Unit 4.5: Using Laplace Transforms for Circuit Analysis

Contents

The preparatory reading for this section is Chapter 4 [Karris, 2012] which presents examples of the applications of the Laplace transform for electrical solving circuit problems. Much of the same material is covered in Section 3.7 D of [Hsu, 2020].

Follow along at cpjobling.github.io/eg-150-textbook/laplace_transform/5/circuit_analysis

Skip to main content

Agenda

We look at applications of the Laplace Transform for circuit analysis. In particular we will consider

- Circuit Transformation from Time to Complex Frequency
- Complex Impedance Z(s)
- Complex Admittance Y(s)
- Examples 12

```
% initialize MATLAB
clearvars
format compact
syms t L R C i_R(t) v_R(t) i_L(t) v_L(t) v_C(t) i_C(t)
```

Circuit Transformation from Time to Complex Frequency

(resistive network)=

Time Domain Model of a Resistive Network

Consider Fig. 50

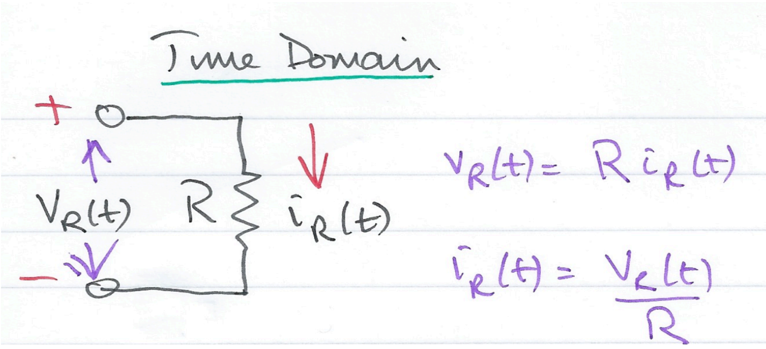

Fig. 50 Time Domain Model of a Resistive Network.

In the time domain

In Fig. 50 the voltage across the resistor $v_R(t)$ is proportional to the current flowing through the resistor $i_R(t)$

$$v_R(t) = Ri_R(t) \tag{18}$$

```
eqvrt = v_R(t) == R * i_R(t)
```

```
eqvrt =
```

```
v_R(t) == R*i_R(t)
```

The current flowing through the resistor is inversely proportional to the voltage across the resistor. This is easily confirmed by rewriting (18) to isolate $i_R(t)$

```
eqirt = isolate(eqvrt,i_R(t))
```

```
eqirt =
```

```
i_R(t) == v_R(t)/R
```

Skip to main content

Rewritten nicely as

$$i_R(t) = \frac{v_R(t)}{R} \tag{19}$$

From these results, which of the following equations represent the Laplace transform of the current flowing through, and the voltage across, the resistor R?

$$V_R(s) = RI_R(s)$$

$$I_R(s) + \frac{V_R(s)}{R}$$

$$V_R(s) = \frac{I_R(s)}{R}$$

$$I_R(s) = RV_R(s)$$

-> Open poll

In the complex frequency domain

We take the Laplace transforms of (18) and (19) to obtain

$$V_R(s) = RI_R(s) \tag{20}$$

$$I_R(s) = \frac{V_R(s)}{R} \tag{21}$$

which we illustrate in Fig. 51.

> **Note**
>
> The current and voltage are transformed but the resitance is unchanged by the transformation.

Skip to main content

Complex Frequency Domain Model of a Resistive Circuit

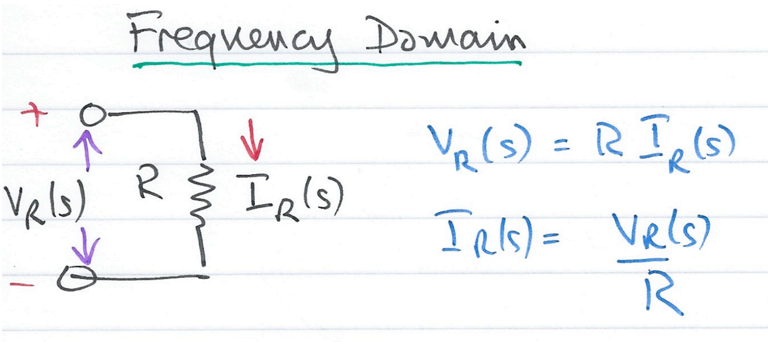

Fig. 51 Complex Frequency Domain Model of a Resistive Circuit

Time Domain Model of an Inductive Network

Consider Fig. 52

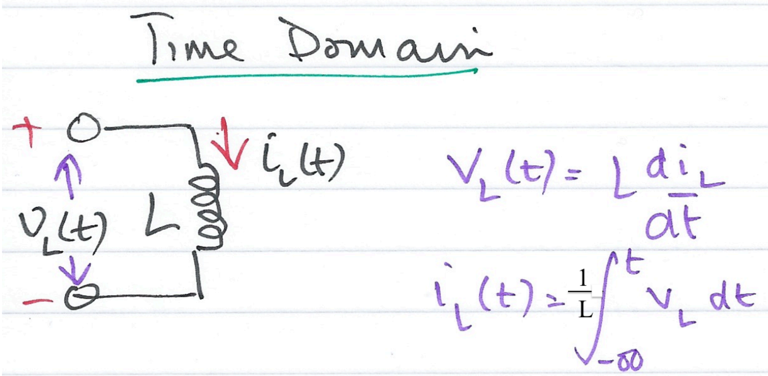

Fig. 52 Time Domain Model of an Inductive Network.

In the time domain

The voltage across the inductor $i_L(t)$ is proportional to the rate of change of the current $i_L(t)$ flowing through the inductor

$$v_L(t) = L\frac{d}{dt}i_L(t) \tag{22}$$

```
eqvlt = v_L(t) == L*diff(i_L(t))
```

Skip to main content

```
eqvlt =
```

```
v_L(t) == L*diff(i_L(t), t)
```

The current flowing through the inductor is inversely proportional to the integral of the voltage across the inductor which is easily confirmed by taking the integral of both sides of (22) and rewriting the equation to isolate $i_L(t)$

```
int(lhs(eqvlt)) == int(rhs(eqvlt));
eqilt = isolate(ans,i_L(t))
```

```
eqilt =
```

```
i_L(t) == int(v_L(t), t)/L
```

Rewritten nicely as

$$i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v_L(t)\,dt \tag{23}$$

From these results, which of the following equations represent the Laplace transform of the current flowing through, and the voltage across, the inductor L?

$$I_L(s) = sLV_L(s) - v_L(0^-)$$

$$I_L(s) = \frac{V_L(s)}{sL} + \frac{v_L(0^-)}{s}$$

$$V_L(s) = sLI_L(s) - i_L(0^-)$$

$$V_L(s) = \frac{I_L(s)}{sL} + \frac{i_L(0^-)}{s}$$

-> Open poll

Complex Frequency Domain Model of an Inductive Network

Consider Fig. 53

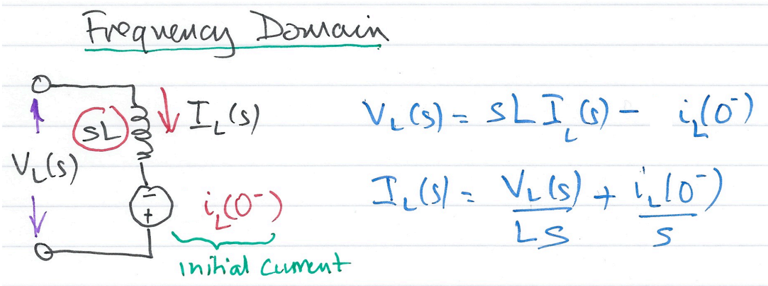

Fig. 53 Time Domain Model of a Resistive Network.

In the complex frequency domain

We take the Laplace transforms of (22) and (23) to obtain

$$V_L(s) = sLI_L(s) - i_L(0^-) \tag{24}$$

$$I_L(s) = \frac{V_L(s)}{sL} + \frac{i_L(0^-)}{s} \tag{25}$$

Note

The current and voltage are transformed but so is the inductance. The complex frequency representation has used the derivative property for the voltage across the inductor and the integration properties for the current through the inductor. The use of the dervative and integration transforms has introduced a term that depends on the initial current flowing through the inductor. Therefore, the initial current would need to be considered in computing the actual voltage and current in the complex frequency domain.

Time Domain Model of a Capacitive Network

Consider Fig. 54

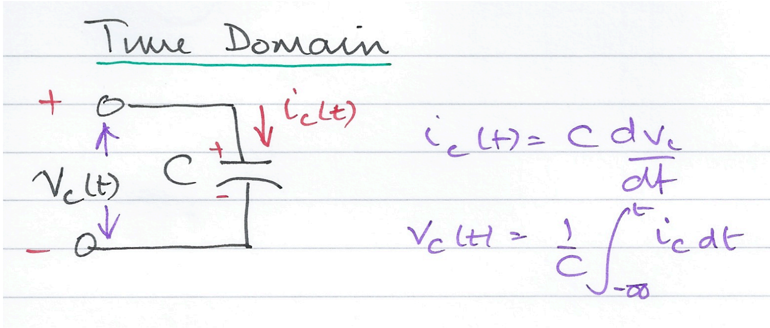

Fig. 54 Time Domain Model of a Capacitive Network.

In the time domain

The current flowing into the capacitor is proportional to the change in voltage across the capacitor

$$i_C(t) = C\frac{d}{dt}v_C(t) \tag{26}$$

```
eqict = i_C(t) == C * diff(v_C(t))
```

```
eqict =
```

```
i_C(t) == C*diff(v_C(t), t)
```

The voltage across the capacitor is inversely proportional to the integral of the current flowing into the capacitor which is easily confirmed by taking the integral of both sides of (26) and rewriting the equation to isolate $v_C(t)$

```
int(lhs(eqict)) == int(rhs(eqict));
eqvct = isolate(ans,v_C(t))
```

```
eqvct =
```

```
v_C(t) == int(i_C(t), t)/C
```

Which can be rewritten nicely as

Skip to main content

$$v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(t)\,dt \tag{27}$$

From the previous results, which of the following equations represent the Laplace transform of the current flowing into, and the voltage across, the capacitor C?

$$V_c(s) = sCI_C(s) - i_C(0^-)$$

$$I_c(s) = sCV_C(s) - v_C(0^-)$$

$$V_c(s) = \frac{I_C(s)}{sC} + \frac{i_C(0^-)}{s}$$

$$I_c(s) = \frac{V_C(s)}{sC} + \frac{v_C(0^-)}{s}$$

-> Open poll

In the complex frequency domain

We take the Laplace transforms of (26) and (27) to obtain

$$I_C(s) = sCV_C(s) - v_C(0^-) \tag{28}$$

$$V_C(s) = \frac{I_C(s)}{sC} + \frac{v_C(0^-)}{s} \tag{29}$$

Note

The current and voltage are transformed but so is the capacitance. The complex frequency representation has used the derivative property for the voltage across the capacitor and the integration property for the current flowing into the capacitor. The use of the dervative and integration transforms has introduced a term that depends on the initial voltage (initial charge) across the capacitor. Therefore, the initial voltage would need to be considered in computing the actual voltage and current introduced by the capacitor in the complex frequency

Complex Frequency Domain of a Capacitive Network

Consider Fig. 55

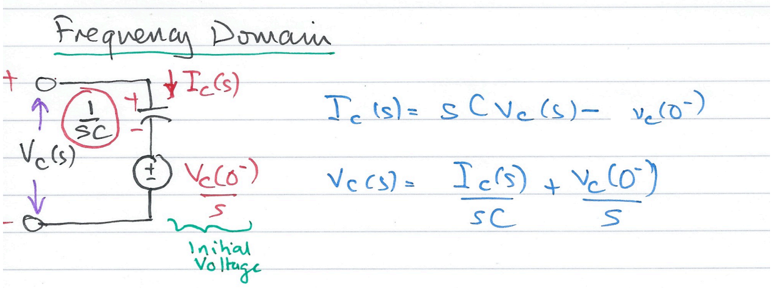

Fig. 55 Time Domain Model of a Capacitive Network.

Complex Impedance $Z(s)$

By analogy with the *resistance* of a resistor R, a component with complex impedance $Z(s)$ satisfies Ohm's law:

$$V(s) = I(s)Z(s)$$

from which

$$Z(s) = \frac{V(s)}{I(s)}$$

Complex impedance of components

For the resistance $R\Omega$, inductance LH and capacitance CF, which of the following represent the complex impedance, $Z(s) = V(s)/I(s)$ of the components?

$$sL$$

$$1/R$$

Skip to main content

$$\frac{1}{sC}$$

$$\frac{1}{sL}$$

$$R$$

-> Open Poll

Consider the s-domain RLC series circuit shown in Fig. 56, where the initial conditions are assumed to be zero.

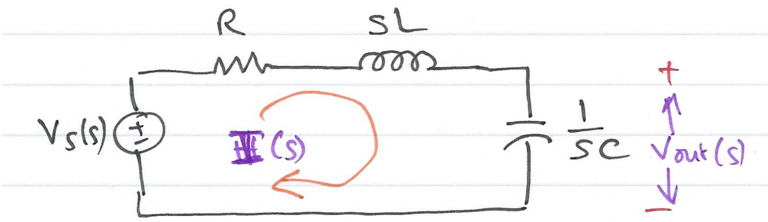

Fig. 56 RLC series circuit

For this circuit, the sum

$$R + sL + \frac{1}{sC}$$

represents that total opposition to current flow.

Then,

$$I(s) = \frac{V_s(s)}{R + sL + 1/(sC)}$$

and defining the ratio $V_s(s)/I(s)$ as $Z(s)$, we obtain

$$Z(s) = \frac{V_s(s)}{I(s)} = R + sL + \frac{1}{sC}$$

The s-domain current $I(s)$ can be found from

$$I(s)=\frac{V_s(s)}{Z(s)}$$

where

$$Z(s)=R+sL+\frac{1}{sC}.$$

Since $s=\sigma+j\omega$ is a complex number, $Z(s)$ is also complex and is known as the *complex input impedance* of this RLC series circuit.

Complex Admittance $Y(s)$

By analogy with the *admittance* of a resistor G, a component with complex admittance $Y(s)$ satisfies Ohm's law:

$$I(s)=V(s)Y(s)$$

from which

$$Y(s)=\frac{I(s)}{V(s)}$$

Complex admittance of components

For the resistance $R\Omega$, inductance LH and capacitance CF, which of the following represent the complex admittance, $Y(s)=I(s)/V(s)$ of the components?

For the resistor $R\Omega$, inductor LH and capacitance CF, which of the following represent the complex admittance of the components?

$$sL$$

$$1/R$$

$$sC$$

$$\frac{1}{sC}$$

$$\frac{1}{sL}$$

$$R$$

-> Open Poll

Consider the s-domain GLC parallel circuit shown in Fig. 57 where the initial conditions are zero.

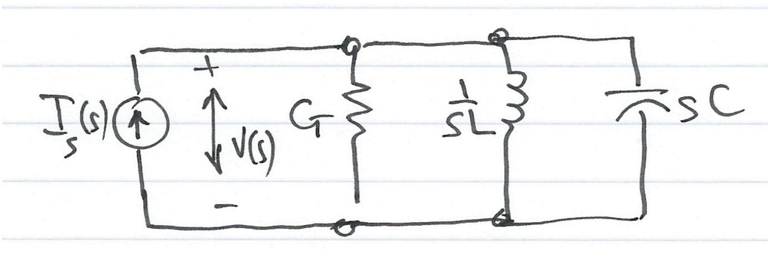

Fig. 57 A GLC parallel circuit

For this circuit

$$GV(s) + \frac{1}{sL}V(s) + sCV(s) = I_s(s)$$

$$\left(G + \frac{1}{sL} + sC\right)V(s) = I_s(s)$$

Defining the ratio $I_s(s)/V(s)$ as $Y(s)$ we obtain

$$Y(s) = \frac{I_s(s)}{V(s)} = G + \frac{1}{sL} + sC = \frac{1}{Z(s)}$$

The s-domain voltage $V(s)$ can be found from

$$V(s) = \frac{I_s(s)}{Y(s)}$$

Skip to main content

$$Y(s) = G + \frac{1}{sL} + sC.$$

$Y(s)$ is complex and is known as the *complex input admittance* of this GLC parallel circuit.

Examples 12

We will work through these in class.

Example 12.1

Note

This is based on Example 4.1 from [Karris, 2012].

Use the Laplace transform method and apply Kirchoff's Current Law (KCL) to find the voltage $v_c(t)$ across the capacitor for the circuit in Fig. 58 given that $v_c(0^-) = 6$ V.

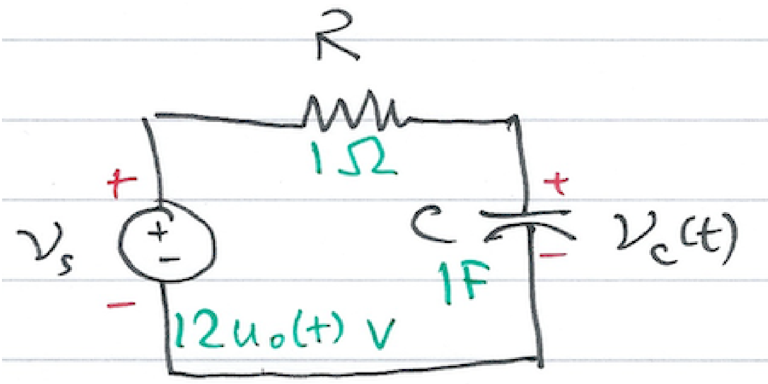

Fig. 58 Circuit for Example 12.1

Example 12.2

Note

This is based on Example 4.2 from [Karris, 2012].

Skip to main content

Use the Laplace transform method and apply Kirchoff's Voltage Law (KVL) to find the voltage $v_c(t)$ across the capacitor for the circuit shown in `fig:12.2` given that $v_c(0^-) = 6$ V.

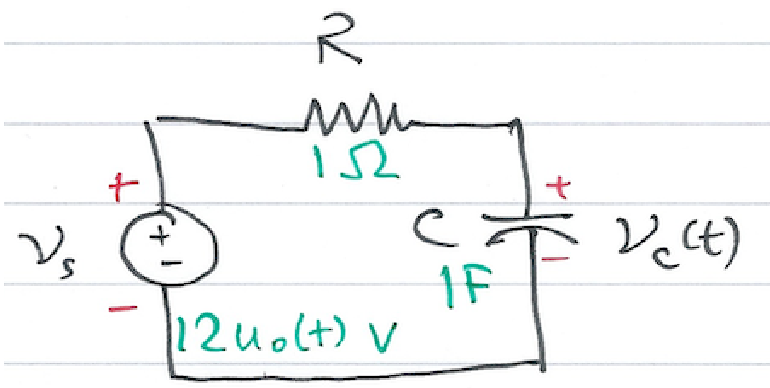

Fig. 59 Circuit for Example 12.2

Example 12.3

MATLAB Example

This is based on Example 4.3 in [Karris, 2012].

We will solve this example by hand and then review the solution in MATLAB lab 5.

In the circuit shown in Fig. 60, switch S_1 closes at $t = 0$, while at the same time, switch S_2 opens. Use the Laplace transform method to find $v_{\text{out}}(t)$ for $t > 0$.

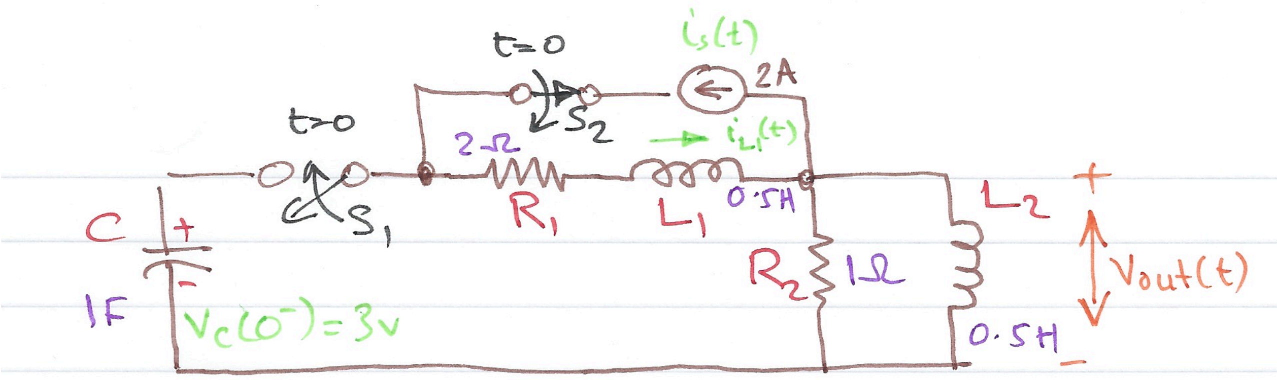

Fig. 60 Circuit for Example 12.3

Skip to main content

We can show how with the assistance of MATLAB (See solution12_3.mlx) that the solution is

$$V_{\text{out}} = \left(1.36e^{-6.57t} + 0.64e^{-0.715t}\cos 0.316t - 1.84e^{-0.715t}\sin 0.316t\right)u_0(t) \quad (30)$$

and we can plot the result (see 4. Complete solution in MATLAB)

Worked Solution: Example 12.3

File Pencast: example12_3.pdf - Download and open in Adobe Acrobat Reader.

The attached PDF gives the solution to Example 12.3 by hand. It's quite a complex, error-prone (as you can see by the crosssings out!) calculation that needs careful attention to detail. This in itself gives justification to my belief that you should use computers wherever possible.

Solution to Example 12.3

We will use a combination of pen-and-paper and MATLAB to solve this.

1. Equivalent Circuit

Draw equivalent circuit at $t = 0$

2. Transform model

Convert to transforms

3. Determine equation

Determine equation for $V_{\text{out}}(s)$.

4. Complete solution in MATLAB

In the lecture we showed that after simplification for Example 12.3

$$V_{\text{out}}(s) = \frac{2s(s+3)}{s^3 + 8s^2 + 10s + 4}$$

We will use MATLAB to factorize the denominator $D(s)$ of the equation into a linear and a quadratic factor.

Find roots of Denominator D(s)

```
p = roots([1, 8, 10, 4])
```

Find quadratic form

```
syms s t
y = expand((s - p(2))*(s - p(3)))
```

Simplify coefficients of s

```
y = sym2poly(y)
```

Complete the Square

Plot result

From equation (30)

```
t=0:0.01:10;
Vout = 1.36.*exp(-6.57.*t)...
   +0.64.*exp(-0.715.*t).*cos(0.316.*t)...
   -1.84.*exp(-0.715.*t).*sin(0.316.*t);
plot(t, Vout); grid
title('Plot of Vout(t) for the circuit of Example 3')
ylabel('Vout(t) V'),xlabel('Time t s')
```

Alternative solution using transfer functions

Skip to main content

```
impulse(Vout)
```

Example 12.4

Consider Fig. 56 and give an expression for $V_c(s)$.

Example 12.5

MATLAB Example

This is based on Example 4.4 from [Karris, 2012].

We will solve this examples by hand and then review the solution in MATLAB lab 5.

For the network shown in Fig. 61, all the complex impedance values are given in Ω (ohms).

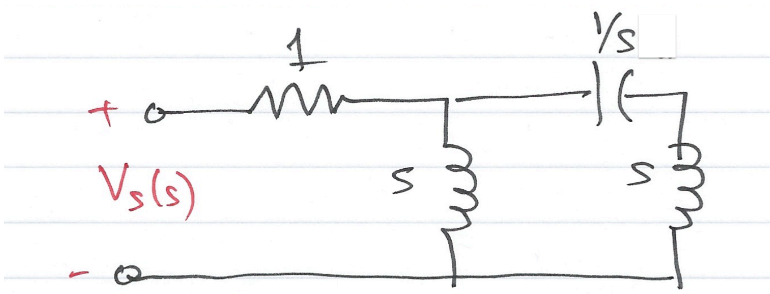

Fig. 61 Circuit for Example 12.5

Find $Z(s)$ using:

1. nodal analysis
2. by successive application of parallel and series combination of impedences

1. Solution by nodal analysis

2. Solution by by successive application of parallel and series combination of impedences

Skip to main content

Example 12.6

MATLAB Example

This is Example 4.5 from [Karris, 2012]

We will solve this examples by hand and then review the solution in MATLAB Lab 5.3.

Compute $Z(s)$ and $Y(s)$ for the circuit shown in Fig. 62. All impedence values are in Ω (ohms). Verify your answers with MATLAB.

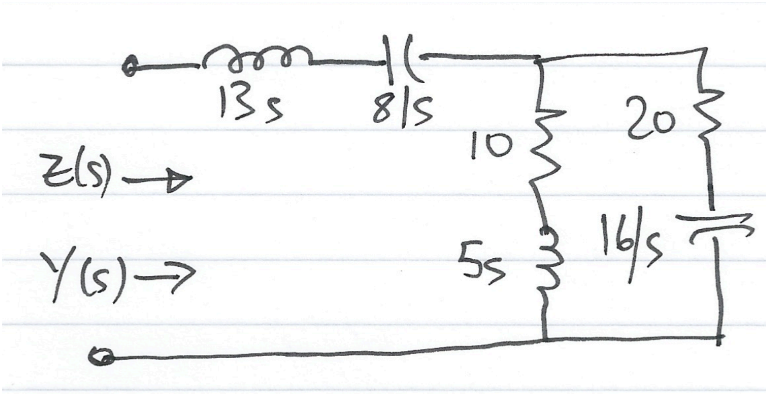

Fig. 62 Circuit for Example 12.6

Answer 12.6

$$Z(s) = \frac{65s^4 + 490s^3 + 528s^2 + 400s + 128}{s(5s^2 + 30s + 16)}$$

$$Y(s) = \frac{1}{Z(s)} = \frac{s(5s^2 + 30s + 16)}{65s^4 + 490s^3 + 528s^2 + 400s + 128}$$

Matlab verification: solution12_6.mlx

Example 12.6: Verification of Solution

```
syms s;
```

Skip to main content

```
z2 = 5*s + 10;
z3 = 20 + 16/s;
```

```
z = z1 + z2 * z3 /(z2 + z3)
```

```
z10 = simplify(z)
```

```
pretty(z10)
```

Admittance

```
y10 = 1/z10;
pretty(y10)
```

Example 12.7

MATLAB Example

This is Exercise 4 from 4.7 Exercises from [Karris, 2012]

This example will be solved as part of in MATLAB Lab 5.3.

For the s-domain circuit shown in Fig. 63

- Compute the admittance $Y(s) = I_1(s)/V_1(s)$
- Compute the time domain value of $i_1(t)$ when $v_1(t) = u_0(t)$ and all intial conditions are zero.

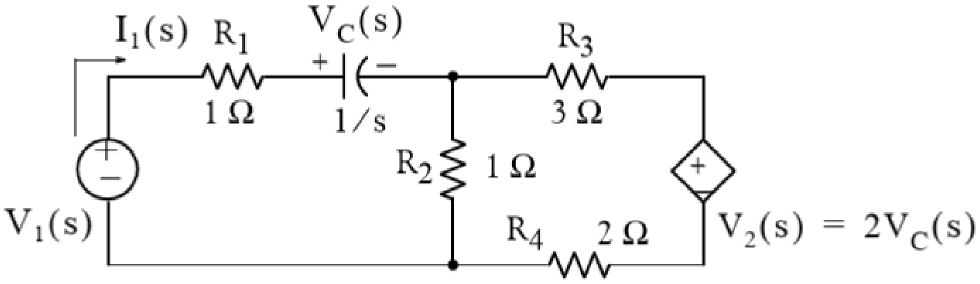

Fig. 63 Circuit for Example 12.7

Skip to main content

Lab Work

In MATLAB Lab 5, we will explore the tools provided by MATLAB for solving circuit analysis problems.

Homework

Complete any exercises that were not covered in the class or follow-up examples class. There are a number of related problems in Solved Problems 3.39—3.41 in [Hsu, 2020] and in section 4.7 Exercises in [Karris, 2012].

Supplementary problems 3.52 and following ([Hsu, 2020]) provide opportunities for extra practice.

Summary

In this section we have looked at the application of the Laplace transform to circuit analysis.

- Circuit Transformation from Time to Complex Frequency
- Complex Impedance Z(s)
- Complex Admittance Y(s)
- Examples 12

Take Aways

Circuit analysis can be performed using Laplace transforms by using the Laplace transform equivalents of the component impedence or admittance. In particular, for impedence, we use R, sL and $1/sC$; for admittance we use $G = 1/R$, $1/sL$, sC. Once the circuit has been reduced to a rational polynomial in s, the inverse laplace transform can be used to determine the time response of the circuit.

When dealing with components using their complex component equivalents, the usual circuit analysis rules, KVL, KCL, voltage-divider rule, etc, can all be used.

Complex impedence of a circuit is the resistance to current flow and is given by the general law $V(s) = Z(s)I(s)$ from which the impendence is given by

Skip to main content

$Z(s) = V(s)/I(s)$. Similarly, the complex admittance of a circuit is given by $Y(s) = I(s)/V(s)$.

Complex admittance is the reciprocal of complex impedence $Y(s) = 1/Z(s)$.

Though not a consequence of the Laplace transform, it is worth noting that the use impedence facilitates the analysis of circuits for which the compoents are commected in series; for circuits with parallel connection of components, the use of admittance facilitates the analsysis.

Next time

We move on to consider

- Unit 4.6: Transfer Functions

References

[Hsu20](1,2,3) Hwei P. Hsu. *Schaums outlines signals and systems*. McGraw-Hill, New York, NY, 2020. ISBN 9780071634724. Available as an eBook. URL: https://www.accessengineeringlibrary.com/content/book/9781260454246.

[Kar12](1,2,3,4,5,6,7,8) Steven T. Karris. *Signals and systems with MATLAB computing and Simulink modeling*. Orchard Publishing, Fremont, CA., 2012. ISBN 9781934404232. Library call number: TK5102.9 K37 2012. URL: https://ebookcentral.proquest.com/lib/swansea-ebooks/reader.action?docID=3384197.

Matlab Solutions

For convenience, single script MATLAB solutions to the examples are provided and can be downloaded from the accompanying MATLAB folder in the GitHub repository.

- Solution 12.3 [solution12_3.mlx]
- Solution 12.6 [solution12_6.mlx]