Advanced Topics Campbell Phalen
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1. Let x and y be real numbers. Show that there exists a positive integer N such that $Nx > y$.

   Proof. Define $A = \{nx | n \in \mathbb{Z}\}$, thus $A \neq \emptyset$. Assume that for all $n \in \mathbb{Z}$ such that $n > 0$, we have $nx \leq y$, thus y is an upper bound of A. As, A is a set of integers, it has an integral largest element, we call that value u. However, it is clear that u cannot be the largest element, as $n(u+1) = un + n \in A$. Thus the set A is unbounded and cannot have any upper bound. Hence, for any $n \in \mathbb{Z}$ such that $n > 0$, we can choose N such that $Nx > y$. □

2. Use the result in problem 1 to show that given any two distinct real numbers a, b, there is always a rational number q that lies between a and b.

   Proof. Given a and b. Consider the set $A = \{\frac{z}{n} | z \in \mathbb{Z}\}$. For each element in A, the difference between that element and the next is $\frac{1}{n}$, in other words, the step size between elements is $\frac{1}{n}$. We can demonstrate this as follows

$$\frac{z+1}{n} - \frac{z}{n} = \frac{(z+1)-z}{n}$$
$$= \frac{1}{n}$$

   Now, using problem 1 we continue. By problem 1, we can pick a N large enough to satisfy the that

$$\frac{1}{b-a} < N$$

   As this is analogous to saying

$$N(b-a) > 1$$

   Now consider the subset $B \subseteq A$, $B = \{\frac{z}{n} \in A | \frac{z}{n} < a\}$. As B is bounded above by a there exists some $\beta = sup(B)$. Because B is entirely integral, we know that $sup(B)$ will also be integral, thus $sup(B) \in B$. Because $\frac{1}{n} > 0$, we know $\frac{1}{n} + \beta > \beta$ and thus $\frac{1}{n} + \beta \notin B$, so $\frac{1}{n} + \beta$ must break the membership role of B, namely $\frac{1}{n} + \beta \geq a$. Furthermore, we compute from our previous determination of N

$$\frac{1}{N} < b - a$$
$$\frac{1}{N} + a < b - a + a$$
$$\frac{1}{N} + a < b$$

   And because, $\beta \leq a$, we also have,

$$\frac{1}{N} + \beta < b$$

   Hence, $\frac{1}{n} + \beta$ belongs to (a, b). Moreover, because $\frac{1}{N}$ is rational and, β must also be rational as $\beta \in B$, we have that $\frac{1}{N} + \beta$ is also rational. □

3. Let $I_j = [a_j, b_j]$ for each $j \in \mathbb{N}$, where for each $j, a_j \leq b_j$ and $I_{j+i} \subseteq I_j$. Show that the intersection $\cap_{j \in \mathbb{N}} I_j$ is not empty. Moreover, if we let $\delta_j = b_j - a_j$, and $\delta_j \to 0$ as $j \to \infty$, show that $\cap_{j \in \mathbb{N}} I_j$ contains exactly one point.

Proof. We first consider the set $A = \{a_j | j \in \mathbb{N}\}$. For any a_i, b_j, we know that $a_i \leq b_j$. There are two cases, first if $a_i \leq a_j$ because $a_j \leq b_j$, we know also $a_i \leq b_j$. Secondly, if $a_j \leq a_i$, then also $b_i \leq b_j$, thus because $a_i \leq b_i$ it follows that $a_i \leq b_j$. Hence, for any a_i, b_j, $a_i \leq b_j$. Thus, it follows that any b_j is an upper bound of A. As A is bounded above, there must exist a $sup(A)$ that we denote α. We know that, by definition, $\alpha \geq a_j$ for all $j \in \mathbb{N}\}$. However, we also know that $\alpha \leq b_j$, as b_j is an upper bound and α is the least upper bound. Thus, we can write

$$a_j \leq \alpha \leq b_j$$

Thus the interval (a_j, b_j) for any $j \in \mathbb{N}\}$ must have at least one element. Also, because $I_{j+i} \subseteq I_j$, all $(a_i, b_i$ for $i < j$ will also have the interval (a_j, b_j) as a subset. Hence, the $\cap_{j \in \mathbb{N}} I_j \neq \emptyset$.

Continuing with this line of reasoning. We want to show that as $\delta_j \to 0$ and $j \to \infty$, $\cap_{j \in \mathbb{N}} I_j$ has only one element. This is of course the case because as $\delta_j \to 0$, we get that $a = b$, thus

$$a_j \leq \alpha \leq b_j$$

Simplifies to

$$a_j = \alpha = b_j$$

Thus, we have only one element in the intersection $\cap_{j \in \mathbb{N}} I_j$. □

4. A sequence a_j of real numbers is said to be *monotonically increasing* if and only if $a_j \leq a_{j+1}$ for each $j \in \mathbb{N}$. Similarly, one defines a *monotonically decreasing* sequence. If a sequence is either monotonically increasing or decreasing, we say that the sequence is *monotone*. Show that a monotone sequence converges if and only if it is bounded.

Proof. Let $A = \{a_1, a_2, ...\}$ be our monotone sequence.

First, given that A is a monotone which is bounded. Without loss of generality let's assume A is monotonically increasing. To show that A converges we need to an α which implies that for any $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that for all $j \in \mathbb{N}$, $j > N$ it is true that $|a_j - \alpha| \leq \epsilon$. Now because A is bounded and thus bounded above, A must then have a least upper bound that we call α. We claim that given any $\epsilon > 0$ there exists $a_j > \alpha - \epsilon$. This must be true, as it were not and $a_j \leq \alpha - \epsilon$ then $\alpha - \epsilon$ is an upper bound of A which is less than α and this is a violation of the definition of least upper bound. We then carry out the following calculation

$$a_j > \alpha - \epsilon$$
$$\epsilon > \alpha - a_j$$

Because $\alpha \geq a_j$, we know $\alpha - a_j \geq 0$, thus $\alpha - a_j = |\alpha - a_j|$

$$\begin{aligned} \epsilon &> \alpha - a_j = |\alpha - a_j| \\ \epsilon &> |\alpha - a_j| = |a_j - \alpha| \\ \epsilon &> |a_j - \alpha| \end{aligned}$$

Furthermore we know that for all a_j such that $i > j$, $a_j \leq a_i$ because A is monotonically increasing. Thus let $N = j$, then $\epsilon > |a_i - \alpha|$ for all ai such that $i > j$. Thus, A converges on α.

Second, we assume that A is a monotone sequence which is converges. As A converges we know that there exists some β such that for any $\epsilon > 0$ we can find an $N \in \mathbb{N}$ which implies that for all $j > N$ we have also that $|a_j - \beta| < \epsilon$. Now, because A isa monotonically increasing sequence, we also have that for any $i < k$, $a_k \leq a_j$. Hence, when $i \leq N$, we have that $a_i \leq a_j \leq \beta$, and for $i > N$ we also have that $a_i \leq \beta$ by the definition of convergence. Combining these two inequalities we get that $a_i \leq \beta$ for all $a_i \in A$. Thus A is bounded above by β. Additionally, A is bounded below by a_1 because, once again, A is monotonically increasing, thus $a_1 \leq a_i$ for all $a_i \in A$. Finally we have, $a_1 \leq a_i \leq \beta$ for all $a_i \in A$ thus A is bounded.

□