Unit 2.2: Periodic, Energy and Power Signals

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We continue with our survey of Signals and Classification of Signals by looking at Periodic and Nonperiodic Signals and Energy and Power Signals.

This section is based on Section 1.2 of [Hsu, 2020].

Follow along at cpjobling.github.io/eg-150-textbook/signals_and_systems/signals/pep_signals

Periodic and Nonperiodic Signals

Periodic signals

A continuous-time signal $x(t)$ is said to be *periodic* with *period* T if there is a positive nonzero value of T for which

$$x(t+T) = x(t) \text{ all } t$$

An example of such a signal is given in Fig. 17.

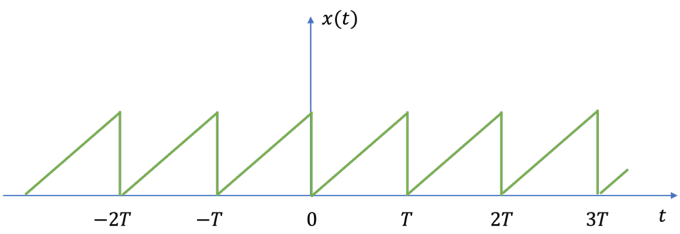


Fig. 17 An example of a periodic signal.

We can use the periodicity to synthesize a periodic signal such as that shown in Fig. 17.

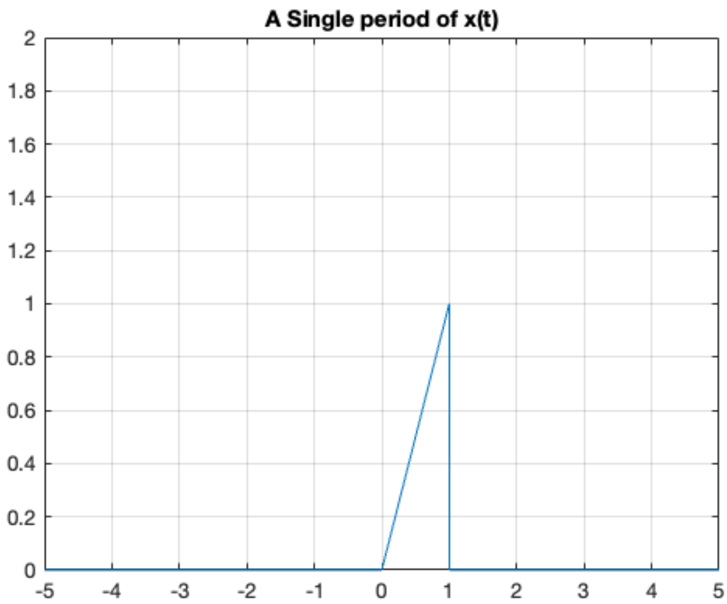
Let's first define the signal over one period. We will use MATLAB and the symbolic toolbox for this example:

Let one period of periodic signal be defined by

$$x\left(t\right) = \begin{cases} t & 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

We can use the Heaviside function (unit step) (MATLAB function `heaviside`: see [The Unit Step Function](#)) to sythesise this signal.

```
syms t
T = 1; % period of periodic signal
x(t) = t*(heaviside(t)-heaviside(t-T));
fplot(x(t)),ylim([0 2]),grid,title('A Single period of x(t)')
```



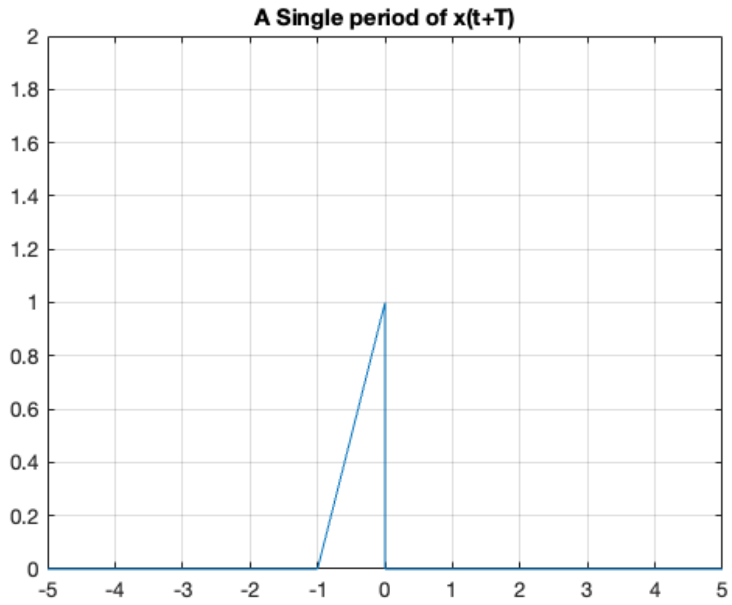
One period earlier:

$$x(t+T)$$

```
signal1 = x(t + T)
fplot(signal1),ylim([0 2]),grid,title('A Single period of x(t+T)')
```

```
signal1 =
```

```
(t + 1)*(heaviside(t + 1) - heaviside(t))
```



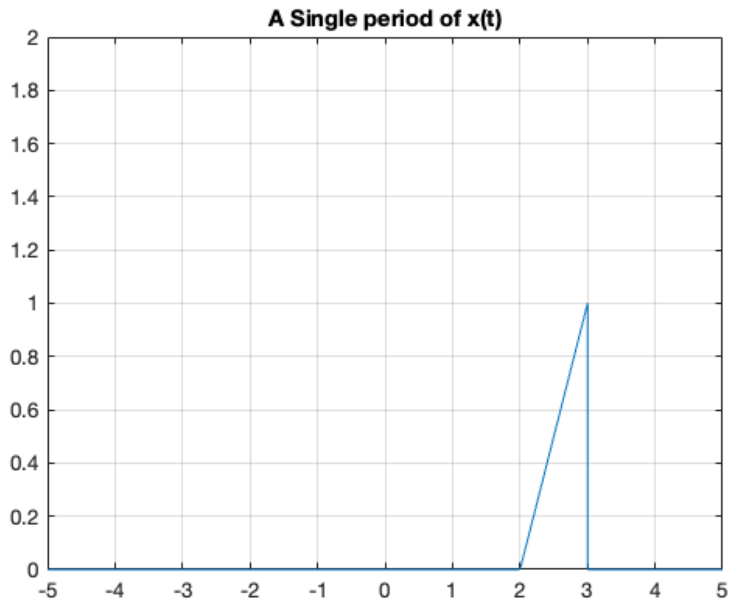
Two periods later:

$$x(t-2T)$$

```
signal2 = x(t-2*T)
fplot(signal2),ylim([0 2]),grid,title('A Single period of x(t)')
```

```
signal2 =
```

```
(heaviside(t - 2) - heaviside(t - 3))*(t - 2)
```



It follows that

$$x(t + mT) = x(t)$$

for all t and any integer m.

Now we use a loop and the definition of periodic function to repeat this signal multiple times

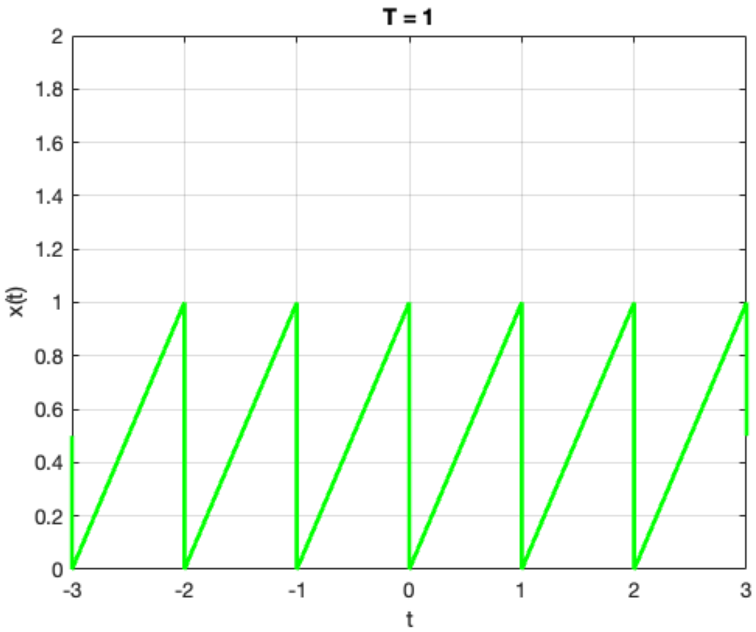
```
periodic_signal = 0;
for m = 5:-1:-5
    periodic_signal = periodic_signal + x(t + m*T);
end
periodic_signal
```

```
periodic_signal =
```

```
(t + 1)*(heaviside(t + 1) - heaviside(t)) + (heaviside(t - 1) -
heaviside(t - 2))*(t - 1) - (heaviside(t + 1) - heaviside(t + 2))*(t
+ 2) + (heaviside(t - 2) - heaviside(t - 3))*(t - 2) - (heaviside(t
+ 2) - heaviside(t + 3))*(t + 3) + (heaviside(t - 3) - heaviside(t -
4))*(t - 3) - (heaviside(t + 3) - heaviside(t + 4))*(t + 4) +
(heaviside(t - 4) - heaviside(t - 5))*(t - 4) - (heaviside(t + 4) -
heaviside(t + 5))*(t + 5) + (heaviside(t - 5) - heaviside(t - 6))*(t
- 5) - t*(heaviside(t - 1) - heaviside(t))
```

Now we plot the result

```
fplot(periodic_signal,'g-',"LineWidth",2),...
    grid,ylabel('x(t)'),xlabel('t'),title('T = 1')
xlim([-3.00 3.00])
ylim([0.00 2.00])
```



Fundamental period

The *fundamental period* T_0 of $x(t)$ is the smallest value of T for which $x(t + mT) = x(t)$ holds.

DC signals

Note that the definition of the *fundamental period* does not hold for a constant signal $x(t)$ (known as a DC signal).

For a constant signal $x(t) = c$ the fundamental period is undefined since $x(t)$ is periodic for any choice of T (and so there is no smallest postive value). See `dc_signal`.

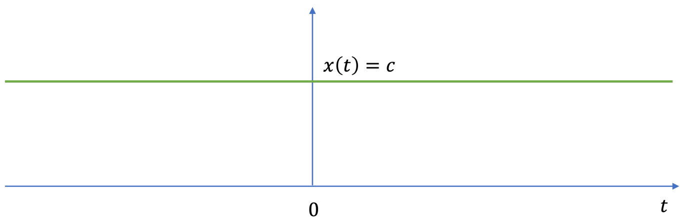


Fig. 18 A DC signal

> **Note**
>
> Note that the sum of two continuous time sigmals may not be periodic (Example {ref}`2.1`)

Nonperiodic signals

Any continuous-time signal which is not periodic is called a *nonperiodic* (or *aperiodic*) sigmal. For example see Fig. 19

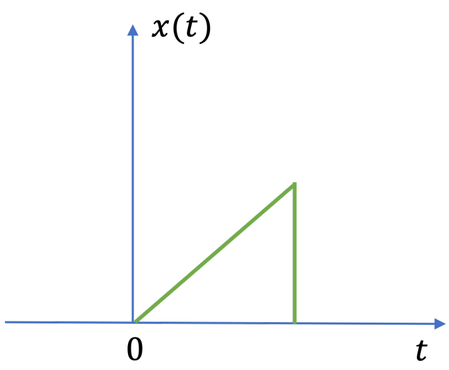


Fig. 19 A nonperiodic signal

Energy and Power Signals

Consider $v(t)$ to be the voltage across a resistor R prtoducing a current $i(t)$. (Fig. 20)

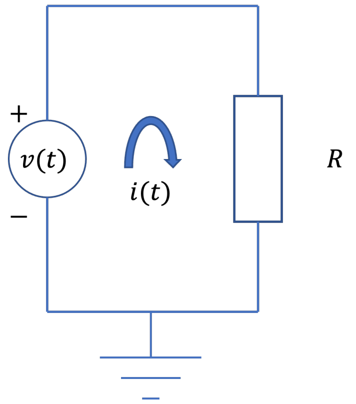


Fig. 20 A simple resistor circuit.

The instantaneous power $p(t)$ per ohm is defined as

$$p(t) = \frac{v(t)i(t)}{R} = i(t)^2$$

Total energy E and average power P on a per-ohm basis are

$$E = \int_{-\infty}^{\infty} i(t)^2\, dt \quad \text{joules}$$

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} i(t)^2\, dt \quad \text{watts}$$

Normalised energy content of a signal

For an arbitrary continuous-time signal $x(t)$, the *normalised energy content* E of $x(t)$ is defined as

$$E = \int_{-\infty}^{\infty} |i(t)|^2\, dt$$

Normalised average power of a signal

The *normalised average power* P of $x(t)$ is defined as

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\, dt \quad \text{watts}$$

Energy and power signals

Based on the previous definitions, the following classes of signals can be defined:

- $x(t)$ is said to be an *energy signal* if and only if $0 < E < \infty$, and so $P = 0$.

- $x(t)$ is said to be an *power signal* if and only if $0 < P < \infty$, thus implying that $E = \infty$.
- Signals that satisfy neither property are referred to as neither energy signals nor power signals.

Note that a periodic signal is a power signal if its energy content per period is finite, and then the average power of this signal need only be calulated over a period (ex:1.18).

Other Measures of Signal Size

There are other measures of signal size that are used:

Mean value

$$M_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, dt$$

For periodic signals with fundamental period T_0

$$M_x = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)\, dt$$

The mean value is also known as the dc value.

Observations

- the mean value corresponds to the arithmetic average
- the signal $x(t) - M_x$ has zero mean

Measures of spread

Root-mean square (RMS)

$$\mathrm{RMS}_x = \sqrt{P}$$

E.g. the power of $x(t) = A\sin(\omega t + \theta)$ is $A^2/2$ hence its RMS value is $A/\sqrt{2}$.

Peak value

$$|x|_{\mathrm{peak}} = \max_t |x(t)|$$

Crest factor (CF)

$$\mathrm{CF}_x = \frac{|x|_{\mathrm{peak}}}{\mathrm{RMS}_x} \geq 1$$

Peak-to-average power ratio (PAPR)

$$\mathrm{PAPR}_x = \frac{|x|^2_{\mathrm{peak}}}{P}$$

Observations

- CF a PAPR measure the dispersion of a signal about its average power.
- To express CF or PAPR we often use decibels (dB). To obtain a measure of the quantity y in dB use $20\log_{10}(y)$.

Examples 2

Example 2.1: Sum of two periodic signals

Let $x_1(t)$ and $x_2(t)$ be periodic signals with fundamental periods T_1 and T_2 respectively. Under which conditions is the sum $x(t) = x_1(t) + x_2(t)$ periodic, and what is the fundamental period of $x(t)$ if it is periodic?

For the answer, refer to the lecture recording or see solved problem 1.14 in [Hsu, 2020].

Example 2.2: Periodic signals

MATLAB Example

We will solve this example by hand and then give the solution in the MATLAB lab.

Determine whether or not each of the following signals is periodic. If a signal is periodic, determine its fundamental period.

a). $x(t) = \cos\left(t + \frac{\pi}{4}\right)$;

b). $x(t) = \sin\frac{2\pi}{3}t$;

c). $x(t) = \cos\frac{\pi}{3}t + \sin\frac{\pi}{4}t$;

d). $x(t) = \cos t + \sin\sqrt{2}t$;

e). $x(t) = \sin^2 t$;

f). $x(t) = e^{j[\pi/2t-1]}$;

For the answer, refer to the lecture recording or see solved problem 1.16 in [Hsu, 2020].

Example 2.3

Show that if $x(t+T) = x(t)$, then

$$\int_{\alpha}^{\beta} x(t)\,dt = \int_{\alpha+T}^{\beta+T} x(t)\,dt$$

$$\int_{0}^{T} x(t)\,dt = \int_{a}^{a+T} x(t)\,dt$$

for any real α, β and a.

For the answer, refer to the lecture recording or see solved problem 1.17 in [Hsu, 2020].

Example 2.4

Show that if $x(t)$ is periodic with fundamental period T_0, the the normalized average power P of $x(t)$ defined by

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left|x(t)\right|^2\,dt$$

is the same as the average power $x(t)$ over any interval of length T_0, that is,

$P = \frac{1}{T_0} \int_0^{T_0} \left|x(t)\right|^2\,dt$.

For the answer, refer to the lecture recording or see solved problem 1.18 in [Hsu, 2020].

Example 2.5: Power and energy signals

> **MATLAB Example**
>
> We will solve this example by hand and then give the solution in the MATLAB lab.

Determine whether the following signals are energy signals, power signals, or neither.

a). $x(t) = e^{-at}u_0(t)$, $a > 0$;

b). $x(t) = A\cos\left(\omega_0 t + \theta\right)$

c). $x(t) = t u_0(t)$

Note $u_0(t)$ is the unit step (or Heaviside) function formally introduced in the next lecture. For the answer, refer to the lecture recording or see solved problem 1.20 in [Hsu, 2020].

Example 2.6

Domestic mains power in the UK is delivered as a sinusoidal signal $x(t) = A\cos(\omega_0 t + \theta)$ with frequency of $50\mathrm{Hz}$ and RMS value of $240\mathrm{V}$.

Calculate the fundamental period T_0, fundamental angular frequency ω_0, average power P, peak voltage $A = x_{\text{peak}}$, crest factor (CF) and peak-to-average power ratio (PAPR) of the power signal provided to the home.

Express CF and PAPR in dB.

Summary

In this lecture we completed our look at signals and the classification of signals.

In particular we have looked at

- Periodic and Nonperiodic Signals
- Energy and Power Signals
- Other Measures of Signal Size

Next Time

- Unit 2.3: Elementary Signals
- System Representation
- Deterministic and Stochastic Systems
- Continuous-Time and Discrete-Time Systems
- aystems_with_memory_and_without_memory
- Causal and Non-Causal Systems
- Linear Systems and Nonlinear Systems
- Linear Time-Invariant Systems
- Stable Systems
- Feedback Systems