Stochastic variables:

- State variable $s \sim$ Bernoulli(π). $P(s = H) = \pi$, $P(s = L) = 1 - \pi$.
- Risk factor payoffs $\tilde{f} \mid s \sim N(\Gamma^{-1}\mu_s, \Sigma_s)$
- Risk factor supply $x \sim N(\overline{x}, \sigma_x I)$
- Private signals $\eta_j \mid s \sim N(z, \Sigma_{\eta_j})$
- Price signal $\eta_p \mid s \sim N(z, \Sigma_p)$

Joint density:

$$P(s, \tilde{f}, x, \eta_j, \eta_p) = P(\tilde{f} \mid s)P(\eta_j \mid s)P(\eta_p \mid s)P(s)P(x)$$

Posterior density:

$$P(s, \tilde{f} \mid x, \eta_j, \eta_p) = \frac{P(x, \eta_j, \eta_p \mid s, \tilde{f})P(s, \tilde{f})}{P(x, \eta_j, \eta_p)}$$

Unknown values:

- $E_j[\tilde{f} - \tilde{p}r \mid H]$
- $E_j[\tilde{f} - \tilde{p}r \mid L]$
- $V_j[\tilde{f} - \tilde{p}r \mid H]$
- $V_j[\tilde{f} - \tilde{p}r \mid L]$
- $P(H \mid \eta_p, \eta_j)$
- $P(L \mid \eta_p, \eta_j)$
- $E_j[\tilde{f} \mid \eta_p, \eta_j]$
- $V_j[\tilde{f} \mid \eta_p, \eta_j]$
- $\tilde{p}$

Portfolio choice problem:

$$U_{2j} = \max_{\tilde{q}_j} \rho E_j[W_j] - \frac{\rho^2}{2} V_j[W_j]$$

Optimal quantity:

$$
\begin{aligned}
\tilde{q}_j &= \frac{1}{\rho}\bigg(P(H)\Sigma_H + P(L)\Sigma_L\bigg)^{-1}\bigg(P(H)E_j[\tilde{f} \mid H] + P(L)E_j[\tilde{f} \mid L] - \tilde{p}r\bigg) \\
&= \frac{1}{\rho}V_j[\tilde{f}]^{-1}(E_j[\tilde{f}] - \tilde{p}r)
\end{aligned}
$$

Ex-ante expected utility:

$$
\begin{aligned}
U_{1j} &= E\bigg[\rho E_j[W_j] - \frac{\rho^2}{2}V_j[W_j]\bigg] \\
&= \pi E\bigg[\rho E_j[W_j \mid H] - \frac{\rho^2}{2}V_j[W_j \mid H]\bigg] \\
&\quad + (1-\pi)E\bigg[\rho E_j[W_j \mid L] - \frac{\rho^2}{2}V_j[W_j \mid L]\bigg] \\
&= \rho r W_0 \\
&\quad + \rho \tilde{q}_j'\bigg(\pi E_j[\tilde{f} - \tilde{p}r \mid H] + (1-\pi)E_j[\tilde{f} - \tilde{p}r \mid L]\bigg) \\
&\quad - \frac{\rho^2}{2}\tilde{q}_j'\bigg(\pi V_j[\tilde{f} - \tilde{p}r \mid H] + (1-\pi)V_j[\tilde{f} - \tilde{p}r \mid L]\bigg)\tilde{q}_j
\end{aligned}
$$

Information choice and state uncertainty

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1 Introduction

A strand of economic literature studies how limited attention and cognitive constraints guide economic choices. In these papers, researchers examine how prices can be used to aggregate private signals observed by attention-constrained investors. Signals usuall[1] take the form of the true payoff with additive Gaussian nois, $\eta_j = \tilde{f} + \epsilon_j$, for investor j, signal η_j, payoff vector $\tilde{f}$, and noise term ϵ_j.

I examine how endogenous information choice with attention limitations can lead investors to choose distinct portfolios when signals inform investors about both the underlying economic state and asset payoffs in those states. Allowing risk averse investors to select signals that are informative about state and payoffs jointly produces substantially different portfolios from standard noisy rational equilibrium models, as well as shifting the conclusions of canonical models of information choice where signals inform investors *only* about payoffs.

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2 Model framework

Much of my notation and model structure follows from Kacperczyk et al. (2016), which introduce attention constraints and information choice to the multiasset noisy rational equilibrium model of Admati (1985).

The model has three periods. In time 1, informed investors allocate their attention across n signals. At time 2, all investors construct portfolios. At time 3, all investors receive payoffs.

I assume, as in Kacperczyk et al. (2016), that there are n risky assets with an arbitrary factor structure given by principal components. Assets $1, 2, \ldots, n$ represent specific assets with idiosyncratic shocks. The key difference between my paper and Kacperczyk et al. (2016) is that the economic state is a stochastic

[1] A noteworthy exception is Breon-Drish (2015), which allows signals and payoff distributions to vary to a greater degree – namely, that the density of the payoff conditional on a private signal ($P(\tilde{f} \mid \eta_j)$, in terms of Kacperczyk et al. (2016)) be a member of the exponential family of distributions.

variable that is not known by investors. The economy is in state $s \in H, L$, where $s = H$ represents a "good" state with probability π and $s = L$ represents a "bad" state with probability $1 - \pi$. The density of s is written

$$P(s) = \begin{cases} \pi & \text{if } s = H \\ 1 - \pi & \text{if } s = L \end{cases}$$

Importantly, asset payoffs are governed by two different stochastic processes conditional on state. Payoffs of the n assets are written

$$f_i = \mu_{i,s} + z_i \tag{1}$$
$$z = [z_1, z_2, \ldots, z_n]' \sim \mathcal{N}(0, \Sigma_s) \tag{2}$$
$$f \mid s \sim \mathcal{N}(\mu_s, \Sigma_s) \tag{3}$$

That is, both the mean payoff vector $\mu_s = [\mu_{1,s}, \mu_{2,s}, \ldots, \mu_{n,s}]'$ and the $n \times n$ variance-covariance matrix of payoff shocks Σ_s are functions of the unobserved economic state. When investors are allowed to receive signals about the underlying shocks z, those same signals will allow investors to assign a probability to the underlying state and the associated payoff structure.[2]

Note that the unconditional payoff density $P(f)$ is a two-component Gaussian mixture distribution with mixture weights π and $1 - \pi$. The density function is written

$$P(f) \sim \pi\mathcal{N}(f \mid \mu_H, \Sigma_H) + (1 - \pi)\mathcal{N}(f \mid \mu_L, \Sigma_L) \tag{5}$$

Gaussian mixture distributions have the conceptual benefit of moving payoffs outside the traditional exponential family of distributions. In principle, a mixture model with more and more components can approximate any complex joint density (Nguyen and McLachlan, 2019). The shift towards a mixture distribution does however present some technical difficulties in that closed-form posterior distributions are not generally available.

As in Admati (1985) and Kacperczyk et al. (2016), I employ CARA utility to abstract from wealth effects. However, the conditions in Kacperczyk et al. (2016) that reduce the investment problem to a mean-variance problem do

[2] Kacperczyk et al. (2016) utilize a transformation of asset payoffs to the corresponding risk factor payoffs – in my case, the eigen-decomposition $\Sigma_s = \Gamma_s \Lambda_s \Gamma_s'$ for $s \in H, L$ yields Arrow-Debreu synthetic securities on risk factors:

$$\tilde{f} \mid s \sim \mathcal{N}(\Gamma_s^{-1}\mu_s, \Lambda_s) \tag{4}$$

Unfortunately, I cannot proceed with the Kacperczyk et al. (2016) solution method, which requires an additional transformation of risk factor prices $\tilde{p} = \Gamma^{-1}p$ and risk factor quantities $\tilde{q} = \Gamma^{-1}q$ for some eigenvector matrix Γ. My model only permits the orthogonalization of the prior variance Σ, but in general the transforms on $\tilde{q}$ and $\tilde{p}$ will remain correlated conditional on state.

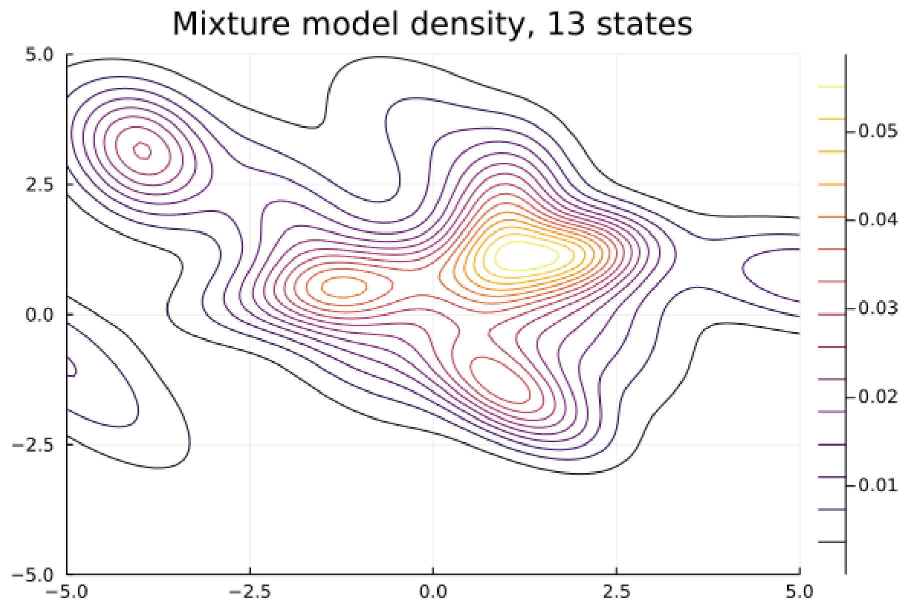

Figure 1: The joint density contour plot of a multimodal Gaussian mixture distribution. Dimensionality is $\mathcal{R}^2$.

not hold in my setting. Namely, the distribution of f is non-Gaussian. Non-Gaussian payoffs invalidate the Taylor expansion of the utility function that is commonly employed to simplify the investment decision.

The economy is populated by atomistic investors j with unit mass ($j \in [0, 1]$). Investors have exponential preferences on final-period wealth W_j, with a risk-aversion coefficient ρ. Expected utilityat time 2 (after receiving private signals) is a function of risk-free rate r, initial wealth W_0, asset quantities q_j, asset payoffs f, and asset prices p.

$$U_{j2} = E_j[\exp\{-\rho W_j\}] \tag{6}$$

for law of motion on wealth $W_j = rW_0 + q_j'(f - pr)$. Since wealth effects do not enter the investment decision for CARA utilities, I follow Kacperczyk et al. (2016) and equalize all initial wealth to W_0.

A portion of investors receive private signals η_j about time 3 payoffs f. Signals take the form of additive Gaussian noise around the true payoff, where the precision of the noise is determined by investor attention allocation. The form of a private signal is

$$\eta_j \sim \mathcal{N}(f, \Sigma_{\eta,j}) \tag{7}$$

The matrix Σ_{η_j} is a diagonal matrix with entries K_{ij}^{-1}. K_{ij} is a *unit* of attention given to signal i by investor j. Higher values of K_{ij} imply higher precision, and thus a more accurate signal of f.

Investors have limited attention, in that they cannot pay attention to all the signals they would like. Concretely, this constraint is written

$$\sum_{i=1}^{n} K_{ij} \leq K_j \tag{8}$$

though for simplicity I equalize attention constraints across investors to $K_j = K$ for informed investors and $K_j = 0$ for uninformed investors. Uninformed investors can only use prices as signals about payoffs, whereas informed investors can use both prices and private signals. The attention constraint utilized here is common in the information choice literature – see Kacperczyk et al. (2016).

Investors have two optimization problems to make. First, if the investor is informed, they must allocate their attention across private signals at time 1. Second, conditional on any information observed in time 1, investors construct portfolios to optimize expected utility at time 2, U_{j2}.

The investor's information choice problem is to maximize expected time-1 utility U_{j1}:

$$\begin{aligned} \underset{K_{ij}}{\text{maximize}} \quad & U_{j1} = E\Big[E_j[\exp\{-\rho W_j\}]\Big] \\ \text{subject to} \quad & \quad W_j = rW_0 + q_j'(f - pr), \\ & \sum_i K_{ij} \leq 1, \\ & \quad K_{ij} \geq 0, \quad \forall i \end{aligned} \tag{9}$$

Next, the time-2 portfolio choice problem is to maximize expected utility U_{j2}:

$$\begin{aligned} \underset{q_j}{\text{maximize}} \quad & U_{j1} = E_j[\exp\{-\rho W_j\} \mid \eta_j, p] \\ \text{subject to} \quad & W_j = rW_0 + q_j'(f - pr) \end{aligned} \tag{10}$$

Transition out

3 Equilibrium

Kacperczyk et al. (2016) work backwards by solving the portfolio allocation problem first, and then using the solution from the portfolio allocation problem to determine the optimal information choice.