Unit 2.3: Elementary Signals

Contents

The preparatory reading for this section is Chapter 1 of [Karris, 2012] which

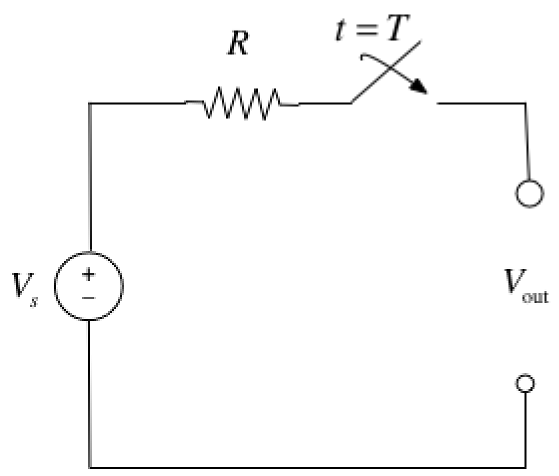
- begins with a discussion of the elementary signals that may be applied to electrical circuits
- introduces the unit step, unit ramp and dirac delta functions
- presents the sampling and sifting properties of the delta function and
- concludes with examples of how other useful signals can be synthesised from these elementary signals.

Additional information has been adapted from Section 1.4 of [Hsu, 2020].

Follow along at cpjobling.github.io/eg-150-textbook/signals_and_systems/elementary_signals

Introduction

Consider the network shown in below where the switch is closed at time $t = T$ and all components are ideal.



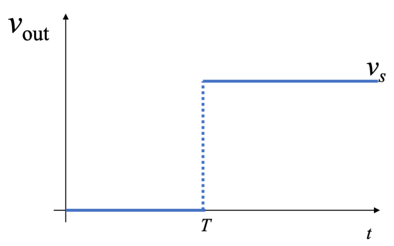
Express the output voltage V_{out} as a function of the unit step function, and sketch the appropriate waveform.

Solution

Before the switch is closed at $t < T$: \begin{equation} V_{\mathrm{out}} = 0. \end{equation}

After the switch is closed for $t > T$: \begin{equation} V_{\mathrm{out}} = V_s. \end{equation}

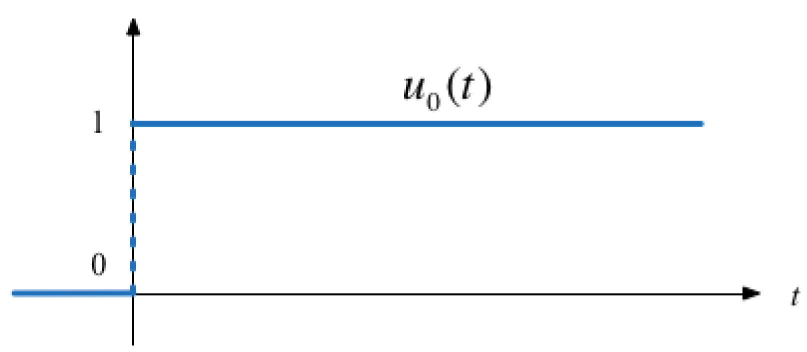
We imagine that the voltage jumps instantaneously from 0 to V_s volts at $t = T$ seconds as shown below.



We call this type of signal a step function.

The Unit Step Function

$$u_0(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$



In Matlab

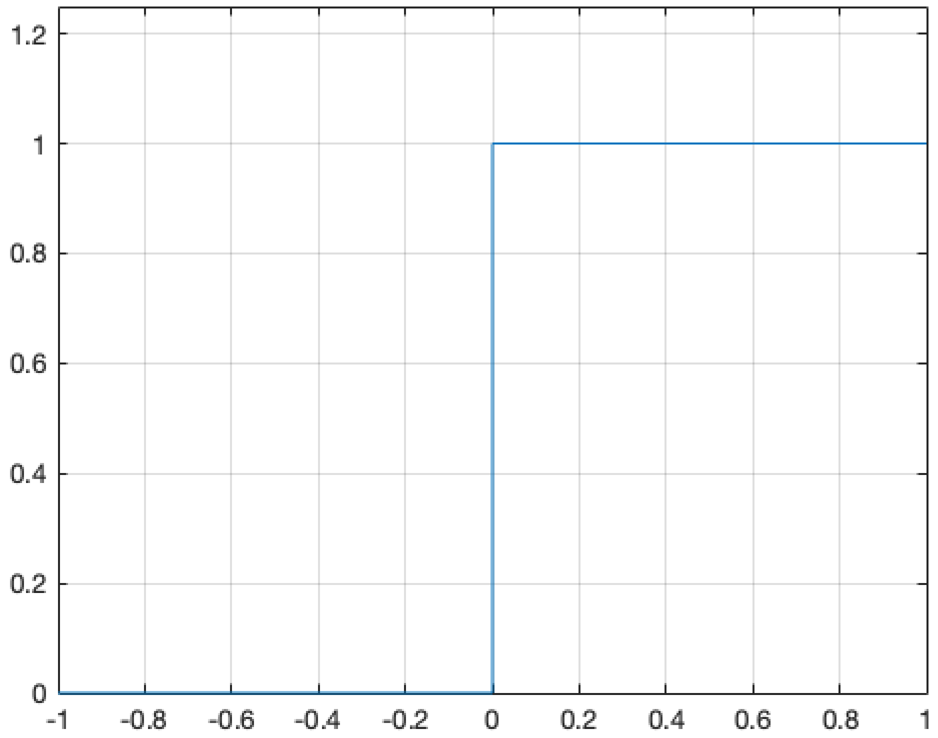
In Matlab, we use the `heaviside` function (named after [Oliver Heaviside](#)).

```
%%file plot_heaviside.m
syms t
fplot(heaviside(t),[-1,1]),grid,ylim([0 1.25])
heaviside(0)
```

```
Created file '/Users/eechris/code/src/github.com/cpjobling/eg-150-
textbook/signals_and_systems/elementary_signals/plot_heaviside.m'.
```

```
plot_heaviside
```

```
ans =
    0.5000
```



Note that, so that it can be plotted, Matlab defines the *Heaviside function* slightly differently from the mathematically ideal unit step:

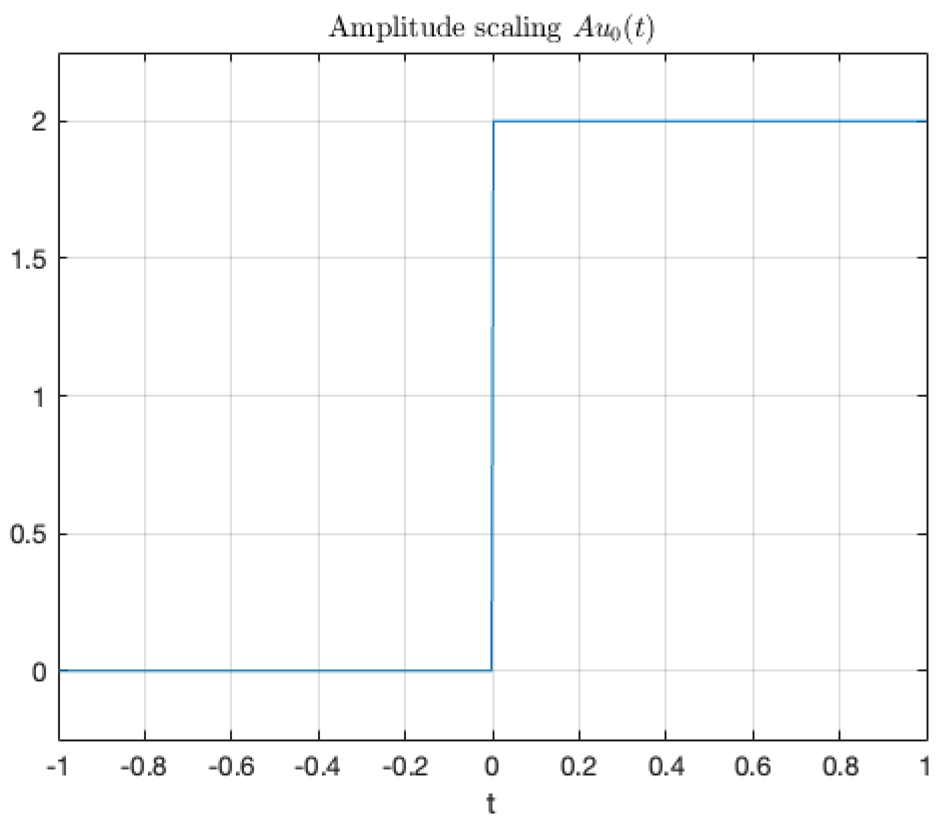
$$\text{heaviside}(t) = \begin{cases} 0 & t < 0 \\ 1/2 & t = 0 \\ 1 & t > 0 \end{cases}$$

Simple Signal Operations

Amplitude Scaling

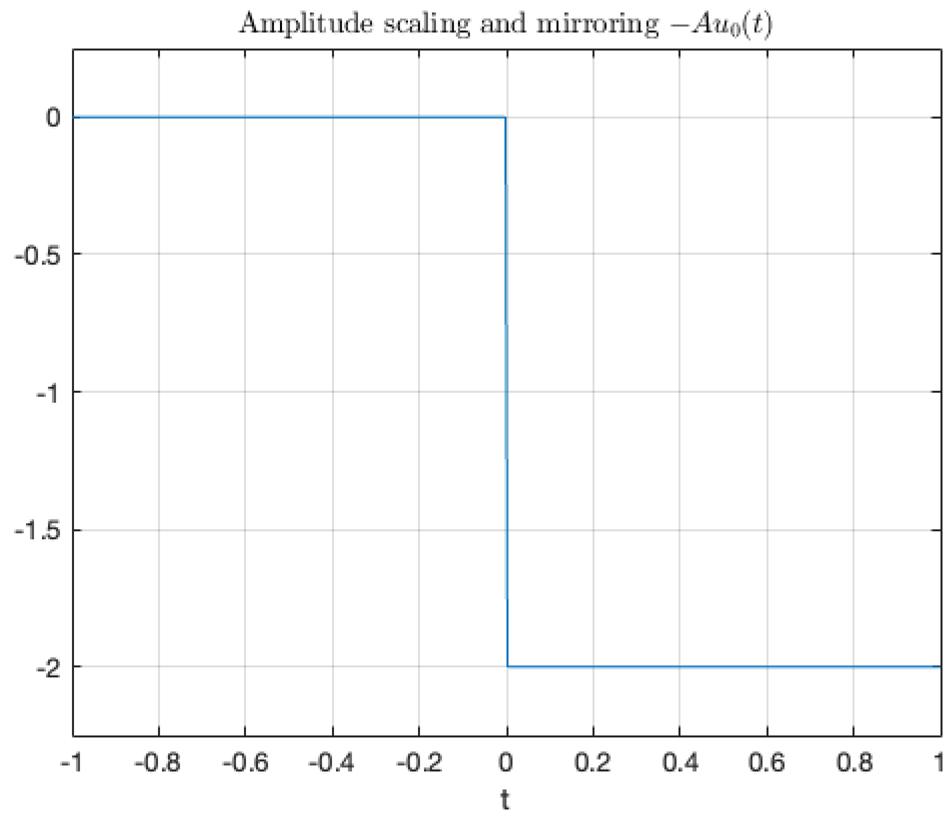
Sketch $Au_0(t)$ and $-Au_0(t)$

```
syms t;
u0(t) = heaviside(t); % rename heaviside function for ease of use
A = 2; % so signal can be plotted
ezplot(A*u0(t),[-1,1]),grid,title('Amplitude scaling
$$Au_0(t)$$','interpreter','latex')
```



Note that the signal is scaled in the y direction.

```
ezplot(-A*u0(t),[-1,1]),grid,title('Amplitude scaling and mirroring $$-Au_0(t)$$','interpreter','latex')
```

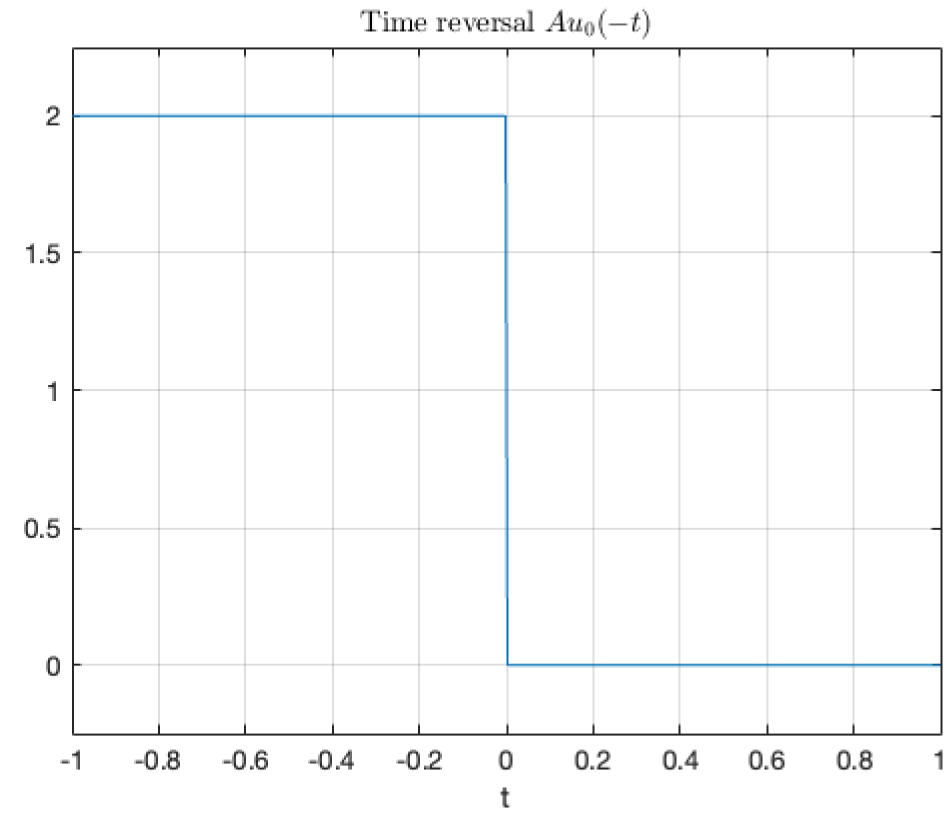


Note that, because of the sign, the signal is mirrored about the x axis as well as being scaled by 2.

Time Reversal

Sketch $u_0(-t)$

```
ezplot(A*u0(-t),[-1,1]),grid,title('Time reversal $$Au_0(-t)$$','interpreter','latex')
```

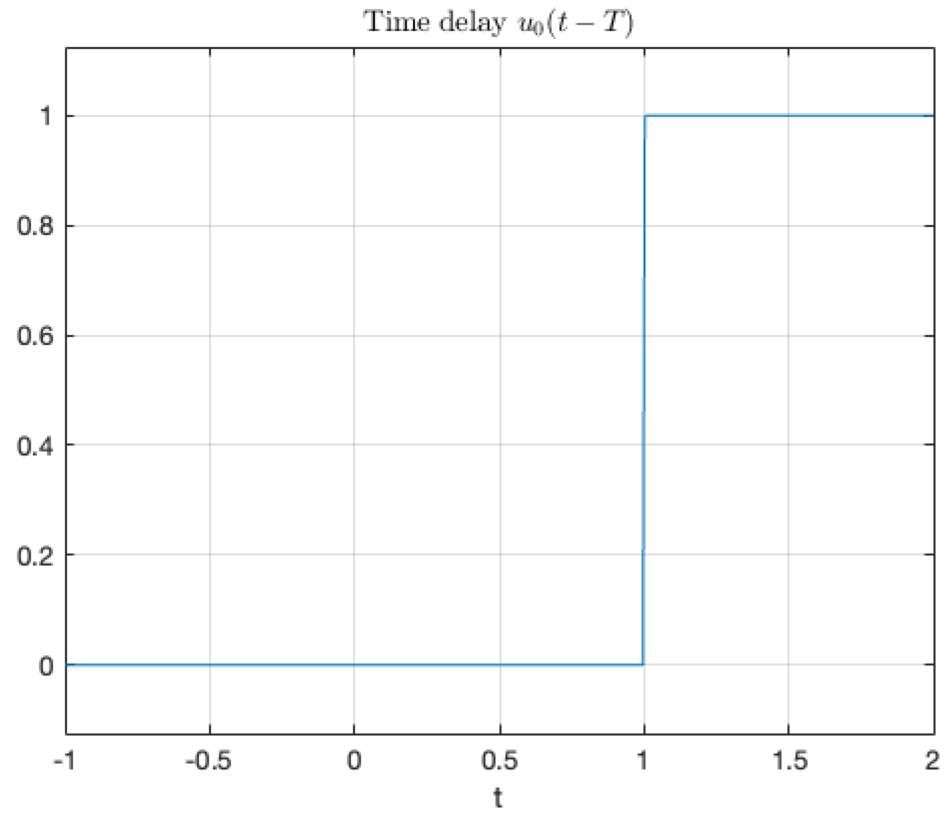


The sign on the function argument $-t$ causes the whole signal to be reversed in time. Note that another way of looking at this is that the signal is mirrored about the y axis.

Time Delay and Advance

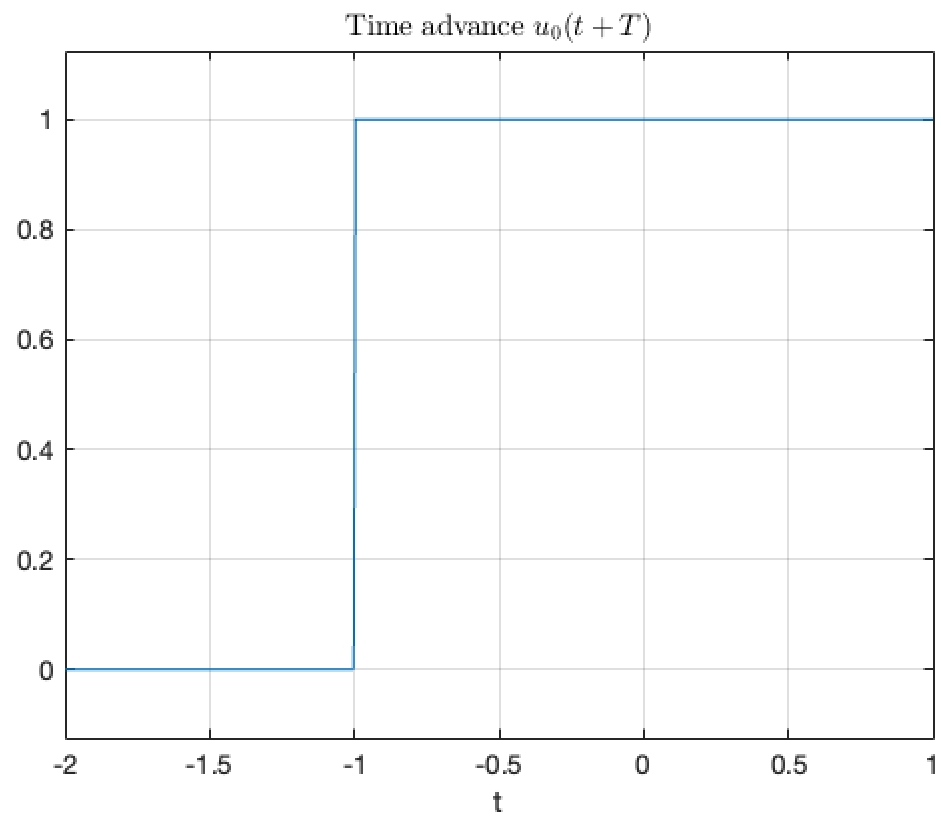
Sketch $u_0(t - T)$ and $u_0(t + T)$

```
T = 1; % again to make the signal plottable.
ezplot(u0(t - T),[-1,2]),grid,title('Time delay $$u_0(t - T)$$','interpreter','latex')
```



This is a *time delay* ... note for $u_0(t-T)$ the step change occurs T seconds **later** than it does for $u_o(t)$.

```
ezplot(u0(t + T),[-2,1]),grid,title('Time advance $$u_0(t +
T)$$','interpreter','latex')
```



This is a *time advance* ... note for $u_0(t+T)$ the step change occurs T seconds **earlier** than it does for $u_o(t)$.

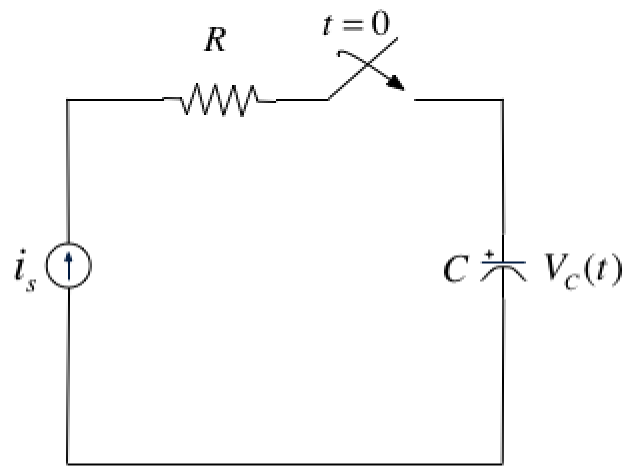
Examples

We will work through some examples in class. See Examples 3.

Synthesis of Signals from the Unit Step

Unit step functions can be used to represent other time-varying functions such as rectangular pulses, square waves and triangular pulses. See Worksheet 3 for the examples that we will look at in class.

The Ramp Function



In the circuit shown above i_s is a constant current source and the switch is closed at time $t = 0$.

When the current through the capacitor $i_c(t) = i_s$ is a constant and the voltage across the capacitor is

$$v_c(t) = \frac{1}{C}\int_{-\infty}^{t} i_c(\tau)\,d\tau$$

where τ is a dummy variable.

Since the switch closes at $t = 0$, we can express the current $i_c(t)$ as

$$i_c(t) = i_s u_0(t)$$

and if $v_c(t) = 0$ for $t < 0$ we have

$$v_c(t) = \frac{i_s}{C}\int_{-\infty}^{t} u_0(\tau)\,d\tau = \underbrace{\frac{i_s}{C}\int_{-\infty}^{0} 0\,d\tau}_{0} + \frac{i_s}{C}\int_{0}^{t} 1\,d\tau$$

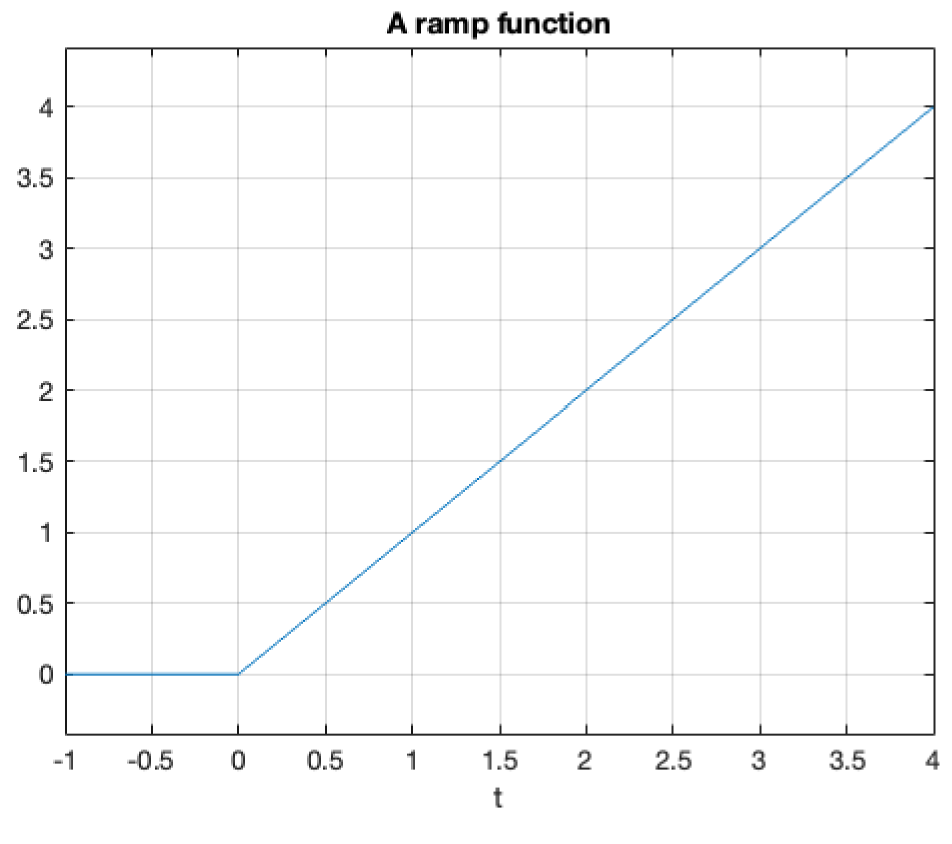
So, the voltage across the capacitor can be represented as

$$v_C(t) = \frac{i_s}{C} t u_0(t)$$

Note that in this as in other examples throughout these notes, and in published tables of transforms, the inclusion of $u_0(t)$ in $v_c(t)$ acts as a “gating function” that limits the definition of the signal to the causal range $0 \le t < \infty$.

To sketch the wave form, let’s arbitrarily let C and i_s be one and then plot with MATLAB.

```
C = 1; is = 1;
vc(t)=(is/C)*t*u0(t);
ezplot(vc(t),[-1,4]),grid,title('A ramp function')
```



This type of signal is called a **ramp function**. Note that it is the *integral* of the step function (the resistor-capacitor circuit implements a simple integrator circuit).

The unit ramp function is defined as

$$u_1(t) = \int_{-\infty}^{t} u_0(\tau)d\tau$$

so

$$u_1(t) = \begin{cases} 0 & t < 0 \\ t & t \geq 0 \end{cases}$$

and

$$u_0(t) = \frac{d}{dt}u_1(t)$$

Note

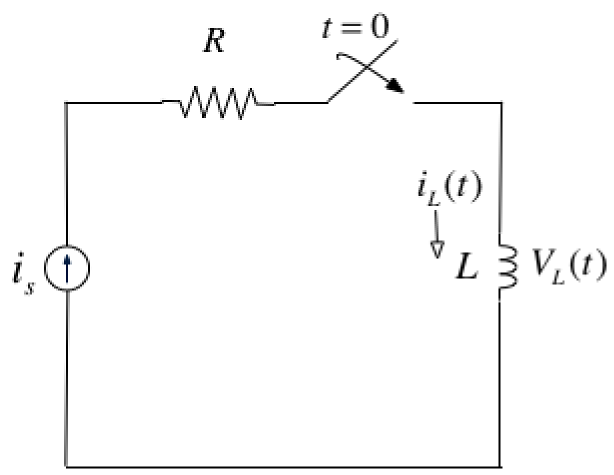
Higher order functions of t can be generated by the repeated integration of the unit step function.

For future reference, you should determine $u_2(t)$, $u_3(t)$ and $u_n(t)$ for yourself and make a note of the general rule:

$$u_{n-1} = \frac{1}{n}\frac{d}{dt}u_n(t)$$

Details are given in equations 1.26—1.29 in Karris.

The Dirac Delta Function



In the circuit shown above, the switch is closed at time $t = 0$ and $i_L(t) = 0$ for $t < 0$. Express the inductor current $i_L(t)$ in terms of the unit step function and hence derive an expression for $v_L(t)$.

Solution

$$v_L(t) = L\frac{di_L}{dt}$$

Because the switch closes instantaneously at $t = 0$

$$i_L(t) = i_s u_0(t)$$

Thus

$$v_L(t) = i_s L\frac{d}{dt}u_0(t).$$

The unit Impulse Function

The unit impulse function $\delta(t)$, is the derivative of the unit step.

$$\delta(t) = \frac{d}{dt} u_0(t)$$

which is tricky to compute because $u_0(t)$ is discontinuous at $t = 0$ but it must have the properties

$$\int_{-\infty}^{t} \delta(\tau) d\tau = u_0(t)$$

To solve this problem we need to invent a function that represents the derivative of the unit step function. This function is called the *unit impulse function* $\delta(t)$, also known as the *Dirac delta* function (named after Paul Dirac).

Traditionally, $\delta(t)$ is often defined as the limit of a suitably chosen conventional function having unity area over an infinitesimal time interval as shown in Fig. Fig. 21.

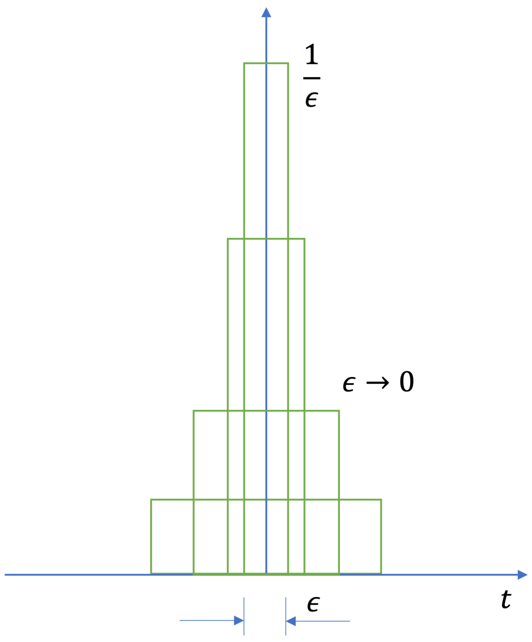


Fig. 21 Visualisation of the Dirac delta function as the limit of a conventional function with unit area.

The Dirac delta posseses the following properties

$$\delta\left(t\right) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

$$\int_{\epsilon}^{-\epsilon} \delta(t)\, dt = 1$$

The unit impulse function plays a fundamental role in systems analysis.

Sketch of the delta function

Continuing the example, and replacing the derivative of the unit step $u_0(t)$ with the unit impulse $\delta(t)$

$$V_{\text{out}}(t) = V_L(t) = i_s L \delta(t)$$

Note when we draw the unit impulse we show the height of $\delta(t)$ as one so the height of the impulse in the figure is $i_s L$.

MATLAB Confirmation

```
syms is L;
vL(t) = is * L * diff(u0(t))
```

```
vL(t) =
```

```
L*is*dirac(t)
```

Note that we can't plot dirac(t) in MATLAB with `fplot`.

Important properties of the delta function

Sampling Property

The *sampling property* of the delta function states that

$$f(t)\delta(t-a) = f(a)\delta(t-a)$$

or, when $a = 0$,

$$f(t)\delta(t) = f(0)\delta(t)$$

Multiplication of any function $f(t)$ by the delta function $\delta(t)$ results in sampling the function at the time instants for which the delta function is not zero.

The study of descrete-time (sampled) systems is based on this property.

You should work through the proof for youself.

Sifting Property

The *sifting property* of the delta function states that

$$\int_{-\infty}^{\infty} f(t)\delta(t-\alpha)dt = f(\alpha)$$

That is, if multiply any function $f(t)$ by $\delta(t-\alpha)$, and integrate from $-\infty$ to $+\infty$, we will get the value of $f(t)$ evaluated at $t = \alpha$.

You should also work through the proof for yourself.

Higher Order Delta Fuctions

the n*th*-order *delta function* is defined as the n*th* derivative of $u_0(t)$, that is

$$\delta^n(t) = \frac{d^n}{dt^n}[u_0(t)]$$

The function $\delta'(t)$ is called the *doublet*, $\delta''(t)$ is called the *triplet* and so on.

By a procedure similar to the derivation of the sampling property we can show that

$$f(t)\delta'(t-a) = f(a)\delta'(t-a) - f'(t)\delta(t-a)$$

Also, derivation of the sifting property can be extended to show that

$$\int_{-\infty}^{\infty} f(t)\delta^n(t-\alpha)dt = (-1)^n \frac{d^n}{dt^n}[f(t)]\bigg|_{t=\alpha}$$

Summary

In this chapter we have looked at some elementary signals and the theoretical circuits that can be used to generate them.

Takeaways

- You should note that the unit step is the *heaviside function* $u_0(t)$.
- Many useful signals can be synthesized by use of the unit step as a "gating function" in combination with other signals
- That unit ramp function $u_1(t)$ is the integral of the step function.
- The *Dirac delta* function $\delta(t)$ is the derivative of the unit step function. We sometimes refer to it as the *unit impulse function*.
- The delta function has sampling and sifting properties that will be useful in the development of *time convolution* and *sampling theory*.

Examples

We will do some of these in class. See Examples 3.

Homework

These are for you to do later for further practice. See Homework 1.

References

See Bibliography

Next Time

Systems and Classification of Systems

- System Representation
- Deterministic and Stochastic Systems
- Continuous-Time and Discrete-Time Systems
- Systems with Memory and without Memory
- Causal and Non-Causal Systems
- Linear Systems and Nonlinear Systems
- Linear Time-Invariant Systems
- Stable Systems
- Feedback Systems

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This page was created by Dr Chris P. Jobling for Swansea University .