A diffusion approximation for a generalized Jackson network with reneging

Josh Reed* and Amy R. Ward

Department of Industrial and Systems Engineering
Georgia Institute of Technology
Atlanta, GA 30332

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Abstract

We consider a generalized Jackson network with reneging customers in heavy traffic. In particular, each customer joining a particular station may abandon the network if his service does not begin within a station-dependent, exponential amount of time. We establish that in heavy traffic this system can be approximated by a multi-dimensional regulated Ornstein-Uhlenbeck process.

1 Introduction

In many practical applications, customers faced with long waiting times abandon the system before receiving service, or renege. We incorporate this behavior into a generalized Jackson network by assuming that each customer joining a particular station reneges from the network if his service does not begin within an exponentially distributed amount of time. Our objective is to study the stationary behavior of the network. Of course, the stationary distribution of a (conventional) Jackson network *without* reneging does not, in general, have an explicit analytical form. Therefore, having little hope of an exact analysis, we develop heavy traffic limit theorems that support stationary distribution approximations.

In contrast to the work of Reiman [8] for conventional Jackson networks, our limit diffusion process is a multi-dimensional regulated Ornstein-Uhlenbeck (O-U) process, rather than a regulated Brownian motion (RBM). Much is known about RBM; see, for example, the work of Harrison and Reiman [5] [4], Harrison and Williams [6], Harrison and Dai [3], and Shen et al [9]. Although work on RO-U processes is more recent, its stationary behavior also appears tractable. To begin, a straightforward adaptation of

*Corresponding author, email: je-reed@isye.gatech.edu

the algorithm of [9] allows for the numeric computation of the stationary distribution of regulated O-U. Further, the current work of Reed [7] establishes conditions under which the stationary distribution of regulated O-U is multivariate normal.

The remainder of this paper is organized as follows. We provide a detailed model formulation in Section 2. Section 3 sets up our heavy traffic asymptotic regime, and Section 4 establishes that a generalized Jackson network with reneging is well approximated by a multi-dimensional regulated O-U in this regime.

2 Model Formulation

We consider a stochastic network with J service stations. Customers arrive from outside the network to station $j = 1, \ldots, J$ according to a renewal process A_j. Service at station j is FIFO, and the server works whenever queue j is non-empty. The renewal process S_j tracks the cumulative number of service completions at station j over the amount of time the server has worked. We form these renewal processes using $2J$ independent sequences of i.i.d. mean 1 random variables $\{u_j(1), u_j(2), \ldots\}$ and $\{v_j(1), v_j(2), \ldots\}$, having respective variances a_j and b_j. Define

$$A_j(t) \equiv \max\left\{n : \sum_{i=1}^{n} u_j(i) \leq \lambda_j t\right\} \text{ and } S_j(t) \equiv \max\left\{n : \sum_{i=1}^{n} v_j(i) \leq \mu_j t\right\}$$

for $j = 1, \ldots, J$ so that the arrival and service rates at station j are $\lambda_j \geq 0$ and $\mu_j \geq 0$ respectively. We assume λ_j is strictly positive for at least one $j = 1, \ldots, J$ so that outside arrivals to the network do occur.

When a customer completes service at station j, with probability p_{jk} the customer departs for station k, and with probability $1 - \sum_{k=1}^{J} p_{jk} > 0$ the customer exits the network. The sub-stochastic routing matrix $P = (p_{ij})$ thus has spectral radius less than one so that our network is open. In other words, each (non-reneging) customer leaves the network after visiting a finite number of stations. We let the J i.i.d. sequences (independent of all other random sequences introduced so far) of J-dimensional random vectors $\{\xi^j(1), \xi^j(2), \ldots\}$ denote the routing of each customer visiting station j. In particular, for $e(k)$ the kth unit vector in $\Re^J$,

$$\xi^j(n) = \begin{cases} e(k) & \text{with probability } p_{jk}, \\ 0 & \text{with probability } 1 - \sum_{k=1}^{J} p_{jk} > 0, \end{cases}$$

Let the vector $R^j(n)$ have components

$$R_k^j(n) = \sum_{m=1}^{n} \xi_k^j(m),\ k = 1, \ldots, J$$

that denote the cumulative number of the first n jobs completing service at station j that are routed to station k. Observe that $n - \sum_{k=1}^{J} R_k^j(n)$ tracks the number of customers that leave the network after completing service at station j.

Each customer arrives to station j with a memoryless "patience" clock, and independently abandons the network without finishing his service at station j (or reneges) if his service is not completed within an exponentially distributed amount of time having mean $\gamma_j^{-1} \geq 0$. For $N_1, \ldots, N_J$ independent standard Poisson processes, the evolution equation for the queue-length process at station j is

$$Q_j(t) \equiv Q_j(0) + A_j(t) + \sum_{k=1}^{J} R_j^k(S_k(B_k(t))) - S_j(B_j(t)) - N_j\left(\gamma_j \int_0^t Q_j(s)ds\right),$$

where $Q_j(0)$ is the initial number of customers at station j and $B_j(t) \equiv \int_0^t \mathbf{1}\{Q_j(s) > 0\}ds$ is the cumulative server j busy time in $[0, t]$ (with $\mathbf{1}$ being the indicator function).

It is convenient for our purposes to represent the queue-length process in terms of the linearly generalized regulator mapping shown in the appendix. Define the centered process

$$\begin{aligned} X_j(t) \equiv\ & Q_j(0) + (A_j(t) - \lambda_j t) + \sum_{k=1}^{J} \left(R_j^k(S_k(B_k(t))) - p_{kj} S_k(B_k(t))\right) \\ & + \sum_{k=1}^{J} p_{kj}(S_k(B_k(t)) - \mu_k B_k(t)) - (S_j(B_j(t)) - \mu_j B_j(t)) \\ & - N_j\left(\gamma_j \int_0^t Q_j(s)ds\right) + \gamma_j \int_0^t Q_j(s)ds + (\lambda_j + \sum_{k=1}^{J} p_{kj}\mu_k - \mu_j)t. \end{aligned} \tag{1}$$

Let the matrix Γ have the vector γ determine its diagonal, and have all other elements be 0. Define $Y_j(t) \equiv \mu_j I_j(t)$, where $I_j(t) = t - B_j(t)$ is the server idletime. The pathwise equation for Q in matrix-vector format is

$$Q(t) = X(t) - \int_0^t \Gamma Q(s)ds + (I - P^T)Y(t).$$

Since conditions (C1)-(C2) for the linearly generalized regulator mapping defined in the appendix are satisfied, Proposition 2 shows

$$(Q, Y) = (\Phi_\Gamma(X), \Psi_\Gamma(X)). \tag{2}$$

3 The Heavy Traffic Asymptotic Regime

Our asymptotic regime is one in which the system experiences a high volume of demand and servers work very quickly. Specifically, we consider a sequence of systems, indexed by n, where n tends to infinity through a strictly increasing sequence of values in $[0, \infty)$. In the nth system, order arrival and service rates are $n\lambda^n$ and $n\mu^n$ respectively, where

$$\lambda^n \to \lambda, \text{ and } \mu^n \to \mu, \tag{3}$$

as $n \to \infty$. Henceforth, when we wish to refer to any process or other quantity associated with the network having order arrival and service rates $n\lambda^n$ and $n\mu^n$, we superscript the appropriate symbol by n.

The following technicalities are needed. All random variables are defined on a common probability space $(\Omega, \mathcal{F}, P)$. For each positive integer d, let $D([0,\infty), \Re^d)$ be the space of right continuous functions with left limits (RCLL) in $\Re^d$ having time domain $[0,\infty)$. We endow $D([0,\infty), \Re^d)$ with the usual Skorokhod J_1 topology, and let M^d denote the Borel σ-algebra associated with the J_1 topology. All stochastic processes are measurable functions from $(\Omega, \mathcal{F}, P)$ into $(D([0,\infty), \Re^d), M^d)$ for some appropriate dimension d. Suppose $\{\xi^n\}_{n=1}^{\infty}$ is a sequence of stochastic processes. The notation $\xi^n \Rightarrow \xi$ means that the probability measures induced by the ξ^n's on $(D([0,\infty), \Re^d), M^d)$ converge weakly to the probability measure on $(D([0,\infty), \Re^d), M^d)$ induced by the stochastic process ξ. For $x \in (D([0,\infty), \Re^d), M^d)$ and $T > 0$, let

$$\|x\|_T \equiv \sup_{0\le t\le T} \max_{j=1,\ldots,d} |x_j(t)|,$$

and note that ξ^n converges almost surely to a continuous limit process ξ in the J_1 topology if and only if

$$\|\xi^n - \xi\|_T \to 0 \ a.s.$$

for every $T > 0$.

For a mapping $f : D([0,\infty), \Re^d) \to D([0,\infty), \Re^d)$, we say f is Lipschitz continuous if for any $T > 0$, there exists a constant κ (which may depend on T) such that for $x_1, x_2 \in D([0,\infty), \Re^d)$

$$\|f(x_1) - f(x_2)\|_T \le \kappa \|x_1 - x_2\|_T.$$

Also, for $x, \tau \in D([0,\infty), \Re^d)$, where τ is positive and increasing, by "$x \circ \tau$", we mean element-by-element function composition, so that $(x \circ \tau)_j(t) \equiv x_j(\tau_j(t)), t \ge 0, j = 1, \ldots, d$. Finally, for two integers i and j, we define

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

It is convenient for our analysis to define the fluid scaled processes for each $j = 1, \ldots, J$

$$\begin{aligned} \overline{A}_j^n(t) &= n^{-1} A_j^n(t) - \lambda_j^n t, \\ \overline{S}_j^n(t) &= n^{-1} S_j^n(t) - \mu_j^n t, \\ \overline{N}_j^n(t) &= n^{-1} N_j^n(nt) - t, \\ \overline{R}_k^{n,j}(t) &= n^{-1} R_k^j(\lfloor nt \rfloor) - p_{jk} t, \ k = 1, \ldots, J. \end{aligned}$$

The functional strong law of large numbers (see, for example Theorem 5.10 in Chen and Yao [2]) guarantees that for any $T > 0$

$$\|\overline{A}^n\|_T \to 0, \ \|\overline{S}^n\|_T \to 0, \ \|\overline{N}^n\|_T \to 0, \text{ and } \|\overline{R}^{j,n}\|_T \to 0, \quad a.s., \tag{4}$$

as $n \to \infty$. We also define the diffusion-scaled processes for each $j = 1, \ldots, J$

$$\begin{aligned}
\tilde{A}_j^n(t) &= \sqrt{n}(n^{-1}A_j^n(t) - \lambda_j^n t) \\
\tilde{S}_j^n(t) &= \sqrt{n}(n^{-1}S_j^n(t) - \mu_j^n t) \\
\tilde{N}_j^n(t) &= \sqrt{n}(n^{-1}N_j^n(nt) - t) \\
\tilde{R}_k^{n,j}(t) &= \sqrt{n}(n^{-1}R_k^j(\lfloor nt \rfloor) - p_{jk}t),\ k = 1, \ldots, J.
\end{aligned}$$

For $B^A, B^S, B^N,$ and $B^{R,j},\ j = 1, \ldots, J$ independent, d-dimensional Brownian motions having respective covariance matrices C^A, C^S, C^N, and $C^{R,j}$, whose (k,l)th element is defined as

$$C_{kl}^A \equiv \lambda_k a_k \delta_{kl},\ C_{kl}^S \equiv \mu_k b_k \delta_{kl},\ C_{kl}^N \equiv \delta_{kl},\ C_{k,l}^{R,j} \equiv p_{jk}(\delta_{kl} - p_{jl}),$$

the weak convergences

$$\tilde{A}^n \Rightarrow B^A,\ \tilde{S}^n \Rightarrow B^S,\ \tilde{N}^n \Rightarrow B^N,\ \tilde{R}^{n,j} \Rightarrow B^{R,j},\ j = 1, \ldots, J, \tag{5}$$

as $n \to \infty$ follow by the functional central limit theorem (see, for example Theorem 5.11 in Chen and Yao [2]). Finally, we observe that the linearly generalized regulator mapping satisfies a scaling property. Therefore, from equation (2), on fluid scale, letting $\overline{Q}^n \equiv n^{-1}Q^n, \overline{Y}^n \equiv n^{-1}Y^n$, and $\overline{X}^n \equiv n^{-1}X^n$,

$$(\overline{Q}^n, \overline{Y}^n) = (\Phi_\Gamma(\overline{X}^n), \Psi_\Gamma(\overline{X}^n)), \tag{6}$$

and, on diffusion scale, letting $\tilde{Q}^n = n^{-1/2}Q^n, \tilde{Y}^n = n^{-1/2}Y^n$, and $\tilde{X}^n = n^{-1/2}X^n$,

$$(\tilde{Q}^n, \tilde{Y}^n) = (\Phi_\Gamma(\tilde{X}^n), \Psi_\Gamma(\tilde{X}^n)). \tag{7}$$

4 Multi-dimensional Regulated Ornstein-Uhlenbeck Approximation

Our first proposition establishes the behavior of a generalized Jackson network with reneging under fluid scaling. We require this proposition, which is interesting in its own right, to prove the weak convergence of the queue-length process to a multi-dimensional regulated O-U process in Theorem 1.

Proposition 1 *Assume* $\max_{j=1,\ldots,J} |n^{-1}Q_j^n(0) - q_j| \to 0$ *a.s., as* $n \to \infty$ *for fixed* $q \geq 0$. *Then, for* $x(t) \equiv q + (\lambda - (I - P^T)\mu)t,\ t > 0$,

$$\overline{Q}^n \to \Phi_\Gamma(x) \text{ and } \overline{Y}^n \to \Psi_\Gamma(x),\ a.s.,$$

as $n \to \infty$.

Proof: For each $j = 1, \ldots, J$, from (1),

$$\begin{aligned}\overline{X}_j^n(t) &= n^{-1}Q_j^n(0) + \overline{A}_j^n(t) + \sum_{k=1}^J \overline{R}_j^{n,k}\left(n^{-1}S_k^n(B_k^n(t))\right) + \sum_{k=1}^J p_{kj}\overline{S}_k^n(B_k^n(t)) \\ &\quad -\overline{S}_j^n(B_j^n(t)) - \overline{N}_j^n\left(\gamma_j \int_0^t \overline{Q}_j^n(s)ds\right) + \left(\lambda_j^n + \sum_{k=1}^J p_{kj}\mu_k^n - \mu_j^n\right)t.\end{aligned}$$

Given $T > 0$, provided we can show there exists C_1 and C_2 (which may be dependent on T) such that for large enough n

$$\|n^{-1}S^n \circ B^n\|_T \leq C_1 \text{ a.s., and } \|\Gamma\overline{Q}^n(t)\|_T \leq C_2 \text{ a.s.,} \tag{8}$$

then, from (4) and assumption (3),

$$\|\overline{X}^n\|_T \to x, \text{ a.s.,}$$

as $n \to \infty$. Therefore, because Proposition 2 shows the linearly generalized regulator mapping is continuous, the representation of $(\overline{Q}^n, \overline{Y}^n)$ in terms of this mapping in (6) establishes the stated a.s. convergence.

To see (8), first observe that for $t > 0$, since the time of the first arrival to station k is of order n^{-1} in the nth system, for large enough n (where how large n must be depends on the sample path), $0 < B_k^n(t) < t$, *a.s.*, since the service discipline is non-idling and server busy time in $[0, t]$ cannot exceed t. Therefore, the strong law for renewal processes guarantees that for any $t > 0$,

$$\frac{S_k^n(B_k^n(t))}{n\mu^n B_k^n(t)} \to 1, \ a.s., \tag{9}$$

as $n \to \infty$. Since $B_k^n(t)$ and S_k^n are non-decreasing in t, and $B_k^n(T) \leq T$, for any $\epsilon > 0$,

$$n^{-1}S_k^n(B_k^n(t)) \leq \frac{S_k^n(B_k^n(T))}{n\mu^n B_k^n(T)}\mu^n B_k^n(T) \leq (\mu + \epsilon)T, \ a.s.,$$

by assumption (3) and the convergence in (9), for large enough n. Finally, the second inequality in (8) follows because for any $\epsilon > 0$, and for large enough n,

$$\overline{Q}_j^n(t) \leq \overline{A}_j^n(t) + \lambda_j^n t \leq (\lambda_j + \epsilon)T, \ j = 1, \ldots, J.$$

□

Remark 1 If $(\lambda - (I - P^T)\mu) \geq 0$, then $\Psi_\Gamma(x) = \mathbf{0}$, and so from standard ordinary differential equation theory,

$$\Phi_\Gamma(x)(t) = \Gamma^{-1}(\lambda - (I - P^T)\mu) + (q - \Gamma^{-1}(\lambda - (I - P^T)\mu))e^{-\Gamma t}, \quad t \geq 0.$$

In particular, the convergence rate to steady-state (which depends on the reneging rate matrix Γ) is exponential, and

$$\lim_{t\to\infty} \Phi_\Gamma(x)(t) = \Gamma^{-1}(\lambda - (I - P^T)\mu).$$

Additionally, because the convergence in Proposition 1 is almost sure, uniformly on compact sets, to a continuous limit process, the limit interchange

$$\Gamma^{-1}(\lambda - (I-P^T)\mu) = \lim_{t\to\infty}\lim_{n\to\infty} \overline{Q}^n(t) = \lim_{n\to\infty}\limsup_{t\to\infty} \overline{Q}^n(t) = \lim_{n\to\infty}\liminf_{t\to\infty} \overline{Q}^n(t) \ a.s.$$

is justified.

To set the stage for our main limit theorem, for a given drift $\alpha \in \Re^J$, (possibly random) initial state $q \geq 0$, square matrix M, and B a zero drift Brownian motion having covariance matrix C, let (Z, Y) be the unique strong solution to the stochastic equation

$$\begin{aligned} Z(t) &= q + \alpha t - \int_0^t \Gamma Z(s)ds + B(t) + MY(t) \geq 0,\ t \geq 0, \\ Y(0) &= 0,\ Y \text{ non-decreasing, and } \int_0^\infty Z_j(t)dY_j(t) = 0,\ j = 1, \ldots, J. \end{aligned} \tag{10}$$

When M has positive diagonal elements, non-positive off-diagonal elements, and a non-negative inverse, Proposition 2 in the appendix guarantees a unique strong solution to (10).

Theorem 1 *Suppose for each $j = 1, \ldots, J$*

$$\sqrt{n}\left(\lambda_j^n + \sum_{j=1}^J p_{kj}\mu_k^n - \mu_j^n\right) \to \theta_j, \tag{11}$$

as $n \to \infty$. Then, assuming $\tilde{Q}^n(0) \Rightarrow q$ in $\Re$ for a given initial state q,

$$(\tilde{Q}, \tilde{Y}) \Rightarrow (Z, Y),$$

where (Z, Y) satisfy the stochastic equation (10) with $\alpha = \theta$, $M = I - P^T$, and covariance matrix C having

$$C_{kl} \equiv \lambda_k a_k \delta_{kl} + \sum_{j=1}^J \mu_j p_{jk}(\delta_{kl} - p_{jl}) + \mu_j b_j (p_{jk} - \delta_{jk})(p_{jl} - \delta_{jl}).$$

Proof: For each $j = 1, \ldots, J$, define $\tau_j^n(t) \equiv \gamma_j \int_0^t \overline{Q}_j^n(s)ds$, and observe from (1) that,

$$\begin{aligned} n^{-1/2}X_j^n(t) &= n^{-1/2}Q_j^n(0) + \tilde{A}_j^n(t) + \sum_{k=1}^{J} \tilde{R}_j^{n,k}(n^{-1}S_k^n(B_k^n(t))) \\ &\quad + \sum_{k=1}^{J} p_{kj}\tilde{S}_k^n(B_k^n(t)) - \tilde{S}_j^n(B_j^n(t)) - \tilde{N}_j^n\left(\tau_j^n(t)\right) \\ &\quad + \sqrt{n}t\left(\lambda_j^n + \sum_{j=1}^{J} p_{kj}\mu_k^n - \mu_j^n\right). \end{aligned} \tag{12}$$

Let $\mathbf{0}$ be the zero process and observe that for any $t > 0$, $\Psi_\Gamma(0)(t) = \Phi_\Gamma(0)(t) = 0$. Then, because (11) implies

$$\lambda_j^n + \sum_{j=1}^{J} p_{kj}\mu_k^n - \mu_j^n \to 0,$$

as $n \to \infty$, Proposition 1 (with $x = \mathbf{0}$) guarantees

$$\overline{Y}^n \to 0 \text{ and } \overline{Q}^n \to 0 \ a.s.,$$

as $n \to \infty$. Therefore, the convergences

$$\tau^n \to 0 \text{ and } B^n \to e,\ a.s.,$$

hold, as $n \to \infty$, where $e(t) = t$ for all $t > 0$. The random time change theorem then shows

$$\tilde{N}^n \circ \tau^n \Rightarrow 0 \text{ and } \tilde{S}^n \circ B^n \Rightarrow B^S, \tag{13}$$

as $n \to \infty$. Furthermore, $n^{-1}S^n \circ B^n \to \mu e$ a.s., and so

$$\tilde{R}_j^{n,k} \circ n^{-1}S_k^n \circ B_k^n \Rightarrow B_j^{R,k} \circ \mu_j e. \tag{14}$$

Therefore, from (5), (11), (12), (13), and (14), we have, in vector form, as $n \to \infty$,

$$\tilde{X}^n \Rightarrow q + B^A + \sum_{j=1}^{J} B^{R,j} \circ \mu e + (P^T - I)B^S + \theta,$$

a Brownian motion with drift θ and covariance matrix C. Because the linearly generalized regulator mapping is continuous by Proposition 2, the continuous mapping theorem and the representation for the queue-length and idle-time processes in (7) establish the stated weak convergence. □

5 Appendix

We show a result, similar to that of Chen [1], that establishes the existence, uniqueness, and continuity of a linearly generalized regulator mapping. For d a positive integer, $x \in D([0,\infty),\Re^d)$ having $x(0) \geq 0$, and Γ, M square matrices of dimension $d \times d$, the linearly generalized regulator mapping

$$(\Phi_\Gamma, \Psi_\Gamma)(x) : D([0,\infty),\Re^d) \to D([0,\infty),[0,\infty)^{2d})$$

is defined by

$$(\Phi_\Gamma, \Psi_\Gamma)(x) \equiv (z, l),$$

where

(C1) $z(t) + \int_0^t \Gamma z(s) ds = x(t) + Rl(t) \geq 0$ for all $t \geq 0$

(C2) $l(0) = 0$, l is non-decreasing, and $\int_0^\infty z_j(t) dl_j(t) = 0,\ j = 1, \ldots, J.$

Observe that if Γ is the zero matrix, we have the conventional reflection mapping discussed in Section 7.2 of Chen and Yao [2]. We write Φ, Ψ, where $\Phi = \Phi_0$ and $\Psi = \Psi_0$ to emphasize when we are referring to the conventional reflection mapping.

The key to establishing the existence, uniqueness, and Lipschitz continuity of Φ_Γ and Ψ_Γ is understanding the properties of the following integral equation

$$u(t) = x(t) - \int_0^t \Gamma \Phi(u)(s) ds. \tag{15}$$

In particular, define the mapping $\mathcal{M} : D([0,\infty),\Re^d) \to D([0,\infty),\Re^d)$ (which exists uniquely by Lemma 1) by $\mathcal{M}(x) \equiv u$, and observe that conditions (C1)-(C2) are satisfied when

$$(\Phi_\Gamma, \Psi_\Gamma)(x) = (\Phi, \Psi)(\mathcal{M}(x)). \tag{16}$$

We first state a lemma establishing basic properties of integral equations having the form (15), and then state the main proposition of the appendix.

Lemma 1 *Suppose $\eta : D([0,\infty),\mathcal{R}^d) \to D([0,\infty),\mathcal{R}^d)$ is Lipschitz continuous. Then for any given $x \in D([0,\infty),\mathcal{R}^d)$, there exists a unique $u \in D([0,\infty),\mathcal{R}^d)$ that satisfies the integral equation*

$$u(t) = x(t) - \int_0^t \eta(u)(s) ds, \tag{17}$$

and has initial condition $u(0) = x(0)$. Furthermore, the mapping $\mathcal{M}_\eta : D([0,\infty),\mathcal{R}^d) \to D([0,\infty),\mathcal{R}^d)$ defined by $\mathcal{M}_\eta(x) \equiv u$ is Lipschitz continuous.

Proposition 2 *Suppose M has positive diagonal elements, non-positive off-diagonal elements, and a non-negative inverse. Then, for each $x \in D([0,\infty),\Re^d)$ having $x(0) \geq 0$, there exists a unique (z, l) satisfying (C1)-(C2). Furthermore, the mappings Φ_Γ and Ψ_Γ are Lipschitz continuous.*

Proof: The proof is immediate from the representation (16), Lemma 1, and Theorem 7.2 of Chen and Yao [2], which establishes the existence, uniqueness, and Lipschitz continuity of the mapping (Φ, Ψ). $\square$

Proof of Lemma 1: For $T > 0$, let κ be such that for $x_1, x_2 \in D([0, \infty), \Re^d)$, the inequality $\|\eta(x_1) - \eta(x_2)\|_T \leq \kappa \|x_1 - x_2\|_T$ holds.

Existence: We use the method of successive approximations to construct a solution to (17) on $[0, T]$. Let $u_0 \equiv 0$ and recursively define

$$u_{n+1}(t) = x(t) - \int_0^t \eta(u_n)(s) ds.$$

Choose δ so that $\kappa\delta < 1$. Partition the interval $[0, T]$ into $\lfloor \delta^{-1} T \rfloor$ intervals of length δ, and one interval of length $T - \lfloor \delta^{-1} T \rfloor \delta$. Let

$$\overline{c} \equiv \|x\|_T + T \|\eta(0)\|_T,$$

and observe that for any $0 < t \leq T$,

$$\|u_1 - u_0\|_t \leq \|x\|_t + t \|\eta(0)\|_t \leq \overline{c}. \tag{18}$$

Furthermore, for $n = 1, 2, \ldots$

$$\|u_{n+1} - u_n\|_\delta \leq \delta\kappa \|u_n - u_{n-1}\|_\delta,$$

and so repeated iteration shows

$$\|u_{n+1} - u_n\|_\delta \leq (\delta\kappa)^n \overline{c}.$$

Now suppose for a given positive integer k,

$$\|u_{n+1} - u_n\|_{j\delta} \leq j n^j (\delta\kappa)^n \overline{c}, \ j = 1, \ldots, k, \tag{19}$$

for all $n = 1, 2, \ldots$, and so

$$\begin{aligned} \|u_{n+1} - u_n\|_{(k+1)\delta} &= \|\sum_{j=1}^{k+1} \int_{(j-1)\delta}^{j\delta} \left(\eta(u_n) - \eta(u_{n-1})\right)(s) ds\|_{(k+1)\delta} \\ &\leq \sum_{j=1}^{k} \delta\kappa \|u_n - u_{n-1}\|_{j\delta} + \delta\kappa \|u_n - u_{n-1}\|_{(k+1)\delta} \\ &\leq k n^k (\delta\kappa)^n \overline{c} + \delta\kappa \|u_n - u_{n-1}\|_{(k+1)\delta}. \end{aligned} \tag{20}$$

From (18) and (20),

$$\|u_2 - u_1\|_{(k+1)\delta} \leq (k+1)\delta\kappa\overline{c}.$$

Repeatedly iterating (20) shows that for all $n = 1, 2, \ldots,$

$$\begin{aligned}\|u_{n+1} - u_n\|_{(k+1)\delta} &\leq \left(k \sum_{i=1}^{n} i^k + 1\right) (\delta\kappa)^n \overline{c} \\ &\leq (kn^{k+1} + 1)(\delta\kappa)^n \overline{c}.\end{aligned}$$

Therefore, by mathematical induction, (19) holds for all positive integers k. In particular, (19) holds for $k = \lceil \delta^{-1} T \rceil$, and so

$$\|u_{n+1} - u_n\|_T \leq \sum_{j=1}^{\lceil \delta^{-1} T \rceil} \delta\kappa \|u_n - u_{n-1}\|_{j\delta} \leq \sum_{j=1}^{\lceil \delta^{-1} T \rceil} jn^j (\delta\kappa)^n \overline{c} \to 0,$$

as $n \to \infty$. We deduce that the sequence of approximating functions u_n tends to the limit function $u^\star \equiv \lim_{n\to\infty} u_n$ in $D([0,T], \Re^d)$ (when viewed as a Banach space under the uniform norm on compact sets). Finally, to see $u^\star$ is a fixed point of the mapping

$$f(u) \equiv x(t) - \int_0^t \eta(u)(s) ds,$$

observe that f is Lipschitz continuous since $\|f(u_1) - f(u_2)\|_T \leq \kappa T \|u_1 - u_2\|_T$.

Uniqueness: Suppose that u and v both satisfy (17). Let

$$\triangle(t) \equiv u(t) - v(t) = \int_0^t (\eta(v) - \eta(u))(s) ds.$$

Then, by the Lipschitz property of η, for any $t > 0$,

$$|\triangle(t)| \leq \int_0^t |(\eta(v) - \eta(u))(s)| ds \leq t\kappa \|\triangle\|_t,$$

which implies $\triangle \equiv 0$ on $[0, \kappa^{-1}]$. For $\kappa^{-1} < t < 2\kappa^{-1}$,

$$|\triangle(t)| \leq \|\triangle\|_{\kappa^{-1}} + (t - \kappa^{-1})\kappa \|\triangle\|_{2\kappa^{-1}} = (t - \kappa^{-1})\kappa \|\triangle\|_{2\kappa^{-1}},$$

which implies $\triangle \equiv 0$ on $[0, 2\kappa^{-1}]$, since $(t - \kappa^{-1})\kappa < 1$. Iterating this argument implies $\triangle$ is the zero function.

Lipschitz continuity: Because η is a Lipschitz mapping,

$$\|\mathcal{M}_\eta(x_1) - \mathcal{M}_\eta(x_2)\|_t \leq \|x_1 - x_2\|_t + t\kappa \|\mathcal{M}_\eta(x_1) - \mathcal{M}_\eta(x_2)\|_t,$$

which implies, for $t \leq (2\kappa)^{-1}$,

$$\|\mathcal{M}_\eta(x_1) - \mathcal{M}_\eta(x_2)\|_t \leq (1 - t\kappa)^{-1} \|x_1 - x_2\|_t. \tag{21}$$

For $(2\kappa)^{-1} < t \le 2(2\kappa)^{-1} = \kappa^{-1}$, use of (21) and the Lipschitz continuity of η shows

$$\begin{aligned}
\|\mathcal{M}_\eta(x_1) - \mathcal{M}_\eta(x_2)\|_t &= \sup_{0\le s\le t}\max_{j=1,\ldots,d}\left|\begin{array}{l} x_1(\kappa^{-1}) - \int_0^{(2\kappa)^{-1}} \eta(\mathcal{M}_\eta(x_1))(s)ds \\ -x_2(\kappa^{-1}) + \int_0^{(2\kappa)^{-1}} \eta(\mathcal{M}_\eta(x_2))(s)ds \\ -x_1(\kappa^{-1}) + x_2(\kappa^{-1}) + x_1(s) - x_2(s) \\ + \int_{(2\kappa)^{-1}}^t \left(\eta(\mathcal{M}_\eta(x_2)) - \eta(\mathcal{M}_\eta(x_1))\right)(s)ds \end{array}\right| \\
&\le (1-(2\kappa)^{-1}\kappa)^{-1}\|x_1 - x_2\|_{(2\kappa)^{-1}} + 2\|x_1 - x_2\|_t \\
&\quad +(t-(2\kappa)^{-1})\kappa\|\mathcal{M}_\eta(x_2) - \mathcal{M}_\eta(x_1)\|_t \\
&\le 4\|x_1 - x_2\|_t + (t-(2\kappa)^{-1})\kappa\|\mathcal{M}_\eta(x_2) - \mathcal{M}_\eta(x_1)\|_t,
\end{aligned}$$

and so the inequality

$$\|\mathcal{M}_\eta(x_1) - \mathcal{M}_\eta(x_2)\|_t \le \frac{4}{1-(t-(2\kappa)^{-1})\kappa}\|x_1 - x_2\|_t$$

follows. Since only a finite number of intervals of length $(2\kappa)^{-1}$ partition the interval $[0, T]$, iterating as above establishes Lipschitz continuity of the mapping $\mathcal{M}_\eta$. $\square$

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