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8. Recall from the discussion on pp.18-20 of Lang that an *open ball* of radius r about $x \in \mathbb{R}^n$ consists of the set of points $y \in \mathbb{R}^n$ such that $\|y - x\| < r$. We denote the open ball of radius r about x by $B(x, r)$. A set U of points in $\mathbb{R}^n$ is said to be *open* if and only if for each $x \in U$ there exists some $r > 0$ such that $B(x, r) \subseteq U$; that is, for some radius $r > 0$ the open ball $B(x, r)$ about x lies entirely in the set U, and as such, contains only points of U.

(a) For $n = 1$, show that the set of open balls in $\mathbb{R}^1 = \mathbb{R}$ is precisely the set of open intervals (a, b), with $a < b$.

We must show that for any one interval there exists a corresponding open ball. If we take any interval (a, b) where $a < b$, then we can also create an analogous open ball. Let the center of the ball $P = \frac{(a+b)}{2}$ so that P is the midpoint of the interval (a, b). Now, let the radius of the ball $r = b - a$. We know that $a < b$ so $b - a$ must express the radius of the interval (a, b). We have thus shown that an open ball $B(P, r)$ can be created for each interval (a, b) in $\mathbb{R}$

(b) Show that an open interval in $\mathbb{R}$ is open, according to the definition above.

Pick any point $p \in (a, b)$ where $a, b \in \mathbb{R}$ and $a < b$ this will be the center of our open ball. Now, P must be closer to one of the two endpoints, so we take $R = min(a - p, b - p)$. Thus, R represents the distance between p and the closer endpoint of the interval (a, b). Hence, we can simply take the radius of our ball $r = \frac{R}{2}$. We can now construct a ball $B(p, r)$ at any point p along any interval (a, b). Thus, any interval (a, b) in $\mathbb{R}$ is open.

(c) Show that, in general, an open ball in $\mathbb{R}^n$ is open, according to the definition above.

Pick some point p such that $p \in B(c, R)$ where c is the center of the open ball in $\mathbb{R}^n$ and R is the radius of the open ball. Now, the distance between p and c must be less than R by the definition of a ball. We can take $d = \|p - c\|$, and we know that $d < R$. Now we simply take the number $r' = |d - R|$ which is the shortest distance between p and the edge of the open ball. Take $r = \frac{r'}{2}$ and we have a distance we know must be less than the distance from p to the edge of the circle. Thus, all points in $B(p, r)$ must also fall within $B(c, R)$ and $B(c, R)$ must be open.

(d) Show that the empty set and $\mathbb{R}^n$ are both open sets in $\mathbb{R}^n$.

Let's begin with the empty set. A set is open if all points within the set can have a ball $B(p, r)$ constructed about them such that all points in $B(p, r)$ are also in the original set. As the empty has not constituents, it satisfies this requirement. There are no elements within the empty set that violate the theorem. Now for $\mathbb{R}^n$, any point $p \in \mathbb{R}^n$ can be matched with and r to create a open ball $B(p, r)$ of which every element is also within $\mathbb{R}^n$. As there is no possible way to construct an open ball $B(p, r)$ where $p \in \mathbb{R}^n$ and r is some real number, $\mathbb{R}^n$ must be open.

(e) Show that a set consisting of a single point in $\mathbb{R}^n$ is not open. More generally, show that a non-empty set containing finitely many points is not open.

For any set of numbers that has finitely many elements, we must be able to select some element p in our set of numbers and some number r that can be infinitely large, such that the open ball $B(p, r)$ will contain elements that are not within the finite set we have been given.

(f) Show that if U_1 and U_2 are open sets in $\mathbb{R}^n$, then $U_1 \cap U_2$ must also be open.

Let's take $W = U_1 \cap U_2$, and some point $P \in W$. Now, as P is in U_1 and U_1 is open, there exists some open ball $B(P, r_1)$ such that all points in $B(P, r_1)$ are also in U_1. Furthermore, as P is is in U_2 and U_2 is open, there also exists some open ball $B(P, r_2)$ such that all points in $B(P, r_2)$ are in U_2. Now, we simply take $r = min(r_1, r_2)$, so that all points in $B(p, r)$ are necessarily in U_1 as well as U_2. Thus, $W = U_1 \cap U_2$ is open. Additionally, if $W = \emptyset$ then W is still open as we've proved that the empty set is open.

(g) Show that the intersection of any finite collection of open sets is also open in $\mathbb{R}^n$.

This is actually fairly simple. We've already shown above that the intersection of any two open sets in $\mathbb{R}^n$ is open. Now, we simple expand to any finite number of open sets. if we have n open sets, we can repeat the process above to get $r_1, r_2, ..., r_{n-1}, r_n$ for different radii of open balls around P. Now we can take $min(r_1, r_2, ..., r_{n-1}, r_n)$ to get the smallest radius r and construct the open ball $B(P, r)$ such that all points in $B(P, r)$ are also in all of the finite open sets and thus in their intersection.

(h) Show that the intersection of infinitely many open sets needs not be open.

Consider $\lim_{n\to 0} x|x \in (-n, n)$. Once we allow for there to be infinitely many open sets, their intersection becomes infinitely specific. In this case the end result will be a set $S = 0$ which would of course be a closed set. Hence, not all intersections of infinitely many open sets need be open.

(i) Show that a line is not open in $\mathbb{R}^2$.

Take some point on the line P with coordinates (a, b). We can write the equation of the line parametrically as $X = P + Mt$. Now we construct any open ball $B(P, r)$ about P. There must exist some point $Z = (a + \frac{r}{2}, b)$. This point cannot exist on the line unless $M = (1, 0)$ or some scalar multiple, if this is the case instead take $Z = (a, b + \frac{r}{2})$. Either way, the point is that we can find a point within $B(P, r)$ for any $r > 0$ that is not on the line. Thus, the line in $\mathbb{R}^2$ must not be open.

(j) Show that a plane is not open in $\mathbb{R}^2$.

Take some point P to be on the plane $ax + by + cz = d$ with normal vector $N = (a, b, c)$. Now, we can construct any open ball $B(P, r)$ of any $r > 0$. We can also create a unit vector $U = \frac{N}{\|N\|}$ so that U is in the direction N. Now we find the point $P' = P + U \cdot (\frac{r}{2})$. This point P‘ cannot be in the given plane because it have been shifted up by a scalar of the normal vector, but it is still within the open ball $B(P, r)$, thus we have shown that for any $r > 0$ there will be a point in $B(P, r)$ not contained within the plane. Hence, the plane is not open.