

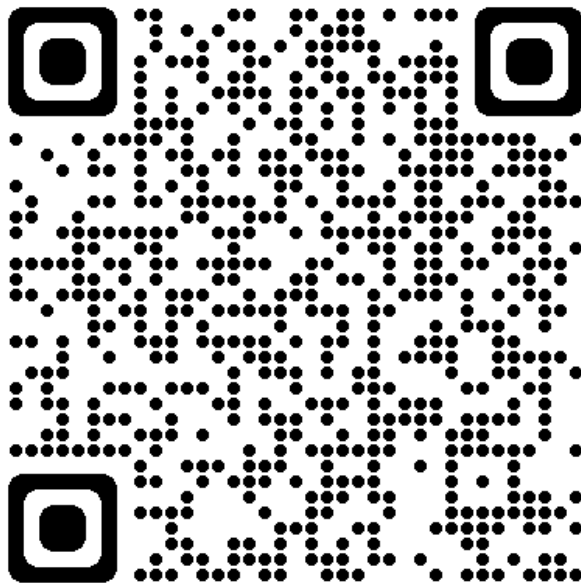
Unit 3.3: Systems Described by Differential Equations

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This section is based on Section 2.5 of [\[Hsu, 2020\]](#)

Follow along at cpjobling.github.io/eg-150-textbook/lti_systems/lti3



Subjects to be covered

We conclude our introduction to continuous-time LTI system by considering

- [Continuous-time LTI systems described by differential equations](#)
- [Examples 8: Systems described by differential equations](#)

Continuous-time LTI systems described by differential equations

- [A. Linear constant-coefficient differential equations](#)
- [B. Linearity](#)
- [C. Causality](#)
- [D. Time-invariance](#)
- [E. Impulse response](#)

A. Linear constant-coefficient differential equations

A general N th-order linear constant-coefficient differential (LCCDE) equation is given by

$$\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^M b_k \frac{d^k x(t)}{dt^k}$$

where the coefficients a_k and b_k are real constants.

The order N refers to the highest derivative of $y(t)$ in the differential equation.

Applications of linear constant-coefficient differential equations

LCCDEs play a central role in describing the input-output relationships of a wide variety of electrical, mechanical, chemical and biological systems.

Illustration: An RC Circuit

For instance, in the RC circuit considered in [Example 4.1: RC Circuit](#), the input $x(t) = v_s(t)$ and the output $y(t) = v_c(t)$ are related by a first-order constant-coefficient differential equation

$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)$$

So, by inspection, $N = 1$, $a_1 = 1$, $a_0 = b_0 = 1/RC$.

General solution of the general linear constant-coefficient differential equation

The general solution of the general linear constant-coefficient differential equation for a particular input $x(t)$ is given by

$$y(t) = y_p(t) + y_h(t)$$

where $y_p(t)$ is a *particular solution* satisfying the linear constant-coefficient differential equation and $y_h(t)$ is a *homogeneous solution* (or *complementary solution*) satisfying the homogeneous differential equation

$$\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = 0$$

The exact form of $y_h(t)$ is determined by N auxiliary conditions.

Note that

$$\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^M b_k \frac{d^k x(t)}{dt^k}$$

does not completely specify the the output $y(t)$ in terms of $x(t)$ unless auxiliary conditions are defined. In general, a set of auxiliary conditions are the values of

$$y(t), \frac{dy(t)}{dt}, \dots, \frac{d^{N-1}y(t)}{dt^{N-1}}$$

at some point in time.

B. Linearity

The system defined by

$$\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^M b_k \frac{d^k x(t)}{dt^k}$$

will be linear only if all the auxilliary conditions are zero (see [Example 8.4](#)).

If the auxiliary conditions are not zero, then the response $y(t)$ of a system can be expressed as

$$y(t) = y_{zi}(t) + y_{zs}(t)$$

where $y_{zi}(t)$ called the *zero-input response*, is the response to the auxilliary conditions, and $y_{zs}(t)$, called the *zero-state response*, is the response of a linear system with zero auxiliary conditions.

This is illustrated in [Fig. 38](#)

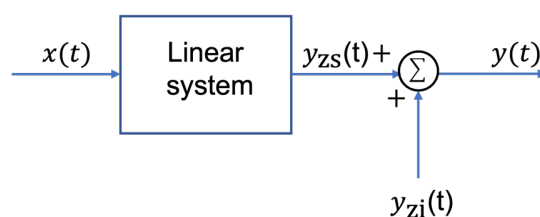


Fig. 38 Zero-state and zero-input responses

Note that $y_{zi}(t) \neq y_h(t)$ and $y_{zs}(t) \neq y_p(t)$ and that in general $y_{zi}(t)$ contains $y_h(t)$ and $y_{zs}(t)$ contains both $y_h(t)$ and $y_p(t)$ (see [Example 8.3](#)).

C. Causality

In order for the linear system described by a linear constant-coefficient differential equation

$$\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^M b_k \frac{d^k x(t)}{dt^k}$$

to be causal, we must assume the condition of *initial rest* (or an *initially relaxed condition*).

That is, if $x(t) = 0$ for $t \leq t_0$, then assume $y(t) = 0$ for $t \leq t_0$ (See [Example 4.6](#)).

Thus, the response for $t > t_0$ can be calculated from the linear constant-coefficient differential equation with the initial conditions

$$y(t_0) = \frac{dy(t_0)}{dt} = \dots = \frac{d^{N-1}y(t_0)}{dt^{N-1}} = 0$$

where

$$\frac{d^k y(t_0)}{dt^k} = \frac{d^k y(t)}{dt^k} \Big|_{t=t_0}$$

Clearly, at initial rest, $y_{zs}(t) = 0$.

D. Time-invariance

For a linear causal system, initial rest also implies time-invariance ([Example 8.6](#)).

E. Impulse response

The impulse response $h(t)$ of a linear constant-coefficient differential equation satisfies the differential equation

$$\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^M b_k \frac{d^k \delta(t)}{dt^k}$$

with the initial rest condition.

Examples of finding impulse responses are given in [Example 8.6](#) to [Example 8.8](#).

A peek into the future

Later in this course, and probably for the rest of your career, you will find the impulse response by using the Laplace transform.

Examples 8: Systems described by differential equations

Example 8.1

The continuous-time system shown in [Fig. 39](#) consists of one integrator and one scalar multiplier. Write the differential equation that relates the output $y(t)$ to the input $x(t)$.

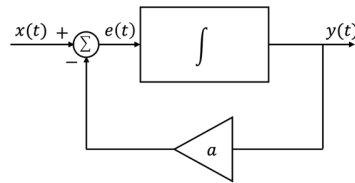


Fig. 39 A one-integrator linear system

For the answer, refer to the lecture recording or see solved problem 2.18 in [\[Hsu, 2020\]](#).

Example 8.2

The continuous-time system shown in [Fig. 40](#) consists of two integrators and two scalar multipliers. Write the differential equation that relates the output $y(t)$ to the input $x(t)$.

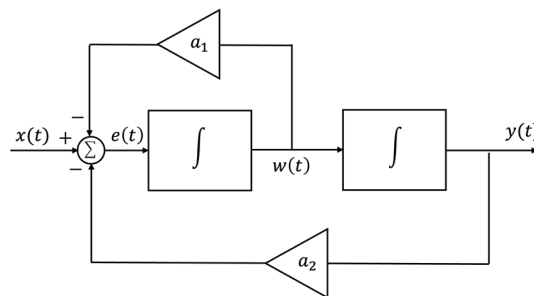


Fig. 40 A one-integrator linear system

For the answer, refer to the lecture recording or see solved problem 2.19 in [\[Hsu, 2020\]](#).

Example 8.8

Consider the system described by

$$\frac{dy(t)}{dt} + 2y(t) = x(t) + \frac{dx(t)}{dt}$$

Find the impulse response $h(t)$ of the system.

For the answer, refer to the lecture recording or see solved problem 2.25 in [\[Hsu, 2020\]](#).

Note

As we will be moving on to show how differential equations can be solved by the Laplace transform, the remaining examples are optional and will not be examined.

Example 8.3

Consider a continuous-time system whose input $x(t)$ and output $y(t)$ are related by

$$\frac{dy(t)}{dt} + ay(t) = x(t)$$

where a is a constant.

(a) Find $y(t)$ with the auxiliary condition $y(0) = y_0$ and

$$x(t) = Ke^{bt}u_0(t)$$

(b) Express $y(t)$ in terms of the zero-input and zero-state responses.

For the answer, see the solved problem 2.20 in [Hsu, 2020].

Example 8.4

Consider the system in [Example 8.3](#).

(a) Show that the system is not linear if $y(0) = y_0 \neq 0$.

(b) Show that the system is linear if $y(0) = 0$.

For the answer, see the solved problem 2.21 in [Hsu, 2020].

Example 8.5

Consider the system in [Example 8.3](#). Show that the initial rest condition $y(0) = 0$ also implies that the system is time-invariant.

For the answer, see the solved problem 2.22 in [Hsu, 2020].

Example 8.6

Consider the system in [Example 8.3](#). Find the impulse response $h(t)$ of the system.

For the answer, see the solved problem 2.23 in [Hsu, 2020].

Example 8.7

Consider the system in [Example 8.3](#) with $y(0) = 0$.

- (a) Find the step response $s(t)$ of the system without using the impulse response $h(t)$.
- (b) Find the step response $s(t)$ of the system with the impulse response $h(t)$ obtained in [Example 8.6](#).
- (c) Find the impulse response $h(t)$ from the step response $s(t)$.

For the answer, see the solved problem 2.24 in [\[Hsu, 2020\]](#).

Summary

In this lecture we have concluded our introduction to LTI systems by looking at linear constant-coefficient differential equations.

Take aways

Continuous-Time LTI Systems Described by Differential Equations

- [A. Linear constant-coefficient differential equations](#)
- [B. Linearity](#)
- [C. Causality](#)
- [D. Time-invariance](#)
- [E. Impulse response](#)

Next Time

We move on to consider

- [Unit 4: Laplace Transforms and their Applications](#)

References

[Hsu20]([1](#),[2](#),[3](#),[4](#),[5](#),[6](#),[7](#),[8](#),[9](#)) Hwei P. Hsu. *Schaums outlines signals and systems*. McGraw-Hill, New York, NY, 2020. ISBN 9780071634724. Available as an eBook. URL: <https://www.accessengineeringlibrary.com/content/book/9781260454246>.

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