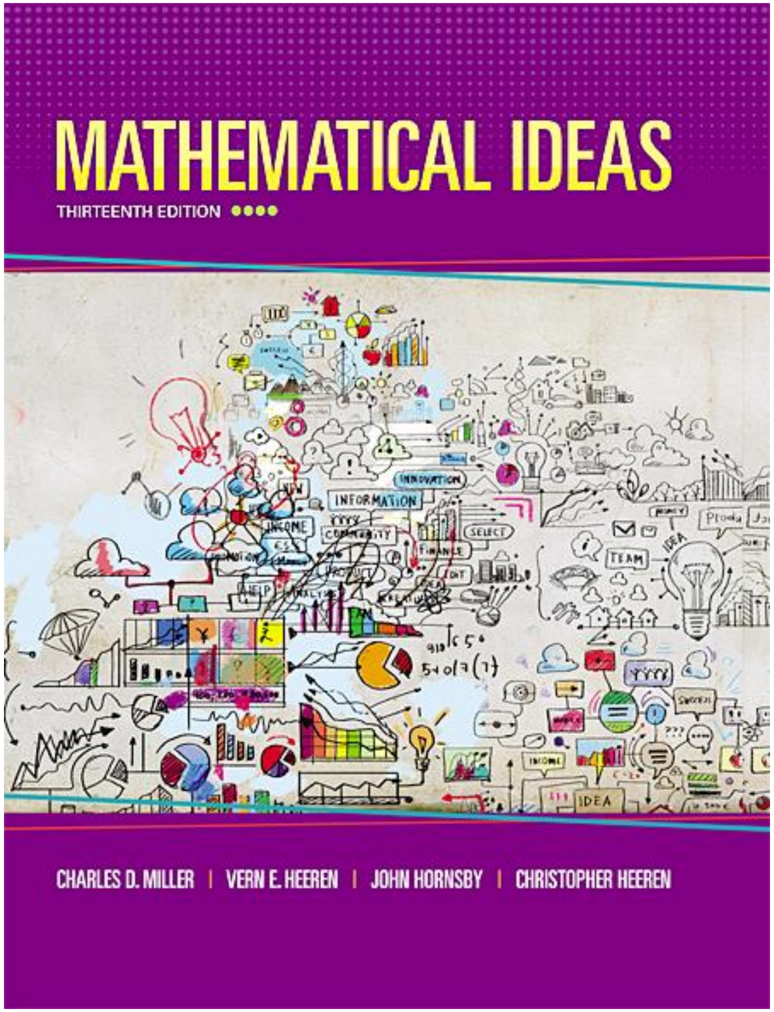
Chapter 12

The Real Numbers and Their Representation



Chapter 12: The Real Numbers and Their Representation

12.1 Rational Numbers and Decimal Representation

12.2 Irrational Numbers and Decimal Representation

12.3 Applications of Decimals and Percents

Section 12-2

Irrational Numbers and Decimal Representation

Irrational Numbers and Decimal Representation

- Understand how irrational numbers differ from rational numbers in their decimal representations.
- Follow the proof that $\sqrt{2}$ is an irrational number.
- Use a calculator to find square roots.
- Apply the product and quotient rules for square roots.
- Rationalize a denominator.
- Understand the relevance of the irrational numbers π, ϕ, and e in mathematics.

Definition: Irrational Numbers

Irrational Numbers

$\{x \mid x$ is a number represented by a nonrepeating, nonterminating decimal$\}$

Irrationality of $\sqrt{2}$

The proof of this fact is a **proof by contradiction**.

We need to consider:

1. When a rational number is written in lowest terms, the greatest common factor of its numerator and denominator is 1.
2. If an integer is even, then it has 2 as a factor and may be written in the form $2k$, where k is an integer.
3. If a perfect square is even, then its square root is even.

Irrationality of $\sqrt{2}$ (Proof)

Statement: $\sqrt{2}$ is an irrational number.

Proof: Assume that $\sqrt{2}$ is a rational number. Then, by definition, $\sqrt{2} = \dfrac{p}{q}$, for some integers p and q.

Furthermore, assume that $\dfrac{p}{q}$ is written in lowest terms, so the greatest common factor of p and q is 1.

$$2 = \frac{p^2}{q^2}$$

$$2q^2 = p^2$$

Irrationality of $\sqrt{2}$ (Proof)

From the last equation, we see that 2 is a factor of p^2, so p^2 is even and so is p. Since p is even it can be written as $2k$ for some integer k.

$$2q^2 = (2k)^2 \quad \text{Replace } p \text{ with } 2k.$$

$$2q^2 = 4k^2$$

$$q^2 = 2k^2$$

Since 2 is a factor of q^2, q^2 and also q must be even.

Irrationality of $\sqrt{2}$ (Proof)

This leads to a contradiction: p and q cannot both be even because they would have a common factor of 2, although it was assumed that their greatest common factor was 1.

Therefore, since the original assumption that $\sqrt{2}$ is rational led to a contradiction, it must be that $\sqrt{2}$ is irrational.

Operations with Square Roots

Multiplication, division, addition and subtraction with square roots will be covered on the next several slides.

Example: Using a Calculator to Approximate Square Roots

Use a calculator to verify $\sqrt{45} \approx 6.708203932$

Solution

Use a calculator, such as the ones found on smartphones, to verify the approximation. Depending on the mode, fewer or more digits may be displayed, and because of different rounding procedures, final digits may differ slightly.

Product Rule for Square Roots

For nonnegative real numbers a and b,

$$\sqrt{a} \cdot \sqrt{b} = \sqrt{a \cdot b}.$$

Conditions for a Simplified Square Root Radical

1. The number under the radical (**radicand**) has no factor (except 1) that is a perfect square.
2. The radicand has no fractions.
3. No denominator contains a radical.

Example: Simplifying a Square Root Radical (Product Rule)

Simplify $\sqrt{32}$.

Solution

$$\begin{aligned}\sqrt{32} &= \sqrt{16 \cdot 2} \\ &= \sqrt{16} \cdot \sqrt{2} \\ &= 4\sqrt{2}\end{aligned}$$

Quotient Rule for Square Roots

For nonnegative real numbers a and positive real numbers b,

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}.$$

Example: Simplifying a Square Root Radical (Quotient Rule)

Simplify $\sqrt{\frac{7}{9}}$.

Solution

$$\sqrt{\frac{7}{9}} = \frac{\sqrt{7}}{\sqrt{9}}$$

$$= \frac{\sqrt{7}}{3}$$

Rationalizing a Denominator

Given $\dfrac{5}{\sqrt{3}}$, to arrive at an equivalent expression with no radical in the denominator, we use a procedure called **rationalizing the denominator** shown below.

$$\frac{5}{\sqrt{3}} = \frac{5}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{5\sqrt{3}}{3} \quad \text{Simplified form}$$

Example: Adding and Subtracting Square Root Radicals

Add or subtract as indicated.

a) $8\sqrt{7}+2\sqrt{7}$ b) $\sqrt{50}-\sqrt{18}$

Solution

a) $(8+2)\sqrt{7}=10\sqrt{7}$

b) $\sqrt{25\cdot 2}-\sqrt{9\cdot 2}$

$=5\sqrt{2}-3\sqrt{2}=2\sqrt{2}$

The Irrational Numbers π, ϕ, and *e*

Pi (π)

Pi represents the ratio of the circumference of a circle to its diameter.

$$\pi \approx 3.14159265358979$$

The Irrational Numbers π, ϕ, and *e*

Phi (ϕ)

Phi is the Golden Ratio and has exact value

$$\frac{1+\sqrt{5}}{2}.$$

$$\phi \approx 1.61803398874989$$

The Irrational Numbers π, ϕ, and e

e

e is a fundamental number is our universe. It is the base of the *natural exponential* and *natural logarithmic* functions.

$$e \approx 2.71828182845904$$