Classial Mechanics Homework 7

Cameron Kimber
Student ID: 61241043s

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Problem 1

Under especially favorable conditions, an ocean current circulating counter-clockwise when viewed from directly overhead was discovered in a well-isolated layer beneath the surface. The period of rotation was 14 hours. At what latitude and which hemisphere was the current detected?

Response

$$a = -2\omega \times v$$

$$\omega = \omega \cos\theta \hat{i} + \omega \sin\theta \hat{k}$$

We can calculate this product

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\cos\theta & 0 & \sin\theta \\ v_x & vy & 0 \end{vmatrix}$$

and so the horizontal acceleration is

$$a_H = -2\omega \sin\theta(-v_y \hat{i} + v_x \hat{j}) = -2\omega_z \hat{k} \times v$$

And this causes a circular motion, so we can say

$$a_H = \frac{v^2}{r} = v\Omega$$

So now

$$\sin\theta = \frac{\Omega}{2\omega} = \frac{6}{7}$$

so

$$\theta = 59^o$$

In the southern hemisphere.

Problem 2

A particle of mass m is acted on by a force whose potential is $V(r)$.

1. Set up the Lagrangian function in a spherical coordinate system which is rotating with angular velocity ω about the z axis.
2. Show that your Lagrangian has the same form as in a fixed coordinate system with the addition of a velocity-dependent potential U which gives the centrifugal and Coriolis forces.
3. Calculate from U the components of the centrifugal and Coriolis forces in the radial r and azimuthal ϕ directions.

Response

1. $$\mathcal{L}(\mathbf{r}, \dot{\mathbf{r}}) = \frac{m}{2}\left[\dot{\mathbf{r}} + \omega \times \mathbf{r}\right]^2 - U(r)$$

   And
   $$(\dot{\mathbf{r}} + \omega \times \mathbf{r})^2 = \dot{r}^2 + 2\omega \cdot (r \times \dot{r}) + (\omega^2 r^2 - (\omega \cdot r)^2)$$

2. This is what we want. We have a Coriolis force and a centrifugal force In the fixed frame we know we will have
   $$\dot{r}^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2 \sin^2\theta$$

   So in the rotating frame the Lagrangian is:
   $$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\varphi}\sin^2\theta + 2\omega r^2\dot{\varphi}\sin^2\theta + \omega^2 r^2 \sin^2\theta\right) - V(r)$$

   Which we can rewrite as
   $$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\varphi}^2\sin^2\theta\right) - U - V$$

   And so the potential due to the rotation is
   $$U = -\frac{1}{2}m\left(2\omega r^2\dot{\varphi}\sin^2\theta + \omega^2 r^2\sin^2\theta\right)$$

3. Getting the forces in the r, θ, and φ directions ain't so hard. We use:
   $$\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_i}\right) - \left(\frac{\partial U}{\partial q_i}\right) = Q_i'$$

   And so I made my high school calculus students take these derivatives for me:
   $$F_r = Q_r = 2m\omega r\dot{\varphi}\sin^2\theta + m\omega^2 r\sin^2\theta$$
   $$F_\theta = 2m\omega r\dot{\varphi}\sin\theta\cos\theta + m\omega^2 r\sin\theta\cos\theta$$
   $$F_\varphi = -2m\omega\dot{r}\sin\theta - 2m\omega r\dot{\theta}\cos\theta$$

Problem 3

A particle of mass m performs a two-dimensional motion in the xy plane under the influence of the potential, while choosing the polar coordinates (r, ϕ) that

$$V(x, y) = V(r) = \frac{\alpha}{2}r^2 + \beta r + \gamma$$

where α, β, and γ are constants.

1. Formulate the Hamiltonian.
2. Find two integrals of motion
3. Derive Hamilton's canonical expressions.

Response

1. The potential is independent of velocity and time, so we can simply write

   $$\mathcal{H} = T + V$$

   So

   $$\mathcal{H} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) + V(r)$$

2. The two integrals of motion are the angular momentum and the LRL vector

3. I think that means the canonical equations are

   $$\frac{P_r}{m} = \dot{r}$$

   $$\frac{P_\varphi}{m} = \dot{\varphi}$$

   $$\dot{P}_r = \alpha r + \beta + mr\dot{\varphi}^2$$