Unit 4.1: The Laplace Transformation

Contents

The preparatory reading for this section is Chapter 2 of [Karris, 2012] and Chapter 3 of [Hsu, 2020].

Follow along at cpjobling.github.io/eg-150-textbook//laplace_transform/1/laplace

Agenda

- The Laplace Transform
- MATLAB Representation
- Region of Convergence
- Properties of the ROC

The Laplace Transform

In Eigenfunctions of Continuous-Time LTI Systems we saw that for a continuous-time LTI system with impulse response $h(t)$, the output of the system in response to a complex input of the form $x(t) = e^{st}$ is

$$y(t) = \mathbf{T}\left\{x(t)\right\} = H(s)e^{st}$$

where

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt$$

Definition

The function $H(s)$ above is referred to as the Laplace transform of $h(t)$.

For a general continuous-time signal $x(t)$, the Laplace transform $X(s)$ is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

The variable s is generally complex valued and is expressed as

$$\sigma + j\omega$$

The Laplace transform defined above is often called the *bilateral* (or *two-sided*) Laplace transform in contrast the the *unilateral* (or *one-sided*) Laplace transform which is defined as

$$X_I(s) = \int_{0^-}^{\infty} x(t)e^{-st}\,dt$$

where $0^- = \lim_{\epsilon\to 0}(0-\epsilon)$.

Clearly the bilateral and unilateral tranforms are equivalent only if $x(t) = 0$ for $t < 0$.

In this course, because we are dealing with causal signals and systems, we will be concerned only with unilateral Laplace transform.

The laplace tranform equation is sometimes considered an operator that transforms a signal $x(t)$ into a function $X(s)$ represented symbolically as

$$X(s) = \mathcal{L}\left\{x(t)\right\}$$

and the signal $x(t)$ and its Laplace transform $X(s)$ are said to form a Laplace transform pair denoted as

$$x(t) \Leftrightarrow X(s)$$

Laplace transform pairs are tabulated (lt_table.md) for ease of reference.

> **Note**
>
> By convention, lower-case symbols are used for continuous-time signals and uppercase symbols for their Laplace tranforms.

MATLAB Representation

The Laplace transform operator is provided in the MATLAB symbolic math toolkit by the function `laplace` and can be used as follows:

```
format compact
syms s t x(t) % define Laplace transform variable and time as
symbols
X(s) = laplace(x(t))
```

```
X(s) =
```

```
laplace(x(t), t, s)
```

Region of Convergence

For a Laplace transfomation to exist, the integral must be bounded. That is $ $\left|\int_0^\infty f(t)e^{-st}dt\right| < \infty$$

The range of values for the complex variables s for which the Laplace tranform converges is called the *region of convergence* (ROC). To illustrate this concept, let us consider some examples.

Solved Problem 1

Consider the signal

$$x(t) = e^{-at}u_0(t) \quad a\,\text{real}$$

By hand

We will work through the analysis in class

MATLAB analysis

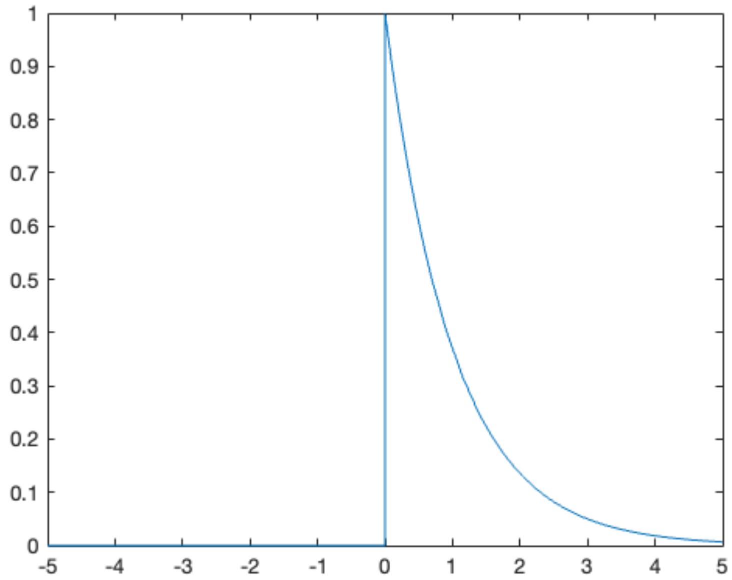
```
% set up
syms s t a
assume(a,'real')
u0(t) = heaviside(t);
```

```
x(t) = exp(-a*t)*u0(t)
```

```
x(t) =
```

```
exp(-a*t)*heaviside(t)
```

```
fplot(subs(x(t),a,1))
```




```
int(x(t)*exp(-s*t),t,0,inf)
```

```
ans =
```

```
piecewise(in(s, 'real') & a + s < 0, Inf, ~in(s, 'real') & a +
real(s) ~= 0, 1/(a + s) - limit(exp(- t*a - t*s), t, Inf)/(a + s),
in(s, 'real') & 0 < a + s | angle(s) in Dom::Interval(-pi/2, pi/2) &
0 < a & s ~= 0, 1/(a + s), (in(s, 'real') | a + real(s) == 0) &
(~in(s, 'real') | 0 <= a + s) & (~in(s, 'real') | a + s <= 0) &
(~angle(s) in Dom::Interval(-pi/2, pi/2) | a <= 0 | s == 0),
int(exp(-t*a)*exp(-t*s), t, 0, Inf))
```

```
assume(s + a > 0)
```

```
int(x(t)*exp(-s*t),t,0,inf)
```

```
ans =
```

```
1/(a + s)
```

```
X(s) = laplace(x(t))
```

```
X(s) =
```

```
1/(a + s)
```

The Laplace transform of $x(t)$

$$X(s) = \int_{-\infty}^{\infty} e^{-at}u_0(t)e^{-st}\,dt = \int_{0^+}^{\infty} e^{-(s+a)t}\,dt$$

$$X(s) = -\frac{1}{s+a}e^{-(s+a)t}\Big|_{0^+}^{\infty} = \frac{1}{s+a} \quad \mathrm{Re}(s) > -a$$

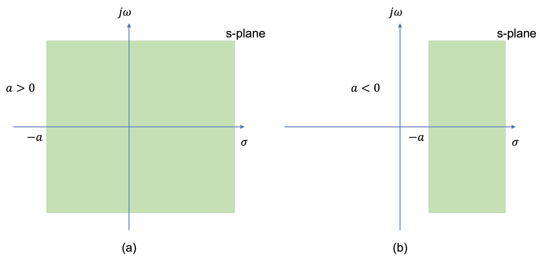


Fig. 41 ROC for Example 1

because $\lim_{t\to\infty} e^{-(s+a)t} = 0$ only if $\mathrm{Re}(s+a) > 0$ or $\mathrm{Re}(s) > -a$.

Thus, the ROC for Solved Problem 1 for Solved Problem 1 is specified as $\mathrm{Re}(s) > -a$ and is illustrated in the complex plane as shown in Fig. 42 by the shaded area to the right of the line $\mathrm{Re}(s)=-a.

In Laplace transform applications, the complex plane is commonly referred to as the s-plane. The horizontal and vertical axes are sometimes referred to as the σ-axis ($\mathrm{Re}(s)$) and the $j\omega$-axis ($\mathrm{Im}(s)$), respectively.

Solved Problem 2

Consider the signal

$$x(t) = -e^{-at}u_0(-t) \quad a\,\mathrm{real}$$

MATLAB Analysis

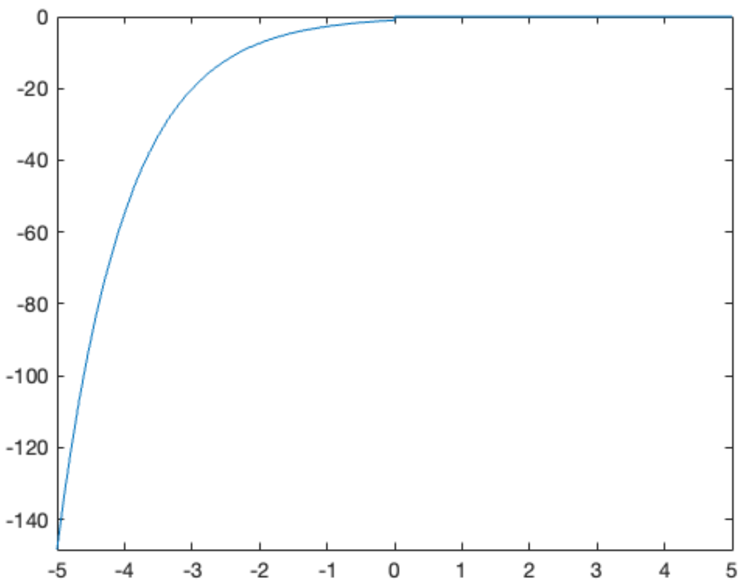
We will work through the analysis in class

```
x(t) = -exp(-a*t)*u0(-t)
```

```
x(t) =
```

```
-heaviside(-t)*exp(-a*t)
```

```
fplot(subs(x(t),a,1))
```



By hand analysis

```
int(x(t)*exp(-s*t),t,-inf,0)
```

```
ans =
```

```
-Inf
```

```
int(x(t)*exp(-s*t),t,-inf,0)
```

```
ans =
```

```
-Inf
```

```
X(s)=laplace(x(t))
```

```
X(s) =
```

```
0
```

Its Laplace transform $X(s)$ is given by ex:9.1

$$X(s) = \frac{1}{s+a} \quad \mathrm{Re}(s) < -a$$

Thus the ROC for Solved Problem 2 is specified as $\mathrm{Re}(s) < -a$ and is illustrated in the complex plane as shown in Fig. 42 by the shaded area to the left of the line $\mathrm{Re}(s) = -a$.

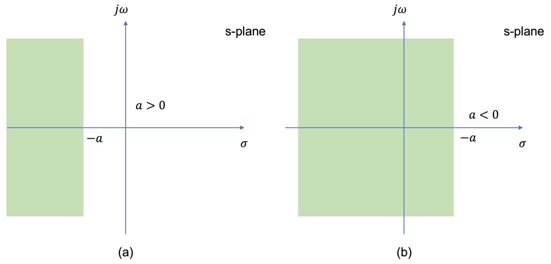


Fig. 42 ROC for Example 2

Comparing the results of Solved Problem 1 and Solved Problem 2, we see that that algebraic expressions for $X(s)$ for these two signals are identical apart from the ROCs.

Therefore, in order for the Laplace transform to be unique for each signal $x(t)$, *the ROC must be specified as part of the transform*.

Poles and Zeros of X(s)

Usually, $X(s)$ will be a rational polynomial in s; that is

$$X(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdot a_1 s + a_0} = \frac{b_m}{a_m}\,\frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)}$$

The coefficients b_k and a_k are real constants, and k, m and n are positive integers.

The transform $X(s)$ is called a *proper* rational function if $n > m$, and an *improper* rational function if $n \leq m$.

The roots of the numerator polynomial, z_k, are called the *zeros* of $X(s)$ because $X(s) = 0$ for those values of s.

Similarly, the zeros of the denominator polynomial, p_k, are called the *poles* of $X(s)$ because $X(s)$ is infinite for those values of s.

Therefore, the poles of $X(s)$ lie outside the ROC since, by definition, $X(s)$ does not converge on the poles.

The zeros, on the other hand, may lie inside or outside the ROC.

Except for the scale factor b_m/a_n, $X(s)$ can be completely specified by its poles and zeros.

Thus a very compact representation of $X(s)$ is the s-plane is to show the locations of the poles and zeros in addition to the ROC.

Traditionally, an "x" is used to indicate each pole and a "o" is used to indicate each zero.

This is illustrated in Fig. 43 for $X(s)$ given by

$$X(s) = \frac{2s+4}{s^2+4s+3} = 2\frac{(s+2)}{(s+1)(s+3)} \quad \mathrm{Re}(s) > -1$$

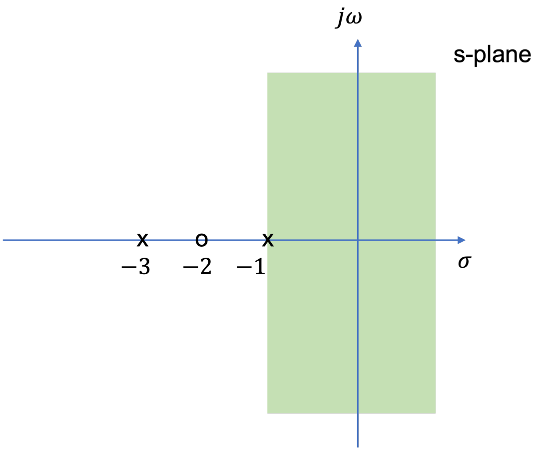


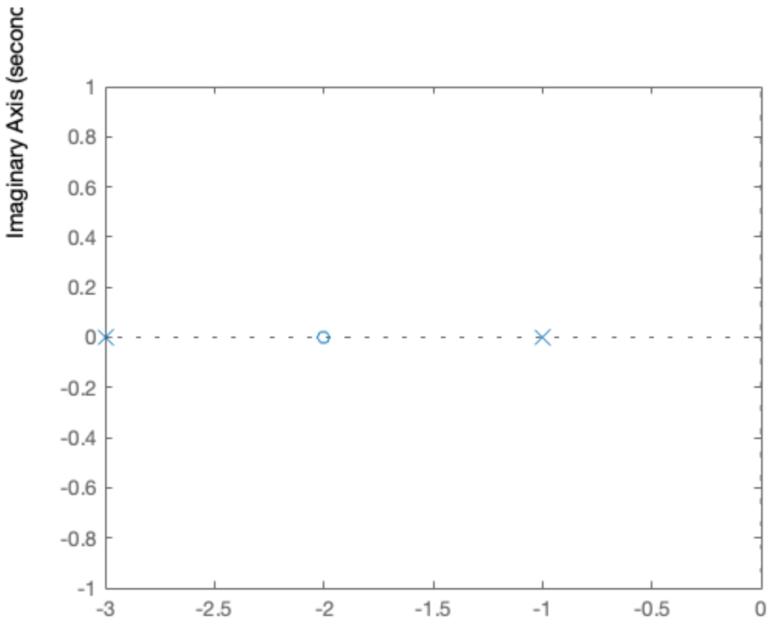
Fig. 43 s-plane representation of $X(s) = (2s^2+4)(s^2+rs+3)$

MATLAB Analysis

```
z =-2; ; p = [-1, -3]; k = 2;
H = zpk(z,p,k)
```

```
H =


    2 (s+2)

  -----------

  (s+1) (s+3)


Continuous-time zero/pole/gain model.
```

```
pzmap(H)
```



Note that $X(s)$ has one zero at $s=-2$ and two poles at $s=-1$ and $s=-3$ with scale factor 2. The ROC is $\mathrm{Re}(s) > -1$.

Properties of the ROC

[not examinable]

The properties of the ROC are summarised in 3.1 D of schaum and as they are not examinable, we leave their study to the interested student.

Examples 9

Example 9.1

Find the Laplace transform of

a). $x(t) = -e^{-at}u_0(-t)$

b). $x(t) = e^{at}u_0(-t)$

Solution

a)

$$X(s) = -\int_{-\infty}^{\infty} e^{-at}u_o(-t)e^{-st}\,dt = -\int_{-\infty}^{0^-} e^{-(s+a)t}$$

$$X(s) = \frac{1}{s+a}e^{-(s+a)t}\Big|_{-\infty}^{0^-} = \frac{1}{s+a} \quad \mathrm{Re}(s) < -a$$

Thus we obtain

$$-e^{-at}u_0(-t) \Leftrightarrow \frac{1}{s+a} \quad \mathrm{Re}(s) < -a$$

b). Similarly

$$X(s) = \int_{-\infty}^{\infty} e^{at}u_o(-t)e^{-st}\,dt = -\int_{-\infty}^{0^-} e^{-(s-a)t}$$

$$X(s) = -\frac{1}{s+a}\Big|_{-\infty}^{0^-} e^{-(s-a)t} = \frac{1}{s-a} \quad \mathrm{Re}(s) < a$$

Thus we obtain

$$e^{at}u_0(-t) \Leftrightarrow \frac{1}{s-a} \quad \mathrm{Re}(s) < a$$

Next Time

We move on to consider

- ../2/laplace_of_common_signals.md

References

[Hsu20] Hwei P. Hsu. *Schaums outlines signals and systems*. McGraw-Hill, New York, NY, 2020. ISBN 9780071634724. Available as an eBook. URL: https://www.accessengineeringlibrary.com/content/book/9781260454246.

[Kar12] Steven T. Karris. *Signals and systems with MATLAB computing and Simulink modeling*. Orchard Publishing, Fremont, CA., 2012. ISBN 9781934404232. Library call number: TK5102.9 K37 2012. URL: https://ebookcentral.proquest.com/lib/swansea-ebooks/reader.action?docID=3384197.

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