MoPuMMAM Lecture Notes: Cohomology for Dummies

A glimpse at the beating heart of modern pure math

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Abstract

In the 1890s, two European mathematicians invented two new approaches to two old problems that *seemed* to have nothing in common, namely:

(1) Quantify mathematically the difference between a beach ball and a donut.

(2) Count the number of 7th-degree polynomials $P(x, y)$ that vanish at three given points $\{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$.

Although it was far from obvious at the time, Henri Poincaré's solution to problem **(1)** ultimately turned out to be closely related to David Hilbert's solution to problem **(2)**: as would become clear in the ensuing decades, both were applications of a common strategy we might roughly characterize as "analyzing complicated objects by decomposing them into simple pieces"—using specific, well-defined algorithms—and the same basic strategy turned out to be useful for attacking tough math problems in several other fields.

When mathematicians in the early and mid-20th century started to figure this out, they got *really* excited—so much so that a certain band of mid-20th-century mathematicians set about promoting these ideas, and a new technical jargon for discussing them, to the status of an independent branch of mathematics in its own right. This new subdiscipline—*homological algebra*—together with the closely related subject of *category theory*, developed around the same time by many of the same people, aspired to amalgamate techniques from disparate branches of mathematics into a single conceptual framework, with the idea being that it's wasteful for mathematicians working in different fields (say, number theory and differential geometry) to develop separate methods and write separate proofs for theorems that are ultimately saying the same thing.

The success of this project to define the style of modern pure mathematics was near total, and today most research papers, seminars, and

textbooks in pure math assume considerable familiarity with category-theoretical reasoning and language. Perhaps this helps pure mathematicians working in diverse fields communicate with each other more effectively than they could without the specialized dialect. If so, however, it comes at the cost of largely excluding the *applied*-math and science and engineering communities, as the new language—with its bandying about of abstruse terms like *derived functors* and *perverse sheaves* and *cohomology*—inevitably seems somewhat private and forbidding to those of us who weren't raised on it.

The goal of these notes is to attempt to break down some of these barriers by presenting some of the central ideas of homological algebra, motivated by some of the historical problems that originally spurred its development, and in a language that will hopefully be more welcoming to the applied-minded thinker. This process will also serve to yield a glimpse into several core research areas in modern mathematics, throughout which homological reasoning seems to thread with unreasonable effectiveness in uniting seemingly disparate notions.[1]

[1] Or does it? To be honest, it's a little hard for an outsider to tell whether all the homological machinery is *really all that necessary*—or whether perhaps instead the pure-math folks just get a little carried away in insisting as much. I don't take an opinion shadedone way or the other and hope only to empower you to make up your own mind.

1 Invitation: Why is the pure-math literature so impenetrable for everyone else?

Most branches of science and engineering share at least the *rudiments* of a common language. A physicist reading a mechanical-engineering papers will usually know at least *roughly* what's going on; an electrical engineer reading a biochemistry paper can generally glean at least some sense of the objectives and methods, even if more specialized nouns are unfamiliar.

In contrast, modern pure mathematics is something of an outlier, having evolved in the 20th century such a private set of terms, concepts, and research that today it's entirely possible—even for an otherwise mathematically sophisticated reader—to spend 15 minutes with a pure-math paper and *still have literally zero understanding of what it is saying.* I speak from experience!

So *why* is modern pure math so forbidding to the rest of the technical world? I can think of several reasons.

Why pure math is hard, take 1: Mysterious research topics

One difficulty is that many of the major research subfields in modern pure math have come to be known by names that are unfamiliar outside the pure-math world and which generally don't indicate much of what the subject is about. Here are some examples.

Algebraic topology. This is about using algebraic tools (multiplication, division, factorization, etc.) to understand exotic high-dimensional shapes that otherwise defy human mental intuition. The heart of the subject is a set of algorithms for assigning certain groups or other algebraic structures to any given topological space, after which topological manipulations become pencil-and-paper algebra exercises.

Differential topology. As far as I can tell, despite the different-sounding name, this term actually refers to the same subject as the previous item, perhaps with the distinction that methods of differential calculus are used to figure out which groups to assign to a space—after which one proceeds to manipulate those groups by the same algebraic operations.

Differential geometry. The distinction between "geometry" and "topology" is roughly that distances between points are relevant in the former case but not the latter—topology studies properties invariant under stretching and squeezing, while geometry is about things that are more rigid. In particular, things may be so rigid that an arrow (say, a compass needle) sticking out of your pocket at point A on a hiking trail is forced by the geometry to bend and twist in a prescribed way as you walk along the trail, so that it points in a different direction when you get to the end—and, moreover, the new direction may depend on which trail you traversed, even if the starting and endpoints are the same. This situation is described by the notion of a *connection on a fiber bundle,* the subject of differential geometry.

Algebraic geometry. You would never guess it from the name, but this subject is all about studying zeros of polynomials—specifically, the points (or lines, or surfaces; that's the "geometry" bit) at which some given set of polynomials all vanish simultaneously.

Number theory. Compared to all of the above, the title of this subject is rather transparent,[2], as are its objectives: to answer quantitative questions about integers using complicated tricks like contour integrals in the complex plane. For example, the number of partitions of an integer n is given by the contour integral

$$p(n) = \frac{1}{2\pi i} \int_C \left\{ \prod_{n=1}^{\infty} (1 - z^m)^{-1} \right\} e^{-2\pi i n z} \, dz$$

which can be evaluated approximately for large n using complex-analysis tricks to yield the wacky formula

$$\xrightarrow{n \to \infty} \frac{1}{4\sqrt{3}(n - \frac{1}{24})} e^{\frac{2\pi}{\sqrt{6}} \sqrt{(n - \frac{1}{24})}}$$

[2]Of course the nitpicker observes that the term "number theory" is still a little imprecise, as it fails to pinpoint *which* particular subspecies of number it concerns. In this sense the Japanese/Chinese word for this subject is superior: 整数論, literally "theory of integers."

Why pure math is hard, take 2: Lots of names for structures

Pure mathematicians, much more than us applied-math people, tend to be obsessed with classifying things, identifying the precise species to which any given mathematical entity belongs, and then calling it by that name. As a result, textbooks and research papers tend to be awash in a large number of common nouns that may be unfamiliar to applied mathematicians—even if the actual *entities* in question are familiar.[3] Here are just a few etxamples.

Sets This one is kind of a default, null term; it means a collection of things with no particular relationship between them, like apples and oranges and bowling balls.

Groups Actually we applied people already know what groups are: in a nutshell, sets in which you can multiply two elements to get a third, but not add or subtract or perform other operations. Groups arise most naturally as the proper structural framework for permutations or other transformations: you can compose ("multiply") two permutations to get a third, but there is no notion of "adding" two permutations.

Rings Rings are enhanced groups in which, in addition to being able to multiply elements, you can also add and subtract them—but *not* necessarily divide. The main examples are integers and polynomials.

Fields Fields are rings in which you *can* divide: examples include the real numbers, the complex numbers, the rational numbers, and the set of integers modulo a prime number (but not the set of integers modulo a non-prime number). Fields arise naturally as the coefficients of polynomials.

Vector spaces We normally think of vector spaces as **(1)** collections of vectors, i.e. numbers stacked on top of one another in multi-slotted towers. Given this definition, there are obvious ways in which **(a)** two vectors can be added together, **(b)** a vector can be multiplied by a scalar.

However, pure mathematicians turn this reasoning on its head, and instead define vector spaces to be collections of things in which properties **(a)** and **(b)** hold. But what's the difference, you ask, between this and property **(1)** above? Nothing—*if the scalar multipliers come from a field*, i.e. they are real or complex or rational numbers, etc.

Modules However, it is also possible to construct sets of elements that satisfy properties **(a)** and **(b)** above with the scalars coming from a *ring*, in which case it is no longer necessarily true that the set can be described by property **(1)** above. In this case the set is not a vector space but rather a *module*.

[3] If pure math were a programming language, we would call it *strongly typed*, and applied-math people getting confused by all the precise naming of structures would be like assembly-language or C programmers trying to program in Java.

Algebras You've heard the general term "algebra," which describes a branch of mathematical technique, but did you know there's also a more specific thing called "an algebra?" An algebra is a vector space with an extra bonus rule for combining two objects to get a third.

Topologies This is similar to "algebra:" there is a branch of mathematical research and technique known as "topology," but there's also a specific mathematical entity known as "a topology," This is basically a choice of subsets of a space that you define to be the "open sets" for the purposes of topological investigations like identifying continuous functions. That's right—*you* get to define what you want to be the open sets, subject to some mild constraints regarding how open sets interact with each other within your definition.

Why pure math is hard, take 3: Homological algebra

If inscrutable research topics and a long organization chart were the only barriers, presumably many more applied-minded readers would be able to read and benefit from textbooks and papers in pure mathematics. But in fact there's another obstruction that makes this difficult—and this one is more formidable than just some unfamiliar terminology.

You see, in the late 19th century two separate European mathematicians, working on two separate problems in two separate countries, stumbled onto (what eventually turned out to be) a common framework for thinking about math problems, which eventually came to be known as *homological algebra.* Over the next few decades, this set of techniques turned out to be of use in an increasingly wide-ranging variety of math research problems—until, sometime around the mid-20th century, the pure math world collectively lost its mind in rapture over the stuff and began rewriting the pure-math lexicon—and training their pure-math graduate students—to emphasize homological language and reasoning, together with the closely-related philosophy and jargon of *category theory*—wherever possible. This trend continued and even accelerated through the ensuing decades, right up to the present day.

Meanwhile, of course, homological language and reasoning has remained almost entirely absent from the science , engineering, and applied-math curriculum, and thus in some ways the pure and applied worlds have now been careening apart for decades. I think *this* must be among the main reasons for why modern pure math seems to mysterious to applied-minded thinkers these days, and the goal of these notes is to attempt an initial stab at a bridging of the gap.

Why pure math is hard, take 4: An obsolete lexicon

Finally, one factor that has definitely been an impediment to me personally in trying to decipher the modern pure-math literature has been the fact that some of the more complicated and subtle notions in the field are saddled with technical terminology that fails to illuminate their meaning. But we will return to this many times.

2 Poincaré's problem: Turning topology into a computational science

Scientists, engineers, applied mathematicians, and even non-technical folks generally have at least a rough sense of what "topology" is about: the study of the properties of objects that remain unchanged under stretching, squeezing, and other elastic deformations—or, in common colloquial lore, it is the branch of mathematics in which a donut is considered equivalent to a coffee cup, but not to a beach ball or to a beverage-flight[4] holder thingy:

Figure 1: As applied mathematicians, we generally have a vague understanding that "topology" is the field in which a coffee cup is somehow equivalent to a donut, but not to a beach ball or a beverage-flight holder. But how do we turn this rough intuition into something rigorous, algorithmic, mathematical?

Now, if you've heard this explanation of what topology is about, you may have found it intuitively reasonable—it's easy to picture how a putty donut might be deformed into the shape of a coffee cup, but not into the shape of a beverage-flight holder, at least not without doing greater violence—but also somewhat non-rigorous and non-mathematical: how exactly would one *quantify* these distinctions? For example, how would one go about programming a computer—that is, devising a computational algorithm—to tell the difference? And how could these ideas be extended to higher-dimensional shapes, for which intuitive mental pictures are of little help to our feeble, low-dimensional brains?

These were the questions in which the great[5] French mathematician Henri Poincarè got interested in the late 19th century.

[4]A *beverage flight* is what you order when a restaurant offers an overwhelming menu of beverage choices and you can't make up your mind; it typically consists of five small glasses, each containing a sample-sized portion of a different beverage, with each glass typically situated in one of the holes of the wooden thingy shown in the figure here.

[5]This is no understatement: Poincarè really was pretty awesome, so much so that his

2.1 A coarse invariant: dimension

One particularly obvious thing that all of the above surfaces have in common is that they are *two-dimensional*, in the sense that an ant crawling over them has, at every point, exactly two orthogonal directions in which to move; another way to think about this is that to specify a unique point on any of these surfaces requires exactly two real numbers (coordinates).[6] The dimension of a surface is a topological invariant, in the sense that it remains unchanged when the surface is stretched or squeezed[7].

Thus, one way to quantify surfaces mathematically is to assign them their dimension, which in all of the cases mentioned above is 2. At the risk of being pedantic, let's pause to record this observation explicitly by tabulating the dimensions of some surfaces:

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invention—pretty much out of whole cloth—of algebraic topology was only *one* of the incredible things he did in 1894. A great way to marvel at the vast breadth of his mathematical contributions is to peruse the two-volume series *The Mathematical Heritage of Henri Poincaré* from the American Mathematical society: `http://www.ams.org/books/pspum/039.1/pspum039.1.pdf` and `http://www.ams.org/books/pspum/039.2/pspum039.2.pdf`.

[6]The technical mathematical way to say this is that the surfaces of the beach ball, donut, and beverage-flight holder are all two-dimensional *manifolds*. As we will explain when we discuss differential geometry, the term "N-dimensional manifold" is just a fancy name for a set in which each element (each point) requires N real numbers (coordinates) to be uniquely specified. Systems of coordinates may be chosen in various different ways, and there may be no single coordinate system that covers the entire manifold (in which case we will need two or more coordinate systems to cover the whole manifold), but all coordinate systems will have in common that they use N real numbers to specify individual points.

Actually, since we supposed to be talking about topology in this section, purists will point out that the definition we just gave of dimension (number of independent coordinates required to specify points uniquely) really belongs, together with the concept of a manifold, to the area of mathematics known as *geometry*, which is distinguished from *topology* by the use of coordinate systems and notions of quantifiable distances between points—things that are clearly not topological invariants, as they change under stretching and squeezing. Within the strict confines of topology we can't talk about coordinates *at all*, and the dimension of a space must instead be defined as the length of a maximal tower of proper subsets.

[7]The technical mathematical term here is *homeomorphic*: two topological spaces are homeomorphic if they can be obtained from each other by stretching or squeezing without tearing holes or otherwise doing violence, and the operation that takes one into the other is a *homeomorphism.* Homeomorphisms are more general than *diffeomorphisms*, the distinction being basically that in the latter case one is not allowed to smooth sharp corners. The 2-sphere (surface of a beach ball) is **(a)** homeomorphic and diffeomorphic to the surface of an ellipsoid, while being **(b)** homeomorphic but not diffeomorphic to the surface of a cube.

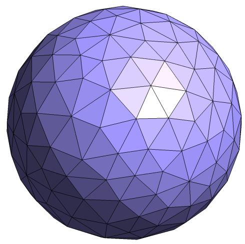
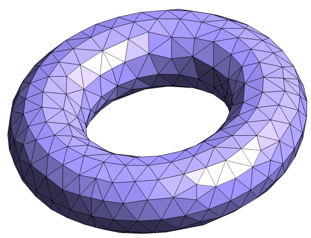
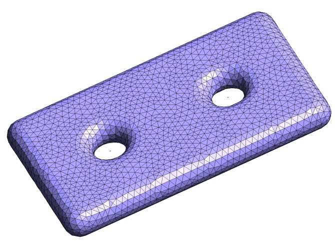
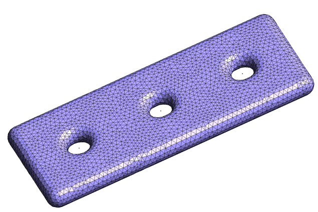
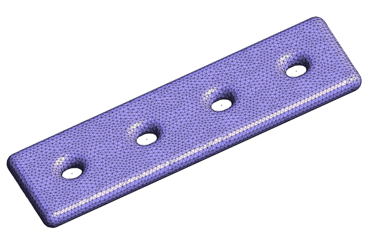
Surface	**Dimension**
	2
	2
	2
	2
	2

Table 1: Various surfaces and the numbers assigned to them by the `Dimension` operation.

The point of Table 2.1 is that assigning dimensions to topological entities doesn't help much in *distinguishing* the various entities here; for that we need to assign a more fine-grained mathematical quantification of structure.

2.2 A finer invariant: Betti numbers

The idea for how to do this, which evolved from various sources in the late 19th century but was crystallized into its modern rigorous form by Poincaré in the 1890s, went something like this: If assigning a *single* integer (the dimension) to each topological space doesn't suffice to distinguish inequivalent spaces, maybe we could do better by trying the next simplest thing: assigning *multiple* integers to each space Poincaré invented an algorithm for assigning, to any given topological space, an array of numbers—which he called the *Betti numbers* of the surface, in honor of the Italian mathematician[8] who anticipated some of the ideas—with the properties that **(a)** like the dimension, the Betti numbers are topological invariants (don't change when the surface is stretched or squeezed), but **(b)** unlike dimension, they can be used to distinguish topologically distinct surfaces from each other. Poincaré described his new algorithm using the term *homology*.

The topological significance of the various Betti numbers, and the actual mechanics of this algorithm for computing them, are a bit complicated—we will sketch the former in Section 2.5 below and discuss the latter in our unit on algebraic topology—but but for now let's just look at the raw *output* of the algorithm: the Betti numbers for the surfaces of Table 2.1 are tabulated below.

[8] Enrico Betti (1823-1892), whose 1871 paper discussed the numbers that would eventually be named after him: `http://www-history.mcs.st-andrews.ac.uk/Biographies/Betti.html`.

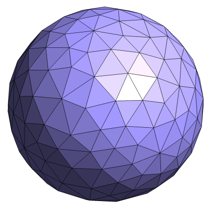
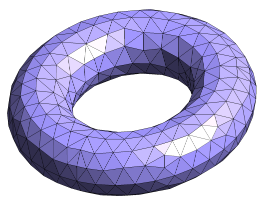
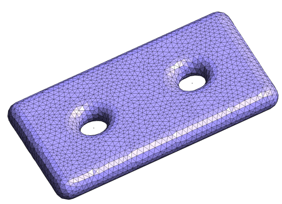
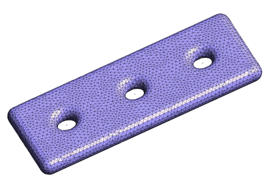
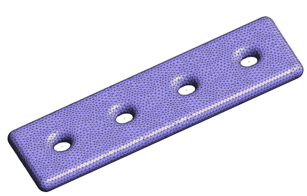
Surface	**Dimension**	**Betti numbers**
	2	[1 0 1]
	2	[1 2 1]
	2	[1 4 1]
	2	[1 6 1]
	2	[1 8 1]

Table 2: Various surfaces and the numbers assigned to them by the `Dimension` and `BettiNumbers` operations. Evidently the Betti numbers do a better job than the dimension at distinguishing inequivalent topological spaces!

2.3 On beyond spheres and donuts: apples, chopsticks, and the real projective plane

Moreover, the notion of Betti numbers is not limited to the simple surfaces pictured above, but can also be applied to more general two-dimensional entities—that is, point sets parameterized by two independent real numbers that may not have manifestations as the surfaces of physical 3D objects.

First example: The Möbius band

A first example that you already know all about is the Möbius band.

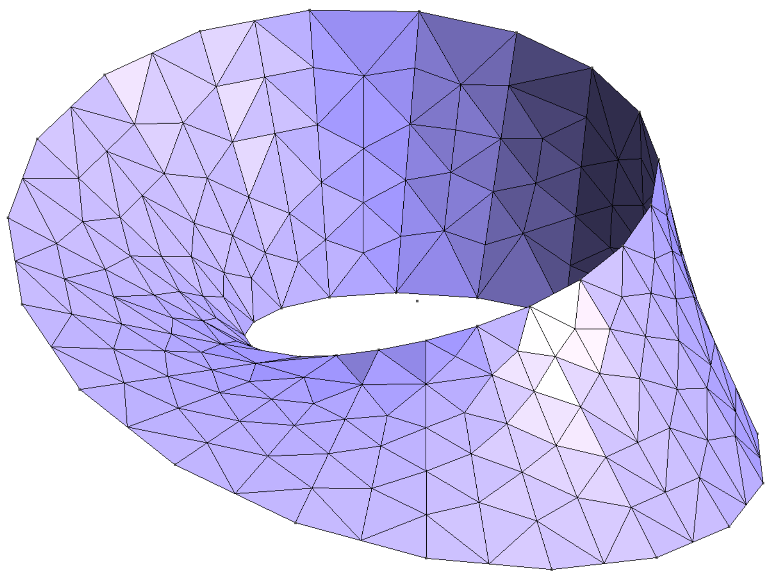


Figure 2: The Möbius band.

This is a two-dimensional topological space that is not the surface of any physical 3D object, and whose Betti numbers may be computed by Poincaré's algorithm, with the result shown in Table 3.

Second example: The real projective plane

Here's another example that illustrates an important point justifying the study of topology for applied mathematicians: The exotic topological spaces whose study is facilitated by homology and other algebraic-topology methods are not necessarily chimerical high-dimensional objects of no practical utility living only in the imaginations of abstract theorists: they may be parameter spaces for configurations of real-life physical objects or engineering systems.

Thus, consider the parameter space of all possible ways to spear an apple with a chopstick such that the chopstick goes through the apple's center. (We take the apple to be perfectly spherical with radius R and the chopstick to be an infinitesimally thin cylindrical rod.)

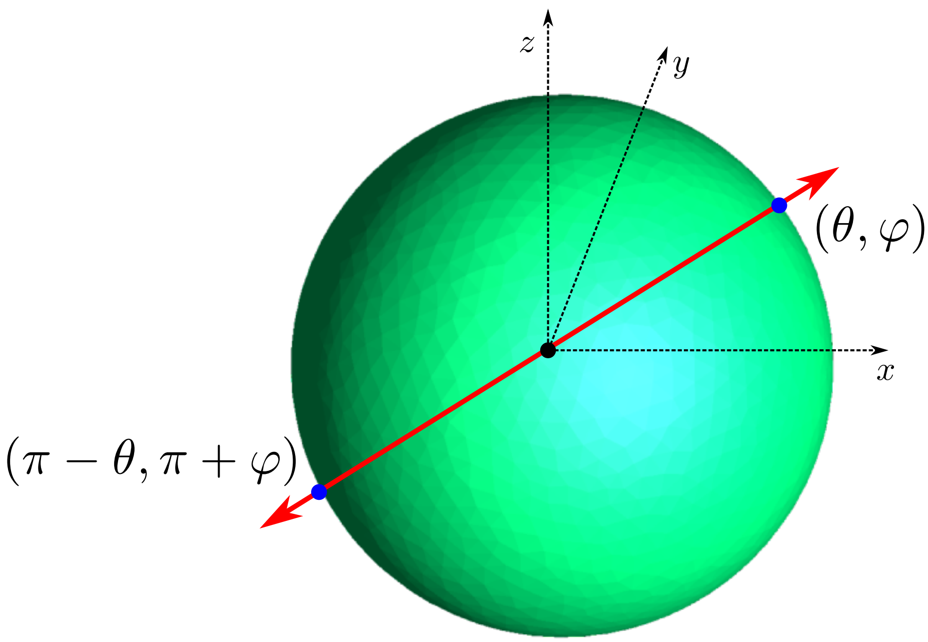


Figure 3: One possible way to drive an (infinitesimally thin) chopstick through the center of a (perfectly spherical) apple.

As is evident from the figure, any such spearing is characterized by two numbers, which we may take to be the polar and azimuthal angles (θ, φ) of the point $\mathbf{x}^{\text{pierce}} = (R, \theta, \varphi)$ on the apple's surface through which we stick the chopstick. So each possible apple-spearing configuration is labeled by a tuple of two independent numbers (θ, φ)—a situation reminiscent of the fact that every point on a sphere can be similarly labeled—and thus, like the sphere, the set of all possible apple spearings is clearly a two-dimensional beast.

Nonetheless, the set of all possible apple spearings is *not* topologically equivalent (homeomorphic) to a sphere. The crucial distinction, visible already in the figure above, is this: If the chopstick pierces the sphere surface at (θ, φ), then it *also* pierces the sphere surface at the antipodally opposite point $(\overline{\theta}, \overline{\varphi}) = (\pi - \theta, \pi + \varphi)$ and there is no way to tell the difference between the (θ, φ) piercing and the $(\overline{\theta}, \overline{\varphi})$ piercing: these two distinct points in the sphere both correspond to the *same* point in the apple-spearing set. So the two sets are not the same.

Thus the apple-spearing set is another example of an entity with dimension 2 that is not the same as the other dimension-2 entities we considered above. If the apple-spearing set could be realized as an actual surface in our 3D world (it can't, for reasons that will become clear below), then an ant crawling over this surface would again have two independent directions in which to move from each point, just as in the case of the sphere. In fact, as long as the ant stays in the upper hemisphere it wouldn't be able to detect that it *isn't* on a sphere. However, as soon as the ant crawls southward past the equator, it instantly magically warps to the opposite side of the globe and finds itself now crawling *northward*! So clearly *this* surface is quite different from any of the other surfaces above, and yet it has the same dimension—again testifying to the inability of dimension to differentiate between distinct entities.

Topologists describe this situation by saying that the apple-spearing set is obtained from the sphere by *identifying* antipodally opposite points, and they have a the 2-sphere with opposite points identified is called the *real projective plane* or *projective two-space* and denoted $\mathbb{P}^2$ or $\mathbb{RP}^2$.

We will have much more to say about projective space later in MoPuMMAM. For the time being let's just make an addendum to Table 2 for the Betti numbers of $\mathbb{RP}^2$:

Space	**Dimension**	**Betti numbers**
Möbius band	2	$[1\,1\;\;0]$
$\mathbb{RP}^2$	2	$[1\,1^*\,0]$

Table 3: Betti numbers for some slightly more exotic two-dimensional spaces, computed by Poincaré's algorithm.

The most important thing to notice here is that—unlike the surfaces of Table 2—the last entry in the Betti-number vectors for both the Möbius band and the real-projective plane is *zero*. As discussed below, this has to do with the fact that the Möbius-band and $\mathbb{RP}^2$ are *non-orientable* surfaces: there is no consistent way to assign an outward-pointing normal vector to the surface at each point.[9]

Another thing to notice is that the second entry in the Betti-number table for $\mathbb{RP}^2$ is starred. This has to do with an interesting phenomenon in homology known as *torsion*, which we will discuss in our unit on algebraic topology.

[9] To see this, suppose such an assignment existed, pick a point $\mathbf{x}$, and trace out the trajectory executed by the normal vector $\hat{\mathbf{n}}(\mathbf{x})$ as you complete one full loop around the Möbius band (or $\mathbb{RP}^2$). When you get back to where you started, the normal vector has flipped sign! So which direction is the outward-pointing normal at $\mathbf{x}$?

2.4 On beyond two dimensions: higher-dimensional spheres and tori

In addition to exotic 2D spaces like $\mathbb{RP}^2$, Poincaré's algorithm for computing Betti numbers works for higher-dimensional spaces as well. For example, the Betti numbers for the first few dimensions' worth of spheres and tori are tabulated in Table 4.[10]

Space	**Symbol**	**Dimension**	**Betti numbers**
1-sphere (circle)	S_1	1	[1 1]
2-sphere	S_2	2	[1 0 1]
3-sphere	S_3	3	[1 0 0 1]
4-sphere	S_4	4	[1 0 0 0 1]
5-sphere	S_5	5	[1 0 0 0 0 1]
2-torus	$\mathbb{T}_2$	2	[1 2 1]
3-torus	$\mathbb{T}_3$	3	[1 3 3 1]
4-torus	$\mathbb{T}_4$	4	[1 4 6 4 1]
5-torus	$\mathbb{T}_5$	5	[1 5 10 10 5 1]

Table 4: Betti numbers of some higher-dimensional topological spaces.

One immediate thing to notice is that the arrays of numbers are longer for higher-dimensional spaces; in fact, the Betti-number array for a d-dimensional space has $d + 1$ entries. Beyond this observation, do you see any patterns here?

Poincaré polynomials and the Kunneth formula

Do you notice anything interesting about the Betti numbers for the $3, 4, 5$-torus in Table 4? The Betti-number arrays for the higher-dimensional tori appear to be tracing out the binomial coefficients—that is, the Betti numbers for the d-dimensional torus are the coefficients of x in the expansion of $(1+x)^d$. Another way to say the same thing is that the polynomial $P_d(x) \equiv (1 + x)^d$ is the *generating function* for the Betti numbers of the d-dimensional torus. The generating function for the Betti numbers of a topological space S is called the *Poincaré polynomial* $P_{\mathrm{S}}(x)$ of the space.

The simple form of the Poincare polynomial for the d-torus is an example of the *Kunneth formula*, which says that, if a topological space S is the product of two subspaces X and Y (a situation mathematicians describe by writing

[10] There is no entry for the 1-torus in Table 4, because the 1-dimensional torus is the same thing as the 1-dimensional sphere (the circle).

$S = X \times Y$), then the Poincaré polynomial for the Betti numbers of S is simply the product—in the usual sense of polynomial multiplication—of the Poincaré polynomials of X and Y:

$$S = X \times Y \qquad \Longrightarrow \qquad P_S(x) = P_X(x) P_Y(x).$$

Now, the d-dimensional torus $\mathbb{T}_d$ is defined to be the product of d copies of the one-dimensional sphere (the circle) S_1, and—from Table 4—the circle S_1 has Betti numbers $[1\,1]$ and thus Poincaré polynomial $P_{S_1}(x) = (1 + x)$. The Kunneth formula then says

$$\mathbb{T}_d = \underbrace{S_1 \times S_1 \times \cdots \times S_1}_{d \text{ factors}} \qquad \Longrightarrow \qquad P_{\mathbb{T}_d}(x) = \Big[P_{S_1}(x)\Big]^d = (1 + x)^d.$$

This explains why the Betti numbers for tori in Table 4 are just binomial coefficients.

2.5 But what do the Betti numbers *mean*?

As noted above, the actual mechanics of Poincare's algorithm for computing Betti numbers are somewhat involved, and we will wait until our unit on algebraic topology to delve into these details. Nonetheless, at least *some* of the entries in the Betti-number arrays are easy enough to describe qualitatively already at this stage.

I will use the notation

$$\mathbf{b} = [b_0\, b_1\, b_2\, \cdots\, b_d]$$

to label Betti numbers; thus $\mathbf{b}$ refers to the full array and b_n refers to its nth entry counting from the left starting at 0.

b_0: Number of connected components

The zeroth Betti number of a topological space is simply the number of disconnected pieces it contains. In all of the examples we considered above, the topological space was just a single connected entity, which explains why the entry in all Betti-number arrays in all of the tables above was 1.

b_1: One-dimensional holes

The first Betti number b_1 of a topological space S is a bit more interesting. One way to think about it, which we will formalize in our unit on algebraic topology, is that it counts the number of essentially distinct ways to draw a closed loop in the space whose interior does *not* live in the space—that is, it counts the number of one-dimensional *holes*, defined roughly as "closed curves in the space that aren't the boundary of any surface in the space."[11]

[11]If you have ever heard of the topological notion of *homotopy*, this discussion of loops in topological spaces might remind you of the procedure for constructing the first homotopy group (fundamental group) of a space. As we will discuss in our unit on algebraic topology, this is no accident; for sneak-preview teaser, see Appendix A below.

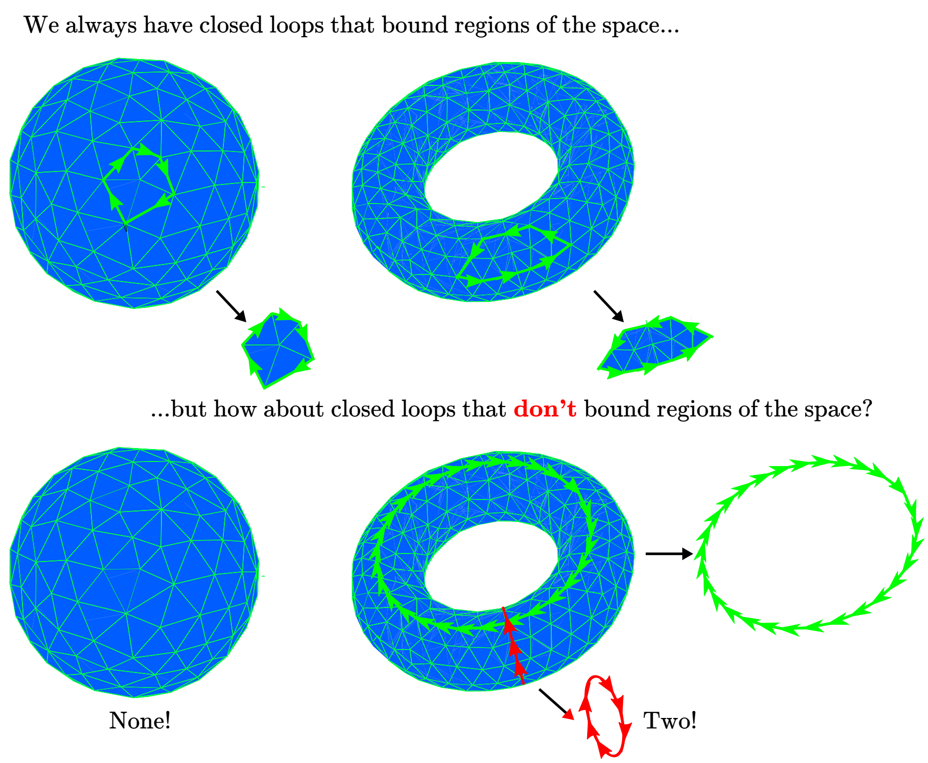


Figure 4: Why the sphere and torus have different values for the innermost entry of the Betti-number table.

b_2: Two-dimensional holes

$b_3, b_4, \cdots, b_{d-1}$: Higher-dimensional holes

Continuing this holes, it is only natural to expect that higher Betti numbers count "higher-dimensional holes"—a notion which is hard to grasp intuitively at first, but which is made rigorous and computable by Poincaré's homology algorithm.

b_d: Orientability

The last entry in the Betti-number array for a topological space encodes information on whether or not the space is *orientable*. For the simplest case of 2D topological spaces (surfaces), orientability means "At each point $\mathbf{x}$ on the surface I can make a consistent choice of outward-pointing normal vector $\hat{\mathbf{n}}(\mathbf{x})$." All of the spaces in Tables 2 and 4 are orientable, while Möbius band and $\mathbb{RP}^2$ are non-orientable. This information is encoded by the fact that the final entries in all Betti-number arrays in Tables 2 and 4 are nonzero, while the final entries in the Betti-number arrays in 3 are zero.

2.6 The most important picture in this section

The details of Poincaré's algorithm for computing Betti numbers, and their significance for the specific mathematical subfield of topology, are interesting in their own right, and we will study them in detail in our unit on algebraic topology. However, our goal *here* is to emphasize the bits that are common to other fields, and for this purpose the only picture you need to have in mind is the one on the following page.

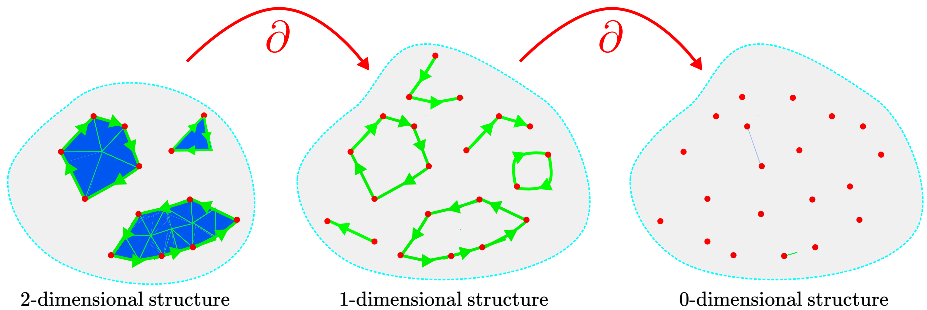


Figure 5: *Homology in algebraic topology:* Schematic depiction of the differential complex and boundary operator relevant for simplicial homology.

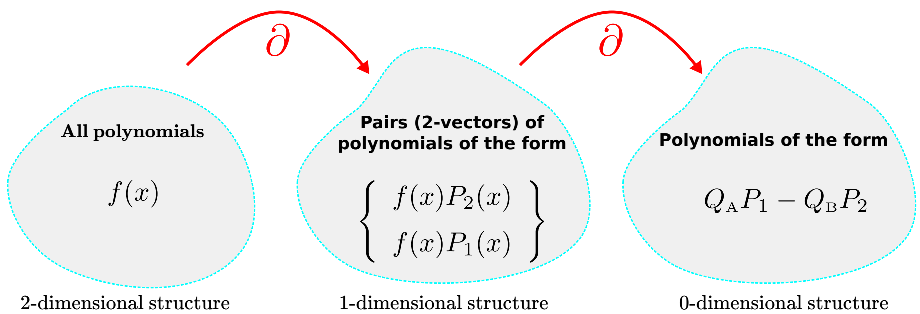


Figure 6: *Homology in algebraic geometry:* Schematic depiction of the Koszul complex for two polynomials $P_{1,2}(x)$ and the corresponding boundary operator.

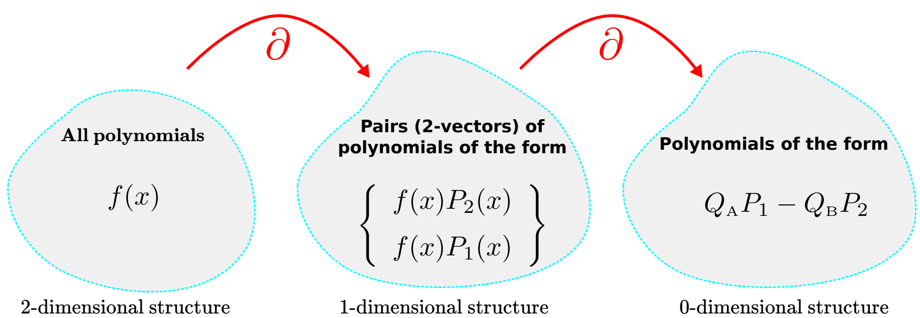


Figure 7: *Homology in differential geometry:* Schematic depiction of the de Rham complex.

3 Hilbert's problem: Counting polynomials that satisfy given constraints

20th-century textbooks and courses on numerical analysis and computational science leave readers and students with the following clear impression: *systems of linear equations* ***good***, *systems of quadratic or higher-degree polynomial equations* ***bad***. In particular, we are generally told that

- for simultaneous systems of *linear* (first-degree polynomial) equations,

  1. there is a simple diagnostic that tells you whether or not solutions exist (the determinant),
  2. if solutions exist there are simple deterministic solution algorithms for computing them (e.g. Gaussian elimination),
  3. these methods are already implemented for you in the form of high-quality free software packages like LAPACK, and
  4. the methods always work.

  On the other hand,

- for simultaneous systems of *quadratic or higher-degree* polynomial equations, all of the above goes out the window:

  1. there is no simple way to determine whether solutions exist,
  2. there are no deterministic algorithms or robust software tools for computing solutions if they do exist, and thus
  3. one's only hope is to cobble together some sort of iterative numerical solver (using Newton-Raphson root-finding or other brute-force nonlinear methods), which
  4. may or may not work.

A central goal of MoPuMMAM is to update this obsolete worldview by introducing applied mathematicians to a beautiful set of techniques, *with* robust existing software implementations, that pure mathematicians use to analyze systems of polynomial equations—and which, bizarrely, turn out to share a common conceptual framework with the topological methods we discussed in the previous section. These methods date back to the work of David Hilbert in the 1890s—almost exactly contemporaneous with Poincaré's work on topology.

Quantifying

Hilbert's work was motivated by the following simple question. Suppose I give you three polynomials in three variables—say, $P_{1,2,3}(x_1, x_2, x_3)$—and I ask you to construct a new polynomial $Q(x_1, x_2, x_3)$ that is guaranteed to vanish whenever P vanishes. That is, I need Q to satisfy

$$Q(\mathbf{x}) = 0 \quad \text{for all points } \mathbf{x} = (x_1, x_2, x_3) \text{ at which} \quad P_1(\mathbf{x}) = P_2(\mathbf{x}) = P_3(\mathbf{x}) = 0. \tag{1}$$

The set of all points $\mathbf{x}$ at which a collection of polynomials vanish simultaneously is called the *zero locus* of the collection,[12] which I will denote $\mathbb{ZL}(P_1, P_2, P_3)$ or $\mathbb{ZL}$ for short. $\mathbb{ZL}$ is just a subset of $\mathbb{R}^3$, which may be discrete or infinite and may take various different shapes—that's the "geometry" bit of algebraic geometry—depending on the choice of Ps; our task may be rephrased as finding polynomials $Q(\mathbf{x})$ that vanish for all $\mathbf{x} \in \mathbb{ZL}(P_1, P_2, P_3)$. (Note that Q may or may not vanish at *other* points outside the zero locus—it is allowed to, but doesn't have to.)

Of course, there will be many such polynomials, so a better way to state our objective is that we want to understand the structure of the set of *all* polynomials $Q(\mathbf{x})$ that vanish on the common zero locus of $P_{1,2,3}$. How big is this set? How many degrees of freedom does it have? And is there a systematic procedure for constructing polynomials that belong to it?

Generators

Well, one obvious way to construct such a polynomial is simply to form a linear combination of $P_{1,2,3}(\mathbf{x})$, with "coefficients" $Q_{1,2,3}(\mathbf{x})$ that are themselves polynomials. That is, any function of the form

$$Q(\mathbf{x}) = Q_1(\mathbf{x})P_1(\mathbf{x}) + Q_2(\mathbf{x})P_2(\mathbf{x}) + Q_3(\mathbf{x})P_3(\mathbf{x}), \tag{2}$$

where the $Q_{1,2,3}$ may be any arbitrary polynomials, is clearly guaranteed[13] to vanish everywhere on $\mathbb{ZL}(P_1, P_2, P_3)$. I will use the symbol $\mathbb{I}$ to denote the set of all functions $Q(\mathbf{x})$ of the form (2); note that the coefficient polynomials $Q_i(\mathbf{x})$ do *not* have to vanish on $\mathbb{ZL}$—they really are arbitrary.

[12] Or at least it would be in a perfect world. In our *fallen* world the zero locus of a set of polynomials is referred to by the galactically stupid term "variety," so the proper pure-math way to state our task here is that we are looking for polynomials Q that vanish identically on the variety of $\{P_1, P_2, P_3.\}$ A great virtue of not being a pure mathematician is that I need feel no compunction about refusing to use this sorry excuse for useful descriptive terminology, but you should at least know that it's out there.

Sadly, this is not even the most useless term in the pure-math lexicon: `http://homerreid.com/teaching/MoPuMMAM/Notes/HallOfShame.pdf`.

[13] Abstracting out the principle at work here, what's really going on is that the property of vanishing on $\mathbb{ZL}(P_1, P_2, P_3)$ is *contagious*: if *either* of two polynomials has this property, then their product has this property, and sums of products have this property, so *the set of polynomials that vanish on* $\mathbb{ZL}(P_1, P_2, P_3)$ *is closed under the operation of forming linear combinations* with arbitrary polynomial "coefficients", i.e. under the operation of equation (2). Now, as we will discuss in detail later, the set of all polynomials (with, say, complex-valued coefficients) is a *ring* (as opposed to being a group or a field or a module or some other algebraic structure) and a subset of a ring that is closed under taking linear combinations with arbitrary coefficients is called an *ideal.* We won't need this terminology here, but will get into it later when we talk about the various types of algebraic structures flying around in modern pure math.

Equation (2) is reminiscent of the construction of vectors in a vector space: given any basis of vectors $\{\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{x}}_3\}$ in $\mathbb{R}^3$ (real three-dimensional space), we can construct *all* vectors in $\mathbb{R}^3$ by taking linear combinations of the basis vectors with real-value coefficients. We say that the elements $\{\mathbf{x}_i\}$ are *generators* of $\mathbb{R}^3$. In equation (2), the $\{P_i\}$ polynomials play the role of basis vectors, the $\{Q_i\}$ polynomials play the role of coefficients, and we say that the $\{P_i\}$ are *generators* of $\mathcal{I}$ (the set of functions that vanish on $\mathbb{ZL}$).

Relations

Syzygies

4 Cohomology for Dummies: Nomadic tribes, migrations, and premature morbidity

4.1 Evolution of nomadic tribes: Differential complexes

4.2 Quantifying premature morbidity: Cohomology

4.3 When all tribe members live to ripe old ages: Exact sequences

5 Cohomology in Modern Pure Mathematics

The idea of MoPuMMAM is to give a flavor of modern pure mathematics by surveying four

With this in mind,

5.1 Cohomology in Number Theory

5.2 Cohomology in Algebraic Topology

5.3 Cohomology in Differential Geometry

5.4 Cohomology in Algebraic Geometry

6 Cohomology in Physics and Engineering

As noted above, the ubiquity of cohomological language and reasoning in modern pure mathematics is mirrored by its almost complete absence in modern applied mathematics, physics, and engineering—an observation which constitutes one of the underlying premises of this course. However, applications of these ideas have *occasionally* been spotted in more applied domains of science. Here are two of which I'm aware.

- *Anomalies and BRST quantization in non-abelian gauge theories.*
- *Classification of finite-element meshes for numerical boundary-value problems.*

  Reference: Matti Pellikka, Saku Suuriniemi and Lauri Kettunen, "Homology in Electromagnetic Boundary Value Problems", *Boundary Value Problems* **2010** 381953 (2010). OI:`http://dx.doi.org/10.1155/2010/381953`

A Homotopy vs. Homology