Crisis Learning

Cameron Pfiffer

June 9, 2021

Howdy hey, gang. The goal of this document is to describe the rough shape of an economy with a simple state-based structure.

1 Model styles

In all models, it is assumed that the initial state distribution is categorically distributed with probabilities $\Pi = [\pi_1, \pi_2, \ldots, \pi_Q]$ for Q states.

The state transition matrix is given by A. It is assumed that all know the transition matrix.

1.1 Simple Gaussian

Figure 1.1 demonstrates an extremely simple economy where a latent state variable S_t evolves over time. In each possible state $S_t \in \mathbb{S}$, the location and scale parameters ($\mu(S_t)$ and $\Sigma(S_t)$) of the Gaussian payoff vector f_t may vary:

1. The degenerate constant mean/variance condition $\Sigma(S_t) = \Sigma$ and a changing mean payoff vector, $\mu(S_t) = \mu$.
2. A constant variance term $\Sigma(S_t) = \Sigma(S_t') = \Sigma$ and a changing mean payoff vector, $\mu(S_t) \neq \mu(S_t')$.
3. A varying variance term $\Sigma(S_t) \neq \Sigma(S_t')$ and a constant mean payoff vector, $\mu(S_t) = \mu$.
4. The "everything changes" case where both $\mu(S_t)$ and $\Sigma(S_t)$ vary.

Regardless of the choice of method above, going forward I will simply denote means and variances as μ and Σ to reduce notational costs, but keep in mind that they are implicit functions of macroeconomic state S_t.

Investors do not observe all elements of f_t simultaneously. Rather, they observe them sequentially — denote a partition of f_t after $n \leq N$ firms have been observed with $f_{t,1:n}$. Write the distributions of observed payoffs $f_{t,A}$ and the payoffs yet to be observed $f_{t,B}$ as the partitions

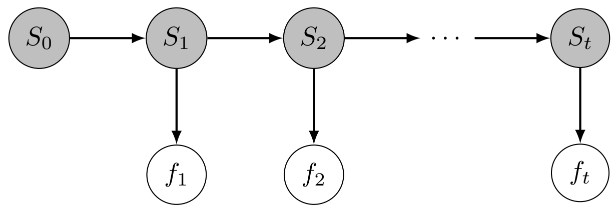

Figure 1: Depiction of the underlying economic process. S_t is a Markov state process drawn from $S_t \sim P(S_t, S_{t-1})$, while f_t is a multivariate Gaussian of dimension N (one for each firm). Payoffs f_t are Gaussian only conditional on S_t, i.e. $f_t \mid S_t \sim \mathcal{N}(\mu(S_t), \Sigma(S_t))$.

$$
\begin{aligned}
f_t &= \begin{bmatrix} f_t^A & f_t^B \end{bmatrix}' \\
f_t^A &\sim \mathcal{N}(\mu^A, \Sigma^A) \\
f_t^B &\sim \mathcal{N}(\mu^B, \Sigma^B)
\end{aligned}
$$

where

$$E[f_t \mid S_t] = \mu = \begin{bmatrix} \mu_A \\ \mu_B \end{bmatrix}$$

The superscripts A and B are used to indicate a subsetting operation where $\mu^A = \mu_{1:n}$ and $\mu_B = \mu_{n+1:N}$. The covariance matrices can be similarly partitioned into a block matrix

$$\text{Var}[f_t \mid S_t] = \underset{N\times N}{\Sigma} = \begin{bmatrix} \underset{n\times n}{\Sigma^A} & \underset{n\times N-n}{\Sigma^{AB}} \\ \underset{N-n\times n}{\Sigma^{BA}} & \underset{N-n\times N-n}{\Sigma^B} \end{bmatrix}$$

Note, however, that f_t^A is observed, and is no longer a stochastic variable. However, it was *drawn* from a distribution correlated to f_t^B, and thus can be used as conditioning information to more precisely determine the distribution of the payoffs to be revealed, f_t^B. It can be shown that, conditional on observing f_t^A, the distribution of f_t^B is

Actually show this?

$$f_t^B \mid f_t^A, S_t \sim \mathcal{N}(\overline{\mu}^B, \overline{\Sigma}^B)$$

for conditional parameters

$$
\begin{aligned}
\overline{\mu}^B &= \mu^B + \Sigma^{BA}(\Sigma^A)^{-1}(f_t^A - \mu^A) \\
\overline{\Sigma}^B &= \Sigma^B + \Sigma^{BA}(\Sigma^A)^{-1}\Sigma^{AB}
\end{aligned}
$$

I assume that, for any state, the covariance matrix of firm payoffs is drawn from an inverse Wishart distribution parameterized by the shape matrix $\mathbf{\Omega}$ and precision ν. The distribution of Σ holds even for cases where the covariance matrix does not change across states[1]. The matrix $\mathbf{\Omega}$ determines the fundamental "shape" of the covariance structure, in that the mean of the distribution of Σ is

$$E[\Sigma \mid S_t] = \frac{\mathbf{\Omega}}{\nu - N - 1}$$

The average *covariance* can vary substantially in terms of scale as ν changes, but the average *correlation* remains the same regardless of ν. Any two draws Σ_1 and Σ_2 can have highly varied behavior. For example, Σ_1 might suggest a negative correlation in the payoffs of two firms, while Σ_2 could suggest a positive correlation. The inverse Wishart distribution is advantageous because the covariance matrix governing firm payoffs can vary meaningfully between states, and the distribution's properties are well-known.

The joint density

The physical joint density of the economy is defined in terms of a particular set of states $\mathbf{S} = S_0, \ldots, S_t$ and a set of payoffs $\mathbf{f} = f_0, \ldots, f_{t-1}, f_t^A$. The chain rule of probability allows us to factor this probability as

$$P(\mathbf{S}, \mathbf{f}, \Sigma, \mu) = P(\mathbf{f} \mid \mathbf{S}) P(\Sigma, \mu \mid \mathbf{S}) P(\mathbf{S})$$

The second part of the term above is a function of A and Π. The first element of a particular path $S_0, \ldots, S_t$ is drawn using probabilities Π, and the underlying state transitions according to the entries in the matrix A.

$$P(S_0, \ldots, S_t) = P(\mathbf{S}) = P(S_0) P(S_1 \mid S_0) \ldots P(S_t \mid S_{t-1})$$

The term $P(S_0)$ is either π_1 or π_2, depending on S_0. Denote this as $\pi(S_0)$. Additionally, denote the transition probability from S_{t-1} to S_t as $A(S_{t-1}, S_t)$. The above equation can then be rewritten as

$$P(\mathbf{S}) = \pi(S_0) \prod_{i=1}^{t} A(S_{i-1}, S_i)$$

The state space of $\mathbf{S}$ is Q^t. Each "path" of states $\mathbf{S}$ maps to an element on a discrete table of probabilities. Computing this table is computationally difficult but can be achieved with robust forward-backward passes.

[1] In models where the covariance matrix changes with state (cases 3 and 4), the inverse Whishart distribution can still be used as the distribution collapses with certainty as $\nu \to \infty$.

Prices

I assume for the moment that there is a representative agent investor with utility

$$U = \sum_{j=0}^{\infty} \rho^j u(C_{t+j})$$

for a utility function

$$u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}$$

One way to model this economy is to allow wealth to differ in each time period, but this is generally a singificant amount of record keeping and seems to result in a particularly complex stochastic discount factor.