BIMS 8601 Homework #1: Due in class Tuesday, February 24, 2022

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1. Problem 1 (25 points):

   (a) Show that the Poisson distribution can be derived as the limit of the binomial distribution as the number of trials, n, approaches infinity and the probability of success on each trial, p approaches zero in such a way that $np = \lambda$. (Hint: For $n \to \infty$ use Sterling's approximation $n! \sim n^n e^{-n}\sqrt{2\pi n}$ and $(1 - \frac{\lambda}{n})^n \to e^{-\lambda}$.)

   (b) Show that the normal distribution can be derived as the limit of the binomial distribution as the number of trials, n, approaches infinity and the probability of success on each trial, p is not close to 0 or 1. Steps of derivation: (1) apply Sterling's approximation (2) collect terms and take the natural log of both sides of the equation (3) use a change of variables $\delta = k - np$ (4) apply the expansion $\ln(1 + x) \sim x - \frac{1}{2}x^2$ (5) assume δ is much smaller than n keeping lowest order terms and (6) apply the exponential function to both sides of the equation.

2. Problem 2 (25 points): Let $Y = \sum_i^n X_i$ where the X_i are independent Bernoulli random variables that take the value 1 and 0 with probability p and $1 - p$, respectively. Calculate the expectation and variance of Y: $E(Y)$ and $Var(Y)$. How do you expect Y to be distributed? Briefly explain your answer and why this makes sense.

3. Problem 3 (25 points): Calculate the expectation and variance of a random variable assumed to follow the (a) Poisson and (b) normal distribution. Feel free to use the moment generating function for the Poisson distribution $M(t) = e^{\lambda(e^t-1)}$ and the normal distribution $M(t) = e^{\mu t}e^{\sigma^2 t^2/2}$.

4. Problem 4 (25 points): Prove the Central Limit Theorem by showing that the moment generating function of $Z_n = S_n/\sigma\sqrt{n}$ (where $S_n = \sum_{i=1}^n X_i$ and the X_i are independent random variables with mean 0 and variance σ^2 and moment generating function $M(t)$) tends to the moment generating function of the standard normal distribution as $n \to \infty$. Execute the following steps:

   (a) Calculate the moment generating function of Z_n, $M_{Z_n}(t)$, in terms of the moment generating function of the X_i, $M(t)$. Use the rules of how to calculate moment generating functions of sums of random variables and linear functions of random variables shown in class.

   (b) Use the fact that $M(s)$ has a Taylor series expansion $M(s) = M(0) + sM'(0) + \frac{1}{2}s^2 M''(0) + \cdots$ and the definition of expectation and variance in terms of $M'(0)$ and $M''(0)$, respectively.

   (c) Use the fact that as $n \to \infty$, $(1 + \frac{\lambda}{n})^n \to e^{\lambda}$.

   (d) Arrive at the final result, $M_{Z_n}(t) \to e^{t^2/2}$ as $n \to \infty$.