Unit 3.1: Response of a Continuous-Time LTI System and the Convolution Integral

Contents

This section is based on Section 2.1 of [Hsu, 2020].

Follow along at cpjobling.github.io/eg-150-textbook/lti_systems/lti1

Subjects to be Covered

A. Impulse Response

The *impulse response* $h(t)$ of a continuous-time LTI system (represented by $\mathbf{T}$) is defined as the response of the system when the input is $\delta(t)$, that is,

$$h(t) = \mathbf{T}\left\{\delta(t)\right\}$$

B. Response to an Arbitrary Input

From the Sifting Property

$$\int_{-\infty}^{\infty} f(t)\delta(t-\alpha)dt = f(\alpha)$$

an arbitrary continuous-time input can be expressed in terms of the Dirac delta function as

$$x(\tau) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t)$$

Since the system is linear, the response $y(t)$ of the system with arbitrary input $x(t)$ can be expressed as

$$y(t) = \mathbf{T}\left\{x(t)\right\} = \mathbf{T}\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau\right\}$$

$$= \int_{-\infty}^{\infty} x(t)\mathbf{T}\left\{\delta(t-\tau)\right\}$$

Since the system is time-invariant, we have

$$h(t-\tau) = \mathbf{T}\left\{\delta(t-\tau)\right\}d\tau$$

Substituting $h(t-\tau)$ into the equation for $y(t)$ gives

$$y(t) = \int_{-\infty}^{\infty} x(t)h(t-\tau)\,d\tau$$

This equation indicates that a continuous-time LTI system is completely characterised by its impulse response $h(t)$.

C. Convolution Integral

The equation

$$y(t) = \int_{-\infty}^{\infty} x(t)h(t-\tau)\,d\tau$$

defines the *convolution* of two continuous-time signals $x(t)$ and $h(t)$ denoted by

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t)h(t-\tau)\,d\tau$$

The equation

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t)h(t-\tau)\,d\tau$$

is commonly called the *convolution integral*.

Thus we have the fundamental result that:

the output of any continuous-time LTI system is the convolution of the input $x(t)$ *with the impulse response* $h(t)$ *of the system*.

Fig. 33 illustrates the definition of the impulse response $h(t)$ and the convolution integral.

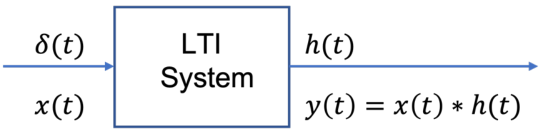


Fig. 33 Continuous-time LTI system

D. Properties of the Convolution Integral

The convolution integral has the following properties.

1. Commutative:

$$x(t) * h(t) = h(t) * x(t)$$

2. Associative:

$$\{x(t) * h_1(t)\} * h_2(t) = x(t) * \{h_1(t) * h_2(t)\}$$

3. Distributive:

$$x(t) * \{h_1(t) + h_2(t)\} = x(t) * h_1(t) + x(t) * h_2(t)$$

E. Convolution Integral Operation

Applying the communitative propery of convolution to the convolution integral, we obtain

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$$

which may at times be easier to evaluate than

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t)h(t-\tau)\,d\tau$$

Graphical Evaluation of the Convolution Integral

The convolution integral is most conveniently evaluated by a graphical evaluation. We give three examples (5.4—5.6) which we will demonstrate in class using a graphical visualization tool developed by Teja Muppirala of the Mathworks and updated by Rory Adams.

The tool: convolutiondemo.m (see license.txt).

We will then work through the examples again in the examples class.

```
clear all
cd matlab/convolution_demo
pwd
```

```
ans =

    '/Users/eechris/code/src/github.com/cpjobling/eg-150-
textbook/lti_systems/matlab/convolution_demo'
```

```
convolutiondemo % ignore warnings
```

```
Warning: The EraseMode property is no longer supported and will
error in a future release.
```

```
> In convolutiondemo>convolutiondemo_LayoutFcn (line 398)
In convolutiondemo>gui_mainfcn (line 1181)
In convolutiondemo (line 44)
```

Summary of Steps

1. The inpulse response $h(\tau)$ is time reversed (that is, reflected about the origin) to obtain $h(-\tau)$ and then shifted by t to form $h(t-\tau) = h\left[-(\tau - t)\right]$, which is a function of τ with parameter t.

1. The signal $x(\tau)$ and $h(t-\tau)$ are multiplied together for all values of τ with t fixed at some value.

1. The product $x(\tau)h(t-\tau)$ is integrated over all τ to produce a single output value $y(t)$.

1. Steps 1 ro 3 are repeated as t varies over $-\infty$ to ∞ to produce the entire output $y(t)$.

Examples of the above convolution integral operation are given in Examples 5.4 to 5.6.

F. Step Response

The *step response* $s(t)$ of a continuous-time LTI system (represented $\mathbf{T}$) is defined by the response of the system when the input is $u_0(t)$; that is,

$$s(t) = \mathbf{T}\left\{u_0(t)\right\}$$

In many applications, the step response $s(t)$ is also a useful characterisation of the system. The step response can be easily determined using the convolution integral; that is,

$$s(t) = h(t) * u_0(t) = \int_{-\infty}^{\infty} h(\tau)u_0(t-\tau)\,d\tau = \int_{-\infty}^{t} h(\tau)\,d\tau$$

Thus the step response $s(t)$ can be obtained by integrating the impulse response $h(t)$.

Impulse response from step response

Differentiating the step response with respect to t, we get

$$h(t) = s'(t) = \frac{ds(t)}{dt}$$

Thus the impulse response $h(t)$ can be determined by differentiating the step response $s(t)$.

Examples 5: Responses of a Continuous-Time LTI System and Convolution

Example 5.1

Verify the following properties of the convolution integral; that is,

(a) $x(t) * h(t) = h(t) * x(t)$

(b) $\{x(t) * h_1(t)\} * h_2(t) = x(t) * \{h_1(t) * h_2(t)\}$

For the answer, refer to the lecture recording or see solved problem 2.1 in in [Hsu, 2020].

Example 5.2

Show that

(a) $x(t) * \delta(t) = x(t)$

(b) $x(t) * \delta(t-t_0) = x(t-t_0)$

(c) $x(t) * u_0(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$

(d) $x(t) * u_0(t-t_0) = \int_{-\infty}^{t_0} x(\tau)\,d\tau$

For the answer, refer to the lecture recording or see solved problem 2.2 in in [Hsu, 2020].

Example 5.3

Let $y(t) = x(t) * h(t)$. Then show that

$$x(t-t_1) * h(t-t_2) = y(t-t_1-t_2)$$

For the answer, refer to the lecture recording or see solved problem 2.3 in in [Hsu, 2020].

Example 5.4

The input $x(t)$ and the impulse response $h(t)$ of a continuous-time LTI system are given by

$$x(t) = u_0(t)$$

$$h(t) = e^{-\alpha t}u_0(t),\ \alpha > 0$$

(a) Compute the output $y(t)$ by using the convolution integral

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t)h(t-\tau)\,d\tau$$

(b) Compute the output $y(t)$ by using the convolution integral

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$$

Solutions

(a) Graphical

Using the *convolutiondemo* tool chose a value for α. I will use $\alpha = 1$.

Then set `f(t)`, which represents $x(t)$, to `heaviside(t)` and `g(t)`. which represents $h(t)$ to `exp(-1*t)`

Manual solution

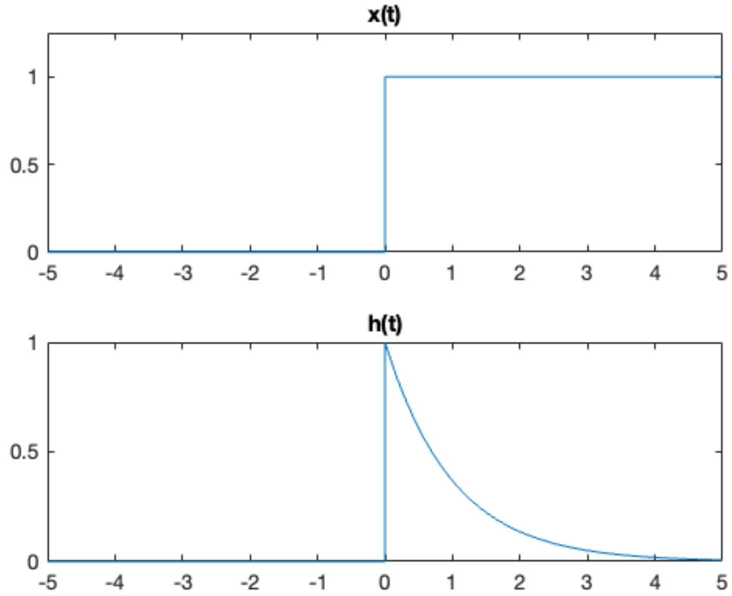
For the manual solution, refer to the lecture recording or see solved problem 2.3 in in [Hsu, 2020].

MATLAB Solution

We can also use the Symbolic Math Toolbox to solve the problem directly:

```
syms t tau alpha
assume(alpha > 0)

x(t) = heaviside(t); % unit step function
subplot(211)
fplot(x(t)),title('x(t)'),ylim([0,1.25])
h(t) = exp(-alpha*t)*heaviside(t);
subplot(212)
fplot(subs(h(t),alpha,1)),title('h(t)')
```



Compute $y(t)$ using the MATLAB function `int` to compute the convolution integral symbolically.

```
y(t) = int(x(tau)*h(t - tau),tau,-Inf,Inf)
```

```
y(t) =
```

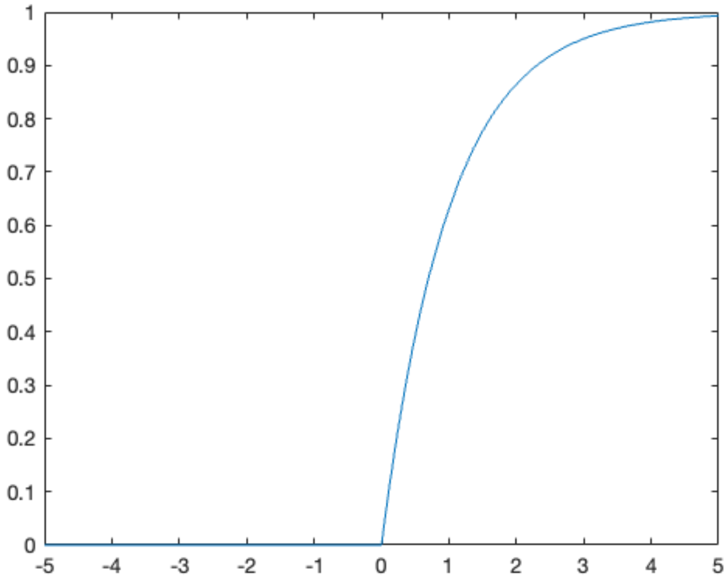
```
-(2*exp(-(alpha*t)/2)*(exp(-(alpha*t)/2)/2 - exp((alpha*t)/2)/2)*
(sign(t)/2 + 1/2))/alpha
```

Plot the result for $\alpha = 1$

```
ya(t) = subs(y(t),alpha,1)
fplot(ya(t))
```

```
ya(t) =
```

```
-2*exp(-t/2)*(exp(-t/2)/2 - exp(t/2)/2)*(sign(t)/2 + 1/2)
```



(b) Graphical

Reverse the settings for `f(t)`and `g(t)` in the *convolutiondemo* tool.

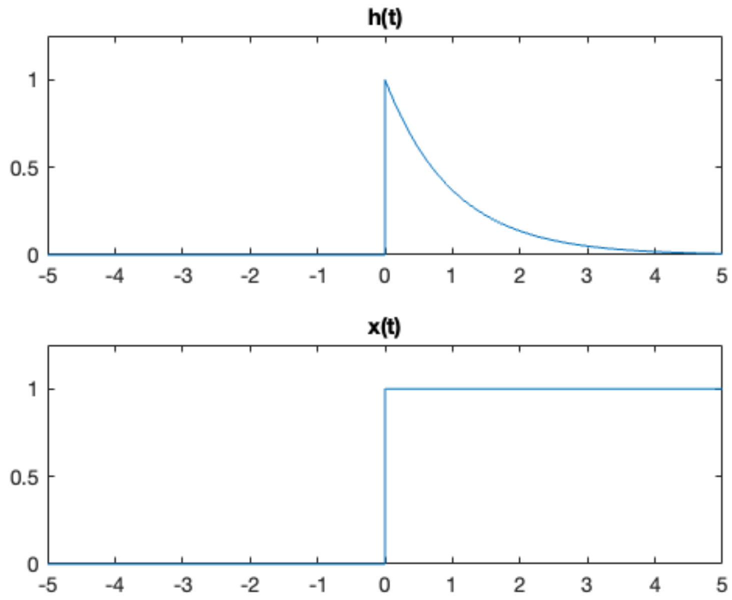
Manual solution

For the manual solution, refer to the lecture recording or see solved problem 2.3 in in [Hsu, 2020].

MATLAB Solution

Reverse the arguments to the `fplot` and `int` functions.

```
subplot(211)
fplot(subs(h(t),alpha,1)),title('h(t)'),ylim([0,1.25])
subplot(212)
fplot(x(t)),title('x(t)'),ylim([0,1.25])
```




```
y(t) = int(h(tau)*x(t - tau),tau,-Inf,Inf)
```

```
y(t) =
```

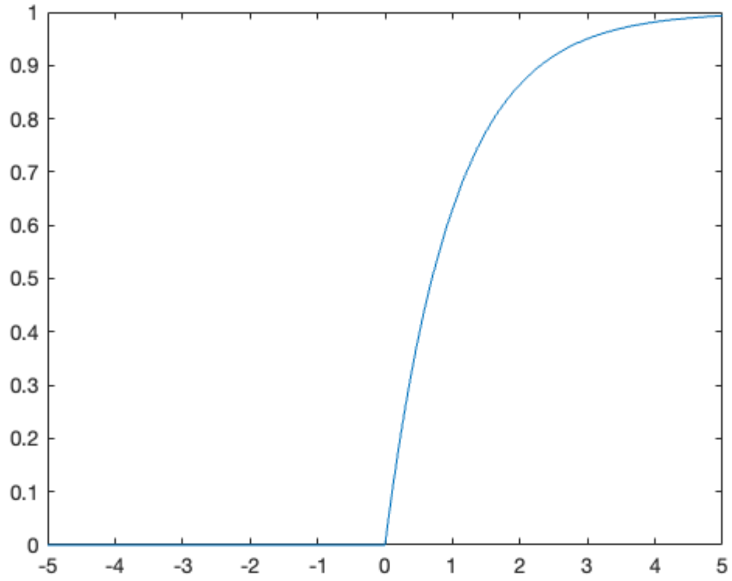
```
-(2*exp(-(alpha*t)/2)*(exp(-(alpha*t)/2)/2 - exp((alpha*t)/2)/2)*
(sign(t)/2 + 1/2))/alpha
```

Plot the result for $\alpha = 1$

```
yb(t) = subs(y(t),alpha,1)
fplot(yb(t))
```

```
yb(t) =
```

```
-2*exp(-t/2)*(exp(-t/2)/2 - exp(t/2)/2)*(sign(t)/2 + 1/2)
```



Example 5.5

Compute the output $y(t)$ for a continuous-time LTI system whose impulse response $h(t)$ and the input $x(t)$ are given by

$$h(t) = e^{-\alpha t}u_0(t)$$

$$x(t) = e^{\alpha t}u_0(-t)$$

$$\alpha > 0.$$

Solutions

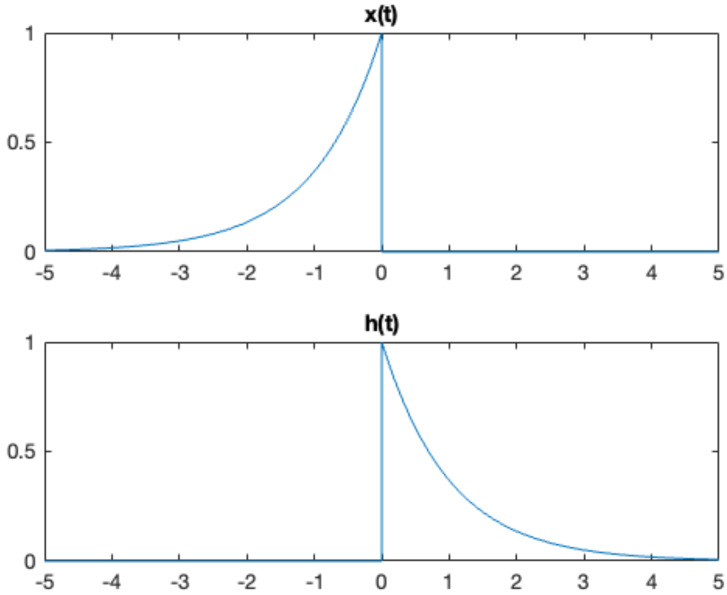
Manual solution

For the manual solution, refer to the lecture recording or see solved problem 2.3 in in [Hsu, 2020].

MATLAB Solution

We can also use the Symbolic Math Toolbox to solve the problem directly:

```
x(t) = exp(t)*heaviside(-t);
subplot(211)
fplot(x(t)),,title('x(t)')
h(t) = exp(-1*t)*heaviside(t);
subplot(212)
fplot(h(t)),title('h(t)')
```



Compute $y(t)$ using the convolution integral

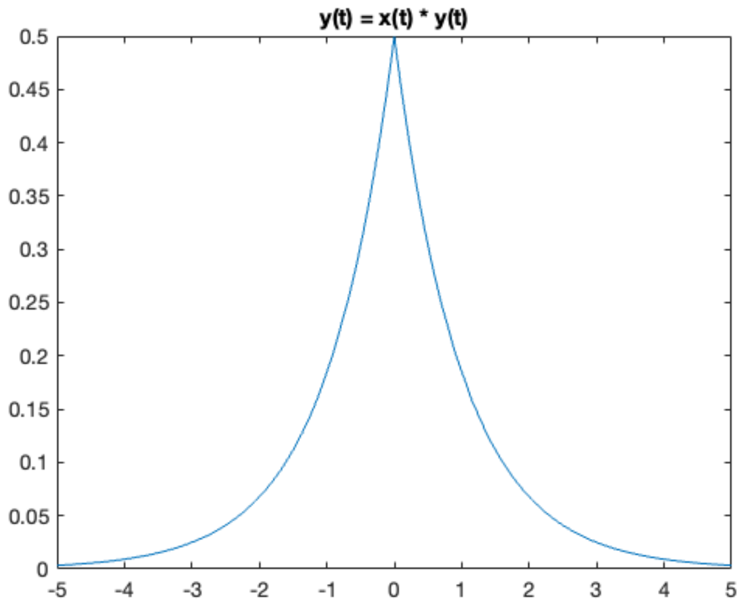
```
y(t) = int(x(tau)*h(t - tau),tau,-Inf,Inf)
```

```
y(t) =
```

```
piecewise(0 <= t, exp(-t)/2, t <= 0, exp(t)/2)
```

Plot the result for $\alpha = 1$

```
fplot(y(t)),title('y(t) = x(t) * y(t)')
```



Example 5.6

Evaluate $y(t) = x(t) * h(t)$, where $x(t)$ and $h(t)$ are shown in Fig. 34, by an alalytical technique, and (b) by a graphical method.

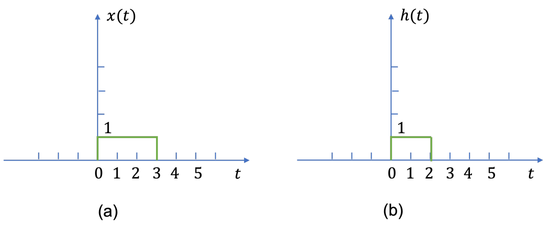


Fig. 34 Signal and system for example 5.6

Solutions

(a) Analytical

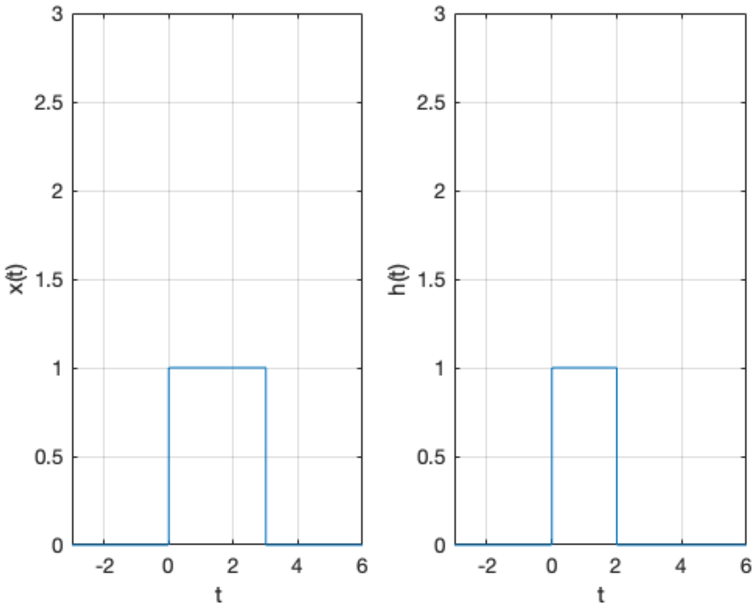
We first express $x(t)$ and $h(t)$ in functional form using the unit step (or *Heaviside* function)

$$x(t) = u_0(t) - u_0(t-3)$$

$$h(t) = u_0(t) - u_0(t-2)$$

We will use the MATLAB Symbolic Math Toolbox:

```
x(t) = heaviside(t)-heaviside(t-3);
h(t) = heaviside(t)-heaviside(t-2);
subplot(121)
fplot(x(t),[-3,6]),grid,ylim([0,3]),ylabel('x(t)'),xlabel('t')
subplot(122)
fplot(h(t),[-3,6]),grid,ylim([0,3]),ylabel('h(t)'),xlabel('t')
```



Compute $y(t)$

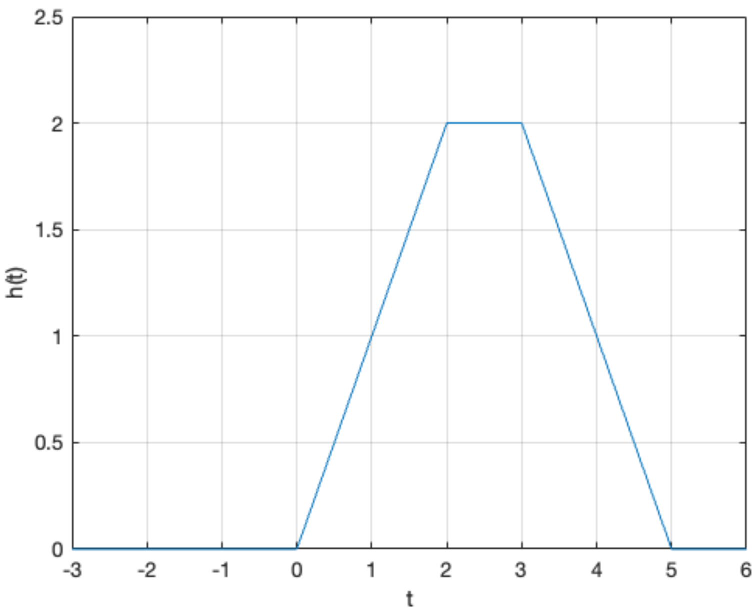
```
y(t) = int(x(tau)*h(t - tau),tau,-Inf,Inf)
```

```
y(t) =
```

```
heaviside(t - 5)*(t - 5) - heaviside(t - 3)*(t - 3) - heaviside(t -
2)*(t - 2) + t*heaviside(t)
```

Plot the result

```
fplot(y(t),[-3,6]),grid,ylim([0,2.5]),ylabel('h(t)'),xlabel('t')
```



(b) Graphical

Since both functions are unity between the limits set by the Heaviside function, graphical solution requires multiple applications of the definate integral

$$\int_{t_0}^{t_1} 1 \times 1\, d\tau = \int_{t_0}^{t_1} 1\, d\tau$$

with different values for the limits t_0 and t_1. The *convolutiondemo* tool can help us discover the limits for the piecewise continuous signal $y(t)$.

For the complete solution to Example 5.2 refer to the lecture recording or see solved problem 2.6 in in [Hsu, 2020].

Summary

In this lecture we have looked at

- A. Impulse Response
- B. Response to an Arbitrary Input
- C. Convolution Integral
- D. Properties of the Convolution Integral
- E. Convolution Integral Operation
- F. Step Response

Unit 3.1: Take Aways

- *Impulse response*: $h(t) = \mathbf{T}\left\{\delta(t)\right\}$
- *Arbitrary system response*: $y(t) = \int_{-\infty}^{\infty} x(t)h(t-\tau)\, d\tau$
- *Convolution itegral*: $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t)h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} x(t=\tau)h(t)\, d\tau$
- *Properties of the convolution integral:

  - *Communitative*: $x(t) * h(t) = h(t) * x(t)$
  - *Associative*: $\{x(t) * h_1(t)\} * h_2(t) = x(t) * \{h_1(t) * h_2(t)\}$
  - *Distributive*: $x(t) * \{h_1(t) + h_2(t)\} = x(t) * h_1(t) + x(t) * h_2(t)$
- The convolution integral can be computed graphically or analytically.

Next Time

We continue our introduction to continuous-time LTI system by considering

- Properties of Continuous-Time LTI Systems
- Eigenfunctions of Continuous-Time LTI Systems

References

[Hsu20](1,2,3,4,5,6,7,8) Hwei P. Hsu. *Schaums outlines signals and systems*. McGraw-Hill, New York, NY, 2020. ISBN 9780071634724. Available as an eBook. URL: https://www.accessengineeringlibrary.com/content/book/9781260454246.

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