Unit 3.3: Systems Described by Differential Equations

Contents

This section is based on Section 2.5 of [Hsu, 2020]

Follow along at cpjobling.github.io/eg-150-textbook/lti_systems/lti3

Subjects to be covered

We conclude our introduction to continuous-time LTI system by considering

- Continuous-time LTI systems described by differential equations
- Examples 8: Systems described by differential equations

Continuous-time LTI systems described by differential equations

A. Linear constant-coefficient differential equations

A general Nth-order linear constant-coefficient differential (LCCDE) equation is given by

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

where the coeefficients a_k and b_k are real constants.

The order N refers to the highest derivative of $y(t)$ in the differential equation.

Applications of linear constant-coefficient differential equations

LCCDEs play a central role in describing the input-output relationships of a wide variety of electrical, mechanical, chemical and biological systems.

Illustration: An RC Circuit

For instance, in the RC circuit considered in Example 4.1: RC Circuit, the input $x(t) = v_s(t)$ and the output $y(t) = v_c(t)$ are related by a first-order constant-coefficient differential equation

$$\frac{dy(t)}{dt} + \frac{1}{RC} y(t) = \frac{1}{RC} x(t)$$

So, by inspection, $N = 1$, $a_1 = 1$, $a_0 = b_0 = 1/RC$.

General solution of the general linear constant-coefficient differential equation

The general solution of the general linear constant-coefficient differential equation for a particular input $x(t)$ is given by

$$y(t) = y_p(t) + y_h(t)$$

where $y_p(t)$ is a *particular solution* satisfying the linear constant-coefficient differential equation and $y_h(t)$ is a *homegeneous solution* (or *complementary solution*) satisfying the homegeneous differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0$$

The exact form of $y_h(t)$ is determined by N auxiliary conditions.

Note that

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

does not completely specify the the output $y(t)$ in terms of $x(t)$ unless auxiliary conditions are defined. In general. a set of auxiliary conditions are the values of

$$y(t), \frac{dy(t)}{dt}, \ldots, \frac{d^{N-1}y(t)}{dt^N - 1}$$

at some point in time.

B. Linearity

The system defined by

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

will be linear only if all the auxilliary conditions are zero (see Example 8.5).

If the auxilliary conditions are not zero, then the response $y(t)$ of a system can be expressed as

$$y(t) = y_{\mathrm{zi}}(t) + y_{\mathrm{zs}}(t)$$

where $y_{\mathrm{zi}}(t)$ called the *zero-input response*, is the response to the aunxilliary conditions, and $y_{\mathrm{zs}}(t)$, called the *zero-state response*, is the response of a linear system with zero auxiliary conditions.

This is illustrated in Fig. 38

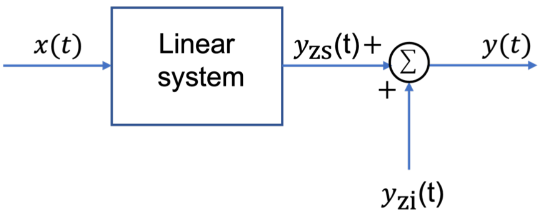

Fig. 38 Zero-state and zero-input responses

Note that $y_{\mathrm{zi}}(t) \neq y_h(t)$ and $y_{\mathrm{zs}}(t) \neq y_p(t)$ and that in general $y_{\mathrm{zi}}(t)$ contains $y_h(t)$ and $y_{\mathrm{zs}}(t)$ contains both $y_h(t)$ and $y_p(t)$ (see Example 8.4).

C. Causality

In order for the linear system described by a linear constant-coefficient differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

to be causal, we must assume the condition of *initial rest* (or an *initially relaxed condition*).

That is, if $x(y) = 0$ for $t \leq t_0$, then assume $y(t) = 0$ for $t \leq t_0$ (See Example 4.6).

Thus, the response for $t > t_0$ can be calculated from the linear constant-coefficient differential equation with the initial conditions

$$y(t_0) = \frac{dy(t_0)}{dt} = \cdots = \frac{d^{N-1}y(t_0)}{dt^{N-1}} = 0$$

where

$$\frac{d^k y(t_0)}{dt^k} = \left.\frac{d^k y(t)}{dt^k}\right|_{t=t_0}$$

Clearly, at initial rest, $y_{\mathrm{zs}}(t) = 0$.

D. Time-invariance

For a linear causal system, initial rest also implies time-invariance (Example 8.7).

E. Impulse response

The impulse response $h(t)$ of a linear constant-coefficient differential equation satisfies the differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k \delta(t)}{dt^k}$$

with the initial rest condition.

Examples of finding impulse responses are given in Example 8.7 to Example 8.3.

A peek into the future

Later in this course, and probably for the rest of your career, you will find the impulse response by using the Laplace transform.

Examples 8: Systems described by differential equations

Example 8.1

The continuous-time system shown in Fig. 39 consists of one integrator and one scalar multiplier. Write the differential equation that relates the output $y(t)$ to the input $x(t)$.

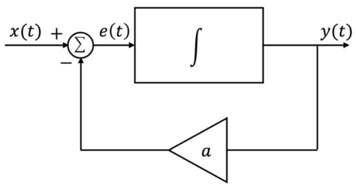

Fig. 39 A one-integrator linear system

For the answer, refer to the lecture recording or see solved problem 2.18 in [Hsu, 2020].

Example 8.2

The continuous-time system shown in Fig. 40 consists of two integrators and two scalar multipliers. Write the differential equation that relates the output $y(t)$ to the input $x(t)$.

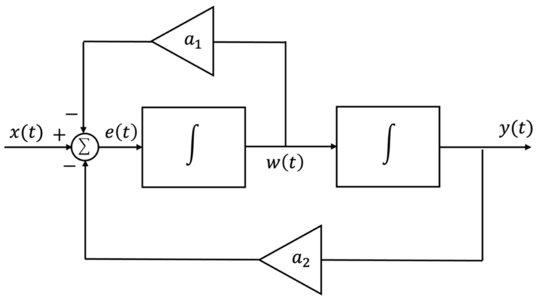

Fig. 40 A one-integrator linear system

For the answer, refer to the lecture recording or see solved problem 2.19 in in [Hsu, 2020].

Example 8.3

Consider the system described by

$$\frac{dy(t)}{dt} + 2y(t) = x(t) + \frac{dx(t)}{dt}$$

Find the impulse response $h(t)$ of the system.

For the answer, refer to the lecture recording or see solved problem 2.25 in [Hsu, 2020].

> **Note**
>
> As we will be moving on to show how differential equations can be solved by the Laplace transform, the remaining examples are optional and will not be examined.

Example 8.4

Consider a continuous-time system whose input $x(t)$ and output $y(t)$ are related by

$$\frac{dy(t)}{dt} + ay(t) = x(t)$$

where a is a constant.

(a) Find $y(t)$ with the auxilliary condition $y(0) = y_0$ and

$$x(t) = Ke^{bt}u_0(t)$$

(b) Express $y(t)$ in terms of the zero-input and zero-state responses.

For the answer, see the solved problem 2.20 in [Hsu, 2020].

Example 8.5

Consider the system in Example 8.4.

(a) Show that the system is not linear if $y(0) = y_0 \neq 0$.

(b) Show that the system is linear if $y(0) = 0$.

For the answer, see the solved problem 2.21 in [Hsu, 2020].

Example 8.6

Consider the system in Example 8.4. Show that the initial rest condition $y(0) = 0$ also implies that the system is time-invariant.

For the answer, see the solved problem 2.22 in [Hsu, 2020].

Example 8.7

Consider the system in Example 8.4. Find the impulse response $h(t)$ of the system.

For the answer, see the solved problem 2.23 in [Hsu, 2020].

Example 8.8

Consider the system in Example 8.4 with $y(0) = 0$.

(a) Find the step response $s(t)$ of the system without using the impulse response $h(t)$.

(b) Find the step response $s(t)$ of the system with the impulse response $h(t)$ obtained in Example 8.7.

(c) Find the impulse response $h(t)$ from the step response $s(t)$.

For the answer, see the solved problem 2.24 in [Hsu, 2020].

Summary

In this lecture we have concluded our introduction to LTI systems by looking at linear constant-coefficient differential equations.

Take aways

Continuous-Time LTI Systems Described by Differential Equations

- A. Linear constant-coefficient differential equations
- B. Linearity
- C. Causality
- D. Time-invariance
- E. Impulse response

Next Time

We move on to consider

- Unit 4: Laplace Transforms and their Applications

References

[Hsu20](**1**,**2**,**3**,**4**,**5**,**6**,**7**,**8**,**9**) Hwei P. Hsu. *Schaums outlines signals and systems*. McGraw-Hill, New York, NY, 2020. ISBN 9780071634724. Available as an eBook. URL: https://www.accessengineeringlibrary.com/content/book/9781260454246.

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