Starting the study of outer length billiards

Luca Baracco, Olga Bernardi, Corentin Fierobe

baracco@math.unipd.it, obern@math.unipd.it, cpef@gmx.de

Abstract

We focus on the outer length billiard dynamics, acting on the exterior of a strictly-convex planar domain. We first show that ellipses are totally integrable. We then provide an explicit representation of first order terms for the formal Taylor expansion of the corresponding Mather's β-function. Finally, we provide explicit Lazutkin coordinates up to order 4.

1 Introduction

The aim of the present paper is starting an accurate study of outer length billiards, first described by P. Albers and S. Tabachnikov in 2024, see [2][Section 3.4]. These billiards are the counterpart of Birkhoff ones since the generating function is the outer length instead of the inner length. They are also called "fourth billiards". In fact, there are two billiards systems –Birkhoff and outer area billiards– which have been extensively studied in literature; we refer respectively to [22] and [21] for exhaustive surveys. Another type of billiards, namely symplectic billiards, whose generating function is the inner area, were introduced in 2018 by P. Albers and S. Tabachnikov [1] and their study started to become more intensive only recently. We refer to [4], [5] and [23] for integrability results and to [7] and [12] for area spectral rigidity results for symplectic billiards. Regarding outer length billiards, to the best of our knowledge, they were not studied yet. However, the seminal idea on the base of the definition of this dynamical system –detecting, in particular, circumscribed polygons to a strictly-convex domain with minimal perimeter– can already be found in some former papers of convex planar geometry, see e.g. [11][Theorem 1] and [10][Section 2].

We first give all the details to introduce this dynamical system, acting on the exterior of a strictly-convex planar domain. We then prove –by arguments of elementary planar geometry– that ellipses are totally integrable, that is the phase-space is fully foliated by continuous invariant curves which are not null-homotopic.

We successively focus on the main topic of the paper, which is providing an explicit representation of first order terms for the formal Taylor expansion of Mather's β-function (or minimal average function) for outer length billiards. In particular, we write these coefficients (up to order 5) by means of the ordinary curvature and length of the boundary of the billiard table. As already noticed, for such a dynamical system, Mather's β-function is related to the minimal perimeter of polygons circumscribed to a strictly-convex domain. These perimeters are special cases (i.e. for periodic trajectories of winding number = 1) of the corresponding marked length spectrum for outer length billiards.

Finally, by using the computations we made to obtain minimal average function's coefficients, we provide explicit Lazutkin coordinates up to order 4 and discuss straightforward facts regarding the existence/non existence of caustics for outer length billiards.

In order to state our results, we proceed with some preliminaries.

2 Twist maps and Mather's β-function

Let $\mathbb{T} \times (a,b)$ be the annulus, where $\mathbb{T} = \mathbb{R}/\mathbb{Z} = [0,1]/\sim$ identifying $0 \sim 1$ and (eventually) $a = -\infty$ and/or $b = +\infty$. Given a diffeomorphism $\Phi : \mathbb{S}^1 \times (a,b) \to \mathbb{S}^1 \times (a,b)$, we denote by

$$\phi : \mathbb{R} \times (a,b) \to \mathbb{R} \times (a,b), \qquad (x_0, y_0) \mapsto (x_1, y_1)$$

a lift of Φ to the universal cover. Then ϕ is a diffeomorphism and $\phi(x+1, y) = \phi(x, y) + (1, 0)$. In the case where a (resp. b) is finite, we assume that ϕ extends continuously to $\mathbb{R} \times \{a\}$ (resp. $\mathbb{R} \times \{b\}$) by a rotation of fixed angle:

$$\phi(x_0, a) = (x_0 + \rho_a, a) \qquad (\text{resp. } \; \phi(x_0, b) = (x_0 + \rho_b, b)). \tag{2.1}$$

Once fixed the lift, the numbers ρ_a, ρ_b are unique. The choice of ρ_a (resp. ρ_b) if $a = -\infty$ (resp. $b = +\infty$) depends on the dynamics at infinity. For example, in the case of outer length billiards where $b = +\infty$, it is natural to set $\rho_b = 1/2$, we refer to point 1. of Section 3 for details.

We recall here below – for reader's convenience – the well-known definition of monotone twist map, see e.g. [18][Page 2].

Definition 1. *A monotone twist map $\phi : \mathbb{R} \times (a,b) \to \mathbb{R} \times (a,b)$, $(x_0, y_0) \mapsto (x_1, y_1)$ is a diffeomorphism satisfying:*

1. *$\phi(x_0 + 1, y_0) = \phi(x_0, y_0) + (1, 0)$.*
2. *ϕ preserves orientations and the boundaries of $\mathbb{R} \times (a,b)$.*
3. *ϕ extends to the boundaries by rotation, as in (2.1).*
4. *ϕ satisfies a monotone twist condition, that is*
$$\frac{\partial x_1}{\partial y_0} > 0. \tag{2.2}$$
5. *ϕ is exact symplectic; this means that there exists a generating function $H \in C^2(\mathbb{R} \times \mathbb{R}; \mathbb{R})$ for ϕ such that*
$$y_1 dx_1 - y_0 dx_0 = dH(x_0, x_1). \tag{2.3}$$

Clearly, $H(x_0 + 1, x_1 + 1) = H(x_0, x_1)$ and, due to the twist condition, the domain of H is the strip $\{(x_0, x_1) : \; \rho_a + x_0 < x_1 < x_0 + \rho_b\}$. Moreover, equality (2.3) reads

$$\begin{cases} y_1 = H_2(x_0, x_1) \\ y_0 = -H_1(x_0, x_1) \end{cases} \tag{2.4}$$

and the twist condition (2.2) becomes $H_{12} < 0$. As a consequence, $\{(x_i, y_i)\}_{i\in\mathbb{Z}}$ is an orbit of ϕ if and only if $H_2(x_{i-1}, x_i) = y_i = -H_1(x_i, x_{i+1})$ for all $i \in \mathbb{Z}$. This means – on a formal level – that the corresponding bi-infinite sequence $x := \{x_i\}_{i\in\mathbb{Z}}$ is a so-called critical configuration of the action functional $\sum_{i\in\mathbb{Z}} H(x_i, x_{i+1})$. In such a setting, minimal orbits play a fundamental role. We recall that a critical configuration x of ϕ is minimal if every finite segment of x minimizes the action functional with fixed end points (we refer to [18][Page 7] for details). Clearly, all these facts remain true if we consider a monotone twist map on $\{(x_0, x_1) : \ u_a(x_0) < x_1 < u_b(x_0)\}$.

For a twist map ϕ generated by H, we finally introduce the rotation number and the Mather's β-function (or minimal average action).

Definition 2. *The rotation number of an orbit $\{(x_i, y_i)\}_{i\in\mathbb{Z}}$ of ϕ is*

$$\rho := \lim_{i\to\pm\infty} \frac{x_i}{i}$$

if such a limit exists.

A relevant class of monotone twist maps are planar billiard maps. In such a setting, the rotation number of a periodic trajectory is the rational

$$\frac{m}{n} = \frac{\text{winding number}}{\text{number of reflections}} \in (0, \frac{1}{2}],$$

we refer to [18][Page 40] for details.

In view of the celebrated Aubry-Mather theory (see e.g. [3]), a monotone twist map possesses minimal orbits for every rotation number ρ inside the so-called twist interval (ρ_a, ρ_b). As a consequence, we can associate to each ρ the average action of any minimal orbit having that rotation number.

Definition 3. *The Mather's β-function of ϕ is $\beta : (\rho_a, \rho_b) \to \mathbb{R}$ with*

$$\beta(\rho) := \lim_{N\to\infty} \frac{1}{2N} \sum_{i=-N}^{N-1} H(x_i, x_{i+1})$$

where $\{x_i\}_{i\in\mathbb{Z}}$ is any minimal configuration of ϕ with rotation number ρ.

In the framework of Birkhoff billiards, A. Sorrentino in [19] gave an explicit representation of the coefficients of the (formal) Taylor expansion at zero of the corresponding Mather's β-function. More recently, J. Zhang in [24] got (locally) an explicit formula for this function via a Birkhoff normal form. Moreover, M. Bialy in [9] obtained an explicit formula for Mather's β-function for ellipses by using a non-standard generating function, involving the support function. Regarding symplectic and outer billiards, the first two authors and A. Nardi in [6] computed explicitly the higher order terms of such an expansion, by using tools from affine differential geometry. As anticipated, one of the target of the present paper is writing explicitly these coefficients (up to order 5) in the case of forth billiards.

3 The dynamical system

Let Ω be a strictly-convex planar domain with smooth boundary $\partial\Omega$. Assume that the perimeter of $\partial\Omega$ is $\ell = |\partial\Omega|$. Fixed the positive counter-clockwise orientation, let $\gamma : \mathbb{T} \to \partial\Omega$ be the smooth arc-length parametrization of $\partial\Omega$. For every $s \in \mathbb{T}$, we denote by $s^* \in \mathbb{T}$ the (unique, by strict-convexity) arc-length parameter such that $T_{\gamma(s)}\partial\Omega = T_{\gamma(s^*)}\partial\Omega$. We refer to

$$\mathcal{P} = \{(s, r) \in \mathbb{T} \times \mathbb{T} : \ s < r < s^*\}$$

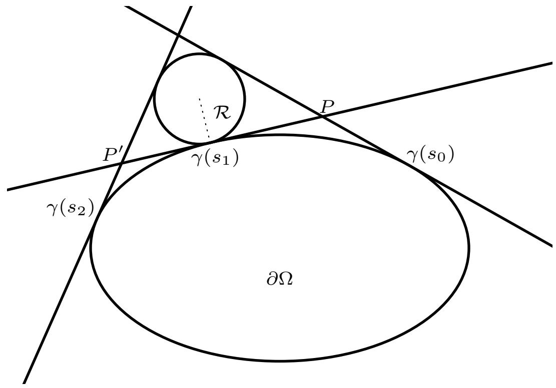

Figure 1: The outer-length billiard map around the domain Ω. It associates the point P to the point P'.

as the (open, positive) phase-space and we define the outer length billiard map as follows (we refer to [2][Section 3.4]).

Since Ω is strictly-convex, to every point $P \in \mathbb{R}^2 \setminus \mathrm{cl}(\Omega)$ can be uniquely associated a pair $(s_0, s_1) \in \mathbb{T} \times \mathbb{T}$ with $s_0 < s_1$ and such that the lines $P\gamma(s_0)$ and $P\gamma(s_1)$ are the (negative and positive) tangents to $\partial\Omega$. Consider the circle in $\mathbb{R}^2 \setminus \Omega$ tangent to $\partial\Omega$ at $\gamma(s_1)$ and to the line $P\gamma(s_0)$. Then then image P' of P is defined as the intersection point between the lines $P\gamma(s_1)$ and the other common tangent line of the circle and $\partial\Omega$ (hence passing through P' and $\gamma(s_2)$):

$$T : \mathcal{P} \to \mathcal{P}, \qquad (s_0, s_1) \mapsto (s_1, s_2).$$

We refer to Figure 1. Setting $\varepsilon_0 = s_1 - s_0$ and

$$\hat{\mathcal{P}} = \{(s, \varepsilon) \in \mathbb{T} \times \mathbb{R} : \; 0 < \varepsilon < s^* - s\},$$

the outer length billiard map can be equivalently defined as

$$T : \hat{\mathcal{P}} \to \hat{\mathcal{P}}, \qquad (s_0, \varepsilon_0) \mapsto (s_1, \varepsilon_1).$$

Here are some properties of the outer length billiard map.

1. T is continuous and can be continuously extended so that $T(s, s) = (s, s)$ and $T(s, s^*) = (s^*, s)$.

2. The function

$$H : \mathcal{P} \to \mathbb{R}, \qquad H(s_0, s_1) := |P\gamma(s_0)| + |P\gamma(s_1)|$$

generates T, that is

$$T(s_0, s_1) = (s_1, s_2) \qquad \Longleftrightarrow \qquad H_2(s_0, s_1) + H_1(s_1, s_2) = 0. \tag{3.1}$$

We refer to [2][Lemma 3.1] for the proof. In view of (3.1), we can equivalently refer to

$$\bar{H} : \mathcal{P} \to \mathbb{R}, \qquad \bar{H}(s_0, s_1) := |P\gamma(s_0)| + |P\gamma(s_1)| - s_1 + s_0$$

as a generating function, which is exactly the Lazutkin parameter of $\partial\Omega$, interpreted as convex caustic for a Birkhoff billiard.

3. T is a twist map preserving the area form $-H_{12}(s_0, s_1)\, ds_0 \wedge ds_1$.

4. By introducing new variables

$$y_0 = -H_1(s_0, s_1), \qquad y_1 = H_2(s_0, s_1),$$

(s, y) are coordinates on $\mathcal{P}$ and the outer length billiard map results a (negative) twist map, since

$$\frac{\partial y_1}{\partial s_0} = H_{12}(s_0, s_1) = -\frac{k(s_0)k(s_1)H(s_0, s_1)}{2\sin^2(\varphi/2)} < 0,$$

where φ is the angle between the tangent lines $P\gamma(s_0)$ and $P\gamma(s_1)$ (see also [2][Page 11]). In these coordinates, the preserved area form is the standard one: $ds \wedge dy$.

5. The marked length spectrum for the outer length billiard is the map $\mathcal{ML}_o(\Omega) : \mathbb{Q} \cap (0, \frac{1}{2}\Big) \to \mathbb{R}$ that associates to any m/n in lowest terms the minimal perimeter of the periodic trajectories having rotation number m/n. We refer to [18][Sections 3.1 and 3.2] for a general treatment of the marked spectrum. Clearly, periodic outer length billiard minimal trajectories (with winding number = 1) correspond to convex polygons realizing the minimal (circumscribed) perimeter, so that:

$$\beta\left(\frac{1}{n}\right) = \frac{1}{n}\mathcal{ML}_o(\Omega)\left(\frac{1}{n}\right). \tag{3.2}$$

3.1 Circles and ellipses

As expected, the outer length billiard on the circle (of center O) is totally integrable: the phase-space is completely foliated by concentric invariant circles. By using as coordinates $(\alpha_0, \alpha_1) \in \mathbb{T} \times \mathbb{T}$, where α_0 and α_1 are respectively the angles of $O\gamma(s_0)$ and $O\gamma(s_1)$ with respect to the positive horizontal direction, the generating function in the case of disk of unit radius is

$$H(\alpha_0, \alpha_1) = 2\tan\left(\frac{\alpha_1 - \alpha_0}{2}\right).$$

Equivalently, in terms of $(\alpha_0, y_0) = (\alpha_0, -H_1(\alpha_0, \alpha_1)) = \left(\alpha_0, 1 + \tan^2\left(\frac{\alpha_1-\alpha_0}{2}\right)\right)$, we have that

$$H(\alpha_0, y_0) = 2\sqrt{y_0 - 1}$$

and the total integrability immediately follows.

An unexpected fact –at least from the authors' point of view, since the billiard dynamics are not invariant by affine transformations– is that also the outer length billiard on the ellipse is totally integrable, as stated in the next proposition.

Proposition 4. *Let $\mathcal{E}$ and Γ be two confocal nested ellipses, $\mathcal{E} \subset \Gamma$. Then Γ is a caustic for the outer-length billiard dynamics outside $\mathcal{E}$.*

The proof of Proposition 4 relies on the following lemma of elementary planar geometry, see Figure 2.

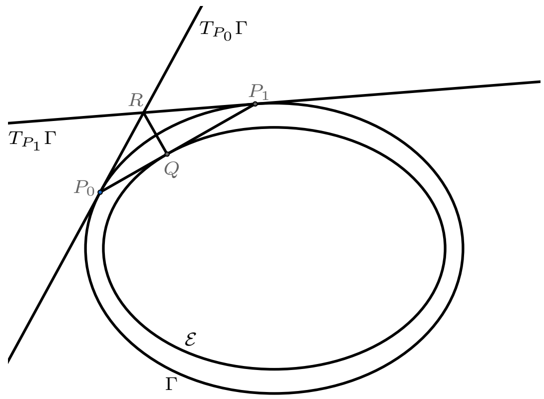

Figure 2: The line RQ is orthogonal to the line P_0P_1

Lemma 5 (Lemma 2.4 in [20]). *Let $P_0, P_1 \in \Gamma$ two distinct points such that the line P_0P_1 is tangent to $\mathcal{E}$ at a point Q. Let R be the intersection point of the tangent lines to Γ at P_0 and P_1. Then the lines P_0P_1 and RQ are orthogonal.*

Proof of Proposition 4. Let a point P_0 on Γ. Consider the positive tangent line to $\mathcal{E}$ at a point Q and passing through P_0. let $P_1 \in \Gamma$ be the intersection point of the latter tangent line P_0Q with Γ, see Figure 3. We need to show that P_1 is the image of P_0 under the outer-length billiard reflection outside $\mathcal{E}$. Consider the point P, such that PP_0 and PP_1 are the two tangent lines to $\mathcal{E}$ passing through P, see Figure 3. Since $\mathcal{E}$ and Γ are confocal, $\mathcal{E}$ is a caustic for the classical billiard in Γ. In particular, the tangent line $T_{P_0}\Gamma$ is a bisector of the angle $\widehat{P_1P_0P}$. With the same argument the tangent line $T_{P_1}\Gamma$ is a bisector of the angle $\widehat{P_0P_1P}$. Hence $T_{P_0}\Gamma$ and $T_{P_1}\Gamma$ intersects at a point R which is the center of the inscribed circle $\mathcal{D}$ to the triangle P_0PP_1. Now –by Lemma 5– the lines RQ and P_0P_1 are orthogonal. In particular $\mathcal{D}$ is tangent to the ellipse $\mathcal{E}$. This implies that P_1 is obtained from P_0 by the outer-length billiard law of reflection. $\square$

We underline that it would be interesting to investigate if these are the unique cases; this fundamental problem (possibly to be studied by an integral inequality à la Bialy [8]) may present non-trivial difficulties, due to the infinite total area of the phase-space.

4 Asymptotic expansions

S. Marvizi and R. Melrose's theory, first stated and proved for Birkhoff billiards [15][Theorem 3.2], can be applied to the general case of (strongly) billiard-like maps, see [13][Section 2.1]. As an outcome, the following expansion at $\rho = 0$ of the corresponding minimal average function holds:

$$\beta(\rho) \sim \beta_1\rho + \beta_3\rho^3 + \beta_5\rho^5 + \dots$$

in terms of odd powers of ρ. It is well-known –see e.g. [15][Section 7] again– that for usual billiards the sequence $\{\beta_k\}$ can be interpreted as a spectrum of a differential operator, see also Remark 2.11 in [1]. The question is widely open for other types of billiards, included outer length billiards.

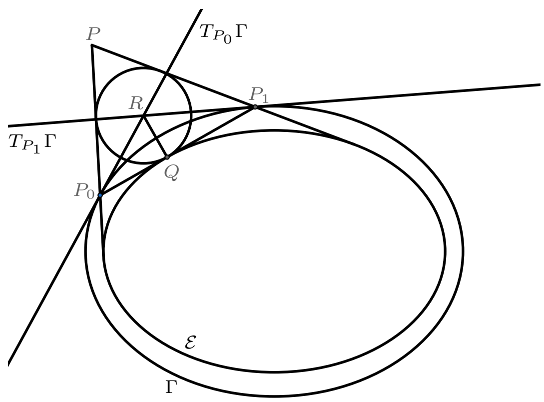

Figure 3: The point $P_0 \in \Gamma$ is reflected to the point $P_1 \in \Gamma$ by the outer-length billiard dynamics around $\mathcal{E}$.

In this section, we gather all the technical results in order to prove the next theorem, providing coefficient β_5 for the outer length billiard map. This result is a refinement of [17][Theorem 1, point (*iii*)]. In fact, in a genuine framework of convex planar geometry, D.E. Vitale and R.A. McClure computed β_3 by using as coordinate the support function and as parameter the angle with respect to a fixed direction.

Theorem 6. *Let Ω be a strictly-convex planar domain with smooth boundary $\partial\Omega$. Suppose that $\partial\Omega$ has everywhere positive curvature. Denote by $k(s)$ the (ordinary) curvature of $\partial\Omega$ with arc-length parameter s. Let ℓ be the length of the boundary and*

$$L := \int_0^\ell k^{2/3}(s)ds.$$

The formal Taylor expansion at $\rho = 0$ of Mather's β-function for the outer length billiard map has coefficients:

$$\begin{aligned} \beta_{2k} &= 0 \text{ for all } k \\ \beta_1 &= \ell \\ \beta_3 &= \frac{L^3}{12} \\ \beta_5 &= L^4 \int_0^\ell \left(\frac{k^{4/3}(s)}{120} + \frac{k^{-\frac{8}{3}}(s)k'^2(s)}{2160} \right) ds. \end{aligned}$$

As expected, a straightforward consequence of the previous result is that –as for other billiards– also for outer length ones, the two coefficients β_1 and β_3 allow to recognize a circle among all strictly-convex planar domains.

Corollary 7. *The coefficients β_1 and β_3 recognize a circle. In particular:*

$$3\beta_3 + \pi^2\beta_1 \le 0$$

with equality if and only if $\partial\Omega$ is a circle.

Proof. We apply Hölder's inequality with $p = 3/2$ and $q = 3$ to obtain

$$L = \int_0^\ell k^{2/3}(s)ds \le \left(\int_0^\ell (k^{2/3}(s))^{3/2}ds\right)^{2/3} \left(\int_0^\ell 1^3 ds\right)^{1/3} = (2\pi)^{2/3}\ell^{1/3} \tag{4.1}$$

since $\int_0^\ell k(s)ds = 2\pi$. Using the expressions of β_1 and β_3 found in Theorem 6, we can write

$$3\beta_3 + \pi^2\beta_1 = \frac{1}{4}L^3 - \pi^2\ell \le \frac{1}{4}(2\pi)^2\ell - \pi^2\ell = 0.$$

In the case of equality, namely if $3\beta_3 + \pi^2\beta_1$, then $L = (2\pi)^{2/3}\ell^{1/3}$, and the case of equality is reached in (4.1). In that case, k is constant. Hence Ω is a disk. $\square$

Remark 8. *Let $\mathcal{P}_n^c$ be the set of all convex polygons with at most n vertices which are circumscribed to Ω. We define*

$$\delta(\Omega; \mathcal{P}_n^c) := \inf\{\ell(P_n) : \; P_n \in \mathcal{P}_n^c\},$$

where $\ell(P_n)$ is the perimeter length of P_n. Clearly, essentially in view of equality (3.2), Theorem 6 gives also the formal expansion of $\delta(\Omega; \mathcal{P}_n^c)$ at $n \to +\infty$.

Since we use the arc-length parametrization of $\partial\Omega$, it is useful to recall that

$$\begin{cases} \gamma'' = kJ\gamma', \quad \gamma''' = -k^2\gamma' + k'J\gamma' \\ \gamma^{(4)} = -3kk'\gamma' + (-k^3 + k'')J\gamma' \\ \gamma^{(5)} = (k^4 - 4kk'' - 3k'^2)\gamma' + (-6k^2k' + k''')J\gamma' \\ \gamma^{(6)} = (10k^3k' - 10k'k'' - 5kk''')\gamma' + (k^5 - 10k^2k'' - 15kk'^2 + k^{(4)})J\gamma' \end{cases} \tag{4.2}$$

where J is the rotation of angle $\pi/2$ in the positive verse.

Proposition 9. *For $0 \le r \le s \le \ell$, it holds*

$$\begin{aligned} H(r,s) = (s-r) + \frac{k^2(r)}{12}(s-r)^3 + \frac{k(r)k'(r)}{12}(s-r)^4 + \\ + \frac{2k^4(r) + 4k'^2(r) + 7k(r)k''(r)}{240}(s-r)^5 + O((s-r)^6) \end{aligned} \tag{4.3}$$

uniformly as $(s-r) \to 0$.

Proof. We start by writing separately the Taylor expansions of numerator and denominator of the generating function

$$H(r,s) = \frac{(\gamma(s) - \gamma(r)) \wedge (\gamma'(s) - \gamma'(r))}{\gamma'(r) \wedge \gamma'(s)}. \tag{4.4}$$

From now on, we omit the dependence on r of γ, k and their derivates; moreover, we indicate $\delta := (s-r)$. The Taylor expansion of the numerator is

$$\begin{gathered} (\gamma(s) - \gamma(r)) \wedge (\gamma'(s) - \gamma'(r)) = \\ = \left(\gamma'\delta + \tfrac{\gamma''}{2}\delta^2 + \tfrac{\gamma'''}{6}\delta^3 + \tfrac{\gamma^{(4)}}{24}\delta^4 + \tfrac{\gamma^{(5)}}{5!}\delta^5 + O(\delta^6)\right) \wedge \left(\gamma''\delta + \tfrac{\gamma'''}{2}\delta^2 + \tfrac{\gamma^{(4)}}{6}\delta^3 + \tfrac{\gamma^{(5)}}{24}\delta^4 + \tfrac{\gamma^{(6)}}{5!}\delta^5 + O(\delta^6)\right) = \\ = k\delta^2 + \tfrac{k'}{2}\delta^3 + \tfrac{1}{6}\left(\tfrac{2k''-k^3}{2}\right)\delta^4 + \left(\tfrac{k'''-3k^2k'}{24}\right)\delta^5 + \left(\tfrac{2k^5-48kk'^2-29k^2k''+6k^{(4)}}{720}\right)\delta^6 + O(\delta^7), \end{gathered}$$

where –in the last equality– we have used formulas (4.2).
Similarly, the Taylor expansion of the denominator is

$$\gamma'(r)\wedge\gamma'(s)=\gamma'\wedge\left(\gamma'+\gamma''\delta+\tfrac{\gamma'''}{2}\delta^2+\tfrac{\gamma^{(4)}}{6}\delta^3+\tfrac{\gamma^{(5)}}{24}\delta^4+\tfrac{\gamma^{(6)}}{5!}\delta^5+O(\delta^6)\right)=$$
$$=k\delta+\tfrac{k'}{2}\delta^2+\left(\tfrac{-k^3+k''}{6}\right)\delta^3+\left(\tfrac{-6k^2k'+k'''}{24}\right)\delta^4+\left(\tfrac{k^5-10k^2k''-15kk'^2+k^{(4)}}{5!}\right)\delta^5+O(\delta^6)=$$
$$=k\delta\left[1-\tfrac{k'}{2k}\delta+\left(\tfrac{2k^4+3k'^2-2kk''}{12k^2}\right)\delta^2-\left(\tfrac{3k'^3-2k'(k^4+2kk'')+k^2k'''}{24k^3}\right)\delta^3+D\delta^4+O(\delta^5)\right]^{-1}$$

where

$$D=\frac{45k'^4-90kk'^2k''+30k^2k'k'''+2k^2(7k^6+10k^3k''+10k''^2-3kk^{(4)})}{720k^4}.$$

By using the above expansions of numerator and denominator, we obtain that

$$H(r,s)=\delta+\tfrac{k^2}{12}\delta^3+\tfrac{kk'}{12}\delta^4+\tfrac{2k^4+4k'^2+7kk''}{240}\delta^5+O(\delta^6),$$

which is the desired result. □

Proposition 10. *The outer length billiard map $T:(s_0,\varepsilon_0)\mapsto(s_1,\varepsilon_1)$ has the following expansion:*

$$\begin{cases}s_1=s_0+\varepsilon_0\\ \varepsilon_1=\varepsilon_0+A(s_0)\varepsilon_0^2+B(s_0)\varepsilon_0^3+C(s_0)\varepsilon_0^4+O(\varepsilon_0^5)\end{cases}\tag{4.5}$$

where

$$A(s)=-\frac{2k'(s)}{3k(s)},\quad B(s)=\frac{10k'^2(s)}{9k^2(s)}-\frac{2k''(s)}{3k(s)}$$

and

$$C(s)=\frac{-24k^4(s)k'(s)-1160k'^3(s)+1200k(s)k'(s)k''(s)-216k^2(s)k'''(s)}{540k^3(s)}.$$

Proof. We start by writing separately the Taylor expansions of numerator and denominator of the radius $\mathcal{R}$ of the circle in $\mathbb{R}^2\setminus\Omega$ tangent to $\partial\Omega$ at $\gamma(s_1)$ and to the line $P\gamma(s_0)$, see Figure 1.

$$\mathcal{R}=\frac{(\gamma(s_1)-\gamma(s_0))\wedge\gamma'(s_1)}{1+\gamma'(s_1)\cdot\gamma'(s_0)}=\frac{(\gamma(s_2)-\gamma(s_1))\wedge\gamma'(s_2)}{1+\gamma'(s_2)\cdot\gamma'(s_1)}.$$

From now on, we indicate –by subscripting 1– the dependence on s_1 of γ, k and their derivates. Moreover, we recall that $\varepsilon_1=s_2-s_1$. The Taylor expansion of the numerator is

$$(\gamma(s_2)-\gamma(s_1))\wedge\gamma'(s_2)=$$
$$=\left(\gamma_1'\varepsilon_1+\tfrac{\gamma_1''}{2}\varepsilon_1^2+\tfrac{\gamma_1'''}{6}\varepsilon_1^3+\tfrac{\gamma_1^{(4)}}{24}\varepsilon_1^4+\tfrac{\gamma_1^{(5)}}{5!}\varepsilon_1^5+O(\varepsilon_1^6)\right)\wedge\left(\gamma_1'+\gamma_1''\varepsilon_1+\tfrac{\gamma_1'''}{2}\varepsilon_1^2+\tfrac{\gamma_1^{(4)}}{6}\varepsilon_1^3+\tfrac{\gamma_1^{(5)}}{24}\varepsilon_1^4+O(\varepsilon_1^5)\right)=$$
$$=\tfrac{k_1}{2}\varepsilon_1^2+\tfrac{k_1'}{3}\varepsilon_1^3+\left(\tfrac{-k_1^3+3k_1''}{24}\right)\varepsilon_1^4+\left(\tfrac{-9k_1^2k_1'+4k_1'''}{120}\right)\varepsilon_1^5+O(\varepsilon_1^6),$$

where –in the last equality– we have used formulas (4.2).
Similarly, the Taylor expansion of the denominator is

$$1+\gamma'(s_2)\cdot\gamma'(s_1)=1+\left(\gamma_1'+\gamma_1''\varepsilon_1+\tfrac{\gamma_1'''}{2}\varepsilon_1^2+\tfrac{\gamma_1^{(4)}}{6}\varepsilon_1^3+\tfrac{\gamma_1^{(5)}}{24}\varepsilon_1^4+O(\varepsilon_1^5)\right)\cdot\gamma_1'=$$
$$=2\left(1-\tfrac{k_1^2}{4}\varepsilon_1^2-\tfrac{k_1k_1'}{4}\varepsilon_1^3+O(\varepsilon_1^4)\right)=2\left(1+\tfrac{k_1^2}{4}\varepsilon_1^2+\tfrac{k_1k_1'}{4}\varepsilon_1^3+O(\varepsilon_1^4)\right)^{-1}.$$

By using the above expansions of numerator and denominator, we obtain that

$$2\mathcal{R} = \frac{k_1}{2}\varepsilon_1^2 + \frac{k_1'}{3}\varepsilon_1^3 + \left(\frac{2k_1^3 + 3k_1''}{24}\right)\varepsilon_1^4 + \left(\frac{16k_1^2k_1' + 4k_1'''}{120}\right)\varepsilon_1^5 + O(\varepsilon_1^6) \tag{4.6}$$

or, equivalently:

$$2\mathcal{R} = \frac{k_1}{2}\varepsilon_0^2 - \frac{k_1'}{3}\varepsilon_0^3 + \underbrace{\left(\frac{2k_1^3 + 3k_1''}{24}\right)}_{:=A_4}\varepsilon_0^4 - \underbrace{\left(\frac{16k_1^2k_1' + 4k_1'''}{120}\right)}_{:=A_5}\varepsilon_0^5 + O(\varepsilon_0^6). \tag{4.7}$$

Substituting the powers of the expansion

$$\varepsilon_1 = \varepsilon_0 + \alpha(s_1)\varepsilon_0^2 + \beta(s_1)\varepsilon_0^3 + \gamma(s_1)\varepsilon_0^4 + O(\varepsilon_0^5)$$

in (4.6), we obtain (we omit in the sequel the dependence on s_0 in α, β and γ):

$$\begin{aligned} 2\mathcal{R} &= \frac{k_1}{2}\varepsilon_0^2 + \left(k_1\alpha + \frac{k_1'}{3}\right)\varepsilon_0^3 + \left(\frac{k_1}{2}(\alpha^2 + 2\beta) + k_1'\alpha + A_4\right)\varepsilon_0^4 + \\ &+ \left(k_1(\alpha\beta + \gamma) + k_1'(\alpha^2 + \beta) + 4\alpha A_4 + A_5\right)\varepsilon_0^5 + O(\varepsilon_0^6). \end{aligned}$$

By equaling the above expansion to (4.7), we have:

$$\alpha(s) = -\frac{2k'(s)}{3k(s)}, \quad \beta(s) = \frac{4k'^2(s)}{9k^2(s)}$$

and

$$\gamma(s) = \frac{-320k'^3(s) + 3k'(s)(-8k^4(s) + 60k(s)k''(s)) - 36k^2(s)k'''(s)}{540k^3(s)}.$$

Finally, from

$$\varepsilon_1 = \varepsilon_0 + \alpha(s_0)\varepsilon_0^2 + (\alpha'(s_0) + \beta(s_0))\varepsilon_0^3 + \left(\frac{\alpha''(s_0)}{2} + \beta'(s_0) + \gamma(s_0)\right)\varepsilon_0^4 + O(\varepsilon_0^5),$$

we obtain the desired result, that is:

$$A(s) = -\frac{2k'(s)}{3k(s)}, \quad B(s) = \frac{10k'^2(s)}{9k^2(s)} - \frac{2k''(s)}{3k(s)}$$

and

$$C(s) = \frac{-24k^4(s)k'(s) - 1160k'^3(s) + 1200k(s)k'(s)k''(s) - 216k^2(s)k'''(s)}{540k^3(s)}.$$

□

Proposition 11. *Let $q \geq 3$. The q-periodic orbits of rotation number $1/q$ for the outer length billiard map have the following expansion:*

$$\begin{cases} s_k = s_0^q + a_0\,(k/q) + \frac{a_1(k/q)}{q} + \frac{a_2(k/q)}{q^2} + O\left(\frac{1}{q^3}\right) \\ \varepsilon_k = \frac{b_1(k/q)}{q} + \frac{b_2(k/q)}{q^2} + \frac{b_3(k/q)}{q^3} + O\left(\frac{1}{q^4}\right) \end{cases} \tag{4.8}$$

where $s_0^q \in \mathbb{R}$ converges to 0 with q, $a_0 : \mathbb{R} \to \mathbb{R}$ is a map such that $a_0(x+1) = a_0(x) + \ell$ for any x and $a_1, a_2, b_1, b_2, b_3 : \mathbb{R} \to \mathbb{R}$ are 1-periodic maps which can be expressed as

$$\begin{cases} a_0^{-1}(s) = \frac{1}{L}\int_0^s k^{2/3}(r)dr := x(s), \qquad L := \int_0^\ell k^{2/3}(r)dr \\ \\ a_1(x) = 0 \\ \\ a_2(x) = k^{-\frac{2}{3}}(a_0(x))\left(\int_0^x L^3\left(\frac{1}{810}\left(9k''k^{-\frac{7}{3}} - 12(k')^2k^{-\frac{10}{3}}\right) + \frac{k^{\frac{2}{3}}}{15}\right)(a_0(t))dt + cx\right) \\ \\ b_1(x) = a_0'(x) = Lk^{-2/3}(a_0(x)) \\ \\ b_2(x) = \frac{a_0''(x)}{2} = -\frac{L^2k'(a_0(x))k^{-7/3}(a_0(x))}{3} \\ \\ b_3(x) = a_2' + \frac{a_0'''}{6}. \end{cases} \tag{4.9}$$

The constant c in the expression of a_2 is such that $L^3\left(\frac{1}{810}\left(9k''k^{-\frac{7}{3}} - 12(k')^2k^{-\frac{10}{3}}\right) + \frac{k^{\frac{2}{3}}}{15}\right) + c$ has zero mean.

Proof. Since the points in the orbits are equidistributed as $q \to +\infty$, for any q we can choose the first point of the orbit s_0^q such as $s_0^q \to 0$ for $q \to +\infty$. To simplify the notations, we omit the dependence of a_i and b_j on k/q.
On one hand, combining the expansions in (4.5), we have

$$\begin{aligned} \varepsilon_{k+1} - \varepsilon_k = A(s_k)\varepsilon_k^2 + B(s_k)\varepsilon_k^3 + C(s_k)\varepsilon_k^4 + O(\varepsilon_k^5) = \\ = \frac{A(s_0^q + a_0)b_1^2}{q^2} + \frac{B(s_0^q+a_0)b_1^3 + A'(s_0^q+a_0)a_1b_1^2 + 2A(s_0^q+a_0)b_1b_2}{q^3} + \frac{F(a_i,b_j)}{q^4} + O\left(\frac{1}{q^5}\right), \end{aligned}$$

where

$$\begin{aligned} F(a_i,b_j) \;&:=\; A(s_0^q+a_0)b_2^2 + 2A(s_0^q+a_0)b_1b_3 + 2A'(s_0^q+a_0)a_1b_1b_2 + A'(s_0^q+a_0)a_2b_1^2 + \\ &+\; A''(s_0^q+a_0)a_1^2b_1^2/2 + 3B(s_0^q+a_0)b_1^2b_2 + B'(s_0^q+a_0)a_1b_1^3 + C(s_0^q+a_0)b_1^4. \end{aligned}$$

Moreover, directly from the second expansion in (4.8), we have

$$\varepsilon_{k+1} - \varepsilon_k = \frac{b_1'}{q^2} + \frac{b_2' + b_1''/2}{q^4} + \frac{b_3' + b_2''/2 + b_1'''/6}{q^5} + O\left(\frac{1}{q^5}\right).$$

Equaling these two expansions, we obtain that a_i and b_j solve

$$\begin{cases} A(s_0^q + a_0)b_1^2 = b_1' \\ B(s_0^q+a_0)b_1^3 + A'(s_0^q+a_0)a_1b_1^2 + 2A(s_0^q+a_0)b_1b_2 = b_2' + b_1''/2 \\ F(a_i,b_j) = b_3' + b_2''/2 + b_1'''/6 \end{cases} \tag{4.10}$$

From the other hand, directly from the first expansion in (4.5), we conclude that

$$s_{k+1} - s_k = \frac{a_0'}{q} + \frac{a_1' + a_0''/2}{q^2} + \frac{a_2' + a_1''/2 + a_0'''/6}{q^3} + O\left(\frac{1}{q^4}\right),$$

which –compared which the second expansion in (4.5)– gives the system

$$\begin{cases} a_0' = b_1 \\ a_1' + a_0''/2 = b_2 \\ a_2' + a_1''/2 + a_0'''/6 = b_3 \end{cases} \tag{4.11}$$

• Expressions of a_0 and b_1. To compute a_0 and b_1, we solve the system

$$\begin{cases} b_1 = a_0' \\ b_1' = A(s_0^q + a_0)b_1^2 \end{cases} \tag{4.12}$$

Replacing b_1 by a_0' in the second equation, it gives

$$a_0'' = (a_0')^2 A(s_0^q + a_0). \tag{4.13}$$

If we denote by $A_1(s) = -\frac{2}{3}\log k(s)$ a primitive of A, then from Equation (4.13) follows

$$\left(a_0' e^{-A_1(s_0^q+a_0)}\right)' = 0.$$

Hence $a_0' e^{-A_1(s_0^q+a_0)}$ is constant. Consider now $A_2(s) = \int_0^s k^{2/3}(r)dr$, which is a primitive of $\exp(-A_1)$. We just proved that $A_2(s_0^q + a_0)$ has constant derivative, hence it must be of the form $A_2(s_0^q + a_0(x)) = ux + v$ for any $x \in \mathbb{R}$, where $u, v \in \mathbb{R}$. Since, by definition, $A_2(s_0^q + a_0(0)) = A_2(s_0^q) = v$, we have $v = A_2(s_0^q)$. The expression of u is given by $u = A_2(s_0^q + a_0(1)) - A_2(s_0^q) = A_2(s_0^q + \ell) - A_2(s_0^q) = \int_0^\ell k^{2/3}(r)dr$. Finally, b_1 follows from $b_1 = a_0'$.
•• Expressions of a_1 and b_2. To compute a_1 and b_2, we solve the system

$$\begin{cases} b_2 = a_1' + a_0''/2 \\ b_2' + b_1''/2 = B(s_0^q + a_0)b_1^3 + A'(s_0^q + a_0)a_1 b_1^2 + 2A(s_0^q + a_0)b_1 b_2. \end{cases} \tag{4.14}$$

Note that the terms note containing a_1 nor b_2 can be computed using the expression of a_0 and b_1 we just obtained. Let us replace in the second equation of (4.14) b_2 by the expression given by the first equation: we obtain an equation for which we split the terms containing a_1 from the others. Namely,

$$a_1'' - 2A(s_0^q + a_0)b_1 a_1' - A'(s_0^q + a_0)a_1 b_1^2 = A(s_0^q + a_0)b_1 a_0'' + B(s_0^q + a_0)b_1^3 - \tfrac{1}{2}b_1'' - \tfrac{1}{2}a_0^{(3)}. \tag{4.15}$$

Replacing a_0 and b_1 by the expressions we just found, the left-hand side of (4.15) can be expressed as

$$a_1'' + \tfrac{4L}{3}k^{-5/3}k' a_1' + \tfrac{2L^2}{3}\left(k^{-7/3}k'' - k^{-10/3}k'^2\right)a_1 = k^{-2/3}(a_1 k^{2/3})'',$$

where it is implicitly understood that k and its derivatives are evaluated in $s_0^q + a_0$. The right-hand side of (4.15) vanishes. Hence equation (4.15) is equivalent to

$$k^{-2/3}(a_1 k^{2/3})'' = 0.$$

Since a_1 is periodic and vanishes at 0, we necessarily have $a_1 = 0$. The expression of b_2 comes from the first equation of (4.14), namely $b_2 = a_0''/2$.
• • • Expressions of a_2 and b_3. Although it will not be employed in the subsequent computations, we

shall derive an explicit expression for the coefficient a_2. By making use of equations (4.10) and (4.11), and taking into account that $a_1 = 0$, we obtain the following system:

$$\begin{cases} b_3 = a_2' + \frac{a_0'''}{6} \\ A'(s_0^q + a_0)b_1^2 a_2 + A(s_0^q + a_0)\left(b_2^2 + 2b_1 b_3\right) + B(s_0^q + a_0)3b_1^2 b_2 + C(s_0^q + a_0)b_1^4 = \frac{b_1'''}{6} + \frac{b_2''}{2} + b_3' \end{cases} \tag{4.16}$$

From the first equation of (4.16) we have $b_3' = a_2'' + \frac{a_0^{(4)}}{6}$ which in turn gives

$$b_3' = a_2'' + \left(\frac{11k'k''k^{-\frac{10}{3}}}{27} - \frac{8(k')^3 k^{\frac{13}{3}}}{27} - \frac{k'''k^{-\frac{7}{3}}}{9} \right).$$

Replacing into the second of (4.16) and grouping all the terms with a_2, we get

$$(k^{\frac{2}{3}} a_2)'' = L^4 \left(\frac{40(k')^3 - 45kk'k'' + 9k^2 k''}{810k^5} + \frac{2k'}{45k} \right). \tag{4.17}$$

The second term is the derivative of

$$L^3 \left(\frac{1}{810} \left(9k''k^{-\frac{7}{3}} - 12(k')^2 k^{-\frac{10}{3}} \right) + \frac{k^{\frac{2}{3}}}{15} \right) + c,$$

where c is a constant such that this function has zero mean. Consequently, at this point we can integrate another time and get

$$a_2(x) = k^{-\frac{2}{3}}(a_0(x)) \left(\int_0^x L^3 \left(\frac{1}{810} \left(9k''k^{-\frac{7}{3}} - 12(k')^2 k^{-\frac{10}{3}} \right) + \frac{k^{\frac{2}{3}}}{15} \right) (a_0(t))dt + cx \right).$$

The value of b_3 can now be easily derived from the first one of (4.16). □

5 Proof of Theorem 6

This section is entirely devoted to the proof of Theorem 6, providing coefficient β_5 for the outer length billiard map.

Proof. We start the computation of the beta function by writing its value at rational points of the form $\frac{1}{q}$, which (by the expansion (4.3) of the generating function H) is

$$\beta\left(\frac{1}{q}\right) = \frac{1}{q} \sum_{n=0}^{q-1} H\left(s_n, s_{n+1}\right) = \frac{1}{q} \sum_{n=0}^{q-1} \varepsilon_n + \frac{k^2}{12}\varepsilon_n^3 + \frac{kk'}{12}\varepsilon_n^4 + \frac{2k^4 + 4k'^2 + 7kk''}{240}\varepsilon_n^5 + O(\varepsilon_n^6). \tag{5.1}$$

Here, the curvature k and its derivates k' and k'' are to be understood as evaluated in s_n.
Now, we substitute in the above formula s_n and ε_n with their corresponding Taylor expansions obtained in Proposition 11. We then proceed to group the various terms according to their order of magnitude q^k.

First, we observe that the summation of ε_n is simply equal to the perimeter ℓ of D, so that $\beta_1 = \ell$.

By inspecting the formula even before performing the substitution, we see that there are no terms of order q^{-2}, so that $\beta_2 = 0$, as expected by Marvizi-Melrose's theory.

The second term of the summation on the right-hand side of (5.1) becomes, after the substitution and after grouping the various powers of q,

$$\begin{aligned}\frac{1}{12}\sum_{n=0}^{q-1} k^2(s_n)\,\varepsilon_n^3 &= \frac{1}{12}\sum_{n=0}^{q-1} k^2\left(a_0+\frac{a_2}{q^2}+O\left(\frac{1}{q^3}\right)\right)\left(\frac{b_1}{q}+\frac{b_2}{q^2}+\frac{b_3}{q^3}\right)^3 = \\ &= \sum_{n=0}^{q-1}\frac{k^2b_1^3}{12}\frac{1}{q^3}+\frac{k^2b_1^2b_2}{4}\frac{1}{q^4}+\frac{2kk'b_1^3a_2+3k^2\left(b_1^2b_3+b_1b_2^2\right)}{12}\frac{1}{q^5}+O\left(\frac{1}{q^6}\right).\end{aligned} \tag{5.2}$$

Similarly, we have that

$$\begin{aligned}\frac{1}{12}\sum_{n=0}^{q-1} k(s_n)\,k'(s_n)\,\varepsilon_n^4 &= \frac{1}{12}\sum_{n=0}^{q-1} kk'\left(\frac{b_1}{q}+\frac{b_2}{q^2}+O\left(\frac{1}{q^3}\right)\right)^4 = \\ &= \sum_{n=0}^{q-1}\frac{kk'b_1^4}{12}\frac{1}{q^4}+\frac{kk'b_1^3b_2}{3}\frac{1}{q^5}+O\left(\frac{1}{q^6}\right).\end{aligned} \tag{5.3}$$

Finally, the last term is

$$\sum_{n=0}^{q-1}\left(\frac{2k^4+4k'^2+7kk''}{240}\right)b_1^5\frac{1}{q^5}+O\left(\frac{1}{q^6}\right). \tag{5.4}$$

We recall that, in the last three formulas, it is implicitly understood that all functions a_i, b_i are evaluated at n/q, and the curvature k and its derivatives k' and k'', where not explicitly specified, are computed at $s_0^q + a_0(n/q)$. To determine β_3, we compute $\lim_{q\to+\infty} q^3\left(\beta\left(\frac{1}{q}\right)-\frac{\ell}{q}\right)$.
From formulas (5.2), (5.3), and (5.4), we simply obtain that

$$\beta_3 = \lim_{q\to+\infty}\frac{1}{12q}\sum_{n=0}^{q-1}\left(k^2b_1^3+O\left(\frac{1}{q}\right)\right)$$

By Proposition 11, we have that $b_1 = Lk^{-\frac{2}{3}} \Rightarrow k^2b_1^3 = L^3$, so that

$$\beta_3 = \frac{1}{12}L^3 = \frac{1}{12}\left(\int_0^\ell k^{2/3}(r)dr\right)^3.$$

Note that the leading part of this limit is constant, while the term denoted by $O(1/q)$ contains only higher-order terms. We will take this into account when analyzing $\beta - \ell/q - \beta_3/q^3$, considering only the terms present in $O(1/q)$.

Regarding the terms of order 4, we obtain the following expression:

$$\sum_{n=0}^{q-1}\left(\frac{k^2b_1^2b_2}{4}+\frac{kk'b_1^4}{12}\right)\frac{1}{q^4}.$$

Since, by Proposition 11, we have that $b_1 = Lk^{-\frac{2}{3}}$ and $b_2 = -\frac{L^2k'k^{-\frac{7}{3}}}{3}$, we immediately conclude (again, as expected by Marvizi-Melrose's theory), that $\beta_4 = 0$.

The terms of order 5 are

$$\sum_{n=0}^{q-1}\left[\frac{2k^4+4k'^2+7kk''}{240}b_1^5+\frac{1}{12}\left(2kk'b_1^3a_2+3k^2\left(b_3b_1^2+b_2^2b_1\right)+4kk'b_1^3b_2\right)\right]\frac{1}{q^5}.$$

Since –by Proposition 11)– $b_3 = a_2' + \frac{a_0'''}{6}$, we substitute it into the previous equation and separate the summation containing the terms with a_2:

$$\underbrace{\sum_{n=0}^{q-1}\frac{1}{12}\left(2kk'b_1^3a_2+3k^2b_1^2a_2'\right)}_{:=S_1}+\underbrace{\sum_{n=0}^{q-1}\left(\frac{2k^4+4k'^2+7kk''}{240}b_1^5+\frac{1}{12}\left(3k^2b_2^2b_1+4kk'b_1^3b_2+\frac{1}{2}k^2a_0'''b_1^2\right)\right)}_{:=S_2}.$$

We remark that the sum S_1 contains a_2 and the sum S_2 doesn't contain a_2. As established earlier, we have

$$\beta_5 = \lim_{q\to+\infty} q^5\left(\beta\left(\frac{1}{q}\right)-\frac{\ell}{q}-\frac{\beta_3}{q^3}\right) = \lim_{q\to+\infty}\frac{1}{q}(S_1+S_2).$$

By studying the limit $\lim_{q\to+\infty}\frac{1}{q}S_1$, we obtain:

$$\begin{aligned}&\frac{1}{12}\lim_{q\to+\infty}\frac{1}{q}\sum_{n=0}^{q-1}\left(2kk'k^{-2}a_2L^3+3k^2k^{-\frac{4}{3}}a_2'L^2\right)=\\&\frac{1}{12}\int_0^1\left(\frac{2k'}{k}\left(a_0(x)\right)a_2(x)L^3+3k^{\frac{2}{3}}\left(a_0(x)\right)a_2'(x)L^2\right)dx\end{aligned}\tag{5.5}$$

where –once again– in the summations we have used the convention that the functions a_i, b_i are evaluated at n/q, while the functions k, k' are evaluated at $a_0(n/q)$. Similarly, in the integral on the right-hand side, a_i, b_i are evaluated at x, and k, k' at $a_0(x)$. By integrating by parts the second term inside the integral, we have

$$\int_0^1 k^{\frac{2}{3}}\left(a_0(x)\right)a_2'(x)L^2dx = k^{\frac{2}{3}}\left(a_0(x)\right)a_2(x)L^2\Big|_0^1-\int_0^1\frac{2k'}{3k}\left(a_0(x)\right)a_2(x)L^3dx. \tag{5.6}$$

By periodicity, the first term is 0. By substituting the remaining expression of (5.6) inside (5.5), we conclude that the first limit is 0.
Let us proceed with the calculation of the limit $\lim_{q\to+\infty}\frac{1}{q}S_2$. Taking into account that $a_0'(x) = Lk^{-\frac{2}{3}}(a_0(x))$, see (4.9), we have that

$$a_0''' = L^3\left(-\frac{2k''}{3k^3}+\frac{14k'^2}{9k^4}\right).$$

Taking into account the expressions of a_i, b_j given in (4.9) and by substituting the previous expression into S_2, we obtain:

$$\lim_{q\to+\infty}\frac{1}{q}S_2=\lim_{q\to+\infty}\frac{1}{q}\sum_{n=0}^{q-1}\frac{2k^4+4k'^2+7kk''}{240}b_1^5+\frac{1}{12}\left(3k^2b_2^2b_1+4kk'b_1^3b_2+\frac{1}{2}k^2a_0'''b_1^2\right)=$$
$$=\lim_{q\to+\infty}\frac{L^5}{q}\sum_{n=0}^{q-1}\left(\frac{k^{2/3}}{120}-\frac{k^{-\frac{10}{3}}k'^2}{540}+\frac{k^{-\frac{7}{3}}k''}{720}\right)=L^5\int_0^1\left(\frac{k^{2/3}}{120}-\frac{k^{-\frac{10}{3}}k'^2}{540}+\frac{k^{-\frac{7}{3}}k''}{720}\right)dx. \tag{5.7}$$

We finally integrate by parts the last term of the integral, obtaining:

$$\int_0^1 L^5k^{-\frac{7}{3}}k''dx=L^4\int_0^1 k^{-5/3}k''\left(k^{-\frac{2}{3}}L\right)dx=L^4\int_0^1 k^{-\frac{5}{3}}\left(k'\right)'dx=\frac{5L^5}{3}\int_0^1 k^{-\frac{10}{3}}k'^2dx.$$

Replacing the expression above in (5.7), we conclude that

$$\lim_{q\to+\infty}\frac{1}{q}S_2=L^5\int_0^1\left(\frac{k^{2/3}}{120}+\frac{k^{-\frac{10}{3}}k'^2}{2160}\right)dx$$

Finally, switching to arc length as the variable of integration, we obtain the desired result:

$$\beta_5=L^4\int_0^\ell\left(\frac{k^{4/3}(s)}{120}+\frac{k^{-\frac{8}{3}}(s)k'^2(s)}{2160}\right)ds.$$

□

6 Lazutkin coordinates and caustics

A consequence of Proposition 10 is that we can compute explicitly Lazutkin coordinates [14] for order 4 in the case of outer length billiards.

Lemma 12 (Lazutkin for outer length billiards)**.** *The coordinates*

$$\begin{aligned} x(s) &= \frac{1}{L}\int_0^s k^{2/3}(r)dr, \qquad L:=\int_0^\ell k^{2/3}(r)dr \\ y(s,\varepsilon) &= x(s+\varepsilon)-x(s) \end{aligned}$$

are so that the outer length billiard dynamics is given by

$$x\mapsto x+y,\qquad y\mapsto y+O(y^4).$$

Proof. Let

$$(s,\varepsilon)\mapsto(x,y):=(f(s),f(s+\varepsilon)-f(s))$$

a change of coordinates so that $(x_k,y_k)\mapsto(x_k+y_k,y_{k+1})$. Then –by using the expansion of ε_1 given in (10)– we have

$$
\begin{aligned}
y_1 &= x_2 - x_1 = f(s_1+\varepsilon_1) - f(s_1) = f'(s_1)\varepsilon_1 + \frac{f''(s_1)}{2}\varepsilon_1^2 + \frac{f'''(s_1)}{6}\varepsilon_1^3 + O(\varepsilon_1^4) = \\
&= \left(f'(s_0) + f''(s_0)\varepsilon_0 + \frac{f'''(s_0)}{2}\varepsilon_0^3\right)\left(\varepsilon_0 + A(s_0)\varepsilon_0^2 + B(s_0)\varepsilon_0^3\right) + \\
&+ \left(f''(s_0) + f'''(s_0)\varepsilon_0\right)\frac{\left(\varepsilon_0 + A(s_0)\varepsilon_0^2 + B(s_0)\varepsilon_0^3\right)^2}{2} + \frac{f'''(s_0)}{6}\varepsilon_0^3 + O(\varepsilon_0^4) = \\
&= \left(f'(s_0)\varepsilon_0 + \frac{f''(s_0)}{2}\varepsilon_0^2 + \frac{f'''(s_0)}{6}\varepsilon_0^3\right) + \left(f''(s_0) + f'(s_0)A(s_0)\right)\varepsilon_0^2 + \\
&+ \left(f'(s_0)B(s_0) + 2f''(s_0)A(s_0) + f'''(s_0)\right)\varepsilon_0^3 + O(\varepsilon_0^4) = \\
&= y_0 + \left(f''(s_0) + f'(s_0)A(s_0)\right)\varepsilon_0^2 + \left(f'(s_0)B(s_0) + 2f''(s_0)A(s_0) + f'''(s_0)\right)\varepsilon_0^3 + O(\varepsilon_0^4).
\end{aligned}
$$

Consequently, if we want to get rid of the ε_0^2 and ε_0^3 terms, we need to choose f solving

$$
\begin{cases}
f''(s_0) + f'(s_0)A(s_0) = 0 \\
f'(s_0)B(s_0) + 2f''(s_0)A(s_0) + f'''(s_0) = 0
\end{cases}
$$

Integrating the first equation, we immediately obtain the desired formula for f, giving (up to normalization):

$$
x(s) = \frac{1}{L}\int_0^s k^{2/3}(r)dr, \qquad L := \int_0^\ell k^{2/3}(r)dr.
$$

Then, by direct computation, it is easy to check that such a function solves also the second equation. $\square$

As a consequence, the outer length billiard map is a small perturbation of the integrable map

$$
(x, y) \mapsto (x+y, y),
$$

satisfying the assumptions of Lazutkin's theorem [14][Theorem 1]. Applying this theorem, the next corollary of Proposition 10 immediately follows.

Theorem 13. *Arbitrary close to the boundary $\partial\Omega$, there exist smooth caustics for the outer length billiard map. The union of these caustics has positive measure.*

On the other hand –regarding the non existence of caustics– we underline that the following outer length billiard version of Mather's theorem still holds.

Theorem 14. *If the curvature of the boundary $\partial\Omega$ vanishes at some point, then the outer length billiard in $\partial\Omega$ has no caustics.*

Proof. We use Mather's necessary analytic condition for the existence of a caustic [16], that is

$$
H_{22}(s_0, s_1) + H_{11}(s_1, s_2) < 0.
$$

By using the general expression of the generating function (4.4), it is easily seen that

$$
\begin{aligned}
H_1(s_1, s_2) = -1 - \frac{(\gamma(s_2) - \gamma(s_1)) \wedge \gamma''(s_1)}{\gamma'(s_1) \wedge \gamma'(s_2)} - \\
\frac{(\gamma(s_2) - \gamma(s_1)) \wedge (\gamma'(s_2) - \gamma'(s_1)) \cdot (\gamma''(s_1) \wedge \gamma'(s_2))}{(\gamma'(s_1) \wedge \gamma'(s_2))^2}.
\end{aligned}
$$

Hence, we have that

$$
\begin{aligned}
H_{11}(s_1,s_2) &= \frac{\gamma'\left(s_1\right)\wedge\gamma''\left(s_1\right)}{\gamma'\left(s_1\right)\wedge\gamma'\left(s_2\right)} - \frac{\left(\gamma\left(s_2\right)-\gamma\left(s_1\right)\right)\wedge\gamma'''\left(s_1\right)}{\gamma'\left(s_1\right)\wedge\gamma'\left(s_2\right)} + \\
&+ 2\frac{\left(\gamma\left(s_2\right)-\gamma\left(s_1\right)\right)\wedge\gamma''\left(s_1\right)}{\left(\gamma'\left(s_1\right)\wedge\gamma'\left(s_2\right)\right)^2}\left(\gamma''\left(s_1\right)\wedge\gamma'\left(s_2\right)\right) + \\
&+ \frac{\gamma''\left(s_1\right)\wedge\gamma'\left(s_2\right)}{\left(\gamma'\left(s_1\right)\wedge\gamma'\left(s_2\right)\right)^2} - \frac{\left(\gamma\left(s_2\right)-\gamma\left(s_1\right)\right)\wedge\left(\gamma'\left(s_2\right)-\gamma'\left(s_1\right)\right)}{\left(\gamma'\left(s_1\right)\wedge\gamma'\left(s_2\right)\right)^2}\left(\gamma'''\left(s_1\right)\wedge\gamma'\left(s_2\right)\right) + \\
&+ 2\frac{\left(\gamma\left(s_2\right)-\gamma\left(s_1\right)\right)\wedge\left(\gamma'\left(s_2\right)-\gamma'\left(s_1\right)\right)}{\left(\gamma'\left(s_1\right)\wedge\gamma'\left(s_2\right)\right)^3}\left(\gamma''\left(s_1\right)\wedge\gamma'\left(s_2\right)\right)^2 .
\end{aligned}
$$

Let us now assume that at a point on the boundary corresponding to the arc-length parameter value s_1, the curvature is zero, that is, $k(s_1) = 0$. Since the set is convex, this condition implies that also $k'(s_1) = 0$. From formulas (4.2), it follows that $\gamma''(s_1) = \gamma'''(s_1) = 0$. Substituting into the previous formula, we see that all the terms composing H_{11} vanish, and a similar argument holds for H_{22}. As a consequence, we have that $H_{11}(s_1, s_2) + H_{22}(s_0, s_1) = 0$ for every s_0, s_2, and therefore no topologically nontrivial invariant curve can exist.

□

References

[1] Albers, A.; Tabachnikov, S. Introducing symplectic billiards. Adv. Math. 333, 822–867, (2018).

[2] Albers, A.; Tabachnikov, S. Monotone twist maps and Dowker-type theorems. Pacific Journal of Mathematics, Vol. 330 (2024), No. 1, 1–24.

[3] Bangert, V. Mather Sets for Twist Maps and Geodesics on Tori. In: Kirchgraber, U., Walther, HO. (eds) Dynamics Reported. Dynamics Reported, vol 1, (1988).

[4] Baracco, L.; Bernardi, O. Totally integrable symplectic billiards are ellipses, Advances in Mathematics, Volume 454, (2024).

[5] Baracco, L.; Bernardi, O.; Nardi, A. Bialy-Mironov type rigidity for centrally symmetric symplectic billiards, Nonlinearity 37 125025, (2024).

[6] Baracco, L.; Bernardi, O.; Nardi, A. Higher order terms of Mather's β-function for symplectic and outer billiards, Journal of Mathematical Analysis and Applications, Volume 537, Issue 2, (2024).

[7] Baracco, L.; Bernardi, O.; Nardi, A. Area spectral rigidity for axially symmetric and Radon domains, ArXiv, https://arxiv.org/abs/2410.12644

[8] Bialy, M. Convex billiards and a theorem by E. Hopf. Math Z 214, 147–154 (1993).

[9] Bialy, M. Mather β-Function for Ellipses and Rigidity. Entropy, 24, 1600, (2022).

[10] Cieślak, W., Góźdź, S. Extremal polygons circumscribed an oval. Acta Math Hung 54, 63–68 (1989).

[11] DeTemple, D. The geometry of circumscribing polygons of minimal perimeter. J Geom 49, 72–89 (1994).

[12] Fierobe, C.; Sorrentino, A.; Vig, Amir. Deformational spectral rigidity of axially-symmetric symplectic billiards. ArXiv, https://arxiv.org/abs/2410.13777

[13] Glutsyuk A. On infinitely many foliations by caustics in strictly convex open billiards. Ergodic Theory and Dynamical Systems. 44(5), 1418-1467, (2024).

[14] Lazutkin, V.F. The existence of caustics for a billiard problem in a convex domain, Mathematics of the USSR-Izvestiya, Volume 7, Issue 1, 185–214, (1973).

[15] Marvizi S.; Melrose R. Spectral invariants of convex planar regions. J. Differential Geom., 17 (3): 475-503, (1982).

[16] Mather, J.N. Glancing billiards, Ergodic Theory Dynam. Systems, Ergodic Theory and Dynamical Systems, 2, 3-4, 397-403, (1982).

[17] McClure, D.E.; Vitale, R.A. Polygonal approximation of plane convex bodies, Journal of Mathematical Analysis and Applications, Volume 51, Issue 2, (1975).

[18] Siburg, K.F. The principle of least action in geometry and dynamics. Lecture Notes Math. 1844. Springer-Verlag, Berlin, (2004).

[19] A. Sorrentino. Computing Mather's β-function for Birkhoff billiards. Discrete and Contin. Dyn. Syst. A, 35, 5055-5082, (2015).

[20] Stachel H. The geometry of billiards in ellipses and their poncelet grids, Journal of Geometry, 40, 112, 3 (2021).

[21] Tabachnikov, S. On the dual billiard problem. Adv. in Math. 115, 221–249, (1995).

[22] Tabachnikov, S. Geometry and billiards. Amer. Math. Soc., Providence, RI, (2005).

[23] Tsodikovich, D. Local rigidity for symplectic billiards, ArXiv, https://arxiv.org/abs/2501.08849

[24] Zhang, J. Spectral invariants of convex billiard maps: a viewpoint of Mather's beta function, SCIENCE CHINA Mathematics, ISSN 1674-7283, (2021).