
Exercise 1. Let X be a set and $A, B \subseteq X$. Show that the following (*called De Morgan's Laws*) are true.

(a) $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$

Proof. Suppose also that $x \in X \setminus (A \cup B)$, then $x \notin A \cup B$. Hence, $x \notin A$ and $x \notin B$. Therefore, $x \in (X \setminus A)$ and $x \in (X \setminus B)$. It then follows that $x \in (X \setminus A) \cap (X \setminus B)$. □

(b) $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$

Proof. Suppose that $x \in X \setminus (A \cap B)$, then $x \notin A$ or $x \notin B$. Therefore, $x \in (X \setminus A)$ or $x \in (X \setminus B)$. Namely, $X \in (X \setminus A) \cup (X \setminus B)$. □

Exercise 2. Let X be a set and $\mathcal{A}$ a family of subsets of X. Prove the following statements.

(a) $X \setminus (\cup \mathcal{A}) = \cap \{X \setminus A | A \in \mathcal{A}\}$

Proof. Assume $x \in X \setminus (\cup \mathcal{A})$. Thereby, $x \notin \cup \mathcal{A}$, so we can take any $A \in \mathcal{A}$, and we know that $x \notin A$. This implies that $x \in X \setminus A$ for all $A \in \mathcal{A}$. In other words, $A \in \cap \mathcal{A}$. □

(b) $X \setminus (\cap \mathcal{A}) = \cup \{X \setminus A | A \in \mathcal{A}\}$

Proof. Take $x \in \cup \{X \setminus A | A \in \mathcal{A}\}$. It follows that $x \notin A$ for all $A \in \mathcal{A}$. This means also that $x \in X \setminus A$ for all $A \in \mathcal{A}$. Namely, $x \in \cap (X \setminus A)$ for all $A \in \mathcal{A}$. Rewriting this slightly, we get $x \in X \setminus (\cap \mathcal{A})$. □

Exercise 3. Let $\{I_j\}_{j \in \mathbb{N}}$ be as in the example above (an indexed family of sets). Describe the followings sets, justify your responses.

(a) $\cup_{j \in \mathbb{N}} I_j$

This is simply a union of every set in the family $\{I_j\}_{j \in \mathbb{N}}$. Every element in some indexed member of the family I_j will be in the union. We know this by the definition of union of a family of sets

$$\cup_{j \in \mathbb{N}} I_j = \{x | x \in I_j \text{ for any } j \in \mathbb{N}\}$$

(b) $\cap_{j \in \mathbb{N}} I_j$

This denotes the intersection of all I_j for all $j \in \mathbb{N}$. This will be the set of all elements which exist in every set of the family $\{I_j\}_{j \in \mathbb{N}}$. This one again follows directly from the definition

$$\cap_{j \in \mathbb{N}} I_j = \{x | x \in I_j \text{ for every } j \in \mathbb{N}\}$$

Exercise 4. Prove the following statements.

(a) If $\mathcal{U}$ is a family of open subsets of $\mathbb{R}^n$, then $\cup \mathcal{U}$ is open.

Proof. Take any $x \in \cup \mathcal{U}$. We know that x lies within at least one open set $U \in \mathcal{U}$. Within this open set U we can construct an open ball $B(r, x)$ around x, such that everything within the open ball is also within U. Now because $U \in \mathcal{U}$, we know that $U \subseteq \cup \mathcal{U}$. Therefore also $B(r, x) \subseteq \cup \mathcal{U}$. Hence, we can take any point $x \in \cup \mathcal{U}$ and construct an open ball with $r > 0$ around x that lies entirely within $\cup \mathcal{U}$. This shows that $\cup \mathcal{U}$ is also open. □

(b) If $\mathcal{A}$ is a family of closed sets of $\mathbb{R}^n$, then $\cap\mathcal{A}$ is closed.

Proof. Take any $A \in \mathcal{A}$. We know that this A is closed by the given statement. By the definition of closed, we know that $\mathbb{R} \setminus A$ is open. From **Exercise 2** (b), we have that

$$\mathbb{R} \setminus (\cap\mathcal{A}) = \cup\{\mathbb{R} \setminus A | A \in \mathcal{A}\}$$

We know that $\mathbb{R} \setminus A$ is open for all $A \in \mathcal{A}$. Furthermore their union $\cup\{\mathbb{R} \setminus A | A \in \mathcal{A}\}$ is then open by (a). Therefore,

$$\mathbb{R} \setminus (\cap\mathcal{A}) \text{ is open}$$

and by the definition of closed in $\mathbb{R}^n$,

$$\cap\mathcal{A} \text{ is closed}$$

□

(c) $\emptyset$ and $\mathbb{R}^n$ are closed in $\mathbb{R}^n$.

Proof. Begin with $\emptyset$. The complement of $\emptyset$ in $\mathbb{R}^n$ is $\mathbb{R}^n \setminus \emptyset = \mathbb{R}^n$. Therefore, as $\mathbb{R}^n$ is open, we know that the complement of $\emptyset$ is open in $\mathbb{R}^n$, and therefore $\emptyset$ is closed in $\mathbb{R}^n$.

Second, for $\mathbb{R}^n$. We calculate the complement of $\mathbb{R}^n$ in $\mathbb{R}^n$ to be $\mathbb{R}^n \setminus \mathbb{R}^n = \emptyset$. We have previously shown that $\emptyset$ is open, therefore by the definition of closed, $\mathbb{R}^n$ is closed in $\mathbb{R}^n$. □