Unit 3.2: Properties and Eigenfunctions of Continuous-Time LTI Systems

Contents

This section is based on Sections 2.3 and 2.4 of [Hsu, 2020]

Follow along at cpjobling.github.io/eg-150-textbook/lti_systems/lti2

Subjects to be covered

We continue our introduction to continuous-time LTI systems by considering:

- Properties of Continuous-Time LTI Systems
- Eigenfunctions of Continuous-Time LTI Systems
- Examples 6: Properties of Continuous-Time LTI Systems
- Examples 7: Eigenfunctions of Continuous-Time LTI systems

Properties of Continuous-Time LTI Systems

- A. Systems with or without memory
- B. Causality
- C. Stability

A. Systems with or without memory

Since the output $y(t)$ of a memoryless system depends only on the current input $x(t)$, then, if the system is also linear and time-invariant, this relationship can only be of the form

$$y(t) = Kx(t)$$

where K is a (gain) constant.

Thus the corresponding impulse response $h(t)$ is simply

$$h(t) = K\delta(t)$$

Therefore, if $h(t_0) \neq 0$ for $t_0 \neq 0$, then the continuous-time LTI system has memory.

B. Causality

Causal continuous-time LTI systems

As discussed in Section Causal and Non-Causal Systems, a causal system does not respond to an input event until that event actually occurs.

Therefore, for a causal LTI system, we have

$$h(t) = 0 \quad t < 0$$

Applying the causality condition to the convolution integral, the output of a causal continuous-time LTI system is expressed as

$$y(t) = \int_{0}^{\infty} h(\tau)x(t = \tau)\,d\tau$$

Alternatively, applying the causality to the convolution integral (defined in Section C. Convolution Integral)

$$y(t) = \int_{-\infty}^{\infty} x(t)h(t-\tau)\,d\tau$$

we have

$$y(t) = \int_{-\infty}^{t} x(t)h(t-\tau)\,d\tau$$

This equation shows that the only values of the input $x(t)$ used to evaluate the output $y(t)$ are those for $\tau \leq t$.

Causal signals

Based on the causality condition, any signal is called *causal* if

$$x(t) = 0 \quad t < 0$$

and is called *anticausal* if

$$x(t) = 0 \quad t > 0$$

Combining the definition of a causal signal with a causal signal, when the input $x(t)$ is causal, the output of a causal continuous-time LTI system is given by

$$y(t) = \int_{0}^{t} h(t)x(t-\tau)\,d\tau = \int_{0}^{t} x(t)h(t-\tau)\,d\tau$$

C. Stability

The BIBO (bounded-input/bounder-output) stability of an LTI system (Section Stable Systems) is readily acertined by the impuse response. It can be shown (Example 6.x) that a continuous-time LTI system is BIBO stable if its impulse response is absolutely integrable; that is,

$$\int_{-\infty}^{\infty} [h(\tau)|\,d\tau < \infty$$

Eigenfunctions of Continuous-Time LTI Systems

In Chapter Systems and Classification of Systems (Example Example 4.7) we saw that the eigenfunctions of continuous-time LTI system represented by the complex exponentials e^{st}, with s a complex variable.

That is

$$\mathbf{T}\left\{e^{st}\right\} = \lambda e^{st}$$

where λ is the eigenvalue of $\mathbf{T}$ associated with e^{st}.

Setting $x(t) = e^{st}$ in the convolution integral, we have

$$y(t) = \mathbf{T}\left\{e^{st}\right\}$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}\,d\tau$$

$$y(t) = \left[\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,d\tau\right]e^{st}$$

$$y(t) = H(s)e^{st} = \lambda e^{st}$$

where

$$\lambda = H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,d\tau$$

Thus, the eigenvalue of a continuous-time LTI system associated with the eigenfunction e^{st} is given by $H(s)$ which is a complex constant whose value is determined by the value of s via the equation

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,d\tau.$$

Note from the equation

$$y(t) = H(s)e^{st}$$

that $y(0) = H(s)$ (see ex:4_7).

Looking Ahead

The above results inderline the definition of the Laplace transform and Fourier transform. The ../laplace_transform/1/laplace.md will be discussed later in this course. The Fourier Transform will be introduced in **EG-247 Digital Signal Processing** next year.

Examples 6: Properties of Continuous-Time LTI Systems

Example 6.1

The signals in Fig. 31(a) and (b) are the input $x(t)$ and the output $y(t)$, respectively, of a certain continuous-time LTI system.

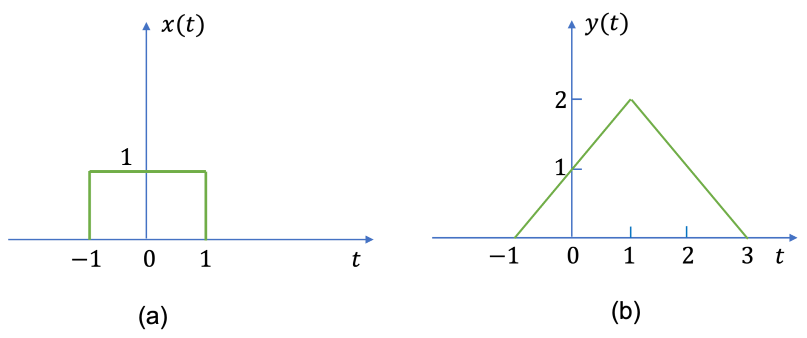

Fig. 31 Input and output signals for a continuous-time LTI system

Sketch the output to the following inputs:

(a)$x(t-2)$;

(b) $\frac{1}{2}x(t)$.

For the answer, refer to the lecture recording or see solved problem 2.9 in in [Hsu, 2020].

Example 6.2

Consider a continuous-time LTI system whose step response is given by

$$y(t) = e^{-t}u_0(t)$$

Determine and sketch the output of the system to the input $x(t)$ shown in Fig. 32.

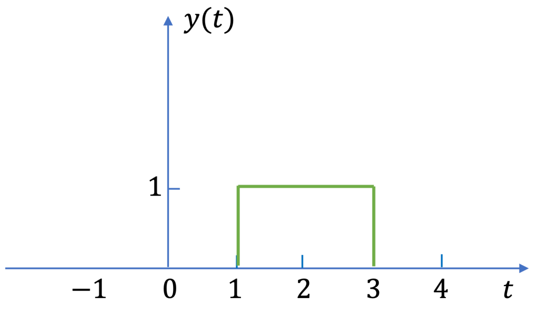

Fig. 32 Signal for Exercise 6.2

For the answer, refer to the lecture recording or see solved problem 2.2 in in {cite}schaum.

Example 6.3

Consider a continuous-time LTI system system described by

$$y(t) = \mathbf{T}\left\{x(t)\right\} = \frac{1}{T}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} x(\tau)\,d\tau$$

(a) Find and sketch the impulse response $h(t)$ of the system.

(b) Is this system causal?

For the answer, refer to the lecture recording or see solved problem 2.11 in in [Hsu, 2020].

Example 6.4

Let $y(t)$ be the output of a continuous-time LTI system with input $x(t)$. Find the output of the system if ths input is $x'(t)$, where $x'(t)$ is the first derivative of $x(t)$.

For the answer, refer to the lecture recording or see solved problem 2.12 in in [Hsu, 2020].

Example 6.5

Verify the BIBO stability condition (C. Stability) for continuous-time LTI systems.

For the answer, refer to the lecture recording or see solved problem 2.13 in [Hsu, 2020].

Example 6.6

The system shown in Fig. 33(a) is formed by connecting two systems in *cascade*. The impulse responses of the two systems are $h_1(t)$ and $h_2(t)$, respectively, and

$$h_1(t) = e^{-2t}u_0(t) \quad h_2(t) = 2e^{-t}u_0(t)$$

(a) Find the impulse response $h(t)$ of the overall system shown in Fig. 32(b).

(b) Determine if the overall system is BIBO stable.

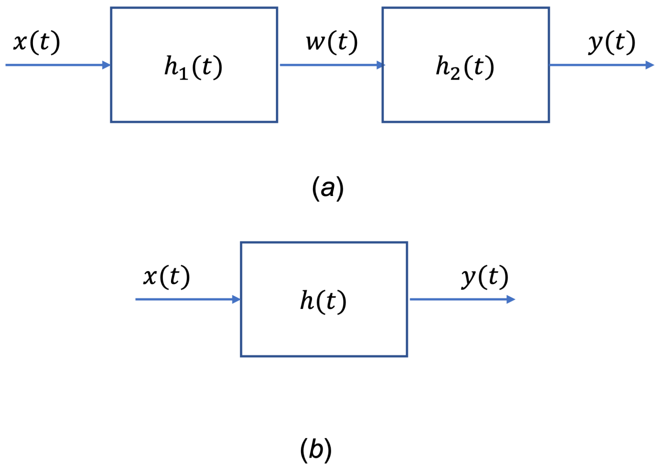

Fig. 33 A cascade connection of two continuous-time LTI systems

For the answer, refer to the lecture recording or see solved problem 2.14 in [Hsu, 2020].

Examples 7: Eigenfunctions of Continuous-Time LTI systems

Example 7.1

Consider a continuous-time LTI system with the input-output reltion given by

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} x(\tau)\, d\tau$$

(a) Find the impulse response $h(t)$ of this system.

(b) Show that the complex exponential e^{st} is an eigenfunction of the system.

(c) Find the eigenvalue of the system corresponding to e^{st} using the impulse response $h(t)$ obtained in part (a).

For the answer, refer to the lecture recording or see solved problem 2.15 in [Hsu, 2020].

Example 7.2

Consider the continuous-time LTI system described by

$$y(t) = \frac{1}{T} \int_{t-\frac{T}{2}}^{t+\frac{T}{2}} x(\tau)\, d\tau$$

(a) Find the eigenvalue of the system corresponding to the eigenfunction e^{st}.

(b) Repeat part (a) by using the impulse response $h(t)$ of the system.

For the answer, refer to the lecture recording or see solved problem 2.16 in [Hsu, 2020].

Example 7.3

Consider a stable continuous-time LTI system with impulse response $h(t)$ that is real and even. Show that $\cos \omega t$ and $\sin \omega t$ are eigenfunctions of this system with the same eigenvalue.

For the answer, refer to the lecture recording or see solved problem 2.17 in [Hsu, 2020].

Summary

We have continued our introdiuction the continuous-time LTI systems by considering

- Properties of Continuous-Time LTI Systems
- Eigenfunctions of Continuous-Time LTI Systems

Next Time

We will conclude our look at continuous-time LTI systems by considering

- Unit 3.3: Systems Described by Differential Equations

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