Unit 2.3: Elementary Signals

Contents

The preparatory reading for this section is Chapter 1 of [Karris, 2012] which

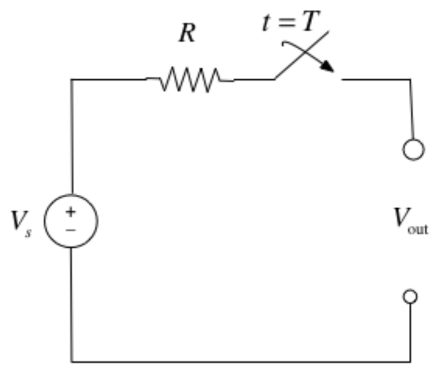
- begins with a discussion of the elementary signals that may be applied to electrical circuits
- introduces the unit step, unit ramp and dirac delta functions
- presents the sampling and sifting properties of the delta function and
- concludes with examples of how other useful signals can be synthesised from these elementary signals.

Additional information has been adapted from Section 1.4 of [Hsu, 2020].

Follow along at cpjobling.github.io/eg-150-textbook/signals_and_systems/elementary_signals

Introduction

Consider the network shown in below where the switch is closed at time $t = T$ and all components are ideal.



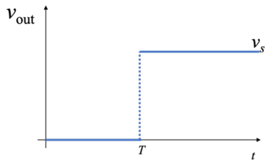
Express the output voltage V_{out} as a function of the unit step function, and sketch the appropriate waveform.

Solution

Before the switch is closed at $t < T$: \begin{equation} V_{\mathrm{out}} = 0. \end{equation}

After the switch is closed for $t > T$: \begin{equation} V_{\mathrm{out}} = V_s. \end{equation}

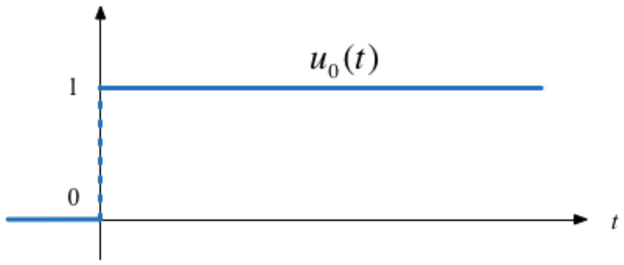
We imagine that the voltage jumps instantaneously from 0 to V_s volts at $t = T$ seconds as shown below.



We call this type of signal a step function.

The Unit Step Function

$$u_0(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$



In Matlab

In Matlab, we use the `heaviside` function (named after Oliver Heaviside).

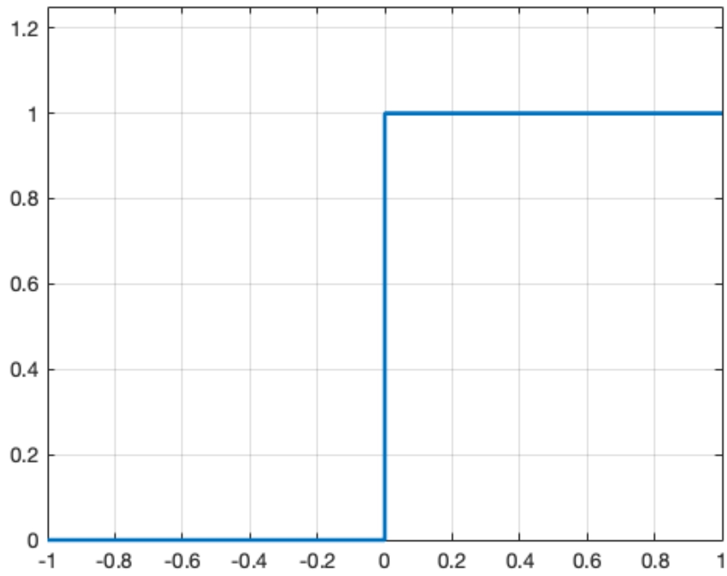
```
%%file plot_heaviside.m
syms t
fplot(heaviside(t),[-1,1],'LineWidth',2),grid,ylim([0 1.25]),...
title('The Heaviside function
$$u_0(t)$$','interpreter','latex'),xlabel('t')
heaviside(0)
```

```
Created file '/Users/eechris/code/src/github.com/cpjobling/eg-150-
textbook/signals_and_systems/elementary_signals/plot_heaviside.m'.
```

```
plot_heaviside
```

```
ans =

    0.5000
```



Note that, so that it can be plotted, Matlab defines the *Heaviside function* slightly differently from the mathematically ideal unit step:

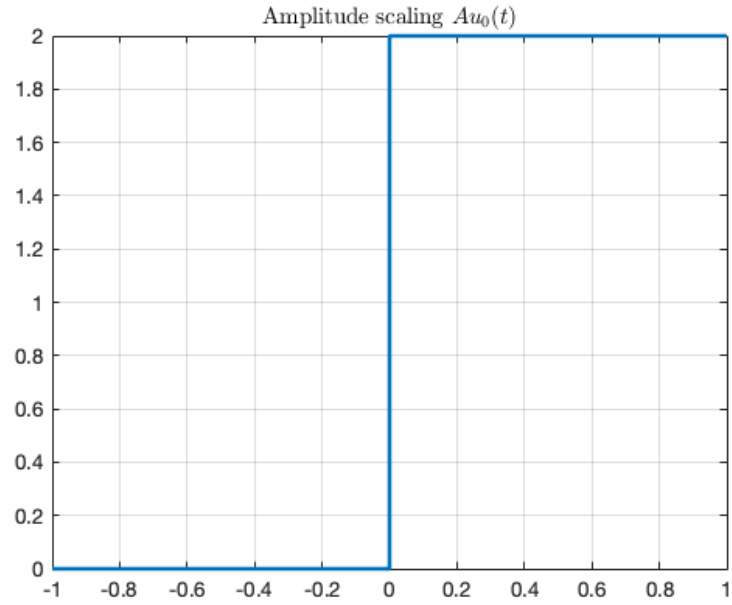
$$\mathrm{heaviside}(t) = \begin{cases} 0 & t < 0 \\ 1/2 & t = 0 \\ 1 & t > 0 \end{cases}$$

Simple Signal Operations

Amplitude Scaling

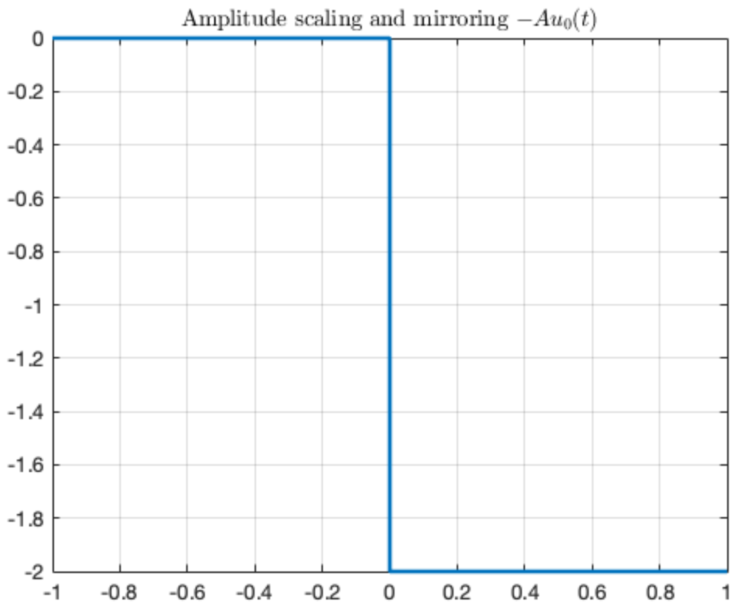
Sketch $Au_0(t)$ and $-Au_0(t)$

```
syms t;
u0(t) = heaviside(t); % rename heaviside function for ease of use
A = 2; % so signal can be plotted
fplot(A*u0(t),[-1,1],'LineWidth',2),grid,title('Amplitude scaling
$$Au_0(t)$$','interpreter','latex')
```



Note that the signal is scaled in the y direction.

```
fplot(-A*u0(t),[-1,1],'LineWidth',2),grid,...
title('Amplitude scaling and mirroring $$-
Au_0(t)$$','interpreter','latex'),...
xlabel('t')
```

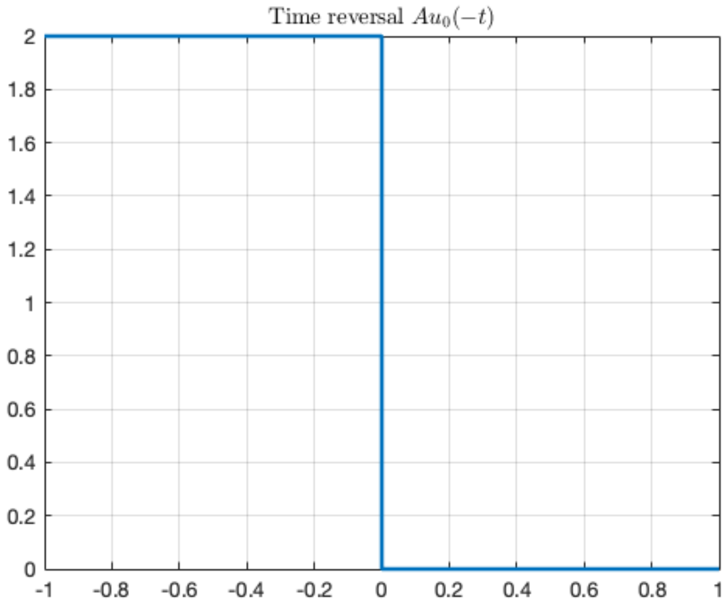


Note that, because of the sign, the signal is mirrored about the x axis as well as being scaled by 2.

Time Reversal

Sketch $u_0(-t)$

```
fplot(A*u0(-t),[-1,1],'LineWidth',2),grid,title('Time reversal
$$Au_0(-t)$$','interpreter','latex'),xlabel('t')
```

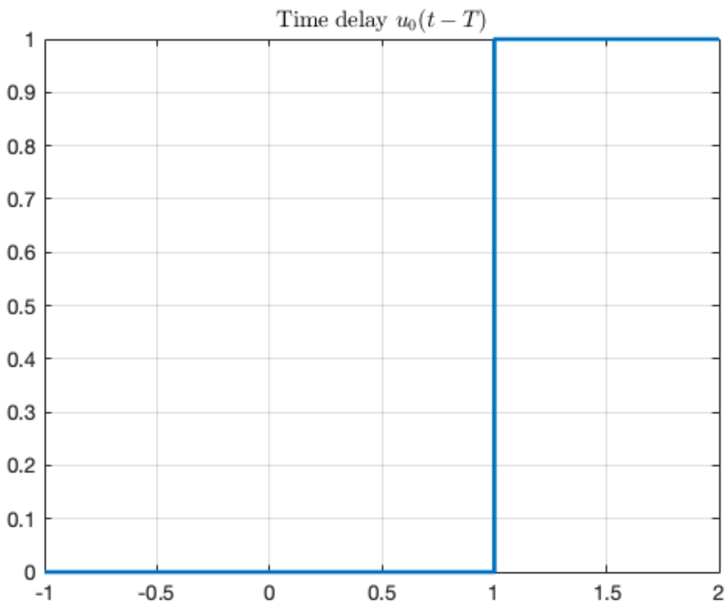


The sign on the function argument $-t$ causes the whole signal to be reversed in time. Note that another way of looking at this is that the signal is mirrored about the y axis.

Time Delay and Advance

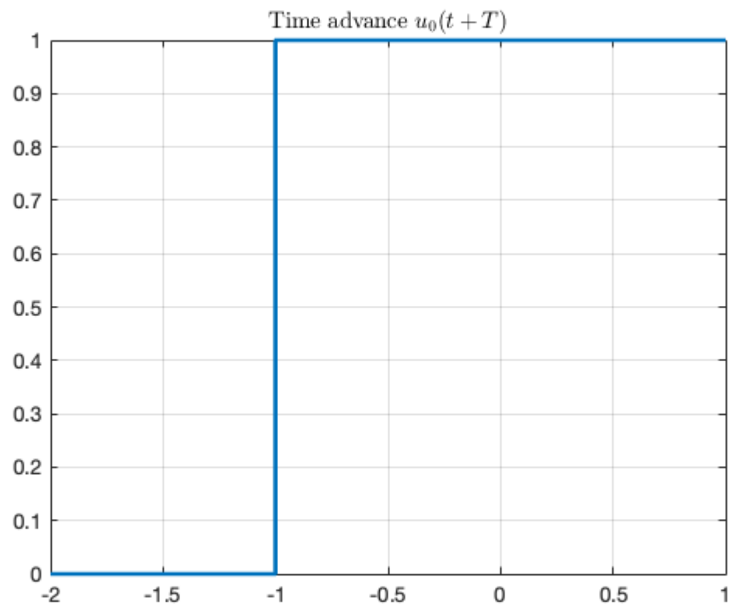
Sketch $u_0(t-T)$ and $u_0(t+T)$

```
T = 1; % again to make the signal plottable.
fplot(u0(t - T),[-1,2],'LineWidth',2),grid,title('Time delay $$u_0(t
- T)$$','interpreter','latex'),xlabel('t')
```



This is a *time delay* ... note for $u_0(t-T)$ the step change occurs T seconds **later** than it does for $u_o(t)$.

```
fplot(u0(t + T),[-2,1],'LineWidth',2),grid,title('Time advance
$$u_0(t + T)$$','interpreter','latex'),xlabel('t')
```



This is a *time advance* ... note for $u_0(t+T)$ the step change occurs T seconds **earlier** than it does for $u_o(t)$.

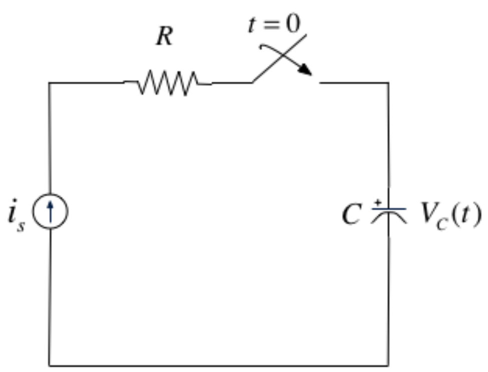
Examples

We will work through some examples in class. See Examples 3.

Synthesis of Signals from the Unit Step

Unit step functions can be used to represent other time-varying functions such as rectangular pulses, square waves and triangular pulses. See [Examples 3](#) for the examples that we will look at in class.

The Ramp Function



In the circuit shown above i_s is a constant current source and the switch is closed at time $t = 0$.

When the current through the capacitor $i_c(t) = i_s$ is a constant and the voltage across the capacitor is

$$v_c(t) = \frac{1}{C}\int_{-\infty}^{t} i_c(\tau)\, d\tau$$

where τ is a dummy variable.

Since the switch closes at $t = 0$, we can express the current $i_c(t)$ as

$$i_c(t) = i_s u_0(t)$$

and if $v_c(t) = 0$ for $t < 0$ we have

$$v_c(t) = \frac{i_s}{C}\int_{-\infty}^{t} u_0(\tau)\, d\tau = \underbrace{\frac{i_s}{C}\int_{-\infty}^{0} 0\ d\tau}_{0} + \frac{i_s}{C}\int_{0}^{t} 1\ d\tau$$

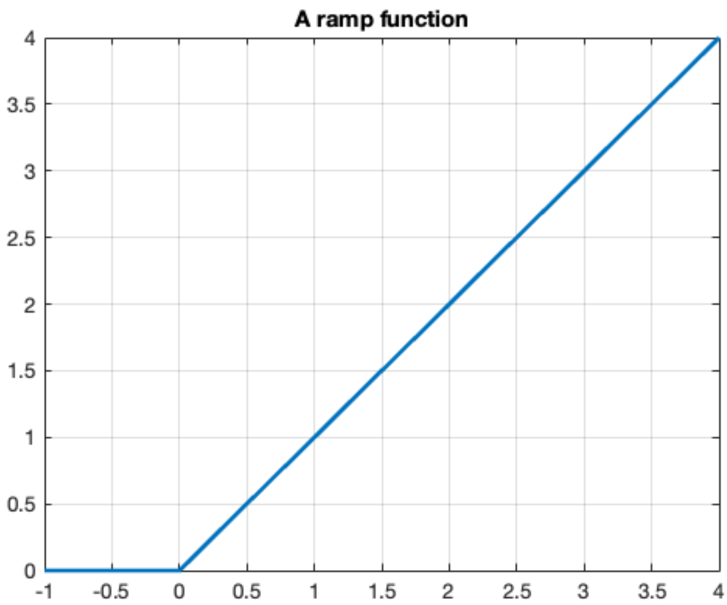
So, the voltage across the capacitor can be represented as

$$v_C(t) = \frac{i_s}{C} t u_0(t)$$

Note that in this as in other examples throughout these notes, and in published tables of transforms, the inclusion of $u_0(t)$ in $v_c(t)$ acts as a "gating function" that limits the definition of the signal to the causal range $0 \le t < \infty$.

To sketch the wave form, let's arbitrarily let C and i_s be one and then plot with MATLAB.

```
C = 1; is = 1;
vc(t)=(is/C)*t*u0(t);
fplot(vc(t),[-1,4],'LineWidth',2),grid,title('A ramp
function'),xlabel('t')
```



This type of signal is called a **ramp function**. Note that it is the *integral* of the step function (the resistor-capacitor circuit implements a simple integrator circuit).

The unit ramp function is defined as

$$u_1(t) = \int_{-\infty}^{t} u_0(\tau)d\tau$$

so

$$u_1(t) = \begin{cases} 0 & t < 0 \\ t & t \geq 0 \end{cases}$$

and

$$u_0(t) = \frac{d}{dt}u_1(t)$$

Note

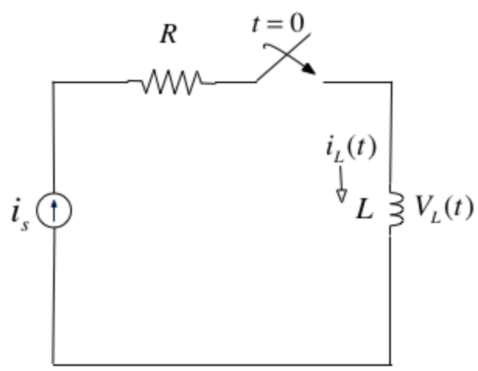
Higher order functions of t can be generated by the repeated integration of the unit step function.

For future reference, you should determine $u_2(t)$, $u_3(t)$ and $u_n(t)$ for yourself and make a note of the general rule:

$$u_{n-1} = \frac{1}{n}\frac{d}{dt}u_n(t)$$

Details are given in equations 1.26—1.29 in Karris.

The Dirac Delta Function



In the circuit shown above, the switch is closed at time $t = 0$ and $i_L(t) = 0$ for $t < 0$. Express the inductor current $i_L(t)$ in terms of the unit step function and hence derive an expression for $v_L(t)$.

Solution

$$v_L(t) = L\frac{di_L}{dt}$$

Because the switch closes instantaneously at $t = 0$

$$i_L(t) = i_s u_0(t)$$

Thus

$$v_L(t) = i_s L \frac{d}{dt} u_0(t).$$

The unit Impulse Function

The unit impulse function $\delta(t)$, is the derivative of the unit step.

$$\delta(t) = \frac{d}{dt} u_0(t)$$

which is tricky to compute because $u_0(t)$ is discontinuous at $t = 0$ but it must have the properties

$$\int_{-\infty}^{t} \delta(\tau) d\tau = u_0(t)$$

To solve this problem we need to invent a function that represents the derivative of the unit step function. This function is called the *unit impulse function* $\delta(t)$, also known as the *Dirac delta* function (named after Paul Dirac).

Traditionally, $\delta(t)$ is often defined as the limit of a suitably chosen conventional function having unity area over an infinitesimal time interval as shown in Fig. Fig. 21.

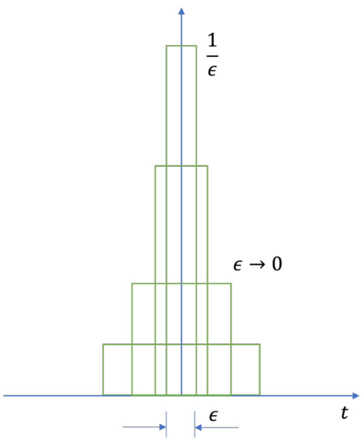


Fig. 21 Visualisation of the Dirac delta function as the limit of a conventional function with unit area.

The Dirac delta posseses the following properties

$$\delta\left(t\right) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

$$\int_{\epsilon}^{-\epsilon} \delta(t)\, dt = 1$$

The unit impulse function plays a fundamental role in systems analysis.

Sketch of the delta function

Continuing the example, and replacing the derivative of the unit step $u_0(t)$ with the unit impulse $\delta(t)$

$$V_{\text{out}}(t) = V_L(t) = i_s L \delta(t)$$

Note when we draw the unit impulse we show the height of $\delta(t)$ as one so the height of the impulse in the figure is $i_s L$.

MATLAB Confirmation

```
syms is L;
vL(t) = is * L * diff(u0(t))
```

```
vL(t) =
```

```
L*is*dirac(t)
```

Note that we can't plot dirac(t) in MATLAB with `fplot`. The best we can do is a stem plot.

```
L = 1; is = 2;
stem(0,L*is),title('Impulse $$v_L(t) =
L*i_s*\delta(t)$$','interpreter','latex'),grid,xlabel('t')
```

Important properties of the delta function

Sampling Property

The *sampling property* of the delta function states that

$$f(t)\delta(t-a) = f(a)\delta(t-a)$$

or, when $a = 0$,

$$f(t)\delta(t) = f(0)\delta(t)$$

Multiplication of any function $f(t)$ by the delta function $\delta(t)$ results in sampling the function at the time instants for which the delta function is not zero.

The study of descrete-time (sampled) systems is based on this property.

You should work through the proof for youself.

Sifting Property

The *sifting property* of the delta function states that

$$\int_{-\infty}^{\infty} f(t)\delta(t-\alpha)dt = f(\alpha)$$

That is, if multiply any function $f(t)$ by $\delta(t-\alpha)$, and integrate from $-\infty$ to $+\infty$, we will get the value of $f(t)$ evaluated at $t = \alpha$.

You should also work through the proof for yourself.

Higher Order Delta Fuctions

the n*th*-order *delta function* is defined as the n*th* derivative of $u_0(t)$, that is

$$\delta^n(t) = \frac{d^n}{dt^n}[u_0(t)]$$

The function $\delta'(t)$ is called the *doublet*, $\delta''(t)$ is called the *triplet* and so on.

By a procedure similar to the derivation of the sampling property we can show that

$$f(t)\delta'(t-a) = f(a)\delta'(t-a) - f'(t)\delta(t-a)$$

Also, derivation of the sifting property can be extended to show that

$$\int_{-\infty}^{\infty} f(t)\delta^n(t-\alpha)dt = (-1)^n \frac{d^n}{dt^n}[f(t)]\bigg|_{t=\alpha}$$

Summary

In this chapter we have looked at some elementary signals and the theoretical circuits that can be used to generate them.

Takeaways

- You should note that the unit step is the *heaviside function* $u_0(t)$.
- Many useful signals can be synthesized by use of the unit step as a "gating function" in combination with other signals
- That unit ramp function $u_1(t)$ is the integral of the step function.
- The *Dirac delta* function $\delta(t)$ is the derivative of the unit step function. We sometimes refer to it as the *unit impulse function*.
- The delta function has sampling and sifting properties that will be useful in the development of *time convolution* and *sampling theory*.

Examples

We will do some of these in class. See Examples 3.

Homework

These are for you to do later for further practice. See Homework 2

References

See Bibliography

Next Time

Systems and Classification of Systems

- System Representation
- Deterministic and Stochastic Systems
- Continuous-Time and Discrete-Time Systems
- Systems with Memory and without Memory
- Causal and Non-Causal Systems
- Linear Systems and Nonlinear Systems
- Linear Time-Invariant Systems
- Stable Systems
- Feedback Systems

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