

8. Recall from the discussion on pp.18-20 of Lang that an *open ball* of radius r about $x \in \mathbb{R}^n$ consists of the set of points $y \in \mathbb{R}^n$ such that $\|y - x\| < r$. We denote the open ball of radius r about x by $B(x, r)$. A set U of points in \mathbb{R}^n is said to be *open* if and only if for each $x \in U$ there exists some $r > 0$ such that $B(x, r) \subseteq U$; that is, for some radius $r > 0$ the open ball $B(x, r)$ about x lies entirely in the set U , and as such, contains only points of U .

- (a) For $n = 1$, show that the set of open balls in $\mathbb{R}^1 = \mathbb{R}$ is precisely the set of open intervals (a, b) , with $a < b$.

We must show that for any one interval there exists a corresponding open ball. If we take any interval (a, b) where $a < b$, then we can also create an analogous open ball. Let the center of the ball $P = (a + b)/2$ so that P is the midpoint of the interval (a, b) . Now, let the radius of the ball $r = b - a$. We know that $a < b$ so $b - a$ must express the radius of the interval (a, b) . We have thus shown that an open ball $B(P, r)$ can be created for each interval (a, b) in \mathbb{R} .

- (b) Show that an open interval in \mathbb{R} is open, according to the definition above.

Pick any point $p \in (a, b)$ where $a, b \in \mathbb{R}$ and $a < b$ this will be the center of our open ball. Now, P must be closer to one of the two endpoints, so we take $R = \min(a - p, b - p)$. Thus, R represents the distance between p and the closer endpoint of the interval (a, b) . Hence, we can simply take the radius of our ball $r = R/2$. We can now construct a ball $B(p, r)$ at any point p along any interval (a, b) . Thus, any interval (a, b) in \mathbb{R} is open.

- (c) Show that, in general, an open ball in \mathbb{R}^n is open, according to the definition above.

Pick some point p such that $p \in B(c, R)$ where c is the center of the open ball in \mathbb{R}^n and R is the radius of the open ball. Now, the distance between p and c must be less than R by the definition of a ball. We can take $d = \|p - c\|$, and we know that $d < R$. Now we simply take the number $r' = |d - R|$ which is the shortest distance between p and the edge of the open ball. Take $r = r'/2$ and we have a distance we know must be less than the distance from p to the edge of the circle. Thus, all points in $B(p, r)$ must also fall within $B(c, R)$ and $B(c, R)$ must be open.

- (d) Show that the empty set and \mathbb{R}^n are both open sets in \mathbb{R}^n .

Let's begin with the empty set. A set is open if all points within the set can have a ball $B(p, r)$ constructed about them such that all points in $B(p, r)$ are also in the original set. As the empty set has no constituents, it satisfies this requirement. There are no elements within the empty set that violate the theorem. Now for \mathbb{R}^n , any point $p \in \mathbb{R}^n$ can be matched with an r to create an open ball $B(p, r)$ of which every element is also within \mathbb{R}^n . As there is no possible way to construct an open ball $B(p, r)$ where $p \in \mathbb{R}^n$ and r is some real number, \mathbb{R}^n must be open.

- (e) Show that a set consisting of a single point in \mathbb{R}^n is not open. More generally, show that a non-empty set containing finitely many points is not open.

For any set of numbers that has finitely many elements, we must be able to select some element p in our set of numbers and some number r that can be infinitely large, such that the open ball $B(p, r)$ will contain elements that are not within the finite set we have been given.

- (f) Show that if U_1 and U_2 are open sets in \mathbb{R}^n , then $U_1 \cap U_2$ must also be open.

Let's take $W = U_1 \cap U_2$, and some point $P \in W$. Now, as P is in U_1 and U_1 is open, there exists some open ball $B(P, r_1)$ such that all points in $B(P, r_1)$ are also in U_1 . Furthermore, as P is in U_2 and U_2 is open, there also exists some open ball $B(P, r_2)$ such that all points in $B(P, r_2)$ are in U_2 . Now,

we simply take $r = \min(r_1, r_2)$, so that all points in $B(p, r)$ are necessarily in U_1 as well as U_2 . Thus, $W = U_1 \cap U_2$ is open. Additionally, if $W = \emptyset$ then W is still open as we've proved that the empty set is open.

- (g) Show that the intersection of any finite collection of open sets is also open in \mathbb{R}^n .

This is actually fairly simple. We've already shown above that the intersection of any two open sets in \mathbb{R}^n is open. Now, we simply expand to any finite number of open sets. If we have n open sets, we can repeat the process above to get $r_1, r_2, \dots, r_{n-1}, r_n$ for different radii of open balls around P . Now we can take $\min(r_1, r_2, \dots, r_{n-1}, r_n)$ to get the smallest radius r and construct the open ball $B(P, r)$ such that all points in $B(P, r)$ are also in all of the finite open sets and thus in their intersection.

- (h) Show that the intersection of infinitely many open sets needs not be open.

Consider $\lim_{n \rightarrow 0} x | x \in (-n, n)$. Once we allow for there to be infinitely many open sets, their intersection becomes infinitely specific. In this case the end result will be a set $S = 0$ which would of course be a closed set. Hence, not all intersections of infinitely many open sets need be open.

- (i) Show that a line is not open in \mathbb{R}^2 .

Take some point on the line P with coordinates (a, b) . We can write the equation of the line parametrically as $X = P + Mt$. Now we construct any open ball $B(P, r)$ about P . There must exist some point $Z = (a + r/2, b)$. This point cannot exist on the line unless $M = (1, 0)$ or some scalar multiple, if this is the case instead take $Z = (a, b + r/2)$. Either way, the point is that we can find a point within $B(P, r)$ for any $r > 0$ that is not on the line. Thus, the line in \mathbb{R}^2 must not be open.

- (j) Show that a plane is not open in \mathbb{R}^2 .

Take some point P to be on the plane $ax + by + cz = d$ with normal vector $N = (a, b, c)$. Now, we can construct any open ball $B(P, r)$ of any $r > 0$. We can also create a unit vector $U = \frac{N}{\|N\|}$ so that U is in the direction N . Now we find the point $P' = P + U * (r/2)$. This point P' cannot be in the given plane because it has been shifted up by a scalar of the normal vector, but it is still within the open ball $B(P, r)$, thus we have shown that for any $r > 0$ there will be a point in $B(P, r)$ not contained within the plane. Hence, the plane is not open.