Adv. Appl. Prob. **16**, 867–886 (1984)
Printed in N. Ireland

BURKE'S THEOREM ON PASSAGE TIMES IN GORDON–NEWELL NETWORKS

HANS DADUNA,* *Technische Universität Berlin*

Abstract

In a closed cycle of exponential queues where the first and the last nodes are multiserver queues while the other nodes are single-server queues, the cycle-time distribution has a simple product form. The same result holds for passage-time distributions on overtake-free paths in Gordon–Newell networks. In brief, we prove Burke's theorem on passage times in closed networks.

CYCLIC QUEUES; MULTISERVER QUEUES; CYCLE TIMES

1. Introduction

From a customer's point of view possibly the most important performance measure in distributed systems (local area networks, telecommunication systems, computer centers, etc.) may be the passage time, i.e. the time between entering the system (or a particular part of it) and the moment of departure. But the problem of determining passage-time distributions in networks of queues (which are the natural models for distributed systems) resists solution insofar as properties beyond expected values are requested. (The latter are easily obtained by Little's theorem when steady-state distributions are known.)

Clearly, passage-time expectations are important in projecting a distributed system, but a typical request of an employer is that 'at most five percent of all executed jobs may need more than a given value of execution time in travelling through the system': this is a question about the distribution of passage times in general networks. But for a long time the famous theorems of Reich (1957), (1963) and Burke (1964), (1968), concerning open tandems of multiserver queues, were the only results available (for a review see Burke (1972)). Their results on passage times can be summarized as follows. In a sequence of exponential systems, the first and last of which are multiserver queues, while all other systems are single-server queues, with jobs arriving in a Poisson stream, the sojourn times at different nodes are independent. So the passage-time distribution can be obtained by simple convolutions.

Received 15 September 1983.

* Present address: Institut für Mathematische Stochastik der Universität Hamburg, Bundesstrasse 55, 2000 Hamburg 13, West Germany.

In 1980 Walrand and Varaiya (1980) generalized this result to overtake-free paths in single-server open Jackson networks, and Chow (1981) found the cycle-time distribution in a two-stage cycle of exponential single-server queues. Chow's result initiated a sequence of papers on cycle-time distributions in closed cycles (Schassberger and Daduna (1983), Boxma and Donk (1982), Boxma, Kelly and Konheim (1982)), and on passage-time distributions in Gordon–Newell networks (Daduna (1982), Kelly and Pollett (1983)). But all these results are concerned only with the case when the paths under consideration are single-server queues; the only exceptions I know of are the papers of Kawashima and Torigoe (1983), who investigated a two-stage cycle of multiserver queues, and of Wong (1979) and Sekino (1972) who investigated a two-stage cycle of multiserver queues where one stage acts as an infinite server.

So Boxma (1983), in his talk at the 44th Session of the International Statistical Institute, posed the problem: 'How far does the analogy with the results of Burke hold concerning the allowance of many-server queues at the beginning and end of the path?' The aim of this paper is to show that the analogy is complete as far as passage-time distributions are involved.

Clearly, independence can not hold as in open tandems of queues, but in the same way as for single-server systems a simple product-form expression can be given for the Laplace–Stieltjes transform (LST) of the passage times which can be inverted easily. But we prove even more: the same results are valid for overtake-free paths in general Gordon–Newell networks, and moreover in Gordon–Newell networks with different job classes.

We restrict our proofs here to the case of one job class only to reduce the amount of unnecessary formalisation which would lead to tedious computations giving no new insight into the problem. (The amount of additional work which would be necessary may be seen by reading the proofs in Daduna (1982).)

The paper is organised as follows. Section 2 introduces the simplest cycle where Boxma's question is meaningful: the three-stage cycle. In Section 3 the LST for the cycle time is reduced to an expression concerning conditional LST of the passage time through the second and third node. This result leads to the first analogy of Burke's theorem in Section 4. In Section 5 we state the general result for Gordon–Newell networks.

2. The model

The system under consideration is shown in Figure 1. We have a closed three-node cycle with $N \geqq 1$ jobs cycling. Node i, $i = 1, 2, 3$, consists of m_i exponential servers each with parameter $\mu_i > 0$. The queueing discipline is first-come–first-served (FCFS), i.e., jobs present at node $i \in \{1, 2, 3\}$ form a queue in order of their arrival, the first m_i jobs receive service, and a job reaching a

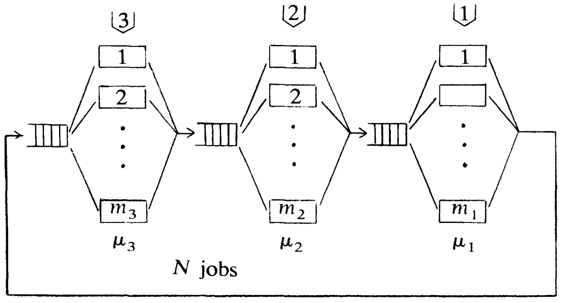

Figure 1

service position instantaneously starts receiving service. Having obtained its service at some node a job proceeds directly to the next node and joins the tail of the queue there. We assume all service times to be independent of each other.

Let $J_i(t)$, $i=1,2,3$, $t \geqq 0$, denote the number of jobs present at node i at time t. Then $J=((J_3(t), J_2(t), J_1(t)): t \geqq 0)$ is a discrete-state Markov process with state space

$$Z(N)=\left\{(n_3, n_2, n_1): n_i \geqq 0, \sum_{i=1}^{3} n_i = N\right\}.$$

We assume J to have right-continuous paths. (Note, that J is an ergodic strong Markov process.)

Our system is a special Gordon–Newell network of queues, therefore the stationary distribution is given by (see Gordon and Newell (1967)):

$$(1) \qquad p(n_3, n_2, n_1)=G(N)^{-1}\prod_{i=1}^{3} \mu_i^{-n_i}\beta_i^{-1}(n_i), \qquad (n_3, n_2, n_1) \in Z(N),$$

where

$$G(N)=\sum_{(n_3,n_2,n_1)\in Z(N)} \prod_{i=1}^{3} \mu_i^{-n_i}\beta_i^{-1}(n_i)$$

and for $i=1,2,3$

$$\beta_i(n_i)=\begin{cases} n_i! & \text{for} \quad n_i \leqq m_i - 1 \\ (m_i-1)!\, m_i^{n_i-m_i+1} & \text{for} \quad n_i \geqq m_i. \end{cases}$$

In our computations we shall follow a test job's journey through the system during a complete cycle, i.e. between two successive arrivals at node 3. Therefore we tag a job at its arrival at node 3 and call it C for easier reference.

The steady-state probability $\tilde{p}(n_3, n_2, n_1)$ for n_i, $i=1,2,3$, untagged jobs to be found in node i just after C's arrival at node 3 is given by

$$(2)\qquad \tilde{p}(n_3, n_2, n_1)) = G(N-1)^{-1}\prod_{i=1}^{3} \mu_i^{-n_i}\beta_i^{-1}(n_i), \qquad (n_3, n_2, n_1) \in Z(N-1),$$

where $Z(N-1)$, $G(N-1)$ are defined as in (1). (See for example Sevcik and Mitrani (1981).) The probability (2) is extensively discussed by Melamed (1982b) and Kelly and Pollett (1983). In particular the following holds:

If the system is started at time $t=0$ with C at the tail of the queue of node 3 and n_i, $i=1,2,3$ other jobs at node i with probability $\tilde{p}(n_3, n_2, n_1)$, $(n_3, n_2, n_1) \in Z(N-1)$, then the Markov process J is not in equilibrium, but the Markov chain obtained by observing J only at C's successive arrival instants at node 3 is in equilibrium.

Describing C's journey through the system between two successive arrivals at node 3 we introduce the following notation for possible intermediate states. We shall say that the system is in state $((\tilde{n}_3, n_3), n_2, n_1) \in Z(N-1; \tilde{n}_3)$, $0 \leqq \tilde{n}_3 \leqq N-1$, if the following holds:

(a) n_i untagged jobs are present at node $i=1,2$;

(b) C is present at node 3 and

(i) for $n_3 \geqq m_3$ there are

—m_3 untagged jobs in service,

—$n_3 - m_3$ untagged jobs waiting before C,

—$\tilde{n}_3$ untagged jobs waiting behind C;

(ii) for $n_3 \leqq m_3 - 1$ there are

—n_3 untagged jobs and C in service,

—$\tilde{n}_3$ untagged jobs waiting.

(Note that $n_3 < m_3 - 1$ implies $\tilde{n}_3 = 0$, due to the FCFS service discipline, which guarantees that possible service of a job starts immediately.)

(c) $\tilde{n}_3 + n_3 + n_2 + n_1 + 1 = N$.

We shall say that the system is in state

$$(n_3, (\tilde{n}_2, n_2), n_1) \in Z'(N-1; \tilde{n}_2), \qquad 0 \leqq \tilde{n}_2 \leqq N-1,$$

if C is present at node 2 and an interpretation similar to that above holds for the distribution of the untagged jobs.

3. Cycle times

We assume the system to be started at time $t=0$ with initial distribution $\tilde{p}$ on $Z(N-1; 0)$, i.e. C is at the tail in the queue of node 3 and n_i, $i=1,2,3$, untagged jobs are present at node i with probability $\tilde{p}(n_3, n_2, n_1)$, $(n_3, n_2, n_1) \in Z(N-1)$. The cycle time of C is the time until C leaves node 1 for the first time. Let

$$f_N(s),\ s \geqq 0,$$

denote the Laplace–Stieltjes transform (LST) of C's cycle time. We then have

$$(3)\qquad f_N(s) = \sum_{(n_3,n_2,n_1)\in Z(N-1)} \tilde{p}(n_3, n_2, n_1) f_N((0, n_3), n_2, n_1)(s), \qquad s \geqq 0,$$

where $f_N((0, n_3), n_2, n_1)(s)$ is the LST of C's cycle time, given that the system is in state $((0, n_3), n_2, n_1)$ at time $t=0$.

Now assume that during C's cycle the system jumps (by any transition) into state $((\tilde{n}_3, n_3), n_2, n_1) \in Z(N-1; \tilde{n}_3)$, and assume a clock is started at this jump instant and stopped when C's cycle expires: the amount of time indicated by this clock is then a random variable the LST of which we denote by

$$(4)\qquad f_N((\tilde{n}_3, n_3), n_2, n_1)(s), \qquad s \geqq 0.$$

In the same way we define the conditional LST

$$(5)\qquad g_N(n_3, (\tilde{n}_2, n_2), n_1)(s), \qquad s \geqq 0,$$

by the distribution of the time determined by a clock which is started when the system jumps (by any transition) into state $(n_3, (\tilde{n}_2, n_2), n_1) \in Z'(N-1; \tilde{n}_2)$, and which is stopped when C's first cycle expires.

The strong Markov property of J implies that a set of first-entrance equations for the conditional LST (4) and (5) is valid, which will be partially solved to obtain (3). Let

$$M((\tilde{n}_3, n_3), n_2, n_1) = \alpha_3(\tilde{n}_3, n_3)\mu_3 + \alpha_2(n_2)\mu_2 + \alpha_1(n_1)\mu_1, \qquad ((\tilde{n}_3, n_3), n_2, n_1) \in Z(N-1; \tilde{n}_3),$$

where

$$\alpha_3(\tilde{n}_3, n_3) = \begin{cases} n_3 + 1 + \tilde{n}_3, & \text{if} \quad n_3 + 1 + \tilde{n}_3 \leqq m_3 - 1, \\ m_3, & \text{if} \quad n_3 + 1 + \tilde{n}_3 \geqq m_3 \end{cases}$$

$$\alpha_i(n_i) = \begin{cases} n_i & \text{if} \quad n_i \leqq m_i - 1, \\ m_i & \text{if} \quad n_i \geqq m_i \end{cases} \qquad i = 1, 2.$$

So, if the system is in state $((\tilde{n}_3, n_3), n_2, n_1)$, $\alpha_3(\tilde{n}_3, n_3) \,.\, \mu_3$ is the service rate at node 3, $\alpha(n_2) \,.\, \mu_2$, $\alpha(n_1) \,.\, \mu_1$ are the service rates at nodes 2, 1, and so on.

Our starting point is the following set of first-entrance equations for the conditional LST:

$$(6)\qquad f_N((\tilde{n}_3, n_3), n_2, n_1)(s)$$

$$(6.1)\qquad = \frac{M((\tilde{n}_3, n_3), n_2, n_1)}{M((\tilde{n}_3, n_3), n_2, n_1) + s} \times$$

$$(6.2)\qquad \times\left[\frac{\alpha_2(n_2)\cdot\mu_2}{M((\tilde n_3, n_3), n_2, n_1)} f_N((\tilde n_3, n_3), n_2-1, n_1+1)(s)\right.$$

$$(6.3)\qquad +\frac{\alpha_1(n_1)\cdot\mu_1}{M((\tilde n_3, n_3), n_2, n_1)} f_N((\tilde n_3+1_{\{n_3\geqq m_3-1\}}, n_3+1_{\{n_3<m_3-1\}}), n_2, n_1-1)(s)$$

$$(6.4)\qquad +\frac{1_{\{n_3\geqq m_3\}}\cdot m_3\cdot\mu_3}{M((\tilde n_3, n_3), n_2, n_1)} f_N((\tilde n_3, n_3-1), n_2+1, n_1)(s)$$

$$(6.5)\qquad +\frac{1_{\{n_3\leqq m_3-1\}}\cdot n_3\cdot\mu_3}{M((\tilde n_3, n_3), n_2, n_1)} f_N((\tilde n_3-1_{\{\tilde n_3\neq 0\}}, n_3-1_{\{\tilde n_3=0\}}), n_2+1, n_1)(s)$$

$$(6.6)\qquad \left. +\frac{1_{\{n_3\leqq m_3-1\}}\cdot\mu_3}{M((\tilde n_3, n_3), n_2, n_1)} g_N(\tilde n_3+n_3, (0, n_2), n_1)(s)\right],$$

$$((\tilde n_3, n_3), n_2, n_1)\in Z(N-1;\tilde n_3),\qquad 0\leqq\tilde n_3\leqq N-1.$$

($1_{\{\cdot\}}$ is the indicator function, $f_N((\tilde n_3, n_3), -1, n_1)(s)$ is formally defined as 0, etc.).

The LHS of the first-entrance equation (6) represents the conditional LST of C's residual cycle time, given that he finds the system in state $((\tilde n_3, n_3), n_2, n_1)$ during his cycle. We have labeled the RHS of (6) for easier reference to the summands. This RHS divides C's residual cycle time into:

(a) the time until the next jump in the system takes place—the LST of this time interval is

$$\frac{M((\tilde n_3, n_3), n_2, n_1)}{M((\tilde n_3, n_3), n_2, n_1)+s}$$

and

(b) C's residual cycle time from that jump on.

Due to the strong Markov property of J, the parts (a) and (b) are independent conditioned on the systems state just after the jump (which is a Markov time).

The different possibilities of the system's evolution after the jump, conditioned on the event that caused the jump, are recorded in the square brackets on the RHS: e.g. (6.6): If the system is in state $((\tilde n_3, n_3), n_2, n_1)$ then $1_{\{n_3\leqq m_3-1\}}=1$ indicates that C is just in service, and with probability

$$\mu_3\,.\,M((\tilde n_3, n_3), n_2, n_1)^{-1}$$

the next jump in the system is caused by C's departure from node 3; leaving node 3 C joins the tail of the queue at node 2 and a clock is started, recording the time until termination of C's cycle. This time has (conditional) LST $g(\tilde n_3+n_3, (0, n_2), n_1)(s)$.

The other summands can be interpreted in the same way:

(6.2) corresponds to all jumps caused by a termination of a service time at node 2; the departing job joins the queue at node 1.

(6.3) corresponds to all jumps caused by a termination of a service time at node 1; the departing job has to wait at node 3 if $n_3 \geqq m_3 - 1$. If $n_3 < m_3 - 1$ the departing job's service immediately commences.

(6.4) and (6.5) correspond to all jumps caused by a departure of untagged jobs from node 3.

From (6) we shall derive three further sets of equations:

(i) For $\tilde{n}_3 = 0$ we multiply both sides of (6) by

$$\left(\prod_{i=1}^{3} \mu_i^{-n_i} \beta_i^{-1}(n_i)\right)(M((0, n_3), n_2, n_1) + s)$$

and sum over $Z(N-1; 0)$. With an additional rearrangement of the sums appearing on the RHS we then obtain

$$\sum_{((0,n_3),n_2,n_1)\in Z(N-1;0)} \prod_{i=1}^{3} \mu_i^{-n_i} \beta_i^{-1}(n_i) f_N((0, n_3), n_2, n_1)(s) \tag{7}$$
$$\times[\alpha_3(0, n_3)\mu_3 + \alpha_2(n_2)\mu_2 + \alpha_1(n_1)\mu_1 + s]$$

$$= \sum_{\substack{((0,n_3),n_2,n_1)\in Z(N-1;0)\\ n_1 \geqq 1}} \prod_{i=1}^{3} \mu_i^{-n_i} \beta_i^{-1}(n_i) f_N((0, n_3), n_2, n_1)(s)[\alpha_1(n_1)\mu_1] \tag{7.1}$$

$$+ \sum_{\substack{((0,n_3),n_2,n_1)\in Z(N-1;0)\\ 1 \leqq n_3 \leqq m_3 - 1}} \prod_{i=1}^{3} \mu_i^{-n_i} \beta_i^{-1}(n_i) f_N((0, n_3), n_2, n_1)(s)[n_3 \,.\, \mu_3] \tag{7.2}$$

$$+ \sum_{\substack{((1,n_3),n_2,n_1)\in Z(N-1;1)\\ n_3 \geqq m_3 - 1}} \left(\prod_{i=1}^{2} \mu_i^{-n_i} \beta_i^{-1}(n_i)\right) \mu_3^{-(n_3+1)} \tag{7.3}$$
$$\times \beta_3^{-1}(n_3+1) f_N((1, n_3), n_2, n_1)(s)[m_3 \,.\, \mu_3]$$

$$+ \sum_{\substack{((0,n_3),n_2,n_1)\in Z(N-1;0)\\ n_2 \geqq 1, n_3 \geqq m_3 - 1}} \prod_{i=1}^{3} \mu_i^{-n_i} \beta_i^{-1}(n_i) f_N((0, n_3), n_2, n_1)(s)[\alpha_2(n_2)\mu_2] \tag{7.4}$$

$$+ \sum_{\substack{((0,n_3),n_2,n_1)\in Z(N-1;0)\\ n_2 \geqq 1, n_3 < m_3 - 1}} \prod_{i=1}^{3} \mu_i^{-n_i} \beta_i^{-1}(n_i) f_N((0, n_3), n_2, n_1)(s)[\alpha_2(n_2 \,.\, \mu_2)] \tag{7.5}$$

$$+ \sum_{\substack{(n_3,(0,n_2),n_1)\in Z'(N-1;0)\\ 0 \leqq n_3 \leqq m_3 - 1}} \prod_{i=1}^{3} \mu_i^{-n_i} \beta_i^{-1}(n_i) g_N(n_3, (0, n_2), n_1)(s) \,.\, [\mu_3]. \tag{7.6}$$

Here (7.1) cancels against $\alpha_1(n_1)\mu_1$ on the LHS, (7.4) and (7.5) cancel against $\alpha_2(n_2)\mu_2$ on the LHS, and (7.2) cancels against $(\alpha_3(0, n_3) - 1)\mu_3$ for $n_3 \leqq m_3 - 1$.

So we have

$$
\begin{aligned}
&\sum_{((0,n_3),n_2,n_1)\in Z(N-1;0)} \prod_{i=1}^{3} \mu_i^{-n_i}\beta_i^{-1}(n_i) f_N((0,n_3),n_2,n_1)(s) \\
&\quad + \sum_{\substack{((0,n_3),n_2,n_1)\in Z(N-1;0)\\ n_3\geqq m_3}} \prod_{i=1}^{3} \mu_i^{-n_i}\beta_i^{-1}(n_i) \\
&\quad \times f_N((0,n_3),n_2,n_1)(s)\,\frac{(m_3-1)\mu_3}{\mu_3+s} \\
&\quad = \sum_{\substack{((1,n_3),n_2,n_1)\in Z(N-1;1)\\ n_3\geqq m_3-1}} \prod_{i=1}^{2} \mu_i^{-n_i}\beta_i^{-1}(n_i)\mu_3^{-(n_3+1)}\beta_3^{-1}(n_3+1) \\
&\quad \times f_N((1,n_3),n_2,n_1)(s)\,\frac{m_3\mu_3}{\mu_3+s} \\
&\quad + \sum_{\substack{((0,n_3),n_2,n_1)\in Z(N-1;0)\\ 0\leqq n_3\leqq m_3-1}} \prod_{i=1}^{3} \mu_i^{-n_i}\beta_i^{-1}(n_i) g_N(n_3,(0,n_2),n_1)(s)\,\frac{\mu_3}{\mu_3+s}.
\end{aligned}
\tag{8}
$$

(ii) For $\tilde{n}_3\geqq 1$ we multiply both sides of (6) by

$$\left(\prod_{i=1}^{2} \mu_i^{-n_i}\beta_i^{-1}(n_i)\mu_3^{-(\tilde{n}_3+n_3)}\beta_3^{-1}(\tilde{n}_3+n_3)\right)(M((\tilde{n}_3,n_3),n_2,n_1)+s)$$

and sum over $\{n_3\geqq m_3-1\}\cap Z(N-1;\tilde{n}_3)$. Rearranging the sums on the RHS we then obtain:

$$
\begin{aligned}
&\sum_{\substack{((\tilde{n}_3,n_3),n_2,n_1)\in Z(N-1;\tilde{n}_3)\\ n_3\geqq m_3-1}} \mu_3^{-(\tilde{n}_3+n_3)}\beta_3^{-1}(\tilde{n}_3+n_3)\prod_{i=1}^{2} \mu_i^{-n_i}\beta_i^{-1}(n_i) \\
&\quad \times f_N((\tilde{n}_3,n_3),n_2,n_1)(s)\times[\alpha_3(\tilde{n}_3,n_3)\mu_3+\alpha_2(n_2)\mu_2+\alpha_1(n_1)\mu_1+s] \\
&= \sum_{\substack{((\tilde{n}_3,n_3),n_2,n_1)\in Z(N-1;\tilde{n}_3)\\ n_3\geqq m_3-1,\,n_1\geqq 1}} \mu_3^{-(\tilde{n}_3+n_3)}\beta_3^{-1}(\tilde{n}_3+n_3)\prod_{i=1}^{2} \mu_i^{-n_i}\beta_i^{-1}(n_i) \\
&\quad \times f_N((\tilde{n}_3,n_3),n_2,n_1)(s)\,.\,[\alpha_1(n_1)\mu_1] \\
&\quad + \sum_{\substack{((\tilde{n}_3+1,n_3),n_2,n_1)\in Z(N-1;\tilde{n}_3+1)\\ n_3\geqq m_3-1}} \mu_3^{-(\tilde{n}_3+n_3+1)}\beta_3^{-1}(\tilde{n}_3+n_3+1)\prod_{i=1}^{2} \mu_i^{-n_i}\beta_i^{-1}(n_i) \\
&\quad \times f_N((\tilde{n}_3+1,n_3),n_2,n_1)(s)[\mu_3 m_3] \\
&\quad + \sum_{\substack{((\tilde{n}_3,n_3),n_2,n_1)\in Z(N-1;\tilde{n}_3)\\ n_3\geqq m_3-1,\,n_2\geqq 1}} \mu_3^{-(\tilde{n}_3+n_3)}\beta_3^{-1}(\tilde{n}_3+n_3)\prod_{i=1}^{2} \mu_i^{-n_i}\beta_i^{-1}(n_i) \\
&\quad \times f_N((\tilde{n}_3,n_3),n_2,n_1)(s)\,.\,[\alpha_2(n_2)\mu_2]+
\end{aligned}
\tag{9}
$$

$$+\sum_{\substack{((\tilde n_3-1,m_3-1),n_2,n_1)\in Z(N-1;\tilde n_3-1)\\ n_2\geqq 1}} \mu_3^{-(\tilde n_3-1+m_3-1)}$$

$$\times\beta_3^{-1}(\tilde n_3-1+m_3-1)\prod_{i=1}^{2}\mu_i^{-n_i}\beta_i^{-1}(n_i)$$

$$\times f_N((\tilde n_3-1,\,m_3-1),\,n_2,\,n_1)(s)\,.\left[\frac{m_3-1}{m_3}\,.\,\alpha_2(n_2)\mu_2\right]$$

$$+\sum_{(\tilde n_3+m_3-1,(0,n_2),n_1)\in Z'(N-1;0)} \mu_3^{-(\tilde n_3+m_3-1)}\beta_3^{-1}(\tilde n_3+m_3-1)\prod_{i=1}^{2}\mu_i^{-n_i}\beta_i^{-1}(n_i)$$

$$\times g_N(\tilde n_3+m_3-1,\,(0,\,n_2),\,n_1)(s)\,.\,[\mu_3].$$

(Empty sums are defined to be 0.)

Cancelling in (9) as in (7) we obtain for $\tilde n_3\geqq 1$

$$\sum_{\substack{((\tilde n_3,n_3),n_2,n_1)\in Z(N-1;\tilde n_3)\\ n_3\geqq m_3-1}} \mu_3^{(-\tilde n_3+n_3)}\beta_3^{-1}(\tilde n_3+n_3)\prod_{i=1}^{2}\mu_i^{-n_i}\beta_i^{-1}(n_i)$$

$$\times f_N((\tilde n_3,\,n_3),\,n_2,\,n_1)(s)$$

$$=\sum_{\substack{((\tilde n_3+1,n_3),n_2,n_1)\in Z(N-1;\tilde n_3+1)\\ n_3\geqq m_3-1}} \mu_3^{-(\tilde n_3+n_3+1)}\beta_3^{-1}(\tilde n_3+n_3+1)\prod_{i=1}^{2}\mu_i^{-n_i}\beta_i^{-1}(n_i)$$

$$\times f_N((\tilde n_3+1,\,n_3),\,n_2,\,n_1)(s)\,\frac{m_3\mu_3}{m_3\mu_3+s}$$

$$+\sum_{\substack{((\tilde n_3-1,m_3-1),n_2,n_1)\in Z(N-1;\tilde n_3-1)\\ n_2\geqq 1}} \mu_3^{-(\tilde n_3-1+m_3-1)} \tag{10}$$

$$\times\beta_3^{-1}(\tilde n_3-1+m_3-1)\prod_{i=1}^{2}\mu_i^{-n_i}\beta_i^{-1}(n_i)$$

$$\times f_N((\tilde n_3-1,\,m_3-1),\,n_2,\,n_1)(s)\,\frac{m_3-1}{m_3}\,\alpha_2(n_2)\mu_2\,\frac{1}{m_3\mu_3+s}$$

$$+\sum_{(\tilde n_3+m_3-1,(0,n_2),n_1)\in Z'(N-1;0)} \mu_3^{-(\tilde n_3+m_3-1)}\beta_3^{-1}(\tilde n_3+m_3-1)\prod_{i=1}^{2}\mu_i^{-n_i}\beta_i^{-1}(n_i)$$

$$\times g_N(\tilde n_3+m_3-1,\,(0,\,n_2),\,n_1)(s)\,\frac{\mu_3}{m_3\mu_3+s}.$$

(iii) For $\tilde n_3\geqq 0$ we multiply both sides of (6) by

$$\left(\prod_{i=1}^{2}\mu_i^{-n_i}\beta_i^{-1}(n_i)\mu_3^{-(\tilde n_3+n_3)}\beta_3^{-1}(\tilde n_3+n_3)\right)(M((\tilde n_3,\,n_3),\,n_2,\,n_1)+s)$$

and sum over $\{n_3\geqq m_3\}\cap Z(N-1;\,\tilde n_3)$. Rearranging the sums on the RHS we then obtain

$$\sum_{\substack{((\tilde n_3,n_3),n_2,n_1)\in Z(N-1;\tilde n_3)\\ n_3\geqq m_3}} \mu_3^{-(\tilde n_3+n_3)}\beta_3^{-1}(\tilde n_3+n_3)\prod_{i=1}^{2}\mu_i^{-n_i}\beta_i^{-1}(n_i)$$

$$\times f_N((\tilde n_3, n_3), n_2, n_1)(s)\,.\,[m_3\mu_3+\alpha_2(n_2)\mu_2+\alpha_1(n_1)\mu_1+s]$$

$$= \sum_{\substack{((\tilde n_3,n_3),n_2,n_1)\in Z(N-1;\tilde n_3)\\ n_3\geqq m_3,n_1\geqq 1}} \mu_3^{-(\tilde n_3+n_3)}\beta_3^{-1}(\tilde n_3+n_3)\prod_{i=1}^{2}\mu_i^{-n_i}\beta_i^{-1}(n_i)$$

$$\times f_N((\tilde n_3, n_3), n_2, n_1)(s)\,.\,[\alpha_1(n_1)\mu_1] \tag{11}$$

$$+ \sum_{\substack{((\tilde n_3+1,n_3),n_2,n_1)\in Z(N-1;\tilde n_3+1)\\ n_3\geqq m_3}} \mu_3^{-(\tilde n_3+n_3+1)}\beta_3^{-1}(\tilde n_3+n_3+1)\prod_{i=1}^{2}\mu_i^{-n_i}\beta_i^{-1}(n_i)$$

$$\times f_N((\tilde n_3+1, n_3), n_2, n_1)(s)\,.\,[m_3\mu_3]$$

$$+ \sum_{\substack{((\tilde n_3,n_3),n_2,n_1)\in Z(N-1;\tilde n_3)\\ n_3\geqq m_3-1,n_2\geqq 1}} \mu_3^{-(\tilde n_3+n_3)}\beta_3^{-1}(\tilde n_3+n_3)\prod_{i=1}^{2}\mu_i^{-n_i}\beta_i^{-1}(n_i)$$

$$\times f_N((\tilde n_3, n_3), n_2, n_1)(s)\,.\,[\alpha_2(n_2)\mu_2].$$

Cancelling again as before it follows for $\tilde n_3\geqq 0$ that

$$\sum_{\substack{((\tilde n_3,n_3),n_2,n_1)\in Z(N-1;\tilde n_3)\\ n_3\geqq m_3}} \mu_3^{-(\tilde n_3+n_3)}\beta_3^{-1}(\tilde n_3+n_3)\prod_{i=1}^{2}\mu_i^{-n_i}\beta_i^{-1}(n_i)$$

$$\times f_N((\tilde n_3, n_3), n_2, n_1)(s)$$

$$= \sum_{\substack{((\tilde n_3+1,n_3),n_2,n_1)\in Z(N-1;\tilde n_3+1)\\ n_3\geqq m_3}} \mu_3^{-(\tilde n_3+n_3+1)}\beta_3^{-1}(\tilde n_3+n_3+1)\prod_{i=1}^{2}\mu_i^{-n_i}\beta_i^{-1}(n_i)$$

$$\times f_N((\tilde n_3+1, n_3), n_2, n_1)(s)\,\frac{\mu_3 m_3}{\mu_3 m_3+s} \tag{12}$$

$$+ \sum_{\substack{((\tilde n_3,m_3-1),n_2,n_1)\in Z(N-1;\tilde n_3)\\ n_2\geqq 1}} \mu_3^{-(\tilde n_3+m_3-1)}\beta_3^{-1}(\tilde n_3+m_3-1)\prod_{i=1}^{2}\mu_i^{-n_i}\beta_i^{-1}(n_i)$$

$$\times f_N((\tilde n_3, m_3-1), n_2, n_1)(s)\,\frac{\alpha_2(n_2)\mu_2}{\mu_3 m_3+s}.$$

Proceeding further, from (12) we obtain after multiplication by $(\mu_3 m_3/(\mu_3 m_3 + s))^{\tilde{n}_3}$, and summation over $\tilde{n}_3 \geqq 0$,

$$\begin{aligned}
&\sum_{\substack{((0,n_3),n_2,n_1)\in Z(N-1;0)\\ n_3\geqq m_3}} \prod_{i=1}^{3} \mu_i^{-n_i}\beta_i^{-1}(n_i) f_N((0,n_3),n_2,n_1)(s) \\
&= \sum_{\tilde{n}_3=0}^{N-1} \sum_{\substack{((\tilde{n}_3,m_3-1),n_2,n_1)\in Z(N-1;\tilde{n}_3)\\ n_2\geqq 1}} \mu_3^{-(\tilde{n}_3+n_3)}\beta_3^{-1}(\tilde{n}_3+n_3)\prod_{i=1}^{2}\mu_i^{-n_i}\beta_i^{-1}(n_i) \\
&\quad \times f_N((\tilde{n}_3,m_3-1),n_2,n_1)(s)\,.\,\frac{\alpha_2(n_2)\mu_2}{\mu_3 m_3+s}\left(\frac{\mu_3 m_3}{\mu_3 m_3+s}\right)^{\tilde{n}_3}.
\end{aligned} \tag{13}$$

On the other hand, multiplication of (10) by $(m_3\mu_3/(m_3\mu_3+s))^{\tilde{n}_3-1}$ and summation over $\tilde{n}_3 \geqq 1$ yields

$$\begin{aligned}
&\sum_{((1,\tilde{n}_3),n_2,n_1)\in Z(N-1;1)} \mu_3^{-(n_3+1)}\beta_3^{-1}(n_3+1)\prod_{i=1}^{2}\mu_i^{-n_i}\beta_i^{-1}(n_i) \\
&\quad \times f_N((1,n_3),n_2,n_1)(s) \\
&= \sum_{\tilde{n}_3=0}^{N-2} \sum_{\substack{((\tilde{n}_3,m_3-1),n_2,n_1)\in Z(N-1;\tilde{n}_3)\\ n_2\geqq 1}} \mu_3^{-(\tilde{n}_3+n_3)}\beta_3^{-1}(\tilde{n}_3+n_3)\prod_{i=1}^{2}\mu_i^{-n_i}\beta_i^{-1}(n_i) \\
&\quad \times f_N((\tilde{n}_3,m_3-1),n_2,n_1)(s)\left(\frac{\mu_3 m_3}{\mu_3 m_3+s}\right)^{\tilde{n}_3}\frac{m_3-1}{m_3}\frac{\alpha_2(n_2)\mu_2}{\mu_3 m_3+s} \\
&\quad + \sum_{\tilde{n}_3=1}^{N-1} \sum_{(\tilde{n}_3+m_3-1,(0,n_2),n_1)\in Z'(N-1;0)} \mu_3^{-(\tilde{n}_3+m_3-1)} \\
&\quad \times \beta_3^{-1}(\tilde{n}_3+m_3-1)\prod_{i=1}^{2}\mu_i^{-n_i}\beta_i^{-1}(n_i) \\
&\quad \times g_N(\tilde{n}_3+m_3-1,(0,n_2),n_1)(s)\frac{\mu_3}{\mu_3 m_3+s}\left(\frac{\mu_3 m_3}{\mu_3 m_3+s}\right)^{n_3-1}.
\end{aligned} \tag{14}$$

Inserting (14) and (13) into (8) yields our main result as follows.

Lemma 1. The LST of a job's cycle time in the system of Figure 1 which is started with initial distribution $\tilde{p}(\cdot)$ fulfils the following recursion:

$$\begin{aligned}
f_N(s) &= \sum_{(n_3,n_2,n_1)\in Z(N-1)} \tilde{p}(n_3,n_2,n_1) f_N((0,n_3),n_2,n_1)(s) \\
&= \sum_{(n_3,n_2,n_1)\in Z(N-1)} \tilde{p}(n_3,n_2,n_1) g_N(n_3,(0,n_2),n_1)(s) \\
&\quad \times\left(\frac{\mu_3}{\mu_3+s}\right)\left(\frac{\mu_3 m_3}{\mu_3 m_3+s}\right)^{(n_3-m_3+1)_+}, \qquad s\geqq 0,
\end{aligned} \tag{15}$$

where $(n_3-m_3+1)_+ = \max(0, n_3-m_3+1)$.

As a first consequence of Lemma 1 assume that service times at server 1 are 0, i.e. we have a closed two-stage tandem with multiple-server queues.

Kawashima and Torigoe (1983) proved the following result.

Theorem 2. The cycle-time LST in a two-stage tandem of multiserver queues having the same parameters as nodes 3 and 2 in Figure 1 is given by

$$f_N^{(ii)}(s) = \sum_{n_2+n_3=N-1} G^{(ii)}(N-1)^{-1} \prod_{i=2}^{3} \mu_i^{-n_i} \beta_i^{-1}(n_i) \times \frac{\mu_i}{\mu_i+s}\left(\frac{\mu_i m_i}{\mu_i m_i+s}\right)^{(n_i-m_i+1)_+}, \qquad s \geqq 0, \tag{16}$$

where $G^{(ii)}(N-1)$ is the norming constant.

Remark. Kawashima and Torigoe proved even more: they computed by reversibility arguments the LST of the joint distributions of the successive sojourn times in nodes 3 and 2. It is (as one would expect) found by replacing s in (16) by s_2, s_3, in factors concerning μ_2, μ_3, respectively.

4. Burke's theorem on cycle times

Let us consider the system of Figure 2, an open three-node tandem consisting of a multiserver, a single-server, and another multiserver node. Jobs arrive at node 3 in a Poisson stream of intensity $\lambda > 0$, which is independent of the independent family of service times, traverse through the system and eventually leave it. For an extensive discussion see for example Burke (1972) or Kelly (1979).

From the point of applications an important byproduct of Burke's and Reich's theorems on such systems is the distribution of a job's passage time through the system. Because the sojourn times at the three nodes are independent of each other the passage time is the convolution of the successive sojourn times.

The proof depends heavily on the fact that jobs which are behind a test job

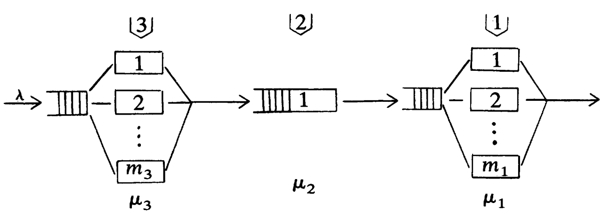

Figure 2

C when C is present in node 2 cannot overtake C and influence C's residual passage time.

The question arises whether such a theorem holds for the closed system of Figure 1, replacing 'passage times' by 'cycle times'. Clearly a convolution formula is impossible, due to the finite population of the system. But only recently it has turned out that for single-server queues a 'convolution-like' formula describes the cycle-time distribution. To be more precise, the following theorem holds.

Theorem 3 (Schassberger and Daduna (1983)). Assume that the system of Figure 1 consists of single-server nodes, i.e., $m_1 = m_2 = m_3 = 1$. Then the cycle-time LST is given by

$$f_N(s) = \sum_{(n_3,n_2,n_1)\in Z(N-1)} G(N-1)^{-1} \prod_{i=1}^{3} \mu_i^{-n_i} \left(\frac{\mu_i}{\mu_i + s}\right)^{n_i+1}, \qquad s \geqq 0. \tag{17}$$

Remark. Indeed Boxma, Kelly and Konheim (1982) computed the LST of the joint distribution of the successive sojourn times, which again lead to a formula obtained from (17) by inserting independent variables s_i for s in factors concerning μ_i, $i = 1, 2, 3$.

We shall give an answer to the question stated above in the following theorem.

Theorem 4. Assume that in the system of Figure 1 the second node has only a single server, i.e. $m_2 = 1$. If the system is started with initial distribution $\tilde{p}(\cdot)$ (see (2)) with a tagged job at the tail of the queue at node 3, then the LST of the tagged job's cycle time is given by

$$\begin{aligned} f_N(s) = \sum_{(n_3,n_2,n_1)\in Z(N-1)} & G(N-1)^{-1} \left(\prod_{i=1}^{3} \mu_i^{-n_i}\right) \beta_3^{-1}(n_3)\beta_1^{-1}(n_1) \left(\prod_{i=1}^{3} \left(\frac{\mu_i}{\mu_i + s}\right)\right) \\ & \times \left(\frac{\mu_1 m_1}{\mu_1 m_1 + s}\right)^{(n_1 - m_1 + 1)_+} \left(\frac{\mu_2}{\mu_2 + s}\right)^{n_2} \left(\frac{\mu_3 m_3}{\mu_3 m_3 + s}\right)^{(n_3 - m_3 + 1)_+}, \qquad s \geqq 0. \end{aligned} \tag{18}$$

Proof. From Lemma 1 we have

$$\begin{aligned} f_N(s) = \sum_{(n_3,n_2,n_1)\in Z(N-1)} & G(N-1)^{-1} \left(\prod_{i=1}^{3} \mu_i^{-n_i}\right) \beta_3^{-1}(n_3)\beta_1^{-1}(n_1) \\ & \times g_N(n_3, (0, n_2), n_1)(s) \frac{\mu_3}{\mu_3 + s} \left(\frac{\mu_3 m_3}{\mu_3 m_3 + s}\right)^{(n_3 - m_3 + 1)_+}. \end{aligned} \tag{15'}$$

This recursion reduces $f_N(s)$ to the $g_N(\cdot)(s)$, which are LST of residual cycle times conditioned on C's arrival at node 2 just before. A little reflection shows that jobs left behind at node 3 can no longer influence C's actual cycle time, i.e.

(19) $g_N(n_3, (0, n_2), n_1)(s) = g_{N-n_3}(0, (0, n_2), n_1)(s), \qquad n_3+n_2+n_1=N-1.$

Rearranging the summation in (15′) yields

$$
\begin{aligned}
f_N(s) &= \sum_{n_3=0}^{N-1} G(N-1)^{-1}G(N-1-n_3)\mu_3^{-n_3}\beta_3^{-1}(n_3)\frac{\mu_3}{\mu_3+s}\left(\frac{\mu_3 m_3}{\mu_3 m_3+s}\right)^{(n_3-m_3+1)_+} \\
&\quad \times \sum_{\substack{(n_2,n_1) \\ n_2+n_1=N-1-n_3}} G(N-1-n_3)^{-1}\prod_{i=1}^{2}\mu_i^{-n_i}\beta_i^{-1}(n_i)g_N(0, (0, n_2), n_1)(s).
\end{aligned} \tag{20}
$$

But from Lemma 1 again (with zero service time at node 3) we conclude (because for fixed $n_3 \in \{0, \cdots, N-1\}$ the last sum in (20) is just the cycle time in a two-stage tandem) that (18) holds.

In order to show that under the condition m_3, $m_1 < N$ the condition $m_2 = 1$ is essential for the product form (18) of the passage-time distribution, let us consider the following simple example:

$$m_1 = m_3 = 1, \qquad m_2 = N \geqq 2,$$

i.e., nodes 1 and 3 are single-server queues, while node 2 acts as infinite server. Lemma 1 yields for the passage-time LST

$$f_N(s) = \sum_{(n_3,n_2,n_1)\in Z(N-1)} G(N-1)^{-1}\prod_{i=1}^{3}\mu_i^{-n_i}\frac{1}{n_2!}g_N(n_3, (0, n_2), n_1)(s)\left(\frac{\mu_3}{\mu_3+s}\right)^{n_3+1}.$$

Now for this simple system we obtain a system of first-entrance equations like (6) for the $g_N(\cdot)(s)$. The arguments leading to it are the same as presented there, so we shall state it here without further comment. For $(n_3, n_2, n_1) \in Z(N-1)$ we set

$$M'(n_3, (0, n_2), n_1) = 1_{\{n_3\neq 0\}}\mu_3 + 1_{\{n_1\neq 0\}}\mu_1 + (n_2+1)\mu_2$$

and have

$$
\begin{aligned}
&g_N(n_3, (0, n_2), n_1)(s) \\
&= \frac{M'(n_3, (0, n_2), n_1)}{M'(n_3, (0, n_2), n_1)+s} \\
&\quad \times \left[\frac{1_{\{n_3\neq 0\}}\mu_3}{M'(n_3, (0, n_2), n_1)} g_N(n_3-1, (0, n_2+1), n_1)(s)\right. \\
&\quad + \frac{1_{\{n_1\neq 0\}}\mu_1}{M'(n_3, (0, n_2), n_1)} g_N(n_3+1, (0, n_2), n_1-1)(s) \\
&\quad + \frac{n_2\mu_2}{M'(n_3, (0, n_2), n_1)} g_N(n_3, (0, n_2-1), n_1+1)(s) \\
&\quad \left. + \frac{\mu_2}{M'(n_3, (0, n_2), n_1)}\left(\frac{\mu_1}{\mu_1+s}\right)^{n_1+1}\right], \ (n_3, n_2, n_1) \in Z(N-1).
\end{aligned}
$$

Solving this system explicitly for $N=2$ yields

$$f_2(s) \neq \sum_{(n_3,n_2,n_1)} G(1)^{-1} \prod_{i=1}^{3} \mu_i^{-n_i} \frac{1}{n_2!} \left(\frac{\mu_1}{\mu_1+s}\right)^{n_1+1} \frac{\mu_2}{\mu_2+s} \left(\frac{\mu_3}{\mu_3+s}\right)^{n_3+1}.$$

The reader should note that starting the cycle at node 2 or 3 yields the desired product-form cycle-time distribution!

5. Gordon–Newell networks

The tandem system is a special case of a Gordon–Newell network (see e.g. Gordon and Newell (1967)) or of a BCMP network (see Baskett et al. (1975)) and (assuming deterministic routing) of a Kelly network (see Kelly (1979)). (The last two classes of networks have different job classes, in contrast to the original Gordon–Newell network.) We give only a very short description of these networks.

We have a set $\tilde{M}=\{1,\dots,M\}$ of nodes, node i consisting of $m_i \geqq 1$ exponential servers each with parameter $\mu_i > 0$. $N \geqq 1$ jobs are travelling through the network, their routes are governed by the routing matrix $R=(r_{ij}: i, j \in \tilde{M})$, where

$$r_{ij} \geqq 0, \qquad \sum_{j=1}^{M} r_{ij} = 1, \qquad i=1,\cdots,M.$$

We assume $\tilde{M}$ to be partitioned by R into disjoint irreducible subsets, and fix throughout the paper a strict positive probability vector $\lambda=(\lambda_1,\cdots,\lambda_M)$ satisfying the traffic equation

$$\lambda = \lambda \,.\, R. \tag{21}$$

The state space for a Markovian description of the system is

$$Z(N)=\left\{(n_M,\cdots,n_1): n_i \geqq 0, \qquad \sum_{i=1}^{M} n_i = N\right\},$$

where n_i is the queue length at node i. The stationary distribution of the describing strong Markov process is given by

$$p(n_M,\cdots,n_1) = G(N)^{-1} \prod_{i=1}^{M} \left(\frac{\lambda_i}{\mu_i}\right)^{n_i} \beta_i^{-1}(n_i), \quad (n_M,\cdots,n_1) \in Z(N) \tag{22}$$

where $\beta_i^{-1}(n_i)$ is defined as in (1), and $G(N)$ is the norming constant. We define a path to be a sequence of nodes $W=(W_K,\cdots,W_1)$ which satisfies $r_{W_k,W_{k-1}} > 0$, $2 \leqq k \leqq K$.

The notion of an overtake-free path in an open Jackson network was independently introduced by Walrand and Varaiya (1980) and Melamed

(1982a). It carries over to closed networks without difficulty (see Daduna (1982)). We state the definition here only for the case of indistinguishable jobs: Let P_{ij} be the set of paths from i to j, $i, j \in \tilde{M}$, i.e.

$$W = (W_K, \cdots, W_1) \in P_{ij} \Leftrightarrow W_K = i,\ W_1 = j,$$

and P_{ij}^{l} be the set of paths from i to j via l; $i, j, l \in \tilde{M}$, i.e.

$$W = (W_K, \cdots, W_1) \in P_{ij}^{l} \Leftrightarrow (W \in P_{ij} \text{ and } W_k = 1 \text{ for some } k \in \{1, \cdots, K\}).$$

A path $W = (W_K, \cdots, W_1)$ of different nodes is overtake-free if and only if

$$P_{W_k W_l} \subseteq P_{W_k W_l}^{W_{k-1}} \quad \text{for} \quad K \geqq k > l \geqq 1$$

holds.

Let us consider any path $W = (W_K, \cdots, W_1)$ and a job which will traverse this path directly. We tag that job at its arrival at node W_K and refer to it henceforth as C. The probability of C finding n_i untagged jobs at node i, when it enters W_K, $i = 1, \cdots, M$, $\sum_{i=1}^{M} n_i = N-1$, is given by

$$(23) \quad \tilde{p}(n_M, \ldots, n_1) = G(N-1)^{-1} \prod_{i=1}^{M} \left(\frac{\lambda_i}{\mu_i}\right)^{n_i} \beta_i^{-1}(n_i), \qquad (n_M, \cdots, n_1) \in Z(N-1),$$

where $Z(N-1)$, $G(N-1)$ are defined as in (22).

Now a tedious but direct generalisation of Lemma 1 yields the following result.

Lemma 5. Given a Gordon–Newell network as described above which is started with initial distribution $\tilde{p}(\cdot)$, i.e. with an arrival of a job at node W_K which will proceed directly through W, the passage time through W of that job has LST

$$(24) \quad \begin{aligned} f_N^W(s) = & \sum_{(n_M, \cdots, n_{W_K}, \cdots, n_{W_{K-1}}, \cdots, n_1) \in Z(N-1)} \tilde{p}(n_M, \cdots, n_{W_K}, \cdots, n_{W_{K-1}}, \cdots, n_1) \\ & \times g_N^W(n_M, \cdots, n_{W_K}, \cdots, (0, n_{W_{K-1}}), \cdots, n_1)(s) \\ & \times \frac{\mu_{W_K}}{\mu_{W_K} + s} \left(\frac{\mu_{W_K} m_{W_K}}{\mu_{W_K} m_{W_K} + s}\right)^{(n_{W_K} - m_{W_K} + 1)_+} \end{aligned}$$

where $g_N(n_M, \cdots, n_{W_K}, \cdots, (0, n_{W_{K-1}}), \ldots, n_1)(s)$ is to be interpreted as the conditional LST of the residual passage time of C when it enters W_{K-1} finding n_i jobs at node $i \in M$.

An immediate consequence of this lemma is the following.

Theorem 6. Given a Gordon–Newell network as described above which is started with initial distribution $\tilde{p}(\cdot)$, i.e. with an arrival of a job at the path $W = (W_K, \cdots, W_1)$, which will proceed directly through W, let $f_N^W(s)$, $s \geqq 0$, denote the LST of this job's passage time through W.

(a) If $W=(W_2, W_1)$, then

$$f_N^W(s)=\sum_{(n_M,\cdots,n_{W_2},\cdots,n_{W_1},\cdots,n_1)\in Z(N-1)} \tilde{p}(n_M,\cdots,n_{W_2},\cdots,n_{W_1},\cdots,n_1)$$
$$\times\prod_{i=1}^{2}\frac{\mu_{W_i}}{\mu_{W_i}+s}\left(\frac{\mu_{W_i}m_{W_i}}{\mu_{W_i}m_{W_i}+s}\right)^{(n_{W_i}-m_{W_i}+1)_+}.$$

(b) If $W=(W_1, W_1)$, then

$$f_N^W(s)=\sum_{(n_M,\cdots,n_{W_1},\cdots,n_1)\in Z(N-1)} \tilde{p}(n_M,\cdots,n_{W_1},\cdots,n_1)$$
$$\times\left[\frac{\mu_{W_1}}{\mu_{W_1}+s}\cdot\left(\frac{\mu_{W_1}m_{W_1}}{\mu_{W_1}m_{W_1}+s}\right)^{(n_{W_1}-m_{W_1}+1)_+}\right]^2.$$

(c) If $W=(W_K, W_{K-1},\cdots, W_2, W_1)$ is an overtake-free path such that nodes W_k are single-server nodes, $k=K-1,\cdots,2$, i.e. $m_k=1$, for $k=K-1,\cdots,2$, then

$$f_N^W(s)=\sum_{(n_M,\cdots,n_1)\in Z(N-1)} \tilde{p}(n_M,\cdots,n_1)\prod_{k=1}^{K}\left(\frac{\mu_{W_k}}{\mu_{W_k}+s}\left(\frac{\mu_{W_k}m_{W_k}}{\mu_{W_k}m_{W_k}+s}\right)^{(n_{W_k}-m_{W_k}+1)_+}\right).$$

(d) If $W=(W_K,\ldots, W_{L+1}, W_L,\ldots, W_{I+1}, W_I,\ldots, W_1)$ can be split into a path $W'=(W_K,\ldots, W_{L+1})$, a path $W''=(W_L,\ldots, W_{I+1})$, and a path $W'''=(W_I,\cdots, W_1)$, such that the following holds:

(i) W'' is overtake-free and $W_{L+1},\cdots, W_{I+2}$ are single-server nodes;

(ii) the nodes of W' and W''' are working like infinite-server nodes, i.e.

$$m_k\geqq N \quad \text{for} \quad k=1,\cdots, I, L+1,\cdots, K;$$

then

$$f_N^W(s)=\left(\prod_{k=1}^{K}\frac{\mu_{W_k}}{\mu_{W_k}+s}\right)\sum_{(n_M,\cdots,n_1)\in Z(N-1)} \tilde{p}(n_M,\cdots,n_1)$$
$$\times\prod_{k=I+1}^{L}\left(\frac{\mu_{W_k}m_{W_k}}{\mu_{W_k}m_{W_k}+s}\right)^{(n_{W_k}-m_{W_k}+1)_+}.$$

Proof. (a) and (b) follow directly from (24) by inserting the known quantities for the conditional LST of the residual passage times. (c) can be proved by induction. Starting from (24) one proceeds in the same way as in Theorem 4 walking through the path and reducing the number of jobs by elimination of those jobs left behind by the test job, and those jobs which leave the part of the network from where they can influence the test job's actual passage through W. (A careful study of the complicated induction structure can be found in Daduna (1982); it carries over to our proof here.) (d) is an obvious generalization of (c) because Lemma 5 reduces the problem easily to the problem of determining the passage-time distribution through (W'', W'''). Here in repeating the proof of (c) we have to change only the last step of the induction.

Remarks. (a) Theorem 6 holds for networks with different customer classes; we have only to change the probabilities $\tilde{p}(\cdot)$. The proof is again obvious after reading the proof of Theorem 5 in Daduna (1982).

(b) 6(a) is a generalisation of the result of Kawashima and Torigoe (1983) on cycle times, and 6(a) and 6(b) solve the problem of passage times for 'two-stations-walks'.

(c) Wong (1979) dealt with a special case of 6(d): a cycle of an infinite server $m_1 \geqq N$ and a multiserver queue with $m_2 < N$.

(d) As Theorems 4 and 6 show the non-overtaking property of the paths under consideration is the core of the passage-time analysis. Moreover Walrand and Varaiya (1980) and Simon and Foley (1979) proved for open networks that paths with overtaking produce dependent sojourn times. On the other hand in an open three-tandem node where the second node is not a single-server node, dependent sojourn times are likewise found. (The same holds for single-server nodes where the queueing discipline allows overtaking, e.g. last-come–first-served or processor-sharing.)

(e) The outline of Burke's work presented in Section 4 leads to the question whether our results of Theorem 6 are valid for open networks with Poisson input streams as well. The answer to this question is non-trivial because of the following observations.

Burke's proof depends heavily on the Poissonian nature of the departure process from node 3. (See Figure 2.) But as Melamed (1979) has shown, the internal customer streams in open networks are in general non-Poissonian. On the other hand, any open network is in a sense the weak limit of a sequence of closed networks, see e.g. Barbour (1982). So it should be possible to extend Theorem 6 by convergence arguments to open networks—and even to mixed networks with different customer classes (Baskett et al. (1975)). Indeed, Theorem 6 is true for general mixed networks with different job classes and deterministic or probabilistic routing (Kelly or BCMP networks). This is proved in exactly the same way as presented for closed networks in this paper: proving a formula similar to (24) with an appropriate interpretation of $\tilde{p}(\cdot)$, $Z(\infty)$ (the number of jobs present in the system is unlimited), we derive the expressions of Theorem 6. In proving (24) for mixed networks there is one additional step to perform. The summations in formulas (8) through (14) provide infinite sums; in particular, the summation done in the RHS of (14) over $\tilde{n}_3 = 0, 1, 2, \cdots$, needs some justification. But the convergence of the series is easily shown by arguments similar to those presented for the single-server case in Daduna (1983).

Let us go back to the closed system of Figure 1. The remarks above seem to prevent results on cycle times in general which go behind the scope of Theorem 6. On the other hand, by 6(d) the assumption $m_3 = N$ leads to a simple expression for the cycle-time LST although it is obvious that jobs which had to wait in node 2 until C or one of $m_2 - 1$ other jobs leave node 2 can influence C's next sojourn time at node 1.

Acknowledgement

I want to thank Professor Schassberger, TU Berlin, for helpful discussions on the subject of this paper. The paper represents work done in the project on DISTRIBUTED SYSTEMS of the Deutsche Forschungsgemeinschaft at the Technical University of Berlin.

References

BARBOUR, A. (1982) Generalized semi-Markov schemes and open queueing networks. *J. Appl. Prob.* **19,** 469–474.

BASKETT, F., CHANDY, K. M., MUNTZ, R. R. AND PALACIOS, F. G. (1975) Open, closed, and mixed networks of queues with different classes of customers. *J. Assoc. Comput. Mach.* **22,** 248–260.

BOXMA, O. J. (1983) Response time distributions in cyclic queues. Paper presented at the 44th Session of the International Statistical Institute, Madrid.

BOXMA, O. J. AND DONK, P. (1982) On response-time and cycle-time distributions in cyclic queues. *Performance Evaluation* **2,** 181–194.

BOXMA, O. J., KELLY, F. P. AND KONHEIM, A. G. (1984) The product form for sojourn time distributions in cyclic exponential queues. *J. Assoc. Comput. Mach.* **31,** 128–133.

BURKE, P. J. (1964) The dependence of delays in tandem queues. *Ann. Math. Statist.* **35,** 874–875.

BURKE, P. J. (1968) The output process of a stationary *M/M/s* queueing system. *Ann. Math. Statist.* **39,** 1144–1152.

BURKE, P. J. (1972) Output processes and tandem queues. In *Proc. Symp. Computer Communications Networks and Teletraffic, Brooklyn* 1972, ed. J. Fox, 419–428.

CHOW, W. M. (1981) The cycle time distribution of exponential cyclic queues. *J. Assoc. Comput. Mach.* **27,** 281–286.

DADUNA, H. (1982) Passage times for overtake-free paths in Gordon–Newell networks. *Adv. Appl. Prob.* **14,** 672–686.

DADUNA, H. (1983) On passage times in Jackson networks: Two stations network and overtake-free paths. *Z. Operat. Res.* **27,** 239–256.

GORDON, W. J. AND NEWELL, G. F. (1967) Closed queueing systems with exponential servers. *Operat. Res.* **15,** 254–265.

KAWASHIMA, T. AND TORIGOE, N. (1983) Cycle time distribution in a central server queuing system with multi-server stations.

KELLY, F. P. (1979) *Reversibility and Stochastic Networks.* Wiley, New York.

KELLY, F. P. AND POLLETT, P. K. (1983) Sojourn times in closed queueing networks. *Adv. Appl. Prob.* **15,** 638–656.

MELAMED, B. (1979) Characterisation of Poisson traffic streams in Jackson queueing networks. *Adv. Appl. Prob.* **11,** 422–438.

MELAMED, B. (1982a) Sojourn times in queueing networks. *Math. Operat. Res.* **7,** 223–244.

MELAMED, B. (1982b) On Markov jump processes imbedded at jump epochs and their queueing-theoretic applications. *Math. Operat. Res.* **7,** 111–128.

REICH, E. (1957) Waiting times when queues are in tandem. *Ann. Math. Statist.* **28,** 768–773.

REICH, E. (1963) Note on queues in tandem. *Ann. Math. Statist.* **34,** 338–341.

SCHASSBERGER, R. AND DADUNA, H. (1983) The time for a round trip in a cycle of exponential queues. *J. Assoc. Comput. Mach.* **30,** 146–150.

SEKINO, A. (1972) Response time distribution of multiprogrammed time-shared computer systems. *Proc. 6th Annual Princeton Conf. Information Science and Systems*, 613–619.

SEVCIK, K. C. AND MITRANI, I. (1981) The distribution of queueing network states at input and output instants. *J. Assoc. Comput. Mach.* **28,** 358–371.

SIMON, B. AND FOLEY, R. D. (1979) Some results on sojourn times in acyclic Jackson networks. *Management Sci.* **25,** 1027–1034.

WALRAND, J. AND VARAIYA, P. (1980) Sojourn times and the overtaking condition in Jacksonian networks. *Adv. Appl. Prob.* **12,** 1000–1018.

WONG, J. (1979) Response time distribution of the $M/M/m/N$ queueing model. *Operat. Res.* **27,** 1196–1202.