PARAMETRIC KAM RIGIDITY IN ANALYTIC BILLIARD MAPS

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Abstract. In this paper,

1. Introduction

Billiards are models that describe the motion of a ray of light inside an empty cavity that is reflected on its boundary, considered as perfectly reflecting mirror, according to the law of reflection *angle of incidence = angle of reflection*. In this paper, we consider the case convex planar billiard, that is when the caivty is represented by a bounded convex domain $\Omega \subset \mathbb{R}^2$ whose boundary $\partial\Omega$ is $\mathscr{C}^r$-smooth, $r \in \mathbb{N}^{\geqslant 2} \cup \{\infty, \omega\}$, and has nowhere vanishing curvature.

The billiard dynamics can be encoded by a map $T : \mathscr{C} \to \mathscr{C}$ where $\mathscr{C} = \mathbb{R}/|\partial\Omega|\mathbb{Z} \times (-\pi, \pi)$ in the following way. Consider an arc-length parametrization $s \mapsto \gamma(s)$ of $\partial\Omega$. For any $(s, \varphi) \in \mathscr{C}$, write

$$T(s, \varphi) = (s_1, \varphi_1)$$

if the angle between the vector $\gamma'(s)$ and the oriented line $\ell = \gamma(s)\gamma(s_1)$ is φ and φ_1 is the angle made between the oriented line ℓ and the vector $\gamma'(s_1) - \varphi_1$ is the angle after the reflection of ℓ at the point $\gamma(s_1)$. In this way, the billiard map T is $\mathscr{C}^{r-1}$-smooth (see [4]).

Lazutkin [6] showed that if r is sufficiently large then T has a large set of invariant curves. More precisely, given two parameters $\gamma, \tau > 0$, he introduced the set

$$D(\gamma, \tau) := \{\omega \in [0, 1] \,:\, \forall m/n \in \mathbb{Q} \quad |n\omega - m| \geqslant \gamma |m| n^{-\tau}\}$$

which is a *Cantor subset* of $[0, 1]\backslash\mathbb{Q}$ of positive measure accumulating to 0 and 1. He showed that for any $\omega \in D(\gamma, \tau)$, T has an invariant curve $\Gamma_\omega \subset \mathscr{C}$ of rotation number ω: this means that $T(\Gamma_\omega) = \Gamma_\omega$ and $T_{|\Gamma_\omega}$ is conjugated to a rotation of rotation number ω. Except 0, the numbers in $D(\gamma, \tau)$ are called *Diophantine numbers*. Morover, there exist $b > 0$, an integer s depending on r and a map

$$\Phi : D(\gamma, \tau) \cap [0, b) \times \mathbb{R}/\mathbb{Z} \to \mathscr{C}$$

which is $\mathscr{C}^s$-smooth in a Whitney sense such that for any $(\omega, \theta) \in D(\gamma, \tau) \times \mathbb{R}/\mathbb{Z}$ the billiard map satisfies

$$T\Phi(\omega, \theta) = \Phi(\omega, \theta + \omega). \tag{1}$$

In particular, $\Gamma_\omega = \operatorname{Im} \Phi(\omega, \cdot)$. Note that Φ conjugate T to the integrable map $A : (\omega, \theta) \mapsto (\omega, \theta + \omega)$:

$$\Phi^{-1} T \Phi = A.$$

In this paper, we show that if the boundary of Ω is analytic then, up to considering a smaller value of b in the definition of Φ, the previous object can be extended to a larger subset in the complex plane, with additional regularity. More precisely, there

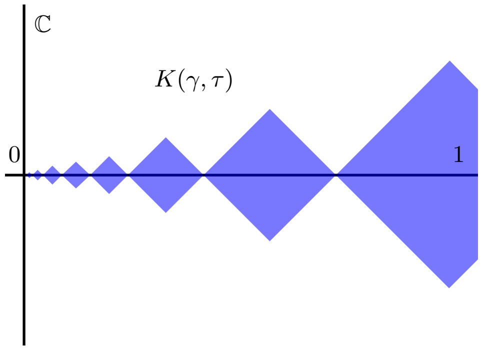

FIGURE 1. Diamonds (in blue) whose vertices lying on the x-axis are the diophantine numbers $D(\gamma,\tau)$. Their complementary set (in white) together with the boundary of the diamonds constitute the set $K(\gamma,\tau)$.

exist $b>0$ and a set $K(\gamma,\tau)\subseteq\mathbb{C}$ containing $D(\gamma,\tau)$ and of non-empty interior – see Figure 1, such that the map Φ described above can be uniquely extended to $K(\gamma,\tau)\cap B(0,b)$, where $B(0,b)=\{z\in\mathbb{C}\mid |z|<b\}$, and has the so-called $\mathscr{C}^\infty$-*holomorphic regularity* introduced in [1]: it roughly means the dependency in θ is analytic, the dependency in ω is analytic in the interior of $K(\gamma,\tau)$ and $\mathscr{C}^\infty$-smooth everywhere else. The $\mathscr{C}^\infty$-holomorphic regularity is an iterated version of the $\mathscr{C}^1$-*holomorphic regularity*, which comes with a specific Banach norm $\|\cdot\|_{\mathscr{C}^1_{\text{hol}}}$, which can be seen as the sup norm of f and its differential. This follows recent works, see [1], which were written for one-parameter families of standard maps.

1.1. **Main theorem.** Let $\mathbb{T}=\mathbb{R}/\mathbb{Z}$. The main result of this paper applies to a wide class of maps, called *billiard-like maps*, including billiard maps. These are maps defined on $\mathbb{T}\times(-b,b)$, where $b>0$, of the form

$$T:(x,y)\mapsto\left(x+y+\mathcal{O}\left(y^m\right),y+\mathcal{O}\left(y^{m+1}\right)\right) \tag{2}$$

for a certain integer $m\geqslant 3$, and satisfying the so-called *intersection property*, which can be defined as follows: given any non homotopically-trivial curve $\mathcal{C}\subset\mathbb{T}\times(-b,b)$, its image $T(\mathcal{C})$ intersects $\mathcal{C}$.

Billiard maps are examples of such maps: Lazutkin [7] exhibited a change of coordinates $(s,\varphi)\mapsto(x,y)$ such that any billiard map admits an expansion of the form (2) in the new coordinates (x,y) with $m=3$. Moreover it preserves the area-form $ydxdy$, and hence satisfies the intersection property – see [5, Chapter 1] for more details.

In the following, given $b,r>0$ and $R:\mathbb{T}_r\times B(0,b)\subseteq\mathbb{C}/\mathbb{Z}\times\mathbb{C}\to\mathbb{C}^2$ we denote by $T_R:\mathbb{T}_r\times B(0,b)\to\mathbb{C}/\mathbb{Z}\times\mathbb{C}$ the map given by

$$T_R:=A+R,\qquad\text{where}\quad A(x,y)=(x+y,y).$$

We say that T_R is *real-analytic* if R is analytic and restricts to $\mathbb{R}/\mathbb{Z}\times(B(0,b)\cap\mathbb{R})$ as a real-valued map.

Theorem 1. *For any $0<\gamma<1\leqslant\tau$ there exists $0<\lambda(\gamma,\tau)<\min\{1,\gamma\}$ such that, for any $0<r\leqslant 1$, any $0<b\leqslant\min\{r^{2(\tau+1)},\frac{r}{2}\}$, and any real analytic map $R:\mathbb{T}_r\times B(0,b)\subseteq\mathbb{C}/\mathbb{Z}\times\mathbb{C}\to\mathbb{C}^2$ of the form*

$$R(x,y)=(y^2R_1(x,y),y^3R_2(x,y)), \tag{3}$$

for which T_R has the intersection property and satisfying

$$\max\{\|R_1\|_{\mathscr{C}^1_{hol}}, \|R_2\|_{\mathscr{C}^1_{hol}}\} \leqslant \varepsilon(\gamma,\tau,b,r) := \lambda(\gamma,\tau) b r^{8(\tau+1)}, [1]$$

there exists a $\mathscr{C}^\infty$-holomorphic map

$$\mathfrak{C} : \Omega \to C^\omega\big(\mathbb{T}_{r/2}, \mathbb{T}_r \times B(0,b)\big), \qquad \Omega := K(\gamma,\tau) \cap \overline{B(0,\tfrac{b}{2})},$$

of the form

$$\mathfrak{C}(y)(x) = (x + \varphi_1(x,y), y + \varphi_2(x,y))$$

and obeying

$$\max\left\{\|\varphi_1\|_{\mathscr{C}^1_{\mathrm{hol}}(\Omega)}, \|\varphi_2\|_{\mathscr{C}^1_{\mathrm{hol}}(\Omega)}\right\} \leqslant \varepsilon^{1/2}, \tag{4}$$

such that the transformation

$$\begin{array}{rccc} \Phi(\mathfrak{C}) : & \mathbb{T}_{r/2} \times \Omega & \to & \mathbb{T}_r \times B(0,b) \\ & (x,y) & \mapsto & \mathfrak{C}(y)(x) \end{array} \tag{5}$$

is diffeomorphic onto its image and T_R satisfies

$$T_R \circ \Phi(x,y) = \Phi(x+y,y), \tag{6}$$

for any $(x,y) \in \mathbb{T}_{r/4} \times \Omega$.

Following the nomenclature introduced in [3], we refer as a *KAM curve* associated with a billiard-like map $T_R : \mathbb{T}_r \times B(0,b) \to \mathbb{C}/\mathbb{Z} \times \mathbb{C}$ to any $\mathscr{C}^1$-holomorphic map $\mathfrak{C} : \Omega \subseteq B(0,1) \cap D^{\mathbb{C}} \to C^\omega(\mathbb{T}_{r'}, \mathbb{T}_r \times B(0,b))$ for which 0 is a density point of $\Omega \cap \mathbb{R}$, the map $\Phi(\mathfrak{C})$ given by (5) is diffeomorphic onto its image, and $\Phi(\mathfrak{C})$ satisfies (6) on $\mathbb{T}_{r''} \times \Omega$, for some $0 < r'' < r' < r$.

Notice that the definition above does not allow us to speak of a *unique* KAM curve associated to a billiard-like map in a strict sense, since we do not define in advance any sort of maximal domain for the set of rotation numbers of the invariant (or translated) curves associated with the map. However, these maps do display an important uniqueness property already highlighted in [3], and which we describe in the following remark.

Remark 1. *Any two KAM curves associated to the same billiard-like map must coincide on the intersection of their domains. Indeed, it is well-known that for twist maps there is at most one invariant curve for a given irrational rotation number. Hence, any two KAM curves for a billiard-like map must coincide in their common domain along the real numbers. Since this intersection has positive Lebesgue measure in $\mathbb{R}$ and the maps are C^1-holomorphic, it follows that the two KAM curves must coincide on the intersection of their domains.*

Moreover, let us point out that any KAM curve completely characterizes the underlying system. More precisely, the following uniqueness property (analogous to the uniqueness property of analytic functions on connected sets of $\mathbb{C}$) holds.

Corollary 2. *Let $b, r, \rho > 0$ and $T_R, T_S : \mathbb{T}_r \times B(0,b) \to \mathbb{C}/\mathbb{Z} \times \mathbb{C}$ be two analytic billiard-like maps. Assume T_R and T_S admit KAM curves $\mathfrak{C}_R : \Omega_R \to C^\omega(\mathbb{T}_\rho, \mathbb{T}_r \times B(0,b))$ and $\mathfrak{C}_S : \Omega_S \to C^\omega(\mathbb{T}_\rho, \mathbb{T}_r \times B(0,b))$ respectively.*

If the set $\{y \in \Omega_R \cap \Omega_S \cap \mathbb{R} \mid \mathfrak{C}_R(y) = \mathfrak{C}_S(y)\}$ has an accumulation point, then $R = S$.

[1] I'll give a brief explanation about the norm in the previous paragraphs.

Proof. Let Φ_R as in (5). By (6) and by considering a smooth extension of Φ_R to an open neighbourhood of $\mathbb{T}_\rho \times \Omega_R$, we may assume that Φ_R is a C^∞ diffeomorphism defined on an open neighborhood of $\mathbb{T} \times \Omega_R$.

Let us assume WLOG that the KAM curves $\mathfrak{C}_R$ and $\mathfrak{C}_S$ coincide along a sequence accumulating at 0. Then,

$$\Phi_R^{-1} \circ T_R \circ \Phi_R = \Phi_R^{-1} \circ T_S \circ \Phi_R + \mathcal{O}^\infty(y), \tag{7}$$

and the Corollary follows by [12, Lemma 2.5] $\square$

Notice that the set of Diophantine numbers in Theorem 1 depends strongly on the norm of the perturbation being considered and, as stated, may not apply to certain billiard-like maps as in (2).

However, by first conjugating the map to obtain a billiard-like map of higher order, we will be able to apply Theorem 1 on appropriate rescaled domains. Furthermore, gluing the functions obtained for different Diophantine constants (see Remark 1) leads to a well-defined $\mathscr{C}^1$-holomorphic function.

Theorem 3. *For any real analytic map $R : \mathbb{T}_r \times B(0,b) \to \mathbb{C}^2$ of the form (3) for which T_R has the intersection property, there exists a continuous map $\rho : [0,1) \times [1,+\infty) \to \left[0, \min\left\{b, \frac{r}{4}\right\}\right)$ obeying*

$$\lim_{\gamma \to 0^+} \rho(\gamma, \cdot) = 0, \qquad \lim_{\tau \to +\infty} \rho(\cdot, \tau) = 0, \qquad \textit{uniformly,} \tag{8}$$

such that, for any $0 < \gamma < 1 \leqslant \tau$, there exists a $\mathscr{C}^\infty$-holomorphic map

$$\mathfrak{C}^{\gamma,\tau} : \Omega(\gamma,\tau) \to C^\omega\big(\mathbb{T}_{r/2}, \mathbb{T}_r \times B(0,b)\big), \qquad \Omega(\gamma,\tau) := K(\gamma,\tau) \cap \overline{B(0,\rho(\gamma,\tau))},$$

such that the transformation

$$\begin{array}{rccc} \Phi^{\gamma,\tau}(\mathfrak{C}) : & \mathbb{T}_{r/2} \times \Omega(\gamma,\tau) & \to & \mathbb{T}_r \times B(0,b) \\ & (x,y) & \mapsto & \mathfrak{C}(y)(x) \end{array}$$

is diffeomorphic onto its image and T_R satisfies

$$T_R \circ \Phi^{\gamma,\tau}(x,y) = \Phi^{\gamma,\tau}(x+y,y),$$

for any $(x,y) \in \mathbb{T}_{r/4} \times \Omega(\gamma,\tau)$.

Moreover, the map

$$\mathfrak{C} : \Omega \to C^\omega\big(\mathbb{T}_{r/2}, \mathbb{T}_r \times B(0,b)\big), \qquad \Omega := \bigcup_{\substack{\tau \geqslant 1 \\ 0<\gamma<1}} \Omega(\gamma,\tau),$$

given by

$$\mathfrak{C}(y) = \mathfrak{C}^{\gamma,\tau}(y), \qquad \textit{for any } 0 < \gamma < 1 \leqslant \tau \textit{ such that } y \in \Omega(\gamma,\tau),$$

is a well-defined $\mathscr{C}^1$-holomorphic map.

Proof. By Theorem 9, we may assume that R_1 and R_2 in (3) are defined on $\mathbb{T}_{r/2} \times B(0,b')$, for some $0 < b' < b$, and satisfy $R_1, R_2 = \mathcal{O}^m(y)$, for $m = 2$.

By Lemma 27, for any $0 < \eta < 1$, we have

$$\|R_i\|_{\mathscr{C}^1_{\mathrm{hol}}(\mathbb{T}_{r/2}\times B(0,\eta b'))} \leqslant \eta^m \|R_i\|_{\mathscr{C}^1_{\mathrm{hol}}(\mathbb{T}_{r/2}\times B(0,b'))}, \qquad i = 1,2.$$

Let us fix $0 < \gamma < 1 \leqslant \tau$. Taking

$$\eta(\gamma,\tau) := \frac{\lambda(\gamma,\tau) b r^{8(\tau+1)}}{2^{8(\tau+1)} \max\left\{1, \|R_1\|_{\mathscr{C}^1_{\mathrm{hol}}(\mathbb{T}_{r/2}\times B(0,b'))}, \|R_2\|_{\mathscr{C}^1_{\mathrm{hol}}(\mathbb{T}_{r/2}\times B(0,b'))}\right\}},$$

where $\lambda(\gamma,\tau)$ is given by Theorem 1, and defining

$$r_* = \frac{r}{2}, \qquad b_* = \eta b',$$

we have

$$\max\left\{\|R_1\|_{\mathscr{C}^1_{\mathrm{hol}}(\mathbb{T}_{r_*}\times B(0,b_*))}, \|R_2\|_{\mathscr{C}^1_{\mathrm{hol}}(\mathbb{T}_{r_*}\times B(0,b_*))}\right\} \leqslant \lambda b_* r_*^{8(\tau+1)}.$$

Define

$$\rho(\gamma,\tau) := \frac{1}{2}\eta(\gamma,\tau) b'.$$

Then, by Theorem 1, there exists $\mathfrak{C}^{\gamma,\tau} : \Omega(\gamma,\tau) \to C^\omega\big(\mathbb{T}_{r/2}, \mathbb{T}_r \times B(0,b)\big)$ of the form

$$\mathfrak{C}^{\gamma,\tau}(y)(x) = (x + \varphi_1^{\gamma,\tau}(x,y), y + \varphi_2^{\gamma,\tau}(x,y))$$

obeying

$$\max\left\{\|\varphi_1^{\gamma,\tau}\|_{\mathscr{C}^1_{\mathrm{hol}}(\Omega)}, \|\varphi_2^{\gamma,\tau}\|_{\mathscr{C}^1_{\mathrm{hol}}(\Omega)}\right\} \leqslant 1, \tag{9}$$

and satisfying the desired conclusions.

Consider the map $\mathfrak{C} : \Omega \to C^\omega\big(\mathbb{T}_{r/2}, \mathbb{T}_r \times B(0,b)\big)$ as defined in the statement. As mentioned in Remark 1, any two KAM curves for the same map must coincide in the intersection of their domains, and therefore the map $\mathfrak{C}$ is well-defined and of the form

$$\mathfrak{C}(y)(x) = (x + \varphi_1(x,y), y + \varphi_2(x,y)),$$

where, for $i = 1, 2$, the map $\varphi_i : \Omega \to \mathbb{C}$ is given by

$$\varphi_i(y) = \varphi_i^{\gamma,\tau}(y), \qquad \text{for any } 0 < \gamma < 1 \leqslant \tau \text{ such that } y \in \Omega(\gamma,\tau).$$

Let us show that $\mathfrak{C}$ is in fact a $\mathscr{C}^1$-holomorphic map.

First, let us show that Ω is a closed set without isolated points. Ω is clearly a set without isolated points since it is the union of set without isolated points. To show that it a closed set, let us consider $(y_n)_{n\geqslant 0} \subseteq \Omega$ such that $y_n \to y \in \mathbb{C}$. Let us assume that $y \neq 0$, since $0 \in \Omega$. By definition of Ω, there exists sequences $(\gamma_n)_{n\geqslant 0}, (\tau_n)_{n\geqslant 0} \subseteq \mathbb{R}_{>0}$ such that $y_n \in \Omega(\gamma_n,\tau_n) \subseteq \overline{B(0,\rho(\gamma_n,\tau_n))}$. Notice that since $y \neq 0$ and ρ satisfies (8), up to considering a subsequence, we may assume that $\gamma_n \to \gamma > 0$ and $\tau_n \to \tau \geqslant 1$, for some $0 < \gamma < 1 \leqslant \tau$. Therefore, since ρ is continuous, it follows that $y \in \Omega(\gamma,\tau) \subseteq \Omega$.

Finally, since by (9) the $\mathscr{C}^1$-holomorphic norms of the maps $\varphi_i^{\gamma,\tau}$ are uniformly bounded, it follows that φ_1, φ_2, are well-defined $\mathscr{C}^1$-holomorphic maps. □

Remark 2. *In Theorem 3, we cannot conclude that the map $\mathfrak{C}$ is $\mathscr{C}^\infty$-holomorphic, since Theorem 1, nor its proof, allows us to bound uniformly arbitrarily high derivatives of the maps $\varphi_i^{\gamma,\tau}$.*

However, let us point out that by increasing the power $8(\tau+1)$ in the smallness condition of Theorem 1, and thus adapting the definitions of the maps ρ and $\mathfrak{C}$ in Theorem 3, we could obtain such control for a fixed number of derivatives and conclude that $\mathfrak{C}$ admits a finite number of $\mathscr{C}^1$-holomorphic derivatives.

1.2. **Application to billiard dynamics.** Given a strongly convex billiard $\Omega \subset \mathbb{R}^2$, denote by T_Ω the billiard map inside to Ω expressed in (s,φ)-coordinates.

We introduce the complex torus of perimeter $\ell > 0$

$$\mathbb{T}_r^\ell = \{z \in \mathbb{C}/\ell\mathbb{Z} \ : \ |\mathrm{Im}\, z| < r\}.$$

Theorem 4. *Assume that Ω has an analytic boundary. There exist $r > 0$, a strictly increasing map $b : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$, with $b(0) = 0$, and a $\mathscr{C}^1$-holomorphic map*

$$\Gamma : K(\tau) \to C^\omega\big(\mathbb{T}_{r/2}, \mathbb{T}_r^{|\partial\Omega|} \times B(0,1)\big), \qquad K(\tau) := \bigcup_{\gamma>0} K(\gamma,\tau) \cap \overline{B(0,b(\gamma))},$$

such that the map $\Phi = \Phi(\Gamma)$ given by (5) *is diffeomorphic onto its image and satisfies* (6) *for any $(x,y) \in \mathbb{T}_{r/2} \times K(\tau)$*

$$\Phi^{-1} T_\Omega \Phi(x,y) = (x+y, y).$$

Using the previously introduced terminology, the map Γ will be called the *KAM curve* of the billiard in Ω.

Corollary 5. *Let $\tau, \rho > 0$, Ω_1, Ω_2 be analytic billiards with respective KAM curves Γ_1 and Γ_2 where for $j \in \{1,2\}$*

$$\Gamma_j : K_j(\tau) \to C^\omega\big(\mathbb{T}_\rho, \mathbb{T}_{r_j} \times B(0,1)\big),$$

with $r_j > 0$ and $K_j(\tau)$ be defined as in Theorem 4. If the set $\{y \in K_1(\tau) \cap K_2(\tau) \mid \Gamma_1(y) = \Gamma_2(y)\}$ has an accumulation point, then Ω_1 and Ω_2 are isometric.

Proof of Theorem 4. Consider an analytic change of coordinates $\Psi : \mathbb{R}/|\Omega|\mathbb{Z} \times (-\pi,\pi) \to \mathbb{R}/\mathbb{Z} \times (-1,1)$, a diffeomorphism onto its image, such that the map $T := \Psi \circ T_\Omega \circ \Psi^{-1}$ satisfies for any (x,y) as $y \to 0$

$$T(x,y) = \left(x + y + \mathcal{O}\left(y^2\right), y + \mathcal{O}\left(y^3\right)\right).$$

We can choose for example Ψ to be Lazutkin change of coordinates. Moreover, the map T has the intersection property, since T_Ω also has: the latter is known to be an area-preserving twist-map (see [5, Chapter 1] for more details).

By analyticity, there is $r > 0$ and a map $R : \mathbb{T}_r \times B(0,b_0) \to \mathbb{C}^2$ such that for any (x,y),

$$R(x,y) = (y^2 R_1(x,y), y^3 R_2(x,y))$$

and $T = A + R$. By shrinking r and b_0, one can assume that the map Ψ^{-1} extends as a diffeomorphism of $\mathbb{T}_r \times B(0,b_0)$ onto its image which is a subset of $\mathbb{T}_r^{|\partial\Omega|} \times B(0,1)$.

We can therefore apply Theorem 3 to T: Given $\tau > 0$, consider the set

$$K(\tau) := \bigcup_{\gamma>0} K(\gamma,\tau) \cap \overline{B(0,b(\gamma))},$$

and the maps $b : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ and $\mathfrak{C} : K(\tau) \to C^\omega\big(\mathbb{T}_{r/2}, \mathbb{T}_r \times B(0,b_0)\big)$ given by Theorem 3. For any $y \in K(\tau)$, denote by $\Gamma(y)$ the map given by

$$\Gamma(y) : \left\{ \begin{array}{ccc} \mathbb{T}_{r/2} & \to & \mathbb{T}_r^{|\partial\Omega|} \times B(0,1) \\ x & \mapsto & \Psi^{-1}(\mathfrak{C}(y)(x)) \end{array} \right.$$

If $\Phi = \Phi(\Gamma)$, then for any (x, y) we have by construction of Γ:

$$T_\Omega \Phi(x,y) = T_\Omega \Psi^{-1}(\mathfrak{C}(y)(x)) = \Psi^{-1} T(\mathfrak{C}(y)(x)) \\ = \Psi^{-1}(\mathfrak{C}(y)(x+y)) = \Phi(x+y, y)$$

and the result follows. □

Proof of Corollary 5. The proof is similar to the one of Corollary 2: the regularity of the maps imposes that Equation 7 still holds, namely

$$\Phi_1^{-1} T_{\Omega_1} \Phi_1 = \Phi_1^{-1} T_{\Omega_2} \Phi_1 + \mathcal{O}\left(y^\infty\right)$$

where $\Phi_1 = \Phi(\Gamma_1)$, and therefore that $T_{\Omega_1} = T_{\Omega_2}$. Now, as shown in [7], when $\varphi \to 0$

$$T_{\Omega_1}(s, \varphi) = \left(s + 2\varrho_1(s)\varphi + \mathcal{O}\left(\varphi^2\right), \varphi + \mathcal{O}\left(\varphi^2\right)\right)$$

and

$$T_{\Omega_2}(s, \varphi) = \left(s + 2\varrho_2(s)\varphi + \mathcal{O}\left(\varphi^2\right), \varphi + \mathcal{O}\left(\varphi^2\right)\right)$$

where ϱ_1 and ϱ_2 are the respective radii of curvature of Ω_1 and Ω_2. Hence $\varrho_1 = \varrho_2$ and the result follows. □

1.3. **Mather's β function.** Let Ω be a strictly convex billiard. Mather's β function is a convex map $\beta_\Omega : [0,1) \to \mathbb{R}$ which can be expressed for any $\omega \in [0,1)$ as

$$\beta_\Omega(\omega) = -\lim_{q \to +\infty} \frac{1}{2q-1} \sum_{j=-q+1}^{q-1} L(s_j, s_{j+1})$$

where $(s_j)_{j \in \mathbb{Z}}$ is a sequence of arc-length coordinates corresponding to an billiard orbit of rotation number ω minimizing the action, and $L(s_j, s_{j+1})$ is the distance between the two impact points of coordinates s_j and s_{j+1}. Roughly, the quantity $\beta_\Omega(\omega)$ expresses the average action of orbits of rotation number ω minimizing the action.

The following result is a version of [2] which can be applied to billiards.

Theorem 6. *Let Ω be a billiard with analytic boundary and $\tau > 0$. Denote by $K(\tau)$ the set introduced in Theorem 4. There is a $\mathscr{C}^\infty$-holomorphic map*

$$\beta^* : K(\tau) \to \mathbb{C}$$

such that $\beta^*_{|[0,1)} = \beta_\Omega$.

Proof. Consider $r, \rho > 0$, as well as the map $b : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ and the set $K(\tau) := \bigcup_{\gamma > 0} K(\gamma, \tau) \cap \overline{B(0, b(\gamma))}$ associated to the KAM curve $\Gamma : K(\tau) \to C^\omega\left(\mathbb{T}_\rho, \mathbb{T}_r^{|\partial\Omega|} \times B(0,1)\right)$ of the billiard in Ω given by Theorem 4.

Write $\Gamma(\omega)(\theta) = (s_\omega(\theta), \varphi_\omega(\theta))$ for any $\omega \in K(\tau)$ and $\theta \in \mathbb{T}_\rho$. If $\omega \in K(\tau) \cap [0,1)$,

$$\Gamma(\omega) : \theta \mapsto (s_\omega(\theta), \varphi_\omega(\theta))$$

restricts to $\mathbb{R}$ as an invariant curve of rotation number ω of the real valued billiard map T_Ω. Hence $\beta_\Omega(\omega)$ admits the following expression

$$\beta_\Omega(\omega) = -\int_0^1 L(s_\omega(\theta), s_\omega(\theta + \omega)) d\theta. \tag{10}$$

This result is a consequence of Birkhoff's ergodic theorem, see [2, Formula (18)] for an analogous formula in the case of the standard map. By analyticity, there is $r' > 0$

such that L is well-defined and analytic on $\left(\mathbb{T}_{r'}^{|\partial\Omega|}\right)^2$. By shrinking ρ we can assume that $r < r'$ so that the right-hand side of formula 10, namely the quantity

$$\beta^*(\omega) = -\int_0^1 L(s_\omega(\theta), s_\omega(\theta + \omega))d\theta,$$

is well-defined also for general $\omega \in K(\tau)$. By composition, β^* is $\mathscr{C}^\infty$-holomorphic and the result follows. $\square$

Corollary 7. *Consider two billiards Ω_1 and Ω_2 with analytic boundary. There exists $\delta \in (0, 1)$ such that if*

$$\beta_{\Omega_1|F} \equiv \beta_{\Omega_2|F}$$

where $F \subset [0, \delta)$ is a positive measure set, then $\beta_{\Omega_1} = \beta_{\Omega_2}$.

2. Billiard and billiard-like maps

The main objects of the paper are what we call billiard-like maps, which are defined as follows:

Definition 8. Let $m \geqslant 2$[2] be an integer. A *billiard-like map* of order m is a map $T : \mathbb{T} \times (-b, b) \to \mathbb{T} \times (-b, b)$, where $b > 0$, of the form (2), where $m \geqslant 2$ is an integer, and satisfying the intersection property.

In fact, m can be made arbitrarily high after an appropriate change of coordinates, as can be deduced from [10, Theorem 3]:

Theorem 9 ([10])**.** *Let $r, b > 0$ and $T : \mathbb{T} \times (-b, b) \to \mathbb{T} \times (-b, b)$ be an analytic billiard-like map of order m which is the restriction of a complex-valued analytic map*

$$T : \mathbb{T}_r \times B(0, b) \to \mathbb{C}^2.$$

For any integer $m' \geqslant m$ and $r' \in (0, r)$, there exist $b' \in (0, b)$ and a real-analytic change of coordinates $\Phi : \mathbb{T}_{r'} \times B(0, b') \to \mathbb{C}^2$ – a diffeomorphism onto its image – such that

$$\Phi^{-1}T\Phi : \mathbb{T}_{r'} \times B(0, b') \to \mathbb{C}^2$$

restricts on $\mathbb{T} \times (-b', b')$ as an analytic billiard-like map of order m'.

The proof works by induction on m, and relies on the idea that if one expands Φ to order $m - 1$ as

$$\Phi(x, y) = (x + y^{m-1}h_1(x) + \ldots, y + y^m h_2(x) + \ldots),$$

then using the intersection property and the form of T given in (2) there is a choice of analytic maps h_1 and h_2 such that

$$\Phi^{-1} \circ T \circ \Phi(x, y) = (x + y + \mathcal{O}\left(y^{m+1}\right), y + \mathcal{O}\left(y^{m+2}\right)).$$

A billiard map T associated to a strictly convex domain with $\mathscr{C}^3$-smooth boundary is – in (s, φ) coordinates – a billiard-like map of order 2: it admits a natural expansion of the form given by (2) with $m = 2$ and it preserves the area form $dsd(-\cos\varphi)$, see [4] for more details. The fact that it is also a billiard-like map of order 3 under a change of coordinates is a consequence of Theorem 9 – for analytic billiards. But this was proven originally by Lazutkin [6], whose statement is reproduced below.

[2] in fact $m = 2$ possible

Theorem 10. *Let Ω be a strictly convex domain with $\mathscr{C}^4$-smooth boundary, whose radius of curvature is denoted by ϱ. Consider the change of cooridnates $\Phi : (s, \varphi) \in \mathscr{C} \mapsto (x, y) \in \mathbb{R}/\mathbb{Z} \times (-1, 1)$ given by*

$$x(s) = C \int_0^s \varrho^{-2/3}(\sigma) d\sigma \qquad \text{and} \qquad y(s, \varphi) = 4C\varrho^{1/3}(s) \sin\left(\frac{\varphi}{2}\right)$$

where $C > 0$ is such that $x(|\partial\Omega|) = 1$. Then the map $\Phi \circ T\Phi^{-1}$ can be expressed as

$$\Phi \circ T\Phi^{-1} : (x, y) \mapsto \left(x + y + \mathcal{O}\left(y^3\right), y + \mathcal{O}\left(y^4\right)\right)$$

when $y \to 0$. Moreover it preserves the area form $ydxdy$, hence has the intersection property.

3. $\mathscr{C}^1$-HOLOMORPHICITY

3.1. **General definition.** Let $K \subseteq \mathbb{C}$ be a closed set without isolated points, $(B, |\cdot|)$ be a Banach space and two maps $f, g : K \to B$. Consider the functions $\delta_f, \Omega_{f,g} : K \times K \to B$ defined, for any $z, z' \in K$, by

$$\delta_f(z, z') = f(z') - f(z),$$

and

$$\Omega_{f,g}(z, z') = \begin{cases} \frac{f(z')-f(z)}{z'-z} - g(z) & \text{if} \quad z \neq z', \\ 0 & \text{if} \quad z = z'. \end{cases}$$

For any $f : U \subseteq \mathbb{C} \to B$, let us denote its sup-norm by

$$|f|_U := \sup_{z \in U} |f(z)|.$$

Definition 11. A map $f : K \to B$ is said to be $\mathscr{C}^1$-*holomorphic* if it is continuous and there exist a continuous map $g : K \to B$, such that $\Omega_{f,g}$ has a continuous extension to $K \times K$.

Since K has no isolated points, there is at most one such g and we denote it by f'. Notice that if f is analytic on an open neighbourhood of K then this map coincides with the restriction to K of the usual complex derivative of f.

Denote by $\mathscr{C}^1(K, B)$ the set of $\mathscr{C}^1$-holomorphic maps from K to B.

Define the norm of f by $\|f\|_{\mathscr{C}^1_{hol}} \in \mathbb{R} \cup \{+\infty\}$ by

$$\|f\|_{\mathscr{C}^1_{hol}} = n_0(f) + n_1(f) + n_2(f),$$

where

$$n_0(f) := |f|_K, \quad n_1(f) := \max\{|f'|_K, |\delta_f|_{K \times K}\}, \quad n_2(f) := |\Omega_{f,f'}|_{K \times K}.$$

Proposition 12. *Endowed with its norm, $\mathscr{C}^1(K, B)$ is a Banach space.*

3.2. **Analyticity versus $\mathscr{C}^1$-holomorphicity.** Let $K \subseteq \mathbb{C}$ be a closed set without isolated points.

A $\mathscr{C}^1$-holomorphic map $f : K \to B$ is necessarily analytic on the interior $\mathrm{int}(K)$, although its derivative $f' : K \to B$ is not necessarily a $\mathscr{C}^1$-holomorphic map.

Conversely, if K is contained in an open set $U \subseteq \mathbb{C}$, and $F : U \to B$ is an analytic map, then $F|_K \in \mathscr{C}^1(K, B)$. Moreover, $F^{(r)}|_K \in \mathscr{C}^1(K, B)$, for any $r \geqslant 0$.

The space of analytic maps $\mathscr{C}^\omega(U, B)$ is complete for the analytic topology which is induced by the sup-norm on every compact of U. However, the restriction of this topology to the space of $\mathscr{C}^1$-holomorphic maps $\mathscr{C}^1(K, B)$ is not complete anymore.

It follows from the properties above that for any $\mathscr{C}^1$-holomorphic map $f : K \to B$ and any $r \geqslant 1$, the restriction of the r-th derivative $f^{(r)}$ of f to any closed set $K' \subseteq \mathrm{int}(K)$ without isolated points is $\mathscr{C}^1$-holomorphic. Let us mention, however, that we will be particularly interested in the regularity properties of these functions when restricted to the real numbers, in particular, when $K \cap \mathbb{R}$ is a subset of the Diophantine numbers (and hence $\mathrm{int}(K) \cap \mathbb{R} = \varnothing$).

In what follows, we will consider sequences of analytic maps $f_n : U_n \subseteq \mathbb{C} \to B$ defined on open subsets U_n of $\mathbb{C}$ all containing the same closed set without isolated points $K \subseteq \cap_n U_n$. We will estimate their $\mathscr{C}^1$-holomorphic norms and consider the behaviour of their restrictions to K, which we denote by $f_n|_K$.

With this in mind, for any $K \subseteq \mathbb{C}$, let us denote

$$K_0 := K, \qquad K_h := \{z \in \mathbb{C} \mid \mathrm{d}(z, K) < h\}, \qquad \text{for any } h > 0,$$

$$K_h^b := K_h \cap B(0, b), \qquad \text{for any } b > 0 \text{ and any } h \geqslant 0,$$

where $B(0, b) \subseteq \mathbb{C}$ denotes the open ball of radius b around 0.

Let us point out that, similarly to the analytic functions setting, estimating the $\mathscr{C}^1$-holomorphic norm of a function on a h-neighbourhood of its domain allows one to estimate the $\mathscr{C}^1$-holomorphic norm of its derivatives. More precisely, we have the following.

Lemma 13. *Let $f \in \mathscr{C}^1(\overline{K_h}, B)$ for some closed set $K \subseteq \mathbb{C}$ without isolated points and some $0 < h < 1$. Then,*

$$\|f^{(r)}\|_{\mathscr{C}^1(K)} \leqslant \frac{C_r}{h^r} \|f\|_{\mathscr{C}^1(\overline{K_h})},$$

for any $r \geqslant 1$, where $C_r > 1$ is a constant depending only on r.

Proof. Fix $r \geqslant 1$. Notice that $f^{(r)}, f^{(r+1)} : K_h \to B$ are analytic. By Cauchy's differentiation formula, there exists $C > 1$, depending only on r, such that

$$\begin{aligned} |f^{(r)}|_K &\leqslant |f^{(r)}|_{K_\delta} \leqslant \frac{C}{(h-\delta)^r} |f|_{K_h}, \\ |f^{(r+1)}|_K &\leqslant |f^{(r+1)}|_{K_\delta} \leqslant \frac{C}{(h-\delta)^r} |f'|_{K_h}, \end{aligned} \tag{11}$$

for any $0 \leqslant \delta < h$.

Notice that for any $z, z' \in K$, we have the following dichotomy. Either $|z - z'| < \frac{h}{2}$ and

$$\begin{aligned}|\Omega_{f^{(r)},f^{(r+1)}}(z,z')| &\leqslant \left|\frac{f^{(r)}(z)-f^{(r)}(z')}{z-z'}\right| + |f^{(r+1)}(z)| \\ &\leqslant |f^{(r+1)}(z'')| + |f^{(r+1)}(z)|,\end{aligned}$$

for some $z'' \in [z, z'] \subseteq K_{h/2}$, where $[z, z']$ denotes the segment joining z and z'. Or, $|z - z'| \geqslant \frac{h}{2}$ and

$$\begin{aligned}|\Omega_{f^{(r)},f^{(r+1)}}(z,z')| &\leqslant \left|\frac{f^{(r)}(z)-f^{(r)}(z')}{z-z'}\right| + |f^{(r+1)}(z)| \\ &\leqslant 2\frac{|f^{(r)}(z)|+|f^{(r)}(z')|}{h} + |f^{(r+1)}(z)|.\end{aligned}$$

Hence, by the previous equations and (11) with $\delta = \frac{h}{2}$, there exists $C' > 1$, depending only on r, such that

$$|\Omega_{f^{(r)},f^{(r+1)}}|_{K\times K} \leqslant \frac{C'}{h^r}|f'|_{K_h}.$$

Therefore,

$$\begin{aligned}\|f^{(r)}\|_{\mathscr{C}^1(K)} &= |f^{(r)}|_K + \max\left\{|f^{(r+1)}|_K, |\delta_{f^{(r)}}|_{K\times K}\right\} + |\Omega_{f^{(r)},f^{(r+1)}}|_{K\times K} \\ &\leqslant |f^{(r)}|_K + \max\left\{\tfrac{C}{h^r}|f'|_{K_h}, 2|f^{(r)}|_K\right\} + \tfrac{C'}{h^r}|f'|_{K_h} \\ &\leqslant \tfrac{3C}{h^r}|f|_{K_h} + \tfrac{C+C'}{h^r}|f^{(r)}|_{K_h} \\ &\leqslant \tfrac{C_r}{h}\|f\|_{\mathscr{C}^1(\overline{K_h})},\end{aligned}$$

where $C_r := 4C + C'$. □

Remark 3. *If the geometry of the set K is not too complicated, for example, if any two points in K can be connected by a piecewise smooth path of length bounded by M, it is possible to obtain bounds of the form $\|f^{(r)}\|_{\mathscr{C}^1(K)} \leqslant \frac{C(r,M)}{h^{r+1}}|f|_{K_h}$, for any $f \in \mathscr{C}^1(\overline{K_h}, B)$ and any $r \geqslant 0$, where $C(r, M)$ is a constant depending only on r and M.*

Similar bounds (with constants depending also on K) can be obtained if K is the union of finitely many connected components having the property mentioned above.

3.3. **Space of 1-periodic maps.** For $r > 0$, we can define the complex torus $\mathbb{T}_r = \{z \in \mathbb{C} \,|\, |\mathrm{Im}\, z| < r\}/\mathbb{Z}$ and the set $B = H_r = H^\omega(\mathbb{T}_r)$ of analytic complex-valued 1-periodic maps with analyticity strip r. To simplify the notations, we will write any $\mathscr{C}^1$-holomorphic map $h \in \mathscr{C}^1(K, H_r)$ in the form

$$h : (x, y) \in \mathbb{T}_r \times K \mapsto h(x, y) \in \mathbb{C}$$

instead of considering the association $h : x \mapsto h(x)$, where $h(x) : y \mapsto h(x)(y)$, as given by the definition.

Let us now define the regularity spaces and the norms associated to the maps we consider. We fix $K \subset \mathbb{C}$ is a closed set with non-empty interior accumulating to 0. It will be made more precise later. Let positive numbers $r, b, h > 0$. The set K_h^b is open, hence we consider analytic maps of the form $\varphi : \mathbb{T}_r \times K_h^b \to \mathbb{C}$, satisfying

$$\varphi(\mathbb{R}/\mathbb{Z} \times (K_h^b \cap \mathbb{R})) \subset \mathbb{R}. \tag{12}$$

We will say that φ is *real analytic*. Since φ is analytic, it is also $\mathscr{C}^1$-holomorphic and we define

$$\|\varphi\|_{r,b,h} = \|\varphi_1\|_{\mathscr{C}^1_{\mathrm{hol}}}.$$

Let us introduce the Banach space of such maps with finite norms:

$$\mathscr{C}^1_{r,b,h}(\mathbb{R}) = \{\varphi \,:\, \|\varphi\|_{r,b,h} < +\infty\}.$$

Given an integer $q > 0$, we can also consider the space of maps with values in $\mathbb{C}^q$

$$\mathscr{C}^1_{r,b,h}(\mathbb{R}^q) = \{\varphi = (\varphi_1, \dots, \varphi_q) \,:\, \|\varphi\|_{r,b,h} < +\infty\}$$

where $\|\varphi\|_{r,b,h} = \max_{j=1,\dots,q} \|\varphi_j\|_{\mathscr{C}^1_{\mathrm{hol}}}$.

Now let us choose an integer $m > 0$ and a map $\varphi : \mathbb{T}_r \times K^b_h \to \mathbb{C}^2$ defined for any (x, y) by

$$\varphi(x, y) = \left(y^m \varphi_1(x, y), y^{m+1}\varphi_2(x, y)\right),$$

where $\varphi_1, \varphi_2 : \mathbb{T}_r \times K^b_h \to \mathbb{C}$ are real analytic maps. The notation φ_1 and φ_2 to refer to the corresponding maps associated to φ will be used all along the paper. Since φ_1 and φ_2 are analytic, they are also $\mathscr{C}^1$-holomorphic and we define

$$\|\varphi\|_{r,b,h,m} = \max(\|\varphi_1\|_{\mathscr{C}^1_{\mathrm{hol}}}, \|\varphi_2\|_{\mathscr{C}^1_{\mathrm{hol}}}).$$

We will drop the index m when there are no ambiguities. The space of real analytic maps defined by

$$\mathscr{C}^1_{r,b,h,m} = \{\varphi \,:\, \|\varphi\|_{r,b,h,m} < +\infty\},$$

together with the norm $\|\cdot\|_{r,b,h,m}$, is a Banach space. In particular if $\varphi \in \mathscr{C}^1_{r,b,h,m}$, then $\varphi_1, \varphi_2 \in \mathscr{C}^1_{r,b,h}(\mathbb{R})$.

In what follows, we are going to solve an equation whose solutions will be defined on a slightly larger set as $\mathbb{T}_r \times K^b_h$. Let us introduce them rigorously. We first consider the linear map $A : \mathbb{C}^2 \to \mathbb{C}^2$ defined for all (x, y) by

$$A(x, y) = (x + y, y).$$

We consider the spaces

$$\mathscr{C}^{1+}_{r,b,h,m} = \{\varphi \in \mathscr{C}^1_{r,b,h,m} \,:\, \varphi \circ A \in \mathscr{C}^1_{r,b,h,m}\}$$

and

$$\mathscr{C}^{1-}_{r,b,h,m} = \{\varphi \in \mathscr{C}^1_{r,b,h,m} \,:\, \varphi \circ A^{-1} \in \mathscr{C}^1_{r,b,h,m}\}$$

together with the norms

$$\|\varphi\|^+_{r,b,h,m} = \max(\|\varphi\|_{r,b,h,m}, \|\varphi \circ A\|_{r,b,h,m}).$$

and

$$\|\varphi\|^-_{r,b,h,m} = \max(\|\varphi\|_{r,b,h,m}, \|\varphi \circ A^{-1}\|_{r,b,h,m}).$$

In a similar way, we can define the spaces $\mathscr{C}^{1+}_{r,b,h}(\mathbb{R})$ and $\mathscr{C}^{1-}_{r,b,h}(\mathbb{R})$ with the corresponding norms $\|\cdot\|^+_{r,b,h}$, $\|\cdot\|^-_{r,b,h}$.

3.4. **Quasianalyticity.** Quasianalyticity is a certain uniqueness property of a subset of functions, which admits different non-equivalent definitions. In this subsection we introduce the notion of quasianalyticity as defined in [9], and we extend Theorem A which can be found in the same paper.

The authors introduce the following notion of quasianalyticity. Denote by $\mathscr{H}^1$ the Hausdorff measure on $\mathbb{C} \simeq \mathbb{R}^2$.

Definition 14 (See Definition 1 in [9])**.** Let B be a complex Banach space, a subset $K \subset \mathbb{C}$, and E be a linear space of maps $K \to B$. The space E is said to be $\mathscr{H}^1$*-quasianalytic* if for any subset $K' \subset K$ of positive Hausdorff measure, the only map of E vanishing identically on K' is the 0 function.

As an example, if K is an open set, then any subspace of the set of analytic maps on U is quasianalytic. In this subsection we prove the two following results:

Proposition 15. *If $U \subset \mathbb{C}$ is a bounded simply connected open set such that ∂U is a Jordan curve. Write $K := \overline{U}$. Then the space $\mathscr{C}^1(K, B)$ is $\mathscr{H}^1$-quasianalytic.*

Corollary 16. *Given $0 < \gamma < 1 \leqslant \tau$ and $b > 0$, write*

$$K = K(\gamma, \tau) \cap B(0, b).$$

the space $\mathscr{C}^1(K, B)$ is $\mathscr{H}^1$-quasianalytic.

Proof of Proposition 15. The proof follows from the proof of Theorem A in [9], and is almost the same. We show that the same setting is the same here and the rest of the proof will follow from [9]. Let $f \in \mathscr{C}^1(K, B)$ and $K' \subset K$ such that $f_{|K'} = 0$. Without loss of generality, after consider compositions of f with linear maps $\ell : B \to \mathbb{C}$, we can assume that $B = \mathbb{C}$. Caratheodory's theorem on conformal mapping implies the existence of a biholomorphic map

$$\varphi : B(0, 1) \to U$$

which extends continuously to a continuous map $\varphi : \overline{B}(0, 1) \to K$. The map $f \circ \phi$ is holomorphic on $B(0, 1)$, continuous on $\overline{B}(0, 1)$ and vanishes on $\gamma = \varphi^{-1}(K')$. The rest of the proof follows from [9]. $\square$

Proof of Corollary 16. Introduce $K^+ = K \cap \{z : \operatorname{Im} z \geqslant 0\}$ and $K^- = K \cap \{z : \operatorname{Im} z \leqslant 0\}$. Since K^+ and K^- intersects on a set of positive Hausdorff measure, it is enough to show that the spaces $\mathscr{C}^1(K^+, B)$ and $\mathscr{C}^1(K^-, B)$ are $\mathscr{H}^1$-quasianalytic. Let us do the proof for $\mathscr{C}^1(K^+, B)$, the proof for $\mathscr{C}^1(K^-, B)$ is the same.

By Proposition 15, it suffices to show that $K^+ = \overline{U}$, where U is an open simply conected set. Indeed, $K^+ = \overline{U}$, where U is the intersection of two simply connected sets of the plane, namely the intersection of $B(0, b) \cap \{z : \operatorname{Im} z \geqslant 0\}$ with $\operatorname{int} K(\gamma, \tau)$. Now U is path-connected: any two point p_0, p_1 can be connected by vertical segments to the boundary of a upper half disk of radius $b - \varepsilon$ for small enough $\varepsilon > 0$. Hence U must be simply connected as a path-connected intersection of two simply connected sets of $\mathbb{R}^2$. Note that this result holds only in the planar case. $\square$

4. Extracted sets and complex diophantine numbers

For any integer $n \neq 0$, consider the map λ_n defined for all $z \in \mathbb{C} \backslash \frac{1}{n}\mathbb{Z}$ by

$$\lambda_n(z) = \frac{z}{e^{2i\pi nz} - 1}. \tag{13}$$

The family formed by all the λ_n's plays a central role in KAM schemes. Notice that the map λ_n is analytic with poles at each rational number of the form m/n with $m \in \mathbb{Z}$. Hence the union of all their poles is dense in $\mathbb{R}$. A classical idea in KAM is to estimates the norms of all the λ_n's on a common subset of $\mathbb{R}$ made of so-called Diophantine numbers, numbers which are far from being rational:

$$D(\gamma, \tau) := \{\omega \in \mathbb{R} \,:\, \forall m/n \in \mathbb{Q} \quad |n\omega - m| \geqslant \gamma |m| n^{-\tau}\} \tag{14}$$

where $\gamma, \tau > 0$ are fixed contants.

A central part of this work, to which this section is devoted, consists of estimating the norms of each λ_n on a common subspace of $\mathbb{C} \backslash \mathbb{Q}$ containing the diophantine numbers $D(\gamma, \tau)$. We show that the norms can be controlled on strictly complex inside cones with vertex on elements of $D(\gamma, \tau)$, defined by

$$D(\gamma, \tau)^{\mathbb{C}} := \{z \in \mathbb{C} \,:\, \exists \omega \in D(\gamma, \tau) \quad |\mathrm{Im}\, z| \geqslant |\mathrm{Re}\, z - \omega|\}. \tag{15}$$

The definition of the set (15) was introduced in [3]; but notice that the set (41) of real Diophantine numbers $D(\gamma, \tau)$ was introduced in [8] and is slightly different than the one in [3] as here 0 is a density point of $D(\gamma, \tau)$ while in [3] $D(\gamma, \tau)$ avoids a neighborhood of 0.

Throughout the paper, $\gamma > 0$ and $\tau > 0$ will be two fixed real numbers and $K = D(\gamma, \tau)^{\mathbb{C}}$.[3]

The main result of this section is the following statement, to be compared to [1, Proposition 6] for a different set.

Proposition 17. *Let $0 < b < 1$, $N \in \mathbb{N}_{>0}$ and $0 < h \leqslant \frac{\gamma}{2\sqrt{2}N^\tau}$. For any $|n| \leqslant N$, the map λ_n given by (13) is $\mathscr{C}^1$-holomorphic on the set K_h^b, and there exists a constant $C_0 > 0$, not depending on any of the parameters, such that*

$$\|\lambda_n\|_{\mathscr{C}^1_{hol}(K_h^b, \mathbb{C})} \leqslant \frac{C_0 n^{2\tau+1}}{\gamma^2}.$$

The proof of Proposition 17 is technical, and to simplify it we will prove intermediate results.

4.1. Approximative diophantine numbers.

Given an integer $N \geqslant 1$, we define

$$D(\gamma, \tau, N) := \{\omega \in \mathbb{R} \,|\, \forall m/n \in \mathbb{Q} \quad |n| \leqslant N \Rightarrow |n\omega - m| \geqslant \gamma |m| n^{-\tau}\}.$$

The construction of $D(\gamma, \tau, N)$ only requires rational numbers with denominators smaller than N; in a bounded region of $\mathbb{R}$, there are only fintely many such rational numbers. For this reason it has non-empty interior. It is important to notice the inclusion $D(\gamma, \tau) \subseteq D(\gamma, \tau, N)$. In particular their extracted sets satisfy the same inclusions $K \subseteq D(\gamma, \tau, N)^{\mathbb{C}}$.

[3] Perhaps move Proposition 7 here? I guess Proposition 5 and Lemma 6 are facts needed in the proof.

Proposition 18. *If* $\gamma' \in (0, \gamma)$ *and* $h \leqslant (\gamma - \gamma')N^{-\tau}/\sqrt{2}$, *then*

$$D(\gamma, \tau)^{\mathbb{C}} \subseteq K_h \subseteq D(\gamma', \tau, N)^{\mathbb{C}}.$$

Proof. The inclusion $D(\gamma, \tau)^{\mathbb{C}} \subseteq K_h$ is quite obvious. For the second inclusion $K_h \subseteq D(\gamma', \tau, N)^{\mathbb{C}}$, we need to define auxiliary sets. Consider the h-neighborhood of $D(\gamma, \tau)$ in $\mathbb{R}$, namely if $D = D(\gamma, \tau)$, set

$$V_h(D) = \{x \in \mathbb{R} \,:\, d(x, D) < h\}$$

and we define the set

$$\tilde{K}_h = \{z \in \mathbb{C} \,:\, \exists x \in V_h(D) \quad |\mathrm{Im}\, z| > |\mathrm{Re}\, z - x|\}.$$

The following result will help us to conclude:

Lemma 19.

$$K_{h/\sqrt{2}} \subset \tilde{K}_h \subset K_h. \tag{16}$$

Proof of Lemma 19. **A FIGURE would be great**

The inclusion $\tilde{K}_h \subset K_h$ is obvious. To prove the other inclusion, let $z \in K_h$. If $z \in D(\gamma, \tau)^{\mathbb{C}}$, then $z \in \tilde{K}_{\sqrt{2}h}$ trivially. If not, let $z' \in D(\gamma, \tau)^{\mathbb{C}}$ and $x' \in D(\gamma, \tau)$ such that

$$|z - z'| \leqslant h \qquad \text{and} \qquad |\mathrm{Im}\, z'| \geqslant |\mathrm{Re}\, z' - x'|.$$

Since $|\mathrm{Im}\, z'| < |\mathrm{Re}\, z' - x'|$, by replacing z' by a complex number on the segment $[z, z']$, we can assume that $|\mathrm{Im}\, z'| = |\mathrm{Re}\, z' - x'|$. Moreover, by minimizing the distance to the lines $L_+ = x' + \mathbb{R}(1 + i)$ and $L_- = x' + \mathbb{R}(1 - i)$, we can assume that z' is the projection of z onto either L_+ or L_-, say L_+. Choose $z_0 \in L_+$ such that $\mathrm{Im}\, z_0 = \mathrm{Im}\, z$. Since L_+ and L_- form an absolute angle of $\pi/4$ with the real line, it follows that the triangle $zz'z_0$ is isoceles with a right angle at z', and by Pythagoras theorem $|zz_0| = \sqrt{2}|zz'| \leqslant \sqrt{2}h$. Now define x as $x = x' + \mathrm{Re}\,(z - z_0)$. It satisfies, by the choice of z_0, the following inequality

$$|\mathrm{Re}\, z - x| = |\mathrm{Re}\, z_0 - x'| \leqslant |\mathrm{Im}\, z_0| = |imz|,$$

and $|x - x'| \leqslant |z - z_0| \leqslant \sqrt{2}h$. $\square$

To finish the proof of Proposition 18, it suffices to show that if $h \leqslant (\gamma - \gamma')N^{-\tau}$, then $\tilde{K}_h \subseteq D(\gamma', \tau, N)^{\mathbb{C}}$. Indeed, if $z \in \tilde{K}_h$, then there is $x \in V_h(E)$ such that $|\mathrm{Re}\, z - x| \leqslant |\mathrm{Im}\, z|$. Let us show that $x \in D(\gamma, \tau, N)$. By construction, there is $x' \in E$ such that$|x' - x| \leqslant h$. Hence for $p/q \in \mathbb{Q}\backslash\{0\}$ with $1 \leqslant q \leqslant N$, we have

$$\left|x - \frac{p}{q}\right| \geqslant \left|x' - \frac{p}{q}\right| - h \geqslant \frac{\gamma|p|}{q^\tau} - \frac{\gamma - \gamma'}{N^\tau}.$$

Since

$$\frac{\gamma - \gamma'}{N^\tau} \leqslant \frac{(\gamma - \gamma')|p|}{q^\tau}$$

we deduce that

$$\left|x - \frac{p}{q}\right| \geqslant \frac{\gamma'|p|}{q^\tau}.$$

From previous inequality, we conclude that $x \in D(\gamma', \tau, N)$, and hence that $z \in D(\gamma', \tau, N)^{\mathbb{C}}$. Whence $\tilde{K}_h \subset D(\gamma', \tau, N)^{\mathbb{C}}$ and by (16), $K_{h/\sqrt{2}} \subset D(\gamma', \tau, N)^{\mathbb{C}}$. The result follows. $\square$

4.2. Proof of Proposition 17.

Lemma 20. *There exist a constant $c > 0$ such that if $z \in \mathbb{C}$ and $\omega \in \mathbb{R}$ satisfy $|Im\, z| \geqslant |Re\, z - \omega|$ then we have the inequality*

$$|e^{2i\pi z} - 1| \geqslant c|\omega - m|$$

where $m \in \mathbb{Z}$ is such that $|\omega - m| = \inf_{m' \in \mathbb{Z}} |\omega - m'|$.

Proof. Let $c_1 > 0$ be the infimum of the map $|e^{2i\pi z} - 1|/|z|$ for $|z| \leqslant 1/2$, and $c_2 > 0$ the infimum of the map $|e^{2i\pi z} - 1|$ for $|\mathrm{Im}\, z| \geqslant 1/2$. Let us now consider a general $z \in \mathbb{C}$, and $m \in \mathbb{Z}$ as in the statement of Lemma 20. In the case when $\mathrm{Im}\, z| \leqslant 1/2$, we can estimate

$$|e^{2i\pi z} - 1| = |e^{2i\pi(z-m)} - 1| \geqslant c_1|z - m|.$$

Now, if one writes $z = x + iy$ with $x, y \in \mathbb{R}$, by assumptions we can compute

$$|z - m|^2 = (x - m)^2 + y^2 \geqslant (x - m)^2 + (x - \omega)^2$$

and the latter expression is minimal when $x = m + \omega$. Thus

$$|z - m|^2 \geqslant \omega^2 + m^2 \geqslant |\omega - m|^2$$

and therefore $|e^{2i\pi z} - 1| \geqslant c_1|\omega - m|$. In the case when $|\mathrm{Im}\, z| \geqslant 1/2$, we can use the fact that $|\omega - m| \leqslant 1/2$ and write

$$|e^{2i\pi z} - 1| \geqslant c_2 \geqslant 2c_2|\omega - m|$$

and the result follow with $c = \min(c_1, 2c_2)$. □

Consider the set $S \subset \mathbb{C}$ defined by

$$S = S_- \cup S_0 \cup S_+ \tag{17}$$

where

$$S_- = \{z = x + iy \,|\, x \geqslant -1/2,\ |y| \geqslant -x - 1/2\}$$
$$S_0 = \{z = x + iy \,|\, -1/2 \leqslant x \leqslant 1/2\}$$
$$S_+ = \{z = x + iy \,|\, x \geqslant 1/2,\ |y| \geqslant x - 1/2\}.$$

Lemma 21. *Given $b > 0$, the map ψ defined for $z \in \mathbb{C}\backslash\mathbb{Z}$ by*

$$\psi(z) = \frac{z}{e^{2i\pi z} - 1}$$

and its derivative are bounded on $S \cap \overline{B}(0, b)$ by a constant $C_S > 0$.

Proof. The map ψ is holomorphic, and 0 is a removable singularity. □

Proof of Proposition 17. Let $n \in \mathbb{Z}$ such that $1 \leqslant |n| \leqslant N$. Given a complex number $z \in K_h^b$, we will distinguish between two cases: i) $nz \in S$ and ii) $nz \notin S$, where S is the set defined in Equation (17). This distinction is due to the following observation on m, the integer from Lemma 20: in case i) m may vanish whereas in case ii) $m \neq 0$.

Notice that for $\gamma' := \gamma/2$, by Proposition 18, $K_h^b \subseteq D(\gamma', \tau, N)^{\mathbb{C}}$, and thus we will use the fact that $z \in D(\gamma', \tau, N)^{\mathbb{C}}$.

Estimates of $n_0(\lambda_n)$. In case i) we write $\lambda_n(z) = n^{-1}\psi(nz)$ where ψ is defined in Lemma 21. Thus $|\lambda_n(z)| \leqslant C_S$. In case ii), let $\omega \in D(\gamma', \tau, N)^{\mathbb{C}}$ such that $|\mathrm{Im}\, z| \geqslant$

$|\mathrm{Re}\, z - \omega|$ and an integer $m \in \mathbb{Z}$ such that $|n\omega - m|$ is minimal. Because $nz \notin S$ we have $m \neq 0$ and hence by Lemma 20

$$|e^{2in\pi z} - 1| \geqslant c|n\omega - m| \geqslant \frac{c\gamma'}{n^\tau} = \frac{c\gamma}{2n^\tau} \tag{18}$$

and therefore

$$|\lambda_n(z)| \leqslant \frac{2bn^\tau}{c\gamma}.$$

<u>*Estimates of* $n_1(\lambda_n)$.</u> The estimates for δ_{λ_n} are straightforward as $|\delta_{\lambda_n}(z,z')| \leqslant 2n_0(\lambda_n)$ for $z, z' \in K_h^b$. The derivative of λ_n can be expressed in case i) as $\lambda_n'(z) = \psi'(nz)$ and hence the latter is bounded by the constant C_S. In the case ii), the same derivative is given by

$$\lambda_n'(z) = \frac{1}{e^{2i\pi nz} - 1} - \frac{2i\pi nze^{2i\pi nz}}{(e^{2i\pi nz} - 1)^2} = \frac{1 - 2i\pi nz}{e^{2i\pi nz} - 1} - \frac{2i\pi nz}{(e^{2i\pi nz} - 1)^2}$$

and by (18) we obtain

$$|\lambda_n'(z)| \leqslant \frac{Cn^{2\tau+1}}{\gamma^2}$$

for some constant $C > 0$.

<u>*Estimates of* $n_2(\lambda_n)$.</u> Let $z, z' \in K_h^b$. Assume first that z, z' both satisfy case i). We can rewrite

$$\Omega_{\lambda_n,\lambda_n'}(z,z') = \frac{\psi(Z') - \psi(Z)}{Z' - Z} - \psi'(Z)$$

where $Z = nz$ and $Z' = nz'$ are in $\overline{B}(0, 1/2)$. This is uniformly bounded by a universal constant depending on the $\mathscr{C}^2$-norm of ψ. In the remaining cases, write $\Omega_{\lambda_n,\lambda_n'}(z,z') = A - \lambda_n'(z)$ where

$$A = \frac{1}{z' - z}\left(\frac{z'}{e^{2i\pi nz'} - 1} - \frac{z}{e^{2i\pi nz} - 1}\right).$$

Since we already have a bound on $|\lambda_n'(z)|$, it is enough to have abound on A. Thus write $A = A_1 - A_2$ where

$$A_1 = \frac{z'e^{2i\pi nz} - ze^{2i\pi nz'}}{(z' - z)(e^{2i\pi nz} - 1)(e^{2i\pi nz'} - 1)}$$

and

$$A_2 = \frac{1}{(e^{2i\pi nz} - 1)(e^{2i\pi nz'} - 1)}.$$

Let us rewrite A_1 as

$$A_1 = -\frac{e^{2i\pi z'} - e^{2i\pi z}}{z' - z}\sum_{k=0}^{n-1} \frac{ze^{2i\pi kz}}{e^{2i\pi nz} - 1}\frac{e^{2i\pi(n-1-k)z'}}{e^{2i\pi nz'} - 1} + \frac{e^{2i\pi nz}}{(e^{2i\pi nz} - 1)(e^{2i\pi nz'} - 1)}$$

hence

$$A = -\frac{e^{2i\pi z'} - e^{2i\pi z}}{z' - z}\sum_{k=0}^{n-1} \frac{ze^{2i\pi kz}}{e^{2i\pi nz} - 1}\frac{e^{2i\pi(n-1-k)z'}}{e^{2i\pi nz'} - 1} + \frac{1}{e^{2i\pi nz'} - 1}.$$

The term $(e^{2i\pi z'} - e^{2i\pi z})/(z' - z)$ is uniformly bounded with a bound depending only on b of order $\mathcal{O}(b)$ as $b \to 0$. If z and z' satisfy case ii), we can use Lemma 20 to bound each term

$$\frac{e^{2i\pi kz}}{e^{2i\pi nz} - 1}\frac{e^{2i\pi(n-1-k)z'}}{e^{2i\pi nz'} - 1}$$

by $Cn^{2\tau}\gamma^{-2}$ with a constant $C > 0$. If z satisfies case i) and z' satisfies case ii), then we can also use the same technique as in the bound for n_0. Since A is unchanged when permuting z and z', the same bounds holds when z satisfies case ii) and z' satisfies case i). □

5. Difference equation

Following the terminology in [6], we call *difference equation* an equation of the form

$$w(x+y,y) - w(x,y) = y \cdot \mathscr{T}_N v(x,y), \tag{19}$$

where v is a given map which is 1-periodic in x and defined in a y-neighborhood of zero. $N > 0$ is an integer and w is a map we need to determine. The symbol $\mathscr{T}_N v$ refers to the map obtained from v by removing all Fourrier modes of rank at least $N+1$, that is

$$\mathscr{T}_N v(x,y) = \sum_{|p|\leqslant N} \widehat{v}_p(y) e^{2i\pi px}.$$

Proposition 22. *Let $0 < r, b, h \leqslant 1$ and $v \in \mathscr{C}^1_{r,b,h}(\mathbb{R})$ such that $\widehat{v}_0 = 0$. Fix an integer $N > 0$ and assume that $h \leqslant \frac{\gamma}{2\sqrt{2}N^\tau}$. Then there is a unique $w \in \mathscr{C}^{1+}_{r,b,h}(\mathbb{R})$ such that $\widehat{w}_0 = 0$ satisfying Equation (19). Moreover, for any $r' \in (0,r)$,*

$$\|w\|^+_{r',b,h} \leqslant \frac{C_1}{(r-r')^{2(\tau+1)}} \|v\|_{r,b,h},$$

where $C_1 = C_1(\gamma,\tau) = \frac{\overline{C_1}(\tau)}{\gamma^2}$ and $\overline{C_1}(\tau) > 1$ is a constant depending only on τ.

Proof. The solution w is given by

$$w(x,y) = \sum_{1\leqslant|n|\leqslant N} \lambda_n(y)\widehat{v}_n(y)e_n(x)$$

where λ_n are defined in Equation (13). Notice that, by Proposition 17 and our assumptions on N and h, this is a well-defined analytic function on $\mathbb{T}_r \times K^h_b$.

Fix $r' \in (0,r)$. Then

$$\|w\|_{r',b,h} \leqslant \sum_{1\leqslant|n|\leqslant N} \|\lambda_n\|_{\mathscr{C}^1(K^b_h,\mathbb{C})} \|v_n\|_{\mathscr{C}^1(K^b_h,\mathbb{C})} e^{2\pi|n|r'}.$$

Now, by [3, Corollary 10],

$$\|\widehat{v}_p\|_{\mathscr{C}^1(K,\mathbb{C})} \leqslant e^{-2\pi|p|r} \|v\|_{\mathscr{C}^1(K,H_r)}.$$

Hence, together with Proposition 17, we obtain

$$\|w\|_{r',b,h} \leqslant \frac{2C_0}{\gamma^2} \|v\|_{r,b,h} \sum_{n=1}^N n^{2\tau+1} e^{-2\pi|n|(r-r')}$$

where C_0 is given in Proposition 17. It follows from Lemma 35, with $\alpha = 2\pi(r-r')$, $\beta = 2\tau+1$ and

$$\overline{C_1} = \frac{2C_0}{(2\pi)^{2(\tau+1)}} C_{2\pi,2\tau+1} \geqslant \frac{2C_0}{(2\pi)^{2(\tau+1)}} C_{\alpha,\beta}$$

that

$$\|w\|_{r',b,h} \leqslant \frac{\overline{C_1}}{\gamma^2(r-r')^{2(\tau+1)}} \|v\|_{r,b,h}.$$

Now note that

$$w \circ A(x,y) = \sum_{1 \leqslant |n| \leqslant N} e_n(y)\lambda_n(y)\widehat{v}_n(y)e_n(x) = -\sum_{1 \leqslant |n| \leqslant N} \lambda_{-n}(y)\widehat{v}_n(y)e_n(x)$$

where we used the equality $e_n(y)\lambda_n(y) = -\lambda_{-n}(y)$ as in [1]. Hence we also conclude that

$$\|w \circ A\|_{r',b,h} \leqslant \frac{\overline{C_1}}{\gamma^2(r-r')^{2(\tau+1)}}\|v\|_{r,b,h},$$

and therefore $w \in \mathscr{C}^{1+}_{r,b,h,m}$. $\square$

Now we will solve a system of equations of the type (19). Let R_1, R_2 be given maps which play the same role as previous v, an integer $N > 0$, and two unknown maps φ_1, φ_2 to be determined according to the relations

$$\left\{ \begin{array}{lcl} \varphi_1(x+y,y) - \varphi_1(x,y) & = & y(\mathscr{T}_N R_1 + \varphi_2(x,y)) \\ \varphi_2(x+y,y) - \varphi_2(x,y) & = & y \cdot \mathscr{T}_N R_2. \end{array} \right. \tag{20}$$

Note that to solve (20), the map R_2 should have zero average in x. In the first equation of (20), by an appropriate choice of φ_2, we can make the right hand-side $\mathscr{T}_N R_1 + \varphi_2$ having zero average.

Corollary 23. *Let $0 < r, b, h \leqslant 1$, $N \in \mathbb{N}_{>0}$ and $R = (R_1, R_2) \in \mathscr{C}^1_{r,b,h}(\mathbb{R}^2)$ such that R_2 has zero average. If $h \leqslant \frac{\gamma}{2\sqrt{2}N^\tau}$, there exist $\varphi = (\varphi_1, \varphi_2) \in \mathscr{C}^{1+}_{r,b,h}(\mathbb{R}^2)$ satisfying (20). Moreover, for any $r' \in (0, r)$,*

$$\|\varphi\|^+_{r',b,h} \leqslant \frac{C_2}{(r-r')^{4(\tau+1)}}\|R\|_{r,b,h},$$

where $C_2 = C_2(\gamma, \tau) = \frac{\overline{C_2}(\tau)}{\gamma^4}$ and $\overline{C_2}(\tau) > 1$ is a constant depending only on τ.

Proof of Corollary 23. Let $r'' = (r + r')/2$. By Proposition 22 we can define $w \in \mathscr{C}^{1+}_{r'',b,h}(\mathbb{R})$ such that $\widehat{w}_0 = 0$ and satisfying

$$w(x+y,y) - w(x,y) = y \cdot \mathscr{T}_N R_2.$$

By construction it satisfies $\mathscr{T}_N w = w$, together with the bound

$$\|w\|^+_{r'',b,h} \leqslant \frac{C_1}{(r-r'')^{2(\tau+1)}}\|R_2\|_{r,b,h},$$

where C_1 is given by Proposition 22. Now consider $\varphi_2 = \mu + w$ where $\mu(y) = -\int_0^1 R_1(x,y)dx$ is the average of $-R_1(\cdot, y)$ for any y. Note that by construction μ is analytic on K^b_h and satisfies $\|\mu\|_{\mathscr{C}^1_{hol}} \leqslant \|R_1\|_{r,b,h}$, whence

$$\|\varphi_2\|^+_{r'',b,h} \leqslant \frac{C_1}{(r-r'')^{2(\tau+1)}}\|R_2\|_{r,b,h} + \|R_1\|_{r,b,h} \leqslant \frac{2^{2(\tau+1)}C_1}{(r-r')^{2(\tau+1)}}\|R\|_{r,b,h} \tag{21}$$

where we used the fact that $C_1 \geqslant 1$ and $r - r' < 1$.

Moreover, the map $R_1 + \varphi_2$ has zero x-average, and again by Proposition 22 there exists $\varphi_1 \in \mathscr{C}^{1+}_{r',b,h}(\mathbb{R})$ satisfying

$$\varphi_1(x+y,y) - \varphi_1(x,y) = y(\mathscr{T}_N R_1 + \varphi_2)$$

and

$$\|\varphi_1\|^+_{r',b,h} \leqslant \frac{C_1}{(r'-r'')^{2(\tau+1)}}\|R_1+\varphi_2\|_{r',b,h} \tag{22}$$
$$\leqslant \frac{2^{2(\tau+1)}C_1}{(r-r')^{2(\tau+1)}}\left(\|R_1\|_{r,b,h}+\|\varphi_2\|_{r',b,h}\right)$$
$$\leqslant \frac{2^{2(\tau+1)}C_1}{(r-r')^{2(\tau+1)}}\|R\|_{r,b,h}+\frac{2^{4(\tau+1)}{C_1}^2}{(r-r')^{4(\tau+1)}}\|R\|_{r,b,h}$$

where the last inequality follows from the fact that $C_1 \geqslant 1 > r-r'$ and Equation (21). □

6. KAM Step

In this section, we describe in detail the iterative KAM step on which the proof of Theorem 1 will rely. For the sake of clarity, we start by introducing the notations and providing an outline of the construction. Precise estimates are formally stated in Proposition 24 and are proven at the end of the section.

Suppose that we are given a map $T : \mathbb{T}_r \times K_h^b \to \mathbb{C}/\mathbb{Z} \times \mathbb{C}$ satisfying the intersection property of the form

$$T = A + R$$

where A is the linear map given for any $(x, y) \in \mathbb{C}/\mathbb{Z} \times \mathbb{C}$ by

$$A(x, y) = (x + y, y)$$

and $R \in \mathscr{C}^1_{r,b,h,m}$, with fixed real numbers $0 < b, r, h \leqslant 1$ and $m \in \mathbb{N}_{\geqslant 2}$.

We describe a procedure to construct a diffeomorphism $\Phi = I + \varphi$ and estimate the norm of the map $T^+ = \Phi^{-1}T\Phi : \mathbb{T}_{r'} \times K_{b'}^{h'} \to \mathbb{C}/\mathbb{Z}\times\mathbb{C}$, for some $r' \in (0, r)$, $b' \in (0, b)$ and $h' \in (0, h)$. We will construct Φ, then explain in which sense the composition $\Phi^{-1}T\Phi$ is well-defined and analytic, and finally give estimates of its $\mathscr{C}^1$-holomorphic norm.

Fix an integer $N > 0$ and let us assume that

$$h \leqslant \tfrac{\gamma}{2N^{\tau+1}}.$$

Let $\sigma \in (0, r/4)$, $\overline{\sigma} \in (0, \min(b, h)/3)$ and $\kappa \in (0, 1)$. Suppose that

$$\|R\|_{r,b,h} \leqslant M\sigma^{4(\tau+1)} \min(\sigma, \overline{\sigma}) \tag{23}$$

with

$$M = \frac{(1-\kappa)^2\kappa}{2(m-1)(2K + C_2)} > 0,$$

where C_2 is as in Corollary 23 and K is as in Proposition 32 with $\kappa_1 = \kappa = \kappa_2$, namely $K = (1-\kappa)^{-2}$. Let

$$\varphi \in \mathscr{C}^{1+}_{r-\sigma,h,b,m-1}$$

satisfying the difference equation

$$\varphi A - A\varphi = \mathscr{T}_N R, \tag{24}$$

which is equivalent to (20) ($\mathscr{T}_N R$ is the trigonometric polynomial with the same Fourier coefficients as R except the ones of order $> N$ which are set to be zero).

The existence of φ is guaranteed by Corollary 23 since R_2 has zero average as ensured by the intersection property of T. We consider the map

$$\Phi = I + \varphi : \mathbb{T}_{r-\sigma} \times K_b^h \to \mathbb{C}/\mathbb{Z} \times \mathbb{C}.$$

Notice that if the C^1-norm of φ is smaller than 1 (in particular, if its $\mathscr{C}^1$-holomorphic norm verifies this), then Φ is a diffeomorphism onto its image. Let us assume for the moment that this is the case, as this will be a consequence of the smallness condition on R together with Corollary 23, and let us consider the inverse of Φ. Condition (23) together with the estimates given in Corollary 23 imply that Assumption (41) of Proposition 32 is satisfied, hence there exists

$$\psi \in \mathscr{C}^{1+}_{r-2\sigma,h-\overline{\sigma},b-\overline{\sigma},m-1}$$

given by Proposition 32 and such that

$$\Phi^{-1} = I + \psi : \mathbb{T}_{r-2\sigma} \times K_{b-\overline{\sigma}}^{h-\overline{\sigma}} \to \mathbb{C}/\mathbb{Z} \times \mathbb{C}.$$

Let us show that the composition $\Phi^{-1}T\Phi$ is well-defined and analytic on the set $\mathbb{T}_{r-3\sigma} \times K_{b-2\overline{\sigma}}^{h-2\overline{\sigma}}$. First, the composition

$$R \circ \Phi : \mathbb{T}_{r-2\sigma} \times K_{b-\overline{\sigma}}^{h-\overline{\sigma}} \to \mathbb{C}^2$$

is well-defined, analytic, and belongs to the space $\mathscr{C}^1_{r-2\sigma,h-\overline{\sigma},b-\overline{\sigma},m}$ as ensured by Proposition 30. Indeed

$$\|\varphi\|_{\mathscr{C}^1_{hol}} \leqslant 2(m-1)b^{m-2}\|\varphi\|_{r-\sigma,b,h,m-1};$$

now the estimates given in Corollary 23 and Condition (23) imply that $\|\varphi\|_{\mathscr{C}^1_{hol}} \leqslant \kappa \min(\sigma, \overline{\sigma})$ and Proposition 30 can be applied to $R \circ \Phi$.

It remains to show that

$$\Phi^{-1}T\Phi : \mathbb{T}_{r-3\sigma} \times K_{b-2\overline{\sigma}}^{h-2\overline{\sigma}} \to \mathbb{C}/\mathbb{Z} \times \mathbb{C}$$

is well-defined and analytic. It suffices to prove it for

$$\psi T\Phi = \psi \circ A(I + \varphi + A^{-1}R\Phi).$$

We first have

$$\|\varphi + A^{-1}R\Phi\|_{\mathscr{C}^1_{hol}} \leqslant 2(m-1)b^{m-2}\|\varphi + A^{-1}R\Phi\|_{r-2\sigma,b-\overline{\sigma},h-\overline{\sigma},m-1}.$$

Now under previous assumptions, Proposition 30 guarantees that

$$\|R \circ \Phi\|_{r-2\sigma,b-\overline{\sigma},h-\overline{\sigma},m} \leqslant K\|R\|_{r,b,h,m}$$

and it follows from Corollary 23 and Condition (23) that

$$\|\varphi + A^{-1}R\Phi\|_{\mathscr{C}^1_{hol}} \leqslant \kappa \min(\sigma, \overline{\sigma}).$$

Applying Proposition 30, this is enough to ensure that $\psi T\Phi$ is analytic on the set $\mathbb{T}_{r-3\sigma} \times K_{b-2\overline{\sigma}}^{h-2\overline{\sigma}}$. [4] As a result

$$T^+ : \mathbb{T}_{r-3\sigma} \times K_{b-2\overline{\sigma}}^{h-2\overline{\sigma}} \to \mathbb{C}/\mathbb{Z} \times \mathbb{C}$$

is well-defined. Denoting

$$T^+ = A + R^+,$$

and using the procedure above, we can deduce the following result.

[4] Here it is essential that ψ AND $\psi \circ A$ belong to the space $\mathscr{C}^1_{r-2\sigma,b-\overline{\sigma},h-\overline{\sigma},m-1}$.

Proposition 24. *Let $b, r, h, m, \sigma, \overline{\sigma}, N, R$ as above. There exist constants $c_* = c_*(\gamma, \tau, m) = \frac{\gamma^4}{(m-1)\overline{C_3}(\tau)}$ and $C_4 = C_4(\gamma, \tau) = \frac{\overline{C_4}(\tau)}{\gamma^8}$, where $\overline{C_3}(\tau), \overline{C_4}(\tau) > 1$ are constants depending only on τ, such that if the map $R \in \mathscr{C}^1_{r,b,h,m}$ satisfies*

$$\|R\|_{r,b,h} \leqslant c_* \sigma^{4(\tau+1)} \min(\sigma, \overline{\sigma}). \tag{25}$$

then there exists a diffeomorphism onto its image

$$\Phi : \mathbb{T}_{r-\sigma} \times K_b^h \to \mathbb{C}/\mathbb{Z} \times \mathbb{C}$$

such that R^+, given by $A + R^+ = \Phi^{-1} T_R \Phi$, is an element of $\mathscr{C}^1_{r-3\sigma, b-2\overline{\sigma}, h-2\overline{\sigma}, m}$ satisfying

$$\|R^+\|_{r-3\sigma, b-2\overline{\sigma}, h-2\overline{\sigma}} \leqslant \frac{e^{-2\pi N \sigma}}{\pi \sigma} \|R\|_{r,b,h} + \frac{C_4}{\sigma^{8(\tau+1)}} \left(\frac{1}{\sigma} + \frac{1}{\overline{\sigma}} \right) \|R\|^2_{r,b,h}.$$

Moreover, we can write the diffeomorphism Φ as $\Phi = I + \varphi$, where φ is an element of $\mathscr{C}^{1+}_{r-\sigma,b,h,m-1}$ satisfying

$$\|\varphi\|_{r-\sigma,b,h} \leqslant \frac{C_2}{\sigma^{4(\tau+1)}} \|R\|_{r,b,h},$$

and its inverse Φ^{-1} as $\Phi^{-1} = I + \psi$, where ψ is an element of $\mathscr{C}^{1+}_{r-2\sigma, b-\overline{\sigma}, h-\overline{\sigma}, m-1}$ satisfying

$$\|\psi\|_{r-2\sigma, b-\overline{\sigma}, h-\overline{\sigma}, m-1} \leqslant \frac{4C_2}{\sigma^{4(\tau+1)}} \|R\|_{r,b,h},$$ [5]

where C_2 is given by Corollary 23.

Proof. We will fix $\kappa = \frac{1}{2}$ in the outline given at the beginning of this section and define $c_* = \min\left\{ \frac{\kappa}{6C_2(m-1)}, \frac{\kappa}{6K(m-1)} \right\} > 0$,[6] where the constants C_2 and K are given in Corollary 23 and Proposition 30 with $\kappa_1 = \kappa_2 = \kappa$.

The existence of φ and ψ as well as their bounds are a consequence of Corollary 23 and Proposition 32. Similarly, it was already shown that the composition $\Phi^{-1} T \Phi$ is well defined on the desired domain. We do not give more details about it as this was already discussed at the beginning of the section.

Thus, it remains to prove the part of Proposition 24 devoted to the estimates of R^+. Following the ideas of Martin, Ramirez-Ros and Sarol in [11], we consider the expansion

$$R^+ = R_a^+ + R_b^+ + R_c^+ + R_d^+,$$

where

$$R_a^+ = R - \mathscr{T}_N R, \qquad R_b^+ = \varphi A - \varphi T \Phi, \qquad R_c^+ = R\Phi - R,$$
$$R_d^+ = (\varphi + \psi) T \Phi.$$

The previous expansion holds since the right hand-side denoted by $\tilde{R}$ simplifies as

$$\tilde{R} = \varphi A - \mathscr{T}_N R + R\Phi + \psi T \Phi$$

and by construction of φ, the term $\varphi A - \mathscr{T}_N R$ can be replaced by $A\varphi$, hence

$$\tilde{R} = A\varphi + R\Phi + \psi T\Phi = T\Phi - A + \psi T \Phi = \Phi^{-1} T \Phi - A = R^+.$$

For each $x \in \{a, b, c, d\}$, let us prove that R_x^+ lies in $\mathscr{C}^1_{r',b',h',m}$ and estimate its norm. The proof is described in the next subsections.

[5] I replaced $K = (1-\kappa)^{-2}$ by 4 since we will assume $\kappa = 1/2$.

[6] Why we do not define it as M above?

6.1. **Estimates of $\|R_a^+\|_{r-\sigma,b,h}$.** Lemma 29 implies that

$$\|R_a^+\|_{r-\sigma,b,h} \leqslant \frac{e^{-2\pi N\sigma}}{\pi\sigma}\|R\|_{r,b,h}.$$

6.2. **Estimates of $\|R_b^+\|_{r-2\sigma,b-\overline{\sigma},h-\overline{\sigma}}$.** We write $-R_b^+$ as

$$-R_b^+ = f(I+\varphi+A^{-1}R\Phi)-f$$

where $f = \varphi\circ A$ is an element of $\mathscr{C}^1_{r-\sigma,b,h,m-1}$ by construction of φ (Corollary 23), and A^{-1} is given by $A^{-1}(x,y)=(x-y,y)$.

We first give estimates on the map $\varphi+A^{-1}R\Phi$:

(1) φ is an element of $\mathscr{C}^1_{r-\sigma,b,h,m-1}$ and Assumption 25 together with Corollary 23 imply that

$$\|\varphi\|_{\mathscr{C}^1_{hol}} \leqslant 2(m-1)b^{m-2}\|\varphi\|_{r-\sigma,b,h,m-1} \leqslant \frac{1}{3}\kappa\min(\sigma,\overline{\sigma}).$$

(2) Using previous estimates we can apply Proposition 30 to $R\Phi$ with $\kappa_1=\kappa_2=\kappa$ to obtain that $R\Phi\in\mathscr{C}^1_{r-2\sigma,b-\overline{\sigma},h-\overline{\sigma},m}$ with

$$\|R\Phi\|_{r-2\sigma,b-\overline{\sigma},h-\overline{\sigma}} \leqslant K\|R\|_{r,b,h} \leqslant \frac{1}{3(m-1)}\kappa\min(\sigma,\overline{\sigma}).$$

The two previous points imply that the map $\varphi+A^{-1}R\Phi$ is an element of $\mathscr{C}^1_{r-2\sigma,b-\overline{\sigma},h-\overline{\sigma},m-1}$ and

$$\|\varphi+A^{-1}R\Phi\|_{\mathscr{C}^1_{hol}} \leqslant 2(m-1)b^{m-2}\|\varphi+A^{-1}R\Phi\|_{r-2\sigma,b-\overline{\sigma},h-\overline{\sigma}} \leqslant \kappa\min(\sigma,\overline{\sigma})$$

Hence by Proposition 31 with $\kappa_1=\kappa_2=\kappa$, $R_b^+\in\mathscr{C}^1_{r-2\sigma,b-\overline{\sigma},h-\overline{\sigma},m}$ and

$$\begin{aligned}\|R_b^+\|_{r-2\sigma,b-\overline{\sigma},h-\overline{\sigma}} \leqslant\ & K\|f\|_{r-\sigma,b,h}\frac{\|\varphi_1+(A^{-1}R\Phi)_1\|_{r-2\sigma,b-\overline{\sigma},h-\overline{\sigma}}}{\sigma}\\ &+K\|f\|_{r-\sigma,b,h}\frac{\|\varphi_2+(A^{-1}R\Phi)_2\|_{r-2\sigma,b-\overline{\sigma},h-\overline{\sigma}}}{\overline{\sigma}}\\ \leqslant\ & K\|f\|_{r-\sigma,b,h}\frac{\|\varphi_1\|_{r-2\sigma,b-\overline{\sigma},h-\overline{\sigma}}+2K\|R\|_{r,b,h}}{\sigma}\\ &+K\|f\|_{r-\sigma,b,h}\frac{\|\varphi_2\|_{r-2\sigma,b-\overline{\sigma},h-\overline{\sigma}}+K\|R\|_{r,b,h}}{\overline{\sigma}}.\end{aligned}$$

It then follows from Proposition 23 that

$$\|R_b^+\|_{r-2\sigma,b-\overline{\sigma},h-\overline{\sigma}} \leqslant KC_2(C_2+2K)\frac{\|R\|^2_{r,b,h}}{\sigma^{8(\tau+1)}}\left(\frac{1}{\sigma}+\frac{1}{\overline{\sigma}}\right).$$

6.3. **Estimates of $\|R_c^+\|_{r-\sigma,b,h}$.** φ is an element of $\mathscr{C}^1_{r-\sigma,b,h,m-1}$ and Assumption 25 together with Corollary 23 imply that

$$\|\varphi\|_{\mathscr{C}^1_{hol}} \leqslant \kappa\min(\sigma,\overline{\sigma}).$$

Hence we can apply Proposition 31 and then the estimates given in Corollary 23 to obtain

$$\begin{aligned}\|R_c^+\|_{r-\sigma,b,h} &\leqslant K\|R\|_{r,b,h}\left(\frac{\|\varphi_1\|_{r-\sigma,b,h}}{\sigma}+\frac{\|\varphi_2\|_{r-\sigma,b,h}}{\overline{\sigma}}\right)\\ &\leqslant KC_2\frac{\|R\|^2_{r,b,h}}{\sigma^{4(\tau+1)}}\left(\frac{1}{\sigma}+\frac{1}{\overline{\sigma}}\right).\end{aligned}$$

6.4. **Estimates of $\|R_d^+\|_{r-3\sigma,b-2\overline{\sigma},h-2\overline{\sigma}}$.** Let us write R_d^+ as

$$R_d^+ = (\varphi + \psi) \circ A(I + \varphi + A^{-1}R\Phi).$$

As discussed before for R_b^+, $\varphi + A^{-1}R\Phi$ is an element of $\mathscr{C}^1_{r-2\sigma,b-\overline{\sigma},h-\overline{\sigma},m-1}$ and

$$\|\varphi + A^{-1}R\Phi\|_{\mathscr{C}^1_{hol}} \leqslant \kappa \min(\sigma, \overline{\sigma}).$$

By Proposition 32, $\phi + \psi \in \mathscr{C}^{1+}_{r-2\sigma,b-\overline{\sigma},h-\overline{\sigma},2m}$ with

$$\|\varphi + \psi\|^+_{r-2\sigma,b-\overline{\sigma},h-\overline{\sigma},2m} \leqslant K^2(\|\varphi\|^+_{r-\sigma,b,h,m})^2 \left(\frac{1}{\sigma} + \frac{1}{\overline{\sigma}}\right).$$

Hence by Proposition 30, R_d^+ is an element of $\mathscr{C}^1_{r-3\sigma,b-2\overline{\sigma},h-2\overline{\sigma},m}$ and

$$\|R_d^+\|_{r-3\sigma,b-2\overline{\sigma},h-2\overline{\sigma},2m} \leqslant K^3(\|\varphi\|^+_{r-\sigma,b,h,m})^2 \left(\frac{1}{\sigma} + \frac{1}{\overline{\sigma}}\right).$$

Using the estimates on φ given in Corollary 23, we obtain

$$\|R_d^+\|_{r-3\sigma,b-2\overline{\sigma},h-2\overline{\sigma},2m} \leqslant K^3{C_2}^2 \frac{\|R\|^2_{r,b,h}}{\sigma^{8(\tau+1)}} \left(\frac{1}{\sigma} + \frac{1}{\overline{\sigma}}\right).$$

This completes the proof of Proposition 24. □

7. Proof of Theorem 1

This whole section concerns the proof of Theorem 1.

Let $0 < r \leqslant 1$ and $0 < b \leqslant r^{2(\tau+1)}$. Define

$$\nu = 16, \qquad \lambda(\gamma, \tau) = \frac{c_*}{6^{\nu(\tau+1)}C_6^2},$$

where $C_6 = C_6(\gamma, \tau) \geqslant 2$ is a constant, depending only on γ, τ, satisfying (29) and (30). We will assume WLOG that

$$c_*^{1/2} \leqslant \min\left\{C_4^{-1}, 2^{(\nu+2)(\tau+1)+1}, 4C\right\}, \tag{26}$$

where C_4 and C are given by Propositions 24 and 31, respectively.

By defining an appropriate iterative scheme relying on Proposition 24, we will show that the conclusions of Theorem 1 hold for any $R \in \mathscr{C}^1_{b,r,3}$ satisfying

$$\|R\|_{b,r,3} \leqslant \lambda b r^{\nu(\tau+1)}.$$

Define

$$\sigma = \frac{r}{6}, \qquad \overline{\sigma} = \frac{b}{C_6^2},$$

and

$$\sigma_n = \frac{\sigma}{2^n}, \qquad \overline{\sigma}_n = \frac{\overline{\sigma}}{2^{2n(\tau+1)}},$$

for any $n \geqslant 0$. Notice that with these definitions $\overline{\sigma} \leqslant \sigma^{\tau+1}$ and

$$\frac{\overline{\sigma}^{1/2}}{\sigma^{\tau+1}} \leqslant \frac{6^{\nu(\tau+1)}}{C_6}. \tag{27}$$

Let $(r_n)_{n\geqslant 0}$, $(b_n)_{n\geqslant 0}$, and $(h_n)_{n\geqslant 0}$ be the decreasing sequences of (positive) real numbers given by

$$r_{n+1} = r_n - 3\sigma_n, \quad b_{n+1} = b_n - 2\overline{\sigma}_n, \quad h_{n+1} = h_n - 2\overline{\sigma}_n,$$

for any $n \geqslant 0$, with $r_0 = r$, $b_0 = b$ and $h_0 = \frac{2\overline{\sigma}}{1-\frac{1}{2^{2(\tau+1)}}}$.

Notice that with such a choice of h_0, the sequence $(h_n)_{n\geqslant 0}$ satisfies

$$h_n = \frac{h_0}{2^{2(\tau+1)n}}, \qquad \text{for any } n \geqslant 0. \tag{28}$$

Let

$$\varepsilon = c_*\overline{\sigma}\sigma^{\nu(\tau+1)}, \qquad \varepsilon_n = \varepsilon^{(3/2)^n}$$

and define

$$N_n = \left\lfloor \frac{|\log(\pi C_4 \varepsilon_n)|}{2\pi\sigma_n} \right\rfloor,$$

for any $n \geqslant 0$, with C_4 as in Proposition 24. We assume that C_6 is large enough so that

$$\frac{\pi C_4 c_*}{C_6^2} \leqslant e^{-4\pi}, \tag{29}$$

and thus $N_n \geqslant 2$, for any $n \geqslant 0$.

As we would like to apply Proposition 24 using the sequences defined above, let us start by proving some simple properties for them.

Lemma 25. *The sequences defined above satisfy the following.*

(1) $r_n, b_n, h_n > 0$, *for any* $n \geqslant 0$. *Moreover,*

$$\lim_{n\to\infty} r_n = \tfrac{r}{2}, \qquad \lim_{n\to\infty} b_n \geqslant \tfrac{b}{2}, \qquad \lim_{n\to\infty} h_n = 0.$$

(2) $\varepsilon_n \leqslant c_*\overline{\sigma}_n\sigma_n^{\nu(\tau+1)}$, *for any* $n \geqslant 0$.

(3) $h_n \leqslant \frac{\gamma}{2N_n^{\tau+1}}$, *for any* $n \geqslant 0$.

(4) *For any* $a, L > 0$,

$$\lim_{n\to\infty} \frac{\varepsilon_n^a}{\overline{\sigma}_n^L} \to 0.$$

Proof. The first assertion follows directly from the definitions.

To prove the second assertion we proceed by induction. Notice that by definition $\epsilon_0 = c_*\overline{\sigma}\sigma^{\nu(\tau+1)}$. Now, assume that for some $n \geqslant 0$ we have $\varepsilon_n \leqslant c_*\overline{\sigma}_n\sigma_n^{\nu(\tau+1)}$. Then, since $\varepsilon_{n+1} = \varepsilon_n^{3/2}$, we have

$$\frac{\varepsilon_{n+1}}{c_*\overline{\sigma}_{n+1}\sigma_{n+1}^{\nu(\tau+1)}} \leqslant \frac{2^{(\nu+2)(\tau+1)}c_*^{3/2}\overline{\sigma}_n^{3/2}\sigma_n^{3\nu(\tau+1)/2}}{c_*\overline{\sigma}_n\sigma_n^{\nu(\tau+1)}} \leqslant c_*^{1/2}2^{(\nu+2)(\tau+1)} \leqslant 1,$$

where in the last inequality we used (26).

Finally, for any $n \geqslant 0$, we have

$$\begin{aligned}
\frac{h_n N_n^{\tau+1}}{\gamma} &\leqslant \frac{h_0}{\gamma 2^{2(\tau+1)n}} \frac{\left(\frac{3}{2}\right)^{n(\tau+1)} \left|\log \pi C_4 c_* \sigma^{\nu(\tau+1)}\overline{\sigma}\right|^{\tau+1}}{\sigma_n^{\tau+1}} \\
&\leqslant \frac{\overline{\sigma}}{\sigma^{\tau+1}} \frac{\left(\frac{3}{2}\right)^{n(\tau+1)} \left(|\log c_*| + |\log \sigma^{\nu(\tau+1)}\overline{\sigma}|\right)^{\tau+1}}{\gamma 2^{n(\tau+1)}} \\
&\leqslant \frac{6^{\nu(\tau+1)}}{\gamma C_6} \left(\frac{3}{4}\right)^{n(\tau+1)} \left(|\log c_*| + 2\overline{\sigma}^{1/2(\tau+1)}|\log \overline{\sigma}|\right)^{\tau+1},
\end{aligned}$$

where in the last line we use (27) and the fact that $\overline{\sigma} \leqslant \sigma^{\tau+1} < 1$.

Notice that $t \mapsto t^{1/2(\tau+1)}|\log t|$ has a local maximum on $(0,1)$ at $t = e^{-2(\tau+1)}$ with value $(\tau+1)e^{-1}$. Thus, assuming that C_6 is sufficiently large, we have

$$\frac{h_n N_n^{\tau+1}}{\gamma} \leqslant \frac{6^{\nu(\tau+1)}}{\gamma C_6}\left(\frac{3}{4}\right)^{n(\tau+1)} \left(|\log c_*| + 2(\tau+1)e^{-1}\right)^{\tau+1} \leqslant 1, \tag{30}$$

where we used the fact that the RHS in the equation above goes to 0 as n goes to infinity. □

Let us denote $R_0 = R$. We claim that by repeteadly applying Proposition 24 to the sequences defined above, we can define sequences $(R_n)_{n\geqslant 0}$, and $(\varphi_n)_{n\geqslant 0}$ of maps such that for any $n \geqslant 0$

$$R_n \in \mathscr{C}^1_{r_n,b_n,h_n,m}, \qquad \varphi_n \in \mathscr{C}^{1+}_{r_n-\sigma_n,b_n,h_n,m-1},$$

the map $\Phi_n := I + \varphi_n$ is a diffeomorphism onto its image, whose inverse can be written as $\Phi_n^{-1} := I + \psi_n$ such that $\psi_n \in \mathscr{C}^{1+}_{r_n-2\sigma_n,b_n-\overline{\sigma}_n,h_n-\overline{\sigma}_n,m-1}$ and

$$A + R_{n+1} = \Phi_n^{-1}(A + R_n)\Phi_n. \tag{31}$$

Moreover, these maps satisfy the inequalities

$$\|R_n\|_{r_n,b_n,h_n} \leqslant \varepsilon_n, \tag{32}$$

$$\|\varphi_n\|_{r_n-\sigma_n,b_n,h_n} \leqslant \overline{\sigma}_n \varepsilon_n^{1/2} \tag{33}$$

$$\|\psi_n\|_{r_n-2\sigma_n,b_n-\overline{\sigma}_n,h_n-\overline{\sigma}_n} \leqslant \overline{\sigma}_n \varepsilon_n^{1/2} \tag{34}$$

for any $n \geqslant 0$.

We will prove by induction that this is indeed possible.

Notice that the initial hypothesis on R implies that

$$\|R_0\|_{r_0,b_0,h_0} \leqslant \varepsilon_0 = c_* \overline{\sigma}_0 \sigma_0^{\nu(\tau+1)},$$

and thus Proposition 24 can be applied with $N = N_0$.

Let us assume, for the sake of induction, that for some $n \geqslant 0$ the map $R_n \in \mathscr{C}^1_{r_n,b_n,h_n,m}$ has been defined and satisfies $\|R_n\|_{r_n,b_n,h_n} \leqslant \varepsilon_n$. We will show that then equations (33), (34) are satisfied and that we can define $R_{n+1} \in \mathscr{C}^1_{r_{n+1},b_{n+1},h_{n+1},m}$ (using Proposition 24) so that $\|R_{n+1}\|_{r_{n+1},b_{n+1},h_{n+1}} \leqslant \varepsilon_{n+1}$.

By Lemma 25, Proposition 24 can be applied to R_n with $N = N_n$, which leads to maps $\varphi_n \in \mathscr{C}^{1+}_{r_n-\sigma_n,b_n,h_n,m-1}$ and $\psi_n \in \mathscr{C}^{1+}_{r_n-2\sigma_n,b_n-\overline{\sigma}_n,h_n-\overline{\sigma}_n,m-1}$ such that $\Phi_n = \mathrm{Id} + \varphi_n$ on $\mathbb{T}_{r_n-\sigma} \times K^{h_n}_{b_n}$ is a diffeomorphism onto its image and its inverse Φ_n^{-1} on $\mathbb{T}_{r_n-2\sigma} \times K^{h_n-\overline{\sigma}_n}_{b_n-\overline{\sigma}_n}$ is given by $\Phi_n^{-1} = \mathrm{Id} + \psi_n$. Moreover, these maps satisfy

$$\|\varphi_n\|_{r_n-\sigma_n,b_n,h_n} \leqslant \frac{C_2}{\sigma_n^{4(\tau+1)}}\|R_n\|_{r_n,b_n,h_n} \leqslant \overline{\sigma}_n \varepsilon_n^{1/2}.$$

$$\|\psi_n\|_{r_n-2\sigma_n,b_n-\overline{\sigma}_n,h_n-\overline{\sigma}_n} \leqslant \frac{4C_2}{\sigma_n^{4(\tau+1)}}\|R_n\|_{r_n,b_n,h_n} \leqslant \overline{\sigma}_n \varepsilon_n^{1/2}.$$

By defining R_{n+1} as the map on $\mathbb{T}_{r_{n+1}} \times K^{h_{n+1}}_{b_{n+1}}$ satisfying (31), by Proposition 24, we have the following bounds.

$$\begin{aligned}
\|R_{n+1}\|_{r_{n+1},b_{n+1},h_{n+1}} &\leqslant \frac{e^{-2\pi N_n\sigma_n}}{\pi\sigma_n}\|R_n\|_{r_n,b_n,h_n} + \frac{C_4}{\sigma_n^{8(\tau+1)}}\left(\frac{1}{\sigma_n}+\frac{1}{\overline{\sigma}_n}\right)\|R_n\|^2_{r_n,b_n,h_n}.\\
&\leqslant \varepsilon_n^2\frac{2C_4}{\sigma_n^{8(\tau+1)}\overline{\sigma}_n} = \varepsilon_{n+1}\varepsilon_n\frac{2^{(\nu+2)(\tau+1)+1}C_4}{\sigma_n^{8(\tau+1)}\overline{\sigma}_n}\\
&\leqslant \varepsilon_{n+1}c_*2^{(\nu+2)(\tau+1)+1}C_4\\
&\leqslant \varepsilon_{n+1},
\end{aligned}$$

where in the last inequality we used (26).

We will now consider the composition of the maps $(\Phi_n)_{n\geqslant 0}$ constructed above.

In the following, let us denote $K^{(\infty)} := \bigcap_{n\geqslant 0} K^{b_n}_{h_n} = K^{b_\infty}_0$. Notice that

$$K^{(\infty)}_{\overline{\sigma}_{n+1}} \subseteq K^{b_n}_{h_n}, \qquad \text{for any } n \geqslant 0. \tag{35}$$

Lemma 26. *Denote*

$$\Phi^{(n+1)} := \Phi_1\circ\ldots\circ\Phi_{n+1} : \mathbb{T}_{r_n-\sigma_n}\times K^{b_n}_{h_n} \to \mathbb{T}_r\times K^h_b, \qquad \Phi^{(n+1)} = Id + \varphi^{(n+1)},$$

for any $n \geqslant 0$.

(1) $\varphi^{(n+1)} \in \mathscr{C}^1_{r_n-\sigma_n,b_n,h_n,m-1}$, *for any* $n \geqslant 0$. *Moreover,*

$$\sup_{n\in\mathbb{N}} \|\varphi^{(n+1)}\|_{r_n-\sigma_n,b_n,h_n} < +\infty.$$

(2) *There exist* $0 < a < 1$ *and* $C > 1$ *such that*

$$\|\varphi^{(n+1)} - \varphi^{(n)}\|_{r_{n+1},h_{n+1},b_{n+1}} \leqslant C\varepsilon_n^a,$$

for any $n \geqslant 1$.

(3) *There exists* $0 < b < a$ *such that for any* $r \geqslant 1$ *there exists* $C' > 1$ *satisfying*

$$\|D^r\varphi^{(n+1)} - D^r\varphi^{(n)}\|_{\mathscr{C}^1_{\mathrm{hol}}(K^{(\infty)})} \leqslant C'\varepsilon_n^b,$$

Proof. Fix $n \geqslant 0$. Recall that, for any $i \geqslant 0$, $\Phi_{i+1} = \mathrm{Id} + \varphi_{i+1}$, for some $\varphi_{i+1} \in \mathscr{C}^{1+}_{r_i-\sigma_i,b_i,h_i,m-1}$. By (33),

$$\|\varphi_{i+1}\|_{r_i-\sigma_i,b_i,h_i} \leqslant \overline{\sigma}_n\sigma_n^2 \leqslant \frac{1}{4^i}, \tag{36}$$

Hence, by Proposition 30,

$$\varphi^{(n+1)} = \varphi_{n+1} + \sum_{i=1}^n \varphi_i\circ(\mathrm{Id} + \varphi_{i+1})\circ\ldots\circ(\mathrm{Id} + \varphi_n) \in \mathscr{C}^1_{r_n-\sigma_n,b_n,h_n,m-1},$$

and

$$\|\varphi^{(n+1)}\|_{r_n-\sigma_n,b_n,h_n} \leqslant \sum_{i=1}^{n+1}\sigma_i^2\prod_{j=i+1}^n\frac{1}{(1-\sigma_j)^2} < +\infty,$$

which proves the first assertion.

Notice that

$$\varphi^{(n+1)} - \varphi^{(n)} = \varphi_{n+1} + \varphi^{(n)}\circ(\mathrm{Id} + \varphi_{n+1}) - \varphi^{(n)}.$$

Thus, the second assertion follows from Proposition 31 and (33).

The last part of the statement follows by Lemma 13 and the second assertion, together with Item (4) in Lemma 25 and (35). ☐

By the lemma above, the map $\Phi := \lim_{n\to\infty} \Phi^{(n)} : \mathbb{T}_{r/2} \times K_0^{b\infty} \to \mathbb{T}_r \times B(0,b)$ is a well defined $\mathcal{C}^\infty$-holomorphic map satisfying

$$\|\Phi - \mathrm{Id}\,\|_{\mathcal{C}^1_{\mathrm{hol}}} < 2C \sum_{n\geqslant 0} \varepsilon_n \leqslant 4C\varepsilon_0 < 4Cc_* < 1,$$

where in the last inequality we used (26).

Moreover, it follows from Equations (31) and (32) that this map satisfies

$$T_R \circ \Phi(x,y) = \Phi \circ A(x,y) \qquad \text{for any } (x,y) \in \mathbb{T}_{r/2} \times K_0^{b\infty}.$$

This finishes the proof of Theorem 1.

Appendix A. Technical results

This section is devoted to study the norms of different $\mathscr{C}^1$-holomorphic maps, obtained for example by compositions of maps, or by truncating a 1-periodic map in x.

A.1. **Restriction.**

Lemma 27. *Let $r, b, \eta > 0$, an integer $m > 0$ and an analytic map $f : \mathbb{T}_r \times B(0,b) \to \mathbb{C}$ be an analytic map of the form*

$$f(x,y) = y^m g(x,y).$$

Then

$$\|f\|_{\mathscr{C}^1_{hol(\mathbb{T}_r\times B(0,\eta b))}} \leqslant \eta^m \|f\|_{\mathscr{C}^1_{hol(\mathbb{T}_r\times B(0,b))}}.$$

Proof. Let $b' \in (\eta b, b)$. First by multiplicity of the norm, we have

$$\|f\|_{\mathscr{C}^1_{\mathrm{hol}(\mathbb{T}_r\times B(0,\eta b))}} \leqslant \|y^m\|_{\mathscr{C}^1_{\mathrm{hol}(\mathbb{T}_r\times B(0,\eta b))}} \|g\|_{\mathscr{C}^1_{\mathrm{hol}(\mathbb{T}_r\times B(0,\eta b))}}.$$

First, $\|y^m\|_{\mathscr{C}^1_{\mathrm{hol}(\mathbb{T}_r\times B(0,\eta b))}} = m(\eta b)^{m-1}$. Now,

$$\|g\|_{\mathscr{C}^1_{\mathrm{hol}(\mathbb{T}_r\times B(0,\eta b))}} \leqslant \|g\|_{\mathscr{C}^1_{\mathrm{hol}(\mathbb{T}_r\times B(0,b'))}}$$

Notice that by analyticity, the maximum principle applies (even in several variables) and we deduce that

$$\|g\|_{\mathscr{C}^1_{\mathrm{hol}(\mathbb{T}_r\times B(0,b'))}} = \|g\|_{\mathscr{C}^1_{\mathrm{hol}(\mathbb{T}_r\times \partial B(0,b'))}}.$$

□

Proposition 28. *Let $r, b > 0$, an integer $m > 0$ and an analytic map $f : \mathbb{T}_r \times B(0,b) \to \mathbb{C}^2$ of the form*

$$f(x,y) = (y^m f_1(x,y), y^{m+1} f_2(x,y)).$$

nsider an additional $\eta \in (0,1)$. Then

$$\|f\|_{r,}$$

A.2. **Truncation.**

Let $r > 0$, $b > 0$ and $f \in \mathscr{C}^1(K, H_r)$ a $\mathscr{C}^1$-holomorphic map from K to H_r where $H_r = H^\omega(S_r)$. Given an integer $N > 0$, define $\mathscr{T}_N f \in \mathscr{C}^1(K, H_r)$ by

$$\mathscr{T}_N f(x,y) = \sum_{|p|\leqslant N} \widehat{f}_p(y) e_p(x).$$

Lemma 29. *Assume that $0 < r' < r$ and $r - r' < 1$. Then*

$$\|\mathscr{T}_N f\|_{\mathscr{C}^1(K,H_{r'})} \leqslant \frac{\pi+1}{\pi(r-r')} \|f\|_{\mathscr{C}^1(K,H_r)},$$

$$\|f - \mathscr{T}_N f\|_{\mathscr{C}^1(K,H_{r'})} \leqslant \frac{e^{-2\pi N(r-r')}}{\pi(r-r')} \|f\|_{\mathscr{C}^1(K,H_r)}.$$

In particular, if $N \geqslant \frac{|\log \varepsilon|}{2\pi(r-r')}$ for some $0 < \varepsilon < 1$, then

$$\|f - \mathscr{T}_N f\|_{\mathscr{C}^1(K,H_{r'})} \leqslant \frac{\varepsilon \|f\|_{\mathscr{C}^1(K,H_r)}}{\pi(r-r')}.$$

Proof. We have

$$\|\mathscr{T}_N f\|_{\mathscr{C}^1(K,H_{r'})} \leqslant \sum_{|p|\leqslant N} \|\widehat{f}_p\|_{\mathscr{C}^1(K,\mathbb{C})} e^{2\pi|p|},$$

$$\|f - \mathscr{T}_N f\|_{\mathscr{C}^1(K,H_{r'})} \leqslant \sum_{|p|>N} \|\widehat{f}_p\|_{\mathscr{C}^1(K,\mathbb{C})} e^{2\pi|p|}.$$

By [3, Corollary 10],

$$\|\widehat{f}_p\|_{\mathscr{C}^1(K,\mathbb{C})} \leqslant e^{-2\pi|p|r} \|f\|_{\mathscr{C}^1(K,H_r)},$$

for any $p \in \mathbb{Z}$. Hence

$$\begin{aligned}
\|\mathscr{T}_N f\|_{\mathscr{C}^1(K,H_{r'})} &\leqslant \|f\|_{\mathscr{C}^1(K,H_r)} \sum_{|p|\leqslant N} e^{-2\pi|p|(r-r')} \\
&\leqslant \|f\|_{\mathscr{C}^1(K,H_r)} \left(1 + 2\sum_{p>0} e^{-2\pi p(r-r')}\right) \\
&\leqslant \|f\|_{\mathscr{C}^1(K,H_r)} \left(1 + \frac{1}{\pi(r-r')}\right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|f - \mathscr{T}_N f\|_{\mathscr{C}^1(K,H_{r'})} &\leqslant \|f\|_{\mathscr{C}^1(K,H_r)} \sum_{|p|>N} e^{-2\pi|p|(r-r')} \\
&\leqslant \|f\|_{\mathscr{C}^1(K,H_r)} \frac{e^{-2\pi N(r-r')}}{\pi(r-r')},
\end{aligned}$$

The last assertion in the lemma follows directly if one replaces N by $\frac{|\log\varepsilon|}{2\pi(r-r')}$ in the RHS of the previous inequality. □

A.3. **Composition estimates.**

Proposition 30. *Let $f \in \mathscr{C}^1_{r,b,h,m}$ and $\varphi \in \mathscr{C}^1_{r',b',h',n}$*[7] *such that*

$$\|\varphi_1\|_{r',b',h'} \leqslant \kappa_1(r-r'), \qquad \|\varphi_2\|_{r',b',h'} \leqslant \kappa_2 \min(h-h', b-b').$$

where $\kappa_1, \kappa_2 \in (0,1)$, $r' \in (0,r)$, $b' \in (0,b)$ and $h' \in (0,h)$. Then the maps $g, g_x, g_y : \mathbb{T}_{r'} \times K^{h'}_{b'} \to \mathbb{C}^2$ defined by $g = f \circ (I + \varphi)$, $g_x = f \circ (I+\varphi)$ and $g_y = f \circ (I + \varphi)$ are respectively elements of $\mathscr{C}^1_{r',b',h',m}$, $\mathscr{C}^1_{r',b',h',m}$ and $\mathscr{C}^1_{r',b',h',m}$ whose norms satisfy

$$\|g\|_{r',b',h',m} \leqslant CK\|f\|_{r,b,h} \tag{37}$$

$$\|g_x\|_{r',b',h',m}, \|g_x\|_{r',b',h',m} \leqslant \frac{CK}{r-r'}\|f\|_{r,b,h}$$

and

$$\|g_y\|_{r',b',h',m-1} \leqslant \frac{CK}{h-h'}\|f\|_{r,b,h}$$

where $C > 0$ is a constant depending uniquely on m and b and

$$K = \frac{1}{(1-\kappa_1)(1-\kappa_2)} > 0.$$

[7] Here we can assume φ defined on r', b', h'? Yes

Proof. CHECK constants Assume first that f is of the form $\mathbb{T}_r \times K_b^{h'} \to \mathbb{C}$ and $g = (\partial_x^p \partial_y^q f) \circ (I + \varphi)$ for some $p, q \geqslant 0$. Then the following Taylor expansion holds for g:

$$g(x,y) = \sum_{u,v \geqslant 0} \frac{1}{u!v!} \partial_x^{p+u} \partial_y^{q+v} f(x,y) y^{n(u+v)+v} \varphi_1^u \varphi_2^v.$$

Combining the Cauchy estimates given in Carminatti-Marmi-Sauzin by

$$\left\| \partial_x^{p+u} \partial_y^{q+v} f \right\|_{r',b',h'} \leqslant \frac{(p+u)!(q+v)!}{(r-r')^{p+u}(h-h')^{q+v}} \|f\|_{r,b,h}{}^{8}$$

and the assumptions on $\|\varphi_1\|_{r',b',h'}$ and $\|\varphi_2\|_{r',b',h'}$ give

$$\|g\|_{r',b',h'} \leqslant \|f\|_{r,b,h} \sum_{u,v \geqslant 0} \frac{1}{u!v!} \frac{(p+u)!(q+v)!}{(r-r')^{p+u}(h-h')^{q+v}} \|\varphi_1\|_{r',b',h'}^u \|\varphi_2\|_{r',b',h'}^v$$

and it follows that

$$\|g\|_{r',b',h',m-1} \leqslant \frac{Kp!q!}{(r-r')^p(h-h')^q} \|f\|_{r,b,h} \tag{38}$$

Now if $f(x,y) = (y^m F_1(x,y), y^{m+1} F_2(x,y))$, the composition $f(I+\varphi)$ is given by

$$f(I+\varphi)(x,y) = \left(y^m \tilde{F}_1(x,y), y^{m+1} \tilde{F}_2(x,y) \right) \tag{39}$$

where

$$\tilde{F}_1(x,y) = (1 + y^n \varphi_2(x,y))^m F_1(I+\varphi)(x,y)$$

and

$$\tilde{F}_2(x,y) = (1 + y^n \varphi_2(x,y))^{m+1} F_2(I+\varphi)(x,y).$$

Inequality (38) applied to $\tilde{F}_1$ and $\tilde{F}_2$ ensure the result for $f(I+\varphi)$. The result holds in a similar way for $\partial_x f(I+\varphi)$ and $\partial_y f(I+\varphi)$. As an exemple, for $\partial_y f(I+\varphi)$, previous computations gives

$$\partial_y f(I+\varphi)(x,y) = \left(y^{m-1} \tilde{F}_1(x,y), y^m \tilde{F}_2(x,y) \right) \tag{40}$$

where

$$\tilde{F}_1(x,y) = Y^{m-1}(mF_1(I+\varphi)(x,y) + Y \partial_y F_1(I+\varphi)(x,y))$$

$$\tilde{F}_2(x,y) = Y^m((m+1)F_2(I+\varphi)(x,y) + Y \partial_y F_2(I+\varphi)(x,y)).$$

and $Y = 1 + y^n \varphi_2(x,y)$ □

Proposition 31. *Under the assumptions of Proposition 30, the map* $h : \mathbb{T}_{r'} \times K_{b'}^{h'} \to B$ *defined by* $h = f(I+\varphi) - f$ *is an element of* $\mathscr{C}^1_{r',b',h',m+n}$ *whose norm satisfies*

$$\|h\|_{r',b',h'} \leqslant K \|f\|_{r,b,h} \left(\frac{\|\varphi_1\|_{r',b',h'}}{r-r'} + \frac{\|\varphi_2\|_{r',b',h'}}{h-h'} \right)$$

where

$$K = \frac{1}{(1-\kappa_1)(1-\kappa_2)} > 0.$$

[8] I think some constants are missing here

Proof. The map h can be written in the following integral form
$$h(x,y) = \int_0^1 df(I + t\varphi) \cdot \varphi \, dt.$$
Now for a fixed $t \in [0,1]$, using expressions (39) and (40), we obtain that
$$df(I+t\varphi)\cdot\varphi = y^n \varphi_1 \partial_x f(I+t\varphi) + y^{n+1}\varphi_2 \partial_y f(I+t\varphi) \in \mathscr{C}^1_{r',b',h',m+n}$$
with correspoding estimates, when we apply Proposition 30. The result follows by integrating in t. $\square$

A.4. **Inverse estimates.**

Proposition 32. *Let Φ be a diffeomorphism of the form $\Phi = I+\varphi$ where $\varphi \in \mathscr{C}^{1+}_{r,b,h,m}$ is a $\mathscr{C}^1$-holomorphic map. Fix $r' \in (0,r)$, $b' \in (0,b)$, $h' \in (0,h)$ and $\kappa \in (0,1)$ and assume that*
$$\|\varphi\|^+_{r,b,h} \leqslant (1-\kappa)^2\kappa\alpha \tag{41}$$
where $\alpha < \min(r-r', h-h', b-b')$. Then we can write $\Phi^{-1} = I + \psi$ where $\psi \in \mathscr{C}^{1+}_{r',b',h',m}$ satifies the estimates
$$\|\psi\|^+_{r',b',h'} \leqslant K\|\varphi\|_{r,b,h} \tag{42}$$
where $K = (1-\kappa)^{-2}$. Moreover $\phi + \psi \in \mathscr{C}^{1+}_{r',b',h',2m}$ with
$$\|\varphi+\psi\|^+_{r',b',h',2m} \leqslant K^2(\|\varphi\|^+_{r,b,h,m})^2\left(\frac{1}{r-r'}+\frac{1}{h-h'}\right). \tag{43}$$

Proof. We found a similar idea in [Ramirez-Roz, Martin, Sarol]. The map $\Psi = I+\psi$ is an inverse of Φ if and only if $\Phi\circ\Psi = I$ which is equivalent to say that ψ is a fixed point of the map F defined by
$$F(h) = -\varphi\circ(I+h).$$
Let $\mathcal{B}$ be the ball of radius $\kappa\alpha$ in the space of $\mathscr{C}^1$-holomorphic map $\mathscr{C}^1_{r',b',h',m}$. Given $h \in \mathcal{B}$, by Proposition 30 $F(h) \in \mathscr{C}^1_{r',b',h',m}$ is a well defined holomorphic map which satisfies
$$\|F(h)\|_{r',b',h'} \leqslant \frac{\|\varphi\|_{r,b,h}}{(1-\kappa)^2} \leqslant \kappa\alpha \tag{44}$$
where the last inequality follows from the assumption that $\|\varphi\|_{r,b,h} \leqslant (1-\kappa)^2\kappa\alpha$. Hence $F(h) \in \mathcal{B}$ and F is a well-defined functionnal from $\mathcal{B}$ to itself.

Let us now show that F is a contracting map. Given $h_1, h_2 \in \mathcal{B}$, we can write
$$F(h_2) - F(h_1) = -\int_0^1 df\circ(I+h_1+t(h_2-h_1))\cdot(h_1-h_1)dt.$$
By Proposition 30, given $t\in[0,1]$, the map $g_t = d\varphi\circ(I+h_1+t(h_2-h_1))\cdot(h_1-h_1) \in \mathscr{C}^1_{r',b',h',2m}$ satisfies
$$\|g_t\|_{r',b',h',2m} \leqslant \frac{2\|\varphi\|_{r,b,h}}{\alpha(1-\kappa)^2}\|h_2-h_1\|_{r',b',h'} \leqslant \kappa\|h_2-h_1\|_{r',b',h'}$$
where the last inequality follows from (41). We deduce the existence of ψ by applying the fixed point theorem. Estimates on its norm follow from the property $\psi = F(\psi)$ together with the first inequality in (44).

Finally, notice that $\varphi+\psi = F(\psi)-F(0)$, hence by Proposition 31, $\varphi+\psi \in \mathscr{C}^1_{r',b',h',2m}$ and the following estimates hold

$$\|\varphi+\psi\|_{r',b',h',2m} \leqslant K\|\varphi\|_{r,b,h}\left(\frac{\|\psi_1\|_{r',b',h'}}{r-r'}+\frac{\|\psi_2\|_{r',b',h'}}{h-h'}\right)$$

which together with Inequality (42) finish the proof. □

A.5. **Change of power.**

Proposition 33. *Let $b \in (0,1)$, B be a Banach space, $K \subset B(0,b)$ a compact set, $\varphi : K \to B$ be a $\mathscr{C}^1$-holomorphic map, and an integer $s \geqslant 1$. Then*

$$\|y^s\varphi\|_{\mathscr{C}^1_{hol}(K,B)} \leqslant 2sb^{s-1}\|\varphi\|_{\mathscr{C}^1_{hol}(K,B)}.$$

Proof. By multiplicativity,

$$\|y^s\varphi\|_{\mathscr{C}^1_{hol}(K,B)} \leqslant \|y^s\|_{\mathscr{C}^1_{hol}(K,\mathbb{C})}\|\varphi\|_{\mathscr{C}^1_{hol}(K,B)}.$$

Hence it is enough to estimate $\|y^s\|_{\mathscr{C}^1_{hol}(K,\mathbb{C})}$. We first consider the following inequality obtained by restriction

$$\|y^s\|_{\mathscr{C}^1_{hol}(K,\mathbb{C})} \leqslant \|y^s\|_{\mathscr{C}^1_{hol}(\overline{B}(0,b),\mathbb{C})}.$$

The sup-norm of f defined for any y by $f(y) = y^s$ on $\overline{B}(0,b)$ is $n_0(f) = |f|_{\overline{B}(0,b)} = b^s$. The computation is the same for the derivative, $|f'|_{\overline{B}(0,b)} \leqslant sb^{s-1}$. Moreover, for any $y_0, y_1 \in \overline{B}(0,b)$,

$$|\delta_f(y_0,y_1)| \leqslant 2b^s$$

which implies that $n_1(f) \leqslant 2sb^{s-1}$. Finally, if $y_0 \neq y_1$, observe that

$$\Omega_{f,f'}(y_0,y_1) = \sum_{k=0}^{q-1} y_0^k y_1^{s-1-k} - sy_0^{s-1}$$

and it follows from the triangular inequality that $|\Omega_{f,f'}(y_0,y_1)| \leqslant 2sb^{s-1}$. Hence $\|f\|_{\mathscr{C}^1_{hol}(\overline{B}(0,b),\mathbb{C})}$ and the result follows. □

Proposition 34. *Let $r,b,h > 0$, $b \in (0,1)$, and integers $m' > m \geqslant 0$. If $R \in \mathscr{C}^1_{r,b,h,m'}(\mathbb{R}^2)$, then*

$$\|R\|_{r,b,h,m} \leqslant 2sb^{s-1}\|R\|_{r,b,h,m'}$$

where $s = m'-m$, and

$$\|R\|_{\mathscr{C}^1_{hol}(K^h_b)} \leqslant 2m'b^{m'-1}\|R\|_{r,b,h,m'}.$$

Proof. For the first one, write

$$R(x,y) = (y^m\tilde{R}_1, y^{m+1}\tilde{R}_2)$$

where $\tilde{R}_1 = y^sR_1$ and $\tilde{R}_2 = y^sR_2$. Since $s \geqslant 1$, following Proposition 33 each $\tilde{R}_j$ satisfies

$$\|\tilde{R}_j\|_{\mathscr{C}^1_{hol}(K^b_h)} \leqslant 2sb^{s-1}\|R_j\|_{\mathscr{C}^1_{hol}(K^b_h)}$$

and the first result follows. The second estimate is similar. □

A.6. **Technical lemmas.** A proof of the following lemma can be found in the proof of [3, Proposition 12].

Lemma 35. *Given* $\alpha > 0$ *and* $\beta > 1$,

$$\Sigma_{\alpha,\beta} := \sum_{n>0} n^\beta e^{-\alpha n} \leqslant \frac{C_{\alpha,\beta}}{\alpha^{\beta+1}},$$

where $C_{\alpha,\beta} = \beta! + \alpha(e^{-1}\beta)^\beta > 0$.

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