Unit 4.2: Laplace Transform of Some Common Signals

Contents

The preparatory reading for this section is Chapter 2.2 of [Karris, 2012] and Chapter 3.4 of [Hsu, 2020].

Follow along at cpjobling.github.io/eg-150-textbook/laplace_transform/2/laplace_of_common_signals

Agenda

A. Unit impulse function $\delta(t)$

$$\mathcal{L}\left\{\delta(t)\right\} = \int_{-\infty}^{\infty} \delta(t)e^{-st}\,dt$$

Using the sifting property of the Dirac delta function

$$\mathcal{L}\left\{\delta(t)\right\} = \int_{-\infty}^{\infty} \delta(t)e^{-st}\,dt = e^{-s(0)} = 1 \quad \text{all } s$$

Thus the Laplace transform pair is

$$\delta(t) \Leftrightarrow 1$$

for all s.

B. Delayed impulse function $\delta(t-a)$

$$\mathcal{L}\left\{\delta(t-a)\right\} = \int_{-\infty}^{\infty} \delta(t-a)e^{-st}\,dt$$

Using the sifting property of the Dirac delta function again

$$\mathcal{L}\left\{\delta(t)\right\} = \int_{-\infty}^{\infty} \delta(t-a)e^{-st}\,dt = e^{-as} \quad \sigma > 0$$

Thus the Laplace transform pair is

$$\delta(t-a) \Leftrightarrow e^{-as}$$

for $\mathrm{Re}(s) > 0$.

C. Unit step function $u_0(t)$

$$\mathcal{L}\left\{u_0(t)\right\} = \int_{-\infty}^{\infty} u_0(t)e^{-st}\,dt = \int_{0^+}^{\infty} e^{-st}\,dt$$

$$X(s) = -\frac{1}{s}e^{-st}\Big|_{0^+}^{\infty} = \frac{1}{s} \quad \mathrm{Re}(s) > 0$$

where $0^+ = \lim_\epsilon \to 0(0+\epsilon)$.

Thus the Laplace transform pair is

$$u_0(t) \Leftrightarrow \frac{1}{s}$$

for $\mathrm{Re}(s) > 0$.

D. Exponential function $x(t) = e^{-at}u_0(t)$

We already showed in Solved Problem 1 that

$$\mathcal{L}\left\{e^{-at}u_0(t)\right\} = \frac{1}{s+a}.$$

Thus the Laplace transform pair is

$$e^{-at}u_0(t) \Leftrightarrow \frac{1}{s+a}$$

for $\mathrm{Re}(s) > -a$.

E. Laplace Transform Pairs for Other Common SIgnals

We can continue to derive the Laplace transforms of the most commonly encoutered signals, and in some cases, e.g. unit ramp $r(t) = u_1(t) = tu_0(t)$, $u_n(t) = t^n u_0(t)$, $t^n e^{-at} u_0(t)$, $\cos \omega t u_0(t)$, $\sin \omega t u_0(t)$ and many others, that we use often, the mathematics can tricky.

Luckily for us, most of the most useful transforms have already been calculated for us and gathered together into tables.

Here are a couple that are on the net for your reference

- Laplace transform (Wikipedia)
- Laplace Transform (Wolfram Alpha)

Every textbook that covers Laplace transforms will provide tables of properties (see laplace_transfer_properties) and the most commonly encountered transforms. You will find such tables in Tables 2.1 and 2.2 of [Karris, 2012] and Table 3.1 and Table 3.3 of [Hsu, 2020].

We reproduce Table 3.1 of [Hsu, 2020] below. You will find a table of Laplace transform properties in the next chapter.

F. Transforms of Common Signals

	$f(t)$	$F(s)$	**ROC**
1	$\delta(t)$	1	All s
2	$\delta(t-a)$	e^{-as}	All s
3	$u_0(t)$	$\frac{1}{s}$	Re(s) > 0
4	$-u_0(-t)$	$\frac{1}{s}$	Re(s) < 0
5	$tu_0(t)$	$\frac{1}{s^2}$	Re(s) > 0
6	$t^n u_0(t)$	$\frac{n!}{s^{n+1}}$	Re(s) > 0
7	$e^{-at}u_0(t)$	$\frac{1}{s+a}$	Re(s) > $-$Re(a)
8	$-e^{-at}u_0(-t)$	$\frac{1}{s+a}$	Re(s)< $-$Re(a)
9	$t^n e^{-at}u_0(t)$	$\frac{n!}{(s+a)^{n+1}}$	Re(s) > $-$Re(a)
10	$-t^n e^{-at}u_0(-t)$	$\frac{n!}{(s+a)^{n+1}}$	Re(s) < $-$Re(a)
11	$\sin(\omega t)u_0(t)$	$\frac{\omega}{s^2+\omega^2}$	Re(s) > 0
12	$\cos(\omega t)u_0(t)$	$\frac{s}{s^2+\omega^2}$	Re(s) > 0
13	$e^{-at}\sin(\omega t)u_0(t)$	$\frac{\omega}{(s+a)^2+\omega^2}$	Re(s) > $-$Re(a)
14	$e^{-at}\cos(\omega t)u_0(t)$	$\frac{s+a}{(s+a)^2+\omega^2}$	Re(s) > $-$Re(a)

Refer to the Chapter 2.3 of [Karris, 2012] if you want to study the proofs of these transforms.

Don’t panic

Tables of Laplace transform properties and transforms will be included with the exam paper.

G. MATLAB Examples

Let's use the MATLAB Symbolic Math Toolbox to prove some of these transforms.

```
format compact
syms s t a omega
assume(a>0)
assume(omega>0)
u0(t) = heaviside(t);
```

Impulse $\delta(t)$

```
laplace(dirac(t))
```

```
ans =
```

```
1
```

Delayed impulse $\delta(t-a)$

```
laplace(dirac(t-a))
```

```
ans =
```

```
exp(-a*s)
```

Unit step $u_0(t)$

```
laplace(u0(t))
```

```
ans =
```

```
1/s
```

For some functions we need to define the ROC and use the integral directly

```
laplace(-u0(-t))
```

```
ans =
```

```
0
```

```
assume(real(s) < 0)
int(-u0(-t)*exp(-s*t),t,-inf,0)
```

```
ans =
```

```
1/s
```

Unit ramp $u_1(t) = tu_0(t)$

```
laplace(t*u0(t))
```

```
ans =
```

```
1/s^2
```

Powers of t

```
syms n integer
assume(n > 1)
laplace(t^n*u0(t))
```

```
ans =
```

```
gamma(n + 1)/s^(n + 1)
```

Exponentials

```
laplace(exp(-a*t)*u0(t))
```

```
ans =
```

```
1/(a + s)
```

```
laplace(-exp(-a*t)*u0(-t))
```

```
ans =
```

```
0
```

Defining the ROC and using `int`

```
assume(s + a < 0)
int(-exp(-a*t)*u0(-t)*exp(-s*t),t,-inf,0)
```

```
ans =
```

```
1/(a + s)
```

```
laplace(t*exp(-a*t)*u0(t))
```

```
ans =
```

```
1/(a + s)^2
```

```
syms n integer
laplace(t^n*exp(-a*t)*u0(t))
```

```
ans =
```

```
piecewise(0 <= n, gamma(n + 1)/(a + s)^(n + 1))
```

Sinusoids

```
laplace(cos(omega*t)*u0(t))
laplace(sin(omega*t)*u0(t))
```

```
ans =
```

```
s/(omega^2 + s^2)
```

```
ans =
```

```
omega/(omega^2 + s^2)
```

Decaying sinusoids

```
laplace(exp(-a*t)*cos(omega*t)*u0(t))
laplace(exp(-a*t)*sin(omega*t)*u0(t))
```

```
ans =
```

```
(a + s)/((a + s)^2 + omega^2)
```

```
ans =
```

```
omega/((a + s)^2 + omega^2)
```

Next Time

We move on to consider

- ../3/laplace_properties.md

References

[Hsu20](1,2,3) Hwei P. Hsu. *Schaums outlines signals and systems*. McGraw-Hill, New York, NY, 2020. ISBN 9780071634724. Available as an eBook. URL: https://www.accessengineeringlibrary.com/content/book/9781260454246.

[Kar12](1,2,3) Steven T. Karris. *Signals and systems with MATLAB computing and Simulink modeling*. Orchard Publishing, Fremont, CA., 2012. ISBN 9781934404232. Library call number: TK5102.9 K37 2012. URL: https://ebookcentral.proquest.com/lib/swansea-ebooks/reader.action?docID=3384197.