Numer Math. 43, 161–173 (1984)

Numerische Mathematik

Convergence of Product Formulas for the Numerical Evaluation of Certain Two-Dimensional Cauchy Principal Value Integrals*

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Summary. We consider product rules of interpolatory type for the numerical approximation of certain two-dimensional Cauchy principal value integrals. We present convergence results which generalize those known in the one-dimensional case.

Subject Classifications: AMS (MOS): 65D32, 41A17; CR: 5.16.

1. Introduction

Very recently [9] we have considered interpolatory type quadrature rules proposed by several authors for the evaluation of one-dimensional Cauchy principal value integrals. These rules have the following form

$$\fint_{-1}^{1} w(x)\frac{f(x)}{x-\lambda}dx=\sum_{i=1}^{n} w_i(\lambda)f(x_i)+R_n^{(1)}(f) \tag{1.1}$$

and are obtained simply by interpolating the function $f(x)$ at n distinct knots $\{x_i\}$ with a polynomial of degree $n-1$. A second class of formulas may be derived by adding the point $x=\lambda$ to the abscissas $\{x_i\}$; in this case the corresponding formula may be written as follows:

$$\begin{aligned}\fint_{-1}^{1} w(x)\frac{f(x)}{x-\lambda}dx&=A(\lambda)f(\lambda)+\sum_{i=1}^{n}\bar{w}_i(\lambda)f(x_i)+R_n^{(2)}(f), \quad \text{if } \lambda\neq x_i, \quad i=1,\dots,n,\\ &=\bar{A}(\lambda)f'(\lambda)+\sum_{i=1}^{n}\bar{w}_i(\lambda)f(x_i)+R_n^{(2)}(f), \quad \text{if } \lambda=x_r. \end{aligned}\tag{1.2}$$

Assume we know the quadrature rule (of interpolatory type)

$$\int_{-1}^{1} w(x)f(x)dx \doteq \sum_{i=1}^{n} H_i f(x_i), \tag{1.3}$$

* Work sponsored by the Ministero della Pubblica Istruzione of Italy

constructed on the same set of nodes $\{x_i\}$ of rules (1.1) and (1.2). We recall that if (1.3) has degree of exactness d ($\geqq n-1$) then the degree of the corresponding rule (1.2) is $d+1$, while rule (1.1) has always, except for a special case that shortly we shall mention, degree of exactness $n-1$. Introduce the polynomial $p_n(x)=\prod_{i=1}^{n}(x-x_i)$ and define the corresponding Cauchy principal value integral

$$q_n(\lambda)=\fint_{-1}^{1} w(x)\frac{p_n(x)}{x-\lambda}\,dx.$$

Then, the following expressions for the coefficients of (1.1) and (1.2) have been given:

$$w_i(\lambda)=\begin{cases}\dfrac{q_n(x_i)-q_n(\lambda)}{p_n'(x_i)(x_i-\lambda)}, & \text{if } x_i\neq\lambda,\\[2ex] \dfrac{q_n'(x_i)}{p_n'(x_i)}, & \text{if } x_i=\lambda,\end{cases}$$

$$A(\lambda)=\frac{q_n(\lambda)}{p_n(\lambda)},\qquad \bar{w}_i(\lambda)=\frac{H_i}{x_i-\lambda},\quad i=1,\dots,n;$$

when in (1.2) $\lambda=x_r$ we have

$$\bar{A}(\lambda)=\frac{q_n(\lambda)}{p_n'(\lambda)},\qquad \bar{w}_r(\lambda)=\frac{q_n'(\lambda)}{p_n'(\lambda)}-\frac{1}{2}\frac{q_n(\lambda)}{p_n'(\lambda)}\frac{p_n''(\lambda)}{p_n'(\lambda)}.$$

The computation of $w_i(\lambda)$ in (1.1) by means of the representation above is not recommended when λ is close to one of the x_i's. This difficulty seems to have been overcome when (1.3) is a Gauss, or Lobatto or Radau type rule; in these cases more efficient representations may be given (see [9]). Also when $w(x)=(1-x)^\alpha(1+x)^\beta$ and $x_i=\cos((i-1)\pi/(n-1))$ a more efficient algorithm for the evaluation of the quadrature sum has been proposed [1].

When λ is close to one of the nodes x_i, the use of rule (1.2) is not recommended, since in this situation the evaluation of the sum in (1.2) would cause numerical cancellation. Formula (1.2) has been used extensively in the numerical solution of singular integral equations with quadrature methods; in these situations the variable λ is "collocated" at the zeros of the function $q_n(\lambda)$. Notice that in these cases formulas (1.1) and (1.2) are identical and furthermore they coincide with rule (1.3) directly applied to the function $f(x)/(x-\lambda)$. In this special circumstance the degree of exactness of both rules coincides with that of rule (1.3) plus one.

The convergence of some formulas of type (1.1) and (1.2) has been examined and proved (see [1, 2, 3, 9]). The study of the convergence of rule (1.2) is quite straightforward if we assume the existence of $f'(\lambda)$; indeed, in this case rule (1.2) is convergent whenever rule (1.3) is. For rules of type (1.1) we have the following results (see [9]):

Assume that all derivatives of $f(x)$ of order $j=0,\dots,p$, $p\geqq 0$,[1] *exist in* $[-1,1]$, *and that $f^{(p)}(x)$ satisfies a Hölder condition of order* $0<\mu\leqq 1$; *further-*

[1] When $p=0$, by the 0th-derivative we mean the function itself

more, assume that the nodes $\{x_i\}$ *are chosen in accordance with one of the following two situations:*

(i) $w(x)=(1-x)^{\alpha}(1+x)^{\beta}$, α, $\beta>-1$, *and the* x_i*'s are the zeros of the* nth-*degree Jacobi polynomial* $P_n^{(\alpha,\beta)}(x)$,

(ii) $w(x)$ *is an arbitrary (nonnegative) weight function and the* x_i*'s are the zeros of* $P_n^{(\alpha,\beta)}(x)$, $\max(\alpha,\beta)\leqq -1/2$, *or* $x_i=\cos((i-1)\pi/(n-1))$, $i=1,\dots,n$; *then*

$$R_n^{(1)}(f)=O\left(\frac{1}{n^{p+\mu-\varepsilon}}\right)$$

for any $0<\varepsilon<\mu$.

In case (i) above for the weights $w_i(\lambda)$ of (1.1) we also have (see [2]) the following result that we shall need later to prove the main theorem:

Lemma 1. *When in* (1.1) $w(x)=(1-x)^{\alpha}(1+x)^{\beta}$, α, $\beta>-1$, *and the nodes* $\{x_i\}$ *are the zeros of the* nth-*degree Jacobi polynomial* $P_n^{(\alpha,\beta)}(x)$, *there exist constants* K_0 *and* K_1, *depending only on* λ, *such that for large* n

$$\sum_{i=1}^{n}|w_i(\lambda)|\leqq K_0+K_1\log n.$$

In the next section we consider the following two-dimensional Cauchy principal value integral

$$⨍_{-1}^{1}⨍_{-1}^{1}(1-x)^{\alpha_1}(1+x)^{\beta_1}(1-y)^{\alpha_2}(1+y)^{\beta_2}\frac{f(x,y)}{(x-x_0)(y-y_0)}\,dx\,dy$$

and use a product of rules of type (1.1) to approximate it. We examine the convergence property of such rules and generalize the above one-dimensional convergence results to cover this new situation.

We ought to remark that when in the chosen basic rules (1.1) the nodes $\{x_i\}$ are the zeros of the Jacobi polynomials corresponding to the indices α_i, β_i of the associated weight functions, convergence results for the resulting product rule have been stated in [11]; however, as remarked also by a reviewer of that paper (see MR 80g:65027), no proofs are given. Furthermore, our results appear slightly different from those presented in [11].

2. Convergence of Product Rules

In this section we examine the numerical evaluation of Cauchy principal value integrals of the form

$$I(x_0,y_0;f)=⨍_{-1}^{1}⨍_{-1}^{1}w_1(x)w_2(y)\frac{f(x,y)}{(x-x_0)(y-y_0)}\,dx\,dy, \tag{2.1}$$

where $w_1(x)=(1-x)^{\alpha_1}(1+x)^{\beta_1}$, $w_2(y)=(1-y)^{\alpha_2}(1+y)^{\beta_2}$, α_i, $\beta_i>-1$, by means of a product of one-dimensional quadrature rules of type (1.1).

The concept of Hölder continuity can be extended to the case of several variables. The following definition and Lemma 2 are taken from [10, pp. 12–16].

Definition. The function $f(x,y)$ satisfies the $H(\lambda_1,\lambda_2)$ condition, where $0<\lambda_i\leqq 1$, if for any given pair of values (x_1,y_1), (x_2,y_2) we have

$$|f(x_1,y_1)-f(x_2,y_2)|\leqq A|x_1-x_2|^{\lambda_1}+B|y_1-y_2|^{\lambda_2}, \tag{2.2}$$

where A and B are constants.

It is clear that if $f(x,y)\in H(\lambda_1,\lambda_2)$ then it satisfies the condition $H(\lambda_1)$ [2] for the variable x uniformly with respect to y, and the condition $H(\lambda_2)$ for the variable y uniformly with respect to x, i.e. the Hölder constants in the conditions $H(\lambda_1)$ and $H(\lambda_2)$ may be chosen independently of y and x, respectively.

It is also clear that if $f(x,y)\in H(\lambda_1,\lambda_2)$ then $f(x,y)\in H(\mu,\mu)$, where $\mu=\min(\lambda_1,\lambda_2)$, i.e.,

$$|f(x_1,y_1)-f(x_2,y_2)|\leqq C[|x_1-x_2|^{\mu}+|y_1-y_2|^{\mu}],$$

where C is a constant.

Lemma 2. *Let $f(x,y)\in H(\lambda_1,\lambda_2)$. The function*

$$\varphi(x,y)=\frac{f(x,y)-f(x,y_0)}{|y-y_0|^{\varepsilon}}, \qquad 0\leqq\varepsilon<\mu=\min(\lambda_1,\lambda_2),$$

satisfies the $H(\mu-\varepsilon)$ condition for the variable x uniformly with respect to y, and the $H(\lambda_2-\varepsilon)$ condition for the variable y uniformly with respect to x.

Now, define

$$s(x)=\int_{-1}^{1} w_2(y)\frac{f(x,y)-f(x,y_0)}{y-y_0}\,dy; \tag{2.3}$$

the above lemma is then sufficient to prove the following:

Lemma 3. *The function $s(x)$ defined by (2.3) satisfies a Hölder condition of order $\mu-\varepsilon$, where $\mu=\min(\lambda_1,\lambda_2)$ and ε is an arbitrary real number such that $0<\varepsilon<\mu$.*

Proof. Consider

$$s(x)-s(x_0)=\int_{-1}^{1} w_2(y)\left[\frac{f(x,y)-f(x,y_0)}{y-y_0}-\frac{f(x_0,y)-f(x_0,y_0)}{y-y_0}\right]dy;$$

we have

$$|s(x)-s(x_0)|\leqq\int_{-1}^{1}\frac{w_2(y)}{|y-y_0|^{1-\varepsilon}}\,\frac{|[f(x,y)-f(x,y_0)]-[f(x_0,y)-f(x_0,y_0)]|}{|y-y_0|^{\varepsilon}}\,dy.$$

However, because of Lemma 2,

$$\frac{|[f(x,y)-f(x,y_0)]-[f(x_0,y)-f(x_0,y_0)]|}{|y-y_0|^{\varepsilon}}\leqq K_0|x-x_0|^{\mu-\varepsilon},$$

where K_0 is a constant; hence

[2] We say that a function $g(x)$ satisfies the condition $H(\mu)$ when $g(x)$ is Hölder continuous of order μ

$$|s(x)-s(x_0)| \leqq K_1 |x-x_0|^{\mu-\varepsilon},$$

where K_1 is a constant (depending only on ε). This proves the lemma.

Having defined the new function $s(x)$, integral (2.1) may then be rewritten as follows:

$$\begin{aligned} I(x_0,y_0;f) &= \int_{-1}^{1} w_1(x)\frac{s(x)-s(x_0)}{x-x_0}dx \\ &+ \left[\fint_{-1}^{1} \frac{w_2(y)}{y-y_0}dy\right] \int_{-1}^{1} w_1(x)\frac{f(x,y_0)-f(x_0,y_0)}{x-x_0}dx \\ &+ \left[s(x_0)+f(x_0,y_0)\fint_{-1}^{1}\frac{w_2(y)}{y-y_0}dy\right]\fint_{-1}^{1}\frac{w_1(x)}{x-x_0}dx. \end{aligned} \tag{2.4}$$

This representation will be used later in the proof of Theorem 1. Incidentally, this expression also shows that the condition $f(x,y)\in H(\lambda_1,\lambda_2)$ is sufficient for the existence of the integral (once we assume the existence of the Cauchy principal value integrals of the weight functions $w_1(x)$ and $w_2(y)$).

To approximate integral (2.1) we consider one-dimensional formulas to type (1.1) and construct product rules of the following type:

$$I(x_0,y_0;f) = \sum_{i=1}^{n}\sum_{j=1}^{m} w_i(x_0)w_j(y_0)f(x_i,y_j) + E_{n,m}(x_0,y_0;f). \tag{2.5}$$

Whenever $f(x,y)$ is a polynomial of degree $n-1$ in x and $m-1$ in y, in (2.5) we have $E_{n,m}(x_0,y_0;f)=0$.

To simplify the notations and the calculation, in (2.5) we assume $m=an$, where a is a real constant. Moreover, when all partial derivatives of $f(x,y)$ of order $j=0,\dots,p$, $p\geqq 0$, exist and are continuous, and each derivative of order p satisfies a Hölder condition of order $0<\mu\leqq 1$, we write $f(x,y)\in H_p(\mu,\mu)$.

Before proceeding further, from [6, Thm. 8, p. 90] we take the following result:

Lemma 4. *Let $f(x,y)\in H_p(\mu,\mu)$. Then, given integers n and m, there exists a sequence of polynomials $\{p_{n,m}(x,y)\}$ of degree $n-1$ in x and $m-1$ in y such that*

$$\max_{\substack{-1\leqq x\leqq 1\\ -1\leqq y\leqq 1}} |f(x,y)-p_{n,m}(x,y)| \leqq M\left(\frac{1}{n^{p+\mu}}+\frac{1}{m^{p+\mu}}\right), \tag{2.6}$$

where M is a constant.

This result, however, is not sufficient for our purposes. Indeed, the proof of the next lemma appear straightforward if we have at our disposal a sequence of polynomials $\{p_{n,m}(x,y)\}$ of the type described in Lemma 4, but with the following extra property:

$$\begin{aligned} |p_{n,m}(x,y)-p_{n,m}(x,y_0)| &\leqq K_3|y-y_0|^{\mu_1} \\ |p_{n,m}(x,y)-p_{n,m}(x_0,y)| &\leqq K_4|x-x_0|^{\mu_1} \end{aligned} \tag{2.7}$$

where $\mu_1=\min(p+\mu,1)$, and K_3 and K_4 are constants (independent of n, m, x and y). In the appendix of this paper we shall explicitly construct (see Theorem 3) such a sequence of polynomials; thus in the following we assume its existence.

Lemma 5. *Let $r_{n,m}(x,y)=f(x,y)-p_{n,m}(x,y)$, where $f(x,y)\in H_p(\mu,\mu)$ and $p_{n,m}(x,y)$ is the polynomial characterized by (2.6) and (2.7). Then, for $0<\nu<\mu_1=\min(p+\mu,1)$ and $y\neq y_0$, we have*

$$\frac{|r_{n,m}(x,y)-r_{n,m}(x,y_0)|}{|y-y_0|^\nu}\leqq\frac{L_1}{n^{p+\mu-\delta}},\qquad \delta=\nu\frac{p+\mu}{\mu_1},\tag{2.8}$$

where L_1 is a constant.

Proof. First we remark that

$$\begin{aligned}|r_{n,m}(x,y)-r_{n,m}(x,y_0)|&\leqq|f(x,y)-f(x,y_0)|+|p_{n,m}(x,y)-p_{n,m}(x,y_0)|\\&\leqq B|y-y_0|^{\mu_1}+K_3|y-y_0|^{\mu_1}\leqq B_0|y-y_0|^{\mu_1},\end{aligned}$$

where $\mu_1=\min(\mathrm{p}+\mu,1)$. Then, for $y\neq y_0$ we obtain

$$\begin{aligned}&\frac{|r_{n,m}(x,y)-r_{n,m}(x,y_0)|}{|y-y_0|^\nu}\\&=|r_{n,m}(x,y)-r_{n,m}(x,y_0)|^{1-\frac{\nu}{\mu_1}}\frac{|r_{n,m}(x,y)-r_{n,m}(x,y_0)|^{\frac{\nu}{\mu_1}}}{|y-y_0|^\nu}\\&\leqq B_1[|r_{n,m}(x,y)|+|r_{n,m}(x,y_0)|]^{1-\frac{\nu}{\mu_1}}\leqq\frac{L_1}{n^{p+\mu-\delta}},\end{aligned}$$

which proves the lemma. Obviously the role of the variables x and y may be interchanged.

This lemma generalizes a result of Kalandiya [4] concerning the corresponding case of a function $f(x)$ of a single variable. Notice also that our proof appear different.

The results presented so far are sufficient to prove the main theorem.

Theorem 1. *Let $f(x,y)\in H_p(\mu,\mu)$. Let the nodes $\{x_i\}$ and $\{y_j\}$ coincide either* (i) *with the zeros of the Jacobi polynomials $P_n^{(\alpha_1,\beta_1)}(x)$ and $P_m^{(\alpha_2,\beta_2)}(y)$, respectively, or* (ii) *with those of $P_n^{(a_1,b_1)}(x)$ and $P_m^{(a_2,b_2)}(y)$, $-1<a_i$, $b_i\leqq-\frac{1}{2}$, or,* (iii) *let $x_i=\cos((i-1)\pi/(n-1))$ and $y_i=\cos((j-1)\pi/(m-1))$. Then, for the remainder term in* (2.5) *we have*

$$E_{n,m}(x_0,y_0;f)=O\left(\frac{1}{n^{p+\mu-\gamma}}\right),$$

where $0<\gamma<\mu$ is a real number, small as we like.

Proof. Consider the polynomial $p_{n,m}(x,y)$ of Lemma 4 with property (2.7), and remark that

$$E_{n,m}(x_0,y_0;f)=E_{n,m}(x_0,y_0;r_{n,m}),\tag{2.9}$$

where $r_{n,m}(x,y)=f(x,y)-p_{n,m}(x,y)$.

From the first part of the proof of Lemma 5 we recall the following inequalities

$$\begin{aligned}|r_{n,m}(x,y)-r_{n,m}(x_0,y)|&\leqq C_1|x-x_0|^{\mu_1},\\ |r_{n,m}(x,y)-r_{n,m}(x,y_0)|&\leqq C_2|y-y_0|^{\mu_1},\end{aligned}\tag{2.10}$$

where the constants C_1 and C_2 are independent of n and m, and $\mu_1 = \min(p+\mu, 1)$. By inspecting the proof of Lemma 2 in [10], because of relations (2.10) we may claim that

$$\varphi_{n,m}(x,y)=\frac{r_{n,m}(x,y)-r_{n,m}(x,y_0)}{|y-y_0|^{\varepsilon}}\in H(\mu_1-\varepsilon,\mu_1-\varepsilon),\tag{2.11}$$

where the Hölder constants are independent of n and m, i.e.

$$\begin{aligned}|\varphi_{n,m}(x,y)-\varphi_{n,m}(x_0,y)|&\leqq D_1|x-x_0|^{\mu_1-\varepsilon}\\ |\varphi_{n,m}(x,y)-\varphi_{n,m}(x,y_0)|&\leqq D_2|y-y_0|^{\mu_1-\varepsilon}.\end{aligned}$$

At this point we introduce the function

$$s_{n,m}(x)=\int_{-1}^{1} w_2(y)\frac{r_{n,m}(x,y)-r_{n,m}(x,y_0)}{y-y_0}\,dy,$$

and remark that, because of Lemma 5,

$$|s_{n,m}(x)|\leqq\frac{L_2}{n^{p+\mu-\delta}},\qquad -1\leqq x\leqq 1.\tag{2.12}$$

Furthermore,

$$\begin{aligned}|s_{n,m}(x)-s_{n,m}(x_0)|&\leqq\int_{-1}^{1}\frac{w_2(y)}{|y-y_0|^{1-\varepsilon}}|\varphi_{n,m}(x,y)-\varphi_{n,m}(x_0,y)|\,dy\\ &\leqq L_3|x-x_0|^{\mu_1-\varepsilon}.\end{aligned}\tag{2.13}$$

Inequalities (2.12) and (2.13) are sufficient to prove a result similar to that of Lemma 5 for the function $s_{n,m}(x)$, that is,

$$\frac{|s_{n,m}(x)-s_{n,m}(x_0)|}{|x-x_0|^{\nu}}<\frac{L_4}{n^{p+\mu-\delta_1}}.\tag{2.14}$$

To prove the theorem, recalling (2.4), (2.5) and (2.9) we write the remainder term $E_{n,m}(x_0,y_0;f)$ as follows

$$\begin{aligned}E_{n,m}(x_0,y_0;f)&=\int_{-1}^{1} w_1(x)\frac{s_{n,m}(x)-s_{n,m}(x_0)}{x-x_0}\,dx\\ &\quad+\left[\fint_{-1}^{1}\frac{w_2(y)}{y-y_0}\,dy\right]\int_{-1}^{1} w_1(x)\frac{r_{n,m}(x,y_0)-r_{n,m}(x_0,y_0)}{x-x_0}\,dx\\ &\quad+\left[s_{n,m}(x_0)+r_{n,m}(x_0,y_0)\fint_{-1}^{1}\frac{w_2(y)}{y-y_0}\,dy\right]\fint_{-1}^{1}\frac{w_1(x)}{x-x_0}\,dx\\ &\quad-\sum_{i=1}^{n}\sum_{j=1}^{m} w_i(x_0)w_j(y_0)r_{n,m}(x_i,y_j).\end{aligned}$$

With the nodes $\{x_i\}$ and $\{y_j\}$ chosen as in (i) we only need to apply Lemmas 1, 4, 5 and inequalities (2.12) and (2.14).

To prove the thesis in cases (ii) and (iii) we need to recall that (see [12, p. 338] for case (ii), and [7] for (iii)), denoting by $l_k(x)$ the Lagrange fundamental polynomials associated with the set of nodes $\{x_i\}$, we have

$$\max_{-1\leqq x\leqq 1} \sum_{k=1}^{n} |l_k(x)| = O(\log n).$$

Let in (1.1) $w(x)=(1-x)^\alpha(1+x)^\beta$; denote by $v_i(\lambda)$ the weights of (1.1) when the nodes $\{x_i\}$ are chosen as in case (ii) or (iii), and by $w_i(\lambda)$ those corresponding to the zeros $\{x_i^{(\alpha,\beta)}\}$ of the Jacobi polynomial of degree n associated with $w(x)$. We have

$$v_k(\lambda) = \fint_{-1}^{1} w(x)\frac{l_k(x)}{x-\lambda}\,dx = \sum_{i=1}^{n} w_i(\lambda)\, l_k(x_i^{(\alpha,\beta)}),$$

hence

$$\sum_{k=1}^{n} |v_k(\lambda)| \leqq \sum_{i=1}^{n} |w_i(\lambda)| \sum_{k=1}^{n} |l_k(x_i^{(\alpha,\beta)})| = O(\log^2 n).$$

This last bound is sufficient to complete the proof of the theorem.

3. Appendix

First we consider the case of a function of a single variable $f(x)$, $-1\leqq x\leqq 1$; we assume that all its derivatives of order $j=0,\dots,p$, $p\geqq 0$, exist, and that $f^{(p)}(x)$ satisfies a Hölder condition of order $0<\mu\leqq 1$. In this case we write $f(x)\in H_p(\mu)$.

Under these hypothesis, a well-known result of Jackson (see [8, p. 57]) tells us that for each integer n there exists a polynomial $p_n(x)$ of degree n such that

$$\max_{-1\leqq x\leqq 1} |f(x)-p_n(x)| = O\left(\frac{1}{n^{p+\mu}}\right). \tag{3.1}$$

However we can claim the existence of a sequence of polynomials of the type above with the following extra property:

$$|p_n(x)-p_n(x_0)| \leqq N|x-x_0|^{\mu_1}, \tag{3.2}$$

where $\mu_1=\min(p+\mu, 1)$ and N is a constant independent of n and x. To construct a sequence of polynomials $p_n(x)$ with properties (3.1) and (3.2) we set $x=\cos\theta$ and consider the 2π-periodic even function

$$g(\theta)=f(\cos\theta)\sim\sum_{n=0}^{\infty} a_n\cos n\theta.$$

It is known that $g(\theta)$ satisfies the same regularity condition of $f(x)$. Then we seek for a summability kernel (see [5, Sect. 2]) $K_n(t)$ such that the trigonometric polynomial

$$\sigma(K_n, g; \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(\theta - t) g(t) dt \tag{3.3}$$

satisfies condition (3.1) and (3.2) with respect to the variable θ. The well-known Fejér and Jackson kernels are not fully suitable because when the function $g(\theta)$ possesses continuous derivatives or is analytic, the corresponding polynomials $\sigma(K_n, g; \theta)$ only produce an approximation of order n^{-1}.

A kernel which satisfies our requirements is, for example, the de la Vallée Poussin kernel $V_n(t)$, related to that of Fejér

$$K_n(t) = \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n+1}\right) e^{ijt} = \frac{1}{n+1} \left[\frac{\sin \frac{n+1}{2} t}{\sin \frac{t}{2}}\right]^2$$

by the relationship

$$V_n(t) = 2K_{2n+1}(t) - K_n(t). \tag{3.4}$$

We recall that if we consider the partial sums $S_0(g; \theta)$, $S_1(g; \theta), \ldots$ of the Fourier expansion of $g(\theta)$, then for the polynomials $\sigma(K_n, g; \theta)$ associated with the Fejér kernel we have the following representation

$$\sigma(K_n, g; \theta) = \frac{1}{n+1} \sum_{i=0}^{n} S_i(g; \theta),$$

while for those associated with the de la Vallée Poussin kernel

$$\sigma(V_n, g; \theta) = \frac{1}{n+1} \sum_{i=0}^{n} S_{n+1+i}(g; \theta);$$

notice that $\sigma(K_n, g; \theta)$ and $\sigma(V_n, g; \theta)$ are trigonometric polynomials of degree n and $2n+1$, respectively.

For the properties of these kernels see, for example, [5, 6, 8]. Here we recall that $\sigma(V_n, g; \theta)$ satisfies (3.1), with respect to the variable θ, and furthermore

$$\|V_n\|_{L^1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |V_n(t)| dt \leqq 3.^3$$

We also recall that given any trigonometric polynomial $t_n(\theta)$ of degree n we have

$$\sigma(V_n, t_n; \theta) = t_n(\theta).$$

Now, consider the space $H(\mu_1)$, where $\mu_1 = \min(p + \mu, 1)$, with the norm

$$\|g\|_{H(\mu_1)} = \sup_{\theta} |g(\theta)| + \sup_{\substack{\theta \\ h \neq 0}} \frac{|g(\theta + h) - g(\theta)|}{|h|^{\mu_1}}.$$

It is known (see [5, Sect. 2]) that if $h \in L^1[-\pi, \pi]$ then

$$h * g = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(t - \theta) g(\theta) d\theta = g * h \in H(\mu_1);$$

furthermore,

$$\|h * g\|_{H(\mu_1)} \leqq \|h\|_{L^1} \cdot \|g\|_{H(\mu_1)}.$$

In our case we have

$$\|V_n * g\|_{H(\mu_1)} \leqq \|V_n\|_{L^1} \cdot \|g\|_{H(\mu_1)} \leqq 3\|g\|_{H(\mu_1)}.$$

This last result allows us to claim the following:

Lemma 6. *Let* $g(\theta) \in H(\mu_1)$; *then*

$$\frac{|\sigma(V_n, g; \theta) - \sigma(V_n, g; \theta_0)|}{|\theta - \theta_0|^{\mu_1}} \leqq 3\|g\|_{H(\mu_1)}. \tag{3.5}$$

A trigonometric polynomial of degree $2n+2$ with the same properties is readily available. Indeed it is sufficient to consider, for example,

$$\sigma(V_n, g; \theta) + \frac{1}{n^{p+\mu+1}} \cdot \cos(2n+2)\theta$$

and recall that

$$\left|\frac{\cos m\theta - \cos m\theta_0}{\theta - \theta_0}\right| \leqq m.$$

Having constructed the trigonometric polynomial $\sigma(V_n, g; \theta)$ it is almost immediate to obtain the corresponding algebraic polynomial satisfying properties (3.1) and (3.2). To this end, we assume, without any restriction, $\theta_0 \in \left(0, \frac{\pi}{2}\right]$, $\theta \in [0, \pi]$. When $\theta \in \left[0, \frac{\pi}{2}\right]$ we have

$$|\cos\theta - \cos\theta_0| = 2\left|\sin\frac{\theta+\theta_0}{2}\sin\frac{\theta-\theta_0}{2}\right| \geqq \frac{2}{\pi^2}\theta_0|\theta - \theta_0|,$$

while when $\theta \in \left(\frac{\pi}{2}, \pi\right]$

$$|\cos\theta - \cos\theta_0| \geqq \frac{\sqrt{2}}{\pi}|\theta - \theta_0|.$$

Hence, for $\theta_0 \in \left(0, \frac{\pi}{2}\right]$ and $\theta \in [0, \pi]$

$$|\cos\theta - \cos\theta_0| \geqq A_0|\theta - \theta_0|,$$

where A_0 is a constant, depending only on θ_0. This last inequality allows us to state that for $\theta \neq \theta_0$

$$\frac{\sigma(V_n, g; \theta) - \sigma(V_n, g; \theta_0)|}{|\theta - \theta_0|^{\mu_1}} \geqq A_1 \frac{|\sigma(V_n, g; \theta) - \sigma(V_n, g; \theta_0)|}{|\cos\theta - \cos\theta_0|^{\mu_1}}.$$

Thus, after setting $x = \cos\theta$, we have:

[3] By $L^1 = L^1[-\pi, \pi]$ we denote the Banach space of the Lebesgue integrable functions on $[-\pi, \pi]$

Theorem 2. *Let $f(x)\in H_p(\mu)$, $0<\mu\leqq 1$. Then there exists a sequence of polynomials $p_n(x)$ of degree $n=0,1,2,\ldots$ such that*

$$\max_{-1\leqq x\leqq 1} |f(x)-p_n(x)|\leqq\frac{M_0}{n^{p+\mu}} \quad \text{and} \quad |p_n(x)-p_n(x_0)|\leqq N_0|x-x_0|^{\mu_1},$$

where $\mu_1=\min(p+\mu,1)$ and n_0 is a constant independent of n and x.

Having this result, the proof of Kalandiya's lemma (see [4, Lemma 2]) is then immediate and can be obtained following the proof of Lemma 5.

Next we want to extend the theorem above to the case of functions of two variables. The same procedure would hold in the case of several variables, however, in order to simplify the notation, we examine with details only the two-dimensional case.

Assume $f(\cos\theta,\cos\varphi)=g(\theta,\varphi)\in H_p(\lambda_1,\lambda_2)$ and write its Fourier expansion with respect to the variable θ; we have

$$g(\theta,\varphi)\sim\sum_{k=0}^{\infty} a_k(\varphi)\cos k\theta,$$

where

$$a_k(\varphi)=\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta,\varphi)\cos k\theta\, d\theta.$$

For any given φ first we construct

$$\sigma(V_n,g;\theta,\varphi)=\sigma_n(V_n,g;\theta,\varphi)=\frac{1}{2\pi}\int_{-\pi}^{\pi} V_n(\theta-t)g(t,\varphi)dt,$$

and then

$$\sigma(V_n,\sigma_n;\theta,\varphi)=\frac{1}{2\pi}\int_{-\pi}^{\pi} V_m(\varphi-u)\sigma_n(V_n,g;\theta,u)du.$$

We set

$$T_{2n+1,2m+1}(g;\theta,\varphi)=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} V_m(\varphi-u)V_n(\theta-t)g(t,u)dt\,du.$$

$T_{2n+1,2m+1}(g;\theta,\varphi)$ is a (pure cosine) trigonometric polynomial of degree $2n+1$ with respect to the variable θ and of degree $2m+1$ with respect to φ. Furthermore, given any trigonometric polynomial $S_{n,m}(\theta,\varphi)$ of degree n in θ and m in φ, we have

$$T_{2n+1,2m+1}(S_{n,m};\theta,\varphi)=S_{n,m}(\theta,\varphi).$$

Let $T^*_{n,m}(g;\theta,\varphi)$ denote the trigonometric polynomial, of degree n in θ and m in φ, of best approximation to $g(\theta,\varphi)$, i.e. so that the quantity

$$\max_{\theta,\varphi}|g(\theta,\varphi)-T^*_{n,m}(g;\theta,\varphi)|$$

is minimum among all possible trigonometric polynomials of degree n and m. We label this minimum value by $E^*_{n,m}(g)$. The following result is then known (see [6, Thm. 6, p. 87]):

Lemma 7. *Let* $g(\theta,\varphi)\in H_p(\lambda_1,\lambda_2)$. *Then*

$$E^*_{n,m}(g)\leqq G\left(\frac{1}{n^{p+\lambda_1}}+\frac{1}{m^{p+\lambda_2}}\right). \tag{3.6}$$

At this point we are ready to generalize Theorem 2 to functions $f(x,y)$ of two variables.

Theorem 3. *Let* $f(x,y)\in H_p(\lambda_1,\lambda_2)$, $0<\lambda_i\leqq 1$. *Then for all integers n and m there exist polynomials* $p_{n,m}(x,y)$ *of degree n in x and m in y such that*

$$\max_{x,y}|f(x,y)-p_{n,m}(x,y)|\leqq M_0\left(\frac{1}{n^{p+\lambda_1}}+\frac{1}{m^{p+\lambda_2}}\right) \tag{3.7}$$

and

$$\begin{aligned}|p_{n,m}(x,y)-p_{n,m}(x_0,y)|&\leqq M_1|x-x_0|^{\lambda_1^*}\\ |p_{n,m}(x,y)-p_{n,m}(x,y_0)|&\leqq M_2|y-y_0|^{\lambda_2^*},\end{aligned} \tag{3.8}$$

where $\lambda_i^*=\min(p+\lambda_i,1)$ *and* M_i *are constants (independent of n, m, x and y).*

Proof. First we prove (3.7):

$$\begin{aligned}&|g(\theta,\varphi)-T_{2n+1,2m+1}(g;\theta,\varphi)|\\ &\quad\leqq|g(\theta,\varphi)-T^*_{n,m}(g;\theta,\varphi)|+|T^*_{n,m}(g;\theta,\varphi)-T_{2n+1,2m+1}(g;\theta,\varphi)|\\ &\quad=|g(\theta,\varphi)-T^*_{n,m}(g;\theta,\varphi)|+|T_{2n+1,2m+1}(g-T^*_{n,m};\theta,\varphi)|\\ &\quad\leqq 10E^*_{n,m}(g)\leqq M_0\left(\frac{1}{n^{p+\lambda_1}}+\frac{1}{m^{p+\lambda_2}}\right).\end{aligned}$$

Next, to derive, for example, the first inequality in (3.8) we consider the quantity

$$\begin{aligned}&\frac{|T_{2n+1,2m+1}(g;\theta+h,\varphi)-T_{2n+1,2m+1}(g;\theta,\varphi)|}{|h|^{\lambda_1^*}}\\ &\quad=\frac{1}{2\pi}\left|\int_{-\pi}^{\pi}V_m(\varphi-u)\frac{\sigma_n(V_n,g;\theta+h,u)-\sigma_n(V_n,g;\theta,u)}{|h|^{\lambda_1^*}}du\right|.\end{aligned}$$

Since for any given u

$$\sigma_n(V_n,g;\theta,u)\in H(\lambda_1^*)$$

(we consider σ_n a function of the single variable θ), we have

$$\|\sigma_n(V_n,g;\theta,u)\|_{H(\lambda_1^*)}\leqq 3\|g\|_{H(\lambda_1^*)}\leqq A_2,$$

where A_2 is a constant, independent of n, θ and u; hence

$$\frac{|\sigma_n(V_n,g;\theta+h,u)-\sigma_n(V_n,g;\theta,u)|}{|h|^{\lambda_1^*}}\leqq A_2,$$

and

$$\frac{|T_{2n+1,2m+1}(g;\theta+h,\varphi)-T_{2n+1,2m+1}(g;\theta,\varphi)|}{|h|^{\lambda_1^*}}\leqq 3A_2.$$

Then, we proceed as in the one-dimensional case, i.e. setting $x = \cos\theta$ and $y = \cos\varphi$. The second inequality in (3.8) is obtained very similarly. Having constructed $T_{2n+1,2m+1}(g;\theta,\varphi)$ it is immediate to construct $T_{2n+2,2m+1}$, $T_{2n+1,2m+2}$ and $T_{2n+2,2m+2}$. To this end it is sufficient to add terms like

$$\frac{1}{n^{p+\lambda_1+1}}\cos(2n+2)\theta, \qquad \frac{1}{m^{p+\lambda_2+1}}\cos(2m+2)\varphi.$$

Acknowledgement. I acknowledge helpful discussions with F. Ricci.

References

1. Chawla, M.M., Kumar, S.: Convergence of quadratures for Cauchy principal value integrals. Computing **23**, 67–72 (1979)
2. Elliott, D., Paget, D.F.: On the convergence of a quadrature rule for evaluating certain Cauchy principal value integrals. Numer. Math. **23**, 311–319 (1975) Addendum: Numer. Math. **25**, 287–289 (1976)
3. Elliott, D., Paget, D.F.: Gauss type quadrature rules for Cauchy principal value integrals. Math. Comput. **33**, 301–309 (1979)
4. Kalandiya, A.I.: On a direct method of solution of an equation in wing theory and its application to the theory of elasticity. Mat. Sb. **42**, 249–272 (1957) (Russian)
5. Katznelson, Y.: An introduction to Harmonic Analysis. John Wiley and Sons, 1968
6. Lorentz, G.G.: Approximation of functions. Holt Rinehart and Winston, 1965
7. McCabe, J.H., Phillips, G.M.: On a certain class of Lebesgue constants. BIT, **13**, 434–442 (1973)
8. Meinardus, G.: Approximation of functions: Theory and Numerical Methods. Berlin, Heidelberg, New York: Springer 1967
9. Monegato, G.: The numerical evaluation of one-dimensional Cauchy principal value integrals. Computing **29**, 337–354 (1982)
10. Muskhelishvili, N.I.: Singular integral equations. Groningen, Holland: Noordhoff 1953
11. Šeško, M.A.: Convergence of cubature processes for a two-dimensional singular integral. Dokl. Akad. Nauk USSR, **23**, 4, 293–297 (1979) (Russian)
12. Szegö, G.: Orthogonal Polynomials. Amer. Math. Soc. Colloquium Publications, 23, Providence R.I., 1975

Received September 23, 1982 / April 25, 1983