Implementing Reed-Solomon

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Recall

- Reed-Solmon represents messages as polynomials and over-samples them for redundancy.
- An $(n, k, n-k+1)$ code has
  - k digit messages
  - n digit codewords
  - $n-k+1$ distance between codewords (at least)
  - $(n-k)/2$ errors before it cannot be decoded
  - $2s = n-k$
- In this presentation, all messages and codewords are over the finite field $GF(2^8)$. This makes byte-oriented implementation easy

Recall

- Generator Polynomial:
  - $g(x) = (x - \alpha)(x - \alpha^2) \cdots (x - \alpha^{n-k})$
  - α is a generator element in $GF(2^8)$
- Encoding Process:
  - m is the message encoded as a polynomial
  - $m' = mx^{2s}$
  - $b = m' \pmod{g}$
    - $m' = qg + b$ for some q
  - $c = m' - b$
- Codewords are multiples of g, *and* are systematic
- Verifying a codeword is valid is a matter of checking for divisibility by g

Decoding Procedure Overview

1. **Calculate Syndromes**
2. **Berlekamp-Massey Algorithm** - calculates the Error Locator Polynomials and Error Evaluator Polynomials
3. **Chien Search** - Finds the error locations using the Error Locator Polynomial
4. **Forney's Formula** - Finds the error magnitudes using the error evaluator polynomial
5. **Correct the Errors**

Decoding (Defining Terms)

- Error Polynomial

$$\begin{aligned} R(x) &= C(x) + E(x) \\ E(x) &= E_0 + E_1 x + \cdots + E_{n-1} x^{n-1} \end{aligned}$$

  - Has at most s coefficients that are non-zero
- Error Positions
  - $j_1, j_2, \cdots j_s$, each a value between 0 and $n-1$
- Error Locations

$$X_i = \alpha^{j_i}$$

- Error Magnitudes

$$Y_i = E_{j_i}$$

- Notice that there are $2s$ unknowns

Decoding (Syndromes)

- Step 1: Calculate the first $2s$ syndromes
- Syndromes are defined for all l:

$$s_l = \sum_{i=1}^{s} Y_i X_i^l$$

- For the first $2s$, it reduces to:

$$s_l = E(\alpha^l) = \sum_{i=1}^{s} Y_i \alpha^{l j_i} \quad 1 \leq l \leq 2s$$

- $s_l = R(\alpha^l) = E(\alpha^l)$ for the first $2s$ powers of α.
- Equivalent to having $2s$ equations with $2s$ unknowns

Decoding (Syndromes)

- Encode the syndromes in a generator polynomial:

$$s(z) = \sum_{i=1}^{\infty} s_i z^i$$

- This can be computed by finding each s_i from the received codeword for the first $2s$ values. That's all we need though.

Berlekamp-Massey Algorithm

- Input: Syndrome polynomial from the last slide
- Output: Error Locator Polynomial $\sigma(z)$ and Error Evaluator Polynomial $\omega(z)$. Defined as:

$$\begin{aligned}
\sigma(z) &= \prod_{i=1}^{s}(1 - X_i z) \\
\omega(z) &= \sigma(z) + \sum_{i=1}^{s} z X_i Y_i \prod_{\substack{j=1 \\ j \neq i}}^{s}(1 - X_j z)
\end{aligned}$$

- Notice that the error locations are the inverse roots of $\sigma(z)$. (Roots are $1/X_1, 1/X_2, \cdots 1/X_s$)

B-M (The Key Equation)

- Observe the following relation:

$$\frac{\omega(z)}{\sigma(z)} = 1 + \sum_{i=1}^{s} \frac{zX_iY_i}{1 - X_iz}$$
$$= \text{...intermediate steps omitted}$$
$$= 1 + s(z)$$

- Key equation thus states:

$$(1 + s(z))\sigma(z) \overset{(\mathrm{mod}\ z^{2s+1})}{=} \omega(z)$$

- $\sigma(z)$ and $\omega(z)$ have degree at most s
- Key Equation represents a set of $2s$ equations and $2s$ unknowns

B-M (procedure)

- B-M iterates $2s$ times
- At each iteration, it produces a pair of polynomials:

$$(\sigma_{(l)}(z), \omega_{(l)}(z))$$

- where the polynomials satisfy that iteration's key equation:

$$(1 + s(z))\sigma_{(l)}(z) \quad \overset{(\text{mod } z^{l+1})}{=} \quad \omega_{(l)}(z)$$

B-M (procedure)

- Once we have
$$(\sigma_{(l)}(z), \omega_{(l)}(z))$$
for some l. If we're lucky, they already satisfy the next key equation:
$$(1 + s(z))\sigma_{(l)}(z) \quad \overset{(\mathrm{mod}\ z^{(l+2)})}{=} \quad \omega_{(l)}(z)$$
in which case we can set $\sigma_{(l+1)}(z) = \sigma_{(l)}(z)$ and similarly for $\omega(z)$
- However, usually we have an unwanted higher-order term:
$$(1 + s(z))\sigma_{(l)}(z) \quad \overset{(\mathrm{mod}\ z^{l+2})}{=} \quad \omega_{(l)}(z) + \Delta_{(l)} z^{l+1}$$

B-M (procedure)

- $\Delta_{(l)}$ is the non-zero coefficient of z^{l+1} in $(1 + s(z))\sigma_{(l)}(z)$
- Basic idea is to iteratively improve estimates of σ and ω
- But since there may be a higher order term, we can't always just extend to $l + 1$ from iteration l
- A complex set of rules determines how to handle different cases
- The next 5 slides describe these cases and how to handle them

B-M (Details)

- $\Delta_{(l)}$ is the non-zero coefficient in $(1 + s(z))\sigma_{(l)}(z)$
- To find the next iteration's polynomials, we introduce two more polynomials $\tau_{(l)}(z)$ and $\gamma_{(l)}(z)$
- They must satisfy:

$$(1 + s(z))\tau_{(l)}(z) \overset{(\mathrm{mod}\ z^{l+1})}{=} \gamma_{(l)}(z) + z^l$$

- And we have the following rules to derive the next σ and ω:

$$\begin{aligned} \sigma_{(l+1)}(z) &= \sigma_{(l)}(z) - \Delta_{(l)} z \tau_{(l)}(z) \\ \omega_{(l+1)}(z) &= \omega_{(l)}(z) - \Delta_{(l)} z \gamma_{(l)}(z) \end{aligned}$$

B-M (Details)

- But how to compute $\tau_{(l+1)}(z)$ and $\gamma_{(l+)}(z)$?
- Use one of the following rules:

(A)
$$\tau_{(l+1)}(z) = z\tau_{(l)}(z)$$
$$\gamma_{(l+1)}(z) = z\gamma_{(l)}(z)$$

(B)
$$\tau_{(l+1)}(z) = \frac{\sigma_{(l)}(z)}{\Delta_{(l)}}$$
$$\gamma_{(l+1)}(z) = \frac{\omega_{(l)}(z)}{\Delta_{(l)}}$$

B-M (Details)

- One of (A) or (B) is chosen each iteration to minimize the degrees of $\tau_{(l+1)}(z)$ and $\gamma_{(l+1)}(z)$
- To choose, define a single value $D_{(l)}$ for each iteration
- Choose rule (A) if $\Delta_{(l)} = 0$ or $D_{(l)} > \frac{l+1}{2}$
- Choose rule (B) if $\Delta_{(l)} \neq 0$ and $D_{(l)} < \frac{l+1}{2}$
- With rule (A) set $D_{(l+1)} = D_{(l)}$
- With rule (B) set $D_{(l+1)} = l + 1 - D_{(l)}$
- These rules and conditions ensure $0 < D_{(l+1)} \leq l + 1$ and the degrees of $\sigma_{(l+1)}$ and $\omega_{(l+1)}$ are upper-bounded by $D_{(l+1)}$ and degrees of $\tau_{(l+1)}$ and $\gamma_{(l+1)}$ are upper-bounded by $l - D_{(l)}$

B-M (Details)

- But what about when $\Delta_{(l)} \neq 0$ and $D_{(l)} = \frac{l+1}{2}$?
- Either rule works, but to do even better, define one last value, a binary value $B_{(l)}$, for each iteration
- When $B_{(l)} = 0$ use rule (A)
- When $B_{(l)} = 1$ use rule (B)
- With rule (A) set $B_{(l+1)} = B_{(l)}$
- With rule (B) set $B_{(l+1)} = 1 - B_{(l)}$
- This keeps the degree inequalities satisfied:

$$\begin{aligned} \text{degree}\, \omega_{(l)}(z) &\leq D_{(l)} - B_{(l)} \\ \text{degree}\, \gamma_{(l)}(z) &\leq l - D_{(l)} - (1 - B_{(l)}) \end{aligned}$$

B-M (Details)

- All those rules ensure the degrees of σ and ω do not grow too large. Each step they satisfy:

$$\begin{aligned} \text{degree}\,\sigma_{(l)} &\leq (l+1)/2 \\ \text{degree}\,\omega_{(l)} &\leq l/2 \end{aligned}$$

- Last piece: the initial conditions:

$$\begin{aligned} \sigma_{(0)}(z) &= 1 \\ \omega_{(0)}(z) &= 1 \\ \tau_{(0)}(z) &= 1 \\ \gamma_{(0)}(z) &= 0 \\ D_{(0)} &= 0 \\ B_{(0)} &= 0 \end{aligned}$$

Decoding: Next Steps

- Now we have the Error Locator Polynomial $\sigma(z)$ and the Error Evaluator Polynomial $\omega(z)$
- Chien's Search takes $\sigma(z)$ and outputs the error locations/positions (X_i and j_i)
- Forney's Formula takes $\omega(z)$ and the array X_i of error locations outputs the error magnitudes (Y_i)

Chien's Procedure

- Recall the definition of $\sigma(z)$:

$$\sigma(z) = \prod_{i=1}^{s}(1 - X_i z)$$

- Now that we have $\sigma(z)$, finding the array of X_i values is simply a matter of solving for the roots
- The Easy Way: since we're working over a small field, just test every value
  1. Let α be a generator
  2. Initialize $\{X_i\}$ to the empty set
  3. For $l = 1, 2, \ldots$
     If $\sigma(\alpha^l) = 0$: add α^{-l} to $\{X_i\}$

Chien's Procedure

- But we can do better than evaluating it 255 times!
- If we have computed the α^lth evaluation, we get:

$$\sigma(\alpha^l) = 1 + \sigma_1\alpha^l + \sigma_2\alpha^{2l} + \sigma_3\alpha^{3l} + \cdots + \sigma_s\alpha^{sl}$$

- Then, computing $\sigma(\alpha^{l+1})$ is an $O(s)$ operation:

$$\sigma(\alpha^{l+1}) = 1 + \sigma_1\alpha^{l+1} + \sigma_2\alpha^{2l+2} + \sigma_3\alpha^{3l+3} + \cdots + \sigma_s\alpha^{sl+s}$$

- The ith term in $\sigma(\alpha^{l+1})$ can be computed from the ith term in $\sigma(\alpha^l)$ by multiplying that term by α^i

Forney's Formula

Using the Error Evaluator Polynomial $\omega(z)$ and the error locations $\{X_i\}$, the error magnitudes $\{Y_i\}$ can be computed

$$\omega(z) = \sigma(z) + \sum_{i=1}^{s} z X_i Y_i \prod_{\substack{j=1 \\ j \neq i}}^{s} (1 - X_j z)$$

Evaluate at X_l^{-1}

$$\omega(X_l^{-1}) = \sigma(X_l^{-1}) + \sum_{i=1}^{s} X_l^{-1} X_i Y_i \prod_{\substack{j=1 \\ j \neq i}}^{s} (1 - X_j X_l^{-1})$$

Forney's Formula

$$\omega(X_l^{-1}) = \sigma(X_l^{-1}) + \sum_{i=1}^{s} X_l^{-1} X_i Y_i \prod_{\substack{j=1 \\ j \neq i}}^{s} (1 - X_j X_l^{-1})$$

Then simplifies to:

$$= Y_l \prod_{\substack{j=1 \\ j \neq l}}^{s} (1 - X_j X_l^{-1})$$

since $\sigma(X_l^{-1}) = 0$

Forney's Formula

$$\omega(X_l^{-1}) = Y_l \prod_{\substack{j=1 \\ j \neq l}}^{s} (1 - X_j X_l^{-1})$$

Can then be solved for Y_l:

$$Y_l = \frac{\omega(X_l^{-1})}{\prod_{\substack{j=1 \\ j \neq l}}^{\cancel{s}\,|X|} (1 - X_j X_l^{-1})}$$

Use the length of X to bound the product if you want to support erasures or erasures+errors.

And that can be directly computed. We know all the values on the right hand side!

Putting it all together

- We know:
  - $\{X_i\}$ The error locations
  - $\{Y_i\}$ The error magnitudes
- Put them together to build the Error Polynomial $E(x)$
- Then subtract to get the codeword!

$$C(x) = R(x) - E(x)$$

Reed-Solomon Implementation

The rest of the presentation is about my implementation

- Done in Python with no external libraries or dependencies
- Implemented a Finite Field class for $GF(2^8)$
- Implemented a Polynomial Class for manipulating polynomials
- Implemented the RS algorithms as described

Finite Fields

- Created a Python class that subclasses int
- Instances are integers, which represent the corresponding finite field element when translated to a polynomial

$$51 = 00110011 = x^5 + x^4 + x + 1$$

- Overwrote addition, subtraction, multiplication, division, and exponentiation for finite field arithmetic
- Multiplication defined using an exponentiation table and a logarithm table, pre-generated

Finite Fields (multiplication)

`exptable = (1, 3, 5, 15, 17, 51, ... 246, 1)`

- This table holds all powers of 3
- `exptable[1] = 3`
- `exptable[255] = 1`

`logtable = (None, 0, 25, 1, 50, 2, ... 112, 7)`

- This table holds all logarithms in base 3
- `logtable[3] = 1`
- `logtable[17] = 4`
  (since $3^4 = 17$)
- `logtable[0]`
  is an error

Finite Fields (multiplication)

```
exptable = (1, 3, 5, 15, 17, 51, ... 246, 1)
logtable = (None, 0, 25, 1, 50, 2, ... 112, 7)
```

- These tables together define multiplication like this:

```
def multiply(a, b):
    x = logtable[a]
    y = logtable[b]
    z = (x + y) % 255
    return exptable[z]
```

Finite Fields (more)

```
exptable = (1, 3, 5, 15, 17, 51, ... 246, 1)
logtable = (None, 0, 25, 1, 50, 2, ... 112, 7)
```

- Exponentiation and multiplicative inverses also use these tables:

```
def power(a, b):
    x = logtable[a]
    z = (x * b) % 255
    return exptable[z]

def inverse(a):
    e = logtable[a]
    return exptable[255 - e]
```

Polynomial Class

- Stores numbers from high degree to low degree
- All coefficient math is done using regular Python operators
- Compatible with both integers and field elements as coefficients
- Supports long division and remainders (essential for RS coding)

Reed Solomon Encoding

Since the polynomial class abstracts polynomial math away, encoding boils down to basically:

```
def encode(m):
    mprime = m * xshift
    b = mprime % g
    c = mprime - b
    return c
```

Reed Solomon Decoding

Decoding is also fairly simple:

```
def decode(r):
    sz = syndromes(r)
    sigma, omega = berlekamp_massey(sz)
    X, j = chien_search(sigma)
    Y = forney(omega, X)

    # There is a loop to build E here
    ...

    return r - E
```

Reed Solomon Decoding

- My implementation of those functions are straight up implementations of the math. Nothing surprising.

```
def syndromes(r):
    s = [GF256int(0)]
    for l in range(1, n-k+1):
        s.append(r.evaluate(GF256int(3)**l))
```

- My Chien Search isn’t actually Chien’s search though, it just evaluates the polynomial 255 times:

```
    p = GF256int(3)
    for l in range(1,256):
        if sigma.evaluate( p**l ) == 0:
            X.append( p**(-l) )
            j.append(255 - l)
```

Implementation Notes

- Message to Polynomial translations
  1. “hello”
  2. 104, 101, 108, 108, 111
  3. $104x^4 + 101x^3 + 108x^2 + 108x^1 + 111$
- Messages are effectively left-padded with null bytes

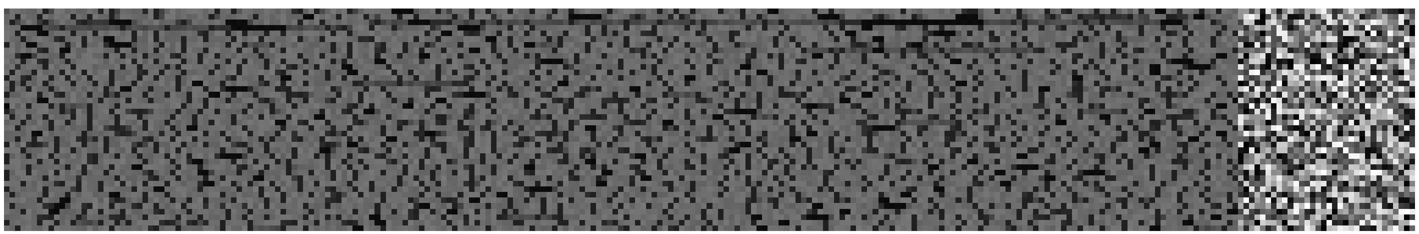
Example

- RS(20,13) code: 13 message bytes and 7 parity bytes. Can correct 3 errors.
- Message: “Hello, world!”
- Codeword: “Hello, world![8d][13][f4][f9][43][10][e5]”
- R: “[00][00][00]lo, world![8d][13][f4][f9][43][10][e5]”
- Decoded: “Hello, world!”

And, to prove this isn’t faked...

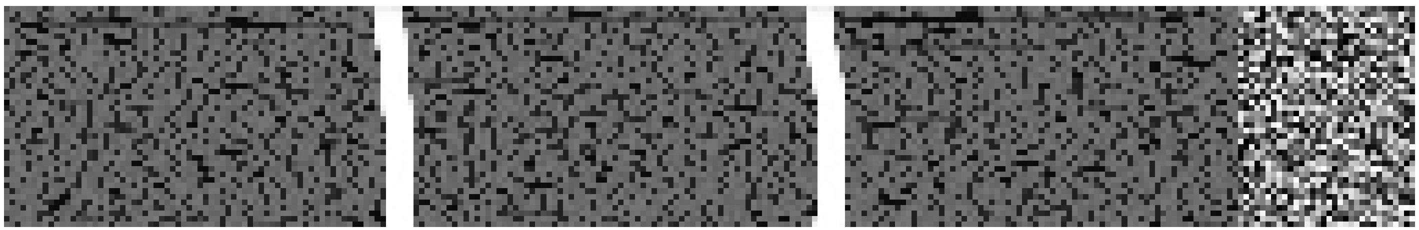
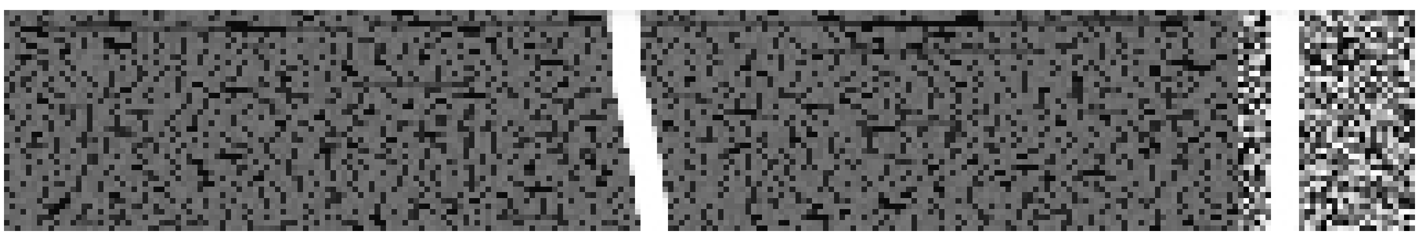
Demo!

As an example, I have written a program that encodes codewords as rows in an image

- Uses RS(255,223)
- Encodes each symbol as a pixel in a grayscale image
- Each row is a codeword
- 
- Decodes to:

```
ALICE'S ADVENTURES IN WONDERLAND
Alice was beginning to get very tired of
sitting by her sister on the ...
```

Demo!

- Since each row is a RS(255,223) codeword, it can handle up to 16 pixel errors per row.
- Drawing 5 px stripes, each of the following still decodes:
- 
- 
-