Homework Chapter 2

1. On the probability space of Example 1.10, define the random variable $X(\omega) := \omega$.

   (a) Find all the outcomes ω that belong to the event $\{\omega : X(\omega) \leq 4\}$.

   (b) Find all the outcomes ω that belong to the event $\{\omega : X(\omega) > 4\}$.

   (c) Compute $\mathsf{P}(X \leq 4)$ and $\mathsf{P}(X > 4)$.

9. Consider the sample space $\Omega := \{-2,-1,0,1,2,3,4\}$. For an event $A \subset \Omega$, suppose that $\mathsf{P}(A) = |A|/|\Omega|$. Define the random variable $X(\omega) := \omega^2$. Find the probability mass function of X.

10. Let $X \sim \text{Poisson}(\lambda)$. Evaluate $\mathsf{P}(X > 1)$; your answer should be in terms of λ. Then compute the numerical value of $\mathsf{P}(X > 1)$ when $\lambda = 2$

12. A class consists of 15 students. Each student has probability $p = 0.2$ of getting an "A" in the course. Find the probability that exactly one student receives an "A." Assume the students' grades are independent.

16. Show that the geometric$_0(p)$ pmf $p_X(k) = (1-p)p^k, k = 0,1,\ldots$ sums to one. Repeat for the geometric$_1(p)$ pmf $p_X(k) = (1-p)p^{k-1}, k = 1,2,\ldots$. *Hint:* Use the geometric series formula from Problem 27 in Chapter 1.

18. Suppose that $X_1,\ldots,X_n$ are independent, geometric$_1(p)$ random variables. Evaluate $\mathsf{P}(\min(X_1,\ldots,X_n) > \ell)$ and $\mathsf{P}(\max(X_1,\ldots,X_n) \leq \ell)$.

21. Let $X \sim \text{geometric}_1(p)$.

    (a) Show that $\mathsf{P}(X > n) = p^n$.

    (b) Compute $\mathsf{P}(\{X > n+k\}|\{X > n\})$. *Hint:* If $A \subset B$, then $A \cap B = A$.

    Remark. Your answer to (b) should not depend on n. For this reason, the geometric random variable is said to have the **memoryless property**. For example, let X model the number of the toss on which the first heads occurs in a sequence of coin tosses. Then given a heads has not occurred up to and including time n, the conditional probability that a heads does not occur in the next k tosses does not depend on n. In other words, given that no heads occurs on tosses $1,\ldots,n$ has no effect on the conditional probability of heads occurring in the future. Future tosses do not remember the past.

25. Let X and Y be jointly discrete, integer-valued random variables with joint pmf

$$p_{XY}(i,j) = \begin{cases} \dfrac{3^{j-1}e^{-3}}{j!}, & i=1,\ j\geq 0, \\ 4\dfrac{6^{j-1}e^{-6}}{j!}, & i=2,\ j\geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Find the marginal pmfs $p_X(i)$ and $p_Y(j)$, and determine whether or not X and Y are independent.

33. If $X \sim \text{geometric}_0(1/2)$, compute $\mathsf{E}[I_{(2,6)}(X)]$.

34. If X is Poisson(λ), compute $\mathsf{E}[1/(X+1)]$.

37. Compute $\mathsf{E}[(X+Y)^3]$ if $X \sim \text{Bernoulli}(p)$ and $Y \sim \text{Bernoulli}(q)$ are independent.

38. Let X be a random variable with mean m and variance σ^2. Find the constant c that best approximates the random variable X in the sense that c minimizes the **mean-squared error** $\mathsf{E}[(X-c)^2]$.

41. Let $X \sim \text{Poisson}(3/5)$. Compute both sides of the Chebyshev inequality,

$$\mathsf{P}(X \geq 2) \leq \frac{\mathsf{E}[X^2]}{4}.$$

44. Show by counterexample that being uncorrelated does not imply independence. *Hint:* Let $\mathsf{P}(X=\pm 1) = \mathsf{P}(X=\pm 2) = 1/4$, and put $Y := |X|$. Show that $\mathsf{E}[XY] = \mathsf{E}[X]\mathsf{E}[Y]$, but $\mathsf{P}(X=1, Y=1) \neq \mathsf{P}(X=1)\,\mathsf{P}(Y=1)$.