Øksendal: Stochastic Differential Equations

Solutions Manual

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Contents

Introduction

This is a solutions manual for Stochastic Differential Equations, 6^{th} edition, by Bernt Øksendal.

It was last updated in January 2026. Errors in the solutions to §3.1 and §3.2 have been corrected. The following problems have been solved to date:

- Chapter 2: Problems #1-17
- Chapter 3: Problems #1-17
- Chapter 4: Problems #1-15
- Chapter 5: Problems #1-17
- Chapter 7: Problems #1-11.

Some Mathematical Preliminaries (§2)

2.1 Suppose $X : \Omega \to \mathbb{R}$ is a function that assumes countably many values $\{a_j\}$ in $\mathbb{R}$.

(a) Note that X is a random variable if and only if it is measurable. If $X : \Omega \to \mathbb{R}$ is measurable, then $U = X^{-1}(\mathbb{R} \setminus a_k) \in \mathcal{F}$ and thus $X^{-1}(a_k) = \Omega \setminus U \in \mathcal{F}$, $\forall k$. On the other hand, if $X^{-1}(a_k) \in \mathcal{F}$, $\forall k$, then Borel set $V \subseteq \mathbb{R}$, $X^{-1}(V) = \bigcup_{a_k \in V} X^{-1}(a_k) \in \mathcal{F}$ and thus X is measurable.

(b) Compute $\mathbb{E}(|X|) = \int_{\mathbb{R}} |x| \, d\mathbb{P}_X = \int_{\bigcup_{k=1}^{\infty}\{a_k\}} |x| \, d\mathbb{P}_X = \sum_{k=1}^{\infty} |a_k| \mathbb{P}(X = a_k)$.

(c) If $\mathbb{E}(|X|) < \infty$, then the series

$$\mathbb{E}(X) = \int_{\mathbb{R}} x \, d\mathbb{P}_X = \int_{\bigcup_{k=1}^{\infty}\{a_k\}} x \, d\mathbb{P}_X = \sum_{k=1}^{\infty} a_k \mathbb{P}(X = a_k)$$

is absolutely convergent and therefore converges.

(d) If f is measurable and $|f|$ is bounded by M, then

$$\mathbb{E}(|f(X)|) = \int_{\mathbb{R}} |f(x)| \, d\mathbb{P}_X \leq \int_{\mathbb{R}} M \, d\mathbb{P}_X = M \int_{\mathbb{R}} d\mathbb{P}_X = M < \infty.$$

Hence,

$$\mathbb{E}(f(X)) = \int_{\mathbb{R}} f(x) \, d\mathbb{P}_X = \int_{\bigcup_{k=1}^{\infty}\{a_k\}} f(x) \, d\mathbb{P}_X = \sum_{k=1}^{\infty} f(a_k) \mathbb{P}(X = a_k)$$

is absolutely convergent and therefore converges.

2.2 Let $F(x) = \mathbb{P}(X \leq x)$ be the distribution function of X.

(a) By monotonicity of $\mathbb{P}$, $0 = \mathbb{P}(\emptyset) \leq \mathbb{P}(X \leq x) \leq P(\mathbb{R}) = 1$. Now, by the Monotone Convergence Theorem,

$$\lim_{n \to \infty} F(n) = \lim_{n \to \infty} \int_{\mathbb{R}} \chi_{(-\infty, n]} \, d\mathbb{P}(x) = \int_{\mathbb{R}} d\mathbb{P}(x) = 1.$$

Similarly, for $G(n) := 1 - F(-n)$, we have

$$\lim_{n \to \infty} G(n) = \lim_{n \to \infty} \int_{\mathbb{R}} (1 - \chi_{(-\infty, -n]}) dP_X(x) = 1.$$

Moreover, F is increasing by monotonicity of P and finally, again by Monotone Convergence,

$$\lim_{h\to 0^+} 1 - F(x+h) + F(x) = \lim_{h\to 0^+} \int_{\mathbb{R}} (1 - \chi_{(x,x+h]})\, d\mathbb{P}(x) = \int_{\mathbb{R}} d\mathbb{P}(x) = 1$$

and so $\lim_{h\to 0^+} F(x+h) = F(x)$, i.e. F is right-continuous.

(b) Compute the expectation

$$\mathbb{E}(g(X)) = \int_{\mathbb{R}} g(x)\, d\mathbb{P}(x) = \int_{\mathbb{R}} g(x)\chi_{(-\infty,x]}\, d\mathbb{P}(x) = \int_{\mathbb{R}} g(x)\, dF(x).$$

(c) Compute the density of B_t^2

$$\begin{aligned} F(u) := \mathbb{P}(B_t^2 \le u) &= \mathbb{P}(-\sqrt{u} \le B_t \le \sqrt{u}) \\ &= 2\int_{[0,\sqrt{u}]} p(y)dy \\ &= 2\int_{[0,u]} \frac{p(\sqrt{u})}{2\sqrt{u}} du \\ &= \int_{(-\infty,u]} \chi_{[0,\infty)} \frac{p(\sqrt{u})}{\sqrt{u}} du. \end{aligned}$$

and so $p(u) = \chi_{[0,\infty)} \frac{p(\sqrt{u})}{\sqrt{u}}$ where $p(u)$ is the density of B_t.

2.3 Since $\mathcal{H}_i$ is a σ-algebra, $\emptyset \in \mathcal{H}_i$, $\forall i \in I$. So $\emptyset \in \mathcal{H} = \cap_{i\in I}\mathcal{H}_i$. If $\{U_j\}_{j\in\mathbb{N}} \subseteq \mathcal{H}$, then $\{U_j\}_{j\in\mathbb{N}} \subseteq \mathcal{H}_i$ for each $i \in I$ and so $\Omega \setminus U_j \in \mathcal{H}_i$ and $\cup_{j\in\mathcal{A}} U_j \in \mathcal{H}_i$, $\forall i \in I$. Conclude that $\Omega \setminus U_j \in \mathcal{H}$ and $\cup_{j\in\mathcal{A}} U_j \in \mathcal{H}$ and $\mathcal{H} = \cap_{i\in I}\mathcal{H}_i$ is also a σ-algebra.

2.4 Let $X : \Omega \mapsto \mathbb{R}$ be a random variable with $\mathbb{E}(|X|^p) < \infty$.

(a) Let $A = \{\omega \in \Omega \,|\, |X| \ge \lambda > 0\}$ and compute

$$\mathbb{E}(|X|^p) = \int_{\Omega} |X|^p\, d\mathbb{P} \ge \int_A |X|^p\, d\mathbb{P} \ge \lambda^p \int_A d\mathbb{P} = \lambda^p \mathbb{P}(|X| \ge \lambda).$$

(b) By Chebychev, $\mathbb{P}(|X| \ge \lambda) = \mathbb{P}(e^{|X|} \ge e^{\lambda}) \le \frac{1}{e^{k\lambda}}\mathbb{E}(e^{k|X|}) = Me^{-k\lambda}$.

2.5 Since the measures are σ-finite, $f(x,y) = xy$ is $\mathbb{P}_X \otimes \mathbb{P}_Y$ measurable and $\mathbb{E}(|XY|) < \infty$, apply Fubini-Tonelli and compute

$$\begin{aligned} \mathbb{E}(XY) &= \int_{\mathbb{R}^2} xy\, d\mathbb{P}_{XY}(x,y) \\ &= \int_{\mathbb{R}^2} xy\, d\mathbb{P}_X(x) \otimes d\mathbb{P}_Y(y) \\ &= \int_{\mathbb{R}} y \left(\int_{\mathbb{R}} x\, d\mathbb{P}_X(x)\right) d\mathbb{P}_Y(y) \\ &= \mathbb{E}(X) \int_{\mathbb{R}} y\, d\mathbb{P}_Y(y) \\ &= \mathbb{E}(X)\mathbb{E}(Y). \end{aligned}$$

2.6 *(Borel-Cantelli)* Let $\{A_k\}_{k=1}^{\infty} \subseteq \mathcal{F}$ and suppose $\sum_{k=1}^{\infty} \mathbb{P}(A_k) < \infty$. Then

$$\mathbb{P}(\cap_{m=1}^{\infty} \cup_{k=m}^{\infty} A_k) \leq \lim_{m\to\infty} \sup_{k\geq m} \mathbb{P}(A_k) = 0$$

by dominated convergence.

2.7 Let $\Omega = \bigsqcup_{i=1}^{n} G_i$.

(a) Note $\emptyset \in \mathcal{G}$ and $\mathcal{G}$ is closed under unions by construction. It is also closed under complements as $\Omega \setminus G_i = \cup_{j\neq i} G_j \in \mathcal{G}$.

(b) Write a new sequence defined by $F_i = G_i \setminus \cup_{j\leq i} F_j$ and $\{F_i\}$ will satisfy (a).

(c) Note that $\{X^{-1}(x \in \mathbb{R})\} \subseteq \mathcal{F}$ is disjoint. So, by (a) and (b), $\mathcal{F}$ is finite if and only if all but finitely many $X^{-1}(x \in \mathbb{R})$ are empty.

2.8 Let B_t be a 1-dimensional Wiener process.

(a) By Equation 2.2.3, since $B_t \sim N(0, t)$,

$$\mathbb{E}(e^{iuB_t}) = \exp\left(-\frac{u^2}{2}\mathbb{V}(B_t) + iu\mathbb{E}(B_t)\right) = e^{-\frac{u^2}{2}}.$$

(b) Comparing power series coefficients, we deduce that

$$\frac{(iu)^{2n}}{(2n)!}\mathbb{E}(B_t^{2n}) = \frac{1}{n!}\left(-\frac{u^2 t}{2}\right)^n,$$

and so $\mathbb{E}(B_t^{2n}) = \frac{(2n)!}{2^n n!} t^n$.

(c) Integrating by parts, compute the n^{th} moment of B_t

$$\begin{aligned}
\mathbb{E}(B_t^{2k}) &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} x^{2k} e^{-\frac{x^2}{2t}}\, dx \\
&= x^{2k-1}\sqrt{\frac{2t}{\pi}} \int^{\frac{x}{\sqrt{2t}}} u e^{-u^2}\, du \Bigg|_{x=-\infty}^{x=\infty} - \int_{\mathbb{R}} (2k-1)x^{2k-2}\sqrt{\frac{2t}{\pi}} \int^{\frac{x}{\sqrt{2t}}} u e^{-u^2}\, du \\
&= -(2k-1)\sqrt{\frac{2t}{\pi}} \int_{\mathbb{R}} x^{2k-2}\left(\frac{-1}{2} e^{-\frac{x^2}{2t}}\right) dx \\
&= (2k-1)t \cdot \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} x^{2k-2} e^{-\frac{x^2}{2t}}\, dx \\
&= (2k-1)t\mathbb{E}(B_t^{2k-2}).
\end{aligned}$$

As $\mathbb{E}(B_t^2) = t$, we have that $\mathbb{E}(B_t^{2k}) = \frac{(2k)!t^{k-1}}{2^k k!} \cdot t = \frac{(2k)!t^k}{2^k k!}$.

(d) Check the base case, $n = 2k = 2$, where $\mathbb{E}(B_t)^2)] = \frac{2!\cdot t}{2\cdot 1!} = t$. If the claim is true for $n = 2k$, then

$$\mathbb{E}(B_t^{2k+2}) = (2k-1)t\mathbb{E}(B_t^{2k}) = (2k+1)t \cdot \frac{(2k)!t^k}{2^k k!} = \frac{(2k+2)!t^{k+1}}{2^{k+1}(k+1)!},$$

and so it is also true for $n = 2(k+1) = 2k+2$, thus completing the induction step.

2.9 Note that $\{X_t\}$ and $\{Y_t\}$ have the same distributions since neither distribution has any atoms and they agree except on a zero set $\forall t \geq 0$. Yet $t \mapsto X_t$ is discontinuous while $t \mapsto Y_t$ is continuous.

2.10 As B_t is Brownian, $B_{t+h} - B_t \sim N(0, h)$. Since h is fixed, $\{B_{t+h} - B_t\}_{h \geq 0}$ have the same distributions $\forall t \geq 0$.

2.11 As $B_0 = \left(B_0^{(1)}, B_0^{(2)}, \ldots B_0^{(n)}\right) = 0$, $B_0^{(j)} = 0$ for all $j \in \{1, \ldots n\}$. B_t is almost surely continuous only if its components are almost surely continuous. Each component is normally distributed with $\mathbb{E}(B_t^j) = 0$ as $\mathbb{E}(B_t) = \vec{0}$ and $\operatorname{Cov}(B_t^{(i)}, B_t^{(j)}) = t\delta_{ij}$ as $\operatorname{Cov}(B_t) = tI$.

2.12 Let $W_t := B_{s+t} - B_s$ where $s \geq 0$ is fixed. Then $W_0 = B_s - B_s = 0$ and W_t is almost surely continuous as the sum of two almost surely continuous stochastic processes. Noting $W_{t_2} - W_{t_1} = B_{s+t_2} - B_{s+t_1}$ is independent of both B_{s+t_1} and B_s, deduce that $W_{t_2} - W_{t_1}$ is independent of $W_{t_1} = B_{s+t_1} - B_s$. The expected value is $\mathbb{E}(W_t) = \mathbb{E}(B_{s+t}) - \mathbb{E}(B_s) = 0$ and the variance is

$$\begin{aligned}
\mathbb{V}(W_t) &= \mathbb{E}((B_{s+t} - B_s)^2) \\
&= \mathbb{E}(B_{s+t}^2) - 2\mathbb{E}(B_s B_{s+t}) + \mathbb{E}(B_s^2) \\
&= \mathbb{E}(B_{s+t}^2) - 2\mathbb{E}(B_s(B_{s+t} - B_s)) - \mathbb{E}(B_s^2) \\
&= \mathbb{E}(B_{s+t}^2) - 2\mathbb{E}(B_s)\mathbb{E}(B_{s+t} - B_s) - \mathbb{E}(B_s^2) \\
&= (s+t) - 0 - s \\
&= t.
\end{aligned}$$

Since W_t is the sum of two normal distributions, it is also normal and $W_t \sim N(0, t)$.

2.13 Compute

$$\mathbb{P}_0(B_t \in D_\rho) = \int_{|x|<\rho} \frac{1}{2\pi t} e^{-\frac{|\vec{x}|^2}{2t}} \, d^2\vec{x} = \frac{2\pi}{2\pi t} \int_0^\rho r e^{-\frac{r^2}{2t}} \, dr = \int_0^{\frac{\rho^2}{2t}} e^{-u} \, du = 1 - e^{-\frac{\rho^2}{2t}}.$$

2.14 Compute

$$\begin{aligned}
\mathbb{E}_x\left(\int_{[0,\infty]} \chi_K(B_t)\,dt\right) &= \int_{[0,\infty]} \mathbb{P}(B_t \in K)\,dt \\
&= \int_{[0,\infty]} \left(\int_K \frac{1}{(2\pi t)^{n/2}} e^{-\frac{|\vec{x}-\vec{y}|^2}{2t}}\, d^n\vec{x}\right) dt \\
&\le \int_{[0,\infty]} \left\| \frac{1}{(2\pi t)^{n/2}} e^{-\frac{|\vec{x}-\vec{y}|^2}{2t}} \right\|_\infty \mu(K)\,dt \\
&= 0
\end{aligned}$$

and deduce that the expected total time spent in K is 0.

2.15 Note that $UU^T = I$, whence $|\det U| = 1$ and the probability measures are identical by change of variables. It follows that both are Brownian.

2.16 Let $W_t = \frac{1}{c}B_{c^2t}$. We have $W_0 = B_0 = 0$ and that W_t is absolutely continuous as a scaling of absolutely continuous B_t. Finally,

$$\begin{aligned}
\mathbb{P}_0(W_t \in U) &= \mathbb{P}_0(B_{c^2t} \in cU) \\
&= \int_{cU} p(c^2t, 0, y)\,dy \\
&= \int_{cU} \frac{1}{c} p(t, 0, y/c)\,dy \\
&= \int_U \frac{1}{c} p(t, 0, y')(cdy') \\
&= \mathbb{P}_0(B_t \in U),
\end{aligned}$$

and so W_t is also a Brownian motion.

2.17 Let $X_t(\cdot)$ be a continuous stochastic process.

(a) Recall that $\mathbb{E}(B_t) = 0$, $\mathbb{E}(B_t^2) = t$ and $\mathbb{E}(B_t^4) = 3t^2$. Then

$$\begin{aligned}
\mathbb{E}\left((\sum_k \left(\Delta B_k^2 - \Delta t_k\right))^2\right) &= \mathbb{E}\left((\sum_k \left(\Delta B_k^2 - \Delta t_k\right)^2)\right) \\
&= \sum_k \left(\mathbb{E}(\Delta B_k^4) - 2\Delta t_k \mathbb{E}(\Delta B_k^2) + \Delta t_k^2\right) \\
&= \sum_k \left(3\Delta t_k^2 - 2\Delta t_k^2 + \Delta t_k^2\right) \\
&= 2\sum_k \Delta t_k^2.
\end{aligned}$$

So $\langle B, B\rangle_t^{(2)}(w) = t$.

(b) Note that the Brownian motion has positive quadratic variation t on $[0,t]$. So

$$\langle B, B\rangle_t^{(1)}(w) \geq \lim_{\|\Delta B_k\|\to 0^+} \frac{\langle B, B\rangle_t^{(2)}(w)}{\|\Delta B_k\|} = \infty.$$

Itô Integrals (§3)

3.1 Using a dyadic mesh, compute

$$\begin{aligned}\int_0^t s\,dB_s &= \lim_{n\to\infty}\left(\frac{\lfloor 2^n t\rfloor}{2^n}\left(B_t - B_{\frac{\lfloor 2^n t\rfloor}{2^n}}\right) + \sum_{j=0}^{\lfloor 2^n t\rfloor -1}\frac{j}{2^n}\left(B_{\frac{j+1}{2^n}} - B_{\frac{j}{2^n}}\right)\right)\\ &= \lim_{n\to\infty}\frac{\lfloor 2^n t\rfloor}{2^n}(B_t - B_{\frac{\lfloor 2^n t\rfloor}{2^n}}) + \lim_{n\to\infty}\frac{\lfloor 2^n t\rfloor - 1}{2^n}B_{\frac{\lfloor 2^n t\rfloor}{2^n}} - \lim_{n\to\infty}\frac{1}{2^n}\sum_{j=0}^{\lfloor 2^n t\rfloor -1} B_{\frac{j}{2^n}}\\ &= tB_t - \int_0^t B_s\,ds,\end{aligned}$$

where the first term vanishes as $B_t - B_{\frac{\lfloor 2^n t\rfloor}{2^n}} \overset{d}{\sim} \mathcal{N}(0, t - 2^{-n}\lfloor 2^n t\rfloor) \xrightarrow{d} 0$.

3.2 Using the same mesh from §3.1, compute

$$\begin{aligned}&\int_0^t B_s^2\,dB_s\\ &= \lim_{n\to\infty}\sum_{j=0}^{\lfloor 2^n t\rfloor -1} B_{j\cdot 2^{-n}}^2\left(B_{(j+1)\cdot 2^{-n}} - B_{j\cdot 2^{-n}}\right)\\ &= \lim_{n\to\infty}\sum_{j=0}^{\lfloor 2^n t\rfloor -1}\left(\frac{1}{3}B_{(j+1)\cdot 2^{-n}}^3 - \frac{1}{3}B_{j\cdot 2^{-n}}^3 - B_{j\cdot 2^{-n}}(B_{(j+1)\cdot 2^{-n}} - B_{j\cdot 2^{-n}})^2 - \frac{1}{3}(B_{(j+1)\cdot 2^{-n}} - B_{j\cdot 2^{-n}})^3\right)\\ &= \lim_{n\to\infty}\left(\frac{1}{3}B_{\lfloor 2^n t\rfloor\cdot 2^{-n}} - \left[\sum_{j=0}^{\lfloor 2^n t\rfloor -1} 2^{-n}B_{j\cdot 2^{-n}}\right] + \mathcal{O}(2^{-n/2})\right)\\ &= \frac{1}{3}B_t^3 - \int_0^t B_s\,ds.\end{aligned}$$

3.3 Let $\{\mathcal{N}_t\}$ be some filtration and let $\{\mathcal{H}_t^{(X)}\}$ be the filtration of process X_t.

(a) Compute

$$\mathbb{E}(X_t \,|\, \mathcal{H}_s^{(X)}) = \mathbb{E}\left(\mathbb{E}(X_t \,|\, \mathcal{N}_s) \,|\, \mathcal{H}_s^{(X)}\right) = \mathbb{E}(H_s \,|\, \mathcal{H}_s^{(X)}) = H_s.$$

(b) Compute

$$\mathbb{E}(X_t) = \mathbb{E}(\mathbb{E}(X_t \,|\, H_0^{(X)})) = \mathbb{E}(X_0).$$

(c) Let $Y \sim \text{Bernoulli}(0.5)$ and fix $X_0 = 2Y - 1$. Then $X_t = t \cdot \text{sgn}(X_0)$ satisfies $\mathbb{E}(X_t) = \mathbb{E}(X_0) = 0$, but $\mathbb{E}(X_t \,|\, \mathcal{F}_s) = t \cdot \text{sgn}(X_0) \neq s \cdot \text{sgn}(X_0)$.

3.4 Compute

$$\begin{aligned}
&\mathbb{E}(B_t + 4t \,|\, \mathcal{F}_s) = B_s + 4t \neq B_s + 4s \\
&\mathbb{E}(B_t^2 \,|\, \mathcal{F}_s) = \mathbb{E}((B_t - B_s)^2 + 2B_s(B_t - B_s) + B_s^2 \,|\, \mathcal{F}_s) = B_s^2 + t - s \neq B_s^2 \\
&\mathbb{E}\left(t^2 B_t - 2\int_0^t uB_u\,du \,|\, F_s\right) = t^2 B_s - 2\int_0^s uB_u\,du - 2\int_s^t uB_s\,du = s^2 B_s - 2\int_0^s uB_u du \\
&\mathbb{E}(B_t^{(1)} B_t^{(2)} \,|\, \mathcal{F}_s) = \mathbb{E}(B_t^{(1)} \,|\, \mathcal{F}_s)\mathbb{E}(B_t^{(2)} \,|\, \mathcal{F}_s) = B_s^{(1)} B_s^{(2)},
\end{aligned}$$

and deduce that only the last two are martingales.

3.5 Verify $\mathbb{E}(|B_t^2 - t|) \leq \mathbb{E}(B_t^2) + t = 2t < \infty$ and compute

$$\mathbb{E}(B_t^2 - t \,|\, \mathcal{F}_s) = \mathbb{E}((B_t - B_s)^2 + 2B_s(B_t - B_s) + B_s^2 - t \,|\, \mathcal{F}_s) = B_s^2 + t - s - t = B_s^2 - s.$$

to deduce that $X_t := B_t^2 - t$ is a martingale.

3.6 Verify $\mathbb{E}(|B_t^3 - 3tB_t|) \leq \sqrt{\mathbb{E}(B_t^2)}(\sqrt{\mathbb{E}(B_t^4)} + 3t) = (3 + \sqrt{3})t^{3/2} < \infty$ and compute

$$\begin{aligned}
\mathbb{E}(B_t^3 - 3tB_t \,|\, \mathcal{F}_s) &= \mathbb{E}((B_t - B_s)^3 + 3B_s(B_t - B_s)^2 + 3B_s^2(B_t - B_s) + B_s^3 - 3tB_s \,|\, \mathcal{F}_s) \\
&= 3B_s(t - s) + B_s^3 - 3tB_s \\
&= B_s^3 - 3sB_s
\end{aligned}$$

to deduce that $Y_t := B_t^3 - 3tB_t$ is a martingale.

3.7 In this question, the formula for Itô iterated integrals is derived.

(a) Note that $\{0 \leq u_1 \cdots \leq u_n\}$ is Borel measurable and $\chi_{0 \leq u_1 \cdots \leq u_n}$ is $\mathcal{F}_t$-adapted. Finally $\mathbb{E}\left(\int_0^T f(t_1, \ldots t_n, \omega)^2\,dt_1 \ldots dt_n\right) \leq T^n < \infty$.

(b) For $n \in \{1, 2, 3\}$

$$\begin{aligned}
&1!\int_0^t dB_u = B_t = t^{1/2} H_1\left(\frac{B_t}{\sqrt{t}}\right) \\
&2!\int_0^t \int_0^v dB_u\,dB_v = 2\int_0^t B_v\,dB_v = B_t^2 - t = tH_2\left(\frac{B_2}{\sqrt{t}}\right) \\
&3!\int_0^t \int_0^w \int_0^v dB_u\,dB_v\,dB_w = 3\int_0^t (B_w^2 - w)\,dB_w = B_t^3 - 3tB_t = t^{3/2} H_3\left(\frac{B_t}{\sqrt{t}}\right).
\end{aligned}$$

(c) Deduce that $d(B_t^3 - 3tB_t) = 3(B_t^2 - t)\,dB_t$ and so $Y_t := B_t^3 - 3tB_t$ is a martingale.

3.8 There exists continuous martingale M_t iff there exists $Y \in L^1$ such that $M_t = \mathbb{E}(Y \mid \mathcal{F}_t)$.

(a) Verify that $\mathbb{E}(|\mathbb{E}(Y \mid \mathcal{F}_t)|) \le \mathbb{E}(\mathbb{E}(|Y| \mid \mathcal{F}_t) = \mathbb{E}(|Y|) < \infty$ and

$$\mathbb{E}(M_t \mid \mathcal{F}_s) = \mathbb{E}(\mathbb{E}(Y \mid \mathcal{F}_t) \mid \mathcal{F}_s) = \mathbb{E}(Y \mid \mathcal{F}_s) = M_s.$$

(b) If M_t is a continuous martingale such that $\sup_{t>0} \mathbb{E}(|X|^p) < \infty$ for $p \in (1, \infty)$, then $\exists M$ such that $\|M_t - M\|_{L^1} \to 0$ as $t \to \infty$. So let $Y = M$ and

$$\begin{aligned}
\lim_{s\to\infty} \int_{\Omega_s} |M_s - \mathbb{E}(M \mid \mathcal{F}_s)|\, d\mathbb{P} &= \lim_{s\to\infty} \int_{\Omega_s} |\mathbb{E}(M_s - M \mid \mathcal{F}_s)|\, d\mathbb{P} \\
&\le \lim_{s\to\infty} \int_{\Omega_s} \mathbb{E}(|M_s - M| \mid \mathcal{F}_s)\, d\mathbb{P} \\
&= \lim_{s\to\infty} \int_{\Omega_s} |M_s - M|\, d\mathbb{P} \\
&= 0.
\end{aligned}$$

3.9 Compute

$$\begin{aligned}
\int_0^T B_t \circ dB_t &= \lim_{n\to\infty} \sum_{j=0}^{\frac{\lfloor nt \rfloor}{t}-1} \frac{1}{2}(B_{\frac{jt}{n}} + B_{\frac{(j+1)t}{n}})(B_{\frac{(j+1)t}{n}} - B_{\frac{jt}{n}}) \\
&= \lim_{n\to\infty} \sum_{j=0}^{\frac{\lfloor nt \rfloor}{t}-1} B_{\frac{jt}{n}}(B_{\frac{(j+1)t}{n}} - B_{\frac{jt}{n}}) + \lim_{n\to\infty} \sum_{j=0}^{\frac{\lfloor nt \rfloor}{t}-1} \frac{1}{2}(B_{\frac{(j+1)t}{n}} - B_{\frac{jt}{n}})^2 \\
&= \frac{1}{2}B_t^2 - \frac{t}{2} + \frac{t}{2} \\
&= \frac{1}{2}B_t^2.
\end{aligned}$$

3.10 If $f(t, \omega)$ varies smoothly in t, then the Itô and Stratonovich integrals coincide. Compute

$$\int_0^T f(t,\omega) \circ dB_t = \int_0^T f(t,\omega)\, dB_t + \frac{1}{2}\langle f(t,\omega), B_t\rangle^{(2)}$$

and

$$\begin{aligned}
\mathbb{E}(\langle f(t,\omega), B_t\rangle^{(2)})^2 &\le \mathbb{E}(\langle B_t, B_t\rangle^{(2)}\mathbb{E}(\langle f(t,\omega), f(t,\omega)\rangle^{(2)} \\
&\le T \lim_{\|\Delta t_k\|\to 0^+} \sup_{|\Delta t_k|} \frac{T}{|\Delta t_k|}(K|\Delta t_k|^{1+\varepsilon}) \\
&= KT^2 \lim_{\|\Delta t_k\|\to 0^+} \|\Delta t_k\|^{\varepsilon} \\
&= 0.
\end{aligned}$$

3.11 Define white noise $W_t^{(N)} = \max\{-N, \min\{W_t, N\}\}$. Since W_t and W_s are independent and identically distributed, it follows that $W_t^{(N)}$ and $W_s^{(N)}$ are as well. If W_t is continuous, then since $|W_t^{(N)}| \leq N$ and by bounded convergence

$$\lim_{t \to s} 2\mathbb{E}(W_t^{(N)})^2 = \lim_{t \to s} \mathbb{E}(|W_t^{(N)} - W_s^{(N)}|^2) = 0.$$

But then $W_t \stackrel{\text{a.s.}}{=} 0$, which is a contradiction.

3.12 Let $\circ dB_t$ denote the Stratonovich differential.

(i) Since $\alpha X_t \circ dB_t = \frac{\alpha^2}{2} X_t dt + \alpha X_t \, dB_t$,

$$dX_t = (\gamma + \frac{\alpha^2}{2}) X_t \, dt + \alpha X_t \, dB_t.$$

Since $(t^2 + \cos(X_t)) \circ dB_t = -\frac{\sin(X_t)}{2}(t^2 + \cos(X_t)) \, dt + (t^2 + \cos(X_t) \, dB_t$,

$$dX_t = \frac{\sin(X_t)}{2}(\cos(X_t) - t^2) \, dt + (t^2 + \cos(X_t)) \, dB_t.$$

(ii) Since $\alpha X_t \, dB_t = \alpha X_t \circ dB_t - \frac{\alpha^2}{2} X_t \, dt$,

$$dX_t = (r - \frac{\alpha^2}{2}) X_t \, dt + \alpha X_t \circ dB_t.$$

Since $X_t^2 \, dB_t = X_t^2 \circ dB_t - X_t^3 \, dt$,

$$dX_t = (2e^{-X_t} - X_t^3) \, dt + X_t^2 \circ dB_t.$$

3.13 Let X_t be continuous in mean square. Calculate

(a) $\lim_{s \to t} \mathbb{E}[(B_t - B_s)^2] = \lim_{s \to t} \mathbb{E}[(B_{t-s})^2] = \lim_{s \to t}(t - s) = 0$

(b) $\lim_{s \to t} \mathbb{E}[(f(B_t) - f(B_s))^2] \leq \lim_{s \to t} C^2 \mathbb{E}[(B_t - B_s)^2] = 0$

(c) and finally by Itô isometry,

$$\begin{aligned}
\lim_{n \to \infty} \mathbb{E}\left[\left(\int_S^T (X_s - \phi_n(s)) \, dB_s\right)^2\right] &= \lim_{n \to \infty} \mathbb{E}\left[\int_S^T (X_s - \phi_n(s))^2 \, ds\right] \\
&= \lim_{n \to \infty} \mathbb{E}\left[\sum_j \int_{t_j^{(n)}}^{t_j^{(n+1)}} (X_t - X_{t_j^{(n)}})^2 \, dt\right] \\
&\leq (T - S) \lim_{n \to \infty} \sup_{1 \leq j \leq n} \mathbb{E}[(X_t - X_{t_j^{(n)}})^2] \\
&= 0.
\end{aligned}$$

3.14 Show that $h(\omega)$ is $\mathcal{F}_t$ measurable if and only if it is the pointwise limit of a sum-product of bounded continuous functions $g(B_{t_j})$.

(a) Assume that h is bounded since $\{h_n(\omega) := h(\omega)\mathbb{1}_{\{|h(\omega)|<n\}}\}$ converges pointwise to h.

(b) Let $\mathcal{H}_n$ be the σ-algebra generated by $B(t_j)$ for $t_j = \frac{j}{2^n} \le t$. Then $\mathcal{F}_t = \sigma(\cup_n \mathcal{H}_n)$ and so by Corollary (C.9), $h = \mathbb{E}[h|\mathcal{F}_n] = \lim_{n\to\infty} \mathbb{E}[h|\mathcal{H}_n]$.

(c) By Doob-Dynkin, $\mathbb{E}[h|\mathcal{H}_n](\omega) = g\left(B_{t_1}, \dots B(t_{\lfloor 2^n t \rfloor})\right)$. Since $C(\mathbb{R}^k)$ is dense in $L^1(\mathbb{R}^k)$ and by Stone-Weierstrass $P(\mathbb{R}^k)$ is dense in $C(\mathbb{R}^k)$, a limiting sequence must exist.

3.15 Suppose $C + \int_S^T f(t,\omega)\, dB_t(\omega) = D + \int_S^T g(t,\omega)\, dB_t(\omega)$. Then we have that

$$C - D = \mathbb{E}[C-D] = \mathbb{E}\left[\int_S^T g(t,\omega)\, dB_t(\omega) - \int_S^T f(t,\omega)\, dB_t(\omega)\right] = 0 \implies C = D,$$

and by Itô isometry,

$$0 = \mathbb{E}\left[\left(\int_S^T g(t,\omega)\, dB_t(\omega) - \int_S^T f(t,\omega)\, dB_t(\omega)\right)^2\right] = \int_S^T \mathbb{E}[(g(t,\omega) - f(t,\omega))^2]\, ds,$$

whence $g(t,\omega) = f(t,\omega)$ almost surely for $(t,\omega) \in [S,T] \times \Omega$.

3.16 By Jensen's inequality, $\mathbb{E}\left[\mathbb{E}[X|\mathcal{H}]^2\right] \le \mathbb{E}\left[\mathbb{E}[X^2|\mathcal{H}]\right] = \mathbb{E}[X^2]$.

3.17 Let $\mathcal{G}$ be a finite σ-algebra with partition $\Omega = \bigsqcup_{i=1}^n G_i$.

(a) Note that $\mathbb{E}[X|\mathcal{G}](\omega) = \sum_{i=1}^n c_i \mathbb{1}_{G_i}(\omega) = c_i$ on G_i.

(b) Show that

$$\int_{G_i} \left(\frac{\int_{G_i} X\, d\mathbb{P}}{\mathbb{P}(G_i)}\, d\mathbb{P}\right) = \frac{\int_{G_i} X\, d\mathbb{P}}{\mathbb{P}(G_i)} \int_{G_i} 1\, d\mathbb{P} = \int_{G_i} X\, d\mathbb{P},\ \forall i \in \{1, \dots n\}.$$

(c) By part (b), $c_i = \frac{\int_{G_i} X\, d\mathbb{P}}{\mathbb{P}(G_i)}$. Show for $\omega \in G_i$ that

$$\begin{aligned}
\mathbb{E}[X|\mathcal{G}](\omega) &= \sum_{i=1}^n \frac{\int_{G_i} X\, d\mathbb{P}}{\mathbb{P}(G_i)} \mathbb{1}_{G_i}(\omega) \\
&= \frac{\int_{G_i} X\, d\mathbb{P}}{\mathbb{P}(G_i)} \\
&= \frac{\sum_{k=1}^m a_k \mathbb{P}(X = a_k, \omega \in G_i)}{\mathbb{P}(G_i)} \\
&= \sum_{k=1}^m a_k \mathbb{P}(X = a_k | G_i).
\end{aligned}$$

The Itô Formula (§4)

4.1 Compute

(a) $dX_t = d(B_t^2) = 2B_t\,dB_t + d[B,B]_t = 2B_t\,dB_t + dt$

(b) $dX_t = d(2 + t + e^{B_t}) = (1 + \frac{1}{2}e^{B_t})\,dt + e^{B_t}\,dB_t$

(c) $dX_t = d\left((B_t^{(1)})^2 + (B_t^{(2)})^2\right) = 2B_t^{(1)}\,dB_t^{(1)} + 2B_t^{(2)}\,dB_t^{(2)} + 2\,dt$

(d) $dX_t = d((t_0 + t, B_t)) = (dt, dB_t)$

(e) and finally

$$\begin{aligned} dX_t &= d((B_t^{(1)} + B_t^{(2)} + B_t^{(3)}, (B_t^{(2)})^2 - B_t^{(1)}B_t^{(3)})) \\ &= (dB_t^{(1)} + dB_t^{(2)} + dB_t^{(3)}, 2B_t^{(2)}\,dB_t^{(2)} + dt - B_t^{(3)}\,dB_t^{(1)} - B_t^{(1)}\,dB_t^{(3)}). \end{aligned}$$

4.2 Using Itô's Lemma, differentiate

$$d\left(\frac{1}{3}B_t^3 - \int_0^t B_s\,ds\right) = B_t^2\,dB_t + B_t\,d[B,B]_t - B_t\,dt = B_t^2 dB_t$$

and deduce that

$$\int_0^t B_s^2\,dB_s = \frac{1}{3}B_t^3 - \int_0^t B_s\,ds.$$

4.3 Let X_t and Y_t be Itô processes. Then, letting $f(t,x,y) = xy$ and by Itô's formula

$$\begin{aligned} d(X_tY_t) &= f_t(t,X_t,Y_t)\,dt + f_x(t,X_t,Y_t)\,dX_t + f_y(t,X_t,Y_t)\,dY_t \\ &\quad + \frac{1}{2}f_{xx}(t,X_t,Y_t)\,d[X,X]_t + f_{xy}(t,X_t,Y_t)\,d[X,Y]_t + \frac{1}{2}f_{yy}(t,X_t,Y_t)\,d[Y,Y]_t \\ &= Y_t\,dX_t + X_t\,dY_t + d[X,Y]_t \end{aligned}$$

and deduce the integration of parts formula

$$\begin{aligned} \int_0^t X_s\,dY_s &= \int_0^t (d(X_sY_s) - Y_s\,dX_s - d[X,Y]_s) \\ &= X_tY_t - X_0Y_0 - \int_0^t Y_s\,dX_s - \int_0^t d[X,Y]_s. \end{aligned}$$

4.4 Let $Z_t = \exp\left(\int_0^t \langle \theta(s,\omega), dB_s\rangle - \frac{1}{2}|\theta(s,\omega)|^2\,ds\right)$.

(a) Then, letting $Z_t = e^{Y_t}$ and by Itô's formula,

$$\begin{aligned} dZ_t &= e^{Y_t}\,dY_t + \frac{1}{2}e^{Y_t}\,d[Y,Y]_t \\ &= Z_t\left(\langle \theta(t,\omega), dB_t\rangle - \frac{1}{2}|\theta(t,\omega)|^2\,dt + \frac{1}{2}\sum_{i,j=1}^n \left[\theta_i(s,\omega)\,dB^{(i)}, \theta_j(s,\omega)\,dB^{(j)}\right]_s\right) \\ &= Z_t\langle \theta(t,\omega), dB_t\rangle. \end{aligned}$$

(b) It suffices to check that

$$\begin{aligned} [\mathbb{E}(|Z_t|)]^2 &= \left[\mathbb{E}\left(\left|\int_0^t dZ_s\right|\right)\right]^2 \\ &= \left[\mathbb{E}\left(\left|\int_0^t Z_s\langle\theta(s,\omega), dB_s\rangle\right|\right)\right]^2 \\ &\le \mathbb{E}\left(\int_0^t \sum_{i=1}^n |Z_s\theta_i(s,\omega)|\,dB_s^{(i)}\right)^2 \\ &= \mathbb{E}\left(\sum_{i,j=1}^n \int_0^t |Z_s\theta_i(s,\omega)||Z_s\theta_j(s,\omega)|\,d[B^{(i)}, B^{(j)}]_s\right) \\ &= \sum_{i=1}^n \mathbb{E}\left(\int_0^t |Z_s\theta_i(s,\omega)|^2\,ds\right) \\ &< \infty. \end{aligned}$$

4.5 Let $\beta_k(t) = \mathbb{E}(B_t^k)$. Then, by Itô's lemma,

$$dB_t^k = kB_t^{k-1}\,dB_t + \frac{1}{2}k(k-1)B_t^{k-2}\,dt$$

and so

$$\beta_k(t) = \mathbb{E}(B_t^k) = \mathbb{E}\left(\int_0^t dB_s^k\right) = \int_0^t \mathbb{E}\left(\frac{1}{2}k(k-1)B_t^{k-2}\right)\,ds = \frac{1}{2}k(k-1)\int_0^t \beta_{k-2}(s)\,ds.$$

Deduce that $\beta_4(t) = 6\int_0^t \beta_2(s)\,ds = 6\cdot\frac{t^2}{2} = 3t^2$ and $\beta_6(t) = 15\int_0^t 3s^2\,ds = 15t^3$.

4.6 Define geometric Brownian motions $X_t = e^{ct+\alpha B_t}$ and $Y_t = e^{ct+\sum_{j=1}^n \alpha_j B_t^{(j)}}$.

(a) Calculate

$$\begin{aligned} dX_t &= ce^{ct+\alpha B_t}\,dt + \alpha e^{ct+\alpha B_t}\,dB_t + \frac{1}{2}\alpha^2 e^{ct+\alpha B_t}\,d[B,B]_t \\ &= X_t\left((c+\frac{\alpha^2}{2})\,dt + \alpha\,dB_t\right). \end{aligned}$$

(b) Calculate

$$\begin{aligned} dY_t &= Y_t \left(c\,dt + \sum_{j=1}^n \alpha_j\,dB_t^{(j)} + \frac{1}{2}\sum_{i,j=1}^n \alpha_i\alpha_j d[B^{(i)}, B^{(j)}]_t \right) \\ &= Y_t \left((c + \frac{1}{2}\sum_{j=1}^n \alpha_i^2)\,dt + \sum_{j=1}^n \alpha_j\,dB_t^{(j)} \right). \end{aligned}$$

4.7 Let X_t solve $dX_t = v(t,\omega)\,dB_t$.

(a) Note that B_t is a martingale while B_t^2 is not.

(b) Define $M_t = X_t^2 - \int_0^t v(s,\omega)^2\,ds$. Then

$$\begin{aligned} dM_t &= 2X_t\,dX_t + [dX, dX]_t - v(t,\omega)^2, dt \\ &= 2X_t v(t,\omega)\,dB_t + \left(v(t,\omega)^2 - v(t,\omega)^2 \right)\,dt \\ &= 2X_t v(t,\omega)\,dB_t. \end{aligned}$$

Moreover,

$$\begin{aligned} \mathbb{E}(|M_t|) &\le \mathbb{E}(X_t^2) + \mathbb{E}\left(\int_0^t v(s,\omega)^2\,ds \right) \\ &= \mathbb{E}\left(\int_0^t v(s,\omega)\,dB_s \right)^2 + \mathbb{E}\left(\int_0^t v(s,\omega)^2\,ds \right) \\ &= 2\mathbb{E}\left(\int_0^t v(s,\omega)^2\,ds \right) \\ &< \infty. \end{aligned}$$

4.8 Let $f(x^{(1)}, \dots x^{(n)})$ be a function of class C^2.

(a) By Itô's lemma,

$$\begin{aligned} d(f(B_t)) &= \sum_{i=1}^n \partial_i f(B_t)\,dB_t^{(i)} + \frac{1}{2}\sum_{i,j=1}^n \partial_{ij}^2 f(B_t)\,d[B^{(i)}, B^{(j)}]_t \\ &= \langle \nabla f(B_t), dB_t \rangle + \frac{1}{2}\Delta f(B_t)\,dt \end{aligned}$$

and so

$$f(B_t) - f(B_0) = \int_0^t d(f(B_s)) = \int_0^t \langle \nabla f(B_s), dB_s \rangle + \frac{1}{2}\int_0^t \Delta f(B_s)\,ds.$$

(b) Assume that g is of class C^1 everywhere, as well as C^2 and uniformly bounded outside of finitely many points with $|g''(z)| \le M$ for $z \notin \{z_1, \dots z_k\}$. Then the set of functions $\{f\}$ of class C^2 uniformly bounded with $|f''(z)| \le M$ are C^k-dense. So we can extract a sequence $\{f_k\}$ such that $f_k \rightrightarrows g$, $f_k' \rightrightarrows g'$ as well as $f_k'' \to g''$ and $|f_k''| \le M$ on $\mathbb{R} \setminus \{z_1, \dots z_k\}$. So

$$\begin{aligned}
&\lim_{k\to\infty} \left| (f_k - g)(B_t) + (f_k - g)(0) + \int_0^t (f_k' - g')\, dB_s + \frac{1}{2}\int_0^t (f_k'' - g'')\, ds \right| \\
&\le \lim_{k\to\infty} |(f_k - g)(B_t)| + |(f_k - g)(0)| + t\|f_k' - g'\|_\infty + \frac{1}{2}\int_0^t |f_k'' - g''|\, ds \\
&= 0,
\end{aligned}$$

where the last term vanishes by bounded convergence.

4.9 Clearly

$$\int_0^t v\frac{\partial g_n}{\partial x}(s, X_s)\chi_{s\le\tau_n}\, dB_s = \int_0^{t\wedge\tau_n} v\frac{\partial g}{\partial x}(s, X_s)\, dB_s$$

and the result follows by Itô's lemma where $dX_t = u\, dt + v\, dB_t$. Since $\mathbb{E}(|X_t|) < \infty$, it follows that $\lim\limits_{n\to\infty} \mathbb{P}(\tau_n > t) = \lim\limits_{n\to\infty} \mathbb{P}(X_t < n) = 1$ and so the identity holds almost surely.

4.10 *(Tanaka)* In this problem, Tanaka's formula for Brownian motion is derived.

(a) Substitute $u \equiv 0$ and $v \equiv 1$ here. Then as $g_\varepsilon''(x) = \frac{1}{\varepsilon}\chi_{|x|<\varepsilon}(x)$

$$\frac{1}{2}\int_0^t \frac{d^2 g_\varepsilon}{dx^2}(B_s)\, ds = \frac{1}{2\varepsilon}\int_0^t \chi_{|B_s|<\varepsilon}\, ds = \frac{1}{2\varepsilon}|\{s \in [0,t] \mid |B_s| < \varepsilon\}|.$$

(b) Differentiate to get

$$\int_0^t g_\varepsilon'(B_s)\chi_{|B_s|<\varepsilon}\, dB_s = \int_0^t \frac{B_s}{\varepsilon}\chi_{|B_s|<\varepsilon}\, dB_s,$$

and apply Itô isometry to get

$$\lim_{\varepsilon\to 0^+} \mathbb{E}\left(\int_0^t \frac{B_s}{\varepsilon}\chi_{|B_s|<\varepsilon}\, dB_s\right)^2 = \lim_{\varepsilon\to 0^+} \mathbb{E}\left(\int_0^t \frac{B_s^2}{\varepsilon^2}\chi_{|B_s|<\varepsilon}\, ds\right) \le \lim_{\varepsilon\to 0^+}\int_0^t \mathbb{P}(|B_s| < \varepsilon)\, ds = 0.$$

(c) As $\varepsilon \to 0$ for $g(x) = x$,

$$\begin{aligned}
|B_t| &= |B_0| + \lim_{\varepsilon\to 0^+}\int_0^t \operatorname{sgn}(B_s)\chi_{|B_s|\ge\varepsilon}\, ds + \lim_{\varepsilon\to 0^+}\frac{1}{2\varepsilon}|\{s \in [0,t] \mid |B_s| < \varepsilon\}| \\
&= |B_0| + \int_0^t \operatorname{sgn}(B_s)\, ds + L_t.
\end{aligned}$$

4.11 Let $X_t = e^{t/2}\cos(B_t)$, $Y_t = e^{t/2}\sin(B_t)$ and $Z_t = (B_t + t)e^{-B_t - t/2}$. Compute

(a) $dX_t = \frac{1}{2}e^{t/2}\cos(B_t)\, dt - e^{t/2}\sin(B_t)\, dB_t + \frac{1}{2}(-e^{t/2}\cos(B_t))\, d[B,B]_t = -e^{t/2}\sin(B_t)\, dB_t$

(b) $dY_t = \frac{1}{2}e^{t/2}\sin(B_t)\, dt + e^{t/2}\cos(B_t)\, dB_t + \frac{1}{2}(-e^{t/2}\sin(B_t))\, d[B,B]_t = e^{t/2}\cos(B_t)\, dB_t$

(c) and finally

$$\begin{aligned}
dZ_t &= e^{-B_t-t/2}d(B_t+t) + (B_t+t)d(e^{-B_t-t/2}) + d[B_t+t, e^{-B_t-t/2}] \\
&= e^{-B_t-t/2}(dt+dB_t) - \frac{1}{2}X_t\,dt - X_t\,dB_t - e^{-B_t-t/2}\,dt + \frac{1}{2}(B_t+t)e^{-B_t-t/2}\,dt \\
&= e^{-B_t-t/2}(1-t-B_t)\,dB_t.
\end{aligned}$$

4.12 The given condition implies $\mathbb{E}(|X_t|) < \infty$. So X_t is a martingale if and only if $\mathbb{E}(X_t \mid \mathcal{F}_s) = X_s$. Then

$$\mathbb{E}(\int_s^t u(r,\omega)\,dr \mid \mathcal{F}_s) = \mathbb{E}(X_t - X_s \mid \mathcal{F}_s) = 0.$$

Moreover by dominated convergence

$$\mathbb{E}(u(t,\omega)\,dr \mid \mathcal{F}_s) = \mathbb{E}(\frac{d}{ds}\int_s^t u(r,\omega)\,dr \mid \mathcal{F}_s) = 0.$$

Then

$$u(t,\omega) = \mathbb{E}(u(t,\omega) \mid \mathcal{F}_t) = \lim_{s\to t^-} \mathbb{E}(u(t,\omega) \mid \mathcal{F}_s) = 0.$$

4.13 Let $dX_t = u(t,\omega)\,dt + dB_t$ where $u(t,\omega) \in \mathcal{V}([0,T])$. Then $Y_t = X_t M_t$ is a martingale, where

$$M_t = \exp\left(-\int_0^t u(r,\omega)\,dB_r - \frac{1}{2}\int_0^t u^2(r,\omega)\,dr\right)$$

since $\mathbb{E}(|M_t|) < \infty$ (see question 4b), $\mathbb{E}(|X_t|) \leq \sqrt{t}\left(\sqrt{\int_0^t u^2(r,\omega)\,dr} + 1\right) < \infty$ and

$$\begin{aligned}
d(X_t M_t) &= M_t dX_t + X_t dM_t + d[X,M]_t \\
&= M_t(u(t,\omega)\,dt + dB_t) + M_t X_t(-u(t,\omega)\,dB_t - \frac{1}{2}u^2(t,\omega)\,dt) \\
&\quad - M_t u(t,\omega)\,dt + \frac{1}{2}M_t X_t u^2(t,\omega)\,dt \\
&= M_t(1 - u(t,\omega)X_t)\,dB_t.
\end{aligned}$$

4.14 In this problem, the martingale representation of stochastic processes is explicitly shown.

(a) Compute $dF_t = dB_t$, $\mathbb{E}(F_T) = 0$ and

$$dF_t - d\mathbb{E}(F_t) = 1\,dB_t \implies f(t,\omega) = 1.$$

(b) Compute $dF_t = B_t\,dt$, $\mathbb{E}(F_T) = 0$ and

$$dF_t - d\mathbb{E}(F_t) = B_t\,dt = d(TB_T) - t\,dB_t = (T-t)\,dB_t \implies f(t,\omega) = T - t.$$

(c) Compute $dF_t = 2B_t\,dB_t + \,dt$, $\mathbb{E}(F_T) = T$ and

$$dF_t - d\mathbb{E}(F_t) = 2B_t\,dB_t + 1\,dt - 1\,dt = 2B_t\,dB_t \implies f(t,\omega) = 2B_t.$$

(d) Compute $dF_t = 3B_t^2\,dB_t + 3B_t\,dt$, $\mathbb{E}(F_T) = 0$ and

$$\begin{aligned} dF_t - d\mathbb{E}(F_t) &= 3B_t^2\,dB_t + 3B_t\,dt \\ &= 3B_t^2 + 3(T-t))\,dB_s \implies f(t,\omega) = 3B_t^2 + 3T - 3t. \end{aligned}$$

(e) Recall that $e^{B_t - t/2}$ is a martingale and compute

$$d(e^{B_t - t/2}) = e^{B_t - t/2}\,dB_t.$$

Deduce that

$$e^{B_T} = e^{T/2}\left(1 + \int_0^T e^{B_t - t/2}\,dB_t\right) \implies f(t,\omega) = e^{B_t + (T-t)/2}.$$

(f) Find martingale $e^{t/2}\sin(B_t)$ and compute

$$d(e^{t/2}\sin(B_t)) = e^{t/2}\cos(B_t)\,dB_t$$

Deduce that

$$\sin(B_T) = e^{-T/2}\int_0^T e^{t/2}\cos(B_t)\,dB_t \implies f(t,\omega) = e^{-(T-t)/2}\cos(B_t).$$

4.15 Define $X_t = (x^{1/3} + \frac{1}{3}B_t)^3$. Then

$$\begin{aligned} dX_t &= 3X_t^{2/3} d(x^{1/3} + \frac{1}{3}B_t) + 3X_t^{1/3} d\left[x^{1/3} + \frac{1}{3}B_t, x^{1/3} + \frac{1}{3}B_t\right] \\ &= X_t^{2/3}\,dB_t + \frac{1}{3}X_t^{1/3}\,dt. \end{aligned}$$

Stochastic Differential Equations (§5)

5.1 Compute

(a) $dX_t = d(e^{B_t}) = e^{B_t}\,dB_t + \frac{1}{2}^{B_t}\,d[B,B]_t = \frac{1}{2}X_t\,dt + X_t\,dB_t$

(b) $dX_t = d\left(\frac{B_t}{1+t}\right) = \frac{1}{1+t}\,dB_t - \frac{B_t}{(1+t)^2}\,dt = \frac{1}{1+t}\,dB_t - \frac{1}{1+t}X_t\,dt$

(c) $dX_t = d(\sin(B_t)) = \cos(B_t)\,dB_t - \frac{1}{2}\sin(B_t)\,dt = \cos(B_t)\,dB_t - \frac{1}{2}X_t\,dt$

(d) $dX_t^{(1)} = dt$ and

$$dX_t^{(2)} = d(e^t B_t) = e^t\,dB_t + e^t B_t\,dt = e^t\,dB_t + X_t^{(2)}\,dt.$$

(e) and finally differentials

$$d(\cosh(B_t)) = \sinh(B_t)\,dB_t + \frac{1}{2}\cosh(B_t)\,dt$$

and

$$d(\sinh(B_t)) = \cosh(B_t)\,dB_t + \frac{1}{2}\sinh(B_t)\,dt$$

to deduce that

$$\begin{pmatrix} dX_t^{(1)} \\ dX_t^{(2)} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} X_t^{(1)} \\ X_t^{(2)} \end{pmatrix}\,dt + \begin{pmatrix} X_t^{(2)} \\ X_t^{(1)} \end{pmatrix}\,dB_t.$$

5.2 Let $X_t^{(1)} = a\cos(B_t)$ and $X_t^{(2)} = b\sin(B_t)$. Then

$$dX_t^{(1)} = -a\sin(B_t)\,dB_t - \frac{a}{2}\cos(B_t)\,dt = -\frac{1}{2}X_t^{(1)}\,dt - \frac{a}{b}X_t^{(2)}\,dB_t$$

and

$$dX_t^{(2)} = b\cos(B_t)\,dB_t - \frac{b}{2}\sin(B_t)\,dt = -\frac{1}{2}X_t^{(2)}\,dt + \frac{b}{a}X_t^{(1)}\,dB_t.$$

5.3 The solution is given by

$$X_t = X_0\exp\left((r - \frac{1}{2}\sum_{k=1}^{n}\alpha_k^2)t + \sum_{k=1}^{n}\alpha_k\,dB_k\right).$$

5.4 In this problem, solutions to stochastic differential equations are found.

(a) The solution to $dX_t^{(1)} = dt + dB_t^{(1)}$ is $X_t^{(1)} = X_0^{(1)} + t + B_t^{(1)}$ and

$$dX_t^{(2)} = X_t^{(1)}\, dB_t^{(2)} = (X_0^{(1)} + t + B_t^{(1)})\, dB_t^{(2)}$$

is

$$X_t^{(2)} = X_0^{(2)} + X_0^{(1)} B_t^{(2)} + \int_0^t (s + B_s^{(1)})\, dB_s^{(2)}.$$

(b) Using integrating factors, solve $dX_t = X_t\, dt + dB_t$ for

$$e^{-t} X_t - X_0 = \int_0^t e^{-s}\, dB_s$$

and deduce that the solution X_t is

$$X_t = e^t X_0 + \int_0^t e^{t-s}\, dB_s.$$

(c) Using integrating factors, solve $dX_t = -X_t\, dt + e^{-t} dB_t$ for

$$e^t X_t - X_0 = \int_0^t dB_s$$

and deduce that the solution X_t is

$$X_t = e^{-t}(X_0 + B_t).$$

5.5 The Langevin equation is given by

$$dX_t - \mu X_t\, dt = \sigma dB_t.$$

(a) Using integrating factors, solve for

$$e^{-\mu t} X_t - X_0 = \int_0^t e^{-\mu s} \sigma\, dB_s$$

and deduce that the solution X_t is

$$X_t = e^{\mu t} X_0 + \sigma \int_0^t e^{\mu(t-s)}\, dB_s.$$

(b) The expected value of X_t is

$$\mathbb{E}(X_t) = e^{\mu t} X_0$$

and, by Itô isometry, the variance of X_t is

$$\mathbb{V}(X_t) = \mathbb{E}\left(\sigma^2 \left(\int_0^t e^{\mu(t-s)}\, dB_s\right)^2\right) = \mathbb{E}\left(\sigma^2 \int_0^t e^{2\mu(t-s)} ds\right) = \frac{\sigma^2}{2\mu}(e^{2\mu t} - 1).$$

5.6 Suppose Y_t is given by

$$dY_t = r\,dt + \alpha Y_t\,dB_t.$$

Using integrating factors, solve for

$$d(e^{-\alpha B_t} Y_t) = e^{-\alpha B_t} Y_t \left(r - \frac{\alpha^2}{2}\right)\,dt$$

and

$$e^{-\alpha B_t + \frac{\alpha^2}{2}t} Y_t - Y_0 = \int_0^t r e^{-\alpha B_s + \frac{\alpha^2}{2}s}\,ds.$$

Deduce that

$$Y_t = e^{\alpha B_t - \frac{\alpha^2}{2}t} Y_0 + r\int_0^t e^{\alpha(B_t - B_s) - \frac{\alpha^2}{2}(t-s)}\,ds.$$

5.7 The Ornstein-Uhlenbeck process is given by

$$dX_t = (m - X_t)\,dt + \sigma\,dB_t.$$

(a) Using integrating factors, solve for

$$e^t X_t - X_0 = \int_0^t e^s m\,ds + \int_0^t e^s \sigma\,dB_s$$

and deduce that the solution X_t is

$$X_t = e^{-t} X_0 + m(1 - e^{-t}) + \sigma\int_0^t e^{s-t}\,dB_s.$$

(b) The expected value of X_t is

$$\mathbb{E}(X_t) = m + e^{-t}(X_0 - m)$$

and the variance of X_t is

$$\mathbb{V}(X_t) = \mathbb{E}\left(\sigma^2\left(\int_0^t e^{s-t}\,dB_s\right)^2\right) = \mathbb{E}\left(\sigma^2\int_0^t e^{2s-2t}\,ds\right) = \frac{\sigma^2}{2}(1 - e^{-2t}).$$

5.8 Consider the stochastic differential equation

$$\begin{pmatrix} dX_t^{(1)} \\ dX_t^{(2)} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} X_t^{(1)} \\ X_t^{(2)} \end{pmatrix}\,dt + \begin{pmatrix} \alpha\,dB_t^{(1)} \\ \beta\,dB_t^{(2)} \end{pmatrix}.$$

By d'Alembert's formula, it has a solution of the form

$$X_t = e^{At} X_0 + \int_0^t e^{A(t-s)} g(s)\, ds,$$

where

$$e^{At} = \begin{pmatrix} 1 & 1 \\ i & -1 \end{pmatrix} \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -1 \end{pmatrix}^{-1} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}.$$

Conclude that the solutions are

$$X_t^{(1)} = X_0^{(1)} \cos(t) + X_0^{(2)} \sin(t) + \alpha \int_0^t \cos(t-s)\, dB_s^{(1)} + \beta \int_0^t \sin(t-s)\, dB_s^{(2)}$$

and

$$X_t^{(2)} = -X_0^{(1)} \sin(t) + X_0^{(2)} \cos(t) - \alpha \int_0^t \sin(t-s)\, dB_s^{(1)} + \beta \int_0^t \cos(t-s)\, dB_s^{(2)}.$$

5.9 Let $dX_t = \ln(1+X_t^2)\, dt + \chi_{\{X_t>0\}} X_t\, dB_t$. It suffices to check that

$$|b(t,x)| + |\sigma(t,x)| = \ln(1+x^2) + \chi_{\{x>0\}}|x| \le \frac{2}{e}(|x|+1) + |x| \le 2(|x|+1),$$

$\mathbb{E}(|X_0|^2) = \alpha^2 < \infty$, and

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le |\ln(x^2) - \ln(y^2)| + |x-y| \le 3|x-y|.$$

Hence, by Theorem 5.2.1, there is a unique strong solution to the stochastic differential equation.

5.10 Calculate

$$\begin{aligned}
\mathbb{E}(X_t^2) &= \mathbb{E}\left(Z + \int_0^t b(s,X_s)\, ds + \int_0^t \sigma(s,X_s)\, dB_s\right)^2 \\
&\le 3\left(\mathbb{E}(Z^2) + \mathbb{E}\left(\int_0^t b(s,X_s)\, ds\right)^2 + \mathbb{E}\left(\int_0^t \sigma(s,X_s)\, dB_s\right)^2\right) \\
&\le 3\left(\mathbb{E}(Z^2) + T\mathbb{E}\left(\int_0^t b(s,X_s)^2\, ds\right) + \mathbb{E}\left(\int_0^t \sigma(s,X_s)^2\, ds\right)\right) \\
&\le 3\mathbb{E}(Z^2) + 6C^2\left(T + \int_0^t \mathbb{E}(|X_s|^2)\, ds\right)(T+1) \\
&= (3\mathbb{E}(Z^2) + 6C^2 T(T+1)) + 6C^2(T+1)\int_0^t \mathbb{E}(|X_s|^2)\, ds.
\end{aligned}$$

and apply Gronwall to derive the result.

5.11 Consider the stochastic process

$$Y_t = a(1-t) + bt + (1-t)\int_0^t \frac{dB_s}{1-s}.$$

Then $Y_0 = a$ and, for $t \in [0,1)$, Y_t solves

$$\begin{aligned}
dY_t &= (b-a)\,dt - \int_0^t \frac{dB_s}{1-s}\,dt + (1-t)\frac{dB_t}{1-t} \\
&= \frac{1}{1-t}\left((b-a)(1-t) - (1-t)\int_0^t \frac{dB_s}{1-s}\right)dt + dB_t \\
&= \frac{1}{1-t}\left(b - a(1-t) - bt - (1-t)\int_0^t \frac{dB_s}{1-s}\right)dt + dB_t \\
&= \frac{b - Y_t}{1-t}\,dt + dB_t.
\end{aligned}$$

Finally by Itô isometry $\mathbb{E}\left((1-t)^2 \int_0^t \frac{dB_s}{1-s}\right)^2 = (1-t)^2 \int_0^t \frac{1}{(1-s)^2}\,ds = (1-t)t \to 0$ as $t \to 1^-$ and so limit $\lim\limits_{t\to 1^-} Y_t \overset{\text{a.s.}}{=} b$.

5.12 Let $y''(t) + (1+\varepsilon W_t)y(t) = 0$ where $W_t = \frac{dB_t}{dt}$ is 1-dimensional white noise.

(a) Rewrite

$$\begin{pmatrix} dy_t \\ d\dot{y}_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} y_t \\ \dot{y}_t \end{pmatrix} dt + \begin{pmatrix} 0 & 0 \\ -\varepsilon & 0 \end{pmatrix}\begin{pmatrix} y_t \\ \dot{y}_t \end{pmatrix} dB_t.$$

(b) Check that, if $y(t) = y(0) + y'(0)t + \int_0^t (r-t)y(r)\,dr + \int_0^t \varepsilon(r-t)y(r)\,dB_r$, then

$$y'(t) = y'(0) - \int_0^t y(r)\,dr - \int_0^t \varepsilon y(r)\,dB_r = y'(0) - \int_0^t y(r)(1+\varepsilon W_r)\,dr$$

and $y''(t) = -(1+\varepsilon W_r)\,dr$.

5.13 Let $x_t'' + a_0 x_t' + w^2 x_t = (T_0 - \alpha_0 x_t')\eta W_t$ where W_t is 1-dimensional white noise. Then

$$\begin{pmatrix} dx_t \\ d\dot{x}_t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -w^2 & -a_0 \end{pmatrix}\begin{pmatrix} x_t \\ \dot{x}_t \end{pmatrix} dt + \begin{pmatrix} 0 & 0 \\ 0 & -\alpha_0\eta \end{pmatrix}\begin{pmatrix} x_t \\ \dot{x}_t \end{pmatrix} dB_t + \begin{pmatrix} 0 \\ T_0\eta \end{pmatrix} dB_t$$

and by d'Alembert's formula the solution is

$$X_t = e^{At}X_0 + \int_0^t e^{A(t-s)}KX_s\,dB_s + \int_0^t e^{A(t-s)}M\,dB_s.$$

The eigenvalues of A satisfy $\lambda^2+a_0\lambda+w^2=0$ and are $\lambda_\pm = -\frac{a_0}{2}\pm\sqrt{w^2-\frac{a_0^2}{4}}i =: -\lambda\pm\xi i$. Then take the exponential of matrix A

$$\begin{aligned}
e^{At} &= \begin{pmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{pmatrix}\begin{pmatrix} e^{\lambda_+ t} & 0 \\ 0 & e^{\lambda_- t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{pmatrix}^{-1} \\
&= \frac{1}{\lambda_- - \lambda_+}\begin{pmatrix} \lambda_- e^{\lambda_+ t} - \lambda_+ e^{\lambda_- t} & e^{\lambda_- t} - e^{\lambda_+ t} \\ -\lambda_-\lambda_+(e^{\lambda_- t} - e^{\lambda_+ t}) & \lambda_- e^{\lambda_- t} - \lambda_+ e^{\lambda_+ t} \end{pmatrix} \\
&= -\frac{1}{2\xi i}\begin{pmatrix} e^{-\lambda t}(-\lambda\cdot 2i\sin(\xi t) - \xi i\cdot 2\cos(\xi t) & e^{-\lambda t}(-2i\sin(\xi t)) \\ -w^2 e^{-\lambda t}(-2i\sin(\xi t)) & e^{-\lambda t}(-\lambda\cdot 2i\sin(\xi t) - \xi i\cdot 2\cos(\xi t) + 2\lambda\cdot 2i\sin(\xi t)) \end{pmatrix} \\
&= \frac{e^{-\lambda t}}{\xi}\begin{pmatrix} \lambda\sin(\xi t) + \xi\cos(\xi t) & \sin(\xi t) \\ -w^2\sin(\xi t) & \lambda\sin(\xi t) + \xi\cos(\xi t) - 2\lambda\sin(\xi t) \end{pmatrix} \\
&= \frac{e^{-\lambda t}}{\xi}\left((\lambda\sin(\xi t) + \xi\cos(\xi t))I + A\sin(\xi t)\right).
\end{aligned}$$

Next, letting $y_s = \dot{x_s}$, $g_t = e^{-\lambda t}\frac{\sin(\xi t)}{\xi}$ and $h_t = e^{-\lambda t}\frac{\xi\cos(\xi t)-\lambda\sin(\xi t)}{\xi}$, compute

$$e^{A(t-s)}KX_s = -\frac{\alpha_0\eta e^{-\lambda(t-s)}}{\xi}\begin{pmatrix} 0 & \sin(\xi(t-s)) \\ 0 & \xi\cos(\xi(t-s)) - \lambda\sin(\xi(t-s)) \end{pmatrix}\begin{pmatrix} x_s \\ \dot{x_s} \end{pmatrix} = \begin{pmatrix} -\alpha_0\eta y_s g_{t-s} \\ -\alpha_0\eta y_s h_{t-s} \end{pmatrix}$$

and

$$e^{A(t-s)}M = \frac{T_0\eta e^{-\lambda(t-s)}}{\xi}\begin{pmatrix} \sin(\xi(t-s)) \\ \xi\cos(\xi(t-s)) - \lambda\sin(\xi(t-s)) \end{pmatrix} = \begin{pmatrix} \eta T_0 g_{t-s} \\ \eta T_0 h_{t-s} \end{pmatrix}.$$

It follows that

$$x_t = \eta\int_0^t (T_0 - \alpha_0 y_s)g_{t-s}\,dB_s$$

and

$$y_t = \eta\int_0^t (T_0 - \alpha_0 y_s)h_{t-s}\,dB_s.$$

5.14 Letting $Z_t = F(\mathbf{B}_t)$, where $\mathbf{B}_t = B_t^{(1)} + iB_t^{(2)}$, calculate

$$\begin{aligned}
dZ_t &= F_x(\mathbf{B}_t)\,dB_t^{(1)} + F_y(\mathbf{B}_t)\,dB_t^{(2)} \\
&\quad + \frac{1}{2}F_{xx}(\mathbf{B}_t)\,d[B^{(1)},B^{(1)}]_t + F_{xy}(\mathbf{B}_t)\,d[B^{(1)},B^{(2)}]_t + F_{yy}(\mathbf{B}_t)\,d[B^{(2)},B^{(2)}]_t \\
&= (u_x + iv_x)\,dB_t^{(1)} + (u_y + iv_y)\,dB_t^{(2)} + \frac{1}{2}(u_{xx} + iv_{xx} + u_{yy} + iv_{yy})\,dt \\
&= \langle F'(\mathbf{B}_t), d\mathbf{B}_t\rangle + \frac{1}{2}(v_{xy} - iu_{xy} + u_{yy} + iv_{yy})\,dt \\
&= \langle F'(\mathbf{B}_t), d\mathbf{B}_t\rangle + \frac{1}{2}(-u_{yy} - iv_{yy} + u_{yy} + iv_{yy})\,dt \\
&= \langle F'(\mathbf{B}_t), d\mathbf{B}_t\rangle.
\end{aligned}$$

5.15 Consider the non-linear stochastic differential equation

$$dX_t = rX_t(K - X_t)\,dt + \beta X_t\,dB_t, \qquad X_0 = x > 0.$$

Comparing to the deterministic Bernoulli equation, do a substitution $Y_t = X_t^{-1}$, then

$$\begin{aligned} dY_t &= -rY_t(K - X_t)\,dt - \beta Y_t\,dB_t + \beta^2 Y_t\,dt \\ &= (-rK + \beta^2)Y_t\,dt - \beta Y_t\,dB_t + r\,dt. \end{aligned}$$

Next do a new change of variables

$$Z_t = Y_t e^{(rK-\beta^2)t}$$

and calculate

$$\begin{aligned} dZ_t &= -\beta Z_t\,dB_t + re^{(rk-\beta^2)t}\,dt \\ \implies Z_t &= e^{-\beta B_t}\left(x^{-1} + r\int_0^t e^{(rk-\beta^2)s+\beta B_s}\,ds\right). \end{aligned}$$

Conclude that

$$X_t = \frac{e^{(rk-\beta^2)t}}{Z_t} = \frac{e^{(rk-\beta^2)t+\beta B_t}}{x^{-1} + r\int_0^t e^{(rk-\beta^2)s+\beta B_s}\,ds}.$$

5.16 Consider the non-linear stochastic differential equation

$$dX_t = f(t, X_t)\,dt + c(t)X_t\,dB_t, \qquad X_0 = x.$$

(a) Let $F_t(\omega) = \exp\left(-\int_0^t c(s)\,dB_s + \frac{1}{2}\int_0^t c(s)^2\,ds\right)$. Then calculate

$$\begin{aligned} d(F_tX_t) &= X_t\,dF_t + F_t\,dX_t + d[F_t, X_t] \\ &= X_t\left[F_t\left(-c(t)\,dB_t - \frac{1}{2}c(t)^2\,dt - \frac{1}{2}c(t)^2\,dt\right)\right] \\ &\quad + [f(t, X_t)F_t\,dt + c(t)X_tF_t\,dB_t] - c(t)^2F_tX_t\,dt \\ &= f(t, X_t)F_t\,dt. \end{aligned}$$

(b) Defining $Y_t = F_tX_t$, deduce that

$$\frac{dY_t}{dt} = F_t(\omega)f(t, F_t^{-1}(\omega)Y_t(\omega)).$$

(c) Consider $dX_t = X_t^{-1} + \alpha X_t\,dB_t$, $X_0 = x > 0$. Then

$$\frac{dY_t}{dt} = e^{-2\alpha B_t + \alpha^2 t}Y_t^{-1},$$

which implies

$$Y_t = \sqrt{Y_0^2 + 2\int_0^t e^{-2\alpha B_t + \alpha^2 s}\, ds}$$

and

$$X_t = e^{\alpha B_t - \frac{\alpha^2}{2}t}\sqrt{x^2 + 2\int_0^t e^{-2\alpha B_t + \alpha^2 s}\, ds}.$$

(d) Consider $dX_t = X_t^\gamma\, dt + \alpha X_t\, dB_t, X_0 = x > 0$. Then

$$\frac{dY_t}{dt} = e^{-(1-\gamma)B_t + (1-\gamma)\frac{\alpha^2}{2}t} Y_t^\gamma,$$

which implies

$$Y_t = \left(Y_0^{1-\gamma} + (1-\gamma)\int_0^t e^{-(1-\gamma)B_s + (1-\gamma)\frac{\alpha^2}{2}s}\, ds\right)^{\frac{1}{1-\gamma}}$$

and

$$X_t = e^{\alpha B_t - \frac{\alpha^2}{2}t}\left(x^{1-\gamma} + (1-\gamma)\int_0^t e^{-(1-\gamma)B_s + (1-\gamma)\frac{\alpha^2}{2}s}\, ds\right)^{\frac{1}{1-\gamma}}.$$

5.17 Let $v \geq 0$ satisfy $v(t) \leq C + A\int_0^t v(s)\, ds$ and consider quantity $w(t) = \int_0^t v(s)\, ds$. Then

$$w'(t) = v(t) \leq C + A\int_0^t v(s)\, ds = C + Aw(t).$$

Then for $f(t) = w(t)e^{-At}$, calculate

$$f'(t) = e^{-At}\left(w'(t) - Aw(t)\right) \leq Ce^{-At}$$

and

$$w(t)e^{-At} \leq \int_0^t Ce^{-As}\, ds = \frac{C}{A}(1 - e^{-At})$$
$$\implies w(t) \leq \frac{C}{A}(e^{At} - 1).$$

Deduce that

$$v(t) \leq C + Aw(t) \leq Ce^{At}.$$

Diffusions: Basic Properties (§7)

7.1 Using Theorem 7.3.3, the generators of the Itô diffusions are

(a) $(Af)(x) = \mu x f'(x) + \frac{1}{2}\sigma^2 f''(x)$ for $f \in C_0^2(\mathbb{R})$

(b) $(Af)(x) = rxf'(x) + \frac{1}{2}\alpha^2 x^2 f''(x)$ for $f \in C_0^2(\mathbb{R})$

(c) $(Af)(y) = rf'(y) + \frac{1}{2}\alpha^2 y^2 f''(y)$ for $f \in C_0^2(\mathbb{R})$

(d) $\mathbf{b} = (1, \mu X)^T$ and $\sigma = (0, \sigma)^T$ to deduce for $f \in C_0^2(\mathbb{R}^2)$

$$(Af)(y_1, y_2) = \frac{\partial f}{\partial y_1} + \mu y_2 \frac{\partial f}{\partial y_2} + \frac{1}{2}\sigma^2 \frac{\partial^2 f}{\partial y_2^2}$$

(e) $(Af)(x_1, x_2) = \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \frac{1}{2} e^{2x_1} \frac{\partial^2 f}{\partial x_2^2}$ for $f \in C_0^2(\mathbb{R}^2)$.

(f) Since $d[B_i, B_j] = \delta_{ij}\, dt$, for $f \in C_0^2(\mathbb{R}^2)$, deduce

$$(Af)(x_1, x_2) = \frac{\partial f}{\partial x_1} + \frac{1}{2}\frac{\partial^2 f}{\partial x_1^2} + \frac{1}{2} x_1^2 \frac{\partial^2 f}{\partial x_2^2}$$

(g) and finally for $\mathbf{X}$ and $f \in C_0^2(\mathbb{R}^n)$

$$\begin{aligned}(Af(\mathbf{x})) &= (rx)_k \cdot \nabla f(\mathbf{x}) + \frac{1}{2}\sum_{j=1}^{n}\sum_{k,\ell=1}^{n} (\alpha_{kj}x_k)(\alpha_{\ell j}x_\ell)\frac{\partial^2 f}{\partial x_k \partial x_\ell} \\ &= (rx)_k \cdot \nabla f(\mathbf{x}) + \frac{1}{2}\sum_{k,\ell=1}^{n} x_k x_\ell \left(\sum_{j=1}^{n} \alpha_{kj}\alpha_{\ell j}\right) \frac{\partial^2 f}{\partial x_k \partial x_\ell}.\end{aligned}$$

7.2 In this problem, the Itô diffusion processes are found.

(a) Since $b = 1, \frac{1}{2}\sigma^2 = 2$, it follows $dX_t = b\, dt + \sigma\, dB_t = dt \pm \sqrt{2}\, dB_t$.

(b) Rewrite

$$(Af)(t, x) = (1, cx) \cdot (Df)(t, x) + \frac{1}{2}\begin{pmatrix} 0 & \alpha x \end{pmatrix} (D^2 f)(t, x) \begin{pmatrix} 0 \\ \alpha x \end{pmatrix}$$

and deduce that

$$dY_t = \begin{pmatrix} dY_1 \\ dY_2 \end{pmatrix} = \begin{pmatrix} 1 \\ cY_2 \end{pmatrix} dt \pm \begin{pmatrix} 0 \\ \alpha Y_2 \end{pmatrix} dB_t.$$

(c) The equation reduces to

$$(Af)(x_1,x_2) = (2x_2, \ln(1+x_1^2+x_2^2)) \cdot (Df)(x_1,x_2) + \frac{1}{2}\begin{pmatrix}1 & 0\end{pmatrix}(D^2f)(x_1,x_2)\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}x_1 & 1\end{pmatrix}(D^2f)(x_1,x_2)\begin{pmatrix}x_1\\1\end{pmatrix}$$

and thus

$$\begin{pmatrix}dX_1\\dX_2\end{pmatrix} = \begin{pmatrix}2X_2\\\ln(1+X_1^2+X_2^2)\end{pmatrix} dt \pm \begin{pmatrix}1\\0\end{pmatrix} dB_t^{(1)} \pm \begin{pmatrix}X_1\\1\end{pmatrix} dB_t^{(2)}.$$

7.3 Let B_t be standard Brownian motion and define $X_t = x \cdot e^{ct+\alpha B_t}$. We prove it for $\mathcal{P}(\mathbb{R})$, which is dense by Stone-Weierstrass in the set of bounded Borel measurable functions. Fix $f(x) = x^n$ and since $\sigma(X_t) = \sigma(B_t)$, we have

$$\begin{aligned}
\mathbb{E}[f(X_{t+h}) \,|\, \mathcal{F}_t] &= \mathbb{E}[x^n \cdot e^{cn(t+h)+\alpha n(B_t+(B_{t+h}-B_t))} \,|\, \mathcal{F}_t] \\
&= x^n e^{cn(t+h)} e^{\alpha n B_t} \mathbb{E}[e^{\alpha n(B_{t+h}-B_t)} \,|\, \mathcal{F}_t] \\
&= x^n e^{cn(t+h)} e^{\alpha n B_t} \mathbb{E}[e^{\alpha n(B_{t+h}-B_t)}] \\
&= x^n e^{cn(t+h)} \mathbb{E}[e^{\alpha n(B_t+(B_{t+h}-B_t))} \,|\, B_t] \\
&= x^n e^{cn(t+h)} \mathbb{E}[e^{\alpha n(B_t+(B_{t+h}-B_t))} \,|\, X_t] \\
&= \mathbb{E}[f(X_{t+h}) \,|\, X_t].
\end{aligned}$$

7.4 Let B_t be Brownian motion starting at $x > 0$. The generator is $(Af)(x) = \frac{1}{2}f''(x)$. Let $\tau = \inf\{t \,|\, B_t = 0\}$.

(a) Consider exit time σ_k from interval $A_k = \{0 < y < k\}$ where $X_0 = x \in (0,k)$. Applying Dynkin's formula for $f(x) = x$, calculate

$$k\mathbb{P}(X_{\sigma_k} = k) = x \implies \mathbb{P}(X_{\sigma_k} = 0) = 1 - \frac{x}{k} \overset{k\to\infty}{\to} 1.$$

(b) Applying Dynkin's formula again for $f(x) = x^2$, derive

$$\begin{aligned}
kx = k^2\mathbb{P}(X_{\sigma_k} = k) &= x^2 + \mathbb{E}[X_{\sigma_k}] \\
&= x^2 + \mathbb{E}[\sigma_k \,|\, X_{\sigma_k} = 0]\mathbb{P}(X_{\sigma_k} = 0) + \mathbb{E}[\sigma_k \,|\, X_{\sigma_k} = k]\mathbb{P}(X_{\sigma_k} = k) \\
&\overset{k\to\infty}{\to} x^2 + \mathbb{E}[\tau],
\end{aligned}$$

and so $\mathbb{E}[\tau] = \infty$.

7.5 By Theorem 5.2.1, there exists a unique t-continuous solution $X_t(\omega)$. Applying Dynkin's formula with $f(\mathbf{x}) = |\mathbf{x}|^2$ and $\tau = \min\{t, \tau_R\}$, calculate

$$
\begin{aligned}
\mathbb{E}[|X_\tau|^2] &= \mathbb{E}[X_t^2]\mathbb{P}(t \leq \tau_R) + R^2\mathbb{P}(\tau_R < t) \\
&= |X_0|^2 + \mathbb{E}\left[\int_0^\tau (Af)(X_s)\,ds\right] \\
&= |X_0|^2 + \mathbb{E}\left[\int_0^\tau \mathbf{b}(s, X_s) \cdot 2X_s + \frac{1}{2}\sigma(s, X_s)(2\mathbf{I}_n)\sigma^T(s, X_s)\,ds\right] \\
&\leq |X_0|^2 + \mathbb{E}\left[\int_0^\tau |\mathbf{b}|^2 + |X_s|^2 + |\sigma|^2\,ds\right] \\
&\leq |X_0|^2 + \mathbb{E}\left[\int_0^\tau C^2(1 + |X_s|)^2 + |X_s|^2\right] \\
&\leq |X_0|^2 + (2C^2 + 1)\mathbb{E}\left[\int_0^\tau (1 + |X_s|^2)\,ds\right].
\end{aligned}
$$

Passing to the limit as $R \to \infty$,

$$\mathbb{E}[X_t^2] \leq |X_0|^2 + (2C^2 + 1)\mathbb{E}\left[\int_0^t (1 + |X_s|^2)\,ds\right].$$

Letting $f(t) := \mathbb{E}[X_t^2] + 1$, $f(t) - f(0) \leq (2C^2 + 1)\int_0^t f(s)\,ds$. By applying Gronwall's Lemma (or checking that $g(t) := f(t)e^{-(2C^2+1)t}$ is decreasing), the proof is complete.

7.6 Let $g(x, \omega) = (f \circ F)(x, t, t + h, \omega)$.

(a) It suffices to check that $x \mapsto F(x, \dots)$ is continuous. Consider two paths starting at $X_t(\omega)$. Then check

$$
\begin{aligned}
&\mathbb{E}[|X_{t+h}^{(1)} - X_{t+h}^{(2)}|^2] \\
&\quad \leq |X_t^{(1)} - X_t^{(2)}|^2 + (2D^2 + 1)\mathbb{E}\left[\int_t^{t+h} |X_u^{(1)} - X_u^{(2)}|^2\,du\right]
\end{aligned}
$$

and apply the previous problem.

(b) Follow the hint.

7.7 Let B_t be n-dimensional Brownian motion starting at x and D be a ball centred at x.

(a) Recall $\widetilde{B}_t = UB_t$ where U is orthogonal is also Brownian. Then for $S \subset \partial D$

$$\mu_D^x(S) = \mathbb{P}(B_{\tau_D} \in S) = \mathbb{P}(\widetilde{B}_{\tau_D} \in U \cdot S) = \mu_D^x(U \cdot S).$$

Hence, the harmonic measure is rotation-invariant.

(b) Calculate

$$\begin{aligned}
u(x) &= \mathbb{E}^x[\phi(B_{\tau_W})] \\
&= \mathbb{E}^x\left[\mathbb{E}^x[\phi(B_{\tau_W}) \mid B_{\tau_D}]\right] \\
&= \mathbb{E}^x\left[\mathbb{E}^{B_{\tau_D}}[\phi(B_{\tau_W})]\right] \\
&= \mathbb{E}^x[u(B_{\tau_D})] \\
&= \int_{\partial D} u(y)\, d\mu_D^x(y) \\
&= \int_{\partial D} u(y)\, d\sigma(y),
\end{aligned}$$

which completes the proof.

7.8 Let $\mathcal{N}_t$ be a right-continuous family of σ-algebras of subsets of Ω, containing all sets of measure zero.

(a) Since $\mathcal{N}_t$ is closed under finite unions and intersections, it follows that $\{\min\{\tau_1, \tau_2\} \leq t\} = \cup_{j=1}^2 \{\tau_j \leq t\} \in \mathcal{N}_t$ and $\{\max\{\tau_1, \tau_2\} \leq t\} = \cap_{j=1}^2 \{\tau_j \leq t\} \in \mathcal{N}_t$.

(b) Check $\{\tau \leq t\} = \lim_{n\to\infty}\{\tau_n \leq t\} = \cup_{j=1}^\infty \{\tau_n \leq t\} \in \mathcal{N}_t$.

(c) F is a $G_{1/n}$ set and $\{\tau_n \leq t\} \in \mathcal{M}_t$ for $\tau_n = \{\inf t > 0 \mid X_t \notin G_{1/n}\}$. Since $G_{1/n} \downarrow F$ and given that τ_n is a decreasing family of stopping times, it follows by part (b) that $\tau_F = \cup_{j=1}^n \{\tau_n \leq t\} \in \mathcal{M}_t$.

7.9 Let X_t be geometric Brownian motion

$$dX_t = rX_t\, dt + \alpha X_t\, dB_t, \quad X_0 = x > 0.$$

(a) Compute $(Af)(x) = rxf'(x) + \frac{1}{2}\alpha^2 x^2 f''(x)$ for $f \in C_0^2(\mathbb{R})$, whence for $f(x) = x^\gamma$

$$(Af)(x) = \left(r + \frac{1}{2}\alpha^2(\gamma - 1)\right)\gamma x^\gamma.$$

(b) Applying Dynkin's formula with $f(x) = x^{\gamma_1}$, $\gamma_1 = 1 - \frac{2r}{\alpha^2}$ and $\sigma_k = \min\{k, \tau\}$, calculate

$$\begin{aligned}
f(x) = x^{\gamma_1} &= x^{\gamma_1} + \mathbb{E}[\int_0^{\sigma_k} Af(X_s)\, ds] \\
&= \mathbb{E}[f(X_{\sigma_k})] \\
&= R^{\gamma_1}\mathbb{P}(\tau \leq k) + \mathbb{E}[f(X_k) \mid k < \tau]\mathbb{P}(k < \tau) \\
&\overset{k\to\infty}{\to} R^{\gamma_1} p,
\end{aligned}$$

and deduce $p = \left(\frac{x}{R}\right)^{\gamma_1}$.

(c) Consider exit time σ_ρ from annulus $A_\rho = \{\rho < y < R\}$ where $X_0 = x \in (\rho, R)$. Using parts (a) and (b), note that

$$\mathbb{P}(X_{\sigma_\rho} = R) = \frac{x^\gamma - \rho^\gamma}{R^\gamma - \rho^\gamma},$$

and thus for exit time σ_ρ from A_ρ, calculate

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{\sigma_\rho} \left(r - \frac{1}{2}\alpha^2\right) ds\right] \\
&\quad = \left(r - \frac{1}{2}\alpha^2\right) \mathbb{E}[\sigma_\rho] \\
&\quad = \left(r - \frac{1}{2}\alpha^2\right) \mathbb{E}[\sigma_\rho \,|\, X_{\sigma_\rho} = R]\mathbb{P}(X_{\sigma_\rho} = R) + \left(r - \frac{1}{2}\alpha^2\right) \mathbb{E}[\sigma_\rho \,|\, X_{\sigma_\rho} = \rho]\mathbb{P}(X_{\sigma_\rho} = \rho) \\
&\overset{\rho\to 0^+}{\to} \left(r - \frac{1}{2}\alpha^2\right) \mathbb{E}[\tau]
\end{aligned}$$

and

$$\mathbb{E}[\ln(X_{\sigma_\rho})] = \ln(R)\mathbb{P}(X_{\sigma_\rho} = R) + \ln(\rho)\mathbb{P}(X_{\sigma_\rho} = \rho) \overset{\rho\to 0^+}{\to} \ln(R).$$

Applying Dynkin's formula with $f(x) = \ln(x)$, deduce

$$\ln(R) = \ln(x) + \left(r - \frac{1}{2}\alpha^2\right) \mathbb{E}[\tau],$$

which completes the proof.

7.10 Let X_t be geometric Brownian motion.

(a) Using the Markov property,

$$\begin{aligned}
\mathbb{E}^x[X_T \,|\, \mathcal{F}_t] &= \mathbb{E}^{X_t}[X_{T-t}] \\
&= X_t + \mathbb{E}^{X_t}\left[\int_0^{T-t} rX_u \, du + \int_0^{T-t} \alpha X_u \, dB_u\right] \\
&= X_t + r\int_0^{T-t} \mathbb{E}^{X_t}[X_u] \, du.
\end{aligned}$$

Letting $f(s) = \mathbb{E}^{X_t}[X_{s-t}]$, we have $f'(s) = rf(s)$ by differentiation and so

$$\mathbb{E}^x[X_T \,|\, \mathcal{F}_t] = f(T-t) = f(0)e^{r(T-t)} = X_t e^{r(T-t)}.$$

(b) The solution is given by $X_t = xe^{rt}M_t$, where $M_t = e^{\alpha B_t - \frac{1}{2}\sigma^2 t}$ is a martingale. So

$$\mathbb{E}^x[X_T \,|\, \mathcal{F}_t] = \mathbb{E}[X_T \,|\, X_t] = xe^{rT}\mathbb{E}[M_T \,|\, X_t] = xe^{rT}M_t = X_t e^{r(T-t)}.$$

7.11 Let τ be a stopping time. By the strong Markov property,

$$\begin{aligned}
\mathbb{E}^x[g(X_\tau)] &= \mathbb{E}^x\left[\mathbb{E}^{X_\tau}\left[\int_0^\infty f(X_u)\,du\right]\right] \\
&= \mathbb{E}^x\left[\int_0^\infty \mathbb{E}^{X_\tau}\left[f(X_u)\right]\,du\right] \\
&= \mathbb{E}^x\left[\int_0^\infty \mathbb{E}\left[f(X_{u+\tau})\,|\,\mathcal{F}_\tau\right]\,du\right] \\
&= \mathbb{E}^x\left[\int_0^\infty f(X_{u+\tau})\,du\right] \\
&= \mathbb{E}^x\left[\int_\tau^\infty f(X_t)\,dt\right],
\end{aligned}$$

which completes the proof.

Other Chapters (§6, §8 –12)

The Filtering Problem (§6)

Other Topics in Diffusion Theory (§8)

Applications to Boundary Value Problems (§9)

Applications to Optimal Stopping (§10)

Applications to Stochastic Control (§11)

Applications to Mathematical Finance (§12)