PHYS*4150: Problem Set 2

Distributed: Friday January 25, 2019
Due: Friday February 8, 2019 at 10:30am

Problem 1 (15 pts): Sommerfeld theory of metals

a) A certain metal has a density of $6.90\,\mathrm{g/cm^3}$ and an atomic mass $A = 173.045\,\mathrm{u}$. Assuming it is a ***tri***valent metal with a spherical Fermi surface, find:

i) The density of free electrons, n.

ii) The Fermi wavevector, k_F.

iii) The Fermi velocity, v_F.

iv) The Fermi energy, ε_F.

v) The Fermi temperature, T_F.

What is the metal?

b) Starting from the $T = 0$ expression for U/N, show that the bulk modulus of the quantum free electron gas can be expressed as:

$$B = \frac{2}{3} n \varepsilon_F. \qquad (1)$$

Using values from Tables 1.1 and 2.1, compute the bulk modulus of copper. Does this change much at finite temperatures (explain only, do not calculate). Compare your calculated value to the measured value in Table 2.2. Suggest one significant source of the discrepancy. Is this the only source of error? Use other values from Table 2.2 to justify your reasoning.

c) Show that the average electron speed (v_0) at finite temperatures differs from the $T = 0$ value by terms of the order $(\frac{k_B T}{\varepsilon_F})^2$.

Problem 2 (10 pts): Free electron model

Answer the following in the context of the Free electron model (FEM), that is, the Drude model with the Sommerfeld corrections.

a) ***Explain*** the three core assumptions of the free electron model.

b) Describe ***three*** phenomena or parameters that the FEM explains well.

c) Describe ***three*** phenomena or parameters that the FEM explains poorly.

d) Describe ***three*** phenomena or parameters that the FEM does not address whatsoever.

Problem 3 (15 pts): Fermi gas in astrophysics

a) Given $M_{sun} = 2 \times 10^{30}\,\mathrm{kg}$ for the mass of the Sun, estimate the number of electrons in the Sun.

b) In a white dwarf star this number of electrons may be ionized and contained in a sphere of radius 2×10^7 m; find the Fermi energy of the electrons in electron volts.

c) The energy of an electron in the relativistic limit $\varepsilon >> mc^2$ is related to the wavevector as $\varepsilon \approx pc = \hbar kc$. Show that the Fermi energy in this limit is roughly

$$\varepsilon_F = 3\hbar c n^{1/3}. \tag{2}$$

d) If the above number of electrons were contained within a pulsar of radius 10 km, show that the Fermi energy would be $\approx 4 \times 10^8$ eV (in the relativistic limit).

This value explains why pulsars are believed to be composed largely of neutrons rather than of protons and electrons, for the energy released in the reaction $n \to p + e^-$ is only 0.8×10^6 eV, which is not large enough to enable many electrons to form a Fermi sea. The neutron decay process proceeds only until the electron concentration builds up enough to create a Fermi level of 0.8×10^6 eV, at which point the neutron, proton and electron concentrations are in equilibrium.

Problem 4 (25 pts): Free electron gas in two dimensions

Ashcroft & Mermin question 2.1

Problem 5 (10 pts): Bravais lattices

Ashcroft & Mermin question 4.1

Problem 6 (10 pts): Close-packed structures

Ashcroft & Mermin question 4.5

Problem 7 (0 pts): Not to be handed in

Complete the Sommerfeld expansion, i.e. show:

$$\int_{-\infty}^{+\infty} \mathcal{H}(\varepsilon) f(\varepsilon)\,\mathrm{d}\varepsilon = \int_{-\infty}^{\mu} \mathcal{H}(\varepsilon)\,\mathrm{d}\varepsilon + \sum_{n=1}^{\infty} (k_B T)^{2n} a_n \left.\frac{\mathrm{d}^{2n-1}}{\mathrm{d}\varepsilon^{2n-1}} \mathcal{H}(\varepsilon)\right|_{\varepsilon=\mu} \tag{3}$$