Electrodynamics HW 1

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Note to the grader- *Hi there! This homework took a LONG time to type into LaTeX. I think next time, I'll just turn it in on paper! Also, all my graphs and figures are included as an appendix after the main body. I tried to balance showing enough algebra, but also not just writing down every single step, I hope that it works out. Cheers!*

Problem 1

Read the wikipedia article about Landau Poles, argue with Chat GPT about it for an hour, then spit out something that sounds relevant to Maxwell's Theory.

Response

The coupling constant of a charge to a field, for example, an electrically charged particle to the electromagnetic field, depends on a number we call the fine structure constant, α. Clearly, this number would be of some significance if we were to investigate how charged particles or massive particles interact with one another. In fact, investigations into this field of study have yielded a very interesting behavior that we call renormalizability. In essence, the answers that we predict using our theory of electricity and magnetism at the particle scale depend on the momentum (or equivalently energy) that the particle carries.
Imagine the whole universe is a single electron. You, from some other plane of existence, with your voltmeter, want to come and make the measurements of the voltage, to find out some things about electrons. First, you'd find even if the whole universe only consists of this one electron, the electric field it makes at some point in space is not constant- this is due to quantum fluctuation or vacuum energy. At some level, it is safe to pretend this as a result of the uncertainty principle: $\Delta E \Delta t \leq \hbar$. Virtual particles with some energy ΔE can come into existence for some time $\Delta t = \hbar/\Delta E$. As stated this fluctuation makes it impossible to get a perfectly clear reading at that distance away from your electron.
You decide now to get closer to the electron, you think perhaps you will get

a more accurate reading on your voltmeter if you're closer- only now the uncertainty principle has made itself seen again. As you localize the electron to a smaller and smaller Δx, it necessarily has some unknown momentum Δp. Because of this, the constant by which it interacts with the virtual particles begins to change. If we want to accurately describe how it interacts with those particles we have to *renormalize* the degree to which it interacts or *couples* with those particles.
The Landau Pole is a problem that arises when this momentum becomes too high. As the momentum reaches a cutoff point, we can no longer renormalize the problem, and as a result, the coupling constant of the charges goes to infinity. This wouldn't be a problem if it only happened in the single electron universe, but it doesn't; according to our best calculations, it is a problem in our universe as well.
This is perhaps a consequence of the way that we solve these problems using perturbation theory. Particularly, the observed value g_{obs} of the charge and the "bare" value of the charge g_{bare} are related in a way very similar to how things are related in special relativity (albeit missing a square root)-

$$g_{bare} = \frac{g_{obs}}{1 - \beta_2 \ln \Lambda/m}$$

So that there's some number β in the denominator times the charge. It is exactly in finding what this number is where the problem arises. The "Landau Pole" is the Λ in the equation, and is the maximum momentum at a given energy scale. As Λ trends towards infinity, g diverges away from a finite value:

$$\Lambda_{Landau} \propto e^{\frac{1}{\beta g_{obs}}}$$

As a matter of fact, there are three competing possibilities to our problem and they are as follows:

1. β goes to zero at a finite value of g, so $g(\mu) \to g^*$ as $\mu \to \infty$
2. $\beta \propto g^a$ with $a \leq 1$ for large g, so $g(\mu) \to \infty$
3. $\beta \propto g^a$ for $a \geq 1$ and therefore $g(\mu)$ diverges at $\mu_0 << \infty$ in which case the Landau Pole is real, but this doesn't work for $\mu > \mu_0$

So, in the end, what does all this have to do with Maxwell's Theory? Running renormalization and the Landau Pole are big questions on the validity of QED, which again is just the application of Maxwell theory to the quantum realm. Ultimately, physics is the tool that we use to quantitatively probe the way matter and energy interact with one another, or more fundamentally, the tool we use to probe action and fields. Tools can be too blunt, they can be unelegant, they can be simply wrong. That isn't to say physics isn't useful, it is without a doubt the most successful of all human endeavors. Let's think for a moment about the biggest, most groundbreaking shakeups to occur in physics- Newton's Laws, Maxwell Theory, the Principle of Least Action, General

Relativity, and Quantum Mechanics. What do they have in common? They are new mathematical frameworks for thinking about nature. In some cases, we find that our previous physics is the consequence of these new frameworks- for example, Newton's Laws being a consequence of the principle of least action. In other cases, we find our "tools" aren't "sharp enough" for the scale necessary- for example Newton's Laws and Einstein's relativity.
All this to say maybe there's some other fundamental law of nature, of which Maxwell Theory is a consequence, or perhaps QED just isn't valid above a certain energy. Maxwell's Equations were figured out by people drawing arrows on paper, running electricity through wires, and measuring magnetic fields with lots of compasses. It's possible the idealizations it makes to be valid are too many to apply entirely to the interactions of fundamental particles. I don't have a good answer, aside from this, but it is the type of problem I like to think about.

Problem 2

Read the appendix of Jackson, calculate the energy density of the electric field produced by surface charge density $\sigma = 1C/m^2$ in SI units and Gaussian units. Write down the Maxwell equations and constituent equations in SI units and Gaussian units.

Response

For starters, our electric field density is given:

$$w = \frac{\epsilon_0}{2}|\mathbf{E}|^2 = \frac{\sigma^2}{2\epsilon_0}$$

In Gaussian Units, $\epsilon_0 = 1$, so it becomes

$$w = \frac{\sigma^2}{2}$$

Equation	SI	Gaussian
$\mathbf{D}$	$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$	$\mathbf{D} = \mathbf{E} + 4\phi\mathbf{P}$
$\mathbf{H}$	$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$	$\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$
Gauss's Law	$\nabla \cdot \mathbf{D} = \rho$	$\nabla \cdot \mathbf{D} = 4\pi\rho$
Faraday's Law	$\nabla \times \mathbf{E} + \frac{\partial B}{\partial t} = 0$	$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial B}{\partial t} = 0$
Ampere's Law	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial J}{\partial t}$	$\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial D}{\partial t}$
Gauss's Law for Magnetism	$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \mathbf{B} = 0$

Problem 3

[Jackson 1.3] Using Dirac Delta Functions, express the following charge distributions as three dimensional charge densities $\rho(x)$

1. In spherical coordinates, charge Q evenly on the surface of shell radius R.
2. In cylindrical coordinates, a charge λ per unit length uniformly distributed over a cylinder of radius b
3. In cylindrical coordinates, a charge Q spread uniformly over a flat circular disk of negligible thickness and radius R.
4. Same as previous, but in spherical coordinates.

Response

In each of these solutions, I'll equate the charge density ρ to an arbitrary function A times the Dirac delta in the "infinitely thin" direction, let the total charge be equal to the integral over space, then solve the result of integration for A.

$$\rho(x) = A\delta(x - x')$$

$$Q = \int_{space} d\tau \rho(x)$$

$$A = Q / \int d\tau$$

Before we begin, we will write down the volume elements needed to solve the problem.

$$dV_{sph} = r^2 \sin\theta dr d\theta d\phi$$

$$dV_{cyl} = r d\theta dr dz$$

1. For the spherical shell radius R:

$$\rho = A\delta(r - R) \implies Q = \int_{space} r^2 d\theta dr d\phi \delta(r - R)$$

$$Q = \int_{space} dr 4\pi r^2 \delta(r - R)$$

$$Q = 4\pi R^2$$

$$\rho(r) = \frac{Q}{4\pi R^2}\delta(r - R)$$

2. For the cylinder with linear charge density λ:

$$\lambda = \frac{Q}{z}$$

$$\rho = A(\lambda)\delta(r-b) \implies Q = \int_{space} r d\theta dr dz \delta(r-b)$$

$$\rho = \frac{Q}{2\pi z b}\delta(r-b)$$

$$\rho = \frac{\lambda}{2\pi b}\delta(r-b)$$

3. For the disk in cylindrical coordinates:

$$\rho = A\delta(z)\Theta(R-r)$$

There's a Heaviside function in here in addition to our Dirac delta in the z-direction.

$$Q = \int_{space} A\delta(z)\theta(R-r) dr dz d\theta$$

$$Q = 2\pi \int r\Theta(R-r), dz\delta(z)$$

$$Q = 2\pi \frac{r^2}{2}|_0^R$$

$$\rho(x) = \frac{Q}{\pi R^2}\delta(z)\Theta(R-r)$$

The Heaviside function really just says says "don't integrate over r from $-\infty$ to $+\infty$, instead do it over 0 to R.

4. Finally, the disk done in spherical coordinates. Again, we'll use a Heaviside function in the r-direction and a Dirac delta in the θ direction.

$$\rho(x) = A\delta(\theta)\Theta(R-r)$$

$$Q = \int_{space} \sin\theta dr d\theta d\phi r^2 A\delta(\theta)\Theta(R-r)$$

$$Q = 2\pi \frac{R^2}{2} Ar$$

$$\rho(x) = \frac{Q}{\pi R^2}\delta(\theta)\Theta(R-r)$$

哇噻 It is the same answer we got before!

Problem 4

[Jackson 1.4] For each of three charged spheres of radius a, use Gauss's Law to find the electric field inside and outside. The first is said to be conducting, the second has uniform charge density in the volume, and finally the third has spherically symmetric charge density proportional to r^n with charge Q.

Response

1. I don't quite understand the book for the first part, but I assume it means a spherical shell with a uniform charge spread on the outside. Let's find the E_{in} and E_{out}.
$$E_{in}4\pi b^2 = 0 \implies E_{in} = 0$$
$$E_{out}4\pi b^2 = Q/\epsilon_0 \implies E_{out} = \frac{Q}{4\pi\epsilon_0 b^2}$$
There's probably a way to write E without needing to split it into E_{in} and E_{out} using a Heaviside function- something like
$$E = \frac{Q}{4\pi\epsilon_0 b^2}H(r-a)$$
but I am not sure I'm doing that correctly.

2. For number two, the charge is distributed evenly throughout the volume of the sphere.
$$E_{in}4\pi b^2 = \frac{4\pi\rho b^3}{3\epsilon_0} \implies E_{in} = \frac{\rho b}{3\epsilon_0}$$
$$E_{out}4\pi b^2 = \frac{Q}{\epsilon_0} \implies E_{out} = \frac{Q}{4\pi\epsilon_0 b^2}$$

3. Finally, let's turn our attention to the case where density is proportional to r^n for $n > -3$. Let $\rho = Ab^n$ (though this perhaps isn't a general enough assumption.)
$$E_{in}4\pi b^2 = \frac{4A\pi b^3 b^n}{3\epsilon_0} \implies E_{in} = \frac{Ab^{n+1}}{3\epsilon_0}$$
$$E_{out}4\pi b^2 = \frac{Q}{\epsilon_0} \implies E_{out} = \frac{Q}{4\pi\epsilon_0 b^2}$$
We can double back now, since we know that
$$\frac{4\pi A a^3}{3} = Q \implies A = \frac{3Q}{4\pi a^3}$$
Let's rewrite the first answer:
$$E_{in} = \frac{Qb^{n+1}}{4\pi\epsilon_0 a^3}$$
As stated in the note above, the graphs are in the appendix.

Problem 5

[Jackson 1.10] Prove the *mean value theorem*: For charge-free space the value of the electrostatic potential at any point is equal to the average of the potential over the surface of *any* sphere centered at that point

Response

Let's write down a Green's Function and Green's theorem.

$$G(x, x') = \frac{1}{|x - x'|}$$

and because I hold on to Griffith's Introduction to Electrodynamics like a baby holds on to a bottle, we can rewrite our Green's Function like this:

$$G = \frac{1}{|\mathcal{R}|}$$

It's a shame the Griffith's "curly R" isn't easy to type in Latex.

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\int_V d^3x' \frac{\rho(x')}{\mathcal{R}} + \frac{1}{4\pi}\oint da' \left[\underbrace{\frac{1}{\mathcal{R}}\frac{\partial\Phi}{\partial n'}}_{Zero} - \Phi\frac{\partial}{\partial n'}\frac{1}{\mathcal{R}}\right]$$

We recognize the Poissonian in the first integral, and rewrite it:

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\int_v d^3x' \nabla\Phi \cdot n' - \frac{1}{4\pi}\oint da' \Phi(x') \left(\frac{-1}{\mathcal{R}^2}\right)$$

We can once again rewrite the first integral:

$$\Phi(x) = \frac{1}{4\pi\epsilon_0 \mathcal{R}} \underbrace{\int_V d^3x - E \cdot n'}_{\to 0} + \frac{1}{4\pi R^2}\oint da' \Phi(x')$$

Where the first integral goes to zero because of our 好朋友 Gauss' Law. Finally, we can see:

$$\Phi(x) = \frac{1}{4\pi\mathcal{R}^2}\oint da' \Phi(x')$$

The scalar potential at some point is equal to the average of the potential around that point, or the sum of the infinitesimal scalar potentials, each divided by the area.

Problem 6

[Jackson 1.11] Use Gauss's Theorem to prove that at the surface of a curved charged conductor, the normal derivative of the electric field is given by:

$$\frac{1}{E}\frac{\partial E}{\partial n} = -\left(\frac{1}{R_1}\frac{1}{R_2}\right)$$

Where R_1 and R_2 are the principal radii of curvature of the surface.

Response

Note:I spent hours on this problem and couldn't make any good progress. I had to get outside help for it.

$$\frac{\partial E}{\partial n} = \lim_{\Delta r \to 0} \frac{E(x + \Delta r\hat{n}) - E(x)}{\Delta r}$$

Gauss's Law:

$$\oint_S da\ \mathbf{E} \cdot \hat{n} = 0 \implies \int da\ \mathbf{E}_{top} \cdot \hat{n} - \int da\ E_{bot} \cdot \hat{n} = 0$$

The sign flips at the conducting surface, so one is negative.

$$E(x + \Delta r\hat{n}) = \frac{E(x)R_1R_2}{(R_1 + \Delta r)(R_2 + \Delta r)}$$

We can plug this in to the definition of $\frac{\partial E}{\partial n}$:

$$\frac{\partial E}{\partial n} = \frac{\left(\frac{R_1R_2}{(R_1+\Delta r)(R_2+\Delta r)} - 1\right)E}{\Delta r}$$

Bada bing bada boom:

$$\frac{1}{E}\frac{\partial E}{\partial n} = -\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

Problem 7

[Jackson 1.12] Prove *Green's Reciprocation Theorem*: if Φ is the potential due to a volume charge density ρ within a volume V and a surface charge density σ on the conducting surface S bounding the volume V, while Φ' is the potential due to another charge distribution ρ' and σ', then

$$\int_V \rho\Phi' d^3x + \int_S \sigma\Phi' da = \int_V \rho'\Phi d^3x + \int_S \sigma'\Phi da$$

Response

This one really isn't too bad- Let's use our noggins for a moment:

$$\rho = -\epsilon_0 \nabla^2 \Phi$$

$$\sigma = \epsilon_0 \frac{\partial \Phi}{\partial n}$$

And for good measure, let's write down Green's Second Theorem:

$$\int_V d^3x \left(\Phi \nabla^2 \Psi - \Psi \nabla^2 \Phi \right) = \oint_S da \left(\Phi \frac{\partial \Psi}{\partial n} - \Psi \frac{\partial \Phi}{\partial n} \right)$$

At this point, this problem just becomes a problem of putting together Legos.

$$\int_V d^3x \left(\Phi \nabla^2 \Phi' - \Phi' \nabla^2 \Phi \right) = \oint_S da \left(\Phi \frac{\partial \Phi'}{\partial n} - \Phi' \frac{\partial \Phi}{\partial n} \right)$$

$$\int_V d^3x \left(\Phi \frac{\rho'}{\epsilon_0} - \Phi' \frac{\rho}{\epsilon_0} \right) = \oint_S da \left(\Phi \partial \frac{\sigma'}{\epsilon_0} - \Phi' \frac{\sigma}{\epsilon_0} \partial n \right)$$

We will 把 all of the ϵ_0 and put them into the 垃圾桶. (See figure in appendix.)

$$\int_v d^3x \left(\Phi \rho' - \Phi' \rho \right) = \oint_S da \left(\Phi \sigma' - \Phi' \sigma \right)$$

And now we have arrived at the result we've been looking for:

$$\int_V \rho \Phi' d^3x + \int_S \sigma \Phi' da = \int_V \rho' \Phi d^3x + \int_S \sigma' \Phi da$$

Problem 8

[Jackson 1.18] Consider a volume V bound by a surface S consisting of several seperate conducting surface S_i. Every conductor except S_1 is held at zero potential, while S_1 is held at unit potential.

1. Show that the potential $\Phi(x)$ anywhere in the volume V *and* on any of the surface S_i can be written

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \oint_{S_1} \sigma_1(\mathbf{x}') G(\mathbf{x}, \mathbf{x}') da'$$

where $\sigma_1(\mathbf{x}')$ is the surface charge density on S_1 and $G(\mathbf{x}, \mathbf{x}')$ is the Green function potential for a point charge in the presence of all surfaces that are held at zero potential (but with S_1 absent). Show also that the electrostatic energy is

$$W = \frac{1}{8\pi\epsilon_0} \oint_{S_1} da \oint_{S_1} da' \sigma(\mathbf{x}) G(\mathbf{x}, \mathbf{x}') \sigma_1(\mathbf{x}')$$

2. Show that the variational expression

$$C^{-1}[\sigma] = \frac{\oint_{S_1} da \oint_{S_1} da' \sigma(x) G(x, x') \sigma(x')}{4\pi\epsilon_0 \left[\oint_{S_1} \sigma(x) da\right]^2}$$

with an arbitrary integrable function $\sigma(x)$ defined on S_1 is stationary for small variations of σ away from σ_1. Use Thomson's theorem to prove that the reciprocal of $C^{-1}[\sigma]$ gives a *lower bound* to the true capacitance of the conductor S_1.

Response

I feel like we can't write Green's Theorem down enough, so let's take a look at it again:

$$\Phi(x) - \frac{1}{4\pi\epsilon_0} \int_V d^3x' \rho(x') G(x, x') + \frac{1}{4\pi} \oint_S da' \left[G(x, x') \frac{\partial \Phi}{\partial n} - \Phi(x') \frac{\partial G(x, x')}{\partial n} \right]$$

And since we're talking about the scalar potential on one surface, we recognize these are Type-1 a.k.a. Dirichlet BCs.

$$\Phi(x) = \frac{1}{4\pi\epsilon_0} \int_V d^3x' \rho(x') G_D(x, x') - \frac{1}{4\pi} \oint_S da' \Phi(x') \frac{\partial G_D}{\partial n'}$$

And we will say that $\Phi(x') = 0$ (because aside from S_1, each conducting surface is held at ground potential), so we can simplify this expression:

$$\Phi(x) = \frac{1}{4\pi\epsilon_0} \int_V d^3x' \rho(x') G(x, x')$$

This is actually just equation 1.23 in Jackson; it's fairly easy to see that because we're dealing with a surface charge and not a volume charge that we've gotten the correct answer, just swap the ρ for a σ and call it a day. The only outstanding thing to say is that the charges on S_1 are on it's surface, so let's change this to a surface integral:

$$\Phi(x) = \frac{1}{4\pi\epsilon_0} \oint_{S_1} da' \sigma_1(x') G(x, x')$$

Electrostatic potential in our case is given by equation 1.53:

$$W = \frac{1}{2} \int d^3x \rho(x') \Phi(x)$$

And we can just insert our potential.

$$W = \frac{1}{8\pi\epsilon_0} \int d^3x \rho(x') \oint_{S_1} da' \sigma_1(x') G(x, x')$$

Again, we are dealing with a surface density, not a volume density, so:

$$W = \frac{1}{8\pi\epsilon_0} \oint da\ \sigma_1(x) \oint_{S_1} da' \sigma_1(x') G(x, x')$$

The density isn't a function of the area, so we can move it to the second integral if we want:

$$W = \frac{1}{8\pi\epsilon_0} \oint da \oint_{S_1} da' \sigma_1(x') \sigma_1(x) G(x, x')$$

For this next part I had to go deep into OpenStax University Physics Vol. 1 and find the equation relating charge, potential, and capacitance. I think it's something like:

$$W = \frac{1}{2} C V^2 = \frac{1}{2} Q V$$

Essentially, by inspection of the problem, we know we're looking to wrangle C^{-1} into the form $2W/Q^2$, so I hired a high school senior to help me with the algebra involved. Let's write it down one time and then let σ vary some:

$$C^{-1} = \frac{2\frac{1}{8\pi\epsilon_0} \oint da\ \ \oint_{S_1} da' \sigma_1(x') \sigma_1(x) G(x, x')}{\left(\oint da\ \sigma_1\right)^2}$$

Clean this mess up a little bit:

$$C^{-1} = \frac{\oint da \oint_{S_1} da' \sigma_1(x') \sigma_1(x) G(x, x')}{4\pi\epsilon_0 \left(\oint da\ \sigma_1\right)^2}$$

Multiply by the denominator, and let $\sigma_1 = \sigma_1 + \delta\sigma_1$:

$$4\pi\epsilon_0 \left(\oint_{S_1} da\ \sigma_1 + \delta\sigma_1\right)^2 C^{-1} = \oint da \oint_{S_1} da'\ (\sigma_1 + \delta\sigma_1 \sigma_1 + \delta\sigma_1) G(x, x')$$

The left hand side can be simplified:

$$= 4\pi\epsilon_0 \left(Q^2 C^{-1} + (\oint da\ \delta\sigma_1)^2 C^{-1}\right)$$

The right hand side can be simplified as well:

$$2 \oint da \oint da' (\sigma_1 + \delta\sigma_1) G(x, x')$$

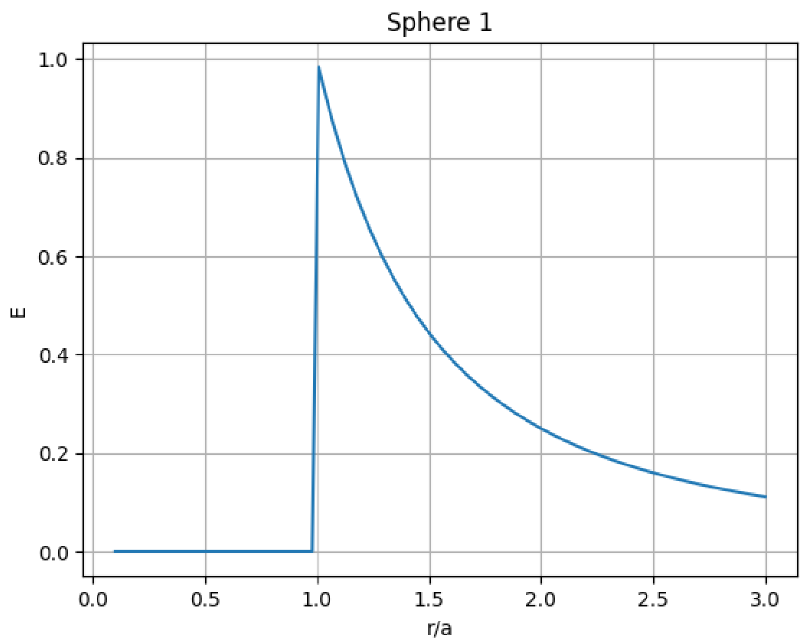
This can turn into:

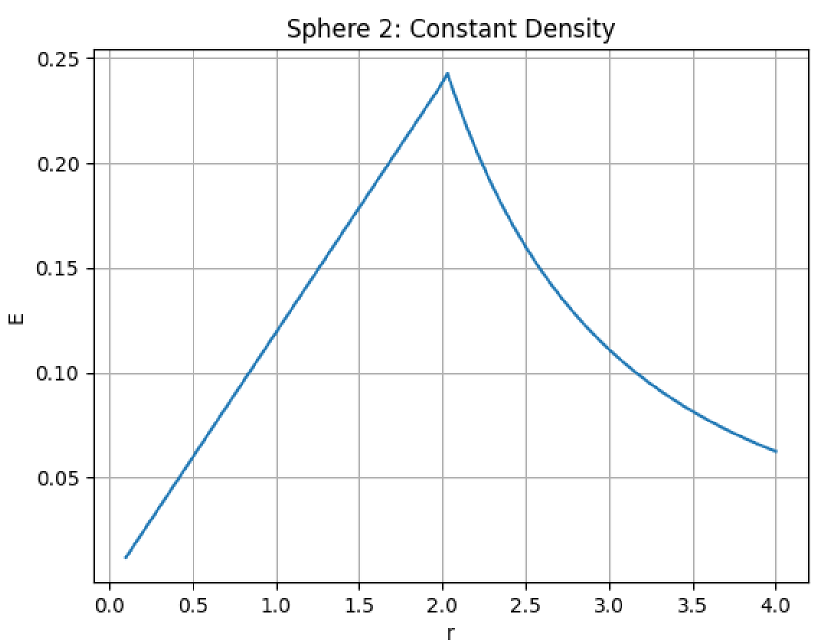
$$4\pi\epsilon_0 (\oint da\ \delta\sigma_1)^2 C^{-1} = 2 \oint da \oint da' (\sigma_1 + \delta\sigma_1) G(x, x') - \frac{4\pi\epsilon_0 Q^2}{C}$$

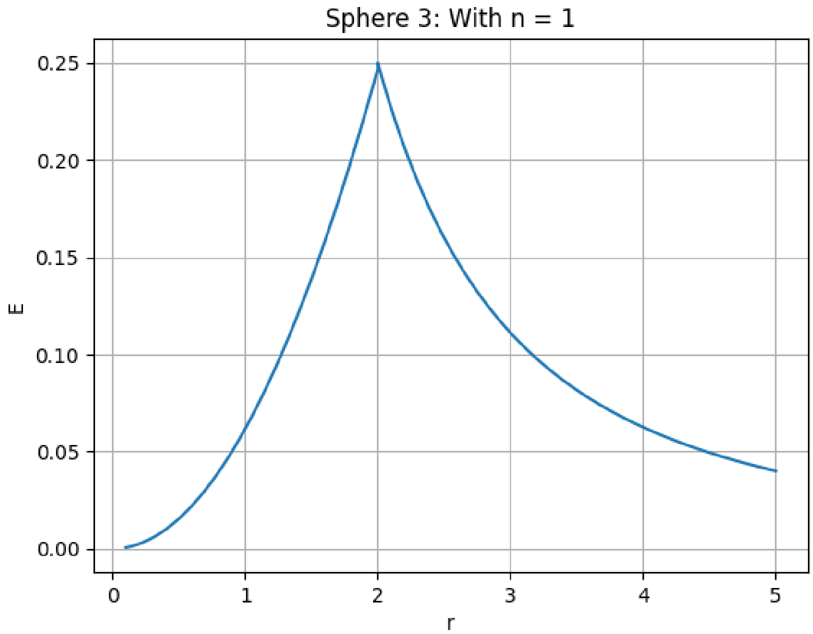
. Because of this, C^{-1} cannot vary.
C^{-1} is actually called electrical elastance, and is related deeply to Thomson's Theorem that a single electrostatic charge cannot be in a stable point on a conducting surface. C^{-1} is the resistance to change in a configuration of charge.

Appendix

Here are all of the relevant graphs and images.







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