Advanced Topics
Analytics PSet #4

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1. A sequence $\{b_k\}$ is a *subsequence* of a sequence $\{a_n\}$ if and only if there exists an increasing function $\varphi : \mathbb{N} \to \mathbb{N}$ such that $b_k = a_{\varphi(k)}$ for every $k \in \mathbb{N}$.

   A sequence a_n is said to be *monotonically increasing* (respectively, *monotonically decreasing*) if and only if $a_n \leq a_{n+1}$ (respectively, $a_{n+1} \leq a_n$) for all $n \in \mathbb{N}$. A sequence is said to be monotone if and only if it is either monotonically increasing or monotonically decreasing.

   Show that every sequence of real numbers has a monotone subsequence.

   Proof. For every $\{x_i\}$ subsequence of $\mathbb{R}^n$, $\{x_i\}$ either has a largest element or not. That is, there exists some x_j such that $x_j \geq x_i$ for all x_i, or there does not exists such x_j.

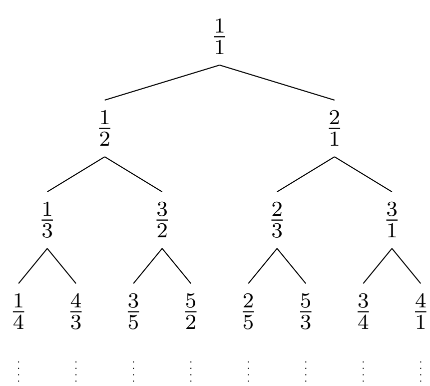
   Case I: such a x_j does not exist, therefore the sequence is unbounded and finding a subsequence is trivial. Simply define

   $$\varphi(i) = \begin{cases} x_i & x_i \geq x_j \text{ for all } j \leq i \\ x_{i-1} & \text{otherwise} \end{cases}$$

   We know this sequence will continue on forever, as $\{x_i\}$ is unbounded above. Thus we can find a monotonically increasing subsequence.

   Case I: such a x_j does exist. Now our argument becomes somewhat recursive. Look at the entire sequence $\{x_i\}$ such that $i > j$. That is, the entire sequence of $\{x_i\}$ after x_j. Now we can repeat the same logic, either there is a maximum of the new sequence or there is not. If there exists no maximum, then b *Case I* we have a monotonically increasing subsequence. If there is a maximum we continue to repeat. If there is always a maximum, what we get is a sequence of maximums we denote $\{m_1, m_2, ..., m_j\}$. However, we know that each maximum must be less than or equal to the previous by the definition of max. Therefore, this sequence of maximums is monotonically decreasing. Hence, we have found a monotonic subsequence of $\{x_i\}$, showing that a sequence of real numbers $\{x_i\}$ will *always* have a monotone subsequence. $\square$

2. The following tree (LaTeX forest!) is constructed inductively using the following rule. Begin with $\frac{1}{1}$ at the first level. Suppose that the m-th level of the tree has been constructed with 2^{m-1} nodes, and each node consists of a fraction of $\frac{i}{j}$. Then each of the 2^{m-1} entries at the m-th level in the tree produces two children, a left child and a right child; the 2^m nodes at the $(m+1)$-th level consist of these. For $\frac{i}{j}$ at the m-th level of the tree, the left child is $\frac{i}{i+j}$ and the right child is $\frac{i+j}{j}$.



Prove that every fraction $\frac{m}{n}$ that appears in the three is in lowest terms that is, $(m, n) = 1$. (Here, (m, n) denotes the *greatest common divisor* of m and n).

Proof. We proceed by induction. First, we note that $\frac{1}{1}$ is in lowest possible terms, as $(1, 1) = 1$. Now, we take our induction hypothesis to be that $\frac{i}{j}$ is in lowest possible terms, and attempt to show that both $\frac{i}{i+j}$ and $\frac{i+j}{j}$ are also in lowest possible terms. First, we know that $(i, j) = 1$, this means that given some $x, y \in \mathbb{R}$

$$\begin{aligned} ix + jy &= 1 \\ ix + jy + (xj - xj) &= 1 \\ ix + xj + jy - xj &= 1 \\ (i + j)x + j(y - x) &= 1 \end{aligned}$$

Therefore $i + j$ and j are coprime, meaning that $(i + j, j) = 0$. Hence, $\frac{i+j}{i}$ is in lowest terms. Additionally we can compute

$$\begin{aligned} ix + jy &= 1 \\ ix + jy + (yi - yi) &= 1 \\ ix - yj + jy + yi &= 1 \\ i(x - y) + (i + j) &= 1 \end{aligned}$$

Once again, we see similarly that i and $i + j$ are coprime and that $(i, i + j) = 1$. Meaning also that $\frac{i}{i+j}$ is in lowest terms.

By induction we can therefore conclude that every element of the tree will be in lowest terms. $\square$