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Ozan Candogan <http://web.mit.edu/candogan/>, Ishai Menache <http://www.mit.edu/~ishai/>,  
Asuman Ozdaglar <http://web.mit.edu/asuman/www/>, Pablo A. Parrilo <http://www.mit.edu/~parrilo/>,

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# Flows and Decompositions of Games: Harmonic and Potential Games

Ozan Candogan

Laboratory for Information and Decision Systems (LIDS), Massachusetts Institute of Technology, Cambridge, Massachusetts 02139,  
[candogan@mit.edu](mailto:candogan@mit.edu), <http://web.mit.edu/candogan/>

Ishai Menache

Microsoft Research New England, Cambridge, Massachusetts 02139, [t-ismena@microsoft.com](mailto:t-ismena@microsoft.com), <http://www.mit.edu/~ishai/>

Asuman Ozdaglar, Pablo A. Parrilo

Laboratory for Information and Decision Systems (LIDS), Massachusetts Institute of Technology, Cambridge, Massachusetts 02139  
[asuman@mit.edu](mailto:asuman@mit.edu), <http://web.mit.edu/asuman/www/>, [parrilo@mit.edu](mailto:parrilo@mit.edu), <http://www.mit.edu/~parrilo/>

In this paper we introduce a novel flow representation for finite games in strategic form. This representation allows us to develop a canonical direct sum decomposition of an arbitrary game into three components, which we refer to as the potential, harmonic, and nonstrategic components. We analyze natural classes of games that are induced by this decomposition, and in particular, focus on games with no harmonic component and games with no potential component. We show that the first class corresponds to the well-known potential games. We refer to the second class of games as harmonic games, and demonstrate that this new class has interesting properties which contrast with properties of potential games. Exploiting the decomposition framework, we obtain explicit expressions for the projections of games onto the subspaces of potential and harmonic games. This enables an extension of the equilibrium properties of potential and harmonic games to “nearby” games.

**Key words:** decomposition of games; potential games; harmonic games; strategic equivalence

**MSC2000 subject classification:** Primary: 91A10, 91A70; secondary: 91B02, 58A14

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**1. Introduction.** Potential games play an important role in game-theoretic analysis due to their desirable static properties (e.g., existence of a pure strategy Nash equilibrium) and tractable dynamics (e.g., convergence of simple player dynamics to a Nash equilibrium); see Monderer and Shapley [33, 34], and Neyman [37]. However, many multi-agent strategic interactions in economics and engineering cannot be modeled as a potential game.

This paper provides a novel flow representation of the preference structure in strategic-form finite games, which allows for delineating the fundamental characteristics in preferences that lead to potential games. This representation enables us to develop a canonical orthogonal decomposition of an arbitrary game into a potential component, a harmonic component, and a nonstrategic component, each with its distinct properties. The decomposition can be used to define the “distance” of an arbitrary game to the set of potential games. We use this fact to describe the approximate equilibria of the original game in terms of the equilibria of the closest potential game.

The starting point is to associate to a given finite game a *game graph*, where the set of nodes corresponds to the strategy profiles and the edges represent the “comparable strategy profiles,” i.e., strategy profiles that differ in the strategy of a single player. The utility differences for the deviating players along the edges define a flow on the game graph. Although this graph contains strictly less information than the original description of the game in terms of utility functions, all relevant strategic aspects (e.g., equilibria) are captured.

Our first result provides a *canonical decomposition* of an arbitrary game using tools from the study of flows on graphs (which can be viewed as combinatorial analogues of vector fields). In particular, we use the *Helmholtz decomposition theorem* (e.g., Jiang et al. [23]), which enables the decomposition of a flow on a graph into three components: globally consistent, locally consistent (but globally inconsistent), and locally inconsistent components (see Theorem 3.1). The globally consistent component represents a gradient flow while the locally consistent flow corresponds to flows around global cycles. The locally inconsistent component represents local cycles (or circulations) around 3-cliques of the graph.

Our game decomposition has three components: *nonstrategic*, *potential*, and *harmonic*. The first component represents the “nonstrategic interactions” in a game. Consider two games in which, given the strategies of the other players, each player’s utility function differs by an additive constant. These two games have the same utility differences, and therefore they have the same flow representation. Moreover, since equilibria are defined in terms of utility differences, the two games have the same equilibrium set. We refer to such games as *strategically equivalent*. We normalize the utilities (by adding constants to the utilities of a player given the other players’ strategies such that their sum is equal to zero), and refer to the utility differences between a game and its normalization as the nonstrategic component of the game. Our next step is to remove the nonstrategic

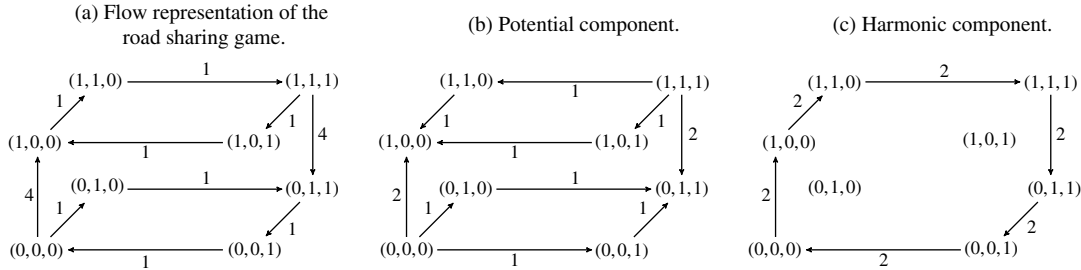


FIGURE 1. Potential-harmonic decomposition of the road-sharing game.

Note. An arrow between two strategy profiles, indicates the improvement direction in the payoff of the player who changes its strategy, and the associated number quantifies the improvement in its payoff.

component and apply the Helmholtz decomposition to the remainder. The flow representation of a game defined in terms of utility functions (as opposed to preferences) does not exhibit local cycles; therefore, the Helmholtz decomposition yields the two remaining components of a game: the *potential component* (gradient flow) and the *harmonic component* (global cycles). The decomposition result is particularly insightful for bimatrix games (i.e., finite games with two players; see §4.3), where the potential component represents the “team part” of the utilities (suitably perturbed to capture the utility matrix differences), and the harmonic component corresponds to a zero-sum game.

The canonical decomposition we introduce is illustrated in the following example.

**EXAMPLE 1.1 (ROAD-SHARING GAME).** Consider a three-player game, where each player has to choose one of the two roads  $\{0, 1\}$ . We denote the players by  $d_1$ ,  $d_2$ , and  $s$ . The player  $s$  tries to avoid sharing the road with other players: its payoff decreases by 2 with each player  $d_1$  and  $d_2$  who shares the same road with it. The player  $d_1$  receives a payoff  $-1$ , if  $d_2$  shares the road with it and 0 otherwise. The payoff of  $d_2$  is equal to the negative of the payoff of  $d_1$ , i.e.,  $u^{d_1} + u^{d_2} = 0$ . Intuitively, player  $d_1$  tries to avoid player  $d_2$ , whereas player  $d_2$  wants to use the same road with  $d_1$ .

In Figure 1(a) we present the flow representation for this game (described in detail in §2.2), where the nonstrategic component has been removed. Figures 1(b) and 1(c) show the decomposition of this flow into its potential and harmonic components. In the figure, each tuple  $(a, b, c)$  denotes a strategy profile, where player  $s$  uses strategy  $a$  and players  $d_1$  and  $d_2$  use strategies  $b$  and  $c$ , respectively.

This example shows that the harmonic component of a game satisfies the flow conservation condition; i.e., the total flow at each node is equal to zero. On the other hand, for the potential component, the total flow around every cycle is equal to zero. These observations highlight a key distinction in the flow representations of harmonic and potential components of a game: the harmonic component can be characterized by the presence of “preference cycles,” while the potential component does not have such cycles.

These components induce a direct sum decomposition of the space of games into three respective subspaces, which we refer to as the *nonstrategic*, *potential*, and *harmonic* subspaces, denoted by  $\mathcal{N}$ ,  $\mathcal{P}$ , and  $\mathcal{H}$ , respectively. We use these subspaces to define classes of games with distinct equilibrium properties. We establish that the set of potential games coincides with the direct sum of the subspaces  $\mathcal{P}$  and  $\mathcal{N}$ ; i.e., potential games are those with no harmonic component. Similarly, we define a new class of games in which the potential component vanishes as *harmonic games*. The classical rock-paper-scissors and matching pennies games are examples of harmonic games. The decomposition then has the following structure:

$$\underbrace{\mathcal{P} \oplus \mathcal{N}}_{\text{Potential games}} \oplus \underbrace{\mathcal{H}}_{\text{Harmonic games}}.$$

It is insightful to provide alternative definitions of potential and harmonic games in terms of the payoff functions of players. A game with a set of players  $\mathcal{M}$ , set of strategies  $E^m$  for all  $m \in \mathcal{M}$ , and a collection of utility functions  $\{u^m\}_{m \in \mathcal{M}}$  is a *potential game* if there exists a potential function  $\phi$  satisfying

$$\phi(\mathbf{p}^m, \mathbf{p}^{-m}) - \phi(\mathbf{q}^m, \mathbf{p}^{-m}) = u^m(\mathbf{p}^m, \mathbf{p}^{-m}) - u^m(\mathbf{q}^m, \mathbf{p}^{-m}), \quad (1)$$

for every  $m \in \mathcal{M}$ ,  $\mathbf{p}^m, \mathbf{q}^m \in E^m$ ,  $\mathbf{p}^{-m} \in E^{-m}$ . The game is a *harmonic game* if for all strategy profiles  $\mathbf{p}$  and strategies  $\mathbf{q}^m \in E^m$ , the utility functions satisfy

$$\sum_{m \in \mathcal{M}} \sum_{\mathbf{q}^m \in E^m} (u^m(\mathbf{p}^m, \mathbf{p}^{-m}) - u^m(\mathbf{q}^m, \mathbf{p}^{-m})) = 0. \quad (2)$$

It can be seen from these definitions that harmonic games are games which satisfy the flow conservation condition illustrated in Example 1.1, and potential games are those for which the total flow around each cycle is equal to zero. The equivalences of subspace definitions of potential and harmonic games, and the definitions in (1) and (2) are established in §§5.1 and 5.2.

Our second set of results establishes properties of potential and harmonic games. Harmonic games can be characterized by the existence of improvement cycles, i.e., cycles in the game graph, where at each step the player that changes its action improves its payoffs. We show that harmonic games generically do not have pure Nash equilibria. Interestingly, for the special case when the number of strategies of each player is the same, a harmonic game satisfies a “multi-player zero-sum property” (i.e., the sum of utilities of all players is equal to zero at all strategy profiles). We also study the mixed Nash and correlated equilibria of harmonic games. We show that the uniformly mixed strategy profile (see Definition 5.2) is always a mixed Nash equilibrium and if there are two players in the game, the set of mixed Nash equilibria generically coincides with the set of correlated equilibria.

Our third set of results focuses on the *projection of a game onto its respective components*. We first define a natural inner product and show that under this inner product the components in our decomposition are orthogonal. We further provide explicit expressions for the closest potential and harmonic games to a game with respect to the norm induced by the inner product. We use the distance of a game to its closest potential game to characterize the approximate equilibrium set in terms of the equilibria of the potential game.

The decomposition framework in this paper leads to the identification of subspaces of games with distinct and tractable equilibrium properties. Understanding the structural properties of these subspaces and the classes of games they induce, provides new insights and tools for analyzing the static and dynamical properties of general noncooperative games; further implications are outlined in §7.

**1.1. Related literature.** Besides the works already mentioned, our paper is also related to several papers in the cooperative and noncooperative game theory literature:

The idea of decomposing a game (using different approaches) into simpler games which admit more tractable equilibrium analysis appeared even in the early works in the cooperative game theory literature. In Von Neumann and Morgenstern [47], the authors propose to decompose games with large numbers of players into games with fewer players. In Gilboa and Schmeidler [15], Marinacci [31], Shapley [42], a different approach is followed: the authors identify cooperative games through the games’ value functions (see Von Neumann and Morgenstern [47]) and obtain decompositions of the value function into simpler functions. By defining the component games using the simpler value functions, they obtain decompositions of games. In this approach, the set of players is not made smaller or larger by the decomposition but the component games have simpler structure. Another method for decomposing the space of cooperative games appeared in Kleinberg and Weiss [27, 28]. In these papers, the algebraic properties of the space of games and the properties of the nullspace of the Shapley value operator (see Shapley [42]) and its orthogonal complement are exploited to decompose games. This approach does not necessarily simplify the analysis of games, but it leads to an alternative expression for the Shapley value (Kleinberg and Weiss [27]). Our work is on decomposition of noncooperative games, and different from the above references since we explicitly exploit the properties of noncooperative games in our framework.

In the context of noncooperative game theory, a decomposition for games in normal form appeared in Sandholm [40]. In this paper, the author defines  $2^M$  component games that are characterized by their sets of active and passive players, and provides a decomposition of normal form games to these components. This decomposition is then used to identify potential games: the original game is a potential game if and only if in each component game the active players have identical payoff functions. We note that our decomposition approach is different from this work in the properties of the component games. In particular, using the global preference structure in games, our approach yields decomposition of games to three components with distinct equilibrium properties, and these properties can be exploited to gain insights about the static and dynamic features of the original game.

A different decomposition of noncooperative games appeared in Başar and Ho [1], Kalai and Kalai [25]. This decomposition relies on identifying zero-sum and identical interest components for a given game, and is used for dealing with cooperation-related issues that emerge in games with strategic players. It is fundamentally different from the decomposition we propose, since one of its components is always zero-sum, whereas this is not the case in our decomposition.

Related ideas of representing finite strategic form games as graphs previously appeared in the literature to study different solution concepts in normal form games (Christodoulou et al. [7], Goemans et al. [16]). In these references, the authors focus on the restriction of the game graph to best-reply paths and analyze the outcomes of games using this subgraph.

In our work, the graph representation of games and the flows defined on this graph lead to a natural equivalence relation. Related notions of strategic equivalence are employed in the game theory literature to generalize the desirable static and dynamic properties of games to their equivalence classes (Germano [14], Hammond [18], Hillas and Kohlberg [19], Hofbauer and Hopkins [20], Kannan and Theobald [26], Mertens [32], Morris and Ui [35], Moulin and Vial [36], Rosenthal [39], Voorneveld [48]). Moulin and Vial [36] refer to games that have the same better-response correspondence as equivalent games and study the equilibrium properties of games that are equivalent to zero-sum games. In Hammond [18] and Hofbauer and Hopkins [20], the dynamic and static properties of certain classes of bimatrix games are generalized to their equivalence classes. Using the best-response correspondence instead of the better-response correspondence, the papers by Morris and Ui [35], Rosenthal [39], Voorneveld [48] define different equivalence classes of games. We note that the notion of strategic equivalence used in our paper implies some of the equivalence notions mentioned above. However, unlike these papers, our notion of strategic equivalence leads to a canonical decomposition of the space of games, which is then used to extend the desirable properties of potential games to “close” games that are not strategically equivalent.

Despite the fact that harmonic games were not defined in the literature before (and thus, the term “harmonic” does not appear explicitly as such), specific instances of harmonic games were studied in different contexts. Hofbauer and Schlag [22] study dynamics in “cyclic games” and obtain results about a class of harmonic games that generalize the matching pennies game. A parametrized version of Dawkins’ battle of the sexes game, which is a harmonic game under certain conditions, is studied in Smith and Hofbauer [43]. Other examples of harmonic games have also appeared in the buyer/seller game of Friedman [11] and the crime deterrence game of Cressman and Morrison [9].

**1.2. Structure of the paper.** The remainder of this paper is organized as follows. In §2, we present the relevant game-theoretic background and provide a representation of games in terms of graph flows. In §3, we state the Helmholtz decomposition theorem which provides the means of decomposing a flow into orthogonal components. In §4, we use this machinery to obtain a canonical decomposition of the space of games. We introduce natural classes of games in §5, namely potential and harmonic games, which are induced by this decomposition and describe the equilibrium properties thereof. In §6, we define an inner product for the space of games, under which the components of games turn out to be orthogonal. Using this inner product and our decomposition framework we propose a method for projecting a given game to the spaces of potential and harmonic games. We then apply the projection to study the equilibrium properties of “near-potential” games. We close in §7 with concluding remarks and directions for future work.

**2. Game-theoretic background.** In this section, we describe the required game-theoretic background. Notation and basic definitions are given in §2.1. In §2.2, we provide an alternative representation of games in terms of flows on graphs. This representation is used in the rest of the paper to analyze finite games.

**2.1. Preliminaries.** A (noncooperative) *strategic-form finite game* consists of:

- A finite set of players, denoted  $\mathcal{M} = \{1, \dots, M\}$ .
- Strategy spaces: A finite set of strategies (or actions)  $E^m$ , for every  $m \in \mathcal{M}$ . The joint strategy space is denoted by  $E = \prod_{m \in \mathcal{M}} E^m$ .
- Utility functions:  $u^m: E \rightarrow \mathbb{R}$ ,  $m \in \mathcal{M}$ .

A (strategic-form) game instance is accordingly given by the tuple  $\langle \mathcal{M}, \{E^m\}_{m \in \mathcal{M}}, \{u^m\}_{m \in \mathcal{M}} \rangle$ , which for notational convenience will often be abbreviated to  $\langle \mathcal{M}, \{E^m\}, \{u^m\} \rangle$ .

We use the notation  $\mathbf{p}^m \in E^m$  for a strategy of player  $m$ . A collection of players’ strategies is given by  $\mathbf{p} = \{\mathbf{p}^m\}_{m \in \mathcal{M}}$  and is referred to as a strategy profile. A collection of strategies for all players but the  $m$ th one is denoted by  $\mathbf{p}^{-m} \in E^{-m}$ . We use  $h_m = |E^m|$  for the cardinality of the strategy space of player  $m$ , and  $|E| = \prod_{m=1}^M h_m$  for the overall cardinality of the strategy space. As an alternative representation, we shall sometimes enumerate the actions of the players, so that  $E^m = \{1, \dots, h_m\}$ .

The basic solution concept in a noncooperative game is that of a *Nash equilibrium* (NE). A strategy profile  $\mathbf{p} \triangleq \{\mathbf{p}^1, \dots, \mathbf{p}^M\}$  is an  $\epsilon$ -equilibrium if

$$u^m(\mathbf{p}^m, \mathbf{p}^{-m}) \geq u^m(\mathbf{q}^m, \mathbf{p}^{-m}) - \epsilon \quad \text{for every } \mathbf{q}^m \in E^m \text{ and } m \in \mathcal{M}. \quad (3)$$

A (pure) Nash equilibrium<sup>1</sup> is an  $\epsilon$ -equilibrium with  $\epsilon = 0$ .

<sup>1</sup> In strategic-form finite games, Nash equilibrium always exists in mixed strategies. However, pure Nash equilibria, defined here need not always exist.



Lemma 2.1 shows that the  $\epsilon$ -equilibria of two games can be related in terms of the differences in utilities.

LEMMA 2.1. Consider two games  $\mathcal{G}$  and  $\hat{\mathcal{G}}$ , which differ only in their utility functions; i.e.,  $\mathcal{G} = \langle \mathcal{M}, \{E^m\}, \{u^m\} \rangle$  and  $\hat{\mathcal{G}} = \langle \mathcal{M}, \{E^m\}, \{\hat{u}^m\} \rangle$ . Assume that  $|u^m(\mathbf{p}) - \hat{u}^m(\mathbf{p})| \leq \epsilon_0$  for every  $m \in \mathcal{M}$  and  $\mathbf{p} \in E$ . Then, every  $\epsilon_1$ -equilibrium of  $\hat{\mathcal{G}}$  is an  $\epsilon$ -equilibrium of  $\mathcal{G}$  for some  $\epsilon \leq 2\epsilon_0 + \epsilon_1$  (and vice versa).

PROOF. Let  $\mathbf{p}$  be an  $\epsilon_1$ -equilibrium of  $\hat{\mathcal{G}}$  and let  $\mathbf{q} \in E$  be a strategy profile with  $\mathbf{q}^k \neq \mathbf{p}^k$  for some  $k \in \mathcal{M}$ , and  $\mathbf{q}^m = \mathbf{p}^m$  for every  $m \in \mathcal{M} \setminus \{k\}$ . Then,

$$u^k(\mathbf{q}) - u^k(\mathbf{p}) \leq u^k(\mathbf{q}) - u^k(\mathbf{p}) - (\hat{u}^k(\mathbf{q}) - \hat{u}^k(\mathbf{p})) + \epsilon_1 \leq 2\epsilon_0 + \epsilon_1,$$

where the first inequality follows since  $\mathbf{p}$  is an  $\epsilon_1$ -equilibrium of  $\hat{\mathcal{G}}$ ; hence  $\hat{u}^k(\mathbf{p}) - \hat{u}^k(\mathbf{q}) \geq -\epsilon_1$ , and the second inequality follows by the lemma's assumption.  $\square$

We turn now to describe a particular class of games that is central in this paper, the class of potential games (Monderer and Shapley [34]).

DEFINITION 2.1 (POTENTIAL GAME). A potential game is a noncooperative game for which there exists a function  $\phi: E \rightarrow \mathbb{R}$  satisfying

$$\phi(\mathbf{p}^m, \mathbf{p}^{-m}) - \phi(\mathbf{q}^m, \mathbf{p}^{-m}) = u^m(\mathbf{p}^m, \mathbf{p}^{-m}) - u^m(\mathbf{q}^m, \mathbf{p}^{-m}), \quad (4)$$

for every  $m \in \mathcal{M}$ ,  $\mathbf{p}^m, \mathbf{q}^m \in E^m$ ,  $\mathbf{p}^{-m} \in E^{-m}$ . The function  $\phi$  is referred to as a *potential* function of the game.

Potential games can be regarded as games in which the interests of the players are aligned with a global potential function  $\phi$ . Games that obey condition (4) are also known in the literature as *exact* potential games, to distinguish them from other classes of games that relate to a potential function (in a different manner). For simplicity of exposition, we will often write “potential games” when referring to exact potential games. Potential games have desirable equilibrium properties as summarized in §5.1.

**2.2. Games and flows on graphs.** In noncooperative games, the utility functions capture the preferences of agents at each strategy profile. Note that a Nash equilibrium is defined in terms of payoff differences, suggesting that actual payoffs in the game are not required for the identification of equilibria, as long as the payoff differences are well defined.

A pair of strategy profiles that differ only in the strategy of a single player will be henceforth referred to as *comparable strategy profiles*. We denote the set (of pairs) of comparable strategy profiles by  $A \subset E \times E$ ; i.e.,  $\mathbf{p}, \mathbf{q}$  are comparable if and only if  $(\mathbf{p}, \mathbf{q}) \in A$ . A pair of strategy profiles that differ only in the strategy of player  $m$  is called a pair of *m-comparable strategy profiles*. The set of pairs of *m-comparable* strategy profiles is denoted by  $A^m \subset E \times E$ . Clearly,  $\bigcup_m A^m = A$ , where  $A^m \cap A^k = \emptyset$  for any two different players  $m$  and  $k$ .

For any given *m-comparable* strategy profiles  $\mathbf{p}$  and  $\mathbf{q}$ , the payoff difference  $[u^m(\mathbf{p}) - u^m(\mathbf{q})]$  will be referred to as their *pairwise comparison*. For any game, we define the *pairwise comparison* function  $X: E \times E \rightarrow \mathbb{R}$  as follows:

$$X(\mathbf{p}, \mathbf{q}) = \begin{cases} u^m(\mathbf{q}) - u^m(\mathbf{p}) & \text{if } (\mathbf{p}, \mathbf{q}) \text{ are } m\text{-comparable for some } m \in \mathcal{M}, \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$

In view of Definition 2.1, a game is an exact potential game if and only if there exists a function  $\phi: E \rightarrow \mathbb{R}$  such that  $\phi(\mathbf{q}) - \phi(\mathbf{p}) = X(\mathbf{p}, \mathbf{q})$  for any comparable strategy profiles  $\mathbf{p}$  and  $\mathbf{q}$ . Note that the pairwise comparisons are uniquely defined for any given game. However, the converse is not true; for instance, any two games for which the utilities differ by a constant have the same pairwise comparisons.

The usual solution concepts in games (e.g., Nash, mixed Nash, correlated equilibria) are defined in terms of pairwise comparisons only. Consequently, games with identical pairwise comparisons share the same equilibrium sets. Thus, in this paper, we refer to games with identical pairwise comparisons as *strategically equivalent* games.<sup>2</sup>

By employing the notion of pairwise comparisons, we can concisely represent any strategic-form game in terms of a *flow* in a graph. We recall this notion next. Let  $G = (N, L)$  be an undirected graph, with a set of nodes  $N$  and a set of links  $L$ . An *edge flow* (or just *flow*) on this graph is a function  $Y: N \times N \rightarrow \mathbb{R}$  such that  $Y(\mathbf{p}, \mathbf{q}) = -Y(\mathbf{q}, \mathbf{p})$  and  $Y(\mathbf{p}, \mathbf{q}) = 0$  for  $(\mathbf{p}, \mathbf{q}) \notin L$  (Jiang et al. [23], Bertsimas and Tsitsiklis [2]). Note that the flow conservation equations are not enforced under this general definition.

<sup>2</sup> Other strategic equivalence definitions can be found in the literature (see the introduction for a review of the relevant literature). We focus on this particular definition to ensure that the flow representations of equivalent games are identical, a feature that will be used when studying decompositions of games. It is also true that under our equivalence notion, equivalent games have identical equilibrium sets (mixed, pure, and correlated).

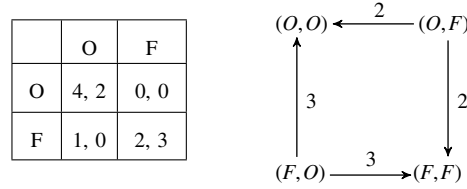


FIGURE 2. Flows on the game graph corresponding to “battle of the sexes.”

Given a game  $\mathcal{G}$ , we define a graph where each node corresponds to a pure strategy profile, and each edge connects two comparable strategy profiles. This undirected graph is referred to as the *game graph* and is denoted by  $G(\mathcal{G}) \triangleq (E, A)$ , where  $E$  and  $A$  are the strategy profiles and pairs of comparable strategy profiles defined above, respectively. Notice that, by definition, the graph  $G(\mathcal{G})$  has the structure of a direct product of  $M$  cliques (one per player), with clique  $m$  having  $h_m$  vertices. The pairwise comparison function  $X: E \times E \rightarrow \mathbb{R}$  defines a flow on  $G(\mathcal{G})$ , as it satisfies  $X(\mathbf{p}, \mathbf{q}) = -X(\mathbf{q}, \mathbf{p})$  and  $X(\mathbf{p}, \mathbf{q}) = 0$  for  $(\mathbf{p}, \mathbf{q}) \notin A$ . This flow may thus serve as an equivalent representation of any game (up to a “nonstrategic” component). It follows directly from the statements above that two games are strategically equivalent if and only if they have the same flow representation and game graph. An example of the game graph representation is given in Example 1.1. Another example can be found below.

EXAMPLE 2.1. The payoff matrix of the “battle of the sexes” game is given in the following table. The game graph has four vertices, corresponding to the direct product of two 2-cliques, and is presented in Figure 2.

The representation of a game as a flow in a graph is natural and useful for the understanding of its strategic interactions, as it abstracts away the absolute utility values and allows for more direct equilibrium-related interpretation. In more mathematical terms, it considers the quotient of the utilities modulo the subspace of games that are “equivalent” to the trivial game (the game where all players receive zero payoff at all strategy profiles), and allows for the identification of “equivalent” games as the same object, a point explored in more detail in later sections. The game graph also contains much structural information. For instance, two games where the directions of arrows are identical (regardless of the flow values) share the same pure Nash equilibria. Our goal in this paper is to use tools from the theory of graph flows to decompose a game into components, each of which admits tractable equilibrium characterization. The next section provides an overview of the tools that are required for this objective.

**3. Flows and Helmholtz decomposition.** The objective of this section is to provide a brief overview of the notation and tools required for the analysis of flows on graphs. The basic high-level idea is that under certain conditions (e.g., for graphs arising from games), it is possible to consider graphs as natural topological spaces with nontrivial homological properties. In what follows, we make this idea precise. For simplicity and accessibility to a wider audience, we describe the methods in relatively elementary linear algebraic language, limiting the usage of algebraic topology notions whenever possible. The main technical tool we use is the Helmholtz decomposition theorem, a classical result from algebraic topology with many applications in applied mathematics, including among others, electromagnetism, computational geometry and data visualization; see e.g., Polthier and Preuß [38], Tong et al. [44]. In particular, we mention the very interesting recent work by Jiang et al. [23], where the Helmholtz/Hodge decomposition is applied to the problem of statistical ranking for sets of incomplete data.

Consider an undirected graph  $G = (E, A)$ , where  $E$  is the set of the nodes, and  $A$  is the set of edges of the graph.<sup>3</sup> Since the graph is undirected,  $(\mathbf{p}, \mathbf{q}) \in A$  if and only if  $(\mathbf{q}, \mathbf{p}) \in A$ . We denote the set of 3-cliques of the graph by  $T = \{(\mathbf{p}, \mathbf{q}, \mathbf{r}) \mid (\mathbf{p}, \mathbf{q}), (\mathbf{q}, \mathbf{r}), (\mathbf{p}, \mathbf{r}) \in A\}$ .

We denote by  $C_0 = \{f \mid f: E \rightarrow \mathbb{R}\}$  the set of real-valued functions on the set of nodes. Recall that the *edge flows*  $X: E \times E \rightarrow \mathbb{R}$  are functions which satisfy

$$X(\mathbf{p}, \mathbf{q}) = \begin{cases} -X(\mathbf{q}, \mathbf{p}) & \text{if } (\mathbf{p}, \mathbf{q}) \in A, \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

Similarly, the *triangular flows*  $\Psi: E \times E \times E \rightarrow \mathbb{R}$  are functions for which

$$\Psi(\mathbf{p}, \mathbf{q}, \mathbf{r}) = \Psi(\mathbf{q}, \mathbf{r}, \mathbf{p}) = \Psi(\mathbf{r}, \mathbf{p}, \mathbf{q}) = -\Psi(\mathbf{q}, \mathbf{p}, \mathbf{r}) = -\Psi(\mathbf{p}, \mathbf{r}, \mathbf{q}) = -\Psi(\mathbf{r}, \mathbf{q}, \mathbf{p}), \quad (7)$$

<sup>3</sup> The results discussed in this section apply to arbitrary graphs. We use the notation introduced in §2 since in the rest of the paper we focus on the game graph introduced there.

and  $\Psi(\mathbf{p}, \mathbf{q}, \mathbf{r}) = 0$  if  $(\mathbf{p}, \mathbf{q}, \mathbf{r}) \notin T$ . Given a graph  $G$ , we denote the set of all possible edge flows by  $C_1$  and the set of triangular flows by  $C_2$ . Notice that both  $C_1$  and  $C_2$  are alternating functions of their arguments. It follows from (6) that  $X(\mathbf{p}, \mathbf{p}) = 0$  for all  $X \in C_1$ .

The sets  $C_0$ ,  $C_1$ , and  $C_2$  have a natural structure of vector spaces, with the obvious operations of addition and scalar multiplication. In this paper, we use the following inner products:

$$\begin{aligned}\langle \phi_1, \phi_2 \rangle_0 &= \sum_{\mathbf{p} \in E} \phi_1(\mathbf{p}) \phi_2(\mathbf{p}), \\ \langle X, Y \rangle_1 &= \frac{1}{2} \sum_{(\mathbf{p}, \mathbf{q}) \in A} X(\mathbf{p}, \mathbf{q}) Y(\mathbf{p}, \mathbf{q}), \\ \langle \Psi_1, \Psi_2 \rangle_2 &= \sum_{(\mathbf{p}, \mathbf{q}, \mathbf{r}) \in T} \Psi_1(\mathbf{p}, \mathbf{q}, \mathbf{r}) \Psi_2(\mathbf{p}, \mathbf{q}, \mathbf{r}).\end{aligned}\quad (8)$$

We shall frequently drop the subscript in the inner product notation, as the respective space will often be clear from the context.

We next define linear operators that relate the above defined objects. To that end, let  $W: E \times E \rightarrow \mathbb{R}$  be an indicator function for the edges of the graph, namely

$$W(\mathbf{p}, \mathbf{q}) = \begin{cases} 1 & \text{if } (\mathbf{p}, \mathbf{q}) \in A, \\ 0 & \text{otherwise.} \end{cases} \quad (9)$$

Notice that  $W(\mathbf{p}, \mathbf{q})$  can be simply interpreted as the adjacency matrix of the graph  $G$ .

The first operator of interest is the *combinatorial gradient operator*  $\delta_0: C_0 \rightarrow C_1$ , given by

$$(\delta_0 \phi)(\mathbf{p}, \mathbf{q}) = W(\mathbf{p}, \mathbf{q})(\phi(\mathbf{q}) - \phi(\mathbf{p})), \quad \mathbf{p}, \mathbf{q} \in E, \quad (10)$$

for  $\phi \in C_0$ . An operator that is used in the characterization of “circulations” in edge flows is the *curl operator*  $\delta_1: C_1 \rightarrow C_2$ , which is defined for all  $X \in C_1$  and  $\mathbf{p}, \mathbf{q}, \mathbf{r} \in E$  as

$$(\delta_1 X)(\mathbf{p}, \mathbf{q}, \mathbf{r}) = \begin{cases} X(\mathbf{p}, \mathbf{q}) + X(\mathbf{q}, \mathbf{r}) + X(\mathbf{r}, \mathbf{p}) & \text{if } (\mathbf{p}, \mathbf{q}, \mathbf{r}) \in T, \\ 0 & \text{otherwise.} \end{cases} \quad (11)$$

We denote the adjoints of the operators  $\delta_0$  and  $\delta_1$  by  $\delta_0^*$  and  $\delta_1^*$ , respectively. Recall that given inner products  $\langle \cdot, \cdot \rangle_k$  on  $C_k$ , the adjoint of  $\delta_k$ , namely  $\delta_k^*: C_{k+1} \rightarrow C_k$ , is the unique linear operator satisfying

$$\langle \delta_k f_k, g_{k+1} \rangle_{k+1} = \langle f_k, \delta_k^* g_{k+1} \rangle_k, \quad (12)$$

for all  $f_k \in C_k$ ,  $g_{k+1} \in C_{k+1}$ .

Using the definitions in (12), (10), and (8), it can be readily seen that the adjoint  $\delta_0^*: C_1 \rightarrow C_0$  of the combinatorial gradient  $\delta_0$  satisfies

$$(\delta_0^* X)(\mathbf{p}) = - \sum_{\mathbf{q} | (\mathbf{p}, \mathbf{q}) \in A} X(\mathbf{p}, \mathbf{q}) = - \sum_{\mathbf{q} \in E} W(\mathbf{p}, \mathbf{q}) X(\mathbf{p}, \mathbf{q}). \quad (13)$$

Note that  $-(\delta_0^* X)(\mathbf{p})$  represents the total flow “leaving”  $\mathbf{p}$ . We shall sometimes refer to the operator  $-\delta_0^*$  as the *divergence operator*, due to its similarity to the standard divergence operator in calculus.

The domains and codomains of the operators  $\delta_0$ ,  $\delta_1$ ,  $\delta_0^*$ ,  $\delta_1^*$  are summarized below:

$$\begin{aligned}C_0 &\xrightarrow{\delta_0} C_1 \xrightarrow{\delta_1} C_2, \\ C_0 &\xleftarrow{\delta_0^*} C_1 \xleftarrow{\delta_1^*} C_2.\end{aligned} \quad (14)$$

We next define the Laplacian operator,  $\Delta_0: C_0 \rightarrow C_0$ , given by

$$\Delta_0 \triangleq \delta_0^* \circ \delta_0, \quad (15)$$

where  $\circ$  represents operator composition. To simplify the notation, we henceforth omit  $\circ$  and write  $\Delta_0 = \delta_0^* \delta_0$ . Note that functions in  $C_0$  can be represented by vectors of length  $|E|$  by indexing all nodes of the graph and constructing a vector whose  $i$ th entry is the function evaluated at the  $i$ th node. This allows us to easily represent these operators in terms of matrices. In particular, the Laplacian can be expressed as a square matrix of size  $|E| \times |E|$ ; using the definitions for  $\delta_0$  and  $\delta_0^*$ , it follows that

$$[\Delta_0]_{\mathbf{p}, \mathbf{q}} = \begin{cases} \sum_{\mathbf{r} \in E} W(\mathbf{p}, \mathbf{r}) & \text{if } \mathbf{p} = \mathbf{q}, \\ -1 & \text{if } \mathbf{p} \neq \mathbf{q} \text{ and } (\mathbf{p}, \mathbf{q}) \in A, \\ 0 & \text{otherwise,} \end{cases} \quad (16)$$



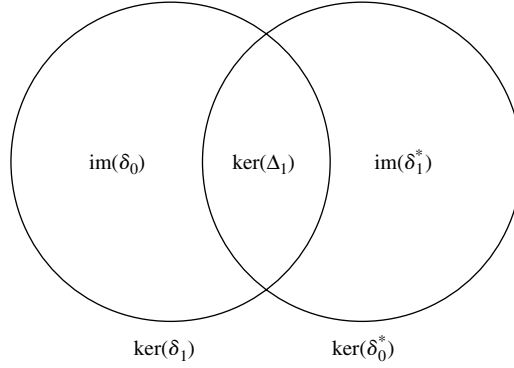


FIGURE 3. Helmholtz decomposition of  $C_1$ .

where, with some abuse of the notation,  $[\Delta_0]_{\mathbf{p},\mathbf{q}}$  denotes the entry of the matrix  $\Delta_0$ , with rows and columns indexed by the nodes  $\mathbf{p}$  and  $\mathbf{q}$ . The above matrix naturally coincides with the definition of a Laplacian of an undirected graph (Chung [8]).

Since the entry of  $\Delta_0\phi$  corresponding to  $\mathbf{p}$  equals  $\sum_{\mathbf{q}} W(\mathbf{p},\mathbf{q})(\phi(\mathbf{p}) - \phi(\mathbf{q}))$ , the Laplacian operator gives a measure of the aggregate “value” of a node over all its neighbors. A related operator is

$$\Delta_1 \triangleq \delta_1^* \circ \delta_1 + \delta_0 \circ \delta_0^*, \quad (17)$$

known in the literature as the vector Laplacian [23].

We next provide additional flow-related terminology which will be used in association with the above defined operators, and highlight some of their basic properties. In analogy to the well-known identity in vector calculus,  $\text{curl} \circ \text{grad} = 0$ , we have that  $\delta_0$  is a closed form; i.e.,  $\delta_1 \circ \delta_0 = 0$ . An edge flow  $X \in C_1$  is said to be *globally consistent* if  $X$  corresponds to the combinatorial gradient of some  $f \in C_0$ , i.e.,  $X = \delta_0 f$ ; the function  $f$  is referred to as the *potential function* corresponding to  $X$ . Equivalently, the set of globally consistent edge flows can be represented as the image of the gradient operator, namely  $\text{im}(\delta_0)$ . By the closedness of  $\delta_0$ , observe that  $\delta_1 X = 0$  for every globally consistent edge flow  $X$ . We define *locally consistent* edge flows as those satisfying  $(\delta_1 X)(\mathbf{p}, \mathbf{q}, \mathbf{r}) = X(\mathbf{p}, \mathbf{q}) + X(\mathbf{q}, \mathbf{r}) + X(\mathbf{r}, \mathbf{p}) = 0$  for all  $(\mathbf{p}, \mathbf{q}, \mathbf{r}) \in T$ . Note that the kernel of the curl operator  $\ker(\delta_1)$  is the set of locally consistent edge flows. The latter subset is generally not equivalent to  $\text{im}(\delta_0)$ , as there may exist edge flows that are globally inconsistent but locally consistent (in fact, this will happen whenever the graph has a nontrivial topology). We refer to such flows as *harmonic flows*. Note that the operators  $\delta_0, \delta_1$  are linear operators; thus their image spaces are orthogonal complements of the kernels of their adjoints, i.e.,  $\text{im}(\delta_0) \perp \ker(\delta_0^*)$  and  $\text{im}(\delta_1) \perp \ker(\delta_1^*)$  [similarly,  $\text{im}(\delta_0^*) \perp \ker(\delta_0)$  and  $\text{im}(\delta_1^*) \perp \ker(\delta_1)$ ] as can be easily verified using (12)].

We state below a basic flow-decomposition theorem, known as the Helmholtz decomposition,<sup>4</sup> which will be used in our context of noncooperative games. The theorem (see Jiang et al. [23]) implies that any graph flow can be decomposed into three orthogonal flows.

**THEOREM 3.1 (HELMHOLTZ DECOMPOSITION).** *The vector space of edge flows  $C_1$  admits an orthogonal decomposition*

$$C_1 = \text{im}(\delta_0) \oplus \ker(\Delta_1) \oplus \text{im}(\delta_1^*), \quad (18)$$

where  $\ker(\Delta_1) = \ker(\delta_1) \cap \ker(\delta_0^*)$ .

Below, we summarize the interpretation of each of the components in the Helmholtz decomposition (see also Figure 3):

- $\text{im}(\delta_0)$ —globally consistent flows.
- $\ker(\Delta_1) = \ker(\delta_1) \cap \ker(\delta_0^*)$ —harmonic flows, which are globally inconsistent but locally consistent. Observe that  $\ker(\delta_1)$  consists of locally consistent flows (that may or may not be globally consistent), while  $\ker(\delta_0^*)$  consists of globally inconsistent flows (that may or may not be locally consistent).
- $\text{im}(\delta_1^*)$  (or equivalently, the orthogonal complement of  $\ker(\delta_1)$ )—locally inconsistent flows.

<sup>4</sup> The Helmholtz decomposition can be generalized to higher dimensions through the Hodge decomposition theorem (see Jiang et al. [23]); however this generalization is not required for our purposes.

We conclude this section with a brief remark on the decomposition and the flow conservation. For  $X \in C_1$ , if  $\delta_0^* X = 0$ , i.e., if for every node, the total flow leaving the node is zero, then we say that  $X$  satisfies the flow conservation condition. Theorem 3.1 implies that  $X$  satisfies this condition only when  $X \in \ker(\delta_0^*) = \text{im}(\delta_0)^{\perp} = \ker(\Delta_1) \oplus \text{im}(\delta_1^*)$ . Thus, the flow conservation condition is satisfied for harmonic flows and locally inconsistent flows but not for globally consistent flows.

**4. Canonical decomposition of games.** In this section we obtain a canonical decomposition of an arbitrary game into basic components, by combining the game graph representation introduced in §2.2 with the Helmholtz decomposition discussed above.

Section 4.1 introduces the relevant operators that are required for formulating the results. In §4.2 we provide the basic decomposition theorem, which states that the space of games can be decomposed as a direct sum of three subspaces, referred to as the *potential*, *harmonic*, and *nonstrategic* subspaces. In §4.3, we focus on bimatrix games, and provide explicit expressions for the decomposition.

**4.1. Preliminaries.** We consider a game  $\mathcal{G}$  with set of players  $\mathcal{M}$ , strategy profiles  $E \triangleq E^1 \times \dots \times E^M$ , and game graph  $G(\mathcal{G}) = (E, A)$ . Using the notation of the previous section, the utility functions of each player can be viewed as elements of  $C_0$ ; i.e.,  $u^m \in C_0$  for all  $m \in \mathcal{M}$ . For given  $\mathcal{M}$  and  $E$ , every game is uniquely defined by its set of utility functions. Hence, the space of games with players  $\mathcal{M}$  and strategy profiles  $E$  can be identified as  $\mathcal{G}_{\mathcal{M}, E} \cong C_0^M$ . In the rest of the paper we use the notations  $\{u^m\}_{m \in \mathcal{M}}$  and  $\mathcal{G} = \langle \mathcal{M}, \{E^m\}, \{u^m\} \rangle$  interchangeably when referring to games.

The pairwise comparison function  $X(\mathbf{p}, \mathbf{q})$  of a game, defined in (5), corresponds to a flow on the game graph, and hence it belongs to  $C_1$ . In general, the flows representing games have some special structure. For example, the pairwise comparison between any two comparable strategy profiles is associated with the payoff of exactly a single player. It is therefore required to introduce *player-specific* operators and highlight some important identities between them, as we elaborate below.

Let  $W^m: E \times E \rightarrow \mathbb{R}$  be the indicator function for  $m$ -comparable strategy profiles, namely

$$W^m(\mathbf{p}, \mathbf{q}) = \begin{cases} 1 & \text{if } \mathbf{p}, \mathbf{q} \text{ are } m\text{-comparable,} \\ 0 & \text{otherwise.} \end{cases}$$

Recalling that any pair of strategy profiles cannot be comparable by more than a single player, we have

$$W^m(\mathbf{p}, \mathbf{q}) W^k(\mathbf{p}, \mathbf{q}) = 0, \quad \text{for all } k \neq m \text{ and } \mathbf{p}, \mathbf{q} \in E, \quad (19)$$

and

$$W = \sum_{m \in \mathcal{M}} W^m, \quad (20)$$

where  $W$  is the indicator function of comparable strategy profiles (edges of the game graph) defined in (9). Note that this can be interpreted as a decomposition of the adjacency matrix of  $G(\mathcal{G})$ , where the different components correspond to the edges associated with different players.

Given  $\phi \in C_0$ , we define  $D_m: C_0 \rightarrow C_1$  such that

$$(D_m \phi)(\mathbf{p}, \mathbf{q}) = W^m(\mathbf{p}, \mathbf{q})(\phi(\mathbf{q}) - \phi(\mathbf{p})). \quad (21)$$

This operator quantifies the change in  $\phi$  between strategy profiles that are  $m$ -comparable. Using this operator, we can represent the pairwise differences  $X$  of a game with payoffs  $\{u^m\}_{m \in \mathcal{M}}$  as follows:

$$X = \sum_{m \in \mathcal{M}} D_m u^m. \quad (22)$$

We define a relevant operator  $D: C_0^M \rightarrow C_1$ , such that  $D = [D_1, \dots, D_M]$ . As can be seen from (22), for a game with collection of utilities  $u = [u^1; u^2; \dots; u^M] \in C_0^M$ , the pairwise differences can alternatively be represented by  $Du$ .

Let  $\Lambda_m: C_1 \rightarrow C_1$  be an operator so that

$$(\Lambda_m X)(\mathbf{p}, \mathbf{q}) = W^m(\mathbf{p}, \mathbf{q}) X(\mathbf{p}, \mathbf{q})$$

for every  $X \in C_1$ ,  $\mathbf{p}, \mathbf{q} \in E$ . From (20), it can be seen that for any  $X \in C_1$ ,  $\sum_{m \in \mathcal{M}} \Lambda_m X = X$ . The definition of  $\Lambda_m$  and (19) imply that  $\Lambda_m \Lambda_k = 0$  for  $k \neq m$ . Additionally, the definition of the inner product in  $C_1$  implies that for  $X, Y \in C_1$ , it follows that  $\langle \Lambda_m X, Y \rangle = \langle X, \Lambda_m Y \rangle$ ; i.e.,  $\Lambda_m$  is self-adjoint.

This operator provides a convenient description for the operator  $D_m$ . From the definitions of  $D_m$  and  $\Lambda_m$ , it immediately follows that  $D_m = \Lambda_m \delta_0$ , and since  $\sum_{m \in \mathcal{M}} \Lambda_m X = X$  for all  $X \in C_1$ ,

$$\delta_0 = \sum_{m \in \mathcal{M}} \Lambda_m \delta_0 = \sum_{m \in \mathcal{M}} D_m.$$

Since  $\Lambda_m$  is self-adjoint, the adjoint of  $D_m$ , which is denoted by  $D_m^*: C_1 \rightarrow C_0$ , is given by:

$$D_m^* = \delta_0^* \Lambda_m.$$

Using (13) and the above definitions, it follows that

$$(D_m^* X)(\mathbf{p}) = - \sum_{\mathbf{q} \in E} W^m(\mathbf{p}, \mathbf{q}) X(\mathbf{p}, \mathbf{q}), \quad \text{for all } X \in C_1, \quad (23)$$

and

$$\delta_0^* = \sum_{m \in \mathcal{M}} D_m^*. \quad (24)$$

Observe that  $D_k^* D_m = \delta_0^* \Lambda_k \Lambda_m \delta_0 = 0$  for  $k \neq m$ . This immediately implies that the image spaces of  $\{D_m\}_{m \in \mathcal{M}}$  are orthogonal; i.e.,  $D_k^* D_m = 0$ . Let  $D_m^\dagger$  denote the (Moore-Penrose) pseudoinverse of  $D_m$ , with respect to the inner products introduced in §3. By the properties of the pseudoinverse, we have  $\ker D_m^\dagger = (\text{im } D_m)^\perp$ . Thus, orthogonality of the image spaces of  $D_k$  operators implies that  $D_k^\dagger D_m = 0$  for  $k \neq m$ .

The orthogonality leads to the following expression for the Laplacian operator:

$$\Delta_0 = \delta_0^* \delta_0 = \sum_{k \in \mathcal{M}} D_k^* \sum_{m \in \mathcal{M}} D_m = \sum_{m \in \mathcal{M}} D_m^* D_m. \quad (25)$$

In view of (21) and (23),  $D_m$  and  $-D_m^*$  are the gradient and divergence operators on the graph of  $m$ -comparable strategy profiles  $(E, A^m)$ . Therefore, the operator  $\Delta_{0,m} \triangleq D_m^* D_m$  is the Laplacian of the graph induced by  $m$ -comparable strategy profiles, and is referred to as the *Laplacian operator of the  $m$ -comparable strategy profiles*. It follows from (25) that

$$\Delta_0 = \sum_{m \in \mathcal{M}} \Delta_{0,m}.$$

The relation between the Laplacian operators  $\Delta_0$  and  $\Delta_{0,m}$  is illustrated in Figure 4.

Similarly,  $\delta_1 \Lambda_m$  is the curl operator associated with the subgraph  $(E, A^m)$ . From the closedness of the curl  $(\delta_1 \Lambda_m)$  and gradient  $(\Lambda_m \delta_0)$  operators defined on this subgraph, we obtain  $\delta_1 \Lambda_m^2 \delta_0 = 0$ . Observing that  $\Lambda_m^2 \delta_0 = \Lambda_m \delta_0 = D_m$ , it follows that

$$\delta_1 D_m = 0. \quad (26)$$

This result also implies that  $\delta_1 D = 0$ ; i.e., the pairwise comparisons of games belong to  $\ker \delta_1$ . Thus, it follows from Theorem 3.1 that the pairwise comparisons do not have a locally inconsistent component. Intuitively, there is no local inconsistency, because only three cliques in the game graph are due to unilateral deviations of a single player, and hence cannot lead to local inconsistency.

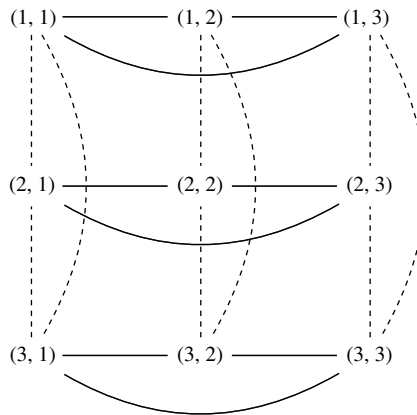


FIGURE 4. A game with two players, each of which has three strategies.

Notes. A node  $(i, j)$  represents a strategy profile in which player 1 and player 2 use strategies  $i$  and  $j$ , respectively. The Laplacian  $\Delta_{0,1}$  ( $\Delta_{0,2}$ ) is defined on the graph whose edges are represented by dashed (solid) lines. The Laplacian  $\Delta_0$  is defined on the graph that includes all edges.

TABLE 1. Notation summary.

$\mathcal{G}$	A game instance $\langle \mathcal{M}, \{E^m\}_{m \in \mathcal{M}}, \{u^m\}_{m \in \mathcal{M}} \rangle$ .
$\mathcal{M}$	Set of players, $\{1, \dots, M\}$ .
$E^m$	Set of actions for player $m$ , $E^m = \{1, \dots, h_m\}$ .
$E$	Joint action space $\prod_{m \in \mathcal{M}} E^m$ .
$u^m$	Utility function of player $m$ . We have $u^m \in C_0$ .
$W^m$	Indicator function for $m$ -comparable strategy profiles, $W^m: E \times E \rightarrow \{0, 1\}$ .
$W$	A function indicating whether strategy profiles are comparable, $W: E \times E \rightarrow \{0, 1\}$ .
$C_0$	Space of utilities, $C_0 = \{u^m \mid u^m: E \rightarrow \mathbb{R}\}$ . Note that $C_0 \cong \mathbb{R}^{ E }$ .
$C_1$	Space of pairwise comparison functions from $E \times E$ to $\mathbb{R}$ .
$\delta_0$	Gradient operator, $\delta_0: C_0 \rightarrow C_1$ , satisfying $(\delta_0 \phi)(\mathbf{p}, \mathbf{q}) = W(\mathbf{p}, \mathbf{q}) (\phi(\mathbf{q}) - \phi(\mathbf{p}))$ .
$D_m$	$D_m: C_0 \rightarrow C_1$ , such that $(D_m \phi)(\mathbf{p}, \mathbf{q}) = W^m(\mathbf{p}, \mathbf{q}) (\phi(\mathbf{q}) - \phi(\mathbf{p}))$ .
$D$	$D: C_0^M \rightarrow C_1$ , such that $D(u^1; \dots; u^M) = \sum_{m \in \mathcal{M}} D_m u^m$ .
$\delta_0^*, D_m^*$	$\delta_0^*, D_m^*: C_1 \rightarrow C_0$ are the adjoints of the operators $\delta_0$ and $D_m$ , respectively.
$\Delta_0$	Laplacian for the game graph, $\Delta_0: C_0 \rightarrow C_0$ ; satisfies $\Delta_0 = \delta_0^* \delta_0 = \sum_{m \in \mathcal{M}} \Delta_{0,m}$ .
$\Delta_{0,m}$	Laplacian for the graph of $m$ -comparable strategy profiles, $\Delta_{0,m}: C_0 \rightarrow C_0$ ; satisfies $\Delta_{0,m} = D_m^* D_m = D_m^* \delta_0$ .
$\Pi_m$	Projection operator onto the orthogonal complement of kernel of $D_m$ , $\Pi_m: C_0 \rightarrow C_0$ ; satisfies $\Pi_m = D_m^\dagger D_m$ .

Lastly, we introduce projection operators that will be useful in the subsequent analysis. Consider the operator,

$$\Pi_m = D_m^\dagger D_m.$$

Since  $D_m$  is a linear operator,  $\Pi_m$  is a projection operator<sup>5</sup> to the orthogonal complement of the kernel of  $D_m$ . Using these operators, we define  $\Pi: C_0^M \rightarrow C_0^M$  such that  $\Pi = \text{diag}(\Pi_1, \dots, \Pi_M)$ ; i.e., for  $u = \{u_m\}_{m \in \mathcal{M}} \in C_0^M$ , we have  $\Pi u = [\Pi_1 u^1; \dots; \Pi_M u^M] \in C_0^M$ . We extend the inner product in  $C_0$  to  $C_0^M$  (by defining the inner product as the sum of the inner products in all  $C_0$  components), and denote by  $D^\dagger$  the pseudoinverse of  $D$  according to this inner product. In Lemma 4.4, we will show that  $\Pi$  is equivalent to the projection operator to the orthogonal complement of the kernel of  $D$ ; i.e.,  $\Pi = D^\dagger D$ .

For easy reference, Table 1 provides a summary of notation. We next state some basic facts about the operators we introduced, which will be used in the subsequent analysis. The proofs of these results can be found in Appendix A.

LEMMA 4.1. *The Laplacian of the graph induced by  $m$ -comparable strategy profiles and the projection operator  $\Pi_m$  are related by  $\Delta_{0,m} = h_m \Pi_m$ , where  $h_m = |E^m|$  denotes the number of strategies of player  $m$ .*

LEMMA 4.2. *The kernels of operators  $D_m$ ,  $\Pi_m$ , and  $\Delta_{0,m}$  coincide, namely,  $\ker(D_m) = \ker(\Pi_m) = \ker(\Delta_{0,m})$ . Furthermore, a basis for these kernels is given by a collection  $\{v_{\mathbf{q}^{-m}}\}_{\mathbf{q}^{-m} \in E^{-m}} \in C_0$  such that*

$$v_{\mathbf{q}^{-m}}(\mathbf{p}) = \begin{cases} 1 & \text{if } \mathbf{p}^{-m} = \mathbf{q}^{-m}, \\ 0 & \text{otherwise.} \end{cases} \quad (27)$$

LEMMA 4.3. *The Laplacian  $\Delta_0$  of the game graph (the graph of comparable strategy profiles) always has eigenvalue 0 with multiplicity 1, corresponding to the constant eigenfunction (i.e.,  $f \in C_0$  such that  $f(\mathbf{p}) = 1$  for all  $\mathbf{p} \in E$ ).*

LEMMA 4.4. *The pseudoinverses of operators  $D_m$  and  $D$  satisfy the following identities:*

- (i)  $D_m^\dagger = (1/h_m) D_m^*$ ,
- (ii)  $(\sum_{i \in \mathcal{M}} D_i)^\dagger D_j = (\sum_{i \in \mathcal{M}} D_i^* D_i)^\dagger D_j^* D_j$ ,
- (iii)  $D^\dagger = [D_1^\dagger; \dots; D_M^\dagger]$ ,
- (iv)  $\Pi = D^\dagger D$ ,
- (v)  $DD^\dagger \delta_0 = \delta_0$ .

**4.2. Decomposition of games.** In this subsection we prove that the space of games  $\mathcal{G}_{\mathcal{M}, E}$  is a direct sum of three subspaces—potential, harmonic, and nonstrategic, each with distinguishing properties.

We start our discussion by formalizing the notion of nonstrategic information. Consider two games  $\mathcal{G}, \hat{\mathcal{G}} \in \mathcal{G}_{\mathcal{M}, E}$  with utilities  $\{u^m\}_{m \in \mathcal{M}}$  and  $\{\hat{u}^m\}_{m \in \mathcal{M}}$  respectively. Assume that the utility functions  $\{u^m\}_{m \in \mathcal{M}}$  satisfy

<sup>5</sup> For any linear operator  $L$ ,  $L^\dagger L$  is a projection operator on the orthogonal complement of the kernel of  $L$  (see Golub and Van Loan [17]).

$u^m(\mathbf{p}^m, \mathbf{p}^{-m}) = \hat{u}^m(\mathbf{p}^m, \mathbf{p}^{-m}) + \alpha(\mathbf{p}^{-m})$  where  $\alpha$  is an arbitrary function. It can be seen that these two games have exactly the same pairwise comparison functions; hence, they are strategically equivalent. To fix a representative for strategically equivalent games, we introduce below a notion of normalization for games.

DEFINITION 4.1 (NORMALIZED GAMES). We say that a game with utility functions  $\{u^m\}_{m \in \mathcal{M}}$  is *normalized* or does not contain nonstrategic information if

$$\sum_{\mathbf{p}^m \in E^m} u^m(\mathbf{p}^m, \mathbf{p}^{-m}) = 0 \quad (28)$$

for all  $\mathbf{p}^{-m} \in E^{-m}$  and all  $m \in \mathcal{M}$ .

It will be shown in the sequel that for each game there is a unique strategically equivalent normalized game; hence, normalized games can be used to identify representatives for strategically equivalent games. Normalization can be made with an arbitrary constant. However, to simplify the subsequent analysis we normalize the sum of the payoffs to zero. Intuitively, in normalized games, given the strategies of a player's opponents, the expected payoff of a player for a uniformly mixed strategy is equal to zero. The following lemma characterizes the set of normalized games in terms of the operators introduced in the previous section.

LEMMA 4.5. Given a game  $\mathcal{G}$  with utilities  $u = \{u^m\}_{m \in \mathcal{M}}$ , the following are equivalent: (i)  $\mathcal{G}$  is normalized, (ii)  $\Pi_m u^m = u^m$  for all  $m$ , (iii)  $\Pi u = u$ , (iv)  $u \in (\ker D)^\perp$ .

PROOF. The equivalence of (iii) and (iv) is immediate since by Lemma 4.4,  $\Pi = D^\dagger D$  is a projection operator to the orthogonal complement of the kernel of  $D$ . The equivalence of (ii) and (iii) follows from the definition of  $\Pi = \text{diag}(\Pi_1, \dots, \Pi_M)$ . To complete the proof we prove (i) and (ii) are equivalent.

Observe that (28) holds if and only if  $\langle u^m, \nu_{\mathbf{q}^{-m}}(\mathbf{p}) \rangle = 0$  for all  $\mathbf{q}^{-m} \in E^{-m}$ , where  $\nu_{\mathbf{q}^{-m}}$  is as defined in (27). Lemma 4.2 implies that  $\{\nu_{\mathbf{q}^{-m}}\}$  are basis vectors of  $\ker D_m$ . Thus, it follows that (28) holds if and only if  $u^m$  is orthogonal to all of the basis vectors of  $\ker D_m$ , or equivalently, when  $u^m \in (\ker D_m)^\perp$ . Since  $\Pi_m = D_m^\dagger D_m$  is a projection operator to  $(\ker D_m)^\perp$ , we have  $u^m \in (\ker D_m)^\perp$  if and only if  $\Pi_m u^m = u^m$ , and the claim follows.  $\square$

Using Lemma 4.5, we next show below that for each game  $\mathcal{G}$  there exists a unique strategically equivalent game which is normalized (contains no nonstrategic information).

LEMMA 4.6. Let  $\mathcal{G}$  be a game with utilities  $\{u^m\}_{m \in \mathcal{M}}$ . Then there exists a unique game  $\hat{\mathcal{G}}$  which (i) has the same pairwise comparison function as  $\mathcal{G}$ , and (ii) is normalized. Moreover, the utilities  $\hat{u} = \{\hat{u}^m\}_{m \in \mathcal{M}}$  of  $\hat{\mathcal{G}}$  satisfy  $\hat{u}^m = \Pi_m u^m$  for all  $m$ .

PROOF. To prove the claim, we show that given  $u = \{u^m\}_{m \in \mathcal{M}}$ , the game with the collection of utilities  $D^\dagger D u = \Pi u$ , is a normalized game with the same pairwise comparisons, and moreover, there cannot be another normalized game that has the same pairwise comparisons.

Since  $\Pi$  is a projection operator, it follows that  $\Pi \Pi u = \Pi u$ , and hence, Lemma 4.5 implies that  $\Pi u$  is normalized. Additionally, using properties of the pseudoinverse we have  $D \Pi u = D D^\dagger D u = D u$ ; thus  $\Pi u$  and  $u$  have the same pairwise comparison.

Let  $v \in C_0^M$  denote the collection of payoff functions of a game which is normalized and has the same pairwise comparison as  $u$ . It follows that  $D v = D u = D \Pi u$ , and hence  $v - \Pi u \in \ker D$ . On the other hand, since both  $v$  and  $\Pi u$  are normalized, by Lemma 4.5, we have  $v, \Pi u \in (\ker D)^\perp$ , and thus  $v - \Pi u \in (\ker D)^\perp$ . Therefore, it follows that  $v - \Pi u = 0$ ; hence,  $\Pi u$  is the collection of utility functions of the unique normalized game, which has the same pairwise comparison function as  $\mathcal{G}$ . By Lemma 4.4,  $\Pi u = \{\Pi_m u^m\}$ ; hence, the claim follows.  $\square$

We are now ready to define the subspaces of games that will appear in our decomposition result.

DEFINITION 4.2. The potential subspace  $\mathcal{P}$ , the harmonic subspace  $\mathcal{H}$ , and the nonstrategic subspace  $\mathcal{N}$  are defined as:

$$\begin{aligned} \mathcal{P} &\triangleq \{u \in C_0^M \mid u = \Pi u \text{ and } D u \in \text{im } \delta_0\}, \\ \mathcal{H} &\triangleq \{u \in C_0^M \mid u = \Pi u \text{ and } D u \in \ker \delta_0^*\}, \\ \mathcal{N} &\triangleq \{u \in C_0^M \mid u \in \ker D\}. \end{aligned} \quad (29)$$

Since the operators involved in the above definitions are linear, it follows that the sets  $\mathcal{P}$ ,  $\mathcal{H}$ , and  $\mathcal{N}$  are indeed subspaces.

Lemma 4.5 implies that the games in  $\mathcal{P}$  and  $\mathcal{H}$  are normalized (contain no nonstrategic information). The flows generated by the games in these two subspaces are related to the flows induced by the Helmholtz decomposition. It follows from the definitions that the flows generated by a game in  $\mathcal{P}$  are in the image space of  $\delta_0$  and the flows generated by a game in  $\mathcal{H}$  are in the kernel of  $\delta_0^*$ . Thus,  $\mathcal{P}$  corresponds to the set of normalized games, which have globally consistent pairwise comparisons. Due to (26), the pairwise comparisons of games do



not have locally inconsistent components; thus Theorem 3.1 implies that  $\mathcal{H}$  corresponds to the set of normalized games, which have globally inconsistent but locally consistent pairwise comparisons. Hence, from the perspective of the Helmholtz decomposition, the flows generated by games in  $\mathcal{P}$  and  $\mathcal{H}$  are gradient and harmonic flows respectively. On the other hand the flows generated by games in  $\mathcal{N}$  are always zero, since  $Du = 0$  in such games.

As discussed in the previous section the image spaces of  $D_m$  are orthogonal. Thus, since by definition  $Du = \sum_{m \in \mathcal{M}} D_m u^m$ , it follows that  $u = \{u^m\}_{m \in \mathcal{M}} \in \ker D$  if and only if  $u^m \in \ker D_m$  for all  $m \in \mathcal{M}$ . Using these facts together with Lemma 4.5, we obtain the following alternative description of the subspaces of games:

$$\begin{aligned}\mathcal{P} &= \left\{ \{u^m\}_{m \in \mathcal{M}} \mid D_m u^m = D_m \phi \text{ and } \Pi_m u^m = u^m \text{ for all } m \in \mathcal{M} \text{ and some } \phi \in C_0 \right\}, \\ \mathcal{H} &= \left\{ \{u^m\}_{m \in \mathcal{M}} \mid \delta_0^* \sum_{m \in \mathcal{M}} D_m u^m = 0 \text{ and } \Pi_m u^m = u^m \text{ for all } m \in \mathcal{M} \right\}, \\ \mathcal{N} &= \left\{ \{u^m\}_{m \in \mathcal{M}} \mid D_m u^m = 0 \text{ for all } m \in \mathcal{M} \right\}.\end{aligned}\quad (30)$$

The main result of this section shows that not only do these subspaces have distinct properties in terms of the flows they generate, but in fact they form a direct sum decomposition of the space of games. We exploit the Helmholtz decomposition (Theorem 3.1) for the proof.

**THEOREM 4.1.** *The space of games  $\mathcal{G}_{\mathcal{M}, E}$  is a direct sum of the potential, harmonic, and nonstrategic subspaces; i.e.,  $\mathcal{G}_{\mathcal{M}, E} = \mathcal{P} \oplus \mathcal{H} \oplus \mathcal{N}$ . In particular, given a game with utilities  $u = \{u^m\}_{m \in \mathcal{M}}$ , it can be uniquely decomposed in three components:*

- Potential Component:  $u_P \triangleq D^\dagger \delta_0^\dagger Du$ ,
- Harmonic Component:  $u_H \triangleq D^\dagger (I - \delta_0 \delta_0^\dagger) Du$ ,
- Nonstrategic Component:  $u_N \triangleq (I - D^\dagger D)u$ ,

where  $u_P + u_H + u_N = u$ , and  $u_P \in \mathcal{P}$ ,  $u_H \in \mathcal{H}$ ,  $u_N \in \mathcal{N}$ . The potential function associated with  $u_P$  is  $\phi \triangleq \delta_0^\dagger Du$ .

**PROOF.** The decomposition of  $\mathcal{G}_{\mathcal{M}, E}$  described above follows directly from pulling back the Helmholtz decomposition of  $C_1$  through the map  $D$ , and removing the kernel of  $D$ ; see Figure 5.

The components of the decomposition clearly satisfy  $u_P + u_H + u_N = u$ . We verify the inclusion properties according to (29). Both  $u_P$  and  $u_H$  are orthogonal to  $\mathcal{N} = \ker D$ , since they are in the range of  $D^\dagger$ .

- For the potential component, let  $\phi \in C_0$  be such that  $\phi = \delta_0^\dagger Du$ . Then, we have  $Du_P \in \text{im}(\delta_0)$ , since

$$Du_P = DD^\dagger \delta_0 \delta_0^\dagger Du = \delta_0 \delta_0^\dagger Du = \delta_0 \phi,$$

where we used the definition of  $u_P$ , the property (v) in Lemma 4.4, and the definition of  $\phi$ , respectively. This equality also implies that  $\phi$  is the potential function associated with  $u_P$ .

- For the harmonic component  $u_H$ , we have  $Du_H \in \ker \delta_0^*$ :

$$\delta_0^* Du_H = \delta_0^* DD^\dagger (I - \delta_0 \delta_0^\dagger) Du = \delta_0^* (I - \delta_0 \delta_0^\dagger) Du = 0,$$

as follows from the definition of  $u_H$ , the property (v) in Lemma 4.4, and properties of the pseudoinverse.

- To check that  $u_N \in \mathcal{N}$ , we have

$$Du_N = D(I - D^\dagger D)u = (D - DD^\dagger D)u = 0.$$

To prove that the direct sum decomposition property holds, we assume that there exists  $\hat{u}_P \in \mathcal{P}$ ,  $\hat{u}_H \in \mathcal{H}$ , and  $\hat{u}_N \in \mathcal{N}$  such that  $\hat{u}_P + \hat{u}_H + \hat{u}_N = 0$ . Observe that  $I - D^\dagger D$  is a projection operator to the kernel of  $D$ . Thus, from the definition of the subspaces  $\mathcal{P}$ ,  $\mathcal{H}$ , and  $\mathcal{N}$ , it follows that  $(I - D^\dagger D)\hat{u}_N = \hat{u}_N$  and  $(I - D^\dagger D)\hat{u}_P = (I - D^\dagger D)\hat{u}_H = 0$ . Similarly,  $\delta_0 \delta_0^\dagger$  is a projection operator to the image of  $\delta_0$ . Since by definition  $D\hat{u}_P \in \text{im} \delta_0$ , and  $D\hat{u}_H \in \ker \delta_0^* = (\text{im} \delta_0)^\perp$ , it follows that  $\delta_0 \delta_0^\dagger D\hat{u}_P = D\hat{u}_P$  and  $\delta_0 \delta_0^\dagger D\hat{u}_H = 0$ .

Using these identities, it follows that

$$\begin{aligned}(D^\dagger \delta_0 \delta_0^\dagger D)(\hat{u}_P + \hat{u}_H + \hat{u}_N) &= \hat{u}_P, \\ D^\dagger (I - \delta_0 \delta_0^\dagger) D(\hat{u}_P + \hat{u}_H + \hat{u}_N) &= \hat{u}_H, \\ (I - D^\dagger D)(\hat{u}_P + \hat{u}_H + \hat{u}_N) &= \hat{u}_N.\end{aligned}$$

Since  $\hat{u}_P + \hat{u}_H + \hat{u}_N = 0$  by our assumption, it follows that  $\hat{u}_P = \hat{u}_H = \hat{u}_N = 0$ , and hence the direct sum decomposition property follows.  $\square$

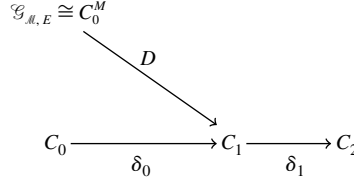


FIGURE 5. The Helmholtz decomposition of the space of flows ( $C_1$ ) can be pulled back through  $D$  to a direct sum decomposition of the space of games ( $\mathcal{G}_{M,E}$ ).

The pseudoinverse of a linear operator  $L$  projects its argument to the image space of  $L$ , and then pulls the projection back to the domain of  $L$ . Thus, intuitively, the potential function  $\phi = \delta_0^\dagger Du$ , defined in the theorem, is such that the gradient flow associated with it ( $\delta_0 \phi$ ) approximates the flow in the original game ( $Du$ ), in the best possible way. The potential component of the game can be identified by pulling back this gradient flow through  $D$  to  $C_0^M$ . The harmonic component can similarly be obtained using the harmonic flow.

Since  $\delta_0 = \sum_{m \in \mathcal{M}} D_m$ , it follows that  $\phi = \delta_0^\dagger Du = (\sum_{m \in \mathcal{M}} D_m)^\dagger \sum_{m \in \mathcal{M}} D_m u^m$ . Thus, Lemma 4.4(ii), and identities  $\Delta_{0,m} = D_m^* D_m$  and  $\Delta_0 = \sum_{m \in \mathcal{M}} \Delta_{0,m}$  imply that

$$\phi = \Delta_0^\dagger \sum_{m \in \mathcal{M}} \Delta_{0,m} u^m.$$

Additionally, from Lemma 4.4(iii) and (iv) it follows that  $D^\dagger \delta_0 = [D_1^\dagger D_1; \dots; D_M^\dagger D_M] = [\Pi_1; \dots; \Pi_M]$  and  $D^\dagger D = \Pi = \text{diag}(\Pi_1, \dots, \Pi_M)$ . Using these identities, the utility functions of components of a game can alternatively be expressed as follows:

- *Potential Component:*  $u_p^m = \Pi_m \phi$ , for all  $m \in \mathcal{M}$ ,
- *Harmonic Component:*  $u_H^m = \Pi_m u^m - \Pi_m \phi$ , for all  $m \in \mathcal{M}$ ,
- *Nonstrategic Component:*  $u_N^m = (I - \Pi_m) u^m$ , for all  $m \in \mathcal{M}$ .

It can be seen that the definitions of the subspaces do not rely on the inner product in  $C_0^M$ . Thus, the direct sum property implies that the decomposition is canonical; i.e., it is independent of the inner product used in  $C_0^M$ . The above expressions provide closed form solutions for the utility functions in the decomposition, without reference to this inner product. We show in §6 that our decomposition is indeed orthogonal with respect to a natural inner product in  $C_0^M$ .

Note that  $\Delta_0: C_0 \rightarrow C_0$ ; whereas,  $\delta_0: C_0 \rightarrow C_1$ . Since  $C_1$  and  $C_0$  are associated with the edges and the nodes of the game graph respectively, in general  $C_1$  is higher dimensional than  $C_0$ . Therefore, calculating  $\Delta_0^\dagger$  is computationally more tractable than calculating  $\delta_0^\dagger$ . Hence, the alternative expressions for the components of a game and the potential function  $\phi$  have computational benefits over using the results of Theorem 4.1 directly.

In Facchini et al. [10] and Voorneveld et al. [49], decompositions of potential games to a “congestion” component (where all players have identical utility functions), and a “dummy” component (which is nonstrategic) were provided. We note that this decomposition is different from the decomposition suggested by Theorem 4.1 even when the original game is a potential game. In particular, in the potential component of a game, players can have different utility functions (as can be seen from the alternative expressions, for utilities in different components of a game). This is not the case for the “congestion component” provided in these works.

We conclude this section by characterizing the dimensions of the potential, harmonic, and nonstrategic subspaces. The proof can be found in the appendix.

**PROPOSITION 4.1.** *The dimensions of the subspaces  $\mathcal{P}$ ,  $\mathcal{H}$ , and  $\mathcal{N}$  are:*

- (i)  $\dim(\mathcal{P}) = \prod_{m \in \mathcal{M}} h_m - 1$ ,
- (ii)  $\dim(\mathcal{H}) = (M - 1) \prod_{m \in \mathcal{M}} h_m - \sum_{m \in \mathcal{M}} \prod_{k \neq m} h_k + 1$ .
- (iii)  $\dim(\mathcal{N}) = \sum_{m \in \mathcal{M}} \prod_{k \neq m} h_k$ .

**4.3. An example: Decomposition of bimatrix games.** We conclude this section by providing an explicit decomposition result for bimatrix games, i.e., finite games with two players. Consider a bimatrix game where the payoff matrix of the row player is given by  $A$ , and that of the column player is given by  $B$ ; that is, when the row player plays  $i$  and the column player plays  $j$ , the row player’s payoff is equal to  $A_{ij}$  and the column player’s payoff is equal to  $B_{ij}$ .

Assume that both the row player and the column player have the same number  $h$  of strategies. It immediately follows from Proposition 4.1 that  $\dim \mathcal{P} = h^2 - 1$ ,  $\dim \mathcal{H} = (h - 1)^2$ , and  $\dim \mathcal{N} = 2h$ . For simplicity, we further

TABLE 2. Generalized RPS game and its components.

	$R$	$P$	$S$
(a) Generalized RPS game			
$R$	0, 0	$-3x, 3x$	$3y, -3y$
$P$	$3x, -3x$	0, 0	$-3z, 3z$
$S$	$-3y, 3y$	$3z, -3z$	0, 0
(b) Nonstrategic component			
$R$	$(x - y), (x - y)$	$(z - x), (x - y)$	$(y - z), (x - y)$
$P$	$(x - y), (z - x)$	$(z - x), (z - x)$	$(y - z), (z - x)$
$S$	$(x - y), (y - z)$	$(z - x), (y - z)$	$(y - z), (y - z)$
(c) Potential component			
$R$	$(y - x), (y - x)$	$(y - x), (x - z)$	$(y - x), (z - y)$
$P$	$(x - z), (y - x)$	$(x - z), (x - z)$	$(x - z), (z - y)$
$S$	$(z - y), (y - x)$	$(z - y), (x - z)$	$(z - y), (z - y)$
(d) Harmonic component			
$R$	0, 0	$-(x + y + z), (x + y + z)$	$(x + y + z), -(x + y + z)$
$P$	$(x + y + z), -(x + y + z)$	0, 0	$-(x + y + z), (x + y + z)$
$S$	$-(x + y + z), (x + y + z)$	$(x + y + z), -(x + y + z)$	0, 0

assume that the payoffs are normalized.<sup>6</sup> Thus, the definition of normalized games implies that  $\mathbf{1}^T A = B\mathbf{1} = 0$ , where  $\mathbf{1}$  denotes the vector of ones. Denote by  $A_P(B_P)$  and  $A_H(B_H)$  respectively, the payoff matrices of the row player (column player) in the potential and harmonic components of the game. Using our decomposition result (Theorem 4.1), it follows that

$$(A_P, B_P) = (S + \Gamma, S - \Gamma), \quad (A_H, B_H) = (D - \Gamma, -D + \Gamma), \quad (31)$$

where  $S = \frac{1}{2}(A + B)$ ,  $D = \frac{1}{2}(A - B)$ ,  $\Gamma = 1/(2h)(A\mathbf{1}\mathbf{1}^T - \mathbf{1}\mathbf{1}^T B)$ . Interestingly, the payoff of each player in these components depends on the payoffs of both players in the original game: The potential component of the game relates to the average of the payoffs in the original game and the harmonic component relates to the difference in payoffs of players. The  $\Gamma$  term ensures that the potential and harmonic components do not contain nonstrategic information. We use the above characterization in the next example for obtaining explicit payoff matrices for each of the game components.

**EXAMPLE 4.1. (GENERALIZED ROCK-PAPER-SCISSORS).** The payoff matrix of the generalized Rock-Paper-Scissors (RPS) game is given in Table 2(a). Tables 2(b), 2(c), and 2(d) include the nonstrategic, potential, and the harmonic components of the game. The special case where  $x = y = z = \frac{1}{3}$  corresponds to the celebrated RPS game. Note that in this case, the potential component of the game is equal to zero.

**5. Potential and harmonic games.** In this section we study the classes of games that are naturally motivated by our decomposition. In particular, we focus on two classes of games:<sup>7</sup> (i) Games with no harmonic component; (ii) Games with no potential component. We show that the first class is equivalent to the well-known class of *potential games*. We refer to the games in the second class as *harmonic games*. Pictorially, we have

$$\underbrace{\mathcal{P}}_{\text{Potential games}} \oplus \underbrace{\mathcal{N} \oplus \mathcal{H}}_{\text{Harmonic games}}.$$

In §§5.1 and 5.2, we establish the equivalence of the subspace definitions of potential and harmonic games, and the utility definitions given in Equations (1) and (2). Additionally we develop and discuss several properties of

<sup>6</sup>Lemmas 4.2 and 4.6 imply that if the payoffs are not normalized, the normalized payoffs can be obtained as  $(A - (1/h)\mathbf{1}\mathbf{1}^T A, B - (1/h)\mathbf{1}\mathbf{1}^T B)$ .

<sup>7</sup>Since the nonstrategic component does not have an impact on the strategic actions of players, we do not analyze its effect separately in this paper. However, we mention that the nonstrategic component affects efficiency in games. For instance, modifying the nonstrategic component of a game properly, it is possible to make all Nash equilibria Pareto optimal (see Candogan et al. [6]).

TABLE 3. Properties of potential and harmonic games.

	Potential games	Harmonic games
Subspaces	$\mathcal{P} \oplus \mathcal{N}$	$\mathcal{H} \oplus \mathcal{N}$
Flows	Globally consistent	Locally consistent but globally inconsistent
Pure NE	Always Exists	Generically does not exist
Mixed NE	Always Exists	—Uniformly mixed strategy is always a mixed NE —Players do not strictly prefer their equilibrium strategies.
Special cases		—(Two players) Set of mixed Nash equilibria coincides with the set of correlated equilibria —(Two players and equal number of strategies) Uniformly mixed strategy is the unique mixed NE

these classes of games, with particular emphasis on their equilibria. Since potential games have been extensively studied in the literature, our main focus is on harmonic games. Potential and harmonic games are related to other well-known classes of games, such as the zero-sum games and identical interest games. In §5.3, we discuss this relation in the context of bimatrix games. As a preview, in Table 3, we summarize some of the properties of potential and harmonic games established in the subsequent sections.

**5.1. Potential games.** Since the seminal paper of Monderer and Shapley [34], potential games have been an active research topic. The desirable equilibrium properties and structure of these games played a key role in this. In this section we explain the relation of the potential games to the decomposition in §4 and briefly discuss their properties.

Recall from Definition 2.1 that a game is a potential game if and only if there exists some  $\phi \in C_0$  such that  $Du = \delta_0 \phi$ . This condition implies that a game is potential if and only if the associated flow is globally consistent. Thus, it can be seen from the definition of the subspaces and Theorem 4.1 that the set of potential games is actually equivalent to  $\mathcal{P} \oplus \mathcal{N}$ . For future reference, we summarize this result in the following theorem.

**THEOREM 5.1.** *The set of potential games is equal to the subspace  $\mathcal{P} \oplus \mathcal{N}$ .*

Theorem 5.1 implies that potential games are games that only have potential and nonstrategic components. Since this set is a subspace, one can consider *projections* onto the set of potential games; i.e., it is possible to find the closest potential game to a given game. We pursue the idea of projection in §6. Using the previous theorem, we next find the dimension of the subspace of potential games (similar results can be found in Sandholm [40] and Monderer and Shapley [34]).

**COROLLARY 5.1.** *The subspace of potential games,  $\mathcal{P} \oplus \mathcal{N}$ , has dimension  $\prod_{m \in \mathcal{M}} h_m + \sum_{m \in \mathcal{M}} \prod_{k \neq m} h_k - 1$ .*

**PROOF.** The result immediately follows from Theorem 5.1 and Proposition 4.1.  $\square$

We next provide a brief discussion of the equilibrium properties of potential games.

**THEOREM 5.2 (MONDERER AND SHAPLEY [34]).** *Let  $\mathcal{G} = \langle \mathcal{M}, \{E^m\}, \{u^m\} \rangle$  be a potential game and let  $\phi$  be a corresponding potential function.*

- (i) *The equilibrium set of  $\mathcal{G}$  coincides with the equilibrium set of  $\mathcal{G}_\phi \triangleq \langle \mathcal{M}, \{E^m\}, \{\phi\} \rangle$ .*
- (ii)  *$\mathcal{G}$  has a pure Nash equilibrium.*

The first result follows from the fact that the games  $\mathcal{G}$  and  $\mathcal{G}_\phi$  are strategically equivalent. Alternatively, the preferences in  $\mathcal{G}$  are aligned with the global objective denoted by the potential function  $\phi$ . The second result is implied by the first one since in finite games the potential function  $\phi$  necessarily has a maximum, and the maximum is a Nash equilibrium of  $\mathcal{G}_\phi$ . These results indicate that potential games can be analyzed by an equivalent game where each player has the same utility function  $\phi$ . The second game is easy to analyze since when agents have the same objective, the game is similar to an optimization problem with objective function  $\phi$ .

**5.2. Harmonic games.** In this section, we focus on games in which the potential component is zero; hence the strategic interactions are governed only by the harmonic component. We refer to such games as *harmonic games*; i.e., a game  $\mathcal{G}$  is a harmonic game if  $\mathcal{G} \in \mathcal{H} \oplus \mathcal{N}$ . We first provide an alternative definition of harmonic games in terms of the payoff functions of players (cf. Equation (2)).

**THEOREM 5.3.** *The set of games which satisfy*

$$\sum_{m \in \mathcal{M}} \sum_{\mathbf{q}^m \in E^m} (u^m(\mathbf{p}^m, \mathbf{p}^{-m}) - u^m(\mathbf{q}^m, \mathbf{p}^{-m})) = 0,$$

*for all strategy profiles  $\mathbf{p}$  and strategies  $\mathbf{q}^m \in E^m$ , is equivalent to the set of harmonic games  $\mathcal{H} \oplus \mathcal{N}$ .*

PROOF. The condition in the theorem statement can equivalently be expressed as  $\delta_0^* Du = 0$ , using the operators introduced in §4. Since the flow associated with a game is given by  $Du$ , a game satisfies this condition if and only if it belongs to  $\mathcal{H} \oplus \mathcal{N}$ , as can be seen from Definition 4.2.  $\square$

This theorem provides a certificate that can be used to check whether a given game is a harmonic game. For instance, using the theorem it immediately follows that the rock-paper-scissors game (for  $x = y = z = \frac{1}{3}$ ) in Example 4.1 is a harmonic game.

The rest of this section studies the properties of equilibria of harmonic games. We first characterize the Nash equilibria of such games, and show that generically they do not have a pure Nash equilibrium. We further consider mixed Nash and correlated equilibria, and show how the properties of harmonic games restrict the possible set of equilibria.

**5.2.1. Pure equilibria.** In this section, we focus on pure Nash equilibria in harmonic games. Additionally, we characterize the dimension of the space of harmonic games,  $\mathcal{H} \oplus \mathcal{N}$ .

We first show that at a pure Nash equilibrium of a harmonic game, all players are indifferent between *all* of their strategies.

LEMMA 5.1. *Let  $\mathcal{G} = \langle \mathcal{M}, \{E^m\}, \{u^m\} \rangle$  be a harmonic game and let  $\mathbf{p}$  be a pure Nash equilibrium. Then,*

$$u^m(\mathbf{p}^m, \mathbf{p}^{-m}) = u^m(\mathbf{q}^m, \mathbf{p}^{-m}) \quad \text{for all } m \in \mathcal{M} \text{ and } \mathbf{q}^m \in E^m. \quad (32)$$

PROOF. By definition, in harmonic games the utility functions  $u = \{u^m\}$  satisfy the condition  $\delta_0^* Du = 0$ . By (13) and (21),  $\delta_0^* Du$  evaluated at  $\mathbf{p}$  can be expressed as,

$$\sum_{m \in \mathcal{M}} \sum_{\mathbf{q} \mid (\mathbf{p}, \mathbf{q}) \in A^m} (u^m(\mathbf{p}) - u^m(\mathbf{q})) = 0. \quad (33)$$

Since  $\mathbf{p}$  is a Nash equilibrium it follows that  $u^m(\mathbf{p}) - u^m(\mathbf{q}) \geq 0$  for all  $(\mathbf{p}, \mathbf{q}) \in A^m$  and  $m \in \mathcal{M}$ . Combining this with (33) it follows that  $u^m(\mathbf{p}) - u^m(\mathbf{q}) = 0$  for all  $(\mathbf{p}, \mathbf{q}) \in A^m$  and  $m \in \mathcal{M}$ . Observing that  $(\mathbf{p}, \mathbf{q}) \in A^m$  if and only if  $\mathbf{q} = (\mathbf{q}^m, \mathbf{p}^{-m})$ , the result follows.  $\square$

Using this result we next prove that harmonic games generically do not have pure Nash equilibria. By “generically,” we mean that it is true for almost all harmonic games, except possibly for a set of measure zero (for instance, the trivial game where all utilities are zero is harmonic, and clearly has pure Nash equilibria).

PROPOSITION 5.1. *Harmonic games generically do not have pure Nash equilibria.*

PROOF. Define  $\mathcal{G}_{\mathbf{p}} \subset \mathcal{H} \oplus \mathcal{N}$  as the set of harmonic games for which  $\mathbf{p}$  is a pure Nash equilibrium. Observe that  $\bigcup_{\mathbf{p} \in E} \mathcal{G}_{\mathbf{p}}$  is the set of all harmonic games that have a pure Nash equilibrium. We show that  $\mathcal{G}_{\mathbf{p}}$  is a lower dimensional subspace of the space of harmonic games for each  $\mathbf{p} \in E$ . Since the set of harmonic games with pure Nash equilibrium is a finite union of lower dimensional subspaces it follows that generically harmonic games do not have pure Nash equilibria.

By Lemma 5.1 it follows that

$$\mathcal{G}_{\mathbf{p}} = (\mathcal{H} \oplus \mathcal{N}) \cap \left\{ \{u^m\}_{m \in \mathcal{M}} \mid u^m(\mathbf{p}) = u^m(\mathbf{q}), \text{ for all } \mathbf{q} \text{ such that } (\mathbf{p}, \mathbf{q}) \in A^m \text{ and } m \in \mathcal{M} \right\}.$$

Hence  $\mathcal{G}_{\mathbf{p}}$  is a subspace contained in  $\mathcal{H} \oplus \mathcal{N}$ . It immediately follows that  $\mathcal{G}_{\mathbf{p}}$  is a lower dimensional subspace if we can show that there exist harmonic games that are not in  $\mathcal{G}_{\mathbf{p}}$ , i.e., in which  $\mathbf{p}$  is not a pure Nash equilibrium.

Assume that  $\mathbf{p}$  is a pure Nash equilibrium in all harmonic games. Since  $\mathbf{p}$  is arbitrary this holds only if all strategy profiles are pure Nash equilibria in harmonic games. If all strategy profiles are Nash equilibria, by Lemma 5.1 it follows that the pairwise ranking function is equal to zero in harmonic games; hence  $\mathcal{H} \oplus \mathcal{N} \subset \mathcal{N}$ . We reach a contradiction since the dimension of  $\mathcal{H}$  is larger than zero.

Therefore,  $\mathcal{G}_{\mathbf{p}}$  is a strict subspace of the space of harmonic games, and thus harmonic games generically do not have pure Nash equilibria.  $\square$

We conclude this section with a dimension result that is analogous to the result obtained for potential games.

THEOREM 5.4. *The set of harmonic games,  $\mathcal{H} \oplus \mathcal{N}$ , has dimension  $(M - 1) \prod_{m \in \mathcal{M}} h_m + 1$ .*

PROOF. The result immediately follows from Theorem 4.1 and Proposition 4.1.  $\square$



**5.2.2. Mixed Nash and correlated equilibria in harmonic games.** In the previous section we showed that harmonic games generically do not have pure Nash equilibria. In this section, we study their mixed Nash and correlated equilibria. In particular, we show that in harmonic games, the mixed strategy profile, in which players uniformly randomize over their strategies is always a mixed Nash equilibrium. Additionally, in the case of two-player harmonic games, mixed Nash and correlated equilibria coincide, and if players have equal numbers of strategies, the uniformly mixed strategy profile is the unique correlated equilibrium of the game. Before we discuss the details of these results, we next provide some preliminaries and notation.

We denote the set of probability distributions on  $E$  by  $\Delta E$ . Given  $x \in \Delta E$ ,  $x(\mathbf{p})$  denotes the probability assigned to  $\mathbf{p} \in E$ . Observe that for all  $x \in \Delta E$ ,  $\sum_{\mathbf{p} \in E} x(\mathbf{p}) = 1$ , and  $x(\mathbf{p}) \geq 0$ . Similarly for each player  $m \in \mathcal{M}$ ,  $\Delta E^m$  denotes the set of probability distributions on  $E^m$  and for  $x^m \in \Delta E^m$ ,  $x^m(\mathbf{p}^m)$  is the probability assigned to strategy  $\mathbf{p}^m \in E^m$ . As before, all  $x^m \in \Delta E^m$  satisfies  $\sum_{\mathbf{p}^m \in E^m} x^m(\mathbf{p}^m) = 1$  and  $x^m(\mathbf{p}^m) \geq 0$ . We refer to the distribution  $x^m \in \Delta E^m$  as a mixed strategy of player  $m \in \mathcal{M}$  and the collection  $x = \{x^m\}_m$  as a mixed strategy profile. Note that  $\{x^m\}_m \in \prod_{m \in \mathcal{M}} \Delta E^m \subset \Delta E$ . Mixed strategies of all players but the  $m$ th one are denoted by  $x^{-m}$ .

With some abuse of the notation, we define the mixed extensions of the utility functions  $u^m: \prod_{m \in \mathcal{M}} \Delta E^m \rightarrow \mathbb{R}$  such that for any  $x \in \prod_{m \in \mathcal{M}} \Delta E^m$ ,

$$u^m(x) = \sum_{\mathbf{p} \in E} u^m(\mathbf{p}) \prod_{k \in \mathcal{M}} x^k(\mathbf{p}^k). \quad (34)$$

Similarly, if player  $m$  uses pure strategy  $\mathbf{q}^m$  and the other players use the mixed strategies  $x^{-m}$ , we denote the payoff of player  $m$  by,

$$u^m(\mathbf{q}^m, x^{-m}) = \sum_{\mathbf{p}^{-m} \in E^{-m}} u^m(\mathbf{q}^m, \mathbf{p}^{-m}) \prod_{k \in \mathcal{M}, k \neq m} x^k(\mathbf{p}^k). \quad (35)$$

Using this notation we can define the solution concepts.

**DEFINITION 5.1 (MIXED NASH/CORRELATED EQUILIBRIUM).** Consider the game  $\langle \mathcal{M}, \{E^m\}, \{u^m\} \rangle$ .

(i) A mixed strategy profile  $x = \{x^m\}_m \in \prod_{m \in \mathcal{M}} \Delta E^m$  is a *mixed Nash equilibrium* if for all  $m \in \mathcal{M}$  and  $\mathbf{p}^m \in E^m$ ,  $u^m(x^m, x^{-m}) \geq u^m(\mathbf{p}^m, x^{-m})$ .

(ii) A probability distribution  $x \in \Delta E$  is a *correlated equilibrium* if for all  $m \in \mathcal{M}$  and  $\mathbf{p}^m, \mathbf{q}^m \in E^m$ ,  $\sum_{\mathbf{p}^{-m} \in E^{-m}} (u^m(\mathbf{p}^m, \mathbf{p}^{-m}) - u^m(\mathbf{q}^m, \mathbf{p}^{-m}))x(\mathbf{p}^m, \mathbf{p}^{-m}) \geq 0$ .

From these definitions it can be seen that every mixed Nash equilibrium is a correlated equilibrium where the corresponding distribution  $x \in \prod_{m \in \mathcal{M}} \Delta E^m \subset \Delta E$  is a product distribution; i.e., it satisfies  $x(\mathbf{p}) = \prod_{m \in \mathcal{M}} x^m(\mathbf{p}^m)$ .

These definitions also imply that similar to Nash equilibria, the conditions for mixed Nash and correlated equilibria can be expressed only in terms of pairwise comparisons. Therefore, these equilibrium sets are independent of the nonstrategic components of games.

We next obtain an alternative characterization of correlated equilibria in normalized harmonic games. This characterization will be more convenient when studying the equilibrium properties of harmonic games, as it is expressed in terms of equalities, instead of inequalities. See the appendix for a proof.

**PROPOSITION 5.2.** Consider a normalized harmonic game,  $\mathcal{G} = \langle \mathcal{M}, \{u^m\}, E^m \rangle$  and a probability distribution  $x \in \Delta E$ . The following are equivalent:

- (i)  $x$  is a correlated equilibrium.
- (ii) For all  $\mathbf{p}^m, \mathbf{q}^m$ , and  $m \in \mathcal{M}$ ,

$$\sum_{\mathbf{p}^{-m} \in E^{-m}} (u^m(\mathbf{p}^m, \mathbf{p}^{-m}) - u^m(\mathbf{q}^m, \mathbf{p}^{-m}))x(\mathbf{p}^m, \mathbf{p}^{-m}) = 0. \quad (36)$$

- (iii) For all  $\mathbf{p}^m, \mathbf{q}^m$ , and  $m \in \mathcal{M}$ ,

$$\sum_{\mathbf{p}^{-m} \in E^{-m}} u^m(\mathbf{q}^m, \mathbf{p}^{-m})x(\mathbf{p}^m, \mathbf{p}^{-m}) = 0. \quad (37)$$

The above proposition implies that the correlated equilibria of harmonic games correspond to the intersection of the probability simplex with a subspace defined by the utilities in the game. Using this result, we obtain the following characterization of mixed Nash equilibria of harmonic games.

**COROLLARY 5.2.** Let  $\mathcal{G} = \langle \mathcal{M}, \{u^m\}, \{E^m\} \rangle$  be a harmonic game. The mixed strategy profile  $x \in \prod_{m \in \mathcal{M}} \Delta E^m$  is a mixed Nash equilibrium if and only if,

$$u^m(x^m, x^{-m}) = u^m(\mathbf{p}^m, x^{-m}) \quad \text{for all } \mathbf{p}^m \in E^m \text{ and } m \in \mathcal{M}. \quad (38)$$

PROOF. Assume that (38) holds. Then clearly all players are indifferent between all their mixed strategies; hence, it follows that  $x$  is a mixed Nash equilibrium of the game.

Let  $x$  be a mixed Nash equilibrium. Since each mixed Nash equilibrium is also a correlated equilibrium, from equivalence of (i) and (ii) of Proposition 5.2 for all harmonic games, it follows that for all  $\mathbf{p}^m, \mathbf{q}^m$ , and  $m \in \mathcal{M}$ ,

$$\begin{aligned} 0 &= \sum_{\mathbf{p}^{-m} \in E^{-m}} (u^m(\mathbf{p}^m, \mathbf{p}^{-m}) - u^m(\mathbf{q}^m, \mathbf{p}^{-m})) x(\mathbf{p}^m, \mathbf{p}^{-m}) \\ &= x^m(\mathbf{p}^m) \sum_{\mathbf{p}^{-m} \in E^{-m}} (u^m(\mathbf{p}^m, \mathbf{p}^{-m}) - u^m(\mathbf{q}^m, \mathbf{p}^{-m})) \prod_{k \neq m} x^k(\mathbf{p}^k) \\ &= x^m(\mathbf{p}^m) (u^m(\mathbf{p}^m, x^{-m}) - u^m(\mathbf{q}^m, x^{-m})). \end{aligned} \quad (39)$$

Since by definition of probability distributions, there exists  $\mathbf{p}^m$  such that  $x^m(\mathbf{p}^m) > 0$  it follows that  $u^m(\mathbf{p}^m, x^{-m}) = u^m(\mathbf{q}^m, x^{-m})$  for all  $\mathbf{q}^m \in E^m$ . Thus,  $u^m(x^m, x^{-m}) = u^m(\mathbf{q}^m, x^{-m})$  for all  $\mathbf{q}^m \in E^m$ . Since  $m$  is arbitrary, the claim follows.  $\square$

It is well known that in mixed Nash equilibria of games, players are indifferent between all the pure strategies in the support of their mixed strategy (see Fudenberg and Tirole [13]); i.e., if  $x \in \prod_{m \in \mathcal{M}} \Delta E^m$  is a mixed Nash equilibrium, then

$$u^m(x^m, x^{-m}) \begin{cases} = u^m(\mathbf{p}^m, x^{-m}) & \text{for all } \mathbf{p}^m \text{ such that } x^m(\mathbf{p}^m) > 0, \\ \geq u^m(\mathbf{p}^m, x^{-m}) & \text{for all } \mathbf{p}^m \text{ such that } x^m(\mathbf{p}^m) = 0. \end{cases} \quad (40)$$

The above corollary implies that at a mixed equilibrium of a harmonic game, each player is indifferent between all its pure strategies, including those that are not in the support of its mixed strategy.

We next define a particular mixed strategy profile, and show that it is an equilibrium in all harmonic games.

DEFINITION 5.2 (UNIFORMLY MIXED STRATEGY PROFILE). The *uniformly mixed strategy* of player  $m$  is a mixed strategy where player  $m$  uses  $x_{\mathbf{q}^m} = 1/h_m$  for all  $\mathbf{q}^m \in E^m$ . Respectively, we define the *uniformly mixed strategy profile* as the one in which all players use uniformly mixed strategies.

Recall that rock-paper-scissors and matching pennies are examples of harmonic games, in which the uniformly mixed strategy profile is a mixed Nash equilibrium. The next theorem shows that this is a general property of harmonic games, and the uniformly mixed strategy profile is always a Nash equilibrium.

THEOREM 5.5. In harmonic games, the uniformly mixed strategy profile is always a Nash equilibrium.

PROOF. Let  $\mathcal{G} = \langle \mathcal{M}, \{u^m\}, \{E^m\} \rangle$  be a harmonic game, and let  $x$  be the uniformly mixed strategy profile. To prove the claim, we first state the following useful identity (see the appendix for a proof), on the utility functions of harmonic games.

LEMMA 5.2. Let  $\mathcal{G} = \langle \mathcal{M}, \{u^m\}, \{E^m\} \rangle$  be a harmonic game. Then for all  $\mathbf{q}^m, \mathbf{r}^m \in E^m$ ,  $m \in \mathcal{M}$ ,  $\sum_{\mathbf{p}^{-m} \in E^{-m}} u^m(\mathbf{r}^m, \mathbf{p}^{-m}) - u^m(\mathbf{q}^m, \mathbf{p}^{-m}) = 0$ .

Using this lemma, and the definition of the uniformly mixed strategy, it follows that

$$\begin{aligned} u^m(\mathbf{q}^m, x^{-m}) - u^m(\mathbf{p}^m, x^{-m}) &= \sum_{\mathbf{p}^{-m} \in E^{-m}} c_m (u^m(\mathbf{q}^m, \mathbf{p}^{-m}) - u^m(\mathbf{p}^m, \mathbf{p}^{-m})) \\ &= c_m \sum_{\mathbf{p}^{-m} \in E^{-m}} (u^m(\mathbf{q}^m, \mathbf{p}^{-m}) - u^m(\mathbf{p}^m, \mathbf{p}^{-m})) \\ &= 0, \end{aligned} \quad (41)$$

where  $c_m = \prod_{k \neq m} x^k(\mathbf{p}^k) = \prod_{k \neq m} (1/h_k)$ . Since  $\mathbf{p}^m$  and  $\mathbf{q}^m$  are arbitrary, (41) implies that

$$u^m(x^m, x^{-m}) = u^m(\mathbf{p}^m, x^{-m}) \quad (42)$$

for all  $\mathbf{p}^m \in E^m$ , and by Corollary 5.2,  $x$  is a mixed strategy Nash equilibrium.  $\square$

We conclude this section by providing an additional characterization of normalized harmonic games that is useful for the study of the equilibrium properties of these games.

THEOREM 5.6. The game  $\mathcal{G}$  with utilities  $u = \{u^m\}_{m \in \mathcal{M}}$  is a normalized harmonic game; i.e., it belongs to  $\mathcal{H}$  if and only if  $\sum_{m \in \mathcal{M}} h_m u^m = 0$  and  $\Pi_m u^m = u^m$  for all  $m \in \mathcal{M}$ , where  $h_m = |E^m|$ .

TABLE 4. Dimensions of subspaces of games and their intersections.

	$\mathcal{Z}$	$\mathcal{I}$	$\mathcal{Z} \oplus \mathcal{I}$
$\mathcal{P} \oplus \mathcal{N}$	$2h - 1$	$h^2$	$h^2 + 2h - 1$
$\mathcal{H} \oplus \mathcal{N}$	$h^2 - 2h + 2$	$1$	$h^2 + 1$
$\mathcal{P} \oplus \mathcal{H} \oplus \mathcal{N}$	$h^2$	$h^2$	$2h^2$

PROOF. By Definition 4.2,  $\mathcal{G} \in \mathcal{H}$  if and only if  $\Pi u = u$  and  $\delta_0^* Du = 0$ . Using the definitions of the operators, these conditions can alternatively be expressed as  $\Pi_m u^m = u^m$  and  $\delta_0^* \sum_{m \in \mathcal{M}} D_m u^m = 0$ . By (24) and the orthogonality of image spaces of operators  $D_m$ , the latter equality implies that  $\sum_{m \in \mathcal{M}} D_m^* D_m u^m = \sum_{m \in \mathcal{M}} \Delta_{0,m} u^m = 0$ . Using Lemma 4.1,  $\Delta_{0,m} = h_m \Pi_m$ , and hence it follows that  $\mathcal{G} \in \mathcal{H}$ , if and only if

$$\sum_{m \in \mathcal{M}} h_m \Pi_k u^k = 0 \quad \text{and} \quad \Pi_k u^k = u^k \text{ for all } k \in \mathcal{M}. \quad (43)$$

The claim follows by replacing  $\Pi_m u^m$  in the summation with  $u^m$ .  $\square$

The above theorem implies that normalized harmonic games, where players have an equal number of strategies, are *zero-sum games*, i.e., in such games the payoffs of players add up to zero at all strategy profiles. This suggests that equilibrium properties of zero-sum games and harmonic games should be related. We study the relation between zero-sum games and harmonic games in §5.3.

For two-player harmonic games, using the above zero-sum characterization, additional properties of equilibria can be established. For instance, in these games sets of correlated equilibria and mixed equilibria are identical. These properties are explored in Appendix B.

**5.3. Zero-sum games and identical interest games.** In this section we present a different decomposition of the space of games, and discuss its relation to our decomposition. To simplify the presentation, we focus on bimatrix games, where each player has  $h$  strategies. Before introducing the decomposition, we define zero-sum games and identical interest games.

DEFINITION 5.3 (ZERO-SUM AND IDENTICAL INTEREST GAMES). Let  $\mathcal{G}$  denote the bimatrix game with payoff matrices  $(A, B)$ .  $\mathcal{G}$  is a *zero-sum game* if  $A + B = 0$ , and  $\mathcal{G}$  is an *identical interest game* if  $A = B$ .

We denote the set of zero-sum games by  $\mathcal{Z}$ , and the set of identical interest games by  $\mathcal{I}$ . Since these sets are defined by equality constraints on the payoff matrices, it follows that they are subspaces.

The idea of decomposing a game to an identical interest game and a zero-sum game was previously mentioned in the literature for two-player games, (Başar and Ho [1]). The following lemma implies that the  $\mathcal{Z}$  and  $\mathcal{I}$  decomposition of the set of games has the direct sum property.

LEMMA 5.3. The space of two-player games  $\mathcal{G}_{\mathcal{M}, E}$  is a direct sum of subspaces of zero-sum and identical interest games; i.e.,  $\mathcal{G}_{\mathcal{M}, E} = \mathcal{Z} \oplus \mathcal{I}$ .

PROOF. Consider a bimatrix game with utilities  $(u^1, u^2)$ . Observe that this game can be decomposed to the games with payoff functions  $((u^1 - u^2)/2, (u^2 - u^1)/2)$  and  $((u^1 + u^2)/2, (u^1 + u^2)/2)$ . Clearly the former game is a zero-sum game, where the latter is an identical interest game. Since the initial game was arbitrary, it follows that any game can be decomposed to a zero-sum game and an identical interest game. The direct sum property follows, since for two-player zero-sum and identical interest games, with utility functions  $(u, -u)$  and  $(v, v)$  respectively, if  $(u + v, u - v) = (0, 0)$ , then  $u = v = 0$ .  $\square$

Note that Theorem 5.6 suggests that two-player normalized harmonic games, where players have equal numbers of strategies are zero-sum.<sup>8</sup> Also, it immediately follows by checking the definitions that identical interest games are potential games. This intuitively suggests that the zero-sum and identical interest game decomposition closely relates to our decomposition. In the following theorem, we establish this relation by characterizing the dimensions of the intersections of the subspaces  $\mathcal{Z}$  and  $\mathcal{I}$ , with the sets of potential and harmonic games. We provide a proof in the appendix.

THEOREM 5.7. Consider two-player games, in which each player has  $h$  strategies. The dimensions of intersections of the subspaces of zero-sum and identical interest games ( $\mathcal{Z}$  and  $\mathcal{I}$ ) with the subspaces of potential and harmonic games ( $\mathcal{P} \oplus \mathcal{N}$  and  $\mathcal{H} \oplus \mathcal{N}$ ) are as in Table 4.

<sup>8</sup> In addition, if the definition of zero-sum is generalized to include multiplayer games where payoffs of all players add up to zero, then it can be seen that normalized harmonic games where players have equal numbers of strategies are still zero-sum games.

TABLE 5. A zero-sum potential game.

	$a$	$b$
(a) Payoffs		
$x$	0, 0	1, -1
$y$	-1, 1	0, 0
(b) Potential function		
$x$	2	1
$y$	1	0

TABLE 6. A potential game  $\mathcal{G}$  and its zero-sum ( $\mathcal{G}_Z$ ) and identical interest components ( $\mathcal{G}_I$ ).

	$a$	$b$
(a) Payoffs in $\mathcal{G}$		
$x$	1, 1	1, -1
$y$	-1, 1	-1, -1
(b) Potential function of $\mathcal{G}$		
$x$	4	2
$y$	2	0
(c) Payoffs in $\mathcal{G}_Z$		
$x$	0, 0	1, -1
$y$	-1, 1	0, 0
(d) Payoffs in $\mathcal{G}_I$		
$x$	1, 1	0, 0
$y$	0, 0	-1, -1

The above theorem suggests that the dimensions of harmonic games and zero-sum games (and similarly identical interest games and potential games) are close to the dimension of their intersections. Thus, zero-sum games are in general closely related to harmonic games, and identical interest games are related to potential games. On the other hand, it is possible to find instances of zero-sum games that are potential games, and not harmonic games (see Table 5).

In general, the identical interest component is a potential game, and it can be used to approximate a given game with a potential game. However, as illustrated in Table 6, this approximation need not yield the closest potential game to a given game. In this example, despite the fact that the original game is a potential game, the zero-sum and identical interest game decomposition may lead to a potential game that is much farther than the closest potential game.

We believe that the decomposition presented in §4 is more natural than the zero-sum identical interest game decomposition, as it clearly separates the strategic ( $\mathcal{P} \oplus \mathcal{H}$ ) and nonstrategic ( $\mathcal{N}$ ) components of games and further identifies components, such as potential and harmonic components, with distinct strategic properties. In addition, it is invariant under trivial manipulations that do not change the strategic interactions, i.e., changes in the nonstrategic component.

**6. Projections onto potential and harmonic games.** In this section, we discuss projections of games onto the subspaces of potential and harmonic games. In §4.2, we defined the subspaces  $\mathcal{P}$ ,  $\mathcal{H}$ ,  $\mathcal{N}$  of potential, harmonic, and nonstrategic components, respectively. We also proved that they provide a direct sum decomposition of the space of all games. In this section, we show that under an appropriately defined inner product in  $\mathcal{G}_{\mathcal{M}, E}$ , the harmonic, potential, and nonstrategic subspaces become *orthogonal*. We use our decomposition result together with this inner product to obtain projections of games to these subspaces; i.e., for an arbitrary game, we present closed-form expressions for the “closest” potential and harmonic games with respect to this inner product.

Let  $\mathcal{G}, \hat{\mathcal{G}}$  be two games in  $\mathcal{G}_{\mathcal{M}, E}$ . We define the inner product on  $\mathcal{G}_{\mathcal{M}, E}$  as

$$\langle \mathcal{G}, \hat{\mathcal{G}} \rangle_{\mathcal{M}, E} \triangleq \sum_{m \in \mathcal{M}} h_m \langle u^m, \hat{u}^m \rangle, \quad (44)$$

where the inner product in the right-hand side is the inner product of  $C_0$  as defined in (8); i.e., it is the inner product of the space of functions defined on  $E$ . Note that it can be easily checked that (44) is an inner product, by observing that it is a weighted version of the standard inner product in  $C_0^M$ . The given inner product also induces a norm that will help us quantify the distance between games. We define the norm on  $\mathcal{G}_{\mathcal{M}, E}$  as follows:

$$\|\mathcal{G}\|_{\mathcal{M}, E}^2 = \langle \mathcal{G}, \mathcal{G} \rangle_{\mathcal{M}, E}. \quad (45)$$

Note that this norm also corresponds to a weighted  $l_2$  norm defined on the space  $C_0^M$ .

Next we prove that the potential, harmonic, and nonstrategic subspaces are orthogonal under this inner product.

**THEOREM 6.1.** *Under the inner product introduced in (44), we have  $\mathcal{P} \perp \mathcal{H} \perp \mathcal{N}$ ; i.e., the potential, harmonic, and nonstrategic subspaces are orthogonal.*

**PROOF.** Let  $\{u_P^m\}_{m \in \mathcal{M}} = \mathcal{G}_P \in \mathcal{P}$ ,  $\{u_H^m\}_{m \in \mathcal{M}} = \mathcal{G}_H \in \mathcal{H}$ , and  $\{u_N^m\}_{m \in \mathcal{M}} = \mathcal{G}_N \in \mathcal{N}$  be arbitrary games in  $\mathcal{P}$ ,  $\mathcal{H}$ , and  $\mathcal{N}$ , respectively. To prove the claim we will first prove  $\mathcal{G}_N \perp \mathcal{G}_H$  and  $\mathcal{G}_N \perp \mathcal{G}_P$ . Secondly we prove  $\mathcal{G}_P \perp \mathcal{G}_H$ . Since the games are arbitrary, the first part will imply that  $\mathcal{N} \perp \mathcal{H}$  and  $\mathcal{N} \perp \mathcal{P}$ , and the second part will imply that  $\mathcal{P} \perp \mathcal{H}$ , proving the claim.

Note that by definition,  $u_N^m \in \ker(D_m)$  for all  $m \in \mathcal{M}$  and  $u_P^m, u_H^m$  are in the orthogonal complement of  $\ker(D_m)$  since  $\Pi_m u_P^m = u_P^m$ ,  $\Pi_m u_H^m = u_H^m$ , and  $\Pi_m$  is the projection operator to the orthogonal complement of  $\ker(D_m)$ . This implies that  $\langle u_P^m, u_N^m \rangle = \langle u_H^m, u_N^m \rangle = 0$  for all  $m \in \mathcal{M}$  and hence, using the inner product introduced in (44), it follows that  $\mathcal{G}_N \perp \mathcal{G}_H$  and  $\mathcal{G}_N \perp \mathcal{G}_P$ .

Next observe that for all  $m \in \mathcal{M}$ ,

$$\langle u_P^m, u_H^m \rangle = \langle D_m^\dagger D_m u_P^m, u_H^m \rangle = \frac{1}{h_m} \langle D_m^* D_m \phi, u_H^m \rangle = \frac{1}{h_m} \langle \phi, D_m^* D_m u_H^m \rangle,$$

where the first equality follows from  $\Pi_m u_P^m = u_P^m$ , and the second equality follows from Lemma 4.1 and the fact that  $D_m u_P^m = D_m \phi$ . The third equality uses the properties of the operators  $D_m$  and  $D_m^*$ . Therefore,

$$\langle \mathcal{G}_P, \mathcal{G}_H \rangle_{\mathcal{M}, E} = \sum_{m \in \mathcal{M}} \langle \phi, D_m^* D_m u_H^m \rangle = \left\langle \phi, \sum_{m \in \mathcal{M}} D_m^* D_m u_H^m \right\rangle = \left\langle \phi, \delta_0^* \sum_{m \in \mathcal{M}} D_m u_H^m \right\rangle = 0,$$

since  $\delta_0^* \sum_{m \in \mathcal{M}} D_m u_H^m = 0$  by the definition of  $\mathcal{H}$ . Here the last equality follows using  $\delta_0^* = \sum_{m \in \mathcal{M}} D_m^*$  and orthogonality of the image spaces of  $D_m$  for  $m \in \mathcal{M}$ . Therefore, we have,  $\mathcal{G}_H \perp \mathcal{G}_P$  as claimed and the result follows.  $\square$

The next theorem provides closed-form expressions for the closest potential and harmonic games with respect to the norm in (45).

**THEOREM 6.2.** *Let  $\mathcal{G} \in \mathcal{G}_{\mathcal{M}, E}$  be a game with utilities  $\{u^m\}_{m \in \mathcal{M}}$ , and let  $\phi = \delta_0^\dagger D u$ . With respect to the norm in (45),*

- (i) *The closest potential game to  $\mathcal{G}$  has utilities  $\Pi_m \phi + (I - \Pi_m) u^m$  for all  $m \in \mathcal{M}$ .*
- (ii) *The closest harmonic game to  $\mathcal{G}$  has utilities  $u^m - \Pi_m \phi$  for all  $m \in \mathcal{M}$ .*

**PROOF.** By Theorem 6.1, the harmonic component of  $\mathcal{G}$  is orthogonal to the space of potential games  $\mathcal{P} \oplus \mathcal{N}$ . Thus, the closest potential game to  $\mathcal{G}$  has utilities  $u^m - u_H^m$ , where  $\{u_H^m\}_{m \in \mathcal{M}}$  is the harmonic component of  $\mathcal{G}$ . Similarly, the potential component of  $\mathcal{G}$  is orthogonal to the space of harmonic games  $\mathcal{H} \oplus \mathcal{N}$ , and thus the closest harmonic game to  $\mathcal{G}$  has utilities  $u^m - u_P^m$ , where  $\{u_P^m\}_{m \in \mathcal{M}}$  is the potential component of  $\mathcal{G}$ . Using the closed-form expressions for  $u_P^m$  and  $u_H^m$  from Theorem 4.1, the claim follows.  $\square$

Note that the utilities in the closest potential game consist of two parts: the term  $\Pi_m \phi$  expresses the preferences that are captured by the potential function  $\phi$ , and  $(I - \Pi_m) u^m$  corresponds to the nonstrategic component of the original game. Similarly, the closest harmonic game differs from the original game by its potential component, and hence has the same nonstrategic and harmonic components with the original game. This implies that the projection decomposes the flows generated by a game to its consistent and inconsistent components and is closely related to the decomposition of flows to the orthogonal subspaces of the space of flows provided in the Helmholtz decomposition.

Analyzing the projection of a game to the space of potential games may provide useful insights for the original game; see §7 for a description of ongoing and future work in this direction. We conclude this section by relating the approximate equilibria of a game to the equilibria of the closest potential game.

**THEOREM 6.3.** *Let  $\mathcal{G}$  be a game, and let  $\hat{\mathcal{G}}$  be its closest potential game. Assume that  $h_m$  denotes the number of strategies of player  $m$ , and define  $\alpha \triangleq \|\mathcal{G} - \hat{\mathcal{G}}\|_{\mathcal{M}, E}$ . Then, every  $\epsilon_1$ -equilibrium of  $\hat{\mathcal{G}}$  is an  $\epsilon$ -equilibrium of  $\mathcal{G}$  for some  $\epsilon \leq \max_m (2\alpha/\sqrt{h_m}) + \epsilon_1$  (and vice versa).*



PROOF. By the definition of the norm, it follows that  $|u^k(\mathbf{p}) - \hat{u}^k(\mathbf{p})| \leq (1/\sqrt{h_k})\|\mathcal{G} - \hat{\mathcal{G}}\|_{\mathcal{M}, E} \leq \max_m(\alpha/\sqrt{h_m})$ , for all  $k \in \mathcal{M}$ ,  $\mathbf{p} \in E$ . Using Lemma 2.1, the result follows.  $\square$

This result implies that the study and characterization of the structure of approximate equilibria in an arbitrary game can be facilitated by making use of the connection between its  $\epsilon$ -equilibrium set and the equilibria of its closest potential game.

**7. Conclusions.** We have introduced a novel and natural direct sum decomposition of the space of games into potential, harmonic, and nonstrategic subspaces. We studied the equilibrium properties of the subclasses of games induced by this decomposition, and showed that the potential and harmonic components of games have quite distinct and appealing equilibrium properties. In particular, there is a sharp contrast between potential games, that always have pure Nash equilibria, and harmonic games, that generically never do. Using the decomposition framework, we obtained closed-form expressions for the projections of games to their corresponding components, enabling the approximation of arbitrary games in terms of potential and harmonic games. This provides a systematic method for characterizing the set of  $\epsilon$ -equilibria by relating it to the equilibria of the closest potential game.

The framework provided in this paper opens up a number of interesting research directions, several of which we are currently investigating. Among them, we mention the following:

**Decomposition and dynamics.** An interesting application of our decomposition framework is in the analysis of adaptive player dynamics in games. For general games, various examples have been provided that show that adaptive dynamics do not necessarily converge to an equilibrium (Fudenberg and Levine [12], Jordan [24]). It is well known that in potential games many natural dynamics, such as best-response and fictitious play, converge to an equilibrium (Candogan et al. [4], Fudenberg and Levine [12], Hofbauer and Sandholm [21], Marden et al. [30], Marden and Shamma [29], Monderer and Shapley [34], Shamma and Arslan [41], Young [50]). The decomposition approach can be used to examine whether the convergence properties of dynamics in potential games can be extended to games that are “close” to potential games, thereby enhancing the predictability of player interactions. Intuitively, dynamics in potential games and dynamics in games that are close (in terms of the payoffs of the players) to potential games should be related. In our companion paper (Candogan et al. [4]), we make this intuition precise by establishing convergence of dynamics to “approximate” equilibrium sets in near-potential games. We use the potential component of the original game to approximately characterize its dynamic properties.

**Dynamics in harmonic games.** While the behavior of player dynamics in potential games is reasonably well understood, there seems to be a number of interesting research questions regarding their harmonic counterpart. In Candogan et al. [4], we made some partial progress in this direction, by showing that in harmonic games, the uniformly mixed strategy profile is the unique equilibrium of the continuous-time fictitious play dynamics and this equilibrium point is locally stable. The global stability of this equilibrium in two-player harmonic games (which are also zero-sum games) follows from known results on convergence of fictitious play dynamics in zero-sum games. Analyzing global stability of equilibrium under fictitious play dynamics for harmonic games with  $M > 2$  players and convergence of other adaptive dynamics are interesting research directions which we plan to pursue in future work.

**Game approximation.** The idea of analyzing an arbitrary game through a “nearby” game with tractable equilibrium properties seems to be a useful approach. In Candogan et al. [5], we have developed this methodology (“near-potential” games), and applied it in the context of pricing in a networking application. We believe that these techniques can be extended to other special classes of games. An extension to generalizations of potential games (weighted and ordinal potential games) was considered in Candogan et al. [3]. Another interesting direction is to study the proximity of an arbitrary game to supermodular games (Topkis [45]) and analyze how properties of supermodular games are inherited in “near-supermodular” games.

**Alternative projections.** In this work, the projections onto the spaces of potential and harmonic games are obtained using a weighted  $l_2$  norm. This norm leads to closed-form expressions for the components, but some problems may require or benefit from projections using different metrics. For instance, finding the closest potential game, by perturbing each of the pairwise comparisons in a minimal way requires a projection using a suitably defined  $\infty$ -norm, and this projection leads to better error bounds when analyzing approximate equilibria and dynamics. Projections under different norms and their properties are left for future research.

**Additional restrictions.** Another interesting extension is to study projections to subsets of potential games with additional properties. For example, in the current projection framework, if we require the potential function to be concave (see Ui [46] for a definition of discrete concavity), it may be possible to project a given game to a set of potential games with a unique Nash equilibrium.

**Appendix A. Additional proofs.** In this section we provide proofs to some of the results from §§4 and 5.

**PROOF OF LEMMAS 4.1 AND 4.2.** The proof relies on the fact that  $D_m^* D_m = \Delta_{0,m}$  is a Laplacian operator defined on the graph of  $m$ -comparable strategy profiles. We show that the kernels of  $D_m$  and  $\Delta_{0,m}$  coincide, and using the spectral properties of the Laplacian and projection matrices we obtain the desired result.

For a fixed  $m$ , it can be seen that strategy profile  $\mathbf{p} = (\mathbf{p}^m, \mathbf{p}^{-m})$  is comparable to strategy profiles  $(\mathbf{q}^m, \mathbf{p}^{-m})$  for all  $\mathbf{q}^m \in E^m$ ,  $\mathbf{q}^m \neq \mathbf{p}^m$ , but to none of the strategy profiles  $(\mathbf{q}^m, \mathbf{q}^{-m})$  for  $\mathbf{q}^{-m} \neq \mathbf{p}^{-m}$ . This implies that the graph over which  $\Delta_{0,m}$  is defined has  $|E^{-m}| = \prod_{k \neq m} h_k$  components (each  $\mathbf{p}^{-m} \in E^{-m}$  creates a different component), each of which has  $|E^m| = h_m$  elements. Note that all strategy profiles in a component are  $m$ -comparable, thus the underlying graph consists of  $|E^{-m}|$  components, each of which is a complete graph with  $|E^m|$  nodes.

The Laplacian of an unweighted complete graph with  $n$  nodes has eigenvalues 0 and  $n$ , where the multiplicity of nonzero eigenvalues is  $n - 1$  (Chung [8]). Each component of  $\Delta_{0,m}$  leads to eigenvalues 0 and  $h_m$  with multiplicities 1 and  $h_m - 1$ , respectively. Therefore,  $\Delta_{0,m}$  has eigenvalues 0 and  $h_m$  where the multiplicity of nonzero eigenvalues is  $(h_m - 1) \prod_{k \neq m} h_k = \prod_{k \in \mathcal{M}} h_k - \prod_{k \neq m} h_k$ . This suggests that the dimension of the kernel of  $\Delta_{0,m}$  is  $\prod_{k \neq m} h_k$ .

Observe that the kernel of  $\Delta_{0,m} = D_m^* D_m$  contains the kernel of  $D_m$ . For every  $\mathbf{q}^{-m} \in E^{-m}$  define  $\nu_{\mathbf{q}^{-m}} \in C_0$  such that

$$\nu_{\mathbf{q}^{-m}}(\mathbf{p}) = \begin{cases} 1 & \text{if } \mathbf{p}^{-m} = \mathbf{q}^{-m}, \\ 0 & \text{otherwise.} \end{cases} \quad (\text{A1})$$

It is easy to see that  $\nu_{\mathbf{p}^{-m}} \perp \nu_{\mathbf{q}^{-m}}$  for  $\mathbf{p}^{-m} \neq \mathbf{q}^{-m}$  and  $D_m \nu_{\mathbf{p}^{-m}} = 0$  for all  $\mathbf{p}^{-m} \in E^{-m}$ . Thus, for all  $\mathbf{q}^{-m}$ ,  $\nu_{\mathbf{q}^{-m}}$  belongs to the kernel of  $D_m$ , and by mutual orthogonality of these functions, the kernel of  $D_m$  has dimension at least  $|E^{-m}| = \prod_{k \neq m} h_k$ . As the dimension of the kernel of  $\Delta_{0,m}$  is  $\prod_{k \neq m} h_k$  and it contains the kernel of  $D_m$ , this implies that the kernels of  $D_m$  and  $\Delta_{0,m}$  coincide.

Thus  $\Delta_{0,m}$  maps any  $\nu \in C_0$  in the kernel of  $D_m$  to zero and scales the  $\nu$  in the orthogonal complement of the kernel by  $h_m$ . On the other hand,  $D_m^* D_m$  is a projection operator and it has eigenvalue 0 for all functions in the kernel of  $D_m$  and 1 for the functions in the orthogonal complement of the kernel of  $D_m$ . This implies that

$$\Delta_{0,m} = h_m D_m^* D_m, \quad (\text{A2})$$

and the kernels of  $\Pi_m$ ,  $D_m$ , and  $\Delta_{0,m}$  coincide as the claim suggests.  $\square$

**PROOF OF LEMMA 4.3.** For a game, the graph of comparable strategy profiles is connected as can be seen from the definition of the comparable strategy profiles. It is known that for a connected graph, the Laplacian operator has multiplicity 1 for eigenvalue 0 (Chung [8]). By (16) it follows that the function  $f \in C_0$  satisfying  $f(\mathbf{p}) = 1$  for all  $\mathbf{p} \in E$ , is an eigenfunction of  $\Delta_0$  with eigenvalue 0, implying the result.  $\square$

**PROOF OF LEMMA 4.4.** For the proof of this lemma, we use the following property of the pseudoinverse

$$A^\dagger = (A^* A)^\dagger A^*, \quad (\text{A3})$$

and the orthogonality properties of the  $D_m$  operators:  $D_m^* D_k = 0$  and  $D_m^\dagger D_k = 0$  if  $m \neq k$ .

(i) Using (A3) with  $A = D_m$  implies that  $D_m^\dagger = (D_m^* D_m)^\dagger D_m^*$ . Since for any linear operator  $L$ ,  $(L^\dagger)^* = (L^*)^\dagger$ , it follows that  $D_m^\dagger = (D_m (D_m^* D_m)^\dagger)^* = (D_m (\Delta_{0,m})^\dagger)^*$ . Hence, using Lemma 4.1 we obtain  $D_m^\dagger = h_m (D_m (\Pi_m)^\dagger)^*$ . Since  $\Pi_m$  is a projection operator to the orthogonal complement of the kernel of  $D_m$ , we have  $\Pi_m^\dagger = \Pi_m$ , and  $D_m \Pi_m = D_m$ . Hence, it follows that  $D_m^\dagger = h_m (D_m \Pi_m)^* = h_m D_m^*$  as claimed.

(ii) The identity in (A3) implies that

$$\left( \sum_{i \in \mathcal{M}} D_i \right)^\dagger = \left( \left( \sum_{i \in \mathcal{M}} D_i \right)^* \left( \sum_{i \in \mathcal{M}} D_i \right) \right)^\dagger \left( \sum_{i \in \mathcal{M}} D_i \right)^*.$$

By the orthogonality of the image spaces of the  $D_i$ , it follows that  $(\sum_{i \in \mathcal{M}} D_i)^* (\sum_{i \in \mathcal{M}} D_i) = \sum_{i \in \mathcal{M}} D_i^* D_i$ , and hence

$$\left( \sum_{i \in \mathcal{M}} D_i \right)^\dagger = \left( \sum_{i \in \mathcal{M}} D_i^* D_i \right)^\dagger \left( \sum_{i \in \mathcal{M}} D_i \right)^*.$$

Right-multiplying the above equation by  $D_j$  and using the orthogonality of the image spaces of the  $D_i$ 's it follows that

$$\left( \sum_{i \in \mathcal{M}} D_i \right)^\dagger D_j = \left( \sum_{i \in \mathcal{M}} D_i^* D_i \right)^\dagger \left( \sum_{i \in \mathcal{M}} D_i \right)^* D_j = \left( \sum_{i \in \mathcal{M}} D_i^* D_i \right)^\dagger D_j^* D_j.$$

(iii) From the definition of pseudoinverse, it is sufficient to show the following four properties to prove the claim: (a)  $(DD^\dagger)^* = DD^\dagger$ , (b)  $(D^\dagger D)^* = D^\dagger D$ , (c)  $DD^\dagger D = D$ , (d)  $D^\dagger DD^\dagger = D^\dagger$ .

Using the identity  $D_m^\dagger D_k = 0$  for  $k \neq m$ , it follows that  $DD^\dagger = \sum_{m \in \mathcal{M}} D_m D_m^\dagger$ , and  $D^\dagger D = \text{diag}(D_1^\dagger D_1, \dots, D_M^\dagger D_M)$ . The pseudoinverse of  $D_m$  satisfies the properties  $D_m^\dagger D_m = (D_m^\dagger D_m)^*$  and  $D_m D_m^\dagger = (D_m D_m^\dagger)^*$ , and the requirements (a) and (b) follow immediately using these properties. The identity  $D_m^\dagger D_k = 0$  also implies that  $DD^\dagger D = [D_1 D_1^\dagger D_1, \dots, D_M D_M^\dagger D_M]$ , and  $D^\dagger DD^\dagger = [D_1^\dagger D_1 D_1^\dagger; \dots; D_M^\dagger D_M D_M^\dagger]$ . Since the pseudoinverse of  $D_m$  also satisfies  $D_m^\dagger D_m D_m^\dagger = D_m^\dagger$ , and  $D_m D_m^\dagger D_m = D_m$ , the requirements (c) and (d) are satisfied and the claim follows.

(iv) Since  $\Pi = \text{diag}(\Pi_1, \dots, \Pi_M)$ , and  $\Pi_m = D_m^\dagger D_m$ , it follows that

$$D^\dagger D = \text{diag}(D_1^\dagger D_1, \dots, D_M^\dagger D_M) = \text{diag}(\Pi_1, \dots, \Pi_M) = \Pi.$$

(v) Using the identities  $D_m^\dagger D_k = 0$  for  $k \neq m$ ,  $\delta_0 = \sum_{m \in \mathcal{M}} D_m$ , it follows that

$$DD^\dagger \delta_0 = DD^\dagger \sum_{m \in \mathcal{M}} D_m = \sum_{m \in \mathcal{M}} D_m D_m^\dagger D_m = \sum_{m \in \mathcal{M}} D_m = \delta_0. \quad \square$$

PROOF OF PROPOSITION 4.1. Lemma 4.2 provides a basis for the kernel of  $D_m$  and  $\dim(\ker(D_m)) = |E^{-m}|$ ; i.e., the cardinality of the basis is equal to  $|E^{-m}|$ . By definition,  $\mathcal{N} = \ker D = \prod_{m \in \mathcal{M}} \ker(D_m)$ ; hence,

$$\dim(\mathcal{N}) = \sum_{m \in \mathcal{M}} \dim(\ker(D_m)) = \sum_{m \in \mathcal{M}} |E^{-m}| = \sum_{m \in \mathcal{M}} \prod_{k \neq m} h_k. \quad (\text{A4})$$

Next consider the subspace  $\mathcal{P}$  of normalized potential games. By definition, the games in this set generate globally consistent flows. Moreover, by Lemma 4.6 it follows that there is a unique game in  $\mathcal{P}$ , which generates a given gradient flow. Thirdly, note that any globally consistent flow can be obtained as  $\delta_0 \phi$  for some  $\phi \in C_0$ , and the game  $\{\Pi_m \phi\}_{m \in \mathcal{M}} \in \mathcal{P}$  generates the same flows as  $\delta_0 \phi$ . These three facts imply that there is a linear bijective mapping between the games in  $\mathcal{P}$  and the globally consistent flows, and hence the dimension of  $\mathcal{P}$  is equal to the dimension of the globally consistent flows.

On the other hand, the dimension of the globally consistent flows is equivalent to  $\dim(\text{im}(\delta_0))$ . Since  $\Delta_0 = \delta_0^* \delta_0$  it follows that  $\ker(\delta_0) \subset \ker(\Delta_0)$ . By Lemma 4.3 it follows that  $\ker(\Delta_0) = \{f \in C_0 \mid f(\mathbf{p}) = c \in \mathbb{R}, \text{ for all } \mathbf{p} \in E\}$ . It follows from the definition of  $\delta_0$  that  $\delta_0 f = 0$  for all  $f \in \ker(\Delta_0)$ . These facts imply that  $\ker(\delta_0) = \ker(\Delta_0)$ , and hence  $\dim(\ker(\delta_0)) = 1$ . Since  $\delta_0$  is a linear operator it follows that  $\dim(\text{im}(\delta_0)) = \dim(C_0) - \dim(\ker(\delta_0)) = |E| - 1 = \prod_{m \in \mathcal{M}} h_m - 1$ .

Finally, observe that  $\dim(\mathcal{G}_{\mathcal{M}, E}) = \dim(C_0^M) = M|E| = M \prod_{m \in \mathcal{M}} h_m$ . Theorem 4.1 implies that  $\dim(\mathcal{G}_{\mathcal{M}, E}) = \dim(\mathcal{P}) + \dim(\mathcal{H}) + \dim(\mathcal{N})$ . Therefore, it follows that  $\dim(\mathcal{H}) = (M-1) \prod_{m \in \mathcal{M}} h_m - \sum_{m \in \mathcal{M}} \prod_{k \neq m} h_k + 1$ .  $\square$

PROOF OF LEMMA 5.2. Let  $X = Du$  denote the pairwise comparison function of the harmonic game. By definition,  $(\delta_0^* X)(\mathbf{p}) = 0$  for all  $\mathbf{p} \in E$ . Thus, for all  $\mathbf{r}^m \in E^m$ , it follows that

$$0 = \sum_{\mathbf{p}^{-m} \in E^{-m}} (\delta_0^* X)(\mathbf{r}^m, \mathbf{p}^{-m}) = \sum_{\mathbf{p} \in S} (\delta_0^* X)(\mathbf{p}), \quad (\text{A5})$$

where  $S = \{(\mathbf{r}^m, \mathbf{p}^{-m}) \mid \mathbf{p}^{-m} \in E^{-m}\}$ . To complete the proof we require the following identity related to the pairwise comparison functions.

LEMMA A.1. For all  $\hat{X} \in C_1$  and set of strategy profiles  $\hat{S} \subset E$ ,  $\sum_{\mathbf{p} \in \hat{S}} (\delta_0^* \hat{X})(\mathbf{p}) = -\sum_{\mathbf{p} \in \hat{S}} \sum_{\mathbf{q} \in \hat{S}^c} \hat{X}(\mathbf{p}, \mathbf{q})$ .

PROOF. It follows from the definition of  $\delta_0^*$  that

$$\sum_{\mathbf{p} \in \hat{S}} (\delta_0^* \hat{X})(\mathbf{p}) = -\sum_{\mathbf{p} \in \hat{S}} \sum_{\mathbf{q} \in E} \hat{X}(\mathbf{p}, \mathbf{q}) = -\sum_{\mathbf{p} \in \hat{S}} \sum_{\mathbf{q} \in \hat{S}^c} \hat{X}(\mathbf{p}, \mathbf{q}) - \sum_{\mathbf{p} \in \hat{S}} \sum_{\mathbf{q} \in \hat{S}} \hat{X}(\mathbf{p}, \mathbf{q}) = -\sum_{\mathbf{p} \in \hat{S}} \sum_{\mathbf{q} \in \hat{S}^c} \hat{X}(\mathbf{p}, \mathbf{q}),$$

since  $\hat{X}(\mathbf{p}, \mathbf{q}) + \hat{X}(\mathbf{q}, \mathbf{p}) = 0$  for any  $\mathbf{p}, \mathbf{q}$  and thus  $\sum_{\mathbf{p} \in \hat{S}} \sum_{\mathbf{q} \in \hat{S}} \hat{X}(\mathbf{p}, \mathbf{q}) = 0$ .  $\square$

Using this lemma in (A5), we obtain

$$0 = -\sum_{\mathbf{p} \in S} \sum_{\hat{\mathbf{p}} \in \hat{S}^c} X(\mathbf{p}, \hat{\mathbf{p}}) = \sum_{\mathbf{p}^{-m} \in E^{-m}} \sum_{\mathbf{p}^m \in E^m} u^m(\mathbf{r}^m, \mathbf{p}^{-m}) - u^m(\mathbf{p}^m, \mathbf{p}^{-m}). \quad (\text{A6})$$

Since  $\mathbf{r}^m$  is arbitrary, it follows that  $\sum_{\mathbf{p}^{-m} \in E^{-m}} u^m(\mathbf{q}^m, \mathbf{p}^{-m}) - u^m(\mathbf{r}^m, \mathbf{p}^{-m}) = 0$  for all  $\mathbf{q}^m, \mathbf{r}^m \in E^m$ .

PROOF OF THEOREM 5.7. Since  $\mathcal{L} \oplus \mathcal{F} = \mathcal{G}_{\mathcal{M}, E}$ , the last column immediately follows from Proposition 5.1 and Theorem 4.1. Below, we present the dimension results for each row of the table and the corresponding entries in the first two columns.

Throughout the proof we denote by  $e$  the  $h$ -dimensional vector of ones. Since  $\mathcal{N} = \ker D$ , the two-player games in  $\mathcal{N}$  take the form  $(ea^T, be^T)$  for some  $a, b \in \mathbb{R}^h$ . We shall make use of this fact in the proof.

$\mathcal{P} \oplus \mathcal{H} \oplus \mathcal{N}$ : Since  $\mathcal{P} \oplus \mathcal{H} \oplus \mathcal{N} = \mathcal{G}_{\mathcal{M}, E}$ , it follows that  $\dim(\mathcal{P} \oplus \mathcal{H} \oplus \mathcal{N}) \cap \mathcal{L} = \dim \mathcal{L}$  and  $\dim(\mathcal{P} \oplus \mathcal{H} \oplus \mathcal{N}) \cap \mathcal{F} = \dim \mathcal{F}$ . Zero-sum games are games with payoff matrices  $(A, -A)$  for some  $A \in \mathbb{R}^{h \times h}$ . Thus, the dimension of the zero-sum games is equivalent to the dimension of possible  $A$  matrices that define zero-sum games, and hence  $\dim \mathcal{L} = h^2$ . Similarly, identical interest games are games with payoff matrices  $(A, A)$  for some  $A \in \mathbb{R}^{h \times h}$ , and hence  $\dim \mathcal{F} = h^2$ .

$\mathcal{P} \oplus \mathcal{N}$ : By Theorem 5.1, it follows that  $\mathcal{P} \oplus \mathcal{N}$  is equivalent to the set of potential games. Observe that all identical interest games are potential games, where the utility functions of players are equal to the potential function of the game. Thus, it follows that  $\dim(\mathcal{P} \oplus \mathcal{N}) \cap \mathcal{F} = \dim \mathcal{F} = h^2$ .

Let  $\mathcal{G}$  denote a zero-sum game in  $\mathcal{P} \oplus \mathcal{N}$ , with payoff matrices  $(A, -A)$ , and denote the matrix corresponding to a potential function of  $\mathcal{G}$  by  $\phi$ . Thus, both the game with payoffs  $(A, -A)$  and  $(\phi, \phi)$  belong to  $\mathcal{N} \oplus \mathcal{P}$ , and  $(A, -A)$  is different from  $(\phi, \phi)$  by its nonstrategic component. Hence, for some  $a, b \in \mathbb{R}^h$ ,  $A = \phi + ea^T$ ,  $-A = \phi + be^T$ , for some  $a, b \in \mathbb{R}^h$  and

$$A - A = \phi + ea^T + \phi + be^T = 2\phi + ea^T + be^T = 0;$$

thus  $-2\phi_{ij} = a_j + b_i$  and  $A_{ij} = \phi_{ij} + a_j = (a_j - b_i)/2$  for all  $i, j \in \{1 \dots n\}$ . Hence,  $a, b \in \mathbb{R}^h$  characterize the possible payoff matrices  $A$ , and it can be seen that the set of these matrices has dimension  $2h - 1$ . Since these matrices uniquely characterize zero-sum games that are also potential games, it follows that the dimension of  $(\mathcal{P} \oplus \mathcal{N}) \cap \mathcal{L}$  is equal to  $2h - 1$ .

$\mathcal{H} \oplus \mathcal{N}$ : The games in this set do not have potential components. If a game in  $\mathcal{H} \oplus \mathcal{N}$  is an identical interest game, then it also belongs to  $\mathcal{P} \oplus \mathcal{N}$ . Due to the direct sum property of  $\mathcal{P} \oplus \mathcal{H} \oplus \mathcal{N}$ , it follows that this game can only have nonstrategic component. Therefore,  $\dim(\mathcal{H} \oplus \mathcal{N}) \cap \mathcal{F} = \dim \mathcal{N} \cap \mathcal{F}$ . Let  $\mathcal{G}$  denote a game in  $\mathcal{N} \cap \mathcal{F}$ . Since  $\mathcal{G}$  has only nonstrategic information it follows that its payoffs are given by  $(ea^T, be^T)$ , for some vectors  $a$  and  $b$ . Then, being an identical interest game implies that  $ea^T = be^T$ , which requires that all entries of payoff matrices are identical; thus  $\dim \mathcal{N} \cap \mathcal{F} = 1$ .

Consider a zero-sum game in  $\mathcal{H} \oplus \mathcal{N}$  with payoff matrices  $(A, -A)$ . Since both players have equal numbers of strategies, the harmonic component of this game is also zero-sum and the payoff matrices in the harmonic component can be denoted by  $(A_H, -A_H)$  for some  $A_H \in \mathbb{R}^{h \times h}$ . Because the original game is in  $\mathcal{H} \oplus \mathcal{N}$ , the payoff matrices satisfy  $A = A_H + ea^T$ ,  $-A = -A_H + be^T$ , where  $(ea^T, be^T)$  corresponds to the nonstrategic component of the game. It follows that  $ea^T + be^T = 0$ , and hence  $ea^T$  and  $-be^T$  are matrices, which have all of their entries identical. Thus, the nonstrategic component of the games in  $(\mathcal{H} \oplus \mathcal{N}) \cap \mathcal{L}$ , forms a one-dimensional subspace. Since the harmonic component is arbitrary, it follows that  $\dim(\mathcal{H} \oplus \mathcal{N}) \cap \mathcal{L} = \dim \mathcal{H} + 1 = (h - 1)^2 + 1 = h^2 - 2h + 2$ .  $\square$

PROOF OF PROPOSITION 5.2. We prove the claim by first showing (i) and (ii) are equivalent and then establishing the equivalence (ii) and (iii).

By the definition of correlated equilibrium, (36) implies that  $x$  is a correlated equilibrium. To see that any correlated equilibrium of  $\mathcal{G}$  satisfies (36), assume  $x \in \Delta E$  is a correlated equilibrium. Since the game is a harmonic game, by definition, the utility functions  $u = \{u^m\}$  satisfy the condition  $\delta_0^* Du = 0$ . Using (13) and (21), this condition can equivalently be expressed as

$$\sum_{m \in \mathcal{M}} \sum_{\mathbf{q}^m \in E^m} u^m(\mathbf{q}^m, \mathbf{p}^{-m}) - u^m(\mathbf{p}^m, \mathbf{p}^{-m}) = 0 \quad \text{for all } \mathbf{p} \in E. \quad (\text{A7})$$

Thus, it follows that

$$\begin{aligned} 0 &= \sum_{\mathbf{p} \in E} x(\mathbf{p}) \sum_{m \in \mathcal{M}} \sum_{\mathbf{q}^m \in E^m} u^m(\mathbf{q}^m, \mathbf{p}^{-m}) - u^m(\mathbf{p}^m, \mathbf{p}^{-m}) \\ &= \sum_{m \in \mathcal{M}} \sum_{\mathbf{q}^m \in E^m} \sum_{\mathbf{p}^m \in E^m} \sum_{\mathbf{p}^{-m} \in E^{-m}} x(\mathbf{p}^m, \mathbf{p}^{-m}) (u^m(\mathbf{q}^m, \mathbf{p}^{-m}) - u^m(\mathbf{p}^m, \mathbf{p}^{-m})). \end{aligned} \quad (\text{A8})$$

Since  $x$  is a correlated equilibrium,  $\sum_{\mathbf{p}^{-m} \in E^{-m}} x(\mathbf{p}^m, \mathbf{p}^{-m}) (u^m(\mathbf{q}^m, \mathbf{p}^{-m}) - u^m(\mathbf{p}^m, \mathbf{p}^{-m})) \leq 0$  for all  $\mathbf{p}^m, \mathbf{q}^m$  and  $m \in \mathcal{M}$ . Hence, (A8) implies that

$$\sum_{\mathbf{p}^{-m} \in E^{-m}} x(\mathbf{p}^m, \mathbf{p}^{-m}) (u^m(\mathbf{q}^m, \mathbf{p}^{-m}) - u^m(\mathbf{p}^m, \mathbf{p}^{-m})) = 0$$

for all  $\mathbf{p}^m, \mathbf{q}^m$  and  $m \in \mathcal{M}$ . Thus, we conclude that (i) and (ii) are equivalent.

To see the equivalence of (ii) and (iii), observe that (iii) immediately implies (ii). Assume (ii) holds. Then writing (36) for two strategies  $\mathbf{r}^m, \mathbf{q}^m \in E^m$ , and subtracting these equations from each other, it follows that  $\sum_{\mathbf{p}^{-m} \in E^{-m}} (u^m(\mathbf{r}^m, \mathbf{p}^{-m}) - u^m(\mathbf{q}^m, \mathbf{p}^{-m}))x(\mathbf{p}^m, \mathbf{p}^{-m}) = 0$ . Since  $\mathbf{r}^m$  and  $\mathbf{q}^m$  are arbitrary it follows that for all  $\mathbf{q}^m \in E^m$ ,

$$\sum_{\mathbf{p}^{-m} \in E^{-m}} u^m(\mathbf{q}^m, \mathbf{p}^{-m})x(\mathbf{p}^m, \mathbf{p}^{-m}) = c_{\mathbf{p}^m}, \quad (\text{A9})$$

for some  $c_{\mathbf{p}^m} \in \mathbb{R}$ . Since the game is normalized, we have  $\sum_{\mathbf{q}^m \in E^m} u^m(\mathbf{q}^m, \mathbf{p}^{-m}) = 0$ . Thus summing (A9) over  $\mathbf{q}^m \in E^m$  it follows that  $c_{\mathbf{p}^m} = 0$ , and hence (ii) implies (iii).

Therefore we conclude that (i), (ii), and (iii) are equivalent for normalized harmonic games.  $\square$

Note that in the above proof, we used the assumption that the game is normalized only when establishing the equivalence of (ii) and (iii). Therefore, it can be seen that (i) and (ii) are equivalent for all harmonic games.

**Appendix B. Equilibria in two-player harmonic games.** In this section we focus on the equilibrium properties of two-player harmonic games. We show that sets of correlated and mixed equilibria are identical in these games.

In the following theorem, we present a basis for two-player normalized harmonic games. The idea behind our construction is to obtain a collection of games in which both players have “effectively” two strategies (the payoffs are equal to zero, if other strategies are played), and ensure that they are linearly independent normalized harmonic games.

**THEOREM B.1.** *Consider the set of two-player games where the first player has  $h_1$  strategies and the second player has  $h_2$  strategies. For any  $i \in \{1, \dots, h_1 - 1\}$  and  $j \in \{1, \dots, h_2 - 1\}$ , define bimatrix games  $\mathcal{G}^{ij}$ , with payoff matrices  $(h_2 A^{ij}, -h_1 A^{ij})$ , where  $A^{ij} \in \mathbb{R}^{h_1 \times h_2}$  is such that*

$$A_{kl}^{ij} = \begin{cases} 1 & \text{if } (k, l) = (i, j) \text{ or } (k, l) = (i + 1, j + 1), \\ -1 & \text{if } (k, l) = (i + 1, j) = (k, l) \text{ or } (k, l) = (i, j + 1), \\ 0 & \text{otherwise.} \end{cases} \quad (\text{B1})$$

*The collection  $\{\mathcal{G}^{ij}\}$  provides a basis of  $\mathcal{H}$ .*

**PROOF.** It can be seen that each  $\mathcal{G}^{ij}$  is normalized, since row and column sums of  $A^{ij}$  are equal to zero. By Theorem 5.6 and (B1), it also follows that  $\mathcal{G}^{ij}$  belongs to  $\mathcal{H}$ . It can be seen from Proposition 4.1 that  $\dim \mathcal{H} = (h_1 - 1)(h_2 - 1)$  is equal to the cardinality of the collection  $\{\mathcal{G}^{ij}\}$ . Thus, to prove the claim, it is sufficient to prove that

$$\sum_{i \in \{1, \dots, h_1 - 1\}} \sum_{j \in \{1, \dots, h_2 - 1\}} \alpha_{ij} A^{ij} = 0, \quad (\text{B2})$$

only if  $\alpha_{ij} = 0$  for all  $i, j$ .

Note that  $A^{11}$  is the only matrix that has a nonzero entry in the first column and the first row. Thus, (B2) implies that  $\alpha_{11} = 0$ . Similarly it can be seen that  $A^{11}$  and  $A^{12}$  are the only matrices which have nonzero entries in the first row and the second column; thus  $\alpha_{12} = 0$ . Proceeding iteratively it follows that if (B2) holds, then  $\alpha_{ij} = 0$  for all  $i, j$ , and the claim follows.  $\square$

The next example uses the basis introduced above to show that in harmonic games, the uniformly mixed strategy profile is not necessarily the unique mixed Nash equilibrium.

**EXAMPLE B.1.** In this example we consider two-player harmonic games, where  $E^1 = \{x, y\}$  and  $E^2 = \{a, b, c\}$ . Using Theorem B.1, a basis for normalized two-player harmonic games is given in Tables B.1(a) and B.1(b). Thus, any harmonic game with these strategy sets, can be expressed as in Table B.1(c). Consider some fixed  $\alpha$  and  $\beta$ . As can be seen from Definition 5.1, the mixed equilibria for this game are given by

$$\left(\frac{1}{2}, \frac{1}{2}\right) \times (\theta_1, \theta_2, \theta_3),$$

where  $\theta_1, \theta_2$ , and  $\theta_3$  are scalars that satisfy  $\theta_1 + \theta_2 + \theta_3 = 1$ ,  $\theta_1, \theta_2, \theta_3 \geq 0$ , and  $\theta_1(6\alpha) + \theta_2(-6\alpha + 6\beta) + \theta_3(-6\beta) = 0$ . Note that since there are two linear equations in three variables, this system has a continuum of solutions. Moreover, since  $(\theta_1, \theta_2, \theta_3) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$  is a solution, it follows that there is a continuum of solutions for which  $\theta_1, \theta_2, \theta_3 \geq 0$ .

Since this is true for any  $\alpha, \beta$ , we conclude that all games in  $\mathcal{H}$  have uncountably many mixed equilibria. Additionally, since the nonstrategic component does not affect the equilibrium properties of a game, it follows that all harmonic games on  $E^1 \times E^2$  (all games in  $\mathcal{H} \oplus \mathcal{N}$ ) have uncountably many mixed Nash equilibria.



TABLE B.1. Basis of  $\mathcal{H}$ .

	$a$	$b$	$c$
(a) Basis element 1			
$x$	$3, -2$	$-3, 2$	$0, 0$
$y$	$-3, 2$	$3, -2$	$0, 0$
(b) Basis element 2			
$x$	$0, 0$	$3, -2$	$-3, 2$
$y$	$0, 0$	$-3, 2$	$3, -2$
(c) A game in $\mathcal{H}$			
$x$	$3\alpha, -2\alpha$	$-3\alpha + 3\beta, 2\alpha - 2\beta$	$-3\beta, 2\beta$
$y$	$-3\alpha, 2\alpha$	$3\alpha - 3\beta, -2\alpha + 2\beta$	$3\beta, -2\beta$

Using this basis, we characterize in the following theorem, the correlated equilibria in two-player harmonic games. Interestingly, our results suggest that in two-player harmonic games, the set of mixed Nash equilibria and correlated equilibria generically coincide.

**THEOREM B.2.** *Consider the set of two-player harmonic games where the first player has  $h_1$  strategies and the second player has  $h_2$  strategies. Without loss of generality assume that  $h_1 \geq h_2$ . Generically,*

- (i) *Every correlated equilibrium is a mixed Nash equilibrium, where the player with minimum number of strategies uses the uniformly mixed strategy.*
- (ii) *The dimension of the set of correlated equilibria is  $h_1 - h_2$ .*

**PROOF.** As discussed earlier, nonstrategic components of games do not affect the equilibrium sets. Thus, to prove that (i) and (ii) are generically true for harmonic games, it is sufficient to prove that they generically hold for normalized harmonic games.

Consider a two-player normalized harmonic game with payoff matrices  $(A, B)$ , where  $A, B \in \mathbb{R}^{h_1 \times h_2}$ . By Theorem 5.6, it follows that  $A = -(h_2/h_1)B$ . Denote by  $e_1$  (similarly  $e_2$ ), the  $h_1$  (similarly  $h_2$ ) dimensional vector, all entries of which are identically equal to 1. Since the game is normalized, it follows that  $e_1^T A = 0$  and  $B e_2 = -(h_1/h_2)A e_2 = 0$ .

Let  $x$  be a correlated equilibrium of this game. For each  $\mathbf{p}^1 \in E^1$ , denote by  $x(\mathbf{p}^1, \cdot) \in \mathbb{R}^{h_2}$  the vector of probabilities  $[x(\mathbf{p}^1, \mathbf{p}^2)]_{\mathbf{p}^2}$ . By Proposition 5.2(iii), it follows that these vectors satisfy the condition

$$Ax(\mathbf{p}^1, \cdot) = 0. \quad (\text{B3})$$

Note that we need to characterize the kernel of the payoff matrix  $A$ , to identify the correlated equilibria. For that reason, we state the following technical lemma:

**LEMMA B.1.** *Consider the set of normalized harmonic games in Theorem B.2. Generically, the payoff matrices of players have their row and column ranks equal to  $h_2 - 1$ .*

**PROOF.** The payoff matrices of the players satisfy  $A = -h_2/h_1 B$ , so they have the same row and column rank. It follows from Theorem B.1 that the collection of matrices  $\{A^{ij}\}$  span the payoff matrices of harmonic games. It can be seen that the matrices in the span of this collection generically have row and column rank equal to  $h_2 - 1$ , and the claim follows.  $\square$

Using this lemma, it follows that generically the kernel of  $A$  is one-dimensional. As shown earlier,  $e_2$  is in the kernel of  $A$ ; thus (B3) implies that generically  $x(\mathbf{p}^1, \cdot)$  has the form  $x(\mathbf{p}^1, \cdot) = c_{\mathbf{p}^1} e_2$ , for some  $c_{\mathbf{p}^1} \in \mathbb{R}$ . Since  $x$  is a probability distribution, the definition of  $x(\mathbf{p}^1, \cdot)$  implies that  $x(\mathbf{p}^1, \mathbf{p}^2) = c_{\mathbf{p}^1} \geq 0$ , and  $\sum_{\mathbf{p}^1 \in E^1} \sum_{\mathbf{p}^2 \in E^2} x(\mathbf{p}^1, \mathbf{p}^2) = h_2 \sum_{\mathbf{p}^1 \in E^1} c_{\mathbf{p}^1} = 1$ . Thus, it follows that  $x(\mathbf{p}^1, \mathbf{p}^2) = c_{\mathbf{p}^1} = \alpha_{\mathbf{p}^1}/h_2$ , for some  $\alpha_{\mathbf{p}^1} \geq 0$  such that  $\sum_{\mathbf{p}^1 \in E^1} \alpha_{\mathbf{p}^1} = 1$ . It can be seen from this description that generically, the correlated equilibria are mixed equilibria where the first player uses the probability distribution  $x^1 = \alpha \triangleq [\alpha_{\mathbf{p}^1}]_{\mathbf{p}^1} \in \Delta E^1$  and the second player uses the distribution  $x^2 = [1/h_2]_{\mathbf{p}^2}$ .

Since the correlated equilibria have this form, it can be seen using Proposition 5.2(iii) for the second player that

$$\sum_{\mathbf{p}^1 \in E^1} u^2(\mathbf{p}^1, \mathbf{q}^2) x(\mathbf{p}^1, \mathbf{p}^2) = \frac{1}{h_2} \sum_{\mathbf{p}^1 \in E^1} u^2(\mathbf{p}^1, \mathbf{q}^2) \alpha_{\mathbf{p}^1} = 0, \quad (\text{B4})$$

where  $\alpha \in \Delta E^1$ . The above condition can be restated using the payoff matrices as follows:

$$\alpha^T B = -\frac{h_1}{h_2} \alpha^T A = 0, \quad (\text{B5})$$

where  $\alpha \in \Delta E^1$ . Since the row rank of  $A$  is  $h_2 - 1$ , the dimension of  $\alpha$  that satisfies (B5) is  $h_1 - h_2 + 1$ . Note that since  $\alpha$  is a probability distribution, it also satisfies the condition  $\alpha^T e_1 = 1$ . Note that since  $e_1^T A = 0$ , this condition is orthogonal to the ones in (B5). Hence, it follows that the dimension of  $\alpha$  that satisfies the correlated equilibrium conditions in (B5) (other than the positivity) is  $h_1 - h_2$ . On the other hand,  $\alpha = (1/h_1)e_1$  gives a correlated equilibrium (by Theorem 5.5); thus the positivity condition does not change the dimension of the set of correlated equilibria, and the dimension is generically  $h_1 - h_2$ .  $\square$

An immediate implication of this theorem is the following.

**COROLLARY B.1.** *In two-player harmonic games where players have equal numbers of strategies, the profile of uniformly mixed strategies is generically the unique correlated equilibrium.*

Note that Theorem B.2 implies that in two-player harmonic games, generically there are no correlated equilibria that are not mixed equilibria. This statement fails when the number of players is more than two, as shown in the following theorem.

**THEOREM B.3.** *Consider a  $M$ -player harmonic game, where  $M > 2$ , and in which each player has  $h$  strategies such that  $h^M > M(h^2 - 1) + 1$ . The set of correlated equilibria is strictly larger than the set of mixed Nash equilibria: The set of correlated equilibria has dimension at least  $h^M - 1 - Mh(h - 1)$ , and the set of mixed equilibria has dimension at most  $M(h - 1)$ .*

**PROOF.** Since each player has  $h$  strategies, the set of mixed strategies has dimension  $M(h - 1)$ , and this is a trivial upper bound on the dimension of the set of mixed equilibria. The set of correlated equilibria, on the other hand, is defined by the equalities in Proposition 5.2. Note that there are  $Mh(h - 1)$  such equalities and the dimension of  $\Delta E$  is  $h^M - 1$ ; hence the dimension of the correlated equilibria is at least  $h^M - 1 - Mh(h - 1)$  (by ignoring possible dependence of the equalities).

The difference in the dimensions implies that the set of correlated equilibria is strictly larger than the set of mixed equilibria.  $\square$

Note that this theorem can be easily generalized to the case when players have different numbers of strategies. An interesting problem is to find the exact dimensions of the set of mixed Nash and correlated equilibria when there are more than two players. However, due to complicated dependence relations of the correlated equilibrium conditions in Proposition 5.2, we do not pursue this question in this paper, and leave it as a future problem.

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