

In this lecture we will see complexity-theoretic lower and upper bounds for SVP and CVP. On the one hand, they are NP-hard (under suitable types of reductions) in their exact versions, and even for small approximation factors. On the other hand, there is good evidence *against* the NP-hardness of their approximate versions for factors  $\gamma \geq \sqrt{n/\log n}$ .

## 1 NP-Hardness of CVP and SVP

### 1.1 NP-Hardness of the Closest Vector Problem

First let us recall the decisional version of the (approximate) Closest Vector Problem.

**Definition 1.1 (CVP, decision version).** For an approximation factor  $\gamma = \gamma(n) \geq 1$ , an instance of  $\text{GapCVP}_1$  is a basis  $\mathbf{B}$  of a lattice  $\mathcal{L} = \mathcal{L}(\mathbf{B})$ , a target point  $\mathbf{t} \in \mathbb{R}^n$ , and a distance  $d \in \mathbb{R}$ . It is a YES instance if  $\text{dist}(\mathbf{t}, \mathcal{L}) \leq d$ , and is a NO instance if  $\text{dist}(\mathbf{t}, \mathcal{L}) > \gamma \cdot d$ .

The problem is equivalent to asking whether the coset  $\mathbf{t} + \mathcal{L}$  has an element of length at most  $d$  or not.

**Theorem 1.2 (van Emde Boas [vEB81]).**  $\text{GapCVP}_1$  is NP-complete.

*Proof.* To show that a problem is NP-complete, we need to show that it is in NP, and also that it is NP-hard. The former is easy: a witness  $w$  for a YES instance  $(\mathbf{B}, \mathbf{t}, d)$  is a lattice vector  $\mathbf{v} \in \mathcal{L}(\mathbf{B})$  such that  $\|\mathbf{t} - \mathbf{v}\| \leq d$ , which by definition exists for a YES instance and does not exist for a NO instance. Clearly, the conditions can be efficiently verified.<sup>1</sup>

Next we need to show NP-hardness, i.e., we need to give a reduction from some NP-hard problem to  $\text{GapCVP}_1$ . We reduce from the subset-sum problem, which is a natural choice since it has a very similar linear structure to lattice problems. Recall that the subset-sum problem is: given  $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$  and  $S \in \mathbb{Z}$ , decide if there exists  $\mathbf{x} \in \{0, 1\}^n$  such that  $\langle \mathbf{a}, \mathbf{x} \rangle = \sum_{i=1}^n a_i x_i = S$ . Our reduction takes a subset-sum instance  $(a_1, \dots, a_n, S)$  as input, and outputs the  $\text{GapCVP}_1$  instance  $(\mathbf{B}, \mathbf{t}, d)$ , where

$$\mathbf{B} = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ 2 & & & \\ & 2 & & \\ & & \ddots & \\ & & & 2 \end{pmatrix} \in \mathbb{Z}^{(n+1) \times n}, \quad \mathbf{t} = \begin{pmatrix} S \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{Z}^{n+1}, \quad d = \sqrt{n}.$$

We need to show that the above is a YES instance of  $\text{GapCVP}_1$  if and only if the given subset-sum instance is a YES instance. In one direction, suppose the subset-sum instance has a solution  $\mathbf{x} \in \{0, 1\}^n$ . Then for the lattice vector  $\mathbf{v} = \mathbf{B}\mathbf{x}$ , we have

$$\mathbf{v} - \mathbf{t} = \begin{pmatrix} S \\ 2x_1 \\ 2x_2 \\ \vdots \\ 2x_n \end{pmatrix} - \begin{pmatrix} S \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \pm 1 \\ \pm 1 \\ \vdots \\ \pm 1 \end{pmatrix},$$

<sup>1</sup>One technical subtlety is that the witness must have bit length which is polynomial in the instance length. This holds because the bit length of  $\mathbf{v}$  is bounded by the sum of those of  $\mathbf{t}$  and  $d$ .

so  $\|\mathbf{v} - \mathbf{t}\| = \sqrt{n}$  and  $(\mathbf{B}, \mathbf{t}, d)$  is a YES instance of  $\text{GapCVP}_1$ .

In the other direction, suppose that there exists some lattice vector  $\mathbf{v} = \mathbf{B}\mathbf{x}$  for  $\mathbf{x} \in \mathbb{Z}^n$  such that  $\|\mathbf{v} - \mathbf{t}\| \leq \sqrt{n}$ . Since the last  $n$  entries of  $\mathbf{v}$  are even, the last  $n$  entries of  $\mathbf{v} - \mathbf{t}$  are odd, and hence must all be  $\pm 1$  because  $\|\mathbf{v} - \mathbf{t}\| \leq \sqrt{n}$ . Therefore,  $\mathbf{x} \in \{0, 1\}^n$ . Moreover, the first entry of  $\mathbf{v} - \mathbf{t}$  must be zero (against because  $\|\mathbf{v} - \mathbf{t}\| \leq \sqrt{n}$ ), so  $\mathbf{x}$  is a solution to the subset-sum instance.  $\square$

The above theorem is presented only for the  $\ell_2$  norm. It is not difficult to generalize it to the  $\ell_p$  norm for any  $p > 1$ , including  $p = \infty$ .

## 1.2 NP-Hardness of the Shortest Vector Problem

One might wonder whether similar methods can be used to prove that the decisional Shortest Vector Problem ( $\text{GapSVP}_1$ ) is NP-complete. For the  $\ell_2$  norm, it turns out to be *much more challenging* to show this—in fact, it was not until 1998 that Ajtai showed NP-hardness, but under a *randomized* reduction [Ajt98]. This means that an efficient (possibly randomized) algorithm for  $\text{GapSVP}_1$  would imply that  $\text{NP} \subseteq \text{RP}$  (but not necessarily that  $\text{NP} = \text{P}$ ). Even today, it is still not known whether  $\text{GapSVP}_1$  in the  $\ell_2$  norm is NP-hard under a *deterministic* reduction!

Here we show a much easier result, that  $\text{GapSVP}_1$  is NP-complete in the  $\ell_\infty$  norm (also known as max norm), defined as  $\|\mathbf{x}\|_\infty = \max_i |x_i|$ .

**Definition 1.3 (SVP in  $\ell_\infty$ , decision version).** For an approximation factor  $\gamma = \gamma(n) \geq 1$ , an instance of  $\text{GapSVP}_\gamma^{(\infty)}$  is a basis  $\mathbf{B}$  of a lattice  $\mathcal{L} = \mathcal{L}(\mathbf{B})$  and a distance  $d \in \mathbb{R}$ . It is a YES instance if the minimum distance of  $\mathcal{L}$  in  $\ell_\infty$  norm is at most  $d$ , i.e., if  $\lambda_1^{(\infty)}(\mathcal{L}) \leq d$ , and is a NO instance if  $\lambda_1^{(\infty)}(\mathcal{L}) > \gamma \cdot d$ .

**Theorem 1.4 (van Emde Boas [vEB81]).**  $\text{GapSVP}_1^{(\infty)}$  is NP-complete.

*Proof.* Membership in NP is easy to see: a witness for instance  $(\mathbf{B}, d)$  is a vector  $\mathbf{v} \in \mathcal{L}(\mathbf{B})$  for which  $\|\mathbf{v}\|_\infty \leq d$ , which can be efficiently checked.

For NP-hardness, we reduce from the NP-hard “weak partition” problem, which is a homogeneous variant of the subset-sum problem. The weak partition problem is: given  $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$ , determine whether there exist disjoint sets  $X, Y \subseteq \{1, \dots, n\}$ , not both empty, such that  $\sum_{i \in X} a_i = \sum_{i \in Y} a_i$ . Equivalently, it asks whether there is a nonzero  $\mathbf{x} \in \{-1, 0, +1\}^n$  such that  $\langle \mathbf{a}, \mathbf{x} \rangle = 0$ . (The indices of the  $-1$  entries of  $\mathbf{x}$  correspond to the elements of  $X$ , and the indices of the  $+1$  entries correspond to the elements of  $Y$ .)

The reduction works as follows: given an instance  $\mathbf{a} = (a_1, \dots, a_n)$  of the weak partition problem, it outputs the following instance  $(\mathbf{B}, d)$  of  $\text{GapSVP}_1^{(\infty)}$ :

$$\mathbf{B} = \begin{pmatrix} 2a_1 & 2a_2 & \cdots & 2a_n \\ 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \in \mathbb{Z}^{(n+1) \times n}, \quad d = 1.$$

We need to show that the given weak partition instance is a YES instance if and only if the above  $\text{GapSVP}_1^{(\infty)}$  instance is a YES instance. In one direction, suppose that there exists a nonzero solution  $\mathbf{x} \in \{0, \pm 1\}^n$  to the weak partition instance, so that  $\langle \mathbf{a}, \mathbf{x} \rangle = 0$ . Then the lattice vector  $\mathbf{B}\mathbf{x} \in \mathcal{L}(\mathbf{B})$  is nonzero and has  $\ell_\infty$  norm  $\|\mathbf{B}\mathbf{x}\|_\infty = 1$ , as desired. In the other direction, suppose there exists a nonzero

lattice vector  $\mathbf{v} = \mathbf{B}\mathbf{x} = \begin{pmatrix} 2\langle \mathbf{a}, \mathbf{x} \rangle \\ \mathbf{x} \end{pmatrix} \in \mathbb{Z}^{n+1}$  for  $\mathbf{x} \in \mathbb{Z}^n$  such that  $\|\mathbf{v}\|_\infty \leq 1$ . The first entry  $2\langle \mathbf{a}, \mathbf{x} \rangle$  of  $\mathbf{v}$  is even, so it must be zero. The remainder of  $\mathbf{v}$  is just the vector  $\mathbf{x}$ , so we must have  $\mathbf{x} \in \{0, \pm 1\}^n$  and  $\mathbf{x} \neq \mathbf{0}$ . This means that  $\mathbf{x}$  is a solution to the weak partition instance, which completes the proof.  $\square$

### 1.3 Other Results

As mentioned above, in 1998 Ajtai [Ajt98] showed that  $\text{GapSVP}_1$  in the  $\ell_2$  norm is NP-complete under a randomized reduction. This has since been substantially improved: over a series of works [Mic98, Kho04, HR07], it has been shown that  $\text{GapSVP}_c$  in  $\ell_2$  is NP-complete (still under randomized reduction) for any *constant* approximation factor  $\gamma = O(1)$ , and even under nearly polynomial factors  $\gamma = 2^{\log^{1-\epsilon} n}$  for any constant  $\epsilon > 0$  if NP cannot be solved in (randomized) quasi-polynomial  $2^{\text{poly}(\log n)}$  time. For CVP, the state of the art is that  $\text{GapCVP}_\gamma$  is NP-complete under deterministic reduction for factors as large as  $\gamma = n^{\Omega(1/\log \log n)}$  [ABSS93], which is “almost polynomial” in  $n$ .

A natural question is, how far might we hope to increase the approximation factors  $\gamma$  for the NP-hardness of  $\text{GapSVP}$  and  $\text{GapCVP}$ ? There are (at least) two answers:

1. Clearly, we should not expect to have NP-hardness for very large factors  $\gamma \geq 2^n$ , because  $\text{GapSVP}_\gamma$  and  $\text{GapCVP}_\gamma$  for such factors can be solved in polynomial time using LLL.
2. More interestingly, we should not expect to have NP-hardness for factors  $\gamma \geq \sqrt{n/\log n}$ . We will show why this is the case in the next section.

## 2 The Goldreich–Goldwasser Protocol

Clearly,  $\text{GapCVP}_\gamma \in \text{NP}$  for any  $\gamma \geq 1$ . To show that  $\text{GapCVP}_\gamma$  is not likely to be NP-complete for  $\gamma \geq \sqrt{n/\log n}$ , Goldreich and Goldwasser [GG98] proved that it belongs to the complexity class coAM. That is, the *complement* problem  $\text{coGapCVP}_\gamma$ —which simply flips the YES and NO instances of  $\text{GapCVP}_\gamma$ —is in the class AM of problems that have “Arthur–Merlin protocols,” defined below.

The Goldreich–Goldwasser result is significant because if  $\text{GapCVP}_\gamma$  was NP-complete for some  $\gamma \geq \sqrt{n/\log n}$ , then it would follow that  $\text{NP} \subseteq \text{coAM}$ .<sup>2</sup> It is known that this would imply the collapse of the polynomial-time hierarchy, which is considered very unlikely. Therefore, this can be considered strong evidence (but not proof!) that  $\text{GapCVP}_{\sqrt{n/\log n}}$  is not NP-complete.

### 2.1 $\text{coGapCVP}_{\sqrt{n/\log n}} \in \text{AM}$

Informally, the complexity class AM consists of decision/promise problems for which an unbounded prover can convince an efficient randomized verifier that an instance is a YES instance, but even a (possibly malicious) unbounded prover cannot reliably convince the verifier on a NO instance.

**Definition 2.1 (AM).** A promise problem  $L = (L_{\text{YES}}, L_{\text{NO}})$  is in AM if there exists a constant-round protocol between a probabilistic polynomial-time Turing machine  $A$  (“Arthur”) and a computationally unbounded Turing machine  $M$  (“Merlin”) with the following properties:

<sup>2</sup>There are some technical subtleties here related to the fact that  $\text{GapCVP}_\gamma$  is a *promise* problem, but the chain of reasoning holds for a wide class of reductions by which  $\text{GapCVP}_\gamma$  might be shown NP-complete.

- *Completeness*: for any YES instance  $x \in L_{\text{YES}}$ , we have that  $\Pr[A(x) \leftrightarrow M(x) \text{ accepts}] = 1$ , i.e.,  $M$  always convinces  $A$  to accept.
- *Soundness*: for any NO instance  $x \in L_{\text{NO}}$  and for any unbounded  $M^*$ , we have that  $\Pr[A(x) \leftrightarrow M^*(x) \text{ accepts}] \leq 1 - 1/\text{poly}(|x|)$ , i.e.,  $A$  rejects with some noticeable probability.

It is straightforward to show that by repeating the protocol in parallel a polynomial number of times, the “soundness error” (i.e., the probability that  $A$  accepts on a NO instance) can be made very small, e.g.,  $2^{-n}$ .

**Theorem 2.2 (Goldreich–Goldwasser [GG98]).**  $\text{coGapCVP}_\gamma \in \text{AM}$  for  $\gamma = \sqrt{n/\log n}$  (or more generally, any  $\gamma = \Omega(\sqrt{n \log n})$ ).

To prove this theorem, we need to give an Arthur–Merlin protocol which causes Arthur to accept whenever the target point  $\mathbf{t}$  is *far* from the given lattice  $\mathcal{L}$ , i.e., when all the vectors in the coset  $\mathbf{t} + \mathcal{L}$  have length more than  $\gamma d$  (these are the YES instances of  $\text{coGapCVP}$ ). On the other hand, when the coset  $\mathbf{t} + \mathcal{L}$  contains a vector of length at most  $d$  (a NO instance of  $\text{coGapCVP}$ ), Arthur should reject with noticeable probability. Note that it’s not obvious how to convincingly prove the *absence* of a short vector in a lattice coset; this is where *interaction* with an unbounded prover helps.

The intuition behind the protocol is as follows. Arthur first flips a fair coin. If it comes up heads, he chooses a “uniformly random” point in the lattice  $\mathcal{L}$ ; if it comes up tails, he chooses a “uniformly random” point in the coset  $\mathbf{t} + \mathcal{L}$ . Let  $\mathbf{w}$  denote the resulting point. Arthur then randomly chooses uniform “noise”  $\mathbf{e}$  from the ball of radius  $(\gamma d)/2$ , and sends  $\mathbf{x} = \mathbf{w} + \mathbf{e}$  to Merlin. Merlin—who, to recall, is computationally unbounded—is supposed to figure out whether Arthur’s coin came up heads or not, i.e., whether  $\mathbf{w} \in \mathcal{L}$  or  $\mathbf{w} \in \mathbf{t} + \mathcal{L}$ . Under what conditions can Merlin always do this, versus necessarily having some noticeable probability of failing?

Notice that if  $\text{dist}(\mathbf{t}, \mathcal{L}) \geq \gamma d$ , then there is *no overlap* between the balls centered at the points of  $\mathcal{L}$  and ones centered at the points of  $\mathbf{t} + \mathcal{L}$ , so Merlin can always give the correct answer. On the other hand, if  $\text{dist}(\mathbf{t}, \mathcal{L}) \leq d$ , we will argue that the *overlap* between the two collections of balls is relatively large, hence Merlin must make a mistake with some noticeable probability. We now formalize the protocol and its analysis to prove the theorem. In particular, we eliminate the (mathematically problematic) need for a “uniformly random” lattice point by working *modulo the lattice*, using the fundamental parallelepiped of the input basis as a fundamental region.

*Proof of Theorem 2.2.* The Arthur–Merlin protocol is as follows. Arthur and Merlin are given some  $\text{coGapCVP}$  instance  $(\mathbf{B}, \mathbf{t}, d)$  as input. Arthur chooses a bit  $b \in \{0, 1\}$  and  $\mathbf{e} \leftarrow r\bar{\mathcal{B}}$  uniformly at random, where  $r = (\gamma d/2)$  and  $\bar{\mathcal{B}}$  is the closed unit ball. Arthur then sends the vector

$$\mathbf{x} := (b \cdot \mathbf{t} + \mathbf{e}) \bmod \mathbf{B}$$

to Merlin, i.e.,  $\mathbf{x}$  is the unique element of  $(b\mathbf{t} + \mathbf{e} + \mathcal{L}(\mathbf{B})) \cap \mathcal{P}(\mathbf{B})$  (which is easy to compute). If  $\text{dist}(\mathbf{x}, \mathcal{L}) \leq r$ , Merlin returns  $b' = 0$ ; otherwise he returns  $b' = 1$ . Arthur accepts if  $b' = b$ .

First we show completeness. Let  $(\mathbf{B}, \mathbf{t}, d)$  be a YES instance of  $\text{coGapCVP}_\gamma$ , so  $\text{dist}(\mathbf{t}, \mathcal{L}) > \gamma d$  where  $\mathcal{L} = \mathcal{L}(\mathbf{B})$ . By the triangle inequality, for *any*  $\mathbf{x} \in \mathbb{R}^n$ , at most one of  $\text{dist}(\mathbf{x}, \mathcal{L}) \leq r$  and  $\text{dist}(\mathbf{x} - \mathbf{t}, \mathcal{L}) \leq r$  can hold. When  $b = 0$ , we have  $\mathbf{x} = \mathbf{e} \bmod \mathbf{B}$ , so  $\text{dist}(\mathbf{x}, \mathcal{L}) = \text{dist}(\mathbf{e}, \mathcal{L}) \leq r$ , hence Merlin correctly returns  $b' = 0$ . Similarly, when  $b = 1$ , we have  $\mathbf{x} = \mathbf{t} + \mathbf{e} \bmod \mathbf{B}$ , so  $\text{dist}(\mathbf{x} - \mathbf{t}, \mathcal{L}) \leq r$ , so  $\text{dist}(\mathbf{x}, \mathcal{L}) > r$  and Merlin correctly return  $b' = 1$ .

Proving soundness is more involved. Let  $(\mathbf{B}, \mathbf{t}, d)$  be a NO instance, so  $\text{dist}(\mathbf{t}, \mathcal{L}) \leq d$  where  $\mathcal{L} = \mathcal{L}(\mathbf{B})$ ; we need to show that Merlin answers incorrectly with some noticeable  $1/\text{poly}(n)$  probability.

To see this, let  $\mathbf{v} \in \mathcal{L}$  be a lattice vector for which  $\|\mathbf{t}'\| \leq d$  where  $\mathbf{t}' = \mathbf{t} - \mathbf{v}$ . Now observe that Arthur's message  $\mathbf{x}$  in the protocol is *identically distributed* to one generated in a slightly different way in the case  $b = 1$ , as  $\mathbf{x} := \mathbf{t}' + \mathbf{e} \bmod \mathbf{B}$  (the case  $b = 0$  is unchanged). This is simply because  $\mathbf{t}' + \mathbf{e} \bmod \mathbf{B}$  equals  $\mathbf{t} + \mathbf{e} \bmod \mathbf{B}$  for any  $\mathbf{v} \in \mathcal{L}$  and  $\mathbf{e} \in \mathbb{R}^n$ . Now let  $I = r\bar{\mathcal{B}} \cap (\mathbf{t}' + r\bar{\mathcal{B}})$  be the intersection of the balls centered at the origin and at  $\mathbf{t}'$ , and observe that for any  $\mathbf{y} \in I$  (even before reducing modulo  $\mathbf{B}$ ), it is equally likely that Arthur chose  $b = 0$  and  $\mathbf{e} = \mathbf{y}$  versus  $b = 1$  and  $\mathbf{e} = \mathbf{y} - \mathbf{t}'$ .<sup>3</sup> So, in this case Merlin cannot do any better than a random guess, which succeeds with probability  $1/2$ . Therefore, the probability that Merlin answers incorrectly is at least half of

$$\frac{\text{vol}(r\bar{\mathcal{B}} \cap (\mathbf{t}' + r\bar{\mathcal{B}}))}{\text{vol}(r\bar{\mathcal{B}})}. \quad (2.1)$$

Therefore, it suffices to give a lower bound on the above quantity, which by rescaling is the fraction of overlap between two  $n$ -dimensional balls of unit radius whose centers are  $\delta \leq 2/\gamma$  apart. It is not hard to see that the intersection contains a cylinder  $C$  with radius  $\sqrt{1 - \delta^2}$  and height  $\delta$ . The volume of an  $n$ -dimensional unit ball is known to be

$$V_n = \frac{\pi^{n/2}}{(n/2)!},$$

where the generalized factorial function satisfies  $0! = 1$ ,  $n! = n(n-1)!$  for all real  $n \geq 1$ , and  $(1/2)! = \sqrt{\pi}$ . We also need the fact that  $(n + \frac{1}{2})!/n! = \Theta(\sqrt{n})$ .

Therefore, the quantity in Equation (2.1) is at least

$$\begin{aligned} \frac{\delta \cdot (1 - \delta^2)^{(n-1)/2} \cdot V_{n-1}}{V_n} &= \frac{\delta \cdot (1 - \delta^2)^{(n-1)/2} \cdot \pi^{(n-1)/2} \cdot (n/2)!}{(n/2 - 1/2)! \cdot \pi^{n/2}} \\ &= (1 - \delta^2)^{(n-1)/2} \cdot \Theta(\delta\sqrt{n}). \end{aligned}$$

Recalling that  $(1 - 1/n)^n = 1/O(1)$  (indeed, it approaches  $1/e$  as  $n$  grows), we have  $(1 - (\log n)/n)^n = 1/O(1)^{\log n} = 1/\text{poly}(n)$ . So for any  $\delta = O(\sqrt{(\log n)/n})$  and hence any  $\gamma = \Omega(\sqrt{n/\log n})$ , the above quantity is  $1/\text{poly}(n)$ , as needed.  $\square$

## 2.2 Summary

To summarize:

- $\text{GapCVP}_{n^{1/\log \log n}}$  is NP-complete.
- $\text{GapCVP}_{\sqrt{n/\log n}} \in \text{coAM}$ , so it is unlikely to be NP-hard.
- A work of Aharonov and Regev [AR04] (which we will cover later in this course) showed that  $\text{GapCVP}_{\sqrt{n}}$  is in coNP, and hence is unlikely to be NP-hard.
- $\text{GapCVP}_{2^n}$  is in P, due to the LLL algorithm.

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<sup>3</sup>This is not quite rigorous, because we are conditioning on an event of probability zero. This can be fixed by instead using measure.

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