

1 Fourier Transform

We begin with some basic definitions.

Definition 1.1 (L^1 function). The function class $L^1(\mathbb{R})$ is the set of all functions $f: \mathbb{R} \rightarrow \mathbb{C}$ for which

$$\int_{\mathbb{R}} |f(x)| dx < \infty.$$

Definition 1.2 (Fourier transform). Given $f \in L^1(\mathbb{R})$, its Fourier Transform $\hat{f}: \mathbb{R} \rightarrow \mathbb{C}$ is defined as

$$\hat{f}(w) := \int_{\mathbb{R}} f(x) e^{-2\pi i x w} dx.$$

Note that

$$|\hat{f}(w)| \leq \int_{\mathbb{R}} |f(x) e^{-2\pi i x w}| dx \leq \int_{\mathbb{R}} |f(x)| \cdot |e^{-2\pi i x w}| dx = \int_{\mathbb{R}} |f(x)| dx < \infty$$

for all $w \in \mathbb{R}$, thus $\hat{f}(w)$ is finite for all w . (The second to last step uses the fact that $|e^{-2\pi i x w}| = 1$.)

Example 1.3. The Fourier transform of the Gaussian function $f(x) = e^{-\pi x^2}$ is

$$\begin{aligned} \hat{f}(w) &= \int_{\mathbb{R}} e^{-\pi x^2} \cdot e^{-2\pi i x w} dx \\ &= \int_{\mathbb{R}} e^{-\pi((x+iw)^2 + w^2)} dx \\ &= e^{-\pi w^2} \int_{\mathbb{R}} e^{-\pi(x+iw)^2} dx \\ &= e^{-\pi w^2} \int_{\mathbb{R}} e^{-\pi x^2} dx^1 \\ &= e^{-\pi w^2} 2 \\ &= f(w), \end{aligned}$$

i.e., the Gaussian is its own Fourier transform.

¹This step follows from applying Cauchy's integral formula to the contour integral around the closed path $R \rightarrow R - iw \rightarrow -R - iw \rightarrow -R \rightarrow R$, as $R \rightarrow \infty$. Since $e^{-\pi x^2}$ is "nice enough" (holomorphic), this integral must be 0. As $R \rightarrow \infty$, the integrals from $[R \rightarrow R - iw]$ and $[-R - iw \rightarrow -R]$ are each bounded in absolute value by $e^{-\pi w R^2}$, hence they both approach 0. Therefore,

$$0 = \lim_{R \rightarrow \infty} \int_{-R}^R e^{-\pi x^2} dx + \int_{R-iw}^{-R-iw} e^{-\pi x^2} dx = \lim_{R \rightarrow \infty} \int_{-R}^R e^{-\pi x^2} dx + \int_R^{-R} e^{-\pi(u+iw)^2} du.$$

The result follows by swapping the bounds of integration in the second integral, and taking $R \rightarrow \infty$.

²This follows from integrating the square in polar coordinates:

$$\int_{\mathbb{R}} e^{-\pi x^2} dx = \sqrt{\int_{\mathbb{R}^2} e^{-\pi x^2} e^{-\pi y^2} dx} = \sqrt{\int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} e^{-\pi r^2} \cdot r dr d\theta} = \sqrt{\int_{\theta=0}^{2\pi} \frac{-1}{2\pi} (0 - 1) d\theta} = 1.$$

Basic facts. We recall some important facts about the Fourier transform.

Fact 1.4 (Linearity). For all $f, g \in L^1(\mathbb{R})$ and all $a \in \mathbb{R}$, $\widehat{f+g} = \hat{f} + \hat{g}$ and $\widehat{a \cdot f} = a\hat{f}$.

Fact 1.5 (Time shift). Let $f \in L^1(\mathbb{R})$ and $h(x) = f(x - c)$ for some $c \in \mathbb{R}$. Then $\hat{h}(w) = e^{-2\pi icw} \cdot \hat{f}(w)$:

$$\hat{h}(w) = \int_{\mathbb{R}} f(x - c) e^{-2\pi i x w} dx = \int_{\mathbb{R}} f(u) e^{-2\pi i(u+c)w} du = e^{-2\pi icw} \cdot \hat{f}(w).$$

Fact 1.6 (Time scale). Let $f \in L^1(\mathbb{R})$ and $h(x) = f(x/s)$ for some $s > 0$. Then $\hat{h}(w) = s\hat{f}(sw)$:

$$\hat{h}(w) = \int_{\mathbb{R}} f(x/s) e^{-2\pi i x w} dx = s \int_{\mathbb{R}} f(u) e^{-2\pi i s u w} du = s \int_{\mathbb{R}} f(u) e^{-2\pi i u (s w)} du = s\hat{f}(s w).$$

Fact 1.7 (Transform inversion). For any $f \in L^1(\mathbb{R})$,

$$f(x) = \int_{\mathbb{R}} \hat{f}(w) e^{2\pi i x w} dw.$$

In words, this says that $f(x)$ is a linear combination of “character” functions of the form $e^{2\pi i x w}$ for $w \in \mathbb{R}$, each weighted by the Fourier coefficient $\hat{f}(w)$. Each component $e^{2\pi i x w}$ is periodic with period $1/w$ (for $w \neq 0$), hence frequency w . Note that [Fact 1.7](#) implies that $\hat{\hat{f}}(x) = f(-x)$.

2 Fourier Series for Periodic Functions

Now consider a function $g: \mathbb{R} \rightarrow \mathbb{C}$ (not necessarily in $L^1(\mathbb{R})$) that is periodic with unit period, i.e., $g(x) = g(x + z)$ for any $z \in \mathbb{Z}$. We call such a function \mathbb{Z} -periodic. Equivalently, we can work with the function $g: (\mathbb{R}/\mathbb{Z}) \rightarrow \mathbb{C}$, since the original g is constant over any fixed coset $c + \mathbb{Z}$. Such an g can be decomposed as a linear combination of character functions with *integer* frequencies.

Definition 2.1 (Fourier series). For $g: (\mathbb{R}/\mathbb{Z}) \rightarrow \mathbb{C}$, its Fourier series $\hat{g}: \mathbb{Z} \rightarrow \mathbb{C}$ is

$$\hat{g}(w) = \int_{\mathbb{R}/\mathbb{Z}} g(x + \mathbb{Z}) e^{-2\pi i x w} dx.$$

Equivalently, the integral can be taken over any fundamental region of \mathbb{Z} ; often it is convenient to use $[0, 1)$.

[Facts 1.4](#) to [1.6](#) also apply to the Fourier series, by essentially the same proofs. But there is a slightly different inversion formula.

Fact 2.2 (Series inversion).

$$g(x + \mathbb{Z}) = \sum_{w \in \mathbb{Z}} \hat{g}(w) e^{2\pi i x w}.$$

Example 2.3. The Fourier series of $g(x) = e^{2\pi i x k}$, where $k \in \mathbb{Z}$ (so g is \mathbb{Z} -periodic), can be derived in two different ways. First, we calculate the Fourier series directly: for any $w \in \mathbb{Z}$,

$$\begin{aligned} \hat{g}(w) &= \int_{[0,1)} e^{2\pi i x k} e^{-2\pi i x w} dx \\ &= \int_{[0,1)} e^{2\pi i x (k-w)} dx. \end{aligned}$$

When $w \neq k$, $e^{2\pi i x(k-w)}$ completes a nonzero integer number of revolutions around the unit circle (as x goes from 0 to 1), and thus the above integral is 0. When $k = w$, the integral is simply $\int_{[0,1)} e^0 dx = 1$. Therefore $\hat{g}(w) = \delta_{k,w}$, where $\delta_{k,w}$ is the Kronecker delta function.

Alternatively, we can match coefficients for each character function $e^{2\pi i x w}$ in the series inversion formula. We observe that if $e^{2\pi i x k} = \sum_{w \in \mathbb{Z}} \hat{g}(w) e^{2\pi i x w}$, we must have $\hat{g}(w) = \delta_{k,w}$.

Example 2.4. The Fourier series of $g(x) = \cos(2\pi x)$ can be obtained by recalling that $\cos(2\pi x) = \frac{1}{2}e^{2\pi i x} + \frac{1}{2}e^{-2\pi i x}$. So by matching coefficients in the series inversion formula, we have that $\hat{g}(1) = \hat{g}(-1) = \frac{1}{2}$, and $\hat{g}(w) = 0$ for $w \notin \{-1, 1\}$.

Periodization. Let $f \in L^1(\mathbb{R})$, so it has a Fourier transform. For a countable set S , define the notation $f(S) = \sum_{x \in S} f(x)$. Now “ \mathbb{Z} -periodize” f by summing all its \mathbb{Z} -translates, i.e., define $g: (\mathbb{R}/\mathbb{Z}) \rightarrow \mathbb{C}$ as

$$g(x + \mathbb{Z}) := f(x + \mathbb{Z}) = \sum_{z \in \mathbb{Z}} f(x + z). \quad (2.1)$$

Lemma 2.5. *The Fourier series of g is $\hat{g}(w) = \hat{f}(w)$.*

Intuitively, this says that periodization by \mathbb{Z} “zeroes out” all the non-integer frequencies, and preserves all the integer ones.

Proof. For any $w \in \mathbb{Z}$, we have

$$\begin{aligned} \hat{g}(w) &= \int_{\mathbb{R}/\mathbb{Z}} g(x + \mathbb{Z}) e^{-2\pi i x w} dx \\ &= \int_{[0,1)} \sum_{z \in \mathbb{Z}} f(x + z) e^{-2\pi i x w} dx \\ &= \int_{[0,1)} \sum_{z \in \mathbb{Z}} f(x + z) e^{-2\pi i (x+z) w} dx && (z, w \text{ are integers}) \\ &= \int_{\mathbb{R}} f(u) e^{-2\pi i u w} du && (u = x + z \text{ runs over } \mathbb{R}) \\ &= \hat{f}(w). \end{aligned} \quad \square$$

Poisson summation. We can use the above to get an interesting formula involving the sum of a function evaluated at the integers.

Theorem 2.6 (Poisson summation formula). *For any $f \in L^1(\mathbb{R})$, we have $f(\mathbb{Z}) = \hat{f}(\mathbb{Z})$.*

Proof. Define $g: (\mathbb{R}/\mathbb{Z}) \rightarrow \mathbb{C}$ to be the \mathbb{Z} -periodization of f , as in Equation (2.1). By Fact 2.2 and Lemma 2.5, it follows that

$$f(\mathbb{Z}) = g(0 + \mathbb{Z}) = \sum_{w \in \mathbb{Z}} \hat{g}(w) e^{2\pi i 0 w} = \sum_{w \in \mathbb{Z}} \hat{g}(w) = \sum_{w \in \mathbb{Z}} \hat{f}(w) = \hat{f}(\mathbb{Z}). \quad \square$$

More generally, to sum a function over a scaling of the integers $s^{-1}\mathbb{Z}$ (i.e., a general one-dimensional lattice) for some real $s > 0$, we can use [Fact 1.6](#) to rescale: defining $h(x) = f(x/s)$, by [Theorem 2.6](#) (Poisson summation) and [Fact 1.6](#), we have

$$f(s^{-1}\mathbb{Z}) = h(\mathbb{Z}) = \hat{h}(\mathbb{Z}) = s\hat{f}(s\mathbb{Z}). \quad (2.2)$$

Example 2.7. Define $f(x) = e^{-\pi x^2}$, and $f_s(x) := f(x/s) = e^{-\pi(x/s)^2}$. We approximate $f_s(\mathbb{Z})$ for somewhat large s :

$$\begin{aligned} f_s(\mathbb{Z}) &= f(s^{-1}\mathbb{Z}) = s\hat{f}(s\mathbb{Z}) && \text{(Equation (2.2))} \\ &= sf(s\mathbb{Z}) && (\hat{f} = f \text{ for the Gaussian}) \\ &\approx sf(0) && (f(sz) = e^{-\pi(sz)^2} \approx 0 \text{ for } z \in \mathbb{Z} \setminus \{0\}) \\ &= s. \end{aligned}$$