Electrodynamics HW 2

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Problem 1

[Jackson 7.6] A plane wave of frequency ω is incident normally from vacuum on a semi-infinite slab of material with a *complex* index of refraction $n(\omega)[n^2(\omega) = \epsilon(\omega)\epsilon_0]$

1. Show that the ratio of reflected power to incident power is

$$R = \left|\frac{1 - n(\omega)}{1 + n(\omega)}\right|^2$$

while the ratio of power transmitted into the medium to the incident power is

$$T = \frac{4\text{Re } n(\omega)}{|1 + n(\omega)|^2}$$

2. Evaluate $Re[i\omega(\vec{E} \cdot \vec{D}^* - \vec{B} \cdot \vec{H}^*)/2]$ as a function of (x, y, z). Show that this rate of change of energy per unit volume accounts for the relative transmitted power T.

3. For a conductor with $n^2 = 1 + i(\sigma/\omega\epsilon_0)$, σ real, write out the results of parts a and b in the limit $\epsilon_0\omega << \sigma$. Express your answer in terms of δ as much as possible. Calculate $\frac{1}{2}Re(\vec{J}^* \cdot \vec{E})$ and compare with the result of part b. Do both enter the complex form of Poynting's Theorem?

Response

1. Incident, refracted, and reflected $\vec{E}$ and $\vec{B}$ fields:

$$\text{Incident: } \mathbf{E} = \mathbf{E_0}e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}$$

$$\mathbf{B} = \sqrt{\mu\epsilon}\frac{\mathbf{k} \times \mathbf{E}}{k}$$

$$\text{Refracted: } \mathbf{E}' = \mathbf{E_0}'e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}$$

$$\mathbf{B}' = \sqrt{\mu'\epsilon'}\frac{\mathbf{k}' \times \mathbf{E}'}{k'}$$

$$\text{Reflected: } \mathbf{E}'' = \mathbf{E_0}'' e^{\mathbf{k}''\cdot\mathbf{x}-i\omega t}$$

$$\mathbf{B}'' = \sqrt{\mu\epsilon}\frac{\mathbf{k}''\times\mathbf{E}''}{k}$$

and the wave numbers have magnitudes:

$$|\mathbf{k}| = |\mathbf{k}''| = k = \omega\sqrt{\mu\epsilon}$$

$$|\mathbf{k}'| = k' = \omega\sqrt{\mu'\epsilon'}$$

If we let $\mathbf{n}$ be complex (i.e. $\mathbf{n} = \mathbf{n}_R + i\mathbf{n}_I$), the exponential in the above equations becomes:

$$e^{ik\mathbf{n}\cdot\mathbf{x}-i\omega t} = e^{-k\mathbf{n}_i\cdot\mathbf{x}}e^{ik\mathbf{n}_R\cdot\mathbf{x}-i\omega t}$$

The Complex Poynting Vector is given:

$$S = \frac{1}{2}\mathbf{E}\times\mathbf{H}*$$

$$\mathbf{S} = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}|E_0|^2\hat{n}$$

And the power in the direction of the Poynting vector is simply

$$P = \mathbf{k}\cdot\mathbf{S} = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}|E_0|^2\mathbf{k}\cdot\hat{n}$$

Because the power transmission is $P \propto |E_0|^2$, the ratio of the reflected and transmitted power can be expressed as ratios of the squares of the E_0 terms, which are given in the following ratios, first for the $\mathbf{E}$ field being perpendicular to the plane of incidence

$$\text{Transmitted: } \frac{E_0'}{E_0} = \frac{2n\cos i}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}$$

$$\text{Reflected: } \frac{E_0''}{E_0} = \frac{n\cos i - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}{n\cos i + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}$$

Where we have used $\cos i$ to express this in terms of angle of incidence. For an electric field parallel to the plane of incidence:

$$\text{Transmitted: } \frac{E_0'}{E_0} = \frac{2nn'\cos i}{\frac{\mu}{\mu'}n'^2\cos i + n\sqrt{n'^2 - n^2\sin^2 i}}$$

$$\text{Reflected: } \frac{E_0''}{E_0} = \frac{\frac{\mu}{\mu'}n'^2\cos i - n\sqrt{n'^2 - n^2\sin^2 i}}{\frac{\mu}{\mu'}n'^2\cos i + n\sqrt{n'^2 - n^2\sin^2 i}}$$

Looking first at the transmitted wave in the perpendicular case, we get the ratio

$$T = \frac{4n^2\cos^2(i)}{\left(\frac{\mu\sqrt{(n')^2 - n^2\sin^2(i)}}{\mu'} + n\cos(i)\right)^2}$$

And for the reflected wave in the perpendicular case:

$$R = \frac{\left(\mu\sqrt{(n')^2 - n^2\sin^2(i)} - n\cos(i)\mu'\right)^2}{\left(\mu\sqrt{(n')^2 - n^2\sin^2(i)} + n\cos(i)\mu'\right)^2}$$

and we can identify

$$n(\omega) = \frac{n\mu'\cos i}{\mu\sqrt{n'^2 - n^2\sin^2 i}}$$

So we are left with the answer

$$R = \left|\frac{1-n(\omega)}{1+n(\omega)}\right|^2$$

And

$$T = \frac{4\text{Re } n(\omega)}{|1+n(\omega)|^2}$$

2. To find $Re[i\omega(\vec{E}\cdot\vec{D}^* - \vec{B}\cdot\vec{H}^*)/2]$, we first note that

$$D = \epsilon(\omega)\mathbf{E} \implies \mathbf{E}\cdot\mathbf{D}^* = \epsilon(\omega)^*|E_0|^2$$

$$H = \frac{1}{\mu_0}\mathbf{B} \implies \mathbf{B}\cdot\mathbf{H}^* = \frac{|\mathbf{k}\cdot\mathbf{E}|^2}{\mu_0\omega^2}$$

And that the electric and magnetic field inside the material is just the transmitted component of the incident fields. So now it looks like:

$$i\omega(\vec{E}\cdot\vec{D}^* - \vec{B}\cdot\vec{H}^*) = i\omega(\epsilon(\omega)^*|E_0'|^2 - \epsilon_0|n(\omega)^2||E_0'|^2)e^{-2\Im(k)z}$$

$$Re[i\omega(\vec{E}\cdot\vec{D}^* - \vec{B}\cdot\vec{H}^*)/2] = \frac{1}{2}\Re\left[i\omega(\epsilon(\omega)^* - \epsilon_0|n(\omega)^2|)|E_0'|^2e^{-2\Im(k)z}\right]$$

$$= sqrt\frac{\epsilon_0}{\mu_0}\text{Re }[n(\omega)\Im[k(\omega)]|E_t|^2e^{-2\Im[k(\omega)]}]$$

And now for the transmitted power per unit area:

$$\frac{P}{A} = \int_0^\infty dz sqrt\frac{\epsilon_0}{\mu_0}\text{Re }[n(\omega)\Im[k(\omega)]|E_t|^2e^{-2\Im[k(\omega)]}]$$

$$= sqrt\frac{\epsilon_0}{\mu_0}\text{Re }[n(\omega)\Im[k(\omega)]|E_t|^2\int_0^\infty dz e^{-2\Im[k(\omega)]}]$$

$$= \epsilon_0|E_t|^2\Re(n)\frac{c}{2}$$

and then T

$$T = \Re\frac{|E_t|^2}{|E_i|^2}$$

$$T = \frac{4\Re(n(\omega))}{|1+n(\omega)|^2}$$

3.

Problem 2

[Jackson 7.16] Plane waves propagate in a homogeneous, non-permeable, but *anisotropic* dielectric. The dielectric is characterized by a tensor ε_{ij}, but if coordinate axes are chosen as the principle axes, the components of displacement along these axes are related to the electric-field components by $D_i = \epsilon_i E_i (i = 1, 2, 3)$, where ϵ_i are the eigenvalues of the matrix ε_{ij}.

1. Show that plane waves with frequency ω and wave vector $\vec{k}$ must satisfy

$$\vec{k} \times (\vec{k} \times \vec{E}) + \mu_0 \omega^2 \vec{D} = 0$$

2. Show that for a given wave vector $\vec{k} = k\vec{n}$ there are two distinct modes of propagation with different phase velocities $v = \omega/k$ that satisfy the Fresnel equation

$$\sum_{i=1}^{3} \frac{n_i^2}{v^2 - v_i^2} = 0$$

where $v_i = 1/\sqrt{\mu_0 \epsilon_i}$ is called a principle velocity, and n_i is the component of $\vec{n}$ along the ith principle axis.

3. Show that $\vec{D_a} \cdot \vec{D_b} = 0$, where $\vec{D_a}$, $\vec{D_b}$ are the displacements associated with the two modes of propagation.

Response

1. Assuming $\vec{E}$ and $\vec{B}$ are harmonic with time:

$$\nabla \times \vec{E} - i\omega \vec{B} = 0$$

$$\frac{1}{\mu_0} \nabla \times \vec{B} + i\omega \vec{D} = 0$$

Taking a transform so that $\nabla \to i\vec{k}$:

$$i\vec{k} \times \vec{E} - i\omega \vec{B} = 0$$

$$i\vec{k} \times \vec{B} + i\mu_0 \omega \vec{D} = 0$$

So we can solve the first equation and plug it into the second:

$$i\vec{k} \times \left(\frac{\vec{k} \times \vec{E}}{\omega} \right) + i\omega\mu_0 \vec{D} = 0$$

$$\vec{k} \times \left(\vec{k} \times \vec{E} \right) + \omega^2 \mu_0 \vec{D} = 0$$

2. We know

$$k^2 \hat{n} \left(\hat{n} \times \vec{E} \right) + \omega^2 \mu_0 \vec{D} = 0$$

And using the $BAC - CAB$ rule:

$$k^2 [\hat{n} \left(\hat{n} \cdot \vec{E} \right) - \vec{E}(\hat{n} \cdot \hat{n})] + \omega^2 \mu_0 \vec{D} = 0$$

Simplifying a bit,

$$n_i\left(n_jE_j\right) - E_i + \frac{\omega^2}{k^2}\mu_0 D_i = 0$$

and since $\mu_0\epsilon_i = 1/v_i^2$ and $D_i = \epsilon_i E_i$

$$n_in_jE_j + \frac{v^2E_i}{v_i^2} = 0$$

We can solve this eigenvalue equation in v.

$$\frac{v^2}{v_1v_2v_3}[n_1^2(v^4+v_2^2v_3^2-v^2v_2^2-v^2v_3^2)+n_2^2(v^4+v^2v_3^2+v_1^2v_3^2-v^2v_1^2)+n_3^2(v^4+v^2v_2^2-v^2v_1^2+v_1^2v_2^2)]=0$$

Which, through the magic of television can be turned into:

$$\frac{v^2}{v_1v_2v_3}\left[\sum_{i=1}^{3}\frac{n_i^2}{v^2-v_i^2}\right] = 0$$

When the term in brackets is not zero, we have

$$\sum_{i=1}^{3}\frac{n_i^2}{v_\pm^2-v_i^2} = 0$$

3.

$$\left(\hat{k}\cdot\vec{E}_a\right)\hat{k} - \vec{E}_a = -\mu_0\frac{\omega^2}{k^2}\epsilon\cdot\vec{E}_a = -v_a^2\vec{D}_a$$

$$\vec{D}_b\cdot\vec{D}_a v_a^2 = \vec{D}_b\cdot\left(\vec{E}_a - (\hat{k}\cdot\vec{E}_a\hat{k})\right) = \vec{D}_b\cdot\vec{E}_a$$

but $\hat{k}\cdot\vec{E} = 0$, so

$$\vec{D}_b\cdot\vec{D}_a(v_a^2-v_b^2) = \vec{D}_b\cdot\vec{E}_a - \vec{D}\cdot\vec{E}_b$$

$$\vec{D}_b\cdot\vec{E}_a = \vec{D}_a\cdot\vec{E}_b$$

so we can say

$$\vec{D}_b\cdot\vec{D}_a(v_a^2-v_b^2) = 0$$

and if the two velocities aren't the same, then

$$\vec{D}_b\cdot\vec{D}_a = 0$$

Problem 3

[Jackson 7.22] Use the Kramers-Kronig relation (7.120) to calculate the real part of $\varepsilon(\omega)$, given the imaginary part of $\epsilon(\omega)$ for positive ω as

1.

$$\Im\epsilon/\epsilon_0 = \lambda[\theta(\omega-\omega_1)-\theta(\omega-\omega_2)], \qquad \omega_2>\omega_1>0$$

2.

$$\Im\epsilon/\epsilon_0 = \frac{\lambda\gamma\omega}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}$$

In each case sketch the behavior of Im $\epsilon(\omega)$ and the result for Re $\epsilon(\omega)$ as functions of ω. Comment on the reasons for similarities or differences of your results as compared with the curves in Fig 7.8. The step function is $\theta(x)=0$, $x<0$ and $\theta(x)=1$, $x>0$.

Response

1. We can write

$$\Re(\epsilon(\omega)/\epsilon_0) = 1+\frac{2}{\pi}P\int_0^\infty d\omega' \frac{\Im(\epsilon(\omega')/\epsilon_0))\omega'}{\omega'^2-\omega^2}$$

$$\Re(\epsilon(\omega)/\epsilon_0) = 1+\frac{2}{\pi}P\int_0^\infty d\omega' \frac{\lambda[\theta(\omega'-\omega_1)-\theta(\omega'-\omega_2)]\omega'}{\omega'^2-\omega^2}$$

$$\Re(\epsilon(\omega)/\epsilon_0) = 1+\frac{2}{\pi}P\int_{\omega_1}^\infty d\omega' \frac{\omega'}{\omega'^2-\omega^2} - \frac{2}{\pi}P\int_{\omega_2}^\infty d\omega' \frac{\omega'}{\omega'^2-\omega^2}$$

$$= 1+\frac{2}{\pi}P\left[1/2\ln(|\omega'^{2-\omega^2}|)\right]_{\omega_1}^\infty - \frac{2}{\pi}P\left[1/2\ln(|\omega'^2-\omega^2|)\right]_{\omega_2}^\infty$$

$$= 1+\frac{\lambda}{\pi}\left[\ln(|\omega_2^2-\omega^2|)-\ln(|\omega_1^2-\omega^2|\right]$$

Problem 4

[Jackson 11.5] A coordinate system K' moves with a velocity $\mathbf{v}$ relative to another system K. In K' a particle has velocity $\mathbf{u}'$ and an acceleration $\mathbf{a}'$. Find the Lorentz transformation law for accelerations, and show that in the system K the components of acceleration parallel and perpendicular to $\mathbf{v}$ are

$$\vec{a}_\parallel = \frac{\left(1-\frac{v^2}{c^2}\right)^{3/2}}{\left(1+\frac{\vec{u}\cdot\vec{u'}}{c^2}\right)^3}\vec{a'}_\parallel$$

$$\vec{a}_\perp = \frac{\left(1-\frac{v^2}{c^2}\right)}{\left(1+\frac{\vec{v}\cdot\vec{u'}}{c^2}\right)^3}\left(\vec{a'}_\perp+\frac{\vec{u}}{c^2}\times(\vec{a'}\times\vec{u'})\right)$$

Response

1. We can just do a boost in one direction like:

$$x^0=\gamma(x^{0'}+\beta x') \qquad x=\gamma(x'+\beta x^{0'}) \qquad y=y' \qquad z=z'$$

Three velocities in the two frames are now:

$$\text{K: } \vec{u}=c\frac{\partial\vec{x}}{\partial x^0}, \qquad \vec{a}=c\frac{\partial\vec{u}}{\partial x^0}$$

$$\text{K: } \vec{u'} = c\frac{\partial \vec{x'}}{\partial x^{0'}}, \qquad \vec{a} = c\frac{\partial \vec{u}}{\partial x^{0'}}$$

And we have the relation

$$\frac{dx^{0'}}{dx^0} = \frac{1}{\gamma(1+\beta u'_x/c)}$$

Now we can say

$$u_x = c\frac{dx}{dx_0} = c\frac{dx^{0'}}{dx^0}\frac{dx}{dx^{0'}}$$

$$= \frac{c}{\gamma(1+\beta u'_x/c)}\frac{d}{dx^{0'}}\gamma(x' + \beta x^{0'})$$

$$= \frac{u'_x + c\beta}{1+\beta u'_x/c}$$

and also

$$u_y = c\frac{dy}{dx^0} = c\frac{dx^{0'}}{dx^0}\frac{dy}{dx^{0'}}$$

$$= \frac{c}{\gamma(1+\beta u'_x/c)}(u'_y/c)$$

$$= \frac{u'_y}{\gamma(1+\beta u'_x/c)}$$

Which now we can write in terms of parallel and perpendicular:

$$\vec{u_\parallel} = \frac{\vec{u'_\parallel + c\vec{\beta}}}{1+\vec{\beta}\cdot\vec{u'}/c}$$

and

$$\vec{u_\perp} = \frac{\vec{u'_\perp}}{\gamma(1+\vec{\beta}\cdot\vec{u'}/c)}$$

And now we can take the time derivatives of these two get the accelerations we want. First we will derive the accelerations in x and y, then change to parallel and perpendicular.

$$a_x = c\frac{du_x}{dx^0} = \frac{c}{\gamma(1+\beta u'_x/c)}\frac{d}{dx^{0'}}\frac{u'_x + c\beta}{1+\beta u'_x/c}$$

$$= \frac{(1-\beta^2)a'_x}{\gamma(1+\beta u'_x/c)^3}$$

$$= \frac{a'_x}{\gamma^3(1+\beta u'_x/c)^3}$$

Now changing this to parallel:

$$\vec{a_\parallel} = \frac{\vec{a_\parallel}'}{\gamma^3(1+\vec{\beta}\cdot\vec{u'}/c)^3}$$

And in the y direction:

$$a_y = c\frac{du_y}{dx^0} = \frac{c}{\gamma(1+\beta u_x'/c)}\frac{d}{dx^{0'}}\frac{u_y'}{\gamma(1+\beta u_x'/c)}$$
$$= \frac{c}{\gamma^2(1+\beta u_x'/c)}\frac{(1+\beta u_x'/c)(a_y'/c) - u_y'(\beta a_x'/c^2)}{(1+\beta u_x'/c)^2}$$
$$= \frac{a_y' + \beta(u_x' a_x' - u_y' a_x')/c}{\gamma^2(1+\beta u_x'/c)^3}$$

We will have to use the $BAC - CAB$ rule to convert this into the perpendicular acceleration:

$$\vec{a_\perp} = \frac{\vec{a}'_\perp + \vec{\beta}\times(\vec{a}'\times\vec{u}')/c}{\gamma^2(1+\vec{\beta}\cdot\vec{u'}/c)^3}$$

Which with one more step is the relation that we wanted to show.

$$\vec{a}_\perp = \frac{\left(1-\frac{v^2}{c^2}\right)}{\left(1+\frac{\vec{v}\cdot\vec{u'}}{c^2}\right)^3}\left(\vec{a'}_\perp + \frac{\vec{u}}{c^2}\times(\vec{a'}\times\vec{u'})\right)$$

Problem 5

[Jackson 11.12] Apply the result of Problem 11.11 to a purely algebraic deviation of (11.116) on Thomas precession.

1. With

$$L = -\frac{\beta\cdot\mathbf{K}(\tanh^{-1}\beta)}{\beta}$$
$$L + \delta L = -\frac{(\beta + \delta\beta_\parallel + \delta\beta_\perp)\cdot\mathbf{K}(\tanh^{-1}\beta)}{\beta'}$$

where $\beta' = \sqrt{(\beta+\delta\beta_\parallel)^2 + (\delta\beta_\perp)^2}$, show that

$$\delta L = -\gamma^2\delta\beta_\parallel\cdot\mathbf{K} - \frac{\delta\beta_\perp\cdot\mathbf{K}(\tanh^{-1}\beta)}{\beta}$$

2. Using the commutation relations for K and S, show that

$$C_1 = [L,\delta L] = -\left(\frac{\tanh^{-1}\beta}{\beta}\right)^2(\beta\times\delta\beta_\perp)$$
$$C_2 = [L, C_1] = (\tanh^{-1}\beta)^2\delta L_\perp$$
$$C_3 = [L, C_2] = (\tanh^{-1}\beta)^2 C_1$$
$$C_4 = [L, C_3] = (\tanh^{-1})^4\delta L_\perp$$

where $\delta L_\perp$ is the term in δL for solving $\delta\beta_\perp$.

3. Sum the series of terms for $A_T = A_2 A_1^{-1}$ to obtain

$$A_T = I - \left(\gamma^2\delta\beta_\parallel + \gamma\delta\beta_\perp\right)\cdot\mathbf{K} - \frac{\gamma^2}{\gamma+1}(\beta\times\delta\beta_\perp)\cdot\mathbf{S}$$

correct to first order in $\delta\beta$.

Response

1.

Problem 6

[Jackson 11.26] In an elastic scattering process the incident particle imparts energy to the stationary target. The energy ΔE lost by the incident particle appears as recoil kinetic energy of the target. In the notation of Problem 11.23, $m_3 = m_1$ and $m_4 = m_2$, while $\Delta E = T_4 = E_4 - m_4$

1. Show that ΔE can be expressed in the following ways,

$$\Delta E = \frac{m_2}{W^2} p_{Lab}^2 (1 - \cos\theta')$$

$$\Delta E = \frac{2m_2 p_{Lab}^2 \cos^2\theta_4}{W^2 + p_{Lab}^2 \sin^2\theta_4}$$

$$\Delta E = \frac{Q^2}{2m_2}$$

   Where $Q^2 = -(p_1 - p_3)^2 = (\mathbf{p_1} - \mathbf{p_3})^2 - (E_1 - E_3)^2$ is the Lorentz invariant momentum transfer (squared)

2. Show that for charged particles other than electrons incident on stationary electrons ($m_1 >> m_2$) the maximum energy loss is approximately

$$\Delta E \approx 2\gamma^2\beta^2 m_e$$

   where γ and β are characteristic of the incident particle and $\gamma << (m_1/m_e)$. Give this result a simple interpretation by considering the relevant collision in the rest frame of the incident particle and then transforming back to the laboratory.

3. For electron-electron collisions, show that the maximum energy transfer is

$$\Delta E^{(e)}_{max} = (\gamma - 1)m_e$$

Response

1. We can say as a result of conservation of momentum that

$$E_2' = E_4'$$

   and now

$$p_2' p_4' = E_2' E_4' - pq\cos\theta' = m_2^2 + p_2^{'2} - p_2^{'2}\cos\theta' = m_2^2 + p_2^2(1 - \cos\theta')$$

   Now to the lab frame

$$p_2 p_4 = m_2 E_4 = m_2^2 + m_2 \Delta E$$

   and because $E_4 = \Delta E + m_2$ and that scalars are invariant under Lorentz transformations:

$$\Delta E = \frac{p_2^2}{m_2}(1 - \cos\theta')$$

Next

$$p_1 + p_2 = p_3 + p_4 \rightarrow p_1 - p_4 = p_3 - p_2$$

So

$$(p_1 - p_4)^2 = (p_3 - p_2)^2$$

Expanding this we see

$$m_1^2 + m_4^2 - 2(E_1E_4 - p_1p_4\cos\Theta') = m_3^2 + m_2^2 - 2(E_2E_3 - p_2p_3)$$

We know, however, that m_2 is at rest before the collision occurs, so its momentum is 0 and so $E_2 = m_2$.

$$E_1E_4 - p_1p_4\cos\Theta' = m_2E_3 \implies p_1p_4\cos\Theta' = E_1E_4 - m_2E_3$$

So using our relation for E_4 we saw before,

$$p_1p_4\cos\Theta' = E_1(\Delta E + m_2) - m_2(E_1 - \Delta E) = \Delta E(E_1 + m_1)$$

IF we square this and let $W^2 = (E_1 + m_2)^2 - p_1^2$

$$p_1^2p_4^2\cos^2\Theta' = \Delta E^2(W^2 + p_1^2)$$

And now

$$\Delta E = \frac{2m_2p_1^2\cos^2\Theta'}{W^2 + p_1^2\sin^2\Theta'}$$

Finally,

$$Q^2 = -(p_1 - p_3)^2 = (p_1 - p_3)^2 - (E_1 - E_2)^2$$

And $|p_1'| = |p_3'| = p'$

$$E_1' - E_3' = \sqrt{m^2 + p_1^2} - \sqrt{m^2 + p_3^{'2}} = 0$$

So

$$Q^2 = 2p^{'2}(1 - \cos\theta')$$

Now we can say

$$\Delta E = \frac{Q^2}{2m_2}$$

2. We know for $m_1 >> m_2$, we can say

$$\Delta E = \frac{m_2p_1^2}{W^2}(1 - \cos\theta')$$

This is at a maximum when $\cos\theta = -1$, so

$$\Delta E_{max} = \frac{2m_2p_1^2}{W^2} = \frac{2m_e\gamma^2\beta^2m_1^2}{W^2}$$

Where

$$W^2 = m_1^2 + m_e^2 + 2m_eE_1$$

and we can say

$$W^2 = m_1^2\left(1 + \frac{m_e^2}{m_1^2} + 2\frac{m_e}{m_1}\frac{\gamma}{\frac{m_1}{m_e}}\right)$$

So we can say

$$\Delta E_{max} \approx 2m_e^2\gamma^2\beta^2$$

3. Finally, for electron collisions, $m_1 = m_2 = m_e$, so

$$\Delta E_{max} = \frac{2m_2 p_1^2}{2m_2 E_1 + m_1^2 + m_2^2} = \frac{m_e p_1^2}{m_e E_1 + m_e^2}$$

$$= \left(\frac{\gamma^2 \beta^2}{\gamma + 1}\right) m_e$$

$$\beta^2 = \frac{\gamma^2 - 1}{\gamma^2}$$

Now we can say

$$\Delta E_{max} = (\gamma - 1) m_e$$