# Unit 3.1: Response of a Continuous-Time LTI System and the Convolution Integral

## Contents


- Subjects to be Covered
- A. Impulse Response
- B. Response to an Arbitrary Input
- C. Convolution Integral
- D. Properties of the Convolution Integral
- E. Convolution Integral Operation
- F. Step Response
- Examples 5: Responses of a Continuous-Time LTI System and Convolution
- Summary
- Next Time


This section is based on Section 2.1 of [Hsu, 2020].

Follow along at cpjobling.github.io/eg-150-textbook/lti_systems/lti1

## Subjects to be Covered

- A. Impulse Response
- B. Response to an Arbitrary Input
- C. Convolution Integral
- D. Properties of the Convolution Integral
- E. Convolution Integral Operation
- F. Step Response
- Examples 5: Responses of a Continuous-Time LTI System and Convolution

## A. Impulse Response

The *impulse response* $h(t)$ of a continuous-time LTI system (represented by $\mathbf{T}$) is defined as the response of the system when the input is $\delta(t)$, that is,

$$h(t) = \mathbf{T}\left\{\delta(t)\right\}$$

## B. Response to an Arbitrary Input

From the Sifting Property

$$\int_{-\infty}^{\infty} f(t)\delta(t-\alpha)dt = f(\alpha)$$

an arbitrary continuous-time input can be expressed in terms of the Dirac delta function as

$$x(\tau) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t)$$

Since the system is linear, the response $y(t)$ of the system with arbitrary input $x(t)$ can be expressed as

$$y(t) = \mathbf{T}\left\{x(t)\right\} = \mathbf{T}\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau\right\}$$
$$= \int_{-\infty}^{\infty} x(t)\mathbf{T}\left\{\delta(t-\tau)\right\}$$

Since the system is time-invariant, we have

$$h(t-\tau) = \mathbf{T}\left\{\delta(t-\tau)\right\}d\tau$$

Substituting $h(t-\tau)$ into the equation for $y(t)$ gives

$$y(t) = \int_{-\infty}^{\infty} x(t)h(t-\tau)\,d\tau$$

This equation indicates that a continuous-time LTI system is completely characterised by its impulse response $h(t)$.

# C. Convolution Integral

The equation

$$y(t) = \int_{-\infty}^{\infty} x(t)h(t-\tau)\,d\tau$$

defines the *convolution* of two continuous-time signals $x(t)$ and $h(t)$ denoted by

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t)h(t-\tau)\,d\tau$$

The equation

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t)h(t-\tau)\,d\tau$$

is commonly called the *convolution integral*.

Thus we have the fundamental result that:

*the output of any continuous-time LTI system is the convolution of the input $x(t)$ with the impulse response $h(t)$ of the system.*

Fig. 30 illustrates the definition of the impulse response $h(t)$ and the convolution integral.

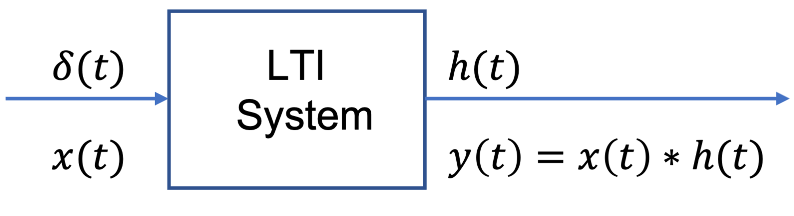



*Fig. 30* Continuous-time LTI system

# D. Properties of the Convolution Integral

The convolution integral has the following properties.

## 1. Commutative:

$$x(t) * h(t) = h(t) * x(t)$$

## 2. Associative:

$$\{x(t) * h_1(t)\} * h_2(t) = x(t) * \{h_1(t) * h_2(t)\}$$

## 3. Distributive:

$$x(t) * \{h_1(t) + h_2(t)\} = x(t) * h_1(t) + x(t) * h_2(t)$$

# E. Convolution Integral Operation

Applying the communitative propery of convolution to the convolution integral, we obtain

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$$

which may at times be easier to evaluate than

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t)h(t-\tau)\,d\tau$$

## Graphical Evaluation of the Convolution Integral

The convolution integral is most conveniently evaluated by a graphical evaluation. We give three examples (5.4—5.6) which we will demonstrate in class using a graphical visualization tool developed by Teja Muppirala of the Mathworks.

The tool: convolutiondemo.m (see license.txt).

We will then work through the examples again in the examples class.

```
clear all
cd matlab/convolutiondemo
pwd
```

```
ans =
    '/Users/eechris/code/src/github.com/cpjobling/eg-150-
textbook/lti_systems/matlab/convolutiondemo'
```

```
convolutiondemo % ignore warnings
```

```
Warning: The EraseMode property is no longer supported and will error in a
future release.
```

```
> In convolutiondemo>convolutiondemo_LayoutFcn (line 398)
In convolutiondemo>gui_mainfcn (line 1188)
In convolutiondemo (line 44)
```

```
Warning: The EraseMode property is no longer supported and will error in a
future release.
```

```
> In convolutiondemo>convolutiondemo_LayoutFcn (line 449)
In convolutiondemo>gui_mainfcn (line 1188)
In convolutiondemo (line 44)
```

```
Warning: The EraseMode property is no longer supported and will error in a
future release.
```

```
> In convolutiondemo>convolutiondemo_LayoutFcn (line 500)
In convolutiondemo>gui_mainfcn (line 1188)
In convolutiondemo (line 44)
```

```
Warning: The EraseMode property is no longer supported and will error in a
future release.
```

```
> In convolutiondemo>convolutiondemo_LayoutFcn (line 551)
In convolutiondemo>gui_mainfcn (line 1188)
In convolutiondemo (line 44)
```

```
Warning: The EraseMode property is no longer supported and will error in a
future release.
```

```
> In convolutiondemo>convolutiondemo_LayoutFcn (line 621)
In convolutiondemo>gui_mainfcn (line 1188)
In convolutiondemo (line 44)
```

```
Warning: The EraseMode property is no longer supported and will error in a
future release.
```

```
> In convolutiondemo>convolutiondemo_LayoutFcn (line 672)
In convolutiondemo>gui_mainfcn (line 1188)
In convolutiondemo (line 44)
```

```
Warning: The EraseMode property is no longer supported and will error in a
future release.
```

```
> In convolutiondemo>convolutiondemo_LayoutFcn (line 723)
In convolutiondemo>gui_mainfcn (line 1188)
In convolutiondemo (line 44)
```

```
Warning: The EraseMode property is no longer supported and will error in a
future release.
```

```
> In convolutiondemo>convolutiondemo_LayoutFcn (line 774)
In convolutiondemo>gui_mainfcn (line 1188)
In convolutiondemo (line 44)
```

## Summary of Steps

1. The inpulse response $h(\tau)$ is time reversed (that is, reflected about the origin) to obtain $h(-\tau)$ and then shifted by $\tau$ to form $h(t-\tau) = h\left[-(\tau - t)\right]$, which is a function of $\tau$ with parameter $t$.

1. The signal $x(\tau)$ and $h(t-\tau)$ are multiplied together for all values of $\tau$ with $t$ fixed at some value.

1. The product $x(\tau)h(t-\tau)$ is integrated over all $\tau$ to produce a single output value $y(t)$.

1. Steps 1 ro 3 are repeated as $t$ varies over $-\infty$ to $\infty$ to produce the entire output $y(t)$.

Examples of the above convolution integral operation are given in Examples 4.1 to 4.3.

# F. Step Response

The *step response* $s(t)$ of a continuous-time LTI system (represented $\mathbf{T}$) is defined by the response of the system when the input is $u_0(t)$; that is,

$$s(t) = \mathbf{T}\left\{u_0(t)\right\}$$

In many applications, the step response $s(t)$ is also a useful characterisation of the system. The step response can be easily determined using the convolution integral; that is,

$$s(t) = h(t) * u_0(t) = \int_{-\infty}^{\infty} h(\tau)u_0(t-\tau)\,d\tau = \int_{-\infty}^{t} h(\tau)\,d\tau$$

Thus the step response $s(t)$ can be obtained by integrating the impulse response $h(t)$.

## Impulse response from step response

Differentiating the step response with respect to $t$, we get

$$h(t) = s'(t) = \frac{ds(t)}{dt}$$

Thus the impulse response $h(t)$ can be determined by differentiating the step response $s(t)$.

# Examples 5: Responses of a Continuous-Time LTI System and Convolution

## Example 5.1

Verify the following properties of the convolution integral; that is,

(a) $x(t) * h(t) = h(t) * x(t)$

(b) $\{x(t) * h_1(t)\} * h_2(t) = x(t) * \{h_1(t) * h_2(t)\}$

For the answer, refer to the lecture recording or see solved problem 2.1 in in [Hsu, 2020].

## Example 5.2

Show that

(a) $x(t) * \delta(t) = x(t)$

(b) $x(t) * \delta(t-t_0) = x(t-t_0)$

(c) $x(t) * u_0(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$

(d) $x(t) * u_0(t-t_0) = \int_{-\infty}^{t_0} x(\tau)\,d\tau$

For the answer, refer to the lecture recording or see solved problem 2.2 in in [Hsu, 2020].

## Example 5.3

Let $y(t) = x(t) * h(t)$. Then show that

$$x(t-t_1) * h(t-t_2) = y(t-t_1-t_2)$$

For the answer, refer to the lecture recording or see solved problem 2.3 in in [Hsu, 2020].

## Example 5.4

The input $x(t)$ and the impulse response $h(t)$ of a continuous-time LTI system are given by

$$x(t) = u_0(t)$$
$$h(t) = e^{-\alpha t}u_0(t),\ \alpha > 0$$

(a) Compute the output $y(t)$ by using the convolution integral

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t)h(t-\tau)\,d\tau$$

(b) Compute the output $y(t)$ by using the convolution integral

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$$

## Solutions

### (a) Graphical

Using the *convolutiondemo* tool chose a value for $\alpha$. I will use $\alpha = 1$.

Then set `f(t)`, which represents $x(t)$, to `heaviside(t)` and `g(t)`. which represents $h(t)$ to `exp(-1*t)`

### Manual solution

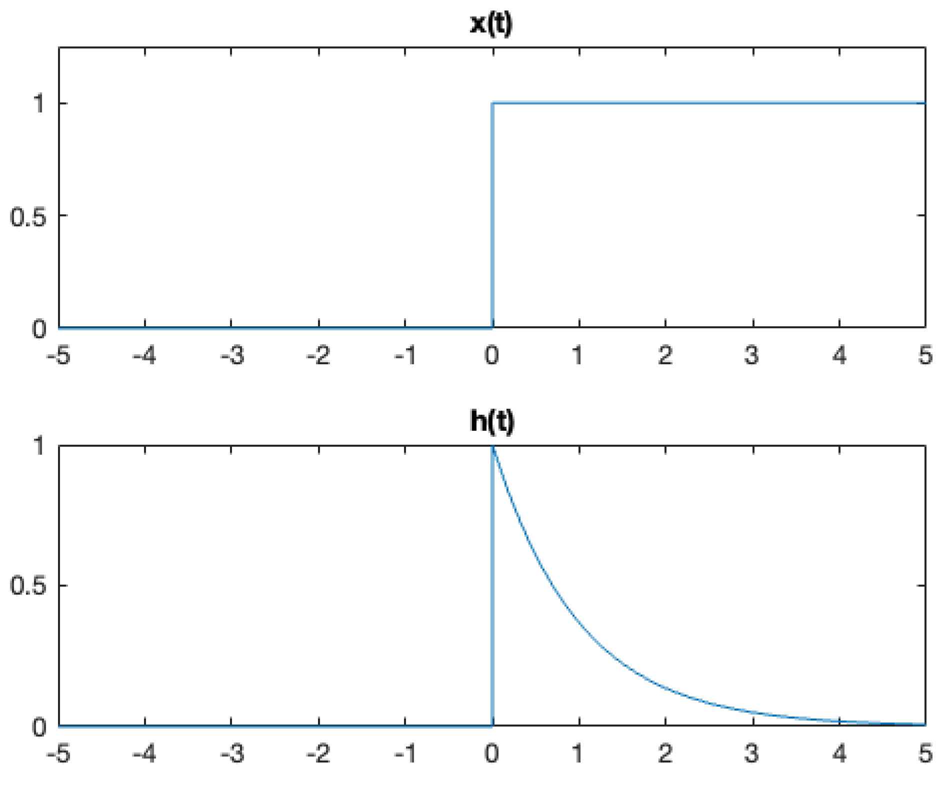
For the manual solution, refer to the lecture recording or see solved problem 2.3 in in [Hsu, 2020].

### MATLAB Solution

We can also use the Symbolic Math Toolbox to solve the problem directly:

```
syms t tau alpha
assume(alpha > 0)

x(t) = heaviside(t); % unit step function
subplot(211)
fplot(x(t)),title('x(t)'),ylim([0,1.25])
h(t) = exp(-alpha*t)*heaviside(t);
subplot(212)
fplot(subs(h(t),alpha,1)),title('h(t)')
```




Compute $y(t)$ using the MATLAB function `int` to compute the convolution integral symbolically.

```
y(t) = int(x(tau)*h(t - tau),tau,-Inf,Inf)
```

```
y(t) =
```

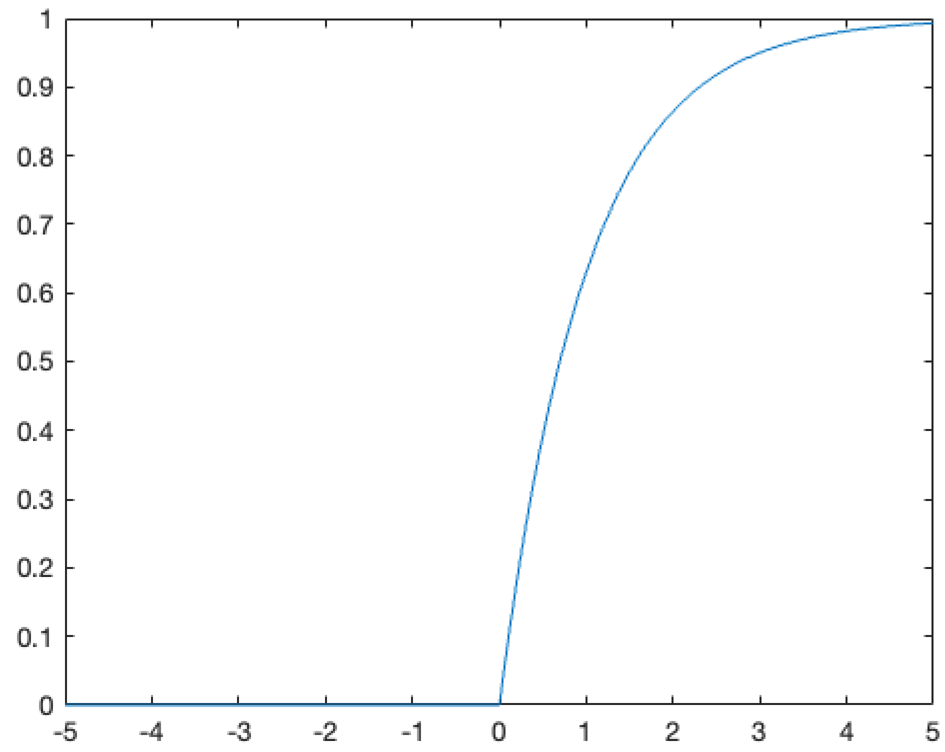
```
-(2*exp(-(alpha*t)/2)*(exp(-(alpha*t)/2)/2 - exp((alpha*t)/2)/2)*(sign(t)/2 +
1/2))/alpha
```

Plot the result for $\alpha = 1$

```
ya(t) = subs(y(t),alpha,1)
fplot(ya(t))
```

```
ya(t) =
```

```
-2*exp(-t/2)*(exp(-t/2)/2 - exp(t/2)/2)*(sign(t)/2 + 1/2)
```



### (b) Graphical

Reverse the settings for `f(t)`and `g(t)` in the *convolutiondemo* tool.

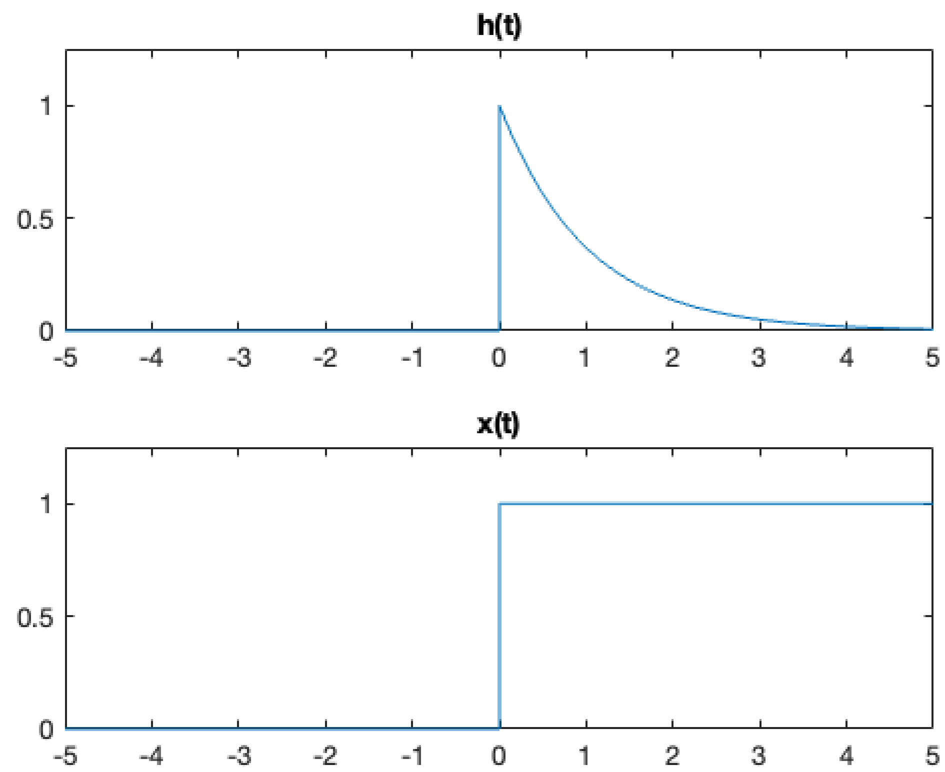
### Manual solution

For the manual solution, refer to the lecture recording or see solved problem 2.3 in in [Hsu, 2020].

### MATLAB Solution

Reverse the arguments to the `fplot` and `int` functions.

```
subplot(211)
fplot(subs(h(t),alpha,1)),title('h(t)'),ylim([0,1.25])
subplot(212)
fplot(x(t)),title('x(t)'),ylim([0,1.25])
```




```
y(t) = int(h(tau)*x(t - tau),tau,-Inf,Inf)
```

```
y(t) =
```

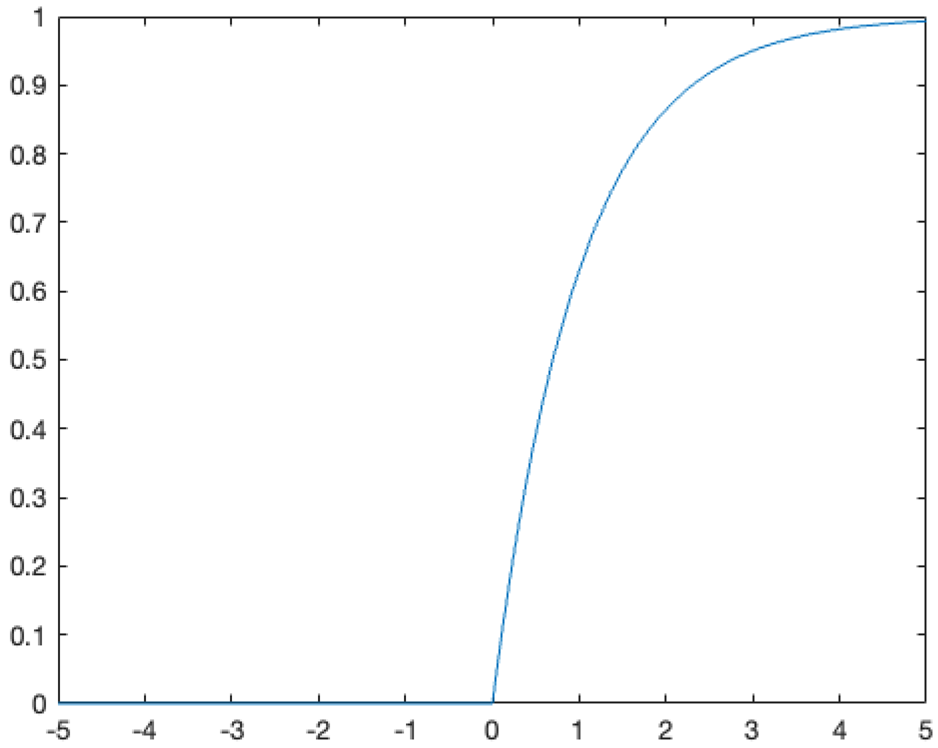
```
-(2*exp(-(alpha*t)/2)*(exp(-(alpha*t)/2)/2 - exp((alpha*t)/2)/2)*(sign(t)/2 +
1/2))/alpha
```

Plot the result for $\alpha = 1$

```
yb(t) = subs(y(t),alpha,1)
fplot(yb(t))
```

```
yb(t) =
```

```
-2*exp(-t/2)*(exp(-t/2)/2 - exp(t/2)/2)*(sign(t)/2 + 1/2)
```




## Example 5.5

Compute the output $y(t)$ for a continuous-time LTI system whose impulse response $h(t)$ and the input $x(t)$ are given by

$$h(t) = e^{-\alpha t}u_0(t)$$

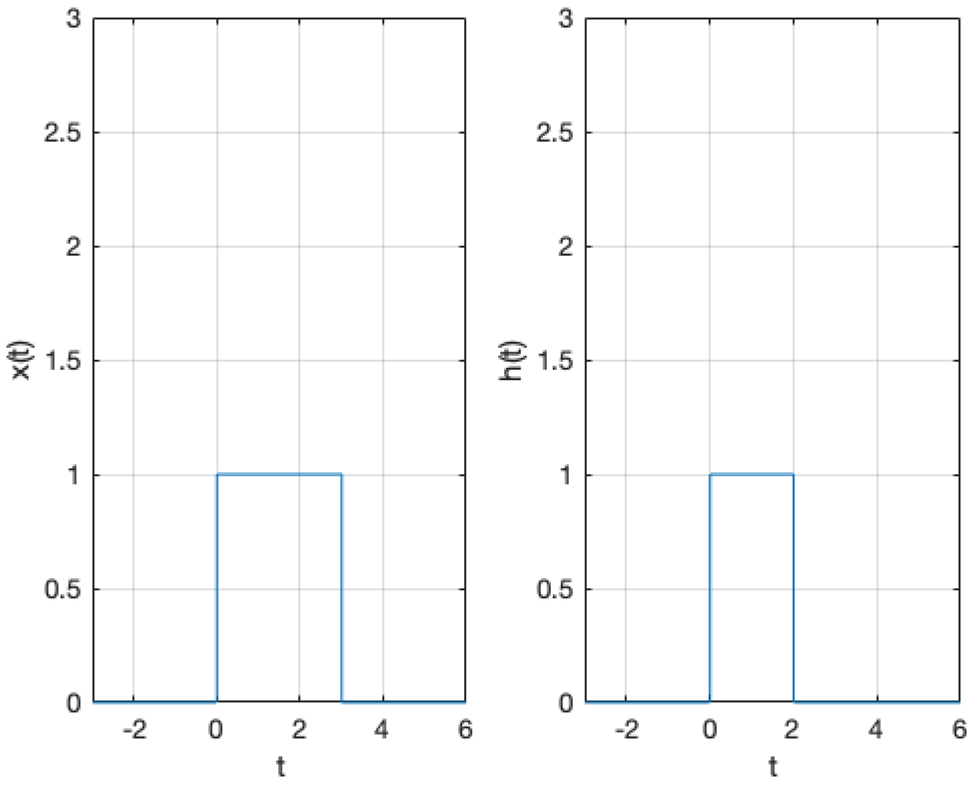
$$x(t) = e^{\alpha t}u_0(-t)$$

You can't use 'macro parameter character #' in math mode

$$h(t) = u_0(t) - u_0(t-2)$$

We will use the MATLAB Symbolic Math Toolbox:

```
x(t) = heaviside(t)-heaviside(t-3);
h(t) = heaviside(t)-heaviside(t-2);
subplot(121)
fplot(x(t),[-3,6]),grid,ylim([0,3]),ylabel('x(t)'),xlabel('t')
subplot(122)
fplot(h(t),[-3,6]),grid,ylim([0,3]),ylabel('h(t)'),xlabel('t')
```




Compute $y(t)$

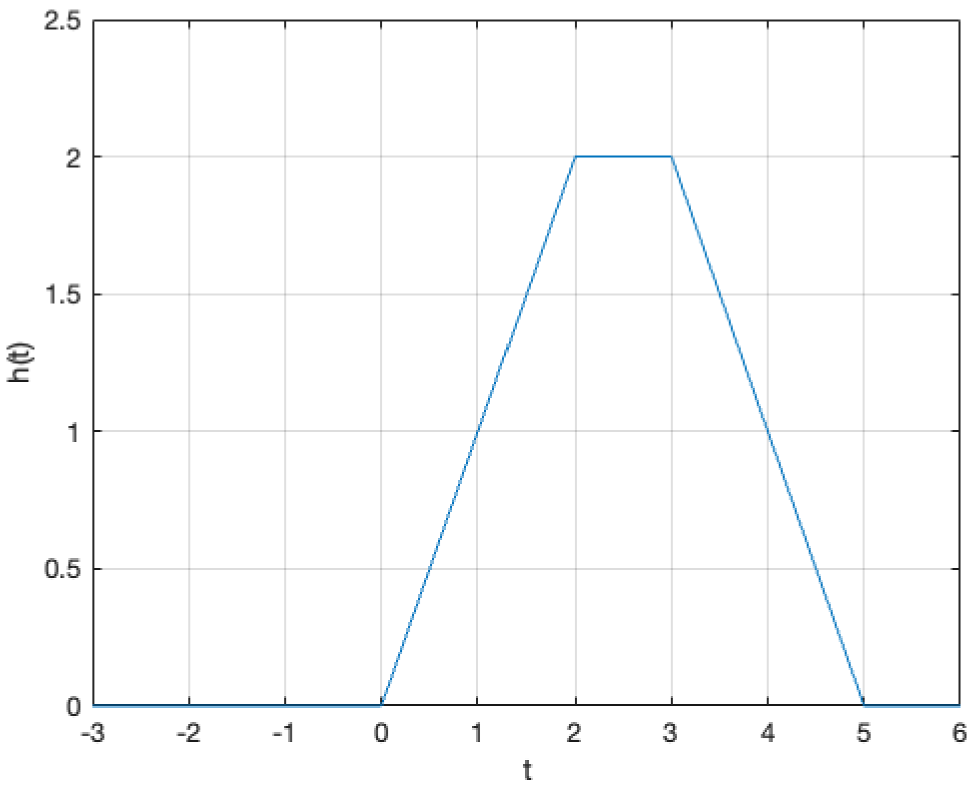
```
y(t) = int(x(tau)*h(t - tau),tau,-Inf,Inf)
```

```
y(t) =
```

```
heaviside(t - 5)*(t - 5) - heaviside(t - 3)*(t - 3) - heaviside(t - 2)*(t -
2) + t*heaviside(t)
```

Plot the result

```
fplot(y(t),[-3,6]),grid,ylim([0,2.5]),ylabel('h(t)'),xlabel('t')
```




We can obtain the signal $y(t)$ analytically by use of the convolution integral as will be shown in the examples class.

## (b) Graphical

Since both functions are unity between the limits set by the Heaviside function, graphical solution requires multiple applications of the definate integral

$$\int_{t_0}^{t_1} 1 \times 1 \, d\tau = \int_{t_0}^{t_1} 1 \, d\tau$$

with different values for the limits $t_0$ and $t_1$. The *convolutiondemo* tool can help us discover the limits for the piecewise continuous signal $y(t)$.

For the complete solution to Example 5.2 refer to the lecture recording or see solved problem 2.6 in in [Hsu, 2020].

## Summary

In this lecture we have looked at

- A. Impulse Response
- B. Response to an Arbitrary Input
- C. Convolution Integral
- D. Properties of the Convolution Integral
- E. Convolution Integral Operation
- F. Step Response

## Next Time

We continue our introduction to continuous-time LTI system by considering

- Properties of Continuous-Time LTI Systems
- Eigenfunctions of Continuous-Time LTI Systems

---

By Dr Chris P. Jobling


This page was created by Dr Chris P. Jobling for Swansea University .