

# 1 Fourier Transform

We begin with some basic definitions.

**Definition 1.1 ( $L^1$  function).** The function class  $L^1(\mathbb{R})$  is the set of all functions  $f: \mathbb{R} \rightarrow \mathbb{C}$  for which

$$\int_{\mathbb{R}} |f(x)| dx < \infty.$$

**Definition 1.2 (Fourier transform).** Given  $f \in L^1(\mathbb{R})$ , its Fourier Transform  $\hat{f}: \mathbb{R} \rightarrow \mathbb{C}$  is defined as

$$\hat{f}(w) := \int_{\mathbb{R}} f(x) e^{-2\pi i x w} dx.$$

Note that

$$|\hat{f}(w)| \leq \int_{\mathbb{R}} |f(x) e^{-2\pi i x w}| dx \leq \int_{\mathbb{R}} |f(x)| \cdot |e^{-2\pi i x w}| dx = \int_{\mathbb{R}} |f(x)| dx < \infty$$

for all  $w \in \mathbb{R}$ , thus  $\hat{f}(w)$  is finite for all  $w$ . (The second to last step uses the fact that  $|e^{-2\pi i x w}| = 1$ .)

*Example 1.3.* The Fourier transform of the Gaussian function  $f(x) = e^{-\pi x^2}$  is

$$\begin{aligned} \hat{f}(w) &= \int_{\mathbb{R}} e^{-\pi x^2} \cdot e^{-2\pi i x w} dx \\ &= \int_{\mathbb{R}} e^{-\pi((x+iw)^2 + w^2)} dx \\ &= e^{-\pi w^2} \int_{\mathbb{R}} e^{-\pi(x+iw)^2} dx \\ &= e^{-\pi w^2} \int_{\mathbb{R}} e^{-\pi x^2} dx^1 \\ &= e^{-\pi w^2} 2 \\ &= f(w), \end{aligned}$$

i.e., the Gaussian is its own Fourier transform.

<sup>1</sup>This step follows from applying Cauchy's integral formula to the contour integral around the closed path  $R \rightarrow R - iw \rightarrow -R - iw \rightarrow -R \rightarrow R$ , as  $R \rightarrow \infty$ . Since  $e^{-\pi x^2}$  is "nice enough" (holomorphic), this integral must be 0. As  $R \rightarrow \infty$ , the integrals from  $[R \rightarrow R - iw]$  and  $[-R - iw \rightarrow -R]$  are each bounded in absolute value by  $e^{-\pi w R^2}$ , hence they both approach 0. Therefore,

$$0 = \lim_{R \rightarrow \infty} \int_{-R}^R e^{-\pi x^2} dx + \int_{R-iw}^{-R-iw} e^{-\pi x^2} dx = \lim_{R \rightarrow \infty} \int_{-R}^R e^{-\pi x^2} dx + \int_R^{-R} e^{-\pi(u+iw)^2} du.$$

The result follows by swapping the bounds of integration in the second integral, and taking  $R \rightarrow \infty$ .

<sup>2</sup>This follows from integrating the square in polar coordinates:

$$\int_{\mathbb{R}} e^{-\pi x^2} dx = \sqrt{\int_{\mathbb{R}^2} e^{-\pi x^2} e^{-\pi y^2} dx} = \sqrt{\int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} e^{-\pi r^2} \cdot r dr d\theta} = \sqrt{\int_{\theta=0}^{2\pi} \frac{-1}{2\pi} (0 - 1) d\theta} = 1.$$

**Basic facts.** We recall some important facts about the Fourier transform.

**Fact 1.4 (Linearity).** For all  $f, g \in L^1(\mathbb{R})$  and all  $a \in \mathbb{R}$ ,  $\widehat{f+g} = \hat{f} + \hat{g}$  and  $\widehat{a \cdot f} = a\hat{f}$ .

**Fact 1.5 (Time shift).** Let  $f \in L^1(\mathbb{R})$  and  $h(x) = f(x - c)$  for some  $c \in \mathbb{R}$ . Then  $\hat{h}(w) = e^{-2\pi icw} \cdot \hat{f}(w)$ :

$$\hat{h}(w) = \int_{\mathbb{R}} f(x - c) e^{-2\pi i x w} dx = \int_{\mathbb{R}} f(u) e^{-2\pi i(u+c)w} du = e^{-2\pi icw} \cdot \hat{f}(w).$$

**Fact 1.6 (Time scale).** Let  $f \in L^1(\mathbb{R})$  and  $h(x) = f(x/s)$  for some  $s > 0$ . Then  $\hat{h}(w) = s\hat{f}(sw)$ :

$$\hat{h}(w) = \int_{\mathbb{R}} f(x/s) e^{-2\pi i x w} dx = s \int_{\mathbb{R}} f(u) e^{-2\pi i s u w} du = s \int_{\mathbb{R}} f(u) e^{-2\pi i u (sw)} du = s\hat{f}(sw).$$

**Fact 1.7 (Transform inversion).** For any  $f \in L^1(\mathbb{R})$ ,

$$f(x) = \int_{\mathbb{R}} \hat{f}(w) e^{2\pi i x w} dw.$$

In words, this says that  $f(x)$  is a linear combination of “character” functions of the form  $e^{2\pi i x w}$  for  $w \in \mathbb{R}$ , each weighted by the Fourier coefficient  $\hat{f}(w)$ . Each component  $e^{2\pi i x w}$  is periodic with period  $1/w$  (for  $w \neq 0$ ), hence frequency  $w$ . Note that [Fact 1.7](#) implies that  $\hat{\hat{f}}(x) = f(-x)$ .

## 2 Fourier Series for Periodic Functions

Now consider a function  $g: \mathbb{R} \rightarrow \mathbb{C}$  (not necessarily in  $L^1(\mathbb{R})$ ) that is periodic with unit period, i.e.,  $g(x) = g(x + z)$  for any  $z \in \mathbb{Z}$ . We call such a function  $\mathbb{Z}$ -periodic. Equivalently, we can work with the function  $g: (\mathbb{R}/\mathbb{Z}) \rightarrow \mathbb{C}$ , since the original  $g$  is constant over any fixed coset  $c + \mathbb{Z}$ . Such an  $g$  can be decomposed as a linear combination of character functions with *integer* frequencies.

**Definition 2.1 (Fourier series).** For  $g: (\mathbb{R}/\mathbb{Z}) \rightarrow \mathbb{C}$ , its Fourier series  $\hat{g}: \mathbb{Z} \rightarrow \mathbb{C}$  is

$$\hat{g}(w) = \int_{\mathbb{R}/\mathbb{Z}} g(x + \mathbb{Z}) e^{-2\pi i x w} dx.$$

Equivalently, the integral can be taken over any fundamental region of  $\mathbb{Z}$ ; often it is convenient to use  $[0, 1)$ .

[Facts 1.4](#) to [1.6](#) also apply to the Fourier series, by essentially the same proofs. But there is a slightly different inversion formula.

**Fact 2.2 (Series inversion).**

$$g(x + \mathbb{Z}) = \sum_{w \in \mathbb{Z}} \hat{g}(w) e^{2\pi i x w}.$$

*Example 2.3.* The Fourier series of  $g(x) = e^{2\pi i x k}$ , where  $k \in \mathbb{Z}$  (so  $g$  is  $\mathbb{Z}$ -periodic), can be derived in two different ways. First, we calculate the Fourier series directly: for any  $w \in \mathbb{Z}$ ,

$$\begin{aligned} \hat{g}(w) &= \int_{[0,1)} e^{2\pi i x k} e^{-2\pi i x w} dx \\ &= \int_{[0,1)} e^{2\pi i x (k-w)} dx. \end{aligned}$$

When  $w \neq k$ ,  $e^{2\pi i x(k-w)}$  completes a nonzero integer number of revolutions around the unit circle (as  $x$  goes from 0 to 1), and thus the above integral is 0. When  $k = w$ , the integral is simply  $\int_{[0,1)} e^0 dx = 1$ . Therefore  $\hat{g}(w) = \delta_{k,w}$ , where  $\delta_{k,w}$  is the Kronecker delta function.

Alternatively, we can match coefficients for each character function  $e^{2\pi i x w}$  in the series inversion formula. We observe that if  $e^{2\pi i x k} = \sum_{w \in \mathbb{Z}} \hat{g}(w) e^{2\pi i x w}$ , we must have  $\hat{g}(w) = \delta_{k,w}$ .

*Example 2.4.* The Fourier series of  $g(x) = \cos(2\pi x)$  can be obtained by recalling that  $\cos(2\pi x) = \frac{1}{2}e^{2\pi i x} + \frac{1}{2}e^{-2\pi i x}$ . So by matching coefficients in the series inversion formula, we have that  $\hat{g}(1) = \hat{g}(-1) = \frac{1}{2}$ , and  $\hat{g}(w) = 0$  for  $w \notin \{-1, 1\}$ .

**Periodization.** Let  $f \in L^1(\mathbb{R})$ , so it has a Fourier transform. For a countable set  $S$ , define the notation  $f(S) = \sum_{x \in S} f(x)$ . Now “ $\mathbb{Z}$ -periodize”  $f$  by summing all its  $\mathbb{Z}$ -translates, i.e., define  $g: (\mathbb{R}/\mathbb{Z}) \rightarrow \mathbb{C}$  as

$$g(x + \mathbb{Z}) := f(x + \mathbb{Z}) = \sum_{z \in \mathbb{Z}} f(x + z). \quad (2.1)$$

**Lemma 2.5.** *The Fourier series of  $g$  is  $\hat{g}(w) = \hat{f}(w)$ .*

Intuitively, this says that periodization by  $\mathbb{Z}$  “zeroes out” all the non-integer frequencies, and preserves all the integer ones.

*Proof.* For any  $w \in \mathbb{Z}$ , we have

$$\begin{aligned} \hat{g}(w) &= \int_{\mathbb{R}/\mathbb{Z}} g(x + \mathbb{Z}) e^{-2\pi i x w} dx \\ &= \int_{[0,1)} \sum_{z \in \mathbb{Z}} f(x + z) e^{-2\pi i x w} dx \\ &= \int_{[0,1)} \sum_{z \in \mathbb{Z}} f(x + z) e^{-2\pi i (x+z) w} dx && (z, w \text{ are integers}) \\ &= \int_{\mathbb{R}} f(u) e^{-2\pi i u w} du && (u = x + z \text{ runs over } \mathbb{R}) \\ &= \hat{f}(w). \end{aligned} \quad \square$$

**Poisson summation.** We can use the above to get an interesting formula involving the sum of a function evaluated at the integers.

**Theorem 2.6 (Poisson summation formula).** *For any  $f \in L^1(\mathbb{R})$ , we have  $f(\mathbb{Z}) = \hat{f}(\mathbb{Z})$ .*

*Proof.* Define  $g: (\mathbb{R}/\mathbb{Z}) \rightarrow \mathbb{C}$  to be the  $\mathbb{Z}$ -periodization of  $f$ , as in Equation (2.1). By Fact 2.2 and Lemma 2.5, it follows that

$$f(\mathbb{Z}) = g(0 + \mathbb{Z}) = \sum_{w \in \mathbb{Z}} \hat{g}(w) e^{2\pi i 0 w} = \sum_{w \in \mathbb{Z}} \hat{g}(w) = \sum_{w \in \mathbb{Z}} \hat{f}(w) = \hat{f}(\mathbb{Z}). \quad \square$$

More generally, to sum a function over a scaling of the integers  $s^{-1}\mathbb{Z}$  (i.e., a general one-dimensional lattice) for some real  $s > 0$ , we can use [Fact 1.6](#) to rescale: defining  $h(x) = f(x/s)$ , by [Theorem 2.6](#) (Poisson summation) and [Fact 1.6](#), we have

$$f(s^{-1}\mathbb{Z}) = h(\mathbb{Z}) = \hat{h}(\mathbb{Z}) = s\hat{f}(s\mathbb{Z}). \quad (2.2)$$

*Example 2.7.* Define  $f(x) = e^{-\pi x^2}$ , and  $f_s(x) := f(x/s) = e^{-\pi(x/s)^2}$ . We approximate  $f_s(\mathbb{Z})$  for somewhat large  $s$ :

$$\begin{aligned} f_s(\mathbb{Z}) &= f(s^{-1}\mathbb{Z}) = s\hat{f}(s\mathbb{Z}) && \text{(Equation (2.2))} \\ &= sf(s\mathbb{Z}) && (\hat{f} = f \text{ for the Gaussian}) \\ &\approx sf(0) && (f(sz) = e^{-\pi(sz)^2} \approx 0 \text{ for } z \in \mathbb{Z} \setminus \{0\}) \\ &= s. \end{aligned}$$