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Retrial Queue With Lattice Distribution of Inter-arrival Times and Constant Retrial Rate

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***Abstract*—In this paper, we extend the result obtained in [6] for retrial queueing system with recurrent arrival flow to the case of periodic arrival process. We consider a single server retrial queueing system with lattice distribution of inter-arrival times, constant retrial rate and exponential service time distribution. For this queue, we derive the stationary distributions of the system states at arrival times and at an arbitrary times and the Laplace-Stieltjes transform of the customer sojourn time distribution. Little's formula for this system is proved. Results can be used for performance evaluation and capacity planning of telecommunication networks where effect of repeated calls is essential**

Keywords–Retrial queue; renewal input; lattice distribution; constant retrial rate; stationary distribution

I. Introduction

Retrial queueing systems differ in the fact that a customer, who does not get service immediately upon arrival, neither enters a finite of infinite buffer nor leaves the system forever. He/she goes to the so-called orbit (the virtual place for repeated calls) from which he/she makes attempts to reach the service facility in a random amount of time. Retrial queues are of great interest among researches in the field of queueing theory and telecommunication. So they are intensively studied in the literature, for references see, e.g., surveys in [1], [3] - [5]. All research papers cited in these surveys assume that the arrival flow of customers is described by the stationary Poisson process or its various Markovian extensions (PH – renewal arrival process with phase type distribution of inter-arrival times, MAP – Markovian Arrival Process, $BMAP$ – Batch Markovian Arrival Process, etc).

We can mention only three papers devoted to the analysis of retrial queues that are free of an assumption about the Markovian nature of the input flow. In [6], a retrial queueing system with renewal arrival process is investigated. Analysis of the system is based on the matrix technique but the final results were obtained in a simple scalar form. The motivation for the consideration of the system, besides of importance of the model itself, was the unsuccessful study of this system in the early paper [7], for more details see [6].

Thus, in [6], the retrial $GI/M/1$ queue with constant retrial rate was investigated. Note that, under classic retrial strategy, the orbital customers make their attempts to reach server independently of each others. In [6], it is assumed that there is a queue in the orbit. An arriving unsuccessful customer is placed at the end of the queue. A repeated attempt can be done only by the customer which stands at the head of the queue. In the recent paper [2], essentially more general than one in [6], $GI/GI/c/K$ type, retrial queue is considered. But the authors restrict their task only to analysis of stability condition for this system. Important problem of computation of stationary distribution is not considered there.

The flaw in results of [6] is that some of them, in particular, the results for the stationary distribution of the system states at an arbitrary times and the Laplace-Stieltjes transform of the sojourn time distribution were derived under an assumption of non-lattice distribution of inter-arrival times. This assumption essentially simplifies derivation of results because it allows to use a simple known expression for distribution of the length of time interval between the arbitrary time moment and the previous (or the next) arrival moment. In some practically important cases, this assumption reduces the range of applicability of the results. In this paper, we extend the results of [6] to the case of non-lattice distribution of inter-arrival times. Note, that fortunately some results coincide with the corresponding results for more simple case of non-lattice distribution. For reader's convenience, we present below these results briefly without proof.

II. Model description

We consider $GI/M/1$ retrial queue with constant retrial rate. The inter-arrival times are independent random variables with lattice distribution $A(t)$, where $A(t)$ is a step function (continuous at the right) whose jump points belong to the set $n\Delta$, $n \geq 0$.

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Let

$$A^*(s) = \int_0^\infty e^{-st} dA(t),\ Re\ s \geq 0,$$

denote the Laplace-Stieltjes transform of the distribution function $A(t)$,

$$a = \int_0^\infty t dA(t)$$

denote the first moment and $\lambda = a^{-1}$ denote the arrival rate. It is assumed that $a < \infty$.

If an arriving (primary) customer meets server being idle, it starts the service immediately. In the opposite case, it goes to the orbit and stands at the end of the queue of the so called repeated customers. The first customer in the queue makes repeated attempts to get service in a random amount of time until it succeeds to enter the service. The intervals between two successive retrials are independent exponentially distributed random variables with parameter γ. The orbit capacity is assumed to be unlimited.

The service times of primary customers and customers from the orbit are independent exponentially distributed random variables with parameter μ.

III. Process of the system states. Stationary distribution of embedded Markov chain

We define the system state at time t as (0) if the system is empty at this time. In the opposite case, the state of the system is defined as pair (i, m) where i is the number of customers in the system (in the orbit and in service, if any), m is equal to 0 if the server is idle and is equal to 1 if the server is busy at time t. Note that the server can be idle because the system is empty or the system is not empty but a customer in the input flow did not arrive and a retrial from the orbit did not occur.

The process of the system states at an arbitrary time is described by the process ζ_t, $t \geq 0$, with the state space

$$X = \{(0);\ \ (i, m),\ i \geq 1,\ m = 0, 1\}.$$

In the case under consideration, the process ζ_t is non-Markovian because the input flow does not possess the Markov property. We intend to study the stationary distribution of this process through the investigation of the embedded Markov chain.

Let t_n denote the instant of the nth arrival. It is readily seen that the process

$$\xi_n^- = \zeta_{t_n - 0},\ n \geq 1,$$

with the state space X is irreducible and aperiodic Markov chain embedded in the process ζ_t over the moments $t_n - 0$, $n \geq 1$.

Hereinafter we use notions $t_n - 0$ and $t_n + 0$ for the moment just before and just after the nth arrival, respectively.

To calculate the transition probabilities of the chain ξ_n^-, $n \geq 1$, we follow the arguments from [6] introducing the notion of generalized service time of a customer from the orbit. The generalized service time consists of two phases: the exponentially distributed with parameter γ time interval, during which the customer reaches the idle server, and a proper service time of this customer by the server.

Then the generalized service time distribution of the orbital customer can be described as phase-type (PH) distribution with the irreducible representation $(\boldsymbol{\beta}, S)$ where $\boldsymbol{\beta} = (1, 0)$ and

$$S = \begin{pmatrix} -\gamma & \gamma \\ 0 & -\mu \end{pmatrix}.$$

Here, the matrix S describes the transition rates of the phases of generalized service that do not lead to service completion. The vector $\boldsymbol{S}_0 = -S\mathbf{e}$, where $\mathbf{e} = (1, 1)^T$, describes the rates of transitions leading to service completion. The vector $\boldsymbol{\beta}$ indicates that the service of the repeated customer always starts from the first phase. For further use we introduce the notation

$$G(z) = S + \boldsymbol{S}_0 \boldsymbol{\beta} z,\ |z| \leq 1.$$

The service time of a primary customer, which found the idle server, can be considered as the second phase of the generalized service time. It is evident that service time of such a customer has PH distribution with the irreducible representation $(\hat{\mathbf{e}}, S)$ where $\hat{\mathbf{e}} = (0, 1)$.

Below in this section, we present theorems whose statements and proofs do not differ from the ones given in [6] for non-lattice distribution $A(t)$.

Let $P(k, t)$, $k \geq 0$, be the matrix entries of which define probability that k renewals occur in the renewal process defined by the above PH distribution and the underlying process of PH makes the corresponding transitions during the time interval of length t. It is well-known that the matrices $P(k, t)$, $k \geq 0$, are defined by the matrix expansion

$$\sum_{k=0}^{\infty} P(k, t) z^k = e^{G(z)t},\ |z| \leq 1. \tag{1}$$

Denote by $P_{i,j}$ transition probability matrix of the two-dimensional Markov chain ξ_n^-, $n \geq 1$, from the states corresponding to the number i of customers in the system to the states corresponding to the number j of customers, $i, j \geq 0$.

Theorem 1. *The transition probability matrix of the chain* ξ_n^-, $n \geq 1$, *has the following block structure*

$$P^- = (P_{i,j})_{i,j \geq 0} = \begin{pmatrix} B_0 & A_0 & 0 & 0 & \dots \\ B_1 & A_1 & A_0 & 0 & \dots \\ B_2 & A_2 & A_1 & A_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where the matrices $A_k,\ B_k,\ k \geq 0$, are defined by formulas

$$A_k = \mathbf{e}\hat{\mathbf{e}} \int\limits_0^\infty P(k,t)dA(t),$$

$$B_k = \mathbf{e}\hat{\mathbf{e}} \sum_{l=k+1}^\infty \int\limits_0^\infty P(l,t)dA(t)\mathbf{e}.$$

Corollary 1. *The process $\xi_n^-, n \geq 1$, is a multi-dimensional Markov chain of the $GI/M/1$ type, see [8].*

Theorem 2. *Stationary distribution of the Markov chain $\xi_n^-,\ n \geq 1$, exists if and only if the inequality*

$$\frac{\mu}{\mu+\gamma}\left\{\frac{\mu}{\mu+\gamma}\left[1 - A^*(\mu+\gamma)\right] + a\gamma\right\} > 1 \qquad (2)$$

is fulfilled.

Denote the stationary state probabilities of the chain ξ_n^- as

$$\pi_0 = \lim_{n\to\infty} P\{\xi_n^- = 0\},$$

$$\pi(i,m) = \lim_{n\to\infty} P\{\xi_n^- = (i,m)\},\ i \geq 1,\ m = 0,1,$$

and combine probabilities corresponding to the values i of the first component of the Markov chain, $i \geq 1$, into row vectors $\boldsymbol{\pi}_i = (\pi(i,0), \pi(i,1))$.

Theorem 3. *The stationary distribution of the Markov chain $\xi_n^-,\ n \geq 1$, has the following form:*

$$\pi_0 = 1 - \sigma, \qquad (3)$$

$$\boldsymbol{\pi}_i = (1-\sigma)\sigma^{i-1}(\sigma_1,\, \sigma_2),\ i \geq 1, \qquad (4)$$

where the row vector (σ_1, σ_2) is found as solution to equation

$$(\sigma_1, \sigma_2) = \hat{\mathbf{e}} \int\limits_0^\infty e^{G(\sigma)t} dA(t)), \qquad (5)$$

and $\sigma = \sigma_1 + \sigma_2$.

IV. Stationary distribution at an arbitrary time

In this section, we calculate the stationary distribution of the non-Markovian process $\zeta_t,\ t \geq 0$, of the system states based on the stationary distribution of the embedded Markov chain in assumption that the arrival process is periodic, i.e., function $A(t)$ is lattice.

Let

$$p_0(x) = \lim_{n\to\infty} P\{\zeta_{n\Delta+x} = (0)\},$$

$$\mathbf{p}_i(x) = (p(i,0,x), p(i,1,x)), \qquad (6)$$

$$i > 0,\ 0 \leq x < \Delta,$$

where

$$p(i,m,x) = \lim_{n\to\infty} P\{\zeta_{n\Delta+x} = (i,m)\},\ i \geq 1, m = 0,1.$$

Denote $\hat{A}(n\Delta)$ the value of the function $A(t)$ jump at the point $n\Delta,\ n \geq 0$.

Limits (6) exist if the embedded Markov chain $\xi_n,\ n \geq 1$, is ergodic and the mean inter-arrival time a is finite. We assume that ergodicity condition (2) for the chain ξ_n holds good and $a < \infty$. This guarantees existence of the stationary distribution of the process $\zeta_t,\ t \geq 0$.

Theorem 4. *The steady state probabilities of the process $\zeta_t,\ t \geq 0$, are calculated as*

$$p_0(x) = 1 - \mathbf{c}(x)\mathbf{e}, \qquad (7)$$

$$\mathbf{p}_i(x) = \sigma^{i-1}(1-\sigma)\mathbf{c}(x),\ i \geq 1. \qquad (8)$$

where

$$\mathbf{c}(x) = \frac{\Delta}{a}(-\sigma_1,\, 1-\sigma_2)(I - e^{G(\sigma)\Delta})^{-1}\, e^{G(\sigma)x},$$

σ_1 and σ_2 are defined by the equation

$$(\sigma_1, \sigma_2) = \hat{\mathbf{e}} \sum_{k=0}^\infty e^{G(\sigma)k\Delta} \hat{A}(k\Delta) \qquad (9)$$

and $\quad \sigma = \sigma_1 + \sigma_2.$

Proof. The vectors $\mathbf{p}_i(x),\ i \geq 1$, are related to the stationary distribution of the chain $\xi_n^-,\ n \geq 1$, as follows:

$$\mathbf{p}_i(x) = \qquad (10)$$

$$\frac{\Delta}{a} \sum_{j=i-1}^\infty \boldsymbol{\pi}_j \mathbf{e}\hat{\mathbf{e}} \sum_{n=0}^\infty P(j-i+1, n\Delta+x)\times$$

$$(1 - \sum_{k=0}^n \hat{A}(k\Delta)).$$

Substituting into (10) expressions (3)-(4) for the vectors $\boldsymbol{\pi}_j$ and transforming the obtained equation, we get relations

$$\mathbf{p}_i(x) = \qquad (11)$$

$$\frac{\Delta}{a} \sum_{j=i-1}^\infty \boldsymbol{\pi}_j \mathbf{e}\hat{\mathbf{e}} \sum_{n=0}^\infty P(j-i+1, n\Delta+x)\times$$

$$(1 - \sum_{k=0}^n \hat{A}(k\Delta)) =$$

$$\frac{\Delta}{a}(1-\sigma)(\sigma_1,\sigma_2)\mathbf{e}\hat{\mathbf{e}} \sum_{j=i-1}^\infty \sigma^{j-1} \sum_{n=0}^\infty P(j-i+1, n\Delta+x)\times$$

$$(1 - \sum_{k=0}^n \hat{A}(k\Delta))$$

$$= \frac{\Delta}{a}(1-\sigma)\sigma^{i-1}\hat{\mathbf{e}}\sum_{n=0}^{\infty}\sum_{j=i-1}^{\infty} P(j-i+1, n\Delta+x)\sigma^{j-i+1}\times$$

$$(1-\sum_{k=0}^{n}\hat{A}(k\Delta)) =$$

$$\frac{\Delta}{a}(1-\sigma)\sigma^{i-1}\hat{\mathbf{e}}\sum_{n=0}^{\infty}(e^{G(\sigma)\Delta})^n\times$$

$$(1-\sum_{k=0}^{n}\hat{A}(k\Delta))\, e^{G(\sigma)x}.$$

Note that in the transformation of the next to last row to the last row in (11) we have used equation (1).

Now, transform the double sum in (11) as follows:

$$\sum_{n=0}^{\infty}(e^{G(\sigma)\Delta})^n(1-\sum_{k=0}^{n}\hat{A}(k\Delta))$$

$$= \sum_{n=0}^{\infty}(e^{G(\sigma)\Delta})^n - \sum_{n=1}^{\infty}(e^{G(\sigma)\Delta})^n\sum_{k=1}^{n}\hat{A}(k\Delta)$$

$$= (I-e^{G(\sigma)\Delta})^{-1} - \sum_{k=1}^{\infty}\hat{A}(k\Delta)\sum_{n=k}^{\infty}(e^{G(\sigma)\Delta})^n$$

$$= (I-e^{G(\sigma)\Delta})^{-1} - \sum_{k=1}^{\infty}\hat{A}(k\Delta)(e^{G(\sigma)\Delta})^k$$

$$\times\sum_{n=0}^{\infty}(e^{G(\sigma)\Delta})^n$$

$$= (I-\sum_{k=1}^{\infty}e^{G(\sigma)k\Delta}\hat{A}(k\Delta))$$

$$\times(I-e^{G(\sigma)\Delta})^{-1}. \qquad (12)$$

Using (12) in (11), we get

$$\mathbf{p}_i(x) = \qquad (13)$$

$$\frac{\Delta}{a}(1-\sigma)\sigma^{i-1}\hat{\mathbf{e}}(I-\sum_{k=1}^{\infty}e^{G(\sigma)k\Delta}\hat{A}(k\Delta))$$

$$\times(I-e^{G(\sigma)\Delta})^{-1}.$$

In periodic case under consideration, equation (5) has the following form:

$$(\sigma_1,\sigma_2) = \hat{\mathbf{e}}\sum_{k=0}^{\infty}e^{G(\sigma)k\Delta}\hat{A}(k\Delta). \qquad (14)$$

Substituting (14) into (13), we get the final expression (8) for $\mathbf{p}_i(x)$, $i\geq 1$. Expression (7) for $p_0(x)$ is found using the normalization condition

$$p_0(x) + \sum_{i=1}^{\infty}\mathbf{p}_i(x)\mathbf{e} = 1.$$

Corollary 2. *The stationary distribution of the process ζ_t, $t\geq 0$, at an arbitrary time is calculated as follows:*

$$p_0 = 1-\lambda\Big(\frac{\sigma_2}{\gamma}+\frac{1}{\mu}\Big),$$

$$\boldsymbol{p}_i = \lambda(1-\sigma)\sigma^{i-1}\Big(\frac{\sigma_2}{\gamma},\,\frac{1}{\mu}\Big),\ i\geq 1. \qquad (15)$$

Proof. The steady state probability vectors $\boldsymbol{p}_i$, $i>0$, are calculated by formula

$$\boldsymbol{p}_i = \frac{\int_0^{\Delta}\boldsymbol{p}_i(x)dx}{\Delta},\ i>0. \qquad (16)$$

Substituting in (16) expressions for $\boldsymbol{p}_i(x)$ from (8) and performing some transformations, we get relation

$$\boldsymbol{p}_i = \frac{1}{\Delta}\int_0^{\Delta}\boldsymbol{p}_i(x)dx =$$

$$\sigma^{i-1}(1-\sigma)\frac{1}{a}(-\sigma_1,\,1-\sigma_2)(I-e^{G(\sigma)\Delta})^{-1}$$

$$\times\int_0^{\Delta}e^{G(\sigma)x}dx$$

$$= \sigma^{i-1}(1-\sigma)\frac{1}{a}(-\sigma_1,\,1-\sigma_2)(I-e^{G(\sigma)\Delta})^{-1}$$

$$\times(-G(\sigma))^{-1}(I-e^{G(\sigma)\Delta}). \qquad (17)$$

It can be verified that

$$(-G(\sigma))^{-1} = \frac{1}{1-\sigma}\begin{pmatrix}\gamma^{-1} & \mu^{-1}\\ \gamma^{-1}\sigma & \mu^{-1}\end{pmatrix},$$

and

$$(-\sigma_1,\,1-\sigma_2)(-G(\sigma))^{-1} = (\gamma^{-1}\sigma,\mu^{-1}). \qquad (18)$$

Substituting (18) into (17), we get (15). Expression for the probability p_0 follows from the normalization condition.

Corollary 3. *Mean number of customers in the system at an arbitrary time is given by*

$$L_{arb} = \frac{\lambda}{1-\sigma}\Big(\frac{\sigma_2}{\gamma}+\frac{1}{\mu}\Big). \qquad (19)$$

V. Laplace-Stieltjes transform of the stationary distribution of the sojourn time

Let $V(t)$ be the stationary distribution function of the sojourn time of an arbitrary customer in the system,

$$v(u) = \int\limits_0^\infty e^{-ut} dV(t), Re\ u \geq 0.$$

Theorem 5. *The Laplace-Stieltjes transform $v(u)$ of the stationary distribution of the sojourn time of an arbitrary customer in the system is calculated as*

$$v(u) = (1-\sigma_2)\frac{\mu}{u+\mu} + (1-\sigma)\frac{\sigma_2}{\sigma}\Phi(\sigma, u) \qquad (20)$$

where the function $\Phi(\sigma, u)$ is given by the following expression:

$$\Phi(\sigma, u) = \sigma\mu \bigg\{ \hat{\mathbf{e}} \Big[I - A^*(uI - G(\sigma) \Big] (1, \sigma)^T \bigg\}^{-1}$$

$$\times \hat{\mathbf{e}} \Big[I - A^*(uI - G(\sigma) \Big]$$

$$\times \Big[(uI - G(\sigma))^{-1} - (u+\mu)^{-1} I \Big] \hat{\mathbf{e}}^T.$$

The proof of the theorem is analogous to the proof of theorem 7 in [6] and we omit it here.

Corollary 4. *The mean value $\bar{v}$ of the sojourn time of an arbitrary customers in the system is calculated as follows:*

$$\bar{v} = \frac{1}{1-\sigma}\left(\frac{\sigma_2}{\gamma} + \frac{1}{\mu}\right). \qquad (21)$$

Proof. We calculate the mean $\bar{v}$ using (20) and well known formula

$$\bar{v} = -v'(0)$$

and applying L'Hospital's rule.

Corollary 5. *For the queue under consideration, Little's formula*

$$\bar{v} = \lambda^{-1} L_{arb}$$

is valid.

Proof follows immediately from comparison of formulas (19) and (21).

VI. Conclusion

In this paper, we investigated the queueing system $GI/M/1$ with constant retrial rate. This research extended the results of [6] to the case of lattice distribution of inter-arrival times. The importance of addressing this problem is due to the fact that class of lattice distributions includes such practically significant distributions as the deterministic one and many distributions that are described by step probability mass functions. We present the condition for existence of the steady state of the system, the stationary distribution of the system states at arrival epochs. We derive the stationary distribution of the system states at an arbitrary moment x belonging the interval of constancy of probability mass functions and at an arbitrary time. We also present the Laplace-Stieltjes transform of the sojourn time in the system and prove the Little's formula.

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References

[1] Artalejo, J. R. and Gomez-Corral, A.: Retrial queueing systems: A computational approach. Berlin-Heidelberg, Springer (2008)

[2] Avrachenkov, K. and Morozov, E.: Stability analysis of $GI/GI/c/K$ retrial queue with constant retrial rate. Mathematical Methods of Operations Research 79, 1– 19 (2014)

[3] Dudin, A. and Klimenok, V.: A Retrial $BMAP/PH/N$ Queueing System with Markov Modulated Retrials. Proceedings of the 2nd Baltic Congress on Future Internet Communications. 2012. No. BCFIC. 1569570919. 1- 6 (2012)

[4] Falin, G.I. and Templeton, J.G.C.: Retrial Queues, Chapman & Hall, London (1997)

[5] Gomez-Corral, A.: A bibliographical guide to the analysis of retrial queues through matrix analytic techniques. Annals of Operations Research 141, 163–191 (2006)

[6] Kim, C.S., Klimenok, V. and Dudin, A.: A $G/M/1$ retrial queue with constant retrial rate. TOP 22, 509-529 (2014)

[7] Lillo, R.E.: A $G/M/1$-queue with exponential retrials. TOP 4, 99-120 (1996)

[8] Neuts, M.F. Matrix-Geometric Solutions in Stochastic Models – An Algorithmic Approach. Johns Hopkins University Press, Baltimore. (1981)