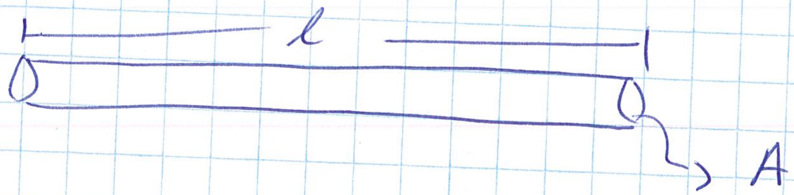
PHYS 450 PS #1 Solutions.

a)



$$E = \rho j \qquad (1)$$

$\therefore E = \frac{V}{\ell} \Rightarrow V = E\ell$

↳ Volts/m.

$j \Rightarrow$ current / area

$j = \frac{I}{A} = I = jA$

$\rho \Rightarrow \frac{\Omega}{m}$ and R should be $\propto \ell\rho$ and $\propto \frac{1}{A}$

$\rho = \frac{RA}{\ell} \Rightarrow R = \frac{\rho \ell}{A}$.

Sub into #1:

$$\frac{V}{\ell} = \frac{RA}{\ell}\frac{I}{A}$$

$\Rightarrow V = IR$

b) $\bar{j} \rightarrow$ current density: current

= charge per unit area per unit time.

charge density: n = charge per volume.

assume charges move w/ average velocity $\bar{v}$.

$\therefore$ $\bar{j}$ & $\bar{v}$ will be parallel (either same or opposite direction).

In time dt, charges will move a distance $\bar{v}dt$

For wire w/ cross-sectional area A, the # of charges moving through surface A in time dt is:

$n\,v\,dt\,A$ $\therefore$ total charge through surface is $qnv\,dt\,A$.

$\therefore$ current density (charge through A per unit time)

$$\bar{j} = qn\bar{v}$$

For electrons:

$$\bar{j} = -en\bar{v}$$

$\bar{v} \rightarrow m \cdot s^{-1}$

$n \rightarrow m^{-3}$

$e \rightarrow C$

$\bar{j} \rightarrow A \cdot m^{-2} = C \cdot m^{-2} \cdot s^{-1}$

<u>LS</u>

$C \cdot m^{-2} \cdot s^{-1}$

<u>RS</u>

$C \cdot m^{-3} \cdot m \cdot s^{-1} = C \cdot m^{-2} \cdot s^{-1}$

c) electron density: $n \left(\frac{\# \bar{e}}{m^3} \right)$

volume per $\bar{e} = \frac{1}{n}$

For copper: $n = 8.47 \times 10^{22}\ cm^{-3}$

$$V = \frac{1}{n} = 1.18 \times 10^{-29}\ m^3$$

$$= 11.8\ \text{Å}^3$$

From table 1.2: $\rho = 1.56\ \mu\Omega \cdot cm$ @ 273K

$\tau = \frac{m}{\rho n e^2}$ $m \to$ mass of $\bar{e} = 9.11 \times 10^{-31}$ Kg.

$e \to$ charge of $\bar{e} = 1.602 \times 10^{-19}$ C

$$\tau = \frac{(9.11 \times 10^{-31}\ Kg)}{(1.56 \times 10^{-11}\ \Omega \cdot m)(8.47 \times 10^{28}\ m^{-3})(1.602 \times 10^{-19}\ C)^2}$$

$$\tau = 2.686 \times 10^{-14}\ s$$

mean free path: $\ell = v_{avg} \tau$

what is V_{avg}? Use classical equipartion (pg. 9 in A & M)

$$\frac{1}{2} m v_{avg}^2 = \frac{3}{2} k_B T \qquad T = 273\ K.$$

$k_B \Rightarrow$ Boltzmann constant
$= 1.381 \times 10^{-23} \frac{m^2 kg}{s^2 K}.$

$$v_{avg} = \sqrt{\frac{3 k_B T}{m}} \approx 1.11 \times 10^5\ m/s \ @ \ 273 K.$$

$$l = \sqrt{\frac{3 k_B T}{m}}\, \tau$$

$$l \approx 2.99\ nm \approx 30\ \text{Å}$$

Copper atomic spacing ≈ 3.6 Å $\rightarrow$ any reasonable est. $\approx$ Å is fine.

$l \approx 10 \times$ ion spacing.

2. Derive: $\frac{d\bar{p}}{dt} = \bar{f}(t) - \frac{\bar{p}}{\tau}$

momentum per $\bar{e}$ is @ time t is $\bar{p}(t)$

Consider small time step dt, what is $\bar{p}(t+dt)$

Note avg time between collisions is τ.

$\therefore$ probability of collision between t and $t+dt$ is $\frac{dt}{\tau}$

$\therefore$ prob. of no collision is $1 - \frac{dt}{\tau}$

Between collisions $\bar{e}$ gains momentum from external force $f(t)$:

$$\bar{f}(t)dt + O(dt)^2$$

$\therefore$ for $\bar{e}$'s that do not collide in time dt. momentum is:

no collision $\quad \bar{p}(t+dt) = \left(1 - \frac{dt}{\tau}\right)\left[\bar{p}(t) + \bar{f}(t)dt + O(dt)^2\right]$

For $\bar{e}$'s that do collide $\to$ they will gain some fraction of $f(t)\,dt$

Prob of collision is $\frac{dt}{\tau}$: momentum gain is order:

$$dt\, f(t)\, dt \to O(dt)^2$$

$\therefore$ $\bar{p}(t+dt)$ contribution from collided e^-'s is $O(dt)^2$ (eqn. unchanged).

$$\therefore \quad \bar{p}(t+dt) = \left(1 - \frac{dt}{\tau}\right)\left[\bar{p}(t) + \bar{f}(t)dt + O(dt)^2\right].$$

$$= \bar{p}(t) - \frac{dt}{\tau}\bar{p}(t) + f(t)dt + O(dt)^2$$

lump all dt^2 terms together

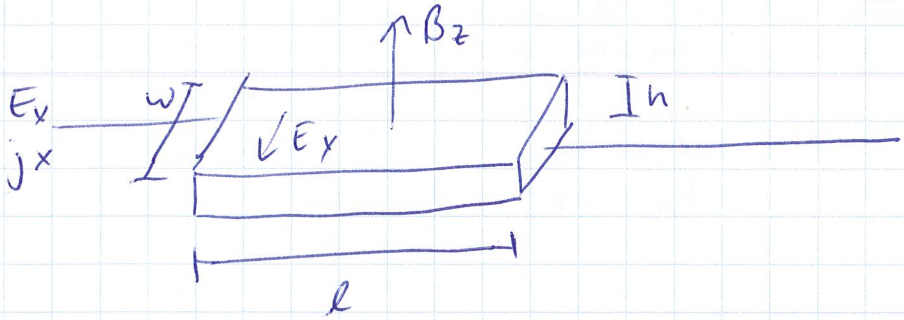
$$\therefore \quad \underbrace{\bar{p}(t+dt) - \bar{p}(t)}_{d\bar{p}(t)} = f(t)dt - \frac{dt}{\tau}\bar{p}(t) + O(dt)^2$$

$\Rightarrow$ take $\frac{\partial}{dt}$

$$\boxed{\therefore \quad \frac{d\bar{p}(t)}{dt} = f(t) - \frac{\bar{p}(t)}{\tau}}$$

3. Hall effect.

$$R_H = \frac{E_y}{j_x B_z} = \frac{1}{nq}$$



need E_y, j_x, B_z.

We control B_z $\Rightarrow$ know it.

<u>Measure</u> $E_y\ (V \cdot m^{-1}) = \frac{V_y}{w}$ $\rightarrow$ Voltage accross plate in y-dir.

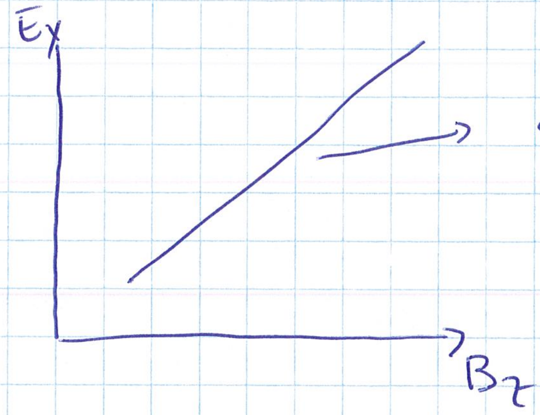
measure V_y $\bar{w}$ voltmeter
w $\bar{w}$ ruler

<u>Measure</u> $j_x\ (A \cdot m^{-2}) = \frac{I_x}{\text{Area}} = \frac{I_x}{hw}$

measure I_x $\bar{w}$ ammeter
h, w $\bar{w}$ ruler.

$\Rightarrow$ rewrite: $E_y = j_x R_H B_z$

$\therefore$ vary B_z & measure E_y as described.



slope $= j_x R_H$

$$R_H = \frac{\text{slope}}{j^x}.$$

Note: length measurements (area) are not actually rq'd since they cancel in slope. Only need thickness of slab, h.

$$B_x = \frac{h}{R_H} \frac{V_y}{I_x}$$

if $R_H > 0$ charge carriers are (+) $\Rightarrow$ "holes"

$R_H < 0$ charge carriers are (−) $\Rightarrow$ "e^-"

$$R_H = \frac{1}{nq}$$

→ can calculate n = carrier density

→ can infer # of charge carriers / atom.

4. Derive $-\nabla^2 E = \frac{\omega^2}{c^2} \epsilon(\omega) \bar{E}$

where $\epsilon(\omega) = 1 + \frac{i\sigma(\omega)}{\omega \varepsilon_0}$ $\Rightarrow$ SI units

Maxwell's Eqn's.

(1) $\bar{\nabla} \cdot \bar{E} = 0$ $\qquad$ $\bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$ (3)

(2) $\bar{\nabla} \cdot \bar{B} = 0$

$\bar{\nabla} \times \bar{B} = \mu_0 \left(\bar{j} + \varepsilon_0 \frac{\partial \bar{E}}{\partial t} \right)$ (4)

Vector identity: $\bar{\nabla} \times \bar{\nabla} \times \bar{A} = -\nabla^2 A + \underbrace{\nabla(\nabla \cdot A)}_{\nabla \cdot E = 0 \ (1)}$

Apply to (3):

$\bar{\nabla} \times (\bar{\nabla} \times \bar{E}) = -\nabla^2 E = -\left(\bar{\nabla} \times \frac{\partial \bar{B}}{\partial t} \right)$

$= -\frac{\partial}{\partial t} \bar{\nabla} \times \bar{B}$

Sub in (4):

$$-\nabla^2 E = -\frac{\partial}{\partial t} \left(\mu_0 \left(\bar{J} + \varepsilon_0 \frac{\partial \bar{E}}{\partial t} \right) \right)$$

Assume solutions of form $\bar{E} = E_0 e^{-i\omega t}$.

$\therefore \frac{\partial E}{\partial t} = -i\omega \bar{E}$

Note: $\bar{J} = \sigma \bar{E}$

$$\therefore\ -\nabla^2 \bar{E} = +i\omega\mu_0\sigma(\omega)\,E(\omega) + \omega^2\mu_0\varepsilon_0\,\bar{E}(\omega)$$

Note $\sigma(\omega)$ & $\bar{E}(\omega)$ are functions of ω.
to do Formally you could have used FT to
explicitly construct freq. dep. functions

speed of light: $c = \sqrt{\dfrac{1}{\mu_0\varepsilon_0}}$

can write $\mu_0 = \dfrac{1}{\varepsilon_0 c^2}$

$$\therefore\ -\nabla^2 \bar{E} = -\frac{i\omega\sigma(\omega)}{\varepsilon_0 c^2}\bar{E}(\omega) - \frac{\omega^2}{c^2}\bar{E}(\omega)$$

$$-\nabla^2 \bar{E} = \frac{\omega^2}{c^2}\left(1 + \frac{i\sigma(\omega)}{\omega\varepsilon_0}\right)\bar{E}(\omega)$$

has the form: $-\nabla^2 \bar{E} = \dfrac{\omega^2}{c^2}\epsilon(\omega)\,\bar{E}$

where $\epsilon(\omega) = 1 + \dfrac{i\sigma(\omega)}{\omega\varepsilon_0}$

4 Joule Heating:

a) Kinetic energy: $E_k = \frac{p^2}{2m}$

$$* \xrightarrow[t]{} *$$

in static $\bar{E}$ field: constant $\bar{F}$ & $\bar{a}$

$$\bar{F} = m\bar{a} = -e\bar{E} \qquad \text{and} \qquad \bar{a} = \frac{1}{m}\frac{d\bar{p}}{dt}$$

time between collisions is t.

$$\frac{d\bar{p}}{dt} = -e\bar{E}$$

$$\bar{p} = \int_0^t -eE \, dt = -eEt.$$

$$\therefore \quad E_k = \frac{p^2}{2m} = \frac{(eEt)^2}{2m}$$

b) Prob. of collision in range $t \rightarrow t+dt$ is $\frac{dt}{\tau} e^{-t/\tau}$

$$\langle E \rangle = \sum_i P_i E_i$$

$E_i \rightarrow$ energy lost / collision

$P_i \rightarrow$ probability of collision.

$\langle E \rangle \rightarrow$ avg. energy lost per collision

$$\Rightarrow \langle E \rangle = \int_0^\infty \underbrace{\frac{dt}{\tau} e^{-t/\tau}}_{\text{Prob. of collision}} \underbrace{\frac{(eEt)^2}{2m}}_{\text{Energy lost per collision (part a)}}$$

$$\therefore \langle E \rangle = \frac{(eE)^2}{2m\tau} \underbrace{\int_0^\infty t^2 e^{-t/\tau} dt}_{= 2\tau^3}$$

$$\therefore \langle E \rangle = \frac{(eE\tau)^2}{m}$$

Energy loss per volume per second

$\langle E \rangle$ = energy loss per $\bar{e}$ per collision:

collisions occur (on avg.) every time τ.

$\bar{e}$ gas density is $n \Rightarrow \frac{\# \bar{e}\text{'s}}{\text{Volume}}$.

$\therefore$ $\langle \dot{E}_v \rangle \Rightarrow$ energy loss per volume per second.

$$\langle \dot{E}_v \rangle = \langle E \rangle \cdot \frac{n}{\tau}$$

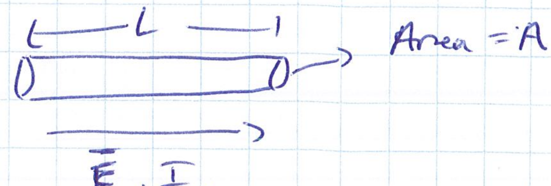
$$= \frac{n e^2 E^2 \tau}{m}$$

Recall $\sigma = \frac{n e^2 \tau}{m}$.

$$\therefore \langle \dot{E}_v \rangle = \sigma E^2$$

Power loss $= \langle \dot{E}_v \rangle \cdot V$ → volume of wire.

$$P = \sigma V E^2$$



$$V = LA$$

Recall: $j = \sigma E$

Current $= A \cdot j = I$

resistance $= \frac{L}{\sigma A} = R$.

$$\therefore \quad P = \frac{L}{AR} \cdot LA \cdot \left(\frac{j}{\sigma}\right)^2$$

$$= \frac{\cancel{L}}{\cancel{A}R} \cdot \cancel{L}\cancel{A} \cdot \left(\frac{I}{\cancel{A}}\right)^2 \left(\frac{\cancel{A}R}{\cancel{L}}\right)^2$$

$$P = I^2 R .$$

6. $f(\varepsilon) \simeq e^{-(\varepsilon-\mu)/k_B T}$

Note: $\frac{1}{n} = \frac{4}{3}\pi r_s^3 \Rightarrow r_s = \left(\frac{3}{4\pi n}\right)^{1/3}$ need n.

$$n = \int_{-\infty}^{\infty} g(\varepsilon) f(\varepsilon)\, d\varepsilon$$

where $g(\varepsilon) = \frac{m}{\hbar^2 \pi^2}\sqrt{\frac{2m}{\hbar^2}}\sqrt{\varepsilon}$

$$n = \frac{m}{\hbar^2\pi^2}\sqrt{\frac{2m}{\hbar^2}} \int_{-\infty}^{\infty} \sqrt{\varepsilon}\, e^{-(\varepsilon-\mu)/k_B T}\, d\varepsilon.$$

$$= \frac{\sqrt{2}\, m^{3/2}}{\pi^2 \hbar^3} e^{\mu/k_B T} \underbrace{\int_{-\infty}^{\infty} \sqrt{\varepsilon}\, e^{-\varepsilon/k_B T}\, d\varepsilon}_{\text{Gamma function}}.$$

$$= (k_B T)^{3/2}\, \Gamma(3/2)$$

$$= \frac{(k_B T)^{3/2}\sqrt{\pi}}{2}$$

$$\therefore\ n = \left(\frac{m k_B T}{\pi}\right)^{3/2} \frac{1}{\sqrt{2}\,\hbar^3}\, e^{\mu/k_B T}$$

$$r_s = \left(\frac{3}{4\pi n}\right)^{1/3}$$

$$= \left[\frac{3\sqrt{2}\,\hbar^3}{4\pi}\left(\frac{\pi}{mk_BT}\right)^{3/2} e^{-\mu/k_BT}\right]^{1/3}$$

$$= \pi^{1/6}\, 3^{1/3}\, e^{-\mu/3k_BT}\sqrt{\frac{\hbar^2}{2mk_BT}}$$

b) $$r_s \gg \left(\frac{\hbar^2}{2mk_BT}\right)^{1/2}$$

Note: de Broglie wavelength: $\lambda = \frac{2\pi\hbar}{p} = \frac{2\pi\hbar}{\sqrt{2mE_{kn}}}$

kinetic energy of particles.

For ideal gas: $E_{kn} = \frac{3}{2}k_BT$.

$$\therefore \lambda \propto \left(\frac{\hbar^2}{2mk_BT}\right)^{1/2}.$$

$\therefore r_s \gg \lambda$ de Broglie wavelength

i.e. the average spacing between $\bar{e}$ must be much greater than their de Boglie wavelength.

c)

$$a_0 = 5.29 \times 10^{-11}\ m.$$

$$\hbar = 1.05 \times 10^{-34}\ m^2\ kg\ s^{-1}$$

$$m = 9.11 \times 10^{-31}\ kg.$$

Want: $\dfrac{r_s}{a_0} \gg \left(\dfrac{10^5\ K}{T}\right)^{1/2}$

$$\Rightarrow \quad r_s \gg a_0 (10^5\ K)^{1/2} \frac{1}{T^{1/2}}$$

$$r_s \gg \left(\frac{\hbar^2}{2 m k_B T}\right)^{1/2}$$

$$\Rightarrow \quad \left(\frac{\hbar^2}{2 m k_B}\right)^{1/2} = 2.1 \times 10^{-8}\ m\ K^{-1/2}$$

$$a_0 (10^5\ K)^{1/2} = (5.29 \times 10^{-11}\ m)(3.16 \times 10^{2}\ K^{-1/2}).$$

$$= 1.67 \times 10^{-8}\ m\ K^{-1/2}$$

same up to order of magnitude.

d) Note: $T_F \equiv \frac{\varepsilon_F}{k_B}$ $\varepsilon_F = \frac{\hbar^2 k_F^2}{2m}$

$k_F = (3\pi^2 n)^{1/3}$

constant: $\frac{m^3}{4\pi^3 \hbar^3}$

$$\therefore \quad T_F = \frac{\hbar^2}{2 m k_B} (3\pi^2 n)^{2/3}$$

$$\Rightarrow \quad \hbar^3 = \left[\frac{2 m k_B T_F}{(3\pi^2 n)^{2/3}}\right]^{3/2} = \frac{(2 m k_B T_F)^{3/2}}{3\pi^2 n}$$

$$\rightarrow \quad = \frac{3\pi^2 m^3 n}{4\pi^3 (2 m k_B T_F)^{3/2}}$$

Rearrange: $= \left(\frac{3\sqrt{\pi}}{4}\right) n \left(\frac{m}{2\pi k_B T_F}\right)^{3/2}$

Note: $f_B(0) = n\left(\frac{m}{2\pi k_B T}\right)^{3/2}$ Boltzmann dist. @ $v=0$ (eq. 2.1).

$$\therefore \quad f_B(0)/f(0) = \left(\frac{4}{3\sqrt{\pi}}\right)\left(\frac{T_F}{T}\right)^{3/2}$$