Advanced Topics Campbell Phalen
Fields and Ordered Fields Exercises October 4, 2018

Exercise 3. Let K be an ordered field and $x, y, z \in K$. Prove the following statements.

(a) $0 \succ x$, if and only if $-x \prec 0$

Suppose that

$$0 \prec x$$

then

$$\begin{aligned} 0 &\prec x \\ 0 - x &\prec x - x \\ -x &\prec 0 \end{aligned}$$

To show the converse, assume

$$-x \prec 0$$

then, we compute

$$\begin{aligned} -x &\prec 0 \\ -x + x &\prec 0 + x \\ 0 &\prec x \end{aligned}$$

Hence we can conclude that $0 \prec x \iff -x \prec 0$

(b) If $x \succ 0$ and $y \prec z$, then $x \cdot z \prec x \cdot y$.

Given that $x \succ 0$ and $y \prec z$, we begin with

$$\begin{aligned} y &\prec z \\ y - y &\prec z - y \\ 0 &\prec z - y \end{aligned}$$

As we know $0 \prec z - y$ and $0 \prec x$, by (FO-4) the following is deduced

$$\begin{aligned} 0 &\prec x(z - y) \\ 0 &\prec x \cdot z - (x \cdot y) \\ 0 + x \cdot y &\prec x \cdot z - (x \cdot y) + x \cdot y \\ x \cdot y &\prec x \cdot z \end{aligned}$$

(c) If $x \prec 0$ and $y \prec z$, then $x \cdot z \prec x \cdot y$.

Given that $x \prec 0$ and $y \prec z$, we know that $(-x) \succ 0$, thus by part (b),

$$\begin{aligned} y \cdot (-x) &\prec z \cdot (-x) \\ -(y \cdot x) &\prec -(z \cdot x) \\ -(y \cdot x) + y \cdot x + z \cdot x &\prec -(z \cdot x) + y \cdot x + z \cdot x \\ z \cdot x &\prec y \cdot x \end{aligned}$$

(d) If $x \neq 0$, then $x^2 \succ 0$; in particular $1 \succ 0$.

By trichotomy (FO-1), we have two cases.

Case 1: $x \succ 0$, then by part (b)

$$0 \prec x$$
$$0 \cdot x \prec x \cdot x$$
$$0 \prec x^2$$

Case 2: $x \prec 0$, then $-x \prec 0$, and by part (b)

$$0 \prec (-x)$$
$$0 \cdot (-x) \prec (-x) \cdot (-x)$$
$$0 \prec (-x)^2$$
$$0 \prec x^2$$

Thus $x^2 \succ 0$. In a special case, if $x = 1$, then $1^2 = 1 \succ 0$. Thus, we know that $1 \succ 0$.

(e) If $0 \prec x \prec y$, then $0 \prec \frac{1}{y} \prec \frac{1}{x}$

We begin by showing that for any $x \succ 0$, $x^{-1} \succ 0$. We show this by counter example, if $x^{-1} \prec 0$, then $x \cdot x^{-1}$ would be negative, but by definition $x * x^{-1} = 1$ and we've just shown that $1 \succ 0$.

Now, as $x, y \succ 0$, $x^{-1}, y^{-1} \succ 0$. Thus, $x^{-1} \cdot y^{-1} > 0$, and we can do the following computation

$$0 \prec x \prec y$$
$$0 \cdot (x^{-1} \cdot y^{-1}) \prec x \cdot (x^{-1} \cdot y^{-1}) \prec y \cdot (x^{-1} \cdot y^{-1})$$
$$0 \prec 1 \cdot y^{-1}) \prec 1 \cdot x^{-1}$$
$$0 \prec y^{-1} \prec x^{-1}$$
$$0 \prec \frac{1}{y} \prec \frac{1}{x}$$

Exercise 4.

(a) Give a definition of **lower bound** for a non-empty subset of an ordered field.

A lower bound for a non-empty subset of A of an ordered field K is any $x \in K$ such that $x \preceq a$ for all $a \in A$.

(b) Define the **greatest least bound** o fa non-empty subset of an ordered field.

The greatest upper bound for a non-empty subset A of an ordered field K is any $\alpha \in K$ such that α is a lower bound of A, and no member of $x \in K$ such that $\alpha \prec x$ is a lower bound of A.

(c) Define what it means for an ordered field to have the **greatest lower bound property**.

An ordered field K is said to have the greatest lower bound property if and only if every non-empty set of K that is bounded below has a greatest lower bound.

Exercise 5.

(a) Let A be a non-empty subset of an ordered field K. Show that if α is a least upper bound of A, then for every $x \in K$ such that $x \prec \alpha$, there is some $a \in A$, such that $x \prec a \preceq \alpha$.

As $x \prec \alpha$ there must be some value, $a \in A$ that lies between x and $alpha$. We know this to be true, because if such a space did not exist, then x would be an upper bound of A, as there is no value in A greater than x. However, x cannot be an upper bound, as $x \prec \alpha$ and α is the least upper bound of A.

(b) Show that if a subset, A of an ordered field K has a least upper bound then the upper bound is unique.

We can prove the above with contradiction. Assume that A has two least upper bounds α and β. Then, $\alpha \prec \beta$ as α is the least of the upper bounds. Similarly, $\beta \prec \alpha$ as β is the least of the upper bounds. These two statements are in contradiction, and thus neither can be true. It thus follows that, by trichotomy, $\alpha = \beta$

Exercise 6. Show that if an ordered field K has the least upper bound property, then it also has the greatest lower bound property.

Given that K has the least upper bound property, assume A is a non-empty set of K which is bounded below. Let L denote the set of lower bounds of A. As L is a non-empty set of K, L has a least upper bound. We proceed by showing that the least upper bound of L must be the greatest lower bound of A. Let α denote the least upper bound of L and let l denote the greatest lower bound of A. If $\alpha \prec l$ then α cannot be the least upper bound of L as $l \in L$ and α must be succeed all elements in L. Likewise, if $l \prec \alpha$, then l is not the greatest lower bound of A, as α is also a lower bound of A, but succeeds l. Thus, again by trichotomy $\alpha = l$ and the greatest lower bound of any non-empty set of K bounded below must have a greatest lower bound. Hence, if K has the least upper bound property, it follows that K also has the greatest lower bound property.