Crisis Learning

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Howdy hey, gang. The goal of this document is to describe the rough shape of an economy with a simple state-based structure.

1 Model styles

In all models, it is assumed that the initial state distribution is categorically distributed with probabilities $\Pi = [\pi_1, \pi_2]$. We'll keep it simple for now and use only two possible states, but in general everything should function with an arbitrary number of states.

1.1 Simple Gaussian

Figure 1.1 demonstrates an extremely simple economy where a latent state variable S_t evolves over time. In each possible state $S_t \in \mathbb{S}$, the location and scale parameters ($\mu(S_t)$ and $\Sigma(S_t)$) of the Gaussian return vector R_t may vary:

1. The degenerate constant mean/variance condition $\Sigma(S_t) = \Sigma$ and a changing mean return vector, $\mu(S_t) = \mu$.
2. A constant variance term $\Sigma(S_t) = \Sigma(S_t') = \Sigma$ and a changing mean return vector, $\mu(S_t) \neq \mu(S_t')$.
3. A varying variance term $\Sigma(S_t) \neq \Sigma(S_t')$ and a constant mean return vector, $\mu(S_t) = \mu$.
4. The "everything changes" case where both $\mu(S_t)$ and $\Sigma(S_t)$ vary.

Regardless of the choice of method above, going forward I will simply denote means and variances as μ and Σ to reduce notational costs, but keep in mind that they are implicit functions of macroeconomic state S_t.

Investors do not observe all elements of R_t simultaneously. Rather, they observe them sequentially — denote a partition of R_t after $n \leq N$ firms have been observed with $R_{t,1:n}$. Write the distributions of observed returns $R_{t,A}$ and the returns yet to be observed $R_{t,B}$ as the partitions

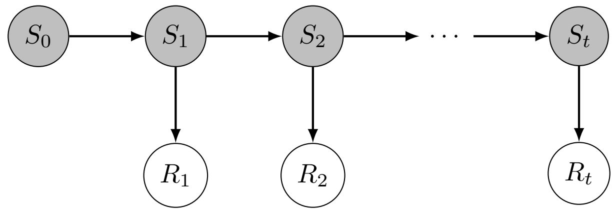

Figure 1: Depiction of the underlying economic process. S_t is a Markov state process drawn from $S_t \sim P(S_t, S_{t-1})$, while R_t is a multivariate Gaussian of dimension N (one for each firm). R_t is Gaussian only conditional on S_t, i.e. $R_t \mid S_t \sim \mathcal{N}(\mu(S_t), \Sigma(S_t))$.

$$
\begin{aligned}
R_t &= \begin{bmatrix} R_t^A & R_t^B \end{bmatrix}' \\
R_t^A &\sim \mathcal{N}(\mu^A, \Sigma^A) \\
R_t^B &\sim \mathcal{N}(\mu^B, \Sigma^B)
\end{aligned}
$$

where

$$E[R_t \mid S_t] = \mu = \begin{bmatrix} \mu_A \\ \mu_B \end{bmatrix}$$

The superscripts A and B are used to indicate a subsetting operation where $\mu^A = \mu_{1:n}$ and $\mu_B = \mu_{n+1:N}$. The covariance matrices can be similarly partitioned into a block matrix

$$\text{Var}[R_t \mid S_t] = \underset{N \times N}{\Sigma} = \begin{bmatrix} \underset{n \times n}{\Sigma^A} & \underset{n \times N-n}{\Sigma^{AB}} \\ \underset{N-n \times n}{\Sigma^{BA}} & \underset{N-n \times N-n}{\Sigma^B} \end{bmatrix}$$

Note, however, that R_t^A is observed, and is no longer a stochastic variable. However, it was *drawn* from a distribution correlated to R_t^B, and thus can be used as conditioning information to more precisely determine the distribution of the returns to be revealed, R_t^B. It can be shown that, conditional on observing R_t^A, the distribution of R_t^B is

Actually show this?

$$R_t^B \mid R_t^A, S_t \sim \mathcal{N}(\overline{\mu}^B, \overline{\Sigma}^B)$$

for conditional parameters

$$
\begin{aligned}
\overline{\mu}^B &= \mu^B + \Sigma^{BA}(\Sigma^A)^{-1}(R_t^A - \mu^A) \\
\overline{\Sigma}^B &= \Sigma^B + \Sigma^{BA}(\Sigma^A)^{-1}\Sigma^{AB}
\end{aligned}
$$

I assume that, for any state, the covariance matrix of firm payoffs is drawn from an inverse Wishart distribution parameterized by the shape matrix $\mathbf{\Omega}$ and precision ν. The distribution of Σ holds even for cases where the covariance matrix does not change across states[1]. The matrix $\mathbf{\Omega}$ determines the fundamental "shape" of the covariance structure, in that the mean of the distribution of Σ is

$$E[\Sigma \mid S_t] = \frac{\mathbf{\Omega}}{\nu - N - 1}$$

The average *covariance* can vary substantially in terms of scale as ν changes, but the average *correlation* remains the same regardless of ν. Any two draws Σ_1 and Σ_2 can have highly varied behavior. For example, Σ_1 might suggest a negative correlation in the payoffs of two firms, while Σ_2 could suggest a positive correlation. The inverse Wishart distribution is advantageous because the covariance matrix governing firm payoffs can vary meaningfully between states, and the distribution's properties are well-known.

[1]In models where the covariance matrix changes with state (cases 3 and 4), the inverse Whishart distribution can still be used as the distribution collapses with certainty as $\nu \to \infty$.