Extra pearls in graph theory

Anton Petrunin

I used these topics together with "Pearls in graph theory" by Nora Hartsfield and Gerhard Ringel [18] to teach an undergraduate course in graph theory at the Pennsylvania State University. I tried to keep clarity and simplicity on the same level.

Hope that someone will find it useful for something.

I want to thank Semyon Alesker, Rostislav Matveyev, Alexei Novikov, Dmitri Panov, and Lukeria Petrunina for help.

Anton Petrunin

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Chapter 1

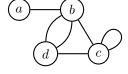
Introduction

Terminology

The diagram on the right may describe regular flights of an airline. It has six flights which serve four airports labeled by a, b, c, and d.

For this and similar type of data, mathematicians use the notion of *pseudograph*.

Formally, a pseudograph is a finite nonempty set of *vertexes* (in our example a vertex is an



airport) and a finite collection of edges; each edge connects two vertexes (in the example above, an edge is a regular flight). A pair of vertexes can be connected by a few edges, such edges are called parallel (in our example, it might mean that the airline makes few flights a day between these airports). Also, an edge can connect a vertex to itself, such an edge is called a loop (we might think of it as a sightseeing flight).

Thus, from a mathematical point of view, the diagram above describes an example of a pseudograph with vertexes a, b, c, d, and six edges, among them is one loop at c and a pair of parallel edges between b and d.

The number of edges coming from one vertex is called its *degree*, the loops are counted twice. In the example above, the degrees of a, b, c, and d are 1, 4, 4, and 3 correspondingly.

A vertex with zero degree is called *isolated* and a vertex of degree one is called an *end vertex*.

A pseudograph without loops is also called a *multigraph*. A multigraph without parallel edges is also called a *graph*. Most of the time we will work with graphs.

If x and y are vertexes of a pseudograph G, we say that x is adjacent to y if there is an edge between x and y. We say that a vertex x is incident with an edge e if x is an end vertex of e.

Wolf, goat, and cabbage

Usually we visualize the vertexes of a graph by points and its edges are represented by a line connecting two vertexes.

However, the vertexes and edges of the graph might have a very different nature. As an example, let us consider the following classic problem.

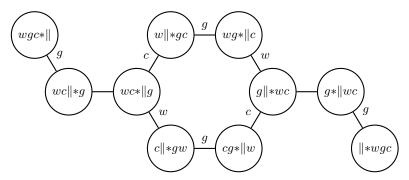
1.1. Problem. A farmer purchased a wolf, a goat, and a cabbage; he needs to cross a river with them. He has a boat, but he can carry only himself and a single one of his purchases: the wolf, the goat, or the cabbage.

If left unattended together, the wolf would eat the goat, and the goat would eat the cabbage.

The farmer has to carry himself and his purchases to the far bank of the river, leaving each purchase intact. How can he do it?

Solution. Let us denote the farmer by *, the river by $\|$ the wolf by w, the goat by g, and the cabbage by c. For example, $wc\|*g$ means that the wolf and cabbage are on the left bank of the river and the goat with the farmer are on the right bank.

The starting position is $wgc*\parallel$; that is, everyone is on the left bank. The following graph describes all possible positions which can be achieved; each edge is labeled by the transported purchase.



This graph shows that the farmer can achieve $\|*wgc$ by legal moves. It solves the problem, and also shows that there are exactly two different solutions, assuming that the farmer does not want to repeat the same position twice.

Often, a graph comes with an extra structure; for example, labeling of edges and/or vertexes as in the example above.

Here is a small variation of another classic problem.

1.2. Problem. Missionaries and cannibals must cross a river using a boat which can carry at most two people, under the constraint that, for both banks, if there are missionaries present on the bank, they cannot be outnumbered by cannibals; otherwise the missionaries will be eaten. The boat cannot cross the river by itself with no people on board.

Let us introduce a notation to describe the positions of the missionaries, cannibals, and the boat on the banks. The river will be denoted by $\|$; let * denotes the boat, we will write the number of cannibals on each side of $\|$, and the number of missionaries by subscript. For example, $4_2^*\|0_2$ means that on the left bank we have four cannibals, two missionaries, and the boat (these two missionaries will be eaten), and on the right bank there are no cannibals and two missionaries.

1.3. Exercise. Assume four missionaries and four cannibals need to cross the river; in other words, the beginning stage is $4_4^*||0_0$. Draw a graph for all possible positions which can be achieved.

Conclude that all of them can not cross the river.

Chapter 2

Probabilistic method

In this chapter we give a general discussion of Ramsey numbers, introduce the probabilistic method, and use it to give a lower bound on Ramsey number.

Ramsey numbers

Recall that the Ramsey number r(m, n) is a least positive integer such that every blue-red coloring of edges in the complete graph $K_{r(m,n)}$ contains a blue K_m or a red K_n .

Switching colors in the definition shows that r(m, n) = r(n, m) for any m and n. Therefore we may assume that $m \leq n$.

Note that r(1,n) = 1 for any positive integer n. Indeed, the one-vertex graph K_1 has no edges; therefore we can say that all its edges are blue (as well as red and deep green-cyan turquoise at the same time).

2.1. Exercise. Show that r(2, n) = n for any positive integer n.

The following table from [30] gives the values of r(m, n); it includes all currently known values for $n \ge m \ge 3$:

m	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8	9
3	1	3	6	9	14	18	23	28	36
4	1	4	9	18	25	?	?	?	?

In order to prove that r(4,4) = 18 we have to prove two inequalities $r(4,4) \ge 18$ and $r(4,4) \le 18$. The inequality $r(4,4) \ge 18$ means that

there is a blue-red coloring of edges of K_{17} that has no monochromatic K_4 . The inequality $r(4,4) \leq 18$ means that in any blue-red coloring of K_{18} there is a monochromatic K_4 .

Binomial coefficients

In this section we review properties of binomial coefficients that will be needed further.

Binomial coefficients will be denoted by $\binom{n}{m}$. They can be defined as unique numbers such that the identity

$$(a+b)^n = \binom{n}{0} \cdot a^0 \cdot b^n + \binom{n}{1} \cdot a^1 \cdot b^{n-1} + \dots + \binom{n}{n} \cdot a^n \cdot b^0$$

holds for any real numbers a, b and integer $n \ge 0$. This identity is called binomial expansion. It can be used to derive some identities on binomial coefficients; for example,

$$\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = (1+1)^n = 2^n.$$

The number $\binom{n}{m}$ plays an important role in combinatorics — it gives the number of ways that m objects can be chosen from n different objects; this value can be found explicitly:

$$\binom{n}{m} = \frac{n!}{m! \cdot (n-m)!}.$$

Note that all $\binom{n}{m}$ different ways to choose m objects from n different objects are falling into two categories: (1) those which include the last object — there are $\binom{n-1}{m-1}$ of them, and (2) those which do not include it — there are $\binom{n-1}{m}$ of them. It follows that

$$\binom{n}{m} = \binom{n-1}{m-1} + \binom{n-1}{m}.$$

This identity will be used in the proof of Theorem 2.2.

Upper bound

Recall that according to Theorem 4.3.2 in [18], the inequality

$$r(m,n) \leqslant r(m-1,n) + r(m,n-1)$$

holds for all integers $m, n \ge 2$.

In other words, any value r(m, n) in the table above can not exceed the sum of values in the cells directly above and on the left from it. The inequality \bullet might be strict; for example,

$$r(3,4) = 9 < 4 + 6 = r(2,4) + r(3,3).$$

2.2. Theorem. For any positive integers m, n we have that

$$r(m,n) \leqslant {m+n-2 \choose m-1}.$$

Proof. Set

$$s(m,n) = {m+n-2 \choose m-1} = \frac{(m+n-2)!}{(m-1)! \cdot (n-1)!},$$

so we need to prove the following inequality:

$$f(m,n) \leqslant s(m,n).$$

Note that from **3**, we get the identity

6
$$s(m,n) = s(m-1,n) + s(m,n-1)$$

which is similar to the inequality **4**.

Further note that s(1,n) = s(n,1) = 1 for any positive integer n. Indeed, $s(1,n) = \binom{n-1}{0}$, and there is only one choice of 0 objects from the given n-1. Similarly $s(n,1) = \binom{n-1}{n-1}$, and there is only one choice of n-1 objects from the given n-1.

The above observations make it possible to calculate the values of s(m,n) recursively. The following table provides some of its values.

m	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8	9
3	1	3	6	10	15	21	28	36	45
4	1	4	10	20	35	56	84	120	165

The inequality \bullet means that any value in this table can not exceed the corresponding value in the table for r(m,n) on page 7. The latter is nearly evident from \bullet and \bullet ; let us show it formally.

Since

$$r(1,n) = r(n,1) = s(1,n) = s(n,1) = 1,$$

the inequality \bullet holds if m=1 or n=1.

Assume the inequality \bullet does not hold for some m and n. Choose a minimal criminal pair (m, n); that is, a pair with minimal value m + n

such that \bullet does not hold. From above we have that $m, n \ge 2$. Since m+n is minimal, we have that

$$r(m-1,n) \leqslant s(m-1,n)$$
 and $r(m,n-1) \leqslant s(m,n-1)$

summing these two inequalities and applying \bullet together with \bullet we get \bullet — a contradiction.

2.3. Corollary. The inequality

$$r(n,n) \leqslant \frac{1}{4} \cdot 4^n$$

holds for any positive integer n.

Proof. By \mathbf{Q} , we have that $\binom{k}{m} \leq 2^k$. Applying Theorem 2.2, we get that

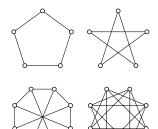
$$\begin{split} r(n,n) &\leqslant {2 \cdot n - 2 \choose n - 1} \leqslant \\ &\leqslant 2^{2 \cdot n - 2} = \\ &= \frac{1}{4} \cdot 4^n. \end{split}$$

Lower bound

In order to show that

$$r(m,n) \geqslant s+1,$$

it is sufficient to color the edges of K_s in red and blue so that it has no red K_m and no blue K_n . Equivalently, it is sufficient to decompose K_s into two subgraphs with no isomorphic copies of K_m in the first one and no isomorphic copies of K_n in the second one.

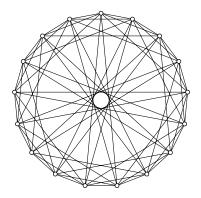


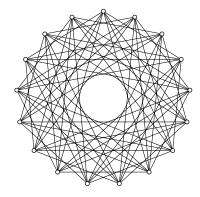
For example, the subgraphs in the decomposition of K_5 on the diagram has no monochromatic triangles; the latter implies that $r(3,3) \ge 6$. We already showed that for any decomposition of K_6 into two subgraphs, one of the subgraphs has a triangle; that is, r(3,3) = 6.

Similarly, to show that $r(3,4) \ge 9$, we need to construct a decomposition of K_8 into two subgraphs G and H such that G contains no triangle K_3 and H contains no K_4 . In

fact, in any decomposition of K_9 into two subgraphs, either the first subgraph contains a triangle or the second contains a K_4 . That is, r(3,4) = 9 [see 18, p. 82–83].

Further, to show that $r(4,4) \ge 18$, we need to construct a decomposition of K_{17} into two subgraphs with no K_4 . (In fact, r(4,4) = 18, but we are not going to prove it.) The corresponding decomposition is given on the diagram. The constructed decomposition is rationally symmetric; the first subgraph contains the chords of angle lengths 1, 2, 4, and 8 and the second contains all the chords of angle lengths 3, 5, 6, and 7.





2.4. Exercise. Show that

- (a) In the decomposition of K_8 above, the left graph contains no triangle, and the right graph contains no K_4 .
- (b) In the decomposition of K_{17} , neither graph contains any K_4 .

Hint: Assuming such a subgraph exists. Fix a vertex v. Note that we can assume that the subgraph contains v; otherwise rotate it. In each case, draw the subgraph induced by the vertexes connected to v. (If uncertain, see the definition of *induced subgraph*.)

For larger values m and n, the problem of finding the exact lower bound for r(m, n) quickly becomes too hard. Even getting a reasonable estimate is challenging. In the next section we will show how to obtain such an estimate by using probability.

Probabilistic method

The probabilistic method makes it possible to prove the existence of graphs with certain properties without constructing them explicitly. The idea is to show that if one randomly chooses a graph or its coloring from a specified class, then the probability that the result is of the needed property is more than zero. The latter implies that a graph with needed property exists.

Despite that this method of proof uses probability, the final conclusion is determined for certain, without any possible error.

2.5. Theorem. Assume that the inequality

$$\binom{R}{n} < 2^{\binom{n}{2} - 1}$$

holds for a pair of positive integers R and n. Then r(n,n) > R.

Proof. We need to show that the complete graph K_R admits a coloring of edges in red and blue such that it has no monochromatic subgraph isomorphic to K_n .

Let us color the edges randomly — color each edge independently with probability $\frac{1}{2}$ in red and otherwise in blue.

Fix a set S of n vertexes. Define the variable X(S) to be 1 if every edge between the vertexes in S has the same color, and otherwise set X(S) = 0. Note that the number of monochromatic n-subgraphs in K_R is the sum of X(S) over all possible n-vertex subsets S.

Note that the expected value of X(S) is simply the probability that all of the $\binom{n}{2} = \frac{n \cdot (n-1)}{2}$ edges in S are the same color. The probability that all the edges with the ends in S are blue is $2^{-\binom{n}{2}}$ and with the same probability all the edges are red. Since these two possibilities exclude each other, the expected value of X(S) is $2 \cdot 2^{-\binom{n}{2}}$.

This holds for any n-vertex subset S of the vertexes of K_R . The total number of such subsets is $\binom{R}{n}$. Therefore the expected value for the sum of X(S) over all n-vertex subsets S is

$$W = 2 \cdot {\binom{R}{n}} \cdot 2^{-\binom{n}{2}}.$$

In other words, W is the expected number of monochromatic K_n 's in a random coloring of K_R . For any coloring, this number has to be an integer. Therefore, if W < 1, then at least one edge-coloring of K_R has no monochromatic K_n . That is, if $\binom{R}{n} < 2^{\binom{n}{2}-1}$, then there is a coloring K_R without any monochromatic n-subgraphs. \square

The following corollary implies that the function $n \mapsto r(n, n)$ grows at least exponentially.

2.6. Corollary. $r(n,n) > \frac{1}{8} \cdot 2^{\frac{n}{2}}$.

Proof. Set $R = \lfloor \frac{1}{8} \cdot 2^{\frac{n}{2}} \rfloor$; that is, R is the largest integer $\leq \frac{1}{8} \cdot 2^{\frac{n}{2}}$. Note that

$$2^{\binom{n}{2}-1} > (2^{\frac{n-3}{2}})^n \geqslant R^n.$$

and

$$\binom{R}{n} = \frac{R \cdot (R-1) \cdots (R-n+1)}{n!} < R^n.$$

Therefore

$$\binom{R}{n} < 2^{\binom{n}{2} - 1}.$$

By Theorem 2.5, we get that r(n,n) > R.

2.7. Exercise. By random coloring we understand a coloring of edges of a given graph in red and blue such that each edge is colored independently in red or blue with equal chances.

Assume the edges of the complete graph K_{100} is colored randomly. Find the expected number of monochromatic Hamiltonian cycles in K_{100} . (You may use factorials in the answer.)

Remark. The answer in the exercise above is a huge number that is bigger than 10^{125} . One might think that this estimate alone is sufficient to conclude that *most* of the colorings have a monochromatic Hamiltonian cycles — let us show that it is not that easy. (It is still true that probability of the existence of a monochromatic coloring is close to 1, but the proof requires more work; it does not follow solely from the given estimate.)

The total number of colorings of K_{100} is $2^{\binom{100}{2}} > 10^{1400}$. Therefore in principle, it might happen that 99.99% of the colorings have no monochromatic Hamiltonian cycles and .01% of the colorings contain all the monochromatic Hamiltonian cycles. To keep the expected value above 10^{125} , this .01% of colorings should have less than 10^{130} of monochromatic cycles in average; the latter does not seem impossible since the total number of Hamiltonian cycles in K_{100} is $99!/2 > 10^{155}$.

Counting proof

In this section, we translate the proof of Theorem 2.5 into a combinatoric language, without the use of probability. We do this to affirm that the probabilistic method provides a real proof, without any possible error.

In principle, any probabilistic proof admits such a translation, but in most cases, the translation is less intuitive.

Proof of 2.5. The graph K_R has $\binom{R}{2}$ edges. Each edge can be colored in blue or red; therefore the total number of different colorings is

$$\Omega = 2^{\binom{R}{2}}.$$

Fix a subgraph isomorphic to K_n in K_R . Note that this graph is red in $\Omega/2^{\binom{n}{2}}$ different colorings and yet in $\Omega/2^{\binom{n}{2}}$ different colorings this subgraph is blue.

There are $\binom{R}{n}$ different subgraphs isomorphic to K_n in K_R . Therefore the total number of monochromatic K_n 's in all the colorings is

$$M = \binom{R}{n} \cdot \Omega \cdot 2/2^{\binom{n}{2}}.$$

If $M < \Omega$, then by the pigeonhole principle, there is a coloring with no monochromatic K_n . Hence the result.

Graph of n-cube

In this section we give another classic application of the probabilistic method. It requires a bit more probability theory.



Let us denote by Q_n the graph of the n-dimensional cube; Q_n has 2^n vertexes, each vertex is labeled by a sequence of length n made up of zeros and ones; two vertexes are adjacent if their labels differ only in one digit.

Graph Q_4 is shown on the diagram. Note that each vertex of Q_n has degree n.

Recall that distance between two vertexes in a graph G is the length of a shortest path connecting the vertexes. The maximal distance between vertexes in G is called diameter of G.

- **2.8.** Exercise. Show that the diameter of Q_n is n.
- **2.9. Problem.** Suppose $\ell(n)$ denotes the maximal number of vertexes in Q_n on a distance more than n/3 from each other. Then $\ell(n)$ grows exponentially in n; moreover, $\ell(n) \ge 1.05^n$.

To solve the problem one has to construct a set with at least 1.05^n vertexes in Q_n that lie far from each other. However, it is hard to construct such a set explicitly. Instead, we will show that if one chooses that many vertexes randomly, then they lie far from each other with a positive probability. To choose a random vertex in Q_n , one can toss a coin n times, each time writing 1 for a head and 0 for a tail and then take the vertex labeled by the obtained sequence.

The following exercise guides you to a solution of the problem. The same argument shows that for any coefficient $k < \frac{1}{2}$, the maximal number of vertexes in Q_n on the distance larger than $k \cdot n$ from each other grows exponentially in n. According to Exercise 2.13, the case $k = \frac{1}{2}$ is very different.

- **2.10. Exercise.** Let P_n denote the probability that randomly chosen vertexes in Q_n lie with the distance $\leq \frac{n}{3}$ between them.
 - (a) Use Claim 2.11 to show that

$$P_n < .95^n$$
.

- (b) Assume k vertexes v_1, \ldots, v_k in Q_n are fixed and v is a random vertex. Show that $k \cdot P_n$ is the average number of vertexes v_i that lies on a distance $\leq \frac{n}{3}$ from v. Conclude that v lies on a distance larger than $\frac{n}{3}$ from each of v_i with probability at least $1 k \cdot P_n$.
- (c) Apply (a) and (b) to show that there are at least 1.05^n vertexes in Q_n on a distance larger than $\frac{n}{3}$ from each other.
- **2.11. Claim.** The probability P_n to obtain less than one third heads after n fair tosses of a coin decays exponentially in n; in fact $P_n < .95^n$ for any n.

In the proof, we will use the following observation.

2.12. Markov's inequality. Suppose Y is a nonnegative random variable and c > 0. Denote by P the probability of the event $Y \ge c$ and by y the expected value of Y. Then

$$P \cdot c \leqslant y.$$

Proof. Consider another random variable \bar{Y} such that $\bar{Y}=c$ if $Y\geqslant c$ and $\bar{Y}=0$ otherwise; denote by \bar{y} its expected value.

Note that $\bar{Y} \leq Y$ and therefore

$$\bar{y} \leqslant y$$
.

The random variable \bar{Y} takes the value c with probability P and 0 with probability 1 - P. Therefore $\bar{y} = P \cdot c$; whence \bullet follows.

Proof of 2.11. Let us introduce independent n random variables $X_1, \ldots X_n$; each X_i returns the number of heads after i-th toss of the coin; in particular, each X_i takes values 0 or 1 with the probability of $\frac{1}{2}$ each. We need to show that the probability P_n of the event $X_1 + \cdots + X_n \leq \frac{n}{3}$ is less than .95ⁿ.

Consider the random variable

$$Y = 2^{-X_1 - \dots - X_n};$$

denote by y its expected value.

Note that P_n is the probability of the event that $Y \ge 2^{-\frac{n}{3}}$. Further note that Y > 0 always. By Markov's inequality, we get that

$$P_n \cdot 2^{-\frac{n}{3}} \leqslant y$$
.

The random variable 2^{-X_i} takes the two values 1 and $\frac{1}{2} = 2^{-1}$ with the probability of $\frac{1}{2}$ each; the expected value of 2^{-X_i} has to be $\frac{1}{2} \cdot (1 + \frac{1}{2}) = \frac{3}{4}$. Note that

$$Y = 2^{-X_1} \cdots 2^{-X_n}.$$

Since the random variables X_i are independent, we have that

$$y = \left(\frac{3}{4}\right)^n$$
.

It follows that

$$P_n \leqslant \left(\frac{3}{4} \cdot 2^{\frac{1}{3}}\right)^n < .95^n.$$

2.13. Advanced exercise.

- (a) Show that Q_n contains at most $2 \cdot n$ vertexes on a distance at least $\frac{n}{2}$ from each other.
- (b) Show that Q_n contains at most n+1 vertexes on a distance larger than $\frac{n}{2}$ from each other.

Remarks

Existence of Ramsey number r(m, n) for any m and n, is the first result in the so called Ramsey theory. A typical theorem in this theory states that any large object of a certain type contains a very ordered piece of a given size. We recommend a book of Matthew Katz and Jan Reimann [24] on the subject.

Corollaries 2.3 and 2.6 imply that

$$\frac{1}{9} \cdot 2^{\frac{1}{2} \cdot n} \leqslant r(n, n) \leqslant \frac{1}{4} \cdot 2^{2 \cdot n}.$$

It is unknown if these inequalities can be essentially improved. More precisely, it is unknown whether there are constants c>0 and $\alpha>\frac{1}{2}$ such that the inequality

$$r(n,n) \geqslant c \cdot 2^{\alpha \cdot n}$$

¹This question might look insignificant from the first sight, but it is considered as one of the major problems in combinatorics [14].

holds for any n. Similarly, it is unknown whether there are constants c and $\alpha < 2$ such that the inequality

$$r(n,n) \leqslant c \cdot 2^{\alpha \cdot n}$$

holds for any n.

The probabilistic method was introduced by Paul Erdős. It finds applications in many areas of mathematics, not only in graph theory.

Note that the probabilistic method is nonconstructive — often when the existence of a certain graph is probed by the probabilistic method, it is still uncontrollably hard to describe a concrete example.

More involved examples of proofs based on the probabilistic method deal with *typical properties* of random graphs.

To describe the concept, let us consider the following random process which generates a graph G_n with n vertexes.

Fix a positive integer n. Consider a graph G_n with the vertexes labeled by $1, \ldots, n$, where existence of edge between every pair of vertexes is decided independently by flipping a coin.

Note that the described process depends only on n, and as a result we can get a graph isomorphic to any given graph with n vertexes.

2.14. Exercise. Let H be a graph with n vertexes. Denote by α the probability that G_n is isomorphic to H. Show that

$$1/2^{\binom{n}{2}} \leqslant \alpha \leqslant n!/2^{\binom{n}{2}}.$$

Fix a property of a graph (for example, connectedness) and denote by α_n the probability that G_n has this property. We say that the property is typical if $\alpha_n \to 1$ as $n \to \infty$.

2.15. Exercise. Show that random graphs typically have a diameter of 2. That is, the probability that G_n is has a diameter of 2 converges to 1 as $n \to \infty$.

Hint: Find the probability that two given vertexes lie on a distance > 2 from each other in G_n ; find the average number of such pairs in G_n ; make a conclusion.

Note that from the exercise above, it follows that in the described random process, the random graphs are *typically connected*.

2.16. Exercise. Show that random graphs typically has a subgraph isomorphic to K_{100} . That is, the probability that G_n has a subgraph isomorphic to K_{100} converges to 1 as $n \to \infty$.

The following theorem gives a deeper illustration of the probabilistic method with the use of typical properties, a proof can be found in [1, Chapter 44].

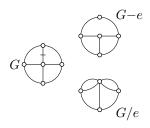
2.17. Theorem. Given positive integers g and k, there is a graph G with girth of at least g and a chromatic number of at least k.

Chapter 3

Deletion and contraction

Definitions

Let G be a pseudograph with a marked edge e. Denote by G-e the pseudograph obtained from G by deleting e, and by G/e the pseudograph obtained from G by contracting the edge e to a point; see the diagram.



Assume G is a graph; that is, G has no loops and no parallel edges. In this case, G-e is also a graph. However, G/e might have parallel edges, but no loops; that is, G/e is a multigraph.

If G is a multigraph, then so is G - e. If the edge e is parallel to f in G, then f in G/e becomes a loop; that is, G/e is a pseudograph in general.

Number of spanning trees

Recall that s(G) denotes the number of spanning trees in the pseudo-graph G.

An edge e in a connected graph G is called the bridge, if deletion of e makes the graph disconnected; in this case, the remaining graph has two connected components which are called banks.

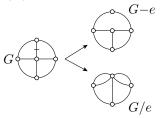
3.1. Exercise. Assume that the graph G contains a bridge between banks H_1 and H_2 . Show that

$$s(G) = s(H_1) \cdot s(H_2).$$

3.2. Deletion-plus-contraction formula. Let e be an edge in the pseudograph G. Assume e is not a loop, then the following identity holds

$$s(G) = s(G - e) + s(G/e).$$

It is convenient to write the identity \bullet using a diagram as on the picture — the arrows point from one multigraph to multigraphs with the same total number of spanning trees; the edge e is marked in G.



Proof. Note that the spanning trees of G can be subdivided into two groups - (1) those which contain the edge e and (2) those which do not. For the trees in the first group, the contraction of e to a point gives a spanning tree in G/e, while the trees in the second group are also spanning trees in G-e.

Moreover, both of the described correspondences are one-to-one. Hence the formula follows. $\hfill\Box$

Note that a spanning tree can not have loops. Therefore if we remove all loops from the pseudograph, the number of spanning trees remains unchanged. Let us state it precisely.

3.3. Claim. If e is a loop in a pseudograph G, then

$$s(G) = s(G - e).$$

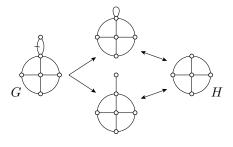
The proof of the following claim uses the deletion-plus-contraction formula.

3.4. Claim. If one removes an end vertex w from a pseudograph G, then in the obtained graph G - w the number of spanning trees remains unchanged; that is,

$$s(G) = s(G - w).$$

Proof. Denote by e the only edge incident to w. Note that the graph G-e is not connected, since the vertex w is isolated. In particular, s(G-e)=0. On the other hand, G/e=G-w. Therefore the deletion-plus-contraction formula \bullet implies \bullet .

On the diagrams, we may use two-sided arrow "\(\to\)" for the graphs with equal number of the spanning trees. For example, using deletion-plus-contraction formula together with the claims, we can draw the diagram, which in particular implies the following identity:



$$s(G) = 2 \cdot s(H)$$
.

Note that the deletion-plus-contraction formula gives an algorithm to calculate the value s(G) for a given pseudograph G. Indeed, for any edge e, both graphs G - e and G/e have smaller number of edges. That is, the deletion-plus-contraction formula reduces the problem of finding the number of the trees to simpler graphs; applying this formula a few times we can reduce the question to a collection of graphs with an evident answer for each. In the next section we will show how it works.

Fans and their relatives

Recall that Fibonacci numbers f_n are defined using the recursive identity $f_{n+1} = f_n + f_{n-1}$ with $f_1 = f_2 = 1$. The sequence of Fibonacci numbers starts as

$$1, 1, 2, 3, 5, 8, 13, \ldots$$

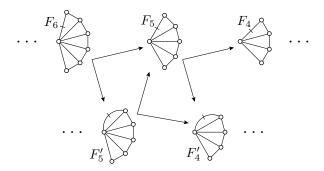
The graphs of the following type are called fans; a fan with n+1

$$F_1$$
 F_2 F_3 F_4 F_5 F_6 \cdots

vertex will be denoted by F_n .

3.5. Theorem.
$$s(F_n) = f_{2 \cdot n}$$
.

Proof. Applying the deletion-plus-contraction formula, we can draw the following infinite diagram. (We ignore loops and end vertexes since they do not change the number of spanning trees.) In addition to the fans F_n we use its variations F'_n , which differ from F_n by an extra parallel edge.



Set $a_n = s(F_n)$ and $a'_n = s(F'_n)$. From the diagram, we get the following two recursive relations:

$$a_{n+1} = a'_n + a_n,$$

 $a'_n = a_n + a'_{n-1}.$

That is, in the sequence

$$a_1, a'_1, a_2, a'_2, a_3 \dots$$

every number starting from a_2 is sum of previous two.

Further note that F_1 has two vertexes connected by a unique edge, and F'_1 has two vertexes connected by a pair of parallel edges. Hence $a_1 = 1 = f_2$ and $a'_1 = 2 = f_3$ and therefore

$$a_n = f_{2 \cdot n}$$

for any n.

Comments. We can deduce a recursive relation for a_n , without using a'_n :

$$a_{n+1} = a'_n + a_n =$$

= $2 \cdot a_n + a'_{n-1} =$
= $3 \cdot a_n - a_{n-1}$.

This is a special case of the so called *constant-recursive sequences*. The general term of constant-recursive sequences can be expressed by a closed formula — read [20] if you wonder how. In our case it is

$$a_n = \frac{1}{\sqrt{5}} \cdot \left(\left(\frac{3 + \sqrt{5}}{2} \right)^n - \left(\frac{3 - \sqrt{5}}{2} \right)^n \right).$$

Since a_n is an integer and $0 < \frac{1}{\sqrt{5}} \cdot (\frac{3-\sqrt{5}}{2})^n < 1$ for any $n \ge 1$, a shorter formula can be written

$$a_n = \left| \frac{1}{\sqrt{5}} \cdot \left(\frac{3 + \sqrt{5}}{2} \right)^n \right|,$$

where $\lfloor x \rfloor$ denotes floor of x; that is, $\lfloor x \rfloor$ is the maximal integer that does not exceed x.

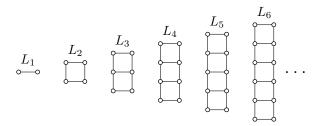
3.6. Exercise. Consider the sequence of zig-zag graphs Z_n of the following type:

$$Z_1$$
 Z_2 Z_3 Z_4 Z_5 Z_6 \ldots

Show that $s(Z_n) = f_{2 \cdot n}$ for any n.

Hint: Use the induction on n and/or mimic the proof of Theorem 3.5.

3.7. Exercise. Let us denote by b_n the number of spanning trees in the n-step ladder L_n ; that is, in the graph of the following type:

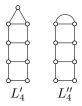


Apply the method we used for the fans F_n to show that the sequence b_n satisfies the following linear recursive relation:

$$b_{n+1} = 4 \cdot b_n - b_{n-1}$$
.

Hint: To construct the recursive relation, in addition to the ladders L_n , you will need two of its analogs $-L'_n$ and L''_n , shown on the diagram.

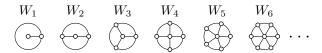
Note that $b_1 = 1$ and $b_2 = 4$; applying the exercise, we could calculate the first numbers of the sequence (b_n) :



$$1, 4, 15, 56, 209, 780, 2911, \dots$$

The following exercise is analogous, but more complicated.

3.8. Advanced exercise. Recall that a wheel W_n is the graph of following type:



Show that the sequence $c_n = s(W_n)$ satisfies the following recursive relation:

$$c_{n+1} = 4 \cdot c_n - 4 \cdot c_{n-1} + c_{n-2}.$$

Using the exercise above and applying induction, one can show that

$$c_n = f_{2 \cdot n + 1} + f_{2 \cdot n - 1} - 2 = l_{2 \cdot n} - 2$$

for any n. The numbers $l_n = f_{n+1} + f_{n-1}$ are called *Lucas numbers*; they pop up in combinatorics as often as Fibonacci numbers.

Remarks

The deletion-plus-contraction formula together with Kirchhoff's rules were used in the solution of the so called squaring the square problem. The history of this problem and its solution are discussed in a book of Martin Gardiner [12, Chapter 17].

The proof of recurrent relation above is given by Mohammad Hassan Shirdareh Haghighi and Khodakhast Bibak [34]; this problem is also discussed in a book of Ronald Graham, Donald Knuth, and Oren Patashnik [15].

Chapter 4

Matrix theorem

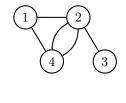
Adjacency matrix

Let us describe a way to encode a given multigraph G with p vertexes by an $p \times p$ matrix. First, enumerate the vertexes of the multigraph by numbers from 1 to p; such a multigraph will be called *labeled*. Consider the matrix $A = A_G$ with the component $a_{i,j}$ equal to the number of edges from the i-th vertex to the j-th vertex of G.

This matrix A is called the *adjacency matrix* of G. Note that A is *symmetric*; that is, $a_{i,j} = a_{j,i}$ for any pair i, j. Also, the diagonal components of A vanish; that is, $a_{i,i} = 0$ for any i.

For example, for the labeled multigraph G shown on the diagram, we get the following adjacency matrix:

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{pmatrix}.$$



4.1. Exercise. Let A be the adjacency matrix of a labeled multigraph. Show that the components $b_{i,j}$ of the n-th power A^n is the number of walks of length n in the graph from vertex i to vertex j.

Hint: Use induction on n.

Kirchhoff minor

In this section we construct a special matrix, called *Kirchhoff minor*, associated with a pseudograph, and we discuss its basic properties. This

matrix will be used in the next section in a formula for the number of spanning trees in a pseudograph G. Since the loops do not change the number of spanning trees, we can remove all of them. In other words, we can (and will) always assume that G is a multigraph.

Fix a multigraph G and consider its adjacency matrix $A = A_G$; it is a $p \times p$ symmetric matrix with zeros on the diagonal.

- 1. Revert the signs of the components of A and exchange the zeros on the diagonal by the degrees of the corresponding vertexes. (The matrix A' is called the *Kirchhoff matrix*, *Laplacian matrix* or *admittance matrix* of the graph G.)
- 2. Delete from A' the last column and the last row; the obtained matrix $M = M_G$ will be called the *Kirchhoff minor* of the labeled pseudograph G.

For example, the labeled multigraph G on the diagram has the following Kirchhoff matrix and Kirchhoff minor:

$$\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$
.

$$A' = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 4 & -1 & -2 \\ 0 & -1 & 1 & 0 \\ -1 & -2 & 0 & 3 \end{pmatrix}, \quad M = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

4.2. Exercise. Show that in any Kirchhoff matrix A' the sum of the components in each row or column vanishes. Conclude that

$$\det A' = 0.$$

4.3. Exercise. Draw a labeled pseudograph with following Kirchhoff minor:

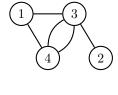
$$\begin{pmatrix} 4 & -1 & -1 & -1 & 0 \\ -1 & 4 & -1 & 0 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & 0 & -1 & 4 & -1 \\ 0 & -1 & -1 & -1 & 4 \end{pmatrix}.$$

4.4. Exercise. Show that the sum of all components in every column of Kirchhoff minor is nonnegative.

Moreover, the sum of all components in the i-th column vanishes if and only if the i-th vertex is not adjacent to the last vertex.

Relabeling. Let us understand what happens with Kirchhoff minor and its determinant as we swap two labels distinct from the last one.

For example, if we swap the labels 2 and 3 in the graph above, we get another labeling shown on the diagram. Then the corresponding Kirchhoff minor will be



$$M' = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 4 \end{pmatrix},$$

which is obtained from M by swapping columns 2 and 3 following by swapping rows 2 and 3.

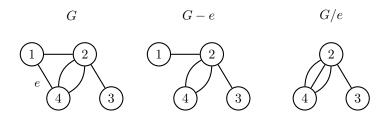
Note that swapping a pair of columns or rows changes the sign of the determinant. Therefore, swapping one pair of rows and one pair of columns does not change the determinant. Summarizing we get the following:

4.5. Observation. Assume G is a labeled graph with p vertexes and M_G is its Kirchhoff minor. If we swap two labels i, j < p, then the corresponding Kirchhoff minor M'_G can be obtained from M_G by swapping columns i and j, and then swapping rows i and j. In particular,

$$\det M'_G = \det M_G$$
.

Deletion and contraction. Let us understand what happens with Kirchhoff minor if we delete or contract an edge in the labeled multigraph. (If after the contraction of an edge we get loops, we remove it; this way we obtain a multigraph.)

Assume an edge e connects the first and the last vertex of a labeled multigraph G as in the following example:



Note that deleting e only reduces the corner component of M_G by one, while contracting it removes the first row and column. That is, since

$$M_G = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 1 \end{pmatrix},$$

we have

$$M_{G-e} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$
 and $M_{G/e} = \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}$.

Summarizing the above discussion we get the following:

- **4.6. Observation.** Assume e is an edge of a labeled multigraph G between the first and last vertex and M_G is the Kirchhoff minor of G. Then
 - (a) the Kirchhoff minor M_{G-e} of G-e can be obtained from M_G by subtracting 1 from the corner element with index (1,1);
 - (b) the Kirchhoff minor $M_{G/e}$ of G/e can be obtained from M_G by removing the first row and the first column in M_G .

In particular, applying the cofactor expansion of a determinant, we get that

$$\det M_G = \det M_{G-e} + \det M_{G/e}.$$

Note that the last formula resembles the deletion-plus-contraction formula. This observation will be a key to the proof of the matrix theorem; see the next section.

Matrix theorem

4.7. Matrix theorem. Let M be the Kirchhoff minor of a labeled multigraph G with at least two vertexes. Then

$$s(G) = \det M,$$

where s(G) denotes the number of spanning trees in G.

Proof. Denote by d the degree of the last vertex in G.

Assume d=0. Then G is not connected and therefore s(G)=0. On the other hand, the sum in each row of M_G vanish (compare to Exercise 4.4). Hence the sum of all columns in M_G vanish; in particular, the columns in M_G are linearly dependent and hence $\det M_G = 0$. Hence the equality \bullet holds if d=0.

As usual, we denote by p and q the number of vertexes and edges in G; by the assumption we have that $p \ge 2$.

Assume p = 2; that is, G has two vertexes and q parallel edges connecting them. Clearly, s(G) = q. Further note that $M_G = (q)$; that is, the Kirchhoff minor M_G is a 1×1 matrix with single component q. In particular, det $M_G = q$ and therefore the equality



In particular, det $M_G = q$ and therefore the equality \bullet holds.

Assume the equality $\mathbf{0}$ does not hold in general; choose a minimal criminal graph G; that is a graph that minimize the value p + q among the graphs violating $\mathbf{0}$.

From above we have that p > 2 and d > 0. Note that we may assume that the first and last vertexes of G are adjacent; otherwise permute pair of labels 1 and some j < p and apply Observation 4.5. Denote by e the edge between the first and last vertex.

Note that the total number of vertexes and edges in the pseudographs G - e and G/e are smaller than p + q. Therefore we have that

$$s(G-e) = \det M_{G-e}, \qquad s(G/e) = \det M_{G/e}.$$

Applying these two identities together with the deletion-plus-contraction formula and Observation 4.6, we get that

$$s(G) = s(G - e) + s(G/e) =$$

$$= \det M_{G-e} + \det M_{G/e} =$$

$$= \det M_G;$$

that is, the identity $\mathbf{0}$ holds for G — a contradiction.

- **4.8.** Exercise. Fix a labeling for each of the following graphs, find its Kirchhoff minor and use the matrix theorem to find the number of spanning trees.
 - (a) $s(K_{3,3})$;
 - $(b) \ s(W_6);$
 - (c) $s(Q_3)$.

 $(Use\ https://matrix.reshish.com/determinant.php,\ or\ any\ other\ matrix\ calculator.)$

Calculation of determinants

In this section we recall key properties of the determinant which will be used in the next section.

Let M be an $n \times n$ -matrix; that is, a table $n \times n$, filled with numbers which are called *components of the matrix*. The determinant $\det M$ is a polynomial of the n^2 components of M, which satisfies the following conditions:

1. The unit matrix has determinant 1; that is,

$$\det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} = 1.$$

2. If we multiply each component of one of the rows of the matrix M multiply by a number λ , then for the obtained matrix M', we have

$$\det M' = \lambda \cdot \det M.$$

3. If one of the rows in the matrix M add (or subtract) term-byterm to another row, then the obtained matrix M' has the same determinant

$$\det M' = \det M$$
.

These three conditions define determinant in a unique way. We will not give a proof of the statement; it is not evident and not complicated (sooner or later you will have to learn it, if it is not done already).

- **4.9.** Exercise. Show that the following property follows from the properties above.
 - 4. Interchanging any pair of rows of a matrix multiplies its determinant by -1; that is, if a matrix M' is obtained from a matrix M by permuting two of its rows, then

$$\det M' = -\det M$$
.

The determinant of $n \times n$ -matrix can be written explicitly as a sum of n! terms. For example,

$$a_1 \cdot b_2 \cdot c_3 + a_2 \cdot b_3 \cdot c_1 + a_3 \cdot b_1 \cdot c_2 - a_3 \cdot b_2 \cdot c_1 - a_2 \cdot b_1 \cdot c_3 - a_1 \cdot b_3 \cdot c_2$$

is the determinant of the matrix

$$M = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}.$$

However, the properties described above give a more convenient and faster way to calculate the determinant, especially for larger values n.

Let us show it on one example which will be needed in the next section.

$$\det\begin{pmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{pmatrix} = \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{pmatrix} =$$

$$= \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix} =$$

$$= 5^{3} \cdot \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix} =$$

$$= 5^{3} \cdot \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$= 5^{3}.$$

Let us describe what we used on each line above:

- 1. property 3 three times we add to the first row each of the remaining rows;
- 2. property 3 three times we add the first row to the each of the remaining three rows;
- 3. property 2 three times;
- 4. property 3 three times we subtract from the first row the remaining three rows;
- 5. property 1.

Cayley formula

Recall that a *complete graph* is a graph where each pair of vertexes is connected by an edge; a complete graph with p vertexes is denoted by K_p .

Note that every vertex of K_p has degree p-1. Therefore the Kirchhoff minor $M=M_{K_p}$ in the matrix formula \bullet for K_p is the following $(p-1)\times(p-1)$ -matrix:

$$M = \begin{pmatrix} p-1 & -1 & \cdots & -1 \\ -1 & p-1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & p-1 \end{pmatrix}.$$

The argument given in the end of the previous section admits a direct generalization:

$$\det \begin{pmatrix} p-1 & -1 & \cdots & -1 \\ -1 & p-1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & p-1 \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ -1 & p-1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & p-1 \end{pmatrix} = \\ = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & p & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & p \end{pmatrix} = \\ = p^{p-2}.$$

That is,

$$\det M = p^{p-2}.$$

Therefore, applying the matrix theorem, we get the following:

4.10. Cayley formula.

$$s(K_n) = p^{p-2};$$

that is, the number of spanning trees in the complete graph K_p is p^{p-2} .

- **4.11.** Exercise. Show $s(K_p e) = (p-2) \cdot p^{n-3}$ for any edge e in K_p .
- **4.12.** Exercise. Use the matrix theorem to show that

$$s(K_{m,n}) = m^{n-1} \cdot n^{m-1}.$$

Remarks

There is strong connection between counting spanning trees of a given graph, calculations of currents in an electric chain and random walks; a good survey is given in the book of Peter Doyle and Laurie Snell [6]. Let us give some examples.

Assume that the graph G describes an electric chain; each edge has resistance one Ohm and a battery is connected to the vertexes a and b. Assume that the total current between these vertexes is one Ampere. The following procedure calculates the current I_e along a given edge e.

Fix an orientation of e. Note that any spanning tree T of G has exactly one the following three properties: (1) the edge e appears on the (necessary unique) path from a to b in T with a positive orientation, (2) the edge e appears on the path from a to b in T with a negative orientation, (3) the edge e does not appear on the path from a to b in T. Denote by s_+ , s_- , and s_0 the number of the trees in each group. Clearly,

$$s(G) = s_+ + s_- + s_0.$$

Then the current I_e can be calculated using the following formula:

$$I_e = \frac{s_+ - s_-}{s(G)} \cdot I.$$

This statement can be proved by checking Kirchhoff's rules for the currents calculated by this formula.

There are many other applications of Kirchhoff's rules to graph theory. For example, in [27], they were used to prove the Euler's formula

$$p-q+r=2$$
,

where p, q, and r denote the number of vertexes, edges and regions of in a plane drawing of graph.

Chapter 5

Graph-polynomials

Counting problems often lead to an organized collection of numbers. Sometimes it is convenient to consider a polynomial with these numbers as coefficients. If it is done in a smart way, then the algebraic structure of the obtained polynomial reflects the original combinatorial structure of the graph.

Chromatic polynomial

Denote by $P_G(x)$ the number of different colorings of the graph G in x colors such that the ends of each edge get different colors.

5.1. Exercise. Assume that a graph G has exactly two connected components H_1 and H_2 . Show that

$$P_G(x) = P_{H_1}(x) \cdot P_{H_2}(x)$$

for any x.

5.2. Exercise. Show that for any integer $n \ge 3$,

$$P_{W_n}(x+1) = (x+1) \cdot P_{C_n}(x),$$

where W_n denotes a wheel with n spokes and C_n is a cycle of length n.

5.3. Deletion-minus-contraction formula. For any edge e in the pseudograph G we have

$$P_G(x) = P_{G-e}(x) - P_{G/e}(x).$$

Proof. The valid colorings of G-e can be divided into two groups: (1) those where the ends of the edge e get different colors — these remain to be valid colorings of G and (2) those where the ends of e get the same color — each of such colorings corresponds to a unique coloring of G/e. Hence

$$P_{G-e}(x) = P_G(x) + P_{G/e}(x),$$

which is equivalent to the deletion-minus-contraction formula $\mathbf{0}$.

Note that if the pseudograph G has loops, then $P_G(x) = 0$ for any x. Indeed, in a valid coloring the ends of a loop should get different colors, which is impossible.

The latter can also be proved using the deletion-minus-contraction formula. Indeed, if e is a loop in G, then G/e = G - e. Therefore $P_{G-e}(x) = P_{G/e}(x)$ and

$$P_G(x) = P_{G-e}(x) - P_{G/e}(x) = 0.$$

Similarly, removing a parallel edge from a pseudograph G does not change the value $P_G(x)$ for any x. Indeed, if e is an edge of G which has a parallel edge f, then in G/e the edge f becomes a loop. Therefore $P_{G/e}(x) = 0$ for any x and by the deletion-minus-contraction formula we get that

$$P_G(x) = P_{G-e}(x).$$

The same identity can be seen directly — any admissible coloring of G - e is also admissible in G — since the ends of f get different colors, so does e.

Summarizing the discussion above: the problem of finding $P_G(x)$ for a pseudograph G can be reduced to the case when G is a graph; that is, G has no loops and no parallel edges. Indeed, if G has a loop, then $P_G(x) = 0$ for all x. Further, removing one of the parallel edges from G does not change $P_G(x)$.

Recall that polynomial P of x is an expression of the following type

$$P(x) = a_0 + a_1 \cdot x + \dots + a_n \cdot x^n,$$

with constants a_0, \ldots, a_n , which are called *coefficients* of the polynomial. The coefficient a_0 is called *free term* of the polynomial. If $a_n \neq 0$, it is called the *leading coefficient* of P; in this case n is the degree of P. If the leading coefficient is 1, then the polynomial is called *monic*.

5.4. Theorem. Let G be a pseudograph with p vertexes. Then $P_G(x)$ is a polynomial with integer coefficients and vanishing free term.

Moreover, if G has a loop, then $P_G(x) \equiv 0$; otherwise $P_G(x)$ is monic and has degree p.

Based on this result we can call $P_G(x)$ the *chromatic polynomial* of the graph G. The deletion-minus-contraction formula will play the central role in the proof.

Proof. As usual, denote by p and q the number of vertexes and edges in G. To prove the first part, we will use the induction on q.

As the base case, consider the null graph N_p ; that is, the graph with p vertexes and no edges. Since N_p has no edges, any coloring of N_p is admissible. We have x choices for each of n vertexes therefore

$$P_{N_p}(x) = x^p.$$

In particular, the function $x \mapsto P_{N_p}(x)$ is given by a monic polynomial of degree p with integer coefficients and vanishing free term.

Assume that the first statement holds for all pseudographs with at most q-1 edges. Fix a pseudograph G with q edges. Applying the deletion-minus-contraction formula for some edge e in G, we get that

$$P_G(x) = P_{G-e}(x) - P_{G/e}(x).$$

Note that the pseudographs G-e and G/e have q-1 edges. By the induction hypothesis, $P_{G-e}(x)$ and $P_{G/e}(x)$ are polynomials with integer coefficients and vanishing free terms. Hence ② implies the same for $P_G(x)$.

If G has a loop, then $P_G(x) = 0$, as G has no valid colorings. It remains to show that if G has no loops, then $P_G(x)$ is a monic polynomial of degree p.

Assume that the statement holds for any multigraph G with at most q-1 edges and at most p vertexes.

Fix a multigraph G with p vertexes and q edges. Note that G - e is a multigraph with p vertexes and q - 1 edges. By the assumption, its chromatic polynomial P_{G-e} is monic of degree p.

Further the pseudograph G/e has p-1 vertexes, and its chromatic polynomial $P_{G/e}$ either vanishes or it has degree p-1. In both cases the difference $P_{G-e} - P_{G/e}$ is a monic polynomial of degree p. It remains to apply ②.

5.5. Advanced exercise. Let G be a graph with p vertexes, q edges, and n connected components. Show that

$$P_G(x) = x^p - a_{p-1} \cdot x^{p-1} + a_{p-2} \cdot x^{p-2} + \dots + (-1)^{p-n} a_n \cdot x^n,$$

where a_n, \ldots, a_{p-1} are positive integers and $a_{p-1} = q$.

Hint: Apply induction on q and use the deletion-minus-contraction formula the same way as in the proof of the theorem.

- **5.6.** Exercise. Use induction and the deletion-minus-contraction formula to show that
 - (a) $P_T(x) = x \cdot (x-1)^q$ for any tree T with q edges;
 - (b) $P_{C_p}(x) = (x-1)^p + (-1)^p \cdot (x-1)$ for the cycle C_p of length p.
 - (c) if G is a graph with p vertexes, then $P_G(x) \ge 0$ for any $x \ge p-1$.
 - (d) $P_{F_n}(x) = x \cdot (x-1) \cdot (x-2)^{n-1}$, where F_n denotes the n-spine fan, defined on page 21.
 - (e) $P_{L_n}(x) = x \cdot (x-1) \cdot (x^2 3 \cdot x + 3)^{n-1}$, where L_n denotes the n-step ladder, defined on page 23.
- **5.7.** Exercise. Show that graph G is a tree if and only if

$$P_G(x) = x \cdot (x-1)^{p-1}$$

for some positive integer p.

5.8. Exercise. Show that

$$P_{K_p}(x) = x \cdot (x-1) \cdot \cdot \cdot (x-p+1).$$

Remark. Note that for any graph G with p vertexes we have

$$P_{K_p}(x) \leqslant P_G(x) \leqslant P_{N_p}(x)$$

for any x. Since both polynomials

$$P_{K_p}(x) = x \cdot (x-1) \cdots (x-p+1),$$
 and $P_{N_p}(x) = x^p,$

are monic of degree p, it follows that so is P_G .

Hence Exercise 5.8 leads to an alternative way to prove the second statement in Theorem 5.4.

5.9. Exercise. Construct a pair of nonisomorphic graphs with equal chromatic polynomials.

Matching polynomial

Recall that a *matching* in a graph is a set of edges without common vertexes.

Given an integer $n \ge 0$, denote by $m_n = m_n(G)$ the number of matchings with n edges in the graph G.

Note that for a graph G with p vertexes and q edges, we have $m_0(G) = 1$, $m_1(G) = q$, and if $2 \cdot n > p$, then $m_n(G) = 0$. The maximal

integer k such that $m_k(G) \neq 0$ is called the matching number of G. The expression

$$M_G(x) = m_0 + m_1 \cdot x + \dots + m_k \cdot x^k$$

is called the matching polynomial of G.

The matching polynomial $M_G(x)$ conviniently organizes the numbers $m_n(G)$ so we can work with all of them simultaneously. For example, the degree of $M_G(x)$ is the matching number of G and the total number of matching in G is its value at 1:

$$M_G(1) = m_0 + m_1 + \dots + m_k.$$

5.10. Exercise. Assume that a graph G has exactly two connected components H_1 and H_2 . Show that

$$M_G(x) = M_{H_1}(x) \cdot M_{H_2}(x)$$

for any x.

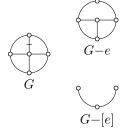
5.11. Exercise. Show that the values

$$\frac{1}{2} \cdot [M_G(1) + M_G(-1)]$$
 and $\frac{1}{2} \cdot [M_G(1) - M_G(-1)]$

equal to the number of matchings with even and odd number of edges correspondingly.

Assume e is an edge in a graph G. Recall that the graph G - e is obtained by deleting e from G. Let us denote by G - [e] the graph obtained by deleting the vertexes of e with all their edges from G; that is, if e connects two vertexes v and w, then

 $G - [e] = G - \{v, w\}.$



The following exercise is analogous to the deletion-contraction formulas 5.3 and 3.2.

- **5.12.** Exercise. Let G be a graph.
 - (a) Show that

$$M_G(x) = M_{G-e}(x) + x \cdot M_{G-[e]}(x)$$

for any edge e in G.

(b) Use part (a) to show that the matching polynomials of complete graphs satisfy the following recursive relation:

$$M_{K_{n+1}}(x) = M_{K_n}(x) + n \cdot x \cdot M_{K_{n-1}}(x).$$

(c) Use (b) to calculate $M_{K_n}(x)$ for $1 \leq n \leq 6$.

Spanning-tree polynomial

Consider a connected graph G with p vertexes; assume $p \ge 2$.

Let us prepare independent variables x_1, \ldots, x_p , one for each vertex of G. For each spanning tree T in G consider the monomial

$$x_1^{d_1-1}\cdots x_p^{d_p-1},$$

where d_i denotes the degree of the *i*-th vertex in T.

The tree T has p-1 edges and therefore $d_1 + \cdots + d_p = 2 \cdot (p-1)$. It follows that the total degree of the monomial is p-2.

The sum of these monomials is a polynomial of degree p-2 of p variables x_1, \ldots, x_p . This polynomial will be called the *spanning-tree* polynomial of G and it will be denoted by $S_G(x_1, \ldots, x_p)$.

For example, the graph G shown on the diagram has three spanning trees; each is obtained by deleting one of the edges in the cycle xyz. Abusing notation slightly, let us use the same label for a vertex in G and for the corresponding variable. The monomial for the tree obtained by deleting the edge xy is $x^2 \cdot y \cdot z$. Indeed, in this tree the vertex x has degree 3, vertexes



y and z have degree 2 and the remaining vertexes u, v, and w have degree 1. The monomials for the other two trees are $x^2 \cdot z^2$ and $x^3 \cdot z$. Therefore

$$S_G(x, y, z, u, v, w) = x^2 \cdot y \cdot z + x^2 \cdot z^2 + x^3 \cdot z.$$

Note that $S_G(x, y, z, u, v, w)$ does not depend on u, v and w since the corresponding vertexes have degree 1 in any spanning tree of G.

Note that $s(G) = S_G(1, ..., 1)$ for any graph G; that is, $S_G(1, ..., 1)$ is the total number of spanning trees in G. Indeed each spanning tree in G contributes one monomial to S_G and each monomial contributes 1 to the value $S_G(1, ..., 1)$. The following exercise shows that, the polynomial S_G keeps a lot more information about spanning trees in G.

- **5.13. Exercise.** Let $S_G(x_1, ..., x_p)$ be the spanning-tree polynomial of a graph G. Show the following:
 - (a) $S_G(0,1,\ldots,1)$ is the number of spanning trees with a leaf at the first vertex.
 - (b) The coefficient of S_G in front of $x_1 \cdots x_{p-2}$ equals to the number of paths of length p-1 connecting (p-1)-th and p-th vertexes.
 - (c) The partial derivative

$$\frac{\partial S_G}{\partial x_1}(0,1,\ldots,1)$$

is the numbers of spanning trees in G with degree 2 at the first vertex.

(d) The two values

$$\frac{1}{2}$$
 $\cdot [S_G(1,1,\ldots,1) \pm S_G(-1,1,\ldots,1)]$

are the numbers of spanning trees in G with odd or correspondingly even degree at the first vertex.

5.14. Theorem.

$$S_{K_p}(x_1,\ldots,x_p) = (x_1 + \cdots + x_p)^{p-2},$$

where K_p is the complete graph with $p \ge 2$ vertexes.

Note that the theorem above generalizes the Cayley formula (4.10). Indeed,

$$s(K_p) = S_{K_p}(1, \dots, 1) = (1 + \dots + 1)^{p-2} = p^{p-2}.$$

In the proof we will use the following algebraic lemma; its proof is left to the reader.

5.15. Lemma. Assume a polynomial $P(x_1, ..., x_n)$ vanish if $x_n = 0$, then P is divisible by x_n ; that is, there is a polynomial $Q(x_1, ..., x_n)$ such that

$$P(x_1,\ldots,x_n)=x_n\cdot Q(x_1,\ldots,x_n).$$

Proof of 5.14. Let us apply induction on p; the base case p=2 is evident.

Assume that the statement holds for p-1; that is,

$$S_{K_{p-1}}(x_1,\ldots,x_{p-1})-(x_1+\cdots+x_{p-1})^{p-3}=0.$$

We need to show that

$$S_{K_p}(x_1,\ldots,x_{p-1},x_p)-(x_1+\cdots+x_{p-1}+x_p)^{p-2}=0.$$

First let us show that the equality holds if $x_p = 0$; that is,

$$S_{K_n}(x_1,\ldots,x_{p-1},0)-(x_1+\cdots+x_{p-1})^{p-2}=0.$$

Indeed, $S_{K_p}(x_1, \ldots, x_{p-1}, 0)$ is the sum of all monomials in S_{K_p} without x_p . Each of these monomials correspond to a spanning tree T in K_p with $d_p = 1$; in other words, T has a leaf at x_p .¹ Note that the tree T is obtained from another tree T' with the vertexes x_1, \ldots, x_{p-1} by adding an edge from x_p to x_i for some i < p.

¹We use x_i as a label of a vertex in K_p and as the corresponding variable.

Note that the monomial in S_{K_p} that corresponds to T equals to the product of x_i times the monomial in $S_{K_{p-1}}$ that corresponds to T'. To get the sum of all monomials in S_{K_p} without x_p , we need to sum up these products for all i < p and all the monomials in $S_{K_{p-1}}$; this way we get

$$S_{K_{p-1}}(x_1,\ldots,x_{p-1})\cdot(x_1+\cdots+x_{p-1}).$$

By **3**, the latter equals to

$$(x_1 + \dots + x_{p-1})^{p-2}$$

which implies **6**.

Now assume \bullet does not hold. Denote by $P(x_1, \ldots, x_p)$ the left hand side in \bullet . Observe that P is a *symmetric homogeneous* polynomial of degree p-2. That is, any permutation of values x_1, \ldots, x_p does not change $P(x_1, \ldots, x_p)$ and each monomial in P has total degree p-2.

By Lemma 5.15 and \bullet , we get that P is divisible by x_p . Since P is symmetric it is divisible by each x_i ; that is, there is a polynomial $Q(x_1, \ldots, x_p)$ such that

$$P(x_1,\ldots,x_p)=x_1\cdots x_p\cdot Q(x_1,\ldots,x_p).$$

Since $P \neq 0$, the total degree of P is at least p. But P has degree p-2, a contraction.

5.16. Exercise. Show that the number of spanning trees in K_p with degree k at the first vertex equals to $\binom{p-2}{k-1} \cdot (p-1)^{p-k-1}$.

Hint: Calculate $N = \frac{\partial^{k-1}}{\partial x_1^{k-1}} S_{K_n}(0,1,\ldots,1)$ using the expression for S_{K_n} given in Theorem 5.14 and determine how much a tree with degree d at the first vertex contributes to the value N.

5.17. Exercise. Assume that the vertexes of the left part of $K_{m,n}$ have corresponding variables x_1, \ldots, x_m and the vertexes in the right part have corresponding variables y_1, \ldots, y_n . Show that

$$S_{K_{m,n}}(x_1,\ldots,x_m,y_1,\ldots,y_n)=(x_1+\cdots+x_m)^{n-1}\cdot(y_1+\cdots+y_n)^{m-1}.$$

Conclude that $s(K_{m,n}) = m^{n-1} \cdot n^{m-1}$.

Hint: Modify the proof of Theorem 5.14.

Remarks

Very good expository papers on chromatic polynomials are written by Ronald Read [31] and Alexandr Evnin [10]. Matching polynomials are discussed in a paper of Christopher Godsil and Ivan Gutman [13].

Our discussion of spanning-tree polynomials is based on a modification of Fedor Petrov [29] of the original proof of Arthur Cayley [4].

Generating functions discussed in Chapter 6 give connections between graph theory and power series; this subject is more challenging, but worth to learn.

Chapter 6

Generating functions

For this chapter, the reader has to be familiar with power series.

Exponential generating functions

The power series

$$A(x) = a_0 + a_1 \cdot x + \frac{1}{2} \cdot a_2 \cdot x^2 + \dots + \frac{1}{n!} \cdot a_n \cdot x^n + \dots$$

is called the exponential generating function of the sequence a_0, a_1, \ldots

If the series A(x) converges in some neighborhood of zero, then it defines a function which remembers all information of the sequence a_n . The latter follows since

$$A^{(n)}(0) = a_n;$$

that is, the *n*-th derivative of A(x) at 0 equals to a_n .

However, without assuming the convergence, we can treat A(x) as a formal power series. We are about to describe how to add, multiply, take the derivative, and do other operations with formal power series.

Sum and product. Consider two exponential generating functions

$$A(x) = a_0 + a_1 \cdot x + \frac{1}{2} \cdot a_2 \cdot x^2 + \frac{1}{6} \cdot a_3 \cdot x^3 + \dots$$

$$B(x) = b_0 + b_1 \cdot x + \frac{1}{2} \cdot b_2 \cdot x^2 + \frac{1}{6} \cdot b_3 \cdot x^3 + \dots$$

We will write

$$S(x) = A(x) + B(x), \quad P(x) = A(x) \cdot B(x)$$

if the power series S(x) and P(x) are obtained from A(x) and B(x) by opening the parentheses of these formulas and combining like terms.

It is straightforward to check that S(x) is the exponential generating function for the sequence

$$s_0 = a_0 + b_0,$$

 $s_1 = a_1 + b_1,$
...
 $s_n = a_n + b_n.$

The product P(x) is also exponential generating function for the sequence

$$p_{0} = a_{0} \cdot b_{0},$$

$$p_{1} = a_{0} \cdot b_{1} + a_{1} \cdot b_{0},$$

$$p_{2} = a_{0} \cdot b_{2} + 2 \cdot a_{1} \cdot b_{1} + a_{2} \cdot b_{0},$$

$$p_{3} = a_{0} \cdot b_{3} + 3 \cdot a_{1} \cdot b_{2} + 3 \cdot a_{2} \cdot b_{1} + a_{3} \cdot b_{0},$$

$$\dots$$

$$p_{n} = \sum_{i=0}^{n} \binom{n}{i} \cdot a_{i} \cdot b_{n-i}.$$

6.1. Exercise. Assume A(x) is the exponential generating function of the sequence a_0, a_1, \ldots Show that $B(x) = x \cdot A(x)$ corresponds to the sequence $b_n = n \cdot a_{n-1}$.

Composition. Once we define addition and multiplication of power series we can also plug in one power series in another. For example, if $a_0 = 0$, then the expression

$$E(x) = e^{A(x)}$$

is another power series which is obtained by plugging A(x) instead of x in the power series of exponent:

$$e^x = 1 + x + \frac{1}{2} \cdot x^2 + \frac{1}{6} \cdot x^3 + \dots$$

It is harder to express the sequence (e_n) corresponding to E(x) in terms of a_n , but it is easy to find the first few terms. Since we assume $a_0 = 0$, we have

$$e_0 = 1,$$

 $e_1 = a_1,$
 $e_2 = a_2 + 2 \cdot a_1^2,$
 $e_3 = a_3 + 6 \cdot a_1 \cdot a_2,$

Derivative. The derivative of

$$A(x) = a_0 + a_1 \cdot x + \frac{1}{2} \cdot a_2 \cdot x^2 + \dots + \frac{1}{n!} \cdot a_n \cdot x^n + \dots$$

is defined as the following formal power series

$$A'(x) = a_1 + a_2 \cdot x + \frac{1}{2} \cdot a_3 \cdot x^2 + \dots + \frac{1}{n!} \cdot a_{n+1} \cdot x^n + \dots$$

Note that A'(x) coincides with the ordinary derivative of A(x) if the latter converges.

Note that A'(x) is the exponential generating function of the sequence

$$a_1, a_2, a_3, \dots$$

obtained from the original sequence

$$a_0, a_1, a_2, \dots$$

by deleting the zero-term and shifting the indexes by 1.

6.2. Exercise. Let A(x) be the exponential generating function of the sequence $a_0, a_1, a_2 \ldots$ Describe the sequence b_n with the exponential generating function

$$B(x) = x \cdot A'(x)$$
.

Calculus. If A(x) converges and

$$E(x) = e^{A(x)},$$

then we have

$$ln E(x) = A(x).$$

Also by taking the derivative of $E(x) = e^{A(x)}$ we get that

$$E'(x) = e^{A(x)} \cdot A'(x) =$$

= $E(x) \cdot A'(x)$.

These identities have a perfect meaning in terms of formal power series and they still hold without assuming the convergence. We will not prove it formally, but this is not hard.

Fibonacci numbers

Recall that Fibonacci numbers f_n are defined using the recursive identity $f_{n+1} = f_n + f_{n-1}$ with $f_0 = 0$, $f_1 = 1$.

6.3. Exercise. Let F(x) be the exponential generating function of Fibonacci numbers f_n .

(a) Show that it satisfies the following differential equation

$$F''(x) = F(x) + F'(x).$$

(b) Conclude that

$$F(x) = \frac{1}{\sqrt{5}} \cdot \left(e^{\frac{1+\sqrt{5}}{2} \cdot x} - e^{\frac{1-\sqrt{5}}{2} \cdot x} \right).$$

(c) Use the identity **1** to derive

$$f_n = \frac{1}{\sqrt{5}} \cdot \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1+\sqrt{5}}{2} \right)^n \right).$$

(This is the so called Binet's formula.)

Exponential formula

Fix a set of graphs S. Denote by $c_n = c_n(S)$ the number of spanning subgraphs of K_n isomorphic to one of the graphs in S. Let

$$C(x) = C_{\mathcal{S}}(x)$$

be the exponential generating function of the sequence c_n .

- **6.4.** Theorem. Let S be a set of connected graphs.
 - (a) Fix a positive integer k and denote by w_n the number of spanning subgraphs of K_n which have exactly k connected components, and each connected component is isomorphic to one of the graphs in S. Then

$$W_k(x) = \frac{1}{k!} \cdot C_{\mathcal{S}}(x)^k$$

where $W_k(x)$ is the exponential generating function of the sequence w_{-}

(b) Denote by a_n the number of all spanning subgraphs of K_n such that each connected component of it is from the class S. Let A(x) be the exponential generating function of the sequence (a_n) . Then

$$1 + A(x) = e^{C_{\mathcal{S}}(x)}.$$

Taking the logarithm and derivative of the formula in (b), we get the following:

6.5. Corollary. Assume A(x) and C(x) as in Theorem 6.4(b). Then

$$ln[1 + A(x)] = C(x)$$
 and $A'(x) = [1 + A(x)] \cdot C'(x)$.

The second formula in this corollary provides a recursive formula for the corresponding sequences which will be important later.

Proof; (a). Denote by v_n the number of spanning subgraphs of K_n which have k ordered connected components and each connected component is isomorphic to one of the graphs in S. Let $V_k(x)$ be the corresponding generating function.

Note that for each graph described above there are k! ways to order its k components. Therefore $w_n = \frac{1}{k!} \cdot v_n$ for any n and

$$W_k(x) = \frac{1}{k!} \cdot V_k(x).$$

Hence it is sufficient to show that

$$V_k(x) = C(x)^k.$$

To prove the latter identity, we apply induction on k and the multiplication formula $\mathbf{0}$ for exponential generating functions. The base case k=1 is evident.

Assuming that the identity 3 holds for k; we need to show that

$$V_{k+1} = V_k(x) \cdot C(x).$$

Assume that a spanning graph with k+1 ordered connected components of K_n is given. Denote by m the number of vertexes in the first k components. There are $\binom{n}{m}$ ways to choose these m vertexes among n vertexes of K_n . For each choice, we have v_m ways to choose a spanning subgraph with k components in it. The last component has m-n vertexes and we have c_{n-m} ways to choose a subgraph from \mathcal{S} . All together, we get

$$\binom{n}{m} \cdot v_m \cdot c_n$$

spanning graphs with k+1 ordered connected components and m vertexes in the first k components. Summing it up for all m, we get the multiplication formula 2 for exponential generating functions; hence 4 follows.

(b). Note that a_n is the sum of numbers of spanning graphs in K_n with $1, 2, \ldots$ components. That is,

$$A(x) = W_1(x) + W_2(x) + \dots =$$

applying part (a), we can continue

$$= C(x) + \frac{1}{2} \cdot C(x)^2 + \frac{1}{6} \cdot C(x)^3 + \dots =$$

since $e^x = 1 + x + \frac{1}{2} \cdot x^2 + \frac{1}{6} \cdot x^3 + \dots$, we can rewrite it as

$$=e^{C(x)}-1.$$

Perfect matchings

Recall that a *perfect matching* is a 1-factor of the graph. In other words, it is a set of isolated edges which covers all the vertexes. Note that if a graph admits a perfect matching, then the number of its vertexes is even.

The double factorial is defined as the product of all the integers from 1 up to some non-negative integer n that have the same parity (odd or even) as n; the double factorial of n is denoted by n!!. For example,

$$9!! = 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1 = 945$$
 and $10!! = 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2 = 3840$.

- **6.6. Exercise.** Let a_n denote the number of perfect matchings in K_n . Show that
 - (a) $a_2 = 1$;
 - (b) $a_n = 0$ for any odd n;
 - (c) $a_{n+1} = n \cdot a_{n-1}$ for any integer $n \ge 2$.
 - (d) Conclude that $a_n = 0$ and $a_{n+1} = n!!$ for any oodd n.

Now we give a more complicated proof of Exercise 6.6(d).

6.7. Problem. Use Theorem 6.4 to show that the number of perfect matching in $K_{2\cdot n}$ is $(2\cdot n-1)!!$.

Solution. Denote by a_n the number of perfect matchings in K_n and let A(x) be the corresponding exponential generating function.

Note that a perfect matching can be defined as a spanning subgraph such that each connected component is isomorphic to K_2 . So we can apply the formula in Theorem 6.4 for the set S consisting of only one graph K_2 .

Note that if K_n contains a spanning subgraph isomorphic to K_2 , then n=2. It follows that $c_2(\mathcal{S})=1$ and $c_n(\mathcal{S})=0$ for $n\neq 2$. Therefore

$$C(x) = C_{\mathcal{S}}(x) = \frac{1}{2} \cdot x^2.$$

By Theorem 6.4(b),

$$1 + A(x) = e^{C(x)} =$$

$$= e^{\frac{1}{2} \cdot x^{2}} =$$

$$= 1 + \frac{1}{2} \cdot x^{2} + \frac{1}{2 \cdot 4} \cdot x^{4} + \frac{1}{6 \cdot 8} \cdot x^{6} + \dots + \frac{1}{n! \cdot 2^{n}} \cdot x^{2 \cdot n} + \dots$$

That is,

$$\frac{1}{(2 \cdot n - 1)!} \cdot a_{2 \cdot n - 1} = 0$$
 and $\frac{1}{(2 \cdot n)!} \cdot a_{2 \cdot n} = \frac{1}{n! \cdot 2^n}$

for any positive integer n. In particular,

$$a_{2 \cdot n} = \frac{(2 \cdot n)!}{n! \cdot 2^n} =$$

$$= \frac{1 \cdot 2 \cdots (2 \cdot n)}{2 \cdot 4 \cdots (2 \cdot n)} =$$

$$= 1 \cdot 3 \cdots (2 \cdot n - 1) =$$

$$= (2 \cdot n - 1)!!$$

That is, $a_n = 0$ for any odd n and $a_n = (n-1)!!$ for even n.

Remark. Note that by Corollary 6.5, we also have

$$A'(x) = [1 + A(x)] \cdot x,$$

which is equivalent to the recursive identity

$$a_{n+1} = n \cdot a_{n-1}$$

in Exercise 6.6(c).

All matchings

Now let S be the set of two graphs K_1 and K_2 . Evidently $c_1(S) = c_2(S) = 1$. Further, we have that $c_n(S) = 0$ for all $n \ge 3$ since K_n contains no spanning subgraphs isomorphic to K_1 or K_2 .

Therefore the exponential generating function of the sequence $c_n(S)$ is a polynomial of degree 2

$$C(x) = x + \frac{1}{2} \cdot x^2.$$

Note that a matching in a graph G can be identified with a spanning subgraph with all connected components isomorphic to K_1 or K_2 . If we denote by a_n the number of all matchings and by A(x) the corresponding exponential generating function, then by Theorem 6.4(b), we get that

$$A(x) = e^{x + \frac{1}{2} \cdot x^2} - 1.$$

Applying Corollary 6.5, we also have

$$A'(x) = [1 + A(x)] \cdot (1 + x).$$

The latter is equivalent to the following recursive formula for a_n :

$$a_{n+1} = a_n + n \cdot a_{n-1}.$$

Since $a_1 = 1$ and $a_2 = 2$, we can easily find first the few terms of this sequence:

$$1, 2, 4, 10, 26, \dots$$

6.8. Exercise. Prove formula \bullet directly — without using generating functions. Compare to Exercise 5.12(b).

Two-factors

Let S be the set of all cycles.

Note that a 2-factor of a graph can be defined as a spanning subgraph with components isomorphic to cycles. Denote by a_n and c_n the number of 2-factors and spanning cycles¹ in K_n respectively. Let A(x) and C(x) be the corresponding exponential generating functions.

6.9. Exercise.

(a) Show that $c_1 = c_2 = 0$ and

$$c_n = (n-1)!/2$$

for $n \geqslant 3$. In particular,

$$c_1 = 0, c_2 = 0, c_3 = 1, c_4 = 3, c_5 = 12, c_6 = 60.$$

(b) Use part (a) and the identity

$$A'(x) = [1 + A(x)] \cdot C'(x)$$

to find a_1, \ldots, a_6 .

- (c) Count the number of 2-factors in K_1, \ldots, K_6 by hand, and compare it to the result in part (b).
- (d) Use part (a) to conclude that

$$C(x) = -\frac{1}{2} \cdot \ln(1-x) - \frac{1}{2} \cdot x - \frac{1}{4} \cdot x^2.$$

(e) Use part (d) and Theorem 6.4(b) to show that

$$A(x) = \frac{1}{e^{\frac{x}{2} + \frac{x^2}{4}} \cdot \sqrt{1 - x}} - 1.$$

¹Also known as *Hamiltonian cycles*.

Counting spanning forests

Recall that a *forest* is a graph without cycles. Assume we want to count the number of spanning forests in K_n ; denote by a_n its number and by c_n the number of connected spanning forests. That is, c_n is the number of spanning trees in K_n . For example, K_3 has the following 7 spanning forests; therefore $a_3 = 7$.

By Corollary 6.5, the following identity

$$A'(x) = [1 + A(x)] \cdot C'(x)$$

holds for the corresponding exponential generating functions. According to the Cayley theorem, $c_n = n^{n-2}$; in particular,

$$c_1 = 1, c_2 = 1, c_3 = 3, c_4 = 16, \dots$$

Applying the product formula \mathbf{Q} , we can use c_n to calculate a_n recurrently:

$$\begin{aligned} a_1 &= c_1 = 1, \\ a_2 &= c_2 + a_1 \cdot c_1 = \\ &= 1 + 1 \cdot 1 = 2, \\ a_3 &= c_3 + 2 \cdot a_1 \cdot c_2 + a_2 \cdot c_1 = \\ &= 3 + 2 \cdot 1 \cdot 1 + 2 \cdot 1 = 7, \\ a_4 &= c_4 + 3 \cdot a_1 \cdot c_3 + 3 \cdot a_2 \cdot c_2 + a_3 \cdot c_1 = \\ &= 16 + 3 \cdot 1 \cdot 3 + 3 \cdot 2 \cdot 1 + 7 \cdot 1 = 38 \end{aligned}$$

It is instructive to check by hand that there are exactly 38 spanning forests in K_4 .

For the general term of a_n , no simple formula is known, however the recursive formula above provides a sufficiently fast way to calculate its terms.

Counting connected subgraphs

Let a_n be the number of all spanning subgraphs of K_n and c_n be the number of connected spanning subgraphs of K_n . All 4 connected spanning subgraphs of K_3 are shown on the diagram; therefore $c_3 = 4$.



Assume A(x) and C(x) are the corresponding exponential generating functions. These two series diverge for all $x \neq 0$; nevertheless, the formula for formal power series in Theorem 6.4(b) still holds, and by Corollary 6.5 we can write

$$A'(x) = [1 + A(x)] \cdot C'(x).$$

Note that $a_n = 2^{\binom{n}{2}}$; indeed, to describe a subgraph of K_n we can choose any subset of $\binom{n}{2}$ edges of K_n , and a_n is the total number of $\binom{n}{2}$ these independent choices. In particular, the first few terms of a_n are

$$a_1 = 1$$
, $a_2 = 2$, $a_3 = 8$, $a_4 = 64$, ...

Applying the product formula \mathbf{Q} , we can calculate the first few terms of c_n :

$$c_1 = a_1 = 1$$

$$c_2 = a_2 - a_1 \cdot c_1 =$$

$$= 2 - 1 \cdot 1 = 1,$$

$$c_3 = a_3 - 2 \cdot a_1 \cdot c_2 - a_2 \cdot c_1 =$$

$$= 8 - 2 \cdot 1 \cdot 1 - 2 \cdot 1 = 4,$$

$$c_4 = a_4 - 3 \cdot a_1 \cdot c_3 - 3 \cdot a_2 \cdot c_2 - 1 \cdot a_3 \cdot c_1 =$$

$$= 64 - 3 \cdot 1 \cdot 4 - 3 \cdot 2 \cdot 1 - 1 \cdot 8 \cdot 1 = 38,$$
...

Note that in the previous section we found a_n from c_n , and now we go in the opposite direction. No closed formula is known for c_n , but the recursive formula gives a sufficiently good way to calculate it.

Remarks

The method of generating functions was introduced and widely used by Leonard Euler; the term *generating function* was coined later by Pierre Laplace. For more on the subject, we recommend a classic book of Frank Harary and Edgar Palmer [17].

Let us mention another application of exponential generating functions.

Assume r_n denotes the number of rooted spanning trees in K_n . (A tree with one marked vertex is called a *rooted tree* and the marked

vertex is called its root). Then it is not hard to see that the exponential generating function of r_n satisfies the following identity

$$R(x) = x \cdot e^{R(x)}.$$

By the Lagrange inversion theorem, formula $\mathbf{6}$ implies that $r_n = n^{n-1}$. Since in any spanning tree of K_n we have n choices for the root, we have that

$$r_n = n \cdot s(K_n).$$

This way we get another proof of the Cayley formula (4.10)

$$s(K_n) = n^{n-2}.$$

Chapter 7

Minimum spanning trees

Optimization problems

We say that a graph G has weighted edges if each edge in G is labeled by a real number, called its weight. In this case the weight of a subgraph H (briefly weight(H)) is defined as the sum of the weights of its edges.

An optimization problem typically asks to minimize weight of a subgraph of a certain type. A classic example is the so called *traveling* salesman problem. It asks to find a Hamiltonian cycle with minimal weight (if it exists in the graph).

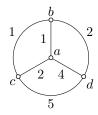
No fast algorithm is known to solve this problem and it is expected that no fast algorithm exists. We say *fast* if the required time depends polynomially on the number of the vertexes in the graph. The best known algorithms require exponential time, which is just a bit better than a brute force checking of all possible Hamiltonian cycles.

In fact the traveling salesman problem is a classic example of the so called NP-hard problems; it means that if you find a fast algorithm to solve it, then you solve the P=NP problem — the most important question in modern mathematics.

The so called *nearest neighbor algorithm* gives a heuristic way to find a Hamiltonian cycle with small (but not necessary smallest) weight. In the following description, we assume that the graph is complete.

- (i) Start at an arbitrary vertex.
- (ii) Find the edge with minimal weight connecting the current vertex v and an unvisited vertex w and move to w.
- (iii) Repeat step (ii) until all vertexes are visited and then return to the original vertex. This way we walked along a Hamiltonian cycle.

The nearest neighbor algorithm is an example of greedy algorithms; in other words, it chooses the cheapest step at each stage. The cheap choices at the beginning might force it to make expansive choices at the end, so it may not find an optimal solution. In fact a greedy algorithm might give the worse solution. For example, if we start from the vertex a in the shown weighted graph, then nearest neighbor al-



gorithm produce the cycle abcd which is the worse — this cycle has total weight 11 = 1 + 1 + 5 + 4 and the other two Hamiltonian cycles have weights 9 and 10.

Let us list a few other classic examples of optimization problems. Unlike the traveling salesman problem, there are fast algorithms which solve them.

- The shortest path problem asks for a path with minimal weight, connecting two given vertexes. It can be thought as optimal driving directions between two locations.
- ♦ The assignment problem asks for a matching of a given size with minimal weight, in a bipartite graph. It can be used to minimize the total cost of work, assuming that available workers charge different prices for each task. This problem is closely related to the subject of the next chapter.
- ⋄ The minimum spanning tree problem asks for a spanning tree with minimal weight. For example, we may think of minimizing the cost to build a computer network between given locations. This problem will be the main subject of the remaining sections of this chapter.

Neighbors of a spanning tree

Let G be a connected graph. Suppose T and T' are spanning trees in G. If

$$T' = T - e + e'$$

for some edges e and e', then we say that T' is a neighbor of T.

- **7.1. Exercise.** Let G be a connected graph that is not a tree. Suppose that any two distinct spanning trees in G are neighbors. Show that G has exactly one cycle.
- **7.2. Theorem.** Let G be a graph with weighted edges. A spanning tree T in G has minimal weight if and only if

$$\operatorname{weight}(T) \leqslant \operatorname{weight}(T')$$

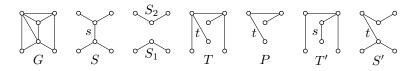
for any neighbor T' of T.

7.3. Exchange lemma. Let S and T be spanning trees in a graph G. Then for any edge s in S that is not in T there is an edge t in T, but not in S such that the subgraphs

$$S' = S - s + t$$
 and $T' = T + s - t$

are spanning trees in G.

Proof. Note that S-s has two components, denote them by S_1 and S_2 . Let P be a path in T that connects the vertexes of s. Note that the path P starts in S_1 and ends in S_2 . Therefore one of the edges of P, say t, connects S_1 to S_2 .



The subgraph T+s has a cycle created by P and s. Removing the edge t from this cycle leaves a connected subgraph T'=T-t+s. Evidently T' and T have the same number of vertexes and edges; since T is a tree, so is T'.

Further $S' = S - s + t = S_1 + S_2 + t$. Since t connects two connected subgraphs S_1 and S_2 , the resulting subgraph S' is connected. Again S' and S have the same number of vertexes and edges; since S is a tree, so is S'.

Proof of 7.2. The "only-if" part is trivial — if T has minimal weight among all spanning trees, then, in particular, it has to be minimal among its neighbors. It remains to show the "if" part.

Suppose that a spanning tree T has the minimal weight among its neighbors. Consider a minimum weight spanning tree S in G; if there are more than one, then we assume in addition that S shares with T the maximal number of edges.

Arguing by contradiction, assume T is not a minimum weight spanning tree; in this case $T \neq S$. Then there is an edge s in S, but not in T. Let t be an edge in T provided by the exchange lemma; in particular, S' = S - s + t and T' = T + s - t are spanning trees.

Since T' is a neighbor of T, we have

$$\operatorname{weight}(T) \leq \operatorname{weight}(T') =$$

$$= \operatorname{weight}(T) + \operatorname{weight}(s) - \operatorname{weight}(t).$$

Therefore weight(s) \geqslant weight(t) and

$$\operatorname{weight}(S') = \operatorname{weight}(S) - \operatorname{weight}(s) + \operatorname{weight}(t) \leq$$

 $\leq \operatorname{weight}(S).$

Since S has minimal weight, we have an equality in the last inequality. That is, S' is another minimum weight spanning tree in G. By construction, S' has one more common edge with T; the latter contradicts the choice of S.

7.4. Exercise. Let G be a graph with weighted edges and T a minimal weight spanning tree in G.

Suppose that for each edge e, we change its weight from w to 2^w . Show that T remains a minimal weight spanning tree in G with the new weights.

Kruskal's algorithm

Kruskal's algorithm finds a minimum weight spanning tree in a given connected graph. If the graph is not connected, the algorithm finds a spanning tree in each component of the graph. Let us describe it informally.

- (i) Suppose G is a graph with weighted edges. Start with the spanning forest F in G formed by all vertexes and no edges.
- (ii) Remove from G an edge with minimal weight. If this edge connects different trees in F, then add it to F.
- (iii) Repeat the procedure while G has any remaining edges.

Note that Kruskal's algorithm is greedy — it chooses the cheapest step at each stage.

7.5. Theorem. Kruskal's algorithm finds a minimum weight spanning tree in any connected pseudograph with weighted edges.

Proof. Evidently the the subgraph F remains to be a forest at all stages. If at the end, F has more than one component, then the original graph G could not have edges connecting these components; otherwise the algorithm would choose it at some stage. Therefore the resulting graph is a tree.

Suppose the spanning tree T produced by Kruskal's algorithm does not have minimal total weight. Then T does not satisfy Theorem 7.2; that is, there is a neighbor T' = T + e' - e of T such that

$$\operatorname{weight}(T) > \operatorname{weight}(T') =$$

$$= \operatorname{weight}(T) + \operatorname{weight}(e') - \operatorname{weight}(e).$$

Therefore weight(e') < weight(e).

Denote by F the forest at the stage right before adding e to it. Note that F is a subgraph of T-e. Since e' connects different trees in T-e, it connects different trees in F. But weight(e') > weight(e'), therefore the algorithm should reject e at this stage — a contradiction.

Other algorithms

Let us describe two more algorithms which find a minimal weight spanning tree in a given connected graph G. For simplicity, we assume that G has distinct weights.

Prim's algorithm.

- (i) Start with a tree T formed by a single vertex v in G.
- (ii) Choose an edge e with the minimal weight that connects T to a vertex not yet in the tree. Add e to T.
- (iii) Repeat step (ii) until all vertexes are in the tree T.

Borůvka's algorithm.

- (i) Start with the forest F formed by all vertexes of G and no edges.
- (ii) For each component T of F, find an edge e with minimal weight that connects T to another component of F; add e to F.
- (iii) Repeat step (ii) until F has one component.
- **7.6. Exercise.** Show that (a) Prim's algorithm and (b) Borůvka's algorithm produce a minimum weight spanning tree of a given connected graph with all different weights.

Hint: Mimic the proof for Kruskal's algorithm.

- **7.7. Exercise.** Compare Kruskal's algorithm, Prim's algorithm, and Borůvka's algorithm. Which one should work faster? When and why?
- **7.8. Exercise.** Assume G is a connected graph with weighted edges and all weights are different. Let T be a minimum weight spanning tree in G.
 - (a) Suppose v is an end vertex in G and e is its adjacent edge. Show that e belongs to T.
 - (b) Suppose C is a cycle in T and f is an edge in C with maximal weight. Show that f does not belong to T.
 - (c) Find a minimum-spanning-tree algorithm that is based on observations (a) and (b).

Remarks

The NP-hardness of the traveling salesman problem follows from a result of Richard Karp [23]. The discussed algorithms are named after Joseph Kruskal, Robert Prim and Otakar Borůvka. Joseph Kruskal also described the so called reverse-delete algorithm [26] which is closely relevant to a solution of Exercise 7.8. Prim's algorithm was first discovered by Vojtěch Jarník, rediscovered independently two times: by Robert Prim and by Edsger Dijkstra.

For more on optimization problems read a book by Dieter Jungnickel [21]. A reader friendly introduction to P=NP problem is given in a book by Thomas Cormen, Charles Leiserson, Ronald Rivest, and Slifford Stein [5, Chapter 34].

Chapter 8

Marriage theorem and its relatives

Alternating and augmenting paths

Recall that a matching in a graph is a set of edges without common vertexes.

Let M be a matching in a graph G. A path P in G is called M-alternating if the edges in P alternate between edges from M and edges not from M.

If an alternating path connects two unmatched vertexes of G, then it is called M-augmenting. An M-augmenting path P can be used to improve the matching M; namely by deleting all the edges of P in M and adding the remaining edges of P, we obtain a new matching M' with more edges. This construction implies the following:

8.1. Observation. Assume G is a graph and M is a maximal matching in G. Then G has no M-augmenting paths.

On the diagrams we denote the edges in M by solid lines and the remaining edges by dashed lines.

8.2. Exercise. Find an augmenting path for the matching on the diagram and use it to construct a larger matching.



Recall that bigraph stands for bipartite graph.

8.3. Exercise. Let M be a matching in a bigraph G. Show that any M-augmenting path connects vertexes from the opposite parts of G.

The following theorem states that the converse to the observation.

8.4. Theorem. Assume M is a matching in a graph G. If M is not maximal, then G contains an M-augmenting path.

This theorem implies that Hungarian algorithm [18, Section 7.2] produces a maximal matching. Indeed, the Hungarian algorithm checks all M-alternating paths in the graph and it will find an M-augmenting path if it exists. Once it is found it could be used to improve the matching M. Therefore, by the theorem, Hungarian algorithm improves any nonmaximal matching; in other words, it can stop only at a maximal matching.

Proof. Suppose M is nonmaximal matching; that is, there is a matching M' with lager number of edges. Consider the subgraph H of G formed by all edges in M and M'.

Note that each component of H is either an M-alternating path or a cycle. Each cycle in H has the same number of edges from M and M'. Since |M'| > |M|, the one component of H has to be a path that starts and ends by an edge in M'. Evidently such path is M-augmenting. \square

Marriage theorem

Assume that G is a graph and S is a set of its vertexes. We say that a matching M of G covers S if any vertex in S is incident to an edge in M.

Given a set of vertexes W in a graph G, the set W' of all vertexes adjacent to at least one of vertexes in W will be called the *set of neighbors* of W. Note that if G is a bigraph and W lies in the left part, then W' lies in the right part.

8.5. Marriage theorem. Let G be a bigraph with the left and right parts L and R. Then G has a matching which covers L if and only if for any subset $W \subset L$ the set $W' \subset R$ of all neighbors of W contains at least as many vertexes as W; that is,

$$|W'|\geqslant |W|.$$

Proof. Assume that a matching M is covering L. Note that for any set $W \subset L$, the set W' of its neighbors includes the vertexes matched with W. In particular,

$$|W'| \geqslant |W|;$$

it proves the "only if" part.

Consider a maximal matching M of G. To prove the "if" part, it is sufficient to show that M covers L. Arguing by contradiction, assume that there is a vertex w in L which is not incident to any edge in M.

Consider the maximal set S of vertexes in G which are reachable from w by M-alternating paths. Denote by W and W' the set of left and right vertexes in S correspondingly.

Since S is maximal, W' is the set of neighbors of W. According to Observation 8.1, the matching M provides a bijection between W-w and W'. In particular,

$$|W| = |W'| + 1;$$

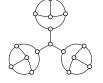
the latter contradicts the assumption.

- **8.6.** Exercise. Assume G is an r-regular bigraph; $r \ge 1$. Show that
 - (a) G admits a 1-factor;
 - (b) the edge chromatic number of G is r; in other words, G can be decomposed into 1-factors.

Remark. If $r = 2^n$ for an integer $n \ge 1$, then G in the exercise above has an Euler's circuit. Note that the total number of edges in G is even, so we can delete all odd edges from the circuit. The obtained graph G' is regular with degree 2^{n-1} . Repeating the described procedure recursively n times, we will end up at a 1-factor of G.

There is a tricky way to make this idea work for arbitrary r, not necessarily a power of 2; it was discovered by Noga Alon [see 2 and also 22].

In Exercise 8.6 one has to assume that G is bipartite. Indeed, the following exercise states that the graph on the diagram has no 1-factor, while it is 3-regular.



- **8.7. Exercise.** Prove that the graph shown on the diagram has no 1-factor.
- **8.8. Exercise.** Children from 25 countries, 10 kids from each, decided to stand in a rectangular formation with 25 rows of 10 children in each row. Show that you can always choose one child from each row so that all 25 of them will be from different countries.
- **8.9. Exercise.** The sons of the king divided the kingdom between each other into 23 parts of equal area one for each son. Later a new son was born. The king proposed a new subdivision into 24 equal parts and gave one of the parts to the newborn son.

Show that each of 23 older sons can choose a part of land in the new subdivision which overlaps with his old part.

8.10. Exercise. A table $n \times n$ is filled with nonnegative numbers. Assume that the sum in each column and each row is 1. Show that one can

choose n cells with positive numbers which do not share columns and rows.

8.11. Advanced exercise. In a group of people, for some fixed s and any k, any k girls like at least k-s boys in total. Show that then all but s girls may get married to the boys they like.

Vertex covers

A set S of vertexes in a graph is called a *vertex cover* if any edge is incident to at least one of the vertexes in S.

8.12. Theorem. In any bigraph, the number of edges in a maximal matching equals the number of vertexes in a minimal vertex cover.

It is instructive to do the following exercise before reading the proof.

8.13. Exercise. Let M be a maximal matching in a bigraph G. Assume two unmatched vertexes l and r lie on the opposite parts of G. Show that no pair of M-alternating paths starting from l and r can have a common vertex.

On the following diagram, a maximal matching is marked by solid lines; the remaining edges of the graph are marked by dashed lines. The vertexes of the cover are marked in black and the remaining vertexes in white; the unmatched vertexes are marked by a cross.

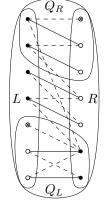
Proof. Fix a bigraph G; denote by L and R its left and right parts. Let M be a matching and S be a vertex cover in G.

By the definition of vertex cover, any edge m in M is incident to at least one vertex in S. Therefore

$$|S| \geqslant |M|$$
.

That is, the number of vertexes in any vertex cover S is at least as large as the number of edges in any matching M.

Now assume that M is a maximal matching. Let us construct a vertex cover S such that |S| = |M|.



Denote by U_L and U_R the set of left and right unmatched vertexes (these are marked by cross on the diagram). Denote by Q_L and Q_R the set of vertexes in G which can be reached by M-alternating paths starting from U_L and from U_R correspondingly.

Note that Q_L and Q_R do not overlap. Otherwise there would be an M-augmenting path from U_L to U_R (compare to Exercise 8.13). Therefore M is not maximal — a contradiction.

Further note that if m is an edge in M, then both of its end vertexes lie either in Q_L or Q_R , or neither.

Let us construct the set S by taking one incident vertex (left or right) of each edge m in M by the following rule: if m connects vertexes in Q_L , then include its right vertex in S; otherwise include its left vertex. Since S has exactly one vertex incident to each edge of M, we have

$$|S| = |M|$$
.

It remains to prove that S is a vertex cover; that is, at least one vertex of any edge e in G is in S.

Note that if the left vertex of e lies in Q_L , then e is an edge on an M-alternating path starting from U_L . Therefore the right vertex of e also lies in Q_L .

Therefore it is sufficient to consider only the following three cases:

- \diamond The edge e has its right vertex in Q_L and its left vertex outside of Q_L . In this case, both vertexes of e lie in S.
- \diamond The edge e connects vertexes in Q_L . In this case, the right vertex of e is in S.
- \diamond The edge e connects vertexes outside of Q_L . In this case, the left vertex of e is in S.

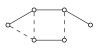
8.14. Exercise. A few squares on a chessboard are marked. Show that the minimal number of rows and columns that cover all marked squares is the same as the maximal number of rooks on the marked squares that do not threaten each other.

Edge cover

A collection of edges N in a graph is called an *edge cover* if every vertex is incident with at least one of the edges in N.

On the diagram, two edge covers of the same graph are marked in solid lines.

8.15. Theorem. Let G be a connected graph with p vertexes and p > 1. Assume that a minimal edge cover N of G contains n edges, and a maximal matching M of G contains m edges. Then



The following exercise will guide you thru the proof.

- **8.16.** Exercise. Suppose G, N, M, p, n, and m are as in the theorem. Denote by m' the the number of components in N.
 - (a) Show that N contains no paths of length 3 and no triangle. Conclude that each component of N is a star; that is, it is isomorphic to $K_{1,k}$ for some k. Use it to show that

$$m' + n = p$$
.

(b) Choose m' edges, one in each component of N. Observe that it is a matching. Conclude that $m' \leq m$. Use (a) to show that

$$m+n\geqslant p$$
.

(c) Consider a subgraph H of G formed by all edges in the matching M and one edge incident to each unmatched vertex in G. Observe that H is a vertex cover with p-m edges. Conclude that

$$m+n \leqslant p$$
.

(d) Observe that (b) and (c) imply the theorem.

Minimal cut

Recall that a *directed graph* (or briefly a *digraph*) is a graph, where the edges have a direction associated with them; that is, an edge in a digraph is defined as an *ordered* pair of vertexes.

8.17. Min-cut theorem. Let G be a digraph. Fix vertexes s and t in G. Then the maximal number of oriented paths from s to t which do not have common edges equals to the minimal number of edges one can remove from G so that there will be no oriented path from s to t.

Proof. Denote by m the maximal number of oriented paths from s to t which do not have common edges. Denote by n the minimal number of edges one can remove from G to disconnect t from s; more precisely, after removing n edges from G, there will be no oriented path from s to t.

Let P_1, \ldots, P_m be a maximal collection of oriented paths from s to t which have no common edges. Note that in order to disconnected t from s, we have to cut at least one edge in each path P_1, \ldots, P_m . In particular, $n \ge m$.

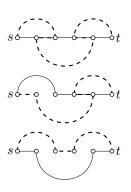
Consider the new orientation on G where each path P_i is oriented backwards — from t to s.

Consider the set S of the vertexes which are reachable from s by oriented paths for this new orientation.

Assume S contains t; that is, there is a path Q from s to t, which can move along P_i only backwards.

(Further, the path Q will be used the same way as the augmenting path in the proof of the marriage theorem. In a sequence of moves, we will improve the collection Q, P_1, \ldots, P_m so that there will be no overlaps. On the diagram, a case with m = 1 that requires two moves is shown; P_1 is marked by a solid line and Q is marked by a dashed line.)

Since P_1, \ldots, P_m is a maximal collection, Q overlaps with some of the paths P_1, \ldots, P_m . Without loss of generality, we can assume that Q first overlaps with P_1 —assume it meets P_1 at the vertex v and leaves it at the vertex w. Let us modify the paths Q and P_1 the following way: Instead of the path P_1 consider the path P_1' that goes along Q from s to v and after that goes along P_1 to t. Instead of the path Q, consider the trail Q' which goes along P_1 from s to w and after that goes along Q to t.

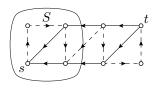


If the constructed trail Q' is not a path (that is, if Q' visits some vertexes several times), then we can discard some circuits from Q' to obtain a genuine path, which we will still denote by Q'.

Note that the obtained collection of paths $Q', P'_1, P_2 \ldots, P_m$ satisfies the same conditions as the original collection. Further, since we discard the part of P_1 from w to v, the total number of edges in $Q', P'_1, P_2 \ldots, P_m$ is smaller than in the original collection $Q, P_1, P_2 \ldots, P_m$. Therefore, by repeating the described procedure several times, we get m+1 paths without overlaps — a contradiction.

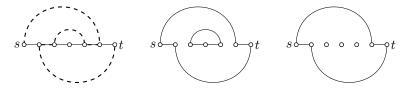
It follows that $S \not\ni t$.

Note that all edges which connect S to the remaining vertexes of G are oriented toward to S. That is, every such edge which comes out of S in the original orientation belongs to one of the paths P_1, \ldots, P_m .



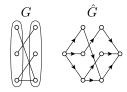
Moreover, for each path P_i there is only one such edge. In other words, if a path P_i leaves S, then it can not come back. Otherwise S could be made larger by moving backwards along P_i . Therefore cutting one such edge in each path P_1, \ldots, P_m makes it impossible to leave S. In particular, we can disconnect t from s by cutting m edges from G; that is, $n \leq m$.

Remark. The described process has the following physical interpretation. Think of each path P_1, \ldots, P_m , and Q like water pipelines from s to t. At each overlap of Q with another path P_i , the water in P_i and Q



runs the opposite directions. So we can cut the overlapping edges and reconnect the open ends of the pipes to each other while keeping the water flow from s to t unchanged. As the result, we get m+1 pipes form s to t with no common edges and possibly some cycles which we can discard. An example of this procedure for two paths P_1 and Q is shown on the diagram; as above, P_1 is marked by a solid line and Q is marked by a dashed line.

8.18. Advanced exercise. Assume G is a bigraph. Let us add two vertexes, s and t, to G so that s is connected to each vertex in the left part of G, and t is connected to each vertex in the right part of G. Orient the graph from left to right. Denote the obtained digraph by \hat{G} .



Give another proof of the marriage theorem for a bigraph G, applying the min-cut theorem to the digraph \hat{G} .

Remarks

The marriage theorem was proved by Philip Hall in [16]; it has many applications in all branches of mathematics. The theorem on vertex cover was discovered by Dénes Kőnig [25], and rediscovered by Jenő Egerváry [7]. The theorem on min-cut was proved by Peter Elias, Amiel Feinstein, and Claude Shannon [8], it was rediscovered by Lester Ford and Delbert Fulkerson [11].

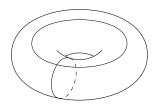
These theorems form a base for the Hungarian algorithm that solves the *assignment problem*; see page 55. This algorithm was found by Harold Kuhn, but (as it was discovered by François Ollivier) essentially the same algorithm was found much earlier by Carl Gustav Jacobi [28].

An extensive overview of the marriage theorem and its relatives is given by Alexandr Evnin in [9].

Chapter 9

Toroidal graphs

Recall that a graph is called *planar* if it can be drawn on a plane with no crossings; the latter is equivalent to the existence of its drawing on a sphere. Here and further, we say *drawing* for *drawing with no crossings*.



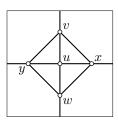
In this chapter we will discuss drawing of graphs on a torus — another surface that can be obtained by revolving a circle about an axis (this is the surface of a donut).

A graph is called *toroidal*, if it can be drawn on a torus with no crossing.

If one cuts a torus along a parallel and a meridian as shown on the diagram, then the

obtained surface can be developed into a square.¹ It gives a convenient way to describe drawings of graphs on the torus which will be called square diagram. One only has to remember that the corresponding points on opposite sides of the square are identified in the torus; in particular, all four vertexes of the square correspond to one point in the torus.

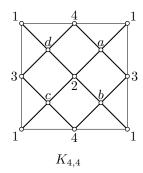
For example, on the given square diagram you see a drawing of the complete graph K_5 ; in particular it shows that K_5 is toroidal. Note that the edge xy, after coming to the right side of the square, reappears at the corresponding point of the left side and goes further to y. Similarly, the edge vw comes to the top and reappears at the bottom. At these points, the edges cross the par-

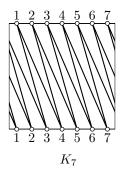


¹Formally speaking it means that there is a continuous map $f: \Box \to T$ from a square to torus such that f(x) = f(y) if and only if x = y or x and y are corresponding points on the opposite sides of the square.

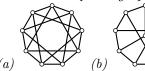
allel and the meridian.

The following square diagrams show that $K_{4,4}$ and K_7 are toroidal graphs as well.





- **9.1.** Exercise. Show that any graph with crossing number 1 is toroidal. That is, if a graph admits a drawing on a sphere with one crossing, then it admits a drawing on a torus with no crossings.
- **9.2. Exercise.** Show that each of the following graphs is toroidal; construct a corresponding square diagram in each case.

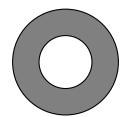




Simple regions

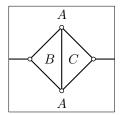
Choose a drawing G of a pseudograph on a torus or sphere; it subdivides the the surface into regions. A region R is called *simple* if its interior can be parameterized by an open plane disc.²

The annulus shown on the diagram is an example of a nonsimple region. On a sphere it may appear only in drawings of nonconnected graphs. That is, if a drawing of a pseudograph G on a sphere has a nonsimple region, then G is not connected. This statement should be intuitively obvious, but the proof is not trivial.

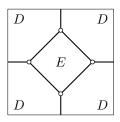


Drawings of connected graphs on a torus may have nonsimple regions. For example, in the first drawing of K_4 below, the regions B and C are simple and the region A

 $^{^2}$ Formally speaking it means that the there is a continuous bijection from interior of R to an open disc in the plane such that its inverse is also continuous.



p = 4, q = 6, r = 3; the regions B, C are simple, and A is not.



p = 4, q = 6, r = 2; both regions D and E are simple.

is not (it contains the meridian of the torus). The second drawing of K_4 has only two regions, D and E, and both of them are simple.

Further we will use the following claim which should be intuitively obvious. We do not present its proof, but it is not hard; a reader familiar with topology may consider it as an exercise.

9.3. Claim. Let G be a drawing of a pseudograph on a torus or sphere and R its simple region. Suppose that a drawing G' obtained from G by adding a new edge e in the region R.



- (a) If both of vertexes of e are in G (the edge e might be a loop), then e divides R into two simple regions.
- (b) If only one of vertexes of e are in G, then e does not divide R and the corresponding region of G' is simple.



9.4. Advanced exercise. Suppose that G is a drawing of a connected nonlanar graph on the torus. Show that each region of G is simple.

Euler's formula

9.5. Theorem. For any drawing G of a connected pseudograph on the torus we have

$$p - q + r \geqslant 0$$
,

where p, q, and r denote the number of vertexes, edges and regions in G respectively.

Moreover, equality holds if all regions of G are simple.

Note that the inequality might be strict. For example, for the first drawing of K_4 above we have

$$p-q+r=4-6+3=1>0.$$

It happens since the region A is not simple. For the second drawing of K_4 we have the equality

$$p - q + r = 4 - 6 + 2 = 0,$$

as it is supposed to be by the second part of the theorem.

In the proof we will use the following lemma which is a simple corollary of Claim 9.3. Given a drawing of pseudograph G with p vertexes, q edges and r regions, set

$$\Sigma_G = p - q + r.$$

9.6. Lemma. Let G be a drawing of a pseudograph on a torus or sphere. Suppose that another drawing G' of a connected pseudograph that is obtained from G by adding a new edge e. Then

$$\Sigma_{G'} \leqslant \Sigma_G.$$

Moreover, if all regions of G are simple, then

- (a) we have equality in **1** and
- (b) all regions of G' are simple as well.

Proof. Since G' is connected, one of the ends of e belongs to G. Denote by R the region of G that contains e.

If the other end of e is not in G, then e does not divide its region. In this case G' has an extra vertex and an extra edge, and the number of regions did not change; that is, p' = p + 1, q' = q + 1, and r' = r. Therefore

$$\Sigma_{G'} = p' - q' + r' = p - q + r = \Sigma_G.$$

If the other end of e is in G, then e may divide R in two regions or may not divide it. According to Claim 9.3, the latter may happen only if R is not simple. Note that after adding e, the number of vertexes did not change. That is, p' = p, q' = q + 1, $r' \leq r + 1$ and the equality holds if R is simple. Therefore

$$\Sigma_{G'} = p' - q' + r' \leqslant p - q + r = \Sigma_G$$

and the equality holds if R is simple.

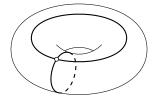
According to Claim 9.3, if R is simple, then the region(s) of G' that correspond to R are simple as well. The rest of the regions did not change. Hence (b) follows.

Proof of 9.5. We need to show that

$$\Sigma_G \leqslant 0$$

for any drawing of connected pseudograph G on the torus T.

Let H be a drawing of the pseudograph formed by one meridian and one parallel as shown. Note that it has 1 vertex, 2 edges and 1 region; therefore



$$\Sigma_H = 1 - 2 + 1 = 0.$$

Without loss of generality, we may assume that the edges of G and H intersect and have only a finite number of points of intersection. The latter can be achieved by perturbing the drawing of G.

Let us subdivide the graphs G and H by adding a new vertex at every crossing point of the graphs. The obtained graphs, say \bar{G} and \bar{H} , are subgraphs of a bigger graph, say W, formed by all edges and vertexes of \bar{G} and \bar{H} . Note that adding a vertex on an edge increases the number of vertexes and edges by 1 and the number of regions stays the same. Since the subdivision is obtained by adding a finite number of extra vertexes, we get that

$$\Sigma_G = \Sigma_{\bar{G}} \quad \text{and} \quad \Sigma_H = \Sigma_{\bar{H}}.$$

By construction, W is connected. If $W \neq G$, then there is an edge e of W that is not in G, but has one of its ends in G. By adding e to G and applying the procedure recursively, we will obtain W in a finite number of steps. Applying Lemma 9.6 at each step, we get

$$\Sigma_W \leqslant \Sigma_{\bar{G}}.$$

Analogously, W can be obtained from H in a finite number of steps by adding one edge at a time. Since the only region of H is simple, the obtained drawings will have only simple regions. Therefore, applying Lemma 9.6 at each step, we get

$$\Sigma_W = \Sigma_{\bar{H}}.$$

Finally **3**, **4**, **5**, and **6** imply **2**; indeed

$$\Sigma_G = \Sigma_{\bar{G}} \geqslant \Sigma_W = \Sigma_{\bar{H}} = \Sigma_H = 0.$$

9.7. Exercise. Suppose that G is a toroidal graph with girth at least 4. Show that

$$q \leqslant 2 \cdot p$$
,

where p and q denote number of vertexes and edges in G.

9.8. Exercise. Is there a drawing of K_5 on torus with only square regions? If "yes", then draw a square diagram; if "no", explain why.

П

The seven color theorem

9.9. Theorem. The chromatic number of any toroidal graph can not exceed 7.

Recall that K_7 is toroidal; see the diagram on page 69. Therefore the theorem gives an optimal bound. In the proof, we will use the following lemma, which is a simple corollary of Euler's formula (9.5).

9.10. Lemma. Let G be a toroidal graph with p vertexes and q edges. Then

$$q \leqslant 3 \cdot p$$
.

Proof. Choose a drawing of G on a torus. By Euler's inequality we have

$$p - q + r \geqslant 0$$
,

where r denotes the number of regions in the drawing.

Note that each region in the drawing has at least 3 sides and each edge of G appears twice as a side of a region. Therefore

$$3 \cdot r \leqslant 2 \cdot q$$
.

These two inequalities imply the corollary.

Proof of 9.9. Suppose that there is a toroidal graph G that requires 8 colors. Choose a critical subgraph H in G with chromatic number 8. Denote by p and q the number of vertexes and edges in H.

By [18, Theorem 2.1.3], each vertex in H has degree of at least 7. By the handshake lemma [18, Theorem 1.1.1], we have

$$7 \cdot p \leq 2 \cdot a$$
.

On the other hand, by Lemma 9.10, we have

$$q \leq 3 \cdot p$$
.

These two inequalities contradict each other.

A remark about forbidden minors

It is straightforward to see that if G is toroidal, then any graph obtained from G by deletion of a vertex or an edge or by contracting an edge is also toroidal.

A graph that can be obtained from a given graph G by applying a sequence of such operations is called a *minior* of G. The pseudograph G is considered to be a minor of itself. The other minors require at least one deletion or contraction; they are called *proper minors*.

Note that the statement above implies that any minor of a toroidal graph is toroidal, or in other words, toridality is inherited to minors.

The following deep result was proved by Neil Robertson and Paul Seymour [33].

9.11. Theorem. Any property of pseudographs that is inherited to minors can be described by a finite set of forbidden minors; that is, a pseudograph meets the property if and only if none of its minors are forbidden.

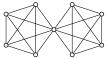
For example, note that deletion and contraction do not create cycles; that is, any minor of a forest is a forest. Forests can be described by one forbidden minor — a pseudograph formed by one loop. Indeed if a pseudograph has a cycle, then by a sequence of deletions and contractions, one can get a single loop from it.

- **9.12.** Exercise. Describe the following classes of graphs by a single forbidden minor.
 - (a) Graphs that do not contain trees with 5 end vertexes.
 - (b) Graphs in which any two cycles have at most one common vertex.

A more complicated example if given by the Pontryagin-–Kuratowski theorem which states that planar graphs are characterized by two forbidden minors: K_5 and $K_{3,3}$.

Since toroidality is inherited to its minors, it can be described by a set of forbidden minors. The complete list of the forbidden minors for this problem has to be huge and it is not yet known.

9.13. Advanced exercise. Show that the graph on the diagram is not toroidal, but every proper minor of this graph is toroidal.

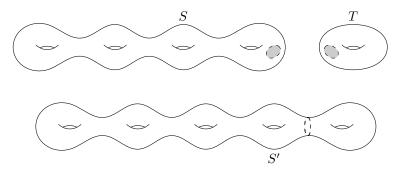


Other surfaces

One may draw graphs on other surfaces, for example, on the so called *surfaces of genus g*. These surfaces can be obtained by attaching g toruses to each other; a surface of genus g = 4 is shown on the picture.



Namely, to construct a surface S' of genus g + 1, start with a surface S of genus g and a torus T, drill a hole in each and reconnect them to each other as shown.³



It is natural to assume that the sphere has genus 0 and torus has genus 1. (In general genus tells how many disjoint closed curves one could draw on the surface so that they do not cut the surface into pieces.)

The simple regions of a drawing on a surface of genus g can be defined the same way as on the torus. Euler's formula given in Theorem 9.5 admits the following straightforward generalization

9.14. Theorem. For any drawing G of a pseudograph on the surface of genus g we have

$$p - q + r \geqslant 1 - 2 \cdot g,$$

where p, q, and r denote the number of vertexes, edges and regions of G. Moreover, equality holds if all regions of G are simple.

The seven color theorem also admits the following straightforward generalization; it was proved by Percy John Heawood [19].

9.15. Theorem. If a graph G admits a drawing on a surface of genus $g \geqslant 1$, then its chromatic number cannot exceed

$$\frac{7+\sqrt{1+48\cdot g}}{2}.$$

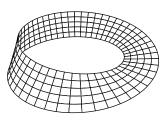
Note that for a sphere (that is, for g = 0) the formula gives 4, which is the right bound for the chromatic number for planar graphs. However this is just a coincidence, the proof of Heawood works only for $g \ge 1$.

The estimate in the last theorem is sharp. The latter was proved by constructing a drawing on the surface of genus g of the complete

³The described construction is called *connected sum*; so we can say that *connected sum of a surface of genus g and a torus is a surface of genus g* + 1.

graphs K_n for any $n \leqslant \frac{7+\sqrt{1+48\cdot g}}{2}$. The final step in this construction was made by Gerhhard Ringel and Ted Youngs [32]. The solution uses the so called *rotations of graphs*; which is a combinativic way to encode a drawing of a graph on a surface. This is the subject of [18, Chapter 10].

One may consider nonoriented surfaces; for example, the so called Möbius strip shown on the diagram. It turns out that it is possible to generalize Euler's inequality and get an exact upper bound for the chromatic numbers for graphs that can be drawn on such surfaces. (The problem is slightly harder for the so called Klein bottle, but still, its difficulty is not comparable with the four color theorem.)



9.16. Exercise. Draw the complete graph K_6 on a Möbius strip (assume it is made from a transparent material).

Chapter 10

Rado graph

In this chapter, we consider one graph with many surprising properties. Unlike most of the graphs we considered so far, this graph has an infinite set of vertexes.

Definition

Recall that a set is *countable* if it can be enumerated by natural numbers $1, 2, \ldots$; it might be infinite or finite.

A countable graph is a graph with a countable set of vertexes; the set of vertexes can be infinite or finite, but it can not be empty since we always assume that a graph has a nonempty set of vertexes.

10.1. Definition. A Rado graph is a countable graph satisfying the following property:

Given two finite disjoint sets of vertexes V and W, there exists a vertex $v \notin V \cup W$ that is adjacent to any vertex in V and nonadjacent to any vertex in W.

The property in the definition will be called the $Rado\ property$; so we can say that for the sets of vertexes V and W in a graph, the $Rado\ property\ holds$ or does not hold.

- 10.2. Exercise. Show that any Rado graph has an infinite number of vertexes.
- 10.3. Exercise. Show that any Rado graph has diameter 2.

Stability

The following exercises show that the Rado property is very stable — small changes can not destroy it.

- **10.4.** Exercise. Let R be a countable graph.
 - (a) Assume e is an edge in R. Show that R-e is a Rado graph if and only if so is R.
 - (b) Assume v is a vertex in R. Show that R v is a Rado graph if so is R.
 - (c) Assume v is a vertex in R. Consider the graph R' obtained from R by replacing each edge from v by a non-edge, and each non-edge from v by an edge (leaving the rest unchanged). Show that R' is a Rado graph if and only if so is R.
- 10.5. Exercise. Assume the set of vertexes of a Rado graph is partitioned into two subsets. Show that the subgraph induced by one of these subsets is Rado.

Hint: Let P and Q be the induced subgraphs in the Rado graph R. Assume P is not Rado; that is, there is a pair of finite vertex sets V and W in P, such that any vertex v in R that meet the Rado property for V and W does not lie in P (and therefore it lies in Q). Use the pair of sets V and W to show that Q is Rado.

10.6. Exercise. Let R be a Rado graph. Assume that Z is the set of all vertexes in R adjacent to a given vertex z. Show that the subgraph induced by Z is Rado.

Existence

10.7. Theorem. There is a Rado graph.

Proof. Let G be a finite graph. Denote by G' the graph obtained from G according to the following rule: for each subset V of vertexes in G add a vertex v and connect it to all the vertexes in V.

Note that if G has p vertexes, then G' has $p+2^p$ vertexes — it has p vertexes of G and 2^p additional vertexes — one for each of 2^p subsets of the p-element set (including the empty set).

The original graph G is an induced subgraph in G'. Note also that G' is finite — it has $p + 2^p$ vertexes.

By construction, the Rado property holds in G' for any two sets V and W of vertexes in G — the required vertex v is the vertex in G' that corresponds to the subset V.

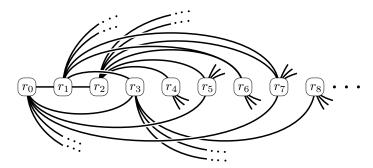


Let G_1 be a graph with one vertex. By repeating the construction, we get a sequence of graphs G_1, G_2, G_3, \ldots , such that $G_{n+1} = G'_n$ for any n. The graphs G_1, G_2, G_3 are shown on the diagram.¹

Since G_n is a subgraph of G_{n+1} for any n, we can consider the union of the graphs in the sequence (G_n) ; denote it by R. By construction, each graph G_n is a subgraph R induced by finitely many vertexes. Moreover, any vertex or edge of R belongs to any G_n with a sufficiently large n.

Note that R is Rado. Indeed, any two finite sets of vertexes V and W belong to G_n for some n. From above, the Rado property holds for V and W in G_{n+1} , and therefore in R.

Another construction. One could also construct a Rado graph by directly specifying which vertexes are adjacent. Namely, consider the graph R as on the diagram with vertexes r_0, r_1, \ldots such that r_i is adja-



cent to r_j for some i < j if the *i*-th bit of the binary representation of j is 1.

For instance, vertex r_0 is adjacent to all r_n with odd n, because the numbers whose 0-th bit is nonzero are exactly the odd numbers. Vertex r_1 is adjacent to r_0 (since 1 is odd) and to all r_n with $n \equiv 2$ or 3 (mod 4); and so on.

¹It would be hard to draw G_4 since it contains $1 + 2^1 + 2^3 + 2^{11} = 2059$ vertexes, and it is impossible to draw G_5 — it has $1 + 2^1 + 2^3 + 2^{11} + 2^{2059}$ vertexes which exceeds by many orders the number of particles in the observable universe.

10.8. Exercise. Show that the described graph is Rado.

Uniqueness

In this section we will prove that any two Rado graphs are isomorphic, so essentially there is only one Rado graph. First, let us prove a simpler statement.

10.9. Theorem. Let R be a Rado graph. Then any countable graph G (finite or infinite) is isomorphic to an induced subgraph of R.

Proof. Enumerate the vertexes of G as v_1, v_2, \ldots (the sequence might be finite or infinite).

It is sufficient to construct a sequence r_1, r_2, \ldots of vertexes in R such that r_i is adjacent to r_j if and only if v_i is adjacent to v_j . In this case, the graph G is isomorphic to the subgraph of R induced by $\{r_1, r_2 \ldots\}$.

We may choose any vertex of R as r_1 . Suppose that the sequence r_1, \ldots, r_n is constructed. If G has n vertexes, then the required sequence is already constructed. Otherwise note that the Rado property implies that there is a vertex r_{n+1} in R that is adjacent to r_i for $i \leq n$ if and only if v_{n+1} is adjacent to v_i .

Clearly, the new vertex r_{n+1} meets all the required properties. Repeating this procedure infinitely many times, or until the sequence (v_n) terminates, produces the required sequence (r_n) .

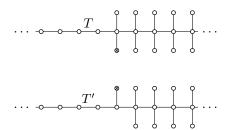
10.10. Exercise. Show that any two vertexes in a Rado graph can be connected by a path of length 10.

10.11. Theorem. Any two Rado graphs R and S are isomorphic. Moreover any isomorphism $f_0: S_0 \to R_0$ between finite induced subgraphs in R and S can be extended to an isomorphism $f: S \to R$.

Note that Theorem 10.9 implies that R is isomorphic to an induced subgraph in S and the other way around — S is isomorphic to an induced subgraph in R. For finite graphs these two properties would imply that the graphs are isomorphic; see Exercise 10.15. As the following example shows, it does not hold for infinite graphs. It is instructive to understand this example before going into the proof.

The first graph T on the diagram has an infinite number of vertexes, non of which has degree 3. The second graph T' has exactly one vertex of degree 3. Therefore these two graphs are not isomorphic.

Deleting the marked vertexes from one graph produces the other one. Therefore T is isomorphic to a subgraph of T' and the other way around.



The proof below uses the same construction as in the proof of Theorem 10.9, but it is applied back and forth to ensure that the constructed subgraphs contain all the vertexes of the original graph.

Proof. Once we have proved the second statement, the first statement will follow if you apply it to single-vertex subgraphs R_0 and S_0 .

Since the graphs are countable, we can enumerate the vertexes of R and S, as r_1, r_2, \ldots and s_1, s_2, \ldots respectively. We will construct a sequence of induced subgraphs R_n in R and S_n in S with a sequence isomorphisms $f_n \colon R_n \to S_n$.

Suppose that an isomorphism $f_n: R_n \to S_n$ is constructed.

If n is even, set m to be the smallest index such that r_m not in R_n . The Rado property guarantees that there is a vertex s_k such that for any vertex r_i in R_n , s_k is adjacent to $f_n(r_i)$ if and only if r_m is adjacent to r_i . Set R_{n+1} to be the graph induced by vertexes of R_n and r_m ; further set S_{n+1} to be the graph induced by vertexes of S_n and s_k . The isomorphism f_n can be extended to the isomorphism $f_{n+1}: R_{n+1} \to S_{n+1}$ by setting $f_{n+1}(r_m) = s_k$.

If n is odd, we do the same, but backwards. Let m be the smallest index such that s_m not in S_n . The Rado property guarantees that there is a vertex r_k which is adjacent to a vertex r_i in R_n if and only if $f_n(r_i)$ is adjacent to s_m . Set R_{n+1} to be the graph induced by vertexes of R_n and r_k ; further set S_{n+1} to be the graph induced by vertexes of S_n and s_m . The isomorphism f_n can be extended to the isomorphism $f_{n+1} \colon R_{n+1} \to S_{n+1}$ by setting $f_{n+1}(r_k) = s_m$.

Note that if $f_n(r_i) = s_j$, then $f_m(r_i) = s_j$ for all $m \ge n$. Therefore we can define $f(r_i) = s_j$ if $f_n(r_i) = s_j$ for some n.

By construction we get that

- $\Leftrightarrow f_n(r_i)$ is defined for any $n > 2 \cdot i$. Therefore f is defined at any vertex of R.
- \diamond s_j lies in the range of f_n for any $n > 2 \cdot j$. Therefore the range of f contains all the vertexes of S.
- $\diamond r_i$ is adjacent to r_j if and only if $f(r_i)$ is adjacent to $f(r_j)$. Therefore $f: R \to S$ is an isomorphism.

- **10.12.** Exercise. Explain how to modify the proof of theorem above to prove the following theorem.
- **10.13. Theorem.** Let R be a Rado graph. A countable graph G is isomorphic to a spanning subgraph of R if and only if, given any finite set V of vertexes of G, there is a vertex w that is not adjacent to any vertex in V.
- **10.14. Exercise.** Let v and w be two vertexes in a Rado graph R. Show that there is an isomorphism from R to itself that sends v to w.
- **10.15.** Exercise. Let G and H be two finite graphs. Assume G is isomorphic to a subgraph of H and the other way around -H is isomorphic to a subgraph of G. Show that G is isomorphic to H.

The random graph

The following theorem explains why a Rado graph is also named random graph.

10.16. Theorem. Assume an infinite countable graph is chosen at random by choosing an edge, independently and with probability $\frac{1}{2}$ for each pair of vertexes. Then, with probability 1, the resulting graph Rado.

Proof. It is sufficient to show that for two given finite sets of vertexes V and W, the Rado property fails with probability 0.

Assume n = |V| + |W|; that is, n is the total number of vertexes in V and W. The probability that a given vertex v outside of V and W satisfies the Rado property for V and W is $\frac{1}{2^n}$. Therefore probability that a given vertex v does not satisfy this property is $1 - \frac{1}{2^n}$.

Note that events that a given vertex does not satisfy the property are independent. Therefore the probability that N different vertexes v_1, \ldots, v_N outside of V and W do not satisfy the Rado property for V and W is

$$\left(1-\frac{1}{2^n}\right)^N$$
.

This value tends to 0 as $N \to \infty$; therefore the event that no vertex is correctly joined has probability 0.

10.17. Exercise. Let $0 < \alpha < 1$. Assume an infinite countable graph is chosen at random by selecting edges independently with probability α from the set of 2-element subsets of the vertex set. Show that with probability 1, the resulting graph is a Rado graph.

Remarks

The Rado graph is also called the Erdős–Rényi graph or random graph; it was first discovered by Wilhelm Ackermann, rediscovered later by Paul Erdős and Alfréd Rényi and yet by Richard Rado. Theorem 10.16 was discovered by Paul Erdős and Alfréd Rényi. A good survey on the subject is written by Peter Cameron [3].

Appendix A

Corrections and additions

Here we include corrections and additions to [18].

Correction to 3.2.2

The proof of Theorem 3.2.2 about decomposition of a cubic graph with a bridge into 1-factors does not explain why "each bank has an odd number of vertexes".

This is true since the 1-factor containing the bridge breaks all the vertexes of each bank into pairs except the end vertex of the bridge.

Correction to 8.4.1

There is an inaccuracy in the proof of Theorem 8.4.1 about stretchable planar graphs. Namely, in the planar drawing of G - h, the region R might be unbounded.

To fix this inaccuracy, one needs to prove a slightly stronger statement. Namely that any planar drawing of the maximal planar graph G can be stretched. That is, given a planar drawing of G, there is a stretched drawing of G and a bijection between the bounded (necessarily triangular) regions such that corresponding triangles have the same edges of G as the sides.

The remaining part of the proof works with no other changes.

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