A Probabilistic Look at Networks of Quasi-Reversible Queues

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Abstract—**Probabilistic arguments are given to explain some recent results on networks of queues. Those results are the product form, the output theorems, the distributions at the jumps, and the Poisson character of the flows. The proposed approach replaces the usual calculations by arguments showing these properties to be consequences of the behavior of the nodes in isolation.**

I. Introduction

NETWORKS of queues are used as models in applications ranging from traffic networks (e.g., [1]) to computer communication systems (e.g., [2]).

The analytically tractable networks belong to the class of so-called product form or quasi-reversible networks. In recent years a vast literature has been devoted to those systems (see, e.g., [3] and [4]).

The history of those models goes back to 1956, when Burke [5] and Reich [6] proved the *output theorem* for M/M/s queues. That result states that in equilibrium the departure process $(D_t, -\infty < t < +\infty)$ of an M/M/s queue is Poisson and, in addition, for any time t its past $\sigma(D_s, s \leqslant t)$ and the present queue length are independent.

That result inspired J. R. Jackson to derive the independence in equilibrium of the states (queue lengths) of the M/M/s queues in an open network with independent routing and Poisson exogeneous arrival processes. Moreover, he showed that the equilibrium distribution of the state of each queue in the network is the same as if its arrival process was Poisson. Such a property is referred to as a *product form* result: the equilibrium distribution of the state of the network (the vector of queue lengths) has a product form.

This knowledge of the equilibrium distribution opens the door to the classical design problems in networks such as capacity assignment and routing optimization (see, e.g., [8]).

In 1966, Gordon and Newell [9] showed that a closed network of M/M/s queues admits the same *invariant probability measure* as an open network built from the same queues. This is a conditioning result: the equilibrium probability of a given state in a closed network with N customers is the equilibrium probability of that state in the open network given that it contains N customers. As a consequence, the equilibrium distribution of the closed network also has a product form.

In recent years the output theorem was shown to hold for more general nodes than the M/M/s queues (e.g., [4], [10]). The product form and the conditioning results were then established for networks of such nodes (e.g., [4], [10]–[13]). The proof of those results involves the verification (particularly elegant in [13]) of the balance equations of the Markov chain describing the network.

It is now known that the arrival process at a node in those networks is in general not Poisson ([14]–[16]). A simple argument will explain this fact in Section V. This makes those product form results surprising and a much less immediate consequence of the output theorem than Jackson himself appeared to believe in his fundamental paper [7]. The verifications of the balance equations do little to alleviate the mystery. Why should such simple results be true?

Another class of remarkable results for product form networks concern the equilibrium distribution of the state seen by a customer when he jumps from one node to another. In open or closed networks the results say that this customer sees the others in equilibrium. Again this is verified by a calculation. The algebra is such that, quite surprisingly, simplifications occur that give the indicated interpretation to the result. (See, e.g., [17] or [18].)

In this paper we propose some probabilistic arguments that establish the above results with little algebra. They may provide a more intuitive justification of those properties.

The paper is organized as follows: Section II reviews the concept of quasi-reversibility. The product form result is explained in Section III. The analysis of the distribution at the jumps is discussed in Section IV. Finally, Section V reviews the results on the Poisson character of flows.

II. Quasi-Reversibility

In this paper we choose to represent a node by a pure jump Markov process. Thus, let $(x_t, t \geqslant 0)$ be a time homogeneous Markov process on some Borel set S, with right continuous piecewise constant paths. That is, the jump times of the process have no finite accumulation point (or explosion). Under these conditions the strong

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Markov property is known to hold (e.g., [19, ch. 15]). As a usual particular case, $(x_t, t \geqslant 0)$ could be a regular Markov chain on a countable state space (e.g., [20, pp. 266]).

One then associates some jumps of (x_t) to arrivals and others to departures of customers at the node described by that Markov process. For instance, the Markov process could be a birth-and-death process and the birth times would be considered as arrivals, while the death times would be considered as departures.

The arrival process is typically some multivariate point process $(A_t, t \geqslant 0)$, where $A_t = \{A_t^c, c \in C\}$ counts the number A_t^c of arrivals in $[0, t]$ of customers of class c for all c in a countable set C. Similarly, one defines the departure process $(D_t, t \geqslant 0)$ with $D_t = \{D_t^c, c \in C\}$.

The quasi-reversibility is, in some sense, an *input–output* property of the node. It is defined below.

Assume that (A_t) is an *independent Poisson arrival process* with rate $\lambda = \{\lambda_c, c \in C\}$, i.e., that at any time t the future $\sigma(A_s - A_t, s \geqslant t)$ of the arrival process and the present state x_t of the node are independent, and the processes $(A_t^c, t \geqslant 0)$ for $c \in C$ are independent Poisson processes with respective rates λ_c.

The node described by (x_t) is then said to be *quasi-reversible* if (x_t) admits an invariant probability distribution π on S under which the departure process (D_t) is such that for all $t > 0$ its past $\sigma(D_s, s \leqslant t)$ and the present state x_t of the node are independent. As a consequence, the process (D_t) is then Poisson and it will be assumed that its rate is λ. Thus, *a node is quasi-reversible if the output theorem applies.*

The terminology can be justified as follows. Assume that a node (x_t) with an independent Poisson arrival process (A_t) and a departure process (D_t) has the following property: in equilibrium, the reversed process (x_{-t}) describes another node with an independent Poisson arrival process. Then the claim is that (x_t) describes a quasi-reversible node. To see that, observe that the jumps of (D_{-t}) correspond to the arrival times in (x_{-t}). Therefore (D_t) is a reversed Poisson process and is then Poisson. Moreover, the future $\sigma(D_{-s} - D_t, s \geqslant -t)$ of the arrival process of (x_{-s}) being independent of the present state x_t, it follows that the past $\sigma(D_s, s \leqslant t)$ of the original departure process and x_t are independent, as claimed.

The M/M/s queue is described by a reversible Markov chain (x_t). It is therefore quasi-reversible. (See [6] and [4] for details.)

Discussions of necessary and sufficient conditions for quasi-reversibility can be found in [4], [18], and [21], as well as in Section IV-A.

III. Product Form

A. The Result

Roughly speaking, the result states that the product form holds for an arbitrary (open, closed, or mixed) network of quasi-reversible nodes. The main contribution of this section is a probabilistic argument that explains the result.

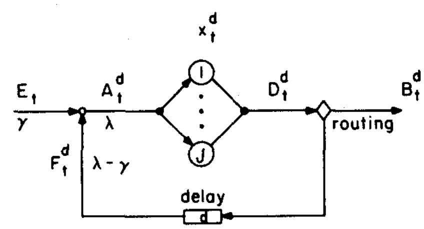

Fig. 1. Network with delay line.

Algebraic proofs are available in [4], [10], [12], [13], and particular examples in [7], [9], and [11].

For $i = 1,\cdots,J$ let $(x_t^i, t \geqslant 0)$ be a pure jump Markov process describing a quasi-reversible node on a state space S_i, with invariant probability measure π_i, independent Poisson arrival process $A_t^i = \{A_t^{ic}, c \in C\}$ of rate $\lambda^i = \{\lambda^{ic}, c \in C\}$, and departure process $D_t^i = \{D_t^{ic}, c \in C\}$, as discussed in Section II.

That is, the J nodes (x_t^i) are quasi-reversible in isolation, i.e., considered individually.

Construct a network with those J nodes as indicated on Fig. 1 (with $d = 0$ for the time being). Specifically, let $E_t = \{E_t^{ic}, (i, c) \in \{1,\cdots,J\} \times C\}$ be an independent multivariate Poisson arrival process with rate $\gamma = \{\gamma^{ic}\}$, where E_t^{ic} is the number of arrivals of customers of class c into node i in $[0, t]$; when a customer of class c leaves node i, he is sent to node j as a customer of class c' with probability $r_{ij}^{cc'}$ or else leaves the network. This sampling of the departure process, *the routing*, is independent of everything else. It will be assumed that $r_{ii}^{cc'} \equiv 0$.

Assume that γ and $R = \{r_{ij}^{cc'}\}$ are such that λ (see above) is a possible set of average rates of flows through the nodes of the network (i.e., that $\lambda^{ic} \equiv \gamma^{ic} + \sum_{jc'}\lambda^{jc'}r_{ji}^{c'c}$, these are the flow conservation equations).

Observe that this network can be open (every customer eventually leaves), closed (in which case $\gamma = 0$ and $\sum_{jc'}r_{ij}^{cc'} \equiv 1$: no one leaves), or mixed (some sets of classes are open and others are closed).

Assume also that the resulting process is pure jump. Let x_t^{0i} denote the state of node $i = 1,\cdots,J$ at time $t \geqslant 0$ and A_t^0 [resp. D_t^0] arrival [resp. departure] process at the nodes.

Theorem: The process $(x_t^0 = (x_t^{01},\cdots,x_t^{0J}), t \geqslant 0)$ admits the invariant measure

$$\pi = \pi_1 \times \pi_2 \times \cdots \times \pi_J$$

on the space $S = S_1 \times S_2 \times \cdots \times S_J$.

Moreover, under π the departure process from the network, i.e., (B_t^0) (see Fig. 1), is Poisson and such that at any time t its past $\sigma(B_s^0, s \leqslant t)$ and the present state x_t are independent.

Observe that this result says that π is an invariant probability measure on S. If the network is not open, then the Markov process (x_t) is generally not irreducible on the space S. Clearly, the invariant distribution on any given irreducible subspace of S is the normalized restriction of π on that subset. See Proposition 2 for similar considerations.

The proof is given in two steps. The main argument is discussed in the next section. It involves a limiting procedure that is detailed in Section III-C.

B. The Main Argument

For $d > 0$, introduce a pure delay d in the links between nodes of the network (see Fig. 1). This delay can be realized by an $M/D/\infty$ queue that will be called the *delay line*.

Denote by x_t^d the resulting state process for the nodes, by A_t^d [resp. D_t^d] the total number of arrivals [resp. departures] at the nodes. Let F_t^d be the output of the delay line.

Assume that x_0^d has distribution π and that $F_{(0,d]}^d = \{F_t, 0 < t \leqslant d\}$ (the content of the delay line at $t = 0$) is Poisson with rate $\lambda - \gamma$ and is independent of x_0^d and $(E_t, t \geqslant 0)$. Then $A_{(0,d]} = F_{(0,d]} + E_{(0,d]}$ is Poisson with rate λ and independent of x_0^d. Thus (x_t^d, A_t^d, D_t^d) will behave for t in $(0, d]$ as if the nodes were in isolation. By quasi-reversibility it follows that $D_{(0,d]}^d$ is Poisson and independent of x_d^d. By independence of the routing, the same is true of $F_{(d,2d]}$. Also, x_t^d has distribution π for t in $(0, d]$. The situation at time d being identical to that at time 0, this argument proves by induction that x_t^d has distribution π for all $t \geqslant 0$.

Hence the presence of an arbitrarily small delay d in the links of the network implies that π is an invariant distribution. Intuitively, it is natural to expect that this will imply that π is also invariant when there is no delay. This will be shown in Section III-C.

For communication network models, notice that there always is a delay in the transmission links (the propagation delay), so that the limiting argument of the next section is not necessary for those systems.

The argument above also shows that the departure process B_t^d is Poisson and such that $\sigma(B_s^d, s \leqslant t)$ and $\sigma\{x_t^d, F_{(t,t+d]}\}$ are independent for all $t > 0$.

The following limiting procedure shows that those properties continue to hold for $d = 0$, i.e., for the original network.

Observe that the presence of a delay d implies that the flows in the network are *piecewise* Poisson in the sense that on any interval $[t, t + d)$ they are distributed as Poisson processes. Notice also that the jump times of those flows in successive intervals of length d will in general not be independent, so that the flows will not be Poisson. For instance, $F_{(0,d]}$ and $D_{(0,d]}$ are likely to be dependent, and therefore the same is true of $F_{(0,d]}$ and $F_{(d,2d]}$. Those considerations will be taken up again in Section V.

C. The Limiting Procedure

The idea of the argument is as follows. Assume that at some time t a customer has just been sent into the delay line d which was otherwise empty. Assume also that the delay d is small enough so that, on the given sample path, no transition of x occurs in $(t, t + d)$. Then the state x_{t+d} of the nodes, just after the customer leaves the delay line, does not depend on d: the only transition between x_t and x_{t+d} is that a customer left some node and was sent into another node.

Now, take a particular sample path and a given time t. If d is small enough, then no transition will occur in the nodes during the finitely many transits of customers in the delay line that take place before time t. As a consequence, the states of the nodes in the networks with and without the delay line will coincide, *except* during the transit epochs. But then, if t is not in a transit epoch, the state values of the two networks at time t will agree. Thus the random variables $x_t^d(\omega)$ can be constructed in such a way that for all realization ω they will eventually agree with $x_t^0(\omega)$ as $d \to 0$, and this for any given t.

Since pathwise convergence implies convergence in distribution, this will show that the law of x_t^0 has to be the same as that of all the x_t^d namely, π.

In fact the same argument shows much more. Consider once again the networks with and without a small delay d. Not only do their state paths agree in $[0, t]$ except for the transit epochs, the realizations of their departure processes in $[0, t]$ also agree. This will imply the Poisson nature of (B_t^0) and also the independence property stated in the theorem.

We now proceed with the formal derivation. The only difficulty is one of notation. The processes $(x_t^d(\omega), t \geqslant 0)$ will be constructed for $d \geqslant 0$ on the same probability space in such a way that $P(B_{[0,t]}^d \equiv B_{[0,t]}^0, x_t^d = x_t^0)$ approaches one as d goes to zero. This will prove the theorem.

Consider the pure jump Markov process (x_t) on S describing the J nodes when $A_t \equiv 0$ and without feedback. If T denotes the first jump time of $(x_t, t \geqslant 0)$, then it is known that for $x \in S$, B Borel in S, and $t > 0$,

$$P_s(T > t, x_T \in B) = e^{-\lambda(x)t} p(x, B),$$

where $\lambda(x) > 0$ and $p(x, \cdot)$ is a probability measure on S. (See, e.g., [19, ch. 15].)

Considering now the process $(x_t, t \geqslant 0)$ corresponding to the J nodes with a Poisson arrival process A_t with jump times $(\sigma_n, n \geqslant 1)$ we can define

$$p(B; x, i, c) = P\left(x_{\sigma_n} \in B \mid x_{\sigma_{n-}} = x, \sigma_n \text{ is an } (i, c)\text{-arrival time}\right).$$

(By an (i, c)-arrival we mean an arrival of a customer of class c into node i, i.e., a jump time of A^{ic}.)

Let

$$\omega = \{x_0, \tau_n(x), y_n(x), y_n(x, i, c), z_n(i, c), a_t, b_t | (x, n, i, c, t) \in H\},$$

where $H = S \times \{1, 2, \cdots,\} \times \{1, \cdots, J\} \times C \times [0, \infty)$, and where the elements are as follows:

- The value x_0 is chosen in S with distribution π.
- The times $\{\tau_n(x), n \geqslant 1\}$ are a realization of the jump times of a Poisson process with rate $\lambda(x)$.
- The sequence $\{y_n(x), n \geqslant 1\}$ [resp. $\{y_n(x, i, c), n \geqslant 1\}$] is a realization of a sequence of independent random variables with distribution $p(x; \cdot)$ [resp. $p(\cdot, x, i, c)$] in S.
- The sequence $\{z_n(i, c), n \geqslant 1\}$ is a realization of a sequence of independent random variables taking the value (i', c') with probability $r_{ii'}^{cc'}$.
- Finally, $(a_t, t \geqslant 0)$ [resp. $(b_t, t \geqslant 0)$] is a realization of a Poisson process with rate γ [resp. $\lambda - \gamma$].

The construction of $x^d(\omega)$ then proceeds as follows: The *initial state* is $x_0^d(\omega) = x_0$. The *initial content of the delay line* is $F^d_{[0,d]}(\omega) \equiv b_{[0,d]}$.

Assume that $x^d_{[0,t]}(\omega)$ has been constructed for some $t \geqslant 0$ and let $x = x_t^d(\omega)$. The next transition will occur at time τ which is the minimum of the following three random variables:

a) $\tau_{n_0} = \min\{\tau_n(x) | \tau_n(x) > t\}$, the end of the *holding time* of x when no arrival occurs before that time;
b) the time of the first exogeneous arrival, which is the first jump time of $\{a_s, s \geqslant t\}$,
c) the time of the first departure from the delay line, which is an endogeneous arrival.

In case a) the next state is $y_{n_0}(x)$. This transition from x to $y_{n_0}(x)$ may correspond to the kth (i, c)-service completion; if this is so, then $z_k(i, c)$ indicates the routing of the customer. If that routing induces an arrival into another node, then the effect of that arrival is the same as if the arrival were exogeneous and it is described below.

In cases b) or c) say that the transition is the mth (i, c)-arrival occurring when the state is x. Then the next state is $y_m(x, i, c)$. This defines $(x_t^d(\omega), t \geqslant 0)$ for $d \geqslant 0$.

By construction it follows that all the processes (x_t^d) have the desired laws. Moreover it is shown below that these processes have the convergence property announced at the beginning of the proof.

Denote by $\{T_n(\omega), n \geqslant 1\}$ the feedback times in $x^0(\omega)$, and let $T_0(\omega) = 0$. For $d > 0$ and $n \ge 1$, let Ω_n^d be the collection of ω's such that $d < T_1(\omega)$, $T_m(\omega) + d \leqslant T_{m+1}(\omega)$ for $m = 1,\cdots,n$ and that neither $x^0(\omega)$ nor $x^d(\omega)$ jump in $\cup_{m=0}^{n}(T_m(\omega), T_m(\omega) + d)$. In particular, in Ω_n^d, the times $T_m(\omega)$ for $m = 1,\cdots,n$ are spaced at least d time units apart.

The first observation is that if $\omega \in \Omega_n^d$, then $x^0(\omega)$ and $x^d(\omega)$ agree in

$$I_n^d(\omega) = [0, T_1(\omega)) \cup [T_1(\omega) + d, T_2(\omega)) \cup \cdots \cup [T_{n-1}(\omega) + d, T_n(\omega)).$$

To see this, notice that the delay line is empty at $t = 0$ so that $x^0(\omega)$ and $x^d(\omega)$ agree on $I_1^d(\omega) = [0, T_1(\omega))$. Next assume that $x^0(\omega)$ and $x^d(\omega)$ agree on $I_k^d(\omega)$ for some $1 \leqslant k < n$. Then the delay line of $x^d(\omega)$ must be empty at $T_k(\omega)$—so that $x^0(\omega)$ and $x^d(\omega)$ must agree at $T_k(\omega) + d$, by construction. Also, the delay line being empty at $T_k(\omega) + d$, it follows that $x^0(\omega)$ and $x^d(\omega)$ will agree on $[T_k(\omega) + d, T_{k+1}(\omega))$, and therefore on $I_{k+1}^d(\omega)$, as claimed.

The second observation is that repeating the argument above with d replaced by $d' < d$ shows that $\Omega_n^d \subset \Omega_n^{d'}$ for all $0 < d' < d$ and $n \geqslant 1$. Similarly, if $\omega \in \Omega_n^d$, then there must be some d' in $(0, d)$ such that $\omega \in \Omega_{n+1}^{d'}$. Also, for any ω there is some $d > 0$ small enough so that $\omega \in \Omega_1^d$.

Therefore, $\lim_{d \to 0} P(\Omega_n^d) = 1$ for $n \ge 1$. Now, for $t > 0$, say that $\omega \in \Omega_t^d$ if $\omega \in \Omega_n^d$ for $n = \min\{m \geqslant 1 | T_m(\omega) > t\}$. Since x^0 is assumed to be pure jump, it follows that $\lim_{n \to \infty} P(T_n > t) = 1$. Hence, $\lim_{d \to 0} P(\Omega_t^d) = 1$ for all $t > 0$.

Since $B^d_{[0,t]}(\omega) \equiv B^0_{[0,t]}(\omega)$ whenever $\omega \in \Omega_t^d$, and $x_t^t(\omega) = x_t^0(\omega)$ if $\omega \in \Omega_t^d$ and $t \notin \cup_{m=1}^{\infty}(T_m(\omega), T_m(\omega) + d)$, it now suffices to prove that $\lim_{d \to 0} P(t \in \cup_{m=1}^{\infty}(T_m(\omega), T_m(\omega) + d)) = 0$. But this follows from the fact that $\lim_{d \to 0} P(t \in (T_n(\omega), T_n(\omega) + d)) = 0$ for every $n \geqslant 1$, by the quasi-left-continuity of the pure jump Markov process x^0 (e.g., [23, p. 68]).

This concludes the proof.

IV. Distribution at the Jumps

To simplify the notation we will consider Markov chain models, i.e., the case of a countable state space S. First we review some simple results on the jumps of a Markov chain.

A. Preliminaries

Let $(x_t, t \geqslant 0)$ be a regular Markov chain on S with generator $Q = \{q(i, j) | i, j \in S\}$ and with invariant distribution π. Fix Θ, a subset of $\{(i, j) | i, j \in S \text{ and } i \neq j\}$ and define $T_0 = 0$ and, for $n \geqslant 0$, $T_{n+1} = \min\{t > T_n | (x_{t^-}, x_t) \in \Theta\}$. Let also, for $t \geqslant 0$, $A_t = \max\{n \geqslant 0 | T_n \leqslant t\}$. Hence A_t is the number of transitions in Θ that x has made in $[0, t]$. For $i \in S$, define $\Theta^i = \{j \in S | (j, i) \in \Theta\}$, and $\Theta_i = \{j \in S | (i, j) \in \Theta\}$.

If t_n, $n \geqslant 1$ are the jump times of $(x_t, t \geqslant 0)$, then it is easy to check that $\{z_n = (x_{t_n}, x_{t_{n+1}}), n \geqslant 1\}$ admits $\mu = \{\mu(i, j) = \alpha\pi(i)q(i, j) | i \neq j\}$ as invariant distribution. (α is a normalizing constant assumed to exist.)

Since $(x_{T_n^-}, x_{T_n})$ is z_n watched in Θ, this implies the following. (See, e.g., [22, p. 133].) (For an alternative derivation of Fact 1 and Fact 3, see also [18, th. 5.1, 5.2].)

Fact 1: The distributions

$$\pi_{T^-}(i) = \frac{\mu(\{i\} \times \Theta_i)}{\mu(\Theta)}, \qquad i \in S, \tag{4.1}$$

and

$$\pi_T(i) = \frac{\mu(\Theta^i \times \{i\})}{\mu(\Theta)}, \qquad i \in S \tag{4.2}$$

are invariant for $\{x_{T_n^-}, n \geqslant 1\}$ and $\{x_{T_n}, n \geqslant 1\}$, respectively.

Observe that

$$\pi_{T^-}(i) = \pi(i)\frac{\lambda(i)}{\lambda}$$

and

$$\pi_T(i) = \lambda^{-1} \sum_{j \in \Theta^i} \pi(j) q(j, i), \tag{4.3}$$

where $\lambda = \sum_{(i,j) \in \Theta} \pi(i) q(i, j)$ is the average rate of (A_t), and $\lambda(i) = \sum_{j \in \Theta_i} q(i, j)$ is the rate of A_t given that $x_t = i$, i.e.,

$$\lambda(i) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} P(A_{t+\epsilon} = A_t + 1 | x_t = i).$$

This leads to the following result.

Fact 2: $\pi_{T^-} \equiv \pi$ if and only if, in equilibrium, $\sigma(A_s - A_t, s \geqslant t)$ is independent of x_t for all $t \geqslant 0$. (As a consequence,

(A_t) is then Poisson.) Indeed, (A_t) has the indicated independence property if and only if $\lambda(i) \equiv \lambda$.

By reversing the time, i.e., by applying the above fact to (x_{-t}), one then obtains the following.

Fact 3: $\pi_T \equiv \pi$ if and only if, in equilibrium, $\sigma(A_s, s \leqslant t)$ is independent of x_t for all $t \geqslant 0$. (As a consequence, (A_t) is then Poisson.)

Remarks 1:

a) The above considerations show that *a node is quasi-reversible if and only if its customers find it and leave it in equilibrium*.

b) In general $\mu_T(i) \neq P_\pi(x_{T_1} = i)$, where P_π indicates that x_0 has the law π.

To see that, consider a Markov chain on $\{0,1\}$ with $q(0,1) = \lambda$, $q(1,0) = \mu = 1 - \lambda$, and $\Theta = \{(0,1),(1,0)\}$. Then $P_\pi(x_{T_1} = 1) = \pi(0) = \mu$, while $\pi_T(1) = \pi(0)q(0,1)\{\pi(0)q(0,1) + \pi(1)q(1,0)\}^{-1} = 1/2$.

B. Jumps in Networks

Consider a network of quasi-reversible nodes as in Section III and let π be an invariant distribution. Denote by A_t the total arrival process of customers of a given class c into a given node j, and by $\{T_n, n \geqslant 1\}$ the jump times of that process. Using the formulas of Fact 1 we could calculate π_T. This would, however, require a more explicit description of the nodes (in order to determine π). It turns out, as has been shown in the literature on various examples (e.g., [17], [18]), that the calculations would lead to a simple interpretation for π_T. This will be explained by an argument based upon the probabilistic interpretation of the quasi-reversibility (i.e., Fact 3 or Remark 1a).)

To simplify the discussion we will denote by $\tilde{x}_{T_n}$ the state at time T_n of the network from which the customer who jumped at time T_n has been removed.

The fundamental result is the following.

Proposition 1: The distribution π is in invariant for $\{\tilde{x}_{T_n}, n \geqslant 1\}$.

Proof: Let λ be the average rate of A_t. Choose $0 < \epsilon < 1$ and modify the network as follows (see Fig. 2). For each $n \geqslant 1$, the customer who jumps at time T_n is (independently of the state of the network) forced to leave the network with probability ϵ, otherwise he is allowed to join node j. This creates a new departure process L_t with jump times $\{S_n, n \geqslant 1\}$. This loss of customers is compensated for by a new independent Poisson arrival process with rate $\lambda\epsilon$ of customers of class c into node j. From the quasi-reversibility results of Section III and Fact 3 it follows that $\pi_s = \pi$, i.e., that π is invariant for $\{x_{s_n}, n \geqslant 1\}$.

To find the invariant distribution $\nu(\cdot)$ for $\{\tilde{x}_{T_n}, n \geqslant 1\}$ in the original network we argue as follows.

Denote by $B(x)$ the set of pairs (y, z) such that a transition from y to z corresponds to a jump time T_n of A with $\tilde{x}_{T_n} = x$. Then using (4.3) gives

$$\nu(x) = \lambda^{-1} \sum_{(y,z)\in B(x)} \pi(y)q(y,z). \tag{4.4}$$

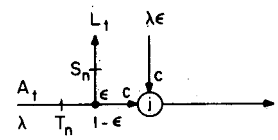

Fig. 2. Modification for distribution at jumps.

Considering now the modified network, we observe that a transition (y, x) with a jump S_n occurs exactly when a transition $(y, z) \in B(x)$ occurs in the original network and when the jumping customer is forced to leave the network. Hence using (4.3) gives

$$\pi_s(x) = (\lambda\epsilon)^{-1} \sum_{(y,z)\in B(x)} \pi(y)\epsilon q(y,z). \tag{4.5}$$

Comparing (4.4) and (4.5) then shows that $\nu = \pi_s$, and since $\pi_s = \pi$ this concludes the proof. Q.E.D.

The same argument applies if A_t counts the customers of class c jumping from node i into node j as customers of class c'.

As in Section III it must be emphasized that the result gives an invariant probability distribution on a state space S which may or may not be irreducible. If the network is closed (or mixed), then $(x_t, t \geqslant 0)$ is generally not irreducible on S. A typical situation is described below.

Proposition 2: Assume that S can be partitioned as $(S_p, p \geqslant 1)$, where each S_p is an irreducible subspace for $(x_{T_n}, n \geqslant 1)$ and such that, for some (p, q), $x_{T_n} \in S_p$ if and only if $\tilde{x}_{T_n} \in S_q$. Then, the distribution

$$\left(\frac{\pi(x)}{\pi(S_q)}, x \in S_q\right)$$

is invariant for $(\tilde{x}_{T_n}, n \geqslant 1)$.

Proof: Proposition 1 implies that

$$\sum_y \pi(y)P(\tilde{x}_{T_{n+1}} = x|\tilde{x}_{T_n} = y) = \pi(x).$$

Now, for $x \in S_q$, $P(\tilde{x}_{T_{n+1}} = x|\tilde{x}_{T_n} = y) \neq 0$ if and only if $y \in S_q$, and this concludes the proof. Q.E.D.

For example, S_p may specify the number of customers of each class in a closed network or the number of customers of the closed classes in a mixed network.

Clearly, in many situations the subspace obtained by specifying only the number of customers of the various classes is not irreducible, and the choice of the subsets S_p must be modified accordingly. See [18], Section VI for related considerations.

Norton's Theorem for network of queues can be explained by an argument similar to that used for Proposition 1 (see [24]).

V. Poisson Flows

The basic argument of [16] is reviewed. It characterizes the Poisson flows in networks of quasi-reversible nodes.

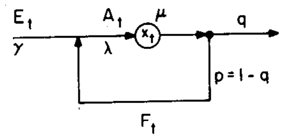

Fig. 3. M/M/1 queue with feedback.

A. An Example

Consider the M/M/1 queue with feedback of Fig. 3. Assume that it is in equilibrium, i.e., that

$$P(x_t = x) = \rho^x(1-\rho), \qquad x \geqslant 0,$$

where $\rho = (\lambda/\mu q) < 1$ and $\lambda = \gamma + \lambda p$ (conservation of flow).

To show that A_t is not Poisson, one can argue as follows.

Assume that t is a jump time of A_t. Then $x_t > 0$, and therefore the conditional rate of A_t is $\gamma + \mu p$ which is strictly larger than λ, the unconditional rate. Hence, having observed a jump of A modifies the law of its future. This shows that A does not have independent increments, and therefore it cannot be Poisson.

Unfortunately this simple argument does not extend directly to a general network. To see this, try to replace the M/M/1 queue of Fig. 3 by two M/M/1 queues *in tandem*.

The central idea used in the proof above is that a jump of A_t tells us exactly that one more customer is present in the queue into which A_t enters. This observation, made precise by Proposition 1, will extend to a general network of quasi-reversible nodes.

B. The Result

Consider once again a network such as in Section III. Assume that it is open, and that all the customers are of the same class. Thus, a customer entering the network via node l ($l = 1,\cdots,J$) will cross a given link (i, j) some finite average number of times m_l. Let A_t be the number of customers who have crossed (i, j) in $[0, t]$. Given that τ is a jump time of A_t, we know that the other customers are in equilibrium (see Proposition 1) and therefore that the conditional law of x_τ is obtained by adding one customer in node j to the equilibrium distribution π. Denote by $\tilde{\pi}$ the resulting distribution.

Assume that for $l = 1,\cdots,J$

$$E_{\tilde{\pi}}|x_t^l| \to E_\pi|x_t^l| = E_\pi|x_0^l| < \infty \qquad \text{as } t \to \infty, \quad (5.1)$$

where $|x_t^l|$ is the number of customers in node l when its state is x_t^l, and where $E_\mu(\cdot)$ denotes expectation with respect to the law of $(x_t, t \geqslant 0)$ given that x_0 has distribution μ.

The technical condition (5.1) is shown in [16] to be satisfied for the usual quasi-reversible nodes with bounded service rates.

Proposition 3: In equilibrium, A_t is Poisson if and only if (i, j) is not part of a loop.

Proof: If (i, j) is not part of a loop, then A_t can be seen to be the departure process from a subnet of the original network. It follows then from Section III that it must be Poisson in equilibrium.

Assume now that A_t is Poisson, so that it has independent increments. Then, having observed a jump at $t = 0$ should not say anything about the future of A_t. Hence, starting the network with the distributions π or $\tilde{\pi}$ should lead to the same law of $(A_t, t \geqslant 0)$. In particular, it must be true that for all $t > 0$

$$E_\pi A_t = E_{\tilde{\pi}} A_t.$$

Now, if one starts the network with $\tilde{\pi}$, then one knows that an extra customer is present in node j at $t = 0$. Therefore, if there is a positive probability for that customer to go from j to i, i.e., if (i, j) is part of a loop, then he should contribute to A_t, and this leads one to suspect that

$$E_\pi A_t < E_{\tilde{\pi}} A_t, \qquad (5.2)$$

so that the process A_t will not be Poisson.

To see that (5.2) indeed holds for some $t > 0$, assume that all the exogeneous arrival processes are stopped after time t, and denote by $\{A'_s, s \geqslant 0\}$ the resulting flow on (i, j) (thus, $A_s = A'_s$ for all $s \leqslant t$). Then one must have for this modified network

$$E_\mu A'_\infty = \sum_{l=1}^{J} m_l\left(E_\mu|x_0^l| + \gamma^l t\right),$$

since $E_\mu|x_0^l| + \gamma^l t$ is the average number of customers who started their stay in the network in node l in the interval $[0, t]$. (γ^l is the rate of the exogeneous arrival process in node l.)

This shows that

$$E_{\tilde{\pi}} A'_\infty = E_\pi A'_\infty + m_j, \qquad (5.3)$$

since there is one more customer in node j at time $t = 0$ under $\tilde{\pi}$ than under π.

On the other hand,

$$E_\mu\left[A'_\infty - A_t | x_t = x\right] = \sum_{l=1}^{J} m_l |x^l|,$$

so that

$$E_\mu\left[A'_\infty - A_t\right] = \sum_{l=1}^{J} m_l E_\mu |x_t^l|.$$

This shows that (5.1) implies that $E_\pi[A'_\infty - A_t]$ and $E_{\tilde{\pi}}[A'_\infty - A_t]$ can be made arbitrarily close by choosing t large enough. Hence (5.3) will imply (5.2) whenever $m_j > 0$, i.e., whenever (i, j) is part of a loop. Q.E.D.

Conclusion

Networks of quasi-reversible nodes have nice properties that can be explained quite simply. In this paper we have proposed an explanation of the product form results by a *forward time* probabilistic argument. This argument shows why the product form and the associated conditioning property follow from the quasi-reversibility. The results on the distribution at the jumps were also give a simple justification and were used to obtain the characterization of Poisson flows. This approach complements the reversibility arguments of Kelly ([4], [13]) and the more algebraic derivations of other authors (e.g., [11], [15], [17], [18]).

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References

[1] A. J. Miller, "A queueing model for road traffic flow," *J. Roy. Stat. Soc.*, Series, B, vol. 23, pp. 64–90, 1961.
[2] L. Kleinrock, *Queueing Systems, Vol. 2.* New York: Wiley-Interscience, 1976.
[3] A. J. Lemoine, "Networks of queues: A survey of equilibrium analysis," *Management Sci.*, vol. 24, pp. 464–481, Dec. 1977.
[4] F. P. Kelly, *Reversibility and Stochastic Networks.* New York: Wiley 1979.
[5] P. J. Burke, "The output of a queueing system," *Oper. Res.*, vol. 4, pp. 699–704, Dec. 1956.
[6] E. Reich, "Waiting times when queues are *in tandem*," *Ann. Math. Stat.*, vol. 28, pp. 527–530, 1957.
[7] J. R. Jackson, "Networks of waiting lines," *Oper. Res.*, vol. 5, pp. 518–521, 1957.
[8] M. Schwartz, *Computer Communication Network Design and Analysis.* Englewood Cliffs, NJ: Prentice-Hall 1977.
[9] W. L. Gordon and G. F. Newell, "Closed queueing systems with exponential servers," *Oper. Res.*, vol. 15, pp. 254–265, 1967.
[10] R. R. Muntz, "Poisson departure processes and queueing networks," *IBM Res. Rep. RC-4145*, IBM T. J. Watson Research Center, Yorktown Heights, NY, Dec. 1972.
[11] F. Baskett, K. M. Chandy, R. R. Muntz, and F. G. Palacios, "Open, closed and mixed networks of queues with different classes of customers," *J. Ass. Comput. Mach.*, vol. 22, pp. 248–260, 1975.
[12] J. Walrand and P. Varaiya, "Interconnections of Markov chains and quasi-reversible queuing networks," *Stoch. Proc. and Appl.*, vol. 10, pp. 209–219, 1980.
[13] F. P. Kelly, "On networks of quasi-reversible nodes," in *Applied Probability-Computer Science: The Interface*, R. L. Disney and T. J. Ott, Eds. Birkhäuser, 1982.
[14] P. J. Burke, "Proof of a conjecture on the interarrival time distribution in an M/M/1 queue with feedback," *IEEE Trans. Comm.*, vol. COM-24, pp. 575–576, May 1976.
[15] B. Melamed, "Characterization of Poisson traffic streams in Jackson queueing networks," *Adv. Appl. Prob.*, vol. 11, pp. 422–438, 1979.
[16] J. Walrand, "Poisson flows in single class open networks of quasi-reversible queues," *Stoch. Proc. and Appl.*, vol. 13, pp. 293–303, 1982.
[17] K. C. Sevcik and I. Mitrani, "The distribution of queueing network states at input and output instants," *J. Ass. Comput. Mach.*, vol. 28, 1981.
[18] B. Melamed, "On Markov jump processes imbedded at jump epochs and their queueing-theoretic applications," *Math. Oper. Res.*, vol. 7, pp. 111–128, 1983.
[19] L. Breiman, *Probability.* New York: Addison Wesley, 1968.
[20] J. L. Doob, *Stochastic Processes.* New York: Wiley, 1953.
[21] J. Walrand and P. Varaiya, "Flows in queueing networks: A martingale approach," *Math. Oper. Res.*, vol. 6, pp. 387–404, 1981.
[22] J. Kemeny, J. Snell, and A. Knapp, *Denumerable Markov Chains.* Princeton, NJ: Van Nostrand, 1966.
[23] R. M. Blumenthal and R. K. Getoor, *Markov Processes and Potential Theory.* New York: Academic, 1968.
[24] J. Walrand, "A note on Norton's theorem for queueing networks," to appear in *J. Appl. Prob.*, 1983.

Partitioned Linear Block Codes for Computer Memory with "Stuck-at" Defects

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***Abstract*—Linear block codes are studied for improving the reliability of message storage in computer memory with stuck-at defects and noise. The case when the side information about the state of the defects is available to the decoder or to the encoder is considered. In the former case, stuck-at cells act as erasures so that techniques for decoding linear block codes for erasures and errors can be directly applied. We concentrate on the complimentary problem of incorporating stuck-at information in the encoding of linear block codes. An algebraic model for stuck-at defects and additive errors is presented. The notion of a "partitioned" linear block code is introduced to mask defects known at the encoder and to correct random errors at the decoder. The defect and error correction capability of partitioned linear block codes is characterized in terms of minimum distances. A class of partitioned cyclic codes is introduced. A BCH-type bound for these cyclic codes is derived and employed to construct partitioned linear block codes with specified bounds on the minimum distances. Finally, a probabilistic model for the generation of stuck-at cells is presented. It is shown that partitioned linear block codes achieve the Shannon capacity for a computer memory with symmetric defects and errors.**

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I. Introduction

LINEAR block codes can be used to improve the reliability of message storage in an imperfect computer memory. We consider a memory that is composed of n cells. Each cell is expected to store one of q symbols. We

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