Two-state neutrino oscillations in the density varying environment of the solar system.

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(Dated: May 1, 2020)

I. INTRODUCTION

While sudden, sometimes spontaneous transformations of one type of particle into another are completely commonplace in the particle physics, it is usually expected that the truly fundamental Standard Model particles (e.g. up quark, electron, etc) will stay as they are. However, in the 1960s experimental measurements of incident solar neutrinos published results that detected far fewer electron neutrinos than theory predicted should be arriving from the Sun [1]. Motivated by this, physicists have proposed that the "flavor" states of the neutrino, ν_e, ν_μ, and ν_τ, are linear combinations of the true energy eigenstates of the neutrino. This is a straightforward quantum mechanical phenomenon, but leads to the surprising result that neutrinos oscillate between different flavors.

II. APPROACH

A. Mass eigenstates

We will consider the simplified case of only two neutrino flavors, electron neutrinos and muon neutrinos. Evidently the basis vectors in the "flavor space" we typically classify them in are

$$|\nu_e\rangle = \begin{pmatrix}1\\0\end{pmatrix}, |\nu_\mu\rangle = \begin{pmatrix}0\\1\end{pmatrix}.$$

However, as we have just discussed, these states do not diagonalize the Hamiltonian and so do not have well defined mass or energy. Instead, they are linear combinations of the states that do, the mass eigenstates ν_1 and ν_2. In the mass basis [2],

$$|\nu_1\rangle = \begin{pmatrix}1\\0\end{pmatrix}, |\nu_2\rangle = \begin{pmatrix}0\\1\end{pmatrix} \rightarrow$$

$$|\nu_e\rangle = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}, |\nu_\mu\rangle = \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}.$$

The coefficients can be considered as unknown, but the motivation for defining them in terms of θ as we have comes from ensuring that the two flavor states are orthonormal [1]. Evidently transformation between the two bases can be achieved by [2]:

$$U_{\mathrm{PMNS}} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}. \quad (1)$$

Physically, θ represents a mixing angle between $|\nu_e\rangle$ and $|\nu_\mu\rangle$ states. We use the experimental input $\theta = 33.6°$.

B. Hamiltonian

The Hamiltonian in the mass basis is of course simply

$$H = \begin{pmatrix}E_1 & 0\\ 0 & E_2\end{pmatrix}.$$

We may rewrite each energy relativistically as

$$E_{1,2} = (p_{1,2}^2c^2 + m_{1,2}^2c^4)^{1/2} = p_{1,2}c\Big(1 + \frac{m_{1,2}^2c^2}{p_{1,2}^2}\Big)^{1/2}$$

$$\approx p_{1,2}c\Big(1 + \frac{m_{1,2}^2c^2}{2p_{1,2}^2}\Big),$$

the last step following from the ultrarelativistic approximation $p_{1,2} >> m_{1,2}c$. Note that we will work in a unit system where E is in eV, m is in eV/c^2, and c is in cm/s. In this limit the mass difference becomes so small in comparison to the momentum and energy that we may write $p_1 \approx p_2 = p$ [2]. As a result, we can rewrite H as

$$H = \begin{pmatrix}pc + m_1^2c^3/2p & 0\\ 0 & pc + m_2^2c^3/2p\end{pmatrix},$$

to which we are free to apply a constant energy shift of $pc - m_1^2c^3/2p$ and rewrite the momentum as $p = E/c$. As a result the Hamiltonian is simply [2]

$$H = \begin{pmatrix}0 & 0\\ 0 & (m_2^2 - m_1^2)c^4/2E\end{pmatrix}. \quad (2)$$

C. Time Evolution

Since electron neutrinos do not diagonalize Equation 2, a neutrino which is initially in the $|\nu_e\rangle$ will evolve in time. Its time evolution is governed by

$$S = e^{-iHt/\hbar} = \begin{pmatrix}e^0 & 0\\ 0 & \exp\left(-i\Delta m^2c^4L/2E\hbar c\right)\end{pmatrix}. \quad (3)$$

The origin of this operator is detailed in Section IV and Reference [2]. In the last step we expressed $\exp(H)$ by exponentiating its eigenvalues, using the shorthand

$\Delta m^2 = m_2^2 - m_1^2$ for the mass difference and using the ultrarelativistic approximation to convert between time and distance using $L = ct$. The end effect is that we can compute the electron neutrino coefficient of a mass basis state $|\nu\rangle$ as a function of the distance it has traveled according to

$$|\nu(L)\rangle = S(L)|\nu(0)\rangle$$

and finally the probability that it is an electron neutrino

$$P(\nu_e) = \left|\langle\nu_e^m|S(L)|\nu(0)\rangle\right|^2 \tag{4}$$

where $|\nu_e^m\rangle$ indicates that the flavor state must be expressed in the mass basis.

III. MSW EFFECT

A. Hamiltonian

Neutrinos propagating through the vacuum conform to the treatment above, but in the presence of matter they are also subject to weak interactions. Specifically, in the two state case we are considering neutrinos experience a weak interaction potential

$$V = G_F\sqrt{2}(N_e + N_\mu)$$

where G_F is the Fermi coupling constant and N_e, N_μ are the number densities of electrons and muons respectively [2]. Since muons are not present in the everyday matter of our universe we may simplify this by letting $N_\mu = 0$. The catch is that V is due to the interaction of electron flavored leptons and thus is expressed in the flavor basis. As a result the Hamiltonian must be expressed in the flavor basis as well, using U_{PMNS} to convert Equation 2 and obtaining

$$H' = U_{\text{PMNS}}\begin{pmatrix}0 & 0\\ 0 & \frac{\Delta m^2 c^4}{2E}\end{pmatrix}U_{\text{PMNS}}^\dagger + \begin{pmatrix}\sqrt{2}G_F V_e & 0\\ 0 & 0\end{pmatrix}. \tag{5}$$

The mass basis states governed by H' are different from the eigenstates of H in that we can no longer use U_{PMNS} to convert to the flavor basis. Instead we will use a transformation matrix consisting of the eigenvectors of H'.

B. Time Evolution

The time evolution operator takes the same form as in the vaccum case,

$$S' = e^{-iH't/\hbar} = \begin{pmatrix}\exp -i\lambda_1 L/\hbar c & 0\\ 0 & \exp -i\lambda_2 L/\hbar c\end{pmatrix}. \tag{6}$$

The matrix representation differs in that the eigenvalues of H' are not known analytically as they were for H so are denoted as λ_1, λ_2. However, we can still determine them computationally to obtain S', then proceed as before to express the probability as

$$P(\nu_e) = \left|\langle\nu_e^m|S'(L,\lambda_1,\lambda_2)|\nu(0)\rangle\right|^2. \tag{7}$$

C. Varying Density

The previous section assumes a constant V_e as $|\nu_e\rangle$ is propagated forward. In reality, a neutrino produced at the center of the Sun will experience varying electron number density $N_e(L)$ as it works its way to the surface. The electron number density of the Sun as a function of its radius is

$$N_e(R) = \frac{10^{26}}{1.67}e^{-4R/R_{\text{Sun}}} \tag{8}$$

and the potential V_e varies accordingly. As a result the hamiltonian H' and its eigenvalues are L dependent, so the time evolution operator can no longer span a distance ΔL over which the Hamiltonian is varying. Instead, we construct it only for a small step dL over which Equation 8 can be considered constant. In other words, it is now

$$S'' = \begin{pmatrix}\exp -i\lambda_1(L)(dL/\hbar c) & 0\\ 0 & \exp -i\lambda_2(L)(dL/\hbar c)\end{pmatrix} \tag{9}$$

where S'' is in the mass basis. Now we may propagate a state $|\nu(0)\rangle$ to $|\nu(L)\rangle$ by continuously constructing time evolution operators, converting them to the flavor basis and stepping forward by dL until L has been reached. Then the probability can similarly be given by

$$P(\nu_e) = \Big|\langle\nu_e^m|S''(L_0,\lambda_1(L_0,dL),\lambda_2(L_0,dL))... \tag{10}$$

$$S''(L_f,\lambda_1(L_f,dL),\lambda_2(L_f,dL))|\nu(0)\rangle\Big|^2.$$

This is the method we will use to approximate the oscillations of a neutrino propagating from the center of the Sun to the Earth.

IV. BENCHMARKING

Instead of the matrix methods for determining neutrino states with and without matter outlined above, we can proceed by solving the time dependent Schrodinger equation for a neutrino state $|\nu(L)\rangle$ which begins in the flavor basis as an electron neutrino, $|\nu(0)\rangle = |\nu_e\rangle$. From the time dependent Schrodinger Equation

$$-i\hbar\frac{\partial}{\partial t}|\nu(t)\rangle = H|\nu(t)\rangle$$

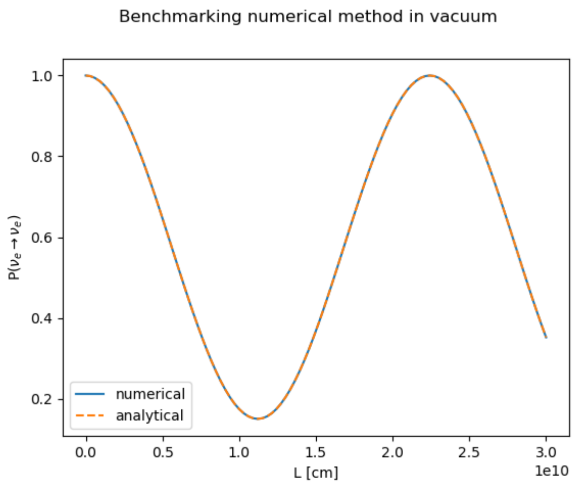


FIG. 1: Comparison of the matrix method for determining $P(\nu_e \to \nu_e)$ as the neutrino propagates a distance L through the vaccum with the analytical result (Equation 12). The two methods show close agreement. The oscillations have wavelength on the order of 10^{10} cm or 1 light-second. The typical distance from the Sun to the Earth is 499 light-seconds.

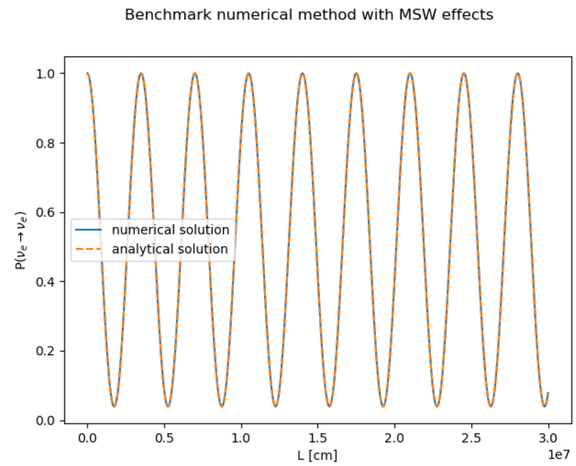


FIG. 2: Comparison of the matrix method for determining $P(\nu_e \to \nu_e)$ in the presence of matter with the analytical result (Equation). The two methods show close agreement. Oscillations have much higher frequency with MSW effects included, with λ on the order of 10^7 cm or $\frac{1}{1000}$ light-seconds.

we can write the solution in the flavor basis in terms of the initial state as

$$|\nu(t)^f\rangle = e^{-iHt/\hbar}|\nu_e\rangle. \tag{11}$$

The exponential term is the time evolution operator S and is where that matrix comes from. Notice that his result is very similar to the classic time dependence $e^{-iEt/\hbar}$ where H is replace by its eigenvalues. Subsequently, the next step is to express Equation 11 in the mass basis so that H is diagonalized and can be evaluated by its eigenvalues. Then we take the inner product with $\langle \nu_e^m|$ to obtain the probability. In vacuum when the Hamiltonian is simply Equation 2 the analytical result is [2]:

$$P(\nu_e) = 1 - \sin^2(2\theta)\sin^2\left(\frac{\Delta m^2 L}{4E\hbar c}\right). \tag{12}$$

This serves as a comparison with which we can benchmark the matrix method (Equation 3). The results are shown in Figure 1.

In addition to benchmarking the numerical determination of $P(\nu_e \to \nu_e)$ in vaccuum, we are also concerned with benchmarking the usage of S' to determine this quantity in the presence of interacting electrons according to Equation 7. Following Referemce [2], by a similar method as before we can determine the analytical dependence on L to be

$$P(\nu_e \to \nu_e) = 1 - \frac{\sin^2 2\theta}{\sin^2 2\theta + C^2}\sin^2\left(\frac{\Delta\tilde{m}^2 c^4 L}{4E\hbar c}\right) \tag{13}$$

where $\Delta\tilde{m}^2$ is a modified mass term defined as

$$\Delta\tilde{m}^2 = \Delta m^2\sqrt{\sin^2 2\theta + C^2}$$

and the constant

$$C = \cos 2\theta - \frac{2V_e E}{\Delta m^2}.$$

By plotting this function of L against the numerical results, we may benchmark the MSW effect matrix method. The results are shown in Figure 2.

Finally, we check the dependability of continuously operating with $S''(L, dL)$ to propagate $|\nu\rangle$ through a medium of varying density. This situation is complex enough that there is no analytical solution. However, we can do a benchmarking of sorts by replacing Equation 8 with a constant so that the analytical solution of Equation 13 is again valid. This at least confirms that the discretizing of the domain into dL steps is valid. This method does succeed in this case, as shown in Figure 3.

V. RESULTS

At this stage, we have all the machinery in place to calculate $P(\nu_e \to \nu_e)$ against L as a neutrino produced in the center of the Sun propagates to Earth. Looking at the analytical solutions, it is obvious that the resulting probability will oscillate countless times on this journey, so a more meaningful quantity than $P(\nu_e \to \nu_e)(L = L_{\text{Earth}})$, which is extremely variable, is the average of this quantity over the whole domain. We have plotted the instantaneous probability as well as a running average of the probabilities in the following figures. This allows us to calculate the desired quantity, which is the average probability of the neutrino being in the $|\nu_e\rangle$ state when it is measured at Earth. If we assume that the neutrino passes only through vacuum, then we only need to act

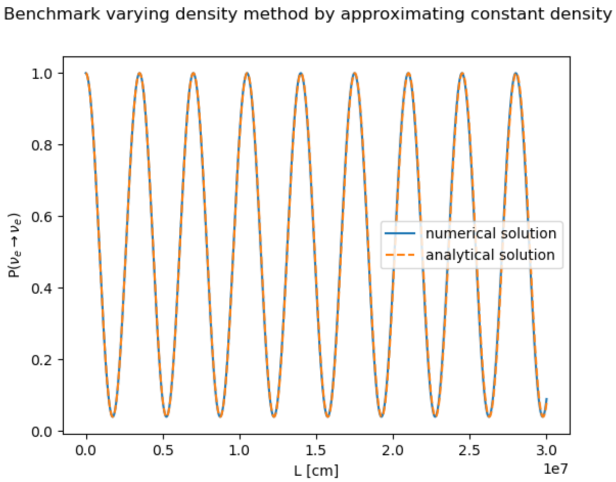


FIG. 3: Comparison of the matrix method for determining $P(\nu_e \to \nu_e)$ as the neutrino propagates through matter of varying density $N_e(L)$, but with this density set to a constant so that the analytical solution remains valid.

with S according to Section II to get the resulting probabilities, which are plotted against L in Figures 4-6 for different neutrino energies. In this case the physics is not very exciting, and the average probability quickly settles down before the neutrino has even exited the Sun. Furthermore, the final probability is not dependent on the initial energy.

The case where MSW effects are included is far more interesting. We have plotted this scenario in Figures 7-9, where we have used the methods of Section III C to propagate this state through regions of varying density. We use a log scale for L in these plots so that behavior inside and outside the Sun can be observed. The MSW effect can be clearly seen for high energy neutrinos especially. The constant MSW Hamiltonian for large N_e in the Sun keeps the probability that the state is an electron neutrino high. As the particle exits the Sun and $N_e \to 0$, the probability is dramatically decreased and the running average slowly falls off as the neutrino approaches Earth.

In each of these cases, the final probability is lower than in the purely-vacuum case. This quantity also falls off as the initial energy is increased and the MSW effects become more pronounced.

[1] D. Griffiths, *Introduction to Elementary Particles, 2nd ed.* (Wiley-VCH, 2008).

[2] L. Boccioli, Clqm spring 2020 neutrino oscillations (lecture notes) (2020).

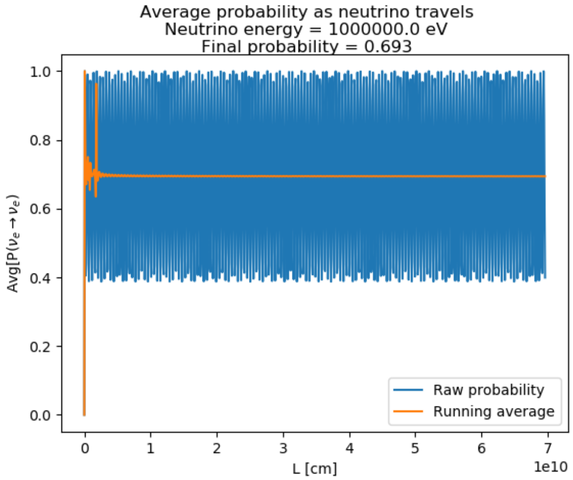


FIG. 4

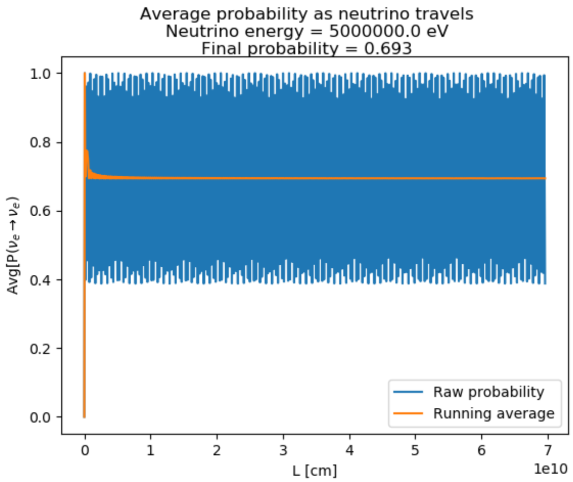


FIG. 5

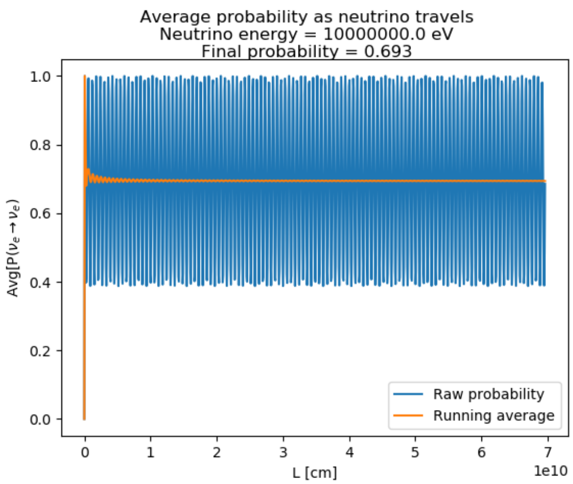


FIG. 6

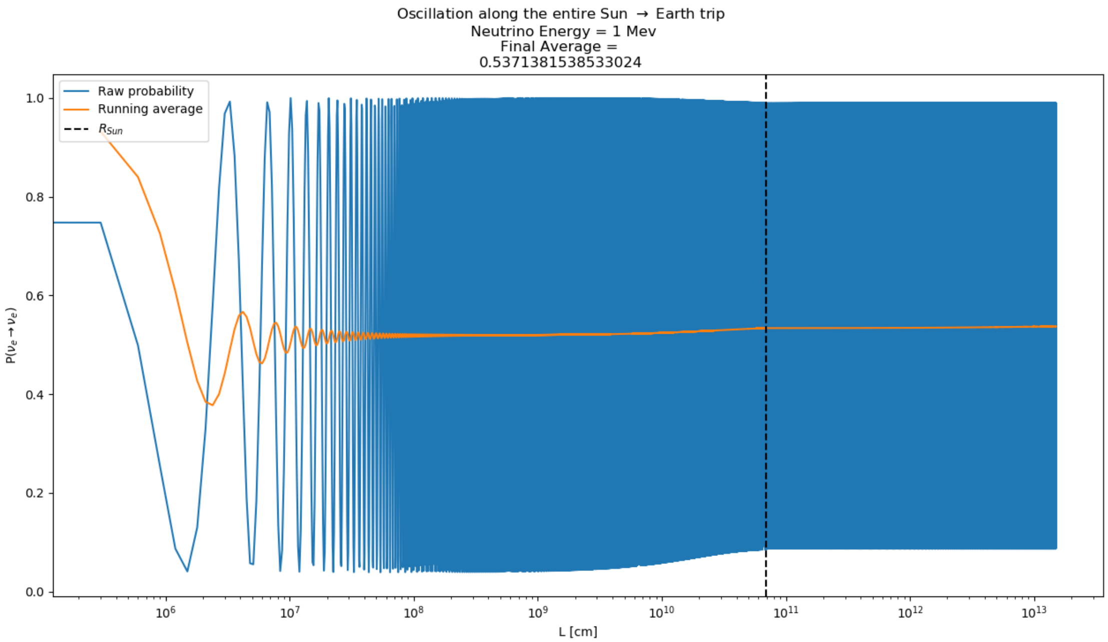

Oscillation along the entire Sun → Earth trip
Neutrino Energy = 1 Mev
Final Average =
0.5371381538533024
Raw probability
Running average
R_{Sun}
$P(\nu_e \to \nu_e)$
L [cm]

FIG. 7

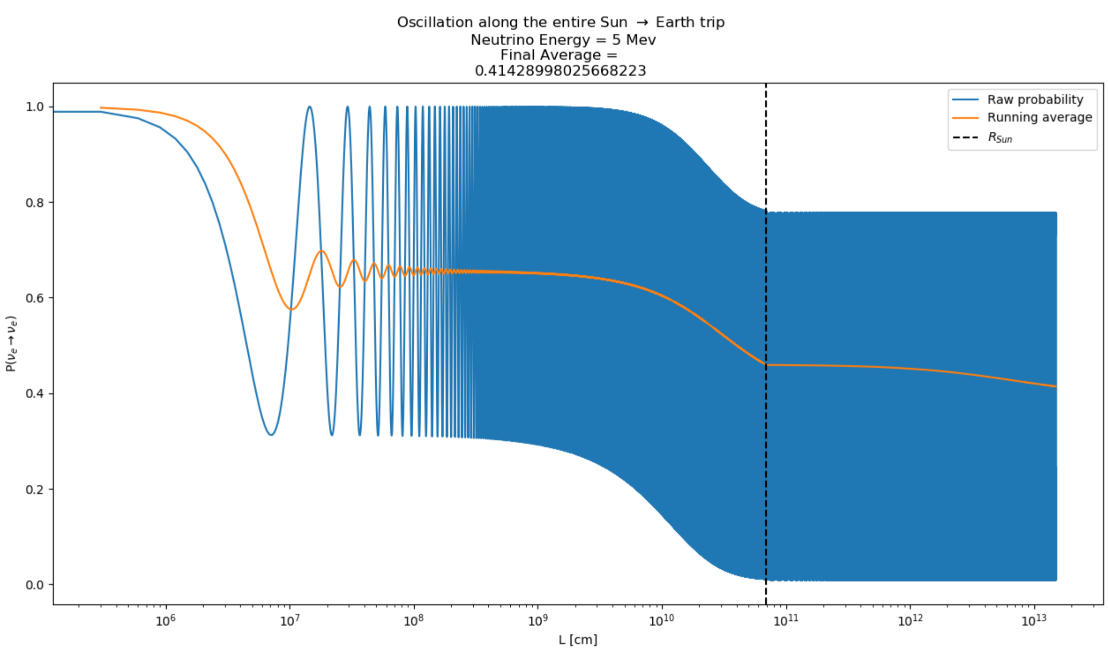
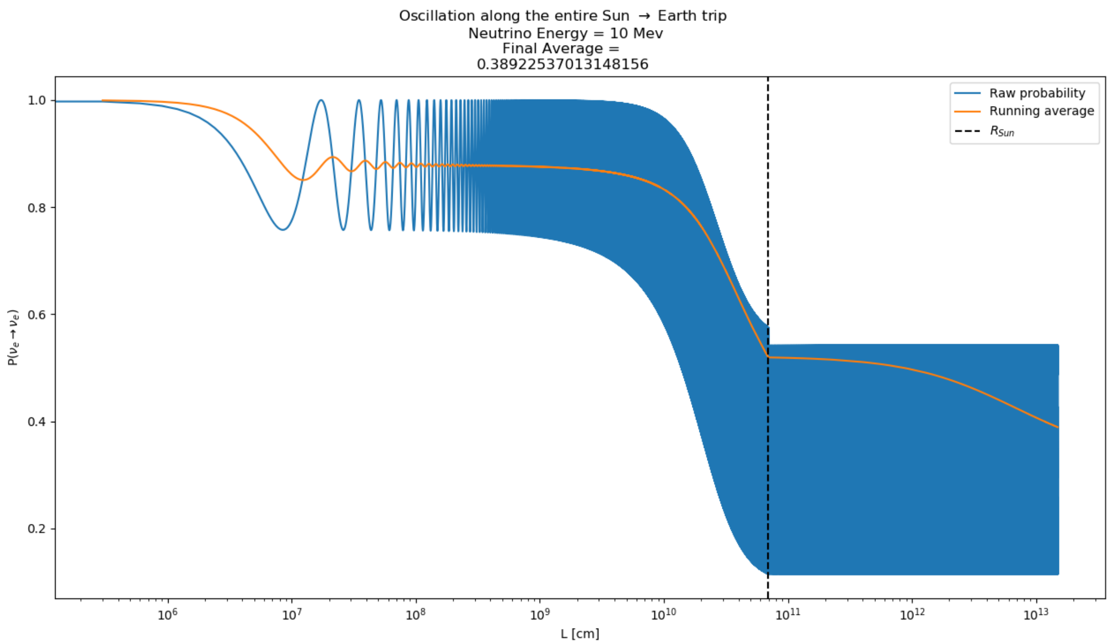

Oscillation along the entire Sun → Earth trip
Neutrino Energy = 5 Mev
Final Average =
0.41428998025668223
Raw probability
Running average
R_{Sun}
$P(\nu_e \to \nu_e)$
L [cm]

FIG. 8



Oscillation along the entire Sun → Earth trip
Neutrino Energy = 10 Mev
Final Average =
0.38922537013148156
Raw probability
Running average
R_{Sun}
$P(\nu_e \to \nu_e)$
L [cm]

FIG. 9