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**Analytics PSet #5** November 3, 2018

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1. A sequence $\{b_k\}$ is a *subsequence* of a sequence $\{a_n\}$ if and only if there exists an increasing function $\varphi : \mathbb{N} \to \mathbb{N}$ such that $b_k = a_{\varphi(k)}$ for every $k \in \mathbb{N}$.

   A sequence $a_n$ is said to be *monotonically increasing* (respectively, *monotonically decreasing*) if and only if $a_n \leq a_{n+1}$ (respectively, $a_{n+1} \leq a_n$) for all $n \in \mathbb{N}$. A sequence is said to be monotone if and only if it is either monotonically increasing or monotonically decreasing.

   Show that every sequence of real numbers has a monotone subsequence.

   *Proof.* For every $\{x_i\}$ subsequence of $\mathbb{R}^n$, $\{x_i\}$ either has a largest element or not. That is, there exists some $x_j$ such that $x_j \geq x_i$ for all $x_i$, or there does not exists such $x_j$.

   *Case I:* such a $x_j$ does not exist.

   Therefore the sequence is unbounded and finding a subsequence is trivial. Simply define

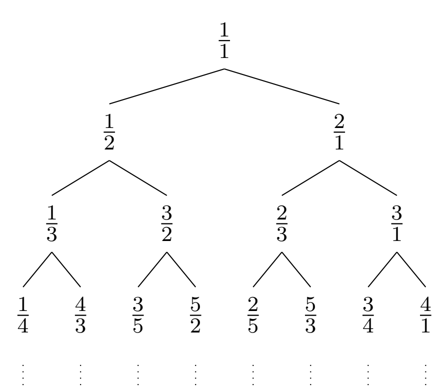
   $$\varphi(i) = \begin{cases} x_i & x_i \geq x_j \text{ for all } j \leq i \\ x_{i-1} & \text{otherwise} \end{cases}$$

   We know this sequence will continue on forever, as $\{x_i\}$ is unbounded above. Thus we can find a monotonically increasing subsequence.

   *Case II:* such a $x_j$ does exist.

   Now our argument becomes somewhat recursive. Look at the entire sequence $\{x_i\}$ such that $i > j$. That is, the entire sequence of $\{x_i\}$ after $x_j$. Now we can repeat the same logic, either there is a maximum of the new sequence or there is not. If there exists no maximum, then b *Case I* we have a monotonically increasing subsequence. If there is a maximum we continue to repeat. If there is always a maximum, what we get is a sequence of maximums we denote $\{m_1, m_2, ..., m_j\}$. However, we know that each maximum must be less than or equal to the previous by the definition of max. Therefore, this sequence of maximums is monotonically decreasing. Hence, we have found a monotonic subsequence of $\{x_i\}$, showing that a sequence of real numbers $\{x_i\}$ will *always* have a monotone subsequence. $\square$

2. The following tree (LaTeX forest!) is constructed inductively using the following rule. Begin with $\frac{1}{1}$ at the first level. Suppose that the $m$-th level of the tree has been constructed with $2^{m-1}$ nodes, and each node consists of a fraction of $\frac{i}{j}$. Then each of the $2^{m-1}$ entries at the $m$-th level in the tree produces two children, a left child and a right child; the $2^m$ nodes at the $(m+1)$-th level consist of these. For $\frac{i}{j}$ at the $m$-th level of the tree, the left child is $\frac{i}{i+j}$ and the right child is $\frac{i+j}{j}$.



Prove the following statements about the tree.

(a) Every fraction $\frac{m}{n}$ that appears in the three is in lowest terms that is, $(m, n) = 1$. (Here, $(m, n)$ denotes the *greatest common divisor* of $m$ and $n$).

*Proof.* We proceed by induction. First, we note that $\frac{1}{1}$ is in lowest possible terms, as $(1, 1) = 1$. Now, we take our induction hypothesis to be that $\frac{i}{j}$ is in lowest possible terms, and attempt to show that both $\frac{i}{i+j}$ and $\frac{i+j}{j}$ are also in lowest possible terms. First, we know that $(i, j) = 1$, this means that given some $x, y \in \mathbb{R}$

$$\begin{aligned} ix + jy &= 1 \\ ix + jy + (xj - xj) &= 1 \\ ix + xj + jy - xj &= 1 \\ (i + j)x + j(y - x) &= 1 \end{aligned}$$

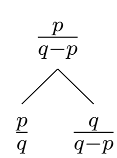
Therefore $i + j$ and $j$ are coprime, meaning that $(i + j, j) = 0$. Hence, $\frac{i+j}{i}$ is in lowest terms. Additionally we can compute

$$\begin{aligned} ix + jy &= 1 \\ ix + jy + (yi - yi) &= 1 \\ ix - yj + jy + yi &= 1 \\ i(x - y) + (i + j) &= 1 \end{aligned}$$

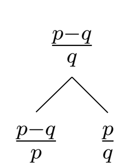
Once again, we see similarly that $i$ and $i + j$ are coprime and that $(i, i + j) = 1$. Meaning also that $\frac{i}{i+j}$ is in lowest terms.
By induction we can therefore conclude that every element of the tree will be in lowest terms. □

(b) Every positive rational number appears somewhere in the tree.

*Proof.* Assume that there were such rational numbers that did not appear somewhere in the tree. If this were the case we pick $q$ to be lowest denominator of any rational number that does not exist in the tree. We then pick $p$ to be the small numerator of numbers which do not appear in the tree that have the denominator $q$. We then have a rational number, $\frac{p}{q}$ which does not exist in the tree. Now, we know that if $\frac{p}{q}$ does not exist in the free, neither can it's parent node, otherwise the parent nodes inclusion in the tree would imply that $\frac{p}{q}$ is in the tree. We then have two cases,
*Case I:* $\frac{p}{q}$ is a left child. Thus, by definition of the tree



However, this poses a contradiction as $\frac{p}{q-p}$ is then also not in the tree, but $q - p < q$ and we defined $q$ to the lowest denominator of any rational number that did not exist on the tree. Therefore $\frac{p}{q-p}$ must exist on the tree after all, implying that $\frac{p}{q}$ exists in the tree as well.
*Case II:* $\frac{p}{q}$ is a right child. Then,



Once again, we find a contradiction as $\frac{p-q}{q}$ must not be in the tree, but $p-q < p$ and we defined $p$ to be the lowest numerator of any rational number with denominator $q$. However, $\frac{p-q}{q}$ also has denominator $q$ and has a smaller numerator, thus $\frac{p-q}{q}$ exists in the tree and so much $\frac{p}{q}$. $\square$

(c) No rational number appears more than once in the tree.

*Proof.* We will proceed by induction. First we have our base case that $\frac{1}{1}$ only appears in the tree once. This is because $i+j > 1$ as $i, j > 1$. Now we take our induction hypothesis to be that $\frac{i}{j}$ only appears in the tree once, and we need to show that $\frac{i}{i+j}$ and $\frac{i+j}{j}$ also only exist in the three once. However, assume that $\frac{i}{i+j}$ did appear in the tree more than once.

*Case I:* $\frac{i+j}{j}$ is a left child.

Then it's parent element, $\frac{i}{j}$ would also appear in the tree more than once, but this violates our induction hypothesis, therefore $\frac{i}{i+j}$ must only appear in the tree once. The same goes for $\frac{i+j}{j}$

*Case II:* $\frac{i+j}{j}$ is a right child. Then,

$$\begin{array}{ccc} & \frac{i+j}{-i} & \\ \swarrow & & \searrow \\ \frac{i}{-i} & & \frac{i+j}{j} \end{array}$$

However, this is also impossible as we are only dealing with the positive rational numbers, therefore $\frac{i+j}{-i}$ cannot exist on the tree. Hence, $\frac{i+j}{j}$ must only appear on the tree once.

Similarly, with $\frac{i}{i+j}$.

*Case I:* $\frac{i}{i+j}$ is a left child.

$$\begin{array}{ccc} & \frac{-j}{i+j} & \\ \swarrow & & \searrow \\ \frac{i}{i+j} & & \frac{-j}{i} \end{array}$$

Once again, this is impossible as this tree contains only the positive rational numbers, therefore $\frac{-j}{i+j}$ cannot be a node of the tree.

*Case II:* $\frac{i}{i+j}$ is a right child.

Then, $\frac{i}{j}$ is the parent node of $\frac{i}{i+j}$. However, if there are multiple occurrences of $\frac{i}{i+j}$, then it follows that there are multiple occurrences of $\frac{i}{j}$, but this violates our induction hypothesis, and is therefore a contradiction. As a result, we know that $\frac{i}{i+j}$ must exist only once in the three.

By this logic we have shown our inductive step that, given $\frac{i}{j}$ appears only once on the tree, it follows that $\frac{i+j}{j}$ and $\frac{i}{i+j}$ also appear only once on the tree. We have thus shown that ever rational number appears only once on the tree. $\square$