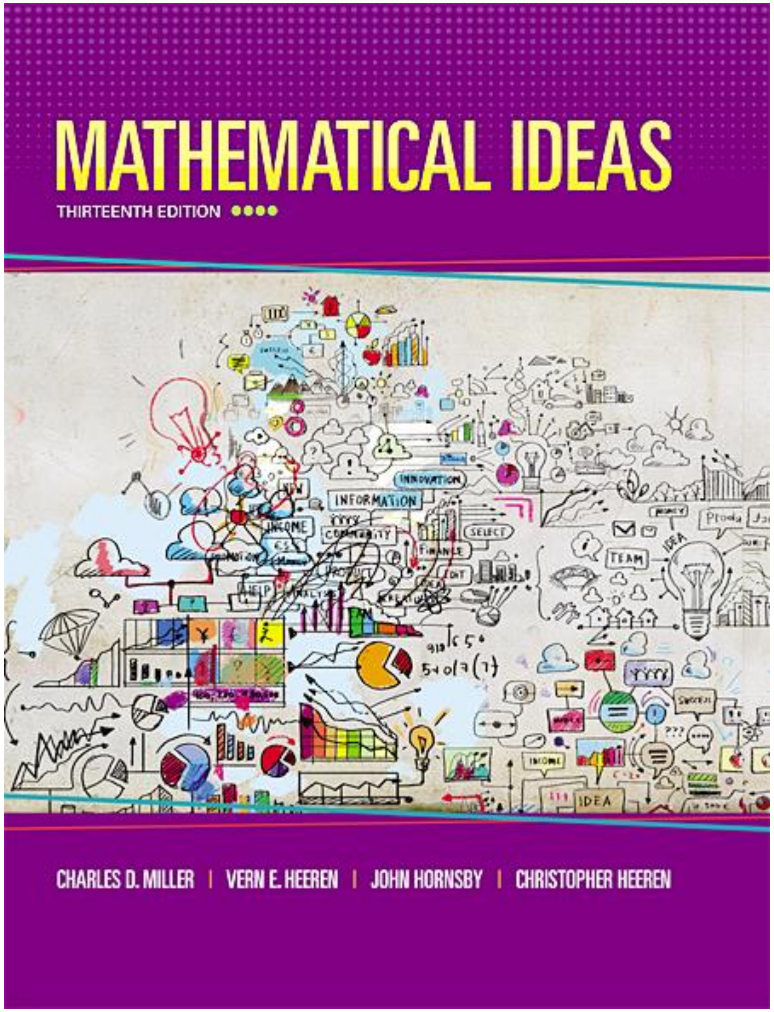
Chapter 10

Number Theory



Chapter 10: Number Theory

Section 5-5

The Fibonacci Sequence and the Golden Ratio

The Fibonacci Sequence and the Golden Ratio

- Work with the Fibonacci sequence.
- Understand the golden ratio.
- See relationships between the Fibonacci sequence and the golden ratio.

The Fibonacci Sequence

A famous problem:

A man put a pair of rabbits in a cage. During the first month the rabbits produced no offspring, but each month thereafter produced one new pair of rabbits. If each new pair thus produced reproduces in the same manner, how many pairs of rabbits will there be at the end of one year?

The Fibonacci Sequence

The solution of the problem leads to the **Fibonacci sequence**. Here are the first thirteen terms of the sequence:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233

Notice the pattern. After the first two terms (both 1), each term is obtained by adding the two previous terms.

Recursive Formula for Fibonacci Sequence

If F_n represents the Fibonacci number in the nth position in the sequence, then

$$F_1 = 1$$

$$F_2 = 1$$

$$F_n = F_{n-2} + F_{n-1}, \quad \text{for } n \geq 3.$$

Example: Observing a Pattern of the Fibonacci Numbers

Find the sum of the squares of the first *n* Fibonacci numbers for *n* = 1, 2, 3, 4, 5, and examine the pattern. Generalize this relationship.

Solution

$$1^2 = 1 \qquad F_1 \cdot F_2$$

$$1^2 + 1^2 = 2 \qquad F_2 \cdot F_3$$

$$1^2 + 1^2 + 2^2 = 6 \qquad F_3 \cdot F_4$$

$$1^2 + 1^2 + 2^2 + 3^2 = 15 \qquad F_4 \cdot F_5$$

$$1^2 + 1^2 + 2^2 + 3^2 + 5^2 = 40 \qquad F_5 \cdot F_6$$

Pattern:

$$F_n \cdot F_{n+1}$$

The Golden Ratio

Consider the quotients of successive Fibonacci numbers and notice a pattern.

$$\frac{1}{1}=1,\quad \frac{2}{1}=2,\quad \frac{3}{2}=1.5,\quad \frac{5}{3}=1.66\ldots,$$

$$\frac{8}{5}=1.6,\quad \frac{13}{8}=1.625,\quad \frac{21}{13}\approx 1.615384$$

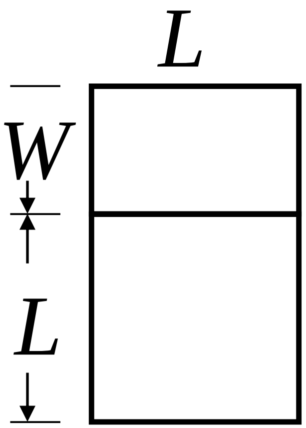
These quotients seem to go toward 1.618.

In fact, they approach $\dfrac{1+\sqrt{5}}{2}$.

This number is known as the **golden ratio**.

Golden Rectangle

A **golden rectangle** is one that can be divided into a square and another (smaller) rectangle, the same shape as the original.

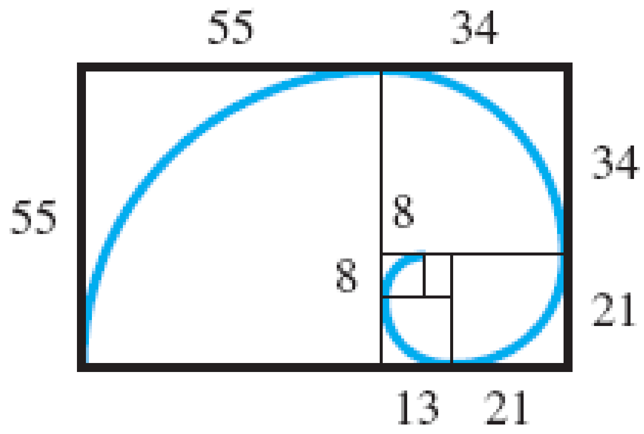


$$\frac{L}{W} = \frac{L+W}{L}$$

Example of Golden Rectangle: Parthenon in Athens

Spiral

Construct the divisions of a (*nearly*) golden rectangle below. Use a smooth curve to connect the vertices of the squares formed. This curve is a *spiral*.



Example of Spiral in Nature: Shell of Chambered Nautilus