Queen's University
Faculty of Arts and Science
Department of Mathematics and Statistics

MTHE 224 – C. Kennedy
December 2024

HAND IN
Answers recorded on exam paper

- You have 3 hours for this final exam.
- No aids other than your calculator (Casio 991 series) are allowed.
- There are 7 questions worth a total of 80 points.
- For full marks, you must show all your work and explain how you arrived at your answers unless explicitly told to do otherwise.
- Proctors cannot respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written.

Do not start the test until your invigilator instructs you to do so.

Problem 1

(10pts) Consider the ordinary differential equation

$$y'(x) = \frac{x}{y} - xy. \tag{1}$$

(a) Substituting $z = y^2(x)$, show that

$$z' + 2xz = 2x. \tag{2}$$

(b) Hence, find the solution to Equation (1) on $(0, \infty)$ when $y(0) = 2$.

Continue your solution of Problem 1 here.

Problem 2

(10pts) Consider the ordinary differential equation

$$y'(x) = f\left(\frac{y}{x}\right), \; x > 0. \tag{3}$$

(a) Substituting $y = xv(x)$, show that

$$\int \frac{dv}{f(v) - v} = \ln(x) + C. \tag{4}$$

(b) Hence, find the solution for $f(v) := v^2 + 3v + 1$ when $y(1) = 1$.

Continue your solution of Problem 2 here.

Problem 3

(10pts) Consider the ordinary differential equation

$$3y^3e^{3xy} - 1 + (2ye^{3xy} + 3xy^2e^{3xy})y'(x) = 0. \tag{5}$$

(a) Show that Equation (5) is exact.

(b) Hence, find the solution to Equation (5) when $y(0) = 1$.

Continue your solution of Problem 3 here.

Problem 4

(10pts) Consider functions $\phi_1(t) = t^3$ and $\phi_2(t) = t^2|t|$.

(a) Show that $\{\phi_1(t), \phi_2(t)\}$ is a linearly dependent set on $(0, \infty)$.

(b) Show that $\{\phi_1(t), \phi_2(t)\}$ is linearly independent set on $(-\infty, \infty)$.

Continue your solution of Problem 4 here.

Problem 5

(10pts) Consider the ordinary differential equation

$$y'' - 4y' - 12y = 36e^{4t}. \tag{6}$$

(a) Find the general solution to

$$y'' - 4y' - 12y = 0. \tag{7}$$

(b) Find the general solution to Equation (6). (*Hint: Use undetermined coefficients.*)

Continue your solution of Problem 5 here.

Problem 6

(15pts) Consider the ordinary differential equation

$$ty'' - (t+1)y' + y = t^2. \tag{8}$$

(a) Without solving Equation (8), find the Wronskian W. (*Hint: Use Abel's identity.*)

(b) Show that $y_1(t) = e^t$ solves the homogeneous equation

$$ty'' - (t+1)y' + y = 0. \tag{9}$$

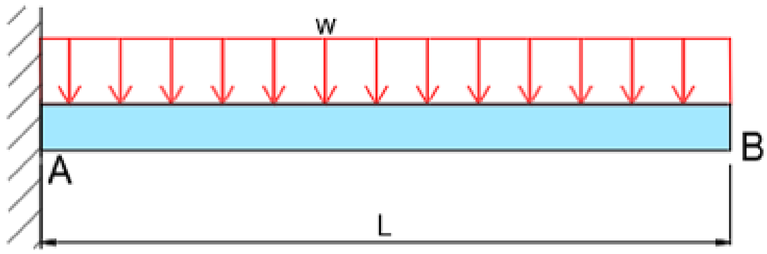
(c) Find the general solution to Equation (9). (*Hint: Use the Wronskian.*)

(d) Find the general solution to Equation (8). (*Hint: Use variation of parameters.*)

Continue your solution of Problem 6 here.

Problem 7

(15pts) Suppose a cantilevered beam has length $L\,[\mathrm{N}]$, a uniform load $w\,[\mathrm{N}]\cdot[\mathrm{m}]^{-1}$ and constant flexural rigidity $EI\,[\mathrm{N}]\cdot[\mathrm{m}]^2$.



The differential equation of the deflection curve is

$$EIy^{(4)}(x) = -w, \tag{10}$$

where $y(x)$ is the height of the beam relative to the fixed end at A.

(a) Find the general solution to Equation (10).

For simplicity, assume that $EI = 1$, $L = 1$ and $w = 24$. Since the end at A is fixed, $y(0) = y'(0) = 0$. Also, since the end at B is free, $y''(L) = y^{(3)}(L) = 0$. The boundary value problem of the deflection curve is

$$\begin{cases} y^{(4)}(x) = -24 \\ y(0) = y'(0) = 0 \\ y''(1) = y^{(3)}(1) = 0. \end{cases} \tag{11}$$

(b) Solve the boundary value problem (11) for $y(x)$.

(c) Find the reaction moment M_A.

(d) Find the reaction force R_A.

Continue your solution of Problem 7 here.

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Useful Formulae

Calculus

$$y'(x) = \lim_{h\to 0} \frac{y(x+h) - y(x)}{h}$$
$$y(x) = (g \circ u)(x) \implies y'(x) = g'(u(x)) \cdot u'(x)$$
$$y(x) = u(x) \cdot v(x) \implies y'(x) = u(x)v'(x) + u'(x)v(x)$$
$$y(x) = \frac{u(x)}{v(x)} \implies y'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2}$$
$$\int^x (f \circ g)(x) g'(x)\, dx = \int^{g(x)} f(u)\, du$$
$$\int u\, dv = uv - \int v\, du$$

First-Order Differential Equations

$$y'(x) = f(x,y) \implies y_{n+1} = y_n + (x_{n+1} - x_n) f(x_n, y_n)$$
$$y'(x) + p(x)y = q(x) \implies \mu(x) = e^{\int^x p(x')\, dx'}$$
$$M(x,y) + N(x,y)y'(x) = 0 \qquad \text{exact} \iff M_y = N_x$$
$$\psi(x,y) = \int^x M\, dx + \int^y N - \partial_y \left(\int^x M\, dx \right) dy \equiv C$$

Higher-Order Differential Equations

$$ay'' + by' + cy = 0 \implies ar^2 + br + c = 0$$
$$r_1, r_2 \in \mathbb{R}, r_1 \neq r_2 \implies y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$
$$r_1 = r_2 = r \implies y(t) = (c_1 + c_2 t)e^{rt}$$
$$r_\pm = \alpha \pm \beta i \implies y(t) = (c_1 \cos(\beta t) + c_2 \sin(\beta t))e^{\alpha t}$$
$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$
$$y'' + p(t)y' + q(t)y = 0 \implies W(y_1, y_2) = e^{-\int^t p(s)\, ds}$$
$$y = y_h + y_p \qquad g(t) = e^{\lambda t} \implies y_p(t) = At^n e^{\lambda t}$$
$$y = y_1 \int \frac{-g \cdot y_2}{W} + y_2 \int \frac{g \cdot y_1}{W}$$
$$A = PDP^{-1} \qquad y = \sum c_j e^{\lambda_j t} \vec{v_j}$$
$$y'' = \frac{M}{EI} \qquad V(x) = \frac{dM}{dx}$$