PHYS 4150 PS#2 Solutions.

1. $\rho_m = 6.90\ g/cm^3$, $A = 173.045$

trivalent.

a) i) $n = N_A \dfrac{Z \rho_m}{A}$

N_A - Avogadro's # $= 6.022 \times 10^{23}$

Z = valence $= 3$

$A = 173.045\ u = g/mol/N_A$

$\rho_m = 6.90\ g/cm^3 = 6.90 \times 10^{3}\ g/m^3$

$n = (6.022 \times 10^{23}) \dfrac{3 \cdot (6.90 \times 10)}{173.045}$

$n = 7.20 \times 10^{28}\ m^{-3} = 7.20 \times 10^{22}\ cm^{-3}$

ii) $k_F = (3n\pi^2)^{1/3}$

use n above.

$k_F = 1.29 \times 10^{10}\ m^{-1} = 1.29\ Å^{-1}$

iii) $\varepsilon_F = \dfrac{\hbar^2 k_F^2}{2m} = 1.01 \times 10^{-18}\ J = 6.3\ eV$

iii) $v_F = \frac{\hbar k_F}{m} = 1.49 \times 10^6 \ m/s$

$= 1.49 \times 10^8 \ cm/s$.

v) $T_F = \varepsilon_F / k_B = 7.31 \times 10^4 \ K$.

The metal is Ytterbium (Yb) #70.

b) $\frac{U}{N} = \frac{3}{5}\varepsilon_F$

Bulk modulus: $B = \frac{1}{K} = -V\frac{\partial P}{\partial V}$

What is P?

$P = -\frac{\partial U}{\partial V}\Big|_N$

note $U \propto \varepsilon_F \propto k_F^2 \propto n^{2/3} \propto \left(\frac{1}{V}\right)^{2/3}$

$\therefore$ $\frac{\partial U}{\partial V}$ yeilds factor of $-\frac{2}{3}\frac{1}{V}$.

$P = \frac{2}{3}\frac{U}{V} \propto V^{-5/3}$

$\therefore$ $B = -V\frac{\partial P}{\partial V} = -V\left(-\frac{5}{3}\frac{P}{V}\right)$

$B = \frac{5}{3}P = \frac{10}{9}\frac{U}{V}$

Note $\frac{U}{N} = \frac{3}{5}\varepsilon_F \Rightarrow \frac{U}{V} = \frac{N}{V}\frac{3}{5}\varepsilon_F = \frac{3}{5}n\varepsilon_F$

$\therefore$ $B = \frac{2}{3}n\varepsilon_F$.

$$B = \frac{2}{3} n \varepsilon_F$$

For $Cu \Rightarrow n = 8.47 \times 10^{22}\ cm^{-3}$

$$\varepsilon_F = 7.00\ eV = 1.12 \times 10^{-18}\ J.$$

$$\therefore\ B = \frac{2}{3}\left(8.47 \times 10^{28}\ m^{-3}\right)\left(1.12 \times 10^{-18}\ J\right)$$

$$B = 6.32 \times 10^{10} \left.\frac{J}{m^3}\right\}\ Pa \rightarrow \text{Pascals.}$$

Note $1\ Pa = 10\ Dyne/cm^2$

From Table 2.2: B measured $= 134.3\ Dy/cm^2$

$$B_{\text{est.}} = 63.2\ Dy/cm^2 \approx \frac{1}{2} B_{\text{meas.}}$$

→ This only accounts for pressure from $\bar{e}$ gas

→ No contribution from Lattice!!

→ Can't be only effect since in Table 2.2

Sometimes $B_{\text{meas.}} > B_{\text{est.}}$

sometimes $B_{\text{meas}} < B_{\text{est.}}$

c) $u = u_0 + \frac{\pi^2}{6}(k_B T)^2 D(\varepsilon_F)$ eq. 2.79.

$u = \frac{U}{V}$ energy density.

$u_0 = \frac{3}{5} n \varepsilon_F = \frac{2}{5} \varepsilon_F^2 D(\varepsilon_F) = U_0 \frac{N}{V}$

write $\Delta\mu = -\frac{\pi^2}{12} \frac{(k_B T)^2}{\varepsilon_F}$. & $D(\varepsilon_F) = \frac{3}{2} \frac{n}{\varepsilon_F}$

$\therefore \quad u = \frac{3}{5} n \left(\varepsilon_F - \left(\frac{5}{3}\right) 3 \Delta\mu \right)$

$= \frac{3}{5} n \varepsilon_F \left(1 - 5 \frac{\Delta\mu}{\varepsilon_F} \right)$

speed $v \Rightarrow \frac{1}{2} m v^2 = \frac{U}{N}$ (energy per particle)
per particle.

$u = \frac{U}{V}$

$\therefore \quad \frac{U}{N} = u \frac{V}{N} = \frac{u}{n}$.

$\therefore \quad \frac{U}{N} = \frac{3}{5} \varepsilon_F \left(1 - 5 \frac{\Delta\mu}{\varepsilon_F} \right)$

$\frac{1}{2} m v^2 = \frac{3}{5} \varepsilon_F \left(1 - 5 \frac{\Delta\mu}{\varepsilon_F} \right)$

Note $\varepsilon_F = \frac{1}{2} m v_F^2$

$$\therefore \quad v^2 = \frac{3}{5} v_F^2 \left(1 - 5\frac{\Delta\mu}{\varepsilon_F}\right)$$

$$v = \sqrt{\frac{3}{5}}\, v_F \underbrace{\left(1 - \frac{5\Delta\mu}{\varepsilon_F}\right)^{1/2}}_{\approx 1 - \frac{5}{2}\frac{\Delta\mu}{\varepsilon_F}}$$

Taylor Exp.

$T = 0$ result is

$$v = \sqrt{\frac{3}{5}}\, v_F$$

$\therefore$ correction is $-\frac{5}{2}\frac{\Delta\mu}{\varepsilon_F}$.

$$= \frac{5}{2}\frac{\pi^2}{12}\left(\frac{k_B T}{\varepsilon_F}\right)^2$$

2. Read Chpt 3. A&M.

3. a) $M_{sun} = 2.0 \times 10^{30}$ Kg.

estimate $N_{\cancel{sun}\,e^-}$

if Sun $\approx$ Hydrogen $\rightarrow$ 1 proton + 1 electron

then $N_H = N_{\cancel{sun}\,e^-} = \dfrac{M_{sun}}{m_H}$

$m_H = 1u = 1.6\cancel{7}6 \times 10^{-27}$ kg.

$\therefore N_{e^-} \approx \dfrac{2.0 \times 10^{30}}{1.6\cancel{6} \times 10^{-27}} = 1.2 \times 10^{57}$ electrons

$\Rightarrow$ can also assume 75% H $m = 1u$, $1e^-$
25% He $m = 4u$, $2e^-$

$\therefore N_{e^-} = M_{sun}\left(\dfrac{0.75}{1u} + \dfrac{2 \cdot 0.25}{4u}\right)$

$N_{e^-} \approx 1.05 \times 10^{57}$ e^-'s

b) $N = 1.2 \times 10^{57}$

WD radius $= 2 \times 10^{7}$ m.

Volume $= \frac{4}{3}\pi r^3 = 3.35 \times 10^{22}$ m^3

$n = \dfrac{N}{V}$

$\varepsilon_F = \dfrac{\hbar^2 k_F^2}{2m} = \dfrac{\hbar^2}{2m}\left(3n\pi^2\right)^{2/3}$

$= 6.34 \times 10^{-15}$ J $=$ 39 keV

c) $\varepsilon = \hbar k c$

$\therefore \varepsilon_F = \hbar k_F c$

$k_F = (3\pi^2 n)^{1/3}$

$\therefore \varepsilon_F = \hbar k_F c = \hbar c \underbrace{(3\pi^2)^{1/3}}_{\approx 3.} n^{1/3}$

$\varepsilon_F \approx 3 \hbar c n^{1/3}$

d) $n = \frac{N}{V}$ $\quad N = 1.2E57$

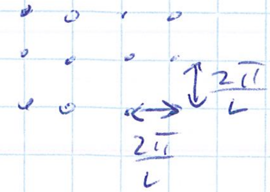
$V = \frac{4}{3}\pi (10 km)^3$

$\therefore \varepsilon_F = 3\hbar c \left(\frac{1.2 \times 10^{57}}{\frac{4}{3}\pi (10^4)^3} \right)^{1/3}$

$\varepsilon_F = 6.3 \times 10^{-11} J \approx 4 \times 10^8 eV = 0.4 GeV$

4. $\bar{e}$ gas in 2D.

a) in 2D density of k-space



$$\Omega_0 = \left(\frac{2\pi}{L}\right)^2 = \frac{4\pi^2}{A} \quad \frac{\text{vol}}{\text{k-state}}$$

or $\Delta k = \dfrac{A}{4\pi^2}$ where $\Omega \Delta k$ = # of states in area Ω.

Fermi circle: $\pi k_F^2 = \Omega_F$

Total states (k-states) is $\dfrac{N}{2}$

$$\therefore \quad N = 2 \cdot \Omega_F \Delta k = 2 \cdot \pi k_F^2 \left(\frac{A}{4\pi^2}\right)$$

$$\therefore \quad N = \frac{1}{2\pi} k_F^2 A$$

or $n = \dfrac{N}{A} = \dfrac{1}{2\pi} k_F^2$

b) In 2D: $\underset{\text{per } \bar{e}}{\text{volume}} = \pi r_s^2 = \dfrac{A}{N} = \dfrac{1}{n}$

$$\therefore \quad \pi r_s^2 = \frac{2\pi}{k_F^2}$$

$$r_s = \frac{\sqrt{2}}{k_F}$$

or $k_F = \dfrac{\sqrt{2}}{r_s}$

c) $D(\varepsilon) = \frac{1}{V} \sum_i^N \delta(\varepsilon - \varepsilon_{k_i})$

$\hookrightarrow$ in 2D V is an Area A.

$\therefore D(\varepsilon) = \frac{1}{A} \cdot 2 \cdot \sum_k \delta(\varepsilon - \varepsilon_k)$

In 2D $\Rightarrow \frac{1}{A} \sum_k F(k)$

$= \frac{1}{A} \frac{A}{4\pi^2} \int F(k)\, d^2k.$

$d^2k = k\, dk\, d\theta$ $\rightarrow$ integrating θ from $0 \rightarrow 2\pi$ yeilds 2π.

$\therefore D(\varepsilon) = \frac{2}{4\pi^2} 2\pi \int_0^\infty k\, \delta(\varepsilon - \varepsilon_k)\, dk.$

$\varepsilon = \frac{\hbar^2 k^2}{2m}$ $\qquad k = \sqrt{\frac{2m\varepsilon}{\hbar^2}}$

$d\varepsilon = \frac{2\hbar^2 k}{2m} dk.$

$dk = \frac{m}{\hbar^2 k} d\varepsilon$

$\therefore D(\varepsilon) = \frac{1}{\pi} \int_0^\infty \frac{m}{\hbar^2} \delta(\varepsilon - \varepsilon_k)\, d\varepsilon = \frac{m}{\pi \hbar^2}$

d) Sommerfeld:

$$\int H(\varepsilon) f(\varepsilon)\, d\varepsilon = \int_{-\infty}^{\mu} H(\varepsilon)\, d\varepsilon + \sum_{n=1}^{\infty} (k_B T)^{2n} a_n \frac{d^{2n-1}}{d\varepsilon^{2n-1}} H(\varepsilon) \Big|_{\varepsilon=\mu}$$

for example:

$$n = \int_{-\infty}^{\infty} \cancel{g}\, D(\varepsilon) f(\varepsilon)\, d\varepsilon \quad \Rightarrow H(\varepsilon) = D(\varepsilon)$$

$$\therefore\ n = \int_0^{\mu} D(\varepsilon)\, d\varepsilon + \sum_{n=1}^{\infty} (k_B T)^{2n} a_n \underbrace{\frac{d^{2n-1}}{d\varepsilon^{2n-1}} D(\varepsilon) \Big|_{\varepsilon=\mu}}_{\to 0}.$$

if $D(\varepsilon)$ is const. all derivative terms $\to 0$.

$$\therefore\ n = \int_0^{\mu} D(\varepsilon)\, d\varepsilon$$

$$n = \frac{m}{\pi\hbar^2}\mu.$$

recall $n = \frac{k_F^2}{2\pi}$ & $\varepsilon_F = \frac{\hbar^2 k_F^2}{2m}.$

$$\therefore\ n = \frac{\cancel{2}m\,\varepsilon_F}{\cancel{2}\pi\hbar^2} = \frac{m}{\pi\hbar^2}\mu$$

$$\therefore\ \varepsilon_F = \mu.$$

e) $D(\varepsilon) = \frac{m}{\pi\hbar^2}$

$$n = \int D(\varepsilon) f(\varepsilon)\, d\varepsilon$$

$$= \frac{m}{\pi\hbar^2} \int \frac{1}{e^{(\varepsilon-\mu)/k_BT}+1}\, d\varepsilon$$

$$= \frac{m}{\pi\hbar^2}\left[\varepsilon - k_BT\left(\ln\left(1+e^{(\varepsilon-\mu)/k_BT}\right)\right)\right]\Bigg|_0^\infty$$

now as $\varepsilon \to \infty$, $\ln\left(1+e^{(\varepsilon-\mu)/k_BT}\right) \simeq \ln\left(e^{(\varepsilon-\mu)/k_BT}\right)$

$$= (\varepsilon-\mu)/k_BT$$

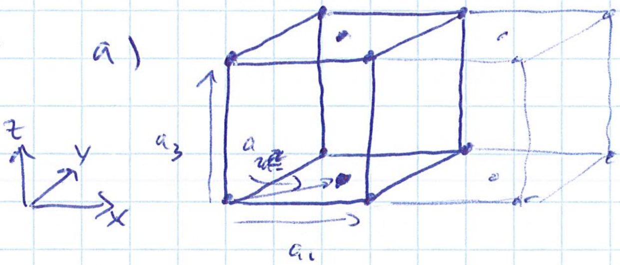
$$\therefore n = \frac{m}{\pi\hbar^2}\left[\left(\cancel{\varepsilon} - \frac{\cancel{k_BT}(\cancel{\varepsilon}-\mu)}{\cancel{k_BT}}\right)\Bigg|_\infty - 0 - k_BT\ln\left(1+e^{-\mu/k_BT}\right)\right]$$

$$n = \frac{m}{\pi\hbar^2}\left(\mu + k_BT\ln\left(1+e^{-\mu/k_BT}\right)\right)$$

$$n = \frac{k_F^2}{2\pi} = \frac{m\varepsilon_F}{\pi\hbar^2}$$

$$\therefore n = \frac{m}{\pi\hbar^2}\varepsilon_F = \frac{m}{\pi\hbar^2}\left(\mu + k_BT\ln\left(1+e^{-\mu/k_BT}\right)\right)$$

5. a)



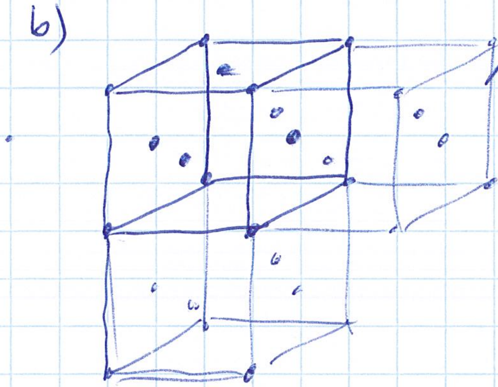
yes, Bravais lattice.

$\bar{a}_1 = a\hat{x}$ or $\frac{a}{2}(\hat{y} - \hat{x})$

$\bar{a}_2 = \frac{a}{2}(\hat{x} + \hat{y})$

$\bar{a}_3 = a\hat{z}$

b)



Not a B.L.

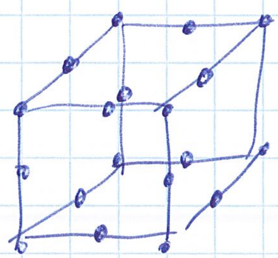
Face pts have 4 n-n.

Corner pts have 8 n-n.

$\bar{a}_1 = a\hat{x}$ $\bar{a}_2 = a\hat{y}$ $\bar{a}_3 = a\hat{z}$

basis => $0, \frac{a}{2}(\hat{x}+\hat{z}), a/2(\hat{y}+\hat{z})$

c)



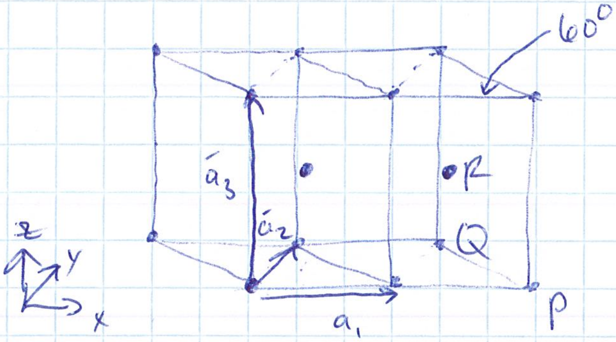
Not B.L.

eg. vert edge pts have diff view than horiz. edge pts.

→ still $a_{1,2,3} = a(\hat{x}, \hat{y}, \hat{z})$

basis => $0, \frac{a}{2}\hat{x}, \frac{a}{2}\hat{y}, a/2\,\hat{z}$

6. hcp.



$\bar{a}_1 = a\hat{x}$

$\bar{a}_2 = \frac{a}{2}(\hat{x} + \sqrt{3}\,\hat{y}) \Rightarrow |a_2| = a$

$\bar{a}_3 = c\hat{z}$

basis: $0, \left(\frac{\bar{a}_1}{3} + \frac{\bar{a}_2}{3} + \frac{\bar{a}_3}{2}\right) = \bar{r}$

For close-pack, envision uniform spheres touching pts P, Q, R.

$|P-Q| = a$ &

$\therefore$ sphere radius $= \frac{a}{2}$

$\therefore |R-P| = a.$

$\therefore$ i.e. the distance between the basis pts must also be $= a$.

$\therefore$ an $\left|\left(\frac{\bar{a}_1}{3} + \frac{\bar{a}_2}{3} + \frac{\bar{a}_3}{2}\right)\right| = a$

$\cancel{\left(\frac{a^2}{3} + \frac{a^2}{4}(1\right.}$

$$\bar{r} = \frac{\bar{a}_1}{3} + \frac{\bar{a}_2}{3} + \frac{\bar{a}_3}{2}$$

$$= \frac{a}{3}\hat{x} + \frac{a}{6}\hat{x} + \frac{\sqrt{3}a}{6}\hat{y} + \frac{c}{2}\hat{z}$$

$$= \frac{a}{2}\hat{x} + \frac{\sqrt{3}}{6}a\,\hat{y} + \frac{c}{2}\hat{z}$$

$$|\bar{r}| = a$$

$$\left(\frac{a^2}{4} + \frac{3}{36}a^2 + \frac{c^2}{4}\right)^{1/2} = a.$$

$$\cancel{\frac{c^2}{4}} \left(\frac{a^2}{3} + \frac{c^2}{4}\right)^{1/2} = a.$$

$$\frac{a^2}{3} - a^2 + \frac{c^2}{4} = 0.$$

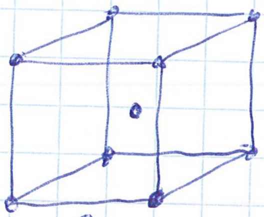
$$c^2 = 4\left(\frac{2}{3}\right)a^2$$

$$c = \sqrt{\frac{8}{3}}\,a$$

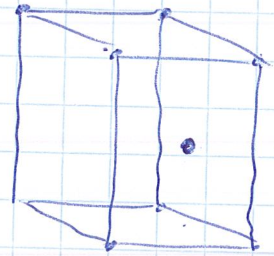
b)

bcc

hcp.



$\longrightarrow$



Constant $n = \frac{N}{V}$.

Volume of cubic bcc

$= a_c^3$ w 2pts.

$V_0 = \frac{a_c^3}{2}$

Volume hcp.

$= a_h^2 \sin(60^\circ) \cdot c$

$= \frac{\sqrt{3}}{2} a_h^2 c$.

w 2 pts. $V_0 = \frac{\sqrt{3}}{4} a_h^2 c$

$a_h \sqrt{\frac{8}{3}} = c$.

$\therefore \frac{a_c^3}{2} = \frac{\sqrt{3}}{4} a_h^2 c$

$a_c^3 = \frac{\sqrt{8}}{2} a_h^3$

$\therefore a_c^3 = \sqrt{2}\, a_h^3$

$a_h = \left(\frac{1}{\sqrt{2}}\right)^{1/3} a_c = 2^{-1/6} a_c$

$a_c = 4.23\text{Å}$ $\therefore a_h = 2^{-1/6}(4.23) = 3.77\text{Å}$.