Economic complexity and information choice

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1   Introduction

A strand of economic literature studies how limited attention and cognitive constraints guide economic choices. In these papers, researchers examine how prices can be used to aggregate private signals observed by attention-constrained investors. Signals usuall[1] take the form of the true payoff with additive Gaussian nois, $\eta_j = \tilde{f} + \epsilon_j$, for investor j, signal η_j, payoff vector $\tilde{f}$, and noise term ϵ_j.

I examine how endogenous information choice with attention limitations can lead investors to choose distinct portfolios when signals inform investors about both the underlying economic state and asset payoffs in those states. Allowing risk averse investors to select signals that are informative about state and payoffs jointly produces ==substantially different== portfolios from standard noisy rational equilibrium models, as well as shifting the conclusions of canonical models of information choice where signals inform investors *only* about payoffs.

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2   Model framework

Much of my notation and model structure follows from Kacperczyk et al. (2016), which introduce attention constraints and information choice to the multiasset noisy rational equilibrium model of Admati (1985).

The model has three periods. In time 1, informed investors allocate their attention across n signals. At time 2, all investors construct portfolios. At time 3, all investors receive payoffs.

I assume, as in Kacperczyk et al. (2016), that there are n risky assets with an arbitrary factor structure given by principal components. Assets $1, 2, \ldots, n$ represent specific assets with idiosyncratic shocks. The key difference between my paper and Kacperczyk et al. (2016) is that the economic state is a stochastic

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[1] A noteworthy exception is Breon-Drish (2015), which allows signals and payoff distributions to vary to a greater degree – namely, that the density of the payoff conditional on a private signal ($P(\tilde{f} \mid \eta_j)$, in terms of Kacperczyk et al. (2016)) be a member of the exponential family of distributions.

variable that is not known by investors. The economy is in state $s \in H, L$, where $s = H$ represents a "good" state with probability π and $s = L$ represents a "bad" state with probability $1 - \pi$. The density of s is written

$$P(s) = \begin{cases} \pi & \text{if } s = H \\ 1 - \pi & \text{if } s = L \end{cases}$$

The good and bad states dictate the mean and covariance of asset payoffs. Payoffs of the n assets, denoted by $n \times 1$ vector f, are written

$$f_i = \mu_{i,s} + z_i \tag{1}$$
$$z = [z_1, z_2, \ldots, z_n]' \sim \mathcal{N}(0, \Sigma_s) \tag{2}$$
$$f \mid s \sim \mathcal{N}(\mu_s, \Sigma_s) \tag{3}$$

The mean payoff vector $\mu_s = [\mu_{1,s}, \mu_{2,s}, \ldots, \mu_{n,s}]'$ and the $n \times n$ variance-covariance matrix of payoff shocks Σ_s are functions of the unobserved economic state. When investors are allowed to receive signals about the underlying shocks z, those same signals will allow investors to assign a probability to the underlying state and the associated payoff structure.[2]

Note that the unconditional payoff density $P(f)$ is a two-component Gaussian mixture distribution with mixture weights π and $1 - \pi$. The density function is written

$$P(f) \sim \pi \mathcal{N}(f \mid \mu_H, \Sigma_H) + (1 - \pi)\mathcal{N}(f \mid \mu_L, \Sigma_L) \tag{5}$$

Gaussian mixture distributions have the conceptual benefit of moving payoffs outside the traditional exponential family of distributions. In principle, a mixture model can approximate any complex joint density as the number of components increase (Nguyen and McLachlan, 2019). I maintain only two components as a proof-of-concept, though much of my analysis expands easily to an arbitrary number of Gaussian components. Additionally, many of the posterior distributions are only marginally more complex than when using traditional normal or log-normal distributions.

As in Admati (1985) and Kacperczyk et al. (2016), I employ CARA utility to abstract from wealth effects. However, the conditions in Kacperczyk et al.

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[2]Kacperczyk et al. (2016) utilize a transformation of asset payoffs to the corresponding risk factor payoffs – in my case, the eigen-decomposition $\Sigma_s = \Gamma_s \Lambda_s \Gamma_s'$ for $s \in H, L$ yields Arrow-Debreu synthetic securities on risk factors:

$$\tilde{f} \mid s \sim \mathcal{N}(\Gamma_s^{-1}\mu_s, \Lambda_s) \tag{4}$$

Unfortunately, I cannot proceed with the Kacperczyk et al. (2016) solution method, which requires an additional transformation of risk factor prices $\tilde{p} = \Gamma^{-1}p$ and risk factor quantities $\tilde{q} = \Gamma^{-1}q$ for some eigenvector matrix Γ. My model only permits the orthogonalization of the prior variance Σ, but in general the transforms on $\tilde{q}$ and $\tilde{p}$ will remain correlated conditional on state.
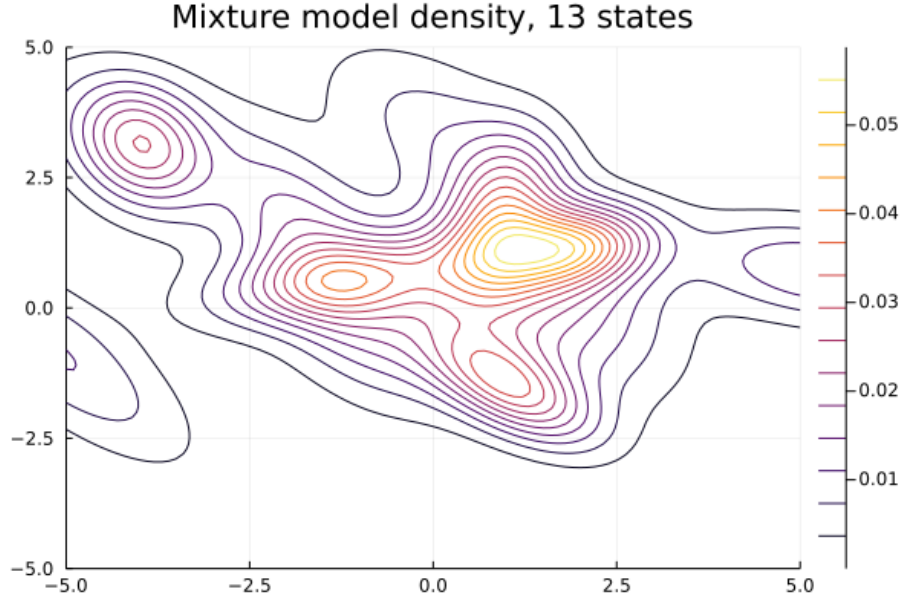
Figure 1: The joint density contour plot of a multimodal Gaussian mixture distribution. Dimensionality is $\mathcal{R}^2$.

(2016) that reduce the investment problem to a mean-variance problem do not hold in my setting without minor modifications. Namely, the distribution of f is unconditionally non-Gaussian. Fortunately, the Gaussian mixture distribution I pay off benefits from the fact that the non-Gaussian density $P(f)$ can be rewritten as a weighted sum of two conditionally Gaussian densities, i.e. $P(f \mid s = H)P(s = H) + P(f \mid s = L)P(s = L)$. Many of my results thus benefit from the simplicity of the Normal distribution without reducing the joint density of payoffs to a simplistic unimodal distribution.

The economy is populated by atomistic investors j with unit mass ($j \in [0, 1]$). Investors have exponential preferences on final-period wealth W_j, with a risk-aversion coefficient ρ. Expected utility at time 2 (after receiving private signals) is a function of risk-free rate r, initial wealth W_0, asset quantities q_j, asset payoffs f, and asset prices p.

$$U_{j2} = E_j[\exp\{-\rho W_j\}] \tag{6}$$

for law of motion on wealth $W_j = rW_0 + q'_j(f - pr)$. Since wealth effects do not enter the investment decision for CARA utilities, I follow Kacperczyk et al. (2016) and equalize initial wealth to W_0 for all investors.

A portion of investors (the *informed*) receive private signals η_j about time 3 payoffs f. Signals take the form of additive Gaussian noise around the true

payoff, where the precision of the noise is determined by investor attention allocation. The form of a private signal is

$$\eta_j \sim \mathcal{N}(f, \Sigma_{\eta,j}), \tag{7}$$
$$\text{or } \eta_j = f + \epsilon_j, \quad \epsilon_j \sim \mathcal{N}(0, \Sigma_{\eta_j}) \tag{8}$$

The matrix Σ_{η_j} is a diagonal matrix with entries K_{ij}^{-1}. K_{ij} is the total amount of attention given to signal i by investor j. Higher values of K_{ij} imply higher precision, and thus a more accurate signal of f.

Investors have limited attention, in that they cannot pay attention to all the signals they would like. Concretely, this constraint is written

$$\sum_{i=1}^{n} K_{ij} \leq K_j \tag{9}$$

though for simplicity I equalize attention constraints across investors to $K_j = K$ for informed investors and $K_j = 0$ for uninformed investors. Uninformed investors can only use prices as signals about payoffs, whereas informed investors can use both prices and private signals. The attention constraint utilized here is common in the information choice literature – see Kacperczyk et al. (2016).

Investors have two optimization problems to make. First, if the investor is informed, they must allocate their attention across private signals at time 1. Second, conditional on any information observed in time 1, investors construct portfolios to optimize expected utility at time 2, U_{j2}.

The investor's information choice problem is to maximize expected time-1 utility U_{j1}:

$$
\begin{aligned}
\underset{K_{ij}}{\text{maximize}} \quad & U_{j1} = E\left[E_j[\exp\{-\rho W_j\}]\right] \\
\text{subject to} \quad & W_j = rW_0 + q_j'(f - pr), \\
& \sum_i K_{ij} \leq 1, \\
& K_{ij} \geq 0, \quad \forall i
\end{aligned}
\tag{10}
$$

Next, the time-2 portfolio choice problem is to maximize expected utility U_{j2}:

$$
\begin{aligned}
\underset{q_j}{\text{maximize}} \quad & U_{j2} = E_j[\exp\{-\rho W_j\} \mid \eta_j, p] \\
\text{subject to} \quad & W_j = rW_0 + q_j'(f - pr)
\end{aligned}
\tag{11}
$$

Finally, markets must clear at price p and quantities q_j, leading to the traditional market clearing condition

$$\int_j q_j(p) = \overline{x} + x \tag{12}$$

Market clearing requires that quantities and prices be such that all supply is allocated to an investor. I now turn to the formal declaration of an equilibrium in my setting, as in Breon-Drish (2015).

Definition 1 *A noisy rational expectations equilibrium (NREE) is a function $p(s, f, x)$ that maps the state s, payoffs f, and asset supply x to a vector of prices for the n assets, such that (a) prices maximize aggregate surplus:*

$$q_j(p, \eta_j) \in \arg\max_{q_j} E_j[\exp\{-\rho W_j\} \mid \eta_j, p], \quad \forall j \in [0, 1] \tag{13}$$

that (b) markets clear, and (c) that all agents are optimizing conditional on their information set $\{p, \eta_j\}$.

Prices providing information to invesotrs is a long-standing proposition with a strong theoretical background like Grossman (1976), Admati (1985), and others. Empirical support for price informativeness is robust as well. The often-conjectured linear price function in noisy rational expectations models are interperetable as Gaussian signals around the true payoffs with noise due to uncertain asset supply. Fortunately, my model permits the use of a linear pricing function and the corresponding Gaussian signal as long as rational expectations hold – equilibrium prices must be a linear function of the true payoffs f.

I conjecture and verify that the pricing function follows the form

$$p = A + Bf + Cx \tag{14}$$

The linear form of p is Gaussian conditional on f, so the closed-form posterior for an investor j is well-defined:

$$
\begin{aligned}
P(f \mid \eta_j, p) &= \frac{P(\eta_j, p \mid f)P(f)}{P(\eta_j, p)} \\
&= \frac{P(\eta_j, p \mid f, s_H)P(f \mid s_H)P(s_H) + P(\eta_j, p \mid f, s_L)P(f \mid s_L)P(s_L)}{\pi P(\eta_j, p \mid s_H) + (1 - \pi)P(\eta_j, p \mid s_L)} \\
&= \frac{\pi P(\eta_j, p \mid f, s_H)P(f \mid s_H) + (1 - \pi)P(\eta_j, p \mid f, s_L)P(f \mid s_L)}{\pi P(\eta_j, p \mid s_H) + (1 - \pi)P(\eta_j, p \mid s_L)}
\end{aligned}
$$

The above density is simply a weighted sum of Gaussian densities in the numerator and denominator. To identify these posteriors, it is useful to note that the three variables of interest to investor j, $m_j = [f, p, \eta_j]'$, is Normal when

conditioned on the latent state s since prices conditional on s are the sum of two Gaussians (payoff shocks z and supply shocks x).

The mean of the payoff, price, and signal vector $m_j \mid s$ is

$$E[m_j \mid s] = [\mu_s, A + B\mu_s + C\bar{x}, \mu_s]$$

with the block variance matrix

$$\text{Var}[m_j \mid s] = \begin{bmatrix} \Sigma_s & B\Sigma_s & \Sigma_s \\ B\Sigma_s & B\Sigma_s B' + C\Sigma_x C' & B\Sigma_s \\ \Sigma_s & B\Sigma_s & \Sigma_s + \Sigma_j \end{bmatrix}$$

The posterior density of interest, $P(f \mid p, \eta_j, s)$, is well-defined by the rules of the multivariate Gaussian. Both the joint, prior, and posterior are all Gaussian when conditioned on state s.

To derive the posterior mean and variance, denote the aggregate signal held by investor j as $y_j = [p, \eta_j]'$ and the corresponding state mean of that signal $\bar{y}_{j,s} = [A + B\mu_s + C\bar{x}, \mu_s]'$. Then the mean $\hat{\mu}_{j,s}$ and variance $\hat{\Sigma}_{j,s}$ of this posterior density are given by the identities

$$\hat{\Sigma}_{j,s} = \Sigma_s - \Omega_{1,2}\Omega_{2,2,j}^{-1}\Omega_{2,1}$$
$$\hat{\mu}_{j,s} = \mu_s + \Omega_{1,2}\Omega_{2,2,j}^{-1}(y_j - \bar{y}_{j,s})$$

where $\Omega_{1,2}$, $\Omega_{2,1}$, and $\Omega_{2,2}$ are block matrices formed from the partition of $\text{Var}[m_j \mid s]$:

$$\Omega_{1,2} = \Omega_{2,1}' = \begin{bmatrix} B\Sigma_s & \Sigma_s \end{bmatrix}$$
$$\Omega_{2,2,j} = \begin{bmatrix} B\Sigma_s B' + C\Sigma_x C' & B\Sigma_s \\ B\Sigma_s & \Sigma_s + \Sigma_j \end{bmatrix}$$

The matrices A, B, and C are calculated from the fixed-point that solves the market clearing condition. At the moment, I solve for these numerically, but I will eventually calculate the analytic forms for A, B, and C.

Derive the joint density of prices

3   Equilibrium

Kacperczyk et al. (2016) work backwards by solving the portfolio allocation problem at time 2 first, and then using this solution to determine the optimal information choice at time 1. I defer to their solution, though with some added complexity due to the change in the densities of payoffs f from multivariate Gaussian to mixture distributed.

$$\begin{aligned} \underset{q_j}{\text{maximize}} \quad & U_{j2} = E_j[\exp\{-\rho W_j\} \mid \eta_j, p] \\ \text{subject to} \quad & W_j = rW_0 + q_j'(f - pr) \end{aligned} \tag{15}$$
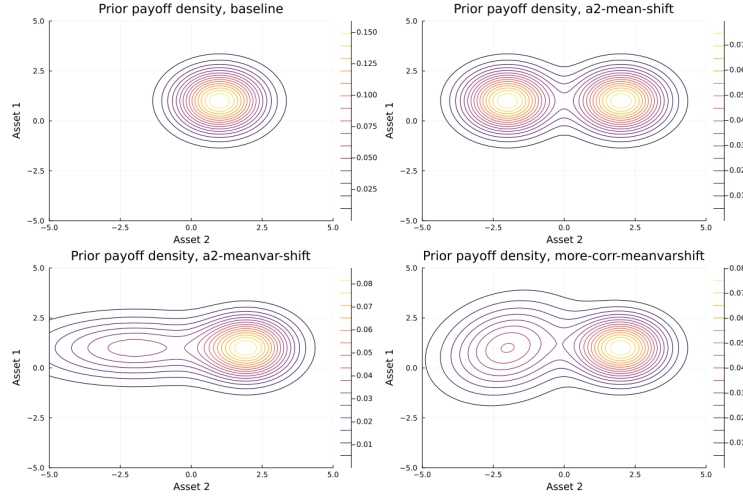
Figure 2: Unconditional joint payoff densities of f in different economies.

The appendix shows that the quantity q_j that solves (15) takes the form

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$$q_j = \frac{1}{\rho}\left(\hat{s}_j \hat{\Sigma}_{H,j} + (1 - \hat{s}_j)\hat{\Sigma}_{L,j}\right)^{-1}\left(\hat{s}_j \hat{\mu}_{H,j} + (1 - \hat{s}_j)\hat{\mu}_{L,j}\right) \tag{16}$$

where $\hat{s}_j$ represents the posterior probability of being in the high state by investor j. It is worth commenting that the form of the optimal quantity is relatively simple and maintains the general form of strict Gaussian case. The only real difference between my q_j and that of papers like Kacperczyk et al. (2016) is that the means and variances are weighted by the posterior state probabilities and summed. The form of q_j demonstrates that it is trivial to add an arbitrary number of Gaussian clusters to the payoff density, should a modeler wish to approximate increasingly complex joint distributions.

I start by defining several types of economies a

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A   Mixture model variance

The variance-covariance matrix of variable X following a Gaussian mixture distribution with means $\mu = [\mu_1, \mu_2, \ldots, \mu_k]'$, covariances $\Sigma = [\Sigma_1, \Sigma_2, \ldots, \Sigma_k]$, and mixture proportions $\pi = [\pi_1, \pi_2, \ldots, \pi_k]'$ is[3]

$$\text{Var}[X] = E[\text{Var}[X \mid k]] + \text{Var}[E[X \mid k]] \tag{17}$$

$$= \sum_k \pi_k \left(\Sigma_k + (\mu_k - \overline{\mu})(\mu_k - \overline{\mu})' \right) \tag{18}$$

for $\overline{\mu} = \sum_k \pi_k \mu_k$.

B   Misc identities

Stochastic variables:

- State variable $s \sim \text{Bernoulli}(\pi)$. $P(s = H) = \pi$, $P(s = L) = 1 - \pi$.

- Risk factor payoffs $\tilde{f} \mid s \sim N(\Gamma^{-1}\mu_s, \Sigma_s)$

- Risk factor supply $x \sim N(\overline{x}, \sigma_x I)$

- Private signals $\eta_j \mid s \sim N(z, \Sigma_{\eta_j})$

- Price signal $\eta_p \mid s \sim N(z, \Sigma_p)$

Joint density:

$$P(s, \tilde{f}, x, \eta_j, \eta_p) = P(\tilde{f} \mid s)P(\eta_j \mid s)P(\eta_p \mid s)P(s)P(x)$$

Posterior density:

$$P(s, \tilde{f} \mid x, \eta_j, \eta_p) = \frac{P(x, \eta_j, \eta_p \mid s, \tilde{f})P(s, \tilde{f})}{P(x, \eta_j, \eta_p)}$$

Unknown values:

- $E_j[\tilde{f} - \tilde{p}r \mid H]$

- $E_j[\tilde{f} - \tilde{p}r \mid L]$

- $V_j[\tilde{f} - \tilde{p}r \mid H]$

- $V_j[\tilde{f} - \tilde{p}r \mid L]$

- $P(H \mid \eta_p, \eta_j)$

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[3]From https://math.stackexchange.com/questions/195911/calculation-of-the-covariance-of-gaussian-mixtures

- $P(L \mid \eta_p, \eta_j)$

- $E_j[\tilde{f} \mid \eta_p, \eta_j]$

- $V_j[\tilde{f} \mid \eta_p, \eta_j]$

- $\tilde{p}$

Portfolio choice problem:

$$U_{2j} = \max_{\tilde{q}_j} \rho E_j[W_j] - \frac{\rho^2}{2} V_j[W_j]$$

Optimal quantity:

$$
\begin{aligned}
\tilde{q}_j &= \frac{1}{\rho} \Big(P(H)\Sigma_H + P(L)\Sigma_L \Big)^{-1} \Big(P(H)E_j[\tilde{f} \mid H] + P(L)E_j[\tilde{f} \mid L] - \tilde{p}r \Big) \\
&= \frac{1}{\rho} V_j[\tilde{f}]^{-1}(E_j[\tilde{f}] - \tilde{p}r)
\end{aligned}
$$

Ex-ante expected utility:

$$
\begin{aligned}
U_{1j} &= E\left[\rho E_j[W_j] - \frac{\rho^2}{2} V_j[W_j] \right] \\
&= \pi E\left[\rho E_j[W_j \mid H] - \frac{\rho^2}{2} V_j[W_j \mid H] \right] \\
&\quad + (1-\pi)E\left[\rho E_j[W_j \mid L] - \frac{\rho^2}{2} V_j[W_j \mid L] \right] \\
&= \rho r W_0 \\
&\quad + \rho \tilde{q}_j' \Big(\pi E_j[\tilde{f} - \tilde{p}r \mid H] + (1-\pi)E_j[\tilde{f} - \tilde{p}r \mid L] \Big) \\
&\quad - \frac{\rho^2}{2} \tilde{q}_j' \Big(\pi V_j[\tilde{f} - \tilde{p}r \mid H] + (1-\pi)V_j[\tilde{f} - \tilde{p}r \mid L] \Big) \tilde{q}_j
\end{aligned}
$$