J. Appl. Prob. 21, 860–869 (1984)
Printed in Israel

THE NON-ERGODIC JACKSON NETWORK

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Abstract

We generalize Jackson's theorem to the non-ergodic case. Here, despite the fact that the entire Jackson network will not achieve steady state, it is still possible to determine the maximal subnetwork that does. We do so by formulating and algorithmically solving a new non-linear throughput equation. These results, together with the ergodic results and the ones for closed networks, completely characterize the large-time behavior of any Jackson network.

NETWORKS OF QUEUES; ASYMPTOTIC BEHAVIOUR; STOCHASTIC DOMINANCE; NEW THROUGHPUT EQUATION

1. Introduction

The Jackson network [3] is an N-node network where the ith node is an $\boldsymbol{M/M/1}$ queue with arrival and service rates $\lambda_i \geqq 0$ and $\mu_i > 0$ respectively. The queues are connected by an $N \times N$ switching matrix $\boldsymbol{P}$, where a customer, after completing service at node i, leaves and joins node j with probability p_{ij}. With probability $q_i = 1 - \Sigma_{j=1}^{N} p_{ij}$ however, it may decide to leave the network entirely. We shall always assume that $p_{ii} = 0$ for all i. Since the Jackson network is formally a collection of $\boldsymbol{M/M/1}$ queues, we take liberties with Kendall notation and henceforth refer to an N-node Jackson network as $(\boldsymbol{M/M/1})^N$. We will say that a given node i can be *filled* if $\lambda_i \neq 0$ or there exists a node j and a positive integer m such that $\lambda_j \neq 0$ and $p_{ji}^{(m)} > 0$, where $p_{ji}^{(m)}$ is the (j, i) entry of $\boldsymbol{P}^m$. We say that node i can be *drained* if $q_i \neq 0$, or there exists a node k and a positive integer n such that $q_k \neq 0$ and $p_{ik}^{(n)} > 0$.

Let $\boldsymbol{Q}(t) = (Q_1(t), \cdots, Q_N(t))$ be the joint queue-length process for $(\boldsymbol{M/M/1})^N$. Jackson's theorem characterizes the large-time behavior of $\boldsymbol{Q}(t)$ for a distinguished class of $(\boldsymbol{M/M/1})^N$ systems, the ergodic ones. We restate Jackson's theorem as follows.

Received 29 June 1983; revision received 10 November 1983.
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Theorem (Jackson). *Given an* $(M/M/1)^N$ *system where every node can be filled and drained, let* $\boldsymbol{\theta} = [\theta_1, \cdots, \theta_N]$ *be the solution to the throughput equation*

$$\boldsymbol{\theta} = \boldsymbol{\theta} \boldsymbol{P} + \boldsymbol{\lambda}$$

where $\boldsymbol{\lambda} = [\lambda_1, \cdots, \lambda_N]$. *If* $\rho_i = \theta_i / \mu_i$ *and* $\theta_i < \mu_i$ *for all* i, *then*

$$\lim_{t \to \infty} \Pr\{Q_1(t) = n_1, \cdots, Q_N(t) = n_N\} = \prod_{i=1}^{N} (1 - \rho_i) \rho_i^{n_i}$$

for all integers $n_i \geqq 0$.

The motivation for Jackson's theorem comes from Burke's theorem for the $M/M/1$ queue (see [1]). It states that whenever $\lambda < \mu$, then the output process for large time is Poisson with parameter λ. In other words, for a system in equilibrium, the 'output' is identical to the 'input'. This in turn inspires the usual formulation of the throughput equation. Unfortunately, Jackson's theorem tells us nothing directly if $\theta_i \geqq \mu_i$ for some i or if some node cannot be drained. In the latter case, the above throughput equation may not even have a solution. Take, for example, an $(M/M/1)^2$ system with non-zero λ_1 or λ_2, and $p_{12} = p_{21} = 1$.

In this paper, we shall generalize Jackson's theorem so that we can describe the large-time behavior for any non-ergodic $(M/M/1)^N$ system. Given existing results however, it is enough for us to treat a reduced class of networks. For any $(M/M/1)^N$ system, consider the class of nodes that can neither be filled nor drained. They form a subnetwork that is a closed Jackson network completely independent, not merely in steady state but for all time, from the rest of the network. For these closed systems, their large-time behavior is well known, see Gordon and Newell [2]. Now consider nodes that can be drained but not filled. They will lose their customers in a finite amount of time and stay empty thereafter. In light of this, we need only concern ourselves with $(M/M/1)^N$ systems all of whose nodes can be filled.

First, we formulate the throughput equation. The equation arises from a more careful examination of Burke's theorem. We can then infer that when $\lambda \geqq \mu$, the output for an $M/M/1$ queue is still Poisson for large time, not at rate λ, but at rate μ. To make an unconditional statement, we can say that for an $M/M/1$ queue, its output process for large time is Poisson with rate $\lambda \wedge \mu = \min(\lambda, \mu)$. This leads us to the following throughput equation for an arbitrary $(M/M/1)^N$ network:

$$\boldsymbol{\theta} = (\boldsymbol{\theta} \wedge \boldsymbol{\mu}) \boldsymbol{P} + \boldsymbol{\lambda}$$

where $\boldsymbol{\theta} \wedge \boldsymbol{\mu}$ is the componentwise minimum of two vectors. This equation was formulated in Massey [6] and independently derived in Schweitzer [7].

In Section 2, we shall show that the throughput equation always has a unique solution for any $(M/M/1)^N$ system where each node can be filled or drained.

Moreover, we can give an efficient algorithm for solving it. In Section 3, we then prove our main theorem.

Theorem 1. *Given an* $(\boldsymbol{M}/\boldsymbol{M}/\mathbf{1})^N$ *system where every node can be filled, let* $\boldsymbol{\theta}$ *be the solution to the throughput equation*

$$\boldsymbol{\theta} = (\boldsymbol{\theta} \wedge \boldsymbol{\mu})\boldsymbol{P} + \boldsymbol{\lambda}$$

where $\boldsymbol{\mu} = [\mu_1, \cdots, \mu_N]$. *If* $\rho_i = \theta_i/\mu_i$ *and* $U = \{i \mid \theta_i < \mu_i\}$ *then*

$$\lim_{t\to\infty} \Pr\{Q_i(t) = n_i\,;\, i \in U\} = \prod_{i\in U} (1-\rho_i)\rho_i^{n_i}$$

for all integers $n_i \geqq 0$ *with* $i \in U$. *Moreover, if* $j \notin U$ *then*

$$\lim_{t\to\infty} \Pr\{Q_j(t) = n\} = 0$$

for all integers $n \geqq 0$.

2. Solving the new throughput equation

Intuitively, the method for solving the new throughput equation is very simple. We can think of θ_i as the net input rate into node i. Consequently, $\theta_i \wedge \mu_i$ is the output rate from node i. When the output rate is θ_i and $\theta_i \neq \mu_i$, we say that node i is *stable*. Otherwise, the rate is μ_i and node i is *unstable*. If we knew which nodes were stable and which were not, then determining the θ_i's would reduce to solving a linear equation. We would know the outputs for the unstable nodes, hence we would know the *external* inputs for the subnetwork of stable nodes. Solving for the θ_i's when i indexes the stable nodes, is exactly the same as solving Jackson's throughput equation. This now determines the output rate for all of the nodes, which in turn gives the net input rates or θ_i's for all of the nodes.

Unfortunately, we do not initially know which nodes are stable or unstable. We can, however, figure this out in a systematic manner. First assume that all of the nodes are unstable. This means that the output rate is μ_i for the ith node. Let $\theta_i(1)$ be our initial guess for θ_i based on this assumption. We then have

$$\theta_i(1) = \lambda_i + \sum_{j=1}^{N} \mu_j p_{ji}.$$

Since the true output rate is $\theta_i \wedge \mu_i$, our guess is at worst an *overestimate* of the true situation, and so $\theta_i(1) \geqq \theta_i$. If $\theta_i(1) \geqq \mu_i$ for all i, then every node is indeed unstable and the $\theta_i(1)$'s solve the throughput equation. But if $\theta_i(1) < \mu_i$ for some i, then this node must be stable since the true θ_i is less than $\theta_i(1)$. Now we solve the throughput equation again, using the *fact* that the nodes indexed by $I(1) = \{i \mid \theta_i(1) < \mu_i\}$ are stable, and *assuming* that the rest are not. Call this solution $\theta_i(2)$. Once again, the assumptions are at worst an overestimate of the true behavior but more conservative than the assumptions made for $\theta_i(1)$. Hence we have $\theta_i \leqq \theta_i(2) \leqq \theta_i(1)$ for all i. We then still have $\theta_i(2) < \mu_i$ for all i in $I(1)$, and if this set equalled $I(2) = \{i \mid \theta_i(2) < \mu_i\}$, we would be done. If not, then $I(1) \subseteq I(2)$, and $I(2)$ now indexes a larger subset of the stable nodes. We then repeat the above procedure with $I(2)$. By induction, the $I(n)$'s will be an increasing chain of sets. This means that there is some first positive integer n_* less than or equal to N such that $I(n_*) = I(n_* + 1)$. Thus the $\theta_i(n_*)$'s are the true solution to the throughput equation. This procedure determines θ_i algorithmically in at most N steps.

Before we proceed proving the above remarks rigorously, let us introduce some useful notation. Let I be a subset of $\{1, \cdots, N\}$, the set of indices for the nodes. If J is the complement of I, and $\boldsymbol{\theta} = [\theta_1, \cdots, \theta_N]$, we may partition $\boldsymbol{\theta}$ into subvectors $\boldsymbol{\theta}_I$ and $\boldsymbol{\theta}_J$ or write $\boldsymbol{\theta} = [\boldsymbol{\theta}_I, \boldsymbol{\theta}_J]$. For our $N \times N$ substochastic matrix $\boldsymbol{P}$, let $\boldsymbol{P}_{II}$, $\boldsymbol{P}_{IJ}$, $\boldsymbol{P}_{JI}$, and $\boldsymbol{P}_{JJ}$ be the obvious submatrices of $\boldsymbol{P}$, and let $\sigma(\boldsymbol{P})$ be its spectral radius. Note that for any I, $\sigma(\boldsymbol{P}_{II}) \leqq 1$ since $\boldsymbol{P}_{II}$ is substochastic. Also, we shall say that $\boldsymbol{\theta} \leqq \boldsymbol{\theta}'$ if $\theta_i \leqq \theta'_i$ for $i = 1, \cdots, N$.

Theorem 2. *For any* $(\boldsymbol{M}/\boldsymbol{M}/\mathbf{1})^N$ *system where each node can be filled or drained, the throughput equation* $\boldsymbol{\theta} = (\boldsymbol{\theta} \wedge \boldsymbol{\mu})\boldsymbol{P} + \boldsymbol{\lambda}$ *has a unique solution that can be determined algorithmically in no more than N steps.*

Proof. Given $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, $\boldsymbol{P}$, and an index set I, we say that $\boldsymbol{\theta} = [\theta_1, \cdots, \theta_N]$ is the unique solution to the I-partition equation if $\sigma(\boldsymbol{P}_{II}) < 1$, and $\boldsymbol{\theta} = [\boldsymbol{\theta}_I, \boldsymbol{\theta}_J]$ where

$$\begin{aligned} \boldsymbol{\theta}_I &= (\boldsymbol{\lambda}_I + \boldsymbol{\mu}_J \boldsymbol{P}_{JI})(\boldsymbol{I} - \boldsymbol{P}_{II})^{-1} \\ \boldsymbol{\theta}_J &= \boldsymbol{\lambda}_J + \boldsymbol{\mu}_J \boldsymbol{P}_{JJ} + \boldsymbol{\theta}_I \boldsymbol{P}_{IJ}. \end{aligned}$$

Notice that $\sigma(\boldsymbol{P}_{II}) < 1$ implies the existence of $(\boldsymbol{I} - \boldsymbol{P}_{II})^{-1}$ and makes it a positive matrix, so $\boldsymbol{\theta}$ is a positive vector.

The algorithm for solving $\boldsymbol{\theta} = (\boldsymbol{\theta} \wedge \boldsymbol{\mu})\boldsymbol{P} + \boldsymbol{\lambda}$ is to construct recursively a sequence of vectors $\boldsymbol{\theta}(n)$ which correspond to a sequence of index sets $I(n)$, where:

1. $I(0) = \varnothing$.
2. $\boldsymbol{\theta}(n)$ solves the $I(n)$-partition equation.
3. $I(n+1) = \{i \mid \theta_i(n) \leqq \mu_i\}$.

What we shall show is that $\boldsymbol{\theta}(n)$ converges to a unique solution $\boldsymbol{\theta}$ in no more than N steps. This follows from an induction argument that proves for each n, $\sigma(\boldsymbol{P}_{I(n)I(n)}) < 1$ and $I(n) \subseteq I(n+1)$. The latter condition ensures that there is some $n_* \leqq N$ with $I(n_*) = I(n_* + 1)$. Then we have $I(n_*) = \{i \mid \theta_i(n_*) \leqq \mu_i\}$ and the solution to the throughput equation is $\boldsymbol{\theta}(n_*)$.

If $n = 0$, then $I(0) = \varnothing$ and these two conditions $\sigma(\boldsymbol{P}_{I(0)I(0)}) < 1$ and $I(0) \subseteq I(1)$, hold vacuously. Now assume that these results hold up to level n. First we show that $\sigma(\boldsymbol{P}_{I(n+1)I(n+1)}) < 1$. For simplicity, let $I = I(n)$, $\boldsymbol{\theta} = \boldsymbol{\theta}(n)$, and $I_* = I(n+1)$. Suppose that $\sigma(\boldsymbol{P}_{I_*I_*}) = 1$, by Perron–Frobenius theory, $\boldsymbol{P}_{I_*I_*}$ has 1 as an eigenvalue which corresponds to a non-zero, positive eigenvector. If $L \subseteq I_*$ indexes the non-zero entries of this eigenvector, then $\boldsymbol{P}_{LL}$ is stochastic. From this it follows that none of the nodes indexed by L can be drained, hence they can all be filled. Let $\boldsymbol{\xi} = [\xi_1, \cdots, \xi_N]$ where

$$\xi_i = \begin{cases} \theta_i, & i \in I \\ \mu_i, & i \notin I. \end{cases}$$

If M is the complement of L, then

$$\boldsymbol{\theta}_L = \boldsymbol{\xi}_L \boldsymbol{P}_{LL} + \boldsymbol{\xi}_M \boldsymbol{P}_{ML} + \boldsymbol{\lambda}_L.$$

Summing over the components of each vector gives

$$\sum_{i \in L} \theta_i = \sum_{i \in L} \xi_i + \sum_{i \in L} \left[\lambda_i + \sum_{j \in M} \xi_j p_{ji} \right]$$

$$\sum_{i \in L} \theta_i - \xi_i = \sum_{i \in L} \left[\lambda_i + \sum_{j \in M} \xi_j p_{ji} \right].$$

By hypothesis, and the definition of $\boldsymbol{\xi}$ and L, the left-hand side of the last equation is non-positive. Equality can then occur only if $\theta_i = \xi_i$, $\lambda_i = 0$, and $\xi_j p_{ji} = 0$ for all $i \in L$ and $j \notin L$. For every $i \in L$ we have $\lambda_i = 0$, so there must exist a $j \in M$ such that $p_{ji} \neq 0$. This implies that $\xi_j = 0$, and since $\mu_j > 0$, we have $\xi_j = \theta_j = 0$. This implies that $\lambda_j = 0$, and so there exists some $k \in M - \{j\}$ with $p_{kj} \neq 0$ and $\xi_k = \theta_k = 0$. By induction, we see that no node in L can be filled. Since the nodes indexed by L can neither be filled nor drained, this is a contradiction. Therefore, $\sigma(P_{I_*I_*}) < 1$ must always hold. This creates a unique solution to the $I(n+1)$-partition equation, $\boldsymbol{\theta}(n+1)$.

By induction hypothesis, $I \subseteq I_*$ so let $K = I_* - I$ and have J_* equal the complement of I_*. We then have the following equations for index sets I and K,

$$\boldsymbol{\theta}_I = \boldsymbol{\lambda}_I + \boldsymbol{\theta}_I \boldsymbol{P}_{II} + \boldsymbol{\mu}_K \boldsymbol{P}_{KI} + \boldsymbol{\mu}_{J_*} \boldsymbol{P}_{J_*I} \tag{2.1}$$

$$\boldsymbol{\theta}_K = \boldsymbol{\lambda}_K + \boldsymbol{\theta}_I \boldsymbol{P}_{IK} + \boldsymbol{\mu}_K \boldsymbol{P}_{KK} + \boldsymbol{\mu}_{J_*} \boldsymbol{P}_{J_*K}. \tag{2.2}$$

By definition, $\boldsymbol{\theta}_K \leqq \boldsymbol{\mu}_K$ so (2.2) is equivalent to

$$\boldsymbol{\mu}_K = (\boldsymbol{\lambda}_K + \boldsymbol{\mu}_K - \boldsymbol{\theta}_K) + \boldsymbol{\theta}_I \boldsymbol{P}_{IK} + \boldsymbol{\mu}_K \boldsymbol{P}_{KK} + \boldsymbol{\mu}_{J_*} \boldsymbol{P}_{J_*K}.$$

Let $\tilde{\boldsymbol{\theta}}_{I_*} = [\boldsymbol{\theta}_I, \boldsymbol{\mu}_K]$ and $\tilde{\boldsymbol{\lambda}}_{I_*} = [\boldsymbol{\lambda}_I, \boldsymbol{\lambda}_K + \boldsymbol{\mu}_K - \boldsymbol{\theta}_K]$, then (2.1) and (2.2) can be written as

$$\tilde{\boldsymbol{\theta}}_{I_*} = \tilde{\boldsymbol{\lambda}}_{I_*} + \boldsymbol{\mu}_{J_*} \boldsymbol{P}_{J_*I_*} + \tilde{\boldsymbol{\theta}}_{I_*} \boldsymbol{P}_{I_*I_*}.$$

Solving for $\tilde{\boldsymbol{\theta}}_{I_*}$ gives

$$\tilde{\boldsymbol{\theta}}_{I_*} = (\tilde{\boldsymbol{\lambda}}_{I_*} + \boldsymbol{\mu}_{J_*}\boldsymbol{P}_{J_*I_*})(\boldsymbol{I} - \boldsymbol{P}_{I_*I_*})^{-1}.$$

Let $\boldsymbol{\theta}^* = \boldsymbol{\theta}(n+1)$. Since $\boldsymbol{\theta}^*$ solves the I_*-partition equation, then

$$\boldsymbol{\theta}^*_{I_*} = (\boldsymbol{\lambda}_{I_*} + \boldsymbol{\mu}_{J_*}\boldsymbol{P}_{J_*I_*})(\boldsymbol{I} - \boldsymbol{P}_{I_*I_*})^{-1}.$$

But $(\boldsymbol{I} - \boldsymbol{P}_{I_*I_*})^{-1}$ is a positive operator, and $\boldsymbol{\lambda}_{I_*} \leqq \tilde{\boldsymbol{\lambda}}_{I_*}$, so $\boldsymbol{\theta}^*_{I_*} \leqq \tilde{\boldsymbol{\theta}}_{I_*}$. From this it follows that $I_* \subseteq \{i \mid \theta^*_i \leqq \mu_i\}$. Consequently, $I(n+1) \subseteq I(n+2)$ and we have proved the induction step.

We now show uniqueness. Suppose $\boldsymbol{\theta}$ and $\tilde{\boldsymbol{\theta}}$ are both solutions and $\boldsymbol{\theta}$ is constructed by the algorithm, then

$$\theta_i - \tilde{\theta}_i = \sum_{j=1}^{N} (\theta_j \wedge \mu_j - \tilde{\theta}_j \wedge \mu_j) p_{ji}.$$

Taking absolute values and then summing over i gives

$$\sum_{i=1}^{N} |\theta_i - \tilde{\theta}_i| \leqq \sum_{i=1}^{N} |\theta_i \wedge \mu_i - \tilde{\theta}_i \wedge \mu_i| \leqq \sum_{i=1}^{N} |\theta_i - \tilde{\theta}_i|.$$

The last inequality holds componentwise, hence for all i

$$|\theta_i \wedge \mu_i - \tilde{\theta}_i \wedge \mu_i| = |\theta_i - \tilde{\theta}_i|.$$

We then have $\theta_i > \mu_i$ if and only if $\tilde{\theta}_i > \mu_i$. This means that $\{i \mid \theta_i \leqq \mu_i\} = \{i \mid \tilde{\theta}_i \leqq \mu_i\}$, but we can construct $\boldsymbol{\theta}$ by using the algorithm. Since $\boldsymbol{\theta}$ and $\tilde{\boldsymbol{\theta}}$ solve the same I-partition equation, $\boldsymbol{\theta} = \tilde{\boldsymbol{\theta}}$.

3. Proving the main theorem

To prove Theorem 1, it is sufficient to prove that

$$\lim_{t\to\infty} \Pr\{Q_i(t) < n_i\,;\, i \in U\} = \prod_{i \in U} (1 - \rho_i^{n_i}) \tag{3.1}$$

and for all $j \notin U$, $\lim_{t\to\infty} \Pr\{Q_j(t) < n\} = 0$. The key problem in proving such a result is that in general, $\{Q_i(t)\}_{i \in I}$ is *not* a Markov process. As a consequence, the ergodic theorems for Markov chains do not apply directly. We will prove Theorem 1 by using a coupling argument to stochastically bound, in the sense of Kirstein, Franken and Stoyan [5], the desired subnetwork above and below by ergodic Markov networks.

Given an arbitrary parameter $\varepsilon \geqq 0$, we shall construct the $(\boldsymbol{M}/\boldsymbol{M}/\mathbf{1})^N$ systems $\boldsymbol{Q}^+(t)$ and $\boldsymbol{Q}^-(t, \varepsilon)$ such that $Q_i^-(t, \varepsilon) \leqq_{\mathrm{st}} Q_i(t) \leqq_{\mathrm{st}} Q_i^+(t)$ for all i by modifying the original process $\boldsymbol{Q}(t)$. We derive $\boldsymbol{Q}^+(t)$ from $\boldsymbol{Q}(t)$ by stopping the flow of customers from the non-U nodes (nodes not indexed by U) into the U nodes (nodes that are). We then substitute into the latter, Poisson inputs equaling the maximal output rate from the non-U nodes. The parameters for $\boldsymbol{Q}^+(t)$ are then given as

$$\lambda_i^+ = \begin{cases} \lambda_i + \sum_{j \notin U} \mu_j p_{ji} & i \in U \\ \lambda_i & i \notin U \end{cases}$$

$$\mu_i^+ p_{ij}^+ = \begin{cases} 0 & i \notin U \text{ and } j \in U \\ \mu_i p_{ij} & \text{otherwise} \end{cases}$$

$$\mu_i^+ q_i^+ = \mu_i q_i.$$

Notice that $\boldsymbol{Q}_U^+(t) = \{\boldsymbol{Q}_i^+(t)\}_{i \in U}$ is an $(\boldsymbol{M}/\boldsymbol{M}/\boldsymbol{1})^{N'}$ system in its own right where $N' = |U|$. Its parameters are the same as for $\boldsymbol{Q}^+(t)$ restricted to U except that $\mu_i^+ q_i^+$ is replaced by $\mu_i(q_i + \Sigma_{j \notin U} p_{ij})$ instead of $\mu_i q_i$. $\boldsymbol{Q}_U^+(t)$ is ergodic and $\boldsymbol{\theta}_U^+$, the solution to $\boldsymbol{Q}_U^+(t)$'s throughput equation, equals $\boldsymbol{\theta}_U$. This holds because $\boldsymbol{\theta}$ also solves the U-partition equation for $\boldsymbol{Q}(t)$.

For $\boldsymbol{Q}^-(t, \varepsilon)$ we take the external rates for $\boldsymbol{Q}(t)$ $(\mu_i q_i)$ for the non-U nodes and increase them until the entire system is ergodic at $\varepsilon > 0$. At $\varepsilon = 0$ however, all of the non-U nodes will not attain steady state. The parameters for $\boldsymbol{Q}^-(t, \varepsilon)$ will then be

$$\lambda_i^-(\varepsilon) = \lambda_i$$

(3.2) $$\mu_i^-(\varepsilon) p_{ij}^-(\varepsilon) = \mu_i p_{ij}$$

$$\mu_i^-(\varepsilon) q_i^-(\varepsilon) = \begin{cases} \mu_i q_i, & i \in U \\ \mu_i q_i + \theta_i - \mu_i + \varepsilon, & i \notin U. \end{cases}$$

If we let $\boldsymbol{\theta}^-(\varepsilon)$ solve the throughput equation for $\boldsymbol{Q}^-(t, \varepsilon)$ then we will show that $\lim_{\varepsilon \downarrow 0} \boldsymbol{\theta}^-(\varepsilon) = \boldsymbol{\theta}$, the throughput vector for $\boldsymbol{Q}(t)$. Notice that $\lim_{\varepsilon \downarrow 0} \boldsymbol{\mu}^-(0) = [\boldsymbol{\mu}_U, \boldsymbol{\theta}_V]$ where V is the complement of U.

These results finish the proof of the theorem since, using Jackson's theorem on $\boldsymbol{Q}_U^+$

$$\liminf_{t \to \infty} \Pr\{Q_i(t) < n_i ; i \in U\} \geqq \liminf_{t \to \infty} \Pr\{Q_i^+(t) < n_i ; i \in U\}$$
$$\geqq \prod_{i \in U} (1 - \rho_i^{n_i}).$$

Similarly,

$$\limsup_{t \to \infty} \Pr\{Q_i(t) < n_i ; i \in U\} \leqq \limsup_{t \to \infty} \Pr\{Q_i^-(t, \varepsilon) < n_i ; i \in U\}$$
(3.3) $$\leqq \prod_{i \in U} (1 - \rho_i^-(\varepsilon)^{n_i}).$$

Letting ε go to 0 in (3.3) gives us (3.1). Notice that we also have for non-U nodes

$$\limsup_{t\to\infty} \Pr\{Q_j(t)<n\} \leqq 1-\rho_j^-(\varepsilon)^n$$

and since $\lim_{\varepsilon\downarrow 0}\rho_j^-(\varepsilon)=1$ for $j\notin U$, we get $\lim_{t\to\infty}\Pr\{Q_j(t)<n\}=0$.

Now we take care of the technicalities. Since $\boldsymbol{\theta}^-(\varepsilon)$ solves the equation $\boldsymbol{\theta}^-(\varepsilon)=(\boldsymbol{\theta}^-(\varepsilon)\wedge\boldsymbol{\mu}^-(\varepsilon))\boldsymbol{P}^-(\varepsilon)+\boldsymbol{\lambda}$, then $|\boldsymbol{\theta}^-(\varepsilon)|_1\leqq|\boldsymbol{\lambda}|_1+|\boldsymbol{\mu}^-(\varepsilon)|_1$ so $\{\boldsymbol{\theta}^-(\varepsilon)\,|\,0\leqq\varepsilon\leqq\varepsilon'\}$ is a bounded set of vectors. This means that we can always find a convergent subsequence $\{\boldsymbol{\theta}^-(\varepsilon_k)\}_{k\geqq 0}$ where $\varepsilon_k\downarrow 0$ as $k\to\infty$. We shall always have $\boldsymbol{\mu}^-(\varepsilon_k)\to\boldsymbol{\mu}^-(0)$ and $\boldsymbol{P}^-(\varepsilon_k)\to\boldsymbol{P}^-(0)$ as $k\to\infty$, so this gives us

$$\lim_{k\to\infty}\boldsymbol{\theta}^-(\varepsilon_k)=\left(\lim_{k\to\infty}\boldsymbol{\theta}^-(\varepsilon_k)\wedge\boldsymbol{\mu}^-(0)\right)\boldsymbol{P}^-(0)+\boldsymbol{\lambda}.$$

Since $\boldsymbol{\theta}^-(0)$ is the unique solution of this equation, we must have $\boldsymbol{\theta}^-(0)=\lim_{k\to\infty}\boldsymbol{\theta}^-(\varepsilon_k)$ for all convergent subsequences hence $\boldsymbol{\theta}^-(0)=\lim_{\varepsilon\downarrow 0}\boldsymbol{\theta}^-(\varepsilon)$.

Now to prove that $\boldsymbol{\theta}^-(0)=\boldsymbol{\theta}$. Solving for $\mu_i^-(0)$ and $p_{ij}^-(0)$ from (3.2) gives us $\boldsymbol{\mu}^-(0)=[\boldsymbol{\mu}_U,\boldsymbol{\theta}_V]$ where V is the complement of U, and

$$p_{ij}^-(0)=\begin{cases} p_{ij}, & i\in U\\ \dfrac{\mu_i}{\theta_i}p_{ij}, & i\notin U.\end{cases}$$

Given $\boldsymbol{\theta}=(\boldsymbol{\theta}\wedge\boldsymbol{\mu})\boldsymbol{P}+\boldsymbol{\lambda}$ it is then immediate that $\boldsymbol{\theta}=\boldsymbol{\theta}\boldsymbol{P}^-(0)+\boldsymbol{\lambda}$ with $\theta_i\leqq\mu_i^-(0)$ for all i.

By Theorem 2, the above argument shows that $\sigma(\boldsymbol{P}^-(0))<1$ and $\boldsymbol{\theta}=\boldsymbol{\lambda}(\boldsymbol{I}-\boldsymbol{P}^-(0))^{-1}$. Now $\mu_i^-(\varepsilon)\geqq\mu_i^-(0)$ since $\mu_i^-(\varepsilon)$ equals μ_i for the U nodes but it equals $\theta_i+\varepsilon$ for i non-U. This means that $p_{ij}^-(\varepsilon)\leqq p_{ij}^-(0)$ for all i and j hence

$$\begin{aligned}\boldsymbol{\theta}^-(\varepsilon)&=(\boldsymbol{\theta}^-(\varepsilon)\wedge\boldsymbol{\mu}^-(\varepsilon))\boldsymbol{P}^-(\varepsilon)+\boldsymbol{\lambda}\\ &\leqq\boldsymbol{\theta}^-(\varepsilon)\boldsymbol{P}^-(0)+\boldsymbol{\lambda}.\end{aligned}$$

So $\boldsymbol{\theta}^-(\varepsilon)\leqq\boldsymbol{\lambda}(\boldsymbol{I}-\boldsymbol{P}^-(0))^{-1}=\boldsymbol{\theta}^-(0)$. Since $\boldsymbol{\theta}^-(0)\leqq\boldsymbol{\mu}^-(0)\leqq\boldsymbol{\mu}^-(\varepsilon)$, we then have $\theta_i^-(\varepsilon)<\mu_i^-(\varepsilon)$ for all i. This shows that $\boldsymbol{Q}^-(t,\varepsilon)$ is ergodic for all $\varepsilon>0$ where $\boldsymbol{\theta}^-(\varepsilon)=\boldsymbol{\lambda}(\boldsymbol{I}-\boldsymbol{P}^-(\varepsilon))^{-1}$.

To prove the stochastic ordering results, we note that $\boldsymbol{Q}(t)$ is a *uniformizable* Markov process, that is the rate of flow out of any state is bounded above by a constant, namely $\Sigma_{i=1}^N\lambda_i+\mu_i$. As such, we can decompose $\boldsymbol{Q}(t)$ into a discrete-time Markov chain that is subordinated to a Poisson process (see Keilson and Kester [4]). This rigorously establishes the following pathwise description of $\boldsymbol{Q}(t)$. Run a Poisson process at rate $\xi=\Sigma_{i=1}^N\lambda_i+\mu_i$. At each transition epoch of the Poisson process, flip an independent many-sided coin and choose one of the following activities for the queueing system to undertake:

$[A_i]$ = Add a customer to the ith node.

$[B_i]$ = Delete a customer from the ith node if possible. Otherwise do nothing.

$[C_{ij}]$ = Transfer a customer from the ith node to the jth node if possible. Otherwise do nothing.

We then select activity $[A_i]$ with probability λ_i/ξ, $[B_i]$ with probability $\mu_i q_i/\xi$, and $[C_{ij}]$ with probability $\mu_i p_{ij}/\xi$. We note that ξ can be larger than $\Sigma_{i=1}^N \lambda_i + \mu_i$. Merely add the activity

$$[D] = \text{Do nothing}$$

and then select $[D]$ with probability

$$1 - \frac{1}{\xi}\sum_{i=1}^{N} \lambda_i + \mu_i.$$

We define $\boldsymbol{Q}^+(t)$ using the above pathwise construction. In fact, we build it on the same Poisson process that we used for $\boldsymbol{Q}(t)$. We then insist that $\boldsymbol{Q}(t)$ and $\boldsymbol{Q}^+(t)$ choose activities with the same probabilities. It will then be clear that if $\boldsymbol{Q}^+(t)$ consistently chooses activities that give it more or the same number of customers in each node compared to $\boldsymbol{Q}(t)$, then $\boldsymbol{Q}^+(t) \geqq \boldsymbol{Q}(t)$ on all sample paths.

We choose $\boldsymbol{Q}^+(t)$ to differ from $\boldsymbol{Q}(t)$ in the following manner. For $i \not\in U$ and $j \in U$, with probability $\mu_i p_{ij}/\xi$, choose activity $[A_j]$ instead of $[C_{ij}]$. A customer is still added to the same node for both processes, but no corresponding customer is deleted for $\boldsymbol{Q}^+(t)$. If $\boldsymbol{Q}^+(0) = \boldsymbol{Q}(0) = (n_1, \cdots, n_N)$, then $\boldsymbol{Q}^+(t) \geqq \boldsymbol{Q}(t)$ pathwise.

Now for comparing $\boldsymbol{Q}^-(t, \varepsilon)$ and $\boldsymbol{Q}(t)$, run a Poisson process at rate

$$\xi = \sum_{i=1}^{N} \lambda_i + \mu_i + \sum_{j \in V} \theta_j - \mu_j + \varepsilon \mid V \mid$$

where $|V|$ is the cardinality of V. Here we can reconstruct $\boldsymbol{Q}(t)$ as before, choosing $[A_i]$ with probability λ_i/ξ and so on but by also choosing $[D]$ with probability $(1/\xi)[\varepsilon \mid V \mid + \Sigma_{j \in V} \theta_j - \mu_j]$. Except for $[D]$, $\boldsymbol{Q}^-(t, \varepsilon)$ chooses the same activities that $\boldsymbol{Q}(t)$ does with the same probabilities. For $j \in V$, it chooses $[B_j]$ with probability $(1/\xi)(\theta_j - \mu_j + \varepsilon)$. From this it follows that $\boldsymbol{Q}^-(t, \varepsilon) \leqq \boldsymbol{Q}(t)$ on all sample paths.

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