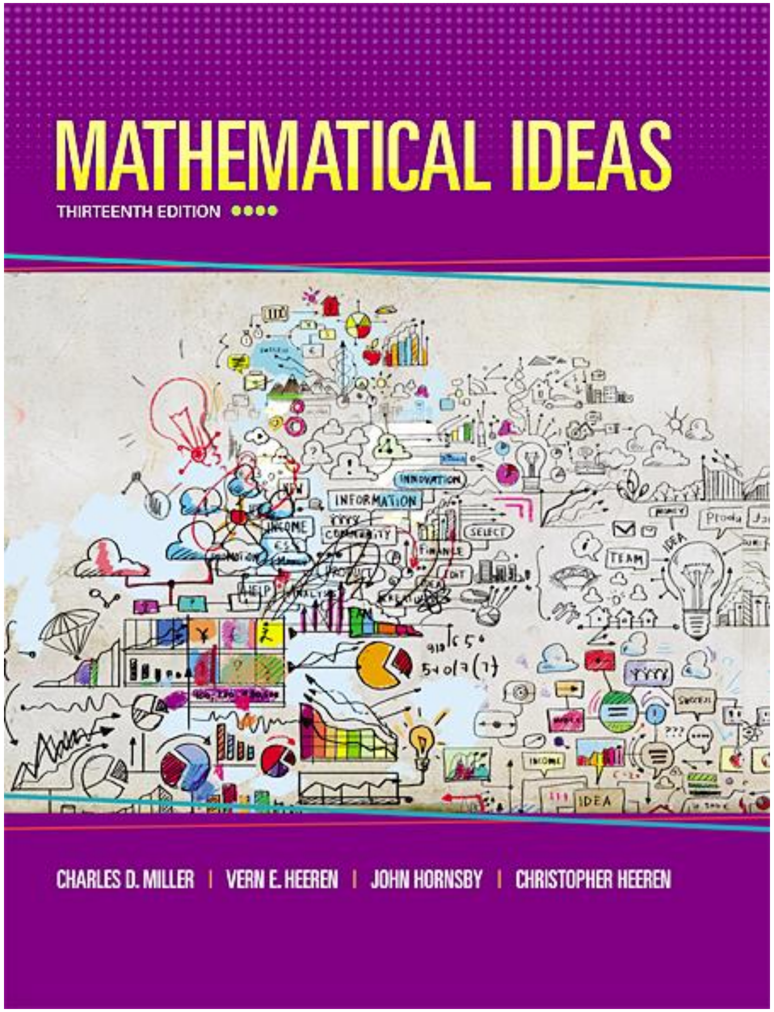
Chapter 10

Number Theory



Chapter 10: Number Theory

Section 10-1

Prime and Composite Numbers

Prime and Composite Numbers

- Identify prime and composite numbers.
- Apply divisibility tests for natural numbers.
- Apply the fundamental theorem of arithmetic.

Number Theory

Number Theory is the branch of mathematics devoted to the study of the properties of the natural numbers.

Natural numbers are also known as **counting numbers** and the **positive integers.**

Divisibility

A counting number is *divisible* by another if the operation of dividing the first by the second leaves a remainder of 0.

Formally: the natural number a is **divisible** by the natural number b if there exists a natural number k such that $a = bk$. If b divides a, then we write $b|a$.

Terminology

If the natural number a is divisible by the natural number b, then b is a **factor** (or **divisor**) of a, and a is a **multiple** of b.

The number 30 equals $6 \cdot 5$; this product is called a **factorization** of 30.

Example: Finding Factors

Find all the natural number factors of each number.

a) 24 b) 13

Solution

a) To find factors try to divide by 1, 2, 3, 4, 5, 6 and so on to get the factors 1, 2, 3, 4, 6, 8, 12, and 24.

b) The only factors are 1 and 13.

Prime and Composite Numbers

A natural number greater than 1 that has only itself and 1 as factors is called a **prime number**. A natural number greater than 1 that is not prime is called **composite**.

Alternative Definition of a Prime Number

A **prime number** is a natural number that has *exactly* two different natural number factors.

The natural number 1 is neither prime nor composite.

Sieve of Eratosthenes

One systematic method for identifying primes is known as the **Sieve of Eratosthenes**. To construct a sieve, list all the natural numbers from 2 through some given natural number. The number 2 is prime, but all multiples of it are composite. Circle the 2 and cross out all other multiples of 2. Continue this process for all primes less than or equal to the square root of the last number in the list. Circle all remaining numbers that are not crossed out.

Sieve of Eratosthenes

TABLE 1 Sieve of Eratosthenes

	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~	11	~~12~~	13	~~14~~
~~15~~	~~16~~	17	~~18~~	19	~~20~~	~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~
29	~~30~~	31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~	41	~~42~~
43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~	~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~
~~57~~	~~58~~	59	~~60~~	61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~	~~81~~	~~82~~	83	~~84~~
~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~	~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~
~~99~~	~~100~~												

Divisibility Tests

Divisibility tests are an aid to determine whether a natural number is divisible by another natural number. Simple tests are given on the next two slides. There are tests for 7 and 11, but they are more involved.

Divisibility Tests

Divisible by	Test
2	Number ends in 0, 2, 4, 6, or 8.
3	Sum of the digits is divisible by 3.
4	Last two digits form a number divisible by 4.
5	Number ends in 0 or 5.
6	Number is divisible by both 2 and 3.

Divisibility Tests (continued)

Divisible by	Test
8	Last three digits form a number divisible by 8.
9	Sum of the digits is divisible by 9.
10	The last digit is 0.
12	Number is divisible by both 3 and 4.

Example: Applying Divisibility Tests

Is the number 2,984,094 divisible by 4?

Solution

The last two digits form the number 94. Since 94 is not divisible by 4, the given number is not divisible by 4.

Example: Applying Divisibility Tests

Is the number 4,355,211 divisible by 3?

Solution

Check: $4 + 3 + 5 + 5 + 2 + 1 + 1 = 21$, which is divisible by 3. Therefore, the given number is divisible by 3.

The Fundamental Theorem of Arithmetic

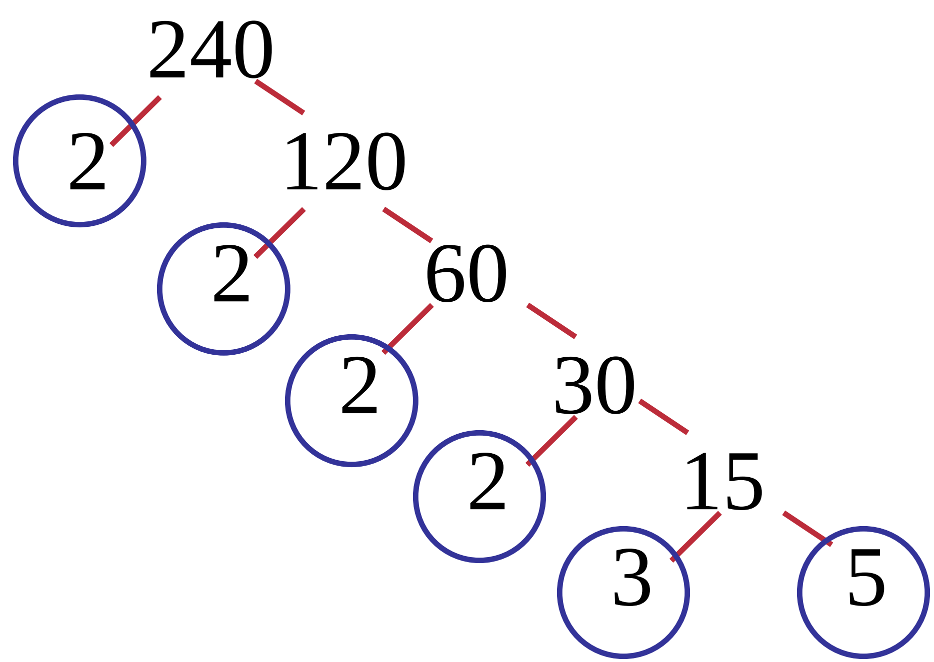
Every natural number can be expressed in one and only one way as a product of primes (if the order of the factors is disregarded). This unique product of primes is called the **prime factorization** of the natural number.

Example: Unique Prime Factorization

Find the prime factorization of 240.

Solution

Using a tree format:



$$240 = 2^4 \cdot 3 \cdot 5$$

Large Prime Numbers

- Understand the infinitude of primes.
- Investigate several categories of prime numbers.
- Learn how large primes are identified.

The Infinitude of Primes

There is no largest prime number. Euclid proved this around 300 B.C.

The Search for Large Primes

Primes are the basis for modern cryptography systems, or secret codes. Mathematicians continue to search for larger and larger primes.

The theory of prime numbers forms the basis of security systems for vast amounts of personal, industrial, and business data.

Mersenne Numbers and Mersenne Primes

For n = 1, 2, 3, …, the **Mersenne numbers** are those generated by the formula:

$$M_n = 2^n - 1.$$

1. If n is composite, then M_n is composite.
2. If n is prime, then M_n may be prime or composite.

The prime values of M_n are called **Mersenne primes**.

Example: Mersenne Numbers

Find the Mersenne number for $n = 5$.

Solution

$M_5 = 2^5 - 1 = 32 - 1 = 31$

Fermat Numbers

Fermat numbers are another attempt at generating prime numbers.

The Fermat numbers are generated by the formula:

$$2^{2^n} + 1.$$

The first five Fermat numbers (through $n = 4$) are prime.

Euler's and Escott's Formulas for Finding Primes

Euler's prime number formula first fails at $n = 41$:

$$n^2 - n + 41$$

Escott's prime number formula first fails at $n = 80$:

$$n^2 - 79n + 1601$$