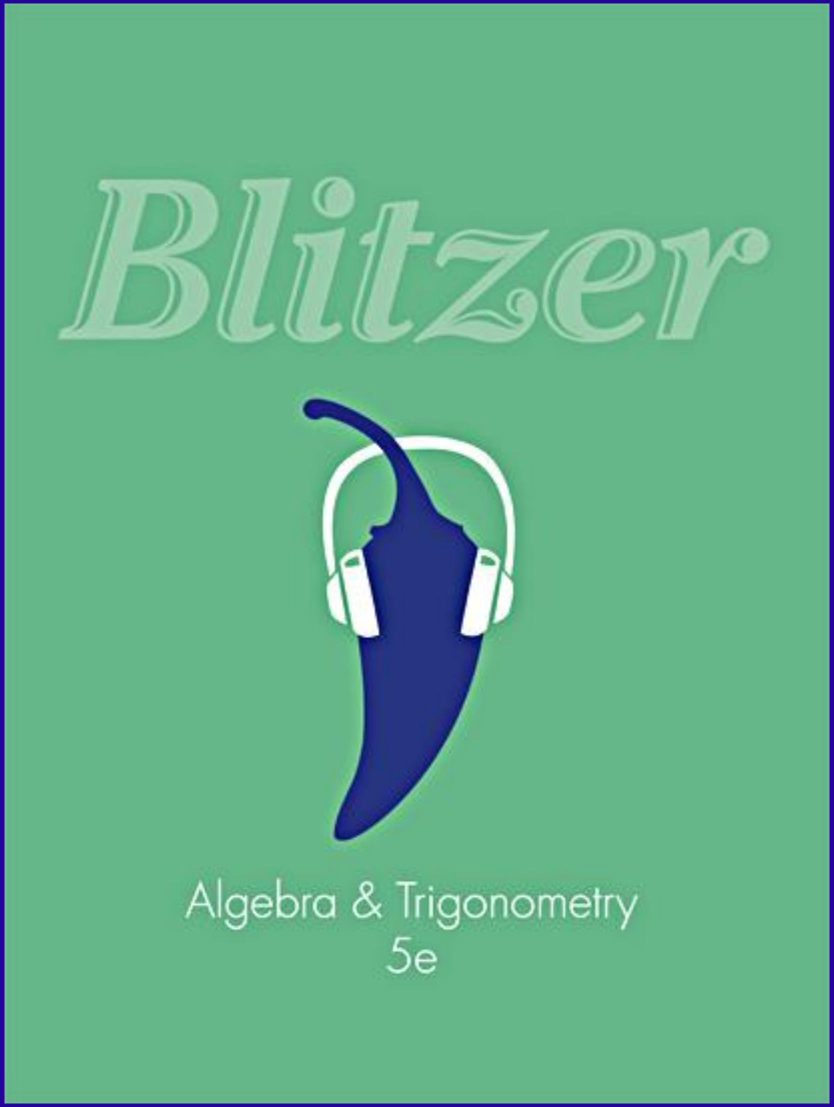
Chapter P

Prerequisites:

Fundamental Concepts of Algebra

P.6 Rational Expressions



Objectives:

- Specify numbers that must be excluded from the domain of a rational expression.
- Simplify rational expressions.
- Multiply rational expressions.
- Divide rational expressions.
- Add and subtract rational expressions.
- Simplify complex rational expressions.

Rational Expressions

A **rational expression** is the quotient of two polynomials. The set of real numbers for which an algebraic expression is defined is the **domain** of the expression. Because rational expressions indicate division and division by zero is undefined, **we must exclude numbers from a rational expression's domain that make the denominator zero.**

Example: Excluding Numbers from the Domain

Find all the numbers that must be excluded from the domain of the rational expression: $\dfrac{7x}{x^2-5x-14}$

To determine the numbers that must be excluded from the domain, examine the denominator.

$$x^2-5x-14=(x-7)(x+2)$$

We set each factor equal to zero to find the excluded values.

$$(x-7)=0\to x=7 \qquad (x+2)=0\to x=-2$$

Example: Excluding Numbers from the Domain *(continued)*

For the rational expression $$\frac{7x}{x^2-5x-14}$$

the values that must be excluded from the domain are $x = 7$ and $x = -2$.

These excluded numbers are often written to the right of a rational expression:

$$\frac{7x}{x^2-5x-14}, x \neq 7, x \neq -2$$

Simplifying Rational Expressions

A rational expression is **simplified** if its numerator and denominator have no common factors other than 1 or –1. The following procedure can be used to simplify rational expressions:

1. Factor the numerator and the denominator completely.
2. Divide both the numerator and the denominator by any common factors.

Example: Simplifying Rational Expressions

Simplify: $\dfrac{x^2-1}{x^2+2x+1}$

Factor the numerator and the denominator:

$$\frac{x^2-1}{x^2+2x+1} = \frac{(x+1)(x-1)}{(x+1)(x+1)}$$

Check for excluded values.

Because the denominator is $(x+1)(x+1), x \neq -1$

Example: Simplifying Rational Expressions *(continued)*

We have factored the numerator and denominator:

$$\frac{x^2 - 1}{x^2 + 2x + 1} = \frac{(x+1)(x-1)}{(x+1)(x+1)}$$

Divide out the common factor, $(x + 1)$

$$\frac{x^2 - 1}{x^2 + 2x + 1} = \frac{(x+1)(x-1)}{(x+1)(x+1)} = \frac{x-1}{x+1}, x \neq -1$$

Multiplying Rational Expressions

1. Factor all numerators and denominators completely.
2. Divide numerators and denominators by common factors.
3. Multiply the remaining factors in the numerators and multiply the remaining factors in the denominators.

Example: Multiplying Rational Expressions

Multiply: $$\frac{x+3}{x^2-4} \cdot \frac{x^2-x-6}{x^2+6x+9}$$

Factor the numerator and the denominator:

$$\frac{x+3}{x^2-4} \cdot \frac{x^2-x-6}{x^2+6x+9} = \frac{(x+3)}{(x+2)(x-2)} \cdot \frac{(x-3)(x+2)}{(x+3)(x+3)}$$

Check for excluded values.

Because the denominator is

$$(x+2)(x-2)(x+3)(x+3), x \neq -2, x \neq 2, x \neq -3$$

Example: Multiplying Rational Expressions *(continued)*

Divide numerators and denominators by common factors:

$$\frac{x+3}{x^2-4} \cdot \frac{x^2-x-6}{x^2+6x+9} = \frac{\cancel{(x+3)}}{\cancel{(x+2)}(x-2)} \cdot \frac{(x-3)\cancel{(x+2)}}{\cancel{(x+3)}(x+3)}$$

$$= \frac{x-3}{(x-2)(x+3)}$$

Example: Multiplying Rational Expressions *(continued)*

Multiply the remaining factors in the numerators and denominators.

$$\frac{x-3}{(x-2)(x+3)} = \frac{x-3}{x^2+x-6}$$

Thus,

$$\frac{x+3}{x^2-4} \cdot \frac{x^2-x-6}{x^2+6x+9} = \frac{x-3}{x^2+x-6}, x \neq 2, x \neq -2, x \neq -3$$

Dividing Rational Expressions

The quotient of two rational expressions is the product of the first expression and the multiplicative inverse, or reciprocal, of the second expression. The reciprocal is found by interchanging the numerator and the denominator. Thus, **we find the quotient of two rational expressions by inverting the divisor and multiplying**.

Example: Dividing Rational Expressions

Divide: $\dfrac{x^2-2x+1}{x^3+x} \div \dfrac{x^2+x-2}{3x^2+3}$

Invert the divisor and multiply.

$$\frac{x^2-2x+1}{x^3+x} \div \frac{x^2+x-2}{3x^2+3} = \frac{x^2-2x+1}{x^3+x} \cdot \frac{3x^2+3}{x^2+x-2}$$

Factor as many numerators and denominators as possible.

$$\frac{x^2-2x+1}{x^3+x} \cdot \frac{3x^2+3}{x^2+x-2} = \frac{(x-1)(x-1)}{x(x^2+1)} \cdot \frac{3(x^2+1)}{(x+2)(x-1)}$$

Example: Dividing Rational Expressions *(continued)*

Check for excluded values.

Because the denominator is

$$x(x^2+1)(x+2)(x-1),\ x\neq 0,\ x\neq -2,\ x\neq 1$$

Divide numerators and denominators by common factors.

$$\frac{\cancel{(x-1)}(x-1)}{x\cancel{(x^2+1)}}\cdot\frac{3\cancel{(x^2+1)}}{(x+2)\cancel{(x-1)}}=\frac{(x-1)\cdot 3}{x(x+2)}$$

Example: Dividing Rational Expressions *(continued)*

Multiply the remaining factors in the numerators and in the denominators.

$$\frac{(x-1)\cdot 3}{x(x+2)} = \frac{3x-3}{x^2+2x}$$

Thus,

$$\frac{x^2-2x+1}{x^3+x} \div \frac{x^2+x-2}{3x^2+3} = \frac{3x-3}{x^2+2x}, x \neq 0, x \neq 1, x \neq -2$$

Adding and Subtracting Rational Expressions with the Same Denominator

To add or subtract rational expressions with the same denominator:

1. Add or subtract the numerators.
2. Place this result over the common denominator.
3. Simplify, if possible.

Reminder: check for excluded values.

Example: Subtracting Rational Expressions with the Same Denominator

Subtract: $\dfrac{x}{x+1} - \dfrac{3x+2}{x+1}$

Check for excluded values: $(x+1) = 0 \rightarrow x = -1$

$$\frac{x}{x+1} - \frac{3x+2}{x+1} = \frac{x-(3x+2)}{x+1}$$

$$= \frac{x-3x-2}{x+1}$$

$$= \frac{-2x-2}{x+1} \qquad = \frac{-2(x+1)}{x+1} \qquad = -2, x \neq -1$$

Finding the Least Common Denominator (LCD)

1. Factor each denominator completely.
2. List the factors of the first denominator.
3. Add to the list in step 2 any factors of the second denominator that do not appear in the list.
4. Form the product of the factors from the list in step 3. This product is the least common denominator (LCD).

Adding and Subtracting Rational Expressions That Have Different Denominators

1. Find the LCD of the rational expressions.
2. Rewrite each rational expression as an equivalent expression whose denominator is the LCD. To do so, multiply the numerator and the denominator of each rational expression by any factor(s) needed to convert the denominator into the LCD.
3. Add or subtract numerators, placing the resulting expression over the LCD.
4. If possible, simplify the resulting rational expression.

Example: Subtracting Rational Expressions with Different Denominators

Subtract: $$\frac{x}{x^2-10x+25}-\frac{x-4}{2x-10}$$

Step 1 Find the least common denominator

$$x^2-10x+25=(x-5)(x-5)$$

$$2x-10=2(x-5)$$

The LCD is $2(x-5)(x-5)$

Example: Subtracting Rational Expressions with Different Denominators *(continued)*

Step 2 Write equivalent expressions with the LCD as denominators

$$\frac{x}{x^2-10x+25}=\frac{x}{(x-5)(x-5)}=\frac{2x}{2(x-5)(x-5)}$$

$$\frac{x-4}{2x-10}=\frac{x-4}{2(x-5)}=\frac{(x-4)(x-5)}{2(x-5)(x-5)}$$

Example: Subtracting Rational Expressions with Different Denominators *(continued)*

Step 3 Subtract numerators, putting this difference over the LCD

$$\frac{x}{x^2-10x+25}-\frac{x-4}{2x-10}=\frac{2x-(x-4)(x-5)}{2(x-5)(x-5)}$$

$$=\frac{2x-(x^2-9x+20)}{2(x-5)(x-5)}=\frac{2x-x^2+9x-20}{2(x-5)(x-5)}$$

$$=\frac{-x^2+11x-20}{2(x-5)(x-5)}$$

Example: Subtracting Rational Expressions with Different Denominators *(continued)*

Step 4 If necessary, simplify.

$$\frac{x}{x^2-10x+25}-\frac{x-4}{2x-10}=\frac{-x^2+11x-20}{2(x-5)(x-5)}$$

Because the numerator is prime, no further simplification is possible.

$$\frac{x}{x^2-10x+25}-\frac{x-4}{2x-10}=\frac{-x^2+11x-20}{2(x-5)(x-5)},x\neq 5$$

Complex Rational Expressions

Complex rational expressions, also called **complex fractions**, have numerators or denominators containing one or more rational expressions. A complex fraction must be simplified so that neither the numerator nor the denominator contains any rational expressions.

Example: Simplifying a Complex Rational Expression

Simplify $\dfrac{\dfrac{1}{x+7}-\dfrac{1}{x}}{7}$

We will use the method of multiplying the numerator and the denominator by the least common denominator for all rational expressions. The least common denominator for all rational expressions is $x(x+7)$.

$$\frac{\dfrac{1}{x+7}-\dfrac{1}{x}}{7}=\left(\frac{\dfrac{1}{x+7}-\dfrac{1}{x}}{7}\right)\frac{x(x+7)}{x(x+7)}$$

Example: Simplifying a Complex Rational Expression *(continued)*

We continue the process of simplifying

$$\frac{\frac{1}{x+7}-\frac{1}{x}}{7}=\left(\frac{\frac{1}{x+7}-\frac{1}{x}}{7}\right)\frac{x(x+7)}{x(x+7)}$$

$$=\frac{x(x+7)\left(\frac{1}{x+7}\right)-x(x+7)\left(\frac{1}{x}\right)}{x(x+7)\cdot 7}\qquad=\frac{x-(x+7)}{7x(x+7)}$$

Example: Simplifying a Complex Rational Expression *(continued)*

We continue the process of simplifying

$$\frac{\frac{1}{x+7}-\frac{1}{x}}{7} = \frac{x-(x+7)}{7x(x+7)} = \frac{x-x-7}{7x(x+7)}$$

$$= \frac{-7}{7x(x+7)} = \frac{-1}{x(x+7)}, x \neq 0, x \neq -7$$