Math 2100 – Final Project

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Observe that

$$\begin{aligned}
S_R' &= \epsilon_{2,1}S_RS_S - \epsilon_{1,2}S_RS_S - \delta_1 S_R - \zeta_1 S_R Z \\
S_S' &= \epsilon_{1,2}S_RS_S + \epsilon_{3,2}S_SS_C - \epsilon_{2,1}S_RS_S - \epsilon_{2,3}S_SS_C - \delta_2 S_S - \zeta_2 S_S Z \\
S_C' &= \epsilon_{2,3}S_SS_C - \epsilon_{3,2}S_SS_C - \delta_3 S_C - \zeta_3 S_S Z \\
Z' &= \zeta_1 S_R Z + \zeta_2 S_S Z + \zeta_3 S_C Z \\
C &= S_R + S_S + S_C + Z
\end{aligned}$$

where C is some constant representing the sum of all populations. Because there is no birth rate and no death rate, C will be constant.

In an equilibrium state, we would see

$$\begin{pmatrix} \epsilon_{2,1}S_RS_S - \epsilon_{1,2}S_RS_S - \delta_1 S_R - \zeta_1 S_R Z \\ \epsilon_{1,2}S_RS_S + \epsilon_{3,2}S_SS_C - \epsilon_{2,1}S_RS_S - \epsilon_{2,3}S_SS_C - \delta_2 S_S - \zeta_2 S_S Z \\ \epsilon_{2,3}S_SS_C - \epsilon_{3,2}S_SS_C - \delta_3 S_C - \zeta_3 S_C Z \\ \zeta_1 S_R Z + \zeta_2 S_S Z + \zeta_3 S_C Z \\ S_R + S_S + S_C + Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ C \end{pmatrix}$$

Solving this system of equations yields a number of non-trivial solutions

$$S_R + S_c = C,\ S_S = 0,\ Z = 0$$

In this equilibrium is when all the population is split between the raiders and the citizens, the zombies are eradicated though.

$$S_R = 0,\ S_S = C,\ S_c = 0,\ Z = 0$$

For this equilibrium, all of the population falls into the suvivialist group, the zombies are again ereadicated.

$$S_R = 0,\ S_S = 0,\ S_c = 0,\ Z = C$$

Here, the entire population is infected and become zombeis.

$$S_R = \frac{C\zeta_2}{\epsilon_{1,2} - \epsilon_{2,1} - \zeta_1 + \zeta_2},\ S_S = \frac{C\zeta_1}{-\epsilon_{1,2} + \epsilon_{2,1} + \zeta_1 - \zeta_2},\ S_C = 0,\ Z = \frac{C(\epsilon_{1,2} - \epsilon_{2,1})}{\epsilon_{1,2} - \epsilon_{2,1} - \zeta_1 + \zeta_2}$$

Perhaps a more intruiging equilibrium, population is split between raider, survivalist, and zombie populations. However, this is actually an invalid equilibrium because it does not match the constraint that $S_R, S_S, S_C, Z \geq 0$. To demonstrate this, observe that the denominator for S_R is

$$\epsilon_{1,2} - \epsilon_{2,1} - \zeta_1 + \zeta_2$$

while the demoninator for S_S is

$$-\epsilon_{1,2} + \epsilon_{2,1} + \zeta_1 - \zeta_2$$

because these are additive inverse, we know that one must be positive and the other negative. Therefore one of S_R or S_S must be negative, meaning we break the constraint of all populations being positive.

$$S_R = \frac{C\zeta_3}{\zeta_3 - \zeta_1},\ S_S = 0,\ S_C = \frac{C\zeta_1}{\zeta_1 - \zeta_3},\ Z = 0$$

This next equilibrium has the same issue, $\zeta_3 - \zeta_1$ and $\zeta_1 - \zeta_3$ are additive inverses so one must be negative, resulting in S_R or S_C being negative.

$$S_R = \frac{C(\epsilon_{2,3} - \epsilon_{3,2})}{\epsilon_{1,2} - \epsilon_{2,1} + \epsilon_{2,3} - \epsilon_{3,2}},\ S_S = 0,\ S_C = \frac{C(\epsilon_{1,2} - \epsilon_{2,1})}{\epsilon_{1,2} - \epsilon_{2,1} + \epsilon_{2,3} - \epsilon_{3,2}},\ Z = 0$$

which is our final equilibrium that is valid (for certain values of $\epsilon_{1,2}, \epsilon_{2,1}, \epsilon_{2,3}, \epsilon_{3,2}$). Not qutie sure yet what those values are or what this equilibrium looks like but we can definitely look into it. Now for stability analysis. The Jacobian will be

$$\begin{pmatrix} \epsilon_{2,1}S_S - \epsilon_{1,2}S_S - \delta_1 - \zeta_1 Z & \epsilon_{2,1}S_R - \epsilon_{1,2}S_R & 0 & \zeta_1 S_R \\ \epsilon_{1,2}S_S - \epsilon_{2,1}S_S & \epsilon_{1,2}S_R + \epsilon_{3,2}S_C - \epsilon_{2,1}S_R - \epsilon_{2,3}S_C - \delta_2 - \zeta_2 Z & \epsilon_{3,2}S_S - \epsilon_{3,2}S_S & \zeta_2 S_S \\ 0 & \epsilon_{2,3}S_C - \epsilon_{3,2}S_C & \epsilon_{2,3}S_S - \epsilon_{3,2}S_S - \delta_3 - \zeta_3 Z & \zeta_3 S_C \\ \zeta_1 Z & \zeta_2 Z & \zeta_3 Z & \zeta_1 S_R + \zeta_2 S_S + \zeta_3 S_C \end{pmatrix}$$

Plugging in

$$S_R = \frac{C(\epsilon_{2,3} - \epsilon_{3,2})}{\epsilon_{1,2} - \epsilon_{2,1} + \epsilon_{2,3} - \epsilon_{3,2}},\ S_S = 0,\ S_C = \frac{C(\epsilon_{1,2} - \epsilon_{2,1})}{\epsilon_{1,2} - \epsilon_{2,1} + \epsilon_{2,3} - \epsilon_{3,2}},\ Z = 0$$

we have

$$\begin{pmatrix} \delta_1 - \zeta_1 Z & \epsilon_{2,1}S_R - \epsilon_{1,2}S_R & 0 & \zeta_1 S_R \\ 0 & \epsilon_{1,2}S_R + \epsilon_{3,2}S_C - \epsilon_{2,1}S_R - \epsilon_{2,3}S_C - \delta_2 - \zeta_2 Z & 0 & 0 \\ 0 & \epsilon_{2,3}S_C - \epsilon_{3,2}S_C & \delta_3 - \zeta_3 Z & \zeta_3 S_C \\ \zeta_1 Z & \zeta_2 Z & \zeta_3 Z & \zeta_1 S_R + \zeta_3 S_C \end{pmatrix}$$

which further simplifies to

$$\begin{pmatrix} -\delta_1 & \frac{C(\epsilon_{1,2}-\epsilon_{2,1})(\epsilon_{2,3}-\epsilon_{3,2})}{\epsilon_{1,2}-\epsilon_{2,1}+\epsilon_{2,3}-\epsilon_{3,2}} & 0 & \frac{C\zeta_1(\epsilon_{2,3}-\epsilon_{3,2})}{\epsilon_{1,2}-\epsilon_{2,1}+\epsilon_{2,3}-\epsilon_{3,2}} \\ 0 & \frac{2C(\epsilon_{1,2}-\epsilon_{2,1})(\epsilon_{2,3}-\epsilon_{3,2})}{\epsilon_{1,2}-\epsilon_{2,1}+\epsilon_{2,3}-\epsilon_{3,2}} - \delta_2 & 0 & 0 \\ 0 & \frac{C(\epsilon_{1,2}-\epsilon_{2,1})(\epsilon_{2,3}-\epsilon_{3,2})}{\epsilon_{1,2}-\epsilon_{2,1}+\epsilon_{2,3}-\epsilon_{3,2}} & -\delta_3 & \frac{C\zeta_3(\epsilon_{1,2}-\epsilon_{2,1})}{\epsilon_{1,2}-\epsilon_{2,1}+\epsilon_{2,3}-\epsilon_{3,2}} \\ 0 & 0 & 0 & C \cdot \frac{\zeta_1(\epsilon_{1,2}-\epsilon_{2,1})+\zeta_3(\epsilon_{2,3}-\epsilon_{3,2})}{\epsilon_{1,2}-\epsilon_{2,1}+\epsilon_{2,3}-\epsilon_{3,2}} \end{pmatrix}$$

this matrix has eigenvalues

$$-\delta_1$$

$$-\delta_3$$

$$\frac{-\delta_2\epsilon_{1,2} + \delta_2\epsilon_{2,1} - \delta_2\epsilon_{2,3} + 2C\epsilon_{1,2}\epsilon_{2,3} - 2C\epsilon_{2,1}\epsilon_{2,3} + \delta_2\epsilon_{3,2} - 2C\epsilon_{1,2}\epsilon_{3,2} + 2C\epsilon_{2,1}\epsilon_{3,2}}{\epsilon_{1,2} - \epsilon_{2,1} + \epsilon_{2,3} - \epsilon_{3,2}}$$

$$\frac{C(\epsilon_{1,2}\zeta_1 - \epsilon_{2,1}\zeta_1 + \epsilon_{2,3}\zeta_3 - \epsilon_{3,2}\zeta_3)}{\epsilon_{1,2} - \epsilon_{2,1} + \epsilon_{2,3} - \epsilon_{3,2}}$$