Notes on Jackson-like queue networks

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1 Notation

Graph $G = (V, E)$, adjacency A, $n \times n$, in number of vertices. For vertices $i, j \in V$, we say i is connected to j, $i \sim j$ if the ordered pair $(i, j) \in E$. Directionality is implied by the ordering.

2 Jackson Networks

A Jackson Network is a network of queues where exogenous jobs arrive according to a Poisson process (exponential interarrival) with rate $\lambda > 0$. For a job completed at vertex i, jobs are routed to vertex j with some probability. For vertices $j \sim i$, jobs move onto another vertex probability $\sum_j p_{ij}$ or leave the network with probability $1 - \sum_j p_{ij}$.

The overall arrival rate at a vertex i, λ_i becomes:

$$\lambda_i = \lambda p_{0i} + \sum_{j=1} \lambda_i p_{ji}$$

for $(j, i) \in E$, and p_{0i} equal to the probability that an exogenous job arrives at i first. Additionally, p_{i0} is equal to the probability that a job item has left the network. The 0'th vertex acts as both source and sink–this is referred to as an *open* Jackson network.

3 Parking Garage Problem

Consider an $M/M/1/1$ queue with Poisson arrival λ and service rate μ (representative of a blockface with 1 parking space). If the single server is busy, any new arrival is automatically rejected (and elect to park in a parking garage). We can observe this is a birth-death process with two states for the queue: busy or idle. Denote these states as either X_0 or X_1.

We wish to obtain the transition probabilities for this continuous time Markov Chain for which all states communicate. Using the Kolmogorov Backwards Equations, we obtain the following system of differential equations:

$$p'_{00} = \lambda p_{10}(t) - \lambda p_{00}(t)] \tag{1}$$

$$p'_{10} = \mu p_{00}(t) - \mu p_{10}(t) \tag{2}$$

Multiplying (1) by μ, (2) by λ, adding, and integrating the resulting equation we obtain:

$$\mu p_{00}(t) + \lambda p_{10}(t) = c \tag{3}$$

To solve for the constant of integration, we assume the initial condition that $p_{00}(0) = 1$, thus $p_{10}(0) = 0$, and $c = \mu$. Substituting (3) into (1):

$$p'_{00}(t) = \mu - (\mu + \lambda)p_0 0(t) \tag{4}$$

Let $h(t) = p_0 0(t) - \frac{\mu}{\mu+\lambda}$. Then $h'(t) = p'_{00}(t) = -(\lambda + \mu)h(t)$

$$\frac{h'(t)}{h(t)} = -(\mu + \lambda) \tag{5}$$

$$\int \frac{h'(t)}{h(t)} = \int -(\mu + \lambda)dt \tag{6}$$

$$log\ h(t) = -(\mu + \lambda)t + c \tag{7}$$

$$h(t) = ke^{-(\mu+\lambda)t} \tag{8}$$

$$p_{00}(t) = ke^{-(\mu+\lambda)t} + \frac{\mu}{\lambda + \mu} \tag{9}$$

$$\tag{10}$$

Using $p_{00}(0) = 1$, $k = \frac{\lambda}{\mu+\lambda}$, therefore

$$p_{00}(t) = \frac{\lambda}{\mu + \lambda} e^{-(\mu+\lambda)t} + \frac{\mu}{\mu + \lambda} \tag{11}$$

$$p_{01}(t) = (1 - p_{00}(t)) \tag{12}$$

What is the distribution of P_{01} and P_{10}? Is the expectation of the convolution to the expected transit time for vehicle?

4 Recovering Steady State

Burke's theorem states that for such an M/M/1 queue in the steady state with arrivals a Poisson process with rate λ, the departure process is also a Poisson process with rate λ. To evaluate the validity of this claim for a finite state birth-death process, we must determine the limiting probabilities from the above section. The existence of the limiting probabilities requires the assumptions

a all states of the Markov chain communicate

b the expected time to return to any state is finite

(a) is already true by assumption, and (b) is true for finite service rates and all reasonable distributions of P_{01} and P_{10}.

$$P_j := \lim_{t\to\infty} p_{ij}(t) \tag{13}$$

Consider the Kolmogorov Forward Equations in the limit:

$$p'_{ij}(t) = \sum_{k\neq j} q_{kj} p_{ik}(t) - v_j p_{ij}(t)$$

$$\lim_{t\to\infty} p'_{ij}(t) = \lim_{t\to\infty} \sum_{k\neq j} q_{kj} p_{ik}(t) - v_j p_{ij}(t)$$

If (7) converges, then $\lim_{t\to\infty} p'_{ij}(t)$ must converge to 0, so:

$$0 = \sum_{k\neq j} q_{kj} P_k - v_j P_j$$

Up to this point no assumptions are made about the nature of the distributions of the p_{ij}'s

For any birth-death process with exit rates λ_i, enter rates μ_i, assuming the limiting probabilities exist, solving for the sum of the balance equations in terms of P_0

$$1 = P_0 + P_0 \sum_{i=0}^{n} \frac{\lambda_{i-1} \dots \lambda_0}{\mu_i \dots \mu_0} \tag{14}$$

The necessary and sufficient condition then for the existence of the limiting probabilities is that the sum in (8) is bounded. For finite states n this is true for reasonable values of $\lambda < \infty$ and $\mu > 0$. For infinite states, $\lambda < \mu$ is also required. In our simple, 2-state case, we have that:

$$P_0 = \frac{1}{1 + \lambda/\mu} \tag{15}$$

$$P_1 = \frac{\lambda/\mu}{1 + \lambda/\mu} \tag{16}$$

In the limiting case, we can recover the number of vehicles then served, and therefore, the number of vehicles rejected and sent to a parking garage.

5 Rejection

We're ultimately concerned with the expected time between cars being sent to a parking garage. As $\lambda >> \mu$, $P_1 \to 1$, w.r.t. the aim of determining traffic flow between queues based on rejection.

Let the inter arrival times $_i$ be exponentially distributed with parameter λ, $S(t)$ be the corresponding Poisson arrival process, and let $E[R] :=$ *expected time* *between rejections*. Because arrivals are independent of the state of queue, we can observe that $E[R|queuefull] = 1/\lambda$.

$$
\begin{aligned}
E[R] &= \int_0^\infty E[R|S(t_0) = 1]S(t)dt && (17) \\
&= \int_0^\infty E[R|S(t_0) = 1, X(t_0) = X_0]P_0(t)S(t)dt + \int_0^\infty E[R|S(t_0) = 1, X(t_0) = X_1]P_1(t)S(t)dt && (18) \\
&= (\frac{1}{\lambda} + E[R])P_0 + \frac{1}{\lambda}P_1 && (19) \\
&= \frac{1}{\lambda}P_0 + \frac{1}{\lambda}P_1 + P_0E[R] && (20) \\
E[R](1 - P_0) &= \frac{1}{\lambda}P_0 + \frac{1}{\lambda}(1 - P_0) && (21) \\
E[R] &= \frac{1}{\lambda}\frac{1}{1 - P_0} && (22) \\
&= \frac{\mu + \lambda}{\lambda^2} && (23)
\end{aligned}
$$