Notes on Jackson-like queue networks

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February 1, 2016

1 Notation

Graph $G = (V, E)$, adjacency A, $n \times n$, in number of vertices. For vertices $i, j \in V$, we say i is connected to j, $i \sim j$ if the ordered pair $(i, j) \in E$. Directionality is implied by the ordering.

2 Jackson Networks

A Jackson Network is a network of queues where exogenous jobs arrive according to a Poisson process (exponential interarrival) with rate $\lambda > 0$. For a job completed at vertex i, jobs are routed to vertex j with some probability. For vertices $j \sim i$, jobs move onto another vertex probability $\sum_j p_{ij}$ or leave the network with probability $1 - \sum_j p_{ij}$.

The overall arrival rate at a vertex i, λ_i becomes:

$$\lambda_i = \lambda p_{0i} + \sum_{j=1} \lambda_i p_{ji}$$

for $(j, i) \in E$, and p_{0i} equal to the probability that an exogenous job arrives at i first. Additionally, p_{i0} is equal to the probability that a job item has left the network. The 0'th vertex acts as both source and sink–this is referred to as an *open* Jackson network.

3 Special Case

We wish to examine a special case where if the queue at a vertex i is full, a job is automatically rerouted to another queue. Once a job reaches the first available queue and the task is completed, the job leaves the network. Each queue in the network is an G/M/n/n queue. Note that although a vertex's exogenous arrivals may be Poissonian, it is not the case that the combination of inter-vertex arrivals plus exogenous arrivals is also Poissonian.

Consider such a network of k queues with uniform service rate μ and exogenous arrival rate λ. We can immediately observe that the expected number of tasks that have exited the system at time t is bounded above by $(kn)\mu t$ (best possible service rate). The number of tasks within the system is at least $(\lambda - \mu)t$. How do I know all of this is true?

Def'n: Call a directed acyclic graph a chain if the first and last vertex has degree 1, all others have degree 2. The longest path in the chain is the graph itself.

Observe that a chain of queues with $p_0 1 = 1$, the steady state behavior of a chain of k M/M/1 queues is equivalent to an M/M/k queue with capacity k.

Claim: A tree of k M/M/1 queues steady state service rate is bounded above by that of an M/M/n queue where n is the length of the longest chain in the tree.

Proof: Moving down the tree of available queues cuts off branches of potentially available queues, congestion at root nodes causes the bounding behavior. How bad?

4 Similarities to Jackson network