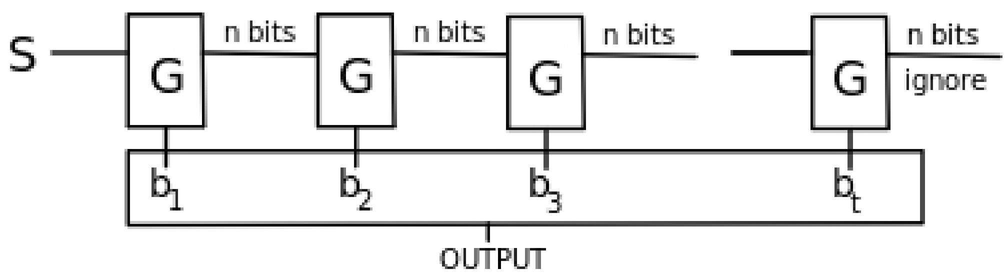
1 Expanding a PRG

In the last lecture we saw the definition of a Pseudorandom Generator (PRG) as a deterministic function that, given a seed of size n, outputs a pseudorandom string of length $\ell(n)$. We can ask the following question: how big can $\ell(n)$ be? Is there any limit on how much pseudorandom data we can generate starting from a seed of a certain size? We will find out that if we can get a PRG with even *one bit* of expansion (i.e., $\ell(n) = n + 1$), then we can get a PRG with *any* polynomial output length.

Theorem 1.1. *Suppose that there exists a PRG G with output of length $\ell(n) = n + 1$. Then for any $t(n) = \mathrm{poly}(n)$ (where $t(n) > n$), there exists a PRG G_t with output length $t(n)$.*

Remark 1.2. The size of the set $\{G_t(s) : s \in \{0,1\}^n\}$ is at most 2^n (because G_t is deterministic), while the number of possible $t(n)$-bit string is $|\{0,1\}^{t(n)}| = 2^{t(n)}$. The ratio of possible output strings that could actually be output by the PRG is at most $2^n/2^{t(n)} = 2^{n-t(n)}$, which is absurdly small when $t(n) \geq 2n$ (and even smaller when $t(n) = n^{10}$, say).

Proof. We will construct $G_t(s)$ from G. Our construction will apply the function $G(\cdot)$ $t(n)$ times, outputting one new bit at each step and reusing the other n bits of the previous step's output as a seed. See the following picture for intuition about the construction.



Formally, G_t is defined as follows (note that it always applies G on string of the same length, n bits):

Algorithm 1 $G_t(s)$

if $t = 0$ **then**
  return ε (the empty string)
else
  let $(x|b) = G(s)$, where $x \in \{0,1\}^n$, $b \in \{0,1\}$
  return $b|G_{t-1}(x)$
end if

By construction, $|G_t(s)| = t$. We want to show that G_t is a PRG. The function clearly runs in polynomial time (each call to G can be resolved in polynomial time and we only use a polynomial number of steps), so what's left is to prove that $\{G_t(U_n)\} \stackrel{c}{\approx} \{U_{t(n)}\}$. We need to be careful: we already know that G is a PRG, but in our construction we are giving a *pseudorandom* seed to G, instead of a truly random seed. We will see that this fact will not affect the pseudorandomness of G_t. Intuitively, because no efficient algorithm can tell a pseudorandom seed apart from a random string, then in particular neither can G.

To prove that G_t is a PRG we will define a set of "hybrid experiments." We will build a sequence of distributions, where the first is equal to our "real" construction $G_t(U_n)$, the last is equal to the "ideal" truly uniform distribution $U_{t(n)}$, and each consecutive pair of distributions are computationally indistinguishable. By the hybrid lemma, we conclude that $\{G_t(U_n)\}$ and $\{U_{t(n)}\}$ are computationally indistinguishable and that G_t is a PRG, thus proving the theorem.

To give some intuition about how we design the hybrid experiments, we imagine that instead of invoking the first G on n uniform bits, what if we replaced its output with $n+1$ truly uniform bits? Intuitively, these two cases should not be distinguishable, because G is a PRG. And then what if we replaced the first two invocations of G, and so on? Eventually, we would end up with $t = t(n)$ truly uniform output bits, as desired.

Formally, the hybrid experiments are defined as follows:

- $H_0 = G_t(U_n)$
- $H_1 = U_1 | G_{t-1}(U_n)$
- In general, $H_i = U_i | G_{t-i}(U_n)$ for $i \in \{0\} \cup [t]$
- $H_t = U_t$

We now show that for all $i \in [t-1]$, $H_i \overset{c}{\approx} H_{i+1}$. We will do this by using the simulation/composition lemma, and the fact that G is a PRG. For each i, we design a PPT "simulator" algorithm $\mathcal{S}_i$ such that $\mathcal{S}_i(G(U_n)) = H_{i-1}$, and $\mathcal{S}_i(U_{n+1}) = H_i$. Since we know that $G(U_n) \overset{c}{\approx} U_{n+1}$ from the fact that G is a PRG, the composition lemma implies that $H_{i-1} \overset{c}{\approx} H_i$.

We define $\mathcal{S}_i$ as follows:

Algorithm 2 $\mathcal{S}_i(y \in \{0,1\}^{n+1})$

parse y as $(x|b)$ for $x \in \{0,1\}^n$, $b \in \{0,1\}$
return $U_{i-1}|b|G_{t-i}(x)$

This algorithm clearly runs in polynomial time. We need to check that it maps $G(U_n)$ to H_{i-1}, and U_{n+1} to H_i. First suppose that the input of $\mathcal{S}_i$ comes from U_{n+1}:

$$\mathcal{S}_i(U_{n+1}) = U_{i-1}|b|G_{t-i}(x) = U_{i-1}|U_1|G_{t-i}(U_n) = U_i|G_{t-i}(U_n) = H_i.$$

Now suppose that the input comes from $G(U_n)$. By the definition of G_t, we can see the following:

$$\mathcal{S}_i(G(U_n)) = U_{i-1}|b|G_{t-i}(x) = U_{i-1}|G_{t-i+1}(U_n) = H_{i-1}.$$

This completes the proof. □

To recap, our proof of Theorem 1.1 took the following path:

- We defined a construction: the actual PRG G_t having a pseudorandom output of polynomial length.
- We defined the sequence of hybrid ("imaginary") experiments. In each step, we replaced *one* "real" invocation of a crypto primitive ($G(U_n)$) with its "ideal" counterpart (U_{n+1}).
- We proved that consecutive pairs of hybrids are computationally indistinguishable, using the composition lemma and the security properties of the underlying primitives (i.e., that G is a PRG).
  - To apply the composition lemma, we defined a "simulator" (reduction) for each pair of adjacent hybrids and analyzed its behavior.

2 Obtaining a PRG

Thanks to Theorem 1.1, we know that all we need is to obtain a PRG with one extra bit of output. We describe a number-theoretic construction, due to Blum and Micali, of such an object.

2.1 Number Theory Background

Theorem 2.1 (Euler's theorem)**.** *Let G be a finite abelian (i.e., commutative) multiplicative group. For every $a \in G$, we have $a^{|G|} = 1$.*

Proof. Consider the set $A = a \cdot G = \{ax \ : \ x \in G\}$. Because G is a group, a is invertible, and we have $A = G$. Taking products over all elements in $A = G$, we have

$$\prod_{x \in G} (ax) = \prod_{x \in G} x.$$

Because G is commutative, the LHS is $a^{|G|} \cdot \prod_{x \in G} x$, and we can multiply by the inverse of the RHS to obtain $a^{|G|} = 1$. □

When $G = \mathbb{Z}_p^*$ for a prime p, we have $|\mathbb{Z}_p^*| = \varphi(p) = p - 1$, so we obtain the following corollary:

Corollary 2.2 (Fermat's "little" theorem)**.** *Let p be a prime. For any $a \in \mathbb{Z}_p^*$, we have $a^{p-1} = 1 \bmod p$.*

The following structural theorem will be very useful. (Its proof is elementary bur rather tedious, so we won't go through it today.)

Theorem 2.3. *Let p be a prime. The multiplicative group $\mathbb{Z}_p^*$ is* cyclic*, i.e., there exists some* generator *$g \in \mathbb{Z}_p^*$ such that $\mathbb{Z}_p^* = \langle g \rangle := \{g^1, g^2, \ldots, g^{p-1} = g^0 = 1\}$. (Equivalently, we can write the group as $\mathbb{Z}_p^* = \{g^0, g^1, \ldots, g^{p-2}\}$.)*

Question 1. Suppose there is a "black box" B that outputs a uniformly random element of $\mathbb{Z}_{p-1}$ for some known prime p. Show that there exists an algorithm that, using B as its only source of randomness, samples a uniformly random element of $\mathbb{Z}_p^*$.

2.2 Discrete Logarithm Problem and One-Way Function

Theorem 2.3 leads naturally to the so-called *discrete logarithm problem*, which is: given $y \in \mathbb{Z}_p^*$ (and prime p and generator g of $\mathbb{Z}_p^*$), find $\log_g y$, i.e., the $x \in \{1, \ldots, p-1\}$ for which $y = g^x \bmod p$. This problem is believed to be infeasible for large values of p.

Conjecture 2.4 (Discrete logarithm assumption)**.** Let $\mathsf{S}(1^n)$ be a PPT algorithm that outputs some prime p and generator g of $\mathbb{Z}_p^*$. For every non-uniform PPT algorithm $\mathcal{A}$,

$$\Pr_{(p,g) \leftarrow \mathsf{S}(1^n), y \leftarrow \mathbb{Z}_p^*} [\mathcal{A}(p, g, y) = \log_g y] = \text{negl}(n).$$

We would like to design a collection of OWFs based on the discrete logarithm assumption. The collection is made up of the functions $f_{p,g} \colon \{1, \ldots, p-1\} \to \mathbb{Z}_p^*$ (for prime p and generator g of $\mathbb{Z}_p^*$), defined as

$$f_{p,g}(x) = g^x \bmod p.$$

Moreover, these functions are even *permutations* if we identify $\{1, \ldots, p-1\}$ with $\mathbb{Z}_p^*$ in the natural way.

It is a tautology that the collection is one-way under the discrete logarithm assumption. It is also clear that we can efficiently sample from the domain of $f_{p,g}$. But we still need to check that $f_{g,p}$ can be evaluated efficiently, and that (p, g) can be generated efficiently.

For the first, we use the standard "repeated squaring" technique for exponentiation, which requires $O(|x|)$ multiplications modulo p. The solution to the second issue is not entirely straightforward. Given only the prime p, it is unknown (in general) how to find a generator g of $\mathbb{Z}_p^*$ efficiently. However, given the *factorization* of $p-1$, which can be generated along with p, it is possible: every element in $\mathbb{Z}_p^*$ has order dividing $p-1$, so g is a generator if and only if $g^{(p-1)/q} \neq 1 \bmod p$ for every prime divisor q of $p-1$. The number of non-generators is at most the sum of $(p-1)/q$ over all prime divisors q of $p-1$, so the density of generators is typically large enough. An often-used special case is $p = 2q+1$ for prime q, for which there are $q = (p-1)/2$ generators. However, it is not even known whether there exist infinitely many such "Sophie Germain" primes of this form! (Empirically, though, they are abundant.)

2.3 Blum-Micali PRG

We now present a PRG that uses the ideas presented in the previous section. From Section 1 we know that if we have a PRG that is able to generate one extra bit of randomness, we can generate a polynomial number of pseudorandom bits. Our goal will be the following: we want to construct a PRG $G_{p,g} \colon \mathbb{Z}_p^* \to \mathbb{Z}_p^* \times \{0,1\}$.[1]

Our solution is a function with the following form:

$$G_{p,g}(x) = (f_{p,g}(x) = g^x \bmod p \quad , \quad h(x)).$$

Note that $f_{p,g}(x)$ performs the modular exponentiation function (which is a one-way under the discrete log assumption), while $h \colon \mathbb{Z}_p^* \to \{0,1\}$ is some function (yet to be defined) that provides the additional bit.

Looking at the function, we can make the following observation: if $x \in \mathbb{Z}_p^*$ is chosen uniformly at random, then also $f_{p,g}(x)$ is uniform (because $f_{p,g}$ is a permutation). We still need to choose the function h, keeping in mind what we want from from the function: $h(x)$ should "look like a random bit," *even given* $f_{p,g}(x)$. That is, $h(x)$ should compute "something about x" that f hides *completely*.

We can think of many possible candidates for h: apply the xor function to all the bit of x; take the least significant bit of x (though you will show that this *does not* meet our requirements!); take the "most significant bit" of x (more precisely, test if $x > \frac{p-1}{2}$); etc.

Let's formalize the security property we want from h. Informally, we want it to be the case that no efficient algorithm, given $f(x)$, should be able to guess $h(x)$ with probability much better than the $\frac{1}{2}$ that is achievable by random guessing.

Definition 2.5 (Hardcore predicate)**.** A predicate $h \colon \{0,1\}^* \to \{0,1\}$ is *hard-core* for f if for all non-uniform PPT algorithms $\mathcal{A}$,

$$\mathbf{Adv}_{f,h}(\mathcal{A}) := \Pr_x[\mathcal{A}(f(x)) = h(x)] - \frac{1}{2} = \text{negl}(n).$$

Next time, we will show that under the discrete log assumption, the "most significant bit" predicate $h(x) = [x > \frac{p-1}{2}]$ is hard-core for $f_{p,g}$.

[1] To be pedantic, $G_{p,g}$ is a "collection" of PRGs, where the input seed comes from a set that depends on the function index (p, g). It it easy to check that our construction from Section 1 is compatible with this collection $G_{p,g}$.

Question 2. Prove that if h is hard-core for a one-way permutation $f\colon D \to D$, then

$$(f(x), h(x)) \stackrel{c}{\approx} (U(D), U_1) \equiv (f(x), b),$$

where $x \leftarrow D$ and $b \leftarrow \{0,1\}$ are independent. This means that $G(x) = (f(x), h(x)) \in D \times \{0,1\}$ is a PRG that expands by one bit. *Hint:* give a reduction that converts any distinguisher for the above two distributions into a predictor of $h(x)$ given $f(x)$.

Answers

Question 1. Suppose there is a "black box" B that outputs a uniformly random element of $\mathbb{Z}_{p-1}$ for some known prime p. Show that there exists an algorithm that, using B as its only source of randomness, samples a uniformly random element of $\mathbb{Z}_p^*$.

Answer. By Theorem 2.3, there exists a generator g of $\mathbb{Z}_p^*$. So, there exists an algorithm A that simply samples an $x \leftarrow \mathsf{B}$ and returns g^x. By Theorem 2.3, we know that $\langle g \rangle = \{g^0, \ldots, g^{p-2}\} = \mathbb{Z}_p^*$, and since $|\mathbb{Z}_p^*| = p-1$, it must be the case that for each $x \in \mathbb{Z}_p^*$, there exists a *unique* $i \in \mathbb{Z}_{p-1}$ such that $x = g^i$. Hence, $\Pr[\mathsf{A} \text{ outputs } x] = \Pr[B \text{ outputs } i] = 1/(p-1)$, as needed.

Question 2. Prove that if h is hard-core for a one-way permutation $f \colon D \to D$, then

$$(f(x), h(x)) \overset{c}{\approx} (U(D), U_1) \equiv (f(x), b),$$

where $x \leftarrow D$ and $b \leftarrow \{0,1\}$ are independent. This means that $G(x) = (f(x), h(x)) \in D \times \{0,1\}$ is a PRG that expands by one bit. *Hint:* give a reduction that converts any distinguisher for the above two distributions into a predictor of $h(x)$ given $f(x)$.

Answer. We prove this by reduction. Let $\mathcal{D}$ be an nuPPT algorithm that seeks to distinguish $(f(x), h(x))$ from $(f(x), b)$, where $x \leftarrow D, b \leftarrow \{0,1\}$ are independent. We construct an nuPPT predictor $\mathcal{A}$ against the hard-core predicate h for f, which uses $\mathcal{D}$ as a black box and has the same advantage. The key idea is that $\mathcal{A}$, given some $y = f(x)$ for unknown x, will simply guess a candidate value b for the hard-core bit $h(x)$, and invoke $\mathcal{D}(y, b)$ to get some indication of whether this guess is correct. Since $\mathcal{D}$ may not be a perfect distinguisher, its answer may not always be "correct," so it requires some care to analyze this reduction.

Formally, we define the predictor $\mathcal{A}$ to work as follows: given input $y \in D$, choose a uniformly random bit $b \leftarrow \{0,1\}$ and invoke $\mathcal{D}(y, b)$. If $\mathcal{D}$ accepts (i.e., outputs 1), then output b as the prediction for $h(f^{-1}(y))$; otherwise, output $\bar{b}$ as the prediction. Clearly, $\mathcal{A}$ is nuPPT.

We now relate $\mathcal{A}$'s advantage in the prediction game to $\mathcal{D}$'s advantage in the distinguishing game. First, observe that $\mathcal{A}$'s advantage is

$$\begin{aligned}
\mathbf{Adv}(\mathcal{A}) &= \Pr_{x \leftarrow D}[\mathcal{A}(f(x)) = h(x)] - 1/2 \\
&= \frac{1}{2}\left(\Pr_{x \leftarrow D}[\mathcal{D}(f(x), h(x)) = 1] + \Pr_{x \leftarrow D}[\mathcal{D}(f(x), \overline{h(x)}) = 0] - 1\right) \\
&= \frac{1}{2}\left(\Pr_{x \leftarrow D}[\mathcal{D}(f(x), h(x)) = 1] - \Pr_{x \leftarrow D}[\mathcal{D}(f(x), \overline{h(x)}) = 1]\right),
\end{aligned}$$

because $\mathcal{A}(f(x))$ outputs the correct bit $h(x)$ if $\mathcal{A}$ chooses $b = h(x)$ and $\mathcal{D}(f(x), b)$ accepts, or if $\mathcal{A}$ chooses $b = \overline{h(x)}$ and $\mathcal{D}(f(x), b)$ rejects. These events are disjoint, so we can add their probabilities, and b is uniform and independent of x, leading to the $1/2$ factor.

Now let

$$\begin{aligned}
p_U &= \Pr_{x \leftarrow D, b \leftarrow \{0,1\}}[\mathcal{D}(f(x), b) = 1] \\
&= \frac{1}{2}\left(\Pr_{x \leftarrow D}[\mathcal{D}(f(x), h(x)) = 1] + \Pr_{x \leftarrow D}[\mathcal{D}(f(x), \overline{h(x)}) = 1]\right),
\end{aligned}$$

because each of $b = h(x)$ and $b = \overline{h(x)}$ occur with probability $1/2$. Similarly, define

$$p_G = \Pr_{x \leftarrow D}[\mathcal{D}(f(x), h(x)) = 1].$$

By definition, $\mathcal{D}$'s advantage is

$$\begin{aligned}\mathbf{Adv}(\mathcal{D}) = p_G - p_U &= \frac{1}{2}\left(\Pr_{x \leftarrow D}[\mathcal{D}(f(x), h(x)) = 1] - \Pr_{x \leftarrow D}[\mathcal{D}(f(x), \overline{h(x)}) = 1]\right) \\ &= \mathbf{Adv}(\mathcal{A}).\end{aligned}$$

Since $\mathcal{A}$'s advantage is negligible by the hypothesis that h is hard-core for f, so is $\mathcal{D}$'s advantage, as desired.