Random Matrix

Dibakar Sigdel

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A simple example:Generating random points on a sphere

Consider a two dimensional unit sphere ($\mathbb{S}^2$) imbeded in the three dimensional Eulidean space ($\mathbb{R}^3$). Let x, y and z be the coordinate axes. Then the equation of sphere is

$$x^2 + y^2 + z^2 = 1. \tag{1}$$

One can parametrize $(x, y, z) \in \mathbb{S}^2$ using spherical coordinates as follows:

$$\begin{aligned} x &= \sin\theta\cos\phi, \\ y &= \sin\theta\sin\phi, \\ z &= \cos\theta. \end{aligned} \tag{2}$$

Where $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$.

Since every surface of dimension $k \leq N$ embeded in an N dimentional Eucledian space inherits a natural Riemannian metric from the underlying Euclidean structure. Taking Euclidean space as $(x_1, ..., x_n)$, the Rimemannian metric g_{ml} on the surface parametrized by coordinates $(q_1, ...q_k; k \leq N)$ is defined from the Eucledian length element according to

$$dS^2 = \sum_{i=1}^{N}(dx_i)^2 = \sum_{i=1}^{N}\left(\sum_{i=1}^{N}\frac{\partial x_i}{\partial q_m}dq_m\right)^2 = \sum_{m,l=1}^{N} g_{mn}dq_m dq_n. \tag{3}$$

Thus for $\mathbb{S}^2$, $x_i = x, y, z$ and $q_m = \theta, \phi$; The length element is geven by

$$dS^2 = \sin^2\theta d\theta^2 + d\phi^2, \tag{4}$$

alongwith

$$g_{mn} = \begin{pmatrix} \sin^2\theta & 0 \\ 0 & 1 \end{pmatrix}, \tag{5}$$

$$|g| = \sin^2\theta. \tag{6}$$

Now, the volume element on $\mathbb{S}^2$ becomes a surface element

$$d\mu = \sqrt{|g|}d\theta d\phi = \sin\theta d\theta d\phi = -d(\cos\theta)d\phi. \tag{7}$$

This is a integration measure $d\mu$ on $\mathbb{S}^2$ induced by Riemannian metric which is called **Haar measure** on $\mathbb{S}^2$.

Clearly one can not generate the the random points uniformly distributed on the sphere simply by generating angles θ and ϕ within the domains defined above. Rather the points appear to be "bunched" close to ploe as shown in figure below.

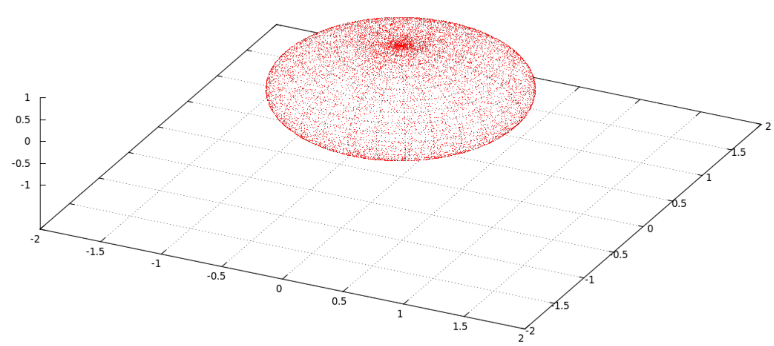

Figure 1: Random points bunched close to poles due to randomly generated θ and ϕ

Hence to obtain the uniform distribution over $\mathbb{S}^2$, One has to pick $\phi \in [0, \pi]$ and $t \in [-1, 1]$ uniformly at random choice and compute θ as follows:

$$\theta = \cos^{-1} t \tag{8}$$

In this way $\cos\theta$ becomes uniformly distributed in [-1,1].

Integrating the surface element in equation **??**, we get the total surface area of the sphere, 4π. Thus the probability of geting a random point within the range θ to $\theta + d\theta$ and ϕ to $\phi + d\phi$ is geven by

$$P(\theta, \phi) d\theta d\phi = \frac{\sin\theta d\theta d\phi}{4\pi} \tag{9}$$

This implies that if we plot the histogram by taking the no of random points on $\mathbb{S}^2$ in between certain range of angle θ then the histogram must look like the plot of probability function; $p(\theta, \phi) = \frac{sin\theta}{4\pi}$.

Complex matrix $GL(n, \mathbb{C})$ as a point in Euclidean $\mathbb{R}^{(2N^2)}$ space

Consider the matrix M of dimension $N \times N$, with complex entries $z_{ij} = x_{ij} + iy_{ij}$. Technically, there are N^2 number of x_{ij} coordinates and N^2 number of y_{ij} coordinates. Thus each such matrix can be conveniently looked at as a point in a $2N^2$-dimensional Euclidean space with real cartesian coordinates x_{ij} and y_{ij} with $i, j = 1, 2, ...N$ The length element in this space is defined in a standard way as:

$$dS^2 = Tr(dMdM^\dagger) = \sum_{ij} dz_{ij} d\bar{z}_{ij} = \sum_{ij} [(dx)^2_{ij} + (dy)^2_{ij}] \tag{10}$$

For example consider a matrix A as below,

$$A = \begin{pmatrix} x_1 + iy_1 & x_2 + iy_2 \\ x_3 + iy_3 & x_4 + iy_4 \end{pmatrix} \tag{11}$$

$$dA = \begin{pmatrix} dx_1 + idy_1 & dx_2 + idy_2 \\ dx_3 + idy_3 & dx_4 + idy_4 \end{pmatrix} \tag{12}$$

$$dA^\dagger = \begin{pmatrix} dx_1 - idy_1 & dx_2 - idy_2 \\ dx_3 - idy_3 & dx_4 - idy_4 \end{pmatrix} \tag{13}$$

$$Tr(dA^\dagger dA) = dx_1^2 + dy_1^2 + dx_2^2 + dy_2^2 + dx_3^2 + dy_3^2 + dx_4^2 + dy_4^2 \tag{14}$$

Random generation of SU(2) matrix

Since every Special Unitary matrix $A \in SU(N)$ has the following property:

$$\begin{aligned} A^{\dagger}A = I \\ |A| = 1. \end{aligned} \tag{15}$$

Due to this property, one can express every SU(2) matrix A as:

$$A = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}. \tag{16}$$

Where a and b are complex numbers and satisfy

$$a^2 + b^2 = 1. \tag{17}$$

One can parametrize the complex numbers a and b using the parameter θ, ϕ and ψ as :

$$\begin{aligned} a = \cos\theta e^{i\phi} \\ b = \sin\theta e^{i\psi} \end{aligned} \tag{18}$$

$$\therefore A = \begin{pmatrix} \cos\theta e^{i\phi} & \sin\theta e^{i\psi} \\ -\sin\theta e^{-i\psi} & \cos\theta e^{-i\phi} \end{pmatrix}. \tag{19}$$

Where, $0 \leq \theta \leq \frac{\pi}{2}$, $0 \leq \phi \leq 2\pi$ and $0 \leq \psi \leq 2\pi$.

Now in the space these parameters, every SU(2) matrix becomes a point with the fixed value of angles.To define the volume elements we need to find the Haar measure of SU(2) in this space. For this we first calculate the line element dS^2 and then the metric g_{mn} of the space.

The length element is given by

$$ds^2 = Tr(dA^{\dagger}dA). \tag{20}$$

Where,

$$dA = \begin{pmatrix} -\sin\theta e^{i\phi}d\theta + i\cos\theta e^{i\phi}d\phi & \cos\theta e^{i\psi}d\theta + i\sin\theta e^{i\psi}d\psi \\ -\cos\theta e^{-i\psi}d\theta + i\sin\theta e^{-i\psi}d\psi & -\sin\theta e^{-i\phi}d\theta - i\cos\theta e^{-i\phi}d\phi \end{pmatrix} \tag{21}$$

and

$$dA^\dagger = \begin{pmatrix} -\sin\theta e^{-i\phi}d\theta - i\cos\theta e^{-i\phi}d\phi & -\cos\theta e^{i\psi}d\theta - i\sin\theta e^{i\psi}d\psi \\ \cos\theta e^{-i\psi}d\theta - i\sin\theta e^{-i\psi}d\psi & -\sin\theta e^{i\phi}d\theta + i\cos\theta e^{i\phi}d\phi \end{pmatrix}. \quad (22)$$

$$\therefore (dS)^2 = Tr(dA^\dagger dA) = 2(d\theta)^2 + 2\cos^2\theta(d\phi)^2 + 2\sin^2\theta(d\psi)^2 \quad (23)$$

From this,

$$g_{mn} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2\cos^2\theta & 0 \\ 0 & 0 & 2\sin^2\theta \end{pmatrix}, \quad (24)$$

$$|g| = 8\sin^2\theta\cos^2\theta = 2(\sin 2\theta)^2. \quad (25)$$

Thus the corresponding integration measure in this case is given by the volume element

$$d\mu = \sqrt{|g|}d\theta d\phi d\psi = \sqrt{2}(\sin 2\theta)d\theta d\phi d\psi = -2\sqrt{2}d(\cos 2\theta)d\phi d\psi. \quad (26)$$

Thus it is clear that in order to generate random SU(2) matrix, one need to generate $t \in [-1, 1]$ such that

$$\theta = \frac{1}{2}\cos^{-1}(t). \quad (27)$$

Where as other parameters ϕ and θ can be generated randomly within the interval$[0, 2\pi]$. For a randomly generated SU(2) matrix, the no of matrix generated within θ to $\theta + d\theta$,ϕ to $\phi + d\phi$ and ψ to $\psi + d\psi$ is given by normalized probability distribution function,

$$P(\theta, \phi, \psi)d\theta d\phi d\psi = \frac{1}{4\pi}\sin 2\theta d\theta d\phi d\psi \quad (28)$$

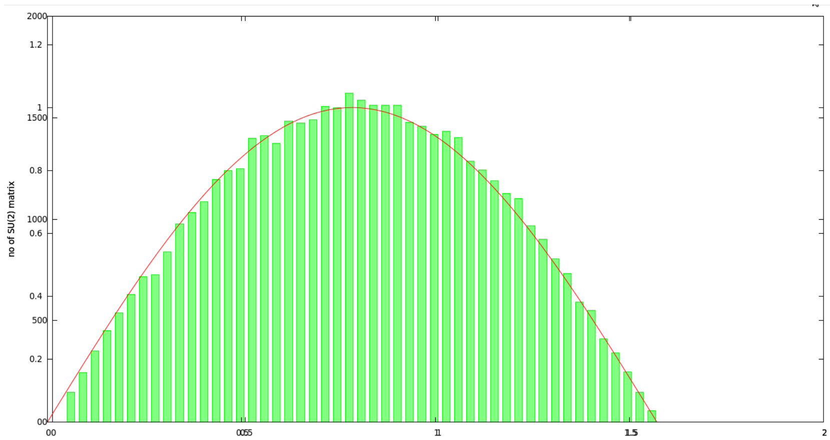

Figure 2: Histogram representing the no of random SU(2) matrix within the range of defined angle $\theta \in [0, \pi/2]$.Red curve is expected distribution function $\sin(2\theta)$.

Another look:SU(2) matrix as a point on $\mathbb{S}^3$.

Since every Special Unitary matrix $A \in SU(N)$ has to have the following property;

$$\begin{aligned} A^{\dagger}A &= I \\ |A| &= 1. \end{aligned} \tag{29}$$

One can express every SU(2) matrix A as:

$$A = \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix} \tag{30}$$

where,

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1. \tag{31}$$

Thus from above equation, it is clear that every SU(2) matrix is a point on the three dimensional unit sphere ($\mathbb{S}^3$) imbeded in the four dimensional Eulidean space ($\mathbb{R}^4$).

One can parametrize the points $(x_1, x_2, x_3) \in \mathbb{S}^3$ using the parameter θ, ϕ and ψ as :

$$x_1 = \sin\theta \sin\phi \sin\psi$$

$$
\begin{aligned}
x_2 &= \sin\theta \sin\phi \cos\psi \\
x_3 &= \sin\theta \cos\phi \\
x_4 &= \cos\theta.
\end{aligned} \tag{32}
$$

Where, $\theta, \phi \in [0, \pi]$ and $\psi \in [0, 2\pi]$

The length element on $\mathbb{S}^3$ in new coordinates θ, ϕ and ψ is given by

$$
dS^2 = (d\theta)^2 + \sin^2\theta(d\phi)^2 + \sin^2\theta \sin^2\phi(d\psi)^2. \tag{33}
$$

alongwith the metric tensor,

$$
g_{mn} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin^2\theta & 0 \\ 0 & 0 & \sin^2\theta \sin^2\phi \end{pmatrix}, \tag{34}
$$

and

$$
|g| = \sin^4\theta \sin^2\phi. \tag{35}
$$

Thus the corresponding integration measure in this case is given by the volume element

$$
d\mu = \sqrt{|g|}d\theta d\phi d\psi = \sin^2\theta \sin\phi d\theta d\phi d\psi. \tag{36}
$$

For a randomly generated SU(2) matrix, the no of matrix generated within θ to $\theta + d\theta$,ϕ to $\phi + d\phi$ and ψ to $\psi + d\psi$ is given by normalized probability distribution function,

$$
P(\theta, \phi, \psi)d\theta d\phi d\psi = \frac{1}{2\pi^2} \sin^2\theta \sin\phi d\theta d\phi d\psi \tag{37}
$$

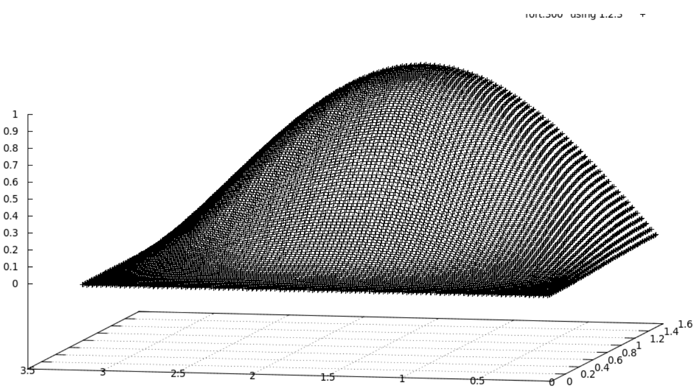

Figure 3: plot of probability distribution function $p(\theta, \phi, \psi) = \sin^2\theta \sin\phi$

Distribution of eigenvalues of random SU(2) matrix in relation: $W = DTD^{\dagger}T^{\dagger}$

Here we are going to generate random matrices W,D and T satisfying the equation $W = DTD^{\dagger}T^{\dagger}$ and we want to plot the histogram for the distribution of eigen values of corresponding matrices.

- W-matrix

Consider $W \in SU(2)$ matrix in the Paramers θ, ϕ, ψ:

$$W = \begin{pmatrix} \cos\theta e^{i\phi} & \sin\theta e^{i\psi} \\ -\sin\theta e^{-i\psi} & \cos\theta e^{-i\phi} \end{pmatrix}; \theta \in [0, \frac{\pi}{2}]; \phi, \psi \in [0, 2\pi]. \tag{38}$$

The eigen value of this SU(2) matrix is given by

$$\eta = cos^{-1}(\cos\theta\cos\alpha); \eta \in [0, 2\pi]] \tag{39}$$

We can diagonalize the matrix W by taking matrix $V \in SU(2)$ such that

$$W = V^{\dagger}\Lambda V \tag{40}$$

The probability distribution function for eigen value of W matrix is given by integration measure

$$d\mu = \sin^2(\eta) d\eta \prod_{i<j}^{2} (\delta v_{ij}) \tag{41}$$

Thus the distribution of eigen values η of random SU(2) matrix must look like $\sin^2 \eta$ function.

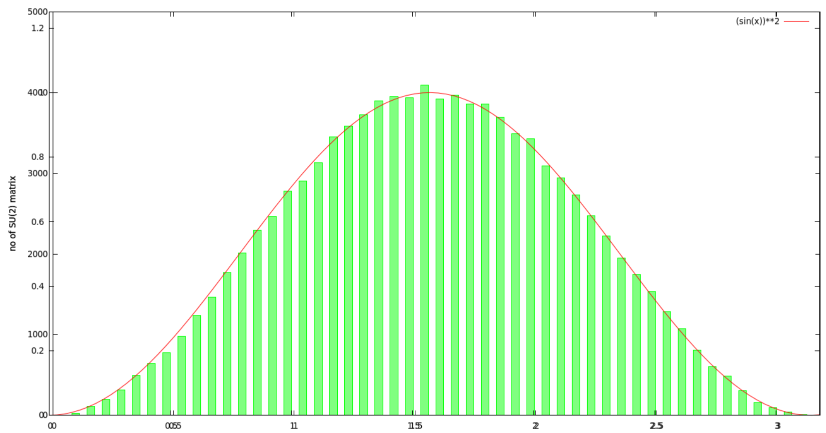

Figure 4: Histogram for distribution of eigen values of random W matrices.The solid curve is distribution function- $\sin^2 \eta$.

- D-matrix

Let the diagonal matix D be

$$D = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}; \alpha \in [0, 2\pi] \tag{42}$$

Solving equation: $W = DTD^{\dagger}T^{\dagger}$ we get

$$e^{-i2\alpha} = \frac{(1 - w_{11}^*)}{(w_{11} - 1)} \tag{43}$$

$$\alpha = \frac{i}{2} \log_e \left[\frac{(1 - w_{11}^*)}{(w_{11} - 1)} \right] \tag{44}$$

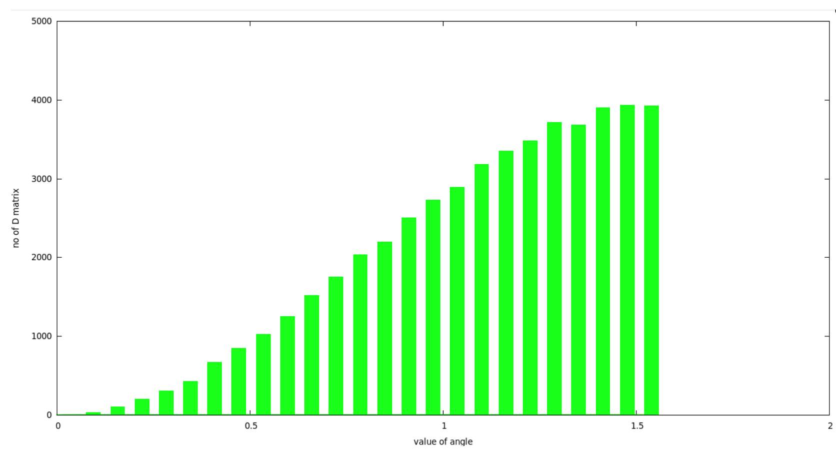

Figure 5: Histogram for distribution of eigen values of random D matrices.

- T-matrix

Similarly the T matrix is given by:

$$T = \frac{1}{\sqrt{(2 - w_{11}^* - w_{11})}} \begin{pmatrix} w_{12}e^{i\xi} & -(1 - w_{11}^*)e^{-i\xi} \\ (1 - w_{11})e^{i\xi} & w_{12}^* e^{-i\xi} \end{pmatrix}, \xi \in [0, 2\pi] \quad (45)$$

The eigenvalue of T matrix is

$$\gamma = \pm \cos^{-1}[\frac{1}{2\sqrt{(2 - w_{11}^* - w_{11})}}(w_{12}e^{i\xi} + w_{12}^* e^{-i\xi})]. \quad (46)$$

The distribution of eigen values of randomly generated T matrix is shown below:

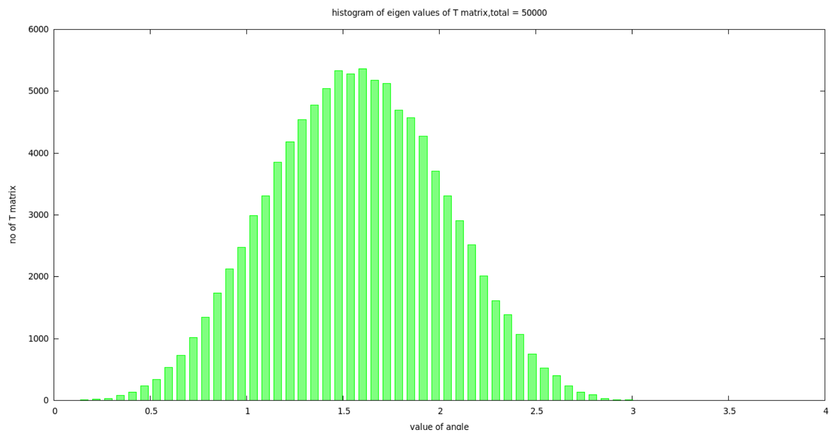

Figure 6: Histogram for distribution of eigen values of random T matrices.

Distribution of eigenvalues of Random SU(3) matrices

Let A and V be SU(3) matrices such that

$$A = V^{\dagger} D V. \tag{47}$$

Where D is a diagonal matix with the diagonal elements as the eigen values of matrix A and columns of matrix V are the corresponding eigen vectors. Let

$$D = \begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{pmatrix}. \tag{48}$$

Where

$$\theta_3 = -\theta_1 - \theta_2 \pm 2\pi k, k = 1, 2, ... \tag{49}$$

We are interested to calculate the line element

$$dS^2 = Tr(dA^{\dagger} dA). \tag{50}$$

For this,

$$dA = dV^{\dagger} DV + V^{\dagger} dDV + V^{\dagger} DdV. \tag{51}$$

since $A^{\dagger} A = I$

$$A^{\dagger}(dA) + A(dA)^{\dagger} = 0, \tag{52}$$

$$dA^\dagger = -A^\dagger (dA) A^\dagger. \tag{53}$$

$$\begin{aligned}\therefore dA^\dagger &= -(V^\dagger D V)^\dagger (dV^\dagger D V + V^\dagger dD V + V^\dagger D dV)(V^\dagger D V)^\dagger \\ &= -(V^\dagger D^\dagger V)(dV^\dagger D V + V^\dagger dD V + V^\dagger D dV)(V^\dagger D^\dagger V).\end{aligned} \tag{54}$$

$$\begin{aligned}\therefore dA^\dagger = -(&V^\dagger D^\dagger V dV^\dagger D V V^\dagger D^\dagger V \\ &+V^\dagger D^\dagger V V^\dagger dD V V^\dagger D^\dagger V \\ &+V^\dagger D^\dagger V V^\dagger D dV V^\dagger D^\dagger V).\end{aligned} \tag{55}$$

using $V^\dagger V = I, D^\dagger D = I$ we get

$$\begin{aligned}dA^\dagger = -(&V^\dagger D^\dagger V dV^\dagger V \\ &+V^\dagger D^\dagger dD D^\dagger V \\ &+V^\dagger D^\dagger D dV V^\dagger D^\dagger V).\end{aligned} \tag{56}$$

$$dA^\dagger dA = -(V^\dagger D^\dagger V dV^\dagger V + V^\dagger D^\dagger dD D^\dagger V + V^\dagger D^\dagger D dV V^\dagger D^\dagger V)(dV^\dagger D V + V^\dagger dD V + V^\dagger D dV). \tag{57}$$

Multiplying term by term we get,

$$\begin{aligned}dA^\dagger dA = -[&V^\dagger D^\dagger V dV^\dagger V dV^\dagger D V \\ &+V^\dagger D^\dagger dD D^\dagger V dV^\dagger D V \\ &+V^\dagger D^\dagger D dV V^\dagger D^\dagger V dV^\dagger D V \\ &+V^\dagger D^\dagger V dV^\dagger V V^\dagger dD V \\ &+V^\dagger D^\dagger dD D^\dagger V V^\dagger dD V \\ &+V^\dagger D^\dagger D dV V^\dagger D^\dagger V V^\dagger dD V \\ &+V^\dagger D^\dagger V dV^\dagger V V^\dagger D dV \\ &+V^\dagger D^\dagger dD D^\dagger V V^\dagger D dV \\ &+V^\dagger D^\dagger D dV V^\dagger D^\dagger V V^\dagger D dV].\end{aligned} \tag{58}$$

Again using $V^\dagger V = I, D^\dagger D = I$ and using the property of trace we get

$$\begin{aligned}Tr(dA^\dagger dA) = -Tr[&V dV^\dagger V dV^\dagger \\ &+dD D^\dagger V dV^\dagger \\ &+D dV V^\dagger D^\dagger V dV^\dagger \\ &+D^\dagger V dV^\dagger dD \\ &+D^\dagger dD D^\dagger dD \\ &+dV V^\dagger D^\dagger dD \\ &+V^\dagger D^\dagger V dV^\dagger D dV\end{aligned}$$

$$+V^\dagger D^\dagger dDdV$$
$$+V^\dagger dVV^\dagger dV]. \qquad (59)$$

Rearranging this we get,

$$\begin{aligned} Tr(dA^\dagger dA) = -Tr[(dDD^\dagger VdV^\dagger + D^\dagger VdV^\dagger dD + D^\dagger dDD^\dagger dD \\ +dVV^\dagger D^\dagger dD + V^\dagger D^\dagger dDdV) \\ +(DdVV^\dagger D^\dagger VdV^\dagger + V^\dagger D^\dagger VdV^\dagger DdV) \\ +(V^\dagger dVV^\dagger dV + VdV^\dagger VdV^\dagger)] \end{aligned} \qquad (60)$$

Now consider the term,

$$Tr[dDD^\dagger VdV^\dagger + D^\dagger VdV^\dagger dD + D^\dagger dDD^\dagger dD + dVV^\dagger D^\dagger dD + V^\dagger D^\dagger dDdV] \qquad (61)$$

since $V^\dagger V = I$,

$$V^\dagger(dV) + V(dV)^\dagger = 0, \qquad (62)$$

$$VdV^\dagger = -V^\dagger(dV). \qquad (63)$$

Also since D is diagonal matrix,

$$[D^\dagger, dD] = 0 \qquad (64)$$

Hence the first two terms and last two terms in expression at eq (**??**) cancel out and we are left with,

$$Tr[D^\dagger dDD^\dagger dD] \qquad (65)$$

Since $D_{kl} = e^{i\theta_k}\delta_{kl}$ and $dD_{kl} = ie^{i\theta_k}\delta_{kl}d\theta_k$,with $k = 1, 2, 3$

$$\therefore Tr[D^\dagger dDD^\dagger dD] = -\sum_{k=1}(d\theta_k)^2. \qquad (66)$$

Now consider the term,

$$Tr[DdVV^\dagger D^\dagger VdV^\dagger + V^\dagger D^\dagger VdV^\dagger DdV] = -Tr[DdVV^\dagger D^\dagger dVV^\dagger + V^\dagger D^\dagger dVV^\dagger DdV]. (67)$$

$$Tr[DdVV^\dagger D^\dagger dVV^\dagger] = \sum_m (DdVV^\dagger D^\dagger dVV^\dagger)_{im}\delta_{im}$$

$$
\begin{aligned}
&= \sum_{kljm} e^{i\theta_i}\delta_{ik}(dVV^\dagger)_{kl}(e^{-i\theta_j})\delta_{lj}(dVV^\dagger)_{jm}\delta_{im} \\
&= \sum_{ij} e^{i\theta_i}(dVV^\dagger)_{ij}(e^{-i\theta_j})(dVV^\dagger)_{ji} \\
&= \sum_{ij} e^{i(\theta_i-\theta_j)}(dVV^\dagger)_{ij}(dVV^\dagger)_{ji}.
\end{aligned} \tag{68}
$$

Similarly,

$$
\begin{aligned}
Tr[V^\dagger D^\dagger dVV^\dagger DdV] &= \sum_m (D^\dagger dVV^\dagger DdVV^\dagger)_{im}\delta_{im} \\
&= \sum_{kljm} e^{-i\theta_i}\delta_{ik}(dVV^\dagger)_{kl}e^{-i\theta_j}\delta_{lj}(dVV^\dagger)_{jm}\delta_{im} \\
&= \sum_{ij} e^{-i(\theta_i-\theta_j)}(dVV^\dagger)_{ij}(dVV^\dagger)_{ji}.
\end{aligned} \tag{69}
$$

Thus

$$
Tr[DdVV^\dagger D^\dagger VdV^\dagger + V^\dagger D^\dagger VdV^\dagger DdV] = -2\sum_{ij}\cos(\theta_i-\theta_j)(dVV^\dagger)_{ij}(dVV^\dagger)_{ji}. \tag{70}
$$

Now consider the term ,

$$
\begin{aligned}
Tr[V^\dagger dVV^\dagger dV + VdV^\dagger VdV^\dagger) &= 2Tr(dVV^\dagger dVV^\dagger) \\
&= 2\sum_i (dVV^\dagger dVV^\dagger)_{ik}\delta_{ik} \\
&= 2\sum_{ij}(dVV^\dagger)_{ij}(dVV^\dagger)_{jk}\delta_{ik} \\
&= 2\sum_{ij}(dVV^\dagger)_{ij}(dVV^\dagger)_{ji}.
\end{aligned} \tag{71}
$$

Thus the sum of above two traces is

$$
2\sum_{ij}(dVV^\dagger)_{ij}(dVV^\dagger)_{ji}(1-\cos(\theta_i-\theta_j)) = 4\sum_{ij}\sin^2\frac{1}{2}(\theta_i-\theta_j)(dVV^\dagger)_{ij}(dVV^\dagger)_{ji}. \tag{72}
$$

Finally the line element is given by

$$
dS^2 = \sum_i d\theta_i^2 + 4\sum_{ij}\sin^2\frac{1}{2}(\theta_i-\theta_j)(dVV^\dagger)_{ij}(dVV^\dagger)_{ji}. \tag{73}
$$

The corresponding integration measure in this space is given by

$$d\mu = \prod_{i<j}(\delta v_{ij}) \prod_i d\theta_i \prod_{i<j} \sin^2 \frac{1}{2}(\theta_i - \theta_j). \tag{74}$$

Consider the term for SU(3),

$$\prod_{i<j} \sin^2 \frac{1}{2}(\theta_i - \theta_j) = \sin^2 \frac{1}{2}(\theta_1 - \theta_2) \sin^2 \frac{1}{2}(\theta_1 - \theta_3) \sin^2 \frac{1}{2}(\theta_2 - \theta_3). \tag{75}$$

using

$$\theta_3 = -\theta_1 - \theta_2 \pm 2\pi k \tag{76}$$

$$\prod_{i<j} \sin^2 \frac{1}{2}(\theta_i - \theta_j) = \sin^2 \frac{1}{2}(\theta_1 - \theta_2) \sin^2 \frac{1}{2}(2\theta_1 + \theta_2) \sin^2 \frac{1}{2}(2\theta_2 + \theta_1). \tag{77}$$

The region of eigenvalues for zero probability of getting SU(3) matrix is given by:

$$\begin{aligned} \theta_1 - \theta_2 &= 0, \\ 2\theta_1 + \theta_2 &= 0, \\ 2\theta_2 + \theta_1 &= 0. \end{aligned} \tag{78}$$

Let

$$\begin{aligned} \alpha &= \frac{1}{\sqrt{2}}(\theta_1 - \theta_2); \\ \beta &= \frac{\sqrt{3}}{\sqrt{2}}(2\theta_1 + \theta_2). \end{aligned} \tag{79}$$

then

$$\begin{aligned} \frac{1}{2}(\theta_1 - \theta_2) &= \frac{\alpha}{\sqrt{2}}, \\ \frac{1}{2}(2\theta_1 + \theta_2) &= \frac{1}{2\sqrt{2}}\alpha + \frac{\sqrt{3}}{2\sqrt{2}}\beta, \\ \frac{1}{2}(2\theta_2 + \theta_1) &= -\frac{1}{2\sqrt{2}}\alpha + \frac{\sqrt{3}}{2\sqrt{2}}\beta. \end{aligned} \tag{80}$$

Now the product becomes

$$\sin^2(\frac{\alpha}{\sqrt{2}})\sin^2(\frac{1}{2\sqrt{2}}\alpha+\frac{\sqrt{3}}{2\sqrt{2}}\beta)\sin^2(-\frac{1}{2\sqrt{2}}\alpha+\frac{\sqrt{3}}{2\sqrt{2}}\beta)$$
$$=\sin^2(\frac{\alpha}{\sqrt{2}})[\cos(\frac{1}{\sqrt{2}}\alpha)-\cos(\frac{\sqrt{3}}{\sqrt{2}}\beta)]^2 \quad . \tag{81}$$

Thus to find the resion of zero probability we need to set,

$$\alpha=0,$$
$$\frac{1}{\sqrt{2}}\alpha=\pm\frac{\sqrt{3}}{\sqrt{2}}\beta\Rightarrow\alpha=\pm\sqrt{3}\beta. \tag{82}$$

Hence altogether there are six region on a plan corresponding to $\alpha=0,+\sqrt{3}\beta,-\sqrt{3}\beta$ as shown in the figure below where α is along the vertical axis.

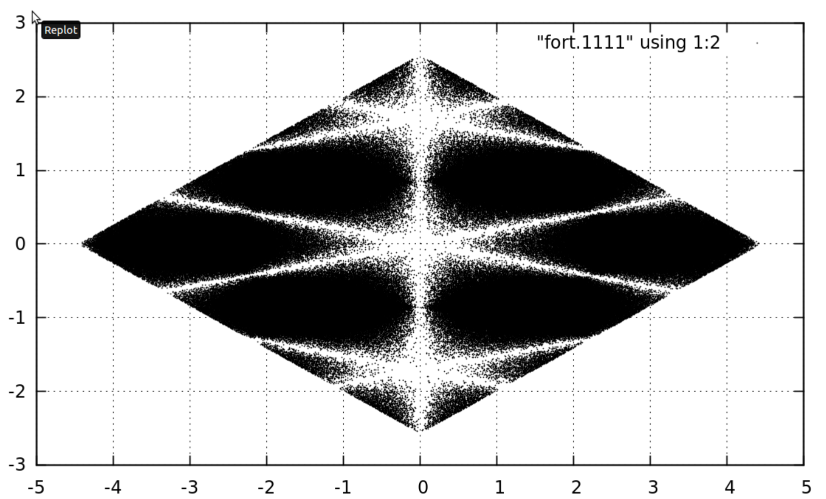

Figure 7: Distribution of eigen values of random SU(3) matrices on a plane using real datas.

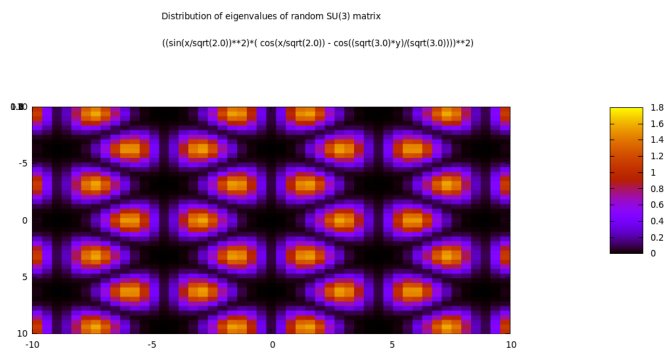

Figure 8: Distribution of eigen values of random SU(3) matrices on a plane.

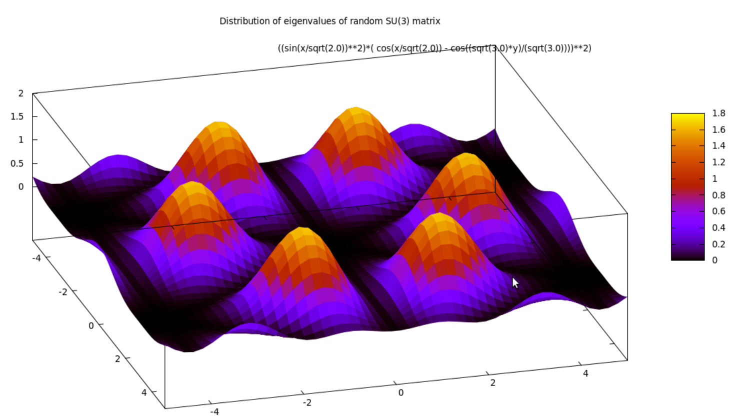

Figure 9: Probability distribution of eigen values of random SU(3) matrices.