{jt}{n}})^3 \right) \\
 &= \frac{1}{3} B_t^3 - \lim_{n \rightarrow \infty} \left(\sum_{j=0}^{\frac{[nt]}{t}-1} \frac{t}{n} B_{\frac{jt}{n}} + \mathcal{O}(t^2/n) \right) \\
 &= \frac{1}{3} B_t^3 - \int_0^t B_s ds.
 \end{aligned}$$

3. Let $\{\mathcal{N}_t\}$ be some filtration and let $\{\mathcal{H}_t^{(X)}\}$ be the filtration of process X_t .

(a) Compute

$$\mathbb{E}(X_t | \mathcal{H}_s^{(X)}) = \mathbb{E}(\mathbb{E}(X_t | \mathcal{N}_s) | \mathcal{H}_s^{(X)}) = \mathbb{E}(H_s | \mathcal{H}_s^{(X)}) = H_s.$$

(b) Compute

$$\mathbb{E}(X_t) = \mathbb{E}(\mathbb{E}(X_t | H_0^{(X)})) = \mathbb{E}(X_0).$$

(c) Let $Y \sim \text{Bernoulli}(0.5)$ and fix $X_0 = 2Y - 1$. Then $X_t = t \cdot \text{sgn}(X_0)$ satisfies $\mathbb{E}(X_t) = \mathbb{E}(X_0) = 0$, but $\mathbb{E}(X_t | \mathcal{F}_s) = t \cdot \text{sgn}(X_0) \neq s \cdot \text{sgn}(X_0)$.

4. Compute

$$\mathbb{E}(B_t + 4t | \mathcal{F}_s) = B_s + 4t \neq B_s + 4s$$

$$\mathbb{E}(B_t^2 | \mathcal{F}_s) = \mathbb{E}((B_t - B_s)^2 + 2B_s(B_t - B_s) + B_s^2 | \mathcal{F}_s) = B_s^2 + t - s \neq B_s^2$$

$$\mathbb{E}\left(t^2 B_t - 2 \int_0^t u B_u du | \mathcal{F}_s\right) = t^2 B_s - 2 \int_0^s u B_u du - 2 \int_s^t u B_s du = s^2 B_s - 2 \int_0^s u B_u du$$

$$\mathbb{E}(B_t^{(1)} B_t^{(2)} | \mathcal{F}_s) = \mathbb{E}(B_t^{(1)} | \mathcal{F}_s) \mathbb{E}(B_t^{(2)} | \mathcal{F}_s) = B_s^{(1)} B_s^{(2)},$$

and deduce that only the last two are martingales.

5. Verify $\mathbb{E}(|B_t^2 - t|) \leq \mathbb{E}(B_t^2) + t = 2t < \infty$ and compute

$$\mathbb{E}(B_t^2 - t | \mathcal{F}_s) = \mathbb{E}((B_t - B_s)^2 + 2B_s(B_t - B_s) + B_s^2 - t | \mathcal{F}_s) = B_s^2 + t - s - t = B_s^2 - s.$$

to deduce that $X_t := B_t^2 - t$ is a martingale.

6. Verify $\mathbb{E}(|B_t^3 - 3tB_t|) \leq \sqrt{\mathbb{E}(B_t^2)}(\sqrt{\mathbb{E}(B_t^4)} + 3t) = (3 + \sqrt{3})t^{3/2} < \infty$ and compute

$$\begin{aligned} \mathbb{E}(B_t^3 - 3tB_t | \mathcal{F}_s) &= \mathbb{E}((B_t - B_s)^3 + 3B_s(B_t - B_s)^2 + 3B_s^2(B_t - B_s) + B_s^3 - 3tB_s | \mathcal{F}_s) \\ &= 3B_s(t - s) + B_s^3 - 3tB_s \\ &= B_s^3 - 3sB_s \end{aligned}$$

to deduce that $Y_t := B_t^3 - 3tB_t$ is a martingale.

7. In this question, the formula for Itô iterated integrals is derived.

(a) Note that $\{0 \leq u_1 \leq \dots \leq u_n\}$ is Borel measurable and $\chi_{0 \leq u_1 \leq \dots \leq u_n}$ is \mathcal{F}_t -adapted. Finally

$$\mathbb{E}\left(\int_0^T f(t_1, \dots, t_n, \omega)^2 dt_1 \dots dt_n\right) \leq T^n < \infty.$$

(b) For $n \in \{1, 2, 3\}$

$$1! \int_0^t dB_u = B_t = t^{1/2} H_1\left(\frac{B_t}{\sqrt{t}}\right)$$

$$2! \int_0^t \int_0^v dB_u dB_v = 2 \int_0^t B_v dB_v = B_t^2 - t = t H_2\left(\frac{B_t}{\sqrt{t}}\right)$$

$$3! \int_0^t \int_0^v \int_0^w dB_u dB_v dB_w = 3 \int_0^t (B_w^2 - w) dB_w = B_t^3 - 3tB_t = t^{3/2} H_3\left(\frac{B_t}{\sqrt{t}}\right).$$

(c) Deduce that $d(B_t^3 - 3tB_t) = 3(B_t^2 - t)dB_t$ and so $Y_t := B_t^3 - 3tB_t$ is a martingale.

8. There exists continuous martingale M_t iff there exists $Y \in L^1$ such that $M_t = \mathbb{E}(Y | \mathcal{F}_t)$.

(a) Verify that $\mathbb{E}(|\mathbb{E}(Y | \mathcal{F}_t)|) \leq \mathbb{E}(\mathbb{E}(|Y| | \mathcal{F}_t)) = \mathbb{E}(|Y|) < \infty$ and

$$\mathbb{E}(M_t | \mathcal{F}_s) = \mathbb{E}(\mathbb{E}(Y | \mathcal{F}_t) | \mathcal{F}_s) = \mathbb{E}(Y | \mathcal{F}_s) = M_s.$$

(b) If M_t is a continuous martingale such that $\sup_{t \geq 0} \mathbb{E}(|X|^p) < \infty$ for $p \in (1, \infty)$, then $\exists M$ such that $\|M_t - M\|_{L^1} \rightarrow 0$ as $t \rightarrow \infty$. So let $Y = M$ and

$$\begin{aligned} \lim_{s \rightarrow \infty} \int_{\Omega_s} |M_s - \mathbb{E}(M | \mathcal{F}_s)| d\mathbb{P} &= \lim_{s \rightarrow \infty} \int_{\Omega_s} |\mathbb{E}(M_s - M | \mathcal{F}_s)| d\mathbb{P} \\ &\leq \lim_{s \rightarrow \infty} \int_{\Omega_s} \mathbb{E}(|M_s - M| | \mathcal{F}_s) d\mathbb{P} \\ &= \lim_{s \rightarrow \infty} \int_{\Omega_s} |M_s - M| d\mathbb{P} \\ &= 0. \end{aligned}$$

9. Compute

$$\begin{aligned} \int_0^T B_t \circ dB_t &= \lim_{n \rightarrow \infty} \sum_{j=0}^{\lfloor \frac{nt}{t} \rfloor - 1} \frac{1}{2} (B_{\frac{jt}{n}} + B_{\frac{(j+1)t}{n}}) (B_{\frac{(j+1)t}{n}} - B_{\frac{jt}{n}}) \\ &= \lim_{n \rightarrow \infty} \sum_{j=0}^{\lfloor \frac{nt}{t} \rfloor - 1} B_{\frac{jt}{n}} (B_{\frac{(j+1)t}{n}} - B_{\frac{jt}{n}}) + \lim_{n \rightarrow \infty} \sum_{j=0}^{\lfloor \frac{nt}{t} \rfloor - 1} \frac{1}{2} (B_{\frac{(j+1)t}{n}} - B_{\frac{jt}{n}})^2 \\ &= \frac{1}{2} B_t^2 - \frac{t}{2} + \frac{t}{2} \\ &= \frac{1}{2} B_t^2. \end{aligned}$$

10. If $f(t, \omega)$ varies smoothly in t , then the Itô and Stratonovich integrals coincide. Compute

$$\int_0^T f(t, \omega) \circ dB_t = \int_0^T f(t, \omega) dB_t + \frac{1}{2} \langle f(t, \omega), B_t \rangle^{(2)}$$

and

$$\begin{aligned} \mathbb{E}(\langle f(t, \omega), B_t \rangle^{(2)})^2 &\leq \mathbb{E}(\langle B_t, B_t \rangle^{(2)}) \mathbb{E}(\langle f(t, \omega), f(t, \omega) \rangle^{(2)}) \\ &\leq T \lim_{\|\Delta t_k\| \rightarrow 0^+} \sup_{|\Delta t_k|} \frac{T}{|\Delta t_k|} (K |\Delta t_k|^{1+\varepsilon}) \\ &= KT^2 \lim_{\|\Delta t_k\| \rightarrow 0^+} \|\Delta t_k\|^\varepsilon \\ &= 0. \end{aligned}$$

11. Define white noise $W_t^{(N)} = \max\{-N, \min\{W_t, N\}\}$. Since W_t and W_s are independent and identically distributed, it follows that $W_t^{(N)}$ and $W_s^{(N)}$ are as well. If W_t is continuous, then since $|W_t^{(N)}| \leq N$ and by bounded convergence

$$\lim_{t \rightarrow s} 2\mathbb{E}(W_t^{(N)})^2 = \lim_{t \rightarrow s} \mathbb{E}(|W_t^{(N)} - W_s^{(N)}|^2) = 0.$$

But then $W_t \stackrel{\text{a.s.}}{=} 0$, which is a contradiction.

12. Let $\circ dB_t$ denote the Stratonovich differential.

(i) Since $\alpha X_t \circ dB_t = \frac{\alpha^2}{2} X_t dt + \alpha X_t dB_t$,

$$dX_t = \left(\gamma + \frac{\alpha^2}{2}\right) X_t dt + \alpha X_t dB_t.$$

Since $(t^2 + \cos(X_t)) \circ dB_t = -\frac{\sin(X_t)}{2}(t^2 + \cos(X_t)) dt + (t^2 + \cos(X_t)) dB_t$,

$$dX_t = \frac{\sin(X_t)}{2}(\cos(X_t) - t^2) dt + (t^2 + \cos(X_t)) dB_t.$$

(ii) Since $\alpha X_t dB_t = \alpha X_t \circ dB_t - \frac{\alpha^2}{2} X_t dt$,

$$dX_t = \left(r - \frac{\alpha^2}{2}\right) X_t dt + \alpha X_t \circ dB_t.$$

Since $X_t^2 dB_t = X_t^2 \circ dB_t - X_t^3 dt$,

$$dX_t = (2e^{-X_t} - X_t^3) dt + X_t^2 \circ dB_t.$$

13. Let X_t be continuous in mean square. Calculate

(a) $\lim_{s \rightarrow t} \mathbb{E}[(B_t - B_s)^2] = \lim_{s \rightarrow t} \mathbb{E}[(B_{t-s})^2] = \lim_{s \rightarrow t} (t - s) = 0$

(b) $\lim_{s \rightarrow t} \mathbb{E}[(f(B_t) - f(B_s))^2] \leq \lim_{s \rightarrow t} C^2 \mathbb{E}[(B_t - B_s)^2] = 0$

(c) and finally by Itô isometry,

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E} \left[\left(\int_S^T (X_s - \phi_n(s)) dB_s \right)^2 \right] &= \lim_{n \rightarrow \infty} \mathbb{E} \left[\int_S^T (X_s - \phi_n(s))^2 ds \right] \\ &= \lim_{n \rightarrow \infty} \mathbb{E} \left[\sum_j \int_{t_j^{(n)}}^{t_j^{(n+1)}} (X_t - X_{t_j^{(n)}})^2 dt \right] \\ &\leq (T - S) \lim_{n \rightarrow \infty} \sup_{1 \leq j \leq n} \mathbb{E}[(X_t - X_{t_j^{(n)}})^2] \\ &= 0. \end{aligned}$$

14. Problem 14

15. Suppose $C + \int_S^T f(t, \omega) dB_t(\omega) = D + \int_S^T g(t, \omega) dB_t(\omega)$. Then we have that

$$C - D = \mathbb{E}[C - D] = \mathbb{E} \left[\int_S^T g(t, \omega) dB_t(\omega) - \int_S^T f(t, \omega) dB_t(\omega) \right] = 0 \implies C = D,$$

and by Itô isometry,

$$0 = \mathbb{E} \left[\left(\int_S^T g(t, \omega) dB_t(\omega) - \int_S^T f(t, \omega) dB_t(\omega) \right)^2 \right] = \int_S^T \mathbb{E}[(g(t, \omega) - f(t, \omega))^2] ds,$$

whence $g(t, \omega) = f(t, \omega)$ almost surely for $(t, \omega) \in [S, T] \times \Omega$.

16. By Jensen's inequality, $\mathbb{E}[\mathbb{E}[X|\mathcal{H}]^2] \leq \mathbb{E}[\mathbb{E}[X^2|\mathcal{H}]] = \mathbb{E}[X^2]$.

17. Let \mathcal{G} be a finite σ -algebra with partition $\Omega = \bigsqcup_{i=1}^n G_i$.

(a) Note that $\mathbb{E}[X|\mathcal{G}](\omega) = \sum_{i=1}^n c_i \mathbb{1}_{G_i}(\omega) = c_i$ on G_i .

(b) Show that

$$\int_{G_i} \left(\frac{\int_{G_i} X d\mathbb{P}}{\mathbb{P}(G_i)} \right) d\mathbb{P} = \frac{\int_{G_i} X d\mathbb{P}}{\mathbb{P}(G_i)} \int_{G_i} 1 d\mathbb{P} = \int_{G_i} X d\mathbb{P}, \forall i \in \{1, \dots, n\}.$$

(c) By part (b), $c_i = \frac{\int_{G_i} X d\mathbb{P}}{\mathbb{P}(G_i)}$. Show for $\omega \in G_i$ that

$$\begin{aligned} \mathbb{E}[X|\mathcal{G}](\omega) &= \sum_{i=1}^n \frac{\int_{G_i} X d\mathbb{P}}{\mathbb{P}(G_i)} \mathbb{1}_{G_i}(\omega) \\ &= \frac{\int_{G_i} X d\mathbb{P}}{\mathbb{P}(G_i)} \\ &= \frac{\sum_{k=1}^m a_k \mathbb{P}(X = a_k, \omega \in G_i)}{\mathbb{P}(G_i)} \\ &= \sum_{k=1}^m a_k \mathbb{P}(X = a_k | G_i). \end{aligned}$$

Chapter 4

The Itô Formula

1. Compute

(a) $dX_t = d(B_t^2) = 2B_t dB_t + d[B, B]_t = 2B_t dB_t + dt$

(b) $dX_t = d(2 + t + e^{B_t}) = (1 + \frac{1}{2}e^{B_t}) dt + e^{B_t} dB_t$

(c) $dX_t = d\left((B_t^{(1)})^2 + (B_t^{(2)})^2\right) = 2B_t^{(1)} dB_t^{(1)} + 2B_t^{(2)} dB_t^{(2)} + 2 dt$

(d) $dX_t = d((t_0 + t, B_t)) = (dt, dB_t)$

(e) and finally

$$\begin{aligned} dX_t &= d((B_t^{(1)} + B_t^{(2)} + B_t^{(3)}, (B_t^{(2)})^2 - B_t^{(1)} B_t^{(3)})) \\ &= (dB_t^{(1)} + dB_t^{(2)} + dB_t^{(3)}, 2B_t^{(2)} dB_t^{(2)} + dt - B_t^{(3)} dB_t^{(1)} - B_t^{(1)} dB_t^{(3)}). \end{aligned}$$

2. Using Itô's Lemma, differentiate

$$d\left(\frac{1}{3}B_t^3 - \int_0^t B_s ds\right) = B_t^2 dB_t + B_t d[B, B]_t - B_t dt = B_t^2 dB_t$$

and deduce that

$$\int_0^t B_s^2 dB_s = \frac{1}{3}B_t^3 - \int_0^t B_s ds.$$

3. Let X_t and Y_t be Itô processes. Then, letting $f(t, x, y) = xy$ and by Itô's formula

$$\begin{aligned} d(X_t Y_t) &= f_t(t, X_t, Y_t) dt + f_x(t, X_t, Y_t) dX_t + f_y(t, X_t, Y_t) dY_t \\ &\quad + \frac{1}{2} f_{xx}(t, X_t, Y_t) d[X, X]_t + f_{xy}(t, X_t, Y_t) d[X, Y]_t + \frac{1}{2} f_{yy}(t, X_t, Y_t) d[Y, Y]_t \\ &= Y_t dX_t + X_t dY_t + d[X, Y]_t \end{aligned}$$

and deduce the integration of parts formula

$$\begin{aligned}\int_0^t X_s dY_s &= \int_0^t (d(X_s Y_s) - Y_s dX_s - d[X, Y]_s) \\ &= X_t Y_t - X_0 Y_0 - \int_0^t Y_s dX_s - \int_0^t d[X, Y]_s.\end{aligned}$$

4. Let $Z_t = \exp \left(\int_0^t \langle \theta(s, \omega), dB_s \rangle - \frac{1}{2} |\theta(s, \omega)|^2 ds \right)$.

(a) Then, letting $Z_t = e^{Y_t}$ and by Itô's formula,

$$\begin{aligned}dZ_t &= e^{Y_t} dY_t + \frac{1}{2} e^{Y_t} d[Y, Y]_t \\ &= Z_t \left(\langle \theta(t, \omega), dB_t \rangle - \frac{1}{2} |\theta(t, \omega)|^2 dt + \frac{1}{2} \sum_{i,j=1}^n [\theta_i(s, \omega) dB^{(i)}, \theta_j(s, \omega) dB^{(j)}]_s \right) \\ &= Z_t \langle \theta(t, \omega), dB_t \rangle.\end{aligned}$$

(b) It suffices to check that

$$\begin{aligned}[\mathbb{E}(|Z_t|)]^2 &= \left[\mathbb{E} \left(\left| \int_0^t dZ_s \right| \right) \right]^2 \\ &= \left[\mathbb{E} \left(\left| \int_0^t Z_s \langle \theta(s, \omega), dB_s \rangle \right| \right) \right]^2 \\ &\leq \mathbb{E} \left(\int_0^t \sum_{i=1}^n |Z_s \theta_i(s, \omega)| dB_s^{(i)} \right)^2 \\ &= \mathbb{E} \left(\sum_{i,j=1}^n \int_0^t |Z_s \theta_i(s, \omega)| |Z_s \theta_j(s, \omega)| d[B^{(i)}, B^{(j)}]_s \right) \\ &= \sum_{i=1}^n \mathbb{E} \left(\int_0^t |Z_s \theta_i(s, \omega)|^2 ds \right) \\ &< \infty.\end{aligned}$$

5. Let $\beta_k(t) = \mathbb{E}(B_t^k)$. Then, by Itô's lemma,

$$dB_t^k = k B_t^{k-1} dB_t + \frac{1}{2} k(k-1) B_t^{k-2} dt$$

and so

$$\beta_k(t) = \mathbb{E}(B_t^k) = \mathbb{E} \left(\int_0^t dB_s^k \right) = \int_0^t \mathbb{E} \left(\frac{1}{2} k(k-1) B_s^{k-2} \right) ds = \frac{1}{2} k(k-1) \int_0^t \beta_{k-2}(s) ds.$$

Deduce that $\beta_4(t) = 6 \int_0^t \beta_2(s) ds = 6 \cdot \frac{t^2}{2} = 3t^2$ and $\beta_6(t) = 15 \int_0^t 3s^2 ds = 15t^3$.

6. Problem 6

(a) Define $X_t = e^{ct+\alpha B_t}$. Then

$$\begin{aligned} dX_t &= ce^{ct+\alpha B_t} dt + \alpha e^{ct+\alpha B_t} dB_t + \frac{1}{2}\alpha^2 e^{ct+\alpha B_t} d[B, B]_t \\ &= X_t \left(\left(c + \frac{\alpha^2}{2} \right) dt + \alpha dB_t \right). \end{aligned}$$

(b) Define $X_t = e^{ct+\sum_{j=1}^n \alpha_j B_t^{(j)}}$. Then

$$\begin{aligned} dX_t &= X_t \left(c dt + \sum_{j=1}^n \alpha_j dB_t^{(j)} + \frac{1}{2} \sum_{i,j=1}^n \alpha_i \alpha_j d[B^{(i)}, B^{(j)}]_t \right) \\ &= X_t \left(\left(c + \frac{1}{2} \sum_{j=1}^n \alpha_j^2 \right) dt + \sum_{j=1}^n \alpha_j dB_t^{(j)} \right). \end{aligned}$$

7. Let X_t solve $dX_t = v(t, \omega) dB_t$.

(a) Note that B_t is a martingale while B_t^2 is not.

(b) Define $M_t = X_t^2 - \int_0^t v(s, \omega)^2 ds$. Then

$$\begin{aligned} dM_t &= 2X_t dX_t + [dX, dX]_t - v(t, \omega)^2 dt \\ &= 2X_t v(t, \omega) dB_t + (v(t, \omega)^2 - v(t, \omega)^2) dt \\ &= 2X_t v(t, \omega) dB_t. \end{aligned}$$

Moreover,

$$\begin{aligned} \mathbb{E}(|M_t|) &\leq \mathbb{E}(X_t^2) + \mathbb{E} \left(\int_0^t v(s, \omega)^2 ds \right) \\ &= \mathbb{E} \left(\int_0^t v(s, \omega) dB_s \right)^2 + \mathbb{E} \left(\int_0^t v(s, \omega)^2 ds \right) \\ &= 2\mathbb{E} \left(\int_0^t v(s, \omega)^2 ds \right) \\ &< \infty. \end{aligned}$$

8. Problem 8

(a) Let $f(x^{(1)}, \dots, x^{(n)})$ be a function of class C^2 . Then

$$\begin{aligned} d(f(B_t)) &= \sum_{i=1}^n \partial_i f(B_t) dB_t^{(i)} + \frac{1}{2} \sum_{i,j=1}^n \partial_{ij}^2 f(B_t) d[B^{(i)}, B^{(j)}]_t \\ &= \langle \nabla f(B_t), dB_t \rangle + \frac{1}{2} \Delta f(B_t) dt \end{aligned}$$

and so

$$f(B_t) - f(B_0) = \int_0^t d(f(B_s)) = \int_0^t \langle \nabla f(B_s), dB_s \rangle + \frac{1}{2} \int_0^t \Delta f(B_s) ds.$$

- (b) Assume that g is of class C^1 everywhere, as well as C^2 and uniformly bounded outside of finitely many points with $|g''(z)| \leq M$ for $z \notin \{z_1, \dots, z_k\}$. Then the set of functions $\{f\}$ of class C^2 uniformly bounded with $|f''(z)| \leq M$ are C^k -dense. So we can extract a sequence $\{f_k\}$ such that $f_k \rightrightarrows g$, $f'_k \rightrightarrows g'$ as well as $f''_k \rightarrow g''$ and $|f''_k| \leq M$ on $\mathbb{R} \setminus \{z_1, \dots, z_k\}$. So

$$\begin{aligned} & \lim_{k \rightarrow \infty} \left| (f_k - g)(B_t) + (f_k - g)(0) + \int_0^t (f'_k - g') dB_s + \frac{1}{2} \int_0^t (f''_k - g'') ds \right| \\ & \leq \lim_{k \rightarrow \infty} |(f_k - g)(B_t)| + |(f_k - g)(0)| + t \|f'_k - g'\|_\infty + \frac{1}{2} \int_0^t |f''_k - g''| ds \\ & = 0, \end{aligned}$$

where the last term vanishes by bounded convergence.

9. Clearly

$$\int_0^t v \frac{\partial g_n}{\partial x}(s, X_s) \chi_{s \leq \tau_n} dB_s = \int_0^{t \wedge \tau_n} v \frac{\partial g}{\partial x}(s, X_s) dB_s$$

and the result follows by Itô's lemma where $dX_t = u dt + v dB_t$. Since $\mathbb{E}(|X_t|) < \infty$, it follows that $\lim_{n \rightarrow \infty} \mathbb{P}(\tau_n > t) = \lim_{n \rightarrow \infty} \mathbb{P}(X_t < n) = 1$ and so the identity holds almost surely.

10. Problem 10

- (a) Substitute $u \equiv 0$ and $v \equiv 1$ here. Then as $g'_\varepsilon(x) = \frac{1}{\varepsilon} \chi_{|x| < \varepsilon}(x)$

$$\frac{1}{2} \int_0^t \frac{d^2 g_\varepsilon}{dx^2}(B_s) ds = \frac{1}{2\varepsilon} \int_0^t \chi_{|B_s| < \varepsilon} ds = \frac{1}{2\varepsilon} |\{s \in [0, t] \mid |B_s| < \varepsilon\}|.$$

- (b) Differentiate to get

$$\int_0^t g'_\varepsilon(B_s) \chi_{|B_s| < \varepsilon} dB_s = \int_0^t \frac{B_s}{\varepsilon} \chi_{|B_s| < \varepsilon} dB_s,$$

and apply Itô isometry to get

$$\lim_{\varepsilon \rightarrow 0^+} \mathbb{E} \left(\int_0^t \frac{B_s}{\varepsilon} \chi_{|B_s| < \varepsilon} dB_s \right)^2 = \lim_{\varepsilon \rightarrow 0^+} \mathbb{E} \left(\int_0^t \frac{B_s^2}{\varepsilon^2} \chi_{|B_s| < \varepsilon} ds \right) \leq \lim_{\varepsilon \rightarrow 0^+} \int_0^t \mathbb{P}(|B_s| < \varepsilon) ds = 0.$$

(c) As $\varepsilon \rightarrow 0$ for $g(x) = x$,

$$\begin{aligned} |B_t| &= |B_0| + \lim_{\varepsilon \rightarrow 0^+} \int_0^t \operatorname{sgn}(B_s) \chi_{|B_s| \geq \varepsilon} ds + \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\varepsilon} |\{s \in [0, t] \mid |B_s| < \varepsilon\}| \\ &= |B_0| + \int_0^t \operatorname{sgn}(B_s) ds + L_t. \end{aligned}$$

11. Problem 11

(a) Let $X_t = e^{t/2} \cos(B_t)$. Then

$$dX_t = \frac{1}{2} e^{t/2} \cos(B_t) dt - e^{t/2} \sin(B_t) dB_t + \frac{1}{2} (-e^{t/2} \cos(B_t)) d[B, B]_t = -e^{t/2} \sin(B_t) dB_t.$$

(b) Let $X_t = e^{t/2} \sin(B_t)$. Then

$$dX_t = \frac{1}{2} e^{t/2} \sin(B_t) dt + e^{t/2} \cos(B_t) dB_t + \frac{1}{2} (-e^{t/2} \sin(B_t)) d[B, B]_t = e^{t/2} \cos(B_t) dB_t.$$

(c) Let $X_t = (B_t + t)e^{-B_t - t/2}$. Then

$$\begin{aligned} dX_t &= e^{-B_t - t/2} d(B_t + t) + (B_t + t) d(e^{-B_t - t/2}) + d[B_t + t, e^{-B_t - t/2}] \\ &= e^{-B_t - t/2} (dt + dB_t) - \frac{1}{2} X_t dt - X_t dB_t - e^{-B_t - t/2} dt + \frac{1}{2} (B_t + t) e^{-B_t - t/2} dt \\ &= e^{-B_t - t/2} (1 - t - B_t) dB_t. \end{aligned}$$

12. The given condition implies $\mathbb{E}(|X_t|) < \infty$. So X_t is a martingale if and only if $\mathbb{E}(X_t | \mathcal{F}_s) = X_s$. Then

$$\mathbb{E}\left(\int_s^t u(r, \omega) dr \mid \mathcal{F}_s\right) = \mathbb{E}(X_t - X_s \mid \mathcal{F}_s) = 0.$$

Moreover by dominated convergence

$$\mathbb{E}(u(t, \omega) dr \mid \mathcal{F}_s) = \mathbb{E}\left(\frac{d}{ds} \int_s^t u(r, \omega) dr \mid \mathcal{F}_s\right) = 0.$$

Then

$$u(t, \omega) = \mathbb{E}(u(t, \omega) \mid \mathcal{F}_t) = \lim_{s \rightarrow t^-} \mathbb{E}(u(t, \omega) \mid \mathcal{F}_s) = 0.$$

13. Let $dX_t = u(t, \omega) dt + dB_t$ where $u(t, \omega) \in \mathcal{V}([0, T])$. Then $Y_t = X_t M_t$ is a martingale, where

$$M_t = \exp\left(-\int_0^t u(r, \omega) dB_r - \frac{1}{2} \int_0^t u^2(r, \omega) dr\right)$$

since $\mathbb{E}(|M_t|) < \infty$ (see question 4b), $\mathbb{E}(|X_t|) \leq \sqrt{t} \left(\sqrt{\int_0^t u^2(r, \omega) dr} + 1 \right) < \infty$ and

$$\begin{aligned} d(X_t M_t) &= M_t dX_t + X_t dM_t + d[X, M]_t \\ &= M_t(u(t, \omega) dt + dB_t) + M_t X_t(-u(t, \omega) dB_t - \frac{1}{2}u^2(t, \omega) dt) \\ &\quad - M_t u(t, \omega) dt + \frac{1}{2}M_t X_t u^2(t, \omega) dt \\ &= M_t(1 - u(t, \omega)X_t) dB_t. \end{aligned}$$

14. Problem 14

(a) Compute $dF_t = dB_t$, $\mathbb{E}(F_T) = 0$ and

$$dF_t - d\mathbb{E}(F_t) = 1 dB_t \implies f(t, \omega) = 1.$$

(b) Compute $dF_t = B_t dt$, $\mathbb{E}(F_T) = 0$ and

$$dF_t - d\mathbb{E}(F_t) = B_t dt = d(TB_T) - t dB_t = (T - t) dB_t \implies f(t, \omega) = T - t.$$

(c) Compute $dF_t = 2B_t dB_t + dt$, $\mathbb{E}(F_T) = T$ and

$$dF_t - d\mathbb{E}(F_t) = 2B_t dB_t + 1 dt - 1 dt = 2B_t dB_t \implies f(t, \omega) = 2B_t.$$

(d) Compute $dF_t = 3B_t^2 dB_t + 3B_t dt$, $\mathbb{E}(F_T) = 0$ and

$$\begin{aligned} dF_t - d\mathbb{E}(F_t) &= 3B_t^2 dB_t + 3B_t dt \\ &= 3B_t^2 + 3(T - t) dB_t \implies f(t, \omega) = 3B_t^2 + 3T - 3t. \end{aligned}$$

(e) Recall that $e^{B_t - t/2}$ is a martingale and compute

$$d(e^{B_t - t/2}) = e^{B_t - t/2} dB_t.$$

Deduce that

$$e^{B_T} = e^{T/2} \left(1 + \int_0^T e^{B_t - t/2} dB_t \right) \implies f(t, \omega) = e^{B_t + (T-t)/2}.$$

(f) Find martingale $e^{t/2} \sin(B_t)$ and compute

$$d(e^{t/2} \sin(B_t)) = e^{t/2} \cos(B_t) dB_t$$

Deduce that

$$\sin(B_T) = e^{-T/2} \int_0^T e^{t/2} \cos(B_t) dB_t \implies f(t, \omega) = e^{-(T-t)/2} \cos(B_t).$$

15. Define $X_t = (x^{1/3} + \frac{1}{3}B_t)^3$. Then

$$\begin{aligned} dX_t &= 3X_t^{2/3} d(x^{1/3} + \frac{1}{3}B_t) + 3X_t^{1/3} d \left[x^{1/3} + \frac{1}{3}B_t, x^{1/3} + \frac{1}{3}B_t \right] \\ &= X_t^{2/3} dB_t + \frac{1}{3}X_t^{1/3} dt. \end{aligned}$$

Chapter 5

Stochastic Differential Equations

1. Question 1

(a) Compute $dX_t = d(e^{B_t}) = e^{B_t} dB_t + \frac{1}{2} e^{B_t} d[B, B]_t = \frac{1}{2} X_t dt + X_t dB_t$.

(b) Compute

$$dX_t = d\left(\frac{B_t}{1+t}\right) = \frac{1}{1+t} dB_t - \frac{B_t}{(1+t)^2} dt = \frac{1}{1+t} dB_t - \frac{1}{1+t} X_t dt.$$

(c) Compute $dX_t = d(\sin(B_t)) = \cos(B_t) dB_t - \frac{1}{2} \sin(B_t) dt = \cos(B_t) dB_t - \frac{1}{2} X_t dt$.

(d) Compute $dX_t^{(1)} = dt$ and

$$dX_t^{(2)} = d(e^t B_t) = e^t dB_t + e^t B_t dt = e^t dB_t + X_t^{(2)} dt.$$

(e) Compute differentials

$$d(\cosh(B_t)) = \sinh(B_t) dB_t + \frac{1}{2} \cosh(B_t) dt$$

and

$$d(\sinh(B_t)) = \cosh(B_t) dB_t + \frac{1}{2} \sinh(B_t) dt.$$

Deduce that

$$\begin{pmatrix} dX_t^{(1)} \\ dX_t^{(2)} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} X_t^{(1)} \\ X_t^{(2)} \end{pmatrix} dt + \begin{pmatrix} X_t^{(2)} \\ X_t^{(1)} \end{pmatrix} dB_t.$$

2. Let $X_t^{(1)} = a \cos(B_t)$ and $X_t^{(2)} = b \sin(B_t)$. Then

$$dX_t^{(1)} = -a \sin(B_t) dB_t - \frac{a}{2} \cos(B_t) dt = -\frac{1}{2} X_t^{(1)} dt - \frac{a}{b} X_t^{(2)} dB_t$$

and

$$dX_t^{(2)} = b \cos(B_t) dB_t - \frac{b}{2} \sin(B_t) dt = -\frac{1}{2} X_t^{(2)} dt + \frac{b}{a} X_t^{(1)} dB_t.$$

3. The solution is given by

$$X_t = X_0 \exp \left(\left(r - \frac{1}{2} \sum_{k=1}^n \alpha_k^2 \right) t + \sum_{k=1}^n \alpha_k dB_k \right).$$

4. Question 4

(a) The solution to $dX_t^{(1)} = dt + dB_t^{(1)}$ is $X_t^{(1)} = X_0^{(1)} + t + B_t^{(1)}$ and

$$dX_t^{(2)} = X_t^{(1)} dB_t^{(2)} = (X_0^{(1)} + t + B_t^{(1)}) dB_t^{(2)}$$

is

$$X_t^{(2)} = X_0^{(2)} + X_0^{(1)} B_t^{(2)} + \int_0^t (s + B_s^{(1)}) dB_s^{(2)}.$$

(b) Using integrating factors, solve $dX_t = X_t dt + dB_t$ for

$$e^{-t} X_t - X_0 = \int_0^t e^{-s} dB_s$$

and deduce that the solution X_t is

$$X_t = e^t X_0 + \int_0^t e^{t-s} dB_s.$$

(c) Using integrating factors, solve $dX_t = -X_t dt + e^{-t} dB_t$ for

$$e^t X_t - X_0 = \int_0^t dB_s$$

and deduce that the solution X_t is

$$X_t = e^{-t}(X_0 + B_t).$$

5. The Langevin equation is given by

$$dX_t - \mu X_t dt = \sigma dB_t.$$

(a) Using integrating factors, solve for

$$e^{-\mu t} X_t - X_0 = \int_0^t e^{-\mu s} \sigma dB_s$$

and deduce that the solution X_t is

$$X_t = e^{\mu t} X_0 + \sigma \int_0^t e^{\mu(t-s)} dB_s.$$

(b) The expected value of X_t is

$$\mathbb{E}(X_t) = e^{\mu t} X_0$$

and, by Itô isometry, the variance of X_t is

$$\mathbb{V}(X_t) = \mathbb{E} \left(\sigma^2 \left(\int_0^t e^{\mu(t-s)} dB_s \right)^2 \right) = \mathbb{E} \left(\sigma^2 \int_0^t e^{2\mu(t-s)} ds \right) = \frac{\sigma^2}{2\mu} (e^{2\mu t} - 1).$$

6. Suppose Y_t is given by

$$dY_t = r dt + \alpha Y_t dB_t.$$

Using integrating factors, solve for

$$d(e^{-\alpha B_t} Y_t) = e^{-\alpha B_t} Y_t \left(r - \frac{\alpha^2}{2} \right) dt$$

and

$$e^{-\alpha B_t + \frac{\alpha^2}{2} t} Y_t - Y_0 = \int_0^t r e^{-\alpha B_s + \frac{\alpha^2}{2} s} ds.$$

Deduce that

$$Y_t = e^{\alpha B_t - \frac{\alpha^2}{2} t} Y_0 + r \int_0^t e^{\alpha(B_t - B_s) - \frac{\alpha^2}{2}(t-s)} ds.$$

7. The Ornstein-Uhlenbeck process is given by

$$dX_t = (m - X_t) dt + \sigma dB_t.$$

(a) Using integrating factors, solve for

$$e^t X_t - X_0 = \int_0^t e^s m ds + \int_0^t e^s \sigma dB_s$$

and deduce that the solution X_t is

$$X_t = e^{-t} X_0 + m(1 - e^{-t}) + \sigma \int_0^t e^{s-t} dB_s.$$

(b) The expected value of X_t is

$$\mathbb{E}(X_t) = m + e^{-t}(X_0 - m)$$

and the variance of X_t is

$$\mathbb{V}(X_t) = \mathbb{E} \left(\sigma^2 \left(\int_0^t e^{s-t} dB_s \right)^2 \right) = \mathbb{E} \left(\sigma^2 \int_0^t e^{2s-2t} ds \right) = \frac{\sigma^2}{2} (1 - e^{-2t}).$$

8. Consider the stochastic differential equation

$$\begin{pmatrix} dX_t^{(1)} \\ dX_t^{(2)} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} X_t^{(1)} \\ X_t^{(2)} \end{pmatrix} dt + \begin{pmatrix} \alpha dB_t^{(1)} \\ \beta dB_t^{(2)} \end{pmatrix}.$$

By d'Alembert's formula, it has a solution of the form

$$X_t = e^{At} X_0 + \int_0^t e^{A(t-s)} g(s) ds,$$

where

$$e^{At} = \begin{pmatrix} 1 & 1 \\ i & -1 \end{pmatrix} \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -1 \end{pmatrix}^{-1} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}.$$

Conclude that the solutions are

$$X_t^{(1)} = X_0^{(1)} \cos(t) + X_0^{(2)} \sin(t) + \alpha \int_0^t \cos(t-s) dB_s^{(1)} + \beta \int_0^t \sin(t-s) dB_s^{(2)}$$

and

$$X_t^{(2)} = -X_0^{(1)} \sin(t) + X_0^{(2)} \cos(t) - \alpha \int_0^t \sin(t-s) dB_s^{(1)} + \beta \int_0^t \cos(t-s) dB_s^{(2)}.$$

9. Let $dX_t = \ln(1 + X_t^2) dt + \chi_{\{X_t > 0\}} X_t dB_t$. It suffices to check that

$$|b(t, x)| + |\sigma(t, x)| = \ln(1 + x^2) + \chi_{\{x > 0\}} |x| \leq \frac{2}{e}(|x| + 1) + |x| \leq 2(|x| + 1),$$

$\mathbb{E}(|X_0|^2) = \alpha^2 < \infty$, and

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq |\ln(x^2) - \ln(y^2)| + |x - y| \leq 3|x - y|.$$

Hence, by Theorem 5.2.1, there is a unique strong solution to the stochastic differential equation.

10. Calculate

$$\begin{aligned} \mathbb{E}(X_t^2) &= \mathbb{E} \left(Z + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s \right)^2 \\ &\leq 3 \left(\mathbb{E}(Z^2) + \mathbb{E} \left(\int_0^t b(s, X_s) ds \right)^2 + \mathbb{E} \left(\int_0^t \sigma(s, X_s) dB_s \right)^2 \right) \\ &\leq 3 \left(\mathbb{E}(Z^2) + T \mathbb{E} \left(\int_0^t b(s, X_s)^2 ds \right) + \mathbb{E} \left(\int_0^t \sigma(s, X_s)^2 ds \right) \right) \\ &\leq 3\mathbb{E}(Z^2) + 6C^2 \left(T + \int_0^t \mathbb{E}(|X_s|^2) ds \right) (T + 1) \\ &= (3\mathbb{E}(Z^2) + 6C^2 T(T + 1)) + 6C^2(T + 1) \int_0^t \mathbb{E}(|X_s|^2) ds. \end{aligned}$$

and apply Gronwall to derive the result.

11. Consider the stochastic process

$$Y_t = a(1-t) + bt + (1-t) \int_0^t \frac{dB_s}{1-s}.$$

Then $Y_0 = a$ and, for $t \in [0, 1)$, Y_t solves

$$\begin{aligned} dY_t &= (b-a) dt - \int_0^t \frac{dB_s}{1-s} dt + (1-t) \frac{dB_t}{1-t} \\ &= \frac{1}{1-t} \left((b-a)(1-t) - (1-t) \int_0^t \frac{dB_s}{1-s} \right) dt + dB_t \\ &= \frac{1}{1-t} \left(b - a(1-t) - bt - (1-t) \int_0^t \frac{dB_s}{1-s} \right) dt + dB_t \\ &= \frac{b - Y_t}{1-t} dt + dB_t. \end{aligned}$$

Finally by Itô isometry $\mathbb{E} \left((1-t)^2 \int_0^t \frac{dB_s}{1-s} \right)^2 = (1-t)^2 \int_0^t \frac{1}{(1-s)^2} ds = (1-t)t \rightarrow 0$ as $t \rightarrow 1^-$ and so $\lim_{t \rightarrow 1^-} Y_t \stackrel{\text{a.s.}}{=} b$.

12. Let $y''(t) + (1 + \varepsilon W_t)y(t) = 0$ where $W_t = \frac{dB_t}{dt}$ is 1-dimensional white noise.

(a) Rewrite

$$\begin{pmatrix} dy_t \\ dy_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_t \\ \dot{y}_t \end{pmatrix} dt + \begin{pmatrix} 0 & 0 \\ -\varepsilon & 0 \end{pmatrix} \begin{pmatrix} y_t \\ \dot{y}_t \end{pmatrix} dB_t.$$

(b) Check that, if $y(t) = y(0) + y'(0)t + \int_0^t (r-t)y(r) dr + \int_0^t \varepsilon(r-t)y(r) dB_r$, then

$$y'(t) = y'(0) - \int_0^t y(r) dr - \int_0^t \varepsilon y(r) dB_r = y'(0) - \int_0^t y(r)(1 + \varepsilon W_r) dr$$

$$\text{and } y''(t) = -(1 + \varepsilon W_r) dr.$$

13. Let $x_t'' + a_0 x_t' + w^2 x_t = (T_0 - \alpha_0 x_t')\eta W_t$ where W_t is 1-dimensional white noise. Then

$$\begin{pmatrix} dx_t \\ d\dot{x}_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -w^2 & -a_0 \end{pmatrix} \begin{pmatrix} x_t \\ \dot{x}_t \end{pmatrix} dt + \begin{pmatrix} 0 & 0 \\ 0 & -\alpha_0 \eta \end{pmatrix} \begin{pmatrix} x_t \\ \dot{x}_t \end{pmatrix} dB_t + \begin{pmatrix} 0 \\ T_0 \eta \end{pmatrix} dB_t$$

and by d'Alembert's formula the solution is

$$X_t = e^{At} X_0 + \int_0^t e^{A(t-s)} K X_s dB_s + \int_0^t e^{A(t-s)} M dB_s.$$

The eigenvalues of A satisfy $\lambda^2 + a_0\lambda + w^2 = 0$ and are $\lambda_{\pm} = -\frac{a_0}{2} \pm \sqrt{w^2 - \frac{a_0^2}{4}}i =: -\lambda \pm \xi i$. Then take the exponential of matrix A

$$\begin{aligned}
e^{At} &= \begin{pmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{pmatrix} \begin{pmatrix} e^{\lambda_+ t} & 0 \\ 0 & e^{\lambda_- t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{pmatrix}^{-1} \\
&= \frac{1}{\lambda_- - \lambda_+} \begin{pmatrix} \lambda_- e^{\lambda_+ t} - \lambda_+ e^{\lambda_- t} & e^{\lambda_- t} - e^{\lambda_+ t} \\ -\lambda_- \lambda_+ (e^{\lambda_- t} - e^{\lambda_+ t}) & \lambda_- e^{\lambda_- t} - \lambda_+ e^{\lambda_+ t} \end{pmatrix} \\
&= -\frac{1}{2\xi i} \begin{pmatrix} e^{-\lambda t}(-\lambda \cdot 2i \sin(\xi t) - \xi i \cdot 2 \cos(\xi t)) & e^{-\lambda t}(-2i \sin(\xi t)) \\ -w^2 e^{-\lambda t}(-2i \sin(\xi t)) & e^{-\lambda t}(-\lambda \cdot 2i \sin(\xi t) - \xi i \cdot 2 \cos(\xi t) + 2\lambda \cdot 2i \sin(\xi t)) \end{pmatrix} \\
&= \frac{e^{-\lambda t}}{\xi} \begin{pmatrix} \lambda \sin(\xi t) + \xi \cos(\xi t) & \sin(\xi t) \\ -w^2 \sin(\xi t) & \lambda \sin(\xi t) + \xi \cos(\xi t) - 2\lambda \sin(\xi t) \end{pmatrix} \\
&= \frac{e^{-\lambda t}}{\xi} ((\lambda \sin(\xi t) + \xi \cos(\xi t))I + A \sin(\xi t)).
\end{aligned}$$

Next, letting $y_s = \dot{x}_s$, $g_t = e^{-\lambda t} \frac{\sin(\xi t)}{\xi}$ and $h_t = e^{-\lambda t} \frac{\xi \cos(\xi t) - \lambda \sin(\xi t)}{\xi}$, compute

$$e^{A(t-s)} K X_s = -\frac{\alpha_0 \eta e^{-\lambda(t-s)}}{\xi} \begin{pmatrix} 0 & \sin(\xi(t-s)) \\ 0 & \xi \cos(\xi(t-s)) - \lambda \sin(\xi(t-s)) \end{pmatrix} \begin{pmatrix} x_s \\ \dot{x}_s \end{pmatrix} = \begin{pmatrix} -\alpha_0 \eta y_s g_{t-s} \\ -\alpha_0 \eta y_s h_{t-s} \end{pmatrix}$$

and

$$e^{A(t-s)} M = \frac{T_0 \eta e^{-\lambda(t-s)}}{\xi} \begin{pmatrix} \sin(\xi(t-s)) \\ \xi \cos(\xi(t-s)) - \lambda \sin(\xi(t-s)) \end{pmatrix} = \begin{pmatrix} \eta T_0 g_{t-s} \\ \eta T_0 h_{t-s} \end{pmatrix}.$$

It follows that

$$x_t = \eta \int_0^t (T_0 - \alpha_0 y_s) g_{t-s} dB_s$$

and

$$y_t = \eta \int_0^t (T_0 - \alpha_0 y_s) h_{t-s} dB_s.$$

14. Letting $Z_t = F(\mathbf{B}_t)$, where $\mathbf{B}_t = B_t^{(1)} + iB_t^{(2)}$, calculate

$$\begin{aligned}
dZ_t &= F_x(\mathbf{B}_t) dB_t^{(1)} + F_y(\mathbf{B}_t) dB_t^{(2)} \\
&\quad + \frac{1}{2} F_{xx}(\mathbf{B}_t) d[B^{(1)}, B^{(1)}]_t + F_{xy}(\mathbf{B}_t) d[B^{(1)}, B^{(2)}]_t + F_{yy}(\mathbf{B}_t) d[B^{(2)}, B^{(2)}]_t \\
&= (u_x + iv_x) dB_t^{(1)} + (u_y + iv_y) dB_t^{(2)} + \frac{1}{2} (u_{xx} + iv_{xx} + u_{yy} + iv_{yy}) dt \\
&= \langle F'(\mathbf{B}_t), dB_t \rangle + \frac{1}{2} (v_{xy} - iu_{xy} + u_{yy} + iv_{yy}) dt \\
&= \langle F'(\mathbf{B}_t), dB_t \rangle + \frac{1}{2} (-u_{yy} - iv_{yy} + u_{yy} + iv_{yy}) dt \\
&= \langle F'(\mathbf{B}_t), dB_t \rangle.
\end{aligned}$$

15. Consider the non-linear stochastic differential equation

$$dX_t = rX_t(K - X_t)dt + \beta X_t dB_t, \quad X_0 = x > 0.$$

Comparing to the deterministic Bernoulli equation, do a substitution $Y_t = X_t^{-1}$, then

$$\begin{aligned} dY_t &= -rY_t(K - X_t)dt - \beta Y_t dB_t + \beta^2 Y_t dt \\ &= (-rK + \beta^2)Y_t dt - \beta Y_t dB_t + r dt. \end{aligned}$$

Next do a new change of variables

$$Z_t = Y_t e^{(rK - \beta^2)t}$$

and calculate

$$\begin{aligned} dZ_t &= -\beta Z_t dB_t + r e^{(rK - \beta^2)t} dt \\ \implies Z_t &= e^{-\beta B_t} \left(x^{-1} + r \int_0^t e^{(rK - \beta^2)s + \beta B_s} ds \right). \end{aligned}$$

Conclude that

$$X_t = \frac{e^{(rK - \beta^2)t}}{Z_t} = \frac{e^{(rK - \beta^2)t + \beta B_t}}{x^{-1} + r \int_0^t e^{(rK - \beta^2)s + \beta B_s} ds}.$$

16. Consider the non-linear stochastic differential equation

$$dX_t = f(t, X_t)dt + c(t)X_t dB_t, \quad X_0 = x.$$

(a) Let $F_t(\omega) = \exp\left(-\int_0^t c(s)dB_s + \frac{1}{2}\int_0^t c(s)^2 ds\right)$. Then calculate

$$\begin{aligned} d(F_t X_t) &= X_t dF_t + F_t dX_t + d[F_t, X_t] \\ &= X_t \left[F_t \left(-c(t)dB_t - \frac{1}{2}c(t)^2 dt - \frac{1}{2}c(t)^2 dt \right) \right] \\ &\quad + [f(t, X_t)F_t dt + c(t)X_t F_t dB_t] - c(t)^2 F_t X_t dt \\ &= f(t, X_t)F_t dt. \end{aligned}$$

(b) Defining $Y_t = F_t X_t$, deduce that

$$\frac{dY_t}{dt} = F_t(\omega) f(t, F_t^{-1}(\omega) Y_t(\omega))$$

(c) Consider $dX_t = X_t^{-1} + \alpha X_t dB_t$, $X_0 = x > 0$. Then

$$\frac{dY_t}{dt} = e^{-2\alpha B_t + \alpha^2 t} Y_t^{-1},$$

which implies

$$Y_t = \sqrt{Y_0^2 + 2 \int_0^t e^{-2\alpha B_t + \alpha^2 s} ds}$$

and

$$X_t = e^{\alpha B_t - \frac{\alpha^2}{2}t} \sqrt{x^2 + 2 \int_0^t e^{-2\alpha B_t + \alpha^2 s} ds}.$$

(d) Consider $dX_t = X_t^\gamma dt + \alpha X_t dB_t$, $X_0 = x > 0$. Then

$$\frac{dY_t}{dt} = e^{-(1-\gamma)B_t + (1-\gamma)\frac{\alpha^2}{2}t} Y_t^\gamma,$$

which implies

$$Y_t = \left(Y_0^{1-\gamma} + (1-\gamma) \int_0^t e^{-(1-\gamma)B_s + (1-\gamma)\frac{\alpha^2}{2}s} ds \right)^{\frac{1}{1-\gamma}}$$

and

$$X_t = e^{\alpha B_t - \frac{\alpha^2}{2}t} \left(x^{1-\gamma} + (1-\gamma) \int_0^t e^{-(1-\gamma)B_s + (1-\gamma)\frac{\alpha^2}{2}s} ds \right)^{\frac{1}{1-\gamma}}.$$

17. Let $v \geq 0$ satisfy $v(t) \leq C + A \int_0^t v(s) ds$ and consider quantity $w(t) = \int_0^t v(s) ds$. Then

$$w'(t) = v(t) \leq C + A \int_0^t v(s) ds = C + Aw(t).$$

Then for $f(t) = w(t)e^{-At}$, calculate

$$f'(t) = e^{-At} (w'(t) - Aw(t)) \leq Ce^{-At}$$

and

$$\begin{aligned} w(t)e^{-At} &\leq \int_0^t Ce^{-As} ds = \frac{C}{A}(1 - e^{-At}) \\ \implies w(t) &\leq \frac{C}{A}(e^{At} - 1). \end{aligned}$$

Deduce that

$$v(t) \leq C + Aw(t) \leq Ce^{At}.$$

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