Øksendal: Stochastic Differential Equations

Solutions Manual

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Chapter 1

Introduction

This is a solutions manual for Stochastic Differential Equations by Bernt Øksendal. This is a working document last updated October 25, 2020. Progress to date:

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- Chapter 3: Problems #1-12 (omitted: #13-17)
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Chapter 2

Some Mathematical Preliminaries

1. Suppose $X : \Omega \to \mathbb{R}$ is a function that assumes countably many values $\{a_j\}$ in $\mathbb{R}$.

   (a) Note that X is a random variable if and only if it is measurable. If $X : \Omega \to \mathbb{R}$ is measurable, then $U = X^{-1}(\mathbb{R} \setminus a_k) \in \mathcal{F}$ and thus $X^{-1}(a_k) = \Omega \setminus U \in \mathcal{F}$, $\forall k$. On the other hand, if $X^{-1}(a_k) \in \mathcal{F}$, $\forall k$, then Borel set $V \subseteq \mathbb{R}$, $X^{-1}(V) = \bigcup_{a_k \in V} X^{-1}(a_k) \in \mathcal{F}$ and thus X is measurable.

   (b) Compute $\mathbb{E}(|X|) = \int_{\mathbb{R}} |x| \, d\mathbb{P}_X = \int_{\bigcup_{k=1}^{\infty}\{a_k\}} |x| \, d\mathbb{P}_X = \sum_{k=1}^{\infty} |a_k| \mathbb{P}(X = a_k)$.

   (c) If $\mathbb{E}(|X|) < \infty$, then the series

   $$\mathbb{E}(X) = \int_{\mathbb{R}} x \, d\mathbb{P}_X = \int_{\bigcup_{k=1}^{\infty}\{a_k\}} x \, d\mathbb{P}_X = \sum_{k=1}^{\infty} a_k \mathbb{P}(X = a_k)$$

   is absolutely convergent and therefore converges.

   (d) If f is measurable and $|f|$ is bounded by M, then

   $$\mathbb{E}(|f(X)|) = \int_{\mathbb{R}} |f(x)| \, d\mathbb{P}_X \leq \int_{\mathbb{R}} M \, d\mathbb{P}_X = M \int_{\mathbb{R}} d\mathbb{P}_X = M < \infty.$$

   Hence,

   $$\mathbb{E}(f(X)) = \int_{\mathbb{R}} f(x) \, d\mathbb{P}_X = \int_{\bigcup_{k=1}^{\infty}\{a_k\}} f(x) \, d\mathbb{P}_X = \sum_{k=1}^{\infty} f(a_k) \mathbb{P}(X = a_k)$$

   is absolutely convergent and therefore converges.

2. Let $F(x) = \mathbb{P}(X \leq x)$ be the distribution function of X.

   (a) By monotonicity of $\mathbb{P}$, $0 = \mathbb{P}(\emptyset) \leq \mathbb{P}(X \leq x) \leq P(\mathbb{R}) = 1$. Now, by the Monotone Convergence Theorem,

   $$\lim_{n \to \infty} F(n) = \lim_{n \to \infty} \int_{\mathbb{R}} \chi_{(-\infty,n]} \, d\mathbb{P}(x) = \int_{\mathbb{R}} d\mathbb{P}(x) = 1.$$

Similarly, for $G(n) := 1 - F(-n)$, we have

$$\lim_{n\to\infty} G(n) = \lim_{n\to\infty} \int_{\mathbb{R}} (1 - \chi_{(-\infty,-n]}) dP_X(x) = 1.$$

Moreover, F is increasing by monotonicity of P and finally, again by Monotone Convergence,

$$\lim_{h\to 0^+} 1 - F(x+h) + F(x) = \lim_{h\to 0^+} \int_{\mathbb{R}} (1 - \chi_{(x,x+h]})\, d\mathbb{P}(x) = \int_{\mathbb{R}} d\mathbb{P}(x) = 1$$

and so $\lim\limits_{h\to 0^+} F(x+h) = F(x)$, i.e. F is right-continuous.

(b) Compute the expectation

$$\mathbb{E}(g(X)) = \int_{\mathbb{R}} g(x)\, d\mathbb{P}(x) = \int_{\mathbb{R}} g(x)\chi_{(-\infty,x]}\, d\mathbb{P}(x) = \int_{\mathbb{R}} g(x)\, dF(x).$$

(c) Compute the density of B_t^2

$$\begin{aligned} F(u) := \mathbb{P}(B_t^2 \le u) &= \mathbb{P}(-\sqrt{u} \le B_t \le \sqrt{u}) \\ &= 2\int_{[0,\sqrt{u}]} p(y) dy \\ &= 2\int_{[0,u]} \frac{p(\sqrt{u})}{2\sqrt{u}} du \\ &= \int_{(-\infty,u]} \chi_{[0,\infty)} \frac{p(\sqrt{u})}{\sqrt{u}} du. \end{aligned}$$

and so $p(u) = \chi_{[0,\infty)} \frac{p(\sqrt{u})}{\sqrt{u}}$ where $p(u)$ is the density of B_t.

3. Since $\mathcal{H}_i$ is a σ-algebra, $\emptyset \in \mathcal{H}_i$, $\forall i \in I$. So $\emptyset \in \mathcal{H} = \cap_{i\in I}\mathcal{H}_i$. If $\{U_j\}_{j\in\mathbb{N}} \subseteq \mathcal{H}$, then $\{U_j\}_{j\in\mathbb{N}} \subseteq \mathcal{H}_i$ for each $i \in I$ and so $\Omega \setminus U_j \in \mathcal{H}_i$ and $\cup_{j\in\mathcal{A}} U_j \in \mathcal{H}_i$, $\forall i \in I$. Conclude that $\Omega \setminus U_j \in \mathcal{H}$ and $\cup_{j\in\mathcal{A}} U_j \in \mathcal{H}$ and $\mathcal{H} = \cap_{i\in I}\mathcal{H}_i$ is also a σ-algebra.

4. Let $X : \Omega \mapsto \mathbb{R}$ be a random variable with $\mathbb{E}(|X|^p) < \infty$.

(a) Let $A = \{\omega \in \Omega \,|\, |X| \ge \lambda > 0\}$ and compute

$$\mathbb{E}(|X|^p) = \int_\Omega |X|^p\, d\mathbb{P} \ge \int_A |X|^p\, d\mathbb{P} \ge \lambda^p \int_A d\mathbb{P} = \lambda^p \mathbb{P}(|X| \ge \lambda).$$

(b) By Chebychev, $\mathbb{P}(|X| \ge \lambda) = \mathbb{P}(e^{|X|} \ge e^\lambda) \le \frac{1}{e^{k\lambda}}\mathbb{E}(e^{k|X|}) = Me^{-k\lambda}$.

5. Since the measures are σ-finite, $f(x,y) = xy$ is $\mathbb{P}_X \otimes \mathbb{P}_Y$ measurable and $\mathbb{E}(|XY|) < \infty$, apply Fubini-Tonelli and compute

$$\begin{aligned}
\mathbb{E}(XY) &= \int_{\mathbb{R}^2} xy\, d\mathbb{P}_{XY}(x,y) \\
&= \int_{\mathbb{R}^2} xy\, d\mathbb{P}_X(x) \otimes d\mathbb{P}_Y(y) \\
&= \int_{\mathbb{R}} y \left(\int_{\mathbb{R}} x\, d\mathbb{P}_X(x) \right) d\mathbb{P}_Y(y) \\
&= \mathbb{E}(X) \int_{\mathbb{R}} y\, d\mathbb{P}_Y(y) \\
&= \mathbb{E}(X)\mathbb{E}(Y).
\end{aligned}$$

6. *(Borel-Cantelli)* Let $\{A_k\}_{k=1}^{\infty} \subseteq \mathcal{F}$ and suppose $\sum_{k=1}^{\infty} \mathbb{P}(A_k) < \infty$. Then

$$\mathbb{P}(\cap_{m=1}^{\infty} \cup_{k=m}^{\infty} A_k) \leq \lim_{m\to\infty} \sup_{k\geq m} \mathbb{P}(A_k) = 0$$

by dominated convergence.

7. Let $\Omega = \bigsqcup_{i=1}^{n} G_i$.

   (a) Note $\emptyset \in \mathcal{G}$ and $\mathcal{G}$ is closed under unions by construction. It is also closed under complements as $\Omega \setminus G_i = \cup_{j\neq i} G_j \in \mathcal{G}$.

   (b) Write a new sequence defined by $F_i = \mathcal{G}_i \setminus \cup_{j\leq i} F_j$ and $\{F_i\}$ will satisfy (a).

   (c) Note that $\{X^{-1}(x \in \mathbb{R})\} \subseteq \mathcal{F}$ is disjoint. So, by (a) and (b), $\mathcal{F}$ is finite if and only if all but finitely many $X^{-1}(x \in \mathbb{R})$ are empty.

8. Let B_t be a 1-dimensional Wiener process.

   (a) By Equation 2.2.3, since $B_t \sim N(0,t)$,

   $$\mathbb{E}(e^{iuB_t}) = \exp\left(-\frac{u^2}{2}\mathbb{V}(B_t) + iu\mathbb{E}(B_t) \right) = e^{-\frac{u^2}{2}}.$$

   (b) Comparing power series coefficients, we deduce that

   $$\frac{(iu)^{2n}}{(2n)!}\mathbb{E}(B_t^{2n}) = \frac{1}{n!}\left(-\frac{u^2 t}{2} \right)^n,$$

   and so $\mathbb{E}(B_t^{2n}) = \frac{(2n)!}{2^n n!} t^n$.

(c) Integrating by parts, compute the n^{th} moment of B_t

$$\begin{aligned}
\mathbb{E}(B_t^{2k}) &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} x^{2k} e^{-\frac{x^2}{2t}}\, dx \\
&= x^{2k-1} \sqrt{\frac{2t}{\pi}} \int^{\frac{x}{\sqrt{2t}}} u e^{-u^2}\, du \Big|_{x=-\infty}^{x=\infty} - \int_{\mathbb{R}} (2k-1) x^{2k-2} \sqrt{\frac{2t}{\pi}} \int^{\frac{x}{\sqrt{2t}}} u e^{-u^2}\, du \\
&= -(2k-1) \sqrt{\frac{2t}{\pi}} \int_{\mathbb{R}} x^{2k-2} \left(\frac{-1}{2} e^{-\frac{x^2}{2t}} \right) dx \\
&= (2k-1) t \cdot \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} x^{2k-2} e^{-\frac{x^2}{2t}}\, dx \\
&= (2k-1) t \mathbb{E}(B_t^{2k-2}).
\end{aligned}$$

As $\mathbb{E}(B_t^2) = t$, we have that $\mathbb{E}(B_t^{2k}) = \frac{(2k)!t^{k-1}}{2^k k!} \cdot t = \frac{(2k)!t^k}{2^k k!}$.

(d) Check the base case, $n = 2k = 2$, where $\mathbb{E}(B_t)^2)] = \frac{2! \cdot t}{2 \cdot 1!} = t$. If the claim is true for $n = 2k$, then

$$\mathbb{E}(B_t^{2k+2}) = (2k-1) t \mathbb{E}(B_t^{2k}) = (2k+1) t \cdot \frac{(2k)!t^k}{2^k k!} = \frac{(2k+2)!t^{k+1}}{2^{k+1}(k+1)!},$$

and so it is also true for $n = 2(k+1) = 2k+2$, thus completing the induction step.

9. Note that $\{X_t\}$ and $\{Y_t\}$ have the same distributions since neither distribution has any atoms and they agree except on a zero set $\forall t \geq 0$. Yet $t \mapsto X_t$ is discontinuous while $t \mapsto Y_t$ is continuous.

10. As B_t is Brownian, $B_{t+h} - B_t \sim N(0, h)$. Since h is fixed, $\{B_{t+h} - B_t\}_{h \geq 0}$ have the same distributions $\forall t \geq 0$.

11. As $B_0 = \left(B_0^{(1)}, B_0^{(2)}, \ldots B_0^{(n)} \right) = 0$, $B_0^{(j)} = 0$ for all $j \in \{1, \ldots n\}$. B_t is almost surely continuous only if its components are almost surely continuous. Each component is normally distributed with $\mathbb{E}(B_t^j) = 0$ as $\mathbb{E}(B_t) = \vec{0}$ and $\text{Cov}(B_t^{(i)}, B_t^{(j)}) = t\delta_{ij}$ as $\text{Cov}(B_t) = tI$.

12. Let $W_t := B_{s+t} - B_s$ where $s \geq 0$ is fixed. Then $W_0 = B_s - B_s = 0$ and W_t is almost surely continuous as the sum of two almost surely continuous stochastic processes. Noting $W_{t_2} - W_{t_1} = B_{s+t_2} - B_{s+t_1}$ is independent of both B_{s+t_1} and B_s, deduce that $W_{t_2} - W_{t_1}$ is independent of $W_{t_1} = B_{s+t_1} - B_s$. The expected value is $\mathbb{E}(W_t) = \mathbb{E}(B_{s+t}) - \mathbb{E}(B_s) = 0$ and the variance is

$$\begin{aligned}
\mathbb{V}(W_t) &= \mathbb{E}((B_{s+t} - B_s)^2) \\
&= \mathbb{E}(B_{s+t}^2) - 2\mathbb{E}(B_s B_{s+t}) + \mathbb{E}(B_s^2) \\
&= \mathbb{E}(B_{s+t}^2) - 2\mathbb{E}(B_s(B_{s+t} - B_s)) - \mathbb{E}(B_s^2) \\
&= \mathbb{E}(B_{s+t}^2) - 2\mathbb{E}(B_s)\mathbb{E}(B_{s+t} - B_s) - \mathbb{E}(B_s^2) \\
&= (s+t) - 0 - s \\
&= t.
\end{aligned}$$

Since W_t is the sum of two normal distributions, it is also normal and $W_t \sim N(0,t)$.

13. Compute

$$\mathbb{P}_0(B_t \in D_\rho) = \int_{|x|<\rho} \frac{1}{2\pi t} e^{-\frac{|\vec{x}|^2}{2t}}\, d^2\vec{x} = \frac{2\pi}{2\pi t}\int_0^\rho r e^{-\frac{r^2}{2t}}\, dr = \int_0^{\frac{\rho^2}{2t}} e^{-u}\, du = 1 - e^{-\frac{\rho^2}{2t}}.$$

14. Compute

$$\begin{aligned}
\mathbb{E}_x\left(\int_{[0,\infty]} \chi_K(B_t)\, dt\right) &= \int_{[0,\infty]} \mathbb{P}(B_t \in K)\, dt \\
&= \int_{[0,\infty]} \left(\int_K \frac{1}{(2\pi t)^{n/2}} e^{-\frac{|\vec{x}-\vec{y}|^2}{2t}}\, d^n\vec{x}\right) dt \\
&\leq \int_{[0,\infty]} \left\| \frac{1}{(2\pi t)^{n/2}} e^{-\frac{|\vec{x}-\vec{y}|^2}{2t}} \right\|_\infty \mu(K)\, dt \\
&= 0
\end{aligned}$$

and deduce that the expected total time spent in K is 0.

15. Note that $UU^T = I$, whence $|\det U| = 1$ and the probability measures are identical by change of variables. It follows that both are Brownian.

16. Let $W_t = \frac{1}{c} B_{c^2 t}$. We have $W_0 = B_0 = 0$ and that W_t is absolutely continuous as a scaling of absolutely continuous B_t. Finally,

$$\begin{aligned}
\mathbb{P}_0(W_t \in U) &= \mathbb{P}_0(B_{c^2 t} \in cU) \\
&= \int_{cU} p(c^2 t, 0, y)\, dy \\
&= \int_{cU} \frac{1}{c} p(t, 0, y/c)\, dy \\
&= \int_U \frac{1}{c} p(t, 0, y')(c\,dy') \\
&= \mathbb{P}_0(B_t \in U),
\end{aligned}$$

and so W_t is also a Brownian motion.

17. Let $X_t(\cdot)$ be a continuous stochastic process.

(a) Recall that $\mathbb{E}(B_t) = 0$, $\mathbb{E}(B_t^2) = t$ and $\mathbb{E}(B_t^4) = 3t^2$. Then

$$\begin{aligned}
\mathbb{E}\left((\sum_k \left(\Delta B_k^2 - \Delta t_k\right))^2 \right) &= \mathbb{E}\left((\sum_k \left(\Delta B_k^2 - \Delta t_k\right)^2) \right) \\
&= \sum_k \left(\mathbb{E}(\Delta B_k^4) - 2\Delta t_k \mathbb{E}(\Delta B_k^2) + \Delta t_k^2 \right) \\
&= \sum_k \left(3\Delta t_k^2 - 2\Delta t_k^2 + \Delta t_k^2 \right) \\
&= 2 \sum_k \Delta t_k^2.
\end{aligned}$$

So $\langle B, B \rangle_t^{(2)}(w) = t$.

(b) Note that the Brownian motion has positive quadratic variation t on $[0, t]$. So

$$\langle B, B \rangle_t^{(1)}(w) \geq \lim_{\|\Delta B_k\| \to 0^+} \frac{\langle B, B \rangle_t^{(2)}(w)}{\|\Delta B_k\|} = \infty.$$

Chapter 3

Itô Integrals

1. Compute

$$\begin{aligned}
\int_0^t s\,dB_s &= \lim_{n\to\infty} \sum_{j=0}^{\frac{\lceil nt\rceil}{t}-1} \frac{jt}{n}(B_{\frac{(j+1)t}{n}} - B_{\frac{jt}{n}}) \\
&= \lim_{n\to\infty} \frac{\lceil nt\rceil}{n} B_{\frac{\lceil nt\rceil}{n}} - \lim_{n\to\infty} \frac{t}{n} \sum_{j=0}^{\frac{\lceil nt\rceil}{t}-1} B_{\frac{jt}{n}} + \lim_{n\to\infty} \frac{t}{n}(B_0 - B_{\frac{\lceil nt\rceil}{n}}) \\
&= tB_t - \int_0^t B_s\,ds.
\end{aligned}$$

2. Compute

$$\begin{aligned}
\int_0^t B_s^2\,dB_s &= \lim_{n\to\infty} \sum_{j=0}^{\frac{\lceil nt\rceil}{t}-1} B^2_{\frac{jt}{n}}(B_{\frac{(j+1)t}{n}} - B_{\frac{jt}{n}}) \\
&= \lim_{n\to\infty} \sum_{j=0}^{\frac{\lceil nt\rceil}{t}-1} \left(\frac{1}{3}B^3_{\frac{(j+1)t}{n}} - \frac{1}{3}B^3_{\frac{j}{n}} - B_{\frac{jt}{n}}(B_{\frac{(j+1)t}{n}} - B_{\frac{j}{n}})^2 - \frac{1}{3}(B_{\frac{(j+1)t}{n}} - B_{\frac{j}{n}})^3\right) \\
&= \frac{1}{3}B_t^3 - \lim_{n\to\infty}\left(\sum_{j=0}^{\frac{\lceil nt\rceil}{t}-1} \frac{t}{n} B_{\frac{jt}{n}} + \mathcal{O}(t^2/n)\right) \\
&= \frac{1}{3}B_t^3 - \int_0^t B_s\,ds.
\end{aligned}$$

3. Let $\{\mathcal{N}_t\}$ be some filtration and let $\{\mathcal{H}_t^{(X)}\}$ be the filtration of process X_t.

   (a) Compute

$$\mathbb{E}(X_t \,|\, \mathcal{H}_s^{(X)}) = \mathbb{E}\left(\mathbb{E}(X_t \,|\, \mathcal{N}_s) \,|\, \mathcal{H}_s^{(X)}\right) = \mathbb{E}(H_s \,|\, \mathcal{H}_s^{(X)}) = H_s.$$

(b) Compute

$$\mathbb{E}(X_t) = \mathbb{E}(\mathbb{E}(X_t \,|\, H_0^{(X)})) = \mathbb{E}(X_0).$$

(c) Let $Y \sim \text{Bernoulli}(0.5)$ and fix $X_0 = 2Y - 1$. Then $X_t = t \cdot \text{sgn}(X_0)$ satisfies $\mathbb{E}(X_t) = \mathbb{E}(X_0) = 0$, but $\mathbb{E}(X_t \,|\, \mathcal{F}_s) = t \cdot \text{sgn}(X_0) \neq s \cdot \text{sgn}(X_0)$.

4. Compute

$$\begin{aligned}
&\mathbb{E}(B_t + 4t \,|\, \mathcal{F}_s) = B_s + 4t \neq B_s + 4s \\
&\mathbb{E}(B_t^2 \,|\, \mathcal{F}_s) = \mathbb{E}((B_t - B_s)^2 + 2B_s(B_t - B_s) + B_s^2 \,|\, \mathcal{F}_s) = B_s^2 + t - s \neq B_s^2 \\
&\mathbb{E}\left(t^2 B_t - 2\int_0^t uB_u\,du \,|\, F_s\right) = t^2 B_s - 2\int_0^s uB_u\,du - 2\int_s^t uB_s\,du = s^2 B_s - 2\int_0^s uB_u du \\
&\mathbb{E}(B_t^{(1)} B_t^{(2)} \,|\, \mathcal{F}_s) = \mathbb{E}(B_t^{(1)} \,|\, \mathcal{F}_s)\mathbb{E}(B_t^{(2)} \,|\, \mathcal{F}_s) = B_s^{(1)} B_s^{(2)},
\end{aligned}$$

and deduce that only the last two are martingales.

5. Verify $\mathbb{E}(|B_t^2 - t|) \leq \mathbb{E}(B_t^2) + t = 2t < \infty$ and compute

$$\mathbb{E}(B_t^2 - t \,|\, \mathcal{F}_s) = \mathbb{E}((B_t - B_s)^2 + 2B_s(B_t - B_s) + B_s^2 - t \,|\, \mathcal{F}_s) = B_s^2 + t - s - t = B_s^2 - s.$$

to deduce that $X_t := B_t^2 - t$ is a martingale.

6. Verify $\mathbb{E}(|B_t^3 - 3tB_t|) \leq \sqrt{\mathbb{E}(B_t^2)}(\sqrt{\mathbb{E}(B_t^4)} + 3t) = (3 + \sqrt{3})t^{3/2} < \infty$ and compute

$$\begin{aligned}
\mathbb{E}(B_t^3 - 3tB_t \,|\, \mathcal{F}_s) &= \mathbb{E}((B_t - B_s)^3 + 3B_s(B_t - B_s)^2 + 3B_s^2(B_t - B_s) + B_s^3 - 3tB_s \,|\, \mathcal{F}_s) \\
&= 3B_s(t - s) + B_s^3 - 3tB_s \\
&= B_s^3 - 3sB_s
\end{aligned}$$

to deduce that $Y_t := B_t^3 - 3tB_t$ is a martingale.

7. In this question, the formula for Itô iterated integrals is derived.

(a) Note that $\{0 \leq u_1 \cdots \leq u_n\}$ is Borel measurable and $\chi_{0 \leq u_1 \cdots \leq u_n}$ is $\mathcal{F}_t$-adapted. Finally $\mathbb{E}\left(\int_0^T f(t_1, \ldots t_n, \omega)^2\,dt_1 \ldots dt_n\right) \leq T^n < \infty$.

(b) For $n \in \{1, 2, 3\}$

$$\begin{aligned}
&1!\int_0^t dB_u = B_t = t^{1/2} H_1\left(\frac{B_t}{\sqrt{t}}\right) \\
&2!\int_0^t \int_0^v dB_u\,dB_v = 2\int_0^t B_v\,dB_v = B_t^2 - t = tH_2\left(\frac{B_2}{\sqrt{t}}\right) \\
&3!\int_0^t \int_0^w \int_0^v dB_u\,dB_v\,dB_w = 3\int_0^t (B_w^2 - w)\,dB_w = B_t^3 - 3tB_t = t^{3/2} H_3\left(\frac{B_t}{\sqrt{t}}\right).
\end{aligned}$$

(c) Deduce that $d(B_t^3 - 3tB_t) = 3(B_t^2 - t)\,dB_t$ and so $Y_t := B_t^3 - 3tB_t$ is a martingale.

8. There exists continuous martingale M_t iff there exists $Y \in L^1$ such that $M_t = \mathbb{E}(Y \mid \mathcal{F}_t)$.

   (a) Verify that $\mathbb{E}(|\mathbb{E}(Y \mid \mathcal{F}_t)|) \leq \mathbb{E}(\mathbb{E}(|Y| \mid \mathcal{F}_t) = \mathbb{E}(|Y|) < \infty$ and
   $$\mathbb{E}(M_t \mid \mathcal{F}_s) = \mathbb{E}(\mathbb{E}(Y \mid \mathcal{F}_t) \mid \mathcal{F}_s) = \mathbb{E}(Y \mid \mathcal{F}_s) = M_s.$$

   (b) If M_t is a continuous martingale such that $\sup_{t>0} \mathbb{E}(|X|^p) < \infty$ for $p \in (1, \infty)$, then $\exists M$ such that $\|M_t - M\|_{L^1} \to 0$ as $t \to \infty$. So let $Y = M$ and
   $$\begin{aligned}\lim_{s\to\infty} \int_{\Omega_s} |M_s - \mathbb{E}(M \mid \mathcal{F}_s)|\,d\mathbb{P} &= \lim_{s\to\infty} \int_{\Omega_s} |\mathbb{E}(M_s - M \mid \mathcal{F}_s)|\,d\mathbb{P} \\ &\leq \lim_{s\to\infty} \int_{\Omega_s} \mathbb{E}(|M_s - M| \mid \mathcal{F}_s)\,d\mathbb{P} \\ &= \lim_{s\to\infty} \int_{\Omega_s} |M_s - M|\,d\mathbb{P} \\ &= 0.\end{aligned}$$

9. Compute
$$\begin{aligned}\int_0^T B_t \circ dB_t &= \lim_{n\to\infty} \sum_{j=0}^{\frac{\lfloor nt \rfloor}{t}-1} \frac{1}{2}(B_{\frac{jt}{n}} + B_{\frac{(j+1)t}{n}})(B_{\frac{(j+1)t}{n}} - B_{\frac{jt}{n}}) \\ &= \lim_{n\to\infty} \sum_{j=0}^{\frac{\lfloor nt \rfloor}{t}-1} B_{\frac{jt}{n}}(B_{\frac{(j+1)t}{n}} - B_{\frac{jt}{n}}) + \lim_{n\to\infty} \sum_{j=0}^{\frac{\lfloor nt \rfloor}{t}-1} \frac{1}{2}(B_{\frac{(j+1)t}{n}} - B_{\frac{jt}{n}})^2 \\ &= \frac{1}{2}B_t^2 - \frac{t}{2} + \frac{t}{2} \\ &= \frac{1}{2}B_t^2.\end{aligned}$$

10. If $f(t, \omega)$ varies smoothly in t, then the Itô and Stratonovich integrals coincide. Compute
$$\int_0^T f(t, \omega) \circ dB_t = \int_0^T f(t, \omega)\,dB_t + \frac{1}{2}\langle f(t, \omega), B_t \rangle^{(2)}$$
and
$$\begin{aligned}\mathbb{E}(\langle f(t, \omega), B_t \rangle^{(2)})^2 &\leq \mathbb{E}(\langle B_t, B_t \rangle^{(2)} \mathbb{E}(\langle f(t, \omega), f(t, \omega) \rangle^{(2)} \\ &\leq T \lim_{\|\Delta t_k\| \to 0^+} \sup_{|\Delta t_k|} \frac{T}{|\Delta t_k|}(K|\Delta t_k|^{1+\varepsilon}) \\ &= KT^2 \lim_{\|\Delta t_k\| \to 0^+} \|\Delta t_k\|^\varepsilon \\ &= 0.\end{aligned}$$

11. Define white noise $W_t^{(N)} = \max\{-N, \min\{W_t, N\}\}$. Since W_t and W_s are independent and identically distributed, it follows that $W_t^{(N)}$ and $W_s^{(N)}$ are as well. If W_t is continuous, then since $|W_t^{(N)}| \leq N$ and by bounded convergence

$$\lim_{t\to s} 2\mathbb{E}(W_t^{(N)})^2 = \lim_{t\to s} \mathbb{E}(|W_t^{(N)} - W_s^{(N)}|^2) = 0.$$

But then $W_t \overset{\text{a.s.}}{=} 0$, which is a contradiction.

12. Problem 12

(i) Since $\alpha X_t \circ dB_t = \frac{\alpha^2}{2} X_t dt + \alpha X_t\, dB_t$,

$$dX_t = (\gamma + \frac{\alpha^2}{2}) X_t\, dt + \alpha X_t\, dB_t.$$

Since $(t^2 + \cos(X_t)) \circ dB_t = -\frac{\sin(X_t)}{2}(t^2 + \cos(X_t))\, dt + (t^2 + \cos(X_t)\, dB_t$,

$$dX_t = \frac{\sin(X_t)}{2}(\cos(X_t) - t^2)\, dt + (t^2 + \cos(X_t))\, dB_t.$$

(ii) Since $\alpha X_t\, dB_t = \alpha X_t \circ dB_t - \frac{\alpha^2}{2} X_t\, dt$,

$$dX_t = (r - \frac{\alpha^2}{2}) X_t\, dt + \alpha X_t \circ dB_t.$$

Since $X_t^2\, dB_t = X_t^2 \circ dB_t - X_t^3\, dt$,

$$dX_t = (2e^{-X_t} - X_t^3)\, dt + X_t^2 \circ dB_t.$$

Chapter 4

The Itô Formula

1. Compute

   (a) $dX_t = d(B_t^2) = 2B_t\,dB_t + d[B,B]_t = 2B_t\,dB_t + dt$

   (b) $dX_t = d(2 + t + e^{B_t}) = (1 + \frac{1}{2}e^{B_t})\,dt + e^{B_t}\,dB_t$

   (c) $dX_t = d\left((B_t^{(1)})^2 + (B_t^{(2)})^2\right) = 2B_t^{(1)}\,dB_t^{(1)} + 2B_t^{(2)}\,dB_t^{(2)} + 2\,dt$

   (d) $dX_t = d((t_0 + t, B_t)) = (dt, dB_t)$

   (e) and finally

$$\begin{aligned} dX_t &= d((B_t^{(1)} + B_t^{(2)} + B_t^{(3)}, (B_t^{(2)})^2 - B_t^{(1)}B_t^{(3)})) \\ &= (dB_t^{(1)} + dB_t^{(2)} + dB_t^{(3)}, 2B_t^{(2)}\,dB_t^{(2)} + dt - B_t^{(3)}\,dB_t^{(1)} - B_t^{(1)}\,dB_t^{(3)}). \end{aligned}$$

2. Using Itô's Lemma, differentiate

$$d\left(\frac{1}{3}B_t^3 - \int_0^t B_s\,ds\right) = B_t^2\,dB_t + B_t\,d[B,B]_t - B_t\,dt = B_t^2 dB_t$$

   and deduce that

$$\int_0^t B_s^2\,dB_s = \frac{1}{3}B_t^3 - \int_0^t B_s\,ds.$$

3. Let X_t and Y_t be Itô processes. Then, letting $f(t,x,y) = xy$ and by Itô's formula

$$\begin{aligned} d(X_tY_t) &= f_t(t,X_t,Y_t)\,dt + f_x(t,X_t,Y_t)\,dX_t + f_y(t,X_t,Y_t)\,dY_t \\ &\quad + \frac{1}{2}f_{xx}(t,X_t,Y_t)\,d[X,X]_t + f_{xy}(t,X_t,Y_t)\,d[X,Y]_t + \frac{1}{2}f_{yy}(t,X_t,Y_t)\,d[Y,Y]_t \\ &= Y_t\,dX_t + X_t\,dY_t + d[X,Y]_t \end{aligned}$$

and deduce the integration of parts formula

$$\begin{aligned}\int_0^t X_s\,dY_s &= \int_0^t (d(X_sY_s) - Y_s\,dX_s - d[X,Y]_s)\\ &= X_tY_t - X_0Y_0 - \int_0^t Y_s\,dX_s - \int_0^t d[X,Y]_s.\end{aligned}$$

4. Let $Z_t = \exp\left(\int_0^t \langle \theta(s,\omega), dB_s\rangle - \frac{1}{2}|\theta(s,\omega)|^2\,ds\right)$.

   (a) Then, letting $Z_t = e^{Y_t}$ and by Itô's formula,

$$\begin{aligned}dZ_t &= e^{Y_t}\,dY_t + \frac{1}{2}e^{Y_t}\,d[Y,Y]_t\\ &= Z_t\left(\langle\theta(t,\omega), dB_t\rangle - \frac{1}{2}|\theta(t,\omega)|^2\,dt + \frac{1}{2}\sum_{i,j=1}^n \left[\theta_i(s,\omega)\,dB^{(i)}, \theta_j(s,\omega)\,dB^{(j)}\right]_s\right)\\ &= Z_t\langle\theta(t,\omega), dB_t\rangle.\end{aligned}$$

   (b) It suffices to check that

$$\begin{aligned}[\mathbb{E}(|Z_t|)]^2 &= \left[\mathbb{E}\left(\left|\int_0^t dZ_s\right|\right)\right]^2\\ &= \left[\mathbb{E}\left(\left|\int_0^t Z_s\langle\theta(s,\omega), dB_s\rangle\right|\right)\right]^2\\ &\le \mathbb{E}\left(\int_0^t \sum_{i=1}^n |Z_s\theta_i(s,\omega)|\,dB_s^{(i)}\right)^2\\ &= \mathbb{E}\left(\sum_{i,j=1}^n \int_0^t |Z_s\theta_i(s,\omega)||Z_s\theta_j(s,\omega)|\,d[B^{(i)}, B^{(j)}]_s\right)\\ &= \sum_{i=1}^n \mathbb{E}\left(\int_0^t |Z_s\theta_i(s,\omega)|^2\,ds\right)\\ &< \infty.\end{aligned}$$

5. Let $\beta_k(t) = \mathbb{E}(B_t^k)$. Then, by Itô's lemma,

$$dB_t^k = kB_t^{k-1}\,dB_t + \frac{1}{2}k(k-1)B_t^{k-2}\,dt$$

   and so

$$\beta_k(t) = \mathbb{E}(B_t^k) = \mathbb{E}\left(\int_0^t dB_s^k\right) = \int_0^t \mathbb{E}\left(\frac{1}{2}k(k-1)B_t^{k-2}\right)\,ds = \frac{1}{2}k(k-1)\int_0^t \beta_{k-2}(s)\,ds.$$

   Deduce that $\beta_4(t) = 6\int_0^t \beta_2(s)\,ds = 6\cdot\frac{t^2}{2} = 3t^2$ and $\beta_6(t) = 15\int_0^t 3s^2\,ds = 15t^3$.

6. Problem 6

   (a) Define $X_t = e^{ct+\alpha B_t}$. Then

$$\begin{aligned} dX_t &= ce^{ct+\alpha B_t}\,dt + \alpha e^{ct+\alpha B_t}\,dB_t + \frac{1}{2}\alpha^2 e^{ct+\alpha B_t}\,d[B,B]_t \\ &= X_t\left((c+\frac{\alpha^2}{2})\,dt + \alpha\,dB_t\right). \end{aligned}$$

   (b) Define $X_t = e^{ct+\sum_{j=1}^n \alpha_j B_t^{(j)}}$. Then

$$\begin{aligned} dX_t &= X_t\left(c\,dt + \sum_{j=1}^n \alpha_j\,dB_t^{(j)} + \frac{1}{2}\sum_{i,j=1}^n \alpha_i\alpha_j d[B^{(i)},B^{(j)}]_t\right) \\ &= X_t\left((c+\frac{1}{2}\sum_{j=1}^n \alpha_i^2)\,dt + \sum_{j=1}^n \alpha_j\,dB_t^{(j)}\right). \end{aligned}$$

7. Let X_t solve $dX_t = v(t,\omega)\,dB_t$.

   (a) Note that B_t is a martingale while B_t^2 is not.

   (b) Define $M_t = X_t^2 - \int_0^t v(s,\omega)^2\,ds$. Then

$$\begin{aligned} dM_t &= 2X_t\,dX_t + [dX,dX]_t - v(t,\omega)^2,dt \\ &= 2X_t v(t,\omega)\,dB_t + \left(v(t,\omega)^2 - v(t,\omega)^2\right)\,dt \\ &= 2X_t v(t,\omega)\,dB_t. \end{aligned}$$

   Moreover,

$$\begin{aligned} \mathbb{E}(|M_t|) &\le \mathbb{E}(X_t^2) + \mathbb{E}\left(\int_0^t v(s,\omega)^2\,ds\right) \\ &= \mathbb{E}\left(\int_0^t v(s,\omega)\,dB_s\right)^2 + \mathbb{E}\left(\int_0^t v(s,\omega)^2\,ds\right) \\ &= 2\mathbb{E}\left(\int_0^t v(s,\omega)^2\,ds\right) \\ &< \infty. \end{aligned}$$

8. Problem 8

   (a) Let $f(x^{(1)},\dots x^{(n)})$ be a function of class C^2. Then

$$\begin{aligned} d(f(B_t)) &= \sum_{i=1}^n \partial_i f(B_t)\,dB_t^{(i)} + \frac{1}{2}\sum_{i,j=1}^n \partial_{ij}^2 f(B_t)\,d[B^{(i)},B^{(j)}]_t \\ &= \langle \nabla f(B_t), dB_t\rangle + \frac{1}{2}\Delta f(B_t)\,dt \end{aligned}$$

and so

$$f(B_t) - f(B_0) = \int_0^t d(f(B_s)) = \int_0^t \langle \nabla f(B_s), dB_s \rangle + \frac{1}{2}\int_0^t \Delta f(B_s)\, ds.$$

(b) Assume that g is of class C^1 everywhere, as well as C^2 and uniformly bounded outside of finitely many points with $|g''(z)| \leq M$ for $z \notin \{z_1, \dots z_k\}$. Then the set of functions $\{f\}$ of class C^2 uniformly bounded with $|f''(z)| \leq M$ are C^k-dense. So we can extract a sequence $\{f_k\}$ such that $f_k \rightrightarrows g$, $f_k' \rightrightarrows g'$ as well as $f_k'' \to g''$ and $|f_k''| \leq M$ on $\mathbb{R} \setminus \{z_1, \dots z_k\}$. So

$$\begin{aligned}
&\lim_{k\to\infty} \left| (f_k - g)(B_t) + (f_k - g)(0) + \int_0^t (f_k' - g')\, dB_s + \frac{1}{2}\int_0^t (f_k'' - g'')\, ds \right| \\
&\leq \lim_{k\to\infty} |(f_k - g)(B_t)| + |(f_k - g)(0)| + t\|f_k' - g'\|_\infty + \frac{1}{2}\int_0^t |f_k'' - g''|\, ds \\
&= 0,
\end{aligned}$$

where the last term vanishes by bounded convergence.

9. Clearly

$$\int_0^t v\frac{\partial g_n}{\partial x}(s, X_s)\chi_{s\leq\tau_n}\, dB_s = \int_0^{t\wedge\tau_n} v\frac{\partial g}{\partial x}(s, X_s)\, dB_s$$

and the result follows by Itô's lemma where $dX_t = u\,dt + v\,dB_t$. Since $\mathbb{E}(|X_t|) < \infty$, it follows that $\lim\limits_{n\to\infty} \mathbb{P}(\tau_n > t) = \lim\limits_{n\to\infty} \mathbb{P}(X_t < n) = 1$ and so the identity holds almost surely.

10. Problem 10

(a) Substitute $u \equiv 0$ and $v \equiv 1$ here. Then as $g_\varepsilon''(x) = \frac{1}{\varepsilon}\chi_{|x|<\varepsilon}(x)$

$$\frac{1}{2}\int_0^t \frac{d^2 g_\varepsilon}{dx^2}(B_s)\, ds = \frac{1}{2\varepsilon}\int_0^t \chi_{|B_s|<\varepsilon}\, ds = \frac{1}{2\varepsilon}|\{s \in [0,t] \mid |B_s| < \varepsilon\}|.$$

(b) Differentiate to get

$$\int_0^t g_\varepsilon'(B_s)\chi_{|B_s|<\varepsilon}\, dB_s = \int_0^t \frac{B_s}{\varepsilon}\chi_{|B_s|<\varepsilon}\, dB_s,$$

and apply Itô isometry to get

$$\lim_{\varepsilon\to 0^+} \mathbb{E}\left(\int_0^t \frac{B_s}{\varepsilon}\chi_{|B_s|<\varepsilon}\, dB_s\right)^2 = \lim_{\varepsilon\to 0^+} \mathbb{E}\left(\int_0^t \frac{B_s^2}{\varepsilon^2}\chi_{|B_s|<\varepsilon}\, ds\right) \leq \lim_{\varepsilon\to 0^+} \int_0^t \mathbb{P}(|B_s| < \varepsilon)\, ds = 0.$$

(c) As $\varepsilon \to 0$ for $g(x) = x$,

$$|B_t| = |B_0| + \lim_{\varepsilon \to 0^+} \int_0^t \operatorname{sgn}(B_s)\chi_{|B_s| \geq \varepsilon}\, ds + \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} |\{s \in [0,t] \,|\, |B_s| < \varepsilon\}|$$
$$= |B_0| + \int_0^t \operatorname{sgn}(B_s)\, ds + L_t.$$

11. Problem 11

(a) Let $X_t = e^{t/2}\cos(B_t)$. Then

$$dX_t = \frac{1}{2} e^{t/2}\cos(B_t)\, dt - e^{t/2}\sin(B_t)\, dB_t + \frac{1}{2}(-e^{t/2}\cos(B_t))\, d[B,B]_t = -e^{t/2}\sin(B_t)\, dB_t.$$

(b) Let $X_t = e^{t/2}\sin(B_t)$. Then

$$dX_t = \frac{1}{2} e^{t/2}\sin(B_t)\, dt + e^{t/2}\cos(B_t)\, dB_t + \frac{1}{2}(-e^{t/2}\sin(B_t))\, d[B,B]_t = e^{t/2}\cos(B_t)\, dB_t.$$

(c) Let $X_t = (B_t + t)e^{-B_t - t/2}$. Then

$$dX_t = e^{-B_t - t/2} d(B_t + t) + (B_t + t) d(e^{-B_t - t/2}) + d[B_t + t, e^{-B_t - t/2}]$$
$$= e^{-B_t - t/2}(dt + dB_t) - \frac{1}{2} X_t\, dt - X_t\, dB_t - e^{-B_t - t/2}\, dt + \frac{1}{2}(B_t + t)e^{-B_t - t/2}\, dt$$
$$= e^{-B_t - t/2}(1 - t - B_t)\, dB_t.$$

12. The given condition implies $\mathbb{E}(|X_t|) < \infty$. So X_t is a martingale if and only if $\mathbb{E}(X_t \,|\, \mathcal{F}_s) = X_s$. Then

$$\mathbb{E}(\int_s^t u(r,\omega)\, dr \,|\, \mathcal{F}_s) = \mathbb{E}(X_t - X_s \,|\, \mathcal{F}_s) = 0.$$

Moreover by dominated convergence

$$\mathbb{E}(u(t,\omega)\, dr \,|\, \mathcal{F}_s) = \mathbb{E}(\frac{d}{ds}\int_s^t u(r,\omega)\, dr \,|\, \mathcal{F}_s) = 0.$$

Then

$$u(t,\omega) = \mathbb{E}(u(t,\omega) \,|\, \mathcal{F}_t) = \lim_{s \to t^-} \mathbb{E}(u(t,\omega) \,|\, \mathcal{F}_s) = 0.$$

13. Let $dX_t = u(t,\omega)\, dt + dB_t$ where $u(t,\omega) \in \mathcal{V}([0,T])$. Then $Y_t = X_t M_t$ is a martingale, where

$$M_t = \exp\left(-\int_0^t u(r,\omega)\, dB_r - \frac{1}{2}\int_0^t u^2(r,\omega)\, dr \right)$$

since $\mathbb{E}(|M_t|) < \infty$ (see question 4b), $\mathbb{E}(|X_t|) \leq \sqrt{t}\left(\sqrt{\int_0^t u^2(r,\omega)\,dr} + 1\right) < \infty$ and

$$\begin{aligned}
d(X_t M_t) &= M_t dX_t + X_t dM_t + d[X,M]_t \\
&= M_t(u(t,\omega)\,dt + dB_t) + M_t X_t(-u(t,\omega)\,dB_t - \frac{1}{2}u^2(t,\omega)\,dt) \\
&\quad - M_t u(t,\omega)\,dt + \frac{1}{2}M_t X_t u^2(t,\omega)\,dt \\
&= M_t(1 - u(t,\omega)X_t)\,dB_t.
\end{aligned}$$

14. Problem 14

(a) Compute $dF_t = dB_t$, $\mathbb{E}(F_T) = 0$ and

$$dF_t - d\mathbb{E}(F_t) = 1\,dB_t \implies f(t,\omega) = 1.$$

(b) Compute $dF_t = B_t\,dt$, $\mathbb{E}(F_T) = 0$ and

$$dF_t - d\mathbb{E}(F_t) = B_t\,dt = d(TB_T) - t\,dB_t = (T-t)\,dB_t \implies f(t,\omega) = T - t.$$

(c) Compute $dF_t = 2B_t\,dB_t +\,dt$, $\mathbb{E}(F_T) = T$ and

$$dF_t - d\mathbb{E}(F_t) = 2B_t\,dB_t + 1\,dt - 1\,dt = 2B_t\,dB_t \implies f(t,\omega) = 2B_t.$$

(d) Compute $dF_t = 3B_t^2\,dB_t + 3B_t\,dt$, $\mathbb{E}(F_T) = 0$ and

$$\begin{aligned}
dF_t - d\mathbb{E}(F_t) &= 3B_t^2\,dB_t + 3B_t\,dt \\
&= 3B_t^2 + 3(T-t))\,dB_s \implies f(t,\omega) = 3B_t^2 + 3T - 3t.
\end{aligned}$$

(e) Recall that $e^{B_t - t/2}$ is a martingale and compute

$$d(e^{B_t - t/2}) = e^{B_t - t/2}\,dB_t.$$

Deduce that

$$e^{B_T} = e^{T/2}\left(1 + \int_0^T e^{B_t - t/2}\,dB_t\right) \implies f(t,\omega) = e^{B_t + (T-t)/2}.$$

(f) Find martingale $e^{t/2}\sin(B_t)$ and compute

$$d(e^{t/2}\sin(B_t)) = e^{t/2}\cos(B_t)\,dB_t$$

Deduce that

$$\sin(B_T) = e^{-T/2}\int_0^T e^{t/2}\cos(B_t)\,dB_t \implies f(t,\omega) = e^{-(T-t)/2}\cos(B_t).$$

15. Define $X_t = (x^{1/3} + \frac{1}{3}B_t)^3$. Then

$$\begin{aligned}
dX_t &= 3X_t^{2/3} d(x^{1/3} + \frac{1}{3}B_t) + 3X_t^{1/3} d\left[x^{1/3} + \frac{1}{3}B_t, x^{1/3} + \frac{1}{3}B_t\right] \\
&= X_t^{2/3}\,dB_t + \frac{1}{3}X_t^{1/3}\,dt.
\end{aligned}$$

Chapter 5

Stochastic Differential Equations

1. Question 1

   (a) Compute $dX_t = d(e^{B_t}) = e^{B_t}\,dB_t + \frac{1}{2}^{B_t}\,d[B,B]_t = \frac{1}{2}X_t\,dt + X_t\,dB_t$.

   (b) Compute
   $$dX_t = d\left(\frac{B_t}{1+t}\right) = \frac{1}{1+t}\,dB_t - \frac{B_t}{(1+t)^2}\,dt = \frac{1}{1+t}\,dB_t - \frac{1}{1+t}X_t\,dt.$$

   (c) Compute $dX_t = d(\sin(B_t)) = \cos(B_t)\,dB_t - \frac{1}{2}\sin(B_t)\,dt = \cos(B_t)\,dB_t - \frac{1}{2}X_t\,dt$.

   (d) Compute $dX_t^{(1)} = dt$ and
   $$dX_t^{(2)} = d(e^t B_t) = e^t\,dB_t + e^t B_t\,dt = e^t\,dB_t + X_t^{(2)}\,dt.$$

   (e) Compute differentials
   $$d(\cosh(B_t)) = \sinh(B_t)\,dB_t + \frac{1}{2}\cosh(B_t)\,dt$$
   and
   $$d(\sinh(B_t)) = \cosh(B_t)\,dB_t + \frac{1}{2}\sinh(B_t)\,dt.$$
   Deduce that
   $$\begin{pmatrix} dX_t^{(1)} \\ dX_t^{(2)} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} X_t^{(1)} \\ X_t^{(2)} \end{pmatrix}\,dt + \begin{pmatrix} X_t^{(2)} \\ X_t^{(1)} \end{pmatrix}\,dB_t.$$

2. Let $X_t^{(1)} = a\cos(B_t)$ and $X_t^{(2)} = b\sin(B_t)$. Then
   $$dX_t^{(1)} = -a\sin(B_t)\,dB_t - \frac{a}{2}\cos(B_t)\,dt = -\frac{1}{2}X_t^{(1)}\,dt - \frac{a}{b}X_t^{(2)}\,dB_t$$
   and
   $$dX_t^{(2)} = b\cos(B_t)\,dB_t - \frac{b}{2}\sin(B_t)\,dt = -\frac{1}{2}X_t^{(2)}\,dt + \frac{b}{a}X_t^{(1)}\,dB_t.$$

3. The solution is given by

$$X_t = X_0 \exp\left((r - \frac{1}{2}\sum_{k=1}^{n} \alpha_k^2)t + \sum_{k=1}^{n} \alpha_k \, dB_k \right).$$

4. Question 4

   (a) The solution to $dX_t^{(1)} = dt + dB_t^{(1)}$ is $X_t^{(1)} = X_0^{(1)} + t + B_t^{(1)}$ and

   $$dX_t^{(2)} = X_t^{(1)} \, dB_t^{(2)} = (X_0^{(1)} + t + B_t^{(1)}) \, dB_t^{(2)}$$

   is

   $$X_t^{(2)} = X_0^{(2)} + X_0^{(1)} B_t^{(2)} + \int_0^t (s + B_s^{(1)}) \, dB_s^{(2)}.$$

   (b) Using integrating factors, solve $dX_t = X_t \, dt + dB_t$ for

   $$e^{-t} X_t - X_0 = \int_0^t e^{-s} \, dB_s$$

   and deduce that the solution X_t is

   $$X_t = e^t X_0 + \int_0^t e^{t-s} \, dB_s.$$

   (c) Using integrating factors, solve $dX_t = -X_t \, dt + e^{-t} dB_t$ for

   $$e^t X_t - X_0 = \int_0^t dB_s$$

   and deduce that the solution X_t is

   $$X_t = e^{-t}(X_0 + B_t).$$

5. The Langevin equation is given by

$$dX_t - \mu X_t \, dt = \sigma dB_t.$$

   (a) Using integrating factors, solve for

   $$e^{-\mu t} X_t - X_0 = \int_0^t e^{-\mu s} \sigma \, dB_s$$

   and deduce that the solution X_t is

   $$X_t = e^{\mu t} X_0 + \sigma \int_0^t e^{\mu(t-s)} \, dB_s.$$

(b) The expected value of X_t is

$$\mathbb{E}(X_t) = e^{\mu t} X_0$$

and, by Itô isometry, the variance of X_t is

$$\mathbb{V}(X_t) = \mathbb{E}\left(\sigma^2 \left(\int_0^t e^{\mu(t-s)}\, dB_s\right)^2\right) = \mathbb{E}\left(\sigma^2 \int_0^t e^{2\mu(t-s)} ds\right) = \frac{\sigma^2}{2\mu}(e^{2\mu t} - 1).$$

6. Suppose Y_t is given by

$$dY_t = r\, dt + \alpha Y_t\, dB_t.$$

Using integrating factors, solve for

$$d(e^{-\alpha B_t} Y_t) = e^{-\alpha B_t} Y_t \left(r - \frac{\alpha^2}{2}\right) dt$$

and

$$e^{-\alpha B_t + \frac{\alpha^2}{2}t} Y_t - Y_0 = \int_0^t r e^{-\alpha B_s + \frac{\alpha^2}{2}s}\, ds.$$

Deduce that

$$Y_t = e^{\alpha B_t - \frac{\alpha^2}{2}t} Y_0 + r \int_0^t e^{\alpha(B_t - B_s) - \frac{\alpha^2}{2}(t-s)}\, ds.$$

7. The Ornstein-Uhlenbeck process is given by

$$dX_t = (m - X_t)\, dt + \sigma\, dB_t.$$

(a) Using integrating factors, solve for

$$e^t X_t - X_0 = \int_0^t e^s m\, ds + \int_0^t e^s \sigma\, dB_s$$

and deduce that the solution X_t is

$$X_t = e^{-t} X_0 + m(1 - e^{-t}) + \sigma \int_0^t e^{s-t}\, dB_s.$$

(b) The expected value of X_t is

$$\mathbb{E}(X_t) = m + e^{-t}(X_0 - m)$$

and the variance of X_t is

$$\mathbb{V}(X_t) = \mathbb{E}\left(\sigma^2 \left(\int_0^t e^{s-t}\, dB_s\right)^2\right) = \mathbb{E}\left(\sigma^2 \int_0^t e^{2s-2t}\, ds\right) = \frac{\sigma^2}{2}(1 - e^{-2t}).$$

8. Consider the stochastic differential equation

$$\begin{pmatrix} dX_t^{(1)} \\ dX_t^{(2)} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} X_t^{(1)} \\ X_t^{(2)} \end{pmatrix} dt + \begin{pmatrix} \alpha\, dB_t^{(1)} \\ \beta\, dB_t^{(2)} \end{pmatrix}.$$

   By d'Alembert's formula, it has a solution of the form

$$X_t = e^{At} X_0 + \int_0^t e^{A(t-s)} g(s)\, ds,$$

   where

$$e^{At} = \begin{pmatrix} 1 & 1 \\ i & -1 \end{pmatrix} \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -1 \end{pmatrix}^{-1} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}.$$

   Conclude that the solutions are

$$X_t^{(1)} = X_0^{(1)} \cos(t) + X_0^{(2)} \sin(t) + \alpha \int_0^t \cos(t-s)\, dB_s^{(1)} + \beta \int_0^t \sin(t-s)\, dB_s^{(2)}$$

   and

$$X_t^{(2)} = -X_0^{(1)} \sin(t) + X_0^{(2)} \cos(t) - \alpha \int_0^t \sin(t-s)\, dB_s^{(1)} + \beta \int_0^t \cos(t-s)\, dB_s^{(2)}.$$

9. Let $dX_t = \ln(1+X_t^2)\, dt + \chi_{\{X_t>0\}} X_t\, dB_t$. It suffices to check that

$$|b(t,x)| + |\sigma(t,x)| = \ln(1+x^2) + \chi_{\{x>0\}}|x| \le \frac{2}{e}(|x|+1) + |x| \le 2(|x|+1),$$

   $\mathbb{E}(|X_0|^2) = \alpha^2 < \infty$, and

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le |\ln(x^2) - \ln(y^2)| + |x-y| \le 3|x-y|.$$

   Hence, by Theorem 5.2.1, there is a unique strong solution to the stochastic differential equation.

10. Calculate

$$\begin{aligned}
\mathbb{E}(X_t^2) &= \mathbb{E}\left(Z + \int_0^t b(s,X_s)\, ds + \int_0^t \sigma(s,X_s)\, dB_s\right)^2 \\
&\le 3\left(\mathbb{E}(Z^2) + \mathbb{E}\left(\int_0^t b(s,X_s)\, ds\right)^2 + \mathbb{E}\left(\int_0^t \sigma(s,X_s)\, dB_s\right)^2\right) \\
&\le 3\left(\mathbb{E}(Z^2) + T\mathbb{E}\left(\int_0^t b(s,X_s)^2\, ds\right) + \mathbb{E}\left(\int_0^t \sigma(s,X_s)^2\, ds\right)\right) \\
&\le 3\mathbb{E}(Z^2) + 6C^2\left(T + \int_0^t \mathbb{E}(|X_s|^2)\, ds\right)(T+1) \\
&= (3\mathbb{E}(Z^2) + 6C^2 T(T+1)) + 6C^2(T+1)\int_0^t \mathbb{E}(|X_s|^2)\, ds.
\end{aligned}$$

   and apply Gronwall to derive the result.

11. Consider the stochastic process

$$Y_t = a(1-t) + bt + (1-t)\int_0^t \frac{dB_s}{1-s}.$$

Then $Y_0 = a$ and, for $t \in [0,1)$, Y_t solves

$$\begin{aligned}
dY_t &= (b-a)\,dt - \int_0^t \frac{dB_s}{1-s}\,dt + (1-t)\frac{dB_t}{1-t} \\
&= \frac{1}{1-t}\left((b-a)(1-t) - (1-t)\int_0^t \frac{dB_s}{1-s}\right)dt + dB_t \\
&= \frac{1}{1-t}\left(b - a(1-t) - bt - (1-t)\int_0^t \frac{dB_s}{1-s}\right)dt + dB_t \\
&= \frac{b - Y_t}{1-t}\,dt + dB_t.
\end{aligned}$$

Finally by Itô isometry $\mathbb{E}\left((1-t)^2\int_0^t \frac{dB_s}{1-s}\right)^2 = (1-t)^2\int_0^t \frac{1}{(1-s)^2}\,ds = (1-t)t \to 0$ as $t \to 1^-$ and so limit $\lim\limits_{t\to 1^-} Y_t \overset{\text{a.s.}}{=} b$.

12. Let $y''(t) + (1+\varepsilon W_t)y(t) = 0$ where $W_t = \frac{dB_t}{dt}$ is 1-dimensional white noise.

    (a) Rewrite

$$\begin{pmatrix} dy_t \\ d\dot{y}_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} y_t \\ \dot{y}_t \end{pmatrix}dt + \begin{pmatrix} 0 & 0 \\ -\varepsilon & 0 \end{pmatrix}\begin{pmatrix} y_t \\ \dot{y}_t \end{pmatrix}dB_t.$$

    (b) Check that, if $y(t) = y(0) + y'(0)t + \int_0^t (r-t)y(r)\,dr + \int_0^t \varepsilon(r-t)y(r)\,dB_r$, then

$$y'(t) = y'(0) - \int_0^t y(r)\,dr - \int_0^t \varepsilon y(r)\,dB_r = y'(0) - \int_0^t y(r)(1+\varepsilon W_r)\,dr$$

    and $y''(t) = -(1+\varepsilon W_r)\,dr$.

13. Let $x_t'' + a_0 x_t' + w^2 x_t = (T_0 - \alpha_0 x_t')\eta W_t$ where W_t is 1-dimensional white noise. Then

$$\begin{pmatrix} dx_t \\ d\dot{x}_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -w^2 & -a_0 \end{pmatrix}\begin{pmatrix} x_t \\ \dot{x}_t \end{pmatrix}dt + \begin{pmatrix} 0 & 0 \\ 0 & -\alpha_0\eta \end{pmatrix}\begin{pmatrix} x_t \\ \dot{x}_t \end{pmatrix}dB_t + \begin{pmatrix} 0 \\ T_0\eta \end{pmatrix}dB_t$$

and by d'Alembert's formula the solution is

$$X_t = e^{At}X_0 + \int_0^t e^{A(t-s)}KX_s\,dB_s + \int_0^t e^{A(t-s)}M\,dB_s.$$

The eigenvalues of A satisfy $\lambda^2+a_0\lambda+w^2=0$ and are $\lambda_\pm=-\frac{a_0}{2}\pm\sqrt{w^2-\frac{a_0^2}{4}}i=:-\lambda\pm\xi i$. Then take the exponential of matrix A

$$\begin{aligned}
e^{At} &= \begin{pmatrix}1 & 1\\ \lambda_+ & \lambda_-\end{pmatrix}\begin{pmatrix}e^{\lambda_+ t} & 0\\ 0 & e^{\lambda_- t}\end{pmatrix}\begin{pmatrix}1 & 1\\ \lambda_+ & \lambda_-\end{pmatrix}^{-1}\\
&= \frac{1}{\lambda_- - \lambda_+}\begin{pmatrix}\lambda_- e^{\lambda_+ t}-\lambda_+ e^{\lambda_- t} & e^{\lambda_- t}-e^{\lambda_+ t}\\ -\lambda_-\lambda_+(e^{\lambda_- t}-e^{\lambda_+ t}) & \lambda_- e^{\lambda_- t}-\lambda_+ e^{\lambda_+ t}\end{pmatrix}\\
&= -\frac{1}{2\xi i}\begin{pmatrix}e^{-\lambda t}(-\lambda\cdot 2i\sin(\xi t)-\xi i\cdot 2\cos(\xi t) & e^{-\lambda t}(-2i\sin(\xi t))\\ -w^2e^{-\lambda t}(-2i\sin(\xi t)) & e^{-\lambda t}(-\lambda\cdot 2i\sin(\xi t)-\xi i\cdot 2\cos(\xi t)+2\lambda\cdot 2i\sin(\xi t))\end{pmatrix}\\
&= \frac{e^{-\lambda t}}{\xi}\begin{pmatrix}\lambda\sin(\xi t)+\xi\cos(\xi t) & \sin(\xi t)\\ -w^2\sin(\xi t) & \lambda\sin(\xi t)+\xi\cos(\xi t)-2\lambda\sin(\xi t)\end{pmatrix}\\
&= \frac{e^{-\lambda t}}{\xi}\left((\lambda\sin(\xi t)+\xi\cos(\xi t))I+A\sin(\xi t)\right).
\end{aligned}$$

Next, letting $y_s=\dot{x}_s$, $g_t=e^{-\lambda t}\frac{\sin(\xi t)}{\xi}$ and $h_t=e^{-\lambda t}\frac{\xi\cos(\xi t)-\lambda\sin(\xi t)}{\xi}$, compute

$$e^{A(t-s)}KX_s=-\frac{\alpha_0\eta e^{-\lambda(t-s)}}{\xi}\begin{pmatrix}0 & \sin(\xi(t-s))\\ 0 & \xi\cos(\xi(t-s))-\lambda\sin(\xi(t-s))\end{pmatrix}\begin{pmatrix}x_s\\ \dot{x}_s\end{pmatrix}=\begin{pmatrix}-\alpha_0\eta y_s g_{t-s}\\ -\alpha_0\eta y_s h_{t-s}\end{pmatrix}$$

and

$$e^{A(t-s)}M=\frac{T_0\eta e^{-\lambda(t-s)}}{\xi}\begin{pmatrix}\sin(\xi(t-s))\\ \xi\cos(\xi(t-s))-\lambda\sin(\xi(t-s))\end{pmatrix}=\begin{pmatrix}\eta T_0 g_{t-s}\\ \eta T_0 h_{t-s}\end{pmatrix}.$$

It follows that

$$x_t=\eta\int_0^t(T_0-\alpha_0 y_s)g_{t-s}\,dB_s$$

and

$$y_t=\eta\int_0^t(T_0-\alpha_0 y_s)h_{t-s}\,dB_s.$$

14. Letting $Z_t=F(\mathbf{B}_t)$, where $\mathbf{B}_t=B_t^{(1)}+iB_t^{(2)}$, calculate

$$\begin{aligned}
dZ_t &= F_x(\mathbf{B}_t)\,dB_t^{(1)}+F_y(\mathbf{B}_t)\,dB_t^{(2)}\\
&\quad+\frac{1}{2}F_{xx}(\mathbf{B}_t)\,d[B^{(1)},B^{(1)}]_t+F_{xy}(\mathbf{B}_t)\,d[B^{(1)},B^{(2)}]_t+F_{yy}(\mathbf{B}_t)\,d[B^{(2)},B^{(2)}]_t\\
&= (u_x+iv_x)\,dB_t^{(1)}+(u_y+iv_y)\,dB_t^{(2)}+\frac{1}{2}(u_{xx}+iv_{xx}+u_{yy}+iv_{yy})\,dt\\
&= \langle F'(\mathbf{B}_t),dB_t\rangle+\frac{1}{2}(v_{xy}-iu_{xy}+u_{yy}+iv_{yy})\,dt\\
&= \langle F'(\mathbf{B}_t),dB_t\rangle+\frac{1}{2}(-u_{yy}-iv_{yy}+u_{yy}+iv_{yy})\,dt\\
&= \langle F'(\mathbf{B}_t),dB_t\rangle.
\end{aligned}$$

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