Unit 2.4: Systems and Classification of Systems

Contents

This section is based on Section 1.5 of [Hsu, 2020].

Follow along at cpjobling.github.io/eg-150-textbook/signals_and_systems/systems

Subjects to be covered

- System Representation
- Deterministic and Stochastic Systems
- Continuous-Time and Discrete-Time Systems
- Systems with Memory and without Memory
- Causal and Non-Causal Systems
- Linear Systems and Nonlinear Systems
- Time-Invariant and Time-Varying Systems
- Linear Time-Invariant Systems
- Stable Systems
- Feedback Systems
- Examples 4

System Representation

A *system* is a mathematical model of a physical process that relates the *input* (or *excitation*) signal to the *output* (or *response*) signal.

Let x and y be the input and output signals, respectively, of a system. Then the system is viewed as a *transformation* (or *mapping*) of x into y. The transformation is represented by the mathematical notation

$$y = \mathbf{T}x$$

where $\mathbf{T}$ is the *operator* representing some well defined rule by which x is transformed into y.

The relationship is depicted graphically as shown in Fig. 25(a).

Multiple input and/or output systems are possible as shown in Fig. 25(b). In this module we will restrict our attention to the single-input, single-output case.

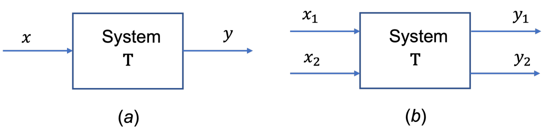

Fig. 25 System with single or multiple inputs and outputs

Deterministic and Stochastic Systems

If the input and output signals x and y are deterministic signals, then the system is called a *deterministic* system.

If the input and output signals x and y are random signals, then the system is called a *stochastic* system.

Continuous-Time and Discrete-Time Systems

If the input and output signals x and y are continuous-time signals, then the system is called a *continuous-time system* (Fig. 26(a)).

If the input and output signals x and y are discrete-time signals or sequences, then the system is called a *discrete-time system* (Fig. 26(b)).

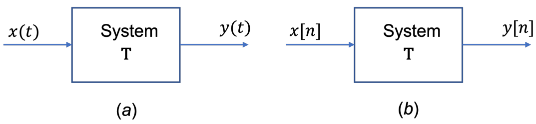

Fig. 26 (a) Continuous-time system; (b) discrete time system.

Note that in a continuous-time system the input $x(t)$ and $y(t)$ are often expressed as a *differential equation* (see Examples 4) and in a discrete-time system $x[n]$ and $y[n]$ are often expressed by a *difference equation*.

Systems with Memory and without Memory

A system is said to be *memoryless* if the output at any time only depends on the input at the same time.

Otherwise the system is said to have *memory*.

A memoryless system

An example of a memoryless system is a resistor R with and the input $x(t)$ taken as the current and the voltage taken as the output $y(t)$.

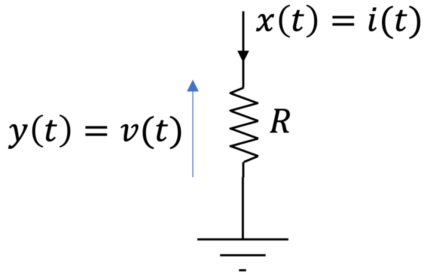

Fig. 27 A memoryless system: a resistor

The input-output relationship (Ohm's law) of a resistor is

$$y(t) = Rx(t)$$

A system with memory

An example of a system with memory is a capacitor C with and the current as the input $x(t)$ taken as the current and the voltage as the output $y(t)$.

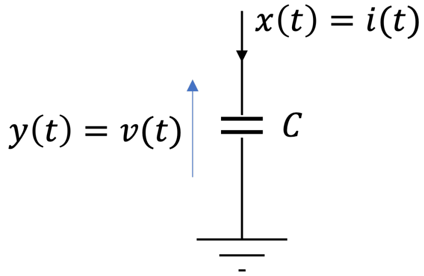

Fig. 28 A system with memory: a capacitor

Then:

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\, d\tau$$

Causal and Non-Causal Systems

A system is called *causal* if its output at the present time depends only on the present and/or past values of the input.

Thus, in a causal system, it is not possible to obtain an output before an input is applied to the system.

A system is called *noncausal* (or *anticipative*) if its output at the present time depends on future values of the input.

An example of a noncausal system is

$$y(t) = x(t+1)$$

Note that all memoryless systems are causal but not all *vice versa*.

Linear Systems and Nonlinear Systems

If an operator $\mathbf{T}$ satisfies the following two conditions, then $\mathbf{T}$ is called a *linear operator* and the system represented by the linear operator $\mathbf{T}$ is called a *linear system*:

Properties of Linear Systems

1. Additivity

Given that $\mathbf{T}\left\{x_1\right\} = y_1$ and $\mathbf{T}\left\{x_2\right\} = y_2$,

then

$$\mathbf{T}\left\{x_1 + x_2\right\} = y_1 + y_2$$

for any signals x_1 and x_2.

2. Homogeneity (or *Scaling*)

$$\mathbf{T}\left\{\alpha x\right\} = \alpha y$$

for any signals x and any scalar α.

Nonlinear systems

Any system that does not satisfy the additivity and homogeneity conditions is classified as a *nonlinear system*.

Superposition property

The additivity and homogeneity conditions can combined in a single condition (known as the *superposition property*) as

$$\mathbf{T}\left\{\alpha_1 x_1 + \alpha_2 x_2\right\} = \alpha_1 y_1 + \alpha_2 y_2$$

where α_1 and α_2 are arbitrary scalars.

Example linear systems

Examples of linear systems are the resistor and capacitor discussed earlier.

Example nonlinear systems

Examples of nonlinear systems are

$$y = x^2$$

$$y = \cos x$$

Zero input property

Note that a consequence of the homegenity (or scaling) property of linear systems is that a *zero input yields a zero output*. This follows readilty by setting $\alpha = 0$ in the equation $\mathbf{T}\left\{\alpha x\right\} = \alpha y$. This is another important property of linear systems.

Time-Invariant and Time-Varying Systems

A system is called *time-invariant* if a time-shift (delay or advance) in the input signal causes the same time-shift in the output signal.

Thus for a continuous-time system, the system is time-invariant if

$$\mathbf{T}\left\{x(t-\tau)\right\} = y(t-\tau)$$

for any real value of τ.

Time-varying system

A system that does not satisfy the equation $\mathbf{T}\left\{x(t-\tau)\right\} = y(t-\tau)$ is called a *time-varying system*.

Testing for time-invariance

To check for time invariance, we can compare the time-shifted output with the output produced by the time-shifted input (See Example 4.2: Capacitor circuit and Example 4.3: Signal modulator).

Linear Time-Invariant Systems

If a system is linear and also time-invariant it is called a *linear time-invariant* (LTI) system.

All ths systems analysed in in the rest of the module and in **EG-247 Digital Signal Processing** and **EG-243 Control Systems** next year will be LTI systems.

Stable Systems

A system is *bounded-input/bounded-output* (BIBO) *stable* if for any bounded input signal x defined by

$$|x| \leq k_1$$

the corresponding output y is also bounded defined by

$$|y| \leq k_2$$

where k_1 and k_2 are finite real constants.

Unstable systems

An *unstable* system is one in which not all bounded inputs lead to a bounded output.

For example, consider the system where output

$$y(t) = tx(t)$$

and input $x(t)$ is the unit step $u_0(t)$

In this case $x(t) = 1$ (so is bounded) but the output $y(t)$ increases without bound as t increases.

Feedback Systems

A special class of systems of great importance consists of systems having *feedback*.

In a *feedback system*, a portion of the output signal is fed back and added to the input as shown in Fig. 29.

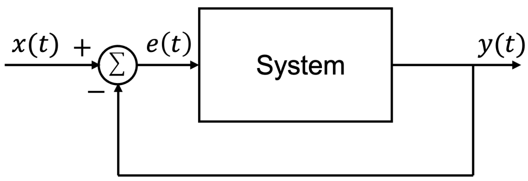

Fig. 29 A feedback system with negative feedback: $e(t) = x(t) - y(t)$.

You will see examples of systems with feedback when you study op-amp circuits in **EG-152 Practical Electronics**, the simple closed-loop systems to be studied in **EG-142 Instrumentation and Control**. Feedback, and its impact on system stability, is also the basis of control theory to be studied next year in **EG-243 Control Systems**.

Examples 4

MATLAB Example

We will solve this example by hand and then give the solution in the MATLAB lab.

Example 4.1: RC Circuit

Consider the RC circuit shown in Fig. 30. Find the relationship between the input $x(t)$ and the output $y(t)$

(a) If $x(t) = v_s(t)$ and $y(t) = v_c(t)$.

(b) If $x(t) = v_s(t)$ and $y(t) = i(t)$.

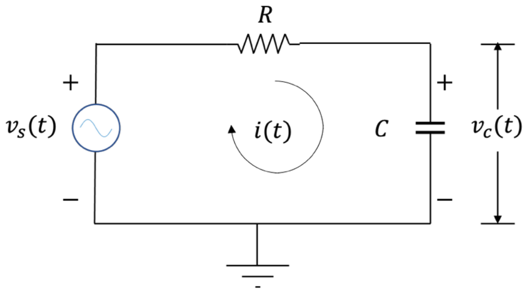

Fig. 30 RC circuit

For the answer, refer to the lecture recording or see solved problem 1.32 in [Hsu, 2020].

Example 4.2: Capacitor circuit

MATLAB Example

We will solve this example by hand and then give the solution in the MATLAB lab.

Consider the capacitor shown in Fig. 31. Let the input $x(t) = i(t)$ and the output $y(t) = v_c(t)$.

(a) Find the input-output relationship.

(b) Determine whether the system is (i) memoryless, (ii) causal, (iii) linear, (iv) time invariant, or (v) stable.

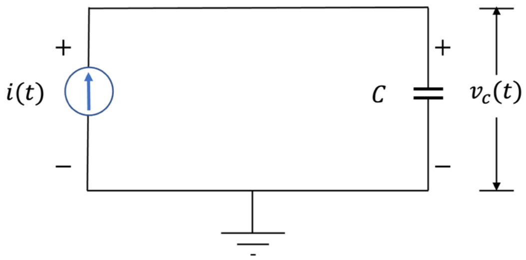

Fig. 31 A capacitor citcuit.

For the answer, refer to the lecture recording or see solved problem 1.33 in [Hsu, 2020].

Example 4.3: Signal modulator

MATLAB Example

We will solve this example by hand and then give the solution in the MATLAB lab.

Consider the system shown in Fig. 32. Determine whether it is (a) memoryless, (b) causal, (c) linear, (d) time invariant, or (e) stable.

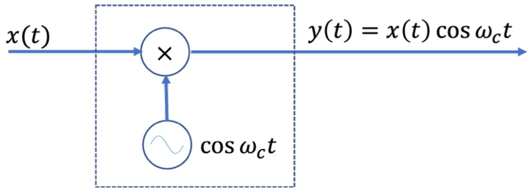

Fig. 32 A signal modulator

For the answer, refer to the lecture recording or see solved problem 1.34 in [Hsu, 2020].

Example 4.4

A system has the input-output relationship given by

$$y = \mathbf{T}\{x\} = x^2$$

Show that this system is nonlinear.

For the answer, refer to the lecture recording or see solved problem 1.35 in [Hsu, 2020].

Example 4.5

Consider the system whose input-output relationship is given by the linear equation

$$y = ax + b$$

where x and y are the input and output respectively and s and b are constant. Is this system linear?

For the answer, refer to the lecture recording or see solved problem 1.40 in [Hsu, 2020].

Example 4.6

(a) Show that the causality for a continuous-time linear system is equivalent to the following statement: For any time t_0 and any input $x(t)$ with $x(t) = 0$ for $t \leq t_0$, the output $y(t)$ is zero for $t \leq t_0$.

(b) Find a nonlinear system that is causal but does not satisfy this condition.

(c) Find a nonlinear system that satisfies this condition but is not causal.

For the answer, refer to the lecture recording or see solved problem 1.43 in [Hsu, 2020].

Example 4.7

Let $\mathbf{T}$ represent a continuous-time LTI system. Then show that

$$\mathbf{T}\left\{e^{st}\right\} = \lambda e^{st}$$

where s is a complex variable and λ is a complex constant.

> **Note**
>
> The solution to this problem is part of a journey that leads us from continuous-time LTI systems to the Laplace transform. I have therefore included the solution in full.

Solution

Let $y(t)$ be the output of the system with input $x(t) = e^{st}$. Then

$$\mathbf{T}\left\{e^{st}\right\} = y(t)$$

Since the system is time-invariant, we have

$$\mathbf{T}\left\{e^{s(t+t_0)}\right\} = y(t+t_0)$$

for arbitrary real t_0. Since the system is linear, we have

$$\mathbf{T}\left\{e^{s(t+t_0)}\right\} = \mathbf{T}\left\{e^{st}e^{st_0}\right\} = e^{st_0}\mathbf{T}\left\{s^{st}\right\} = e^{st_0}y(t)$$

Hence,

$$y(t+t_0) = e^{st_0}y(t)$$

Setting $t = 0$, we obtain

$$y(t_0) = y(0)e^{st_0}$$

Since t_0 is arbitrary, by changing t_0 to t, we can rewrite $y(t_0) = y(0)e^{st_0}$ as

$$y(t) = y(0)e^{st} = \lambda e^{st}$$

or

$$\mathbf{T}\left\{e^{st}\right\} = \lambda e^{st}$$

where $\lambda = y(0)$.

Summary

In this lecture we have started our look at systems and the classification of systems.

In particular we have looked at

- System Representation

$$y(t) = \mathbf{T}\left\{x(t)\right\}$$

- Deterministic and Stochastic Systems
- Continuous-Time and Discrete-Time Systems
- Systems with Memory and without Memory
- Causal and Non-Causal Systems
- Linear Systems and Nonlinear Systems

A continuous-time system is linear if superposition holds. That is

$$\mathbf{T}\left\{\alpha_1 x_1 + \alpha_2 x_2\right\} = \alpha_1 y_1 + \alpha_2 y_2$$

where α_1 and α_2 are arbitrary scalars.

- Time-Invariant and Time-Varying Systems

A continuous-time system is time-invariant if

$$\mathbf{T}\left\{x(t-\tau)\right\} = y(t-\tau)$$

for any real value of τ.

- Linear Time-Invariant Systems

For continuous-time LTI system represented by $\mathbf{T}$

$$\mathbf{T}\left\{e^{st}\right\} = \lambda e^{st}$$

where s is a complex variable and λ is a complex constant. In this formulation e^{st} is called an *eigenfunction* of $\mathbf{T}$ and λ is the *eigenvalue*. See Eigenfunctions of Continuous-Time LTI Systems for the further development of this idea.

- Stable Systems

A system is *bounded-input/bounded-output* (BIBO) *stable* if for any bounded input signal x defined by

$$|x| \le k_1$$

the corresponding output y is also bounded defined by

$$|y| \le k_2$$

where k_1 and k_2 are finite real constants.

- Feedback Systems

Next time

- Unit 3.2: Properties and Eigenfunctions of Continuous-Time LTI Systems

References

[Hsu20] (1,2,3,4,5,6,7) Hwei P. Hsu. *Schaums outlines signals and systems*. McGraw-Hill, New York, NY, 2020. ISBN 9780071634724. Available as an eBook. URL: https://www.accessengineeringlibrary.com/content/book/9781260454246.

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