Unit 5.1: Fourier Analysis

Contents

The preparatory reading for this section is Chapter 7 of [Karris, 2012] and Chapter 5 of [Hsu, 2020].

Follow along at cpjobling.github.io/eg-150-textbook/fourier_series/1/trig_fseries

Introduction

Any periodic waveform with *fundamental frequency* $\Omega_0 = 2\pi F_0$ can be approximated by a DC component (which may be 0) and the sum of sinusoidal waveforms at the fundamental and *integer multiples* of the fundamental frequency.

These integer multiples of the fundamental frequency $2\Omega_0$, $3\Omega_0$, $4\Omega_0$, $\ldots,\ \Omega_N$ are called the *harmonic frequencies*.

The approximation of a periodic waveform by a sum of *harmonic waveforms*, is known as *Fourier analysis*.

Fourier analysis has important applications in many branches of electronics but is particularly crucial for signal processing and communications.

Agenda

Periodic Signals

In Periodic signals we defined a continuous-time signal $x(t)$ to be periodic if there is a positive nonzero value of T for which

$$x(t+nT) = x(t) \qquad \text{all } t \tag{44}$$

The *fundamental period* T_0 of $x(t)$ is the smallest positive value of T for which Eq. (44) is satisfied, and $1/T_0 = f_0$ is referred to as the *fundamental frequency*.

Two basic examples of periodic signals are the real sinusoidal signal

$$x(t) = \cos\left(\Omega_0 t + \phi\right) \tag{45}$$

and the complex exponential signal

$$x(t) = e^{j\Omega t} \tag{46}$$

where $\Omega_0 = 2\pi/T_0 = 2\pi f_0$ is called the *fundamental angular frequency*.

Motivating Examples

This Fourier Series demo, developed by Members of the Center for Signal and Image Processing (CSIP) at the School of Electrical and Computer Engineering at the Georgia Institute of Technology, shows how periodic signals can be synthesised by a sum of sinusoidal signals.

It is here used as a motivational example in our introduction to Fourier Series [Wikipedia]. (See also Fourier Series from Wolfram MathWorld)

To install this example, download the zip file and unpack it somewhere on your MATLAB path.

Demo 1

Building up wave forms from sinusoids.

```
fseriesdemo
```

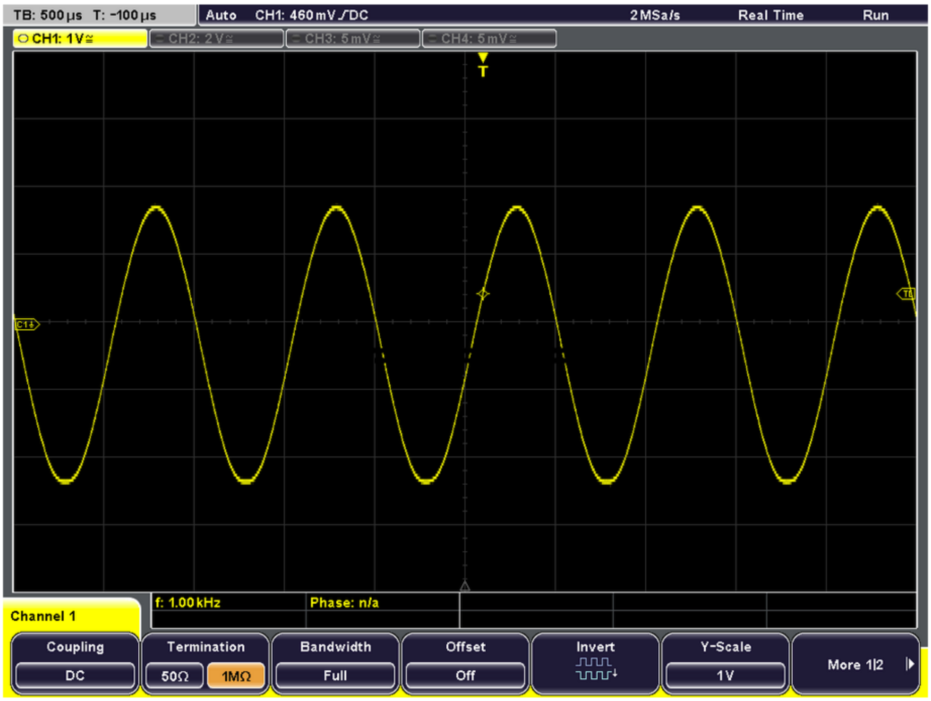
Demo 2

Actual measurements

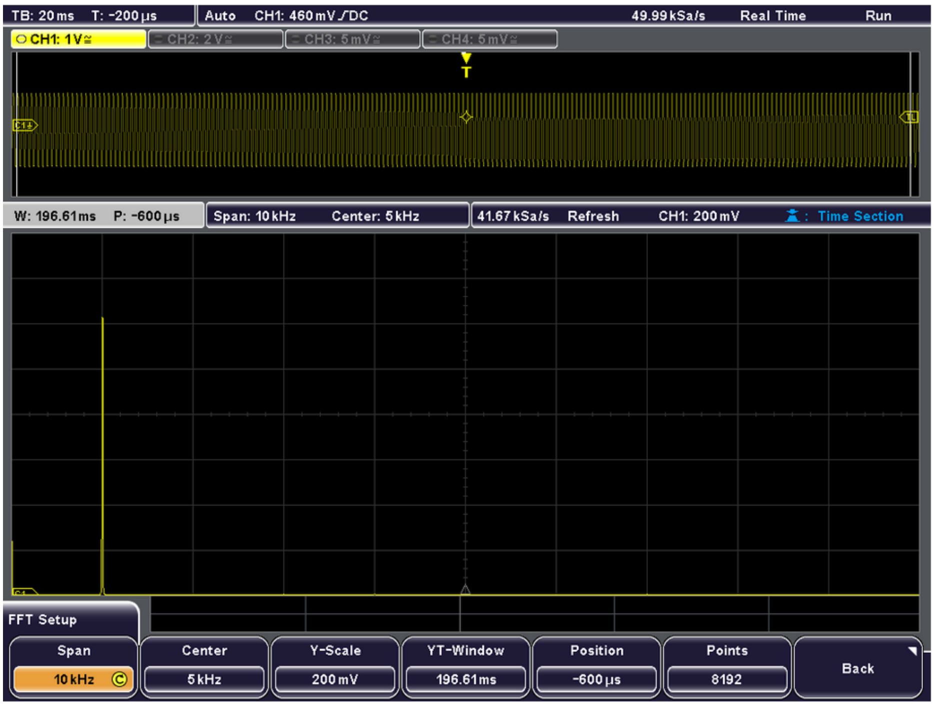
Taken by Dr Tim Davies with a Rhode&Schwarz Oscilloscope.

Note all spectra shown in these slides are generated numerically from the input signals by sampling and the application of the Fast Fourier Transform (FFT).

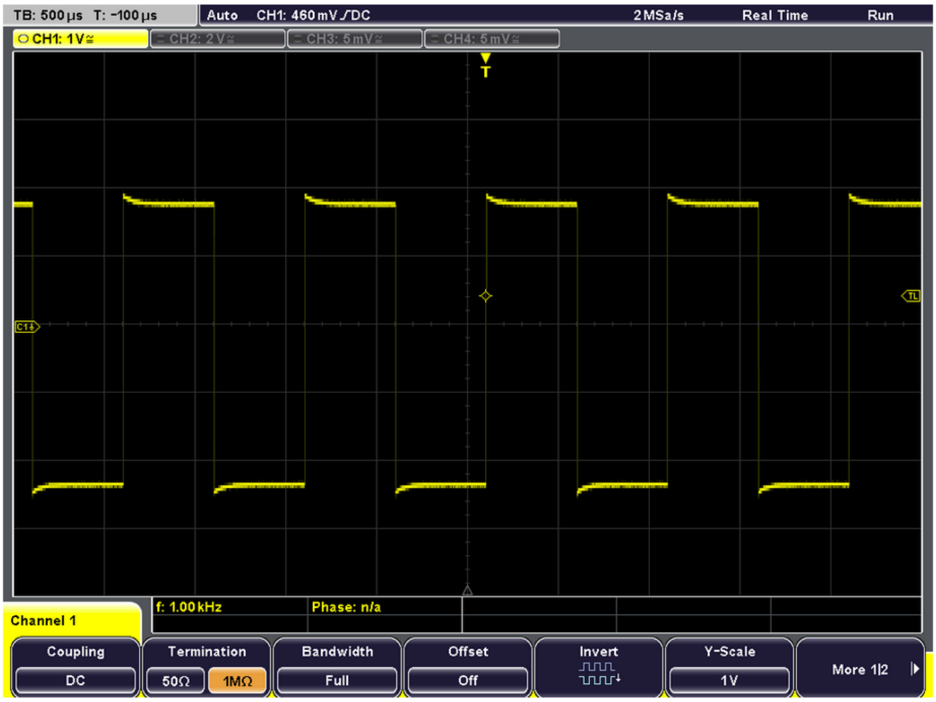
1 kHz Sinewave



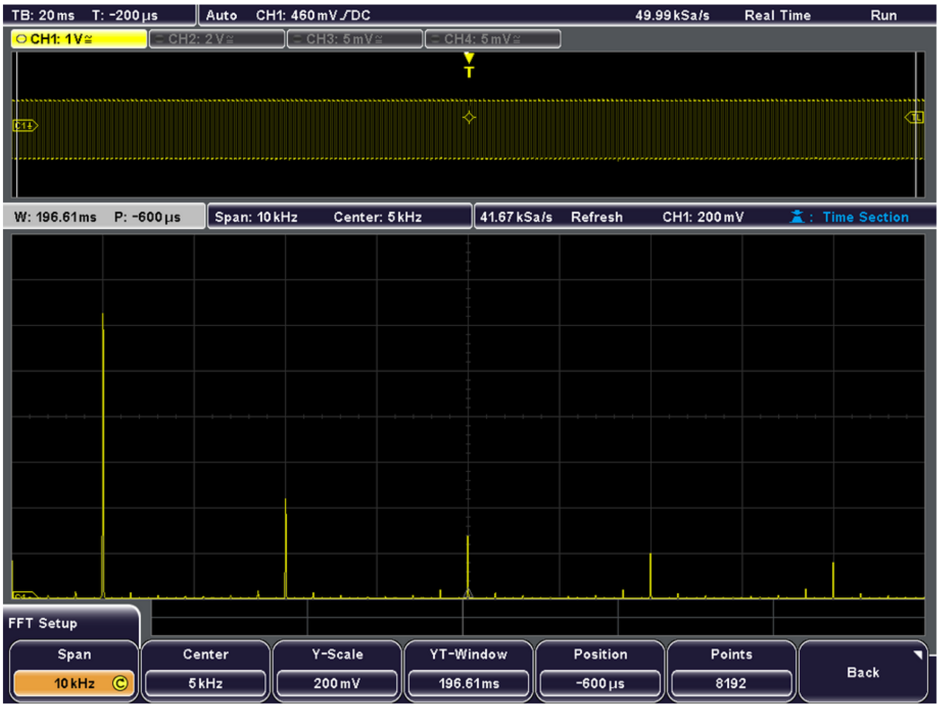
Spectrum of 1kHz sinewave



1 kHz Squarewave

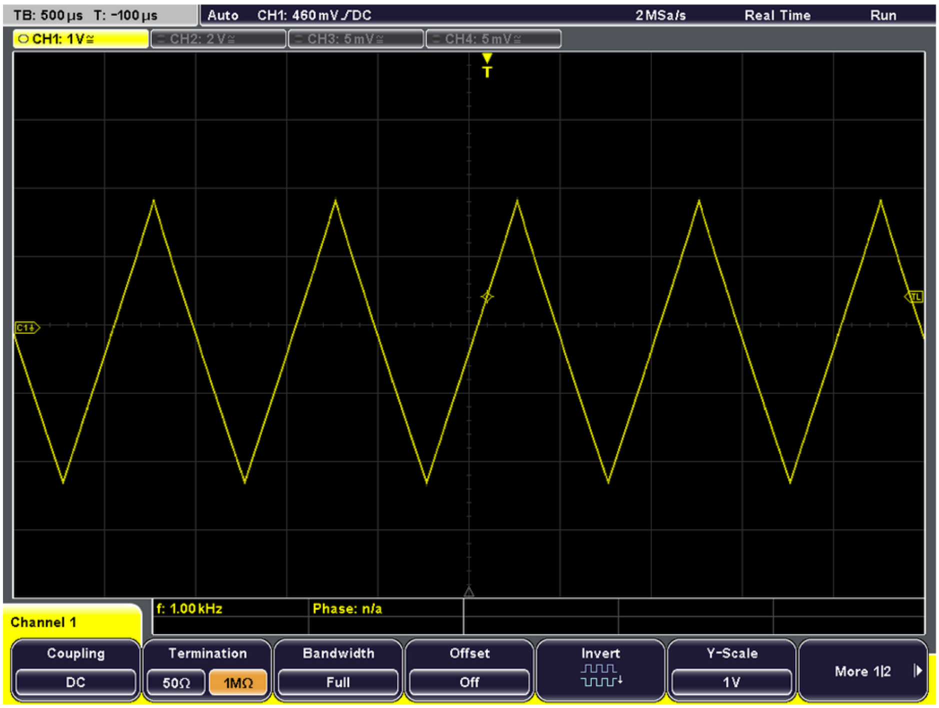


Spectrum of 1kHz square wave

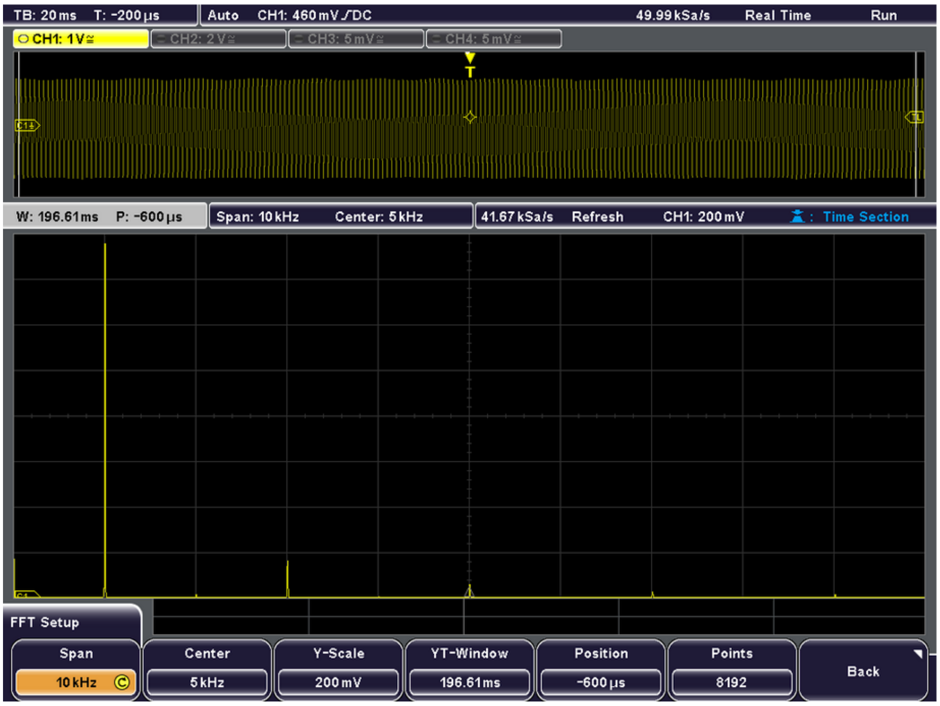


Clearly showing peaks at fundamental, 1/3, 1/5, 1/7 and 1/9 at 3rd, 5th and 7th harmonic frequencies. Note for the square wave, harmonics decline in amplitude as the reciprocal of the harmonic number n.

1 kHz triangle waveform



Spectrum of 1kHz triangle waveform



Clearly showing peaks at fundamental, 1/9, 1/25, 1/7 and 1/49 at 3rd, 5th and 7th harmonic frequencies. Note for the triangle waveform, harmonics decline in amplitude as the reciprocal of the square of n.

Wave Analysis

- Jean Baptiste Joseph Fourier (21 March 1768 – 16 May 1830) discovered that any ***periodic*** signal could be represented as a series of *harmonically related* sinusoids.
- An *harmonic* is a frequency whose value is an integer multiple of some *fundamental frequency*
- For example, the frequencies 2 MHz, 3 Mhz, 4 MHz are the second, third and fourth harmonics of a sinusoid with fundamental frequency 1 Mhz.

The Trigonometric Fourier Series

Any periodic waveform $f(t)$ can be represented as

$$\begin{aligned} f(t) = {} & \frac{1}{2}a_0 + a_1\cos\Omega_0 t + a_2\cos 2\Omega_0 t + a_3\cos 3\Omega_0 t + \cdots + a_n \cos n \\ & + b_1\sin\Omega_0 t + b_2\sin 2\Omega_0 t + b_3 \sin 3\Omega_0 t + \cdots + b_n \sin n\Omega_0 t + \cdots \end{aligned} \tag{47}$$

or equivalently (if more confusingly)

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos n\Omega_0 t + b_n \sin n\Omega_0 t\right) \tag{48}$$

where Ω_0 rad/s is the *fundamental frequency*.

Notation

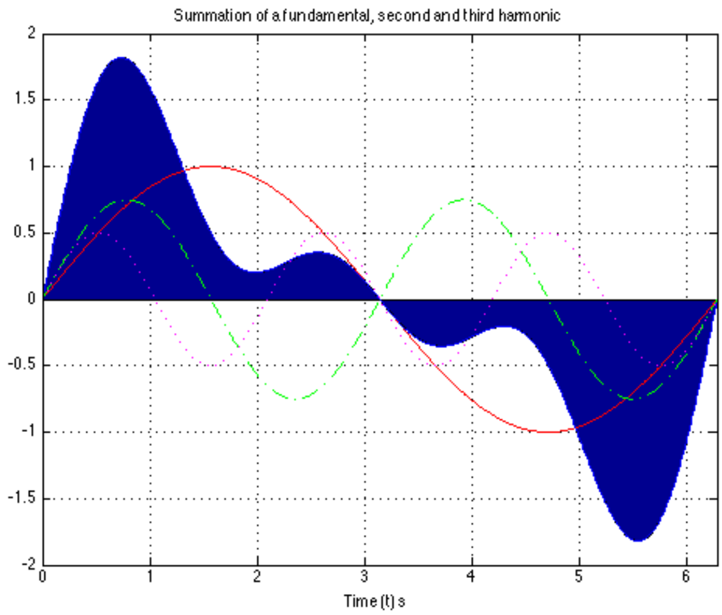
- The first term $a_o/2$ is a constant and represents the DC (average) component of the signal $f(t)$
- The terms with coefficients a_1 and b_1 together represent the fundamental frequency component of $f(t)$ at frequency Ω_0.
- The terms with coefficients a_2 and b_2 together represent the second harmonic frequency component of $f(t)$ at frequency $2\Omega_0$.

And so on.

Since any periodic function $f(t)$ can be expressed as a Fourier series, it follows that the sum of the DC, fundamental, second harmonic and so on must produce the waveform $f(t)$.

Sums of sinusoids

In general, the sum of two or more sinusoids does not produce a sinusoid as shown below.



To generate this picture use fourier_series1.m.

Evaluation of the Fourier series coefficients

The coefficients are obtained from the following expressions (valid for any periodic waveform with fundamental frequency Ω_0 so long as we integrate over one period $0 \to T_0$ where $T_0 = 2\pi/\Omega_0$), and $\theta = \Omega_0 t$:

$$\frac{1}{2}a_0 = \frac{1}{T_0}\int_0^{T_0} f(t)dt = \frac{1}{\pi}\int_0^{2\pi} f(\theta)d\theta \tag{49}$$

$$a_n = \frac{1}{T_0}\int_0^{T_0} f(t)\cos n\Omega_0 t\,dt = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta \tag{50}$$

$$b_n = \frac{1}{T_0}\int_0^{T_0} f(t)\sin n\Omega_0 t\,dt = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\sin n\theta\,d\theta \tag{51}$$

Odd, Even and Half-wave Symmetry

Odd and even symmetry

- An *odd* function is one for which $f(t) = -f(-t)$. The function $\sin t$ is an *odd* function.
- An *even* function is one for which $f(t) = f(-t)$. The function $\cos t$ is an *even* function.

Half-wave symmetry

- A periodic function with period T is a function for which $f(t) = f(t+T)$
- A periodic function with period T, has *half-wave symmetry* if $f(t) = -f(t+T/2)$

Symmetry in Trigonometric Fourier Series

There are simplifications we can make if the original periodic properties has certain properties:

- If $f(t)$ is odd, $a_0 = 0$ and there will be no cosine terms so $a_n = 0\ \forall n > 0$
- If $f(t)$ is even, there will be no sine terms and $b_n = 0\ \forall n > 0$. The DC may or may not be zero.
- If $f(t)$ has *half-wave symmetry* only the odd harmonics will be present. That is a_n and b_n is zero for all even values of n (0, 2, 4, …)

Some simplifications that result from symmetry

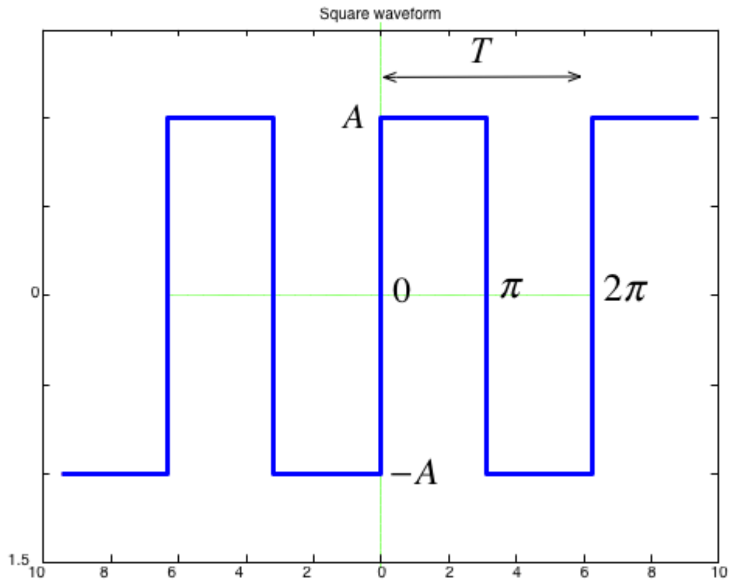
- The limits of the integrals used to compute the coefficents a_n and b_n of the Fourier series are given as $0 \to 2\pi$ which is one period T
- We could also choose to integrate from $-\pi \to \pi$
- If the function is *odd*, or *even* or has *half-wave symmetry* we can compute a_n and b_n by integrating from $0 \to \pi$ and multiplying by 2.
- If we have *half-wave symmetry* we can compute a_n and b_n by integrating from $0 \to \pi/2$ and multiplying by 4.

(For more details see page 7-10 of [Karris, 2012])

Computing coefficients of Trig. Fourier Series in MATLAB

The computation of the coefficients of the trig. fourier series is a painstaking, error-prone process and we need to use a computer.

As an example let's take a square wave with amplitude $\pm A$ and period T.



Solution

```
clear all
cd ../matlab
format compact
```

```
syms t n A pi
n = [1:11];
```

DC component

```
half_a0 = 1/(2*pi)*(int(A,t,0,pi)+int(-A,t,pi,2*pi))
```

```
half_a0 =
```

```
0
```

Compute harmonics

```
ai = 1/pi*(int(A*cos(n*t),t,0,pi)+int(-A*cos(n*t),t,pi,2*pi));
bi = 1/pi*(int(A*sin(n*t),t,0,pi)+int(-A*sin(n*t),t,pi,2*pi));
```

Reconstruct $f(t)$ from harmonic sine functions

```
ft = half_a0;
for k=1:length(n)
    ft = ft + ai(k)*cos(k*t) + bi(k)*sin(k*t);
end;
```

Make numeric

```
ft_num = subs(ft,A,1.0);
```

Print using 4 sig digits

```
ft_num = vpa(ft_num, 4)
```

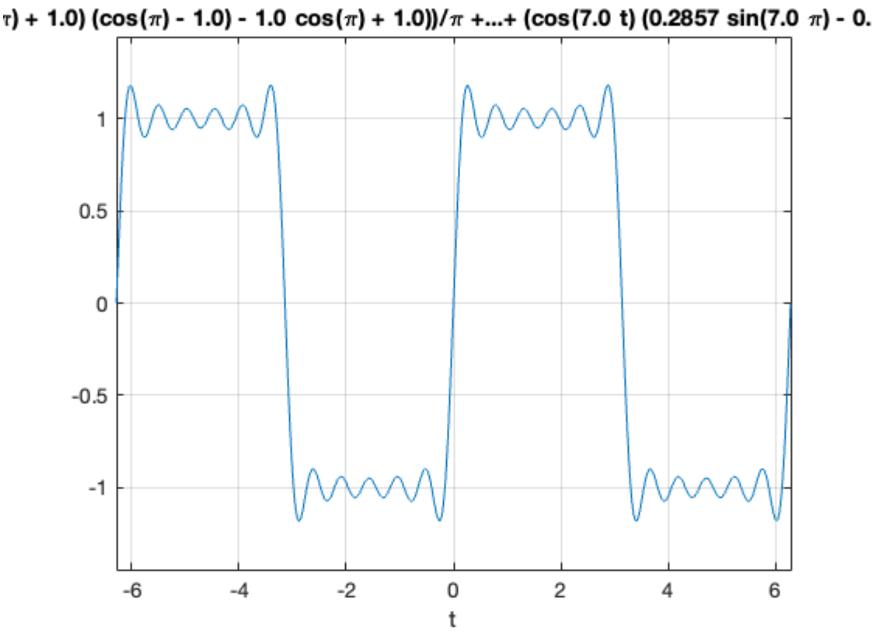
```
ft_num =
```

```
(sin(t)*((2.0*cos(pi) + 1.0)*(cos(pi) - 1.0) - 1.0*cos(pi) +
1.0))/pi + (cos(4.0*t)*(0.5*sin(4.0*pi) - 0.25*sin(8.0*pi)))/pi +
(cos(8.0*t)*(0.25*sin(8.0*pi) - 0.125*sin(16.0*pi)))/pi +
(sin(5.0*t)*(0.2*cos(10.0*pi) - 0.2*cos(5.0*pi) +
0.4*sin(2.5*pi)^2))/pi + (sin(10.0*t)*(0.1*cos(20.0*pi) -
0.1*cos(10.0*pi) + 0.2*sin(5.0*pi)^2))/pi + (sin(9.0*t)*
(0.1111*cos(18.0*pi) - 0.1111*cos(9.0*pi) +
0.2222*sin(4.5*pi)^2))/pi + (sin(11.0*t)*(0.09091*cos(22.0*pi) -
0.09091*cos(11.0*pi) + 0.1818*sin(5.5*pi)^2))/pi + (sin(3.0*t)*
(0.3333*cos(6.0*pi) - 0.3333*cos(3.0*pi) + 0.6667*sin(1.5*pi)^2))/pi
+ (sin(6.0*t)*(0.1667*cos(12.0*pi) - 0.1667*cos(6.0*pi) +
0.3333*sin(3.0*pi)^2))/pi + (sin(7.0*t)*(0.1429*cos(14.0*pi) -
0.1429*cos(7.0*pi) + 0.2857*sin(3.5*pi)^2))/pi + (sin(2.0*t)*
(sin(pi)^2 + sin(pi)^2*(4.0*sin(pi)^2 - 3.0)))/pi + (sin(4.0*t)*
(0.25*cos(8.0*pi) - 0.25*cos(4.0*pi) + 0.5*sin(2.0*pi)^2))/pi +
(sin(8.0*t)*(0.125*cos(16.0*pi) - 0.125*cos(8.0*pi) +
0.25*sin(4.0*pi)^2))/pi + (cos(9.0*t)*(0.2222*sin(9.0*pi) -
0.1111*sin
```

```
(18.0*pi)))/pi + (cos(5.0*t)*(0.4*sin(5.0*pi) -
0.2*sin(10.0*pi)))/pi + (cos(10.0*t)*(0.2*sin(10.0*pi) -
0.1*sin(20.0*pi)))/pi + (cos(2.0*t)*(0.5*sin(2.0*pi) +
0.5*sin(2.0*pi)*(4.0*sin(pi)^2 - 1.0)))/pi + (cos(11.0*t)*
(0.1818*sin(11.0*pi) - 0.09091*sin(22.0*pi)))/pi + (cos(3.0*t)*
(0.6667*sin(3.0*pi) - 0.3333*sin(6.0*pi)))/pi + (cos(6.0*t)*
(0.3333*sin(6.0*pi) - 0.1667*sin(12.0*pi)))/pi + (cos(t)*(sin(pi) -
1.0*sin(pi)*(2.0*cos(pi) - 1.0)))/pi + (cos(7.0*t)*
(0.2857*sin(7.0*pi) - 0.1429*sin(14.0*pi)))/pi
```

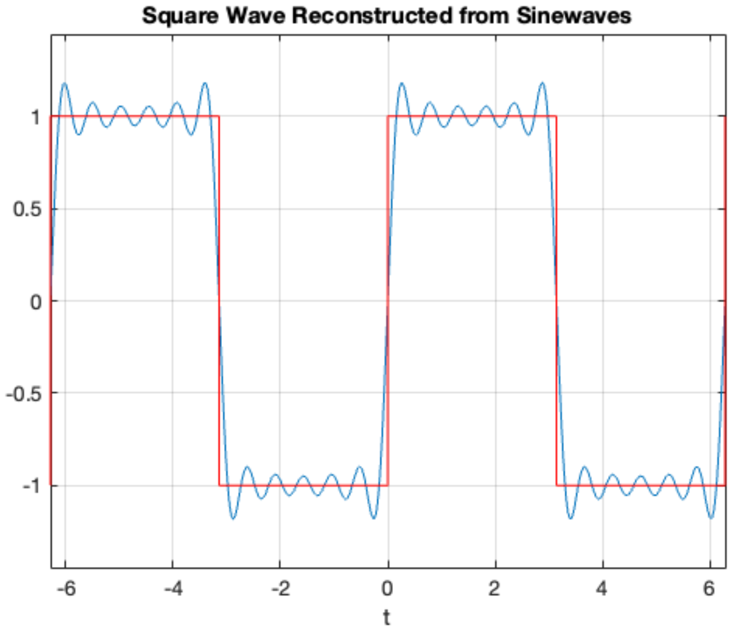
Plot result

```
ezplot(ft_num),grid
```



Plot original signal (we could use `heaviside` for this as well)

```
ezplot(ft_num)
hold on
clear pi
t = [-3,-2,-2,-2,-1,-1,-1,0,0,0,1,1,1,2,2,2,3]*pi;
f = [-1,-1,0,1,1,0,-1,-1,0,1,1,0,-1,-1,0,1,1];
plot(t,f,'r-')
grid
title('Square Wave Reconstructed from Sinewaves')
hold off
```



To run the full solution yourself download and run square_ftrig.mlx.

The Result confirms that:

- $a_0 = 0$
- $a_i = 0$: function is odd
- $b_i = 0$: for i even - half-wave symmetry

```
ft =

(4*A*sin(t))/pi + (4*A*sin(3*t))/(3*pi) + (4*A*sin(5*t))/(5*pi) +
(4*A*sin(7*t))/(7*pi) + (4*A*sin(9*t))/(9*pi) +
(4*A*sin(11*t))/(11*pi)
```

Note that the coefficients match those given in the textbook (Section 7.4.1).

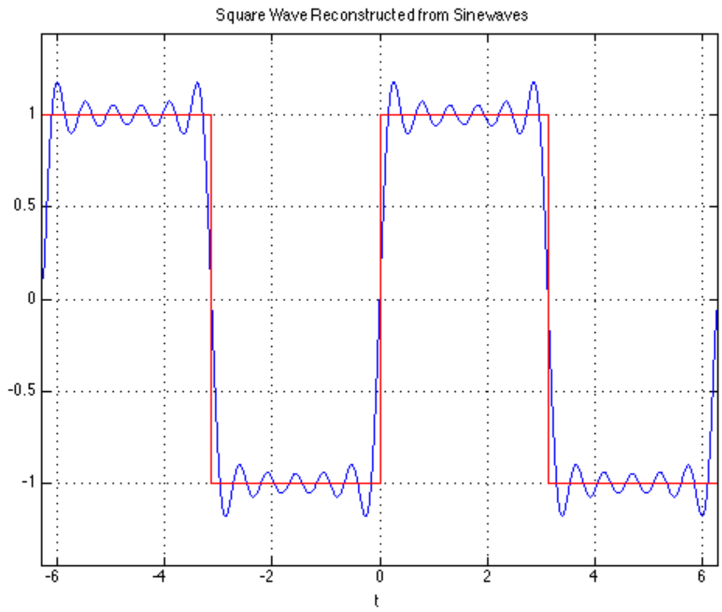
$$f(t) = \frac{4A}{\pi}\left(\sin\Omega_0 t + \frac{1}{3}\sin 3\Omega_0 t + \frac{1}{5}\sin 5\Omega_0 t + \cdots\right) = \frac{4A}{\pi}\sum_{n=\text{odd}}\frac{1}{n}\sin n\Omega$$

Gibbs Phenomenon

In an earlier slide we found that the trigonometric for of the Fourier series of the square waveform is

$$f(t) = \frac{4A}{\pi}\left(\sin\Omega_0 t + \frac{1}{3}\sin 3\Omega_0 t + \frac{1}{5}\sin 5\Omega_0 t + \cdots\right) = \frac{4A}{\pi}\sum_{n=\mathrm{odd}}\frac{1}{n}\sin n\Omega$$

This figure shows the approximation for the first 11 harmonics:



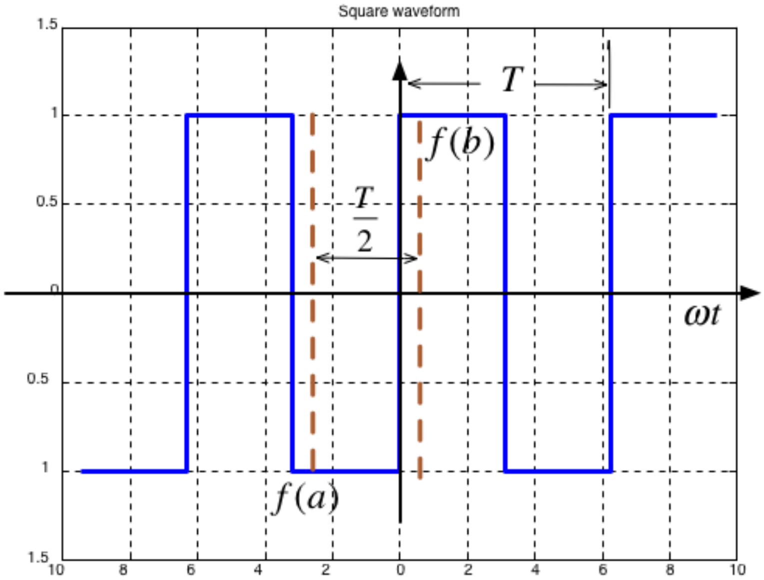
As we add more harmonics, the sum looks more and more like a square wave. However the crests do not become flattened; this is known as *Gibbs Phenomenon* and it occurs because of the discontinuity of the perfect sqare waveform as it changes from $+A$ to $-A$ and *vice versa*.

Example 16.1: Symmetry in Common Waveforms

To reproduce the following waveforms (without annotation) publish the script waves.m.

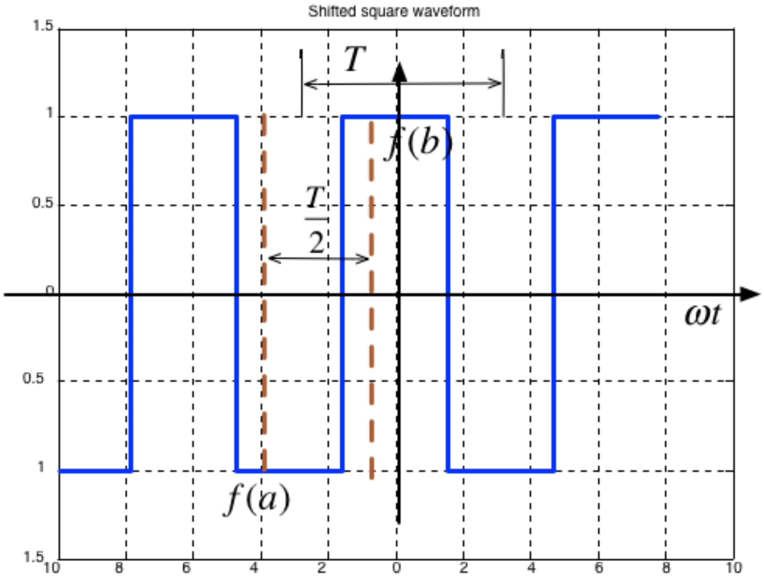
For each of the following, determine the average value of the waveform over 1 period, state whether it is even or odd, determine if the waveform has halfwave symmetry $f(t) = -f(t + T/2)$.

a) Squarewave



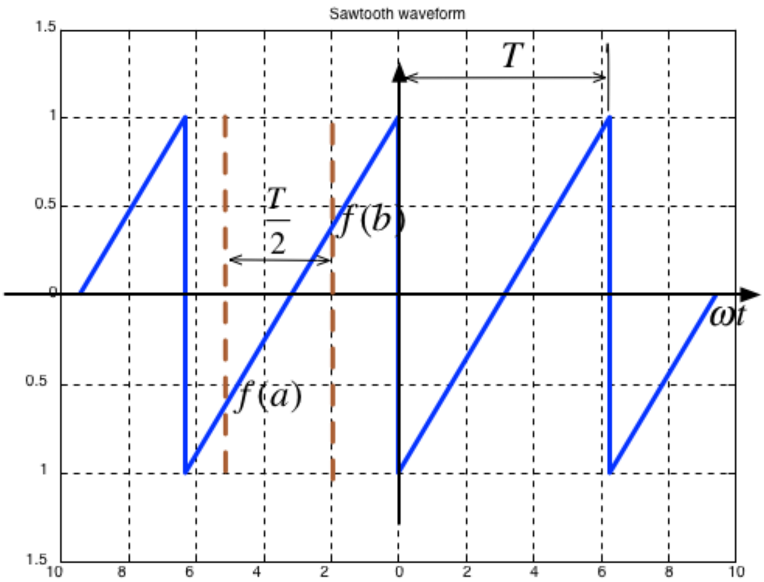
- Average value over period T is ...?
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry $f(t) = -f(t + T/2)$?

b) Shifted Squarewave



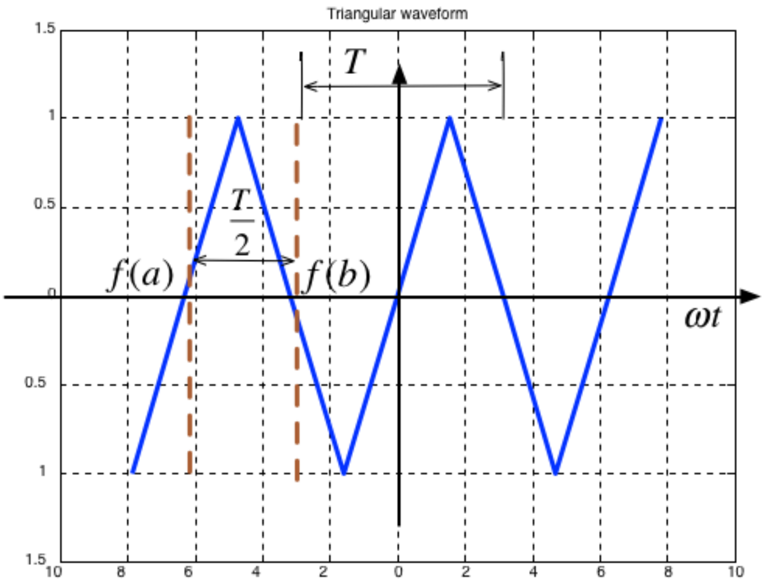
- Average value over period T is
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry $f(t) = -f(t + T/2)$?

c) Sawtooth



- Average value over period T is
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry $f(t) = -f(t + T/2)$?

Triangle



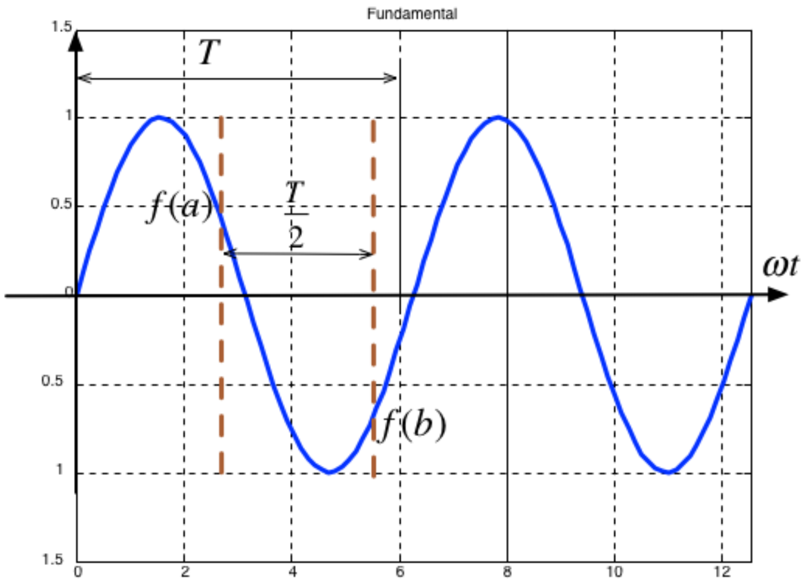
- Average value over period T is
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry $f(t) = -f(t + T/2)$?

Example 16.2: Symmetry in fundamental, Second and Third Harmonics

In the following, $T/2$ is taken to be the half-period of the fundamental sinewave.

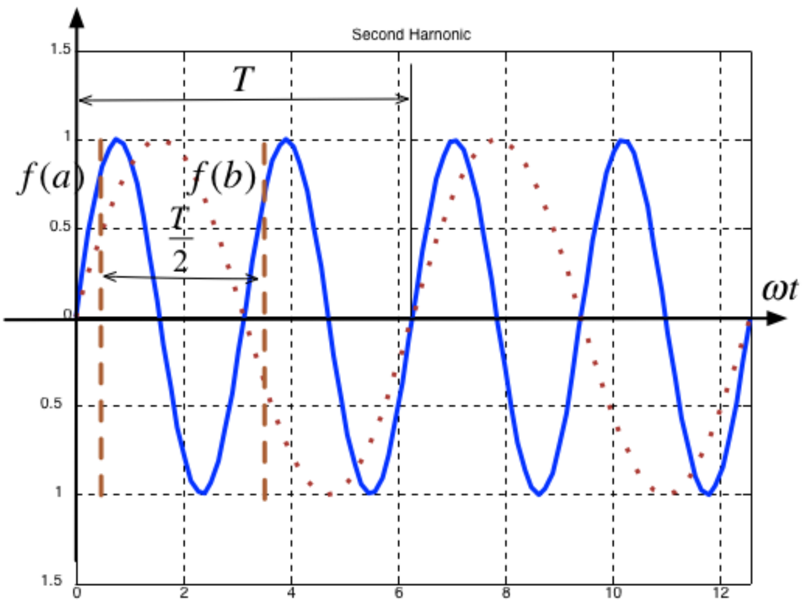
Evaluate the symmetry of the following fundamental and harmonic frequencies.

a) Fundamental



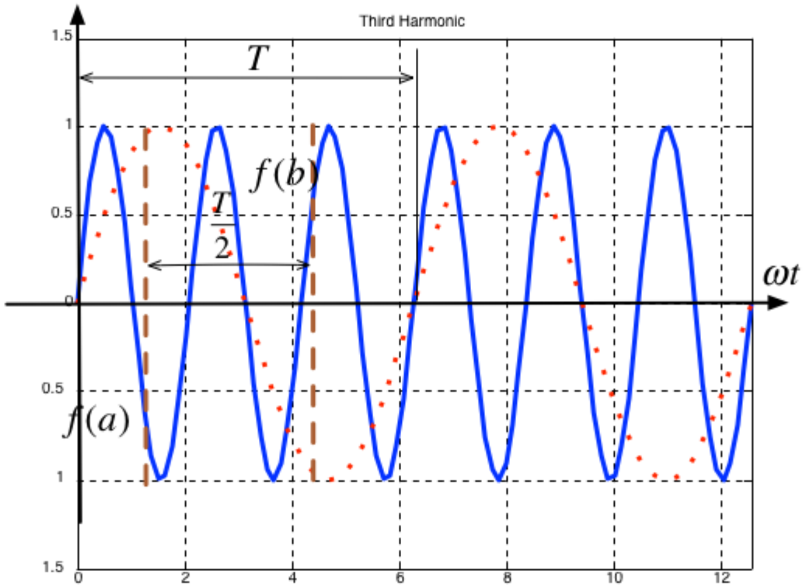
- Average value over period T is
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry $f(t) = -f(t+T/2)$?

b) Second Harmonic



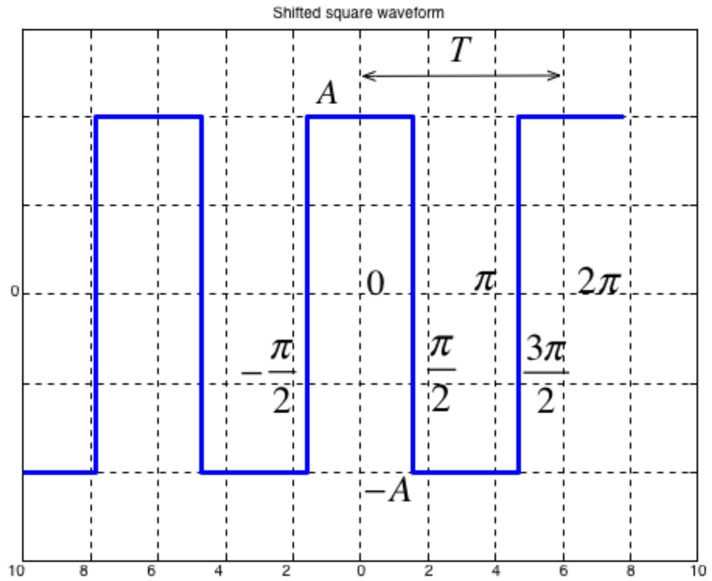
- Average value over period T is
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry $f(t) = -f(t+T/2)$?

b) Third Harmonic



- Average value over period T is
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry $f(t) = -f(t + T/2)$?

Example 16.3: Using symmetry - computing the Fourier series coefficients of the shifted square wave



Calculation of Fourier coefficients for Shifted Square Wave Exploiting half-wave symmetry. This is almost the same procedure as illustrated in Computing coefficients of Trig. Fourier Series in MATLAB.

You can confirm the results by downloading and executing this file: shifted_sq_ftrig.mlx.

```
clear all
syms t n A pi
```

Define harmonics

```
n = [1:11];
```

DC component

```
half_a0 = 0
```

```
half_a0 =
     0
```

Compute harmonics - use half-wave symmetry

```
ai = 4/pi*int(A*cos(n*t),t,0,pi/2);
```

```
bi = zeros(size(n));
```

Reconstruct f(t) from harmonic sine functions

```
ft = half_a0;
for k=1:length(n)
    ft = ft + ai(k)*cos(k*t) + bi(k)*sin(k*t);
end
```

Make numeric and print to 4 sig. figs.

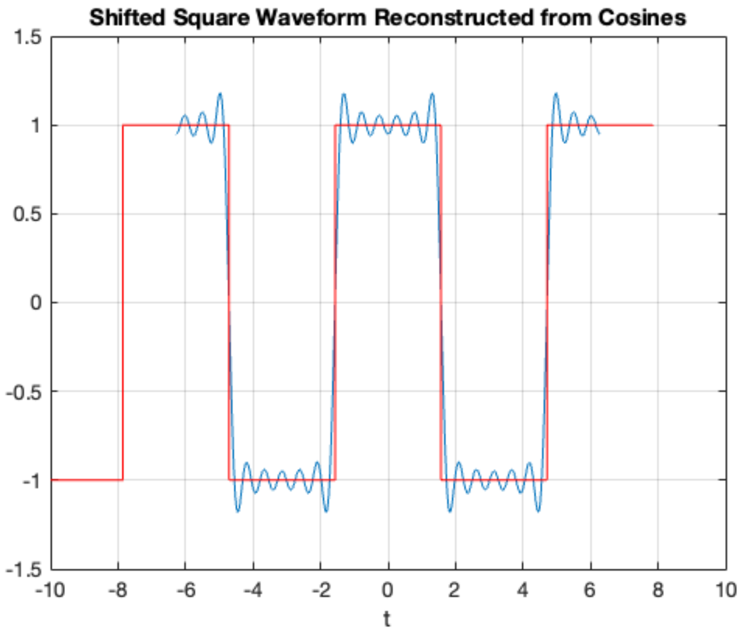
```
ft_num = subs(ft,A,1.0);
ft_num = vpa(ft_num, 4)
```

```
ft_num =
```

```
(cos(4.0*t)*sin(2.0*pi))/pi + (0.5*cos(8.0*t)*sin(4.0*pi))/pi +
(0.4444*cos(9.0*t)*sin(4.5*pi))/pi + (0.8*cos(5.0*t)*sin(2.5*pi))/pi
+ (0.4*cos(10.0*t)*sin(5.0*pi))/pi +
(0.3636*cos(11.0*t)*sin(5.5*pi))/pi +
(1.333*cos(3.0*t)*sin(1.5*pi))/pi +
(0.6667*cos(6.0*t)*sin(3.0*pi))/pi +
(0.5714*cos(7.0*t)*sin(3.5*pi))/pi + (4.0*sin(0.5*pi)*cos(t))/pi +
(2.0*cos(2.0*t)*sin(pi))/pi
```

plot result and overlay original signal (we could use `heaviside` for this as well.

```
clear pi
ezplot(ft_num)
hold on
t = [-3,-2,-2,-2,-1,-1,-1,0,0,0,1,1,1,2,2,2,3]*pi;
f = [-1,-1,0,1,1,0,-1,-1,0,1,1,0,-1,-1,0,1,1];
plot(t-pi/2,f,'r-')
axis([-10,10,-1.5,1.5])
grid
title('Shifted Square Waveform Reconstructed from Cosines')
hold off
```



- As before $a_0 = 0$
- We observe that this function is even, so all b_k coefficents will be zero
- The waveform has half-wave symmetry, so only odd indexed coefficents will be present.
- Further more, because it has half-wave symmetry we can just integrate from $0 \to \pi/2$ and multiply the result by 4.

Note that the coefficients match those given in the {cite|`karris` (Section 7.4.2).

$$f(t) = \frac{4A}{\pi}\left(\cos\Omega_0 t - \frac{1}{3}\cos 3\Omega_0 t + \frac{1}{5}\cos 5\Omega_0 t - \cdots\right) = \frac{4A}{\pi}\sum_{n=\mathrm{odd}}(-1)^{\frac{n-1}{2}}$$

Summary

In this unit we ...

- Introduction
- Periodic Signals
- Motivating Examples
- Wave Analysis
- Odd, Even and Half-wave Symmetry
- Computing coefficients of Trig. Fourier Series in MATLAB
- Gibbs Phenomenon
- Examples 16

Takeaways

Next Time

We move on to consider

- Unit 5.2: Exponential Fourier Series

References

[Hsu20] Hwei P. Hsu. *Schaums outlines signals and systems*. McGraw-Hill, New York, NY, 2020. ISBN 9780071634724. Available as an eBook. URL: https://www.accessengineeringlibrary.com/content/book/9781260454246.

[Kar12](1,2) Steven T. Karris. *Signals and systems with MATLAB computing and Simulink modeling*. Orchard Publishing, Fremont, CA., 2012. ISBN 9781934404232. Library call number: TK5102.9 K37 2012. URL: https://ebookcentral.proquest.com/lib/swansea-ebooks/reader.action?docID=3384197.
