Economic complexity and information choice

Cameron Pfiffer

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1 Introduction

A strand of economic literature studies how limited attention and cognitive constraints guide economic choices. In these papers, researchers examine how prices can be used to aggregate private signals observed by attention-constrained investors.

The general conclusion of these papers, such as Kacperczyk et al. (2016) or Peng and Xiong (2006), is that ration investors will allocate more attention to a common component of variation in payoffs as the variance of that common component rises. A shortcoming in these papers is that the economies are extraordinarily simplistic and may misunderstand the role of information choice can be when investors are confronted with complex economic environments.

The joint density of asset payoffs is complex. The economy-wide density depends on laws, competition, natural disasters, expectations, monetary policy, and any number of other stochastic values. Importantly, investors generally have (or can conjecture) priors about what might happen to the probabilities of a given set of payoffs in response to a change in fundamental economic conditions. The complexity of joint payoffs suggests that attention should be precisely allocated to assets that inform investors the most about the joint density.

Consider a relatively small industry that has exposure to trade tariffs where it is uncertain whether tariffs will be implemented. Tariffs have a high magnitude effect on the industry but a diffuse effect on all payoffs. Is it rational for the investor to allocate attention to representative firms in this industry to better form forecasts of the tariffs? Certainly, doing so would allow the investor to understand the industry by paying the opportunity cost to understand some other signal that might tell the investor more about all payoffs with a greater precision. The investor's ultimate goal is to form a portfolio of assets that satisfy their utility functions, so they must consider the universe of potential investments and how their payoffs relate to one another.

I use a distributional assumption (the Gaussian mixture distribution) to approximate complex joint densities while maintaining the ease of a single joint-Gaussian framework as in Kacperczyk et al. (2016). Extending the joint density of payoffs to a more complex distribution that incorporates state probabilities allows me to analyze how complexity drives attention allocation decisions, as well as the resulting portfolio choice problem.

The information and portfolio choice problems essentially reduce to a single question: how useful is knowledge about the state? First, the usefulness of state knowledge depends on the choices of other investors. Public information contained in prices can be used to infer the information sets of other investors (Grossman (1977)). Second, the usefulness of state information depends critically on how different the states are in their conditional densities. If the good state and bad state are extremely different, it may pay for an investor to be asymmetrically informed about state even if they are less informed about specific payoffs.

I examine how endogenous information choice with attention limitations can lead investors to choose distinct portfolios when signals inform investors about both the underlying economic state and asset payoffs in those states. Allowing risk averse investors to select signals that are informative about state and payoffs jointly produces substantially different portfolios from standard noisy rational equilibrium models, as well as shifting the conclusions of canonical models of information choice where signals inform investors *only* about payoffs.

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2 Model framework

Much of my notation and model structure follows from Kacperczyk et al. (2016), which introduce attention constraints and information choice to the multiasset noisy rational equilibrium model of Admati (1985).

The model has three periods. In time 1, informed investors allocate their attention across n signals. At time 2, all investors construct portfolios. At time 3, all investors receive payoffs.

I assume, as in Kacperczyk et al. (2016), that there are n risky assets. I add the assumption that there is an arbitrary factor structure to payoffs. Assets $1, 2, \ldots, n$ represent specific assets with idiosyncratic shocks. The key difference between my paper and Kacperczyk et al. (2016) is that the economic state[1] is a stochastic variable that is not known by investors. The economy is in state $s \in H, L$, where $s = H$ represents a "good" state with probability π and $s = L$ represents a "bad" state with probability $1 - \pi$. The density of s is written

$$P(s) = \begin{cases} \pi & \text{if } s = H \\ 1 - \pi & \text{if } s = L \end{cases}$$

The good and bad states dictate the mean and covariance of asset payoffs. Payoffs of the n assets, denoted by $n \times 1$ vector f, are written

[1] I employ two-state mixture model, but in principle any number of mixture components can be employed with only minor modifications to the model.

$$f_i = \mu_{i,s} + z_i \tag{1}$$
$$z \mid s = [z_1, z_2, \ldots, z_n]' \sim \mathcal{N}(0, \Sigma_s) \tag{2}$$
$$f \mid s \sim \mathcal{N}(\mu_s, \Sigma_s) \tag{3}$$

The mean payoff vector $\mu_s = [\mu_{1,s}, \mu_{2,s}, \ldots, \mu_{n,s}]'$ and the $n \times n$ variance-covariance matrix of payoff shocks Σ_s are functions of the unobserved economic state. When investors are allowed to receive signals about the underlying shocks z, those same signals will allow investors to assign a probability to the underlying state and the associated payoff structure.[2]

Note that the unconditional payoff density $P(f)$ is a two-component Gaussian mixture distribution with mixture weights π and $1 - \pi$. The density function is written

$$P(f) \sim \pi \mathcal{N}(f \mid \mu_H, \Sigma_H) + (1 - \pi)\mathcal{N}(f \mid \mu_L, \Sigma_L) \tag{5}$$

Gaussian mixture distributions have the conceptual benefit of moving payoffs outside the traditional exponential family of distributions. In principle, a mixture model can approximate any complex joint density as the number of components increase (Nguyen and McLachlan, 2019). I maintain only two components as a proof-of-concept, though much of my analysis expands easily to an arbitrary number of Gaussian components. Additionally, many of the posterior distributions are only marginally more complex than when using traditional normal or log-normal distributions.

As in Admati (1985) and Kacperczyk et al. (2016), I employ CARA utility to abstract from wealth effects. However, the conditions in Kacperczyk et al. (2016) that reduce the investment problem to a mean-variance problem do not hold in my setting without modifications. Namely, the distribution of f is unconditionally non-Gaussian. Fortunately, the Gaussian mixture distribution benefits from the fact that the non-Gaussian density $P(f)$ can be rewritten as a weighted sum of two conditionally Gaussian densities, i.e. $P(f \mid s = H)P(s = H) + P(f \mid s = L)P(s = L)$. Many of my results come from the simplicity of the Normal distribution without reducing the joint density of payoffs to a simplistic unimodal distribution.

The economy is populated by atomistic investors j with unit mass ($j \in [0, 1]$). Investors have exponential preferences on final-period wealth W_j, with

[2]Kacperczyk et al. (2016) utilize a transformation of asset payoffs to the corresponding risk factor payoffs – in my case, the eigen-decomposition $\Sigma_s = \Gamma_s \Lambda_s \Gamma_s'$ for $s \in H, L$ yields Arrow-Debreu synthetic securities on risk factors:

$$\tilde{f} \mid s \sim \mathcal{N}(\Gamma_s^{-1}\mu_s, \Lambda_s) \tag{4}$$

Unfortunately, I cannot proceed with the Kacperczyk et al. (2016) solution method, which requires an additional transformation of risk factor prices $\tilde{p} = \Gamma^{-1}p$ and risk factor quantities $\tilde{q} = \Gamma^{-1}q$ for some eigenvector matrix Γ. My model only permits the orthogonalization of the prior variance Σ, but in general the transforms on $\tilde{q}$ and $\tilde{p}$ will remain correlated conditional on state.

a risk-aversion coefficient ρ. Expected utility at time 2 (after receiving private signals) is a function of risk-free rate r, initial wealth W_0, asset quantities q_j, asset payoffs f, and asset prices p.

$$U_{j2} = E_j[\exp\{-\rho W_j\}] \tag{6}$$

for law of motion on wealth $W_j = rW_0 + q_j'(f - pr)$. Since wealth effects do not enter the investment decision for CARA utilities, I follow Kacperczyk et al. (2016) and equalize initial wealth to W_0 for all investors.

A portion of investors (the *informed*) receive private signals η_j about time 3 payoffs f. Signals take the form of additive Gaussian noise around the true payoff, where the precision of the noise is determined by investor attention allocation. The form of a private signal is

$$\eta_j \sim \mathcal{N}(f, \Sigma_{\eta,j}), \tag{7}$$

$$\text{or } \eta_j = f + \epsilon_j, \quad \epsilon_j \sim \mathcal{N}(0, \Sigma_{\eta_j}) \tag{8}$$

The matrix Σ_{η_j} is a diagonal matrix with entries K_{ij}^{-1}. K_{ij} is the total amount of attention given to signal i by investor j. Higher values of K_{ij} imply lower variance of ϵ_{ij}, and thus a more accurate signal of f_i, as well as more precise signals of whichever assets are correlated with asset i.

Investors have limited attention, in that they cannot pay infinite attention to all the signals they would like. Concretely, this constraint is written

$$\sum_{i=1}^{n} K_{ij} \leq K_j \tag{9}$$

though for simplicity I equalize attention constraints across investors to $K_j = K$ for informed investors and $K_j = 0$ for uninformed investors. Uninformed investors can only use prices as signals about payoffs, whereas informed investors can use both prices and private signals. The attention constraint utilized here is common in the information choice literature – see Kacperczyk et al. (2016).

Investors have two optimization problems to solve. First, if the investor is informed, they must allocate their attention across private signals at time 1. Second, conditional on any information observed in time 1, investors construct portfolios to optimize expected utility at time 2, U_{j2}.

The investor's information choice problem is to maximize expected time-1 utility U_{j1}:

$$
\begin{aligned}
\underset{K_{ij}}{\text{maximize}} \quad & U_{j1} = E\Big[E_j[\exp\{-\rho W_j\}]\Big] \\
\text{subject to} \quad & \quad W_j = rW_0 + q_j'(f - pr), \\
& \sum_i K_{ij} \le 1, \\
& \quad K_{ij} \ge 0, \quad \forall i
\end{aligned}
\tag{10}
$$

Next, the time-2 portfolio choice problem is to maximize expected utility U_{j2}:

$$
\begin{aligned}
\underset{q_j}{\text{maximize}} \quad & U_{j2} = E_j[\exp\{-\rho W_j\} \mid \eta_j, p] \\
\text{subject to} \quad & W_j = rW_0 + q_j'(f - pr)
\end{aligned}
\tag{11}
$$

Finally, markets must clear at price p and quantities q_j, leading to the traditional market clearing condition

$$
\int_j q_j(p) = \overline{x} + x \tag{12}
$$

Market clearing requires that quantities and prices be such that all supply is allocated to an investor. I now turn to the formal declaration of an equilibrium in my setting, as in Breon-Drish (2015).

Definition 1 *A noisy rational expectations equilibrium (NREE) is a function $p(s, f, x)$ that maps the state s, payoffs f, and asset supply x to a vector of prices for the n assets, such that (a) prices maximize aggregate surplus:*

$$
q_j(p, \eta_j) \in \arg\max_{q_j} E_j[\exp\{-\rho W_j\} \mid \eta_j, p], \quad \forall j \in [0, 1] \tag{13}
$$

that (b) markets clear, and (c) that all q_j^* is optimal conditional on investor j's information set $\{p, \eta_j\}$.

I conjecture and verify that an equilibrium pricing function follows the form

$$
p = A + Bf + Cx \tag{14}
$$

The linear pricing function above is common in prior works – see Kacperczyk et al. (2016), Admati (1985), and others. The often-conjectured linear price function in noisy rational expectations models are interperetable as Gaussian signals around the true payoffs with noise due to uncertain asset supply. Fortunately, my model permits the use of a linear pricing function and the corresponding Gaussian signal as long as rational expectations hold – equilibrium prices must be a linear function of the true payoffs f.

The linear form of p is Gaussian conditional on f, so the closed-form posterior for an investor j is well-defined:

$$
\begin{aligned}
P(f \mid \eta_j, p) &= \frac{P(\eta_j, p \mid f)P(f)}{P(\eta_j, p)} \\
&= \frac{\pi P(\eta_j, p \mid f, s_H)P(f \mid s_H) + (1-\pi)P(\eta_j, p \mid f, s_L)P(f \mid s_L)}{\pi P(\eta_j, p \mid s_H) + (1-\pi)P(\eta_j, p \mid s_L)}
\end{aligned}
$$

The above density is simply a weighted sum of Gaussian densities in the numerator and denominator. To identify this posterior, it is useful to note that the three variables of interest to investor j, $m_j = [f, p, \eta_j]'$, is Normal when conditioned on the latent state s since prices conditional on s are the sum of two Gaussians (payoff shocks z and supply shocks x).

The mean of the payoff, price, and signal vector $m_j \mid s$ is

$$
E[m_j \mid s] = \begin{bmatrix} f \\ p \\ \eta_j \end{bmatrix} = \begin{bmatrix} \mu_s \\ A + B\mu_s + C\bar{x} \\ \mu_s \end{bmatrix}
$$

with the $3n \times 3n$ block conditional variance matrix

$$
\mathrm{Var}[m_j \mid s] = \begin{bmatrix} \Sigma_s & B\Sigma_s & \Sigma_s \\ B\Sigma_s & B\Sigma_s B' + C\Sigma_x C' & B\Sigma_s \\ \Sigma_s & B\Sigma_s & \Sigma_s + \Sigma_j \end{bmatrix}
$$

The posterior density of interest, $P(f \mid p, \eta_j, s)$, is well-defined by the rules of the multivariate Gaussian. Both the joint, prior, and posterior are all Gaussian when conditioned on state s.

To derive the posterior mean and variance, denote the aggregate signal held by investor j as $y_j = [p, \eta_j]'$ and the corresponding state mean of that signal $\bar{y}_{j,s} = [A + B\mu_s + C\bar{x}, \mu_s]'$. Then the mean $\hat{\mu}_{j,s}$ and variance $\hat{\Sigma}_{j,s}$ of this posterior density are given by the identities

$$
\begin{aligned}
\hat{\Sigma}_{j,s} &= \Sigma_s - \Omega_{1,2}\Omega_{2,2,j}^{-1}\Omega_{2,1} \\
\hat{\mu}_{j,s} &= \mu_s + \Omega_{1,2}\Omega_{2,2,j}^{-1}(y_j - \bar{y}_{j,s})
\end{aligned}
$$

where $\Omega_{1,2}$, $\Omega_{2,1}$, and $\Omega_{2,2}$ are block matrices formed from a partition of $\mathrm{Var}[m_j \mid s]$:

$$
\begin{aligned}
\Omega_{1,2} = \Omega_{2,1}' &= \begin{bmatrix} B\Sigma_s & \Sigma_s \end{bmatrix} \\
\Omega_{2,2,j} &= \begin{bmatrix} B\Sigma_s B' + C\Sigma_x C' & B\Sigma_s \\ B\Sigma_s & \Sigma_s + \Sigma_j \end{bmatrix}
\end{aligned}
$$

The matrices A, B, and C are calculated from the fixed-point that solves the market clearing condition. At the moment, I solve for these numerically, but

I will eventually calculate the analytic forms for A, B, and C. The numerical limitation prevents me from deriving the full equilibrium where information choice is endogenized, but the portfolio choice problem is easily solved.

3 Equilibrium

Kacperczyk et al. (2016) work backwards by solving the portfolio allocation problem at time 2 first, and then using this solution to determine the optimal information choice at time 1. I defer to their solution, though with some added complexity due to the change in the densities of payoffs f from multivariate Gaussian to mixture distributed.

The appendix shows that the optimal quantity q_j^* that solves (11) takes the form

Add to appendix

$$q_j^* = \frac{1}{\rho}\big(\hat{s}_j\hat{\Sigma}_{H,j} + (1-\hat{s}_j)\hat{\Sigma}_{L,j}\big)^{-1}\big(\hat{s}_j\hat{\mu}_{H,j} + (1-\hat{s}_j)\hat{\mu}_{L,j}\big) \tag{15}$$

where $\hat{s}_j$ represents the posterior probability of being in the high state by investor j. It is worth commenting that the form of the optimal quantity is relatively simple and maintains the general form of strict Gaussian case. The only substantive difference between my q_j and that of papers like Kacperczyk et al. (2016) is that the posterior means and variances are weighted by the posterior state probabilities. The form of q_j demonstrates that it is trivial to add an arbitrary number of Gaussian clusters to the payoff density, should a modeler wish to approximate increasingly complex joint distributions.

I start by defining several types of economies and solving prices conditional on the mean and covariance relationships of each economy. The four types I have cover the qualitative space of two-state economies, in that payoffs differ two states by the combination of means, variances, and correlations. Figure 1 plots the joint density of payoffs for each economy.

- **Economy 1: the baseline**. The baseline economy is a simple multivariate Gaussian economy to which the other economies are compared. It is equivalent to the case studied in Kacperczyk et al. (2016). Assets one and two have unit payoffs and an identity covariance matrix in both states, so that both states are identical.

- **Economy 2: mean shift**. The mean shift economy allows the mean payoff for asset 2 to move from $+2$ (good state) to -2 (bad state), while asset 1 has the same mean payoff of 1 in each state. The covariance matrix of payoffs is the identity matrix in both states.

- **Economy 3: mean and variance shift**. Economy 3 is intended to capture cases where an asset varies in boht mean and variance between states. Asset 2 has a mean of $+2$ and a variance of 1 in the good state, and a mean of -2 and a variance of 5 in the bad state. Asset 1 remains a state-independent asset.

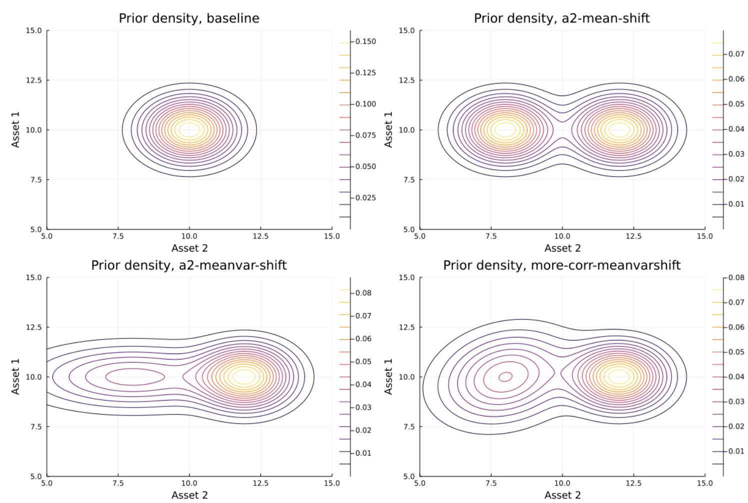

Figure 1: Unconditional joint payoff densities of f in different economies.

- **Economy 4: mean, variance, and correlation shift**. Economy 4 makes asset 1 and 2 payoffs positively correlated in the bad state. Asset two maintains its $+2/-2$ means and $1/5$ variances.

At present, I am most interested in how different attention allocation impacts investor beliefs and asset prices. I fix attention allocation and observe the time-2 portfolio construction problem faced by investors. Later sections will allow agents to optimize their information choices. I generate J agents, half of which allocate 90% of their attention to the safer asset 1, and the other half allocates 90% of their attention to the riskier asset 2. Next, I assign each agent a private signal η_j based on their attention allocation. I fix the stochastic supply to its average ($x = \bar{x}$) to more precisely observe equilibrium conditions.

Prices are determined through numerical minimization of the squared difference of the economy from market clearing. Denote the total quantity demanded by investors at price p as $\mathbf{q}(p) = q_1(p) + q_2(p) + \dots q_J(p)$.

$$\begin{aligned} \underset{A,B,C}{\text{minimize}} \quad & \big(\mathbf{q}(p) - x\big)'\big(\mathbf{q}(p) - x\big) \\ \text{subject to} \quad & p = A + Bf + Cx \end{aligned} \tag{16}$$

The optimizer will conjecture price parameters A, B, and C, and then calculate all investor's posterior beliefs about the means, variances, and state probabilities conditional on the particular draw of price parameters. If the price parameters are such that net quantities $\mathbf{q}(p) - x$ is highly positive or negative, the conjectured price function cannot be the market clearing price as there exists some assets which are not held by anyone. I use the LBFGS algorithm with

Find cite for LBFGS

automatic differentiation for the gradient[3]. Typically, the numerical approach yields a squared quantity disjoint around $1e^{-22}$, which is approximately exact for most numerical purposes.

[3] It has historically been more common to use finite difference gradients, but advances in computation have enabled the faster and more accurate automatic differentiation. cite Chris' paper?

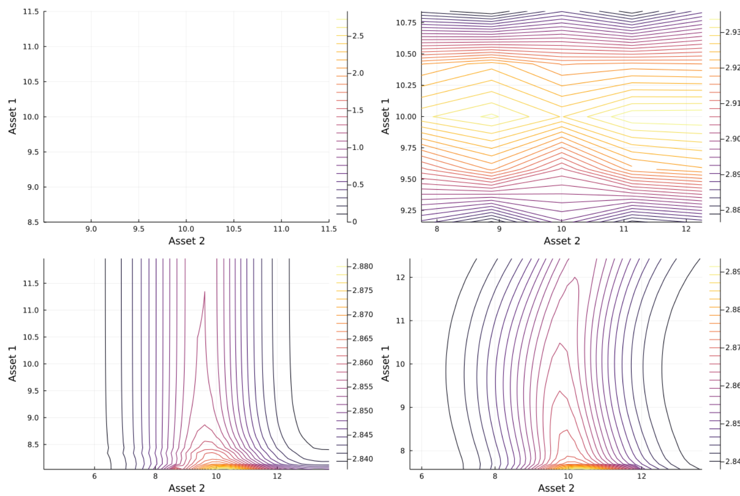

Figure 2: Distributions of the lower bound on posterior entropy for different observations of f.

A Figures

B Mixture model variance

The variance-covariance matrix of variable X following a Gaussian mixture distribution with means $\mu = [\mu_1, \mu_2, \ldots, \mu_k]'$, covariances $\Sigma = [\Sigma_1, \Sigma_2, \ldots, \Sigma_k]$, and mixture proportions $\pi = [\pi_1, \pi_2, \ldots, \pi_k]'$ is[4]

$$
\begin{aligned}
\mathrm{Var}[X] &= E[\mathrm{Var}[X \mid k]] + \mathrm{Var}[E[X \mid k]] && (17)\\
&= \sum_k \pi_k \Big(\Sigma_k + (\mu_k - \overline{\mu})(\mu_k - \overline{\mu})' \Big) && (18)
\end{aligned}
$$

for $\overline{\mu} = \sum_k \pi_k \mu_k$.

C Misc identities

Stochastic variables:

- State variable $s \sim \text{Bernoulli}(\pi)$. $P(s = H) = \pi$, $P(s = L) = 1 - \pi$.
- Risk factor payoffs $\tilde{f} \mid s \sim N(\Gamma^{-1}\mu_s, \Sigma_s)$

[4]From https://math.stackexchange.com/questions/195911/calculation-of-the-covariance-of-gaussian-mixtures

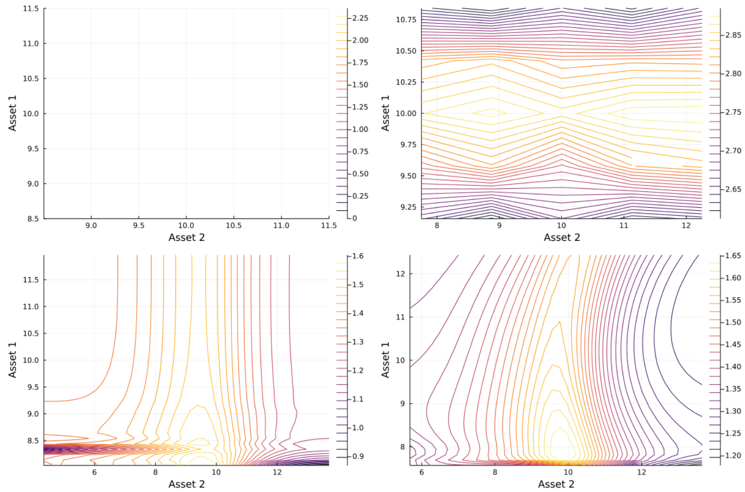

Figure 3: Distributions of the upper bound on posterior entropy for different observations of f.

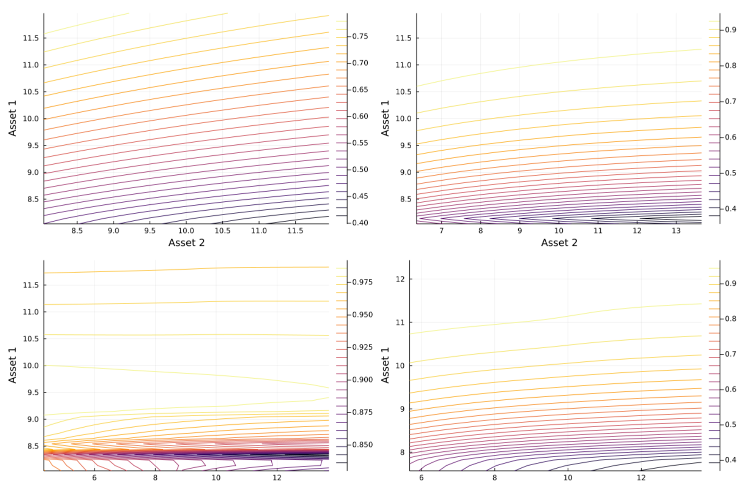

Figure 4: Distributions of payoff coefficient matrix $B_{1,1}$ for different observations of f.

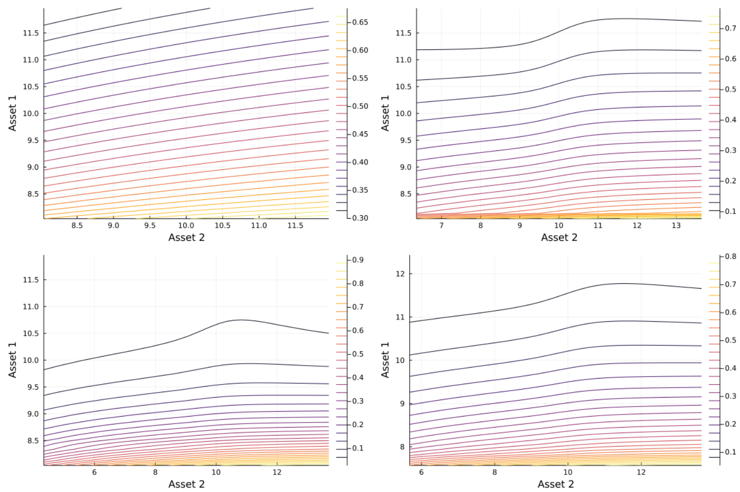

Figure 5: Distributions of payoff coefficient matrix $B_{2,2}$ for different observations of f.

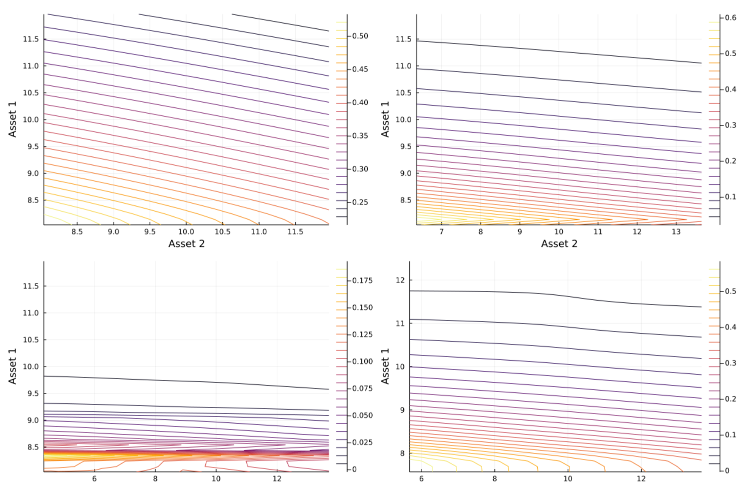

Figure 6: Distributions of payoff coefficient matrix $B_{1,2}$ for different observations of f.

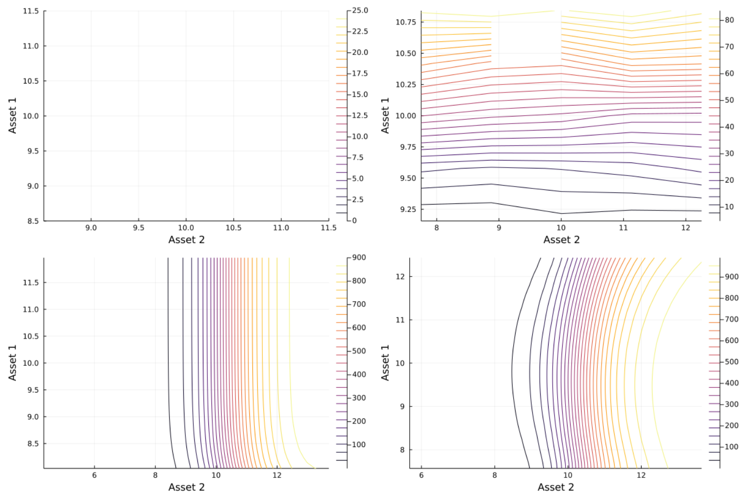

Figure 7: Distributions of disagreement ($\sum \hat{s}_j^2$) for different observations of f.

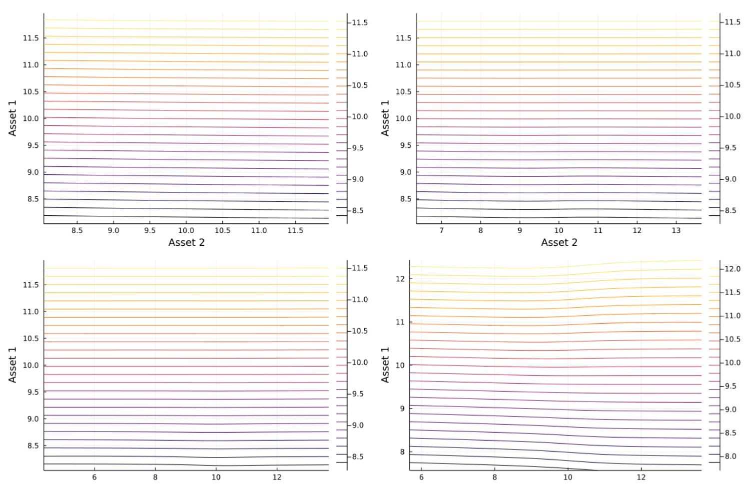

Figure 8: Distributions of the asset 1 price for different observations of f.

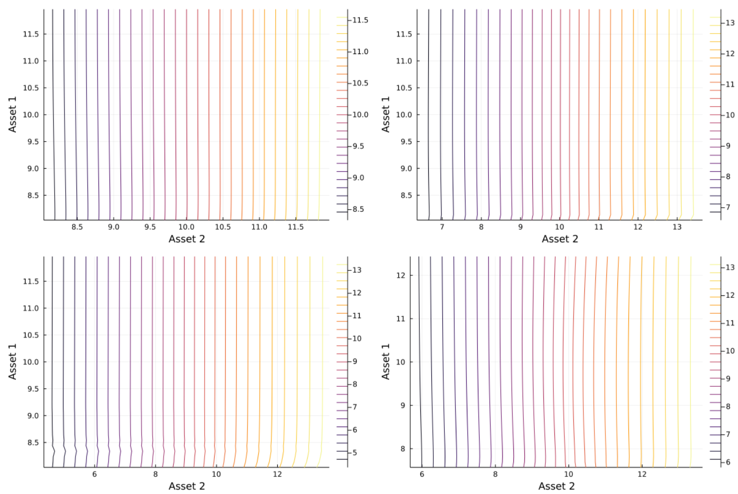

Figure 9: Distributions of the asset 2 price for different observations of f.

- Risk factor supply $x \sim N(\overline{x}, \sigma_x I)$
- Private signals $\eta_j \mid s \sim N(z, \Sigma_{\eta_j})$
- Price signal $\eta_p \mid s \sim N(z, \Sigma_p)$

Joint density:

$$P(s, \tilde{f}, x, \eta_j, \eta_p) = P(\tilde{f} \mid s)P(\eta_j \mid s)P(\eta_p \mid s)P(s)P(x)$$

Posterior density:

$$P(s, \tilde{f} \mid x, \eta_j, \eta_p) = \frac{P(x, \eta_j, \eta_p \mid s, \tilde{f})P(s, \tilde{f})}{P(x, \eta_j, \eta_p)}$$

Unknown values:

- $E_j[\tilde{f} - \tilde{p}r \mid H]$
- $E_j[\tilde{f} - \tilde{p}r \mid L]$
- $V_j[\tilde{f} - \tilde{p}r \mid H]$
- $V_j[\tilde{f} - \tilde{p}r \mid L]$
- $P(H \mid \eta_p, \eta_j)$
- $P(L \mid \eta_p, \eta_j)$
- $E_j[\tilde{f} \mid \eta_p, \eta_j]$

- $V_j[\tilde{f} \mid \eta_p, \eta_j]$
- $\tilde{p}$

Portfolio choice problem:

$$U_{2j} = \max_{\tilde{q}_j} \rho E_j[W_j] - \frac{\rho^2}{2} V_j[W_j]$$

Optimal quantity:

$$\begin{aligned}
\tilde{q}_j &= \frac{1}{\rho}\Big(P(H)\Sigma_H + P(L)\Sigma_L\Big)^{-1}\Big(P(H)E_j[\tilde{f} \mid H] + P(L)E_j[\tilde{f} \mid L] - \tilde{p}r\Big) \\
&= \frac{1}{\rho} V_j[\tilde{f}]^{-1}(E_j[\tilde{f}] - \tilde{p}r)
\end{aligned}$$

Ex-ante expected utility:

$$\begin{aligned}
U_{1j} &= E\bigg[\rho E_j[W_j] - \frac{\rho^2}{2} V_j[W_j]\bigg] \\
&= \pi E\bigg[\rho E_j[W_j \mid H] - \frac{\rho^2}{2} V_j[W_j \mid H]\bigg] \\
&\quad + (1-\pi) E\bigg[\rho E_j[W_j \mid L] - \frac{\rho^2}{2} V_j[W_j \mid L]\bigg] \\
&= \rho r W_0 \\
&\quad + \rho \tilde{q}_j' \Big(\pi E_j[\tilde{f} - \tilde{p}r \mid H] + (1-\pi) E_j[\tilde{f} - \tilde{p}r \mid L]\Big) \\
&\quad - \frac{\rho^2}{2} \tilde{q}_j' \Big(\pi V_j[\tilde{f} - \tilde{p}r \mid H] + (1-\pi) V_j[\tilde{f} - \tilde{p}r \mid L]\Big) \tilde{q}_j
\end{aligned}$$